FORBIDDEN MINORS: FINDING THE FINITE FEW

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Abstract. The Graph Minor Theorem of Robertson and Seymour asserts that any graph property, whatsoever, is determined by an associated finite lists of graphs. We view this as an impressive generalization of Kuratowski’s theorem, which characterizes planarity in terms of two forbidden subgraphs, $K_5$ and $K_{3,3}$. Robertson and Seymour’s result empowers students to devise their own Kuratowski type theorems; we propose several undergraduate research projects with that goal. As an explicit example, we determine the seven forbidden minors for a property we call strongly almost–planar (SAP). A graph is SAP if, for any edge $e$, both deletion and contraction of $e$ result in planar graphs.

1. Introduction

Kuratowski’s Theorem [16], a highlight of undergraduate graph theory, classifies a graph as planar in terms of two forbidden subgraphs, $K_5$ and $K_{3,3}$. (We defer a precise statement to the next paragraph.) We will write $\text{Forb}(\mathcal{P}) = \{K_5, K_{3,3}\}$ where $\mathcal{P}$ denotes the planarity property. We can think of the Graph Minor Theorem of Robertson and Seymour [22] as a powerful generalization of Kuratowski’s Theorem. In particular, their theorem, which has been called the “deepest” and “most important” result in all of graph theory [15], implies that each graph property $\mathcal{P}$, whatsoever, is characterized by a corresponding finite list of graphs. This scaffolding allows students to devise their own Kuratowski type theorems. As an example, we will determine the seven forbidden minors for a property that we call strongly almost–planar.

![Figure 1. Edge contraction.](image)

To proceed, we must define graph minor, which is a generalization of subgraph. We will assume familiarity with the basic terminology of graph theory; West’s [26] book is a good reference at the undergraduate level. While Diestel [8] is at a higher level, it includes an accessible approach to graph minor theory. For us, graphs are simple (no loops or double edges) and not directed. We can define the notion of a minor using graph operations. We obtain subgraphs through the operations of edge and vertex deletion. For minors, we allow an additional operation: edge contraction. As in Figure 1 when we contract edge $ab$ in $G$, we replace the pair...
of vertices with a single vertex \( \bar{a} \) that is adjacent to each neighbor of \( a \) or \( b \). The resulting graph \( G' \) has one less vertex and at least one fewer edge than \( G \). (If \( a \) and \( b \) share neighbors, even more edges are lost.) A minor of graph \( G \) is any graph obtained by contracting (zero or more) edges in a subgraph of \( G \). Recall that \( K_5 \) is the complete graph on five vertices and \( K_{3,3} \) the complete bipartite graph with two parts of three vertices each. We can now state Kuratowski’s Theorem, using the formulation in terms of minors due to Wagner.

**Theorem 1.1** (Kuratowski-Wagner [16, 25]). A graph \( G \) is planar if and only if it has no \( K_5 \) nor \( K_{3,3} \) minor.

Robertson and Seymour’s theorem can be stated as follows.

**Theorem 1.2** (The Graph Minor Theorem [22]). In any infinite sequence of graphs \( G_1, G_2, G_3, \ldots \) there are indices \( i \neq j \) such that \( G_i \) is a minor of \( G_j \).

This yields two important corollaries, which we now describe.

Planarity is an example of a property that is minor closed: If \( H \) is a minor of a planar graph \( G \), then \( H \) must also be planar. If \( P \) is minor closed, the Graph Minor Theorem implies a finite set of forbidden minors.

**Corollary 1.3.** Let \( P \) be a graph property that is minor closed. Then there is a finite set of forbidden minors \( \text{Forb}(P) \) such that \( G \) has \( P \) if and only if it has no minor in \( \text{Forb}(P) \).

In honor of the theorem for planar graphs, we call \( \text{Forb}(P) \) the Kuratowski set for \( P \).

Even if \( P \) is not minor closed, the Graph Minor Theorem determines a finite set. For this, note that \( K_5 \) and \( K_{3,3} \) are minor minimal nonplanar; each is nonplanar with every proper minor planar. More generally, for graph property \( P \), a graph \( G \) is minor minimal for \( P \) or \( \text{MMP} \) if \( G \) has \( P \), but no proper minor does.

**Corollary 1.4.** Let \( P \) be a graph property. The set of \( \text{MMP} \) graphs is finite.

Every \( P \) graph has a \( \text{MMP} \) minor, but the converse may fail if the negation, \( \neg P \), is not minor closed. The \( \text{MMP} \) list constitutes a test for \( \neg P \); a graph with no \( \text{MMP} \) minor definitely does not have property \( P \).

In the next section we summarize the graph properties \( P \) with known \( \text{MMP} \) or Kuratowski set. In Section 3 we illustrate how students might develop their own Kuratowski type theorem through an explicit example: we determine \( \text{Forb}(P) \) for a property that we call strongly almost–planar. In the final section we propose several concrete research projects and provide some suggestions about how to choose graph properties \( P \).

Throughout the paper, we present a list of ‘challenges’ and ‘project ideas.’ Challenges are warm up exercises for talented undergraduates. In most cases, the solution is known and can be found through a web search or in the references at the end of the paper. Project ideas, on the other hand, are generally open problems (as far as we know). Some are quite difficult, but, we hope, all admit openings. Indeed, we see this as a major theme in this area of research. Even if we know \( \text{MMP} \) and Kuratowski sets are finite, a complete enumeration is often elusive. However, it is generally not too hard to capture graphs that belong to the set. These problems, then, promise a steady diet of small successes along the way in our hunt to catch all of the finite few.
2. Properties with known Kuratowski set

In this section we summarize the graph properties with known MM or Kuratowski set. First, an important caveat. While the Graph Minor Theorem ensures these sets are finite, the proof is not at all constructive and gives no guidance about their size. It makes for nice alliteration to talk of the ‘finite few,’ but some finite numbers are really rather large. A particularly dramatic cautionary tale is Y\nY reducibility (we omit the definition) for which Yu \cite{yu27} has found more than 68 billion forbidden minors.

On the other hand, bounding the order (number of vertices) or size (number of edges) of a graph is a minor closed condition. For example, whatever property you may be interested in, appending the condition “of seven or fewer vertices,” ensures that the set of MM graphs is no larger than 1044, the number of order seven graphs. In general, it is quite difficult to predict the size of a Kuratowski set in advance and researchers in this area often do resort to restricting properties by simple graph parameters such as order, size, or connectivity.

We will focus on results that generalize planarity in various ways. However, we briefly mention graphs of bounded tree-width as another important class of examples. Let \( T_k \) denote the graphs of tree-width at most \( k \). For the sake of brevity we omit the definition of tree-width (which can be found in \cite{8}, for example) except for noting that \( T_1 \) is the set of forests, i.e., graphs whose components are trees. For small \( k \), the obstructions are quite simple: \( \text{Forb}(T_1) = \{K_3\} \), \( \text{Forb}(T_2) = \{K_4\} \), and \( \text{Forb}(T_3) \) has four elements, including \( K_5 \) \cite{5, 24}. However, for \( k \geq 4 \) the Kuratowski set for \( T_k \) is unknown.

**Project Idea 1.** Find graphs in \( T_k \) for \( k \geq 4 \). Is \( K_{k+2} \) always forbidden? Is there always a planar graph in \( \text{Forb}(T_k) \)?

We can think of a planar graph as a ‘spherical’ graph since it can be embedded on a sphere with no edges crossing. More generally, the set of graphs that embed on a particular compact surface (orientable or not) is also minor closed. However, to date, (in addition to the sphere) only the Kuratowski set for embeddings on a projective plane is known; there are 35 forbidden minors \cite{2, 12, 21}. The next step would be toroidal graphs, those that embed on a torus; Gagarin, Myrvold, and Chambers remark that there are at least 16 thousand forbidden minors \cite{11}. In the same paper they show only four of them have no \( K_{3,3} \) minor. This is a good example of how a rather large \( \text{Forb}(P) \) can be tamed by adding conditions to the graph property \( P \). While observing that it’s straightforward to determine the toroidal obstructions of lower connectivity, Mohar and Škoda \cite{20} find there are 68 forbidden minors of connectivity two. Explicitly listing the forbidden minors of lower connectivity would be a nice challenge for a strong undergraduate.

**Challenge 1.** Determine the forbidden minors of connectivity less than two for embedding in the torus. Find those for a surface of genus two.

The Kuratowski sets for more complicated surfaces are likely even larger than the several thousand known for the torus.

Outerplanarity is a different way to force smaller Kuratowski sets. A graph is outerplanar (or has property \( OP_l \)) if it can be embedded in the plane with all vertices on a single face. The set of forbidden minors for this property, \( \text{Forb}(OP_l) \) is well known and perhaps best attributed to folklore (although, see \cite{7}).
Challenge 2. Determine $\text{Forb}(\text{OP})$. (Use Kuratowski’s theorem!)

Similarly, one can define outerprojective planar or outertoroidal graphs as graphs that admit embeddings into those surfaces with all vertices on a single face. There are 32 forbidden minors for the outerprojective planar property [3].

Challenge 3. Find forbidden minors for the outerprojective planar property.

Project Idea 2. Find forbidden minors for the outertoroidal property.

Apex vertices also lead to minor closed properties. Let $v \in V(G)$ be a vertex of graph $G$. We will use $G - v$ to denote the subgraph obtained from $G$ by deleting $v$ (and all its edges). Given property $P$, we say that $G$ has $P'$ (or is apex-$P$) if there is a vertex $v$ (called an apex) such that $G - v$ has $P$. If $P$ is minor closed, then $P'$ is as well. For example, Ding and Dziobiak determined the 57 graphs in $\text{Forb}(\text{OP}')$ [4]. In the same paper, they report that there are at least 396 graphs in the Kuratowski set for apex-planar.

Project Idea 3. Find graphs in $\text{Forb}(P')$ when $P$ is toroidal with no $K_{3,3}$, $T_1$, $T_2$, $T_3$, or for some other property with small Forb($P$).

The set of linklessly embeddable graphs are closely related to those that are apex-planar. We say a graph is linklessly embeddable (or has property $L$) if there is an embedding in $\mathbb{R}^3$ that contains no pair of nontrivially linked cycles. (See [4] for a gentle introduction to this idea). An early triumph of graph minor theory was the proof that $\text{Forb}(L)$ has exactly seven graphs [23]. An apex-planar graph is also $L$ and, as part of an undergraduate research project, we showed that $\text{Forb}(L) \subset \text{Forb}(P_l')$ [6]. The related idea of knotlessly embeddable (which, like $L$, is minor closed) has more than 240 forbidden minors [13].

As a final variation on properties related to planarity, rather than vertex deletion (which gives apex properties), let’s think about the other two operations for graph minors, edge deletion and contraction. For graph $G$ and edge $ab \in E(G)$, let $G - ab$ denote the subgraph resulting from deletion and $G/ab$ the minor obtained by edge contraction. Unlike apex properties, in general edge operations do not preserve closure under taking minors. This is why we frame some results below in terms of MMP sets.

As we’ve mentioned, there are at least several hundred graphs in $\text{Forb}(P_l')$. In an undergraduate research project [17] we found that there are also large numbers of graphs that are not simply an edge away from planar. Call a graph $G$ NE (not edge apex) if there is no edge $ab$ with $G - ab$ planar and similarly NC (not contraction apex) if no $G/ab$ is planar. We showed that the are at least 55 MMNE and 82 MMNC graphs. On the other hand, if we switch from the existential to the universal quantifier, we obtain properties that are minor closed with reasonably small Kuratowski sets; in the next challenge, each $\text{Forb}(P)$ has at most ten elements. Say that a graph $G$ is CA (completely apex) if $G - v$ is planar for every vertex $v$, CE (completely edge apex) if every $G - ab$ is planar, and CC (completely contraction apex) if every $G/ab$ is planar.

Challenge 4. For $P = CA$, show that $P$ is minor closed and determine $\text{Forb}(P)$. Repeat for $P = CE$ and $CC$.

Instead of flipping quantifiers, we can think about combing operations with other logical connectives. For example, Gubser [14] calls $G$ almost-planar if, for every $ab \in E(G)$, $G - ab$ or $G/ab$ is planar. There are six forbidden minors for this
property \textsuperscript{10}. In the next section we determine the Kuratowski set for a property that we call \textit{strongly almost–planar} or SAP: for every \(ab \in E(G)\), both \(G - ab\) and \(G/ab\) are planar. Note that every strongly almost–planar graph is almost–planar.

3. \textbf{Strongly almost–planar graphs}

In this section we model how a research project in this area might play out through an explicit example, the strongly almost–planar or SAP property: \(G\) is SAP if, \(\forall ab \in E(G)\), both \(G - ab\) and \(G/ab\) are planar. Note that every strongly almost–planar graph is almost–planar.

Our first task is to determine whether or not this property is minor closed. If not, we would target the list of MMSAP graphs. However, as we will now show, SAP is minor closed, meaning our goal is instead \(\text{Forb}(\text{SAP})\).

\textbf{Lemma 3.1.} \(\text{SAP is minor closed}\)

\textit{Proof.} It is enough to observe that SAP is preserved by the three operations used in constructing minors, vertex or edge deletion and edge contraction.

Suppose \(G\) is SAP and \(v \in V(G)\). Let \(G' = G - v\). We must show that for each \(ab \in E(G')\), both \(G' - ab\) and \(G'/ab\) are planar. Since \(V(G') \subset V(G)\), we can think of \(ab\) as an edge in \(E(G)\). Then it’s easy to identify \(G' - ab\) as a subgraph of the planar graph \(G - ab\), which shows \(G' - ab\) is also planar. Similarly, we’ll know that \(G'/ab\) is planar once we show that it is a subgraph of \(G/ab\). There are a few cases to think about (Is \(a\) or \(b\) or both adjacent to \(v\)?) but it always turns out that \(G'/ab = (G/ab) - v\).

For this property, the argument for edge contraction and deletion is quite simple. For any \(ab \in E(G)\), by assumption \(G - ab\) and \(G/ab\) are planar. Then any minor of these graphs is again planar, including those given by deleting or contracting an edge.

Next, we must generate examples of forbidden minors, meaning graphs that are minor minimal for not SAP. We are looking for graphs \(G\) that are just barely not SAP: although \(G\) is not SAP, every proper minor is. Most likely, there’s only a single edge \(ab\) with \(G - ab\) or \(G/ab\) nonplanar. And that graph is probably minor minimal nonplanar, so one of the Kuratowski graphs \(K_5\) or \(K_{3,3}\). We will use \(K\) to represent a generic Kuratowski graph, that is \(K \in \{K_5, K_{3,3}\}\). In summary, we are looking for graphs of the form \(K\) ‘plus an edge,’ where adding an edge includes the idea of reversing an edge contraction.

We encourage you to take a minute to see what graphs you can discover that have the form \(K\) ‘plus an edge’. Hopefully, you will find five \(G\) for which \(G - ab\) is nonplanar. Perhaps you have even more? Remember we want minor minimal examples, so check if any pair are minors one of the other.

Since edge contraction may be a new idea for the reader, let’s delve a little further into examples where \(G/ab\) is nonplanar. The reverse operation of edge contraction is called a \textit{vertex split} and defined as follows. Replace a vertex \(\bar{a}\) with two vertices \(a\) and \(b\) connected by an edge. Each neighbor of \(\bar{a}\) becomes a neighbor of at least one of \(a\) and \(b\).

Suppose \(G/ab = K_{3,3}\). There are essentially two ways to make a vertex split and recover \(G\). One is to make one of the new vertices, say \(a\), adjacent to no neighbor of \(\bar{a}\) and the other, \(b\), adjacent to all three. Then, in \(G\), \(a\) has degree one (its only neighbor is \(b\)) and \(b\) will have degree four. The other option is to make \(a\) adjacent to one neighbor of \(\bar{a}\) and let \(b\) have the other two. There are other possibilities.
since we may choose to make both $a$ and $b$ adjacent to one of $\bar{a}$’s neighbors; but such graphs will have one of the two we described earlier as a minor.

Figure 2. The graph $K_{3,3} + 2e$.

If $G/ab = K_5$, there are three ways to split the degree four vertex $\bar{a}$. Two are similar to the ones just described for $K_{3,3}$ where we make $a$ adjacent to zero or to one neighbor of $\bar{a}$. The third option, split up the four neighbors of $\bar{a}$ by making $a$ and $b$ adjacent to two each, results in the graph $K_{3,3} + 2e$ shown in Figure 2. However, you should observe that this graph has a proper subgraph among those found by adding an edge to $G - ab = K_{3,3}$.

A little experimentation along these lines should lead to the seven graphs of Figure 3. Note that the six graphs in the top two rows occur in pairs where we perform similar operations on $K_5$ and $K_{3,3}$. We’ll write $K \cup K_2$, $K \dot{\cup} K_2$, and $\bar{K}$ for the pairs at left, center, and right, respectively and $K_{3,3} + e$ for the seventh graph at the bottom of the figure. The five graphs with $G - ab$ nonplanar are $K_{3,3} + e$, and the pairs $K \cup K_2$ and $K \dot{\cup} K_2$. The graphs obtained from $G/ab = K$ where $a$ is made adjacent to a single neighbor of $\bar{a}$ are the $\bar{K}$ pair. When $a$ shares no neighbors with $\bar{a}$, we construct the graphs $K \dot{\cup} K_2$ for a second time. The graph $K_{3,3} + 2e$ (see Figure 2), obtained from $K_5$ by splitting a vertex so that $a$ and $b$ are each adjacent to two neighbors of $\bar{a}$, is not SAP. But, it has another non SAP graph, $\bar{K}_{3,3} + e$, as a proper subgraph and cannot be minor minimal. The others are both non SAP and minor minimal, as we now verify.

Lemma 3.2. The seven graphs of Figure 3 are minor minimal for not SAP.

Proof. As we noticed, the two $\bar{K}$ graphs become Kuratowski graphs after an edge contraction and the rest have an edge deletion that leaves a nonplanar graph. This shows that none of the graphs are SAP.

It remains to show that every proper minor of each graph is SAP. Since SAP is minor closed, it’s enough to verify this for the three basic operations vertex or edge deletion and edge contraction. Actually, since none of our graphs has an isolated vertex, we need only check edge deletion and contraction. For once we have those in hand, then any graph of the form $G - a$ is automatically SAP as it’s a subgraph of one of the $G - ab$ graphs formed by deleting an edge on $a$. (Recall that we just proved that SAP is closed under taking minors.)

Note that planar graphs are SAP, so we can reduce to the case where an edge deletion or contraction gives a nonplanar graph. Let $G$ be a graph in the figure and suppose $G'$ is a nonplanar minor obtained by an edge deletion or contraction.
Observe that, up to isolated vertices, $G'$ is simply a Kuratowski graph $K$. In particular, $E(G') = E(K)$. Since $K$ is minor minimal nonplanar, any further edge deletion or contraction leaves a planar graph, which shows $G'$ is SAP, as required. This completes the argument that the seven graphs in the figure are in $\text{Forb}(\text{SAP})$.

We will argue that there are no other graphs in $\text{Forb}(\text{SAP})$. We begin with graphs that are not connected.
Lemma 3.3. If \( G \in \text{Forb}(\text{SAP}) \) is not connected, then \( G = K \sqcup K_2 \) with \( K \in \{K_5, K_{3,3}\} \).

Proof. Let \( G = G_1 \sqcup G_2 \) in \( \text{Forb}(\text{SAP}) \) be the disjoint union of (nonempty) graphs \( G_1 \) and \( G_2 \). Since planar graphs are SAP, at least one of \( G_1 \) and \( G_2 \), say \( G_1 \), is not planar.

We first observe that \( G_2 \) must have an edge, \( E(G_2) \neq \emptyset \). Otherwise, since \( G \) is not SAP, there is an edge \( ab \in E(G) \) and a nonplanar minor \( G' \), formed by deleting or contracting \( ab \). Since \( G_2 \) has no edges, it is planar and \( ab \in E(G_1) \). If follows that deleting or contracting \( ab \) in \( G_1 \) already gives a nonplanar graph. That is, \( G_1 \) is a proper minor of \( G \) that is not SAP. This contradicts our assumption that \( G \) is minor minimal not SAP.

By Kuratowski’s theorem, \( G_1 \) has a Kuratowski graph minor, \( K \). We claim that \( G_1 = K \). Further, since \( G_2 \) has an edge, we must have \( G_2 = K_2 \). For, if either of these fail, the graph \( K \sqcup K_2 \), which is not SAP by the previous lemma, is a proper minor of \( G \). This contradicts our assumption that \( G \) is minor minimal for not SAP. \( \square \)

We can now complete the argument.

Theorem 3.4. The seven graphs of Figure 3 are precisely the elements of \( \text{Forb}(\text{SAP}) \).

Proof. Using the previous two lemmas, it remains only to verify that if \( G \) is connected and in \( \text{Forb}(\text{SAP}) \), then it is one of the five connected graphs in the figure. Suppose \( G \) is connected and minor minimal not SAP. Since \( G \) is not SAP, there is an \( ab \in E(G) \) such that \( G' \), a minor formed by deleting or contracting \( ab \), is not planar. Then \( G' \) has a Kuratowski graph \( K \) as a minor. In fact \( K \) must appear as a subgraph of \( G' \). If not, one of the two \( K \) graphs is a minor of \( G' \) and, hence also of \( G \). This contradicts our assumption that \( G \) is minor minimal for not SAP.

Suppose that the nonplanar \( G' \) is formed by edge deletion: \( G' = G - ab \). There are several cases depending on the size of \( V(K) \cap \{a, b\} \). If there is no common vertex, then \( G \) has a \( K \sqcup K_2 \) minor. Since we assumed \( G \) is minor minimal for not SAP, \( G = K \sqcup K_2 \), but this contradicts our assumption that \( G \) is connected. Suppose there is one vertex in the intersection. Then \( G \) has a \( K \sqcup K_2 \) minor. By minor minimality, \( G = K \sqcup K_2 \) and appears in Figure 3 as required. Finally, if \( \{a, b\} \subset V(K) \), then \( K \) must be \( K_{3,3} \) and, by minor minimality, \( G = K_{3,3} + e \) is one of the graphs in the figure.

If instead \( G' = G/ab \), let \( \bar{a} \) denote the vertex that results from identifying \( a \) and \( b \). If \( \bar{a} \in V(K) \), there are two possibilities. It may be that \( G \) has one of the \( K \) or \( K \sqcup K_2 \) graphs of Figure 3 as a minor. But then, by minor minimality, \( G \) is one of those graphs in the figure, as required. The other possibility is \( G \) has the \( K_{3,3} + 2e \) graph of Figure 2 as a minor. Then, \( K_{3,3} + e \) is a proper minor, contradicting the minor minimality of \( G \). On the other hand, if \( \bar{a} \notin V(K) \), \( G \) must have a \( K \sqcup K_2 \) minor. By minor minimality, \( G = K \sqcup K_2 \), which contradicts our assumption that \( G \) is connected. \( \square \)

While it’s difficult to convey the hard work that went into finalizing the list of seven graphs, we hope this account gives some of the flavor of a project in this area. This argument is, in fact, not so different from what appears in (soon to be) published research, see [10, 17]. Recall that CA, CE, and CC all have Kuratowski sets with at most 10 members (see Challenge 4). We can think of almost–planar...
as CE or CC and SAP as CE and CC. This suggests that other combinations of the three C properties are also likely to be minor closed with a small number of forbidden minors. For example, here are two ways to combine CA and CE.

**Project Idea 4.** Say graph $G$ has property CACE if, for every edge $ab$ and every vertex $v \not\in \{a, b\}$, either $G - v$ or $G - ab$ is planar. Determine whether or not CACE is minor closed and find the Kuratowski set or MMCACE set. Repeat for strongly CACE, which requires both $G - v$ and $G - ab$ planar.

### 4. Additional project ideas

In this section we propose several additional project ideas along with general strategies to develop even more.

Let $\mathcal{E}_k$ denote the graphs of size $k$ or less. We have mentioned that this property is minor closed. It’s straightforward to verify what happens when $k = 0$.

**Challenge 5.** Determine the Kuratowski set for edge-free graphs, Forb($\mathcal{E}_0$) and that for the corresponding apex property, Forb($\mathcal{E}_0'$).

However, $\mathcal{E}_1$ is already interesting and general observations about higher $k$ would be worth pursuing.

**Project Idea 5.** Find graphs in Forb($\mathcal{E}_1$). Find forbidden minors for $\mathcal{E}_k$ when $k \geq 2$. Can you formulate any conjectures about Forb($\mathcal{E}_k$)?

In a different direction, if $\mathcal{P}$ is minor closed, then so too are all $\mathcal{P}^{(k)}$ where $\mathcal{P}^{(k+1)} = (\mathcal{P}^{(k)})'$.

**Project Idea 6.** Find graphs in Forb($\mathcal{E}_0''$). We might call $\mathcal{E}_0''$ graphs 2-apex edge free. Any conjectures about $k$-apex edge free?

How about working with order instead of size?

**Project Idea 7.** Find forbidden minors for graphs of order at most $k$. What about apex versions of these Kuratowski sets? Any conjectures?

Naturally, one can combine these. What is the Kuratowski set for graphs that have at least two edges and three vertices? What of graphs that have either an edge or four vertices?

These project ideas encourage you to formulate your own conjectures. As examples of the kinds of conjectures that might arise, we refer to Project Idea 1. There we noticed that $K_{k+2}$ is a forbidden minor in $\mathcal{T}_k$ for $k = 1, 2, 3$, which led us to ask if the pattern persists. That project idea also includes a guess about planar graphs, again based on what is known for small $k$. Recently, we made similar observations about forbidden minors for $\mathcal{P}^{(k)}$ which is also called $k$-apex [19]. While proving that $K_{k+5} \in \text{Forb}(\mathcal{P}^{(k)})$ we were unable to confirm a stronger conjecture that all graphs in the $K_{k+5}$ family are forbidden. (Please refer to [19] for the definition of a graph’s family.) We have a similar conjecture for graphs in the family of $K_{3, 2, 1^k}$, a $k + 2$-partite graph with two parts of three vertices each and the remainder having only one vertex.

**Project Idea 8.** Prove the conjecture of [19]: The $K_{k+5}$ and $K_{3, 2, 1^k}$ families are in Forb($k$ - apex).

So far we have focused attention on graph properties that are minor closed and most of the discussion in Section 2 described techniques for generating such
properties. The meta-problem of finding additional minor closed graph properties is also worthwhile.

**Project Idea 9.** Find a minor closed graph property \( \mathcal{P} \) different from those described to this point. Find graphs in \( \text{Forb}(\mathcal{P}) \).

A survey by Archdeacon [1] includes a listing of several more problems on forbidden graphs; many of them would be great undergraduate research projects.

As in Corollary 1, minor closed \( \mathcal{P} \) are attractive because \( \text{Forb}(\mathcal{P}) \) then precisely characterizes graphs with the property. On the other hand, Corollary 2 shows that even if \( \mathcal{P} \) is not minor closed, there is a finite list of MMP\( \mathcal{P} \) graphs that can be used to rule out the property. The possible projects in this direction are virtually endless. Take your favorite graph invariant (e.g., chromatic number, girth, diameter, minimum or maximum degree, degree sequence, etc.) and see how many MMP\( \mathcal{P} \) graphs you can find for specific values of the invariant. Of course, if you choose a graph property at random, you run the risk of stumbling onto a MMP\( \mathcal{P} \) list that, while finite, is rather large. In that case, you can simply restrict by graph order or size, for example.

If you’re fortunate enough to be working with a student with some computer skills, you might let her loose on the graph properties that are built into many computer algebra systems. With computer resources, even the 300 thousand or so graphs of order nine or less are not out of the question, see for example [18].

Finally, let us note that the recent vintage of the Graph Minor Theorem and the rather specific interests of graph theorists leave a virtually untouched playing field open to those of us working with undergraduates. To date, serious researchers have focused on finding forbidden minors for a fairly narrow range of properties deemed important in the field. For those of us who needn’t worry overly about the significance of the result, there is tremendous freedom to pursue pretty much any idea that comes to mind and see where it takes us. These are early days in this area and whichever path you choose to follow, there’s an excellent chance of capturing a Kuratowski type theorem of your very own.

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