Some examples of rigid representations*

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To my mother

Abstract

Consider the Deligne-Simpson problem: give necessary and sufficient conditions for the choice of the conjugacy classes $C_j \subset \text{GL}(n, \mathbb{C})$ (resp. $c_j \subset \text{gl}(n, \mathbb{C})$) so that there exist irreducible $(p + 1)$-tuples of matrices $M_j \in C_j$ (resp. $A_j \in c_j$) satisfying the equality $M_1 \ldots M_{p+1} = I$ (resp. $A_1 + \ldots + A_{p+1} = 0$). The matrices $M_j$ and $A_j$ are interpreted as monodromy operators and as matrices-residua of fuchsian systems on Riemann’s sphere.

We give new examples of existence of such $(p + 1)$-tuples of matrices $M_j$ (resp. $A_j$) which are rigid, i.e. unique up to conjugacy once the classes $C_j$ (resp. $c_j$) are fixed. For rigid representations the sum of the dimensions of the classes $C_j$ (resp. $c_j$) equals $2n^2 - 2$.

1 Fuchsian linear systems and the Deligne-Simpson problem

Consider the fuchsian system (i.e. with logarithmic poles) of $n$ linear differential equations

$$
\frac{dX}{dt} = \left( \sum_{j=1}^{p+1} A_j/(t - a_j) \right) X, \quad t \in \mathbb{C}P^1 = \mathbb{C} \cup \infty
$$

$A_j \in \text{gl}(n, \mathbb{C})$ being its matrices-residua. Assume that it has no pole at infinity, i.e.

$$A_1 + \ldots + A_{p+1} = 0. \quad (1)$$

Fix a base point $a_0 \in S := \mathbb{C}P^1 \setminus \{a_1, \ldots, a_{p+1}\}$ and the value $B \in \text{GL}(n, \mathbb{C})$ of the solution $X$ for $t = a_0$. Every pole $a_j$ defines a conjugacy class $\gamma_j$ in the fundamental group $\pi_1(S, a_0)$. The class $\gamma_j$ is represented by a closed contour consisting of a segment $[a_0, a_j']$ (where the point $a_j'$ is close to $a_j$), of a circumference centered at $a_j$ and of radius $|a_j - a_j'|$ (containing inside no pole of the system other than $a_j$ and circumventing $a_j$ counterclockwise) and of the segment $[a_j', a_0]$. One enumerates the segments so that the index increases when one turns around $a_0$ clockwise.

Hence, $\pi_1(S, a_0)$ admits the presentation

$$\langle \gamma_1, \ldots, \gamma_{p+1} | \gamma_{p+1} \ldots \gamma_1 = e \rangle.$$ 

The monodromy operator $M_j$ defined by the class $\gamma_j$ is the one mapping the solution with initial data $X|_{t=a_0} = B$ onto the value at $a_0$ of its analytic continuation along the contour defining $\gamma_j$ (i.e. $X \mapsto XM_j$). The monodromy operators of the system generate its monodromy group which is an antirepresentation $\pi_1(S, a_0) \rightarrow \text{GL}(n, \mathbb{C})$ because the monodromy operator corresponding to the class $\gamma_i\gamma_j$ equals $M_jM_i$. Thus for the matrices $M_j$ one has

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Remark 1 If there are no non-zero integer differences between the eigenvalues of \( A_j \), then the operator \( M_j \) is conjugate to \( \exp(2\pi i A_j) \).

Remark 2 Fuchsian systems are a particular case of regular systems, i.e. linear systems whose solutions when restricted to sectors centered at the poles \( a_j \) grow no faster than some power of the distance to the pole \( a_j \). Their monodromy groups are defined in the same way.

The Deligne-Simpson problem (DSP) is formulated like this: give necessary and sufficient conditions for the choice of the conjugacy classes \( C_j \subset GL(n, \mathbb{C}) \) (resp. \( c_j \subset gl(n, \mathbb{C}) \)) so that there exist irreducible \((p+1)\)-tuples of matrices \( M_j \in C_j \) satisfying (2) (resp. of matrices \( A_j \in c_j \) satisfying (3)). In the multiplicative version (i.e. for matrices \( M_j \)) it is stated by P. Deligne\(^1\) and C. Simpson was the first to obtain a significant result towards its resolution, see [5].

The problem is formulated without using the notions of fuchsian system and monodromy operator, yet they explain the interest in the problem. The multiplicative version is more important because the monodromy operators are invariant under the changes \( X \mapsto W(t)X \) (where \( W \) depends meromorphically on \( t \) and \( \det W \neq 0 \)) while the matrices-residua are not. In the multiplicative version the problem admits the interpretation: for which \((p+1)\)-tuples of local monodromies \( c_j \) does there exist an irreducible monodromy group with such local monodromies?

The paper is structured as follows. In the next section we recall the basic results announced in [Ko1] and proved in [Ko2]. In Section 3 we define the case of rigid \((p+1)\)-tuples. In Section 4 we give some new examples of existence of rigid \((p+1)\)-tuples of diagonalizable matrices. In Section 5 we describe all rigid \((p+1)\)-tuples of such matrices in which the multiplicities of all eigenvalues of one of the matrices are \( \leq 2 \). In Section 6 we explain how the examples from the previous two sections give rise to other examples in which the matrices are not necessarily diagonalizable.

2 The Deligne-Simpson problem for generic eigenvalues

Definition 3 A Jordan normal form (JNF) of size \( n \) is a family \( J^n = \{ b_{i,l} \} \) \((i \in I_l, I_l = \{1, \ldots, s_l\}, l \in L)\) of positive integers \( b_{i,l} \) whose sum is \( n \). Here \( L \) is the set of indices of the eigenvalues \( \lambda_l \) (all distinct) and \( I_l \) is the set of indices of Jordan blocks with eigenvalue \( \lambda_l \); \( b_{i,l} \) is the size of the \( i \)-th block with this eigenvalue. We assume that for each \( l \) fixed one has \( b_{1,l} \geq \ldots \geq b_{s_l,l} \). An \( n \times n \)-matrix \( Y \) has the JNF \( J^n \) (notation: \( J(Y) = J^n \)) if to its distinct eigenvalues \( \lambda_l, l \in L \), there belong Jordan blocks of sizes \( b_{i,l} \).

In what follows we presume the necessary condition \( \prod \det(C_j) = 1 \) (resp. \( \sum \text{Tr}(c_j) = 0 \)) to hold. In terms of the eigenvalues \( \sigma_{k,j} \) (resp. \( \lambda_{k,j} \)) of the matrices from \( C_j \) (resp. \( c_j \)) repeated with their multiplicities, this condition reads

\[
\prod_{k=1}^{n} \prod_{j=1}^{p+1} \sigma_{k,j} = 1, \quad \text{resp.} \quad \sum_{k=1}^{n} \sum_{j=1}^{p+1} \lambda_{k,j} = 0.
\]

\(^1\)It seems that the author of the present paper was the first to state the problem in the additive version
**Definition 4** An equality of the form
\[ \prod_{j=1}^{p+1} \prod_{k \in \Phi_j} \sigma_{k,j} = 1, \quad \text{resp.} \quad \sum_{j=1}^{p+1} \sum_{k \in \Phi_j} \lambda_{k,j} = 0, \]
is called a non-genericity relation; the sets \( \Phi_j \) contain one and the same number \( k \) \((1 < k < n)\) of indices for all \( j \). Eigenvalues satisfying none of these relations are called generic. Reducible \((p+1)\)-tuples exist only for non-generic eigenvalues. Indeed, a reducible \((p+1)\)-tuple can be conjugated to a block upper-triangular form and the eigenvalues of the restriction of the \((p+1)\)-tuple to each diagonal block must satisfy condition \([2]\) or \([1]\) which is a non-genericity relation.

For a conjugacy class \( C \) in \( GL(n, C) \) or \( gl(n, C) \) denote by \( d(C) \) its dimension. Remind that \( d(C) \) is always even. For a matrix \( Y \) from \( C \) set \( r(C) := \min_{\lambda \in \mathbb{C}} \text{rank}(Y - \lambda I) \). The integer \( n - r(C) \) equals the maximal number of Jordan blocks of \( J(Y) \) with one and the same eigenvalue.

Set \( d_j := d(C_j) \) (resp. \( d(c_j) \)), \( r_j := r(C_j) \) (resp. \( r(c_j) \)). The quantities \( r(C) \) and \( d(C) \) depend only on the JNF \( J(Y) = J^n \), not on the eigenvalues and we write sometimes \( r(J^n) \) and \( d(J^n) \).

**Proposition 5** (C. Simpson, see [8].) The following couple of inequalities is a necessary condition for the existence of irreducible \((p+1)\)-tuples of matrices \( M_j \) satisfying \([2]\):
\[ d_1 + \ldots + d_{p+1} \geq 2n^2 - 2 \quad (\alpha_n) \]
\[ \text{for all } j, \quad r_1 + \ldots + r_j + \ldots + r_{p+1} \geq n \quad (\beta_n). \]

**Definition 6** Denote by \( \{ J^n_j \} \) a \((p+1)\)-tuple of JNFs, \( j = 1, \ldots, p + 1 \). We say that the DSP is solvable (resp. is weakly solvable) for a given \( \{ J^n_j \} \) and given eigenvalues if there exists an irreducible \((p+1)\)-tuple (resp. a \((p+1)\)-tuple with a trivial centralizer) of matrices \( M_j \) satisfying \([2]\) or of matrices \( A_j \) satisfying \([1]\), with \( J(M_j) = J^n_j \) or \( J(A_j) = J^n_j \) and with the given eigenvalues. By definition, the DSP is solvable and weakly solvable for \( n = 1 \).

**Theorem 7** The DSP is solvable for conjugacy classes \( C_j \) or \( c_j \) with generic eigenvalues and satisfying the condition
\[ (r_1 + \ldots + r_{p+1}) \geq 2n \quad (\omega_n). \]

For a given \( \{ J^n_j \} \) with \( n > 1 \), which satisfies conditions \((\alpha_n)\) and \((\beta_n)\) and doesn’t satisfy condition \((\omega_n)\) set \( n_1 = r_1 + \ldots + r_{p+1} - n \). Hence, \( n_1 < n \) and \( n - n_1 \leq n - r_j \). Define the \((p+1)\)-tuple \( \{ J^{n_1}_j \} \) as follows: to obtain the JNF \( J^{n_1}_j \) from \( J^n_j \) one finds one of the eigenvalues of \( J^n_j \) with greatest number \( n - r_j \) of Jordan blocks, then decreases by 1 the sizes of the \( n - n_1 \) smallest Jordan blocks with this eigenvalue and deletes the Jordan blocks of size 0. We use the notation \( \Psi : \{ J^n_j \} \mapsto \{ J^{n_1}_j \} \).

**Theorem 8** Let \( n > 1 \). The DSP is solvable for the conjugacy classes \( C_j \) or \( c_j \) (with generic eigenvalues, defining the JNFs \( J^n_j \) and satisfying conditions \((\alpha_n)\) and \((\beta_n)\)) if and only if either \( \{ J^n_j \} \) satisfies condition \((\omega_n)\) or the construction \( \Psi : \{ J^n_j \} \mapsto \{ J^{n_1}_j \} \) iterated as long as it is defined stops at a \((p+1)\)-tuple \( \{ J^{n'}_j \} \) either with \( n' = 1 \) or satisfying condition \((\omega_{n'}).\)
In the case of diagonalizable matrices $M_j$ or $A_j$ the JNF of $M_j$ or $A_j$ is completely defined by the multiplicity vector (MV) of its eigenvalues. This is a vector $\Lambda^n_j$ with positive integer components equal to the multiplicities of the eigenvalues of $M_j$ or $A_j$; hence, their sum is $n$.

**Remark 9** Set $\Lambda^n_j = (m_{1,j}, \ldots, m_{i,j})$ where $m_{1,j} \geq \ldots \geq m_{i,j}$. Hence, one has $r_j = m_{2,j} + \ldots + m_{i,j}$ and $d_j = n^2 - \sum_{i=1}^j (m_{i,j})^2$. In particular, the MV with greatest value of $d_j$ is $(1, \ldots, 1)$, with $d_j = n^2 - n$.

Call polymultiplicity vector (PMV) the $(p + 1)$-tuple of MVs $\Lambda^n := (\Lambda^n_1, \ldots, \Lambda^n_p)$. 

**Remark 10** In the particular case of diagonalizable matrices $M_j$ or $A_j$ the mapping $\Psi$ is defined by the following rule (to be checked directly):

The MV $\Lambda^{n_1}_{j_1}$ defining the JNF $J^{n_1}_{j_1}$ equals $(m_{1,j} - n + n_1, m_{2,j}, m_{3,j}, \ldots, m_{i,j})$.

### 3 The case of rigid $(p + 1)$-tuples

**Definition 11** The case when $d_1 + \ldots + d_{p+1} = 2n^2 - 2$ is called rigid. Such $(p + 1)$-tuples of matrices $A_j$ satisfying $[\mathbb{P}]$ or of matrices $M_j$ satisfying $[\mathbb{P}]$ or of JNFs or of PMVs are also called rigid.

A priori, if in the rigid case for a certain $(p+1)$-tuple of conjugacy classes the DSP is solvable, then up to conjugacy it has only finitely many solutions.

**Proposition 12** (see [Si] and [Ka]) If for a given $(p + 1)$-tuple of conjugacy classes $C_j \subset GL(n, \mathbb{C})$ with generic eigenvalues and with $d_1 + \ldots + d_{p+1} = 2n^2 - 2$ the DSP is solvable for matrices $M_j$, then its solution is unique up to conjugacy.

**Proposition 13** Suppose that for a given $(p + 1)$-tuple of conjugacy classes $c_j \subset gl(n, \mathbb{C})$ with generic eigenvalues and with $d_1 + \ldots + d_{p+1} = 2n^2 - 2$ the DSP is solvable for matrices $A_j$. Then its solution is unique up to conjugacy.

The proposition is proved at the end of the section.

**Remark 14** Rigid representations in the multiplicative case are studied in [Ka] where an algorithm is given which tells whether the DSP is solvable for given conjugacy classes $C_j$ and the construction of rigid $(p + 1)$-tuples of matrices $M_j \in C_j$ is explained. The algorithm of Katz is based on a middle convolution functor in the category of pervers sheaves. The same functor is defined in a purely algebraic way in [DR]. The algorithm in [Ka] also results in the construction $\Psi : \{J^p_j\} \mapsto \{J^{n_1}_j\}$ but in the case of rigid representations one never encounters $(p + 1)$-tuples $\{J^{n_1}_j\}$ satisfying condition $(\omega_{n'})$. In fact, there holds the following lemma (see [Ko1] and [Ko2]):

**Lemma 15** The quantity $2n^2 - \sum_{j=1}^{p+1} d_j$ is invariant for the construction $\Psi : \{J^p_j\} \mapsto \{J^{n_1}_j\}$.

The lemma implies that it is sufficient to check that condition $(\alpha_n)$ holds for $\{J^p_j\}$ (see Theorem 8) but for $\{J^{n_1}_j\}$. If $n' > 1$ and condition $(\omega_{n'})$ holds, then for generic eigenvalues the DSP is solvable for the JNFs $J^{n'}_j$, see [Ko1], hence, the necessary condition $(\alpha_{n'})$ holds – it is a strict inequality. If $n' = 1$, then condition $(\alpha_{n'})$ is an equality (this is the rigid case). Hence, in
both cases condition \((\alpha_{n'})\) holds and a posteriori one knows that in fact it is not necessary to check it.

**Proof of Proposition 13:**

10. One can assume that for every \(j\) there is no non-zero integer difference between two eigenvalues of the matrix \(A_j\) (otherwise this can be achieved by a multiplication of the matrices by \(c \in \mathbb{C}^*\)). Hence,

1) the monodromy operators \(M_j\) of a fuchsian system with residues \(A_j\) equal up to conjugacy \(\exp(2\pi i A_j)\), see Remark 1;

2) the eigenvalues of the matrices \(M_j\) are generic.

By Proposition 12, the \((p+1)\)-tuple of matrices \(M_j\) is unique up to conjugacy. Indeed, denote by \(C_j\) the conjugacy class of \(M_j\). Then \(d(C_j) = d_j\) (see 1)) and \(d(C_1) + \ldots + d(C_{p+1}) = 2n^2 - 2\).

20. Suppose that there are at least two \((p+1)\)-tuples of matrices \(A_j \in C_j\) (denoted by \(A^1_j\), \(A^2_j\)) non conjugate to one another which are solutions to the DSP. Denote by \((F_1)\), \((F_2)\) two fuchsian systems with residues equal respectively to \(A^1_j\), \(A^2_j\) and with one and the same poles. Then these systems have one and the same monodromy group, see 10. Hence, there exists a meromorphic change \(X \mapsto W(t)X\) bringing \((F_1)\) to \((F_2)\). (The fact that \(W\) can have a priori at most poles as singularities follows from the regularity of \((F_1)\) and \((F_2)\)).

30. For \(t \neq a_j\), \(j = 1, \ldots, p+1\), the matrix \(W\) is holomorphic and holomorphically invertible. Indeed, it equals \(X_2(X_1)^{-1}\) where \(X_i\) is some fundamental solution to \((F_i)\). Prove that \(W\) has no pole at \(a_j\).

Suppose it has. Set \(W = \sum_{k=-l}^{\infty} W_k(t-a_j)^k, l \in \mathbb{N}^*\). In a neighbourhood of \(a_j\) one has

\[
(F_i) : dX/dt = (A^1_j/(t-a_j) + O(1))X, i = 1, 2.
\]

Then one has

\[
-W^{-1}dW/dt + W^{-1}(A^1_j/(t-a_j) + O(1))W = (A^2_j/(t-a_j) + O(1)), \quad i.e.
\]

\[
-dW/dt + (A^1_j/(t-a_j) + O(1))W = W(A^2_j/(t-a_j) + O(1)) \quad \text{and, hence,}
\]

\[
-lW_{-l} + A^1_jW_{-l} - W_{-l}A^2_j = 0.
\]

This implies that \(W_{-l} = 0\), i.e. \(W\) has no pole at \(a_j\). (Indeed, the eigenvalues of the linear operator \(\cdot \mapsto -l(\cdot) + A^1_j(\cdot) - \cdot A^2_j\) acting on \(gl(n, \mathbb{C})\) are of the form \(\eta = -l + \lambda_1 - \lambda_2\) where \(\lambda_i\) is eigenvalue of \(A^1_j\). Their set is one of the same for \(i = 1, 2\) and by 10, one has \(\eta \neq 0\).)

But then \(W\) is holomorphic on \(\mathbb{C}P^1\), hence, constant, i.e. \(W \in GL(n, \mathbb{C})\) which means that the two \((p+1)\)-tuples (of matrices \(A^1_j\) and \(A^2_j\)) are conjugate.

The proposition is proved.

### 4 Some series of rigid representations

In this section we list several series of rigid representations with diagonalizable matrices \(M_j\) or \(A_j\) by means of their PMVs. Their existence follows from Theorem 8 and Remark 11. (The eigenvalues are presumed generic.) In Section 8 we explain how to deduce from their existence the one of other rigid series with generic eigenvalues in which at least one of the matrices \(M_j\) or \(A_j\) is not diagonalizable.
For $p = 2$ we define several series of PMVs. We avoid the letters $A$ and $M$ which denote already matrices and the notation should not be mixed up with similar notation for singularities or Lie algebras:

\[
\begin{align*}
W_k & : (k, k, k + 1), \quad (k, k, k + 1), \quad (k, k, k + 1) \\
B_k & : (k, k, k - 1), \quad (k, k, k - 1), \quad (k, k, k - 1) \\
C_k & : (k, k, k), \quad (k, k, k), \quad (k, k + 1, k - 1) \\
D_k & : (k, k, k, k + 1), \quad (k, k, k, k + 1), \quad (2k, 2k + 1) \\
E_k & : (k, k, k, k - 1), \quad (k, k, k, k - 1), \quad (2k, 2k - 1) \\
F_k & : (k, k, k, k), \quad (k, k, k, k), \quad (2k + 1, 2k - 1) \\
\Phi_k & : (k, k, k + 1, k - 1), \quad (k, k, k), \quad (2k, 2k) \\
G_k & : (k, k, k + 1, k + 1), \quad (k, k, k + 1, k + 1), \quad (2k + 1, 2k + 1) \\
H_k & : (k, k, k, k, k + 1), \quad (3k, 3k + 1), \quad (2k, 2k, 2k + 1) \\
I_k & : (k, k, k, k, k, k - 1), \quad (3k, 3k - 1), \quad (2k, 2k, 2k - 1) \\
J_k & : (k, k, k, k, k, k), \quad (3k + 1, 3k - 1), \quad (2k, 2k, 2k) \\
K_k & : (k, k, k, k, k), \quad (3k, 3k), \quad (2k, 2k + 1, 2k - 1) \\
L_k & : (k, k, k, k, k + 1, k - 1), \quad (3k, 3k), \quad (2k, 2k, 2k) \\
V_k & : (k, k, k, k, k + 1, k + 1), \quad (3k + 1, 3k + 1), \quad (2k, 2k + 1, 2k + 1) \\
N_k & : (k, k, k, k + 1, k + 1, k + 1), \quad (3k + 1, 3k + 2), \quad (2k + 1, 2k + 1, 2k + 1) \\
F_k & : (k, k, k, k, k - 1, k - 1), \quad (3k - 1, 3k - 1), \quad (2k, 2k - 1, 2k - 1)
\end{align*}
\]

Here $k \in \mathbb{N}$ or $k \in \mathbb{N}^+$ according to the case. Each of these PMVs satisfies Conditions $(\alpha_n)$ and $(\beta_n)$ (to be checked directly). Moreover, $(\alpha_n)$ is equality everywhere.

The series $W_k$, $B_k$ and $C_k$ were discovered by O.Gleizer (see [4]). We don’t use his result but deduce their existence from Theorem 8 and Remark 10 (partly because we need to prove the existence of rigid triples from other series as well and partly because he claims in [Gl] the non-existence of the rigid series $\beta$ and ($\beta$)

\[
OG_k : (2, \ldots, 2, 1, 1, 1) \quad (2, \ldots, 2, 1) \quad (2k - 1, 1, 1)
\]

which contradicts Theorem 8, we deduce the existence of this series at the end of the section.

To prove the existence of these rigid series it suffices to explicit the sequence of PMVs $\Lambda^n$, $\Lambda^{n_1}$, $\ldots$, $\Lambda^{n_\alpha}$ occurring when the construction $\Psi$ from Section 2 is iterated, see Theorem 8 and Remark 10, we set $n_s = n'$. For $\Lambda^n = W_k$ this sequence equals $W_k$, $B_k$, $W_{k-1}$, $B_{k-1}$, $\ldots$, $W_0$. Write it symbolically in the form

\[
W_k \to B_k \to W_{k-1} \to B_{k-1} \to \ldots \to W_0
\]

All requirements of Theorem 8 and Remark 10 are met which implies the existence of irreducible triples with PMV $W_k$ (and $B_k$ as well if one deletes the first term of the sequence).

One finds by analogy (for $\Lambda^n = C_k$) the sequence

\[
C_k \to B_k \to W_{k-1} \to B_{k-1} \to \ldots \to W_0
\]

which differs from the previous one only in its first term. Hence, there exist irreducible triples with $\Lambda^n = C_k$. In the same way one obtains the sequences

\[
D_k \quad \text{or} \quad F_k \quad \text{or} \quad \Phi_k \to E_k \to G_{k-1} \to D_{k-1} \to E_{k-1} \to G_{k-2} \to \ldots \to G_0 \to D_0
\]

\[
H_k \quad \text{or} \quad J_k \quad \text{or} \quad K_k \quad \text{or} \quad L_k \to I_k \to P_k \to N_{k-1} \to V_{k-1} \to H_{k-1} \to I_{k-1} \to \ldots \to H_0
\]
from which one deduces the existence of irreducible triples with \( \Lambda^n \) equal to any of the other PMVs listed above.

For \( p = 3 \) we define two series:
- \( R_k : (k, k), (k, k), (k, k), (k + 1, k - 1) \);
- \( S_k : (k + 1, k), (k + 1, k), (k + 1, k), (k + 1, k) \).

The corresponding sequence equals

\[ S_k \text{ or } R_k \to S_{k-1} \to S_{k-2} \to \ldots \to S_0 \]

For \( p = 4 \) we define the series

\[ T_k : (2k + 1, 2k - 1), (3k, k), (3k, k), (3k, k), (3k, k) \]

The PMV \( \Lambda^{n_1} \) equals \((2k - 1), (k, k - 1), (k, k - 1), (k, k - 1), (k, k - 1)\). This means that the matrix \( A_1 \) must be scalar and the PMV of the other four matrices equals \( S_{k-1} \). Thus, the existence of irreducible quintuples follows from the existence of irreducible quadruples with PMV \( S_{k-1} \).

Finally, we recall the existence of other four series discovered by C.Simpson (the first three, see [Si]) and by O.Gleizer (see [Gl]):

\[
\begin{align*}
HG_n & : (n-1, 1) \quad (1, \ldots, 1) \quad (1, \ldots, 1) \quad \text{hypergeometric} \\
OF_n & : ((n + 1)/2, (n - 1)/2) \quad ((n - 1)/2, (n - 1)/2, 1) \quad (1, \ldots, 1) \quad \text{odd family} \\
EF_n & : (n/2, n/2) \quad (n/2, (n - 2)/2, 1) \quad (1, \ldots, 1) \quad \text{even family} \\
FF_n & : (2, 1, \ldots, 1) \quad (2, \ldots, 2, 1, \ldots, 1) \quad (n - 2, 2) \quad \text{finite family} \\
& \quad (n - 4 \text{ times } 2) \quad n = 5, 6, 7, 8
\end{align*}
\]

For the series \( OG_k \) defined above one obtains the sequence

\[ OG_k \to OG_{k-1} \to \ldots \to OG_1 \to HG_2 \to HG_1 \]

Note that \( OG_1 = HG_3 \).

The existence of the series

\[ [n - 1, 1] : (n - 1, 1), \ldots, (n - 1, 1) \quad (n + 1 \text{ times}) \]

follows from \([n - 1, 1] \to (1), \ldots, (1)\).

**Remark 16** In the series \( W_k = P_k \) the multiplicities of the eigenvalues are equal to (or differ by no more than 2 from) \( n/s_1, n/s_2, n/s_3 \) where \((s_1, s_2, s_3) \in (\mathbb{N}^*)^3\) is a solution to the equation

\[ 1/s_1 + 1/s_2 + 1/s_3 = 1 \]

(these solutions are \((3, 3, 3), (4, 4, 2)\) and \((6, 3, 2)\) up to permutation). One can consider the series \( OF_n \) and \( EF_n \) (resp. \( HG_n \)) as corresponding to the “generalized” solution \((2, 2, \infty)\) (resp. \((1, \infty, \infty)\)) of the above equation.

**Remark 17** C.Simpson has shown in [Si] that the three series \( OF_n, EF_n \) and \( HG_n \) include all rigid triples of diagonalizable matrices \( M_j \) in which one of them has distinct eigenvalues. Hence, this is the case of matrices \( A_j \) as well because the criterion for existence of irreducible \((p + 1)\)-tuples (i.e. Theorem [3]) is the same in the additive and in the multiplicative situation.
5 Rigid representations with an upper bound on the multiplicities of the eigenvalues of the first matrix

5.1 Formulation of the problem

In the present section we consider the problem:

*Give the complete list of PMVs for which there exist rigid irreducible $(p+1)$-tuples of diagonalizable matrices $M_j$ satisfying \( (3) \) (resp. of diagonalizable matrices $A_j$ satisfying \( (3) \)), with generic eigenvalues, in which the multiplicities of all eigenvalues of $M_1$ (resp. of $A_1$) are $\leq u$ for some $u \in \mathbb{N}^*$.  

We solve the problem for $u = 2$. In what follows we set $m_{1,1} = u = 2$. (If $m_{1,1} = 1$, then $u = 1$ and in this case the answer to the problem is given by Remark [17]). The techniques can be used to solve the problem for any given $u$. We assume that no MV equals $(n)$ in which case the corresponding matrix $A_j$ or $M_j$ must be scalar. We also assume that no MV is of the form $(1,\ldots,1)$ (see Remark [7]).

**Remark 18** The cases $u = 1$ and $u = 2$ are exceptional in the following sense – whenever one finds a rigid PMV satisfying condition $(\beta_j)$, there exist rigid $(p+1)$-tuples of diagonalizable matrices with this PMV. (For $u = 3$ this is not true, see Example [21].) More generally, there holds

**Theorem 19** If $u \leq 2$, then conditions $(\alpha_n)$ and $(\beta_n)$ are necessary and sufficient for the existence for generic eigenvalues of irreducible $(p+1)$-tuples of matrices $M_j$ satisfying \( (3) \) or of matrices $A_j$ satisfying \( (3) \).

The theorem is proved in Section [6]. It generalizes Simpson’s result from [Si]: if one of the matrices $M_j$ has distinct eigenvalues, then for generic eigenvalues conditions $(\alpha_n)$ and $(\beta_n)$ are necessary and sufficient for the existence of irreducible $(p+1)$-tuples of matrices $M_j$ satisfying \( (3) \). In the above theorem condition $(\alpha_n)$ is not presumed to be an equality, i.e. the theorem does not consider only the rigid case.

**Example 20** For $p = 2$, $u = 3$, $n = 6m + 3$, $m \in \mathbb{N}^*$ the PMV $(3,\ldots,3,2,1,\ldots,1)$ ($m$ times 3, $3m + 1$ units), $(3m + 1,3m + 1,1)$, $(3m + 1,3m + 1,1)$ is rigid and satisfies condition $(\beta_n)$ but the PMV obtained from it after applying the construction $\Psi$ (one has $n_1 = n - 2$) does not satisfy condition $(\beta_{n-2})$.

5.2 The results

The basic result is contained in Theorems [21], [22] and [23]. In the next subsection we explain the method of proof.

**Theorem 21** If $u = 2$, $p = 3$ and $\Lambda_3^n = (n-1,1)$, then

1) one has $d_1 + \ldots + d_4 \geq 2n^2 - 2$ in all cases except in

**Case $\Omega$:** $n$ is even, $r_2 = n/2$, $r_3 = n/2 - 1$, $\Lambda_1^n = (2,\ldots,2)$, $\Lambda_2^n = (n/2,n/2)$, $\Lambda_3^n = (n/2+1,n/2-1)$; in **Case $\Omega$** one has $d_1 + d_2 + d_3 + d_4 = 2n^2 - 4$;

2) the only PMVs of rigid quadruples for $n$ even are

| $\Xi_n$ | $(2,\ldots,2)$ | $(n/2,n/2)$ | $(n/2,n/2)$ | $(n-1,1)$ |
|-------------|----------------|----------------|----------------|----------------|
| $\Theta_n$ | $(2,\ldots,2,1,1)$ | $(n/2,n/2)$ | $(n/2+1,n/2-1)$ | $(n-1,1)$ |
| $\Psi_6$   | $(2,2,2)$      | $(3,3)$       | $(4,1,1)$      | $(5,1)$       |
and the only ones for \( n \) odd are

\[
\begin{align*}
\Pi_n &: (2, \ldots, 2, 1) \quad ((n+1)/2, (n-1)/2) \quad ((n+1)/2, (n-1)/2) \quad (n-1, 1) \\
\Delta_n &: (2, \ldots, 2, 1) \quad (2, \ldots, 2, 1) \quad (n-1, 1) \quad (n-1, 1)
\end{align*}
\]

The theorem is proved in Subsection 5.4.

**Theorem 22** If \( u = 2 \), then for a rigid \((p + 1)\)-tuple one has \( p \leq 3 \). If \( p = 3 \), then one of the MVs of a rigid quadruple equals \((n-1, 1)\).

The theorem is proved in Subsection 5.3.

**Theorem 23** If \( u = 2 \) and \( p = 2 \), then with the exception of finitely many cases with \( n \leq 21 \) the only PMVs for which there exist rigid triples are the following ones:

For \( n \) even:

\[
\begin{align*}
1a) \quad \Gamma_1^n &: (2, \ldots, 2) \quad (2, \ldots, 2, 1,1,1,1,1) \quad (n-2, 2) \\
1b) \quad \Gamma_2^n &: (2, \ldots, 2, 1,1) \quad (2, \ldots, 2, 1,1,1,1) \quad (n-2, 2) \\
1c) \quad \Gamma_3^n &: (2, \ldots, 2, 1,1) \quad (2, \ldots, 2, 1,1) \quad (n-2, 1,1) \\
1d) \quad \Gamma_4^n &: (2, \ldots, 2) \quad (2, \ldots, 2, 1,1,1,1) \quad (n-2, 1,1) \\
1e) \quad Y_1^n &: (2, \ldots, 2, 1,1,1,1) \quad (m, m, 2) \quad (m+1, m+1) \\
1f) \quad Y_2^n &: (2, \ldots, 2, 1,1) \quad (m, m, 1) \quad (m+1, m+1) \\
1g) \quad Y_3^n &: (2, \ldots, 2, 1,1,1,1) \quad (m+2, m, 1,1) \quad (m+2, m+2) \\
1h) \quad Y_4^n &: (2, \ldots, 2, 1,1,1,1,1) \quad (m+2, m, 2) \quad (m+2, m+2) \\
i) \quad Y_5^n &: (2, \ldots, 2, 1,1) \quad (m+1, m, 1) \quad (m+1, m, 1) \\
j) \quad Y_6^n &: (2, \ldots, 2, 1,1,1,1) \quad (m, m, 1) \quad (m+2, m) \\
k) \quad Y_7^n &: (2, \ldots, 2, 1,1,1,1,1) \quad (m, m, 2) \quad (m+2, m)
\end{align*}
\]

For \( n \) odd:

\[
\begin{align*}
2a) \quad X_1^n &: (2, \ldots, 2, 1,1,1,1,1) \quad (2, \ldots, 2, 1) \quad (n-2, 2) \\
2b) \quad X_2^n &: (2, \ldots, 2, 1,1,1,1) \quad (2, \ldots, 2, 1,1,1) \quad (n-2, 2) \\
2c) \quad O_{G_1}^n &: (2, \ldots, 2, 1) \quad (2, \ldots, 2, 1,1,1) \quad (n-2, 1,1) \\
2d) \quad Z_1^n &: (2, \ldots, 2, 1) \quad (m, m, 1) \quad (m, m, 1) \\
2e) \quad Z_2^n &: (2, \ldots, 2, 1,1,1,1,1) \quad (m, m-1, 2) \quad (m+1, m) \\
2f) \quad Z_3^n &: (2, \ldots, 2, 1,1,1,1) \quad (m, m-1, 1,1) \quad (m+1, m) \\
2g) \quad Z_4^n &: (2, \ldots, 2, 1,1,1,1) \quad (m, m, 1) \quad (m+1, m-1, 1)
\end{align*}
\]

where \( m \in \mathbb{N} \) or \( m \in \mathbb{N}^* \).

The theorem is proved in Subsection 5.4. We do not explicit the exceptional cases with \( n \leq 21 \). The reader can do this by iterating the construction \( \Psi \) from Section 2 backward.

### 5.3 The method of proof

The method of proof consists in trying to minimize the quantities \( d_j \) for \( r_j \) fixed. Denote these minimal possible values of \( d_j \) by \( d_j^* \) and the PMVs realizing these minimal values by \( \Lambda^m \). (A posteriori they turn out to be unique up to permutation of the components of their MVs.)

The PMVs \( \Lambda^m \) in part of the cases turn out to be rigid and then we prove the existence of the corresponding \((p + 1)\)-tuples of matrices by means of Theorem 8. In another part of the cases one finds out that \( d_1^* + \ldots + d_{p+1}^* > 2n^2 - 2 \), i.e. no rigid \((p + 1)\)-tuples exist for such
quantities \( r_j \). Finally, in the remaining part of the cases one has \( d'_1 + \ldots + d'_{p+1} < 2n^2 - 2 \) (i.e. no irreducible \((p+1)\)-tuples of matrices exist for the PMVs \( \Lambda'^n \)) and one finds out how to change the PMVs in order to have \( d'_1 + \ldots + d'_{p+1} = 2n^2 - 2 \), without changing the quantities \( r_j \); after this one proves the existence of rigid \((p+1)\)-tuples from the new PMVs.

The following lemmas explain how this is done in more details. Recall that we denote by \( u \) the component \( m_{1,1} \) of \( \Lambda^n_1 \) and that \( m_{1,1} \geq \ldots \geq m_{i_1,1} \).

**Lemma 24** If \( r_j \leq n/2 \) is fixed, then \( d_j \) is minimal if and only if \( \Lambda^n_j = (n - r_j, r_j) \).

**Proof:**
One has \( d_j = n^2 - (m_{1,j})^2 - \sum_{k=2}^{i_j} (m_{k,j})^2 \) where \( m_{1,j} = (n - r_j) \geq n/2 \), see Remark 9. The sum \( \sum_{k=2}^{i_j} (m_{k,j})^2 \) is maximal if and only if \( i_j = 2 \), \( m_{2,j} = r_j \).

**Definition 25** Recall that \( \Lambda^n_j = (m_{1,j}, \ldots, m_{i_j,j}) \), \( m_{1,j} \geq \ldots \geq m_{i_j,j} \). If \( m_{1,j} = \ldots = m_{\mu,j} > m_{\mu+1,j}, \mu + 1 < i_j \), then the change \( m_{\mu+1,j} \mapsto m_{\mu+1,j} + 1, m_{i_j,j} \mapsto m_{i_j,j} - 1 \) is called a passage. Its inverse is called an antipassage. A passage preserves \( r_j \) and decreases \( d_j \) (to be checked directly). If after the change one has \( m_{i_j,j} = 0 \), then one deletes the last component of \( \Lambda^n_j \) and sets \( i_j \mapsto i_j - 1 \).

**Lemma 26** If \( r_j > n/2 \) is fixed, then \( d_j \) is minimal if and only if \( \Lambda^n_j = (m, m, \ldots, m, q) \) where \( 1 \leq q \leq m = n - r_j \).

**Proof:**
Perform passages as long as they are defined. No matter what the components \( m_{i,j} \) are at the beginning, at the end one has \( \Lambda^n_j = (m_{1,j}, \ldots, m_{1,j}, q) \).

**Corollary 27** If \( u = 2 \), then \( d_1 \) is minimal if and only if \( \Lambda^n_1 = (2, \ldots, 2) \) for \( n \) even (and, hence, \( d_1 = n^2 - 2n \)) or \( \Lambda^n_1 = (2, \ldots, 2, 1) \) for \( n \) odd and \( d_1 = n^2 - 2n + 1 \).

The corollary is direct.

**Remark 28** Suppose that two of the MVs equal \((\alpha, \beta), (v, w)\) with \( \alpha > \beta, v \geq w, \beta \geq w \) (hence, \( \alpha \leq v \)) and \( \beta + 1 \leq n/2 \). Hence, their quantities \( d_j \) equal respectively \( 2\alpha \beta, 2vw \). Their quantities \( r_j \) equal respectively \( \beta, w \).

Change the two MVs to \((\alpha - 1, \beta + 1), (v + 1, w - 1)\). Hence, their new quantities \( r_j \) are \( \beta + 1, \) \( w - 1 \), i.e. their sum does not change. The new quantities \( d_j \) are \( \alpha \beta + \alpha - \beta - 1, vw + w - v - 1 \), their sum changes by \( \alpha - \beta + w - v - 2 = (\alpha - v) + (w - \beta) - 2 < 0 \), i.e. their sum decreases.

**Lemma 29** If \( u = 2 \), then one has \( r_2 + \ldots + r_{p+1} = n \) or \( n + 1 \).

Indeed, if \( u = 2 \), then \( r_1 = n - 2 \). For rigid \((p+1)\)-tuples condition \((\beta_n)\) holds while condition \((\omega_n)\) does not. This leaves only the two possible values \((n \) and \( n + 1 \)) for \( r_2 + \ldots + r_{p+1} \).
5.4 Proof of Theorem 21

To prove the theorem we consider all cases in which for given quantities \( r_j \) the corresponding quantities \( d_j \) are minimal. They are given by Lemmas 24, 26 and Corollary 27. We prove that among these cases Case \( \Omega \) is the only one in which condition \( (\alpha_n) \) does not hold. This is part 1 of the theorem. We also find all rigid cases among them (this is part 2)).

There are two possible cases: \( r_2 + r_3 = n - 1 \) or \( n \) (Lemma 29).

Case 1) \( r_2 + r_3 = n \).

Subcase 1.1 \( r_2 = r_3 = n/2 \) (i.e. \( n \) is even).

One has \( \Lambda_2 = \Lambda_3 = (n/2, n/2) \), \( d_2 = d_3 = n^2/2 \) (Lemma 24), \( d_4 = 2n - 2 \) and \( d_1 \geq n^2 - 2n \) (Corollary 27). Hence, to have rigid quadruples the last inequality must be equality and we have the series \( \Xi_n \).

Subcase 1.2) \( r_2 > r_3 \).

One has \( r_2 > n/2, r_3 < n/2 \) and by Lemmas 26 and 24 \( d_2, d_3 \) are minimal if and only if \( \Lambda_2 = (m, \ldots, m, s), \Lambda_3 = (n - m, m) \) where \( n = lm + s, l \in \mathbb{N}, 1 \leq s \leq m, r_2 = (l - 1)m + s, r_3 = m \).

Hence, \( l \geq 2 \) (otherwise \( r_2 \leq n/2 \)). One has

\[
d_1 \geq n^2 - 2n, \quad d_2 = l(l - 1)m^2 + 2lm s, \quad d_3 = 2m((l - 1)m + s) \quad \text{and} \quad d_4 = 2n - 2.
\]

Set \( \Delta = d_1 + d_2 + d_3 + d_4 - (2n^2 - 2) \). Hence,

\[
\Delta \geq -n^2 - 2n + 2l(l - 1)m^2 + 2lm s + 2m((l - 1)m + s) + 2n - 2 = \\
= -(ml + s)^2 + l(l - 1)m^2 + 2lm s + 2m(l - 1)m + s = \\
= (l - 2)m^2 + 2ms - s^2 = (l - 2)m^2 + ms + s(m - s) > 0.
\]

This means that rigid \( (p + 1) \)-tuples with \( r_2 > n/2, r_3 < n/2 \) and \( r_2 + r_3 = n \) do not exist.

Case 2) \( r_2 + r_3 = n - 1 \).

Subcase 2.1) \( r_2 = r_3 = (n - 1)/2 \) (i.e. \( n \) is odd).

One has \( d_2 = d_3 = (n^2 - 1)/2, d_4 = 2n - 2 \) and \( d_1 \geq n^2 - 2n + 1 \) with equality if and only if \( \Lambda_2 = (2, \ldots, 2, 1) \) (Corollary 27). Hence, to have a rigid quadruple the last inequality must be equality and we have the series \( \Pi_n \).

Subcase 2.2) \( n \) is odd and \( r_2 > r_3 \).

One has \( r_2 > n/2, r_3 < n/2 \) and by Lemmas 26 and 24 \( d_2, d_3 \) are minimal if and only if \( \Lambda_2 = (m, \ldots, m, s), \Lambda_3 = (n - m + 1, m - 1) \) where \( n = lm + s, l \in \mathbb{N}, 1 \leq s \leq m, r_2 = (l - 1)m + s, r_3 = m - 1 \).

Hence, \( l \geq 2 \), (otherwise \( r_2 < n/2 \); note that \( l = 1, s = m \) is impossible because \( n \) is odd) and \( m > 1 \) (otherwise \( A_3 \) or \( M_3 \) must be scalar). One has

\[
\Delta \geq -n^2 - 2n + 2l(l - 1)m^2 + 2lm s + 2m((l - 1)m + s + 1) + 2n - 2 = \\
= -(ml + s)^2 + l(l - 1)m^2 + 2lm s + 2m(l - 1)m + s + 1 = \\
= (l - 2)m^2 + 2ms - s^2 - 2(l - 1)m - 2s - 2 + 2m = (l - 2)m^2 + ms + s(m - s) - 2(l - 2)m - 2s - 2 = \\
= (l - 2)m(m - 2) + s(m - s) + (m - 2)s - 2 > 0
\]

for \( m > 2 \) because either \( l > 2 \) or \( l = 2 \) and \( m \geq s \geq 1 \). Hence, there are no rigid quadruples in this case. If \( m = 2 \), then \( \Lambda_3 = \Lambda_4 = (n - 1, 1) \) – this gives the series \( \Delta_n \).

Subcase 2.3) \( n \) is even, \( r_2 > r_3, r_2 > n/2 \) and \( r_3 < n/2 \).
Like in Subcase 2.2) we show that no rigid quadruples exist (it is impossible to have \( l = 1, s = m \) because in this case \( r_2 = n/2 \)).

**Subcase 2.4** \( n \) is even, \( r_2 = n/2, r_3 = n/2 - 1 \) and \( \Lambda_1^n = (2, \ldots, 2), \Lambda_2^n = (n/2, n/2), \Lambda_3^n = (n/2 + 1, n/2 - 1) \).

One has \( d_1 + d_2 + d_3 + d_4 = 2n^2 - 4 \). This is precisely Case \( \Omega \). In this case to have an irreducible representation one cannot choose for all three matrices \( A_1, A_2, A_3 \) (or \( M_1, M_2, M_3 \)) the Jordan normal forms defined by the MVs \( \Lambda_1^n, \Lambda_2^n, \Lambda_3^n \).

All conjugacy classes are even-dimensional. To have a rigid quadruple one has to choose only for one of the indices \( j = 1, 2, 3 \) a conjugacy class of dimension \( \rho \) next after the minimal one \( \rho_{\min} = d_j \) and one must have \( \rho = \rho_{\min} + 2 \) (because \( d_1 + \ldots + d_4 \) has to increase by 2).

For \( n \geq 8 \) this can be done only for \( j = 1 \) and this gives the series \( \Theta_n \). For \( n = 6 \) one can choose \( j = 3 \) as well (but not \( j = 2 \)) and this gives the case \( \Psi_6 \). For \( n = 4 \) the only possibility is \((2, 1, 1), (2, 2), (3, 1), (3, 1)\) which is the case \( \Theta_4 \).

3°. Prove that rigid quadruples from the five cases \( \Xi_n, \Theta_n, \Psi_6, \Pi_n \) and \( \Delta_n \) really exist. Use the notation from Section II. One has

\[
\Xi_{n+1} \to \Pi_n \text{ and } \Pi_n \to \Pi_{n-2} \to \ldots \to \Pi_3 = [2, 1],
\]

this proves the existence of the rigid series \( \Xi_n \) and \( \Pi_n \). One also has

\[
\Theta_n \to \Theta_{n-2} \to \ldots \to \Theta_4 \to HG_2 \text{ and } \Psi_6 \to \Theta_4 \to HG_2.
\]

This proves the existence of the series \( \Theta_n \) and of \( \Psi_6 \). The one of the series \( \Delta_n \) follows from \( \Delta_n \to \Delta_{n-2} \) and \( \Delta_3 = [2, 1] \).

The theorem is proved.

### 5.5 Proof of Theorem 22

1°. Recall that the change of two MVs \((n-r_j, r_j), (n-r_i, r_i)\) to \((n-r_j-1, r_j+1), (n-r_i+1, r_i-1)\) (provided that \( r_i \leq r_j \leq n/2 \) and \( r_j + 1 \leq n/2 \)) does not change the sum \( r_j + r_i \) and decreases the sum \( d_j + d_i \), see Remark 28. In what follows when such a change is performed and after it a MV becomes equal to \((n)\) we delete it because the corresponding matrix \( A_j \) or \( M_j \) must be scalar.

Remind that MVs like the above ones give the minimal value of \( d_j \) when \( r_j \) is fixed and \( r_j \leq n/2 \), see Lemma 24.

2°. Consider only these \((p+1)\)-tuples \((p \geq 4)\) in which the MVs provide minimal possible values for \( d_j \) when \( r_j \) is fixed (see Lemmas 24 and 26 and Corollary 27). For all of them we show that condition \((a_n)\) holds and is a strong inequality. Hence, it is strong for all other possible MVs with these values of \( r_j \), i.e. no rigid \((p+1)\)-tuples exist for \( p \geq 4 \).

As a result of suitably chosen changes of MVs like in 1° one comes to the case \( p = 3 \), \( \Lambda_1^n = (n - 1, 1) \). In this case one has \( d_1 + \ldots + d_4 < 2n^2 - 2 \) only in Case \( \Omega \), see Theorem 2, when one has \( d_1 + \ldots + d_4 = 2n^2 - 4 \).

Hence, if starting with a \((p+1)\)-tuple one comes as a result of such changes of MVs to the case \( p = 3, \Lambda_4^n = (n - 1, 1) \), but not to Case \( \Omega \), then the \((p+1)\)-tuple is not rigid, see Remark 28.

3°. So consider only the \((p+1)\)-tuples which after a change like in 1° become the quadruple from Case \( \Omega \). This means that either \( p = 4 \) or \( p = 3 \) (as a result of a change of MVs no more than one MV of the form \((n)\) can appear).

We show in 4° why the case \( p = 3 \) needs not to be considered. If \( p = 4 \), then there are only two possibilities:
1) $\Lambda^n_1 = (2, \ldots, 2)$, $\Lambda^n_2 = \Lambda^n_3 = (n/2 + 1, n/2 - 1)$, $\Lambda^n_4 = \Lambda^n_5 = (n - 1, 1)$
2) $\Lambda^n_1 = (2, \ldots, 2)$, $\Lambda^n_2 = (n/2, n/2)$, $\Lambda^n_3 = (n/2 + 2, n/2 - 2)$, $\Lambda^n_4 = \Lambda^n_5 = (n - 1, 1)$.

One has respectively $d_1 + \ldots + d_5 = 2n^2 + 2n - 8$ and $2n^2 + 2n - 12$. Hence, the first possibility never gives a rigid quintuple (one has $n \geq 4$). The second can give a rigid quintuple only for $n = 5$, but $n$ must be even. Note that for $n = 4$ the MV $\Lambda^n_5$ from 2) equals (4), so this is in fact a quadruple, not a quintuple.

4). If as a result of changes of MVs a $(p + 1)$-tuple with $p \geq 4$ becomes first a quadruple different from the one of Case $\Omega$ and then the one from Case $\Omega$, then it cannot be rigid – each change decreases $d_1 + \ldots + d_{p+1}$ by at least 2 and in Case $\Omega$ this sum equals $2n^2 - 4$.

The theorem is proved.

5.6 Proof of Theorem 23

Set $\Delta = d_1 + d_2 + d_3 - (2n^2 - 2)$. Irreducible (resp. rigid) triples can exist only for $\Delta \geq 0$ (resp. $\Delta = 0$), see condition $(\alpha_n)$. Like in the proof of Theorem 21 we consider all cases in which for given quantities $r_j$ the corresponding quantities $d_j$ are minimal, see Lemmas 24 and 26 and Corollary 27. We assume that no MV equals $(1, \ldots, 1)$, see Remark 17.

Case 1) $r_2 + r_3 = n$.

Subcase 1.1) $r_2 > n/2$, $r_3 < n/2$.

1). The quantities $d_2$ and $d_3$ are minimal if and only if one has $\Lambda^n_2 = (m, \ldots, m, s)$, $\Lambda^n_3 = (n - m, m)$, $n = lm + s$, $l \in \mathbb{N}$, $1 \leq s \leq m$, $r_2 = (l - 1)m + s$, $r_3 = m$, see Lemmas 24 and 26.

For such $\Lambda^n_2$, $\Lambda^n_3$ one has $d_2 = l(l - 1)m^2 + 2lms$, $d_3 = 2m(l(l - 1)m + s)$. Hence,

\[
\Delta \geq -n^2 - 2n + 2 + l(l - 1)m^2 + 2lms + 2m(l - 1)m + s =
\]

\[
= -(ml + s)^2 - 2ml - 2s + 2 + l(l - 1)m^2 + 2lms + 2ml(l - 1)m + s =
\]

\[
= (l - 2)m^2 + 2lms - s^2 - 2lms - 1 = (l - 2)m(m - 2) + s(2m - s - 2) - 4m + 2.
\]

One has $m \geq 2$, otherwise $\Lambda^n_2 = (1, \ldots, 1)$. Hence, $s(2m - s - 2) \geq 0$.

2). If $l \geq 3$, $m \geq 6$ or $l \geq 4$, $m \geq 4$, then $(l - 2)m(m - 2) - 4m > 0$ and the triple cannot be rigid. On the other hand $l \geq 2$, otherwise $r_2 \leq n/2$. Hence, rigid triples exist only for $l = 2$ or 3 or for $m = 2$ or 3.

3). If $m = 2$, then $\Delta < 0$ (for $s = 1$ or 2). The PMVs for which the minimal value of $\Delta$ is attained are:

1) $(2, \ldots, 2)$, $(2, \ldots, 2)$, $(n - 2, 2)$ for $n$ even ; $\Delta = -6$ ;
2) $(2, \ldots, 2, 1)$, $(2, \ldots, 2, 1)$, $(n - 2, 2)$ for $n$ odd ; $\Delta = -4$ .

Find all rigid triples with such values of $r_2, r_3$ (i.e. $n - 2, 2$). To this end one has to replace 3 multiplicities equal to 2 for $n$ even (resp. 2 multiplicities equal to 2 for $n$ odd) by couples of multiplicities 1,1. (Indeed, the biggest component of $\Lambda^n_1$ is $\leq 2$, the ones of $\Lambda^n_2$ and $\Lambda^n_3$ do not change because they define $r_2$ and $r_3$.) Each change of 2 by 1,1 increases $\Delta$ by 2.

The possibilities (up to permutation of $\Lambda^n_1$ and $\Lambda^n_2$) for $n$ even are $1a) - 1d$, for $n$ odd they are $2a) - 2c)$. Possibility $2c)$ is the series $OG_k$ introduced in the previous section.

4). If $m = 3$, $l \geq 5$, then $\Delta > 0$, see 1). Hence, for $m = 3$ rigid triples can exist only for $n \leq 15$. 

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5. If \( l = 3 \), then \( \Delta = m(m-2) + s(2m-s-2) - 4m + 2 \) and \( \Delta > 0 \) if \( m \geq 5 \) or \( m = 4, s = 2, 3, 4 \) (to be checked directly). Hence, rigid triples with \( l = 3 \) exist only for \( n \leq 13 \).

6. If \( l = 2 \), then \( \Delta = s(2m-s-2) - 4m + 20 \) and if \( 4 \leq s \leq m-2 \), then \( \Delta > 0 \). Hence, rigid triples can exist only for \( s = 1, 2, 3, m-1, m \).

If \( s = 3 \) and \( m \geq 7 \), then \( \Delta > 0 \), i.e. with \( s = 3 \) rigid triples can exist only for \( n \leq 21 \).

If \( s = m \), then for \( m \geq 6 \) one has \( \Delta > 0 \), i.e. such rigid triples can exist only for \( n \leq 20 \).

If \( s = m-1 \), then again for \( m \geq 6 \) one has \( \Delta > 0 \), i.e. such rigid triples can exist only for \( n \leq 17 \).

If \( s = 1 \), then we have \( \Lambda^1 = (2, \ldots, 2, 1) \), \( \Lambda^2 = (m, m, 1) \), \( \Lambda^3 = (m+1, m) \). One has \( \Delta = -(n-1) \), i.e. for \( n > 1 \) one cannot choose these MVs to have rigid triples. Give the list of the MVs with the same quantities \( r_j \) for which \( \Delta = 0 \). They are obtained from the given ones as a result of one or several antipassages, see Subsection 5.3.

The MV \( \Lambda^2 \) after one antipassage becomes \((m, m-1, 2)\) (and \( d_2 \) increases by \((n-1) - 4\)) or \((m, m-1, 1, 1)\) (and \( d_2 \) increases by \((n-1) - 2\)). The MV \( \Lambda^3 \) after one antipassage becomes \((m+1, m-1, 1)\) and \( d_3 \) increases by \((n-1) - 2\). To increase \( d_1 \) by 2 one has to make \( s \) antipassages in which a component 2 is replaced by a couple of units. However, we avoid to have \( \Lambda^1 = (1, \ldots, 1) \) which case was considered in Section 3. Therefore for \( n \geq 22 \) the only PMVs which give rigid triples for \( s = 1 \) are \( Z_n^2, Z_n^3 \) and \( Z_n^1 \).

If \( s = 2 \), one gets the series \( \Lambda^1 = (2, \ldots, 2, 2) \), \( \Lambda^2 = (m, m, 2) \), \( \Lambda^3 = (m+2, m) \) with \( \Delta = -6 \). The only ways to increase \( \Delta \) by 6 for \( m \geq 10 \) are to make three antipassages changing a component 2 by two components 1.1. This yields possibilities 1j) and 1k).

**Subcase 1.2** \( r_2 = r_3 = n/2 \) (\( n \) is even).

We assume that \( n \geq 22 \). The PMV which minimizes the sum \( d_1 + d_2 + d_3 \) equals \((2, \ldots, 2), (n/2, n/2), (n/2, n/2)\) and one has \( \Delta = -2n + 2 \). Hence, to obtain irreducible triples one has to choose another PMV, in which at least one MV defines a JNF giving a greater value of the corresponding quantity \( d_j \).

For the PMV as above one has \( d_1 = n^2 - 2n, d_2 = d_3 = n^2/2 \). By replacing consecutively components equal to 2 of \( \Lambda^2 \) by couples of units one can obtain as values of \( d_1 \) all even numbers from \( n^2 - 2n \) to \( n^2 - n \).

Hence, one cannot increase enough \( \Delta \) by changing only \( \Lambda^1 \). If one changes \( \Lambda^2 \) and/or \( \Lambda^3 \) without changing \( r_2 \) and \( r_3 \), the new choices have to be among the following MVs, otherwise \( \Delta \) increases by more than \( 2n - 2 \):

1) \((n/2, n/2 - 1, 1)\) \( d_j = n^2/2 + n - 2 \);
2) \((n/2, n/2 - 2, 2)\) \( d_j = n^2/2 + 2n - 8 \);
3) \((n/2, n/2 - 2, 1, 1)\) \( d_j = n^2/2 + 2n - 6 \).

If one uses possibility 2) or 3), then the only cases in which \( \Delta = 0 \) are 1g) and 1h). If one uses possibility 1), then this leads to case 1i) or to the series \( EF_n \), see Section 4.

**Case 2** \( r_2 + r_3 = n + 1 \).

**Subcase 2.1** \( r_2 > n/2, r_3 \leq n/2 \).

The quantities \( d_2 \) and \( d_3 \) are minimal if and only if one has

\[ \Lambda^2 = (m, \ldots, m, s), \Lambda^3 = (n-m-1, m+1), n = lm+s, l \in \mathbb{N}, 1 \leq s \leq m, m \geq 2, r_2 = (l-1)m+s, r_3 = m+1 \],

see Lemmas 24 and 25. For such \( \Lambda^2, \Lambda^3 \) one has

\[ d_2 = l(l-1)m^2 + 2lm, d_3 = 2(m+1)((l-1)m+s-1) = 2m(l-1)m+s + \delta \]

where \( \delta = 2(l-1)m + 2s - 2 - 2m = 2(l-2)m + 2s - 2 \). Like in 10 one finds
$$\Delta \geq (l - 2)m(m - 2) + s(2m - s - 2) - 4m + 2 + \delta = (l - 2)m^2 + s(2m - s) - 4m$$

(the difference in the estimation of $d_3$ w.r.t. 1$^0$ equals $\delta$). For $l > 3$ one has $\Delta > 0$. The same is true for $l = 3$ except for $m = 2$. In the latter case one has $n = 7$ or 8.

For $l = 2$ one does not have $\Delta > 0$ only if $s = 1, 2, 3$ or 4; if $s = 4$, then $m = s = 4$ and $n = 12$; if $s = 3$, then $m = 3$ or 4, resp. $n = 9$ or 11.

The case $l = 2$, $s = 1$ is impossible because then one has $r_2 = m + 1$, $r_3 = m$ and $r_2 + r_3 = n < n + 1$.

If $l = 2$, $s = 2$, then $n = 2m + 2$ is even and for $\Lambda_1^n = (2, \ldots, 2)$, $\Lambda_2^n = (m, m, 2)$, $\Lambda_3^n = (m + 1, m + 1)$ one has

$$\Delta = (n^2 - 2n) + 2(n/2 - 1)^2 + 4(n - 2) + n^2/2 - 2n^2 + 2 = -4.$$ 

If $m \leq 9$, then $n \leq 20$. If $m \geq 10$, i.e. $n \geq 22$, then it is possible to increase $\Delta$ by 4 (without changing $r_1$, $r_2$, $r_3$) only by replacing the PMV by one of the PMVs from 1e) or 1f).

For all other choices of $\Lambda_j^n$ with $r_1 = n - 2$, $r_2 = m + 2$, $r_3 = m + 1$ one has $\Delta > 0$. Hence, no rigid triples exist for such PMVs.

**Subcase 2.2)** $r_2 > n/2$, $r_3 > n/2$.

Necessarily $n$ is odd ($n = 2m + 1$) and the minimal possible value of $d_1 + d_2 + d_3$ is attained for and only for $\Lambda_1^n = (2, \ldots, 2)$, $\Lambda_2^n = (m, m, 1)$, $\Lambda_3^n = (m, m, 1)$ (see Lemma 26). Such triples are rigid. They give possibility 2d).

Prove the existence of the listed series. With the notation from Section 4 one has

$$\Gamma_n^i \rightarrow \Gamma_{n-2}^i, \quad \Gamma_6^1 \rightarrow X_5^1 \rightarrow Y_4^1 = \Gamma_4^2 = \Gamma_4^4 \rightarrow HG_3, \quad \Gamma_4^3 \rightarrow HG_2,$$

which proves the existence of the series $\Gamma_n^i$, $i = 1, 2, 3, 4$. One also has

$$X_n^i \rightarrow X_{n-2}^i, \quad X_5^1 \rightarrow \Gamma_4^2, \quad X_3^2 = HG_3$$

which proves the existence of the series $X_n^i$. Next,

for $n > 4$ $Y_1^i \rightarrow Z_{n-1}^2 \rightarrow Z_{n-3}^2 \rightarrow \ldots \rightarrow Z_5^2 \rightarrow \Gamma_4^4$,

hence, the series $Y_1^i$ and $Z_2^i$ also exist. From $Z_n^i \rightarrow Z_{n-2}^i$, $Z_3^i = HG_3$, $i = 3, 4$ there follows the existence of the series $Z_3^3$, $Z_3^4$. From

$$Z_n^1 \rightarrow Y_{n-1}^5 \rightarrow Y_{n-3}^5 \rightarrow \ldots \rightarrow Y_2^5 = HG_2$$

follows the existence of $Z_1^3$ and $Y_{n-1}^5$. From $Y_n^2 \rightarrow Z_{n-1}^3$ follows the one of $Y_n^2$. One has

$$Y_n^3 \rightarrow Y_{n-2}^3 \rightarrow Y_{n-4}^3 \rightarrow Y_{n-6}^3 \rightarrow \ldots,$$

hence, $Y_n^6$ and $Y_n^3$ also exist (we let the reader prove the existence of $Y_4^3$; one has $Y_4^6 = HG_4$).

Finally, one has

$$Y_n^4 \rightarrow Y_{n-2}^7 \rightarrow Y_{n-4}^7 \rightarrow Y_{n-6}^7 \rightarrow \ldots$$

which proves the existence of $Y_n^4$ and $Y_n^7$ (the reader has to prove the existence of $Y_6^4$ and $Y_6^7$).

The theorem is proved.
6 The case of arbitrary (not necessarily diagonal) Jordan normal forms

Definition 30 For a given JNF \( J^n = \{ b_{i,l} \} \) define its corresponding diagonal JNF \( J'^n \). (We say that \( J^n \) and \( J'^n \) are corresponding to one another.) A diagonal JNF is a partition of \( n \) defined by the multiplicities of the eigenvalues. For each \( l \) the family \( \{ b_{i,l} \} \) is a partition of \( \sum_{i \in I_l} b_{i,l} \) and \( J'^n \) is the disjoint sum of the dual partitions.

Example 31 If a JNF is defined by the family \( B = \{ b_{i,l} \} \) where \( l = 1, 2 \) and \( B = \{ 4, 2, 2 \} \{ 5, 1 \} \), i.e. there are two eigenvalues, the first (resp. the second) with three Jordan blocks, of sizes 4, 2, 2 (resp. with two Jordan blocks, of sizes 5, 1), then the corresponding diagonal JNF is defined by the MV \( (3, 3, 1, 1, 4, 2) \) (or, better, by the MV with non-increasing components \( (4, 3, 3, 2, 1, 1) \)). Indeed, \( (3, 3, 1, 1) \) (resp. \( (4, 2) \)) is the partition dual to \( (4, 3, 3, 2, 1, 1) \).

The following theorem explains why it is sufficient to know (for generic eigenvalues) the solution to the DSP only in the case of diagonalizable matrices. The theorem is announced in [Ko1] and proved in [Ko2].

Theorem 32 If for some eigenvalues the DSP is weakly solvable for a given \( \{ J^n_j \} \) (resp. for \( \{ J'^n_j \} \)), then it is solvable for \( \{ J'^n_j \} \) (resp. for \( \{ J^n_j \} \)) for any generic eigenvalues.

Thus if one knows that the DSP is solvable for a certain PMV \( \Lambda^n \) for generic eigenvalues, then one knows that it is solvable (for generic eigenvalues) for all \( (p + 1) \)-tuples of JNFs \( \{ J^n_j \} \) such that the JNF defined by \( \Lambda^n_j \) corresponds to \( J^n_j \). This allows one to construct new series of \( (p + 1) \)-tuples of JNFs (not all of which diagonal) for which there exist rigid \( (p + 1) \)-tuples of matrices \( A_j \) or \( M_j \). One should know, however, that for certain \( (p + 1) \)-tuples of JNFs one cannot have generic eigenvalues.

Example 33 Consider the series \( C_k \) from Section 4 for matrices \( A_j \). A possible triple of JNFs corresponding to the diagonal ones defined by the PMV is the following one: \( J^n_1 \) and \( J^n_2 \) are the same as before, i.e. diagonalizable, with MVs of the eigenvalues equal to \( (k, k, k) \) while \( J^n_j \) has a single eigenvalue with Jordan blocks of sizes \( (1, 2, 3, \ldots, 3) \). Hence, the multiplicities of all eigenvalues are divisible by \( k \). The sum of all eigenvalues counted with multiplicities \( k \) times smaller equals 0 and this is a non-genericity relation.

Consider the same example for matrices \( M_j \). The product of all eigenvalues with multiplicities \( k \) times smaller is a root of unity of order \( k \). If this root is non-primitive, then again a non-genericity relation holds and there exist no such generic eigenvalues. In this case the set of possible eigenvalues with these JNFs is a reducible variety with \( k \) connected components each of which corresponds to one of the roots of unity. The eigenvalues from the components corresponding to non-primitive roots are all non-generic.

7 Proof of Theorem 19

10. Theorem 32 allows one to prove the theorem only in the case of diagonalizable matrices. For \( n \leq 3 \) the reader can check the theorem oneself, so suppose that \( n \geq 4 \).

It suffices to prove that the PMV \( \Lambda^n_{n_1} \) obtained from \( \Lambda^n \) after applying \( \Psi \) (see Section 4) satisfies condition \( (\beta_{n_1}) \). (The PMV \( \Lambda^n_{n_1} \) satisfies condition \( (\alpha_{n_1}) \) if and only if \( \Lambda^n \) satisfies \( (\alpha_1) \), see Lemma 15.)
If one of the MVs is of the form \((1, \ldots, 1)\) and conditions \((\alpha_n), (\beta_n)\) hold, then in the case of matrices \(M_j\) the answer to the DSP is positive, see [5], hence, it is positive for matrices \(A_j\) as well (for generic eigenvalues the criterion is the same in the case of matrices \(A_j\) or \(M_j\)). Therefore we assume that for all \(j\) one has \(m_{1,j} \geq 2\).

**Remark 34** Remind that

1) the maximal value of \(d_j\) equals \(n^2 - n\) and it is attained only for a MV of the form \((1, \ldots, 1)\);

2) for the MV \((n-1,1)\) the quantity \(d_j\) equals \(2n - 2\); hence, if \(p = 2\) and one of the MVs equals \((n-1,1)\), then \((\alpha_n)\) holds only if the other two equal \((1, \ldots, 1)\);

3) for the MVs \((n/2, n/2)\) and \((n/2, n/2 - 1, 1)\) the values of \(d_j\) equal respectively \(n^2/2\) and \(n^2/2 + n - 2\).

\[0\] Set \(\rho_j := r_1 + \ldots + r_j + \ldots + r_{p+1}\). One has \(r_1 = n - 1\) or \(n - 2\), therefore for \(j \neq 1, p \geq 3\) one has \(\rho_j \geq n - 2 + p - 1 \geq n\); this is true for \(p = 2\) as well because no MV equals \((n-1,1)\), otherwise \((\alpha_n)\) does not hold, see Remark [14]. Therefore we check only that after performing the construction \(\Psi\) from Section 2 one has \(\rho_1 \geq n_1\).

One has \(n - n_1 \leq 2\), see Remark [10]. If \(n - n_1 = 1\), then every quantity \(r_j\) remains the same or decreases by 1. The second possibility takes place only if \(r_j \geq n/2\) and \(\Lambda^n\) has two equal greatest components. Denote by \(l\) the number of indices \(j\) for which \(r_j \geq n/2\). Hence, \(j = 1\) is always among them. Three cases are possible:

**Case 1\) \(l \leq 2\).**

Condition \((\beta_n)\) satisfied by \(\Lambda^n\) implies that \(\Lambda^n\) satisfies condition \((\beta_{n_1})\) because for \(j > 1\) either all \(r_j\) remain the same or only one decreases by 1 when \(\Psi\) is performed.

**Case 2\) \(l \geq 3\) and \(p \geq 3\).**

After applying \(\Psi\) in the sum \(\rho_1\) there are two quantities \(r_j\) which are \(\geq n/2 - 1\) and one which is \(\geq 1\), so \(\Lambda^n\) satisfies condition \((\beta_{n_1})\).

**Case 3\) \(p = 2\) and \(l = 3\).**

The sum \(\rho_1\) can become \(< n - 1\) after applying \(\Psi\) only if \(\Lambda^3_\beta = \Lambda^3_\alpha = (n/2, n/2)\) and \(n\) is even. But in this case condition \((\alpha_n)\) does not hold for any MV \(\Lambda^n_1\) (see Remark [24]), hence, the case has to be excluded. In all other cases the sum \(\rho_1\) decreases by 1 and the PMV \(\Lambda^n\) satisfies condition \((\beta_{n_1})\).

\[1\] Let \(n - n_1 = 2\). Like in the case \(n - n_1 = 1\), for \(j \neq 1\) the sum \(\rho_j\) is \(\geq n\). Indeed, if \(p > 2\), then such a sum contains \(r_1 \geq n - 2\) and two more quantities \(r_j\) which are \(\geq 1\). If \(p = 2\) and \(u = 2\), then no MV is of the form \((n-1,1)\) because condition \((\alpha_n)\) would not hold, see Remark [34]. Hence, except \(r_1 \geq n - 2\), \(\rho_j\) contains \(r_2 \geq 2\) or \(r_3 \geq 2\), i.e. \(\rho_j \geq n\). So there remains to check that after applying \(\Psi\) one has \(\rho_1 \geq n - 2\).

\[2\] Denote by \(s_j\) the difference \(m_{1,j} - m_{2,j}\) and by \(\kappa\) the number of quantities \(s_j\) which are \(\leq 1\). Hence, \(s_1\) is always one of them. Four cases are possible:

**Case 4\) \(\kappa = 1\) or 2.**

At most one quantity \(r_j\) from \(\rho_1\) decreases by at most 2, so \(\Lambda^n\) satisfies condition \((\beta_{n_1})\).

**Case 5\) \(\kappa \geq 4\).**

After performing \(\Psi\) one has \(r_j \geq n/2 - 2\) for three indices \(j > 1\), hence, \(\rho_1 \geq 3n/2 - 6 \geq n - 2\) because \(n \geq 4\).

**Case 6\) \(\kappa = 3, p \geq 3\).**

In this case after performing \(\Psi\) one has \(r_j \geq n/2 - 2\) for two indices \(j > 1\) and \(r_j \geq 1\) for another one, so \(\rho_1 \geq n - 3\) with equality only if two MVs \(\Lambda^n_j\) with \(j > 1\) equal \((n/2, n/2)\) and a third equals \((n-1,1)\). But in such a case \(n - n_1 = 1\), so the case has to be excluded.
Case 7) \( \kappa = 3, \ p = 2. \)

After performing \( \Psi \) one has \( \rho_1 < n - 2 \) only if \( n \) is even and either both \( \Lambda_n^2, \ \Lambda_n^3 \) are of the form \((n/2, n/2)\) or one is of this form while the other equals \((n/2, n/2 - 1, 1)\). In the first case condition \((\alpha_n)\) does not hold for any \( \Lambda_n \), see Remark \([\text{Ko1}]\). In the second it holds only for \( \Lambda_n = (1, \ldots, 1) \), but in this case \( n - n_1 = 1 \), so both cases have to be excluded.

The theorem is proved.

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