Abstract
Assuming only ordinal preferences as transparent, we study players interacting under the veil of ignorance, that cannot produce beliefs as probability measures but rather have coarse beliefs represented as subsets of opponents’ actions. These players follow either max max or max min decision criteria. The criteria can be identified as optimistic and pessimistic attitudes, respectively, which we refer to as “tropical”. Explicitly formalizing these attitudes and how players reason under ignorance, we characterize the behavioral implications related to common belief in these events: while optimism is related to Point Rationalizability, a new algorithm—Wald Rationalizability—captures pessimism. Our characterizations allow us to uncover novel connections and results: (i) regarding optimism, we prove that dropping the (implicit) assumption that whatever a player believes is also true allows to capture wishful thinking à la Yildiz (2007), thus reversing an existence failure described therein; (ii) by studying how pessimism and optimism relate to Börgers dominance, we shed light on the appropriate notion of rationality in ordinal games; (iii) finally, with respect to pessimism, our analysis identifies Wald Rationalizability as the limit point of players becoming infinitely risk averse, hence, clarifying the conceptual underpinnings behind a discontinuity in the analysis of Rationalizability in presence of varying risk attitudes hinted in Weinstein (2016).

Keywords: Ordinal Preferences, Optimism/Pessimism, Interactive Epistemology, Wishful Thinking, Börgers Dominance, Rationalizability.

JEL Classification Number: C63, C72, C73.

“... une nature [...] qui tient [...] aux Tropiques par la violence illogique de ses passions...”
—Honoré de Balzac, “Le contrat de mariage”

1. Introduction
1.1 Motivation & Results
Games played under the veil of ignorance are all those strategic interactions where players are not able to describe their beliefs via any form of probability measure (and the like). This essentially implies that these strategic interactions lie outside the realm of the Bayesian paradigm à la Savage (1954) or the models on ambiguity stemming from Schmeidler (1989) and Gilboa & Schmeidler (1989).2 As a result, whenever we, as analysts, study games under ignorance, we lack all those—nice—mathematical properties coming from the assumption of working with cardinal utility that, starting from von Neumann & Morgenstern (1944) and Nash (1950, 1951),3 considerably generalize the scope of game theory: indeed, lacking probability measures (and the like) essentially results in not being able to perform expected utility computations.

In light of this, it seems natural to ask why there should be any interest in games under ignorance, where players’ beliefs are—rather coarsely—represented simply via collections of their preferences.

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3See Milnor (1954), Luce & Raiffa (1957, Chapter 13), Arrow & Hurwicz (1977), and Kelsey & Quigg (1992).
opponents’ actions. Indeed, one might wonder if there is any hope of obtaining reasonable predictions for strategic interactions under ignorance\(^4\) when these nice mathematical structures are not applicable. Bypassing the—given the appropriate circumstances—descriptive accuracy provided by assuming to be in presence of a strategic interaction under ignorance, we show that extremely strong predictions are still possible even though the assumptions underlying our analysis are naturally rather weak. Indeed, since probability measures play no role and—as a result—risk attitudes are out of place, the study of games under ignorance essentially corresponds to the study of ordinal games. That is, in our analysis only ordinal preferences over the outcomes of the game are transparent between the players.\(^5\) This in turn implies that the predictions we obtain in this context cover all those strategic interactions where we, as analysts, simply assume to have knowledge on our side of the ordinal preferences of the players and their transparency\(^6\) between them, along with the characterizing assumption that players’ beliefs are coarsely represented via collections of their opponents’ actions. As such, this endeavor can be considered in line with the so called Wilson doctrine going back to Wilson (1987), that asks for a relaxation of the common knowledge assumptions in game theory.\(^7\)

By focusing on games under ignorance, any strategic analysis immediately runs into a conceptual issue. Whereas there is general agreement on the notion of rationality—whenever explicitly posited—that should be employed in the analysis of ‘standard’ games,\(^8\) namely, that of Bayesian rationality as subjective expected utility maximization, the same cannot be stated regarding games under ignorance—as ordinal games. We bypass this conceptual issue by focusing on two classic decision criteria under ignorance, namely, max max and max min, building on their decision-theoretic foundation going back to Wald (1950), Milnor (1954), and Arrow & Hurwicz (1977).\(^9\) As a matter of fact, these decision criteria are the very reason behind our choice of calling our players “tropical”, in light of what is known as tropical algebra, which studies the algebraic properties of structures where max and min are possible operators.\(^10\) Thus, informally, a tropical player—with her rather coarse beliefs—can have the following different attitudes concerning her play:

- she can be optimistic, in which case she is going to assume that, for every action of hers, her opponents choose their actions (from the actions she contemplates as possible), to maximize her utility and—consequently—she is going to choose the action that gives her the highest utility accordingly (hence, proceeding according to the max max criterion);
- she can be pessimistic, in which case she is going to assume that, for every action of hers, her opponents choose their actions (from the actions she contemplates as possible), to minimize her utility and—consequently—she is going to choose the action that gives her the highest utility accordingly (thus, proceeding according to the max min criterion).

Moving from our tropical decision criteria, we study what are the behavioral implication that we (or the players themselves) can expect to obtain when we (or they) assume that players are optimistic (resp., pessimistic), that there is optimism and mutual belief in optimism (resp., pessimism), and so on up to optimism and common belief in optimism (resp., pessimism). Thus, to provide an explicit analysis of strategic reasoning under ignorance, we perform our investigation by employing the tools of epistemic game theory. With respect to this point, first of all, we identify in Definition 2.1 the framework appropriate for our analysis, which happens to be that of epistemic possibility structures of Mariotti et al. (2005, Sections 2 & 3). The main difference between epis-

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\(^4\)See Section 7.1 for a discussion of this point.

\(^5\)See Bomanno (2018) for a textbook focusing on this variety of games. It has to be observed that, throughout this work, we use the word “variety” to refer to games played under ignorance rather than the word “class”. Indeed, assuming the presence of ignorance does not have any impact on the actual description of the primitives of the games, which is what should be effected in order to distinguish between different classes of games.

\(^6\)That is, common knowledge in the informal sense of the expression.

\(^7\)With the understanding that—implicitly—our analysis relies on assuming common knowledge of a situation under ignorance.

\(^8\)That is, games where players have cardinal utilities and have beliefs in the form of probability measures.

\(^9\)The max min decision criterion goes back to Wald (1950, Chapter 1.4.2, p.18). The max max criterion can be obtained by replacing the convexity axiom in Milnor (1954) with a concavity axiom.

\(^10\)The exotic name of this branch of algebra was chosen by French mathematicians to honor Imre Simon, the first—Hungarian-Brazilian—mathematician who worked extensively on the topic. See Speyer & Sturmfels (2009) and Footnote 21.
tenic possibility structures and (the more commonly used) epistemic type structures\footnote{See Dekel & Siniscalchi (2015) for a comprehensive survey of epistemic game theory or Perea (2012) for a textbook completely devoted to the topic.} is that in epistemic possibility structures beliefs are represented exactly in the coarse way described above, i.e., as subsets of the space of uncertainty.

With epistemic possibility structures at our disposal, we proceed by explicitly defining those epistemic events that correspond to a player being optimistic or pessimistic (respectively, in Equation (2.1) and Equation (2.2)), thus formalizing the—informal—description of these attitudes provided above. Also, by employing modal operators capturing belief and common belief (as it is standard in the epistemic game theory literature), we define in Definition 2.2 the events in epistemic possibility structures of Optimism/Pessimism and Common Belief in Optimism/Pessimism. Having established the epistemic events of interest, we proceed by providing a characterization of their behavioral implications. In Definition 3.1, we give a definition in our language of Point Rationalizability of Bernheim (1984), that algorithmically characterizes the behavior under Optimism and Common Belief in Optimism (as established in Theorem 2). Furthermore, in Definition 3.2, we introduce a new procedure with a Rationalizability flavor, called Wald Rationalizability, that— as we show in Theorem 3—algorithmically characterizes the behavioral implications of Pessimism and Common Belief in Pessimism. Regarding a comparison of the two algorithmic procedures, Proposition 4 shows that the set of actions that survive Point Rationalizability is a subset of the set of actions that survive Wald Rationalizability. In other words, Pessimism and Common Belief in Pessimism has less identification power through observed behavior than its optimistic counterpart.

Taking into accounts the peculiarities of working with epistemic possibility structures, our characterization theorems follow arguments that are now common in epistemic game theory literature. Nevertheless, there is an interesting detail in Theorem 2 and Theorem 3, namely, our language allows us to provide one proof for the two characterizations (as in Appendix A.2). However, we do see the main contribution of the present paper—obviously, beyond the introduction of the tropical decision criteria in a game theoretical context—to be conceptual in providing a precise language to address questions for games under ignorance. To appreciate this last statement, we use our main characterizations to shed light on some existing—and seemingly unrelated—results.

First of all, in Section 4 we compare our notion of Optimism to the notion of Wishful Thinking introduced in Yildiz (2007). Quite naturally, our Optimism and Common Belief in Optimism is essentially equivalent to Wishful Thinking and Common Belief in Wishful Thinking. However, we illustrate that the ‘wishful thinking’-oriented algorithm of Yildiz (2007, Section 3, p.326) crucially, but somewhat implicitly, relies on knowledge instead of belief. It is well known that the key difference between these modal attitudes is the Truth axiom, which asserts that whatever a payer believes is also true. In the realm of static ‘standard’ games the belief-knowledge distinction is often inconsequential,\footnote{See Section 7.4 for a more detailed discussion on this point. For dynamic games it is well known that the distinction might matter as argued in Samet (2013).} but this is not the case for games under ignorance. An important consequence of this conceptual remark is that Wishful Thinking and Common Belief in Wishful Thinking is never empty: an important property that is orthogonal to the existence failure of Wishful Thinking and Common Knowledge in Wishful Thinking as established in Yildiz (2007, Example 5). Furthermore, our analysis illustrates that, in contrast to Yildiz (2007), who employs probability measures and (transparency of) players’ risk attitudes, the rather weak assumptions described above are sufficient to study wishful thinking in strategic environments.

In Section 5, rather naturally given that we work in an ‘ordinal’ setting, we investigate the relation between our notions of Rationalizability and a form of Rationalizability built on Börgers Dominance, in light of the latter being a notion specifically designed for ordinal games. Thus, first of all, it has to be recalled that Börgers Dominance has been introduced\footnote{As “Pure Strategy Dominance” in the title, whereas in the body of the article it is simply called “dominance”.} in Börgers (1993) to capture in ordinal games the notion of justifiability,\footnote{Börgers (1993) actually does not use the word “justifiability”: rather, he calls an action rational if it satisfies the condition described in the main body. The recent literature on epistemic game theory distinguishes rationality and justifiability in the following way: an action is justifiable, while an action-type pair is rational (see Battigalli et al. (Work in Progress)). Since our contribution is related to the epistemic game theory literature, we employ this recent terminology.} which in the original article means that for every justifiable action we can produce a probability measure and a von Neumann-Morgenstern utility function that agrees with the player’s ordinal preferences according to which the action is a
maximizer. Thus, as such, this dominance notion is—by definition—directly linked to the standard realm of the usual Bayesian paradigm. Given this, starting from a notion of rationality defined as choosing an action that is weakly undominated by a pure action relative to the opponents’ actions that are deemed possible.\footnote{See Section 5 and Section 7.2 for a thorough discussion of this notion of rationality.} Bonanno & Tsakas (2018, Theorem 1) show that Rationality and Common Belief in Rationality (as defined above) is algorithmically characterized by an algorithmic procedure that iteratively eliminates actions that are Börgers dominated. To be able to compare our notions with the one in Bonanno & Tsakas (2018), in Equation (5.3) we define in our language the notion of rationality of Bonanno & Tsakas (2018, Definition 2, Section 3), that we—rather naturally given its definition—call “Admissibility,”\footnote{Where we define it for pure actions only as in Luce & Raiffa (1957, Chapter 13.3, p.287). Brandenburger et al. (2008, Definition 3.1, p.320) is the corresponding version in presence of mixed actions.} and we replicate Bonanno & Tsakas (2018, Theorem 1) in our setting, where the focus is on the players’ perspective rather than on that of an outside analyst, by showing that Admissibility and Common Belief in Admissibility is algorithmically characterized by an appropriately defined version of Börgers Rationalizability (as in Definition 5.1). Armed with this result as a benchmark, we compare Börgers Rationalizability to Point Rationalizability and Wald Rationalizability: while we can state in Proposition 9 that Point Rationalizability always selects a subset of the profiles of actions selected by Börgers Rationalizability, we show that there is no inclusion relation between Börgers and Wald Rationalizability. Thus, for ordinal games, the predictions based on a clear decision-theoretic optimality criterion for situations under ignorance might be distinct from those obtained when the baseline assumption of behavior is derived either from a Bayesian notion or, equivalently, from a particular dominance notion. However, interestingly, we establish in Proposition 10 that, in generic games, Börgers Rationalizability always selects a superset of the profiles of actions selected by Wald Rationalizability. Therefore, we can conclude that, in generic ordinal games, common belief in either tropical attitudes refines Börgers Rationalizability. That is, our two decision-theoretic notions under ignorance provide sharper predictions than the one based on the Bayesian approach.

Finally, in Section 6, we focus on the relation between the present setting and that of Weinstein (2016), a recent important contribution that—among the other things—studies the behavior of the Rationalizability correspondence as players become infinitely risk averse or risk seeking, considering the payoffs of a given game as monetary payoffs. As a matter of fact, this section is actually related to Section 5, since here as well Börgers Rationalizability happens to enter the stage. Indeed, Weinstein (2016, Proposition 3) shows that the set of (standard) rationalizable action profiles converges to the—opportunistically defined—Börgers rationalizable action profiles as players become infinitely risk averse, whereas point rationalizable action profiles are the result of players becoming infinitely risk seeking (as shown in Weinstein (2016, Proposition 2)). Focusing on the corresponding limit points, while Point Rationalizability is the limit point of players being infinitely risk seeking, Börgers Rationalizability does not coincide with the limit point of players being infinitely risk averse. This actually corresponds to the discontinuity hinted in Weinstein (2016, p.1891). Indeed, it is Wald Rationalizability that does coincide with the limit point of players being infinitely risk averse. Thus, by identifying this along with the epistemic characterization of Wald Rationalizability given in Theorem 3, we provide a conceptual foundation to this phenomenon. As a matter of fact, this hides an interesting conceptual twist, namely that the discontinuity along with the epistemic characterization seem to be a symptom of the fact that, even if defined for ordinal games, Börgers dominance is fundamentally related to the standard Bayesian framework, a point which is consistent with the rationale behind this notion of dominance.

1.2 Related Literature

This paper fits various streams of literature. On one side, it belongs to those studies that focus on games where only ordinal preferences are assumed to be transparent between players: as such, it is related to Börgers (1993) and Bonanno & Tsakas (2018). In using the tools of epistemic game theory by starting from explicitly defined assumptions concerning the players, it is related to the literature on the topic—broadly—as in Perea (2012) or Dekel & Siniscalchi (2015) and—more precisely—to Mariotti (2003) and Mariotti et al. (2005). It is related to the two latter works also in how beliefs are coarsely represented as subset of actions (profiles). As a matter of fact, with respect to this point, it is also related to Aumann (1999), Samet (2010), Bonanno (2015), and—
taking into account a different stream of literature—Chen & Micali (2015) and Jakobsen (2020). Finally, regarding the fact that here we investigate ‘extreme’ players’ attitudes, it is related to Yildiz (2007) and Weinstein (2016), where the relation with the latter arises in the way in which these attitudes are identified as polar opposites. In Section 7, we address in a more detailed way the relation between our work an the aforementioned contributions. Furthermore, Brunner et al. (2021) experimentally find that players more often play according to what we call tropical attitudes relative to Nash equilibrium behavior. Finally, Eichberger & Kelsey (2014) study optimism and pessimism in games, but in a setting of ambiguity and equilibrium: therefore, their contribution is distinct and complementary to our approach.

1.3 Synopsis

Summarizing Section 1.1 in a more compact way, this paper is structured as follows. In Section 2, we introduce the variety of games we study and the epistemic structures appropriate for our analysis along with our events of interest. In Section 3, we define the solution concepts that algorithmically characterize the behavioral implications of the events which are the focus of our analysis. In Section 4, we study the relation between our notion of optimism and that of wishful thinking as introduced by Yildiz (2007). In Section 5, we relate our work to the notion of Börgers dominance, in light of the relation between this notion and the implications of rational behavior in ordinal games. In Section 6, we show how our work relates to Weinstein (2016). Finally, in Section 7, we further discuss various aspects of our work and how our results relate to the existing literature. All the proofs of the results established in the paper are relegated to Appendix A.

2. Primitive Objects

The primitive objects of our analysis are finite ordinal games. In particular, a finite ordinal game (henceforth, game) is a tuple

\[ \Gamma := (I, (A_i, \succsim_i)_{i \in I}) \]

where, for every \( i \in I \), \( A_i \) is player \( i \)'s finite set of actions, with \( A_{-i} := \prod_{j \in I \setminus \{i\}} A_j \) and \( A := \prod_{i \in I} A_i \), and \( \succsim_i \subseteq A \times A \) is player \( i \)'s complete and transitive preference relation over action profiles. As usual, any complete and transitive preference relation \( \succsim_i \) can be represented by a utility function \( u_i : A \to \mathbb{R} \), which is unique up to monotone transformation. Fixing for every player one of these utility functions induces a standard game \( \Lambda(\Gamma) := (I, (A_i, u_i)_{i \in I}) \). To ease notation, we slightly abuse it by using \( \Gamma := (I, (A_i, u_i)_{i \in I}) \) for an ordinal game, where it is understood that \( u_i \) is one possible representation of \( \succsim_i \). It is clear that all definitions and results about ordinal games refer to \( \succsim \) and do not depend on the choice of the utility functions representing the preferences. However, this makes the notation slightly less involved and should not lead to any confusion. With this in mind, we call a game (ordinal or standard) generic if, for every \( i \in I \) and \( a_i, a'_i \in A_i \), the fact that \( a_i \neq a'_i \) implies that \( u_i(a_i, a_{-i}) \neq u_i(a'_i, a_{-i}) \), for every \( a_{-i} \in A_{-i} \). Finally, for every ordinal game \( \Gamma \) there exists an equivalence class of standard games induced by it, i.e., there exists an equivalence relation \( \sim \) such that \( \Lambda(\Gamma) \sim \Lambda'(\Gamma) \), for every two induced standard games \( \Lambda(\Gamma) \) and \( \Lambda'(\Gamma) \).

In what follows, every topological space is assumed to be compact Hausdorff, where in the case of finite spaces this is a consequence of endowing them—as we do—with the discrete topology. Thus, given an arbitrary space \( X \), we let \( \mathcal{X}(X) \) denote the family of all its nonempty, compact subsets endowed with the Hausdorff topology, which makes it compact Hausdorff, whenever \( X \) is compact Hausdorff.\textsuperscript{19}

\textsuperscript{17}They do not consider play according to (iterated) mutual belief of the tropical attitudes, but—within our language—their setting would correspond to the assumption that players consider all the opponents’ actions as possible (which would correspond to a form of full-support assumption).

\textsuperscript{18}Clearly, we could have started with standard games as primitive objects by defining ordinal games as equivalence classes of \( \sim \). However, since our main focus is on ordinal games, we opted for using ordinal games as our primitive objects.

\textsuperscript{19}Recall that the Hausdorff topology is the topology generated by all subsets of the form \( \{ \kappa \in \mathcal{X}(X) \mid \kappa \subseteq G \} \) and \( \{ \kappa \in \mathcal{X}(X) \mid \kappa \cap G \neq \emptyset \} \) with \( G \) open in \( \mathcal{X}(X) \).

\textsuperscript{20}See Section 7.5 for a discussion of this technical assumption.
Definition 2.1 (Epistemic Possibility Structure). Given a game $\Gamma := \langle I, (A_i, u_i)_{i \in I} \rangle$, an epistemic possibility structure (henceforth, possibility structure) appended on $\Gamma$ is a tuple

$$\mathcal{P} := \langle I, (A_i, T_i, \pi_i)_{i \in I} \rangle$$

where, for every $i \in I$, $T_i$ is her compact Hausdorff set of epistemic types (henceforth, types) and $\pi_i : T_i \to \mathcal{X}(A_{-i} \times T_{-i})$ is her continuous possibility function.

To ease the reading, we introduce the function $\varphi_i : T_i \to \mathcal{X}(A_{-i})$ defined as $\varphi_i(t_i) := \text{proj}_{A_{-i}} \pi_i(t_i)$, for every $t_i \in T_i$ (where $\text{proj}$ denotes the—continuous—projection operator as canonically defined), which captures what an arbitrary player $i$ considers possible regarding only the actions of the remaining players, i.e., her first-order beliefs. For every $i \in I$, we let $\Omega_i := A_i \times T_i$, with $\Omega := \Pi_{i \in I} \Omega_i$ being the state space associated to the possibility structure and $\Omega_{-i} := \Pi_{j \in I \setminus \{i\}} \Omega_j$. We call events only the following objects, that have by construction a product structure: $E_i \in \mathcal{X}(\Omega_i)$, $E_{-i} := \Pi_{j \in I \setminus \{i\}} E_j \in \mathcal{X}(\Omega_{-i})$, and $E := \Pi_{i \in I} E_i \in \mathcal{X}(\Omega)$ (e.g., $E_i$ is an event concerning player $i$). That is, all our events of interest are assumed to have a product structure relative to the player indices.

Remark 2.1 (Universality). For every game $\Gamma$, we let $\mathcal{P}^* := \langle I, (A_i, T^*_i, \pi^*_i)_{i \in I} \rangle$ denote the canonical hierarchical structure appended on $\Gamma$ that is constructed as the space that comprises all the players’ infinite hierarchies of beliefs that satisfy a coherency requirement (see Mariotti et al. (2005, Section 3)). The canonical hierarchical structure $\mathcal{P}^*$ is a possibility structure in its own rights that is universal according to the terminology introduced by Siniscalchi (2008), that is:

- it is terminal, since every other possibility structure can be uniquely embedded in it,
- and belief-complete, since the possibility function $\pi_i$ is surjective, for every $i \in I$.

Thus, we call $\mathcal{P}^*$ the universal possibility structure.

Given a possibility structure $\mathcal{P}$ with state space $\Omega$, interactive reasoning is captured by means of opportune modal operators acting on $\Omega$. In particular, the belief operator $\mathcal{B}_i$ of player $i$ is defined as

$$\mathcal{B}_i(E_{-i}) := \{ (a_i, t_i) \in A_i \times T_i \mid \pi_i(t_i) \subseteq E_{-i} \},$$

for every $E_{-i} \in \mathcal{X}(\Omega_{-i})$, with $\mathcal{B}(E) := \Pi_{i \in I} \mathcal{B}_i(E_{-i})$.

The basic events we want to formalize concerning behavior in a possibility structure are those that capture a player being either pessimistic or optimistic. Thus, we let

$$O_i := \left\{ (a^*_i, t_i) \in A_i \times T_i \mid a^*_i \in \arg \max_{a_i \in A_i, \tilde{a}_{-i} \in \varphi_i(t_i)} u_i(a_i, \tilde{a}_{-i}) \right\} \quad (2.1)$$

be the event in $\Omega_i$ that captures player $i$ being optimistic. On the contrary, we let

$$P_i := \left\{ (a^*_i, t_i) \in A_i \times T_i \mid a^*_i \in \arg \min_{a_i \in A_i, \tilde{a}_{-i} \in \varphi_i(t_i)} u_i(a_i, \tilde{a}_{-i}) \right\} \quad (2.2)$$

be the event in $\Omega_i$ that captures player $i$ being pessimistic. It is the reliance on the max and min operators of these decision criteria that lead us to call our players tropical, since in tropical algebra the usual addition is replaced with the max or min operators. As a result, much in the

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Formally, a monoid is an algebraic structure $(M, \sqcup)$, where $\sqcup$ is a binary operation defined over a set $M$ which is associative, i.e., for every $a, b, c \in M$, $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$, with identity element $\mathbb{1} \in M$, i.e., for every $a \in M$, $a \sqcup \mathbb{1} = \mathbb{1} \sqcup a = a$. A monoid is commutative if the binary operation is also commutative. Given that $\mathbb{R}^+ := \mathbb{R} \cup \{ -\infty \}$, a max tropical semiring is a tuple $(\mathbb{R}, \sqcup, \ominus)$, where:

- for every $x, y \in \mathbb{R}$, $x \sqcup y := \max\{x, y\}$ and $x \ominus y := x + y$,
- $(\mathbb{R}^+, \sqcup, \ominus)$ is a commutative monoid with identity element $-\infty$,
- $(\mathbb{R}^+, \ominus, \sqcup)$ is a commutative monoid with identity element $0$.

Similarly, one can define a min tropical semiring $(\mathbb{R}^+, \sqcup, \ominus)$ related to our definition of pessimism, with $\mathbb{R}^+ := \mathbb{R} \cup \{ +\infty \}$ and $x \ominus y := \min\{x, y\}$.
same spirit in which expected utility is a (weighted) sum of the underlying utilities of outcomes in a standard Bayesian framework, for tropical players our decision criteria involve ‘(tropical) sums’ again.

**Example 1 (Leading Example).** Consider the game represented in Figure 1 with two players, namely, Ann (viz., $a$) and Bob (viz., $b$).

|       | $L$ | $C$ | $R$ |
|-------|-----|-----|-----|
| $T$   | 2.3 | 3.2 | 1.1 |
| $M$   | 4.3 | 1.1 | 4.0 |
| $D$   | 2.0 | 2.2 | 1.1 |

Figure 1: A 3 × 3 game.

To see the events we have introduced at work, we append on it a possibility structure. In particular, we focus on Ann, with $T_a := \{t_a, t'_a, t''_a\}$ and $ϕ_a(t_a) := \{L\}$, $ϕ_a(t'_a) := \{C\}$, $ϕ_a(t''_a) := A_b$.

Then it is straightforward to observe that $O_a := \{(M, t_a), (T, t'_a), (M, t''_a)\}$, $P_a := \{(M, t_a), (T, t'_a), (T, t''_a), (M, t''_a), (D, t''_a)\}$.

Crucially, the difference between Ann’s attitude arises when she contemplates the idea that Bob can play more than one action, i.e., when her type is $t''_a$. If she is optimistic, she is going to expect Bob to play $L$ or $R$, because both those actions can give her the highest utility, thus she is going to play $M$ (indeed, in both cases she can get 4); if she is pessimistic, she is indifferent between $T$, $M$, and $D$, since 1 is the lowest possible payoff she could get given $L$, $C$, or $R$. ⋄

Having defined what it means for a player to be either optimistic or pessimistic by opportune events in $Ω_i$, the natural next step is to investigate the implications of having players involved in a game interactively reason about each others. First, we have that $O := \prod_{i \in J} O_i$ and $P := \prod_{i \in J} P_i$. Given this, we let $CB^0(O) := O$ and $CB^0(P) := P$ denote the events that all players are optimistic and pessimistic, respectively. Concerning interactive reasoning, we then define inductively for every $m \in \mathbb{N}$ the corresponding (correct) $m^{th}$-order mutual belief events:\(^2^2\)

$CB^m(O) := O \cap B(CB^{m-1}(O))$,

$CB^m(P) := P \cap B(CB^{m-1}(P))$.

The role in the rest of the analysis of the events concerning common belief is such that they deserve their own definition.

**Definition 2.2 (Optimism/Pessimism and Common Belief in Optimism/Pessimism).** Given a game $Γ$ and a possibility structure $P$ with state space $Ω$, the epistemic condition Optimism and Common Belief in Optimism is captured by the event

$OCBO := CB^∞(O) := \bigcap_{m \geq 0} CB^m(O)$.

\(^2^2\)Note that, as usual in models without introspective beliefs, we do impose correct beliefs to restrict behavior in addition to restricting (higher-order) beliefs. Even though they consider a standard Bayesian framework, the arguments made in the first paragraph in Dekel & Siniscalchi (2015, Section 3.2) apply verbatim to our framework. This is not to be confused with imposing the Truth Axiom, which would impose correct beliefs for all possible events. Such an additional assumption would change our results, as discussed in more detail in Section 4. With this in mind, in what follows we do not employ the word “correct” unless explicitly needed.
while

\[
\text{PCBP} := \text{CB}^\infty(P) := \bigcap_{m \geq 0} \text{CB}^m(P)
\]

is the event that captures the condition Pessimism and Common Belief in Pessimism.

Having established our events of interest, a crucial step whenever involved in an epistemic analysis is to establish that those events are actually epistemic ‘events’ for the players. That is, we just defined \text{OCBO} and \text{PCBP}, but are those events part of the language of the players? This is a crucial problem, since we want our players to reason about these very events. This is exactly what we achieve next.

**Proposition 1.** Given a possibility structure \( \mathfrak{P} \) with state space \( \Omega \) appended on a game \( \Gamma \):

i) for every \( n \in \mathbb{N} \), \( \text{CB}^n(O) \in \mathcal{K}(\Omega) \) and \( \text{CB}^n(P) \in \mathcal{K}(\Omega) \);

ii) \( \text{OCBO} \in \mathcal{K}(\Omega) \) and \( \text{PCBP} \in \mathcal{K}(\Omega) \).

The reason why Proposition 1 is enough to establish this point is that, rather informally, given our topological assumptions, these results amount to stating that the relevant sets are events in the measurable sense of the term.\(^{23}\)

### 3. Capturing Tropical Attitudes

Having formalized the epistemic framework that we append on an ordinal game, it is natural to ask ourselves if we can algorithmically characterize the behavioral implications of the epistemic events of interest, with a particular attention to those defined in Definition 2.2. The following two subsections provide such characterization.

#### 3.1 The Optimistic Player

We want to formalize a notion of best-reply that captures the max max decision criterion (in presence of coarse beliefs) that lies behind our notion of optimism. Thus, given a game \( \Gamma \), a player \( i \in I \), and a \( \kappa_i \in \mathcal{K}(A_{-i}) \), we let

\[
\rho^\text{max}_i(\kappa_i) := \arg\max_{a_i \in A_i} \max_{\tilde{a}_{-i} \in \kappa_i} u_i(a_i, \tilde{a}_{-i})
\]

denote the set of optimistic best-replies to belief \( \kappa_i \in \mathcal{K}(A_{-i}) \). We deem justifiable all those actions belonging to \( \rho^\text{max}_i(\kappa_i) \) for a given \( \kappa_i \in \mathcal{K}(A_{-i}) \).

**Remark 3.1 (Singleton Beliefs).** For every \( \kappa_i \in \mathcal{K}(A_{-i}) \), if \( \kappa_i \) is a singleton, then

\[
\rho^\text{max}_i(\{a_{-i}\}) = \arg\max_{a_i \in A_i} u_i(a_i, a_{-i})
\]

for \( a_{-i} \in A_{-i} \) such that \( \kappa_i = \{a_{-i}\} \).

Building on the notion of optimistic best-replies, we can now define a solution concept which is essentially a formulation based on our language of Point Rationalizability, as introduced in Bernheim (1984, Section 3(b)).

**Definition 3.1 (Point Rationalizability).** Fix a game \( \Gamma := (I, (A_i, u_i)_{i \in I}) \) and consider the following procedure, for every \( i \in I \) and \( m \in \mathbb{N} \):

- (Step \( m = 0 \)) \( P^0_i := A_i \);

---

\(^{23}\)See Appendix A.1 for a formalization of this point along with the proof of the result.
(Step $m > 0$) Assume that $P^m := P_i^m \times P_j^m$ has been defined and let

$$P_i^{m+1} := \left\{ a_i^* \in P_i^m \mid \exists \kappa_i \in \mathcal{K}(A_{-i}) \exists a_{-i}^* \in P_{-i}^m : 
\begin{align*}
1. & \kappa_i = \{ a_{-i}^* \}, \\
2. & a_i^* \in \rho^{\max}_i(\kappa_i)
\end{align*}
\right\}.$$ 

Thus, for every $m \in \mathbb{N}$, we let $P_i^m$ denote the set of actions of player $i$ that survive the $m$-th iteration of the Point Rationalizability procedure. Finally,

$$P_i^\infty := \bigcap_{m \geq 0} P_i^m$$

is the set of actions of player $i$ that survive the Point Rationalizability procedure, with $P^\infty := \prod_{j \in I} P_j^\infty$ denoting the set of point rationalizable action profiles.

Before seeing Point Rationalizability at work, it is important to recall that its nonemptiness has been established in Bernheim (1984, Proposition 3.1). Thus, we now go back to our leading example to see what are the behavioral predictions we obtain there via Point Rationalizability.

Example 1 (Leading Example, Continued). To see Definition 3.1 at work, we consider the game in Figure 1. There we have that $P_a^I = \{ \Gamma, M \}$ and $P_b^I = \{ \Lambda, C \}$ and then $P_a^2 = P_a^1$ and $P_b^2 = \{ \Lambda \}$. As a result, $P_a^3 = \{ M \} = P_a^\infty$ and $P_b^3 = \{ \Lambda \} = P_b^\infty$.\[\Box\]

We can now tackle the problem of the algorithmic characterization of the behavioral implications of Optimism and Common Belief in Optimism. As a matter of fact, the result that we state next settles the issue.

Theorem 2 (Tropical Foundation of Point Rationalizability). Fix a game $\Gamma$.  

i) If $\mathcal{P}$ is an arbitrary possibility structure appended on it, then

$$\text{proj}_A \text{CB}^n(0) \subseteq P_i^{n+1},$$

for every $n \in \mathbb{N}$, and

$$\text{proj}_A \text{OCBO} \subseteq P_i^\infty.$$  

(3.2)

ii) Given the universal possibility structure $\mathcal{P}^*$,

$$\text{proj}_A \text{CB}^n(0) = P_i^{m+1},$$

for every $n \in \mathbb{N}$, and

$$\text{proj}_A \text{OCBO} = P_i^\infty.$$  

(3.4)

3.2 The Pessimistic Player

We now formalize a notion of best-reply that captures the max min decision criterion (in presence of coarse beliefs) that lies behind our notion of pessimism, in the same spirit of what we did in the previous section. Thus, given a game $\Gamma$, a player $i \in I$, and a $\kappa_i \in \mathcal{K}(A_{-i})$, we let

$$\rho^{\min}_i(\kappa_i) := \arg \max_{a_i \in A_i} \min_{a_{-i} \in \kappa_i} u_i(a_i, a_{-i})$$

denote the set of pessimistic best-replies to belief $\kappa_i \in \mathcal{K}(A_{-i})$. As for the optimistic best-replies, we deem justifiable as well all those actions belonging to $\rho^{\min}_i(\kappa_i)$, for a given $\kappa_i \in \mathcal{K}(A_{-i})$.

Remark 3.2 (Singleton Beliefs & Optimism I). For every $\kappa_i \in \mathcal{K}(A_{-i})$, if $\kappa_i$ is a singleton, then

$$\rho^{\min}_i(\{ a_{-i} \}) = \max_{a_i \in A_i} u_i(a_i, a_{-i}) = \rho^{\max}_i(\{ a_{-i} \})$$

for $a_{-i} \in A_{-i}$ such that $\kappa_i = \{ a_{-i} \}$.
We can now introduce our algorithmic procedure that captures interactive pessimism in static games, that we call Wald Rationalizability in honor of Abraham Wald’s celebrated decision criterion introduced in Wald (1950).

**Definition 3.2 (Wald Rationalizability).** Fix a game \( \Gamma := (I, (A_i, u_i)_{i \in I}) \) and consider the following procedure, for every \( i \in I \) and \( m \in \mathbb{N} \):

- (Step \( m = 0 \)) \( W_0^i := A_i \);
- (Step \( m > 0 \)) Assume that \( W^m := W^m_i \times W^m_{-i} \) has been defined and let

\[
W^{m+1}_i := \left\{ a^*_i \in W^m_i \mid \begin{array}{l}
\exists \kappa_i \in K(A_{-i}) \exists \tilde{A}_{-i} \subseteq W^m_{-i} : \\
1. \kappa_i = \tilde{A}_{-i}, \\
2. a^*_i \in \rho_i^{\min}(\kappa_i)
\end{array} \right\}.
\]

Thus, for every \( m \in \mathbb{N} \), we let \( W^m_i \) denote the set of actions of player \( i \) that survive the \( m \)-th iteration of the Wald Rationalizability procedure. Finally,

\[
W^\infty_i := \bigcap_{m \geq 0} W^m_i
\]

is the set of actions of player \( i \) that survive the Wald Rationalizability procedure, with \( W^\infty := \prod_{j \in I} W^\infty_j \) denoting the set of Wald rationalizable action profiles.

Mirroring the structure of Section 3.1, we now state a crucial property of Wald Rationalizability (implied by Proposition 4 below).

**Remark 3.3 (Nonemptiness).** For every game \( \Gamma \), \( W^\infty \neq \emptyset \).

Again, we go back to our leading example to see how Wald Rationalizability performs there.

**Example 1 (Leading Example, Continued).** To see Definition 3.1 at work, we consider again the game in Figure 1. Now using Section 3.2 gives \( W^1_a = A_a \) and \( W^1_b = \{L, C\} \). As a matter of fact, the algorithm stops here. Thus, we have that \( W^\infty_a = A_a \) and \( W^\infty_b = \{L, C\} \).

As we did for Optimism and Common Belief in Optimism, we now solve the issue of providing an algorithmic characterization for the behavioral implications of Pessimism and Common Belief in Pessimism.

**Theorem 3 (Tropical Foundation of Wald Rationalizability).** Fix a game \( \Gamma \).

i) If \( \mathcal{P} \) is an arbitrary possibility structure appended on it, then

\[
\text{proj}_A \mathbb{C}B^n(\mathcal{P}) \subseteq W^{n+1},
\]

for every \( n \in \mathbb{N} \), and

\[
\text{proj}_A \mathbb{P}C\mathbb{B} \subseteq W^\infty.
\]

ii) Given the universal possibility structure \( \mathcal{P}^* \),

\[
\text{proj}_A \mathbb{C}B^n(\mathcal{P}) = W^{n+1},
\]

for every \( n \in \mathbb{N} \), and

\[
\text{proj}_A \mathbb{P}C\mathbb{B} = W^\infty.
\]

24 It is obvious that this can be formulated by simply dropping Condition (1) and one of the existential statements. We stick to this formulation, because it makes more perspicuous the formulation of ‘one proof for two results’ hinted in Section 1.1.
3.3 Relation between the Algorithms

Having formalized procedures that, as shown, algorithmically characterize the behavior corresponding to the epistemic events of interests, it is natural to investigate what is the relation between the two solutions concepts just introduced. Our Example 1 already shows that $W^\infty \nsubseteq P^\infty$. But what about the reverse inclusion? Can we say that $P^\infty$ is a refinement of $W^\infty$? On intuitive grounds, this should be the case and the following result formally establishes exactly this point.

**Proposition 4.** Given a game $\Gamma$, $P^n \subseteq W^n$, for every $n \in \mathbb{N}$.

The proof is equally intuitive: if a strategy is an optimistic best-reply, then it a pure best-reply to the player’s favorite opponent’s action, but then it also a pessimistic best-reply to the singleton belief considering only this opponent’s strategy as possible. In other words, as pointed out in Remark 3.2, for singleton beliefs the two notions coincide and for the optimistic case it is without loss to consider such singleton beliefs.\(^{25}\) Conversely, a pessimistic best-reply might need a non-singleton belief. Therefore, there are occasions in which the inclusion is strict.

4. Wishful Thinking Revisited

Yildiz (2007) proposes a model of wishful thinking in strategic environments to which our notion of optimism shares its behavioral attitude along with its mathematical formalization as in Equation (2.1). However, there are some crucial differences between our approach and that of Yildiz (2007). Most obviously, the algorithm in Yildiz (2007, Section 3) differs from Point Rationalizability, since the former deletes action profiles, while the latter actions.

**Definition 4.1 (Wishful Thinking Procedure).** Fix a game $\Gamma := \langle I, (A_i, u_i)_{i \in I} \rangle$ and consider the following procedure, for every $m \in \mathbb{N}$:

- (Step $m = 0$) $Y^0 := A$;
- (Step $m > 0$) Assume that $Y^m$ has been defined and let

$$Y^{m+1} := \left\{ a^* \in Y^m \mid \forall i \in I \exists a_{-i} \in A_{-i} : \begin{array}{l}
1. (a_i^*, a_{-i}) \in Y^m,
2. a_i^* \in \rho_i^{\max}(\{a_{-i}\}),
3. u_i(a_i^*, a_{-i}) \geq \max_{a_i \in A_i} u_i(a_i, a_i^*)
\end{array} \right\}$$

Then, for every $m \in \mathbb{N}$, $Y^m$ denotes the set of action profiles that survive the $m$-th iteration of the Wishful Thinking procedure. Finally,

$$Y^\infty := \bigcap_{m \geq 0} Y^m$$

is the set of Wishful Thinking action profiles.

Yildiz (2007) defined the algorithm in terms of deleting action profiles on each round. In contrast, our Definition 4.1 is defined as collecting actions that are justifiable by means of wishful thinking taking as given those profiles that are deemed justifiable in the previous rounds. Of course, the difference is just a change in quantifiers, but we opted for the current version to facilitate the comparison with the procedures introduced before. In particular, the current definition makes clear the connection to Point Rationalizability (as in Definition 3.1). Indeed, the Wishful Thinking procedure is a refinement of Point Rationalizability.

**Proposition 5.** Given a game $\Gamma$, $Y^n \subseteq P^n$, for every $n \in \mathbb{N}$.

\(^{25}\)We elaborate on this in Section 7.3.
Furthermore, Yildiz (2007, Example 5) illustrates an existence failure of his model, whereas Point Rationalizability is always nonempty. Thus, the inclusion in Proposition 5 might be strict and, as a result, the behavioral implications of Optimism and Common Belief in Optimism differ from those obtained via the Wishful Thinking procedure. For illustration purposes, we now show an example of the Wishful Thinking procedure selecting a strict subset of action profiles of those selected via Point Rationalizability.

**Example 2 (Battle of the Sexes).** To see the difference, consider the leading example of Yildiz (2007), which happens to be the Battle of the Sexes.

|   |   |   |
|---|---|---|
| a | T | R |
|   | 2.1 | 0.0 |
| D | 0.0 | 1.2 |

Figure 2: Battle of the Sexes.

Clearly, $P^\infty = A_a \times A_b$. However, the algorithm in Yildiz (2007) deletes the action profile $(D, L)$. To justify the profile $(D, L)$ under optimism/wishful thinking, Ann must believe that $L$ is impossible, which *ipso facto* is a wrong belief, which is not allowed in the model of Yildiz (2007), but is allowed in our framework. Indeed, as pointed out in Yildiz (2007, Section 1, p.321), by focusing—without loss of generality—on Ann, it is not possible for her to indulge in wishful thinking, play $D$, and believe that $L$ is possible, since then she would believe that $L$ will happen.$^{26}$

Given this example and the fact that the baseline assumptions about players’ behavior are virtually the same, it is natural to ask ourselves why this difference arises with respect to behavioral predictions. As already hinted in Example 2, the crucial issue lies in the modal operators employed: we use the belief operator, while Yildiz (2007) uses the knowledge operator. It is well known that knowledge differs from belief in that knowledge satisfies the Truth Axiom, which states that whatever is known must be true. $^{27}$ Since belief does not satisfy this axiom, a player in our model might believe an event that is actually wrong. $^{28}$

We now formalize the informal argument sketched in the paragraph above. Thus, first of all, we introduce the appropriate epistemic model to work with knowledge and then proceed by providing a proper comparison between the two approaches.$^{29}$

**Definition 4.2 (Knowledge Structure).** Given a game $\Gamma := \langle I, (A_i, u_i)_{i \in I} \rangle$, a knowledge structure appended on $\Gamma$ is a tuple

$$\mathcal{R} := \langle I, \Psi, (T_i, \Pi_i)_{i \in I} \rangle$$

where

1. for every player $i \in I$, $T_i$ is a finite set of types for each player,
2. $\Psi \subseteq \prod_{i \in I} (A_i \times T_i)$ is the state space,
3. for every $i \in I$, $\Pi_i$ is a partition of $\Psi$ with $\Pi_i(\omega) \subseteq \Psi$ denoting the cell containing $\omega \in \Psi$, and
4. for every $i \in I$, $\Pi_i$ satisfies the following properties:
   (a) (Introspection) for every $\omega \in \Psi$, $\text{proj}_{A_i \times T_i} \Pi_i(\omega) = \{\text{proj}_{A_i \times T_i} \omega\}$, and
   (b) (Independence) for every $\omega, \omega' \in \Psi$, if $\text{proj}_{T_i} \omega = \text{proj}_{T_i} \omega'$, then $\text{proj}_{A_{-i} \times T_{-i}} \Pi_i(\omega) = \text{proj}_{A_{-i} \times T_{-i}} \Pi_i(\omega')$.

$^{26}$An anonymous referee suggested this statement that is clearer than the statement we had in an earlier version of this paper. We are thankful for this suggestion.

$^{27}$See for example Osborne & Rubinstein (1994, Section 5.1.2, p.70).

$^{28}$Samet (2013, Section 3.2) provides a detailed discussion of the differences within the framework of belief structures.

$^{29}$To simplify notation, in this section we focus on finite epistemic models. For the purpose of a meaningful comparison between the approaches, this restriction is without loss of generality.
For readers familiar with usual definitions of knowledge structures as introduced in Aumann (1976) (henceforth, *Aumann structures*), Definition 4.2 might look a bit obscure and therefore some remarks are in order. In contrast to Aumann structures, in our knowledge structures, states are not completely abstract, but rather are comprised of action-type pairs of every player similarly to states in our possibility structures. Indeed, this is the main reason why we use this definition of knowledge structure, as it makes the comparison to our possibility structures more transparent. However, in contrast to our possibility structures, but exactly as in Aumann structures, Definition 4.2 allows for a state space that does not have a product structure. Finally, partitions in our knowledge structures are assumed to satisfy two properties: Condition 4(a) states that a player is introspective in the sense of knowing his own action-type pair. Additionally, we impose an independence condition as stated in Condition 4(b): this is due to the aforementioned intrinsic meaning of states in our formalization and is related to the arguments made in Stahmker (1998, Footnote 5, p.35). This independence condition itself is reminiscent of the *AI condition* of Dekel & Siniscalchi (2015, Definition 12.15), studied in Guarino & Tsakas (2021).

With this in mind, any knowledge structure naturally gives rise to a possibility structure by using the same type spaces for every player and by defining the possibility functions as $\pi_i(t_i) := \text{proj}_{A_i \times T_i} \Pi_i(\omega)$ for every $\omega \in \Psi$ such that $\omega = (a_i, t_i, a_{-i}, t_{-i})$, where this construction is well-defined thanks to the independence condition. The state space $\Omega$ associated with the resulting possibility structure might in general be larger than the state space associated with the knowledge structure $\Psi$: in particular, if $\Psi$ has a non-product structure, then $\Psi \not\subseteq \Omega$. Conversely, starting from a possibility structure, it might not always be possible to construct a knowledge structure. Naturally, one would try to construct partitions with cells of the form $\{(a_i, t_i)\} \times \pi_i(t_i)$, which would satisfy introspection and independence. However, it is well known that such a construction does not yield a partition unless more restrictions are placed on the possibility functions $\pi_i$.

We can now introduce the operator of interest in this framework, namely, the *knowledge operator* $\mathbb{K}_i$ of player $i$, defined as $\mathbb{K}_i(E) := \{ \omega \in \Psi \mid \Pi_i(\omega) \subseteq E \}$, for a (possibly non-product) $E \in \mathcal{X}(\Psi)$. Naturally, $\mathbb{K}(E) := \cap_{i \in J} \mathbb{K}_i(E)$ and the iterated application of the operator gives rise to $\mathbb{K}^\infty(E)$. Hence, $\mathbb{K}^\infty(E)$ denotes the *common knowledge operator* applied on an arbitrary event $E \in \mathcal{X}(\Psi)$.

It is important to observe that we do not need to define a correct knowledge operator. Indeed, the operator $\mathbb{K}$ satisfies the so called *Truth Axiom*, i.e., $\mathbb{K}_i(E) \subseteq E$, for every (possibly non-product) $E \in \mathcal{X}(\Omega)$. In other words, whatever a player knows is also true. Hence, a correct knowledge operator would be essentially redundant, since knowledge implies being correct. This difference is critical for the dichotomy optimism/wishful thinking and illustrates the discrepancies in the behavioral implications for the Battle of the Sexes.

**Example 2 (Battle of the Sexes, Continued).** Consider again the game depicted in Figure 2. We append a possibility structure to it with $T_a := \{(t_a, t'_a), (T, t_a)\}$, $T_b := \{(t_b, t'_b)\}$, and

$$
\begin{align*}
\pi_a(t_a) &= \{(L, t_b), (R, t'_b)\}, & \pi_a(t'_a) &= \{(R, t'_b)\}, \\
\pi_b(t_b) &= \{(T, t_a)\}, & \pi_b(t'_b) &= \{(T, t_a), (D, t'_a)\}.
\end{align*}
$$

Within this possibility structure, we have $O_a = \{(T, t_a), (D, t'_a)\}$ and $O_b = \{(L, t_b), (R, t'_b)\}$. Because these states are the only ones which are considered possible by the players, there is optimism and common belief in optimism. In particular, note that the behavioral implications correspond to $P^\infty = A_a \times A_b$. Now, let us have a close look at the state $((D, t'_a), (L, t_b)) \in \Omega_CBO$. At this state, since $\varphi_a(t'_a) = \{(R, t'_b)\}$, Ann clearly holds a wrong belief. Therefore, Ann cannot know $\{(R, t'_b)\}$ at this state as this would violate the Truth Axiom. Thus, any event she knows at this state has

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30See Section 7.6 for a discussion of the need to consider state spaces without a product structure.

31See Section 7.6 for an alternative formulation of Introspection. In Aumann structures, Introspection is essentially captured via measurability assumptions.

32See Section 7.6 for a discussion of the conceptual reasons behind the decision of imposing the independence condition.

33In particular, an appropriate version of *reflexive* and *Euclidean* possibility functions would be needed to obtain a partition (see Battigalli & Bonanno (1999, Section 2) for the related definitions).

34In contrast, when working with possibility structures, correctness has to be imposed for some particular events to restrict behavior. Footnote 22 discusses this point in more detail.

13
to be a strict superset of \{ (R_i, t_i^*) \} and—in particular—has to include Bob’s action \( L \). Wishful thinking in Yildiz (2007) is defined with respect to knowledge. Therefore, at this state she cannot choose \( D \) as a wishful thinker à la Yildiz (2007). This argument generalizes leading to a removal according to the algorithm in Yildiz (2007). 

We can now translate in our language Yildiz (2007, Proposition 1) in terms of common knowledge of optimism.\(^{35}\)

**Theorem 6.** Fix a game \( \Gamma \).

i) If \( K \) is an arbitrary knowledge structure appended on it, then

\[
\text{proj}_\Pi \mathbb{K}_n(O) \subseteq \mathbb{Y}^{n+1},
\]

for every \( n \in \mathbb{N} \), and

\[
\text{proj}_\Pi \mathbb{K}_\infty(O) \subseteq \mathbb{Y}^\infty.
\]

ii) There exists a knowledge structure \( K \) such that,

\[
\text{proj}_\Pi \mathbb{K}_n(O) = \mathbb{Y}^{n+1},
\]

for every \( n \in \mathbb{N} \), and

\[
\text{proj}_\Pi \mathbb{K}_\infty(O) = \mathbb{Y}^\infty.
\]

5. Relation to Börgers Dominance

We now compare the behavior of Point Rationalizability and Wald Rationalizability to a form of Rationalizability built upon the notion of Börgers Dominance, introduced in Börgers (1993).

Given a game \( \Gamma \) and a player \( i \in I \), action \( a_i \in A_i \) is weakly dominated relative to \( \tilde{A}_{-i} \) for player \( i \) by action \( a_i^* \in A_i \) if \( u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}) \) for every \( a_{-i} \in \tilde{A}_{-i} \) and there exists an action \( a_{-i}^* \in \tilde{A}_{-i} \) such that \( u_i(a_i^*, a_{-i}^*) > u_i(a_i, a_{-i}) \).\(^{36}\) Thus, action \( a_i^* \in \tilde{A}_i \) is admissible if it is not weakly dominated relative to \( \tilde{A}_{-i} \), where we let \( A_i(\tilde{A}_{-i}) \) denote the set of actions of player \( i \) that are admissible. Even if for our purposes it is enough to define admissible actions, it is instructive to recall that an action \( a_i \in \tilde{A}_i \) is Börgers dominated with respect to \( \tilde{A}_{-i} \) if \( a_i \notin A_i(\tilde{A}_{-i}) \), for every subset \( \tilde{A}_{-i} \subseteq \tilde{A}_{-i} \).

Armed with this definition, we want to formalize in our language based on ‘coarse’ beliefs a notion of Rationalizability based on this dominance notion. To achieve this result, given a game \( \Gamma \), a player \( i \in I \), and a belief \( \kappa_i \in \mathcal{K}(\tilde{A}_{-i}) \), we let

\[
\rho_i^\Pi(\kappa_i) := A_i(\kappa_i)
\]

denote the set of admissible best-replies to belief \( \kappa_i \in \mathcal{K}(\tilde{A}_{-i}) \).

**Remark 5.1 (Singleton Beliefs & Optimism II).** For every \( \kappa_i \in \mathcal{K}(\tilde{A}_{-i}) \), if \( \kappa_i \) is a singleton, then

\[
\rho_i^\Pi(\{a_{-i}\}) = \arg \max_{a_i \in A_i} u_i(a_i, a_{-i}) = \rho_i^{\max}(\{a_{-i}\})
\]

for \( a_{-i} \in A_{-i} \) such that \( \kappa_i = \{a_{-i}\} \).

\(^{35}\)Pedantically, we should also define a new event corresponding to optimism in presence of knowledge functions, since possibility functions enter in the definition of optimism as in Equation (2.1). For a knowledge structure, the corresponding event would be

\[
\{ \omega^* = (a_i^*, t_i^*, a_{-i}^*, t_{-i}^*) \in \Psi \mid a_i^* \in \arg \max_{a_i \in A_i} \max_{a_{-i} \in \text{proj}_\Pi \mathbb{K}_\infty(O)} u_i(a_i, a_{-i}) \}
\]

which we do not assign a new symbol to avoid further notational clutter.

\(^{36}\)It has to be observed that—typically—it is necessary to specify also a subset \( \tilde{A}_i \subseteq A_i \) of actions of player \( i \) with respect to which admissibility is defined. Since for our purposes this is not necessary, we omit it to lighten the terminology and the notation.
Much in the same spirit of the procedures we defined in the previous sections, this is really everything we need to formalize in our language Börgers Rationalizability, stated next.

**Definition 5.1 (Börgers Rationalizability).** Fix a game \( \Gamma := \langle I, (A_i, u_i)_{i \in I} \rangle \) and consider the following procedure, for every \( i \in I \) and \( k \in \mathbb{N} \):

- (Step \( m = 0 \)) \( B_i^0 := A_i \);
- (Step \( m > 0 \)) Assume that \( B_i^m := B_i^m \times \mathbb{B}_i^m \) has been defined and let

\[
B_i^{m+1} := \{ a_i^* \in B_i^m \mid \exists \kappa_i \in \mathcal{K}(A_{-i}) : \kappa_i \subseteq B_i^m, a_i^* \in p_i^A(\kappa_i) \}.
\]

(5.2)

Thus, for every \( m \in \mathbb{N} \), \( B_i^m \) denotes the set of actions of player \( i \) that survive the \( n \)-th iteration of Börgers Rationalizability. Finally,

\[
B_i^\infty := \bigcap_{m \geq 0} B_i^m
\]

is the set of actions of player \( i \) that survive Börgers Rationalizability, with \( B^\infty := \prod_{i \in I} B_i^\infty \) denoting the set of action profiles surviving Börgers Rationalizability.

It has to be observed that Börgers undominance as defined above is clearly not captured in Equation (5.1), but rather in Equation (5.2), where the necessary union across all subsets of the \( \kappa_i \in \mathcal{K}(A_{-i}) \) under scrutiny is taken. Given this, it is well known that \( B^\infty \) is nonempty.

Now we can proceed by providing the epistemic foundation to this algorithmic procedure in our epistemic framework based on possibility structures. Before doing so, we want to highlight that already our definition of the procedure is built on having admissibility as the relevant notion of individual behavior and Börgers (un)dominance is only a behavioral manifestation of admissibility across all possible types. Thus, for any possibility structure \( \mathcal{P} \) appended to a game \( \Gamma \) we let\(^{37}\)

\[
A_i := \{ (a_i^*, t_i) \in A_i \times T_i \mid a_i^* \in A_i(\mathcal{P}_i(t_i)) \}.
\]

(5.3)

denote the event that captures those states in \( \Omega \) where player \( i \) does choose an admissible action given her beliefs (as captured via types). Observe that, in contrast to \( O_i \) and \( P_i \), the event \( A_i \) is not defined as an optimal choice for a decision criterion, but rather directly based on a dominance notion. That is, whereas our notions of optimism and pessimism are based on classic decision criteria under ignorance, admissibility is fundamentally a notion of (un)dominance.

With the event \( A_i \) at our disposal we need to make sure the related events are measurable. For this define \( A \) and \( \mathbb{C} \mathbb{B}^n(A) \) for every \( n \in \mathbb{N} \) similar to the definition about optimism and pessimism. Then, all (common belief) events about admissibility are measurable.

**Proposition 7.** Given a possibility structure \( \mathcal{P} \) with state space \( \Omega \) appended on a game \( \Gamma \):

i) for every \( n \in \mathbb{N} \), \( \mathbb{C} \mathbb{B}^n(A) \in \mathcal{K}(\Omega) \);

ii) \( ACBA \in \mathcal{K}(\Omega) \).

Now, it is straightforward to proceed with an epistemic foundation of Börgers Rationalizability, as we do next.

**Theorem 8 (Foundation of Börgers Rationalizability).** Fix a game \( \Gamma \).

i) If \( \mathcal{P} \) is an arbitrary possibility structure appended on it, then

\[
\text{proj}_A \mathbb{C} \mathbb{B}^n(A) \subseteq B^{n+1},
\]

for every \( n \in \mathbb{N} \), and

\[
\text{proj}_A ACBA \subseteq B^\infty.
\]

\(^{37}\)Bonanno & Tsakas (2018) use a similar notion based on introspection.
ii) Given the universal possibility structure \( \mathcal{P}^* \),

\[
\text{proj}_A \mathbb{C}^n(A) = \mathbb{B}^{n+1},
\]

for every \( n \in \mathbb{N} \), and

\[
\text{proj}_A \mathbb{ACBA} = \mathbb{B}^\infty.
\]

Our characterization can be seen as taking the perspective of the players. Within a different framework, Bonanno & Tsakas (2018, Theorem 1) state a seemingly similar result, but provide a different proof. The difference can be interpreted as their analysis taking the perspective of an (outside) analyst. Therefore, we see Theorem 8 as complementary to Bonanno & Tsakas (2018, Theorem 1).\(^{38}\)

As the—well known—result that follows establishes, it is rather easy to show that there exists an immediate relation between Point Rationalizability and Börgers Rationalizability. Like in Proposition 4, the argument follows from the coincidence of the two best-replies for singleton beliefs as in Remark 5.1.\(^{39}\)

**Proposition 9.** Given a game \( \Gamma \), \( P^n \subseteq B^n \), for every \( n \in \mathbb{N} \).

However, as the two examples that follow show, it is not possible to establish an inclusion relation between Börgers Rationalizability and Wald Rationalizability.

**Example 1 (Leading Example (\( W^\infty \not\subseteq B^\infty \)), Continued).** Consider again the game depicted in Figure 1, where the only payoffs represented are those of Ann. It is easy to observe that \( D \not\in B^1_a \). Indeed, for every singleton \( \{a\} \in A_a \), there exists an action in \( A_a \) that strictly dominates \( D \) (e.g., \( T \) strictly dominates \( D \) with respect to \( C \); also, \( T \) weakly dominates \( D \) with respect to \( \{L,C\} \) and \( \{C,R\} \); \( M \) strictly dominates \( D \) with respect to \( \{L,R\} \); finally, \( T \) strictly dominates \( D \) with respect to \( A_b \). However, as we already observed, \( W^1_a = A_a \), since \( A_a = \arg \max_{a \in A_a} \rho^\min_a(\kappa_a) \) for \( \kappa_a = A_b \).

**Example 3 (\( B^\infty \not\subseteq W^\infty \)).** Consider the following game, with two players, namely, Ann (viz., \( a \)) and Bob (viz., \( b \)), where only Ann’s payoffs are represented.

\[
\begin{array}{c|cc}
   & L & R \\
\hline
a & 6 & 1 \\
M & 5 & 2 \\
D & 4 & 3 \\
\end{array}
\]

Figure 3: A game showing that \( B^\infty \not\subseteq W^\infty \).

It is easy to observe that \( B^1_a = A_a \). However, \( M \not\in W^1_a \). Indeed, \( \rho^\min_a(\kappa_a) = \{T\} \) with \( \kappa_a = \{L\} \), while \( \rho^\min_a(\kappa'_a) = \{D\} \) with \( \kappa'_a = \{R\} \) or \( \kappa'_a = \{L,R\} \).

However, if the game is generic, things change and Börgers rationalizable actions result in being a superset of the Wald rationalizable ones. The reason is simple: in generic games, Börgers dominance is the same as strict dominance by a pure action. A strategy that is strictly dominated by a pure action cannot be pessimistic best-reply either, but, as Example 3 shows, the converse does not hold. More generally, in non-trivial (i.e. when the opponent has more than one action available) games, the number of actions satisfying the max min-criterion is bounded by the number of nonempty subsets of \( A_{-i} \), whereas no such bound exists for Börgers-undominated actions. The following proposition summarizes this discussion formally.

\(^{38}\)See Friedenberg & Keisler (Forthcoming, Sections 2.2–2.4) for a thorough discussion of these two interpretations.

\(^{39}\)We provide a direct and simple proof in the appendix without reference to standard best-replies/Rationalizability. However, this result is obvious given the well known implications of point-best-replies being best-replies, which in turn are Börgers-undominated actions. For the latter, see Section 6.
Proposition 10. Given a generic game $\Gamma$, $W^n \subseteq B^n$, for every $n \in \mathbb{N}$.

6. Relation to Rationalizability

Although one of our motivations is to study interactions under ignorance, the notions of optimism and pessimism could be seen as decision criteria under extreme risk seeking and risk aversion, respectively, when players do have beliefs in form of probability measures. Among other things, Weinstein (2016) studies the predictions of the standard Rationalizability algorithm (as in Osborne & Rubinstein (1994, Definition 54.1, Chapter 4.1)—henceforth, Rationalizability) when players’ risk attitudes vary. In particular, he characterizes the limits of the algorithm if risk attitudes tend to either extremes: while Rationalizability converges to Point Rationalizability in the limiting case of extreme risk seeking behavior, Rationalizability converges to Börgers Rationalizability in the limiting case of extreme risk aversion. Now, it has to be observed that our definitions of Point and Wald Rationalizability can be seen as the limit points of the convergence process described above once the opponents’ actions a players considers as possible are those that belong to the support of her probabilistic belief in the standard model. With this association in mind, focusing on the most interesting case, Pessimism and Common Belief in Pessimism can be interpreted as extreme risk aversion as commonly believed among players. Thus, to clarify why Proposition 10 is not puzzling after all, they are simply—as anticipated in Section 1.1—a manifestation of a discontinuity.

In light of this observation, an analysis of the relation between Wald Rationalizability and Rationalizability might be of interest for applications. However, it has to be pointed out that Rationalizability crucially relies on beliefs in the usual sense of probability measures or, equivalently due to Pearce (1984, Lemma 3), on strict dominance by possibly mixed actions. Either way, such constructs are ruled out in our setting. Hence, it is conceptually inappropriate to compare our algorithms to Rationalizability. Nonetheless, given this caveat, we proceed with this comparison in a mechanical fashion for the potential applications that could arise. Thus, we let $R^\infty$ denote the set of rationalizable actions and $R^1_i$ the collection of payer $i$’s actions surviving the first iteration of the Rationalizability algorithm. Given that $R^1_i \subseteq B^1_i$ from Börgers (1993, Proposition, p.427) and that induction provides the inclusion for further rounds of the procedures (as noted by Weinstein (2016, p. 1885)), we have that $R^\infty \subseteq B^\infty$ and—as a result—the discussion in Section 5 does not provide further guidance on the relationship with $W^\infty$ for nongeneric games. As a matter of fact, there is no relationship even for generic games, as the next two examples show.

Example 3 ($R^\infty \nsubseteq W^\infty$, Continued). In the generic game of Figure 3, it easy to see that $R^1_a = B^1_a = A_a$, but $M \not\in W^1_a$ as argued before.

Example 4 ($W^\infty \nsubseteq R^\infty$). Consider the following game, with two players, namely, Ann (viz., $a$) and Bob (viz., $b$), where only Ann’s payoffs are represented.

\[
\begin{array}{cc}
T & R \\
L & 3 & 0 \\
M & 1 & 1 \\
D & 0 & 3
\end{array}
\]

Figure 4: A generic game showing that $W^\infty \nsubseteq R^\infty$.

Here, $M$ is the only strategy of Ann which is strictly dominated (by a mixture of $T$ and $D$). Hence, $M \not\in R^1_a$. However, $M \in W^1_a$, because $M \in \rho^\in\inf_a(\kappa_a)$ with $\kappa_a = \{L, R\}$.

Example 3 might suggest a failure of upper hemicontinuity of the Rationalizability correspondence taking the limit to extreme risk aversion. To appreciate this point, we recall a definition

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40 Battigalli et al. (2016) extend this analysis to allow for ambiguity by appropriately extending Pearce (1984, Lemma 3).
from Weinstein (2016, Section 2), stated next, where, as usual, $\Delta(A)$ denotes the set of all (correlated) mixed action profiles and $\text{supp}\, \mu$ denotes the support of an arbitrary probability measure $\mu \in \Delta(A)$, i.e., the set of all $a \in A$ such that $\mu[a] > 0$.

**Definition 6.1.** Fix an (ordinal) game $\Gamma = (I, (A_i, u_i)_{i \in I})$. An indexed family of induced standard games $\Lambda^{-r}(\Gamma) := (I, (A_i, u_i^{-r})_{i \in I})$, with $r \in (0, \infty)$, is unboundedly concave if

1. for every $r > s$ and $i \in I$, $u_i^{-r} = f_{i,r,s} \circ u_i^{-s}$ for an increasing and concave function $f_{i,r,s}$,
2. for every $\pi, \pi' \in \Delta(A)$, if
   $\min_{a \in \text{supp}(\pi)} u_i(a) > \min_{a \in \text{supp}(\pi')} u_i(a),$
   then there exists a $\tilde{r} \in (0, \infty)$ such that $\sum_{a \in A} u_i^{-r}(a)\pi[a] > \sum_{a \in A} u_i^{-\tilde{r}}(a)\pi'[a]$, for every $r > \tilde{r}$.

To simplify notation we suppress the explicit reference to the ordinal game in the indexed family. The context should make the underlying ordinal game clear. Thus, we just write $\Lambda^{-r}$. Similarly, we write $B^\infty(\Gamma)$ and $W^\infty(\Gamma)$ for the corresponding ordinal game to denote the action profile that are Börgers and Wald rationalizable, respectively.

For a given unboundedly concave family, Weinstein (2016, Proposition 3) proves that $R^\infty(\Lambda^{-r})$ is increasing in $r$ (by set-inclusion) and, loosely speaking, $\lim_{r \to \infty} R^\infty(\Lambda^{-r}) = B^\infty(\Gamma)$. That is, as players become more risk averse, the set of rationalizable action profiles increases and in the limit the set approaches Börgers Rationalizability action profiles. However, Weinstein (2016, p.1886) also observes that the limiting game itself corresponds to a game with preferences given by the max min criterion. Thus, if the limit is taken before Rationalizability is applied to the game, we could expect Wald Rationalizability to be the appropriate solution concept, because, after all, the limiting game is one in which players have Pessimism and Common Belief in Pessimism. Equivalently, but staying informal, one would expect $W^\infty(\Gamma) = R^\infty(\lim_{r \to \infty} \Lambda^{-r})$. Now, Example 3 illustrates that $W^\infty(\Gamma) \subseteq B^\infty(\Gamma)$ or, in this informal language, that $R^\infty(\lim_{r \to \infty} \Lambda^{-r}) \subset \lim_{r \to \infty} R^\infty(\Gamma^{-r})$, which—seemingly—corresponds to a failure of upper hemicontinuity mentioned above. In particular, along the sequence, $M$ is always rationalizable, but $M \notin W^1_a$.

However, there is a problem with this informal argument. Indeed, Example 3 does not show a failure of upper hemicontinuity, because $\lim_{r \to \infty} \Lambda^{-r}$ might be ill-defined. As already pointed out by Weinstein (2016, p.1892),\(^ {41}\) we might have unbounded payoffs and, therefore, the sequence of games might not have a convergent subsequence: this is exactly what happens in Example 3. To remedy this problem, it suffices to additionally impose normalized payoffs in $\Lambda^{-r}$, for every $r \in (0, \infty)$: e.g., $\min_{a} u_i^{-r}(a) = 0$ and $\max_{a} u_i^{-r}(a) = 1$, for every $i \in I$. With this normalization, the limiting game $\Lambda^{-\infty} := \lim_{r \to \infty} \Lambda^{-r}$ is well-defined. In what immediately follows, we use again Example 3 to show this point.

**Example 3 (Limiting Game, Continued).** Starting from Figure 3, Figure 5 shows the corresponding limiting game $\Lambda^{-\infty}$ for any unboundedly concave family with payoffs normalized to lie within $[0, 1]$.

![Figure 5](image_url)

Figure 5: Limiting game of extreme risk aversion of Figure 3.

Clearly, we have that $W_1^a = A_a$, thus, restoring upper hemicontinuity.

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\(^ {41}\)Observe that, although his argument is made for Nash equilibrium, it applies to the Rationalizability correspondence as well.
Given the above, we can show that Rationalizability fails lower hemicontinuity, i.e., we can find a game such that even with this normalization in place we have.

\[
W^\infty(\Lambda^{-\infty}) := R^\infty \left( \lim_{r \to \infty} \Lambda^{-r} \right) \supset \lim_{r \to \infty} R^\infty(\Gamma^{-r}) = B^\infty(\Gamma).
\]

**Example 1 (Limiting Game, Continued).** Consider the limiting game \(\Lambda^{-\infty}\) corresponding to the game in Figure 1. Focusing on Bob, Figure 6 depicts his payoffs in this limiting game.

|   |   |   |
|---|---|---|
|   |   |   |
|   |   |   |
|   |   |   |

Figure 6: Limiting game of extreme risk aversion of Figure 1.

In this limiting game, we have \(W^1_b = A_b\) and, in particular, \(R \notin W^1_b\). However, along the sequence \(R\) will be always strictly dominated by \(C\) and therefore \(R\) cannot be an element of the limit of the upper-hemicontinuous Rationalizability correspondence.\(^{42}\)

Note that in both examples, the limiting game \(\Lambda^{-\infty}\) is *not* induced from the ordinal game \(\Gamma\) we started from, i.e., \(\Lambda^{-\infty} \neq \Lambda^{-r}(\Gamma)\), for every \(r \in (0, \infty)\). Thus, it would be inappropriate to set \(W^\infty(\Gamma) := R^\infty \left( \lim_{r \to \infty} \Lambda^{-r}(\Gamma) \right)\). As a result, our analysis of PCBP along with the introduction of Wald Rationalizability clarifies the conceptual underpinnings behind the discontinuity hinted in Weinstein (2016, p.1891) and formally illustrated in the previous example.\(^{43}\) Thus, whether one takes \(W^\infty\) or \(B^\infty\) as the appropriate solution concept depends on the application. If the interactive situation is best captured by PCBP, then our analysis shows that \(W^\infty(\Gamma)\) is the right solution concept. When the question is what are the behavioral implications of (common belief in) extreme risk aversion, then \(W^\infty(\Lambda^{-\infty})\) should be used. Finally, if the analyst wants to study the limiting behavior of extreme risk aversion in a situation of (common belief) of (Bayesian) rationality,\(^{44}\) then \(B^\infty(\Gamma)\) is the suitable solution concept as shown by Weinstein (2016, Proposition 3).

7. **Discussion**

7.1 On the Interpretation of the word “Ignorance”

In contemporary philosophy, the notion of ignorance is declined in various ways. For example, it is possible to identify the following forms of ignorance: *factual* ignorance, that arises whenever there is lack of knowledge of some fact; *object* ignorance, related to lack of knowledge of an object; *technical* ignorance, which manifests itself whenever there is lack of knowledge of how to perform something.\(^{45}\)

In Section 1.1, we implicitly interpreted the notion of ignorance along the lines of the original contributors to the topic in the economic literature. Thus, our original narrative covers the first two forms of ignorance described above: indeed, a player who is unable to provide a belief via a probability measure (and the like) can support this either in terms of lack of knowledge of some facts (*e.g.*, facts concerning her opponents which preclude her to form beliefs in a measure-theoretic...
way), or via lack of her knowledge of her own belief (as a probability measure—and the like).\footnote{This is related to the problem of belief elicitation. See, for example, Schotter & Trevino (2014, Section 3), where it is written that—without belief elicitation—people could proceed in a game without trying to predict their opponents’ behavior or might use various forms of heuristics not based on beliefs (understood as represented via probability measures).}

However, it is also possible to interpret the notion of ignorance as technical ignorance: that is, a player can be (technically) ignorant in the sense that, even if she is actually able to describe her belief via some form of probability measure (and the like), she is going to find herself unable to make the appropriate related computations (i.e., her expected utility). We feel this interpretation to be not at all farfetched, in particular when players happen to be undergraduate students in the role of subjects of an experiment based on a static game. Thus, when we embrace this interpretation of the word “ignorance”, our results can be seen as an analysis of rules-of-thumbs used by players in static games that lack the mathematical skills to perform expected utility computations, but that—nonetheless—are perfectly capable of performing strategic interactive reasoning.

7.2 Rationality in Ordinal Games

As we mentioned at the end of Section 1.1, there is no agreement on what is the ‘right’ notion of rationality for players in ordinal games that do not hold beliefs in the form of probability measures (and the like). This can be seen from the fact that two different notions have been proposed in the literature, each leading to different behavioral predictions when we impose them along with common belief in them.

On one side, there is the notion of rationality as in Equation (5.3), that we call Admissibility. This notion goes back to Hillas & Samet (2014, Definition 5) and it is called “Weak Dominance Rationality” in Bonanno & Tsakas (2018, Definition 2). As we show in Theorem 8, Admissibility and Common Belief in Admissibility epistemically characterizes the iterative elimination of actions that are Börgers dominated (a result established in Bonanno & Tsakas (2018, Theorem 1) for a different framework, as we already mentioned in various instances).

On the other side, it is possible to provide a different notion of rationality as in Bonanno (2015, Definition 3), call it rationality∗, according to which an action a∗i of a player i is rational∗ at a state if it is not the case that there exists another action ai that yields a strictly higher payoff than a∗i against all the action profiles of the other players that player i considers possible at that state. If we focus on this notion of rationality, then Bonanno (2015, Proposition 1) establishes that Rationality∗ and Common Belief in Rationality∗ is algorithmically characterized by the iterative elimination of actions that are strictly dominated by pure actions.

Given that these notions are all based on a dominance criterion, optimism and pessimism can be seen as alternatives to the aforementioned notions of rationality, in particular in light of their solid decision-theoretic foundation.

7.3 Relation to Mariotti (2003)

Mariotti (2003) epistemically characterizes Point Rationalizability using possibility structures, like in this contribution. However, in contrast to our approach, he focuses on players that choose best-replies to pure actions of the opponents without explicitly modeling—as we do—how a player chooses an action when her type considers possible actions of the opponents.

To see the difference, consider a game Γ with an appended possibility structure Ω and a player i ∈ I. Now, define an action a∗i ∈ Ai to be point-justifiable given a ˜a−i ∈ A−i, if a∗i ∈ arg maxai∈Ai uι(ai, ˜a−i).\footnote{Mariotti (2003) uses the word “justifiable” instead. We use the expression “point-justifiable”, since we employ the word “justifiable” in a—slightly—different way (see Section 3).} Thus, action a∗i ∈ Ai is point-justifiable if the set

\[ M_i(a^*_i) := \{ \tilde{a}_{-i} \in A_{-i} \mid a^*_i \in \arg \max_{a_i \in A_i} u_i(a_i, \tilde{a}_{-i}) \} \]

is nonempty. With this definition about behavior, he proceeds by defining an epistemic event that relates the choice of player i’s point-justifiable actions to player i’s types (and related possibility functions) as

\[ \mathcal{M}_i := \{ (a^*_i, t_i) \in A_i \times T_i \mid M_i(a^*_i) \neq \emptyset, \varphi_i(t_i) \subseteq M_i(a^*_i) \}. \]
Contrary to our approach based on the notion of optimism, $M_i$ not only restricts player $i$’s behavior, but also her epistemic state. Intuitively, $M_i$ can be interpreted as capturing two assumptions at once:

i) player $i$ chooses an action which is a best-reply to all opponents’ actions she deems possible;

ii) player $i$’s possibilities are restricted in such a way that an optimal-for-all action exists.\(^{48}\)

Our approach, on the contrary, distinguishes assumptions about behavior and epistemic attitudes. Indeed, $O_i$ is only a restriction on how player $i$ chooses an action, since in our model every type has an ‘optimistic’ action available and no types need to be ruled out to ensure existence.

Taken into account the discussion above, it has to be observed that the behavioral implications of both events $M_i$ and $O_i$ are—of course—the same: considering only types with singleton $\varphi_i(t_i)$ does not change the behavioral implications of either event, but under this restriction optimistic choices are clearly point-justifiable and vice versa. However, it has to be pointed out that the goals of the two papers are different: the explicit goal of Mariotti (2003) is to epistemically characterize Point Rationalizability via possibility structures, while our aim, rather than to provide a foundation for Point Rationalizability per se, is to study the behavioral implications of—optimism and common belief in—optimism (and the same for pessimism) starting with an explicit formalization of these notions.

Nevertheless, we can provide a more direct epistemic foundation for Point Rationalizability as follows. First of all, we define the event in a possibility structure that an arbitrary player $i \in I$ has point beliefs:\(^{49}\)

$$D_i := \{(a_i, t_i) \in A_i \times T_i \mid \exists (a_{-i}^*, t_{-i}^*) \in A_{-i} \times T_{-i} : \pi_i(t_i) = \{(a_{-i}^*, t_{-i}^*)\}\}.$$  

With this definition, the promised foundation—stated next—obtains as a corollary of Theorem 2.\(^{50}\)

**Corollary 11 (Direct Foundation of Point Rationalizability).** Fix a game $\Gamma$.

i) If $\mathcal{P}$ is an arbitrary possibility structure appended on it, then

$$\text{proj}_A \mathcal{CB}^n(O \cap D) \subseteq P^{n+1},$$

for every $n \in \mathbb{N}$, and

$$\text{proj}_A \mathcal{CB}^\infty(O \cap D) \subseteq P^\infty.$$

ii) Given the universal possibility structure $\mathcal{P}^*$,

$$\text{proj}_A \mathcal{CB}^n(O \cap D) = P^{n+1},$$

for every $n \in \mathbb{N}$, and

$$\text{proj}_A \mathcal{CB}^\infty(O \cap D) = P^\infty.$$

7.4 Common Belief vs. Common Knowledge & Algorithmic Procedures

**Bonanno & Tsakas (2018)** show how Admissibility (as in Equation (5.3)—of course, in their language and terminology, where it is called “Weak Dominance Rationality”) and Common Belief vs. Common Knowledge in Admissibility are algorithmically characterized by two different procedures. **Bonanno & Tsakas (2018, Theorem 1)** (similar to our Theorem 8) proves that Admissibility and Common Belief in Admissibility is algorithmically characterized by the iterative elimination of actions that are Borgers dominated, indeed a procedure based on the elimination of actions. Interestingly, and clearly related to the differences between Point Rationalizability and the Wishful Thinking procedure in Yildiz (2007), Admissibility and Common Knowledge in Admissibility is algorithmically characterized by an elimination of *action profiles* known as Iterated Deletion of Inferior Profiles, introduced in Stalnaker (1994, Section 3, p.62).

\(^{48}\)Formally, this would correspond to a model of decision making with incomplete preferences due to multiple point beliefs. Ziegler & Zuazo-Garin (2020) use a similar model in the realm of multiple beliefs to provide a foundation for iterated admissibility.

\(^{49}\)That is, $\pi_i(t_i)$ being a singleton set. Within a Bayesian framework the same can be accomplished by imposing degenerate distributions as allowable beliefs.

\(^{50}\)Corollary 11(ii) can be established under the weaker condition of an appropriately defined *degenerately belief-complete possibility structure* similar to Friedenberg (2019, Section 8).
7.5 (Seemingly) Technical Assumptions

Given that we focus on finite games, one might wonder if we need the generality provided by our topological assumption of type sets being compact Hausdorff. Indeed, this assumption might be overly general and we could work with type sets that are, for example, compact metrizable. Our characterization results rely on the existence of the universal possibility structure and Mariotti et al. (2005) provide a canonical construction of such an object based on the topological assumption of compact Hausdorffness. Since our goal in this paper is to study the behavioral implications of tropical attitudes and common belief therein and to provide a deeper understanding of ordinal games more generally, we opted to use their ready-made construction instead of providing yet another canonical construction that exactly fits our framework where the underlying space of uncertainty is finite. However, we want to stress that, even in our case, topological assumptions need to be imposed. If not, Brandenburger (2003, Proposition 1) and Mariotti et al. (2005, Lemma 1) illustrate that any such construction needs to fail because it would contradict Cantor’s Theorem. As pointed out in Brandenburger & Keisler (2006, Section 11), there is a sense in which all constructions of large structures have in common some sort of topological assumptions.

The second parts of our characterization results (i.e., Theorem 2, Theorem 3, and Theorem 8) do rely on the existence of a rich possibility structure as constructed by Mariotti et al. (2005). As a matter of fact, these parts of the theorems can be made stronger by only requiring a belief-complete possibility structure as defined in Brandenburger (2003) and noted in Remark 2.1. However, such a characterization would raise the question on whether such a type structure does indeed contain all hierarchies of beliefs and how the answer would depend on topological assumptions. Friedenberg (2010) addresses this question for standard type structures based on probabilistic beliefs, but it remains an open question for possibility structures. Since our formal results use the canonical construction of Mariotti et al. (2005), we bypass the issue by establishing the results relying on an object that contains all hierarchies of beliefs by construction.51

7.6 Introspection, Independence, and Knowledge in Product Structures

Concerning the knowledge structures in Section 4, we can restrict our analysis to finite structures, since the result of Yildiz (2007) (and its translation to our framework as stated in Theorem 6) does not rely on the existence of a sufficiently large (e.g., belief-complete) structure. However, we add some extra generality by allowing for knowledge structures with state spaces that do not have a product structure. Whereas in models without knowledge (and without introspection) the product structure seems—at least to us—quite natural, the product structure would impose severe restrictions on the knowledge structure. This is an implication of the Truth Axiom, which imposes cross-player restrictions as illustrated by Example 2. Indeed, imposing a product structure would render the players’ knowledge as trivial. Thus, before showing the nature of the severe restrictions mentioned above, two points are in order regarding Introspection and Independence, since they play a role in what comes next.

Regarding Introspection, it is important to observe that is equivalent to requiring \( \mathcal{K}_i([a_i, t_i]) = \{[a_i, t_i] \} \) for every \((s_i, t_i) \in \mathcal{A}_i \times T_i\), where \([a_i, t_i] := \{ \omega \in \Psi : \text{proj}_{A_i \times T_i} \omega = (a_i, t_i) \} \).

Concerning Independence, we want to highlight that, without such an assumption, conceptual problems arise concerning the interpretation of types. Indeed, in presence of Independence, a type of a player captures exactly the—interactive—knowledge that player does have, whereas this is not the case when Independence is lacking. In particular, consider the game Example 2 where we argued already that \((D, L)\) cannot be played under Wishful Thinking. Now, we consider the following structure: for both players let \(T_i := \{ t_i \} \) with state space \( \Psi := \{ (D, R), (D, L), (T, L) \} \), where type labels do not play an important role here and are therefore omitted, and set

- \( \Pi_a(D, L) := \{(D, R)\} \) and \( \Pi_b(D, L) := \{(T, L)\} \),
- \( \Pi_a(D, R) := \{(D, L)\} \) and \( \Pi_b(D, R) := \{(D, R)\} \),
- \( \Pi_a(T, L) := \{(T, L)\} \) and \( \Pi_b(T, L) := \{(D, L)\} \).

51 Furthermore, because we use their construction, our results cannot just be reinterpreted as situations where players do have well-formed probabilistic beliefs, but have preferences where only the support of these beliefs matter. Mariotti et al. (2005, Section 4.2) discuss this point in more detail.
Note that, as just defined, $\Pi_i$ does not satisfy $\omega \in \Pi_i(\omega)$ or Independence, for every $i \in I$. As a result, this is not quite a knowledge structure. However, when considering the knowledge structure generated from the types only, we get a knowledge structure that satisfies introspection (for the type only) and forms a—trivial—partition. However, here we would have the state $(D, L)$ being consistent with Wishful Thinking, because, once players get informed of their own action, they are delusional. In any case, these sort of structures are ruled out by requiring $\omega \in \Pi_i(\omega)$ and, as such, the independence assumption does not play a crucial role in Theorem 6. Indeed, Theorem 6 can be strengthened to allow for knowledge structures (potentially) not satisfying Independence in the first part of the theorem and to specify that there exists a knowledge structure satisfying Independence in second part of the theorem. Nevertheless, we opt to impose Independence on any knowledge structure considered here for the conceptual points raised above.

We can now go back to state more formally concerning the trivial nature of the knowledge that players would hold in a product state space. Thus, consider a knowledge structure $\mathcal{K}$ with a product state space $\Psi = \prod_{i \in I} A_i \times T_i$ and, for every player $i \in I$, let $\Psi_i := \text{proj}_{A_i \times T_i} \Psi$. Additionally, fix a player $i \in I$ and, for every $E_{-i} \in \mathcal{K}(\Psi, i)$, define $[E_{-i}] := \Psi_i \times E_{-i}$ as the corresponding interactive event (for player $i$). Interestingly, given our assumption that the knowledge structure satisfies Introspection and Independence, the following remark states that the only interactive event a player knows is the full state space.

**Remark 7.1.** Given a game $\Gamma$ and an appended knowledge structure $\mathcal{K}$ such that $\Psi = \prod_{i \in I} A_i \times T_i$. For every player $i \in I$,

$$\mathcal{K}_i([E_{-i}]) \neq \emptyset \iff E_{-i} = \Psi_{-i},$$

for every event $E_{-i} \in \mathcal{K}(\Psi, i)$.

For one, just note that $E_{-i} = \Psi_{-i}$ says that $[E_{-i}] = \Psi$ and then $\mathcal{K}_i([E_{-i}]) = \mathcal{K}_i(\Psi) = \Psi$. For the converse, assume that $E_{-i} \neq \Psi_{-i}$, i.e., $E_{-i} \subsetneq \Psi_{-i}$. By Introspection and Independence, every partition cell for a given $\omega = (\omega_i, \omega_{-i}) \in \Psi$ is of the form $\Pi_i(\omega) = \{\omega_i\} \times \tilde{E}_{-i}$, for a $\tilde{E}_{-i} := \text{proj}_{T_i} \omega$ and a $\tilde{E}^{c}_{-i} \in \mathcal{K}(\Psi_{-i})$. Thus, we can index the partition cells by $\omega_i$ only, which implies, given the definition of a partition, that we also need $\bigcup_{\omega_i \in \Psi_i} \Pi_i(\omega) = \Psi$. This, in turn, implies that every cell needs to be of the form $\Pi_i(\omega) = \{\omega_i\} \times \tilde{E}_{-i}$, i.e., they are cylinder sets. Thus, for every $\omega \in \Psi$, $\Pi_i(\omega) = \{\omega_i\} \times \tilde{E}_{-i} \subseteq [E_{-i}]$, i.e., $\mathcal{K}_i([E_{-i}]) = \emptyset$.

Although the construction of a product structure mentioned above is somewhat standard (see—for example—Zamir (2009, p. 429)), the remark just stated about product structures is—even if simple—new to the best of our knowledge, where—of course—our insistence on requiring Introspection and Independence comes from the very fact that we want to relate knowledge structures to our possibility structures.

### 7.7 Applications to Mechanism Design

Much of the original work in implementation theory has focused on *ordinal* implementation, as in Hurwicz & Schmeidler (1978) and Hurwicz (1979). More recently, Chen & Micali (2015) use possibilistic beliefs to study implementation of single-good auction formats. Among other things, they obtain a positive full implementation for their revenue benchmark using a solution concept based on strict dominance by possibly mixed actions. In our complete information setting, this dominance relation is the same as if player’s are standard Bayesian players with cardinal utilities and probabilistic beliefs (as in Pearce (1984)), which we discussed already in Section 6. Their implementation notion only requires mutual belief of players not choosing undominated actions. Clearly, Optimism and Mutual Belief in Optimism is behaviorally more selective then their requirement, which would make full implementation easier. In particular, their revenue benchmark can be implemented fully under Optimism and Mutual Belief in Optimism. On the other hand, as illustrated with Example 3 (in Section 6) and Example 4, no connection of their procedure with Pessimism and Mutual Belief in Pessimism can be established in general. Therefore, it remains an open question whether their benchmark can be fully implemented if players are pessimistic. We leave this question and, more generally, extending our framework to incomplete information to analyze mechanism design problems for future research.
Jakobsen (2020) studies a principal-agent problem, where the agent is boundedly rational and—in our language—is in the state of ignorance about what mechanism she is facing. However, the agent could (potentially) refine the mechanisms she considers possible. If after this deliberation the agent still cannot single out a unique mechanism, then she behaves according to the maximin criterion and as a result she is pessimistic about what mechanism she is actually in. Effectively, the principal-agent model of Jakobsen (2020) reduces to a decision problem of a single agent, whereas our framework by its very nature is appropriate for multiplayer strategic interactions.

In particular, one could interpret the iterations of mutual beliefs in our framework as an agent refining the opponents’ actions he considers possible. As such, it might be interesting to use our framework and results to extend Jakobsen (2020) to a mechanism design setting, where multiple players interact in a mechanism. Similarly, Nikzad (2021) studies a persuasion problem where the receiver is pessimistic in our sense. Here, too, our paper could provide a language to extend his model to an information design setting with multiple receivers as introduced by Taneva (2019).

**Appendix**

A. Proofs

A.1 Proofs of Section 2

Given an arbitrary topological space $X$, $\mathcal{B}(X)$ denotes its Borel $\sigma$-algebra and $|X|$ its cardinality. Given our topological assumptions spelled out in Section 2, we can state the following remark.

**Remark A.1 (Measurability).** If $X$ is compact Hausdorff, then $\mathcal{K}(X) \subseteq \mathcal{B}(X)$.

We provide a unique proof for all the results in Section 2. Thus, in the following—joint—proof, we let

$$(S, \rho, E) \in \{ (P, \rho^{\max}, O), (W, \rho^{\min}, P) \}.$$

**Remark A.2.** For every $E \in \{O, P\}$ and $i \in I$,

$$\text{proj}_{\Omega_i} CB^n(E) = \text{proj}_{\Omega_i} CB^{n-1}(E) \cap B_i \left( \text{proj}_{\Omega_{-i}} CB^{n-1}(E) \right),$$

(A.1)

for every $n \in \mathbb{N}$.

**Proof of Proposition 1.** Fix a game possibility structure $\mathcal{F}$ with state space $\Omega$ appended on a game $\Gamma$, a tuple $(S, \rho, E)$, and a player $i \in I$.

i) We now proceed with the proof of part (i). In light of Remark A.2, we are going to establish the truth of Equation (A.1). To do so, we proceed by induction on $n \in \mathbb{N}$.

- $(n = 0)$ First of all, notice that, since

$$\text{proj}_{\Omega_i} CB^0(E) = E_i = \bigcup_{a_i \in A_i} \left( \{a_i\} \times \text{proj}_{T_i}(E_i \cap \{a_i\} \times T_i) \right),$$

we have to prove that $\text{proj}_{T_i}(E_i \cap \{a_i\} \times T_i)$ is closed for an arbitrary $a_i \in A_i$. Now, we have that

$$\text{proj}_{T_i}(E_i \cap \{a_i\} \times T_i) = \pi_i^{-1}\left( \{ \xi_i \in \mathcal{K}(A_{-i} \times T_{-i}) \mid a_i \in \rho_i \left( \text{proj}_{A_{-i}} \xi_i \right) \} \right).$$

Thus, since $\pi_i$ is continuous by assumption, we simply have to show that the set

$$\{ \xi_i \in \mathcal{K}(A_{-i} \times T_{-i}) \mid a_i \in \rho_i \left( \text{proj}_{A_{-i}} \xi_i \right) \}$$

is closed. Let $(\tilde{\xi}_i)_{\ell \in \mathbb{N}} \subseteq \Omega_{-i}$ be a sequence such that $a_i \in \rho_i \left( \text{proj}_{A_{-i}} \tilde{\xi}_i \right)$ for every $\ell \in \mathbb{N}$ and assume that $\tilde{\xi}_i \to \tilde{\xi}_i$. Thus, we need to prove that $a_i \in \rho_i \left( \text{proj}_{A_{-i}} \tilde{\xi}_i \right)$. Now, for every $\ell \in \mathbb{N}$,
observe that we have \( \tilde{\xi}_i \subseteq A_{-i} \) with \( A_{-i} \) finite and—by assumption—endowed with the discrete topology. Also, recall that convergence of a sequence in the discrete topology means that there exists a \( k \in \mathbb{N} \) such that, for every \( m > k \), \( \text{proj}_{A_{-i}} \tilde{\xi}_i^m = \text{proj}_{A_{-i}} \tilde{\xi}_i^k \). Thus, we have—a fortiori—also that \( \text{proj}_{A_{-i}} \tilde{\xi}_i = \text{proj}_{A_{-i}} \tilde{\xi}_i^k \). Hence, it follows that \( a_i^* \in \rho_i \left( \text{proj}_{A_{-i}} \tilde{\xi}_i \right) \).

- \((n \geq 1)\) Assume the result holds for \( n \in \mathbb{N} \). Thus, we have to prove that \( \mathbb{C} \mathbb{B}^{n+1}(E) \in \mathcal{K}(\Omega) \). Let \( i \in I \) be arbitrary and, focusing on Equation (A.1), observe that we have \( \text{proj}_{\Omega_i} \mathbb{C} \mathbb{B}^n(E) \in \mathcal{K}(\Omega) \), from the induction hypothesis. Thus, it remains to prove that \( B_i \left( \text{proj}_{\Omega_i} \mathbb{C} \mathbb{B}^n(E) \right) \in \mathcal{K}(\Omega_i) \). Now, notice that

\[
B_i \left( \text{proj}_{\Omega_{-i}} \mathbb{C} \mathbb{B}^n(E) \right) = \pi_i^{-1} \left( \{ \xi_i \in \mathcal{K}(A_{-i} \times T_i) \mid \xi_i \subseteq \text{proj}_{\Omega_{-i}} \mathbb{C} \mathbb{B}^n(E) \} \right).
\]

Thus, since \( \pi_i \) is continuous by assumption, we simply have to show that the set

\[
\{ \xi_i \in \mathcal{K}(A_{-i} \times T_i) \mid \xi_i \subseteq \text{proj}_{\Omega_{-i}} \mathbb{C} \mathbb{B}^n(E) \}
\]

is closed, which is immediately established by noticing that \( \text{proj}_{\Omega_{-i}} \mathbb{C} \mathbb{B}^n(E) \) is closed from the induction hypothesis.

- ii) Regarding part (ii), the result follows immediately from part (i), since a countable intersection of closed set is a closed set.

This establishes the result. \( \blacksquare \)

### A.2 Proofs of Section 3

As for the results in the previous section, we provide a unique proof for all the results in Section 3. Thus, in the following—joint—proof, we again let

\[
(S, \rho, E) \in \left\{ (P, \rho^{\text{max}}, O), (W, \rho^{\text{min}}, P) \right\}.
\]

We divide the proof of Theorem 2/Theorem 3 in two parts for clarity of exposition. Of course we start from part (i) and then move to part (ii). Concerning part (i), we need additional notation. That is, given an action-type pair \( (a_i^*, \tilde{t}_i) \in A_i \times T_i \), we let \( \kappa_i^\tilde{t}_i \in \mathcal{K}(A_{-i}) \) be defined as

\[
\kappa_i^\tilde{t}_i := \begin{cases} 
(a_{-i}^*: a_{-i}^* \in \arg \max_{a_{-i} \in \phi_i(\tilde{t}_i)} u_i(a_{-i}^*, a_{-i}), \quad & \text{if } (S, \rho, E) = (P, \rho^{\text{max}}, O), \\
\varnothing, & \text{otherwise}
\end{cases}
\]

where in both cases we have by construction that \( \kappa_i^\tilde{t}_i \subseteq \phi_i(\tilde{t}_i) \).

**Proof of Theorem 2/Theorem 3(i).** We divide the proof in two parts. We proceed by proving Equation (3.1)/Equation (3.5) first and then move to prove Equation (3.2)/Equation (3.6). Fix a tuple \((S, \rho, E)\).

- Regarding the proof of Equation (3.1)/Equation (3.5), we proceed by induction on \( n \in \mathbb{N} \).
  - \((n = 0)\) Let \( (a^*, \tilde{t}) \in E \) and \( i \in I \) be arbitrary. Let \( \kappa_i^\tilde{t} \in \mathcal{K}(A_{-i}) \) be defined as in Equation (A.2). From our assumption, \( a_i^* \in \rho_i \left( \kappa_i^\tilde{t} \right) \). Hence, \( a_i^* \in S_i^1 \).
  - \((n \geq 1)\) Fix a \( n \in \mathbb{N} \), assume the result holds for \( n - 1 \), and let \( (a^*, \tilde{t}) \in \mathbb{C} \mathbb{B}^n(E) \) and \( i \in I \) be arbitrary. Hence, \( \pi_i(\tilde{t}_i) \subseteq \text{proj}_{\Omega_{-i}} \mathbb{C} \mathbb{B}^{n-1}(E) \). Let \( \kappa_i^\tilde{t} \in \mathcal{K}(A_{-i}) \) be defined as in Equation (A.2). From the induction hypothesis, \( \kappa_i^\tilde{t} \subseteq S_i^{n-1} \). Thus, since—a fortiori—we have that \( (a_i^*, \tilde{t}_i) \in E_i \), it is the case that \( a_i^* \in \rho_i \left( \kappa_i^\tilde{t} \right) \). Hence, it follows that \( a_i^* \in S_i^{n+1} \), because \( \kappa_i^\tilde{t} \subseteq S_i^{n-1} \).

- Equation (3.2)/Equation (3.6) immediately follow from Equation (3.1)/Equation (3.5), the finiteness assumption, and the nonemptiness of the solution concepts. \( \blacksquare \)
Proof of Theorem 2/Theorem 3(ii). Let $\mathcal{P}^*$ be the universal possibility structure. Fix a tuple $(S, \rho, E)$.

- We now prove Equation (3.3)/Equation (3.7). Clearly, one side of the result has already been established in the proof of part (i). Thus, we establish the other side of the result by proceeding again by induction on $n \in \mathbb{N}$.

  - $(n = 0)$ Fix a profile of actions $a^* \in S^1$ and let $i \in I$ be arbitrary. Then there exists a $\kappa_i \in \mathcal{X}(A_{-i})$ such that $a^*_i \in \rho_i(\kappa_i)$. From the belief-completeness of $\mathcal{P}^*$, there exists a type $\tilde{t}_i \in T_i$ such that $\pi_i(\tilde{t}_i) = \kappa_i \times T_{-i}$. Thus, it follows that $(a^*_i, \tilde{t}_i) \in E_i$ by construction. Since the player $i$ was chosen arbitrarily, the result follows.

  - $(n \geq 1)$ Fix a $n \in \mathbb{N}$, assume the result holds for $n - 1$, and fix a profile of actions $a^* \in S^{n+1}$. Let $i \in I$ be arbitrary. Then there exists a $\kappa_i \in \mathcal{X}(A_{-i})$ with $\kappa_i \subseteq S^{n_i}$ such that $a^*_i \in \rho_i(\kappa_i)$. From the induction hypothesis, for every $a_{-i} \in \kappa_i$ there exists a type $\ell_{-i}^0 \in T_{-i}$ such that $(a_{-i}, \ell_{-i}^0) \in \text{proj}_{\kappa_i} \mathcal{CB}^{n-1}(E)$. Hence, from the belief-completeness of $\mathcal{P}^*$, there exists a type $\tilde{t}_i \in T_i$ such that

$$\pi_i(\tilde{t}_i) := \{ (a_{-i}, \ell_{-i}^0) \in A_{-i} \times T_{-i} \mid a_{-i} \in \kappa_i \}$$

and—by construction—we have that $(a^*_i, \tilde{t}_i) \in \text{proj}_i \mathcal{CB}^n(E)$. Since player $i$ was chosen arbitrarily, the result follows.

We now prove Equation (3.4)/Equation (3.8), where—again—we already established one side in the proof above. Thus, first of all, observe that $\mathcal{CB}^\infty(E) \neq \emptyset$. This is a consequence of the fact that $S^n \neq \emptyset$ for every $n \in \mathbb{N}$ and that $T$ is compact Hausdorff by assumption. Hence, $(\mathcal{CB}^n(E))_{m \geq 0}$ is a nested family of nonempty closed sets having the finite intersection property. Let $2 := \min \left\{ n \in \mathbb{N} \mid S^n = S^{n+1} = S^\infty \right\}$. Let $a^* \in S^2 = S^\infty$ be arbitrary. Let

$$M^\ell(\underline{n}, a^*) := \begin{cases} \{a^*\} \times T, & \text{if } \underline{n} = 0, \\
\mathcal{CB}^{\underline{n}+1+\ell}(E) \cap (\{a^*\} \times T), & \text{otherwise,} \end{cases}$$

for every $\ell \geq 0$. Notice that this definition induces a sequence of sets. Since every $M^\ell(\underline{n}, a^*)$ is nonempty and closed and the sequence of sets is decreasing, it has the finite intersection property. Hence, there exists a $t^* \in T$ such that $(a^*, t^*) \in \bigcap_{\ell \geq 0} M^\ell(\underline{n}, a^*) \subseteq \mathcal{CB}^\infty(E)$.

This completes the proof of part (ii).

Proof of Proposition 4. We proceed by induction on $n \in \mathbb{N}$.

- $(n = 0)$ Trivial.

- $(n \geq 1)$ Fix a $n \in \mathbb{N}$ and assume the result holds for $n - 1$. Let $a^* \in S^n$ and $i \in I$ be arbitrary. Hence, there exists a $a_{-i} \in S^{n-1}$ such that $a^*_i \in \rho_i^{\max}(\kappa_i)$, with $\kappa_i := \{a_{-i}\}$. Let $A_{-i} := \kappa_i$. Then, a fortiori also $a^*_i \in \rho_i^{\min}(\kappa_i)$.

This completes the proof.

A.3 Proofs of Section 4

Proof of Proposition 5. We fix a game $\Gamma$ and proceed by induction on $n \in \mathbb{N}$.

- $(n = 0)$ Trivial

- $(n \geq 1)$ Fix a $n \in \mathbb{N}$ and assume the result holds for $n - 1$. Let $a^* \in S^{m+1}$ and $i \in I$ be arbitrary. Hence, there exists a $a_{-i} \in A_{-i}$ such that $(a^*_i, a_{-i}) \in Y^m \subseteq P^m$ from the induction hypothesis and $a^*_i \in \rho_i^{\max}(\{a_{-i}\})$. Thus, the conclusion follows since $(a^*_i, a_{-i}) \in Y^m \subseteq P^m$ gives a profile $a_{-i} \in P^{m_{-i}}$.

This completes the proof.
Theorem 6 can be easily obtained by ‘translating’ the proof of Yildiz (2007, Proposition 1) in Yildiz (2007, pp.341–342) to our framework, with the understanding that the event $W_i$ in Yildiz (2007) is equivalent to our $O_i$ (in particular, as defined in Footnote 35). Some care has to be taken when constructing the relevant state space by adding the appropriate types that are explicit in our framework. Furthermore, the statement in Yildiz (2007, Proposition 1) is slightly different from ours since a (possibly distinct) knowledge structure is constructed for every step $m \geq 0$. To obtain our statement, one can just take the union across all the knowledge structures as the relevant knowledge structure. The reason the arguments in Yildiz (2007, pp.341–342) can be translated in the present setting is that, for most of the arguments therein, the Bayesian framework with probabilistic beliefs employed does not play a crucial role. In particular, many steps rely on the existence of point-beliefs, which we do have in our non-probabilistic framework too as observed in Remark 3.1. Only, Yildiz (2007, Lemma 2, p.341) uses properties of expectations: as a result, we need to prove the corresponding result in our framework (stated below) in a slightly different fashion.

**Lemma 1.** For every $F \subseteq \Psi$, $i \in I$, and $\tilde{a} := (\tilde{a}_i, \tilde{a}_{i-}) \in \text{proj}_A K_i(F) \cap O_i$, there exists a $(\tilde{a}_i, a_{i-}) \in \text{proj}_A F$ such that:

(i) $\tilde{a}_i \in \rho^\text{max}_i(\{a_{i-}\})$,

(ii) $u_i(\tilde{a}_i, a_{i-}) \geq \max_{a_i \in A_i} u_i(a_i, \tilde{a}_{i-})$.

**Proof.** Let $\tilde{a} := (\tilde{a}_i, \tilde{a}_{i-}) \in \text{proj}_A K_i(F) \cap O_i$ be arbitrary, with $\omega$ be a corresponding state in $K_i(F) \cap O_i$. By Introspection, $\{\tilde{a}_i\} = \text{proj}_A \Pi_i(\omega)$. Therefore, for every $\omega' \in \Pi_i(\omega)$, we need to have the same action $\tilde{a}_i$ prescribed for player $i$. Thus, let

$$a^\omega_{i-} \in \arg \max_{a_{i-} \in \text{proj}_{A_{i-}} \Pi_i(\omega)} u_i(\tilde{a}_i, a_{i-})$$

and note that $(\tilde{a}_i, a^\omega_{i-}) \in \arg \max_{a_{i-} \in \text{proj}_{A_{i-}} \Pi_i(\omega)} u_i(a)$. Now, since $\omega \in K_i(F)$, we have that $\Pi_i(\omega) \subseteq F$. Therefore, $\text{proj}_A \Pi_i(\omega) \subseteq \text{proj}_A F$. Thus, by Wishful Thinking, $\tilde{a}_i \in \arg \max_{a_i \in A_i} u_i(a_i, a^\omega_{i-})$. Therefore, $\tilde{a}_i \in \rho^\text{max}_i(\{a^\omega_{i-}\})$. Finally, since $\omega \in \Pi_i(\omega)$, we know that $\tilde{a}_{i-} \in \text{proj}_{A_{i-}} \Pi_i(\omega)$.

Thus,

$$u_i(\tilde{a}_i, a^\omega_{i-}) = \max_{a_i \in A_i} u_i(a_i, a^\omega_{i-}) = \max_{a_i \in A_i} \max_{a_{i-} \in \text{proj}_{A_{i-}} \Pi_i(\omega)} u_i(a_i, a_{i-}) \geq \max_{a_i \in A_i} u_i(a_i, \tilde{a}_{i-}).$$

A.4 Proofs of Section 5

Regarding the measurability as in Proposition 7 of CBPA(A), for every $m \geq 0$, and ACBA, the proofs in Appendix A.1 apply verbatim with $(S, \rho, E) = (B, \rho^B, A)$, where the same applies to the proof of Theorem 8 as proved in Appendix A.2 with $\kappa^m_i \in \mathcal{K}(A_{i-})$ be defined as $\kappa^m_i := \varphi_i(\tilde{t}_i) \subseteq A_{i-}$.

**Proof of Proposition 9.** We fix a game $\Gamma$ and proceed by induction on $n \in \mathbb{N}$.

- $(n = 0)$ Trivial
- $(n \geq 1)$ Fix a $n \in \mathbb{N}$ and assume the result holds for $n - 1$. Let $a^* \in \mathbb{P}^n$ and $i \in I$ be arbitrary. Hence, there exists a $\kappa_i \in \mathcal{K}(A_{i-})$ such that $a^*_i \in \rho^\text{max}_i(\kappa_i)$, with $\kappa_i := \{\tilde{a}_{i-}\}$ for a $\tilde{a}_{i-} \in \mathbb{P}^{n-1}$. From the induction hypothesis, $\tilde{a}_{i-} \in \mathcal{B}^{n-1}_i$. Hence, $a^*_i \in \rho^A(\kappa_i)$.

This completes the proof.

**Proof of Proposition 10.** In generic games, given an arbitrary player $i \in I$, an action $a_i \in A_i$ is B-dominated if and only if it is strictly dominated by a pure action.\footnote{See Weinstein (2016, Footnote 5, p.1884).} Hence, this establishes the result, for every $n \in \mathbb{N}$.\footnote{See Weinstein (2016, Footnote 5, p.1884).}
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