The higher sharp IV: the higher levels

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Abstract

We establish the descriptive set theoretic representation of the mouse \( M_\#^n \), which is called \( 0^{(n+1)#} \). This part deals with the case \( n > 3 \).

1 Introduction

This is the final part of a series starting with [8]. In this paper, we generalize the previous three papers to the higher levels in the projective hierarchy. Section 2 makes the purely syntactical definitions on trees of uniform cofinalities and descriptions that will show up in the higher levels. Section 3 writes down all the inductive definitions and hypotheses under \( \Delta^1_{2n} \)-determinacy. Section 4 proves a part of the inductive hypotheses in Section 3 under \( \Pi^1_{2n+1} \)-determinacy. Section 5 proves the rest of inductive hypotheses under \( \Delta^1_{2n+2} \)-determinacy, thereby finishing a cycle of the induction.

As introduced in [10, Section 1.1], a technical component in the level-2 and level-3 analysis is a self-similar stack of definitions. This stack grows to the higher levels. The whole picture is in Fig. 1. Every node has a distinguished name in this diagram and denotes a tree of uniform cofinality. The number denotes the level of the tree, e.g. 5a denotes a level-5 (or level \( \leq 5 \), to be exact) tree of uniform cofinality. An arrow stands for a factoring
Figure 1: The longer stack of definitions
map. A solid line stands for membership, e.g. 4a is the tree component of an entry of 5a. The stack of definitions goes in the following order:

- 1c-description,
- (1a, 1c)-factoring map,
- (2a, 1c)-description,
- (1b, 2a, 1c)-factoring map,
- (2b, 2a, 1c)-description,
- (2c, 2b, 2a)-factoring map,
- (3c, 2b, 2a)-description,
- (3a, 3c, 2b)-factoring map,
- (4a, 3c, 2b)-description,
- (3b, 4a, 3c)-factoring map,
- (4b, 4a, 3c)-description,
- (4c, 4b, 4a)-factoring map,
- (5c, 4b, 4a)-description,
- (5a, 5c, 4b)-factoring map,
- (6a, 5c, 4b)-description,
- (5b, 6a, 5c)-factoring map,
- (6b, 6a, 5c)-description,
- (6c, 6b, 6a)-factoring map,
- (7c, 6b, 6a)-description,
- (7a, 7c, 6b)-factoring map,
- ....
2  Syntactical definitions at the higher levels

The definitions related to trees of uniform cofinalities are purely syntactical. They will be defined inductively. The base of the inductive definition are in \([8–10]\). For notational consistency, a level-1 tree \(P\) will be identified with a function on \(P\) with constant value \(\emptyset\). A level \(\leq 1\) tree is \(P' = (P)\) where \(P\) is a level-1 tree. If \(P' = (P)\) is a level \(\leq 1\) tree, put \(P'' = \overline{1}P'\) and put \(\text{dom}(P') = \{(1, p) : p \in P\}\). \(P''\) is regarded as a function on \(\text{dom}(P'')\) with constant value \(\emptyset\). A partial level \(\leq 1\) tree \((P, p)\) will be identified with \(((P), (d, p, \emptyset))\), where \(d = 0\) if \(p = -1\), \(d = 1\) if \(p \neq -1\). A potential partial level \(\leq 1\) tower \((P, (p_i)_{i \leq k})\) will be identified with \(((P), ((d, p_i, \emptyset)_{i \leq k})\) where \(d_i = 1\) for any \(i < k\), \(d_k = 0\) if \(p_k = -1\), \(d_k = 1\) if \(p_k \neq -1\).

A level \(\leq 2n + 1\) tree \(P\) is \textit{regular} iff \(2n+1P\) is regular and if \(2n+1P = \emptyset\), then \(\text{dom}(P) = \{(2i, \emptyset) : 1 \leq i \leq n\}\). A \textit{partial level} \(\leq 2n + 1\) tree is a pair \((P, (d, p, Z))\) such that \(P\) is a finite regular level \(\leq 2n + 1\) tree and either

1. \((d, p) \notin \text{dom}(P)\), there is a regular level \(\leq 2n + 1\) tree \(P'\) extending \(P\) such that \(\text{dom}(P') = \text{dom}(P) \cup \{(d, p)\}\) and \(\overline{d}P''\text{tree}(p) = Z\), or

2. \(2n+1P \neq \emptyset\), \(d = 0\), \(p = -1\), \(Z = \emptyset\).

The \textit{degree} of \((P, (d, p, Z))\) is \(d\). Put \(\text{dom}(P, (d, p, Z)) = \text{dom}(P) \cup \{(d, p)\}\). If \(d > 0\), a \textit{completion} of \((P, (d, p, Z))\) is a level \(\leq 2n + 1\) tree \(P'\) extending \(P\) such that \(\text{dom}(P') = \text{dom}(P) \cup \{(d, p)\}\) and \(\overline{d}P''\text{tree}(p) = Z\). The \textit{uniform cofinality} of \((P, (d, p, Z))\) is

\[
\text{ucf}(P, (d, p, Z)),
\]

defined as follows:

1. \(\text{ucf}(P, (d, p, Z)) = \text{ucf}(\leq 2nP, (d, p, Z))\) if \(d \leq 2n\);
2. \(\text{ucf}(P, (d, p, Z)) = (0, -1)\) if \(n = 1\), \(d = 0\);
3. \(\text{ucf}(P, (d, p, Z)) = (1, p^-)\) if \(n = d = 1\);
4. \(\text{ucf}(P, (d, p, Z)) = (2n + 1, (p', Z, (e, z, \overline{Q})))\) if \(n \leq 1\), \(d = 2n + 1\), \(2n+1P[p'] = (Z, (e, z, \overline{Q})),\) and \(p'\) is the \(<_{BK}\)-least element of \(2n+1\{p, +, Z\}\);
5. \(\text{ucf}(P, (d, p, Z)) = (2n + 1, (p^-, Z, (e, z, \overline{Q})))\) if \(n \leq 1\), \(d = 2n + 1\), \(p \neq ((0)), 2n+1P[p^-] = (Z^-, (e, z, \overline{Q})),\) and \(2n+1\{p, +, Z\} = \emptyset\);
6. \(\text{ucf}(P, (d, p, Z)) = (2n + 1, \emptyset)\) if \(n \geq 1\), \(d = 2n + 1\), \(p = ((0))\).
A partial level $\leq 2n + 1$ tower of discontinuous type is a nonempty finite sequence $\langle P_i \rangle_{i \leq k} = (P_i, (d_i, p_i, Z_i))$ such that each $(P_i, (d_i, p_i, Z_i))$ is a partial level $\leq 2n + 1$ tree, and $P_{i+1}$ is a completion of $(P_i, (d_i, p_i, Z_i))$. Its signature is $((d_i, p_i))_{i \leq k}$. Its uniform cofinality is $\operatorname{ucf}(P_k, (d_k, p_k, Z_k))$. A partial level $\leq 2n + 1$ tower of continuous type is $\langle P_i \rangle_{i \leq k} = (P_i, (d_i, p_i, Z_i))$ such that either $k = 0 \wedge 2^i + 1 P_i = \emptyset$ or $(P_i, (d_i, p_i, Z_i))$ is a partial level $\leq 2n + 1$ tower of discontinuous type $\land P_*$ is a completion of $(P_{k-1}, (d_{k-1}, p_{k-1}, Z_{k-1}))$. Its signature is $(d_i, p_i)_{i \leq k}$. When $k > 0$, its uniform cofinality is $\langle (1, q_k) \rangle_{i \leq k}$ if $d_k = 1$, $(d_{k-1}, (p_{k-1})^* \downarrow d_k, p_{k-1})$ if $d_{k-1} > 1$.

A potential partial level $\leq 2n + 1$ tower is $\langle P_i \rangle_{i \leq k}$ such that for some $P = (P_i)_{i \leq k}$, either $P_* = P_k \wedge (\overline{P_i \langle d, p, Z \rangle})$ is a partial level $\leq 2n + 1$ tower of discontinuous type or $(\overline{P_i \langle d, p, Z \rangle})^* (P_*)$ is a potential partial level $\leq 2n + 1$ tower of continuous type. The signature, (dis-)continuity type, uniform cofinality of $(P_*, \overline{P_i \langle d, p, Z \rangle})$ are defined according to the partial level $\leq 2n + 1$ tree generating $(P_*, \overline{P_i \langle d, p, Z \rangle})$.

$$\operatorname{ucf}(P_*, \overline{P_i \langle d, p, Z \rangle})$$

denotes the uniform cofinality of $(P_*, \overline{P_i \langle d, p, Z \rangle})$. If $(P_* \langle d_i, p_i, Z_i \rangle)_{i \leq k}$ is a potential partial level $\leq 2n + 1$ tower of discontinuous type, then $P^*$ is one of its completions iff $P^*$ is a completion of $(P_*, (d_k, p_k, Z_k))$.

A level-$2n + 2$ tree is a function $Q$ such that $\operatorname{dom}(Q)$ is a tree of level-1 trees, $\emptyset \in \operatorname{dom}(Q)$ and for any $q \in \operatorname{dom}(Q)$, $(Q(q \cup i))_{i \leq \ell(q)}$ is a partial level $\leq 2n + 1$ tower of discontinuous type. In particular, $Q(\emptyset) = (P, (2n+1, (0)), Z))$ where $\operatorname{dom}(P) = \operatorname{dom}(Z) = \{(2i, \emptyset) : 1 \leq i \leq n\}$. If $Q(q) = (P_q, (d_q, p_q, Z_q))$, we denote $Q_{\text{tree}}(q) = P_q$, $Q_{\text{node}}(q) = (d_q, p_q)$, $Q[q] = (P_q, (d_{q*}, p_{q*}, Z_{q*}))_{i \leq \ell(q)}$. So $Q[q]$ is a potential partial level $\leq 2n + 1$ tower of discontinuous type. If $P$ is a completion of $Q(q)$, put $Q[q, P] = (P, (d_{q*}, p_{q*}, Z_{q*}))_{i \leq q}$, which is a potential partial level $\leq 2n + 1$ tower of continuous type. For $q \in \operatorname{dom}(Q)$, put $Q[q] = \{a \in \omega^\omega : q^<(a) \in \operatorname{dom}(Q)\}$, which is a level-1 tree; if $P$ is a level $\leq 2n + 1$ tree, put $Q[q, P] = \{a \in Q[q] : Q_{\text{tree}}(q^<(a)) = P\}$.

For $Q$ a level-$2n + 2$ tree, let $\operatorname{dom}^*(Q) = \operatorname{dom}(Q) \cup \{q^<(a) : q \in \operatorname{dom}(Q)\}$. If $q \neq \emptyset$, denote $Q[q, -] = \{q^<(a) : q_\emptyset^< a < BK q(\ell(q) - 1)\}$, $Q[q, +, P] = \{q^<(a) : Q_{\text{tree}}(q^<(a)) = P \land a > BK q(\ell(q) - 1)\}$. A $Q$-description is a triple

$$q = (q, P, \overline{P_i \langle d, p, Z \rangle})$$

such that $q \in \operatorname{dom}^*(Q)$ and either $q$ is of discontinuous type $\land (P, \overline{P_i \langle d, p, Z \rangle}) = \emptyset$.
$Q[q]$ or $q$ is of continuous type $\land(P, (d, p, Z)) = Q[q^-, P]$. A $Q$-description $(q, P, (d, p, Z))$ is of (dis-)continuous type iff $q$ is of (dis-)continuous type. The constant $Q$-description is $(\emptyset)^{-} Q[\emptyset]$. If a $Q$-description $q = (q, P, (d, p, Z))$ is of discontinuous type and $P^+$ is a completion of $Q(q)$, then $q^{-} (-1, P^+) = (q^{-} (-1), P^+, (d, p, Z))$. $Q$ is $\Pi^1_{2n+2}$-wellfounded iff

1. $\forall q \in \text{dom}(Q) \ Q\{q\} \in \Pi^1_{1}$-wellfounded,

2. $\forall y \in [\text{dom}(Q)] \ Q(y) \equiv \cup_n \cup_{n<\omega} Q_{\text{tree}}(y | n)$ is not $\Pi^1_{2n+1}$-wellfounded.

A level $\leq 2n+2$ tree is a tuple $Q = (4Q, \ldots, 2^{n+2} Q)$ such that $4Q$ is a level-$d$ tree for $1 \leq d \leq 2n+2$. $4^Q$ always stands for the level-$d$ component of a level $\leq 2n+2$ tree $Q$. $4^Q$ denotes the level $\leq d$ tree $(4Q, \ldots, 4Q)$. $\text{dom}(Q) = \cup_d (\{d\} \times \text{dom}^*(4Q))$. $Q$ is regarded as a function sending $(d, q) \in \text{dom}(Q)$ to $4^Q(q)$. $\text{dom}^*(Q) = \cup_d (\{d\} \times \text{dom}^*(4Q))$. $\text{desc}(Q) = \cup_d (\{d\} \times \text{desc}^*(4Q))$ is the set of $Q$-descriptions. $(d, q) \in \text{desc}(Q)$ is of continuous type iff $d \geq 2$ and $q$ is of continuous type; otherwise, $(d, q)$ is of discontinuous type. $Q$ is $\Pi^1_{2n+2}$-wellfounded iff $4^{\leq n+1} Q$ is $\Pi^1_{2n+1}$-wellfounded and $4^{n+2} Q$ is $\Pi^1_{2n+2}$-wellfounded.

Suppose $Q$ is a level $\leq 2n+2$ tree. An extended $Q$-description is either a $Q$-description or of the form $(d, (q, P, (e, p, Z)))$ such that $(d, (q^{-} (-1), P, (e, p, Z)))$ is a $Q$-description of continuous type. $\text{desc}^*(Q)$ is the set of extended $Q$-descriptions. $(d, q) \in \text{desc}^*(Q)$ is regular iff either $(d, q) \in \text{desc}(Q)$ of discontinuous type or $(d, q) \notin \text{desc}(Q)$.

A partial level $\leq 2n+2$ tree is a pair $(Q, (d, q, P))$ such that $Q$ is a finite level $\leq 2n+2$ tree, and either

1. $(d, q) \notin \text{dom}(Q)$, there is a level $\leq 2n+2$ tree $Q^+$ extending $Q$ such that $\text{dom}(Q^+) = \text{dom}(Q) \cup \{(d, q)\}$ and $4^Q_{\text{tree}}(q) = P$, or

2. $(d, q, P) = (0, -1, \emptyset)$.

The degree of $(Q, (d, q, P))$ is $d$. If $d > 0$, a completion of $(Q, (d, q, P))$ is a level $\leq 2n+2$ tree $Q^+$ extending $Q$ such that $\text{dom}(Q^+) = \text{dom}(Q) \cup \{(d, q)\}$ and $4^Q_{\text{tree}}(q) = P$. When $n = 1$, the uniform cofinality of $(Q, (d, q, P))$ has been defined in [10]. When $n \geq 1$, the uniform cofinality of $(Q, (d, q, P))$ is

$$\text{ucf}(Q, (d, q, P)),$$

defined as follows:

1. $\text{ucf}(Q, (d, q, P)) = \text{ucf}(\leq 2n Q, (d, q, P))$ if $d \leq 2n$;

2. $\text{ucf}(Q, (d, q, P)) = (2n+2, (\emptyset)^{-} 2n+2 Q[\emptyset])$ if $d = 2n+1, q = \max_{<B_k} (\text{dom}(2n+1 Q))$.
3. \( \text{ucf}(Q, (d, q, P)) = (d, (q', P, \overline{(e, p, Z)})) \) if \( 2n + 1 \leq d \leq 2n + 2 \), \( 4Q[q'] = (P, (e, p, Z)) \), and \( q' \) is the \( <_{BK} \)-least element of \( Q\{q, +, P\}, q' \neq q^-; \)

4. \( \text{ucf}(Q, (d, q, P)) = (d, (q^-, P, \overline{(e, p, Z)})) \) if \( 2n + 1 \leq d \leq 2n + 2 \), \( 4Q[q^-] = (P^-, (e, p, Z)) \), and \( 4Q\{q, +, P\} = \{q^-\}. \)

A partial level \( \leq 2n + 2 \) tower of discontinuous type is a nonempty finite sequence \( (Q_i, (d_i, q_i, P_i))_{1 \leq i < k} \) such that \( \text{dom}(Q_1) = \{(2i, \emptyset) : 1 \leq i \leq n + 1\} \), each \( (Q_i, (d_i, q_i, P_i)) \) is a partial level \( \leq 2n + 2 \) tree, and each \( Q_{i+1} \) is a completion of \( (Q_i, (d_i, q_i, P_i)) \). Its signature is \( (d_i, q_i)_{1 \leq i < k} \). Its uniform cofinality is \( \text{ucf}(Q_k, (d_k, q_k, P_k)) \). A partial level \( \leq 2n + 2 \) tower of continuous type is \( (Q_i, (d_i, q_i, P_i))_{1 \leq i < k} \) such that either \( k = 0 \land Q_* \) is the level \( \leq 2n + 2 \) tree with domain \( \{(2j, \emptyset) : 1 \leq j \leq n + 1\} \) or \( (Q_i, (d_i, q_i, P_i))_{1 \leq i < k} \) is a partial level \( \leq 2n + 2 \) tower of discontinuous type \( \land Q_* \) is a completion of \( (Q_{k-1}, (d_{k-1}, q_{k-1}, P_{k-1})) \). Its signature is \( (d_i, q_i)_{1 \leq i < k} \). If \( k > 0 \), its uniform cofinality is \( (1, q_{k-1}) \) if \( d_{k-1} = 1 \), \( (d_{k-1}, (q_{k-1} \overline{\ell_{d_{k-1}} - Q[q_{k-1}]} )) \) if \( d_{k-1} > 1 \). A potential partial level \( \leq 2n + 2 \) tower is \( (Q_i, (d, q, P)) \) such that for some \( \tilde{Q} = (Q_i)_{1 \leq i < k} \), either \( Q_* = Q_k \setminus (\tilde{Q} (d, q, P)) \) is a partial level \( \leq 2n + 2 \) tower of discontinuous type or \( (\tilde{Q} (d, q, P))^{-1} (Q_*) \) is a partial level \( \leq 2n + 2 \) tower of continuous type. The signature, (dis-)continuity type, uniform cofinality of \( (Q_*, (d, q, P)) \) are defined according to the partial level \( \leq 2n + 2 \) tree generating \( (Q_*, (d, q, P)) \).

\[
\text{ucf}(Q_*, (d, q, P))
\]
denotes the uniform cofinality of \( (Q_*, (d, q, P)) \).

A level-(2n+3) tree is a function \( R \) such that \( \emptyset \notin \text{dom}(R), \text{dom}(R) \cup \{\emptyset\} \) is a tree of level-1 trees and for any \( r \in \text{dom}(R), (R(r \mid l))_{1 \leq l \leq \ell_R(r)} \) is a partial level \( \leq 2n + 2 \) tower of discontinuous type. If \( R(r) = (Q_r, (d_r, q_r, P_r)) \), we denote \( R_{\text{tree}}(r) = Q_r, R_{\text{node}}(r) = (d_r, q_r), R[r] = (Q_r, (d_{\vec{r}}, q_{\vec{r}}, P_{\vec{r}}))_{1 \leq l \leq \ell_R(r)} \). \( R[r] \) is a potential partial level \( \leq 2n + 2 \) tower of discontinuous type. If \( Q \) is a completion of \( R(r) \), put \( R[r, Q] = (Q, (d_{\vec{r}}, q_{\vec{r}}, P_{\vec{r}}))_{1 \leq l \leq \ell_R(r)} \), which is a potential partial level \( \leq 2n + 2 \) tower of continuous type. For \( r \in \text{dom}(R) \cup \{\emptyset\} \), put \( R\{r\} = \{a \in \omega^{<\omega} : r^{-1}(a) \in \text{dom}(R)\} \), which is a level-1 tree. For \( r \in \text{dom}(R) \) and a level \( \leq 2n + 2 \) tree \( Q \), put \( R\{r, Q\} = \{a \in R\{r\} : R_{\text{tree}}(r^{-1}(a)) = Q\} \). \( R \) is regular iff \( \{(1)\} \notin \text{dom}(R) \).

Suppose \( R \) is a level-(2n+3) tree. Let \( \text{dom}^*(R) = \text{dom}(R) \cup \{r^{-1}(-1) : r \in \text{dom}(R)\} \). For \( r \in \text{dom}^*(R) \), \( r \) is of continuous type if \( r \in \text{dom}(R) \) \( \land \) \( \text{dom}(R) \). For \( r \in \text{dom}(R), \) put \( R\{r, -\} = \{r^{-1}(-1)\} \cup \{r^{-1}(a) : R_{\text{tree}}(r^{-1}(a)) = R_{\text{tree}}(r), a <_{BK} r(\ell_R(r) - 1)\} \). For
$r \in \text{dom}(R)$ and a level $\leq 2n + 2$ tree $Q$, put $R\{r, +, Q\} = \{r^\dashv\} \cup \{r^\dashv(a) : R_{\text{tree}}(r^\dashv(a)) = Q, a >_{BK} r(\ell h(r) - 1)\}$. The constant $R$-description is $\emptyset$, which is of discontinuous type. An $R$-description is either the constant $R$-description or a triple

$$r = (r, Q, (d, q, P))$$

such that $r \in \text{dom}^+(R)$ and either $r$ is of discontinuous type $\land (Q, (d, q, P)) = R[r]$ or $r$ is of continuous type $\land (Q, (d, q, P)) = R[r^\dashv, Q]$. A non-constant $R$-description $(r, Q, (d, q, P))$ is of (dis-)continuous type iff $r$ is of (dis-)continuous type. If an $R$-description $r = (r, Q, (d, q, P))$ is of discontinuous type and $Q^+$ is a completion of $R(r)$, then $r^\dashv(−1, Q^+) = (r^\dashv(−1), Q^+, (d, q, P))$. An extended $R$-description is either an $R$-description or a triple $(r, Q, (d, q, P))$ such that $(r^\dashv(−1), Q, (d, q, P))$ is an $R$-description of continuous type. desc$^*(R)$ is the set of extended $R$-descriptions. An extended $R$-description $r$ is regular iff either $r \in \text{desc}(R)$ of discontinuous type or $r \notin \text{desc}(R)$. A generalized $R$-description is either $(\emptyset, \emptyset, \emptyset)$ or of the form

$$A = (r, \pi, T)$$

so that $r = (r, Q, (d, q, P)) \in \text{desc}(R) \setminus \{\emptyset\}$, $T$ is a finite level $\leq 2n + 2$ tree, $\pi$ factors $(Q, T)$. desc$^{**}(R)$ is the set of generalized $R$-descriptions.

$R$ is $\Pi^1_{2n+3}$-wellfounded iff

1. $\forall r \in \text{dom}(R) \cup \{\emptyset\} \ R\{r\}$ is $\Pi^1_{2n+3}$-wellfounded, and
2. $\forall z \in [\text{dom}(R)] \ R(z) =_{\text{DEF}} \cup_{n<\omega} (R_{\text{tree}}(z | n))_{1 \leq n < \omega}$ is not $\Pi^1_{2n+2}$-wellfounded.

If $R$ is a level-$((2n + 3)$ tree, $L^R$ is the language $\{\leq_r, c_r : r \in \text{dom}(R)\}$, and $L^{z,2}$ is the language $L^R \cup \{z\}$.

A level $\leq 2n + 3$ tree is a tuple $R = (1R, \ldots, 2n+3R)$ such that $\downarrow dR$ is a level-$d$ tree for any $d$. $\leq dR$ stands for the level-$d$ component of $R$. $\leq dR$ stands for the level $\leq d$ tree $(1R, \ldots, 4R)$. If $Z$ is a level $\leq 2n + 2$ tree and $W$ is a level-$2(n + 3)$ tree, then $Z \oplus W$ denotes the level $\leq 2n + 3$ tree $(1Z, \ldots, 2n+2Z, W)$. A level $\leq 2n + 3$ tree $R$ is $\Pi^1_{2n+3}$-wellfounded iff $\leq 2n+3R$ is $\Pi^1_{2n+3}$-wellfounded and $2n+3R$ is $\Pi^1_{2n+3}$-wellfounded. If $R$ is level $\leq 2n + 3$ tree, define $\text{dom}(R) = \cup_d\{d\} \times \text{dom}(\downarrow dR), \text{desc}(R) = \cup_d\{d\} \times \text{desc}(\downarrow dR)$. $R$ is regarded as a function sending $(d, r)$ to $\downarrow dR(r)$.

If $R$ is a level $\leq m$ tree and $(d, r) \in \text{dom}(R)$, $d > 1$, put $R[d, r] = 4R[r]$.

Suppose $\sigma$ factors level-1 trees $(P, W)$. If $\sigma \in P$, then $(\sigma, W)$ is continuous at $(1, p)$ iff either $\sigma(p) = \min(\downarrow W)$ or pred$\downarrow W(\sigma(p)) \in \text{ran}(\sigma)$; otherwise $(\sigma, W)$ is discontinuous at $(1, p)$.
Suppose $Q, T$ are level-$m$ trees. A map $\pi$ is said to factor $(Q, T)$ iff $\text{dom}(\pi) = \text{dom}(Q)$, $q <_{BK} q' \iff \pi(q) <_{BK} \pi(q')$, $q \subseteq q' \iff \pi(q) \subseteq \pi(q')$, and for any $q \in \text{dom}(Q)$, $Q(q) = T(\pi(q))$. If $\pi$ factors $(Q, T)$, $\pi$ is allowed to act on extended $Q$-descriptions as well. If $m > 1$ and $q = (q, P, (d, p, Z)) \in \text{desc}^*(Q)$ then

$$\pi(q) = \begin{cases} (\pi(q), P, (d, p, Z)) & \text{if } q \text{ is of discontinuous type,} \\ (\pi(q)^{-}\rightarrow(-1), P, (d, p, Z)) & \text{otherwise.} \end{cases}$$

Suppose $Q, T$ are level $\leq m$ trees. $\pi$ is said to factor $(Q, T)$ iff $\text{dom}(\pi) = \text{dom}(Q)$ and there is $(\ell_{n})_{1 \leq d \leq m}$ such that for any $d$, $\ell_{n}$ factors $(\ell_{d}, T)$ and $\pi(d, q) = (d, \ell_{n}(q))$ for any $q \in \text{dom}(\ell_{Q})$. If $\pi$ factors $(Q, T)$, $\ell_{n}$ has this fixed meaning. Suppose $Q, T$ are both finite and suppose $\pi$ factors $(Q, T)$, $\pi$ is allowed to act on extended $Q$-descriptions as well. If $(d, q) \in \text{desc}^*(Q)$, then $\pi(d, q) = (d, \ell_{n}(q))$. level-$m$ tree isomorphisms and level $\leq m$ tree isomorphisms have obvious definitions. If $Q$ is a level-$m$ or a level $\leq m$ tree, $\text{id}_Q$ is the identity tree isomorphism between $Q$ and itself. If $\pi$ factors $(Q, T)$ and $\beta = (\beta_{d, t})_{d, t} \in \text{dom}(T)$ is a tuple indexed by $\text{dom}(T)$, then $\beta_{d, t} = (\beta_{d, t}(q))_{q \in \text{dom}(Q)}$, where $\beta_{d, t}(q) = \beta_{d, t}(q)_{q \in \text{dom}(Q)}$.

Suppose $Q, T$ are level $\leq m$ trees and $\pi$ factors $(Q, T)$. $(\pi, T)$ is said to be discontinuous at $(0, -1)$. Suppose $(d, q) \in \text{desc}^*(Q)$ is regular. $(\pi, T)$ is continuous at $(d, q)$ iff one of the following holds:

1. $d = 1$, either $\ell_{n}(q) = \min(q^{T})$ or $\text{pred}_{\pi} (\ell_{n}(q)) \in \text{ran}(\pi)$.
2. $d = 2i$, $q = (\emptyset, \ldots) \in \text{desc}^*(Q)$, either $2i - 1 = \emptyset$ or $\text{max}_{BK} (\text{dom}(2i - 1)) \in \text{ran}(\ell_{n})$.
3. $d > 1$, $q = (q, \ldots) \notin \text{desc}^*(Q)$, $q \neq \emptyset$, and letting $t' = \max_{BK} (\ell_{n}(q) \cup \{-1\})$, either $t' = \ell_{n}(q)^{-}\rightarrow(-1)$ or $t' \in \text{ran}(\ell_{n})$.
4. $d > 1$, $q = (q, P, \ldots) \notin \text{desc}^*(Q)$, and letting $a = \max_{BK} \ell_{n}(q) \cup \{-1\})$, then either $a = -1$ or $\ell_{n}(q)(a) \in \text{ran}(\ell_{n})$.

Otherwise, $(\pi, T)$ is discontinuous at $(d, q)$. If $(\pi, T)$ is discontinuous at $(d, q)$, the decomposition of $(\pi, T)$ is $(\pi^{+}, Q^{+})$ such that $Q^{+}$ is a level $\leq m$ tree extending $Q$, $\pi^{+}$ factors $(Q^{+}, T)$, $\pi^{+}$ extends $\pi$, and

1. if $d = 1$, then $\text{dom}(Q^{+}) \setminus \text{dom}(Q) = \{(1, q^{+})\}$, $q = q^{+} \upharpoonright 1Q$, $\ell_{n}(q^{+}) = \text{succ}_{\pi}(\ell_{n}(\text{pred}_{\pi}(q^{+})))$;
2. if $d = 2$ and $q = (\emptyset, \emptyset, ((0)))$, then $\text{dom}(Q^{+}) \setminus \text{dom}(Q) = \{(1, q^{+})\}$, $q = q^{+} \upharpoonright 1Q$, $\ell_{n}(q^{+}) = \min_{\pi}(a : \forall r \in \text{dom}(Q) \ell_{n}(r) \not\in r^T a)$.
3. if \( d = 2i > 2 \) and \( q = (\emptyset, \ldots) \in \text{desc}(2^{n}Q) \), then \( \text{dom}(Q^{+}) \setminus \text{dom}(Q) = \{(2i - 1, q^{+})\} \), \( \text{lh}(q^{+}) = 1 \), \( \emptyset = q^{+}(0) \upharpoonright 2^{i-1}Q(\emptyset), \ 2^{i-1} \pi^{+}(q^{+}) = T(t^{+}), \ \text{lh}(t^{+}) = 1 \), \( t^{+}(0) = \min_{\leq 2^{i-1}T} \{a : \forall r \in \text{dom}(2^{i-1}Q) \ 2^{i-1} \pi(r) \prec 2^{i-1}T(\emptyset) (a)\}; \)

4. if \( d > 1 \) and \( q = (q, P, \ldots) \in \text{desc}(4^{n}Q), q \neq \emptyset \), then \( \text{dom}(Q^{+}) \setminus \text{dom}(Q) = \{(d, q^{+})\}, q^{+} = \max_{<_{BK}^{\pi}(q, -)}^{Q^{+}} \{q, -\}, \text{ and } d_{\pi^{+}}(q^{+}) = d_{\pi}(q^{+}) \prec (a), a = \min_{<_{BK}^{\pi}} \{b : d_{Q_{\text{tree}}}(q^{+}(a)) = P \wedge \forall r \in Q(q, -) \{q^{+}(a)\} d_{\pi}(r) <_{BK} d_{\pi}(q^{+}(b))\}. \)

5. if \( d > 1 \) and \( q = (q, P, \ldots) \notin \text{desc}(4^{n}Q) \), then \( \text{dom}(Q^{+}) \setminus \text{dom}(Q) = \{(d, q^{+})\}, q^{+} = q^{\prec} \{\max_{<_{BK}^{\pi}} d_{Q^{+}}(q)\}, d_{\pi^{+}}(q^{+}) = d_{\pi}(q^{+}) \prec (a), a = \min_{<_{BK}^{\pi}} \{b : d_{Q_{\text{tree}}}(q^{+}(a)) = P \wedge \forall c \in Q(q, P) d_{\pi}(q^{+}(c)) <_{BK} d_{\pi}(q^{+}(b))\}. \)

If \((\pi, T)\) is discontinuous at \((d, q)\), then \(\text{pred}(\pi, T, (d, q))\) is a node in \(\text{dom}(T)\) defined as follows:

1. If \( d = 1 \), then \(\text{pred}(\pi, T, (d, q)) = (1, \text{pred}_{<_{TR}}(1_{\pi}(q))).\)

2. If \( d = 2i \) and \( q = (\emptyset, \ldots) \in \text{desc}(2^{n}Q) \), then \(\text{pred}(\pi, T, (d, q)) = (2i - 1, \max_{<_{BK}^{\pi}} \text{dom}(2^{n-1}T)).\)

3. If \( d > 1 \) and \( q = (q, \ldots) \in \text{desc}(Q), q \neq \emptyset \), then \(\text{pred}(\pi, T, (d, q)) = (d, \max_{<_{BK}^{\pi}} d^{T}(\beta_{\pi}(q), -)).\)

4. If \( d > 1 \) and \( q = (q, P, \ldots) \notin \text{desc}(Q) \), then \(\text{pred}(\pi, T, (d, q)) = (d, q^{\prec}(a)), a = \max_{<_{BK}^{\pi}} d^{T}(\beta_{\pi}(q), P).\)

If \((d, q) = (d, (q, \ldots)) \in \text{desc}(Q)\) then put \(\text{pred}(\pi, T, (d, q)) = \text{pred}(\pi, T, (d, q)).\)

Suppose \(R\) is a finite level \(\leq 2n+1\) tree. For \(A = (r, \pi, T) \in \text{desc}^{*}(2^{n+1}R)\), define its uniform cofinality

\[
\text{ucf}(A)
\]

as follows: If \( r = \emptyset \) then \(\text{ucf}(A) = A\). Suppose \( r = (r, Q, (d, q, P)) \neq \emptyset \), \(\text{lh}(r) = k\).

1. If \( r \) is of continuous type and \((\pi, T)\) is continuous at \((d_{k-1}, q_{k-1})\), then \(\text{ucf}(A) = (r^{-}, R_{\text{tree}}(r^{-}), (d, q, P)).\)

2. If \( r \) is of continuous type and \((\pi, T)\) is continuous at \((d_{k-1}, q_{k-1})\), then \(\text{ucf}(A) = (r^{-}, Q, (d, q, P)).\)

3. If \( r \) is of discontinuous type and \((\pi, T)\) is continuous at \((d_{k-1}, q_{k-1})\), then \(\text{ucf}(A) = r\).
4. If \( r \) is of discontinuous type and \((\pi, T)\) is continuous at \((d_{k-1}, q_{k-1})\), then \(\text{ucf}(A) = (r, Q^+, (d, q, P))\), where \((Q^+, \pi^+)\) is the decomposition of \((\pi, T)\).

We fix the notation for the trivial level \(\leq 2n\) tree:

- \(Q_0^{(2n)}\) is the level \(\leq 2n\) tree with domain \(\{(2i, \emptyset) : 1 \leq i \leq n\}\).

To be consistent with the higher levels, we rename some definitions in [10] concerning descriptions.

Suppose \(Q\) is a level \(\leq 2\) tree and \(W\) is a level-1 tree. Put \(W' = (W)\), a level \(\leq 1\) tree. Then \(\text{desc}(Q, W', \ast) = \text{desc}(Q, W)\) is the set of \((Q, W', \ast)-descriptions\). Suppose \(D \in \text{desc}(Q, W)\). If \(\text{sign}(D) = (w_i)_{i<k}\) viewing \(D\) as a \((Q, W)\)-description, then the signature of \(D\) is \(\text{sign}(D) = ((1, w_i))_{i<k}\). If \(\text{ucf}(D) = -1\) viewing \(D\) is a \((Q, W)\)-description then \(\text{ucf}(D) = (0, -1)\). If \(\text{ucf}(D) = w_\ast\) viewing \(D\) is a \((Q, W)\)-description then \(\text{ucf}(D) = (1, w_\ast)\).

Inductively, we make the syntactical definitions of descriptions. In the rest of this section, we suppose that \(n_1, n_2, n_3\) are consecutive entries of the following list:

\[
1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, \ldots
\]

Suppose \(Y\) is a level \(\leq n_3\) tree and \(T\) is a level \(\leq n_2\) tree. Then a \((Y, T, -1)\)-description is a \((\leq^2 Y, \leq^1 T, \ast)\)-description. The constant \((Y, T, \ast)\)-description is

1. \((n_3, (\emptyset, \emptyset))\), if \(n_3\) is odd;
2. \((n_3, ((\emptyset)^{\ast n_3} Y[0], \tau))\), if \(n_3\) is even, \(\tau\) factors \((^{n_3} Y_{\text{tree}}(\emptyset), T, \ast)\).

Suppose \((Q, (d, q, P))\) is a potential partial level \(\leq n_1\) tree, \(n_1 < n_1\). If \(n_1 < n_1\), then a \((Y, T, (Q, (d, q, P)))\)-description is a \((\leq^n Y, \leq^n T, (Q, (d, q, P)))\)-description. Suppose now \(n_1 = n_1\). Put \((d, q, P) = (d_i, q_i, P_i)_{0 \leq i < m}\), where \(m_0 = 0\) if \(n_1\) is odd, \(m_0 = 1\) if \(n_1\) is even. Suppose \(m > 0\). A \((Y, T, (Q, (d, q, P)))\)-description is of the form

\[
\mathcal{B} = (b, (y, \pi))
\]

such that

1. If \(n_2\) is odd then \(b \in \{n_3 - 1, n_3\}\). If \(n_2\) is even then \(d = n_3\).
2. \(y \in \text{desc}(Y)\) is not the constant \(Y\)-description. Put \(y = (y, X_i(e, x, W))\), \(\text{lh}(y) = k\). If \(d\) is even, put \((e, x, W) = (e_i, x_i, W_i)_{i < \text{lh}(x)}\). If \(d\) is odd, put \((e, x, W) = (e_i, x_i, W_i)_{1 \leq i \leq \text{lh}(x)}\).
3. If \( d = n_3 \) then \( \sigma \) factors \((X, T, Q)\). If \( d = n_3 - 1 \) then \( \sigma \) factors \((X, \leq n T, Q)\).

4. The contraction of \((\text{sign}(\pi(e_i, x_i)))_{i \leq k}\) is the signature of \((Q, (d, q, P))\).
   (When \((e_0, x_0)\) is undefined, the contraction of \((\text{sign}(\pi(e_i, x_i)))_{i \leq k}\) simply means the contraction of \((\text{sign}(\pi(e_i, x_i)))_{1 \leq i \leq k}\).

5. If \( y \) is of continuous type and \((e_{k-1}, x_{k-1})\) does not appear in the contraction of \((\text{sign}(\pi(e_i, x_i)))_{i \leq k}^{-}\)(\(\text{sign}(\pi(e_k, x_k)^{-}\)), then \(\pi(e_{k-1}, x_{k-1})\) is of discontinuous type.

6. Put \( \text{ucf}(X, (e, x, W)) = (e_*, x_*) \).
   (a) If \( e_* = 0 \) then \( d_m = 0 \).
   (b) If \( e_* = 1 \) then \( \text{ucf}(\pi(1, x_*)) = \text{ucf}(Q, (d, q, P)) \).
   (c) If \( e_* > 1, x_* = (x_*, \ldots) \in \text{desc}(X) \), then \( \text{ucf}(\pi(e_*, x_*)) = \text{ucf}(Q, (d, q, P)) \).
   (d) If \( e_* > 1, x_* = (x_*, W_*, \ldots) \notin \text{desc}(X) \), then \( \text{ucf}^{0*}(\pi(e_*, x_*)) = \text{ucf}(Q, (d, q, P)) \).

We often abbreviate \((b, (y, \pi))\) by \((b, y, \pi)\). If \( Q' \) is a level \( \leq n'_1 \) tree, a \((Y, T, Q')\)-description is a \((Y, T, (Q', (d'_i, q'_i, P'_i))\)-description for some potential partial level \( \leq n'_1 \) tower \((Q', (d'_i, q'_i, P'_i))\) of discontinuous type. A \((Y, T, *)\)-description is either the constant \((Y, T, *)\)-description or a \((Y, T, Q')\)-description, where either \( Q' = -1 \) or \( Q' \) is a level \( \leq n'_1 \) tree, \( n'_1 \leq n_1 \). \( \text{desc}(Y, T, (Q, (d, q, P))) \), \( \text{desc}(Y, T, Q) \), \( \text{desc}(Y, T, *) \) denote the sets of relevant descriptions.

Suppose \((Q, (d, q, P))\) is a potential partial level \( \leq n_1 \) tower of discontinuous type, \((d, q, P) = (d_i, q_i, P_i)_{m_0 \leq i \leq m} \), and \( B \in \text{desc}(Y, T, (Q, (d, q, P))) \), \( y = (y, X, (e, x, W)), \text{lh}(y) = k, (e, x, W) = (e_i, x_i, W_i) \). The signature of \( B \) is

\[
\text{sign}(B) = \text{the contraction of } (\text{sign}^0(\pi(e_i, x_i)))_{i \leq k-1}.
\]

**\( B \) is of continuous type** if \( y \) is of continuous type and \( \pi(e_{i-1}, x_{i-1}) \) is of \( *\)-\( Q \)-continuous type. Otherwise, \( B \) is of _discontinuous type_. The uniform cofinality of \( B \) is

\[
\text{ucf}(B),
\]

defined as follows:

1. If \( \text{ucf}(X, (e, x, W)) = (0, -1) \) then \( \text{ucf}(B) = (0, -1) \).
2. If \( \text{ucf}(X, (e, x, W)) = (1, x_*) \) then \( \text{ucf}(B) = \text{ucf}^{0*}(\pi(e_*, x_*)) \).
3. If \( \text{ucf}(X, \langle e, x, W \rangle) = (e_*, x_*) \), \( e_* > 1 \), \( x_* = (x_*, \ldots) \), then \( \text{ucf}(B) = \text{ucf}^Q(\pi(e_*, x_*)) \).

If \( d_m > 0 \), then \( B \) is said to be of \textit{plus-discontinuous type}, and if \( \text{ucf}(X, \langle e, x, W \rangle) = (e_*, x_*) \), \( x_* = (x_*, \ldots) \) if \( e_* = 1 \), \( x_* = (x_*, \ldots) \) if \( e_* > 1 \), \( Q^+ \) is a completion of \( (Q, (d_m, q_m, P_m)) \), put

\[
\text{ucf}^Q(B) = \text{ucf}^Q(\pi(e_*, x_*))
\]

The \( \ast \)-signature of \( B \) is

\[
\text{sign}_*(B) = \begin{cases} 
((d, y \upharpoonright i))_{1 \leq i \leq k-1} & \text{if } y \text{ is of continuous type,} \\
((d, y \upharpoonright i))_{1 \leq i \leq k} & \text{if } y \text{ is of discontinuous type.}
\end{cases}
\]

\( B \) is of \( \ast \)-\textit{(dis-)continuous type} iff \( \pi \) is \( T \otimes Q \)-\textit{(dis-)continuous at} \( \text{ucf}(X, \langle e, x, W \rangle) \).

The \( \ast \)-\textit{T-uniform cofinality} of \( B \) is

\[
\text{ucf}_T^*(B),
\]

defined as follows. If \( y \) is of continuous type,

1. if \( B \) is of \( \ast \)-\textit{T-continuous type}, then \( \text{ucf}_T^*(B) = (d, (y^-, Y_{\text{tree}}(y^-), \langle e, x, W \rangle)) \);
2. if \( B \) is of \( \ast \)-\textit{T-discontinuous type}, then \( \text{ucf}_T^*(B) = (d, (y^-, X, \langle e, x, W \rangle)) \).

If \( y \) is of discontinuous type,

1. if \( B \) is of \( \ast \)-\textit{T-continuous type}, then \( \text{ucf}_T^*(B) = (d, y) \);
2. if \( B \) is of \( \ast \)-\textit{T-discontinuous type}, then \( \text{ucf}_T^*(B) = (d, (y, X^+, \langle e, x, W \rangle)) \),

\( X^+ \) is a completion of \( Y(y) \), \( X^+(Y_{\text{node}}(y)) = T \otimes Q(\pi(Y(y))) \).

Suppose \( (\vec{Q}, (d, q, P)) = (Q, (d_i, q_i, P_i))_{m_0 \leq i \leq m} \) is a partial level \( \leq n_1 \)
tower and \( B = (b, y, \pi) \in \text{desc}(Y, T, (Q_m, (d, q, P))) \). Define \( \text{lh}(B) = \text{lh}(q) \).

Define

\[
B \prec B'
\]

iff \( B = B' \upharpoonright \tilde{m} \) for some \( \tilde{m} < \text{lh}(B') \). Define \( \prec^{Y,T} = \sqsubseteq \text{desc}(Y, T, \ast) \).

Suppose \( y = (y, X, \langle e, x, W \rangle) \), \( 0 < \tilde{m} < m \). Then

\[
B \upharpoonright (Y, T, Q_m) \in \text{desc}(Y, T, (Q_m, (d, q, P)_{m_0 \leq i \leq m}))
\]

is defined by the following: letting \( l \) be the least such that \( \pi(e_l, x_l) \notin \text{desc}(T, Q_m, \ast) \), letting \( \text{ucf}(X(e_l, x_l)) = (e_*, x_*) \), \( x_* = (x_*, \ldots) \) if \( e_* = 1 \), \( x_* = (x_*, \ldots) \) if \( e_* > 1 \), and letting \( C \in \text{desc}(T, Q_m, \ast) \) be such that \( C = \pi(e_*, x_*) \upharpoonright (T, Q_m) \), then
1. if $C \neq \pi(e_*, x_*)$, then $B \upharpoonright (Y, T, Q_m) = (y \upharpoonright l \leftarrow (-1, X^+), \bar{\pi})$, where $\bar{\pi}$ and $\pi$ agree on $\text{Y-tree}(y \upharpoonright l)$, $\bar{\pi}(e_i, x_i) = C$, $\pi$ factors $(X^+, T, Q_m)$;

2. if $C = \pi(e_*, x_*)$, then $B \upharpoonright (Y, T, Q_m) = (y \upharpoonright l, \pi \mid \text{Y-tree}(y \upharpoonright l))$.

Given a $(Y, T, \ast)$-description $B = (b, y, \pi)$, define

$$<B> = (b, \pi \oplus y),$$

where $\emptyset \oplus y = \emptyset$ and if $y = (y, X, \overrightarrow{(e, x, W)})$, $\text{lh}(y) = k$ then

$$\pi \oplus y = \begin{cases} (\pi(e_0, x_0), y(0)), \ldots, (\pi(e_{k-1}, x_{k-1}), y(k-1)) & \text{if } b \text{ is even}, \\
(y(0), \pi(e_1, x_1), y(1)), \ldots, (\pi(e_{k-1}, x_{k-1}), y(k-1)) & \text{if } b \text{ is odd}. \end{cases}$$

Define

$$B < B'$$

iff $<B> <_{BK} <B'>$, the ordering on coordinates in $\text{desc}(T, Q, \ast)$ for some $T, Q$ again according to $\prec$. Define $<Y, T> = \prec \text{desc}(Y, T, \ast)$.

Define

$$n_3 \otimes n_2 = \begin{cases} n_3 & \text{if } n_2 \text{ is even}, \\
n_2 & \text{if } n_2 \text{ is odd}. \end{cases}$$

Suppose $R$ is a level $\leq n_3 \otimes n_2$ tree, $Y$ is a level $\leq n_3$ tree and $T$ is a level $\leq n_2$ tree. Suppose $\rho: \text{dom}(R) \cup \{(n_3 \otimes n_2, \emptyset)\} \to \text{desc}(Y, T, \ast)$ is a function. $\rho$ factors $(R, Y, T)$ iff

1. $\rho(n_3 \otimes n_2, \emptyset)$ is the constant $(Y, T, \ast)$-description.

2. For any $(d, r) \in \text{dom}(R)$, $\rho(d, r) \in \text{desc}(Y, T, \overrightarrow{dR[r]})$.

3. For any $r, r' \in \overrightarrow{1R}$, if $r <_{BK} r'$ then $\rho(1, r) < \rho(1, r')$.

4. For any $d > 1$, any $r \prec(a), r \prec(b) \in \text{dom}(\overrightarrow{dR})$, if $a <_{BK} b$ and $\overrightarrow{dR_{\text{tree}}(r \prec(a))} = \overrightarrow{dR_{\text{tree}}(r \prec(b))}$ then $\rho(d, r \prec(a)) < \rho(d, r \prec(b))$.

5. For any $d > 1$, any $r \in \text{dom}(\overrightarrow{dR}) \setminus \{\emptyset\}$, $\rho(d, r \prec) = \rho(d, r) \mid (Y, T, \overrightarrow{dR_{\text{tree}}(r \prec)})$.

$\rho$ factors $(R, Y, \ast)$ if $\rho$ factors $(R, Y, T')$ where $T'$ is some level $\leq n_2'$ tree, $n_2' \leq n_2$. If $n_2$ is even and $Y$ is a level $\leq n_3$ tree, then

$$\text{id}_{Y, \ast}$$

factors $(Y, Y, \ast)$ where $\text{id}_{Y, \ast}(d, y) = (d, (y, X, (e, x, W)), \text{id}_{*, X})$ for $\overrightarrow{dY[y]} = (X, (e, x, W))$. If $n_2 = n_2$ and $T$ is a level $\leq n_2$ tree, then

$$\text{id}_{\ast, T}$$

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factors \((T, Q_0^{(n_2)}, T)\), defined as follows: If \(n_2 = 2\) then \(\text{id}_{*,T}\) has been defined in [10]. If \(n_2 > 2\) then \(\text{id}_{*,T}\) extends \(\text{id}_{*,n_2-2}\) and for \(d \in \{n_2 - 1, n_2\}\), \(\text{id}_{*,T}(d, t) = (n_2, q^d_t, \tau^d_t)\), where \(q^d_t = ((-1), P^d_t, (n_2 - 1, ((0))))\), \(\tau^d_t\) factors \((P^d_t, T, d[T])\), \(\tau^d_t(n_2 - 1, ((0))) = (d, t, \text{id}_{\text{tree}(t),*})\).

Suppose \(Y\) is a level \(\leq n_3\) tree and \(T\) is a level \(\leq n_2\) tree. A representation of \(Y \otimes T\) is a pair \((R, \rho)\) such that \(R\) is a level \(\leq n_3 \otimes n_2\) tree, \(\rho\) factors \((R, Y, T)\), and \(\text{ran}(\rho) = \text{desc}(Y, T, *)\). Representations of \(Y \otimes T\) are clearly mutually isomorphic. We shall regard

\[Y \otimes T\]

itself as a level \(\leq n_3 \otimes n_2\) tree whose level-\(d\) component has domain \(\{(y, \pi) : (d, (y, \pi)) \in \text{desc}(Y, T, *)\}\) and if \(d > 1\) then \(Y \otimes T[d, (y, \pi)]\) is the unique \((Q, (d, q, P^1))\) for which \((d, (y, \pi))\) is a \((Y, T, (Q, (d, q, P)))\)-description. Suppose \(m_1, n_2, n_3\) are consecutive entries of the following list:

\[1, 2, 2, 3, 4, 4, 4, 5, 6, 6, 6, 7, 8, 8, 9 \ldots\]  (2)

Then

\[(n_3 \otimes n_2) \otimes m_1 = n_3 \otimes (n_2 \otimes m_1) = \text{def} n_3 \otimes n_2 \otimes m_1.\]

If \(Q\) is a level \(\leq m_1\) tree, then \((Y \otimes T) \otimes Q\) is regarded as a level \(\leq n_3 \otimes n_2 \otimes m_1\) tree. There is a natural isomorphism

\[\iota_{Y,T,Q}\]

between “level \(\leq n_3 \otimes n_2 \otimes m_1\) trees” \((Y \otimes T) \otimes Q\) and \(Y \otimes (T \otimes Q)\), defined as follows: If \(n_3 \leq 2\), \(\iota_{Y,T,Q}\) has been defined in [9]. Suppose now \(n_3 > 2\). Let \(m_0, m_1, n_2, n_3\) be consecutive entries in the list \([2]\). Then \(\iota_{Y,T,Q}\) extends \(\iota_{n_2 \leq Y, \leq n_3, T, \leq m_0 Q}\) and

1. if \((n_3, y, \pi) \in \text{desc}(Y, T, (Z, (d, z, N))), A = (n_3 \otimes n_2, ((y, \pi), Z, (d, z, N))), \psi) \in \text{desc}(Y \otimes T, Q, U), \text{then} \iota_{Y,T,Q}(A) = (n_3, y, \iota_{T,Q,U}^1 \circ (T \otimes \psi) \circ \pi);\]

2. if \((n_3, y, \pi) \in \text{desc}(Y, T, (Z, (d, z, N))), A = (n_3 \otimes n_2, ((y, \pi)^{-1}, Z^+, (d, z, N))), \psi) \in \text{desc}(Y \otimes T, Q, U), (d, z, N) = (d_1, z_i, N_i)_{0 \leq i \leq t}, y = (y, X, (e_i, t_i, W_i)_{i_0 \leq i \leq k})\), then

(a) if \(y\) is of discontinuous type, then \(\iota_{Y,T,Q}(A) = (n_3, y^{-1}, X^+, \psi \ast_0 \pi), \text{where} X^+ \text{ is a completion of} \psi \ast_0 \pi \text{ factors} (X^+, T \otimes Q, U), \psi \ast_0 \pi \text{ extends} \iota_{T,Q,U}^{-1}(T \otimes \psi) \circ \pi, \psi \ast_0 \pi(e_k, x_k) = \iota_{T,Q,U}^{-1}(n_2, t_0, \tau), t_0 = ((-1), S_0, (b_0, s_0)), \tau \text{ factors} (S_0, Q, U), \tau(b_0, s_0) = \psi(d_2, z_i);\)

\section*{5}
Inductively, we can show that $\iota_{Y,T,Q}$ is a level $\leq n_3 \otimes n_2 \otimes m_1$ tree isomorphism between $(Y \otimes T) \otimes Q$ and $Y \otimes (T \otimes Q)$. The base case $(n_3, n_2, m_1) = (2, 2, 1)$ is in [9], whose idea is easily modified to the general case. $\iota_{Y,T,Q}$ justifies the associativity of the $\otimes$ operator acting on level $(\leq n_3, \leq n_2, \leq m_1)$ trees.

The identity map $\text{id}_Y \otimes T$ factors $(Y \otimes T, Y, T)$. $\rho$ factors $(R, Y \otimes T)$ iff $\rho$ factors $(R, Y \otimes T)$. If $(d, y) \in \text{dom}(Y)$, $y = (y, x, (e, x, W)) \in \text{desc}(\pi)$, $Y \otimes (d, y) \otimes T$

is the level $\leq n_3$ subtree of $Y \otimes T$ whose domain is $\text{dom}(Y \otimes Q^0)$ plus all the $(Y, T, \ast)$-descriptions of the form $(d, y, \tau)$. If $\pi$ factors level $\leq n_2$ trees $(T, Q)$, then

$$Y \otimes \pi$$

factors $(Y \otimes T, Y \otimes Q)$, where $Y \otimes \pi(y, \psi) = (y, (\pi \otimes U) \circ \psi)$ for $(y, \psi) \in \text{desc}(Y, T, U)$. If $\rho$ factors level $\leq n_3$ trees $(R, Y)$, then

$$R \otimes Y$$

factors $(R \otimes T, Y \otimes T)$, where $\rho \otimes T(d, r, \psi) = (d, \rho(r), \psi)$.

Suppose $T$ is a proper level $\leq n_2$ subtree of $T'$, both trees are finite, $(Q_i, (d_i, q_i, P_i))_{i \leq i \leq l'}$ is a partial level $\leq n_1$ tower, $l \leq l'$, $B \in \text{desc}(Y, T, (Q_i, (d_i, q_i, P_i))_{i \leq l})$, and $B' \in \text{desc}(Y, T', (Q'_{i'}, (d_i, q_i, P_i))_{i \leq l'} \setminus \text{desc}(Y, T, (Q_i, (d_i, q_i, P_i))_{i \leq l}))$.

Define

$$B = B' \upharpoonright (Y, T)$$

iff $B' \preceq B$ and $\bigcup_{l \leq l'} \{B^* \in \text{desc}(Y, T, (Q_i, (d_i, q_i, P_i))_{i \leq l}) : B' \preceq B^* \preceq B\} = \emptyset$. Inductively, we can show that $B = B' \upharpoonright (Y, T)$ iff both $B, B'$ are of degree $n_3$ and letting $B = (n_3, (y, X, (e, x, W^y)), \pi)$, $B' = (n_3, (y', X', (e', x', W'^{y'})), \pi')$, $\text{lh}(y) = k, (e, x, W) = (e_i, x_i, W_i), \text{ucf}(X, (e, x, W)) = (e_s, x_s), e_s = 1 \rightarrow x_s = x_s, e_s > 1 \rightarrow x_s = (x_s, \ldots)$, then either

1. $y$ is of continuous type, $y \mid k - 1 = y' \mid k - 1, \pi \mid n\text{Ytree}(y^-) \subseteq \pi'$, $\pi(b_{k-1}, p_{k-1}) = \pi '(b_{k-1}, p_{k-1}) \mid (T, Q_l)$, or

2. $y$ is of discontinuous type, $B \preceq B'$, $\pi(e_s, x_s) = \pi(b_k, p_k) \mid (T, Q_l).$
Suppose \( Y \) is a proper level \( \preceq n_3 \) subtree of \( Y' \), both finite. Suppose \( T \) is a level \( n_2 \) tree. For \( B \in \text{desc}(Y, T, *) \), \( B' \in \text{desc}(Y', T, *) \), define

\[
B = B' \upharpoonright (Y, T)
\]

iff \( B' \prec B \) and \( \{ B^* \in \text{desc}(Y, T, *) : B' \prec B^* \prec B \} = \emptyset \). Putting

\[
B = (d, y, \pi), B' = (d', y', \pi'), y = (y, X, (e, x, W)), y' = (y', X', (e', x', W')),
\]

\[
\text{lh}(y) = k, (e, x, W) = (e_i, x_i, W_i)_{k_0 \leq i \leq k}, (e', x', W') = (e'_i, x'_i, W'_i)_{k'_0 \leq i \leq k'},
\]

\[
\text{ucf}(X, (e, x, W)) = (e_s, x_s), e_s = 1 \rightarrow x_s = x_s, e_s > 1 \rightarrow x_s = (x_s, \ldots),
\]

inductively, we can show that \( B = B' \upharpoonright (Y, T) \) iff one of the following holds:

1. \( d, d' \leq n_s < n_3, n_s \) is even, \( B = B' \upharpoonright (\leq^{n_s} Y, \leq^n T) \).

2. \( n_3 - 1 \leq d = d' = n_3, B \in \text{desc}(Y, T, Q) \) is of continuous type, \( y \upharpoonright k - 1 = y' \upharpoonright k - 1, \pi \upharpoonright \delta Y_{\text{tree}}(y^{-}) \subseteq \pi', \delta Y_{\{y \upharpoonright k, +, X'_k\}} = \{y^{-}\} \), either

\[
e_k - 1 = 0 \text{ or } \pi'(e_{k-1}, x_{k-1}) = \pi(e_{k-1}, x_{k-1}) \upharpoonright (T, Q).
\]

3. \( n_3 - 1 \leq d = d' = n_3, B \in \text{desc}(Y, T, Q) \) is of discontinuous type, \( y = y' \upharpoonright k, \pi \subseteq \pi', \delta Y_{\{y \upharpoonright k + 1, +, X'_{k+1}\}} = \{y\}, \pi'(e_k, x_k) = \pi(e_s, x_s) \upharpoonright (T, Q) \).

4. \( d' = n_3 - 1, d = n_3, n_3 \) is even, \( \emptyset = y(0) \upharpoonright \delta Y_{\{\emptyset\}} \), \( (\pi, T \otimes Q) \) is continuous at \((e_0, x_0)\).

If \( n_3 \) is odd, \( R, Y \) are level-\( n_3 \) trees, \( T \) is a level \( n_2 \) tree, then \( \rho \) factors \((R, Y, T)\) iff \( \rho \) extends to some \( \rho' \) which factors \((Q_0^{(n_2)} \oplus R, Q_0^{(n_2)} \oplus Y, T)\).

3 The induction hypotheses

From now on until the end of this paper, we assume \( \Delta_{2n}^1 \)-determinacy, where \( n < \omega \). This section lists the inductive definitions and hypotheses for any \( 0 \leq m < n \). The base of the induction is in \([10]\). Define \( E(0) = 1, E(k+1) = \omega^{E(i)} \) in ordinal exponentiation. Define \( u_i^{(1)} = u_i \) for \( i \leq \omega \).

\( \nu_{2m+1} \) denotes the \( \mathbb{L}[T_{2m+1}] \)-club filter on \( \delta_{2m+1}^1 \), i.e., \( A \in \nu_{2m+1} \) iff \( A \in \mathbb{L}[T_{2m+1}] \) and there is a club \( C \subseteq \delta_{2m+1}^1 \) such that \( C \in \mathbb{L}[T_{2m+1}] \) and \( C \subseteq A \).

Assume by induction that:

\( (1:m) \) If \( m > 0 \) and \( A \subseteq u_{E(2m-1)}^{(2m-1)} \), then \( A \in \mathbb{L}[\delta_{2m+1}^1, T_{2m}] \) iff \( A \in \mathbb{L}[T_{2m+1}] \).

\( (\Pi_m) \) ensures that the \( \mathbb{L}[\delta_{2m+1}^1, T_{2m}] \)-measures induced by level \( \leq 2n \) trees are indeed \( \mathbb{L}[T_{2m+1}] \)-measures. Assume we have defined by induction the level-(\( 2m + 1 \)) sharp operator \( x \mapsto x^{(2m+1)} # \) for \( x \in \mathbb{R} \) with the following property:
(2:m) \( x^{(2m+1)\#} \) is many-one equivalent to \( M^{\#}_{2m}(x) \), the many-one reductions being independent of \( x \).

Define

\[ A \in \mu^P \]

iff \( A \subseteq \prod_{i \leq m} \nu_{2i+1} \) and there is \( \tilde{C} = (C_i)_{i \leq m} \in \prod_{i \leq m} \nu_{2i+1} \) such that \( \tilde{C} \cup A \subseteq A \). If \( W \) is a finite level-(2m + 1) tree, let \( A \in \mu^W \) iff

\[ \prod_{i \leq m} Q^{(2m+1)\#}_{i} \cup A \in \mu^{Q^{(2m+1)\#}_{i,W}}. \]

We assume by induction that:

(3:m) Suppose \( P \) is a finite level \( \leq 2m + 1 \). Then

(a) \( \mu^P \) and \( \mu^{2m+1P} \) are both \( \mathbb{L}[T_{2m+1}] \)-measures.

(b) If \( m > 0 \), then \( \mu^P \) is the product \( \mathbb{L}[T_{2m+1}] \)-measure of \( \mu^{2mP} \) and \( \mu^{2m+1P} \), i.e., \( A \in \mu^P \) iff there exist \( B \in \mu^{2mP} \) and \( C \in \mu^{2m+1P} \) such that \( B \cup C \subseteq A \), where \( B \cup C = \{ \tilde{\alpha} \cup \tilde{\beta} : \tilde{\alpha} \in B, \tilde{\beta} \in C \} \), \( \tilde{\alpha} \cup \tilde{\beta} = \gamma \) where \( \tilde{\alpha} = \tilde{\beta} = \gamma \)

(c) The set of \( \mathbb{L}[j^P(T_{2m+1})] \)-cardinals in the interval \( \delta^P_{2m+1} \) is the closure of \( \{ \text{seed}^P_{2m+1,A} : A \in \text{desc}^*(2m+1) \} \).

If \( P \) is a finite level \( \leq 2m + 1 \) tree or level-(2m + 1) tree, Let

\[ j^P_\mu = j^P_{\mathbb{L}[T_{2m+1}]} : \mathbb{L}[T_{2m+1}] \rightarrow \mathbb{L}[j^P(T_{2m+1})] \]

be the induced restricted ultrapower map. For any real \( x \), \( j^P \) is elementary from \( \mathbb{L}[T_{2m+1}, x] \) to \( \mathbb{L}[j^P(T_{2m+1}), x] \). If \( P \) is a subtree of \( P' \), both finite, then \( j^{P,P'} \) is the factor map from \( \mathbb{L}[j^P(T_{2m+1})] \) to \( \mathbb{L}[j^{P'}(T_{2m+1})] \). If \( \sigma \) factors finite level \( \leq 2m + 1 \) trees \( (P,W) \), then \( \sigma^W \) is the induced factor map from \( \mathbb{L}[j^P(T_{2m+1})] \) to \( \mathbb{L}[j^W(T_{2m+1})] \), i.e., \( \sigma^W([h]_{\mu^P}) = [h^\sigma]_{\mu^W} \), where \( h^\sigma(\tilde{\alpha}) = h(\tilde{\alpha}) \). By (3;m) \( [m]_m \rightarrow m - 1 \), \( j^P \cup \delta^P_{2m+1} = j^{2mP} \cup \delta^P_{2m+1} \), and similarly for \( j^{P,P'} \), \( \sigma^W \). Define \( j^\text{sup}(\alpha) = \text{sup}(j^P)^\alpha \alpha \), and similarly for \( j^{\text{sup},P} \), \( \sigma^W \).

Assume by induction that:

(4:m) Suppose \( (P_i)_{i < \omega} \) is a level-(2m + 1) tower and \( P_\omega = \bigcup_{i < \omega} P_i \). Then \( P_\omega \)

is \( \Pi^1_{2m+1} \)-wellfounded iff the direct limit of \( (j^{P_i,P'_i})_{i < \omega} \) is wellfounded.

Suppose \( A \in \text{desc}^*(2m+1) \). Then

\[ \text{seed}^P_{(2m+1,A)} \]

is represented modulo \( \mu^P \) by the function \( \tilde{\alpha} \mapsto 2m+1 \alpha_A \). Similarly define \( \text{seed}^P_{(d,p)} \) for \( (d,p) \in \text{desc}^*(P) \) and \( \text{seed}^P_{(d,p)} \) for \( (d,p) \in \text{dom}(P) \).
If $P$ is an infinite $\Pi^1_{2m+1}$-wellfounded level $\leq 2m + 1$ tree, $j^P$ is the direct limit map and seed$^P_{\langle 2m+1, A \rangle}$, seed$^P_{\langle 2m+1, r \rangle}$, seed$^P_{\langle 2m+1, r \rangle}$ are the images under appropriate tails of the direct limit map.

Suppose $m > 0$, $W$ is a finite level $\leq 2m + 1$ tree, $Z$ is a finite level $\leq 2m$ tree and $B \in \text{desc}(W, Z, *)$. Then

$$\text{seed}^W_Z$$

is the element represented modulo $\mu^W$ by $\text{id}^W_Z$.

Suppose $P$ is another level $\leq 2m + 1$ tree and $\sigma$ factors $(P, W, Z)$. Let

$$\text{seed}_{\sigma}^{W,Z} = [\text{id}_{\sigma}^{W,Z}]_{\mu^W}.$$ 

So $\text{seed}_{\sigma}^{W,Z} = (\text{seed}_{\sigma(d,w)}^{W,Z})_{(d,w) \in \text{dom}(W)}$. If $W, Z$ are (possibly infinite) $\Pi^1_{2m+1}$-wellfounded trees, seed$^W_Z$ and seed$^W_Z$ make sense as the images under appropriate tails of direct limit maps.

We assume by induction that:

(5: $m$) Suppose $P, W$ are finite level $\leq 2m+1$ trees, $Z$ is a finite level $\leq 2m$ tree and $\sigma$ factors $(P, W, Z)$. Then for any $A \in \mu^P$, seed$^W_Z \in j^W \circ j^Z(A)$.

We can then define

$$\sigma^{W,Z} : L[j^P(T_{2m+1})] \rightarrow L[j^W \circ j^Z(T_{2m+1})]$$

by sending $j^P(f)(\text{seed}^P)$ to $j^W \circ j^Z(f)(\text{seed}^{W,Z})$.

We assume by induction that:

(6: $m$) Suppose $W$ is a finite level $\leq 2m + 1$ tree and $Z$ is a finite level $\leq 2m$ tree. Then $(\text{id}_W \circ Z)^{W,Z}$ is the identity on $j^W \circ j^Z(\delta_{2m+1}^1) + 1$.

(7: $m$) Suppose $P$ is a finite level $\leq 2m + 1$ tree. Then the set of uncountable $L[j^P(T_{2m+1})]$-regular cardinals in the interval $[\delta_{2m+1}^1, j^P(\delta_{2m+1}^1)]$ is $\{\text{seed}_r^P : r \in \text{desc}^*(P) \text{ is regular} \}$.

By (14: $k$) for $k \leq m$ and (15: $k$) (16: $k$) (18: $k$) (27: $k$) for $k < m$, if $P$ is a finite level $\leq 2m + 1$ tree, then the set of uncountable $L[j^P(T_{2m+1})]$-cardinals below $j^P(\delta_{2m+1}^1)$ is the closure of

$$\{u_\xi^{(2k+1)} : k < m, 0 < \xi \leq E(2k + 1)\} \cup \{\text{seed}_{\langle 2m+1, A \rangle}^P : A \in \text{desc}^*(\leq 2m+1)P)\},$$

and the set of uncountable $L[j^P(T_{2m+1})]$-regular cardinals is

$$\{\text{seed}_{\langle d,p \rangle}^P : (d,p) \in \text{desc}^*(P) \text{ is regular} \}.$$
1. $u_{(2m+1)}^{(2n+1)} = j^P(\delta_{2m+1}^1)$ when $\xi < E(2m + 1)$, $P$ is a $\Pi^1_{2m+1}$-wellfounded level $\leq 2m + 1$ tree and $\llbracket \emptyset \rrbracket_{2m+1} = \kappa$.

2. If $0 < \xi \leq E(2m + 1)$ is a limit, then $u_{(2m+1)}^{(2n+1)} = \sup_{\eta < \xi} u_{\eta}^{(2m+1)}$.

A level-1 sharp code is a usual sharp code for an ordinal below $u_w$. If $m > 0$, a level-$(2m + 1)$ sharp code is a pair $\langle (\tau, x^{(2m+1)}#) \rangle$ where $\tau$ is an $\mathcal{L}^{\beta} R^{(2m+1)}_{E(2m+1)}$-Skolem term for an ordinal without free variables. For $0 < \xi \leq E(2m + 1)$, $WO_{(2m+1)}^{E(2m+1)}$ is the set of level-$(2m + 1)$ sharp codes $\langle (\tau, x^{(2m+1)}#) \rangle$ such that $\tau$ is an $\mathcal{L}^{\beta} R^{(2m+1)}_{E(2m+1)}$-Skolem term. The ordinal coded by $\langle (\tau, x^{(2m+1)}#) \rangle$ is

$$\left| \langle (\tau, x^{(2m+1)}#) \rangle \right| = \tau^{\beta} R^{(2m+1)}_{E(2m+1)}(\nu_{2m+1}(x)) \text{ seed } \tau^{(2m+1)}.$$ 

Assume by induction that:

(8: m) $WO_{(2m+1)}^{E(2m+1)}$ is $\Pi^1_{2m+2}$ for $0 < \xi \leq E(2m + 1)$, uniformly in $\xi$. The following relations are all $\Delta^1_{2m+1}$:

(a) $v, w \in WO_{(2m+1)}^{E(2m+1)} \land |v| = |w|.$

(b) $v, w \in WO_{(2m+1)}^{E(2m+1)} \land |v| < |w|.$

(c) $k < m \land v \in WO_{(2m+1)}^{E(2m+1)} \land w \in WO_{(2k+1)}^{E(2k+1)} \land |v| = |w|.$

If $\Gamma$ is a pointclass, say that $A \subseteq u_{(2m+1)}^{E(2m+1)} \times R$ is in $\Gamma$ iff $\{ (v, x) : v \in WO_{(2m+1)}^{E(2m+1)} \land (|v|, x) \in A \}$ is in $\Gamma$. $\Gamma$ acting on subsets of product spaces is defined in the obvious way. Assume that we have constructed by induction the level-$(2m + 2)$ Martin-Solovay tree $T_{2m+2}$ on $2 \times u_{(2m+1)}^{E(2m+1)}$. The construction should ensure that $T_{2m+2}$ is $\Delta^1_{2m+3}$ in the codes and projects to $\{ x^{(2m+1)}# : x \in R \}$. Let $\kappa^{e}_{2m+3}$ be the least $(T_{2m+2}, x)$-admissible ordinal.

Suppose $W$ is a finite level $\leq 2m+1$ tree, $\overrightarrow{(d, w)} = (d_i, w_i)_{i< k}$ is a distinct enumeration of a subset of $\text{dom}(W)$. Suppose $f : [\alpha^{\delta_{2i+1}}_{i\leq m}]^{W^+} \rightarrow \delta^1_{2m+1}$ is a function which lies in $\mathcal{L}[T_{2m+1}]$. The signature of $f$ is $(d, w)$ iff there is $\overrightarrow{C} = (C_i)_{i \leq m}$ such that $C_i \in \nu_{2i+1}$ for any $i$ and

1. for any $\overrightarrow{\alpha}, \overrightarrow{\beta} \in [\overrightarrow{C}]^{W^+}$, if $(d_{0} \alpha_{w_0}, \ldots, d_{k-1} \alpha_{w_{k-1}}) <_{BK} (d_{0} \beta_{w_0}, \ldots, d_{k-1} \beta_{w_{k-1}})$ then $f(\overrightarrow{\alpha}) < f(\overrightarrow{\beta});$

2. for any $\overrightarrow{\alpha}, \overrightarrow{\beta} \in [\overrightarrow{C}]^{W^+}$, if $(d_{0} \alpha_{w_0}, \ldots, d_{k-1} \alpha_{w_{k-1}}) = (d_{0} \beta_{w_0}, \ldots, d_{k-1} \beta_{w_{k-1}})$ then $f(\overrightarrow{\alpha}) = f(\overrightarrow{\beta}).$
Suppose the signature of \( f \) is \((d_\nu, w)_{\nu<k}\) and \( k \geq 0 \), \( d_0 = 2m + 1 \). \( f \) is essentially continuous iff for \( \mu^W \)-a.e. \( \tilde{\alpha} \), \( f(\tilde{\alpha}) = \sup\{f(\tilde{\beta}) : (d_0, \beta_{w_0}, \ldots, d_{k-1}\beta_{w_{k-1}}) \leq (d_0\alpha_{w_0}, \ldots, d_{k-1}\alpha_{w_{k-1}})\} \). Otherwise, \( f \) is essentially discontinuous. Put \([\tilde{\beta}]^{W_\uparrow(0, -1)} = [\tilde{\beta}]^{W_\uparrow} \times \omega\). For \((d^*, \omega^*) \in \text{desc}^*(W)\) regular, put \([\tilde{\beta}]^{W_\uparrow(d^*, \omega^*)} = \{ (\tilde{\beta}, \gamma) : \tilde{\beta} \in [\tilde{\beta}]^{W_\uparrow}, \gamma < d^*\beta_{\omega^*} \} \). Say that the uniform cofinality of \( f \) is \((d^*, \omega^*)\) iff there is \( g : ([\delta^1_{2m+1}]_{i \leq m})^{W_\uparrow(d^*, \omega^*)} \to \delta^1_{2m+1}\) such that \( g \in L[T_{2m+1}] \) and for \( \mu^W \)-a.e. \( \tilde{\alpha} \), \( f(\tilde{\alpha}) = \sup\{g(\tilde{\alpha}, \beta) : (\tilde{\alpha}, \beta) \in ([\delta^1_{2m+1}]_{i \leq m})^{W_\uparrow}\} \) and the function \( \beta \mapsto g(\tilde{\alpha}, \beta) \) is order preserving. Let \((P_\nu, (d_\nu, p, R_\nu))_{\nu<k} \prec (P_k)\) be the partial level \( \leq 2m + 1 \) tower of continuous type and let \( \sigma \) factor \((P_k, W)\) such that \( \sigma(d_\nu, p_\nu) = (d_\nu, w_\nu) \) for each \( \nu < k \). Note that \((d_\nu, w_\nu) \prec \omega^2 \) \( (d_0, w_0) \) for \( 0 < \nu < k \), so each \( P_\nu \) is indeed a regular level \( \leq 2m + 1 \) tree. \( \tilde{P} = (P_\nu)_{\nu \leq m} \) is called the level \( \leq 2m + 1 \) tower induced by \( f \), and \( \sigma \) is called the factoring map induced by \( f \). Note that \( \sigma \upharpoonright P_\nu \) factors \((P_\nu, W)\) for each \( \nu \). The potential partial level \( \leq 2m + 1 \) tower induced by \( f \) is

1. \((P_k, (d_\nu, p_\nu, R_\nu))_{\nu<k}\), if \( f \) is essentially continuous;

2. \((P_k, (d_\nu, p_\nu, R_\nu))_{\nu<k} \prec (0, -1, \emptyset)\), if \( f \) is essentially discontinuous and has uniform cofinality \((0, -1)\);

3. \((P_k, (d_\nu, p_\nu, R_\nu))_{\nu<k} \prec (d^+, x^+, R^+)\), if \( f \) is essentially discontinuous and has uniform cofinality \((d_\nu, \omega^*)\), \( d_\nu > 0 \), \((P_k, (d^+, x^+, R^+))\) is a partial level \( \leq 2m + 1 \) tree with uniform cofinality \((d_\nu, \omega^*)\).

The approximation sequence of \( f \) is \((f_i)_{i<k}\) where \( \text{dom}(f_i) = ([\delta^1_{2m+1}]_{i \leq m})^{P_{\uparrow}} \); \( f_0 \) is the constant function with value \( \delta^1_{2m+1} \); \( f_i(\tilde{\alpha}) = \sup\{f(\tilde{\beta}) : \tilde{\beta} \in [\omega]^W, (d_{i+1}, p_0, \ldots, d_{i-1}, \tilde{\alpha}, p_{i-1})\} \) for \( 1 \leq i \leq k \). In particular, \( f_0(\tilde{\beta}_0) = f(\tilde{\beta}) \) for \( \mu^W \)-a.e. \( \tilde{\beta} \).

Suppose \( \delta^1_{2m+1} \leq \beta = [f]_{\mu^W} \prec j^W(\delta^1_{2m+1}) \). Suppose the signature of \( f \) is \((d_\nu, w_\nu)_{\nu<k}\), the approximation sequence of \( f \) is \((f_i)_{i<k}\), the level \( \leq 2m + 1 \) tower induced by \( f \) is \((P_i)_{i<k}\), the factoring map induced by \( f \) is \( \sigma \). Then the \( W \)-signature of \( \beta \) is \((d_\nu, w_\nu)_{\nu<k}\), the \( W \)-approximation sequence of \( \beta \) is \((f_i)_{i<k}\), \( \beta \) is \( W \)-essentially continuous iff \( f \) is essentially continuous. The \( W \)-uniform cofinality of \( \beta \) is \( \omega \) if \( f \) has uniform cofinality \((0, -1)\), seed\(^W\) if \( f \) has uniform cofinality \((d_\nu, \omega^*)\). The \( W \)-(potential) partial level \( \leq 2m + 1 \) tower and \( W \)-factoring map induced by \( \beta \) are the (potential) partial level \( \leq 2m + 1 \) tower and factoring map induced by \( f \) respectively. Assume by induction that:

\((9m)\) The \( W \)-partial level \( \leq 2m + 1 \) tower induced by \( \beta \) and the \( W \)-approximation sequence of \( \beta \) are uniformly \( \Delta^4_{2m} \) definable from \((W, \beta)\).
Suppose $Q$ is a level-$(2m+2)$ tree. The *ordinal representation* of $Q$ is the set
\[
\text{rep}(Q) = \{\vec{\beta} \oplus Q q : q \in \text{dom}(Q), \vec{\beta} \text{ respects } Q_{\text{tree}}(q)\}
\cup \{\vec{\beta} \oplus Q q^-(-1) : q \in \text{dom}(Q), \vec{\beta} \text{ respects } Q(q)\}.
\]
Here for $q \in \text{dom}^*(Q)$ of length $k$, \(\vec{\beta} \oplus Q q = (\beta_{Q_{\text{node}}(q)}, q(0), \ldots, \beta_{Q_{\text{node}}(q)}, q(k-1))\). \text{rep}(Q) is endowed with the \(\prec_{BK}\) ordering
\[
\prec^Q = \prec_{BK}[\text{rep}(Q)].
\]
Assume by induction that:

$(10:m)$ Suppose $Q$ is a level-$(2m+2)$ tree. Then $Q$ is $\Pi^1_{2m+2}$-wellfounded iff \(\prec^Q\) is a wellordering.

Suppose $B \in L[T_{2m+1}]$. Define
\[
f \in B^Q_{\uparrow^*}
\]
iff $f \in L[T_{2m+1}]$ is an order preserving function from $\text{rep}(Q)$ to $B$. If $f \in [\delta_{2m+1}^1]^Q_{\uparrow^*}$, then for any $q \in \text{dom}(Q)$, \(f_q\) is a function on $(\delta_{2m+1}^1)^{Q_{\text{tree}}(q)\uparrow^*}$ that sends $\vec{\alpha}$ to $f(\vec{\alpha} \oplus Q q)$, and $f$ represents a tuple of ordinals
\[
[j]^Q = ([j]^Q_q)_{q \in \text{dom}(P)}
\]
where $[j]^Q_q = [f_q]_{\mu Q_{\text{tree}}(q)}$ for $q \in \text{dom}(Q)$. Let
\[
[B]^Q_{\uparrow^*} = \{[j]^Q : f \in B^Q_{\uparrow^*}\}.
\]
A tuple of ordinals $\vec{\beta} = (\beta_q)_{q \in \text{dom}(Q)}$ respects $Q$ iff $\vec{\beta} \in [\delta_{2m+1}^1]^Q_{\uparrow^*}$. $\vec{\beta}$ weakly respects $Q$ iff $\beta_0 = \delta_{2m+1}$ and for any $p, p' \in \text{dom}(Q)$, if $p$ is a proper initial segment of $p'$, then $f_{Q_{\text{tree}}(p), Q_{\text{tree}}(p)'}(\beta_q) > \beta_{q'}$.

Suppose now $Q$ is a finite level $\leq 2m+2$ tree. Then $\text{rep}(Q) = \cup_d \{d\} \times \text{rep}^{(Q)}$. Suppose $\vec{B} = (B_i)_{i \leq m+1} \in L[\delta_{2m+1}^1[T_{2m+2}]$. Define $f \in \vec{B}^Q_{\uparrow^*}$ iff $L[\delta_{2m+1}^1[T_{2m+2}]$ is an order preserving function from $\text{rep}(Q)$ to $\cup_i B_i$ such that for any $i$, ran($f$) \(\subseteq B_i$. Define $[\vec{B}]^Q_{\uparrow^*} = \{[j]^Q : f \in \vec{B}^Q_{\uparrow^*}\}$. $\vec{\beta} = (\beta_q)_{q \in \text{dom}(Q)}$ respects $Q$ iff $\vec{\beta} \in [(\delta_{2m+1}^1)_{i \leq m}]^Q_{\uparrow^*}$. If $f \in [((\delta_{2m+1}^1)_{i \leq m})^Q_{\uparrow^*}$ and $q = (d, (q, P, \ldots)) \in \text{desc}^*(Q)$, $d > 1$, then $d_f q$ is the function on $[(\delta_{2m+1}^1)_{i \leq m}]^P_{\uparrow^*}$ defined as follows: $d_f q = d_f q$ if $(d, q) \in \text{desc}(Q)$; $d_f q(\vec{\alpha}) = \alpha_f q(\vec{\alpha} \oplus Q_{\text{tree}}(q))$ if $(d, q) \notin \text{desc}(Q)$. If $\vec{\beta} = (\beta_{(d, q)})_{(d, q) \in \text{dom}(Q)} = (\beta_q)_{(d, q) \in \text{dom}(Q)} \in [(\delta_{2m+1}^1)_{i \leq m}]^Q_{\uparrow^*}$, we define $\beta_{(d, q)} = \beta_f q$ for $(d, q) \in \text{desc}^*(Q)$: if $d > 1$, $q = (P, \ldots)$, put $\beta_f q = [d_f q]_{\mu P}$ where $\vec{\beta} = [f]^Q$. Clearly, $\beta_f q = \beta_f q$ if $(d, q) \in \text{desc}(Q)$ of discontinuous type, $\beta_f q = j_{\text{Qtree}(q), P}(\beta_f q)$ if $(d, q) \notin \text{desc}(Q)$. The next induction hypothesis computes the remaining case when $q \in \text{desc}(Q)$ is of continuous type, justifying that $\beta_f q$ does not depend on the choice of $f$. 

\[
22
\]
Suppose $Q$ is a level $\leq 2m + 2$ tree. Suppose $\vec{\beta} = (d_q)_{(d,q) \in \text{dom}(Q)} \in [\delta_{2m+1}]_{i \leq m}^{\mathcal{Q}_i}$, $(d, q) = (d, (q, P, \ldots)) \in \text{desc}(Q)$ is of continuous type, $P^- = Q_{\text{tree}}(q^-)$, then $d_q \beta = J_{\text{sup}}(\vec{\beta} q)$. Suppose by induction that:

Then $\vec{\beta}$ respects $Q$ iff all of the following holds:

(a) $\leq 2m+1 \vec{\beta}$ respects $\leq 2m+1 Q$, where $\leq 2m+1 \vec{\beta} = \vec{\beta} \rightarrow \text{dom}(\leq 2m+1 Q)$.

(b) For any $q \in \text{dom}(2m+2 Q)$, the $2m+2 Q_{\text{tree}}(q)$-potential partial level $\leq 1$ tower induced by $\beta_q$ is $2m+2 Q[q]$, and the $2m+2 Q_{\text{tree}}(q)$-approximation sequence of $2m+2 \beta_q$ is $(2m+2 \beta_q)_{\leq \text{lh}(q)}$.

(c) If $2m+2 Q_{\text{tree}}(q^-) (a)$ is $2m+2 Q_{\text{tree}}(q^-) (b)$ and $a < BK b$ then $2m+2 \beta_q^- (a) < 2m+2 \beta_q^- (b)$.

Moreover, if $\vec{C} = (C_i)_{i \leq m} \in \prod_{i \leq m} \nu_{2i+1}$ is a sequence of clubs, then $\vec{\beta} \in [\vec{C}]^{\mathcal{Q}_i}$ iff $\vec{\beta}$ respects $Q$, $\leq 2m+1 \vec{\beta} \in [\vec{C}]^{\leq 2m+1 \mathcal{Q}_i}$, and letting $C'$ be the set of limit points of $C$, then $2m+2 \beta_q \in j_{2m+2 \mathcal{Q}_{\text{tree}}(q)}(C')$ for each $q \in \text{dom}(2m+2 Q)$.

We assume by induction the level-$2m+2$ Becker-Kechris-Martin theorem:

For each $A \subseteq u_{(2m+1) E}(2m+1) \times \mathbb{R}$, the following are equivalent.

(a) $A$ is $\Pi^1_{2m+3}$.

(b) There is a $\Sigma^1_1$ formula $\varphi$ such that $(\alpha, x) \in A$ iff $L_{\kappa_{2m+3}^{(2m+1)}} [T_{2m+2}, x] \models \varphi(T_{2m+2}, \alpha, x)$.

If $\beta < u_{(2m+1) E}(2m+1) \ni A \subseteq \mathbb{R}$ is $\beta\Pi^1_{2m+3} (x)$ iff there is a $\Pi^1_{2m+3} (x)$ set $B \subseteq u_{(2m+1) E}(2m+1) \times \mathbb{R}$ such that $A = \text{Diff} B$. $\beta\Pi^1_{2m+3} (x)$ acting on product spaces of $\omega$ and $\mathbb{R}$ is defined in the obvious way. Lightface $\beta\Pi^1_{2m+1}$ and boldface $\beta\Pi^1_{2m+1}$ have the obvious meanings.

Define

$$\mathcal{O}^{T_{2m+2}} = \{ (\vec{\varphi}, \alpha) : \varphi \text{ is a } \Sigma_1 \text{-formula}, \alpha < u_{(2m+1) E}(2m+1), L_{\kappa_{2m+3}^{(2m+1)}} [T_{2m+2}, x] \models \varphi(T_{2m+2}, x, \alpha) \},$$

$$\mathcal{M}^{(2m+2)}_x = \{ (\vec{\varphi}, \vec{\psi}) : \exists \alpha < u_x^{(2m+1)}) (\vec{\varphi}, \alpha) \notin \mathcal{O}^{T_{2m+2}} x \land \forall \eta < \alpha (\vec{\psi}, \eta) \in \mathcal{O}^{T_{2m+2}} x \},$$

$$\mathcal{F}^{(2m+2)}_x = \{ (k, \vec{\varphi}, \vec{\psi}) : k < \omega \land (\vec{\varphi}, \vec{\psi}) \in \mathcal{O}^{T_{2m+2}} x \}.$$
where \( F(1, k) = k, \ F(l + 1, k) = \omega^F(l, k) \) in ordinal arithmetic.

Assume by induction that:

\[(14:m) \quad \text{x}^{(2m+2)\#} \text{ is many-one equivalent to } M_{2m+1}^\#(x), \text{ the many-one reductions being independent of } x.\]

\[(15:m) \quad \text{If } A \subseteq \delta_{2m+1}^1, \text{ then } A \in L[T_{2m+1}] \iff A \in L_{\delta_{2m+3}^1}[T_{2m+2}].\]

\[(15) m \text{ ensures that the } L[T_{2m+1}]-\text{measures on } \delta_{2m+1}^1 \text{ induced by level } \leq 2m + 1 \text{ trees are indeed } L_{\delta_{2m+3}^1}[T_{2m+2}]-\text{measures.}\]

\[(13) m \quad \text{enables the generalization of Silver’s dichotomy on } \Pi_{2m+3}^1(x) \text{ equivalence relations: If } E \text{ is a thin } \Delta_{2m+3}^1(x) \text{ equivalence relation on } \mathbb{R}, \text{ then } E \text{ is } \Delta_{2m+3}^1(x)\text{-reducible to } =_{u_{E(2m+1)}}, \text{ where } \alpha = u_{E(2m+1)}, \beta \text{ if } \alpha = \beta < u_{E(2m+1)}.\]

As a corollary, if \( \leq^* \text{ is a } \Delta_{2m+3}^1(x) \text{ prewellordering on } \mathbb{R} \text{ and } A \text{ is a } \Sigma_{2m+3}^1(x) \text{ subset of } \mathbb{R}, \text{ then } |\leq^*| \text{ and } \{|y|_{\leq^*} : y \in A\} \text{ are both } \Delta_1\text{-definable over } L_{\kappa_{2m+3}^1}[T_{2m+2}, M_{2m+1}^\#(x)] \text{ from parameters in } \{T_{2m+2}, M_{2m+1}^\#(x)\}. \text{ This proves an effective version of the Harrington-Kechris theorem (cf. [2, SG.21]):}\]

\[
\text{If } \leq \text{ and } \leq' \text{ are two } \Delta_{2m+3}^1 \text{ prewellorderings of } \mathbb{R}, \text{ then the relation } \|x\|_{\leq} = \|y\|_{\leq'} \text{ is } \Delta_{2m+4}^1 \text{ and is absolute in } M \text{ whenever } M \text{ is a } \Sigma_{2m+3}^1\text{-correct transitive model of ZFC and } \mathbb{R}^M \text{ is closed under the } M_{2m+1}^\#\text{-operator.}
\]

Consequently, if \( \Gamma \) is a pointclass containing \( \Delta_{2m+4}^1 \) and is closed under recursive preimages, then \( \Gamma \) acting on spaces of the form \( (\delta_{2m+3}^1)^k \times \mathbb{R}^l \) is independent of the choice of the \( \Pi_{2m+3}^1 \)-coding of ordinals in \( \delta_{2m+3}^1 \). That is, if \( \varphi \) is a regular \( \Pi_{2m+3}^1 \)-norm on a good universal \( \Pi_{2m+3}^1 \) set, then for any \( A \subseteq \delta_{2m+3}^1 \times \mathbb{R}, \{ (v, x) : (\varphi(v), x) \in A \} \) is in \( \Gamma \) iff \( \{ (v, x) \} \in \Gamma \).

\( \delta_{2m+1}^1 \) is said to have the level\((2m+2)\) strong partition property iff for every finite level \( \leq 2m + 2 \) tree \( Q \), for every \( A \in L_{\delta_{2m+3}^1}[T_{2m+2}] \), there is an club \( C_i \in V_{2i+1} \) for \( i \leq m \) such that either \( [(C_i)_{i \leq m}]^{Q_\uparrow} \subseteq A \) or \( [(C_i)_{i \leq m}]^{Q_\uparrow} \cap A = \emptyset \). We assume by induction that:

\[(16:m) \quad \text{\( \delta_{2m+1}^1 \) has the level\(-(2m+2)\) strong partition property.}\]

\[(17:m) \quad \text{Suppose } (Q_k)_{1 \leq k < \omega} \text{ is an infinite level\(-(2m+2)\) tower and } Q_\omega = \cup_{k < \omega} Q_k. \text{ Then } Q_\omega \text{ is } \Pi_{2m+2}^1\text{-wellfounded iff the direct limit of } (j^{Q_k}Q_{k'})_{k \leq k' < \omega} \text{ is wellfounded.}\]

If \( Q \) is a finite level \( \leq 2m + 2 \) tree, define

\[
A \in \mu^Q
\]
iff $A \subseteq [(\delta_{2i+1})_{i \leq m}]^{Q\uparrow}$, $A \in \mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$ and there is $\bar{C} = (C_i)_{i \leq m}$ such that $\bar{C} \in \prod_{i \leq m} \nu_{2i+1}$ and $[\bar{C}]^{Q\uparrow} \subseteq A$. By (15m), $\mu^Q$ is an $\mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$-measure and $\mu^Q$ is product measure of its level $\leq 2m$ component, its level-$(2m + 1)$ component and its level-$(2m + 2)$ component. Let

$$j^Q : \mathbb{L}_{\delta_{2m+3}}[T_{2m+2}] \to \mathbb{L}_{\sup(j^Q)^{\nu}\delta_{1m+3}}[j^Q(T_{2m+2})]$$

be the associated $\mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$-ultrapower map. Assume by induction that:

(18:m) Suppose $Q$ is a finite level $\leq 2m + 2$ tree. Then $j^Q(\alpha) < \delta_{2m+3}$ for any $\alpha < \delta_{2m+3}$. $j^Q(T_{2m+2}) \in \mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$.

So $\sup(j^Q)^{\nu}\delta_{1m+3} = \delta_{2m+3}$. If $Q$ is a subtree of $Q$, $j^QQ'$ is the induced factor map. If $\pi$ factors $(Q, T)$, $\pi^T$ is the induced factor map. By (15m), $j^Q|\delta_{2m+1} = j^{<2m\uparrow}|\delta_{2m+1}$, and similarly for $j^QQ'$ and $\pi^T$. Define $\sup(j^Q)^Q$, $\pi^T_{\sup}$ as usual. As advertised in the end of [10], we need to prove that every $\mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$ is equivalent to $\mu^Q$ for some finite level $\leq 2m + 2$ tree $Q$ and from this, establish a $\Delta_{1m+3}^1$ coding of subsets of $u_{E(2m+1)}^{(2m+1)}$ in $\mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$.

We assume by induction that:

(19:m) Suppose $P$ is a finite level $\leq (2m+1)$ tree and $\mu$ is a nonprincipal $\mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$-measure on $j^P(\delta_{2m+1})$. Then there are functions $g,h \in \mathbb{L}[j^P(T_{2m+1})]$, a finite level $\leq 2m+2$ tree $Q$, nodes $(d_1, q_1), \ldots, (d_k, q_k) \in \text{dom}(Q)$, such that $h : j^P(\delta_{2m+1}) \to (j^P(\delta_{2m+1}))^k$, $g : (j^P(\delta_{2m+1}))^k \to j^P(\delta_{2m+1})$, $h$ is 1-1 a.e. ($\nu$), $g$ is 1-1 a.e. ($\mu_{\delta_{2m+1}}^Q$), $g = h^{-1}$ a.e. ($\nu$), and

$$A \in \mu^Q_{(d,q)} \iff (h^{-1})^Q A \in \nu.$$

(20:m) There is a $\Delta^1_{2m+3}$ set $X \subseteq \mathbb{R} \times u_{E(2m+1)}^{(2m+1)}$ such that every subset of $u_{E(2m+1)}^{(2m+1)}$ in $\mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$ is equal to some $X_v = \text{DEF} \{a : (v, a) \in X\}$.

Renaming the second coordinate in (20m), we fix a $\Delta^1_{2m+3}$ set

$$X^{(2m+3)} \subseteq \mathbb{R} \times (V_\omega \cup u_{E(2m+1)}^{(2m+1)})^{\omega}$$

such that every subset of $(V_\omega \cup u_{E(2m+1)}^{(2m+1)})^{\omega}$ in $\mathbb{L}_{\delta_{2m+3}}[T_{2m+2}]$ is equal to some $X^{(2m+3)}_v = \text{DEF} \{a : (v, a) \in X^{(2m+3)}\}$. Let $v \in \text{LO}^{(2m+3)}$ iff $X_v^{(2m+3)}$ is a linear ordering of $u_{E(2m+1)}^{(2m+1)}$, $v \in \text{WO}^{(2m+3)}$ iff $X_v$ is a wellordering of $u_{E(2m+1)}^{(2m+2)}$. $\text{LO}^{(2m+3)}$ is $\Delta^1_{2m+3}$, $\text{WO}^{(2m+3)}$ is $\Pi^1_{2m+3}$. For $v \in \text{WO}^{(2m+3)}$, put $|v| = \text{o.t.}(X_v)$. Pointclasses are allowed to act on spaces of the form $(\delta_{2m+3}^1)^k \times \mathbb{R}^l$ via this coding.
Suppose $Q$ is a level $\leq 2m+2$ tree and $W$ is a level $\leq 2m+1$ tree. If $m = 0$, the notations related to $(Q,W,\ast)$-descriptions have been defined in [9]. Suppose $m > 0$. Suppose $D = (d,q,\sigma) \in \text{desc}(Q,W,\ast)$. For $g \in ((\delta_{2i+1})_{i \leq m})^{Q\uparrow}$, let

$$g^W_D$$

be the function on $[(\delta_{2i+1})_{i \leq m}]^{W\uparrow}$ as follows:

1. If $d \leq 2m+1$, then $g^W_D(\vec{\alpha}) = (\leq^{2m+1}g)^{D^{2m}}(\vec{\alpha})$.

2. If $d = 2m+2$ and $D \in \text{desc}(Q,W,Z)$, then $g^W_D(\vec{\alpha}) = j^Z(2m+2q_\eta) \circ id^W_{\eta,Z}(\vec{\alpha})$, or equivalently, $g^W_D([f]^W) = [2m+2g_\eta \circ f^Z_{\sigma}]_{\mu,z}$. Here $id^W_{\eta,Z}$ and $f^Z_{\sigma}$ have already been defined by induction and $id^W_{\eta,Z}([f]^W) = [f^Z_{\sigma}]_{\mu,z}$.

In particular, if $D$ is the constant $(Q,\ast)$-description, then $g^W_D$ is the constant function with value $\delta_{2m+1}$. Suppose additionally that $Q$ is finite. Let

$$id^Q_D$$

be the function $[g]^Q \mapsto [g^W_D]_{\mu,w}$, or equivalently, $\vec{\beta} \mapsto id^Q_D(\vec{\beta})$ if $d \leq 2m+1$, $\vec{\beta} \mapsto g^W_D(\vec{\beta})$ otherwise.

$$seed^Q_D$$

is the element represented modulo $\mu^Q$ by $id^Q_D$. In particular, if $d = 1$ then $seed^Q_D = seed^Q_{(d,q)}$; if $d > 1$, $q = (q,W,id_W)$, then $seed^Q_D = seed^Q_{(d,q)}$. We assume by induction that:

(21:m) Suppose $Q$ is a finite level $\leq 2m+2$ tree, $W$ is a finite level $\leq 2m+1$ tree, $Z$ is a finite level $\leq 2m$ tree, and suppose $D, D' \in \text{desc}(Q,W,Z)$, $D \prec^Q_W D'$, $D = (d,q,\sigma)$, $D' = (d',q',\sigma')$. Then for any $g \in ((\delta_{2i+1})_{i \leq m})^{Q\uparrow}$, for any $f \in ((\delta_{2i+1})_{i \leq m})^{W\uparrow}$, for any $\vec{\eta} \in [(\delta_{2i+1})_{i \leq m}]^{Z\uparrow}$,

$$d'g_\eta \circ f^Z_{\sigma}(\vec{\eta}) < d'g_{\vec{\eta}} \circ f^Z_{\sigma}(\vec{\eta})$$

(22:m) Suppose $Q$ is a finite level $\leq 2m+2$ tree, $W,W'$ are finite level $\leq 2m+1$ trees, $W$ is a proper subtree of $W'$, $D = (d,q,\sigma) \in \text{desc}(Q,W,Z)$, $D' = (d',q',\sigma') \in \text{desc}(Q',W',Z')$, $D = D'|Q,W)$. Suppose $E = (E_i)_{i \leq m} \in \prod_{i \leq m} \nu_{2i+1}$, each $E_i$ is a club, $\eta \in E_i$ iff $E_i \cap \eta$ has order type $\eta$, $E^\eta = (E_i)_{i \leq m}$. Then for any $g \in ((\delta_{2i+1})_{i \leq m})^{Q\uparrow}$, for any $\vec{\alpha} \in [E^\eta]^{W\uparrow}$,

$$j^{Z,W'} \circ g^W_D(\vec{\alpha}) = \sup\{g^{W'}_{D'}(\vec{\beta}) : \vec{\beta} \in [E]^{W\uparrow}, \vec{\beta} \text{ extends } \vec{\alpha}\}.$$
(23:m) Suppose $Q, Q'$ are finite level $\leq 2m + 2$ trees, $Q$ is a proper subtree of $Q'$, $W$ is a finite level $\leq 2m + 1$ tree and $D \in \text{desc}(Q, W, *)$, $D' \in \text{desc}(Q', W, *)$, $D = D' \upharpoonright (Q, W)$. Suppose $\bar{E} = (E_i)_{i \leq m} \in \prod_{i \leq m} \nu_{2i+1}$, each $E_i$ is a club, $\eta \in E_i'$ if $E_i \cap \eta$ has order type $\eta$, $\bar{E}' = (E_i)_{i \leq m}$. Then for any $\bar{\beta} \in [\bar{E}]^{Q\uparrow}$,

$$\text{id}^{Q,W}_D(\bar{\beta}) = \sup \{\text{id}^{Q',W}_D(\bar{\gamma}) : \bar{\gamma} \in [\bar{E}]^{Q\uparrow}, \bar{\gamma} \text{ extends } \bar{\beta} \}. $$

Suppose $S$ is a level $\leq 2m + 1$ tree and $\tau$ factors $(S, Q, *)$. For $g \in (\delta^1_{2m+1})^{Q\uparrow}$, let

$$g^W_\tau$$

be the function sending $\alpha$ to $(g^W_\tau(\alpha))(d,s)\in \text{dom}(S)$. If $W$ is finite, let

$$\text{id}^{Q,W}_\tau$$

is the map sending $[g]^Q$ to $[g^W_\tau]^{W\uparrow}$. So $\text{id}^{Q,W}_\tau(\bar{\beta}) = (\text{id}^{Q,W}_\tau(\bar{\beta}))(d,s)\in \text{dom}(S)$. If $Q, W$ are both finite, put

$$\text{seed}^{Q,W}_\tau = [\text{id}^{Q,W}_\tau]_{\mu^Q}. $$

We assume by induction that:

(24:m) Suppose $Q$ is a finite level $\leq 2m + 2$ tree, $S, W$ are finite level $\leq 2m + 1$ trees, $\tau$ factors $(S, Q, W)$. Then for any $A \in \mu^S$, $\text{seed}^{Q,W}_\tau \in j^Q \circ j^W(A)$.

We define

$$\tau^{Q,W} : \mathbb{I}_{\delta^1_{2m+3}}[j^S(T_{2m+2})] \rightarrow \mathbb{I}_{\delta^1_{2m+3}}[j^Q \circ j^W(T_{2m+2})]$$

by sending $j^S(h)(\text{seed}^S)$ to $j^Q \circ j^W(h)(\text{seed}^{Q,W}_\tau)$. Assume by induction that:

(25:m) Suppose $Q$ is a finite level $\leq 2m + 2$ tree and $W$ is a finite level $\leq 2m + 1$ tree. Then

(a) $(\text{id}_{Q \otimes W})^{Q,W}$ is the identity on $j^Q \circ j^W(\delta^1_{2m+1} + 1)$.

(b) Suppose $W'$ is another finite level $\leq 2m + 1$ tree and $\sigma$ factors $(W, W')$. Then $j^Q(\sigma^{W'} \upharpoonright j^W(\delta^1_{2m+1} + 1)) = (Q \otimes \sigma)^{Q \otimes W'} \upharpoonright (j^{Q \otimes W}(\delta^1_{2m+1} + 1))$.

(c) Suppose $Q'$ is another finite level $\leq 2m + 2$ tree and $\pi$ factors $(Q, Q')$. Then $\pi^{Q'} \upharpoonright (j^Q \circ j^W(\delta^1_{2m+1} + 1)) = (\pi \otimes W)^{Q' \otimes W} \upharpoonright j^{Q' \otimes W}(\delta^1_{2m+1} + 1)$. 

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(26:m) Suppose $x \in \mathbb{R}$ and $Q$ is a finite level $\leq 2m+2$ tree. Then $j^Q(\kappa_{2m+3}^x) = \kappa_{2m+3}^x$ and $j^Q | \kappa_{2m+3}^x$ is $\Delta_1$-definable over $L_{\kappa_{2m+3}^x}[T_{2m+2}, x]$ from $\{T_{2m+2}, x\}$, uniformly in $(Q, x)$.

(27:m) Suppose $Q$ is a finite level $\leq 2m+2$ tree. Then the set of uncountable $L_{\delta_{2m+3}^1}[j^Q(T_{2m+2})]$-cardinals in the interval $[\delta_{2m+1}^1, \delta_{2m+3}^1]$ is

$$\{u^{(2m+1)}_\xi: 0 < \xi \leq E(2m + 1)\},$$

and the set of uncountable $L_{\delta_{2m+3}^1}[j^Q(T_{2m+2})]$-regular cardinals in the interval $[\delta_{2m+1}^1, \delta_{2m+3}^1]$ is

$$\{\text{seed}^Q_{(d, \mathbf{q})}: 2m + 1 \leq d \leq 2m + 2, (d, \mathbf{q}) \in \text{desc}^*(Q) \text{ is regular}\}.$$

In particular, the set of uncountable $L_{\delta_{2m+3}^1}[T_{2m+2}]$-regular cardinals in the interval $[\delta_{2m+1}^1, \delta_{2m+3}^1]$ is

$$\{j^P(\delta_{2m+1}^1): P \text{ level-(2k + 1) tree, } \text{card}(P) \leq 1\}.$$

By (11k)(15k)(18k)(27k) for $k \leq m$, if $Q$ is a finite level $\leq 2m+2$ tree, then the set of uncountable $L_{\delta_{2m+3}^1}[j^Q(T_{2m+2})]$-cardinals is

$$\{u^{(2k+1)}_\xi: k \leq m, 0 < \xi \leq E(2k + 1)\},$$

and the set of uncountable $L_{\delta_{2m+3}^1}[j^Q(T_{2m+2})]$-regular cardinals is

$$\{\text{seed}^Q_{(d, \mathbf{q})}: (d, \mathbf{q}) \in \text{desc}^*(Q) \text{ is regular}\}.$$

Suppose $Q$ is a finite level $\leq 2m+2$ tree, $(d, \mathbf{q}) = (d_i, q_i)_{1 \leq i < k}$ is a distinct enumeration of a subset of $Q$ and such that for each $d > 1$, $\{q_i : d_i = d\} \cup \{0\}$ forms a tree on $\omega^{<\omega}$. Suppose $F : [\delta_{2i+1}^1]_{i \leq m}^{Q^\uparrow} \rightarrow \delta_{2m+3}^1$ is a function which lies in $L_{\delta_{2m}^1}[T_{2m+2}]$. The signature of $F$ is $(d, \mathbf{q})$ iff there is $\bar{E} \in \prod_{i \leq m} \nu_{2i+1}$ such that

1. for any $\bar{\beta}, \bar{\gamma} \in [E]^{Q^\uparrow}$, if $(d_{\alpha_{q_0}}, \ldots, d_{k-1}\gamma_{q_{k-1}}) <_{BK} (d_{0}\beta_{q_0}, \ldots, d_{k-1}\beta_{q_{k-1}})$ then $f(\bar{\beta}) < f(\bar{\gamma})$;

2. for any $\bar{\beta}, \bar{\gamma} \in [E]^{Q^\uparrow}$, if $(d_{\alpha_{q_0}}, \ldots, d_{k-1}\gamma_{q_{k-1}}) = (d_{0}\beta_{q_0}, \ldots, d_{k-1}\beta_{q_{k-1}})$ then $f(\bar{\beta}) = f(\bar{\gamma})$. 

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Clearly the signature of $F$ exists and is unique. In particular, $F$ is constant on a $\mu^Q$-measure one set iff the signature of $F$ is $\emptyset$.

Suppose the signature of $F$ is $(d^*, q^*) = ((d_i, q_i))_{1 \leq i < k}$. $F$ is essentially continuous iff for $\mu^Q$-a.e. $\vec{\beta}$, $F(\vec{\beta}) = \sup\{F(\vec{\gamma}) : \vec{\gamma} \in [(\delta_{2i+1})_{i \leq m}]^{Q\uparrow}, (d_{\alpha - \gamma q_0}, \ldots, d_{\alpha - \gamma q_{k-1}}) < BK (d_{\vec{\beta} q_0}, \ldots, d_{\vec{\beta} q_{k-1}})\}$. Otherwise, $F$ is essentially discontinuous. Put $[\vec{B}]^{Q\uparrow}(0, -1) = [\vec{B}]^{Q\uparrow} \times \omega$. For $(d, q) \in \text{desc}^*(Q)$ regular, put $[\vec{B}]^{Q\uparrow(d, q)} = \{(\vec{\beta}, \gamma) : \vec{\beta} \in [\vec{B}]^{Q\uparrow}, \gamma < q_0\}$. For $(d, q)$ either $(0, -1)$ or in $\text{desc}^*(Q)$ regular, say that the uniform cofinality of $F$ is $\text{ucf}(F) = (d, q)$ iff there is $G : [(\delta_{2i+1})_{i \leq m}]^{Q\uparrow(d, q)} \rightarrow \delta_{2m+3}^1$ such that $G \in \mathbb{L}_{\delta_{2m+3}^1}[T_{2m+2}]$ and for any for $\mu^Q$-a.e. $\vec{\beta}$, $F(\vec{\beta}) = \sup\{G(\vec{\beta}, \gamma) : (\vec{\beta}, \gamma) \in [(\delta_{2i+1})_{i \leq m}]^{Q\uparrow(d, q)}\}$ and the function $\gamma \mapsto G(\vec{\beta}, \gamma)$ is order preserving. Let $(X_i, (d_i, x_i, W_i)) \sim (X_k)$ be the partial level $\leq 2m + 2$ tower of continuous type and let $\pi$ factor $(X_k, Q)$ such that $\pi(d_i, x_i) = (d_i, q_i)$ for each $1 \leq i < k$. The potential partial level $\leq 2m + 2$ tower induced by $F$ is

1. $(X_k, (d_i, x_i, W_i))_{1 \leq i < k}$, if $F$ is essentially continuous;
2. $(X_k, (d_i, x_i, W_i)) \sim (0, -1, 0)$, if $F$ is essentially discontinuous and has uniform cofinality $(0, -1)$;
3. $(X_k, (d_i, x_i, W_i))_{1 \leq i < k} \sim (d^+, x^+, W^+)$, if $F$ is essentially discontinuous and has uniform cofinality $(d^*, q^*)$, $d^* > 0$, $(X_k, (d^+, x^+, W^+))$ is a partial level $\leq 2m + 2$ tree with uniform cofinality $(d^*, x^*)$, $d^*\pi(x^*) = q^*$.

The approximation sequence of $F$ is $(F_i)_{1 \leq i \leq k}$ where $F_i$ is a function on $[(\delta_{2i+1})_{i \leq m}]^{X_{i\uparrow}}$, $F_i(\vec{\beta}) = \sup\{F(\vec{\gamma}) : \vec{\gamma} \in [(\delta_{2i+1})_{i \leq m}]^{Q\uparrow}, (d_{\gamma q_0}, \ldots, d_{\gamma q_{k-1}}) = (d_{\vec{\beta} q_0}, \ldots, d_{\vec{\beta} q_{k-1}})\}$ for $1 \leq i \leq k$.

Suppose $\delta_{2m+1}^1 \leq \gamma < \delta_{2m+3}^1$ is a limit ordinal. Suppose $Q$ is a finite level $\leq 2m + 2$ tree, $\gamma = [F]_{\mu^Q}$, the signature of $F$ is $(d_i, q_i)_{1 \leq i < k}$, the approximation sequence of $F$ is $(F_i)_{1 \leq i \leq k}$. Then the $Q$-signature of $\beta$ is $(d_i, q_i)_{1 \leq i < k}$, the $Q$-approximation sequence of $\gamma$ is $([F]_{\mu^Q})_{1 \leq i \leq k}$; $\gamma$ is $Q$-essentially continuous iff $F$ is essentially continuous. The $Q$-uniform cofinality of $\gamma$ is $\omega$ if $F$ has uniform cofinality $(0, -1)$, seed $^Q_{(d, q)}$ if $F$ has uniform cofinality $(d, q) \in \text{desc}^*(Q)$. The $Q$-(potential) partial level $\leq 2m + 2$ tower induced by $\gamma$ and the $Q$-factoring map are the potential partial level $\leq 2m + 2$ tower induced by $F$ and the factoring map induced by $F$ respectively. Assume by induction that:

(28:m) If $x \in \mathbb{R}$ and $\gamma < \kappa_{2m+3}$, then the $Q$-potential partial level $\leq 2m + 2$ tower induced by $\gamma$ and the $Q$-approximation sequence of $\gamma$ are uniformly $\Delta_1$-definable over $L_{\kappa_{2m+3}^x}[T_{2m+2}, x]$ from $(T_{2m+2}, x, Q, \gamma)$.
Suppose \( T, Q \) are level \( \leq 2m + 2 \) trees and \( C = (d, t, \tau) \in \operatorname{desc}(T, Q, \ast) \). For \( h \in (\langle \delta_{2i+1} \rangle_{i \leq m})_{T^*} \),
\[
h^Q_C
\]
is the function on \( \langle (\delta_{2i+1})_{i \leq m} \rangle^{Q^*} \) defined as follows:

1. If \( d \leq 2m + 1 \), then \( h^Q_C(\vec{\beta}) = (\leq 2m+1h)^{\leq 2mQ}_C(\vec{\beta}) \).
2. If \( d = 2m + 2 \), then \( h^Q_C(\vec{\beta}) = j^W(2m+2h_t) \circ \operatorname{id}^Q_W(\vec{\beta}) \), or equivalently, \( h^Q_C([g]^Q) = [2m+2h_t \circ g^W \tau]^{\mu} \).

Suppose additionally that \( T \) is finite. Let
\[
\operatorname{id}^{T, Q}_C
\]
be the function \( [g]^T \mapsto [g^Q]_{\nu^Q, \ast} \), or equivalently, \( \xi \mapsto \operatorname{id}^{\leq 2m+1\nu^T, \leq 2mQ}(\xi) \) if \( d \leq 2m + 1 \), \( \xi \mapsto \tau^Q_W(\xi) \) otherwise.

\[
\operatorname{seed}^{T, Q}_C
\]
is the element represented modulo \( \mu^T \) by \( \operatorname{id}^{T, Q}_C \). By (25) and induction hypotheses at lower levels, we have:

1. If \( d = 1 \), then \( \operatorname{seed}^{T, Q}_C = \operatorname{seed}^T_C \).
2. If \( d > 1 \) and \( C \in \operatorname{desc}(T, Q, W) \), then \( \operatorname{seed}^{T, Q}_C = \operatorname{seed}^{Q \otimes W}_C = \operatorname{seed}^{(Q \otimes W)}_C \).

We assume by induction that:

(29) Suppose \( T, Q \) are finite level \( \leq 2m + 2 \) trees, \( W \) is a finite level \( \leq 2m + 1 \) tree, and suppose \( C, C' \in \operatorname{desc}(T, Q, W) \), \( C <_{T, Q} C', C = (d, t, \tau), C' = (d', t', \tau') \). Then for any \( h \in (\langle \delta_{2i+1} \rangle_{i \leq m})_{T^*} \), for any \( f \in (\langle \delta_{2i+1} \rangle_{i \leq m})^{Q^*} \), for any \( \vec{\alpha} \in (\langle \delta_{2i+1} \rangle_{i \leq m})^{W^*} \),
\[
d_{g_{t'}^W} f_{t^W}^W(\vec{\alpha}) \leq d_{g_{t'}^W} f_{t^W}^W(\vec{\alpha}).
\]

(30) Suppose \( T, Q, Q' \) are finite level \( \leq 2m + 2 \) trees, \( Q \) is a proper subtree of \( Q' \), \( C = (d, t, \tau) \in \operatorname{desc}(T, Q, W) \), \( C' = (d', t', \tau') \in \operatorname{desc}(T', Q', W') \), \( C \cap (T, Q) \). Suppose \( \vec{E} = (E_i)_{i \leq m} \in \prod_{i \leq m} E_{\nu_{2i+1}} \), each \( E_i \) is a club, \( \eta \in E_i \) iff \( E_i \cap \eta \) has order type \( \eta \), \( \vec{E}' = (E_i)_{i \leq m} \). Then for any \( h \in (\langle \delta_{2i+1} \rangle_{i \leq m})_{T^*} \), for any \( \vec{\alpha} \in (\vec{E}')^{Q^*} \),
\[
j^{W', W} \circ h^Q_C(\vec{\alpha}) = \sup \{ h^{Q'}_{C'}(\vec{\beta}) : \vec{\beta} \in [\vec{E}]^{Q^*}, \vec{\beta} \text{ extends } \vec{\alpha} \}.\]
(31:m) Suppose \( T, T', Q \) are finite level \( \leq 2m + 2 \) trees, \( T \) is a proper subtree of \( T' \) and \( C \in \text{desc}(T, Q, *) \), \( C' \in \text{desc}(T', Q, *) \), \( C = C' \upharpoonright (T, Q) \). Suppose \( \vec{E} = (E_i)_{i \leq m} \in \prod_{i \leq m} \nu_{2i+1} \), each \( E_i \) is a club, \( \eta \in E_i' \) iff \( E_i \cap \eta \) has order type \( \eta \), \( \vec{E}' = (E_i)_{i \leq m} \). Then for any \( \vec{\xi} \in [\vec{E}']^{T'} \),
\[
\text{id}_{T,Q}^{\vec{\xi}} = \sup \{ \text{id}_{T',Q}^{\vec{\eta}}(\vec{\eta}) : \vec{\eta} \in [\vec{E}]^{T}, \vec{\eta} \text{ extends } \vec{\xi} \}.
\]
Suppose \( X \) is a level \( \leq 2m + 2 \) tree and \( \pi \) factors \( (X, T, Q) \). For \( h \in ((\delta_{2i+1})_{i \leq m})^{T'} \), let
\[
h_{\pi}^T
\]
be the function sending \( \overrightarrow{\beta} \) to \( (h_{\pi_{(d,x)}}(\overrightarrow{\beta}))_{(d,x) \in \text{dom}(X)} \). If \( Q \) is finite, let
\[
\text{id}_{T,Q}^{\vec{\xi}}
\]
be the map sending \( [h]^T \) to \( [h_{\pi}^{\mu_Q}]_{\mu_Q} \). So \( \text{id}_{T,Q}^{\vec{\xi}}(\vec{\xi}) = (\text{id}_{T,Q}^{\vec{\eta}}(\vec{\eta}))_{(d,x) \in \text{dom}(X)} \). If \( T, Q \) are both finite, put
\[
\text{seed}_{T,Q}^{\vec{\xi}} = [\text{id}_{T,Q}^{\vec{\xi}}]_{\mu_T}.
\]
We assume by induction that:

(32:m) Suppose \( X, T, Q \) are finite level \( \leq 2m + 2 \) trees. Then for any \( A \in \mu^X \),
\[
\text{seed}_{T,Q}^{\vec{\xi}} \in j^T \circ j^Q(A).
\]
We define
\[
\pi_{T,Q} : \mathbb{L}_{\delta_{2m+1}^1}[j^X(T_{2m+2})] \to \mathbb{L}_{\delta_{2m+3}^1}[j^T \circ j^Q(T_{2m+2})]
\]
by sending \( j^X(h)(\text{seed}^X) = j^T \circ j^Q(h)(\text{seed}_{T,Q}^{\vec{\xi}}) \).
We assume by induction that:

(33:m) Suppose \( X, T, Q \) are finite level \( \leq 2m + 2 \) trees and \( \pi \) factors \( (X, T) \). Then
\[
(\text{a}) \quad j^T \circ j^Q = j^{T \otimes Q}.
\]
\[
(\text{b}) \quad j^Q(\pi^{T \upharpoonright a}) = (Q \otimes \pi)^{Q \otimes T} \upharpoonright j^Q(a) \quad \text{for any } a \in \mathbb{L}_{\delta_{2m+1}^1}[T_{2m+2}];
\]
\[
(\text{c}) \quad \pi^{T \upharpoonright \mathbb{L}_{\delta_{2m+3}^1}[j^X \otimes Q(T_{2m+2})]} = (\pi \otimes Q)^{T \otimes Q}.
\]
Suppose \( Q \) is a finite level \( \leq 2m + 2 \) tree. Suppose \( (d, q) \in \text{desc}^+(Q) \), and if \( d > 1 \) then \( q = (q, P, \overrightarrow{p}) \). Put
\[
[(d, q)]_Q = \begin{cases} 
\| (1, (q)) \|_q & \text{if } d = 1, \\
[\overrightarrow{\alpha} \mapsto \| (d, \overrightarrow{\alpha} \oplus q, q) \|_q]_{\mu_P} & \text{if } d > 1.
\end{cases}
\]
To save ink, put \([d, q]_Q = [(d, q)]_Q\). If in addition, \(d > 1\) and \(q = (q, \ldots) \in \text{desc}(Q)\) of discontinuous type, put \([d, q]_Q = [d, q]_Q\). If \(\pi\) factors level \(\leq 2m + 2\) trees \((Q, T)\), then \(\pi\) is said to minimally factor \((Q, T)\) iff \(X, T\) are both \(\Pi^1_{2m+2}\)-wellfounded and for any \((d, q) \in \text{dom}(Q)\), \([d, q]_Q = [\pi(d, q)]_T\).

Assume by induction that:

(34) Suppose \(X, T\) are \(\Pi^1_{2m+2}\)-wellfounded level \(\leq 2m + 2\) trees. Then there exist a \(\Pi^1_{2m+2}\)-wellfounded level \(\leq 2n\) tree \(Q\) and a map \(\pi\) minimally factoring \((X, T \otimes Q)\).

Suppose \(R\) is a level-(2\(+3\)) tree. The ordinal representation of \(R\) is the set

\[
\text{rep}(R) = \{ \vec{\alpha} +_R r : r \in \text{dom}(R), \vec{\alpha} \text{ respects } R_{\text{tree}}(r) \} \\
\cup \{ \vec{\alpha} +_R r \uparrow (-1) : r \in \text{dom}(R), \vec{\alpha} \text{ respects } R(r) \}.
\]

Here for \(r \in \text{dom}^*(R)\) of length \(k\), \(\vec{\alpha} +_R r = (r(0), \alpha_{\text{node}(r)}, r(1), \ldots, \alpha_{\text{node}(r)}, r(k - 1))\). \(\text{rep}(R)\) is endowed with the \(<_{BK}\) ordering

\[
<^R = <_{BK}|\text{rep}(R).
\]

Assume by induction that:

(35) A level-(2\(+3\)) tree \(R\) is \(\Pi^1_{2m+3}\)-wellfounded iff \(<^R\) is a wellordering.

Suppose \(B \in \mathbb{L}_{\delta^1_{2m+2}}[T_{2m+2}]\). Define

\[
F \in B^{R^T}
\]

iff \(F \in \mathbb{L}_{\delta^1_{2m+3}}[T_{2m+2}]\) is an order preserving function from \(\text{rep}(R)\) to \(B\). If \(F \in (\delta^1_{2m+3})^{R^T}\), then for any \(r \in \text{dom}(R)\), \(F_r\) is a function on \([\delta^1_{2m+3}]_{\leq m}^{R_{\text{tree}}(r)}\|_{\text{tree}}\) that sends \(\vec{\beta}\) to \(F(\vec{\beta} +_R r)\), and \(F\) represents a tuple of ordinals

\[
[F]^R = ([F]^R_{r \in \text{dom}(R)}
\]

where \([F]^R_r = [F]^R_{r}\|_{\text{tree}(r)}\) for \(r \in \text{dom}(R)\). Let

\[
[B]^R = \{ [F]^R : F \in B^{R^T}\}.
\]

A tuple of ordinals \(\vec{\gamma}(r) \in \text{dom}(R)\) respects \(R\) iff \(\vec{\gamma} \in [\delta^1_{2m+3}]{\|_{\text{tree}}}R^T\). \(\vec{\gamma}\) weakly respects \(R\) iff for any \(r, r' \in \text{dom}(R)\), if \(r\) is a proper initial segment of \(r'\), then \(j^{R_{\text{tree}}(r), R_{\text{tree}}(r')}(\gamma_r) > \gamma_{r'}\). If \(r = (r, Q, \ldots) \in \text{desc}^*(R)\) and \(F \in (\delta^1_{2m+3})^{R^T}\), define \(F_r\) to be a function on \([\delta^1_{2m+3}]_{\leq m}^{Q_{\text{tree}}(r)}\|_{\text{tree}}\) if \(r \in \text{desc}(R)\), then \(F_r = F_r\); if \(r \notin \text{desc}(R)\), then \(F_r(\vec{\beta}) = F_r(\vec{\beta} | R_{\text{tree}}(r))\). If \(\vec{\gamma}(r) \in \text{dom}(R)\)
\[ \delta_{1,2m+3}^T \text{, put } \gamma_r = [F_r]_{\mathcal{R}}. \text{ If } r \in \text{desc}(R) \text{ and } A = (r, \pi, T) \in \text{desc}^{**}(R), \text{ put } \gamma_A = \pi^T(\gamma_r). \text{ Put } \gamma_0 = \gamma_0(0, \delta, 0) = \delta_{2m+3}^T. \text{ Thus, if } r \in \text{desc}(R) \text{ is of discontinuous type, then } \gamma_r = \gamma_r \text{; if } r \notin \text{desc}(R), \text{ then } \gamma_r = j_{\text{tree}(r), Q}(\gamma_r) = \gamma_{(r, \text{id}_{\text{tree}(r), Q})}. \] 

The next induction hypothesis computes the remaining case when \( r \in \text{desc}(R) \) is of continuous type, justifying that \( \gamma_r \) does not depend on the choice of \( F \).

\[(36:m) \text{ Suppose } R \text{ is a level-(2m + 3) tree, } \bar{\gamma} \in [\delta_{1,2m+3}^T]^R, r = (r, Q, (d, q, P)) \in \text{desc}(R) \text{ is of continuous type. Then } \gamma_r = j_{\text{tree}(r), Q}(\gamma_{r-}). \]

We assume by induction that:

\[(37:m) \text{ Suppose } R \text{ is a level-(2m + 3) tree and } \bar{\gamma} = (\gamma_{r})_{r \in \text{dom}(R)} \text{ is a tuple of ordinals in } \delta_{1,2m+3}^T. \text{ Then } \bar{\gamma} \text{ respects } R \text{ iff the following holds:}

\begin{itemize}
  \item[(a)] For any \( r \in \text{dom}(R) \), the \( R_{\text{tree}}(r) \)-potential partial level \( \leq 2m + 2 \) tower induced by \( \gamma_r \) is \( R[r] \), and the \( R_{\text{tree}}(r) \)-approximation sequence of \( \gamma_r \) is \( (\gamma_{\mathcal{R}}^t)_{1 \leq t \leq h(r)} \).
  \item[(b)] If \( R_{\text{tree}}(r^-(a)) = R_{\text{tree}}(r^-(b)) \) and \( a <_{B^K} b \) then \( \gamma_{p^-(a)} < \gamma_{p^-(b)} \).
\end{itemize}

Moreover, if \( B \in \mathcal{L}_{\delta_{1,2m+3}}^T(T_{2m+2}) \) is a closed set, \( B' \) is the set of limit points of \( B \), then \( \bar{\gamma} \in [B]^R \) iff \( \bar{\gamma} \) respects \( R \) and for each \( r \in \text{dom}(R) \), \( \gamma_r \in j_{R_{\text{tree}}(r)}(B') \).

Define \( C_{2m+3}^* = \{ \xi < \delta_{1,2m+3}^T : \text{for any finite level } \leq 2m + 2 \text{ tree } Q, j_Q^T(\xi) = \xi \} \). By \((26,m)\), \( C_* \cap \kappa_{2m+3}^* \) has order type \( \kappa_{2m+3}^* \), and hence \( C_* \) has order type \( \delta_{1,2m+3}^T \). Suppose \( R \) is a level-(2m + 3) tree. \( \bar{\gamma} \) strongly respects \( R \) iff \( \bar{\gamma} \in [C_*]^R \). The function \( A \mapsto (A) \) is defined exactly as in \([9]\). So are the relations \( A \prec_R A', r \prec_R A, r \sim_R A, \text{ etc. for } r \in \text{desc}^*(R), r \in \text{dom}(R). \) Define \( \prec_* = \prec \text{ desc}^*(R), \sim_* = \sim \text{ desc}^*(R) \). If \( A = (r, \pi, T) \) and \( \bar{\gamma} \) respects \( R \), let \( \gamma_A = \pi^T(\gamma_r) \). Assume by induction that:

\[(38:m) \text{ Suppose } R \text{ is a level-(2m + 3) tree, } A, A' \in \text{desc}^*(R), \bar{\gamma} \text{ strongly respects } R \text{. Then } A \prec_R A' \text{ iff } \gamma_A < \gamma_{A'}; A \sim_R A' \text{ iff } \gamma_A = \gamma_{A'}. \]

Suppose now \( R \) is a finite level \( \leq 2m + 3 \) tree. Then \( \text{rep}(R) = \cup_i \{d_i \times \text{rep}(d_i)\} \). Suppose \( B = (B_i)_{i \leq m+1} \in \mathcal{L}[T_{2m+3}] \). Define \( f \in \mathcal{B}^{R_{\uparrow}^T} \) iff \( f \in \mathcal{L}[T_{2m+3}] \) is an order preserving function from \( \text{rep}(R) \) to \( \cup_i B_i \) such that for any \( i \), \( \text{ran}(f) \subseteq B_i \). Define \( [B]^{R_{\uparrow}^T} = \{ [f]^R : f \in \mathcal{B}^{R_{\uparrow}^T} \} \). \( \alpha = (\alpha_{(d,r)})_{(d,r) \in \text{dom}(R)} = (\alpha_{r})_{r \in \text{dom}(R)} \) respects \( R \) iff \( \bar{\alpha} \in [\delta_{1,2m+1}]^{R_{\uparrow}^T} \).

Suppose \( Y \) is a level \( \leq 2m + 3 \) tree, \( T \) is a level \( \leq 2m + 2 \) tree, and \( B = (d, y, \pi) \in \text{desc}(Y, T, *) \), \( F \in ((\delta_{2i+1})_{i \leq m+1})^{\uparrow} \). Then

\[ F_B^T : [\omega_1]^{\uparrow} \to \delta_3 \]
is the function that sends \( \xi \) to \( j^Q(dF_y) \circ \text{id}_T^Y(\xi) \), or equivalently, sends \( [h]^T \) to \( [dF_y \circ h^Q]^T_\mu Q \).

\[
\text{id}_B^Y \quad \text{id}_B^T 
\]

is the function \( [F]^Y \mapsto [F_B]^T_\mu \), or equivalently, \( \vec{\gamma} \mapsto \pi^T Q(\vec{\gamma}_y) \).

\[
\text{seed}_B^Y \quad \text{seed}_B^T
\]

is the element represented modulo \( \mu^Y \) by \( \text{id}_B^Y \). We assume by induction that:

\((39: m)\) Suppose \( Y \) is a finite level \( \leq 2m + 3 \) tree, \( T, Q \) are finite level \( \leq 2m + 2 \) trees, and suppose \( B, B' \in \text{desc}(Y, T, Q), B \prec^Y T B', B = (d, y, \pi), B' = (d', y', \pi'). \) Then for any \( F \in (((\delta_{2i+1})_{i \leq m+1})^Y, \) for any \( h \in (((\delta_{2i+1})_{i \leq m})^Y)_{Q^T}, \) \( Y, \) \( Y, Y' \) is a proper subtree of \( E \), \( E \) is a level \( \leq 2m + 3 \) tree, and suppose \( B, B' \in \text{desc}(Y', T, Q), B = (d', y', \pi'), B' = (d, y, \pi), B' = (d', y', \pi'). \) Then for any \( F \in (((\delta_{2i+1})_{i \leq m+1})^Y \uparrow T, \) for any \( \vec{\beta} \in (((\delta_{2i+1})_{i \leq m})^Y)_{Q^T}, \)

\[
F_y \circ h^Q_B(\vec{\beta}) < d'F_{y'} \circ h^Q_{B'}(\vec{\beta}).
\]

\((40: m)\) Suppose \( Y \) is a finite level \( \leq 2m + 3 \) tree, \( T, T' \) are finite level \( \leq 2m + 2 \) trees, \( T \) is a proper subtree of \( T', \) \( B = (y, \pi) \in \text{desc}(Y, T, Q), B' = (y', \pi') \in \text{desc}(Y', T', Q'), B = B' \uparrow (Y, T). \) Suppose \( E = (E_i)_{i \leq m} \in \prod_{i \leq m} \nu_{2i+1}, \) each \( E_i \) is a club, \( \eta \in E_i \) iff \( E_i \cap \eta \) has order type \( \eta, \)

\[
\vec{E} = (E_i)_{i \leq m}. \) \) Then for any \( F \in (((\delta_{2i+1})_{i \leq m+1})^Y \uparrow T, \) \( \vec{F} \in [E]^T \uparrow, \)

\[
\text{sup}\{F_{B'}^T(\vec{\eta}) : \vec{\eta} \in [E]^T \uparrow, \vec{\eta} \text{ extends } \vec{\xi}\}.
\]

\((41: m)\) Suppose \( Y, Y' \) are finite level \( \leq 2m + 3 \) trees, \( Y \) is a proper subtree of \( Y', T \) is a finite level \( \leq 2m + 2 \) tree and \( B \in \text{desc}(Y', T, *), \)

\[
B' \in \text{desc}(Y', T', *), B = B' \uparrow (Y, T). \) Suppose \( E = (E_i)_{i \leq m+1} \in \prod_{i \leq m+1} \nu_{2i+1}, \) each \( E_i \) is a club, \( \eta \in E_i \) iff \( E_i \cap \eta \) has order type \( \eta, \)

\[
\vec{E} = (E_i)_{i \leq m+1}. \) \) Then for any \( \vec{\gamma} \in [E]^Y \uparrow, \)

\[
\text{id}_B^Y(\vec{\gamma}) = \text{sup}\{\text{id}_B^Y(\vec{\delta}) : \vec{\delta} \in [E]^Y \uparrow, \vec{\delta} \text{ extends } \vec{\gamma}\}.
\]

Suppose \( \rho \) is a level \( \leq 2m + 3 \) tree and \( \rho \) factors \( (R, Y, T). \) For \( F \in (((\delta_{2i+1})_{i \leq m+1})^Y \uparrow, \) let

\[
F_{\rho}^Y
\]

be the function sending \( \vec{\gamma} \) to \( (F_{\rho(d,r)}^T(\vec{\gamma}))(d,r) \in \text{dom}(R). \) If \( T \) is finite, let

\[
\text{id}_{\rho}^Y
\]

34
be the map sending \([ F ]^Y \) to \([ F ]^T_{\rho,R} \). So \( \text{id}^Y_T(\gamma) = (\text{id}^Y_T(\gamma))_{(d,r) \in \text{dom}(R)} \).

Suppose \( R \) is \( \Pi^1_{2m+3} \)-wellfounded. Put \( [2m+3,\emptyset]_R = \text{o.t.}(\langle R \rangle) \). For \((d,r) \in \text{desc}^*(R)\), \( r = (r,Q,(d,q,P)) \), put

\[
\llbracket d,r \rrbracket_R = \llbracket \beta \mapsto \|d,\beta \oplus_R r\|_{\llbracket\mu\rrbracket}\rrbracket_R.
\]

If \((d,r) \in \text{desc}(R)\) is of discontinuous type, put \( \llbracket d,r \rrbracket_R = \llbracket d,r \rrbracket_R \). We say that \( \rho \) minimally factors \((R,Y)\) iff \( \rho \) factors \((R,Y)\), \( R,Y \) are both \( \Pi^1_{2m+3} \)-wellfounded and \( \llbracket d,r \rrbracket_R = \llbracket \rho(d,r) \rrbracket_Y \) for any \( r \in \text{dom}(R) \). Assume the induction hypothesis:

(42:m) Suppose \( R,Y \) are \( \Pi^1_{2m+3} \)-wellfounded level \( \leq 2m+3 \) trees and \( [2m+3,\emptyset]_R \leq [2m+3,\emptyset]_Y \). Then there exist a \( \Pi^1_{2m+2} \)-wellfounded level \( \leq 2m+2 \) tree \( T \) and a map \( \rho \) minimally factoring \((R,Y \otimes T)\). If \( [2m+3,\emptyset]_R < [2m+3,\emptyset]_Y \), we can further assume that for some \( B \in \text{desc}(Y,T,\ast) \) we have \( \text{lh}(B) = 1 \) and \( [2m+3,\emptyset]_R = [B]_Y \).

If \( R \) is a level-\((2m+3)\) tree, then \( \llbracket r \rrbracket_R = [2m+3]_{R'} \) and \( \llbracket r \rrbracket_R = [2m+3]_{Y'} \), where \( R' = Q^0_0(2m+2) \oplus R \). If \( R,Y \) are level-\((2m+3)\) trees, then \( \rho \) minimally factors \((R,Y)\) iff \( \rho \) extends to \( \rho' \) which minimally factors \((Q^0_0(2m+2) \oplus R,Q^0_0(2m+2) \oplus Y)\).

Suppose \( R \) is a level-\((2m+3)\) tree and \( \text{dom}(R) = \{(0)\} \), \( R_{\text{node}}(((0))) = (d,q) \), \( Q \) is a completion of \( R(((0))) \). We say that \( \epsilon \) is \( Q \)-represented by \( T \) iff \( Q \) is a subtree of \( T \) and \( \llbracket d,q \rrbracket_T = \epsilon \). Suppose \( Q \) is a finite level \( \leq 2m+2 \) tree and \( \tilde{\epsilon} = (\tilde{\epsilon}_t)_{(d,t) \in \text{dom}(Q)} \) is a tuple of ordinals indexed by \( \text{dom}(Q) \). We say that \( \tilde{\epsilon} \) is represented by \( Q' \) iff \( Q \) is a subtree of \( Q' \), \( Q' \) is \( \Pi^1_{2m+2} \)-wellfounded and \( \tilde{\epsilon} = ([d,q]_{(d,t) \in \text{dom}(Q')}) \). Similarly define a tuple \( \tilde{\epsilon} = (\tilde{\epsilon}_t)_{r \in \text{dom}(R)} \) being represented by a level-\((2m+3)\) tree \( R \). Assume by induction that:

(43:m) Suppose \( R \) is a level-\((2m+3)\) tree and \( \text{dom}(R) = \{(0)\} \), \( R(((0))) = (Q^0_0(2m+2),(d,q),P) \), \( 2k+1 \leq d < 2k+3 \), \( Q \) is a completion of \( R(((0))) \). Then cofinally many ordinals in \( j^R(d^1_{2k+1}) \) are \( Q \)-represented by some level \( \leq 2m+2 \) tree.

(44:m) Suppose \( Q \) is a finite level \( \leq 2m+2 \) tree and \( \tilde{\epsilon} \) respects \( Q \). Then there is \( Q' \) extending \( Q \) such that \( \tilde{\epsilon} \) is represented by \( Q' \).

(45:m) Suppose \( R \) is a finite level-\((2m+3)\) tree and \( \tilde{\epsilon} \) respects \( R \). Then there is \( R' \) extending \( R \) such that \( \tilde{\epsilon} \) is represented by \( R' \).

To every ordinal \( \xi < E(2m+3) \), we assign \( \hat{\xi} \) as follows:

1. If \( \xi < E(2m+1) \), then \( \hat{\xi} \) has been defined by induction.

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2. If $0 < \eta = \omega^{E(2m)+\eta_1} \cdots + \omega^{E(2m)+\eta_k} + \chi < E(2m + 2), E(2m + 1) > \eta_1 \geq \cdots \geq \eta_k, E(2m + 1) > \chi$, then $\omega^\eta = u_{\eta_1+1}^{(2m+1)} \cdots u_{\eta_k+1}^{(2m+1)} \cdot \hat{\xi}$.

3. If $0 < \xi = \omega^{\eta_1} \cdots + \omega^{\eta_k}, E(2m + 2) > \eta_1 \geq \cdots \geq \eta_k$, then $\hat{\xi} = \omega^{\eta_1} + \cdots + \omega^{\eta_k}$.

Let $R_{E(2m+3)}^{(2m+3)}$ be the unique (up to an isomorphism) level-$(2m + 3)$ tree such that

1. for any finite level-$(2m + 3)$ tree $Y$, there exists $\rho$ which minimally factors $(Y, R_{E(2m+3)}^{(2m+3)})$;
2. if $r \in \text{dom}(R_{E(2m+3)}^{(2m+3)})$ then there exist a finite $Y$ and $\rho$ which minimally factors $(Y, R)$ such that $r \in \text{dom}(\rho)$.

We fix the following representation of $R_{E(2m+3)}^{(2m+3)}$, whose domain consists of finite tuples of ordinals in $E(2m + 3)$:

1. $(\xi_1) \in \text{dom}(R_{E(2m+3)}^{(2m+3)})$ iff $0 < \xi_1 < E(2m + 3)$. $R_{E(2m+3)}^{(2m+3)}((\xi_1))$ is the $Q_0^{(2m)}$-partial level $\leq 2m + 2$ tree induced by $\hat{\xi}_1$.
2. If $r = (\xi_1, \ldots, \xi_{k-1}) \in \text{dom}(R_{E(2m+3)}^{(2m+3)})$, then $r^{-}((\xi_k)) \in \text{dom}(R_{E(2m+3)}^{(2m+3)})$ iff $\xi_k < E(2m + 3)$ and there exists a completion $Q^+$ of $R_{E(2m+3)}^{(2m+3)}(r)$ such that the $Q^+$-approximation sequence of $\hat{\xi}_k$ is $(\hat{\xi}_1)_{1 \leq i \leq k}$; if $r^{-}((\xi_k)) \in \text{dom}(R_{E(2m+3)}^{(2m+3)})$ and $Q^+$ is the unique such completion, then $R_{E(2m+3)}^{(2m+3)}(r^{-}(\xi_k))$ is the $Q^+$-partial level $\leq 2m + 2$ tree induced by $\hat{\xi}_k$.

Therefore, $\llbracket \emptyset \rrbracket_R = u_{E(2m+1)}^{(2m+1)}$ and if $r = (\xi_1, \ldots, \xi_k) \in \text{dom}(R_{E(2m+3)}^{(2m+3)})$, then $\llbracket r \rrbracket_R = \hat{\xi}_k$. If $Y$ is a finite level-$3$ tree, then the map $y \mapsto r_y$ minimally factors $(Y, R)$, where if $(\llbracket y \rrbracket_R y)_{1 \leq i \leq \text{lh}(y)} = (\hat{\xi}_1, \ldots, \hat{\xi}_{\text{lh}(y)})$ then $r_y = (\xi_1, \ldots, \xi_{\text{lh}(y)})$. For $0 < \xi < E(2m + 3)$, let $R_{\xi}^{(2m+3)} = R_{E(2m+3)}^{(2m+3)} | (\xi)$.

## 4  The level-$(2n + 1)$ sharp

From now on, we assume $\Pi^1_{2n+1}$-determinacy. This is equivalent to “$\forall x \in \mathbb{R}$ there is an $(\omega_1, \omega_1)$-iterable $M_{2n}^\mathbb{R}$ by Neeman [3, 4] and Woodin [1, 5, 7].” We shall show the induction hypotheses $(\Pi n)$-$(\Pi 2 n)$.

$(\Pi n)$ follows from Steel’s computation of $L_{\delta^{1+}_{2n+1}}[T_{2n+1}] = M_{2n, \infty}$. We proceed with the definition of the operator $x \mapsto x^{(2n+1)\#}$. It will be basically copying the arguments in [9].
4.1 The equivalence of $M^{\#}_{2n}$ and $0^{(2n+1)^\#}$

Suppose $C \subseteq \delta^{1}_{2n+1}$. $C$ is firm iff every member of $C$ is additively closed, the set $\{ \xi : \xi = \text{o.t.}(C \cap \xi) \}$ has order type $\delta^{1}_{2n+1}$ and $C \cap \xi \in \mathbb{L}_{\delta^{1}_{2n+1}}[T_{2n}]$ for all $\xi < \delta^{1}_{2n+1}$. $C$ is a set of potential level-$(2n+1)$ indiscernibles for $M^{-}_{2n,\infty}$ iff for any level-$(2n + 1)$ tree $R$, for any $F, G \in C^{R} \cap \mathbb{L}_{\delta^{1}_{2n+1}}[T_{2n}]$,

$$(M^{\#}_{2n}; [F]^{R}) \equiv (M^{-}_{2n}; [G]^{R}).$$

Say that $\delta$ is an $M_{2n-2}$-Woodin cardinal iff $M_{2n-2}(V_{\delta}) \models \text{“}\delta\text{ is Woodin”}$. By a theorem of Woodin, putting $\kappa = \omega_{E(2n-1)}$, then $M^{-}_{2n-2,\infty}(x) \models \text{“}\kappa\text{ is the least }M_{2n-2}\text{-Woodin cardinal”}$. By $\Pi^{1}_{2n+1}$-determinacy, if $C$ is the set of $M_{2n-2}$-Woodin cardinals in $M^{-}_{2n}(M^{\#}_{2n})$ and their limits, then $C$ is a firm set of potential level-$(2n+1)$ indiscernibles for $M^{-}_{2n,\infty}$. We define

$0^{(2n+1)^\#}$
to be a map sending a level-$(2n + 1)$ tree $R$ to $0^{(2n+1)^\#}(R)$, where $\forall \varphi \in 0^{(2n+1)^\#}(R)$ iff $\varphi$ is an $L^{R}$-formula and for all $\bar{\varphi} \in [C]^{R}$,

$$(M^{-}_{2n,\infty}; \bar{\varphi}) \models \varphi.$$

If $R$ is a finite level-$(2n + 1)$ tree, we have:

1. $0^{(2n+1)^\#}(R)$ is a $\mathcal{D}([\emptyset]^{R}_{R} + \omega)\cdot \Pi^{1}_{2n+1}$ real, and
2. the universal $\mathcal{D}([\emptyset]^{R}_{R} - \Pi^{1}_{2n+1})$ real is many-one reducible to $0^{(2n+1)^\#}(R)$, uniformly in $R$.

Hence, by induction hypotheses $(13 \ n - 1)(1\ n - 1)$, $0^{(2n+1)^\#} \equiv_{m} M^{\#}_{2n}$. Relativizing to any real,

$x^{(2n+1)^\#} \equiv_{m} M^{\#}_{2n}(x)$, uniformly in $x$.

This verifies $(2n)$.

4.2 Syntactical properties of $0^{(2n+1)^\#}$

By $(13 \ n - 1)(2\ n - 1)$, if $Q, T$ are finite level $\leq 2n$ trees and $\pi$ factors $(Q, T)$, then $j^{Q} \upharpoonright \delta^{1}_{2n+1}$ and $\pi^{T} \upharpoonright \delta^{1}_{2n+1}$ are $\Sigma^{1}_{2n+2}$ in the codes. We make the following informal symbols that will occur in a level-$(2n + 1)$ EM blueprint:

1. If $Q$ is a finite level $\leq 2n$ tree, $j^{Q}(a) = b$ iff for any $\xi$ cardinal cutpoint such that $\{a, b\} \in K|\xi$, the $\text{Coll}(\omega, \xi)$-generic extension satisfies $j^{Q}(\pi_{K|\xi,\infty}(a)) = \pi_{K|\xi,\infty}(b)$.
2. If \( \pi \) factors finite level \( \leq 2n \) trees \((X, T)\), \( \pi^T(a) = b \) iff for any \( \xi \) cardinal cutpoint such that \( \{a, b\} \in K|\xi \), the \( \Coll(\omega, \xi) \)-generic extension satisfies \( \pi^T(\pi_{K|\xi, \infty}(a)) = \pi_{K|\xi, \infty}(b) \).

3. If \( Q \) is a level \( \leq 2n \) subtree of \( Q' \), \( Q' \) is finite, then \( j^{Q, Q'} = (\id_Q)^{Q, Q'} \).

4. Corresponding to items 1-3, \( j^Q, \pi^T, j^Q_{\sup} \) stand for functions on ordinals that send \( \alpha \) to \( \sup(j^Q)^{\alpha}, \sup(\pi^T)^{\alpha}, \sup(j^{Q})^{\alpha} \) respectively.

5. If \( k \) is a definable class function and \( W \) is a definable class, then \( k(W) = \bigcup \{k(W \cap V_\alpha) : \alpha \in \Ord\} \).

6. If \( X, T, T' \) are finite level \( \leq 2n \) trees, \( T \) is a subtree of \( T' \), \( a \in j^X(V) \), \( R \) is a level-\((2n + 1)\) tree, \( \dom(R) = \{((0))\} \), then
   
   \( B^T_{X,a} = \{\pi^{T \otimes Q}(a) : Q \text{ finite level } \leq 2n \text{ tree, } \pi \text{ factors } (X, T \otimes Q)\} \);
   
   \( H^T_{X,a} \) is the transitive collapse of the Skolem hull of \( B^T_{X,a} \cup \ran(j^T) \) in \( j^T(V) \) and \( \phi^T_{X,a} : H^T_{X,a} \to j^T(V) \) is the associated elementary embedding;
   
   \( j^T_{X,a} = (\phi^T_{X,a})^{-1} \circ j^T \);
   
   \( j^{T, T'}_{X,a} = (\phi^T_{X,a})^{-1} \circ j^{T, T'} \circ \phi^T_{X,a} \);
   
   \( B^T_{R,a} = B^T_{\phi_{Q}^{(2n), a}} \cup \bigcup \{B^Q_{\sup} : Q \text{ is a completion of } R(((0)))\} \).
   
   \( H^T_{R,a} \) is the transitive collapse of the Skolem hull of \( B^T_{R,a} \cup \ran(j^T) \) in \( j^T(V) \) and \( \phi^T_{R,a} : H^T_{R,a} \to j^T(V) \) is the associated elementary embedding;
   
   \( j^T_{R,a} = (\phi^T_{R,a})^{-1} \circ j^T \);
   
   \( j^{T, T'}_{R,a} = (\phi^T_{R,a})^{-1} \circ j^{T, T'} \circ \phi^T_{R,a} \).

7. Suppose \( R \) is a level-\((2n + 1)\) tree.
   
   \( c_r = \{c_r(\cdot) : \} \) is the informal \( \cal L^R \)-symbol whose interpretation is
   
   \[
   c_r = \begin{cases} 
   j^{R, \sup}_{\sup}(r^{-})(c_r) & \text{if } r \in \desc(R) \text{ of continuous type,} \\
   c_r & \text{if } r \in \desc(R) \text{ of discontinuous type,} \\
   j^{R, \tree}_{\tree}(r, Q)(c_r) & \text{if } r \notin \desc(R).
   \end{cases}
   \]
(b) If $T, U$ are finite level $\leq 2n$ trees and $B = (r, \pi) \in \text{desc}(R, T, U)$, $r \neq \emptyset$, then $c^T_B$ is the informal $\mathcal{L}^R$-symbol which stands for $\pi^T_U(c^r_B)$.

(c) If $A = (r, \pi, T) \in \text{desc}^*(R)$, $r \neq \emptyset$, then $c^A_A$ is the informal $\mathcal{L}^R$-symbol which stands for $\pi^T_T(c^r_A)$.

**Definition 4.1.** A pre-level-$(2n + 1)$ blueprint is a function $\Gamma$ sending any finite level-$(2n + 1)$ tree $Y$ to a complete consistent $\mathcal{L}^Y$-theory $\Gamma(Y)$ which contains all of the following axioms:

1. ZFC + there is no inner model with $2n$ Woodin cardinals + $V = K +$ there is no strong cardinal + $V$ is closed under the $M^#_{2n-1}$-operator.

2. Suppose $X, T, Q, Z$ are finite level $\leq 2n$ trees, $\pi$ factors $(X, T)$, $\psi$ factors $(T, Z)$.

   (a) $j^T_X : V \to j^T_X(V)$ is $\mathcal{L}$-elementary. $j^{Q(2n)}_{2n + 1}$ is the identity map on $V$.

   (b) $\pi^T : j^X(V) \to j^T(V)$ is $\mathcal{L}$-elementary. $j^{Q(2n)}_{2n + 1} = j^T$. $j^T \circ j^X$ is the identity map on $j^T(V)$.

   (c) $(\psi \circ \pi)^Z = \psi^Z \circ \pi^T$.

   (d) $j^T \circ j^Q = j^{T \circ Q}$.

   (e) $j^Q(\pi^T) = (Q \circ \pi)^{Q \circ T}$.

   (f) $\pi^T \upharpoonright (j^X \circ Q)(V) = (\pi \circ Q)^{T \circ Q}$.

3. If $\xi$ is a cardinal and strong cutpoint, then $V^{\text{Coll}(\omega, \xi)}$ satisfies the following: If $U$ is a $\Pi^1_{2n}$-wellfounded level $\leq 2n$ tree, then $K|\xi$ and $(j^U_K)(K|\xi)$ are countable $\Pi^1_{2n+1}$-iterable mice and $(j^U_K) \upharpoonright (K|\xi)$ is essentially an iteration map from $K|\xi$ to $(j^U_K)(K|\xi)$. Here $(j^U_K)$ stands for the direct limit map of $(j^{Z,Z'})^K$ for $Z, Z'$ finite subtrees of $U$, $Z$ a finite subtree of $Z'$.

4. For any $y \in \text{dom}(Y)$, “$c^y_y \in \text{Ord}$” is an axiom.

5. If $y \sim^Y y'$, then “$c^y_y < c^{y'}_{y'}$” is an axiom; if $y \sim^Y y'$, then “$c^y_y = c^{y'}_{y'}$” is an axiom.

A level-$(2n + 1)$ EM blueprint is a pre-level-$(2n + 1)$ EM blueprint satisfying the coherency property: if $R$ is a finite level-$(2n + 1)$ tree, $Y, T$ are finite level $\leq 2n$ trees, $\rho$ factors $(R, Y, T)$, then for each $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n)$, for each $r_1, \ldots, r_n \in \text{dom}(R)$,

$$\varphi(c^r_1, \ldots, c^r_n) \in \Gamma(R)$$
iff
\[ \Gamma^T(V) \models \varphi(c^T_{\rho(r_1)}, \ldots, c^T_{\rho(r_n)}) \in \Gamma(Y). \]

If \( \Gamma \) is a level-\((2n + 1)\) EM blueprint and \( Y \) is a level-\((2n + 1)\) tree, the EM model \( \mathcal{M}_{\Gamma,Y} \) is defined. If \( \rho \) factors \((R,Y)\), then \( \rho^Y_T \) is defined. If \( T \) is a level \( \leq 2n \) tree and \( \rho \) factors \((R,Y,T)\), the notations \( \mathcal{M}^Y_{\Gamma,Y}, j^Y_{\Gamma,Y}, \rho^Y_{\Gamma,Y} \), etc. are defined.

A level-\((2n + 1)\) EM blueprint is iterable iff for any \( \Pi^1_{2n+1} \)-wellfounded level-(2\(n + 1\)) tree, \( \mathcal{M} \) is a \( \Pi^1_{2n+1} \)-iterable mouse. Unboundedness, weak remarkability and level \( \leq 2n \) correctness are defined as in [9], only replacing every occurrence of “level-3 tree”, “level \( \leq 2 \) tree”, “\( \Pi^1_2 \)-wellfounded” by “level-(\(2n + 1\)) tree”, “level \( \leq 2n \) tree”, “\( \Pi^1_{2n} \)-wellfounded” respectively. If \( \Gamma \) is a weakly remarkable level-\((2n + 1)\) EM blueprint, for a level-\((2n + 1)\) tree, the full model \( \mathcal{M}_{\Gamma,Y} \) is defined. Define \( \mathcal{M}_{\Gamma,Y} \) for any limit ordinal \( \gamma < \delta^1_{2n+1} \). \( \Gamma \) is remarkable iff \( \Gamma \) is weakly remarkable and

1. If \( R \) is a level-\((2n + 1)\) tree, \( \text{dom}(R) = \{(0)\} \), \( \mathcal{M}((0)) \) is of degree 0, then \( \Gamma(R) \) contains the axiom “\( \text{c}_{(0)} \) is not measurable”.

2. If \( R \) is a level-\((2n + 1)\) tree, \( \text{dom}(R) = \{(0)\}, \mathcal{M}((0)) = \langle Q^{(0)}_0, (d, q, P) \rangle \), then \( \Gamma(R) \) contains the following axiom: if \( \xi \) is a cardinal and strong cutpoint, \( c = c_{(0)} \), \( b^Q = (Q^R_{\xi(c)})^{-1}(c) \) for \( Q \) a completion of \( R((0)) \), then \( V^{\text{Coll}(\omega, \xi)} \) satisfies the following:

(a) If \( Q \) is a completion of \( R((0)) \) and \( \alpha \) is \( Q \)-represented by both \( T \) and \( T' \), then \( (\langle J^T_{\xi(c)} \rangle^K(K|\xi), \langle J^Q_{\xi(c)} \rangle^K(b^Q)) \sim_{D_J} (\langle J^T_{\xi(c)} \rangle^K(K|\xi), \langle J^Q_{\xi(c)} \rangle^K(b^Q)) \).

(b) Let \( F(\alpha) = \pi_{\langle J^T_{\xi(c)} \rangle^K(K|\xi), \langle J^Q_{\xi(c)} \rangle^K(b^Q)}(\langle J^Q_{\xi(c)} \rangle^K(b^Q)) \) for \( \alpha \) \( Q \)-represented by \( T \) and \( Q \) a completion of \( R((0)) \). Then \( \sup_{0 < \alpha < (\delta^1_{2n+1})} F(\alpha) = \pi_{K|\xi, \omega}(c) \) where \( 2k + 1 \leq d < 2k + 3 \).

By appealing to (45: \( n - 1 \)), if \( \Gamma \) is an iterable, weakly remarkable level-\((2n + 1)\) EM blueprint, then:

1. The following are equivalent:

(a) \( \Gamma \) is remarkable.
(b) The map \( \gamma \mapsto c_{\Gamma, \gamma} \) is continuous.
(c) If $R$ is a level-$(2n+1)$ tree with domain $\{((0))\}$, $R((0)) = (Q_0^{(2n)}, (d, q, P))$, $2k + 1 \leq d < 2k + 3$, then there is $\gamma_R$ such that $\text{cf}^{L^{\delta_2^{2n+1}}(\mathbb{I}_{2n+2})}_{2k+1}(\gamma_R) = j^P(\delta_{2k+1})$ and $\mathcal{C}_R, \gamma_R = \{\mathcal{C}_R, \beta : \beta < \gamma_R\}$.

2. The following are equivalent:

(a) $\Gamma$ is level $\leq 2n$ correct.

(b) For any potential partial level $\leq 2n$ tower $(X, (e, x, W))$ of continuous type, if $F \in ((\delta_{2n+1})_{\leq n})^{(X, (e, x, W))}$, then

$$\mathcal{C}_{\Gamma, X, F} = \left[\vec{a} \mapsto \mathcal{C}_{\Gamma, F}(\vec{a})\right]_{\mu_X}.$$

(c) For any potential partial level $\leq 2n$ tower $(X, (e, x, W))$ of continuous type, there exists $F \in ((\delta_{2n+1})_{\leq n})^{(X, (e, x, W))}$ satisfying

$$\mathcal{C}_{\Gamma, X, F} = \left[\vec{a} \mapsto \mathcal{C}_{\Gamma, F}(\vec{a})\right]_{\mu_X}.$$

$0^{(2n+1)}\#$ is the unique iterable, remarkable, level $\leq 2n$ correct level-$(2n+1)$ EM blueprint. $c_Q^{(2n+1)}$, $c_\gamma^{(2n+1)}$ are defined as usual. $I^{(2n+3)}$ is the set of level-$(2n + 1)$ indiscernibles for $M_{2n, \infty}$. $0^{(2n+1)}\#$ contains the universality of level $\leq 2n$ ultrapowers axiom:

If $\alpha$ is a limit ordinal and $\xi > \alpha$ is a cardinal and cutpoint, then $V^{\text{Coll}(\omega, \xi)}$ satisfies $\pi_{K|^{\infty}}(\alpha) = \sup\{\pi^{(j T)K}(K|^{\xi}, \infty)(\beta) : T$ is $\Pi_{2n}^1$-wellfounded, $\beta < (j^{T}K)(\alpha)\}$.

If $\mathcal{N}$ is a structure that satisfies Axioms $\mathbb{H} \mathbb{H}$ in Definition $\mathbb{L} \mathbb{L}$ and the universality of level $\leq 2n$ ultrapowers axiom, then

$\mathcal{G}_N$ is the direct system consisting of models $\mathcal{N}^T$ for which $T$ is a $\Pi_{2n}^1$-wellfounded level $\leq 2n$ tree and maps $\pi_{\mathcal{N}^T, \mathcal{N}^{T'}} : \mathcal{N}^T \rightarrow \mathcal{N}^{T'}$ for $\pi$ minimally factoring $T, T'$. Define

$$\mathcal{N}_{\infty} = \text{dirlim} \mathcal{G}_N,$$

$$\pi_{\mathcal{N}_{\infty}} : \mathcal{N}^T \rightarrow \mathcal{N}_{\infty}$$

is tail of the direct limit map.

If in addition, $\mathcal{N}$ is countable $\Pi_{2n+1}^1$-iterable mouse, then $\mathcal{G}_N$ is a dense subsystem of $\mathcal{I}_N$, so there is no ambiguity in the notation $\mathcal{N}_{\infty}$. If $Q$ is a
finite level $\leq 2n$ tree, $a \in \mathcal{N}$, $\vec{b} = (d_{\beta, x})_{(d, x) \in \text{dom}(Q)}$ is represented by both $T$ and $T'$, then $\pi_{\mathcal{N}, \infty} \circ j_{\mathcal{N}, T}^Q (a) = \pi_{\mathcal{N}, \infty} \circ j_{\mathcal{N}, T'}^Q (a)$. We can define

$$\pi_{\mathcal{N}, \mathcal{Q}, \vec{b}, \infty} (a) = \pi_{\mathcal{N}, \infty} \circ j_{\mathcal{N}, T}^Q (a)$$

for $\vec{b}$ represented by $T$. So

$$\mathcal{N}_\infty = \{ \pi_{\mathcal{N}, \mathcal{Q}, \vec{b}, \infty} (a) : a \in \mathcal{N}, Q \text{ finite level } \leq 2n, \vec{b} \text{ respects } Q \}.$$ 

If $\vec{c}$ respects a level-(2n + 1) tree $R$, define

$$c_{\vec{c}} = (c_{(R_x, \vec{c}, x)} r \in \text{dom}(R))$$

which strongly respects $R$. The general remarkable of $0^{(2n+1)\#}$ is as follows:

Suppose $\vec{c}$ and $\vec{c}'$ both respect a finite level-(2n + 1) tree $R$. Suppose $r \in \text{dom}(R)$ and for any $s \prec_R r$, $\gamma_s = \gamma'_s$. Then for any $\mathcal{L}$-Skolem term $\tau$,

$$M_{2n, \infty} \models \tau (c_{\vec{c}}) < c_{(R_x, \vec{c}, x)} \rightarrow \tau (c_{\vec{c}}) = \tau (c_{\vec{c}'}).$$

For any $c_{(2n+1)}^{(2n+1)} < \xi \in I^{(2n+1)}$, there is an $\mathcal{L}$-Skolem term $\tau$ such that $M_{2n, \infty} (0^{(2n+1)\#}) \models \tau (\sup(I^{(2n+1)} \cap \xi), \cdot)$ is a surjection from $\sup(I^{(2n+1)} \cap \xi)$ onto $\xi$. For any $u_{(2n+1)}^{(2n+1)} < \xi < c_{(2n+1)}^{(2n+1)}$, there is an $\mathcal{L}$-Skolem term $\tau$ such that $M_{2n, \infty} (0^{(2n+1)\#}) \models \tau (u_{(2n+1)}^{(2n+1)} \cap \xi, \cdot)$ is a surjection from $u_{(2n+1)}^{(2n+1)}$ onto $\xi$. For notational convenience, if $X$ is a finite level $\leq 2n$ tree and $\gamma = [F]_{\mu X}$ is a limit ordinal, define $c_{X, \gamma}^{(2n+1)} = (\vec{a} \mapsto c_{X, \gamma}^{(2n+1)} |_{\mu X})$; define $c_{(2n+1)}^{(2n+1)} = \delta_2^{2n+1}$. Ordinals of the form $c_{X, \gamma}^{(2n+1)}$ when $X \neq \emptyset$ are definable from elements in $I^{(2n+1)}$ over $M_{2n, \infty}$. Define $\overline{I}^{(2n+1)}$ = the closure of $I^{(2n+1)}$ under the order topology. Every ordinal in $\overline{I}^{(2n+1)}$ is of the form $c_{X, \gamma}^{(2n+1)}$ where $X$ is finite and $\gamma < \delta_2^{2n+1}$ is a limit. Thus, if $A = (r, \pi, T) \in \text{desc}^{**} (R)$ and $\vec{c}$ strongly respects $R$, then $c_{(2n+1)}^{(2n+1)} \in \overline{I}^{(2n+1)}$ and is a limit point of $I^{(2n+1)}$.

We define the level $\leq 2n$ indiscernibles below $u_{(2n+1)}^{(2n+1)}$ as in the last section of [10]. This leads to a closed-below-$u_{(2n+1)}^{(2n+1)}$ set $I^{(\leq 2n)} \subseteq u_{(2n+1)}^{(2n+1)}$ enumerated as $b_{\beta}^{(\leq 2n)}$ for $\beta < u_{(2n+1)}^{(2n+1)}$ in the increasing order. We get clubs $\vec{C} \in \prod_{i \leq n} P_i$ such that if $P$ is a finite level $\leq 2n - 1$ tree and $\beta = [f]_{\mu P} < u_{(2n+1)}^{(2n+1)}$, then $b_{\beta}^{(\leq 2n)} = [\vec{a} \mapsto b_{\beta}^{(\leq 2n)}]_{\mu P}$. As in the last section of [10], for any $\min(I^{(\leq 2n)}) < \xi \in I^{(\leq 2n)}$, there is an $\mathcal{L}$-Skolem term $\tau$ such that
$M_{2n,\infty}^{\lt}(0(2n+1)\#) \models \langle \tau(\sup(I(\leq 2n) \cap \xi),\cdot) \rangle$ is a surjection from $\sup(I(\leq 2n) \cap \xi)$ onto $\xi^\circ$; for any $\xi < \min(I(\leq 2n))$, there is an $\mathcal{L}$-Skolem term $\tau$ such that $M_{2n,\infty}^{\lt}(0(2n+1)\#) \models \langle \tau(\cdot) \rangle$ is a surjection from $\omega$ onto $\xi^\circ$.

All the above notions relativize to an arbitrary real. The relevant notations are $c_r^{(2n+1)}$, $c_{x,Q,\gamma}'$, $I_x^{(2n+1)}$, etc. If $x \in \mathbb{R}$, a level $\leq 2n+1$ code for an ordinal in $\delta_{2n+1}^1$ relative to $x$ is of the form

$$(x, R, \bar{\gamma}, Q, \bar{\beta}, \sigma^\gamma)$$

such that $R$ is a finite level-$(2n+1)$ tree, $\bar{\gamma}$ respects $R$, $Q$ is a finite level $\leq 2n$ tree, $\bar{\beta}$ respects $Q$, and $\sigma$ is an $\mathcal{L}^R$-Skolem term for an ordinal. It codes the ordinal

$$\langle x, R, \bar{\gamma}, Q, \bar{\beta}, \sigma^\gamma \rangle = \pi_{\mathcal{M}_x^{(2n+1)}\#, R, c_r}^{\mathcal{M}_x^{(2n+1)}\#, R}(\sigma^{\mathcal{M}_x^{(2n+1)}\#, R}(c_r))_{r \in \text{dom}(R)}).$$

For any $x$, every ordinal in $\delta_{2n+1}^1$ has a level $\leq 2n+1$ code relative to $x$.

### 4.3 Level-$(2n+1)$ uniform indiscernibles

Suppose $R$ is a finite level-$(2n+1)$ tree, $\tau$ is an $\mathcal{L}$-Skolem term, $\bar{\gamma}$ strongly respects $R$. Suppose $A = (r, \pi, T) \in \text{desc}^{\ast\ast}(R)$. Then $\tau_{\mathcal{M}_x^{(n)}}(c_\gamma') < c_r^{(2n+1)} \rightarrow \tau_{\mathcal{M}_x^{(n)}}(c_\gamma') < \min(I^{(n)}(\sup(c_r^{(2n+1)} : A' \prec^R T, \gamma))).$

Suppose $R$ is a finite level-$(2n+1)$ tree and $A = (r, \pi, T) \in \text{desc}^{\ast\ast}(R)$, $r \neq \emptyset$, $\bar{\gamma}$ strongly respects $R$. Then $\tau_{\mathcal{M}_x^{(n)}}(c_\gamma')$ is a cardinal in $M_{2n,\infty}^{\lt}$.

The theory of level-$(2n+1)$ sharps imply $(3n)$. $(4n)$ follows from $(3n)$. $(5n)$ follows from $(5n-1)(39n-1)10n-1$ and L"os. $(6n)$ follows from the theory of level-$(2n+1)$ sharps.

Recall the lemma in [10] on the well-definedness of $\gamma_r$ for $r$ an $R$-description when $R$ is a level-3 tree.

**Lemma 4.2.** Suppose $R$ is a level-3 tree, $\bar{\gamma} \in [\delta_3^1]^R$, $r = (r, Q, \langle d, q, P \rangle) \in \text{desc}(R)$ is of continuous type. Then $\gamma_r = f_{\text{seed}(r)}^{\text{desc}(r)}(\gamma_r)$.

It has an obvious generalization to the higher levels.

**Lemma 4.3.** Suppose $R$ is a finite level-$(2n+1)$ tree, $A \in \text{desc}^{\ast\ast}(R)$, $A \neq (\emptyset, \emptyset, \emptyset)$. Then $\text{cf}^{\mathcal{L}[R](\mathcal{T}_{2n+1})}(\text{seed}^R_A) = \text{seed}^R_{\text{uf}(A)}$.

**Proof.** By the higher analog of proof of Lemma [4.2] for any real $x$, $x^{3\#}(R)$ contains the axiom “$\text{cf}(c_\mathcal{A}) = \text{cf}(c_{\text{uf}(A)})$”. So $\text{cf}^{\mathcal{L}[R](\mathcal{T}_{2n+1})}(\text{seed}^R_A) = \text{cf}^{\mathcal{L}[R](\mathcal{T}_{2n+1})}(\text{seed}^R_{\text{uf}(A)})$. 43
So the $\mathbb{L}[j^R(T_{2n+1})]$-regular cardinals in the interval $[\delta_{2n+1}^1, j^R(\delta_{2n+1}^1)]$ is a subset of $\{\text{seed}^R_r : r \in \text{desc}^*(R) \text{ is regular}\}$. It remains to show that for any $r = (r, Q, (d, q, P)) \in \text{desc}^*(R)$ regular, $\text{seed}^R_r$ is regular in $\mathbb{L}[j^R(T_{2n+1})]$. Suppose towards a contradiction that $\text{seed}^R_r$ is the least counterexample and for some real $x$, $\text{cf}^{M_{2n+\infty}(x)}(\text{seed}^R_r) = \delta < \text{seed}^R_r$ and $\delta \in \mathbb{L}[j^R(T_{2n+1})]$-regular.

Case 1: $\delta = \text{seed}^R_r$, $r' \prec^R r$.

So $x^{(2n+1)\#}(R)$ contains the axiom “$\text{cf}(c_r) = c_{r'}$”. However, we can easily construct a one-node extension $S$ of $R$ such that $s \in \text{dom}(S) \setminus \text{dom}(R)$, $s \prec^S r$ and $A \prec^S s$ for any $A \in \text{desc}^{**}(R)$. Let $\tau$ be an $\mathcal{L}^R$-Skolem term such that $x^{(2n+1)\#}(R)$ contains the formula “$\tau(\cdot)$ is a cofinal map from $c_r$ to $c_{r'}$”. By general remarkability, $x^{(2n+1)\#}(S)$ contains the formula “$\tau''c_r \subseteq c_{r'}$”. This is a contradiction.

Case 2: $\delta < \delta_{2n+1}^1$.

By $(1.k) (3.k)$ for $k \leq n$, $(15.k) (27.k)$ for $k \leq n - 1$, either $\delta = \omega$ or $\delta = j^R(\delta_{2k+1}^1)$ satisfying $k < n$, $P$ is a level-$(2k + 1)$ tree, $\text{card}(P) \leq 1$. $\delta$ is definable in $M_{2n}(x)$, allowing us to proceed with the proof of Case 1. $\square$

Lemma 4.3 implies $(7.n)$. The proof of $(8.n)$ is basically in $(10)$.

We prove $(9.n)$. Recall that a level-$(2n + 1)$ sharp code is based on the tree $R^{E(2n+1)}_{E(2n+1)}$. One can define the “$R^{E(2n+1)}_{E(2n+1)}$-signature”, “$R^{E(2n+1)}_{E(2n+1)}$-approximation sequence”, etc. in a completely parallel way. Appealing to $(2.n)$ for uniform cofinality when necessary, the $R^{E(2n+1)}_{E(2n+1)}$-partial level $\leq 2n + 1$ tower induced by $\beta$ and the $R^{E(2n+1)}_{E(2n+1)}$ approximation sequence of $\beta$ are uniformly $\Delta^1_{2n+3}$ definable from $\beta$. The issue is to reduce $R^{E(2n+1)}_{E(2n+1)}$ to finite trees. If $\beta = [f]_\mu^W$, $W$ is a finite level-$(2n + 1)$ tree, and $f(\bar{\alpha}) = \tau^{M_{2n+\infty}(x)}(\bar{\alpha})$ for $\mu$-a.e. $\bar{\alpha}$, $\rho$ factors $(W, R^{E(2n+1)}_{E(2n+1)})$, then $\rho^{E(2n+1)}_{E(2n+1)}(\beta) = \left(\langle \tau(x, \langle c_{\rho(w)}\rangle_{w \in \text{dom}(W)}) \rangle_{x \in (2n+1)\#} \right)$. It suffices to show that $\rho^{E(2n+1)}_{E(2n+1)} \upharpoonright j^W(\delta_{2n+1}^1)$ is $\Delta^1_{2n+3}$ uniformly in $(W, \rho)$. By looking at the $R^{E(2n+1)}_{E(2n+1)}$-signature, $(\rho^{E(2n+1)}_{E(2n+1)})^* j^W(\delta_{2n+1}^1)$ is uniformly $\Delta^1_{2n+3}$. Hence, $\rho^{E(2n+1)}_{E(2n+1)}(\langle \langle \tau(\sigma), y(2n+1)\# \rangle \rangle) = \langle \langle \chi, z(2n+1)\# \rangle \rangle$ iff (some or equivalently, for any) countable transitive model $M$ of ZFC containing $y, z$ such that $\mathbb{R}^M$ is closed under $M_{2n+\#}^\#$, $M$ satisfies that $\rho^{E(2n+1)}_{E(2n+1)}(\langle \langle \tau(\sigma), y(2n+1)\# \rangle \rangle) = \langle \langle \chi, z(2n+1)\# \rangle \rangle$. This is a $\Delta^1_{2n+3}$ definition of $\rho^{E(2n+1)}_{E(2n+1)} \upharpoonright j^W(\delta_{2n+1}^1)$.

The level-$(2n + 2)$ Martin-Solovay tree $T_{2n+2}$ is defined as follows. Let $T^{(2n+1)}$ be a recursive tree so that $z \in [T] \iff z$ is a remarkable level-$(2n + 1)$ EM blueprint over a real. Let $(\tau_i)_{1 \leq i < \omega}$ be an effective enumeration of $\text{dom}(R^{E(2n+1)}_{E(2n+1)})$ and let $(\tau_k)_{k < \omega}$ be an effective enumeration of all the $\mathcal{L}$-Skolem
terms for an ordinal, where $\tau_k$ is $(f(k)+1)$-ary. $T_{2n+2}$ is the tree on $2 \times u^{(2n+1)}_{E(2n+1)}$ where

$$(t, \vec{\alpha}) \in T_{2n+2}$$

iff $t \in T^{(2n+1)}$ and

1. if $\xi \leq \eta < E(2n+1)$, $r_1, \ldots, r_{f(k)} \in \text{dom}(R^{(2n+1)}_{\xi})$, $r_1, \ldots, r_{f(\eta)} \in \text{dom}(R^{(2n+1)}_{\eta}), \rho$ factors $(R^{(2n+1)}_{\xi}, R^{(2n+1)}_{\eta})$,

(a) if “$\tau_k(\vec{x}, c_{\rho(r_1)}, \ldots, c_{\rho(r_{f(k)})}) = \tau_1(\vec{x}, c_{r_1}, \ldots, c_{r_{f(\eta)})})$” is true in $t$, then $\rho^{R^{(2n+1)}_{\eta}}(\alpha_k) = \alpha_\xi$;

(b) if “$\tau_k(\vec{x}, c_{\rho(r_1)}, \ldots, c_{\rho(r_{f(k)})}) < \tau_1(\vec{x}, c_{r_1}, \ldots, c_{r_{f(\eta)})})$” is true in $t$, then $\rho^{R^{(2n+1)}_{\eta}}(\alpha_k) < \alpha_\xi$;

2. if $\xi < E(2n+1)$, $r_1, \ldots, r_{f(k),f(\eta)} \in \text{dom}(R^{(2n+1)}_{\xi}), Q, Q'$ are finite level $\leq 2n+1$ trees, $Q$ is a subtree of $Q'$, “$j_Q Q'(\tau_k(\vec{x}, c_{r_1}, \ldots, c_{r_{f(k)})}) = \tau_1(\vec{x}, c_{r_1}, \ldots, c_{r_{f(\eta)})})$” is true in $t$, then $j_q^{R^{(2n+1)}_{E(2n+1)} \otimes Q, R^{(2n+1)}_{E(2n+1)} \otimes Q'}(\alpha_k) = \alpha_\xi$.

We have:

- $p[T_{2n+2}] = \{x^{(2n+1)}\# : x \in \mathbb{R}\}$.
- $T_{2n+2}$ is $\Delta^1_{2n+3}$ in the level-$(2n+1)$ sharp codes.
- For any $x \in \mathbb{R}$, $(T_{2n+2})_{x^{(2n+1)}\#}$ has an honest leftmost branch

$$(\tau_k^{R^{(2n+1)}_{E(2n+1)}(M_{2n,\infty}(x))} (x, \text{seed}_{[\tau_1]}, \ldots, \text{seed}_{[\tau_{f(k)}]}))_{k<\omega}.$$ 

By analyzing the representative functions of ordinals below $u^{(2n+1)}_{E(2n+1)}$, we conclude the following in parallel to the level-1 scenario:

**Lemma 4.4.** Suppose $\delta^1_{2n+1} \leq \beta < j^W(\delta^1_{2n+1})$ is a limit ordinal. Then:

1. The $W$-uniform cofinality of $\beta$ is equal to $\text{ucf}^{L[j^W(T_{2n+1})]}(\beta)$.

2. Suppose the $W$-signature of $\beta$ is $(d_i, w_i)_{i<k}$, the $W$-approximation sequence of $\beta$ is $(\gamma_i)_{i<k}$, the $W$-partial level $\leq 2n+1$ tower induced by $\beta$ is $([P_i])$. Then:

(a) For $i < l < k$, $j^W_{P_i}(\gamma_i) < \gamma_i < j^W_{P_l}(\gamma_i)$.

(b) For $i < k$, $(\sigma \upharpoonright \text{dom}(P_i))^{W(\gamma_i)}_{\sup} \leq \gamma_k < (\sigma \upharpoonright \text{dom}(P_i))^{W(\gamma_i)}$. 

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Lemma 4.5. Suppose $\pi$ factors finite level $\leq 2n+1$ trees $(W,W')$. Suppose $\gamma < j^P(\delta_{2n+1}^{\pi})$ and $\pi''^W(\gamma) < \gamma' < \pi^W(\gamma)$. Let $((d_k,w_k))_{k<v}$, $(\gamma_k)_{k<v}$, $(P,(d,p,R))$ be the $W$-signature, $W$-approximation sequence and $W$-potential partial level $\leq 2n+1$ tower induced by $\gamma$ respectively. Let $((d_k',w_k'))_{k<v'}$, $(\gamma_k')_{k<v'}$, $(P',(d',p',R'))$ be the $W'$-signature, $W'$-approximation sequence and $W'$-potential partial level $\leq 2n+1$ tower induced by $\gamma'$ respectively. Let $\text{cf}^{L,j^W(T_{2n+1})}(\gamma) = \text{seed}^{W}_{(d,w)}$.

1. $v < v'$, $\pi(d_k,w_k) = (d_k',w_k')$ and $\gamma_k = \gamma_k'$ for any $k < v$. $\gamma$ is essentially discontinuous $\rightarrow \gamma_v = \gamma_v'$. $\gamma$ is essentially continuous $\rightarrow \gamma_v < \gamma_v'$.

2. $l_k' \notin \text{ran}(\pi)$ for $v \leq k < v'$.

3. $P$ is a proper subtree of $P'$ and $\bar{p}$ is an initial segment of $\bar{p}'$. Moreover, if $\gamma' < \gamma'' < \pi''^W(\gamma)$ and $(\gamma_k')_{k<v''}$ is the $W'$-approximation sequence of $\gamma''$, then $\gamma_v < \gamma_v''$.

Lemma 4.6. Suppose $P,W$ are finite level $\leq 2n+1$ trees and $\sigma$ factors $(P,W)$. Suppose $\gamma < j^P(\delta_{2n+1}^{\pi})$ and $\text{cf}^{L,j^P(T_{2n+1})}(\gamma) = \text{seed}^{P}_{(d,p)}$, $(d,p) \in \text{desc}^\pi(P)$.

1. $\sigma$ is continuous at $\gamma$ iff $(\sigma,W)$ is continuous at $(d,p)$.

2. Suppose $(\sigma,W)$ is discontinuous at $(d,p)$. Let $(P^+,\sigma^+)$ be the $W$-decomposition of $\pi$. Then $\sigma''^W(\gamma) = (\sigma^+)W \circ j^P_{P^+}(\gamma)$.

Lemma 4.7. Suppose $(P^-, (d,p,R))$ is a partial level $\leq 2n+1$ tree and $P$ is a completion of $(P^-, (d,p,R))$. Suppose $W$ is a level $\leq 2n+1$ tree and $\sigma$, $\sigma'$ both factor $(P,W)$, $\sigma$ and $\sigma'$ agree on dom$(P^-)$, $\sigma'(d,p) = \text{pred}(\sigma(W,(d,p)))$.

Then for any $\beta < j^P(\delta_{2n+1}^{\pi})$ such that $\text{cf}^{L,j^P(T_{2n+1})}(\beta) = \text{seed}^{P^-}_{\text{ucf}(P^-, (d,p,R))}$, we have

$$\sigma^W \circ j^P_{P^-}(\beta) = (\sigma')^W \circ j^P_{P^-}(\beta).$$

Lemma 4.8. Suppose $(P,(d,p,R))$ is a partial level $\leq 2n+1$ tree, $\text{ucf}(P,(d,p,R)) = (d^*,p^*)$ and $\sigma$ factors $(P,W)$. Suppose $\beta < j^P(\delta_{2n+1}^{\pi})$ and either

1. $d = 0$, $P^+ = P$, $\sigma' = \sigma$, $\text{cf}^{L,j^P(T_{2n+1})}(\beta) = \omega$, or
2. $d > 0$, $P^+$ is a completion of $P$, $\sigma'$ factors $(P^+, W)$, $\sigma = \sigma' \upharpoonright \text{dom}(P)$, $\sigma'(d, p) = \text{pred}(\sigma, T, (d^*, p^*))$, $\text{cf}^{l_{2n+1}}(T_{2n+2})(\beta) = \text{seed}^P_{d^*, p^*}$.

Then

$$\sigma^W(\beta) = (\sigma')^W \circ j^P_P(\beta).$$

(10) follows from (4). The proofs of (11) and (12) generalize their level-1 versions, appealing to the general remarkability of level-$(2n+1)$ sharps when necessary.

5. The level-$(2n+2)$ sharp

From now on, we assume $\Delta^1_{2n+3}$-determinacy. We will prove (13) and (15).

For $x \in \mathbb{R}$, a putative $(2n+1)$-sharp is a remarkable, level $\leq 2n$ correct level-$(2n+1)$ EM blueprint over $x$ that satisfies the universality of level $\leq 2n$ ultrapowers axiom. Suppose $x^*$ is a putative $x$-$3$-sharp. For any limit ordinal $\alpha < \delta^+_n$, we can build an EM model

$$M^*_{x^*, \alpha}$$

as follows. Let $R$ be a level-$(2n+1)$ tree such that $[\emptyset]_R = \alpha$. Then $M^*_{x^*, \alpha} = (M^*_{x^*, R})^\infty$. This definition is independent of the choice of $R$. We say that $x^*$ is $\alpha$-iterable iff $\alpha$ is in the wellfounded part of $M^*_{x^*, \alpha}$.

A putative level-$(2n+1)$ sharp code for an increasing function is $w = \langle \langle \tau^*, x^* \rangle \rangle$ such that $x^*$ is a putative $x$-$(2n+1)$-sharp, $\tau$ is a unary $\mathcal{L}_2$-Skolem term and

$$\forall v, v' ((v, v' \in \text{Ord} \land v < v') \rightarrow (\tau(v) \in \text{Ord} \land \tau(v) < \tau(v')))$$

is true in $x^*(\emptyset)$. The statement " $\langle \langle \tau^*, x^* \rangle \rangle$ is a putative level-$(2n+1)$ sharp code for an increasing function, $x^*$ is $\alpha$-iterable, $r$ codes the order type of $\tau^* M^*_{x^*, \alpha}(\alpha)$" about $\langle \langle \tau^*, x^* \rangle, r \rangle$ is $\Sigma^1_{2n+1}$ in the code of $\alpha$. In addition, when $x^* = x^{(2n+1)+}$, $\langle \langle \tau^*, x^* \rangle \rangle$ is called a (true) level-$(2n+1)$ sharp code for an increasing function.

The proof of (13) is basically a copy of the arguments in [10].

By (13), every subset of $u^{(2n+1)}_{E(2n+1)}$ in $\mathbb{L}_{\delta^+_1[T_{2n+2}]}$ is $\Delta^1_{2n+3}$. We use this and Moschovakis Coding Lemma [2] to prove (15). Suppose $A \subseteq \delta^+_1$ is in $\mathbb{L}_{\delta^+_1[T_{2n+2}]}$. Suppose $B, C \subseteq \mathbb{R}$ are $\Pi^1_{2n+2}(x)$ subsets of $\mathbb{R}^2$ such that $(w \in \text{WO}^{(2n+1)} \land |w| \in A) \iff \exists z((w, z) \in B) \iff \exists z((w, z) \in C)$. By Moschovakis Coding Lemma, there is a real $y$ and a $\Sigma^1_{2n+2}(y)$ set $D \subseteq \mathbb{R}^2$ satisfying:

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• If \((w, z) \in D\) then \(w \in \text{WO}^{2n+1}\) and there is \(w' \in \text{WO}^{(2n+1)}\) such that 
\[\|w\| = \|w'\|\] and \((w', z) \in B \cup C\).

• If \(w \in \text{WO}^{(2n+1)}\) then there is \(w' \in \text{WO}^{(2n+1)}\) and \(z\) such that \((w', z) \in B \cup C\).

Then \(A\) is \(\Sigma^1_{2n+2}(x, y)\) and hence \(A \in L[T_{2n+1}, x, y]\).

\((16)\) is a simple generalization of the level-2 partition property of \(\omega_1\) in \([9]\). The idea of partially iterable level-\((2n - 1)\) sharps is used in the proof. 

\((17)\) follows from \((10)\). \((18)\) is a simple generalization of the \(n = 0\) case, using \((12)\) when necessary.

We now prove \((19)\). To save notations, we prove the case \(n = 1\). The statement is:

Suppose \(R\) is a \(\Pi^1_3\)-wellfounded level-3 tree. Suppose \(\mu\) is a non-principal \(L_\delta[T_3]\)-measure on \(j^R(\delta^3_3)\). Then there are functions 
\[g, h \in L[j^R(T_3)],\] a finite level \(\leq 4\) tree \(Q\), nodes \((d_1, q_1), \ldots, (d_k, q_k) \in \text{dom}(Q)\), such that \(h : j^P(\delta^3_3) \to (j^P(\delta^3_1))^k\), \(g : (j^P(\delta^3_1))^k \to j^P(\delta^3_3)\), \(h\) is 1-1 a.e. (\(\nu\)), \(g\) is 1-1 a.e. (\(\mu^{Q \to \delta^3_3}_{d,q}\)), \(g = h^{-1}\) a.e. (\(\nu\)), and 
\[A \in \mu^{Q \to \delta^3_3}(x) \iff (h^{-1})^\nu A \in \nu.\]

Take the restricted ultrapower 
\[j : L[j^R(T_3)] \to \text{Ult}(L[j^R(T_3)], \mu) = L[j \circ j^R(T_3)].\]

\(\beta < j \circ j^R(\delta^3_3)\) is called (only in this proof) a uniform indiscernible iff \(\beta\) is represented by some \(f\) in this ultrapower such that for any \(x \in \mathbb{R}\), for \(\nu\)-a.e. \(\xi\), \(f(\xi) \in j^R(I^{(\leq 3)}_x)\).

**Claim 5.1.** The set of uniform indiscernibles is closed below \(j \circ j^R(\delta^3_3)\).

**Proof.** Suppose that \(\beta < j^R(\delta^3_3)\) is not a uniform indiscernible and \(\beta = [f]_\nu\). Pick \(x\) such that for \(\nu\)-a.e. \(\xi\), \(f(\xi) \notin j^R(I^{(\leq 3)}_x)\). Let \(g(\xi) = \max(f(\xi) \cap j^R(I^{(\leq 3)}_x))\). Then \([g]_\nu < \beta\) and any \(\gamma \in ([g]_\nu, \beta)\) is not a uniform indiscernible. So \(\beta\) cannot be a limit of uniform indiscernibles. \(\square\)

**Claim 5.2.** If \([f]_\nu < j \circ j^R(\delta^3_3)\) is not a uniform indiscernible, then there is a real \(x\) such that putting \(g(\xi) = \max(f(\xi) \cap j^R(I^{(\leq 3)}_x))\), either \([g]_\nu = 0\) or \([g]_\nu\) is a uniform indiscernible.

**Proof.** Suppose not. Pick \(x_0\) such that for \(\nu\)-a.e. \(\xi\), \(f(\xi) \notin j^R(I^{(\leq 3)}_{x_0})\). Let \(f_0(\xi) = \max(f(\xi) \cap j^R(I^{(\leq 3)}_{x_0}))\). Then \([f_0]_\nu\) is not a uniform indiscernible and 
\[0 < [f_0]_\nu < [f]_\nu.\]
Continuing this way, we obtain a descending chain of ordinals 
\([f]_\nu > [f_0]_\nu > [f_1]_\nu > \ldots\). \(\square\)
Claim 5.3. If $\alpha < j \circ j^R(\delta^1_3)$ then there is a real $z$, an $\mathcal{L}$-Skolem term $\tau$ and uniform indiscernibles $\beta_1, \ldots, \beta_k \leq \alpha$ such that
\[
\alpha = \tau^{L[j \circ j^R(T_3), z]}(z, \beta_1, \ldots, \beta_k).
\]

Proof. Suppose without loss of generality that $\alpha$ is not a uniform indiscernible. We show by induction on $\alpha$. Let $\alpha$ the smallest uniform indiscernible, by Claim 5.2, there is $x$ such that for $\nu$-a.e. $\xi$, $f(\xi) < \min(I_x^{\leq 3})$. By the analysis of level $\leq 2$ indiscernibles, there is a term $\tau$ such that $\tau^{L[j \circ j^R(T_3), x^3\#]}(x^3\#, \cdot)$ defines a surjection from $\omega$ onto $I_x^{\leq 3}$. So there is $l < \omega$ such that
\[
\alpha = \tau^{L[j \circ j^R(T_3), x^3\#]}(x^3\#, l)
\]
and we are done.

If $\beta$ is the largest uniform indiscernible below $\alpha$, by Claims 5.1-5.2, there is $x$ and $g(\xi) = \max(f(\xi) \cap j^R(I_x^{\leq 3}))$ such that $[g]_{\nu}$ is the largest uniform indiscernible below $\alpha$. By the analysis of level $\leq 2$ indiscernibles and level-3 indiscernibles, there is a term $\tau$ such that $\tau^{L[j \circ j^R(T_3), x^3\#]}(x^3\#, g(\xi), \cdot)$ defines a surjection from $g(\xi)$ onto $\min(j^R(I_x^{\leq 3}) \setminus f(\xi))$. So there is $\gamma < [g]_{\nu}$ such that $\alpha = \tau^{L[j \circ j^R(T_3), x^3\#]}(x^3\#, g(\xi), \gamma)$.

By induction, $\gamma$ can be represented as
\[
\gamma = \sigma^{L[j \circ j^R(T_3), z]}(z, \beta_1, \ldots, \beta_k)
\]
for uniform indiscernibles $\beta_1, \ldots, \beta_k \leq \gamma$. Now combine the last two formulas together.

Let $\beta_1, \ldots, \beta_k \leq [id]_{\nu}$ be uniform indiscernibles and $z, \tau$ be given by Claim 5.3 such that
\[
[id]_{\nu} = \tau^{L[j \circ j^R(T_3), z]}(z, \beta_1, \ldots, \beta_k).
\]
Let $\beta_i = [f_i]_{\nu}$. Let $h : j^R(\delta^1_3) \to (j^R(\delta^1_3))^k$ be
\[
h(\xi) = (f_1(\xi), \ldots, f_k(\xi)).
\]
Let $g(\gamma_1, \ldots, \gamma_k) = \tau^{L[j \circ j^R(T_3), z]}(z, \gamma_1, \ldots, \gamma_k)$. Clearly $g \circ h = \text{id}$ a.e. ($\nu$). There is a unique finite level $\leq 4$ tree $Q$ and nodes $(d_1, q_1), \ldots, (d_k, q_k) \in \text{dom}(Q)$.
such that \( \text{dom}(Q) \) is the upward closure of \( \{(d_1, q_1), \ldots, (d_k, q_k)\} \) and for \( \nu \)-a.e. \( \xi \), there is \( \delta \) respecting \( Q \) such that \( \delta_{(d_i, q_i)} = f_i(\xi) \). If \( \mu_{Q,d,q}(A) = 1 \), take clubs \( C \subseteq \delta^1_3 \) and \( E \subseteq \omega_1 \), both in \( \mathbb{L}[T_3] \) such that \( [E, C]^{Q^+}_{(d,q)} \subseteq A \). Since \( f_i \) is a uniform indiscernible, we have \( f_i(\xi) \in (\cup_{P \text{ finite}} j^P(E)) \cup \{u_\omega\} \cup j^R(C) \) for \( \nu \)-a.e. \( \xi \). Thus \( (h^{-1})^n A \in \nu \). Thus \( h \circ g = \text{id} \) a.e. \( (\mu_{Q,d,q}^{-1}) \) from [3] Fact 3.4.

This finishes the proof of (19) for \( n = 1 \).

We then use the proof of (19) to show (20). Again we assume \( n = 1 \). A set \( A \subseteq j^R(\delta^1_3) \) is \( R \)-simple iff there are clubs \( E \subseteq \omega_1, C \subseteq \delta^1_3 \), both in \( \mathbb{L}[T_3] \), a finite level \( \leq 4 \) tree \( Q \), nodes \( (d_1, q_1), \ldots, (d_k, q_k) \) and an \( F : (j^R(\delta^1_3))^m \rightarrow j^R(\delta^1_3) \) such that

1. \( F \) is 1-1 on \( [E, C]^{Q^+}_{(d,q)} \),
2. \( A = F''[E, C]^{Q^+}_{(d,q)} \),
3. \( F \in \mathbb{L}[j^R(T_3)] \).

Every subset of \( j^R(\delta^1_3) \) in \( \mathbb{L}[\delta^1_3][T_3] \) is \( \Delta^1_3 \). Let \( G \subseteq \mathbb{R}^2 \) be a universal \( \Pi^1_1 \) set. Apply the proof of [3] Section 3.3 but change \( h : \mathbb{R} \rightarrow \mathcal{P}(\lambda) \) in the proof of Lemma 3.7 to \( h(x) = G_x \). We see that every \( A \in \mathcal{P}(j^R(\delta^1_3)) \cap \mathbb{L}[\delta^1_3][T_3] \) is a countable union of \( R \)-simple sets. Applying everything above to \( R = R^{(3)} \), we get a \( \Delta^1_3 \) coding of subsets of \( u_\omega^{(3)} \) in \( \mathbb{L}[\delta^1_3][T_3] \).

(21) simply follows from definitions. \( (22) \) follows from \( \Pi^1_1(n - 1) \) and \( \Pi^1_2(k) \) for \( k < n, (12) \) for \( k \leq n \). \( (23) \) follows from Lemmas 4.7, 4.8 and \( \Pi^1_2(k) \) for \( k < n, (12) \) for \( k \leq n \). \( (24) \) follows from Loś, (21) \( (22) n - 1 \) and \( (33) \) for \( k < n, (12) k \) for \( k \leq n \).

The proof of (25) generalizes the \( n = 0 \) case in [10]. We explain some of the details for the \( n > 0 \) case. Parts (b)(c) follow from part (a) and Loś. We prove part (a). Let \( Q, W \) be as given. By \( (33) n - 1 \), \( (\text{id}_{\leq 2n - 2} Q \otimes \leq 2n - 2 W) \) is the identity on \( \delta^1_{2n+1} \) and agrees with \( (\text{id}_{Q \otimes W})^{Q^W} \). By \( (33) n \), the set of \( L[j^{Q \otimes W}(T_{2n+1})] \)-cardinals in the interval \( [\delta^1_{2n+1}, j^{Q \otimes W}(\delta^1_{2n+1})] \) is the closure of \( \{\text{seed}_{(2n+1, A)}^{Q \otimes W} : A \in \text{desc}^{**}(2n+1, Q \otimes W)\} \).

We prove by induction on \( \|A\|_{\leq 2n+2} \) that \( (\text{id}_{Q \otimes W})^{Q^W} \upharpoonright (\text{seed}_{(2n+1, A)}^{Q \otimes W} + 1) \) is the identity. By elementarity, it suffices to prove that \( (\text{id}_{Q \otimes W})^{Q, W} \) is continuous at \( \text{seed}_{(2n+1, A)}^{Q \otimes W} \). We prove the typical case when \( \leq 2n+1 Q = \emptyset \) and \( \leq 2n+3 W \neq \emptyset \).

Case 1: \( \|A\|_{\leq 2n+1} = 0 \).

Then \( A = (\langle q, \sigma \rangle, \text{id}_T, T) \) where putting \( D = (2n+2, q, \sigma) \in \text{desc}(Q, W, *) \), we have \( \text{lh}(D) = 1, Q \otimes W(D) = (T, (c, t, S)) \), \( D \prec Q \otimes W D' \) whenever
lh(D') = 1 and D \neq D'. Let Q' be the extension of Q by adding the node (2n+1, ((0))) into its domain such that Q'(2n+1, ((0))) = (T, (c, t, S)). Given any g such that [g]_\mu \prec (id\otimes\nu)^Q,W(\delta_{2n+1}^1), we partition functions f \in (\delta_{2n+1}^1)_{i \leq m}Q^+ according to whether or not 2n+1[f]_{((0))}^Q \leq g([f | rep(Q)]^Q). We obtain, by (16n) and the assumption on g, clubs \(\bar{E} = (E_i)_{i \leq m} \subseteq \prod i \leq m \nu_{2n+1}\) such that for any f \in \(E^Q, 2n+1[f]_{((0))}^Q > g([f | rep(Q)]^Q). This implies that [g]_\mu \prec \text{the } u^{E(2n-1)}\text{-th element of } E_m. So (id\otimes\nu)^Q,W is continuous at \(\delta_{2n+1}^1 = \text{seed}((2n+1), A).\)

Case 2: \(\|A\|_{Q \otimes W} > 0.\)

We prove the case when A = ((q, \sigma), \pi, T), where putting D = (2n + 2, q, \sigma), we have D \subseteq \text{desc}(Q, W, \ast) and q is of discontinuous type. The other cases when A = ((q, \sigma)\prec (-1), \pi, T) or q is of continuous type are similar. Put q = (q, \pi, \{d_i, p_i, R_i\}_{i \leq k+1}). If \(\|A\|_{Q \otimes W} = \eta + 1\) is a successor, we must have that \(d_{k+1} = 0\) and \(\pi, T\) is discontinuous at \(\langle d_k, p_k \rangle\). Let \(A' = ((q, \sigma), \pi', T)\) where \(\pi'\) and \(\pi\) agree on \text{dom}(Q) \setminus \{\{d_k, p_k\}\}, \pi'(d_k, p_k) = \text{pred}(\pi, T, (d_k, p_k)). Then \(\|A'\|_{Q \otimes W} = \eta\). Given any g such that [g]_\mu \prec (id\otimes\nu)^Q,W(\text{seed}((2n+1), A)), we partition function f \in (\delta_{2n+1}^1)_{i \leq m}Q^+ according to whether or not \([f]_{\pi'(d_k, p_k)}^T \leq g([f]_{\pi}^T). The homogeneous side must satisfy \(>\), yielding that [g]_\mu \prec \text{seed}((2n+1), A).\) If \(\|A\|_{Q \otimes W} > \eta\) is a limit, we must have \(d_{k+1} > 0\) and we obtain \((A_i, \pi_i, T_i)_{i < \omega}\) such that \(\sup_{i < \omega} \|A_i\|_{Q \otimes W} = \eta\) and for any \(i, A_i = ((q, \sigma)\prec (-1), \pi_i, T_i) \subseteq \text{desc}^\ast(Q \otimes W).\) We may further assume that: T_0 is a one-node extension of T, T_i+1 is a one-node extension of T_i, i_0 \in \text{dom}(T_0) \setminus \text{dom}(T), t_{i+1} \in \text{dom}(T_{i+1}) \setminus \text{dom}(T_i), \ d_i = i_0, p_i = t_i and \(d_k \geq 1\) implies that \(t_i = t_0\) and \(d_i t_i [t_i] = d_i t_i [t]_0.\) Let T_\omega = \cup_i T_i and work with partition arguments based on T_\omega.

(26n) follows from (25n) and Los'.

(27n) is proved as follows. Suppose Q is a finite level \(\leq 2n + 2\) tree. By (18n), \(j^P(T_{2n+1}) \in \mathbb{L}_{\beta_{2n+3}}[T_{2n+2}]\) for any finite level-(2n + 1)-tree P, and hence by (3n), the set of \(\mathbb{L}_{\beta_{2n+3}}[j^Q(T_{2n+2})]\)-cardinals in the interval \([\delta_{2n+1}^1, \delta_{2n+3}^1]\) is a subset of \(\{u^Q_{2n+1} : 0 < \xi \leq E(2n + 1)\}.\) But every \(u^Q_{2n+1}\) is an \(\mathbb{L}_{\beta_{2n+3}}[j^Q(T_{2n+2})]\)-cardinal by an easy adaption of Martin’s proof that under AD, if \(\kappa\) has the strong partition property and \(\mu\) is an ultrafilter on \(\kappa,\) then \(j^\mu(\kappa)\) is a cardinal. The part on \(\mathbb{L}_{\beta_{2n+3}}[j^Q(T_{2n+2})]\)-regular cardinals is an easy generalization of the \(n = 0\) case in (10).

(28n) is a simple computation, using (27n) for the part concerning uniform cofinality.

(29n) simply follows from definitions. (30n) follows from (23n) and (37k) for \(k < n, (12k)\) for \(k \leq n. (31n)\) follows from Lemmas 4.7, 1.8 and
acting on level $(\leq (37: k) \text{ for } k < n$, $(12: k) \text{ for } k \leq n$. $(32: n)$ follows from Loś, $(29: n)$, $(30: n)$ and $(37: k) \text{ for } k < n$, $(12: k) \text{ for } k \leq n$.

$(33: n)$ follows from $(25: n)$, $(27: n)$ and the associativity of the $\otimes$-operator acting on level $(\leq 2n + 2$, $\leq 2n + 2$, $\leq 2n + 1$) trees.

We outline the proof of $(33: n)$. Let $\theta : \text{rep}(X) \rightarrow \text{rep}(T)$ be an isomorphism. For $(e, x) \in \text{dom}(X)$ and $e > 1$, let $X_{\text{tree}}(e, x) = W_{(e, x)}$. Let $E = (E_i)_{i \leq n} \subseteq \prod_{i \leq n} \nu_{2i+1}$, $(d_{(e, x)}, t_{(e, x)}) \in \text{desc}(T)$, $t_{(e, x)} = (t_{e,x}, S_{e,x}, \ldots)$ when $d_{(e, x)} > 1$, and let $\theta(e, x) \in \mathbb{L}_{32n+3}[T_{2n+2}]$ be such that

- $d_{(e, x)} = 1$ implies $e = 1$ and $\theta(1, (x)) = (d_{(1, x)}, (t_{(1,x)}))$,
- $d_{(e, x)} > 1$ implies that for any $\vec{\alpha} \in [\vec{E}]^{W_{(e, x)}} \uparrow$, $\theta_{(e, x)}(\vec{\alpha} \oplus q x) = (d_{(e, x)}, \theta_{(e, x)}(\vec{\alpha} \oplus q x)_T, t_{(e, x)})$.

If $d_{(e, x)} > 1$, let $\vec{\beta}_{(e, x)} = (\beta_{(e, x), (a,s)})_{(a,s) \in \text{dom}(S_{(e, x)})}$, $\vec{\beta}_{(e, x)} = [\theta_{(e, x)}]_{\mu}^{W_{(e, x)}}$, for $k \leq n$,

$$B_{(e, x)}^{(2k+1)} = \{(a, s) \in \text{dom}(S_{(e, x)}): a = 2k + 1, \beta_{(e, x), (a,s)} < \delta_{2k+1}^1\},$$

$$B_{(e, x)}^{(2k+2)} = \{(a, s) \in \text{dom}(S_{(e, x)}) \setminus B_{(e, x)}^{(2k+1)}: 2k + 1 \leq a \leq 2k + 2\}.$$}

For $e > 1$ and $2 \leq k \leq 2n + 2$, let

$$D_{(e, x)}^{(k)} = \{(a, s) \in B_{(e, x)}^{(k)}: \beta_{(e, x), (a,s)} \text{ is } S_{(e, x)}\text{-essentially continuous}\},$$

$$E_{(e, x)}^{(k)} = B_{(e, x)}^{(k)} \setminus D_{(e, x)}^{(k)}.$$}

If $(a, s) \in B_{(e, x)}^{(k)}$ and $a > 1$, let $(S_{(e, x)})_{\text{tree}}(a, s) = U_{(e, x), (a,s)}$, let $\Phi_{(e, x), (a,s)}$ be the $\leq_{k-1}(\nu(W_{(e, x)} \otimes U_{(e, x), (a,s)})$-potential partial level $\leq k - 1$ tower induced by $\beta_{(e, x), (a,s)}$, let $\text{lh}(\Phi_{(e, x), (a,s)})$ be the length of the second coordinate of $\Phi_{(e, x), (a,s)}$, let $(B_{(e, x), (a,s), i})_{\leq \text{lh}(\Phi_{(e, x), (a,s)})}$ be the $\leq_{k-1}(\nu(W_{(e, x)} \otimes U_{(e, x), (a,s)})$-signature of $\beta_{(e, x), (a,s)}$, let $\sigma_{(e, x), (a,s)}$ be $\leq_{k-1}(\nu(W_{(e, x)} \otimes U_{(e, x), (a,s)})$-factoring map induced by $\beta_{(e, x), (a,s)}$, and let $(\gamma_{(e, x), (a,s), i})_{\leq \text{lh}(\Phi_{(e, x), (a,s)})}$ be the $\leq_{k-1}(\nu(W_{(e, x)} \otimes U_{(e, x), (a,s)})$-approximation sequence of $\beta_{(e, x), (a,s)}$.

Let

$$\phi^1 : \{(\beta_{(e, x), (a,s)}: (e, x) \in \text{dom}(X), (a, s) \in B_{(e, x)}^{(1)} \} \rightarrow Z^1$$

be a bijection such that $Z^1$ is a level-1 tree and $v < v' \rightarrow \phi^1(v) < \beta_{BK} \phi^1(v')$.

For $2 \leq k \leq 2n + 2$, let

$$\phi^k : \{(B_{(e, x), (a,s), i}, \gamma_{(e, x), (a,s), i})_{i < l} : (e, x) \in \text{dom}(X), (a, s) \in B_{(e, x)}^{(k)}, l < \text{lh}(\Phi_{(e, x), (a,s)})\} \rightarrow Z^k \cup \{\emptyset\}$$

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be a bijection such that $Z^k$ is a tree of level-1 trees and $v \subseteq v' \leftrightarrow \phi^k(v) \subseteq \phi^k(v')$, $v <_{BK} v' \leftrightarrow \phi^k(v) <_{BK} \phi^k(v')$. Let
\[
Q = (1^Q, \ldots, 2n^Q)
\]
be a level $\leq 2n + 2$ tree where $\text{dom}^k(Q) = Z^k$ and for $2 \leq k \leq 2n + 2$,
\[
k^Q[\phi^k((w(e,x),(a,s),i, \gamma(e,x),(a,s),i)_{i<\text{lh}(\Phi(e,x),(a,s))})^{−1}] = \Phi(e,x),(a,s) \text{ when } (a, s) \in D^k_{(e,x)},
\]
\[
k^Q[\phi^k((w(e,x),(a,s),i, \gamma(e,x),(a,s),i)_{i<\text{lh}(\Phi(e,x),(a,s))})] = \Phi(e,x),(a,s) \text{ when } (a, s) \in E^k_{(e,x)}.
\]
Let $\pi$ factor $(X, T, Q)$, where $\pi(1, x) = (1, t(e,x), \emptyset)$ if $d_{(e,x)} = 1$, $\pi(e,x) = (d_{(e,x)}, t(e,x), \tau(e,x))$ if $d_{(e,x)} > 1$, where $\tau(e,x)$ factors $(S_{(e,x)}, Q, W_{(e,x)}, \tau_{(e,x)}(a,s))$ is equal to
\begin{itemize}
  \item $(1, \phi^1(\beta(e,x),(a,s)), \emptyset)$ if $(a, s) \in B^1_{(e,x)},$
  \item $(k, (\phi^k(B(e,x),(a,s),i, \gamma(e,x),(a,s),i)_{i<\text{lh}(\Phi(e,x),(a,s))})^{−1})^{−1} \Phi(e,x),(a,s), \sigma(e,x),(a,s))$ if $(a, s) \in D^k_{(e,x)}, k > 1,$
  \item $(k, (\phi^k(B(e,x),(a,s),i, \gamma(e,x),(a,s),i)_{i<\text{lh}(\Phi(e,x),(a,s))})^{−1} \Phi(e,x),(a,s), \sigma(e,x),(a,s))$ if $(a, s) \in E^k_{(e,x)}, k > 1.$
\end{itemize}

The fact that $\theta$ is an isomorphism implies that $\pi$ minimally factors $(X, T, Q)$.

By analyzing the representative functions, we obtain the following lemmas in parallel to Lemmas 4.6-4.8.

**Lemma 5.4.** Suppose $Q, T$ are finite level $\leq 2n + 1$ trees and $\pi$ factors $(Q, T)$. Suppose $\gamma < j^Q(\delta^1_{2n+1})$ and $\text{cf}^1j^Q_{2n+3}[j^Q(Z_{2n+2})](\gamma) = \text{seed}^Q_{(d,q)}$, $(d, q) \in \text{desc}^*(Q)$ is regular. Then

1. $\pi^T$ is continuous at $\gamma$ iff $(\pi, T)$ is continuous at $(d, q)$.

2. Suppose $(\pi, T)$ is discontinuous at $(d, q)$. Let $(Q^+, \pi^+)$ be the $T$-decomposition of $\pi$. Then $\pi^T_{\text{sup}}(\gamma) = (\pi^+)T \circ j^Q_{\text{sup}}(\gamma)$.

**Lemma 5.5.** Suppose $(Q^-, (d, q, P))$ is a partial level $\leq 2n + 2$ tree and $Q$ is a completion of $(Q^-, (d, q, P))$. Suppose $T$ is a level $\leq 2n + 2$ tree and $\pi, \pi'$ both factor $(Q, T)$, $\pi$ and $\pi'$ agree on $\text{dom}(Q^-)$, $\pi(d, q) = \text{pred}(\pi(T, (d, q)))$.

Then for any $\gamma < \delta^1_{2n+3}$ such that $\text{cf}^1\delta^1_{2n+3}[j^Q(T_{2n+2})](\gamma) = \text{seed}^{Q^-}_{(d,q)}$, we have
\[
\pi^T \circ j^Q_{\text{sup}}(\gamma) = (\pi')^T_{\text{sup}} \circ j^Q_{\text{sup}}(\gamma).
\]
Lemma 5.6. Suppose $(Q, (d, q, P))$ is a partial level $\leq 2n+2$ tree, $\text{ucf}(Q, (d, q, P)) = (d^*, q^*)$ and $\pi$ factors $(Q, T)$. Suppose $\gamma < \delta^1_{2n+3}$ and either

1. $d = 0$, $Q^+ = Q$, $\pi' = \pi$, $\text{cf}^{\uparrow}j^{Q(T_{2n+2})}([\gamma]) = \omega$, or

2. $d > 0$, $Q^+ = Q$, $\pi'$ factors $(Q^+, T)$, $\pi = \pi' \upharpoonright \text{dom}(Q)$, $\pi'(d, q) = \text{pred}(\pi, T, (d^*, q^*))$, $\text{cf}^{\uparrow}j^{\delta^1_{2n+3}}(j^{Q(T_{2n+2})})([\gamma]) = \text{seed}^Q_{(d^*, q^*)}$.

Then

$$\pi^T(\gamma) = (\pi')^{\text{sup}} \circ j^{Q, Q^+}(\gamma).$$

(35)n follows from (17)n. The proof of (36)n generalizes Lemma 4.2 using (25)n when necessary. The proof of (37)n generalizes the lower levels in an obvious way, using (27)n when necessary.

The proof of (38)n is an easy generalization of the lower levels, using Lemmas 5.3, 5.6 when necessary.

(39)n simply follows from definitions. (40)n follows from (31)n and (12)k.(37)k for $k \leq n$. (41)n follows from Lemmas 5.5, 5.6 and (12)k.(37)k for $k \leq n$.

The proof of (42)n is similar to (33)n.

We outline the proof of (43)n. Suppose $R, (d, q, P), k$ are as given. The case $k < n$ follows from (43)k. The case $d = 2n + 1$ follows from (45)n - 1.

Assume now $d = 2n + 2$. Ordinals of the form $\tau^{j^P(M_{2n+2, \infty}^\omega)(x, \text{seed}^P_{(0)})}$ are cofinal in $j^P(\delta^1_{2n+1})$. Fix such an $\alpha = \tau^{j^P(M_{2n+2, \infty}^\omega)(x, \text{seed}^P_{(0)})}(x, \text{seed}^P_{(0)})$ and we build a $\Pi^1_{2n+2}$-wellfounded level-(2n + 2) tree $T$ such that $T_{\text{tree}}(((0)))) = P$ and $\text{cf}^{\uparrow}(\text{dom}(T)) = \alpha$. Indeed, we build $T$ satisfying $T_{\text{tree}}(((0)))) = P$ and for any $z \in [\text{dom}(T)]$, $T(z) = \text{def} \cup_i T(z \upharpoonright i)$ is a level $\leq 2n+1$ tree as the “join” of $(W_{(z), (z), (z)\upharpoonright i})_{1 \leq i < \omega}$, where $(z)\upharpoonright _1$ is computable from $(z)\upharpoonright_0$ so that if $(z)\upharpoonright_0$ codes $(x, P, \gamma, Q, \beta, \tau\upharpoonright_0)$, a level $\leq 2n+1$ code for an ordinal in $\delta^1_{2n+1}$ relative to $x$, then $(z)\upharpoonright _1$ codes $(x, P, \gamma, Q_{(0)^n}, \beta(2n), \tau\upharpoonright 1)$, $\beta(2n)$ respects $Q_{(0)^n}$, and $W(v, v')$ is $\Pi^1_{2n+1}$-wellfounded iff $v, v'$ code $v, v'$, level $\leq 2n+1$ codes for ordinals in $\delta^1_{2n+1}$ relative to $x$, such that $|\bar{v}| > |\bar{v}'|$.

(44)n follows from (43)n by a straightforward generalization of the corresponding arguments in (9)(10). The reader who can follow us this far should have no problem figuring out the details. The proof of (45)n is similar to (44)n.

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