GEOMETRIC COMBINATORIAL ALGEBRAS: CYCLOHEDRON AND SIMPLEX

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Abstract. In this paper we report on results of our investigation into the algebraic structure supported by the combinatorial geometry of the cyclohedron. Our new graded algebra structures lie between two well known Hopf algebras: the Malvenuto-Reutenauer algebra of permutations and the Loday-Ronco algebra of binary trees. Connecting algebra maps arise from a new generalization of the Tonks projection from the permutohedron to the associahedron, which we discover via the viewpoint of the graph associahedra of Carr and Devadoss. At the same time that viewpoint allows exciting geometrical insights into the multiplicative structure of the algebras involved. Extending the Tonks projection also reveals a new graded algebra structure on the simplices.

1. Introduction

1.1. Background. In 1998 Loday and Ronco found an intriguing Hopf algebra of planar binary trees lying between the Malvenuto-Reutenauer Hopf algebra of permutations \[ \text{Perms} \] and the Solomon descent algebra of Boolean subsets \[ \text{Desc} \]. They also described natural Hopf algebra maps which neatly factor the descent map from permutations to Boolean subsets. Their first factor turns out to be the restriction (to vertices) of the Tonks projection from the permutohedron to the associahedron. Chapoton made sense of this latter fact when he found larger Hopf algebras based on the faces of the respective polytopes \[ \text{Polytopes} \]. Here we study several new algebraic structures based on polytope sequences, including the cyclohedra, \( \mathcal{W}_n \), and the simplices, \( \Delta^n \). In Figure 1 we show the central polytopes, in three dimensions.

![Figure 1. The main characters, left to right: \( \mathcal{P}_4, \mathcal{W}_4, \mathcal{K}_4, \) and \( \Delta^3 \).](image)

The cyclohedron \( \mathcal{W}_n \) of dimension \( n-1 \) for \( n \) a positive integer was originally developed by Bott and Taubes \[ \text{Bott-Taubes} \], and received its name from Stasheff. The name points out the close connection to Stasheff’s associahedra, which we denote \( \mathcal{K}_n \) of dimension \( n-1 \). The former authors described the facets of \( \mathcal{W}_n \) as being indexed by subsets of \( [n] = 1, 2, \ldots, n \) of cardinality \( \geq 2 \) in cyclic order. Thus there are \( n(n-1) \) facets. All the faces can be indexed by cyclic bracketings of the string.
123...n, where the facets have exactly one pair of brackets. The vertices are complete bracketings, enumerated by \(\binom{2(n-1)}{n-1}\).

**Figure 2.** The cyclohedron \(\mathcal{W}_3\) with various indexing.

The space \(\mathcal{W}_n \times S^1\), seen as the compactification of the configuration space of \(n\) distinct points in \(\mathbb{R}^3\) which are constrained to lie upon a given knot, is used to define new invariants which reflect the self linking of knots \([3]\). Since their inception the cyclohedra have proven to be useful in many other arenas. They provide an excellent example of a right operad module (over the operad of associahedra) as shown in \([12]\). Devadoss discovered a tiling of the \((n-1)\)-torus by \((n-1)!\) copies of \(\mathcal{W}_n\) in \([7]\). Recently the cyclohedra have been used to look for the statistical signature of periodically expressed genes in the study of biological clocks \([13]\).

The faces of the cyclohedra may also be indexed by the centrally symmetric subdivisions of a convex polygon of \(2n\) sides, as discovered by Simion \([17]\). In this indexing the vertices are centrally symmetric triangulations, which allowed Hohlweg and Lange to develop geometric realizations of the cyclohedra as convex hulls \([9]\). This picture is related to the work of Fomin, Reading and Zelevinsky, who see the cyclohedra as a generalization of the associahedra corresponding to the \(B_n\) Coxeter diagrams \([16]\). From their perspective the face structure of the cyclohedron is determined by the sub-cluster structure of the generators of a finite cluster algebra.

In contrast, for Devadoss the cyclohedra arise from truncating simplex faces corresponding to sub-diagrams of the \(\tilde{A}_n\) Coxeter diagram, or cycle graph \([4]\). In this paper we will work from the point of view taken by Devadoss and consider the faces as indexed by tubings of the cycle graph on \(n\) vertices. Given a graph \(G\), the **graph associahedron** \(KG\) is a convex polytope generalizing the associahedron, with a face poset based on the full connected subgraphs, or tubes of \(G\). For instance, when \(G\) is a path, a cycle, or a complete graph, \(KG\) results in the associahedron, cyclohedron, and permutohedron, respectively. In \([8]\), a geometric realization of \(KG\) is given, constructing this polytope from truncations of the simplex. In \([4]\) the motivation for the development of \(KG\) is that it appears in tilings of minimal blow-ups of certain Coxeter complexes, which themselves are natural generalizations of the moduli spaces \(\overline{\mathcal{M}}_{0,n}(\mathbb{R})\).

The value of the graph associahedron to algebraists, as we hope to demonstrate here, is twofold. First, a unified description of so many combinatorial polytopes allows useful generalizations of the known algebraic structures on familiar polytope sequences. Here we generalize the well known algebras based on associahedra and permutohedra to new ones on cyclohedra and simplices, leaving for future investigation many potential algebras and coalgebras based on novel sequences of graph associahedra. Second, the recursive structure described by Carr and Devadoss in a general way for graph associahedra turns out to lend new geometrical meaning to the graded algebra structures of both the Malvenuto-Reutenauer and Loday-Ronco algebras. The product of two vertices, from
terms \(i\) and \(j\) of a given sequence of polytopes, is described as a sum of vertices of the term \(i + j\) to which the operands are mapped. The summed vertices in the product are the images of classical inclusion maps composed with our new extensions of the Tonks projection.

1.2. **Summary.** We will be referring to the algebras and maps discussed in [2] and [1]. In these two papers, Aguiar and Sottile make powerful use of the weak order on the symmetric groups and the Tamari order on binary trees. By leveraging the Möbius function of these two lattices they provide clear descriptions of the antipodes and of the geometric underpinnings of the Hopf algebras. The also demonstrate cofreeness, characterize primitives of the coalgebras, and, in the case of binary trees, demonstrate equivalence to the non-commutative Connes-Kreimer Hopf algebra from renormalization theory. The Hopf algebras of permutations, binary trees, and Boolean subsets are denoted respectively \(\mathfrak{S}Sym\), \(\mathfrak{Y}Sym\) and \(\mathfrak{Q}Sym\). Note that some of our sources, including Loday and Ronco’s original treatment of binary trees, actually deal with the dual graded algebras. The larger algebras of faces of permutohedra, associahedra and cubes are denoted \(\mathfrak{S}\tilde{S}ym\), \(\mathfrak{Y}\tilde{S}ym\) and \(\mathfrak{Q}\tilde{S}ym\). The new algebras of cyclohedra vertices and faces are denoted \(W\Sym\) and \(W\tilde{S}ym\). The new algebra of vertices of the simplex is denoted \(\Delta\Sym\).

1.2.1. **Main Results.** In Theorem 5.3 we demonstrate that \(W\Sym\) is an associative graded algebra. In Theorem 6.4 we extend this structure to the full poset of faces to describe the associative graded algebra \(W\tilde{S}ym\). In Theorem 7.3 we demonstrate an associative graded algebra structure on \(\Delta\Sym\). Before discussing algebraic structures, however, we build a geometric combinatorial foundation. We show precisely how the graph associahedra are all cellular quotients of the permutohedra (Lemma 3.2), and how the associahedron is a quotient of any given connected graph associahedron (Theorem 3.5). Results similar to these latter statements are implied by the work of Postnikov in [15], and were also reportedly known to Tonks in the cases involving the cyclohedron [4].

Theorems 4.1, 4.3 and 6.1 and Remark 5.5 point out that the multiplications in all the algebras studied here can be understood in a unified way via the recursive structure of the face posets of associated polytopes. To summarize, the products can be seen as a process of projecting and including of faces. The coproducts of \(\mathfrak{Y}Sym\) and \(\mathfrak{S}Sym\) are also understandable as projections of the polytope geometry, as mentioned in Remarks 4.2 and 4.4.

The various maps arising from factors of the Tonks projection are shown to be algebra homomorphisms in Theorems 5.6, 5.7 and Lemma 7.5. Corollary 5.9 points out the implication of module structures on \(W\Sym\) over \(\mathfrak{S}Sym\); and on \(\mathfrak{Y}Sym\) over \(W\Sym\).

1.2.2. **Overview of subsequent sections.** Section 2 describes the posets of connected subgraphs which are realized as the graph associahedra polytopes. We also review the cartesian product structure of their facets. Section 3 shows how the Tonks cellular projection from the permutohedron to the associahedron can be factored in multiple ways so that the intermediate polytopes are graph associahedra for connected graphs. We also show that the Tonks projection can be extended to cellular projections to the simplices, again in such a way that the intermediate polytopes are graph associahedra. In particular, we focus on a factorization of the Tonks projection which has the cyclohedron (cycle graph associahedra) as an intermediate quotient polytope between permutohedron and associahedron. In Section 4 we use the viewpoint of graph associahedra to redescribe the products in \(\mathfrak{S}Sym\) and \(\mathfrak{Y}Sym\), and to point out their geometric interpretation. The latter is based upon our new cellular projections as well as classical inclusions of (cartesian products of) low dimensional associahedra and permutohedra as faces in the higher dimensional polytopes of their respective sequences. In Section 5 we begin our exploration of new graded algebras with the vertices of the cyclohedron. Then we show that the linear projections following from the factored
Tonks projection (restricted to vertices) are algebra maps. We extend these findings in Section 6 to the full algebras based on all the faces of the polytopes involved. Finally in Section 7 we generalize our discoveries to the sequence of edgeless graph associahedra. This allows us to build a graded algebra based upon the vertices of the simplices.

2. Review of some geometric combinatorics

We begin with definitions of graph associahedra; the reader is encouraged to see [4, Section 1] and [6] for further details.

**Definition 2.1.** Let $G$ be a finite connected simple graph. A tube is a set of nodes of $G$ whose induced graph is a connected subgraph of $G$. Two tubes $u$ and $v$ may interact on the graph as follows:

1. Tubes are nested if $u \subset v$.
2. Tubes are far apart if $u \cup v$ is not a tube in $G$, that is, the induced subgraph of the union is not connected, or none of the nodes of $u$ are adjacent to a node of $v$.

Tubes are compatible if they are either nested or far apart. We call $G$ itself the universal tube. A tubing $U$ of $G$ is a set of tubes of $G$ such that every pair of tubes in $U$ is compatible; moreover, we force every tubing of $G$ to contain (by default) its universal tube. By the term $k$-tubing we refer to a tubing made up of $k$ tubes, for $k \in \{1, \ldots, n\}$.

When $G$ is a disconnected graph with connected components $G_1, \ldots, G_k$, an additional condition is needed: If $u_i$ is the tube of $G$ whose induced graph is $G_i$, then any tubing of $G$ cannot contain all of the tubes $\{u_1, \ldots, u_k\}$. However, the universal tube is still included despite being disconnected. Parts (a)-(c) of Figure 3 from [6] show examples of allowable tubings, whereas (d)-(f) depict the forbidden ones.

![Figure 3](image-url)

**Figure 3.** (a)-(c) Allowable tubings and (d)-(f) forbidden tubings, figure from [6].

**Theorem 2.2.** [4, Section 3] For a graph $G$ with $n$ nodes, the graph associahedron $KG$ is a simple, convex polytope of dimension $n-1$ whose face poset is isomorphic to the set of tubings of $G$, ordered such that $U \prec U'$ if $U$ is obtained from $U'$ by adding tubes.

The vertices of the graph associahedron are the $n$-tubings of $G$. Faces of dimension $k$ are indexed by $(n-k)$-tubings of $G$. Many of the face vectors of graph associahedra for path-like graphs have been found, as shown in [14]. This source also contains the face vectors for the cyclohedra. There are many open questions regarding formulas for the face vectors of graph associahedra for specific types of graphs.

**Example 2.3.** Figure 4, partly from [6], shows two examples of graph associahedra. These have underlying graphs a path and a disconnected graph, respectively, with three nodes each. These turn out to be the 2 dimensional associahedron and a square. The case of a three node complete graph, which is both the cyclohedron and the permutohedron, is shown in Figure 2.

To describe the face structure of the graph associahedra we need a definition from [4, Section 2].
Definition 2.4. For graph $G$ and a collection of nodes $t$, construct a new graph $G^*(t)$ called the \textit{reconnected complement}: If $V$ is the set of nodes of $G$, then $V - t$ is the set of nodes of $G^*(t)$. There is an edge between nodes $a$ and $b$ in $G^*(t)$ if either $\{a, b\}$ or $\{a, b\} \cup t$ is connected in $G$.

Example 2.5. Figure 5 illustrates some examples of graphs along with their reconnected complements.

For a given tube $t$ and a graph $G$, let $G(t)$ denote the induced subgraph on the graph $G$. By abuse of notation, we sometimes refer to $G(t)$ as a tube.

Theorem 2.6. [4, Theorem 2.9] Let $V$ be a facet of $KG$ and let $t$ be the single, non-universal, tube of $V$. The face poset of $V$ is isomorphic to $KG(t) \times KG^*(t)$.

A pair of examples is shown in Figure 6. The isomorphism described in [4, Theorem 2.9] is called $\hat{\rho}$. Since we will be using this isomorphism more than once as an embedding of faces in polytopes, then we will specify it, according to the tube involved, as $\hat{\rho}_t : KG(t) \times KG^*(t) \hookrightarrow KG$. In fact, often there will be more than one tube involved, as we now indicate:

Corollary 2.7. Let $\{t_1, \ldots, t_k, G\}$ be an explicit tubing of $G$, such that each pair of non-universal tubes in the list is far apart. Then the face of $KG$ associated to that tubing is isomorphic to $KG(t_1) \times \cdots \times KG(t_n) \times KG^*(t_1 \cup \cdots \cup t_k)$.

We will denote the embedding by:

$\hat{\rho}_{t_1 \ldots t_k} : KG(t_1) \times \cdots \times KG(t_n) \times KG^*(t_1 \cup \cdots \cup t_k) \hookrightarrow KG$.

Proof. This follows directly from Theorem 2.6. Note that the reconnected complement with respect to the union of several tubes $t_1, \ldots, t_k$ is the same as taking successive iterated reconnected complements with respect to each tube in the list. That is,

$G^*(t_1 \cup \cdots \cup t_k) = ((G \ast (t_1)) \ast (t_2)) \ast \cdots \ast (t_k)$.
3. Factoring and extending the Tonks projection.

3.1. Loday and Ronco’s Hopf algebra map. The two most important existing mathematical structures we will use in this paper are the graded Hopf algebra of permutations, $\mathcal{S} \text{Sym}$, and the graded Hopf algebra of binary trees, $\mathcal{Y} \text{Sym}$. The $n^{\text{th}}$ component of $\mathcal{S} \text{Sym}$ has basis the symmetric group $S_n$, with number of elements counted by $n!$. The $n^{\text{th}}$ component of $\mathcal{Y} \text{Sym}$ has basis the collection of binary trees with $n$ interior nodes, and thus $n + 1$ leaves, denoted $\mathcal{Y}_n$. These are counted by the Catalan numbers.

The connection discovered by Loday and Ronco between the two algebras is due to the fact that a permutation on $n$ elements can be pictured as a binary tree with $n$ interior nodes, drawn so that the interior nodes are at $n$ different vertical heights from the base of the tree. This is called an ordered binary tree. The interior nodes are numbered left to right. We number the leaves $0, 1, \ldots, n - 1$ and the nodes $1, 2, \ldots, n - 1$. The $i^{\text{th}}$ node is “between” leaf $i - 1$ and leaf $i$ where “between” might be described to mean that a rain drop falling between those leaves would be caught at that node. Distinct vertical levels of the nodes are numbered top to bottom. Then for a permutation $\sigma \in S_n$ the corresponding tree has node $i$ at level $\sigma(i)$. The map from permutations to binary trees is achieved by forgetting the levels, and just retaining the tree. This surjection, denoted $\tau : S_n \to \mathcal{Y}_n$, was first noticed by Loday. An example is in Figure 7.

In [18] Tonks found a corresponding cellular projection from permutohedron to associahedron: $\Theta : P_n \to K_n$. In the cellular projection a face of the permutohedron, which is an ordered tree, is taken to its underlying tree, which is a face of the associahedron. Figure 8 shows an example. The new revelation of Loday and Ronco is that the map $\tau$ gives rise to a Hopf algebraic projection $\tau : \mathcal{S} \text{Sym} \to \mathcal{Y} \text{Sym}$, so that the algebra of binary trees is seen to be embedded in the algebra of permutations.

3.2. Tubings, permutations, and trees. Our new approach to the Tonks projection is made possible by the recent discovery of Devadoss in [8] that the graph-associahedron of the complete graph on $n$ vertices is precisely the $n^{\text{th}}$ permutohedron $P_n$. Each of its vertices corresponds to a permutation of $n$ elements. Its faces in general correspond to ordered partitions of $[n]$. Keep in mind that for a permutation $\sigma \in S_n$, the corresponding ordered partition of $[n]$ is $\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(n)\}$.

Here is how to describe a bijection from the $n$-tubings on the complete graph to the permutations, as found by Devadoss in [8]. First a numbering of the $n$ nodes must be chosen. Given an $n$-tubing,
since the tubes are all nested we can number them starting with the innermost tube. Then the permutation \( \sigma \in S_n \) pictured by our \( n \)-tubing is such that node \( i \) is within tube \( \sigma(i) \) but not within any tube nested inside of tube \( \sigma(i) \). Figure 7 shows an example.

It is easy to extend this bijection between \( n \)-tubings and permutations to all tubings of the complete graph and ordered partitions of \([n]\). Given a \( k \)-tubing of the complete graph, each tube contains some numbered nodes which are not contained in any other tube. These subsets of \([n]\), one for each tube, make up the \( k \)-partition, and the ordering of the partition is from innermost to outermost. Recall that an ordered \( k \)-partition of \([n]\) corresponds to an ordered tree with \( n + 1 \) leaves and \( k \) levels, numbered top to bottom, at which lie the internal nodes. Numbering the \( n \) spaces between leaves from left to right (imagine a raindrop falling into each space), we see that the raindrops collecting at the internal nodes at level \( i \) represent the \( i^{th} \) subset in the partition. We denote our bijection by:

\[
f : \{ \text{ordered trees with } n \text{ leaves} \} \to \mathcal{K}(\text{complete graph on } n - 1 \text{ nodes}).
\]

Figure 8 shows an example.

The binary trees with \( n + 1 \) leaves (and \( n \) internal nodes) correspond to the vertices of the \((n - 1)\)-dimensional associahedron, or Stasheff polytope \( \mathcal{K}_n \). In the world of graph-associahedra, these vertices correspond to the \( n \)-tubings of the path graph on \( n \) nodes. Carr and Devadoss realized that in fact the path graph associahedron is precisely the Stasheff associahedron [4]. Thus for any tree with \( n + 1 \) leaves we can bijectively determine a tubing on the path graph. This correspondence can be described by creating a tube of the path graph for each internal node of the tree. We number the leaves from left to right \( 0, 1, \ldots, n \) and the nodes of the path from left to right \( 1, \ldots, n \). The tube we create contains the same numbered nodes of the path graph as all but the leftmost leaf of
the subtree determined by the internal node. This bijection we denote by:
\[ g : \{ \text{trees with } n \text{ leaves} \} \to \mathcal{K}(\text{path on } n - 1 \text{ nodes}). \]

Figures 7 and 8 show examples.

3.3. Generalizing the Tonks projection. The fact that every graph of \( n \) nodes is a subgraph of the complete graph leads us to a grand factorization of the Tonks cellular projection through all connected graph associahedra. Incomplete graphs are formed simply by removing edges from the complete graph. As a single edge is deleted, the tubing is preserved up to connection. That is, if the nodes of a tube are no longer connected, it becomes two tubes made up of the two connected subgraphs spanned by its original set of nodes.

**Definition 3.1.** Let \( G \) be a graph on \( n \) nodes, and let \( e \) be an edge of \( G \). Let \( G - e \) denote the graph achieved be the deletion of \( e \), while retaining all \( n \) nodes. We define a cellular projection \( \Theta_e : \mathcal{K}G \to \mathcal{K}(G - e) \). First, allowing an abuse of notation, we define \( \Theta_e \) on individual tubes. For \( t \in T \) such that \( t \) is not a tube of \( G - e \), then let \( t', t'' \) be the tubes of \( G - e \) such that \( t' \cup t'' = t \). Then:
\[ \Theta_e(t) = \begin{cases} t, & \text{if } t \text{ is a tube of } G - e \\ \{t', t''\}, & \text{otherwise.} \end{cases} \]

Now given a tubing \( T \) on \( G \) we define its image as follows:
\[ \Theta_e(T) = \bigcup_{t \in T} \Theta_e(t). \]

See Figures 9, 10 and 11 for examples.

\[ \text{Figure 9. The Tonks projection performed on (1243), factored by graphs.} \]

**Lemma 3.2.** For a graph \( G \) with an edge \( e \), \( \Theta_e \) is a cellular surjection of polytopes \( \mathcal{K}G \to \mathcal{K}(G - e) \).

**Proof.** By Theorem 2.2 we have that the face posets of the polytopes are isomorphic to the posets of tubings. The map takes a tubing on \( G \) to a tubing on \( G - e \) with a greater or equal number of tubes. This establishes its projective property. Also, for two tubings \( U \prec U' \) of \( G \) we see that either \( \Theta_e(U) = \Theta_e(U') \) or \( \Theta_e(U) \prec \Theta_e(U') \). Thus the poset structure is preserved by the projection.

Finally, the map is surjective, since given any tubing \( T \) on \( G - e \) we can find a (maximal) preimage \( T' \) as follows: First consider all the tubes of \( T \) as a candidate tubing of \( G \). If it is a valid tubing, we have our \( T' \). If not, then it must be that some pairs of tubes \( t', t'' \in T \) are adjacent via the edge \( e \). Then let \( T' \) be the result of replacing each such pair in \( T \) with the single tube \( t = t' \cup t'' \). Note that not only \( T' \) but any tubing \( T'' \prec T' \) will be a preimage of \( T \).

Composition of these cellular projections is commutative.

**Lemma 3.3.** Let \( e, e' \) be edges of \( G \). Then \( \Theta_e \circ \Theta_{e'} = \Theta_{e'} \circ \Theta_e \).

Proof. Consider the image of a tubing of \( G \) under either composition. A tube of \( G \) that is a tube of both \( G - e \) and \( G - e' \) will persist in the image. Otherwise it will be broken, perhaps twice. The same smaller tubes will result regardless of the order of the breaking. □

By Lemma 3.3 we can unambiguously use the following notation:

**Definition 3.4.** For any collection \( E \) of edges of \( G \) we denote by \( \Theta_E : \mathcal{K}G \to \mathcal{K}(G - E) \) the composition of projections \( \{ \Theta_e \mid e \in E \} \).

Now the Tonks projection from ordered trees to trees can be described in terms of the tubings. Beginning with a tubing on the complete graph with numbered nodes, we achieve one on the path graph by deleting all the edges of the complete graph except for those connecting the nodes in consecutive order from 1 to \( n \). See Figures 7, 8 and 9 for a picture to accompany the following:

**Theorem 3.5.** Let \( e_{i,k} \) be the edge between the nodes \( i, i + k \) of the complete graph \( G \) on \( n \) nodes. Let \( P = \{ e_{i,k} \mid i \in \{1, \ldots, n - 2\} \text{ and } k \in \{2, \ldots, n - i\} \} \). These are all but the edges of the path graph. Then the following composition gives the Tonks map:

\[
g^{-1} \circ \Theta_P \circ f = \Theta.
\]

*Proof. We begin with an ordered \( j \)-partition of \( n \) drawn as an ordered tree \( t \) with \( n + 1 \) leaves numbered left to right, and \( j \) levels numbered top to bottom. The bijection \( f \) tells us how to draw \( t \) as a tubing of the complete graph \( K_n \) with numbered nodes.

Consider the internal nodes of \( t \) at level \( i \). In \( f(t) \) there corresponds a tube \( u_i \) containing the nodes of \( K_n \) with the same numbers as all the spaces between leaves of the subtrees rooted at the internal nodes at level \( i \). The set in the partition which is represented by the internal nodes at level \( i \) is the precise set of nodes of \( u_i \) which are not contained by any other tube in \( f(t) \). The relative position of this subset of \([n]\) in our ordered partition is reflected by the relative nesting of the tube \( u_i \).

The map \( \Theta_P \) has the following action on tubings. Let \( u \) be a tube of \( f(t) \). We partition \( u \) into subsets of consecutively numbered nodes such that no union of two subsets is consecutively numbered. Then the tubing \( \Theta_P(f(t)) \) contains the tubes given by these subsets.

Now we claim that the non-ordered tree \( g^{-1}(\Theta_P(f(t))) \) has the same branching structure as \( t \) itself. First, for any interior node of \( t \) there is a tube of consecutively numbered nodes of the path graph, arising from the original tube of \( f(t) \). This then becomes a corresponding interior node of our non-ordered tree, under the action of \( g^{-1} \). Secondly, if any interior node of \( t \) lies between the root of \( t \) and a second interior node, then the same relation holds in the non-ordered tree between the corresponding interior nodes. This follows from the fact that the action of any of our maps \( \Theta_{e_{i,k}} \) will preserve relative nesting. That is, for two tubes \( u \subset v \) we have that in the image of \( \Theta_{e_{i,k}} \) any tube resulting from \( u \) must lie within a tube resulting from \( v \). □

To sum up, there is a factorization of the Tonks cellular projection through various graph-associahedra. An example on an \( n \)-tubing is shown in Figure 9, and another possible factorization of the projection in dimension 3 is demonstrated in Figure 10.

**3.4. Disconnected graph associahedra.** The special case of extending the Tonks projection to graphs with multiple connected components will be useful. Consider a partition \( S_1 \sqcup \cdots \sqcup S_k \) of the \( n \) nodes of a connected graph \( G \), chosen such that we have connected induced subgraphs \( G(S_i) \). Let \( E_S \) be the set of edges of \( G \) not included in any \( G(S_i) \). Thus the graph \( G - E_S \) formed by deleting these edges will have the \( k \) connected components \( G(S_i) \). In this situation \( \Theta_{E_S} \) will be a generalization of the Tonks projection to the graph associahedron of a disconnected graph.

In Figure 11 we show the extended Tonks projections in dimension 2.
Figure 10. A factorization of the Tonks projection through 3 dimensional graph associahedra. The shaded facets correspond to the shown tubings, and are collapsed as indicated to respective edges. The first, third and fourth pictured polytopes are above views of \( P_4, W_4 \) and \( K_4 \) respectively.

Remark 3.6. As shown in [8], the graph associahedron of the graph consisting of \( n \) nodes and no edges is the \((n - 1)\)-simplex \( \Delta^{(n - 1)} \). Thus the graph associahedron of a graph with connected components \( G_1, G_2, \ldots, G_k \) is actually equivalent to the polytope \( KG_1 \times \cdots \times KG_k \times \Delta^{(k - 1)} \).

Lemma 3.7. For a disconnected graph \( G \) with multiple connected components \( G_1, G_2, \ldots, G_k \), there is always a cellular surjection from \( KG \) to \( KG_1 \times \cdots \times KG_k \).

Proof. Given a tubing \( T \) of \( G \), define \( \eta(T) = (T_1, \ldots, T_k) \) where \( T_i \) is the tubing of \( G_i \) which contains each tube of \( T \) that lies in \( G_i \) as well as the universal tube \( G_i \) itself. The map \( \eta : KG \to KG_1 \times \cdots \times KG_k \) clearly preserves the poset structure. It is projective since in fact it only forgets which of the connected components of \( G \) are originally tubes of \( T \). The map \( \eta \) is onto since any collection of tubings \( (T_1, \ldots, T_k) \) has a maximal preimage given by \( (T_1 - G_1) \sqcup \cdots \sqcup (T_k - G_k) \sqcup \{G\} \). \( \square \)

Figure 11. Extending the Tonks projection to the 2-simplex. The highlighted edges are collapsed to the respective vertices.

4. Geometrical view of \( S\text{Sym} \) and \( Y\text{Sym} \)

Before proving the graded algebra structures on graph associahedra which our title promised, we motivate our point of view by showing how it will fit with the well known graded algebra structures on permutations and binary trees.

4.1. Review of \( S\text{Sym} \). Let \( S\text{Sym} \) be the graded vector space over \( \mathbb{Q} \) with the \( n^{th} \) component of its basis given by the permutations \( S_n \). An element \( \sigma \in S_n \) is given by its image \((\sigma(1), \ldots, \sigma(n))\), often without commas. We follow [1] and [2] and write \( F_u \) for the basis element corresponding to \( u \in S_n \) and 1 for the basis element of degree 0. A graded Hopf Algebra structure on \( S\text{Sym} \) was
discovered by Malvenuto and Reutenauer in [11]. First we review the product and coproduct and then show a new way to picture those operations.

Recall that a permutation \(\sigma\) is said to have a descent at location \(p\) if \(\sigma(p) > \sigma(p + 1)\). The \((p,q)\)-shuffles of \(S_{p+q}\) are the \((p + q)\) permutations with at most one descent, at position \(p\). We denote this set as \(S^{(p,q)}\). The product in \(\mathfrak{S} Sym\) of two basis elements \(F_u\) and \(F_v\) for \(u \in S_p\) and \(v \in S_q\) is found by summing a term for each shuffle, created by composing the juxtaposition of \(u\) and \(v\) with the inverse shuffle:

\[
F_u \cdot F_v = \sum_{\sigma \in S^{(p,q)}} F_{(u \times v) \cdot \sigma^{-1}}.
\]

Here \(u \times v\) is the permutation \((u(1), \ldots, u(p), v(1) + p, \ldots, v(q) + p)\).

### 4.2. Geometry of \(\mathfrak{S} Sym\)

The algebraic structure of \(\mathfrak{S} Sym\) can be linked explicitly to the recursive geometric structure of the permutohedra. In \(\mathfrak{S} Sym\) we may view our operands (a pair of permutations) as a vertex of the cartesian product of permutohedra \(P_p \times P_q\). Then their product is the formal sum of images of that vertex under the collection of inclusions of \(P_p \times P_q\) as a facet of \(P_{p+q}\). An example is in Figure 12, where the product is shown along with corresponding pictures of the tubings on the complete graphs. To make this geometric claim precise we use the facet isomorphism \(\hat{\rho}_t\) which exists by Theorem 2.6.

**Theorem 4.1.** The product in \(\mathfrak{S} Sym\) of basis elements \(F_u\) and \(F_v\) for \(u \in S_p\) and \(v \in S_q\) may be written:

\[
F_u \cdot F_v = \sum_{\iota : [p] \rightarrow [p+q]} F_{\hat{\rho}_t \iota(u,v)}
\]

where the sum is over all order preserving injections \(\iota\), and where the tube \(t_i\) contains the nodes numbered by \(\iota([p])\).

![Figure 12](image-url)
\textbf{Proof.} We first choose an order preserving injection \( \iota : [p] \to [p+q] \). There is then automatically an order preserving injection \( \hat{i} : [q] \to [p+q] \) such that \( \iota([p]) \cup \hat{i}([q]) = [p+q] \). These together play the role of the shuffle:
\[
\sigma(x) = \begin{cases} 
\iota(x), & x \in [p] \\
\hat{i}(x), & x \in \{p+1, \ldots, p+q\}.
\end{cases}
\]

Our term of \( F_u \cdot F_v \) is the \((p+q)\)-tubing on \( K_{p+q} \) given by including for each tube \( t \) of \( u \) the tube with nodes the image \( \iota(t) \). Denote the resulting tubes by \( \overline{\iota(u)} \). For each tube \( s \) of \( v \) we include the tube formed by \( \iota([p]) \cup \hat{i}(s) \). We denote the resulting tubes as \( \overline{\hat{i}(v)} \). This transfer of tubes has the effect of creating \( u \times v \). Thus we can rewrite the definition of our product in \( \mathcal{S}\text{Sym} \) as follows:
\[
F_u \cdot F_v = \sum_{\substack{\iota([p]) \cup \hat{i}(v) \subset [p+q]}} F_{\overline{\iota(u) \cup \hat{i}(v)}}.
\]

From Theorem 2.6 we have that for each tube \( t \) of the graph, the corresponding facet of \( KG \) is isomorphic to \( KG^*(t) \times KG(t) \). In the case of the complete graph \( G = K_{p+q} \), for any tube \( t \) of \( p \) nodes, we have the facet inclusion \( \hat{\rho}_t : \mathcal{P}_q \times \mathcal{P}_p \to \mathcal{P}_{p+q} \). For \( t = \iota([p]) \) the definition of \( \hat{\rho}_t(u,v) \), as described in the proof of [4, Theorem 2.9], is precisely the same as our \( \overline{\iota(u) \cup \hat{i}(v)} \).

To sum up, in our view of \( \mathcal{S}\text{Sym} \) each permutation is pictured as a tubing of a complete graph with numbered nodes. Since any subset of the nodes of a complete graph spans a smaller complete graph, we can draw the terms of the product in \( \mathcal{S}\text{Sym} \) directly. Choosing a \((p,q)\)-shuffle is accomplished by choosing \( p \) nodes of the \((p+q)\)-node complete graph \( K_{p+q} \). First the permutation \( u \) (as a \( p \)-tubing) is drawn upon the induced \( p \)-node complete subgraph, according to the ascending order of the chosen nodes. Then the permutation \( v \) is drawn upon the subgraph induced by the remaining \( q \) nodes—with a caveat. Each of the tubes of \( v \) is expanded to also include the \( p \) nodes that were originally chosen. This perspective will generalize nicely to the other graphs and their graph associahedra.

In [1] and [2] the authors give a related geometric interpretation of the products of \( \mathcal{S}\text{Sym} \) and \( \mathcal{Y}\text{Sym} \) as expressed in the Moebius basis. An interesting project for the future would be to apply that point of view to \( W\text{Sym} \).

\textbf{Remark 4.2.} The coproduct of \( \mathcal{S}\text{Sym} \) can also be described geometrically in terms of the graph tubings. The coproduct is usually described as a sum of all the ways of splitting a permutation \( u \in S_n \) into two permutations \( u_i \in S_i \) and \( u_{n-i} \in S_{n-i} \):
\[
\Delta(F_u) = \sum_{i=0}^{n} F_{u_i} \otimes F_{u_{n-i}}
\]

where \( u_i = (u(1) \ldots u(i)) \) and \( u_{n-i} = (u(i+1) - i \ldots u(n) - i) \).

Given an \( n \)-tubing \( u \) of the complete graph on \( n \) vertices we can find \( u_i \) and \( u_{n-i} \) via the extended Tonks projection. Let \( G([i]) \) denote the subgraph spanned by the nodes \( p \leq i \) and \( G([n] - [i]) \) the subgraph spanned by the remaining nodes \( q > i \). Let
\[
E_i = \{ e \text{ an edge of } G \mid e \text{ connects a node } p \leq i \text{ to a node } q > i \}.
\]

Then \( \Theta_{E_i} : KG \to KG(G - E_i) \) denotes the composition of projections. From Lemma 3.7 we have a map \( \eta_i \) from \( KG(G - E_i) \) to \( KG([i]) \times KG([n] - [i]) \), Then \( (u_i, u_{n-i}) = \eta_i(\Theta_{E_i}(u)) \).

In other words, the coproduct is formed by summing projections of the permutation onto faces of quotient polytopes; faces which are equivalent to cartesian products of smaller permutohedra. An example is in Figure 13.
4.3. Review of $\mathcal{YSym}$. The product and coproduct of $\mathcal{YSym}$ are described by Aguiar and Sottile in terms of splitting and grafting binary trees [2]. We can vertically split a tree into smaller trees at each leaf from top to bottom—as if a lightning strike hits a leaf and splits the tree along the path to the root. We graft trees by attaching roots to leaves (without creating a new interior node at the graft.) The product of two trees with $n$ and $m$ interior nodes (in that order) is a sum of $\binom{n+m}{n}$ terms, each with $n+m$ interior nodes. Each is achieved by vertically splitting the first tree into $m+1$ smaller trees and then grafting them to the second tree, left to right. A picture is in Figure 14.

4.4. Geometry of $\mathcal{YSym}$. Since Loday and Ronco demonstrated that the Tonks projection, restricted to vertices, gives rise to an algebra homomorphism $\tau: \mathfrak{SSym} \to \mathcal{YSym}$, it is no surprise that the processes of splitting and grafting have geometric interpretations. Grafting corresponds to certain face inclusions of associahedra, and splitting corresponds to the extension of Tonks’s projection to disconnected graphs. To see the latter, note that splitting at leaf $i$ is the same as deleting the edge from node $i$ to node $i+1$. Thus the product in $\mathcal{YSym}$ can be described with the language of path graph tubings, using a combination of facet inclusion and the extended Tonks projection. An example is shown in Figure 14.

Let $U$ be the path graph with $p$-tubing $u$ and $V$ the path graph with $q$-tubing $v$. Given an order preserving injection $\iota: [p] \to [p+q]$, we can partition the nodes of $U$ into the preimages $s_1, \ldots, s_k$ of the connected components $\iota(s_1), \ldots, \iota(s_k)$ of the possibly disconnected subgraph induced by the nodes $\iota([p])$ on the $(p+q)$-path. Let $E_\iota$ be the edges of $U$ not included in the subgraphs $U(s_i)$ induced by our partition. For short we denote the extended Tonks projection $\Theta_{E_\iota}$ as simply $\Theta_\iota$. Now $\Theta_\iota(u)$ is the projection of $u$ onto the possibly disconnected graph $U - E_\iota$. Recall from Lemma 3.7 that a vertex of $\mathcal{K}(U - E_\iota)$ may be mapped to a vertex of $\mathcal{K}U(s_1) \times \cdots \times \mathcal{K}U(s_k)$. We call this map $\eta_\iota$.

**Theorem 4.3.** The product in $\mathcal{YSym}$ can be written:

$$F_u \cdot F_v = \sum_{\iota: [p] \to [p+q]} F_{\rho_\iota(\Theta_\iota(u)), v}.$$ 

Where $\rho_\iota$ is shorthand notation for the isomorphism $\bar{\rho}_{\iota(s_1) \ldots \iota(s_k)}$ from Corollary 2.7.

**Proof.** We will explain how the splitting and grafting of trees in a term of the product may be put into the language of tubings, and then argue that the term thus described is indeed the image of projections and inclusions as claimed. The product in $\mathcal{YSym}$ of two basis elements $F_u$ and $F_v$ for $u \in \mathcal{Y}_p$ and $v \in \mathcal{Y}_q$ is found by summing a term for each order preserving injection $\iota: [p] \to [p+q]$. We draw $u$ and $v$ in the form of tubings on path graphs of $p$ and $q$ nodes, respectively.
Figure 14. The product in $\mathcal{Y}Sym$. The circled vertices of $\mathcal{K}_4$ which are at the upper end of highlighted edges are the fifth, third and second terms of the product, in that order respectively from bottom to top in the picture.

Here is the non-technical description: first the $p$-tubing $u$ is drawn upon the induced subgraph of the nodes $\iota([p])$ according to the ascending order of the chosen nodes. However, each tube may need to be first broken into several subtubes. Then the $q$-tubing $v$ is drawn upon the subgraph induced by the remaining $q$ nodes. In this last step, each of the tubes of $v$ is expanded to also include any of the previously drawn tubes that its nodes are adjacent to.

To be precise, we first choose an order preserving injection $\iota : [p] \to [p+q]$. There is then automatically an order preserving injection $\hat{\iota} : [q] \to [p+q]$ such that the images of $\iota$ and $\hat{\iota}$ do not overlap. Our term of $F_u \cdot F_v$ is the $(p+q)$-tubing on $(p+q)$-path given by the following:

First, let $t$ be a tube of $u$. We partition $\iota(t)$ into subsets of consecutively numbered nodes such that no union of two subsets is consecutively numbered. Then our term of $F_u \cdot F_v$ contains the tubes given by these subsets. Let $\iota(u)$ denote the tubing constructed thus far. After this step in terms of trees, we have performed the splitting and chosen where to graft the (non-trivial) subtrees. In terms of trees the splitting occurs at leaves labeled by $\hat{\iota}(i) - i$ for $i \in [q]$.

Second, let $s$ be a tube of $v$. We include in our term of the product the tube formed by $\hat{\iota}(s) \cup \{ j \in t' \in \iota(u) | t' \textrm{ is adjacent to } \hat{\iota}(s) \}$.

Let $\iota(v)$ denote the tubes added in this second step. Now we have completed the grafting operation, and can restate the product as:

$$F_u \cdot F_v = \sum_{e : [p] \to [p+q]} F_{\iota(u) \cup \hat{\iota}(v)}.$$  

Now we just point out that $\hat{\rho}_e(\eta_i(\Theta_i(u), v))$ is precisely the same as $\iota(u) \cup \iota(v)$. The splitting of tubes is accomplished by $\Theta_i$; and $\eta_i$ simply recasts the result as an element of the appropriate
cartesian product. Then \( \hat{\rho}_{i} \), as defined in [4], performs the inclusion of that element paired together with \( v \) just as we describe above using the notation \( \hat{\iota}(u) \cup \hat{\iota}(v) \).

To summarize, the product in \( \mathcal{Y} \text{Sym} \) can be seen as a process of splitting and grafting, or equivalently of projecting and including. From the latter viewpoint, we see the product of two associahedra vertices being achieved by projecting the first onto a cartesian product of smaller associahedra, and then mapping that result paired with the second operand into a large associahedron via face inclusion. Notice that the reason the second path graph tubing \( v \) can be input here is that any reconnected complement of a path graph is another path graph.

Remark 4.4. The coproduct of \( \mathcal{Y} \text{Sym} \) can also be described geometrically in terms of the graph tubings. The coproduct is usually described as a sum of all the ways of splitting a binary tree \( u \in \mathcal{Y}_{n} \) along leaf \( i \) into two trees: \( u_{i} \in \mathcal{Y}_{i} \) and \( u_{n-i} \in \mathcal{Y}_{n-i} \):

\[
\Delta(F_{u}) = \sum_{i=0}^{n} F_{u_{i}} \otimes F_{u_{n-i}}.
\]

Given an \( n \)-tubing \( u \) of the path graph on \( n \) vertices we can find \( u_{i} \) and \( u_{n-i} \) via the extended Tonks projection. Let \( e_{i} \) be the edge from node \( i \) to node \( i + 1 \). Then \( \Theta_{e_{i}} : \mathcal{K}G \rightarrow \mathcal{K}(G - e_{i}) \) denotes the extended Tonks projection. From Lemma 3.7 we have a map \( \eta_{i} \) from \( \mathcal{K}(G - e_{i}) \) to \( \mathcal{K}_{i} \times \mathcal{K}_{n-i} \), Then \( (u_{i}, u_{n-i}) = \eta_{i}(\Theta_{e_{i}}(u)) \).

In other words, the coproduct is formed by summing projections of the tree onto faces of quotient polytopes; faces which are equivalent to cartesian products of smaller associahedra. An example is in Figure 15.

![Figure 15. The coproduct in \( \mathcal{Y} \text{Sym} \).](image)

5. The algebra of the vertices of cyclohedra

Recall that the \((n-1)\)-dimensional cyclohedron \( \mathcal{W}_{n} \) has vertices which are indexed by \( n \)-tubings on the cycle graph of \( n \)-nodes. We will define a graded algebra with a basis which corresponds to the vertices of the cyclohedra, and whose grading respects the dimension (plus one) of the cyclohedra.

Definition 5.1. Let \( \mathcal{W} \text{Sym} \) be the graded vector space over \( \mathbb{Q} \) with the \( n^{th} \) component of its basis given by the \( n \)-tubings on the cycle graph of \( n \) cyclically numbered nodes. By \( \mathcal{W}_{n} \) we denote the set of \( n \)-tubings on the cycle graph \( C_{n} \). We write \( F_{u} \) for the basis element corresponding to \( u \in \mathcal{W}_{n} \) and 1 for the basis element of degree 0.

5.1. Graded algebra structure. Now we demonstrate a product which respects the grading on \( \mathcal{W} \text{Sym} \) by following the example described above for \( \mathcal{S} \text{Sym} \). The product in \( \mathcal{W} \text{Sym} \) of two basis elements \( F_{u} \) and \( F_{v} \) for \( u \in \mathcal{W}_{p} \) and \( v \in \mathcal{W}_{q} \) is found by summing a term for each order preserving injection \( \iota : [p] \rightarrow [p + q] \). First the \( p \)-tubing \( u \) is drawn upon the induced subgraph of the nodes \( \iota([p]) \) according to the ascending order of the chosen nodes. However, each tube may need to be first broken into several subtubes. Then the \( q \)-tubing \( v \) is drawn upon the subgraph induced by
the remaining \( q \) nodes. However, each of the tubes of \( v \) is expanded to also include any of the previously drawn tubes that its nodes are adjacent to.

To be precise, we first choose an order preserving injection \( \iota : [p] \to [p+q] \). There is then automatically an order preserving injection \( \hat{i} : [q] \to [p+q] \) such that the images of \( \iota \) and \( \hat{i} \) do not overlap. Our term of \( F_u \cdot F_v \) is the \((p+q)\)-tubing on \( C_{p+q} \) given by the following. First, let \( t \) be a tube of \( u \). We partition \( \iota(t) \) into subsets of consecutively numbered nodes \((\text{mod } p+q)\) such that no union of two subsets is consecutively numbered \((\text{mod } p+q)\). Then our term of \( F_u \cdot F_v \) contains the tubes given by these subsets. Let \( \iota(u) \) denote the tubing constructed thus far. Second, let \( s \) be a tube of \( v \). We include in our term of the product the tube formed by \( \hat{i}(s) \cup \{ j \in t' \in \iota(u) \mid t' \text{ is adjacent to } \hat{i}(s) \} \). Let \( \iota(v) \) denote the tubes added in this second step.

**Definition 5.2.**

\[
F_u \cdot F_v = \sum_{\iota : [p] \to [p+q]} F_{\iota(u) \cup \iota(v)}.
\]

An example is shown in Figure 16. For an example of finding a term in the product given a specific map \( \iota \), see the second part of Figure 18.

**Theorem 5.3.** The product we have just described makes \( \text{WSym} \) into an associative graded algebra.

**Proof.** First we must check that the result of the product is indeed a sum of valid \((p+q)\)-tubings of the cycle graph \( C_{p+q} \). We claim that in each term the new tubes we have created are pairwise compatible. This being shown, we will be able to deduce that since \( p+q \) tubes were used in the construction, then the resulting term will necessarily have \( p+q \) tubes as well. To check our claim we compare pairs of tubes in one or both of \( \iota(u) \) and \( \iota(v) \). There are six cases:

1. By our method of construction, any tube of \( \iota(u) \) is either nested within some of or far apart from all of the tubes of \( \iota(v) \).
2. If two tubes of \( \iota(u) \) are made up of nodes in \( \iota(t) \) and \( \iota(t') \) respectively for nested tubes \( t \) and \( t' \) of \( u \) then they will be similarly nested.
3. Two tubes from \( \iota(u) \) might both be made up of nodes in \( \iota(t) \) for a single tube \( t \) of \( u \). In that case they are guaranteed to be far apart, since their respective nodes together cannot be a consecutive string \((\text{mod } p+q)\).
4. If two tubes of \( \iota(u) \) are made up of nodes in \( \iota(t) \) and \( \iota(t') \) respectively for far apart tubes \( t \) and \( t' \) of \( u \), then we claim they will be far apart. This is true since if two nodes \( a \) and \( b \) are nonadjacent in the cycle graph \( C_p \) then the two nodes \( \iota(a) \) and \( \iota(b) \) will be nonadjacent in \( C_{p+q} \).
5. If two tubes of \( \iota(v) \) contain some nodes in \( \iota(t) \) and \( \iota(t') \) respectively for nested tubes \( t \) and \( t' \) of \( u \) then they will be similarly nested. This follows from the fact that \( \iota(t) \) will only be adjacent to nodes that \( \iota(t') \) is also adjacent to.
6. Finally, if two tubes of \( \iota(v) \) contain some nodes in \( \iota(t) \) and \( \iota(t') \) respectively for far apart tubes \( t \) and \( t' \) of \( u \), then we claim they will be far apart. This final case depends on a special property which the cycle graphs exemplify. Given any subset of \( k \) of the nodes of \( C_n \), the reconnected complement of that subset is the cycle graph \( C_{n-k} \). Specifically the reconnected complement of \( C_{p+q} \) with respect to the nodes \( \iota(p) \) is the graph \( C_q \). Thus even the expanded tubes of \( \iota(v) \) remain far apart as long as their components from \( \iota(q) \) were far apart; and this last property is guaranteed since \( \iota \) preserves the cyclic order.

Thus we have shown that the result of multiplying two basis elements is again a basis element, and that this multiplication respects the grading. That this multiplication is associative is a corollary.
of the following result regarding how the multiplication is preserved under a map from $S_{\text{Sym}}$, specifically a corollary of Theorem 5.6.

$\begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{ccc}
2 & 3 & 4 \\
\end{array}
\end{array}
+ 
\begin{array}{c}
\begin{array}{ccc}
3 & 4 & 5 \\
\end{array}
\end{array}
+ \ldots
\end{array}$

Figure 16. A product of cycle graph tubings (10 terms total).

Remark 5.4. Note that the cases (1) and (2) are explainable simply by the fact that we start with two valid tubings and multiply them in the given order. Cases (3)-(6) however can be jointly explained based upon the fact that given any subset of $k$ of the nodes of $C_n$, the reconnected complement of that subset is the cycle graph $C_{n-k}$. Cases (3)-(5) specifically rely on the fact that the reconnected complement of $C_{p+q}$ with respect to the nodes $\hat{i}(q)$ is the graph $C_p$. This property is true of many other graph sequences, including the complete graphs and the path graphs. The property is also more simply stated as follows: the reconnected complement of $G_i$ with respect to any single node is $G_{i-1}$.

Remark 5.5. Once again we can interpret the product geometrically. The entire contents of the proof of Theorem 4.3 apply here, with the term “path” everywhere replaced with the term “cycle.” Thus a term in the product can be seen as first projecting the cyclohedron vertex $u$ onto a collection of sub-path-graphs of the cycle. We then map the vertex of this cartesian product of associahedra, paired with the second vertex of the cyclohedron represented by $v$, into the large cyclohedron via the indicated face inclusion. The usual picture is in Figure 17.

5.2. Algebra homomorphisms. Next we consider the map from the permutohedra to the cyclohedra which is described via the deletion of edges—from the complete graphs to the cycle graphs—and point out that this is an algebra homomorphism. Recall from Theorem 3.5 that $\Theta_P$ is the Tonks projection viewed from the graph associahedra point of view; via deletion of the edges of the complete graph except for the path connecting the numbered nodes in order. Let $\Theta_c$ be the map defined just as $\Theta_P$ but without deleting the edge from node $n$ to node 1. Thus we will be deleting all the edges except those making up the cycle of numbered nodes in cyclic order. Define a map from $S_{\text{Sym}}$ to $W_{\text{Sym}}$ on basis elements by:

$$\hat{\Theta}_c(F_u) = F_{\Theta_c(u)}.$$ 

Theorem 5.6. The map $\hat{\Theta}_c$ is an algebra homomorphism from $S_{\text{Sym}}$ onto $W_{\text{Sym}}$.

Proof. For $u \in S_p$ and $v \in S_q$ we compare $\hat{\Theta}_c(F_u \cdot F_v)$ with $\hat{\Theta}_c(F_u) \cdot \hat{\Theta}_c(F_v)$. Each of the multiplications results in a sum of $\binom{p+q}{p}$ terms. It turns out that comparing the results of the two operations can be done piecewise. Thus we check that for a given injection $\iota : [p] \to [p+q]$ the respective terms of our two operations agree: we claim that

$$\Theta_c(\iota(u) \cup \iota(v)) = \iota(\Theta_c(u)) \cup \iota(\Theta_c(v)).$$

Here $\iota(u)$ and $\iota(v)$ are as described in the proof of Theorem 4.1, and the righthand side of the equation is using the notation of Definition 5.2. The justification is a straightforward comparison of the indicated operations on individual tubes. There are two cases:
Figure 17. The product in \( \mathcal{W}_{\text{Sym}} \). The second and fifth terms of the product are the ones that use a Tonks projection; they are found as the two vertices at the tips of included (highlighted) edges.

(1) For \( t \) a tube of \( u \), the right-hand side first breaks \( t \) into several smaller tubes by deleting certain edges of the \( p \)-node complete graph, then takes each of these to their image under \( \iota \), breaking them again whenever they are no longer connected. The left-hand side takes \( t \) to \( \iota(t) \), a tube of the complete graph on \( p + q \) nodes, and then breaks \( \iota(t) \) into the tubes that result from deleting the specified edges of the \( (p + q) \)-node complete graph. In this last step, by Lemma 3.3, we can delete first those edges that also happen to lie in the complete subgraph induced by \( \iota([p]) \), and then the remaining specified edges. Thus we have duplicated the left-hand side and get the same final set of tubes on either side of the equation.

(2) For \( s \) a tube of \( v \), the right-hand side first breaks \( s \) into several smaller tubes by deleting certain edges of the \( q \)-node complete graph, then takes each of these to their image under \( \hat{i} \), expanding them to include tubes of \( \iota(\Theta_c(u)) \). The left-hand side takes \( s \) to \( \hat{i}(s) \cup \iota([p]) \), and then breaks the result into the tubes that result from deleting the specified edges of the \( (p + q) \)-node complete graph. Again we can delete first those edges that also happen to lie in the complete subgraph induced by \( \iota([p]) \), and then the remaining specified edges, duplicating the left-hand process.

An illustration of the two sides of the equation is in Figure 18. The surjectivity of \( \hat{\Theta}_c \) follows from the surjectivity of the generalized Tonks projection, as shown in Lemma 3.2.

Let \( C_n \) be the cycle graph on \( n \) numbered nodes and let \( w \) be the edge from node 1 to node \( n \). We can define a map from \( \mathcal{W}_{\text{Sym}} \) to \( \mathcal{Y}_{\text{Sym}} \) by:

\[
\hat{\Theta}_w(F_u) = F_{\Theta_w(u)},
\]

for \( u \in \mathcal{W}_n \).

Theorem 5.7. \( \hat{\Theta}_w \) is a surjective homomorphism of graded algebras \( \mathcal{W}_{\text{Sym}} \to \mathcal{Y}_{\text{Sym}} \).
Proof. In [10] it is shown that the map we call $\tau$ is an algebra homomorphism from $\mathcal{S}_{Sym}$ to $\mathcal{Y}_{Sym}$. In the previous theorem we demonstrated that the map $\hat{\Theta}_c$ is a surjective algebra homomorphism from $\mathcal{S}_{Sym}$ to $\mathcal{W}_{Sym}$. Now, since $\tau$ is the same map on vertices as our $\Theta_P$ (from Theorem 3.5), then the relationship of these three is:

$$\tau = \hat{\Theta}_w \circ \hat{\Theta}_c.$$ 

Thus $\hat{\Theta}_w$ is an algebra homomorphism from $\mathcal{S}_{Sym}$ onto $\mathcal{W}_{Sym}$. \qed

Remark 5.8. The existence of a surjective algebra homomorphism from $\mathcal{S}_{Sym}$ not only allows us a shortcut to demonstrating associativity, but leads to an alternate description of the product in the range of that homomorphism. This may be achieved in three steps:

1. Lifting our $p$ and $q$-tubings in $\mathcal{W}_{Sym}$ to any preimages of the generalized Tonks projection $\Theta_c$ on the complete graphs on $p$ and $q$ nodes.
2. Performing the product of these complete graph tubings in $\mathcal{S}_{Sym}$,
3. and finding the image of the resulting terms under the homomorphism $\hat{\Theta}_c$.

This description is independent of our choices of preimages in $\mathcal{S}_{Sym}$ due to the homomorphism property.

The following corollary follows directly from the properties of the surjective algebra homomorphisms of Theorems 5.6 and 5.7.

Corollary 5.9. $\mathcal{W}_{Sym}$ is a left $\mathcal{S}_{Sym}$-module under the definition $F_u \cdot F_{\Theta_c(v)} = \hat{\Theta}_c(F_u \cdot F_v)$ and a right $\mathcal{S}_{Sym}$-module under the definition $F_{\Theta_c(u)} \cdot F_v = \hat{\Theta}_c(F_u \cdot F_v)$. $\mathcal{Y}_{Sym}$ is a left $\mathcal{W}_{Sym}$-module under the definition $F_u \cdot F_{\Theta_w(v)} = \hat{\Theta}_w(F_u \cdot F_v)$ and a right $\mathcal{W}_{Sym}$-module under the definition $F_{\Theta_w(u)} \cdot F_v = \hat{\Theta}_w(F_u \cdot F_v)$.
6. EXTENSION TO THE FACES OF THE POLYTOPES

Chapoton was the first to point out the fact that in studying the Hopf algebras based on vertices of permutahedra, associahedra and cubes, one need not restrict their attention to just the zero-dimensional faces of the polytopes. He has shown that the Loday Ronco Hopf algebra $\mathcal{YSym}$, the Hopf algebra of permutations $\mathfrak{S}Sym$, and the Hopf algebra of quasisymmetric functions $\mathcal{QSym}$ are each subalgebras of algebras based on the trees, the ordered partitions, and faces of the hypercubes respectively [5]. Furthermore, he has demonstrated that these larger algebras of faces are bi-graded and possess a differential. Here we point out that the cyclohedra based algebra $\mathcal{WSym}$ can be extended to a larger algebra based on all the faces of the cyclohedra as well, and conjecture the additional properties.

Chapoton’s product structure on the permutahedra faces is given in [5] in terms of ordered partitions. Let $\mathfrak{S}\mathcal{Sym}$ be the graded vector space over $\mathbb{Q}$ with the $n$th component of its basis given by the ordered partitions of $[n]$. We write $F_u$ for the basis element corresponding to the $m$-partition $u : [n] \to [m]$, for $0 \leq m \leq n$, and 1 for the basis element of degree 0. The product in $\mathfrak{S}\mathcal{Sym}$ of two basis elements $F_u$ and $F_v$ for $u : [p] \to [k]$ and $v : [q] \to [l]$ is found by summing a term for each shuffle, created by composing the juxtaposition of $u$ and $v$ with the inverse shuffle:

$$F_u \cdot F_v = \sum_{\sigma \in S(p,q)} F_{(u \times v) \circ \sigma^{-1}}.$$

Here $u \times v$ is the ordered $(k + l)$-partition of $[p + q]$ given by:

$$(u \times v)(i) = \begin{cases} u(i), & i \in [p] \\ v(i-p) + k, & i \in \{p+1, \ldots, p+q\}. \end{cases}$$

The bijection between tubings of complete graphs and ordered partitions allows us to write this product geometrically.

**Theorem 6.1.** The product in $\mathfrak{S}\mathcal{Sym}$ may be written as:

$$F_u \cdot F_v = \sum_{\iota : [p] \to [p+q]} F_{\hat{\rho}_t}(u,v)$$

where the sum is over all order preserving injections $\iota$, and where the tube $t_\iota$ contains the nodes numbered by $\iota([p])$.

**Proof.** Recall that we found the bijection between tubings of complete graphs and ordered partitions by noting that each tube contains some numbered nodes which are not contained in any other tube. These subsets of $[n]$, one for each tube, make up the partition, and the ordering of the partition is from innermost to outermost tube. Now the Carr-Devadoss isomorphism $\hat{\rho}$ is a bijection of face posets. With this in mind, the same logic applies as in the proof of Theorem 4.1. □

In other words, we view our operands (a pair of ordered partitions) as a face of the cartesian product of permutahedra $\mathcal{P}_p \times \mathcal{P}_q$. Then the product is the formal sum of images of that face under the collection of inclusions of $\mathcal{P}_p \times \mathcal{P}_q$ as a facet of $\mathcal{P}_{p+q}$. An example is in Figure 19.

We leave to the reader the by now straightforward tasks of finding the geometric interpretations of the coproduct on $\mathfrak{S}\mathcal{Sym}$ and of the Hopf algebra structure on the faces of the associahedra, $\mathcal{YSym}$. Each one can be done simply by repeating earlier definitions using tubings, but with all sizes of $k$-tubings as operands.

Recall that the faces of the cyclohedra correspond to tubings of the cycle graph. We will define a graded algebra with a basis which corresponds to the faces of the cyclohedra, and whose grading respects the dimensions (plus one) of the cyclohedra.
Definition 6.2. Let $\tilde{WSym}$ be the graded vector space over $\mathbb{Q}$ with the $n^{th}$ component of its basis given by all the tubings on the cycle graph of $n$ numbered nodes. By $W_n = \mathcal{K}C_n$ we denote the poset of tubings on the cycle graph $C_n$ with a cyclic numbering of nodes. We write $F_u$ for the basis element corresponding to $u \in W_n$ and $1$ for the basis element of degree $0$.

The product in $\tilde{WSym}$ of two basis elements $F_u$ and $F_v$ for $u \in W_p$ and $v \in W_q$ is found by summing a term for each order preserving injection $\iota : [p] \rightarrow [p + q]$. First the $l$-tubing $u$ is drawn upon the induced subgraph of the nodes $\iota([p])$ according to the ascending order of the chosen nodes. However, each tube may need to be first broken into several subtubes. Then the $m$-tubing $v$ is drawn upon the subgraph induced by the remaining $q$ nodes. However, each of the tubes of $v$ is expanded to also include any of the previously drawn tubes that its nodes are adjacent to.

To be precise, we first choose an order preserving injection $\iota : [p] \rightarrow [p + q]$. There is then automatically an order preserving injection $\hat{i} : [q] \rightarrow [p + q]$ such that the images of $\iota$ and $\hat{i}$ do not overlap. Our term of $F_u \cdot F_v$ is the tubing on $C_{p+q}$ given by the following. First, let $t$ be a tube of $u$. We partition $\iota(t)$ into subsets of consecutively numbered nodes (mod $p + q$) such that no union of two subsets is consecutively numbered (mod $p + q$). Then our term of $F_u \cdot F_v$ contains the tubes given by these subsets. Let $\iota(u)$ denote the tubing constructed thus far. Second, let $s$ be a tube of $v$. We include in our term of the product the tube formed by $\hat{i}(s) \cup \{j \in t' \in \iota(u) \mid t' \text{ is adjacent to } \hat{i}(s)\}$. Let $\hat{i}(v)$ denote the tubes added in this second step.

Definition 6.3.

$$F_u \cdot F_v = \sum_{\iota : [p] \rightarrow [p + q]} F_{\iota(u) \cup \hat{i}(v)}.$$ 

Again we leave to the reader the relatively simple task of writing this product in the geometrical terms of cellular projections and facet inclusion.

Theorem 6.4. The product we have just defined makes $\tilde{WSym}$ into an associative graded algebra.

Proof. The proof is precisely the same as for the algebra of vertices of the cyclohedra, $WSym$. The only difference is that the tubings do not always have the maximum number of tubes. First we
must check that the result of the product is indeed a sum of valid tubings of the cycle graph $C_{p+q}$.
We claim that in each term the new tubes we have created are pairwise compatible. The cases to be checked and the reasoning for each are exactly as shown in the proof of Theorem 5.3. Associativity is shown by lifting the tubings to be multiplied to tubings on the complete graphs which are preimages of the extended Tonks projection, and performing the multiplication in Chapoton’s algebra. The fact that the extended Tonks projection does preserve the structure here is again an easy check of corresponding terms, following the same pattern as for the $n$-tubings in Theorem 5.6.

\[ \square \]

Remark 6.5. Chapoton has shown the existence of a differential graded structure on the algebras of faces of the permutohedra, associahedra, and cubes \([5]\). The basic idea is simple, to define the differential as a signed sum of the bounding subfaces of a given face. Here we leave for future investigation the possibility of extending this differential to algebras of graph associahedra.

7. Algebras of simplices

We close with a curious new graded algebra whose $n^{th}$ component has dimension $n$. We denote it $\Delta Sym$. In fact $\Delta Sym$ may be thought of as a graded algebra whose basis is made up of all the standard bases for Euclidean spaces.

The graph associahedron of the edgeless graph on $n$ vertices is the $(n-1)$-simplex $\Delta^{n-1}$ of dimension $n - 1$. Thus the final range of our extension of the Tonks projection is the $(n-1)$-simplex (see Figure 11.) The product of vertices of the permutohedra can be projected to a product of vertices of the simplices. First we define this product by analogy to the previously defined products of tubings of graphs. Then we will show the product to be associative via a homomorphism from the algebra of permutations. Finally we will give a formula for the product using positive integer coefficients and standard Euclidean basis elements.

Definition 7.1. Let $\Delta Sym$ be the graded vector space over $\mathbb{Q}$ with the $n^{th}$ component of its basis given by the $n$-tubings on the edgeless graph of $n$ numbered nodes. By $D_n$ we denote the set of $n$-tubings on the edgeless graph. We write $F_u$ for the basis element corresponding to $u \in D_n$ and $1$ for the basis element of degree 0.

7.1. Graded algebra structure on vertices. Now we demonstrate a product which respects the grading on $\Delta Sym$ by following the example described above for $\mathfrak{S}Sym$. The product in $\Delta Sym$ of two basis elements $F_u$ and $F_v$ for $u \in D_p$ and $v \in D_q$ is found by summing a term for each order preserving injection $\iota : [p] \rightarrow [p+q]$. Let $\hat{i} : [q] \rightarrow [p+q]$ be the order preserving injection such that the images of $\iota$ and $\hat{i}$ do not overlap. Note that $u$ consists of $p-1$ of the $p$ nodes (these are the tubes) together with the universal tube, and likewise for $v$. Our term of $F_u \cdot F_v$ will be a $p + q$ tubing of the edgeless graph on $p + q$ nodes. Its tubes will include all the nodes numbered by $\iota([p])$. We denote these by $\iota(u)$. In addition we include all the tubes $\hat{i}(t)$ for non-universal $t \in v$, and the universal tube. We denote these by $\hat{i}(v)$.

Definition 7.2.

\[ F_u \cdot F_v = \sum_{\iota([p] \rightarrow [p+q]}} F_{\overline{\iota(u)}} \cdot \overline{\hat{i}(v)}. \]

An example is shown in Figure 20.

Theorem 7.3. The product we have just described makes $\Delta Sym$ into an associative graded algebra.

Proof. First we must check that the result of the product is indeed a sum of valid $(p + q)$-tubings of the edgeless graph on $p + q$ nodes. This is clearly true, since there will always be only one node
There are clearly algebra homomorphisms 

\[ \text{Remark 7.6.} \]

\[ \text{product of two standard basis vectors of varying dimension: } e \]

\[ \text{Formula for the product structure.} \]

\[ \text{described by the extended Tonks projections from associahedra and cyclohedra to the simplices.} \]

\[ \text{Theorem 7.7.} \]

\[ \text{Lemma 7.5.} \]

\[ \text{for } u \text{ an } n\text{-tubing of } K_n. \]

\[ \text{Remark 7.6.} \]

\[ \text{Formula for the product structure.} \]

\[ \text{there are clearly algebra homomorphisms } \mathcal{Y} \text{Sym} \to \Delta \text{Sym} \text{ and } \mathcal{W} \text{Sym} \to \Delta \text{Sym} \text{ described by the extended Tonks projections from associahedra and cyclohedra to the simplices.} \]

\[ \text{7.2. Formula for the product structure.} \]

\[ \text{There is a simple bijection from } n\text{-tubings of the edgeless graph on } n \text{ nodes to standard basis elements of } \mathbb{Q}^n. \]

\[ \text{Let } e_m^n \text{ be the column vector of } \mathbb{Q}^n \text{ with all zero entries except for a } 1 \text{ in the } m^{th} \text{ position.} \]

\[ \text{Associate } e_j^n = e_u \text{ with the } n\text{-tubing } u \text{ whose nodes are all tubes except for the } j^{th} \text{ node.} \]

\[ \text{Then use the product of } \Delta \text{Sym} \text{ to define a product of two standard basis vectors of varying dimension: } e_u \cdot e_v \text{ is the sum of all } e_w \text{ for } F_w \text{ a term in the product } F_u \cdot F_v. \]

\[ \text{Theorem 7.7.} \]

\[ \text{Proof.} \]

\[ \text{Let } e_v^n = e_v \text{ for the associated tubing } v. \]

\[ \text{The only node not a tube of } v \text{ is node } l. \]

\[ \text{We need only keep track of where } l \text{ lands under } i : [q] \to [p + q], \text{ where } i \text{ is as in Definition 7.2.} \]

\[ \text{The only possible images of } l \text{ are from } l \text{ to } p + l, \text{ thus the limits of the summation.} \]

\[ \text{When } i(l) = \hat{i} \text{ there are several ways this could have occurred. } \hat{i} \text{ must have mapped } [l - 1] \text{ to } [i - 1] \text{ and the set } \{l + 1, \ldots, q\} \text{ to } \{i + 1, \ldots, p + q\}. \]

\[ \text{The ways this can be done are enumerated by the combinations in the sum.} \]
Example 7.8. Consider the example product performed in Figure 20. Here the formula gives the observed quantities:

\[ e_2^2 \cdot e_2^3 = 3e_2^5 + 4e_3^5 + 3e_4^5. \]

7.3. Faces of the simplex. Note that a product of tubings on the edgeless graph (with any number of tubes) is well defined by direct analogy. Tubings on the edgeless graphs label all the faces of the simplices.

Let \( u \) be a \( j \)-tubing of the edgeless graph on \( p \) nodes and \( v \) a \( k \)-tubing of the edgeless graph on \( q \) nodes. Our term of \( F_u \cdot F_v \) will be a \( p+k \) tubing of the edgeless graph on \( p+q \) nodes. Its tubes will include all the nodes numbered by \( i([p]) \). In addition we include all the tubes \( i(t) \) for non-universal \( t \in v \), and the universal tube.

Remark 7.9. The number of faces of the \( n \)-simplex, including the null face and the \( n \)-dimensional face, is \( 2^n \). By adjoining the null face here we thus have a graded algebra with \( n \)th component of dimension \( 2^n \). It would be of interest to compare this with other algebras of similar dimension, such as \( QSym \).

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