A NOTE ON LINEABILITY

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Abstract. In this note we answer a question concerning lineability of the set of non-absolutely summing operators.

1. Introduction and main result

A subset $A$ of an infinite-dimensional vector space $V$ is $\mu$-lineable if $A \cup \{0\}$ contains an infinite-dimensional subspace of dimension $\mu$. Let $\aleph_0$ be the countable cardinality and $\aleph_1$ be the cardinality of $\mathbb{R}$. From now on $E$ and $F$ denote Banach spaces, the space of absolutely $p$-summing linear operators from $E$ to $F$ will be denoted by $\Pi_p(E; F)$, the space of bounded linear operators from $E$ to $F$ will be represented by $L(E; F)$ and the space of compact operators from $E$ to $F$ is represented by $K(E; F)$. For details on the theory of absolutely summing operators we refer to [3].

In recent papers [1, 5] it was shown that under certain circumstances $L(E; F) \setminus \Pi_p(E; F)$ is $\aleph_0$-lineable. In [1] there is a question from the anonymous referee, asking about the possibility of proving that the set is $\mu$-lineable, for $\mu > \aleph_0$. Our next result shows that an adaptation of the proof of [1] answers this question in the positive:

**Theorem 1.1.** Let $p \geq 1$ and $E$ be superreflexive. If $E$ contains a complemented infinite-dimensional subspace with unconditional basis or $F$ contains an infinite unconditional basic sequence then $K(E; F) \setminus \Pi_p(E; F)$ (hence $L(E; F) \setminus \Pi_p(E; F)$) is $\aleph_1$-lineable.

**Proof.** Assume that $E$ contains a complemented infinite-dimensional subspace $E_0$ with unconditional basis $(e_n)_{n=1}^{\infty}$. First consider

\[(1.1) \quad \mathbb{N} = A_1 \cup A_2 \cup \cdots \]

a decomposition of $\mathbb{N}$ into infinitely many infinite pairwise disjoint subsets $(A_j)_{j=1}^{\infty}$. Since $\{e_n; n \in \mathbb{N}\}$ is an unconditional basis, it is well known that $\{e_n; n \in A_j\}$ is an unconditional basic sequence for every $j \in \mathbb{N}$. Let us denote by $E_j$ the closed span of $\{e_n; n \in A_j\}$. As a subspace of a superreflexive space, $E_j$ is superreflexive as well, so from [2, Theorem] it follows that for each $j$ there is an operator

$$u_j: E_j \rightarrow F$$

belonging to $K(E_j; F) \setminus \Pi_p(E_j; F)$. From the proof of [1] we know that each projection $P_i: E_0 \rightarrow E_i$ is continuous and has norm $\leq \rho$ (the constant of the unconditional
basis of \( E_0 \). This also implies that each \( E_i \) is a complemented subspace of \( E_0 \). If \( \pi_0 : E \to E_0 \) denotes the projection onto \( E_0 \), for each \( j \in \mathbb{N} \) we can define the operator
\[
\tilde{u}_j : E \to F, \quad \tilde{u}_j := u_j \circ P_j \circ \pi_0.
\]
Since \( (P_j \circ \pi_0)(x) = x \) for every \( x \in E_j \), it is plain that \( \tilde{u}_j \) belongs to \( \mathcal{K}(E; F) \setminus \Pi_p(E; F) \).

There is no loss of generality in supposing \( \| \tilde{u}_j \| = 1 \) for every \( j \).

Now, consider the map
\[
T : \ell_1 \to \mathcal{K}(E; F)
\]
\[
T((a_n)_{n=1}^{\infty}) = \sum_{j=1}^{\infty} a_j \tilde{u}_j.
\]

Since the supports of the \( \tilde{u}_n \) are disjoint it is clear that \( T \) is an injective linear operator, such that
\[
T(\ell_1) \subset (\mathcal{K}(E; F) \setminus \Pi_p(E; F)) \cup \{0\}.
\]
And therefore \((\mathcal{K}(E; F) \setminus \Pi_p(E; F)) \cup \{0\}\) contains a vector space with the same dimension of \( \ell_1 \) (and it is well-known that \( \dim \ell_1 = \aleph_1 \)).

Now, suppose that \( F \) contains a subspace \( G \) with unconditional basis \( \{ e_n; n \in \mathbb{N} \} \) with unconditional basis constant \( \rho \). Still considering the subsets \( (A_n) \) of \( \mathbb{N} \) as above, define \( F_j \) as the closed span of \( \{ e_n; n \in A_j \} \) and let \( P_j : G \to F_j \) be the corresponding projections. Proceeding as above we conclude that \( \| P_j \| \leq \rho \). From [2, Theorem] we know that for each \( j \) there is an operator \( u_j : E \to F_j \) belonging to \( \mathcal{K}(E; F_j) \setminus \Pi_p(E; F_j) \). Now by \( \tilde{u}_j \) we mean the composition of \( u_j \) with the inclusion from \( F_j \) to \( F \). Once again consider the map
\[
T : \ell_1 \to \mathcal{K}(E; F)
\]
\[
T((a_n)_{n=1}^{\infty}) = \sum_{j=1}^{\infty} a_j \tilde{u}_j.
\]

Since the projections \( P_j : G \to F_j \) are continuous and have norm \( \leq \rho \), it follows that
\[
\| T((a_n)_{n=1}^{\infty}) (x) \| \geq \rho^{-1} \| a_j \tilde{u}_j (x) \|
\]
for every \( j \in \mathbb{N} \). It is clear that \( T \) is a linear and injective. It also follows from (1.2) that
\[
T(\ell_1) \subset (\mathcal{K}(E; F) \setminus \Pi_p(E; F)) \cup \{0\}.
\]
\[\square\]

**Remark 1.2.** It is not difficult to show that
\[
\dim \mathcal{L}(\ell_p; \ell_q) = \aleph_1
\]
so, for example, for \( E = \ell_p \) \( (p > 1) \) and \( F = \ell_q \) the result of the previous theorem is optimal, i.e., we cannot improve the result to \( \mu \)-lineable for \( \mu > \aleph_1 \).
2. Lineability of the set of norm attaining-operators

Next we show that the same idea of the proof of Theorem 1.1 can be adapted to extend a result from [4] concerning norm-attaining operators.

In what follows $\mathcal{NA}^{x_0}(E; F)$ denotes the set of continuous linear operators from $E$ to $F$ that attain their norms at $x_0$.

**Proposition 2.1.** Let $E$ and $F$ be Banach spaces so that $E$ contains an isometric copy of $\ell_q$ for some $1 \leq q < \infty$, and let $x_0 \in S_E$. Then $\mathcal{NA}^{x_0}(E; F)$ is $\aleph_1$-lineable in $\mathcal{L}(E; F)$.

**Proof.** The beginning of the proof follows the lines of the similar result from [4]. It suffices to prove for $F = \ell_q$. We can write the set of positive integers $\mathbb{N}$ as

$$\mathbb{N} = \bigcup_{k=1}^{\infty} A_k,$$

where each

$$A_k := \{ a_1^{(k)} < a_2^{(k)} < \ldots \}$$

has the same cardinality as $\mathbb{N}$ and the sets $A_k$ are pairwise disjoint. For each positive integer $k$, we define

$$\ell_q^{(k)} := \{ x \in \ell_q : x_j = 0 \text{ if } j \notin A_k \}.$$

For each $k$ we can find operators $u^{(k)}$ on $\mathcal{NA}^{x_0}(E; \ell_q^{(k)})$. By composing these operators with the inclusion of $\ell_q^{(k)}$ into $\ell_q$ we get a vector (and we maintain the same notation for the sake of simplicity) on $\mathcal{NA}^{x_0}(E; \ell_q)$. Consider the map

$$T : \ell_1 \to \mathcal{NA}^{x_0}(E; \ell_q)$$

$$T((a_n)_{n=1}^{\infty}) = \sum_{j=1}^{\infty} a_j u^{(j)}.$$

It is clear that $T$ is linear and injective. We also have that (due the disjoint supports of the $u^{(j)}$)

$$T(\ell_1) \subset \mathcal{NA}^{x_0}(E; \ell_q).$$

Since $T$ is injective, it follows that $T(\ell_1)$ is an infinite-dimensional space and its basis has the same cardinality of the basis of $\ell_1$. Recall that $\dim(\ell_1) = \aleph_1$. \qed

**References**

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