REALISTIC EXPERIMENTS FOR MEASURING THE WAVE FUNCTION OF A SINGLE PARTICLE

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Abstract

We suggest scattering experiments which implement the concept of “protective measurements” allowing the measurement of the complete wave function even when only one quantum system (rather than an ensemble) is available. Such scattering experiments require massive, slow, projectiles with kinetic energies lower than the first excitation of the system in question. The results of such experiments can have a (probabilistic) distribution (as is the case when the Born approximation for the scattering is valid) or be deterministic (in a semi-classical limit).
I. Introduction.

Since the very early days of quantum mechanics, it is generally believed\textsuperscript{1} that the wave function of a single quantum system, say, a hydrogen atom in its ground state, is not a directly measurable quantity. Rather $|\psi(x)|^2 \, dV$ is simply the probability that the electron will be found within a volume element $dV$ around the point $x$. Since localization of the electron does destroy the wave function, we need an ensemble of many identical quantum systems - say many hydrogen atoms in the ground state - to measure the wave function. The latter is therefore believed to be only a mathematical symbol devoid of direct physical reality\textsuperscript{2} at the one-system (atom) level. Recently,\textsuperscript{3} it has been pointed out, however, that the standard framework of quantum mechanics does allow measuring the wave function of a single particle in a bound, stationary, state. The “protective” measurements required for this purpose involve weak, adiabatic interactions of the particle with the measuring device. This suggestions was discussed quite extensively in the literature. Refs. 4-6 contain some critical comments which were answered in Ref.7. Refs. 8-10 address peculiar features of the proposed novel measurements. Recent further elaboration of the basic idea in the context of the two-state vector formalism\textsuperscript{11} were presented in Refs.12,13.

In the protective measurements the “pointer” indicates not an eigenvalue, $a$, of the measured operator $A$, as is in the case in the standard ‘impulsive’ measurements, but rather the expectation value:

$$\bar{A} = \langle \psi_0 \mid A \mid \psi_0 \rangle$$

of $A$ in the state $|\Psi_0\rangle$ of interest.

In the case of impulsive measurements the initial wave function $|\psi_0\rangle$ is reduced (“collapsed”) into one of the eigenstates $|a\rangle$ of the operator $A$, specifically to the state whose eigenvalue is indicated by the pointer. The probability of a specific outcome, $a$, is:

$$P(a) = |\langle a \mid \psi_0 \rangle|^2$$

Thus if $A$ is the position operator $X$ the probability of finding particle at $X = x$
is

\[ | \langle x | \psi_0 \rangle |^2 \equiv | \psi_0(x) |^2 \]  

with \( \psi_0(x) \) the wave function in the usual coordinate basis.

The recently suggested protective measurements are radically different:

(a) First, after the protective measurement of the operator \( A \) is completed, the particle stays in the same initial state \( \psi_0 \) with a probability that can be made arbitrarily close to 1. Specifically, the excitation to other states, assuming these are at a minimal (energy) distance \( \Delta E \) away, can be made exponentially small\(^{14}\)

\[ P_{\text{exc}} \approx e^{-\Delta E \cdot T} \]  

with \( T \) the duration of the adiabatically switched interaction.

(b) Second, the result of the measurement, i.e. the pointer position is deterministic (rather than probabilistic) and equals the average of \( A \) in the state \( \psi_0 \). The accuracy of this last statement depends on the validity of first order perturbation in the interaction \( V_I \) coupling the particle and measuring device, i.e. on the assumption that during the measurement process the wave function is only weakly distorted.

Using feature (a) we can repeat the protective measurement for \( N \) different operators \( A_i \). Knowledge of \( \bar{A}_i = \langle \psi_0 | A_i | \psi_0 \rangle \) allows inferring the initial wave function with precision \( O(1/N) \). We also need \( N \) impulsive, position measurements in order to sample the probability \( P(x) = \psi_0^2(x) \) at \( N \) points \( x_i, i = 1, ..., N \) and get \( \psi_0^2(x) \) to a requisite \( O(1/N) \) precision. There is, however, one crucial difference: Our original wave function is not destroyed by the repeated protective measurements so that one bound electron is all that is needed.

The purpose of the present paper is two-fold. First, we would like to point out (almost) “real experiments” in which we can, by repeated scattering on the same quantum system, say hydrogen in the ground state, map out the complete wave function, without ever ionizing or exciting the system. This is facilitated simply by using massive slow projectiles such that on the one hand their momentum is large enough to allow probing the atomic wave-function of interest and on the
other their kinetic energy is smaller than the energy gap $\Delta E$ between the ground state and the first excited state. Our second result is that the two aspects (a) and (b) of the recently suggested protective measurements—namely the non-destruction of the wave function and the deterministic rather than probabilistic result of the measurements—are really independent. Thus we could have scattering experiments involving heavy projectiles which can never destroy the state of interest, i.e. which are completely elastic, where the elastic scattering is probabilistic. This in particular is the case when the Born approximation is applicable. The result of individual scattering could also be deterministic — e.g., when an “eikonal approximation” involving the exchange of many quanta is applicable.

The plan of the paper is the following. In Section II we briefly review the protective measurements of Aharonov, Anandan, and Vaidman (Ref. 3). Section III recalls the general connection between measurements and scattering experiments. It also explains why generically such experiments are of the “impulsive” type and tend to break the bound state. Section IV contains our main result namely the description of scattering experiments involving heavy projectiles so that they do not excite the target yet map out the wave function. For concreteness we assume that the scattering can be described in Born approximation. Section V extends the discussion to a case where many quanta are exchanged in typical scattering. The resultant deflection is then (for a given initial impact parameter $b$) practically deterministic. This corresponds more closely to the originally suggested scheme of Aharonov and Vaidman and Aharonov, Anandan and Vaidman.3

Finally, in the summary in Section VI, we speculate on real experiments and discuss various putative heavy projectiles, which can implement these experiments.

II. The General von Neumann Formalism and the Novel Protective Measurements.

Let us first briefly review the basic von Neumann formalism15 underlying all quantum measurements, impulsive and protective, alike. The system of a particle
of mass $m$, coordinate $\vec{r}$, momentum $\vec{p}$ and the measurement apparatus (“pointer”) with coordinate $Q$, momentum $P$ and mass $M \gg m$ are initially non-interacting. The Hamiltonian then separates into a sum of two parts pertaining to the particle and pointer respectively:

$$H_0 = \left[ \frac{\vec{p}^2}{2m} + V_B(\vec{r}) \right] + \left[ \frac{\vec{P}^2}{2M} + V(Q) \right]$$

(5)

with $V_B(\vec{r})$, $V(Q)$ binding potentials. The corresponding initial state factorizes into two parts. The particle is assumed to be initially in a state $|\psi_0\rangle$ which is a stationary bound state, say the ground state, in the potential $V_B(\vec{r})$. The pointer is assumed to be initially in some wave packet state, say a Gaussian: $\Phi_0(Q) = exp\left[-\frac{Q^2}{2(\Delta Q)^2}\right]$ centered around the origin in coordinate space [or alternatively as the Fourier transformed Gaussian $\tilde{\Phi}_0(P) = exp\left[-\frac{P^2}{2(\Delta P)^2}\right]$ in momentum space, with $\Delta P\Delta Q = \hbar/2$.]. The measurement process of an operator $A(\vec{r},\vec{p})$ consists of turning on, an interaction of the form:

$$H_I(t) = g(t)A(\vec{r},\vec{p}) \cdot P.$$

(6)

between the system and pointer. The time profile of switching the interaction on and off is normalized by

$$\int_{-\infty}^{\infty} g(t)dt = 1.$$

(7)

In the impulsive measurements $g(t)$ differs from zero only over a short time interval $\delta$. During this time $H_I \approx \delta^{-1}$ is very large and we can neglect $H_0$ in evaluating the evolution of the system

$$U(t, t + \delta) = exp[-i \int_{t}^{t+\delta} g(t)A(\vec{r},\vec{p}) \cdot P].$$

(8)

The initial state of the particle is decomposed in the basis of $A$ eigenstates:

$$|\psi_0\rangle = \sum a |a\rangle \langle a | \psi_0\rangle.$$ Upon applying $U(t, \delta, t)$ each component $|a\rangle$ acquires the appropriate phase:

$$U(t, t + \delta)\{ |\psi_0\rangle \times |\Phi_0\rangle \} = \sum_a e^{iaP} |\Phi_0\rangle \langle a | \psi_0\rangle$$

$$= \sum_a |a\rangle \Phi(a) \langle a | \psi_0\rangle$$

(9)
where $|\Psi(a)\rangle$ is the pointer wave function shifted via the $e^{iaP}$ translation operator from the initial position around $Q = 0$, to a similar Gaussian packet centered around $Q = a$ (so that the pointer indicates the specific eigenvalue $a$ measured).

Let us next consider the protective measurements suggest by AV and AAV.\(^3\) In these measurements $g(t)$ is turned on for a long time $T$. Eq. (7) implies then a very weak $H_I$: $g(t) \approx \frac{1}{T} = \epsilon$. If the turning on and off of the interaction is sufficiently smooth then we can use

1. the adiabatic approximation for treating the effect of $H_I$. The weakness of $H_I$ allows us, furthermore, to do this

2. in a first order perturbation for $H_I$.

Thus (i) implies that the original state $\psi_0$ gets continuously deformed to $\psi_0(t)$ with $\psi_0(t)$ being the instantaneous eigenfunction of the time-dependent Hamiltonian $H(t) = H_0 = H_0 + H_I(t)$ with eigenvalue $E_0(t)$:

$$H(t) | \psi_0(t)\rangle = E_0(t) | \psi_0(t)\rangle$$

and we have:

$$\psi_0(t) \rightarrow_{t \rightarrow \pm \infty} \psi_0; \quad E_0(t) \rightarrow_{t \rightarrow \pm \infty} E_0.$$  \hspace{1cm} (11)

The probability of leaving the system in some excited final state is exponentially small: \(^{14}\)

$$P_{exc} \approx e^{-\Delta E T}.$$ \hspace{1cm} (12)

The second feature implies (ii) that the energy shift $\delta E_0(t) = E_0(t) - E_0$ can be evaluated by first order perturbation in the weak interaction $H_I$:

$$\delta E_0(t) = \langle \psi_0 | H_I(t) | \psi_0 \rangle = g(t)P \langle \psi_0 | A | \psi_0 \rangle \equiv g(t)P \bar{A}$$ \hspace{1cm} (13)

The corresponding time evolution of the pointer will be given by

$$e^{-i \int \delta E_0(t) dt} | Q_0 \rangle = e^{-i \int dt g(t) P \bar{A}} | Q_0 \rangle = | Q_0 + \bar{A} \rangle$$ \hspace{1cm} (14)

i.e. the Gaussian wave packet of the pointer will shift by

$\bar{A} \equiv \langle \psi_0 | A | \psi_0 \rangle$. Such a shift can be discerned if it is larger than the original
width (in configuration space) of the Q wave packet, i.e. if

\[ A \geq \langle \Delta Q \rangle. \]  

(15)

Using Eq. (12), the fact that the transition frequencies to all the higher levels exceed \( \Delta E \), and completeness:

\[ \sum_n |\langle \psi_n | A | \psi_0 \rangle|^2 = \langle \psi_0 | A^2 | \psi_0 \rangle \]  

(16)

it can be shown that the sum of all excitation probabilities to any bound (or unbound) excited state satisfies

\[ \sum_{n \neq 0} P(\psi_0 \rightarrow \psi_n) < C \cdot e^{-\Delta ET}. \]  

(17)

Thus such a measurement indeed “protects” the initial state and maintains it for subsequent experiments of the same protective nature, where other operators \( A_i, i = 1, \ldots, N \) are measured.

In particular, AV and AA\( V^3 \) choose the operators \( A_i \) to be \( P(\delta v_i) \), the projections onto many small space regions around points \( \vec{r}_i \). Since the ground state wave function is real and nodeless, measuring \( \langle P\delta v_i \rangle = \int_{\delta v_i} \psi_0^2(\vec{r}) \) fixes \( \psi_0(\vec{r}_i) \). Letting \( N \rightarrow \infty \) recuperates the complete wave function.

### III. Measurement as Scattering Experiments.

The formal von Neumann pointer-system interaction is similar to usual scattering. Let us scatter form the electron, bound to infinitely heavy “proton” \( P \), at the origin, a projectile particle, \( X \). The latter is initially moving with velocity \( V \) along the \( z \) axis, say. The scattered projectile particle plays here the role of the pointer. The interaction shifts its momentum by \( \Delta \vec{P} \) and deflects it by an angle \( \Delta \theta \approx |\Delta \vec{P}| / P \). This deflection is finally translated, in the subsequent free evolution of the scattered wave packet, into a lateral displacement on a far-away screen.

A large class of \( \gamma - \text{ray} \), neutron or electron scattering, measure the “Form Factor” of quantum systems. The differential scattering cross section of electrons,
say, in the Born approximation, is proportional to $|F|^2$ with $F$ the “Form Factor”:

$$F(\Delta \vec{P}) \approx \int d\vec{r} e^{i\Delta \vec{P} \cdot \vec{r}} \rho(\vec{r})$$  \hspace{1cm} (18)$$

and $\rho$ the charge density. If the charge density is contributed by one electron in the $\psi_0(\vec{r})$ bound state, then $\rho(\vec{r})$ is given by:

$$\rho(\vec{r}) = q |\psi_0(\vec{r})|^2, \hspace{1cm} (19)$$

Once $d\sigma/d\Omega$ is measured, we can infer $|F(\Delta \vec{P})|^2$. For the case $\rho(\vec{r}) = \rho(-\vec{r})$, $F$ is real and $F(\Delta \vec{P})$ can also be determined. By (inverse) Fourier transformation, we would then find $\psi_0(\vec{r})$. However, in general, this measurement cannot be done by scattering on one single atom and an ensemble of atoms is required. This is related to the dual significance of the form factor $F(\Delta \vec{P})$ in impulsive scattering; namely that $|F(\Delta P)|^2$ is then also the probability of not destroying the system in any, specific, scattering event. Indeed, for collision time short in comparison with atomic periods, the “sudden” approximation is appropriate. The transfer of momentum $\Delta \vec{P} = \vec{P}_f - \vec{P}_i$ transforms the initial wave function via:

$$\psi_0 \rightarrow \psi'_0 = e^{i\Delta \vec{P} \cdot \vec{r}} \psi(\vec{r}).$$ \hspace{1cm} (20)$$

The probability of staying in the ground state is then the square of the form factor:

$$P_{\psi_0 \rightarrow \psi'_0} = |\langle \psi_0 | \psi'_0 \rangle|^2 = | \int d\vec{r} e^{i\Delta \vec{P} \cdot \vec{r}} \psi_0(\vec{r}) |^2 = |F(\Delta \vec{P})|^2$$ \hspace{1cm} (21)$$

The normalization condition $\int d\vec{r} \psi_0^2(\vec{r}) = 1$ implies that $F(\Delta \vec{P} = 0) = 1$. However,

$$|F[\Delta \vec{P} \neq 0]| < 1$$  \hspace{1cm} (22)$$

The probability that the initial atom stays in the ground state after $N$ repeated scattering with momentum transfers $\Delta \vec{P}_1, \Delta \vec{P}_2, \ldots, \Delta \vec{P}_i \ldots \Delta \vec{P}_N$, is therefore exponentially small:

$$P_{\psi_0 \rightarrow \psi_0} \text{ (after N collisions)} = \prod_{i=1}^{N} |F(\Delta \vec{P}_i)|^2 \approx e^{-cN}. \hspace{1cm} (23)$$
[To show this recall that the geometric mean of $N$ positive numbers is smaller than the arithmetic mean. Hence:

$$\prod_{i=1}^{N} F^2(\Delta \vec{P}_i) \leq \left[ \frac{1}{N} \sum_{i=1}^{N} F^2(\Delta \vec{P}_i) \right]^N \approx (\bar{F}^2)^N \approx e^{-cN} \quad (24)$$

with $\bar{F}^2$ an effective average of $F^2$ obtained by the experimental sampling.] Consequently, we need $\approx N$ atoms if we want to measure the form factor at $N$ points.

IV. “Protective” Scattering: The Born Approximations

We have seen that typical elastic scattering experiments are similar to impulsive von Neumann measurements. We proceed next to discuss adiabatic scattering corresponding to protective measurements.

If $|\Delta \vec{P}| \approx P\sin \theta$ is small in comparison with the initial momentum $\vec{P} = MV \hat{e}_z$ of the projectile, then the deflection angle is small:

$$\theta = |\Delta \vec{P}|/P \approx \frac{|\Delta \vec{P}|}{MV} \ll 1. \quad (25)$$

The effective collision time, estimated via the classical traversal time, is then:

$$\delta t_{col} \approx a_0/V \quad (26)$$

with $a_0$ the effective size of the state $\psi_0$. By the uncertainty principle the average (rms) momentum of the electron satisfies

$$p \geq \frac{\hbar}{a_0} \quad (27)$$

If $m$ is the electrons’ mass, then:

$$v_e \equiv \frac{p}{m} \approx \frac{\hbar}{ma_0} \quad (28)$$

$$\Delta E \approx \frac{mv^2}{2} = \frac{\hbar^2}{2ma_0^2} \quad (29)$$

$$t_{atom} \approx \frac{a_0}{v} \approx \frac{ma_0^2}{\hbar} \approx \frac{\hbar}{\Delta E} \quad (30)$$
are the electron’s “velocity”, the energy difference to the first excited state, and the atomic period respectively. The sudden approximation used above was predicated on the assumption that

$$\delta t_{col} \ll t_{atom}$$

(31)
or, using Eqs. (30) and (26) on:

$$V \gg v[\text{sudden approximation}].$$

(32)

If, further, \(M \geq m\), this implies a kinetic energy of the projectile, \(KE = \frac{1}{2}MV^2\), which considerable exceed \(\Delta E \approx \frac{1}{2}mv^2\) and the atom can be readily excited or ionized we are however interested. In the opposite, adiabatic case,

$$\delta t_{col} \gg t_{atom}$$

(33)
or

$$v \gg V[\text{adiabatic approximation}]$$

(34)

In order to be able to effectively probe \(F(\Delta \vec{P})\) over the whole relevant range of \(|\Delta \vec{P}| \approx \hbar/a \approx mv\), we need that:

$$|\Delta \vec{P}| \approx MV\theta > p \approx mv$$

(35)
or

$$MV \gg mv$$

(36)

Clearly, Eq. (36) is consistent with Eq. (34), providing that the projectile-electron mass ratio is large enough:

$$f \equiv M/m \gg 1.$$  

(37)

Let us impose next a stronger version of the adiabatic condition namely \(V \leq v/\sqrt{f}\) so that

$$\frac{1}{2}MV^2 \leq mv^2 \approx \Delta E.$$

(38)

In this case, the complete kinetic energy of the projectile is insufficient for exciting the atom. We then have only elastic scattering with the atom staying
always in the ground state.\textsuperscript{16} We also take the mass of the nucleus to be so large
that its recoil velocity and recoil energy are completely negligible.

For the rest of this section, we will assume that the projectile-electron interaction $V_{\text{int}}(\vec{r} - \vec{R}(t))$ is weak enough so that the scattering cross section can be computed in the Born approximation:

$$\frac{d\sigma}{d\Omega} \approx |f_B(\theta)|^2 \approx |F(\Delta \vec{P})|^2 (G(\Delta P))^2$$

(39)

with $F(\Delta \vec{P})$ the form factor defined in Eq. (18) above and $G(\Delta P)$ the interaction potential in Fourier space:

$$G(\Delta \vec{P}) \equiv \int e^{i \Delta \vec{P} \cdot \vec{\rho}} V_{\text{int}}(\vec{\rho}) d\vec{\rho}.$$  

(40)

(Thus for Coulomb interaction $G(\Delta \vec{P}) \approx \alpha_{\text{em}} / |\Delta P|^2$). By measuring $d\sigma/d\Omega$, we can then infer $F(\Delta \vec{P})$ and $\psi_0^2(\vec{r})$. Since the atom is never excited, we need just one atom from which we repeatedly scatter (a beam of) many projectiles. It is amusing to ask how much time is required in such an experiment in order to map out the ground state wave function to within an accuracy of $10^{-3}$. For an incident flux $\Phi$ the rate of collision will be $dN/dt = \Phi \sigma_{\text{el}}$ with $\sigma_{\text{el}} \approx \epsilon a_0^2$ the elastic cross section. (The parameter $\epsilon \ll 1$ expresses the weakness of the interaction.)

In order to obtain an $O(1/N)$ precision in the determination of $F(\Delta \vec{P})$ (or $\psi_0(\vec{r})$), we need $N$ scattering. The “experiment” should therefore be carried over a time $\tau$ such that:

$$\Phi \sigma \tau = N$$  

(41)

Two considerations limit the flux $\Phi = nV$ with $n$ the number density of projectiles in the beam. First, from Eq. (38) $V \leq \sqrt{m/Mv} = v/\sqrt{f}$. Also we take the number density of projectiles $n \leq a_0^{-3}$, to avoid simultaneous scattering of several projectiles. Thus, we find

$$\tau \geq \frac{a_0 \sqrt{f}}{V \eta \epsilon} \geq \frac{N a_0 \sqrt{f} \tau_{\text{atom}}}{\eta \epsilon v} = \frac{N \sqrt{f \tau_{\text{atom}}}}{\eta \epsilon}$$

(42)

with $\eta$ a measure of how far we are from the “maximal density” $n \approx a_0^{-3}$. Even for a large $f$, a small $\epsilon$ (guaranteeing the validity of the Born approximation) and
a tiny $\eta$:

$$f = \frac{M}{m} = 10^4$$  \hspace{1cm} (43)

$$\epsilon \approx 10^{-3}$$ \hspace{1cm} (44)

$$\eta \approx 10^{-6}$$ \hspace{1cm} (45)

We need “only” $\tau \simeq 0.01 \text{sec}$ in order to obtain

$$N \approx 10^3.$$ \hspace{1cm} (46)

This is due to the very short atomic period, $t_{\text{atom}}$, which for hydrogen, is $10^{-16}$ sec.

**V. Protective Scattering: The Semi-classical Approximation.**

In the previous section we have shown that scattering of slow massive projectiles, when treated in the Born approximation, can allow finding the full form factor $F(\Delta \vec{P})$ and hence the complete wave function $\psi_0(r)$ of the atom, even if only a single atom is available. While this demonstrates the measurability of the wave function of a single system, the scattering considered differ form the protective measurements of Aharonov and Vaidman and Aharonov, Anandan, and Vaidman\(^3\) in one crucial respect. The outcome of any single scattering was probabilistic with the probability for scattering with momentum transfer $\Delta \vec{P}$ being proportional to $|F(\Delta \vec{P})|^2$ -whereas in the AV and AAV protective measurements, the pointer deflection [i.e. scattering angle] was deterministic.

As will be shown next, the latter, deterministic case, can also be modeled by semi-classical scattering. To this end we will maintain the conditions for protective scattering. (i) The kinetic energy of the projectile satisfies $\frac{1}{2} M V^2 \leq \Delta E$ so that inelastic scattering cannot happen, though $M V \sqrt{\frac{h}{a}} \approx m v$. (ii) The projectile-electron interaction is weak enough so that the original wave function is not distorted. We will however relax the assumption that scattering can be treated in the Born approximation. For convenience we will assume a coulombic interaction between the electron of change $q$ and projectile of change $q'$:

$$V_I(\vec{r} - \vec{R}) = \frac{qq'}{|\vec{r} - \vec{R}|}.$$ \hspace{1cm} (47)
[Taking q and \( q' \) to have the same sign disallows rearrangement reactions where the electron is picked up by the projectile.] We really need not specify the binding potential \( V_B(r) \). However, to make the discussion more concrete, we also take a Coulombic binding potential:

\[
V_B(\vec{r}) = -\frac{Zq^2}{|\vec{r}|}
\]  

(48)

Our requirement \( V_B \gg V_I \) is implemented by having

\[
Zq \gg q'
\]

(49)

To simplify the treatment, we assume that there is no direct projectile-nucleus interaction. We assume as in (ii) above that \( V_I(\vec{r} - \vec{R}) \) constitutes a small perturbation for the electron. We will not assume that it is weak insofar as the projectile motion is concerned. The weaker projectile (X particle) – electron interaction as compared with electron-proton interaction is conveyed in Fig. 1a by the larger number of virtual quanta (photons exchanged) between \( e \) and \( P \).

**Figure 1:** Scattering of a heavy projectile \( X \) on the light electron \( e \) bound strongly to a super heavy proton \( P \). The thickness of the lines conveys the information on the masses of the particles. (a) The scattering in a semi-classical, many photon exchange, approximation. (b) The scattering in the Born, one photon exchange, approximation.
The phase picked up by the projectile as it moves along a straight line parallel to the z axes at an impact parameter with fixed velocity \( V \):

\[
\vec{R}(t) = \vec{b} + Vt\hat{e}_z
\]

is approximately given by:

\[
\phi \approx \frac{1}{\hbar} \int_{\text{path}} dt V_I(\vec{r} - \vec{R}(t)) = \frac{qq'}{\hbar V} \ln(\frac{\Lambda}{b})
\]  

(50)

with \( \Lambda \) a cut-off length. By having

\[
\frac{qq'}{\hbar V} \equiv \eta \gg 1
\]  

(51)

we make the projectile electron interaction strong. The large Coulomb phase and the large angular momentum

\[
L = \frac{MVb}{\hbar} \approx \frac{MVa}{\hbar} \approx \frac{MV}{mv} \gg 1
\]  

(52)

suggest that we can treat the projectiles’ motion semi-classically. The Feynman diagrammatic expression of this is the fact that the momentum transfer \( \Delta P \) is built up by exchanging a large number \( n_\gamma \approx \eta \gg 1 \) of photons, see Fig. 1a, rather than via a single photon exchange as in the case of the Born approximation, see Fig 1b. A semi-classical computation of the deflection, assuming still, for convenience, that \( \theta \)-the scattering angle is small, can be done as follows. The electrostatic force exerted by the (spherically symmetric!) charge distribution of the electron on the projectile is

\[
\vec{F} = \frac{Q(R)q'\hat{e}_b}{R^2}
\]  

(53)

where \( Q(R) \) is the total electric charge within a sphere of radius \( R = |\vec{R}(t)| \)

\[
Q(R) = 4\pi q \int_0^R r^2 dr (\psi_0(r))^2.
\]  

(54)

[We note that the use of the average value (for the charge inside R) in the state \( \psi_0 \) parallels the general discussion in Section II above where the average

\( \langle \psi_0 | A | \psi_0 \rangle \) determined the deflection of the pointer.]  The momentum transfer is then:

\[
\Delta \vec{P} = \int_{-\infty}^{\infty} \vec{F} dt = q' \int_{-\infty}^{\infty} \frac{dz Q(R = \sqrt{z^2 + b^2})}{(z^2 + b^2)^{3/2}} b\hat{e}_b
\]  

(55)
where we used $dt = dz/V$ and the fact that the net momentum transfer in this approximation [with the projectile moving with uniform velocity parallel to the $z$ axes] is purely transverse i.e. along $\hat{e}_b$. The order of magnitude of $\Delta P$ is readily estimated:

$$\Delta P \approx F \delta t_{col} \approx \frac{1}{\hbar} \frac{qq' a_0}{a_0^2} \frac{\hbar}{V} \approx \eta \frac{\hbar}{a_0}. \tag{56}$$

Such a $\Delta P$ is indeed expected if, as conjectured before, we have $n_\gamma \approx \eta$ photons exchanged, each contribution a typical momentum transfer of $= \hbar/a_0 \approx mv$. This multiple exchange make the total momentum transfer larger by a factor $\eta$ and easier to measure. Also the large number of quanta involved tends to make the deflection almost deterministic so that the above classical treatment of the scattering is justified. [Parenthetically we note that a small deflection:

$$\theta = \frac{\Delta P}{P} = \frac{\eta mv}{MV} < 1 \tag{57}$$

requires a stronger version of Eq. (36).

$$MV > \eta mv; \eta \equiv \frac{qq'}{\hbar V} \gg 1 \tag{58}$$

While the last equation implies that

$$MV^2 > \frac{qq'vm}{\hbar} \approx \frac{qq'}{a_0} \tag{59}$$

the latter does not conflict with the basic requirement

$$kE = \frac{1}{2} MV^2 < \Delta E = \frac{Zq^2}{a_0} \tag{60}$$

proving that the ratio $Zq/q'$ is large enough. This however is precisely our Eq. (49) above.]

To measure the wave function, it suffices to find $Q(r)$ of Eq. (54) since $\psi_0(r)$ is given by:

$$\psi_0(r) \approx \frac{1}{r} \sqrt{\frac{dQ(r)}{dr}}. \tag{61}$$

Scattering projectiles with various initial impact parameters $b$ and measuring in each case the momentum transfer $\Delta P(b)$ given via Eq. (55), we can solve for
\[ Q(r) \text{. [In practice also such classical scattering, where } \Delta P \text{ is uniquely determined for a given } b \text{ by a calculable function } \Delta P(b), \text{ are carried out by using an incident uniform beam of projectiles and measuring } d\sigma/d\Omega.^{17} \]

VI. Summary and Discussion.

The arguments of this paper, complementing and slightly extending the original work of AAV and AV\(^3\) definitely show that there is no quantum mechanics subtlety hinders the measurement of the wave function of a single atom. In all cases we require a projectile which is much heavier than the particle say electron whose wave function is being measured. The set-ups considered were still somewhat idealized. Thus in Section V we assumed that there is a nucleus-electron and electron-projectile interaction, but no direct interaction between projectile and nucleus. These interactions could be Coulomb-like only if the electron carried two types of charges: \( q\epsilon U(1) \), and \( q'\epsilon U'(1) \) sharing with the nucleus one type of charge and with the projectile the other type. While the absence of projectile nucleus interaction is not an absolute must, it excludes the possibility of the projectile being captured around the nucleus. Ideally, the projectile should also:

(a) be point like;
(b) have long lifetimes;
(c) have minimal interaction with the nucleus

Clearly (a) - (c) are a “convenience” issue. Thus in principle we could use whole atoms or ions as massive slow projectiles. The separation of the target wave function from the effects due to the structure of the projectile, would then be more difficult. The muon is point like and \( m_\mu = 200m_e \). However, to satisfy the basic condition (37):

\[ E_\mu = \frac{1}{2}m_\mu\beta_\mu^2 \leq \Delta E_{\text{Hydrogen}} \approx 10eV \]  

(62)

we need very slow muons. The muons’ lifetime is only \( 2.10^{-6} \) seconds and these slow muons propagate only a few centimeters before decaying. The task of generating intense, monochromatic, collimated muon beams, scattering them and later, detecting the scattered muons seems very difficult.
We could also utilize neutrons which are heavy and fairly long lived. However, the low energy neutrons \( (E_n \leq 10eV) \) often have strong nuclear interactions which could mask the effects of the electron-neutron scattering.

Amusingly, the scattering of hypothetical WIMPs (“Weakly Interacting Massive Particles”) of mass, say 200GeV on heavy nuclei \( (A \geq 200) \), provides a simple example of a form factor appearing when the interaction is purely elastic. For \( M \approx 200GeV \), the kinetic WIMP energy: \( E_{WIMP} = \frac{1}{2}MV^2 \approx 100KeV \) (for \( V \approx 10^{-3}c \), typical virial velocity) is below nuclear excitation. Thus our basic no excitation requirement (Eq. (38a)) is readily satisfied. At the same time the maximal momentum transfer \( \Delta P \approx 200MeV \) is appreciable and \( \Delta PR(A;Z) \gg 1 \) so that \( F(\Delta P) \) could be significantly smaller than unity.

In conclusion, it is amusing to note that quantum systems are actually more rigid than classical systems. The latter have continuous spectra and can suffer arbitrarily small excitation. Any scattering from a classical target would leave the latter slightly deformed. This however is not the case for quantum mechanical system

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[1] This statement is explicit or implicit in just about any quantum mechanics textbook. The probabilistic interpretation of the wave function is due to Max Born, Zeits. F. Physik 37, 863 (1926); Nature, 119, 354 (1927).

[2] This interpretation which stripped the wave function of its full physical meaning, as in the case of a classical wave, was strongly bemoaned by E. Schrödinger, see, e. g., M. Jammer The Conceptual Development of Quantum Mechanics, McGraw Hill (1966).

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[16] Since we have a wave packet of finite length, $\delta t$ there is really no absolute protection of the ground state against excitation in scattering beyond the small $e^{-\Delta E \Delta t}$ probability of excitation with $\Delta E$ indicating the finite energy width of the wave packets.

[17] See, e.g., H. Goldstein, Classical Mechanics, Addison-Wesley (1950).