The short ruler on the torus

Peter Stadler

To cite this article: Peter Stadler (2019) The short ruler on the torus, Journal of Difference Equations and Applications, 25:9-10, 1382-1403, DOI: 10.1080/10236198.2019.1647186

To link to this article: https://doi.org/10.1080/10236198.2019.1647186
The short ruler on the torus

Peter Stadler

Department of Mathematics, University of Innsbruck, Innsbruck, Austria

ABSTRACT

We can shorten any path that links two given points by applying short ruler transforms iteratively. In this article we take a closer look at a short ruler process on the torus. The torus is a compact Riemannian manifold and at least a subsequence of the process converges to a geodesic between the two points. However, on compact Riemann manifolds there might exist different limit geodesics (with the same length). On the torus, the geodesics with the same length are isolated and the limit geodesic is unique.

ARTICLE HISTORY

Received 28 February 2019
Accepted 16 July 2019

KEYWORDS

Curve shortening; short ruler; torus; geodesics; iteration

2010 MATHEMATICS SUBJECT CLASSIFICATIONS

53C22; 53B20; 26A18; 39B12; 97I70

1. Introduction to the short ruler processes

On a curved surface, it is difficult to find shortest connections between two points far apart. But, we can shorten any given path linking the two points. For clarity, we first consider the plane $V = \mathbb{R}^2$: There the shortest connections are just straight lines and we could take an infinitely long ruler to connect any two points. But, what if the ruler has finite length $L > 0$ and is too short to connect two given points? Then we can establish the following shortening process:

1. We select an arbitrary curve $\alpha$ linking the two points.$^1$
2. Using the short ruler, we construct a polygonal line along the curve (see Figure 1) by setting its nodes $p_k := \alpha(kL)$ for all integers $k$. Hereby, we re-parameterize $\alpha$ by arc length and extend it constantly beyond its ends.
3. We connect consecutive midpoints of the polygonal line with the short ruler (see Figure 2). The supporting points are transformed by the linear operator

$$S: V^{\mathbb{Z}} \rightarrow V^{\mathbb{Z}}, \quad (Sp)_k = \frac{p_{k-1} + p_k}{2}.$$ 

If we iterate this shortening transformation, the polygonal line with supporting points $Sp$ converges to the straight line (cf. Figure 3).

If we replace the Euclidean plane $\mathbb{R}^2$ by an arbitrary normed vector space $(V, \|\cdot\|)$, the shortening process still converges to the straight line for every short ruler of length $L > 0$.
[see 12, Section 1, pp. 218–221]. This is true although there can be other shortest connections than the straight line, e.g. we can connect $\text{(0, 0)}$ with $\text{(1, 1)}$ directly and via $\text{(1, 0)}$ and both connections have the same length in the 1-norm:

$$\|\text{(1, 1)}\|_1 = 2 = \|\text{(1, 0)}\|_1 + \|\text{(0, 1)}\|_1 .$$

If the space is not flat, other problems arise besides the uniqueness of the shortest connection, e.g. we have to adapt the short ruler to the curvature of the space. Can we establish a similar shortening process as above and is it converging to a shortest connection, too? So far, I used the vague notion of a curved space. We concretize that after a short excursus on functional equations.
Another motivation for the question above arises if we generalize linear functions \( f : \mathbb{R} \rightarrow V \) in a normed vector space \((V, \| \cdot \|)\). They are additive, i.e. they fulfill the following functional equation for all \( x \) and \( y \) in \( \mathbb{R} \):

\[
  f(x + y) = f(x) + f(y).
\]

There exist other solutions than the linear functions. For the simplest vector space \( V = \mathbb{R} \), the additive functions are either linear, or they have a dense graph in the plane [2, Theorem 3, p. 14]:

\[
  \text{graph}(f) = \mathbb{R}^2.
\]

We can exclude the latter case by imposing additional assumptions on \( f \) as done by János Aczél [1, Section 2.1, pp. 44 et seq.] or Marek Kuczma [9, Section 5.2, pp. 128 et seq.]. For example an additive function \( f : \mathbb{R} \rightarrow V \) is linear iff:

- \( f \) is continuous (at one point),
- \( f \) is locally bounded, or
- \( f \) is Lebesgue measurable.

Then \((f)\) is a shortest connection, i.e. for all \( s < t \) the length of the restriction is

\[
  \ell(f|_{[s,t]}) = \|f(s) - f(t)\|.
\]

We can search for continuous additive functions on other topological groups than normed vector spaces, too. For example, a Lie group \( G \) is a manifold with smooth group operations. The continuous additive functions \( f : \mathbb{R} \rightarrow G \) are geodesics according to the left- or right-invariant connection [11, Sections 3–6 of chapter 6, p. 70–76]. \( G \) is a Riemannian manifold if the connection is bi-invariant. On every Riemannian manifold, we can measure lengths, and a geodesic \( \gamma \) is a locally shortest connection fulfilling the following system of ordinary differential equations:

\[
  \ddot{\gamma}^k + \sum_{i,j=1}^{m} \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0 \quad \text{for } 1 \leq k \leq m.
\]

Hereby \( \Gamma^k_{ij} \) are the Christoffel symbols of the 2nd kind and \( m \in \mathbb{N} \) is the dimension of the manifold’s connected component containing \( \gamma \). From here on, we leave the group structure behind and search for geodesics linking two points on an arbitrary Riemannian manifold. With these concepts we can concretize the above question: \textit{Can we establish a shortening process on Riemannian manifolds and is it converging to a geodesic?}

A standard method to approximate geodesics that link two given points is the shooting method:

- You choose a tangent vector at the starting point, and solve the above system of geodesic differential equations using this initial conditions.
- In general, the resulting geodesic will miss the end point. Then you adjust the initial tangent vector and recalculate the geodesic in order to get closer to the end point.
Figure 4. Reduced process (see Definition 2.3). (a) Piecewise geodesic along a curve: $R_0^0 \alpha$. (b) Reduced transformation: $R_1^0 \alpha$.

- Repeating this procedure works quite well if you can estimate how to adjust the tangent vector in each step. This is the case if starting point and end point are close to each other.

Other methods start with an arbitrary curve that connects two given points and transform it (sometimes iteratively) in order to get a geodesic linking the two points. Examples are the Pilgerschritt transform based on an idea of Roman Liedl [see 4, 5] or the disk flow [see 6].

In this article, we assume that we have a short ruler, i.e. a method to calculate geodesics linking two points that are sufficiently close to each other. The above shooting method is such a short ruler, I used it to draw the geodesics on the torus. Roman Liedl proposed a shortening process to get a long ruler from a short ruler at the ECIT 87 [10, Section 5, p. 250 et seq.]:

(1) We select some curve linking the two points.³
(2) Using the short ruler, we construct a piecewise geodesic along the curve.
(3) We use the short ruler again to cut off the corners repeatedly. The new piecewise geodesic is shorter (triangle inequality).

A similar process for closed curves was used by George D. Birkhoff [3, Section 7 of Chapter V., pp. 135–139].

If we use the same shortening process as in normed vector spaces on complete Riemannian manifolds, we get one more (different) node of the piecewise geodesic by each shortening transformation. This makes it unhandy and we modify it to get a reduced transformation (cf. Figure 4): If we apply the shortening transformation $S$ two times, we get two more points. But, then we can skip the first and the last midpoint using the short ruler (triangle inequality). The resulting piecewise geodesic has the same number of nodes as the one we started with. This reduced process has at least one limit point, and all limit points are geodesics with the same length linking the same two points [13, Theorem 3.9, p. 34]. It arises the following question: Has the reduced process a unique limit geodesic?

The torus is a compact Riemannian manifold: the reduced process is defined, and a subsequence of it is converging to a geodesic. We will review the results for compact Riemannian manifolds in the following Section 2. Then we will focus on the following
question: How many different geodesics are on the torus that link two given points and have the same length? The answer leads to the main result: There are only finitely many such geodesics (Corollary 3.5) and the whole reduced process—not only a subsequence—converges to a geodesic (Theorem 3.7).

2. Review of the reduced process on compact riemannian manifolds

With a view to the torus, we summarize the previous considerations regarding the reduced process on compact Riemannian manifolds in a different notation [cf. 8, Lemma 5.5.2, p. 206 et seq.].

Definition 2.1 (Midpoint): A point \( m \in M \) is a midpoint between the points \( p \) and \( q \) in a metric space \( (M, d) \) iff

\[
d(p, m) = \frac{d(p, q)}{2} = d(m, q).
\]

NB: There neither must exist a midpoint, nor has it to be unique.

Lemma 2.2 (Injectivity Radius): For a point \( (p) \) on a compact Riemannian manifold \( ((M, g)) \), we get the injectivity radius as

\[
i(p) := \sup \{ r \text{ For all } q \in M \text{ with } d(p, q) < r \text{ there a unique midpoint } m \text{ exists} \} > 0.
\]

The injectivity radius of \( (M) \) is as usual

\[
i(M) := \inf_{p \in M} \{ i(p) \} > 0.
\]

Definition 2.3 (Reduced Process): For a compact Riemannian manifold \( (M, g) \) and a short ruler with length

\[
L \in [0, i(M)],
\]

the midpoint \( (m(p, q)) \) between two points \( (p) \) and \( (q) \) on \( (M) \) with \( (d(p, q) \leq L) \) is unique (Lemma 2.2).

(1) We define the reduced transformation as

\[
R_L : \ M_L \longrightarrow M_L,
\]

\[
\begin{pmatrix}
p_0 \\
p_1 \\
\vdots \\
p_{n-1} \\
p_n
\end{pmatrix}
\longmapsto
\begin{pmatrix}
p_0 \\
m(m(p_0, p_1), m(p_1, p_2)) \\
\vdots \\
m(m(p_{n-2}, p_{n-1}), m(p_{n-1}, p_n)) \\
p_n
\end{pmatrix}
\]

for an arbitrary \( (n \in \mathbb{N}) \) and the set

\[
M_L := \{ p \in \mathbb{M}^{n+1} \text{ such that } d(p_{k-1}, p_k) \leq L \text{ for } 1 \leq k \leq n \}.
\]
(2) We assume that a curve \( \alpha: [a, b] \rightarrow M \) is parameterized by arc length, i.e. we have
\[
d(\alpha(s), \alpha(t)) \leq \ell(\alpha|[s,t]) \leq L
\]
for all \((a < s < t < b)\) with \((t - s \leq L)\). If we set
\[
n := \left\lceil \frac{\ell(\alpha)}{L} \right\rceil \quad \text{and} \quad R^0_L \alpha := \begin{pmatrix}
\alpha(a) \\
\alpha(a + L) \\
\vdots \\
\alpha(a + (n - 1)L) \\
\alpha(b)
\end{pmatrix} \in M_L,
\]
we can define the reduced transformation \( R_L \) for the curve \( \alpha \) by
\[
R_L \alpha := R_L \left( R^0_L \alpha \right).
\]
(3) The reduced process for the curve \( \alpha \) is the iteration of the transformation:
\[
\left( R^t_L \alpha \right)_{t \in \mathbb{N}}
\]
In order to simplify the notation, we designate the \((k)\)th supporting point of the reduced process after \(t\) steps with
\[
R^t_L \alpha_k := \left( R^t_L \alpha \right)_k.
\]
We interpret a tupel in \((M_L)\) as supporting points of a piecewise geodesic. The reduced process creates a sequence of piecewise geodesics with \((n + 1)\) supporting points each (cf. Figure 4).

**Example 2.4:** In the Euclidean space \((\mathbb{R}^m)\) the reduced transformation is the linear operator
\[
R: p \mapsto \left( p_0, \frac{p_0 + 2p_1 + p_2}{4}, \ldots, \frac{p_{n-2} + 2p_{n-1} + p_n}{4}, p_n \right)^T.
\]
For every short ruler with length \( L > 0 \) and any curve \( \alpha: [a, b] \rightarrow \mathbb{R}^m \) that is parameterized by arc length the reduced process converges to consecutively equidistant points on the straight line between the starting point \( \alpha(a) \) and the end point \( \alpha(b) \):
\[
R^t_L \alpha \xrightarrow{t \to \infty} \left( \frac{n-k}{n} \alpha(a) + \frac{k}{n} \alpha(b) \right)_k^{n} \quad \text{with} \quad n = \left\lceil \frac{\ell(\alpha)}{L} \right\rceil.
\]

**Theorem 2.5 (Continuity):** For a compact Riemannian manifold \((M, g)\) and a short ruler with length \((0 < L < i(M))\) the reduced transformation \((R_L: M_L \rightarrow M_L)\) is continuous.
Figure 5. Reduced process: 10 steps (see Theorem 2.6).

Theorem 2.6 (Limit Points): For a curve $\alpha: [a, b] \to M$ on a compact Riemannian manifold $(M, g)$ and for a short ruler with length

$$0 < L < i(M)$$

the reduced process has a convergent subsequence (cf. Figure 5)

$$R^t_L \alpha_{t \to \infty} \gamma.$$  \hspace{1cm} (1)

(1) Any limit point $\gamma$ is a geodesic with the same starting point and end point as $\alpha$.
(2) All limit points $\gamma$ of the reduced process have the same length:

$$\ell(\gamma) = \lim_{t \to \infty} \ell(R^t_L \alpha).$$

The above definitions and statements can be generalized for an arbitrary curve $\alpha: [a, b] \to M$ on any complete Riemannian manifold $(M, g)$–even if it is not compact [see 13, Section 3, pp. 30–36].

3. The torus

Definition 3.1 (Torus): The torus with aspect ratio $(a > 1)$ is defined as (see Figure 6)

$$T := \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left( \sqrt{x^2 + y^2} - a \right)^2 + z^2 = 1 \right\}.$$  \hspace{1cm} (2)

It is a connected and compact (2)-dimensional Riemannian manifold, which is embedded isometrically in the Euclidean space $\mathbb{R}^3$. 
Proposition 3.2 (Geodesics): The torus $T$ with aspect ratio $a > 1$ is a surface of revolution with the isometric cover (see Figure 7)

$$F: \mathbb{R}^2 \rightarrow T, \left(\begin{array}{c} \psi \\ \psi \end{array}\right) \mapsto \left(\begin{array}{c} (a + \cos \psi) \cos \varphi \\ (a + \cos \psi) \sin \varphi \\ \sin \psi \end{array}\right).$$

For a geodesic $\gamma: \mathbb{R} \rightarrow T, t \mapsto F(\varphi (t), \psi (t))$ that is parametrized by arc length, we review the results of Mark L. Irons in a slightly different notation [cf. 7]. There is a constant $c \in \mathbb{R}$ such that

$$\dot{\varphi} = \frac{c}{(a + \cos \psi)^2} \quad \text{and} \quad \dot{\psi} = \pm \sqrt{(a + \cos \psi)^2 - c^2} \frac{a + \cos \psi}{a + \cos \psi}.$$

The geodesic $\gamma$ is determined by its starting point $\gamma (0)$, the constant $c$ and the sign of $\dot{\psi} (0)$. We classify the geodesics by $c$ into five families:

1. $c = 0$: the geodesic $\gamma$ is a meridian.
2. $0 < |c| < a - 1$: the geodesic $\gamma$ loops like a spiral around the torus.
3. $|c| = a - 1$: the geodesic $\gamma$ is either the inner parallel circle or it is approaching it infinitesimally.
4. $a - 1 < |c| < a + 1$: the geodesic $\gamma$ stays on the outer part of the torus $T$ bouncing between the two parallel circles with radius $|c|$.
5. $|c| = a + 1$: the geodesic $\gamma$ is the outer parallel circle.
Proof: The metric components are
\[
g_{\varphi\varphi} = \left\| \frac{\partial F}{\partial \varphi} \right\|^2 = (a + \cos \psi)^2, \quad g_{\varphi\psi} = \left\| \frac{\partial F}{\partial \psi} \right\| = 0, \\
g_{\psi\varphi} = 0 \quad \text{and} \quad g_{\psi\psi} = \left\| \frac{\partial F}{\partial \psi} \right\|^2 = \sin^2 \psi + \cos^2 \psi = 1.
\]

With \( g^{ij} = (g^{-1})_{ij} \), the Christoffel symbols are
\[
\Gamma^\varphi_{\varphi\varphi} = \frac{1}{2} g^{\varphi\varphi} \left( \frac{\partial g_{\varphi\varphi}}{\partial \varphi} + \frac{\partial g_{\varphi\varphi}}{\partial \varphi} - \frac{\partial g_{\varphi\varphi}}{\partial \varphi} \right) = 0, \\
\Gamma^\varphi_{\varphi\psi} = \Gamma^\psi_{\varphi\varphi} = \frac{1}{2} g^{\psi\psi} \left( \frac{\partial g_{\psi\phi}}{\partial \psi} + \frac{\partial g_{\psi\phi}}{\partial \psi} - \frac{\partial g_{\psi\psi}}{\partial \psi} \right) = -\frac{2}{(a + \cos \psi)^2} \sin \psi = -\sin \psi, \\
\Gamma^\psi_{\psi\psi} = \Gamma^\psi_{\varphi\varphi} = \frac{1}{2} g^{\psi\psi} \left( \frac{\partial g_{\varphi\psi}}{\partial \varphi} + \frac{\partial g_{\varphi\psi}}{\partial \varphi} - \frac{\partial g_{\psi\psi}}{\partial \varphi} \right) = 0, \\
\Gamma^\psi_{\varphi\varphi} = \Gamma^\varphi_{\psi\psi} = \frac{1}{2} g^{\psi\psi} \left( \frac{\partial g_{\varphi\psi}}{\partial \varphi} + \frac{\partial g_{\varphi\psi}}{\partial \varphi} - \frac{\partial g_{\varphi\varphi}}{\partial \varphi} \right) = 0 \\
\Gamma^\varphi_{\psi\psi} = \Gamma^\psi_{\varphi\psi} = \frac{1}{2} g^{\psi\psi} \left( \frac{\partial g_{\varphi\psi}}{\partial \varphi} + \frac{\partial g_{\varphi\psi}}{\partial \varphi} - \frac{\partial g_{\psi\varphi}}{\partial \varphi} \right) = \frac{1}{2} \cdot \frac{2}{(a + \cos \psi)} \sin \psi = \frac{2}{\sin^2 \psi + \cos^2 \psi}.
\]

A geodesic \( \gamma = F(\varphi, \psi) \) fulfills:
\[
\ddot{\psi} = -\left. \frac{\partial^2 F}{\partial \varphi^2} \right|_{\psi=0} \dot{\varphi}^2 - 2 \left. \frac{\partial^2 F}{\partial \varphi \partial \psi} \right|_{\psi=0} \dot{\varphi} \dot{\psi} - \left. \frac{\partial^2 F}{\partial \psi^2} \right|_{\psi=0} \dot{\psi}^2 = \frac{2 \sin \psi}{a + \cos \psi} \dot{\psi} \dot{\psi} = \frac{d}{dt} \ln \left( (a + \cos \psi)^2 \right) \dot{\psi}.
\]

There is a constant \( c \in \mathbb{R} \) such that \( \dot{\varphi} = c/(a + \cos \psi)^2 \).

As \( \gamma \) is parametrized by arc length it has the speed
\[
1 = \| \dot{\gamma} \|^2 = g_{\varphi\varphi} \dot{\varphi}^2 + 2 g_{\varphi\psi} \dot{\varphi} \dot{\psi} + g_{\psi\psi} \dot{\psi}^2 = \frac{2 \sin \psi}{a + \cos \psi} \dot{\varphi} \dot{\psi} + \frac{(a + \cos \psi)^2}{(a + \cos \psi)^2} \dot{\psi}^2.
\]

Therefore, we have
\[
\dot{\psi} = \pm \sqrt{\frac{(a + \cos \psi)^2 - c^2}{a + \cos \psi}}.
\]

(1) \( c = 0 \): We get that \( \varphi \) is constant (\( \dot{\varphi} = 0 \)) and \( \dot{\psi} = 1 \). The geodesic \( \gamma = F(\varphi, \psi) \) is equal to the meridian on the torus \( T \):
\[
\gamma : \mathbb{R} \rightarrow T, \ t \mapsto F(\varphi, t + \psi (0)).
\]
(2) $0 < |c| < a - 1$: The geodesic $\gamma = F(\varphi, \psi)$ spirals around the torus, because both $\phi$ as well as $\psi$ stay away from 0:

$$|\psi| \geq \frac{|c|}{(a + 1)^2} > 0 \text{ and } |\psi| \geq \frac{\sqrt{(a - 1)^2 - c^2}}{a + 1} > 0.$$ 

(3) $|c| = a - 1$: If $\cos \psi$ reaches $-1$ at some time $s$, we have $\sin \psi(s) = 0$. The geodesic $\gamma = F(\varphi, \psi)$ is equal to the inner parallel circle

$$\gamma: \mathbb{R} \rightarrow T, t \mapsto F \left( \frac{\operatorname{sign}(c)}{a - 1} (t - s) + \varphi(s), \psi(s) \right)$$

since they both solve the system of geodesic equations for the same initial values:

$$\ddot{\varphi} = -2 \Gamma^\varphi_{\psi \psi} \dot{\varphi} \dot{\psi} = \frac{2 \sin \psi}{a + \cos \psi} \dot{\psi} = 0 \text{ with given } \varphi(s) \text{ and } \dot{\psi}(s) = \frac{\operatorname{sign}(c)}{a - 1},$$

$$\ddot{\psi} = -\Gamma^\psi_{\varphi \varphi} \dot{\psi}^2 = -(a + \cos \psi) \sin \psi \dot{\psi}^2 = 0 \text{ with given } \psi(s) \text{ and } \dot{\psi}(s) = 0.$$ 

If $\cos \psi$ never reaches $-1$, the map $\dot{\psi}(t)$ cannot change its sign, $t \mapsto \dot{\psi}(t)$ is bounded and monotonic, and the following limits exist:

$$r := \lim_{t \to \infty} (a + \cos \psi(t)) \text{ and } \lim_{t \to \infty} \dot{\psi}(t) = 0.$$ 

The geodesic $\gamma = F(\varphi, \psi)$ is approaching the inner parallel circle as it is $r = a - 1$:

$$0 = \lim_{t \to \infty} |\dot{\psi}(t)| = \lim_{t \to \infty} \frac{\sqrt{(a + \cos \psi(t))^2 - c^2}}{a + \cos \psi(t)} = \frac{\sqrt{r^2 - (a - 1)^2}}{r}.$$ 

(4) $a - 1 < |c| < a + 1$: The geodesic $\gamma = F(\varphi, \psi)$ has the speed

$$1 = \|\dot{\gamma}\|^2 = (a + \cos \psi)^2 \dot{\varphi}^2 + \dot{\psi}^2 \geq \frac{c^2}{(a + \cos \psi)^2}.$$ 

The radius of the geodesic $\gamma$ fulfills $a + \cos \psi \geq |c|$ and $\gamma$ stays on the outer part of the torus $T$ between the two parallel circles with radius $|c|$. If the radius would not reach $|c|$, the following limits exist analogously as in the previous case:

$$\lim_{t \to \infty} (a + \cos \psi(t)) = |c| \text{ and } \lim_{t \to \infty} \dot{\psi}(t) = 0 = \lim_{t \to \infty} \dot{\psi}(t).$$ 

This contradicts that

$$|\ddot{\psi}(t)| = (a + \cos \psi) \sin \psi \dot{\psi}^2 \xrightarrow{t \to \infty} |c| \sqrt{1 - (|c| - a)^2} \frac{c^2}{|c|^4} > 0.$$ 

The radius reaches $|c|$ at some times $s$. There again we have $\ddot{\psi}(s) \neq 0$ and

$$|\dot{\psi}(s)| = \frac{\sqrt{(a + \cos \psi(s))^2 - c^2}}{a + \cos \psi(s)} = 0.$$ 

So, $\psi$ has an extremum at time $s$ and the geodesic $\gamma = F(\varphi, \psi)$ bounces back from the parallel circle with radius $|c|$.
(5) \(|c| = a + 1\): Similar to the former case, the geodesic \(\gamma = F(\varphi, \psi)\) has the radius

\[a + \cos \psi \geq |c| = a + 1.\]

So, \(\psi\) must be constant and the geodesic \(\gamma\) is equal to the outer parallel circle

\[\gamma: \mathbb{R} \rightarrow T, t \mapsto F\left(\frac{\text{sign}(c)}{a - 1} t + \varphi(0), \psi\right).\]

3.1. Lengths of the geodesics

**Proposition 3.3 (Length):** On the torus \(T\) with aspect ratio \(a\), we can calculate the length of a geodesic \(\gamma_c: [0, l] \rightarrow T\) that is parameterized by arc length—if it is not a parallel arc—by integrating

\[\ell'_c : r \mapsto \frac{r}{\sqrt{1 - (r - a)^2 \sqrt{r^2 - c^2}}},\]

between the radius of the starting point \(a + \cos \psi(0)\) and of the end point \(a + \cos \psi(l)\). Hereby, we have to split the integral at every point, where the radius of the geodesic \(a + \cos \psi(t)\) reaches the inner parallel radius \(a - 1\), the outer parallel radius \(a + 1\) or the radius \(|c|\) and bounces back.

**Proof:** Using the identity \(|\sin \psi| = \sqrt{1 - (\varphi + \cos \psi(t) - \varphi)^2}\) and the formula for \(\dot{\psi}\) from Proposition 3.2, we get:

\[\ell (\gamma_c) = \int_0^l \frac{\sin \psi(t) \dot{\psi}(t)}{\sin \psi(t) \dot{\psi}(t)} \, dt = \int_0^l \frac{|\sin \psi(t) \dot{\psi}(t)|}{\sqrt{1 - (a + \cos \psi(t) - a)^2}} \cdot \frac{(a + \cos \psi(t))}{\sqrt{(a + \cos \psi(t))^2 - c^2}} \, dt.\]

Between the outer parallel radius \(a + 1\) and the radius \(|c|\) respective the inner parallel radius \(a - 1\) (depending on whether the constant \(|c|\) is smaller than \(a - 1\) or not), the map \(t \mapsto a + \cos \psi(t)\) is injective, and we can substitute it by \(r\) in the integral:

\[\ell (\gamma_c) = \left| \int_{a + \cos \psi(0)}^{a + \cos \psi(l)} \ell'_c (r) \, dr \right| \cdot \frac{(a + \cos \psi(t))}{\sqrt{(a + \cos \psi(t))^2 - c^2}} \, dt.\]

Lemma 3.4 (Length Integrals): For all \(a, x\) and \(y\) with \(0 < a - 1 \leq x < y \leq a + 1\), we look at the following two length integrals as functions depending on \(c\). It holds true:

\([0, x] \ni c \mapsto \int_x^y \ell'_c (r) \, dr\) is strictly increasing, while

\([a - 1, a + 1] \ni c \mapsto \int_c^{a + 1} \ell'_c (r) \, dr\) is strictly decreasing.

Furthermore, both functions are strictly convex.
Proof: For all $c < r$, the integrand $c \mapsto \ell'_c (r) = \frac{r}{\sqrt{r^2 - c^2} \sqrt{1 - (r-a)^2}}$ is positive and strictly increasing. Its derivative is

$$\frac{d \ell'_c (r)}{dc} = \frac{rc}{\left( \sqrt{r^2 - c^2} \right)^3 \sqrt{1 - (r-a)^2}}$$

strictly increasing, too. The interval $[x, y]$ is not a null set, and the function

$$c \mapsto \int_x^y \ell'_c (r) \, dr$$

is strictly increasing and strictly convex.

For the second integral, we use a substitution for changing the limits of the integral to 0 and 1. The new integrand $f_c$ is positive, strictly decreasing and strictly convex in $c$. The according calculations are a bit lengthy: In the following five steps, we derive the new integrand $f_c$, split the negative of the derivative $\partial c f_c$ into two parts, and show that each part is positive and strictly decreasing.

1. We substitute $r = (1 - s) c + s (a + 1)$ and get:

$$\int_{c}^{a+1} \ell'_c (r) \, dr = \int_{r(0)}^{r(1)} \ell'_c (r) \, dr = \int_{0}^{1} f_c (s) \, ds$$

with the new integrand

$$f_c (s) = \frac{(as + s - cs + c) (a + 1 - c)}{\sqrt{1 - (as + s - cs - a + c)^2 \sqrt{(as + s - cs + c)^2 - c^2}}}.$$

2. The negative of its derivative with respect to $c$ is:

$$- \frac{df_c}{dc} = \frac{1}{\sqrt{vw} \sqrt{s} (1 - s)} \cdot \frac{u}{vw}$$

with the three terms

$$u = (1 - s) (a - 1 + s) c + s^2 (a + 1),$$
$$v = (c - a + 1) (1 - s) + 2s \text{ and}$$
$$w = c (2 - s) + s (a + 1),$$

which are positive for $a + 1 \geq c \geq a - 1 > 0$ and $s \in ]0,1[.$

3. Additionally, $v$ and $w$ are strictly increasing in $c$. So, the first part

$$\frac{1}{\sqrt{vw} \sqrt{s} (1 - s)}$$

is positive and strictly decreasing in $c$. 


As the functions $u$, $v$ and $w$ are linear in $c$, the derivative of the second part is

$$\frac{d}{dc} \left( \frac{u}{v \, w} \right) = \frac{u' \, v \, w - u \, v' \, w - u \, v \, w'}{v^2 \, w^2} = \frac{A \, c^2 + B \, c + C}{v^2 \, w^2}$$

with the coefficients

$$A = -u' \, v' \, w', \quad B = -2 \, u(0) \, v' \, w' \quad \text{and} \quad C = u' \, v(0) \, w(0) - u(0) \, v' \, w(0) - u(0) \, v(0) \, w'.$$

We have

$$u' = (1 - s) \, (a - 1 + s), \quad u(0) = s^2 \, (a + 1),$$

$$v' = 1 - s, \quad v(0) = a \, s - a + s + 1,$$

$$w' = 2 - s, \quad w(0) = s \, (a + 1),$$

and the discriminant of the numerator is

$$B^2 - 4 \, A \, C = 4 \, v' \, w' \left( u(0) \, v' - u' \, v(0) \right) \cdot \left( u(0) \, w' - u' \, w(0) \right)$$

$$= 4 \, (1 - s) \, (2 - s) \, \left( (1 - s) \, (a \, s - a + 1) \, (1 - a) \right)$$

$$\cdot \left( s \, (a + 1) \, (a \, s - a + 1) \right)$$

$$= -4 \, (1 - s)^2 \, (2 - s) \, (a \, s - a + 1)^2 \, (a - 1) \, s \, (a + 1) \leq 0.$$

Thus, there exists at most one (real) zero of the derivative

$$\frac{d}{dc} \left( \frac{u}{v \, w} \right) = \frac{A \, c^2 + B \, c + C}{v^2 \, w^2}$$

For this real zero, the discriminant would be 0, and the zero would be

$$c = \frac{-B}{2 \, A} = -\frac{2 \, v' \, w' \, u(0)}{2 \, v^2 \, w^2 \, u'} = -\frac{s^2 \, (a + 1)}{(1 - s) \, (a - 1 + s)} < 0.$$

This contradicts the assumption that $c \geq a - 1 > 0$. The derivative of the second part cannot change the sign for $c > 0$. For $c \to \infty$, it holds true that

$$A \, c^2 + B \, c + C \to -\infty.$$

So, it is negative for $c > 0$, and the second part

$$\frac{u}{v \, w}$$

is positive and strictly decreasing.
(5) The negative derivative from step 2

\[-\frac{df_c}{dc} = \frac{1}{\sqrt{vw} \sqrt{s (1 - s)}} \cdot \frac{u}{vw}\]

is positive and strictly decreasing with respect to \(c\), since the first part

\[\frac{1}{\sqrt{vw} \sqrt{s (1 - s)}}\]

is positive and strictly decreasing (see step 3) as well as the second part (see step 4):

\[\frac{u}{vw}\]

So, the function \(c \mapsto f_c(s)\) is strictly decreasing and convex, and the length integral

\[c \mapsto \int_c^{a+1} \ell'(r) \, dr = \int_0^1 f_c(s) \, ds\]

is strictly decreasing and convex, too. \(\blacksquare\)

**Corollary 3.5:** There are finitely many geodesics up to reparametrization that connect two given points on the torus \(T\) and have the same length (see Figure 8).

**Proof:** Geodesics \(\gamma_c\) are uniquely determined up to reparametrization by its starting point, the sign of \(\dot{\psi}\) and the value \(c\) (Proposition 3.2). We assume WLOG that the geodesic \(\gamma_c\) is parametrized by arc length, that \(\dot{\psi} \geq 0\) and \(c \geq 0\). Allowing negative values would yield at most four times as many possibilities for geodesics with a given length \(l\). This does not affect the finiteness.

A geodesic \(\gamma_c\) that starts in a point with radius \(x \geq c\) and ends in a point with radius \(y \geq c\) can have the following lengths \(\ell(\gamma_c)\) (cf. Proposition 3.3). We assume WLOG that it is \(x \leq y\) and denote the length integrals by

\[\ell^t_s := \int_s^t \ell'(r) \, dr.\]

1st case: \(c = a+1\): The geodesic \(\gamma_c\) lies completely on the outer parallel circle (see Figure 9). Its length cannot be calculated using the radius, which is the constant \(c\). But, the lengths of the outer parallel arcs that are linking two points differ by a multiple of \(2\pi (a + 1)\). There can be at most one arc \(\gamma_c\) that has a given length \(\ell(\gamma_c) = l\).

![Figure 8](image-url)  
*Figure 8.* Length of geodesics \(\gamma_c\) depending on their parameter \(c\) (see Corollary 3.5).
Figure 9. A geodesic with $c = a + 1$ lies on the outer parallel circle (see 1st case in the proof of Corollary 3.5).

Figure 10. A geodesic with $c = a - 1$ lies on the inner parallel circle or converges to it (see 2nd case in the proof of Corollary 3.5).

2nd case: $c = a - 1$: Either the geodesic $\gamma_c$ lies completely on the inner parallel circle or never reaches it (see figure 10). For the former case, the lengths of the inner parallel arcs that are linking two points differ by a multiple of $2\pi (a - 1)$. So, there is at most one arc $\gamma_c$ that has a given length $\ell (\gamma_c) = l$. For the latter case, we have $x, y > c$ and can use the calculations from the 3rd case (only 1 and 2 can occur). There are finitely many geodesics $\gamma_c$ that have a given length $\ell (\gamma_c) = l$. 3rd

3rd case: $a - 1 < c < a + 1$: The geodesic $\gamma_c$ can cross the outer parallel circle reaching the radius $a + 1$ and/or bounce back reaching the radius $c$ (see Figure 11).

(1) If the geodesic $\gamma_c$ neither crosses the outer parallel circle nor bounces back, it has the length

$$\ell (\gamma_c) = \ell^y_x.$$

(2) If the geodesic $\gamma_c$ crosses the outer parallel circle, but does not bounce back, it has the length

$$\ell (\gamma_c) = \ell^{a+1}_x + \ell^{a+1}_y.$$

(3) If the geodesic $\gamma_c$ does not cross the outer parallel circle, but bounces back, it has the length

$$\ell (\gamma_c) = \ell^x_c + \ell^y_c.$$

(4) If the geodesic $\gamma_c$ crosses the outer parallel circle before it bounces back once, it has the length

$$\ell (\gamma_c) = \ell^{a+1}_x + \ell^{a+1}_c + \ell^y_c.$$
Figure 11. A geodesic with $a-1 < c < a+1$ is part of a curve that bounces between the two parallel circles with radius $c$ (see 3rd case in the proof of Corollary 3.5).
(5) If the geodesic $\gamma_c$ crosses the outer parallel circle after it bounced back once, it has the length
\[
\ell(\gamma_c) = \ell_c^x + \ell_c^{a+1} + \ell_y^{a+1}.
\]

(*) If the geodesic bounces back $n+1$ times, the length between the bouncing points $2n\ell_c^{a+1}$ must be added to the length of case (4), (5) or (*) depending on what happens before the first and after the last bouncing point.

Since the integrals fulfill $\ell_c^x = \ell_c^{a+1} - \ell_x^{a+1}$ and $\ell_c^{a+1} - \ell_y^{a+1} = \ell_y^y$, we can combine the cases to
\[
\ell(\gamma_c) \in \ell_c^{a+1} \mathbb{N} \pm \ell_y^y \text{ or } \ell(\gamma_c) \in \ell_c^{a+1} \mathbb{N} \pm \left( \ell_x^{a+1} + \ell_y^{a+1} \right).
\]

The length of the arc between a bouncing point and the outer parallel circle is at least
\[
\ell_c^{a+1} = \int_c^{a+1} \frac{r \, dr}{\sqrt{r^2 - c^2} \sqrt{1 - (r-a)^2}} \geq \frac{(a+1-c) c}{\sqrt{(a+1-c)^2 - c^2} \sqrt{1 - (c-a)^2}} \geq \frac{a}{2 \sqrt{a+1}} =: A > 0 \text{ for } c > a.
\]

The map $\ell_c^{a+1}$ is strictly decreasing with respect to $c$ (Lemma 3.4), and it is
\[
\ell_c^{a+1} \geq A \text{ for all } a - 1 \leq c < a + 1.
\]

For a given length $l$, there is a natural number $m$ such that $m \cdot A > l$. For a geodesic $\gamma_c$ with length $\ell(\gamma_c) = l$, it holds true that
\[
l \in \ell_c^{a+1} \{0, 1, \ldots m\} \pm \ell_x^y \text{ or } l \in \ell_c^{a+1} \{0, 1, \ldots m\} \pm \left( \ell_x^{a+1} + \ell_y^{a+1} \right).
\]

The length integrals
\[
\ell_x^{a+1}, \ell_y^{a+1} \text{ and } \ell_x^y
\]
are strictly increasing and strictly convex with respect to $c$; the length integral $\ell_c^{a+1}$ is strictly increasing and strictly convex with respect to $c$ (Lemma 3.4). For every $k \in \{0, 1, \ldots m\}$, there are at most two preimages of the value $l$ for each of the following functions since
\[
c \mapsto \ell_c^{a+1} k + \ell_x^y \text{ is strictly convex,}
\]
\[
c \mapsto \ell_c^{a+1} k - \ell_x^y \text{ is strictly decreasing,}
\]
\[
c \mapsto \ell_c^{a+1} k + \ell_x^{a+1} + \ell_y^{a+1} \text{ is strictly convex and}
\]
\[
c \mapsto \ell_c^{a+1} k - \ell_x^{a+1} - \ell_y^{a+1} \text{ is strictly decreasing.}
\]

So, there can be only finitely many geodesics $\gamma_c$ with $a-1 < c < a+1$ and length $\ell(\gamma_c) = l$.

4th case: $0 \leq c < a - 1$: The geodesic $\gamma_c$ can cross the outer parallel circle reaching the radius $a+1$ and/or the inner parallel circle reaching the radius $a-1$ (see Figures 12 and 13).
Figure 12. A geodesic with $0 < c < a-1$ is part of a spiral around the torus (see 4th case in the proof of Corollary 3.5).
Figure 13. A geodesic with $c = 0$ is part of a meridian circle (see 4th case in the proof of Corollary 3.5).

The length of the arc between the inner and the outer parallel circle is at least:

$$
\ell^{a+1}_{a-1} = \int_{a-1}^{a+1} \frac{r \, dr}{\sqrt{r^2 - c^2}} \sqrt{1 - \left(r - a\right)^2} \geq \frac{2 (a - 1)}{a + 1} =: B > 0.
$$

For a given length $l$, there is a natural number $m$ such that $m \cdot B > l$. Analogously to the 3rd case, we split the length $\ell (\gamma_c)$ into length integrals. For a geodesic $\gamma_c$ with length $\ell (\gamma_c) = l$ hold true that

$$
\begin{align*}
&l \in \ell^x_{a-1} + \ell^y_{a-1} + \ell^{a+1}_{a-1} \{0, 1, \ldots, m\}, & l \in \ell^x_{a-1} + \ell^{a+1}_{a-1} + \ell^{a+1}_{a-1} \{0, 1, \ldots, m\}, \\
&l \in \ell^y_{a-1} + \ell^y_{a-1} + \ell^{a+1}_{a-1} \{0, 1, \ldots, m\}, & l \in \ell^y_{a-1} + \ell^{a+1}_{a-1} + \ell^{a+1}_{a-1} \{0, 1, \ldots, m\} \\
&\text{or } l = \ell^x_{a-1}.
\end{align*}
$$

All these length integrals are strictly increasing (Lemma 3.4). Again, there are only finitely many geodesics $\gamma_c$ with $0 \leq c < a - 1$ and length $\ell (\gamma_c) = l$. Altogether, there are only finitely many geodesics $\gamma_c$ that are parametrized by arc length, start in the same point, end in the same point and have the given length $\ell (\gamma_c) = l$.

\section*{3.2. The reduced process}

Remark 3.6: The torus $T$ is a compact Riemannian manifold and we can define the reduced process (see Figure 14)

$$
\left( R^L_t \alpha \right)_{t \in \mathbb{N}}
$$

for each short ruler of length $0 < L < i (T)$ and for any curve $\alpha: [a, b] \longrightarrow T$ that is parameterized by arc length as set out in Definition 2.3.

Theorem 3.7 (Convergence): For a curve $\alpha: [a, b] \longrightarrow T$ on the torus $T$ and for a short ruler with length

$$
0 < L < i (T),
$$

the reduced process converges to a geodesic $\gamma$ (see Figure 15):

$$
R^L_t \alpha \xrightarrow{t \to \infty} \gamma.
$$
Figure 14. The reduced process on the torus: 0, 1 and 2 steps.

Figure 15. The reduced process on the torus converges: 9 steps (see Theorem 3.7).

**Proof:** The reduced process \((R^t \alpha)_t\) has a limit point \(\gamma\), every limit point is a geodesic that connects the starting point of \(\alpha\) with its end point, and all limit points have the same length (Theorem 2.6)

\[
\ell(\gamma) = \lim_{t \to \infty} \ell(R^t \alpha) \in [0, \infty[.
\]

On the torus \(T\) there exist only finitely many such geodesics up to reparametrization (Corollary 3.5), and these geodesics are isolated.

If the reduced process \((R^t \alpha)_t\) would not be convergent, it would have another limit point that is not a reparametrization of \(\gamma\). We choose the limit point \(\eta\) with the smallest distance to \(\gamma\) (they are isolated), and set

\[
\varepsilon := \frac{d(\eta, \gamma)}{2} > 0.
\]
Hereby, the distance between the piecewise geodesics with \( n+1 \) supporting points is defined as the maximum of the distances of their supporting points.\(^4\) The reduced transformation \( R_L \) is continuous in \( \gamma \) (Theorem 2.5), i.e. there is a \( \delta > 0 \) such that

\[
\text{d}(R_L \beta, R_L \gamma) < \varepsilon \quad \text{for all } \beta \text{ with } \text{d}(\beta, \gamma) < \delta < \varepsilon.
\]

(1)

As \( \gamma \), respectively, \( \eta \) are limit points of the reduced process, for every \( p \) there exist \( s \geq p \) and \( r > s \) with

\[
\text{d}(R_s^r \alpha, \gamma) < \delta < \varepsilon \text{ respectively } \text{d}(R_r^s \alpha, \eta) < \varepsilon.
\]

Using the triangle inequality, we get

\[
\text{d}(R_s^r \alpha, \gamma) \geq \text{d}(\eta, \gamma) - \text{d}(R_r^s \alpha, \eta) > \varepsilon.
\]

So, there is a step \( t \geq p \), when the reduced process moves away from \( \gamma \):

\[
\text{d}(R_t^r \alpha, \gamma) < \delta \text{ and } \text{d}(R_t^{t+1} \alpha, \gamma) \geq \delta.
\]

With respect to Equation (1), it holds true that

\[
\text{d}(R_t^{t+1} \alpha, \gamma) < \varepsilon.
\]

Therefore, for all \( p \) there exists a \( t \geq p \) such that

\[
R_t^{t+1} \alpha \in \overline{B}_\varepsilon (\gamma) \setminus B_\delta (\gamma).
\]

A limit point \( \beta \) of the reduced process \( R_t^r \alpha \) is in that ring, i.e. it has the distance

\[
\delta \leq \text{d}(\beta, \gamma) \leq \varepsilon.
\]

This means that \( \beta \) is different from \( \gamma \) and it is closer to \( \gamma \) than \( \eta \), which contradicts the assumption that the distance \( \text{d}(\eta, \gamma) \) is minimal. Therefore, the reduced process cannot have different limit points and must converge as a whole to the geodesic:

\[
R_t^r \alpha \to \gamma.
\]

Corollary 3.8 (Generalization): For a curve \( \alpha : [a, b] \to M \) on a compact Riemannian manifold \((M, g)\), we assume that all geodesics that have the same length and connect its starting point \( \alpha (a) \) with its end point \( \alpha (b) \) are isolated. Then, for any short ruler with length \( 0 < L < i (M) \), the whole reduced process converges to a geodesic \( \gamma \):

\[
R_t^L \alpha \xrightarrow{t \to \infty} \gamma.
\]

Proof: The proof of the latter Theorem 3.7 can be used directly. Only the sentence that states the assumption of this corollary for the torus \( T \) has to be omitted.

This result can be generalized further for an arbitrary curve \( \alpha : [a, b] \to M \) on any complete Riemannian manifold \((M, g)\)–even if it is not compact.
Notes

1. For example, we can restrict an approximation of the Hilbert curve such that its end points are closer to the two points than \( L \). Using the short ruler, we extend this restriction such that it connects the two points. This curve has finite length, and we reparameterize it by arc length.
2. The shortest connections are unique iff the norm is strict (a.k.a. rotund), i.e. iff the triangle inequality is sharp for all linearly independent points.
3. There is always such a path, e.g. we can restrict to a space filling curve accordingly.
4. We could also use another metric, e.g. assuming that \( \gamma \) and \( \eta \) are reparametrized proportional to their arc length to the interval \([0, 1]\):
   \[
   d(\eta, \gamma) := \max_{t \in [0, 1]} d(\eta(t), \gamma(t)).
   \]

Disclosure statement

No potential conflict of interest was reported by the author.

References

[1] J. Aczél, *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften: Mathematische Reihe Vol. 25, Birkhäuser, Basel, 1961.
[2] J. Aczél and J.G. Dhombres, *Functional Equations in Several Variables*, Vol. 31, Cambridge University Press, Cambridge, 1989.
[3] G.D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, Vol. IX, American Mathematical Society, Amsterdam, 1927.
[4] W. Förg-Rob, *Zentrale Ähnlichkeit auf Gruppen und Mannigfaltigkeiten mit linearem Zusammenhang*, Ph.D. diss., University of Innsbruck, 1980.
[5] W. Förg-Rob, *The pilgerschritt (Liedl) transform on manifolds*, in Applications of Nonlinear Analysis, T.M. Rassias, ed., The Pilgerschritt (Liedl) Transform on Manifolds, Springer International Publishing, Cham, 2018, pp. 335–353.
[6] J. Hass and P. Scott, *Shortening curves on surfaces*, Topology 33 (1994), pp. 25–43.
[7] M.L. Irons, *The geodesics of the torus* (2005). Available at http://www.rdrop.com/half/math/torus/geodesics.xhtml.
[8] J. Jost, *Riemannian Geometry and Geometric Analysis*, Universitext, Springer, Berlin, 1995.
[9] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*, 2nd ed., Birkhäuser, Basel, 2009.
[10] R. Liedl, N. Netzer, *Group theoretic and differential geometric methods for solving the translation equation*, European Conference on Iteration Theory (ECIT 87), World Scientific Publishing, Caldes De Malavella, 1989, pp. 240–252.
[11] M.M. Postnikov, *Geometry VI: Riemannian Geometry*, Vol. 6, Springer, Berlin, 2001.
[12] P. Stadler, *Curve shortening by short rulers*, ESAIM 46 (2014), pp. 217–232.
[13] P. Stadler, *Curve shortening by short rulers*, J. Differ. Equ. Appl. 22 (2016), pp. 22–36.