Resistively-shunted superconducting quantum point contacts

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We have studied the Josephson dynamics of resistively-shunted ballistic superconducting quantum point contacts at finite temperatures and arbitrary number of conducting modes. Compared to the classical Josephson dynamics of tunnel junctions, dynamics of quantum point contacts exhibits several new features associated with temporal fluctuations of the Josephson potential caused by fluctuations in the occupation of the current-carrying Andreev levels.

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Andreev levels with energies below the superconducting energy gap $\Delta$ are known to carry the stationary supercurrent in ballistic superconducting point contacts \cite{1,2}, and are also responsible for the dc and ac current flow at small bias voltages $V \ll \Delta/e$ \cite{3,4}. Occupation factors of these states vary randomly, either due to the thermal fluctuations in the contact electrodes \cite{5,6}, or due to the probabilistic nature of the Landau-Zener transitions \cite{7} induced by the bias voltage in contacts with finite reflection coefficients \cite{8}. These random fluctuations in the occupation factors of the sub-gap states lead to a large supercurrent noise. At finite bias voltages, the noise can be interpreted as the shot noise of the large charge quanta of magnitude $2\Delta/e$ \cite{9,10}. In a point contact with an external environment of finite impedance, the supercurrent noise leads to fluctuations of the voltage across the contact, and affects the dynamics of the Josephson phase difference $\varphi$. In the case of quantum point contacts with few propagating electron modes, for example those fabricated with the controllable break junction technique \cite{11,12}, the typical external impedance is much smaller than the contact resistance. In this case the effects of the external impedance are significant only at small bias voltages, where dynamics of $\varphi$ can be described within the adiabatic approach \cite{13}. The aim of this work is to study adiabatic finite-temperature dynamics of superconducting quantum point contacts, taking into account the supercurrent noise caused by fluctuations in the occupation factors of the Andreev levels.

We consider the standard model of a point contact, characterized as a short ballistic constriction supporting $N$ electron modes propagating between two identical bulk superconductors. At small bias voltages, $V \ll \Delta/e$, the current through such a constriction is carried by the two levels (per each electron mode) with energies $\pm \epsilon(\varphi)$, where $\epsilon(\varphi) = \Delta \cos(\varphi/2)$. We assume that the external impedance of the contact reduces to the frequency-independent constant $R$ in the relevant frequency range around the typical Josephson oscillation frequency of $2eI_cR/h$, where $I_c = N\Delta/e$ denotes the zero-temperature critical current of the contact.

The basic result of the adiabatic theory \cite{14} of dynamics of superconducting point contacts is that each occupied Andreev state contributes energy $\epsilon(\varphi)$ or $-\epsilon(\varphi)$ to the Josephson coupling energy of the point contact, while the occupation probability $w$ of the state is determined by the quasiparticle transitions between this state and the electrodes of the contact. These transitions are described by the simple rate equation:

$$ \dot{w} = \gamma(\epsilon)[f(\epsilon) - w], \tag{1} $$

where $f(\epsilon)$ is the equilibrium Fermi distribution, $\epsilon = \epsilon(\varphi(t))$, and the transition rate $\gamma(\epsilon)$ depends on the inelastic or pairbreaking effects which give rise to the finite subgap density of states in the electrodes \cite{15}. Equation (1) shows that the role of quasiparticle transitions is to drive the occupation probabilities of Andreev states towards the equilibrium.

The time evolution of the Josephson phase difference $\varphi(t)$ affects the occupation probabilities of Andreev states through the phase dependence of their energies $\pm \epsilon(\varphi)$. At finite temperatures the phase dynamics is diffusion along the Josephson potential driven by the thermal noise generated by the external resistance $R$. Such a diffusion is governed by the following set of Fokker-Planck equations for the probability density $\sigma$:

$$ \dot{\sigma}_k(\varphi, t) + \frac{\partial j_k}{\partial \varphi} = \Gamma \{ \sigma_k \}, \tag{2} $$

$$ j_k(\varphi) = -(\frac{2e}{h})^2R \left[ T \frac{\partial \sigma_k}{\partial \varphi} + \sigma_k \frac{\partial U^{(k)}}{\partial \varphi} \right], $$

where $k = -N, -N + 1, ..., N$ is the difference of the numbers of the occupied Andreev levels that carry positive and negative currents. It characterizes different realizations of the Josephson potential, $U^{(k)} = -\epsilon(\varphi) - (\hbar I_0/2e)\varphi$, where $I_0$ is the external bias current (see Fig. 1). The relaxation term $\Gamma \{ \sigma_k \}$ in eq. (2) describes the transitions from one realization of Josephson potential to another, induced by the quasiparticle transitions \cite{16} between the Andreev states and the electrodes of the point contact.
contact. Taking a sum over transitions from, and into, all of $2N$ Andreev levels we get for $\Gamma$: \[
\Gamma\{\sigma_k\} = \gamma(\epsilon)f(\epsilon)\left[(N + k + 1)\sigma_{k+1} - (N + k)\sigma_k\right] + 
\gamma(\epsilon)f(-\epsilon)\left[(N - k + 1)\sigma_{k-1} - (N - k)\sigma_k\right]. \quad (3)
\]

Since the dependence of all “internal” properties of the point contact (for instance, Andreev state energies $\pm \epsilon(\phi)$) on $\phi$ is periodic with the period $2\pi$, it is convenient to restrict $\phi$ to the interval $[0, 2\pi]$. The diffusion equation (2) should then be complemented with the boundary conditions for $\sigma$’s at the ends of this interval, $\phi = 0, 2\pi$. The boundary conditions are somewhat complicated by the fact that when the phase $\phi$ approaches 0 or $2\pi$ the energies of the Andreev state hit the gap edges $\pm \Delta$. As a result, the quasiparticle transitions between these states and electrodes become “absolutely” efficient, leading to instantaneous equilibration of the occupation probabilities of Andreev states $\sigma_k$. (More accurately, one should say that this equilibration process has a time scale that is negligible in comparison to the period of the Josephson oscillations.) Since different Andreev states are occupied independently one from another, this implies for the probability densities:

\[
\begin{align*}
\sigma_k(\phi = 0) &= A[f(\Delta)]^N[k][f(-\Delta)]^N \sigma_{k+N}^{N+1} C_{2N}^{N+k}, \\
\sigma_k(\phi = 2\pi) &= A'[f(\Delta)]^N[k][f(-\Delta)]^N \sigma_{k-N}^{N+k} C_{2N}^{N+k}.
\end{align*} \quad (4, 5)
\]

where $A, A'$ are some constants, and $C_{n}^{m} = n!/[m!(n - m)!]!$.

Relations (3) and (4) can serve as the boundary conditions for the diffusion equations (2). One more condition follows from the requirement that the components $\sigma_k(\phi)$ of the probability density diffusing on the different branches $U^{(k)}(\phi)$ of the Josephson potential be continuous. Taking into account that the states of the point contact with $\phi$ and $\phi + 2\pi$ should be identical, we can write this condition as $A' = A$. Combining this result with eqs. (3) and (4), we see that one of these equations can be replaced with the following boundary condition:

\[
\sigma_k(\phi = 2\pi) = \sigma_{-k}(\phi = 0). \quad (6)
\]

Equations (3) and (6), together with the usual normalization condition

\[
\sum_k \int_0^{2\pi} d\phi \sigma_k(\phi) = 1,
\]

determine completely the solution of the diffusion equation (2), and thus allow us to find all the characteristics of the point contact. Particularly, in the steady-state case, eq. (2) reduces to the equation $\partial j_k/\partial \phi = \Gamma(\sigma_k)$, and can be solved explicitly to obtain the dc voltage $V$ across the point contact:

\[
V = \frac{\pi h}{e} \sum_k j_k.
\]

Equations (3) – (6) show that the time evolution of the $\phi$ depends, in a non-trivial way, on the number $N$ of the propagating electron modes in the point contact. Below we analyze two opposite limits of such a dependence, $N = 1$ and $N \gg 1$. We also assume that the subgap density of states in the electrodes of the point contact is small and we can neglect the quasiparticle exchange term $\Gamma$ in eq. (3). We start with the case of a single-mode quantum point contact with $N = 1$. We have then three diffusion equations (2) with $k = 0; \pm 1$ (see Fig. 1). The boundary conditions (3) and (6) are:

\[
\begin{align*}
\sigma_1(0) &= \frac{1}{2} \sigma_0(0)e^{\Delta/T}, & \sigma_{-1}(0) &= \frac{1}{2} \sigma_0(0)e^{-\Delta/T}, \\
\sigma_{\pm 1}(2\pi) &= \sigma_{\mp 1}(0), & \sigma_0(2\pi) &= \sigma_0(0).
\end{align*} \quad (9, 10)
\]

The stationary version of eq. (3), $j_k(\phi) = \text{const}$, can be integrated directly, as in the case of classical Josephson junctions [11]. Taking into account the boundary conditions (3) and (6) we find then the $\sigma$’s:

\[
\begin{align*}
\sigma_{\pm 1}(\phi) &= \frac{\sigma_0}{2} \exp\left(-\frac{U^{(\pm 1)}(\phi)}{T}\right)\left[1 - \frac{1}{K_{\pm}}(1 - e^{-\pi h I_0/2eT})\right] \\
&\quad \times \int_0^{\phi} d\phi' \exp\left(\frac{U^{(\pm 1)}(\phi')}{T}\right),
\end{align*} \quad (11)
\]
where $K_\pm = \int_0^{2\pi} d\varphi \exp\{U^{(\pm)}(\varphi)/T\}$, and $\sigma_0$ is a constant density $\sigma_0(\varphi)$ that is determined by the normalization condition

$$2\pi\sigma_0 + \sum_{\pm 1} \int_0^{2\pi} d\varphi \sigma_{\pm 1}(\varphi) = 1.$$  

(12)

In terms of $\sigma_0$, the dc voltage $V$ across the point contact is:

$$V = 2\pi\sigma_0 I_0 R \left[ 1 + \frac{eT}{\hbar I_0} \left( \frac{1}{K_+} + \frac{1}{K_-} \right) (1 - e^{-\pi\hbar I_0/eT}) \right].$$  

(13)

Figure 2 shows the current $I$ flowing through the point contact, $I = I_0 - V/R$ (relation between $I$ and $I_0$ is illustrated by the equivalent circuit in Fig. 1), as a function of the voltage $V$ calculated from eqs. (11)–(13) for several temperatures. The zero-temperature $I-V$ characteristic (the upper curve in Fig. 2) can be represented analytically by the following dependence of the voltage $V$ on the bias current $I_0$:

$$V = \frac{\pi R(I_0^2 - I_c^2)^{1/2}}{4 \arctan(\sqrt{(I_0 + I_c)/(I_0 - I_c)})}, \quad I_0 > I_c, \quad (14)$$

where $I_c = e\Delta/\hbar$ is the zero-temperature supercurrent of the point contact. This equation describes the transition from the supercurrent at $V = 0$ to a constant current $2I_c/\pi$ at $V \gg R I_c$. The large-voltage dc current arises from the strongly non-equilibrium occupation of the Andreev levels, and is a hallmark of multiple Andreev reflections in a ballistic point contact.

Both the zero-voltage supercurrent and the large-voltage current are suppressed by temperature. For instance, in the limit of large temperatures, $T \gg \Delta$, we get from eqs. (11)–(13):

$$I = I_0 \left( \frac{\Delta}{2T} \right)^2 \left[ \frac{1}{1 + i^2} + \frac{4i^2}{\pi(1 + i^2)^2} \coth\left( \frac{\pi i}{2} \right) \right] \equiv \frac{\hbar I_0}{eT}. \quad (15)$$

This equation describes well the lowest curve in Fig. 2. Transition from low to high temperatures can be traced analytically at large bias currents, $I_0 \gg I_c$, when eqs. (11)–(13) give:

$$I = \frac{2}{\pi} I_c \tanh\left( \frac{\Delta}{2T} \right), \quad (16)$$

in agreement with Fig. 1 and large-current limit of eq. (13).

At small bias currents $I_0 < I_c$, and small temperatures $T \ll \Delta$, the point contact is in the supercurrent state and the bias current flows almost completely through the contact, $I \simeq I_0$. The voltage across the contact is exponentially suppressed and is associated with the rare

“phase-slip” events $\varphi \to \varphi + 2\pi$. In this regime, and for bias currents on the order of the critical current $I_c$, we get from eqs. (11)–(13):

$$V = \frac{R I_c}{2} (1 - i^2)^{1/2} \exp\left\{ -\frac{2\Delta}{T} (1 - i^2)^{1/2} \right\} - i\left( \frac{\pi}{2} - \sin^{-1}(i) \right), \quad \equiv \frac{I}{I_c}. \quad (17)$$

This equation corresponds to the classical thermal activation over the maximum of the Josephson potential, similar to the one found in classical Josephson junctions.

Now we consider point contacts with large number of transverse modes, $N \gg 1$. The probability density $\sigma_k$ is strongly peaked in this case as a function of $k$ at its average value: $k = (2w - 1)N$, where $w$ is the occupation probability of the Andreev states carrying positive current. This means that we can neglect fluctuations of the Josephson potential around the average potential:

$$U(\varphi) = N \epsilon(\varphi)(1 - 2w). \quad (18)$$

The probability $w$, which determines the shape of the potential (18), depends sensitively on the equilibration rate $\gamma$ in the rate equation (I). We start by considering the case of an ideal BCS superconductor in which $\gamma$ is completely suppressed in the relevant sub-gap energy range, and equilibration of the Andreev states occurs only at
\( \varphi = 0, 2\pi \). In this case the probability \( w \) is equal to \( f(-\Delta) \) or \( f(\Delta) \) depending on whether the phase diffuses into the interval \([0, 2\pi]\) through one \((\varphi = 0)\), or the other \((\varphi = 2\pi)\), end of this interval. This means that although we neglect fluctuations of the Josephson potential, we still have two branches of this potential due to the non-equilibrium occupation of Andreev states. The potential is \( E_J \cos(\varphi/2) \) on one branch, and \(-E_J \cos(\varphi/2)\) on the other, where \( E_J = N\Delta \tanh(\Delta/2T) \). The phase diffusion along these two branches can be described with the two equations in (2) with \( k = \pm 1 \), if we replace the energy gap \( \Delta \) in \( U(\varphi) \) with \( E_J \). However, the boundary conditions for \( \sigma_{\pm 1} \) are now quite different from those we used for the single-mode junction — see eqs. (9) and (10).

As we discussed above, the phase diffuses along one or the other branch of the potential depending on whether it enters the interval \([0, 2\pi]\) through one or the other end. This dynamics is accounted for by the following boundary conditions:

\[
\sigma_1(2\pi) = \sigma_{-1}(0) = 0, \quad \sigma_{-1}(2\pi) = \sigma_1(0). \tag{19}
\]

Equations (2) with the boundary conditions (19) give:

\[
\begin{align*}
\sigma_1(\varphi) &= \frac{\sigma}{K_+} \exp\left(-\frac{U^{(1)}(\varphi)}{T}\right) \int_{\varphi}^{2\pi} d\varphi' \exp\left(\frac{U^{(1)}(\varphi')}{T}\right), \\
\sigma_{-1}(\varphi) &= \frac{\sigma}{K_-} e^{-\pi h I_0/eT} \exp\left(-\frac{U^{(-1)}(\varphi)}{T}\right) \\
&\quad \times \int_{0}^{\varphi} d\varphi' \exp\left(-\frac{U^{(-1)}(\varphi')}{T}\right),
\end{align*}
\]

and

\[
V = \frac{4\pi eR}{h} \sigma T \left( \frac{1}{K_+} - \frac{e^{-\pi h I_0/eT}}{K_-} \right), \tag{22}
\]

where \( \sigma \) is a constant that is determined by the normalization condition (3).

Figure 3 shows the \( I - V \) characteristics of the multi-mode point contact between two BCS superconductors at several temperatures. For discussion see text.

At high temperatures, \( T \gg E_J \), the current \( I \) can be found analytically from eqs. (20)–(22):

\[
I = \frac{2I_c}{\pi} \frac{e^2}{1 + i^2} \coth(\frac{\pi i}{2}), \quad i \equiv \frac{h I_0}{eT}. \tag{23}
\]

This equation agrees with the lowest curve in Fig. 3. At low temperatures, \( T \ll E_J \), and small bias currents \( I_0 < I_c \), the contact is in the supercurrent state and the voltage is generated only by the rare thermal activation over the Josephson energy barrier as in the single mode case. For not-too-small bias currents, the voltage is given by the same eq. (14) with \( \Delta \) replaced by \( E_J \). At very small currents, \( I_0/I_c \ll (T/E_J)^{1/2} \), the voltage is:

\[
V = 2I_c Re^{-2E_J/T} \sinh(\frac{\pi h I_0}{eT}). \tag{24}
\]

Finally, we qualitatively discuss the effect of the finite rate \( \gamma \) of quasiparticle exchange between the Andreev states and contact electrodes. At small voltages \( V \ll h\gamma/e \) the phase evolves slowly and the quasiparticle transitions maintain equilibrium occupation of Andreev states. This means that the Josephson potential (13) reduces to

\[
U(\varphi) = -N\epsilon(\varphi) \tanh(\frac{e(\varphi)}{2T}). \tag{25}
\]

The main result of such a reduction is that now the potential depends only on the instantaneous value of \( \varphi \), and therefore has only one branch instead of two as in absence of the quasiparticle relaxation. The exact shape of the potential (25) depends on temperature, changing from \(-N\Delta \cos(\varphi/2)\) \( I_c \ll \Delta \) to \(-N\Delta^2/4T\cos \varphi \) at \( T \gg \Delta \). In this temperature range, \( T \approx \Delta \), the temperature is still negligible on the scale of the Josephson potential, and we can use the zero-temperature version of eqs. (3) to describe the dynamics of \( \varphi \). Combining this equation with the boundary condition \( \sigma(2\pi) = \sigma(0) \),
appropriate for the situation with one branch of the potential, we get for the voltage $V$:

$$V = 2\pi R \left[ \int_0^{2\pi} \frac{d\varphi}{I_0 - I_s(\varphi)} \right]^{-1},$$  \hspace{1cm} (26)$$

where $I_s \equiv (2e/\hbar)dU(\varphi)/d\varphi$. In the two limits $T \gg \Delta$ and $T \ll \Delta$, eq. (26) gives the standard result of the resistively-shunted-junction (RSJ) model, $V = R(I_0^2 - I_c^2)^{1/2}$, where $I_c = \max_{\varphi} I_s(\varphi)$. It can be shown by numerical integration of eq. (26) that deviations from this result are smaller than a few percent for arbitrary $\Delta/T$ ratio.

When the temperature becomes nonvanishing on the scale of Josephson potential, it is already much larger than $\Delta$ and the potential (23) becomes the regular Josephson potential $-E_J \cos \varphi$. This means that the quasi-equilibrium dynamics of the multi-mode ballistic contacts can be described with the classical RSJ model [14][15]. When the voltage $V$ across the contact is sufficiently large, $V \gg \hbar\gamma/e$, this quasi-equilibrium dynamics goes over into the non-equilibrium one, described by Fig. 3 and eqs. (14), (23).

In summary, we have presented a new approach to the description of the classical Josephson dynamics of superconducting quantum point contacts with arbitrary number of conducting quantum modes in a resistive environment. New characteristic features of the Josephson dynamics in these contacts include fluctuations of the Josephson potential caused by fluctuations in the occupation of the current-carrying Andreev levels, and low-voltage excess current associated with the non-equilibrium occupation of these levels.

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