Borel Summable Solutions to 1D Schrödinger Equation

Stefan Giller† and Piotr Milczarski‡

Theoretical Physics Department II, University of Łódź,
Pomorska 149/153, 90-236 Łódź, Poland

e-mail: † sgiller@krysia.uni.lodz.pl
‡ jezykmil@krysia.uni.lodz.pl

Abstract

It is shown that so called fundamental solutions the semiclassical expansions of which have been established earlier to be Borel summable to the solutions themselves appear also to be the unique solutions to the 1D Schrödinger equation having this property. Namely, it is shown in this paper that for the polynomial potentials the Borel function defined by the fundamental solutions can be considered as the canonical one. The latter means that any Borel summable solution can be obtained by the Borel transformation of this unique canonical Borel function multiplied by some $\hbar$-dependent and Borel summable constant. This justifies the exceptional role the fundamental solutions play in 1D quantum mechanics and completes the relevant semiclassical theory relied on the Borel resummation technique and developed in our other papers.

PACS number(s): 03.65.-W, 03.65.Sq, 02.30.Lt, 02.30.Mv

Key Words: fundamental solutions, semiclassical expansion, Borel summability
I. Introduction

The semiclassical approximation is one of the most widely used approximate methods in physics, particularly, in quantum mechanics. In fact it is not limited only to pure power series expansions in the Planck constant $\hbar$ but it is used also in all problems which can be formulated semiclassically. The method can be applied in this way to, say, the quartic oscillator perturbation theory from the one hand \[1, 2\] and to a variety of problems with so called large-N expansions from the other \[3, 4, 5\]. Therefore, independently of the expansion parameter we shall consider all such asymptotic series expansions as semiclassical.

The method can be stated in the Schrödinger wave function formulation of quantum mechanics \[6, 7\] as well as in the Feynman path integral form of the latter \[8, 9\]. Its main ingredient as the approximation method is to represent considered quantities by a limited number of first terms of the corresponding infinite series expansions, knowing usually that the series is typically asymptotic i.e. divergent. Therefore, contrary to the case of convergent series, such a representation of the expanded quantities is of a rather limited value.

First, it can not be done arbitrarily accurate by enlarging a number of kept terms i.e. such an approximation can be only the best one in which the case the finite rest is exponentially small (in the parameter of the expansion) in comparison with the main contribution.

Secondly, it is just these exponentially small differences which can become dominating in other domains of the expansion parameter being arbitrarily close to the original one i.e. such a finite representation of quantities by their corresponding asymptotic series are strongly limited to the original domain and it can not give any information about the analytic properties of the quantities considered as functions of the expansion parameter. In particular this finite sum can not be continued analytically outside the original domain of obtaining it.

The latter means that the considered quantities can not be recovered in some simple way by the knowledge of a finite number of terms of their semiclassical expansions even if the series are abbreviated at their least terms. (In the latter case the approximation is considered to be best). In fact these analytic properties are determined rather by the behaviour of so called large order terms of the series. It is just the properties of these large order terms (considered as functions of their order) which allow us actually to reconstruct quantities represented by such semiclassical series. In particular if these terms grow with their order not faster than factorially then the Borel method of summation of the diverging series can be used in such cases. We shall call such semiclassical series Borel summable.

In the case of the Borel summable semiclassical expansions the Borel method of summation can be used in the following way.

First the Borel function is determined approximately after the knowledge of a limited number of first terms of the asymptotic series expansion of the considered quantity. Namely, by its definition the approximate Borel function is obtained as a sum of these known first terms divided by the corresponding factorials. The sum represents in this way the abbreviated Taylor series expansion corresponding to the exact Borel function. (The latter is obtained if all the terms of the asymptotic series are used to construct the Taylor series in the above way). The last series has a finite radius of convergence and therefore an additional knowledge even approximate of the singularity structure of the corresponding Borel function is still necessary for an approximate recovery of this function from its abbreviated Taylor series. These knowledge can be extracted to the known extent from the detailed knowledge of the large order behaviour of the considered semiclassical series. Having (or assuming) however this knowledge a function with the desired singularity structure can be constructed and its
Taylor series expansion can be compared with the known abbreviated Taylor series so that the free parameters of the assumed singularities can be determined. Finally by the Borel transformation of the Borel function the original quantity is reproduced approximately in this way.

The above reproduction can still be performed in the spirit of asymptotic expansions and on different levels of accuracy. The lowest level is obtained when the Borel integral is substituted by its best asymptotics. It means that the quantity considered is represented again by a finite sum a definite number of first terms of which coincide exactly with the original terms used to construct the (approximate) Borel function. However this sum can now contain much more terms since it ends on the least term of the asymptotic series corresponding to the Borel integral. Therefore it can approximate the considered quantity much better than the sum of the original terms the method has started with.

However, since the method contains still additional information about the singularity structure of the Borel function then one step further can be done in getting still better level of accuracy by extracting from the Borel integral so called exponentially small contributions. Such computations are known as the hyperasymptotic ones. The finite semiclassical sum is then completed by the exponentially small contributions the forms and numbers of which are determined by the known (assumed) singularity structure of the Borel function. The latter means that the Borel summability allows us to realize the principle of resurgence i.e. to recover the information contained in the divergent tails of the semiclassical series.

It should be noted also, however, that the exponentially small contributions are of their own importance since in many cases of quantities considered these contributions are dominant. Among the latter cases the most well known one is the difference between the energy levels of different parities in the symmetric double well. But these are also the cases of transition probabilities in the tunnelling phenomena or their adiabatic limits in the time dependent problem of transitions between two (or more) energy levels (see references cited there) or the exponential decaying of resonances in the weak electric field (see references cited there).

The applicability of the Borel resummation to the semiclassical expansions in quantum mechanics has been proved by many authors. Particularly, the 1-D quantum mechanics offers a possibility of constructing a full semiclassical theory relied on the Borel resummation. Namely, several years ago one of the authors of the present paper discovered that for a large family of analytical potentials including all the polynomial ones there are solutions to 1D stationary Schrödinger equation for which their well defined semiclassical expansions are Borel summable to the solutions themselves. These solutions appearing for polynomial potentials in a finite number were called fundamental because of their completeness for solving any one-dimensional problem. Their Borel summability property played an essential role in many of their applications. In particular this property allowed us to prove the Borel summability of energy levels for most of the polynomial potentials.

On the other hand it is easy to construct solutions to the Schrödinger equation (in fact, infinitely many of them) with well defined Borel summable semiclassical expansions but with results of such Borel resummations not coinciding with the initial solutions generating the series. However the results of the Borel resummations are again solutions to the Schrödinger equation since in general each successful Borel resummation of any semiclassical series always leads to some solution to the Schrödinger equation.

In this paper we want to demonstrate an exceptional role the fundamental solutions
mentioned above play with respect to the Borel summability property showing that they provide a general scheme for a construction of Borel summable solutions to the 1D stationary Schrödinger equation at least for polynomial potentials. A main ingredient of such a scheme is an observation that the Borel function of some fundamental solution is not only such a function for any other fundamental solution but it is also a Borel function allowing us to construct any Borel summable solution to a given 1D Schrödinger equation with polynomial potential.

The latter conclusion means that in all the semiclassical problems in 1-D quantum mechanics in which the Borel resummation method is to be applied the fundamental solutions should be used preferably.

Our way of considering the problem of the Borel summability in 1D quantum mechanics makes use of the global features of the fundamental solutions and the Stokes graphs related to them and as such is to some extent complementary to the way utilized by Delabaere et al. [13, 14] making use of rather local properties of the considered quantities.

Our method can be also used to analyze the adiabatic limits considered by Joye et al. [18] at least in the case of two energy levels [36]. The cases of several levels need, however, a generalization of our method since these cases are described by systems of the linear equations in numbers larger than two.

To make the original results of our paper more transparent we have formulated them in many places in the forms of theorems or lemmas equipped with the corresponding proofs. However we do not consider our paper to pretend to a full formal mathematical rigor supposing most of the presented ideas to be sufficiently obvious and clear by presented proofs or when confronted with our earlier papers or with the papers of other authors mentioned.

The paper is organized as follows.

In the next section we remind a construction and basic properties of the fundamental solutions and Borel functions corresponding to them as well.

In Sec. 3 we show that the Borel functions corresponding to different fundamental solutions are only different branches of the same unique Borel function and can be recovered by the Borel transformations performed along suitably chosen paths on the 'Borel plane'. We show also here that there are two ways of integrations in the Borel plane providing us with the Borel summable solutions to the Schrödinger equation which, on their own, coincide each, up to $\hbar$-dependent multiplicative constants, with the corresponding fundamental solutions.

In Sec. 4 we consider in some details a general expression for the semiclassical expansions to the Schrödinger equation and introduce there also their standardized forms. We point out in this section an essential difference between the forms of the latter for the Borel summable and non-summable quantities.

In Sec. 5 we show the existence almost at each point of the $x$-plane two pairs of the base solutions to the Schrödinger equation with well defined Borel summable semiclassical asymptotic but not summed to the solutions themselves. The semiclassical expansions of the solutions considered in this section and their Borel resummations are a particular illustration of our main thesis that a result of any such a resummation is always some fundamental solution.

In Sec. 6 we generalize the results of Sec. 4 and show that the Borel function defined by the fundamental solutions can be considered as canonical in a sense that up to a multiplicative $\hbar$-dependent constant any Borel summable solution to the Schrödinger equation can be obtained by the Borel transformation of this canonical Borel function. This means that each Borel summable solution has to be essentially some of the fundamental solutions.
Sec. 7 is a discussion of the results of the paper.

II. Fundamental solutions to 1D stationary Schrödinger equation

Let us remind shortly basic lines in defining fundamental solutions [1, 5].

A set of fundamental solutions is attached in a unique way to a so called Stokes graph corresponding to a given polynomial potential \( V(x) \) of \( n \)th degree. Each Stokes graph is a collection of lines (Stokes lines) in the complex \( x \)-plane which are a loci of points where the real parts of action functions defined by the following \( n \) integrals:

\[
W_i(x, E) = \int_{x_i}^{x} \sqrt{q(y, E)}dy
\]

\[
q(x, E) = 2V(x) - 2E
\]

vanish. In (3) \( E \) is the energy of the system and \( x_i, i = 1, 2, ..., n \), are roots of \( q(x) \).

The fundamental solutions are defined in infinite connected domains called sectors with boundaries of the latter consisting of Stokes lines and \( x_i \)'s, see Fig.1a.

In a sector \( S_k \) a corresponding fundamental solution \( \psi_k \) to the Schrödinger equation:

\[
\psi''(x) - \hbar^{-2}q(x)\psi(x) = 0
\]

has Dirac’s form:

\[
\psi_k(x) = q^{-\frac{1}{4}}(x) \cdot e^{\frac{\pi i}{2}\int_{x}^{x_k} \sqrt{q(y, E)}dy} \cdot \chi_k(x)
\]

Fig.1a A general form of the Stokes graph for the polynomial potential with \( x_k \) lying at the boundary of \( S_k \) and with a sign \( \sigma_k (= \pm 1) \), (which we shall call a signature

Fig.1b The \( \xi \)-variable map of Fig.1a containing the sectors \( S_1 \) and \( S_3 \)
of the solution $\chi_k(x, h)$ chosen in such a way to have:

$$\Re (\sigma_k W_k(x)) < 0$$

(4)

The amplitude factor $\chi_k(x, h)$ in (3) has the following Fröman and Fröman’s form [26]:

$$\chi_k(x, h) = 1 + \sum_{n \geq 1} \left( \frac{\sigma_k h}{2} \right)^n \int_{\infty}^{x} dy_1 \int_{\infty}^{y_1} dy_2 \cdots \int_{\infty}^{y_{n-1}} dy_n \omega(y_1) \omega(y_2) \cdots \omega(y_n) \times$$

(5)

$$\left( 1 - e^{-\frac{2\pi k}{\hbar}(W_k(y_1)-W_k(y_2))} \right) \cdots \left( 1 - e^{-\frac{2\pi k}{\hbar}(W_k(y_{n-1})-W_k(y_n))} \right)$$

with

$$\omega(x) = \frac{1}{4} q''(x) - \frac{5}{16} q'^2(x) = -q^{-\frac{1}{4}}(x) \left( q^{-\frac{1}{4}}(x) \right)''$$

(6)

and with integration paths in (5) chosen to be canonical [1, 5] i.e. on such paths the following condition is satisfied:

$$\sigma_k \Re (W_k(y_j) - W_k(y_{j+1})) \geq 0$$

(7)

for any ordered pair of integration variables (with $y_0 = x$). The condition (7) ensures the solution (3) to vanish at the infinity $\infty_k$ of the sector $S_k$.

A domain $D_k (\supset S_k)$ where $\chi_k(x)$ can be represented by (3) with the canonical integration paths is called canonical. In each $D_k$ the following semiclassical expansion for $\chi_k(x)$ can be deduced from (3) by standard methods (see also the next section):

$$\chi_k(x, h) \sim \chi_k^{as}(x, h) = \sum_{n \geq 0} \left( \frac{\sigma_k h}{2} \right)^n \chi_k,n(x)$$

$$\chi_k,n(x) = \int_{\infty}^{x} dy q^{-\frac{1}{4}}(y) \left( q^{-\frac{1}{4}}(y) \chi_k,n-1(y) \right)'' = \int_{\infty}^{x} dy_n q^{-\frac{1}{4}}(y_n)$$

(8)

$$\times \left( q^{-\frac{1}{4}}(y_n) \int_{\infty}^{y_n} dy_{n-1} q^{-\frac{1}{4}}(y_{n-1}) \left( \cdots q^{-\frac{1}{4}}(y_2) \int_{\infty}^{y_2} dy_1 q^{-\frac{1}{4}}(y_1) \left( q^{-\frac{1}{4}}(y_1) \right)'' \cdots \right)'' \right)''$$

$$n = 1, 2, \ldots, \; \chi_{k,0}(x) \equiv 1$$

What has been said above assumed real and positive value of $\lambda \equiv h^{-1}$ (we prefer to use rather $\lambda$ as a more convenient variable). However when considering Borel summability properties of $\chi_k(x, \lambda)$ it is unavoidable to complexify $\lambda$. If it is done the only change in the above descriptions of properties of Fröman and Fröman solutions to Schrödinger equation is to substitute $W(x, E)$ in the conditions (4) and (7) by $e^{i\phi}W(x, E)$ where $\phi = \arg \lambda$. Of course, the domains $D_k$ as well as Stokes graph itself depend then on $\phi$. In particular, all the Stokes lines rotate then around the corresponding turning points $x_i$, $i = 1, 2, \ldots, n$, they emanate from by the angle $-2\phi/3$. For $\phi = \pm \pi$ Stokes graph comes back to its initial position.
i.e. a dependence of Stokes graph on \( \phi \) is periodic with \( \pi \) as its period. Such a full rotation of Stokes graph we shall call cyclic. We can use the cyclic rotations to enumerate all the sectors according to the order they come into each other by the subsequent cyclic rotations starting from the one chosen arbitrarily. We shall assume from now on such a convention for the sector ordering with the numbers attached to sectors increasing anticlockwise.

By a cyclic rotation a solution \( \psi_k(x, \lambda) \) from a sector \( S_k \) transforms into a solution \( \psi_{k-1}(x, \lambda) \) or \( \psi_{k+1}(x, \lambda) \) (modulo \( n+2 \), the last number being the total number of sectors for a polynomial potential of the \( n \)th degree) according to whether the rotation of Stokes graph is clockwise or anticlockwise respectively. Of course, for a fixed \( x \) after at most two subsequent cyclic rotation (in the same direction) the path of integration in (5) stops to be canonical if it was as such before the rotation operations. Let us note also that making, say clockwise, \( n+2 \) subsequent cyclic rotations a solution \( \psi_k(x, \lambda) \) does not come back exactly to its initial form (3) but acquires rather an additional phase factor which in the case of even \( n \) is equal to \((-1)^n \exp(\lambda \sigma_k \oint_{K'} \sqrt{q(x, E)} \, dx)\) where the (closed) contour \( K' \) encloses (anticlockwise) all \( n \) roots of the potential \( V(x) \). In the case of odd \( n \) one needs to surround all the roots twice as much to close the corresponding path of analytical continuation of \( \chi_k(x, \lambda) \) in the \( x \)-plane with the result analogous with the even case. It means of course that as a function of \( \lambda \) a solution \( \psi_k(x, \lambda) \) branches infinitely around the points \( \lambda = 0, \infty \) of the \( \lambda \)-plane [1].

As we have mentioned earlier it was shown in [1] that in sector \( S_k \) the series (8) can be Borel summed to \( \chi_k(x, \lambda) \) itself. To be a little bit detailed it was shown in [2] that when \( x \in S_k \):

1\(^{\text{st}}\) the size of a sector in the \( \lambda \)-plane where the expansion (8) is valid is larger than \( 2\pi \);

and

2\(^{\text{nd}}\) the rate of grow of \( \chi_{k,n}(x) \) in (8) with \( n \) is factorial.

The last property which was established by an application of the Bender-Wu formula [4] ensured that the following Borel series:

\[
\sum_{n \geq 0} \chi_{k,n}(x) \frac{(-s)^n}{n!} \tag{9}
\]

was convergent in a circle: \( |s| < |W_k(x, E)| \).

On its turn the property 1\(^{\text{st}}\) above ensured that the series (9) define Borel functions \( \tilde{\chi}_k(x, s) \) holomorphic in the halfplane: \( \Re s < -\sigma_k \Re W_k(x, E) \) allowing to recover \( \chi_k(x, \lambda) \) from the series (8) by the following Borel transformation of \( \tilde{\chi}_k(x, s) \):

\[
\chi_k(x, \lambda) = 2\lambda \int_{C_\phi} e^{2\lambda s} \tilde{\chi}_k(x, s) \, ds \tag{10}
\]

where \( C_\phi \) is a halfline in the Borel halfplane \( \Re s < -\sigma_k \Re W_k(x, E) \) starting at the infinity and ending at \( s = 0 \) with \( \phi \) as its declination angle \( (\pi/2 \leq \phi \leq 3\pi/2) \).

However, for the latter transformations to exist it is neccessary for the functions \( \tilde{\chi}_k(x, s) \) to be holomorphic only in some vicinity of a ray \( \arg s = \phi_0 \) along which the transformation (10) can be taken [27]. Such a limiting situation appears when \( \chi_k(x, \lambda) \) is continued from the sector \( S_k \) to other domains of Stokes graph so that such a continuation generates singularities of \( \tilde{\chi}_k(x, s) \) in the half plane \( \Re s < 0 \) close to the ray \( \arg s = \phi_0 \). A mechanism of such singularity generations has been described by one of the present authors [12]. Some of these singularities are fixed and the others are moving with their positions in the \( s \)-plane depending on \( x \).
to get $\chi_k(x, \lambda)$ disappears at the moment when two of the moving singularities which are localized close to the ray $\arg s = \phi_0$ pinch the latter. It is clear that such cases depend continuously on $x$ i.e. for a given $\phi_0$ in the domain $D_k(\phi_0)$ of the $x$-plane there is a maximal domain $B_k(\phi_0)$ ($D_k(\phi_0) \supset B_k(\phi_0) \supset S_k(\phi_0)$) inside which the series (8) for $\chi_k(x, \lambda)$ is Borel summable along $C_{\phi_0}$ to $\chi_k(x, \lambda)$ itself. To find a boundary of $B_k(\phi_0)$ one can use Voros’ technique [24] of rotating of the reduced Stokes graph (i.e. the one obtained in the limit $|\lambda| \to \infty$) with changing of $\arg \lambda$ (see also [1]): when $x \in \partial B_k(\phi_0)$ the total change of $\arg \lambda$ preserving the canonicness of the integration path in (8) running from $\infty_k(\phi_0)$ to $x$ cannot be greater than $\pi$. Let us note also that for $x \in B_k(\phi_0)$ but close to $x_0 \in \partial B_k(\phi_0)$ the Borel transformation of $\tilde{\chi}_k(x, s)$ along the ray $\arg s = \phi_0$ provides us with $\chi_k(x, \lambda)$ defined for $\pi/2 - \phi_0 \leq \arg \lambda \leq 3\pi/2 - \phi_0$.

III. Properties of the Borel functions $\tilde{\chi}_k(x, s)$

In this section we shall establish properties of the fundamental solutions and their corresponding Borel functions not discussed in our papers quoted in the previous sections.

First let us note that we can drop the subscribe $k$ at the Borel functions $\tilde{\chi}_k(x, s)$ because in fact all these functions define one and the same Borel function. This property is the subject of the following theorem.

Theorem 1 Let $\tilde{\chi}(x, s)$ coincide with $\tilde{\chi}_1(x, s)$ when $x \in S_1$ and $|s| < |\xi(x)|$ and where $\xi(x) \equiv -\sigma_1 W_1(x, E)$. Then

a) $\tilde{\chi}(x, s)$ coincides with the Borel functions $\tilde{\chi}_k(x, s)$, $k = 2, ..., n + 2$, corresponding to the remaining fundamental solutions;

b) each fundamental solution can be obtained from $\tilde{\chi}(x, s)$ when $x \in S_1$ by the Borel transformation with the integration path obtained by a suitable homotopic deformation of the path $C_1$ used to recover $\chi_1(x, \lambda)$.

Proof.

The part (a) of the theorem follows directly from the definitions of the Borel functions $\tilde{\chi}_k(x, s)$ by [3]. Namely, for $x \in K' \cap S_1$ whilst $|s| < |\xi(x)|$, we can transform the coefficients $\chi_{1,n}(x)$ defining $\tilde{\chi}(x, s)$ by the series [8] into the corresponding coefficients $\chi_{k,n}(x)$. To do it, it is enough to continue analytically the infinite limit $\infty_1$ of all the integrations in (8) from the sector $S_1$ to the sector $S_k$ to achieve the infinity $\infty_k$ of the sector $S_k$. Of course this is a deformation of the integration path $\gamma_1(x)$ in (8) into the $\gamma_k(x)$ one but this does not affect the integrations if none of the turning points is touched by the deformed path what is assumed and seen on Fig.1a. In other words such a deformation should be homotopic. Due to this operation we have, of course, $\chi_{1,n}(x) \equiv \chi_{k,n}(x), n \geq 1$ and, by [9] $\tilde{\chi}_k(x, s) \equiv \tilde{\chi}(x, s)$, $k = 2, ..., n + 2$.

Before going to the proof of the part (b) of the theorem let us make the following comments to the proof done so far.

First let us call as the standard paths the integration paths $\gamma_k(x)$ which appear in this way in (8) linking the infinity $\infty_k$, $k = 2, ..., n + 2$, with the point $x, x \in K' \cap S_1$.

Let us note further that continuing the infinite tail of the standard path by moving by the subsequent sectors around the closed contour $K'$ (to close the contour $K'$ in the case of the odd $n$ it is necessary to round the infinity point two times) we have to come back again to the sector $S_1$ when crossing the 'last' $S_{n+2}$ one. The consistency condition which follows
then from (8) demands that the integral \( \oint_{K'} q^{-\frac{1}{4}}(x)(q^{-\frac{1}{4}}(x)\chi_{1,n}(x))''dx \) should then vanish for each \( n \geq 0 \). One can easily check that it happens indeed for all the polynomial potentials.

For the factor \( \chi_k(x, \lambda) \), however, as given by (5) if \( k \neq 2, n + 2 \) the standard path is of course non canonical (see Fig.1a). But this means merely that \( \chi_k(x, \lambda) \) obtained in this way is an effect of its analytical continuation along \( K' \) from the sector \( S_k \), where it could be initially defined, to the sector \( S_1 \). This of course means also that \( \chi_k(x, \lambda) \) cannot be obtained from (10) by the integration along a halfline but rather by the corresponding integration along some more complicated path described below.

To restore, however, the Borel function corresponding to \( \chi_k(x, \lambda) \) when \( x \in S_k \) it is necessary only to continue \( \tilde{\chi}(x, s) \) analytically by moving the point \( x \) from the sector \( S_1 \) to \( S_k \) along the contour \( K' \). Of course at the end of this continuation the standard path linking \( \infty_k \) with the continued \( x \) is then found completely in the sector \( S_k \) being there a typical canonical path for the integrations in (4) and (8).

On the Borel plane the latter analytical continuation of \( \tilde{\chi}(x, s) \) corresponds to a motion of the branch point of at \( s = \xi(x) = -\sigma_1 \int_{x_1}^{x} \sqrt{q(y,E)}dy \) shown in Fig.3 along the line \( \tilde{K}' \) which is the image of \( K' \) on the \( s \)-Riemann surface given by the map \( s = \xi(x) \). During this motion the argument of this branch point changes by \( (k-1)\pi \).

Let us finish these comments of the part (a) of the theorem by noticing that this part can be proved also using the results of the discussion performed in Appendix 1 and preceding Theorem 6.

The validity of the part (b) follows easily just from Theorem 6 of Appendix 1. Namely, to prove this part consider first \( \tilde{\chi}(x, s) \) when \( x \) is continued from the sector \( S_1 \) to the sector \( S_3 \) along the contour \( K' \) shown in Fig.1a whilst \( s \) is kept fixed. A pattern of the first sheet of the \( s \)-Riemann surface is then shown on Fig.10b. To recover \( \chi_3(x, \lambda) \) by (10) we have to integrate \( \tilde{\chi}(x, s) \) over the negative real halfaxis. Let us now move all the branch points shown on Fig.10b back according to moving back the point \( x \) on the contour \( K' \) to its original position in the sector \( S_1 \). The point \( s = 0 \) of this sheet is then left from the right. This motion causes the integration path just mentioned to be deformed into the one shown in Fig.2a. If we apply now to the cut emerging from the branch point \( \xi - \zeta_1 \) an operation of rotating it clockwise by \( \pi \) which is reversed to the unscreening operation described in Theorem 6 of Appendix 1 we obtain the situation drawn on Fig.2b. This figure shows in details why in these positions of the considered cuts the integration in (10) along the path \( C_3 \) provides us with the factor \( \chi_3(x, \lambda) \) which corresponds to the sector \( S_3 \) (i.e. the infinite integration limit in (3) coincides with \( \infty_3 \)), but is continued on the \( x \)-plane to the sector \( S_1 \) along the contour \( K' \).

Let us note further that in the positions of the cuts shown on Fig.2b we can obtain the factors \( \chi_1(x, \lambda) \) and \( \chi_2(x, \lambda) \) as well integrating along the left and right real halfaxes.
Fig. 2a The position of the branch points at \( \xi - \zeta_1, \ldots, \xi - \zeta_2 \) and of the contour \( C_3 \) for \( \tilde{\chi}(x, s) \) when \( x \in K' \cap S_2 \).

Fig. 2b Same as in Fig. 2a after uprighting the cut emerging from the point \( \xi - \zeta_1 \).

We can repeat the above analysis starting from the cut pattern shown in Fig. 11b and corresponding to the Borel function \( \Phi(\xi(x), s) (\equiv \tilde{\chi}(x, s)) \) continued to the sector \( S_k \) along the contour \( K' \). In this position of \( \xi \) the factor \( \chi_k(x, \lambda) \) is recovered by (10) by integrating \( \Phi(\xi, s) \) along, say, the negative real halfaxis (assuming \( k \) is odd). Next, moving the point \( x \) back to its original position in the sector \( S_1 \) and applying to the consecutive cuts emerging from the branch points at \( \xi - \zeta_{ik-1}, \xi - \zeta_{ik-2}, \ldots, \xi - \zeta_2, \xi - \zeta_1 \) the operations reversed to the uprighting ones described in Theorem 6 of Appendix 1 we achieve the pattern of Fig. 3. It follows from the figure that the above operations deform merely homotopically the integration contour \( C_k \) from its original position when it coincides with the negative real halfaxis to the one shown in this figure where it has a spiral form allows it avoiding all the branch points.
Let us note that as it follows from the analysis performed in Appendix 1 the fixed branch points lie on the lower sheets of the $s$-Riemann surface being always screened by the moving ones and therefore not participating in the deformation of the contour $C_k$.

Note also that this deformation of the contour $C_k$ does not affect the convergence of the integral (10) since as we have shown in [12] and mentioned in Appendix 1 on each sheet of the $s$-Riemann surface the divergence of $\Phi(\xi, s)$ at infinity is at most exponential.

In this way we have however completed the proof of the theorem. QED.

It is certainly worth to stress that a net result which follows from Theorem 1 is that to obtain the subsequent $\chi_k(x, \lambda)$'s, $k = 2, 3, \ldots, n + 2$ it is enough (according to our enumeration convention) to deform $C_1$ homotopically anticlockwise by making its infinite tail to rotate by the angles $\pi, 2\pi, \ldots, (n + 1)\pi$ so that to coincide eventually with the real halfaxes, positive or negative, on the corresponding sheets, see Fig.3. We get in this way the sequence of paths $C_2, C_3, \ldots, C_{n+2}$ integrations on which according to the formula (10) provide us with the corresponding $\chi_k(x, \lambda)$'s, $k = 2, \ldots, n + 2$. But these latter $\chi$-factors are exactly the ones which were obtained in Sec. II by applying to $\chi_1(x, \lambda)$ the subsequent cyclic rotations since the latter correspond to (opposite) rotations on the $\lambda$-Riemann surface by the angles $-\pi, -2\pi, \ldots, -(n + 1)\pi$. It follows however from the formula (10) that such cyclic rotations have to be accompanied then by the compensating rotations of the integration path on the $s$-Riemann surface to maintain the convergence of the formula.

We see therefore that the cyclic rotation property of transforming the fundamental solution $\psi_1(x, \lambda)$ into $\psi_k(x, \lambda)$ can be realized in the following two equivalent ways:

1$^{\text{st}}$ on the $x$-plane by deforming the integration path $\gamma_1(x)$ in the formula (3) (when it defines the solution $\psi_1(x, \lambda)$, with $x \in S_1$) into the corresponding standard path linking the sector $S_1$ with the sector $S_k$; and
2^{n}$ on the Borel plane by deforming the the path $C_1$ in the way described above.

We know however that after such $n+2$ cyclic rotations we do not come back to exactly the same factor $\chi_1(x, \lambda)$ but the latter acquires rather an additional phase factor which for the even degree polynomials is equal to $(-i)^n \exp(-\lambda \sigma_1 \int_{k'} \sqrt{q(x, E)} dx)$. Therefore deforming the path $C_{n+2}$ once more in the above way to a path $C_{n+3}$ and integrating $\tilde{\Phi}(\xi, s)$ along this path we get as a result again $\chi_1(x, \lambda)$ but multiplied by the phase factor just mentioned i.e. for the even degree polynomials we have:

$$\int_{C_{n+3}} e^{2\lambda s} \tilde{\chi}(x, s) ds = e^{-\lambda \sigma_1 \int_{k'} \sqrt{q(x, E)} dy - m \frac{\pi}{2}} \int_{C_1} e^{2\lambda s} \tilde{\chi}(x, s) ds$$  \hspace{1cm} (11)

Deforming $C_1$ appropriately clockwise we obtain of course the corresponding integration paths $C'_{k'}$, $k' = 2, 3, \ldots$, providing us with $\chi_{k'}(x, \lambda)$'s ordered in the opposite way i.e. with $k = n - k' + 4 = n + 2, n + 1, \ldots, 2$. For the path $C'_{n+3}$ we get an identity similar to (11) but with the opposite sign at the exponent of the proportionality coefficient. This confirms that the $s$-Riemann surface of $\tilde{\chi}(x, s)$ is in general infinitely sheeted. The only obvious case with the finite six sheeted $s$-Riemann surface is provided by the linear potential [12].

Let us discuss still in some details the deformation procedure of the path $C_1$ described above.

The singularity pattern of Fig.3 which corresponds to $x \in S_1$ shows that to fall on the corresponding sheets in order to approach eventually the chosen direction of the real axis the paths $C_k$ have to avoid in general the existing singularities of $\tilde{\chi}(x, s)$ on its $s$-Riemann surface. According to Fig.3 such necessary deformations have to be applied for example to the path $C_3$ and to the subsequent ones but not to $C_2$.

The integration in (11) along $C_3$ provides us with $\chi_3(x, \lambda)$ but since $x \in S_1$ the corresponding integration path in (11) cannot be then canonical i.e. for $\lambda \to \infty \psi_3(x, \lambda)$ does not behave according to its JWKB factor in (11). The obvious reason for that is just the (branch point) singularity of $\tilde{\chi}(x, s)$ at $s = \xi$ (with $\Re \xi > 0$) which causes $\chi_3(x, \lambda)$ calculated in this way to diverge as $e^{2\lambda \xi}$ in the semiclassical limit.

To restore, therefore, the proper canonical behaviour of $\chi_3(x, \lambda)$ in this limit given by (11) we would have to move the singularity at $s = \xi$ to the left halfplane of Fig.3 i.e. to move the corresponding variable $x$ from the sector $S_1$ to $S_2$ along the contour $K'$. This is just the procedure described in the course of the proof of the theorem.

Earlier we have distinguished the canonical paths of integrations in (11) as the ones which ensured that the $\chi$-factors in (11) had well defined semiclassical expansions given by (11). A part of these paths penetrating the domains $B_k$'s mentioned in the previous section ensured also that the fundamental solutions defined by them were Borel summable and these resummations were achieved by the Borel transforms of the Borel function $\tilde{\chi}(x, s)$ along halflines running from the infinity of the Borel 'plane' and ending at its center. Let us call canonical also these latter paths on the Borel 'plane'.

However we could notice above that it is possible to generalize substantially the notion of the Borel transformation by integrating in the Borel 'plane' along the paths $C_k$ described above and recovering in this way the fundamental solutions obtained by the deformations of the canonical paths in the formulae (11) due to the cyclic rotations of the Stokes graph.

It is therefore worthwhile to distinguish also these new types of the Borel transformation paths and these non canonical paths in the $x$-plane as well which appear as a result of the homotopic deformations of the canonical paths by the cyclic rotations. Namely, we shall call further such paths both on the $s$- and on the $x$-plane as the standard paths in common.
Although it should be obvious that the Borel transformation of the Borel function $\tilde{\chi}(x,s)$ along any standard path should always provide us with the corresponding $\chi$-factor of the Dirac form (3) of the fundamental solutions we show this fact explicitly in Appendix 2.

It is a good moment of our discussion to mention an old problem of the semiclassical theory known as the connection problem [28, 29, 30, 31]. In the context of our considerations it arises when we are interested in the semiclassical behaviour of $\psi_3(x,\lambda)$ whilst $x$ is kept in $S_1$ (i.e. $x \in S_1$). In such a case we can deform the standard path $C_3$ into two paths, a path $C'_3$ surrounding the cut generated by the singularity at $s = \xi$ (see Fig.4) and again the canonical path $C_1$. By multiplying (10) (with $C_3$ as the integration path) by $q^{-1/4}e^{-\lambda\xi}$ we obtain $\psi_3(x,\lambda)$ to be represented in this way by the following linear combination of two solutions to Schrödinger equation (2):

$$\psi_3(x,\lambda) = \psi_1(x,\lambda) + C(\lambda)\psi_2(x,\lambda)$$

(12)

![Fig.4 The path $C_3$ splitted into the paths $C_1$ and $C_{cut}(\xi - \zeta_1)$](image)

Of course, $\psi_1(x,\lambda)$ is generated by the $C_1$ part of $C_3$. The fact that the cut integration part of $\psi_3(x,\lambda)$ is just proportional to $\psi_2(x,\lambda)$ can be easily seen by pushing $\xi$ to infinity along the cut what corresponds to approaching by $x(\xi)$ the infinity of the sector $S_2$. The cut integral (multiplied by $q^{-1/4}e^{-\lambda\xi}$) vanishes however in this limit (since $\Re\xi \to -\infty$) what proves our assertion. In other words we have:

$$q^{-\frac{1}{4}}(x,E)e^{-\lambda} \int_{x_1}^{x} \sqrt{\nu(y,E)}dy 2\lambda \int_{C_{cut}(\xi)} e^{2\lambda s} \tilde{\chi}(x,s)ds = C(\lambda)\psi_2(x,\lambda)$$

(13)

It is easily seen that the last relation is independent of such critical forms of the Stokes graph as the ones corresponding to coinciding of two (or more) turning points. However,
when the Stokes graph is built only by the simple turning points then this relation can be established by the standard methods, i.e. by continuing the fundamental solutions along the canonical paths on the $x$-'plane', which give \(^1\)\(^2\):

$$C(\lambda) = -i\chi_{3 \to 1}(\lambda)$$  \hspace{1cm} (14)

where $\chi_{3 \to 1}(\lambda) = \lim_{x \to \infty_1} \chi_3(x, \lambda) = \chi_{1 \to 3}(\lambda)$ is calculated by \(^3\) along the canonical path $\gamma_{1 \to 3}$ (see Fig.1b).

Writing further $\psi_2(x, \lambda)$ in its Dirac's form \(^3\) we get:

$$\chi_2(x, \lambda) = i\chi_{3 \to 1}(\lambda) e^{-2i\lambda s} \int_{C_{\text{cut}}(\xi)} e^{2i\lambda \xi} \tilde{\chi}(x, s) ds$$

$$= i\chi_{3 \to 1}(\lambda) 2\lambda \int_{C_{\text{cut}}(0)} e^{2i\lambda \xi} \tilde{\chi}(x, s + \xi) ds$$  \hspace{1cm} (15)

Taking now into account that $\chi_2(x, \lambda)$ is given by the Borel transformation along the canonical path $C_2$ on Fig.4 whilst $\chi_{3 \to 1}(\lambda)(\equiv \chi_{1 \to 3}(\lambda))$ can be obtained analogously by the integration $\tilde{\chi}_{\text{can}}(\infty_3, s)(\equiv \lim_{x \to \infty_3} \tilde{\chi}(x, s))$ along $C_1$ when $\xi(x)$ on Fig.1b goes from the sector $S_1$ to $\infty_3$ of the sector $S_2$ along the canonical path we get the following relation for $\tilde{\chi}(x, s)$ and its jump $\Delta_{s_0} \tilde{\chi}(x, s)$ through the cut emerging from the branch point $s = s_0 = \xi(x)$:

$$\tilde{\chi}(x, s) = i\chi_{\text{can}}^{x \to 1}(\xi_3, s) * \Delta_{0} \tilde{\chi}(x, -s + \xi(x))$$  \hspace{1cm} (16)

The '*' symbols in (16) denote the convolution operations (see Appendix, formula (56)).

One easily recognizes in (16) the fundamental solution version of the analytical bootstrap property of the Borel function $\tilde{\chi}(x, s)$ discovered by Voros \(^24\). It is the aim of this paper, however, to show that in the case of the polynomial potentials there are no other versions of the realization of the analytical bootstrap idea since in this case the Borel function $\tilde{\chi}(x, s)$ is unique (up to an irrelevant constant) being uniquely defined by the fundamental solutions.

The same comments as above are valid of course with respect to the results of the integrations along the subsequent standard paths $C_k$, $k = 4, 5, ...$, i.e. these paths provide us with the corresponding $\chi_k(x, \lambda)$'s calculated along the non canonical standard paths on the $x$-'plane' obtained by the continuation of the variable $x$ from $S_k$ to $S_1$ along the contour $K'$. Such a form does not allows us for estimating easily its semiclassical limit. To recover this limit properly we have to deform $C_k$'s keeping its infinite tail along the appropriate real halfaxis. This deformation splits $C_k$ into $C_1$ or $C_2$ (the latter choice depends on a sign of $\Re\lambda$) and into a number of paths surrounding some cuts (the cuts have to run to the left halfplanes for $\Re\lambda > 0$ or to the right ones in the opposite case). Each such a cut contribution represents a solution to Schrödinger equation (see Appendix 2) being proportional to some fundamental solution. The identification of these solutions can be performed by considering the limit of the latter when $\xi \to \infty$ along the appropriate cuts (or their elongations) (the solutions have to vanish in this limit) and following parallelly the corresponding paths drawn by $x(\xi)$ on the Stokes graph. Again, if the Stokes graph considered is determined only by simple turning points then a total number of the fundamental solutions engaged in the above splitting operation is limited only to those of them which can contact canonically with the sector $S_1$ where the variable $x/\xi$ ($\xi = \xi(x)$) actually is and the proportionality coefficients of the cut contributions to appropriate fundamental solutions can be calculated by the standard methods \(^1\)\(^3\).
Let \( x \) be fixed somewhere on \( K' \) (see Fig.1a).

We shall call a cut path each path surrounding a halfline cut of the \( s \)-Riemann surface running from its infinity and ending at some of its moving or fixed branch points.

Together with the result of Appendix 2 the net results of the above discussion can be summarize as the following two theorems.

**Theorem 2**

a) The Borel function \( \tilde{\chi}(x, s) \) when Borel transformed along a standard or a cut path and multiplied by the JWKB factor always provides us with a solution to the Schrödinger equation (12) having the Dirac form (13): 

\[
\tilde{\Phi}((-1)^{k-1}\xi(x'), s) = \tilde{C}_k(s, s_0) \ast \Delta_{s_0} \tilde{\Phi}(\xi, \pm s + s_0) 
\]

where the RHS of (17) is the convolution of \( \tilde{C}_k(s, s_0) \) which is the Borel function corresponding to the invers of the constant \( C_k(\lambda) \) multiplied by \( e^{2\lambda s_0} \) and of \( \tilde{\Phi}(\xi, s) \) shifted respectively whilst \( \tilde{\Phi}((-1)^{k-1}\xi(x'), s) \) denotes \( \tilde{\chi}(x, s) \) continued analytically along \( K' \) from the point \( x \in K' \cap S_1 \) to the point \( x' \in K' \cap S_k \) such that \( \xi(x) = (-1)^{k-1}\xi(x') \). The \('+\) signs at the variable \( s \) in (17) takes into account that Borel integrations along the cut and the canonical path \( C_{k_{\text{can}}} \) to recover \( \chi_k(x, \lambda) \) can go to the same \('+\) sign) or to the opposite \('-\) sign) infinities of \( \Re s \).

The existence of \( \tilde{C}_k(s, s_0) \) and the holomorphicity of \( \Delta_{s_0} \tilde{\Phi}(\xi(x), \pm s + s_0) \) at \( s = 0 \) is assumed.

**Proof of the theorem**

The part a) of the theorem is obvious by noticing that any Borel transformation along the standard/cut path with the Borel function \( \tilde{\chi}(x, \lambda) \) defined by (11) satisfies the linear differential equation defining \( \chi(x, \lambda)'s \) (see Appendix 2).

The part b) of the theorem when concerning the standard paths is a repetition of the corresponding result of Theorem 1. With respect, however, to the cut paths it follows as a conclusion summarizing the discussion preceding the formulation of this theorem.

The part c) is the direct consequence of the hypothesis of this part of the theorem written explicitly as:

\[
q^{-\frac{1}{4}}(x') e^{\sigma_k \lambda \xi(x')} 2\lambda \int_{C_{\text{cut}}(s_0)} e^{2\sigma_k \lambda s} \tilde{\Phi}(\xi(x), s) ds = C(\lambda) q^{-\frac{1}{4}}(x') e^{\sigma_k \lambda \xi(x')} \chi_k(x', \lambda) 
\]

\[
= C(\lambda) q^{-\frac{1}{4}}(x') e^{\sigma_k \lambda \xi(x')} 2\lambda \int_{C_{\text{can}}^{k}} e^{-2\sigma_k \lambda s} \tilde{\Phi}(\xi, s) ds
\]

where \( C_{\text{can}}^{k} \) denotes the canonical path on the Borel 'plane' used for recovering \( \chi_k(x, \lambda) \), \( \sigma_k \) is a signature of the latter and \( \sigma = \pm 1 \) is taken as \( +1 \) for the integration along the cut \( C_{\text{cut}}(s_0) \) to \(-\infty\) of \( \Re s \) and as \(-1\) in the opposite case.

Since \( \chi_k(x', \lambda) \) in (13) is recovered by the integration \( \tilde{\Phi}(\xi(x'), s) \) along the canonical path \( C_{\text{can}}^{k} \) then it means that this latter function results as its analytical continuation from the first sheet where it determines \( \chi_1(x, \lambda) \) (by the integration along a canonical path \( C_1 \) on this
sheet) to the sheet considered. This continuation has to be performed along the path $\tilde{K}'$, the image of $K'$ on the Borel 'plane', as long as $\xi$ acquires the argument equal to $(k - 1)\pi$ which put $\xi$ in the position $(-1)^{k-1}\xi$ on the final sheet. This corresponds of course to a point $x'$ on $K'$ such that $\xi(x) = (-1)^{k-1}\xi(x')$.

Now, if $\sigma_k = \sigma$ (this case is the only possible when $s_0$ is the fixed branch point) then we get the relation \[(17)\] with '+' at the variable $s$ and with '-' in the opposite case.

This latter conclusion ends, however, the proof of the theorem. QED.

Let us note, as a comment to the part (b) of the above theorem, that the proportionality constants can be always calculated independently when all turning points of considered polynomial problems are simple. We can use then the powerful method of analytical continuation of the fundamental solutions along the canonical paths which guarantees the full control of the semiclassical properties of calculated quantities at each stage of such calculations. The considerations preceding the above theorem are the good illustration of the possibilities of the method.

**Theorem 3** The connection problem i.e. the analytical continuation of the fundamental solutions throughout the $x$-plane along the contour $K'$ of Fig.1a can be solved by performing this continuation on the Borel plane. By such a continuation the original Borel integration along the deformed path has to be splitted into integrations along standard and cut paths the latter emerging from the branch points of $\tilde{\chi}(x,s)$ pinching the deformed path.

**Proof.** The validity of the theorem follows directly from the preceding discussion.

Another important property of the fundamental solutions which distinguishes these solutions among other possible Borel summable solutions can be formulated as the following theorem.

**Theorem 4** Let $x_0 (= x(\xi_0))$ be an arbitrary point of the $x$-plane not coinciding with a root of $q(x, E)$. Then there is a (non empty) subset $N(x_0)$ of fundamental solutions of both signatures with the following properties:

1° The point $x_0$ is canonical for every member of $N(x_0)$;

2° Every element of $N(x_0)$ can be obtained by the formula \[(17)\] integrating along a corresponding standard path.

We shall assume $N(x_0)$ to collect all such fundamental solutions.

**Proof of the theorem**

The validity of this theorem can be easily seen by considering the topology of sectors with respect to the chosen $x$ on the Riemann surface of the action variable $\xi(\equiv \xi(x))$ substituting the variable $x$ (see Fig.5 and \[1\]). For real $\lambda$ the Stokes lines on the surface are now parallel to imaginary axes and the sectors are left and right halfplanes not containing (the images of) turning points on each sheet of the surface \[1\]. The $\lambda$-rotations of Stokes graph make Stokes lines on the $\xi$-Riemann surface rotating around the images of the turning points preserving their parallelness.
Fig.5a The sectors $S_{a+1}$ and $S_a$ the fundamental solutions of which communicate canonically with the point $\xi(x_0)$

Fig.5b The sectors $S_b$ and $S_{b+1}$ the fundamental solutions of which communicate canonically with the same point $\xi(x_0)$

We shall distinguish the following four cases for the position of $x_0$ for real $\lambda$:

1. $x_0$ does not lie on any Stokes line but
   
   a. $x_0$ belongs to some sector,
   
   b. $x_0$ does not belong to any sector;

2. $x_0$ lies on some Stokes line and
   
   c. this Stokes line is finite i.e. it emerges from some turning point and ends on the other,
   
   d. this Stokes line is infinite i.e. it emerges from some turning point and runs to infinity of the $x(\xi)$-plane.

The above possibilities exhaust of course all the possible positions of $x_0$ with respect to the Stokes lines.

Consider the case 1\textsuperscript{a}.

First let us note that for real $\lambda$ the Stokes lines are parallel to the imaginary axis as it is shown on Fig.5. If $x_0$ is in some sector, say $S_a$, then obviously the fundamental solution $\psi_a$ defined in this sector belongs to $N(x_0)$. There are however another two fundamental solutions which belong to $N(x_0)$ too. Namely, these are the two neighbours of $\psi_a$, i.e. $\psi_{a+1}$ and $\psi_{a-1}$ (according to our enumeration convention). Both of them have their signatures opposite to $\psi_a$. Depending on a position of $x_0$ in $S_a$ there is always one of these two neighbour solutions which is Borel summable at $x_0$ simultaneously with $\psi_a$ and at the same time can communicate with it canonically.

Consider next the case 1\textsuperscript{b}.

In this case $x_0$ is in some infinite vertical strip on the $\xi$-Riemann surface (see Fig.5) bounded from each side by two chains of Stokes lines parallel to each other. Following each of these chains of lines whilst keeping on the strip we find on them their extremal turning points i.e. the ones from which Stokes lines emerge running to imaginary infinities, positive or negative. Consider such a Stokes line which bounds the strip from the right and runs
to positive imaginary infinity. This Stokes line bounds simultaneously from the left a strip neighbour to the one just considered. This new strip is again bounded from the right by a chain of Stokes lines which structure is similar to the Stokes line chains already considered. So there is again in this chain a Stokes line ending it and running to positive imaginary infinity. We can repeat this procedure of moving to the right of the $\xi$-'plane' to finish eventually with a 'strip' which is a right halfplane of the $\xi$-plane sheet considered. This procedure has to be finite since there is a finite number of extremal turning points met in this way due to the polynomial potential. It is clear also that this halfplane corresponds to a sector, say $S_\alpha$, of the Stokes graph. The way of its finding proves that the fundamental solution $\psi_\alpha$ defined in it belongs to the set $N(x_0)$. Mutatis mutandis to this set belongs also the solution $\psi_{\alpha+1}$ corresponding to the sector $S_{\alpha+1}$, the next one to $S_\alpha$ and having an opposite signature. By the same resonings but moving down the $\xi$-sheet we find two others sectors, call them $S_b$ and $S_{b+1}$, the fundamental solutions of which are both Borel summable at $x_0$ having of course opposite signatures. Therefore there are at least four fundamental solutions in $N(x_0)$ for this case. Besides, within the pairs $\psi_\alpha, \psi_b$ and $\psi_{\alpha+1}, \psi_{b+1}$ both the fundamental solutions can communicate with themselves canonically and can be simultaneously Borel summable. This last result is a conclusion from the fact that these pairs of fundamental solutions are related by the cyclic rotations.

Consider now the case $2^0c$.

In this case we can consider both the sides of the Stokes line on which $x_0$ is placed as pieces of two strips for which this line is their bound. The strips can be identified as the wholes by the procedure similar to the one used previously. Namely, we move along a chain of Stokes lines directing to, say, the positive imaginary infinity choosing the most right Stokes line each time we meet some turning point. Continuing this motion we meet finally the last such turning point from which the most right Stokes line emerges running directly to the positive imaginary infinity. From this moment we can repeat the arguments of the previous point when concluding that we can find in the right halfplane on a certain $\xi$-sheet a sector, call it $S_\alpha$ again, a fundamental solution of which is Borel summable at $x_0$ i.e. this solution belongs to $N(x_0)$.

However, this case differ a little bit from the previous one by the fact that the neighbour sector $\psi_{\alpha+1}$ is not Borel summable at $x_0$ in the actual position of this variable. Nevertheless, if we rotate the variable $\lambda$ anticlockwise by an arbitrary small angle we immediately satisfy the Watson-Sokal-Nevanlinna conditions for the Borel resummation of $\psi_{\alpha+1}$. Making a cyclic rotation in the same direction we transform the last solution into $\psi_\alpha$ and this rotation can still be continued a little bit further. This proves that $\psi_\alpha$ indeed satisfies the conditions mentioned above to be Borel summable at $x_0$ in its actual position.

Again mutatis mutandis we can prove moving down the $\xi$-sheet the existence of another fundamental solution in the right half of the sheet which is Borel summable at $x_0$. Let us call it $\psi_b$.

Repeating the procedure and keeping the most left Stokes lines whilst moving to the imaginary infinities in both directions, positive and negative, we find still another two respective solutions $\psi_d$ and $\psi_c$. Collecting them into the following pairs $\psi_a, \psi_c$ and $\psi_b, \psi_d$ we find that the solutions in each pair communicate canonically with each other and are Borel summable simultaneously at $x_0$.

In this way we have proved however our theorem. QED.

Theorem 4 is, to some extend, a generalization of Theorem 1. Namely, we have:

**Corollary**
In the assertion (b) of Theorem 1 we can take any regular point of the $x$-plane in which we want to obtain any fundamental solution continued to this point by the cyclic rotation operations defined in Section II.

Proof

Our reasoning is the following.

Theorem 4 tells us that for a given (regular) $x_0$ there is always a fundamental solution, say $\psi_a(x,\lambda)$, Borel summable at this point. This means however that the $\chi$-factor of this solution is recovered from the Borel function $\tilde{\chi}(x,s)$ by the Borel transformation along the path $C_a$ coinciding with the left/right real halfaxis of the Borel plane. The latter plane is of course a sheet of the $s$-Riemann surface obtained from the one on Fig.7 by the unscreening operation. Depending on the actual position of $x_0$ the representation of branch points on this sheet can be richer than in the cases considered in Theorem 1 because of the closest environment of $x_0$ which can be richer in turning points. This structure can be analyse in a way similar to that in Appendix 1 giving typical branch point pattern with both moving and fixed branch points on this sheet lying above or below the path $C_a$ but allowing to perform the Borel transformation along the path.

We can now start to deform homotopically the path $C_a$ exactly in the same way as we did with the path $C_1$ in Theorem 1 taking its infinite end and rotating it by the angles $\pm \pi, \pm 2\pi, \ldots$, to appropriate positions along the left/right halfaxis on the sheets corresponding to subsequent fundamental solutions starting from $\psi_{a+1}(x,\lambda)$ in the clockwise direction of the deformation or from $\psi_{a-1}(x,\lambda)$ in the opposite case. This procedure could be disturbed only when the deformed path $C_a$ met on its way a chain of branch points elongating to an infinity. This is not possible however since such possible chains of branch points are always screened by the moving cuts. A good illustration of the described situation is provided by the harmonic oscillator case (see [12]).

We have already mentioned in the course of the proof of the Theorem 1 that the suitable deformations of the path $C_1$ to $C_2, \ldots, C_{n+2}$ described in this theorem to restore all the fundamental solutions when $x \in S_1$ strictly corresponded to the recovering these solutions from $\psi_1(x,\lambda)$ defined in $S_1$ by the cyclic rotations. Exactly the same relation connects the above deformations of the path $C_a$ with the cyclic rotations of the Stokes graph from its position just considered.

The last conclusion finishes the proof of Corollary. QED.

IV. General form of semiclassical expansion for $\chi$-factors

Let us note that the $\chi$-factors entering the Dirac forms (3) are the solutions of the following two second order linear differential equations obtained by the substitution (3) into the Schrödinger equation:

$$-q^{-\frac{1}{4}}(x)\left(q^{-\frac{1}{4}}(x)\chi(x)\right)'' + 2\sigma\lambda\chi'(x) = 0 \quad (19)$$

The equations (19) provide us with a general form of semiclassical expansions for the $\chi$-factors if such expansions exists. Namely, assuming the latter we can substitute into (19) the semiclassical expansion for $\chi$:

$$\chi(x,\lambda) \sim \sum_{n \geq 0} \left(\frac{\sigma}{2\lambda}\right)^n \chi_n(x) \quad (20)$$
to get the following recurrent relations for $\chi_n(x)$:

$$
\chi_n(x) = C_n + \int_{x_n}^{x} q^{-\frac{1}{4}}(y) \left( q^{-\frac{1}{4}}(y) \chi_{n-1}(y) \right)'' dy , \quad n \geq 1
$$

(21)

$$
\chi_0(x) \equiv C_0
$$

where $x_n, \ n \geq 1$, are arbitrary chosen regular points of $\omega(x)$ and $C_n, \ n \geq 0$, are arbitrary constants. It is, however, easy to show that choosing all the points $x_n$ to be the same, say $x_0$, merely redefines the constants $C_n$. Assuming this we get for $\chi_n(x)$:

$$
\chi_n(x) = \sum_{k=0}^{n} C_{n-k} I_k(x, x_0)
$$

$$
I_0(x, x_0) \equiv 1
$$

$$
I_k(x, x_0) = \int_{x_0}^{x} d\xi_k q^{-\frac{1}{4}}(\xi_k) \left( q^{-\frac{1}{4}}(\xi_k) \int_{x_0}^{\xi_k} d\xi_{k-1} q^{-\frac{1}{4}}(\xi_{k-1}) \left( q^{-\frac{1}{4}}(\xi_{k-1}) \int_{x_0}^{\xi_{k-2}} d\xi_{k-2} \cdots \right)'' \right)''
$$

$$
\times \left( q^{-\frac{1}{4}}(\xi_2) \int_{x_0}^{\xi_2} d\xi_1 q^{-\frac{1}{4}}(\xi_1) \left( q^{-\frac{1}{4}}(\xi_1) \right)'' \cdots \right)''
$$

(22)

$$
k = 1, 2, \ldots
$$

Substituting (22) into (20) we get finally for the expansion:

$$
\chi(x, \lambda) \sim \sum_{n \geq 0} \left( \frac{\sigma}{2\lambda} \right)^n C_n \sum_{k \geq 0} \left( \frac{\sigma}{2\lambda} \right)^k I_k(x, x_0)
$$

(23)

In this way we have proven the following lemma

**Lemma 1**

An arbitrary semiclassical expansion (20) which follows from (19) can be given the form (23) with an arbitrarily chosen regular point $x_0$ and arbitrary constants $C_n, \ n \geq 0$.

We shall call (23) the standard form of the expansion (20).

Of course, for a given $\chi$ the choice of $x_0$ determines the constants i.e. the latter depend on it. However, if such a $\chi$ is given a choice of $x_0$ cannot be arbitrary. The reasons for that are that if $\chi$ considered can be semiclassically expanded then a domain of the $x$-plane for such an expansion is strictly determined. Good examples of the latter statement are provided just by the fundamental solutions. Each of the latter possesses as we have discussed it in Sec. 2 its allowed canonical domain of the semiclassical expansion (23). Therefore each $\chi$ possesses its own domain $D_\chi$ of the existence of the corresponding semiclassical expansion $\chi^{as}$. Such a domain can however be also empty (see below).
Suppose $D_\chi$ to be not empty and let $x, x_0 \in D_\chi$. Then we can expand $\chi$ semiclassically and this expansion has the form (23). Let us assume a little bit more about $\chi$, namely that there is a domain $B_\chi \subset D_\chi$ in which $\chi$ is Borel summable and let $x, x_0 \in B_\chi$. Then both $\chi(x, \lambda)$ and $\chi(x_0, \lambda)$ can be restored by the Borel transformation of the corresponding Borel functions and, respectively, along the negative real halfaxis (by assumption) of the Borel plane. Their semiclassical expansions (23) can be obtained then by substituting simply into the Borel integral the Borel series (9) with the respective arguments $x$ and $x_0$. But it means, of course, that we can obtain $\chi$ as $(x_0, \lambda)$ simply from $\chi$ as $(x, \lambda)$ by putting $x = x_0$ in the latter. Doing this in (23) we see that it takes in this case the following form

$$\chi(x, \lambda) \sim \chi^{as}(x, \lambda) = \chi^{as}(x_0, \lambda) \sum_{k \geq 0} \left( \frac{\sigma}{2\lambda} \right)^k I_k(x, x_0)$$

(24)

Therefore the following lemma has been proven

**Lemma 2**

If $\psi(x, \lambda)$ is a solution to the Schrödinger equation (2) given in some domain $B$ in the Dirac form (3) with the corresponding factor $\chi(x, \lambda)$ having in $B$ the standard semiclassical expansion (23) which is Borel summable in $B$ to the factor $\chi(x, \lambda)$ itself then this semiclassical expansions takes in $B$ the form (24) where $x_0 \in B$.

The above formula shows explicitly the way of determining the series of the constants $C_n$ in the case just discussed. However, we shall show below that in general the form (24) can not be valid i.e. the series of constants in (23) is not a semiclassical expansion of $\chi(x, \lambda)$ at $x = x_0$ even if the corresponding semiclassical expansions exist in both of the points.

Nevertheless, the formula (24) can be certainly applied to the fundamental solution $\chi$-factors $\chi_k(x, \lambda)$ with $\chi^{as}_k(x, \lambda)$ and $\chi^{as}_k(x_0, \lambda)$ defined by (8) when $x, x_0 \in B_k \subset D_k$, with $D_k$ being the canonical domain of $\chi_k(x, \lambda)$. In these latter cases the formula (24) can be derived directly from (8) by noticing that

$$\chi_{k,n}(x) = \sum_{p=0}^n \chi_{k,p}(x_0) I_{n-p}(x, x_0)$$

(25)

and by multiplying both sides of (25) by $(2\sigma \lambda)^{-n}$ and summing over $n$ (from 0 to $\infty$).

**V. Other solutions with well defined Borel summable semiclassical asymptotics**

In this section we shall show that at each point of the $x$-plane not coinciding with the root of $q(x, E)$ there are two pairs of base solutions to (19) each of which can be expanded semiclassically in some well defined domain. These expansions are Borel summable in corresponding domains although not to the solutions themselves.

1. Fröman and Fröman construction of solutions to Schrödinger equation

A construction of the solutions just mentioned is the following (see App. B in [25] and [26]).
In the $x$-plane we choose any point $x_0$ (being not a root of $q(x)$ however). The point distinguishes a line $\Re W_k(x, E) = \Re W_k(x_0, E)$ (it is independent of $k = 1, 2, \ldots, n$) on which it lies so that $\Re W_k(x, E)$ increases on one side of the line and decreases on the other. On each side of the line we can define two independent solutions each having the form (3) with the following formulae for the $\chi$-factors (see App. B in [25] and [26]):

$$\chi_1^\sigma(x, x_0) = 1 + \sum_{n \geq 1} \left( \frac{\sigma}{2\lambda} \right)^n \int_{x_0}^x d\xi_1 \int_{x_0}^{\xi_1} d\xi_2 \cdots \int_{x_0}^{\xi_{n-1}} d\xi_n \omega(\xi_1)\omega(\xi_2)\ldots\omega(\xi_n)$$

(26) and

$$\chi_2^\sigma(x, x_0) = \frac{\sigma}{2\lambda q^2(x_0)} \int_{x_0}^{\xi_{n-1}} d\xi_n \omega(\xi_1)\omega(\xi_2)\ldots\omega(\xi_n)$$

(27)

where $\sigma = +1$ for $x$ on the side of increasing $\Re W_k(x, E)$ and $\sigma = -1$ in the opposite case so that all integrations in (26) and (27) run from $x_0$ to $x$ along the canonical paths, finite this time. Due to that both the solutions to Schrödinger equation obtained by multiplying the $\chi$-factors (26) and (27) by the corresponding WKB-factors increase exponentially in the semiclassical limit.

The $\chi$-factors of (26) and (27) satisfy the following 'initial' conditions:

$$\chi_1^\sigma(x_0, x_0) = \chi_2^\sigma(x_0, x_0) = 1 \quad \text{and} \quad \chi_1^\sigma(x_0, x_0) = \chi_2^\sigma(x_0, x_0) = 0$$

(28)

2. Semiclassical expansions for $\chi_1(x, \lambda)$ and $\chi_2(x, \lambda)$

Consider now the solutions (26) and (27) defined at a vicinity of some point $x_0$. We shall show below that if $x$ can be linked with $x_0$ by a canonical path the solutions can be expanded semiclassically having the corresponding forms (23) where $x_0$ means now the 'initial' point for the solutions.

To formulate the corresponding lemma let us first invoke Theorem 4 of the previous section to note that when $x_0$ is chosen then there are always at least two fundamental solutions of opposite signatures belonging to $N(x_0)$ which are Borel summable at the point $x_0$ and communicate with themselves canonically. Let us choose these two fundamental solutions to be $\psi_a(x, \lambda)$ and $\psi_b(x, \lambda)$.

For the solution $\psi_1(x, \lambda)$ to the Schrödinger equation (2) defined by $\chi_1(x, \lambda)$ and the fundamental solutions $\psi_a(x, \lambda)$ and $\psi_b(x, \lambda)$ we have:

$$\psi_1(x, \lambda) = \alpha(x_0, \lambda)\psi_a(x, \lambda) + \beta(x_0, \lambda)\psi_b(x, \lambda)$$

(29)

due to the linear independence of the latter. For definitness we shall assume further that $\sigma_1 = \sigma_a = -1$ and $\sigma_2 = \sigma_b = -1$ in the corresponding formulae for the solutions so that $\Re \xi(x) < \Re \xi(x_0)$.
if $x$ can be linked with $x_0$ by a canonical path. The coefficients $\alpha$ and $\beta$ in (28) can be easily calculated according to the general rules described in [1, 5], for example. We have:

$$\alpha(x_0, \lambda) = \frac{\chi_b(x_0, \lambda)}{\chi_{a\to b}(\lambda)} \exp \left( \lambda \int_{x_a}^{x_0} \sqrt{q(x, E)} \, dx \right)$$

$$\beta(x_0, \lambda) = \frac{1}{\chi_b(x_0, \lambda)} \left( 1 - \frac{\chi_a(x_0, \lambda) \chi_b(x_0, \lambda)}{\chi_{a\to b}(\lambda)} \right) \exp \left( -\lambda \int_{x_b}^{x_0} \sqrt{q(x, E)} \, dx \right)$$  \hspace{1cm} (30)

where the condition (28) for $\chi_1(x, \lambda)$ has been used as well as the following relation

$$\chi_1(\infty_b, \lambda) = \chi_b(x_0, \lambda)$$  \hspace{1cm} (31)

The last relation generalizes a little bit a relation $\chi_i \to j = \chi_j \to i$ valid for any pair of fundamental solutions communicating canonically [3].

We shall prove the following lemma.

**Lemma 3**

a) The factors $\chi_{1,2}$ given by (26) and (27) respectively can be expanded in corresponding domains $D_{1,2} = \{ x : \Re \xi(x) < \Re \xi(x_0) \}$ into the semiclassical series determined by the following formulæ

$$\chi_1^{as}(x, \lambda) = \frac{\chi_0^{as}(x_0, \lambda) \chi_a^{as}(x, \lambda)}{\chi_{a\to b}(\lambda)} = \frac{\chi_a^{as}(x_0, \lambda) \chi_b^{as}(x_0, \lambda)}{\chi_{a\to b}(\lambda)} \sum_{n \geq 0} \left( \frac{\sigma}{2\lambda} \right)^n I_n(x, x_0)$$  \hspace{1cm} (32)

and

$$\chi_2^{as}(x, \lambda) = \left( 1 - \frac{\chi_a^{as}(x_0, \lambda) \chi_b^{as}(x_0, \lambda)}{\chi_{a\to b}(\lambda)} \right) \frac{\chi_0^{as}(x_0, \lambda)}{\chi_a^{as}(x_0, \lambda)} \sum_{n \geq 0} \left( \frac{\sigma}{2\lambda} \right)^n I_n(x, x_0)$$  \hspace{1cm} (33)

b) The domains $D_{1,2}$ are maximal for the above expansions to be valid and are contained in the canonical domain $D_a$ of the fundamental solution $\chi_a$.

c) The asymptotic series (32) and (33) can be Borel summed with the following results

$$[\chi_1^{as}(x, x_0, \lambda)]_{a, BS}^{BS} = C_a(x_0, \lambda) \frac{\chi_a(x, \lambda)}{\chi_a(x_0, \lambda)}$$  \hspace{1cm} (34)

and

$$[\chi_2^{as}(x, x_0, \lambda)]_{a, BS}^{BS} = (1 - C_a(x_0, \lambda)) \frac{\chi_a(x, \lambda)}{\chi_a(x_0, \lambda)}$$  \hspace{1cm} (35)

where the Borel sum $C_a(x_0, \lambda) \equiv [\frac{\chi_a^{as}(x_0, \lambda) \chi_a^{as}(x_0, \lambda)}{\chi_{a\to b}(\lambda)}]_{BS}$ is defined below.

d) The representations (34) and (35) are not unique.

**Proof of the lemma**

To prove the part a) of the lemma let us first divide both the sides of (23) by $q^{-\frac{1}{2}}(x) \exp(-\lambda \int_{x_a}^{x} \sqrt{q(y)} \, dy)$ to get

$$\chi_1(x, \lambda) = \frac{\chi_b(x_0, \lambda)}{\chi_{a\to b}(\lambda)} \chi_a(x, \lambda) + \left( 1 - \frac{\chi_a(x_0, \lambda) \chi_b(x_0, \lambda)}{\chi_{a\to b}(\lambda)} \right) \exp \left( 2\lambda \int_{x_0}^{x} \sqrt{q(y)} \, dy \right) \frac{\chi_b(x, \lambda)}{\chi_b(x_0, \lambda)}$$  \hspace{1cm} (36)
Next we note that the term in (36) proportional to $\chi_b(x, \lambda)$ is exponentially small in the semiclassical limit when compared with the first one. Therefore pushing $\lambda$ to infinity in (36) we get (32).

It is now easy to find the semiclassical series (33) for $\chi_1(x, \lambda)$. To this end let us note that $\chi_{1,2}(x, x_0)$ are linear independent solutions of (19) satisfying the conditions (28) so that we can write for $\chi_a(x, \lambda)$:

$$\chi_a(x, \lambda) = \chi_a(x_0, \lambda)\chi_1(x, x_0, \lambda) + \chi'_a(x_0, \lambda)\chi_2(x, x_0, \lambda) \quad (37)$$

Getting asymptotics of both the sides of (37) and solving the obtained equation with respect to $\chi_2(x, \lambda)$ we obtain (33).

The thesis b) of the lemma follows from the fact that both the solutions $\chi_{1,2}(x, \lambda)$ diverge exponentially for $\Re x > \Re x_0$ when $\lambda \to \infty$ (the property which follows directly when the considered pair of solutions is expressed by the second pair of them defined by (26) and (27) with the opposite signature) and from the fact that the condition $\Re x < \Re x_0$ defines also a (proper) part of the canonical domain $D_a$ of the fundamental solution $\chi_a$.

To prove the part c) of the lemma it is necessary to invoke the exponential representation of the fundamental solution $\chi$-factors \[5\]. By this representation the following is meant

$$\chi_{a,b}(x, \lambda) = \exp \left( \mp \int_{\infty_{a,b}}^{x} \rho^-(y, \lambda)dy + \int_{\infty}^{x} \rho^+(y, \lambda)dy \right) \quad \chi_{a,b}^as(x, \lambda) = \exp \left( \mp \int_{\infty_{a,b}}^{x} \rho_{as}^-(y, \lambda)dy + \int_{\infty}^{x} \rho_{as}^+(y, \lambda)dy \right) \quad (38)$$

where the coefficients $\rho^\pm_n(y)$, $n \geq 0$, have been calculated explicitly in Ref. \[5\](see Sec.2.3 of this reference, where the roles of $\rho^\pm(x, \lambda)$ are played by the corresponding $\chi^\pm(x, E, \lambda)$-functions). The important properties of the coefficients as well as of the asymptotic series in (38) they constitute are \[1\] \[5\] (see App.2 in Ref. \[5\]):

- a. They are point (path independent) functions of $y$, i.e. they are universal, sector independent functions;
- b. $\rho^+_n(y)$ have square root singularities at every turning point;
- c. $\rho^-_n(y)$ are meromorphic at each turning point with vanishing residues at the points (i.e. $\int \rho^-_n(y)dy = 0$ around any turning point);
- d. Both the series in (38) are Borel summable.

This is the property c. which causes the $\rho_{as}^-$-integral in (38) to be again the point function of $y$ i.e. it is sector independent.

The property d. which follows from the corresponding property of fundamental solutions \[1\] generates two Borel functions $\rho^\pm(x, s)$:

$$\tilde{\rho}^-(x, s) = \sum_{n \geq 0} \frac{\rho^-_n(x)}{(2n)!} s^{2n} \quad \tilde{\rho}^+(x, s) = -\sum_{n \geq 0} \frac{\rho^+_n(x)}{(2n+1)!} s^{2n+1} \quad (39)$$

which can be Borel transformed along any standard path $\tilde{C}$ in the Borel plane providing us each time with the corresponding Borel sums $\rho^\pm_C(x, \lambda)$ of the series in (39). If we performed
a Borel resummation of the first formula in (39) along such a path \( \tilde{C} \) we get:

\[
\chi_{\tilde{C}}(x, \lambda) = e^{-\int_\infty^x (-\rho_{\tilde{C}}(y, \lambda)+\rho_C(y, \lambda))dy} 
\]

where \( \chi_{\tilde{C}}(x, \lambda) \) is a \( \chi \)-function of some fundamental solution, rotated possibly in the \( \lambda \)-plane. The minus sign in (40) has been chosen for definitness.

Noticing further, that:

\[
\chi_{a}(a_{x_{0}}, \lambda_{0}) = e^{\int_{a_{x_{0}}}^{\infty} \rho_{a}(y, \lambda)dy} \chi_{a}(x, \lambda) \]

we can sum a la Borel both the equations (32) and (33) along the path \( \tilde{C}_a \) recovering the factor \( \chi_{a}(x, \lambda) \) to obtain the formulae (34) and (35). In these formulae \( \tilde{C}_a(x_{0}, \lambda) \) is therefore the following Borel sum

\[
\tilde{C}_a(x_{0}, \lambda) = \exp\left( 2\int_{\infty}^{x_{0}} \rho_{a}(y, \lambda)dy \right) 
\]

The representations (34) and (35) are not unique since, in general, there can be other fundamental solutions in \( \mathcal{N}(x_{0}) \) with the same signature as the solutions \( \chi_{1,2} \) have which can substitute the solution \( \chi_{a} \) in our considerations. However, having the same signatures these other fundamental solutions can provide us with the representations (34) and (35) which, in this case, differ between themselves only by exponentially small contributions.

The last statements ends our proof of Lemma 3. QED.

One can easily identify the coefficients in front of the sum in the RHS’s of (32) and (33) as the corresponding series of constants in the standard expansions (23). It is important to note that none of them is equal to asymptotic series corresponding to \( \chi_1(x_{0}, \lambda) \equiv 1 \) and \( \chi_2(x_{0}, \lambda) \equiv 0 \) respectively. This confirms of course our earlier statement that the semiclassical series (22) and (33) cannot be Borel summed to the respective factors \( \chi_1(x, \lambda) \) and \( \chi_2(x, \lambda) \). A reason for that is the presence of exponential terms \( e^{-2\lambda\sigma(W_k(x)-W_k(x_{0}))} \) in the asymptotic formulae for (26) and (27) (when \( \Re(W_k(x)-W_k(x_{0})) = 0 \)) which breaks the necessary conditions for the Watson-Sokal theorem [27] to be applied. Note that these exponential terms are absent in the case of fundamental solutions which are obtained in the limit \( x_{0} \rightarrow \infty_k \) taken along a cannonical path, for any \( k = 1, 2, ..., 2n + 2 \).

VI. Uniqueness of fundamental solutions as Borel summable solutions

Let \( \psi(x, \lambda) \) be any solution to the Schrödinger equation (2) given at some domain \( D \) of the \( x \)-plane. Let us choose in \( D \) a point \( x_{0} \) which is not a root of \( q(x, E) \) (i.e. which is regular for \( \omega(x) \) as given by (1)). The solution \( \psi(x, \lambda) \) can always be given each of the two Dirac forms (3) with the corresponding \( \chi \)-factors satisfying the equation (19). Let us write these forms in the following way:

\[
\psi(x, \lambda) = C_{\pm}(\lambda)q^{-\frac{1}{4}}(x)e^{\pm\lambda\int_{x_{0}}^{x} \sqrt{q(y)}dy} \chi_{\pm}(x, \lambda) 
\]

We shall say that \( \psi(x, \lambda) \) has well defined semiclassical expansion in \( D \) if there is a choice of a sign in (33) and an accompanied constant \( C_{\pm}(\lambda) \) such that the \( \chi \)-factor \( \chi_{\pm}(x, \lambda) \)
corresponding to this choice can be expanded semiclassically in the standard way given by (23). It follows directly from the above definition that only one of the two possible choices can satisfy it.

We shall also say that \( \psi(x, \lambda) \) is Borel summable in \( D \) if it has well defined semiclassical expansion there and the corresponding series (23) is Borel summable to the uniquely chosen \( \chi \)-factor \( \chi(x, \lambda) \) of \( \psi(x, \lambda) \).

We shall prove below the following main theorem of this paper:

**Theorem 5** Let a solution \( \psi(x, \lambda) \) given at some vicinity \( D \) of \( x_0 \) (\( x_0 \) does not coincide with any turning point) be Borel summable in \( D \). Then this solution must coincide with one of the fundamental solutions up to some \( \lambda \)-dependent constant.

**Proof.**

To prove the theorem we could utilize the solutions (26) and (27) and all their properties which we have established in Lemma 3 of the previous section. It can however be quite instructive to prove the theorem not invoking for the latter solutions since it makes the main arguments supporting the theorem (which have worked implicitly also in proving Lemma 3 of the previous section) to be more transparent.

According to Theorem 4 of Section III., for \( x_0 \) chosen we can always find in the set \( N(x_0) \) a number of fundamental solutions of the same signatures as the respective \( \chi(x, \lambda) \) corresponding to \( \psi(x, \lambda) \) has. Let \( \chi_a(x, \lambda) \) be one of them. It is Borel summable at \( x_0 \) and in some of its vicinity. Then using (24) both for \( \chi(x, \lambda) \) and \( \chi_a(x_0, \lambda) \) we have:

\[
\frac{\chi^{as}(x, \lambda)}{\chi^{as}(x_0, \lambda)} = \sum_{n \geq 0} \left( \frac{\sigma}{2\lambda} \right)^n I_n(x, x_0) = \frac{\chi^{as}_a(x, \lambda)}{\chi^{as}_a(x_0, \lambda)}
\]  

(44)

It follows from (44) that the outer parts of this equality having the same semiclassical expansions have to have also the same Borel function. Since \( \chi^{as}(x, \lambda) \) and \( \chi^{as}(x_0, \lambda) \) are both Borel summable in \( D \) they can be summed along the same standard path \( \tilde{C} \) on their corresponding Borel planes if \( x \) is chosen to be sufficiently close to \( x_0 \). It is, however, easy to check (see App. 3) that under the latter condition the same standard path \( \tilde{C} \) can be chosen to sum the quotient on the LHS of (44) since its corresponding Borel function is holomorphic around this path. However, the same must be true for the RHS quotient i.e. the corresponding Borel functions of its two factors can be integrated also along \( \tilde{C} \) lying in their Borel planes. Let us sum therefore a la Borel both the outer sides of (44) along this path. We get

\[
\chi(x, \lambda) = \chi(x_0, \lambda) \frac{2\lambda \int_{\tilde{C}} e^{2\lambda s} \chi_a(x, s) ds}{2\lambda \int_{\tilde{C}} e^{2\lambda s} \chi_a(x_0, s) ds}
\]  

(45)

The last equation, however, ends the proof of the theorem. QED.

As a comment to the last theorem we would like to stress that it summarizes a particular property of the semiclassical theory of the 1D Schrödinger equation with the polynomial potentials. Namely, this is that the standard semiclassical expansion (23) is constructed basically by the series \( \sum_{n \geq 0} \left( \frac{\sigma}{2\lambda} \right)^n I_n(x, x_0) \) which can be Borel summable and the Borel function of which, by (44), coincides up to a \( \lambda \)-dependent multiplicative constant with the one of the fundamental solutions and, also by (44), with the Borel function of any Borel
summable solution. This means that we can consider the Borel function of the fundamental solutions as the canonical one. The latter can be uniquely defined by the condition of being equal to unity at $s = 0$ on the 'first sheet' of the corresponding Riemann surface which the condition it satisfies actually.

**VII. Conclusions and discussion**

Theorem 5 of the previous section shows that in the case of the Schrödinger equation with the polynomial potentials its Borel summable solutions are the fundamental ones. The Borel function generated by these solutions is, up to analytical continuation, the unique one. This property justifies our earlier use of the fundamental solutions to investigate the problem of the Borel summability of energy levels and matrix elements in 1D quantum mechanics [1]. It shows also that only the fundamental solutions can be invoked when any problem connected with the Borel resummation is considered and conditions for such resummations are satisfied [27].

The latter objection is important since not all the results we obtain for the case of polynomial potentials can be immediately extended to other cases of potentials. These are, for example, the rational potentials being the next class of potentials of the modeling importance. In particular, the universality of the Borel function in the later case of potentials seems to be not satisfied [32]. Nevertheless, the role of the corresponding fundamental solutions as the unique Borel summable ones seems to be maintained not only in the case of rational potentials but also in the case of other meromorphic potentials such as the Pöschl-Teller one, for example.

The fundamental solutions we have discussed in Sec. 2 can be given another forms when each of the factors in (3) becomes a complicated function of $\lambda$ [33]. These generalized representations however preserve all the Borel summing features of the original fundamental solutions being only a partial Borel resummation of the latter [25].

Finally, we would like to note that the result obtained in the present paper completes the ones obtained in our other papers [1, 2, 25]. Namely, all these results show that the semiclassical theory in 1D quantum mechanics can be completely formulated on the base of the Borel method of resummation. This is certainly true in the case of the polynomial potentials and it seems to be true with some modifications for meromorphic potentials as well [32]. In the formulation of such a theory the essential role as we have shown in the present paper is played by the fundamental solutions (see also [1, 2, 25]). The theory allows us to construct the simplest semiclassical approximations as well as to complete the latter by the exponentially small contributions up to a desired level of accuracy [12]. In such a theory even a change of variable in the Schrödinger equation the procedure which is used very frequently as a way of improving the semiclassical approximations is also a result of the proper Borel resummation operation [25, 33].

**Acknowledgments**

Stefan Giller has been supported by the KBN grant 2PO3B 07610 and Piotr Milczarski by the Lódź University Grant No 795.
Appendix 1

Here we would like to draw some basic conclusions which follow for the Borel function \( \tilde{\chi}(x, s) \) from its representation given by the topological expansion developed in our recent paper (see [12]) and not discussed there.

First of all let us recapitulate shortly basic elements of this representation. Namely, we have shown in [12] that the Borel function defined originally in some sector, say \( S_1 \), can be represented in this sector as the following series \( (\xi = x, q = \int_{x_1}^x \sqrt{q(y, E)}dy) \):

\[
\tilde{\Phi}(\xi, s) = \sum_{q \geq 0} \tilde{\Phi}^{(q)}(\xi, s)
\]

where \( \tilde{\Phi}(\xi, s) \equiv \tilde{\chi}(x, s) \) and the terms \( \tilde{\Phi}^{(q)}(\xi, s), q \geq 0 \) of the series are given by the formulae:

\[
\tilde{\Phi}^{(0)}(\xi, s) = I_0 \left( \sqrt{4s\Omega(\xi)} \right)
\]

\[
\tilde{\Phi}^{(2q)}(\xi, s) = \int_{c(s)} d\eta_1 \int_{c(\eta_1)} d\eta_2 \cdots \int_{c(\eta_{q-1})} d\eta_q \frac{\xi - \eta_1}{\xi_1} \frac{\xi - \eta_2}{\xi_2} \cdots \frac{\xi - \eta_q}{\xi_q} \frac{\xi_{q+1}}{\xi_{q+1}}
\]

\[
\tilde{\omega}(\xi_1 + \eta_1)\tilde{\omega}(\xi_1 + \eta_2) \cdots \tilde{\omega}(\xi_q + \eta_q)\tilde{\omega}(\xi_q)(2s - 2\eta_1)^2 I_{2q}(\frac{\sqrt{z_{2q}}}{z_{2q}})
\]

\[
z_{2q} = 4(s - \eta_1)\Omega(\xi) + 8(s - \eta_1) \sum_{pq=1}^q \left( \Omega(\xi_p + \eta_{p+1}) - \Omega(\xi_q + \eta_p) \right),
\]

\[
\eta_{q+1} \equiv 0, \quad q = 1, 2, \ldots
\]

\[
\tilde{\Phi}^{(2q+1)}(\xi, s) = \int_{c(s)} d\eta_1 \cdots \int_{c(\eta_q)} d\eta_{q+1} \tilde{\omega}(\xi - \eta_1 + \eta_2) \frac{\xi - \eta_1}{\xi_1} \frac{\xi - \eta_2}{\xi_2} \cdots \frac{\xi - \eta_{q+1}}{\xi_{q+1}} \frac{\xi_{q+2}}{\xi_{q+2}}
\]

\[
\tilde{\omega}(\xi_1 + \eta_1)\tilde{\omega}(\xi_1 + \eta_3) \cdots \tilde{\omega}(\xi_q + \eta_{q+1})\tilde{\omega}(\xi_q)(2s - 2\eta_1)^2 I_{2q+1}(\frac{\sqrt{z_{2q+1}}}{z_{2q+1}})
\]

\[
z_{2q+1} = 4(s - \eta_1)\Omega(\xi) + 8(s - \eta_1) \sum_{pq=0}^q \left( \Omega(\xi_p + \eta_{p+2}) - \Omega(\xi_{p+1} + \eta_{p+1}) \right),
\]

\[
\xi_0 \equiv \xi, \quad \eta_{q+2} \equiv 0, \quad q = 0, 1, 2, \ldots
\]

where \( \tilde{\omega}(\xi(x)) \equiv \omega(x)q^{-\frac{1}{2}}(x), \Omega = \int_{\infty}^\xi \tilde{\omega}(\eta)d\eta \) and the functions \( I_q(x), q \geq 0 \), in (47) are the modified Bessel functions (of the first kind, [34], p.5, formula (12)).

The formulae (47) can be obtained from the following recurrences:

\[
\tilde{\Phi}^{(2q+2)}(\xi, s) = - \int_{c(s)} d\eta \int_{c(\eta)} d\eta' \frac{\xi}{\xi_1} \tilde{\omega}(\xi_1)\tilde{\omega}(\xi_1 - \eta')(2s - 2\eta)
\]
\[
\tilde{\Phi}_1^{(2q)}(\xi_1 - \eta', \eta - \eta') \frac{I_1 \left( \sqrt{4(s - \eta)(\Omega(\xi) - 2\Omega(\xi_1) + \Omega(\xi_1 - \eta'))} \right)}{\sqrt{4(s - \eta)(\Omega(\xi) - 2\Omega(\xi_1) + \Omega(\xi_1 - \eta'))}}
\]

(48)

\[
\tilde{\Phi}_1^{(2q+1)}(\xi, s) = - \int_{\tilde{C}(s)} \frac{d\eta}{\tilde{C}(\eta)} \tilde{\omega}(\xi - \eta') \tilde{\Phi}_1^{(2q)}(\xi - \eta', \eta - \eta') I_0 \left( \sqrt{-4(s - \eta)(\Omega(\xi) - \Omega(\xi - \eta'))} \right)
\]

\[q = 0, 1, 2, \ldots\]

where \(\tilde{\Phi}_1^{(0)}(\xi, s)\) is given by (47).

Note that (48) can be obtained from (47) and vice versa by applying the following relations:

\[
\int_0^1 dx I_m(\sqrt{\alpha x}) I_m(\sqrt{\beta(1-x)}) (\alpha x)^{\frac{m}{2}} (\beta(1-x))^{\frac{n}{2}} = \frac{2\alpha^m \beta^n I_{m+n+1}(\sqrt{\alpha + \beta})}{(\sqrt{\alpha + \beta})^{m+n+1}}
\]

(49)

The \(\xi\)-integrations in (47) and (48) run over some \(\xi\)-Riemann surfaces of the subintegral functions starting from the infinite points of these surfaces which are the corresponding images of the infinite point of the sector \(S_1\). The \(\eta\)-integrations, contrary to the \(\xi\)-ones, are finite and run over the \(s\)-Riemann surfaces. All the latter integrations ends at \(s = 0\).

For these integrations the most important are the branch point structures of the Riemann surfaces corresponding to the functions \(\tilde{\omega}(\xi)\) and \(\Omega(\xi)\) and the shifts of these surfaces by some complex number \(\eta\). The two latter surfaces corresponds to the functions \(\tilde{\omega}(\xi - \eta)\) and \(\Omega(\xi - \eta)\).

Since every of these four surfaces has complicated topology (defined by its branch points) we decided not trying to sew them suitably together when these functions are integrated simultaneously but rather to consider them, for safeness, separately. These topologies are determined, of course, by the singularities of the respective functions \(\tilde{\omega}(\xi)\) and \(\Omega(\xi)\). Besides, since the latter of these two functions is defined as the integral over the former then the corresponding Riemann surface on which \(\Omega(\xi)\) is defined is a map of the surface corresponding to \(\tilde{\omega}(\xi)\), i.e. there is a well defined relation between these two surfaces.

For the polynomial potentials with simple roots all singularities of \(\tilde{\omega}(\xi)\) and \(\Omega(\xi)\) are cubic root branch points corresponding to turning points. Therefore, the four surfaces discussed above acquires a suitable cut pattern each. The corresponding \(\xi\)-integration runs over the sheets of these surfaces which are unambiguously related to each others (by the above shift or by a map) so that the integration paths on this sheets look the same running from \(\infty_1\) to some finite point, the same on each sheet.

It is necessary to stress that every subsequent integration in (47) or every subsequent step in the recurrent formulae (48) changes the structure of the Riemann surfaces corresponding to functions resulting from these integrations. Namely, these surfaces become still more complicated preserving all the branch points of the previous stage and acquiring new ones as a result of the last integration(s).
Nevertheless, these structures look relatively simple if we consider them on definite sheets of the Riemann surfaces we want to stay considering the properties of the Borel functions $\tilde{\Phi}(\xi, s)$. Namely, starting from the sheets corresponding to the sector $S_1$ we shall keep the variable $x$ (or $\xi(x)$) changing along the contour $K'$ of Fig.1a. This corresponds to a path $\gamma_1(x)$ on the $x$-plane which begins in the sector $S_1$ and crosses on its way all the Stokes lines running to the infinity of the plane (but each line only once) penetrating subsequent sectors in their cyclic (clockwise or anticlockwise) ordering introduced in Section II. We shall call such a path the outer path.

Under the above condition the final pattern of the branch points of $\tilde{\Phi}(\xi, s)$ viewed from the relevant sheets is quite simple. First let us note that, as it follows from (47) and (48), under the above circumstances we can deform homotopically the infinite end of the outer path $\gamma$ from the sector $S_1$ to any of the subsequent sectors $S_2, \ldots, S_{n+1}$ (in this or in the reversed orders). This is the consequence of the fact that the initial condition (7) put on the path $\gamma$ to make it canonical is no longer valid since all the dangerous exponentials in the formula (5) enforcing this condition disapeared on the way of passing to the formulae (47) and (48).

Fig.6 shows the corresponding result of such an operation for $x \in K' \cap S_3$. This proves that under the above condition (47) and (48) define the same unique Borel function $\tilde{\Phi}(\xi, s)$ for all the $n+1$ fundamental solutions (3). This deformation can be done keeping the end point $x$ ($\xi(x)$) of the outer path $\gamma$ in any of the sectors $S_1, \ldots, S_{n+1}$. The latter property means, of course, that if $\xi$ is in $S_k$ and the infinite end of $\gamma$ is in $S_l$ then $\tilde{\Phi}(\xi, s)$ represents the Borel function of the fundamental solution $\psi_k$ (defined in $S_k$) analytically continued to the sector $S_l$ and the formulae (47) and (48) define then this continuation explicitly.

The following theorem can now be proved inductively.
Theorem 6 Let the end point $x$ go around the contour $K'$ of Fig. 1a starting from the sector $S_1$ and passing consecutively by the sectors $S_2, \ldots, S_{n+1}$. Then the sheets from which the $\chi$-factors corresponding to the subsequent sectors are recovered by the Borel transformations over the Borel function $\tilde{\chi}(x, s)$ along the suitable real halfaxes have the branch point structures shown in Fig. 11.

Proof.

Let us consider first a sheet of the $s$-Riemann surface corresponding to the sector $S_1$. Let the point $x$ then be on $K'$ in the sector $S_1$ so that $\Re \xi(x) > 0$.

Consider $\tilde{\Phi}^{(0)}(\xi, s)$. Since $I_0(\sqrt{4s\Omega(\xi)})$ is the holomorphic function of its argument then $\tilde{\Phi}^{(0)}(\xi, s)$ is an entire function of $s$ (i.e. holomorphic in the whole $s$-plane) so that its corresponding $s$-Riemann surface coincides with the $s$-plane. What concerns its $\xi$-Riemann surface structure it coincides with the one of $\Omega(\xi)$ since for each natural power of the latter its $\xi$-Riemann structure is the same and $I_0(\sqrt{4s\Omega(\xi)})$ determines $\tilde{\Phi}^{(0)}(\xi, s)$ as the holomorphic function of $\xi$ in each non-singular point of $\Omega(\xi)$. Therefore $\tilde{\Phi}^{(0)}(\xi, s)$ has on the corresponding first sheet in the $\xi$-plane a unique branch point at $\zeta_1$, as it is shown in Fig. 7a, if the corresponding cut emerges from this branch point vertically down. There are no other branch points visible then on the sheet. These another branch points, however, are on the sheets lying below the first sheet and the closest ones at $\zeta_2, \zeta_{i_1}, \ldots, \zeta_{i_l}$ and $\zeta_n$ are shown in Fig. 7a with the dashed lines of cuts emerging from them. The full thin paths on the figure emerging from the point $\xi$ show how to approach these last branch points starting from $\xi$. We shall adopt this convention for the remaining figures too.

Fig. 7b shows the $\xi$-Riemann surface for the shifted function $\tilde{\Phi}^{(0)}(\xi - \eta, s)$.

Next consider $\tilde{\Phi}^{(1)}(\xi, s)$. It is given by the second of the formulae (48) for $q = 0$. In this formula we have to integrate first over the variable $\eta'$. As it follows from the formula the branch point structure of the $\eta'$-Riemann surface is determined by the functions $\tilde{\omega}(\xi - \eta')$, $\Omega(\xi - \eta')$ and $\tilde{\Phi}^{(0)}(\xi - \eta', \eta' - \eta)$ and its first and the second sheets are shown on Fig. 8a.

The integration over $\eta'$ leads us to a function defined on the $\xi$-Riemann function shown in Fig. 7b. The cuts shown there are a result of the end point (EP) mechanism of the singularity
producing ([12], see also [35]). Namely, the \( \eta' \)-integration is perturbed if the moving branch points of Fig.8a can approach the fixed end points \( \eta' = 0 \) and \( \eta' = \eta \) of the integration path \( C(\eta) \). For example, to generate the branch point at \( \zeta_1 (= 0) \) we simply move the branch point \( \xi - \zeta_1 \) against the end point \( \eta' = 0 \) of \( C(\eta) \) to touch it finally. To produce the branch point at \( \zeta_2 \) on Fig.8b we have to move \( \xi - \zeta_1 \) down avoiding the end point \( \eta' \) from the left and below and next moving it to the right in such a way to make the screened branch point at \( \xi - \zeta_2 \) coinciding with the end point \( \eta' \).

In the way described above we can generate the branch points at the position shown in Fig.8b. A convention adopted on this figure for the cut designing is to draw a full thin path emerging from the point \( \xi \) and if the path crosses a cut it becomes dotted. If it crosses the next appropriate cut it is doubly dotted and so on. This way of designing cuts allows us to establish the sheets they are distributed on. The figure in parentheses at the different branch points indicate the multiplicity of the latter i.e. their appearing on different sheets at the same positions. However, for the sake of transparency of the figures not all paths showing the distributions of the branch points on the sheets have been shown.

It is worth to note that this multiplication of branch points with the same coordinates but lying on different sheets on this figure is necessary to keep fixed the relative distribution of the branch points on Fig.8a when the moving ones on Fig.8b change their positions. In fact the branch points in this figure with the same coordinate substitute each other during such a motion. This is because the branch point \( \xi - \zeta_1 \) on Fig.8a has to be always accompanied by the branch points at \( \xi - \zeta_2 \) and \( \xi - \zeta_n \) lying on the next two lower sheets. For example, if the branch points \( \zeta_1 + \eta \) and \( \zeta_2 + \eta \) move to the left so that they pass the branch point at \( \zeta_1 (= 0) \) having it between themselves then the point \( \zeta_2 + \eta \) on the sheet opened by the first of these moving points is screened in some moment by the cut emerging from \( \zeta_1 = 0 \) lying on the first sheet in Fig.8b. But then the branch point at \( \zeta_2 + \eta \) on the sheet opened just by this latter cut becomes unscreened substituting its copy on the sheet we started with. Of course, all three copies of the branch points at \( \zeta_2 + \eta \) are mapped by the relation \( \xi - \eta = \zeta_k, k = 1, 2, 3 \) into the one copy of them at \( \xi - \zeta_2 \) on the \( \eta' \)-plane of Fig.8a.

The final integration over \( \eta \) repeats only the steps done earlier introducing nothing new.
to the distribution of the branch points on the first sheet not modifying the lower sheets as well so that the final branch point structures of the first sheets look again as in the figures 8a-b where we have to substitute $\eta'$ and $\eta$ by $s$ on both the figures.

Consider next $\tilde{\Phi}^{(2)}(\xi, s)$. It is given by the first of the formulae (48) for $q = 0$. The initial $\xi$-Riemann surface structure is similar to that shown on Fig. 7 and look as in Fig. 9a-b. The first $\xi_1$-integration in the corresponding formula (48) provides us this time with both the types of singularities on the $\eta'$-Riemann surface i.e. generated by the E-mechanism, which makes a replica of the branch point structures of Fig. 8a, and by the pinch (P) mechanisms (discussed in [12], see also [35]). A singularity generated by the second mechanism arises when, for example, the branch point at $\zeta_1 + \eta'$ on Fig. 9b moves against the one at $\zeta_1 = 0$ of Fig. 9a pinching the integration path $\tilde{\gamma}(\xi)$. It can be done by making a tour around the end $\xi$ of the path $\tilde{\gamma}(\xi)$ in both the directions i.e. clockwise anticlockwise. In this example we will produce branch point singularities on both the lower sheets of the $\eta'$-Riemann surface shown if Fig. 9c at $\eta' = 0$. We can pinch the path $\tilde{\gamma}(\xi)$ by $\xi_1 + \eta'$ also in this way but against the branch points at $\zeta_2$ and $\zeta_n$. This needs only to cross the cut emerging from $\zeta_1(= 0)$ to reach the points mentioned i.e. the first one by crossing this cut from the left whilst the second - from the right. It means that $\eta'$ itself has to do the same on Fig. 9c crossing the cuts emerging from $\eta' = 0$ on the corresponding sheets. The positions of the branch points generated in this way are shown on Fig. 9c where only a closest part of them is shown. It follows from the figure that some of the closest fixed branch points are at $\eta'$ on the two sheets opened by the branch point at $\xi - \zeta_1$. The other ones lie on the sheets opened by the the branch points at $\xi - \zeta_2, \xi - \zeta_i, \ldots, \xi - \zeta_i$ and $\xi - \zeta_n$ and by the fixed branch points generated in this way i.e. still on the lower sheets.

A discussion of the remaining two $\eta'$- and $\eta$-integrations goes along the same lines as in the previous discussion on calculating $\tilde{\Phi}^{(1)}(\xi, s)$ with the similar results obtained accordingly to Fig. 9c. The only appearing difference is that in both these integrations the P-mechanism of the moving branch point singularity generation on the $\xi$- Riemann surface becomes active since, except the moving singularities, there are also the fixed ones on the corresponding sheets of the $\eta'$- and $\eta$-Riemann surfaces (see Fig. 9c). Therefore, the final first sheet structures are again a replica of those shown in Fig. 8b for the $\xi$-Riemann surface and in Fig. 9c for the $s$-Riemann one (with the $\eta, \eta'$-variables substituted suitably by the $s$-one). Of course, because of the reason of transparency only the branch points on the first three sheets are shown on the figures. A comment made previously on the proliferation of the moving branch points with the same coordinates but lying on different sheets is also still valid and Fig. 8b reproduces this fact correctly.
Now consider the cases $\tilde{\Phi}^{(2q+1)}(\xi, s)$ and $\tilde{\Phi}^{(2q+2)}(\xi, s)$ assuming that the first sheets of the $\xi, s$-Riemann surface structure of $\tilde{\Phi}^{(2q)}(\xi, s)$ is given by Fig.8b and Fig.9c (with the suitable $\eta, \eta' \rightarrow s$ substitutions on the figures). It is clear that we can repeat all the previous analyses and conclusions without any changes if the considered structure is limited only to the sheets defined by the formulae $\{15\}$ and the branch points shown on the figures 8a, 9c, i.e. the structure on these figures is reproduced also for the functions $\tilde{\Phi}^{(2q+1)}(\xi, s)$ and $\tilde{\Phi}^{(2q+2)}(\xi, s)$. Since the series $\{17\}$ is convergent this structure is the same also for $\tilde{\Phi}(\xi, s)$ itself.

It is important to note that the fixed branch points on the $s$-Riemann surface are gen-
erated in the scheme of the topological expansion (47) on lower sheets.

Having \( \tilde{\Phi}(\xi,s) \) defined in the above way we can restore the \( \chi \)-factors defined in the sectors \( S_1 \) or \( S_2 \) by the Borel transformations of \( \tilde{\Phi}(\xi,s) \) along the left halfaxis to get \( \chi_1(x,\lambda) \) or along the right one to get \( \chi_2(x,\lambda) \). Clearly, the signs of \( \lambda \) in both these integrations are different.

To prove the assertion of the theorem about the structure of the \( \xi,s \)-Riemann surface for \( \tilde{\Phi}(\xi,s) \) continued to the sector \( S_k \) along the contour \( K' \) on Fig. 1a we should perform this continuation on the \( \xi,s \)-Riemann surface corresponding to \( \tilde{\Phi}(\xi,s) \) changing \( \xi \) respectively and drawing cuts properly. Such an operation, however, needs the detailed knowledge of the structure of many lower sheets of the \( \xi,s \)-Riemann surface, a task which seems to be in general hopeless. However, we have already noticed that such continuation can be easily performed with the help of the formulae (47)-(48) by changing the infinite ends of the \( \xi \)-integrations in these formulae i.e. moving them to the appropriate sectors when the variable \( \xi(=\xi(x)) \) itself is continued to these sectors when \( x \) changes along the contour \( K' \). For definiteness let \( \xi \) be continued to the sector \( S_3 \). Then we can continue the mentioned infinite ends to the same sector. According to the Stokes graph on Fig. 1a the corresponding pattern of the sheets on which \( \tilde{\omega}(\xi) \) and \( \Omega(\xi) \) are defined are shown on Fig.6. Therefore, taking this figure as the original one and using again the formulae (48) we can repeat once again the analyses done above. The only difference are introduced by additional branch points which can appear according to the Stokes graph on Fig. 1a. Then forgetting about the branch points which lie on the lower sheets in Fig.6 (such as \( \zeta_3 \)) we get as the final structure of the first sheets for \( \tilde{\Phi}(\xi,s) \) the one shown in Fig.10.

The pattern on Fig.10 is the one which has also to follow if we would continue the pattern of Fig.8b (with the \( \eta \to s \) substitution) by moving anticlockwise the variable \( \xi \) around the branch point \( \zeta_1 \) and uprighting the cut emerging from this point by rotating it in the same anticlockwise direction. Then the cut emerging from \( \zeta_2 \) on the figure is unscreened and uprighting it again anticlockwise as well as all the other consecutive cuts met by \( \xi \) on its way to the sector \( S_3 \) we have to reveal by these *uprighting operations* of the cuts the pattern of Fig.10a. Comparing the latter with Fig.8b we see that these operations do their job properly. The proliferation of the branch points with the same coordinates on different sheets plays an essential role in this operation allowing us to recover the cuts which are screened by the uprighting operations made over the cuts emerging from \( \xi = \zeta_1 \) and \( \xi = \zeta_2 \).

On the \( s \)-Riemann surface of \( \tilde{\Phi}(\xi,s) \) shown on Fig.9c (with \( \eta' \to s \) on the axes) the corresponding operation with cuts are of course reversed i.e. each consecutive cut which is unscreened by the anticlockwise upside-down rotation of its predecessor originating by the cut emerging from the branch point \( \xi - \zeta_1 \) on Fig.9c has also to be rotated in the same way. This process stops on the cut emerging from the last branch point at \( \xi - \zeta_2 \) being unscreened.

We can then move this new pattern first down leaving the origin of the first sheet to the right and next move to the right leaving the origin of the sheet above all the moved branch points. The final position has to coincide then with that shown in Fig.10b.
We shall call the above operations with cuts leadind us to uncovering the desired sheets of the $\xi, s$-Riemann surface the unscreening operations.

It is now clear that we can follow the above way considering the $\xi, s$-Riemann surface structure corresponding to $\tilde{\Phi}(\xi, s)$ when $\xi$ is in $S_k$ being continued along the contour $K'$ on Fig.1a. The tour of along the contour $K'$ is mapped properly on the $\xi$-Riemann surface of $\tilde{\Phi}(\xi, s)$ where $\xi$ moves anticlockwise avoiding all met branch points from the left and putting upside-down the crossed cuts emerging from them realizing in this way the unscreening operation. The same unscreening operations of putting the properly chosen cuts upside-down are applied on the $s$-Riemann surface. We start from the pattern of Fig.9b and repeat the procedure described above $k$ times. The final pattern has to have the form shown on Fig.11a.b. Its detailed structure shown on the figure can be obtained from the formulae (48) by the analyses described above.

Borel transforming along the left halfaxis of Fig.11b we recover the $\chi$-factor corresponding to the sector $S_k$ (if $k$ is odd) or the one corresponding to the sector $S_{k+1}$ when the Borel
transformation is performed along the right half axis. QED.

To finish the above discussion let us note yet that as it follows from our estimation of the convergence of the series \( \text{(47)} \) made in \([12]\) (see Appendix A.3 there) its divergence on each sheet of its \( s \)-Riemann surface (when \( \xi \) is fixed) is no faster then exponential one.

Appendix 2

We shall show here that the Borel transformation \([10]\) of \( \tilde{\chi}(x,s) \) (as given by \([8]\) with \( \chi_{k,n}(x) \) in the latter satisfying the recurrent relations \([21]\) (in its differential form)) along any standard path satisfies the differential equation \((19)\).

To this end write \((21)\) in its differential form:

\[
\chi'_{n+1}(x) = q^{-\frac{1}{4}}(x) \left( q^{-\frac{1}{4}}(x) \chi_n(x) \right)' , \quad n \geq 0
\]  

(50)

Next multiply both the sides of \((50)\) by \((-s)^n/n!\) and sum them over \( n \geq 0 \) to get:

\[
\frac{\partial^2 \tilde{\chi}(x,s)}{\partial s \partial x} + q^{-\frac{1}{4}}(x) \frac{\partial^2}{\partial x^2} \left( q^{-\frac{1}{4}}(x) \tilde{\chi}(x,s) \right) = 0
\]  

(51)

Finally, multiply \((51)\) by \( 2\lambda e^{2\sigma \lambda s} \) and integrate (by parts) along a standard/cut path \( C \) to have:

\[
2\sigma \lambda \left( 2\lambda \int_C dse^{2\sigma \lambda s} \tilde{\chi}(x,s) \right)' + q^{-\frac{1}{4}}(x) \left( q^{-\frac{1}{4}}(x) 2\lambda \int_C dse^{2\sigma \lambda s} \tilde{\chi}(x,s) \right)'' = 0
\]

\[
\lambda > 0 , \quad \sigma = \begin{cases} +1 & \text{for infinity of } \Re C < 0 \\ -1 & \text{for infinity of } \Re C > 0 \end{cases}
\]  

(52)

According to \((10)\) the equation \((52)\) coincides with \((19)\).

Appendix 3

We shall show here that if \( x \) is sufficiently close to \( x_0 \) then the Borel function of the quotient of \( \chi^{as}(x,\lambda) \) and \( \chi^{as}(x_0,\lambda) \) (with its factors corresponding to \( \chi(x,\lambda) \) and \( \chi(x_0,\lambda) \) respectively) can be integrated along the same standard path \( \tilde{C} \) along which both the factors of the quotient can be summed too. It means that all the three Borel functions, the quotient and its two factors, are holomorphic in a common strip containing \( \tilde{C} \).

To show this let us note that it is certainly true for the Borel functions \( \tilde{\chi}(x,s) \) and \( \tilde{\chi}(x_0,s) \) of the two quotient factors considered separately from the Borel function of the quotient itself. This is the result of the analytical dependence on \( x \) of singularities of the Borel functions of both these factors \([12]\). Therefore there is a strip \( \tilde{S} \) on the Borel planes of \( \tilde{\chi}(x,s) \) and \( \tilde{\chi}(x_0,s) \) containing a standard path \( \tilde{C} \) along which these functions can be integrated to reproduce the corresponding \( \chi \)-factors \( \chi(x,\lambda) \) and \( \chi(x_0,\lambda) \). It is now elementary to show that if \( \tilde{\chi}(x_0,s) \) is holomorphic in \( \tilde{S} \) then the Borel function of \( \chi^{-1}(x_0,\lambda) \) is also. This latter conclusion follows from the semiclassical expansion of \( \chi^{-1}(x_0,\lambda) \). Namely, we have for this expansion

\[
\left( \frac{1}{\chi(x_0,\lambda)} \right)^{as} = \frac{1}{\chi^{as}(x_0,\lambda)} = \sum_{n \geq 0} \frac{1}{C_0^{n+1}} (C_0 - \chi^{as}(x_0,\lambda))^n
\]  

(53)
where for \( \chi^{as}(x_0, \lambda) \) we have assumed

\[
\chi^{as}(x_0, \lambda) = \sum_{n \geq 0} \frac{C_n}{(2\lambda)^n} \tag{54}
\]

The expansion (53) follows of course from the identical (in form) expansion of \( \chi^{-1}(x_0, \lambda) \) itself valid for \( |\arg| \leq \pi/2 \) when \( \lambda \) is sufficiently large.

The Borel function corresponding to the expansion (53) is therefore

\[
\frac{1}{\tilde{\chi}(x, \lambda)} = C_0 + \sum_{n \geq 1} \frac{(C_0 - \tilde{\chi}(x, \lambda))^n}{C_0^{n+1}} = C_0 + \sum_{n \geq 1} \frac{(C_0 - \tilde{\chi})^n(x, s)}{C_0^{n+1}}
\]

\[
= C_0 + \sum_{n \geq 1} \frac{(-1)^{n+1}}{C_0^{n+1}} \int_0^s ds_1 (C_0 - \tilde{\chi}(x, s - s_1)) \int_0^{s_1} ds_2 \tilde{\chi}'(x, s_1 - s_2) \ldots \tag{55}
\]

\[
\ldots \int_0^{s_{n-3}} ds_{n-2} \tilde{\chi}'(x, s_{n-3} - s_{n-2}) \int_0^{s_{n-2}} ds_{n-1} \tilde{\chi}'(x, s_{n-2} - s_{n-1})
\]

where the prime at \( \tilde{\chi}'(x, s) \) means the differentiation over \( s \) and where the following definition of the star (convolution) operation has been used

\[
(\tilde{f} \ast \tilde{g})(s) = \frac{d}{ds} \int_0^s \tilde{f}(s')\tilde{g}(s - s')ds' \tag{56}
\]

From the representation (56) it follows easily that the strip \( \tilde{S} \) of the holomorphicity of \( \tilde{\chi}(x, s) \) is also such a strip for the Borel function of \( \chi^{-1}(x, \lambda) \) since the series in (55) is uniformly convergent in \( \tilde{S} \).

References

[1] Giller S., *Acta Phys. Pol.* B23 457-511 (1992)
[2] Bender C.M. and Wu T.T., *Phys.Rev.* D7 1620 (1973)
[3] Yaffe L.G., *Rev. Mod. Phys.* 54 407 (1982)
[4] Le Guillou J.-C., Zinn-Justin J. Eds. *Current Physics Sources and Comments* Vol.7: Large-Order Behaviour of Perturbation Theory (Amsterdam: North - Holland 1990)
[5] Giller S., *J. Phys. A: Math. Gen.* 22 2965 (1989)
  Giller S., *Acta Phys. Pol.* B21 675-709 (1990)
[6] Landau L. D., Lifshitz E. M., *Quantum Mechanics. Nonrelativistic Theory* (Oxford, New York: Pergamon Press 1965)
[7] Dirac P.A.M., *The Principle of Quantum Mechanics*, Oxford: Claredon Press 1958
[8] Gutzwiller M. C. *Chaos in Classical and Quantum Mechanics* (New York: Springer 1990)
[9] Schulman L.S. "Techniques and Applications of Path Integration" (New York: John Wiley 1981)

[10] Berry M.V. and Howls C.J., Proc. R. Soc. Lond. A430 653-667 (1990) Proc. R. Soc. Lond. A434 (1991) 657-675

[11] Daalhuis Olde A.B. Proc. R. Soc. Lond. A445 1-29 (1998)

[12] Giller S., J. Phys. A: Math. Gen. 22 1543-1580 (2000)

[13] Delabaere E., Dillinger H. and Pham F., J. Math. Phys. 38 6126-6184 (1997)

[14] Delabaere E. and Pham F., Ann. Inst. H. Poincare Phys. Theor. 71 1-94 (1999)

[15] Ecalle J., "Cinq application des fonctions resurgentes" Publ. Math. D'Orsay, Universite Paris-Sud, 84T 62, Orsay
"Weighted products and parametric resurgence", Analyse Algebrique des Perturbations Singulieres I: Methodes Resurgentes, Travaux en Course (Hermann, Paris, 1994) pp. 7-49

[16] Sternin B. and Shatalov V., "Borel-Laplace Transform and Asymptotic Theory", (CRC Press, Boca Raton, FL, 1996)

[17] Dingle R.B., Asymptotic expansions: their derivation and interpretation Academic Press: New York and London, 1973

[18] Joye A., Pfister C.-H., J. Math. Phys. 34 454-479 (1993)

[19] Hagedorn G.A., Joye A., Ann. Inst. Henri Poincare. Physique Theorique 68 85-134 (1998)

[20] Combes J.-M., Hislop P.D., Commun. Math. Phys. 140 331-320 (1991)

[21] Bentosela F., Grecchi V., Commun. Math. Phys. 142 169-192 (1991)

[22] Simon B., Dicke A. Ann. Phys. (N.Y.) 58 76 (1970)

[23] Loeffel J. J., Martin A., Simon B., Wightman A.S. Phys. Lett. 30B 656 (1969)

[24] Voros A., Ann Inst. Henri Poincare A 39 211 (1983)

[25] Giller S. and Milczarski P., J. Phys. A: Math. Gen. 32 955-976 (1999)

[26] Fröman N. and Fröman P.O., JWKB Approximation. Contribution to the Theory, North-Holland, Amsterdam 1965

[27] Watson G.N., Philos. Trans. Soc. London Ser.A 211 (1912) 279 Sokal A.D., J. Math. Phys. 21 (1980) 261-263

[28] Berry M. V., Proc. Roy. Soc. Lond. A422 7 (1988)

[29] Berry M. V., Proc. Roy. Soc. Lond. A427 241 (1990)

[30] Silverstone H., J. Phys. Rev. Lett. 55 2523-2526 (1985)

[31] Fröman N. and Fröman P.O., J. Math. Phys. 39 4417-4429 (1998)
[32] Giller S. and Milczarski P., in preparation

[33] Giller S., *J. Phys. A: Math. Gen.* **21** 909 (1988)

[34] Bateman H., *Higher Transcendental Functions*, McGraw-Hill, New York, 1953

[35] Eden R.J., Landshoff P.V., Olive D.I. and Polkinghorn J.C., *The Analytic S-Matrix*, CUP, Cambridge 1966

[36] Giller S., Gonera C., *Fundamental solution method applied to time evolution of two energy level systems: exact and adiabatic limit results*, lanl arXiv: quant-ph/0009108