Effects of Quantum Stress Tensor Fluctuations with Compact Extra Dimensions

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Abstract

The effects of compact extra dimensions upon quantum stress tensor fluctuations are discussed. It is argued that as the compactification volume decrease, these fluctuations increase in magnitude. In principle, this would have the potential to create observable effects, such as luminosity fluctuations or angular blurring of distant sources, and lead to constraints upon Kaluza-Klein theories. However, the dependence of the four-dimensional Newton’s constant upon the compactification volume causes the gravitational effects of the stress tensor fluctuations to be finite in the limit of small volume. Consequently, no observational constraints upon Kaluza-Klein theories are obtained.

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I. INTRODUCTION

Theories with extra spacetime dimensions were introduced into physics long ago by Kaluza [1] and by Klein [2]. The original Kaluza-Kaluza theory postulated a fifth spacetime dimension as part of an attempt to unify gravity and electromagnetism. In order that we not directly observe the fifth dimension, it is assumed to be compact with a very small compactification length. Modern versions of Kaluza-Kaluza theories come in a variety of forms [3]. Among the higher dimensional models to attract considerable attention in recent years are the “braneworlds” models [4] and Randall-Sundrum type models [5]. The former postulate that all fields except gravity are confined to a four-dimensional brane, whereas gravity is free to propagate in two or more extra dimensions. All of these higher dimensional models require an explanation as to why the world we observe appears to have only four spacetime dimensions.

In Kaluza-Kaluza theories, the extra dimensions are compact, although they may be either flat or curved. Thus if the size of the extra dimensions is less than the smallest scale at which we can probe experimentally, then the extra dimensions should not be seen. If the largest compact dimension has a size \( L \), then an energy of the order of \( 1/L \) is required to excite momenta in this dimension. The conventional view is that if we have performed scattering experiments at an energy scale of \( E \) and not seen any effects of extra dimensions, then their sizes are constrained to be less than about \( 1/E \).

However, there is a possible loophole in this reasoning. If there are quantum fields propagating in all spacetime dimensions, then the fluctuation effects of these fields will grow, not diminish, as the size of the extra dimensions decreases. This can be seen as a consequence of the uncertainty principle: a quantum system confined to a smaller volume of configuration space must exhibit increased fluctuations in the conjugate variables. A simple illustration of this comes from the way Casimir energy densities scale; in \( d \) spacetime dimensions, a quantum field confined by boundaries or periodicity on a scale \( L \) will have a Casimir energy density of the order of \( L^{-d} \). That Casimir energy creates a potential problem for Kaluza-Kaluza theories has long been recognized. There are two approaches which have been taken to dealing with this large energy density. One is to worry only about the projection of the stress tensor into the four dimensional uncompactified spacetime. Here Lorentz symmetry requires this part of the stress tensor to be proportional to the four-dimensional metric tensor, that is, to be of the form of the cosmological constant. In this case, it can be absorbed by a cosmological constant renormalization. This is not a completely satisfactory solution, however, as it does not address the effects of the large stress components in the compact dimensions. If one wants the Einstein equations in \( d \) dimensions to be satisfied, there is a nontrivial stabilization problem. Another approach which has been proposed involves cancelling the Casimir energies of various fields, such as bosons and fermions [6].

Even if renormalization or cancellation between different quantum fields succeed in making the net expectation value of the stress tensor, \( \langle T^{\mu\nu} \rangle \), acceptably small, there is no guarantee that the fluctuations around the mean value will be small. A cosmological constant counter term can only shift the mean value, \( \langle T^{\mu\nu} \rangle \), but has no effect on the fluctuations of the stress tensor operator around this mean. Similarly, the fluctuations of distinct quantum fields should be uncorrelated, so cancellation of
the mean values of the stress tensor does not imply any cancellation of the fluctuations. Thus if we live in a higher dimensional world with compact extra dimensions, it is possible for the mean stress tensor of quantum fields in our four dimensional subspace to be zero, but for there to be large stress tensor fluctuations around that mean which will produce large passive metric fluctuations. These metric fluctuations could in turn produce observable effects, such as the blurring of the images of distant objects. Fluctuations of the stress tensor induce Ricci curvature fluctuations, which in turn cause the expansion parameter of a bundle of geodesics to undergo fluctuations. This was discussed in Ref. [7], henceforth I, where the Raychaudhuri equation was treated as a Langevin equation. It was shown how the dispersion in the expansion \( \theta \) may be calculated as an integral of the Ricci tensor correlation function. It was also argued that the product \( s \theta_{rms} \) is of the order of the expected angular blurring and of the fractional luminosity fluctuations of the source. Here \( \theta_{rms} \) is the root-mean-square value of \( \theta \) and \( s \) is the distance to a source. Thus sufficiently large stress tensor fluctuations would have already been observed and can be constrained by observation. The purpose of this note is to explore the extent to which this can be used to constrain theories with compact extra dimensions. A possibility of observable effects arising from the active fluctuations of the quantized gravitational field was discussed by Yu and Ford [8].

II. COMPACT EXTRA DIMENSIONS

Consider a flat spacetime with \( 4 + n \) spacetime dimensions. The derivation of the expansion fluctuations given in I for four dimensional spacetime is essentially unchanged, apart from the fact that the Newton’s constant that appears in the Einstein equations is that for \( 4 + n \) dimensions, \( G_{4+n} \). Consider a null geodesic with tangent vector \( k^\mu \), and let \( f(x) \) be a sampling function which describes a world-tube centered on that geodesic. Then the mean squared expansion fluctuation is

\[
\langle (\Delta \theta)^2 \rangle = \int d^{4+n}x \int d^{4+n}x' f(x) f(x') C_{\mu \nu \alpha \beta}(x, x') k^\mu(x) k^\nu(x) k^\alpha(x') k^\beta(x')
\]

where \( C_{\mu \nu \alpha \beta}(x, x') \) is the Ricci tensor correlation function. We take \( k^\mu \) to be a constant vector. The contraction of the correlation function into this tangent vector is

\[
C_{\mu \nu \alpha \beta}(x, x') k^\mu k^\nu k^\alpha k^\beta = 128 \pi^2 G^2_{4+n} (k^\mu \Delta x_\mu)^4 (D)'^2,
\]

where \( D \) is the the appropriate two-point function.

Here we are interested in the vacuum state for a spacetime with periodic compactification in each of the \( n \) extra space dimensions. However, let us first consider uncompactified \( 4 + n \) dimensional Minkowski spacetime, for which the vacuum two point function is

\[
D_0(x - x') = \frac{\Gamma(\frac{d}{2} + 1)}{2 (2\pi)^{\frac{d}{2} + 2}} \frac{1}{\sigma^{\frac{d}{2} + 1}}.
\]

Here

\[
\sigma = \frac{1}{2} \left( -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 + \sum_{j=1}^{n} \Delta w_j^2 \right)
\]
is the invariant interval and $\Delta w_j$ is the coordinate separation in the $j$-th extra dimension.

We now compactify this spacetime with periodicity lengths $L_j$ in the $j$-th extra dimension. The vacuum two point function now becomes

$$D(x - x') = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{2(2\pi)^{\frac{n}{2}+2}} \sum_{m_1 = -\infty}^{\infty} \cdots \sum_{m_n = -\infty}^{\infty} \frac{1}{\sigma_{m_j}^{n/2+1}}, \tag{5}$$

where

$$\sigma_{m_j} = \frac{1}{2} \left[ -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 + \sum_{j=1}^{n} (\Delta w_j + m_j L_j)^2 \right] \tag{6}$$

and $m_j$ is the winding number in the $j$-th extra dimension. The quantity $D''$ is formed by taking the second derivative of $D$ with respect to $\sigma_{m_j}$ inside the summation:

$$D'' = \frac{\Gamma\left(\frac{n}{2} + 3\right)}{2(2\pi)^{\frac{n}{2}+2}} \sum_{m_1 = -\infty}^{\infty} \cdots \sum_{m_n = -\infty}^{\infty} \frac{1}{\sigma_{m_j}^{n/2+3}}. \tag{7}$$

We are primarily interested in the limit in which the compactification lengths are all small compared to the length scales which characterize the bundle of rays in the uncompactified dimensions. In this case, we can replace the summations by integrations over continuous variables:

$$\sum_{m_j = -\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty} dm_j = \frac{1}{L_j} \int_{-\infty}^{\infty} d\xi_j, \tag{8}$$

where $\xi_j = \Delta w_j + m_j L_j$. In this limit, we can write

$$D'' = \frac{\Gamma\left(\frac{n}{2} + 3\right) 2^{\frac{n}{2}+3}}{2(2\pi)^{\frac{n}{2}+2}} V_C \int_{0}^{\infty} \frac{d\rho \, \rho^{n-1}}{(\rho^2 + r^2 - uv)^{\frac{n}{2}+3}}, \tag{9}$$

Here

$$V_C = \prod_{j=1}^{n} L_j \tag{10}$$

is the volume of the compactified subspace, $u = \Delta t - \Delta x$, $v = \Delta t + \Delta x$, $\rho^2 = \sum_{j=1}^{n} \xi_j^2$, and $r$ is the distance from the central line of the bundle. In addition, we have used

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} d\xi_j = S_n \int_{0}^{\infty} d\rho \, \rho^{n-1}, \tag{11}$$

where

$$S_n = \frac{2 \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{12}$$

The integral in Eq (9) may be explicitly evaluated to yield

$$D'' = \frac{2}{\pi^2 V_C (r^2 - uv)^3}. \tag{13}$$
Note that the explicit dependence upon $n$, the number of compact dimensions drops out, apart from the factor of $V_C$.

We can now follow the same procedure as used in I to find the contribution of the pure vacuum term in four dimensional Minkowski spacetime, with the result

$$\langle (\Delta \theta)^2 \rangle = \frac{256 a^2 G_{4+n}^2}{5 \pi^2 V_C^2} \left\langle \frac{1}{r^8} \right\rangle. \quad (14)$$

The quantity $\left\langle \frac{1}{r^8} \right\rangle$ is an average taken over the bundle of rays. It can be defined by an integration by parts procedure, and will have a value determined by the geometric parameters of the bundle. In Kaluza-Klein theories, the Newton’s constant in higher dimensions, $G_{4+n}$, is related to the effective Newton’s constant in four dimensions by

$$G_{4+n} = V_C G_4. \quad (15)$$

This relation follows from the Einstein action in $4+n$ dimensions, and an assumption that we can trivially integrate over the extra dimension to obtain the effective four-dimensional action:

$$S = \frac{1}{16 \pi G_{4+n}} \int d^{4+n}x \sqrt{-g} R = \frac{V_C}{16 \pi G_{4+n}} \int d^4x \sqrt{-g} R = \frac{1}{16 \pi G_4} \int d^4x \sqrt{-g} R. \quad (16)$$

However, if we use the relation between $G_{4+n}$ and $G_4$, the dependence upon $V_C$ cancels, and we have exactly the same result as for four dimensional Minkowski spacetime in the vacuum state:

$$\langle (\Delta \theta)^2 \rangle = \frac{256 a^2 \ell_P^2}{5 \pi^2} \left\langle \frac{1}{r^8} \right\rangle. \quad (17)$$

Because $\left\langle \frac{1}{r^8} \right\rangle$ is of order $1/b^8$, where $b$ is the transverse dimension of the bundle of rays, this quantity is very small [7].

### III. DISCUSSION

We argued that quantum fluctuations for a system confined in a small spatial volume should grow as the volume decreases. This is indeed reflected in Eq. (14), where $\langle (\Delta \theta)^2 \rangle \to \infty$ as $V_C \to 0$ for fixed $G_{4+n}$. The special feature of Kaluza-Klein theories which permits them to avoid exhibiting large focusing fluctuations in the four-dimensional world is the relation Eq. (15). If we let $V_C \to 0$ with fixed $G_4$, then the gravitational coupling vanishes at just the rate needed to mask the effects of the extra dimensions. Thus one cannot place any constraints on Kaluza-Klein theories from phenomena such as focusing fluctuations which scale as powers of $G_{4+n}/V_C$. 

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