Central limit theorem for functionals of a generalized self-similar Gaussian process

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Abstract

We consider a class of self-similar, continuous Gaussian processes that do not necessarily have stationary increments. We prove a version of the Breuer-Majó theorem for this class, that is, subject to conditions on the covariance function, a generic functional of the process increments converges in law to a Gaussian random variable. The proof is based on the Fourth Moment Theorem. We give examples of five non-stationary processes that satisfy these conditions.

1 Introduction

We consider a centered Gaussian process \( X = \{X_t, t \geq 0\} \) that is self-similar of order \( \beta \in (0, 1) \). That is, the process \( \{a^{-\beta}X_{at}, t \geq 0\} \) has the same distribution as the process \( X \) for any \( a > 0 \). Consider the function \( \phi : [1, \infty) \to \mathbb{R} \) given by

\[
\phi(x) = \mathbb{E}[X_1X_x].
\]

This function characterizes the covariance function. Indeed, for \( 0 < s \leq t \), we have

\[
R(s, t) = \mathbb{E}[X_sX_t] = s^{2\beta} \mathbb{E}[X_1X_{t/s}] = s^{2\beta} \phi\left(\frac{t}{s}\right).
\]

The best known self-similar Gaussian process is the fractional Brownian motion (fBm), where

\[
R(s, t) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}),
\]

and the self-similarity exponent \( \beta \) is the Hurst parameter \( H \in (0, 1) \). For the fBm,

\[
\phi(x) = \frac{1}{2} (1 + x^{2H} - (x - 1)^{2H}), \quad x \geq 1.
\]

We will impose conditions on the function \( \phi \) such that \( \mathbb{E}[(X_{t+s} - X_t)^2] \sim s^\alpha \) as \( s \to 0 \), for some constant \( \alpha \in (0, 2\beta) \) that we call the increment exponent. For example, \( \alpha = 2H \) for the fBm, since \( \mathbb{E}[(X_{t+s} - X_t)^2] = s^{2H} \). However, there are examples where \( \alpha < 2\beta \).

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Our goal in this paper is to identify a set of conditions on $\alpha, \beta$ and $\phi$, such that we can establish a central limit theorem for functionals of the increments of $X$. More precisely, let $\gamma = \mathcal{N}(0, 1)$ and consider a function $f \in L^2(\mathbb{R}, \gamma)$, which has an expansion of the form

$$f(x) = \sum_{q=d}^{\infty} c_q H_q(x),$$

where $d \geq 1$, $c_d \neq 0$, and $H_q(x)$ is the $q$th Hermite polynomial, defined as

$$H_q(x) = (-1)^q e^{x^2} \frac{d^q}{dx^q} e^{-x^2}.$$

The index $d$ is called the Hermite rank of $f$.

For integers $n \geq 2$ and $j \geq 0$ define

$$\Delta X_{n}^j = X_{j+1}^n - X_{j}^n \quad \text{and} \quad Y_{j,n} = \frac{\Delta X_{n}^j}{\|\Delta X_{n}^j\|_{L^2(\Omega)}}.$$

We consider the stochastic process defined by

$$F_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor - 1} f(Y_{j,n}), \quad [nt] \geq 1,$$

and $F_n(t) = 0$ if $[nt] < 1$. It is well known that if the process $X$ has stationary increments, the convergence to a normal law for the sequence of random variables $F_n(t)$ can be deduced from the following central limit theorem proved by Breuer and Major in [3].

**Theorem 1.1.** Suppose $\{Y_j, j \geq 1\}$ is a centered stationary Gaussian sequence with unit variance, and denote by $\rho(k) = \mathbb{E}(Y_n Y_{n+k})$ its covariance function. Consider a function $f \in L^2(\mathbb{R}, \gamma)$ with Hermite rank $d \geq 1$. Then the functional

$$V_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(Y_j)$$

converges in distribution to a normal law $\mathcal{N}(0, \sigma^2)$ as $n$ tends to infinity, provided $\sum_{k \in \mathbb{Z}} |\rho(k)|^d < \infty$, and in this case $\sigma^2 = \sum_{q=d}^{\infty} q! c_q^2 \sum_{k \in \mathbb{Z}} \rho(k)^q$.

For example, in the case of the fBm, the sequence $\{Y_{j,n}, 0 \leq j \leq n-1\}$ defined by

$$Y_{j,n} = n^H \left( B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H \right),$$

is stationary, and the Breuer-Major theorem implies that if $d \geq 2$ and $H < 1 - \frac{1}{2d}$, the sequence of random variables

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f \left( n^H \left( B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H \right) \right)$$

##
converges in law as \( n \) tends to infinity to a Gaussian random variable, with mean zero and variance given by

\[
\sigma^2 = \sum_{q=d}^{\infty} c_q^2 2^q \sum_{m \in \mathbb{Z}} (|m + 1|^{2H} - 2|m|^{2H} + |m - 1|^{2H})^q.
\]

(6)

See [3] and Theorem 7.4.1 of [13].

Our main result (see Theorem 3.4) says that if the covariance of \( X \) satisfies certain conditions and the increment exponent \( \alpha \) satisfies \( 0 < \alpha < 2 - \frac{1}{d} \), then the finite dimensional distributions of processes \( \{F_n(t), t \geq 0\} \) defined in [3] converge in law to those of a Brownian motion with scaling given by (6), where \( 2H \) is replaced by \( \alpha \). Notice that the sequence of scaled increments \( \{Y_{j,n}, j \geq 0\} \) is not necessarily stationary and we cannot deduce this result from the Breuer-Major theorem. On the other hand, the relevant parameter in the limit theorem is the increment exponent \( \alpha \), instead of the self-similarity parameter \( \beta \).

The convergence in law of the finite-dimensional distributions in Theorem 3.4 follows from the Fourth Moment Theorem [15, 16], which represents a drastic simplification of the method of moments to show the convergence to a normal distribution. In order to establish convergence, it is sufficient to show the convergence of the variances, and that a condition involving the contraction operator is satisfied. In this paper, we use extended versions of the Fourth Moment Theorem [9, 17].

For the particular case of a single Hermite polynomial, that is \( f = H_q \) for some \( q \geq 2 \), one can show the convergence in total variation of the marginal distributions and a functional central limit theorem (see Section 3.4).

We show examples of known processes that satisfy the required conditions, including:

(a) bifractional Brownian motion (see [8]),

(b) subfractional Brownian motion (see [2]),

(c) an ‘arcsine’ Gaussian process introduced in a paper by Jason Swanson [20], and

(d) two self-similar Gaussian processes that form the decomposition of a process discussed in a paper by Durieu and Wang [6].

In examples (a), (b) and (d), the self-similarity and increment exponents are the same, that is, \( \alpha = 2\beta \). This is not true in example (c), where \( \alpha < 2\beta \).

The outline of this paper is as follows. In Section 2, we give definitions and background needed to use the Fourth Moment Theorem. In Section 3, we introduce a set of covariance conditions on \( X \), and we state Theorem 3.4 which shows that the finite dimensional distributions of \( F_n \) converge in law when these conditions are met. Some particular applications of Theorem 3.4 are discussed. Section 4 discusses the examples (a) - (d) above, and Section 5 contains the proofs of two technical lemmas.

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2 Theoretical background

Following is a brief description of some identities that will be used. The reader may refer to \[13\] \[14\] for detailed coverage of this topic. Let \( Z = \{ Z(h), h \in \mathcal{H} \} \) be an isonormal Gaussian process on a probability space \((\Omega, \mathcal{F}, P)\), indexed by a real separable Hilbert space \( \mathcal{H} \). That is, \( Z \) is a family of Gaussian random variables such that \( \mathbb{E}[Z(h)] = 0 \) and \( \mathbb{E}[Z(h)Z(g)] = \langle h, g \rangle_{\mathcal{H}} \) for all \( h, g \in \mathcal{H} \). We will always assume that \( \mathcal{F} \) is the filtration generated by \( Z \).

For integers \( q \geq 1 \), let \( H^\otimes q \) denote the \( q \)th tensor product of \( H \), and let \( \mathcal{H}^\otimes q \) denote the subspace of symmetric elements of \( H^\otimes q \). Let \( \{ e_n, n \geq 1 \} \) be a complete orthonormal system in \( H \). For functions \( f, g \in \mathcal{H}^\otimes q \) and \( p \in \{ 1, \ldots, q \} \), we define the \( p \)th-order contraction of \( f \) and \( g \) as that element of \( H^\otimes 2(q-p) \) given by

\[
 f \otimes_p g = \sum_{i_1,\ldots,i_p=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{H^\otimes p} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{H^\otimes p},
\]

where \( f \otimes_0 g = f \otimes g \) by definition and, if \( f, g \in \mathcal{H}^\otimes q \), \( f \otimes_q g = \langle f, g \rangle_{H^\otimes q} \). In particular, if \( f, g \) are symmetric functions in \( H^\otimes 2 = L^2(\mathbb{R}^2, \mathcal{B}^2, \mu^2) \) for a measure \( \mu \), then we have

\[
 f \otimes_1 g = \int_{\mathbb{R}} f(s,t_1)g(s,t_2) \mu(ds).
\]

Let \( \mathcal{H}_q \) be the \( q \)th Wiener chaos of \( Z \), that is, the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{ H_q(Z(h)), h \in \mathcal{H}, \| h \|_\mathcal{H} = 1 \} \), where \( H_q(x) \) is the \( q \)th Hermite polynomial. It can be shown (see \[13\], Proposition 2.2.1) that if \( Z, Y \sim \mathcal{N}(0, 1) \) are jointly Gaussian, then

\[
 \mathbb{E} [ H_p(Z)H_q(Y) ] = \begin{cases} 
 p! (\mathbb{E} [ZY])^p & \text{if } p = q \\
 0 & \text{otherwise} 
\end{cases}.
\]

For \( q \geq 1 \), it is known that the map

\[
 I_q(h^\otimes q) = H_q(Z(h))
\]

provides a linear isometry between \( \mathcal{H}^\otimes q \) (equipped with the modified norm \( \sqrt{q!} \| \cdot \|_{\mathcal{H}^\otimes q} \)) and \( \mathcal{H}_q \). The random variable \( I_q(\cdot) \) is the generalized Wiener-Itô stochastic integral (see \[13\], Theorem 2.2.7). By convention, \( \mathcal{H}_0 = \mathbb{R} \) and \( I_0(x) = x \).

It is well known that \( L^2(\Omega) \) can be decomposed into an orthogonal sum of the spaces \( \mathcal{H}_q \). Hence, any \( F \in L^2(\Omega) \) has a Wiener chaos expansion

\[
 F = \sum_{q=0}^{\infty} I_q(f_q),
\]

where \( f_0 = \mathbb{E}[F] \) and the \( f_q \in \mathcal{H}^\otimes q, q \geq 1 \) are uniquely determined by \( F \) (see Theorem 1.1.2 of \[14\]).
The purpose of the above discussion is to provide sufficient background to use the Fourth Moment Theorem. This theorem, first published in 2005, has inspired an extensive body of literature, and provided solution techniques to a new class of problems. This first version of the theorem was proved in [16]. Since then, other equivalent conditions have been added [13, 15]. A key advantage of this theorem is that, unlike the method of moments, it is sufficient to check the convergence of the moments of up to order four.

**Theorem 2.1** (Fourth Moment Theorem). Fix an integer \( q \geq 2 \). For integers \( n \geq 1 \), let \( F_n = I_q(f_n) \) be a sequence of random variables belonging to the \( q \)th Wiener chaos of \( X \), so that \( f_n \in \mathcal{H}^q \). Assume that \( \mathbb{E}[F_n^2] \to \sigma^2 \geq 0 \) as \( n \to \infty \). Then the following are equivalent:

(a) As \( n \to \infty \), \( F_n \) converges in distribution to \( N(0, \sigma^2) \).

(b) \( \lim_{n \to \infty} \mathbb{E}[F_n^4] = 3\sigma^4 = \mathbb{E}[N^4] \).

(c) For each integer \( 1 \leq r \leq q - 1 \), \( \lim_{n \to \infty} \|f_n \otimes_r f_n\|_{\mathcal{H} \otimes 2(q - r)} = 0 \).

We have this multidimensional extension due to Peccati and Tudor:

**Theorem 2.2** ([17]). For an integer \( k \geq 1 \), let \( F_n = (F_n^1, \ldots, F_n^k) \) be a sequence of random vectors, and let \( 1 \leq q_1 \leq \cdots \leq q_k < \infty \) be integers such that for each \( j = 1, \ldots, k \), \( F_n^j = I_{q_j}(f_n^j) \) for some kernel \( f_n^j \in \mathcal{H}^{q_j} \). Moreover, assume that each \( \mathbb{E}[(F_n^j)^2] \to \sigma_j^2 \geq 0 \) as \( n \) tends to infinity, and that \( \lim_{n \to \infty} \mathbb{E}[F_n^j F_n^\ell] = 0 \) for all \( j \neq \ell \). Then the following are equivalent:

(a) \( F_n \) converges in distribution as \( n \to \infty \) to \( N(0, \Sigma) \), where \( \Sigma \) is the \( k \times k \) diagonal matrix with entries \( \sigma_1^2, \ldots, \sigma_k^2 \).

(b) For each \( j = 1, \ldots, k \), \( F_n^j \) converges in distribution as \( n \to \infty \) to \( N(0, \sigma_j^2) \).

For a general sequence of square integrable random variables, the convergence to the normal distribution can be deduced from the Wiener chaos expansion. Using Theorem 2.2, one can show that if every projection on the Wiener chaos satisfies the hypotheses of the Fourth Moment Theorem, then their limits are independent and a central limit theorem holds for the global sequence. This phenomenon can be described as a chaotic central limit theorem.

**Theorem 2.3** ([9]). Let \( \{F_n\} \) be a sequence in \( L_2(\Omega) \) such that \( \mathbb{E}[F_n] = 0 \) for all \( n \). Write each \( F_n \) in the form

\[
F_n = \sum_{q=1}^{\infty} I_q(f_n,q),
\]

and suppose that the following conditions hold:

(a) For each \( q \geq 1 \), \( \lim_{n \to \infty} q! \|f_n,q\|_{\mathcal{H}^{\otimes q}}^2 = \sigma_q^2 \), for some \( \sigma_q^2 \geq 0 \).

(b) \( \sigma^2 = \sum_{q=1}^{\infty} \sigma_q^2 < \infty \).

(c) For each \( q \geq 2 \) and \( r = 1, \ldots, q - 1 \), \( \lim_{n \to \infty} \|f_n,q \otimes_r f_n,q\|_{\mathcal{H} \otimes 2(q - r)} = 0 \).
(d) \( \lim_{N \to \infty} \sup_{n \geq 1} \left( \sum_{q=N+1}^{\infty} q! \| f_{n,q} \|_{\mathcal{H}^q}^2 \right) = 0. \)

Then as \( n \) tends to infinity, \( F_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2) \).

In the following sections, the symbol \( C \) denotes a generic positive constant, which may change from line to line. The value of \( C \) might depend on \( T \) and the properties of the process \( X \).

3 Central limit theorem for variations of a self-similar Gaussian process

3.1 Defining characteristics of the process

Let \( X = \{ X_t, t \geq 0 \} \) denote a centered self-similar Gaussian process with self-similarity parameter \( \beta \in (0, 1) \). We introduce the following conditions on the function \( \phi \) defined in (I), where \( \alpha \in (0, 2\beta] \):

(H.1) \( \phi \) has the form \( \phi(x) = -\lambda(x-1)^\alpha + \psi(x) \), where \( \lambda > 0 \), \( \psi(x) \) is twice-differentiable on an open set containing \([1, \infty)\), and there is a constant \( C \geq 0 \) such that for any \( x \in (1, \infty) \)

(i) \( |\psi'(x)| \leq Cx^{\alpha-1} \);

(ii) \( |\psi''(x)| \leq Cx^{\alpha-1}(x-1)^{\alpha-1} \); and

(iii) \( \psi'(1) = \beta \psi(1) \), when \( \alpha \geq 1 \).

The lemma below shows that \( \alpha \) satisfies \( \mathbb{E} [(X_{t+s} - X_t)^2] \sim s^\alpha \), and we call this value the increment exponent.

Lemma 3.1. Under (H.1) for \( 0 < s \leq t \) we have

\( \mathbb{E} [(X_{t+s} - X_t)^2] = 2\lambda t^{2\beta-\alpha} + g_1(t,s), \)

where \( |g_1(t,s)| \leq Cst^{2\beta-1} \) if \( \alpha < 1 \) and \( |g_1(t,s)| \leq Cs^2t^{2\beta-2} \) if \( \alpha \geq 1 \), for some constant \( C \).

Proof. We can write

\[
\mathbb{E} [(X_{t+s} - X_t)^2] = (t+s)^{2\beta} \phi(1) + t^{2\beta} \phi(1) - 2t^{2\beta} \phi \left( 1 + \frac{s}{t} \right) \\
= \phi(1) \left( (t+s)^{2\beta} - t^{2\beta} \right) + 2t^{2\beta} \left( \phi(1) - \phi \left( 1 + \frac{s}{t} \right) \right) \\
= \phi(1) \left( (t+s)^{2\beta} - t^{2\beta} \right) + 2t^{2\beta} \left( \psi(1) + \lambda \left( \frac{s}{t} \right)^\alpha - \psi \left( 1 + \frac{s}{t} \right) \right) \\
= 2\lambda t^{2\beta-\alpha} s^\alpha + \psi(1) \left( (t+s)^{2\beta} - t^{2\beta} \right) - 2t^{2\beta} \int_1^{1+\frac{s}{t}} \psi'(y) \, dy \\
= 2\lambda t^{2\beta-\alpha} s^\alpha + g_1(t,s).
\]
If $\alpha < 1$, it follows from the Mean Value Theorem and the estimate on $|\psi'(x)|$ that $|g_1(t, s)| \leq Cst^{2\beta-1}$. If $\alpha \geq 1$ we use property (iii) to get

$$g_1(t, s) = 2t^{2\beta} \int_1^{1+\frac{t}{s}} [\psi'(1)y^{2\beta-1} - \psi'(y)]dy,$$

which implies that $|g_1(t, s)| \leq Cs^2t^{2\beta-2}$ due to the estimate on $|\psi''(x)|$.

Notice that Lemma 3.1 implies that for $0 \leq s \leq t$,

$$|g_1(t, s)| \leq Cs^{\alpha+\varepsilon} t^{2\beta-\alpha-\varepsilon},$$

for any $\varepsilon > 0$ such that $1 - \alpha - \varepsilon > 0$ if $\alpha < 1$ and $2 - \alpha - \varepsilon > 0$ if $1 \leq \alpha < 2$.

**Lemma 3.2.** Assume condition (H.1). Then,

(a) For $0 < 2s \leq t$, we have

$$\mathbb{E} [(X_{t+s} - X_t)(X_t - X_{t-s})] = (2^{\alpha} - 2) \lambda t^{2\beta-\alpha}s^{\alpha} + g_2(t, s),$$

where $|g_2(t, s)| \leq Cs^2(t - s)^{2\beta-2} + Cs^{\alpha+1}(t - s)^{2\beta-\alpha-1}$.

(b) For $0 < 2s \leq \frac{t}{3} \leq r \leq t - 2s$,

$$\mathbb{E} [(X_t - X_{t-s})(X_r - X_{r-s})] = \lambda (r - s)^{2\beta-\alpha} [(t - r - s)^{\alpha} + (t - r + s)^{\alpha} - 2(t - r)\alpha] + g_3(r, t, s),$$

where $|g_3(r, t, s)| \leq Cs^2 (r - s)^{2\beta-\alpha - 1} (t - r - s)^{\alpha - 1} + Cs^2 (r - s)^{2\beta-2}$.

**Proof.** For (a), we can write $\mathbb{E} [(X_{t+s} - X_t)(X_t - X_{t-s})]$ as

$$\mathbb{E} [X_{t+s}X_t - X_t^2 - X_{t+s}X_{t-s} + X_tX_{t-s}]$$

$$= t^{2\beta} \left( \phi \left( 1 + \frac{s}{t} \right) - \phi(1) \right) - (t - s)^{2\beta} \left( \phi \left( 1 + \frac{2s}{t - s} \right) - \phi \left( 1 + \frac{s}{t - s} \right) \right)$$

$$= -\lambda t^{2\beta-\alpha}s^{\alpha} + (t - s)^{2\beta-\alpha} \lambda (2s)^{\alpha} - \lambda s^{\alpha}$$

$$+ t^{2\beta} \int_0^{\frac{s}{t}} \psi'(1 + y) dy - \lambda (2^{\alpha} - 2)t^{2\beta-\alpha}s^{\alpha} + (1 - 2\alpha)\lambda s^{\alpha}(t^{2\beta-\alpha} - (t - s)^{2\beta-\alpha})$$

$$= \lambda (2^{\alpha} - 2)t^{2\beta-\alpha}s^{\alpha} + (1 - 2\alpha)\lambda s^{\alpha}(t^{2\beta-\alpha} - (t - s)^{2\beta-\alpha})$$

$$+ (t^{2\beta} - (t - s)^{2\beta}) \int_0^1 \psi'(1 + y) dy$$

$$+ (t - s)^{2\beta} \int_0^{\frac{s}{t}} \psi'(1 + y) - \psi'(1 + \frac{s}{t - s} + y) dy$$

$$- (t - s)^{2\beta} \int_0^{\frac{s}{t}} \psi'(1 + \frac{s}{t - s} + y) dy$$

$$= (2^{\alpha} - 2)\lambda s^{\alpha}t^{2\beta-\alpha} + g_2(t, s),$$
where, given conditions on $\psi$ and its derivatives, we have that

$$|g_2(t, s)| \leq C s^2 (t-s)^{2\beta-2} + C s^{\alpha+1} (t-s)^{2\beta-\alpha-1}.$$  

Next, for (b),

$$E[(X_t - X_{t-s})(X_r - X_{r-s})]$$

$$= r^{2\beta} \left( \phi \left( \frac{t}{r} \right) - \phi \left( \frac{t-s}{r} \right) \right) - (r-s)^{2\beta} \left( \phi \left( \frac{t}{r-s} \right) - \phi \left( \frac{t-s}{r-s} \right) \right)$$

$$= \lambda r^{2\beta-\alpha} ( (t-r-s)^{\alpha} - (t-r)^{\alpha} ) - \lambda (r-s)^{2\beta-\alpha} ((t-r)^{\alpha} - (t-r+s)^{\alpha})$$

$$+ r^{2\beta} \int_0^r \psi' \left( \frac{t-s}{r} + y \right) \, dy - (r-s)^{2\beta} \int_0^{r-s} \psi' \left( \frac{t-s}{r-s} + y \right) \, dy$$

$$= \lambda (r-s)^{2\beta-\alpha} ((t-r-s)^{\alpha} + (t-r+s)^{\alpha} - 2(t-r)^{\alpha})$$

$$- \lambda (r-s)^{2\beta-\alpha} ((t-r)^{\alpha} - (t-r-s)^{\alpha})$$

$$+ r^{2\beta} \int_0^r \psi' \left( \frac{t-s}{r} + y \right) \, dy - (r-s)^{2\beta} \int_0^{r-s} \psi' \left( \frac{t-s}{r-s} + y \right) \, dy$$

$$= \lambda (r-s)^{2\beta-\alpha} ((t-r-s)^{\alpha} + (t-r+s)^{\alpha} - 2(t-r)^{\alpha}) + g_3(r, t, s),$$

where

$$g_3(r, t, s) = \lambda \left( (r-s)^{2\beta-\alpha} - r^{2\beta-\alpha} \right) ((t-r)^{\alpha} - (t-r-s)^{\alpha})$$

$$+ (r^{2\beta} - (r-s)^{2\beta}) \int_0^{r-s} \psi' \left( \frac{t-s}{r-s} + y \right) \, dy$$

$$- r^{2\beta} \int_0^r \left[ \psi' \left( \frac{t-s}{r} + y \right) - \psi' \left( \frac{t-s}{r} + y \right) \right] \, dy$$

$$- r^{2\beta} \int_{r-s}^r \psi' \left( \frac{t-s}{r} + y \right) \, dy.$$  

Using the Mean Value Theorem, the fact that $\frac{r}{2} \leq r-s \leq r$ and bounds on the derivatives of $|\psi|$, we have that, for a constant $C$,

$$|g_3(r, t, s)| \leq C s^2 (r-s)^{2\beta-\alpha-1} (t-r-s)^{\alpha-1} + C s^2 (r-s)^{2\beta-2} \left( \frac{t-s}{r-s} \right)^{\alpha-1}$$

$$+ C s^2 (r-s)^{2\beta-1} \left( \frac{t-s}{r-s} \right)^{-1} \left( \frac{t-r}{r-s} \right)^{\alpha-1} \left( \frac{t-s}{r(r-s)} \right)$$

$$+ C s^2 (r-s)^{2\beta-2} \left( \frac{t-s}{r-s} \right)^{\alpha-1}.$$  

If $\alpha < 1$, then $(t-s)^{\alpha-1} \leq (t-r-s)^{\alpha-1}$ and $(t-r)^{\alpha-1} \leq (t-r-s)^{\alpha-1}$. Therefore, we have the bound

$$|g_3(r, t, s)| \leq C s^2 (r-s)^{2\beta-\alpha-1} (t-r-s)^{\alpha-1}.$$
In the case $\alpha \geq 1$, then $r \geq t/3$ implies $(t-s)/(r-s) \leq 5$. Using this inequality and $t-r \leq 2(t-r-s)$, we obtain

$$|g_3(r,t,s)| \leq Cs^2(r-s)^{2\beta-\alpha-1}(t-r-s)^{\alpha-1} + Cs^2(r-s)^{2\beta-2}.$$ 

Then, the proof of part (b) is complete.

**Example 3.3.** Let $B^{H} = \{B^{H}_t, t \geq 0\}$ denote a fractional Brownian motion with Hurst parameter $H$. Then condition (H.1) is satisfied with $\alpha = 2\beta = 2H$, $\lambda = \frac{1}{2}$. In this case, we obtain

$$\mathbb{E}[(B^{H}_t - B^{H}_{t-s})^2] = s^{2H},$$

$$\mathbb{E}[(B^{H}_{t+s} - B^{H}_t)(B^{H}_t - B^{H}_{t-s})] = \frac{1}{2}(2^{2H} - 2)s^{2H},$$

$$\mathbb{E}[(B^{H}_t - B^{H}_{t-s})(B^{H}_r - B^{H}_{r-s})] = \frac{1}{2}((t-r+s)^{2H} + (t-r-s)^{2H} - 2(t-r)^{2H}),$$

which means that $g_1 = g_2 = g_3 = 0$ in Lemmas 3.1 and 3.2. This means that, in the general case, we can think of $X$ as a process that is similar to the fBm, but with an additional, lower-order correction term on the covariance. In Section 4 we give examples where the terms $g_i$, $i = 1, 2, 3$, are nonzero.

We will make use of the following additional condition on the behavior of the first to derivatives of $\phi$ at infinity, which cannot be deduced from condition (H.1).

(H.2) There are constants $C > 0$ and $1 < \nu \leq 2$ such that for all $x \geq 2$,

(i) $|\phi'(x)| \leq \begin{cases} C(x-1)^{-\nu} & \text{if } \alpha < 1 \\ C(x-1)^{\alpha-2} & \text{if } \alpha \geq 1, \end{cases}$

(ii) $|\phi''(x)| \leq \begin{cases} C(x-1)^{-\nu-1} & \text{if } \alpha < 1 \\ C(x-1)^{\alpha-3} & \text{if } \alpha \geq 1. \end{cases}$

### 3.2 Central limit theorem

We are now ready to state the main result.

**Theorem 3.4.** Suppose a self-similar Gaussian process $(X, \phi)$ satisfies (H.1) and (H.2) above. For $n \geq 2$, consider the stochastic process defined in (3), that is,

$$F_n(t) = \begin{cases} \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor - 1} f(Y_{j,n}), & \text{if } \lfloor nt \rfloor \geq 1, \\ 0 & \text{if } \lfloor nt \rfloor < 1. \end{cases}$$

and $F_n(t) = 0$ if $\lfloor nt \rfloor < 1$. We assume that $f \in L^2(\mathbb{R}, \gamma)$ has the expansion (2) with Hermite rank $d \geq 2$. Then, if $\alpha < 2 - \frac{1}{d}$, the finite dimensional distributions of the
processes \( \{ F_n, n \geq 2 \} \) converge in law to those of a Brownian motion with scaling given by \( \sigma^2 = \sum_{q=d}^{\infty} c_q^2 \sigma^2_q \), where

\[
\sigma^2_q = 2^{-q} q! \sum_{m \in \mathbb{Z}} (|m + 1|^\alpha + |m - 1|^\alpha - 2|m|^\alpha)^q. \tag{13}
\]

### 3.3 Proof of Theorem 3.4

We show the convergence of the finite dimensional distributions using Theorem 2.2 and Theorem 2.3. For an integer \( p \geq 2 \), choose times \( 0 < t_1 < \cdots < t_p < \infty \), and define \( t_0 = 0 \). For \( i = 1, \ldots, p \), define \( G_n(t_i) = F_n(t_i) - F_n(t_{i-1}) \). We want to show that each \( G_n(t_i) \) converges in law to \( \mathcal{N}(0, \sigma^2(t_i - t_{i-1})) \) applying Theorem 2.3. Without loss of generality, it is enough to prove this for \( G_n(t_1) = F_n(t_1) \).

The projection of \( F_n(t_1) \) on the Wiener chaos of order \( q \) is

\[
F_{n,q}(t_1) = \frac{c_q}{\sqrt{n}} \sum_{j=0}^{|nt_1|-1} H_q(Y_{j,n}).
\]

By (10), we can write \( F_{n,q} \) in terms of the stochastic integral operator \( I_q \)

\[
F_{n,q}(t_1) = \frac{c_q}{\sqrt{n}} \sum_{j=0}^{|nt_1|-1} \xi_{j,n}^{-q} I_q \left( \frac{\partial_j}{n} \right),
\]

where we use the notation

\[
\xi_{j,n} = \| \Delta X_{\frac{j}{n}} \|_{L^2(\Omega)}.
\]

The symbol \( \partial_j/n \) denotes the indicator function of the interval \( \left[ \frac{j}{n}, \frac{j+1}{n} \right] \), that is \( \partial_j/n = 1_{\left[ \frac{j}{n}, \frac{j+1}{n} \right]} \). To verify the conditions of Theorem 2.3 we adopt the following Hilbert space notation. The indicator function \( 1_{[0,t]} \) is an element of the Hilbert space \( \mathcal{H} \), defined as the closure of the set of step functions with respect to the inner product

\[
\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = \mathbb{E} \left[ X_s X_t \right], \quad s, t \geq 0.
\]

With this representation, we can write \( F_{n,q}(t_1) = I_q(f_{n,q}) \), where

\[
f_{n,q} = f_{n,q}(t_1) = \frac{c_q}{\sqrt{n}} \sum_{j=0}^{|nt_1|-1} h_{j,n}^\otimes q,
\]

where \( h_{j,n} = \xi_{j,n}^{-q} \partial_j/n \). It is apparent that \( f_n \in \mathcal{H}^\otimes q \).

Now we proceed to verify the conditions of Theorem 2.3.

**Proof of condition (a):** We want to show that for any \( q \geq d \), \( \mathbb{E} \left[ F_{n,q}(t_1)^2 \right] \) converges to \( \sigma^2_q t_1 \), where \( \sigma^2_q \) is given by (13). This is the contents of Lemma 5.2 whose proof is given in Section 5.
Proof of condition (b): This is obvious by definition.

Proof of condition (c): We wish to show that for each \( r = 1, 2, \ldots, q - 1 \),
\[
\lim_{n \to \infty} \| f_{n,q} \otimes_r f_{n,q} \|_{\mathbb{H}^{2(q-r)}}^2 = 0.
\] (15)

By (8) we have for each \( r = 1, 2, \ldots, q - 1 \)
\[
f_{n,q} \otimes_r f_{n,q} = c^q_{q,r} \frac{n^{2|nt_1| - 1}}{n^2} \sum_{j,k=0}^{n^2} \langle h_{j,n}, h_{k,n} \rangle^r_{\mathbb{H}} \langle h_{j,n} \otimes h_{k,n} \rangle^q_{\mathbb{H}}.
\]
which is an element of \( \mathbb{H}^{2(q-r)} \). It follows that
\[
\| f_{n,q} \otimes_r f_{n,q} \|_{\mathbb{H}^{2(q-r)}}^2 = c^q_{q,r} \frac{n^{2|nt_1| - 1}}{n^2} \sum_{j,k,\ell,m=0}^{n^2} \langle h_{j,n}, h_{k,n} \rangle^r_{\mathbb{H}} \langle h_{\ell,n}, h_{m,n} \rangle^r_{\mathbb{H}} \langle h_{j,n}, h_{\ell,n} \rangle^q_{\mathbb{H}} \langle h_{k,n}, h_{m,n} \rangle^q_{\mathbb{H}}.
\] (16)

We proceed in a manner similar to the proof of the convergence (1.3) in [5]. Fix an integer \( M \geq 1 \). First we decompose the set of multi-indexes \( D = [0, [nt_1] - 1]^4 \) into \( D_1, M \cup D_2, M \), where
\[
D_1, M = \{ (j,k,\ell,m) \in D : |j - k| \leq M, |\ell - m| \leq M, |j - \ell| \leq M \},
\]
\[
D_2, M = D_3, M \cup D_4, M \cup D_5, M,
\]
\[
D_3, M = \{ (j,k,\ell,m) \in D : |j - k| > M \},
\]
\[
D_4, M = \{ (j,k,\ell,m) \in D : |\ell - m| > M \},
\]
and
\[
D_5, M = \{ (j,k,\ell,m) \in D : |j - \ell| > M \}.
\]

Taking into account that \( \| h_{j,n} \|_{\mathbb{H}} = 1 \) and using Cauchy-Schwarz inequality, it follows that
\[
\frac{1}{n^2} \sum_{(j,k,\ell,m) \in D_1, M} |\langle h_{j,n}, h_{k,n} \rangle^r_{\mathbb{H}} \langle h_{\ell,n}, h_{m,n} \rangle^r_{\mathbb{H}} \langle h_{j,n}, h_{\ell,n} \rangle^q_{\mathbb{H}} \langle h_{k,n}, h_{m,n} \rangle^q_{\mathbb{H}}| \leq \frac{CM^3}{n},
\]
which converges to zero as \( n \) tends to infinity, for any fixed \( M \). It suffices to handle the sum over one of the sets \( D_i, M \) for \( i = 3, 4, 5 \). The analysis is the same for each of them and we consider only the case \( i = 3 \). Set
\[
A_{n,M} = \frac{1}{n^2} \sum_{(j,k,\ell,m) \in D_3, M} |\langle h_{j,n}, h_{k,n} \rangle^r_{\mathbb{H}} \langle h_{\ell,n}, h_{m,n} \rangle^r_{\mathbb{H}} \langle h_{j,n}, h_{\ell,n} \rangle^q_{\mathbb{H}} \langle h_{k,n}, h_{m,n} \rangle^q_{\mathbb{H}}|.
\]
By Hölder’s inequality, we can write

\[
|A_{n,M}| \leq \frac{1}{n^2} \left( \sum_{(j,k,\ell,m) \in D_{3,M}} \left| \langle h_{j,n}, h_{k,n} \rangle \right|^q \left| \langle h_{\ell,n}, h_{m,n} \rangle \right|^q \right)^{\frac{1}{q}}
\]

\[
\times \left( \sum_{(j,k,\ell,m) \in D_{3,M}} \left| \langle h_{j,n}, h_{\ell,n} \rangle \right|^q \left| \langle h_{k,n}, h_{m,n} \rangle \right|^q \right)^{1-\frac{1}{q}}
\]

\[
\leq \left( \frac{1}{n} \sum_{|j-k|>M} \left| \langle h_{j,n}, h_{k,n} \rangle \right|^q \right)^{\frac{1}{q}} \left( \frac{1}{n} \sum_{\ell,m=0}^{n} \left| \langle h_{\ell,n}, h_{m,n} \rangle \right|^q \right)^{2-\frac{1}{q}}
\]

Using the same arguments as in the proof of Lemma 5.2, one can show that the \( \lim \sup \) as \( n \) tends to infinity of the above expression is bounded by

\[
\left( 2^{-q} \sum_{m \in \mathbb{Z}, |m|>M} \left| |m+1|^\alpha + |m-1|^\alpha - 2|m|^\alpha \right|^q \right)^{\frac{1}{q}}
\]

\[
\times \left( 2^{-q} \sum_{m \in \mathbb{Z}} \left| |m+1|^\alpha + |m-1|^\alpha - 2|m|^\alpha \right|^q \right)^{2-\frac{1}{q}}
\]

which converges to zero as \( M \) tends to infinity. This completes the proof of property (c).

**Proof of condition (d):** Condition (d) follows from the proof of Lemma 5.2. In fact, we know that the series \( \sum_q c_q^d \) is convergent, and it suffices to take into account the estimates (24), (26) and Remark 5.3 after the proof of Lemma 5.2

With conditions (a) - (d) of Theorem 2.3 satisfied, it follows that for \( i = 1, \ldots, p \), \( G_n(t_i) \) converges in law to a Gaussian random variable with mean zero and variance given by \( \sigma^2(t_i-t_{i-1}) \). We next want to show that \( \lim_{n \to \infty} E \left[ G_n(t_i)G_n(t_j) \right] = 0 \) when \( t_i \neq t_j \). Without loss of generality, it is enough to show that

\[
\sup_{q \geq d} \left| \frac{1}{n} \sum_{k=0}^{\left\lfloor nt_1 \right\rfloor-1} \sum_{j=\left\lfloor nt_1 \right\rfloor}^{\left\lfloor nt_2 \right\rfloor-1} \xi_{k,n}^{-q} \xi_{j,n}^{-q} \left( E \left[ \Delta X_{\frac{n}{n}} \Delta X_{\frac{k}{n}} \right] \right)^q \right|
\]

tends to zero as \( n \) tends to infinity. Let \( M_n = (nt_2)^{\frac{1}{4}} \), we can decompose the above sum into

\[
\frac{1}{n} \sum_{(j,k) \in D_1,n} \xi_{k,n}^{-q} \xi_{j,n}^{-q} \left( E \left[ \Delta X_{\frac{n}{n}} \Delta X_{\frac{k}{n}} \right] \right)^q + \frac{1}{n} \sum_{(j,k) \in D_2,n} \xi_{k,n}^{-q} \xi_{j,n}^{-q} \left( E \left[ \Delta X_{\frac{n}{n}} \Delta X_{\frac{k}{n}} \right] \right)^q
\]

where

\[
D_{1,n} = \{ (j,k) \in D_n : |j-k| \leq M_n \}, \quad D_{2,n} = \{ (j,k) \in D_n : |j-k| > M_n \},
\]
and
\[ D_n = \{(j, k) : 0 \leq k \leq \lfloor nt_1 \rfloor - 1, \lfloor nt_2 \rfloor \leq j \leq \lfloor nt_2 \rfloor - 1\}. \]

By Lemma 3.1 and Cauchy-Schwarz, the first sum is bounded by \( C n^{\frac{1}{4}} \). For the second sum, using arguments from Step 4 and Step 5 in the proof of Lemma 5.2 (see (28)), the limit sup as \( n \) tends to infinity is bounded by
\[ C \lim_{n \to \infty} \sum_{m = M_n}^{\infty} ((m + 1)^\alpha + (m - 1)^\alpha - 2m^\alpha)^q \leq C \lim_{n \to \infty} M_n^{q(\alpha - 2) + 1} = 0, \]
where the limit follows since \( M_n = C n^{\frac{1}{4}} \) and \( q(\alpha - 2) + 1 < 0 \).

Based on the above, Theorem 2.2 implies that the vector sequence \( (G_n(t_1), \ldots, G_n(t_p)) \) converges in distribution to \( \mathcal{N}(0, \Sigma) \) as \( n \) tends to infinity, where \( \Sigma \) is the diagonal matrix with entries \( \sigma^2(t_i - t_{i-1}), i = 1, \ldots, p \). Taking \( F_n(t_i) = \sum_{k=1}^{i} G_n(t_k) \), we have that
\[ (F_n(t_1), \ldots, F_n(t_p)) \xrightarrow{\mathcal{L}} (F(t_1), \ldots, F(t_p)) \]
as \( n \to \infty \), where each \( F(t_i) \sim \mathcal{N}(0, t_i \sigma^2) \). This completes the proof of Theorem 3.4.

3.4 Theorem 3.4 examples

In this section we consider some particular cases where the function \( f \) can be represented as a single Hermite polynomial, or a linear combination of finitely many Hermite polynomials.

Example 3.5. For integer \( p \geq 1 \), let \( f(x) = x^{2p} - \mathbb{E}[Z^{2p}] \), where \( Z \) has a \( \mathcal{N}(0, 1) \) distribution. The Stroock formula \[19\] for a Hermite expansion gives:
\[ f(Z) = \mathbb{E}[f(Z)] + \sum_{q=1}^{2p} \frac{1}{q!} \mathbb{E}[f^{(q)}(Z)] H_q(Z) \]
\[ = \sum_{q=1}^{2p} \frac{(2p)!}{q!(2p - q - 1)!} \mathbb{E}[Z^{2p-q}] H_q(Z). \]

Since \( 2p - 1 \) is odd, \( f \) has Hermite rank 2, and Theorem 3.4 can be applied. In this case, \( F_n \) converges in law to a Gaussian random variable with variance \( \sigma^2 = \sum_{q=2}^{2p} c_q^2 \sigma_q^2 \), where
\[ c_q = \frac{(2p)!}{q!(2p - q - 1)!} \mathbb{E}[Z^{2p-q}] = \frac{(2p)!(2p - q - 1)!!}{q!(2p - q - 1)!!}, \]
for even \( q = 2, \ldots, 2p \), and \( c_q = 0 \) for all odd integers and \( q \) greater than \( 2p \).

For the odd integer case, let \( f(x) = |x|^{2p+1} - \mathbb{E}[|Z|^{2p+1}] \). Again, since \( \mathbb{E}[f'(Z)] = 0 \), the Hermite rank is 2 and we can apply Theorem 3.4. In this case, \( \sigma^2 = \sum_{q=2}^{2p} c_q^2 \sigma_q^2 \), where
\[ c_q = \frac{(2p + 1)!}{q!(2p - q)!} \mathbb{E}[|Z|^{2p-q+1}] = \frac{(2p + 1)!(2p - q)!! \sqrt{2}}{q!(2p - q)! \sqrt{\pi}}, \]
for even \( q = 2, \ldots, 2p \), and \( c_q = 0 \) for all odd integers \( q \) and all \( q \) greater than \( 2p \).
**Example 3.6.** Suppose \( f = H_q \) for a single Hermite polynomial of order \( q \geq 2 \). In this case, one can show that for fixed \( t \), \( F_n = F_n(t) \) converges in total variation. Let \( N \) denote a random variable with the \( \mathcal{N}(0, t\sigma_q^2) \) distribution. In [12] it is proved that

\[
d_{TV}(F_n, N) \leq \frac{2}{t\sigma_q^2} \sqrt{\text{Var} \left( \frac{1}{q} \| DF_n \|_{\mathcal{H}}^2 \right)},
\]

where \( d_{TV} \) denotes total variation distance and \( DF_n \) is the Malliavin derivative of \( F_n \). We can write \( F_n = I_q(f_{n,q}) \), where \( f_{n,q} \) is given by (20) with \( t_1 = t \). From [13, Lemma 5.2.4], we have

\[
\text{Var} \left( \frac{1}{q} \| DF_n \|_{\mathcal{H}}^2 \right) = \frac{1}{q^2} \sum_{r=1}^{q-1} r^2 r! \left( \frac{q}{r} \right)^4 (2q - 2r)! \left\| f_{n,q} \overset{\circ}{\otimes} r f_{n,q} \right\|_{\mathcal{H}^{2(q-r)}}^2
\]

where \( f_{n,q} \overset{\circ}{\otimes} r f_{n,q} \) denotes the symmetrization of \( f_{n,q} \otimes r f_{n,q} \). Therefore, by condition (c) in the proof of Theorem 3.4 and using the identity

\[
\left\| f_{n,q} \overset{\circ}{\otimes} r f_{n,q} \right\|_{\mathcal{H}^{2(q-r)}}^2 \leq \left\| f_{n,q} \otimes r f_{n,q} \right\|_{\mathcal{H}^{2(q-r)}}^2
\]

for each \( r = 1, \ldots, q - 1 \), we obtain

\[
\lim_{n \to \infty} d_{TV}(F_n, N) = 0.
\]

On the other hand, in this example we can show a functional central limit theorem. Indeed, by (27) in the proof of Lemma 5.2 and Remark 5.3 following, we can show that for sufficiently large \( n \),

\[
\mathbb{E} \left[ (F_n(t_i) - F_n(t_{i-1}))^2 \right] \leq C \left( \frac{|nt_i| - |nt_{i-1}|}{n} \right).
\]

Moreover, using the fact that all the \( p \)-norms are equivalent on a fixed Wiener chaos, for arbitrary \( 0 \leq s_1 < s < s_2 \leq T \) we deduce that for \( n \) large enough

\[
\mathbb{E} \left[ (F_n(s) - F_n(s_1))^2 (F_n(s_2) - F_n(s))^2 \right]
\leq \left( \mathbb{E} \left[ (F_n(s) - F_n(s_1))^4 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ (F_n(s_2) - F_n(s))^4 \right] \right)^{\frac{1}{2}}
\leq C \left( \frac{|ns_2| - |ns_1|}{n} \right)^2.
\]

This implies that the laws of the processes \( \{F_n, n \geq 2\} \) are tight in the Skorohod space \( \mathbf{D}[0, \infty) \) (see Billingsley [11, Theorem 13.5]). As a consequence, from Theorem 3.4 we deduce that the laws of \( F_n \) converge in the topology of \( \mathbf{D}[0, \infty) \) to a Brownian motion with scaling \( \sigma^2 \).
4 Examples of suitable processes

4.1 Bifractional Brownian motion

Bifractional Brownian motion is a generalization of fBm, first introduced by Houdré and Villa in [8]. It is defined as a centered, Gaussian process $B = \{ B_t^{H,K}, t \geq 0 \}$ with covariance

$$
E[B_sB_t] = \frac{1}{2K} \left[ (s^{2H} + t^{2H})^K - |t-s|^{2HK} \right],
$$

where $H \in (0,1)$ and $K \in (0,1]$. Note that if $K = 1$, then $B$ is an ordinary fBm. The reader may refer to [10, 13] for a discussion of properties.

The covariance can be expressed in terms of $\phi$ with $\beta = HK$ and

$$
\phi(x) = \frac{1}{2K} \left[ (1 + x^{2H})^K - (x - 1)^{2HK} \right].
$$

As stated in Section 1, it is well known that for ordinary fBm, $F_n(t)$ converges in distribution for all $H < 1 - \frac{1}{2d}$. Since the case $K = 1$ is well known, we will assume below that $K < 1$. We now verify the properties (H.1) – (H.2).

For (H.1), we write

$$
\phi(x) = -\frac{1}{2K}(x-1)^{2HK} + \frac{1}{2K}(1 + x^{2H})^K,
$$

which means we have $\lambda = 2^{-K}$ and $\alpha = 2\beta = 2HK$. Then $\psi(x) = 2^{-K}(1 + x^{2H})^K$, and the bounds on $|\psi'|$ and $|\psi''|$ follow. Moreover, $\psi'(1) = HK = \beta \psi'(1)$.

For (H.2),

$$
\phi'(x) = 2^{1-K} HK \left( x^{2H-1}(1 + x^{2H})^{K-1} - (x - 1)^{2HK-1} \right).
$$

We can write

$$
(1 + x^{2H})^{K-1} = (x^{2H})^{K-1} + (K-1) \int_0^1 (x^{2H} + y)^{K-2} dy,
$$

so that

$$
\phi'(x) = 2^{1-K} HK \left( x^{2HK-1} - (x - 1)^{2HK-1} + (K-1)x^{2H-1} \int_0^1 (x^{2H} + y)^{K-2} dy \right).
$$

Hence, we can write $|\phi'(x)| \leq C(x-1)^{2HK-2} + C(x-1)^{2HK-2}$. If $\alpha < 1$, then we take $\nu = \min \{1 + 2H - 2HK, 2 - 2HK \} > 1$, and if $\alpha \geq 1$ then $2 - 2HK \leq 2 + 2H - 2HK$ implies $|\phi'(x)| \leq C(x-1)^{2HK-2}$. Continuing, we have

$$
\phi''(x) = 2^{1-K} HK(2HK-1) \left( x^{2HK-2} - (x - 1)^{2HK-2} \right)
+ 2^{2-K} H^2 (K-1)x^{2H-2} \int_0^1 (x^{2H} + y)^{K-2} dy
+ 2^{2-K} H^2 (K-1)(K-2)x^{2H-2} \int_0^1 (x^{2H} + y)^{K-3} dy,
$$
so it follows that

\[ |\phi''(x)| \leq C(x-1)^{2HK-3} + C(x-1)^{2HK-2H-2} \leq C(x-1)^{-M}, \]

where

\[ M = \begin{cases} 
\min\{2 + 2H - 2HK, 3 - 2HK\} = \nu + 1 & \text{if } \alpha < 1 \\
3 - 2HK & \text{if } \alpha \geq 1 
\end{cases}. \]

4.2 Subfractional Brownian motion

Another variant on the fBm is the process known as sub-fractional Brownian motion (sfBm). This is a centered Gaussian process \( \{S_t, t \geq 0\} \), with covariance defined by:

\[ R(s, t) = s^{2H} + t^{2H} - \frac{1}{2} \left[ (s + t)^{2H} + |s - t|^{2H} \right], \]

with real parameter \( H \in (0, 1) \). Some properties of the sfBm are discussed in [2, 4]. Note that \( H = 1/2 \) is a standard Brownian motion, and also note the similarity of \( R(s, t) \) to the covariance of fBm. Indeed, in [4] it is shown that sfBm may be decomposed into an fBm with Hurst parameter \( H \) and another centered Gaussian process.

Let \( S = \{S_t, t \geq 0\} \) denote a sub-fractional Brownian motion with \( 0 < H < 1 \). In this case we have \( \lambda = \frac{1}{2}, \alpha = 2\beta = 2H \) and

\[ \phi(x) = 1 + x^{2H} - \frac{1}{2} \left( (x+1)^{2H} + (x-1)^{2H} \right) = -\lambda(x-1)^{2H} + \psi(x), \]

where \( \psi(x) = 1 + x^{2H} - \frac{1}{2}(x+1)^{2H} \). Clearly, \( |\psi'(x)| \leq Cx^{2H-1} \) and \( |\psi''(x)| \leq Cx^{2H-2} \). Moreover, \( \psi'(1) = H(2-2^{2H-1}) = \beta \psi(1) \).

To check (H.2), we have

\[ |\phi'(x)| = H |(x+1)^{2H-1} - 2x^{2H-1} + (x-1)^{2H-1}| \leq C(x-1)^{2H-2}, \]

so (H.2)(i) is satisfied for \( x \geq 2 \), and (H.2)(ii) can be shown by a second derivative.

4.3 A Gaussian process introduced by J. Swanson

We consider the centered Gaussian process \( Y = \{Y_t, t \geq 0\} \) with covariance given by

\[ \mathbb{E}[Y_sY_t] = \sqrt{st} \sin^{-1} \left( \frac{s \wedge t}{\sqrt{st}} \right). \]

Then \( Y \) can be characterized as a self-similar Gaussian process with \( \beta = 1/2 \) and

\[ \phi(x) = \sqrt{x} \sin^{-1} \left( \frac{1}{\sqrt{x}} \right). \]

This process was studied by Jason Swanson in a 2007 paper [20], and it arises as the limit of normalized empirical quantiles of a system of independent Brownian motions. The
properties of this process were also considered in [7]. Unlike the fBm family processes in Sections 4.1 and 4.2, this process is an example of the case $\alpha < 2\beta$.

This process satisfies (H.1), with $\beta = 1/2$ and $\alpha = 1/2$. We can write this as

$$\phi(x) = -\sqrt{x-1} + \left(\sqrt{x} \sin^{-1}\left(\frac{1}{\sqrt{x}}\right) + \sqrt{x-1}\right),$$

which gives $\lambda = 1$ and

$$\psi(x) = \sqrt{x} \sin^{-1}\left(\frac{1}{\sqrt{x}}\right) + \sqrt{x-1}.$$ 

Then

$$|\psi'(x)| = \frac{x^{-\frac{1}{2}}}{2} \left|\sin^{-1}\left(\frac{1}{\sqrt{x}}\right) - \frac{\sqrt{x-1}}{\sqrt{1-x}}\right| \leq Cx^{-\frac{1}{2}}.$$ 

And for the second derivative,

$$\psi''(x) = -\frac{x^{-\frac{3}{2}}}{4} \left(\sin^{-1}\left(\frac{1}{\sqrt{x}}\right) - \frac{\sqrt{x-1}}{\sqrt{1-x}}\right) - \frac{x^{-\frac{1}{2}}}{4\sqrt{x-1}} \left(\frac{1}{\sqrt{x}(\sqrt{x}+1)}\right),$$

so

$$|\psi''(x)| \leq Cx^{-\frac{3}{2}} \left(1 + (x-1)^{-\frac{1}{2}}\right) \leq Cx^{-1}(x-1)^{-\frac{1}{2}}.$$ 

To check condition (H.2), we return to the original expression $\phi(x) = \sqrt{x} \sin^{-1}\left(x^{-\frac{1}{2}}\right)$.

We have

$$\phi'(x) = \frac{1}{2\sqrt{x}} \left(\sin^{-1}\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{\sqrt{x-1}}\right)$$

$$= \frac{1}{2\sqrt{x}} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x-1}}\right) + \frac{1}{2\sqrt{x}} \left(\sin^{-1}\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{\sqrt{x}}\right).$$

From a Taylor expansion of $\sin t$, we have for $0 < t < 1$

$$\sin t = t - \frac{t^3}{3!} + \frac{h^5}{5!},$$

for some $0 \leq h \leq t$, and it follows that

$$t = \sin^{-1}\left(t - \frac{t^3}{3!} + \frac{h^5}{5!}\right).$$

We then set $t = x^{-\frac{1}{2}}$, and use a second Taylor expansion on $\sin^{-1}$, to conclude that for $x > 2$,

$$\left|\frac{1}{\sqrt{x}} - \sin^{-1}\left(\frac{1}{\sqrt{x}}\right)\right| \leq \frac{C}{x \sqrt{x-1}}.$$ 

Hence, we have

$$|\phi'(x)| \leq \frac{1}{2\sqrt{x}} \left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x-1}}\right| + \frac{C}{x^2 \sqrt{x-1}} \leq C(x-1)^{-2},$$
Hence, \( \nu = 2 \) since \( \alpha = \frac{1}{2} < 1 \).

Similarly,

\[
\phi''(x) = -\frac{1}{4} x^{-\frac{3}{2}} \left( \sin^{-1} \left( \frac{1}{\sqrt{x}} \right) - \frac{1}{\sqrt{x-1}} \right) - \frac{1}{4} x^{-\frac{1}{2}} \left( \frac{1}{x\sqrt{x-1}} - (x-1)^{-\frac{3}{2}} \right)
\]

\[
= -\frac{1}{4} x^{-\frac{3}{2}} \left( \sin^{-1} \left( \frac{1}{\sqrt{x}} \right) - \frac{1}{\sqrt{x-1}} \right) - \frac{1}{4\sqrt{x(x-1)}} \left( \frac{1}{x} - \frac{1}{x-1} \right),
\]

Hence, \( |\phi''(x)| \leq C(x-1)^{-3} = C(x-1)^{-\nu-1} \) for \( x > 2 \).

### 4.4 Two smooth processes with \( \alpha < 1 \)

For \( 0 < \alpha < 1 \), we consider the centered Gaussian processes \( Z_1(t), Z_2(t) \), with covariances given by:

\[
\mathbb{E}[Z_1(s)Z_1(t)] = \Gamma(1 - \alpha) ((s + t)^\alpha - \max(s, t)^\alpha)
\]

\[
\mathbb{E}[Z_2(s)Z_2(t)] = \Gamma(1 - \alpha) (s^\alpha + t^\alpha - (s + t)^\alpha).
\]

These processes are discussed in a recent paper by Durieu and Wang [6], where it is shown that the process \( Z = Z_1 + Z_2 \) (where \( Z_1, Z_2 \) are independent) is the limit in law of a discrete process studied by Karlin. The process \( Z_1 \) is new, but the process \( Z_2 \), with a different scaling constant, was first described in Lei and Nualart [10].

For \( Z_1 \), we can write \( \mathbb{E}[Z_1(s)Z_1(t)] = s^\alpha \phi_1(t/s) \), where \( \phi_1(x) = \Gamma(1 - \alpha) ((x + 1)^\alpha - x^\alpha) \). We can also write \( \phi_1 \) in the form

\[
\phi_1(x) = -\Gamma(1 - \alpha) (x-1)^\alpha + \Gamma(1 - \alpha) ((x + 1)^\alpha - (x-1)^\alpha - x^\alpha),
\]

so that \( \lambda = \Gamma(1 - \alpha) \) and \( \psi_1(x) = \Gamma(1 - \alpha) ((x + 1)^\alpha + (x-1)^\alpha - x^\alpha) \). Note that by a Taylor expansion for \( x \geq 1 \),

\[
(x-1)^\alpha = x^\alpha - \alpha x^{\alpha-1} + \frac{\alpha(\alpha - 1)}{2} x^{\alpha-2} + O(x^{\alpha-3}),
\]

so that we have \( |\psi_1''(x)| \leq C x^{\alpha-1} \) and \( |\psi_2''(x)| \leq C x^{\alpha-2} \), and \( Z_1 \) satisfies (H.1). For (H.2), note that we can write

\[
\phi_1(x) = \alpha \Gamma(1 - \alpha) \int_0^1 (x+u)^{\alpha-1} du,
\]

so for \( x \geq 2 \)

\[
|\phi_1'(x)| = \alpha |\alpha - 1| \Gamma(1 - \alpha) \int_0^1 (x+u)^{\alpha-2} du \leq 2x^{\alpha-2},
\]

which satisfies (H.2)(i) with \( \nu = 2 - \alpha > 1 \); and it can also be seen that \( |\phi_1''(x)| \leq 4x^{\alpha-3} = 2x^{-\nu-1} \), so that (H.2)(ii) is satisfied.

For \( Z_2 \) we have

\[
\phi_2(x) = \Gamma(1 - \alpha) (1 + x^\alpha - (x + 1)^\alpha)
\]

\[
= -\Gamma(1 - \alpha) (x - 1)^\alpha + \Gamma(1 - \alpha) (1 + x^\alpha + (x - 1)^\alpha - (x + 1)^\alpha)
\]

\[
= -\Gamma(1 - \alpha) (x - 1)^\alpha + \psi_2(x),
\]
where again we take \( \lambda = \Gamma(1 - \alpha) \). By a computation similar to that for \( Z_1 \) above, it can be seen that \( \psi_2 \) satisfies (i) and (ii) of (H.1). The computations for (H.2) are also similar to those for \( Z_1 \) above. We write

\[
\phi_2(x) = \Gamma(1 - \alpha) - \alpha \Gamma(1 - \alpha) \int_0^1 (x + u)^{\alpha - 1} du,
\]

so that (H.2) conditions (i) and (ii) are satisfied with \( \nu = 2 - \alpha > 1 \).

**Remark 4.1.** Both processes satisfy \( \mathbb{E}[Z_i(t)^2] = Ct^\alpha \), so the increment exponent at 0 is \( \alpha \). On the other hand, for \( t > 0 \), \( \mathbb{E}[(Z_i(t+s) - Z_i(t))^2] \approx c_ts \) for a constant \( c_t \), and the increment exponent for \( t > 0 \) is 1. The renormalization in Theorem 3.4 is given by \( \alpha \), which is also the self-similarity parameter.

### 5 Some technical lemmas

We begin with a technical lemma that gives upper bounds on certain covariance terms.

**Lemma 5.1.** Let \( n \geq 6 \) be an integer, and let \( j, k \geq 1 \) be integers satisfying \( 3k \leq j \). Then under (H.2), there is a constant \( C > 0 \) such that

\[
\left| \mathbb{E} \left[ \Delta X_{kn} \Delta X_{kn} \right] \right| \leq \begin{cases} 
Cn^{-2\beta}k^{2\beta - \alpha}(j-k)^{-\alpha} & \text{if } \alpha < 1 \\
Cn^{-2\beta}k^{2\beta - \alpha}(j-k)^{\alpha - 2} & \text{if } \alpha \geq 1 
\end{cases},
\]

where the exponent \( 1 < \nu \leq 2 \) is defined in (H.2).

**Proof.** We have

\[
\mathbb{E} \left[ \Delta X_{kn} \Delta X_{kn} \right] = n^{-2\beta}(k+1)^{2\beta} \left( \phi \left( \frac{j+1}{k+1} \right) - \phi \left( \frac{j}{k+1} \right) \right) \\
- n^{-2\beta}k^{2\beta} \left( \phi \left( \frac{j+1}{k} \right) - \phi \left( \frac{j}{k} \right) \right) \\
= n^{-2\beta} \left( (k+1)^{2\beta} - k^{2\beta} \right) \left( \phi \left( \frac{j+1}{k+1} \right) - \phi \left( \frac{j}{k+1} \right) \right) \\
+ n^{-2\beta}k^{2\beta} \left[ \phi \left( \frac{j+1}{k+1} \right) - \phi \left( \frac{j}{k+1} \right) - \phi \left( \frac{j+1}{k} \right) + \phi \left( \frac{j}{k} \right) \right].
\]

Condition \( j \geq 3k \) and (H.2) imply that there exists a constant \( C > 0 \) such that for each \( x \in \left[ \frac{j}{k+1}, \frac{j+1}{k+1} \right] \)

\[
|\phi'(x)| \leq \begin{cases} 
Ck^\nu(j-k)^{-\nu} & \text{if } \alpha < 1 \\
Ck^{2-\alpha}(j-k)^{\alpha - 2} & \text{if } \alpha \geq 1 
\end{cases}
\]

and for each \( x \in \left[ \frac{j}{k+1}, \frac{j+1}{k} \right] \),

\[
|\phi''(x)| \leq \begin{cases} 
Ck^{\nu + 1}(j-k)^{-\nu - 1} & \text{if } \alpha < 1 \\
Ck^{3-\alpha}(j-k)^{\alpha - 3} & \text{if } \alpha \geq 1 
\end{cases}
\]

Then the desired estimate follows easily from the Mean Value Theorem. \( \square \)
Lemma 5.2. Let $t > 0$. Under above definitions with $\alpha < 2 - \frac{1}{q}$,

$$\lim_{n \to \infty} \mathbb{E} \left[ F_{n,q}^2(t) \right] = \sigma^2_q,$$

where $\sigma^2_q$ is given by (13).

Proof. The proof will be done in several steps.

Step 1. It follows from (9) that

$$\mathbb{E} \left[ F_{n,q}^2(t) \right] = \frac{c^2_q}{n} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[ H_q(Y_j,n) H_q(Y_k,n) \right] \left( \mathbb{E} \left[ \Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}} \right] \right)^q.$$

Fix $\gamma \in (0, 1/2)$ and decompose the above double sum into two terms, that is, $\mathbb{E} \left[ F_{n,q}^2 \right] = A_{1,n} + A_{2,n}$, where

$$A_{i,n} = \frac{q! c^2_q}{n} \sum_{j,k \in D_i} \| \Delta X_{\frac{j}{n}} \|_{L^2(\Omega)}^{-q} \| \Delta X_{\frac{k}{n}} \|_{L^2(\Omega)}^{-q} \left( \mathbb{E} \left[ \Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}} \right] \right)^{q}.$$

and $D_1 = \{ j : 0 \leq j \leq n^\gamma \cap (\lfloor nt \rfloor - 1) \}$ and $D_2 = \{ j : n^\gamma < j \leq \lfloor nt \rfloor - 1 \}$.

The term $A_{1,n}$ can be bounded, using Cauchy-Schwarz inequality, by $q! c^2_q n^{2\gamma - 1}$ and it converges to zero as $n$ tends to infinity. So, it suffices to consider the term $A_{2,n}$. We recall the notation $\xi_{j,n} = \| \Delta X_{\frac{j}{n}} \|_{L^2(\Omega)}$.

Step 2. From (12) we can write for $j \in D_2$

$$\xi_{j,n}^2 = 2\lambda j^{2\beta - \alpha} n^{-2\beta} \left[ 1 + \eta_{j,n} \right], \quad (20)$$

where

$$|\eta_{j,n}| = \frac{1}{2\lambda} j^{\alpha - 2\beta} n^{-2\beta} \left| g_1 \left( \frac{j}{n}, \frac{1}{n} \right) \right| \leq C n^{-\gamma \epsilon}. \quad (21)$$

Consider the following decomposition:

$$A_{2,n} = \frac{q! c^2_q}{n} \sum_{j,k \in D_2} \xi_{j,n}^{-q} \xi_{k,n}^{-q} \left( \mathbb{E} \left[ \Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}} \right] \right)^{q} + \frac{q! c^2_q}{n} \sum_{j,k \in D_2, |k-j|=1} \xi_{j,n}^{-q} \xi_{k,n}^{-q} \left( \mathbb{E} \left[ \Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}} \right] \right)^{q} + \frac{q! c^2_q}{n} \sum_{j,k \in D_2, |k-j|\geq 2} \xi_{j,n}^{-q} \xi_{k,n}^{-q} \left( \mathbb{E} \left[ \Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}} \right] \right)^{q}.$$

The first term clearly converges to $q! c^2_q t$. We denote the second and third term by $B_{2,n}$ and $B_{3,n}$, respectively.
Step 3. Let us consider the term $B_{2,n}$. Using Lemma 3.2(a) we can write
\[ \mathbb{E} \left[ \Delta X_{n} \Delta X_{n-1} \right] = \lambda (2 \alpha - 2) j \beta n^{-2 \beta} + g_2 \left( \frac{j}{n}, \frac{1}{n} \right). \]
Multiplying this expression by $\xi_{j,n}^{-1} \xi_{j-1,n}^{-1}$ and using (20) yields
\[ \xi_{j,n}^{-1} \xi_{j-1,n}^{-1} \mathbb{E} \left[ \Delta X_{n} \Delta X_{n-1} \right] = \left[ 1 + \eta_{j,n} \right]^{- \frac{q}{2}} \left[ 1 + \eta_{j-1,n} \right]^{- \frac{q}{2}} \times \left( \frac{2 \alpha - 1}{j - 1} \right)^{\frac{q}{2}(2 \beta - \alpha) + R_{j,n}} \right), \] (22)
where
\[ R_{j,n} = (2 \lambda)^{-1} n^{2 \beta} (j(j-1))^{- \frac{q}{2}(2 \beta - \alpha)} g_2 \left( \frac{j}{n}, \frac{1}{n} \right). \]
Applying Lemma 3.2(a) and assuming $n \gamma \geq 2$, this term can be bounded as follows.
\[ |R_{j,n}| \leq C_j^{-(2 \beta - \alpha + \delta)}, \] (23)
where $\delta = \min(2 - 2 \beta, 1 + \alpha - 2 \beta)$. We claim that there exist $\rho \in (0, 1)$ such that for $n$ large enough and for all $j \in D_2$,
\[ \left| \xi_{j,n}^{-1} \xi_{j-1,n}^{-1} \mathbb{E} \left[ \Delta X_{n} \Delta X_{n-1} \right] \right| < \rho < 1. \] (24)
This follows from the estimates (21), (23) and the fact that $|2^{\alpha-1} - 1| < 1$ and $2 \beta - \alpha + \delta > 0$. Finally, from the expression (22) and the estimates (21) and (23) it follows that
\[ \lim_{n \to \infty} B_{2,n} = q! c_2^2 \lim_{n \to \infty} \frac{1}{n \left\lfloor \frac{nt}{3} \right\rfloor - 1} \sum_{j \in D_2} \sum_{k=1}^{\left\lfloor j/3 \right\rfloor} \xi_{j,n}^{-q} \xi_{k,n}^{-q} \mathbb{E} \left[ \Delta X_{n} \Delta X_{k} \right]^{q} \]
\[ \times \left( \frac{2 \alpha - 1}{j - 1} \right)^{\frac{q}{2}(2 \beta - \alpha) + R_{j,n}} \right) \]
\[ = q! c_2^2 (2^{\alpha-1} - 1)^q t. \] (25)
Step 4. Let us consider the term $B_{3,n}$. First we show that the terms with $k \leq \left\lfloor j/3 \right\rfloor$ or $j \leq \left\lfloor k/3 \right\rfloor$ do not contribute to the limit. That is, we claim that the following expression converges to zero as $n$ tends to infinity
\[ C_n = \frac{1}{n} \sum_{j=3}^{\left\lfloor nt-1 \right\rfloor} \sum_{k=1}^{\left\lfloor j/3 \right\rfloor} \xi_{j,n}^{-q} \xi_{k,n}^{-q} \mathbb{E} \left[ \Delta X_{n} \Delta X_{k} \right]^{q}. \]
To do this, we consider two cases.
Case 1. When $\alpha < 1$, Lemma 5.1 gives
\[ C_n \leq C n^{-1} \sum_{j=3}^{\left\lfloor nt-1 \right\rfloor} \sum_{k=1}^{\left\lfloor j/3 \right\rfloor} k^{q(\alpha + \nu - 2)} (j - k)^{-q \nu}. \]
Note that \(1 \leq k \leq j/3\) implies \((j - k)^{-\nu} \leq C j^{-\nu}\), so it follows that

\[
C_n \leq \frac{C}{n} \sum_{j=3}^{\lfloor nt \rfloor - 1} j^{-q \nu} \sum_{k=1}^{j/3} k^{q(\alpha + \nu - 2)} \leq \frac{C}{n} \sum_{j=3}^{\lfloor nt \rfloor - 1} j^{q(\alpha - 2) + 1}
\]

which converges to zero as \(n\) tends to infinity because \(q(\alpha - 2) + 1 < 0\) since \(\alpha < 2 - \frac{1}{d}\) and \(q \geq d\).

**Case 2:** Assume \(1 \leq \alpha < 2 - \frac{1}{q}\). For this case, by the other part of Lemma 5.1

\[
C_n \leq \frac{C}{n} \sum_{j=3}^{\lfloor nt \rfloor - 1} j^{q(\alpha - 2) + 1},
\]

which converges to zero as \(n\) tends to infinity because \(q(\alpha - 2) + 1 < 0\) since \(\alpha < 2 - \frac{1}{d}\) and \(q \geq d\).

**Step 5.** Finally, it remains to study the following term

\[
D_n = \frac{2q!}{n^q} \sum_{j,k \in D_2,|j/3| < k \leq j-2} \xi_{j,n}^{-q} \xi_{k,n}^{-q} \left( \mathbb{E} \left[ \Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}} \right] \right)^q.
\]

We have by Lemma 3.2(b),

\[
\mathbb{E} \left[ \Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}} \right] = \lambda n^{-2\beta} k^{2\beta - \alpha} A_{j,k} + g_3 \left( \frac{k + 1}{n}, \frac{j + 1}{n}, \frac{1}{n} \right),
\]

where

\[
A_{j,k} = |j - k + 1|^{\alpha} + |j - k - 1|^{\alpha} - 2|j - k|^\alpha.
\]

Multiplying this expression by \(\xi_{j,n}^{-1} \xi_{k,n}^{-1}\) and using (20) yields

\[
\xi_{j,n}^{-1} \xi_{k,n}^{-1} \left[ \lambda n^{-2\beta} k^{2\beta - \alpha} A_{j,k} + g_3 \left( \frac{k + 1}{n}, \frac{j + 1}{n}, \frac{1}{n} \right) \right] = [1 + \eta_{j,n}]^{-\frac{1}{2}} [1 + \eta_{k,n}]^{-\frac{1}{2}}
\]

\[
\times \left( 2^{-1} \left( k/j \right)^{\frac{1}{2}(2\beta - \alpha)} A_{j,k} + R_{j,k,n} \right),
\]

where

\[
R_{j,k,n} = (2\lambda)^{-1} n^{2\beta} (kj)^{-\frac{1}{2}(2\beta - \alpha)} g_3 \left( \frac{k + 1}{n}, \frac{j + 1}{n}, \frac{1}{n} \right).
\]
We can write
\[
\left(2^{-\frac{1}{2}(2\beta-\alpha)}A_{j,k} + R_{j,k,n}\right)^q = 2^{-q(k/j)^{\frac{1}{2}(2\beta-\alpha)}}A_{j,k}^q + \sum_{r=1}^q \left(\frac{q}{r}\right)2^{-(q-r)(k/j)^{\frac{1}{2}(2\beta-\alpha)}}A_{j,k}^{q-r}R_{j,k,n}^r
\]
=: \Phi_{1,j,k,n} + \Phi_{2,j,k,n}.

From the estimate for \(|g_3|\) from Lemma 3.2(b), we have
\[
|R_{j,k,n}| \leq C j^{-\frac{1}{2}(2\beta-\alpha)}k^{\frac{1}{2}(2\beta-\alpha)-1}(j-k-1)^{\alpha-1} + C j^{-\frac{1}{2}(2\beta-\alpha)}k^{\frac{1}{2}(2\beta-\alpha)+\alpha-2}.
\]

It follows that we can write
\[
|\Phi_{2,j,k,n}| \leq \sum_{r=1}^q \left(\frac{q}{r}\right)C r^{2^{-(q-r)}A_{j,k}^{q-r}(k/j)^{\frac{1}{2}(2\beta-\alpha)}(k^{-r}(j-k-1)^{\alpha-1} + k^r(\alpha-2))}
\]

Using that \(k/j \leq 1, |A_{j,k}| \leq C|j-k-1|^{\alpha-2}\) for \(|j-k| \geq 2, k^{-r} \leq C r^{j-k} \leq C r^{(j-k-1)^{\alpha-2}}\), and similarly that \(k^{\alpha-2} \leq C(j-k-1)^{\alpha-2}\), we obtain
\[
|\Phi_{2,j,k,n}| \leq C^q (j-k-1)^{q(\alpha-2)}
\]
for some constant \(C > 1\). This implies
\[
\frac{2q!c_2^2}{n} \sum_{j=\lfloor n^\gamma \rfloor}^{\lfloor nt \rfloor-1} \sum_{k=\lfloor j/3 \rfloor+1}^{j-2} \left[1 + \eta_{j,n}\right]^{-\frac{2}{3}} \left[1 + \eta_{k,n}\right]^{-\frac{2}{3}}|\Phi_{2,j,k,n}| \leq \frac{2q!c_2^2}{n} C_q \sum_{j=\lfloor n^\gamma \rfloor}^{\lfloor nt \rfloor-1} j^{q(\alpha-2)+1}.
\]

which converges to zero as \(n\) tends to infinity. Moreover, there exists \(\rho \in (0,1)\) such that
\[
\frac{1}{n} C_q \sum_{j=\lfloor n^\gamma \rfloor}^{\lfloor nt \rfloor-1} j^{q(\alpha-2)+1} \leq \rho
\]
for \(n\) large enough.

Therefore,
\[
\lim_{n \to \infty} D_n = q!c_2^2 \lim_{n \to \infty} \frac{2}{n} \sum_{j,k \in D_1, |j - k| < j - 2} 2^{-q(k/j)^{\frac{1}{2}(2\beta-\alpha)}A_{j,k}^q}.
\]

Since we know from Step 1 and Step 4 that the terms with \(j, k \in D_1\) and \(k \leq \lfloor j/3 \rfloor\) do not contribute to the limit, we can add these terms and write
\[
\lim_{n \to \infty} D_n = q!c_2^2 \lim_{n \to \infty} \frac{2}{n} \sum_{j=3}^{\lfloor nt \rfloor-1} \sum_{k=1}^{j-2} 2^{-q(k/j)^{\frac{1}{2}(2\beta-\alpha)}A_{j,k}^q}.
\]

We use the change of index \(m = j - k\) to write
\[
\lim_{n \to \infty} D_n = q!c_2^2 \lim_{n \to \infty} \frac{2}{n} \sum_{k=1}^{\lfloor nt \rfloor-1} \sum_{m=0}^{k-1} 2^{-q} \left(\frac{k}{k+m}\right)^{\frac{1}{2}(2\beta-\alpha)} ((m+1)^{\alpha} + (m-1)^{\alpha} - 2m^\alpha)^q,
\]
\[
\text{(27)}
\]
which leads to
\[
\lim_{n \to \infty} D_n = 2^{1-q} q! c_q^2 t \sum_{m=2}^{\infty} ((m + 1)^\alpha + (m - 1)^\alpha - 2m^\alpha)^q .
\]  
(28)

Thus, the limit follows from results (25) and (28), and Lemma 5.2 is proved.

\begin{remark}
Notice that the right-hand side of (27) is bounded by
\[
2q! c_q^2 t \sum_{m=2}^{\infty} 2^{-d} ((m + 1)^\alpha + (m - 1)^\alpha - 2m^\alpha)^d \rho_m^{q-d} ,
\]
where \( \sup_{m \geq 2} \rho_m < 1 \).
\end{remark}

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