Internal time, test clocks and singularity resolution in dust-filled quantum cosmology

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The problem of time evolution in quantum cosmology is studied in the context of a dust-filled, spatially flat Friedmann-Robertson-Walker universe. In this model, two versions of the commonly-adopted notion of internal time can be implemented in the same quantization, and are found to yield contradictory views of the same quantum state: with one choice, the big-bang singularity appears to be resolved, but with another choice it does not. This and other considerations lead to the conclusion that the notion of internal time as it is usually implemented has no satisfactory physical interpretation. A recently proposed variant of the relational-time construction, using a test clock that is regarded as internal to a specific observer, appears to provide an improved account of time evolution relative to the proper time that elapses along the observer’s worldline. This construction permits the derivation of consistent joint probability densities for observable quantities, which can be viewed either as evolving with proper time or as describing correlations in a timeless manner. Section IV reveals whether the singularity is resolved or not.

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I. INTRODUCTION

It has long been appreciated that time evolution in generally-covariant theories such as general relativity is, on the face of it, a gauge transformation and, contrary to everyday experience, should therefore be unobservable. Reviews of this ‘problem of time’ are given, for example, in [1-3], and textbook discussions may be found in [4, 5]. In recent years, detailed studies of quantized models of cosmology (see, e.g. [6-10] for reviews) have demanded a practical solution to this problem, and a certain notion of ‘internal time’ introduced by Rovelli [11-13] and further developed in, for example, [14-17] has been quite widely adopted. The ‘evolving constant of the motion’ construction advocated in these papers is a particular implementation of the idea of relational time, according to which time evolution in covariant theories can be described only relative to the values assumed by some physical quantity that is chosen to serve as a clock. In simple cosmological models, the clock is typically a scalar field, say \( \phi \), and in the quantum theory one obtains a wavefunction \( \psi(v, \phi, \tau) \), evolving with a time parameter \( \tau \) which is not a clock reading, but corresponds classically to the geometrical proper time that elapses along the observer’s worldline. This new wavefunction yields a joint probability density, evolving in textbook fashion with proper time \( \tau \), for \( v \) and \( \phi \), which are now genuine observable quantities.

In this paper, we first wish to investigate whether, despite the above-mentioned difficulties, the internal-time wavefunction \( \psi(v, \phi) \) can be interpreted as expressing a correlation between two quantities (the volume and scalar field), both of which are observable in some suitably broadened sense. Specifically, we ask whether \( |\psi(v, \phi)|^2 \) can be regarded as a conditional probability density for the volume, given that the scalar field has been determined to have the value \( \phi \). To that end, we study a simple model of an homogeneous universe filled with pressureless matter, described, following Brown and Kuchař [19], by a single scalar field. The classical version of this model is introduced in section IV and we find that two complementary notions of internal time can be straightforwardly defined, using either the volume or the scalar field as a clock. These two internal times carry over to the quantized theory, as discussed in section V, and we use them in section VI to compute two corresponding probability densities. If the conditional-probability interpretation is feasible, then these ought to represent, on

\[ \text{1 Later on, we will adjust the notation in which various wavefunctions are expressed, so as to maintain some important distinctions, which will be made precise in due course.} \]
the one hand the conditional probability density for the volume given some value of the scalar field and, on the other hand, the conditional probability for the scalar field given some value of the volume. We find, however, that these probabilities are inconsistent. In fact, the two notions of internal time lead to two mutually contradictory views of time evolution: if the volume is used as internal time, the quantum theory appears to reproduce the geometrical proper time $t$ that elapses along these worldlines. We consider the case of an homogeneous, spatially flat Friedmann-Robertson-Walker universe, with metric $g = \text{diag}(-N^2, a^2, a^2, a^2)$. To be concrete, we take spatial sections of this universe to be compact, with coordinate volume $\int d^3x = 1$. Homogeneity implies that spatial derivatives of $\phi$ and $Z^k$ vanish, and variation with respect to $W_k$ leads also to $\delta_0 Z^k = 0$. In this special case, therefore, the dust is modeled by a single scalar field $\phi$, along with the Lagrange multiplier $M$:

$$L_D = \frac{1}{2} N a^3 M [N^{-2}(\partial_k \phi)^2 - 1],$$

(2.2)

where $s$ is an arbitrary time coordinate. The momentum conjugate to $\phi$ is $p_\phi = N^{-1} a^3 M \partial_\phi \phi$, and variation with respect to $M$ yields the second-class constraint $p_\phi^2 - (a^3 M)^2 = 0$. Up to a sign, this constraint is trivially solved for $M$, and we can construct the unconstrained dust Hamiltonian

$$H_D = N p_\phi.$$  

(2.3)

Evidently, $p_\phi$ is the total energy content of the dust, and we resolve the sign ambiguity by requiring this to be non-negative.

In the standard way, the Einstein-Hilbert action leads to a gravitational Hamiltonian, which we write in terms of the volume $v = a^3$ and its conjugate momentum $p_v = -(12\pi G)^{-1} N^{-1} \partial_t v / v$ as

$$H_{\text{grav}} = -N (6\pi G) v p_v^2 =: N C_{\text{grav}},$$

(2.4)

$G$ being the usual gravitational constant. The function $C_{\text{grav}}$ defined by this equation is the gravitational contribution to the Hamiltonian constraint.

Taking the lapse function $N(s)$ to be a strictly positive, but otherwise arbitrary function, we introduce the invariant proper time

$$t(s) = \int_0^s N(s') ds'$$

(2.5)

and express the Hamilton equations of motion as

$$\dot{\phi} = N^{-1} \partial H_0 / \partial p_\phi = 1$$

(2.6)

$$\dot{v} = N^{-1} \partial H_0 / \partial p_v = -(12\pi G) v p_v$$

(2.7)

$$\dot{p}_v = -N^{-1} \partial H_0 / \partial v = (6\pi G) p_v^2,$$

(2.8)

where the overdot denotes differentiation with respect to $t$. The total Hamiltonian here is $H_0 = H_{\text{grav}} + H_D$ (distinguished by its subscript from the Hamiltonian of an extended model to be considered later) and the energy $p_\phi$ is a constant of the motion. Finally, the Hamiltonian constraint

$$C_0 := \partial H_0 / \partial N = -(6\pi G) v p_v^2 + p_\phi = 0$$

(2.9)

reproduces the Friedmann equation.
B. Dirac observables and internal time

The equations of motion (2.6)-(2.8) are easily solved. Denoting by \( \tilde{\phi}(v,p_\nu,\phi,p_\phi; t) \), etc. the phase-space trajectory that passes through \((v,p_\nu,\phi,p_\phi) \) at \( t = 0 \), we have

\[
\begin{align*}
\tilde{\phi}(v,p_\nu,\phi,p_\phi; t) &= \phi + t \\
\tilde{v}(v,p_\nu,\phi,p_\phi; t) &= v(1 - 6\pi G p_\nu t)^2 \\
\tilde{p}_v(v,p_\nu,\phi,p_\phi; t) &= p_v(1 - 6\pi G p_\nu t)^{-1}.
\end{align*}
\]

Heuristically, given initial conditions that satisfy the constraint (2.10), these solutions provide a complete description of the evolution of this simple universe relative to the proper time \( t \) elapsed along the worldline of a comoving observer. Classically, this is possible because the equations of motion were derived before imposing the constraint. In a quantum-mechanical treatment, this time evolution cannot be reproduced, because the Hamiltonian \( \hat{C}_0 \) that is supposed to generate it vanishes when acting on a physically allowed state. The same difficulty arises classically in a more formal treatment of the constrained Hamiltonian dynamics. Here, one has to recognize that, while the proper time defined in (2.5) is invariant under reparametrizations of the coordinate \( s \) with \( N'(s') = N(s)ds/ds' \) (the remnant, in this symmetry-reduced model, of the general coordinate invariance of general relativity), it is not invariant under an arbitrary change in the undetermined lapse function \( N(s) \) unless this is accompanied by a compensating reparametrization. From this perspective, a change in \( N(s) \), and hence evolution with respect to \( t \), is a gauge transformation generated by the constraint function \( \hat{C}_0 \). One has the option of fixing a gauge, by specifying once and for all a definite function \( N(s) \). Classically, at least, the content of (2.10)-(2.12) is independent of the actual function chosen, and can be regarded as physically meaningful. But in a manifestly gauge-independent approach, which we follow here, genuine physical information is carried only by gauge-invariant (Dirac) observables, which commute, in the sense of Poisson brackets, with \( \hat{C}_0 \), and are therefore constants of the motion.

A construction due to Rovelli\textsuperscript{11,13} allows us to obtain a 1-parameter family of gauge-invariant quantities—an ‘evolving constant of the motion’—as follows. Denote by \( t_\phi \), the time at which \( \tilde{\phi} \) in (2.10) has the value \( \phi \). Then the quantity

\[
V(\varphi) := \tilde{v}(t_\phi) = v[1 - 6\pi G p_\nu(\varphi - \phi)]^2,
\]

in which we suppress the dependence on the phase-space coordinates \((v,p_\nu,\phi,p_\phi) \), can be interpreted classically as ‘the volume at the time when the scalar field has the value \( \varphi \)’. It is easy to check explicitly that \( V(\varphi, \hat{C}_0) = 0 \) for each value of the parameter \( \varphi \).

A parameter such as \( \varphi \) is commonly referred to in the literature as a ‘relational’, ‘emergent’ or ‘internal’ time, or simply as ‘time’, the idea being that the scalar field serves as a physical clock, and the function \( V(\varphi) \) describes the evolution of the volume with respect to the readings of this clock. In the present case, this may seem especially apt, in view of the linear dependence (2.10) of the scalar field on \( t \). In \[19\], indeed, this idea is enshrined in the notation: these authors use \( T \) for the scalar field that we denote by \( \phi \). For reasons that will become clear later, however, we wish to maintain a clear distinction between the geometrical proper time defined by (2.7) and a scalar field \( \phi \) whose equation of motion \textit{happens to have the solution} (2.10).

We now wish to define a second family of Dirac observables, taking the volume, rather than the scalar field, as an internal time. This entails solving (2.11) for the time \( t_\varphi \) at which the volume has the value \( \nu \), and requires a little care with regard to signs. First, we take the volume always to be positive, in contrast to the loop-quantum-gravity-inspired treatment described in \[21,22\], where the physical volume is the absolute value of a more fundamental variable, whose sign reflects the orientation of a co-triad. Next, it follows from the equation of motion (2.8) that the evolution preserves the sign of \( p_\nu \), so the quantity \( \sigma \) that we provisionally identify as \( \sigma = -\text{sgn}(p_\nu) \), is a constant of the motion. Inspection of the solutions (2.11) and (2.12) then reveals that trajectories on which \( p_\nu \) is negative correspond to an expanding universe, starting from an initial singularity at \( t = -(6\pi G p_\nu)^{-1} \), while those on which \( p_\nu \) is positive correspond to a contracting universe and terminate at a final singularity at \( t = (6\pi G p_\nu)^{-1} \). Consequently, we can take the square root of (2.11), with \( \tilde{v} = \nu \), to obtain

\[
\nu^{1/2} = v^{1/2} - 6\pi G v^{1/2} p_\nu t_\nu,
\]

the sign of the square root being determined unambiguously by the requirement that \( \nu \) is an increasing function of \( t_\nu \) when \( p_\nu \) is negative. Thus, the Dirac observable that represents ‘the value of the scalar field when the volume is \( \nu \)’ is

\[
\Phi(\nu) := \tilde{\phi}(t_\nu) = \phi + (6\pi G p_\nu)^{-1} - (6\pi G v^{1/2} p_\nu)^{-1} \nu^{1/2}.
\]

Again, one may check that \( \{\Phi(\nu), C_0\} = 0 \) for every value of \( \nu \).

Finally, with a view to quantization, we define the Dirac observables

\[
\begin{align*}
V &:= V(0) = v + 12\pi G v p_\nu \phi + (6\pi G)^2 v p_\nu^2 \phi^2 \\
Y &:= -(12\pi G)^{-1} V'(0) = v p_\nu + 6\pi G v p_\nu^2 \phi \\
\Phi &:= \Phi(0) = \phi + (6\pi G p_\nu)^{-1}
\end{align*}
\]

and we note that \( C_{\text{grav}} \), defined in (2.14), is a constant of the motion, and hence also a Dirac observable. We have the Poisson-bracket relations

\[
\begin{align*}
\{V,Y\} &= V, \quad \{V, C_{\text{grav}}\} = -12\pi G Y, \\
\{Y, C_{\text{grav}}\} &= C_{\text{grav}}, \quad \{\Phi, C_{\text{grav}}\} = -1
\end{align*}
\]

and the evolving observables can be expressed as

\[
\begin{align*}
V(\varphi) &= V - 12\pi G Y \varphi - 6\pi G C_{\text{grav}} \varphi^2 \\
\Phi(\nu) &= \Phi + \sigma(-6\pi G C_{\text{grav}})^{-1/2} \nu^{1/2},
\end{align*}
\]
where we now identify
\[ \sigma := -\text{sgn}(v^{1/2}p_v). \] (2.22)

The factor \( v^{1/2} \) here, which follows from (2.15) appears inessential, but it will prove convenient to retain it.

We now have two notions of evolution with respect to internal time, and it will be useful to identify the generators of these evolutions. Indeed, we easily discover from (2.19)-(2.21) that
\[ \frac{dV(\varphi)}{d\varphi} = \{V(\varphi), H_\varphi\} \] (2.23)
\[ \frac{d\Phi(\nu)}{dv^{1/2}} = \{\Phi(\nu), H_\nu\} \] (2.24)

with
\[ H_\varphi := C_{\text{grav}} \] (2.25)
\[ H_\nu := \sigma(-C_{\text{grav}}/6\pi G)^{1/2}. \] (2.26)

On any classical trajectory, \( \Phi(\nu) \) is just the inverse of \( V(\varphi) \), but we shall see that this inverse relationship is not preserved in the quantum theory.

### III. QUANTUM DUST-FILLED COSMOLOGY

We consider a quantization scheme of the Wheeler-de Witt type in which, in the first instance, the canonical coordinates \( (v, p_v, \phi, p_\phi) \) are promoted to operators acting in an auxiliary (or kinematical) vector space. A convenient representation is that in which the operators \( \hat{v} \) and \( \hat{p}_\phi \) act by multiplication on wavefunctions \( \Psi(v, \epsilon_D) \), while their conjugate variables act by differentiation:
\[ \hat{v} \Psi(v, \epsilon_D) = v \Psi(v, \epsilon_D), \]
\[ \hat{p}_\phi \Psi(v, \epsilon_D) = \epsilon_D \Psi(v, \epsilon_D) \] (3.1)
\[ \hat{v} \hat{p}_\phi \Psi(v, \epsilon_D) = -i\hbar \hat{\partial}_v \Psi(v, \epsilon_D) \]
\[ \hat{p}_\phi \hat{v} \Psi(v, \epsilon_D) = -i\hbar \hat{\partial}_\phi \Psi(v, \epsilon_D) \]

The notation \( \epsilon_D \) reflects the fact that \( \hat{p}_\phi \) corresponds to the energy content of the dust. As in [18], we follow the authors of [21, 22] in choosing for the gravitational constraint the operator ordering
\[ \hat{C}_{\text{grav}} = -6\pi G\hat{p}_v \hat{v} \hat{p}_v. \] (3.2)

Then the Hamiltonian constraint equation (2.9) reads
\[ [(4/\lambda^2)\partial_v v \partial_v + \epsilon_D] \Psi(v, \epsilon_D) = 0, \] (3.3)

where
\[ \lambda := \left( \frac{2}{3\pi G\hbar^2} \right)^{1/2}. \] (3.4)

With the definition
\[ z := \lambda \epsilon_D^{1/2} v^{1/2}, \] (3.5)

the general solution to this equation can be expressed as
\[ \Psi(v, \epsilon_D) = \psi_+(\epsilon_D)H_+(z) + \psi_-(-\epsilon_D)H_-(z), \] (3.6)

where, in terms of the usual Hankel functions, we write
\[ H_+(z) = -iH_0^{(2)}(z) \text{ and } H_-(z) = iH_0^{(1)}(z). \]

The classical phase space consists of two disjoint regions, distinguished by the discrete variable (2.22), containing expanding (\( \sigma = 1 \)) and contracting (\( \sigma = -1 \)) trajectories. (We ignore, for now, the hyperplane \( p_v = 0 \), on which the volume is constant, but see the comment following (4.14).) The corresponding operator is
\[ \hat{\sigma} = \text{sgn}\left( \frac{i\hbar \lambda \epsilon_D^{1/2}}{2} \partial_z \right), \] (3.7)

and we see from the integral representation
\[ H_\pm(z) = \frac{2}{\pi} \int_0^\infty d\xi \exp(\mp iz \cosh \xi) \] (3.8)

that \( H_+(z) \) is a linear superposition of eigenfunctions of \( i\partial_z \) with positive eigenvalues, while \( H_-(z) \) is a superposition of eigenfunctions with negative eigenvalues. Moreover, since \( H_+(z) = H_-(-z) \), these two components of \( \Psi \) are orthogonal if, for example, we choose the inner product
\[ (\Psi_1, \Psi_2) = i \int_0^\infty d\epsilon_D \int_0^\infty d\nu \overline{\Psi}_1(v, \epsilon_D) \overline{\partial_v} \Psi_2(v, \epsilon_D). \] (3.9)

While we will not make direct use of this inner product, these considerations provide the heuristic motivation for our actual choice of the physical Hilbert space \( \mathcal{H}_{\text{phys}} \). Clearly, a solution of the constraint equation is specified by a pair of functions \( \psi_\pm(\epsilon_D) \), and we take \( \mathcal{H}_{\text{phys}} \) to be the direct sum of two copies of \( L^2(\mathbb{R}_+, d\epsilon_D) \), with the inner product
\[ (\psi_1, \psi_2)_{\text{phys}} = \int_0^\infty d\epsilon_D \left[ |\psi_+(\epsilon_D)|^2 + |\psi_-(-\epsilon_D)|^2 \right]. \] (3.10)

Then the operator \( \hat{\sigma} \), which is a constant of the motion, has a well-defined action in \( \mathcal{H}_{\text{phys}} \), namely
\[ \hat{\sigma} \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) = \left( \begin{array}{c} \psi_+ \\ -\psi_- \end{array} \right). \] (3.11)

With the exception of \( H_\nu \), defined in (2.20), the observables we need to consider act independently in the two subspaces, and we will use the shorthand \( \hat{O} \psi(\epsilon_D) \) to represent an action of the form
\[ \hat{O} \left( \begin{array}{c} \psi_+(\epsilon_D) \\ \psi_-(-\epsilon_D) \end{array} \right) = \left( \begin{array}{cc} \hat{O} \psi_+ \psi(\epsilon_D) \\ \hat{O} \psi_-(-\epsilon_D) \end{array} \right). \] (3.12)

The operators \( \hat{v}, \hat{p}_v \) and \( \hat{\phi} \) defined in (3.1) do not have any well-defined action in \( \mathcal{H}_{\text{phys}} \); that is, they do not act on a linear combination of \( H_+(z) \) and \( H_-(z) \) with
z-independent coefficients to produce another such function. As one might expect, however, we can convert the classical Dirac observables $V, Y$ and (with slightly more difficulty) $\Phi$ given in (2.18), (2.19) into operators that do act in $\mathcal{H}_{\text{phys}}$. For $\hat{V}$ and $\hat{Y}$ we choose the operator orderings

$$\hat{V} = \hat{v} + 6\pi G(\hat{p}_v + \hat{p}_v)\hat{\phi} - 6\pi G\hat{C}_{\text{grav}}\hat{\phi}^2$$

(3.13)

$$\hat{Y} := \frac{i}{2}(\hat{v}\hat{p}_v + \hat{p}_v\hat{\phi}) - \hat{C}_{\text{grav}}\hat{\phi},$$

(3.14)

with $\hat{C}_{\text{grav}}$ given by (3.2). These three operators have the commutation relations $[\hat{A}, \hat{B}] = i\hbar\{\hat{A}, \hat{B}\}$, with the Poisson brackets shown in (2.19), and it follows that the Heisenberg equation of motion (2.23) is promoted to

$$\frac{i\hbar}{d\tau} \frac{d}{d\tau} \hat{V}(\varphi) = [\hat{V}(\varphi), \hat{H}_\varphi]$$

(3.15)

with $\hat{H}_\varphi = \hat{C}_{\text{grav}}$ and $\hat{V}(\varphi)$ the operator version of (2.20). With the use of Bessel’s equation $(z\partial_z z\partial_z + z^2)H_z(z) = 0$, it is straightforward to find the action of these operators in $\mathcal{H}_{\text{phys}}$:

$$\hat{V}(\psi(\epsilon_D)) = -\frac{4}{\lambda^2} \frac{\partial}{\partial \epsilon_D} \epsilon_D \frac{\partial}{\partial \epsilon_D} \psi(\epsilon_D)$$

(3.16)

$$\hat{Y}(\psi(\epsilon_D)) = i\hbar \frac{\lambda^2}{2} \frac{\partial}{\partial \epsilon_D} \epsilon_D^{1/2} \psi(\epsilon_D)$$

(3.17)

$$\hat{H}_\varphi(\varphi) = \hat{C}_{\text{grav}} \psi(\epsilon_D) = -\epsilon_D \psi(\epsilon_D).$$

(3.18)

With the inner product (3.10), $\hat{V}$ and $\hat{Y}$ are clearly symmetric, and $\hat{H}_\varphi$, which acts by multiplication, is self-adjoint.

Construction of an operator $\hat{\Phi}$ corresponding to (2.18) is not quite straightforward, because $\hat{p}_v = -i\hbar \partial_v$ does not have a well-defined inverse. Consider, however, a wavefunction $\Psi(v, \epsilon_D) = \psi(\epsilon_D)C_0(z)$, where $C_0$ is any Bessel function of order $0$. We define an operator $\hat{p}_v^{-1}$ by writing

$$\hat{\Phi}(v, \epsilon_D) = i\hbar \frac{\partial \Psi(v, \epsilon_D)}{\partial \epsilon_D} + \frac{i\hbar}{4} \int_{v_0}^{v} \Psi(v', \epsilon_D)dv'$$

$$= i\hbar \frac{\partial \psi(\epsilon_D)}{\partial \epsilon_D} C_0(z) + \frac{i\hbar}{2\epsilon_D} \psi(\epsilon_D) \Delta(z),$$

where

$$\Delta(z) = z\partial_z C_0(z) + \int_{z_0}^{z} C_0(z')z'dz',$$

$z_0$ is a constant, and $v_0 = \frac{z_0^2}{\lambda^2 \epsilon_D}$. By virtue of Bessel’s equation, we have

$$\Delta(z) = z\partial_z C_0(z) - \int_{z_0}^{z} \partial_{z'} [z'\partial_{z'} C_0(z')] dz'$$

$$= z_0 C_0'(z_0),$$

and this vanishes, provided that we choose $z_0$ to be a zero of $zC_0'(z)$, which is at a complex infinity in the case of the Hankel functions. In this way, we obtain

$$\hat{\Phi}(\epsilon_D) = i\hbar \partial_{\epsilon_D} \psi(\epsilon_D).$$

(3.19)

This operator is symmetric under the inner product (3.10), and has the commutator $[\hat{\Phi}, \hat{C}_{\text{grav}}] = -i\hbar$, in agreement with the last Poisson-bracket relation of (2.19). Consequently, the equation of motion (2.24) becomes the operator equation

$$i\hbar \frac{d\hat{\Phi}(\nu)}{d\nu^{1/2}} = [\hat{\Phi}(\nu), \hat{H}_\nu],$$

(3.20)

where

$$\hat{H}_\nu = \hat{\delta} G_\nu, \quad \hat{G}_\nu(\psi(\epsilon_D)) = \left( \frac{2\epsilon_D}{3\pi G} \right)^{1/2} \psi(\epsilon_D)$$

(3.21)

is self-adjoint, since it acts independently by multiplication in each subspace.

**IV. INTERPRETATION OF INTERNAL TIME**

**A. Conditional and joint probabilities**

We argued that, while an operator such as $\hat{V}(\varphi)$ is a perfectly good Dirac observable, the internal time parameter $\varphi$ cannot bear the interpretation that one would like to place on it (and which indeed is placed on it, in a significant part of the literature).

At the level of the classical equations of motion (2.6)-(2.8) and their solutions (2.10)-(2.12), it seems quite feasible to say that $V(\varphi)$ is the volume ‘when’ the scalar field has the value $\varphi$, provided that one does not enquire too closely about the instant of time ‘when’ this pair of values is realized. But even classically, if one follows the systematic procedure of constructing a reduced, physical phase space, whose points are gauge orbits in the constraint manifold, one finds that this physical phase space is 2-dimensional. The corresponding configuration space is 1-dimensional: there do not exist two independent physical quantities which might simultaneously be determined to have the values $\varphi$ and $V(\varphi)$.

Quantum-mechanically, the configuration space on which states in $\mathcal{H}_{\text{phys}}$ are defined is again 1-dimensional. The parameter $\varphi$, which labels the family of Dirac observables $V(\varphi)$, cannot be construed as a value obtained by observation of a physical clock (the scalar field), because there is no operator acting in $\mathcal{H}_{\text{phys}}$, independent of $V$, to represent any such observable clock. In particular, the operator $\hat{\Phi}$ cannot serve serve this purpose, for two related reasons. First, $\hat{\Phi}$ does not commute with $V(\varphi)$ for any value of $\varphi$, so the rules of quantum mechanics do not allow simultaneous measurements of the quantities represented by these two operators. Second, the reason these operators do not commute is that they were constructed through the ‘evolving constant of the motion’ algorithm, and are thus quite different from the kinematical operators $\hat{v}$ and $\hat{\phi}$. Classically, to interpret $\varphi$ as a value of $\Phi$ is to interpret $V(\varphi)$ as “the volume at the time when the value $\varphi$ is assumed by ‘the scalar field at the time when the volume is zero’”. This is, of course,
incoherent, and is made no less so by the transition to quantum mechanics. Similar remarks apply, of course to the parameter $\nu$ that labels the family of observables $\hat{\Phi}(\nu)$: it cannot be construed as the result of a measurement of the volume. Given that these parameters cannot be construed as values obtained from measurements, it is hard to see that they have any physical meaning at all.

We now wish to strengthen this conclusion by considering the possibility that, notwithstanding the arguments just given, the internal-time formalism might be construed as yielding a joint probability distribution that represents a correlation between observable quantities, in this case the volume and the scalar field. That is to say, we will try to extend the notion of ‘observables’ by dropping the requirement that they be represented by mutually commuting operators in $\mathcal{H}_{\text{phys}}$. This is, in particular, a timeless interpretation, in which one might decide to ignore conundrums concerning the times at which specific values of the observables are realized. We will show, though, that in general no such probability distribution exists.

The idea is this. If $\hat{V}(\varphi)$ is regarded as the Heisenberg-picture operator associated with evolution in an internal time $\varphi$, generated by the Hamiltonian $\hat{H}_\varphi$, then we can construct the corresponding Schrödinger-picture wavefunction

$$\tilde{\psi}(\tilde{\nu}, \varphi) = \exp(-i\hat{H}_\varphi \tilde{\nu} / \hbar) \psi(\tilde{\nu}, 0),$$

where $\tilde{\psi}(\tilde{\nu}, 0)$ is a suitable transform of $\psi(\epsilon_1)$ on which $\hat{V}$ (that is, $\hat{V}(0)$) acts by multiplication: $\hat{V} \tilde{\psi}(\tilde{\nu}, 0) = \tilde{\nu} \tilde{\psi}(\tilde{\nu}, 0)$. According to the usual interpretation, the object

$$\tilde{\mathcal{P}}(\tilde{\nu}; \varphi) := |\tilde{\psi}_+(\tilde{\nu}, \varphi)|^2 + |\tilde{\psi}_-(\tilde{\nu}, \varphi)|^2$$

is the time-dependent probability density for obtaining the value $\tilde{\nu}$ from a measurement of the volume performed at ‘time’ $\varphi$. In particular, this probability has the time-independent normalization $\int_0^\infty \tilde{\mathcal{P}}(\tilde{\nu}; \varphi) d\tilde{\nu} = 1$. According to the foregoing discussion, the problem with this is that we have no idea what is meant by ‘performing the measurement at time $\varphi$’.

The problem might be alleviated if we could reinterpret (4.2) as a conditional probability density $\tilde{\mathcal{P}}(\tilde{\nu} | \varphi)$ for the volume, given that some other quantity (which we hope to identify with the scalar field) has been determined to have the value $\varphi$, even though that other quantity does not appear explicitly in our formalism. An interpretation of that kind requires the existence of a joint probability density $\mathcal{P}(\tilde{\nu}, \varphi)$, with the normalization

$$\int_0^\infty d\tilde{\nu} \int_{-\infty}^\infty d\varphi \mathcal{P}(\tilde{\nu}, \varphi) = 1,$$

such that

$$\tilde{\mathcal{P}}(\tilde{\nu} | \varphi) = \frac{\mathcal{P}(\tilde{\nu}, \varphi)}{\int_0^\infty \mathcal{P}(\tilde{\nu}, \varphi) d\tilde{\nu}}.$$  

If this joint probability density does exist, we can use it to find the conditional probability density

$$\mathcal{P}(\varphi | \tilde{\nu}) = \frac{\mathcal{P}(\tilde{\nu}, \varphi)}{\int_0^\infty \mathcal{P}(\tilde{\nu}, \varphi) d\tilde{\nu}}$$

for the scalar field, given that the volume has been determined to have the value $\tilde{\nu}$.

For the model under consideration, it happens that we already have such a probability density to hand, namely

$$\mathcal{P}(\varphi | \nu) := |\psi_+(\varphi, \nu)|^2 + |\psi_-(\varphi, \nu)|^2,$$

where $\psi(\varphi, \nu)$ is the wavefunction evolved in the internal-time parameter $\nu$, and expressed in the representation where $\hat{\Phi}$ acts by multiplication: $\hat{\Phi} \psi(\varphi, \nu) = \varphi \psi(\varphi, \nu)$. More precisely, if the expression on the right of (4.2) can be interpreted as a conditional probability, then it must be possible to interpret (4.6) in the same way. Up to this point, we have maintained a notational distinction between quantities that prima facie have quite different meanings, namely $\{\nu, \varphi\}$, which are internal-time parameters, and $\{\tilde{\nu}, \varphi\}$, which are configuration-space coordinates. However, if the internal-time formalism is to be interpreted as expressing an observable correlation between, in this case, the volume and the scalar field, then it must be possible to use these variables interchangeably, and we will now do so, until further notice. In particular, we would like to identify the probability density (4.6) with the one displayed in (4.3).

In the light of the arguments summarized at the beginning of this section, one might expect the conditional probability interpretation to present some difficulty. Let us, indeed, attempt to construct the required joint probability $\mathcal{P}(\nu, \varphi)$ from the conditional probabilities $\mathcal{P}(\nu | \varphi)$ and $\mathcal{P}(\varphi | \nu)$, which can be calculated once a wavefunction is specified. Denote by $f(\varphi)$ the denominator in (4.1) and by $g(\nu)$ the denominator in (4.5). Then we have

$$f(\varphi) \mathcal{P}(\nu | \varphi) = \mathcal{P}(\nu, \varphi) = g(\nu) \mathcal{P}(\varphi | \nu).$$

Up to constants $f_0 := f(0)$ and $g_0 := g(0)$, which are fixed by normalization, the unknown functions are determined by

$$f(\varphi) = \frac{g_0 \mathcal{P}(\varphi | 0)}{\mathcal{P}(0 | \varphi)},$$

$$g(\nu) = \frac{f_0 \mathcal{P}(\nu | 0)}{\mathcal{P}(0 | \nu)}.$$  

We see that the joint probability density is well defined (the first and last expressions in (4.7) are consistent) only if

$$R := \frac{f_0 \mathcal{P}(\varphi | \nu) \mathcal{P}(\nu | 0) \mathcal{P}(\varphi | 0)}{g_0 \mathcal{P}(\nu | \varphi) \mathcal{P}(\nu | 0) \mathcal{P}(\varphi | 0)} = 1.$$  

In the following, we will calculate the ratio $R$ for a specific state, and find that it is not equal to 1. This counterexample is sufficient to demonstrate that $R$ is not equal to 1 in general, but we will also argue that our example is not especially atypical.
B. Inconsistent probabilities

We begin by constructing the wavefunction \( \tilde{\psi}(\nu, \varphi) \) that appears in (4.11) and (4.12). To simplify matters, we consider a state for which \( \tilde{\psi}_- = 0 \), a condition that is preserved by evolution with respect to both \( \hat{H}_\varphi \) and \( \hat{H}_\nu \). That is, we focus on the sector that corresponds classically to an expanding universe, and we will drop the subscript + that indicates this explicitly. The representation in which the volume operator \( \hat{V} \) acts by multiplication is obtained by the Hankel transform

\[
\tilde{\psi}(\nu, \varphi) = \frac{\lambda}{2} \int_0^\infty d\epsilon_D J_0(\lambda\epsilon_D^{1/2} \nu^{1/2}) e^{i\nu\varphi/\hbar} \psi(\epsilon_D),
\]

(4.11)

where \( J_0 \) is the Bessel function of the first kind. Bessel’s equation \( z\partial_z z\partial_z + z^2 J_0(z) = 0 \) implies that the operator \( \hat{V} \) defined in (6.10) does indeed act on this function by multiplication, if \( \psi(\epsilon_D, \varphi) = e^{i\nu\varphi/\hbar} \psi(\epsilon_D) \) belongs to the domain on which \( \hat{V} \) is symmetric. Provided that \( \psi(\epsilon_D, \varphi) \) possesses an invertible Hankel transform, it is straightforward to verify that this transform preserves the normalization, \( \int_0^\infty |\tilde{\psi}(\nu, \varphi)|^2 d\nu = \int_0^\infty |\psi(\epsilon_D)|^2 d\epsilon_D \).

We consider normalized states of the form

\[
\psi(\epsilon_D) = \left[ (2\beta)^{2n+1}/(2n)! \right]^{1/2} \epsilon_D^n e^{-\beta\epsilon_D},
\]

(4.12)

where \( n \) is a positive integer and \( \beta \) a real, positive constant, which satisfies both of these requirements. Our main motivation for this choice is that the Hankel transform can be calculated analytically. In these states, the mean energy of the dust is \( \bar{\epsilon}_D = (n + \frac{1}{2}) \beta^{-1} \) and its dispersion is \( \Delta\epsilon_D = (2n + 1)^{-1/2} \epsilon_D \), so for large values of \( n \) the energy distribution becomes quite sharply peaked. However, the analytic expressions for \( \tilde{\psi}(\nu, \varphi) \) become very cumbersome for large \( n \), so we focus on the case \( n = 1 \).

In that case, we find

\[
\tilde{\mathcal{P}}(\nu|\varphi) = (\lambda^2/\beta \gamma^3)(\gamma - \bar{\nu} + \frac{1}{4} \bar{\varphi}^2)e^{-\bar{\nu}/\gamma},
\]

(4.13)

with

\[
\bar{\nu} := \lambda^2 \nu/2\beta, \quad \gamma := 1 + (\varphi/\beta \hbar)^2.
\]

It is easy to check that \( \int_0^\infty \tilde{\mathcal{P}}(\nu|\varphi) d\nu = 1 \).

Evolution in the internal time \( \nu^{1/2} \) is generated by \( \hat{H}_\nu \), whose action on the \( \sigma = +1 \) subspace, and in the \( \epsilon_D \) representation is multiplication by \( (2\epsilon_D/3\pi G)^{1/2} = \lambda \epsilon_D^{1/2} \).

The representation in which \( \hat{\Phi} \) acts by multiplication is obtained by Fourier transform:

\[
\psi(\varphi, \nu) = (2\pi\hbar)^{-1/2} \int_0^\infty d\epsilon_D e^{i\nu\varphi/\hbar} e^{-i\lambda\epsilon_D^{1/2} \nu^{1/2}} \psi(\epsilon_D).
\]

(4.14)

Again, one may easily check that \( \hat{\Phi}\tilde{\psi}(\varphi, \nu) = \psi\tilde{\psi}(\varphi, \nu) \), provided that \( \psi(\epsilon_D) = 0 \) when \( \epsilon_D = 0 \) or \( \infty \), which is the condition for the operator \( (6.19) \) to be symmetric. We note in passing that classically, when the constraint is satisfied, \( \epsilon_D = 0 \) implies \( \rho = 0 \). On this hypersurface, the volume is constant and the Dirac observable (2.15) is not well defined. The need to restrict attention here to wavefunctions that vanish at \( \epsilon_D = 0 \) is therefore not surprising.

Using the wavefunction (4.12) with \( n = 1 \), we obtain

\[
\mathcal{P}(\varphi|\nu) = \frac{2\beta^3}{\pi\hbar} \frac{|\chi(\rho)|^2}{(\beta^2 + \varphi^2/\hbar^2)^2},
\]

(4.15)

where \( \rho := \lambda^2 \nu/(\beta - i\varphi/\hbar) \) and

\[
|\chi(\rho)| := \frac{1}{\sqrt{8}} \left[ 8 - 2\rho + i\sqrt{\pi}(\rho - 6)e^{-\rho/4} \left[ 1 - \text{erf} \left( \frac{i\sqrt{\rho}}{2} \right) \right] \right].
\]

(4.16)

We do not know (and neither does any computer-algebra package available to us) how to compute the integral \( \int_{-\infty}^\infty \mathcal{P}(\varphi|\nu) d\varphi \) analytically, but numerical evaluation yields the value 1 for randomly selected values of \( \nu \).

With these probabilities in hand, we find the ratio (4.10) to be

\[
R = \frac{\gamma e^{-\bar{\nu}/(\beta \gamma)}}{|\chi| \chi(2\bar{\nu})^2 (\gamma - \bar{\nu} + \frac{1}{4} \bar{\varphi}^2)}.
\]

(4.17)

While \( R \) is equal to 1 by construction when \( \varphi = 0 \) or \( \nu = 0 \), it is not equal to 1 elsewhere. Consequently, the functions \( \widetilde{\mathcal{P}}(\nu|\varphi) \) and \( \mathcal{P}(\varphi|\nu) \) cannot consistently be interpreted as conditional probability densities arising from some underlying joint probability distribution. Certainly, this conclusion is based on a single counterexample, but it seems that our sample wavefunction has no special pathological feature, and the inconsistency is very likely to be generic. This will become a little clearer on examination of the kinds of evolution that are associated with the two internal times \( \varphi \) and \( \nu \).

C. Singularity resolution

It is of considerable interest to see exactly what is implied by the two probability distributions. To this end, we evaluate them using a wavefunction of the form (4.12) with \( n = 4 \). The resulting analytic expressions are lengthy and unilluminating, but the somewhat more sharply peaked energy distribution leads to probability densities whose nature is more readily apparent to the eye.

Figure 4 shows the probability density \( \tilde{\mathcal{P}}(\bar{\nu}; \varphi) \) for the volume, evolved with the internal-time parameter \( \varphi \). (We revert to the notation of (4.2), since the conditional-probability notation has proved to be inappropriate.) Clearly, the classical singularity at \( \nu = 0 \) is resolved, in the sense that it has been replaced by a bounce in the quantum theory. (Since the generator \( \hat{H}_\varphi \) of evolution in \( \varphi \) is independent of \( \sigma \), the probability density for the corresponding state in the classically contracting sector \( \sigma = -1 \) is exactly the same.) This agrees qualitatively
with the results of Amemiya and Koike (23), who studied a similar model, adopting the Brown-Kuchař scalar field as an internal time, though the quantization schemes they considered differ in detail from ours. In loop-quantum-gravity-inspired treatments, such as those described in [21, 22, 24], the singularity is also seen to be resolved, but the mechanism appears to be different. In particular, it is found in [21] (where the matter content is a conventional massless scalar field, which is also used as internal time) that the minimum volume at the bounce corresponds to a density \( \rho_{\text{crit}} = 3/(16\pi^2\zeta^3G^2\hbar) \), \( \zeta \) being the Barbero-Immirzi parameter, independent of the details of the quantum state. In the present treatment, by contrast, the minimum value of \( \langle V(\varphi) \rangle \) is just \( \langle V \rangle \), which is given by \( 2\beta/\lambda^2 \) in any state of the form \( |12\rangle \). (That the bounce occurs at \( \varphi = 0 \) is a consequence of choosing \( \beta \) to be real in \( |12\rangle \); it apparent from \( |11\rangle \) that an imaginary part of \( \beta \) simply shifts \( \varphi \) by a constant.) This gives a maximum density \( \rho_{\text{bounce}} = \lambda^2\zeta^2/(2n + 1) \). Physically, a universal critical density of the order of the Planck density seems more reasonable, so it is worth emphasizing that our purpose here is to explore the nature of time evolution, not to construct an optimal model of cosmology, or an optimal quantization scheme.

Figure 2 shows the probability density \( P(\hat{\varphi}; \nu) \) for the scalar field, evolved with the internal-time parameter \( \nu \). It clearly indicates a universe expanding from the initial singularity: the scalar field, which classically increases linearly with proper time, here increases with the volume. In the same sense, a state in the \( \sigma = -1 \) sector would be seen to contract towards a final singularity. (Specifically, replacing \( 1/(2D) \) with \(-1/(2D) \) in \( |14\rangle \) leads to the mirror image of figure 2.) From this perspective, the two sectors of the quantum theory reproduce the expanding and contracting regions of the classical phase space. A similar result is found in [21] for a Wheeler-de Witt quantization of the model whose matter content is a massless scalar field, and more recently in [22], where the same model is treated in a consistent-histories approach.

It is obvious from these figures that the two probability densities they depict cannot arise as conditional probabilities from an underlying joint probability density. We emphasize that they are computed from exactly the same quantum state; the difference arises solely from the two different notions of internal time used to construct the families of Dirac observables \( \hat{V}(\varphi) \) and \( \hat{\Phi}(\nu) \) and their associated Schrödinger-picture wavefunctions. It seems clear that these two notions of internal time ‘see’ the expanding and contracting sectors of the theory very differently, and that the difference is unlikely to be attributable to specific features of our chosen states.

### D. Difficulties with internal time

The probability densities shown in figures 1 and 2 underline in a rather striking way (which we did not anticipate at the outset of this investigation) the difficulties of interpretation of the internal-time formalism discussed in section IV A. It seems to us that the two notions of internal time must stand or fall together, since they are merely two different implementations of the same algorithm. That is to say, since the two families of Dirac observables are constructed in the same way, one cannot consistently maintain that \( \hat{V}(\varphi) \) represents the volume when the scalar field has the value \( \varphi \) without accepting that, by the same token, \( \hat{\Phi}(\nu) \) must represent the scalar field when the volume is \( \nu \). Since the two associated probability distributions, which arise in exactly the same quantum state are seen to be in gross conflict, we
conclude that both viewpoints cannot simultaneously be correct, and therefore that neither of them is correct. As argued above, the underlying reason for this is that neither \( \varphi \) nor \( \nu \) can be interpreted as a value assumed by any physically observable quantity.

Some might wish to claim that the conflict should be resolved in favour of \( \varphi \) as a preferred internal time, because the model can be ‘deparametrized’ in this variable: that is, the constraint \( \Theta(v,p) \) can be solved to obtain \( p_\varphi = -\Theta(v,p) \), where the function \( \Theta \) is independent of \( \varphi \), and serves as the generator of evolution in the internal time \( \varphi \). We do not think that this is a good argument in itself, but defer discussion of this point to section VI.

A pressing question raised by the above results is, of course, does the quantum theory resolve the classical singularity or not? More generally, since in this example the answer seems to depend on an arbitrary choice of the variable that is to serve as internal time, do we have a reliable way of deciding whether or not the singularity is resolved in the context of any cosmological model and quantization scheme? A possible answer is that quantum mechanics is actually ambivalent on this question. It could be, for example, that given some definite quantum state, different classes of observer will unavoidably disagree on whether that state involves a singularity or not. We proposed in [18], following earlier work in [20], a variant of the idea of relational time that seems to avoid the difficulties of interpretation we have stressed up to now, by introducing a ‘test clock’, which provides a preferred notion of time evolution from the point of view of a specific observer. As described in the following section, this preferred description, from the point of view of a comoving observer internal to the model universe, effectively coincides with that furnished by \( \varphi \) as an internal time.

V. DUST-FILLED COSMOLOGY WITH A TEST CLOCK

A. Quantization of an extended model

As explained at greater length in [18], we supplement the model studied in previous sections with a rough-and-ready description of a small clock, which we consider to be internal to a specific observer, and thus localized on that observer’s worldline. In a complete description, the coordinate functions, say \( x^\mu(s) \), that locate this worldline should appear as extra phase-space coordinates (as should a complete set of metric and matter fields, \( g_{\mu\nu}(x) \) and \( \phi_0(x) \)), but in the spirit of simplified cosmological models of the sort considered here, we assume that these degrees of freedom can be neglected. If the observer is comoving, then the proper time along the worldline is the same \( t \) as appears in [23]. From the set of variables \( q_i \) that describe the internal workings of the clock, we suppose that a function \( r(q) \) can be constructed, which constitutes the reading of the clock. Crucially, however, we take the view that this reading, being internal to the observer in question, is in principle inaccessible to that observer, and for that reason need not feature in the physical phase space that describes the universe from that observer’s point of view. Rather, it provides the context for observations of other physical quantities to be made. Solution of the equations of motion for the variables \( \bar{q}_i(q,t) \) yields the clock reading \( r(q,t) := r(\bar{q}_i(q,t)) \) at the proper time \( t \). This reading need not be linear in \( t \), but, if the clock is fit for our purpose, there must be, classically, a unique function \( t_0(q) \), such that \( r(q,t) = t_0(q) \). That is, \( t_0(q) \) is the proper time at which the clock reads 0. It is not hard to show that \( \{ t_0, h \} = -1 \), where \( Nh \) is the clock’s Hamiltonian. Because the system described by \( h \) is (ideally) localized on a single worldline, \( h \) and \( t_0 \) are independent of the cosmological variables \( (v,p) \). The total Hamiltonian constraint is

\[
C := C_0 + h = C_{\text{grav}} + p_\varphi + h
\]  
and, as shown in [18], the quantity

\[
V(\tau) := \bar{v}(v,p,\varphi,p_\varphi; t_0 + \tau)
\]
is, for each \( \tau \), a Dirac observable, \( \{ V(\tau), C \} = 0 \). So, of course, are \( \Phi(\tau) \), etc., defined in the same way. Thus, we obtain a new set of evolving constants of the motion, which can be interpreted classically as the volume, etc., when a proper time \( \tau \) has elapsed along the observer’s worldline since the clock read 0. The second crucial feature of this construction is that \( \tau \) is not to be interpreted as a reading obtained from observation of a physical clock. From the point of view of the dynamical system governed by the constraint (5.1), it is an external, ‘heraclitian’ time, in the sense of Unruh and Wald [27].

We quantize this enlarged model using essentially the same scheme as in section III, adopting a kinematical representation in which \( \bar{v}, \bar{p}_\varphi \) and \( h \) act by multiplication on wavefunctions \( \Psi(v,\epsilon_D,\epsilon_e) \), where \( \epsilon_e \) is the energy of the clock. The kinematical operator representing the fiducial proper time \( t_0 \) is then \( t_0 = -i\hbar\partial/\partial\epsilon_e \). Since we are no longer using the notion of evolution in the internal-time parameter \( \nu \), no operator of interest distinguishes the two sectors \( \sigma = \pm 1 \), and we will focus on a single sector, writing a solution to the constraint equation as

\[
\Psi(\epsilon_D,\epsilon_e) = \psi(\epsilon,\epsilon_D)C_0(\lambda \epsilon^{1/2}\nu^{1/2}),
\]
where \( \epsilon := \epsilon_D + \epsilon_e \) is the total energy, and \( C_0 \) is any Bessel function of order 0. The physical Hilbert space is now \( \mathcal{H}_{\text{phys}} = L^2(\mathbb{R}^4, d\epsilon_D d\epsilon_e) \), with the inner product

\[
(\psi_1,\psi_2)_{\text{phys}} = \int_0^\infty d\epsilon_D \int_0^\infty d\epsilon_e \bar{\psi}_1(\epsilon_D + \epsilon_e,\epsilon_D)\psi_2(\epsilon_D + \epsilon_e,\epsilon_D).
\]

\[
= \int_0^\epsilon d\epsilon \int_0^\epsilon d\epsilon_D \bar{\psi}_1(\epsilon,\epsilon_D)\psi_2(\epsilon,\epsilon_D).
\]
The intention is that $\epsilon_c$ should be small compared with $\epsilon_D$ in the same sense that the biological system responsible for an astronomer’s circadian rhythm, say, is small compared with the energy content of the visible universe, and we implement this by restricting attention to states such that $|\psi|^2$ is very small unless $\epsilon_c \ll \epsilon_D$. The resulting limitations on the resolution with which $\tau$-dependent observables might be determined are examined in some detail in [18] for a universe whose matter content is a massless scalar field, and we will not repeat the analysis for the present model.

Acting in $\mathcal{H}_{\text{phys}}$, we find families of Dirac observables

\begin{align}
\hat{V}(\tau) &= \hat{V} - 8(h\lambda)^{-2} \hat{\mathcal{Y}} \tau - 4(h\lambda)^{-2} \hat{C}_{\text{grav}}^2 \tau^2 \quad (5.5) \\
\hat{\Phi}(\tau) &= \hat{\Phi} + \tau, \quad (5.6)
\end{align}

where, acting on $|\psi(\epsilon, \epsilon_D)\rangle$,

\begin{align}
\hat{V} &= -(4/\lambda^2) \partial_\epsilon \epsilon \partial_\epsilon \\
\hat{\mathcal{Y}} &= i\hbar \epsilon^{1/2} \partial_\epsilon \epsilon^{1/2} \\
\hat{C}_{\text{grav}}^2 &= -\epsilon \\
\hat{\Phi} &= i\hbar \partial_\epsilon \epsilon_0.
\end{align}

Note, in particular, that in $[5.10]$, the derivative is with respect to $\epsilon_D$ keeping the total energy $\epsilon$ fixed. Treating $\epsilon_D$ and $\epsilon_c$ as independent variables, we have $\hat{\Phi} = \hat{\phi} + i\hat{\mathcal{Y}} = i\hbar(\partial_\epsilon \epsilon_0 - \partial_\epsilon \epsilon_c)$. Consequently, the observables $\hat{V}(\tau)$ and $\hat{\Phi}(\tau)$ now commute for every $\tau$ and, according to the usual rules of quantum mechanics, can simultaneously be assigned measured values $\nu$ and $\varphi$. A domain on which all of these operators are symmetric is provided by the boundary conditions

\begin{equation}
\psi(\epsilon, \epsilon) = \psi(\epsilon, 0) = \partial_\epsilon \psi(\epsilon, \epsilon)|_{\epsilon = \epsilon_D} = 0. \quad (5.11)
\end{equation}

Of course, $\epsilon = \epsilon_D$ means that the clock energy $\epsilon_c$ vanishes, and we always take $\psi(\epsilon, \epsilon_D)$ to vanish fast enough at large arguments to ensure square integrability.

**B. Joint probability density**

Evolution of the Heisenberg-picture operators $\hat{V}(\tau)$ and $\hat{\Phi}(\tau)$ is generated by the self-adjoint Hamiltonian $\hat{H}_\tau = \hat{C}_{\text{grav}}^2 + \hat{\mathcal{Y}} = \epsilon_D - \epsilon$, and states satisfying the boundary conditions $(5.11)$ can be expressed in a representation in which both $\hat{V}$ and $\hat{\Phi}$ act by multiplication, through combined Fourier and Hankel transformation. Thus, we can now define a *bona fide* joint probability density, which evolves in the usual way with proper time $\tau$, namely

\begin{equation}
\mathcal{P}(\nu, \varphi; \tau) = |\psi(\nu, \varphi; \tau)|^2, \quad (5.12)
\end{equation}

where

\begin{equation}
\psi(\nu, \varphi; \tau) = \frac{\lambda}{2\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \left[ \int_{\epsilon_D}^{\infty} d\epsilon_D e^{i\nu \epsilon_D/\hbar} e^{i(\epsilon - \epsilon_D)\tau/\hbar} \right] J_0(\lambda \epsilon^{1/2} \nu^{1/2}/\hbar) \psi(\epsilon, \epsilon_D). \quad (5.13)
\end{equation}

In particular, this probability density has the $\tau$-independent normalization $\int_0^\infty d\nu \int_{-\infty}^{\infty} d\varphi \mathcal{P}(\nu, \varphi; \tau) = 1$.

To make contact with the probabilities discussed in section IV, consider a state that factorizes as

\begin{equation}
\psi(\epsilon, \epsilon_D) = \psi(\epsilon)\psi_c(\epsilon_c), \quad (5.14)
\end{equation}

where $\psi_c$ is a wavefunction for the clock and, as above, $\epsilon_c = \epsilon - \epsilon_D$ is the clock’s energy. (Recall, though, that the clock is *not* represented by any operator independent of $\hat{V}$ and $\hat{\Phi}$ acting in $\mathcal{H}_{\text{phys}}$, and $\tau$ is *not* a value obtained from observation of the clock). The wavefunction $(5.13)$ becomes

\begin{equation}
\psi(\nu, \varphi; \tau) = \frac{\lambda}{2} \int_0^\infty d\epsilon_c e^{i\nu \epsilon_c/\hbar} J_0(\lambda \epsilon_c^{1/2} \nu^{1/2}/\hbar) \psi(\epsilon) \times \frac{1}{\sqrt{2\pi \hbar}} \int_0^\infty d\epsilon_c e^{i\nu (\epsilon - \epsilon_c)/\hbar} \psi_c(\epsilon_c). \quad (5.15)
\end{equation}

The idea of a test clock described previously implies that $\psi(\epsilon)$ should be peaked around a value $\epsilon$ of the order of the energy content of the visible universe, while $\psi_c(\epsilon_c)$ should be peaked around a value $\epsilon_c$ somewhat smaller than the mass of an astronomer. Under these circumstances, it is an excellent approximation to extend the upper limit of the $\epsilon_c$ integral to infinity, in which case the wavefunction $(5.15)$ and the probability density $(5.12)$ also factorize. In fact, we have

\begin{equation}
\mathcal{P}(\nu, \varphi; \tau) \approx \mathcal{P}(\nu)(\varphi; \tau), \quad (5.16)
\end{equation}

where $\mathcal{P}(\nu)(\varphi)$ coincides with the probability density defined in $[4.2]$, except that we do not now need to distinguish the sectors $\sigma = \pm 1$. Here, however, the arguments $\nu$ and $\varphi$ genuinely stand for values obtained from observation of the quantities represented by the commuting operators $\hat{V}$ and $\hat{\Phi}$, and $\mathcal{P}(\nu)(\varphi)$ is a genuine conditional probability. The function

\begin{equation}
\mathcal{P}(\varphi; \tau) := \frac{1}{2\pi \hbar} \left[ \int_0^\infty d\epsilon_c e^{i\nu (\epsilon - \epsilon_c)/\hbar} \psi_c(\epsilon_c) \right]^2, \quad (5.17)
\end{equation}

is just the probability density for obtaining the value $\varphi$ from a measurement of the scalar field at proper time $\tau$. Given a suitable choice of $\psi_c(\epsilon_c)$, it will be sharply peaked on a trajectory of the form $\varphi = \varphi_0 + \tau$, consistent with the solution $(5.6)$ of the Heisenberg equation of motion.

**C. Timeless interpretation**

We have emphasized that the parameter $\tau$, which labels the families of Dirac observables $\hat{V}(\tau)$ and $\hat{\Phi}(\tau)$ is an unobservable, external time parameter. Classically, it coincides with the arc length of an observer’s worldline, on which the fiducial event, that the unobserved clock reads 0, serves to define an origin. Naturally, the time-dependent observations recorded by an astronomer who wishes to test a theoretical expression such as $(5.13)$ will...
not refer either to \( \tau \) or to the internal, unobserved clock that we have pictured, for the sake of argument, as a biological clock. Instead, they will refer to the observed readings of some time-keeping device, which we will call (taking a cautious view of the generosity of the relevant funding agency) a ‘wristwatch’.

From an operational point of view (more or less in the sense of Bridgman\cite{28}) substantive physics is contained only in correlations between measured values of observable quantities. The idea that a solution to the problem of time in constrained systems is to be sought in such correlations is explicitly developed in, for example, \[29–31\], and is implicit in much of the recent literature. We have argued that the required correlations are not directly provided by a wavefunction such as \[14\], but in the model considered here, it is straightforward to see that this wavefunction does indirectly lead to an estimate of the desired correlation, to an approximation that might be extremely good. In principle, we would like to study correlations between observed values of the volume, the scalar field and the astronomer’s wristwatch. Since the wristwatch is another small clock, which classically follows, for practical purposes, the same worldline as the unobserved test clock, we could incorporate it into our model by adding its energy \( h_w \) to the total constraint:

\[
C_{\text{total}} := C_{\text{grav}} + p_\phi + h_w + h. \tag{5.18}
\]

This presents no technical difficulty, but since, for this particular model, \( p_\phi \) and \( h_w \) appear in the same way in \( C_{\text{total}} \), it is clear that we can usefully economize on clocks by deleting \( h_w \) and treating the scalar field as the astronomer’s timekeeping device. (Here, we profit from the enormous simplification that results from the assumption of homogeneity. In a more general setting, we envisage that Dirac observables depending on \( \tau \) can be constructed only from fields in the observer’s immediate locality. Technical difficulties aside, this is no significant limitation, since the local fields include, for example, the CMBR photons entering the astronomer’s telescope.)

Suppose, then, that simultaneous measurements of volume and scalar field are performed at a sequence of constraint surfaces. Without further ado, we expect to obtain an expression similar to \[15\], where \( \epsilon = \epsilon_w + \epsilon_c \) is the total energy of the wristwatch and the unobserved clock, while \( \gamma \) collectively denotes the remaining metric and matter variables. Operators on \( \mathcal{H}_{\text{phys}} \) representing Dirac observables include

\[
\hat{C}_{\text{grav} + \text{matt}} = -\epsilon, \tag{5.22}
\]

\[
\hat{r}_w = i\hbar \frac{\partial}{\partial \epsilon_w}, \tag{5.23}
\]

as in \[18\] and \[19\]. Taking a wave function that factorizes as \( \psi(\gamma, \epsilon, \epsilon_w) = \psi(\gamma, \epsilon)\psi_c(\epsilon_w) \), we will obtain a

\[ P(\gamma, \epsilon) \sim \tilde{P}(\gamma, \epsilon)P(\epsilon), \tag{5.20} \]

where \( P(\epsilon) = \int_{-\infty}^{\infty} P(\gamma, \epsilon, \tau)\eta(\tau)\,d\tau \) is the probability for finding the measurement of the scalar field on a randomly selected occasion to have yielded the value \( \epsilon \). For the state studied in section \[IV\], \( \tilde{P}(\gamma, \epsilon) \) is precisely the function depicted in figure \[11\] but the introduction of an unobserved test clock now allows us to interpret this function as a \textit{bona fide} conditional probability. In that indirect and approximate sense, we identify \( \epsilon \) as a preferred internal time, and conclude that the singularity is resolved in this quantum theory—or at least that the observer whose unobserved clock is represented by \( h \) will discover herself to be living in a bouncing universe.

In part, this conclusion is specific to our somewhat artificial model, in which the matter content of the universe is provided by the Brown-Kuchař scalar field. More generally, consider a Hamiltonian constraint of the form

\[
C := C_{\text{grav} + \text{matt}} + p_w + h = 0. \tag{5.21}
\]

In this expression, \( C_{\text{grav} + \text{matt}} \) is the contribution of metric and matter fields, \( h \) is the Hamiltonian of an unobserved test clock, and \( p_w \) is the momentum conjugate to \( r_w \) of a wristwatch following essentially the same worldline as the test clock. By taking the Hamiltonian of the wristwatch to be just \( p_w \), we model a time-keeping device that is manufactured so as to supply a linear measure of the proper time along its worldline.

Following the same steps as before, we expect to obtain a physical Hilbert space of functions \( \psi(\gamma, \epsilon, \epsilon_w) \), where \( \epsilon = \epsilon_w + \epsilon_c \) is the total energy of the wristwatch and the unobserved clock, while \( \gamma \) collectively denotes the remaining metric and matter variables. Operators on \( \mathcal{H}_{\text{phys}} \) representing Dirac observables include

\[
\hat{C}_{\text{grav} + \text{matt}} = -\epsilon, \tag{5.22}
\]

\[
\hat{r}_w = i\hbar \frac{\partial}{\partial \epsilon_w}, \tag{5.23}
\]

as in \[18\] and \[19\]. Taking a wave function that factorizes as \( \psi(\gamma, \epsilon, \epsilon_w) = \psi(\gamma, \epsilon)\psi_c(\epsilon_w) \), we will obtain a

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Schrödinger-picture wavefunction analogous to \( \psi(\gamma, r_w; \tau) \), of the form
\[
\psi(\gamma, r_w; \tau) = \int_0^\infty d\epsilon e^{i\epsilon r_w / \hbar} \psi(\gamma, \epsilon) \times \int_0^\epsilon d\epsilon_c e^{i\epsilon_c (\tau - r_w) / \hbar} \psi_c(\epsilon_c). \tag{5.24}
\]

From this, we obtain a joint probability \( P(\gamma, r_w; \tau) = |\psi(\gamma, r_w; \tau)|^2 \) and, if we wish, the timeless version \( P(\gamma, r_w) = \int P(\gamma, r_w; \tau) \eta(\tau) d\tau \), which describe correlations between observed values of the cosmological quantities \( \gamma \) and the wristwatch pointer \( r_w \). However, because the energy \( \epsilon = \epsilon_w + \epsilon_c \) is now the total energy of two small clocks, the wide separation of energy scales which led to the factorization in (5.16) is no longer present, and the cosmological field \( \phi \) is replaced by the wristwatch pointer \( r_w \). Generically, therefore, we do not automatically recover a conditional-probability interpretation of any internal time chosen from among the cosmological variables \( \gamma \).

VI. DISCUSSION

We argued in \([18]\) that the notion of internal time, especially as commonly implemented in models of quantum cosmology, is unsatisfactory, for two reasons. First, it offers no account of the passage of time as it is ordinarily conceived. Ordinarily, there seems to be a clear sense in which a well-constructed clock reads ‘10s’ seven seconds after it read ‘3s’, and this does not appear merely to result from a conspiracy amongst the manufacturers of time-pieces. That is, a time-keeping device can be said to work accurately (or not), because there is a time to be kept, not merely because its readings tend to agree (or not) with those of other devices of the same sort. In general relativity, this notion of time is provided, for some specific observer, by the proper time that elapses along that observer’s worldline, but it is not recovered in a treatment that describes evolution by correlating, for example, the volume of a spatial region with the value of a scalar field. Second, as summarized in section \( V \), the meaning of a parameter that serves as an ‘internal time’ is unclear; in particular, it cannot be construed as a value obtained from the observation of any physical quantity that is to be regarded as a clock, because no such quantity is represented by any operator acting in the physical Hilbert space \( \mathcal{H}_{\text{phys}} \).

The textbook interpretations of quantum mechanics are not, of course, designed to deal with dynamics generated by a Hamiltonian constraint, and some modification might well be needed to deal with that situation. In particular, one should perhaps reconsider the usual statements that observable quantities are represented by symmetric operators on \( \mathcal{H}_{\text{phys}} \), and that the values of two such observables can be determined simultaneously only if their corresponding operators commute. Concretely, for models of the kind considered here, it is certainly possible to compute a wavefunction \( \psi(\nu, \phi) \), which ostensibly evolves with respect to internal time \( \phi \), and one would like to be able to interpret it as expressing a correlation between observed values of the volume and the scalar field. In view of the \( \phi \)-independent normalization \( \int |\psi(\nu, \phi)|^2 d\nu = 1 \), the quantity \( |\psi(\nu, \phi)|^2 \) cannot be interpreted as a joint probability density, but it might be possible to regard it as a conditional probability density for the volume, given that the scalar field has the value \( \phi \). If so, then (i) there must be some underlying joint probability density, not directly supplied by the usual interpretation of the wavefunction, and (ii) there must exist a complementary conditional probability density for the scalar field, given some value for the volume.

To investigate this possibility, we have studied a simple model and quantization scheme, whose main virtue is that it admits the explicit construction within the same quantization scheme of two complementary internal-time evolutions. We found, though, that the two complementary probability distributions, computed for the same quantum state are inconsistent. In fact, these two notions of internal time lead to qualitatively different physical interpretations of the same quantum state: with the scalar field as internal time, the universe appears to bounce (figure 1), whereas with the volume as internal time it appears to expand from an initial singularity (figure 2). In view of this inconsistency, we conclude that neither of these two notions of internal time can be correct. If the operator \( \hat{V}(\phi) \) really represents the volume when the scalar field is \( \phi \), then it must also be true that \( \hat{\Phi}(\nu) \) is the scalar field when the volume is \( \nu \), and that cannot be so.

We now wish to conclude further that the notion of internal time, as implemented by the evolving constant of the motion construction, is not tenable in general. To be sure, we have discovered a serious inconsistency only in the context of one particular model and one particular quantization scheme for which the required computations are straightforward, but if the idea were sound in general, then it ought to apply to this example.

As described in section \( VI \), a variant of the relational-time construction proposed in \([13, 20]\) is capable of circumventing these difficulties. We augmented the model by including an idealized description of a small test clock, which we imagine to be internal to a specific observer, in this case a co-moving observer. The time parameter \( \tau \) that labels families of Dirac observables is, classically, at least, the proper time that elapses along the observer’s worldline. Its value is not to be regarded as obtained from observation of any physical clock and, from the point of view of the observer in question, plays the same role as the external time in textbook Newtonian or quantum mechanics. The test clock itself we take to be unobservable in principle by the observer in question, by virtue of being internal to that observer. Using this construction, we could obtain bona fide joint probabilities, which describe genuine correlations between cosmological ob-
servables and whatever time-keeping the observer might use in the course of recording observations.

In the case that the observer uses the cosmological scalar field as a (large) clock, the resulting conditional probability density for the volume essentially coincides with the evolving probability density obtained using the scalar field as an internal time. This is a special feature of the Brown-Kuchař field, whose contribution to the Hamiltonian constraint (2.24) is just its canonical momentum. For the same reason, this model is deparametrizable in the variable $\phi$, in effect, by substituting $p_\phi \rightarrow -i\hbar \partial_x$, one converts the constraint equation into a Schrödinger-like equation that governs evolution in the internal time $\varphi$. We do not think, however, that deparametrizability is in itself sufficient to identify a preferred internal time, though it does, of course, make the implementation straightforward. In general, a model is deparametrizable in a variable $\phi$ if the constraint $C(\phi, p_\phi, \omega) = 0$ can be solved to obtain $p_\phi = -\Theta(\omega)$, where $\Theta$ is independent of $\phi$, and $\omega$ denotes the remaining canonical variables. Again, one can substitute $p_\phi \rightarrow -i\hbar \partial_x$ to obtain a Schrödinger-like equation, but, again, the parameter $\varphi$ cannot be interpreted as the observed value of some physical quantity, and it is hard to see what other meaning it might have. Moreover, this strategy suffers from what Kuchař calls the ‘Hilbert space problem’. That is, the Schrödinger-like evolution can be implemented only if the inner product on $\mathcal{H}_\text{phys}$ is chosen so as to make $\Theta$ self-adjoint. The same inner product will not necessarily confer self-adjointness on the generators of evolution with respect to other candidates for an internal time, and it seems somewhat unreasonable that the inner product, and hence the quantum theory as a whole, should depend on this arbitrary choice of an internal time. In the present example, $\phi$ has a preferred status not because of deparametrizability as such, but rather because its contribution to the constraint is linear in $p_\phi$ with a constant coefficient. In fact, it is worth noting that the canonical transformation $u = v^{1/2}$ and $p_u = 2v^{1/2}p_v$ brings the constraint (2.9) to the form $C_0 = -\lambda \hbar^{-2} p_u^2 + p_\phi = 0$, so that this model is also deparametrizable in the variable $u$. (Some authors contrast a ‘relativistic’ deparametrization in $u$, meaning that $C_0$ is quadratic in $p_u$, with a ‘non-relativistic’ deparametrization in $\phi$, meaning that $C_0$ is linear in $p_\phi$.) This simple form of $C_0$, depending only on momenta, explains why it is straightforward, in this model, to obtain the two notions of evolution with respect to internal time (2.23)-(2.26), with generators that can simultaneously be made self-adjoint in the quantum theory.

We believe that the idea of a test clock, internal to some specific observer, as implemented here provides an improved notion of time evolution, but it would be at best premature to suggest that it yields a definitive solution to the problem of time in general. Among the limitations of the proposal as we have so far described it are the following. (1) While the construction appears to be successful in spatially homogeneous models with a single constraint, it does not follow that a similar construction will work in more general spacetimes. (2) In particular, we have bypassed any explicit description of the observer’s worldline by considering only a comoving observer in a Friedmann-Robertson-Walker universe. (3) The time parameter $\tau$ survives the passage to quantum mechanics as a $\tau$-number parameter, but it is not obvious that this parameter can be unambiguously described in the quantum theory as the arc length of a worldline. (4) In the context of simple cosmological models, at least, it seems to be an inevitable consequence of the constraint that some object to which one is inclined to attribute a real physical existence turns out to be unobservable. We think it is plausible that a clock which is internal to some observer should turn out to be unobservable in a description of the universe ‘from that observer’s point of view’. However, this plausible form of words is not directly mandated by the formalism, and some other way of understanding the unobservability of whatever quantity is eliminated by solution of the constraint may turn out to be better founded. We plan to address these issues in future work.

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