Tri-bimaximal mixing, discrete family symmetries, and a conjecture connecting the quark and lepton mixing matrices

Catherine I Low and Raymond R Volkas
School of Physics, Research Centre for High Energy Physics, The University of Melbourne, Victoria 3010, Australia
(Dated: March 25, 2022)

Neutrino oscillation experiments (excluding the LSND experiment) suggest a tri-bimaximal form for the lepton mixing matrix. This form indicates that the mixing matrix is probably independent of the lepton masses, and suggests the action of an underlying discrete family symmetry. Using these hints, we conjecture that the contrasting forms of the quark and lepton mixing matrices may both be generated by such a discrete family symmetry. This idea is that the diagonalisation matrices out of which the physical mixing matrices are composed have large mixing angles, which cancel out due to a symmetry when the CKM matrix is computed, but do not do so in the MNS case. However, in the cases where the Higgs bosons are singlets under the symmetry, and the family symmetry commutes with $SU(2)_L$, we prove a no-go theorem: no discrete unbroken family symmetry can produce the required mixing patterns. We then suggest avenues for future research.

PACS numbers: 12.15.Ff, 14.60.Pq

I. INTRODUCTION

Experimental observations of neutrino oscillations point to a mixing matrix of the form

$$U_{MNS} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{1}{2} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix},$$

(1)

where the flavour eigenstates are related to the mass eigenstates via $(\nu_e, \nu_\mu, \nu_\tau)^T = U_{MNS}(\nu_1, \nu_2, \nu_3)^T$. Such a mixing pattern has been termed “tri-bimaximal mixing” (Majorana phases have not been included in the above mixing matrix as they do not lead to observable effects in oscillations). A mixing matrix of this form was first investigated by Wolfenstein in 1978 (with degenerate mass eigenstates $\nu_1$ and $\nu_3$), and proposed more recently in the light of the new experimental observations by Harrison, Perkins and Scott and He and Zee. The generation of small deviations from tri-bimaximal mixing has been investigated by Xing.

The tri-bimaximal form is a very special case of the general mixing matrix parameterised in the usual way by the Euler angles $\theta_{ij}$ where $i, j = 1, 2, 3$. The angle $\theta_{23}$, extracted from atmospheric neutrino experiments, takes the best fit value of $\sin^2 \theta_{23} = \frac{1}{2}$ (Majorana phases have not been included in the above mixing matrix as they do not lead to observable effects in oscillations). A mixing matrix of this form was first investigated by Wolfenstein in 1978 (with degenerate mass eigenstates $\nu_1$ and $\nu_3$), and proposed more recently in the light of the new experimental observations by Harrison, Perkins and Scott and He and Zee. The generation of small deviations from tri-bimaximal mixing has been investigated by Xing.

The tri-bimaximal form is a very special case of the general mixing matrix parameterised in the usual way by the Euler angles $\theta_{ij}$ where $i, j = 1, 2, 3$. The angle $\theta_{23}$, extracted from atmospheric neutrino experiments, takes the best fit value of $\sin^2 \theta_{23} = \frac{1}{2}$. Solar neutrino results are accommodated in Eq. (1) through the choice $\sin^2 \theta_{12} = \frac{1}{2}$, which is in the middle of the allowed “large mixing angle regions” denoted LMA-I and LMA-II. The third mixing angle, measured by the non-observation of $\nu_e$ disappearance, is taken as the current best fit $\theta_{13} = 0$. Note that $\theta_{23}$ takes the maximum possible value, while $\theta_{13}$ takes the minimum possible value.

A. Mathematics suggested by tri-bimaximal mixing

If these special mixing angle values are indeed the correct ones, then it is unlikely that they arise from a random choice of parameters. This encourages one to look for exact or approximate symmetries of nature, operative even at low energy scales, that enforce the special tri-bimaximal form (or something close to it).
1. Mixing angles independent of masses

The elements of the tri-bimaximal mixing matrix are square roots of fractions, whereas the charged lepton masses appear to have no precise fractional relationships, and neither do the preferred neutrino $\Delta m^2$ parameters. This motivates the construction of models where the mixing angles, though precisely defined, are independent of the mass eigenvalues. Such an approach is to be contrasted with the often considered alternative proposal that relates mixing angles to mass ratios $[24, 25, 26, 27]$.

2. Abelian symmetries

Harrison, Perkins and Scott $[2]$ proposed weak basis mass matrices for charged leptons and neutrinos that generate tri-bimaximal mixing. An attractive feature of the proposed mass matrices is that they can be generated by discrete Abelian symmetries acting on the three generations of charged leptons and neutrinos. These symmetries dictate the form of the mixing matrix, but leave the masses as free parameters (see above discussion). The utility of these mass matrices suggests that Abelian generation symmetries are interesting candidates for the new symmetries that might explain the neutrino mixing pattern.

B. Aims of this paper

1. Quark and lepton mixing matrices derived from a symmetry

In Sec. II the Harrison, Perkins and Scott proposal $[2]$ will be reviewed. We will then extend their ideas by conjecturing that the underlying symmetries might simultaneously produce a quark mixing matrix that is almost the identity matrix and a leptonic analogue that has the very different tri-bimaximal form. While we find this an attractive hypothesis, it is not so easy to implement in a completely well-defined extension of the standard model. As we shall see, this proposal requires that left-handed charged leptons and left-handed neutrinos transform differently. But to have the symmetry group $G_{SM}$ of the standard model extended to $G_{SM} \otimes G_H$, where $G_H$ is a discrete horizontal or generation symmetry, the left-handed charged leptons and left-handed neutrinos must transform in the same way under the symmetry, as they are members of the same $SU(2)_L$ doublet.

2. Form-diagonalisable matrices

Section III will define a class of matrices that are invariant under a symmetry and where the unitary matrices that diagonalise them are independent of the eigenvalues. We dub matrices such as these “form-diagonalisable” and propose them as good candidates for lepton mass matrices because they generate mixing angles that are independent of the eigenvalues. This section will look at some interesting mathematics that relates the symmetry group to the diagonalisation matrices.

3. No-go theorem

Motivated by the symmetries proposed by Harrison, Perkins and Scott $[2]$, Sec. IV will investigate the possibility of using such symmetries to extend the standard model. We assume left-handed neutrinos transform under the symmetry in the same way as left-handed charged leptons, and that the Higgs bosons are singlets. Given these assumptions, we find that tri-bimaximal mixing, or any other form that is both phenomenologically acceptable and predictive, cannot be generated by an unbroken family symmetry.

4. Further symmetries to investigate

Ways around the no-go theorem will be briefly discussed in Sec. V. Either or both of the assumptions of the theorem – that the Higgs fields are singlets and that the symmetry is unbroken – must be relaxed. The generation symmetry can be extended to the Higgs sector by introducing a number of generations of Higgs fields that transform under the symmetry. Majorana neutrinos have different couplings to the Higgs fields from the Dirac charged leptons. As a
result a symmetry that transforms Higgs fields could potentially explain the differences between the mixing matrices of the leptons and the quarks. Vacuum expectation values of the Higgs fields can break the symmetry and result in different mixing matrices from those of the exact symmetry cases. Work along these lines is in progress. For some recent efforts, see for instance [28, 29, 30, 31, 32].

II. DISCRETE SYMMETRIES CONSTRAIN MIXING MATRICES

Many theories have been constructed using symmetries to generate preferred mass patterns and mixing angles. For example, democratic mass matrices can be generated from an $S_{3L} \times S_{3R}$ generation symmetry [33, 34, 35], $L_e - L_\mu - L_\tau$ symmetry leads to bimaximal mixing (disfavoured by the current data) [36, 37, 38], an $S_2$ permutation symmetry acting on $\nu_\mu$ and $\nu_\tau$ results in maximal atmospheric mixing [39, 40, 41].

A. How symmetries constrain mixing matrices

The mixing matrix is related to the charged lepton mass matrix $M_\ell$ and the neutrino mass matrix $M_\nu$ in any weak basis by the unitary diagonalisation matrices $U_\ell^L$ and $U_\nu^\ast$. We use

$$\text{Diag}(m_e, m_\mu, m_\tau) = U_{\ell L}^\dagger M_\ell U_{\ell R}, \quad \text{Diag}(m_1, m_2, m_3) = U_{\nu}^\dagger M_\nu U_{\nu}^\ast, \quad (2)$$

to extract the lepton mixing matrix via

$$U_{\text{MNS}} = U_{\ell L}^\dagger U_{\nu}. \quad (3)$$

The symmetries of the standard model do not dictate the form of the mass matrices. The charged lepton mass matrix $M_\ell$ can be any $3 \times 3$ matrix, and if neutrinos are Majorana, then $M_\nu$ must be symmetric, but is otherwise unconstrained. As a result the mixing matrix can be of any unitary form, and the masses are unrestricted by the standard model symmetries. However, if a generation symmetry holds, the form of the mass matrices – and hence the mixing matrix – are constrained. For the Lagrangian to be invariant under transformations of the three generations of Majorana neutrinos, the left-handed charged leptons and the right handed charged leptons,

$$\nu \to X_\nu \nu, \quad \ell_L \to X_L \ell_L, \quad \ell_R \to X_R \ell_R, \quad (4)$$

the mass matrices must obey the restrictions,

$$M_\nu = X_L^\dagger M_\nu X_\nu^\ast, \quad M_\ell = X_L^\dagger M_\ell X_R, \quad (5)$$

where $X_\nu$, $X_L$ and $X_R$ are $3 \times 3$ unitary matrices. The special case of the vector-like symmetry would have left and right-handed fields transforming identically, with $X_L = X_R$.

B. Harrison, Perkins and Scott’s proposed symmetries

Harrison, Perkins and Scott [2] suggested mass matrices of the form

$$M_\ell = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \quad M_\nu = \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix}, \quad (6)$$

where the parameters $a, b, c$ are related to the three charged lepton masses, and $x, y, z$ provide three independent neutrino masses. The charged lepton mass matrix is of circulant form and can be generated by a cyclic permutation ($C_3$) symmetry. An $S_2 \times S_2$ symmetry generates the neutrino mass matrix.

The unitary transformation matrices are

$$X_{L1} = X_{R1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad X_{L2} = X_{R2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad X_{\nu 1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad X_{\nu 2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7)$$
The proposed mass matrices are diagonalised by
\[
U_{tL} = U_{tR} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix}, \quad U_{\nu} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{8}
\]
where \(\omega \equiv e^{2\pi i/3}\), which combine to give tri-bimaximal mixing.

C. Using the symmetry to constrain quark mixing to small angles: a conjecture

Harrison, Perkins and Scott’s idea can be extended to include the quarks, and produce small quark mixing. We conjecture that the up-type quarks and the down-type quarks transform under the \(C_3\) generation symmetry in the same way as the charged leptons transform above. This will force both quark mass matrices into circulant form
\[
M_u = \begin{pmatrix} a_u & b_u & c_u \\ c_u & a_u & b_u \\ b_u & c_u & a_u \end{pmatrix}, \quad M_d = \begin{pmatrix} a_d & b_d & c_d \\ c_d & a_d & b_d \\ b_d & c_d & a_d \end{pmatrix}. \tag{9}
\]
These mass matrices are diagonalised by the same matrix \(U_u = U_d\), resulting in \(U_{\text{CKM}} = U_u^\dagger U_d = I\), corresponding to no quark mixing. As with the leptons, all quark masses are unrestricted by the symmetry.

The unbroken symmetry produces \(U_{\text{CKM}} = I\), and \(U_{\text{MNS}}\) to be of tri-bimaximal form. Small symmetry breaking can be introduced to generate off-diagonal terms in the quark mixing matrix. This breaking may also deviate the lepton mixing matrix away from tri-bimaximal form.

The quarks transform together, whereas the neutrinos transform independently of the charged leptons. This accounts for the differences between the quark and the lepton mixing matrices.

Under the symmetry the neutrinos transform in a different way from all the other fermions. This may be associated with other special characteristics of the neutrinos, for example, the Majorana nature of the neutrino, or the lack of electric charge.

D. \(SU(2)_L\) constraint on standard model extensions

The conjecture outlined above shows that discrete generation symmetries can produce tri-bimaximal lepton mixing and small quark mixing. However, these symmetries cannot be incorporated into an extension of the standard model with the structure \(SU(2)_L \otimes G_H\), where the \(G_H\) is the discrete horizontal or family symmetry. The symmetries of Eq. 7 do not commute with \(SU(2)_L\), as the left-handed neutrinos transform under the symmetry in a different way from the left-handed charged leptons, whereas a symmetry that is an extension to the standard model should preserve the standard model symmetry \(SU(2)_L\), by having members of the same \(SU(2)_L\) doublet transform together. \(SU(2)_L\) is not violated by the quark transformations as the up and down-type quarks transform in the same way.

This constraint makes it difficult to find any symmetry that gives rise to tri-bimaximal mixing. Sec. IV investigates whether it is possible for any discrete family symmetry to predict tri-bimaximal mixing when the \(SU(2)_L\) constraint is included.

III. FORM-DIAGONALISABLE MATRICES

A. Definition

A form-diagonalisable matrix is a matrix that is invariant under a symmetry, and with diagonalisation matrices whose elements depend on the form of the original matrix only. As a result the diagonalisation matrices are independent of the matrices’ eigenvalues.

An \(n \times n\) form-diagonalisable matrix is defined by
\[
F = \sum_{i=1}^{k} \alpha_i \lambda_i \tag{10}
\]
where:
• $\lambda_i$ are $n \times n$ matrices of pure numbers, and $\alpha_i$ are $n$ complex parameters;

• $\lambda_i$ are simultaneously diagonalisable by two unitary matrices $U_L$ and $U_R$, where $U_L^\dagger \lambda_i U_R$ is diagonal for all $i$;

• $\lambda_i$ are invariant under a group transformation: $\lambda_i = X_L^\dagger \lambda_i X_R$;

• $k \leq n$.

Note that for $k < n$, only $k$ eigenvalues are independent.

These conditions result in the masses being linear combinations of $\alpha_i$, and the diagonalisation matrices, $U_L$ and $U_R$, being independent of these masses.

B. Examples of form-diagonalisable matrices with Abelian symmetries

Equation (6) has two examples of form diagonalisable mass matrices, with the symmetries being the Abelian groups $C_3$ and $S_2 \times S_2$.

The form of the mass matrices is dependent not only on the symmetry group, but also on the representation of the group that the transformation matrices $X_L$ and $X_R$ take.

1. Regular representation of Abelian groups

An interesting relationship occurs between the symmetry group and the diagonalisation matrix when the symmetry is an Abelian group in the regular representation. The regular representation of a group of order $n$, is a set of $n$ matrices $X_i$. The matrices are unitary, have size $n \times n$, and their elements are 0 or 1. A matrix $M$ is considered to be invariant under the regular representation of a group when $M = X_i^\dagger M X_i$ for all $i$.

For Abelian symmetries the mass matrix that is invariant under the regular representation is a linear combination of all the representation matrices themselves, i.e. $\lambda_i$ of Eq. (10) are the $X_i$. This is shown in App. A.

The matrix $U$ that diagonalises the mass matrix $M$ can be simply derived from the $n$ one-dimensional representations of the group $G$: Each column of the diagonalisation matrix is made up of a normalised list of the elements of the one-dimensional representations, and each column corresponds to a different one-dimensional representation. As all the irreducible representations of Abelian groups are one dimensional, the character table lists these representations, and the diagonalisation matrix can be read directly off the table.

2. $C_3$ example

This relationship between the regular representation and the diagonalisation matrices is illustrated by the $C_3$ symmetry of the charged leptons outlined in Sec. II B.

The charged lepton mass matrix of Eq. (6) is invariant under the regular representation of $C_3$ which is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$  

(11)

The mass matrix is made up of a linear combination of invariant matrices $\lambda_i$. In this case the $\lambda_i$ are the representation matrices themselves, forming the mass matrix $M_{\ell}$ of Eq. (6). The diagonalisation matrix is

$$U_{\ell} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix}$$  

(12)

where $\omega = e^{2\pi i/3}$, $\omega^*$, 1 are the cube roots of unity. This diagonalisation matrix can be constructed using the one-dimensional representations of $C_3$ which are $\{1, 1, 1\}$, $\{1, \omega , \omega^*\}$, $\{1, \omega^*, \omega\}$. Each column of the diagonalisation matrix is made up of a one-dimensional representation, and the matrix is normalised.

Representations other than the regular representations can also produce form-diagonalisable mass matrices. An example of this is the $S_2 \times S_2$ symmetry which generates the mass matrix $M_\nu$ of Eq. (6). In cases other than the regular representation, the relationship between the representation of the symmetry and the diagonalisation matrix is not clear.
IV. NO-GO THEOREM FOR DISCRETE FAMILY SYMMETRIES

Individual lepton number symmetry \( U(1)_{L_e} \otimes U(1)_{L_\mu} \otimes U(1)_{L_\tau} \) is a symmetry of the standard model with massless neutrinos, and is known to be broken by neutrino oscillations. However, if a discrete subgroup of this symmetry is unbroken by the neutrino mass term, this will constrain the form of the mixing matrix.

The success of the symmetries in Eq. (7) in generating tri-bimaximal mixing, and the idea that a subgroup of \( U(1)_{L_e} \otimes U(1)_{L_\mu} \otimes U(1)_{L_\tau} \) may still remain unbroken with massive neutrinos motivates the systematic study of discrete Abelian group symmetries, with the added constraint of having the left-handed charged leptons transform in the same way as the left-handed neutrinos.

This section shows that discrete unbroken generation symmetries (Abelian and non-Abelian) with the \( SU(2)_L \) constraint and the other assumptions stated below cannot generate tri-bimaximal mixing. In fact, the only mixing matrix that falls within experimental bounds and is generated by a symmetry, is the mixing matrix that is completely unrestricted by the symmetry. In this section we assume that the Higgs bosons are singlets of the symmetry.

Section IV B shows that discrete non-Abelian generation symmetries give rise to degenerate charged leptons, proving that non-Abelian symmetries cannot produce mass and mixing schemes that agree with experiment.

Section IV C considers how Abelian groups can constrain the charged lepton Dirac mass matrix. Exactly how the transformations alter the neutrino mass matrix depends on the type of mass term, because Majorana mass terms are constrained by the symmetry in a different way from Dirac mass terms. Because of this the no-go theorem for Abelian groups is segmented into three cases; Majorana neutrinos (Sec. IV D), Dirac neutrinos (Sec. IV E), and Majorana neutrinos when the mass term is generated by the seesaw mechanism (Sec. IV F). In the seesaw case we assume that the right-handed Majorana mass matrix is invertible.

We show that in all three cases all mixing schemes that can be produced by Abelian symmetries are not allowed by experiment, except for the case where the mixing is not constrained by the symmetry at all.

A. Equivalent representations yield identical mixing

The matrices \( X_{Li} \) and \( X_{Ri} \) of Eq. (5) that transform the leptons are representations of the symmetry group. Different representations of the same symmetry group provide different restrictions on the mass matrices. As there are three generations of leptons we are interested in three dimensional representations only. A given symmetry group has an infinite number of three dimensional representations, but only a finite number of inequivalent representations.

Two different representations \( X_i \) and \( Y_i \), are considered to be equivalent if they are related by a similarity transformation

\[ Y_i = V^\dagger X_i V, \]  

where \( V \) is any unitary matrix.

Appendix B shows that two equivalent transformation matrices restrict the mixing matrix in an identical way. This is because the weak basis leptons \((\nu_{LX}, l_{LX})^T \) in the case where the representation \( X_i \) is chosen, are related to the weak basis leptons \((\nu_{LY}, l_{LY})^T \) in the \( Y_i \) case by a basis change, as per

\[ \left( \begin{array}{c} \nu_{LX} \\ l_{LX} \end{array} \right) \rightarrow \left( \begin{array}{c} \nu_{LY} \\ l_{LY} \end{array} \right) = V^\dagger \left( \begin{array}{c} \nu_{LX} \\ l_{LX} \end{array} \right). \]  

Since the mixing matrix is associated with the mass basis of the leptons, not the weak basis, the two equivalent representations will restrict the mixing matrix in an identical way. As there are only a finite number of inequivalent representations of any discrete group it is possible to find all mixing matrices that can be generated by a given group.

All Abelian representations are equivalent to a diagonal representation – a representation where all matrices are diagonal. The converse is also true; no non-Abelian representation has matrices that are all diagonal (as diagonal matrices commute). This provides a convenient way of analysing many groups at once. First we will consider at non-Abelian groups by examining how non-diagonal transformations affect mass matrices and mixing, and then we consider Abelian representations by looking at diagonal representations.

B. Non-Abelian groups

Non-Abelian groups have Abelian (for example the trivial representation) and non-Abelian representations. Abelian representations of non-Abelian groups are not faithful, and are also representations of Abelian groups. This section
shows that non-Abelian representations constrain some charged leptons to be degenerate. Abelian representations are covered by sections [I-V-D] and [I-V-E]. As explained in Sec. [I-V-A] two equivalent representations correspond to two different bases. So if the mass matrices are invariant under some non-Abelian transformation, there exists a non-Abelian representation of the group that corresponds to the charged lepton mass basis $M_\ell = \text{Diag}(m_e, m_\mu, m_\tau)$. As this representation is non-Abelian, there is at least one matrix that is not diagonal.

Mass degeneracy can be concluded by considering just one non-diagonal transformation matrix. For example a block diagonal unitary matrix

$$X_L = \begin{pmatrix} x & 0 & 0 \\ 0 & y & w \\ 0 & z & v \end{pmatrix},$$

(15)

constrains $M_\ell M_\ell^\dagger$ by

$$M_\ell M_\ell^\dagger = X_L^\dagger M_\ell M_\ell^\dagger X_L = \begin{pmatrix} x^* & 0 & 0 \\ 0 & y^* & z^* \\ 0 & w^* & v^* \end{pmatrix} \begin{pmatrix} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & y & w \\ 0 & z & v \end{pmatrix},$$

(16)

$$= \begin{pmatrix} m_e^2 |x|^2 & 0 & 0 \\ 0 & m_\mu^2 |y|^2 + m_\tau^2 |z|^2 & m_\mu^2 y^* w + m_\tau^2 z^* v \\ 0 & m_\mu^2 y w^* + m_\tau^2 z v^* & m_\mu^2 |y|^2 + m_\tau^2 |v|^2 \end{pmatrix}. $$

(17)

The $2 \times 2$ block in $X_L$ rotates $m_\mu^2$ and $m_\tau^2$, so the diagonal mass matrix will only be invariant under this transformation if $m_\mu^2 = m_\tau^2$. An $X_L$ that is not in block diagonal form will result in three degenerate charged leptons.

The same argument also applies when the $X_R$ transformation is non-Abelian. In this case the $X_R$ transformation constrains $M_\ell^\dagger M_\ell = \text{Diag}(m_e^2, m_\mu^2, m_\tau^2)$ by $M_\ell^\dagger M_\ell = X_R^\dagger M_\ell^\dagger M_\ell X_R$, also resulting in degenerate masses.

C. Abelian representations and charged lepton mass matrices

In the case of Abelian groups, every representation is equivalent to a diagonal matrix representation, so to find out all the mixing matrices that can be produced by an Abelian group, we can restrict the study to how mass matrices can be constrained by diagonal representations.

The diagonal representations

$$X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}), \quad X_R = \text{Diag}(e^{i\sigma_1}, e^{i\sigma_2}, e^{i\sigma_3}),$$

(18)

constrain the charged lepton mass matrix $M_\ell$ by $M_\ell = X_L^\dagger M_\ell X_R$, or, more explicitly,

$$M_\ell = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} = \begin{pmatrix} e^{-i(\phi_1 - \sigma_1)} & se^{-i(\phi_1 - \sigma_2)} & te^{-i(\phi_1 - \sigma_3)} \\ ue^{-i(\phi_2 - \sigma_1)} & ve^{-i(\phi_2 - \sigma_2)} & we^{-i(\phi_2 - \sigma_3)} \\ xe^{-i(\phi_3 - \sigma_1)} & ye^{-i(\phi_3 - \sigma_2)} & ze^{-i(\phi_3 - \sigma_3)} \end{pmatrix}. $$

(19)

Not all of the information contained in the mass matrix is required in order to find the masses and the mixing matrix. One may simply compute the hermitian squared mass matrix $M_\ell M_\ell^\dagger$ and then diagonalise it via the left-handed matrix $U_L$ only, as per $U_L^\dagger M_\ell M_\ell^\dagger U_L = \text{Diag}(m_e^2, m_\mu^2, m_\tau^2)$. Now, $M_\ell M_\ell^\dagger$ is restricted by the $X_L$ transformation by

$$M_\ell M_\ell^\dagger = \begin{pmatrix} a & b & c \\ b^* & d & f \\ c^* & f^* & g \end{pmatrix} = X_L^\dagger M_\ell M_\ell^\dagger X_L = \begin{pmatrix} a & be^{-i(\phi_1 - \phi_2)} & ce^{-i(\phi_1 - \phi_3)} \\ b^* e^{i(\phi_1 - \phi_2)} & d & fe^{-i(\phi_2 - \phi_3)} \\ c^* e^{i(\phi_1 - \phi_3)} & f^* e^{i(\phi_2 - \phi_3)} & g \end{pmatrix}. $$

(20)

The $X_L$ transformation constrains the hermitian squared mass matrix in the following way: The diagonal elements of $M_\ell M_\ell^\dagger$ are unrestricted by the symmetry; when $\phi_i = \phi_j$, the $ij$th term in $M_\ell M_\ell^\dagger$ is unrestricted by the symmetry; otherwise the $ij$th element will be zero.

Note that $M_\ell M_\ell^\dagger$ can also be constrained by the $X_R$ matrix. For example, if $X_L = I$ and $X_R = -I$ then $M_\ell = M_\ell M_\ell^\dagger = 0$, even though the $X_R$ transformation does not constrain the mass matrix.

To make the no-go theorem simpler, we look first at how $U_{\text{MNS}}$ can be constrained by the $X_L$ transformation, before analysing how the $X_R$ transformation alters the situation. For nearly all choices of $X_L$, the $X_L$ transformation constrains $M_\ell M_\ell^\dagger$ and $M_\nu$ in such a way to force the mixing matrix $U_{\text{MNS}}$ into a form that has been ruled out experimentally. In these cases the $X_R$ transformations are irrelevant, the symmetry having been ruled out for all possible choices of $X_R$. 
D. Abelian representations and Majorana neutrinos

The left-handed transformation $X_L$ restricts the Majorana neutrino mass matrix by

$$M_{
u} = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} = \begin{pmatrix} A e^{-2i\phi_1} & B e^{-i(\phi_1 + \phi_2)} & C e^{-i(\phi_1 + \phi_3)} \\ B e^{-i(\phi_1 + \phi_2)} & D e^{-2i\phi_2} & E e^{-i(\phi_2 + \phi_3)} \\ C e^{-i(\phi_1 + \phi_3)} & E e^{-i(\phi_2 + \phi_3)} & F e^{-2i\phi_3} \end{pmatrix}. \quad (21)$$

The $X_L$ transformation multiplies each element of the mass matrix by a phase. If the phase equals 1, then the element is unconstrained by the symmetry. If the phase is not equal to 1, then the matrix element is forced to be zero. If $e^{i\phi_i} = \pm 1$, then the $i$th element of the matrix will be unrestricted by the symmetry. If $e^{i\phi_i} = e^{-i\phi_j}$ then the $j$th element will be unrestricted. Otherwise the elements will be zero.

We have performed an exhaustive analysis of all possible forms of lepton mixing matrices that can be produced by an Abelian generation symmetry. The mixing matrices are listed below. Interchanging columns corresponds to relabeling neutrino mass eigenstates.

In the following matrices $s \equiv \sin \theta$ and $c \equiv \cos \theta$, where $\theta$ is unconstrained by the symmetry. The phases $e^{i\delta_i}$ are not necessarily physical.

| Mixing matrix $U_{MNS}$ | Form of $X_L$ required for all $X_L$ |
|-------------------------|--------------------------------------|
| $U_{MNS_1}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_1}, 1)$ |
| $U_{MNS_2}$ | $X_L = \text{Diag}(1, 1, e^{i\phi_2})$ |
| $U_{MNS_3}$ | $X_L = \text{Diag}(1, 1, 1, e^{i\phi_3})$ |
| $U_{MNS_4}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{-i\phi_1}, e^{-i\phi_1})$ |
| $U_{MNS_5}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_1}, e^{-i\phi_1})$ |
| $U_{MNS_6}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{-i\phi_2}, e^{i\phi_1})$ |
| $U_{MNS_7}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{-i\phi_1}, e^{-i\phi_3})$ |
| $U_{MNS_8}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{-i\phi_3})$ |
| $U_{MNS_9}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{-i\phi_3})$ |
| $U_{MNS_{10}}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ |
| $U_{MNS_{11}}$ | $X_L = \pm I$ |
In cases $U_{\text{MNS}_{4,5,6}}$, $m_1 = -m_2$ and $m_3 = 0$. In cases $U_{\text{MNS}_{7,8,9}}$, the two mixed neutrinos have $m_i = -m_j$.

Except for the case where the mixing is unrestricted by the symmetry, none of the above mixing matrices fall within experimental bounds. In the unrestricted case $U_{\nu}$ is unrestricted, so although right-handed charged lepton transformations can alter $U_{\ell_L}$, the mixing matrix $U_{\text{MNS}} = U_{\ell_L}^\dagger U_{\nu}$ will remain unconstrained by the symmetry.

E. Abelian representations and Dirac neutrinos

An Abelian symmetry constrains the neutrino Dirac mass matrix in the same way as the charged lepton Dirac mass matrix, Eq. (11), except that the right-handed neutrino may transform in a different way to the right-handed charged leptons.

Dirac neutrino mass matrices are diagonalised by $\text{Diag}(m_1, m_2, m_3) = U_{\nu_L}^\dagger M_\nu U_{\nu_R}$, and the mixing matrix incorporates only the left diagonalisation matrices. $U_{\nu_L}$ can be obtained from $M_\nu M_\nu^\dagger$ which is restricted by the $X_L$ transformation by $M_\nu M_\nu^\dagger = X_L^\dagger M_\nu M_\nu^\dagger X_L$.

The possible $U_{\text{MNS}}$ matrices obtainable by the left-handed transformation are listed below. It is possible that the right-handed transformations will be able to further restrict the mixing matrices.

| Mixing matrix | Form of $X_L$ required for all $X_L$ |
|---------------|-------------------------------------|
| $U_{\text{MNS}_1}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ (33) |
| $U_{\text{MNS}_2}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ (34) |
| $U_{\text{MNS}_3}$ | $X_L = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_1})$ (35) |
| $U_{\text{MNS}_4}$ | $X_L = e^{i\phi_1} I$ (36) |

The only $U_{\text{MNS}}$ that fits in with experiment is the one that is unrestricted by $X_L$, which occurs when $X_L = e^{i\phi} I$. In this case both $U_{\ell_L}$ and $U_{\nu_L}$ are unconstrained by the $X_L$ transformation. However, $U_{\ell_L}$ and $U_{\nu_L}$ can be restricted by the right-handed transformations $X_{\ell_R}$ and $X_{\nu_R}$. If one or both of the two diagonalisation matrices remains unrestricted under the right-handed transformations, then $U_{\text{MNS}} = U_{\ell_L}^\dagger U_{\nu_L}$ will be unrestricted, independent of how the second diagonalisation matrix is restricted by the symmetry.

The transformation

$$ X_{\ell_R} = \text{Diag}(e^{i\sigma_1}, e^{i\sigma_2}, e^{i\sigma_3}) \quad (38) $$

restricts the charged lepton mass matrix by

$$ M_\ell = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} $$

$$ = X_L^\dagger M_\ell X_{\ell_R} = \begin{pmatrix} e^{-i(\phi - \sigma_1)_R} & e^{-i(\phi - \sigma_2)_R} & e^{-i(\phi - \sigma_3)_R} \\ e^{-i(\phi - \sigma_1)_U} & e^{-i(\phi - \sigma_2)_U} & e^{-i(\phi - \sigma_3)_U} \\ e^{-i(\phi - \sigma_1)_X} & e^{-i(\phi - \sigma_2)_X} & e^{-i(\phi - \sigma_3)_X} \end{pmatrix}. \quad (40) $$

Either the $i$th column is unrestricted by the symmetry, $(\phi = \sigma_i)$, or the symmetry constrains column $i$ to be a column of zeros $(\phi \neq \sigma_i)$. A matrix that has one column of zeros has one massless charged lepton. A matrix that has no columns of zeros is completely unconstrained by the symmetry, and will give an unrestricted $U_{\ell_L}$. 

Therefore, in the case where \( X_L = e^{i\phi} I \), \( U_{\text{MNS}} \) is unrestricted unless one or more of the charged leptons are massless. As there are no massless charged leptons, we can conclude that for Dirac neutrinos no mixing matrix is compatible with experiment, except for when \( U_{\text{MNS}} \) is completely unconstrained by the symmetry.

In fact, if the electron is taken to be massless (corresponding to a single column of zeros), we are convinced that \( U_{\text{MNS}} \) is also completely general, and hence, the mixing matrix is unrestricted by the symmetry. In this case \( U_{\text{MNS}} \) has the same number of free parameters as a completely unconstrained diagonalisation matrix. This has been backed up by numerical calculations. The right-handed diagonalisation matrix \( U_R \), however, is restricted by the right-handed transformation.

\[ F. \text{ Abelian representations and Seesaw neutrinos} \]

Majorana neutrino mass matrices that are generated by the seesaw mechanism can be expressed as

\[ M_\nu = M_d^T M_M^{-1} M_d, \tag{41} \]

where \( M_d \) is the Dirac mass matrix, and \( M_M \) is the right-handed Majorana mass matrix. This equation is valid when \( M_M \) is invertible. In this section we assume that \( M_M \) is invertible. (If the Majorana mass matrix was not invertible, and had rank \( n > 3 \), the physical particles would be \( n \) ultralight neutrinos, \( n \) heavy neutrinos and \( 2n - 6 \) neutrinos whose masses are naturally the same size as the other fermions \([43, 44]\).

Under the \( X_L \) transformations \( M_\nu \) is restricted by

\[ M_\nu = X_L^\dagger M_\nu X_L^\ast, \tag{42} \]

the same as when the neutrinos are Majorana but do not have mass terms generated by the seesaw mechanism. Section \([\text{IVD}]\) lists all the ways that \( X_L \) can restrict the mixing matrix. Again, the only mixing matrix that fits with experiment is the mixing matrix that is restricted by the symmetry, which occurs when \( X_L = \pm I \). In this case the diagonalisation matrices \( U_L \) and \( U_\nu \) are both unrestricted by the \( X_L \) transformation, but can be further restricted by right-handed transformations.

The right-handed charged lepton transformation restricts the mass matrix by

\[ M_{\ell} = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} = X_L^\dagger M_{\ell} X_L = \pm \begin{pmatrix} e^{i\sigma_1 r} & e^{i\sigma_2 s} & e^{i\sigma_3 t} \\ e^{i\sigma_1 u} & e^{i\sigma_2 v} & e^{i\sigma_3 w} \\ e^{i\sigma_1 x} & e^{i\sigma_2 y} & e^{i\sigma_3 z} \end{pmatrix}, \tag{43} \]

The argument in the Dirac neutrino section is applicable here also. Either a column of the mass matrix is unrestricted by the symmetry, or it is zero. If all columns are unrestricted, \( U_L \) is unrestricted by the symmetry, giving a mixing matrix that is unconstrained by the symmetry. For each column that is constrained to be zero, there is a corresponding massless charged lepton which is not seen in nature. If one charged lepton is taken to be massless, the mixing is still unconstrained by the symmetry. Therefore, the only mixing matrix that can be generated by a discrete unbroken symmetry, and is consistent with experiments is the mixing matrix that is completely unconstrained by the symmetry.

\[ V. \text{ CONCLUSIONS AND FUTURE WORK} \]

It is tantalising to suppose that a family symmetry could simultaneously explain both the lepton and the quark mixing matrices. We have shown however, that given certain assumptions, unbroken symmetries acting on the generations of the fermions cannot produce a lepton mixing matrix of tri-bimaximal form, or anything approaching this form. Relaxing the assumptions of this no-go theorem may make it possible for a symmetry to generate an experimentally allowed mixing matrix.

An option for trying to generate non-trivial mixing in the lepton sector, while still including the \( SU(2)_L \) restriction, is to utilise the different mass generation mechanisms for the neutrinos and charged leptons. Charged lepton masses come from Yukawa couplings with the standard model Higgs doublet. Majorana neutrinos will gain masses from another mechanism, possibly using the same Higgs doublets in the seesaw mechanism, or by interaction with a Higgs triplet, or by a different mechanism.
If the Higgs sector is extended by introducing a number of generations of Higgs fields, these Higgs fields can also transform under the symmetry. Since the action of the Higgs fields in creating mass matrices is different for neutrinos compared to charged leptons, different restrictions for the two mass matrices will in general result. This in turn will lead to the diagonalisation matrices for neutrinos being different from that of the charged leptons, possibly resulting in phenomenologically acceptable lepton mixing.

Since both up-like and down-like quarks are Dirac particles, the action of the Higgs fields in creating their mass matrices is similar for both sectors. It might be possible, then, to construct a model whereby these mass matrices are sufficiently similar so as to yield very similar left-diagonalisation matrices. The resulting \( U_{\text{CKM}} \) may then be approximately diagonal, in agreement with the observed form of this matrix. This kind of setting—models with a non-minimal Higgs sector—may the appropriate one in which to realise our conjecture (see Sub-Sec. II C) within a complete and consistent standard model extension, despite its original inspiration coming from the rather different Harrison, Perkins and Scott proposal.

Acknowledgments

RRV would like to thank Tony Zee and Lincoln Wolfenstein for useful discussions during the Neutrinos: Data, Cosmos, and Planck Scale workshop held at the Kavli Institute for Theoretical Physics at the University of California—Santa Barbara. This work was supported in part by the Australian Research Council, and in part by The University of Melbourne.

[1] L. B. Auerbach et al. (LSND), Phys. Rev. C64, 065501 (2001), hep-ex/0105068.
[2] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B530, 167 (2002), hep-ph/0202074.
[3] L. Wolfenstein, Phys. Rev. D18, 958 (1978).
[4] P. F. Harrison and W. G. Scott, Phys. Lett. B535, 163 (2002), hep-ph/0203209.
[5] P. F. Harrison and W. G. Scott, Phys. Lett. B557, 76 (2003), hep-ph/0302025.
[6] X.-G. He and A. Zee, Phys. Lett. B560, 87 (2003), hep-ph/0301092.
[7] X.-G. He and A. Zee (2003), hep-ph/0302201.
[8] Z.-Z. Xing, Phys. Lett. B533, 85 (2002), hep-ph/0204049.
[9] Y. Fukuda et al. (Kamiokande), Phys. Lett. B335, 237 (1994).
[10] W. W. M. Allison et al. (Soudan-2), Phys. Lett. B449, 137 (1999), hep-ex/9901024.
[11] M. Ambrosio et al. (MACRO), Phys. Lett. B434, 451 (1998), hep-ex/9807005.
[12] R. Becker-Szendy et al., Nucl. Phys. Proc. Suppl. 38, 331 (1995).
[13] G. L. Fogli et al., Phys. Rev. D67, 073002 (2003), hep-ph/0212127.
[14] K. Eguchi et al. (KamLAND), Phys. Rev. Lett. 90, 021802 (2003), hep-ex/0212021.
[15] Q. R. Ahmad et al. (SNO), Phys. Rev. Lett. 89, 011301 (2002), nucl-ex/0204008.
[16] M. H. Ahn et al. (K2K), Phys. Rev. Lett. 90, 041801 (2003), hep-ex/0212007.
[17] S. Fukuda et al. (Super-Kamiokande), Phys. Lett. B539, 179 (2002), hep-ex/0205075.
[18] J. N. Abdurashitov et al. (SAGE), J. Exp. Theor. Phys. 95, 181 (2002).
[19] B. T. Cleveland et al., Astrophys. J. 496, 505 (1998).
[20] C. M. Cattadori (GNO), Nucl. Phys. Proc. Suppl. 110, 311 (2002).
[21] G. L. Fogli et al. (2002), hep-ph/0212127.
[22] M. Apollonio et al. (CHOOZ), Phys. Lett. B466, 415 (1999), hep-ex/9907037.
[23] A. de Gouvea and H. Murayama (2003), hep-ph/0301050.
[24] S. Weinberg, Trans. New York Acad. Sci. 38, 185 (1977).
[25] H. Fritzsch, Phys. Lett. B70, 436 (1977).
[26] H. Harari and Y. Nir, Phys. Lett. B195, 586 (1987).
[27] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B147, 277 (1979).
[28] J. Kubo, A. Mondragon, M. Mondragon, and E. Rodriguez-Jauregui (2003), hep-ph/0302196.
[29] W. Grimus and L. Lavoura (2003), hep-ph/0305046.
[30] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B552, 207 (2003), hep-ph/0206292.
[31] E. Ma and G. Rajasekaran, Phys. Rev. D64, 113012 (2001), hep-ph/0106291.
[32] E. Ma, Mod. Phys. Lett. A17, 2361 (2002), hep-ph/0211393.
[33] H. Fritzsch and Z.-Z. Xing, Phys. Lett. B372, 265 (1996), hep-ph/9509389.
[34] H. Fritzsch and Z.-Z. Xing, Phys. Lett. B440, 313 (1998), hep-ph/9808272.
[35] M. Fukugita, M. Tanimoto, and T. Yanagida, Phys. Rev. D57, 4429 (1998), hep-ph/9709388.
[36] K. S. Babu and R. N. Mohapatra, Phys. Lett. B532, 77 (2002), hep-ph/0201176.
[37] L. Lavoura, Phys. Rev. D62, 093011 (2000), hep-ph/0005321.
APPENDIX A: REGULAR REPRESENTATIONS OF ABELIAN GROUPS

For a group of rank $n$, the regular representation involves $n \times n$ matrices, with elements 0 and 1. Each row or column contains one 1. The $ij$th term equals 1 for one and only one matrix in the representation. One of the matrices is the identity.

$M$ is invariant under the regular representation of an Abelian group if $M$ commutes with all $X$:

$$M = X^a_a^T M X_a$$ for all $a$, \hspace{1cm} (A1)

where $a$ labels the $X$ matrices, or for each element

$$M_{ij} = \sum_{kl} (X^a_{ik}) M_{kl} X^a_{lj} \text{ for all } a.$$ \hspace{1cm} (A2)

As the group is Abelian, all the $X$ matrices commute with each other, so an arbitrary linear combination of the $X$ matrices will also commute with all $X$. The following argument shows that if $M$ commutes with $X$, the most general $M$ must be a linear combination of the $X$ matrices.

The restriction forces the diagonal elements of $M$ to be equal:

$$M_{11} = \sum_{kl} (X^a_{ik}) M_{kl} X^a_{lj} \text{ for all } a.$$ \hspace{1cm} (A3)

Since there exists a matrix $X$ such that $X_{j1} = 1$ for all $j$, all the diagonal elements are equal. The diagonal elements of $M$ can be written as $M_{jj}$.

By looking just at the restrictions placed on the mass matrix by an $X$ that has $X_{ij} = 1$, we show that if $M$ commutes with $X$, the $ij$th element of the mass matrix must be equal to the $ij$th element, $M_{ij} = M_{kl}$.

Let us take the $X$ that has $X_{12} = 1$.

$$M_{12} = \sum_{kl} (X^a_{ik}) M_{kl} X^a_{lj} = \sum_{k} (X^a_{ik}) M_{k1} X_{lj} \text{ choose the } X \text{ that has } X_{12} = 1.$$ \hspace{1cm} (A4)

Repeating this will show that the restrictions from the $X$ that has $X_{12} = 1$, ensure that $M_{12} = M_{ij}$ if $X_{ij} = 1$. $M_{12} X$ describes the $ij$ terms of the mass matrix, where $X_{ij} = 1$.

The same argument can be made for any $M$ element. If $X_{ij} = X_{kl}$ for a given $X$, then $M_{ij} = M_{kl}$, showing that the $kl$th elements of $M$ can be expressed as $M_{ij} X$. Therefore $M$ is a linear combination of the $X$ matrices.

APPENDIX B: PROOF THAT TWO EQUIVALENT REPRESENTATIONS CONSTRAIN THE MIXING MATRIX IN AN IDENTICAL WAY

This proof assumes that Higgs bosons are singlets of the generation symmetry, and that the generation symmetry commutes with $SU(2)_L$ meaning $\nu_L$ transforms in the same way as $\ell_L$. The seesaw section assumes that the right-handed Majorana mass matrix is invertible.
1. Charged leptons

$A_{Li}$ and $B_{Li}$ are equivalent representations which will transform the left-handed leptons. Each matrix is labelled by an index $i$. $A_{Li}$ and $B_{Li}$ are also equivalent representations which transform the right-handed charged leptons:

$$U_1^i A_{Li} U_1 = B_{Li}, \quad U_2^i A_{Li} U_2 = B_{Li}. \quad (B1)$$

The two different representations restrict the charged lepton mass matrix by

$$M_{\ell A} = A^\dagger_{Li} M_{\ell A} A_{Li} \text{ for all } i, \quad M_{\ell B} = B^\dagger_{Li} M_{\ell B} B_{Li} \text{ for all } i$$

$$= U_1^i A^\dagger_{Li} U_1 M_{\ell B} U_2^i A_{Li} U_2. \quad (B2)$$

$U_1 M_{\ell B} U_2^\dagger$ has the same restrictions as $M_{\ell A}$. As we assume that the mass matrices are completely unconstrained apart from the generation symmetry constraints, we can set

$$U_1 M_{\ell B} U_2 = M_{\ell A}. \quad (B3)$$

$M_{\ell}$ is diagonalised by $U_{\ell L}$ and $U_{\ell R}$ via

$$\text{Diag}(m_e, m_\mu, m_\tau) = U_{\ell L A}^\dagger M_{\ell A} U_{\ell R A} = U_{\ell L B}^\dagger M_{\ell B} U_{\ell R B},$$

so $U_{\ell L B} = U_{\ell L A}^\dagger U_{\ell R A}$ and $U_{\ell R B} = U_{\ell R A}^\dagger U_{\ell L A}$.

2. Majorana neutrinos

The two representations restrict the neutrino mass matrix by

$$M_{\nu A} = A^\dagger_{Li} M_{\nu A} A_{Li} \text{ for all } i, \quad M_{\nu B} = B^\dagger_{Li} M_{\nu B} B_{Li} \text{ for all } i$$

$$= U_1^i A^\dagger_{Li} U_1 M_{\nu B} U_2^i A_{Li} U_1. \quad (B5)$$

$U_1 M_{\nu B} U_2^\dagger$ has the same restrictions as $M_{\nu A}$, and we can equate $U_1 M_{\nu B} U_2^\dagger = M_{\nu A}$.

$M_{\nu}$ is diagonalised by $U_{\nu}$ via

$$\text{Diag}(m_1, m_2, m_3) = U_{\nu A}^\dagger M_{\nu A} U_{\nu A}^* = U_{\nu B}^\dagger M_{\nu B} U_{\nu B}^* \quad (B6)$$

So $U_{\nu B} = U_{\nu A}^\dagger U_{\nu A}$. Combining this result with the charged lepton results we see

$$U_{\text{MNSB}} = U_{\ell L B}^\dagger U_{\nu B} = U_{\ell L A}^\dagger U_{\nu A}^\dagger U_{\ell L A} U_{\nu A} = U_{\ell L A} U_{\nu A} = U_{\text{MNSA}} \quad (B7)$$

showing that representation $A$ gives the same mixing matrix restrictions as representation $B$.

3. Dirac neutrinos

The right-handed neutrinos transform by the representations $A_{\nu R i}$ and $B_{\nu R i}$ which are related by

$$U_3^i A_{\nu R i} U_3 = B_{\nu R i}. \quad (B8)$$

An identical argument to App. B1 shows $U_1 M_{\nu B} U_3^\dagger$ has the same restrictions as $M_{\nu A}$, enabling us to set $U_1 M_{\nu B} U_3^\dagger = M_{\nu A}$. so $U_{\nu L B} = U_{\nu L A}^\dagger U_{\nu L A}$. $U_{\nu R B} = U_{\nu R A}^3 U_{\nu R A}$.

Combining this with the charged lepton result we see that the mixing matrix for $A$ is the same as the mixing matrix for $B$:

$$U_{\text{MNSB}} = U_{\ell L B} U_{\nu B} = U_{\ell L A} U_{\nu A}^\dagger U_{\ell L A} U_{\nu A} = U_{\ell L A} U_{\nu A} = U_{\text{MNSA}}, \quad (B9)$$

showing that the two equivalent representations restrict the mixing in the same way.
4. Seesaw neutrinos

This section assumes that the Majorana mass matrix is invertible, so the resultant light neutrino mass matrix is given by \( M_\nu = M_d^T M_M^{-1} M_d \).

From App. B3, \((U_1 M_{dB} U_3^T)\) has the same restrictions as \( M_{dA} \), so set them to be equal.

From App. B2, the right-handed Majorana mass term constraints show \((U_3^* M_{MB} U_3^T)\) has the same restrictions as \( M_{MA} \), so they can be set equal.

The resultant light neutrino mass term has the restrictions

\[
M_{\nu A} = M_{dA} M_{dA}^{-1} M_{dA}^T \\
= (U_1 M_{dB} U_3^T)(U_3 M_{MB} U_3^T)(U_3 M_{dB} U_3^T) \\
= U_1 M_{dB} M_{dA}^{-1} M_{dB}^T U_1^T \\
= U_1 M_{\nu B} U_1^T.
\] (B10)

So \( M_{\nu A} \) and \( M_{\nu B} \) are related by a basis change - the same as the case with non-seesaw Majorana neutrinos.

Diagonalising:

\[
\text{Diag}(m_1, m_2, m_3) = U_{\nu A}^T M_{\nu A} U_{\nu A}^* \\
= U_{\nu A}^T U_1 M_{\nu B} U_1^T U_{\nu A}^* \\
= U_{\nu B}^T M_{\nu B} U_{\nu B}^* \\
\] (B11)

So \( U_{\nu B} = U_1^T U_{\nu A} \).

So the mixing matrices for the two representations are

\[
U_{\text{MNSB}} = U_{1L}^T U_{\nu B} = U_{1L}^T U_1 U_{\nu A} = U_{1L}^T U_{\nu A} = U_{\text{MNSA}} \] (B12)

Therefore, two different, but equivalent, representations restrict the mixing matrix in the same way.