Multiphoton resonance and chiral transport in the generalized Rabi model

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The generalized Rabi model (gRM) with both one- and two-photon coupling terms has been successfully implemented in circuit quantum electrodynamics systems. In this paper, we examine theoretically multiphoton resonances in the gRM and derive their effective Hamiltonians. With different detunings in the system, we show that all three- to six-photon resonances can be achieved by involving two intermediate states. Furthermore, we study the interplay between multiphoton resonance and chiral transport of photon Fock states in a resonator junction with broken time-reversal symmetry. Depending on the qubit-photon interaction and photon-hopping amplitude, we find that the system can demonstrate different short-time dynamics.

I. INTRODUCTION

A two-level atom interacting with a quantized electromagnetic field is one of the most basic and oldest problems in quantum optics. The system is described by the celebrated quantum Rabi model [1]. When the atom-photon interaction is not very strong, the rotating-wave approximation (RWA) applies and leads to the Jaynes-Cummings model (JCM) [2]. Although counter-rotating (excitation-number-non-conserving) processes are eliminated by the RWA, the JCM had been a widely accepted model in quantum optics for a long time [3]. After the breakthrough in fabricating circuit quantum electrodynamics (cQED) systems, it has become feasible to reach the ultrastrong coupling (USC) [4, 5] or even the deep-strong-coupling (DSC) regimes of the atom-photon interaction [6, 7]. This achievement has reignited a large amount of studies of the original Rabi model. From a mathematical perspective, the model was shown to be quantum integrable [8, 9] and various attempts were made to construct the analytic solution [9–14]. At the same time, previous work shown that the counter-rotating terms could lead to many novel physical phenomena which may find different applications [15–47].

Recently, a nonlinear two-photon coupling term with a two-level system was implemented in cQED setups [48, 49] and cold-atom systems [50]. As compared to the original Rabi model, the two-photon Rabi model can demonstrate squeezing of light [51, 52] and interaction-induced spectral collapse [49, 53–55]. Instead of just having the single-photon or two-photon term, a generalized Rabi model (gRM) with both terms has emerged in the last decade [56, 57]. In addition, it was suggested that both terms can be tuned in experimentally accessible systems. Surprisingly, the gRM also opens a door to simulate particle dynamics in (1+1)-dimensional curved spacetime [58]. The simultaneous presence of both one-photon and two-photon terms breaks the $\mathbb{Z}_2$ symmetry in the original Rabi model. On one hand, the absence of symmetry makes it more complicated to study the solutions of the gRM. Analytic solutions of the gRM were discussed very recently [59, 60]. On the other hand, it opens the door to achieve new multiphoton resonances which are forbidden in the original Rabi model. Our current manuscript is motivated by the latter viewpoint.

Since the discovery of quantum Hall effect, the concept of topology in condensed matter system has been studied for a long time [61–63]. The external magnetic field breaks time-reversal symmetry (TRS) and leads to a chiral transport at the edge of a quantum Hall fluid [64, 65]. It was suggested that quantum Hall effect of light can be realized in metamaterials [66, 67]. Meanwhile, the seminal work by Haldane shown that topological phases can also occur without applying an external magnetic field [68]. Instead, the breaking of TRS in the quantum anomalous Hall state is achieved by having a suitable complex hopping amplitude for electrons in the lattice. Inducing such an effective magnetic flux in photonic systems is more challenging as photons and atoms are neutral. Recently, this task has become practical as synthetic gauge field (both Abelian and non-Abelian) for photons [69–71] and cold atoms [72–75] have been realized. This advancement has started the burgeoning research field in topological photonics [77–80] and the simulation of topological physics in cold atom systems [80–83].

In this paper, we study multiphoton resonances in the generalized Rabi model with different detunings between the atomic transition frequency and the photon frequency. Also, we discuss the effect of multiphoton resonance on the quantum dynamics of a resonator junction with broken TRS. Our manuscript is organized as follows. First, we review the generalized Rabi model and outline the third-order perturbation theory for analyzing multiphoton resonances in Sec. II. Then, we examine the three-photon resonance, and compare the results with the original Rabi model in Sec. III. The possibility of achieving four- to six-photon resonances in the gRM will be discussed in Sec. IV. In Sec. V, we discuss the quantum dynamics of transferring photons in a TRS-broken resonator junction. Depending on the ratio between effective multiphoton-resonant coupling strength and photon-hopping amplitude, the short-time dynamics of the system can transit from a chiral transfer of multiple photons to a suppression of photon transfer. Lastly, we conclude our work in Sec. VI.
II. REVIEW OF THE GENERALIZED RABI MODEL AND PERTURBATION THEORY

The generalized Rabi model with both one-photon and two-photon coupling terms is described by the Hamiltonian:

\[ H = \frac{\omega_a}{2} \sigma_z + \omega_c a^\dagger a + \lambda (a + a^\dagger) \sigma_x + \kappa \left[ a^2 + (a^\dagger)^2 \right] \sigma_z. \]  

(1)

In this paper, we set \( \hbar = 1 \) unless specified. The transition frequency of the two-level atom is denoted by \( \omega_a \). A photon with frequency \( \omega_c \) is annihilated (created) by the operator \( a \) (\( a^\dagger \)). We denote the ground state and the excited state of the two-level atom as \( |g\rangle \) and \( |e\rangle \), respectively. Then, the Pauli matrices are given by \( \sigma_z = |e\rangle \langle e| - |g\rangle \langle g| \) and \( \sigma_x = |e\rangle \langle g| + |g\rangle \langle e| \). The one- and two-photon coupling terms with the atom have coupling strengths \( \lambda \) and \( \kappa \), respectively.

In the absence of the two-photon coupling term (i.e., when \( \kappa = 0 \)), it is possible to define a parity operator for the quantum Rabi model [9]:

\[ \Pi_1 = \exp \left[ i \pi \left( \frac{1+\sigma_z}{2} + a^\dagger a \right) \right]. \]

(2)

The parity operator satisfies \( [H, \Pi_1] = 0 \). By acting \( \Pi_1 \) on the bare states \( |g, n\rangle \) and \( |e, n\rangle \), it can only take eigenvalues of \( \pm 1 \). Similarly, one may define a corresponding \( Z_4 \)-symmetry operator for the two-photon Rabi model (i.e., when \( \lambda = 0 \) ) [84]:

\[ \Pi_2 = \exp \left[ i \pi \left( \frac{1+\sigma_z}{2} + a^\dagger a \right) \right]. \]

(3)

Every eigenstate of the Hamiltonian must be a linear superposition of bare states in the same subspace with the same eigenvalue of \( \Pi_1 \) or \( \Pi_2 \). However, the discrete symmetry is broken in the generalized Rabi model with both one- and two-photon terms. Thus it becomes more difficult to study the analytic solutions of the model. At the same time, it opens the door to new multiphoton resonances, which are forbidden in the original Rabi model. We will review the third-order perturbation theory for multiphoton resonances in a largely detuned Rabi model.

A. Perturbation theory for multiphoton resonance

In this paper, we study multiphoton resonances between two bare states \( |i\rangle \) and \( |f\rangle \). We only focus on resonances which involve two intermediate states. The two bare states are \( |i\rangle = |g, n_0 + n\rangle \) and \( |f\rangle = |e, n_0\rangle \). In the absence of atom-photon interaction, these two states are degenerate when \( \omega_c = \omega_a/n \). An effective Hamiltonian to describe the resonance can be obtained by eliminating the intermediate states. This can be done by several approaches, such as adiabatic elimination [31], generalized James’ effective Hamiltonian approach [85], and third-order perturbation theory [34]. The derivation based on the second approach is discussed in Appendix A. The end result will be in the following form:

\[ H_{\text{eff}}^{n\text{-ph}} = (E_i + \Delta E_i)|i\rangle \langle i| + (E_f + \Delta E_f)|f\rangle \langle f| + \Omega_{\text{eff}}^{n\text{-ph}}|i\rangle \langle f| + |f\rangle \langle i|. \]

(4)

Here, the unperturbed energy for the state \( |i\rangle \) is denoted by \( E_i \). Due to the atom-photon interaction, the energy levels are Stark shifted. Consequently, the required photon frequency of achieving the \( n \)-photon resonance (or equivalently, the resonant frequency) is perturbed away from \( \omega_c = \omega_a/n \). The perturbed resonant frequency \( \omega_c \) can be obtained by equating the diagonal elements of \( H_{\text{eff}}^{n\text{-ph}} \), that is solving \( E_i + \Delta E_i = E_f + \Delta E_f \). At the resonance, the two nearly-degenerate energy levels develop an avoided crossing with a gap \( 2\Omega_{\text{eff}}^{n\text{-ph}} \).

In the following discussion, we will directly use the result from second-order perturbation theory to obtain the leading order terms in \( \Delta E_i \):

\[ \Delta E_i = \sum_{\alpha} \frac{|\langle \alpha | V | i \rangle|^2}{E_i - E_\alpha}. \]

(5)

Here, the symbol \( V \) denotes the atom-photon interaction terms in the gRM, i.e., \( V = \lambda (a + a^\dagger) \sigma_x + \kappa (a^2 + (a^\dagger)^2) \sigma_x \). All intermediate bare states which can be connected to \( |i\rangle \) by \( V \) are labelled as \( |\alpha\rangle \). The result of \( \Delta E_i/\omega_a \) and \( \omega_c/\omega_a \) from Eq. (5) will be accurate to the order of \( (\lambda/\omega_a)^2 \) and \( (\kappa/\omega_a)^2 \). For the effective coupling strength \( \Omega_{\text{eff}}^{n\text{-ph}} \), it can be deduced from the third order perturbation theory [34]:

\[ \Omega_{\text{eff}}^{n\text{-ph}} = \sum_{\alpha,\beta} \frac{\langle f | V | \beta \rangle \langle \beta | V | \alpha \rangle \langle \alpha | V | i \rangle}{(E_i - E_\alpha)(E_i - E_\beta)}. \]

(6)

The symbol \( |\beta\rangle \) denotes another intermediate state. All results obtained from Eqs. (5) and (6) are consistent with the generalized James’ effective Hamiltonian approach discussed in Appendix A.

III. THREE-PHOTON RESONANCE IN THE GENERALIZED RABI MODEL

In the original Rabi model (i.e., \( \kappa = 0 \)), a three-photon resonance can occur when the frequency of the photon field is tuned to \( \omega_c \approx \omega_a/3 \) [31]. Physically, this is possible due to two reasons. First, the two bare states \( |g, 3\rangle \) and \( |e, 0\rangle \) are nearly degenerate. Second, these two bare states can be connected by intermediate states via counter-rotating processes. Thus an avoided crossing is formed between the two energy levels. Away from the avoided crossing, the corresponding eigenvectors of these two levels have high overlaps with the two bare states. In the framework of the generalized Rabi model, we revisit the three-photon resonance. The three different schemes for achieving the resonance in the gRM are illustrated in Fig. 1.
FIG. 1: Three different coupling schemes for the three-photon resonance in the gRM in Eq. (1). Here, counter-rotating processes are labeled by dashed lines. Single (double) lines with arrows indicate coupling between the bare states by the single-photon (two-photon) term in the generalized Rabi model. In contrast to the original Rabi model (i.e., when $\kappa = 0$), three-photon resonance is possible even when RWA applies as illustrated in (a). The coupling scheme shown in (c) was introduced in Ref. [31]. The values near to the arrowed lines are the matrix elements $\langle \alpha_1, n_1 | V | \alpha_2, n_2 \rangle$.

A. General consideration without rotating-wave approximation

For simplicity, we only consider the three-photon resonance between the bare states $|g, 3\rangle$ and $|e, 0\rangle$ in details. It is straightforward to generalize the discussion to any pair of $|g, n_0 + 3\rangle$ and $|e, n_0\rangle$. To the leading order, the one-photon and two-photon terms in the gRM couple $|g, 3\rangle$ to $|e, 5\rangle$, $|e, 4\rangle$, $|e, 2\rangle$, and $|e, 1\rangle$. Similarly, $|e, 0\rangle$ is coupled to $|g, 2\rangle$ and $|g, 1\rangle$. From the discussion in Sec. II A, the resonant frequency is determined as

$$\frac{\omega'}{\omega_a} = \frac{1}{3} + 3 \left( \frac{\lambda}{\omega_a} \right)^2 + 12 \left( \frac{\kappa}{\omega_a} \right)^2 . \tag{7}$$

The same result can be obtained from Eq. (A11) with $n = 3$ and $n_0 = 0$. By employing Eq. (6) and summing the contributions from all three possible coupling schemes in Fig. 1, we obtain the effective coupling strength for the three-photon resonance:

$$\Omega_{3-ph}^{\text{eff}} = - \left[ \frac{18 \sqrt{6} \lambda \kappa^2}{\omega_a^2} + \frac{9 \sqrt{6} \lambda^2 \kappa^2}{\omega_a^2} + \frac{9 \sqrt{6} \lambda^3}{4 \omega_a^2} \right] .$$

Note that the second term in the final result agrees with the result for three-photon resonance in the original Rabi model [31]. We compare the results from perturbation theory and numerical diagonalization.

In Fig. 2, we show the percentage difference between the resonant frequency from Eq. (7) and numerical diagonalization. Also, we obtain numerically the three-photon Rabi splitting at the resonance, i.e., $\Delta = 2 \left| \Omega_{3-ph}^{\text{eff}} \right|$, and
FIG. 3: The energy splitting at the three-photon resonance as a function of $\kappa/\omega_a$. Here, we fix $\lambda/\omega_a = 0.05$. The results for the general case without RWA are denoted by blue solid line (perturbation theory) and red dotted line (numerical diagonalization). Other two lines present results with the application of RWA. The gray dashed line and purple crosses denote results from perturbation theory and numerical simulation, respectively.

its percentage difference from Eq. (8). For $\lambda/\omega_a < 0.1$ and $\kappa/\omega_a < 0.1$, the approximate result for the resonant frequency has a percentage error of 5% or smaller in most of the region. At the same time, there is a considerable region where the approximate energy splitting has a percentage error smaller than 10%. For another illustration, we plot $\Delta/\omega_a$ as a function of $\kappa/\omega_a$ with $\lambda/\omega_a = 0.05$ in Fig. 3. For comparison, both results from numerical simulation and Eq. (6) are included.

B. Rotating-wave approximation

When the atom-photon interaction is weak, the RWA leads to the Jaynes-Cummings-type Hamiltonian:

$$H_{\text{RWA}} = \frac{\omega_a}{2} \sigma_z + \omega_c a^\dagger a + \lambda (a \sigma^+ + a^\dagger \sigma^-) + \kappa \left( a^2 \sigma^+ + a^{+2} \sigma^- \right).$$

(9)

Here, we define the symbols $\sigma^+ = |e\rangle \langle g|$ and $\sigma^- = |g\rangle \langle e|$. From the lowest panel in Fig. 1, it is observed that a three-photon resonance can also occur in the gRM when the RWA applies. This is drastically different from the situation in the original Rabi model, where the resonance can only happen with the presence of counter-rotating terms [31]. From Eq. (6), we deduce the effective coupling strength of the three-photon resonance under RWA as

$$\Omega_{3-\text{ph}, \text{RWA, eff}} = -\frac{18\sqrt{6}\kappa^2 \lambda}{\omega_a^2}. \quad (10)$$

The corresponding resonant frequency is

$$\left( \frac{\omega'}{\omega_a} \right)_{\text{RWA}}^{3-\text{ph}} = \frac{1}{3} + 2 \left( \frac{\lambda}{\omega_a} \right)^2 + 8 \left( \frac{\kappa}{\omega_a} \right)^2. \quad (11)$$

FIG. 4: Three different coupling schemes for the four-photon resonance in the gRM in Eq. (1). Here, counter-rotating processes are labelled by dashed lines. Single (double) lines with arrows indicate coupling between the bare states by the single-photon (two-photon) term in the generalized Rabi model. The values near to the arrowed lines are the matrix elements $\langle \alpha_1, n_1 | V | \alpha_2, n_2 \rangle$.

IV. MULTIPHOTON RESONANCES WITH FOUR TO SIX PHOTONS

The additional two-photon term in the generalized Rabi model leads to the possibility of realizing multiphoton resonances with larger numbers of photons. By limiting to resonances involving two intermediate states, four- to six-photon resonances can be achieved.

A. Four-photon resonance

We begin by considering the four-photon resonance between the bare states $|g, 4\rangle$ and $|e, 0\rangle$. The resonance is interesting for two reasons. First, the two bare states belong to different symmetry classes under $\Pi_1$ and $\Pi_2$. Thus it is impossible to realize the resonance in the Rabi model with only one- or two-photon term. A possible solution is introducing a parity-violating term in the Hamiltonian. This is achievable in circuit QED systems [5]. Nevertheless, a higher-order coupling involving three intermediate states is required. The second novel feature of the four-photon resonance is the necessity of having counter-rotating terms. This is demonstrated in the three
different coupling schemes in Fig. 4. In other words, the resonance disappears when RWA applies. Using Eqs. (5) and (6), we determine the resonant frequency
\[
\left( \frac{\omega_c}{\omega_a} \right)^{4-\text{ph}} = \frac{1}{4} + \frac{8}{3} \left( \frac{\lambda}{\omega_a} \right)^2 + 12 \left( \frac{\kappa}{\omega_a} \right)^2 .
\]
and the effective coupling as
\[
\Omega_{\text{eff}}^{4-\text{ph}} = -\frac{128\sqrt{6}}{9} \left( \frac{\lambda}{\omega_a} \right)^2 \kappa .
\]
In Fig. 5, we plot the energies of fourth excited and fifth excited states of the gRM at \( \omega_c \approx \omega_a / 4, \lambda / \omega_a = 0.05, \) and \( \kappa / \omega_a = 0.01. \) Percentage differences between the results from perturbation theory and numerical diagonalization are shown in Fig 6.

FIG. 5: An avoided crossing in the energy levels of the fourth and fifth excited states in the gRM when \( \omega_c / \omega_a \approx 1/4. \) The Dirac kets show the major components of the eigenstates away from the four-photon resonance. The resonant frequency is determined numerically as \( \omega_c^\prime / \omega_a \approx 0.258, \) with a corresponding energy splitting \( \Delta / \omega_a \approx 1.57 \times 10^{-3}. \) Here, we set \( \lambda / \omega_a = 0.05 \) and \( \kappa / \omega_a = 0.01. \) Note that both \( \lambda \) and \( \kappa \) must be non zero to achieve the resonance.

B. Five- and six-photon resonance

By involving two intermediate states, multiphoton resonances involving five and six photons can also be realized in the gRM. First, consider the five-photon resonance between \( |e, 0\rangle \) and \( |g, 5\rangle. \) To the leading order, these two states can be connected in three different ways:
\[
|e, 0\rangle \rightarrow |g, 1\rangle \rightarrow |e, 3\rangle \rightarrow |g, 5\rangle ,
\]
\[
|e, 0\rangle \rightarrow |g, 2\rangle \rightarrow |e, 3\rangle \rightarrow |g, 5\rangle ,
\]
\[
|e, 0\rangle \rightarrow |g, 2\rangle \rightarrow |e, 4\rangle \rightarrow |g, 5\rangle .
\]
Using Eqs. (5) and (6), it is straightforward to determine the resonant frequency
\[
\left( \frac{\omega_c^\prime}{\omega_a} \right)^{5-\text{ph}} = \frac{1}{5} + \frac{5}{2} \left( \frac{\lambda}{\omega_a} \right)^2 + \frac{40}{3} \left( \frac{\kappa}{\omega_a} \right)^2 .
\]
and the effective coupling strength
\[
\Omega_{\text{eff}}^{5-\text{ph}} = -\frac{125\sqrt{30}\kappa^2\lambda}{9\omega_a^2} .
\]
Similarly, the following leading-order coupling
\[
|e, 0\rangle \rightarrow |g, 2\rangle \rightarrow |e, 4\rangle \rightarrow |g, 6\rangle
\]
leads to a possible six-photon resonance between \( |e, 0\rangle \) and \( |g, 6\rangle. \) The corresponding resonant frequency and the effective coupling are determined as
\[
\left( \frac{\omega_c^\prime}{\omega_a} \right)^{6-\text{ph}} = \frac{1}{6} + \frac{12}{5} \left( \frac{\lambda}{\omega_a} \right)^2 + \frac{15}{2} \left( \frac{\kappa}{\omega_a} \right)^2 .
\]
and

$$\Omega_{\text{eff}}^{\text{ph}} = \frac{-27\sqrt{3} \kappa^3}{\omega_a^2}. \quad (19)$$

In principle, one can obtain similar plots for the percentage difference between the results from perturbation theory and numerical diagonalization. As the number of photons being involved increases, third-order perturbation theory is accurate in a much smaller region of the parameter space. Nevertheless, the multiphoton resonances do exist. At the same time, the perturbation theory approach is still valid given that the atom-photon interaction is sufficiently weak.

Note that different applications of multiphoton resonances have been proposed, such as production of coherent photons [31], simultaneous excitation of several atoms [33], frequency conversion [36], and preparation of different entangled photon states [42, 45]. We expect the four- to six-photon resonances can be applied similarly.

V. EFFECTS OF MULTIPHOTON RESONANCE ON CHIRAL TRANSPORT

It was suggested by Koch and his collaborators [86] that time reversal symmetry can be broken in photonic systems by employing a circuit-QED architecture. An artificial magnetic flux can be realized by inserting simple superconducting circuits into resonator junctions. Specifically, a chiral transfer of photons was predicted in a junction with three resonators. A similar system with the presence of qubits coupled to the resonators was also studied [87]. Nevertheless, the coupling was assumed to be weak, such that the qubit-photon system can be described by the Jaynes-Cummings model. In this section, we investigate how counter-rotating processes may modify the quantum dynamics and photonic transport in the system. In particular, we are interested to examine the interplay between multiphoton resonances and chiral photon transfer in the ultrastrong coupling regime.

Following the original work, we consider a junction formed by three resonators as shown in Fig. 7. Each microwave resonator is coupled to a superconducting qubit. We do not employ RWA here. We further assume that each qubit-resonator system is described by the generalized Rabi model in Eq. (1). In addition, we tune the optical frequencies for all three resonators to the multiphoton-resonant frequency $\omega'_n$, as predicted from the perturbation theory approach in Sec. II A. The value of $\omega'_n$ depends on which multiphoton resonance one wants to study.

Here, the symbol $H_j^\text{RM}$ denotes the generalized Rabi Hamiltonian for the $j$-th resonator. It takes the form in Eq. (1). In addition, we tune the optical frequencies for all three resonators to the multiphoton-resonant frequency $\omega'_n$, as predicted from the perturbation theory approach in Sec. II A. The value of $\omega'_n$ depends on which multiphoton resonance one wants to study.

![FIG. 7: Schematic diagram of the system with a junction connected to three microwave resonators. Each resonator is coupled to a qubit as shown in the inset. Time reversal symmetry is broken by introducing a synthetic magnetic flux in the system. An effective Aharonov-Bohm phase of $3\theta$ is gained when a photon hops around the system.](image)

Before solving the Hamiltonian exactly, we can understand and predict some features of the short-time dynamics of the system. When the dynamics is completely dominated by photon hopping, it suffices to neglect the qubits and focus on the following Hamiltonian

$$H_{\text{hop}} = \sum_{j=1}^{3} \omega_j a_j^\dagger a_j + J \sum_{j=1}^{3} \left( a_{j+1}^\dagger a_j e^{-i\theta} + \text{H.c.} \right). \quad (21)$$

This Hamiltonian can be easily diagonalized. Suppose the TRS is broken by choosing $\theta = \pi/6$. Then, a chiral transfer of photon was predicted if the system has one photon in any one of the resonators initially [86]. It requires a time $t_H = T_H/3 = 2\pi/(3\sqrt{3}J)$ for the photon to hop to the neighbouring resonator. Here, $T_H$ is the period of photon hopping around the whole system. Different from the original work, our initial state has $n$ photons in one of the resonators. If the dynamics of the system is dominated by photon hopping, a chiral transfer of $n$ photons is expected.

On the other hand, one may consider the opposite limit by assuming the photon-hopping effect is negligible. Thus the photons are trapped in the same resonator. Suppose we choose an initial state: $|g,n\rangle$ for the first resonator and $|g,0\rangle$ for the other two resonators. In this case, a multiphoton resonance is expected in the first resonator due to the coupling between the bare states $|g,n\rangle$ and $|e,0\rangle$. When both $\lambda/\omega_a \ll 1$ and $n/\omega_a \ll 1$ are satisfied, the multiphoton resonance can be approximated by

$$H = \sum_{j=1}^{3} H_j^\text{RM} + J \sum_{j=1}^{3} \left( a_{j+1}^\dagger a_j e^{-i\theta} + \text{H.c.} \right). \quad (20)$$
the effective Hamiltonian in Eq. (4). Then, the quantum state of the first resonator evolves approximately as

$$|\psi(t)\rangle \approx e^{-iEt} \left[ \cos(\Omega_{\text{eff}} t) |g, n\rangle + i \sin(\Omega_{\text{eff}} t) |e, 0\rangle \right].$$ (22)

Here, $E = E_i + \Delta E_i = E_f + \Delta E_f$ for the two bare states with the energy corrected by Stark shift. The form of $|\psi(t)\rangle$ suggests that a time interval $t_R = T_R/2 = \pi/(2\Omega_{\text{eff}})$ is required for the qubit to be excited by the photons. We emphasize that both $E$ and $\Omega_{\text{eff}}$ are approximate results from perturbation theory. Thus Eq. (22) has neglected all possibly small but non-zero projections on other bare states. These projections also affect the quantum dynamics of the system, and being not negligible when the qubit-photon interaction becomes sufficiently strong.

From the above discussion, one can define a scaleless parameter $\mu = t_R/\tau_H = 3\sqrt{3}\lambda/(4\Omega_{\text{eff}})$. Depending on $\mu$, the short-time dynamics of the system can transit from a chiral photon transfer to a suppression of photon transfer.

In the following discussion, we use the four-photon resonance as a demonstration. The two extreme limits $\mu \gg 1$ and $\mu \ll 1$ will be discussed separately.

### A. $\mu \gg 1$: chiral transfer of photons

We first consider the scenario when $t_R \gg \tau_H$. In this case, it does not have enough time for the qubit to be excited before the photons are transferred to the next resonator. Hence, the short-time dynamics of the system is dominated by photon hopping. From the previous discussion, a chiral transfer of four photons and unexcited qubits is predicted. Nevertheless, the actual dynamics of the system will be modified by the qubit-photon interaction.

We set $\lambda/\omega_a = 0.05$ and $\kappa/\omega_a = 0.01$ for the strengths of one-photon and two-photon terms (same parameters as in Fig. 5). As a result, $\omega_j \approx 0.25\omega_a$ is predicted for the optical frequency to achieve a four-photon resonance. We set our initial state as $|\Psi(0)\rangle = |g, 4\rangle_1 \otimes |g, 0\rangle_2 \otimes |g, 0\rangle_3$. By setting $\mu = 10$ and $\theta = \pi/6$, we simulate the time evolution of the system numerically. Using the result, we evaluate both $\langle \Psi(t)|a_j^+a_j|\Psi(t)\rangle$ and $\langle \Psi(t)|\sigma_j^+\sigma_j|\Psi(t)\rangle$. For convenience, these quantities are abbreviated as $\langle a_j^+a_j \rangle$ and $\langle \sigma_j^+\sigma_j \rangle$ in the following discussion. The numerical results are shown in Fig. 8.

For the short-time dynamics, the system shows a chiral transfer of photons between the resonators with a period of $T_H$. At the same time, $\langle a_j^+a_j \rangle$ decrease at the modulation from four-photon Rabi oscillations. By increasing $\mu$, the peaks can attain values closer to four. Also, the chiral transfer of photons can persist for a longer period of time. Note that the eigenstates of the qubit-photon system at the multiphoton resonance are not perfectly given by $|\pm\rangle = (|e, 0\rangle \pm |g, 4\rangle)/\sqrt{2}$. There are small projections on other bare states, such as $|e, 1\rangle$, $|e, 2\rangle$, etc. The possibility of exciting the qubits to these bare states also contributes to the decreasing $\langle a_j^+a_j \rangle$ and non-vanishing $\langle \sigma_j^+\sigma_j \rangle$. Their contributions are small and the corresponding oscillations should be much faster.

To investigate the transition out of the short-time dynamics, we simulated the time-evolution of the system for $t \leq 50T_H$. Within this period of time, $\langle \sigma_j^+\sigma_j \rangle$ remain small and do not go beyond 0.1. It indicates that the qubits are nearly unexcited. For a better illustration, we only show $\langle a_j^+a_j \rangle$ with $t \leq 25T_H$ in the lowest panel in Fig. 8. From the figure, we identify the occurrence of transition at $t \approx 11T_H$. Although the amplitudes of $\langle a_j^+a_j \rangle$ are decreasing, photons are transferred among the resonators chirally before the transition. However, this chiral transfer of photons breaks down and the dynamics of the system becomes complicated after the transition.

### B. $\mu \ll 1$: Multiphoton Rabi oscillation and suppression of photon transfer

By changing $\mu$ to 0.1 and keeping all parameters unchanged, we examine the dynamics in the opposite limit. The short-time dynamics of the system is dominated by multiphoton resonance. Once the rotating-wave approximation is made, there will be no multiphoton resonance (except the three-photon resonance when $\kappa \neq 0$). Therefore, the dynamics strongly depends on the counter-rotating processes in the generalized Rabi model.

Using the same initial state as before, we simulate the time evolution of the system for $0 \leq t \leq T_H$. In the short-time regime, four-photon resonance in the first resonator is anticipated. Since $\mu = 0.1$, we have $T_H = 15T_R$. Hence, approximately 15 four-photon Rabi oscillations between $|g, 4\rangle$ and $|e, 0\rangle$ should be observed. Our numerical results are shown in Fig. 9 which support our prediction. Small projections on other bare states and the tiny probability of photon transfer out of the resonator forbid the occurrence of transition $t \approx 7T_H$. Their contributions are small and do not go beyond 0.1. At the same time, the same figure clearly confirms the absence of four-photon resonance under the RWA. Since the optical frequency is largely detuned from the transition frequency of the qubit, the probability of exciting the qubit by one-photon and two-photon co-rotating processes is small. This is demonstrated by the slight modulation of $\langle a_1^+a_1 \rangle$ and the small-amplitude rapid oscillation in $\langle \sigma_1^+\sigma_1^- \rangle$ under RWA.

Another important feature in the short-time dynamics is the small magnitudes of $\langle a_2^+a_2 \rangle$ and $\langle a_3^+a_3 \rangle$. Naively, one may expect photon transfer across the resonator junction would occur within a time scale $t \approx T_H$. However, our numerical result for the dynamics in a longer time regime in Fig. 10 disproves the idea. The figure shows that $\langle a_2^+a_2 \rangle$ and $\langle a_3^+a_3 \rangle$ remain small even when $t \approx 10T_H$. Instead of the original time scale $T_H = T_R/3$, a much longer time $t \approx 30T_H$ is required for photon trans-
FIG. 8: Time evolution of the expectation values of (a) photon numbers $\langle a_j^+ a_j \rangle$ and (b) qubit excitations $\langle \sigma_j^+ \sigma_j^- \rangle$ in the short-time regime. The transition out of the short-time dynamics is illustrated in (c). Here, the initial state of the system is $|\Psi(0)\rangle = |g, 4\rangle_1 \otimes |g, 0\rangle_2 \otimes |g, 0\rangle_3$. The red solid line, green dashed line, and blue dotted line display the values for the first, second, and third resonators, respectively. Here, the parameters are $\omega_c/\omega_a \approx 0.258$, $\lambda/\omega_a = 0.05$, $\kappa/\omega_a = 0.01$, $\mu = 10$, and $\theta = \pi/6$.

FIG. 9: Time evolution of the expectation values of (a) photon numbers $\langle a_j^+ a_j \rangle$ and (b) qubit excitations $\langle \sigma_j^+ \sigma_j^- \rangle$ in the short-time regime. The red solid line, green dashed line, and blue dotted line display the values for the first, second, and third resonators, respectively. For comparison, the expectation values $\langle a_1^+ a_1 \rangle$ and $\langle \sigma_1^+ \sigma_1^- \rangle$ with RWA are shown by the purple fuzzy lines. The same initial state of the system and parameters in Fig. 8 are adopted, except $\mu$ is changed to 0.1.

fer between different resonators. In other words, the process is strongly suppressed by the four-photon resonance in the first resonator. At the same time, there is no preferred chirality in the photon transfer.

C. Possible implication for energy transfer

Suppose the first and the third resonators in Fig. 7 are coupled to heat baths with temperatures $\Theta_h$ and $\Theta_l$, respectively. This coupling introduces dissipation from photon leakage and qubit decay, with the respective decay rates $\gamma_p$ and $\gamma_a$. The non-zero temperature $\Theta$ of the heat bath modifies the effective decay rates by the Bose-Einstein distribution $n_B(\Theta)$. Here, the dissipation is assumed to be weak such that
FIG. 10: Time evolution of the expectation values of (a) photon numbers $\langle a_j^\dagger a_j \rangle$ and (b) qubit excitations $\langle \sigma_j^+ \sigma_j^- \rangle$ in a longer time regime. The same initial state of the system and parameters in Fig. 9 are adopted.

$[n_B(\Theta)\gamma_p]^{-1}, [n_B(\Theta)\gamma_a]^{-1} \gg \min (T_H, T_R)$. Consider the case when $\Theta_h > \Theta_l$, which heat and energy flow in the system. This flow of energy relies on the photon-hopping process between different resonators. Based on the previous results, we discuss qualitatively possible implication for energy transfer in the system.

The non-equilibrium distribution of photon number in each resonator can be very complicated. However, the chiral transfer of photons is affected only when the qubit frequency and photon frequency match the condition of multiphoton resonance. Suppose the condition is satisfied, a chiral transfer of photons is still expected when $\mu \gg 1$. Thus we expect heat flows in the system with a preferred chirality same as the photon hopping discussed in Sec. V A. On the other hand, a similar argument and the result in Sec. V B suggest the suppression of energy flow in the $\mu \ll 1$ regime. A quantitative analysis can be performed by employing the generalized master equation developed in Ref. [41]. This will be addressed elsewhere.

Furthermore, it is tempting to study the effect of photon-atom interaction on energy transport in other photonic systems. For example, a chiral flow of thermal current was reported in a bosonic Hofstadter square lattice without qubits [88]. This flow was also shown to be robust against disorder. It is unclear whether this chiral flow persists or not when photon-atom interaction exists. At the same time, our result may simply suggest that the flow persists whenever the period of photon hopping is much shorter than the period of Rabi oscillation.

VI. CONCLUSIONS

To summarize our work, we studied new kinds of multiphoton resonances originate from the additional two-photon term in the generalized Rabi model. Using third-order perturbation theory, we successfully derived effective Hamiltonians for three- to six-photon resonances. When the atom-photon interaction is sufficiently weak, the results agree with exact diagonalization. Also, we studied photon transfer in a resonator junction with ultrastrong photon-qubit interactions. The time reversal symmetry in the system was broken, which allows photon hopping with a preferred chirality. From the competition between photon hopping and multiphoton Rabi oscillation, we predicted qualitatively the dynamics of the system by identifying the periods of the two processes. When one of the periods is much longer, we shown that either a chiral transfer of photons or a suppression of photon transfer is preferred. Furthermore, we discussed possible implications of our results in heat transport at non-zero temperature. The possibility of realizing cooperative effects from multiphoton resonance and photon hopping will be left for future work.

Our manuscript suggests the interplay between multiphoton resonances and chiral transport in a simple quantum-optical system. This scenario is a unique feature of the Rabi model, which counter-rotating processes cannot be neglected. Many open problems are waiting for future exploration. Generally, can we control chiral transfer of photons and energy in different optical and optomechanical systems by tuning the light-matter interaction? In addition, other topological phases in optical system have been reported. Two examples are quantum spin Hall effect of light [89] and photonic topological insulators [90, 91]. This opens up the question of what kind of symmetry-protected topological (SPT) phases can be realized in optical systems? Also, how will these SPT phases be affected by light-matter interaction?

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Appendix A: An alternative derivation of effective Hamiltonians for multiphoton resonances

In this Appendix, we derive the effective Hamiltonian for multiphoton resonances in the generalized Rabi model by employing the James’ effective Hamiltonian approach [85]. By doing so, we verify the resonant frequency and effective coupling strengths in the main text.

We start by rewriting the gRM Hamiltonian in the interaction picture with frequency and effective coupling strengths in the main text.

We approach (85). By doing so, we verify the resonant frequency and effective coupling strengths in the main text.

Using the above results, it is straightforward to deduce that

\[ H_I(t) = \lambda \left[ a e^{i(\omega_a - \omega_c) t} + a^\dagger e^{i(\omega_a + \omega_c) t} \right] \sigma_+ + \text{H.c.} + \kappa \left[ a^2 e^{i(\omega_a - 2\omega_c) t} + a^{\dagger 2} e^{i(\omega_a + 2\omega_c) t} \right] \sigma_+ + \text{H.c.} \]  

(A4)

Here, \( H_I(t) \) denotes the Hamiltonian for the generalized Rabi model in the interaction picture. For the \( n \)-photon resonance, we set \( \omega_a = n\omega_c \). Then, we have

\[ H_I(t) = \lambda \left[ a e^{i(n-1)\omega_c t} + a^\dagger e^{i(n+1)\omega_c t} \right] \sigma_+ + \text{H.c.} + \kappa \left[ a^2 e^{i(n-2)\omega_c t} + a^{\dagger 2} e^{i(n+2)\omega_c t} \right] \sigma_+ + \text{H.c.} \]  

(A5)

Based on the James’ effective Hamiltonian approach, the effective Hamiltonian for multiphoton resonance can be obtained as [85]:

\[ H_{\text{eff}}(t) = H_{\text{eff}}^{(2)}(t) + H_{\text{eff}}^{(3)}(t) + \cdots \]  

(A6)

Since we only focus on multiphoton resonances involving two intermediate states, a third-order perturbation theory is sufficient. Explicitly, the second order and third order correction terms are

\[ H_{\text{eff}}^{(2)} = \frac{1}{i} \frac{\partial H_I(t)}{\partial t'} H_I(t') dt', \]

(A7)

\[ H_{\text{eff}}^{(3)} = -H_I(t) \int_{t_1}^{t_2} H_I(t_1) dt_2 dt_3. \]

(A8)

For convenience, we denote the photon-number operator \( \hat{N} = a^\dagger a \). By only keeping terms which do not have oscillating phase factors, we obtain

\[ H_{\text{eff}}^{(2)} = \frac{\lambda^2}{(n^2 - 1)\omega_c} \left[ 2n\hat{N} + (n + 1) \right] \sigma_+ \sigma_- + \frac{\lambda^2}{(n^2 - 1)\omega_c} \left[ 2n\hat{N} + (n - 1) \right] \sigma_- \sigma_+ + \frac{2\kappa^2}{(n^2 - 4)\omega_c} \left[ (n + 2) + (n + 4)\hat{N} + n\hat{N}^2 \right] \sigma_+ \sigma_- + \frac{2\kappa^2}{(n^2 - 4)\omega_c} \left[ (n - 2) + (n - 4)\hat{N} + n\hat{N}^2 \right] \sigma_- \sigma_+ \]  

(A9)

For \( H_{\text{eff}}^{(3)} \), the non-oscillating terms depend on the value of \( n \). For example, we choose \( n = 3 \) and obtain the third-order effective Hamiltonian for the three-photon resonance:

\[ H_{\text{eff}}^{(3)} = -\left( \frac{\lambda^3 a^3}{4\omega_c^3} + \frac{\kappa^2 \lambda a^4}{8\omega_c^2} \right) \sigma_- - \left( \frac{\lambda^3 a^3}{4\omega_c^3} + \frac{\kappa^2 \lambda a^4}{8\omega_c^2} \right) \sigma_+ \]  

(A10)

By choosing different values of \( n \), one may also obtain \( H_{\text{eff}}^{(3)} \) for the four-, five-, and six-photon resonances. All results here are consistent with the calculation by Eq. (6) in the main text.

1. Resonant frequency

Consider the reduced Hilbert space formed by bare states \( \left| e, n_0 \right> \) and \( \left| g, n_0 + n \right> \). Then, Eq. (A9) gives a \( 2 \times 2 \) diagonal Hamiltonian. An effective Hamiltonian can be obtained by rotating the result back to the laboratory frame. Then, the resonant frequency for the \( n \)-photon resonance between the bare states \( \left| e, n_0 \right> \) and \( \left| g, n_0 + n \right> \) is derived by equating the two diagonal matrix elements:

\[ \left. \begin{array}{l} \omega_{\frac{n}{n_0}} \\ \omega_{\frac{n}{n_0}} \end{array} \right\}^{n-\text{ph}} = \frac{1}{n} + \frac{2n(n + n + 1)}{n^2 - 1} \left( \frac{\lambda}{\omega_a} \right)^2 + \frac{2n}{n^2 - 4} \left( \frac{\kappa}{\omega_a} \right)^2. \]  

(A11)

Physically, the quadratic correction terms come from the Stark shift in the energy levels. All equations for the resonant frequencies (without RWA) in the main text can be reproduced from Eq. (A11). Similarly, one can deduce the resonant frequency for the three-photon resonance between \( \left| e, n_0 \right> \) and \( \left| g, n_0 + 3 \right> \) under RWA:

\[ \left. \begin{array}{l} \omega_{\frac{n}{n_0}}^{3-\text{ph}} \end{array} \right\}^{3-\text{ph}} = \frac{1}{3} + (n_0 + 2) \left( \frac{\lambda}{\omega_a} \right)^2 + 2(n_0 + 2)^2 \left( \frac{\kappa}{\omega_a} \right)^2. \]  

(A12)

By setting \( n_0 = 0 \), Eq. (11) in the main text is reproduced.
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