Some Properties of Dirac–Einstein Bubbles

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Abstract
We prove smoothness and provide the asymptotic behavior at infinity of solutions of Dirac–Einstein equations on \( \mathbb{R}^3 \), which appear in the bubbling analysis of conformal Dirac–Einstein equations on spin 3-manifolds. Moreover, we classify ground state solutions, proving that the scalar part is given by Aubin–Talenti functions, while the spinorial part is the conformal image of \(-\frac{1}{2}\)-Killing spinors on the round sphere \( S^3 \).

Keywords Dirac–Einstein equations · Ground state solutions · Killing spinors · Conformally covariant equations

Mathematics Subject Classification 53C27 · 58J90 · 81Q05

1 Introduction

Let \((M, g)\) be a three-dimensional closed oriented Riemannian manifold. It is well known [25, page 87] that such manifold admits a spin structure \( \sigma \).

Consider the energy functional \( \mathcal{E} \)

\[
\mathcal{E}(g, \psi) = \int_M R_g \, dv_g + \int_M \langle D_g \psi, \psi \rangle - \langle \psi, \psi \rangle \, dvol_g,
\]

where \( g \) is a Riemannian metric on \( M \), \( \psi \) is a spinor in the spin bundle \( \Sigma M \) on \( M \), \( R_g \) is the scalar curvature, \( D_g \) is the Dirac operator and \( \langle \cdot, \cdot \rangle \) is the compatible Hermitian metric on \( \Sigma M \). The functional \( \mathcal{E} \) generalizes the classical Hilbert–Einstein functional
and it is invariant under the group of diffeomorphisms of $M$ as well. In fact, as a physical model, the functional describes the interaction of spin particles (fermions) with a gravitational field. This model was investigated in [4,10,23], where the authors study it in its full generality and provide some properties of the critical points.

In [28] the restriction of this functional to a conformal class of the metric was studied. This conformal restriction leads to the functional

$$E(u, \psi) = \frac{1}{2} \left( \int_M u L_g u + \langle D_g \psi, \psi \rangle - |u|^2 |\psi|^2 \, dvol_g \right),$$

(1.1)

where $L_g$ is the conformal Laplacian of the metric $g$. This energy functional can also be seen as the three-dimensional version of the super-Liouville equation investigated in [20–22]. One then can easily see that the critical points of this functional satisfy the following Euler–Lagrange equation

$$\begin{cases}
L_g u = |\psi|^2 u \\
D_g \psi = |u|^2 \psi
\end{cases} \quad \text{on } M. \quad (1.2)$$

This system is actually critical since the functional is conformally invariant. This conformal invariance results in a bubbling phenomenon as detailed in [28], where the authors proved the following:

**Theorem 1.1** Let us assume that $M$ has a positive Yamabe constant $Y_g(M)$ and let $(u_n, \psi_n)$ be a Palais–Smale sequence for $E$ at level $c$. Then there exist $u_\infty \in C^{2,\alpha}(M)$, $\psi_\infty \in C^{1,\beta}(\Sigma M)$ such that $(u_\infty, \psi_\infty)$ is a solution of (1.2), $m$ sequences of points $x_1^n, \ldots, x_m^n \in M$ such that $\lim_{n \to \infty} x^n_k = x_k \in M$, for $k = 1, \ldots, m$ and $m$ sequences of real numbers $R_1^n, \ldots, R_m^n$ converging to zero, such that

i) $u_n = u_\infty + \sum_{k=1}^m v^n_k + o(1)$ in $H^1(M)$,

ii) $\psi_n = \psi_\infty + \sum_{k=1}^m \phi^n_k + o(1)$ in $H^\frac{1}{2}(\Sigma M)$,

iii) $E(u_n, \psi_n) = E(u_\infty, \psi_\infty) + \sum_{k=1}^m E_{\mathbb{R}^3}(U^k_\infty, \Psi^k_\infty) + o(1),$

Thanks to a symmetry trick as in [27,29], it is possible to show the existence of sign-changing solutions. This relies mainly on the fact that every non-trivial solution $(U, \Psi)$ on $\mathbb{R}^3$ (or on the sphere by stereographic projection) satisfies the following lower bound, proved in [28]:

$$E_{\mathbb{R}^3}(U, \Psi) \geq \tilde{Y}(S^3, g_0) := Y(S^3, g_0) \lambda^+(S^3, g_0),$$

(1.3)

where $Y(S^3, g_0)$ is the Yamabe invariant of the round 3-sphere, and

$$\lambda^+(S^3, g_0) := \inf_{g \in [g_0]} \lambda_1(S^3, g) \text{vol}(S^3, g)^{1/3}.$$
is the Hijazi–Bär–Lott invariant [11, Sect. 8.5]. Here $\lambda_1(S^3, g)$ denotes the first positive Dirac eigenvalue on $(S^3, g)$ and $[g_0]$ is the conformal class of the round metric.

In order for one to have a general existence theorem, one needs to understand these bubbles and classify them as is the case of the Yamabe problem in [9].

### 1.1 Main Results

We are then interested in the following coupled system:

\[
\begin{aligned}
-c_3 \Delta u &= |\psi|^2 u, & \text{on } \mathbb{R}^3, \\
\mathcal{D}\psi &= |u|^2 \psi
\end{aligned}
\] (1.4)

with $c_3 = 6$, that describes blow-up profiles stated above since $-c_3 \Delta = L_{g\mathbb{R}^3}$. The first main result of the paper deals with regularity and asymptotic behavior of distributional solution to (1.4).

**Theorem 1.2** Let $(u, \psi) \in L^6(\mathbb{R}^3) \times L^3(\Sigma \mathbb{R}^3)$ be a distributional solution to (1.4), then

\[(u, \psi) \in C^\infty(\mathbb{R}^3) \times C^\infty(\Sigma \mathbb{R}^3).\]

Moreover, the following complete asymptotic expansions hold.

There exist $z_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^3$, where at least one $z_\alpha \neq 0$, such that for any $M \in \mathbb{N}$ there is a $C_M < \infty$ such that

\[
\left| u(x) - |x|^{-1} \sum_{|\alpha| \leq M} \frac{x_\alpha}{|x|^{|\alpha|}} z_\alpha \right| \leq C_M |x|^{-2-M} \quad \text{for all } |x| \geq 1, \tag{1.5}
\]

There are $\zeta_\alpha \in \Sigma \mathbb{R}^3$, $\alpha \in \mathbb{N}_0^3$, where at least one $\zeta_\alpha \neq 0$, such that for any $M \in \mathbb{N}$ there is a $C_M < \infty$ such that

\[
\left| \psi(x) - |x|^{-2} \mathcal{U}(x) \sum_{|\alpha| \leq M} \frac{x_\alpha}{|x|^{|\alpha|}} \zeta_\alpha \right| \leq C_M |x|^{-3-M} \quad \text{for all } |x| \geq 1, \tag{1.6}
\]

where $x_\alpha = x_1^\alpha x_2^\alpha x_3^\alpha$ and $|\alpha|_1 = |\alpha_1| + |\alpha_2| + |\alpha_3|$, and the matrix $\mathcal{U}(x)$ is defined as

\[
\mathcal{U}(x) = \begin{pmatrix}
0 & -i \frac{x}{|x|} \\
\frac{x}{|x|} \cdot \sigma & 0
\end{pmatrix}, \quad x \in \mathbb{R}^3 \setminus \{0\},
\]

By the above theorem one immediately gets the following decay estimates.

**Corollary 1.3** Under the assumptions of Theorem 1.2 there holds

\[
|u(x)| \leq \frac{C}{1 + |x|}, \quad |\psi(x)| \leq \frac{C}{1 + |x|^2}, \quad x \in \mathbb{R}^3,
\]
for some \( C > 0 \). Moreover, if \( u \geq 0 \), then there exists \( \alpha \geq 0 \) such that

\[
\lim_{|x| \to \infty} (1 + |x|)u(x) = \alpha,
\]

and if \( \alpha = 0 \) then \( u = 0 \) and \( \psi = 0 \).

In fact the second assertion of the corollary follows from the strong maximum principle applied to the \( u_K \), the solution obtained after inversion (see Sect. 3.2).

Define the spaces

\[
D^1(\mathbb{R}^3) := \{ u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3) \}
\]

and

\[
D^{1/2}(\Sigma \mathbb{R}^3) := \{ \psi \in L^3(\Sigma \mathbb{R}^3) : |\xi|^{1/2}|\hat{\psi}(\xi)| \in L^2(\mathbb{R}^3) \},
\]

\( \hat{\psi} \) denoting the Fourier transform of \( \psi \).

We now consider solutions to (1.4), \( (u, \psi) \in D^1(\mathbb{R}^3) \times D^{1/2}(\Sigma \mathbb{R}^3) \), corresponding to critical points of the functional at infinity

\[
E_{\mathbb{R}^3}(u, \psi) = \frac{1}{2} \left( \int_{\mathbb{R}^3} c_3 |\nabla u|^2 + \langle D\psi, \psi \rangle - |u|^2 |\psi|^2 \, dx \right).
\]  

As recalled in the next section, the functional (1.7) and the equations (1.4) are conformally covariant, so that we can equivalently consider them on the round sphere \( (S^3, g_0) \). Then the functional reads

\[
E_{S^3}(u, \psi) = \frac{1}{2} \left( \int_{S^3} uL_{g_0}u + \langle D_{g_0}\psi, \psi \rangle - |u|^2 |\psi|^2 \, dvol_{g_0} \right),
\]

while the equations are given by

\[
\begin{cases}
L_{g_0}u = |\psi|^2 u, \\
D_{g_0}\psi = |u|^2 \psi
\end{cases}
\quad \text{on } S^3.
\]

**Definition 1.4** We say that a non-trivial solution \( (u, \psi) \in D^1(\mathbb{R}^3) \times D^{1/2}(\Sigma \mathbb{R}^3) \) is a ground state solution if equality in (1.3) holds, that is

\[
E_{S^3}(u, \psi) = \tilde{Y}(S^3, g_0).
\]  

**Theorem 1.5** Let \( (u, \psi) \in H^1(S^3) \times H^{1/2}(\Sigma_{g_0}S^3) \) be a ground state solution to (1.4), and assume that \( u \geq 0 \). Then, up to a conformal diffeomorphism, \( u \equiv 1 \) and \( \psi \) is a \((-\frac{1}{2})\)-Killing spinor. More precisely, there exists a \((-\frac{1}{2})\)-Killing spinor \( \Psi \) on \( (S^3, g_0) \) and a conformal diffeomorphism \( f \in \text{Conf}(S^3, g_0) \) such that

\[
u = (\text{det}(df))^{\frac{1}{6}}.
\]
and
\[ \psi = (\det(df))^{1/3} \beta_f^* g_0^1 (f^* \Psi), \]
where \( \beta_f^* g_0^1 \) is the spinor identification for conformal metrics.

Corollary 1.6 Let \((u, \psi) \in H^1(\mathbb{R}^3) \times H^{1/2}(\Sigma g_0^1 \mathbb{R}^3)\) be a ground state solution to (1.4), with \( u \geq 0 \). Then there exists \( \Phi_0 \in \Sigma \mathbb{R}^3 \) and \( x_0 \in \mathbb{R}^3, \lambda > 0 \) such that
\[ u(x) = \left( \frac{2\lambda}{\lambda^2 + |x - x_0|^2} \right)^{1/2}, \quad x \in \mathbb{R}^3 \quad (1.10) \]
and
\[ \psi(x) = \left( \frac{2\lambda}{\lambda^2 + |x - x_0|^2} \right)^{3/2} \left( \mathbb{I} - \gamma \left( \frac{x - x_0}{\lambda} \right) \right) \tilde{\Phi}_0, \quad x \in \mathbb{R}^3, \quad (1.11) \]
where \( \gamma(\cdot) \) denotes the Clifford multiplication.

These ground state solutions are of extreme importance, since one can show that they are non-degenerate, and as a consequence obtain existence of solutions to the perturbed problem on the sphere (see [14] for more details). We remark that the classification of all solutions of (1.4) having \( u \geq 0 \) remains an open problem though. Ground state bubbles for conformally invariant Dirac equations have been recently classified in [8]. Such equations appear in the blow-up analysis of critical Dirac equations on manifolds [19] and in the study of the spinorial Yamabe problem and related question, see, e.g., [2,13] and references therein. Recently, two-dimensional critical Dirac equations also attracted a considerable interest as effective models in mathematical physics, see, e.g., [5,6].

2 Some Preliminaries

In this section we recall some notions useful in the sequel, for the convenience of the reader. We refer, in particular, to [25] for more details on spin structures and the Dirac operator.

2.1 Spin Structure and the Dirac Operator

Let \((M, g)\) be an oriented Riemannian manifold, and let \( P_{SO}(M, g) \) be its frame bundle.

Definition 2.1 A spin structure on \((M, g)\) is a pair \((P_{Spin}(M, g), \sigma)\), where \( P_{Spin}(M, g) \) is a Spin\((n)\)-principal bundle and \( \sigma : P_{Spin}(M, g) \rightarrow P_{SO}(M, g) \) is a 2-fold covering map, which is the non-trivial covering \( \lambda : Spin(n) \rightarrow SO(n) \) on each fiber.

In other words, the quotient of each fiber by \( \{-1, 1\} \simeq \mathbb{Z}_2 \) is isomorphic to the frame bundle of \( M \), so that the following diagram commutes.
A Riemannian manifold \((M, g)\) endowed with a spin structure is called a spin manifold. In particular, the euclidean space \((\mathbb{R}^n, g_{\mathbb{R}^n})\) and the round sphere \((S^n, g_0)\), with \(n \geq 2\), admit a unique spin structure.

**Definition 2.2** The complex spinor bundle \(\Sigma M \to M\) is the vector bundle associated to the \(\text{Spin}(n)\)-principal bundle \(P_{\text{Spin}}(M, g)\) via the complex spinor representation of \(\text{Spin}(n)\).

The complex spinor bundle \(\Sigma M\) has rank \(N = 2\left\lfloor \frac{n}{2} \right\rfloor\). It is endowed with a canonical spin connection \(\nabla\) (which is a lift of the Levi-Civita connection, denoted by the same symbol) and a Hermitian metric \(g\) which will be abbreviated as \(\langle \cdot, \cdot \rangle\) if there is no confusion.

In particular, the spinor bundle of the euclidean space \(\mathbb{R}^n\) is trivial, so that we can identify spinors with vector-valued functions \(\psi: \mathbb{R}^n \to \mathbb{C}^N\).

The Clifford multiplication \(\gamma: TM \to \text{End}_\mathbb{C}(\Sigma M)\) verifies the Clifford relation

\[
\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X, Y)\text{Id}_{\Sigma M},
\]

for any tangent vector fields \(X, Y \in \Gamma(TM)\), and is compatible with the bundle metric \(g\).

Locally, taking an oriented orthonormal tangent frame \((e_j)\) the Dirac operator is defined as

\[
\mathcal{D}^g \psi := \gamma(e_j)\nabla^g_{e_j} \psi, \quad \psi \in \Gamma(\Sigma M).
\]

Given \(\alpha \in \mathbb{C}\), a non-zero spinor field \(\psi \in \Gamma(\Sigma M)\) is called \(\alpha\)-Killing if

\[
\nabla^g_X \psi = \alpha \gamma(X)\psi, \quad \forall X \in \Gamma(TM).
\]

For more information on Killing spinors we refer the reader to [11, Appendix A]. In the paper [26] Killing spinors on the round sphere \((S^n, g_0)\) are explicitly computed in spherical coordinates. On \((S^n, g_0)\) \(\alpha\)-Killing spinors only exist for \(\alpha = \pm 1/2\), and via the stereographic projection the pullback of the \(\pm 1/2\)-Killing spinors has the form

\[
\Psi(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{n}{2}} (I \pm \gamma_{g_0}(x)) \Phi_0, \quad (2.1)
\]

where \(I\) denotes the identity endomorphism of the spinor bundle \(\Sigma g_{\mathbb{R}^n} \mathbb{R}^n\) and \(\Phi_0 \in \mathbb{C}^N\), see, e.g., [2].
2.2 Conformal Covariance

In this section we recall known formulas that relate the Dirac and conformal Laplace operators for conformally equivalent metrics, see, e.g., [11,18].

To this aim we explicitly label the various geometric objects with the metric $g$, e.g., $\Sigma gM$, $\nabla g$, $Dg$, etc.

Let $f \in C^\infty(M)$ and consider the conformal metric $g_f = e^{2f}g$. This induces an isometric isomorphism of spinor bundles

$$\beta \equiv \beta_{g,g_f} : (\Sigma gM, g) \rightarrow (\Sigma g_n M, g_f).$$

There holds

$$Dg_f \beta (e^{-n-1/2} f \psi) = e^{-n+1/2} f \beta (Dg \psi),$$
$$Lg_f (e^{-n-2/2} f u) = e^{-n+2/2} f Lg u.$$

The following quantities are conformally invariant. Setting

$$\varphi := \beta (e^{-n-1/2} f \psi), \quad v := e^{-n-2/2} f u$$

there holds

$$\int_M uL_g u \, dvol_g = \int_M vL_{g_f} v \, dvol_{g_f},$$
$$\int_M \langle Dg \psi, \psi \rangle \, dvol_g = \int_M \langle Dg_f \varphi, \varphi \rangle \, dvol_{g_f},$$
$$\int_M |u|^6 \, dvol_g = \int_M |v|^6 \, dvol_{g_f},$$
$$\int_M |\psi|^3 \, dvol_g = \int_M |\varphi|^3 \, dvol_{g_f},$$
$$\int_M |u|^2 |\psi|^2 \, dvol_g = \int_M |v|^2 |\varphi|^2 \, dvol_{g_f}.$$

Consequently the action (1.1) is conformally invariant, and hence also equation (1.2).

In particular, by a conformal change of metric (1.4) can be reinterpreted as an equation on the round sphere $S^3$, where $N \in S^3$ is the north pole, we have $(\pi^{-1})^* g_0 = 4 (1 + |x|^2)^{-2} g_{\mathbb{R}^3}$ with $x \in \mathbb{R}^3$. Then if $(u, \psi) \in D^1(\mathbb{R}^3) \times D^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ is a weak solution to (1.2), consider $(u, \varphi)$ defined in by (2.3) with $e^{2f} = 4 (1 + |x|^2)^{-2} \pi$. Such pair is in $H^1(S^3) \times H^{1/2}(\Sigma g_0 S^3)$ by (2.4), and it is a weak solution to (1.8). Notice that a priori $(v, \varphi)$ is a weak solution only on $S^3 \setminus \{N\}$. However, the removability of the singularity at the north pole $N$ can be proved by a cut-off argument similar to the one contained in Sect. 3.2, see also [1, Theorem 5.1].
Moreover, by (2.4) the functional (1.1) is also conformally invariant, i.e.,

\[ E_{\mathbb{R}^3}(u, \psi) = E_{S^3}(v, \varphi). \]

### 3 Regularity and Asymptotics

#### 3.1 Regularity

We first need the following (see, e.g., [7,12]) Liouville-type result.

**Lemma 3.1** Let \( p, q \geq 1 \) and assume that \( (u, \psi) \in L^p(\mathbb{R}^3) \times L^q(\Sigma \mathbb{R}^3) \) satisfies

\[
\begin{aligned}
- c_3 \Delta u &= 0, \\
\mathcal{D} \psi &= 0
\end{aligned}
\]

on \( \mathbb{R}^3 \) in the sense of distributions. Then \( u \equiv 0 \) and \( \mathcal{D} \psi \equiv 0 \).

We now use the above lemma to rewrite the first equation in (1.4) in integral form. The Green’s functions of \(-c_3 \Delta\) and \(\mathcal{D}\) are given, respectively, by

\[
G(x - y) = -\frac{1}{4\pi |x - y|}, \quad \Gamma(x - y) = \frac{i}{4\pi} \alpha \cdot \frac{x - y}{|x - y|^3},
\]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and the \( \alpha_j \) are \( 4 \times 4 \) Hermitian matrices satisfying the anticommutation relations

\[
\alpha_j \alpha_k + \alpha_k \alpha_j = 2 \delta_{j,k}, \quad 1 \leq j, k \leq 3.
\]  

We choose the \( \alpha_j \) of a particular block-antidiagonal form, namely, let \( \sigma_1, \sigma_2, \sigma_3 \) be \( 2 \times 2 \) Hermitian matrices satisfying analogous anticommutation relations as in (3.1), that is,

\[
\sigma_j \sigma_k + \sigma_k \sigma_j = 2 \delta_{j,k}, \quad 1 \leq j, k \leq 3.
\]

Then the matrices

\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad 1 \leq j \leq 3,
\]

satisfy (3.1) and we shall work in the following with this choice. Such matrices exist and form a representation of the Clifford algebra of the euclidean space \( \mathbb{R}^3 \), so that

\[
\mathcal{D} := -i \alpha \cdot \nabla = -i \sum_{j=1}^{3} \alpha_j \partial_{x_j}.
\]
Lemma 3.2 If \((u, \psi) \in L^6(\mathbb{R}^3) \times L^3(\Sigma \mathbb{R}^3)\) solves (1.4) in the sense of distributions, then
\[
G * (|\psi|^2 u), \quad \Gamma * (|u|^2 \psi).
\] (3.2)

Proof We note the Green functions have the following weak-Lebesgue integrability, namely, \(G \in L^{3,\infty}\) and \(\Gamma \in L^{3/2,\infty}\). Since \(\psi \in L^3\) and \(u \in L^6\), we have \(|\psi|^2 u \in L^{6/5}\) and therefore by the weak Young inequality, the function
\[
\tilde{u} := G * (|\psi|^2 u)
\]
satisfies
\[
\tilde{u} \in L^6(\mathbb{R}^3).
\]
Moreover, it is easy to see that
\[
-c_3 \Delta \tilde{u} = |\psi|^2 u \quad \text{in } \mathbb{R}^3
\]
in the sense of distributions. This implies that
\[
-\Delta (u - \tilde{u}) = 0 \quad \text{in } \mathbb{R}^3
\]
in the sense of distributions and therefore, by Lemma 3.1, \(u = \tilde{u}\), as claimed. With obvious modifications the argument applies to \(\psi\), thus concluding the proof. \(\Box\)

We remark that regularity of weak solutions to (1.4) does not follow by standard bootstrap arguments, as we are dealing with critical equations. In the following proposition we follow the strategy of [7], where the authors deal with critical Dirac equations.

Proposition 3.3 Any distributional solution \((u, \psi) \in L^6(\mathbb{R}^3) \times L^3(\Sigma \mathbb{R}^3)\) to (1.4) is smooth.

Proof Since the non-linearity in (1.4) is smooth, it suffices to prove that \((u, \psi) \in L^{\infty}(\mathbb{R}^3) \times L^{\infty}(\Sigma \mathbb{R}^3)\), as standard elliptic regularity then applies to give smoothness.

We only deal with \(u\) as the argument for the spinorial part \(\psi\) follows along the same lines, with straightforward modifications.

Fix \(r > 6\). We claim that \(u \in L^r(\mathbb{R}^3)\) for \(6 \leq r < \infty\). To this aim, we prove that there exists \(C > 0\) such that for all \(M > 0\) there holds
\[
S_M := \sup \left\{ \left| \int_{\mathbb{R}^3} \bar{v} u \, dx \right| : \|v\|_{r'} \leq 1, \|v\|_{6/5} \leq M \right\} \leq C,
\]
so that
\[
\sup \left\{ \left| \int_{\mathbb{R}^3} \bar{v} u \, dx \right| : \|v\|_{r'} \leq 1, v \in L^{6/5} \right\} \leq C.
\]
and then by density and duality, \( u \in L^r \).

Now fix \( M > 0 \) and let \( \varepsilon > 0 \) be a constant to be chosen later. The function

\[
f_\varepsilon := |\psi|^2 \mathbf{1}_{\{\delta \leq |\psi| \leq \mu\}}
\]

is bounded and supported on a set of finite measure. Moreover, we have

\[
\| |\psi|^2 - f_\varepsilon \|^{3/2}_{3/2} = \int \{|\psi|^3 \mathbf{1}_{\{|\psi|^3 \leq \mu\}} \text{ d}x < \varepsilon
\]

for suitable \( \delta, \mu > 0 \), as \( \psi \in L^3 \). Define \( g_\varepsilon := |\psi|^2 - f_\varepsilon \) and take \( v \in L^{r'} \cap L^{6/5} \), with \( \|v\|_{r'} \leq 1 \) and \( \|v\|_{6/5} \leq M \).

By (3.2), there holds

\[
\int_{\mathbb{R}^3} v u \text{ d}x = \int_{\mathbb{R}^3} v(G * (f_\varepsilon u)) \text{ d}x + \int_{\mathbb{R}^3} v(G * (g_\varepsilon u)) \text{ d}x.
\]

Using Fubini’s theorem we can rewrite the second integral on the right-hand side, as follows:

\[
\int_{\mathbb{R}^3} v(G * (g_\varepsilon u)) \text{ d}x = \int_{\mathbb{R}^3} \text{ d}x \int_{\mathbb{R}^3} \text{ d}y (G(x - y)g_\varepsilon(y)) u(y)
\]

\[
= \int_{\mathbb{R}^3} \text{ d}y \int_{\mathbb{R}^3} \text{ d}x (G(x - y)\overline{v}(x)) g_\varepsilon(y) u(y)
\]

\[
= \int_{\mathbb{R}^3} (G * \overline{v}) g_\varepsilon u \text{ d}y.
\]

Applying again the same argument, and since \( u = G * (|\psi|^2)u \), we can further rewrite the last integral to get

\[
d \int_{\mathbb{R}^3} \overline{v} u \text{ d}x = \int_{\mathbb{R}^3} \overline{v}(G * (f_\varepsilon u)) \text{ d}x + \int_{\mathbb{R}^3} \overline{h_\varepsilon u} \text{ d}x,
\]

where

\[
h_\varepsilon := |\psi|^2 G * (g_\varepsilon (G * v)) .
\]

We now estimate the two terms in (3.4). Choosing \( s := \frac{3r}{3 + 2} \), the first integral in (3.4) can be bounded using the Hölder and Young inequalities

\[
\left| \int_{\mathbb{R}^3} \overline{v}(G * (f_\varepsilon u)) \text{ d}x \right| \leq \|v\|_{r'} \|G * (f_\varepsilon u)\|_{r} \leq \|v\|_{r'} \|G\|_{3, \infty} \|f_\varepsilon u\|_{s}
\]

\[
\leq \|v\|_{r'} \|G\|_{3, \infty} \|f_\varepsilon\|_{6/5} \|u\|_{6} \leq C_\varepsilon ,
\]

where the constant \( C_\varepsilon \) depends on \( \varepsilon, r, \psi, u \) but not on \( M \).
We now turn to the second integral on the right-hand side of (3.4). By (3.5), Hölder and Young inequalities give

\[\| h_\epsilon \|_{r'} \leq \| | \psi |^2 \|_{3/2} \| G \ast (g_\epsilon (G \ast v)) \|_{s'} \leq \| | \psi |^2 \|_{3/2} \| G \|_{3, \infty} \| g_\epsilon (G \ast v) \|_{r'} \]

\[\leq \| | \psi |^2 \|_{3/2} \| G \|_{3, \infty} \| g_\epsilon \|_{3/2} \| v \|_{r'} \leq C' \| g_\epsilon \|_{3/2} \| v \|_{r'},\]

where the constant \( C' > 0 \) depends on \( \psi \). Similarly, we get

\[\| h_\epsilon \|_{6/5} \leq \| | \psi |^2 \|_{3/2} \| G \ast (g_\epsilon (G \ast v)) \|_{6} \leq \| | \psi |^2 \|_{3/2} \| G \|_{3, \infty} \| g_\epsilon (G \ast v) \|_{6} \]

\[\leq \| | \psi |^2 \|_{3/2} \| G \|_{3, \infty} \| g_\epsilon \|_{3/2} \| (G \ast v) \|_{6} \leq \| | \psi |^2 \|_{3/2} \| G \|_{3, \infty} \| g_\epsilon \|_{3/2} \| v \|_{6/5} \]

\[\leq C' \| g_\epsilon \|_{3/2} \| v \|_{6/5},\]

(3.7)

The estimates (3.7) and (3.8) imply that

\[\left| \int_{\mathbb{R}^3} \bar{h}_\epsilon u\, dx \right| \leq C' \| g_\epsilon \|_{3/2} S_M \leq C' \varepsilon S_M,\]

by (3.3). Then taking \( \varepsilon = (2C')^{-1} \) and combining the above estimate with (3.6) we get

\[\left| \int_{\mathbb{R}^3} \bar{u} \, dx \right| \leq C'' + \frac{1}{2} S_M,\]

where \( C'' \) is the constant \( C_\varepsilon \) for \( \varepsilon = (2C')^{-1} \). Taking the supremum over all \( v \) we have

\[S_M \leq C'' + \frac{1}{2} S_M \implies S_M \leq 2C'',\]

and the claim is proved. A similar argument works for \( \psi \), so that we conclude that

\[(u, \psi) \in L^r(\mathbb{R}^3) \times L^s(\Sigma \mathbb{R}^3), \quad \text{for all } r \geq 6, s \geq 3,\]

so that \(|\psi|^2 u \in L^t(\mathbb{R}^3)\) for all \( t > 3/2 \). By (3.2), we have

\[u(x) = \int_{\mathbb{R}^3} G(x - y)|\psi(y)|^2 u(y) \, dy,\]

and then writing \( G \) as the sum of a function in \( L^p \) and one in \( L^q \), for some \( 1 < p < 3 < q \), the Hölder inequality gives \( u \in L^\infty \). Arguing along the same lines one proves that \( \psi \in L^\infty \), and the proof is concluded. \( \square \)
3.2 Existence of Asymptotics

The existence of an asymptotic expansion at infinity for solutions to (1.4) follows by the regularity results of the previous section, via the Kelvin transform (see [7, Section 4] for the spinorial case). Given a function \( u \) and a spinor \( \psi \) on \( \mathbb{R}^3 \), it is defined as

\[
\begin{aligned}
  u_K(x) &= |x|^{-1} u(x/|x|^2), \\
  \psi_K(x) &= |x|^{-2} U(x) \psi(x/|x|^2)
\end{aligned}
\]

where the unitary matrix \( U(x) \) is defined as

\[
U(x) = \begin{pmatrix} 0 & -i \frac{x}{|x|} \cdot \sigma \\ i \frac{x}{|x|} \cdot \sigma & 0 \end{pmatrix}, \quad x \in \mathbb{R}^3 \backslash \{0\},
\]

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) (see previous Section).

Let \( (u, \psi) \in L^6(\mathbb{R}^3) \times L^3(\Sigma \mathbb{R}^3) \) be a distributional solution to (1.4). Then, as proved in the previous section, \( (u, \psi) \in C^\infty(\mathbb{R}^3) \times C^\infty(\Sigma \mathbb{R}^3) \) and it solves the equations in the classical sense. The transformed couple \( (u_K, \psi_K) \) is smooth as well, and there holds

\[
\Delta u_K = |x|^{-4} (-\Delta u)_K, \quad D \psi_K = |x|^{-2} (D \psi)_K.
\]

The computation for the scalar part is a classical one from potential theory [15, Sect. 1.8], while the spinorial one can be found in [7, Sect. 4]. Using (3.9) and (3.10) it is not hardy to see that,

\[
\int_{\mathbb{R}^3} u_K(-\Delta u_K) \, dx = \int_{\mathbb{R}^3} u(-\Delta u) \, dx,
\]

\[
\int_{\mathbb{R}^3} (D \psi_K, \psi_K) \, dx = \int_{\mathbb{R}^3} (D \psi, \psi) \, dx
\]

\[
\int_{\mathbb{R}^3} |u_K|^6 \, dx = \int_{\mathbb{R}^3} |u|^6 \, dx,
\]

\[
\int_{\mathbb{R}^3} |\psi_K|^3 \, dx = \int_{\mathbb{R}^3} |\psi|^3 \, dx
\]

\[
\int_{\mathbb{R}^3} |u_K|^2 |\psi_K|^2 \, dx = \int_{\mathbb{R}^3} |u|^2 |\psi|^2 \, dx.
\]

Moreover, by (3.10), one can easily verify that \( (u_K, \psi_K) \) is smooth and solves (1.4) on \( \mathbb{R}^3 \backslash \{0\} \), and thus is a distributional solution on that set. We now prove that the equation is actually verified in distributional sense on the whole \( \mathbb{R}^3 \), that is, the origin is a removable singularity.
We only work out the argument for the scalar part $u_K$, as a similar argument applies to $\psi_K$. We aim at showing that
\[
\int_{\mathbb{R}^3} (-\Delta f) u_K \, dx = \int_{\mathbb{R}^3} f u_K |\psi_K|^2 \, dx, \quad \forall f \in C_c^\infty(\mathbb{R}^3). \tag{3.11}
\]
Let $\eta \in C_c^\infty(\mathbb{R}^3)$ be (real-valued) a cut-off such that $\eta \equiv 1$ outside the unit ball and $\eta \equiv 0$ near the origin, and define $\eta_\varepsilon(x) := \eta(x/\varepsilon)$. Since $\eta_\varepsilon f \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$, there holds
\[
\int_{\mathbb{R}^3} -\Delta(\eta_\varepsilon f) u_K \, dx = \int_{\mathbb{R}^3} \eta_\varepsilon f u_K |\psi_K|^2 \, dx, \quad \forall f \in C_c^\infty(\mathbb{R}^3). \tag{3.12}
\]
We have that $\eta_\varepsilon f u_K |\psi_K|^2 \in L^1$, $|\eta_\varepsilon| \leq \|\eta\|_\infty$ and $\eta_\varepsilon \to 1$ almost everywhere, as $\varepsilon \to 0$. Then, by dominated convergence, the right-hand side of (3.12) converges to the second integral in (3.11). Observe that
\[
-\Delta(\eta_\varepsilon f) = \eta_\varepsilon (-\Delta f) - 2\nabla \eta_\varepsilon \cdot \nabla f - (\Delta \eta_\varepsilon) f. \tag{3.13}
\]
Arguing as above, one sees that the contribution to (3.12) of the first term on the right-hand side of (3.13) converges to the first integral in (3.11). The other two terms go to zero, as $\varepsilon \to 0$. Indeed, there holds
\[
\left| \int_{\mathbb{R}^3} \nabla \eta_\varepsilon \cdot \nabla u_K f \, dx \right| = \left| \int_{\{|x| < \varepsilon\}} \nabla \eta_\varepsilon \cdot \nabla u_K f \, dx \right| \leq \|f\|_\infty \|\nabla \eta_\varepsilon\|_\infty \|\nabla u_K\|_2 \|\{|x| < \varepsilon\}\|^{1/2} \lesssim \sqrt{\varepsilon},
\]
and
\[
\left| \int_{\mathbb{R}^3} (-\Delta \eta_\varepsilon) u_K f \, dx \right| = \left| \int_{\{|x| < \varepsilon\}} (-\Delta \eta_\varepsilon) u_K f \, dx \right| \leq \|f\|_\infty \|\Delta \eta_\varepsilon\|_\infty \|u_K\|_6 \|\{|x| < \varepsilon\}\|^{5/6} \lesssim \sqrt{\varepsilon},
\]
Then by the regularity result of the previous section we conclude that $(u_K, \psi_K)$ is a smooth solution of (1.4) on $\mathbb{R}^3$. Formulas (1.5) and (1.6) follow by Taylor expanding $u_K$ and $\psi_K$ at the origin and taking the inverse Kelvin transform. Observe that at least one of the coefficients $z_\alpha$ in (1.5) must be non-zero, and the same holds for the $\zeta_\alpha$ in (1.6), as by unique continuation principle a non-trivial solution to (1.4) cannot have derivative of arbitrary order vanishing at a point (see [3] for the Laplacian and [24] concerning the Dirac operator).

4 Classification of Ground States

Proof of Theorem 1.5 Let $(u, \psi) \in H^1(S^3) \times H^{1/2}(\Sigma_{g_0} S^3)$ be a ground state solution to (1.8), with $u \geq 0$. By the strong maximum principle one infers that actually $u > 0$. Springer
Then consider the conformal change of metric
\[ g := \frac{4}{9} u^4 g_0, \quad \text{on } S^3, \]
and the corresponding isometry of spinor bundles \( \beta : \Sigma_{g_0} S^3 \to \Sigma_g S^3 \)
By formulas (2.2)
\[ v := \sqrt{\frac{3}{2}}, \quad \varphi := \frac{3}{2} u^{-2} \beta(\psi), \]
solves equations (1.8) in the metric \( g \), that is
\[
\begin{aligned}
L_g v &= |\varphi|^2 v, \\
D_g \varphi &= \frac{3}{2} \varphi.
\end{aligned}
on S^3.
\]
By the definition of the Yamabe invariant, we get
\[
Y(S^3, g_0) \left( \int_{S^3} u^6 \, dvol_{g_0} \right)^{1/3} \leq \int_{S^3} \bar{u} L_{g_0} u \, dvol_{g_0} = \int_{S^3} u^2 |\varphi|^2 \, dvol_{g_0},
\]
and since \((u, \psi)\) is a ground state, using (1.9) we find
\[
\left( \int_{S^3} u^6 \, dvol_{g_0} \right)^{1/3} \leq \lambda^+ (S^3, g_0).
\]
Observe that the volume of the metric \( g \) is
\[
\text{vol}(S^3, g) = \frac{8}{27} \int_{S^3} u^6 \, dvol_{g_0}
\]
The Hijazi’s inequality [16,17] gives
\[
\frac{3}{8} Y(S^3, g_0) \leq \frac{9}{4} \text{vol}(S^3, g)^{2/3} = \left( \int_{S^3} u^6 \, dvol_{g_0} \right)^{2/3} \leq \lambda^+ (S^3, g_0)^2.
\]
But since the last and the first term of these inequalities coincide, we have equality in the Hijazi inequality, and thus the spinor \( \varphi \) is Killing, with constant \(-\frac{1}{2}\), by the second equation in (1.8).
Combining this with the second equation in (1.8), we conclude that \( \varphi \) is a twistor spinor, so that by [11, Prop. A.2.1]
\[
\frac{9}{4} \varphi = (D_g)^2 \varphi = \frac{3}{8} R_g \varphi,
\]
and the scalar curvature of \( g \) is then \( R_g = R_{g_0} = 6 \). By a result of Obata [30] there exists an isometry
\[
f : (S^3, g) \to (S^3, g_0),
\]
such that $f^*g_0 = g = \frac{4}{9}u^4g_0$. Then we obtain

$$
\text{dvol } f^*g_0 = \det(df) = \left(\frac{8}{27}u^6\right) \text{dvol } g_0 \implies u = \sqrt{\frac{3}{2}} \det (df)^{1/6}.
$$

Recall that the isometry $f$ induces an isometry of spinor bundles $F$, so that the following diagram is commutative

$$(\Sigma f^*g_0 \mathbb{S}^3, f^*g_0) \xrightarrow{F} (\Sigma g_0 \mathbb{S}^3, g_0),
$$

$$(\mathbb{S}^3, f^*g_0) \xrightarrow{f} (\mathbb{S}^3, g_0),
$$

where the vertical arrows are the projections defining the spinor bundles.

Then the spinor $\Psi := F \circ \varphi \circ f^{-1}$ is $-\frac{1}{2}$-Killing with respect to the round metric $g_0$, as this property is clearly preserved by isometries. Thus we can rewrite

$$
\varphi = F^{-1} \circ \Psi \circ f \equiv f^*\Psi,
$$

and

$$
\psi = \frac{2}{3}u^2\beta^{-1}(\varphi) = \det(df)^{1/3} \beta^{-1}(f^*\Psi).
$$

□

**Proof of Corollary 1.6** Formula (1.10) is the well-known one for standard bubbles for the classical Yamabe problem [9].

So we only deal with the spinorial part of the solution. As recalled in (2.1), the pullback on $\mathbb{R}^3$ of $-\frac{1}{2}$-Killing spinors on the round sphere is given by

$$
\Psi(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{3}{2}} \left(1 - \gamma_{\mathbb{R}^3}(x)\right) \Phi_0, \quad x \in \mathbb{R}^3, \tag{4.1}
$$

where $\Phi_0 \in \mathbb{C}^4$. The other solutions to the second equation in (1.2) are obtained from those of the form (4.1) by applying conformal transformations of the euclidean space $(\mathbb{R}^3, g_{\mathbb{R}^3})$.

We deal first with the composition of translation and a scaling. For fixed $x_0 \in \mathbb{R}^3, \lambda > 0$, consider the map $f_{x_0, \lambda} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$
f_{x_0, \lambda}(x) := \frac{x - x_0}{\lambda}, \quad x \in \mathbb{R}^3.
$$

Then $f_{x_0, \lambda}^* g_{\mathbb{R}^3} = \lambda^{-2} g_{\mathbb{R}^3}$. Recall that the frame bundle is trivial, i.e., $P_{SO}(\mathbb{R}^3, g_{\mathbb{R}^3}) = \mathbb{R}^3 \times SO(3)$, so that $f_{x_0, \lambda}$ induces a map $\tilde{f}_{x_0, \lambda} : \mathbb{R}^3 \times SO(3) \rightarrow \mathbb{R}^3 \times SO(3)$, with

$$
\tilde{f}_{x_0, \lambda}(x, v_1, v_2, v_3) = (f_{x_0, \lambda}(x), v_1, v_2, v_3),
$$
acting as the identity on \( SO(3) \), which then lifts to a map on \( P_{Spin}(\mathbb{R}^3, g_{\mathbb{R}^3}) = \mathbb{R}^3 \times Spin(3) \) which also acts as the identity on the second component. Since also the spinor bundle is trivial, that is \( \Sigma \mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{C}^4 \), we finally get a map \( F_{x_0, \lambda} : \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{R}^3 \times \mathbb{C}^4 \), and the transformation on \( \Psi \) is given by

\[
\psi(x) = \beta_{\lambda^{-2} g_{\mathbb{R}^3}, g_{\mathbb{R}^3}} F_{x_0, \lambda}^{-1} \Psi(f_{x_0, \lambda}(x)) = \left( \frac{2\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{3}{2}} \beta_{\lambda^{-2} g_{\mathbb{R}^3}, g_{\mathbb{R}^3}} F_{x_0, \lambda}^{-1} \left( \mathbb{1} - \gamma_{\mathbb{R}^3} \left( \frac{x - x_0}{\lambda} \right) \right) \Phi_0,
\]

where \( \beta_{\lambda^{-2} g_{\mathbb{R}^3}, g_{\mathbb{R}^3}} \) is the conformal identification of spinors induced by the conformal change of metric.

By the above discussion we can take \( \beta_{\lambda^{-2} g_{\mathbb{R}^3}, g_{\mathbb{R}^3}} F_{x_0, \lambda}^{-1} \) to be the identity map, so that (1.11) holds.

To conclude the proof we need to prove that the transformations on spinors of the form (1.11), induced by a rotation on \( \mathbb{R}^3 \), give another spinor of the same form suitably choosing new parameters. Let \( R \in SO(3) \) and consider the induced map \( F_R : \Sigma \mathbb{R}^3 \to \Sigma \mathbb{R}^3 \). Taking a spinor of the form (1.11) we get

\[
F_R^{-1}(\psi(Rx)) = \left( \frac{2\lambda}{\lambda^2 + |Rx - x_0|^2} \right)^{\frac{3}{2}} \beta_{\lambda^{-2} g_{\mathbb{R}^3}, g_{\mathbb{R}^3}} F_R^{-1} \left( \mathbb{1} - \gamma_{\mathbb{R}^3} \left( \frac{Rx - x_0}{\lambda} \right) \right) \Phi_0 = \left( \frac{2\lambda}{\lambda^2 + |x - R^{-1}x_0|^2} \right)^{\frac{3}{2}} \left( \mathbb{1} - \gamma_{\mathbb{R}^3} \left( \frac{x - R^{-1}x_0}{\lambda} \right) \right) F_R^{-1}(\Phi_0),
\]

where we have used the fact that, given \( v \in \mathbb{R}^3, \psi \in \Sigma \mathbb{R}^3 \), there holds \( F_R \left( \gamma_{\mathbb{R}^3}(v) \psi \right) = \gamma_{\mathbb{R}^3}(Rv) F_R(\psi) \). Then the claim follows, as (4.2) gives a spinor of the form (1.11), with parameters \( R^{-1}x_0 \in \mathbb{R}^3, \lambda > 0 \) and \( F_R^{-1}(\Phi_0) \in \Sigma \mathbb{R}^3 \).

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**References**

1. Ammann, B.: The smallest Dirac eigenvalue in a spin-conformal class and cmc immersions. Commun. Anal. Geom. 17, 429–479 (2009)
2. Ammann, B., Humbert, E., Morel, B.: Mass endomorphism and spinorial Yamabe type problems on conformally flat manifolds. Commun. Anal. Geom. 14, 163–182 (2006)
3. Aronszajn, N.: A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, Tech. rep., KANSAS UNIV LAWRENCE, (1956)
4. Belgun, F.A.: The einstein-dirac equation on sasakian 3-manifolds. J. Geometry Phys. 37, 229–236 (2001)
5. Borrelli, W.: Stationary solutions for the 2D critical Dirac equation with Kerr nonlinearity. J. Differ. Equ. 263, 7941–7964 (2017)
6. Borrelli, W.: Weakly localized states for nonlinear Dirac equations. Calc. Var. Partial Differ. Equ. 57, 155 (2018)
7. Borrelli, W., Frank, R.L.: Sharp decay estimates for critical Dirac equations. Trans. Am. Math. Soc. 373, 2045–2070 (2020)
8. Borrelli, W., Malchiodi, A., Wu, R.: Ground state dirac bubbles and killing spinors, arXiv preprint arXiv:2003.03949, (2020)
9. Caffarelli, L.A., Gidas, B., Spruck, J.: Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth. Commun. Pure Appl. Math. 42, 271–297 (1989)
10. Finster, F., Smoller, J., Yau, S.-T.: Particlelike solutions of the einstein-dirac equations. Phys. Rev. D 59, 104020 (1999)
11. Ginoux, N.: The Dirac Spectrum. Lecture Notes in Mathematics, vol. 1976. Springer, Berlin (2009)
12. Greene, R.E., Wu, H.: Integrals of subharmonic functions on manifolds of nonnegative curvature. Invent. Math. 27, 265–298 (1974)
13. Grosse, N.: Solutions of the equation of a spinorial Yamabe-type problem on manifolds of bounded geometry. Commun. Partial Differ. Equ. 37, 58–76 (2012)
14. Guidi, C., Maalaoui, A., Martino, V.: Existence results for the conformal dirac-einstein system, arXiv preprint arXiv:2001.07412, (2020)
15. Helms, L.L.: Potential Theory, Universitext, 2nd edn. Springer, London (2014)
16. Hijazi, O.: A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors. Commun. Math. Phys. 104, 151–162 (1986)
17. Hijazi, O.: Première valeur propre de l’opérateur de dirac et nombre de yamabe, Comptes rendus de l’Académie des sciences. Série 1, Mathématique, 313, 865–868 (1991)
18. Hitchin, N.: Harmonic spinors. Adv. Math. 14, 1–55 (1974)
19. Isobe, T.: Nonlinear Dirac equations with critical nonlinearities on compact Spin manifolds. J. Funct. Anal. 260, 253–307 (2011)
20. Jevnikar, A., Malchiodi, A., Wu, R.: Existence results for a super-liouville equation on compact surfaces, arXiv preprint arXiv:1909.12260, (2019)
21. Jost, J., Wang, G., Zhou, C.: Super-liouville equations on closed riemann surfaces. Commun. Partial Differ. Equ. 32, 1103–1128 (2007)
22. Jost, J., Wang, G., Zhou, C., Zhu, M.: Energy identities and blow-up analysis for solutions of the super liouville equation. J. Math. et Appl. 92, 295–312 (2009)
23. Kim, E.C., Friedrich, T.: The einstein-dirac equation on riemannian spin manifolds. J. Geometry Phys. 33, 128–172 (2000)
24. Kim, Y.M.: Carleman inequalities for the Dirac operator and strong unique continuation. Proc. Am. Math. Soc. 123, 2103–2112 (1995)
25. Lawson, H.B., Michelsohn, M.-L.: Spin Geometry, Princeton Mathematical Series, vol. 38. Princeton University Press, Princeton, NJ (1989)
26. Lü, H., Pope, C.N., Rahmfeld, J.: A construction of Killing spinors on $S^n$. J. Math. Phys. 40, 4518–4526 (1999)
27. Maalaoui, A.: Infinitely many solutions for the spinorial Yamabe problem on the round sphere. NoDEA Nonlinear Differ. Equ. Appl. 23, 25 (2016)
28. Maalaoui, A., Martino, V.: Characterization of the Palais-Smale sequences for the conformal Dirac-Einstein problem and applications. J. Differ. Equ. 266, 2493–2541 (2019)
29. Maalaoui, A., Martino, V., et al.: Changing-sign solutions for the cr-yamabe equation. Differ. Integral Equ. 25, 601–609 (2012)
30. Obata, M.: The conjectures on conformal transformations of Riemannian manifolds. J. Differ. Geometry, 6, 247–258 (1971/72)

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