A Self-Index on Block Trees

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Abstract

The Block Tree is a recently proposed data structure that reaches compression close to Lempel-Ziv while supporting efficient direct access to text substrings. In this paper we show how a self-index can be built on top of a Block Tree so that it provides efficient pattern searches while using space proportional to that of the original data structure. We also show how the structure can be extended to perform document listing and range-restricted pattern matching.

1 Introduction

The Block Tree (BT) [1] is a novel data structure for representing a sequence, which reaches a space close to its LZ77-compressed [23] space. Given a string $S[1,n]$ over alphabet $[1,\sigma]$, on which the LZ77 parser produces $z$ phrases (and thus an LZ77 compressor uses $z \log n + O(z \log \sigma)$ bits, where $\log$ is the logarithm in base 2), the BT on $S$ uses $O(z \log(n/z) \log n)$ bits (also said to be $O(z \log(n/z))$ space). This is the same space obtained with the grammar compressors that offer the best bounds [21, 3, 22, 11, 12]. In exchange for using more space than LZ77 compression, the BT offers fast extraction of substrings: a substring of length $\ell$ can be extracted in time $O((1+\ell/\log\sigma) \log(n/z))$.

Kreft and Navarro [14] introduced a self-index based on LZ77 compression, which proved to be extremely space-efficient on highly repetitive text collections [5]. A self-index on $S$ is a data structure that offers direct access to any substring of $S$ (and thus it replaces $S$), and at the same time offers indexed searches. Their self-index uses $3z \log n + O(z \log \sigma) + o(n)$ bits (that is, about 3 times the size of the compressed text) and finds all the occ occurrences of a pattern of length $p$ in time $O(p^2h + (p + occ) \log z)$, where $h \leq \sqrt{n}$ is the maximum number of times a symbol is successively copied along the LZ77 parsing. A string of length $\ell$ is extracted in $O(h\ell)$ time.

Experiments on repetitive text collections [14, 5] show that this LZ77-index is smaller than any other alternative and is competitive when searching for patterns, especially on the short ones where the term $p^2h$ is small and occ is large, so that the low time to report each occurrence dominates. On longer patterns, however, the index is significantly slower. The term $h$ can reach the hundreds on repetitive collections [14], and thus it poses a significant penalty (and a poor worst-case bound).

In this paper we design the BT-index, a self-index that builds on top of BTs instead of on LZ77 compression. Given a BT with $u = O(z \log(n/z))$ pointers (and thus using $u \log n + O(u)$ bits), the BT-index uses $3u \log n + O(u \log \sigma)$ bits, and it searches for a pattern in time $O(p^2 \log(n/z) + (p + occ) \log n)$, which solves the main problem in the search complexity of the LZ77-index. If we increase the space to $4u \log n + O(u \log \sigma)$ bits, which is still $O(z \log(n/z))$ space, the time becomes $O(p^2 \log(n/z) + occ \log n)$. The result also promises faster searches in practice: In regular texts, the $O(\log(n/z))$ factor is around 3–4, and this raises to 8–10 on highly repetitive texts [14]; both are much lower than the typical values of $h$. 

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The self-indexes that build on grammar compression \cite{7,8} can use the same asymptotic space of our BT-index, \( g = O(z \log(n/z)) \), and their best search time is \( O(p^2 \log \log n + (p + \text{occ}) \log g) \). Belazzougui et al. \cite{1}, however, show that in practice Block Trees are much faster to access than grammar-compressed representations, and use about the same space if the text is highly repetitive. Thus we expect that our self-index will be better in practice than those based on grammar compression too.

We also show how the BT-index can be enhanced to support range-restricted searches (i.e., report occurrences of \( P \) that lie within a range of \( S \)) and document listing (i.e., list the documents of a collection where \( P \) appears) in a time that is competitive with current proposals designed for highly repetitive collections \cite{6}.

# 2 Block Trees

Given a string \( S[1, n] \) over an alphabet \([1, \sigma]\), whose LZ77 parse produces \( z \) phrases, a BT is defined as follows. At the top level, numbered \( l = 0 \), we split \( S \) into \( z \) blocks of length \( b_0 = n/z \). Now each block is recursively split into two, so that if \( b_l \) is the length of the blocks at level \( l \) it holds \( b_{l+1} = b_l/2 \), until reaching blocks of one symbol after \( \log(n/z) \) levels. At each level, every consecutive pair of blocks \( S[i..j] \) that does not appear earlier as a substring of \( S[1..i-1] \) is marked. Blocks that are not marked are replaced by a pointer to their first occurrence in \( S \) (which, by definition, must be a marked block or overlap a pair of marked blocks). For every level \( l \geq 0 \), a bitvector \( D_l \) with one bit per block sets to 1 the positions of marked blocks. In level \( l+1 \) we consider and subdivide only the blocks that were marked in level \( l \). In this paper, this subdivision is carried out up to the last level, where the marked blocks correspond to the first occurrence of each symbol, which is stored associated with each marked block.

We say that a block \emph{exists} in level \( l \) if \( l = 0 \) or all the blocks containing it in levels \(< l \) are marked. The blocks that exist in level \( l \) are either marked or unmarked, and the descendants of those unmarked do not exist in higher levels. We call \( w_l \) the number of existing blocks at level \( l \), of which \( m_l \) are marked and \( u_l \) unmarked. Then \( |D_l| = w_l = m_l + u_l \), \( w_0 = z \), and \( w_{l+1} = 2m_l \). The pointers are stored in an array \( \text{ptr}[1..u_l] \), with one entry per unmarked block. In the last level we store the distinct symbols in a small array \( \text{symb}[1..\sigma] \).

To extract a single symbol \( S[i] \), we compute the block \( j = \lfloor i/b_0 \rfloor \) where \( i \) falls and see if \( j \) is marked, that is, if \( D_0[j] = 1 \). If so, we map \( i \) to a position in the next level, which only contains the marked blocks of this level: \( i \leftarrow (\text{rank}_1(D_0, j) - 1) \cdot b_0 + ((i - 1) \mod b_0) + 1 \). Function \( \text{rank}_c(D_l, j) \) counts the number of occurrences of bit \( c \in \{0, 1\} \) in \( D_l[1..j] \). Bitvector \( D_l \) can be represented in \( w_l + o(w_l) \) bits so that queries \( \text{rank}_c \) can be computed in constant time \cite{4}. Therefore, if \( i \) falls in a marked block, we translate the problem to the next level in constant time. If, instead, \( i \) is in an unmarked block, we take the pointer stored for that block, and replace \( i \leftarrow i - \text{ptr}[\text{rank}_0(D_l, j)] \), because array \( \text{ptr} \) stores the distance towards the first occurrence of the unmarked block. Now \( i \) is again on a marked block, and we can move on to the next level as described. In the last level, the symbol is simply \( \text{symb}[\text{rank}_1(D_l, i)] \). The total time to extract a symbol is then \( O(\log(n/z)) \).

The total number of nodes in the BT is \( w = \sum_i w_i \); the total number of marked blocks is \( m = \sum_i m_i \) and of unmarked blocks is \( u = \sum_i u_i \), with \( w = m + u \). The space used by the BT is \( w + o(w) \) bits for the bitvectors \( D_l \), \( \sigma \log \sigma \) bits for the array \( \text{symb} \), and \( u \log n \) bits for the arrays \( \text{ptr} \). Note that \( w = \sum_i w_i = \sum_i (w_i - m_i) = \sum_i (w_i - w_{i+1}/2) = w - (w - z)/2 = w/2 + z/2 \), and so \( m = w/2 - z/2 \). Since \( z \leq w \), it holds \( w/2 \leq u \leq w \) and \( 0 \leq m \leq w/2 \); therefore the total space is
$u \lg n + O(u + \sigma \log \sigma)$ bits. Since, at every level, a pair of blocks that is not marked cannot strictly contain a Lempel-Ziv phrase\(^1\) it holds $u = O(z \log(n/z))$, and the total space is $O(z \log(n/z) \log n)$ bits [1].

3 A Self-Index

Our self-index structure is made up of two main components: the first finds all the pattern positions that cross block boundaries, whereas the second finds the positions that are copied onto unmarked blocks. The main property that we exploit is the following.

**Lemma 1** The occurrences of a given string $P$ in $S$, with $|P| \geq 2$, either overlap two existing blocks at some level, or are completely inside an unmarked block at some level.

**Proof.** We proceed by induction on the BT block size. Consider the level $l = 0$, where all the blocks exist. If the occurrence overlaps two blocks or it is completely inside an unmarked block, we are done. If, instead, it is completely inside a marked block, then this block is split into two blocks that exist in the next level. If we concatenate all the existing blocks of the next level, we have a new sequence where the occurrence appears, and we use a smaller block size, so by the inductive hypothesis, the property holds. The base case is the leaf level, where the blocks are of length 1. □

We exploit the lemma in the following way. We will define an occurrence of $P$ as *primary* if it overlaps two consecutive blocks at some level. The occurrences that are completely contained in an unmarked block are *secondary* (this is a variant of the classical idea used in all the LZ-based indexes [13]). Secondary occurrences are found by detecting primary occurrences within the areas from where unmarked blocks are copied. We will use a data structure to find the primary occurrences and another to detect the copies. For the case $|P| = 1$, the only primary occurrence will be the block of length 1 in the last level that equals $P$ (these positions are stored in an array of $\sigma \lg u$ bits, where $l$ is the last level).

3.1 The Data Structures

We describe the data structures used by our index. They require $3u \lg n + O(u + \sigma \log u)$ bits, and replace the pointers $ptr_l$ used by the original structure. We also retain the bitvectors $D_l$ and array $symb$, which add $O(u + \sigma \log \sigma)$ bits. Later we will add optional speed-up structures using $O(u \log \sigma)$ bits. Since $u \geq z \geq \sigma$, the total space becomes $3u \lg n + O(u \log \sigma)$ bits in this case.

**Primary occurrences.** Our structure to find the primary occurrences is a two-dimensional discrete grid $G$ of $u \times u$, storing $u$ points $(x, y)$ as follows. Let $B_i \cdot B_{i+1}$ two existing (marked or unmarked) blocks at some level, corresponding to the substrings $S[j..j+\ell-1] \cdot S[j+\ell..j+2\ell-1]$. Then we collect the reverse block $B_i^{rev} = S[j+\ell-1] \cdot S[j+\ell-2] \cdots S[j]$ in the multiset $Y$ and the suffix $S[j+\ell..n]$ in the multiset $X$. If the same suffix $S[j+\ell..n]$ turns out to be paired with preceding blocks of different lengths (at different levels), we choose only the longest of those preceding

\(^1\)To ensure this, one should also mark every descendant of a marked block unless it is not pointed from later blocks; we omit those details here as they are not essential for the correctness of our description.
blocks. Therefore, each marked block in BT induces one distinct suffix, when it is split into two in the next level, and there is a total of $z + \sum_i m_i = z + m = w/2 + z/2 = u$ suffixes indexed. We lexicographically sort $X$ and $Y$, to obtain the strings $X_1, X_2, \ldots, X_u$ and $Y_1, Y_2, \ldots, Y_u$, respectively. The grid has a point at $(x, y)$ for each $X_x$ and $Y_y$ such that $X_x$ corresponds to some reversed block $B_i^rev$ and $Y_k$ to the suffix of $S$ starting with $B_{i+1}$.

We represent $G$ using a wavelet tree [10][9][19], so that it takes $u \log u + o(u)$ bits and can report all the $y$-coordinates of the occ points lying inside any rectangle of the grid in time $O((occ+1) \log u)$. We spend other $u \log n$ bits in an array $R[1..u]$ that gives the position $j + \ell$ in $S$ corresponding to each point $(x, y)$, sorted by $y$-coordinate.

**Secondary occurrences.** If an unmarked block $B_i[1..\ell]$ points to its first occurrence at $S[k..k + \ell - 1]$, we say that $[k..k + \ell - 1]$ is the source of $B_i$. For each level $l$, we store two structures to find the secondary occurrences. The first is a bitvector $B_l[1..n + u_l]$ built as follows: We traverse from $S[1]$ to $S[n]$. For each $S[k]$, we add a 0 to $B_l$, and then as many 1s as sources start at position $k$. The second structure is a permutation $\pi_l$ on $[u_l]$ where $\pi_l(i)$ is the primary occurrence of $i$ if the source of the $i$th unmarked block is signaled by the $j$th 1 in $B_l$ (ties are broken arbitrarily).

Each bitvector $B_l$ can be represented in $u_l \log(n/u_l) + O(u_l)$ bits so that operation $select_e(B_l, r)$ can be computed in constant time [20]. This operation finds the $r$th occurrence of the bit $e$ in $B_l$. On the other hand, we represent $\pi_l$ using a representation [16] that uses $u_l \log u_l + O(u_l)$ and computes any $\pi_l(i)$ in constant time and any $\pi_l^{-1}(j)$ in time $O(\log u_l)$. Added over all the levels, these structures use at most $u \log n + O(u)$ bits.

### 3.2 Extraction

Let us describe how we extract a symbol $S[i]$ using our representation. We first compute the block $j \leftarrow [i/b_0]$ where $i$ falls. If $D_0[i] = 0$, then the block $j$ is not marked. Its rank among the unmarked blocks of this level is $r_0 = rank_0(D_0, i)$. The position of the 1 in $B_0$ corresponding to its source is $p_0 = select_1(B_0, \pi_0(r_0))$. This means that the source of the block $j$ starts at $p_0 - \pi_0(r_0)$. Since block $j$ starts at position $i_0 \leftarrow (j - 1) \cdot b_0 + (i - 1) \mod b_0 + 1$, we set $i \leftarrow i - (i_0 - p_0)$ and recompute $j \leftarrow [i/b_0]$, knowing that the new $S[i]$ is the same as the original one. If, instead, $D_0[i]$ was originally 1, we retain the values $i$ and $j$.

Now that $i$ is inside a marked block $j$, we move to the next level. To compute the position of $i$ in the next level, we do $i \leftarrow (rank_1(D_0, j) - 1) \cdot b_0 + ((i - 1) \mod b_0) + 1$, and continue in the same way from level 1. In the last level we find the symbol stored explicitly. The total time to extract a symbol is $O(\log(n/z))$.

### 3.3 Queries

**Primary occurrences.** To search for a pattern $P[1..p]$, we first find its primary occurrences using $G$ as follows. For each partition $P_\prec = P[1..k]$ and $P_\succ = P[k + 1..p]$, for $1 \leq k < m$, we binary search $Y$ for $P_\prec rev$ and $X$ for $P_\succ$. To compare $P_\prec rev$ with a string $Y_i$, since $Y_i$ is not stored, we extract the consecutive symbols of $S[R[i] - 1]$, $S[R[i] - 2]$, and so on, until the lexicographic comparison can be decided. Thus each comparison requires $O(p \log(n/z))$ time. To compare $P_\succ$ with a string $X_i$, since $X_i$ is also not stored, we extract the only point of the range $[i, i] \times [1, u]$ (or, in terms of the wavelet tree, we extract the $r$th stored $y$-coordinate), in time $O(\log u)$. This yields the point $X_j$. Then we compare $P_\succ$ with the successive symbols of $S[R[j]]$, $S[R[j] + 1]$, and so on.
Such a comparison then costs $O(\log u + p \log(n/z))$. The $p$ binary searches require $p \log u$ binary search steps, for a total cost of $O(p^2 \log u \log(n/z) + p \log^2 u)$.

Each couple of binary searches identifies ranges $[x_1, x_2] \times [y_1, y_2]$, inside which we extract every point. The $p$ range queries cost $O(p \log u)$ time. Further, each point $(x, y)$ extracted costs $O(\log u)$ and it identifies a primary occurrence at $S[R[y] - k..R[y] - k + p - 1]$. Therefore the total cost with $\text{occ}_p$ primary occurrences is $O(p^2 \log u \log(n/z) + p \log^2 u + \text{occ}_p \log u)$.

**Secondary occurrences.** Let $S[i..i + p - 1]$ be a primary occurrence. This is already a range $[i_0..i_0 + p - 1] = [i..i + p - 1]$ at level $l = 0$. We track the range down to positions $[i_1..i_1 + p - 1]$ at all the levels $l > 0$, using the position tracking mechanism described in Section 3.2. $i_{t+1} = (rank_1(D_t, i_t) - 1) \cdot b_t + ((i_t - 1) \mod b_t) + 1$, until the range is not completely contained in one or two consecutive marked blocks. Note that we only need to consider levels $l$ where the block length is $b_l \geq p$, as with shorter blocks there cannot be secondary occurrences. So we only consider the levels $l = 0$ to $l = \log(n/z) - \log p$.

For each valid range $[i_1..i_1 + p - 1]$, we determine the sources that contain the range, as their target will contain a secondary occurrence. Those sources must start between positions $k_l = i_l + p - b_l$ and $k'_l = i_l$. We find the positions $q_l = \text{select}_0(B_l, k_l)$ and $q'_l = \text{select}_0(B_l, k'_l + 1)$, thus the sources of interest are $\pi_i^{-1}(t)$, for $l = q_l - k_l + 1$ to $t = q'_l - k'_l - 1$.

To report the occurrence inside each such block $r = \pi_i^{-1}(t)$, we first find its position in the corresponding unmarked block in its level. The block starts at $pos = (\text{select}_0(D_l, r) - 1) \cdot b_l + 1 + i_l - (\text{select}_1(B_l, t) - t)$ in level $l$. We must project the position $pos$ upwards until reaching the level $l = 0$, where the positions correspond to those in $S$. To project $pos$ to level $l - 1$, we compute the block number $j = \lfloor pos/b_{l-1} \rfloor$, and update $pos \leftarrow (j - 1) \cdot b_{l-1} + ((pos - 1) \mod b_{l-1}) + 1$.

Therefore, the $\text{occ}_s$ secondary occurrences are reported from the $\text{occ}_p$ primary occurrences in time $O(\text{occ}_p(\log(n/z) - \log p) + \text{occ}_s(\log u + \log(n/z) - \log p)) = O(\text{occ}_p \log(n/z) + \text{occ}_s \log n)$\(^2\).

### 3.4 Space and Time

As described, our space is $3u \log n + O(u + \sigma \log u)$ bits and the total query cost, for the $\text{occ}$ primary and secondary occurrences, is $O(p^2 \log u \log(n/z) + p \log^2 u + \text{occ} \log n) = O(p^2 \log^2 n + \text{occ} \log n)$.

A way to reduce this cost is to replace the binary searches on $X$ and $Y$ with Patricia trees \cite{15}, so that the search time is not $O(p \log u \log(n/z))$ but just $O(p \log(n/z))$. A Patricia tree on $u$ strings can be represented in compact form using $O(u \log \sigma)$ bits \cite{14}. Therefore, the space raises to $3u \log n + O(u \log \sigma)$ bits and the query time decreases to $O(p^2 \log(n/z) + p \log u + \text{occ} \log n) = O(p^2 \log(n/z) + (p + \text{occ}) \log n)$.

If we do not care about the constants, we can also store a mapping from $X$ to $Y$ order in $G$. This raises the space to $4u \log n + O(u \log \sigma)$ bits, and the time $O(p \log u)$ disappears, leading to a search time complexity of $O(p^2 \log(n/z) + \text{occ} \log n)$.

**Theorem 1** A string $S[1..n]$ where the LZ77 parse produces $z$ phrases can be represented in $O(z \log(n/z))$ space so that any substring of length $\ell$ can be extracted in time $O(\ell \log(n/z))$ and the $\text{occ}$ occurrences of a pattern $P$ can be obtained in time $O(p^2 \log(n/z) + \text{occ} \log n)$.

Finally, we have assumed that the BT reaches single characters at the leaves. Note that, as soon as the blocks are of length $b_l = \log_{\sigma} u_l$, a pointer in $ptr_l$ uses the same space as storing the

\[\text{occ}_p \log(n/z) + \text{occ}_s \log n\].

\[\text{occ}_p \log(n/z) + \text{occ}_s \log n\].

\[O(\log \log(n/z))\].
text of the block directly. Thus we could take as the leaves of the BT those in the smallest level \(l\) where \(b_l \leq \log_\sigma u_l\). In this leaf level there are at most \(o^h \leq u_l\) marked blocks (recall that in the leaf level we mark only the blocks that do not appear earlier, not the block pairs). Those marked blocks \(B^1, B^2, \ldots\) are concatenated into a string \(S' = B^1\$B^2\$\ldots\), where \(\$\) is a special symbol, and we build an FM-index \([2]\) on \(S'\) that takes \(u_l \log \sigma (1 + o(1)) + O(u_l)\) bits, which replaces a similar term in our space complexity. To find the primary occurrences of \(P\) inside this level, we use the FM-index, which in time \(O(p + occ_p \log n)\) finds the \(occ_p\) primary occurrences inside leaf blocks, and then proceed similarly as before.

4 More Complex Queries

The copying structure of the BTs is simpler than that of LZ parsings: first, a block that is copied cannot be a source of other blocks. Second, within a level, all the sources are of the same length. We can take advantage of this more regular structure to perform other complex searches. In the worst case, we will return each answer in time proportional to \(occ_p\), the number of primary occurrences of \(P\). In highly repetitive texts, when there are many occurrences, most of them are secondary, so we expect \(occ_p\) to be small or significantly smaller than \(occ\).

4.1 Range-Restricted Searches

Assume the query not only specifies \(P[1..p]\), but also a range \([s..e+p−1]\) of \(S\) where the occurrences must appear. To carry out this search efficiently, we use a different structure to represent the permutations \(\pi_{1..u_l}\). Instead of that of Munro et al. \([10]\), we use a grid represented with a wavelet tree. The grid, of \(u_l \times u_l\), stores the points \((x, y) = (i, \pi_l(i))\). The space is as before, but both \(\pi_l(i)\) and \(\pi_l^{-1}(j)\) are obtained in time \(O(\log u_l) = O(\log u)\) \([19]\). A negative consequence of using this representation is that extracting a string of length \(\ell\) from \(S\) takes time \(O(\ell \log(n/z) \log u)\), and the search time related to \(p^2\) becomes \(O(p^2 \log(n/z) \log u)\), even if using Patricia trees.

The advantage of this representation is that, once we determine the range of source positions \([q_l − k_l + 1, q'_l − k'_l - 1]\) covering a primary occurrence \([i_l, i_l + p−1]\) in level \(l\), we can efficiently extract the targets that might be of interest due to the range restriction. In level \(l = 0\), the positions of interest are \([s_0, e_0] = [s, e]\). At any level, the range of blocks of interest is \([c_l, d_l] = [(s_l/b_l), (e_l/b_l)]\). Given \(s_l\), the starting position of interest in the next level is \(s_{l+1} = \text{rank}_1(D_l, c_0) \cdot b_l + 1\) if \(D_l[c_0] = 0\) and \(s_{l+1} = (\text{rank}_1(D_l, c_0) - 1) \cdot b_l + (s_l - 1 \mod b_l) + 1\) otherwise; \(e_{l+1}\) is computed analogously except that \(e_{l+1} = \text{rank}_1(D_l, d_0) \cdot b_l + 1\) if \(D_l[d_0] = 0\). Those ranges are computed once, in time \(O(\log(n/z))\).

Given a primary occurrence \([i, i + p−1]\), we report it if \(s \leq i \leq e\). Then we project it downwards to all the intervals \([i_l, i_l + p−1]\). For each such interval, we compute \(q_l, q'_l, k_l,\) and \(k'_l\) as for unrestricted queries, and then query the two-dimensional structure \(\pi_l\) for the points in \([q_l - k_l + 1, q'_l - k'_l - 1] \times [c_l, d_l]\). Every such point is converted into the initial position of a secondary occurrence, which is checked to be included in \([s_l, e_l]\). If it passes the check, the position is projected upwards towards the root. Note that, for each primary occurrence at each level, we may work on at most two occurrences that at the end are outside of the range: those that are copied to the blocks \(c_l\) and \(d_l\) (if they are unmarked). Therefore, we pay \(O(occ_p \log(n/z) \log u)\) to query the wavelet tree for each primary occurrence at each level, and this also absorbs the time to check the occurrences that are not reported. For each reported occurrence, instead, we pay \(O(\log u + \log(n/z)) = O(\log n)\).
time to extract it from the wavelet tree and project it upwards.

Overall, using Patricia trees our time is \(O((p^2 + \text{occ}_p) \log(n/z) \log u + \text{occ}_i \log n)\), where \(\text{occ}_i\) is the number of occurrences reported to be within the range.

### 4.2 Document Listing

Document listing is the problem of indexing a text collection formed by \(D\) documents (strings) and, given a pattern \(P[1..p]\), listing the \(\text{occ}_d\) distinct document ids where \(P\) occurs. Assume we concatenate those documents into a string \(S[1..n]\), using separators $. The problem can be solved in optimal time \(O(m + \text{occ}_d)\) using \(O(n \log n)\) bits of space \([17]\), and many attempts have been done solve it using less space \([18]\). For repetitive collections, Claude and Munro \([6]\) describe a technique based on grammar compression whose time depends linearly on the number of so-called primary occurrences of \(P\) (a concept similar to ours).

We show how our structure can be adapted to this problem as well. Let us use the same structures of Section 4.1. The wavelet trees that represent the permutations \(\pi_l\) can also return the leftmost occurrence in a range, in time \(O(\log u_l) = O(\log u)\) \([19]\). We first obtain the \(\text{occ}_p\) primary occurrences of \(P\), \([i, i + p - 1]\), project each of them to each level \(l\), obtaining \([i, i + p - 1]\], and also compute the corresponding values \(q_l, q'_l, k_l,\) and \(k'_l\).

Now we take the leftmost primary occurrence in \(S\), \(x\), and compute its document \(d\) by binary search on an array that indicates the starting position of each document in the concatenation \(S\). We then output \(d\) and find the last position \(s - 1\) of document \(d\) in \(S\). Note that we are only interested in further occurrences that start at \(s\) or later. We then project the interval of interest \([s, n]\) downwards to all the levels, obtaining \([s_l, e_l]\) and \([c_l, d_l]\) at level \(l\). At each level, for each primary occurrence mapped to that level with parameters \(q_l, q'_l, k_l,\) and \(k'_l\), we obtain the leftmost point in \([q_l - k_l + 1, q'_l - k'_l + 1] \times [c_l, d_l]\). The obtained point is checked to be in \([s_l, e_l]\). If it is before \(s_l\), we take instead the leftmost point in \([q_l - k_l + 1, q'_l - k'_l + 1] \times [c_l + 1, d_l]\). Then the point is mapped upwards to convert it to a position in \(S\). This is the next relevant occurrence stemming from that primary occurrence at that level. From all those, we take the smallest, report its document, and so on.

Each document we report requires recalculating all the leftmost relevant occurrences from all primary occurrences at all levels, so the time per reported document is \(O(\text{occ}_p \log(n/z) \log u)\). The overall time, with \(\text{occ}_d\) reported documents, is then \(O((p^2 + \text{occ}_p \text{occ}_d) \log(n/z) \log u)\). We can easily add a restriction on a range of documents where we want to perform document listing, by starting the range searches from the initial position \(s\) of the first of those documents, and finishing as soon as we report a document past the range.

### 5 Conclusions

We have proposed a way to build a self-index on the Block Tree \([1]\) data structure. The block tree obtains a compression related to the LZ77-parse of the string. If the parse uses \(z\) phrases, the Block Tree uses \(O(z \log(n/z))\) space, compared to a LZ77-compressor, which uses \(O(z)\) space. Our self-index, within the same asymptotic space, finds all the \(\text{occ}\) occurrences of a pattern \(P[1..p]\) in time \(O(p^2 \log(n/z) + \text{occ} \log n)\). We have also shown how to perform range-restricted searches and document listing with an efficiency comparable to that of current proposals.
The obvious future work is to implement the proposal and determine how efficient it is compared to current implementations [14, 7, 8, 5, 6].

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