HOMOGENEOUS HIGGS AND CO-HIGGS BUNDLES ON HERMITIAN SYMMETRIC SPACES

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ABSTRACT. We define homogeneous principal Higgs and co-Higgs bundles over irreducible Hermitian symmetric spaces of compact type. We provide a classification for each type of object up to isomorphism, which in each case can be interpreted as defining a moduli space.

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1. Introduction

In contrast to parabolic Higgs bundles and other types of meromorphic Higgs bundles, co-Higgs bundles are an extension of the theory of Higgs bundles to the Fano end of the Kodaira spectrum that does not require considering open varieties. As in [Ra1], a co-Higgs bundle on a complex variety $X$ is a holomorphic bundle $E$ with a co-Higgs field $\theta : E \to E \otimes T^{1,0}X$ such that the section $\theta \wedge \theta \in H^0(X, \operatorname{End}(E) \otimes \wedge^2 T^{1,0}X)$ vanishes identically.

These objects bear a formal similarity to the Higgs bundles of [Hi1, Hi2, Si]. For a Higgs bundle, there is a Higgs field $\theta : E \to E \otimes \Omega^{1,0}X$ with $\theta \wedge \theta = 0$. The symmetry condition is required in higher dimensions and is the analogue of the integrability condition.
introduced in [Si]. Higgs bundles have been investigated over the past thirty years in what is by now a large body of work. The interest in co-Higgs bundles is more recent. While Higgs bundles originate from considerations of invariance and self-duality in 4D Yang-Mills gauge theory [Hi1], co-Higgs bundles arose out of generalized complex geometry [Gu, Hi3] as a kind of limit of generalized holomorphic bundles when generalized complex structures become ordinary complex ones. Co-Higgs bundles and their moduli have been investigated on $\mathbb{P}^1$ [Ra2, Ra4, RS, BG3]; on $\mathbb{P}^2$ [Ra3]; on $\mathbb{P}^1 \times \mathbb{P}^1$ [VC]; on the moduli space of stable vector bundles on a complex curve of genus at least 2 [BR], which is a compact Fano variety (motivated by constructions in [FW]); on Calabi-Yau manifolds [BB]; and in various settings and levels of generality by [Py, HMS, BH1, BH2, Co1, Co2].

In the present work, we examine both Higgs and co-Higgs bundles on compact Hermitian symmetric spaces $M$. Investigations in this direction for Higgs bundles had commenced in [BG1, BG1]. To continue this, we take advantage of the identification of each such $M$ with the quotient $G/K$ where $G$ is the simply-connected covering of the group of holomorphic isometries of $M$ and $K$ is the isotropy subgroup of a fixed base point $x_0 \in M$. We can recover $M$ as the orbit of the base point $x_0 \cong K/K$ under the canonical action of $G$. We define natural notions of Higgs and co-Higgs bundles adapted to such a space. These are the homogeneous Higgs bundles (respectively, homogeneous co-Higgs bundles) in which the underlying bundle is a homogeneous holomorphic principal $H$-bundle $E_H$, where $H$ is a connected complex group, and where $\theta$ is regarded as a global 1-form-twisted section (respectively, vector-field-twisted section) of the adjoint bundle of $E_H$ that is furthermore invariant under the action of $G$. We proceed to classify $G$-holomorphic structures on smooth homogeneous principal $H$-bundles $E_H \to M \cong G/K$. For us, $G$-holomorphic means that the self-map of the total space of $E_H$ induced by the action of each $g \in G$ is holomorphic. We refer to these simply as invariant holomorphic structures. In particular, we prove that every homogeneous bundle $E_H$ has a canonical invariant holomorphic structure and that this structure arises from a particular invariant connection on $E_H$. This is Theorem (4.1). We subsequently provide a Lie-algebraic characterization (in terms of the vanishing of a natural tri-linear map on $\mathfrak{h} = \text{Lie}(H)$) for when an invariant almost complex structure on the bundle is integrable (see Proposition (4.2)). Taken together, these statements results in a completely Lie-theoretic characterization of isomorphism classes of (based) homogeneous, invariant holomorphic principal-$H$ bundles in Theorem (4.3).

This analysis of invariant connections and integrable holomorphic structures on smooth bundles $E_H$ enables us to classify homogeneous principal (co-)Higgs bundles on $G/K$ up to isomorphism using exclusively Lie-theoretic data (see Theorem (5.1), Corollary (5.2), Proposition (6.1) and Corollary (6.2). The resulting descriptions can be interpreted as equations for defining algebraic moduli varieties within linear spaces arising directly from $G$ and $H$.
2. Homogeneous bundles on Hermitian symmetric spaces

Let $M$ be an irreducible Hermitian symmetric space of compact type. Fix a base point $x_0 \in M$. Let $G$ denote the simply-connected covering of the group of holomorphic isometries of $M$. Therefore, $G$ has a tautological action on $M$. Let $K < G$ be the isotropy of the point $x_0$. Consequently, we have $M = G/K$, and the point $x_0$ corresponds to $K/K \in G/K$. We shall identify $(M, x_0)$ with $(G/K, K/K)$ by considering $M$ as the orbit of $x_0$ under the tautological action of $G$ on $M$. This identification between $(M, x_0)$ and $(G/K, K/K)$ takes the action of any $g \in G$ on $M$ to the left–translation action

$$t_g : G/K \to G/K, \quad g'K \mapsto gg'K. \quad (2.1)$$

Note that $t_g$ is an holomorphic isometry.

The quotient map

$$q : G \to G/K \quad (2.2)$$

defines a $C^\infty$ principal $K$–bundle over $G/K$. This principal $K$–bundle over $G/K$ will be denoted by $\mathbb{G}$.

**Definition 2.1.** Let $\mathcal{H}$ be a Lie group. A homogeneous $C^\infty$ principal $\mathcal{H}$–bundle over $G/K$ is a pair of the form $(E_\mathcal{H}, \rho)$, where $f : E_\mathcal{H} \to G/K$ is a $C^\infty$ principal $\mathcal{H}$–bundle and

$$\rho : G \times E_\mathcal{H} \to E_\mathcal{H}$$

is a $C^\infty$ left–action of the group $G$ on $E_\mathcal{H}$ satisfying the following two conditions:

1. $f(\rho(g, z)) = t_g(f(z))$, for all $g \in G$, $z \in E_\mathcal{H}$, where $t_g$ is the automorphism of $G/K$ in (2.1), and
2. the actions of $G$ and $\mathcal{H}$ on $E_\mathcal{H}$ commute.

Therefore, a homogeneous $C^\infty$ principal $\mathcal{H}$–bundle $E_\mathcal{H}$ is equipped with an action of $G \times \mathcal{H}$, with $G$ acting on the left of $E_\mathcal{H}$ and $\mathcal{H}$ acting on its right.

An isomorphism between two homogeneous $\mathcal{H}$–bundles

$$f : E_\mathcal{H} \to G/K \quad \text{and} \quad f' : E'_\mathcal{H} \to G/K$$

is a diffeomorphism

$$\delta : E_\mathcal{H} \to E'_\mathcal{H}$$

that satisfies the following two conditions:

- $\delta$ is $G \times \mathcal{H}$–equivariant for the actions of $G \times \mathcal{H}$ on $E_\mathcal{H}$ and $E'_\mathcal{H}$, and
- $f' \circ \delta = f$.

Two homogeneous $C^\infty$ principal bundles are called isomorphic if there is an isomorphism between them.

A based homogeneous $C^\infty$ principal $\mathcal{H}$–bundle over $G/K$ is a homogeneous $C^\infty$ principal $\mathcal{H}$–bundle $(E_\mathcal{H}, \rho)$ over $G/K$ together with a point $z \in (E_\mathcal{H})_{K/K}$ in the fiber of $E_\mathcal{H}$ over the point $K/K$. An isomorphism between two based homogeneous $C^\infty$ principal
$H$–bundles $(E_H, \rho, z)$ and $(E'_H, \rho', z')$ is an isomorphism of homogeneous $C^\infty$ principal bundles

$$\delta : E_H \rightarrow E'_H$$

such that $\delta(z) = z'$.

Note that the left–translation action of $G$ on itself makes the principal $K$–bundle $G$ in (2.2) a homogeneous principal $K$–bundle. The identity element $e$ of $G = G$ makes it a based homogeneous principal $K$–bundle.

**Lemma 2.2.** The based homogeneous principal $H$–bundles on $G/K$ are in bijection with the homomorphisms from $K$ to $H$.

**Proof.** First take a homomorphism $\eta : K \rightarrow H$. Consider the principal $K$–bundle $G$ in (2.2). Let $G_\eta$ be the principal $H$–bundle on $G/K$ obtained by extending the structure group of $G$ using $\eta$. So, $G_\eta$ is the quotient of $G \times H$ where two elements $(g_1, h_1), (g_2, h_2) \in G \times H$ are identified if there is an element $k \in K$ such that $g_2 = g_1 k$ and $h_2 = \eta(k)^{-1} h_1$.

Consider the action of $G$ on $G \times H$ given by the left–translation action of $G$ on itself and the trivial action of $G$ on $H$. This action of $G$ on $G \times H$ produces an action of $G$ on $G_\eta$. The resulting action of $G$ on $G_\eta$ makes $G_\eta$ a homogeneous principal $H$–bundle on $G/K$.

The image in $G_\eta$ of the point $e \times e_H \in G \times H$, where $e_H$ is the identity element of $H$, is a point in the fiber of $G_\eta$ over $K/K$, so $(G_\eta, e \times e_H)$ is a based homogeneous principal $H$–bundle.

For the converse, take a homogeneous principal $H$–bundle $(E_H, \rho)$ on $G/K$ together with a point $z_0$ in the fiber of $E_H$ over the point $K/K \in G/K$. For any element $k \in K$, let $\eta(k) \in H$ be the unique element that satisfies the equation

$$\rho(k, z_0) = z_0 \eta(k);$$

note that since $\rho(k, z_0)$ lies in the fiber of $E_H$ over the point $K/K \in G/K$, there is a unique such $\eta(k)$. Now, for $k, k' \in K$, we have

$$z_0 \eta(kk') = \rho(kk', z_0) = \rho(k, \rho(k', z_0)) = \rho(k, z_0 \eta(k')) = \rho(k, z_0 \eta(k') \eta(k')) = \rho(k, z_0 \eta(k')) \eta(k').$$

This implies that the map

$$\eta : K \rightarrow H, \ k \mapsto \eta(k),$$

is a homomorphism of groups.

For the above homomorphism $\eta$ consider the based principal $H$–bundle $(G_\eta, e \times e_H)$ constructed earlier from $\eta$. We shall show that $(G_\eta, e \times e_H)$ is identified with $(E_H, z_0)$. For this consider the map

$$\eta' : G \times H \rightarrow E_H$$

that sends any $(g, h) \in G \times H$ to $\rho(g, z_0 h)$. It can be shown that $\eta'$ descends to a map

$$\eta'' : G_\eta \rightarrow E_H$$

(2.5)
from the quotient $\mathbb{G}_\eta$ of $G \times \mathcal{H}$. Indeed, for any $(k, g, h) \in K \times G \times \mathcal{H}$, we have
\[
\eta'(gk, \eta(k)^{-1}h) = \rho(gk, z_0\eta(k)^{-1}h) = \rho(g, \rho(k, z_0))\eta(k)^{-1}h
\]
\[
= \rho(g, z_0\eta(k))\eta(k)^{-1}h = \rho(g, z_0)\eta(k)\eta(k)^{-1}h = \rho(g, z_0)h = \rho(g, z_0h) = \eta'(g, h).
\]
Therefore, $\eta'$ descends to a map $\eta''$ as in (2.5). This map $\eta''$ in (2.5) is an isomorphism of homogeneous principal $\mathcal{H}$–bundles. Note that $\eta''$ clearly sends $e \times e_\mathcal{H}$ to $z_0$.

Conversely, take a homomorphism $\eta : K \to \mathcal{H}$. Let $(E_\mathcal{H}, \rho)$ be the based homogeneous principal $\mathcal{H}$–bundle constructed as above using it. Then the homomorphism $K \to \mathcal{H}$ constructed as in (2.3) for this based homogeneous principal $\mathcal{H}$–bundle clearly coincides with $\eta$.

Consequently, the above two constructions, between the space of homomorphisms from $K$ to $\mathcal{H}$ and the space of based homogeneous $\mathcal{H}$–bundles, are inverses of each other. □

3. A Tautological Connection

A connection on a principal $\mathcal{H}$–bundle $f : E_\mathcal{H} \to G/K$ is a $C^\infty$ $\mathcal{H}$–invariant distribution $\mathcal{D} \subset T E_\mathcal{H}$ such that the natural homomorphism $\mathcal{D} \oplus \ker(df) \to T E_\mathcal{H}$ is an isomorphism, where $df : T E_\mathcal{H} \to f^*T(G/K)$ is the differential of $f$. Let $\mathcal{D}$ be the orthogonal complement of $\ker(df)$.

Take a homogeneous principal $\mathcal{H}$–bundle $(E_\mathcal{H}, \rho)$ on $G/K$. For any $g \in G$, let $\rho_g$ be the diffeomorphism of $E_\mathcal{H}$ defined by $z \mapsto \rho(g, z)$. This $\rho_g$ is $\mathcal{H}$–equivariant; more precisely, it is an automorphism of the principal $\mathcal{H}$–bundle $E_\mathcal{H}$ over the biholomorphism $t_g$ of $G/K$ defined in (2.1). Let $C(E_\mathcal{H})$ denote the space of all connections on the principal $\mathcal{H}$–bundle $E_\mathcal{H}$. The group $G$ acts on $C(E_\mathcal{H})$ as follows: the action of any $g \in G$ sends the connection defined by a distribution $\mathcal{D} \subset T E_\mathcal{H}$ to the connection
\[
d\rho_g(\mathcal{D}) \subset T E_\mathcal{H},
\]
where $d\rho_g : T E_\mathcal{H} \to T E_\mathcal{H}$ is the differential of the map $\rho_g$. Let
\[
C(E_\mathcal{H})^G \subset C(E_\mathcal{H})
\]
be the fixed point locus for this action of $G$ on $C(E_\mathcal{H})$.

Consider the homogeneous principal $K$–bundle $\mathbb{G}$ in (2.2). We shall show that it has a tautological $G$–invariant connection.

Let $\mathfrak{g}$ (respectively, $\mathfrak{k}$) be the Lie algebra of $G$ (respectively, $K$). Both $\mathfrak{g}$ and $\mathfrak{k}$ are $K$–modules by the adjoint action. The Killing form on $\mathfrak{g}$ is non-degenerate. Let
\[
\mathfrak{p} := \mathfrak{k}^\perp \subset \mathfrak{g}
\]
be the orthogonal complement of $\mathfrak{k}$ for the Killing form on $\mathfrak{g}$. The adjoint action of $K$ on $\mathfrak{g}$ preserves $\mathfrak{p}$, because the Killing form on $\mathfrak{g}$ is $K$–invariant. Therefore, the natural homomorphism
\[
\mathfrak{k} \oplus \mathfrak{p} \to \mathfrak{g}
\]
is an isomorphism of $K$–modules.
Now the translations of $\mathfrak{p}$ by the left–translation action of $G$ on itself define a distribution

$$D \subset TG.$$

This $D$ is preserved by the right–translation action of $K$ on $G$ because the decomposition in (3.2) is an isomorphism of $K$–modules. From this it follows that $D$ defines a connection on the principal $K$–bundle $G$ in (2.2). This connection on $G$ will be denoted by $\nabla^0$. Since $D$ is preserved by the left–translation action of $G$ on itself, we conclude that

$$\nabla^0 \in C(G)^G$$

(see (3.1)).

Consider the Lie bracket operation composed with the projection to the direct summand $\mathfrak{k}$ in (3.2)

$$\mathfrak{p} \otimes \mathfrak{p} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{k}.$$

(3.4)

The tangent bundle $T(G/K)$ is the vector bundle over $G/K$ associated to the principal $K$–bundle $G$ in (2.2) for the adjoint action of $K$ on $\mathfrak{p}$. Therefore, the composition homomorphism in (3.4), which is $K$–equivariant, define a $C^\infty$ two–form on $G/K$ with values in the adjoint vector bundle $\text{ad}(G)$. This $\text{ad}(G)$–valued two–form on $G/K$ is the curvature of the above connection $\nabla^0$. We shall denote the curvature of $\nabla^0$ by $\mathcal{K}(\nabla^0)$.

The center of $K$ will be denoted by $Z_K$; it is isomorphic to $\text{U}(1)$, because the Hermitian symmetric space $G/K$ is irreducible. Consider the action of $Z_K$ on the complexification of $\mathfrak{p}$. Let

$$\mathfrak{p}^C := \mathfrak{p} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

be the isotypical decomposition for the action of $Z_K$ on $\mathfrak{p}^C$. Note that

$$\mathfrak{p}^C = T_{K/K}(G/K) \otimes_\mathbb{R} \mathbb{C}.$$

The type decomposition given by the complex structure on $G/K$

$$T_{K/K}(G/K) \otimes_\mathbb{R} \mathbb{C} = T_{K/K}^{1,0}(G/K) \oplus T_{K/K}^{0,1}(G/K)$$

coincides with the decomposition in (3.5); the complex subspace $\mathfrak{p}_+$ (respectively, $\mathfrak{p}_-$) of $\mathfrak{p} \otimes_\mathbb{R} \mathbb{C}$ coincides with $T_{K/K}^{1,0}(G/K)$ (respectively, $T_{K/K}^{0,1}(G/K)$) by the above isomorphism $\mathfrak{p}^C = T_{K/K}(G/K) \otimes_\mathbb{R} \mathbb{C}$. (See [He] for the details.)

The complexification of the composition in (3.4) vanishes on $\mathfrak{p}_+ \otimes \mathfrak{p}_+$ and $\mathfrak{p}_- \otimes \mathfrak{p}_-$. From this it follows immediately that both the $(2, 0)$ and $(0, 2)$ type components of the curvature $\mathcal{K}(\nabla^0)(K/K)$ of $\nabla^0$ vanish (at the point $K/K \in G/K$).

Note that from the fact that the connection $\nabla^0$ is $G$–invariant (see (3.3)) it follows immediately that the curvature $\mathcal{K}(\nabla^0)$ is preserved by the action of $G$. Hence $\mathcal{K}(\nabla^0)$ is an $\text{ad}(G)$–valued form of of Hodge type $(1, 1)$ on $G/K$, because it is of type $(1, 1)$ at the point $K/K \in G/K$ and the action of $G$ on $G/K$ is transitive.
4. Invariant holomorphic structures

Let $H$ be a connected complex Lie group. The Lie algebra of $H$ will be denoted by $\mathfrak{h}$. A holomorphic structure on a $C^\infty$ principal $H$–bundle

$$f : E_H \to G/K \quad (4.1)$$

is a complex structure on the manifold $E_H$ such that the projection $f$ is holomorphic and the map $E_H \times H \to E_H$ giving the action of $H$ on $E_H$ is holomorphic. A holomorphic principal $H$–bundle is a $C^\infty$ principal $H$–bundle equipped with a holomorphic structure.

Now let $E_H$ be homogeneous. A holomorphic structure on $E_H$ is called invariant if for every $g \in G$ the self-map of $E_H$ given by the action of $g$ on it is holomorphic.

**Theorem 4.1.** Any $C^\infty$ homogeneous principal $H$–bundle $E_H \to G/K$ has a tautological invariant connection $\nabla^{E_H} \in C(E_H)^G$ (defined in (3.1)). This connection $\nabla^{E_H}$ produces an invariant holomorphic structure on $E_H$.

**Proof.** The action of $G$ on $E_H$ produces a Lie algebra homomorphism

$$\delta : \text{Lie}(G) =: \mathfrak{g} \to C^\infty(E_H, TE_H)$$

to the $C^\infty$ vector fields on $E_H$. Let

$$\mathcal{D} := \delta(p) \subset TE_H \quad (4.2)$$

be the distribution defined by the image of the subspace $\mathfrak{p}$ in (3.2). Since the actions of $G$ and $H$ on $E_H$ commute, for any $v \in \mathfrak{g}$, the above defined vector field $\delta(v)$ on $E_H$ is preserved by the action of $H$ on $E_H$. Consequently, the distribution $\mathcal{D}$ in (4.2) is also preserved by the action of $H$ on $E_H$. Clearly, $\mathcal{D}$ is transversal to the fibers of the projection $f$ in (4.1). Hence $\mathcal{D}$ defines a connection on the principal $H$–bundle $E_H$; this connection on $E_H$ will be denoted by $\nabla^{E_H}$. We have

$$\nabla^{E_H} \in C(E_H)^G,$$

because the distribution $\mathcal{D}$ is preserved by the action of $G$ on $E_H$.

Consider the connection $\nabla^0$ on the principal $K$–bundle $G$ constructed in (3.3). From the proof of Lemma 2.2 we know that the principal $H$–bundle $E_H$ is the extension of the structure group of the principal $K$–bundle $G$ in (2.2) using a homomorphism $K \to H$. Consequently, the connection $\nabla^0$ on $G$ induces a connection on $E_H$. From the construction of the above connection $\nabla^{E_H}$ on $E_H$ it is evident that $\nabla^{E_H}$ coincides with the connection on $E_H$ induced by $\nabla^0$.

It was shown in Section 3 that the curvature $\mathcal{K}(\nabla^0)$ of $\nabla^0$ is of type $(1, 1)$. This implies that the curvature of the induced connection $\nabla^{E_H}$ on $E_H$ is also of type $(1, 1)$, because the curvature of the induced connection $\nabla^{E_H}$ is induced by the curvature of $\nabla^0$. Therefore, $\nabla^{E_H}$ produces a holomorphic structure on the principal $H$–bundle $E_H$ [Ko, p. 9, Proposition 3.7]. This holomorphic structure on $E_H$ is invariant, because $\nabla^{E_H} \in C(E_H)^G$. □
We will classify the space of all invariant holomorphic structures on a homogeneous principal $H$–bundle. For that purpose we need to consider invariant almost holomorphic structures.

An almost holomorphic structure on $E_H$ is a $C^\infty$ automorphism $J : TE_H \rightarrow TE_H$ of vector bundles such that

- $J \circ J = -\text{Id}_{TE_H}$,
- the projection $f$ in (4.1) intertwines $J$ and the almost complex structure on $G/K$, meaning $f$ is an almost holomorphic map, and
- the map $E_H \times H \rightarrow E_H$ giving the action of $H$ on $E_H$ is almost holomorphic.

Note that each fiber of $f$ is identified with $H$ up to left-translations, and hence each fiber of $f$ has a complex structure given by the complex structure of $H$. The above third condition implies that the complex structure on any fiber of $f$ is the restriction of $J$.

So a holomorphic structure on the principal $H$–bundle $E_H$ is an integrable almost holomorphic structure on $E_H$. An almost holomorphic structure $J$ on $E_H$ will be called invariant if the action of $G$ on $E_H$ preserves the automorphism $J$. Note that an invariant holomorphic structure on $E_H$ is an integrable invariant almost holomorphic structure on $E_H$.

Giving an almost holomorphic structure on $E_H$ is equivalent to giving a complex distribution

$$D_J \subset TE_H \otimes \mathbb{C}$$

satisfying the following two conditions:

- the differential

  $$df \otimes \mathbb{C} : TE_H \otimes \mathbb{C} \rightarrow f^*T(G/K) \otimes \mathbb{C}$$

  of the projection $f$ in (4.1) maps $D_J$ isomorphically to $T^{0,1}(G/K) \subset T(G/K) \otimes \mathbb{C}$, where $df$ is the differential of $f$, and

- the distribution $D_J$ is preserved by the action of $H$ on $E_H$.

Note that the first condition implies that $D_J \cap \ker(df \otimes \mathbb{C}) = 0$ and $\dim D_J = \dim \mathbb{C} G/K$.

Given a distribution $D_J$ satisfying the above two conditions, consider

$$D_J \oplus \ker(df \otimes \mathbb{C})^{0,1} \subset TE_H \otimes \mathbb{C},$$

where $\ker(df \otimes \mathbb{C}) = \ker(df \otimes \mathbb{C})^{1,0} \oplus \ker(df \otimes \mathbb{C})^{0,1}$ is the type decomposition corresponding to the complex structure on the fibers of $f$. Then there is a unique almost complex structure on $E_H$ such that the corresponding complex distribution

$$T^{0,1}E_H \subset TE_H \otimes \mathbb{C}$$

is $D_J \oplus \ker(df \otimes \mathbb{C})^{0,1}$. 
Let $\text{ad}(E_H) := E_H \times^H \mathfrak{h}$ be the adjoint bundle associated to $E_H$ for the adjoint action of $H$ on its Lie algebra $\mathfrak{h}$. So sections of $\text{ad}(E_H)$ over an open subset $U \subset G/K$ are identified with the $H$–invariant sections of $\text{kernel}(df)$ over $f^{-1}(U)$.

Let $D_J \subset TE_H \otimes \mathbb{C}$ be a distribution giving an almost complex structure on $E_H$. For any $C^\infty$ section $s$ of $T^{0,1}(G/K)$ defined over an open subset $U \subset G/K$, let $\tilde{s}$ be the unique $C^\infty$ section of $D_J$ over $f^{-1}(U) \subset E_H$ such that $(df \otimes \mathbb{C})(\tilde{s}) = f^*s$. Then for any two $C^\infty$ sections $s$ and $t$ of $T^{0,1}(G/K)$ over $U$, the section

$$\mathcal{K}(D_J)(s, t) := [\tilde{s}, \tilde{t}] - [s, t] \quad (4.3)$$

is an $H$–invariant section of $\text{kernel}(df)$ over $f^{-1}(U)$. Therefore, $\mathcal{K}(D_J)(s, t)$ produces a section of $\text{ad}(E_H)|_U$. It is straightforward to check that $\mathcal{K}(D_J)(\psi \cdot s, t) = \psi \cdot \mathcal{K}(D_J)(s, t)$ for any locally defined $C^\infty$ function $\psi$ on $G/K$, and $\mathcal{K}(D_J)(s, t) = -\mathcal{K}(D_J)(t, s)$. Consequently, $\mathcal{K}(D_J)$ is a $C^\infty$ section of $\Omega^{0,2}G/K \otimes \text{ad}(E_H)$.

Note that the above distribution $D_J$ is integrable if and only if $\mathcal{K}(D_J) = 0$. The almost complex structure on $E_H$ given by $D_J$ is integrable if and only if $\mathcal{K}(D_J) = 0$.

The space of almost holomorphic structures on $E_H$ is an affine space for the vector space $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})$. Note that from Theorem 4.1 we know that the space of almost holomorphic structures on $E_H$ is nonempty. The actions of $G$ on $E_H$ and $G/K$ together produce an action of $G$ on $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})$. The space of invariant almost holomorphic structures on $E_H$ is an affine space for the vector space $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$ of $G$–invariants in $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})$; note that the space of invariant almost holomorphic structures on $E_H$ is nonempty, because the almost holomorphic structures on $E_H$ given by Theorem 4.1 is invariant.

Since the translation action of $G$ on $G/K$ is transitive, the evaluation map

$$\epsilon : C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G \longrightarrow (\text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})_{K/K}, \quad s \longmapsto s(K/K) \quad (4.4)$$

is injective.

Fix a point

$$z_0 \in (E_H)_{K/K}, \quad (4.5)$$

so $(E_H, z_0)$ is a based homogeneous bundle. The map

$$H \longmapsto (E_H)_{K/K}, \quad h \longmapsto z_0h$$

identifies the fiber $(E_H)_{K/K}$ with $H$. Recall that the fiber $\text{ad}(E_H)_{K/K}$ is a quotient of $(E_H)_{K/K} \times \mathfrak{h}$. The map $\mathfrak{h} \rightarrow \text{ad}(E_H)_{K/K}$ that sends any $v \in \mathfrak{h}$ to the element of $\text{ad}(E_H)_{K/K}$ given by $(z_0, v)$, where $z_0$ is the element in (4.5), identifies $\mathfrak{h}$ with $\text{ad}(E_H)_{K/K}$. On the other hand, the two vector spaces $(T^{0,1}_K(G/K))^* = (\Omega^{0,1}_{G/K})_{K/K}$ and $T^{1,0}_K(G/K)$ are identified using the Kähler form on $G/K$, and hence $(\Omega^{0,1}_{G/K})_{K/K}$ is identified with $\mathfrak{p}_+$ defined in (3.5) (recall that $T^{1,0}_K(G/K)$ is identified with $\mathfrak{p}_+$). Therefore, the map $\epsilon$ is (4.4) is in fact an injective map

$$\epsilon : C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G \longrightarrow \mathfrak{h} \otimes \mathfrak{p}_+. \quad (4.6)$$
From Theorem 4.1 we know that $E_H$ has a tautological invariant holomorphic structure. Since the space of almost holomorphic structures (respectively, invariant almost holomorphic structures) on $E_H$ is an affine space for $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})$ (respectively, $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$), using this holomorphic structure on $E_H$ given by Theorem 4.1 as the base point, the space of almost holomorphic structures (respectively, invariant almost holomorphic structures) on $E_H$ gets identified with $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$ (respectively, $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$).

The Lie algebra operation $\mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ and the exterior multiplication $p_+ \otimes p_+ \rightarrow \bigwedge^2 p_+$ together define a homomorphism

$$m_+ : (\mathfrak{h} \otimes p_+)^{\otimes 2} \rightarrow \mathfrak{h} \otimes \bigwedge^2 p_+. \quad (4.7)$$

**Proposition 4.2.** Take an invariant almost holomorphic structure

$$\beta \in C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G.$$  

Then $\phi$ is integrable if and only if

$$m_+(\epsilon(\beta) \otimes \epsilon(\beta)) = 0,$$

where $\epsilon$ and $m_+$ are constructed in (4.6) and (4.7) respectively.

**Proof.** Let $\eta : K \rightarrow H$ be the homomorphism constructed as in (2.4) for the based homogeneous principal $H$–bundle $(E_H, z_0)$. As before, the center of $K$ is denoted by $Z_K$. Consider the action of $Z_K$ on $\mathfrak{h}$ obtained by combining $\eta$ with the adjoint action of $H$ on $\mathfrak{h}$; in other words, this action is given by the following composition of maps

$$K \xrightarrow{\eta} H \xrightarrow{\text{ad}} \text{Aut}(\mathfrak{h}).$$

Let

$$\mathfrak{h} = \bigoplus_{\lambda \in (Z_K)^*} \mathfrak{h}_\lambda \quad (4.8)$$

be the corresponding isotypical decomposition. Since $Z_K$ commutes with $K$, the action of $K$ on $\mathfrak{h}$, constructed as above using $\eta$ and the adjoint action of $H$ on $\mathfrak{h}$, preserves the decomposition in (4.8). Let

$$\mathcal{G}(\mathfrak{h}_\lambda) := \mathcal{G} \times^K \mathfrak{h}_\lambda \rightarrow G/K$$

be the vector bundle over $G/K$ associated to the principal $K$–bundle $\mathcal{G}$ (constructed in (2.2)) for the $K$–module $\mathfrak{h}_\lambda$ in (4.8). From the decomposition in (4.8) we have the decomposition

$$\text{ad}(E_H) = \bigoplus_{\lambda \in (Z_K)^*} \mathcal{G}(\mathfrak{h}_\lambda) \quad (4.9)$$

which is in fact a holomorphic decomposition.
Note that $Z_K$ acts on $(\Omega^0_{\mathcal{G}/K})_{\mathcal{K}/K} = p_+$ through a single character. This character of $Z_K$, through which it acts on $p_+$, will be denoted by $\chi$. The character $\chi$ is actually nontrivial; indeed, this follows from the fact that $G/K$ does not have any nonzero holomorphic one-form.

Take an invariant section $\beta \in C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^0_{\mathcal{G}/K})^G$ as in the statement of the proposition. Therefore, the element

$$\beta(K/K) \in (\text{ad}(E_H) \otimes \Omega^0_{\mathcal{G}/K})_{K/K}$$

is fixed by the action of $Z_K$ on $(\text{ad}(E_H) \otimes \Omega^0_{\mathcal{G}/K})_{K/K}$ (in fact it is fixed by $K$). Since $\beta(K/K)$ is fixed by the action of $Z_K$, it can be shown that

$$\beta(K/K) \in (\mathcal{G}(\mathfrak{h}_x^{-1}) \otimes \Omega^0_{\mathcal{G}/K})_{K/K} \subset (\text{ad}(E_H) \otimes \Omega^0_{\mathcal{G}/K})_{K/K}, \tag{4.10}$$

where $\mathcal{G}(\mathfrak{h}_x^{-1})$ is the direct summand in (4.9) for the character $\chi^{-1}$ defined above. Indeed, this follows immediately from the fact that $Z_K$ acts on the cotangent space $(\Omega^0_{\mathcal{G}/K})_{K/K}$ as multiplication by $\chi$. Since $\beta$ is $G$–invariant, and the action of $G$ on $G/K$ is transitive, from (4.10) we conclude that

$$\beta \in C^\infty(G/K, \mathcal{G}(\mathfrak{h}_x^{-1}) \otimes \Omega^0_{\mathcal{G}/K})^G. \tag{4.11}$$

Let $\overline{\partial}^0_{\text{ad}(E_H)} : \text{ad}(E_H) \rightarrow \text{ad}(E_H) \otimes \Omega^0_{\mathcal{G}/K}$ be the Dolbeault operator on $\text{ad}(E_H)$ induced by the tautological holomorphic structure on $E_H$ (see Theorem 4.1). Since the decomposition in (4.9) is holomorphic, from (4.11) it follows that

$$\overline{\partial}^0_{\text{ad}(E_H)}(\beta) \in C^\infty(G/K, \mathcal{G}(\mathfrak{h}_x^{-1}) \otimes \Omega^0_{\mathcal{G}/K})^G.$$

So $\overline{\partial}^0_{\text{ad}(E_H)}(\beta)(K/K)$ is a $K$–invariant element of $\mathfrak{h}_x^{-1} \otimes \bigwedge^2 p_+$. But $Z_K$ acts on $\mathfrak{h}_x^{-1} \otimes \bigwedge^2 p_+$ as multiplication via the nontrivial character $\chi^{-1} \cdot \chi^2 = \chi$. Hence we conclude that

$$\overline{\partial}^0_{\text{ad}(E_H)}(\beta) = 0. \tag{4.12}$$

Let

$$D_\beta \subset TE_H \otimes \mathbb{C}$$

be the distribution corresponding to the almost complex structure $\beta$ on $E_H$. Let

$$D_0 \subset TE_H \otimes \mathbb{C}$$

be the distribution corresponding to the tautological almost complex structure on $E_H$ (see Theorem 4.1); note that the tautological almost complex structure on $E_H$ corresponds to the identically zero section of $\text{ad}(E_H) \otimes \Omega^0_{\mathcal{G}/K}$. We have

$$\mathcal{K}(D_\beta) = \mathcal{K}(D_0) + \overline{\partial}^0_{\text{ad}(E_H)}(\beta) + m_+(\epsilon(\beta) \otimes \epsilon(\beta)), \tag{4.13}$$

where $\mathcal{K}(D_\beta)$ and $\mathcal{K}(D_0)$ are constructed as in (4.3) for $D_\beta$ and $D_0$ respectively. Now

$$\mathcal{K}(D_0) = 0$$

because the tautological almost holomorphic structure on $E_H$ is integrable. Hence using (4.12), from (4.13) we conclude that $\mathcal{K}(D_\beta) = 0$ if and only if $m_+(\epsilon(\beta) \otimes \epsilon(\beta)) = 0$. 
Since the almost complex structure on $E_H$ corresponding to $\beta$ is integrable if and only if $\mathcal{K}(D_\beta) = 0$ (see [Ko, p. 9, Proposition 3.7]), the proposition follows. \hfill \Box

\textbf{Theorem 4.3.} There is a natural bijection between the following two:

1. Isomorphism classes of based homogeneous principal $H$–bundles with a invariant holomorphic structure.
2. Pairs of the form $(\eta, \beta)$, where $\eta : K \rightarrow H$ is a homomorphism and $\beta \in (\mathfrak{h} \otimes \mathfrak{p}_+)^K$ with $\mathfrak{m}_+ (\beta \otimes \beta) = 0$, where $\mathfrak{m}_+$ is defined in (1.7).

\textbf{Proof.} Let $(E_H, \rho)$ be a homogeneous holomorphic principal $H$–bundle with a base point $z_0 \in (E_H)_{K/K}$. For any $k \in K$, let $\eta(k) \in H$ be the unique element that satisfies the equation

$$\rho(k, z_0) = z_0 \eta(k).$$

It was shown in the proof of Lemma 2.2 that $\eta$ is a homomorphism of groups.

Recall that the space of all $G$–invariant almost complex structures on the underlying $C^\infty$ homogeneous principal $H$–bundle $E_H$ is an affine space for vector space $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$ of $G$–invariants in $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})$. Therefore, using the tautological holomorphic structure on $E_H$ given by Theorem 4.1 the space of all $G$–invariant almost complex structures on the $C^\infty$ homogeneous principal $H$–bundle $E_H$ gets identified with $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$. Now let

$$\tilde{\beta} \in C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$$

be the element corresponding to the given holomorphic structure on $E_H$.

The fiber $(\text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})_{K/K}$ is identified with $\mathfrak{h} \otimes \mathfrak{p}_+$ using the base point $z_0$. Hence

$$\beta := \tilde{\beta}(K/K) \in \mathfrak{h} \otimes \mathfrak{p}_+.$$ 

From Proposition 4.2 it follows that $\mathfrak{m}_+(\beta \otimes \beta) = 0$.

To prove the converse, take any pair $(\eta, \beta)$, where $\eta : K \rightarrow H$ is a homomorphism and $\beta \in (\mathfrak{h} \otimes \mathfrak{p}_+)^K$ with $\mathfrak{m}_+(\beta \otimes \beta) = 0$. Let $E_H$ be the homogeneous $C^\infty$ principal $H$–bundle over $G/K$ given by $\eta$ using Lemma 2.2.

Since $(\text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})_{K/K} = \mathfrak{h} \otimes \mathfrak{p}_+$ and $\beta \in (\mathfrak{h} \otimes \mathfrak{p}_+)^K$, there is a unique invariant section

$$\tilde{\beta} \in C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$$

such that $\tilde{\beta}(K/K) = \beta$. As mentioned above, $C^\infty(G/K, \text{ad}(E_H) \otimes \Omega^{0,1}_{G/K})^G$ is identified with the space of all $G$–invariant almost complex structures on the $C^\infty$ homogeneous principal $H$–bundle $E_H$. Equip $E_H$ with the $G$–invariant almost complex structure corresponding to the above section $\tilde{\beta}$. Since $\mathfrak{m}_+(\beta \otimes \beta) = 0$, from Proposition 4.2 it follows that this almost complex structure is integrable. \hfill \Box

Take any two pairs $(\eta, \beta)$ and $(\eta', \beta')$ as in Theorem 4.3. They will be called equivalent if there is an element $h \in H$ such that
• \( \eta'(g) = h^{-1} \eta(g) h \) for all \( g \in K \), and
• \( \beta' = (\text{ad}(h) \otimes \text{Id}_{\mathfrak{p}})(\beta) \).

**Corollary 4.4.** There is a natural bijection between the following two:

1. Isomorphism classes of principal \( H \)–bundles with a invariant holomorphic structure.
2. Equivalence classes of pairs of the form \((\eta, \beta)\), where \( \eta : K \to H \) is a homomorphism and \( \beta \in (\mathfrak{h} \otimes \mathfrak{p})^K \) with \( m_+(\beta \otimes \beta) = 0 \).

**Proof.** Let \((E_H, \rho)\) be a homogeneous holomorphic principal \( H \)–bundle with a base point \( z_0 \in (E_H)_{K/K} \). As in the proof of Theorem 4.3, for any \( k \in K \), let \( \eta(k) \in H \) be the unique element that satisfies the equation

\[ \rho(k, z_0) = z_0 \eta(k). \]

Now take any \( h_0 \in H \) and set \( z_0 h \) to be the new base point on \( E_H \). Let \( \eta' : K \to H \) be the homomorphism corresponding to the base point \( z_0 h \), so \( \rho(k, z_0 h) = z_0 h \eta'(k) \) for all \( k \in K \). Now, for any \( k \in K \), we have

\[ z_0 h \eta'(k) = \rho(k, z_0 h) = \rho(k, z_0) h = z_0 \eta(k) h. \]

This implies that \( \eta'(k) = h^{-1} \eta(k) h \). Now it is straightforward to deduce the corollary from Theorem 4.3. \(\square\)

### 5. Homogeneous co-Higgs bundles

We treat co-Higgs bundles first and then ordinary Higgs bundles in the next section.

Let \((E_H, \rho)\) be a homogeneous holomorphic principal \( H \)–bundle over \( G/K \). The action of \( G \) on \( E_H \) induces an action of \( G \) on \( \text{ad}(E_H) \). The actions of \( G \) on \( G/K \) and \( \text{ad}(E_H) \) together produce an action of \( G \) on the holomorphic vector bundle \( \text{ad}(E_H) \otimes T^{1,0}(G/K) \).

Take a holomorphic section \( \theta \in H^0(G/K, \text{ad}(E_H) \otimes T^{1,0}(G/K)) \).

Using the Lie algebra structure of the fibers of \( \text{ad}(E_H) \), we have

\[ \theta \wedge \theta \in H^0(G/K, \text{ad}(E_H) \otimes \bigwedge^2 T^{1,0}(G/K)) \).

An invariant co-Higgs field on \( E_H \) is a holomorphic section

\[ \theta \in H^0(G/K, \text{ad}(E_H) \otimes T^{1,0}(G/K)) \]

such that

1. \( \theta \wedge \theta = 0 \), and
2. the action of \( G \) on \( \text{ad}(E_H) \otimes T^{1,0}(G/K) \) fixes the section \( \theta \).
A homogeneous co-Higgs $H$–bundle is a homogeneous holomorphic principal $H$–bundle equipped with an invariant co-Higgs field. Two homogeneous co-Higgs bundles $(E_H', \rho', \theta')$ and $(E_H'', \rho'', \theta'')$ are isomorphic if there is a holomorphic isomorphism of principal $H$–bundles $\alpha : E_H' \rightarrow E_H''$ that satisfies the following two conditions:

- $\alpha$ intertwines the actions of $G$ on $E_H'$ and $E_H''$,
- the isomorphism $\text{ad}(E_H') \otimes T^{1,0}(G/K) \rightarrow \text{ad}(E_H'') \otimes T^{1,0}(G/K)$ constructed using $\alpha$ takes the section $\theta'$ to $\theta''$.

A based homogeneous co-Higgs $H$–bundle on $G/K$ is a homogeneous co-Higgs $H$–bundle $(E_H', \rho', \theta', z_0)$ equipped with a base point $z_0 \in (E_H')_{K/K}$ over $K/K$. Two based homogeneous co-Higgs bundles $(E_H', \rho', \theta', z_0)$ and $(E_H'', \rho'', \theta'', z_0')$ are isomorphic if there is an isomorphism between $(E_H', \rho', \theta', z_0)$ and $(E_H'', \rho'', \theta'', z_0')$ that takes $z_0'$ to $z_0''$.

Using the Lie algebra operation on $\mathfrak{h}$ we define the homomorphism

$$m : (\mathfrak{h} \otimes \mathfrak{p}_+) \otimes (\mathfrak{h} \otimes \mathfrak{p}_+) \rightarrow \mathfrak{h} \otimes \mathfrak{p}_+ \otimes \mathfrak{p}_+.$$

(5.1)

**Theorem 5.1.** There is a natural bijection between the following two:

1. Isomorphism classes of based homogeneous principal co-Higgs $H$–bundles on $G/K$.
2. Triples of the form $(\eta, \beta, \varphi)$, where $\eta : K \rightarrow H$ is a homomorphism and $\beta, \varphi \in (\mathfrak{h} \otimes \mathfrak{p}_+)^K$

such that

$$m_+ (\beta \otimes \beta) = m_+ (\varphi \otimes \varphi) = 0 = m (\beta \otimes \varphi),$$

where $m_+$ and $m$ are constructed in (4.7) and (5.1) respectively.

**Proof.** Let $(E_H, \rho, \theta, z_0)$ be a based homogeneous co-Higgs $H$–bundle. From Theorem 4.3 we know that the based homogeneous holomorphic principal $H$–bundle $(E_H, \rho, z_0)$ gives a pair $(\eta, \beta)$, where $\eta : K \rightarrow H$ is a homomorphism and $\beta \in (\mathfrak{h} \otimes \mathfrak{p}_+)^K$ with $m_+ (\beta \otimes \beta) = 0$.

Consider

$$\varphi = \theta(K/K) \in (\text{ad}(E_H) \otimes T^{1,0}(G/K))_{K/K} = \mathfrak{h} \otimes \mathfrak{p}_+;$$

as before, the fiber $\text{ad}(E_H)_{K/K}$ is identified with $\mathfrak{h}$ using $z_0$, while the identification between $T^{1,0}_{K/K}(G/K)$ and $\mathfrak{p}_+$ is the one in Section 3. Since the section $\theta$ is $G$–invariant, it follow that $\varphi \in (\mathfrak{h} \otimes \mathfrak{p}_+)^K$. The condition that $\theta \wedge \theta(K/K) = 0$ is equivalent to the condition that $m_+ (\varphi \otimes \varphi) = 0$.

Next it will be shown that

$$m (\beta \otimes \varphi) = 0.$$

For that, first note the Dolbeault operator $\overline{\partial}$ for the holomorphic vector bundle $\text{ad}(E_H) \otimes T^{1,0}(G/K)$ satisfies the equation

$$\overline{\partial} = \overline{\partial}_0 + \beta,$$

(5.2)
where $\partial'_{\theta}$ denotes the Dolbeault operator on $\text{ad}(E_H) \otimes T^{1,0}(G/K)$ corresponding to the tautological connection on the homogeneous principal $H$–bundle $E_H$ obtained in Theorem 4.1. Now the given condition that the section $\theta$ is holomorphic implies that $\partial'_{\theta}(\theta) = 0$, and hence from (5.2) we have

$$ (\partial'_{\theta} + \beta)(\theta) = 0. \quad (5.3) $$

On the other hand, we have

$$ \partial'_{\theta}(\theta) = 0 \quad (5.4) $$

because the section $\theta$ is invariant; recall that the tautological holomorphic structure has the property that any invariant section is holomorphic. Now combining (5.3) and (5.4) we conclude that $m(\beta \otimes \varphi) = 0$.

To prove the converse, take a triple $(\eta, \beta, \varphi)$ satisfying the conditions in the statement of the theorem. From Theorem 4.3 we know that the pair $(\eta, \beta)$ gives a holomorphic homogeneous principal $H$–bundle $(E_H, \rho)$. Since $\varphi$ is $K$–invariant, there is a unique $G$–invariant $C^\infty$ section $\theta \in C^\infty(G/K, \text{ad}(E_H) \otimes T^{1,0}(G/K))^G$ such that $\theta(K/K) = \varphi$.

The evaluation $\theta \wedge \theta(K/K) \in (\text{ad}(E_H) \otimes \bigwedge^2 T^{1,0}(G/K))_{K/K} = \mathfrak{h} \otimes \bigwedge^2 \mathfrak{p}_+$ coincides with $m_+(\varphi \otimes \varphi)$. Since $\varphi$ is $G$–equivariant, from the given condition that $m(\beta \otimes \varphi) = 0$ we conclude that $\theta \wedge \theta = 0$.

Next consider

$$ \mathfrak{d}(\theta) \in C^\infty(G/K, \text{ad}(E_H) \otimes T^{1,0}(G/K) \otimes \Omega_{G/K}^{0,1}), $$

where $\mathfrak{d}$ is the Dolbeault operator corresponding to the holomorphic structure on the vector bundle $\text{ad}(E_H) \otimes T^{1,0}(G/K)$. This section $\mathfrak{d}(\theta)$ is $G$–invariant, because $\theta$ is $G$–invariant, and the holomorphic structure is preserved by the action of $G$. Let

$$ \mathfrak{d}(\theta)(K/K) \in (\text{ad}(E_H) \otimes T^{1,0}(G/K) \otimes \Omega_{G/K}^{0,1})_{K/K} = \mathfrak{h} \otimes \mathfrak{p}_+ \otimes \mathfrak{p}_+ $$

be the evaluation of this section at the point $K/K \in G/K$. Combining (5.2) with the fact that $\partial'_{\theta}(\theta) = 0$ it follows that

$$ \mathfrak{d}(\theta)(K/K) = m(\beta \otimes \varphi). \quad (5.5) $$

Since $\mathfrak{d}(\theta)$ is $G$–invariant, and $m(\beta \otimes \varphi) = 0$, from (5.5) we conclude that $\mathfrak{d}(\theta) = 0$. In other words, the section $\theta$ is holomorphic.

Take any two triples $(\eta, \beta, \varphi)$ and $(\eta', \beta', \varphi')$ satisfying the conditions in Theorem 5.1. They will be called equivalent if there is an element $h \in H$ such that

- $\eta'(g) = h^{-1}\eta(g)h$ for all $g \in K$,
- $\beta' = (\text{ad}(h) \otimes \text{Id}_{\mathfrak{p}_+})(\beta)$, and,
- $\varphi' = (\text{ad}(h) \otimes \text{Id}_{\mathfrak{p}_+})(\varphi)$.
The following analogue of Corollary 4.4 is a straightforward consequence of Theorem 5.1.

**Corollary 5.2.** There is a natural bijection between the following two:

1. Isomorphism classes of homogeneous co-Higgs $H$–bundles.
2. Equivalence classes of triples $(\eta, \beta, \varphi)$, where $\eta : K \to H$ is a homomorphism and
   
   $$\beta, \varphi \in (\mathfrak{h} \otimes \mathfrak{p}_+)^K$$

   such that
   
   $$m_+(\beta \otimes \beta) = m_+(\varphi \otimes \varphi) = 0 = m(\beta \otimes \varphi),$$

   where $m_+$ and $m$ are constructed in (4.7) and (5.1) respectively.

6. **Homogeneous Higgs bundles**

   As before, $(E_H, \rho)$ is a homogeneous holomorphic principal $H$–bundle over $G/K$. The actions of $G$ on $G/K$ and $\text{ad}(E_H)$ together produce an action of $G$ on the holomorphic vector bundle $\text{ad}(E_H) \otimes \Omega^{1,0}_{G/K}$. For any $\theta \in H^0(G/K, \text{ad}(E_H) \otimes \Omega^{1,0}_{G/K})$, we have
   
   $$\theta \wedge \theta \in H^0(G/K, \text{ad}(E_H) \otimes \wedge^2 \Omega^{1,0}_{G/K}),$$
   
   which is defined using the Lie algebra structure of the fibers of $\text{ad}(E_H)$. In analogy to the preceding section, an *invariant Higgs field* on $E_H$ is a holomorphic section
   
   $$\theta \in H^0(G/K, \text{ad}(E_H) \otimes \Omega^{1,0}_{G/K})$$

   such that

   1. $\theta \wedge \theta = 0$, and
   2. the action of $G$ on $\text{ad}(E_H) \otimes \Omega^{1,0}_{G/K}$ fixes the section $\theta$.

   Then, a *homogeneous* Higgs $H$–bundle is a homogeneous holomorphic principal $H$–bundle equipped with an invariant Higgs field. Based homogeneous Higgs $H$–bundles on $G/K$ and isomorphisms between them are defined in exactly the same way as for the co-Higgs case.

   Now, the Lie algebra operation $\mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ and the exterior multiplication
   
   $$\mathfrak{p}_- \otimes \mathfrak{p}_- \to \wedge^2 \mathfrak{p}_-$$

   together define a homomorphism
   
   $$m_- : (\mathfrak{h} \otimes \mathfrak{p}_-)^\otimes \to \mathfrak{h} \otimes \wedge^2 \mathfrak{p}_-.$$ (6.1)

**Proposition 6.1.** There is a natural bijection between the following two:

1. Isomorphism classes of based homogeneous principal Higgs $H$–bundles on $G/K$. 


(2) Triples of the form \((\eta, \beta, \varphi)\), where \(\eta : K \rightarrow H\) is a homomorphism, \(\beta \in (\mathfrak{h} \otimes p_+)^K\) and \(\varphi \in (\mathfrak{h} \otimes p_-)^K\) such that
\[
\mathbf{m}_+(\beta \otimes \beta) = 0 = \mathbf{m}_-(\varphi \otimes \varphi) = \mathbf{m}(\beta \otimes \varphi),
\]
where \(\mathbf{m}_+, \mathbf{m}_-\) and \(\mathbf{m}\) are constructed in \((4.7)\) \((6.1)\) and \((5.1)\) respectively.

**Proof.** Consider the fiber \((\Omega^1_{1,0})_{K/K}\) of \((\Omega^1_{1,0})_{G/K}\) over \(K/K \in G/K\). We note that
\[
(\Omega^1_{1,0})_{K/K} = (T^1_{1,0}(G/K))^* = p^+_1 = p^-.
\]
Now the proof is similar to that of Theorem \((5.1)\). We omit the details. \(\square\)

Proposition \((6.1)\) has the following analog of Corollary \((5.2)\).

**Corollary 6.2.** There is a natural bijection between the following two:

1. Isomorphism classes of homogeneous principal Higgs \(H\)-bundles on \(G/K\).
2. Equivalence classes of triples \((\eta, \beta, \varphi)\), where \(\eta : K \rightarrow H\) is a homomorphism, \(\beta \in (\mathfrak{h} \otimes p_+)^K\) and \(\varphi \in (\mathfrak{h} \otimes p_-)^K\) such that
\[
\mathbf{m}_+(\beta \otimes \beta) = 0 = \mathbf{m}_-(\varphi \otimes \varphi) = \mathbf{m}(\beta \otimes \varphi),
\]
where \(\mathbf{m}_+, \mathbf{m}_-\) and \(\mathbf{m}\) are constructed in \((4.7)\) \((6.1)\) and \((5.1)\) respectively.

7. **Remark on moduli spaces**

The classifications above do not distinguish between (semi)stable and unstable objects in the sense of geometric invariant theory. That being said, Proposition \((6.1)\) and Theorem \((5.1)\) endow the sets of isomorphism classes of based homogeneous principal Higgs and co-Higgs \(H\)-bundles on \(G/K\) with the structures of algebraic varieties. In the Higgs case, this space is the subspace of
\[
\text{Hom}(K, H) \times (\mathfrak{h} \otimes p_+)^K \times (\mathfrak{h} \otimes p_-)^K
\]
consisting of all \((\eta, \beta, \varphi)\) satisfying the algebraic equations
\[
\mathbf{m}_+(\beta \otimes \beta) = 0 = \mathbf{m}_-(\varphi \otimes \varphi) = \mathbf{m}(\beta \otimes \varphi).
\]
The space of isomorphism classes of homogeneous based principal co-Higgs \(H\)-bundles on \(G/K\) is the subspace of
\[
\text{Hom}(K, H) \times (\mathfrak{h} \otimes p_+)^K \times (\mathfrak{h} \otimes p_-)^K
\]
defined by the locus of solutions \((\eta, \beta, \varphi)\) of the algebraic equations
\[
\mathbf{m}_+(\beta \otimes \beta) = \mathbf{m}_+(\varphi \otimes \varphi) = 0 = \mathbf{m}(\beta \otimes \varphi).
\]
In either case, the subspace can be regarded as a moduli space for the associated classification problem.
Corollary [5.2] (respectively, Corollary [6.2]) yields the moduli space of isomorphism classes of homogeneous principal co-Higgs (respectively, Higgs) bundles.

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