VERLINDE BUNDLES OF FAMILIES OF HYPERSURFACES AND THEIR JUMPING LINES

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Abstract

Verlinde bundles are vector bundles $V_k$ arising as the direct image $\pi_* (\mathcal{L} \otimes \tau)$ of polarizations of a proper family of schemes $\pi : X \rightarrow S$. We study the splitting behavior of Verlinde bundles in the case where $\pi$ is the universal family $X \rightarrow |\mathcal{O}(d)|$ of hypersurfaces of degree $d$ in $\mathbb{P}^n$ and calculate the cohomology class of the locus of jumping lines of the Verlinde bundles $V_{d+1}$ in the cases $n = 2, 3$.

1 Introduction

Let $\pi : X \rightarrow S$ be a proper family of schemes with a polarization $\mathcal{L}$. For $k \geq 1$, if the sheaf $\pi_* (\mathcal{L} \otimes \tau)$ is locally free, we call it the $k$-th Verlinde bundle of the family $\pi$.

For example ([Iye13]), let $C \rightarrow T$ be a smooth projective family of curves of fixed genus. Consider the relative moduli space $\pi : \text{SU}(r) \rightarrow T$ of semistable vector bundles of rank $r$ and trivial determinant. This family is equipped with a polarization $\Theta$, the determinant bundle. The Verlinde bundles $\pi_* (\Theta)$ of this family are projectively flat ([Hit90],[ADPW91]), and their rank is given by the Verlinde formula.

In this article, we study the example of the universal family $\pi : X \rightarrow |\mathcal{O}(d)|$ of hypersurfaces of degree $d$ in the complex projective space $\mathbb{P}^n$, with $n > 1$. This family comes equipped with the polarization $\mathcal{L}$ given by the pullback of $\mathcal{O}(1)$ along the projection map $X \rightarrow \mathbb{P}^n$. For $k \geq 1$, the sheaf $\pi_* (\mathcal{L} \otimes \tau)$ is locally free, as can be seen by considering the structure sequence of an arbitrary hypersurface of degree $d$ in $\mathbb{P}^n$. For $k \geq 1$, we denote the $k$-th Verlinde bundle of the family $\pi$ by $V_k$.

To better understand $V_k$ we study its splitting type when restricted to lines in $|\mathcal{O}(d)|$.

Let $T \subseteq |\mathcal{O}(d)|$ be a line. On $T = \mathbb{P}^1$, we define the vector bundle $V_{k,T} := V_k|_T$. The splitting type of $V_{k,T}$ is the unique non-increasing tuple $(b_1, \ldots, b_{r(k)})$ of size $r(k) := \text{rk} V_k$ such that $V_{k,T} \cong \bigoplus_i \mathcal{O}(b_i)$. 

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The sequence (2.1) puts constraints on the $b_i$: they are all non-negative and they sum up to $d^k := \deg(V_k)$. The set of such tuples $(b_i)$ can be ordered by defining the expression $(b_i') \geq (b_i)$ to mean

$$
\sum_{i=1}^{s} b_i' \geq \sum_{i=1}^{s} b_i \text{ for all } s = 1, \ldots, r.
$$

With this definition, smaller types are more general: the vector bundle $O(b_i)$ on $P^1$ specializes to $O(b_i')$ in the sense of [Sha76] if and only if $(b_i') \geq (b_i)$.

If $d^k \leq r^k$, then the most generic possible type has thus the form $(1, \ldots, 1, 0, \ldots, 0)$. We call this the generic splitting type. A computation shows that $d^k \leq r^k$ if $k \leq 2d$.

We have the following result on the cohomology class of the set of jumping lines

$$
Z := \{ T \in \text{Gr}(1, |O(d)|) \mid V_{d+1, T} \text{ has non-generic type} \}
$$

in the Grassmannian of lines in $|O(d)|$:

**Theorem 1.1.** Let $n \leq 3$, let $Z$ be set of jumping lines of $V_{d+1}$, and let $[Z]$ be the class of $Z$ in the Chow ring $\text{CH}(\text{Gr}(1, |O(d)|))$. We have

$$
\dim Z = n + 1 + \binom{d - 1 + n}{n}.
$$

Furthermore, let $b$ range over the integers with the property $0 \leq b < \dim Z - \frac{\dim Z}{2}$ and define $a = \dim Z - b$, $a' = a + \frac{\dim Z - \dim Z}{2}$, $b' = b + \frac{\dim Z - \dim Z}{2}$.

(i) If $\dim Z$ is odd or $n = 2$, we have

$$
[Z] = \sum_{a,b} \left( \binom{a + 1}{n} \binom{b + 1}{n} - \binom{a + 2}{n} \binom{b}{n} \right) \sigma_{a',b'}.
$$

(ii) If $\dim Z$ is even and $n = 3$, we have

$$
[Z] = \sum_{a,b} \left( \binom{a + 1}{n} \binom{b + 1}{n} - \binom{a + 2}{n} \binom{b}{n} \right) \sigma_{a',b'} + \left( \frac{\dim Z}{2} + 2 \right) \left( \frac{\dim Z}{2} \right) \sigma_{\dim Z, \dim Z}.
$$

The computation is carried out by the method of undetermined coefficients, leading into various calculations in the Chow ring of the Grassmannian. The assumption $n \leq 3$ is needed for a certain dimension estimation.

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2 Attained splitting types

There exists a short exact sequence of vector bundles on $|\mathcal{O}(d)|$

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k - d)) \xrightarrow{M} \mathcal{O} \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \to V_k \to 0,$$  

(2.1)
as can be seen by taking the pushforward of a twist of the structure sequence of $X$ on $\mathbb{P}^n \times |\mathcal{O}(d)|$. The map $M$ is given by multiplication by the section

$$\sum_I \alpha_I \otimes x^I \in H^0(|\mathcal{O}(d)|, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)).$$

In particular, we have

$$r^{(k)} = \binom{k+n}{n} - \binom{k+n-d}{n}$$

and

$$d^{(k)} = \binom{k+n-d}{n}.$$

**Lemma 2.1.** Let $\mathcal{E}$ be a free $\mathcal{O}_{\mathbb{P}^1}$-module of finite rank, and let

$$0 \to \mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{E}'' \to 0$$

be a short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules. Given a splitting $\mathcal{E}'' = \mathcal{E}_1'' \oplus \mathcal{O}$, we may construct a splitting $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{O}$ such that the image of $\varphi$ is contained in $\mathcal{E}_1$. *Proof.* Define $\mathcal{E}_1 := \ker(\text{pr}_2 \circ \psi)$, which is a locally free sheaf on $\mathbb{P}^1$. By comparing determinants in the short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{O} \to 0$ we see that $\mathcal{E}_1$ is free, hence by an Ext$^1$ computation the sequence splits. The property $\text{im}(\varphi) \subseteq \mathcal{E}_1$ follows from the definition. $\square$

**Proposition 2.2.** Let $f_1, f_2 \in |\mathcal{O}(d)|$ span the line $T \subseteq |\mathcal{O}(d)|$ and let $p$ be the number of zero entries in the splitting type of $V_{k,T}$. Let $U := H^0(\mathbb{P}^n, \mathcal{O}(k - d))$. We have

$$p = \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1U + f_2U).$$

*Proof.* Define $E_1 := \ker(\text{pr}_2 \circ \psi)$, which is a locally free sheaf on $\mathbb{P}^1$. The map $M|_T$ sends a local section $\xi \otimes \theta$ to $s\xi \otimes f_1\theta + t\xi \otimes f_2\theta$. In particular, the image of $\mathcal{O}(-1) \otimes U$ is contained in $\mathcal{O} \otimes (f_1U + f_2U)$. It follows that $p \geq \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1U + f_2U)$. To prove the other inequality, consider the induced sequence

$$0 \to \mathcal{O}(-1) \otimes U \xrightarrow{M|_T} \mathcal{O} \otimes (f_1U + f_2U) \to \mathcal{E}'' \to 0$$

and assume for a contradiction that $\mathcal{E}'' \cong \mathcal{E}_1'' \oplus \mathcal{O}$. By Lemma 2.1, we have a splitting $\mathcal{O} \otimes (f_1U + f_2U) \cong \mathcal{E}_1 \oplus \mathcal{O}$ such that $\text{im}(M|_T) \subseteq \mathcal{E}_1$. Consider the map $\widetilde{M}|_T: (\mathcal{O} \otimes U) \oplus (\mathcal{O} \otimes U) \to \mathcal{O} \otimes (f_1U + f_2U)$ defined by

$$\widetilde{M}|_T(a \otimes \theta, b \otimes \theta) = a \otimes f_1\theta + b \otimes f_2\theta.$$
We obtain the matrix description of $\tilde{M}|_T$ from the matrix description of $M|_T$ as follows. If $M|_T$ is represented by the matrix $A$ with coefficients $A_{i,j} = \lambda_{i,j}s + \mu_{i,j}t$, then $\tilde{M}|_T$ is represented by a block matrix

$$B = \left( \begin{array}{c|c} A' & A'' \end{array} \right)$$

with $A'_{i,j} = \lambda_{i,j}$ and $A''_{i,j} = \mu_{i,j}$.

The property $\text{im}(M|_T) \subseteq E_1$ implies that after some row operations, the matrix $A$ has a zero row. By the construction of $\tilde{M}|_T$, the same row operations lead to the matrix $B$ having a zero row, but this is a contradiction, since the map $\tilde{M}|_T$ is surjective.

Corollary 2.3. Let $T \subseteq |\mathcal{O}(d)|$ be a line spanned by the polynomials $f_1, f_2$. Assume that $d^{(k)} \leq r^{(k)}$. Let $\theta$ range over a monomial basis of $H^0(\mathbb{P}^n, \mathcal{O}(k - d))$. The bundle $V_{k,T}$ has the generic splitting type if and only if $\langle \alpha g_1, \alpha g_2 \rangle$ is a linearly independent set in $H^0(\mathbb{P}^n, \mathcal{O}(k))$.

Corollary 2.4. Let $T \subseteq |\mathcal{O}(d)|$ be a line spanned by the polynomials $f_1, f_2$, and let $d^{(k)} \leq r^{(k)}$. The bundle $V_{k,T}$ has not the generic type if and only if $\deg(\gcd(f_1, f_2)) \geq 2d - k$. In particular, if $d^{(k)} \leq r^{(k)}$ but $k > 2d$ then the generic type never occurs.

Proof. By Corollary 2.3, the bundle $V_{k,T}$ has non-generic type if and only if there exist linearly independent $g_1, g_2 \in H^0(\mathbb{P}^n, \mathcal{O}(k - d))$ such that $g_1f_1 + g_2f_2 = 0$. Let $h := \gcd(f_1, f_2)$ and $d' := \deg h$.

If $d' \geq 2d - k$ then $\deg(f_i/h) \leq k - d$ and we may take $g_1, g_2$ to be multiples of $f_i/h$ and $f_2/h$, respectively.

On the other hand, given such $g_1$ and $g_2$, we have $f_1 \mid g_2f_2$, which implies $f_1/h \mid g_2$, hence $d - d' \leq k - d$.

Proposition 2.5. Let $k = d + 1$. No types of $V_k$ other than $(1,\ldots,1,0,\ldots,0)$ and $(2,1,\ldots,1,0,\ldots,0)$ occur.

Proof. Assume that the type of $V_k$ at some line $(f_1, f_2)$ is other than the two above. Then the type has at least two more zero entries than the general type. By Proposition 2.2, we have $\dim \langle f_1\theta, f_2\theta \mid \theta \rangle \leq 2d^{(k)} - 2$, so we find $g_1, g_2, g'_1, g'_2 \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ and two linearly independent equations

$$g_1f_1 + g_2f_2 = 0$$

$$g'_1f_1 + g'_2f_2 = 0,$$

with both sets $(g_1, g_2), (g'_1, g'_2)$ linearly independent. From the first equation it follows that $f_1 = g_2h$ and $f_2 = -g_1h$, for some common factor $h$. Applying this to the second equation, we find $g'_1g_2 = g'_2g_1$, hence $g'_1 = \alpha g_1$ and $g'_2 = \alpha g_2$ for some scalar $\alpha$, a contradiction.

Corollary 2.6. Let $k = d + 1$, let $T \subset |\mathcal{O}(d)|$ be a line spanned by $f_1$ and $f_2$. The type $(2,1,\ldots,1,0,\ldots,0)$ occurs if and only if $\deg(\gcd(f_1, f_2)) \geq d - 1$. 

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3 The cohomology class of the set of jumping lines

**Definition 3.1.** Let \( k \geq 1 \) and \((b_i)\) be a splitting type for \( V_k \). We define the set \( Z_{(b_i)} \) of all points \( t \in \text{Gr}(1, |O(d)|) \) such that \( V_{k,t} \) has splitting type \((b_i)\). For the set of points \( t \) where \( V_{k,t} \) has generic splitting type we also write \( Z_{\text{gen}} \), and define the set of jumping lines \( Z := \text{Gr}(1, |O(d)|) \setminus Z_{\text{gen}} \).

Now let \( k = d + 1 \). By Corollary 2.6, \( Z \) is the subvariety given as the image of the finite, generically injective multiplication map

\[
\varphi : \text{Gr}(1, |O(1)|) \times |O(d-1)| \to \text{Gr}(1, |O(d)|)
\]

sending the tuple \(((sg_1 + tg_2)_{(s,t)} \in \mathbb{P}^1, h)\) to the line \((shg_1 + thg_2)_{(s,t)} \in \mathbb{P}^1\).

To perform calculations in the Chow ring \( A \) of \( \text{Gr}(1, |O(d)|) \), we follow the conventions found in [EH16]. Let \( N := \dim H^0(\mathcal{O}(d)) = \binom{n+d}{n} \). For \( N - 2 \geq a \geq b \), we have the Schubert cycle

\[
\Sigma_{a,b} := \{ T \in \text{Gr}(1, |O(d)|) : T \cap H \neq \emptyset, T \subset H' \},
\]

where \((H \subset H')\) is a general flag of linear subspaces of dimension \( N - a - 2 \) resp. \( N - b - 1 \) in the projective space \( |O(d)| \). The ring \( A \) is generated by the Schubert classes \( \sigma_{a,b} \) of the cycles \( \Sigma_{a,b} \). The class \( \Sigma_{a,b} \) has codimension \( a + b \), and we use the convention \( \sigma_a := \sigma_{a,0} \).

**Proof of Theorem 1.1.** We have \( \dim Z = n + 1 + \binom{d-1+n}{n} \) since \( Z \) is the image of the generically injective map \( \varphi \).

Let \( Q \subset |O(d)| \) be the image of the multiplication map

\[
f : |O(1)| \times |O(d-1)| \to |O(d)|.
\]

The map \( f \) is birational on its image, since a general point of \( Q \) has the form \( gh \) with \( h \) irreducible of degree \( d - 1 \). The Chow group \( A^{\text{codim} Z} \) is generated by the classes \( \sigma_{a',b'} \) with \( N - 2 \geq a' \geq b' \geq \frac{\text{codim} Z}{2} \) and \( a' + b' = \text{codim} Z \), while the complementary group \( A^{\text{dim} Z} \) is generated by the classes \( \sigma_{\text{dim} Z - b,b} \) with \( b = 0, \ldots, \lfloor \frac{\text{dim} Z}{2} \rfloor \). Write

\[
[Z] = \sum_{a',b'} \alpha_{a',b'} \sigma_{a',b'}.
\]

We have \( \sigma_{a',b'} \sigma_{a,b} = 1 \) if \( b' - b = \lfloor \frac{\text{codim} Z}{2} \rfloor \) and 0 else. Hence, multiplying the above equation with the complementary classes \( \sigma_{a,b} \) and taking degrees gives

\[
\alpha_{a',b'} = \deg([Z] \cdot \sigma_{a,b}).
\]

Using Giambelli’s formula \( \sigma_{a,b} = \sigma_a \sigma_b - \sigma_{a+1} \sigma_{b-1} \) [EH16, Prop. 4.16], we reduce to computing \( \deg([Z] \cdot \sigma_a \sigma_b) \) for \( 0 \leq b \leq \lfloor \frac{\text{dim} Z}{2} \rfloor \). By Kleiman transversality, we have

\[
\deg([Z] \cdot \sigma_a \sigma_b) = |\{ T \in Z : T \cap H \neq \emptyset, T \cap H' \neq \emptyset \}|,
\]
where $H$ and $H'$ are general linear subspaces of $|O(d)|$ of dimension $N - a - 2$ and $N - b - 2$, respectively.

To a point $p = g_ph_p \in Q$ with $g_p \in |O(1)|$ and $h_p \in |O(d - 1)|$, associate a closed reduced subscheme $\Lambda_p \subset Q$ containing $p$ as follows. If $h_p$ is irreducible, let $\Lambda_p$ be the image of the linear embedding $|O(1)| \times \{h_p\} \rightarrow |O(d)|$ given by $g \mapsto gh_p$.

If $h_p$ is reducible, define the subscheme $\Lambda_p$ as the union $\bigcup_h \text{im}(|O(1)| \times \{h\} \rightarrow |O(d)|)$, where $h$ ranges over the (up to multiplication by units) finitely many divisors of $p$ of degree $d - 1$.

Note that for all points $p$, the spaces $\text{im}(|O(1)| \times \{h\} \rightarrow |O(d)|)$ meet exactly at $p$.

By the definition of $Z$, all lines $T \in Z$ lie in $Q$. Furthermore, if $T$ meets the point $p$, then $T \subseteq \Lambda_p$. For $H \subseteq |O(d)|$ a linear subspace of dimension $N - a - 2$, define $Q' := H \cap Q$. For general $H$, the subscheme $Q'$ is a smooth subvariety of dimension $b - n + 1$ such that for a general point $p = gh$ of $Q'$ with $h \in |O(d)|$, the polynomial $h$ is irreducible.

Next, we consider the case $n = 2$ or dim $Z$ odd.

**Claim 3.1.1.** For general $H$, for each point $p \in Q'$ we have $\Lambda_p \cap H = \{p\}$.

**Proof.** Let $H$ denote the Grassmannian $\text{Gr}(\dim H + 1, N)$. Define the closed subset $X \subseteq Q \times H$ by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$ 

The fibers of the induced map $X \rightarrow H$ have dimension at least one. Hence, to prove that the desired condition on $H$ is an open condition, it suffices to prove $\dim(X) \leq \dim(H)$.

The fiber of the map $X \rightarrow Q$ over a point $p$ consists of the union of finitely many closed subsets of the form $X'_p = \{H \in H : \dim(H \cap \Lambda'_p) \geq 1\}$, where $\Lambda'_p \simeq \mathbb{P}^n \subseteq |O(d)|$ is one of the components of $\Lambda_p$. The space $X'_p$ is a Schubert cycle

$$\Sigma_{\dim Q - b, \dim Q - b} = \{H \in \text{Gr}(\dim H + 1, N) : \dim(H \cap H_{n+1}) \geq 2\},$$

with $H_{n+1}$ an $(n+1)$-dimensional subspace of $H^0(O(d))$. The codimension of the cycle is $2(\dim Q - b)$, hence also $\text{codim}(X'_p) = 2(\dim Q - b)$. Finally, we have $\dim(H) - \dim(X) = \text{codim}(X'_p) - \dim(Q) = \dim(Q) - 2b$.

If $\dim Z$ is odd, then $\dim Q - 2b \geq \dim Q - \dim Z + 1 = 3 - n \geq 0$. If $n = 2$, we instead estimate $\dim Q - 2b \geq \dim Q - \dim Z = 2 - n \geq 0$.

Next, let

$$\Lambda := \bigcup_{p \in Q'} \Lambda_p = f(|O(1)| \times \text{pr}_2 f^{-1}(Q'))$$

and

$$\Lambda'' := |O(1)| \times \text{pr}_2 f^{-1}(Q').$$
By the choice of $H$, the map $f^{-1}(Q') \to Q'$ is birational and the map $f^{-1}(Q') \to \text{pr}_2^{-1}(Q')$ is even bijective. It follows that $\Lambda''$ and hence $\Lambda$ have dimension $b + 1$.

The intersection of $\Lambda$ with a general linear subspace $H'$ of dimension $N - b - 2$ is a finite set of points. For each point $p \in Q'$, the linear subspace $H'$ intersects each component $\Lambda'_p$ of $\Lambda_p$ in at most one point. For each point $p' \in H' \cap \Lambda$ there exists a unique $p$ such that $p' \in \Lambda_p$.

The only line $T \in Z$ meeting both $p$ and $H'$ is the one through $p$ and $p'$. If the intersection $H' \cap \Lambda_p$ is empty, then there will be no line meeting $p$ and $H'$. Hence, $\deg([Z] \cdot \sigma_a \sigma_b)$ is the number of intersection points of $\Lambda$ with a general $H'$.

Finally, the pre-image $f^{-1}(Q') = f^{-1}(H)$ is smooth for a general $H$ by Bertini’s Theorem. If $\zeta$ is the class of a hyperplane section of $|O(d)|$ we have $f^*(\zeta) = \alpha + \beta$, where $\alpha$ and $\beta$ are classes of hyperplane sections of $|O(1)|$ and $|O(d)|$, respectively. Since $\text{pr}_2$ and $f$ have degree one, we compute

$$[\Lambda''] = [\text{pr}_2^{-1}\text{pr}_2 f^{-1}(H)] = \text{pr}_2^* f^*[H] = \left(\frac{\text{codim}\: H}{n}\right)_{\beta} \text{codim}\: H - n.$$ 

Hence, by the push-pull formula:

$$\deg([\Lambda] \cdot H') = \deg([\Lambda''] \cdot (\alpha + \beta)^\text{codim}\: H') = \left(\frac{\text{codim}\: H}{n}\right) \left(\frac{\text{codim}\: H'}{n}\right) = \left(\frac{a + 1}{n}\right) \left(\frac{b + 1}{n}\right).$$

We then use Giambelli’s formula to obtain Equation (1.1).

In case $n = 3$ and $\dim Z$ even, we show that for $b = \dim Z/2$ we have $\deg([Z] \cdot \sigma_{b,b}) = 0$. In this case, the hyperplanes $H$ and $H'$ have the same dimension $N - b - 2$.

For $p \in Q$, the set $\Lambda_p$ is defined as before.

**Claim 3.1.2.** For general $H$ of dimension $N - b - 2$, we have $\dim(\Lambda_p \cap H) = 1$.

**Proof.** Define as before the closed subset $X \subseteq Q \times H$ by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$ 

The generic fiber of the projection map $\varphi : X \to H$ is one-dimensional, hence we have $\dim \varphi(X) = \dim(X) - 1 = \dim H$. The last equation holds with $n = 3$ and $2b = \dim Z$. Hence for all $H \in \mathcal{H}$ we have $\dim(\Lambda_p \cap H) \geq 1$.

On the other hand, the equality $\dim(\Lambda_p \cap H) = 1$ is attained by some, and hence by a general, $H$. Indeed, Define the closed subset $X \subseteq Q \times H$ by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$ 

By a similar argument as before, one needs to show that $\dim(\mathcal{H}) - \dim(X) + 1 \geq 0$. The fiber $X_p$ is a Schubert cycle of codimension $3(\dim Q - b + 1)$. Lastly, a computation shows $\dim(\mathcal{H}) - \dim(X) + 1 = \text{codim}(X_p) - \dim(Q) + 1 = \frac{1}{2}(2 \dim Q + 18 - 5n) \geq 0$. □
Now, define $\Lambda''$ as above. We have $\dim \Lambda'' = \dim |\mathcal{O}(1)| + \dim \text{pr}_2 f^{-1}(Q') = b$. Since $f$ is generically of degree one, we still have $\dim \Lambda'' = \Lambda$, hence $\dim \Lambda + \dim H' = N - 2 < \dim |\mathcal{O}(d)|$. It follows that a generic $H'$ does not meet any of the lines $T \subset Z$, hence $\sigma_b \sigma_b \cdot [Z] = 0$. \hfill \qed

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