POSITIVE FACTORIZATION S OF PSEUDOPERIODIC AUTOMORPHISMS

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ABSTRACT. The main result of this paper is that a pseudoperiodic automorphism on an oriented surface with boundary such that a power of it is a composition of positive Dehn twists around disjoint simple closed curves including all boundary components, admits itself a positive factorization. The main ingredients of the proof are a solution for the problem in families of strongly pseudoconvex manifolds with bounds in the appropriate Lipschitz topology by Greene and Krantz, deformation theory for pseudoconvex manifolds as developed by Laufer, Bogomolov and De Oliveira, and a topological characterization of the monodromies on links of isolated complex surface singularities by Anne Pichon. This generalizes previous results by Honda, Kazez and Matić on the punctured torus and greatly narrows the difference between the monoid of right-veering mapping classes and the monoid of mapping classes admitting positive factorizations. We define a new invariant of automorphisms of surfaces that detects how far is an automorphism from admitting a positive factorization. Finally we apply the main theorem to give a sufficiency criterion for certain pseudoperiodic automorphisms to admit a positive factorization.

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Date: January 24, 2020.
2010 Mathematics Subject Classification. 37E30, 32S25 (primary), and 14B07, 32S30 (secondary).
1. Introduction

Although the main result of this paper has its greatest impact in the theory of mapping class groups, its original motivation lies in the field of singularity theory. We describe this motivation first.

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a representative of a holomorphic map germ having an isolated singularity at the origin. Then for a generic linear form $\ell : \mathbb{C}^2 \to \mathbb{C}$ and $\epsilon'$ a real number with $|\epsilon'|$ small enough, we have that

$$\tilde{f} := f + \epsilon'\ell : \mathbb{C}^2 \to \mathbb{C}$$

is what is usually known as a morsification of $f$. That is, a holomorphic map that only has Morse-type singularities with distinct critical values all of them close to $0 \in \mathbb{C}$. On one hand, we have that

$$f^{-1}(\partial D_\delta) \cap B_\epsilon \to \partial D_\delta$$

(1.1)

is a locally trivial fibration for $B_\epsilon \subset \mathbb{C}^2$ a closed ball of small radius $\epsilon > 0$ and $D_\delta \subset \mathbb{C}$ a disk of small radius $0 < \delta << \epsilon$. This is known as the Milnor fibration on the tube [Mil68].

Since transversality is an open condition,

$$\tilde{f}^{-1}(\partial D_\delta) \cap B_\epsilon \to \partial D_\delta$$

defines an equivalent fibration to that of eq. (1.1) for $\epsilon'$ small enough. Since $\tilde{f}$ has only singular points near the origin $0 \in \mathbb{C}^2$, one can easily deduce that the monodromy associated with $f$ can be written as a composition of right-handed Dehn twists. This classical line of argument is known as Picard-Lefschetz theory and works well when the ambient space is a smooth Stein manifold.

Let us change $\mathbb{C}^2$ by a singular ambient space. Take $X'$ to be an isolated complex surface singularity. Let $f' : X' \to \mathbb{C}$ be an element of its ring of functions, that is an holomorphic map. Assume that $f'$ is reduced as an element on the structure ring of $X'$. Associated with $f'$ there is also a locally trivial fibration

$$f'^{-1}(\partial D_\delta) \cap X' \cap B_\epsilon \to \partial D_\delta$$
on a tube. This one is known as the Milnor-Lê fibration [Trà77]. Now we observe that a similar game does not work since a small perturbation $f' + \epsilon'\ell$ has some Morse singularities on the smooth part of $(f' + \epsilon'\ell)^{-1}(\partial D_\delta) \cap X' \cap B_\epsilon$ and the singularity defined by $\ell$ at the singular point of $X'$. To see more on this we recommend the original work by Siersma [Sie95]. So, by these methods, we cannot conclude that the monodromy defined by $f'$ can be written as a composition of right-handed Dehn twists. To decide if these monodromies admit (or not) positive factorizations is the original motivation of this paper.

We argument as follows to solve this problem. First we consider the minimal resolution $X \to X'$ of $X'$. This gives us a strongly pseudoconvex surface $X$ with a
non-trivial exceptional set without curves of the first kind. For these kind of surfaces, Laufer [Lau81, Lau80] and later Bogomolov and De Oliveira [BD97] developed methods to prove the existence of a deformation $\omega : X \to Q$ of $X$ over a disk in such a way that generic fibers are Stein. As a $C^\infty$-fibration, this deformation is a trivial fibration. We lift $f'$ to a holomorphic map on the resolution $f : X \to \mathbb{C}$ and extend this map to the fibers $X_t$ of the fibration as complex valued (but not holomorphic) smooth maps $f_t : X_t \to \mathbb{C}$.

Now we ask the question: Can we correct $f_t$ so that it becomes holomorphic? Note that $\partial f_t$ is small in fibers close to the central fiber of the deformation. So we need to solve what is classically known as the $\partial$-problem and get a nice enough bound for the solution. More concretely, we seek for $u_t : X_t \to \mathbb{C}$ such that

$$\partial f_t = \partial u_t$$

and such that $u_t$ takes small values. It turns out that the PDE’s methods initiated by Kohn and Nirenberg [KN65] and followed Hörmander [Hör65] are not enough. These methods give only bounds of the $L^2$-norm of $u$ which does not assure us that $u$ takes small values. A few years later, Henkin [Hen70] developed integral representation methods to improve these results. His work was later generalized by Kerzman’s [Ker71] manifolds that can be covered by strongly pseudoconvex open domains by patching Henkin’s solution. This explicit integral solutions allowed Greene and Krantz to produce a version of this result for families of strongly pseudo-convex manifolds (where the complex structure varies smoothly) as in [GK82, Section 4]. This is the result that we invoke.

In the end we are able to construct such $u_t : X_t \to \mathbb{C}$ with $f_t - u_t$ a holomorphic map that defines a locally trivial fibration equivalent to the one defined by the original $f$ on $f'^{-1}(\partial D_\delta) \cap X' \cap B_\varepsilon$. Since $X_t$ is Stein can apply Picard-Lefschetz theory to conclude that our original monodromy admits a positive factorization. This is what we prove in Theorem 4.2 which is the main result.

In [Pic01], A. Pichon proved, using a previous result by Winters [Win74], a purely topological characterization of the monodromies that appear in links of isolated complex surface singularities associated with reduced holomorphic map germs. This class coincides with the class of pseudoperiodic diffeomorphisms of surfaces with boundary $\phi : \Sigma \to \Sigma$ such that $\phi^n$ is a composition of powers of right handed Dehn twist around disjoint simple closed curves including all boundary components (à torsades négatives in [Pic01]). So proving that these monodromies admit positive factorizations has highly unexpected consequences in the theory of mapping class groups. As a direct corollary we find, for example, that all freely periodic diffeomorphisms with positive fractional Dehn twist coefficients automatically admit positive factorizations. Observe that Honda, Kazez and Matić, [HKM08] proved this for the punctured torus.

Next we describe the organization of the paper.
Outline of the paper. We start in section 2 by recalling some theory and definitions about pseudoconvex manifolds and their deformations. Then we discuss solutions to the $\bar{\partial}$-problem on Stein manifolds. First we introduce a theorem by Kerzman which proves existence and uniform bounds for the solution in the precise setting in which we are interested and finally we state a version of a theorem (Theorem 2.6 in this work) by Greene and Krantz that gives a solution to the $\bar{\partial}$-problem for $(0,1)$ forms in smooth families of complex manifolds.

Next, in section 3, we fix notation and conventions about certain topics of the theory of mapping class groups. This is always a necessary step if one wants to make precise statements because conventions highly vary from one author to another. We focus on the notions of fractional Dehn twist coefficient and screw numbers associated with a pseudoperiodic automorphism of a surface. We end the section by stating Anne Pichon’s characterization of monodromies on links of isolated complex surface singularities.

In a first read of this work, an expert reader might skip one or the previous two sections and go straight to the proof of the main theorem.

In section 4 we start by proving a transversality proposition and then apply the theory developed in the previous sections to prove our main result Theorem 4.2 following the reasoning described previously in the introduction.

Finally, in section 5 we are devoted to explore the consequences of our main result. We start by exploiting a result by Baykur, Monden and Van Horn-Morris in Proposition 5.1 to prove that the pure mapping class group (diffeomorphisms up to isotopy free on the boundary) is generated as a semi-group by positive Dehn twists except in a few degenerate cases. This leads to the definition of two invariants (Definitions 5.4 and 5.10) that measure the failure of a pseudoperiodic automorphism to be in $\operatorname{Dehn}_{g,r}^+$. We use this together with Theorem 4.2 to give a sufficiency criterion (in Theorem 5.9) that tells us, in many cases, if a pseudoperiodic automorphism admits a positive factorization. The criterion roughly says that if the fractional Dehn twist coefficients are very big with respect to the screw numbers of the separating curves of the automorphism, then the corresponding pseudoperiodic automorphism admits a positive factorization.

Acknowledgments

I wish to thank Xavier Gómez-Mont for inspiring conversations. I am also very thankful to Mohammad Jabbari who is an expert in the $\bar{\partial}$-problem and gave me many useful references that lead me to find the Greene and Krantz result that I ended up using.

Finally, I thank Baldur Sigurðsson who read carefully an early version of this manuscript and pointed out a gap in a lemma that was placed instead of current Proposition 4.1. His critics and comments have helped me greatly improve the final manuscript.
2. Preliminary theory on complex geometry and singularity theory

We start by reviewing some theory on pseudoconvex complex manifolds that we need to state the results. Let $X$ be a complex manifold. Denoting its tangent bundle by $TX$ we can see its complex structure as an endomorphism $J : TX \to TX$ satisfying $J^2 = -\text{id}$ and an integrability condition. Let $d\rho$ denote the exterior derivative of $\rho$ for any smooth real valued function $\rho$. The complex structure $J$ allows us to define the complex exterior derivative as $d^C \rho := d\rho \circ J$. Let $\partial$ and $\bar{\partial}$ be the complex and complex conjugate parts of the exterior derivative. We can write $d = \partial + \bar{\partial}$ and $d^C = i(\partial - \bar{\partial})$.

Let $\rho : X \to \mathbb{R}$ be a smooth real valued function. We define a $(1, 1)$-form $\omega_\rho := -dd^C \rho$, a symmetric bilinear form $g_\rho := \omega_\rho(\cdot, J\cdot)$ and a Hermitian form $h_\rho = g_\rho - i\omega_\rho$. Under suitable circumstances these turn, respectively, into a symplectic form, a Riemannian metric and Hermitian metric. This motivates the following definition.

**Definition 2.1.** Let $X$ be a complex manifold and let $A \subset X$ be an open subset. We say that a smooth function $\rho : X \to \mathbb{R}$ is **strongly plurisubharmonic** (abbreviated spsh) on $A$ if the metric $g_\rho$ is positive-definite (a Riemannian metric) on $A$. If $g_\rho$ is positive-definite on all $X$, we simply say that $\rho$ is strongly plurisubharmonic.

We say that a smooth function $\rho : X \to \mathbb{R}$ is an **exhaustion function** if it is proper and bounded from below.

This leads us to the following important definitions.

**Definition 2.2.** We say that a complex manifold $X$ is **strongly pseudoconvex** if it admits an exhaustion function $\rho : X \to \mathbb{R}$ that is spsh outside a compact set. We say that $X$ is a **Stein manifold** if it admits an exhaustion function which is spsh on all $X$.

We say that $X$ is a **strongly pseudoconvex manifold with boundary** if it is a compact complex manifold with smooth boundary that admits an exhaustion function $\rho : \bar{X} \to \mathbb{R}$ which is spsh outside a compact analytic set and such that $\partial \bar{X}$ is the level set $\{x \in \bar{X} : \rho = 0\}$ of $\rho$. We say that $\bar{X}$ is a **Stein domain** if it is a compact complex manifold with smooth boundary that admits an exhaustion spsh function $\rho : \bar{X} \to \mathbb{R}$ such that $\partial \bar{X} = \{x \in \bar{X} : \rho = 0\}$.

If a complex analytic space $\bar{Y}$ with boundary $\partial \bar{Y}$ satisfies that its boundary is the level set of a spsh function defined on a neighborhood of it, we say that $\bar{Y}$ has a **strongly pseudoconvex boundary** or that its boundary is strongly pseudoconvex.

Observe that if $X$ is a Stein manifold with spsh exhaustion function $\rho : X \to \mathbb{R}$ and $\{x \in X : \rho = 0\}$ is a regular level set, then $\bar{X} := \{x \in X : \rho(x) \leq 0\}$ is a Stein domain by definition. Actually, it can be shown that all Stein domains can be obtained like this.
Remark 2.3. The difference between Stein manifolds and strongly pseudoconvex manifolds is that the latter might have a non-trivial exceptional set. And that is the only difference: a strongly pseudoconvex manifold that does not contain compact analytic sets of dimension greater than 0 is a Stein manifold. A typical example of a strongly pseudoconvex manifold that is not a Stein manifold is the resolution space of an isolated singularity.

For each pair of non negative integers \( p, q \geq 0 \) we denote by \( C^\infty_{p,q}(X) \) the global differential forms of type \((p,q)\) with coefficients smooth complex valued functions. Then the operator \( \bar{\partial} \) acts

\[
\bar{\partial} : C^\infty_{p,q}(X) \to C^\infty_{p,q+1}(X)
\]
as the complex conjugate part of the exterior derivative and satisfies \( \bar{\partial}^2 = 0 \).

Observe that \( C^\infty_{0,0}(X) \) coincides with the set of smooth complex valued functions and that for an element \( f \in C^\infty_{0,0}(X) \) the equation \( \bar{\partial}f = 0 \) is satisfied precisely when \( f \) is a holomorphic function.

The \( \bar{\partial} \)-problem in families of strongly pseudoconvex surfaces. Given an element \( g \in C^\infty_{p,q+1}(X) \) it is a classical problem to determine the existence (or lack thereof) of solutions \( u \in C^\infty_{p,q}(X) \) to the equation

\[
\bar{\partial}u = g
\]
and to bound the value of some norm on \( u \) by the value of some norm on \( g \). This problem is known as the \( \bar{\partial} \)-problem.

In this work we wish to control the growth of the partial derivatives of \( u \) under small smooth perturbations of the complex structure \( J \). For this purpose it is necessary to introduce several norms that are used in the theorems that are invoked related to this questions. We warn the reader that there does not seem to be a consensus in the notation used for the different norms. We follow \([GK82]\) which is the work from where we cite the main result used. The definitions are made for domains in \( \mathbb{C}^n \) and all of them have straightforward generalizations to complex valued functions defined on complex manifolds by patching local charts.

Let \( D \subset \mathbb{C}^n \) be an open set. For a complex valued function \( u : D \to \mathbb{C} \) we define the \textit{supremum norm} by

\[
||u||_\infty := \sup_{x \in X} |u(x)|
\]

where \( | \cdot | \) denotes the usual complex modulus. The next is called the \( C^j \) norm and controls the behavior of the partial derivatives of \( u \) up to the \( j \)-th order.
by the usual multi-index notation and $|\alpha|$ and $|\beta|$ are their total orders. We define $C^j(D) := \{u : D \to \mathbb{C} : ||u||_{C^j} < \infty\}$.

Kerzman, in [Ker71] proved the existence of solutions in the Stein manifold case with an uniform bound of the solution. He actually proved this bounds for the so-called Henkin solution previously discovered by Henkin in [Hen70]. Kerzman first proves the result for strongly pseudoconvex domains (which he defines to be open sets in $\mathbb{C}^n$ which are Stein manifolds) [Ker71, Theorem 1.2.1]. Then he notes that this result is as well valid for the case of strictly pseudoconvex domains in Stein manifolds [Ker71, Theorem 1.2.1", pg. 309].

The original idea for that result emanates from Henkin’s work [Hen70] for (0, 1)-forms in strongly pseudoconvex domains. A natural generalization of this estimate consists of the study of the regularity of the solutions to the $\bar{\partial}$-problem in terms of the regularity of a family of complex structures. In other words: if we consider a family of complex structures $J_t$ that moves continuously (in $t$) in a certain topology defined by some norm, in which topologies do the solutions (with respect to those complex structures) also move continuously?

This question is solved in different settings in [GK82]. In this work we only use their estimates for the Henkin solution for (0, 1)-forms with respect to a smooth family of complex structures. This is gathered in [GK82, Section 4]. As they explain in the introduction to Section 4 therein, the Henkin solution moves continuously in the $C^j$ topology provided that the complex structures varies sufficiently smoothly. From the results proven in that section we can extract the following useful result.

**Theorem 2.6.** Let $X$ be a strongly pseudoconvex manifold with boundary, let $g$ be a $\bar{\partial}$-closed $(0,1)$-form with $C^1$ coefficients and let $u$ be the Henkin solution to the problem $\bar{\partial}u = g$. Then, a small $C^k$ perturbation of the complex structure results in a $C^j$ variation of the solution $u$. Where $j$ can be expressed as a function depending only on $k$ and the dimension $n$ of $X$ in such a way that if $k$ is big enough then $j$ is as big as we want. In particular, we can assure that $j \geq 1$ for $k >> 1$.

Since the statement in the cited paper is expressed in a different language, we make the following observations.

**Remark 2.7.** The previous theorem is contained in the cited paper as follows. First, we have [GK82, Theorem 4.14, eq. 4.14.3'] which proves it for strongly pseudoconvex domains (relatively compact open sets with strongly pseudoconvex boundary) in $\mathbb{C}^n$. In the remark at the end of that same page it is explained that, with the help of Sobolev embedding theorems and similar techniques as the ones used by Kerzman in [Ker71], one can prove similar bounds for strongly pseudoconvex manifolds.
by patching the solutions (observe that a strongly pseudoconvex manifold can be defined as a manifold such that each point has a strictly convex open neighborhood).

Also note that [GK82, Theorem 4.14] is stated for perturbations of the boundary of the strongly pseudoconvex domain but at the end of the statement, the authors remark that similar bounds are true for variations of the complex structure (which they denote by $\prod_{1,0}$).

Finally, we make clear that we do not care about sharp bounds in this work. Actually, in our situation we only require that the first partial derivatives of the solution vary continuously (i.e. that the solution varies continuously in the $C^1$ topology) provided that the complex structure varies smoothly enough. So we do not investigate more in this direction and that is why we have just stated the weakest version of the result that is used here.

**Singularity theory and deformations of strongly pseudoconvex surfaces.** We introduce the notion of deformation of a complex manifold and state the result on deformations of strongly pseudoconvex surfaces proven by Bogomolov and De Oliveira exploiting a previous result by Laufer.

**Definition 2.8.** Let $X$ be a strongly pseudoconvex manifold (maybe with smooth boundary). We say that a map $\omega : X \to Q$ is a deformation of $X$ if

1. $X$ and $Q$ are complex manifolds (maybe with smooth boundary).
2. The map $\omega$ is a flat submersion.
3. The central fiber $\omega^{-1}(0)$ is biholomorphic to $X$.

Suppose that we start with a strongly pseudoconvex manifold with smooth boundary $\bar{X}$. Then, it is known that if one wishes to keep track of the boundary of $X$, the theory of deformations of $X$ becomes infinite-dimensional [BSW78, Corollary 4.2]. What Laufer did in his series of papers [Lau80, Lau81] to avoid this fact is to fix a compact analytic set $A \subset X$ and study deformations of $X$ near $A$. This allowed him to apply Kodaira-Spencer techniques to prove his results. It turns out that this setting is enough for many purposes, in particular for proving the existence of a versal deformation space [Lau80].

The next is a result obtained by Bogomolov and De Oliveira [BD97, Theorem (2)] that extends a previous result by Laufer [Lau81, Theorem 3.6] by iteratively applying it.

**Theorem 2.9** (Laufer & Bogomolov-De Oliveira). Let $\bar{X}$ be a strongly pseudoconvex surface with boundary that has a 1-dimensional exceptional set without curves of the first kind. Then there exists a deformation $\omega : \bar{X} \to Q$ over a small complex disk $Q \subset \mathbb{C}$ such that the fibers above $t \neq 0$ do not contain compact analytic curves.

The theorem above is another key ingredient for the proof of Theorem 4.2.
3. Preliminary theory on mapping class groups

We turn now our attention to the theory of mapping class groups. The purpose of this section is to collect the results on mapping class group that we use throughout the rest of the text as well as to fix notation and conventions.

**Notation 3.1.** The symbol $\Sigma_{g,r}$ denotes an oriented compact surface of genus $g$ and $r$ boundary components. When it is clear from the context or unnecessary in the discussion, the subindex is dropped and we just write $\Sigma$.

We write $\text{Mod}_{g,r}$ for the **mapping class group** of the surface $\Sigma_{g,r}$, that is, the group of diffeomorphisms of $\Sigma_{g,r}$ that restrict as the identity to $\partial \Sigma$ up to isotopy fixing the boundary pointwise. We write $\text{PMod}_{g,r}$ for what is usually known as the **pure mapping class group**, that is, the group of diffeomorphisms of a surface that leave each boundary component invariant, up to isotopy free on the boundary. Observe that there is a natural surjective homomorphism of groups

$$\text{Mod}_{g,r} \twoheadrightarrow \text{PMod}_{g,r}.$$ 

For $\phi \in \text{Mod}_{g,r}$ we denote by $\hat{\phi}$ its image in $\text{PMod}_{g,r}$ by the above projection. It is a known result that these mapping class groups do not depend on whether we consider homeomorphisms or diffeomorphisms (see for example [FM12, Section 1.4]). For this reason we choose to speak of **automorphisms** to refer to a particular representative of a mapping class.

The letter $\phi$ denotes either an automorphism or a mapping class. It is always clear from the context which is the case.

Given a simple closed curve $\gamma \subset \Sigma$ we denote by

$$t_\gamma : \Sigma \to \Sigma$$

a **right-handed Dehn twist** around $\gamma$ or its mapping class in $\text{Mod}_{g,r}$. The support of $t_\gamma$ is concentrated in a tubular neighborhood of $\gamma$. Also, the mapping class of $t_\gamma$ only depends on the isotopy class of the simple closed curve $\gamma$.

Given a boundary component $B \subset \partial \Sigma$ we denote by $t_B$ the **right-handed boundary Dehn twist** around $B$, which is by definition a right-handed Dehn twist around a simple closed curve parallel to $B$. The set of mapping classes that admit a factorization consisting only of right-handed Dehn twists is clearly a monoid and we denote it by $\text{Dehn}^+_g$. We also call these factorizations **positive**.

The following is a landmark result in mapping class group which we state to motivate the next definition.

**Theorem 3.2** (See [Thu88] and Corollary 13.3 from [FM12]). Let $\phi : \Sigma \to \Sigma$ be an orientation preserving automorphism that restricts to the identity on $\partial \Sigma$. Then there exists $\phi'$ isotopic to $\phi$ and a collection $\mathcal{C}$ of disjoint simple closed curves such that:

(i) The collection of curves is invariant by $\phi'$, i.e. $\phi'(C) = C$.
(ii) The automorphism $\phi'$ restricted to each connected component of $\Sigma \setminus C$ (and its iterations by $\phi'$) is either periodic or pseudo-Anosov.

The decomposition given by Theorem 3.2 is called \textbf{Nielsen-Thurston decomposition} and is unique up to isotopy if the system of curves $C$ is minimal. This decomposition leads to the following definition.

\textbf{Definition 3.3.} Let $\phi \in \text{Mod}_{g,r}$, we say that $\phi$ is \textit{pseudoperiodic} if only periodic pieces appear in its Nielsen-Thurston decomposition.

Pseudoperiodic automorphisms are of special importance in Singularity Theory: they are the only type of automorphism that appear in this world. This is almost equivalent to the fact the links of isolated complex surface singularities are graph manifolds. Actually, only a restricted type of pseudoperiodic automorphisms shows up in this area, to be able to describe precisely this class we introduce a couple of famous notions in mapping class group.

\textbf{Notation 3.4.} Let $\phi \in \text{Mod}_{g,r}$ and let $B_1, \ldots, B_r$ be the connected components of $\partial \Sigma_{g,r}$. We denote the fractional Dehn twist coefficient of $\phi$ at $B_i$ by $\text{fr}(\phi, B_i)$. It is a rational number and we use the sign convention (same as in [HKM07]) that a right-handed Dehn twist around a boundary parallel curve has fractional Dehn twist coefficient equal to $+1$.

If we assume that $\phi \in \text{Mod}_{g,r}$ is pseudoperiodic, the fractional Dehn twist coefficient can be defined as follows. Let $B_i \subset \partial \Sigma_{g,r}$ and let $n \in \mathbb{N}$ be such that $\phi^n$ is freely isotopic to the identity on the periodic piece containing $B_i$. Then, in a collar neighborhood of $B_i$, the automorphism $\phi^n$ isotopic to $t^{k}_{B_i}$ for some $k \in \mathbb{Z}$. Then

$$\text{fr}(\phi, B_i) := \frac{k}{n}.$$ 

Note that $k$ might be negative meaning that $\phi^n$ is a power of a left-handed boundary Dehn twist.

Similarly we can define the notion of \textit{screw number} for each orbit of separating curves of the Nielsen-Thurston decomposition of a pseudoperiodic automorphism $\phi$. Let $C_1, \ldots, C_a$ be an orbit of curves in $C$ (as in Theorem 3.2) so $\phi(C_i) = C_{i+1}$ for all $i = 1, \ldots, a - 1$ and $\phi(C_a) = C_1$. Let $n$ be such that $\phi^n$ is the identity on the two periodic pieces (which might be the same) adjacent to the orbit. Then near any of the curves $C_i$ in the orbit, $\phi$ is isotopic to $t^{k}_{C_i}$ for some $k \in \mathbb{Z}$. We define the screw number of $\phi$ at the orbit, as

$$\text{sc}(\phi, C_i) := \frac{ka}{n}.$$ 

This rational number measures the amount of twisting of $\phi^n$ around $C_i$ and it is the same for all curves in the same orbit so it is well defined the orbit itself or for any particular curve in $C$. 

We warn the reader that our conventions may differ from those of other authors. For example, Montesinos and Matsumoto [MMA11] have the same criterion that we do in the sign of fractional Dehn twist coefficients but have the opposite in screw numbers. Our choice of signs makes that *turning to the right around a curve is positive* no matter if the curve is boundary parallel or a non separating simple closed curve in $C$.

The notion of fractional Dehn twist coefficient has been present in the literature for a long time and it is difficult to attribute it to a particular author. For example, Gabai [Gab97] already used these quantities to measure the difference between an automorphism admitting a pseudo-Anosov representative and this representative. More recently, Honda, Kazez and Matić [HKM07] used fractional Dehn twist coefficients as a topological tool to detect tight open books from the monodromy. More concretely, they recovered the notion of an automorphism being right-veering (see [HKM07, Definition 2.1]) and defined the monoid Veer$_{g,r}$ consisting of all automorphisms of the surface $\Sigma_{g,r}$ that are right-veering. The following theorem contains just a couple of results of [HKM07] relevant for this work.

**Theorem 3.5.** Let $\phi \in \text{Mod}_{g,r}$ then

(i) $\text{Dehn}^+_{g,r} \subset \text{Veer}_{g,r}$ ([HKM07, Lemma 2.5]).

(ii) A contact structure on a 3-manifold is tight if and only if all the supporting open books have right-veering monodromy ([HKM07, Theorem 1.1]).

Also, we have the following theorem.

**Theorem 3.6** (Loi-Piergallini [LP01], Giroux [Gir02]). A contact structure is holomorphically fillable if and only if there exists an open book supporting it with monodromy in $\text{Dehn}^+_{g,r}$.

The above two theorems emphasize the importance that it has in Contact Topology to be able to the determine the gap between Veer$_{g,r}$ and $\text{Dehn}^+_{g,r}$.

As we said in the introduction, this work is concerned with automorphisms that appear as monodromies of reduced holomorphic map germs defined on isolated complex surface singularities. Anne Pichon proved [Pic01, Théorème 5.4], using a deep result by Winters [Win74, Theorem 4.3], the following purely topological characterization of these automorphisms. This result is another key ingredient for the proof of Theorem 4.2.

**Theorem 3.7** (Anne Pichon). Let $\phi : \Sigma \to \Sigma$ be an automorphism of a connected and oriented surface with non-empty boundary. Then there exists an isolated complex surface singularity $X$ and a reduced holomorphic map germ $f : X \to \mathbb{C}$ whose associated monodromy is $\phi$ if and only if some power $\phi^n$ of the automorphism is a composition of powers of right-handed Dehn twists around disjoint simple closed curves including curves parallel to all boundary components. This property is equivalent to all fractional Dehn twists and screw numbers being strictly positive.
We name this important class of pseudoperiodic automorphisms once and for all:

**Definition 3.8.** If $\phi \in \text{Mod}_{g,r}$ is a pseudoperiodic automorphism with strictly positive fractional Dehn twist coefficients and screw numbers, we say that it is **fully right-veering**.

One can see in [HKM07] that the definition of right-veering automorphism refers only to the behavior of $\phi$ at the boundary of $\Sigma$. Roughly, a fully-right veering pseudoperiodic automorphism is a right-veering pseudoperiodic automorphism whose restrictions to each periodic piece are also right-veering.

4. **Main theorem**

In this section we prove that all fully right-veering pseudoperiodic automorphisms admit positive factorizations. We start by proving a proposition that has an interest of its own and that plays a key role in the proof of Theorem 4.2.

**Proposition 4.1.** Let $\omega : \bar{X} \rightarrow Q$ be a deformation of a strongly pseudoconvex manifold $\bar{X} := \omega^{-1}(0)$ with smooth boundary, where $Q \subset \mathbb{C}$ is a disk. And let $f : \bar{X} \rightarrow \mathbb{C}$ be a holomorphic map. Then there exists a neighborhood $Q' \subset Q$ of $0$ and a $C^1$ map $F : \omega^{-1}(Q') \rightarrow \mathbb{C}$ such that $F|_{\omega^{-1}(0)}$ is holomorphic for all $t \in Q'$ and $F|_{\omega^{-1}(0)} = f$.

**Proof.** We observe that the map $\omega : \bar{X} \rightarrow Q$ is actually a $C^\infty$ locally trivial fibration and, since $Q$ is contractible, it is actually a trivial fibration. So there exists a diffeomorphism $\bar{X} \times Q \rightarrow \bar{X}$ which induces diffeomorphisms $T_t : \bar{X}_t \rightarrow \bar{X}$ with $T_0 = \text{id}$. This allows us to see the deformation $\omega : \bar{X} \rightarrow Q$ as a family of complex structures $J_t$ on $\bar{X}$ varying smoothly on $t \in Q$. For instance, if $J'_t$ is the complex structure of $\bar{X}_t$, then one can consider $J_t := T_{t*}J'_t$ as the corresponding complex structure on $\bar{X}$. And by letting $\bar{\partial}_t$ be the complex conjugate derivative in $\bar{X}_t$ and $\bar{\partial}_t$ be the corresponding one in $\bar{X}$ with the complex structure $J_t$, we find that if $g, f : X \rightarrow \mathbb{C}$ satisfy the equation

$$\bar{\partial}_tg = \bar{\partial}_tf$$

then, by precomposing, the equation

$$\bar{\partial}'_t(g \circ T_t) = \bar{\partial}'_t(f \circ T_t)$$

holds in the cohomology ring of $\bar{X}_t$ because $(\bar{X}, J_t)$ and $(\bar{X}_t, J'_t)$ are isomorphic as complex manifolds via $T_t$. This discussion tells us that we can solve our problem in the central fiber of our deformation equipped with different complex structures varying smoothly on $t$ and that this is equivalent to solving the extension problem in each fiber of the deformation as the statement of the proposition requires.

By definition, $\bar{\partial}_tf$ is a $\bar{\partial}$-closed $(0,1)$-form for all $t$. Since $f$ is smooth, we find that $\bar{\partial}_tf$ has in particular $C^1$ coefficients. Therefore, we can apply Greene and Krantz...
result (Theorem 2.6) and get that there exists a family of solutions \( \{ u_t : X \to \mathbb{C} \} \) for the \( \overline{\partial} \) problems
\[
\overline{\partial} u_t = \overline{\partial} f.
\]
Moreover, since the family of complex structures is \( C^\infty \), we can assure that our family of solutions is at least \( C^1 \) on the parameter \( t \) for \( t \) varying in a small disk \( Q' \subset Q \).

The Henkin solution in the central fiber is 0 since \( f \) is holomorphic, i.e. \( u_0 = 0 \). Finally we define the sought \( F : \omega^{-1}(Q') \to \mathbb{C} \) fiberwise by \( F|_{\tilde{X}_t} := f \circ T_t - u_t \circ T_t = (f - u_t) \circ T_t \). By construction, \( \overline{\partial} F|_{\tilde{X}_t} = 0 \) (so each restriction is holomorphic). Also by construction \( F|_{\tilde{X}_0} = f \) because \( T_0 = \text{id} \) and \( u_0 = 0 \). And finally, \( F \) is at least \( C^1 \) on the \( t \) parameter because \( u_t \) is \( C^1 \) and because \( f \) and \( T_t \) are smooth. \( \square \)

Now we prove the main theorem of the paper.

**Theorem 4.2.** The following equivalent classes of pseudoperiodic automorphisms \( \phi : \Sigma \to \Sigma \) admit positive factorizations:

(i) Fully right-veering automorphisms.

(ii) Monodromies associated with reduced holomorphic map germs defined on isolated complex surface singularities.

(iii) Pseudoperiodic automorphisms such that there exists \( n \in \mathbb{N} \) such that \( \phi^n \) is a composition of powers of right-handed Dehn twists around disjoint simple closed curves including all boundary components.

**Proof.** First of all we recall that, indeed, the three classes of automorphisms coincide. Class (iii) is the definition of (i) (see Definition 3.8). And that (ii) is the same class of automorphisms as (iii) is the content of Anne Pichon’s result (Theorem 3.7).

So given such a pseudoperiodic automorphism \( \phi : \Sigma \to \Sigma \), there exists an isolated complex surface singularity \( X' \) and a holomorphic function \( f' : X' \to \mathbb{C} \) such that the monodromy of the corresponding Milnor-Lê fibration on the tube
\[
f'_{|f'^{-1}(\partial D_\delta) \cap X' \cap B_\epsilon} : f'^{-1}(\partial D_\delta) \cap X' \cap B_\epsilon \to \partial D_\delta
\]
has fiber diffeomorphic to \( \Sigma \) and monodromy conjugate to \( \phi \) in \( \text{Mod}_{g,r} \).

Let \( X' \) be a Milnor representative of \( X' \), that is, \( X' := X' \cap B_\epsilon \) where \( B_\epsilon \) is a closed ball of small radius that intersects \( X' \) transversely. We can assume as well that \( \partial X' \) is strongly pseudoconvex (recall Definition 2.2). The minimal resolution \( X \to X' \) induces a map \( \pi : X \to X' \) where \( X \) is a strongly pseudoconvex surface with smooth boundary that contains a nontrivial 1-dimensional exceptional set \( A = \cup_j A_j \) such that \( A_j \cdot A_j < -1 \) for all \( j \). By Laufer and Bogomolov-De Oliveira’s result (Theorem 2.9) there exists a deformation \( \omega : \tilde{X} \to Q \) of \( \tilde{X} = \omega^{-1}(0) \) over a small disk such that each fiber \( \tilde{X}_t := \omega^{-1}(t) \) does not contain compact analytic curves. By taking the deformation disk \( Q \) small enough, we can ensure that the boundaries \( \partial \tilde{X}_t \) are strongly pseudoconvex. Indeed, the central fiber has a strongly pseudoconvex
boundary and this is a property which is stable under small perturbations. Hence, we can assume that $\bar{X}_t$ is a Stein domain for all $t \neq 0$.

Since the morphism $\pi : \bar{X} \to \bar{X}'$ induced by the minimal resolution is an isomorphism outside the singular point of $X'$, the Milnor-Lê fibration of eq. (4.3) lifts to an equivalent fibration

$$f_{|f^{-1}(\partial D_\delta) \cap \bar{X}} : f^{-1}(\partial D_\delta) \cap \bar{X} \to \partial D_\delta$$

Now we apply Proposition 4.1 to our deformation $\omega$ and we get a smaller disk $Q' \subset Q$ and a map $F : \omega^{-1}(Q') \to Q'$ that extends $f : X \to \mathbb{C}$. Moreover, that same proposition tells us that the restriction of $F$ to $\bar{X}_t$ is holomorphic for all $t \in Q'$ and the first order partial derivatives of $F$ in directions tangent to the fibers of $\omega$ vary continuously with $t$ because $F$ is $C^1$.

Let $f_t := F|_{\bar{X}_t}$. Since transversality is an open condition and the partial derivatives of the functions $f_t$ vary continuously on $t$ we find that there exists a (maybe smaller) disk $Q'' \subset Q$ containing 0 such that the maps

$$f_{t|f^{-1}_{t}(\partial D_\delta) \cap \bar{X}_t} : f^{-1}_{t}(\partial D_\delta) \cap \bar{X}_t \to \partial D_\delta$$

are locally trivial fibrations equivalent to eq. (4.4) (and so to eq. (4.3)) for all $t \in Q''$.

Again, by maybe shrinking a little bit more $Q''$, we can assert that $f_t$ has no critical points on $\partial \bar{X}_t$ because $f$ did not have critical points on $\partial \bar{X}$ and being a submersion is a stable property. Since $\bar{X}_t$ is a Stein domain, it cannot contain compact analytic curves. So the map from eq. (4.5) only has isolated critical points in a smooth ambient space. Each of these isolated critical points can be locally seen as an isolated plane curve singularity. We conclude by a direct application of Picard-Lefschetz theory that the monodromy of eq. (4.5) admits a positive factorization since it is a composition of monodromies of isolated plane curve singularities. Then the monodromy of the equivalent fibrations eq. (4.4) and eq. (4.3) also admit positive factorizations. \hfill \Box

In [HKM08], Honda, Kazez and Matić proved that every right-veering and freely periodic automorphism of the punctured torus admits a positive factorization. This proof is carried out by checking each case since the list of freely periodic automorphisms of the torus is short. Next we observe that the previous theorem is a vast generalization of this fact, in particular we obtain the following straightforward corollary.

**Corollary 4.6.** Freely periodic automorphisms with positive fractional Dehn twist coefficients admit positive factorizations.

5. Consequences of the main theorem

In this section we define an invariant of a mapping class $\phi \in \text{Mod}_{g,r}$ for all $g, r$ except for a small family of degenerate cases. This invariant is a non-negative integer
that measures how far is a mapping class from being in $\text{Dehn}^+_{g,r}$. This together with our main result (Theorem 4.2) gives sufficient conditions for a pseudoperiodic automorphism to be in $\text{Dehn}^+_{g,r}$.

For all $g \geq 0$ and $r = 0$ all the elements in the pure mapping class group $\text{PMod}_{g,r}$ (recall Notation 3.1) admit a positive factorization. A proof of this statement can be found in [FM12, page 124] under the name of “A strange fact”. In the next proposition we observe that a similar argument together with a recent result from [BMV17] implies a much more general result.

First we introduce a notion that we use in this section. For a surface $\Sigma_{g,r}$ denote by $B$ the mapping class resulting of the composition of a single right-handed Dehn twist around each boundary component, that is, $B := t_{B_1} \cdots t_{B_r}$. We call $B$ a boundary multitwist.

**Proposition 5.1 (A stranger fact).** Every element in the pure mapping class group $\text{PMod}_{g,r}$ can be written as a composition of right-handed Dehn twists along non-separating simple closed curves for all $g \geq 1$ and $r \geq 1$ except for the cases when $g = 1$ and $r > 9$.

**Proof.** We show that one can use the same trick used by Farb and Margalit in [FM12, page 124]. Observe that [BMV17, Theorem A and B] implies that for all $g \geq 1, r \geq 1$ (except in the exceptional cases mentioned in the statement) there exists in $\text{Mod}_{g,r}$ a positive factorization of a power of a boundary multitwist.

A boundary multitwist $B$ is the identity when seen in $\text{PMod}_{g,r}$. So we find that the identity $\hat{\text{id}} \in \text{PMod}_{g,r}$ can be expressed as a (non-empty) product of right-handed Dehn twists along non-separating simple closed curves $\gamma_1, \ldots, \gamma_k$. That is

$$\hat{t}_{\gamma_k} \cdots \hat{t}_{\gamma_1} = \hat{\text{id}}.$$

Therefore, multiplying both sides on the right by $\hat{t}_{\gamma_1}^{-1}$ yields a positive factorization of a left-handed Dehn twist around a non-separating simple closed curve. This together with the Change of Coordinates Principle [FM12, pg 37] which says that all non-separating simple closed curves are the same up to conjugation and the fact that $\text{PMod}_{g,r}$ is generated by (right and left-handed) Dehn twists along non-separating simple closed curves gives the result. Just observe that we can substitute any left handed Dehn twist in a factorization in $\text{PMod}_{g,r}$ by a composition of right-handed Dehn twists. \hfill $\square$

As a consequence we obtain the following corollary, which shows that composing with enough positive boundary Dehn twists stabilizes the whole mapping class group $\text{Mod}_{g,r}$ into $\text{Dehn}^+_{g,r}$.

**Corollary 5.2.** Let $\phi \in \text{Mod}_{g,r}$ and let $t_{B_1}, \ldots, t_{B_r}$ be right-handed boundary Dehn twists. Then for $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}$ big enough the element

$$t_{B_r}^{n_r} \cdots t_{B_1}^{n_1} \phi$$
is in $\text{Dehn}^+_{g,r}$ except in the cases $g = 1$ and $r > 9$.

**Proof.** Suppose that $\phi$ cannot be written as a composition of right-handed Dehn twists. Take its image $\hat{\phi}$ in $\text{PMod}_{g,r}$. By Proposition 5.1, $\hat{\phi}$ can be expressed as a product of right-handed Dehn twists in $\text{PMod}_{g,r}$.

Take the composition of these right-handed Dehn twists in $\text{Mod}_{g,r}$ and denote the resulting mapping class by $\tilde{\phi}$. Observe that $\phi \circ \tilde{\phi}^{-1}$ is a composition of powers of boundary Dehn twists and some of them are left-handed Dehn twists. Just compose $\phi$ with the inverse of these powers. □

**Remark 5.3.** It is worth noting that it is a trivial observation that derives from the definition of right-veering automorphism in [HKM07] that composing with enough right-handed boundary Dehn twists, one can get any automorphism to be right-veering. The above corollary says that, except in the degenerate cases, composing with enough right-handed Dehn twists puts any monodromy in $\text{Dehn}^+_{g,r}$.

The previous corollary contains the idea that there is an obstruction for an automorphism to be in $\text{Dehn}^+_{g,r}$ localized at the fractional Dehn twist coefficients of $\phi$. So, by means of Corollary 5.2 the following is well-defined.

**Definition 5.4.** Let $\Sigma_{g,r}$ be a surface such that $g > 1$ or with $g = 1$ and $r \leq 9$. Let $\phi \in \text{Mod}_{g,r}$, we define $N_\phi$ as the minimal natural number such that

$$B^{N_\phi} \phi \in \text{Dehn}^+_{g,r}.$$ We call $N_\phi$ the **correcting exponent** of $\phi$.

**Definition 5.5.** We say that a pseudoperiodic mapping class $\phi \in \text{Mod}_{g,r}$ is an **essential** mapping class $\text{Mod}_{g,r}$ if $|\text{fr}(\phi, B_i)| < 1$ and $|\text{sc}(\phi, C_i)| < 1$ for all boundary components $B_i$ and all orbits of invariant curves $C$ in its Nielsen-Thurston decomposition.

**Lemma 5.6.** Let $\Sigma_{g,r}$ be a surface with $g, r > 0$ and let $\phi \in \text{Mod}_{g,r}$ be pseudoperiodic. Assume that $\phi$ has exactly $s$ different orbits of invariant curves in its Nielsen-Thurston decomposition and let $C_1, \ldots, C_s$ be a curve for each orbit. Then there exist unique integers $n_1, \ldots, n_r, m_1, \ldots, m_s$ such that

1. The mapping class $\tilde{\phi} := \phi \cdot t_{B_1}^{n_1} \cdots t_{B_r}^{n_r} t_{C_1}^{m_1} \cdots t_{C_s}^{m_s}$ is essential.
2. If $\text{fr}(\tilde{\phi}, B_i) \neq 0$, its sign is the same as $\text{fr}(\phi, B_i)$.
3. If $\text{sc}(\tilde{\phi}, C_i) \neq 0$, its sign is the same as $\text{sc}(\phi, C_i)$.

**Proof.** Let $x \in \mathbb{Q}$ and denote $Z(x) \in \mathbb{Z}$ its integer part. Then we just define $n_i := -Z(\text{fr}(\phi, B_i))$ and $m_i := -Z(\text{sc}(\tilde{\phi}, C_i))$. From these it is straightforward to check that (i), (ii) and (iii) are satisfied. □

After the previous lemma, the following is a natural definition.
Definition 5.7. For a pseudoperiodic mapping class \( \phi \), we call the conjugacy class of the mapping class of \( \tilde{\phi} \) defined by Lemma 5.6 its essential part.

Remark 5.8. The screw numbers and fractional Dehn twist coefficients are conjugacy and isotopy invariants. Since the statement of Lemma 5.6 does not impose conditions on the curves that we are selecting for each orbit, it only makes sense to define the essential part associated to a given pseudoperiodic automorphism as a conjugacy class.

Observe that \( \phi \) is essential if and only if \( \phi \) is conjugate to \( \tilde{\phi} \). Putting together the results of this section yields the following sufficiency criterion that applies to certain pseudoperiodic mapping classes and decides if they are in Dehn\( ^+_{g,r} \).

Theorem 5.9. Let \( \phi \in \text{Mod}_{g,r} \) be a pseudoperiodic automorphism with \( \text{fr}(\phi, B_i) > 0 \) for all \( i = 1, \ldots, r \). Let us denote by \( sc_1, \ldots, sc_s < 0 \) its negative screw numbers and assume that the corresponding orbits of curves are not separating. Let \( k = 1 \) if \( g = 1, r < 9 \) or \( g \geq 2, r \leq 2g - 4 \) and \( k = 2 \) if \( g \geq 2, r > 2g - 4 \). If

\[
\sum_{i=1}^{s} k(|Z(sc_i)| + 1) \leq \min_{j \in \{1, \ldots, r\}} \text{fr}(\phi, B_j)
\]

then \( \phi \) admits a positive factorization.

Proof. First observe that composing \( \phi \) with a right-handed Dehn twist around any curve of an orbit of \( C \) increases the screw number of that orbit by 1.

Putting together the proof of Proposition 5.1 with [BMV17, Theorems (A) and (B))] we get that given a non-separating simple closed curve \( \gamma \), there exist \( c \) non-separating simple closed curves \( \alpha_1, \ldots, \alpha_c \) such that

\[
B^k t_{\gamma}^{-1} = t_{\alpha_1} \cdots t_{\alpha_c}
\]

holds in \( \text{Mod}_{g,r} \) where \( k = 1 \) if \( g = 1, r < 9 \) or \( g \geq 2, r \leq 2g - 4 \) and \( k = 2 \) if \( g \geq 2, r \leq 2g - 4 \) (this numerical conditions are contained in the cited theorems in [BMV17]). That is, we can say that a left-handed Dehn twist, costs a boundary multitwist if \( g = 1, r < 9 \) or \( g \geq 2, r \leq 2g - 4 \) and that it costs 2 boundary multitwists if \( g \geq 2, r \leq 2g - 4 \).

Boundary multitwists commute with each other element in \( \text{Mod}_{g,r} \) because their support is disjoint from any other simple closed curve. So let \( \gamma \) be a curve in \( C \) contained in an orbit with negative screw number and consider \( B^{-k} t_{\gamma} \phi \) (where \( k \) is as in the hypothesis). This automorphism is like \( \phi \) but with one fractional Dehn twist coefficient one smaller than \( \phi \) and the screw number at the orbit of \( \gamma \) one bigger. If the inequality of the hypothesis is satisfied we can iterate this process and get to a fully right-veering automorphism. Then Theorem 4.2 applies. \qed

We finish the article by introducing a natural refinement of the correcting exponent (recall Definition 5.4) which we think it could be useful for future exploration of the set \( \text{Veer}_{g,r} \setminus \text{Dehn}_{g,r}^+ \).
Let $\phi \in \text{Mod}_{g,r}$ and let $\hat{\phi}$ be its image in $\text{PMod}_{g,r}$. We define
\[
\Delta_{\phi} := \left\{ \prod t_{\gamma_i} \prod \hat{t}_{\gamma_i} = \hat{\phi} \right\}
\]
that is, the collection of all preimages in $\text{Mod}_{g,r}$ of positive factorizations of $\hat{\phi}$ in $\text{PMod}_{g,r}$. We know that this is non-empty in the majority of cases by Corollary 5.2.

Fix an order of the boundary components of $\Sigma_{g,r}$. That is, we have a labeling $B_1, \ldots, B_r$.

**Definition 5.10.** Let $\phi \in \text{Mod}_{g,r}$, we define the lattice
\[
L_{\phi} := \left\{ (a_1, \ldots, a_r) \in \mathbb{Z}^r | (a_1, \ldots, a_r) = \text{fr}(\phi \circ \Delta^{-1}) \text{ for some } \Delta \in \Delta_{\phi} \right\}
\]
Which is well defined since the rational parts of $\text{fr}(\Delta)$ and $\text{fr}(\phi)$ coincide because $\hat{\Delta} = \hat{\phi}$. We call $L_{\phi}$ the **correcting lattice** of $\phi$.

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