STOCHASTIC ACCELERATION OF LOW-ENERGY ELECTRONS IN COLD PLASMAS

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ABSTRACT

We investigate the possibility of stochastic acceleration of background low-energy electrons by turbulent plasma waves. We consider the resonant interaction of the charged particles with all branches of the transverse plasma waves propagating parallel to a uniform magnetic field. Numerical results and asymptotic analytic solutions valid at nonrelativistic and ultrarelativistic energies are obtained for the acceleration and scattering times of electrons. These times have a strong dependence on plasma parameter \( \alpha = \omega_{pl}/\Omega_p \) (the ratio of electron plasma frequency to electron gyrofrequency) and on the spectral index of plasma turbulence. It is shown that particles with energies above a certain critical value may interact with higher frequency electromagnetic plasma waves, and this interaction is allowed only in plasmas with \( \alpha < 1 \). We show that for nonrelativistic and semirelativistic electrons in low-\( \alpha \) plasmas, the ratio of the acceleration time to the scattering time can be less than unity for a wide range of energies. From this we conclude that the transport equation derived for cosmic rays that requires this ratio to be much larger than unity is not applicable at these energies. An approximate “critical” value of particle energy above which the dynamics of charged particles may be described by this transport equation is determined as a function of plasma parameters. We propose new transport equation for the opposite limit (energies less than this critical value) when the acceleration rate is much faster than the pitch angle scattering rate. This equation is needed to describe the electron dynamics in plasmas with \( \alpha \lesssim 0.1 \).

Subject headings: acceleration of particles — cosmic rays — plasmas — waves

1. INTRODUCTION

The acceleration and propagation of charged particles in magnetized plasmas via their stochastic interaction with turbulent plasma waves is a problem of wide interest in different astrophysical areas. The interaction of the particles with plasma turbulence causes diffusion of the energetic particles in phase space and leads to stochastic acceleration, which is a second-order Fermi process. The rates of these interactions are controlled by the magnetic field \( B_0 \) and other physical properties of the background plasma such as the background particle density \( n \), the energy density and spectrum of the turbulence, and, if in thermal equilibrium, the background temperature \( T \).

The influence of different modes of plasma waves on the dynamics of high-energy charged particles has been discussed in the literature for different plasma conditions. Whistler waves for electrons in an electron-proton background plasma for solar flare conditions have been considered by Steinacker & Miller (1992), Miller (1991), Miller & Ramaty (1987), Hamilton & Petrosian (1992), and Achatz et al. (1993) for interplanetary conditions. Benz (1977) and Melrose (1980) consider Langmuir (plasma) waves, Achterberg (1981) and de la Beaujardiere & Zweiles (1989) deal with magnetosonic waves, while Benz & Smith (1987) describe interaction with lower hybrid waves. The transport equation for transverse Alfvén waves in a cold background medium has been considered by Schlickeiser (1989), and this consideration have been extended for electromagnetic branch in general astrophysical plasmas by Dung & Petrosian (1994, hereafter DP). Most of these and other works investigating particle acceleration deal with relativistic particles and assume initial source of nonthermal high-energy particles.

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We are interested in the investigation of acceleration of the particles present in the background. These particles usually have a Maxwellian distribution with \( kT \ll mc^2 \) and therefore have low energies. The general problem is formulated in DP, and some results are given for high and intermediate energies. In this paper we continue this investigation of interaction with all modes of transverse plasma waves propagating parallel and/or antiparallel to the ordered magnetic field with emphasize on low-energy electrons. In § 2 we give an overview of the basic equations of turbulent plasma theory used in this paper and define a ratio of scattering to acceleration times in terms of Fokker-Planck coefficients, which are discussed in § 3. In § 4 we evaluate this ratio for different plasmas and particles of different energies and suggest a new transport scenario for low-energy particles. In § 5 and § 6 we derive approximate analytic expressions for the above ratio and the acceleration time for nonrelativistic and extremely relativistic electrons, respectively. A brief summary and our conclusions are presented in § 7.

2. BASIC EQUATIONS

We consider the behavior of the energetic charged particles in a background plasma with uniform magnetic field of strength \( B \) and superposed plasma waves. The gyrophase-averaged phase-space density \( f(z, t, p, \mu) \) then obeys the differential Fokker-Planck equation (Schlickeiser 1989):

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} = \frac{1}{p^2} \frac{\partial}{\partial p} p^2 \left[ D_{pp} \frac{\partial}{\partial p} + D_{pp} \frac{\partial}{\partial \mu} \right] f + \frac{\partial}{\partial \mu} \left[ D_{pp} \frac{\partial}{\partial \mu} + D_{pp} \frac{\partial}{\partial \mu} \right] f + S .
\]

(1)

Here \( z \) is the distance along the field lines, \( S \) is the source function, \( \mu \) denotes the cosine of the pitch angle, \( v = p/(mv) \)
is the velocity, and \( \gamma = [1 + p^2/(mc)^2]^{1/2} \) is the Lorentz factor of the particle with momentum \( p \). The interaction of the particles with plasma waves is described by the three Fokker-Planck coefficients \( D_{\mu\mu}^{-1}, D_{\mu p}^{-1} = D_{p\mu}^{-1}, \) and \( D_{pp}^{-1} \), which depend on the properties of the turbulence.

Solving the differential equation (1) in general is not simple, so one usually considers the solution for certain physical conditions when it can be simplified. In cases when the pitch angle scattering timescale \( \tau_{sc}(\mu) \simeq D_{\mu\mu}^{-1} \) is much shorter than the traverse time \( \tau_{tr} \approx L/v \), where \( L \) is the size of the turbulent plasma region, the pitch angle distribution will be nearly isotropic so that the anisotropic part of the phase-space density \( g(z, t, p, \mu) = f(z, t, p, \mu) - F(z, t, p) \) is much smaller.

For cases when this equation is inapplicable. Here we will deal only with waves propagating parallel or antiparallel to the magnetic field. The influence of transverse plasma waves propagating perpendicular to the magnetic field on the dynamics of charged particles will be considered in a subsequent paper.

\[ F(z, t, p) = \frac{1}{2} \int_{-1}^{+1} d\mu f(z, t, p, \mu). \]  

In addition, if the ratios
\[
R_1(\mu, p) = \frac{D_{pp}/p^2}{D_{\mu\mu}} \ll 1 \quad \text{and} \quad R_2(\mu, p) = \frac{D_{pp}/p}{D_{\mu\mu}} \ll 1 ,
\]  

then equation (1) can be reduced to the diffusion-convection equation, which is also known as the transport equation for the cosmic rays (Jokipii 1966; Kirk, Schlickeiser, & Schneider 1988; Schlickeiser 1989; DP):

\[
\frac{\partial F}{\partial t} + \frac{\partial F}{\partial z} = \frac{\partial}{\partial z} \left( \frac{p\mu}{v} \right) \frac{\partial F}{\partial \mu} + \frac{1}{p} \frac{\partial}{\partial p} \left( \frac{p^2 \partial F}{\partial \mu} \right) + Q(z, t, p) .
\]  

Here \( Q(z, t, p) \) is the pitch-angle–averaged source term, and the three transport coefficients are defined as
\[
\kappa_1 = \frac{v^2}{8} \int_{-1}^{+1} d\mu \frac{(1 - \mu)^2}{D_{\mu\mu}} , \quad \kappa_2 = \frac{1}{4} \int_{-1}^{+1} d\mu (1 - \mu^2) R_2 , \quad \kappa_3 = \frac{1}{2} \int_{-1}^{+1} d\mu D_{\mu\mu}(R_1 - R_2) .
\]  

The first four terms of the right-hand side of equation (4) represent spatial diffusion, adiabatic acceleration/deceleration, spatial convection, and momentum diffusion. If the above equations are satisfied we can define averaged scattering, acceleration, and spatial diffusion times as \( \tau_{sc} = 8\kappa_1/v^2, \tau_{ac} = 1/\kappa_3, \tau_{diff} = L^2/\kappa_1 = 8\tau_{sc}/v \), respectively.

The assumptions that lead to transport equation (4) are not always valid. Although the condition \( \tau_{sc} \ll \tau_{tr} \) holds for a wide range of particle energy in most astrophysical plasmas, the requirements in equation (3) are not always satisfied. As shown below, these ratios can exceed unity for low-energy background particles.

The goal of this paper is to investigate the range of validity of equation (4) and suggest a different transport scenario for cases when this equation is inapplicable. Here we will deal only with waves propagating parallel or antiparallel to the magnetic field. The influence of transverse plasma waves propagating perpendicular to the magnetic field on the dynamics of charged particles will be considered in a subsequent paper.

3. FOKKER–PLANCK COEFFICIENTS

Following DP (see also Kennel & Engelmann 1966; Lerche 1968) and assuming a power-law distribution of plasma turbulent energy density as a function of wave vector, \( \delta(k) = (q - 1)\delta_{tot} K_{min}^{-q} K^{-q} \) (for \( K \geq K_{min} \) and \( q > 1 \)), it can be shown that the Fokker-Planck coefficients necessary for evaluation of the ratios in equation (3) can be written as
\[
D_{\mu\mu}^{-1} = \frac{1}{\tau_{tr}^2} \left( 1 - \mu^2 \right) \sum_{j=1}^{N} \left[ 1 - \mu \frac{\beta_{ph}(k_j)}{\beta} \right] \chi(k_j) ,
\]  
\[
D_{pp}^{-1} = \frac{1}{\tau_{tr}^2} \left( 1 - \mu^2 \right) \sum_{j=1}^{N} \frac{\beta_{ph}(k_j)}{\beta} \left( 1 - \mu \frac{\beta_{ph}(k_j)}{\beta} \right) \chi(k_j) ,
\]  
\[
D_{pp}^{-1} = \frac{1}{\tau_{tr}^2} \left( 1 - \mu^2 \right) \sum_{j=1}^{N} \left( \frac{\beta_{ph}(k_j)}{\beta} \right)^2 \chi(k_j) ,
\]  

where
\[
\chi(k_j) = \frac{|k_j|^{-q}}{|\beta - \beta_{ph}(k_j)|} .
\]  

\[ \beta = v/c, \text{ and the dimensionless wavevector } k_j \text{ is one of the roots (maximum of four) of the resonant condition} \]
\[ \omega(k_j) - \mu k_j + 1/\gamma = 0 , \quad k_j = K_j/c/\Omega_e . \]

Here and in what follows the upper and lower signs refer to the right-hand and left-hand polarized plasma modes. The wave frequency \( \Omega_e \) in units of electron gyrofrequency \( \Omega_e = \omega(k_j)/\Omega_e \) is determined from the dispersion relation (see, e.g., Sturrock 1994)
\[
\frac{k^2}{\omega^2} = 1 - \frac{\alpha^2(1 + \delta)}{(\omega + 1)(\omega + \delta)} , \quad \omega(k_j) = \Omega_e \frac{k_j}{\Omega_e} ,
\]  

where \( \delta = m_i/m_e \) is the ratio of electron to proton masses and
\[
\alpha = \omega_{pe}/\Omega_e = 3.2(n_e/10^{10} \text{ cm}^{-3})^{1/2}(B/100 \text{ G})^{-1} \]

is the ratio of electron plasma frequency to gyrofrequency, which is simply related to the Alfvén velocity \( \beta_A \) expressed in units of speed of light as
\[
\beta_A = \sqrt{\frac{\delta}{\alpha}} .
\]  

The phase and group velocities of these waves (in units of speed of light) \( \beta_{ph}(k_j) = \omega_j/k_j \) and \( \beta_{gr}(k_j) = d\omega_j/dk_j \) respectively, can be obtained from relations (12) and (13). The parameter \( \tau_p \), which is a typical timescale in the turbulent plasma, is defined as
\[
\tau_p^{-1} = \frac{\pi}{2} \Omega_e \left[ \frac{\delta_{tot}}{B^2/(8\pi)} \right] (q - 1) k_{min}^{-q} .
\]  

The parameters important for our problem are \( \Omega_e, \alpha, q, k_{min} \) and the ratio of plasma turbulent density to magnetic energy density, \( f_{turb} = (8\pi\delta_{tot}/B^3) \). The above equations hold for both electrons and protons. In what follows, we consider only interaction and acceleration of electrons.

3.1. Critical Angles

As described in DP, in general, four values of \( k_j \) contribute to the Fokker-Planck coefficients, except for \( \mu = 0 \) and \( \gamma < \delta^{-1} \) and for the pitch angles between the two critical
values when only two values of \( k_j \) are involved. For non-relativistic electrons, one of these roots is due to resonant interaction with ion-cyclotron waves with \( k_j \gg 1 \), so that its contribution to the coefficients is negligible \([\chi(k_j) \ll 1] \). The main contribution to the Fokker-Planck coefficients then comes from the electron's interaction with whistler mode and higher frequency electromagnetic branch. The last interaction, allowed only in low-\( \alpha \) plasmas (\( \alpha < 1 \)), occurs for pitch angles greater than some critical angle \( \mu_{e\text{cr}} \) at which the group velocity of electromagnetic wave becomes equal to the component of the electron velocity along the magnetic field, \( \beta_{e\text{cr}} = \mu_{e\text{cr}} \). Similarly, there exists another critical angle \( \mu_{e\text{cr}} > \mu_{e\text{cr}} \) for interaction of electrons with electron-cyclotron waves \([\text{the high-} k \text{ end of the branch, which is commonly referred to as whistler branch at } \omega(k) < 1] \). It is well known that for these angles the quasilinear approximation breaks down and the Fokker-Planck coefficients become infinite. This has a minor consequence for the general process considered here (Steinacker & Miller 1992; DP). However, in order to clarify some of the behaviors of the coefficients, we give a brief description of these angles.

The dependence of these two critical angles on electron velocity \( \beta \) and plasma parameter \( \alpha \) are shown in Figure 1. The upper and lower curves correspond to \( \mu_{e\text{cr}} \) and \( \mu_{e\text{cr}} \), respectively. Thus, for \( 0 < \mu < \mu_{e\text{cr}} \), there are two roots coming from the interaction with forward (two roots) and backward (one root) moving electron-cyclotron waves. Similarly, for \( 1 > \mu > \mu_{e\text{cr}} \), there are two roots, two from the electromagnetic branch and one from the backward moving electron-cyclotron branch. But for \( \mu_{e\text{cr}} > \mu > \mu_{e\text{cr}} \), only the later root exists. The fourth and unimportant root from the (backward) ion-cyclotron branch mentioned above is common to all three cases. Same situation holds for negative values of \( \mu \) with the role of the forward and backward waves reversed.

For each value of \( \alpha \) there exist a critical velocity \( \beta_{e\text{cr}} \) or kinetic energy \( E_{e\text{cr}} \) below which \( \mu_{e\text{cr}} = 1 \) and interaction of electrons with electromagnetic branch is not allowed. Figure 2 shows the variation of \( E_{e\text{cr}} \) with the plasma parameter \( \alpha \). For high values of \( \alpha \) (high plasma density, low magnetic field) \( \beta_{e\text{cr}} \rightarrow 1 \). In the opposite case of \( \alpha \rightarrow 0 \) (very high magnetic field and/or low plasma density), the two curves merge to the solid line on Figure 1, which is described by the expression \( \mu_{e\text{cr}} = [1 - (1 - \beta^2)^{1/2}]/\beta \).

The effect of this behavior can be seen on Figure 3, where we show the dependence of one of the Fokker-Planck coefficients, \( D_{pp} \), on electron pitch angle in a plasma with \( \alpha = 0.2 \). As we approach one of the critical values of the pitch angle \( \mu_{e\text{cr}} \), the coefficients become infinite, as indicated by the sharp cusps. For each energy, the first and second peaks occur at \( \mu_{e\text{cr}} \) and \( \mu_{e\text{cr}} \), respectively.

4. RATIOS \( R_1 \) AND \( R_2 \)

Using equations (8)–(10), the ratios defined in equation (3) become

\[
R_1 = \frac{\sum_{j=1}^{N} \beta_{ph}(k_j) \chi(k_j)}{\sum_{j=1}^{N} (\beta - \mu \beta_{ph}(k_j))^2 \chi(k_j)},
\]

\[
R_2 = \frac{\sum_{j=1}^{N} \beta_{ph}(k_j) (\beta - \mu \beta_{ph}(k_j))^2 \chi(k_j)}{\sum_{j=1}^{N} (\beta - \mu \beta_{ph}(k_j))^4 \chi(k_j),}
\]  \( (17) \)

which depend on the electron pitch angle and velocity, and on the plasma parameters \( \alpha \) and \( q \). Often, in particular for...
\[ \mu_{\text{cr}} > \mu > \mu_{\text{cr}}^\infty, \] only one root has significant contribution. In this case the above expressions simplify to \( R_1 = R_2^2 = \omega^2/(\beta k - \mu \omega)^2 \).

Since the derivation of transport equation (4) requires these ratios to be much less than 1, we want to determine the plasma conditions and energy range for which this requirement is satisfied. It is well known that relativistic electrons (and all protons) interact mainly with Alfvén waves. In this case (see also § 6 below) \( R_1 = (\beta_\alpha/\beta)^2 = 1 \) if \( \beta_\alpha < 1 \) for relativistic electrons (or \( \beta < \beta_\alpha \) for protons). This condition tends to breakdown as we go to the lower energies, where the interaction of the electrons with other plasma waves become more significant than the interaction with Alfvén and ion-cyclotron waves.

### 4.1. Electrons with 90° Pitch Angles

The above equations can be solved analytically for \( \mu = 0 \) when the details of the distribution \( \delta(k) \) are not important. For this case the resonant condition (12) is simplified to \( \omega(k) = \pm \gamma/\gamma \) and the ratio \( R_1 = \beta_\alpha^2 \langle k \rangle / \beta^2 \). The ratio \( R_2 = 0 \) because \( D_{pp} = 0 \). Substitution of this in equation (13) gives four symmetric roots (two from the Alfvén branch and two from the whistler branch), which differ only in their signs of \( \omega_j \) or \( k_j \). We then have the analytic relation

\[ R_1 = \frac{\gamma^2(1 + \gamma \delta)}{(1 + \gamma)(\gamma^2 \alpha^2 + \gamma - 1)}, \quad R_2 = 0 . \]  

In the extreme relativistic regime \( \gamma \gg \delta^{-1}, R_1 = \delta/(\gamma^2 + \delta) \) and for \( 1 \ll \gamma \leq \delta^{-1}, R_1 = \gamma^{-1}(\gamma^2 + \delta)^{-1} \ll 1 \), but for non-relativistic electrons \( R_1 \approx 1/2\alpha^2 \), which can exceed unity if \( \alpha^2 < 1/2 \). Figure 4 shows the relation between the electron kinetic energy and the plasma parameter for five different values of this ratio. As evident, for small \( \alpha \) the condition (3) required for validity of equation (4) is violated up to very high energies (note that for \( \alpha \leq \delta_1 = 0.024 \) this is true at all energies). For example, for plasmas with \( \alpha \gg 0.1 \) the diffusion approximation becomes valid only for electrons with kinetic energy exceeding few MeV. This means that for low-energy electrons in low-\( \alpha \) plasmas we cannot use equation (4), and we must revert back to the original Fokker-Planck equation (1).

In the opposite case of \( R_1 \gg 1 \) (low energies, \( \alpha^2 \ll 1 \)) the scattering time becomes much greater than the acceleration time, so that the electron pitch angle will not change significantly in times on the order of the acceleration timescale. We can, therefore, neglect the small \( D_{pp} \) and \( D_{pp} \) terms in equation (1) and obtain the simple diffusion equation

\[ \frac{\partial f}{\partial t} = \frac{1}{\mu^2} \frac{\partial}{\partial p} p^2 D_{pp} \frac{\partial f}{\partial p} + S, \quad \mu = 0 . \]  

Thus for this special case of \( \mu = 0 \) (i.e., 90° pitch angle) the Fokker-Planck equation reduces to the pure diffusion equation in momentum (or energy) space. From inspection of Figure 4 we can see that this equation must be used up to several hundreds of keV for \( \alpha \leq 0.3 \).

### 4.2. Electrons with \( \mu \neq 0 \)

The situation is more complicated for electrons with \( \mu \neq 0 \) because the resonant condition (12) is no longer simple and there are multiple roots \( k_j \) contributing in the summations in equations (17). Analytic approximation for the ratios is not possible in this case, and we have to evaluate these ratios numerically. Figures 5–8 show variation of \( R_1 \) with energy for different values of \( \mu \) in plasmas with four different values of \( \alpha \). As evident, these ratios vary consider-

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**Fig. 4.** The dependence of the electron kinetic energy \( E \) on plasma parameter \( \alpha \) for five different ratios \( R_1 \), for electrons with pitch angle \( \mu = 0 \).

**Fig. 5.** The dependence of the ratio \( R_1 \) on electron energy for four different values of pitch angle in the plasma with parameter \( \alpha = 0.1 \) and spectral index \( q = 1.6 \).

**Fig. 6.** Same as Fig. 5, with \( \alpha = 0.1 \) and \( q = 3 \).
ably with \( \mu \), but in general they tend to increase with decreasing energy. Furthermore, they depend on the plasma parameter \( \alpha \) and the spectral index \( q \). The ratios \( R_1 \) and \( R_2 \) tend to increase with decreasing \( \alpha \) (high magnetic field, low plasma density), but they are not very sensitive to \( q \) (cf. Figs. 5 and 6). The discontinuous behavior of the curves arises from the existence of the two critical pitch angles discussed in § 3.1. For given \( \alpha \) and \( \mu \) there exists a maximum of three and a minimum of one energy for which this value of \( \mu \) is critical and leads to a discontinuity (except \( \mu = 0 \), which is not critical for any energy). Unlike the Fokker-Planck coefficients, their ratios do not become infinite at critical values of \( \mu \) or energy. In § 5 we will derive the asymptotic expression for the ratios \( R_1 \) and \( R_2 \) as a function of \( \alpha \) for low-energy electrons.

Following the same argument used in the case of zero pitch angle electrons, we can conclude that for low-energy electrons the distribution function can be obtained from the solution of the following equation:

\[
\frac{\partial f^\mu}{\partial t} + v\mu \frac{\partial f^\mu}{\partial z} = \frac{1}{p^2} \frac{\partial}{\partial \varphi} p^2 D_{\varphi \varphi} \frac{\partial f^\mu}{\partial \varphi} + S^\mu,
\]

where \( D_{\varphi \varphi}^\mu \) is given by equation (10) for each \( \mu \). The advective term (second on the left-hand side) can also be neglected if the traverse time \( L/v \) is much larger than the acceleration time \( p^2/D_{\varphi \varphi}^\mu \) and if \( D_{\varphi \varphi}^\mu \) and the source term \( S^\mu \) are independent of \( z \) (homogeneous plasma). Equation (20) then becomes identical to equation (19) for zero-pitch-angle electrons. Once the electron achieve higher energies and moves above the line \( R_1 = 1 \), so that the conditions (3) are satisfied, then we revert to equation (4). In the transition region neither equation (4) nor equation (20) is valid, and the resulting equation for isotropic pitch angle distribution is almost as complicated as the original equation (1).

5. NONRELATIVISTIC APPROXIMATIONS

For low-energy electrons, \( \gamma \rightarrow 1 + \beta^2/2 \), and the resonance condition (12) combined with dispersion relation (13) gives

\[
\mu \beta k^3 - (1 + 2\mu^2) \beta^2 k^2 - \mu \beta (1 - \alpha^2) k + \alpha^2 = 0,
\]

where we have assumed \( \alpha^2 \gg \beta \).

It can be shown then that \( \mu_{\text{crit}}^\alpha = \alpha \beta^2/6(3)^{1/2} \ll 1 \) and \( \mu_{\text{crit}}^\alpha \rightarrow 1 \) so that the single root solution is applicable for almost all pitch angles with the root

\[
k_j \approx -\beta^{-1/3} \alpha^2 \mu^{-1/3} \mu > \mu_{\text{crit}}^\alpha.
\]

In the limit of \( \mu \rightarrow 0 \) equation (22) has three roots one of which is very large \((k \rightarrow \infty)\) compare to the other two roots, which are

\[
k_j \approx \pm \frac{2\alpha}{\beta}, \quad \mu < \mu_{\text{crit}}^\alpha.
\]

This expression is nearly identical to the value of \( k_j \) obtained from equation (22) at \( \mu = \mu_{\text{crit}}^\alpha \), so that joining these two relations at \( \mu = \mu_{\text{crit}}^\alpha \) will give an approximate description for all \( \mu \).

Now substitution of these values for \( k_j \) in equation (12) gives the values of the resonant frequencies as

\[
\omega_j - 1 \approx \begin{cases} \sigma(\mu \beta)^{2/3} + \alpha(\beta^2), & \mu_{\text{crit}}^\alpha < \mu < 1, \\ \sigma(\beta^3), & \mu < \mu_{\text{crit}}^\alpha. \end{cases}
\]

Then to the first order the phase velocity of the wave is obtained from equations (22)–(24) to be \( \beta_{\text{ph}} = \omega_j/k_j \approx 1/k_j \).

Combining this with the dispersion relation (13) we obtain the group velocity

\[
\beta_{\text{gr}} = \frac{d\omega}{dk} \approx \frac{2\alpha^2}{k^3} \approx \frac{2\mu \beta + \alpha(\beta^2)}{\beta^3} \frac{\mu_{\text{crit}}^\alpha < \mu < 1}{\mu < \mu_{\text{crit}}^\alpha}.
\]

Using the above expressions we can now evaluate the Fokker-Planck coefficients and their ratios in the nonrelativistic limit. For the single root case these ratios are simplified to \( R_1 = R_{2,3} = \omega^2/(\beta k - \mu \omega)^2 \) and in the small range of \( \mu < \mu_{\text{crit}}^\alpha \) we can use the asymptotic relation of equation (18).

Substituting equations (23)–(25) into equation (17) gives the nonrelativistic values of these ratios as

\[
R_1 \approx \begin{cases} 1/2\alpha^2, & \mu < \mu_{\text{crit}}^\alpha \\ \mu^{2/3}, & \mu_{\text{crit}}^\alpha < \mu < 1 \end{cases}
\]

These values agree well with the numerical results shown in Figures 5–8 at low energies.

5.1. Acceleration Time

For nonrelativistic electrons and especially for \( \alpha < 1 \) the ratios \( R_1 \) and \( R_2 \) exceed unity, and the acceleration process...
is described by equation (20). We can therefore define an acceleration time \( \tau_a = p^2/D_{pp} \) (Note that this is different from \( \tau_{ac} = 1/\kappa_a \) defined for relativistic regime in combination with eq. [4]; see below.) Figures 9 and 10 show the dependence of this time (in units of \( \tau_p \)) on \( \mu \) at three different values of kinetic energy in a plasma with \( q = 5/3 \) and \( \alpha = 1 \) and \( \alpha = 0.2 \), respectively. For \( \alpha \geq 1 \) the critical pitch angle \( \alpha_{crt}^m = 1 \), and the electrons interact mainly with the whistler and electron-cyclotron waves and not with the electromagnetic branch. Over the wide range of \( \mu \) the resonant interaction occurs only with backward- (for \( \mu > 0 \)) or forward- (\( \mu < 0 \)) moving whistler waves. For \( \alpha = 0.2 \) we have \( \alpha_{crt}^e < 1 \) (Fig. 10), and there are three distinct regions on the plot with two discontinuous changes. Note that the acceleration time of electrons with pitch angles \( \mu > \alpha_{crt}^m \) becomes large for all energies because of the dependence \( D_{pp}(1 - \mu^2) \).

Using values of resonant wavevectors obtained in equation (8), we get the following expressions for the acceleration time:

\[
\tau_a(\mu) = \left\{ \begin{array}{ll}
\mu^{(1-q)/3} & \alpha \geq 1 \\
\frac{1 - \mu^2}{2(q+1/2)a_{cr} \beta^3 - \nu_{ac}} & \mu < \alpha_{crt}^m \leq \mu_{crt}^m \approx 1
\end{array} \right.
\]

(27)

To demonstrate the dependence of the acceleration time on electron energy alone we plot in Figure 11 the averaged acceleration time:

\[
\langle \tau_a \rangle = \frac{2p^2}{\int \langle 1 \rangle d\mu D_{pp}(\mu)}.
\]

(28)

From equations (27) we also obtain an approximate analytic expression

\[
\langle \tau_a \rangle \approx \frac{(2 + q)(8 + q)}{18} \mu^2 \mu_{crt}^e \beta^3 - \nu_{ac}^{(2-q+1/3) \beta^3 - \nu_{ac}}
\]

(29)

where we have ignored the small contribution from electrons with \( \mu > \mu_{crt}^m \) because in most cases \( \mu_{crt}^m \) is close to 1. This approximation agrees with 10% precision with numerical results shown on Figure 11 for values of \( \alpha \geq 0.3 \) and electron energies on the order of keV.

Simple analytic solutions are not possible for the plasma conditions when a significant fraction of electrons is able to interact with the electromagnetic branch. As can be seen from Figure 11, in the case of small \( \alpha \) the averaged acceleration time is a complicated function of energy and plasma parameters. In order to determine the highest energy for which the approximate analytic expression (29) is valid, one should refer to Figure 2. For example, for \( \alpha = 0.3 \) we obtain \( E_{ac} \approx 10 \) keV. This is in agreement with Figure 11, which shows the correct power-law dependence below 10 keV.

6. RELATIVISTIC APPROXIMATIONS

To complete our investigation of the Fokker-Planck equation (1) in application to the electrons in a turbulent plasma in this section we consider the case of the relativistic electrons for which \( R_1,2(\mu, p) \ll 1 \) and the transport equation (4) is applicable. For this regime the equations (5) and (7) define the average scattering time \( \tau_{ac} \) and, now the longer acceleration time \( \tau_{ac} \) as

\[
\tau_{ac} = \int_{-\infty}^{1} d\mu (1 - \mu^2)^2 / D_{pp} \, ,
\]

(30)

\[
\tau_{ac} = \frac{2}{\int_{-\infty}^{1} d\mu D_{pp} R_1 - R_2^2}.
\]

(31)

Figures 12 and 13 show the variation with energy of these times (in units of \( \tau_p \)) obtained from numerical integration of
the above equations for different values of the plasma parameter $x$ and spectral index $q$ of the plasma turbulence. The scattering time is similar to the scattering time $\tau_{sc} = 1/\langle D_{pp} \rangle$ one would define in the nonrelativistic regime. Thus, the curves in Figure 12 provide a good estimate of the scattering timescale at all energies. However, the acceleration time $\tau_{ac}$ in equation (31) is different from the low-energy definition $\tau_{ac} = p^2/D_{pp}$ used in § 5.1. These two definitions differ by the presence of the $R_q$ term in the denominator of equation (31). It turns out that for relativistic electrons in plasmas with small Alfvén velocity $R_q \approx R_1 < 1$, so that $R_q \ll R_1$, and the two equations give very similar results. This can also be seen from comparison of Figures 11 and 13. As we have shown in § 4, this condition is not true in the nonrelativistic regime, so that equation (31) when extended to low energies is in error by several orders of magnitude.

From Figures 11, 12, and 13 we see several distinct regions of energy with different behavior of the curves.

1. Extremely relativistic energies. $\gamma \gg \delta^{-1} \beta_x \ln \beta_x^{-1}$. The power-law energy dependence $\gamma^{2-q}$ of $\tau_{ac}$ and $\tau_{ac}$ seen in these figures at high energies is well known in the literature (e.g., Schlickeiser 1989; Schlickeiser, Dung, & Jaekel 1991). Relativistic electrons interact with Alfvén waves that have a simple dispersion relation $\omega = \beta_x k$. Recall that $\delta = m_e/m_i$ and $\beta = \delta^{1/2}/c$ is the Alfvén speed in units of speed of light, which we assume to be less than one. These electrons also interact with long-wavelength whistler waves that, to the first order in $\beta_x$, have the same dispersion relation as Alfvén waves. For both waves the resonant frequencies $\omega_R \ll \delta$.

Below we derive analytic expressions which show the above power-law dependence, as well as the dependence of $\tau_{ac}$ and $\tau_{ac}$ on the plasma parameters $x$ and $q$. Using the above simplified dispersion relation and the resonant condition (12) we get two resonant values for the frequency and the wavevector:

\[ k_1 \approx -\frac{1}{\gamma} \frac{1}{\mu + \beta_x}, \quad \omega_1 \approx -\frac{1}{\gamma} \frac{\beta_x}{\mu + \beta_x}, \]  

\[ k_2 \approx -\frac{1}{\gamma} \frac{1}{\mu - \beta_x}, \quad \omega_2 \approx -\frac{1}{\gamma} \frac{\beta_x}{\mu - \beta_x}. \]  

Note that the demarcation energy for this case comes from the condition $\omega_R \ll \delta$ and is different from the usually assumed limit $\gamma \gg \delta^{-1}$. Substitution of the above solutions in equations (30) and (31) gives the asymptotic expressions for scattering and acceleration times

\[ \frac{\tau_{ac}}{\tau_p} \approx \frac{1}{4} \gamma^{2-q} I_{ac}(\beta_x), \quad \frac{\tau_{ac}}{\tau_p} \approx -\frac{1}{16} \gamma^{2-q} \frac{1}{\beta_x^2 I_{ac}(\beta_x)}, \]  

where

\[ I_{ac}(\beta_x) = \int_0^1 \frac{(1 - \mu^2) d\mu}{|\mu + \beta_x|^1 + |\mu - \beta_x|^1}, \]  

and the power indices $s_{ac} = q - 1, s_{ac} = 1 - q$ for scattering and acceleration times, respectively.

One can show that in the case of small $\beta_x$ the function $I_{ac}(\beta_x)$ simplifies to

\[ I_{ac}(\beta_x) \approx \begin{cases} (2 \beta_x)^{1-s} \frac{1}{0} \frac{(1 - t)^{s-2} + (1 + t)^{s-2}}{1 + t^s} dt, & s > 1, \\ -\frac{1}{2} \ln \left( \frac{\beta_x}{4} \right), & s = 1, \\ \frac{1}{(1 - s)(3 - s)} + o(2 \beta_x)^{1-s}, & s < 1. \end{cases} \]  

Now we can use the above results in equations (34) to obtain the dependence of relativistic times on plasma parameters. We will consider spectral indices in the range $1 < q < 4$. In the case of $q > 2$, the scattering time for electrons of relativistic energies has a power-law dependence on plasma parameter $\tau_{ac} \propto x^{2-q}$. In the opposite case of $q < 2$, we have almost no dependence of scattering time on $x$. We can see these behavior in Figure 12 for $q = 3/2$ and $q = 3$.

To the highest order in $\beta_x$ and for all $q$ in the above range, $\tau_{ac} \propto x^{2-q}$. Thus, the relativistic acceleration time is proportional to $x^2$, and this dependence can be seen on Figure 13. The above results indicate that the ratio $R_1 \approx \tau_{ac}/\tau_{ac}$ has a strong dependence on $x$ and $q$, which leads to the fact that in low-$x$ plasmas with spectral index $q < 2$ this ratio is of order 1 even for relativistic electrons and equation (4) is not valid.

2. Intermediate regime. $1 \ll \gamma \ll \delta^{-1} \beta_x \ln \beta_x^{-1}$. As can be seen from Figures 12 and 13 at $\gamma \approx \delta^{-1} \beta_x \ln \beta_x^{-1}$ the curves begin to deviate from the described power law because the
interaction with whistler mode with the higher wavevectors begin to dominate. In this case \( 1 \gg \omega_x \gg \delta \), and the simple expressions such as equations (32) and (34) are not possible, and one has to consider different combinations of plasma parameters in order to get analytic expressions for scattering and acceleration times correspond to that particular plasma conditions. Different cases of interaction of electrons with whistlers have been considered in several papers (e.g., Steinacker & Miller 1992; Hamilton & Petrosian 1992).

3. Nonrelativistic energies.—(\( \gamma - 1 \)) \( \ll 1 \). Last, for nonrelativistic electrons considered in § 5, the resonant frequency \( \omega_x \rightarrow 1 \) and electrons interact most effectively with short-wavelength whistlers. As we described in § 3.1, in plasmas with \( \alpha < 1 \), electrons also interact with lower frequency electromagnetic branch of plasma waves. We see the corresponding change of behavior of the curves in Figures 11 and 12 as we approach nonrelativistic energies. The interaction with electromagnetic waves is allowed only for electrons with energy greater than the critical energy \( E_{cr} \) shown in Figure 2. In this regime the dependence of both scattering and acceleration times on energy deviates significantly from a power law. For energies less than \( E_{cr} \), equation (28) becomes valid, and we again can see in Figure 11 the power-law dependence of the acceleration time on kinetic energy of electron.

7. SUMMARY

The importance of the stochastic acceleration of high-energy charged particles by turbulent plasma waves is well known. In this work we investigate the possibility of acceleration of the low-energy background (often thermal) electrons by this process. We use the well-known formalism developed over the years and especially the formalism proposed by Schlickeiser and DP. In this paper we consider interaction of electrons with plasma waves propagating along the magnetic field lines. At all energies the plasma parameters that determine the acceleration rate of the electrons are the value of the magnetic field, the energy density of the plasma turbulence, the spectral index \( q \) of the waves, and, most important, the plasma parameter \( \alpha \), which is equal to the ratio of plasma frequency to gyrofrequency of electrons. At low energies interaction with the electron cyclotron and electromagnetic branch of plasma waves become important. The second interaction is possible only when \( \alpha < 1 \) and above the critical energies shown in Figure 2. The electrons that do not have enough energy to interact with the electromagnetic waves may still be in resonance with whistler and/or electron-cyclotron modes.

From the above consideration, we show that the ratio of the pitch angle to momentum (or energy) diffusion rates or, alternatively, the ratio of the acceleration to scattering times (eq. [3]) varies strongly with the plasma parameters and the pitch angle and energy of the electron. We give an asymptotic analytic expression for these ratios showing their dependence on these parameters. In particular, we show that this ratio becomes greater than unity at lower energies, which indicates that the usual transport equation derived for cosmic rays is not applicable for electrons for a wide range of energies. An approximate “critical” value of electron energy above which the dynamics of the electrons may be described by this transport equation is determined as a function of plasma parameter. We propose a new transport equation for nonrelativistic electrons in low-\( \alpha \) plasmas.

We also show numerically and through asymptotic analytic expressions that when the above ratio is greater than unity, the acceleration timescale decreases with decreasing energy and could be very short for a magnetic field of 100 G and turbulent energy density of less than \( \sim 10^{-4} \) of that of magnetic field energy density. To complete the discussion we also consider the diffusion rates for relativistic electrons for which the above-mentioned ratio is less than one. We give analytic and numerical results on the acceleration and scattering timescales for different energies and plasma parameters.

Based on these results, we suggest the following scenario for acceleration of the background plasma electrons. The low-energy (background) electrons can be accelerated by whistler, high-frequency electromagnetic, or electron-cyclotron waves without a significant change in their pitch angle over timescale on the order of the acceleration time \( \tau_a = p^2/D_{\alpha}(\mu) \). This process holds until electrons reach an energy where the scattering time becomes comparable to the acceleration time. Then intensive scattering along with acceleration takes place, and for this stage we have to take into account all of the terms in the Fokker-Planck equation to describe the time evolution of the distribution function. When electrons accelerate to high enough energies (typically energies of tens of MeV), their scattering time becomes much less than all other timescales, the distribution function becomes nearly isotropic, and the well-known transport equation (4) becomes applicable. For the two limiting cases with simple transport equations analytic expressions for scattering and acceleration times as a function of energy, plasma parameter and turbulence spectral index can be used.

In this paper we have considered interaction of electrons with plasma waves propagating along the magnetic field lines, in a “cold” background plasma. In subsequent papers we will consider waves with oblique propagation angles and include the effects of the finite temperature of the background plasma.

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