Theory and Satellite Experiment for Critical Exponent $\alpha$ of $\lambda$-Transition in Superfluid Helium

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On the basis recent seven-loop perturbation expansion for $\nu^{-1} = 3/(2 - \alpha)$ we perform a careful reinvestigation of the critical exponent $\alpha$ governing the power behavior $|T_c - T|^{-\alpha}$ of the specific heat of superfluid helium near the phase transition. With the help of the field theoric $\nu$ extremely well with the most recent theoretical determination of $\alpha$, therefore extremely welcome since it comes much closer to previous theoretical values. In fact, it turned out to agree with the space shuttle experimental value $\alpha = -0.01056 \pm 0.00038$.

1. The critical exponent $\alpha$ characterizing the power behavior $|T_c - T|^{-\alpha}$ of the specific heat of superfluid helium near the transition temperature $T_c$ is presently the best-measured critical exponent of all. A microgravity experiment in the Space Shuttle in October 1992 rendered a value with amazing precision

$$\alpha^{ss} = -0.01056 \pm 0.00038.$$  \hfill (1)

This represents a considerable change and improvement of the experimental number found a long time ago on earth by G. Ahlers [2]:

$$\alpha = -0.026 \pm 0.004,$$  \hfill (2)

in which the sharp peak of the specific heat was broadened to $10^{-6}$ K by the tiny pressure difference between top and bottom of the sample. In space, the temperature could be brought to within $10^{-8}$ K close to $T_c$ without seeing this broadening.

The exponent $\alpha$ is extremely sensitive to the precise value of the critical exponent $\nu$ which determines the growth of the coherence length when approaching the critical temperature, $\xi \propto |T - T_c|^{-\nu}$. Since $\nu$ lies very close to $2/3$, and $\alpha$ is related to $\nu$ by the scaling relation $\alpha = 2 - 3\nu$, a tiny change of $\nu$ produces a large relative change of $\alpha$. Ahlers’ value was for many years an embarrassment to quantum field theorists who never could find $\alpha$ quite as negative — the field theoretic $\nu$-value came usually out smaller than $\nu_{\text{Ah}} = 0.6753 \pm 0.0013$. The space shuttle measurement was therefore extremely welcome since it comes much closer to previous theoretical values. In fact, it turned out to agree extremely well with the most recent theoretical determination of $\alpha$ by strong-coupling perturbation theory [3] based on the recent seven-loop power series expansions of $\nu$ [4,7,8], which gave

$$\alpha^{sc} = -0.0129 \pm 0.0006.$$  \hfill (3)

The purpose of this Letter is to present yet another resummation of the perturbation expansion for $\nu^{-1}$ and for $\alpha = 2 - 3\nu$ by variational perturbation theory, applied in a different way than in [3]. Since it is a priori unclear which of the two results should be more accurate, we combine them to the slightly less negative average value with a larger error

$$\alpha^{sc} = -0.01126 \pm 0.0010.$$  \hfill (4)

Before entering the more technical part of the paper, a few comments are necessary on the reliability of error estimates for any theoretical result of this kind. They can certainly be trusted no more than the experimental numbers. Great care went into the analysis of Ahlers’ data [2]. Still, his final result [2] does not accommodate the space shuttle value [1]. The same surprise may happen to theoretical results and their error limits in papers on resummation of divergent perturbation expansions, since there exists so far no safe way of determining the errors. The expansions in powers of the coupling constant $g$ are strongly divergent, and one knows accurately only the first seven coefficients, plus the leading growth behavior for large orders $k$ like $\gamma(-\alpha)^k k! \Gamma(k + b)$. The parameter $b$ is determined by the number of zero modes in a solution to a classical field equation, $a$ is the inverse energy of this solution, and $\gamma$ the entropy of its small oscillations.

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large-order behavior of the expansion coefficients is precisely the virtue of variational perturbation theory, which we have therefore chosen for the resummation of which naturally incorporates his power behavior should converge faster than those which ignore it. This incorporation

\[ \alpha \]

we reexpand the series

\[ \phi \]

critical exponent \[ \alpha \]

closest distance to the expansion point

studies of divergent series [6].

be made much smaller than the distance between the last two approximations, as has been verified in many model studies of divergent series [6].

In the previous paper [5] we have done so by choosing the nonleading parameters

[54x352]h

critical exponent of approach to scaling

By fitting the expansion coefficients with the theoretical large-order behavior (5), this series has been extended to higher orders as follows [5]

\[ \Delta \nu^{-1} = 15.75313406543747 g_0^8 - 35.2944 g_0^9 + 82.690091520064 g_0^{10} - 202.094 g_0^{11} + 514.3394395526179 g_0^{12} \]

\[ - 1361.42 g_0^{13} + 3744.242656157152 g_0^{14} - 10691.7 g_0^{15} + \ldots . \]

(7)

The seven-loop power series expansion for \( \nu \) in powers of the unrenormalized coupling constant of O(2)-invariant \( \phi^4 \)-theory which lies in the universality class of superfluid helium reads [6]

\[ \nu^{-1} = 2 - 0.4 g_0 + 0.4681481481482289 g_0^2 - 0.66739 g_0^3 + 1.079261838589703 g_0^4 - 1.91274 g_0^5 \]

\[ + 3.644347291527398 g_0^6 - 7.37808 g_0^7 + \ldots . \]

(6)

By fitting the expansion coefficients with the theoretical large-order behavior (5), this series has been extended to higher orders as follows [6]

\[ s = \frac{d \log g(g_0)}{d \log g_0} = 1 - g_0 + \frac{947 g_0^2}{675} - 2.322324349407407 g_0^3 + 4.276203609026057 g_0^4 \]

\[ - 8.51611440473227 g_0^5 + 18.05897631325589 g_0^6 + \ldots . \]

A similar best fit of these by the theoretical large-order behavior extends this series by
\[ \Delta s = 40.38657228730114 \alpha_9^7 + 94.6453399123477 \alpha_9^8 - 231.3922442162566 \alpha_9^9 + 588.3206172579102 \alpha_9^{10} - 1552.116358404217 \alpha_9^{11} + 4242.372685080157 \alpha_9^{12} - 12001.18866491822 \alpha_9^{13} + 35115.23006646194 \alpha_9^{14} - 106234.4643086436 \alpha_9^{15} + 332239.2175082959 \alpha_9^{16} + \ldots. \] (9)

Scaling implies that \( g(\alpha_9) \) becomes a constant for \( \alpha_9 \to \infty \), implying that the power \( s \) goes to zero in this limit. By inverting the expansion for \( s \), we obtain an expansion for \( \nu^{-1} \) in powers of \( \nu \equiv 1 - s \) as follows:

\[
\nu^{-1}(\nu) = 2 - 0.4 \nu - 0.093037 \nu^2 + 0.000485012 \nu^3 - 0.0139286 \nu^4 + 0.007349 \nu^5 - 0.0140478 \nu^6 + 0.0159545 \nu^7 - 0.029175 \nu^8 + 0.0521537 \nu^9 - 0.102226 \nu^{10} + 0.224026 \nu^{11} - 0.491045 \nu^{12} + 1.22506 \nu^{13} - 3.00608 \nu^{14} + 8.29528 \nu^{15} - 22.5967 \nu^{16}. \] (10)

This series has to be evaluated at \( \nu = 1 \). For estimating the systematic errors of our resummation, we also calculate from (11) a series for \( \alpha = 2 - 3\nu \)

\[
\alpha(\nu) = 0.5 - 0.3 \nu - 0.129778 \nu^2 - 0.0395474 \nu^3 - 0.0243203 \nu^4 - 0.0032498 \nu^5 - 0.0121091 \nu^6 + 0.00749308 \nu^7 - 0.0194876 \nu^8 + 0.0320172 \nu^9 - 0.0651726 \nu^{10} + 0.14422 \nu^{11} - 0.315055 \nu^{12} + 0.802395 \nu^{13} - 1.95455 \nu^{14} + 5.49143 \nu^{15} - 14.8771 \nu^{16} + \ldots. \] (11)

3. In order to get a rough idea about the behavior of the reexpansions in powers of \( \nu \), we plot their partial sums at \( \nu = 1 \) in the upper row of Fig. 1.

![Figure 1](image-url)

**FIG. 1.** Upper plots: Results of partial sums of series (10) for \( \nu^{-1} \) up to order \( N \), once plotted as \( \nu_N = 1/\nu_N^{-1} \), and once as \( \alpha_N = 2 - 3\nu_N \). The third plot shows the corresponding partial sums of the series for \( \alpha \). The dotted line is the experimental space shuttle value \( \alpha^{exp} \) of Eq. (11). Lower plots: The corresponding resummed values and a fit of them by \( c_0 + c_1/N^2 + c_2/N^4 \). The constant \( c_0 \) is written on top, together with the seventh-order approximation (in parentheses). The square brackets on top of the left-hand plot for \( \nu \) show the corresponding \( \alpha \)-values.

After an initial apparent convergence, these show the typical divergence of perturbation expansions.

A rough resummation is possible using Padé approximants. The results are shown in Table 4. The highest Padé approximants yield

\[
\alpha^{Pad} = -0.0123 \pm 0.0050. \quad (12)
\]

The error is estimated by the distance to the next lower approximation.

4. We now resum the expansions \( \nu^{-1}(\nu) \) and \( \alpha(\nu) \) by variational perturbation theory. This is applicable to divergent perturbation expansions.
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{13} \]

which behave for large \( x \) like

\[ f(x) = x^{p/q} \sum_{m=0}^{\infty} b_m x^{-2m/q} \tag{14} \]

It is easy to adapt our function to this general behavior. Plotting the successive truncated power series for \( \nu^{-1}(h) \) against \( h \) in Fig. 2, we see that this function will have a zero somewhere above \( h = h_0 = 3 \).

![FIG. 2. Successive truncated expansions of \( \nu^{-1}(h) \) of orders \( N = 2, \ldots, 12 \).](image)

We therefore go over to the variable \( x \) defined by \( h = h(x) \equiv h_0 x / (h_0 - 1 + x) \), in terms of which \( f(x) = \nu^{-1}(h(x)) \) behaves like \( \left[ \frac{1}{1 + A} \right] \) with \( p = 0 \) and \( q = 2 \), and has to be evaluated at \( x = 1 \). The large-\( x \) behavior is imposed upon the function with the expansion \( \left[ \frac{1}{1 + A} \right] \) as follows. We insert an auxiliary scale parameter \( \kappa \) and define the truncated functions

\[ f_N(x) = \kappa^p \sum_{n=0}^{N} a_n \left( \frac{x}{\kappa^q} \right)^n. \tag{15} \]

The parameter \( \kappa \) will be set equal to 1 at the end. Then we introduce a variational parameter \( K \) by the replacement

\[ \kappa \to \sqrt{K^2 + \kappa^2 - K^2}. \tag{16} \]

The functions \( f_N(x) \) are so far independent of \( K \). This is changed by expanding the square root in \( \left[ \frac{1}{1 + A} \right] \) in powers of \( \kappa^2 - K^2 \), thereby treating this difference as a quantity of order \( x \). This transforms the terms \( \kappa^p x^n / \kappa^{qn} \) in \( \left[ \frac{1}{1 + A} \right] \) into polynomials of \( r \equiv (\kappa^2 - K^2) / K^2 \):

\[ \kappa^{n} x^n / \kappa^{qn} \to K^{p} x^n / K^{qn} \left[ 1 + \left( \frac{(p - qn)/2}{1} \right) r + \left( \frac{(p - qn)/2}{2} \right) r^2 + \ldots + \left( \frac{(p - qn)/2}{N-n} \right) r^{N-n} \right], \tag{17} \]

Setting now \( \kappa = 1 \), and replacing the variational parameter \( K \) by \( v \) defined by \( K^2 \equiv x/v \), we obtain from \( \left[ \frac{1}{1 + A} \right] \) at \( x = 1 \) the variational expansions

\[ f_N(v) = \sum_{n=0}^{N} a_n v^{q_n-p/2} [1 + (v-1)]^{(p-qn)/2}_{N-n}, \tag{18} \]

where the symbol \([1 + A]^{(p-qn)/2}_{N-n}\) is a short notation for the binomial expansion of \((1 + A)^{(p-qn)/2}\) in powers of \( A \) up to the order \( A^{N-n} \).

The variational expansions are optimized in \( v \) by minima for odd, and by turning points for even \( N \), as shown in Fig. 3. The extrema are plotted as a function of the order \( N \) in the lower row of Fig. 3. The left-hand plot shows directly the extremal values of \( \nu^{-1}(v) \), the middle plot shows the \( \alpha \)-values \( \alpha_N = 2 - 3 \nu_N \) corresponding to these. The right-hand plot, finally, shows the extremal values of \( \alpha_N(v) \). All three sequences of approximations are fitted very well by a large \( N \) expansion \( c_0 + c_1 / N^2 + c_2 / N^4 \), if we omit the lowest five data points which are not yet very regular. The inverse powers 2 and 4 of \( N \) in this fit are determined by starting from a more general ansatz \( c_0 + c_1 / N^{p_1} + c_2 / N^{p_2} \) and varying \( p_1, p_2 \) until the sum of the square deviations of the fit from the points is minimal.
FIG. 3. Successive variational functions $\nu_{N}^{-1}(h)$ and $\alpha_{N}(h)$ with $N = 3, \ldots, 12$ of Table I plotted for $h = x = 1$ against the variational parameter $K = \sqrt{x/v}$, together with their minima for odd $N$, or turning points for even $N$. These points are plotted against $N$ in the lower row of Fig. 1, where they are extrapolated to $N \to \infty$, yielding the critical exponents.

The highest-order data point is taken to be the one with $N = 12$ since, up to this order, the successive asymptotic values $c_{0}$ change monotonously by decreasing amounts. Starting with $N = 13$, the changes increase and reverse direction. In addition, the mean square deviations of the fits increasing drastically, indicating a decreasing usefulness of the extrapolated expansion coefficients in (7) and (9) for the extrapolation $N \to \infty$. From the parameter $c_{0}$ of the best fit for $\alpha$ which is indicated on top of the lower right-hand plot in Fig. 1, we find the critical exponent $\alpha = -0.01126$ stated in Eq. (4), where the error estimate takes into account the basic systematic errors indicated by the difference between the resummation of $\alpha = 2 - 3 \nu$, and of $\nu^{-1}$, which by the lower middle plot in Fig. 1 yields $\alpha = -0.01226$. It also accommodates our earlier seven-loop strong-coupling result (3) of Ref. [5]. The dependence on the choice of $h_{0}$ is negligible as long as the resummed series $\nu^{-1}(x)$ and $\alpha(x)$ do not change their Borel character. Thus $h_{0} = 2.2$ leads to results well within the error limits in (4).

Our number as well as many earlier results are displayed in Fig. 4. The entire subject is discussed in detail in the textbook H. Kleinert and V. Schulte-Frohlinde, Critical Exponents from Five-Loop Strong-Coupling $\phi^{4}$-Theory in $4-\epsilon$ Dimensions, World Scientific, Singapore, 2000 (http://www.physik.fu-berlin.de/~kleinert/re.html#b8).

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Note added in proof:
A recent calculation of $\alpha$ by an improved high-temperature expansion yields the exponent $\alpha = -0.0150(17)$ [M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 61, 5905 (2000)].

FIG. 4. Survey of experimental and theoretical values for $\alpha$. The latter come from resummed perturbation expansions of $\phi^{4}$-theory in $4 - \epsilon$ dimensions, in three dimensions, and from high-temperature expansions of XY-models on a lattice. The sources are indicated below.
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Note that in the journal version, the expansion for $\eta_m = 2 - \nu^{-1}$ in Eq. (61) of the first paper contains a misprinted sign of the $\hat{g}^2$-term, which must be alternating.

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TABLE II. Variational reexpansions of $\nu_N^{-1}(h)$ and $\alpha_N(h)$ for $N = 2, \ldots, 9$ at $h = x = 1$ which are plotted in Fig. 3 and whose minima and turning points are extrapolated to $N = \infty$ in the lower left- and right-hand plots of Fig. 3. The lists are carried only to $N = 9$, to save space, whereas the plots are for $N = 3, \ldots, 12$.

| $\nu_2^{-1}$ | $= 2 - 1.2v + 0.69067v^2$ |
| $\nu_3^{-1}$ | $= 2 - 1.8v + 2.07200v^2 - 0.72036v^3$ |
| $\nu_4^{-1}$ | $= 2 - 2.4v + 4.14400v^2 - 2.88145v^3 + 0.53412v^4$ |
| $\nu_5^{-1}$ | $= 2 - 3.0v + 6.90667v^2 - 7.20363v^3 + 2.67060v^4 + 0.28949v^5$ |
| $\nu_6^{-1}$ | $= 2 - 3.6v + 10.36000v^2 - 14.4073v^3 + 8.01180v^4 + 1.73692v^5 - 2.96286v^6$ |
| $\nu_7^{-1}$ | $= 2 - 4.2v + 14.50400v^2 - 25.2127v^3 + 18.6942v^4 + 6.07922v^5 - 20.7401v^6 + 11.1835v^7$ |
| $\nu_8^{-1}$ | $= 2 - 4.8v + 19.3387v^2 - 40.3403v^3 + 37.3884v^4 + 16.2113v^5 - 82.9602v^6 + 89.4683v^7 - 36.9575v^8$ |
| $\nu_9^{-1}$ | $= 2 - 5.4v + 24.86400v^2 - 60.5105v^3 + 67.2992v^4 + 36.4753v^5 - 248.881v^6 + 402.607v^7 - 332.617v^8 + 121.914v^9$ |

| $\alpha_2$ | $= 0.5 - 0.90v + 0.33830v^2$ |
| $\alpha_3$ | $= 0.5 - 1.35v + 1.13490v^2 - 0.23997v^3$ |
| $\alpha_4$ | $= 0.5 - 1.80v + 2.26800v^2 - 1.07989v^3 + 0.025254v^4$ |
| $\alpha_5$ | $= 0.5 - 2.25v + 3.83000v^2 - 2.69972v^3 + 0.126271v^4 + 0.57604v^5$ |
| $\alpha_6$ | $= 0.5 - 2.70v + 5.74500v^2 - 5.39945v^3 + 0.378812v^4 + 3.45629v^5 - 2.19244v^6$ |
| $\alpha_7$ | $= 0.5 - 3.15v + 8.04300v^2 - 9.44903v^3 + 0.883895v^4 + 12.0970v^5 - 15.3471v^6 + 6.89011v^7$ |
| $\alpha_8$ | $= 0.5 - 3.60v + 10.724v^2 - 15.1184v^3 + 1.76779v^4 + 32.2587v^5 - 61.3884v^6 + 55.1208v^7 - 21.5704v^8$ |
| $\alpha_9$ | $= 0.5 - 4.05v + 13.788v^2 - 22.6777v^3 + 3.182020v^4 + 72.5821v^5 - 184.165v^6 + 248.044v^7 - 194.134v^8 + 70.781v^9$ |