Localization and Large N reduction on $S^3$
for the Planar and M-theory limit

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Abstract
We show a large N reduction on $S^3$ in a BPS sector for a broad class of theories: $\mathcal{N} \geq 2$ supersymmetric Chern-Simons theory with any number of adjoint and bi-fundamental chiral multiplets. We show that a localization method can be applied to the reduced model and the path integral can be written by a multi-contour integral. By taking a particular localization configuration, we also show that the large N equivalence between the original theory on $S^3$ and the reduced model holds for the free energy and the expectation value of BPS Wilson loops. It turns out that the large N reduction on $S^3$ holds also for the M-theory limit.

Keywords: Large N reduction, M-theory, Localization, Matrix model

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1. Introduction

Large $N$ gauge theory is one of key ingredient for exploring non-perturbative aspects of gauge theory and string theory. For example, the $1/N$ expansion [1] has been a useful tool for understanding the phase diagram of QCD. Furthermore, the gauge/gravity duality [2, 3, 4] and the matrix model [5, 6, 7] have suggested that many large $N$ gauge theories are related to string theories. While large $N$ limit often make analysis of gauge theory simpler, it is generally difficult to solve the large $N$ limit of the gauge theories.

However, a drastic simplification occurs by using the large $N$ reduction [8] for some large $N$ gauge theories. It asserts that the planar large $N$ limit of gauge theories can be studied by their reduced models with some assumptions, which can be obtained by dimensional reduction. The original idea does not work in general because of the spontaneous breaking of the $U(1)^D$ symmetry in the reduced model [8], which led to various proposals [9, 10, 11, 12, 13, 14, 15].

A naive question is “how is the large $N$ reduction generalized to curved space?” This generalization was firstly proposed in [16] for $S^3$ and then generalized to the case for semi-simple compact group manifolds [17] and their coset spaces [18]. This proposal is based on correspondence of each Feynman diagram and lifting flat directions up due to mass terms. If such a generalization is possible, this can give an insight to emergent geometry in matrix model and non-perturbative regularization of field theories on curved space [4]. However, suf-

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1 There are some proposals in this direction [19, 20, 21, 22].
ficient conditions for the correspondence has not been established yet and there are only few examples of nontrivial test [23, 24, 25, 26, 27, 28].

In this paper, we consider the large N reduction on $S^3$ for a broad class of theories: three dimensional $\mathcal{N} \geq 2$ supersymmetric quiver Chern-Simons matter theories (CSM) with any number of adjoint and bi-fundamental chiral multiplets. For example, such a class of theory includes the ABJM theory as a special case, which is the leading candidate of low-energy effective theory of M2-branes. While many supersymmetric quiver CSM theories in the planar limit have been conjectured to be dual to superstring theories on certain backgrounds, the gauge/gravity duality suggests that these theories are also dual to M-theories on certain backgrounds for another large N limit called “M-theory limit”. Although the large N reduction has been considered only for the planar limit so far, here we ask a question: “Does such a drastic simplification occur also for the M-theory limit?” This question is highly nontrivial in the following reasons. First of all, we do not well understand general properties of the field theory in the M-theory limit although there are recently a few developments [34, 35]. Secondly we cannot use any perturbative arguments in the M-theory limit. Finally, it is nontrivial whether an usual large N factorization as for the planar limit occurs or not in this limit. Therefore, we expect that usual arguments by Schwinger-Dyson equation [8] and coherent state [36, 15] are not also useful. Thus, we need a non-perturbative method in order to answer the question. In this paper, we adopt a localization method to study non-perturbative aspects of the theories as such a method.

Localization methods have been played important roles and brought many exact analyses in (topologically twisted) supersymmetric theories. Generically, it is difficult to evaluate path integrals exactly in quantum field theories or even in reduced models with finite degrees of freedom such as instanton partition functions. When a theory possesses supersymmetry and one can apply a localization formula in the theory, the path integral reduces to a multi-contour integral (matrix model) or a summation. For example, the path integrals for instanton partition functions in four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are calculated exactly by an equivariant localization formula and reduce to finite summation labeled by Young diagrams [37]. The analyses by the localization formula reproduce the results obtained by analyzing infrared structure of coulomb moduli spaces [38].

Recently, there have been many progresses in a localization method for four dimensional $\mathcal{N} \geq 2$ rigid supersymmetric field theories on spheres attributed to [39]. For instance, it is shown that the expectation value of the circular BPS Wilson loop in the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory on $S^4$ can be described by a Gaussian matrix model. This originally has been conjectured in [40, 41] in the context of AdS$_5$/CFT$_4$ correspondence.

The localization method can be also applied to supersymmetric CSM theo-
ries on $S^3$ and has expected to be useful for quantitative tests of the AdS$_4$/CFT$_3$ correspondence. In fact, the authors of [12] constructed off-shell $\mathcal{N} = 2$ supersymmetry on $S^3$ and showed that the expectation values of the BPS Wilson loops and the partition functions in the $\mathcal{N} = 2$ supersymmetric CSM theories can be described by certain matrix models. This is generalized to the general $R$-charge assignments for matter chiral multiplets in [43, 44]. Especially, the ABJM matrix model is analytically continued to the CS matrix model on the lens space $S^3/\mathbb{Z}_2$ [45]. Large N-duality between the pure CS theory on the lens space and topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ [46] enables to derive large 't Hooft coupling behavior of the BPS Wilson loops [15] and the degrees of freedom of multi-parallel M2-branes [48]. Many other application based on localization methods in three dimensional supersymmetric theories are achieved, for example see [49, 43, 44, 34, 50, 51, 52].

In this paper, we consider the large N reduction on $S^3$ for any $N \geq 2$ supersymmetric quiver CSM theories. In this class of theory, we show via the localization method that the large N reduction on $S^3$ for a kind of BPS operators holds both in the planar and M-theory limit.

This article is organized as follows. In section 2, we briefly review the large N reduction on $S^3$, and the detail, see [25].

\section{Review of large $N$ reduction on $S^3$}

In this section, we review the large N reduction on $S^3$. For the detail, see [25].

\subsection{Large $N$ reduction on $S^3$}

In order to give intuitive understanding of the large N reduction for readers, we first consider the large N reduction on $S^3$ as the simplest example [53]. Let us consider the matrix quantum mechanics on $S^3$ with the radius $l$, whose action is

\[ S = \frac{1}{g^2} \int_0^{2\pi l} dx \text{Tr} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{\xi^2}{2l^2} \phi^2 + \frac{1}{4} \phi^4 \right), \]  

(1)

where $\phi(x)$ is an $N \times N$ hermitian matrix valued and $\xi$ is the dimensionless mass. Making the Fourier transformation $\phi(x) = \sum_{n=-\infty}^{\infty} \tilde{\phi}^{(n)} e^{inx/R}$, the action in the

\footnote{The free energy in the ABJM theory is also numerically studied for arbitrary rank and level in [15].}
momentum representation is given by
\[
S = \frac{V_{S^1}}{g^2} \text{Tr} \left[ \frac{1}{2l^2} \sum_n (n^2 + \xi^2) \phi^{(n)} \phi^{(-n)} + \frac{1}{4} \sum_{n_1, n_2, n_3, n_4} \delta_{n_1+n_2+n_3+n_4,0} \phi^{(n_1)} \phi^{(n_2)} \phi^{(n_3)} \phi^{(n_4)} \right],
\]
(2)
where \(V_{S^1} = 2\pi l\) is the volume of \(S^1\). Let us consider the free energy in the \'t Hooft limit:\n\[
N \to \infty \quad \text{with} \quad \lambda = g^2 N = \text{fixed}.
\]
(3)
The planar contribution at 2-loop level is
\[
\frac{F^{2\text{-loop}}_{\text{planar}}}{V_{S^1}} = \frac{1}{2g^2} \sum_{n_1, n_2, n_3, n_4} \delta_{n_1+n_2+n_3+n_4,0} \phi^{(n_1)} \phi^{(n_2)} \phi^{(n_3)} \phi^{(n_4)}
\]
\[
= \frac{1}{2g^2} \left( \frac{g^2 l^2}{V_{S^1}} \right)^2 \sum_{n_1, n_3} \delta_{n_1 n_3} \delta_{\phi_{ab} \phi_{cd}}
\]
\[
= \frac{N^2 l^4}{2V_{S^1}^2} \sum_{n_1, n_2} \frac{1}{(n_1^2 + \xi^2)(n_2^2 + \xi^2)},
\]
(4)
where \(\phi^{(n_1)} \phi^{(n_2)}\) denotes the propagator.

In order to obtain the reduced model, we apply the following rule:
\[
\phi(x) \to e^{ilP_x} \phi e^{-ilP_x}, \quad g \to g_r,
\]
(5)
where \(\phi\) is an \(M \times M\) constant hermitian matrix and \(P\) is the diagonal matrix taking the form
\[
P = \frac{1}{l} \text{diag} \left( \frac{-\nu + 1}{2}, \frac{-\nu + 3}{2}, \ldots, \frac{\nu - 1}{2} \right) \otimes 1_N \quad \text{with} \quad \nu N = M.
\]
(6)
Then the action of the reduced model is
\[
S_r = \frac{V_{S^1}}{g_r^2} \text{Tr}_M \left( -\frac{1}{2} [P, \phi]^2 + \frac{\xi^2}{2l^2} \phi^2 + \frac{1}{4} \phi^4 \right),
\]
(7)
where \(\text{Tr}_M\) stands for the trace over \(M \times M\) matrices. If we decompose \(\phi\) into a \(N \times N\) matrix \(\phi^{(s,t)}\) \((s, t = 1, 2, \ldots, \nu)\) as
\[
\phi = \begin{pmatrix}
\phi^{(1,1)} & \ldots & \phi^{(1,\nu)} \\
\vdots & \ddots & \vdots \\
\phi^{(\nu,1)} & \ldots & \phi^{(\nu,\nu)}
\end{pmatrix},
\]
(8)
then we can rewrite the action as
\[
S_r = \frac{V_{S^1}}{g_r^2} \text{Tr} \left[ \frac{1}{2l^2} \sum_{s,t} ((P_s - P_t)^2 + \xi^2) \phi^{(s,t)} \phi^{(t,s)} + \frac{1}{4} \sum_{s,t,u,v} \phi^{(s,t)} \phi^{(t,u)} \phi^{(u,v)} \phi^{(v,s)} \right],
\]
(9)
where \( P_s = -\frac{1}{4} (s-1) \). Now let us compute the free energy in the reduced model at 2-loop level and take the \('t Hooft limit:
\[
N \to \infty, \quad \nu \to \infty \quad \text{with} \quad \lambda_r = g_r^2 N = \text{fixed}. \quad (10)
\]
The planar contribution at 2-loop level is
\[
\frac{F_{\text{2-loop}}^{\text{planar}}}{V_{S^1}} = \frac{1}{2g_r^2} \sum_{s,t,u,v} \phi_{ab}^{(s,t)} \phi_{bc}^{(t,u)} \phi_{cd}^{(u,v)} \phi_{da}^{(v,s)}
\]
\[
= \frac{1}{2g_r^2} \left( \frac{g_r^2 l^2}{V_{S^1}} \right)^2 \sum_{s,t,u,v} \delta_{st} \delta_{tu} \delta_{vu} \left( \frac{\delta_{rt} \delta_{rs} \delta_{vu}}{(P_s - P_t)^2 + \xi^2((P_u - P_v)^2 + \xi^2)} \right)
\]
\[
= N^2 \nu \cdot \frac{\lambda_r l^4}{2V_{S^1}^2} \delta_{n_1 n_2} \frac{1}{(n_1^2 + \xi^2)(n_2^2 + \xi^2)}. \quad (11)
\]

Therefore, if we identify \( \lambda_r = \lambda \), we find
\[
\frac{F_{\text{2-loop}}^{\text{planar}}}{N^2} = \frac{F_{\text{2-loop}}^{\text{planar}}}{N^2 \nu}. \quad (12)
\]

Although the non-planar diagrams do not correspond with each other, these are relatively suppressed by the order of \( O(1/N^2) \) against the planar diagrams. This correspondence is based on coincidence of all planar diagrams. Intuitively, the constant matrix \( P \) supplies the “missing Kaluza-Klein momenta” along the \( S^1 \)-direction associated with the dimensional reduction. Such a mechanism occurs only for the planar diagrams in general. From this point of view, we can regard the role of the parameter \( \nu \) as the UV cutoff in the theory.

2.2. Large \( N \) reduction on \( S^3 \)

In this subsection, we briefly review the large \( N \) reduction on \( S^3 \). Let us consider the scalar field theory on \( S^3 \) with the radius \( l \), whose action is
\[
S = \frac{V_{S^3}}{g^2} \int \frac{d\Omega_3}{2\pi^2} \text{Tr} \left( -\frac{2}{l^2} (\mathcal{L}_i \phi)^2 + \frac{2\xi^2}{l^2} \phi^2 + \frac{1}{4} \phi^4 \right), \quad (13)
\]
where \( V_{S^3} = 2\pi^2 l^3 \) is the volume of \( S^3 \) and \( \mathcal{L}_i \) \((i = 1, 2, 3)\) is the Killing vector on the unit \( S^3 \). In order to obtain the action in the angular momentum representation, we make the spherical harmonics expansion as
\[
\phi(\Omega_3) = \sum_J \sum_{m, \bar{m} = -J} \phi_{Jm\bar{m}} Y_{Jm\bar{m}}(\Omega_3) \equiv \sum_J \phi_J Y_J(\Omega_3), \quad (14)
\]
and use the identities
\[
\mathcal{L}_J^2 Y_J(\Omega_3) = J(J + 1) Y_J(\Omega_3), \quad (15)
\]
\[
\int \frac{d\Omega_3}{2\pi^2} Y_{J1}^* (\Omega_3) Y_{J2} (\Omega_3) = \delta_{J1, J2} \delta_{m_1, m_2} \delta_{\bar{m}_1, \bar{m}_2}, \quad (16)
\]
where \( Y_J^J(\Omega_3) \) is given by \( Y_J^J(\Omega_3) \equiv (-1)^{m-\bar{m}} Y_J^J(\Omega_3) \) with \( J^* = (J, -m, -\bar{m}) \).

Then we can rewrite the action as

\[
S = \frac{V_{S^3}}{g^2} \text{Tr} \left( \frac{2}{T^2} \sum_{J} (-1)^{m-\bar{m}} (J(J + 1) + \xi^2) \phi_J \phi_{J^*} + \frac{1}{4} \sum_{J_1, J_2, J_3, J_4} V_{J_1 J_2 J_3 J_4} \phi_{J_1} \phi_{J_2} \phi_{J_3} \phi_{J_4} \right),
\]

(17)

where \( V_{J_1 J_2 J_3 J_4} \) is the 4-point vertex

\[
V_{J_1 J_2 J_3 J_4} = \int \frac{d\Omega_3}{2\pi^2} Y_{J_1}(\Omega_3) Y_{J_2}(\Omega_3) Y_{J_3}(\Omega_3) Y_{J_4}(\Omega_3).
\]

(18)

Let us consider the free energy in the 't Hooft limit (\( N \to \infty \) with \( \lambda = g^2 N \) fixed) again. The planar contribution at 2-loop level is

\[
\frac{F_{\text{planar}}^{2-\text{loop}}}{V_{S^3}} = \frac{1}{2g^2} \sum_{J_1, J_2, J_3, J_4} V_{J_1 J_2 J_3 J_4}^4 \bar{\phi}_{J_1, ab} \phi_{J_2, bc} \cdot \bar{\phi}_{J_3, cd} \phi_{J_4, da}
\]

\[
= \frac{1}{2g^2} \left( \frac{g^2 T^2}{4V_{S^3}} \right)^2 \sum_{J_1, J_3} V_{J_1 J_2 J_3 J_4}^4 \delta_{ac} \delta_{bb} \delta_{ca} \delta_{dd} \frac{(-1)^{m_1-\bar{m}_1}}{J_1(J_1 + 1) + \xi^2} \frac{(-1)^{m_3-\bar{m}_3}}{J_3(J_3 + 1) + \xi^2}
\]

\[
= N^2 \frac{\lambda^4}{8V_{S^3}} \sum_{J_1, J_2} V_{J_1 J_2 J_1 J_2}^4 (-1)^{m_1-\bar{m}_1} (-1)^{m_2-\bar{m}_2} \frac{(-1)^{m_3-\bar{m}_3}}{J_1(J_1 + 1) + \xi^2} \frac{(-1)^{m_4-\bar{m}_4}}{J_2(J_2 + 1) + \xi^2}
\]

(19)

In order to obtain the reduced model, we apply the following rule:

\[
\phi(\Omega_3) \to G^{-1} \phi G, \quad g \to g_r,
\]

(20)

where \( \phi \) is an \( M \times M \) constant hermitian matrix again and \( G \) is the representation matrix of \( SU(2) \) in the \( M \)-dimensional representation whose generator is\(^4\)

\[
L_i = \bigoplus_{s=1}^{\nu} L_i^{(n_s)} \otimes 1_N \quad \text{with} \quad n_s = n + s - \frac{\nu + 1}{2},
\]

(21)

where \( L_i^{(n)} \) denotes the \( n \)-dimensional irreducible representation of \( SU(2) \).

Then the action of the reduced model is

\[
S_r = \frac{V_{S^3}}{g_r^2} \text{Tr}_M \left( -\frac{2}{T^2} |L_i| \phi|^2 + \frac{\xi^2}{2T^2} \phi^2 + \frac{1}{4} \phi^4 \right).
\]

(22)

If we decompose \( \phi \) into a \( n_N \times n_N \) matrix \( \phi^{(s,t)} \) \((s, t = 1, 2, \cdots, \nu)\) again as

\[
\phi = \begin{pmatrix} \phi^{(1,1)} & \cdots & \phi^{(1,\nu)} \\ \vdots & \ddots & \vdots \\ \phi^{(\nu,1)} & \cdots & \phi^{(\nu,\nu)} \end{pmatrix},
\]

(23)

\(^4\) In [17], the authors have been proposed another representation of the generator \( L_i = \bigoplus_{s=1}^{\nu} L_i^{(s)} \otimes 1_s \otimes 1_N \). Here we do not consider the background.
and expand \( \phi^{(s,t)} \) in terms of the scalar fuzzy sphere harmonics\(^5\):

\[
\phi^{(s,t)} = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{m=-J}^{J} \phi^{(s,t)}_{jm} \otimes \hat{Y}_{jm(j_s,j_t)} = \sum_{J} \phi^{(s,t)}_{J} \otimes \hat{Y}_{jst},
\]

(24)

where \( \phi^{(s,t)}_{jm} \) is a \( N \times N \) matrix. In order to rewrite the action in a convenient form, we use the identity

\[
(L_{0})^2 \hat{Y}_{jst} = J(J+1)\hat{Y}_{jst},
\]

(25)

and the orthogonal relation

\[
\frac{1}{n} \text{tr} \left( \hat{Y}_{jst}^\dagger \hat{Y}_{jst} \right) = \delta_{J_1 J_2} \delta_{m_1 m_2},
\]

(26)

where \( \hat{Y}_{jst}^\dagger \) is given by \( \hat{Y}_{jst}^\dagger = (-1)^{m-j_s-j_t}Y_{jst}^\dagger \) with \( J^\dagger = (J, -m, j_t, j_s) \) and \( \text{tr} \) stands for the trace over \((2j_t+1) \times (2j_t+1)\) matrices. Then we can rewrite the action as

\[
S_r = \frac{V_{S^3}}{g_r^2} \left[ \frac{2}{l^2} \sum_{s,t} \sum_{J} (-1)^{m-j_s-j_t} (J(J+1)+\xi^2) \text{Tr} \left( \phi^{(s,t)}_{J} \phi^{(t,s)}_{J} \right) \right. \\
+ \frac{1}{4} \sum_{s,t,u,v} \sum_{J_1,J_2,J_3,J_4} \hat{V}_{J_1,J_2,J_3,J_4} \text{Tr} \left( \phi^{(s,t)}_{J_1} \phi^{(t,u)}_{J_2} \phi^{(u,v)}_{J_3} \phi^{(v,s)}_{J_4} \right),
\]

(27)

where \( \hat{V}_{J_1,J_2,J_3,J_4} \) is the 4-point vertex

\[
\hat{V}_{J_1,J_2,J_3,J_4} = \frac{1}{n} \text{tr} \left( \hat{Y}_{jst} \hat{Y}_{jst} \hat{Y}_{jst} \hat{Y}_{jst} \right). 
\]

(28)

Let us consider the free energy in the following limit:

\[
N \to \infty, \quad \nu \to \infty, \quad n \nu \to \infty \quad \text{with} \quad \lambda_r = \frac{g_r^2 N}{n} \quad \text{fixed},
\]

(29)

which is the counter part of the 't Hooft limit in the original theory.

The planar contribution at 2-loop level is

\[
\frac{F_{r,\text{planar}}^{2\text{-loop}}}{V_{S^3}} = \frac{n}{2g_r^2} \sum_{s,t,u,v} \sum_{J_1,J_2,J_3,J_4} \hat{V}_{J_1,J_2,J_3,J_4} \phi^{(s,t)}_{J_1} \phi^{(t,u)}_{J_2} \phi^{(u,v)}_{J_3} \phi^{(v,s)}_{J_4}
\]

\[
= n \left( \frac{g_r^2}{4nV_{S^3}} \right)^2 \sum_{s,t,u,v} \sum_{J_1,J_2,J_3,J_4} \hat{V}_{J_1,J_2,J_3,J_4} \delta_{J_1 J_2} \delta_{J_3 J_4} \delta_{m_1 m_2} \delta_{m_3 m_4} (J_1(J_1+1)+\xi^2) J_2(J_2+1)+\xi^2
\]

\[
= \frac{N^2 \lambda_r^4}{8V_{S^3}} \sum_{s,t,u,v} \sum_{J_1,J_2,J_3,J_4} \hat{V}_{J_1,J_2,J_3,J_4} \delta_{J_1 J_2} \delta_{J_3 J_4} \delta_{m_1 m_2} \delta_{m_3 m_4} (J_1(J_1+1)+\xi^2) J_2(J_2+1)+\xi^2.
\]

\[(30)\]

\(^5\) For details of the fuzzy sphere harmonics, see Appendix B.
In the limit \( m_1 = j_s - j_t, m_2 = j_s - j_v, m_3 = j_t - j_v \) by reading from the relation \([16]\), we can identify as
\[
\tilde{\lambda}_r = \lambda_r.
\]

Therefore, if we make the identification \( \hat{\lambda}_r = \lambda_r \), we obtain
\[
F^{2-\text{loop}}_{\text{planar}} = \frac{F^{2-\text{loop}}_{\text{planar}}}{N^2 V S^3} = \frac{1}{8} \lambda_r l_4 \sum_{m_1 \neq m_2} V^{(4)}_{J_1 J_2 J_3 J_4} \frac{(-1)^{m_1 - (j_s - j_t)} (-1)^{m_2 - (j_s - j_v)}}{J_1 (J_1 + 1) + \xi^2 J_2 (J_2 + 1) + \xi^2}
\]

Although the case for \( S^3 \) seems more complicated than the case for \( S^1 \), essential features are same with each other. Similarly for the \( S^1 \) case, the parameters \( n \) and \( \nu \) play the role as the UV cutoff of the angular momentum along \( S^1 \). While we have demonstrated large N equivalence at two-loop level only for the free energy, we can also see perturbative coincidence for correlation functions \([25]\).

2.3. Construction of Large N reduced model for supersymmetric quiver CSM on \( S^3 \)

In this subsection, we construct large N reduced models for supersymmetric quiver CSM theories on \( S^3 \) \([19]\). In general, the action of supersymmetric quiver CSM theory is decomposed as
\[
S = S_{\text{CS}} + S_{\text{YM}} + S_{\text{matter}},
\]
where \( S_{\text{CS}} \), \( S_{\text{YM}} \) and \( S_{\text{matter}} \) are the action of the Chern-Simons, Yang-Mills and matter part, respectively. In the following, we construct the reduced model of each part.

\textit{CS part}

The Chern-Simons action for the \( \mathcal{N} = 2 \) vector multiplet is given by
\[
S_{\text{CS}} = \frac{i k}{4\pi} \int \text{Tr} \left[ A \wedge dA - \frac{2}{3} i A \wedge A + (-\bar{\lambda} \lambda + 2\sigma D)\sqrt{g}d^3\mathbf{x} \right],
\]
where \( k \) is the Chern-Simons level. In order to apply the prescription \([20]\), we rewrite the derivative term in terms of the Killing vector. Expanding the gauge

\[\text{[Footnote 6]}\]
field as $A = X^i e^i$ and using the Meurer-Cartan equation, we derive
\[
dA = \frac{2}{l}(J_j X_i) e^j \wedge e^i + \frac{1}{l} \epsilon_{ijk} X_i e^j \wedge e^k
\]
\[
= \frac{2}{l} \left( iJ_i X_j + \frac{1}{2} \epsilon_{ijk} X_k \right) e^i \wedge e^j,
\] (35)
where $J_i$ is the Killing vector on the unit sphere. In this way, we obtain
\[
A \wedge dA - \frac{2}{3} iA \wedge A \wedge A = \left\{ \frac{2}{l} X_i \left( iJ_j X_k + \frac{1}{2} \epsilon_{jkl} X_l \right) - i \frac{2}{3} X_i X_j X_k \right\} e^i \wedge e^j \wedge e^k
\]
\[
= \left\{ i \frac{2}{l} \epsilon_{ijk} X_i X_j X_k + \frac{2}{l} X_i^2 - i \frac{2}{3} \epsilon_{ijk} X_i X_j X_k \right\} l^2 d\Omega_3. \] (36)

The prescription for constructing the reduced model is
\[
\Phi(\Omega_3) \rightarrow G^{-1} \Phi G, \quad \frac{k}{4\pi} \rightarrow \frac{1}{g_{\text{CS},r}}, \] (37)
where $\Phi$ represents the collection of the components fields in the theory. Then, the action of the reduced model is
\[
S_{\text{CS}}^r = -\frac{i V_{S^2}}{g_{\text{CS},r}} \text{Tr} \left[ i \frac{2}{l} \epsilon_{ijk} X_i [L_j, X_k] + \frac{2}{l} X_i^2 - i \frac{2}{3} \epsilon_{ijk} X_i X_j X_k - \bar{\lambda} \lambda + 2\sigma D \right]. \] (38)

Here note that we can absorb $-\frac{2}{l} L_i$ into $X_i$ as
\[
-\frac{2}{l} L_i + X_i \rightarrow X_i. \] (39)
After the absorbing this factor, we obtain the simpler action as
\[
S_{\text{CS}}^r = -\frac{i V_{S^2}}{g_{\text{CS},r}} \text{Tr} \left[ \frac{2}{l} X_i^2 - i \frac{2}{3} \epsilon_{ijk} X_i X_j X_k - \bar{\lambda} \lambda + 2\sigma D \right], \] (40)
which corresponds to the dimensional reduction of the original action (34). The equation of motion for $X_i$ is
\[
[X_i, X_j] = -i \frac{2}{l} \epsilon_{ijk} X_k, \] (41)
which can be solved as
\[
X_i = -\frac{2}{l} \bigoplus_{i=1}^{\nu} L_i^{(n_i)} \otimes 1_{N_i}, \] (42)
where $L_i^{(n_i)}$ denotes the $n_i$-dimensional irreducible representation of $SU(2)$ with $\sum_i n_i N_i = M$. Since this solution includes $-\frac{2}{l} L_i$ as the special case, we can realize the action (38) if we expand $X_i$ around the solution as
\[
X_i \rightarrow -\frac{2}{l} L_i + X_i. \] (43)
Thus, the rule for generating the reduced model of the gauge theory on $S^3$ becomes simpler as follows

$$\Phi(\Omega_3) \rightarrow \Phi, \quad J_i \Phi(\Omega_3) \rightarrow 0, \quad \frac{k}{4\pi} \rightarrow \frac{1}{g_{CS,r}}.$$  \hspace{1cm} (44)

In order to realize the original theory, we have to expand the gauge field as $X_i \rightarrow -\frac{2}{l}L_i + X_i$.

**YM part**

The action of the $\mathcal{N} = 2$ SYM on $S^3$ is given by

$$S_{YM} = \frac{V_{S^3}}{g_{YM}^2} \int \frac{d\Omega_3}{2\pi^2} \text{Tr} \left[ \frac{1}{4} F_{ij}^2 + \frac{1}{2} (D_i \sigma)^2 + \frac{1}{2} \left( D + \frac{\sigma}{l} \right)^2 
+ \frac{i}{2} \bar{\lambda} \gamma^i D_i \lambda + \frac{i}{2} \bar{\lambda} \sigma \lambda - \frac{1}{4l} \bar{\lambda} \lambda \right],$$

where $D_i \sigma = \partial_i \sigma - i [ A_i, \sigma ]$. Similarly for the Chern-Simons term, we rewrite the field strength as

$$F = dA - i A \wedge A
= \frac{1}{2} \epsilon_{ijk} \left\{ \frac{2}{l} i \epsilon_{klm} J_l X_m + \frac{2}{l} X_k - \frac{i}{2} \epsilon_{klm} [ X_l, X_m ] \right\} e^i \wedge e^j.$$

The covariant derivative of the fermion is

$$\gamma^i D_i \lambda = \gamma^a e^i_a (\partial_i + \frac{1}{4} \omega^a_{i\alpha} \lambda - i \gamma^i [ A_i, \lambda ]
= \frac{2i}{l} \gamma^a J^a \lambda + \frac{3i}{2l} \lambda - i \gamma^i [ A_i, \lambda ]$$

Applying the rule

$$\Phi(\Omega_3) \rightarrow \Phi, \quad J_i \Phi(\Omega_3) \rightarrow 0, \quad g_{YM} \rightarrow g_{YM,r},$$

the action of the reduced model is

$$S'_{YM} = \frac{V_{S^3}}{g_{YM,r}^2} \text{Tr} \left[ \frac{1}{2} \left( \frac{2}{l} X_i - \frac{i}{2} \epsilon_{ijk} [ X_j, X_k ] \right)^2 - \frac{1}{2} \left( D_i \sigma \right)^2 + \frac{1}{2} \left( D + \frac{\sigma}{l} \right)^2 
+ \frac{i}{2} \bar{\lambda} \gamma^i [ X_i, \lambda ] + \frac{i}{2} \bar{\lambda} \sigma \lambda - \frac{1}{4l} \bar{\lambda} \lambda \right].$$

**Matter sector (bi-fundamental)**

The action of the chiral multiplet with the bi-fundamental representation under the gauge group $G = U(N_1) \times U(N_2)$ is given by

$$S_{\text{matter}} = \frac{V_{S^3}}{g_{YM,r}^2} \int d\Omega_3 \left( L_{\text{kin}} + L_{\text{pt}} \right),$$

Note that this case also includes an adjoint matter representation under $G = U(N)$ as a special case.
where $L_{\text{kin}}$ and $L_{\text{pt}}$ are a kinetic term and a potential term with higher powers of the matter fields, respectively. $L_{\text{kin}}$ is given by

$$
L_{\text{kin}} = \text{Tr} \left[ D_i \bar{\phi} D^i \phi + \bar{\phi}(\sigma_A - \sigma_B)^2 \phi + \frac{i(2q - 1)}{l} \bar{\phi}(\sigma_A - \sigma_B) \phi \\
+ \frac{q(2 - q)}{l^2} \bar{\phi} \phi + i \bar{\phi}(D_A - D_B) \phi + \bar{F} F \\
- i \bar{\psi} \gamma^i D_i \psi + i \bar{\psi}(\sigma_A - \sigma_B) \psi - \frac{2q - 1}{2l} \bar{\psi} \psi + i \bar{\psi}(\lambda_A - \lambda_B) \phi - i \bar{\phi}(\bar{\lambda}_A - \bar{\lambda}_B) \psi \right],
$$

(49)

where $q$ is the dimension and R-charge of $\phi$. Each matter field is $N_1 \times N_2$ matrix and $D_i \phi = \partial_i \phi - i A_i \phi + i \phi B_i$.

The reduced model is given by

$$
L_{\text{kin}} = \text{Tr} \left[ (X_i \phi - \phi Y_i)^i (X_i \phi - \phi Y_i) + \bar{\phi}(\sigma_A - \sigma_B)^2 \phi + \frac{i(2q - 1)}{l} \bar{\phi}(\sigma_A - \sigma_B) \phi \\
+ \frac{q(2 - q)}{l^2} \bar{\phi} \phi + i \bar{\phi}(D_A - D_B) \phi + \bar{F} F \\
- \bar{\psi} \gamma^i (X_i \psi - \psi Y_i) + i \bar{\psi}(\sigma_A - \sigma_B) \psi - \frac{q - 2}{l} \bar{\psi} \psi + i \bar{\psi}(\lambda_A - \lambda_B) \phi - i \bar{\phi}(\bar{\lambda}_A - \bar{\lambda}_B) \psi \right],
$$

(50)

where $(X_i, \sigma_A, D_A, \lambda_A) : M_1 \times M_1$ matrix, $(Y_i, \sigma_B, D_B, \lambda_B) : M_2 \times M_2$ matrix and $(\phi, F, \psi) : M_1 \times M_2$ matrix.

3. Localization

We briefly explain the concept of the localization formula in our interest. The partition function of a supersymmetric theory is written in schematic way as

$$
Z = \int D\Phi \exp(-S[\Phi]),
$$

(51)

where $\Phi$ represents the collection of the components fields in the theory. $S[\Phi]$ is the action invariant under the nilpotent supercharge $Q$. We choose one of the supercharges $Q$ and deform the action by one parameter family of $Q$-exact term as $S[\Phi] + tQ \cdot V[\Phi]$. We assume $Q \cdot V$ respects the symmetry of the theory and its bosonic part is positively semi-definite. The $Q$-invariance requires that the expectation value of $Q$-closed operator $O(\Phi)$ and th partition function are

---

8Since $L_{\text{pt}}$ is irrelevant in the context of this paper, we do not write down this explicitly.
independent of the coupling parameter $t$. When we take the limit $t \to \infty$, the path integral is exactly evaluated with the quadratic order of fluctuation fields, namely one-loop of $Q \cdot V[\Phi]$ around the localization field configurations $Q \cdot V(\Phi_0) = 0$. Then, the localization configuration only contributes to the action at classical level. Then the expectation value is formally written as

$$\langle O \rangle = \frac{\sum_{\Phi_0} O(\Phi_0) \exp(-S[\Phi_0]) Z_{1\text{-loop}}(\Phi_0)}{\sum_{\Phi_0} \exp(-S[\Phi_0]) Z_{1\text{-loop}}(\Phi_0)}.$$

Here $\sum_{\Phi_0}$ stands for the summation over the configurations with $Q \cdot V(\Phi_0) = 0$. As we will see later, it is actually not summation rather multi-contour integrals of the scalar $\sigma$ in the vector multiplet. $Z_{1\text{-loop}}$ is the one-loop determinant of the fluctuations around the localization configurations. In this section, we evaluate the one-loop determinant of the dimensional reduced $\mathcal{N} = 2$ supersymmetric CSM theories.

3.1. Gauge sector

Localized configuration

Since the reduced $\mathcal{N} = 2$ SYM action $S_{YM}$ itself is rewritten as (For the derivation, see Appendix C)

$$\bar{\epsilon} \epsilon S_{YM}^r = \delta \bar{\epsilon} \epsilon \text{Tr}_{M} \left[ \frac{1}{2} \bar{\lambda} \lambda - 2D\sigma \right],$$

we can choose the deformation term $Q \cdot V$ as the reduced $\mathcal{N} = 2$ YM action $S_{YM}^r$. From the action (47), the localized configuration is determined by the equation

$$[ X_i, X_j ] = -i \frac{2}{l} \epsilon_{ijk} X_k, \quad [ X_i, \sigma ] = 0, \quad D + \frac{\sigma}{l} = 0, \quad \lambda = \bar{\lambda} = 0.$$

This can be solved as

$$X_i = -\frac{2}{l} \bigotimes_{s=1}^{\nu} L_{i}^{(n_s)} \otimes 1_{N_s}, \quad \sigma = \bigotimes_{s=1}^{\nu} 1_{n_s} \otimes \sigma_{N_s} (:= \bar{\sigma}), \quad D = -\bar{\sigma} \frac{l}{\nu}$$

where $L_{i}^{(n_s)}$’s denote the $n_s$-dimensional irreducible representation of $SU(2)$ with $\sum_s n_s N_s = M$. Note that there is an important difference from the original theory. There exists the nontrivial configuration of the gauge field at the localization points contrary to the case for the original theory.

In order to realize the field theory on $S^3$, we specify the representation as $n_s = n + s - \frac{s+1}{2}$, $N_s = N$, namely, $X_i = -\frac{2}{l} L_i$ and $\sigma_{N_s} = \sigma_0$. Substituting (55) into the reduced $\mathcal{N} = 2$ supersymmetric Chern-Simons action (41), the CS action on the localized field configurations becomes

$$S_{CS}^r = \frac{i V_{S^3} n_\nu}{g^2_{CS,r}} \text{Tr} \sigma_0^2,$$

up to an irrelevant constant. Here the trace “Tr” is taken over $N \times N$ matrix.
1-loop determinant

Here, we evaluate the one-loop determinant of the $\mathcal{N} = 2$ SYM action around the localization points \([55]\). In order to obtain the action at the quadratic order of fluctuation fields, we expand the fields around the localized configuration as

\[
X_i \to -\frac{2}{l} L_i + \frac{1}{\sqrt{t}} \tilde{X}_i, \quad \sigma \to \tilde{\sigma} + \frac{1}{\sqrt{t}} \tilde{\sigma}, \quad D \to -\frac{1}{l} \sigma_0 + \frac{1}{\sqrt{t}} \tilde{D}, \quad \lambda \to \frac{1}{\sqrt{t}} \tilde{\lambda},
\]

\[\text{(57)}\]

Then, the quadratic action for the reduced $\mathcal{N} = 2$ SYM is given by

\[
S_{\text{YM}}|_{\text{Gauss}} = \text{Tr}_M \left[ \frac{1}{2} \left( \frac{2}{l} \right)^2 \left( \tilde{X}_i + i e_{ijk} [ L_j, \tilde{X}_k ] \right)^2 - \frac{1}{2} \left( -\frac{2}{l} [ L_i, \tilde{\sigma} ] + [ \tilde{X}_i, \sigma_0 ] \right)^2 \right.
\]

\[
- \frac{1}{l} \bar{\lambda} \gamma^i [ L_i, \lambda ] + \frac{i}{2} \bar{\lambda} [ \tilde{\sigma}, \lambda ] - \frac{1}{l} \bar{\lambda} \lambda D^2 \right].
\]

\[\text{(58)}\]

Since $\tilde{D}$ has the Gaussian form, this is trivially integrated out. In order to perform the path integral over the fluctuation fields, we introduce the vector, scalar and spinor fuzzy sphere harmonics: $\hat{Y}^\rho_{Jm(j_s,j_t)i}$, $\hat{Y}^\rho_{Jm(j_s,j_t)}$ and $\hat{Y}^\kappa_{Jm(j_s,j_t)x}$, respectively. Then the field are expanded as follows;

\[
\tilde{X}_i = \sum_{s,t} \sum_{\rho=-1}^{1} \sum_{Q=|j_s-j_t|}^{J} \sum_{m=-Q}^{Q} \hat{Y}^\rho_{Jm(j_s,j_t)i} \otimes X^{(s,t)}_{Jm\rho},
\]

\[
\tilde{\sigma} = \sum_{s,t} \sum_{J=|j_s-j_t|}^{J} \sum_{m=-J}^{J} \hat{Y}_{Jm(j_s,j_t)} \otimes \sigma^{(s,t)}_{Jm},
\]

\[
\lambda_\alpha = \sum_{s,t} \sum_{\kappa=\pm \frac{1}{2}} \sum_{\tilde{U}=|j_s-j_t|}^{U} \sum_{m=-U}^{U} \hat{Y}^\kappa_{Jm(j_s,j_t)x} \otimes \lambda^{(s,t)}_{Jm\alpha},\]

\[\text{(59)}\]

where $Q = J + \frac{(1+\rho)\rho}{2}$, $\tilde{Q} = J - \frac{(1-\rho)\rho}{2}$, $U = J + \frac{1+\kappa}{2}$ and $\tilde{U} = J + \frac{1-\kappa}{2}$. From the properties of the fuzzy sphere harmonics \([13,30]\) in Appendix B, the quadratic

\[9\] More explicitly, these are given by

\[
Q|_{\rho=+1} = J + 1, \quad \tilde{Q}|_{\rho=+1} = J, \quad Q|_{\rho=-1} = J, \quad \tilde{Q}|_{\rho=-1} = J + 1, \quad U|_{\kappa=+1} = J + \frac{1}{2}, \quad \tilde{U}|_{\kappa=+1} = J, \quad U|_{\kappa=-1} = J, \quad \tilde{U}|_{\kappa=-1} = J + \frac{1}{2}.
\]
action can be rewritten as

\[
S_{\text{YM}}|_{\text{Gauss}} = \sum_{\rho, \tilde{Q}, m} \rho^2 (J + 1)^2 \text{Tr} \left( X_{J\rho m}^{(s,t)} \right) - \sum_{\rho, \tilde{Q}, m} \text{Tr} \left[ \left[ \sigma_0, X_{J\rho m} \right]^{(s,t)} \right] - 2 \frac{J(J + 1)}{J} \text{Tr} \left[ \left[ \sigma_{Jm}^{(s,t)}, \sigma_{Jm} \right]^{(s,t)} \right].
\]

Next, we introduce the Cartan-Weyl basis \((H_i, E_{\alpha}, E_{-\alpha})\) satisfying the relations:

\[
[H_i, H_j] = 0, \quad [H_i, E_{\alpha}] = \alpha_i \cdot E_{\alpha}, \quad [E_{\alpha}, E_{-\alpha}] = \frac{2}{|\alpha|^2} \alpha_i H_i
\]

\[E_{\alpha}^\dagger = E_{-\alpha}, \quad \text{Tr}(E_{\alpha}E_{\beta}) = \delta_{\alpha+\beta,0}.\]  

Then, we expand each \(N \times N\) matrix \(X\) in terms of the Cartan-Weyl basis as

\[
X = X_i H_i + \sum_{\alpha \in \Delta_+} (X_{\alpha} E_{\alpha} + X_{-\alpha} E_{-\alpha})
\]  

and we choose the gauge for the localization configuration \(\sigma_0\) as \(\sigma_0 = \text{diag}(\sigma_1, \cdots, \sigma_N)\).

\[\text{In terms of the basis, the action becomes}^{11}\]

\[
S_{\text{YM}}|_{\text{Gauss}} = \sum_{J,m,\alpha} \text{Tr} \left[ \left( \sigma_{Jm}^{(s,t)} \right)^\dagger \right] C_J \left( \frac{\sigma_{Jm}^{(s,t)}}{X_{Jm0}^{(s,t)}} \right) + \sum_{\rho=\pm 1} \sum_{\tilde{Q},m,\alpha} ((J + 1)^2 + (\alpha \cdot \sigma)^2) \text{Tr} \left[ X_{J\rho m}^{(s,t)} \right].
\]

where \(C_J\) is the \(2 \times 2\) kinematic matrix of \(X_{Jm0}^{(s,t)} : \sigma_{Jm}^{(s,t)}\), whose component is

\[
C_J = \left( \begin{array}{cc} \left( \frac{2}{J(J + 1)} \right)^2 & \frac{2}{J(J + 1)}(\alpha \cdot \sigma) \\ \frac{2}{J(J + 1)}(\alpha \cdot \sigma) & (\alpha \cdot \sigma)^2 \end{array} \right).
\]

In order to find the eigenvalues, we have to diagonalize the matrix \(C_J\). After the straightforward calculation, we find that the eigenvalues of the matrix are 0 and \((\frac{2}{J(J + 1)} \alpha \cdot \sigma)^2\). Since the eigenmodes \((\alpha \cdot \sigma)\sigma_{Jm}^{(s,t)} + \frac{2}{J(J + 1)} \alpha \cdot \sigma\) associated to zero eigenvalue are gauge modes, it can be removed by the BRST procedure as we will see later. Hence, the one-loop determinant associated to the fields \((\sigma_{Jm}^{(s,t)}, X_{Jm0}^{(s,t)})\) is \(\det^{-1/2} \Delta_{\sigma,X}^2\) with

\[
\det \Delta_{\sigma,X}^2 = \prod_{J,m,\alpha} \left( \frac{2}{J(J + 1)} \right)^2 (J(J + 1) + (\alpha \cdot \sigma)^2).
\]

\[^{10}\text{For } U(N) \text{ case, } \sum_{\alpha \in \Delta} f(\alpha \cdot \sigma) = \sum_{1 \leq \alpha \neq \beta \leq N} f(\sigma_\alpha - \sigma_\beta).\]

\[^{11}\text{Here we drop the terms independent of } \sigma \text{ since these terms become only irrelevant overall constant.}\]
However, this factor is exactly canceled to the factors coming from the one-loop determinant of ghosts and gauge fixing delta functions in BRST procedure (See Appendix D). Therefore, the one-loop determinants of the transverse parts $X_{\rho}=\pm 1$ give the bosonic part of the one-loop determinant of the Yang-Mills action $\det^{-1/2} \Delta^2_{X_{\rho}=\pm 1}$ with

$$\det \Delta^2_{X_{\rho}=+1} = \prod_{s,t} \prod_{\alpha \in \Delta} \prod_{J=|j_s-j_t|}^{J+1} \left\{ \left( \frac{2}{l} \right)^2 (J+1)^2 + (\alpha \cdot \sigma)^2 \right\},$$

$$\det \Delta^2_{X_{\rho}=-1} = \prod_{s,t} \prod_{\alpha \in \Delta} \prod_{J=|j_s-j_t|-1}^{J} \left\{ \left( \frac{2}{l} \right)^2 (J+1)^2 + (\alpha \cdot \sigma)^2 \right\}. \tag{66}$$

Next, we calculate the contribution from the fermionic fields. The Lagrangian of the gaugino at quadratic order is expanded by the spinor fuzzy sphere harmonics as

$$-\frac{1}{l} \lambda_i \gamma^i \left[ L_i, \lambda \right] + i \lambda \left[ \sigma_0, \lambda \right] - \frac{1}{l} \bar{\lambda} \lambda = \sum_{\kappa,U,m} \left\{ -\frac{1}{l} \kappa^i \left( J + \frac{3}{4} \right) + i \alpha \cdot \sigma \right\} \left[ \lambda^{(s,t)\alpha} \lambda_{j_{ms}^\alpha} \right]. \tag{67}$$

By integrating out $\bar{\lambda}^{(s,t)}$, $\lambda^{(s,t)}$, one can obtain the fermionic part of the one-loop determinant $\Delta_{\lambda} |_{\kappa=\pm 1}$ as

$$\det \Delta_{\lambda} |_{\kappa=+1} = \prod_{s,t} \prod_{\alpha \in \Delta} \prod_{J=|j_s-j_t|}^{J+1/2} \left\{ \left( \frac{2}{l} \right)(J+1) - i (\alpha \cdot \sigma) \right\},$$

$$\det \Delta_{\lambda} |_{\kappa=-1} = \prod_{s,t} \prod_{\alpha \in \Delta} \prod_{J=|j_s-j_t|-1/2}^{J} \left\{ \left( \frac{2}{l} \right)(-J - \frac{1}{2}) - i (\alpha \cdot \sigma) \right\}. \tag{68}$$

3.2. Matter sector

Localized configuration

Since the reduced matter action $S_{\text{kin}}^r$ itself is rewritten as

$$\bar{\psi} \psi + 2i \bar{\phi} (\sigma_A - \sigma_B) \phi + \frac{2(q-1)}{l} \bar{\phi} \phi,$$

we can choose the deformation term $Q \cdot V$ as the reduced action $S_{\text{kin}}^r$. From (50), the localized configurations are trivial:

$$\phi = \bar{\phi} = \psi = \bar{\psi} = F = \bar{F} = 0. \tag{69}$$
Here we calculate the one-loop determinant of the $U(M_1) \times U(M_2)$ bi-fundamental matter. We rescale all the matter component fields as $\Phi \to \frac{1}{\sqrt{t}} \Phi$ and substitute the localization configuration into (50). Then, in the large $t$ limit, only the quadratic part with respect to the matter component survive as follows

$$L_{\text{kin}}^{\Gamma_{\text{Gauss}}} = \text{Tr}_{M_2} \left[ \left( \frac{2}{l} \right)^2 (L_X^i \phi - \phi L_Y^i) (L_X^i \phi - \phi L_Y^i) + \bar{\phi}(\bar{\sigma}^A - \bar{\sigma}^B)^2 \phi ight]$$

$$+ \frac{i(2q-2)}{l} \bar{\phi}(\bar{\sigma}^A - \bar{\sigma}^B) \phi + \frac{q(2-q)}{l^2} \bar{\phi} \phi$$

$$+ \frac{2}{l} \bar{\psi} \gamma_i (L_X^i \psi - \psi L_Y^i) + i\bar{\psi}(\bar{\sigma}^A - \bar{\sigma}^B) \psi - \frac{q-2}{l} \bar{\psi} \psi + \bar{F} F \right].$$

Here $L_X$ and $L_Y$ are the classical field configuration of reduced gauge fields $X^i$ and $Y^i$, respectively;

$$L_X = \bigoplus_{s=1}^{\nu} L_i^{(n_s)} \otimes 1_{N_1}, \quad L_Y = \bigoplus_{s=1}^{\nu} L_i^{(n_s)} \otimes 1_{N_2}. \quad (71)$$

$\bar{\sigma}^A$ and $\bar{\sigma}^B$ are $N_1 \times N_1$ and $N_2 \times N_2$ classical scalar field configurations in the vector multiplets.

Expanding $\phi$ and $\psi$ in terms of the fuzzy sphere harmonics as

$$\phi = \sum_{s,t} \sum_{j_s+j_t} \sum_{J} \sum_{m=-J}^{J} \hat{Y}_{J m(j_s,j_t)} \otimes \phi_{J m}^{(s,t)},$$

$$\psi_\beta = \sum_{s,t} \sum_{\kappa=\pm} \sum_{U} \sum_{U} \sum_{m=-U}^{U} \hat{Y}_{J m(j_s,j_t)\alpha} \otimes \psi_{J m\kappa}^{(s,t)}, \quad (72)$$

and integrating out $\phi_{J m}^{(s,t)}, \psi_{J m\kappa}^{(s,t)}$, the 1-loop determinants of each field are eval-
uated in the similar manner to the vector multiplet as

\[ \det \Delta_\phi^2 = \prod_{s,t=1}^{N_1} \prod_{a=1}^{N_2} \prod_{j=|j_s-j_t|}^{J} \prod_{m=-J}^{J} \left[ \left( \frac{2}{l} \right)^2 J(J+1) + (\sigma_a - \tilde{\sigma}_b)^2 + \frac{i(2q-2)}{l}(\sigma_a - \tilde{\sigma}_b) + \frac{q(2q-1)}{l^2} \right], \]

\[ \det \Delta_\psi|_{\kappa=+1} = \prod_{s,t=1}^{N_1} \prod_{a=1}^{N_2} \prod_{j=|j_s-j_t|}^{J+1/2} \prod_{m=-J}^{J+1/2} \left[ \frac{2}{l} J + i(\sigma_a - \tilde{\sigma}_b) - \frac{q-2}{l} \right], \]

\[ \det \Delta_\psi|_{\kappa=-1} = \prod_{s,t=1}^{N_1} \prod_{a=1}^{N_2} \prod_{j=|j_s-j_t|-1/2}^{J} \prod_{m=-J}^{J} \left[ \frac{2}{l} \left( -J - \frac{3}{2} \right) + i(\sigma_a - \tilde{\sigma}_b) - \frac{q-2}{l} \right], \]

(73)

where we take the diagonal gauge \( \sigma_0^A = \text{diag}(\sigma_1, \cdots, \sigma_{N_1}) \) and \( \sigma_0^B = \text{diag}(\tilde{\sigma}_1, \cdots, \tilde{\sigma}_{N_2}) \).

The one-loop determinant of the bi-fundamental matter multiplet is given by

\[ \frac{\det \Delta_\psi|_{\kappa=+1}}{\det \Delta_\psi|_{\kappa=-1}}. \]

(74)

4. Planar Large N reduction

In this section, we show the large N reduction for the free energy and the BPS Wilson loops in the planar limit.

4.1. Free energy

Here we consider the large N equivalence for the free energy. We consider the \( G = \prod_{I=1}^{A} U(M_I)g_I \) reduced quiver CSM theories with any number of the bi-fundamental and adjoint matters. For a later convenience, we introduce \( g_I := g_{CS,rI} V_{S^3}^{1/2} \), where \( g_{CS,rI} \) is the coupling constant of the \( U(M_I) \) reduced CS theory. Let \( m_{IJ} \) be the number of the chiral multiplets with the bi-fundamental representation under \( U(M_I) \times U(M_J) \) for \( I \neq J \) or with adjoint representation for \( I = J \). We take the following limit

\[ N_I \to \infty, \nu \to \infty, \frac{n}{\nu} \to \infty \text{ with } \lambda_I = \frac{g_I^2 N_I}{n} = \frac{4\pi N_I}{k_I} \text{ fixed}, \frac{N_I}{N_J} \text{ fixed}, \]

(75)

which corresponds to the 't Hooft limit in the original theory. If we define the free energy in the original theory and the reduced model as \( F^{3d} = \log Z^{3d} \) and \( F^r = \log Z^r \), respectively, the statement of the large N reduction on \( S^3 \) is

\[ F^{3d}|_{\text{planar}} = \frac{F^r}{\nu}|_{\text{planar}}. \]

(76)
the saddle point equation of the reduced model as

$$0 = \frac{\partial S^\nu}{\partial \sigma^I}(\sigma^*) = \nu \left( \partial_{\sigma^I} \log \Delta(\sigma^*) \right) + \frac{2iN_I}{\lambda_I} \rho^I_a \sigma^I_a. \tag{82}$$

For the $U(N)$ pure CS theory, this relation is shown by using the Feynman diagram technique [54, 24]. In the limit $\nu \to \infty$, $n/\nu \to \infty$, combining (66) and (68), the one-loop determinant for the vector multiplet is given by\(^{12}\)

$$\frac{(\alpha \cdot \sigma)^2 \det \Delta_{\kappa=+1} \cdot \det \Delta_{\kappa=-1}}{\sqrt{\det \Delta^2 \mid_{\rho=+1} \cdot \sqrt{\det \Delta^2 \mid_{\rho=-1}}}} \approx \prod_{\alpha \in \Delta_+} \{(\alpha \cdot \sigma)^2\}^{2\nu} \prod_{n=1}^{\infty} \left\{ \left( \frac{n}{\nu} \right)^2 + (\alpha \cdot \sigma)^2 \right\}^{2\nu} = \prod_{1 \leq a < b \leq N_I} 2^{2\nu} \sinh^{2\nu}(\pi l(\sigma^I_a - \sigma^I_b)). \tag{77}$$

From (73), the contribution from the bi-fundamental matter is

$$\frac{\det \Delta_{\kappa=+1} \det \Delta_{\kappa=-1}}{\det \Delta^2} \approx \prod_{a,b} \prod_{a=1}^{\infty} \left[ \frac{n+1}{n} - \frac{2}{n} i(\sigma^I_a - \sigma^I_b) \right] \prod_{a,b} \prod_{a=1}^{\infty} s_b(iq - l(\sigma^I_a - \sigma^I_b)), \tag{78}$$

where $s_b(x)$ is the double sine function defined by

$$s_b(x) := \prod_{m,n=0}^{\infty} \frac{mb + nb^{-1} + Q + ix}{mb + nb^{-1} + Q + ix}, \quad \text{with} \quad Q := b + 1 \frac{1}{b}. \tag{79}$$

Here we note that the one-loop determinants of the original $U(N_I) \, N = 2$ super Yang-Mills theory and the $U(N_I) \times U(N_J)$ bi-fundamental chiral multiplet are $\prod_{1 \leq a < b \leq N_I} 2^{2\nu} \sinh^{2\nu}(\pi l(\sigma^I_a - \sigma^I_b))$ and $\prod_{a=1}^{N_I} \prod_{ij=1}^{N_J} \Pi_a(iq - l(\sigma^I_a - \sigma^I_b)).$ Therefore, the one-loop determinants of the reduced model are the $\nu$-th power of the original one. From (77) and (78), the partition function of the $G = \prod_{I=1}^{A} U(M_I)_{g_I}$ reduced supersymmetric quiver Chern-Simons matter theory is

$$Z^\nu = \int \prod_{a=1}^{A} \prod_{I=1}^{N_I} d\sigma^I_a \Delta^\nu(\sigma) \exp \left( \nu \sum_{I=1}^{A} \sum_{a=1}^{N_I} \frac{iN_I}{\lambda_I} \rho^I_a \sigma^I_a \right), \tag{80}$$

where the $\Delta(\sigma)$ is same as the one-loop determinant of the $G = \prod_{I=1}^{A} U(N_I)_{k_I}$ three dimensional quiver supersymmetric CSM given by

$$\Delta(\sigma) = \prod_{I} \prod_{a < b} 2^{2\nu} \sinh^{2\nu}(\pi l(\sigma^I_a - \sigma^I_b)) \prod_{I,J} \prod_{a,b} s^{m_{IJ}}(i \cdot \sigma^I_a - \sigma^I_b)). \tag{81}$$

In order to discuss the correspondence for the planar free energy, we write down the saddle point equation of the reduced model as
where $S_{\text{eff}}^{r}$ is the effective action given by

$$S_{\text{eff}}^{r} = \nu \left( \log \Delta(\sigma) + \sum_{l=1}^{A} \sum_{a=1}^{N_{l}} \frac{iN_{l}}{\lambda_{l}} \sigma_{a}^{l2} \right). \quad (83)$$

The planar free energy is dominated by the solutions of the saddle point equation

$$F^{r}|_{\text{planar}} = S_{\text{eff}}^{r}(\sigma^{*}). \quad (84)$$

In order to write down the planar free energy more explicitly, we introduce the eigenvalue density $\rho_{l}(x)$ for $\sigma_{l}^{I} = \text{diag}(\sigma_{1}^{I}, \ldots, \sigma_{N_{l}}^{I})$

$$\rho_{l}(x) := \frac{1}{N_{l}} \sum_{a=1}^{N_{l}} \langle \delta(x - \sigma_{a}^{I^{*}}) \rangle. \quad (85)$$

Here $\sigma_{a}^{I^{*}}$ are the solutions of (82). In the large $N_{l}$ limit, $\rho_{l}(x)$ approaches to the continuous distribution with the normalization condition

$$\int_{C_{l}} dx \rho_{l}(x) = 1. \quad (86)$$

Since $S_{\text{eff}}^{r}$ is proportional to the original one, the saddle point equations for the reduced model (82) are same for the $U(N_{l})$ quiver CSM matrix model. Therefore the resolvents and the eigenvalue densities of the two theories are identical with each other. In the $N_{l} \to \infty$ limit, the summation is goes to integrand

$$\lim_{N_{l} \to \infty} \frac{1}{N_{l}^{2}} \sum_{a=1}^{N_{l}} f(\sigma_{a}^{I}) = \int_{C_{l}} dx f(x). \quad (87)$$

Thus, the planar free energy is described by the effective action $S_{\text{eff}}^{r}$ with respect to the saddle point (82) as

$$F^{r}|_{\text{planar}} = \nu \left[ \sum_{l} \int_{C_{l} \times C_{l}} dx dx' \log 2 \sinh^{2}(\pi l(x - x')) \rho_{l}(x) \rho_{l}(x') \right]$$

$$+ \sum_{l} \int_{C_{l} \times C_{l}} dx dx' \log 2 \sinh^{2}(\pi l(x - x')) \rho_{l}(x) \rho_{l}(x')$$

$$+ \sum_{l,J} N_{l} N_{J} \int_{C_{l} \times C_{l}} dx dx' \log s_{l,J}^{m} \rho_{l}(x) \rho_{J}(x')$$

$$= \nu F^{3d}|_{\text{planar}}, \quad (88)$$

which is nothing but (76). Thus we show the large N equivalence for the planar free energy.

4.2. BPS Wilson loops

Here we consider the large N equivalence for the BPS Wilson loops.
**BPS Wilson loop preserving 2 supercharges**

We consider the following type of the Wilson loop

\[ W_{R}(C) := \frac{1}{\text{dim } R} \text{tr}_R P \exp \left( \oint_C d\tau (iX_\mu(\Omega_3) e^\mu_\nu(\tau) dx^\nu + \sigma|\dot{x}(\tau)|) \right). \] (89)

Here \( R \) denotes the representation under the \( U(N) \) gauge group and \( \text{tr}_R \) is taken over the representation \( R \). When the integration contour is the great circle of \( S^3 \), \( W_{R}(C) \) preserves two supercharges \([55, 56, 57]\). For example, if we embed \( W_{R}(C) \) into the ABJ(M) model \([29, 58]\) and integrate out auxiliary scalar \( \sigma \), we can regard the Wilson loop \((89)\) as 1/6-BPS Wilson loop \([59]\). It has been conjectured that the Wilson loop with the different representation \( R \) corresponds to a different quantity on the gravity side. For instance, the gauge/gravity duality states that the Wilson loops with the fundamental, symmetric and anti-symmetric representation are dual to the string, D2-brane and D6-brane worldvolume, respectively \([55]\).

If the contour is the great circle, then we can choose angular velocity \(|\dot{x}(\tau)| = 1\) without loss of generality. The expectation value of the Wilson loop is defined by

\[ \langle W_{R}(C) \rangle := \frac{1}{Z_{d3}} \int D\mu W_{R}(C) \exp(-S_{d3}). \] (90)

Here \( D\mu \) is the integration measure of the supersymmetric CSM theory. Correspondingly, we consider the following operator in the reduced model

\[ \hat{W}_{\hat{R}}(C) := \frac{1}{\text{dim } \hat{R}} \text{Tr}_{\hat{R}} P \exp \left( \oint_C d\tau (iX^\mu(\tau) dx^\mu + \sigma|\dot{x}(\tau)|) \right). \] (91)

Here the representation \( \hat{R} \) is defined by \( \hat{R} := \bigoplus_{i=1}^{\nu} 1_{n_i} \otimes R \). The dimension of \( \hat{R} \) is \( \text{dim } \hat{R} = n\nu \text{ dim } R \). In \([23]\), it has been shown that this operator corresponds to the angular average of the original BPS Wilson loop operator in the ’t Hooft limit. The operator at localization point can be written as

\[ \hat{W}_{\hat{R}}(C) = \frac{1}{\text{dim } \hat{R}} \text{Tr}_{\hat{R}} \exp \left( i \bigoplus_{s=1}^{\nu} L_i^{(s)}(s) \otimes 1_N \oint_C d\tau e_i^\mu(\tau) dx^\mu + 2\pi i \bigoplus_{l=1}^{\nu} 1_{n_l} \otimes \sigma_0 \right) \]  
\[ = \frac{1}{\text{dim } \hat{R}} \text{Tr}_{\hat{R}} \exp \left( 2\pi i \bigoplus_{l=1}^{\nu} 1_{n_l} \otimes \sigma_0 \right) \]  
\[ = \frac{1}{\text{dim } \hat{R}} \text{tr}_{\hat{R}} \exp \left( 2\pi i \lambda_0 \right). \] (92)

Note that this is same as the Wilson loop in the original theory at the localization configuration \([42]\). The expectation value of the Wilson loop is defined by

\[ \langle \hat{W}_{\hat{R}}(C) \rangle := \frac{1}{Z'} \int dX_i d\lambda \cdots \hat{W}_{\hat{R}}(C) \exp(-S') \] (93)
From the localization argument, the expectation value of the BPS Wilson loop is given by

$$\langle \hat{W}_R(C) \rangle = \frac{1}{\text{dim}R^Z} \int \prod_{a} \prod_{I} d\sigma_a^I \Delta^I(\sigma) tr_R \exp \left( 2\pi l\sigma_0 \right) \exp \left( \nu \sum_{I=1}^{N_I} \sum_{a=1}^{iN_{I}} i\frac{N_I}{\lambda_I} \sigma_a^I \right).$$ \hspace{1cm} (94)$$

In the large $N_I$ limit, the expectation value of the Wilson loop can be also evaluated by the solutions of the saddle points equation and becomes

$$\langle \hat{W}_R(C) \rangle = \frac{1}{\text{dim}R} \int dx \rho(x) tr_R \exp \left( 2\pi lx \right) = \langle W_R(C) \rangle .$$ \hspace{1cm} (95)$$

Thus we show the large $N$ correspondence for the Wilson loop in the planar limit.

1/2-BPS Wilson loop in the ABJ(M) theory

In [60], the authors construct the 1/2-BPS Wilson loop in the ABJ(M) theory, whose form is given by

$$SW_R(C) := \frac{1}{\text{dim}R} tr_R P \exp \left( \oint_C d\tau L \right),$$ \hspace{1cm} (96)$$

where $L$ is

$$L := \begin{pmatrix} iX_4(\Omega_3)e_\mu^I(\tau)\dot{x}^\mu + \sigma_A(\Omega_3)|\dot{x}(\tau)| & i\sqrt{\frac{2\pi}{k}}|\dot{x}(\tau)|\eta^I_A(\tau)\bar{\psi}^I_A(\Omega_3) \\ i\sqrt{\frac{2\pi}{k}}|\dot{x}(\tau)|\psi^I_A(\Omega_3)\bar{\eta}^I_A(\tau) & i\sqrt{\frac{2\pi}{k}}|\dot{x}(\tau)|\psi^I_A(\Omega_3)\bar{\eta}^I_A(\tau) \end{pmatrix}.$$ \hspace{1cm} (97)$$

Although $\eta^I_A(\tau)$ and $\bar{\eta}^I_A(\tau)$ are the parameters determined by requiring supersymmetry, these are irrelevant on the localized configuration. Correspondingly, we consider the operator in the reduced model:

$$S\hat{W}_R(C) := \frac{1}{\text{dim}R} tr_R P \exp \left( \oint_C d\tau \hat{L} \right),$$ \hspace{1cm} (98)$$

where $\hat{L}$ is given by

$$\hat{L} := \begin{pmatrix} iX_4e_\mu^I(\tau)\dot{x}^\mu + \sigma_A|\dot{x}(\tau)| & i\sqrt{\frac{2\pi}{k}}|\dot{x}(\tau)|\eta^I_A(\tau)\bar{\psi}^I_A \\ i\sqrt{\frac{2\pi}{k}}|\dot{x}(\tau)|\psi^I_A|\bar{\eta}^I_A(\tau) & i\sqrt{\frac{2\pi}{k}}|\dot{x}(\tau)|\psi^I_A|\bar{\eta}^I_A(\tau) \end{pmatrix}.$$ \hspace{1cm} (99)$$

This operator at the localization point can be written as

$$\hat{W}_R(C) = \frac{1}{\text{dim}R} \text{Str}_R \left( e^{2\pi l\sigma_0^A} \begin{pmatrix} 0 & 0 \\ 0 & -e^{2\pi l\sigma_0^B} \end{pmatrix} \right).$$ \hspace{1cm} (100)$$

\footnote{We assume the rank of the representation is small and the saddle points are unaffected by the insertion of Wilson loop operator.}
which is same as the one in the original theory at the localized configur ation [60]. From the localization argument, the expectation value of the BPS Wilson loop

\[ \langle S \hat{W}_R(C) \rangle = \frac{1}{\dim R \mathcal{Z}} \int \prod_{l=1}^{N_l} \prod_{a=1}^{N} d\sigma^I_a \Delta_\nu(\sigma) \text{Str}_R \left( \begin{array}{cc} e^{2\pi i \sigma^I_a} & 0 \\ 0 & -e^{2\pi i \sigma^I_a} \end{array} \right) \times \exp \left( \nu \sum_{l=1}^{N_l} \sum_{a=1}^{N} \frac{i N_l}{\lambda^I_l} \sigma^I_a \right) \]  

(101)

In the planar limit, the expectation value of the Wilson loop can be also evaluated by the solutions of the saddle points equation, which is same as the original one. Therefore, we conclude that this is same for the expectation value of the Wilson loop in three dimensions.

5. M-theoretic Large N reduction

So far, we have considered the large N reduction on $S^3$ for the planar limit: $N_l \to \infty$, $N/k_I$ fixed. In this limit, many supersymmetric quiver Chern-Simons theories have been conjectured to be dual to the type-IIA string theory on certain backgrounds. However, the gauge/gravity duality suggests that these theories are also dual to M-theory on certain backgrounds for the M-theory limit: $N_l \to \infty$, $k_I$ fixed. Let us consider the large N reduction on $S^3$ in the M-theory limit. This is highly nontrivial in the following reason. First of all, we do not well understand general properties of the field theory in the M-theory limit although there are recently a few developments [34, 35]. Secondly we emphasize that any perturbative argument is unavailable for justifying the large N reduction in the M-theory limit. Finally, it is nontrivial in this limit whether an usual large N factorization as for the planar limit occurs or not. Therefore, we expect that an usual argument by Schwinger-Dyson equation [8] is not also useful. Now we can perform non-perturbative argument thanks to the localization method. Let us consider the following limit in the reduced model:

\[ N \to \infty, \quad \nu \to \infty, \quad \frac{n}{\nu} \to \infty \quad \text{with} \quad \frac{\lambda^2}{n} = \frac{4\pi}{k_I} = \text{fixed}, \]  

(102)

which is the counterpart of the M-theory limit in the original theory. In this limit, the partition function of the reduced model becomes

\[ Z^r = \int \prod_{l=1}^{A} \prod_{a=1}^{N_l} d\sigma^I_a \Delta_\nu(\sigma) \exp \left( \nu \sum_{l=1}^{A} \sum_{a=1}^{N_l} \frac{i N_l}{g^2_l} \sigma^I_a \right) \]  

(103)

which is formally same as the planar partition function (80) if we replace $g^2 = \lambda^I_I$. As in [34], we assume that a saddle point for the integral exists in the limit (102). The saddle point equation is

\[ \partial_{\sigma^I_a} \log \Delta(\sigma^*) + \frac{2 i}{g^2_l} \sigma^I_a \sigma^I_a = 0, \]  

(104)
which is same as the one of the original theory\(^{14}\) in the M-theory limit \(^{34}\) if we substitute \(g^2 = \frac{4\pi}{N}\). Therefore, we conclude that the relation of the free energy between the reduced model and the original model is

\[
\frac{F_r}{\nu} = F^{3d},
\]

also for the M-theory limit. Similarly, the correspondence also holds for the BPS Wilson loops \((89)\) and \((96)\) in the limit since the saddle points are same with each other.

6. Conclusion

In this paper, we show the large N reduction on \(S^3\) in the BPS sector for the general \(\mathcal{N} \geq 2\) supersymmetric quiver Chern-Simons theories by using the localization method. In particular, we calculate the free energy, expectation values of 1/6-BPS Wilson loops and 1/2-BPS Wilson loops. Remarkably, it turns out that the large N reduction holds even for the M-theory limit. Although we have shown that formally, we ask “What does this mean physically?”. We are suspicious to that this is related to a smooth connection between the planar and M-theory limit, which has been recently observed for the free energy \(^{48, 34, 61, 62, 55, 47}\) and BPS Wilson loops \(^{45}\) in the ABJM theory. These results imply that the free energy and BPS Wilson loops in the strong ’t Hooft coupling regime after taking the ’t Hooft limit are same as the ones in the M-theory limit if we simply replace \(\lambda = N/k\). Therefore, we expect that such a smooth connection between the ’t Hooft and M-theory limit is one of sufficient conditions for the large N reduction in the M-theory limit.

We remark on an important point that the reasons why our localization in the reduced model works well are following. When we performed dimensional reduction of the theories on \(S^3\), flat directions of the reduced gauge fields in the \(Q\)-exact term disappear and emerge the non-trivial fuzzy sphere solution at the localization points. This is quite different from the localization of the reduced models on flat space which suffer from the divergence coming from the flat directions. We expect that our localization method is useful to show the large N reduction in other theories defined on \(S^3\); In \(^{62}\), the large N equivalence between the \(\mathcal{N} = 4\) super Yang-Mills theory on \(\mathbb{R} \times S^3\) and the plane wave (BMN) matrix model \(^{64}\) on \(\mathbb{R}\) has been proposed. The localization of this system will be studied in our future work.

Applications to the same class of theory on different space would be also interested. For instance, the theories on \(S^1 \times S^2\) and squashed \(S^3\) are studied in \(^{55, 66, 51}\). In particular, since the squashed \(S^3\) is neither compact semi-simple group manifold nor its coset space whose reduced model is constructed in \(^{17, 18}\), it may give some insights to emergent geometry.

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\(^{14}\) While a standard analysis in the planar limit assumes that edges of eigenvalue distributions is \(O(1)\), we must consider that edges of eigenvalue distributions in the M-theory limit depends on \(N_f\) in order to obtain a nontrivial solution. See the \(^{34}\) for detail.
Note added
When our paper was ready for submission to the arXiv, there appeared a paper \cite{67} which has overlap with ours.

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Appendix A. $S^3$ as $SU(2)$ group manifold

In this appendix, we summarize properties of $S^3$. Let us start with the parametrization of the unit $S^3$ by an element $g$ of the Lie group $SU(2)$:

\[
g = e^{i\alpha \gamma_3} e^{i\theta \gamma_2} e^{i\beta \gamma_3} = \begin{pmatrix} e^{i(\alpha+\beta)} \cos \theta & -e^{i(\alpha-\beta)} \sin \theta \\ -e^{-i(\alpha-\beta)} \sin \theta & e^{-i(\alpha+\beta)} \cos \theta \end{pmatrix},
\]

where $\gamma^i (i = 1, 2, 3)$ are Pauli matrices and $0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \alpha \leq \pi, 0 \leq \beta \leq 2\pi$. The left invariant 1-form $\mu$ is defined by

\[
g^{-1} dg = i \mu^i \gamma^i,
\]

which satisfies the Maurer-Cartan equation $d\mu^i = \epsilon^{ijk} \mu^j \wedge \mu^k$. The left invariant 1-form $\mu$ gives the metric of $S^3$ with the radius $l$ as

\[
ds^2 = \frac{1}{2} l^2 \text{tr} (dg g^{-1}) = l^2 \mu^i \mu^i = l^2 \left[ d\theta^2 + d\alpha^2 + (d\beta + \cos 2\theta d\alpha)^2 \right].
\]

Defining the dreibein in the left-invariant frame as $e^i = l \mu^i$, each component of $e$ is explicitly given by

\[
e^1 = l (-\sin 2\beta d\theta + \cos 2\beta \sin 2\theta d\alpha),
\]

\[
e^2 = l (\cos 2\beta d\theta + \sin 2\beta \sin 2\theta d\alpha),
\]

\[
e^3 = l (d\beta + \cos 2\theta d\alpha),
\]

(A.5)
where \( e^i \) satisfies \( de^i = \frac{1}{4} \epsilon^{ijk} e^j \wedge e^k \) from the Maurer-Cartan equation. The spin connection in this frame is

\[
\omega^{ij} = \frac{1}{4} \epsilon^{ijk} e^k.
\]

(A.6)

The Killing vector \( J_i \) dual to \( e^i \) is

\[
J_i = \frac{i}{2l} e^a_i \partial_a
\]

(A.7)

where \( a = \theta, \alpha, \beta \) and \( e^a_i \) is the inverse of \( e_i^a \). The explicit form of each \( J_i \) is

\[
\begin{align*}
J_1 &= \frac{1}{2l} \left( -\sin 2\beta \partial_\theta + \frac{\cos 2\beta}{\sin 2\theta} \partial_\alpha - \cot 2\theta \cos 2\beta \partial_\beta \right) \\
J_2 &= \frac{1}{2l} \left( \cos 2\beta \partial_\theta + \frac{\sin 2\beta}{\sin 2\theta} \partial_\alpha - \sin 2\theta \cos 2\beta \partial_\beta \right) \\
J_3 &= \frac{1}{2l} \partial_\beta
\end{align*}
\]

(A.8)

and these satisfy \( SU(2) \) algebra

\[
[ J_i, J_j ] = i \epsilon^{ijk} J_k.
\]

(A.9)

Appendix B. Fuzzy sphere harmonics

In this section we briefly review the definition of the fuzzy sphere harmonics and their basic properties. Let \( L_i \) \((i = 1, 2, 3)\), \( L_i^{(n)} \) and \(|j, m\rangle\), \((m = -j, -j + 1, \ldots, j)\) be the generator of \( SU(2) \), the \( n = 2j + 1 \) dimensional irreducible representation and a basis of the representation space, respectively.

Then the tensor products \(|j, m\rangle\langle j', m'|\) span a basis of \((2j + 1) \times (2j' + 1)\)-dimensional representation. \( L_i \) acts on \(|j, m\rangle\langle j', m'|\) as

\[
L_i \circ |j, m\rangle\langle j', m'| := L_i^{(n)} |j, m\rangle\langle j', m'| - |j, m\rangle\langle j', m'| L_i^{(n')},
\]

(B.1)

where \( n = 2j + 1 \) and \( n' = 2j' + 1 \). The scalar fuzzy sphere harmonics is defined by

\[
\hat{Y}_{Jm(jj')} := \sqrt{n} \sum_{r, r'} (-1)^{-j + j'} C_{jjj'-r' \to r}^{Jm} |j, m\rangle\langle j', m'|.
\]

(B.2)

Here \( C_{jjj'-r' \to r}^{Jm} \) are the Clebsch-Gordan coefficients. The fuzzy sphere harmonics also spans a basis of the tensor representation. The vector and spinor spherical harmonics are defined by

\[
\begin{align*}
\hat{Y}_{Jm(jj')}^\rho &:= \hat{V}_m^{Qm} \hat{Y}_{Qp(jj')}, \\
\hat{Y}_{Jm(jj')}^\kappa &:= \hat{U}_m^{Qm} \hat{Y}_{Qp(jj')}.
\end{align*}
\]

(B.3)

(B.4)
Here $\alpha = 1, 2$ and $Q = J + \frac{(1+\rho)\kappa}{2}$, $\bar{Q} = J - \frac{(1-\rho)\kappa}{2}$, $U = J + \frac{(1+\rho)\kappa}{4}$, $\bar{U} = J + \frac{(1+\rho)\kappa}{4}$ with $\rho = -1, 0, 1$ and $\kappa = \pm 1$. $V$ is $3 \times 3$ matrix defined by
\[
V = \begin{pmatrix}
-1 & 0 & 1 \\
-i & 0 & -i \\
0 & \sqrt{2} & 0 \\
\end{pmatrix}.
\] (B.5)

The fuzzy sphere harmonics satisfies the following useful formula,
\[
L_{\pm} \circ \hat{Y}_{j m(jj')} = \sqrt{(J \mp m)(J \mp m + 1)} \hat{Y}_{j m \pm 1(jj')},
\]
\[
L_{3} \circ \hat{Y}_{j m(jj')} = m \hat{Y}_{j m(jj')},
\]
\[
L_{i} \circ L_{i} \circ \hat{Y}_{j m(jj')} = J(J + 1) \hat{Y}_{j m(jj')},
\]
\[
L_{i} \circ \hat{Y}_{j m(jj')} = \sqrt{J(J + 1)} \hat{Y}_{j m(j) j' i},
\]
\[
L_{i} \circ \hat{Y}_{j m(jj') i} = \sqrt{J(J + 1)} \delta_{i0} \hat{Y}_{j m(jj')},
\]
\[
ie_{\alpha i} L_{k} \circ \hat{Y}_{j m(jj') m} + \hat{Y}_{j m(j) j' i} = \rho(J + 1) \hat{Y}_{j m(j) j' i}.
\]
\[
\left( \gamma^{i}_{\alpha \beta} L_{i} + \frac{3}{4} \delta_{\alpha \beta} \right) \hat{Y}_{j m(jj') \beta} = \kappa(J + \frac{3}{4}) \hat{Y}_{j m(j) j' \alpha} \] (B.6)

Here $\gamma^{i}, (i = 1, 2, 3)$ are the Pauli matrices. The orthogonal relations of the fuzzy sphere harmonics are
\[
\frac{1}{n} \mathrm{tr} \left( \hat{Y}^{\dagger}_{j m_{1}(jj')} \hat{Y}_{j m_{2}(jj')} \right) = \delta_{j_{1}j_{2}} \delta_{m_{1}m_{2}},
\]
\[
\frac{1}{n} \mathrm{tr} \left( \hat{Y}^{\rho}_{j m_{1}(jj')} \hat{Y}^{\rho}_{j m_{2}(jj')} \right) = \delta_{\rho_{1} \rho_{2}} \delta_{j_{1}j_{2}} \delta_{m_{1}m_{2}},
\]
\[
\frac{1}{n} \mathrm{tr} \left( \hat{Y}^{\kappa}_{j m_{1}(jj')} \hat{Y}^{\kappa}_{j m_{2}(jj')} \right) = \delta_{\kappa_{1} \kappa_{2}} \delta_{j_{1}j_{2}} \delta_{m_{1}m_{2}}.
\] (B.7)

Appendix C. Supersymmetry

In this appendix, we summarize the supersymmetric transformations of the $\mathcal{N} = 2$ super CSM theories on $S^3$ and their reduced versions. These theories consist of the $\mathcal{N} = 2$ vector multiplets and the matter chiral multiplets.

Appendix C.1. Gauge sector

The $\mathcal{N} = 2$ CS action $S_{CS}$ and SYM action $S_{YM}$ on $S^3$ are invariant under the supersymmetric transformation \[12\] :
\[
\delta A_{a} = -\frac{i}{2} (\epsilon \gamma_{a} \lambda - \bar{\lambda} \gamma_{a} \epsilon),
\]
\[
\delta \sigma = \frac{1}{2} (\epsilon \lambda - \bar{\lambda} \epsilon),
\]
\[
\delta \lambda = \frac{1}{2} \gamma^{ab} \epsilon F_{ab} - D \epsilon + i \gamma^{a} \epsilon D_{a} \sigma + 2i \sigma \bar{\epsilon},
\]
\[
\delta \bar{\lambda} = \frac{1}{2} \gamma^{ab} \epsilon F_{ab} + D \bar{\epsilon} - i \gamma^{a} \bar{\epsilon} D_{a} \sigma - i \sigma \bar{\epsilon},
\]
\[
\delta D = -\frac{i}{2} \bar{\epsilon} \gamma^{a} D_{a} \lambda - \frac{i}{2} D_{a} \bar{\lambda} \gamma^{a} \epsilon + \frac{i}{2} [\epsilon \lambda + \bar{\lambda} \epsilon, \sigma] + \frac{i}{2} (\bar{\epsilon} \lambda - \lambda \bar{\epsilon}).
\] (C.1)
where $\epsilon, \tilde{\epsilon}$ satisfy the killing spinor equation:

$$\tilde{\epsilon} = \frac{i}{2l}\epsilon, \quad \gamma^a D_a \tilde{\epsilon} = -\frac{3}{4l^2} \epsilon. \quad (C.2)$$

Correspondingly, the reduced $\mathcal{N} = 2$ CS action $S_{CS}^r$ and SYM action $S_{YM}^r$ are invariant under the transformation:

$$\delta X_i = -\frac{i}{2}(\tilde{\epsilon} \gamma_i \lambda - \bar{\lambda} \gamma_i \epsilon),$$

$$\delta \sigma = \frac{1}{2}(\epsilon \lambda - \bar{\lambda} \epsilon),$$

$$\delta \lambda = \frac{1}{2} \gamma^{ij} \epsilon f_{ij} - D \epsilon + \gamma^i \epsilon [X_i, \sigma] + \frac{1}{l} \epsilon \sigma \epsilon,$$

$$\delta \bar{\lambda} = \frac{1}{2} \gamma^{ij} \bar{\epsilon} f_{ij} + D \bar{\epsilon} - \gamma^i \bar{\epsilon} [X_i, \sigma] + \frac{1}{2l} \epsilon \bar{\sigma} \epsilon,$$

$$\delta D = -\frac{1}{2} \bar{\epsilon} \gamma^i [X_i, \lambda] - \frac{1}{2} [X_i, \bar{\lambda}] \gamma^i \epsilon + \frac{i}{2} [\bar{\epsilon} \lambda + \bar{\lambda} \epsilon, \sigma] + \frac{1}{l} (\bar{\epsilon} \lambda + \bar{\lambda} \epsilon). \quad (C.3)$$

where $f_{ij}$ is the reduced version of the field strength

$$f_{ij} = \frac{1}{2} \epsilon_{ijk} \left[ \frac{2}{l} X_k - \frac{i}{2} \epsilon_{klm} [X_l, X_m] \right]. \quad (C.4)$$

Note that this supersymmetry is off-shell and therefore the localization works well. Moreover, the reduced $\mathcal{N} = 2$ SYM action is written in the following $Q$-exact form

$$\tilde{\epsilon} \epsilon S_{YM}^r = \delta \tilde{\epsilon} \epsilon \text{Tr} \left[ \frac{1}{2} \bar{\lambda} \lambda - 2D \sigma \right]. \quad (C.5)$$

**Appendix C.2. Matter sector**

Next we consider the supersymmetric transformation of the chiral multiplet $(\phi, \psi, F)$ on $S^3$. The transformation is constructed in [42] for the canonical $R$-charge assignment and extended to general $R$-charge in [43, 44]. We suppose that the matter chiral multiplet is in the bi-fundamental representation under the $U(N_1) \times U(N_2)$ gauge group. The Lagrangian is

$$\mathcal{L}_{kin} = \text{Tr} \left[ D_i \tilde{\phi} D^i \phi + \tilde{\phi} (\sigma_A - \sigma_B)^2 \phi + \frac{i(2q - 1)}{l} \tilde{\phi} (\sigma_A - \sigma_B) \phi 
+ \frac{q(2 - q)}{l^2} \tilde{\phi} \phi + i \bar{\phi} (D_A - D_B) \phi + \bar{F} F 
- i \bar{\psi} \gamma^i D_i \psi + i \bar{\psi} (\sigma_A - \sigma_B) \psi - \frac{2q - 1}{2l} \bar{\psi} \psi + i \bar{\psi} (\lambda_A - \lambda_B) \phi - i \bar{\phi} (\bar{\lambda}_A - \bar{\lambda}_B) \psi \right]. \quad (C.6)$$
The action is invariant under the supersymmetric transformation:

\[
\begin{align*}
\delta \phi &= \bar{\epsilon} \psi, \\
\delta \bar{\phi} &= \epsilon \bar{\psi}, \\
\delta \psi &= i \gamma^a \epsilon D_a \phi + i \epsilon \sigma_{AB} \phi + 2 q i \bar{\epsilon} \phi + \bar{\epsilon} F, \\
\delta \bar{\psi} &= i \gamma^a \epsilon D_a \bar{\phi} + i \bar{\epsilon} \sigma_{AB} \epsilon + 2 q i \bar{\epsilon} \bar{\phi} + \bar{F} \epsilon, \\
\delta F &= \epsilon \left( i \gamma^a D_a \psi - i \sigma_{AB} \psi - i \lambda \phi \right) - i (2q - 1) \bar{\epsilon} \psi, \\
\delta \bar{F} &= \bar{\epsilon} \left( i \gamma^a D_a \bar{\psi} - i \bar{\psi} \sigma_{AB} - i \bar{\phi} \lambda \right) - i (2q - 1) \bar{\epsilon} \bar{\psi},
\end{align*}
\] (C.7)

where \( \sigma_{AB} \equiv \sigma_A - \sigma_B \) and \( q \) is \( R \)-charge of scalar field \( \phi \).

The action of the reduced model for the matter is

\[
S_{\text{kin}} = \text{Tr} \left[ (X_i \phi - \phi Y_i)^\dagger (X_i \phi - \phi Y_i) + \bar{\phi} (\sigma_A - \sigma_B)^2 \phi + \frac{i(2q - 1)}{l} \bar{\phi} (\sigma_A - \sigma_B) \phi \\
+ \frac{q(2 - q)}{l^2} \bar{\phi} \phi + i \bar{\phi} (D_A - D_B) \phi + \bar{F} F \\
- \bar{\psi} \gamma^i (X_i \psi - \psi Y_i) + i \bar{\psi} (\sigma_A - \sigma_B) \psi - \frac{q - 2}{l} \bar{\psi} \psi + i \bar{\psi} (\lambda_A - \lambda_B) \phi - i \bar{\phi} (\bar{\lambda}_A - \bar{\lambda}_B) \psi \right].
\] (C.8)

This is invariant under the following supersymmetric transformation

\[
\begin{align*}
\delta \phi &= \bar{\epsilon} \psi, \\
\delta \bar{\phi} &= \epsilon \bar{\psi}, \\
\delta \psi &= \gamma^i \epsilon D^i \phi + i \epsilon \sigma_{AB} \phi - \frac{q}{l} \epsilon \phi + \bar{\epsilon} F, \\
\delta \bar{\psi} &= \gamma^i \epsilon D^i \bar{\phi} + i \bar{\epsilon} \sigma_{AB} \epsilon - \frac{q}{l} \bar{\epsilon} \epsilon + \bar{F} \epsilon, \\
\delta F &= \epsilon \left( \gamma^i D^i \psi - i \sigma_{AB} \psi - i \lambda \phi \right) + \frac{q - 2}{l} \epsilon \psi, \\
\delta \bar{F} &= \bar{\epsilon} \left( \gamma^i D^i \psi - i \psi \sigma_{AB} - i \bar{\phi} \lambda \right) + \frac{q - 2}{l} \bar{\epsilon} \bar{\psi}.
\end{align*}
\] (C.9)

where we define \( D^i \phi := X_i \phi - \phi Y_i \).

The reduced matter action is also written in \( Q \)-exact form as

\[
\bar{\epsilon} \epsilon S_{\text{kin}} = \delta_\epsilon \delta_\bar{\epsilon} \text{Tr} \left[ \bar{\psi} \psi - 2 \bar{\phi} (\sigma_{AB}) \phi + \frac{2(q - 1)}{l} \bar{\phi} \phi \right].
\] (C.10)

### Appendix D. Gauge fixing

There are zero-eigenvalue in the matrix (64) associated to gauge modes. In this appendix, we consider gauge-fixing of these modes. The BRST transforma-
tion is defined by
\[
\delta \left( -\frac{2}{l} L_i + X_i \right) = -i \left[ -\frac{2}{l} L_i + X_i, c \right], \quad (D.1)
\]
\[
\delta (\bar{\sigma} + \phi) = -i \left[ \bar{\sigma} + \phi, c \right], \quad (D.2)
\]
\[
\delta c = i \{ c, c \}, \quad (D.3)
\]
\[
\delta b = B, \quad (D.4)
\]
\[
\delta B = 0. \quad (D.5)
\]

The gauge-fixing action plus the ghost action is
\[
S_{GF+FP} = \delta \text{Tr}(bG) = \text{Tr} (BG + b\delta G). \quad (D.6)
\]

Here \( G \) is the gauge-fixing function
\[
G = \left[ \sigma_0, \phi \right] + \frac{2}{l} \left[ L_i, X_i \right], \quad (D.7)
\]
which is same as the eigenmodes associated to the zero eigenvalue. Integration over the \( B \) field generates the delta function constraint \( \delta (G) \) and actually fix the gauge modes. However, the integration measure of the gauge mode is normalized as \( \int dG/\sqrt{G} \), the normalization the delta-function \( \delta (G) = \frac{1}{\sqrt{G}} \delta (G/\sqrt{G}) \) gives additional factor:
\[
\prod_{s,t} \prod_{\alpha \in \Delta} j_{s+t} \prod_{J} \left\{ \left( \frac{2}{l} \right)^2 J(J+1) + (\alpha \cdot \sigma)^2 \right\}^{1/2} = \frac{1}{\sqrt{\det \Delta^2_{\sigma,X^0}}} \quad (D.8)
\]

Next we evaluate the Fadeev-Popov determinant. The BRST transformation of \( G \) is
\[
\delta G = -i [ \sigma_0, [ \sigma_0, c ] ] + i \left( \frac{2}{l} \right)^2 [ L_i, [ L_i, c ] ] \quad (D.9)
\]
and the ghost action can be written as
\[
\text{Tr} (b\delta G) = i \text{Tr} \left( -b [ \sigma_0, [ \sigma_0, c ] ] + \left( \frac{2}{l} \right)^2 b [ L_i, [ L_i, c ] ] \right) \quad (D.10)
\]
\[
= i \text{Tr} \left( [ \sigma_0, b ] [ \sigma_0, c ] + \left( \frac{2}{l} \right)^2 b [ L_i, [ L_i, c ] ] \right). \quad (D.11)
\]

The ghost fields are also expanded in terms of the scalar fuzzy sphere harmonics:
\[
c = \sum_{s,t} c^{(s,t)} = \sum_{s,t} \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{m=-J}^{J} \hat{Y}_{jm(s,t)} \otimes c^{(s,t)}_{jm}, \quad (D.12)
\]
\[
b = \sum_{s,t} b^{(s,t)} = \sum_{s,t} \sum_{J=|j_s-j_t|}^{j_s+j_t} \hat{Y}_{jm(s,t)} \otimes b^{(s,t)}_{jm}. \quad (D.12)
\]
Then we obtain the ghost action:

$$\text{Tr } (b\delta G) = i \sum_{s,t} \text{Tr} \left[ \left[ \sigma_0, b \right]^{(s,t)} \left[ \sigma_0, c \right]^{(t,s)} + \left( \frac{2}{l} \right)^2 b^{(s,t)} L_i \circ L_i \circ c^{(t,s)} \right]$$

$$= i \sum_{s,t} \sum_{\alpha,J,m} \left[ (\alpha \cdot \sigma)^2 b^{\alpha(s,t)} c^{\alpha(s,t)} + \left( \frac{2}{l} \right)^2 J(J+1) b^{\alpha(s,t)} c^{\alpha(s,t)} \right]$$

(D.13)

Integrating out $b^{\alpha(s,t)}$ and $c^{\alpha(s,t)}$, the 1-loop determinant for the ghosts is

$$\det \Delta_{\text{ghosts}} = \prod_{s,t} \prod_{\alpha \in \Delta} \prod_{j_s + j_t} \prod_{J} \left\{ \left( \frac{2}{l} \right)^2 J(J+1) + (\alpha \cdot \sigma)^2 \right\}$$

$$= \det \Delta_{\sigma,X}^2.$$  (D.14)

We can see that the one-loop determinant of $(\sigma,X^0)$, (D.8) and (D.14) are canceled out with each other.

**Appendix E. Detailed calculation of the 1-loop determinants**

In this appendix, we present the detail explanation for degeneracy counting of the one-loop determinants and derivation of (77) and (78).

**Appendix E.1. Gauge sector**

First of all, we estimate the one-loop determinant of the reduced YM action. The bosonic one-loop determinant consists of two parts.

- $X_{(s,t)}^{(s,t)} |_{\rho=+1}$

$$\det \Delta_{\rho=+1} = \prod_{s,t} \prod_{\alpha \in \Delta} \prod_{j_s + j_t} \prod_{J} \left\{ \left( \frac{2}{l} \right)^2 J(J+1) + (\alpha \cdot \sigma)^2 \right\}$$

$$= \prod_{\alpha \in \Delta} \prod_{s,t} \prod_{J=j_s-j_t/2} \left\{ \left( \frac{2}{l} \right)^2 J(J+1) + (\alpha \cdot \sigma)^2 \right\}^{2(2J+3)}.$$  (E.1)

If we take the limit $\frac{n}{\nu} \to \infty$, the determinant becomes

$$\det \Delta_{\rho=+1} = \prod_{\alpha \in \Delta} \prod_{s,t} \prod_{J=j_s-j_t/2} \left\{ \left( \frac{2}{l} \right)^2 J(J+1) + (\alpha \cdot \sigma)^2 \right\}^{2(2J+3)}.$$  (E.1)

In order to simplify the products, we count the degeneracy number of $(s,t)$ giving a same value of $J$. First, note that possible values of $(s-t)$ giving a same $J$ is $-2J, -2J+2, \cdots, 2J$. Next, the number of $(s,t)$ giving a fixed
value of $|s-t| = m$ is $(\nu - m)$. Thus, we obtain the degeneracy number as
\[
\sum_{m=-2J, m: \text{even}}^{2J} (\nu - m) = (2J + 1)\nu - 2J(J + 1) \quad \text{for} \quad 2J = \text{even}
\]
\[
\sum_{m=-2J, m: \text{odd}}^{2J} (\nu - m) = (2J + 1)\nu - 2J(J + 1) \quad \text{for} \quad 2J = \text{odd}.
\]  
(E.2)

Therefore, a part of the bosonic one-loop determinant is rewritten as
\[
\det \Delta^2_{X, s,t} |_{\rho=+1} = \prod_{\alpha \in \Delta_+} \prod_{J=0, 2J+1 \in \mathbb{Z}} \left\{ \left( \frac{2\nu}{l} \right)^{2(J+1)^2 + (\alpha \cdot \sigma)^2} \right\}^{2(2J+1)(\nu - 2J(J+1))}
\]
\[
= \prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} \left\{ \left( \frac{n+1}{l} \right)^2 + (\alpha \cdot \sigma)^2 \right\}^{2(n+2)\nu - (n+1)(n+1)/2}
\]
\[
= \prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} \left\{ \left( \frac{n+1}{l} \right)^2 + (\alpha \cdot \sigma)^2 \right\}^{2n+2(n+2)\nu - (n+1)(n+1)(n+1) - \nu - n(n+1)(n+2)}
\]  
(E.3)

The other part can be evaluated in a similar way.

- $X_{J,m,\rho=1}^{(s,t)}$

\[
\det \Delta^2_{X, s,t} |_{\rho=-1} = \prod_{s,t} \prod_{\alpha \in \Delta_+} \prod_{J=0, J+1 \in \mathbb{Z}} \prod_{m=-J}^{J} \left\{ \left( \frac{2\nu}{l} \right)^{2(J+1)^2 + (\alpha \cdot \sigma)^2} \right\}
\]
\[
= \prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} \left\{ \left( \frac{n-1}{l} \right)^2 + (\alpha \cdot \sigma)^2 \right\}^{2n+2(n-2)\nu - (n+1)(n-2)(n+1) - \nu - n(n+1)(n+2)}
\]  
(E.4)

Next we count the degeneracy of the fermionic one-loop determinant.
\( \lambda_{|\kappa=+1} \)

\[
\det \Delta_{\lambda_{|\kappa=+1}} = \prod_{s, t} \prod_{\alpha \in \Delta} \prod_{J=|j_s - j_t| m=-J+1/2}^{j_s + j_t - 1/2} \left\{ \frac{2}{7} \left( J - \frac{1}{2} \right) - i(\alpha \cdot \sigma) \right\}^{J+1/2}
\]

\[
= \prod_{s, t} \prod_{\alpha \in \Delta} \prod_{n=1}^{\infty} \left\{ \frac{1}{\left( \frac{n}{\ell} \right)^2} + (\alpha \cdot \sigma)^2 \right\}^{1/2} \prod_{\alpha \in \Delta_+} \prod_{J=|\alpha_+ - \alpha_-|}^{\infty} \left\{ \frac{1}{J} \right\}^{1/2} \prod_{n=1}^{\infty} \left\{ \frac{1}{\left( \frac{n}{\ell} \right)^2} + (\alpha \cdot \sigma)^2 \right\}^{\nu(n+1) - n^2(n+2)/2} \quad (E.5)
\]

\( \lambda_{|\kappa=-1} \)

\[
\det \Delta_{\lambda_{|\kappa=-1}} = \prod_{s, t} \prod_{\alpha \in \Delta} \prod_{J=|j_s - j_t|}^{j_s + j_t - 1/2} \prod_{m=-J}^{j_s + j_t - 1/2} \left\{ \frac{2}{7} \left( J - \frac{1}{2} \right) - i(\alpha \cdot \sigma) \right\}^{J+1/2}
\]

\[
= \prod_{s, t} \prod_{\alpha \in \Delta} \prod_{n=1}^{\infty} \left\{ \frac{1}{\left( \frac{n}{\ell} \right)^2} + (\alpha \cdot \sigma)^2 \right\} \prod_{\alpha \in \Delta_+} \prod_{J=|\alpha_+ - \alpha_-|}^{\infty} \left\{ \frac{1}{\left( \frac{n}{\ell} \right)^2} + (\alpha \cdot \sigma)^2 \right\}^{n(n+1) - n^2(n+2)/2} \quad (E.6)
\]

Combining these result \( E.3 \), \( E.4 \), \( E.5 \) and \( E.6 \) we obtain in the \( \nu \to \infty \) limit:

\[
\frac{\det \Delta_{\lambda_{|\kappa=+1}} \cdot \det \Delta_{\lambda_{|\kappa=-1}}}{\sqrt{\det \Delta_{\lambda_{|\rho=+1}} \cdot \sqrt{\det \Delta_{\lambda_{|\rho=-1}}}}} = \prod_{\alpha \in \Delta_+} \left\{ (\alpha \cdot \sigma)^2 \right\}^{2\nu} \prod_{n=1}^{\infty} \left\{ \frac{1}{\left( \frac{n}{\ell} \right)^2} + (\alpha \cdot \sigma)^2 \right\}^{2\nu}
\]

\[
= \prod_{\alpha \in \Delta_+} \left\{ (\alpha \cdot \sigma)^2 \right\}^{2\nu} \prod_{n=1}^{\infty} \left( \frac{1}{\left( \frac{n}{\ell} \right)^2} + (\alpha \cdot \sigma)^2 \right)^{2\nu}
\]

\[
= \prod_{\alpha \in \Delta_+} \left\{ (\alpha \cdot \sigma)^2 \right\}^{2\nu} (2\pi l)^{2\nu} \left\{ \frac{\sinh (\pi l (\alpha \cdot \sigma))}{\pi l (\alpha \cdot \sigma)} \right\}^{2\nu}
\]

\[
= \prod_{\alpha \in \Delta_+} (2 \sinh (\pi l (\alpha \cdot \sigma)))^{2\nu}. \quad (E.7)
\]
From the second line to the third line in the above equations, we used the following formula:

\[
\prod_{n=1}^{\infty} \left( \frac{n^2 + x^2}{n^2} \right) = \frac{\sinh(\pi x)}{\pi x},
\]  
(E.8)

\[
\prod_{n=1}^{\infty} n^2 = e^{2\zeta'(0)} = 2\pi,
\]  
(E.9)

\[
\prod_{n=1}^{\infty} c = e^{\zeta(0) \log c} = \frac{1}{\sqrt{c}}.
\]  
(E.10)

**Appendix E.2. Matter sector**

Next, we study the matter sector. The degeneracy counting for the matter one-loop determinant is parallel to the vector multiplet. The calculation for bosonic part is following:

- \((\phi, \bar{\phi})\)

\[
\det \Delta_\phi^2 = \prod_{s,t} \prod_{l_1, l_2} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{m=-J}^{J} \left[ \left( \frac{2}{l} \right)^2 J(J+1) + (\sigma I_1 - \tilde{\sigma} I_2)^2 + \frac{i(2q-2)}{l}(\sigma I_1 - \tilde{\sigma} I_2) + \frac{q(2-q)}{l^2} \right]
\]

\[
= \prod_{s,t} \prod_{l_1, l_2} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{m=-J}^{J} \left[ \left( \frac{2}{l} \right)^2 J(J+1) + (\sigma I_1 - \tilde{\sigma} I_2 + i\frac{q}{l}) \left( \sigma I_1 - \tilde{\sigma} I_2 + i\frac{q}{l} \right) \right]
\]

\[
= \prod_{l_1, l_2} \prod_{n=1}^{\infty} \left[ \left( n + \frac{1}{l} \right) + i(\sigma I_1 - \tilde{\sigma} I_2) \right] \left( n - \frac{1}{l} - i(\sigma I_1 - \tilde{\sigma} I_2) \right) \frac{n^2 - n(n-1)(n+1)/2}{l^2}
\]  
(E.11)

The contributions from the fermionic part are given by

- \((\psi, \bar{\psi})\)_{\kappa=+1}

\[
\det \Delta_\psi|_{\kappa=+1} = \prod_{s,t} \prod_{l_1, l_2} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{m=-(J-1/2)}^{J+1/2} \left\{ \left( \frac{2}{l} \right)^2 J + i(\sigma I_1 - \tilde{\sigma} I_2) - \frac{q-2}{l} \right\}
\]

\[
= \prod_{l_1, l_2} \prod_{n=1}^{\infty} \left\{ \left( \frac{n + 1}{l} \right) + i(\sigma I_1 - \tilde{\sigma} I_2) - \frac{q}{l} \right\}^{n(n+1)-(n-1)(n+1)/2}
\]  
(E.12)

- \((\psi, \bar{\psi})\)_{\kappa=-1}

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$$\det \Delta \psi \mid_{\kappa=-1} = \prod_{s,t} \prod_{\alpha \in \Delta} \prod_{J=|j_s-j_t|-1/2} \prod_{m=-J}^{J} \left( \frac{2}{t} \left( -\frac{3}{2} \right) + i(\sigma_{I_1} - \tilde{\sigma}_{I_2}) - \frac{q-2}{t} \right)$$

$$= \prod_{I_1,I_2} \prod_{n=1}^{\infty} \left( \frac{n-1}{t} + i(\sigma_{I_1} - \tilde{\sigma}_{I_2}) - \frac{q}{t} \right)^{n\nu(n-1)-(n-1)^2(n+1)/2}$$

(E.13)

Thus, in the $\nu \to \infty$, we obtain

$$\frac{\det \Delta \psi \mid_{\kappa=+1} \det \Delta \psi \mid_{\kappa=-1}}{\det \Delta \psi} = \prod_{I_1,I_2} \prod_{n=1}^{\infty} \left[ \frac{n+1}{t} - \frac{1}{2} + i(\sigma_{I_1} - \tilde{\sigma}_{I_2}) \right]^{n\nu}$$

$$= \prod_{I_1,I_2} s_{I_1}(i - iq - l(\sigma_{I_1} - \tilde{\sigma}_{I_2})).$$

(E.14)

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