Neutrino masses and mixing in $A_5$ with flavour antisymmetry

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Abstract

We discuss consequences of assuming that the (Majorana) neutrino mass matrix $M_ν$ and the charged lepton mass matrix $M_l$ satisfy, $S_ν^T M_ν S_ν = -M_ν$, $T_l^† M_l T_l = M_l M_l^†$ with respect to some discrete groups $S_ν, T_l$ contained in $A_5$. These assumptions lead to a neutrino mass spectrum with two degenerate and one massless neutrino and also constrain mixing among them. We derive possible mixing patterns following from the choices $S_ν = Z_2, Z_2 \times Z_2$ and $T_l = Z_2, Z_2 \times Z_2, Z_3, Z_5$ as subgroups of $A_5$. One predicts the maximal atmospheric neutrino mixing angle $θ_{23}$ and $μ-τ$ reflection symmetry in large number of cases but it is also possible to obtain non-maximal values for $θ_{23}$. Only the third column of the neutrino mixing matrix can be obtained at the leading order due to degeneracy in masses of two of the neutrinos. We take up a specific example within $A_5$ group and identify Higgs vacuum expectations values which realize the above assumptions. Non-leading terms present in this example are shown to lead to splitting among degenerate pairs and a consistent description of both neutrino masses and mixing angles.

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I. INTRODUCTION

Two decades of neutrino oscillation experiments have determined five of the key parameters describing oscillations of three neutrinos. These are three mixing angles $\theta_{ij}$, $(i, j = 1, 2, 3; i < j)$ and two (mass)$^2$ differences $\Delta_{\odot}$ and $\Delta_{A}$ controlling the oscillations of the solar and the atmospheric neutrinos respectively. Overall neutrino mass scale and three CP violating phases still remain to be determined. There already exists hint that the CP phase $\delta$ may be nearly maximal.

Theoretical frameworks describing neutrino masses and mixing angles try to understand the values of the observed parameters and aim to predict the unknown ones. Flavour symmetries provide concrete framework to do this. A systematic approach based on flavour symmetries has evolved in last several years, see reviews [1–5] and references therein. This is based on the observation that patterns of neutrino masses and mixing is intimately linked to the residual symmetries of the neutrino and the charged lepton mass matrices [6–8]. These residual symmetries of mass matrices can be related to the full symmetry $G_f$ of the underlying theory by assuming that former symmetries are contained in $G_f$. This provides a direct link between the group theoretical structure of $G_f$ and the observed mixing angles. This approach has been used to predict various mixing patterns consistent with observations in large number of cases with many different discrete symmetry groups $G_f$ [1–5, 9].

The above approach is also generalized to link both the mass and the mixing patterns of neutrinos to some underlying symmetries. Three possible neutrino mass patterns provide a good zeroth order approximation to the observed neutrino mass spectrum, fully degenerate spectrum, quasi degenerate spectrum with two degenerate neutrinos and a spectrum with two massive and one massless neutrinos. A systematic procedure is evolved to relate these patterns to underlying discrete symmetries. A general analysis is presented for three classes of groups, the discrete von-Dyck groups in case of the degenerate and quasi degenerate spectrum [10], all possible discrete subgroups of $SU(3)$ having 3 dimensional irreducible representation in case of the quasi degenerate neutrinos [11] and a large class of discrete subgroups of $U(3)$ in case of one massless neutrino [12–14].

The basic assumption in the above approaches is that the underlying theory is invariant under some discrete group $G_f$ but the Higgs vacuum expectation value (vev) determining neutrino mass matrix $M_\nu$ and the Hermitian combination of the charged lepton mass matrix $M_lM_l^\dagger$ remain invariant under smaller subgroups $G_\nu$ and $G_l$ of $G_f$. The structure of these groups and their embedding in $G_f$ is sufficient for the determination of mixing patterns without the knowledge of the detailed dynamics. A different dynamical possibility was studied in [15]. Here it is assumed that the Higgs vacuum expectation values breaking flavour group $G_f$ lead to a neutrino mass matrix which displays antisymmetry. Specifically, $M_\nu$ satisfies

$$S_\nu^T M_\nu S_\nu = -M_\nu$$

(1)

for some subgroups $S_\nu$ of $G_f$. This assumption was shown [15] to constrain not just the mixing angles but also the neutrino mass spectrum which could be determined purely from
the group theoretical arguments. Detailed mixing and mass patterns allowed within the
discrete subgroups $\Delta(3N^2)$ and $\Delta(6N^2)$ and a specific dynamical realization of the basic
idea in case of the group $A_4 \equiv \Delta(3.2^2)$ was discussed in [15]. Also it was shown in a specific
example that the antisymmetry of the mass matrix can arise from the minimization of some
suitable potential. Here we pursue this idea further and apply it to the symmetry group
$A_5$. We discuss mass patterns and all the mixing patterns possible within $A_5$ using the idea
of flavour antisymmetry of neutrino mass matrix. $A_5$ has been used in the past [16–19]
to predict the neutrino mixing patterns assuming flavour symmetry. The mixing patterns
predicted here are quite different compared to these cases.

Detailed analysis of $A_5$ also becomes interesting from a related point of view. It was
shown [20] that all the discrete subgroups of $O(3)$ can lead to universal prediction $\theta_{23} = \frac{\pi}{4}$
and $|\delta| = \frac{\pi}{2}$ when $G_r$ is chosen as $Z_2 \times Z_2$ or $Z_m$ and $G_l$ is chosen as $Z_n$, $m, n \geq 3$. As
we will see, the same predictions also follow when neutrino mass matrix possesses residual
antisymmetry instead of symmetry.

We review in the next section some of the properties of the group $A_5$ relevant for our study.
We introduce the idea of flavour antisymmetry in section 3 and discuss its consequences.
Section 4 is devoted to a detailed discussion of various mixing patterns possible within
the group $A_5$ under the assumption of the flavour antisymmetry. Section 5 discusses explicit
realization of the ideas discussed in the previous section. The last section summarizes the
findings.

II. $A_5$ AND ITS ABELIAN SUBGROUPS

Group theory of $A_5$ is discussed in several papers [16–19, 21]. We summarize here the
features which we require for subsequent analysis. The $A_5$ group has sixty elements and five
conjugacy classes. The group can be represented in terms of three generators $E, f_1, H$,

$$H = 1/2 \begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix}; \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad f_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad (2)$$

with $\mu_{\pm} = 1/2(-1 \pm \sqrt{5})$ which provide a faithful 3 dimensional irreducible representation.
The above equation defines the basis of the representation labeled as $3_1$ and we will refer to
this basis as symmetry basis. Multiple products of these generate all the sixty elements of
$A_5$. It is convenient for our purpose to discuss these elements in terms of the $Z_n$ subgroups
they form. We list them and their required properties below.

- $Z_2$: 15 $Z_2$ subgroups of $A_5$ are generated by the elements:

$$O_2 \equiv (f_a, H, f_aHf_a, EH E^{-1}, E^{-1} HE, Ef_aHf_aE^{-1}, E^{-1} f_aHf_aE) , \quad (3)$$

where $a = 1, 2, 3$, $f_2 = E^2f_1E$, $f_3 = E^2f_2E$ and $f_1$ is given by eq.(2). One also
needs the matrices which diagonalize the elements in $O_2$ when $Z_2$ is used as a residual
symmetry. These get determined by a matrix $V_H$ which diagonalizes $H$. Let $V_H$ be such matrix then

$$V_H^\dagger H V_H = \text{diag}(1, -1, -1) .$$

(4)

Explicitly,

$$V_H = \begin{pmatrix}
\frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\
\frac{\mu}{2} & \frac{\mu}{2\sqrt{3}} & \frac{\mu}{\sqrt{3}} \\
\frac{\mu}{2} & \frac{-\mu}{2\sqrt{3}} & \frac{-\mu}{\sqrt{3}}
\end{pmatrix}$$

(5)

The above $V_H$ is arbitrary up to a unitary rotation in the 23 plane. We shall use the above explicit form for the subsequent analysis. We can express all the elements of $A_5$ in the form $QPQ^{-1}$. This simplifies their diagonalization since $U_{QPQ^{-1}} = QU_P$ where, $U_g$ diagonalizes the element $g$. Using this, the matrices diagonalizing all the 15 elements in $O_2$ can be expressed in terms of $V_H$ and are given by the following set

$$U_2 \equiv (I, V_H, f_a V_H, E V_H, E^{-1} V_H, E f_a V_H, E^{-1} f_a V_H) .$$

(6)

Respective entries of this set correspond to matrices which diagonalize the corresponding elements of $O_2$.

- $Z_2 \times Z_2$: Not all the fifteen elements in $O_2$ commute among themselves. But one can find five sets of three commuting elements among $O_2$. These three along with identity form a $Z_2 \times Z_2$ subgroup of $A_5$. These subgroups are listed in Table I. Since $S_1$ and $S_2$

| $S_1$ | $S_2$ | $S_3$ | $U_c$ |
|-------|-------|-------|-------|
| $f_1$ | $f_3$ | $f_2$ | $I$   |
| $H$   | $E^{-1} f_2 H f_2 E$ | $E f_3 H f_3 E^{-1}$ | $V_H R_{\mu}$ |
| $f_1 H f_1$ | $E^{-1} f_1 H f_1 E$ | $E f_1 H f_1 E^{-1}$ | $f_1 V_H R_{\mu}$ |
| $f_2 H f_2$ | $E^{-1} f_3 H f_3 E$ | $E H E^{-1}$ | $f_2 V_H R_{\mu}$ |
| $f_3 H f_3$ | $E^{-1} H E$ | $E f_2 H f_2 E^{-1}$ | $f_3 V_H R_{\mu}$ |

TABLE I. Elements of the five $Z_2 \times Z_2$ subgroups of $A_5$ along with their combined diagonalizing matrices $U_c$ defined in the text. $S_1, S_2, S_3$ together with identity form a $Z_2 \times Z_2$ subgroup of $A_5$. In the table commute, they can be simultaneously diagonalized by a matrix $U_c$. We shall define $U_c$ as

$$U_c^\dagger S_1 U_c = f_1 = \text{diag}(1, -1, -1) ,$$

$$U_c^\dagger S_2 U_c = f_3 = \text{diag}(-1, -1, 1) .$$

(7)

The same matrix $U_c$ also puts $S_3 = S_1 S_2$ into a diagonal form $f_2$. As before, the matrix $U_c$ can also be expressed in terms of $V_H$ diagonalizing $H$ and a real rotation
$R_{\mu}$ in the 23 plane

$$R_{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sin \theta_\mu & \cos \theta_\mu \\ 0 & \cos \theta_\mu & \sin \theta_\mu \end{pmatrix},$$

where

$$\tan \theta_\mu = \mu_- - 1.$$ 

$U_c$ for all five subgroups are given in Table I.

• $Z_3$ subgroups: The 20 elements generating $Z_3$ subgroups of $A_5$ are given by the set

$$O_3 = (E^m, f_aE^m f_a, A^m, E A^m E^{-1}, E^{-1} A^m E, A E^{-1} E^{-1} A, A f_{2,3} E^m f_{2,3} A^{-1}) \text{.}$$

where $m = 1, 2, a = 1, 2, 3$ and the matrix $A \equiv H f_1$. The matrices diagonalizing these elements can be expressed in terms of the matrices $U_\omega$ and $U_A$ which diagonalize $E$ and $A$ respectively:

$$U_3 = (U_\omega, f_a U_\omega, U_A, E U_A, E^{-1} U_A, A U_\omega, A f_{2,3} U_\omega).$$

$U_\omega$ is given by

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$\omega = e^{\frac{2\pi i}{3}} \text{ and } U_A \text{ can be found in the Appendix of the reference [20].}$$

• $Z_5$ subgroups: There are 24 different $Z_5$ subgroups within $A_5$. Their generating elements can be expressed in terms of $T \equiv f_1 E H$, $E$ and $f_{1,2,3}$ as follows:

$$O_5 = (T^p, f_2 T^p f_2, E T^p E^{-1}, E^{-1} T^p E, E f_2 T^p f_2 E^{-1}, E^{-1} f_2 T^p f_2 E),$$

where $p = 1, 2, 3, 4$. This set is diagonalized by

$$U_5 = (U_T, f_2 U_T, E U_T, E^{-1} U_T, E f_2 U_T, E^{-1} f_2 U_T),$$

where $U_T$ is a matrix diagonalizing $T$. Its explicit form is given in the Appendix of [20].

The elements in the sets $O_{2,3,5}$ along with the identity constitute all the sixty elements of $A_5$. We note that all the matrices diagonalizing set $O_3$ and $O_5$ possess the following general form as explicitly shown in [20].

$$U = \begin{pmatrix} x_1 & z_1 & z_1^* \\ x_2 & z_2 & z_2^* \\ x_3 & z_3 & z_3^* \end{pmatrix},$$

where $x_1, x_2, x_3$ are real. We shall use this form to derive properties of the mixing matrix in the following.
III. FLAVOUR ANTISYMMETRY AND NEUTRINO MASS TEXTURES

We first briefly review the implications of the assumption of the flavour antisymmetry [15] represented by eq. (1) where $S_\nu$ is assumed to be any $3 \times 3$ matrix belonging to a discrete subgroup of $SU(3)$. The very assumption of flavour antisymmetry implies that (at least) one of the neutrinos remains massless. This simply follows by taking the determinant of eq. (1) and noting that $\text{Det}(S_\nu) = 1$. Other implications of eq. (1) become clear in a basis with diagonal $S_\nu$. Let $S_\nu$ be diagonalized by a unitary matrix $V_{S_\nu}$ as:

$$V_{S_\nu}^\dagger S_\nu V_{S_\nu} = D_S \equiv \text{diag} (\lambda_1, \lambda_2, \lambda_3) ,$$

with $\lambda_1 \lambda_2 \lambda_3 = 1$. Unitarity of $S_\nu$ implies that $\lambda_1, \lambda_2, \lambda_3$ are some roots of unity. It was argued [15] that only two possible forms of $D_S$ can lead to a neutrino mass matrix with two massive neutrinos. These are given by

$$D_{1S} = \text{diag} (\lambda, -\lambda^*, -1) ,$$
$$D_{2S} = \text{diag} (\pm i, \mp i, 1) ,$$

and their permutations. $\lambda^{2p} = 1$ for some integer $p$. The group generated by the residual symmetry $S_\nu$ having diagonal form $D_{1S}$ ($D_{2S}$) is $Z_{2p}(Z_4)$. Define

$$\tilde{M}_\nu = V_{S_\nu}^\dagger M_\nu V_{S_\nu} .$$

Then the allowed textures of $\tilde{M}_\nu$ get determined by the allowed forms of $D_S$. There exists only four allowed textures for $\tilde{M}_\nu$ which correspond to one massless and a degenerate or non-degenerate pair of neutrinos. If $\lambda = 1$ then the relevant texture is given by:

$$\tilde{M}_\nu = m_0 \begin{pmatrix} 0 & c_\nu & s_\nu e^{i\beta_\nu} \\ c_\nu & 0 & 0 \\ s_\nu e^{i\beta_\nu} & 0 & 0 \end{pmatrix} ,$$

where $c_\nu = \cos \theta_\nu$, $s_\nu = \sin \theta_\nu$. $\tilde{M}_\nu$ describes a massless and a degenerate pair of neutrinos. Other three textures are possible for other values of $\lambda$ but as we shall see only the case given in eq.(18) can get realized in $A_5$.

A. The allowed residual symmetries in $A_5$

We now discuss possible residual symmetries of the leptonic mass matrices within $A_5$ and the resulting mixing patterns. The choices of residual antisymmetry of $M_\nu$ within $A_5$ are restricted. These can be obtained simply from the characters $\chi$ of all the sixty elements. $\chi$ is real for all the elements. In this case, the eigenvalues of any element are given by

$$\left(1, \frac{1}{2} \left( \chi - 1 + \sqrt{(\chi - 1)^2 - 4} \right), \frac{1}{2} \left( \chi - 1 - \sqrt{(\chi - 1)^2 - 4} \right) \right) .$$

Note that eq.(1) requires that $S_{2p}^\dagger S_{2p} = 1$ if $M_\nu$ is not identically zero and $S_\nu$ has finite order. This translates to $\lambda^{2p} = 1$. 

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These eigenvalues must have the form displayed in one of the two equations given in (16) in order for an element with character $\chi$ to be able to be a viable antisymmetry operator. Elements belonging to $Z_3$ and $Z_5$ subgroups have $\chi = 0$ and $(-\mu_+, -\mu_-)$. Their eigenvalues following from above do not have these forms. Thus only viable choice for the antisymmetry operator $S_\nu$ can be any element in the set $O_2$ having character -1 and eigenvalues $(1, -1, -1)$. We shall require that at least one of the symmetries of $M_\nu$ acts according to eq.(1). We will thus consider two possible choices of the residual neutrino symmetries (1) $S_\nu = Z_2$ as antisymmetry and (2) $S_\nu = Z_2 \times Z_2$ where one of the $Z_2$ transforms $M_\nu$ into it’s negative and the other leaves it invariant. In contrast, the eigenvalues of the residual symmetry of $M_\nu M_\nu^T$ is not restricted and we can take any of the $Z_n$ of $A_5$ as the residual symmetry $T_1$. We shall consider the following choices for $T_1$ (a) $(Z_3, Z_5)$ groups generated by $(O_3, O_5)$ (b) five $Z_2 \times Z_2$ subgroups or (c) elements of the $Z_3$ subgroups contained in $O_2$. Possible choices of $S_\nu$ and $T_1$ determine the leptonic mixing matrix.

Elements in $O_2$ when used as antisymmetry operator lead to a unique form for the neutrino mass matrix $\tilde{M}_\nu$ given in eq.(18). This texture describes a pair of degenerate and one massless neutrino. Residual antisymmetry in this case is $Z_2$. The neutrino mass matrix in eq.(18) can be diagonalized by a matrix $V_\nu$:

$$V_\nu^T \tilde{M}_\nu V_\nu = \text{diag.}(m_0, m_0, 0)$$  \hspace{1cm} (20)

where,

$$V_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{e^{-i\beta_\nu}}{\sqrt{2}} & \frac{e^{i\beta_\nu}}{\sqrt{2}} & -s_\nu \\ \frac{e^{-i\beta_\nu}}{\sqrt{2}} & \frac{e^{i\beta_\nu}}{\sqrt{2}} & c_\nu e^{-i\beta_\nu} \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \hspace{1cm} (21)$$

The arbitrary rotation by an angle $\psi$ originates due to degeneracy in masses. It follows from eqs.(17,20) that the matrix $M_\nu$ is diagonalized by the product $V_{S_\nu} V_\nu$. Thus the neutrino mixing matrix with the residual antisymmetry $Z_2$ in the symmetry basis is given by

$$U_\nu^T = V_{S_\nu} V_\nu. \hspace{1cm} (22)$$

Note that the $U_\nu^T$ gets determined by the structure of $S_\nu$ and essentially two unknown angles $\theta_\nu$ and $\beta_\nu$. The unknowns can be fixed if the residual symmetry is chosen as $Z_2 \times Z_2$. Consider the $Z_2 \times Z_2$ groups generated by $S_{1\nu} = S_1$ and $S_{2\nu} = S_2$ where $S_1, S_2$ are as in Table I. They satisfy

$$S_1^T M_\nu S_1 = -M_\nu; \hspace{1cm} S_2^T M_\nu S_2 = M_\nu. \hspace{1cm} (23)$$

As discussed in the previous section, both $S_1$ and $S_2$ are diagonalized by $U_c$ as given in Table I. The structure of the neutrino mass matrix in this case becomes transparent in the basis with diagonal $S_1, S_2$ Let

$$M_\nu' = U_c^T M_\nu U_c. \hspace{1cm} (24)$$

Eq.(23) reduces in the prime basis to

$$f_1^T M_\nu' f_1 = -M_\nu'; \hspace{1cm} f_3^T M_\nu' f_3 = M_\nu'. \hspace{1cm} (25)$$
The first of this equation implies the form (18) for \( M'_\nu \). The second imposed on this then leads to the restriction \( s_\nu = 0, c_\nu = 1 \). The final \( M'_\nu \) is determined by an overall scale \( m_0 \) and is diagonalized by \( U_{12} \equiv R_{12}(\frac{\pi}{4})\text{diag}(1, i, 1) \). It follows from this and eq.(24) that \( M_\nu \) is diagonalized by

\[
U_{\nu}^{\text{II}} = U_\nu U_{12} = U_\nu R_{12}(\frac{\pi}{4})\text{diag}(1, i, 1) .
\] (26)

The matrix \( U_l \) diagonalizing \( M_l M_l^\dagger \) also gets determined by its symmetry. Assume that \( T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger \). This implies that \( T_l \) commutes with \( M_l M_l^\dagger \). Hence the matrix \( U_{T_l} \) diagonalizing \( T_l \) can be taken to be the matrix which diagonalizes \( M_l M_l^\dagger \) also. Three possible choices of \( T_l \) referred as (a),(b),(c) above lead to specific forms of \( U_l^a, U_l^b, U_l^c \):

\[
U_l^a = U_{3,5} ,
\]
\[
U_l^b = U_c ,
\]
\[
U_l^c = U_2 U_{23} .
\] (28)

Here, \( U_{3,5} \) are given by any matrix in the set, \( U_3 \) , eq.(10) and \( U_5 \) , eq.(13) when \( T_l \) belongs to \( O_3 \) or \( O_5 \) respectively. \( U_c \) is given in Table I for \( T_l \) belonging to \( Z_2 \times Z_2 \). There is some arbitrariness in the choice of \( U_l \) when \( T_l \) is chosen as any of the element \( O_2 \) forming a \( Z_2 \). These elements have eigenvalues \( (1, -1, -1) \) and matrix diagonalizing \( T_l \) is arbitrary up to a unitary rotation in the 23 plane. This rotation can be taken without the loss of generality to \( U_{23} \equiv \text{diag}(1, 1, e^{i\beta_l})R_{23}(\theta_l) \). Various combinations of \( U_l^{a,b,c} \) and \( U_{\nu}^{I,II} \) give all possible \( U \equiv U_l^I U_\nu \) in \( A_5 \).

IV. MIXING PATTERNS IN \( A_5 \)

As discussed, all possible structure of the PMNS matrix \( U \) in \( A_5 \) with flavour antisymmetry are given by

\[
U \sim U_l^{a,b,c} U_{\nu}^{I,II} .
\] (29)

Not all of these give viable mixing pattern for neutrinos as we will show. Before discussing individual choices, we first derive a fairly general property of the mixing matrix with flavour antisymmetry. If (a) the neutrino mass matrix shows flavour antisymmetry eq.(1), with \( S_\nu^2 = 1 \) and a real mixing matrix \( V_\nu \) or (b) if it has residual symmetry structure \( Z_2 \times Z_2 \) as in eq.(23) and if the charged lepton matrix \( M_l M_l^\dagger \) is invariant under a residual symmetry \( Z_3 \) or \( Z_5 \) within \( A_5 \) then the mixing matrix can be chosen to have the property

\[
|U_{\mu i}| = |U_{\tau i}| , \ (i = 1, 2, 3) .
\] (30)

This property known as the \( \mu - \tau \) reflection symmetry [22] or generalized \( \mu-\tau \) symmetry was derived [23] using a generalized definition of CP. The same result was derived from more
general assumptions in case of the non-degenerate neutrinos [20, 24] as well as for a pair of degenerate neutrinos [10, 20]. The basic assumption in these cases was the existence of a real residual symmetry. The same result also follows when the symmetry is replaced by antisymmetry as we discuss below.

The equality $|U_{\mu 3}| = |U_{\tau 3}|$ implies the maximal atmospheric mixing angle. The equality $|U_{\mu 2}| = |U_{\tau 2}|$ then leads to the maximal CP phase $|\delta| = \frac{\pi}{2}$ if neutrinos are non-degenerate and $s_{13} \neq 0$. For the degenerate solar pair, the first two columns of $U$ depend on an unknown mixing angle $\psi$ as given in eq. (21). But by considering $\psi$ invariant combination of the observables, it was argued [10] that one instead gets $|\delta - \kappa| = \frac{\pi}{2}$ where $\kappa$ is the Majorana phase.

The proof of eq. (30) is straightforward and follows the proof given in [20] in case of the flavour symmetry. Assume that neutrino mass matrix $\tilde{M}_\nu$ has the structure (18) with $\beta_\nu = 0$. Then it is diagonalized by $U_\nu^I = U_2 V_\nu$. Here $U_2$ belonging to the set $U_2$ is real. For $\beta_\nu = 0$, one therefore gets $U_\nu^I = O_\nu P$, with $O_\nu$ being a real orthogonal matrix and $P = \text{diag}(1, i, 1)$. A similar structure of $U_\nu$ also holds in case II with $Z_2 \times Z_2$ symmetry since in this case, the neutrino mixing matrix $U_\nu$ is given by $U_\nu^{II}$, eq. (26) which also can be written as an orthogonal matrix times a phase matrix because of the reality of $U_c$. The charged lepton mixing matrix on the other hand has a general structure specified by eq. (14) when the residual charge lepton symmetry is either $Z_3$ or $Z_5$. It is then easy to see that $U_l$ as in eq. (14) and $U_\nu$ as $O_\nu$ times a diagonal phase matrix leads to eq. (30). This result does not follow when the residual symmetry of the charged leptons is $Z_2$ or $Z_2 \times Z_2$ since in this case $U_l$ does not have the form given in eq. (14).

Let us now discuss individual choices of residual symmetries and their viability or otherwise. We will work out various mixing patterns for various choices and confront them with the results of the global fits as given in [25–27]. For definiteness, we shall use the results given in [27]. The structures of $U_\nu^{I,II}$ appearing in eq. (29) are determined only up to a rotation in the 12 plane and the solar angle remains undetermined at the leading order. The third column of $U$ is however independent of the unknown angle and can be predicted group theoretically at the zeroeth order. We shall thus concentrate on the prediction of $\theta_{13}$ and $\theta_{23}$ determined by the third column of $|U|$. Also, the ordering of eigenvalues of $T_i$ cannot be determined group theoretically. Change in this ordering permutes the rows of $U$. Thus any of the entries of the third column $|U_{i3}|$ may be identified with the physical mixing elements $|U_{a3}|$ ($\alpha = e, \mu, \tau$). In view of this, we shall consider different orderings which can give viable mixing patterns.

A. $S_\nu = Z_2$ and $T_l = Z_3$ or $Z_5$

There are 15 different choices of $Z_2$ and 20+24 choices of the $Z_3 + Z_5$ symmetry within $A_5$. Specific forms of $U_l$ and $U_\nu$ as discussed before can be used to obtain $|U_{i3}|$ in all these cases. They are determined by the unknown angles $\theta_\nu$ and $\beta_\nu$. While the dependence of $|U_{i3}|$ on these are different for different choices of residual symmetries all the choices share
the following features

- If $\beta_\nu = 0$ then eq.(14) holds for the specific ordering of eigenvectors of $T_1$ as given in eq.(30). The atmospheric mixing angle is predicted to be maximal for all the values of $\theta_\nu$. In this case, $|U_{e3}|$ is to be identified with the 13 element of $|U|$ since $|U_{23}| = |U_{33}|$. In all these cases, $|U_{e3}|$ depends on $\theta_\nu$ which can be chosen to obtain the correct $s_{13}^2$.

- If $\beta_\nu \neq 0$ then any of $|U_{e3}|$ can be identified with $|U_{e3}|$. It is possible in this case to choose two unknowns $\theta_\nu$ and $\beta_\nu$ to obtain correct $\theta_{13}$ and $\theta_{23}$. Let us discuss a specific example with $S_\nu = E^2 f_1 H f_1 E$ and $T_1 = E$ as illustration. They respectively generate $Z_2$ antisymmetry in $M_\nu$ and $Z_3$ symmetry in $M_l M_l^\dagger$. The mixing matrix is given by $U = U_{\nu}^2 E^2 f_1 V_H V_\nu$ with $V_H$ as in eq.(5) and $U_\nu$ as in (11). The third column of the mixing matrix is then given by

$$ |U_{13}|^2 = \frac{1}{9} c_\nu (1 + 2 \mu_+) - s_\nu e^{i \beta_\nu} |^2 ,$$

$$ |U_{23}|^2 = \frac{1}{36} | - 2 c_\nu (1 - \omega \mu_+) + s_\nu e^{i \beta_\nu} (\mu_+ + 3 \omega + \omega^2 \mu_-) |^2 ,$$

$$ |U_{33}|^2 = \frac{1}{36} | - 2 c_\nu (1 - \omega^2 \mu_+) + s_\nu e^{i \beta_\nu} (\mu_+ + 3 \omega^2 + \omega \mu_-) |^2 .$$

(31)

For $\beta_\nu = 0$, one gets $|U_{23}|^2 = |U_{33}|^2$ in accordance with the general result discussed above. In this case, identification of $|U_{13}|^2$ with $|U_{e3}|^2$ leads to the result $\theta_{23} = \frac{\pi}{2}$, $\theta_\nu = 0.959$ then leads to $s_{13}^2 \sim 0.024$. Any of $|U_{13}|^2$ can be identified with $|U_{e3}|^2$ when $\beta_\nu$ is non-zero, e.g. the choice $\beta_\nu = -1.076, \theta_\nu = -0.801$ leads to $|U_{e3}|^2 = (0.444, 0.024, 0.532)$. In this case, $|U_{23}|^2$ plays the role of $|U_{e3}|^2$. This specific ordering in $U$ can be obtained by exchanging the first and the second column of $U_\omega$.

B. $S_\nu = Z_2 \times Z_2$ and $T_1 = Z_3$ or $Z_5$

In this case, $S_\nu$ can be chosen in five different ways corresponding to five different $Z_2 \times Z_2$ subgroups. The corresponding neutrino mixing matrix $U_\nu$ is given by eq.(26). As before $T_1$ can be chosen in 44 different ways with $U_1$ either in $U_5$ or $U_6$. Unlike in the previous case, both $U_\nu$ and $U_1$ get completely fixed group theoretically. This case also predicts the maximal atmospheric mixing angle as already outlined. Possible values of $\theta_{13}$ are also fixed. Explicit evaluation of various cases reveal that in all the cases one either gets $\theta_{13} = 0$ or $s_{13}^2 > 0.1$. The zero value for $\theta_{13}$ occurs for example when $S_1 = H, S_2 = E^2 f_2 H f_2 E$ and $T_1 = f_3 E f_3$. One would require relatively large perturbations in this case to get $\theta_{13}$ within its $3\sigma$ range.

C. $S_\nu = Z_2 \times Z_2$ and $T_1 = Z_2$

This case is characterized by completely determined $U_\nu = U_\nu^{II}$ and $U_1 = U_1^c$ containing two unknowns $\theta_1, \beta_1$. The explicit form of $U_1^c$ is given in eq.(28) while $U_c$ can be any of the
five forms given in Table I. $U_1$ in this case does not have the general form given in eq.(14). As a result, one does not obtain eq.(30) corresponding to the $\mu$-$\tau$ reflection symmetry and the atmospheric mixing angle is not predicted to be maximal. But this case has the following interesting feature. Explicit evaluation of $U = U_1^T U_\nu^{II}$ reveals that one of the entries in the third column of $U$ is independent of the unknown angles $\theta_1, \theta_2$ and can be predicted group theoretically. The third column of the mixing matrix $U$ in this case is given by

$$|U_{13}|^2 = |(U_{1}^T U_{\nu}^{II})_{13}|^2,$$

$$|U_{23}|^2 = |c_1(U_{T_1}^T U_{\nu}^{II})_{23} + s_{1e^{-i\beta}}(U_{T_1}^T U_{\nu}^{II})_{33}|^2,$$

$$|U_{33}|^2 = | - s_{1}(U_{T_1}^T U_{\nu}^{II})_{23} + c_{1}e^{-i\beta}(U_{T_1}^T U_{\nu}^{II})_{33}|^2$$

(32)

where $T_1$ belongs to the set $O_2$ and $U_{T_1} \rightarrow U_2$. We get interesting pattern when we identify $T_1$ with $S_{1\nu} = S_1$ residing in $Z_2 \times Z_2$. There exists five such choices and in all these cases, the mixing matrix $U$ is independent of the explicit form of $U_{T_1}$. One gets from eq.(26) and eq.(28)

$$U = U_{23} R_{\mu} R_{12} (\frac{\pi}{4}) \text{diag} (1, i, 1).$$

The neutrino mass matrix $M_{\nu f} \equiv U_1^T M_\nu U_1$ in the flavour basis has the following form in this case:

$$M_{\nu f} = m_0 \begin{pmatrix}
0 & e^{i\beta} c_\mu s_\mu - c_\mu s_\mu & e^{i\beta} c_\mu s_\mu + s_\mu s_\mu \\
0 & 0 & 0 \\
e^{i\beta} c_\mu s_\mu & 0 & 0
\end{pmatrix},$$

(33)

where $c_\mu = \cos \theta_\mu, s_\mu = \sin \theta_\mu, \ldots$ etc. This form can be obtained by imposing $L_e - L_\mu - L_\tau$ symmetry on $M_{\nu f}$ as has been done in the past. Here, this symmetry arises as an effective symmetry of $M_{\nu f}$ from a very different set of basic symmetries. This symmetry leads to a degenerate pair of neutrinos and vanishing $\theta_{13}$. The atmospheric mixing angle is determined as $\tan^2 \theta_{23} = |e^{i\beta} c_{\mu} s_\mu + s_\mu s_\mu|^2$. Perturbations to this symmetry have been studied in the past [28–30]. It is possible to simultaneously generate the correct solar scale, solar angle and $\theta_{13}$ with suitable but relatively large perturbations. Consider perturbing the zero entries in eq.(33) by,

$$\delta M_{\nu f} = m_0 \begin{pmatrix}
\epsilon_1 & 0 & 0 \\
0 & \epsilon_2 & \epsilon_4 \\
0 & \epsilon_4 & \epsilon_3
\end{pmatrix},$$

(34)

Parameters $|\epsilon|$ are assumed less than the dominant entry of $M_{\nu f}$. We give here one example of perturbations which reproduces the observed spectrum within $3\sigma$:

$$\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} = \{-0.284497, 0.284497, -0.0748816, 0.182915\}$$

(35)

leading to

$$\{\frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2}, s^2_{12}, s^2_{13}, s^2_{23}\} = \{0.0339706, 0.358739, 0.0243674, 0.443736\}.$$
We have taken $\beta_l = 0$ and $\cos(\theta_l - \theta_\mu) \approx -0.69$. The values of parameters required to get above values is quite large and the solar angle is also near to it’s 3$\sigma$ limit. We have verified by randomly varying the parameters over a large range that this is a general feature of this case. Relatively large perturbation to the basic symmetry may come from some soft breaking as discussed for example in [30].

We get a non-zero $|U_{13}|^2$ when $T_l$ is not identified with $S_1$. One could determine these values for different choices of $T_l$. The predicted $|U_{13}|^2$ is found from the explicit evaluation of various cases to take one of the three values $(0.095, 0.25, 0.65)$. Of these, only the last value provides a good leading order prediction. $|U_{13}|^2 \sim 0.65$ can be identified in this case either with $|U_{\mu 3}|^2$ or $|U_{\tau 3}|^2$ as this gives $s_{23}^2$ close to its 3$\sigma$ range $0.38 - 0.64$. [27]. This amounts to reordering of the eigenvectors of $T_l$. An example of this choice is provided by $S_1 = f_3 H f_3, S_2 = E^{-1} HE$ and $T_l = f_1 H f_1$. $|U_{13}|^2$ are given in this case by

$$|U_{13}|^2 = \frac{1}{4} (2 + \mu) \approx 0.654,$$

$$|U_{23}|^2 = \frac{|\mu + i\beta_l| + 2(1 + \mu) s_l e^{-i\beta_l}|^2}{12(2 + \mu)},$$

$$|U_{33}|^2 = \frac{|-\mu + s_l + 2(1 + \mu) c_l e^{-i\beta_l}|^2}{12(2 + \mu)}.$$  \hspace{1cm} (37)

One could identify either the second or the third entries with $s_{13}^2$ and determine $\theta_l$ accordingly, e.g. $\theta_l \sim 1.6488, \beta_l = 0$ leads to $s_{13}^2 \equiv |U_{13}|^2 \sim 0.024$ giving $s_{23}^2 c_{13}^2 \equiv |U_{13}|^2 \sim 0.654$. The resulting $\sin^2 \theta_{23}$ is given by 0.67. Small perturbation to this case can lead to $\theta_{23}$ within 3$\sigma$ range and also split the degeneracy.

\[\text{D. } S_\nu = Z_2 \text{ and } T_l = Z_2 \times Z_2\]

In this case, the $Z_2$ can be generated by any of the fifteen elements in $O_2$ while $T_l$ is generated by $T_{l1} \equiv S_1$ and $T_{l2} \equiv S_2$, where $S_1, S_2$ form any of the five $Z_2 \times Z_2$ subgroups listed in Table I. The PMNS matrix in this case is given by $U = U_{ll}^H U_{S_\nu} V_{\nu}$. Just as in the previous case, the atmospheric mixing angle is not predicted to be maximal but now unlike it, both the angles $s_{13}^2$ and $s_{23}^2$ depend on the unknown parameters $\theta_\nu, \beta_\nu$. Not all the choices of the residual symmetries leads to a viable values of $\theta_{13}, \theta_{23}$ in spite of the presence of the two unknowns. We determine the allowed patterns by fitting $\theta_\nu, \beta_\nu$ to the observed values of $\theta_{13}, \theta_{23}$. This allows us to identify cases which provide viable patterns of the mixing angles. One finds essentially three patterns this way. Examples of the residual symmetries, the patterns and best fit values of $\theta_\nu, \beta_\nu$ in each of these cases are listed below.

$$S_\nu = f_3 H f_3 : \theta_\nu = 1.42417, \beta_\nu = 1.84521, s_{13}^2 = 0.024, s_{23}^2 = 0.455,$$

$$S_\nu = f_2 H f_2 : \theta_\nu = -0.487, \beta_\nu = 0, s_{13}^2 = 0.0244, s_{23}^2 = 0.676 ,$$

$$S_\nu = H : \theta_\nu = -0.6716, \beta_\nu = -1.1620, s_{13}^2 = 0, s_{23}^2 = 0.455.$$ \hspace{1cm} (38)

All the above cases occur for the choice $T_{l1} = H$ and $T_{l2} = E^{-1} f_2 H f_2 E$. Similar results follow for different choices of $Z_2 \times Z_2$ as $T_l$ but with alternative choices of $S_\nu$. The first
case given above reproduces the observed values of the mixing angles $\theta_{13}, \theta_{23}$. The second choice gives a $\theta_{23}$ on the verge of its $3\sigma$ value but correct $s_{13}^2$. Thus small perturbation to this case can lead to a viable pattern. The third choice corresponding to $s_{13}^2 = 0$ would need significant corrections from the perturbations and is analogous to the case already discussed in section (IIIC).

E. $S_\nu = Z_2 \times Z_2$ and $T_l = Z_2 \times Z_2$

In this case, the residual symmetries of neutrinos and the charged leptons correspond to (different) $Z_2 \times Z_2$ groups. Due to the presence of two $Z_2$ groups, there are no undetermined parameters and the mixing angles $\theta_{13}, \theta_{23}$ get predicted group theoretically. Since we have five different $Z_2 \times Z_2$ subgroups, there are twenty different choices which would lead to a non-trivial mixing matrix $U$. None of these correspond to even a good zeroth order values. The predicted third column of $|U|^2$ in all these cases is

$$|U_{3i}|^2 = \begin{pmatrix} 0.0954915 \\ 0.25 \\ 0.654508 \end{pmatrix}. \quad (39)$$

and its permutations. These predictions are quite far from the observed mixing angles.

V. EXPLICIT REALIZATION WITH $A_5 \times Z_3$ SYMMETRY

We now discuss a realization of the above group theoretical discussion choosing a specific examples of $S_\nu$ and $T_l$. We discuss necessary Higgs fields and vacuum structure needed to implement above symmetries. The model presented here is very similar to the one presented in [20]. The main differences being that the neutrino symmetry considered in this reference is replaced by the neutrino antisymmetry. Implementation of antisymmetry needs imposition of an additional discrete symmetry which we choose as $Z_3$. We use supersymmetry as basic ingredient.

Irreducible representations (IR) of $A_5$ are: $1 + 3_1 + 3_2 + 4 + 5$ where $3_1$ and $3_2$ are non-equivalent IR. We assign $l_L, l^c$ to $3_1$ which is explicitly generated by $E, H, f_1$ given in eq.(2). It follows from the product rule

$$3_1 \times 3_1 = (1 + 5)_{symm} + 3_{antisym}$$

that the symmetric neutrino mass matrix can arise from $1 + 5$ and the charged lepton masses can arise from all three IR. The neutrino masses are generated from a 5-plet $\eta_{\nu}$ of flavon. Various fields transform under $Z_3$ as

$$(l_L, \eta_{\nu}) \rightarrow \omega (l_L, \eta_{\nu}), \quad l^c \rightarrow \omega^2 l^c.$$  

The standard Higgs fields $H_u, H_d$ and Higgs triplet $\Delta$ are singlets of $A_5 \times Z_3$. 

13
The charged lepton masses are generated by three additional flavons, a singlet $\eta_{ll}$, a 5-plet $\eta_5$ and a 3-plet $\eta_3$, all transforming trivially under $Z_3$. The relevant superpotential is

$$W_I = \frac{1}{\Lambda} [h_d(l_L H_d^{T})_1 \eta_{ll} + h_5(l_L H_d^{T})_5 \eta_5 + h_3(l_L H_d^{T})_3 \eta_3] .$$

The $Z_3$ symmetry separates the neutrino and the charged lepton sectors and does not allow flavons of one sector to couple to the other sector at the leading order.

We specialize to a particular choice of symmetries already discussed in section (IIIA). This corresponds to $T_1 = E$ and $S_{ll} = E^2 f_y H_f E$. The above $S_{ll}$ can serve as an antisymmetry of the neutrino mass matrix if the 5-plet $\eta_5$ has antisymmetric vacuum expectation value:

$$S_{ll}(5) \langle \eta_5 \rangle_{5v} = - \langle \eta_5 \rangle_{5v} .$$

$S_{ll}(5)$ in eq.(42) corresponds to the five dimensional representation of $S_{ll}$. This representation can be obtained from the basic generators defined as $a, b, c$ in [19] by noting the correspondence $E = b, f_3 = a$ and $H = b c$. This leads to

$$S_{ll}(5) = \left( \begin{array}{cccc} \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{4} \\ \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{4} \end{array} \right)$$

(43)

Antisymmetry of $\langle \eta_5 \rangle_{5v}$ together with $A_5$ symmetry in $W_I$ results in the flavour antisymmetric mass matrix. It is worth noting that unlike in the case of symmetry, eq.(42) breaks the symmetry $S_{ll}$ completely and it does not remain as a residual symmetry. But just as in the case with symmetry, a broken solution given in eq.(42) may also arise from the minimization of suitable superpotential but would need enlargement in the model. This is explicitly demonstrated [15] in a simpler case of the group $A_4$.

Denoting the vev $\langle \eta_5 \rangle$ as $(s_1, s_2, s_3, s_4, s_5)^T$, eq.(42) is solved for

$$s_2 = s_3 - s_1 \, , \, s_4 = \sqrt{2} s_3 - \frac{3s_1}{\sqrt{2}} \, , \, s_5 = -\sqrt{\frac{3}{2}} s_1 .$$

(44)

Inserting this solution in eq.(40), we get the neutrino mass matrix

$$M_{\nu}^0 = m_0 \left( \begin{array}{ccc} \frac{-3+\sqrt{5}+y(1-\sqrt{5})}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{y}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -2\sqrt{5}+y(1+\sqrt{5}) & \frac{y-1}{\sqrt{2}} \\ \frac{y}{\sqrt{2}} & \frac{y-1}{\sqrt{2}} & \frac{3+\sqrt{5}-2y}{2\sqrt{2}} \end{array} \right) .$$

(45)
which satisfies the flavour antisymmetry, eq. (1) with respect to \( S_\nu = E^2 f_1 H f_1 E \). This matrix has only one complex parameter \( y \equiv \frac{a_3}{s_1} \) apart from an overall scale. In particular, \( \tilde{M}_\nu \equiv V^T_{S_\nu} M_\nu V_{S_\nu} \) has the form given in eq. (18) with

\[
\tan \theta_\nu e^{i\beta_\nu} = \frac{1 + \mu_+ \frac{a_3}{s_1}}{(\mu_+ - \mu_-)} + \mu - \frac{a_3}{s_1},
\]

(46)

where \( V_{S_\nu} = E^2 f_1 V_H \) diagonalizes \( S_\nu = E^2 f_1 H f_1 E \). The neutrino mixing matrix is then given by \( U_\nu = E^2 f_1 V_H V_\nu \) with \( V_\nu \) as given in eq. (21) and \( \theta_\nu, \beta_\nu \) given by eq. (46) in terms of \( \frac{a_3}{s_1} \). The charged lepton mixing matrix is analogously determined by the form of \( M_l \) obtained from \( W_l \). \( W_l \) and the residual symmetry \( T_l = E \) coincide with the one already discussed in \([20] \). The \( T_l \) invariant vacuum configuration discussed in \([20] \) leads to the following charged lepton mass matrix

\[
M_l = \begin{pmatrix}
m_0 & m_1 - m_2 & m_1 + m_2 \\
m_1 + m_2 & m_0 & m_1 - m_2 \\
m_1 - m_2 & m_1 + m_2 & m_0
\end{pmatrix},
\]

(47)

where \( m_{0,1,2} \) respectively label the singlet, triplet and 5-plet contributions to \( M_l \). \( M_l M_l^\dagger \) is diagonalized by the matrix (11) which also diagonalizes \( T_l \):

\[
U_\omega^\dagger M_l M_l^\dagger U_\omega = \text{diag.}(m_1^2, m_2^2, m_3^2)
\]

with eigenvalues

\[
\lambda_1^2 = m_0^2 + 4|m_1|^2 + 4m_1 R m_0, \\
\lambda_2^2 = m_0^2 + |m_1|^2 + 3|m_2|^2 + 2\sqrt{3} \text{ Im}(m_1 m_2^*) - 2m_0(m_1 R + \sqrt{3} m_2 I), \\
\lambda_3^2 = m_0^2 + |m_1|^2 + 3|m_2|^2 - 2\sqrt{3} \text{ Im}(m_1 m_2^*) - 2m_0(m_1 R - \sqrt{3} m_2 I)
\]

(48)

Here, \( m_{1R,2R} \) and \( m_{1I,2I} \) respectively denote the real and imaginary parts of \( m_{1,2} \). \( m_0 \) is assumed real without loss of generality.

The identification of eigenvalues \( \lambda_{1,2,3}^2 \) with the physical charged lepton masses \( m_{e,\mu,\tau}^2 \) depends on the choice of parameters \( m_{0,1,2} \). In particular, one can choose these parameters in a way that gives \( \lambda_2^2 = m_e^2, \lambda_1^2 = m_\mu^2 \) and \( \lambda_3^2 = m_\tau^2 \). With this identification,

\[
U_l = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
\omega & 1 & \omega^2 \\
\omega^2 & 1 & \omega
\end{pmatrix}.
\]

(49)

This \( U_l \) together with \( U_\nu = E^2 f_1 V_H V_\nu \) gives the mixing matrix \( U \) which is already worked out in eq. (31). The above form of \( U_l \) leads to the identification \( |U_{23}|^2 = s_{13}^2, |U_{13}|^2 = c_{13}^2 s_{23}^2 \). Values of \( \theta_\nu, \beta_\nu \) giving correct \( s_{13}^2, s_{23}^2 \) were already determined in section IIIA. This translates to the following values of the model parameter \( \frac{a_3}{s_1} \) when eq. (46) is used:

\[
\frac{a_3}{s_1} \approx 0.9979 e^{-0.7181 i}.
\]

(50)
Non-zero neutrino masses remain degenerate at the leading order. They can be split and the solar angle can be determined by perturbations which break antisymmetry at the non-leading order. A simple perturbation can be generated by introducing a singlet flavon \( \eta_{1\nu} \) transforming as \( \eta_{1\nu} \rightarrow \omega^2 \eta_{1\nu} \) under \( Z_3 \). This flavon leads to a non-leading term

\[
\frac{h_{1\nu}}{2\Lambda^2} (I_L \Delta m_{LL})_{1} \eta_{1\nu}^2
\]

in \( W_\nu \). This generates a diagonal perturbation which can be parameterized as

\[
M_\nu = m_0 (\hat{M}_0^\nu + \epsilon I)
\]

with \( \hat{M}_0^\nu \equiv \frac{M_0^\nu}{m_0} \) and \(|\epsilon| << 1\). This simple perturbation is enough to generate the solar splitting without disturbing the zeroeth order values of \( s_{13}^2, s_{23}^2 \) significantly. One could vary \( s_{13} \) around the zeroeth order value determined in eq.(50) and find the region of parameters which fits the data with \(|\epsilon| < 0.1\). This procedure leads to a solution close to the best fit values of all parameters. For example, \( s_{13} = 1.00019 e^{-0.711498i} \) and \( \epsilon = 0.0168241 \) leads to

\[
\sin^2 \theta_{12} = 0.295455, \quad \sin^2 \theta_{13} = 0.0235172, \\
\sin^2 \theta_{23} = 0.449634, \quad \frac{\Delta m^2_{\text{sol}}}{\Delta m^2_{\text{atm}}} = 0.0285398.
\]

\[ (51) \]

VI. SUMMARY

We have studied consequences of an ansatz of flavour antisymmetry in the context of the flavour group \( A_5 \) assuming that \( S_\nu \) in eq.(1) and \( T_l \) in eq.(27) are contained in the group \( A_5 \). These assumptions constraint the mixing pattern which we have determined in various cases. The use of flavour antisymmetry in the context of the \( A_5 \) group necessarily leads to a degenerate pair of neutrinos in addition to a massless one. This is a good zeroeth order prediction. Small perturbations splitting the degeneracy can lead to a viable neutrino masses. The predicted neutrino mass hierarchy is inverted.

We have considered discrete subgroups \( Z_2 \) and \( Z_2 \times Z_2 \) of \( A_5 \) as residual symmetries of \( M_\nu \) and discrete groups \( Z_3, Z_5 \) and \( Z_2 \times Z_2 \) contained in \( A_5 \) as symmetries of \( M_l M_l^\dagger \) and worked out the resulting mixing patterns at the leading order in all the cases. The third column of the mixing matrix and hence the angles \( \theta_{13}, \theta_{23} \) get determined at this order. Various predictions discussed in section III can be summarized as follows:

- It is possible to get a universal prediction of the maximal atmospheric mixing angle with the choice \( S_\nu \) as \( Z_2 \) or \( Z_2 \times Z_2 \) and \( T_l \) as any element in \( Z_3, Z_5 \). For \( S_\nu = Z_2 \), one can also get the correct \( \theta_{13} \) at the leading order while the case of \( S_\nu = Z_2 \times Z_2 \) predicts either \( \theta_{13} = 0 \) or large \( s_{13}^2 \geq 0.1 \).

- The case \( T_l = Z_2 \) and \( S_\nu = Z_2 \times Z_2 \) does not predict maximal \( \theta_{23} \) but can be used to predict one of the entries of the third column. The other entry gets determined by an
unknown angle inherent with the use of the $Z_2$ groups. The viable predictions within this case are either $\theta_{13} = 0$ or $s_{23}c_{13}^2 = 0.65$. The former requires large perturbation and we have presented a typical set of such perturbation which lead to correct description of masses and mixing angle.

- The case $S_\nu = Z_2$ and $T_l = Z_2 \times Z_2$ involves an unknown angle and a phase. Not all possible choices of $S_\nu, T_l$ in this category can lead to correct mixing in spite of the presence of two unknowns. We have identified cases which lead to correct description of the mixing angles $\theta_{13}, \theta_{23}$.

- The case of both $S_\nu$ and $T_l$ belonging to different $Z_2 \times Z_2$ subgroups of $A_5$ is fully predictive without any unknowns. But none of the possible cases within this category lead even to a good zeroeth order prediction.

We have supplemented the group theoretical derivation of the mixing patterns in $A_5$ with a concrete example. We have determined the Higgs content and the required vacuum pattern which realizes one of the viable cases discussed group theoretically. The use of a concrete model also allows a systematic discussion of possible perturbations and we have given an example of a perturbation within the model which can be used to split the degeneracy of neutrinos and which can give the correct descriptions of all mixing angles and masses.

VII. ACKNOWLEDGEMENT

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