Generalized Witt and Witt $n$–algebras, Virasoro algebras and constraints, and KdV equations from $\mathcal{R}(p, q)$–deformed quantum algebras

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We perform generalizations of Witt and Virasoro algebras, and derive the corresponding Korteweg-de Vries equations from known $\mathcal{R}(p, q)$–deformed quantum algebras previously introduced in J. Math. Phys. 51, 063518, (2010). Related relevant properties are investigated and discussed. Besides, we construct the $\mathcal{R}(p, q)$–deformed Witt $n$– algebra, and determine the Virasoro constraints for a toy model, which play an important role in the study of matrix models. Finally, as matter of illustration, explicit results are provided for main particular deformed quantum algebras known in the literature.

Keywords: $\mathcal{R}(p, q)$–calculus, Witt algebra, Witt $n$–algebra, Virasoro algebra, Korteweg-de Vries equation, Virasoro constraints.

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I. INTRODUCTION

The Virasoro algebra plays an important role in physics. Kupershmidt investigated the nature of this algebra, and discussed its applications in mathematics and physics, namely in conformal field theory and string theory. Various deformations and generalizations of this algebra were studied in the literature. For instance, Hounkonnou et al. generalized Kupershmidt’s work, relatively to their left-symmetry structure, derived related algebraic and some hydrodynamic properties. The realizations of Witt and Virasoro algebras, and their link with integrable equations were addressed in. Besides, the link between the Korteweg-de Vries (KdV) equation and the Virasoro algebra was pointed out by Gervais and Kupershmidt.

The $q$–analogue of the Lie algebras plays an important role in the study of integrable quantum field theories, solvable statistical models, string models, and related topics in physics and mathematics. Sato investigated the $q$– version of the deformed Virasoro algebra introduced by Curtright and Zachos. New results on the central extension and the operator product expansion were presented and the link between the $q$– Virasoro algebra and the Volterra Poisson bracket algebra was discussed.

The $q$– deformed KdV equation corresponding to the $q$– deformed Virasoro algebra was obtained by Chaichian et al. This equation can be considered as a lattice system which is a particular discretization of KdV and a deformation of conformal field theory.

Chaichian et al. also described the $q$– deformations of the realisations of the conformal algebra depending on the conformal dimension $\Delta$. The $q$– deformed Jacobi identity, $q$– deformed central extension, $q$– deformed energy-momentum tensor corresponding to the conformal dimension $\Delta = 2$, and the transformation properties consistent with the $q$– deformed central extension term were also considered in their work. Chaichian and Presnajder realized the $q$– deformed Virasoro algebra using the bosonic annihilation and creation operators of the $q$– deformed infinite Heisenberg algebra, and expressed the generators of this new algebra as a Sugawara construction. They also presented the fermionic annihilation and creation operators associated to the $q$– deformed infinite Heisenberg superalgebras.

Furthermore, the central extension of the $q$– deformed Witt algebra and the realization of the $q$– deformed Virasoro algebra using the $q$– deformed operator product expansion were studied in. From their side, Wang et al. presented the $q$– deformed Witt algebra.
and performed their \( n \)-algebras. The super \( q \)-deformed Virasoro \( n \)-algebra for \( n \) even and a toy model for the \( q \)-deformed Virasoro constraints were investigated. See also the work by Nedelina and Zabzine on the \( q \)-Virasoro constraints for a toy model\(^{25}\).

Chakrabarty and Jagannathan\(^9\) analyzed a \((p, q)\)-deformation of the Virasoro algebra with conformal dimension \( \Delta \), and defined the comultiplication rule for the generating functional for the case \( \Delta = 0, 1 \). The central charge term for the Virasoro algebra associated to the Jagannathan-Srinivasa deformation\(^2\) was described, and the corresponding \((p, q)\)-deformed nonlinear equation, also called \((p, q)\)-Korteweg-de Vries equation, for the case \( \Delta = 0 \) was derived.

Recently, we investigated generalizations of \((p, q)\)-deformed Heisenberg algebras, called \( \mathcal{R}(p, q) \)-deformed quantum algebras, where \( \mathcal{R} \) is a meromorphic function\(^{17}\). Furthermore, we characterized the \( \mathcal{R}(p, q) \)-deformed conformal Virasoro algebra, deduced the \( \mathcal{R}(p, q) \)-Korteweg-de Vries equation for a conformal dimension \( \Delta = 1 \), and discussed the energy-momentum tensor induced by the \( \mathcal{R}(p, q) \)-deformed quantum algebras for the conformal dimension \( \Delta = 2 \),\(^{20}\).

Before dealing with the main results, let us fix, in this section, the notations and briefly recall some definitions and known results useful in the sequel. Let \( p \) and \( q \) be two positive real numbers such that \( 0 < q < p \leq 1 \). We consider a meromorphic function \( \mathcal{R} \) defined on \( \mathbb{C} \times \mathbb{C} \) by\(^{47}\)

\[
\mathcal{R}(u, v) = \sum_{s, t = -l}^{\infty} r_{st} u^s v^t, \tag{1}
\]

with an eventual isolated singularity at the zero, where \( r_{st} \) are complex numbers, \( l \in \mathbb{N} \cup \{0\} \), \( \mathcal{R}(p^n, q^n) > 0, \forall n \in \mathbb{N} \), and \( \mathcal{R}(1, 1) = 0 \) by definition. We denote by \( \mathbb{D}_R \) the bidisk

\[
\mathbb{D}_R := \prod_{j=1}^{2} \mathbb{D}_{R_j} = \left\{ w = (w_1, w_2) \in \mathbb{C}^2 : |w_j| < R_j \right\},
\]

where \( R \) is the convergence radius of the series \( (\Pi) \) defined by Hadamard formula as follows:

\[
\lim_{s+t \to \infty} \sup_{s+t} \sqrt[|r_{st}|R_1 R_2] = 1.
\]

For the proof and more details see\(^{26}\). Let us also consider \( \mathcal{O}(\mathbb{D}_R) \) the set of holomorphic functions defined on \( \mathbb{D}_R \).
Define the $\mathcal{R}(p, q)$–deformed numbers\textsuperscript{16}:

$$[n]_{\mathcal{R}(p, q)} := \mathcal{R}(p^n, q^n), \quad n \in \mathbb{N} \cup \{0\}, \quad (2)$$

the $\mathcal{R}(p, q)$–deformed factorials

$$[n]!_{\mathcal{R}(p, q)} := \begin{cases} 1 & \text{for } n = 0 \\ \mathcal{R}(p, q) \cdots \mathcal{R}(p^n, q^n) & \text{for } n \geq 1, \end{cases}$$

and the $\mathcal{R}(p, q)$–deformed binomial coefficients

$$\binom{m}{n}_{\mathcal{R}(p, q)} := \frac{[m]!_{\mathcal{R}(p, q)}}{[n]!_{\mathcal{R}(p, q)} [m-n]!_{\mathcal{R}(p, q)}}, \quad m, n \in \mathbb{N} \cup \{0\}, \quad m \geq n$$

satisfying the relation

$$\binom{m}{n}_{\mathcal{R}(p, q)} = \binom{m}{m-n}_{\mathcal{R}(p, q)}, \quad m, n \in \mathbb{N} \cup \{0\}, \quad m \geq n.$$

Consider the following linear operators defined on $\mathcal{O}(\mathbb{D}_R)$, (see\textsuperscript{17} for more details),

$$Q : \Psi \mapsto Q\Psi(z) := \Psi(qz),$$

$$P : \Psi \mapsto P\Psi(z) := \Psi(pz),$$

$$D_{p,q} : \Psi \mapsto D_{p,q}\Psi(z) := \frac{\Psi(pz) - \Psi(qz)}{z(p-q)},$$

and the $\mathcal{R}(p, q)$–derivative

$$\partial_{\mathcal{R}(p, q)} := \partial_{p,q} \frac{p-q}{P-Q} \mathcal{R}(P, Q) = \frac{p-q}{p^P - q^Q} \mathcal{R}(P^P, Q^Q) \partial_{p,q}. \quad (3)$$

The algebra associated with the $\mathcal{R}(p, q)$–deformation is a quantum algebra, denoted $\mathcal{A}_{\mathcal{R}(p, q)}$, generated by the set of operators $\{1, A, A^\dagger, N\}$ satisfying the following commutation relations:

$$AA^\dagger = [N + 1]_{\mathcal{R}(p, q)}, \quad A^\dagger A = [N]_{\mathcal{R}(p, q)}.$$ 

$$[N, A] = -A, \quad [N, A^\dagger] = A^\dagger$$

with the realization on $\mathcal{O}(\mathbb{D}_R)$ given by:

$$A^\dagger := z, \quad A := \partial_{\mathcal{R}(p, q)}, \quad N := z\partial_z,$$
where \( \partial_z := \frac{\partial}{\partial z} \) is the derivative on \( \mathbb{C} \).

The Witt algebra \( \mathcal{W} \) is the Lie algebra, which consists of derivations on the Laurent polynomial ring \( \mathbb{K}[z, z^{-1}] \), given by

\[
\mathcal{W} = \mathbb{K}[z, z^{-1}] \frac{d}{dz}.
\]

Setting \( l_n := -z^{n+1} \frac{d}{dz} \), then

\[
\mathcal{W} = \bigoplus_{n \in \mathbb{Z}} \mathbb{K} l_n,
\]

and these generators satisfy the commutation relations:

\[
[l_n, l_m] = (n - m) l_{n+m}.
\]

The Virasoro algebra

\[
\mathcal{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{K} L_n \oplus \mathbb{K} C
\]

is the Lie algebra which satisfies the following commutation relations:

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0} C,
\]

\[
[\mathcal{Vir}, C] = \{0\},
\]

where \( \delta_{i,j} \) denotes the Kronecker delta and \( C \) the central charge.

Let us finish this series of remindings by pointing out the connection between the Virasoro algebra and the Korteweg-de Vries (KdV) equation. For that, we use the relation (4). In order to give the realization of the algebra \( \mathcal{V} \) in terms of the currents which satisfy the Korteweg-de Vries (KdV) equation, Chaichian et al define the current as follows:

\[
u(x) := 6c \sum_{\infty}^{\infty} l_n e^{-inx} - \frac{1}{4}.
\]

The commutation relation (4) yields the relation:

\[
\left[ u(x), u(y) \right] = P \delta(x - y) = \left( \partial_{xxx} + 2u\partial_x + u_x \right) \delta(x - y),
\]

where \( P \) can be considered as the Hamiltonian operator for which the Schouten bracket vanishes, and hence it can be used to construct Hamiltonian systems. Indeed, defining the so-called second Hamiltonian as:

\[
H = \frac{1}{2} \int u^2(x) \, dx
\]
leads to the Hamiltonian equation

\[ u_t = P \text{grad} H = u_{xxx} + 6 u u_x \]

which is the Korteweg-de Vries equation.

In this paper, we construct the Witt algebra, its \( n \)-version, the Virasoro algebra and its constraints for a toy model, and the Korteweg-de Vries equation induced by the \( \mathcal{R}(p,q) \)-deformed quantum algebras\(^{16} \). We derive particular cases associated to quantum algebras developed in the literature, and discuss their main properties.

This paper is organized as follows. In section 2, we investigate the construction of the \( \mathcal{R}(p,q) \)-deformed Witt algebra and derive its main properties. Section 3 is dedicated to study the \( \mathcal{R}(p,q) \)-deformed Virasoro algebra and associated Korteweg-de Vries (KdV) equation. We deduce particular cases corresponding to deformed quantum algebras known in the literature. In section 4, we investigate the related Witt \( n \)-algebra and a toy model, and deduce relevant specific cases. We end with the concluding remarks in section 5.

II. \( \mathcal{R}(p,q) \)-DEFORMED WITT ALGEBRA

In this section, we study the \( \mathcal{R}(p,q) \)-deformed Witt algebra by using the \( \mathcal{R}(p,q) \)-derivative \(^{13} \)

\[ \partial_{\mathcal{R}(p,q)} \varphi(z) := \frac{1}{z} [z \partial_z]_{\mathcal{R}(p,q)} \varphi(z) \]

which generalizes particular derivatives known in the literature as follows:

(i) \( q \)-Heine derivative\(^{15} \)

\[ \mathcal{R}(1,q) = 1 \quad \text{and} \quad \partial_q \varphi(z) = \frac{1}{z} [z \partial_z]_q \varphi(z); \]

(ii) \( q \)-Quesne derivative\(^{27} \)

\[ \mathcal{R}(1,q) = \frac{1 - q^{-1}}{q - 1} \quad \text{and} \quad \partial_q \varphi(z) = \frac{1}{z} [z \partial_z]_q \varphi(z); \]

(iii) \( (p,q) \)-Jagannathan-Srinivasa derivative\(^{23} \)

\[ \mathcal{R}(p,q) = 1 \quad \text{and} \quad \partial_{p,q} \varphi(z) = \frac{1}{z} [z \partial_z]_{p,q} \varphi(z); \]
(iv) $(p^{-1}, q)$– Chakrabarty - Jagannathan derivative$^{19}$

\[
\mathcal{R}(p, q) = 1 \quad \text{and} \quad \partial_{p^{-1}, q} \varphi(z) = \frac{1}{z} [z \partial_z]_{p^{-1}, q} \varphi(z);
\]

(v) Hounkonnou-Ngompe generalization of $q$– Quesne derivative$^{18}$

\[
\mathcal{R}(p, q) = \frac{pq - 1}{(q - p^{-1})q} \quad \text{and} \quad \partial_{p, q} \varphi(z) = \frac{1}{z} [z \partial_z]_{p, q} \varphi(z);
\]

Let us introduce the model deformation structure functions $\epsilon_i, i \in \{1, 2\}$, depending on the deformation parameters $p$ and $q$, which allow to redefine the $\mathcal{R}(p, q)$–numbers \( \frac{2}{2} \) as

\[
[n]_{\mathcal{R}(p, q)} := \frac{\epsilon_1^n - \epsilon_2^n}{\epsilon_1 - \epsilon_2}, \quad \epsilon_1 \neq \epsilon_2
\]

from which known particular cases can be deduced, namely:

(i) $q$– Arick-Coon-Kuryskin deformation$^{2}$

\[
\epsilon_1 = 1, \quad \epsilon_2 = q \quad \text{and} \quad [n]_q = \frac{1 - q^n}{1 - q};
\]

(ii) $q$– Quesne deformation$^{27}$

\[
\epsilon_1 = 1, \quad \epsilon_2 = q^{-1} \quad \text{and} \quad [n]_q = \frac{1 - q^{-n}}{q - 1};
\]

(iii) $(p, q)$– Jagannathan-Srinivasa deformation$^{23}$

\[
\epsilon_1 = p, \quad \epsilon_2 = q \quad \text{and} \quad [n]_{p, q} = \frac{p^n - q^n}{p - q};
\]

(iv) $(p^{-1}, q)$– Chakrabarty -Jagannathan deformation$^{9}$

\[
\epsilon_1 = p^{-1}, \quad \epsilon_2 = q \quad \text{and} \quad [n]_{p^{-1}, q} = \frac{p^{-n} - q^n}{p^{-1} - q};
\]

(v) Hounkonnou-Ngompe generalization of $q$– Quesne deformation$^{18}$

\[
\epsilon_1 = p, \quad \epsilon_2 = q^{-1} \quad \text{and} \quad [n]_{p, q} = \frac{p^n - q^{-n}}{q - p^{-1}};
\]
This $\mathcal{R}(p,q)$-deformed Witt algebra is spanned by the generators $e_n^\mathcal{R}(p,q)$ acting on a holomorphic function $\varphi$ as

$$e_n^\mathcal{R}(p,q) \varphi(z) := z^{n+1} \partial_\mathcal{R}(p,q) \varphi(z),$$

or, equivalently, from the relation (5),

$$e_n^\mathcal{R}(p,q) \varphi(z) = \left[z \partial_z - n\right]_\mathcal{R}(p,q) z^n \varphi(z).$$

**Lemma II.1** The generators $e_n^\mathcal{R}(p,q)$ satisfy the commutation relation:

$$[e_n^\mathcal{R}(p,q), e_m^\mathcal{R}(p,q)]_{s,t} = \frac{1}{\epsilon_1 - \epsilon_2} \left(\epsilon_1 z \partial_z \left(s \epsilon_1^{-n} - t \epsilon_1^{-m}\right) - \epsilon_2 z \partial_z \left(s \epsilon_2^{-n} - t \epsilon_2^{-m}\right)\right) e_{n+m}^\mathcal{R}(p,q),$$

where

$$[[A, B]]_{u,v} = u AB - v BA,$n

$u$ and $v$ are arbitrary real (or complex) numbers. For $s = 1$ and $t = \epsilon_2^{m-n}$, we obtain:

$$\left[e_n^\mathcal{R}(p,q), e_m^\mathcal{R}(p,q)\right]_{1,\epsilon_2^{m-n}} = [m-n]_\mathcal{R}(p,q) \epsilon_1 z \partial_z - m \epsilon_1^{n} e_{n+m}^\mathcal{R}(p,q).$$

**Proof.** It stems from a straightforward computation. \hfill \Box

This suggests to define new $\mathcal{R}(p,q)$-deformed generators

$$l_n^\mathcal{R}(p,q) \varphi(z) := \epsilon_1^{-z \partial_z} \left[z \partial_z - n\right]_\mathcal{R}(p,q) z^n \varphi(z)$$

whose the commutator obeys the conventional Witt algebra structure given by the next Theorem II.1.

**Theorem II.1**

$$\left[l_n^\mathcal{R}(p,q), l_m^\mathcal{R}(p,q)\right]_{\epsilon_1^{m-n}, \epsilon_2^{m-n}} = [m-n]_\mathcal{R}(p,q) l_{n+m}^\mathcal{R}(p,q) \quad (8)$$

and

$$\left[l_n^\mathcal{R}(p,q), l_m^\mathcal{R}(p,q)\right] = [m-n]_\mathcal{R}(p,q) \epsilon_1^{-z \partial_z + n} \epsilon_2^{z \partial_z - m} l_{n+m}^\mathcal{R}(p,q) \quad (9)$$
The merit of the relations (8) and (9) consists of obtaining the $\mathcal{R}(p, q)$–deformed $su(1, 1)$ subalgebra:

$$\left[ l^\mathcal{R}(p, q)_0, l^\mathcal{R}(p, q)_1 \right]_{\epsilon_1, \epsilon_2} = l^\mathcal{R}(p, q)_1,$$

$$\left[ l^\mathcal{R}(p, q)_{-1}, l^\mathcal{R}(p, q)_0 \right]_{\epsilon_1, \epsilon_2} = l^\mathcal{R}(p, q)_{-1},$$

and

$$\left[ l^\mathcal{R}(p, q)_1, l^\mathcal{R}(p, q)_{-1} \right] = [2] l^\mathcal{R}(p, q) \epsilon_1^{-1} z \partial_z^{-1} \epsilon_2^{-1} l^\mathcal{R}(p, q)_0.$$

Note that:

- taking $\mathcal{R}(p, q) = 1$ and $\epsilon_1 = p^{-1}$, we recover the $(p, q)$–deformed Witt algebra displayed in \cite{20} with the generators $l^{p, q}_n$ acting as follows:

$$l^{p, q}_n \varphi(z) = p^z \varphi_{\epsilon_1} \left[ z \partial_z - n \right]_{p, q} z^n \varphi(z)$$

and satisfying the commutation relations

$$\left[ l^{p, q}_n, l^{p, q}_m \right]_{p^n - m, q^{-m} - n} = \left[ m - n \right]_{p, q} l^{p, q}_{n + m},$$

$$\left[ l^{p, q}_n, l^{p, q}_m \right] = \left[ m - n \right]_{p, q} p^z \varphi_{\epsilon_1} \left[ z \partial_z - n \right]_{p, q} z^n \varphi(z)$$

with

$$[x]_{p, q} = \frac{q^x - p^{-x}}{q - p^{-1}}; \tag{10}$$

- the $q$–deformed Witt algebra given in \cite{7} can be obtained by putting $\epsilon_1 = q$ and $\epsilon_2 = q^{-1}$, with the generators $l^q_n$:

$$l^q_n \varphi(z) = q^{-z \partial_z} \left[ z \partial_z - n \right]_q z^n \varphi(z)$$

obeying the law

$$\left[ l^q_n, l^q_m \right]_{q^n - m, q^{-m} - n} = \left[ m - n \right]_q l^q_{n + m},$$

$$\left[ l^q_n, l^q_m \right] = \left[ m - n \right]_q q^{-z \partial_z + n} q^{z \partial_z + m} l^q_{n + m},$$

and

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$
Remark II.1 We can readily deduce the Witt algebras for classes of known deformations as follows:

(i) Taking \( R(1, q) = 1 \), which implies \( \epsilon_1 = 1 \) and \( \epsilon_2 = q \), we obtain the \( q \)-deformed generators:

\[
q_n \phi(z) = [z \partial_z - n]_q z^n \phi(z)
\]

satisfying the relations:

\[
\begin{align*}
[l_q^n, l_m]_{q^{m-n}} &= [m-n]_q l_{n+m}^q, \\
[l_q^n, l_m^q] &= [m-n]q z \partial_z - m q^n l_{n+m}^q.
\end{align*}
\]

(ii) The Witt algebra associated to the Jagannathan - Srinivasa deformation is obtained with \( \epsilon_1 = p, \epsilon_2 = q \), and

\[
p^{p,q}_n \phi(z) = p^{-z \partial_z} [z \partial_z - n]_{p,q} z^n \phi(z),
\]

satisfying

\[
\begin{align*}
[p_{n}^{p,q}, l_{m}^{p,q}]_{p^{m-n},q^{m-n}} &= [m-n]_{p,q} p_{n+m}^{p,q}, \\
[l_{n}^{p,q}, l_{m}^{p,q}] &= [m-n]_{p,q} p^{-z \partial_z + n} q^{z \partial_z - m} l_{n+m}^{p,q}.
\end{align*}
\]

(iii) For \( \epsilon_1 = p^{-1} \) and \( \epsilon_2 = q \), we get the Witt algebra associated to the Chakrabarty-Jagannathan deformation, with

\[
p_{n}^{p^{-1},q} \phi(z) = p^{z \partial_z} [z \partial_z - n]_{p^{-1},q} z^n \phi(z),
\]

and

\[
\begin{align*}
[p_{n}^{p^{-1},q}, l_{m}^{p^{-1},q}]_{p^{m-n},q^{m-n}} &= [m-n]_{p^{-1},q} p_{n+m}^{p^{-1},q}, \\
[l_{n}^{p^{-1},q}, l_{m}^{p^{-1},q}] &= [m-n]_{p^{-1},q} p^{z \partial_z - n} q^{z \partial_z - m} p_{n+m}^{p^{-1},q}.
\end{align*}
\]

Moreover, the \((p^{-1}, q)\)-deformed \( su(1, 1) \) subalgebra is here furnished by:

\[
\begin{align*}
[p_{0}^{p^{-1},q}, p_{1}^{p^{-1},q}]_{p^{-1},q} &= p_{1}^{p^{-1},q}, \\
[p_{0}^{-1}, q, p_{0}^{-1}, q]_{p^{-1},q} &= p_{-1}^{p^{-1},q}, \\
[p_{1}^{-1}, q, p_{1}^{-1}, q] &= [2]_{p^{-1},q} p^{z \partial_z + 1} q^{z \partial_z - 1} l_{0}^{p^{-1},q}.
\end{align*}
\]
(iv) The Witt algebra corresponding to Hounkonnou-Ngompe generalization of $q$-Quesne deformation is built by taking $\epsilon_1 = p$, $\epsilon_2 = q^{-1}$,
\[ l_{n}^{p,q} \varphi(z) = p^{-z\partial_z - 1} q^n [z\partial_z - n]_{p,q} z^n \varphi(z), \]
with the relations:
\[
\begin{align*}
[l_{n}^{p,q}, l_{m}^{p,q}]_{p^{n-m}, q^{m-n}} &= qp^{-1} [m - n]_{p,q} l_{n+m}^{p,q}, \\
[l_{n}^{p,q}, l_{m}^{p,q}] &= qp^{-1} [m - n]_{p,q} p^{-z\partial_z + n} q^{-z\partial_z + m} l_{n+m}^{p,q}.
\end{align*}
\]
Furthermore, its $su(1,1)$ subalgebra is realized as follows:
\[
\begin{align*}
[l_{0}^{p,q}, l_{1}^{p,q}]_{p^{-1}} &= qp^{-1} l_{1}^{p,q}, \\
[l_{-1}^{p,q}, l_{0}^{p,q}]_{p^{-1}} &= qp^{-1} l_{-1}^{p,q}, \\
[l_{1}^{p,q}, l_{-1}^{p,q}] &= qp^{-1} [2]_{p,q} p^{-z\partial_z - 1} q^{-z\partial_z + 1} l_{0}^{p,q}.
\end{align*}
\]
The $R(p, q)$-deformed Witt algebra satisfies the Jacobi identity:
\[
\sum_{(i,j,l) \in \mathcal{C}(n,m,k)} \frac{1}{(\epsilon_1 \epsilon_2)^l} [2i]^{R(p,q)} [i^{R(p,q)}, [j^{R(p,q)}, l^{R(p,q)}]_{x_i j, y_{j+l}}] x_{(i+j+l), y_{(j+l)}} = 0,
\]
where $n$, $m$ and $k$ are natural numbers, and $\mathcal{C}(n,m,k)$ refers to the cyclic permutation of $(n,m,k)$.

**Remark II.2** Particular cases of the $R(p,q)$-deformed Jacobi identity are deduced as follows:

(i) For the $q$-deformed algebra in $^6$:
\[
\sum_{(i,j,l) \in \mathcal{C}(n,m,k)} \frac{[2i]_q}{[i]_q} [i_q, [j_q, l_q]_{x_{i(j+l)}, y_{j(l+l)}}] x_{(i+j+l), y_{(j+l)}} = 0.
\]

(ii) For the Arick-Coon $q$-deformation $^2$:
\[
\sum_{(i,j,l) \in \mathcal{C}(n,m,k)} \frac{1}{q} [2i]_{[i]_q} [i_q, [j_q, l_q]_{x_{i(j+l)}, y_{j(l+l)}}] x_{(i+j+l), y_{(j+l)}} = 0.
\]
(iii) For the \((p, q)\)– deformed algebra in \(\mathfrak{g}\):

\[
\sum_{(i, j, l) \in \mathcal{C}(n, m, k)} \left( \frac{p}{q} \right)^l \frac{[2i]_{p, q}}{[i]_{p, q}} \left( [p, q, l]_{p, q} \right)_{x_{i(j+i)}, y_{i(j+i)}} = 0.
\]

(iv) For the \textit{Jagannathan-Srinivasa} deformation:\n
\[
\sum_{(i, j, l) \in \mathcal{C}(n, m, k)} \frac{1}{(pq)^l} \frac{[2i]_{p^{-1}, q}}{[i]_{p^{-1}, q}} \left( [p^{-1}, q, l]_{p^{-1}, q} \right)_{x_{i(j+i)}, y_{i(j+i)}} = 0.
\]

(v) For the \textit{Chakrabarty} and \textit{Jagannathan} deformation:\n
\[
\sum_{(i, j, l) \in \mathcal{C}(n, m, k)} \frac{1}{(pq)^l} \frac{[2i]_{p^{-1}, q}}{[i]_{p^{-1}, q}} \left( [p^{-1}, q, l]_{p^{-1}, q} \right)_{x_{i(j+i)}, y_{i(j+i)}} = 0.
\]

(vi) For the \textit{Hounkonnou-Ngompe} generalization of \(q\)– \textit{Quesne} deformation, corresponding to \(\epsilon_1 = p\) and \(\epsilon_2 = q^{-1}\):

\[
\sum_{(i, j, l) \in \mathcal{C}(n, m, k)} \frac{1}{(pq)^l} \frac{[2i]_{p^{-1}, q}}{[i]_{p^{-1}, q}} \left( [p^{-1}, q, l]_{p^{-1}, q} \right)_{x_{i(j+i)}, y_{i(j+i)}} = 0.
\]

### A. Some properties of the \(\mathcal{R}(p, q)\)-deformed Witt algebra

This section is devoted to a list of some remarkable identities pertaining to the \(\mathcal{R}(p, q)\)– deformed Witt algebra. For \(n \neq m\), the \(\mathcal{R}(p, q)\)– deformed Witt generator product yields

\[
[r^{\mathcal{R}(p, q)}_n, r^{\mathcal{R}(p, q)}_m] = D^{\mathcal{R}(p, q)}_n r^{\mathcal{R}(p, q)}_{n+m},
\]

where

\[
D^{\mathcal{R}(p, q)}_n := \epsilon_1^{-z \partial_z + n} [z \partial_z - n]^{\mathcal{R}(p, q)}.
\]

The \(\mathcal{R}(p, q)\)– deformed Witt algebra equipped with the natural operator product \(\cdot\cdot\) is a nonassociative algebra with associator given by the relation:

\[
\left( r^{\mathcal{R}(p, q)}_n, r^{\mathcal{R}(p, q)}_m, r^{\mathcal{R}(p, q)}_k \right) := l^{\mathcal{R}(p, q)}_n \left( r^{\mathcal{R}(p, q)}_m, r^{\mathcal{R}(p, q)}_k \right) - \left( l^{\mathcal{R}(p, q)}_m, r^{\mathcal{R}(p, q)}_k \right) r^{\mathcal{R}(p, q)}_n = R^{n}_{km} - R^{n}_{k(m+n)},
\]

where

\[
R^{i}_{ij} = \epsilon_1^{-2z \partial_z + i+j} [z \partial_z - i]^{\mathcal{R}(p, q)} [z \partial_z - j]^{\mathcal{R}(p, q)}.
\]
The case \( n = 0 \) leads to an associative algebra as required with the trivial associator:
\[
\left( \tau_0^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right) = 0,
\]
for all \( m \) and \( k \).

**Property II.1**
\[
\left[ \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)} \right] = - \left[ \tau_m^{\mathcal{R}(p,q)}, \tau_n^{\mathcal{R}(p,q)} \right].
\]

**Property II.2**
\[
\sum_{(i,j,l) \in C(n,m,k)} \left( \tau_i^{\mathcal{R}(p,q)}, \tau_j^{\mathcal{R}(p,q)}, \tau_l^{\mathcal{R}(p,q)} \right) = \sum_{(i,j,l) \in C(n,k,m)} \left( \tau_i^{\mathcal{R}(p,q)}, \tau_j^{\mathcal{R}(p,q)}, \tau_l^{\mathcal{R}(p,q)} \right)
= \tau_n^{\mathcal{R}(p,q)} + \tau_k^{\mathcal{R}(p,q)} + \tau_m^{\mathcal{R}(p,q)},
\]
where \( C(n,m,k) \) denotes the cyclic permutation of \((n,m,k)\) and
\[
\tau_{ij} = \epsilon_1^{-2z\partial z + i+j[z\partial z - i - j]}R_{(p,q)} \left( \epsilon_1^i [z\partial z - i]R_{(p,q)} - \epsilon_1^j [z\partial z - j]R_{(p,q)} \right) \tau_{i+j+l}^{\mathcal{R}(p,q)}.
\]

**Property II.3**
\[
\left[ \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right] = \tau_n^{\mathcal{R}(p,q)} \left[ \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right] + \left[ \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)} \right] \tau_k^{\mathcal{R}(p,q)}
+ \mathcal{R}_{k(m+n)} - \mathcal{R}_{k(m+n)} + \mathcal{R}_{nk}^m - \mathcal{R}_{nk}^m.
\]
The \( \mathcal{R}(p,q) \)– derivation (Leibniz rule) should be satisfied if
\[
\mathcal{R}_{k(m+n)}^m - \mathcal{R}_{k(m+n)}^n = \mathcal{R}_{n(m+k)}^m - \mathcal{R}_{nk}^m.
\]

**Property II.4**
\[
\left( \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right) = \left( \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right) + \tau_{nk}^m.
\]

Therefore, the \( \mathcal{R}(p,q) \)– left symmetry property would require \( \tau_{nk}^m = 0 \).

**Property II.5 The Nambu 3– bracket defined by:**
\[
\left[ \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right] := \tau_n^{\mathcal{R}(p,q)} \left[ \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right] + \tau_m^{\mathcal{R}(p,q)} \left[ \tau_n^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right]
+ \tau_k^{\mathcal{R}(p,q)} \left[ \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)} \right],
\]
yields
\[
\left[ \tau_n^{\mathcal{R}(p,q)}, \tau_m^{\mathcal{R}(p,q)}, \tau_k^{\mathcal{R}(p,q)} \right] = \mathcal{R}_{km}^n + \mathcal{R}_{nk}^m + \mathcal{R}_{mn}^k.
\]
Property II.6

\[
\left[ \begin{bmatrix} l_n^R(p,q), l_m^R(p,q), l_k^R(p,q), l_s^R(p,q) \end{bmatrix} \right] - \left[ \begin{bmatrix} l_n^R(p,q), l_m^R(p,q), l_k^R(p,q) \end{bmatrix} \right] l_s^R(p,q) - l_k^R(p,q) \left[ \begin{bmatrix} l_n^R(p,q), l_m^R(p,q), l_k^R(p,q) \end{bmatrix} \right] = \mathcal{D}_{n+m+k}^R - \tau_{n,k}^m.
\]

Proposition II.1 The following identities, where the left and right multiplication operators are defined, respectively, by:

\[
L_a(b) := a.b \quad \text{and} \quad R_a(b) := b.a,
\]

hold:

(i)

\[
\left[ \begin{bmatrix} L_{l_n^R(p,q)}, L_{l_m^R(p,q)} \end{bmatrix} l_k^R(p,q) \right] = L_{l_n^R(p,q)} \left[ l_m^R(p,q), l_k^R(p,q) \right] + \tau_{n,m,k}^n, \quad \forall k,
\]

(ii)

\[
\left[ \begin{bmatrix} L_{l_n^R(p,q)}, R_{l_m^R(p,q)} \end{bmatrix} l_k^R(p,q) \right] = R_{l_n^R(p,q)} \left[ l_m^R(p,q), l_k^R(p,q) \right] + \tau_{n,m,k}^m - \tau_{m,n,k}^{n,m}.
\]

(iii)

\[
\left[ \begin{bmatrix} R_{l_n^R(p,q)}, l_k^R(p,q) \end{bmatrix} l_m^R(p,q) \right] = R_{l_n^R(p,q)} \left[ l_m^R(p,q), l_k^R(p,q) \right] - \tau_{n,m,k}^m + \tau_{n,m,k}^{m,n}.
\]

(iv)

\[
\left[ \begin{bmatrix} R_{l_n^R(p,q)} R_{l_m^R(p,q)} + R_{l_n^R(p,q)} R_{l_m^R(p,q)} \end{bmatrix} l_k^R(p,q) \right] = R_{l_n^R(p,q)} \left[ l_m^R(p,q) + R_{l_n^R(p,q)} R_{l_m^R(p,q)} \right] - \tau_{n,k}^m + \tau_{n,m,k}^{m,n}.
\]
III. $\mathcal{R}(p,q)$—DEFORMED VIRASORO ALGEBRA

In this section, the $\mathcal{R}(p,q)$-deformed Virasoro algebra is realized as the central extension of the Witt algebra \(\mathcal{W}\). It is generated by operators acting as follows

\[
L_n^{\mathcal{R}(p,q)}\varphi(z) := \epsilon_1 z^{\partial_z} \left[ z\partial_z - n \right]_{\mathcal{R}(p,q)} z^n \varphi(z)
\]

obeying the commutation relations

\[
\left[ L_n^{\mathcal{R}(p,q)}, L_m^{\mathcal{R}(p,q)} \right]_{\epsilon_1^{m-n}, \epsilon_2^{m-n}} = [m - n]_{\mathcal{R}(p,q)} L_n^{\mathcal{R}(p,q)} + \delta_{n+m,0} C_{\mathcal{R}(p,q)}(p,q),
\]

\[
\left[ L_k^{\mathcal{R}(p,q)}, C_{\mathcal{R}(p,q)}(n) \right]_{\epsilon_1^{k-n}, \epsilon_2^{k-n}} = 0,
\]

with the $\mathcal{R}(p,q)$—deformed central term $C_{\mathcal{R}(p,q)}(p,q)$ given by

\[
C_{\mathcal{R}(p,q)}(n) = C(p,q)(\epsilon_1 \epsilon_2)^{-2n} \frac{[n]_{\mathcal{R}(p,q)}}{[2n]_{\mathcal{R}(p,q)}} [n - 1]_{\mathcal{R}(p,q)} [n]_{\mathcal{R}(p,q)} [n + 1]_{\mathcal{R}(p,q)},
\]

where $C(p,q)$ is an arbitrary function of $(p,q)$.

It is worth noticing the realization of $\mathcal{R}(p,q)$-deformed Virasoro algebras in terms of concrete difference operators corresponding to known deformed quantum algebras as follows:

(i) For $\epsilon_1 = 1$ and $\epsilon_2 = q$, we obtain the Virasoro algebra associated to the Arick and Coon deformation \(^2\) with the generators $L_n^q$,

\[
L_n^q\varphi(z) = [z\partial_z - n]_q z^n \varphi(z)
\]

satisfying the commutation relations

\[
\left[ L_n^q, L_m^q \right]_{1,q^{m-n}} = [m - n]_q L_n^q + \delta_{n+m,0} C_q(n),
\]

\[
\left[ L_k^q, C_q(n) \right]_{1,q^{-k}} = 0,
\]

with

\[
C_q(n) = C(q)q^{-2n} \frac{[n]_q}{[2n]_q} [n - 1]_q [n]_q [n + 1]_q.
\]
(ii) The Virasoro algebra associated to the *Jagannathan-Srinivasa* deformation is derived by taking $\epsilon_1 = p$ and $\epsilon_2 = q$, giving

$$L^{p,q}_n \varphi(z) = \frac{p - z\partial_z}{p} \left[ z\partial_z - n \right]_{p,q} z^n \varphi(z),$$

which induces the commutation relations:

$$\left[ L_{n}, L_{m} \right]_{p-q, q^{-m-q}} = [m - n]_{p,q} L^{p,q}_{n+m} + \delta_{n+m,0} C_{p,q}(n),$$

$$\left[ L_{k}, C_{p,q}(n) \right]_{p-k, q^{-k-q}} = 0,$$

with

$$C_{p,q}(n) = C(p,q)(pq)^{-2n} \frac{[n]_{p,q}}{[2n]_{p,q}} [n - 1]_{p,q} [n]_{p,q} [n + 1]_{p,q}.$$  

(ii) Taking $\epsilon_1 = p^{-1}$ and $\epsilon_2 = q$, we deduce the Virasoro algebra associated to the *Chakrabarty- Jagannathan* deformation with the generator action :

$$L^{p^{-1},q}_n \varphi(z) = \frac{p - z\partial_z}{p^{-1}} \left[ z\partial_z - n \right]_{p^{-1},q} z^n \varphi(z),$$

giving the commutation relations

$$\left[ L^{p^{-1},q}_n, L^{p^{-1},q}_m \right]_{p^{-m-q}, q^{-m-q}} = [m - n]_{p^{-1},q} L^{p^{-1},q}_{n+m} + \delta_{n+m,0} C_{p^{-1},q}(n),$$

$$\left[ L^{p^{-1},q}_k, C_{p^{-1},q}(n) \right]_{p^{-k}, q^{-k-q}} = 0,$$

with

$$C_{p^{-1},q}(n) = C(p,q)(pq^{-1})^{-2n} \frac{[n]_{p^{-1},q}}{[2n]_{p^{-1},q}} [n - 1]_{p^{-1},q} [n]_{p^{-1},q} [n + 1]_{p^{-1},q}.$$  

(iv) The Virasoro algebra associated to the *Hounkonnou-Ngompe* generalization of $q-$Quesne deformation can be obtained by putting $\epsilon_1 = p$ and $\epsilon_2 = q^{-1}$ yielding :

$$L^{p,q}_n \varphi(z) = \frac{p - z\partial_z}{p} \left[ z\partial_z - n \right]_{p,q} z^n \varphi(z),$$

with the commutation relations:

$$\left[ L^{p,q}_n, L^{p,q}_m \right]_{p^{-m-q}, q^{-m-q}} = p q^{-1} [m - n]_{p,q} L^{p,q}_{n+m} + \delta_{n+m,0} C_{p,q}(n),$$

$$\left[ L^{p^{-1},q}_k, C_{p,q}(n) \right]_{p^{-k}, q^{-k-q}} = 0,$$

and

$$C_{p,q}(n) = C(p,q)(pq^{-1})^{-2n} \frac{[n]_{p,q}}{[2n]_{p,q}} [n - 1]_{p,q} [n]_{p,q} [n + 1]_{p,q}.$$
In addition, let us mention that the \( q \)– deformed Virasoro algebra given in 6 is easily recovered by taking \( \varepsilon_1 = q \) and \( \varepsilon_2 = q^{-1} \) as follows:

\[
L^q_n \varphi(z) = q \left[ z \partial_z - n \right]_q z^n \varphi(z),
\]

with

\[
\left[ L^q_n, L^q_m \right]_{q^n-m,q^m-n} = [m-n]_q L^q_{n+m} + \delta_{n+m,0} C_q(n),
\]

\[
\left[ L^q_k, C_q(n) \right]_{q^k,q^{-k}} = 0
\]

and

\[
C_q(n) = C(q) \frac{[n]_q}{[2n]_q} [n-1]_q [n]_q [n+1]_q,
\]

while the \((p,q)\)– deformed Virasoro algebra obtained in 7 is deduced by setting \( \varepsilon_1 = p \) and \( \varepsilon_2 = q \) with the characteristics:

\[
L^{p,q}_n \varphi(z) = p^{-z\partial_z} \left[ z \partial_z - n \right]_{p,q} z^n \varphi(z),
\]

\[
\left[ L^{p,q}_n, L^{p,q}_m \right]_{p^n-m,q^m-n} = [m-n]_{p,q} L^{p,q}_{n+m} + \delta_{n+m,0} C_{p,q}(n),
\]

\[
\left[ L^{p,q}_k, C_{p,q}(n) \right]_{p^k,q^{-k}} = 0,
\]

and

\[
C_{p,q}(n) = C(p,q)(p^{-1})^{-2n} \frac{[n]_{p,q}}{[2n]_{p,q}} [n-1]_{p,q} [n]_{p,q} [n+1]_{p,q}.
\]

A. \( R(p,q) \)– deformed Korteweg-de Vries equation

Let us now derive the \( R(p,q) \)– deformed Korteweg-de Vries (KdV) equation, and deduce its particular cases corresponding to known deformed quantum algebras. For that, we redefine the \( R(p,q) \)– deformed generators as follows:

\[
\mathcal{L}^{R(p,q)}_n \varphi(z) := \varepsilon_2^{-z\partial_z} \left[ z \partial_z - n \right]_{R(p,q)} z^n \varphi(z),
\]

with the commutator

\[
\left[ \mathcal{L}^{R(p,q)}_n, \mathcal{L}^{R(p,q)}_m \right] = [m-n]_{R(p,q)} \varepsilon_1^{z\partial_z-m} \varepsilon_2^{-z\partial_z+n} \mathcal{L}^{R(p,q)}_{n+m} + \delta_{n+m,0} C_{R(p,q)}(n),
\]

(15)
where
\[ C_{\mathcal{R}(p,q)}(n) = C(p,q)(\epsilon_1 \epsilon_2)^{-2n} \epsilon_1^{\partial_x} \frac{[n]_{\mathcal{R}(p,q)}}{[2n]_{\mathcal{R}(p,q)}} [n-1]_{\mathcal{R}(p,q)} [n]_{\mathcal{R}(p,q)} [n+1]_{\mathcal{R}(p,q)}. \]

The $\mathcal{R}(p,q)$- deformed current can thus be expressed by:
\[ w(\tau) := \sum_{n \in \mathbb{Z}} \mathcal{L}^\mathcal{R}(p,q) e^{i n \tau}. \]

We then arrive at the next statement.

**Theorem III.1** The $\mathcal{R}(p,q)$- deformed Korteweg-de Vries equation is written as:
\[
2 \sin \tau \frac{dv}{dt} = \Theta \left( e^{-2\tau \partial_x} v^2(x) - v(x) e^{2\tau \partial_x} u(x) \right) - 2 C_{\mathcal{R}(p,q)} \Theta^3 \sinh 2\tau \partial_x v(x),
\]
where $\Theta = (\epsilon_1 \epsilon_2)^{-1/2}$, $\Delta = \left( \frac{\epsilon_2}{\epsilon_1} \right)^{1/2}$.

**Proof.** From the commutation relation (15),

\[
[w(b), e^{-2\tau \partial_x} \delta(a-b)] = e^{-2\tau \partial_x} w(a) \delta(a-b)
\]

and setting $\Theta = (\epsilon_1 \epsilon_2)^{-1/2}$ and $\Delta = \left( \frac{\epsilon_2}{\epsilon_1} \right)^{1/2}$, we obtain:
\[
\left[ w(a), w(b) \right] = P \delta(a-b)
\]
\[
= \frac{2\pi i \Theta}{2 \sin \tau} \left( e^{-2\tau \partial_x} w(a) - w(a) e^{2\tau \partial_x} \right) \Delta^{-2N} \delta(a-b)
\]
\[
- \Theta^3 C_{\mathcal{R}(p,q)} \frac{\sinh \tau \partial_a \sinh \tau (\partial_a + i) \sinh \tau \partial_a \sinh \tau (\partial_a - i)}{\sinh 2\tau \partial_a \sinh^3 \tau}
\]
\[
\times \Delta^{-2N} \delta(a-b),
\]

where $P$ stands for the Hamiltonian operator. Then,
\[
\frac{dw}{dt} = \frac{\Theta}{4 \sin \tau} \left( e^{-2\tau \partial_x} w(a) - w(a) e^{2\tau \partial_x} \right) \left( \Delta^{-2N} w(a) + w(a) \Delta^{-2N} \right)
\]
\[
- \frac{\Theta^3}{2} C_{\mathcal{R}(p,q)} \frac{\sinh \tau \partial_a \sinh \tau (\partial_a + i) \sinh \tau \partial_a \sinh \tau (\partial_a - i)}{\sinh 2\tau \partial_a \sinh^3 \tau}
\]
\[
\times \left( \Delta^{-2N} w(a) + w(a) \Delta^{-2N} \right).
\]
Setting \( w(x) = \Delta^{2N} v(x) \), we expand (16) as
\[
\frac{dw}{dt} = \frac{\Theta}{4 \sin \tau} \left( e^{-2 \tau \partial_x} w(x) - w(x) e^{2 \tau \partial_x} \right) \left( v(x) + w(x) \Delta^{-2N} \right) \\
- \frac{\Theta^3}{2} C_{R(p,q)} \sinh \tau \partial_x \sinh \tau (\partial_x + i) \sinh \tau \partial_x \sinh \tau (\partial_x - i) \sinh \tau \partial_x \\
\times \left( v(x) + w(x) \Delta^{-2N} \right) \\
= \frac{\Theta \Delta^{2N}}{4 \sin \tau} \left( e^{-2 \tau \partial_x} v(x) - v(x) e^{2 \tau \partial_x} \right) \left( v(x) + \Delta^{2N} v(x) \Delta^{-2N} \right) \\
- \frac{\Theta^3}{2} C_{R(p,q)} \sinh \tau \partial_x \sinh \tau (\partial_x + i) \sinh \tau \partial_x \sinh \tau (\partial_x - i) \sinh \tau \partial_x \\
\times \left( v(x) + \Delta^{2N} v(x) \Delta^{-2N} \right) \\
= \frac{\Theta \Delta^{2N}}{2 \sin \tau} \left( e^{-2 \tau \partial_x} v(x) - v(x) e^{2 \tau \partial_x} \right) v(x) \\
- \frac{\Theta^3}{2} C_{R(p,q)} \sinh \tau \partial_x \sinh \tau (\partial_x + i) \sinh \tau \partial_x \sinh \tau (\partial_x - i) \sinh \tau \partial_x \frac{v(x)}{\sinh \tau} \\
\]
yielding the \( R(p,q) \)- deformed Korteweg-de Vries equation
\[
2 \sin \tau \frac{dv}{dt} = \Theta \left( e^{-2 \tau \partial_x} v^2(x) - v(x) e^{2 \tau \partial_x} v(x) \right) - 2 C_{R(p,q)} \Theta^3 \sinh 2 \tau \partial_x v(x). 
\]

For \( \epsilon_1 = q \) and \( \epsilon_2 = q^{-1} \), we recover the \( q \)- deformed KdV equation (7):
\[
2 \sin \tau \frac{dv}{dt} = e^{-2 \tau \partial_x} v^2(x) - v(x) e^{2 \tau \partial_x} v(x) - 2 C_q \sinh 2 \tau \partial_x v(x). 
\]
The \( (p, q) \)-deformed KdV equation given in (8) can be obtained by taking \( \epsilon_1 = q \) and \( \epsilon_2 = p^{-1} \):
\[
\frac{dw}{dt} = \frac{\Theta}{4 \sin \tau} \left( e^{-2 \tau \partial_a} w(a) - w(a) e^{2 \tau \partial_a} \right) \left( \Delta^{-2N} w(a) + w(a) \Delta^{-2N} \right) \\
- \frac{\Theta^3}{2} C_{p,q} \sinh \tau \partial_a \sinh \tau (\partial_a + i) \sinh \tau \partial_a \sinh \tau (\partial_a - i) \sinh \tau \partial_a \\
\times \left( \Delta^{-2N} w(a) + w(a) \Delta^{-2N} \right) \\
\text{with } \Theta = \left( q p^{-1} \right)^{1/2} \text{ and } \Delta = \left( q p \right)^{1/2}. 
\]
Now, we can easily deduce relevant particular new Korteweg-de Vries (KdV) equations associated with deformations spread in the literature as follows:

(i) For \( \epsilon_1 = 1 \) and \( \epsilon_2 = q \), we deduct the KdV equation associated to the Arick and Coon deformation (2):
\[
2 \sin \tau \frac{dv}{dt} = \Theta \left( e^{-2 \tau \partial_x} v^2(x) - v(x) e^{2 \tau \partial_x} v(x) \right) v(x) - 2 C_q \Theta^3 \sinh 2 \tau \partial_x v(x) \\
\text{where } \Theta = q^{-1/2}. 
\]
(ii) The Korteweg-de Vries equation associated to the Jagannathan-Sirinivasa deformation\(^{23}\) can be established by putting \(\epsilon_1 = p\) and \(\epsilon_2 = q\):

\[
2 \sin \tau \frac{dv}{dt} = \Theta \left( e^{-2\tau \partial_x} v^2(x) - v(x) e^{2\tau \partial_x} v(x) \right) - 2 C_{p,q} \Theta^3 \sinh 2\tau \partial_x v(x)
\]

with \(\Theta = \left( \frac{p}{q} \right)^{-1/2}\).

(iii) Taking \(\epsilon_1 = p^{-1}\) and \(\epsilon_2 = q\), we obtain the KdV equation associated to the Chakrabarty-Jagannathan deformation\(^{10}\):

\[
2 \sin \tau \frac{dv}{dt} = \Theta \left( e^{-2\tau \partial_x} v^2(x) - v(x) e^{2\tau \partial_x} v(x) \right) - 2 C_{p^{-1},q} \Theta^3 \sinh 2\tau \partial_x v(x)
\]

with \(\Theta = \left( \frac{p^{-1}q}{\|} \right)^{-1/2}\).

(iv) The KdV equation corresponding to the Hounkonnou-Ngompe generalization of Quesne\(^{18}\) is deduced by taking \(\epsilon_1 = p\) and \(\epsilon_2 = q^{-1}\):

\[
2 \sin \tau \frac{dv}{dt} = \Theta \left( e^{-2\tau \partial_x} v^2(x) - v(x) e^{2\tau \partial_x} v(x) \right) - 2 C_{p,q}^{Q} \Theta^3 \sinh 2\tau \partial_x v(x)
\]

where \(\Theta = \left( \frac{p}{q^{-1}} \right)^{-1/2}\).

IV. \(\mathcal{R}(p,q)-\) DEFORMED WITT \(n\)- ALGEBRA

Let us consider the operators defined by:

\[
\mathcal{T}_m^{\mathcal{R}(p^\alpha,q^\alpha)} := -\mathcal{D}_{\mathcal{R}(p^\alpha,q^\alpha)} z^{m+1}, \quad (17)
\]

where \(\mathcal{D}_{\mathcal{R}(p^\alpha,q^\alpha)}\) is the \(\mathcal{R}(p,q)-\) deformed derivative given by:

\[
\mathcal{D}_{\mathcal{R}(p^\alpha,q^\alpha)}(\phi(z)) = \frac{p^\alpha - q^\alpha}{p^\alpha P - q^\alpha Q} \mathcal{R}(p^\alpha P, q^\alpha Q) \frac{\phi(p^\alpha z) - \phi(q^\alpha z)}{p^\alpha - q^\alpha}.
\]

From (2), the operators (17) take the form

\[
\mathcal{T}_m^{\mathcal{R}(p^\alpha,q^\alpha)} = -[m + 1]_{\mathcal{R}(p^\alpha,q^\alpha)} z^m.
\]
Proposition IV.1 The operators \([17]\) satisfy the product relation
\[
\mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)} \cdot \mathcal{T}_n^{\mathcal{R}(p^\beta,q^n)} = \frac{(\epsilon_1^{\alpha+\beta} - \epsilon_2^{\alpha+\beta}) \alpha^m \beta}{(\epsilon_1^\alpha - \epsilon_2^\alpha)(\epsilon_1^\beta - \epsilon_2^\beta)} \mathcal{T}_m^{\mathcal{R}(p^{\alpha+\beta},q^{\alpha+\beta})} + \frac{\epsilon_2^{(n+1)\beta}}{\epsilon_1^\beta - \epsilon_2^\beta} \mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)}
\]
and the commutation relation
\[
[\mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)}, \mathcal{T}_n^{\mathcal{R}(p^\beta,q^n)}] = \frac{\epsilon_1^{-\alpha} \epsilon_2^{-\beta}}{\epsilon_1 - \epsilon_2} \mathcal{T}_m^{\mathcal{R}(p^{\alpha+\beta},q^{\alpha+\beta})} - \frac{\epsilon_2^{(m+n+1)\beta}}{\epsilon_1 - \epsilon_2} \mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)} + \frac{\epsilon_2^{(m+n+1)\alpha}}{\epsilon_1 - \epsilon_2} \mathcal{T}_m^{\mathcal{R}(p^\beta,q^n)}. \tag{18}
\]
Taking \(\alpha = \beta = 1\), we obtain:
\[
[\mathcal{T}_m^{\mathcal{R}(p,q)}, \mathcal{T}_n^{\mathcal{R}(p,q)}] = \frac{[-n + 1]}{(\epsilon_1 - \epsilon_2)} \mathcal{T}_m^{\mathcal{R}(p^2,q^2)} - \frac{\epsilon_2^{m+n+1}}{\epsilon_1 - \epsilon_2} \left( (\epsilon_1^{-n} - \epsilon_2^{-m}) - (\epsilon_1^{-m} - \epsilon_2^{-n}) \right) \mathcal{T}_m^{\mathcal{R}(p,q)}. \tag{19}
\]
Note that for \(\mathcal{R}(q,1) = 1\), involving \(\epsilon_1 = q\) and \(\epsilon_2 = q\), we obtain the result given in\(^{25}\):
\[
[\mathcal{T}_m^q, \mathcal{T}_n^q] = ([m + 1]_q - [n + 1]_q) \mathcal{T}_m^{q^n} = ([n]_q - [m]_q) ([2]_q \mathcal{T}_m^{q^n} - \mathcal{T}_m^{q^n}).
\]
We consider the \(n\)– bracket defined by:
\[
[\mathcal{T}_m^{\mathcal{R}(p^{\alpha_1},q^{\alpha_1})}, \ldots, \mathcal{T}_m^{\mathcal{R}(p^{\alpha_n},q^{\alpha_n})}] := \Gamma_{1\ldots n}^{i_1\ldots i_p} \mathcal{T}_m^{\mathcal{R}(p^{\alpha_1},q^{\alpha_1})} \ldots \mathcal{T}_m^{\mathcal{R}(p^{\alpha_n},q^{\alpha_n})}
\]
where \(\Gamma_{1\ldots n}^{i_1\ldots i_p}\) is the Lévi-Civitá symbol given by:
\[
\Gamma_{i_1\ldots i_p}^{j_1\ldots j_p} = \text{det} \begin{pmatrix} \delta_{i_1}^{j_1} & \ldots & \delta_{i_1}^{j_p} \\ \vdots & \ddots & \vdots \\ \delta_{i_p}^{j_1} & \ldots & \delta_{i_p}^{j_p} \end{pmatrix}.
\]
Focussing on the case with the same \(\mathcal{R}(p^\alpha, q^n)\) leads to
\[
[\mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)}, \ldots, \mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)}] = \Gamma_{1\ldots n}^{i_1\ldots i_p} \mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)} \ldots \mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)}.
\]
Taking \(\alpha = \beta\) in the relation\(^{19}\), we obtain:
\[
[\mathcal{T}_m^{\mathcal{R}(p^\alpha,q^n)}, \mathcal{T}_n^{\mathcal{R}(p^\alpha,q^n)}] = \frac{(\epsilon_1^{-\alpha n} - \epsilon_2^{-\alpha n})}{(\epsilon_1^\alpha - \epsilon_2^\alpha)} [2]_{\mathcal{R}(p^\alpha,q^n)} \mathcal{T}_m^{\mathcal{R}(p^{2\alpha},q^{2\alpha})}.
\]
The $n-$ bracket can then be rewritten as follows:

$$
\left[ \mathcal{T}_{m_1}^{\mathcal{R}(p^\alpha,q^\alpha)}, \ldots, \mathcal{T}_{m_n}^{\mathcal{R}(p^\alpha,q^\alpha)} \right] = \frac{(-1)^{n+1}}{(\epsilon_1^\alpha - \epsilon_2^\alpha)^{n-1}} \left( H_n^\alpha [n]_{\mathcal{R}(p^\alpha,q^\alpha)} \prod_{\sum_{i=1}^{n} m_i + 1}^{\mathcal{R}(p^\alpha,q^\alpha)} - [n - 1]_{\mathcal{R}(p^\alpha,q^\alpha)} \right),
$$

where

$$
H_n^\alpha = \epsilon_1^{-\alpha(n-1) \sum_{i=1}^{n} m_i} \left( (\epsilon_1^\alpha - \epsilon_2^\alpha)^{\binom{n}{2}} \prod_{1 \leq j < k \leq n} (m_j)_{\mathcal{R}(q^\alpha)} - (m_j)_{\mathcal{R}(p^\alpha,q^\alpha)} \right)
+ \prod_{1 \leq j < k \leq n} (\epsilon_2^\alpha m_k - \epsilon_2^\alpha m_j).
$$

and

$$
M_n^\alpha = \epsilon_2^{-\alpha(n-1) \sum_{i=1}^{n} m_i} \left( (\epsilon_1^\alpha - \epsilon_2^\alpha)^{\binom{n}{2}} \prod_{1 \leq j < k \leq n} (m_j)_{\mathcal{R}(q^\alpha)} - (m_j)_{\mathcal{R}(p^\alpha,q^\alpha)} \right) + (-1)^{n-1} \prod_{1 \leq j < k \leq n} (\epsilon_2^\alpha m_k - \epsilon_2^\alpha m_j).
$$

Note that the $q-$ Witt $n-$ algebra obtained in\(^{30}\) corresponds to the particular case of $\mathcal{R}(q,1) = 1$:

$$
\left[ \mathcal{T}_{m_1}^{q^\alpha}, \ldots, \mathcal{T}_{m_n}^{q^\alpha} \right] = \frac{(-1)^{n+1}}{(q^\alpha - 1)^{n-1}} \left( [n]_{q^\alpha} \prod_{\sum_{i=1}^{n} m_i + 1}^{\mathcal{T}_{m_1}^{\mathcal{R}(q^\alpha)}} - [n - 1]_{q^\alpha} \prod_{\sum_{i=1}^{n} m_i + 1}^{\mathcal{T}_{m_1}^{\mathcal{R}(q^\alpha)}} \right),
$$

with

$$
H_n^\alpha = (q^\alpha - 1)^{\binom{n}{2}} q^{-\alpha(n-1) \sum_{i=1}^{n} m_i} \prod_{1 \leq j < k \leq n} \left( (m_j)_{q^\alpha} - (m_j)_{q^\alpha} \right).
$$

Taking $n = 3$ in the relation\(^{20}\), we obtain the $\mathcal{R}(p,q)-$ Witt 3- algebra:

$$
\left[ \mathcal{T}_{m_1}^{\mathcal{R}(p^\alpha,q^\alpha)}, \mathcal{T}_{m_2}^{\mathcal{R}(p^\alpha,q^\alpha)}, \mathcal{T}_{m_3}^{\mathcal{R}(p^\alpha,q^\alpha)} \right] = \frac{1}{(\epsilon_1^\alpha - \epsilon_2^\alpha)^3} \left( H_3^3 [3]_{\mathcal{R}(p^\alpha,q^\alpha)} \prod_{\sum_{i=1}^{n} m_i + 1}^{\mathcal{R}(p^\alpha,q^\alpha)} - [n - 1]_{\mathcal{R}(p^\alpha,q^\alpha)} \right).
$$

where

$$
H_3^3 = (\epsilon_1^\alpha - \epsilon_2^\alpha)^3 \left( (m_2)_{\mathcal{R}(p^\alpha,q^\alpha)} - (m_1)_{\mathcal{R}(p^\alpha,q^\alpha)} \right)
\times \left( (m_3)_{\mathcal{R}(p^\alpha,q^\alpha)} - (m_1)_{\mathcal{R}(p^\alpha,q^\alpha)} \right) \left( (m_3)_{\mathcal{R}(p^\alpha,q^\alpha)} - (m_2)_{\mathcal{R}(p^\alpha,q^\alpha)} \right)
+ \epsilon_1^{-2\alpha(m_1+m_2+m_3)} \left( \epsilon_2^\alpha m_2 - \epsilon_2^\alpha m_1 \right) \left( \epsilon_2^\alpha m_3 - \epsilon_2^\alpha m_1 \right) \left( \epsilon_2^\alpha m_3 - \epsilon_2^\alpha m_2 \right).
$$
and
\[ M_\alpha^3 = (\epsilon_1^\alpha - \epsilon_2^\alpha)^3 \epsilon_2^{-2\alpha(m_1+m_2+m_3)} \left( [m_2]_{\mathcal{R}(p^\alpha, q^\alpha)} - [m_1]_{\mathcal{R}(p^\alpha, q^\alpha)} \right) \times \left( [m_3]_{\mathcal{R}(p^\alpha, q^\alpha)} - [m_1]_{\mathcal{R}(p^\alpha, q^\alpha)} \right) \left( [m_3]_{\mathcal{R}(p^\alpha, q^\alpha)} - [m_2]_{\mathcal{R}(p^\alpha, q^\alpha)} \right)
+ \epsilon_2^{-2\alpha(m_1+m_2+m_3)} (\epsilon_1^\alpha m_2 - \epsilon_1^\alpha m_1) (\epsilon_1^{\alpha m_3} - \epsilon_1^{\alpha m_1}) (\epsilon_1^{\alpha m_3} - \epsilon_1^{\alpha m_2}). \]

A. A toy model for \( \mathcal{R}(p, q) - \text{Virasoro constraints} \)

In this section, we study a toy model for the \( \mathcal{R}(p, q) - \text{deformed Virasoro constraints} \), which play an important role in the study of matrix models. Let us consider the generating function with infinitely many parameters given as follows:

\[ Z_{\text{toy}}(t) = \int x^\gamma \exp \left( \sum_{s=0}^\infty t_s x^s \right) dx, \]

which encodes many different integrals. The following property holds for the \( \mathcal{R}(p, q) - \text{deformed derivative} \)

\[ \int_{\mathbb{R}} \mathcal{D}_{\mathcal{R}(p^\alpha, q^\alpha)} f(x) dx = h(p^\alpha, q^\alpha) \left( \int_{-\infty}^{+\infty} f(\epsilon_1^\alpha x) dx - \int_{-\infty}^{+\infty} f(\epsilon_2^\alpha x) dx \right) = 0, \]

where

\[ h(p^\alpha, q^\alpha) = \frac{p^\alpha - q^\alpha}{p^{\alpha n} - q^{\alpha n}} \mathcal{R}(p^{\alpha n}, q^{\alpha n}). \]

For \( f(x) = x^{m+\gamma+1} \exp \left( \sum_{s=0}^\infty \frac{t_s}{s!} x^s \right) \), we have

\[ \int_{-\infty}^{+\infty} \mathcal{D}_{\mathcal{R}(p^\alpha, q^\alpha)} \left( x^{m+\gamma+1} \exp \left( \sum_{s=0}^\infty \frac{t_s}{s!} x^s \right) \right) dx = 0. \]

Considering the following expansion:

\[ \exp \left( \sum_{s=0}^\infty \frac{t_s}{s!} x^s \right) = \sum_{n=0}^\infty B_n(t_1, \cdots, t_n) \frac{x^n}{n!}, \]

where \( B_n \) is the Bell polynomials, we get:

\[ \mathcal{D}_{\mathcal{R}(p^\alpha, q^\alpha)} \left( x^{m+\gamma+1} \exp \left( \sum_{s=0}^\infty \frac{t_s}{s!} x^s \right) \right) = \left( [m + 1 + \gamma]_{\mathcal{R}(p^\alpha, q^\alpha)} \frac{x^m}{\epsilon_1^{\alpha m}} + h(p^\alpha, q^\alpha) \frac{\epsilon_2^{\alpha (m+1+\gamma)}}{\epsilon_1^\alpha - \epsilon_2^\alpha} \sum_{k=1}^{\infty} B_k(t_{1\alpha}, \cdots, t_{k\alpha}) \frac{x^{k+m}}{k!} \right) x^\gamma \exp \left( \sum_{s=0}^\infty \frac{t_s}{s!} x^s \right), \]

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where \( t_k^\alpha = (\epsilon_1^k - \epsilon_2^k) t_k \). Then, using the constraints on the partition function,

\[
\mathcal{T}_m^{\mathcal{R}(p^\alpha, q^\alpha)} Z^{(t^o)}(t) = 0, \quad m \geq 0,
\]

we obtain

\[
\mathcal{T}_m^{\mathcal{R}(p^\alpha, q^\alpha)} = [m + 1 + \gamma]_{\mathcal{R}(p^\alpha, q^\alpha)} m! \epsilon_1^{-m} \frac{\partial}{\partial t_m} + h(p^\alpha, q^\alpha) \sum_{k=1}^{\infty} \frac{(k + m)!}{k!} B_k(t_1^\alpha, \ldots, t_k^\alpha) \frac{\partial}{\partial t_{k+m}}.
\]

Setting \( \bar{m} = m + 1 + \gamma \), \( \bar{n} = n + 1 + \gamma \), and using the substitution,

\[
n! \frac{\partial}{\partial t_n} \rightarrow x^n,
\]

we get

\[
\mathcal{T}_m^{\mathcal{R}(p^\alpha, q^\alpha)}, \mathcal{T}_n^{\mathcal{R}(p^\beta, q^\beta)} = \left[ \frac{\mathcal{T}_m^{\mathcal{R}(p^\alpha, q^\alpha)} \mathcal{T}_n^{\mathcal{R}(p^\beta, q^\beta)}}{\mathcal{T}_{m+n}^{\mathcal{R}(p^\alpha, q^\alpha)}} \right] \times \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} B_k(t_1^\alpha, \ldots, t_k^\alpha) B_l(t_1^\beta, \ldots, t_l^\beta) \frac{1}{k! l!} x^{k+l+m+n} + \frac{h(p^\alpha, q^\alpha) h(p^\beta, q^\beta) \epsilon_2^\alpha \bar{m}^\alpha \bar{n}}{(\epsilon_1^\alpha - \epsilon_2^\alpha)(\epsilon_1^\beta - \epsilon_2^\beta)} \sum_{k=1}^{\infty} B_k(t_1^\alpha, \ldots, t_k^\alpha) \frac{1}{k!} x^{m+k+n} + \frac{h(p^\alpha, q^\alpha) h(p^\beta, q^\beta) \epsilon_2^\alpha \bar{m}^\alpha \bar{n}}{(\epsilon_1^\alpha - \epsilon_2^\alpha)(\epsilon_1^\beta - \epsilon_2^\beta)} \sum_{k=1}^{\infty} B_k(t_1^\beta, \ldots, t_k^\beta) \frac{1}{k!} x^{n+k+m} - \frac{h(p^\alpha, q^\alpha) h(p^\beta, q^\beta) \epsilon_2^\alpha \bar{m}^\alpha \bar{n}}{(\epsilon_1^\alpha - \epsilon_2^\alpha)(\epsilon_1^\beta - \epsilon_2^\beta)} \sum_{k=1}^{\infty} B_k(t_1^\alpha, \ldots, t_k^\alpha) \frac{1}{k!} x^{k+m+n}.
\]

After computation, we get

\[
\mathcal{T}_m^{\mathcal{R}(p^\alpha, q^\alpha)}, \mathcal{T}_n^{\mathcal{R}(p^\beta, q^\beta)} \sim \left( \frac{(\alpha + \beta - \epsilon_1^\alpha - \epsilon_2^\beta)}{\epsilon_1^{\alpha + \beta} (\epsilon_1^\alpha - \epsilon_2^\beta)(\epsilon_1^\beta - \epsilon_2^\beta)} \right)^{m \beta} \mathcal{T}_{m+n}^{\mathcal{R}(p^{\alpha + \beta}, q^{\alpha + \beta})} - \frac{\epsilon_2^{(n+1)\beta}}{\epsilon_1^{\alpha + \beta} (\epsilon_1^\beta - \epsilon_2^\beta)} \mathcal{T}_{m+n}^{\mathcal{R}(p^{\alpha}, q^{\alpha})}
\]

where \( \sim \) denotes the equivalence.

**B. relevant particular cases**

1. **Witt \( n \)– algebra associated to the Arick and Coon deformation\(^2\)**

Putting \( \mathcal{R}(q, 1) = 1 \), we obtain the \( q \)– deformed Witt \( n \)– algebra associated to the **Arick** and **Coon** deformation\(^2\). We define the \( q \)– deformed operators as follows:

\[
\mathcal{T}_m^{q^\alpha} := -D_{q^\alpha} z^{m+1},
\]

(21)
where $\mathcal{D}_{q^\alpha}$ is the $q$– deformed derivative given by:

$$
\mathcal{D}_{q^\alpha}(\phi(z)) = \frac{\phi(qz) - \phi(q^{-1} z)}{q - q^{-1}}
$$

and the $q$– deformed number is

$$
[n]_{q^\alpha} = \frac{q^n - q^{-\alpha n}}{q^\alpha - q^{-\alpha}}.
$$

Thus, the operators (21) are reduced to

$$
T_{q^\alpha}^m = -[m + 1]_{q^\alpha} z^m
$$

and satisfy the relation

$$
[T_{q^\alpha}^m, T_{q^\beta}^n] = [m + 1]_{q^\alpha} T_{q^\beta}^{m+n} - [n + 1]_{q^\beta} T_{q^\alpha}^{m+n}.
$$

Using the $q$– number, we obtain

$$
T_{q^\alpha}^m . T_{q^\beta}^n = \frac{(q^{\alpha + \beta} - q^{-\alpha - \beta})q^{-\beta m}}{(q^\alpha - q^{-\alpha})(q^\beta - q^{-\beta})} T_{q^\alpha + \beta}^{m+n} + \frac{q^{-\beta(n+1)}}{q^\beta - q^{-\beta}} T_{q^\alpha}^{m+n} + \frac{q^{-\alpha(m+n+1)-\beta m}}{q^\alpha - q^{-\alpha}} T_{q^\beta}^{m+n}
$$

and the commutation relation

$$
[T_{q^\alpha}^m, T_{q^\beta}^n] = \frac{(q^{\alpha + \beta} - q^{-\alpha - \beta})(q^{-\alpha n} - q^{-\beta m})}{(q^\alpha - q^{-\alpha})(q^\beta - q^{-\beta})} T_{q^\alpha + \beta}^{m+n} + \frac{q^{-\beta(n+1)}}{q^\beta - q^{-\beta}} T_{q^\alpha}^{m+n} + \frac{q^{-\alpha(m+n+1)-\beta m}}{q^\alpha - q^{-\alpha}} T_{q^\beta}^{m+n}.
$$

(22)

Taking $\alpha = \beta = 1$ in (22), we obtain

$$
[T_{q^1}^m, T_{q^1}^n] = \frac{(q^{-n} - q^{-m})}{q - q^{-1}} \left( [2]_{q^{-1}} T_{q^1}^{m+n} + q^{-1} (1 - q^{-1}) T_{q^1}^{m+n} \right),
$$

which can be rewritten as:

$$
[T_{q^1}^m, T_{q^1}^n] = \frac{q^m}{q + 1} \{m - n\}_q \left( [2]_{q^1} T_{q^1}^{m+n} - q^{-1} (q - 1) \{m - n\}_q T_{q^1}^{m+n} \right),
$$

(23)

where

$$
\{x\} = \frac{q^x - 1}{q - 1}.
$$

Taking the limit $q \rightarrow 1$, the algebra (23) gives the Witt algebra.

The $n$– bracket with the same $q^\alpha$ is given by:

$$
[T_{q^\alpha}^{m_1}, \cdots, T_{q^\alpha}^{m_n}] = \Gamma^{1 \cdots n}_{1 \cdots n} T_{q^\alpha}^{m_1} \cdots T_{q^\alpha}^{m_n}.
$$
For \( \beta = \alpha \), the relation (22) takes the following form:

\[
[T^{q\alpha}_{m_1}, T^{q\alpha}_{m_2}] = \frac{-(q^{-\alpha m_1} - q^{-\alpha m_2})}{q^{\alpha} - q^{-\alpha}} (2) q^{q^{3\alpha}} T^{q^{3\alpha}}_{m_1 + m_2} - q^{-\alpha} (1 - q^{-\alpha (m_1 + m_2)}) T^{q^{\alpha}}_{m_1 + m_2}.
\]

By induction, we deduce the \( n \)- bracket as:

\[
[T^{q\alpha}_{m_1}, \ldots, T^{q\alpha}_{m_n}] = \frac{(-1)^{n+1}}{(q^{\alpha} - q^{-\alpha})^{n-1}} (H^{n}_{\alpha} [n] q^{\alpha} T^{q^{n\alpha}}_{m_1 + \ldots + m_n} - [n - 1] q^{\alpha} \times q^{-\alpha (\sum_{i=1}^{n} m_i + 1)} (H^{n}_{\alpha} + M^{n}_{\alpha}) T^{q^{(n-1)\alpha}}_{m_1 + \ldots + m_n}), (24)
\]

where

\[
H^{n}_{\alpha} = q^{-\alpha (n-1) \sum_{s=1}^{n} m_s} (q^{\alpha} - q^{-\alpha})^{n-2} \prod_{1 \leq j < k \leq n} ([m_k] q^{\alpha} - [m_j] q^{\alpha}) + \prod_{1 \leq j < k \leq n} (q^{-\alpha m_k} - q^{-\alpha m_j})
\]

and

\[
M^{n}_{\alpha} = q^{\alpha (n-1) \sum_{s=1}^{n} m_s} (q^{\alpha} - q^{-\alpha})^{n-2} \prod_{1 \leq j < k \leq n} ([m_k] q^{\alpha} - [m_j] q^{\alpha}) + (-1)^{n-1} \prod_{1 \leq j < k \leq n} (q^{\alpha m_k} - q^{\alpha m_j}).
\]

Taking \( n = 3 \) in (24), we obtain the Witt 3- algebra related to the Arick and Coon deformation\(^2\):

\[
[T^{q\alpha}_{m_1}, T^{q\alpha}_{m_2}, T^{q\alpha}_{m_3}] = \frac{1}{(q^{\alpha} - q^{-\alpha})^2} (H^{3}_{\alpha} [3] q^{\alpha} T^{q^{3\alpha}}_{m_1 + \ldots + m_3} - [n - 1] q^{\alpha} \times q^{-\alpha (\sum_{i=1}^{3} m_i + 1)} (H^{3}_{\alpha} + M^{3}_{\alpha}) T^{q^{2\alpha}}_{m_1 + \ldots + m_3}),
\]

where

\[
H^{3}_{\alpha} = (q^{\alpha} - q^{-\alpha})^3 q^{-2\alpha (m_1 + m_2 + m_3)} (m_2) q^{\alpha} - [m_1] q^{\alpha}) (m_3) q^{\alpha} - [m_1] q^{\alpha}) (m_3) q^{\alpha} - [m_2] q^{\alpha}) + q^{-2\alpha (m_1 + m_2 + m_3)} (q^{-\alpha m_2} - q^{-\alpha m_1}) (q^{-\alpha m_3} - q^{-\alpha m_1}) (q^{-\alpha m_3} - q^{-\alpha m_2})
\]

and

\[
M^{3}_{\alpha} = (q^{\alpha} - q^{-\alpha})^3 q^{-2\alpha (m_1 + m_2 + m_3)} (m_2) q^{\alpha} - [m_1] q^{\alpha}) (m_3) q^{\alpha} - [m_1] q^{\alpha}) (m_3) q^{\alpha} - [m_2] q^{\alpha}) + q^{-2\alpha (m_1 + m_2 + m_3)} (q^{-\alpha m_2} - q^{-\alpha m_1}) (q^{-\alpha m_3} - q^{-\alpha m_1}) (q^{-\alpha m_3} - q^{-\alpha m_2}).
\]
Proposition IV.2 The operators $T_m^{q^α}$ are given by:

$$T_m^{q^α} = [m + 1 + \gamma]q^α m! q^{-αm} \frac{\partial}{\partial t_m} + \frac{q^{-α(m+1+\gamma)}}{q^α - q^{-α}} \sum_{k=1}^{\infty} \frac{(k + m)!}{k!} B_k(t_1^α, \ldots, t_k^α) \frac{\partial}{\partial t_{k+m}}$$

and satisfy the product relation

$$T_m^{q^α} T_n^{q^β} \sim \frac{(q^{α+β} - q^{-α-β})q^{-m β}}{q^{α+β} n (q^α - q^{-α})(q^β - q^{-β})} T_{m+n}^{q^{α+β}} - \frac{q^{-n(1+β)}}{q^{α+β} n (q^β - q^{-β})} T_m^{q^α}$$

$$- \frac{q^{-m (1+β)} q^{-m(n+1)α}}{q^{α+β} n (q^α - q^{-α})} T_m^{q^β}.$$  

2. Witt n– algebra corresponding to the Jagannathan-Srinivassa deformation\(^{33}\)

The Witt n– algebra and properties corresponding to Jagannathan and Srinivassa deformation\(^{23}\) can be obtained by taking $R(p, q) = 1$. We consider the operators defined as:

$$T_m^{p^α, q^α} := -D_{p^α, q^α} z^{m+1}, \quad (25)$$

where $D_{p^α, q^α}$ is the $(p, q)$– deformed derivative:

$$D_{p^α, q^α} \left( \phi(z) \right) = \frac{\phi(p^α z) - \phi(q^α z)}{p^α - q^α}.$$  

From the $(p, q)$– number \(^7\), the operators \(^{25}\) take the form

$$T_m^{p^α, q^α} = -[m+1]_{p^α, q^α} z^m.$$  

The operators \(^{25}\) satisfy

$$T_m^{p^α, q^α} T_n^{p^β, q^β} = -\frac{(p^{α+β} - q^{-α-β})p^{-m β}}{(p^α - q^α)(p^β - q^β)} T_{m+n}^{p^{α+β}, q^{α+β}} + \frac{q^{(n+1)β}}{p^β - q^β} T_m^{p^β, q^α} + \frac{p^{-m β} q^{(m+n+1)α}}{p^α - q^α} T_m^{p^β, q^β}$$

and the commutation relation

$$[T_m^{p^α, q^α}, T_n^{p^β, q^β}] = \frac{(p^{α+β} - q^{-α+β})(p^{-n α} - p^{-m β})}{(p^α - q^α)(p^β - q^β)} T_{m+n}^{p^α, q^α} - \frac{q^{(m+n+1)β}(p^{-n α} - q^{-m β})}{p^β - q^β} T_{m+n}^{p^α, q^α}$$

$$+ \frac{q^{(m+n+1)α}(p^{-m β} - q^{-n α})}{p^α - q^α} T_{m+n}^{p^β, q^β}. \quad (26)$$

Taking $α = β = 1$ in the above relation \(^{26}\), we have:

$$[T_m^{p, q}, T_n^{p, q}] = \frac{(p^{-n} - p^{-m})}{(p - q)} [2]_{p, q} T_{m+n}^{p^2, q^2} - \frac{q^{m+n+1}}{p - q} \left( (p^{-n} - q^{-m}) - (p^{-m} - q^{-n}) \right) T_{m+n}^{p, q}.$$  

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The $n-$ bracket is defined by:

$$\left[\mathcal{T}^{\alpha_1,\alpha_1}_{m_1}, \ldots, \mathcal{T}^{\alpha_n,\alpha_n}_{m_n}\right]:= \Gamma^{i_1 \cdots i_n}_{1 \cdots n} \mathcal{T}^{\alpha_{i_1},\alpha_{i_1}}_{m_{i_1}} \cdots \mathcal{T}^{\alpha_{i_n},\alpha_{i_n}}_{m_{i_n}}$$

where $\Gamma^{i_1 \cdots i_n}_{1 \cdots n}$ is the Lévi-Civitá symbol given by:

$$\Gamma^{i_1 \cdots i_p}_{i_1 \cdots i_p} = \det \begin{pmatrix} \delta^{i_1}_{i_1} & \cdots & \delta^{i_1}_{i_p} \\ \vdots & \ddots & \vdots \\ \delta^{i_p}_{i_1} & \cdots & \delta^{i_p}_{i_p} \end{pmatrix}.$$ 

The $n-$ bracket with the same $(\alpha, q)$ is deduced as:

$$\left[\mathcal{T}^{\alpha,\alpha}_{m_1}, \ldots, \mathcal{T}^{\alpha,\alpha}_{m_n}\right] = \Gamma^{1 \cdots n}_{1 \cdots n} \mathcal{T}^{\alpha,\alpha}_{m_1} \cdots \mathcal{T}^{\alpha,\alpha}_{m_n}.$$ 

Putting $\alpha = \beta$ in the relation (26), we obtain:

$$\left[\mathcal{T}^{\alpha,\alpha}_{m_1}, \ldots, \mathcal{T}^{\alpha,\alpha}_{m_n}\right] = \left(\frac{(p^{-\alpha} - p^{-\alpha})}{(p^\alpha - q^\alpha)}\right) [2]_{p^\alpha, q^\alpha} \mathcal{T}^{\alpha,\alpha}_{m_1} \cdots \mathcal{T}^{\alpha,\alpha}_{m_n}$$

and the $n-$ bracket is rewritten as follows:

$$\left[\mathcal{T}^{\alpha,\alpha}_{m_1}, \ldots, \mathcal{T}^{\alpha,\alpha}_{m_n}\right] = \frac{(-1)^{n+1}}{(p^\alpha - q^\alpha)^{n-1}} \left[ H^{n}_{\alpha}[n]_{\alpha,\alpha} \mathcal{T}^{\alpha,\alpha}_{m_1} \cdots \mathcal{T}^{\alpha,\alpha}_{m_n} \right] - \left[ n - 1 \right]_{\alpha,\alpha}$$

where

$$H^{n}_{\alpha} = (p^\alpha - q^\alpha) \left( \begin{array}{c} n \\ 2 \end{array} \right) p^{-\alpha(n-1)} \sum_{i=1}^{n} \prod_{1 \leq j < k \leq n} \left( [m_k]_{\alpha,\alpha} - [m_j]_{\alpha,\alpha} \right)$$

and

$$M^{n}_{\alpha} = (p^\alpha - q^\alpha) \left( \begin{array}{c} n \\ 2 \end{array} \right) q^{-\alpha(n-1)} \sum_{i=1}^{n} \prod_{1 \leq j < k \leq n} \left( [m_k]_{\alpha,\alpha} - [m_j]_{\alpha,\alpha} \right)$$

Taking $n = 3$ in the relation (27), we obtain the $(p, q)-$ Witt 3- algebra:

$$\left[\mathcal{T}^{\alpha,\alpha}_{m_1}, \mathcal{T}^{\alpha,\alpha}_{m_2}, \mathcal{T}^{\alpha,\alpha}_{m_3}\right] = \left(\frac{1}{(p^\alpha - q^\alpha)^2}\right) \left[ H^3_{\alpha}[3]_{\alpha,\alpha} \mathcal{T}^{\alpha,\alpha}_{m_1} \cdots \mathcal{T}^{\alpha,\alpha}_{m_3} \right] - \left[ n - 1 \right]_{\alpha,\alpha}$$

$$\times q^{\alpha} \left( \sum_{i=1}^{3} m_i + 1 \right) \left( \begin{array}{c} H^3_{\alpha} + M^3_{\alpha} \end{array} \right) \mathcal{T}^{\alpha,\alpha}_{m_1} \cdots \mathcal{T}^{\alpha,\alpha}_{m_3},$$

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where

\[ H_\alpha^3 = (p^\alpha - q^\alpha) \left( \frac{3}{2} \right) p^{-2\alpha} \sum_{s=1}^{3} m_s \prod_{1 \leq j < k \leq 3} \left[ (m_k)_{p^\alpha, q^\alpha} - [m_j]_{p^\alpha, q^\alpha} \right] + p^{-2\alpha} \sum_{s=1}^{3} m_s \prod_{1 \leq j < k \leq 3} (q^\alpha m_k - q^\alpha m_j) \]

and

\[ M_\alpha^3 = (p^\alpha - q^\alpha) \left( \frac{3}{2} \right) q^{-2\alpha} \sum_{s=1}^{3} m_s \prod_{1 \leq j < k \leq 3} \left[ (m_k)_{p^\alpha, q^\alpha} - [m_j]_{p^\alpha, q^\alpha} \right] + q^{-2\alpha} \sum_{s=1}^{3} m_s \prod_{1 \leq j < k \leq 3} (p^\alpha m_k - p^\alpha m_j). \]

**Proposition IV.3** The operators \( T_{m}^{\alpha, q^\alpha} \) are given by

\[ T_{m}^{\alpha, q^\alpha} = [m + 1 + \gamma]_{p^\alpha, q^\alpha} m! p^{-\alpha} \frac{\partial}{\partial t_m} + q^{\alpha(m+1+\gamma)} \sum_{k=1}^{\infty} \frac{(k+m)!}{k!} B_k(t_1^\alpha, \ldots, t_k^\alpha) \frac{\partial}{\partial t_{k+m}} \]

and satisfy the product relation:

\[ T_{m}^{\alpha, q^\alpha} T_{n}^{\beta, q^\beta} \sim \frac{(p^{\alpha+\beta} q^{\alpha+\beta}) p^{-m} \beta}{p^{\alpha} m + \beta n (p^\alpha - q^\alpha)} T_{m+n}^{\alpha+\beta, q^\alpha+\beta} - \frac{q^{(n+1)\beta}}{p^{\alpha} m + \beta n (p^\beta - q^\beta)} T_{m+n}^{\alpha, q^\alpha} \]

\[ - \frac{p^{-m} \beta}{p^{\alpha} m + \beta n (p^\alpha - q^\alpha)} T_{m+n}^{\beta, q^\beta}. \]

3. **Witt n– algebra associated to the Chakrabarty and Jagannathan deformation**

Taking \( \epsilon_1 = p^{-1} \) and \( \epsilon_2 = q \), we obtain the \((p^{-1}, q)\)– deformed Witt n– algebra. The \((p^{-1}, q)\)– deformed derivative is defined by

\[ D_{p^{-\alpha}, q^\alpha}(\phi(z)) := \frac{\phi(p^{-\alpha} z) - \phi(q^\alpha z)}{p^{-\alpha} - q^\alpha} \]

and the operators \( T_{m}^{p^{-\alpha}, q^\alpha} \) by:

\[ T_{m}^{p^{-\alpha}, q^\alpha} := -D_{p^{-\alpha}, q^\alpha} z^{m+1}. \quad (28) \]

From the \((p^{-1}, q)\)– number (7), the operators (28) take the form

\[ T_{m}^{p^{-\alpha}, q^\alpha} = -[m + 1]_{p^{-\alpha}, q^\alpha} z^m. \]
The operators \(28\) satisfy
\[
\mathcal{T}_{m}^{p^{-\beta}q^\alpha}, \mathcal{T}_{n}^{p^{-\beta}q^\beta} = \frac{(p^{-\alpha} - q^{\alpha})(p^{-\beta} - q^{\beta})}{(p^{-\alpha} - q^{\alpha})(p^{-\beta} - q^{\beta})} \mathcal{T}_{m+n}^{p^{-\alpha}q^\alpha} + \frac{q^{(n+1)\beta}}{p^{-\beta} - q^{\beta}} \mathcal{T}_{m+n}^{p^{-\alpha}q^\alpha} \]
and the commutation relation
\[
\left[ \mathcal{T}_{m}^{p^{-\alpha}q^\alpha}, \mathcal{T}_{n}^{p^{-\beta}q^\beta} \right] = \frac{q^{(m+n+1)\beta}}{p^{-\beta} - q^{\beta}} \mathcal{T}_{m+n}^{p^{-\alpha}q^\alpha} + \frac{q^{(m+n+1)\beta}}{p^{-\beta} - q^{\beta}} \mathcal{T}_{m+n}^{p^{-\alpha}q^\alpha}. \quad (29)
\]
Taking \(\alpha = \beta = 1\) in the above relation \(29\), we have:
\[
\left[ \mathcal{T}_{m}^{p^{-1}q}, \mathcal{T}_{n}^{p^{-1}q} \right] = \frac{(p^n - p^m)}{(p^{-1} - q)} [2]_{p^{-1}q}^{p^{-2}q^2} - \frac{q^{m+n+1}}{p^{-1} - q} \left( (p^n - q^{-m}) - (p^{-m} - q^{-n}) \right) \mathcal{T}_{m+n}^{p^{-1}q}.
\]
We consider the \(n\)- bracket defined by:
\[
\left[ \mathcal{T}_{m_1}^{p^{-\alpha_1}q^\alpha_1}, \ldots, \mathcal{T}_{m_n}^{p^{-\alpha_n}q^\alpha_n} \right] := \Gamma_{1\cdots n}^{i_1\cdots i_n} \mathcal{T}_{m_1}^{p^{-\alpha_1}q^\alpha_1} \ldots \mathcal{T}_{m_n}^{p^{-\alpha_n}q^\alpha_n}
\]
where \(\Gamma_{1\cdots n}^{i_1\cdots i_n}\) is the Lévi-Civitá symbol given by:
\[
\Gamma_{1\cdots i_p}^{j_1\cdots j_p} = \det \begin{pmatrix} \delta_{i_1}^{j_1} & \cdots & \delta_{i_p}^{j_p} \\ \vdots & \ddots & \vdots \\ \delta_{i_p}^{j_1} & \cdots & \delta_{i_p}^{j_p} \end{pmatrix}.
\]
Our study is focussed on the case with the same \(p^{-\alpha}, q^\alpha\). Thus, we have
\[
\left[ \mathcal{T}_{m_1}^{p^{-\alpha}q^\alpha}, \ldots, \mathcal{T}_{m_n}^{p^{-\alpha}q^\alpha} \right] = \Gamma_{1\cdots n}^{i_1\cdots i_n} \mathcal{T}_{m_1}^{p^{-\alpha}q^\alpha} \ldots \mathcal{T}_{m_n}^{p^{-\alpha}q^\alpha}.
\]
Putting \(\alpha = \beta\) in the relation \(29\), we obtain:
\[
\left[ \mathcal{T}_{m}^{p^{-\alpha}q^\alpha}, \mathcal{T}_{n}^{p^{-\alpha}q^\alpha} \right] = \frac{(p^{n\alpha} - p^{m\alpha})}{(p^{-\alpha} - q^{\alpha})} [2]_{p^{-\alpha}q^\alpha}^{p^{-2\alpha}q^2} - \frac{q^{(m+n+1)\alpha}}{p^{-\alpha} - q^{\alpha}} \left( (p^{n\alpha} - p^{m\alpha}) + (q^{n\alpha} - q^{-m\alpha}) \right) \mathcal{T}_{m+n}^{p^{-\alpha}q^\alpha}.
\]
Then the \(n\)- bracket is given by:
\[
\left[ \mathcal{T}_{m_1}^{p^{-\alpha}q^\alpha}, \ldots, \mathcal{T}_{m_n}^{p^{-\alpha}q^\alpha} \right] = \frac{(-1)^n+1}{(p^{-\alpha} - q^{\alpha})^{n-1}} \left( \sum_{i=1}^{n} H_{i}^{a} \left[n\right]_{p^{-\alpha}q^{\alpha}}^{p^{-\alpha}q^{\alpha}} \mathcal{T}_{m_1 + \ldots + m_n}^{p^{-\alpha}q^\alpha} - [n-1]_{p^{-\alpha}q^{\alpha}} \mathcal{T}_{m_1 + \ldots + m_n}^{p^{-\alpha}q^\alpha} \right.
\]
\[
\times \left. q^{\alpha} \left( \sum_{i=1}^{n} m_{i+1} \right) \left( H_{a}^{n} + M_{a}^{n} \right) \mathcal{T}_{m_1 + \ldots + m_n}^{p^{-\alpha}q^\alpha}, \right) \quad (30)
\]
where
\[
H_{a}^{n} = (p^{-\alpha} - q^{\alpha})^{(n+1)/2} p^{\alpha(n-1)} \sum_{s=1}^{n} m_{s} \prod_{1 \leq j < k \leq n} \left( [m_{k}]_{p^{-\alpha}q^\alpha} - [m_{j}]_{p^{-\alpha}q^\alpha} \right)
\]
and satisfy the product relation:

\[ + p^{\alpha(n-1) \sum_{s=1}^{n} m_s} \prod_{1 \leq j < k \leq n} (q^{\alpha m_k} - q^{\alpha m_j}) \]

and

\[ M_n^\alpha = (p^{-\alpha} - q^{\alpha}) (2) q_2^{-\alpha(n-1) \sum_{s=1}^{n} m_s} \prod_{1 \leq j < k \leq n} \left( [m_k]_{p^{-\alpha}q^{\alpha}} - [m_j]_{p^{-\alpha}q^{\alpha}} \right) \]

\[ + (-1)^{n-1} q^{-\alpha(n-1) \sum_{s=1}^{n} m_s} \prod_{1 \leq j < k \leq n} (p^{-\alpha m_k} - p^{-\alpha m_j}) \]

Setting \( n = 3 \) in the relation \([30]\), we obtain the \((p, q)\)–Witt 3–algebra:

\[ \left[ T_{m_1}^{-\alpha, q^{\alpha}}, T_{m_2}^{-\alpha, q^{\alpha}}, T_{m_3}^{-\alpha, q^{\alpha}} \right] = \frac{1}{(p^{-\alpha} - q^{\alpha})^2} \left( H_3^3 \right)_{p^{-\alpha, q^{\alpha}} T_{m_1}^{-3\alpha, q^{3\alpha}} - [n - 1]_{p^{-\alpha, q^{\alpha}}} \times q^{\alpha} \left( \sum_{l=1}^{3} m_l \right) (H_3^3 + M_3^\alpha) T_{m_1}^{-2\alpha, q^{2\alpha}} ) \]

where

\[ H_3^3 = (p^{-\alpha} - q^{\alpha}) (2) q_2^{2\alpha \sum_{s=1}^{3} m_s} \prod_{1 \leq j < k < l \leq 3} \left( [m_k]_{p^{-\alpha}q^{\alpha}} - [m_j]_{p^{-\alpha}q^{\alpha}} \right) \]

\[ + p^{2\alpha \sum_{s=1}^{3} m_s} \prod_{1 \leq j < k < l \leq 3} (q^{\alpha m_k} - q^{\alpha m_j}) \]

and

\[ M_3^\alpha = (p^{-\alpha} - q^{\alpha}) (2) q_2^{-2\alpha \sum_{s=1}^{3} m_s} \prod_{1 \leq j < k < l \leq 3} \left( [m_k]_{p^{-\alpha}q^{\alpha}} - [m_j]_{p^{-\alpha}q^{\alpha}} \right) \]

\[ + q^{-2\alpha \sum_{s=1}^{3} m_s} \prod_{1 \leq j < k < l \leq 3} (p^{-\alpha m_k} - p^{-\alpha m_j}) \]

**Proposition IV.4** The operators \( T_{m}^{-\alpha, q^{\alpha}} \) are given by

\[ T_{m}^{-\alpha, q^{\alpha}} = [m + 1 + \gamma]_{p^{-\alpha}q^{\alpha}} m! \frac{\partial}{\partial t_m} + \frac{q^{\alpha(m+1+\gamma)}}{p^{-\alpha} - q^{\alpha}} \sum_{k=1}^{\infty} \frac{(k+m)!}{k!} B_k(t_1^n, \ldots, t_k^n) \frac{\partial}{\partial t_{k+m}} \]

and satisfy the product relation:

\[ T_{m}^{-\alpha, q^{\alpha}} \cdot T_{n}^{-\beta, q^{\beta}} \sim \frac{(p^{-\alpha - \beta} q^{\alpha + \beta}) p^{-\beta} \beta}{p^{-\alpha m - \beta n} (p^{-\alpha} - q^{\alpha}) (p^{-\beta} - q^{\beta})} T_{m+n}^{-\alpha, q^{\alpha}} + \frac{q^{(m+1)\beta}}{p^{-\alpha m - \beta n} (p^{-\beta} - q^{\beta})} T_{m+n}^{-\alpha, q^{\alpha}} \]

\[ - \frac{p^{-m\beta} q^{(m+n+1)\alpha}}{p^{-\alpha m - \beta n} (p^{-\alpha} - q^{\alpha})} T_{m+n}^{-\beta, q^{\beta}}. \]
4. Witt $n$– algebra induced by the Hounkonnou-Ngompe generalization of $q$– Quesne deformation

The Witt $n$– algebra and properties are obtained by taking $\epsilon_1 = p$ and $\epsilon_2 = q^{-1}$. We define the operators $\mathcal{T}^{\alpha\beta}_{m,n}$ as follows:

$$\mathcal{T}^{\alpha\beta}_{m,n} := -D_{\alpha\beta} z^{m+1}, \quad (31)$$

where $D_{\alpha\beta}$ is the derivative

$$D_{\alpha\beta}(\phi(z)) = \frac{\phi(p^\alpha z) - \phi(q^{-\alpha} z)}{q^{-\alpha} - p^\alpha}.$$

The operators (31) satisfy

$$\mathcal{T}^{\alpha\beta}_{m,n} \mathcal{T}^{\alpha\beta}_{n,m} = -\frac{(p^\alpha + q^{-\alpha})p^{-m\beta}}{(p^\alpha - q^{-\alpha})(p^\beta - q^{-\beta})} \mathcal{T}^{\alpha\beta}_{m+n} + \frac{q^{-(m+1)\alpha}}{p^\beta - q^{-\beta}} \mathcal{T}^{\alpha\beta}_{m+n}$$

and the commutation relation

$$\left[ \mathcal{T}^{\alpha\beta}_{m,n}, \mathcal{T}^{\gamma\delta}_{n,m} \right] = \frac{(p^\alpha + q^{-\alpha})p^{-m\beta}}{(p^\alpha - q^{-\alpha})(p^\beta - q^{-\beta})} \mathcal{T}^{\alpha\beta\gamma\delta}_{m+n} - \frac{q^{-(m+1)\alpha} p^{-m\beta}}{p^\beta - q^{-\beta}} \mathcal{T}^{\alpha\beta}_{m+n}$$

Putting $\alpha = \beta = 1$ in relation (32), yields:

$$\left[ \mathcal{T}^{p,q}_{m,n}, \mathcal{T}^{p,q}_{n,m} \right] = \frac{q(p^{-n} - p^{-m})}{p(p - q^{-1})} \mathcal{T}^{p,q}_{m+n} - \frac{q^{(m+n+1)}}{p - q^{-1}} \left( (p^{-n} - q^{m}) - (p^{-m} - q^{n}) \right) \mathcal{T}^{p,q}_{m+n}.$$

We consider the $n$– bracket defined by:

$$\left[ \mathcal{T}^{\alpha\beta}_{m_1,n_1}, \ldots, \mathcal{T}^{\alpha\beta}_{m_n,n_n} \right] := \Gamma^{i_1\ldots i_m}_{i_1\ldots i_m} \mathcal{T}^{\alpha\beta}_{m_1,n_1} \ldots \mathcal{T}^{\alpha\beta}_{m_n,n_n}$$

where $\Gamma^{i_1\ldots i_m}_{i_1\ldots i_m}$ is the Lévi-Civitá symbol given by:

$$\Gamma^{i_1\ldots i_p}_{i_1\ldots i_p} = \det \begin{pmatrix} \delta^{i_1}_{i_1} & \ldots & \delta^{i_1}_{i_p} \\ \vdots & & \vdots \\ \delta^{i_p}_{i_1} & \ldots & \delta^{i_p}_{i_p} \end{pmatrix}.$$

Our study is focussed on the case with the same $(p^\alpha, q^\alpha)$. Thus, we have

$$\left[ \mathcal{T}^{\alpha\beta}_{m_1,n_1}, \ldots, \mathcal{T}^{\alpha\beta}_{m_n,n_n} \right] = \Gamma^{i_1\ldots i_m}_{i_1\ldots i_m} \mathcal{T}^{\alpha\beta}_{m_1,n_1} \ldots \mathcal{T}^{\alpha\beta}_{m_n,n_n}.$$
Putting $\alpha = \beta$ in the relation (32), we obtain:

$$\left[ T_{m, q}^{\alpha} , T_{n, q}^{\alpha} \right] = \frac{q^\alpha (p^{-n\alpha} - p^{-m\alpha})}{p^\alpha (p^\alpha - q^{-\alpha})} [2]_{p^\alpha, q^\alpha} T_{m+n}^{2\alpha, q^{2\alpha}}$$

$$- \frac{q^{-(m+n+1)\alpha}}{p^\alpha - q^{-\alpha}} \left( (p^{-n\alpha} - p^{-m\alpha}) + (q^{n\alpha} - q^{m\alpha}) \right) T_{m+n}^{\alpha, q^\alpha}. \quad (33)$$

From the above relation (33), we deduce the $n-$ bracket as follows:

$$\left[ T_{m_1, q}^{\alpha} , \ldots , T_{m_n, q}^{\alpha} \right] = \frac{(-1)^{n+1} (qp^{-1})^\alpha}{(p^\alpha - q^{-\alpha})^{n-1}} \left( H_n^\alpha [n]_{p^\alpha, q^\alpha} T_{m_1+\ldots+m_n}^{\alpha, q^{n\alpha}} - [n-1]_{p^\alpha, q^\alpha} \right.$$

$$\times \left. q^{-\alpha} \left( \sum_{i=1}^{n-1} m_i + 1 \right) (H_n^\alpha + M_n^\alpha) T_{m_1+\ldots+m_n}^{\alpha, q^{(n-1)\alpha}} \right), \quad (34)$$

where

$$H_n^\alpha = (p^\alpha - q^{-\alpha})^n_{(2)} p^{-\alpha(n-1)\sum_{i=1}^{n-1} m_i} \prod_{1 \leq j < k \leq n} (qp^{-1})^\alpha \left( [m_k]_{p^\alpha, q^\alpha} - [m_j]_{p^\alpha, q^\alpha} \right)$$

$$+ p^{-\alpha(n-1)\sum_{i=1}^{n-1} m_i} \prod_{1 \leq j < k \leq n} \left( q^{-\alpha} m_k - q^{-\alpha} m_j \right)$$

and

$$M_n^\alpha = (p^\alpha - q^{-\alpha})^n_{(2)} q^{\alpha(n-1)\sum_{i=1}^{n-1} m_i} \prod_{1 \leq j < k \leq n} (qp^{-1})^\alpha \left( [m_k]_{p^\alpha, q^\alpha} - [m_j]_{p^\alpha, q^\alpha} \right)$$

$$+ (-1)^{n-1} q^{\alpha(n-1)\sum_{i=1}^{n-1} m_i} \prod_{1 \leq j < k \leq n} \left( p^\alpha m_k - p^\alpha m_j \right).$$

Taking $n = 3$ in the relation (34), we obtain the Witt 3- algebra corresponding to the Hounkonnou-Ngompe generalization of $q-$ Quesne deformation:

$$\left[ T_{m_1, q}^{\alpha} , T_{m_2, q}^{\alpha} , T_{m_3, q}^{\alpha} \right] = \frac{(qp^{-1})^\alpha}{(p^\alpha - q^{-\alpha})^2} \left( H_3^\alpha [3]_{p^\alpha, q^\alpha} T_{m_1+\ldots+m_3}^{\alpha, q^{3\alpha}} - [2]_{p^\alpha, q^\alpha} \right.$$

$$\times \left. q^{-\alpha} \left( \sum_{i=1}^{2} m_i + 1 \right) (H_3^\alpha + M_3^\alpha) T_{m_1+\ldots+m_3}^{\alpha, q^{(3-1)\alpha}} \right),$$

where

$$H_3^\alpha = (p^\alpha - q^{-\alpha})^3_{(2)} p^{-2\alpha\sum_{i=1}^{3} m_i} \prod_{1 \leq j < k \leq 3} (qp^{-1})^\alpha \left( [m_k]_{p^\alpha, q^\alpha} - [m_j]_{p^\alpha, q^\alpha} \right)$$

$$+ p^{-2\alpha\sum_{i=1}^{3} m_i} \prod_{1 \leq j < k \leq 3} \left( q^{-\alpha} m_k - q^{-\alpha} m_j \right)$$

and

$$M_3^\alpha = (p^\alpha - q^{-\alpha})^3_{(2)} q^{2\alpha\sum_{i=1}^{3} m_i} \prod_{1 \leq j < k \leq 3} (qp^{-1})^\alpha \left( [m_k]_{p^{-\alpha}, q^\alpha} - [m_j]_{p^{-\alpha}, q^\alpha} \right)$$

$$+ q^{-2\alpha\sum_{i=1}^{3} m_i} \prod_{1 \leq j < k \leq 3} \left( p^\alpha m_k - p^\alpha m_j \right).$$
Proposition IV.5  The operators $\mathcal{T}_m^{\alpha, q}$ are given by

$$
\mathcal{T}_m^{\alpha, q} = \frac{(qp^{-1})^\alpha}{p^\alpha m} [m + 1 + \gamma] q^{-\alpha m} \frac{\partial}{\partial t_m} + \frac{q^{-\alpha (m+1+\gamma)}}{p^\alpha - q^{-\alpha}} \sum_{k=1}^{\infty} \frac{(k+m)!}{k!} B_k(t_1^\alpha, \ldots, t_k^\alpha) \frac{\partial}{\partial t_{k+m}}
$$

and satisfy the product relation:

$$
\mathcal{T}_m^{\alpha, q} \cdot \mathcal{T}_n^{\beta, q} \sim \frac{(p^{\alpha+\beta} - q^{-\alpha-\beta}) p^{-m \beta}}{p^{\alpha+\beta} n (p^{\alpha} - q^{-\alpha}) (p^{\beta} - q^{-\beta})} \mathcal{T}_m^{\alpha+\beta, q^{\alpha+\beta}} - \frac{q^{-(n+1)\beta}}{p^{\alpha+\beta} n (p^{\beta} - q^{-\beta})} \mathcal{T}_m^{\alpha, q^{\alpha}} - \frac{p^{-m \beta}}{p^{\alpha+\beta} n (p^{\alpha} - q^{-\alpha})} \mathcal{T}_m^{\beta, q^{\beta}}.
$$

V. CONCLUDING AND REMARKS

We have developed a unified framework for deforming Witt and Virasoro algebras from quantum algebras and their generalizations, as well as for establishing deformed quantum Korteweg-de Vries equations. Furthermore, we have extended this study to the construction of the deformed Witt $n-$ algebra, and derived deformed Virasoro constraints for matrix models. Interesting particular quantum deformations have been investigated and discussed.

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