Fast Approximation Algorithms for Near-optimal Large-scale Network Monitoring

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Abstract—We study the problem of optimal traffic prediction and monitoring in large-scale networks. Our goal is to determine which subset of \( K \) links to monitor in order to "best" predict the traffic on the remaining links in the network. We consider several optimality criteria. This can be formulated as a combinatorial optimization problem, belonging to the family of subset selection problems. Similar NP-hard problems arise in statistics, machine learning and signal processing. Some include subset selection for regression, variable selection, and sparse approximation. Exact solutions are computationally prohibitive. We present both new heuristics as well as new efficient algorithms implementing the classical greedy heuristic - commonly used to tackle such combinatorial problems. Our approach exploits connections to principal component analysis (PCA), and yields new types of performance lower bounds which do not require submodularity of the objective functions. We show that an ensemble method applied to our new randomized heuristic algorithm, often outperforms the classical greedy heuristic - commonly used to tackle such combinatorial problems. Our approach exploits connections to principal component analysis (PCA), and yields new types of performance lower bounds which do not require submodularity of the objective functions. We show that an ensemble method applied to our new randomized heuristic algorithm, often outperforms the classical greedy heuristic.

We consider several design problems from the perspective of network prediction (known also as kriging). In a static context, the traffic loads \( x = \{x(t)\}_{t \in \mathbb{N}} \) follow a stationary time series with certain marginal mean vector \( \mu_x \) and covariance matrix \( \Sigma_x \). The temporal dependence here will not be taken into consideration. Consequently by (1), the traffic loads on all links in the network \( y = \{y(t)\}_{t \in \mathbb{N}} \) form a stationary time series with mean \( \mu_y = \mathbf{A} \mu_x \) and covariance matrix \( \Sigma_y = \mathbf{A} \Sigma_x \mathbf{A}^T \).

In a dynamic context, in addition to the information about the marginal distributions of the traffic loads \( x(t) \) and \( y(t) \) a concrete temporal dependence structure will be imposed or derived from a spatio–temporal traffic model. In this paper, we concentrate only on the static context. Therefore, from hereafter we drop the dependence on time for notational convenience.

In the network prediction problem, sometimes referred as kriging, one observes the traffic loads \( y_u = \{y(t)\}_{t \in \mathcal{O}} \) on a set of links \( \mathcal{O} \subseteq \mathcal{L} := \{1, \ldots, L\} \). The goal is to predict the traffic loads \( y_u = \{y(t)\}_{t \in \mathcal{U}} \), where \( \mathcal{U} := \{1, \ldots, L\} \setminus \mathcal{O} \). Depending on the available information, one can apply the classical standard, ordinary or universal kriging approaches from spatial statistics. Without loss of generality, we consider the above design problem in the simplest possible case where \( x \sim \mathcal{N}(\mu_x, I) \) is a standard multivariate normal random vector with known mean \( \mu_x \). Therefore \( y \sim \mathcal{N}(\mu_y, \mathbf{A} \Sigma_x \mathbf{A}^T) \), where \( \mu_y = \mathbf{A} \mu_x \). Hence, the traffic means \( \mu_y \) and covariances \( \Sigma_y \) are known. Then ordinary kriging provides the best linear unbiased predictors (BLUP) for \( y_u \) via \( y_o \):

\[
\hat{y}_u = \mu_u + \Sigma_u \Sigma_{uo}^{-1} (y_o - \mu_o),
\]

where
\[
\mu_y = \begin{pmatrix} \mu_u \\ \mu_o \end{pmatrix}
\text{ and } \Sigma_y = \begin{pmatrix} \Sigma_{uu} & \Sigma_{uo} \\ \Sigma_{ou} & \Sigma_{oo} \end{pmatrix}
\]

are the partitions of the mean and the covariance into blocks corresponding to the unobserved (\( u \)) and observed (\( o \)) links.

1 Note, that in the general case when \( \Sigma_y = \mathbf{A} \Sigma_x \mathbf{A}^T \) we can express the covariance (hence, positive definite) matrix \( \Sigma_x \) as \( \Sigma_x = \Sigma_x^{1/2} \Sigma_y^{1/2} \). This gives \( \Sigma_y = \mathbf{A} \Sigma_x^{1/2} \Sigma_y^{1/2} \mathbf{A}^T = \tilde{\mathbf{A}} \Sigma_y^{1/2} \) with \( \tilde{\mathbf{A}} = \mathbf{A} \Sigma_x^{1/2} \) being the matrix imposed to be used in the following analysis.
If no information about the mean is available, then ordinary kriging may be applied where the traffic means across different links are assumed to be equal to an unknown constant. Alternatively, if additional network-specific information is available, a universal kriging type approach based on modeling the network-wide means can be adopted. The model can be fit by using measurements at a few links and then used for prediction. See [1] for more details.

In the dynamic context, different sets of links may be observed at different times, e.g. in a ‘round robin’ fashion. Then, in the context of a judicious model for the spatio–temporal dependence over the network, one can consider the static context outlined earlier and suppose that the traffic means \( \mu_y \) and covariances \( \Sigma_y \) are known. Thus, the ordinary kriging estimator in (2) provides the BLUP for \( y_u \) via \( y_o \).

**Problem 1** (monitoring design). Given that more than \( K = |O| \) links can be observed, determine the ‘optimal’ collection of links to monitor \( O \). Different ‘optimality’ criteria are possible. Let \( \Sigma_{err} := \mathbb{E}(\hat{y}_a - y_a)(\hat{y}_a - y_a)^T \), be the prediction mean squared error matrix, where \( \hat{y}_a := \mathbb{E}(y_a | y_o) \). Determine \( O \), which minimizes:

- \( \text{trace}(\Sigma_{err}) := \sum_{i \in L} \text{Var}(y_i | y_o) = \mathbb{E}\|\hat{y}_a - y_a\|^2 \), where \( \| \| \) stands for the Euclidean norm.
- \( \| \Sigma_{err} \|_2 \), where \( \| \|_2 \) stands for the spectral matrix norm, i.e., largest eigenvalue of \( \Sigma_{err} \), henceforth written as \( \rho(\Sigma_{err}) \).

The optimization formulation for our optimal monitoring design problem is to find the optimal set \( O^* \) such that:

\[
Z(O^*) = \min_{O \subseteq L} f(O) \quad \text{s.t.} \quad |O| = K, \tag{3}
\]

where \( L = \{1, 2, \ldots, L\} \) is the set of network links, \( Z(O) \) is the prediction error when monitoring links in set \( O \subseteq L \) and function \( f(\cdot) \) is the optimality criterion (i.e., trace or spectral norm). For example, for the trace case, \( f(O) := \text{trace}(\Sigma_{err}(O)) \).

In experimental design lingo, the former criterion is known as A-optimality and the latter as E-optimality. Another classical criterion encountered in experimental design is D-optimality, defined as \( \det(\Sigma_{err}) \), but for space considerations we will not consider it in this paper. The motivation behind considering many different criteria comes from applications. For example, one natural criterion is the trace of the error covariance matrix, which equals the total sum of the variances of the estimates of the unobserved links. Another point of view involves the minimal volume confidence sets for \( y_u \).

These correspond to level-sets of the conditional density function. E.g., under normality, we have

\[
y_u | y_o \sim \mathcal{N}(\hat{y}_u, \Sigma_{uu} - \Sigma_{uo} \Sigma_{oo}^{-1} \Sigma_{ou}), \tag{4}
\]

where \( \hat{y}_u \) is given by (2). Therefore, the minimum volume confidence set for \( y_u \) in this case is of the form

\[
\hat{y}_u + \{y \in \mathbb{R}^d : y^T \Sigma_{err}^{-1} y \leq c\}
\]

for some constant \( c > 0 \) and \( d := |U| \).

It may be of interest to find a design (i.e. a set \( O \)) that minimizes for example the diameter, the volume or some other characteristic of the above confidence set, given a fixed confidence level \( \alpha \). This gives rise to the E-optimal and D-optimal criteria, respectively.

The optimal monitoring design problem (3) is NP-hard, in general. The problem can be reduced to the NP-complete Set Cover problem. An alternative reduction and proof for NP-hardness is given in [2]. The complexity of the covariance matrix \( \Sigma_y \) corresponding to real, large-scale networks makes this problem computationally intractable in practice. An exhaustive search implementation in CPLEX ran on our institution’s cluster required more than two days to converge to the optimal solution for a small-scale network of 9 nodes and 26 links. Other methods, such as the mixed-integer program with constraint linearization and the branch-and-bound algorithm presented in [3] that require submodularity for the objective function, are also computationally expensive and impractical for even moderate-scale networks. Furthermore, even though the trace objective function can be submodular (see [4]), other criteria of interest such as E-optimality and D-optimality are not. These challenges show the need for approximate but fast heuristic algorithms for solving optimization problem (3).

A well-known algorithm to quickly find approximate solutions to Problem (1) is to perform the following forward selection procedure. Start from the empty set, and add elements one at a time, adding at each step the link that minimizes the prediction error \( Z \) the most (or equivalently, maximizes the error reduction the most). We call this, the greedy heuristic. Let \( \delta_j(O) = Z(O) - Z(O \cup \{j\}) \) be the error reduction when adding element \( j \) to the set \( O \). Below, we give its formal description as described by Nemhauser et al. [5].

**Greedy Heuristic.** The steps of the algorithm are:

1) Let \( O^0 = \emptyset, N^0 = L \) and set \( k = 1 \).
2) At iteration \( k \), select \( i_k \in N^{k-1} \) such that

\[
i_k = \arg \max_{i \in N^{k-1}} \delta_i(O^{k-1}) \tag{5}
\]

with ties settled arbitrarily.

3) Set \( \delta_{k-1} = \delta_{i_k}(O^{k-1}) \). If \( \delta_{k-1} \leq 0 \), stop. The selected set is \( O^{k-1} \). Otherwise, set \( O^k = O^{k-1} \cup \{i_k\} \) and \( N^k = N^{k-1} \setminus \{i_k\} \).
4) If \( k = K \) stop and output \( O^K \). Otherwise, set \( k = k + 1 \) and go to Step 2).

In this paper, we propose a new heuristic for Problem (1) as well as new, efficient implementations of the classical greedy heuristic. Notably, the naive implementation of the greedy heuristic is not feasible for large-scale networks since it involves (in our case) the inversion of high-dimensional matrices. We exploit the natural geometric structure in the problem, which is related to principal component analysis (PCA) \( \text{(PCA)} \). This structure motivates new, projection-based heuristics and also leads to exact, efficient implementation of the classical greedy heuristic, that avoids matrix inversions. The connections to PCA also yield lower bounds on the error reduction in Problem (1), which are of independent interest. The efficiency of our algorithms allows us to apply them to very large-scale networks. By using multiple, parallelized executions of randomized versions of our algorithms, we arrive at non-greedy solutions through the method of ensemble. For small and medium-scale networks this can often come close and, in fact, match the optimal solution of Problem (1).

The combinatorial nature of the optimal monitoring design belongs to the family of subset selection problems, and is encountered in scientific areas ranging from computer science and engineering to statistics and linear algebra. In [2], a bank location problem is formulated as an integer program and algorithms based on branch and bound are employed to retrieve the exact solution. The problem, however, is NP-hard so exact algorithms are computationally inefficient. Hence, in [3] approximation algorithms are studied that exploit the submodularity property of the objective function. (In words, submodularity is for set functions the analog of what convexity is for real number functions.) Polynomial-time, greedy heuristics along the lines of the one shown above are offered that yield a solution within \((1 - 1/e)\) of the optimum, where \( e \) is the base of natural logarithm.

In [2], near-optimal sensor placement for temperature monitoring is examined. Temperatures at different locations are modeled with a multivariate Gaussian joint assumption at other locations. The objective is to maximize the number of sensors so as to optimally predict the temperatures at various, where \( e \) is the natural exponential function. The structure of the covariance matrix is also exploited in [7] where the problem of subset selection for regression is studied. The setup includes a large number of variables \( x_i \) that can be observed, and the interest is on the value of the predictor variable \( z \). The problem is to select the \( k \)-subset of variables \( x_i \) that “best” predict \( z \) in the sense of minimizing the mean squared prediction error. This corresponds to Problem (1) under the trace criterion in the special case of searching for the subset of links (variables \( x_i \)) that predict a specific link (variable \( z \)). The authors propose fast, exact algorithms based on dynamic programming. These algorithms may be employed only in the special cases of “low-bandwidth” and “tree covariance” graphs. Unfortunately, such special cases do not arise very often in our problem due to complex network structures and heterogenous traffic sources. Thus, the algorithms in [2], [7] are not practical for real-world instances of our optimal monitoring problem.

In mathematics and signal processing community the, so called, sparse approximation problem is also similar to (3). In that context, the goal is to find a “sparse” subset of a dictionary \( D = \{ \phi_i \}_{i \in [1, \ldots, D]} \) of unit vectors that span \( \mathbb{R}^N \), so that a linear combination of the selected vectors best approximates a given signal \( A \in \mathbb{R}^N \). Two common approaches, namely matching-pursuit and orthogonal matching pursuit are discussed in [5] and a new heuristic with an exponential speedup over those two approaches is proposed.

Another category of subset selection combinatorial problems is the problem of variable selection [9], [10], which seeks the “best” subset of exactly \( K \) columns from a \( m \times n \) matrix \( A \). The problem has been extensively studied by the linear algebra and statistics community. One commonly used technique, namely the Lasso method [11], relaxes the constraint on the number of variables sampled by using an alternate one on the regression coefficients vector. By doing so, the integrality constraint vanishes, and the problem can then be solved using standard quadratic programming algorithms.

The remainder of the paper is organized as follows: in Section II we describe, analyse and give the intuition behind our approximation algorithms. In addition, we discuss the PCA-based lower bound on the performance of our heuristics, and elaborate on the idea of ensembling for enhancing the performance of our algorithms. We evaluate our methods in Section III in a variety of scenario, including very-large scale and real-world networks. We draw our conclusions in Section IV.

II. APPROXIMATION ALGORITHMS

A naive implementation of the described greedy algorithm in both A- and E-optimality contexts has a complexity of \( O(KL \ln^3) \) where \( n = \max\{L, J\} \). This is because a matrix inversion \( O(n^3) \) computations) is involved whenever \( \delta_i(O^{k-1}) \) is calculated. This section

\[ \Sigma_{ij} = 0 \text{ when } |j - i| > \beta. \]
presents fast, approximation algorithms that substantially reduce the computational complexity for solving Problem (I).

Let \( \Sigma_y = A A^T \) be the covariance matrix for the model \( \mathbf{y} = Ax \). The error covariance matrix from (2) (including both the observed and unknown links) equals, \( \Sigma_{\text{err}}(O) = \Sigma - \Sigma_{\text{o}} \Sigma_{\text{o}} \Sigma_{\text{err}}(O) \), where \( \Sigma_{\text{o}} = A A_o^T \), \( \Sigma_{\text{o}} o = A_o A_o^T \), and the partition matrix \( A_o = (a_{ij}) \in O, j \in E \). One can also write the latter matrix as follows:

\[
\Sigma_{\text{err}}(O) = A(I_J - A_o^T (A_o A_o^T)^{-1} A_o) A^T = (I_J - P_O) A^T, 
\]

where \( P_O := A_o^T (A_o A_o^T)^{-1} A_o \). The matrix \( P_O \) is the projection matrix onto the space \( W_O := \text{Range}(A_o^T) \), i.e., the space spanned by the column vectors \( \{ \mathbf{a}_i : \mathbf{a}_i \in \mathbb{R}^J, i \in O \} \) of \( A_o^T \), where \( O \) is the set of the observed links. Therefore, \( (I_J - P_O) \) is the projection matrix onto the orthogonal complement \( \text{Range}(A_o^T)^\perp \).

One can show that \( \| A - P_O A \|_F^2 = \text{tr}(A(I_J - P_O) A^T) \) and \( \| A - P_O A \|_F^2 = \rho(A(I_J - P_O) A^T) \). Consequently, Problem (I) is no different than the combinatorial problem of finding the set \( O \) such that:

\[
O := \arg \min_{|O| = K} \| A - P_O A \|_F^2, \tag{7}
\]

where \( \| \cdot \|_F \) is the spectral norm when \( \xi = 2 \) and the Frobenius norm when \( \xi = F \).

In a geometric perspective, (7) seeks the “optimal” space \( W_O := \text{Range}(A_o^T) \) such that the distances, under the Frobenius norm or spectral norm, of the column vectors \( \mathbf{a}_i \in \mathbb{R}^J, i \in L \) to the space in question are minimized. A known result from PCA (see [12], [10] and Appendix A) is that the optimal space \( W_{PCA} \) is given by \( P_O = V_K V_K^T \) where \( V_K = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K) \) is the matrix of the top \( K \) right singular values of \( A \), or equivalently, the \( K \) principal eigenvectors (components) of \( A^T A \). This geometric observation gives a lower bound for the error and also suggests geometric heuristics. The spaces \( W_O \) in (7) should be spanned by the vectors \( \mathbf{a}_i, i \in L \) corresponding to the \( K \) observed links, so in principle, the PCA-based lower bound cannot be achieved. However, we can think of a sequential “greedy” method that picks the space \( W_O \) that has smallest “angle” with the space \( W_{PCA}^T \).

One idea is to first pick a link \( i_1 \in L \) for which the vector \( \mathbf{a}_{i_1} \) is “closest” to the first PCA component \( \mathbf{v}_1 \). If \( K = 1 \), by (6) this would be the optimal choice where the notions of “close” and “optimal” depend on the type of norm or optimality criterion. If \( K > 1 \), we “subtract” the effect of the chosen link \( \mathbf{a}_{i_1} \) by considering only the orthogonal projections of the remaining links onto the space span\{\mathbf{a}_*\}, where \( \mathbf{a}_* := \mathbf{a}_{i_1}^{(k-1)} \). Thus, obtaining matrix \( A(k) \), as follows:

\[
A(k) = A(k-1) \left( I_J - \frac{\mathbf{a}_* \mathbf{a}_*^T}{\| \mathbf{a}_* \|^2} \right), \tag{10}
\]

with \( A(0) = A \). The following proposition shows that the prediction error covariance matrix can be updated sequentially by using projections. This sequential projection feature is the key behind the computational efficiency of the proposed heuristics presented in the sequel since it allows us to avoid expensive operations such as matrix inversions. Its proof is given in Appendix B.

**Proposition 1.** Let \( O := O_k \) be the set of selected links at the end of step \( k \). Then

\[
A(k)^T A(k)^T \text{err}(O) \equiv A(I_J - A_o^T (A_o A_o^T)^{-1} A_o) A^T, \tag{9}
\]

with \( A(k) \) updated as shown in Eq. (8).

A. A-optimality (trace) criterion

In our first heuristic, we implement the geometric idea discussed above in which, at step \( k \), we select the link \( i_k \in L \) whose vector \( \mathbf{a}_{i_k} \) is closest to the principal component of the current version of \( A \). We can choose between two options for vector proximity; the smallest angle or the longest projection. We decided to work with the latter. We obtain the principal component using the power method [12]. The steps of the algorithm are:

**Algorithm 1** (PCA Projection Heuristic - CAPH). Let \( O_0 = \emptyset \) and \( A(0) = A \). Set \( k = 1 \).

1. **Find the principal component** (i.e., the principal eigenvector), namely \( \mathbf{v}_1 \) of \( A(k-1)^T A(k-1) \). Make use of the power method (see [12]) to obtain a fast approximation of \( \mathbf{v}_1 \).

2. **At iteration** \( k \), choose:

\[
i_k \in \arg \max_{i \in O_{k-1}} |\mathbf{v}_1^T \mathbf{a}_{i_k}^{(k-1)}| \]

where \( \mathbf{a}_{i_k}^{(k-1)} \in \mathbb{R}^J, i \in \{1, 2, \ldots, L\} \) are the column vectors of \( A(k-1)^T \). Put \( i_k \) in the list of links to be monitored, i.e., \( O_k = O_{k-1} \cup \{i_k\} \).

3. **Consider the orthogonal projection of the columns** of \( A(k-1)^T \) onto the orthogonal complement of span\{\mathbf{a}_*\}, where \( \mathbf{a}_* := \mathbf{a}_{i_k}^{(k-1)} \). Thus, obtaining matrix \( A(k) \), as follows:

\[
A(k) = A(k-1) \left( I_J - \frac{\mathbf{a}_* \mathbf{a}_*^T}{\| \mathbf{a}_* \|^2} \right), \tag{10}
\]
4) Set \( k = k + 1 \). If \( k < K \), go to step 1).

One can show that the error reduction at each step is equal to:
\[
R(k) = \sum_{i=1}^{L} \| P_k(a^{(k-1)}_i) \|^2, \quad \text{where } P_k \text{ is the projection matrix to span}\{a^{(k-1)}_i\}.\]

This heuristic does not always pick the link that reduces the error the most. Instead, at each iteration, it picks a vector (link) that is closest to the PCA principal component. As shown in Section III this strategy usually yields a monitoring projection matrix to span\{a^{(k-1)}_i\}. This heuristic, however, does not always pick the link that reduces the error the most. It requires \( m \) times and requires \( O(LJ) \) operations.

Lemma 1. The complexity of the PCA Projection Heuristic is \( O(mKLJ) \).

One may try instead to pick row vectors of \( A \), \( a_{i1}, \ldots, a_{iK} \), such that \( a_{ik} \) is “closest” to the \( k \)th principal component direction. This heuristic, however, does not yield good results for large \( k \) due to accumulation of errors and since the PCA provides only a lower bound.

The next algorithm, is a fast implementation of the classical greedy algorithm. It avoids calculating the inverse of the covariance matrix \( \Sigma_{oo} \). Instead, at each iteration we seek the column vector that maximizes the error reduction the most. This is equivalent to finding the vector that maximizes the squares of the projections of the remaining vectors onto itself. This step is accomplished by the first step of the algorithm. The second step, updates the matrix \( A(k) \).

Algorithm 2 (Fast Greedy Exact - FGE). Let \( O_0 = \emptyset \) and \( A(0) = A \). Set \( k = 1 \).

1) At iteration \( k \), choose:
\[
i_k \in \arg \max_{i \in \mathcal{L}\setminus O_{k-1}} \sum_{j=1}^{L} \left| \frac{a^{(k-1)^T}_j a^{(k-1)}_i}{\|a^{(k-1)}_i\|^2} \right|^2 \tag{11}
\]
where \( a^{(k-1)}_i \in \mathbb{R}^J, i \in \{1, 2, \ldots, L\} \) are the column vectors of \( A^{(k-1)^T} \).

2) Do Step 3) of Algorithm 7 i.e.,
\[
A(k) = A^{(k-1)} \left( I_J - \frac{a^{(k-1)}_i a^{(k-1)^T}_i}{\|a^{(k-1)}_i\|^2} \right) A^{(k-1)^T}, \quad \text{with } a^* := a^{(k-1)}_i. \tag{12}
\]

3) Set \( k = k + 1 \). If \( k < K \), go to step 1).

Step 1) is a “greedy step” since it seeks to pick the link that reduces the error the most. It requires \( O(JL^2) \) operations. Step 2) requires \( O(LJ) \) operations after suitably rearranging the order of operations. The following lemma holds.

Lemma 2. The computational complexity of the Fast Greedy Exact algorithm is \( O(KIJ^2) \).

\[\text{B. E-optimality (spectral norm) criterion}\]

For E-optimality, we present two fast implementations of the greedy heuristic. Unlike the classical greedy heuristic, relation (6) and Proposition 1 allow us to avoid computationally expensive operations like matrix inversion and singular value decomposition. Therefore, our two greedy heuristics perform drastically better than the naive implementation of greedy. In addition to these two heuristics, we emphasize that a similar heuristic as Algorithm 1 could also be implemented. For space considerations, we skip this description (since it is exactly the same as the one presented for the trace case), and present two new greedy algorithms that demonstrate how the aforementioned expensive operations could be avoided. Again, we utilize the power method to lead us to the largest eigenvalue of the error covariance matrix.

Algorithm 3 (Fast Greedy Power - FGP). Let \( O_0 = \emptyset \) and \( A(0) = A \). Set \( k = 1 \). The steps of the algorithm are:

1) At iteration \( k \), choose:
\[
i_k \in \arg \max_{i \in \mathcal{L}\setminus O_{k-1}} \| \Sigma_{err} \|_2, \tag{13}
\]
where \( \Sigma_{err} = A^{(k-1)} \left( I_J - \frac{a^{(k-1)}_i a^{(k-1)^T}_i}{\|a^{(k-1)}_i\|^2} \right) A^{(k-1)^T}, \) and \( a^{(k-1)}_i \in \mathbb{R}^J, i \in \{1, 2, \ldots, L\} \) are the column vectors of \( A^{(k-1)^T} \). Use the power method to obtain the principal eigenvector of \( \Sigma_{err} \). The largest eigenvalue, namely \( \lambda_1 \), follows from its definition (i.e., \( \lambda_1 = v^T \Sigma_{err} v \)). Set \( O_{k} = O_{k-1} \cup \{i_k\} \).

2) Do Step 3) of Algorithm 7.

3) Set \( k = k + 1 \). If \( k < K \), go to step 1).

The computational complexity is analyzed next. In Step 1), note that the calculation of the error covariance matrix, \( \Sigma_{err} \), can be simplified as follows (we symbolize \( A^{(k-1)} \) simply as \( A \) due to space limitation):
\[
\Sigma_{err}^{(k-1)}(i) = A \left( I_J - \frac{a_i a_i^T}{\|a_i\|^2} \right) A^T = A A^T - \frac{(A^T a_i a_i^T)}{\|a_i\|^2}, \tag{14}
\]
To find \( i_k \) in Set 1), we sequentially search all links \( i \in \mathcal{L}, \) to find the one that minimizes the error the most. The first term of (14) costs \( O(JL^2) \) operations and needs to be calculated only once per iteration \( k \) since is independent of the link choice. The second term requires \( O(JL) \) operations. The power method is run \( m \) times and requires \( O(LJ) \) operations. Thus, Step 1) requires \( O(mL^2J) \) operations in total. In Step 2), costs \( O(JL) \) operations.

Lemma 3. The complexity of the Fast Greedy Power algorithm is \( O(mKL^2J) \).
The second algorithm for the spectral norm criterion is also based on relation (10) and Proposition 1. Moreover, we utilize the definition of the largest eigenvalue:

$$\rho(\Sigma_{\text{err}}) := \lambda_1(\Sigma_{\text{err}}) := \max_{x \in \mathbb{R}^L \mid \|x\|=1} x^T \Sigma_{\text{err}} x. \quad (15)$$

**Algorithm 4** (Fast Greedy Randomized - FGR). Let $O_0 = \emptyset$ and $A^{(0)} = A$. Set $k = 1$.

1) Generate $m$ independent, normally distributed random vectors $x_i \in \mathbb{R}^L, i = 1, 2, \ldots, m$ from $N(0, 1)$ and set $x_i := x_i / \|x_i\|_2$.

2) At iteration $k$, $k = 1, 2, \ldots, K$, calculate:

$$c_i := (x_i^T A^{(k-1)}) \cdot (A^{(k-1)} x_i), \forall i = 1, 2, \ldots, m. \quad (16)$$

3) At iteration $k$, select:

$$j_k \in \arg \min_{j \in \mathcal{L}(O_{k-1})} \left\{ \max_{i} \left[ \frac{x_i^T b_j^{(k-1)} - b_j^{(k-1)} x_i}{\|b_j^{(k-1)}\|^2} \right] \right\},$$

where $b_j^{(k-1)} := A^{(k-1)} a_j^{(k-1)}$ and $a_j^{(k-1)} \in \mathbb{R}^L, j \in \{1, 2, \ldots, L\}$ are the column vectors of $A^{(k-1)^T}$. This corresponds to finding the link $j$ that minimizes the error $\|A^{(k)}(I - a_j a_j^T / \|a_j\|^2) A^{(k)}\|_2^2$. Set $O_k = O_{k-1} \cup \{i_k\}$.

4) Do Step 3) of Algorithm 2.

5) Set $k = k + 1$. If $k < K$, go to step 2).

First, we randomly sample $m$ unit vectors $x_i : x_i \in \mathbb{R}^L, i = 1, 2, \ldots, m, \|x_i\| = 1$. We use these vectors to approximate the largest eigenvalue. Note, that in Step 2) we save the $c_i$ values so as to omit unnecessary repetitions of the same quantity. We multiply the terms in parentheses first, to avoid the expensive matrix by matrix multiplication. In Step 3), we choose the vector (link) that minimizes the error the most. Finally, we update matrix $A$ for use in the next iteration. Step 2) requires $O(mLJ)$ operations and Step 3) $O(mLJL)$.

**Lemma 4.** The computational complexity of the Fast Greedy Randomized algorithm is $O(mKL^2J)$.

**C. Ensemble method for randomized algorithms**

Algorithms PCAPH, FGP and FGR can be characterized as randomized algorithms. PCAPH and FGP require the usage of the power method which utilizes a random vector for its initial operation. FGR uses $m$ random vector to approximate the largest eigenvalue. The main idea of the ensemble method is to pick a small $m$ (since by Lemmas 1,3 and 4 $m$ affects the running time of the algorithms) and start several, say $r$, independent and parallel instances of the algorithms. Eventually, we keep the solution set that yields the minimum prediction error. By using multiple, parallelized executions of randomized versions of our algorithms, we arrive at non-greedy solutions. For small and medium-scale networks this can often come close and, in fact, match the optimal solution of Problem [1] (see Figure 1(e)).

**III. PERFORMANCE EVALUATION**

**A. Greedy Vs. Exact algorithms**

We use the real-world network, namely Internet2, to evaluate the solution that greedy heuristics give against the exact solution obtained via exhaustive search. We also calculate and demonstrate the PCA lower bound. Internet2 involves a network with $L = 26$ links and $J = 72$ flows (see [13]). We assume flows with co-variance matrix $\Sigma_x = I_J$. In Figure 2(a) we examine the trace criterion and in Figure 2(b) the spectral norm one. The exhaustive search required several hours to converge to a solution (the implementation was done in Matlab using CPLEX). On the contrary, the greedy algorithms terminate in seconds and the prediction error at the solution set they converge to is very close to the minimum prediction error obtained by the exact algorithm. Observe also that the prediction error of the PCA projection heuristic is only marginally different from the one we get with the greedy algorithm.

Ensembling is depicted in Figure 2(c). Here we compare the FGR algorithm against the classical greedy. We verify that running $r = 128$ parallel implementations of FGR with just $m = 20$ computationally outperforms classical greedy and yields solutions with prediction error closer to the exact solution.

**B. Comparisons of Approximation Algorithms**

In this section we juxtapose the computational performance of our fast approximation algorithms against the classical greedy heuristic. We run our simulations on a moderate network of $N = 100$ nodes generated using preferential attachment, as described in [14]. We assume $J$ source-destination $(s, d)$ flows with identical traffic, such that $x_j \sim \mathcal{N}(\mu, 1), j \in \{1, 2, \ldots, J\}$. The route for each $(s, d)$ pair is chosen using shortest-path routing. We created a network with a routing matrix $A$ of $L = 195$ links and $J = 500$ flows.

Figure 2 illustrates the results for A-optimality. Our proposed heuristics are notably faster than the naive implementation of the greedy algorithm. Our PCA projection heuristic significantly outperforms all other algorithms at the expense of having a slightly larger prediction error. Figure 3 depicts the case of E-optimality. Again, our algorithms perform faster than the naive implementation of the greedy heuristic. We used $m = 100$ iterations for the power method of the FGP algorithm and $m = 100$ random vectors for the FGR algorithm. The difference in the prediction error is due to the approximation techniques used for the calculation of the largest eigenvalue. We use the PCA lower bound to assess the solution quality of the algorithms, since obtaining the exact solution is not computationally feasible.
In this section, we evaluate our algorithms in a large-scale network constructed via the network generation tool \textit{Inet} \cite{gummadi2003inferring}. The generated network has \( N = 3050 \) nodes, and yields a matrix \( A \) with \( L = 4800 \) links and \( J = 1000 \) flows. The naive implementation of greedy was not executed for this scenario; based on its inferior performance observed earlier, such execution would be impractical. Indeed, several hours were needed even for the execution of the greedy heuristics FGE, FGP and FGR, that were shown to be much faster than the classical greedy. On the other hand, the PCA projection heuristic (ran with \( m = 100 \)) terminated in less than a minute. This suggest that this heuristic consists an appealing, advantageous and robust option for extremely large networks.

IV. DISCUSSION

Classical greedy algorithms are impractical for solving large-scale combinatorial problems like the one studied in this paper. Such problems arise in optimal network monitoring, variable selection and sparse approximation. In this paper, we present fast heuristics that outperform the classical greedy and are amenable for use in large-scale problems. Our approach can be also applied in the case of a weighted monitoring design problem (see Appendix C). That is, links are assigned weights which represent their significance; links with higher significance should be included in the monitoring set with higher priority than others. Our algorithms exploit connections to principal component analysis (PCA), and yields new types of performance lower bounds which do not require submodularity of the objective functions. However, submodularity offers \((1 – 1/e)-approximation\) algorithms and, thus, our future plans include investigation of upper bounds on the performance of our algorithms. Moreover, we plan to examine how the routing structure imposed by matrix \( A \) appears in the optimal design under each criterion, and interpret the differences in the solutions obtained from the different optimality criteria.

APPENDIX A

PRINCIPAL COMPONENTS ANALYSIS LOWER BOUNDS

Let \( \Sigma_y = AA^T \) be the covariance matrix for the model \( y = Ax \). Let the singular value decomposition for \( A \) be \( A = UD^{1/2}VT \), with \( D^{1/2} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p) \), \( p = \min\{L, J\} \), the singular values of \( A \). Thus, \( \lambda_1 \geq 0 \).
\[ \cdots \geq \lambda_k \geq \lambda_{k+1} \geq \cdots \lambda_p \] are the eigenvalues of the \( J \times J \) matrix \( A^T A \). The then the following theorems apply:

**Theorem 1** (Geometric Interpretation of PCA bound for trace). A standard result on the properties of the PCA implies that

\[ \min_{\substack{\text{sub-space } W \\ \dim(W) = k}} \sum_{\ell=1}^{L} \text{dist}(a_\ell, W) = \sum_{j=k+1}^{J} \lambda_j, \]

and the space \( W \) achieving the lower bound is spanned by the eigenvectors corresponding to the largest \( k \) eigenvalues.

**Proof:** Recall that the error covariance matrix can be written as

\[ \Sigma_{err}(O) = A(I - P_O)A^T, \]

where \( P_O := A_o^T (A_o A_o^T)^{-1} A_o \). The vector \( (I - P_O) a_\ell \) is the perpendicular dropped from \( a_\ell \) to \( \text{Range}(A_o^T) \), and hence \( \| (I - P_O) a_\ell \| \) is the distance from \( a_\ell \) to \( \text{Range}(A_o^T) = \text{span}(a_\ell, \ell \in O) \), where \( A_o^T = (a_\ell)_{\ell \in O} \).

The projection matrix \( (I - P_O) \) is symmetric and indempotent (i.e., \( (I - P_O) = (I - P_O)^2 \)), so we have

\[ \Sigma_{err} = [(I - P_O)A^T]^T [(I - P_O)A^T], \]

and therefore,

\[ \text{trace}(\Sigma_{err}(O)) = \sum_{\ell=1}^{L} \| (I - P_O) a_\ell \|^2 = \sum_{\ell=1}^{L} \text{dist}(a_\ell, W) \]

where \( W := \text{span}(a_\ell, \ell \in O) \) is the sub-space spanned by the vectors \( a_\ell \) corresponding to the observed links. Using Proposition 1 of \([1]\) we see that the sum of (18) is minimized when the projection matrix \( P_O \) equals \( P_k = V_k V_k^T \), where the columns of the matrix \( V_k \) are the top \( k \) right singular vectors of \( A \), and the minimum value is \( \sum_{j=k+1}^{J} \lambda_j \).

**Lemma 5.** Using Theorem 2, the following lower bound holds:

\[ \text{tr}(\Sigma_{err}(O)) = \text{tr}[A(I - P_O)A^T] \geq \| A - AP_k \|_F^2 = \sum_{i=k+1}^{J} \lambda_i, \]

where \( P_k = V_k V_k^T \).

A similar result holds for the spectral norm case. It is known from PCA analysis (see \([12]\), Theorem 2.5.3, and \([10]\)) that:

**Theorem 2.**

\[ \min_{\text{rank}(B) = k} \| A - B \|_2 = \| A - AP_k \|_2 \]

where \( P_k \) is the projection matrix \( P_k = V_k V_k^T \). The columns of matrix \( V_k \) are the top \( k \) right singular vectors of \( A \), i.e., \( P_k = V \text{diag}(A_o^T A_o)^{-1} A_o) \).

\[ 5 \] We symbolize the vector of ones of dimension \( k \) as \( \mathbf{1}_k \) and the vector of zeros as \( \mathbf{0}_k \).

**Lemma 6.** Using Theorem 2, the following lower bound holds:

\[ \rho(\Sigma_{err}(O)) = \rho(A(I - P_O)A^T) \geq \| A - AP_k \|_2^2 = \lambda_{k+1}. \]

**Proof:** We first calculate the lower bound when \( P_k = V_k V_k^T \). We use the SVD of \( A \), \( A = U D^{1/2} V^T \) and the projection matrix \( I - P_k = V \text{diag}(0_k^T, 1_{L-k}^T)V^T \). Thus, we have

\[ A - AP_k = U D^{1/2} V \text{diag}(0_k^T, 1_{L-k}^T)V^T = U \text{diag}(0_k^T, \sigma_{k+1}, \ldots, \sigma_L)V^T. \]

By the definition of spectral norm:

\[ \| A - AP_k \|_2^2 = \rho \left( U \text{diag}(0_k^T, \sigma_{k+1}, \ldots, \sigma_L)V^T \right) \cdots \rho \left( U \text{diag}(0_k^T, \sigma_{k+1}, \ldots, \sigma_L)V^T \right) \]

\[ = \rho \left( U \text{diag}(0_k^T, \sigma_{k+1}, \ldots, \sigma_L)V^T \right) = \lambda_{k+1}. \]

Consequently,

\[ \rho(\Sigma_{err}(O)) = \rho \left[ A(I - P_O)A^T \right] = \rho \left[ (A - AP_O)(A - AP_O)^T \right] \]

\[ = \rho \left[ (A - AP_O)^T(A - AP_O) \right] \]

\[ \geq \| A - AP_k \|_2^2 = \lambda_{k+1}. \]

For the last inequality we used the result of Theorem 2 and the lower bound obtained above.

**APPENDIX B**

**PROPOSITION: IDENTICAL ERROR REDUCTION**

**Proof of Proposition 7** We will show by induction that

\[ A^{(k)} A^{(k)^T} = A(I_j - A_o^T A_o A_o^T)^{-1} A_o) A^T. \]

**Induction Basis:** For \( k = 2 \) we will prove that the following is true:

\[ \Sigma_{err}^{(k)} := A^{(1)}(I - \frac{a_{12}^{(1)} a_{12}^{(1)^T}}{\| a_{12}^{(1)} \|^2}) A^{(1)^T} \]

\[ = A(I - A_o^T A_o A_o^T)^{-1} A_o) A^T =: \Sigma_{err} \]

where \( A_o^T = (a_{1}, a_{2}) \). For brevity, we define the projection \( P_k \) as \( P_k = I - \frac{a_{1}^{(k-1)} a_{1}^{(k-1)^T}}{\| a_{1}^{(k-1)} \|^2} \).

We have:

\[ A^{(1)}(I - \frac{a_{12}^{(1)} a_{12}^{(1)^T}}{\| a_{12}^{(1)} \|^2}) A^{(1)^T} = A^{(1)} P_2 A^{(1)^T} \]

\[ = A^{(0)} P_1 P_2 A^T A^{(0)^T} = A P_1 P_2 A^T. \]
It suffices to show that \( I - A_k^T (A_o A_o^T)^{-1} A_o = P_l P_2 P_1 \).
Indeed, \( P_l P_2 P_1 = P_{a_1} P_{a_1} P_{a_2} = P_{\text{span} \{a_1, a_2\}}^+ \). We have used the identity
\[
P_{\text{span} \{u_1, u_2, \ldots, u_r\}}^+ = P_{u_1^+} P_{u_2^+} \ldots P_{u_r^+}
\]
where \( u_1, u_2, \ldots, u_r \) are orthonormal vectors. Note that \( \text{span} \{a_1, a_2\} = \text{span} \{a_2, a_1\} \) because \( a_2 = a_1 - P_{a_1}(a_1) \). Therefore,
\[
P_l P_2 P_1 = P_{\text{span} \{a_1, a_2\}}^+ = I - A_2^T (A_o A_o^T)^{-1} A_o
\]
by the definition of a projection matrix. This completes the proof of the basis of induction.

**Induction Hypothesis:** Assume the following is true for step \( k \):
\[
A(k)^T(I - A_k(k-1))A(k)^T = A(k-1) - A(k-1) A_o A_o^T A(k-1)
\]
which means that \( A(k)^T A(k) = A(k-1) - A(k-1) A_o A_o^T A(k-1) \). Hence,
\[
P_l P_2 P_1 = P_{\text{span} \{a_1, a_2\}}^+ = I - A_2^T (A_o A_o^T)^{-1} A_o
\]
implies \( \text{span} \{a_{ik+1}\} = \text{span} \{a_{ik+1} - P_{a_{ik}}(a_{ik+1})\} \).

**Appendix C**

**Weighted Link Monitoring**

Recall that in the trace criterion we minimize:
\[
tr(E[(\hat{y} - y)(\hat{y} - y)^T]) = E[tr((\hat{y} - y)(\hat{y} - y)^T)]
\]
\[
= E[||\hat{y} - y||^2]
\]
\[
= E[(\hat{y} - y)^T(\hat{y} - y)].
\]

The weighted monitoring design problem aims to minimize \( E[(\hat{y} - y)^T G(\hat{y} - y)] \). This implies that if we apply our algorithms to the imposed routing matrix \( A^G := G^{1/2} A \), we obtain the optimal predictor for \( \hat{y}^G \). The optimal predictor for the original vector of links, \( \hat{y} \), is simply obtained by \( \hat{y} = G^{-1/2} \hat{y}^G \).

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