Emergent Phenomena in Nature: A Paradox with Theory?

Christiaan J. F. van de Ven

Received: 28 September 2022 / Accepted: 1 August 2023 / Published online: 25 August 2023
© The Author(s) 2023

Abstract
The existence of various physical phenomena stems from the concept called asymptotic emergence, that is, they seem to be exclusively reserved for certain limiting theories. Important examples are spontaneous symmetry breaking (SSB) and phase transitions: these would only occur in the classical or thermodynamic limit of underlying finite quantum systems, since for finite quantum systems, due to the uniqueness of the relevant states, such phenomena are excluded by Theory. In Nature, however, finite quantum systems describing real materials clearly exhibit such effects. In this paper we discuss these apparently “paradoxical” phenomena and outline various ideas and mechanisms that encompass both theory and reality, from physical and mathematical points of view.

Keywords Spontaneous symmetry breaking · Asymptotic emergence · Algebraic quantum theory · Quantum spin system · Schrödinger operator

1 Introduction
Much of modern mathematical physics concerns the search for and formalization of mathematical structures concerning the occurrence of physical phenomena in Nature. Examples include spontaneous symmetry breaking, the theory of phase transitions, Bose-Einstein condensation, and so on. This can be done at various levels of rigor. In any event, unless perhaps in very simple cases, one can not avoid making assumptions and thus will consider a certain model that approaches reality.¹ This for example already happens in the study of the hyperfine structure of the spectrum of the Hydrogen atom, where one often “neglects” the relativistic interactions which are strictly speaking always present. Depending on the purpose of the study such assumptions are usually well founded and therefore they (often) do not affect the description of the physical system of interest.

¹ The precise meaning of “approximation of reality” is an important point in of discussion the philosophy of physics [1–3]. Our starting point is the one due to Butterfield [4], clarified below.
However, describing Nature by means of a suitable theoretical model does not always have to give a correct result consistent with experimental predictions: it can create a serious mismatch between theory and reality. A particular case for which this mismatch leads to significant ambiguity is the manifestation of emergence, that is, according to Butterfield’s first claim [4], a theoretical framework that describes phenomena as having behavior that is new and robust with respect to some comparison class. In this paper we rather focus on asymptotic emergence in the context of limits, in which emergent behavior occurs in a “higher-level” theory being a limit of a sequence of “lower-level” theories, typically as some parameter, e.g. $N$ corresponding to the size of a lattice describing quantum statistical mechanics goes to infinity, or in the spirit of the classical limit, Planck’s constant $\hbar$ to zero. This concept allows studying emergent phenomena inherent in nature in a mathematically rigorous way, namely by the theory of continuous field of $C^*$-algebras and in view of the classical limit, by strict deformation quantization [6].

On the one hand, we emphasize that this notion of emergence may be viewed as a form of reduction (in the sense of deduction), in that, considering $N \to \infty$ (resp. $\hbar \to 0$) enables one to deduce novel and robust behavior compared to that of systems described for finite $N$ (resp. non-zero $\hbar$). This is confirmed by two important examples one encounters in physics, namely spontaneous symmetry breaking (SSB) and phase transitions. Indeed, theoretically speaking SSB and phase transitions can only show up in classical or infinite quantum theories seen as a classical, respectively thermodynamic limit of an underlying finite quantum theory; and these features can be deduced after taking the limit of some parameter, explained in more detail in §2.2 (see also [7–10]).

On the other hand, as we know from decades of experience and experimental results, emergent phenomena such as SSB or phase transitions are observed in real materials, and therefore in finite systems despite the fact that due to uniqueness principles in Theory, they seem forbidden in such systems. This does therefore not show that the limit at $N = \infty$ is “physically real”. The solution to this paradox is based on Butterfield’s second claim [4]

“There is a weaker, yet still vivid, novel and robust behavior that occurs before we get to the limit, i.e. for finite $N$. And it is this weaker behavior which is physically real.”

This can be interpreted in the sense that a robust behavior of the physical phenomenon should occur before the pertinent limit, i.e. in systems described with a finite (respectively, non-zero) parameter, meaning large but finite $N$ in the case of quantum spin systems or tiny but positive values of Planck’s constant $\hbar$ in the case of classical systems. This principle, what we call Butterfield’s Principle, is taken to be our definition of emergence.

This paper is structured in the following way. In Sect. 1.1 we introduce the concept of phase transitions and spontaneous symmetry breaking, first from a more physical point of view often used in condensed matter physics followed by a

---

2 We also refer to the recent book by Palacios [5] for an overview of the topics of emergence and reduction in the philosophy of physics.
mathematical approach common in algebraic quantum theory. In Sect. 2 the algebraic formalism suitable to describe classical and quantum theories is introduced. Using this language in §2.1 the definition of the classical limit is given. In §2.2 the connection with asymptotic emergence is explained and two examples (Examples 2.3 and 2.4) are provided. In Sect. 3 the relation between SSB in Theory versus Nature is discussed. A solution to the above paradox compatible with Butterfield’s Principle is given by a mathematical mechanism and is illustrated by means of numerical simulations. Finally, in §3.3 the analogy between our approach and the measurement problem is outlined. The paper is concluded by posing some open problems.

1.1 Phase Transitions and Spontaneous Symmetry Breaking

Phase transitions are everywhere in nature. They lay the foundation of many physical phenomena, like the formation of Bose-Einstein condensates, superconductivity of metals and so on. The general belief behind a phase transition is the occurrence of discontinuities in thermodynamic functionals and singularities in response functions. From a theoretical point of view these only arise in the relevant limit of underlying finite quantum theories.

Practically speaking, incipient discontinuities of the relevant thermodynamic functionals are already observed in experiments, e.g. as the number of particles increases, most of the change in the magnetization occurs more and more steeply, i.e. it occurs in a smaller and smaller interval around the applied field being zero [11]. Thus the magnetic susceptibility, defined as the derivative of magnetization with respect to magnetic field, is, in the neighborhood of zero, larger for larger $N$, and tends to infinity as $N \to \infty$. However, at finite $N$ no such singularity is present. This confirms Butterfield’s Principle: the singularities which appear from experiment to exist in finite systems are not in fact singularities, but correspond to “robust behavior” encoded by steep maxima (incipient discontinuities) that precede the singular behavior the theory demonstrates in infinite systems.

Another more abstract picture, common in algebraic quantum theory to study phase transitions is based on the celebrated Kubo-Martin-Schwinger (KMS) condition describing thermal equilibrium at fixed inverse temperature $\beta$ [12]. A phase transition is now reformulated by saying that the associated $\beta$-KMS states are non-unique, that is, the coexistence of more than one $\beta$-KMS state for a given dynamics. At positive temperature, however, KMS states are always unique for any finite system: no phase transitions take place [13]. This rather abstract point of view is the usual one in modern mathematical physics; and it applies in general, i.e., it is not model-dependent. There is however also a drawback: this approach studies properties of states, and generally does not consider steep maxima. Consequently, slowly emerging singular behavior is not taken into account and therefore it is not clear at all how to restore Butterfield’s Principle. We will come back to this point in Sect. 3.

Like phase transitions, symmetry and symmetry-breaking effects affect numerous areas and aspects of mathematical and theoretical physics, e.g. it plays a role in
the Higgs mechanism providing mass generation of $W$ and $Z$ bosons, i.e. the gauge bosons that mediate the weak interaction [14]. The general and common concept behind spontaneous symmetry breaking, originating in the field of condensed matter physics where one typically considers the limit of large particle numbers, often also called thermodynamic limit, is based on the idea that if a collection of quantum particles becomes larger, the symmetry of the system as a whole becomes increasingly unstable against small perturbations [15, 16]. A similar statement can be made for quantum systems in their classical limit, where sensitivity against small perturbations now should be understood to hold in the relevant semi-classical regime, meaning that a certain parameter (e.g. $\hbar$) approaches zero,\(^3\) [6, 8].

To show the occurrence of SSB in a certain particle system various approaches of different levels of rigor are used. There are however some differences between the mathematical physics approach to SSB and the methods used in theoretical and condensed matter physics. In the latter, based mainly on the ideas of Bogoliubov, the idea that the system in question becomes sensitive to small perturbations is generally implemented by adding a so-called infinitesimal symmetry-breaking field term to the Hamiltonian, from which one then wants to show that the limit of large particle numbers or the thermodynamic limit becomes “singular” [18, 19], at least at the level of states, e.g. the ground state. If this happens, one says that the symmetry of the limiting system is spontaneously broken. To illustrate this we consider the quantum Ising Hamiltonian $H_{NI}^Q$ with $N$ sites, i.e.

$$H_{NI}^Q : \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \rightarrow \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2; \quad (1.1)$$

$$H_{NI}^Q = -J \sum_{j=1}^{N} \sigma_3(j)\sigma_3(j+1) - B \sum_{j=1}^{N} \sigma_1(j). \quad (1.2)$$

Here $\sigma_k(j)$ stands for $I_2 \otimes \cdots \otimes \sigma_k \otimes \cdots \otimes I_2$, where $\sigma_k$ denotes the $k$-th spin Pauli matrix ($k = 1, 2, 3$) occupying the $j$-th slot, and $J, B \in \mathbb{R}$ are given constants defining the strength of the spin-spin coupling and the (transverse) external magnetic field, respectively. The Hamiltonian $H_{NI}^Q$ is invariant under $\mathbb{Z}_2$-parity symmetry. For finite $N$, it is a well-known fact that the ground state eigenvector is (up to a phase) unique and therefore $\mathbb{Z}_2$-invariant: no SSB occurs. In order to observe SSB in the limit $N \to \infty$ we take $B \in (0, 1)$ and the constant $J$ then may be chosen as one. The symmetry-breaking term is typically taken to be

---

\(^3\) The physical meaning of the limit in Planck’s constant has several interpretations. Mathematically, this parameter is an element of the base space corresponding to a continuous $C^*$-bundle [17]. In the context of the classical limit, the zero-limit of this parameter corresponds to a classical theory, encoded by a commutative $C^*$-algebra. The general and correct formalism to deal with the classical limit is called strict deformation quantization, also called $C^*$-algebraic deformation quantization.
In this approach one argues that the correct order of the limits should be \( \lim_{\epsilon \to 0} \lim_{N \to \infty} \), which gives SSB by one of the two pure ground states on the quasi-local algebra \( \mathcal{A} \cong M_2(\mathbb{C})^{\otimes \infty} \), one of them defined by tensor products of \( | \uparrow \rangle \) and the other by tensor products of \( | \downarrow \rangle \), where \( \sigma_3 | \uparrow \rangle = | \uparrow \rangle \) and \( \sigma_3 | \downarrow \rangle = -| \downarrow \rangle \). The the sign of \( \epsilon \) then determines the direction of symmetry breaking (i.e. which of the limiting states is produced). In contrast, the opposite order \( \lim_{N \to \infty} \lim_{\epsilon \to 0} \) gives a symmetric but mixed ground state on \( \mathcal{A} \). The symmetry is then said to be broken spontaneously if there is a difference in the order of the limits, as exactly happens in this example. In this sense, if SSB occurs, the limit \( N \to \infty \) can indeed be seen as singular [18, 19]. However, this approach gives no insight in the physical mechanism of SSB and has been quite rightly challenged in the philosophical literature of physics [4]. Without going into the mathematical details this paper instead is based on a definition of SSB that is standard in algebraic quantum theory, i.e. it is encapsulated in terms of an algebraic formulation of symmetries and ground states [13, 6]. The advantage of this approach is that it equally applies to finite and infinite systems, and to classical and quantum systems, namely it states that the ground (or equilibrium) state, suitably defined of a system with \( G \)-invariant dynamics (where \( G \) is some group, usually a discrete group or a Lie group) is either pure but not \( G \)-invariant, or \( G \)-invariant but mixed. \(^4\) Subsequently, in the spirit of the above discussion the “singularity” of the thermodynamic limit of systems with SSB is the fact that the exact pure ground state of a finite quantum system converges to a mixed state on the limit system, explained in detail in [7–10]. The use of the word “singularity” here should thus not be confused with a discontinuous or singular limit. All we mean is the inability to decompose the finite quantum state (pure state), where in the limit, however, this is possible.

This definition is therefore compatible with the general belief that SSB only occurs in the relevant limit. However, despite being mathematically correct, this approach does not confirm Butterfield’s Principle, nor does it achieve the main idea that spontaneous symmetry breaking must be related to instability and sensitivity of the system to small perturbations in the relevant regime, manifested by the presence of non-invariant pure states rather than the unstable or non-physically mixed state. It is precisely the aim of this article to provide a natural mechanism that is mathematically correct and physically relevant, combining these (seemly different) notions of symmetry breaking into a self-contained theory.

\(^4\) Even though at first sight both approaches to SSB might seem different, it turns out that in some cases the former approach is related to the algebraic one (see e.g. [21, Prop. 1.6, Lemma 1.7] and [22, Ch. 6]). Whichever approach to symmetry breaking one chooses, each has its pros and cons and which method is chosen depends on the specific purpose.
2 Algebraic Formulation

To answer the paradox in a mathematically rigorous way we need to introduce a $C^*$-algebraic framework necessary to describe states and observables. To this end we focus on the interplay between classical and quantum physics. The major advantage of this algebraic approach is to have a unified rigorous formalism suitable for both classical and quantum theories. Indeed, classical and quantum physics turn out to share the following mathematical structure:

- The observables constitute the (self-adjoint part of) a $C^*$-algebra $\mathfrak{A}$.
- The states are positive linear functionals $\omega : \mathfrak{A} \to \mathbb{C}$ of norm one on $\mathfrak{A}$.
- Any pure state $\omega$ has no nontrivial convex decomposition, i.e., if $\omega = t\omega_1 + (1-t)\omega_2$ for some $t \in (0, 1)$ and certain states $\omega_1$ and $\omega_2$, then $\omega_1 = \omega_2 = \omega$.

The main difference between classical and quantum observables lies in the non-commutativity of the latter.

2.1 Classical Observables

Classical observables are generally considered to be elements of the $C^*$-algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space $X$, denoted by $\mathfrak{A}^c = C_0(X)$, where $X$ is usually a symplectic or more generally a Poisson manifold playing the role of the underlying phase space.\(^5\) The algebraic operations are defined pointwise, i.e. $(fg)(x) = f(x)g(x)$, involution is complex conjugation, and the norm is the supremum-norm denoted by $|| \cdot ||_\infty$. As a result of Riesz Representation Theorem, states $\omega : \mathfrak{A}^c \to \mathbb{C}$ correspond to probability measures $\mu$ on $X$. Pure states on $X$ in turn correspond to Dirac measures $\delta_x$, $x \in X$ defined as $(\delta_x f) = f(x)$, and therefore are in bijection with points in $X$.

2.2 Quantum Observables

Quantum observables are typically elements of a non-commutative $C^*$ algebra, denoted by $\mathfrak{A}^q$. In the case of many situations of physical interest, $\mathfrak{A}^q$ usually corresponds to a a subalgebra of $B(\mathcal{H})$, the bounded operators on a certain Hilbert space $\mathcal{H}$. For example, $\mathfrak{A}^q$ could be the $C^*$-algebra of compact operators $\mathfrak{A}^q = \mathfrak{B}_\infty(L^2(\mathbb{R}^n))$ on the Hilbert space $L^2(\mathbb{R}^n)$, the latter denoting the square integrable functions on $\mathbb{R}^n$. The algebraic relations between operators may be added and multiplied in the obvious way; involution is hermitian conjugation, and the norm is the usual operator norm. Let us for simplicity consider the case where $\mathfrak{A}^q = \mathfrak{B}_\infty(L^2(\mathbb{R}^n))$. Then,

\(^5\) This choice is a matter of mathematically convenience. More options are possible to define a classical $C^*$-algebra of observables. In any case, as all commutative $C^*$-algebras are $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space $X$, there is only one canonical classical observable algebra.
states \( \omega : \mathcal{B}_\infty(L^2(\mathbb{R}^n)) \to \mathbb{C} \) bijectively correspond to density matrices \( \rho \) through
\[
\rho(\alpha) = \text{Tr}(\rho \alpha).
\]
Since a unit vector \( \Psi \) defines a density matrix \( |\Psi\rangle\langle\Psi| \), pure states \( \phi \) on \( \mathcal{B}_\infty(L^2(\mathbb{R}^n)) \) bijectively correspond to unit vectors \( \Psi \) (up to a phase), via
\[
\phi(\alpha) = \langle \Psi, \alpha \Psi \rangle.
\]
(2.1)

A particular choice of such a unit vector is \( \Psi^{(0)} \), which corresponds to the lowest eigenvalue of a \( h \)-dependent quantum Hamiltonian \( H_h \) (assuming its point spectrum is non-empty). Its corresponding algebraic state defined by \( (2.1) \) is called a ground state.

### 2.3 Classical Limit in Algebraic Quantum Theory

The theory of quantum mechanics provides an accurate description of systems containing microscopic particles, but principally quantum mechanics can be applied to any physical system. As known from long-time experience, classical physics, in turn, is a theory that works well with large objects. One may therefore expect that if quantum mechanics is applied to such objects, it reproduces classical results. Roughly speaking, this is what we call the classical limit, and refers to a way of connecting quantum with classical theories,\(^6\) rather than with infinite (non-commutative) quantum systems [23, 24]. With perhaps abuse of notation the classical limit refers to a way of connecting quantum with classical theories,\(^6\)

6 From a \( C^* \)-algebraic point of view central to algebraic quantum theory, the classical limit refers to a rigorous and correct way connecting non-commutative \( C^* \)-algebras \( \mathfrak{A}_h \) (describing quantum theories) with commutative \( C^* \)-algebras \( \mathfrak{A}_0 \) (encoding classical theories) by means of convergence of algebraic states with respect to so-called quantization maps.

Let us carefully introduce this concept from an algebraic point of view starting with the famous example of phase space \( X = \mathbb{R}^{2n} \). We consider the Schrödinger coherent state \( \Psi^{(q,p)} \) labeled by a point \( (q,p) \in \mathbb{R}^{2n} \):
\[
\Psi^{(q,p)}(x) := e^{-\frac{1}{\hbar} p \cdot q / \hbar} e^{ip \cdot x / \hbar} e^{-(x-q)^2 / (2\hbar)} / (\pi \hbar)^{n/4}, \quad (x \in \mathbb{R}^n, \ h > 0).
\]
(2.2)

It is not difficult to prove that \( \Psi^{(q,p)} \) is a unit vector in \( L^2(\mathbb{R}^n, dx) \). The Schrödinger coherent states induce a family of quantization maps, \( f \mapsto Q^B_h(f) \) with \( f \in C_0(\mathbb{R}^{2n}) \) and \( Q^B_h(f) \in \mathcal{B}_\infty(L^2(\mathbb{R}^n)) \) [25], defined by
\[
Q^B_h(f) := \int_{\mathbb{R}^{2n}} f(q,p)|\Psi^{(q,p)}_h\rangle\langle\Psi^{(q,p)}_h| \frac{dq dp}{(2\pi \hbar)^n}, \quad (h > 0);
\]
(2.3)

If \( (\omega_h)_h \) is a family of quantum states defined on \( \mathcal{B}_\infty(L^2(\mathbb{R}^n)) \) and indexed by \( h \) (with \( h \in (0,1] \)), we say that the quantum states have a classical limit if for all (or a subclass) \( f \in C_0(\mathbb{R}^{2n}) \), \( \lim_{h \to 0} \omega_h(Q^B_h(f)) \) exists and defines a classical state on \( C_0(\mathbb{R}^{2n}) \), say \( \omega_0 \). In other words, the family of quantum states \( (\omega_h)_h \) converges to a classical state \( \omega_0 \), if
\[
\lim_{\hbar \to 0} \omega_{\hbar}(Q^B_\hbar(f)) = \omega_0(f),
\]

for all (or a sub-class) \( f \in C_0(\mathbb{R}^{2n}) \). An important class of states one often encounters in physics are vector states induced by eigenvectors \( \psi_h \) of certain Hamiltonians \( H_\hbar \) parameterized by a semi-classical parameter \( \hbar \). Using these states, by Riesz’ Representation Theorem the statement in (2.4) now means that for all \( f \in C_0(\mathbb{R}^{2n}) \) one has

\[
\mu_0(f) = \lim_{\hbar \to 0} \int_{\mathbb{R}^{2n}} f(q,p) d\mu_{\psi_\hbar}(q,p),
\]

where \( \mu_0 \) is the probability measure corresponding to the state \( \omega_0 \) and \( \mu_{\psi_\hbar} \), with \( \hbar > 0 \), is a probability measure on \( \mathbb{R}^{2n} \) with density \( B_{\psi_\hbar}(q,p) := |\langle \Psi_\hbar^{(q,p)} , \psi_\hbar \rangle|^2 \) (where \( \Psi_\hbar^{(q,p)} \) is the Schrödinger coherent state vector) also called the Husimi density function associated to the unit vector \( \psi_\hbar \). In other words \( \mu_{\psi_\hbar} \) is given by

\[
d\mu_{\psi_\hbar}(q,p) = \langle \Psi_\hbar^{(q,p)} , \psi_\hbar \rangle^2 d\mu_{\psi_\hbar}(q,p), \quad ((q,p) \in \mathbb{R}^{2n}),
\]

with \( \mu_{\psi_\hbar} = \frac{dqdp}{(2\pi\hbar)^n} \) the scaled Lebesgue measure on \( \mathbb{R}^{2n} \).

**Remark 2.1** This approach is highly suitable for solving convergence issues regarding the semi-classical behavior of vector states induced by eigenvectors of certain operators. Indeed, such vectors itself do often not have a limit in the pertinent Hilbert space. As the next basic example already shows this problem can be circumvented by the algebraic approach introduced above. \( \blacksquare \)

**Example 2.2** (Classical limit of ground states of the harmonic oscillator) Consider \( V(x) = 1/2 \omega^2 x^2 \) in the quantum Schrödinger Hamiltonian (with mass \( m = 1/2 \)),

\[
H_\hbar = -\hbar^2 \frac{d^2}{dx^2} + V(x),
\]

densely defined on \( L^2(\mathbb{R}) \). It is well known that the ground state is unique and that its wave-function \( \Psi_h^{(0)}(x) \) is a Gaussian peaked above \( x = 0 \). Using the language of the above formalism, it can be shown that the sequence of algebraic ground states \( \omega_h^{(0)}(\cdot) = \langle \Psi_h^{(0)} \cdot \Psi_h^{(0)} \rangle \) converges to the Dirac measure concentrated in the single point point \((0,0) \in \mathbb{R}^{2n}\). However, the ensuing limit of the eigenvector itself corresponds to a Dirac distribution which is not square integrable. \( \blacksquare \)

### 2.4 Spontaneous Symmetry Breaking and Phase Transitions as Emergent Phenomena

We shall now discuss the role of spontaneous symmetry breaking (SSB) in quantum systems and their ensuing limits. As already elucidated in §1.1, the crucial point is that for finite quantum systems the ground state of almost any physically relevant
Hamiltonian is unique\textsuperscript{7} and hence invariant under whatever symmetry group \(G\) it may have \([15, 20]\). Hence, the possibility of SSB, in the usual sense of having a family of asymmetric pure ground states related by the action of \(G\), seems to be reserved for infinite systems. Using the analogy between the thermodynamic limit and the classical limit (Section §2.1), an exact similar situation occurs in quantum mechanics of finite systems (SSB typically absent), versus classical mechanics, which allows it. Therefore, generally speaking, spontaneous symmetry breaking can be seen as natural \textit{emergent phenomenon}\textsuperscript{8} meaning that it only occurs in the limit (be it an infinite quantum or a classical system) of an underlying finite quantum theory.

Let us give two examples of SSB occurring in the classical limit. The first example concerns a certain class of Schrödinger operators where emphasis is put to the classical limit \(\hbar \to 0\) appearing in front of the Laplacian, the second example is the quantum Curie-Weiss model where the parameter \(\hbar \to 0\) is interpreted as the limit of large number of particles \(N\).

\textbf{Example 2.3} (Schrödinger operator) Consider the Schrödinger operator \(H_\hbar\) suitably defined on a dense domain of \(L^2(\mathbb{R}^n)\) by

\[
H_\hbar = -\hbar^2 \Delta + V
\]

where \(\Delta\) is the Laplacian and \(V\) a potential acting by multiplication. We take \(V\) to be of the form \(V(x) = (x^2 - 1)^2\), describing particularly a double well \((n = 1)\) or Mexican hat \((n = 2)\) potential. Clearly \(V\) is invariant under the natural \(SO(n)\)-action, and this property transfers to \(H_\hbar\). Furthermore, by standard results \(H_\hbar\) has compact resolvent \([27]\), and as a result of the (infinite version of the) Perron-Frobenius Theorem, the ground state eigenvector \(\Psi_\hbar(0)\) of the corresponding Schrödinger operator \(H_\hbar\) is (up to phases) \textit{unique}. The following facts are valid.

- The algebraic (pure) ground state \(\omega_\hbar^{(0)}(\cdot) = \langle \Psi_\hbar^{(0)}, \Psi_\hbar^{(0)} \rangle\) is unique and \(SO(n)\)-invariant. Hence, no SSB occurs for any \(\hbar \neq 0\).
- In a similar fashion as in Example 2.2 the classical limit \(\omega_0 = \lim_{\hbar \to 0} \omega_\hbar^{(0)}\) exists; it yields a \(SO(n)\)-invariant mixed state \(\omega_0^{(0)}\), given by

\[
\omega_0^{(0)}(f) = \int_G f(g\sigma_0) d\mu_G(g);
\]

where \(\mu_G\) denotes the associated Haar measure, and \(\sigma_0 \in \mathbb{R}^{2n}\) is an arbitrary point in the level set \(\Sigma_0 := \{\sigma = (q,p) \mid p = 0, ||q|| = 1\}\). In fact, the quantum Hamiltonian \(H_\hbar\) on \(L^2(\mathbb{R}^n)\) has a classical counter part, namely the function \(h_0^{Sch}\) defined on phase space \(\mathbb{R}^{2n}\) by

\footnotesize
\textsuperscript{7} Perhaps the physically most famous exception to this idea is the ferromagnetic quantum Heisenberg model, where the ground state for any finite \(L\) is degenerate.

\textsuperscript{8} We refer to \([26]\) for a detailed discussion on this topic.

\lchk Springer
\[ h_{0}^{\text{Sch}}(q,p) = p^2 + V(q), \ (q,p) \in \mathbb{R}^{2n} \]  

(2.10)

It is precisely the set \( \Sigma_0 \) that corresponds to the minima of \( h_{0}^{\text{Sch}} \), i.e. 
\[ \Sigma_0 = (h_{0}^{\text{Sch}})^{-1}(\{0\}) . \]

- Since the state \( \omega_0^{(0)} \) (cf. (2.9)) defines a \( SO(n) \)-invariant, but mixed ground state one can prove that the \( SO(n) \) symmetry is spontaneously broken.\(^9\)

\[ \text{Example 2.4 (Curie-Weiss model)} \] Consider the quantum Curie-Weiss Hamiltonian \( H_{CW}^N \) with \( N \) sites

\[ H_{CW}^N : \mathbb{C}^2 \bigotimes \cdots \bigotimes \mathbb{C}^2 \rightarrow \mathbb{C}^2 \bigotimes \cdots \bigotimes \mathbb{C}^2 ; \]  

(2.11)

\[ H_{CW}^N = -\frac{J}{2N} \sum_{i,j=1}^{N} \sigma_3(i)\sigma_3(j) - B \sum_{j=1}^{N} \sigma_1(j) . \]  

(2.12)

In order to observe spontaneous symmetry breaking in the relevant limit we take \( B \in (0,1) \) and then we may choose \( J = 1 \). The following holds.

- The (pure) ground state \( \omega_0^{(0)}(\cdot) = \langle \Psi_N^{(0)}, \cdot \Psi_N^{(0)} \rangle \) is unique and \( \mathbb{Z}_2 \)-invariant. Again, no SSB occurs for any \( N < \infty \).
- Similar as in Example 2.3 the classical limit \( \omega_0 = \lim_{h \rightarrow 0} \omega_h^{(0)} \) exists;\(^10\) it yields a \( \mathbb{Z}_2 \)-invariant, but mixed ground state \( \omega_0^{(0)} \in C(B^3) \), given by

\[ \omega_0^{(0)}(f) = \frac{1}{2}(f(x_-) + f(x_+)) , \]  

(2.13)

where \( x_{\pm} \in B^3 \) correspond to the minima of the function \( h_{0}^{\text{CW}}(x,y,z) = -(\frac{J}{2}z^2 + Bx) \), and \( B^3 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \} \) denotes the closed unit-ball \( \mathbb{R}^3 \). Again, following the algebraic definitions it is not difficult to see that the \( \mathbb{Z}_2 \)-symmetry is spontaneously broken.

In view of the previous discussion, we conclude that SSB shows up as “emergent phenomenon” when passing from the quantum realm to the classical world, i.e. by considering a classical limit, \( h \rightarrow 0 \) in Example 2.3 and \( N \rightarrow \infty \) in Example 2.4. This should be understood as emergence in the sense of deduction, as the novel and robust behavior (i.e. SSB) arises only after taking the limit in the relevant parameter.

\(^9\) We refer to \([8, 10]\) for a mathematical discussion on ground states, spontaneous symmetry breaking and the classical limit in algebraic quantum theory, where the latter concept is strongly related to the theory of strict deformation quantization.

\(^10\) The way proving this is to use macroscopic averages which define quantization maps as well \([7, 10]\).
In this algebraic fashion a similar study may be performed for phase transitions. In that case, one must show that the KMS states at inverse temperature, which are unique for finite systems, coexist in infinite systems. This is much harder problem than the study of SSB, since in general no additional symmetry is imposed on the relevant model. One usually tries to show that the quantum free energy function converges, followed by characterizing its minima and analyzing its continuity properties. This goes beyond the scope of this work, but we examine a simple yet illustrative case in §3.2.

3 Theory Versus Nature

The previous examples have shown that SSB is a natural phenomenon emerging in the limit \( N \to \infty \) (where \( N \) typically plays the role of the number of lattice sites) or in the context of Schrödinger operators, in Planck’s constant \( \hbar \to 0 \) (which may be interpreted as a limit in large mass). Indeed, a pure \( G \)-invariant (quantum) ground state typically converges to a mixed \( G \)-invariant (classical) ground state. For phase transitions instead this is a much harder question. We will therefore mainly focus in SSB and step-by-step show how this algebraic approach in a rigorous way leads to the fulfillment Butterfield’s Principle, thereby solving the paradox (cf. §3.1).

First of all, even though this notion of SSB is theoretically correct, it is not the entire story, since according to the general definitions, SSB also occurs whenever pure (or more generally, extreme) ground states are not invariant under the pertinent symmetry group \( G \). It is precisely the latter notion of SSB that one observes in Nature: the limiting ground state is typically measured to be asymmetric in the sense that it is pure but not \( G \)-invariant. This state is therefore also called the “physical” ground state. Starting from this physical idea that the limiting ground state must be pure but not \( G \)-invariant, to prove SSB one should introduce a kind of quantum ground state that converges to a pure but not \( G \)-invariant, hence physical ground state, instead of to the unphysical \( G \)-invariant mixture predicted by the theory.

To get an idea of what this means we first go back to Example 2.3 for Schrödinger operators \( H_\hbar \) with a \( SO(n) \)-invariant potential of the form \( V(x) = (x^2 - 1)^2 \), \( (x \in \mathbb{R}^n) \), i.e. we consider the operator

\[
H_\hbar = -\hbar^2 \frac{d^2}{dx^2} + (x^2 - 1)^2, \tag{3.1}
\]

The symmetry group \( SO(n) \) acts on \( \mathbb{R}^n \) by rotation. This induces the obvious action on the classical phase space \( \mathbb{R}^{2n} \), i.e., \( R(p, q) = (Rp, Rq) \), \( ((p, q) \in \mathbb{R}^{2n}) \), as well as a unitary map \( U \) on the Hilbert space \( \mathcal{H} := \mathcal{H}_\hbar = L^2(\mathbb{R}^n) \) defined by \( U_R \psi(x) = \psi(R^{-1}x) \). Furthermore, as we have seen in Example 2.3, for any \( \hbar > 0 \) and \( n \in \mathbb{N} \) the ground state of this Hamiltonian is unique and hence invariant under the \( SO(n) \) action; with an appropriate phase choice it can be chosen to be real and strictly positive. Yet the associated classical Hamiltonian

\[
h_0^{ch}(p, q) = p^2 + (q^2 - 1)^2, \tag{3.2}
\]
defined on the classical phase space $\mathbb{R}^{2n}$, has a degenerate ground state depending on the dimension $n$. For convenience we focus on two cases, $n = 1$ and $n = 2$. The case $n = 1$, corresponding to the symmetric double well potential with $\mathbb{Z}_2$-symmetry, the minima of $V$ are obviously two-fold degenerate and given by the Dirac measures $\omega_{\pm}$ defined by

$$\omega_\pm(f) = f(p = 0, q \pm 1),$$  \hspace{1cm} (3.3)

where the point(s) $(p = 0, q = \pm 1)$ are the (absolute) minima of $h_{0}^{sch}$. These (pure) states are clearly not $\mathbb{Z}_2$-invariant. Indeed, the $\mathbb{Z}_2$-action on $\mathbb{R}^2$ induced by the non-trivial element of $\mathbb{Z}_2$ is defined by the map $\zeta : x \mapsto -x$, so that $\omega_\pm \mapsto \omega_\mp$.

The case $n = 2$ is the famous Mexican hat potential with $SO(2)$ rotation symmetry. As we have seen before the level set $\Sigma_0 = (h_{0}^{sch})^{-1}(\{0\})$ corresponds to the $SO(2)$-orbit through the points $(|p|| = 0$ and $||q|| = 1)$. This set of “classical” ground states is now connected and forms a circle in phase space. This is the idea of the (classical) Goldstone Theorem, a particle can freely move in this circle at no cost of energy, i.e., the motion in the orbit $\Sigma_0$ is free. Similarly, since each point in phase space is rotated under the $SO(2)$ action, no point in $\Sigma_0$ is $SO(2)$-invariant.

To explain the issue about the existence of spontaneous symmetry breaking in Nature rather than Theory, then translates to the quest for a mechanism that reproduces Dirac measures in the relevant limit. To do this, we summarize two important approaches used in literature.

### 3.1 Anderson’s Approach

A commonly used mechanism to accomplish localization, originating with Anderson (1952) [28, 26], is based on forming symmetry-breaking linear combinations of low-lying states (sometimes called “Anderson’s tower of states” [29]) whose energy difference vanishes in the pertinent limit. In the limit (i.e. either $\hbar \to 0$ or $N \to \infty$) these low-lying eigenstates, still defining a pure state, converge to some symmetry-breaking pure ground state on the relevant limit system. In view of the quantum Ising model introduced in section §1.1 this limiting state corresponds to one of the tensor product states induced by the vectors $|\uparrow\rangle$ or $|\downarrow\rangle$.

In view of Example 2.3 with $SO(n)$ symmetry, the case $n = 1$, precisely yields two eigenstates, namely the ground state $\Psi^{(0)}_h$ and the first excited state $\Psi^{(1)}_h$ with eigenenergies $E^{(0)}_h$ and $E^{(1)}_h$, respectively, such that the difference $\Delta_h := |E^{(0)}_h - E^{(1)}_h|$ vanishes as $\hbar \to 0$. In contrast to the classical limit of the pure algebraic vector state $\omega^{(0)}_h$ corresponding to the exact ground state $\Psi^{(0)}_h$ (the same is true for the first excited state $\omega^{(1)}_h$), viz. the mixed state $\omega^{0} = \frac{1}{2}(\omega^{+}_0 + \omega^{-}_0)$, the low-lying eigenstates defined by $\Psi^{\pm}_h := \frac{\Psi^{(0)}_h \pm \Psi^{(1)}_h}{\sqrt{2}}$ which are now localized, but define pure states $\omega^{\pm}_0$ as well, converge to pure classical states:

$$\lim_{\hbar \to 0} \omega^{\pm}_h = \omega^\pm_0.$$  \hspace{1cm} (3.4)
In this simple example this is clear, because $\Psi_{\pm}$ has a single peak above $\pm 1$ since neither $\Psi_{+}$ nor $\Psi_{-}$ is an energy eigenstate (whereas their limits $\omega_{0}^{+}$ and $\omega_{0}^{-}$ are energy eigenstates, in the classical sense of being fixed points of the Hamiltonian flow). The reason is that the energy difference $\Delta_{\pm}$ vanishes exponentially as $\pm \rightarrow 0$, so that in the classical limit $\Psi_{+}$ and $\Psi_{-}$ approximately do become energy eigenstates.

A similar situation occurs for the Mexican hat potential, i.e. $n = 2$. As opposed to the double well potential, we now have an infinite number of low-lying eigenstates $\{\Psi_{n}(n), n \in \mathbb{Z}\}$, in the sense that they are all eigenfunctions of the operator $H_{\pm}$ and their eigenenergies all coincide in the limit $\pm \rightarrow 0$. One then define the localized wave function by

$$\Psi_{h,\theta}^{N} = \frac{1}{\sqrt{2N + 1}} \sum_{n=-N}^{N} U_{\theta} \psi_{n}(n),$$

where $\theta \in [0, 2\pi)$ parameterizes the $SO(2)$-action on $L^{2}(\mathbb{R}^{2})$. The limit

$$\lim_{N \rightarrow \infty} \Psi_{h,\theta}^{N},$$

does not exists in $L^{2}(\mathbb{R}^{2})$. Instead, the idea is to exploit the techniques explained in §2.1, which makes the unit vectors $\Psi_{h,\theta}^{N}$ converge to some probability measure $\mu_{h,\theta}$ on $\mathbb{R}^{2n}$ as $N \rightarrow \infty$. In the subsequent limit $h \rightarrow 0$, one should then obtain a probability measure $\mu_{0,\theta}$ concentrated on a suitable point in the orbit of classical ground states $\{(p, q) \in \mathbb{R}^{4} | p = 0, |q| = 1\}$ [6].

To summarize, these two examples illustrating Anderson’s approach show that the right physical ground state is obtained in the pertinent limit. Clearly the pertinent pure (not $SO(n)$-invariant) limiting state breaks the $SO(n)$-symmetry.

3.2 Solving the Paradox: case of SSB

Even though Anderson’s approach may yield the right “physical” ground state in the pertinent limit, the low-lying tower of eigenstates itself is not an eigenstate of the original operator, and breaks the pertinent symmetry already for any finite $h$ (resp. any $N$). This approach therefore is rather artificial: apart from the fact that this approach forces a separated limit to be taken, moreover, it still does not account for the fact that in Nature real and hence large and finite materials evidently display spontaneous symmetry breaking whereas on microscopic scale no SSB should be present, i.e. the pure state should be $G$-invariant in such regimes. Another more sophisticated mechanism that extends Anderson’s approach and is compatible with theory as well as nature is introduced below.

3.2.1 Perturbative Approach

In order to provide a natural mechanism compatible with symmetry breaking in Nature, we recall Butterfield’s Principle [4]: “There is a weaker, yet still vivid, novel and robust behavior that occurs before we get to the limit, i.e. for finite $N$. And it is
this weaker behavior which is physically real.” Despite the fact that this was originally phrased in the context of phase transitions, namely that robust behavior for finite systems is not singular but corresponds to steep maxima (what Lavis et al., call incipient singularities [11]), this can can be interpreted for systems exhibiting SSB as well. This means that some approximate and robust form of symmetry breaking should already occur before the pertinent limit (viz. finite $N$ or non-zero $\hbar$), despite the fact that uniqueness of the ground state seems to forbid this. To accomplish this, it must be shown that for finite $N$ or $\hbar > 0$ the system is not in its exact ground state, but in some other state having the property that as $N \to \infty$ or $\hbar \to 0$, it converges in a suitable sense (see the discussion in §2.1 and [9]) to a symmetry-broken ground state of the limit system, which is either an infinite quantum system or a classical system. Since the symmetry of a state is preserved under the limits in question, this implies that the actual physical state at finite $N$ or $\hbar$ must already break the symmetry.

A natural setting taking this into account is based on some form of perturbation theory [30, 20]. Although in a different context, this approach, first introduced by Jona-Lasinio et al. [31] and later called the “Flea on the elephant” by Simon [32], is based on the fact that the exact ground state of a slightly perturbed Hamiltonian approximates the right physical (symmetry-broken) state in such a way that this perturbed ground state is $G$-invariant for relatively large values of $\hbar > 0$ (or similarly, for small values of $N$), but when approaching the relevant limiting regime (i.e. $\hbar \ll 1$ or $N \gg 0$) the perturbed ground state loses its $G$-invariance, and therefore breaks the symmetry already before the ensuing limit. The physical intuition behind this mechanism is that these tiny perturbations should arise naturally and might correspond either to imperfections of the material or contributions to the Hamiltonian from the environment. This is therefore a version of the physical idea that the symmetry of the system in question should become highly sensitive to small perturbations in the relevant regime (viz. Section 1).

To see what this means in a very simple context we again consider the Schrödinger operator defined by (3.1) for $n = 1$. As already mentioned, for any $\hbar > 0$ the (normalized) ground state $\Psi^{(0)}_\hbar \in L^2(\mathbb{R})$ of this Hamiltonian is unique and hence invariant under the $\mathbb{Z}_2$-symmetry $\psi(x) \mapsto \psi(-x)$; with an appropriate phase choice it is real, strictly positive, and doubly peaked above $x = \pm 1$. The classical Hamiltonian $h^{sch}_0 \in C(\mathbb{R}^2)$ has a two-fold degenerate ground state, given by the Dirac measures $\omega_\pm$ defined by (3.3). As predicted by Theory, the symmetric mixed state $\omega_0 = \frac{1}{2}(\omega_+ + \omega_-)$ turned out to be the classical limit of the exact algebraic ground state $\omega_\hbar$ of (3.1) as $\hbar \to 0$. In view of the previous discussions we perturb (3.1) by adding an asymmetric term $\delta V$ (i.e., the “flea”), which, however small it is, under reasonable assumptions localizes the ground state $\Psi^{(\delta)}_\hbar$ of the perturbed state $\Psi^{(0)}_\hbar$ in $L^2([-2, 2])$.

11 We remind the reader that for any $\hbar > 0$ or $N < \infty$ the unperturbed ground state of a generic Hamiltonian is typically $G$-invariant, so that no SSB occurs. Therefore, strictly speaking any approach to symmetry breaking in Nature ($\hbar > 0$ or $N < \infty$) is explicit rather than spontaneous.

12 To visualize the situation properly we restrict the Schrödinger operator to $L^2([-2, 2])$. 
Hamiltonian $H_{\delta} = \delta V$ in such a way that $\omega_{\delta} \to \omega_+ \text{ or } \omega_-$ (rather than the unphysical mixture $\omega_0$) depending on the sign and location of $\delta V$.\textsuperscript{13} According to the results proved in [32] to get localization the perturbed Hamiltonian should have the following properties:

**Property 3.1**

- $\delta V$ is smooth, bounded and vanishes in a neighbourhood of the minima of $V$.
- The support of $\delta V$ is localized at positive distance from the minima of $V$, but not too “far”, meaning that its support is located not farther way than the Agmon distance between the two minimal points (see [32, 20] for technical details).

**Remark 3.2** The first condition of Property 3.1 means that the perturbation should not have any support at the minimum set itself. The physical interpretation is that tiny perturbations should induce symmetry breaking for small values of $\hbar$, but the symmetry of the minimum (or more generally, the level) set itself should be untouched to not affect the physical situation. The second condition means that the perturbation is indeed “effective”. For example, if the perturbation is such that the

\textsuperscript{13} In this approach there is therefore no need for “low-lying eigenstates”, i.e. the exact ground state of the perturbed model already reflects the physical situation as is clearly visible from Fig. 2. Besides, no separate limit has to be taken since localization follows from $\hbar \to 0$. 

![Fig. 1](image_url) The $\mathbb{Z}_2$-symmetric double well potential $V$ with asymmetric perturbation $\delta V$ whose support is indicated by the black circle. The perturbation $\delta V_{b,c,d}$ is defined for the parameters, $b = 0.65$, $c = 0.2$, $d = -0.1$. 

support is located at a distance larger than the distance between both minima, no effect is observed: the ground state eigenvector will remain $\mathbb{Z}_2$-invariant for any $\hbar$. □

Let us consider an example. A natural perturbation $\delta V$ for the Schrödinger operator $H_\hbar$ is

$$\delta V_{b,c,d}(x) = \begin{cases} d \exp \left( \frac{1}{c^2} - \frac{1}{c^2(x-b)^2} \right), & \text{if } |x - b| < c; \\ 0, & \text{if } |x - b| \geq c, \end{cases} \quad (3.7)$$

where the parameters $(b, c, d)$ represent the location of its center $b$, its width $2c$ and its height $d$, respectively. Using this function as perturbation for a suitable choice of these parameters it can be shown that indeed all conditions of Property 3.1 can be
met. This is confirmed in Figs. 1-2, where the perturbed potential $V + \delta V$ and the ground state eigenvector of $H^{(\delta)}$ for several values of $\hbar$ are plotted. From Fig. 2 it is clear that the ground state eigenfunction already localizes for relatively small, but positive values of $\hbar$, whilst for relatively large values of $\hbar$ the state is $\mathbb{Z}_2$-invariant. For such small values of $\hbar$ the ground state is definitely not $\mathbb{Z}_2$-invariant anymore, and therefore it corresponds to the physical (symmetry broken) ground state yielding a Dirac measure in the subsequent limit $\hbar \to 0$. This precisely confirms the essence of the argument and the flea mechanism: SSB is already foreshadowed in quantum mechanics for small yet positive $\hbar$.

**Remark 3.3** It has been numerically checked that for the values of $\hbar$ used to compute the ground state eigenfunction $\Psi^{(\delta)}$ of $H^{(\delta)}$, the ground state energy $E_0^{(\delta)}$ is indeed non-degenerate: no degeneracy effects due to numerical noise has played a role. Therefore, for these values of $\hbar$ (obviously, they depend on the machine precision of the computer used to perform calculations) the simulations accurately reflect the uniqueness of the ground state, proved from general theory on Schrödinger operators.

**Remark 3.4** We stress that this “flea mechanism” has been independently verified for mean-field quantum spin systems in the thermodynamic limit $N \to \infty$ in [20]. In this

---

14 The ground state wave function of the perturbed Hamiltonian (which has two peaks for $\delta V = 0$) localizes in a direction in which the perturbation assumes a relative minimum. For example, if $\delta V$ is positive and is localized to the right, then the relative energy in the left-hand part of the double well is lowered, so that localization will be to the left. If $\delta V$ is negative and localized on the right, then the localization will be to the right, as is the case in Fig. 2.
work a suitable perturbation in spin space has been defined having the same properties as perturbation (3.7). As a result the exact pure ground state of the perturbed quantum spin Hamiltonian converges to a pure state on the (in this case) phase space $B^3 := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$.

The final example for which we show the validity of this mechanism is a $2d$ solid periodic metal. In order to represent a realistic situation, we will not use potentials with vertical walls, but rather a smoothly varying function taken to be a Gaussian potential [33] assuming the form

$$V(x, y) = V_0 e^{-\left(\frac{(x-x_0)^2}{\alpha_x a^2} + \frac{(y-y_0)^2}{\alpha_y a^2}\right)}$$, \hspace{1cm} (3.8)

where $V_0$ represents the maximum depth of the well, $(x_0, y_0)$ are the coordinates of the center of the well, and $\alpha_x$ and $\alpha_y$ are measures of the range of the well in either direction. For the purpose of this paper we focus on the cases $\alpha_x = \alpha_y = a = 1$ and $x_0 = y_0 = 0$. The potential (3.8) represents a single cell and is translated in each direction to model the overall potential of the metal (see Fig. 3). The symmetry group corresponding to such potential is the discrete group $\mathbb{Z}_N$ acting in the obvious way (i.e. by addition) and $N^2$ denotes the total number of atoms. In what follows we consider the $2d$ Schrödinger operator induced by the potential (3.8), i.e.

$$H_h = -\hbar^2 \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) + V(x, y),$$ \hspace{1cm} (3.9)

and we impose periodic boundary conditions and consider a finite domain. By Bloch’s Theorem it follows that solutions to the eigenvalue problem $H_h \psi_h = E_h \psi_h$
exist and take the form of a plane wave modulated by a periodic function. In a similar fashion as Example 2.3 and Example 2.4 one can show that the algebraic vector state induced by the ground state eigenvector is $\mathbb{Z}_N$-invariant and localizes in each single well, as $\hbar \to 0$.

To move on our discussion regarding localization we perturb the potential (3.8) with a “flea-like” perturbation (cf. Property 3.1). In the same spirit as for the double well potential our perturbation is chosen to be strictly positive and bounded away from each of the minima. To avoid numerical complications we have taken the perturbation to be $\delta = 0.1$ precisely on single grid points. An upper view of the perturbed potential is depicted in Fig. 4.

Similarly to the double well potential one can think of this perturbation as having the property that the relative energy of the lattice is minimized exactly towards one cell, but the absolute minima are untouched. This is visible from Fig. 4, the center cell corresponding to $x = y = 0$ is relatively lower in energy then all other cells since the perturbation is positive. It is clear from Fig. 5 that the perturbed ground state localizes at the point $(0, 0)$, as $\hbar \to 0$. Clearly the associated state (a Dirac measure) breaks the relevant symmetry.

As before, it has been numerically checked that for the values of $\hbar$ used to compute the ground state eigenfunction $\Psi^{(\delta)}_\hbar$ of $H^{(\delta)}_\hbar$, the ground state energy $E^{(\delta)}_\hbar$ is non-degenerate. Therefore, localization is not caused by “numerical noise” that typically does play a role when the (strictly speaking) non-degenerate eigenenergies differ by an order of magnitude comparable with the machine precision.

**Remark 3.5** We note that this perturbative approach generalizes Anderson’s mechanism. Indeed, the “flea”-mechanism that causes a pure state to arise in the classical limit has the same effect as the limit of the tower of states: it simply generates it! ■
3.3 Paradox: Case of Phase Transitions

Despite the fact that Butterfield’s Principle has been confirmed in the context of phase transitions, namely by viewing incipient singularities as robust behavior foreshadowing the singular behavior evidenced in infinite systems, the physical (and mathematical correct) idea of the “flea” illustrated above has yet to be found.

Let us say a few words about the difficulties associated with phase transitions, characterized by coexisting states. Similar as for SSB, one may try to introduce suitable “flea”-type perturbations that favor one of these states, in such a way that before the pertinent limit, this “selection process” already takes place. The main issue with phase transitions is that they might occur without the presence of additional symmetry and worse: all excited states may come into play. It is therefore a much harder issue to find suitable “asymmetric” perturbations. To illustrate the latter, we limit ourselves to the $\mathbb{Z}_2$–invariant Curie-Weiss model (Example 2.4), but now for non-zero temperature $\beta$. In a similar fashion as elucidated in Sect. 2.1 the relevant Gibbs state induced by the CW-Hamiltonian (2.12) converges to

$$\omega_0^{(\beta)}(f) = \begin{cases} \frac{1}{2}(\delta_{(0,0,m_\beta/2)}(f) + \delta_{(0,0,-m_\beta/2)}(f)), & (\beta > 4); \\ \delta_{(0,0,0)}, & (\beta \leq 4), \end{cases}$$

(3.10)

where $m_\beta$ is the temperature dependent magnetization, and $\delta_x(f) = f(x)$. The above probability measure is definitely $\mathbb{Z}_2$–invariant, in complete analogy to the case of SSB. Nonetheless, we expect Butterfield’s Principle to be restored through the introduction of asymmetric “flea”-type perturbations to $H_N^{CW}$ that are localized in spin configuration space, although at nonzero temperature all excited states (rather than just the first) will start to play a role; the precise details of the “flea” scenario remain therefore to be settled.

3.4 Comparison with the Measurement Problem

In this last part, we outline the analogy between the flea mechanism and the measurement problem (we refer to [24] for a detailed discussion). In broad terms, the measurement problem consists in the fact that the Schrödinger equation of quantum mechanics fails to predict that measurements have single outcomes. Instead, it empirically predicts unacceptable “superpositions” thereof. The obstacle here is that at first sight such superpositions seem to survive the classical limit, as shown above by the ground state of the double-well potential in the limit $\hbar \to 0$ or by the ground state of the Curie-Weiss model in the limit $N \to \infty$. More generally, the measurement problem arises whenever in the classical limit a pure quantum state converges to a mixed classical state, since in that case quantum theory fails to predict a single measurement outcome; it rather suggests there are many outcomes. Consequently, the real problem is to show that under realistic measurement conditions pure quantum states actually have pure classical limits. As shown in [24] this is brought about precisely by the “flea”-mechanism. Indeed, In the presence of a small perturbation, the perturbed ground state will localize as $\hbar \to 0$, resulting in a pure state (Dirac
measure). Since this depends only on the support of the perturbation, but not on its size, the tiniest of perturbations may cause collapse in the classical limit. This “collapse” of the wave-function in the limit is thus compatible with the measurement problem.

4 Discussion

In this paper we have discussed the concept of spontaneous symmetry breaking and the theory of phase transitions from various points of view.

Without going into technical details we have seen that the notion of the quantization map and more generally algebraic quantum theory is an rigorous framework to study these natural phenomena. Even though mathematically correct, the results proved by such approaches do not always reflect reality; and in particular they violate Butterfield’s Principle. Several methods and mechanisms we discussed are compatible with the physical idea of symmetry breaking in Nature, namely that the limiting state should be pure and not $G$-invariant. Probably the mechanism that best reflects this is the “flea-”mechanism. Indeed, a small perturbation of the Hamiltonian yields the correct “physical” ground state already before reaching the relevant regime which therefore saves Butterfield’s Principle. Without a mathematical proof, this mechanism has been generalized and confirmed by means of a simulation in case of a 2$d$ periodic potential.

However, it is not clear how such a flea should look like for more complex symmetries (e.g. continuous symmetry groups). Attempts have been made by introducing a similar flea perturbation in case of the Mexican hat potential but they did not seem to work: the perturbed ground state does not localize for any $\hbar$. In order to get localization, it is probably not sufficient to use a perturbation of the type (3.7), i.e. its support is defined on larger sets. This raises the question whether or not such a perturbation is “natural” in the sense that it should resemble a tiny perturbation of the material.

In the case of phase transitions at non-zero temperature the same issue remains: due to the in general absence of additional symmetry and the interplay of all excited states it is not clear how to precisely define the flea “scenario”. Whether this approach can resolve the paradox between Nature and Theory therefore remains an open question.

Acknowledgements The research was financially supported by a postdoctoral fellowship granted by the Alexander von Humboldt Foundation (Germany). The author particularly thanks the referee for providing useful and fruitful comments, as well as Nicolò Drago for giving feedback on technical issues.

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is
References

1. Ardourel, V.: The infinite limit as an eliminable approximation for phase transitions. Stud. Hist. Philos. Sci. Part B 62, 71–84 (2018)
2. Norton, J.D.: Infinite idealizations. In: European Philosophy of Science-Philosophy of Science in Europe and the Viennese Heritage, pp. 197–210 (2014)
3. Wu, J.: Explaining universality: infinite limit systems in the renormalization group method. Synthese 199(5–6), 14897–14930 (2020)
4. Butterfield, J.: Less is different: emergence and reduction reconciled. Found. Phys. 41, 1065–1135 (2011)
5. Palacios, P.: Emergence and Reduction in Physics. Cambridge University Press, Cambridge (2022)
6. Landsman, K.: Foundations of Quantum Theory: From Classical Concepts to Operator Algebras. Springer (2017). Open Access at http://www.springer.com/gp/book/9783319517766
7. Landsman, K., Moretti, V., van de Ven, C.J.F.: Strict deformation quantization of the state space of $M_4(C)$ with applications to the Curie-Weiss model. Rev. Math. Phys. 32(10), 2050031 (2020)
8. Moretti, V., van de Ven, C.J.F.: The classical limit of Schrödinger operators in the framework of Berezin quantization and spontaneous symmetry breaking as emergent phenomenon. Int. J. Geom. Methods Mod. Phys. 19(01), 2250003 (2022)
9. van de Ven, C.J.F.: The classical limit and spontaneous symmetry breaking in algebraic quantum theory. Expos. Math. 40(3), 101016 (2022)
10. van de Ven, C.J.F.: The classical limit of mean-field quantum theories. J. Math. Phys. 61, 121901 (2020)
11. Lavis, D.A., Kühn, R., Frigg, R.: Becoming Large, Becoming Infinite: The Anatomy of Thermal Physics and Phase Transitions in Finite Systems. Found. Phys. 51(5), 1–69 (2021)
12. Haag, R., Hugenholtz, N.M., Winnink, M.: On the equilibrium states in quantum statistical mechanics. Commun. Math. Phys. 5(3), 215–236 (1967)
13. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics. Vol. 1, II: Equilibrium States, Models in Statistical Mechanics. Springer, New York (1981)
14. Ruelle, D.: Natural nonequilibrium states in quantum statistical mechanics. J. Stat. Phys. 98, 57–75 (2000)
15. Landsman, N.P.: Spontaneous symmetry breaking in quantum systems: emergence or reduction? Stud. Hist. Philos. Sci. Part B 44(4), 379–394 (2013)
16. van Wezel, J., van den Brink, J.: Spontaneous symmetry breaking in quantum mechanics. Am. J. Phys. 75, 635 (2007). https://doi.org/10.1119/1.2730839
17. Dixmier, J.: C*-algebras. North-Holland (1977)
18. Batterman, R.: The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence. Oxford University Press, Oxford (2002)
19. Berry, M.V.: Singular limits. Phys. Today 55, 10–11 (2002)
20. van de Ven, C.J.F., Groenenboom, G.C., Reuvers, R., Landsman, N.P.: Quantum spin systems versus Schrödinger operators: A case study in spontaneous symmetry breaking. SciPost (2019)
21. Ueltzchi, D.: Quantum spin systems and phase transitions, Marseille lectures, (2013). Available at http://www.ueltschi.org/publications.php
22. Ruelle, D.: Statistical Mechanics: Rigorous Results. World Scientific, London (1999)
23. Landsman, N.P.: Between classical and quantum, Handbook of the Philosophy of Science Vol. 2: Philosophy of Physics. In: Butterfield, J., Earman, J. (eds.) Part A, pp. 417-554. Elsevier, Amsterdam (2008)
24. Landsman, N.P.: A flea on Schrödinger’s cat. Found. Phys. 43, 373–407 (2013)
25. Berezin, F.A.: General concept of quantization. Commun. Math. Phys. 40, 153–174 (1975)
26. Anderson, P.W.: Basic Notions of Condensed Matter Physics. Taylor & Francis, New York (1984)
27. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, vol. IV. Academic Press, Washington, DC (1975)
28. Anderson, P.W.: An approximate quantum theory of the antiferromagnetic ground state, 811 Phys. Rev. 86, 694 (1952)
29. Tasaki, H.: Long-range order, “Tower” of states, and symmetry breaking in lattice quantum systems. J. Stat. Phys. 174, 735–761 (2019)
30. Fraser, J.D.: Spontaneous symmetry breaking in finite systems. Philos. Sci. 83(4), 585–605 (2016)
31. Jona-Lasinio, G., Martinelli, F., Scoppola, E.: New approach to the semiclassical limit of quantum mechanics. Commun. Math. Phys. 80, 223 (1981)
32. Simon, B.: Semiclassical analysis of low lying eigenvalues. IV. The flea on the elephant. J. Funct. Anal. 63, 123 (1985)
33. Pavelich, R.L., Marsiglio, F.: Calculation of 2D electronic band structure using matrix mechanics. Am. J. Phys. 84(12), 924–935 (2016)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.