On the Life and Work of S. Helgason

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Abstract. This article is a contribution to a Festschrift for S. Helgason. After a biographical sketch, we survey some of his research on several topics in geometric and harmonic analysis during his long and influential career. While not an exhaustive presentation of all facets of his research, for those topics covered we include reference to the current status of these areas.

Preface

Sigurður Helgason is known worldwide for his first book *Differential Geometry and Symmetric Spaces*. With this book he provided an entrance to the opus of Élie Cartan and Harish-Chandra to generations of mathematicians. On this the occasion of his 85th birthday we choose to reflect on the impact of Sigurður Helgason’s sixty years of mathematical research. He was among the first to investigate systematically the analysis of differential operators on reductive homogeneous spaces. His research on Radon-like transforms for homogeneous spaces presaged the resurgence of activity on this topic and continues to this day. Likewise he gave a geometrically motivated approach to harmonic analysis of symmetric spaces. Of course there is much more - eigenfunctions of invariant differential operators, propagation properties of differential operators, differential geometry of homogeneous spaces, historical profiles of mathematicians. Here we shall present a survey of some of these contributions, but first a brief look at the man.

1. Short Biography

Sigurður Helgason was born on September 30, 1927 in Akureyri, in northern Iceland. His parents were Helgi Skúlason (1892-1983) and Kara Sigurðardóttir Briem (1900-1982), and he had a brother Skúli Helgason (1926-1973) and a sister Sigriður Helgadóttir (1933-2003). Akureyri was then the second largest city in Iceland with about 3,000 people living there, whereas the population of Iceland was about 103,000. As with other cities in northern Iceland, Akureyri was isolated, having only a few roads so that horses or boats were the transportation of choice. Its schools, based on Danish traditions, were good. The Gymnasium in Akureyri
was established in 1930 and was the second Gymnasium in Iceland. There Helgason studied mathematics, physics, languages, amongst other subjects during the years 1939-1945. He then went to the University of Iceland in Reykjavík where he enrolled in the school of engineering, at that time the only way there to study mathematics. In 1946 he began studies at the University of Copenhagen from which he received the Gold Medal in 1951 for his work on Nevanlinna-type value distribution theory for analytic almost-periodic functions. His paper on the subject became his master’s thesis in 1952. Much later a summary appeared in [H89].

Leaving Denmark in 1952 he went to Princeton University to complete his graduate studies. He received a Ph.D. in 1954 with the thesis, *Banach Algebras and Almost Periodic Functions*, under the supervision of Salomon Bochner.

He began his professional career as a C.L.E. Moore Instructor at M.I.T. 1954-56. After leaving Princeton his interests had started to move towards two areas that remain the main focus of his research. The first, inspired by Harish-Chandra’s ground breaking work on the representation theory of semisimple Lie groups, was Lie groups and harmonic analysis on symmetric spaces; the second was the Radon transform, the motivation having come from reading the page proofs of Fritz John’s famous 1955 book *Plane Waves and Spherical Means*. He returned to Princeton for 1956-57 where his interest in Lie groups and symmetric spaces led to his first work on applications of Lie theory to differential equations, [H59]. He moved to the University of Chicago for 1957-59, where he started work on his first book [H62]. He then went to Columbia University for the fruitful period 1959-60, where he shared an office with Harish-Chandra. In 1959 he joined the faculty at M.I.T. where he has remained these many years, being full professor since 1965. The periods 1964-66, 1974-75, 1983 (fall) and 1998 (spring) he spent at the Institute for Advanced Study, Princeton, and the periods 1970-71 and 1995 (fall) at the Mittag-Leffler Institute, Stockholm.

He has been awarded a degree Doctoris Honoris Causa by several universities, notably the University of Iceland, the University of Copenhagen and the University of Uppsala. In 1988 the American Mathematical Society awarded him the Steele Prize for expository writing citing his book *Differential Geometry and Symmetric Spaces* and its sequel. Since 1991 he carries the Major Knights Cross of the Icelandic Falcon.

## 2. Mathematical Research

In the Introduction to his selected works, [Sel], Helgason himself gave a personal description of his work and how it relates to his published articles. We recommend this for the clarity of exposition we have come to expect from him as well as the insight it provides to his motivation. An interesting interview with him also may be found in [S09]. Here we will discuss parts of this work, mostly those familiar to us. We start with his work on invariant differential operators, continuing with his work on Radon transforms, his work related to symmetric spaces and representation theory, then a sketch of his work on wave equations.

### 2.1. Invariant Differential Operators

Invariant differential operators have always been a central subject of investigation by Helgason. We find it very informative to read his first paper on the subject [H59]. In retrospect, this shines a beacon to follow through much of his later work on this subject. Here we find a
lucid introduction to differential operators on manifolds and the geometry of homogeneous spaces, reminiscent of the style to appear in his famous book \([H62]\). Specializing to a reductive homogeneous space, he begins the study of \(\mathbb{D}(G/H)\), those differential operators that commute with the action of the group of isometries. The investigation of this algebra of operators will occupy him through many years. What is the relationship of \(\mathbb{D}(G/H)\) to \(\mathbb{D}(G)\) and what is the relationship of \(\mathbb{D}(G/H)\) to the center of the universal enveloping algebra? Harish-Chandra had just described his isomorphism of the center of the universal enveloping algebra with the Weyl invariants in the symmetric algebra of a Cartan subalgebra, so Helgason introduces this to give an alternative description of \(\mathbb{D}(G/H)\). But the goal is always to understand analysis on the objects, so he investigates several problems, variations of which will weave throughout his research.

For symmetric spaces \(X = G/K\) the algebra \(\mathbb{D}(X)\) was known to be commutative, and Godement had formulated the notion of harmonic function in this case obtaining a mean value characterization. Harmonic functions being joint eigenfunctions of \(\mathbb{D}(X)\) for the eigenvalue zero, one could consider eigenfunctions for other eigenvalues. Indeed, Helgason shows that the zonal spherical functions are also eigenfunctions for the mean value operator. When \(X\) is a two-point homogeneous space, and with Ásgeirsson’s result on mean value properties for solutions of the ultrahyperbolic Laplacian in Euclidean space in mind, Helgason formulates and proves an extension of it to these spaces. Here \(\mathbb{D}(X)\) has a single generator, the Laplacian, for which he constructs geometrically a fundamental solution, thereby allowing him to study the inhomogeneous problem for the Laplacian. This paper contains still more. In many ways the two-point homogeneous spaces are ideal generalizations of Euclidean spaces so following F. John \([J55]\) he is able to define a Radon like transform on the constant curvature ones and identify an inversion operator. Leaving the Riemannian case, Helgason considers harmonic Lorentz space \(G/H\). He shows \(\mathbb{D}(G/H)\) is generated by the natural second order operator; he obtains a mean value theorem for suitable solutions of the generator and an explicit inverse for the mean value operator. Finally, he examines the wave equation on harmonic Lorentz spaces and shows the failure of Huygens principle in the non-flat case.

Building on these results he subsequently examines the question of existence of fundamental solutions more generally. He solves this problem for symmetric spaces as he shows that every \(D \in \mathbb{D}(X)\) has a fundamental solution, \([H64]\ Thm. 4.2]. Thus, there exists a distribution \(T \in C^\infty_c(X)'/\) such that \(DT = \delta_{x_0}\). Convolution then provides a method to solve the inhomogeneous problem, namely, if \(f \in C^\infty_c(X)\) then there exists \(u \in C^\infty(X)\) such that \(Du = f\). Those results had been announced in \([H63c]\]. The existence of the fundamental solution uses the deep results of Harish-Chandra on the aforementioned isomorphism as well as classic results of Hörmander on constant coefficient operators. It is an excellent example of the combination of the classical theory with the semisimple theory. Here is a sketch of his approach.

In his classic paper \([HC58]\) on zonal spherical functions, Harish-Chandra introduced several important concepts to handle harmonic analysis. One was the appropriate notion of a Schwartz-type space of \(K\) bi-invariant functions, there denoted \(I(G)\). \(I(G)\) with the appropriate topology is a Fréchet space, and having \(C^\infty_c(X)/K\) as a dense convolution subalgebra.
Another notion from [HC58] is the Abel transform
\[ F_f(a) = a^\rho \int_N f(an) \, dn. \]

Today this is also called the \( \rho \)-twisted Radon transform and denoted \( R_\rho \). Eventually Harish-Chandra showed that this gives a topological isomorphism of \( S(A)^W \), the Weyl group invariants in the Schwartz space on the Euclidean space \( A \). Furthermore, the Harish-Chandra isomorphism \( \gamma : \mathbb{D}(X) \rightarrow \mathbb{D}(A) \) interacts compatibly in that
\[ R_\rho(Df) = \gamma(D)R_\rho(f). \]

One can restate this by saying that the Abel transform turns invariant differential equations on \( X \) into constant coefficient differential equations on \( A \approx a \simeq \mathbb{R}^{\text{rank}X} \). It follows then that \( R_\rho^L : S'(A)^W \rightarrow I(G)' \) is also an isomorphism. This can then be used to pull back the fundamental solution for \( \gamma(D) \) to a fundamental solution for \( D \).

The article [H64] continues the line of investigation from [HC59] into the structure of \( \mathbb{D}(X) \). If we denote by \( U(g) \) the universal enveloping algebra of \( g^C \), then \( U(g) \) is isomorphic to \( \mathbb{D}(G) \). Let \( Z(G) \) be the center of \( \mathbb{D}(G) \). This is the algebra of bi-invariant differential operators on \( G \). The algebra of invariant differential operators on \( X \) is isomorphic to \( \mathbb{D}(G)^K/\mathbb{D}(G)^K \cap \mathbb{D}(G)\mathfrak{t} \) and therefore contains \( Z(G) \) as an Abelian subalgebra.

Let \( \mathfrak{h} \) be a Cartan subalgebra in \( g \) extending \( a \) and denote by \( W_\mathfrak{h} \) its Weyl group. The subgroup \( W_\mathfrak{h}(a) = \{ w \in W_\mathfrak{h} | w(a) = a \} \) induces the little Weyl group \( W \) by restriction. It follows that the restriction \( p \mapsto p|_a \) maps \( S(\mathfrak{h})^W \) into \( S(a)^W \). Now \( Z(G) \simeq S(g^C)^G \simeq S(\mathfrak{h})^W \), and \( \mathbb{D}(X) \simeq S(a)^W \simeq S(\mathfrak{s})^K \), \( s \) a Cartan complement of \( \mathfrak{t} \). The structure of these various incarnations is given in cf. [H64 Prop. 7.4] and [H92 Prop 3.1]. See also the announcements in [H62a, H63c]:

**Theorem 2.1.** The following are equivalent:
1. \( \mathbb{D}(X) = Z(G) \).
2. \( S(\mathfrak{h})^W|_a = S(a)^W \).
3. \( S(\mathfrak{g})^G|_a = S(\mathfrak{s})^K \).

A detailed inspection showed that (2) was always true for the classical symmetric spaces but fails for some of the exceptional symmetric spaces. Those ideas played an important role in [OW11] as similar restriction questions were considered for sequences of symmetric spaces of increasing dimension.

The final answer, prompted by a question from G. Shimura, is [H92]:

**Theorem 2.2.** Assume that \( X \) is irreducible. Then \( Z(G) = \mathbb{D}(X) \) if and only if \( X \) is not one of the following spaces \( E_6/SO(10)T \), \( E_6/F_4 \), \( E_7/E_6T \) or \( E_8/E_7SU(2) \). Moreover, for any irreducible \( X \) any \( D \in \mathbb{D}(X) \) is a quotient of elements of \( Z(G) \).

### 2.2. The Radon Transform on \( \mathbb{R}^n \)

The Radon transform as introduced by J. Radon in 1917 [RaGes] associates to a suitable function \( f : \mathbb{R}^2 \rightarrow \mathbb{C} \) its integrals over affine lines \( L \subset \mathbb{R}^2 \)
\[ R(f)(L) = \widehat{f}(L) := \int_{x \in L} f(x) \, dx \]
for which he derived an inversion formula. This groundbreaking article appeared in a not easily available journal (one can find the reprinted article in [HS0]), and
and both the Radon transform and its dual are
been considered by F. John [J55]. Later, the application of integration over affine lines in three dimensions played an important role in the three dimensional X-ray transform. We refer to [E03, GGG00, H80, H11, N01] for information about the history and the many applications of the Radon transform and its descendants.

Helgason first displayed his interest in the Radon transform in that basic paper [H59]. There he considers a transform associated to totally geodesic submanifolds in a space of constant curvature and produces an inversion formula. To use it as a tool for analysis one needs to determine if there is injectivity on some space of rapidly decreasing functions and compatibility with invariant differential operators, just as Harish-Chandra had done for the map $F$. In [H65] Helgason starts on his long road to answering such questions, and, in the process recognizing the underlying structure as incidence geometry, he is able to describe a vast generalization.

As he had previously considered two-point homogeneous spaces he starts there, but to this he extends Radon’s case to affine $p$-planes in Euclidean space. We summarize the results in the important article [H65].

Denote by $H(p, n)$ the space of $p$-dimensional affine subspaces of $\mathbb{R}^n$. Let $f \in C_c^\infty(\mathbb{R}^n)$ and $\xi \in H(p, n)$. Define

$$\mathcal{R}(f)(\xi) = \hat{f}(\xi) := \int_{x \in \xi} f(x) \, d\xi x$$

where the measure $d\xi x$ is determined in the following way. The connected Euclidean motion group $E(n) = SO(n) \ltimes \mathbb{R}^n$ acts transitively on both $\mathbb{R}^n$ and $H(p, n)$. Take basepoints $x_o = 0 \in \mathbb{R}^n$ and $\xi_o = \{(x_1, \ldots, x_p, 0, \ldots, 0)\} \in H(p, n)$ and take $d\xi x$ Lebesgue measure on $\xi_o$. For $\xi \in H(p, n)$ choose $g \in E(n)$ such that $\xi = g \cdot \xi_o$.

Then $d\xi x = g^* d\xi x_o$ or

$$\int_{\xi} f(x) d\xi x = \int_{\xi_o} f(g \cdot x) \, dx.$$  

For $x \in \mathbb{R}^n$ the set $x^\vee := \{\xi \in H(p, n) \mid x \in \xi\}$ is compact, in fact isomorphic to the Grassmanian $G(p, n) = SO(n)/[O(p) \times O(n-p)]$ of all $p$-dimensional subspaces of $\mathbb{R}^n$. Therefore each of these carries a unique $SO(n)$-invariant probability measure $d\xi x$ which provides the dual Radon transform. Let $\varphi \in C_c(\Xi)$ and define

$$\varphi^\vee(x) = \int_{x^\vee} \varphi(\xi) \, d\xi x.$$  

We have the Parseval type relationship

$$\int_{\Xi} \hat{\varphi}(\xi) \varphi(\xi) \, d\xi = \int_{\mathbb{R}^n} f(x) \varphi^\vee(x) \, dx$$

and both the Radon transform and its dual are $E(n)$ intertwining operators.

If $p = n - 1$ every hyperplane is of the form $\xi = \xi(u, p) = \{x \in \mathbb{R}^n \mid \langle x, u \rangle = p\}$ and $\xi(u, p) = \xi(v, q)$ if and only if $(u, p) = \pm (v, q)$. Thus $H(p, n) \simeq S^{n-1} \times \mathbb{Z}_2 \mathbb{R}$. We now have the hyperplane Radon transform considered in [H65]. This case had been considered by F. John [J55] and he proved the following inversion formulas

consequently was not well known. Nevertheless, its true worth is easily determined by the many generalizations of it that have been made in geometric analysis and representation theory, some already pointed out in Radon’s original article.

An important milestone in the development of the theory was F. John’s book [J55].
for suitable functions $f$:

$$f(x) = \frac{1}{2} \frac{1}{(2\pi i)^{n-1}} \Delta_{x,x}^{n-1} \int_{S^{n-1}} \hat{f}(u, \langle u, x \rangle) \, du, \quad n \text{ odd}$$

$$f(x) = \frac{1}{(2\pi i)^n} \Delta_x^{n-2} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{\partial^2 \hat{f}(u, p)}{p - \langle u, x \rangle} \, dp \, du, \quad n \text{ even}.$$

The difference between the even and odd dimensions is significant, for in odd dimensions inversion is given by a local operator, but not in even dimension. This is fundamental in Huygens’ principle for the wave equation to be discussed subsequently.

For Helgason the problem is to show the existence of suitable function spaces on which these transforms are injective and to show they are compatible with the $E(n)$ invariant differential operators. One shows that the Radon transform extends to the Schwartz space $S(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$ and it maps that space into a suitably defined Schwartz space $S(\mathbb{E})$ on $\mathbb{E}$. Denote by $\mathbb{D}(\mathbb{R}^n)$, respectively $\mathbb{D}(\mathbb{E})$, the algebra of $E(n)$-invariant differential operators on $\mathbb{R}^n$, respectively $\mathbb{E}$. Furthermore, define a differential operator $\Box$ on $\mathbb{E}$ by $\Box f(u, r) = \partial^2_x f(u, r)$.

However a new feature arises whose existence suggests future difficulties in generalizations. Let

$$S^*(\mathbb{R}^n) = \{ f \in S(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} f(x)p(x) \, dx = 0 \text{ for all polynomials } p(x) \}$$

and

$$S^*(\mathbb{E}) = \{ \varphi \in S(\mathbb{E}) \mid \int_{\mathbb{R}} \varphi(u, r)q(r) \, dr = 0 \text{ for all polynomials } q(r) \}.$$

Finally, let $S_H(\mathbb{E})$ be the space of rapidly decreasing function on $\mathbb{E}$ such that for each $k \in \mathbb{Z}^+$ the integral $\int \varphi(u, r)r^k \, dr$ can be written as a homogeneous polynomial in $u$ of degree $k$. Then we have the basic theorem for this transform and its dual:

**Theorem 2.3.** [H65] The following hold:

1. $\mathbb{D}(\mathbb{R}^n) = \mathbb{C}[\Delta]$ and $\mathbb{D}(\mathbb{E}) = \mathbb{C}[\Box]$.
2. $\Delta \hat{f} = \Box \hat{f}$.
3. The Radon transform is a bijection of $S(\mathbb{R}^n)$ onto $S_H(\mathbb{E})$ and the dual transform is a bijection $S_H(\mathbb{E})$ onto $S(\mathbb{R}^n)$.
4. The Radon transform is a bijection of $S^*(\mathbb{R}^n)$ onto $S^*(\mathbb{E})$ and the dual transform is a bijection $S^*(\mathbb{E})$ onto $S^*(\mathbb{R}^n)$.
5. Let $f \in S(\mathbb{R}^n)$ and $\varphi \in S^*(\mathbb{E})$. If $n$ is odd then

$$f = c \Delta^{(n-1)/2}(\hat{f})^\vee \text{ and } \varphi = c \Box^{(n-1)/2}(\varphi^\wedge)^\wedge$$

for some constant independent of $f$ and $\varphi$.
6. Let $f \in S(\mathbb{R}^n)$ and $\varphi \in S^*(\mathbb{E})$. If $n$ is even then

$$f = c_1 J_1(\hat{f})^\vee \text{ and } \varphi = c_2 J_2(\varphi^\wedge)^\wedge$$

where the operators $J_1$ and $J_2$ are given by analytic continuation

$$J_1 : f(x) \mapsto \text{an.cont}\mid_{\alpha=1-2n} \int_{\mathbb{R}^n} f(y) \| y \|^n dy$$

and

$$J_2 : \varphi \mapsto \text{an.cont}\mid_{\beta=-n} \int_{\mathbb{R}} \varphi(u, r) \| s-r \|^\beta dr$$
and $c_1$ and $c_2$ are constants independent of $f$ and $g$.

In [H80] it was shown that the map

$$f \mapsto \Box^{(n-1)/4} \hat{f}$$

extends to an isometry of $L^2(\mathbb{R}^n)$ onto $L^2(\Xi)$.

Needed for the proof of the theorem is one of his fundamental contributions to the subject in the following support theorem in [H65]. An important generalization of this theorem will be crucial for his later work on solvability of invariant differential operators on symmetric spaces.

**Theorem 2.4 (Thm 2.1 in [H65]).** Let $f \in C^\infty(\mathbb{R}^n)$ satisfy the following conditions:

1. For each integer $k \geq 0$, $\|x\|^k |f(x)|$ is bounded.
2. There exists a constant $A > 0$ such that $\hat{f}(\xi) = 0$ for $d(0,\xi) > A$.

Then $f(x) = 0$ for $\|x\| > A$.

An important technique in the theory of the Radon transform, which also plays an important role in the proof of Theorem 2.3, uses the Fourier slice formula: Let $r > 0$ and $u \in S^{n-1}$ then

$$\mathcal{F}(f)(ru) = c \int_\mathbb{R} \hat{f}(u,s)e^{-irs} \, ds.$$  \hspace{1cm} (2.1)

So that if $f$ is supported in a closed ball $B^n_r(0)$ in $\mathbb{R}^n$ of radius $r$ centered at the origin, then by the classical Paley-Wiener theorem for $\mathbb{R}^n$ the function

$$r \mapsto \mathcal{F}(f)(ru)$$

extends to a holomorphic function on $\mathbb{C}$ such that

$$\sup_{z \in \mathbb{C}} (1 + |z|)^n e^{-r|\text{Im}z|} |\mathcal{F}(f)(zu)| < \infty$$

Let $C^\infty_{r,H}(\Xi)$ be the space of $\varphi \in S(\Xi)$ such that $p \mapsto \varphi(u,p)$ vanishes for $p > r$. Then the Classical Paley-Wiener theorem combined with (2.1) shows that the Radon transform is a bijection $C^\infty(\mathbb{R}^n) \sim C^\infty_{r,H}(\Xi)$, [H65, Cor. 4.3]. (2.1) also played an important role in Helgason’s introduction of the Fourier transform on Riemannian symmetric spaces of the noncompact type.

**2.3. The Double Fibration Transform.** The Radon transform on $\mathbb{R}^n$ and the dual transform are examples of the double fibration transform introduced in [H66b, H70]. Recall that both $\mathbb{R}^n$ and $H(p,n)$ are homogeneous spaces for the group $G = E(n)$. Let $K = \text{SO}(n)$, $L = \text{SO}(p) \times \text{O}(n-p)$ and $N = \{(x_1, \ldots, x_p, 0, \ldots 0) \mid x_j \in \mathbb{R}\} \simeq \mathbb{R}^p$ and $H = L \rtimes N$. Then $\mathbb{R}^n \simeq G/K$, $H(p,n) \simeq G/H$ and $L = K \cap H$. Hence we have the double fibration

$$\begin{array}{ccc}
G/L & & \\
\pi & \searrow & \nearrow \\
X = G/K & & \Xi = G/H \\
\end{array}$$  \hspace{1cm} (2.2)
where \( \pi \) and \( p \) are the natural projections. If \( \xi = a \cdot \xi_o \in \Xi \) and \( x = b \cdot x_o \in X \)
then
\[
\hat{f}(\xi) = \int_{H/L} f(ah \cdot x_o) \, dH/L(hL) \quad (2.3)
\]
and
\[
\varphi^\vee(x) = \int_{K/L} \varphi(bk \cdot \xi_o) \, dK/L(kL) \quad (2.4)
\]
for suitable normalized invariant measures on \( H/L \simeq N \) and \( K/L \).

More generally, using Chern’s formulation of integral geometry on homogeneous spaces as incidence geometry [C42]. Helgason introduced the following double fibration transform. Let \( G \) be a locally compact Hausdorff topological group and \( K, H \) two closed subgroups giving the double fibration in [2.2]. We will assume that \( G, K, H \) and \( L := K \cap H \) are all unimodular. Therefore each of the spaces \( X = G/K, \Xi = G/H, G/L, K/L \) and \( H/L \) carry an invariant measure.

We set \( x_o = eK \) and \( \xi_o = eH \). Let \( x = aK \) \( \in X \) and \( \xi = bH \in \Xi \). We say that \( x \) and \( \xi \) are incident if \( aK \cap bH \neq \emptyset \). For \( x \in X \) and \( \xi \in \Xi \) we set
\[
\hat{x} = \{ \eta \in \Xi \mid x \text{ and } \xi \text{ are incident} \}
\]
and similarly
\[
\xi^\vee = \{ x \in X \mid \xi \text{ and } x \text{ are incident} \}.
\]

Assume that if \( a \in K \) and \( aH \subset KH \) then \( a \in H \) and similarly, if \( b \in H \) and \( bK \subset KH \) then \( b \in K \). Thus we can view the points in \( \Xi \) as subsets of \( X \), and similarly points in \( X \) are subsets of \( \Xi \). Then \( x^\vee \) is the set of all \( \xi \) such that \( x \in \xi \) and \( \hat{x} \) is the set of points \( x \in X \) such that \( x \in \xi \). We also have
\[
\hat{x} = p(\pi^{-1}(x)) = aK \cdot \xi_o \simeq H/L \text{ and } \xi^\vee = \pi(p^{-1}(\xi)) = bH \cdot x_o \simeq K/L.
\]

Under these conditions the Radon transform \( (2.3) \) and its dual \( (2.4) \) are well defined at least for compactly supported functions. Moreover, for a suitable normalization of the measures we have
\[
\int_{\Xi} \hat{f}(\xi) \varphi(\xi) \, d\xi = \int_X f(x) \varphi^\vee(x) \, dx.
\]

Helgason [H66b] p.39 and [GGA] p.147 proposed the following problems for these transforms \( f \to \hat{f}, \varphi \to \varphi^\vee \):

1. Identify function spaces on \( X \) and \( \Xi \) related by the integral transforms \( f \to \hat{f} \) and \( \varphi \to \varphi^\vee \).
2. Relate the functions \( f \) and \( \hat{f} \) on \( X \), and similarly \( \varphi \) and \( (\varphi^\vee)^\vee \) on \( \Xi \), including an inversion formula, if possible.
3. Injectivity of the transforms and description of the image.
4. Support theorems.
5. For \( G \) a Lie group, with \( \mathbb{D}(X) \), resp. \( \mathbb{D}(\Xi) \), the algebra of invariant differential operators on \( X \), resp. \( \Xi \). Do there exist maps \( D \to \hat{D} \) and \( E \to E^\vee \) such that
\[
(Df)^\vee = \hat{D} \hat{f} \text{ and } (E\varphi)^\vee = E^\vee \varphi^\vee.
\]

There are several examples where the double fibration transform serves as a guide, e.g. the Funk transform on the sphere \( S^n \), see [F16] for the case \( n = 2 \), and more generally [R02]; and the geodesic X-ray transform on compact symmetric spaces, see [H11, R04]. Other uses of the approach can be found in [K11]. We refer the reader to [E03] and [H11] for more examples.
2.4. Fourier analysis on $X = G/K$. From now on $G$ will stand for a non-compact connected semisimple Lie group with finite center and $K$ a maximal compact subgroup. We take an Iwasawa decomposition $G = KAN$ and use standard notation for projections on to the $K$ and $A$ component. Set $X = G/K$ as before and denote by $x_o$ the base point $eK$. Given Helgason’s classic presentation of the structure of symmetric spaces [H62] there is no good reason for us to repeat it here, so we use it freely and we encourage those readers new to the subject to learn it there.

In this section we introduce Helgason’s version of the Fourier transform on $X$, see [H65a, H68, H70]. At first we follow the exposition in [OS08] which is based more on representation theory, i.e. à la von Neumann and Harish-Chandra, rather than geometry as did Helgason. For additional information see the more modern representation theory approach of [OS08], although we caution the reader that in some places notation and definitions differ.

The regular action of $G$ on $L^2(X)$ is $\ell_g f(y) = f(g^{-1} \cdot y)$, $g \in G$ and $y \in X$. For an irreducible unitary representation $(\pi, V_\pi)$ of $G$ and $f \in L^1(X)$ set

$$\pi(f) = \int_G f(g)\pi(g)\,dg.$$ 

Here we have pulled back $f$ to a right $K$-invariant function on $G$. If $\pi(f) \neq 0$ then $V_\pi^K = \{ v \in V_\pi \mid (\forall k \in K)\pi(k)v = v \}$ is nonzero. Furthermore, as $(G, K)$ is a Gelfand pair we have $\dim V_\pi^K = 1$, in which case $(\pi, V_\pi)$ is called spherical.

Fix a unit vector $e_\pi \in V_\pi^K$. Then $\text{Tr}(\pi(f)) = (\pi(f)e_\pi, e_\pi)$ and $\|\pi(f)\|_{HS} = \|\pi(f)e_\pi\|$. Note that both $(\pi(f)e_\pi, e_\pi)$ and $\|\pi(f)e_\pi\|$ are independent of the choice of $e_\pi$. Let $\hat{G}_K$ be the set of equivalence classes of irreducible unitary spherical representations of $G$. Then as $G$ is a type one group, there exists a measure $\hat{\mu}$ on $\hat{G}_K$ such that

$$f(g \cdot x_o) = \int_{\hat{G}_K} (\pi(f)e_\pi, \pi(g)e_\pi)\,d\hat{\mu}(\pi) \quad \text{and} \quad \|f\|_2^2 = \int_{\hat{G}_K} \|\pi(f)e_\pi\|_{HS}^2\,d\hat{\mu}(\pi). \quad (2.5)$$

Harish-Chandra, see [HC54, HC57, HC58, HC66], determined the representations that occur in the support of the measure in the decomposition (2.5), as well as an explicit formula for the Plancherel measure for the spherical Fourier transform defined by him.

Helgason’s formulation is motivated by “plane waves”. First we fix parameters. Let $(\lambda, b) \in a^*_C \times K/M$ and define an “exponential function” $e_{\lambda,b} : X \to \mathbb{C}$ by $e_{\lambda,b}(x) = e_{b}(x^{\lambda^{-1}})$, where $e_{b}(x) = a(x^{-1}b)$ from the Iwasawa decomposition. Let $\mathcal{H}_\lambda = L^2(K/M)$ with action $\pi_\lambda(g)f(b) = e_{\lambda,b}(g \cdot x_o)f(g^{-1} \cdot b)$.

It is easy to see that $\pi_\lambda$ is a representation with a $K$-fixed vector $e_\lambda(b) = 1$ for all $b \in K/M$; there is a $G$-invariant pairing

$$\mathcal{H}_\lambda \times \mathcal{H}_{-\lambda} \to \mathbb{C}, \quad (f,g) := \int_{K/M} f(b)\overline{g(b)}\,db; \quad (2.6)$$
and it is unitary if and only if \( \lambda \in i\alpha^* \) and irreducible for almost all \( \lambda \) [K75, H76].

With \( \bar{f}_\lambda := \pi_\lambda(f)e_\lambda \) we have

\[
\bar{f}_\lambda(b) = \pi_\lambda(f)e_\lambda(b) = \int_G f(g)\pi_\lambda(g)e_\lambda(b) \, dg = \int_X f(x)e_{\lambda,b}(x) \, dx .
\]  

(2.7)

Then \( \bar{f}(\lambda, b) := \bar{f}_\lambda(b) \) is the Helgason Fourier transform on \( X \), see [H65a] Thm 2.2.

Recall the little Weyl group \( W \). The representation \( \pi_{w,\lambda} \) is known to be equivalent with \( \pi_\lambda \) for almost all \( \lambda \in \alpha^*_c \). Hence for such \( \lambda \) there exists an intertwining operator

\[ A(w, \lambda): \mathcal{H}_\lambda \to \mathcal{H}_{w\lambda} . \]

The operator is unique, up to scalar multiples, by Schur’s lemma. We normalize it so that \( A(w, \lambda)e_\lambda = e_{w\lambda} \). The family \( \{A(w, \lambda)\} \) depends meromorphically on \( \lambda \) and \( A(w, \lambda) \) is unitary for \( \lambda \in i\alpha^*_c \). Our normalization implies that

\[ A(w, \lambda)f_\lambda = \bar{f}_{w\lambda} . \]

(2.8)

We can now formulate the Plancherel Theorem for the Fourier transform in the following way, see [H65a] Thm 2.2] and also [H70] p. 118].

First let

\[ c(\lambda) = \int_N a(n)^{-\lambda \cdot \rho} \, d\bar{n} \]

be the Harish-Chandra \( c \)-function, \( \lambda \) in a positive chamber. The Gindikin–Karpelevich formula for the \( c \)-function [GK62] gives a meromorphic extension of \( c \) to all of \( \alpha^*_c \). Moreover \( c \) is regular and of polynomial growth on \( i\alpha^*_c \).

To simplify the notation let \( d\mu(\lambda, kM) \) be the measure \( (\#W|c(\lambda)|)^{-1} \, d\lambda d(kM) \) on \( i\alpha^* \times K/M : \)

**Theorem 2.5 ([H65a]).** The Fourier transform establishes a unitary isomorphism

\[ L^2(X) \simeq \int_{i\alpha^*/W} (\pi_\lambda, \mathcal{H}_\lambda) \frac{d\lambda}{c(\lambda)^2} . \]

Furthermore, for \( f \in C^\infty_c(X) \) we have

\[ f(x) = \int_{i\alpha^* \times K/M} \bar{f}_\lambda(b)e_{-\lambda,b}(x) \, d\mu(\lambda, b) . \]

Said more explicitly, the Fourier transform extends to a unitary isomorphism

\[ L^2(X) \to L^2(\pi_{i\alpha^*} \pi_{\mu}, L^2(K/M)) \]

\[ = \{ \varphi \in L^2(\pi_{i\alpha^*} \pi_{\mu}, L^2(K/M)) \mid (\forall w \in W) A(w, \lambda) \varphi(\lambda) = \varphi(w\lambda) \} . \]

To connect it with Harish-Chandra’s spherical transform notice that if \( f \) is left \( K \)-invariant, then \( b \mapsto \bar{f}_\lambda(b) = \bar{f}(\lambda) \) is independent of \( b \) and the integral 2.7 can be written as

\[ \bar{f}(\lambda) = \int_X \int_{K/M} e_{\lambda,b}(x) \, db \, dx = \int_X f(x)\varphi_\lambda(x) \, dx \]

(2.9)

where \( \varphi_\lambda \) is the spherical function

\[ \varphi_\lambda(x) = \int_K a(g^{-1}k)^{\lambda \cdot \rho} \, dk . \]
Then \(2.9\) is exactly the Harish-Chandra spherical Fourier transform \(\text{HC58}\) and the proof of Theorem \(2.5\) can be reduced to that formulation.

Since \(\varphi_\lambda = \varphi_\mu\) if and only if there exists \(w \in W\) such that \(w\lambda = \mu\), the spherical Fourier transform \(\tilde{f}(\lambda)\) is \(W\) invariant. The Plancherel Theorem reduces to

**Theorem 2.6.** The spherical Fourier transform sets up an unitary isomorphism

\[
L^2(X)^K \simeq L^2\left(ia^*/W, \frac{d\lambda}{|c(\lambda)|^2}\right).
\]

If \(f \in C_c^\infty(X)^K\) then

\[
f(x) = \frac{1}{\#W} \int_{ia^*} \tilde{f}(\lambda)\varphi_{-\lambda}(x) \frac{d\lambda}{|c(\lambda)|^2}.
\]

A very related result is the Paley-Wiener theorem which describes the image of the smooth compactly supported functions by the Helgason Fourier Transform. For \(K\)-invariant functions in \(\text{H66}\) Helgason formulated the problem and solved it modulo an interchange of a specific integral and sum. The justification for the interchange was provided in \(\text{G71}\); a new proof was given in \(\text{H70}\) Ch.II Thm. 2.4]. The Paley-Wiener theorem for functions in \(C^\infty_c(X)\) was announced in \(\text{H73a}\) and a complete proof was given in \(\text{H73b}\) Thm. 8.3]. Later, Torasso \(\text{T77}\) produced another proof, and Dadok \(\text{D79}\) generalized it to distributions on \(X\).

There are many applications of the Paley-Wiener Theorem and the ingredients of its proof. For example an alternative approach to the inversion formula can be obtained \(\text{R77}\). The Paley-Wiener theorem was used in \(\text{H73b}\) in the proof of surjectivity discussed in the next section, and in \(\text{H76}\) to prove the necessary and sufficient condition for the bijectivity of the Poisson transform for \(K\)-finite functions on \(K/M\) to be discussed subsequently. The Paley-Wiener theorem plays an important role in the study of the wave equation on \(X\) as will be discussed later.

For the group \(G\), an analogous theorem, although much more complicated in statement and proof, was finally obtained by Arthur \(\text{A83}\), see also \(\text{CD84, CD90, vBS05}\). In \(\text{D05}\) the result was extended to non \(K\)-finite functions. The equivalence of the apparently different formulations of the characterization can be found in \(\text{vBS06}\). For semisimple symmetric spaces \(G/H\) it was done by van den Ban and Schlichtkrull \(\text{vBS06}\). The local Paley-Wiener theorem for compact groups was derived by Helgason’s former student F. Gonzalez in \(\text{G01}\) and then for all compact symmetric spaces in \(\text{BÖP05, C06, ÖS08, ÖS10, ÖS11}\).

## 2.5. Solvability for \(D \in \mathcal{D}(X)\)

We come to one of Helgason’s major results: a resolution of the solvability problem for \(D \in \mathcal{D}(X)\). We have seen the existence of a fundamental solution allows one to solve the inhomogeneous equation: given \(f \in C^\infty_c(X)\) does there exists \(u \in C^\infty(X)\) with \(Du = f\)? But what if \(f \in C^\infty(X)\)? This is much more difficult. Given Helgason’s approach outlined earlier it is natural that once again he needs a Radon-type transform but more general than for \(K\) bi-invariant functions.

The Radon transform on symmetric spaces of the noncompact type is, as mentioned in the earlier section, an example of the double fibration transform and probably one of the motivating examples for S. Helgason to introduce this general framework. Here the double fibration is given by
and the corresponding transforms are for compactly supported functions:

\[ \tilde{f}(g \cdot \xi_o) = \int_N f(gn \cdot x_o) \, dn \] and \[ \varphi^\vee(g \cdot x_o) = \int_K \varphi(gk \cdot \xi_o) \, dk. \]

As mentioned before, in the K bi-invariant setting this type of Radon transform had already appeared (with an extra factor \( a^\rho \)) in the work of Harish-Chandra \[ HC58 \] via the map \( f \mapsto F_f \). It also appeared in the fundamental work by Gelfand and Graev \[ GG59, GG62 \] where they introduced the “horospherical method”.

In this section we introduce the Radon transform on \( X \) and discuss some of its properties. It should be noted that Helgason introduced the Radon transform in \[ H63a, H63b \] but the Fourier transform only appeared later in \[ H65a \], see also \[ H66b \].

We have seen that the Fourier transform on \( X \) gives a unitary isomorphism

\[ L^2(X) \simeq \int_{\mathfrak{a}^+} (\pi_\lambda, H_\lambda) \frac{d\lambda}{|c(\lambda)|^2} \]

whereas the Fourier transform in the \( A \)-variable gives a unitary isomorphism

\[ L^2(\Xi) \simeq \int_{\mathfrak{a}^+} (\pi_\lambda, H_\lambda) d\lambda. \]

As the representations \( \pi_\lambda \) and \( \pi_w \lambda, w \in W, \) are equivalent this has the equivalent formulation

\[ L^2(\Xi) \simeq (\#W)L^2(X). \]

In hindsight we could construct an intertwining operator from the following sequence of maps

\[ L^2(X) \to L^2 \left( K/M \times i\mathfrak{a}^*, \frac{d\lambda}{|c(\lambda)|^2} \right) \to L^2(K/M \times i\mathfrak{a}^*, d\lambda) \to L^2(\Xi) \]

obtained with \( b = k \cdot b_o \) from the sequence:

\[ f \mapsto \tilde{f}_\lambda(b) \mapsto \frac{1}{c(\lambda)} \tilde{f}(\lambda, b) \mapsto \mathcal{F}_A^{-1} \left( \frac{1}{c(\cdot)} \tilde{f}(\cdot, b) \right)(a) =: \Lambda(f)(ka \cdot \xi_o). \]

This idea plays a role in the inversion of the Radon transform, but instead we start with the Fourier transform on \( X \) given by (2.7). Then using \( b = k \cdot b_o \) we have

\[
\tilde{f}(\lambda, b) = \int_X f(x) e_{\lambda, b}(x) \, dx \\
= \int_X f(g \cdot x_o) a\left(g^{-1}l\right)^{\lambda-\rho} \, dg \\
= \int_X f(lg \cdot x_o) a\left(g^{-1}l\right)^{\lambda-\rho} \, dg \\
= \int_A \int_N f(lan \cdot x_o) a^{-\lambda+\rho} \, dnda \\
= \mathcal{F}_A((\cdot)^\rho \mathcal{R}(f)(l(\cdot))(\lambda)).
\]
Here $\mathcal{R}(f) = \hat{f}$ is the Radon Transform from before. Thus we obtain that the factorization of the unitary $G$ map discussed above, namely the Fourier transform on $L^2(X)$ is followed by the Radon transform, which is then followed by the Abelian Fourier transform on $A$, all this modulo the application of the pseudo-differential operator $J$ corresponding to the Fourier multiplier $1/c(\lambda)$. Following [H65a] and [H70] p. 41 and p. 42 we therefore define the operator $\Lambda$ by

$$\Lambda(f)(ka \cdot \xi_o) = a^{-\rho} J_a (a^\rho f(ka \cdot \xi_o)).$$

We then get [H65a] Thm. 2.1 and [H70]:

**Theorem 2.7.** Let $f \in C_c^\infty(X)$. Then

$$\#W \int_X |f(x)|^2 dx = \int_\Xi |\Lambda \mathcal{R}(f)(\xi)|^2 d\xi$$

and $f \mapsto \frac{1}{\#W} \Lambda \mathcal{R}(f)$ extends to an isometry into $L^2(X)$. Moreover, for $f \in C_c^\infty(X)$

$$f(x) = \frac{1}{\#W} (\Lambda \Lambda^* \hat{f})^\vee(x).$$

With inversion in hand, in [H63b] and [H73b] Helgason obtains the key properties of the Radon transform needed for the analysis of invariant differential operators on $X$. First we have the compatibility with a type of Harish-Chandra isomorphism:

**Theorem 2.8.** There exists a homorphism $\Gamma : \mathbb{D}(X) \to \mathbb{D}(\Xi)$ such that for $f \in C_c(X)$ we have $\mathcal{R}(D f) = \Gamma(D) \mathcal{R}(f)$.

Then using the Paley-Wiener Theorem for the symmetric space $X$ Helgason generalizes his earlier support theorem.

**Theorem 2.9 ([H73b]).** Let $f \in C_c^\infty(X)$ satisfy the following conditions:

1. There is a closed ball $V$ in $X$.
2. The Radon transform $\hat{f}(\xi) = 0$ whenever the horocycle $\xi$ is disjoint from $V$.

Then $f(x) = 0$ for $x \notin V$.

He now has all the pieces of the proof of his surjectivity result.

**Theorem 2.10.** [H73b] Thm. 8.2 Let $D \in \mathbb{D}(X)$. Then

$$DC^\infty(X) = C^\infty(X).$$

The support theorem has now been extended to noncompact reductive symmetric spaces by Kuit [K11].

### 2.6. The Poisson Transform.

On a symmetric space $X$ the use of the Poisson transform has a long and rich history. But into this story fits a very precise and important contribution - the “Helgason Conjecture”. In this section we recall briefly the background from Helgason’s work leading to this major result.

Let $g \in L^2(K/M)$ and $f \in C_c^\infty(X)$. Recall from Theorem 2.5 that the Fourier transform can be viewed as having values in $L^2(\mathbb{R}^* \frac{d\lambda}{\#W(\mathbb{R}^*)^2}, L^2(K/M))^W$. Denote the Fourier transform on $X$ by $\mathcal{F}_X(f)(\lambda) = \tilde{f}_\lambda$ and by $\mathcal{F}_X^*$ its adjoint. Then we
evaluate $F^*_X$ as follows

$$\langle F^*_X(f), g \rangle = \langle f, F^*_X(g) \rangle = \int_X f(x) \int_{\mathfrak{a}^*} \left( \int_{K/M} e^{-\lambda(b)(x)} g(b) \, db \right) \frac{d\lambda}{|c(\lambda)|^2} \, dx.$$ 

The function inside the parenthesis is the Poisson transform

$$P_\lambda(g)(x) := \int_{K/M} e^{-\lambda(b)(x)} g(b) \, db. \quad (2.11)$$

Helgason had made the basic observation that the functions $e_{\lambda,b}$ are eigenfunctions for $D(X)$, i.e., there exists a character $\chi_\lambda : D(X) \to \mathbb{C}$ such that

$$De_{\lambda,b} = \chi_\lambda(D)e_{\lambda,b}.$$ 

Indeed, they are fundamental to the construction of the Helgason Fourier transform. Here they form the kernel of the construction of eigenfunctions.

Let

$$E_\lambda(X) := \{ f \in C^\infty(X) \mid (\forall D \in D(X)) \, Df = \chi_\lambda(D)f \}. \quad (2.12)$$

Since $D \in \mathbb{D}(X)$ is invariant the group $G$ acts on $E_\lambda$. This defines a continuous representation of $G$ where $E_\lambda$ carries the topology inherited from $C^\infty(X)$. We have $P_\lambda g \in E_\lambda$ and $P_\lambda : \mathcal{H}_K^\infty \to E_\lambda$ is an intertwining operator.

In the basic paper [H59] we have seen that various properties of joint solutions of operators in $D(X)$ are obtained. In hindsight, one might speculate about eigenvalues different than 0 for operators in $\mathbb{D}(X)$, and what properties the eigenspaces might have. In fact, such a question is first formulated precisely in [H70] where several results are obtained. Are the eigenspaces irreducible? Do the eigenspaces have boundary values? What is the image of the Poisson transform on various function spaces?

In [H70] Helgason observed that, as $b \mapsto e_{-\lambda,b}(x)$ is analytic, the Poisson transform extends to the dual $\mathcal{A}'(K/M)$ of the space $\mathcal{A}(K/M)$ of analytic functions on $K/M$.

Recall the Harish-Chandra c-function $c(\lambda)$ and denote by $\Gamma_X(\lambda)$ the denominator of $c(\lambda)c(-\lambda)$. The Gindikin- Karpelevich formula for the c-function gives an explicit formula for $\Gamma_X(\lambda)$ as a product of $\Gamma$-functions. An element $\lambda \in \mathfrak{a}_C^*$ is simple if the Poisson transform $P_\lambda : C^\infty(K/M, \cdots) \to E_\lambda(X)$ is injective.

**Theorem 2.11 (Thm. 6.1 [H76]).** $\lambda$ is simple if and only if the denominator of the Harish-Chandra c-function is non-singular at $\lambda$.

This result was used by Helgason for the following criterion for irreducibility:

**Theorem 2.12 (Thm 9.1, Thm. 12.1, [H76]).** The following are equivalent:

1. The representation of $G$ on $E_\lambda(X)$ is irreducible.
2. The principal series representation $\pi_\lambda$ is irreducible.
3. $\Gamma_X(\lambda)^{-1} \neq 0$.

In [H76] p.217 he explains in detail the relationship of this result to [K75]. With irreducibility under control, Helgason turns to the range question. In [H76] for all symmetric spaces of the non-compact type, generalizing [H70] Thm. 3.2 for rank one spaces, he proves
Theorem 2.13. Every K-finite function in \( E_\lambda(X) \) is of the form \( P_\lambda(F) \) for some K-finite function on \( K/M \).

In [H70] Ch. IV, Thm. 1.8 he examines the critical case of the Poincaré disk. Utilizing classical function theory on the circle he shows that eigenfunctions have boundary values in the space of analytic functionals. This, coupled with the aforementioned analytic properties of the Poisson kernel allow him to prove

Theorem 2.14. \( E_\lambda(X) = P_\lambda(A'(\mathbb{T})) \) for \( \lambda \in i\mathfrak{a}^* \)

Those results initiated intense research related to finding a suitable compactification of \( X \) compatible with eigenfunctions of \( D(X) \); to hyperfunctions as a suitable class of objects on the boundary to be boundary values of eigenfunctions; to the generalization of the Frobenius regular singular point theory to encompass the operators in \( D(X) \); and finally to the analysis needed to treat the Poisson transform and eigenfunctions on \( X \). The result culminated in the impressive proof by Kashiwara, Kowata, Minemura, Okamoto, Oshima and Tanaka [KKMOOT78] that the Poisson transform is a surjective map from the space of hyperfunctions on \( K/M \) onto \( E_\lambda(X) \), referred to as the "Helgason Conjecture".

2.7. Conical Distributions. Let \( X \) be the upper halfplane \( \mathbb{C}^+ = \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} = \text{SL}(2, \mathbb{R})/\text{SO}(2) \). A horocycle in \( \mathbb{C} \) is a circle in \( X \) meeting the real line tangentially or, if the point of tangency is \( \infty \), real lines parallel to the \( x \)-axis. It is easy to see that the horocycles are the orbits of conjugates of the group

\[
N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.
\]

This leads to the definition for arbitrary symmetric spaces of the noncompact type:

Definition 2.15. A horocycle in \( X \) is an orbit of a conjugate of \( N \).

Denote by \( \Xi \) the set of horocycles. Using the Iwasawa decomposition it is easy to see that the horocycles are the subsets of \( X \) of the form \( gN \cdot x_0 \). Thus \( G \) acts transitively on \( \Xi \) and \( \Xi = G/MN \). As we saw before

\[
L^2(\Xi) \simeq \int_{i\mathfrak{a}^*} (\pi_\lambda, \mathcal{H}_\lambda) \, d\lambda \simeq (\#W)L^2(X) \tag{2.13}
\]

the isomorphism being given by

\[
\hat{\phi}_\lambda(g) := \int_A |a^\alpha \varphi(ga \cdot \xi_\alpha)|^\alpha \, da = \mathcal{F}_A(\langle \cdot \rangle^\alpha \ell_{g^{-1}} \varphi)|_A(\lambda).
\]

The description of \( L^2(\Xi) \simeq (\#W)L^2(X) \) suggests the question of relating \( K \) invariant vectors with \( MN \) invariant vectors. But, as \( MN \) is noncompact, it follows from the theorem of Howe and Moore [HM79] that the unitary representations \( \mathcal{H}_\lambda \), \( \lambda \in i\mathfrak{a}^* \) do not have any nontrivial \( MN \)-invariant vectors. But they have \( MN \)-fixed distribution vectors as we will explain.

Let \((\pi, V_\pi)\) be a representation of \( G \) in the Fréchet space \( V_\pi \). Denote by \( V_\pi^\infty \) the space of smooth vectors with the usual Fréchet topology. The space \( V_\pi^\infty \) is invariant under \( G \) and we denote the corresponding representation of \( G \) by \( \pi^\infty \). The conjugate linear dual of \( V_\pi^\infty \) is denoted by \( V_{-\pi}^\infty \). The dual pairing \( V_{-\pi}^\infty \times V_\pi^\infty \to \mathbb{C} \), is denoted \( \langle \cdot, \cdot \rangle \). The group \( G \) acts on \( V_{-\pi}^\infty \) by

\[
\langle \pi^\infty(a)\Phi, \phi \rangle := \langle \Phi, \pi^\infty(a^{-1})\phi \rangle.
\]
The reason to use the conjugate dual is so that for unitary representations $(\pi, V_\pi)$ we have canonical $G$-equivariant inclusions

$$V_\pi^\infty \subset V_\pi \subset V_\pi^{-\infty}.$$  

For the principal series representations we have more generally by (2.16) $G$-equivariant embeddings $\mathcal{H}_\lambda \subset \mathcal{H}_{-\lambda}^{-\infty}$. 

Assume that there exists a nontrivial distribution vector $\Phi \in (V_\pi^{-\infty})^{MN}$. Then we define $T_\Phi : V_\pi^\infty \to C^\infty(\Xi)$ by $T_\Phi(v; g \cdot \xi_\circ) = \langle \pi^{-\infty}(g)\Phi, v \rangle$. Similarly, if $T : V_\pi^\infty \to C^\infty(\Xi)$ is a continuous intertwining operator we can define a $MN$-invariant distribution vector $\Phi_T : V_\pi^\infty \to \mathbb{C}$ by $\langle \Phi_T, v \rangle = T(v; \xi_\circ)$. Clearly those two maps are inverse to each other. The decomposition of $L^2(\Xi)$ in (2.13) therefore suggests that for generic $\lambda$ we should have $\dim(\mathcal{H}_\lambda^{-\infty})^{MN} = \# W$.

As second motivation for studying $MN$-invariant distribution vectors is the following. Let $(\pi, V_\pi)$ be an irreducible unitary representation of $G$ (or more generally an irreducible admissible representation) and let $\Phi, \Psi \in (V_\pi^{-\infty})^{MN}$. If $f \in C^\infty_c(\Xi)$ then $\pi^{-\infty}(f)\Phi$ is well defined and an element in $V_\pi^\infty$. Hence $\langle \Psi, \pi^{-\infty}(f)\Phi \rangle$ is a well defined $MN$-invariant distribution on $\Xi$ and all the invariant differential differential operators on $\Xi$ coming from the center of the universal enveloping algebra act on this distribution by scalars.

A final motivation for Helgason to study $MN$-invariant distribution vectors is the construction of intertwining operators between the representations $(\pi_\lambda, \mathcal{H}_\lambda)$ and $(\pi_{w,\lambda}, \mathcal{H}_{w,\lambda}), w \in W$. This is done in Section 6 in [H70] but we will not discuss this here but refer to [S68, KS71, KS80, VW90] for more information.

We now recall Helgason’s construction for the principal series represenations $(\pi_\lambda, \mathcal{H}_\lambda)$. For that it is needed that $\mathcal{H}_\lambda = L^2(K/M)$ is independent of $\lambda$ and $\mathcal{H}_\lambda^\infty = C^\infty(K/M)$. Let $m^* \in N_{K(a)}$ be such that $m^*M \in W$ is the longest element. Then the Bruhat big cell, $\tilde{N}m^*AMN$, is open and dense. Define

$$\psi_\lambda(g) = \begin{cases} a_\lambda^{\lambda - \rho} & \text{if } g = n_1m^*aman_2 \in Nm^*AMN \\ 0 & \text{otherwise} \end{cases} \quad (2.14)$$

If $\Re \lambda > 0$ then $\psi_\lambda \in \mathcal{H}_{-\lambda}^{-\infty}$ is a $MN$-invariant distribution vector. Helgason then shows in Theorem 2.7 that $\lambda \mapsto \psi_\lambda \in \mathcal{H}_{-\lambda}^{-\infty}$ extends to a meromorphic family of distribution vectors on all of $\mathfrak{a}_c^\ast$. Similar construction works for the other $N$-orbits $NwMAN, w \in W$, leading to distribution vectors $\psi_{w,\lambda}$.

Denote by $\mathbb{D}(\Xi)$ the algebra of $G$-invariant differential operators on $\Xi$. Then $H \mapsto D_H$ extends to an isomorphisms of algebras $S(\mathfrak{a}) \simeq \mathbb{D}(\Xi)$, see [H70] Thm. 2.2.

DEFINITION 2.16. A distribution $\Psi$ (conjugate linear) on $G$ is conical if

1. $\Psi$ is $MN$-biinvariant.
2. $\Psi$ is an eigendistribution of $\mathbb{D}(\Xi)$.

The distribution vectors $\psi_{w,\lambda}$ then leads to conical distributions $\Psi_{w,\lambda}$ and it is shown in [H70, H76] that those distributions generate the space of conical distributions for generic $\lambda$.

For $\lambda \in \mathfrak{a}_c^\ast$ let $C^\infty_c(\Xi)_\lambda'$ (with the relative strong topology) denote the joint distribution eigenspaces of $\mathbb{D}(\Xi)$ containing the function $g : \xi_\circ \mapsto a(x)^{\lambda - \rho}$. Then $G$ acts on $C^\infty_c(\Xi)_\lambda'$ and according to [H70] Ch. III, Prop. 5.2 we have:
Theorem 2.17. The representation on $C_t^\infty(\Xi)_\lambda$ is irreducible if and only if $\pi_\lambda$ is irreducible.

2.8. The Wave Equation. Of the many invariant differential equations on $X$ the wave equation frequently was the focus Helgason’s attention. We shall discuss some of this work, but will omit his later work on the multitemporal wave equation \cite{H98, HS99}.

Let $\Delta_{\mathbb{R}^n} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denote the Laplace operator on $\mathbb{R}^n$. The wave-equation on $\mathbb{R}^n$ is the Cauchy problem
\[
\Delta_{\mathbb{R}^n} u(x, t) = \frac{\partial^2}{\partial t^2} u(x, t) \quad u(x, 0) = f(x), \quad \frac{\partial}{\partial t} u(x, 0) = g(x)
\] (2.15)
where the initial values $f$ and $g$ can be from $C_t^\infty(X)$ or another “natural” function space. Assume that $f, g \in C_t^\infty(\mathbb{R}^n)$ with support contained in a closed ball $B_R(0)$ of radius $R > 0$ and centered at zero. The solution has a finite propagation speed in the sense that $u(x, t) = 0$ if $\|x\| - R \geq |t|$. The Huygens’ principle asserts that $u(x, t) = 0$ for $|t| \geq \|x\| + R$. It always holds for $n > 1$ and odd but fails in even dimensions. It holds for $n = 1$ if $g \in C_t^\infty(\mathbb{R})$ with mean zero.

This equation can be considered for any Riemannian or pseudo-Riemannian manifold. In particular it is natural to consider the wave equation for Riemannian symmetric spaces of the compact or noncompact type. Helgason was interested in the wave equation and the Huygens’ principle from early on in his mathematical career, see \cite{H64, H77, H84a, H86, H92a, H98}. One can probably trace that interest to his friendship with L. Ásgeirsson, an Icelandic mathematician who studied with Courant in Göttingen and had worked on the Huygens’ principle on $\mathbb{R}^n$.

One can assume that in (2.15) we have $f = 0$ and for simplicity assume that $g$ is $K$-invariant. Then $u$ can also be taken $K$-invariant. It is also more natural to consider the shifted wave equation
\[
(\Delta_X + \|\rho\|^2)u(x, t) = \frac{\partial^2}{\partial t^2} u(x, t) \quad u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = g(x)
\] (2.16)
There are three main approaches to the problem. The first is to use the Helgason Fourier transform to reduce (2.16) to the differential equation
\[
\frac{d^2}{dt^2} \tilde{u}(\lambda, t) = -\|\lambda\|^2 \tilde{u}(\lambda, t), \quad \tilde{u}(\lambda, 0) = 0 \quad \text{and} \quad \frac{d}{dt} \tilde{u}(\lambda, 0) = \tilde{g}(\lambda)
\] (2.17)
for $\lambda \in i\mathbb{R}$. From the inversion formula we then get
\[
u(x, t) = \frac{1}{\|\lambda\|^2} \int_{i\mathbb{R}} \tilde{g}(\lambda) \varphi_\lambda(x) \frac{\sin \|\lambda\|t}{\|\lambda\|} \frac{d\lambda}{|c(\lambda)|^2}.
\]
One can then use the Paley-Wiener Theorem to shift the path of integration. Doing that one might hit the singularity of the $c(\lambda)$ function. If all the root multiplicities are even, then $1/c(\lambda)c(-\lambda)$ is a $W$-invariant polynomial and hence corresponds to an invariant differential operator on $X$.

Another possibility is to use the Radon transform and its compatibility with invariant operators
\[
\mathcal{R}((\Delta + \|\rho\|^2)f)|_A = \Delta_A \mathcal{R}(f)|_A
\]
then use the Helgason Fourier transform, and finally the Euclidean result on the Huygens’ principle. This was the method used in \cite{OS92}.
Finally, in [H92a] Helgason showed that
\[
\sin \| \lambda \| t = \int_X e^{i\lambda, b}(x) \, d\tau_t(x) = \int_X \varphi_{-\lambda}(x) \, d\tau_t(x)
\]
for certain distribution \( \tau_t \) and then proving a support theorem for \( \tau_t \).

The result is [ÖS92, H92a].

**Theorem 2.18.** Assume that all multiplicities are even. Then Huygens’s principle holds if \( \text{rank} X \) is odd.

It was later shown in [BÓS95] that in general the solution has a specific exponential decay. In [BÓ97] it was shown, using symmetric space duality, that the Huygens’ principle holds locally for a compact symmetric spaces if and only if it holds for the noncompact dual. The compact symmetric spaces were then treated more directly in [BÓP05].

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