Notes on the (2+1)-Dimensional
Wheeler-DeWitt Equation

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Abstract

In contrast to other approaches to (2+1)-dimensional quantum gravity, the Wheeler-DeWitt equation appears to be too complicated to solve explicitly, even for simple spacetime topologies. Nevertheless, it is possible to obtain a good deal of information about solutions and their interpretation. In particular, strong evidence is presented that Wheeler-DeWitt quantization is not equivalent to reduced phase space quantization.

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In the continuing search for a realistic quantum theory of gravity, it has often proven useful to explore simpler models that share the basic conceptual features of general relativity. Gravity in 2+1 dimensions is one such model, providing a fully diffeomorphism-invariant theory of spacetime geometry that nevertheless avoids many of the technical difficulties of realistic (3+1)-dimensional gravity.

The goal of this paper is to explore the ramifications of one popular approach to quantum gravity, the Wheeler-DeWitt equation, in this simple setting. We shall see that even with the simplifications of 2+1 dimensions, Wheeler-DeWitt quantization is considerably more complicated than one might guess, and that at least in its simplest interpretations, it is not equivalent to other known approaches to quantization.

1. Canonical Gravity in 2+1 Dimensions

Before trying to construct a quantum theory, let us briefly review the canonical formalism for (2+1)-dimensional gravity, as described by Moncrief [1] and Hosoya and Nakao [2]. We begin with a spacetime with the topology \( \mathbb{R} \times \Sigma \), where \( \Sigma \) is a closed surface. The standard (2+1)-dimensional ADM variables are then a spatial metric \( g_{ij} \) and its canonical momentum \( \pi_{ij} \).

By a classical result of Riemann surface theory, any metric on \( \Sigma \) can be written in the form
\[
g_{ij} = e^{2\lambda} f^* \bar{g}_{ij},
\]
where \( f \) is a spatial diffeomorphism and \( \bar{g}_{ij} \) is a “standard” metric of constant curvature 1 (if \( \Sigma \) is a sphere), 0 (if \( \Sigma \) is a torus), or \(-1\) (if \( \Sigma \) is a surface of genus \( g > 1 \)). For a surface of genus \( g > 1 \), the standard metrics \( \bar{g}_{ij} \) comprise a \((6g-6)\)-dimensional family; for the torus, the family is two-dimensional, and we can choose
\[
d\bar{s}^2 = \tau_2^{-1} |dx + \tau d\bar{y}|^2,
\]
where \( x \) and \( y \) are angular coordinates with period 1 and \( \tau = \tau_1 + i\tau_2 \) is a complex parameter, the modulus. Corresponding to the decomposition (1.1), the momentum \( \pi^{ij} \) can be written as
\[
\pi^{ij} = e^{-2\lambda} \sqrt{\bar{g}} \left( \bar{p}^{ij} + \frac{1}{2} \bar{g}^{ij} \pi / \sqrt{\bar{g}} + \bar{\nabla}^i \bar{Y}^j + \bar{\nabla}^j \bar{Y}^i - \bar{g}^{ij} \bar{\nabla}^k \bar{Y}^k \right),
\]
where \( \bar{\nabla}_i \) is the covariant derivative for the connection compatible with \( \bar{g}_{ij} \), indices are now raised and lowered with \( \bar{g}_{ij} \), and \( \bar{p}^{ij} \) is a transverse traceless tensor with respect to \( \bar{\nabla}_i \) (in the language of Riemann surface theory, a holomorphic quadratic differential). Roughly speaking, \( \bar{p}^{ij} \) is canonically conjugate to \( \bar{g}_{ij} \), \( \pi \) to \( \lambda \), and \( \bar{Y}^i \) to \( f \). More precisely, if we consider a cotangent vector \( \delta g_{ij} \) in the space of metrics,
\[
\delta g_{ij} = \nabla_i \delta \xi_j + \nabla_j \delta \xi_i + 2e^{2\lambda} \bar{g}_{ij} \delta \lambda + e^{2\lambda} \delta \bar{g}_{ij},
\]
the symplectic structure can be read off from the expression
\[
\int_{\Sigma} \delta g_{ij} \delta \pi^{ij} = \int_{\Sigma} \left\{ \sqrt{\bar{g}} \delta \bar{g}_{ij} \delta \bar{p}^{ij} + 2 \delta \lambda \delta \pi - 2 \delta \xi^i \left[ e^{-2\lambda} \sqrt{\bar{g}} \left( \Delta + \frac{1}{2} \bar{R} \right) \right] \delta Y_i + \frac{1}{2} \bar{\nabla}_i \left( e^{-2\lambda} \delta \pi \right) \right\},
\]
where \( \bar{R} \) is the scalar curvature of \( \bar{g}_{ij} \).
Once again — and for the remainder of this paper — indices are raised and lowered with \( \bar{g}_{ij} \). The last term in (1.5) tells us that \( \xi^i \) and \( Y_i \) are not quite a conjugate pair, a fact that will cause us considerable grief in the next section.

In terms of the decompositions (1.1) and (1.4), the constraints of canonical gravity become relatively simple. The momentum constraints are

\[
\sqrt{\bar{g}}(\bar{\Delta} + \frac{1}{2}\bar{R})Y_i + \frac{1}{2}e^{2\lambda}\nabla_i\left(e^{-2\lambda}\pi\right) = 0
\]  

(1.6)

— note the similarity to the last term of (1.3) — while the Hamiltonian constraint becomes

\[
\mathcal{H} = -\frac{1}{2}\frac{1}{\sqrt{\bar{g}}}e^{-2\lambda}\pi^2 + \sqrt{\bar{g}}e^{-2\lambda}\left[p^{ij}p_{ij} + 2p_{ij}(PY)^{ij} + (PY)_{ij}(PY)^{ij}\right] + 2\sqrt{\bar{g}}\left[\bar{\Delta}\lambda - \frac{1}{2}\bar{R}\right] = 0, 
\]

(1.7)

where

\[
(PY)_{ij} = \nabla_iY_j + \nabla_jY_i - \bar{g}_{ij}\nabla_kY^k. 
\]

Moncrief has shown that in the classical theory, these constraints fix \( \lambda \) uniquely as a function of \( \bar{g}_{ij} \) and \( p^{ij} \), thus determining a finite-dimensional reduced phase for (2+1)-dimensional gravity. A description of dynamics on this reduced phase space depends on a choice of time slicing; for a foliation by surfaces of constant mean curvature \( TrK = T \), it may be shown \( [1, 3] \) that the Hamiltonian is

\[
H = \int d^2x e^{2\lambda}, 
\]

(1.9)

which can in turn be written as a (complicated) function of \( \bar{g}_{ij} \) and \( p^{ij} \).

## 2. Quantization and the Wheeler-DeWitt Equation

To quantize this system, we may either solve the constraints classically and quantize the resulting reduced phase space, or else quantize first and impose the constraints as conditions to determine physical states. The first alternative is discussed in reference \( [3] \). If \( \Sigma \) has the topology of a torus, the constraints can be solved explicitly; in particular, in the York time slicing, \( \pi e^{-2\lambda}/\sqrt{\bar{g}} = T \), the momentum constraints imply that \( Y_i \) vanishes, and the Hamiltonian constraint has as its solution

\[
e^{2\lambda} = T^{-1}\left(2p_{ij}p^{ij}\right)^{1/2}. 
\]

(2.1)

The system is described by the effective Hamiltonian (1.9), and it is an easy exercise to find the corresponding Schrödinger equation:

\[
i\frac{\partial\psi}{\partial T} = T^{-1}\Delta_0^{1/2}\psi, 
\]

(2.2)
where
\[ \Delta_0 = -\tau_2^2 \left( \frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) \]  
(2.3)
is the Laplacian on the torus moduli space with the standard Poincaré metric \( d\tau d\bar{\tau}/\tau_2^2 \). The square root in (2.2) is ambiguous, of course; it is customary to take it to be the unique positive square root, corresponding physically to an expanding universe.

In Wheeler-DeWitt quantization, on the other hand, the Hamiltonian constraint is not solved classically, but is instead imposed as a Klein-Gordon-like equation of motion constraining physical states. In this approach, we no longer have the luxury of choosing a time slicing, and in particular cannot require that \( \text{Tr} K \) be constant on each slice. As a consequence, the momentum constraints no longer require \( Y_i \) to vanish, but imply instead that
\[ Y_i = -\frac{1}{2} \tilde{\Delta}^{-1} \left[ \frac{1}{\sqrt{g}} e^{2\lambda} \nabla_i \left( e^{-2\lambda} \pi \right) \right]. \]  
(2.4)
To obtain the Wheeler-DeWitt equation, we are instructed to insert this expression into the Hamiltonian constraint and to make the further substitutions
\[ \pi = -\frac{i}{2} \delta \delta \lambda, \quad p^{ij} = -\frac{i}{\sqrt{g}} \delta \delta \bar{g}_{ij}. \]  
(2.5)
For the torus, we find that
\[ \left\{ \frac{1}{8} \delta \delta \lambda e^{-2\lambda} \frac{\partial^2}{\partial \lambda^2} + \frac{1}{2} e^{-2\lambda} \Delta_0 + 2 \tilde{\Delta} \lambda - 2 e^{-2\lambda} Y^i[\pi] \bar{\Delta} Y_i[\pi] \right. \]  
\[ + 2 e^{-2\lambda} \nabla_i \left[ \left( 2p^{ij} + \nabla^i Y^j[\pi] + \nabla^j Y^i[\pi] - \bar{g}^{ij} \nabla_k Y^k[\pi] \right) Y_j[\pi] \right] \} \Psi[\lambda, \tau] = 0, \]  
(2.6)
with \( Y_i[\pi] \) given by (2.4). The operator ordering in the first term of (2.6) is the simplest for which the Hamiltonian constraint is at least formally Hermitian with respect to the inner product defined by the “measure” \([d\lambda]\). This is a natural choice, but it is by no means unique; we shall return briefly to this issue in section 5. As promised, the \( Y_i \) dependence makes equation (2.6) complicated and nonlocal, reflecting the fact that the symplectic structure mixes \( Y_i \) and \( \pi \) — only in the York time slicing do the momentum and Hamiltonian constraints disentangle.

3. Naive Schrödinger Interpretation

It is evident from equation (2.6) that explicit solutions of the Wheeler-DeWitt equation will be difficult to find, even in this simple model. The fundamental simplifying feature of 2+1 dimensions is that the physical phase space, parametrized by \( \bar{g}_{ij} \) and \( p^{ij} \), is finite dimensional, effectively reducing quantum field theory to quantum mechanics. This simplification is evident in the \( p_{ij} p^{ij} \) term of the Hamiltonian constraint, which reduces to a
Laplacian on the two-dimensional moduli space of the torus. But unlike other approaches to quantization, the Wheeler-DeWitt equation retains a complicated dependence on the scale factor \( \lambda \), reflecting the fact that the time slicing has been left arbitrary.

Even without an exact solution to the Wheeler-DeWitt equation, we can try to extract some useful information. To do so, however, we must first decide on an interpretation of the “wave functions” \( \Psi[\lambda, \tau] \). This is a highly nontrivial problem: solutions of (2.6) do not come with a ready-made inner product, and the choice of something like a Hilbert space structure is necessary to give wave functions a quantum mechanical interpretation.

One natural guess, the “naive Schrödinger interpretation” [4], is that \(|\Psi[\lambda, \tau]|^2\) is simply the relative probability of finding a spatial slice of the universe with scale factor \( \lambda \) and modulus \( \tau \). To compare Wheeler-DeWitt and reduced phase space quantization, it is more useful to consider the relative probability of finding a slice with extrinsic curvature \( K = e^{-2\lambda} \pi \) and modulus \( \tau \); this can be obtained from a functional Fourier transform

\[
\tilde{\Psi}[K, \tau] = \int [d\lambda] e^{i\int d^2x K e^{2\lambda}} \Psi[\lambda, \tau].
\] (3.1)

In particular, the wave function on a York time slice \( K = T \) is

\[
\tilde{\psi}(T, \tau) = \int [d\lambda] e^{iT\int d^2x e^{2\lambda}} \Psi[\lambda, \tau],
\] (3.2)

giving the relative probability of finding a geometry with modulus \( \tau \) on a slice of constant mean curvature \( T \).

We can now ask whether this wave function is equivalent to the wave function (2.2) obtained by fixing the same time slicing before quantization. Observe first that by (2.4),

\[
\int [d\lambda] e^{iT\int d^2x e^{2\lambda}} Y_i[\pi] \Psi[\lambda, \tau] = 0
\] (3.3)

since functional integration by parts can be used to move the action of \( \pi \) to the exponential. The functional Fourier transform of the Wheeler-DeWitt equation (2.6) then reduces to

\[
\int [d\lambda] e^{iT\int d^2x e^{2\lambda}} \int d^2x N(x) \left\{ -T^2 e^{2\lambda} + e^{-2\lambda} \Delta_0 + 4 \bar{\Delta} \lambda \right\} \Psi[\lambda, \tau] = 0
\] (3.4)

where \( N(x) \) is an arbitrary function. The expression in brackets is essentially the classical Hamiltonian constraint for the torus, with \( p_{ij}p^{ij} \) replaced by \( \Delta_0 \). Now, however, the constraint need not be obeyed everywhere, but only inside a functional integral. In particular, there is no reason to suppose that the wave function \( \Psi[\lambda, \tau] \) has its support only on solutions of this constraint.

To obtain further information about \( \tilde{\psi}(T, \tau) \), we can set \( N = 1 \) in (3.4). We find that

\[
iT^2 \frac{\partial \psi}{\partial T}(T, \tau) = -\Delta_0 \int [d\lambda] \left\{ e^{iT\int d^2x e^{2\lambda}} \int d^2x e^{-2\lambda} \right\} \Psi[\lambda, \tau],
\] (3.5)
or, differentiating again,
\[
\left[ \frac{\partial}{\partial T} T^2 \frac{\partial}{\partial T} + \Delta_0 \right] \psi(T, \tau) = \int [d\lambda] \left\{ e^{iT \int d^2 x e^{2\lambda}} \left[ 1 - \int d^2 x e^{2\lambda} \int d^2 x e^{-2\lambda} \right] \right\} \Delta_0 \Psi[\lambda, \tau].
\]

(3.6)

Up to minor operator ordering ambiguities, the left-hand side of this expression is the square of the Schrödinger equation (2.2) obtained from solving the constraints before quantizing, and is equivalent to the “gauge-fixed Wheeler-DeWitt equation” obtained by Hosoya and Nakao \[2\] and Martinec \[5\]. The right-hand side thus measures the departure of the full Wheeler-DeWitt equation from these alternative approaches to quantization.

In particular, the right-hand side of (3.6) vanishes when \(\lambda\) is spatially constant, as it is for any classical solution of the constraints. As we have seen above, however, there is no reason to expect the wave function \(\Psi[\lambda, \tau]\) to have its support only on such solutions. Moreover, the right-hand sides of (3.5) and (3.6) depend not only on \(\tilde{\psi}(T, \tau)\), but on the full Wheeler-DeWitt wave function \(\Psi[\lambda, \tau]\), so our would-be Schrödinger equation must be interpreted as one of an infinite family of equations — somewhat analogous to the Schwinger-Dyson equations in quantum field theory — for functions
\[
\tilde{\psi}_n(T, \tau) = \int [d\lambda] \left\{ e^{iT \int d^2 x e^{2\lambda}} \int d^2 x e^{2n\lambda} \right\} \Psi[\lambda, \tau].
\]

(3.7)

While the particular form of equation (3.6) depends on a choice of operator ordering in the Hamiltonian constraint, I have been unable to find any alternative ordering in which the Wheeler-DeWitt equation reduces to a single Schrödinger- or Klein-Gordon-like equation for \(\tilde{\psi}(T, \tau)\).

The details of this argument depend on a specific model, but the general features seem likely to extend to realistic (3+1)-dimensional gravity. In particular, the trace of the extrinsic curvature in 3+1 dimensions is canonically conjugate to \(\sqrt{g}\), so the wave function at fixed \(K\) should take the form
\[
\tilde{\psi}_{(3+1)}[T, \tilde{g}_{ij}] \sim \int [d(\sqrt{g})] e^{iTV[\sqrt{\tilde{g}}]} \Psi_{(3+1)}[\sqrt{\tilde{g}}, \tilde{g}_{ij}],
\]

(3.8)

where \(V = \int d^3x \sqrt{g}\) is the volume of a spatial slice and \(\tilde{g}_{ij}\) symbolizes the remaining metric components. If we take two time derivatives of \(\tilde{\psi}_{(3+1)}\), the integrand will be multiplied by a nonlocal factor \(V^2[\sqrt{\tilde{g}}]\). But just as in our (2+1)-dimensional model, the action of the local part of the Hamiltonian constraint can generate at most a single spatial integral in the integrand. Unless the nonlocal terms in the Hamiltonian constraint conspire to remove this mismatch in the number of integrations — as they fail to do in (2+1) dimensions — it would seem difficult to obtain a simple Klein-Gordon-like equation for the wave function at constant \(K\). The inequivalence of Dirac and reduced phase space quantization in a theory with quadratic constraints has been widely studied — see, for example, \[3, 4, 8, 9\] — but I believe this particular feature is new.
4. Gauge-Fixing the Inner Product

The absence of a simple Schrödinger or Klein-Gordon equation for $\tilde{\psi}(T, \tau)$ should not in itself be viewed as a reason for rejecting Wheeler-DeWitt quantization. There are many possible quantum theories of gravity in 2+1 dimensions \[10\], and we do not yet know how to choose among them. The naive Schrödinger interpretation has a somewhat more serious problem, however — there is no reason to expect the inner product $\langle \tilde{\psi}(T, \tau)|\tilde{\phi}(T, \tau) \rangle$ of two states to be conserved, so the relationship to ordinary quantum mechanics is problematic. This difficulty can be resolved, at least formally, by appealing to a suggestion by Woodard \[11\] for redefining the inner product.

Woodard’s proposal is that all invariances of the action, including those generated by the Hamiltonian constraint $H$, should be gauge-fixed in the inner product of two Wheeler-DeWitt wave functions. For transformations generated by the diffeomorphism constraints, this procedure is uncontroversial: such transformations take a metric on a spatial slice $\Sigma$ to a physically equivalent metric, and must clearly be factored out to avoid overcounting. The proper treatment of the group of transformations generated by the Hamiltonian constraint, on the other hand, is less obvious. Such transformations mix metrics on $\Sigma$ with the corresponding canonical momenta — in the language of geometric quantization, they do not preserve the polarization — and they have no simple interpretation as transformations of the argument of the wave function.

One argument in favor of Woodard’s approach comes from examining wave functions determined by path integration. Consider two Hartle-Hawking wave functions $\Psi_1[g]$ and $\Psi_2[g]$ on a surface $\Sigma$, obtained from path integrals on manifolds $M_1$ and $M_2$ with boundaries $\partial M_1 \approx \partial M_2 \approx \Sigma$. On $M_1$, say, the action is invariant under the transformations

$$\delta g_{ij} = [\epsilon H, g_{ij}]$$

$$\delta \pi^{ij} = [\epsilon H, \pi^{ij}]$$

only when $\epsilon = 0$ on $\Sigma$ \[12\], so only this subset of transformations will be gauge-fixed in the path integral for $\Psi_1[g]$. A similar statement holds for $\Psi_2[g]$. The inner product, on the other hand, can be naturally defined as the path integral over the new manifold $M$ obtained by attaching $M_1$ and $M_2$ along $\Sigma$. In this path integral, however, the transformations \[11,12\] must be gauge-fixed no matter what the value of $\epsilon$ on $\Sigma$; this additional gauge-fixing leads directly to Woodard’s formalism.

To apply this approach to our (2+1)-dimensional model, we begin with the inner product

$$\langle \Psi|\Phi \rangle = \int [d\lambda] \int \frac{d^2\tau}{\tau^2} \Psi^*[\lambda, \tau]|\Phi[\lambda, \tau],$$

(4.2)

gauge-fixed to the York time slicing

$$\chi = \pi - T e^{2\lambda} = 0.$$  

(4.3)
The path integral over $\lambda$ can be evaluated à la Faddeev and Popov by inserting a factor of unity in the form

$$1 = \int [d\epsilon] \nu^{*}[\lambda^\epsilon, \tau^\epsilon] \delta[\chi^\epsilon] \nu[\lambda^\epsilon, \tau^\epsilon],$$

(4.4)

where $\lambda^\epsilon$, for instance, denotes the transformed value of $\lambda$,

$$\lambda^\epsilon = \lambda - i [\epsilon H, \lambda].$$

(4.5)

A simple computation shows that

$$\nu[\lambda, \tau] = \det^{1/2} \left| \Delta - \frac{1}{2} T^2 e^{2\lambda} - \frac{i}{2} e^{-2\lambda} \Delta_0 \right|_{\chi=0},$$

(4.6)

up to operator ordering ambiguities coming from the dependence of $\chi$ on the canonical momentum $\tau$. Note that unlike the more familiar Faddeev-Popov determinants of gauge theory, $\nu[\lambda, \tau]$ is an operator, depending explicitly on the Laplacian $\Delta_0$ on moduli space.

Finding the gauge-fixed form of the inner product (4.2) is now straightforward: it is easy to check that

$$\langle \Psi | \Phi \rangle = \int \frac{d^2 \tau}{\tau_2} \hat{\psi}^*(T, \tau) \hat{\phi}(T, \tau)$$

(4.7)

with

$$\hat{\psi}(T, \tau) = \int [d\lambda] \nu[\lambda, \tau] e^{iT\int d^2x e^{2\lambda} \Psi[\lambda, \tau]}.$$  

(4.8)

The wave function $\hat{\psi}(T, \tau)$ is almost identical to the naive Schrödinger wave function $\tilde{\psi}(T, \tau)$ of the previous section, differing only by the presence of the determinant $\nu[\lambda, \tau]$. Now, however, the inner product (4.7) is guaranteed to be conserved: the parameter $T$ merely labels an arbitrary choice in the gauge-fixing of the original path integral (1.2), and the final result must be independent of any such choice.

It is natural to ask how the the added factor $\nu[\lambda, \tau]$ affects the arguments of the previous section. Unfortunately, this is a hard question: equation (4.6) is complicated enough that I have been unable to find a useful explicit expression for the determinant. Observe, for instance, that the nonlocal terms in the Wheeler-DeWitt equation (2.6) can no longer be neglected. Moreover, $\nu[\lambda, \tau]$ depends on the modulus $\tau$, and does not commute with the Laplacian $\Delta_0$ in the Hamiltonian constraint; to obtain the analog of the would-be Klein-Gordon equation (3.6), we would have to know this dependence explicitly.

Nevertheless, we can at least obtain some qualitative information about the determinant $\nu[\lambda, \tau]$ without too much difficulty. Recall that the naive Schrödinger wave function $\psi(T, \tau)$ failed to obey a simple Klein-Gordon equation because the Wheeler-DeWitt functional $\Psi[\lambda, \tau]$ had support away from solutions of the classical constraints. It is therefore interesting to note that $\nu[\lambda, \tau]$ is itself peaked around solutions of the constraints. Indeed, let us write

$$D = -\Delta + \frac{1}{2} T^2 e^{2\lambda} + \frac{1}{2} e^{-2\lambda} \Delta_0 = D_0 + (V - V_0)$$

(4.9)
with
\[ V = \frac{1}{2} T^2 e^{2\lambda} + \frac{1}{2} e^{-2\lambda} \Delta_0, \quad V_0 = \int d^2x \, V, \quad D_0 = -\Delta + V_0. \] (4.10)

Note that \( V_0 \) is always positive, so \( D_0 \) is invertible. It is not hard to show that
\[ \nu[\lambda, \tau] = \det^{1/2} |D| = \det^{1/2} |D_0| \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n+1}{2n} \Tr \left[ D_0^{-1} (V - V_0) \right]^n \right\}. \] (4.11)

But \( D_0 \) has eigenfunctions and eigenvalues
\[ |mn\rangle = e^{2\pi i (mx + ny)}, \quad \ell_{mn} = \frac{4\pi^2}{\tau_2 \Delta_0} |n - m\tau|^2 + V_0, \] (4.12)

so
\[ \nu[\lambda, \tau] = \det^{1/2} |D_0| \exp \left\{ -\frac{1}{4} \sum \ell_{mn} \ell_{pq} \langle mn | V - V_0 | pq \rangle^2 + \cdots \right\}, \] (4.13)

which is clearly peaked at \( \langle mn | V - V_0 | pq \rangle = 0 \), i.e., at spatially constant values of \( \lambda \). Moreover, for \( \lambda \) constant, the derivative
\[ \frac{\partial}{\partial \lambda} \det^{1/2} |D_0| = \left( T^2 e^{2\lambda} - e^{-2\lambda} \Delta_0 \right) \cdot \frac{\partial}{\partial V_0} \det^{1/2} |D_0| \] (4.14)

vanishes precisely at the solutions (2.1) of the Hamiltonian constraint, now interpreted as an operator equation on moduli space. One can further check that in any regularization in which \( \Tr D_0^{-1} \) is positive, this extremum is in fact a maximum.

Woodard’s proposal thus leads to wave functions at fixed \( \Tr K \) that receive their main contribution from configurations that nearly satisfy the classical constraints. In particular, we see from (4.13) that deviations from a constant scale factor are exponentially suppressed. Note that for \( \lambda \) near its classical value, \( V_0 \sim T \Delta_0^{1/2} \), so the eigenvalues \( \ell_{mn} \) will be smallest — and the suppression strongest — at small values of \( T \), i.e., late times. The analog of the would-be Klein-Gordon equation (3.6) for Woodard’s wave functions should therefore be closer to the true Klein-Gordon equation of reduced phase space quantization, especially at late times.

5. Next Steps

Can the results of the last section be made more precise? Perhaps. The two key problems are to understand the determinant \( \nu[\lambda, \tau] \) and the operator ordering in the Hamiltonian constraint \( \mathcal{H} \). For any fixed \( \lambda \), \( \nu[\lambda, \tau] \) is a modular invariant function on the torus moduli

\[ \text{In an expanding (2+1)-dimensional universe, the value of } \Tr K \text{ on a constant mean curvature slice decreases from infinity at the initial singularity to zero in the far future.} \]
space, and as such can be expanded in terms of eigenfunctions of the Laplacian $\Delta_0$. For instance, it is not hard to show that the zeta function for the operator $D_0$ is

$$\zeta_{D_0}(s) = V_0^{-s} + (4\pi^2)^{-s} \sum_{k=0}^{\infty} \frac{\Gamma(s + k)}{\Gamma(s)\Gamma(k + 1)} \left(-\frac{V_0}{4\pi^2}\right)^k E(\tau, s + k),$$

(5.1)

where $E(s, k)$ is the Eisenstein series \[13, 14\]

$$E(\tau, s) = \sum_{m,n \in \mathbb{Z}}^{\prime} \frac{\tau_2^s}{|m + \tau n|^s}. \quad (5.2)$$

(The prime means that the point $m=n=0$ is omitted from the sum.) Such series have been studied quite extensively, and equation (5.2) may yield some useful information. Elizalde has pointed out that the sum may be further simplified by an analog of the Chowla-Selberg relation \[15\]. Similarly, the exponent in (4.11) can be expressed in terms of another Eisenstein series \[14\],

$$E_{x,y}(\tau, s) = \sum_{m,n \in \mathbb{Z}}^{\prime} e^{2\pi i(mx+ny)} \frac{\tau_2^s}{|m + \tau n|^s}. \quad (5.3)$$

These quantities are complicated, but perhaps not completely intractable.

As for operator ordering, one possible approach is to search for an ordering such that $[\mathcal{H}(x), \mathcal{H}(x')] = 0$. This is a highly nontrivial task, however, even without the nonlocal term in (2.6). For example, the commutator of $\Delta_0$ and $\Delta\lambda$ is a complicated expression involving single derivatives with respect to the modulus $\tau$, and there seems to be no simple ordering which eliminates these terms. Once again, the Wheeler-DeWitt equation proves to be more complex than one might have anticipated from other approaches to (2+1)-dimensional gravity.

Using this model to investigate other interpretations of the Wheeler-DeWitt equation would also be of considerable interest. Wald \[16\] has recently suggested an intriguing approach to defining a Hilbert space structure, based on the analogy between the Hamiltonian formulation of general relativity and that of the relativistic particle. This approach requires a solution of the Cauchy problem for the Wheeler-DeWitt equation (2.6), however, a task made quite difficult by the presence of nonlocal terms in the Hamiltonian constraint.

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