ON THE VOLUME OF SECTIONS OF THE CUBE

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Abstract. We study the properties of the maximal volume $k$-dimensional sections of the $n$-dimensional cube $[-1, 1]^n$. We obtain a first order necessary condition for a $k$-dimensional subspace to be a local maximizer of the volume of such sections, which we formulate in a geometric way. We estimate the length of the projection of a vector of the standard basis of $\mathbb{R}^n$ onto a $k$-dimensional subspace that maximizes the volume of the intersection. We find the optimal upper bound on the volume of a planar section of the cube $[-1, 1]^n$, $n \geq 2$.

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1. Introduction

The problem of volume extrema of the intersection of the standard $n$-dimensional cube $\square^n = [-1, 1]^n$ with a $k$-dimensional linear subspace $H$ has been studied intensively. The tight lower bound for all $n \geq k \geq 1$ was obtained by J. Vaaler [Vaa79], he showed that

$$\text{vol}_k \square^k \leq \text{vol}_k (\square^n \cap H).$$

A. Akopyan and R. Karasev [AHK19] gave a new proof of Vaaler’s inequality in terms of waists. A deep generalization of Vaaler’s result for $\ell_p^n$ balls was made by M. Meyer and A. Pajor [MP88]. K. Ball in [Bal89], using his celebrated version of the Brascamb–Lieb inequality, found the following upper bounds

$$\text{vol}_k (\square^n \cap H) \leq \left( \frac{n}{k} \right)^{k/2} \text{vol}_k \square^k \quad \text{and} \quad \text{vol}_k (\square^n \cap H) \leq \left( \sqrt{2} \right)^{n-k} \text{vol}_k \square^k. \tag{1.1}$$

The leftmost inequality here is tight if and only if $k | n$ (see [Iva19]), and the rightmost one is tight whenever $2k \geq n$. Thus, if $k$ does not divide $n$ and $2k < n$, the maximal volume of a section of $\square^n$ remains unknown. For the hyperplane case $k = n - 1$, the rightmost inequality in (1.1) was generalized to certain product measures which include Gaussian type measures by A. Koldobsky and H. König [KK12].

In [Iva18a], a tight bound on the volume of a section of $\square^n$ by a $k$-dimensional linear subspace was conjectured for all $n > k \geq 1$. Namely, let $C_\square(n, k) 2^k$ be the maximum volume of a section of $\square^n$ by a $k$-dimensional subspace $H$ such that $\square^n \cap H$ is an affine cube.

Conjecture 1. The maximal volume of a section of the cube $\square^n$ by a $k$-dimensional subspace $H$ is attained on subspaces such that the section is an affine cube, i.e.

$$\text{vol}_k (\square^n \cap H) \leq C_\square(n, k) \text{vol}_k \square^k.$$ 

It is not hard to show that

$$C_\square(n, k) = \left[ \begin{array}{c} n \\ k \end{array} \right]^{-k[n/k]} \left[ \begin{array}{c} n \\ k \end{array} \right]^{k-(n-k[n/k])}. \tag{1.2}$$

We give a complete description of the set of $k$-dimensional subspaces of $\mathbb{R}^n$ on which $C_\square(n, k)$ is attained and satisfies identity (1.2) in Lemma 2.2.
In this paper, we continue our study of maximizers of
\[(1.3) \quad G(H) = \text{vol}_k(\Box^n \cap H), \quad H \in \text{Gr}(n, k) \text{ with } n \geq k \geq 2.\]

Using the approach of [Iva18b], which is described in detail below, we get a geometric first order necessary condition for \(H\) to be a local maximizer of (1.3).

**Theorem 1.1.** Let \(H\) be a local maximizer of (1.3), \(v\) be the projection of the \(i\)-th vector of the standard basis of \(\mathbb{R}^n\) onto \(H, i \in [n]\). Denote \(P = \Box^n \cap H\), we understand \(P\) as a \(k\)-dimensional polytope in \(H\). Then

1. \(P = \bigcap \{x \in H : |\langle x, v_i \rangle| \leq 1\}\).
2. For every \(i \in [n]\), \(v_i \neq 0\) and the intersection of \(P\) with the hyperplane \(\{x : \langle x, v_i \rangle = 1\}\) is a facet of \(P\).
3. For every \(i \in [n]\), the line \(\text{span}\{v_i\}\) intersects the boundary of \(P\) in the centroid of a facet of \(P\).
4. Let \(F\) be a facet of \(P\). Denote \(P_F = \text{co}\{0, F\}\). Then
\[\frac{\text{vol}_k P_F}{\text{vol}_k P} = \frac{1}{2} \sum_i |v_i|^2,\]
where the summation is over all indices \(i \in [n]\) such that the line \(\text{span}\{v_i\}\) intersects \(F\) in its centroid.

One of the arguments used by K. Ball to prove the rightmost inequality in (1.1) is that the projection of a vector of the standard basis onto a maximizer of (1.3) for \(2k \geq n\) has length at least \(\sqrt{2}\). We prove the following extension of this result.

**Theorem 1.2.** Let \(n > k \geq 1\) and \(H\) be a global maximizer of (1.3), \(v\) be the projection of a vector of the standard basis of \(\mathbb{R}^n\) onto \(H\). Then
\[
\frac{k}{n + k} \leq |v|^2 \leq \frac{k}{n - k}.
\]

Using these results and some additional geometric observations, we prove the following.

**Theorem 1.3.** Conjecture 1 is true for \(n > k = 2\). That is, for any two-dimensional subspace \(H \subset \mathbb{R}^n\) the following inequality holds
\[
\text{Area}(\Box^n \cap H) \leq C(\Box(n, 2) \text{ vol}_2 \Box^2 = 4 \sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor}.
\]
This bound is optimal and is attained if and only if \(\Box^n \cap H\) is a rectangle with the sides of lengths \(2\sqrt{\left\lfloor \frac{n}{2} \right\rfloor}\) and \(2\sqrt{\left\lfloor \frac{n}{2} \right\rfloor}\).

2. Definitions and Preliminaries

For a positive integer \(n\), we refer to the set \(\{1, 2, \ldots, n\}\) as \([n]\). The standard \(n\)-dimensional cube \([-1, 1]^n\) is denoted by \(\Box^n\). We use \(\langle p, x \rangle\) to denote the standard inner product of vectors \(p\) and \(x\) in \(\mathbb{R}^n\). For vectors \(u, v \in \mathbb{R}^n\), their tensor product (or, diadic product) is the linear operator on \(\mathbb{R}^n\) defined as \(\langle u \otimes v \rangle x = \langle u, x \rangle v\) for every \(x \in \mathbb{R}^d\). The linear hull of a subset \(S\) of \(\mathbb{R}^n\) is denoted by \(\text{span} S\). For a \(k\)-dimensional linear subspace \(H\) of \(\mathbb{R}^n\) and a body \(K \subset H\), we denote by \(\text{vol}_k K\) the \(k\)-dimensional volume of \(K\). The two-dimensional volume of a body \(K \subset \mathbb{R}^2\) is denoted by \(\text{Area} K\). We denote the identity operator on a linear subspace \(H \subset \mathbb{R}^n\) by \(I_H\). If \(H = \mathbb{R}^k\), we use \(I_k\) for convenience.

For a non-zero vector \(v \in \mathbb{R}^k\), we denote by \(H_v^+\) the affine hyperplane \(\{x \in \mathbb{R}^k : \langle x, v \rangle = 1\}\), and by \(H_v^+\) and \(H_v^-\) the half-spaces \(\{x \in \mathbb{R}^k : \langle x, v \rangle \leq 1\}\) and \(\{x \in \mathbb{R}^k : \langle x, v \rangle \geq -1\}\), respectively.
It is convenient to identify a section of the cube with a convex polytope in $\mathbb{R}^k$. Let $\{v_1, \ldots, v_n\}$ be the projections of the standard basis of $\mathbb{R}^n$ onto $H$. Clearly,

$$\Box^n \cap H = \bigcap_{i=1}^n \{x \in H : |\langle x, v_i \rangle| \leq 1\}.$$ 

That means that a section of $\Box^n$ is determined by the set of vectors $\{v_i\}_1^n \subset H$, which are the projections of the orthogonal basis. Such sets of vectors have several equivalent description and names.

**Definition 1.** We will say that an ordered $n$-tuple of vectors $\{v_1, \ldots, v_n\} \subset H$ is a tight frame (or forms a tight frame) in a vector space $H$ if

$$(2.1) \quad \left\| \sum_{i=1}^n v_i \otimes v_i \right\|_H = I_H,$$

where $I_H$ is the identity operator in $H$ and $A|_H$ is the restriction of an operator $A$ onto $H$. We use $\Omega(n, k)$ to denote the set of all tight frames with $n$ vectors in $\mathbb{R}^k$.

**Definition 2.** An $n$-tuple of vectors in a linear space $H$ that spans $H$ is called a frame.

In the following trivial lemma we understand $\mathbb{R}^k \subset \mathbb{R}^n$ as the subspace of vectors, whose last $n-k$ coordinates are zero. For convenience, we will consider $\{v_i\}_1^n \subset \mathbb{R}^k \subset \mathbb{R}^n$ as $k$-dimensional vectors.

**Lemma 2.1.** The following assertions are equivalent:

1. the vectors $\{v_1, \ldots, v_n\} \subset \mathbb{R}^k$ form a tight frame in $\mathbb{R}^k$;
2. there exists an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$ such that $v_i$ is the orthogonal projection of $f_i$ onto $\mathbb{R}^k$, for any $i \in [n]$;
3. $\text{span}\{v_1, \ldots, v_n\} = \mathbb{R}^k$ and the Gram matrix $\Gamma$ of vectors $\{v_1, \ldots, v_n\} \subset \mathbb{R}^k$ is the matrix of the projection operator from $\mathbb{R}^n$ onto the linear hull of the rows of the matrix $M = (v_1, \ldots, v_n)$.
4. the $k \times n$ matrix $M = (v_1, \ldots, v_n)$ is a sub-matrix of an orthogonal matrix of order $n$.

It follows that the tight frames in $\mathbb{R}^k$ are exactly the projections of orthonormal bases onto $\mathbb{R}^k$. This observation allows us to reformulate the problems in terms of tight frames and associated polytopes in $\mathbb{R}^k$. Indeed, identifying $H$ with $\mathbb{R}^k$, we identify the projection of the standard basis onto $H$ with a tight frame $\{v_1, \ldots, v_n\} \subset \Omega(n, k)$. Thus, we identify $\Box^n \cap H$ with the intersection of slabs of the form $H^+_v \cap H^-_v$, $i \in [n]$. Vice versa, assertion (3) gives a way to reconstruct $H$ from a given tight frame $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^k$.

**Definition 3.** We will say that an $n$-tuple $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^k$ generates

1. the polytope

$$(2.2) \quad \Box(S) = \bigcap_{i \in [n]} \left( H^+_v \cap H^-_v \right),$$

which we call the section of the cube generated by $S$;
2. the matrix $\sum_{i \in [n]} v_i \otimes v_i$. We use $A_S$ to denote this matrix.

To sum up, the global extrema of (1.3) coincide with that of

$$(2.3) \quad F(S) = \text{vol}_k \Box(S), \quad S \in \Omega(n, k) \quad \text{with} \quad n \geq k > 1.$$ 

Moreover, it was shown in [Iva18b] that the local extrema of (2.3) coincide with that of (1.3). However, we note that there is an ambiguity when we identify $H$ with $\mathbb{R}^k$. Any choice of orthonormal basis of $H$ gives its own tight frame in $\mathbb{R}^k$, all of them are isometric but different from each other. It is not a problem as there exists a one-to-one correspondence between $\text{Gr}(n, k)$
and $\Omega(n,k) \cap \Omega(k)$, where $O(k)$ is the orthogonal group in dimension $k$. And, clearly, $F(S_1) = F(S_2)$ whenever $S_2 = U(S_1)$ for some $U \in O(k)$.

In the following lemma, we give a complete description of the set $\mathcal{H}$ of $k$-dimensional subspaces of $\mathbb{R}^n$ such that $\square^n \cap H$ is an affine cube and $\operatorname{vol}_k (\square^n \cap H) = 2^k C_{\square}(n,k)$ for $H \in \mathcal{H}$. Since it was proven in [Iva18a], we provide a sketch of its proof in Appendix A.

**Lemma 2.2.** Constant $C_{\square}(n,k)$ is given by (1.2) and is attained on the subspaces given by the following rule.

1. We partition $[n]$ into $k$ sets such that the cardinalities of any two sets differ by at most one.
2. Let $\{i_1, \ldots, i_\ell\}$ be one of the sets of the partition. Then, choosing arbitrary signs, we write the system of linear equations
   $$\pm x[i_1] = \ldots = \pm x[i_\ell],$$
   where $x[i]$ denotes the $i$-th coordinate of $x$ in $\mathbb{R}^n$.
3. Our subspace is the solution of the system of all equations written for each set of the partition at step (2).

From now on, we will study properties of the maximizers of (2.3) and work with tight frames.

### 3. Operations on frames

The following approach to our problem was proposed in [Iva18b] and used in [Iva18a] to study the properties of projections of the standard cross-polytope.

The main idea is to transform a given tight frame $S$ into a new one $S'$ and compare the volumes of the sections of the cube generated by them. Since it is not very convenient to transform a given tight frame into another one, we add an intermediate step: we transform a tight frame $S$ into a frame $\tilde{S}$, and then we transform $\tilde{S}$ into a new tight frame $S'$ using a linear transformation. The main observation here is that we can always transform any frame $\tilde{S} = \{v_1, \ldots, v_n\}$ into a tight frame $S'$ using a suitable linear transformation $L$: $S' = L \tilde{S} = \{Lv_1, \ldots, Lv_n\}$. Equivalently, any non-degenerate centrally symmetric polytope in $\mathbb{R}^k$ is an affine image of a section of a high dimension cube.

For a frame $S$ in $\mathbb{R}^k$, by definition put
$$B_S = A_S^{-\frac{1}{2}} = \left(\sum_{v \in S} v \otimes v\right)^{-\frac{1}{2}}.$$

The operator $B_S$ is well-defined as the condition $\operatorname{span} S = \mathbb{R}^k$ implies that $A_S$ is a positive definite operator. Clearly, $B_S$ maps any frame $S$ to a tight frame:
$$\sum_{v \in S} B_S v \otimes B_S v = B_S \left(\sum_{v \in S} v \otimes v\right) B_S^T = B_S A_S B_S = I_k.$$

We obtain the following necessary and sufficient condition for a tight frame to be a maximizer of (2.3).

**Lemma 3.1.** The maximum of (2.3) is attained at a tight frame $S \in \Omega(n,k)$ iff for an arbitrary frame $\tilde{S}$ in $\mathbb{R}^k$ inequality

$$\frac{\operatorname{vol}_k \square(\tilde{S})}{\operatorname{vol}_k \square(S)} \leq \frac{1}{\sqrt{\det A_{\tilde{S}}}} \quad (3.1)$$

holds.
Proof. For any frame $\tilde{S}$, we have that $B_{\tilde{S}}\tilde{S}$ is a tight frame and $\text{vol}_k \square(B_{\tilde{S}}\tilde{S}) = \text{vol}_k \square(\tilde{S}) / \det B_{\tilde{S}}$. The maximum of (2.3) is attained at a tight frame $S$ if $\text{vol}_k \square(B_{\tilde{S}}\tilde{S}) \leq \text{vol}_k \square(S)$ for an arbitrary frame $\tilde{S}$. Hence

$$1 \geq \frac{\text{vol}_k \square(B_{\tilde{S}}\tilde{S})}{\text{vol}_k \square(S)} = \frac{\text{vol}_k \square(\tilde{S})}{\text{vol}_k \square(S)} \frac{1}{\det B_{\tilde{S}}} = \frac{\text{vol}_k \square(\tilde{S})}{\text{vol}_k \square(S)} \sqrt{\det A_{\tilde{S}}}$$

Dividing by $\sqrt{\det A_{\tilde{S}}}$, we obtain inequality (3.1). \hfill \Box

Clearly, if $\tilde{S}$ in the assertion of Lemma 3.1 is close to $S$, then the tight frame $B_{\tilde{S}}\tilde{S}$ is close to $S$ as well. Therefore, inequality (3.1) gives a necessary condition for local maximizers of (2.3). Let us illustrate how we will use it.

Let $S$ be an extremizer of (2.3), and $T$ be a map from a subset of $\Omega(n,k)$ to the set of frames in $\mathbb{R}^k$. In order to obtain properties of extremizers, we consider a composition of two operations:

$$S \xrightarrow{T} \tilde{S} \xrightarrow{B_{\tilde{S}}} S',$$

where $B_{\tilde{S}}$ is as defined above. For example, see Figure 1, where $T$ is the operation of replacing a vector $v$ of $S$ by the origin.

![Figure 1](image)

**Figure 1.** Here we map one vector to zero.

Choosing a simple operation $T$, we may calculate the left-hand side of (3.1) in some geometric terms. We consider several simple operations: Scaling one or several vectors, mapping one vector to the origin, mapping one vector to another. On the other hand, the determinant in the right-hand side of (3.1) can be calculated for the operations listed above.

In particular, the following first-order approximation of the determinant was obtained by the author in [Iva18b, Theorem 1.2]. We provide a sketch of its proof in Appendix A.

**Lemma 3.2.** Let $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^k$ be a tight frame and the $n$-tuple $\tilde{S}$ be obtained from $S$ by substitution $v_i \to v_i + tx_i$, where $t \in \mathbb{R}$, $x_i \in \mathbb{R}^k$, $i \in [n]$. Then

$$\sqrt{\det A_{\tilde{S}}} = 1 + t \sum_{i=1}^n \langle x_i, v_i \rangle + o(t).$$

We state as lemmas several technical facts from linear algebra that will be used later.

**Lemma 3.3.** Let $A$ be a positive definite operator on $\mathbb{R}^k$. For any $u \in \mathbb{R}^k$, we have

$$\det(A \pm u \otimes u) = (1 \pm |A^{-1/2}u|^2) \det A.$$

Proof. We have

$$\det(A \pm u \otimes u) = \det A \cdot \det \left( I_k \pm A^{-1/2}u \otimes A^{-1/2}u \right).$$

Diagonalizing the operator $I_k \pm A^{-1/2}u \otimes A^{-1/2}u$, we see that its determinant equals $1 \pm |A^{-1/2}u|^2$. This completes the proof. \hfill \Box

For any $n$-tuple $S$ of vectors of $\mathbb{R}^k$ with $v \in S$, we use $S \setminus v$ to denote the $(n-1)$-tuple of vectors obtained from $S$ by removing the first occurrence of $v$ in $S$. 
Lemma 3.4. Let $S \in \Omega(n, k)$ and $v \in S$ be a vector such that $|v| < 1$. Then $S \setminus v$ is a frame in $\mathbb{R}^k$ and $B_{S \setminus v}$ is the stretch of $\mathbb{R}^k$ by the factor $(1 - |v|^2)^{-1/2}$ along span{v}. In particular, for any $u \in \mathbb{R}^k$, we have $|B_{S \setminus v}u| \geq |u|$. 

Proof. Since $A_{S \setminus v} = I_k - v \otimes v > 0$, we have that $S \setminus v$ is a frame in $\mathbb{R}^k$. Clearly, $A_{S \setminus v}$ stretches the space by the factor $(1 - |v|^2)$ along span{v}. Therefore, the operator $B_{S \setminus v} = A_{S \setminus v}^{-1/2}$ stretches the space by the factor $(1 - |v|^2)^{-1/2}$ along the same direction. □

4. Properties of a global maximizer

Theorem 1.2 is formulated in terms of subspaces. For the sake of convenience, we introduce its equivalent reformulation in terms of tight frames.

Theorem 4.1 (Frame version of Theorem 1.2). Let $S \in \Omega(n, k)$ be a global maximizer of (2.3) for $n > k \geq 1$, let $v \in S$. Then

$$\frac{k}{n+k} \leq |v|^2 \leq \frac{k}{n-k}.$$  

(4.1)

Proof. The definition of tight frame implies that $\sum_{v \in S} |v|^2 = k$. Hence there is a vector $u \in S$ such that $|u|^2 \leq k/n$ and a vector $w \in S$ such that $|w|^2 \geq k/n$.

We start with the rightmost inequality in (4.1). Let $\tilde{S}$ be the $n$-tuple obtained from $S$ by substitution $u \rightarrow v$. Since $|u|^2 \leq k/n < 1$, we have $A_{\tilde{S}} = I_k - u \otimes u + v \otimes v \geq I_k - u \otimes u > 0$. Hence $\tilde{S}$ is a frame in $\mathbb{R}^k$. By identity (2.2), the inclusion $\Box(S) \subset \Box(\tilde{S})$ holds. Therefore, $\vol_k \Box(S) \leq \vol_k \Box(\tilde{S})$. Using Lemma 3.1 and Lemma 3.3, we obtain

$$1 \geq \det A_{\tilde{S}} = (1 + |B_{\tilde{S} \setminus v}|^2)(1 - |u|^2).$$

By Lemma 3.4, the right-hand side of this inequality is at least $(1 + |v|^2)(1 - |u|^2)$. Therefore,

$$|v|^2 \leq \frac{|u|^2}{1 - |u|^2} \leq \frac{k}{n-k}.$$  

Let us prove the leftmost inequality in (4.1). There is nothing to prove if $|v|^2 \geq k/n$. Assume that $|v| < k/n$. Let $\hat{S}$ be the $n$-tuple obtained from $S$ by substitution $v \rightarrow w$. Since $A_{\hat{S}} = I - v \otimes v + u \otimes u > 0$, $\hat{S}$ is a frame in $\mathbb{R}^k$. By identity (2.2), the inclusion $\Box(S) \subset \Box(\hat{S})$ holds. Therefore, $\vol_k \Box(S) \leq \vol_k \Box(\hat{S})$. Using Lemma 3.1 and Lemma 3.3, we obtain

$$1 \geq \det A_{\hat{S}} = (1 + |B_{\hat{S} \setminus w}|^2)(1 - |v|^2).$$

Again, by Lemma 3.4, the right-hand side of this inequality is at least $(1 + |w|^2)(1 - |v|^2)$. It follows that

$$|v|^2 \geq \frac{|w|^2}{1 + |w|^2} \geq \frac{k}{n+k}.$$  

This completes the proof. □

Clearly, Theorem 4.1 implies Theorem 1.2.

These theorems can be sharpened in the planar case.

Lemma 4.1. Let $S \in \Omega(n, 2)$ be a maximizer of (2.3) for $k = 2$ and $n \geq 3$, let $v \in S$. Then

$$\frac{2}{n+2} \leq |v|^2 \leq \frac{2}{n-1}.$$  

(4.2)

Proof. By Lemma 2.2, we have

$$\text{Area} \Box(S) \geq 4\sqrt{\frac{1}{n}}$$  

(4.3)

Let us prove the leftmost inequality in (4.2). It is trivial if $|v| \geq 2/n$. Assume that $|v| < 2/n$. Let $S' \in \Omega(n - 1, 2)$ be a maximizer of (2.3). By Ball’s inequality (1.1), we have

$$\text{Area} \Box(S') \leq 2(n-1).$$  

(4.4)
Consider $S \setminus v$. It is a frame by Lemma 3.4. Then, by Lemma 3.1 and Lemma 3.3, we get
\[
\frac{\text{Area} \Box(S \setminus v)}{\text{Area} \Box(S')} \leq \frac{1}{\sqrt{\det A_{S \setminus v}}} = \frac{1}{\sqrt{1 - |v|^2}}.
\]
By identity (2.2), we have $\Box(S) \subset \Box(S \setminus v)$. By this and by inequalities (4.4) and (4.3), we get
\[
\frac{\text{Area} \Box(S \setminus v)}{\text{Area} \Box(S')} \geq \frac{\text{Area} \Box(S)}{\text{Area} \Box(S')} \geq \frac{4\sqrt{|n/2| |n/2|}}{2(n - 1)}.
\]
Combining the last two inequalities, we obtain
\[
|v|^2 \geq 1 - \left(\frac{\text{Area} \Box(S')}{\text{Area} \Box(S)}\right)^2 \geq 1 - \left(\frac{n - 1}{2\sqrt{|n/2| |n/2|}}\right)^2 \geq \frac{2}{n + 1}.
\]
We proceed with the rightmost inequality in (4.2). Let $S' \in \Omega(n + 1, 2)$ be a maximizer of (2.3). By Ball’s inequality (1.1), we have
\[
\text{Area} \Box(S') \leq 2(n + 1).
\]
We use $\hat{S}$ to denote the $(n + 1)$-tuple obtained from $S$ by concatenating $S$ with the vector $v$. Since $S$ is a frame in $\mathbb{R}^2$, $\hat{S}$ is a frame in $\mathbb{R}^2$ as well. By Lemma 3.1 and Lemma 3.3, we get
\[
\frac{\text{Area} \Box(\hat{S})}{\text{Area} \Box(S')} \leq \frac{1}{\sqrt{\det A_{\hat{S}}}} = \frac{1}{\sqrt{1 + |v|^2}}.
\]
By identity (2.2), we have $\Box(S) = \Box(\hat{S})$. By this and by inequalities (4.5) and (4.3), we get
\[
\frac{\text{Area} \Box(S)}{\text{Area} \Box(S')} = \frac{\text{Area} \Box(\hat{S})}{\text{Area} \Box(S')} \geq \frac{4\sqrt{|n/2| |n/2|}}{2(n + 1)}.
\]
Combining the last two inequality, we obtain
\[
|v|^2 \leq \left(\frac{\text{Area} \Box(S')}{\text{Area} \Box(S)}\right)^2 - 1 \leq \left(\frac{n + 1}{2\sqrt{|n/2| |n/2|}}\right)^2 - 1 \leq \frac{2}{n - 1}.
\]

Remark 1. It is possible to sharpen inequality (4.1) for $k > 2$ and $n > 2k$ using the same approach as in Lemma 4.1. The idea is to remove $n \mod k$ from or add $n - (n \mod k)$ vectors to a maximizer and compare the volume of a section of the cube generated by the new frame with the Ball bound (1.1). However, it doesn’t give a substantial improvement.

5. Local properties

In this section, we prove some properties of the local maximizers of (2.3). We will perturb facets of $\Box(S)$ of a local maximizer $S$ (that is, we will perturb the vectors of $S$ in a specific way corresponding to a perturbation of some facets of the polytope $\Box(S)$). To this end, we need to recall some general properties of polytopes connected to perturbations of a half-space supporting a polytope in its facet.

5.1. Properties of polytopes. Recall that a point $c$ is the centroid of a facet $F$ of a polytope $P \subset \mathbb{R}^k$ if
\[
c = \frac{1}{\text{vol}_{k-1} F} \int_F xd\lambda,
\]
where $d\lambda$ is the standard Lebesgue measure on the hyperplane containing $F$.

For a set $W \subset \mathbb{R}^k$, we use $P(W)$ to denote the polytopal set $\bigcap_{w \in W} H^+_w$. Let $W$ be a set of pairwise distinct vectors such that
- the set $P(W)$ is a polytope;
• for every \( w \in W \), the hyperplane \( H_w \) supports \( P(W) \) in a facet of \( P(W) \).

That is, \( W \) is the set of scaled outer normals of \( P(W) \). Denote \( P = P(W) \). We fix \( w \in W \) and the facet \( F = P \cap H_w \) of \( P \). Let \( c \) be the centroid of \( F \).

**Transformation 1.** We will “shift” a facet of a polytope parallel to itself. Let \( W' \) be obtained from \( W \) by substitution \( w \to w + h \frac{w}{|w|} \), where \( h \in \mathbb{R} \). Denote \( P' = P(W') \). That is, the polytopal set \( P' \) is obtained from \( P \) by the shift of the half-space \( H_w^+ \) by \( h \) in the direction of its outer normal. By the celebrated Minkowski existence and uniqueness theorem for convex polytopes (see, for example, [Gru07, Theorem 18.2]), we have

\[
(5.2) \quad \text{vol}_k P' - \text{vol}_k P = h \text{vol}_{k-1} F + o(h).
\]

**Figure 2.** Parallel shift of the facet \( F \) by the vector \( u = h w/|w| \).

**Transformation 2.** We will rotate a facet around a codimension two subspace. Let \( u \) be a unit vector orthogonal to \( w \). Define \( c_w = H_w \cap \text{span}\{w\} \) and \( L_u = H_w \cap (u^\perp + c_w) \). Note that \( L_u \) is a codimension two affine subspace of \( \mathbb{R}^k \) and an affine hyperplane in \( H_w \). Clearly, for any non-zero \( t \in \mathbb{R} \), \( L_u = H_w \cap H_{w + tu} = \{ x \in H_w \mid \langle x - c(w), u \rangle = 0 \} \). Thus, \( L_u \) divides \( F \) into two parts

\[
F^+ = F \cap \{ x \in H_w \mid \langle x - c(w), u \rangle \geq 0 \} \quad \text{and} \quad F^- = F \cap \{ x \in H_w \mid \langle x - c(w), u \rangle \leq 0 \}
\]

(one of the sets \( F^+ \) or \( F^- \) is empty if \( c(w) \notin F \)). Let \( \alpha \in (-\pi/2, \pi/2) \) be the oriented angle between hyperplanes \( H_w \) and \( H_{w + tu} \) such that \( \alpha \) is positive for positive \( t \).

Let \( W' \) be obtained from \( W \) by substitution \( w \to w + tu \), where \( t \in \mathbb{R} \) and \( u \) is a unit vector orthogonal to \( w \). Denote \( P' = P(W') \). Thus, for a sufficiently small \( |t| \), the polytopal set \( P' \) is a polytope obtained from \( P \) by the rotation of the half-space \( H_w^+ \) around the codimension two affine subspace \( L_u \) by some angle \( \alpha = \alpha(t) \). Clearly, in order to calculate the volume of \( P' \), we need to subtract from \( \text{vol}_k P \) the volume of the subset of \( P \) that is above \( H_{w + tu} \) and add to \( \text{vol}_k P \) the volume of the subset of \( P' \) that is above \( H_w \). Formally speaking, denote

\[
Q^+ = \begin{cases} 
P' \cap (\mathbb{R}^k \setminus H_w^+) & \text{for } \alpha \geq 0 \\
\quad P \cap (\mathbb{R}^k \setminus H_{w+tu}^+) & \text{for } \alpha < 0
\end{cases}
\quad \text{and} \quad
Q^- = \begin{cases} 
P \cap (\mathbb{R}^k \setminus H_{w+tu}^+) & \text{for } \alpha \geq 0 \\
\quad P' \cap (\mathbb{R}^k \setminus H_w^+) & \text{for } \alpha < 0
\end{cases}
\]

Then, we have (see Figure 3)

\[
(5.3) \quad \text{vol}_k P' - \text{vol}_k P = \text{sign } \alpha \left( \text{vol}_k Q^+ - \text{vol}_k Q^- \right).
\]

There is a nice approximation for \( \text{vol}_k Q^+ - \text{vol}_k Q^- \). Let \( C^+_\alpha \) (resp., \( C^-_\alpha \)) be the set swept out by \( F^+ \) (resp., \( F^- \)) while rotating around \( L_u \) by the angle \( \alpha \). By routine,

\[
\text{vol}_k C^+_\alpha = |\alpha| \int_0^{+\infty} r \text{vol}_{k-2} (F \cap (L_u + ru)) \, dr
\]
\[
\left(\text{resp., } \text{vol}_k C^-_\alpha = |\alpha| \int_{-\infty}^{0} r \text{vol}_{k-2} (F \cap (L_u - ru)) \, dr = -|\alpha| \int_{0}^{+\infty} r \text{vol}_{k-2} (F \cap (L_u + ru)) \, dr \right).
\]

We claim here without proof that
\[
\text{vol}_k Q^+ = \text{vol}_k C^+_\alpha + o(\alpha) \quad \text{and} \quad \text{vol}_k Q^- = \text{vol}_k C^-_\alpha + o(\alpha).
\]

By this and identity (5.3), we obtain
\[
\text{vol}_k P' - \text{vol}_k P = \alpha (\text{vol}_k C^+_\alpha - \text{vol}_k C^-_\alpha) + o(\alpha) = \alpha \int_{\mathbb{R}} r \text{vol}_{k-2} (F \cap (L_u + ru)) \, dr + o(\alpha).
\]

By this and by (5.1), we get
\[
\text{vol}_k P' - \text{vol}_k P = \alpha \langle c - c_w, u \rangle \text{vol}_{k-1} F + o(\alpha).
\]

Since \(w\) and \(u\) are orthogonal, we have
\[
\alpha = \arctan \frac{|u|}{|w|} t = \frac{|u|}{|w|} t + o(t).
\]

Finally, we obtain
\[
(5.4) \quad \text{vol}_k P' - \text{vol}_k P = \frac{\text{vol}_{k-1} F}{|w|} \langle c - c_w, u \rangle t + o(t).
\]

5.2. Local properties of sections of the cube. Let \(S\) be a frame in \(\mathbb{R}^k\). For every \(v \in S\), we denote the set \(H_v \cap \square(S)\) by \(F_v\). We say that \(v \in S\) corresponds to a facet \(F\) of \(\square(S)\) if either \(F = F_v\) or \(F = -F_v\). Clearly, if some vectors of \(S\) correspond to the same facet of \(\square(S)\), then they are equal up to a sign. For a given frame \(S\) in \(\mathbb{R}^k\) and \(u \in \mathbb{R}^k\), a facet \(F\) of \(\square(S)\) and a vector \(u \in \mathbb{R}^k\), we define an \(F\)-substitution in the direction \(u\) as follows:

- each vector \(v\) of \(S\) such that \(F \subset H_v\) is substituted by \(v + u\);
- each vector \(v\) of \(S\) such that \(-F \subset H_v\) is substituted by \(v - u\);
- all other vectors of \(S\) remain the same.

In order to prove Theorem 1.1, we will use \(F\)-substitutions.

At first, we simplify the structure of a local maximizer.

Lemma 5.1. Let \(S\) be a local maximizer of (2.3) and \(v \in S\). Then \(F_v\) is a facet of \(\square(S)\).

Proof. Let \(K\) be a convex body in \(\mathbb{R}^k\), then its polar body is defined by
\[
\{ y \in \mathbb{R}^k : \langle y, x \rangle \leq 1 \text{ for all } x \in K \}.
\]

Since \(\square(S)\) is the intersection of half-spaces of the form \(\{ \langle w, x \rangle \leq 1 \}\) with \(w \in \pm S\), we have that \(\text{co}\{\pm S\}\) is polar to \(\square(S)\). By the duality argument, it suffices to prove that \(v \in S\) is a vertex of the polytope \(\text{co}\{\pm S\}\). Assume that \(v\) is not a vertex of \(\text{co}\{\pm S\}\).
Clearly, \( v \in \text{co}\{\pm(S \setminus v)\} \) and \( v \) is not a vertex of the polytope \( \text{co}\{\pm(S \setminus v)\} \). Therefore we have that \( \text{span}\{S \setminus v\} = \text{span} S = \mathbb{R}^k \). That is, \( S \setminus v \) is a frame in \( \mathbb{R}^k \). Since \( B_{S\setminus v} \) is a nondegenerate linear transformation, \( B_{S\setminus v}v \) is not a vertex of the polytope \( \text{co}\{\pm B_{S\setminus v}(S \setminus v)\} \). By this and by the triangle inequality, there is a vertex \( u \) of \( \text{co}\{\pm S\} \) such that \( u \in S \) and \( |B_{S\setminus v}v| < |B_{S\setminus v}u| \).

Denote by \( \tilde{S} \) the \( n \)-tupled obtained from \( S \) by substitution \( v \rightarrow v + t(u - v) \), where \( t \in (0, 1] \). Since \( A_{\tilde{S}} \geq A_{S\setminus v} > 0 \), \( \tilde{S} \) is a frame in \( \mathbb{R}^k \). By the choice of \( u \) and identity (2.2), we have \( \square(\tilde{S}) = \square(S) \). Hence \( \text{vol}_k \square(\tilde{S}) = \text{vol}_k \square(S) \). Lemma 3.1 implies that \( \det A_{\tilde{S}} \leq 1 \).

On the other hand, by Lemma 3.3, we have
\[
\det A_{\tilde{S}} = (1 + |B_{S\setminus v}(v + t(u - v))|^2)(1 - |v|^2).
\]
Inequality \( |B_{S\setminus v}v| < |B_{S\setminus v}u| \) implies that \( |B_{S\setminus v}v| < |B_{S\setminus v}(v + t(u - v))| \). By this and by Lemma 3.4, we conclude that \( \det A_{\tilde{S}} > 1 \). This is a contradiction. Thus, \( v \) is a vertex of \( \text{co}\{\pm S\} \). The lemma is proven. \( \square \)

As an immediate corollary of Lemma 5.1 and by the standard properties of polytopes, we have the following statement.

**Corollary 5.1.** Let \( S \) be a local maximizer of (2.3) and \( v \in S \). Let \( \tilde{S} \) be the \( n \)-tupled obtained from \( S \) by \( F_v \)-substitution in the direction \( tu \) with \( t \in \mathbb{R} \) and \( u \in \mathbb{R}^k \). Then, for a sufficiently small \( |t| \), \( \tilde{S} \) is a frame, the vertex \( v + at \) corresponds to a facet of \( \square(\tilde{S}) \) and the polytopes \( \square(S) \) and \( \square(\tilde{S}) \) have the same combinatorial structure. Moreover, \( \text{vol}_k \square(\tilde{S}) \) is a smooth function of \( t \) at \( t = 0 \).

In the following two lemmas, we will perturb a local maximizer by making \( F \)-substitutions. Geometrically speaking, making an \( F \)-substitution in the direction \( tu \) with \( u \in \mathbb{R}^k \) and \( t \in \mathbb{R} \), we move the opposite facets \( F \) and \( -F \) of a local maximizer in a symmetric way. Thus, for a sufficiently small \( t \), perturbations of the facets \( F \) and \( -F \) are independent.

**Lemma 5.2.** Let \( S \) be a local maximizer of (2.3). Let \( v \in S \) and \( d \) be the number of the vectors of \( S \) that correspond to \( F_v \). Then
\[
\frac{2}{|v|} \text{vol}_{k-1} F_v = d|v|^2 \text{vol}_k \square(S).
\]

**Proof.** Denote by \( \tilde{S} \) the \( n \)-tupled obtained from \( S \) by \( F_v \)-substitution in the direction \( tv \) with \( t \in \mathbb{R} \). Thus, we apply Transformation 1 to the facets \( \pm F_v \) of \( \square(S) \). By Lemma 3.1, we have
\[
\frac{\text{vol}_k \square(\tilde{S})}{\text{vol}_k \square(S)} \leq \frac{1}{\sqrt{\det A_{\tilde{S}}}}.
\]

By Lemma 3.2 and Corollary 5.1, both sides of this inequality are smooth as functions of \( t \) in a sufficiently small neighborhood of \( t_0 = 0 \). Consider the Taylor expansions of both sides of inequality (5.6) as functions of \( t \) about \( t_0 = 0 \).

By Lemma 3.3, \( \det A_{\tilde{S}} = 1 + d(2t + t^2)|v|^2 \). Hence
\[
\frac{1}{\sqrt{\det A_{\tilde{S}}}} = 1 - td|v|^2 + o(t).
\]

Geometrically speaking, we shift the half-space \( H^+_v \) (resp., \( H^-_v \)) by
\[
h = \frac{1}{|(1 + t)v|} - \frac{1}{|v|} = \frac{t}{|v|} + o(t)
\]
in the directions of its outer normal. By this and by (5.2), we obtain
\[
\text{vol}_k \square(\tilde{S}) - \text{vol}_k \square(S) = -\frac{2t}{|v|} \text{vol}_{k-1} F_v + o(t).
\]
Using identities (5.8) and (5.7) in (5.6), we get

\[ 1 - \frac{2t \text{vol}_{k-1} F_v}{|v| \text{vol}_k \square(S)} \leq 1 - d|v|^2 t + o(t). \]

Since \( \tilde{S} = S \) for \( t = 0 \) and the previous inequality holds for all \( t \in (-\varepsilon, \varepsilon) \) for a sufficiently small \( \varepsilon \), the coefficients of \( t \) in both sides of the previous inequality coincide. That is,

\[ \frac{2 \text{vol}_{k-1} F_v}{|v| \text{vol}_k \square(S)} = d|v|^2. \]

This completes the proof. \( \square \)

**Lemma 5.3.** Let \( S \) be a local maximizer of (2.3) and \( v \in S \). Then the line \( \text{span}\{v\} \) intersects the hyperplane \( H_v \) in the centroid of the facet \( F_v \).

**Proof.** Denote the centroid of \( F_v \) by \( c \) and let \( c_v = \text{span}\{v\} \cap H_v \). Fix a unit vector \( u \) orthogonal to \( v \). Denote by \( \tilde{S} \) the \( n \)-tuple obtained from \( S \) by \( F_v \)-substitution in the direction \( tu \) with \( t \in \mathbb{R} \). Thus, we apply Transformation 2 to the facets \( \pm F_v \) of \( \square(S) \).

By Lemma 3.1, we have

\[ \frac{\text{vol}_k \square(\tilde{S})}{\text{vol}_k \square(S)} \leq \frac{1}{\sqrt{\det A_S}}. \tag{5.9} \]

By Lemma 3.2 and Corollary 5.1, both sides of this inequality are smooth as functions of \( t \) in a sufficiently small neighborhood of \( t_0 = 0 \). Consider the Taylor expansions of both sides of inequality (5.6) as functions of \( t \) about \( t_0 = 0 \).

By (5.4), we obtain

\[ \text{vol}_k \square(\tilde{S}) - \text{vol}_k \square(S) = C \langle c - c_v, u \rangle t + o(t), \]

where \( C = 2 \text{vol}_{k-1} F_v/|v| > 0 \). By Lemma 3.2, \( \sqrt{\det A_S} = 1 + o(t) \). Therefore, inequality (5.9) takes the following form

\[ 1 + \frac{C}{\text{vol}_k \square(S)} \langle c - c_v, u \rangle t + o(t) \leq 1 + o(t). \]

Since \( \tilde{S} = S \) for \( t = 0 \) and the previous inequality holds for all \( t \in (-\varepsilon, \varepsilon) \) for a sufficiently small \( \varepsilon \), the coefficients of \( t \) in both sides of the previous inequality coincide. That is, we conclude

\[ \langle c - c_v, u \rangle = 0. \]

Since \( c, c_v \in H_v \) and the last identity holds for all unit vectors parallel to \( H_v \), it follows that \( c = c_v \). The lemma is proven. \( \square \)

As a simple consequence of Lemma 5.3, we obtain the following result for the planar case.

**Theorem 5.1.** Let \( S \in \Omega(n, 2) \) be a local maximizer of (2.3) for \( k = 2 \). Then, the polygon \( \square(S) \) is cyclic. That is, there is a circle that passes through all the vertices of \( \square(S) \).

**Proof.** Denote the origin by \( o \). Let \( ab \) be an edge of \( \square(S) \) and \( oh \) be the altitude of the triangle \( abo \). By Lemma 5.3, \( h \) is the midpoint of \( ab \). Hence, the triangle \( abo \) is isosceles and \( ao = bo \). It follows that \( \square(S) \) is cyclic. \( \square \)

We are ready to give a proof of Theorem 1.1.

**Proof of Theorem 1.1.** Recall that any identification of \( H \) with \( \mathbb{R}^k \) identifies the projections of the standard basis \( \{v_1, \ldots, v_n\} \) with a tight frame, denoted by \( S \), that is a local maximizer of (2.3).

Next, assertion 1 is trivial and holds for any section of the cube. Assertion 2 and assertion 3 are equivalent to Lemma 5.1 and Lemma 5.3, respectively.
By Lemma 5.1, all vectors \( v \in S \) such that \( \text{span } v \) intersects \( F \) correspond to \( F \) and have the same length that we denote by \( |v| \). Then by Lemma 5.3, the span of each of these vectors intersects \( F \) in its centroid. Since the length of the altitude of the pyramid \( P_F \) is \( 1/|v| \), we have

\[
\text{vol}_k P_F = \frac{1}{k} \frac{\text{vol}_{k-1} F}{|v|}.
\]

Hence assertion 4 follows from Lemma 5.2. \( \square \)

6. Proof of Theorem 1.3

We use the setting of tight frames developed in the previous sections to prove the theorem. More precisely, we use the obtained necessary conditions for a tight frame in \( \mathbb{R}^2 \) that maximizes (2.3) for \( n > k = 2 \) to prove that the section of the cube generated by the tight frame is a rectangle of area

\[
4 \sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor}
\]

with the sides of lengths \( 2 \sqrt{\left\lfloor \frac{n}{2} \right\rfloor} \) and \( 2 \sqrt{\left\lceil \frac{n}{2} \right\rceil} \).

First, let us introduce the notation. Let \( S = \{ v_1, \ldots, v_n \} \in \Omega(n, 2) \) be a global maximizer of (2.3) for \( k = 2 \) and \( n > 2 \). Clearly, \( \Box(S) \) is a centrally symmetric polygon in \( \mathbb{R}^2 \). The number of edges of \( \Box(S) \) is denoted by \( 2f \). Clearly, \( f \leq n \). By Theorem 5.1, the polygon \( \Box(S) \) is cyclic; and we denote its circumradius by \( R \). Let \( F_1, \ldots, F_{2f} \) be the edges of \( \Box(S) \) enumerated in clockwise direction (that is, edges \( F_i \) and \( F_{i+f} \) are opposite to each other, \( i \in [f] \)). We reenumerate the vectors of \( S \) in such a way that the vector \( v_i \) corresponds to the edge \( F_i \) for every \( i \in [f] \). The central angle subtended by the edge \( F_i \) is denoted by \( 2\varphi_i \), \( i \in [f] \).

Clearly, we have the following identities (see Figure 4)

\[
\varphi_1 + \cdots + \varphi_f = \frac{\pi}{2}, \quad (6.1)
\]

\[
R \cos \varphi_i = \frac{1}{|v_i|}, \quad \forall i \in [f], \quad (6.2)
\]

and

\[
\text{Area } \Box(S) = R^2 \sum_{i=1}^{f} \sin 2\varphi_i, \quad (6.3)
\]

Also, we note here that

\[
\text{Area } \Box(S) \geq 4C_\Box(n, 2) = 4 \sqrt{\left\lceil \frac{n^2}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor}. \quad (6.4)
\]

There are several steps in the proof. We explain the main steps briefly. In fact, we want to show that the number of edges of a local maximizer is \( 2f = 4 \). Using the discrete isoperimetric inequality (see below), we obtain an upper bound on \( \text{Area } \Box(S) \) in terms of \( f \). This upper bound yields the desired result for \( n \geq 8 \) (the bound is less than conjectured volume \( 4C_\Box(n, 2) \) for \( n \geq 8 \)). Finally, we deal with the lower-dimension cases using the necessary conditions obtained earlier.

![Figure 4. Notation for \( \Box(S) \)](image-url)
The discrete isoperimetric inequality for cyclic polygons says that among all cyclic \( f \)-gons with fixed circumradius there is a unique maximal area polygon – the regular \( f \)-gon. We will use a slightly more general form. Namely, fixing one or several central angles of a cyclic polygon, its area is maximized when all other central angles are equal. For example, this follows from concavity of the sine function on \([0, \pi]\). In our notation fixing the central angle \( \phi_i \) and its vertically opposite, we have

\[
R^2 f \sin \frac{\pi}{f} \geq R^2 \left( \sin 2\phi_i + (f - 1) \sin \frac{\pi - 2\phi_i}{f - 1} \right) \geq 4C_{\square}(n, 2). \tag{6.5}
\]

6.1. Step 1.

Claim 1. The area of \( \square(S) \) such that \( f = 2 \) is at most \( 4C_{\square}(n, 2) \). The bound is attained when \( \square(S) \) is a rectangle with the sides of lengths \( 2\sqrt{\lceil n/2 \rceil} \) and \( 2\sqrt{\lfloor n/2 \rfloor} \).

Proof. Since \( f = 2 \), the polygon \( \square(S) \) is an affine square. Hence the claim is an immediate consequence of Lemma 2.2. □

Thus, it suffices to prove that \( f = 2 \) for any \( n > 2 \).

6.2. Step 2.

Claim 2. For any \( n > 2 \) the following inequality holds

\[
R^2 \leq \frac{n + 1}{2} \frac{1}{\cos^2 \frac{\pi}{2f}}. \tag{6.6}
\]

Proof. Let \( \phi_1 \) be the smallest central angle. By identity (6.1), we have \( \cos \phi_1 \geq \cos \frac{\pi}{2f} \). Combining this with the leftmost inequality in (4.2) and identity (6.2), we obtain

\[
R^2 = \frac{1}{|v_1|^2 \cos^2 \phi_1} \leq \frac{n + 1}{2} \frac{1}{\cos^2 \frac{\pi}{2f}}. \tag*{□}
\]

Claim 3. For any \( n > 2 \) the following inequality holds

\[
f \tan \frac{\pi}{2f} \geq \frac{4}{n + 1} \sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}. \tag{6.7}
\]

Proof. By the discrete isoperimetric inequality (6.5), we have

\[
R^2 f \sin \frac{\pi}{f} \geq 4C_{\square}(n, 2).
\]

Combining this with inequalities (6.6) and (6.4), we obtain

\[
f \frac{\sin \frac{\pi}{f}}{\cos^2 \frac{\pi}{2f}} \geq \frac{2}{n + 1} C_{\square}(n, 2) = \frac{8}{n + 1} \sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}.
\]

The claim follows. □

Claim 4. The following bounds on \( f \) hold:

1. \( f = 2 \) if \( n \geq 8 \);
2. \( f \leq 3 \) if \( n = 7 \);
3. \( f \leq 4 \) if \( n = 5 \).

Proof. We consider the functions in the left- and right-hand sides of (6.7) as functions of \( f \) and \( n \) respectively. Set \( g(f) = f \tan \frac{\pi}{2f} \) and \( h(n) = \frac{4}{n + 1} \sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor} \). Thus, inequality (6.7) takes the form \( g(f) \geq h(n) \). By routine analysis, we have that \( g \) is strictly decreasing and \( h \) is increasing on \( \{n \in \mathbb{N} : n \geq 2\} \). The first two assertions of the claim follows from this and the identity \( g(3) = h(7) \). Inequality \( f \leq 4 \) for \( n = 5 \) follows from the direct computations of \( g(5), g(4) \) and \( h(5) \) (see Figure 5). □
Theorem 1.3 is proven for \( n \geq 8 \). We proceed with the lower-dimensional cases.

### 6.3. Step 3.

**Claim 5.** For \( n = 7 \), we have that \( f = 2 \).

**Proof.** We showed that \( f \leq 3 \) for \( n = 7 \). Assume that \( f = 3 \). We see that inequality (6.7) with such values turns into an identity. It follows that \( \varphi_1 = \varphi_2 = \varphi_3 = \pi/6 \) and \( \square(S) \) is a regular hexagon. Hence the vectors of \( S \) are of the same length. Since \( \sum_{v \in S} |v|^2 = \text{tr} I_2 = 2 \), we conclude that \( |v|^2 = 2/7 \) and \( R^2 = \frac{1}{2|\cos^2 \pi/6|} = 14/3 \). However, the volume of such a hexagon is strictly less than \( 4C\square(7, 2) \). We conclude that \( f = 2 \) for \( n = 7 \). \( \square \)

**Claim 6.** For \( n \in \{3, 4, 6\} \), we have that \( f = 2 \).

**Proof.** For \( n = 3 \), the statement is a simple exercise (see [Zon06]). Conjecture 1 was confirmed in [Iva19] for any \( n > k \geq 1 \) such that \( k|n \), in particular, for \( k = 2 \) and \( n \in \{4, 6\} \). \( \square \)

**Remark 2.** The inequality on the area for \( n \in \{4, 6\} \) and \( k = 2 \) is a special case of the leftmost inequality in (1.1) originally proved by K. Ball [Bal89]. In [Iva19], the equality cases in this Ball’s inequality are described.

### 6.4. Step 4.

**Claim 7.** Let \( n = 5 \) and either \( f = 3 \) or \( f = 4 \). Then \( \varphi_i \leq \pi/4 \) for every \( i \in [f] \).

**Proof.** Assume that there is \( i \in [f] \) such that \( \varphi_i > \pi/4 \). Thus, \( \cos \varphi_i < 1/\sqrt{2} \). Using identity (6.2) for \( i \) and for any \( j \in [f] \), we have that

\[
\frac{\cos \varphi_j}{\cos \varphi_i} = \frac{|v_i|}{|v_j|} \leq \sqrt{\frac{n + 1}{n - 1}} = \sqrt{\frac{3}{2}},
\]

where the inequality follows from Lemma 4.1. Hence \( \cos \varphi_j \leq \sqrt{3/2} \cos \varphi_i < \sqrt{3/2} \) and, therefore, \( \varphi_j > \pi/6 \). This contradicts identity (6.1):

\[
\frac{\pi}{2} = \varphi_1 + \cdots + \varphi_f > \frac{\pi}{4} + (f - 1) \frac{\pi}{6} > \frac{\pi}{2}.
\]

Thus, \( \varphi_i \leq \pi/4 \) for every \( i \in [f] \). \( \square \)

**Claim 8.** Let \( n = 5 \) and either \( f = 3 \) or \( f = 4 \). Then \( \varphi_i \geq \pi/10 \) for every \( i \in [f] \).

**Proof.** Fix \( i \in [f] \). By identity (6.2) and Lemma 4.1, we have \( R^2 \leq \frac{3}{2 \cos^2 \varphi_i} \). By inequality (6.5), we get

\[
\frac{3}{\cos^2 \varphi_i} \left( \sin 2\varphi_i + (f - 1) \sin \frac{\pi - 2\varphi_i}{f - 1} \right) \geq 4C\square(5, 2) = 4\sqrt{6}.
\]

The function of \( \varphi_i \) in the left-hand side of this inequality is increasing on \( [0, \pi/2] \). Since the inequality does not hold for \( \varphi_i = \pi/10 \), we conclude that \( \varphi_i \) is necessarily at least \( \pi/10 \). \( \square \)

**Claim 9.** For \( n = 5 \), we have that \( f = 2 \).

**Proof.** Assume that either \( f = 3 \) or \( f = 4 \). Denote by \( d_i \) the number of vectors in \( S \) that correspond to \( F_i \). We want to rewrite the inequality of Lemma 5.2 using the circumradius and the center angle. Since the length of edge \( F_i \) is \( 2R \sin \varphi_i \) and by identity (6.2), identity (5.5) takes the form

\[
2R^2 \sin 2\varphi_i = \frac{d_i}{R^2 \cos^2 \varphi_i} \text{Area} \square(S).
\]
Set $q(\varphi) = \cos^2 \varphi \sin 2\varphi$. Then for all $i, j \in [f]$, we have

$$\frac{d_i}{d_j} = \frac{q(\varphi_i)}{q(\varphi_j)}.$$ 

Clearly, there are $i, j \in [f]$ such that $d_i = 2$ and $d_j = 1$. Therefore, $q(\varphi_i)/q(\varphi_j) = 2$. By Claim 7 and Claim 8, we have that $\varphi_i, \varphi_j \in [\pi/10, \pi/4]$. By simple computations, the maximum of $q$ on the segment $[\pi/10, \pi/4]$ is $q(\pi/6) = 3\sqrt{3}/8$ and the minimum is $q(\pi/4) = 1/2$. Hence

$$\max_{\varphi, \psi \in [\pi/10, \pi/4]} q(\varphi)/q(\psi) = 3\sqrt{3}/4 < 2 \text{ and we come to a contradiction. Thus, } f = 2. \quad \square$$

Thus, we have proved that for a maximizer of (2.3), then $f = 2$. By Claim 1, the conjectured upper bound for the area of a planar section holds and also is tight. The proof of Theorem 1.3 is complete.

**Remark 3.** We used the Ball inequality (1.1) to prove Theorem 1.3 for $n \in \{3, 4, 6\}$. However, it can be done by using our approach without the Ball inequality. The proof is technical. Since it is not of great interest, we do not give a proof.

**APPENDIX A.**

**Sketch of the proof of Lemma 2.2.** Let $H$ be a $k$-dimensional subspace of $\mathbb{R}^n$ such that $C(\overline{n}, k)$ is attained. Denote $P = \square^n \cap H$. Since $P$ is an affine $k$-dimensional cube, there are vectors $\{a_1, \ldots, a_k\}$ such that $P = \bigcap_{i \in [k]} (H_{a_i}^+ \cap H_{a_i}^-)$.

Let $\{v_1, \ldots, v_n\}$ be the projection of the vectors of the standard basis onto $H$. By the same arguments as in Lemma 5.1, the hyperplane $H_v$ meets the polytope $P$ in a facet of $P$ for every $v \in [n]$. Thus, $v_i$ coincides with $\pm a_j$ for a proper sign and $j \in [k]$. Or, equivalently, we partition $[n]$ into $k$ sets and $H$ is the solution of a proper system of linear equations constructed as in (2) and (3), except we have not proved that (1) holds yet. Let us prove this assertion. Let $d_i$ vectors of the standard basis of $\mathbb{R}^n$ project onto a pair $\pm a_i$. Therefore, a $k$-tuple of vectors $\{d_i a_i\}_{i \in [k]}$ is a tight frame. Identifying $H$ with $\mathbb{R}^k$ and by the assertion (4) of Lemma 2.1, we conclude that $a_i$ and $a_j$ are orthogonal whenever $i \neq j$. Therefore, $|a_i|^2 = \frac{1}{d_i}$ and

$$\text{vol}_k \square^n \cap H = 2^k \sqrt{d_1 \cdots d_k}.$$ (A.1)

Suppose $d_i \geq d_j + 2$ for some $i, j \in [k]$. Then $d_i \cdot d_j \leq (d_i - 1)(d_j + 1)$. By this and by (A.1), we showed that (1) holds.

It is easy to see that there are exactly $n - k\lfloor n/k \rfloor$ of $d_i$’s equal $\lfloor n/k \rfloor$ and all others $k - (n - k\lfloor n/k \rfloor)$ are equal to $\lfloor n/k \rfloor$. That is, $C(\overline{n}, k)$ is given by (1.2). This completes the proof.

**Sketch of the proof of Lemma 3.2.** Recall that the cross product of $k - 1$ vectors $\{x_1, \ldots, x_{k-1}\}$ of $\mathbb{R}^k$ is the vector $x$ defined by

$$\langle x, y \rangle = \det(x_1, \ldots, x_{k-1}, y) \quad \text{for all } y \in \mathbb{R}^k.$$

For an ordered $(k-1)$-tuple $L = \{i_1, \ldots, i_{k-1}\} \in [n]^{k-1}$ and a frame $S = \{v_1, \ldots, v_n\}$, we use $[v_L]$ to denote the cross product of $v_{i_1}, \ldots, v_{i_{k-1}}$.

We claim the following property of the tight frames.

Let $S = \{v_1, \ldots, v_n\}$ be a tight frame in $\mathbb{R}^k$. Then the set of vectors $\{[v_L]\}_{L \in [n]^{k-1}}$ is a tight frame in $\mathbb{R}^k$.

We use $\Lambda^k(\mathbb{R}^n)$ to denote the space of exterior $k$-forms on $\mathbb{R}^n$. By assertion (2) of Lemma 2.1, there exists an orthonormal basis $\{f_1\}^n_{i=1}$ of $\mathbb{R}^n$ such that $v_i$ is the orthogonal projection of $f_i$ onto $\mathbb{R}^k$, for any $i \in [n]$. Then the $(k-1)$-form $v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}$ is the orthogonal projections of the $(k-1)$-form $f_{i_1} \wedge \cdots \wedge f_{i_{k-1}}$ onto $\Lambda^{k-1}(\mathbb{R}^k) \subset \Lambda^{k-1}(\mathbb{R}^n)$, for any ordered $(k-1)$-tuple $L = \{i_1, \ldots, i_{k-1}\} \in [n]^{k-1}$. By Lemma 2.1 and since the $(k-1)$-forms
\{f_i \wedge \cdots \wedge f_{i-1}\}_{\{i_1,\ldots,i_{k-1}\} \in \binom{[n]}{k-1}} \text{ form an orthonormal basis of } \Lambda^{k-1}(\mathbb{R}^n), \text{ we have that the set of}
\{v_1 \wedge \cdots \wedge v_{i-1}\}_{\{i_1,\ldots,i_{k-1}\} \in \binom{[n]}{k-1}} \text{ is a tight frame in } \Lambda^{k-1}(\mathbb{R}^k). \text{ Finally, the Hodge star operator maps } v_1 \wedge \cdots \wedge v_{i-1} \text{ to the cross product of vectors } v_1,\ldots,v_{i-1}. \text{ Since the Hodge star is an isometry, the set of cross products } \{[v_L]\}_{L \in \binom{[n]}{k-1}} \text{ is a tight frame. The claim is proven.}

By linearity of the determinant, it is enough to prove the lemma for \( \tilde{S} = \{v_1 + tx, v_2, \ldots, v_n\} \). Denote \( v_i' = v_i + tx \) and \( v_i'' = v_i \), for \( 2 \leq i \leq n \).

By the Cauchy–Binet formula, we have
\[
\det A_{\tilde{S}} = \det \left( \sum_{i=1}^{n} v_i' \otimes v_i' \right) = \sum_{Q \in \binom{[n]}{k}} \det \left( \sum_{i \in Q} v_i' \otimes v_i' \right).
\]

By the properties of the Gram matrix, we have
\[
\det \left( \sum_{i=1}^{k} v_i' \otimes v_i' \right) = (\det (v_1', \ldots, v_n'))^2.
\]

By this, by the definition of cross product and by identity (A.2), we obtain
\[
\det A_{\tilde{S}} = 1 + 2t \sum_{Q \in \binom{[n]}{k-1}, i \in Q} \langle v_1, [v_{Q \setminus 1}] \rangle \langle [v_{Q \setminus 1}], x \rangle + o(t).
\]

Since \( \langle v_1, [v_J] \rangle = 0 \) for any \( J \in \binom{[n]}{k-1} \) such that \( 1 \in J \), we have that the linear term of the Taylor expansion of \( \det A_{\tilde{S}} \) equals
\[
2t \sum_{Q \in \binom{[n]}{k-1}, i \in Q} \langle v_1, [v_{Q \setminus 1}] \rangle \langle [v_{Q \setminus 1}], x \rangle = 2t \sum_{L \in \binom{[n]}{k-1}} \langle v_1, [v_L] \rangle \langle [v_L], x \rangle.
\]

Since \( \{[v_L]\}_{L \in \binom{[n]}{k-1}} \) is a tight frame in \( \mathbb{R}^k \), we have that
\[
\sum_{L \in \binom{[n]}{k-1}} \langle v_1, [v_L] \rangle \langle [v_L], x \rangle = \langle v_1, x \rangle.
\]

Therefore,
\[
\sqrt{\det A_{\tilde{S}}} = \sqrt{\det A_S + 2t \langle v_1, x \rangle + o(t)} = 1 + t \langle v_1, x \rangle + o(t).
\]

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References

[AHK19] Arseniy Akopyan, Alfredo Hubard, and Roman Karasev, Lower and upper bounds for the waists of different spaces, Topological Methods in Nonlinear Analysis 53 (2019), no. 2, 457–490.

[Bal89] Keith Ball, Volumes of sections of cubes and related problems, Geometric aspects of functional analysis (1989), 251–260.

[Gru07] Peter M. Gruber, Convex and discrete geometry, Springer Berlin Heidelberg, 2007.

[Iva18a] Grigory Ivanov, On the volume of projections of cross-polytope, arXiv preprint arXiv:1808.09165 (2018).

[Iva18b] Grigory Ivanov, Tight frames and related geometric problems, arXiv e-prints (2018).

[Iva19] Grigory Ivanov, On the volume of the John–Löwner ellipsoid, Discrete & Computational Geometry (2019).

[KK12] Hermann König and Alexander Koldobsky, On the maximal measure of sections of the n-cube, Geometric Analysis, Mathematical Relativity, and Nonlinear Partial Differential Equations, Contemp. Math 599 (2012), 123–155.
[MP88] Mathieu Meyer and Alain Pajor, *Sections of the unit ball of* $\ell_p^n$, Journal of functional analysis 80 (1988), no. 1, 109–123.

[Vaa79] Jeffrey Vaaler, *A geometric inequality with applications to linear forms*, Pacific Journal of Mathematics 83 (1979), no. 2, 543–553.

[Zon06] Chuanming Zong, *The cube: A window to convex and discrete geometry*, Cambridge Tracts in Mathematics, vol. 168, Cambridge University Press, 2006.

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