Some mixed character sums

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Abstract

In this paper we consider a variety of mixed character sums. In particular we extend a bound of Heath-Brown and Pierce to the case of squarefree modulus, improve on a result of Chang for mixed sums in finite fields, we show in certain circumstances we may improve on some results of Pierce for multidimensional mixed sums and we extend a bound for character sums with products of linear forms to the setting of mixed sums.

1 Introduction

Let $q$ be an integer, $\chi$ be a primitive multiplicative character mod $q$ and let $F$ be a polynomial of degree $d$ with real coefficients. We consider a variety of character sums mixed with terms of the form $e^{2\pi i F(n)}$. The simplest example of such sums are given by

$$\sum_{M<n\leq N+M} \chi(n)e^{2\pi i F(n)}.$$ 

(1)

For $q$ prime, these sums were first studied by Enflo [7] who outlines an argument which gives the bound

$$|S(\chi, F)| \leq N^{1-1/2^d} q^{(r+1)/2^d+2r/2},$$

where $r$ is the number of variables in $F$.
for integer \( r \geq 1 \) and \( N \leq H^{3/4+1/4r} \), which is nontrivial provided \( H > q^{1/4+o(1)} \) (see [10, Theorem 1.1]). This bound was improved by Chang [6] who showed that
\[
|S(\chi, F)| \ll Nq^{-\varepsilon}, \tag{2}
\]
when \( N \geq q^{1/4+\delta} \) and
\[
\varepsilon = \frac{\delta^2}{4(1+2\delta)(2+(d+1)^2)}.
\]
In the same paper, Chang also considered a generalisation of the sums (1) to arbitrary finite fields. More specifically, let \( q \) be prime, \( n \) an integer, \( \chi \) and \( \psi \) multiplicative and additive characters of \( \mathbb{F}_{q^n} \) respectively and let \( F \) be a polynomial of degree \( d \) with coefficients in \( \mathbb{F}_{q^n} \). Let \( \omega_1, \ldots, \omega_n \) be a basis for \( \mathbb{F}_{q^n} \) over \( \mathbb{F}_q \) and let \( \mathcal{B} \) denote the box
\[
\mathcal{B} = \{ \omega_1 h_1 + \cdots + \omega_n h_n : 1 \leq h_i \leq H \}.
\]
Then Chang showed that
\[
\sum_{h \in \mathcal{B}} \chi(h)\psi(F(h)) \ll H^n q^{-\varepsilon}, \tag{3}
\]
when \( H \geq q^{1/4+\delta} \) and
\[
\varepsilon = \frac{\delta^2 n}{4(1+2\delta)(2n+(d+1)^2)}.
\]
Recently, Heath-Brown and Pierce have improved on the bound of Chang (2) for prime fields showing that, subject to some conditions on \( r \) related to Vinogradov’s mean value theorem, we have
\[
\left| \sum_{M<n\leq M+N} \chi(n)e^{2\pi i F(n)} \right| \leq N^{1-1/r} q^{(r+1-d(d+1)/2)/4r(r-d(d+1)/2)}, \tag{4}
\]
which can be compared directly with the result of Chang by noting that for small \( \delta \) and \( N \geq q^{1/4+\delta} \), we have
\[
N^{1-1/r} q^{(r+1-d(d+1)/2)/4r(r-d(d+1)/2)} \leq Nq^{-\varepsilon},
\]
where \( \varepsilon \) behaves like (see [10, Section 4.2])

\[
\left( \frac{2\delta}{1 + \sqrt{1 + 2d(d + 1)\delta}} \right)^2.
\]

Pierce has also considered a multidimensional version of the sums (1). Let \( q_1, \ldots, q_n \) be primes, \( \chi_i \) a multiplicative character mod \( q_i \) and \( F \) a polynomial of degree \( d \) in \( n \) variables. In [13] Pierce has given a number of different bounds for sums of the form

\[
\sum_{N_i<h_i\leq N_i+H_i} \chi_1(h_1) \cdots \chi_n(h_n)e^{2\pi i F(h_1,\ldots,h_n)},
\]

and in the same paper Pierce also mentioned the following problem: Let \( L_1, \ldots, L_n \) be \( n \) linear forms in \( n \) variables which are linearly independent mod \( q \) and let \( F \) be a polynomial of degree \( d \) in \( n \) variables. Then consider giving an upper bound for the sums

\[
\sum_{1 \leq h_i \leq H} \chi\left(\prod_{j=1}^{n} L_j(h_1, \ldots, h_n)\right)e^{2\pi i F(h_1,\ldots,h_n)}.
\]

The sums (6) without the factor \( e^{2\pi i F(h_1,\ldots,h_n)} \) were first considered by Burgess [4] whose bound was later improved in general by Bourgain and Chang [1].

In this paper we consider giving bounds for a variety of mixed character sums. We first consider the problem of extending the bound of Heath-Brown and Pierce [4] to squarefree modulus. The main obstacle in doing this is bounding the double mean value

\[
\int_0^1 \cdots \int_0^1 \sum_{\lambda=1}^{\varphi(q)} \left| \sum_{1 \leq v \leq V} \beta_v \chi(\lambda + v)e^{2\pi i (\alpha_1 v + \cdots + \alpha_d v^d)} \right|^{2r} d\alpha_1 \cdots d\alpha_d,
\]

which for the case of prime modulus, as done by Heath-Brown and Pierce [10], relies on the Weil bounds for complete sums and Vinogradov’s mean value theorem. For the case of squarefree modulus, we can use the Chinese remainder theorem, as done by Burgess [2] for pure sums, so that we may apply the Weil bounds, although there are extra complications in incorporating bounds for Vinogradov’s mean value theorem. Doing this we end up with
a bound weaker than for prime modulus, although in certain cases we can get something just as sharp, in particular when \( q \) does not have many prime factors.

We give an improvement on the bound (2) of Chang for boxes over finite fields. We deal with the factor \( \psi(F(h)) \) in a similar fashion to the case of squarefree modulus. Our argument also relies on Konyagin’s bound on the multiplicative energy of boxes in finite fields [11], Vinogradov’s mean value theorem and the Weil bounds for complete sums.

We show in certain cases we may improve on the results of Pierce for the sums (5). The argument of Pierce relies on a multidimensional version of Vinogradov’s mean value theorem due to Parsell, Prendiville and Wooley [12]. Our improvement comes from averaging the sums (5) in a suitable way so we end up applying the classical Vinogradov mean value theorem rather than the multidimensional version. Although in order to do this, we need the range of summation in each variable not to get too short and each of the \( q_i \) in (5) not to be too small, so our result is less general.

Finally, we consider the problem mentioned by Pierce in [13], of bounding the sums (6). We obtain a result almost as strong as Bourgain and Chang [1] for the case of pure sums. An essential part of our proof is the bound of Bourgain and Chang on multiplicative energy of systems of linear forms.

Our arguments use a different approach to that of Heath-Brown and Pierce [10]. The technique we use to deal with the factor \( e^{2\pi i F(n)} \) can be though of an a generalisation of an idea of Chamizo [5], who gave a simple proof of the Burgess bound for incomplete Gauss sums, which in our case corresponds to mixed sums of degree 1. We also note that our method is capable of reproducing the results of Heath-Brown and Pierce [10]. We briefly indicate our technique for dealing with mixed sums in a general setting. Let \( F(x, y) \) be a polynomial of degree \( d \) with real coefficients, \( \Phi(k, v) \) a sequence of complex numbers and consider the bilinear form

\[
W = \sum_{1 \leq k \leq K} \sum_{1 \leq v \leq V} \gamma_k \beta_v \Phi(k, v) e^{2\pi i F(k, v)}.
\]
We have

\[ W \leq \sum_{1 \leq k \leq K} |\gamma_k| \sum_{1 \leq v \leq V} \beta_v \Phi(k, v) e^{2\pi i F(k, v)} \]

\[ \leq \sum_{1 \leq k \leq K} |\gamma_k| \max_{\alpha_1, \ldots, \alpha_d \in \mathbb{R}} \left| \sum_{1 \leq v \leq V} \beta_v \Phi(k, v) e^{2\pi i (\alpha_1 v + \cdots + \alpha_d v^d)} \right|. \]

For \( i = 1, \ldots, d \), we let

\[ \delta_i = \frac{1}{4V_i}, \]

and define the functions \( \phi_i(v) \) by

\[ 1 = \phi_i(v) \int_{-\delta_i}^{\delta_i} e^{2\pi i x v^d} dx, \]

so that for \( 1 < v \leq V \) we have

\[ \phi_i(v) = \frac{2\pi iv^i}{\sin(2\pi \delta_i v^i)} \ll \frac{1}{\delta_i} \ll v_i, \]

and

\[ W \leq \int_{-\delta_i}^{\delta_i} \cdots \int_{-\delta_d}^{\delta_d} \sum_{1 \leq k \leq K} |\gamma_k| \max_{\alpha_1, \ldots, \alpha_d \in \mathbb{R}} \left| \sum_{1 \leq v \leq V} \beta_v' \Phi(k, v) e^{2\pi i (\alpha_1 v + \cdots + \alpha_d v^d)} \right| dx, \]

where

\[ \beta_v' = \beta_v \prod_{i=1}^{d} \phi_i(v). \]

Applying Hölder’s inequality gives

\[ W^{2r} \leq V^{-(2r-1)d(d+1)/2} \left( \sum_{1 \leq k \leq K} |\gamma_k|^{2r/(2r-1)} \right)^{2r-1} \]

\[ \times \left( \sum_{1 \leq k \leq K} \int_{-\delta_i}^{\delta_i} \cdots \int_{-\delta_d}^{\delta_d} \max_{\alpha_1, \ldots, \alpha_d \in \mathbb{R}} \left| \sum_{1 \leq v \leq V} \beta_v' \Phi(k, v) e^{2\pi i ((\alpha_1 x_1 v + \cdots + \alpha_d x_d v^d)}} \right|^{2r} dx \right). \]
By extending the range of integration we may remove the condition \( \max_{\alpha_1, \ldots, \alpha_d \in \mathbb{R}} \), since
\[
\sum_{1 \leq k \leq K} \int_{-\delta_1}^{\delta_1} \ldots \int_{-\delta_d}^{\delta_d} \max_{\alpha_1, \ldots, \alpha_d \in \mathbb{R}} \left| \sum_{1 \leq v \leq V} \beta'_v \Phi(k, v) e^{2\pi i((\alpha_1 + x_1)v + \cdots + (\alpha_d + x_d)v^d)} \right| dx^n \ll \sum_{1 \leq k \leq K} \int_{[0,1]^d} \left| \sum_{1 \leq v \leq V} \beta'_v \Phi(k, v) e^{2\pi i(x_1v + \cdots + x_dv^d)} \right|^{2r} dx.
\]
At this point we may try and estimate the last double mean value by combining Vinogradov’s mean value theorem with techniques for estimating the sum
\[
\sum_{1 \leq k \leq K} \left| \sum_{1 \leq v \leq V} \beta'_v \Phi(k, v) \right|^{2r},
\]
or we may note that for some \( \beta''_v \) we have
\[
\sum_{1 \leq k \leq K} \int_{[0,1]^d} \left| \sum_{1 \leq v \leq V} \beta'_v \Phi(k, v) e^{2\pi i(x_1v + \cdots + x_dv^d)} \right|^{2r} dx \leq \sum_{1 \leq k \leq K} \left| \sum_{1 \leq v \leq V} \beta''_v \Phi(k, v) \right|^{2r}.
\]
Although our approach is different to that of Heath-Brown and Pierce, we also rely on bounds for Vinogradov’s mean value theorem. For integers \( r, d, V \), we let \( J_{r,d}(V) \) denote the number of solutions to the system of equations
\[
v_1^i + \cdots + v_r^i = v_{r+1}^i + \cdots + v_{2r}^i, \quad 1 \leq i \leq d, \quad 1 \leq v_j \leq V.
\]
Then it is conjectured that for any \( r, d, V \) we have
\[
J_{r,d}(V) \leq (X^r + X^{2r-d(d+1)/2})X^{o(1)}.
\] (7)
Recently, Wooley [15, 16] has made significant progress towards this conjecture. We state our main results in terms of the smallest integer \( r_d \) such that we have a bound
\[
J_{r,d}(V) \leq X^{2r-d(d+1)/2+o(1)},
\]
valid for all \( r \geq r_d \). Our results may then be combined with those of Wooley [15, 16] to give admissible values of \( r \) for which our bounds hold.
2 Main Results

In what follows, \( r_d \) will be defined as in the introduction. We also let \( D = d(d+1)/2 \). Our first two Theorems consider mixed sums to square-free modulus.

**Theorem 1.** Let \( q \) be squarefree and \( \chi \) a primitive character \( \mod q \). Let \( M, N, r \) be integers such that \( r \geq r_d \) and \( N \leq q^{1/2+1/4(r-D/2)} \). For any polynomial \( F(x) \) of degree \( d \) with real coefficients, we have

\[
\left| \sum_{M < n \leq M + N} \chi(n) e^{2\pi i F(n)} \right| \leq N^{1-1/r} q^{1/4r+D/8r(r-D/2)+1/4r(r-D/2)+o(1)}.
\]

Theorem 1 is slightly worse than the bound of Heath-Brown and Pierce \(^4\) for prime modulus. Although in certain cases we can get something almost as strong (except for the conditions on \( r \)).

**Theorem 2.** Let let \( s \) be an integer, \( q \) be squarefree with at most \( s \) prime factors and \( \chi \) a primitive character \( \mod q \). Let \( M, N, r \) be integers with \( N \leq q^{1/2+1/4(r-D)} \) and \( r \geq r_d + s + 1 \). For any polynomial \( F(x) \) of degree \( d \) with real coefficients, we have

\[
\left| \sum_{M < n \leq M + N} \chi(n) e^{2\pi i F(n)} \right| \leq N^{1-1/r} q^{(r+1-D)/4r(r-D)+o(1)}.
\]

Our next Theorem improves the bound of Chang for mixed sums in finite fields \(^6\). Before we state our result we introduce some notation. Let \( \omega_1, \ldots, \omega_n \) be a basis for \( \mathbb{F}_q^n \) over \( \mathbb{F}_q \) and let \( F \) be a polynomial of degree \( d \) in \( n \) variables with real coefficients. For \( x \in \mathbb{F}_q^n \) we define \( F(x) \) by

\[
F(x) = F(h_1, \ldots, h_n),
\]

where

\[
x = h_1 \omega_1 + \cdots + h_n \omega_n.
\]

**Theorem 3.** Let \( q \) be prime, \( n \) an integer and \( \chi \) be a multiplicative character of \( \mathbb{F}_q^n \). Let \( \omega_1, \ldots, \omega_n \) be a basis for \( \mathbb{F}_q^n \) as a vector space over \( \mathbb{F}_q \). For integer \( H \) let \( B \) denote the box

\[
B = \{h_1 \omega_1 + \cdots + h_n \omega_n : 0 < h_i \leq H\}.
\]
Let $F$ be a polynomial of degree $d$ in $n$ variables with real coefficients. Then if $H \leq q^{1/2}$ and $r \geq r_d$ we have

$$\left| \sum_{x \in B} \chi(x) e^{2\pi i F(x)} \right| \leq (\#B)^{1-1/r} q^{n(r-D+1)/4r(r-D)+o(1)}.$$ 

We note that the sums in Theorem 3 are slightly more general than those considered by Chang [6], since any additive character $\psi$ of $\mathbb{F}_{q^n}$ is of the form

$$\psi(x) = e^{2\pi i \text{Tr}(ax)/q},$$

for some $a \in \mathbb{F}_{q^n}$.

Our next Theorem improves on some results of Pierce [13] in certain circumstances.

**Theorem 4.** Let $q_1, \ldots, q_n$ be primes, which may not be distinct, and let $\chi_i$ be a multiplicative character mod $q_i$. Let $F$ be a polynomial of degree $d$ in $n$ variables with real coefficients and let $B$ denote the box

$$B = \{(h_1, \ldots, h_n) : M_i < h_i \leq M_i + H_i\}.$$ 

For any integer $r \geq r_d$, if for each $i$ we have $q_i > q^{1/2(r-D)}$ and $q^{1/2(r-D)} \leq H_i \leq q_i^{1/2+1/4(r-D)}$, then we have

$$\left| \sum_{x \in B} \chi_1(x_1) \cdots \chi_n(x_n) e^{2\pi i F(x)} \right| \leq (\#B)^{1-1/r} q^{(r-D+n)/4r(r-D)+o(1)},$$

where $q = q_1 \ldots q_n$.

Our final Theorem extends a bound of Bourgain and Chang [1] to the setting of mixed character sums.

**Theorem 5.** Let $q$ be prime and $\chi$ a multiplicative character mod $q$. Let $L_1, \ldots, L_n$ be linear forms with integer coefficients in $n$ variables which are linearly independent mod $q$. Let $B$ denote the box

$$B = \{(h_1, \ldots, h_n) : 1 < h_i \leq H\},$$

and let $F$ be a polynomial of degree $d$ in $n$ variables with real coefficients. Then if $H \leq q^{1/2}$ and $r \geq r_d$ we have

$$\left| \sum_{x \in B} \chi \left( \prod_{i=1}^n L_i(x) \right) e^{2\pi i F(x)} \right| \leq (\#B)^{1-1/r} q^{n(r-D+1)/4r(r-D)+o(1)}. $$

8
3 Preliminary results

The following can be thought of a multidimensional version of a technique from the proof of [9, Theorem 1].

Lemma 6. Let \( n = (n_1, \ldots, n_r) \) and \( G(n) \) be any complex valued function on the integers. Let \( B \) and \( B_0 \) denote the boxes

\[
B = \{(n_1, \ldots, n_r) \in \mathbb{Z}^k : 1 \leq n_i \leq N_i, 1 \leq i \leq r \},
\]

\[
B_0 = \{(n_1, \ldots, n_r) \in \mathbb{Z}^k : -N_i \leq n_i \leq N_i, 1 \leq i \leq r \}.
\]

Let \( U_1, \ldots, U_n \) and \( V \) be positive integers such that \( U_i V \leq N_i \) and let \( U \subset \mathbb{Z}^r \) be any set such that if \((u_1, \ldots, u_n) \in U\) then \( 1 \leq u_i \leq U_i \). Then for some \( \alpha \in \mathbb{R} \) we have

\[
\left| \sum_{n \in B} G(n) \right| \ll \frac{\log N_1 \cdots \log N_r}{V \# U} \sum_{n \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} G(n + vu) e^{2\pi i \alpha v} \right|.
\]

Proof. For \( 1 \leq i \leq r \) let

\[
f_i(x) = \begin{cases} 
\min(x - 1, 1, N_i - x), & \text{if } 1 \leq x \leq N_i, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
f(x) = \prod_{i=1}^{r} f_i(x_i).
\]

Let \( g(y) \) denote the Fourier transform of \( f \), so that

\[
g(y) = \frac{1}{(2\pi i)^r} \int_{\mathbb{R}^k} f(x) e^{-2\pi i \langle x, y \rangle} \, dx.
\]

Integrating the above integral by parts in each dimension gives

\[
|g(y)| \ll \prod_{i=1}^{k} \min \left( N_i, \frac{1}{|y_i|}, \frac{1}{|y_i|^2} \right). \tag{8}
\]

For \( u \in U \) and \( 1 \leq v \leq V \) we have

\[
\sum_{n \in B} G(n) = \sum_{n \in B_0} f(n + vu) G(n + vu),
\]
hence by Fourier inversion

\[
\sum_{n \in B} G(n) = \sum_{n \in B_0} f(n + v u) G(n + v u)
\]

\[
= \frac{1}{(2\pi i)^r} \sum_{n \in B_0} \int_{\mathbb{R}^r} g(y) G(n + v u) e^{2\pi i <n + v u, y>} dy.
\]

For \( u = (u_1, \ldots, u_r) \) we let \( |u| = u_1 \ldots u_r \) and \( u^{-1} = (u_1^{-1}, \ldots, u_r^{-1}) \). Then the change of variable \( y = u^{-1} x \) in the above integral gives

\[
\sum_{n \in B} G(n) = \frac{1}{(2\pi i)^r} \sum_{n \in B_0} \sum_{u \in U} \sum_{1 \leq v \leq V} \int_{\mathbb{R}^r} \frac{1}{|u|} g(u^{-1} x) G(n + v u) e^{2\pi i <n + v u, x>} dx,
\]

so that averaging over \( uv \) with \( u \in U \) and \( 1 \leq v \leq V \) we get

\[
\sum_{n \in B} G(n) = \frac{1}{(2\pi i)^r \# U \# V} \sum_{n \in B_0} \sum_{u \in U} \sum_{1 \leq v \leq V} \int_{\mathbb{R}^r} \frac{1}{|u|} g(u^{-1} x) G(n + v u) e^{2\pi i <n + v u, x>} dx,
\]

hence by (8)

\[
\sum_{n \in B} G(n) e^{2\pi i F(n)} \ll \frac{1}{V \# U} \sum_{n \in B_0} \sum_{u \in U} \int_{\mathbb{R}^r} \frac{1}{|u|} g(u^{-1} x) \left| \sum_{1 \leq v \leq V} G(n + v u) e^{2\pi i <v, x>} \right| dx
\]

\[
\ll \frac{1}{V \# U} \int_{\mathbb{R}^r} \prod_{i=1}^r \min \left( N_i, \frac{1}{|x_i|}, \frac{U_i}{|x_i|^2} \right)
\]

\[
\times \sum_{n \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} G(n + v u) e^{2\pi i <v, x>} \right| dx
\]

\[
\ll \max_{\beta \in \mathbb{R}} \frac{1}{V \# U} \sum_{n \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} G(n + v u) e^{2\pi i \beta v} \right|
\]

\[
\times \prod_{i=1}^r \left( \int_{-\infty}^{\infty} \min \left( N_i, \frac{1}{|y|}, \frac{U_i}{|y|^2} \right) dy \right).
\]

Since \( U_i \leq N_i \), we see that

\[
\prod_{i=1}^r \left( \int_{-\infty}^{\infty} \min \left( N_i, \frac{1}{|y|}, \frac{U_i}{|y|^2} \right) dy \right) \ll \log N_1 \ldots \log N_r,
\]
and the result follows by letting $\alpha$ be defined by

$$\max_{\beta \in \mathbb{R}} \sum_{n \in \mathcal{B}_0} \sum_{\mathcal{u} \in \mathcal{U}} \left| \sum_{1 \leq v \leq V} G(n + vu) e^{2\pi i v} \right| = \sum_{n \in \mathcal{B}_0} \sum_{\mathcal{u} \in \mathcal{U}} \left| \sum_{1 \leq v \leq V} G(n + vu) e^{2\pi i v} \right|.$$ 

\[\square\]

4 Mean value estimates

We keep notation as in the introduction and we recall that $J_{r,d}(V)$ denotes the number of solutions to the system of equations

$$v_1^i + \cdots + v_r^i = v_{r+1}^i + \cdots + v_{2r}^i, \quad 1 \leq i \leq d, \quad 1 \leq v_j \leq V.$$ 

The following is due to Burgess and is a special case of [2, Lemma 7], although since the statement of Burgess is weaker than what the argument implies, we reproduce the proof.

Lemma 7. Let $q$ be squarefree, $\chi$ a primitive character mod $q$, let $v = (v_1, \ldots, v_{2r})$ be a $2r$-tuple of integers such that at least $r + 1$ of the $v_i$’s are distinct and let

$$A_i(v) = \prod_{j=1, j\neq i}^{2r} (v_i - v_j).$$

Then for any $1 \leq i \leq 2r$ such that $A_i(v) \neq 0$ we have

$$\sum_{\lambda=1}^{q} \chi \left( \frac{(\lambda + v_1) \cdots (\lambda + v_r)}{(\lambda + v_{r+1}) \cdots (\lambda + v_{2r})} \right) \leq (q, A_i(v))^{1/2} q^{1/2 + o(1)}. \quad (9)$$

Proof. Let

$$q = p_1 \cdots p_k,$$

be the prime factorization of $q$, then by the Chinese remainder theorem there exists primitive characters

$$\chi_j \mod p_j, \quad 1 \leq j \leq k,$$

such that

$$\chi = \chi_1 \cdots \chi_k,$$
and
\[
\sum_{\lambda=1}^{q} x \left( \frac{(\lambda + v_1) \cdots (\lambda + v_r)}{(\lambda + v_{r+1}) \cdots (\lambda + v_{2r})} \right) = \prod_{j=1}^{k} \left( \sum_{\lambda=1}^{p_j} x_j \left( \frac{(\lambda + v_1) \cdots (\lambda + v_r)}{(\lambda + v_{r+1}) \cdots (\lambda + v_{2r})} \right) \right).
\]

We note that since at least \( r + 1 \) of the \( v_j \) are distinct there exists an \( i \) such that \( A_i(v) \neq 0 \), hence from [2, Lemma 1] we have
\[
\sum_{\lambda=1}^{p_j} x_j \left( \frac{(\lambda + v_1) \cdots (\lambda + v_r)}{(\lambda + v_{r+1}) \cdots (\lambda + v_{2r})} \right) \ll (p_j, A_i(v))^{1/2} p_j^{1/2}, \quad (10)
\]
which by the above gives
\[
\left| \sum_{\lambda=1}^{q} x \left( \frac{(\lambda + v_1) \cdots (\lambda + v_r)}{(\lambda + v_{r+1}) \cdots (\lambda + v_{2r})} \right) \right| \leq (q, A_i(v))^{1/2} q^{1/2+o(1)}.
\]

The following will be used in the proof of Theorem [11]

Lemma 8. Let \( q \) be squarefree, \( \chi \) a primitive character \( \operatorname{mod} q \), \( \beta_v \) be a sequence of complex numbers with \(|\beta_v| \leq 1\) and let
\[
W = \int_0^1 \cdots \int_0^1 \sum_{\lambda=1}^{q} \beta_v \chi(\lambda + v) e^{2\pi i (\alpha_1 v + \cdots + \alpha_k v^k)} |^{2r} d\alpha_1 \cdots d\alpha_k.
\]
(11)

Then we have
\[
W \leq (qV^r + q^{1/2} J_{r,k}(V)^{1/2} V^r) q^{o(1)}.
\]

Proof. Let \( J_{r,k}(V) \) denote the set of all \((v_1, \ldots, v_{2r})\) such that
\[
v_{1}^{j} + \cdots + v_{r}^{j} = v_{r+1}^{j} + \cdots + v_{2r}^{j}, \quad 1 \leq j \leq k, \quad 1 \leq v_i \leq V,
\]
then expanding the \( 2r \)-th power in the definition of \( W \) and interchanging summation and integration gives
\[
W \leq \sum_{(v_1, \ldots, v_{2r}) \in J_{r,k}(V)} \left| \sum_{\lambda=1}^{q} x \left( \frac{(\lambda + v_1) \cdots (\lambda + v_r)}{(\lambda + v_{r+1}) \cdots (\lambda + v_{2r})} \right) \right|.
\]
We break $\mathcal{J}_{r,k}(V)$ into sets $\mathcal{J}_{r,k}'(V)$ and $\mathcal{J}_{r,k}''(V)$, where

$\mathcal{J}_{r,k}'(V) = \{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}(V) : \text{at least } r+1 \text{ of the } v_i's \text{ are distinct}\}$,

$\mathcal{J}_{r,k}''(V) = \{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}(V) : (v_1, \ldots, v_{2r}) \not\in \mathcal{J}_{r,k}'(V)\}$,

so that $\# \mathcal{J}_{r,k}''(V) \ll V^r$ and by Lemma 7 we have

$$W \ll qV^r + q^{1/2+o(1)} \left( \sum_{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}'(V)} \sum_{A_i(v) \neq 0} (A_i(v), q)^{1/2} \right)$$

$$= qV^r + q^{1/2+o(1)} \left( \sum_{i=1}^{2r} \sum_{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}'(V)} (A_i(v), q)^{1/2} \right).$$

For $1 \leq i \leq 2r$ let

$$W_i = \sum_{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}'(V)} (A_i(v), q)^{1/2},$$

so that by the Cauchy-Schwartz inequality we have

$$W_i \leq \left( \sum_{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}'(V)} 1 \right)^{1/2} \left( \sum_{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}'(V)} (A_i(v), q) \right)^{1/2}$$

$$\leq \# \mathcal{J}_{r,k}(V)^{1/2} \left( \sum_{v_1, \ldots, v_{2r}} (A_i(v), q) \right)^{1/2}.$$
then since
\[ A_i(v) = \prod_{j=1}^{2r} (v_i - v_j), \]
we see there are \( q^{o(1)} \) choices for the numbers \((v_i - v_1), \ldots, (v_i - v_{2r})\) and choosing \( v_i \) determines \( v_1, \ldots, v_{2r} \), uniquely. Since there are \( V \) choices for \( v_i \) and each \( A_i(v) \ll V^{2r-1} \) we get
\[
\sum_{\substack{(v_1, \ldots, v_{2r}) \in \mathbb{Z}^{2r} \setminus A_i(v) \neq 0}} (A_i(v), q) \ll q^{o(1)} \sum_{d|q} \sum_{1 \leq A \ll V^{2r-1}} dV \\
\ll q^{o(1)} V^{2r} \sum_{d|q} 1 \ll q^{o(1)} V^{2r},
\]
which gives
\[
W \leq (qV^r + q^{1/2} J_{r,k}(V)^{1/2} V^r) q^{o(1)}.
\]

The following will be used in the proof of Theorem 2 and improves on Lemma 8 provided the number of prime factors of \( q \) is bounded.

**Lemma 9.** Let \( s \) be an integer and let \( q \) be squarefree such that the number of prime factors of \( q \) is less than \( s \). Let \( \chi \) a primitive character \( \mod q \), \( \beta_v \) be any sequence of complex numbers with \( |\beta_v| \leq 1 \) and for \( r \geq s + 1 \) let
\[
W = \int_0^1 \ldots \int_0^1 \sum_{\lambda=1}^q \left| \sum_{1 \leq v \leq V} \beta_v \chi(\lambda + v) e^{2\pi i (v_1 \omega + \cdots + v_d \omega^d)} \right|^{2r} \, d\alpha_1 \ldots d\alpha_d. \tag{12}
\]
Then we have
\[
W \leq (qV^r + q^{1/2} J_{r,s-1,d}(V)V^{2s+2}) q^{o(1)}.
\]

**Proof.** We keep the same notation from the proof of Lemma 8 so that following the same argument gives
\[
W \ll qV^r + q^{1/2+o(1)} \left( \sum_{i=1}^{2r} W_i \right),
\]
where
\[ W_i = \sum_{(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)} (A_i(v), q)^{1/2}. \] (13)

We consider only \( W_1 \), the same argument applies to the remaining \( W_i \). Let \( q = q_1 \ldots q_s \) be the prime factorization of \( q \) and for each subset \( S \subseteq \{1, \ldots, s\} \) we partition \( S \) into \( 2r - 1 \) sets
\[ S = \bigcup_{j=2}^{2r} U_j, \quad \text{where} \quad U_i \cap U_j = \emptyset \quad \text{if} \quad i \neq j, \] (14)
where some \( U_j \) may be empty. We have
\[ W_1 \leq \sum_{S \subseteq \{1, \ldots, s\}} \sum_{U_2, \ldots, U_{2r}} \sum_{(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)} (A_i(v), q)^{1/2}, \] (15)
where the sum over \( U_2, \ldots, U_{2r} \) satisfies (14). Hence it is sufficient to show that for fixed \( S \) and fixed \( U_2, \ldots, U_{2r} \) satisfying (14) we have
\[ \sum_{(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)} (A_i(v), q)^{1/2} \leq J_{r-s-1,k}(V)V^{2s+2}q^{o(1)}. \]
Considering values of \( j \) such that \( U_j \neq \emptyset \), each value of \( v_1 \) determines \( v_j \) with \( \ll V/\prod_{i \in U_j} q_i \) possibilities. Since there are are most \( s \) values of \( j \) such that \( U_j \neq \emptyset \), we may choose two sets \( \mathcal{V}_1, \mathcal{V}_2 \) such that
\[ \mathcal{V}_1 \subseteq \{1, \ldots, r\}, \quad \#\mathcal{V}_1 = r - s - 1, \]
\[ \mathcal{V}_2 \subseteq \{r + 1, \ldots, 2r\}, \quad \#\mathcal{V}_2 = r - s - 1, \]
and integers \( \alpha_1, \ldots, \alpha_k \) such that
\[ \sum_{(v_1, \ldots, v_r) \in J'_{r,k}(V)} (A_i(v), q)^{1/2} \ll V^{2s+2}J(\mathcal{V}_1, \mathcal{V}_2, \alpha_1, \ldots, \alpha_k), \]
where \( J(V_1, V_2, \alpha_1, \ldots, \alpha_k) \) denotes the number of solutions to the system of equations

\[
\sum_{j \in V_1} v_i^j - \sum_{j \in V_2} v_i^j = \alpha_i, \quad 1 \leq i \leq d, \quad 1 \leq v_j \leq V.
\]

Since \( J(V_1, V_2, \alpha_1, \ldots, \alpha_k) \leq J_{r-s,d}(V) \), we see that

\[
\sum_{(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)} q^{1/2} \leq V^{2s+2} J_{r-s,d}(V),
\]

so that

\[
W_1 \leq V^{2s+2} J_{r-s,d}(V) q^{o(1)},
\]

which completes the proof.

\[\square\]

**Lemma 10.** Let \( q_1, \ldots, q_n \) be primes, \( \chi_i \) a multiplicative character \( \mod q_i \), \( \beta_v \) be a sequence of complex numbers with \( |\beta_v| \leq 1 \) and let

\[
W = \int_0^1 \ldots \int_0^1 \sum_{\lambda_1=1}^{q_1} \left| \sum_{1 \leq v \leq V} \beta_v \prod_{i=1}^n \chi_i(\lambda_i + v) e^{2\pi i (\lambda_1 v + \cdots + \lambda_k v^k)} \right|^{2r} d\alpha_1 \ldots d\alpha_k.
\]

Then if \( V \leq q_i \) for each \( i \) we have

\[
W \leq (q^{1/2} J_{r,k}(V)) q^{o(1)},
\]

where \( q = q_1 \ldots q_n \).

**Proof.** With notation as in the proof of Lemma 8 following the same argument gives

\[
W \ll q^{V^r} + \left( \sum_{(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)} \prod_{i=1}^n \sum_{\lambda_i=1}^{q_i} \chi_i \left( \frac{(\lambda + v_1) \ldots (\lambda + v_{r})}{(\lambda + v_{r+1}) \ldots (\lambda + v_{2r})} \right) \right).
\]

We claim that if \( (v_1, \ldots, v_{2r}) \in J'_{r,k}(V) \) then for each \( 1 \leq i \leq n \) the function

\[
\chi_i \left( \frac{(\lambda + v_1) \ldots (\lambda + v_{r})}{(\lambda + v_{r+1}) \ldots (\lambda + v_{2r})} \right),
\]

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is not constant. Supposing for some \(i\) this were false and letting \(d\) denote the order of \(\chi_i\), then this implies that the rational function

\[
\frac{(\lambda + v_1) \ldots (\lambda + v_r)}{(\lambda + v_{r+1}) \ldots (\lambda + v_{2r})},
\]

is a \(d\)-th power mod \(q_i\), so that at most \(r + 1\) of the \(v_1, \ldots, v_{2r}\) are distinct mod \(q_i\), and since \(V < q_i\) this implies that at most \(r + 1\) of the \(v_1, \ldots, v_{2r}\) are distinct, contradicting the definition of \(\mathcal{J}'_{r,k}(V)\). Hence from the Weil bound for complete character sums [14, Theorem 2C', pg 43] we have

\[
\sum_{\lambda=1}^{q_i} \chi_i \left( \frac{(\lambda + v_1) \ldots (\lambda + v_r)}{(\lambda + v_{r+1}) \ldots (\lambda + v_{2r})} \right) \ll q_i^{1/2},
\]

provided \((v_1, \ldots, v_{2r}) \in \mathcal{J}'_{r,k}(V)\). Hence we get

\[
W \ll q^{r^2} + q^{1/2 + o(1)} \# \mathcal{J}'_{r,k}(V) \leq (q^{r^2} + q^{1/2} J_{r,d}(V)) q^{o(1)}.
\]

\[\Box\]

**Lemma 11.** Let \(q\) be prime, \(n\) an integer and \(\chi\) a multiplicative character of \(\mathbb{F}_q^n\), \(\beta_v\) any sequence of complex numbers satisfying \(|\beta_v| \leq 1\) and let

\[
W = \int_{[0,1]^d} \sum_{\lambda \in \mathbb{F}_q^n} \left| \sum_{1 \leq v \leq V} \beta_v \chi(\lambda + v) e^{2\pi i (\alpha_1 v_1 + \cdots + \alpha_d v_d)} \right|^{2r} \, d\alpha_1 \ldots d\alpha_d.
\]

Then for any integer \(r\) we have

\[
W \ll q^{n^2} V^r + q^{n/2} J_{r,d}(V).
\]

**Proof.** Arguing as in the proof of Lemma 8, let \(\mathcal{J}_{r,k}(V)\) denote the set of all \((v_1, \ldots, v_{2r})\) such that

\[
v_1^j + \cdots + v_i^j = v_{r+1}^j + \cdots + v_{2r}^j, \quad 1 \leq j \leq k, \quad 1 \leq v_i \leq V.
\]

Expanding the \(2r\)-th power in the definition of \(W\) and interchanging summation and integration gives

\[
W \leq \sum_{(v_1, \ldots, v_{2r}) \in \mathcal{J}_{r,k}(V)} \left| \sum_{\lambda=1}^{q} \chi \left( \frac{(\lambda + v_1) \ldots (\lambda + v_r)}{(\lambda + v_{r+1}) \ldots (\lambda + v_{2r})} \right) \right|.
\]
As in Lemma 8 we break the $J_{r,k}(V)$ into sets $J'_{r,k}(V)$ and $J''_{r,k}(V)$, where

$J'_{r,k}(V) = \{ (v_1, \ldots, v_{2r}) \in J_{r,k}(V) : \text{at least } r + 1 \text{ of the } v_i \text{ are distinct} \}$,

$J''_{r,k}(V) = \{ (v_1, \ldots, v_{2r}) \in J_{r,k}(V) : (v_1, \ldots, v_{2r}) \notin J'_{r,k}(V) \}$,

so that

$$W \ll q^n V^r + \sum_{(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)} \left| \sum_{\lambda = 1}^{q} \chi \left( \frac{(\lambda + v_1) \ldots (\lambda + v_r)}{\lambda + v_{r+1} \ldots (\lambda + v_{2r})} \right) \right|.$$ 

From [14, Theorem 2C', pg 43], we have if $(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)$ then

$$\sum_{\lambda = 1}^{q} \chi \left( \frac{(\lambda + v_1) \ldots (\lambda + v_r)}{\lambda + v_{r+1} \ldots (\lambda + v_{2r})} \right) \ll q^{n/2};$$

so that

$$W \ll q^n V^r + \sum_{(v_1, \ldots, v_{2r}) \in J'_{r,k}(V)} q^{n/2},$$

and the result follows since $\# J'_{r,k}(V) \leq J_{r,d}(V)$. 

## 5 Multiplicative energy of certain sets

The following follows from the proof of [8, Lemma 7].

**Lemma 12.** Let $M, N, U, q$ be integers with

$$NU \leq q,$$

and let $U$ denote the set

$$U = \{ 1 \leq u \leq U : (u, q) = 1 \}.$$ 

Then the number of solutions to the congruence

$$n_1 u_1 \equiv n_2 u_2 \mod q, \quad M < n_1, n_2 \leq M + N, \quad u_1, u_2 \in U$$

is bounded by $NUq^{o(1)}$. 

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The following is due to Konyagin [11, Lemma 1].

**Lemma 13.** Let $q$ be prime and let $\omega_1, \ldots, \omega_n \in \mathbb{F}_{q^n}$ be a basis for $\mathbb{F}_{q^n}$ as a vector space over $\mathbb{F}_q$. Let $B_1$ and $B_2$ denote the boxes

$$B_1 = \left\{ h_1 \omega_1 + \cdots + h_n \omega_n : 1 \leq h_i \leq H \right\},$$

$$B_2 = \left\{ h_1 \omega_1 + \cdots + h_k \omega_n : 1 \leq h_i \leq U \right\},$$

and suppose that $H, U \leq p^{1/2}$. Then the number of solutions to the equation

$$x_1 x_2 \equiv x_3 x_4, \quad x_1, x_3 \in B_1, \quad x_2, x_4 \in B_2,$$

is $\ll (UH)^n \log q$.

The following is due to Bourgain and Chang [1].

**Lemma 14.** Let $q$ be prime, $L_1(x), \ldots, L_n(x)$ be linear forms in $n$ variables which are linearly independent $\mod q$ and let $B_1$ and $B_2$ denote the boxes

$$B_1 = \left\{ h = (h_1, \ldots, h_n) : 1 \leq h_i \leq H \right\},$$

$$B_2 = \left\{ h = (h_1, \ldots, h_n) : 1 \leq h_i \leq U \right\}.$$

Then if $H, U \leq p^{1/2}$ the number of solutions to the system of congruences

$$L_i(x_1)L_i(x_2) \equiv L_i(x_3)L_i(x_4) \mod q, \quad x_1, x_3 \in B_1, \quad x_2, x_4 \in B_2, \quad 1 \leq i \leq n,$$

is bounded by $(NH)^n p^o(1)$.

### 6 Proof of Theorem 1

We define the integers

$$U = \left\lfloor \frac{N}{q^{1/2}(r-d(d+1)/4)} \right\rfloor, \quad V = \left\lfloor q^{1/2(r-d(d+1)/4)} \right\rfloor,$$

(17)

and the set

$$U = \left\{ 1 \leq u \leq U : (u, q) = 1 \right\},$$

so that

$$\#U = U q^o(1).$$

(18)
By Lemma [6] we have
\[
\left| \sum_{M < n \leq M + N} \chi(n) e^{2\pi i F(n)} \right| \leq \frac{q^{o(1)}}{\#U V} \sum_{M - N < n \leq M + N} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} \chi(n + u v) e^{2\pi i F(n + u v)} e^{2\pi i \alpha v} \right|,
\]
for some \( \alpha \in \mathbb{R} \). Let
\[
W = \sum_{M - N < n \leq M + N} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} \chi(n + u v) e^{2\pi i F(n + u v)} e^{2\pi i \alpha v} \right|,
\]
then since the polynomial \( F \) has degree \( d \), we see that
\[
W \leq \sum_{M - N < n \leq M + N} \max_{u \in U} \left| \sum_{1 \leq v \leq V} \chi(n + u v) e^{2\pi i (\alpha_1 v + \cdots + \alpha_d v^d)} \right|,
\]
where \( I(\lambda) \) denotes the number of solutions to the congruence
\[
n u^* \equiv \lambda \pmod{q}, \quad M - N < n \leq M + N, \quad u \in U.
\]
For \( i = 1, \ldots, d \), let
\[
\delta_i = \frac{1}{4V^i},
\]
and define the functions \( \phi_i(v) \) by
\[
1 = \phi_i(v) \int_{-\delta_i}^{\delta_i} e^{2\pi i x v^i} dx,
\]
so that for \( 1 < v \leq V \) we have
\[
\phi_i(v) = \frac{2\pi i v^i}{\sin(2\pi \delta_i v^i)} \ll \frac{1}{\delta_i} \ll V^i.
\]
and let $C(\delta)$ denote the rectangle
\[ [-\delta_1, \delta_1] \times \cdots \times [-\delta_d, \delta_d], \]
then we have
\[
W \leq \sum_{\lambda=1}^{q} I(\lambda) \max_{\alpha \in [0,1]^d} \left| \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \int_{C(\delta)} \chi(\lambda + v) e^{2\pi i <\alpha + x, v>} \, dx \right|,
\]
where $<\, ,\, >$ denotes the standard inner product on $\mathbb{R}^d$. Hence
\[
W \leq \sum_{\lambda=1}^{q} \int_{C(\delta)} I(\lambda) \max_{\alpha \in [0,1]^d} \left| \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i <\alpha + x, v>} \right| \, dx.
\]
Two applications of the Hölder inequality give
\[
|W|^{2r} \ll \left( \prod_{i=1}^{d} \delta_i \right)^{2r-1} \left( \sum_{\lambda=1}^{q} I(\lambda) \right)^{2r-2} \left( \sum_{\lambda=1}^{q} I(\lambda)^2 \right) \times \left( \sum_{\lambda=1}^{q} \max_{\alpha \in [0,1]^d} \int_{C(\delta)} \left| \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i <\alpha + x, v>} \right|^{2r} \, dx \right).
\]
Since we have
\[
\sum_{\lambda=1}^{q} I(\lambda) \ll UV,
\]
and the term
\[
\sum_{\lambda=1}^{q} I(\lambda)^2,
\]
is equal to the number of solutions to the congruence
\[
n_1 u_1 \equiv n_2 u_2 \mod q, \quad 1 \leq n_1, n_2 \leq N, \quad u_1, u_2 \in U,
\]
we have by Lemma 12
\[
\sum_{\lambda=1}^{q} I(\lambda)^2 \leq N U q^{o(1)},
\]
so that
\[ W \leq \left( \prod_{i=1}^{d} \delta_i \right)^{2r-1} (NU)^{2r-1} q^{o(1)} \]
\[ \times \left( \sum_{\lambda=1}^{q} \max_{\alpha \in [0,1]^d} \int_{C(\delta)} \left| \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i \langle \alpha + x, v \rangle} \right|^{2r} dx \right). \]

Let
\[ W_1 = \sum_{\lambda=1}^{q} \max_{\alpha \in [0,1]^d} \int_{C(\delta)} \left| \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i \langle \alpha + x, v \rangle} \right|^{2r} dx, \]
so that by (20)
\[ W^{2r} \leq V^{-(2r-1)d(d+1)/2} (NU)^{2r-1} q^{o(1)} W_1. \] (21)

We have
\[ W_1 = \sum_{\lambda=1}^{q} \max_{\alpha \in [0,1]^d} \int_{C(\delta)} \left| \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i \langle \alpha + x, v \rangle} \right|^{2r} dx \]
\[ \ll \sum_{\lambda=1}^{q} \int_{[0,1]^d} \left| \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i \langle \alpha + x, v \rangle} \right|^{2r} dx. \]
By (20), for each \( 1 \leq v \leq V \) we have
\[ \prod_{i=1}^{d} \phi_i(v) \ll V^{d(d+1)/2}, \]
hence by Lemma 8
\[ W_1 \ll V^{rd(d+1)} \left( qV^r + q^{1/2} V^{2r-d(d+1)/4} \right) q^{o(1)}, \]
so that by (19)
\[ \left| \sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \right|^{2r} \leq \frac{(NU)^{2r-1} \left( qV^r + q^{1/2} V^{2r-d(d+1)/4} \right) V^{d(d+1)/2} q^{o(1)}}{U^{2r} V^{2r}}. \]
Recalling the choices of \( U \) and \( V \) gives
\[ \left| \sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \right|^{2r} \leq N^{2r-2} q^{1/2 + d(d+1)/8(r-d(d+1)/4)+1/2(r-d(d+1)/4)+o(1)}. \]
7 Proof of Theorem 2

Let \( U = \left\lfloor \frac{N}{q^{1/2(r-d(d+1)/2)}} \right\rfloor \), \( V = \lfloor q^{1/2(r-d(d+1)/2)} \rfloor \), and let \( \phi_i(v) \) be defined as in the proof of Theorem 1. Then following the proof of Theorem 1 we have

\[
\sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \bigg| \sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \bigg|^{2r} = \frac{V^{-(2r-1)d(d+1)/2} (NU)^{2r-1}}{V^{2r} U^{2r}} W_1 q^o(1),
\]

where

\[
W_1 = \sum_{\lambda=1}^{q} \int_{[0,1]^d} \sum_{v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i (x_1 v + \cdots + x_d v^d)} dx.
\]

By Lemma 9 we have

\[
W_1 \ll V^{r(d+1)} (qV^r + qV^{2r-d(d+1)/2}) q^o(1),
\]

so that

\[
\sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \bigg| \sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \bigg|^{2r} \leq V^{d(d+1)/2} (NU)^{2r-1} (qV^r + q^{1/2} V^{2r-d(d+1)/2}) q^o(1),
\]

so that recalling the choice of \( U, V \) gives

\[
\sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \bigg| \sum_{M < n \leq M+N} \chi(n) e^{2\pi i F(n)} \bigg|^{2r} \leq N^{2r-2} q^{(r+1-d(d+1)/2)/(r-d(d+1)/2) + o(1)}.
\]

8 Proof of Theorem 3

Let \( U = \left\lfloor \frac{N}{q^{n/2(r-d(d+1)/2)}} \right\rfloor \), \( V = \lfloor q^{n/2(r-d(d+1)/2)} \rfloor \), and let \( \mathcal{U} \) denote the box

\[
\mathcal{U} = \{ u_1 \omega_1 + \cdots + u_n \omega_n : 0 < u_i \leq U \}.
\]
Then with notation as in Lemma 6 we have
\[ \left| \sum_{x \in B} \chi(x)e^{2\pi i F(x)} \right| \leq \frac{q^{\rho(1)}}{VU^n} \sum_{x \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} \chi(x + uv)e^{2\pi i F(x + uv + 2\pi i v)} \right|. \]

Let
\[ W = \sum_{x \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} \chi(x + uv)e^{2\pi i F(x + uv + 2\pi i v)} \right|, \tag{22} \]
so that expanding \( F(x + uv) \) as a polynomial in \( v \) gives
\[ F(x + uv) = \sum_{i=0}^{d} F_i(x, u)v^i, \]
for some real numbers \( F_i(x, u) \). Hence we have
\[
W \leq \sum_{x \in B_0} \sum_{u \in U} \max_{(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d} \left| \sum_{1 \leq v \leq V} \chi(x + uv)e^{2\pi i (\alpha_1 v + \ldots + \alpha_d v^d)} \right| \\
= \sum_{\lambda \in \mathbb{F}_{q^n}} I(\lambda) \max_{(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v)e^{2\pi i (\alpha_1 v + \ldots + \alpha_d v^d)} \right|,
\]
where \( I(\lambda) \) denotes the number of solutions to the equation in \( \mathbb{F}_{q^n} \)
\[ xu^{-1} = \lambda, \quad x \in B_0, \quad u \in U. \]

With \( \phi_i(v), \delta_i \) and \( C(\delta) \) as in Theorem 1 and \( v = (v, \ldots, v^d) \) we have
\[
W \leq \sum_{\lambda \in \mathbb{F}_{q^n}} \int_{C(\delta)} I(\lambda) \max_{\alpha \in \mathbb{R}^d} \left| \sum_{1 \leq v \leq V} \prod_{i=1}^{d} \phi_i(v)\chi(\lambda + v)e^{2\pi i <\alpha + y, v>} \right| dy.
\]

By two applications of Hölder’s inequality, we get
\[
W^{2r} \leq V^{-(2r-1)d(d+1)/2} \left( \sum_{\lambda \in \mathbb{F}_{q^n}} I(\lambda) \right)^{2r-2} \left( \sum_{\lambda \in \mathbb{F}_{q^n}} I(\lambda)^2 \right) \times \left( \sum_{\lambda \in \mathbb{F}_{q^n}} \int_{C(\delta)} \max_{\alpha \in \mathbb{R}^d} \left| \sum_{1 \leq v \leq V} \prod_{i=1}^{d} \phi_i(v)\chi(\lambda + v)e^{2\pi i <\alpha + y, v>} \right| dy \right)^2. \]
We have
\[ \sum_{\lambda \in \mathbb{F}_{q^n}} I(\lambda) \ll (HU)^n, \]
and the term
\[ \sum_{\lambda \in \mathbb{F}_{q^n}} I(\lambda)^2, \]
is equal to the number of solutions to the equation over \( \mathbb{F}_{q^n} \)
\[ x_1 u_1 = x_2 u_2, \quad x_1, x_2 \in B_0, \quad u_1, u_2 \in U, \]
so that by Lemma 13
\[ \sum_{\lambda \in \mathbb{F}_q} I(\lambda)^2 \leq (HU)^n q^{o(1)}, \]
hence we get
\[ W_{2r} \leq V^{-(2r-1)d(d+1)/2} (HU)^{(2r-1)n} q^{o(1)} \]
\[ \times \left( \sum_{\lambda \in \mathbb{F}_q} \int_{\mathbb{C}(\delta)} \max_{\alpha \in \mathbb{R}^d} \left| \sum_{1 \leq v \leq V} \prod_{i=1}^d \phi_i(v) \chi(\lambda + v) e^{2\pi i \langle \alpha + y, v \rangle} \right|^{2r} dy \right). \]

Let
\[ W_1 = \sum_{\lambda \in \mathbb{F}_q} \int_{\mathbb{C}(\delta)} \max_{\alpha \in \mathbb{R}^d} \left| \sum_{1 \leq v \leq V} \prod_{i=1}^d \phi_i(v) \chi(\lambda + v) e^{2\pi i \langle \alpha + y, v \rangle} \right|^{2r} dy, \]
so that
\[ W_1 = \sum_{\lambda \in \mathbb{F}_q} \max_{\alpha \in \mathbb{R}^d} \left| \sum_{1 \leq v \leq V} \prod_{i=1}^d \phi_i(v) \chi(\lambda + v) e^{2\pi i \langle \alpha + y, v \rangle} \right|^{2r} dy \]
\[ \ll \sum_{\lambda \in \mathbb{F}_q} \int_{[0,1]^d} \left| \sum_{1 \leq v \leq V} \prod_{i=1}^d \phi_i(v) \chi(\lambda + v) e^{2\pi i \langle y, v \rangle} \right|^{2r} dy, \]
hence by Lemma 11 we have
\[ W_1 \ll V^{rd(d+1)} \left( q^n V^r + q^{n/2} V^{2r-d(d+1)/2} \right), \]
which gives

\[ \left| \sum_{x \in B} \chi(x) e^{2\pi i F(x)} \right|^{2r} \leq V^{d(d+1)/2} \frac{(HU)^{(2r-1)n}}{V^{2r} U^{2rn}} \left( q^n V^r + q^{n/2} V^{2r-d(d+1)/2} \right)^{q^o(1)}. \]

Recalling the choices of \( U, V \) we get

\[ \left| \sum_{x \in B} \chi(x) e^{2\pi i F(x)} \right|^{2r} \leq H^{(2r-2)n} q^{(nr-nd(d+1)/2)(r-d(d+1)/2)+o(1)}. \]

### 9 Proof of Theorem 4

Let \( q = q_1 \ldots q_n \) and define the integers

\[ V = \left[ q^{1/2(r-d(d+1)/2)} \right], U_i = \left[ \frac{H_i}{q^{1/2(r-d(d+1)/2)}} \right], \]

and the box

\[ U = \{ (u_1, \ldots, u_n) : 1 \leq u_i \leq U_i \}, \]

so that by Lemma 6 we have for some \( \alpha \in \mathbb{R} \)

\[ \left| \sum_{x \in B} \chi_1(x_1) \ldots \chi_n(x_n) e^{2\pi i F(x)} \right| \leq \frac{q^o(1)}{V U_1 \ldots U_n} \sum_{x \in B_0} \sum_{u \in U} \sum_{1 \leq v \leq V} \chi_1(x_1 + u_1 v) \ldots \chi_n(x_n + u_n v) e^{2\pi i (F(x+uv)+\alpha v)}. \]

Writing

\[ W = \sum_{x \in B_0} \sum_{u \in U} \sum_{1 \leq v \leq V} \chi_1(x_1 + u_1 v) \ldots \chi_n(x_n + u_n v) e^{2\pi i (F(x+uv)+\alpha v)}, \]

and letting \( I(\lambda_1, \ldots, \lambda_n) \) denote the number of solutions to the system of congruences

\[ x_i u_i^{-1} \equiv \lambda_i \mod q_i, \quad N_i - H_i < x_i \leq N_i + H_i, 1 \leq u_i \leq U_i, \quad 1 \leq i \leq n, \]
we see that
\[
W \leq \sum_{\lambda_i=1}^{q_i} \frac{I(\lambda_1, \ldots, \lambda_n)}{r} \sum_{1 \leq i \leq n} \chi_1(\lambda_1 + v) \cdots \chi_n(\lambda_n + v) e^{2\pi i (F(x+uv)+ov)}
\]
\[
\leq \sum_{\lambda_i=1}^{q_i} \frac{I(\lambda_1, \ldots, \lambda_n)}{r} \max_{\alpha_1, \ldots, \alpha_d} \left| \sum_{1 \leq i \leq n} \chi(\lambda_1 + v) \cdots \chi_n(\lambda_n + v) e^{2\pi i (\alpha_1 v + \ldots + \alpha_d v^d)} \right|.
\]

With notation as in the proof of Theorem 11, we see that
\[
W \leq \sum_{\lambda_i=1}^{q_i} \int_{C(\delta)} I(\lambda_1, \ldots, \lambda_n) \max_{\alpha \in [0,1]^d} \left| \sum_{1 \leq v \leq V} \left( \prod_{i=1}^d \phi_i(v) \right) \chi(\lambda_1 + v) \cdots \chi_n(\lambda_n + v) e^{2\pi i \langle \alpha + x, v \rangle} \right| dx.
\]

Two applications of Hölder’s inequality give
\[
W^{2r} \leq V^{-(2r-1)d(d+1)/2} \left( \sum_{\lambda_i=1}^{q_i} I(\lambda_1, \ldots, \lambda_n) \right)^{2r-2} \left( \sum_{\lambda_i=1}^{q_i} I(\lambda_1, \ldots, \lambda_n)^2 \right) W_1,
\]
where
\[
W_1 = \sum_{\lambda_i=1}^{q_i} \int_{C(\delta)} \max_{\alpha \in [0,1]^d} \left| \sum_{1 \leq v \leq V} \left( \prod_{i=1}^d \phi_i(v) \right) \chi(\lambda_1 + v) \cdots \chi_n(\lambda_n + v) e^{2\pi i \langle \alpha + x, v \rangle} \right|^{2r} dx.
\]

As in the proof of Theorem 11, we have
\[
W_1 \ll \sum_{\lambda_i=1}^{q_i} \int_{[0,1]^d} \left| \sum_{1 \leq v \leq V} \left( \prod_{i=1}^d \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i (x_1 v + \ldots + x_d v^d)} \right|^{2r} dx,
\]
hence by Lemma 10
\[
W_1 \leq V^{rd(d+1)} \left( qV^r + q^{1/2}V^{2r-d(d+1)/2} \right) q^{\alpha(1)}.
\]

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We have
\[ \sum_{\lambda_i=1}^{q_i} I(\lambda_1, \ldots, \lambda_n) \leq H_1 \ldots H_n U_1 \ldots U_n, \]
and the term
\[ \sum_{\lambda_i=1}^{q_i} I(\lambda_1, \ldots, \lambda_n)^2, \]
is equal to the number of solutions to the system of equations
\[ x_{i,1} u_{i,1} \equiv x_{i,2} u_{i,2} \mod q_i, \quad N_i - H_i < x_{i,1}, x_{i,2} \leq N_i + H_i, 1 \leq u_{i,1}, u_{i,2} \leq U_i, \quad 1 \leq i \leq n, \]
hence by Lemma \[12\] we have
\[ \sum_{\lambda_i=1}^{q_i} I(\lambda_1, \ldots, \lambda_n)^2 \leq H_1 \ldots H_n U_1 \ldots U_n q^{o(1)}, \]
which gives
\[ \left| \sum_{x \in B} \chi_1(x_1) \ldots \chi_n(x_n) e^{2\pi i F(x)} \right|^{2r} \leq V^{d(d+1)/2} (H_1 \ldots H_n)^{2r-1} \frac{q V^r + q^{1/2} V^{2r-d(d+1)/2}}{U_1 \ldots U_n} q^{o(1)}. \]
Recalling the choices of \( V, U_1, \ldots, U_n \), we get
\[ \left| \sum_{x \in B} \chi_1(x_1) \ldots \chi_n(x_n) e^{2\pi i F(x)} \right|^{2r} \leq (H_1 \ldots H_n)^{2r-2} q^{(r-d(d+1)/2+n)/2} q^{o(1)}. \]

10 Proof of Theorem [5]

We define the integers
\[ U = \left\lfloor \frac{N}{q^{1/2(r-d(d+1)/2)}} \right\rfloor, \quad V = \left\lfloor q^{1/2(r-d(d+1)/2)} \right\rfloor, \]
and let \( \mathcal{U} \) and let denote the box
\[ \mathcal{U} = \{ (u_1, \ldots, u_n) : 1 \leq u_i \leq U \}, \]
so that from Lemma 6 we have
\[
\left| \sum_{x \in B} \chi \left( \prod_{i=1}^{n} L_i(x) \right) e^{2\pi i F(x)} \right| \leq \frac{q^{1(1)}}{V U^n} \sum_{x \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} \chi \left( \prod_{i=1}^{n} (L_i(x) L_i(u)^{-1} + v) \right) e^{2\pi i F(x) + 2\pi i ov} \right|,
\]
and since each \( L_i \) is linear this gives
\[
\left| \sum_{x \in B} \chi \left( \prod_{i=1}^{n} L_i(x) \right) e^{2\pi i F(x)} \right| \leq \frac{q^{1(1)}}{V U^n} \sum_{x \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} \chi \left( \prod_{i=1}^{n} (L_i(x) L_i(u)^{-1} + v) \right) e^{2\pi i F(x) + 2\pi i ov} \right|.
\]
Let
\[
W = \sum_{x \in B_0} \sum_{u \in U} \left| \sum_{1 \leq v \leq V} \chi \left( \prod_{i=1}^{n} (L_i(x) L_i(u)^{-1} + v) \right) e^{2\pi i F(x) + 2\pi i ov} \right|,
\]
and let \( I(\lambda_1, \ldots, \lambda_n) \) denote the number of solutions to the system of equations
\[
L_i(x) L_i^{-1}(u) \equiv \lambda_i \mod q, \quad x \in B_0, \quad u \in U, \quad 1 \leq i \leq n,
\]
then we have from the techniques of the preceding arguments
\[
W^{2r} \leq V^{-(2r-1)d(d+1)/2} \left( \sum_{\lambda_i=1}^{q} I(\lambda_1, \ldots, \lambda_n) \right)^{2r-2} \left( \sum_{\lambda_i=1}^{q} I(\lambda_1, \ldots, \lambda_n)^2 \right)
\times \left( \sum_{\lambda_i=1}^{q} \int_{[0,1]^d} \left| \sum_{1 \leq v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi ((\lambda_1 + v) \ldots (\lambda_n + v)) e^{2\pi i (x_1 v + \cdots + x_d v^d)} \right| \right)^{2r} \, dx.
\]
We have
\[
\sum_{\lambda=1}^{q} I(\lambda) \ll (HU)^n,
\]
and by Lemma 14
\[
\sum_{\lambda=1}^{q} I(\lambda)^2 \leq (HU)^n q^{1(1)}.
\]
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By Lemma 10

\[ \sum_{\lambda=1}^{q} \int_{[0,1]^d} \left| \sum_{1 \leq v \leq V} \left( \prod_{i=1}^{d} \phi_i(v) \right) \chi(\lambda + v) e^{2\pi i (x_1 v + \cdots + x_d v^d)} \right|^2 \, dx \]

\[ \ll V^{rd(d+1)} \left( q^n V^r + q^{n/2} V^{2r-d(d+1)/2} \right), \]

so that by the above

\[ \left| \sum_{x \in B} \chi \left( \prod_{i=1}^{n} L_i(x) \right) e^{2\pi i F(x)} \right|^2 \leq V^{d(d+1)/2} \frac{H^{(2r-1)n}}{V^{2r} U^n} \left( q^n V^r + q^{n/2} V^{2r-d(d+1)/2} \right), \]

Recalling the choice of \( U \) and \( V \) gives

\[ \left| \sum_{x \in B} \chi \left( \prod_{i=1}^{n} L_i(x) \right) e^{2\pi i F(x)} \right|^2 \leq H^{(2r-2)n} q^{n(r-D+1)/2(r-D)+o(1)}. \]

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