Density of smooth functions in Musielak–Orlicz spaces

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Abstract
We provide necessary and sufficient conditions for the space of smooth functions with compact supports \( C^\infty_c(\Omega) \) to be dense in Musielak–Orlicz spaces \( L^\Phi(\Omega) \) where \( \Omega \) is an open subset of \( \mathbb{R}^d \). In particular, we prove that if \( \Phi \) satisfies condition \( \Delta_2 \), the closure of \( C^\infty_c(\Omega) \cap L^\Phi(\Omega) \) is equal to \( L^\Phi(\Omega) \) if and only if the measure of singular points of \( \Phi \) is equal to zero. This extends the earlier density theorems proved under the assumption of local integrability of \( \Phi \), which implies that the measure of the singular points of \( \Phi \) is zero. As a corollary we obtain analogous results for Musielak–Orlicz spaces generated by double phase functional and we recover the well-known result for variable exponent Lebesgue spaces.

Keywords Musielak–Orlicz spaces · Variable exponent spaces · Density of smooth functions in Musielak–Orlicz space

Mathematics Subject Classification 46B42 · 46E30 · 46E15

We study here the problem of density of the space \( C^\infty_c(\Omega) \) of smooth functions with compact supports in a subspace of order continuous functions \( E^\Phi(\Omega) \) of Musielak–Orlicz spaces \( L^\Phi(\Omega) \) over an open set \( \Omega \subset \mathbb{R}^d \). This is a standard problem in function spaces and it has been considered before in \( L^\Phi(\Omega) \) under some restrictions on the function \( \Phi \). For rearrangement invariant spaces, and in particular Orlicz spaces, it is well known and standard to prove that smooth functions are dense in the subspace of order continuous elements, which is equivalent to that simple functions are...
dense in this subspace. In Musielak–Orlicz spaces, the situation is different. First, we notice that not every simple function belongs to $E^\Phi(\Omega)$. However, it is possible to show that the set of simple functions in $E^\Phi(\Omega)$ is big enough to be dense in this space. On the other hand, the case of smooth functions is different. There exist functions $\Phi$ such that the smooth functions are not dense in $E^\Phi(\Omega)$. In fact, we will present here necessary and sufficient condition for $\Phi$ to a subspace of the space of smooth functions $C^\infty_c(\Omega)$ to be dense in $E^\Phi(\Omega)$.

In the recent decade, the Musielak–Orlicz spaces and their particular examples of the variable exponent Lebesgue spaces or generated by double-phase functionals have gained special attention in the context of study solutions of PDE belonging to the spaces \([4, 5, 7, 8, 15]\). The knowledge when smooth functions with compact supports are dense in the space is basic for this research.

Let $\mathbb{R}^d$ be the $d$-dimensional Euclidean space equipped with the Lebesgue measure $|\cdot|$. Given a Lebesgue measurable set $A \subset \mathbb{R}^d$, the set of all Lebesgue measurable complex-valued functions on $A$ will be denoted as $L^0_0(A)$. For $f \in L^0(A)$, its support is defined as the set

$$\text{supp}(f) = \{x \in A : f(x) \neq 0\}.$$  

The set of all simple, complex-valued functions on $A$ is denoted as

$$S(A) = \{f \in L^0(A) : |\text{supp} f| < \infty \text{ and } f \text{ has finitely many values}\}.$$  

For any open set $\Omega \subset \mathbb{R}^d$ and any $f \in L^0(\Omega)$, we define the essential support of $f$ as

$$\text{ess supp}(f) = \overline{\Omega \setminus \bigcup \{U \subset \Omega : U \text{ is open and } f = 0 \text{ a.e. on } U\}}.$$  

Notice that $\text{ess supp}(f)$ is a closed subset of $\Omega$.

Throughout this paper, $\Omega \subset \mathbb{R}^d$ stands always for an open set. Recall a function $f : \Omega \mapsto \mathbb{C}$ is said to be smooth if it possesses all derivatives, and the set of all smooth functions is denoted by $C^\infty(\Omega)$. Notice also that, for $f \in C^\infty(\Omega)$,

$$\text{ess supp}(f) = \overline{\text{supp}(f)}.$$  

By $C^\infty_c(\Omega)$ we denote the set of all smooth compactly supported functions defined on $\Omega$, that is,

$$C^\infty_c(\Omega) = \{f \in C^\infty(\Omega) : \text{ess supp}(f) \text{ is compact}\}.$$  

For any point $(x_1, \ldots, x_d) = x \in \mathbb{R}^d$ and $l > 0$, we define the open cube of center $x$ and side length $l$ as

$$Q(x, l) = \left\{(y_1, \ldots, y_d) \in \mathbb{R}^d : |x_n - y_n| < \frac{l}{2}, \text{ for } n = 1, \ldots, d\right\}.$$  

Similarly, for any point $x \in \mathbb{R}^d$ and any $r > 0$, we define the open ball of center $x$ and radius $r$ as

$$B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}.$$
A general open ball or cube often will be given without specifying their centers and radii or side lengths. In those cases, an open ball will be usually denoted by $B$ and an open cube as $Q$.

**Definition 1** Let $A \subset \mathbb{R}^d$ be Lebesgue measurable. A function $\Phi : A \times [0, \infty) \rightarrow [0, \infty)$ is called a Musielak–Orlicz function (MO function) on $A$ if

(a) for every $x \in A$, $t \mapsto \Phi(x, t)$ is convex,
(b) for every $x \in A$, $\Phi(x, t) = 0$ if and only if $t = 0$,
(c) for every $t \in [0, \infty)$, $x \mapsto \Phi(x, t)$ is Lebesgue measurable on $A$.

A MO function $\Phi$ is said to satisfy the $\Delta_2$ condition if there exist a constant $C > 0$ and a positive function $h \in L^1(A)$, that is

$$\int_A h(x) \, dx < \infty,$$

such that

$$\Phi(x, 2t) \leq C\Phi(x, t) + h(x), \quad t \geq 0, \text{ a.a. } x \in A.$$

Given a measurable set $A \subset \mathbb{R}^d$ and MO function $\Phi$, define the modular $I_\Phi$ on $L^0(A)$ by

$$I_\Phi(f) = \int_A \Phi(x, |f(x)|) \, dx.$$

The **Musielak–Orlicz space** (MO space) $L^\Phi(A)$ is defined as

$$L^\Phi(A) = \{ f \in L^0(A) : \exists \lambda > 0 \ I_\Phi(\lambda f) < \infty \},$$

and its **subspace of finite elements** as

$$E^\Phi(A) = \{ f \in L^0(A) : \forall \lambda > 0 \ I_\Phi(\lambda f) < \infty \}.$$

The functional

$$\|f\|_\Phi = \inf \{ \lambda > 0 : I_\Phi(f / \lambda) \leq 1 \}, \quad f \in L^0(A),$$

is a norm on the space $L^\Phi(A)$, called the Luxemburg norm. For extensive information on Musielak–Orlicz spaces, the reader is sent to [3–5, 7, 9, 10, 12–14, 16]. Recall the following auxiliary facts on MO spaces.

**Theorem 2** [14, 16] Let $A$ be a Lebesgue measurable subset of $\mathbb{R}^d$ and $\Phi$ a MO function on $A$. The following statements hold.

1. A sequence $\{f_n\}_{n=1}^{\infty}$ is convergent in norm to $f$ in $L^\Phi(A)$, if and only if for every $\lambda > 0$, $\lim_{n \to \infty} I_\Phi(\lambda(f - f_n)) = 0$.
2. The MO space $L^\Phi(A)$ equipped with the Luxemburg norm $\| \cdot \|_\Phi$ is a Banach function space and $E^\Phi(A)$ is a closed subspace of $L^\Phi(A)$.
3. If $\{f_n\}_{n=1}^{\infty} \subset L^0(A)$ converges to $0$ in the norm, then it converges to $0$ in measure on sets of finite measure.
(4) \( L^\Phi(A) = E^\Phi(A) \) if and only if \( \Phi \) satisfies \( \Delta_2 \) condition.

(5) The set \( S(A) \cap E^\Phi(A) \) is dense in \( (E^\Phi(A), \| \cdot \|_\Phi) \).

It is well known that the space of smooth functions with compact supports is dense in the Lebesgue and Orlicz spaces on open subset of \( \mathbb{R}^d \) [1]. In the case of Musielak–Orlicz spaces \( L^\Phi(\Omega) \), there exist similar results but under some restrictions on \( \Phi \). Recall that \( \Phi \) is said to be locally integrable whenever

\[
\int_K \Phi(x, t) \, dx < \infty
\]

for each \( t \geq 0 \) and every compact set \( K \subset \Omega \). Notice that under the above assumption, \( C^\infty_c(\Omega) \subset L^\Phi(\Omega) \). In [15, Theorem 1], it has been proved that \( C^\infty_c(\Omega) \) is dense in \( E^\Phi(\Omega) \) whenever \( \Phi \) is locally integrable. In Theorem 4.5 in [8], a similar result was obtained under the assumption of so called (A0) condition, which in fact implies local integrability of \( \Phi \).

Our main goal here is to prove a similar result, namely density of \( C^\infty_c(\Omega) \cap E^\Phi(\Omega) \) in \( E^\Phi(\Omega) \) for Musielak–Orlicz functions \( \Phi \) without any additional assumptions. Observe that in general \( C^\infty_c(\Omega) \) is not contained in \( E^\Phi(\Omega) \). Applying Lemma 6, we see that we have enough simple functions belonging to \( E^\Phi(\Omega) \) to get condition (5) of Theorem 2. However for getting the density of smooth functions, we will need additional assumption that \( |\text{Sing } \Phi| = 0 \).

A simple example of an MO function not locally integrable is the function \( \Phi_1 : \mathbb{R} \times [0, \infty) \to [0, \infty) \) given by the formula

\[
\Phi_1(x, t) = \begin{cases} \frac{t}{|x|}, & t \geq 0, \ x \neq 0 \\ 0, & t \geq 0, \ x = 0. \end{cases}
\]

Clearly, it is not locally integrable on any compact set \( K \) containing 0 as an interior point.

A slightly more involved function that is not locally integrable can be constructed as follows. Let \( \{r_n\}_{n=1}^\infty \) be an enumeration of rational numbers from the interval \((0, 1)\). For any natural number \( n \) and any \( x \in (0, 1) \) we set

\[
g_n(x) = \begin{cases} \frac{1}{x-r_n}, & x > r_n \\ 0, & x \leq r_n. \end{cases}
\]

Notice that for any \( n \in \mathbb{N} \), the function \( g_n \) is not integrable on any open interval containing \( r_n \), but the function \( \sqrt{g_n} \) is an element of \( L^1(0, 1) \) and \( \| \sqrt{g_n} \|_1 = \int_0^1 \sqrt{g_n(x)} \, dx \leq 2 \) for every \( n \in \mathbb{N} \). Hence, \( g(x) = \sum_{n=1}^\infty \frac{\sqrt{g_n(x)}}{2^n} \) is an element of \( L^1(0, 1) \). Therefore, there exists a subset \( B \) of \((0, 1)\) of measure 0 such that for any \( x \in (0, 1) \setminus B \), \( g(x) \) is finite. Notice now that for any \( x \in (0, 1) \setminus B \), we have that

\[
w(x) := \sum_{n=1}^\infty \frac{g_n(x)}{4^n} \leq g(x)^2 < \infty.
\]
We define now the function \( \Phi_2 : \mathbb{R} \times [0, \infty) \to [0, \infty) \) by the formula

\[
\Phi_2(x, t) = \begin{cases} 
    tw(x) & x \in (0, 1) \setminus B, \ t \geq 0 \\
    0 & x \in B, \ t \geq 0 \\
    t & x \in \mathbb{R} \setminus (0, 1), \ t \geq 0.
\end{cases}
\]

By construction, the function \( \Phi_2 \) is not locally integrable MO function. In fact, \( w \) is not integrable on any open subinterval of \( (0, 1) \). It is clear that both \( \Phi_1 \) and \( \Phi_2 \) satisfy the \( \Delta_2 \) condition. Hence by Theorem 2 (5), simple functions that belong to \( L^{\Phi_i}(\mathbb{R}) \) are dense in \( L^{\Phi_i}(\mathbb{R}) \), for \( i = 1, 2 \). But what about density of compactly supported smooth functions belonging to the space? It turns out that they are dense in \( L^{\Phi_1}(\mathbb{R}) \), but not in \( L^{\Phi_2}(\mathbb{R}) \). The discussion below explains why that is the case and characterizes those MO functions for which elements of \( C^\infty_c(\Omega) \cap E^{\Phi}(\Omega) \) are dense in \( E^{\Phi}(\Omega) \).

We start with the following definition.

**Definition 3** Let \( \Omega \subset \mathbb{R}^d \) be an open set and \( \Phi \) be an MO function defined on \( \Omega \). We define the set \( \text{Sing } \Phi \), called the set of singular points of \( \Phi \), as

\[
\text{Sing } \Phi = \{ x \in \mathbb{R}^d : \forall r > 0 \ \exists t_r > 0 \ \int_{B(x, r) \cap \Omega} \Phi(y, t_r) \, dy = \infty \}.
\]

**Lemma 4** If \( \Phi \) on an open set \( \Omega \subset \mathbb{R}^d \) is locally integrable then \( | \text{Sing } \Phi | = 0 \).

**Proof** Let \( x \in \Omega \cap \text{Sing } \Phi \). Then there exists \( r > 0 \) such that the closed ball \( \overline{B(x, r)} \) centered at \( x \) with radius \( r \) is contained in \( \Omega \). By definition of \( \text{Sing } \Phi \), there is \( t_r > 0 \) such that \( \int_{B(x, r) \cap \Omega} \Phi(y, t_r) \, dy = \infty \), which implies that \( \int_{B(x, r)} \Phi(y, t_r) \, dy = \infty \). However, the latter contradicts the assumption of local integrability of \( \Phi \).

If \( x \in \partial(\Omega \cap \text{Sing } \Phi) \), the boundary of \( \Omega \cap \text{Sing } \Phi \), then there exists a sequence \( x_n \in \Omega \cap \text{Sing } \Phi \) such that \( x_n \to x \) as \( n \to \infty \). Consequently for every \( n \in \mathbb{N} \), there are \( r_n > 0 \) and \( t_n > 0 \) so that \( B(x_n, r_n) \subset \Omega \) and \( \int_{B(x_n, r_n)} \Phi(y, t_n) \, dy = \infty \), which is a contradiction with local integrability of \( \Phi \).

If \( x \notin \Omega \cap \text{Sing } \Phi \), then there exists a ball \( B(x, r) \subset \mathbb{R}^n \setminus \Omega \) and thus for every \( t > 0 \), \( \int_{B(x, r) \cap \Omega} \Phi(y, t) \, dy = 0 \).

It follows from the above that \( | \text{Sing } \Phi | = 0 \). \( \square \)

The converse implication of Lemma 4 is not satisfied in view of \( | \text{Sing } \Phi_1 | = 0 \) and \( \Phi_1 \) being not locally integrable.

We also notice that for \( \Phi_1 \) and \( \Phi_2 \), we have

\[
\text{Sing } \Phi_1 = \{ 0 \},
\]

\[
\text{Sing } \Phi_2 = [0, 1],
\]

which implies that both of those sets are closed. This is not a coincidence and the following proposition holds.
Proposition 5 For any open set $\Omega \subset \mathbb{R}^d$ and MO function $\Phi$ defined on $\Omega$, the set $\text{Sing } \Phi$ is a closed subset of $\mathbb{R}^d$.

Proof Let $x \in \mathbb{R}^d$, $\{x_n\}_{n=1}^{\infty} \subset \text{Sing } \Phi$ and $\lim_{n \to \infty} x_n = x$. We will show that $x \in \text{Sing } \Phi$. For any $r > 0$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x - x_n| < \frac{r}{2}$. Notice that, for each $n > N$, we have $B(x_n, \frac{r}{2}) \subset B(x, r)$. Since $x_n \in \text{Sing } \Phi$, there exists $t^{(n)}_r > 0$ such that

$$
\int_{B(x, r) \cap \Omega} \Phi(y, t^{(n)}_r) \, dy = \infty;
$$

therefore for each $n > N$,

$$
\int_{B(x, r) \cap \Omega} \Phi(y, t^{(n)}_r) \, dy \geq \int_{B(x_n, \frac{r}{2}) \cap \Omega} \Phi(y, t^{(n)}_r) \, dy = \infty.
$$

Since $r$ was arbitrarily chosen, we conclude that $x \in \text{Sing } \Phi$. \qed

Now we show that for density of $C_{\infty}^c(\Omega) \cap E^\Phi(\Omega)$ in $E^\Phi(\Omega)$, the set $\text{Sing } \Phi$ has to be of zero measure. We will need the following result.

Lemma 6 [11, p. 64], [16] Let $A \subset \mathbb{R}^d$ be measurable and $\Phi$ be an MO function on $A$. There exists a sequence of pairwise disjoint sets $\{A_n\}_{n=1}^{\infty} \subset A$ such that for each $n \in \mathbb{N}$, $|A_n| < \infty$, $|A \setminus \bigcup_{n=1}^{\infty} A_n| = 0$, and

$$
\sup_{x \in A_n} \Phi(x, t) < \infty, \quad n \in \mathbb{N}, \quad t \geq 0.
$$

Consequently $L^\infty(A|_{A_n}) \subset E^\Phi(A|_{A_n})$ for every $n \in \mathbb{N}$.

Theorem 7 Let $\Omega \subset \mathbb{R}^d$ be open and $\Phi$ be an MO function on $\Omega$. If $|\text{Sing } \Phi| > 0$, then $C_{\infty}^c(\Omega) \cap E^\Phi(\Omega)$ is not dense in $E^\Phi(\Omega)$.

Proof We argue by contradiction. Assume that $C_{\infty}^c(\Omega) \cap E^\Phi(\Omega)$ is dense in $E^\Phi(\Omega)$ and $|\text{Sing } \Phi| > 0$. Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of sets such that $\Omega_n = A_n$, where $\{A_n\}_{n=1}^{\infty}$ is the sequence of sets from the conclusion of Lemma 6. Since, $|\Omega \setminus \bigcup_{n=1}^{\infty} \Omega_n| = 0$, there exists $n_0$ such that $|\text{Sing } \Phi \cap \Omega_{n_0}| > 0$. By inner regularity of the Lebesgue measure, there exists a compact set $K \subset \text{Sing } \Phi \cap \Omega_{n_0}$ such that $0 < |K| < |\text{Sing } \Phi \cap \Omega_{n_0}|$. For any $t > 0$,

$$
\int_{K} \Phi(x, t) \, dx \leq \int_{\Omega_{n_0}} \Phi(x, t) \, dx < \infty;
$$

therefore, $\chi_K \in E^\Phi(\Omega)$. By density of $C_{\infty}^c(\Omega) \cap E^\Phi(\Omega)$ in $E^\Phi(\Omega)$, there exists a sequence of functions $f_n \in C_{\infty}^c(\Omega) \cap E^\Phi(\Omega)$ such that
Therefore, by Theorem 2 (3), \( \{ \chi_K - f_n \}_{n=1}^{\infty} \) converges to 0 in measure for any set \( A \subset \Omega \) with \( |A| < \infty \). It follows that there exists a subsequence \( \{ n_k \}_{k=1}^{\infty} \subset \mathbb{N} \) such that \( \lim_{k \to \infty} \chi_K(x) - f_{n_k}(x) = 0 \) for a.a. \( x \in \Omega \). For convenience, let us rewrite \( f_{n_k} \) as \( f_k \).

Since, \( |K| > 0 \) there exists \( x_0 \in K \) such that \( \lim_{k \to \infty} f_k(x_0) = 1 \). Take now \( k_0 \) such that

\[
\frac{1}{2} < f_{k_0}(x_0) < \frac{3}{2}.
\]

By continuity of \( f_{k_0} \), there exists a ball \( B(x_0, r) \), such that for any \( y \in B(x_0, r) \),

\[
-\frac{1}{4} < f_{k_0}(y) - f_{k_0}(x_0) < \frac{1}{4}.
\]

Combining both of the above inequalities, we get that for any \( y \in B(x_0, r) \),

\[
\frac{1}{4} < |f_{k_0}(y)| < \frac{7}{4};
\]

therefore, \( \frac{1}{4} \chi_{B(x_0, r)} < |f_{k_0}| \), so \( \chi_{B(x_0, r)} \in E^\Phi(\Omega) \). Hence for each \( \lambda > 0 \), we have

\[
\int_{B(x_0, r)} \Phi(y, \lambda)dy < \infty.
\]

On the other hand, \( x_0 \in K \subset \text{Sing } \Omega \), and therefore there exists a \( t_r > 0 \) such that

\[
\int_{B(x_0, t_r)} \Phi(y, t_r) = \infty,
\]

and so \( \chi_{B(x_0, r)} \notin E^\Phi(\Omega) \), a contradiction. \( \square \)

To prove that the condition \( |\text{Sing } \Phi| = 0 \) is sufficient for density of \( C_c^\infty(\Omega) \) functions in \( E^\Phi(\Omega) \), we need the following fact about compact sets.

**Lemma 8** Let \( K \) be a compact subset of \( \mathbb{R}^d \). There exists a sequence of open, bounded sets \( \{ U_n \}_{n=1}^{\infty} \) such that, for each \( n \in \mathbb{N} \),

(1) \( K \subset U_n \),

(2) \( \frac{|U_n|}{|U_{n+1}|} \leq |K| + \frac{1}{n} \),

(3) \( U_{n+1} \subset U_n, |U_{n+1}| < |U_n| \).

**Proof** Let \( K \subset \mathbb{R}^d \) be compact. We will construct the sequence \( \{ U_n \}_{n=1}^{\infty} \) by induction.

Let \( \{ Q_{1,n} \}_{n=1}^{\infty} \) be a sequence of open cubes such that \( \bigcup_{n=1}^{\infty} Q_{1,n} \subset |K| + 1 \) and \( K \subset \bigcup_{n=1}^{\infty} Q_{1,n} \). By compactness of \( K \), there exists \( M_1 \) cubes, \( \{ Q_{1,n} \}_{n=1}^{M_1} \), such that
$K \subset \bigcup_{n=1}^{M_1} Q_{1,n}$. Notice that $\bigcup_{n=1}^{M_1} Q_{1,n}$ can be written as a finite union of disjoint cuboids $\{R_j\}_{j=1}^{J_1}$ (a cuboid is a finite intersection of open cubes),

$$\bigcup_{n=1}^{M_1} Q_{1,n} = \bigcup_{j=1}^{J_1} R_j.$$  

Moreover, for each $1 \leq j \leq J_1$, $|\overline{R_j}| = |R_j|$. Therefore

$$\bigg| \bigcup_{n=1}^{M_1} Q_{1,n} \bigg| = \bigg| \bigcup_{j=1}^{J_1} R_j \bigg| = \bigg| \bigcup_{j=1}^{J_1} \overline{R_j} \bigg| = \sum_{j=1}^{J_1} |R_j| = \bigg| \bigcup_{j=1}^{J_1} R_j \bigg| = \bigg| \bigcup_{n=1}^{M_1} Q_{1,n} \bigg|,$$

so

$$\bigg| \bigcup_{n=1}^{M_1} Q_{1,n} \bigg| = \bigg| \bigcup_{n=1}^{M_1} Q_{1,n} \bigg|.$$  

Defining $U_1 = \bigcup_{n=1}^{M_1} Q_{1,n}$, we have $K \subset U_1$, $|\overline{U_1}| = |U_1|$ and $|U_1| < |K| + 1$.

Assume that the set $U_n$ is constructed. Now, we will construct the set $U_{n+1}$. First, notice that $\overline{U_n}$ is compact; therefore the boundary of $U_n$, $\text{bd}(U_n)$ is also compact.

Let $f(x) = \text{dist}(x, \text{bd}(U_n))$ be the distance of $x$ from $\text{bd}(U_n)$. The function $f$ is continuous on $\mathbb{R}^d$ and for any $x \in K$, $f(x) > 0$. By compactness of $K$, there exists $x_0 \in K$ such that $\min_{x \in K} f(x) = f(x_0) = \alpha > 0$, so for any $x \in K$,

$$\text{dist}(x, \text{bd}(U_n)) \geq \alpha.$$  

Let us introduce the following family of open cubes,

$$Q_d(K) = \left\{ Q(x, l) : x \in K, \ l \leq \frac{\alpha}{\sqrt{d}} \right\}.$$  

Notice that, if $Q \in Q_d(K)$, then $\overline{Q} \subset U_n$. Indeed, let $Q = Q(x, l)$ for some $(x_1, \ldots, x_d) = x \in K$ and $l \leq \frac{\alpha}{\sqrt{d}}$. If $(y_1, \ldots, y_d) = y \in \overline{Q}$, then for any $i = 1, \ldots, d$ we have

$$|x_i - y_i| \leq \frac{l}{2},$$

and so

$$|x - y| \leq \frac{l\sqrt{d}}{2} \leq \frac{\alpha}{2}.$$  

For any $z \in \mathbb{R}^d \setminus U_n$, we have

$$|z - x| \geq \text{dist}(x, \text{bd}(U_n)) \geq \alpha,$$

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and thus
\[ |z - y| \geq |z - x| - |x - y| \geq |z - x| - |x - y| \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}. \]

Therefore, \( y \notin \mathbb{R}^d \setminus U_n \) and so \( y \in U_n \), i.e., \( \overline{Q} \subset U_n \). Since \( \overline{Q} \) is closed and \( U_n \) is open, we have that \( \overline{Q} \cap \text{bd}(U_n) = \emptyset \). For each \( x \in K \), denote
\[ Q_a(x) = \left\{ Q(x, l) : l \leq \frac{\alpha}{\sqrt{d}} \right\}. \]

Clearly \( Q_a(x) \subset Q_a(K) \) and \( Q_a(x) \) forms a topological basis for neighborhoods of \( x \) in the Euclidean topology of \( \mathbb{R}^d \). Take now any open set \( U \), such that \( K \subset U \). For each \( x \in K \), there exists \( l_x > 0 \) with \( Q(x, l_x) \subset Q_a(x) \), such that \( Q(x, l_x) \subset U \).

Let \( U \) be an open set, such that \( K \subset U \) and \( |U| < |K| + \frac{1}{n+1} \). Then there exists a family \( \{ Q(x, l_x) \}_{x \in K} \subset Q_a(K) \) such that \( \bigcup_{x \in K} Q(x, l_x) \subset U \). By compactness of \( K \), there exists \( M_{n+1} \in \mathbb{N} \) and a finite subfamily \( \{ Q(x_m, l_{x_m}) \}_{m=1}^{M_{n+1}} \) such that \( K \subset \bigcup_{m=1}^{M_{n+1}} Q(x_m, l_{x_m}) \). Setting \( Q_{m,n+1} := Q(x_m, l_{x_m}) \), \( K \subset \bigcup_{n=1}^{M_{n+1}} Q_{m,n+1} \) and \( \bigcup_{m=1}^{M_{n+1}} Q_{m,n+1} \) \( < |U| < |K| + \frac{1}{n+1} \). Since \( Q_{m,n+1} \subset Q_a(K) \), we have
\[ \overline{Q_{m,n+1}} \cap \text{bd}(U_n) = \emptyset. \]

Define \( U_{n+1} = \bigcup_{m=1}^{M_{n+1}} Q_{m,n+1} \). Arguing as in case \( n = 1 \), we get that
\[ |U_{n+1}| = |U_{n+1}|. \]

For every \( m = 1, \ldots, M_{n+1} \), we have \( \overline{Q_{m,n+1}} \subset U_n \). Therefore
\[ U_{n+1} = \bigcup_{m=1}^{M_{n+1}} Q_{m,n+1} \subset U_n. \]

Recall that for any open set \( U \subset \mathbb{R}^d \) with \( |U| < \infty \) and any compact set \( F \subset U \), we have that \( |F| < |U| \). Hence,
\[ |U_{n+1}| < |U_n|. \]

By induction, we have constructed a sequence of open, bounded sets \( \{ U_n \}_{n=1}^{\infty} \) with the desired properties. \( \square \)

We will need further the following classical result.

**Theorem 9** [6, Theorem 2.6.1] [2] Let \( \Omega \subset \mathbb{R}^d \) be open. For every open \( U \subset \Omega \) and compact \( K \subset U \) there exists \( f \in C_\infty^c(\Omega) \), \( f : \Omega \to [0,1] \) such that \( f|_K \equiv 1 \) and \( \text{supp} f \subset U \).

Now, we can prove that if \( |\text{Sing} \Phi| = 0 \) and for some compact set \( K \) the function \( \chi_K \in E^\Phi(\Omega) \) then \( \chi_K \) can be approximated by elements of \( C_\infty^c(\Omega) \).
**Theorem 10** Let $\Phi$ be an MO function defined on an open $\Omega \subset \mathbb{R}^d$. If $|\text{Sing } \Phi| = 0$, then for any compact $K \subset \Omega$ such that $\chi_K \in E^\Phi(\Omega)$, there exists a sequence \( \{f_n\}_{n=1}^\infty \subset C_c^\infty(\Omega) \), such that $\lim_{n \to \infty} \|f_n - \chi_K\|_\Phi = 0$.

**Proof** Let $K \subset \Omega$ be a compact set such that $\chi_K \in E^\Phi(\Omega)$. Then there exists an open and bounded set $U$ such that $K \subset U$. Define

$$ (\text{Sing } \Phi)_U = \overline{\text{Sing } \Phi} \cap U. $$

By Proposition 5, $\text{Sing } \Phi$ is closed. Hence,

$$ (\text{Sing } \Phi)_U = \overline{\text{Sing } \Phi} \cap U = \text{Sing } \Phi \cap \overline{U}, $$

and $(\text{Sing } \Phi)_U$ is compact. By Lemma 8, there exists a sequence of open sets \( \{U_n\}_{n=1}^\infty \), such that for every $n \in \mathbb{N}$,

1. $(\text{Sing } \Phi)_U \subset U_n$,
2. $|U_n| < \frac{1}{n}$,
3. $U_{n+1} \subset U_n$ and $|U_{n+1}| < |U_n|$.

Take any $n \in \mathbb{N}$ and define $K_n = K \setminus U_n$. Clearly, $K_n$ is compact and $K = (K \cap U_n) \cup K_n$. Setting $V_n = U \setminus U_{n+1}$, we have that $K_n \subset V_n$ and $V_n$ is open.

First, we will show that $\chi_{V_n} \in E^\Phi(\Omega)$. Notice that

$$ H \setminus \chi_{V_n} \subset (U \setminus \text{Sing } \Phi) \subset U \setminus \text{Sing } \Phi. $$

Hence, $V_n \cap \text{Sing } \Phi \subset (U \setminus U_{n+2}) \cap \text{Sing } \Phi = \emptyset$. By definition of $\text{Sing } \Phi$, for each $x \in U \setminus U_{n+2}$ there exists $r_x > 0$ such that for all $\lambda > 0$,

$$ \int_{B(x, r_x) \cap \Omega} \Phi(y, \lambda) \, dy < \infty. $$

On the other hand, since $V_n \subset U \setminus U_{n+2}$ and $U \setminus U_{n+2}$ is closed, then $\overline{V_n} \subset U \setminus U_{n+2}$. Hence, for every $x \in \overline{V_n}$ there exists $r_x > 0$ such that for all $\lambda > 0$,

$$ \int_{B(x, r_x) \cap \Omega} \Phi(y, \lambda) \, dy < \infty. $$

The family $\{B(x, r_x)\}_{x \in \overline{V_n}}$ is an open cover of $\overline{V_n}$, so by compactness of $\overline{V_n}$, there exists a finite subcover $\{B(x_i, r_{x_i})\}_{i=1}^k$ of $\overline{V_n}$. Then for any $\lambda > 0$,

$$ \int_{\Omega} \Phi(x, \lambda \chi_{V_n}(x)) \, dx \leq \frac{1}{k} \sum_{i=1}^k \int_{B(x_i, r_{x_i}) \cap \Omega} \Phi(y, \lambda k) \, dy < \infty, $$

and therefore $\chi_{V_n} \in E^\Phi(\Omega)$. Since $\chi_{K_n} \leq \chi_{V_n}$, we have $\chi_{K_n} \in E^\Phi(\Omega)$. Recall that $K_n = K \setminus U_n$ and $|U_n| < \frac{1}{n}$, and hence $\lim_{n \to \infty} \chi_K - \chi_{K_n} = 0$ in measure. We can find
a subsequence \( \{n_k\}_{k=1}^\infty \subset \mathbb{N} \), such that \( \lim_{k \to \infty} \chi_{K_{n_k}} = \chi_K \) a.e.. Without loss of generality, assume that \( \{ \chi_{K_{n_k}} \}_{k=1}^\infty = \{ \chi_{K_{n}} \}_{n=1}^\infty \). For any \( \lambda > 0 \) we have \( I_{\Phi}(\lambda \chi_K) < \infty \) and

\[
\Phi(x, \lambda(\chi_K(x) - \chi_{K_{n}}(x))) \leq \Phi(x, \lambda \chi_K(x)) \text{ for a.e. } x \in \mathbb{R}^d.
\]

Hence by the Lebesgue Dominated Convergence Theorem, \( \lim_{n \to \infty} I_{\Phi}(\lambda(\chi_K - \chi_{K_{n}})) = 0 \) so in view of Theorem 2 (1),

\[
\lim_{n \to \infty} \| \chi_K - \chi_{K_{n}} \|_{\Phi} = 0. \tag{1}
\]

For a fixed \( n \in \mathbb{N} \), by Lemma 8, there exists a sequence of open sets \( \{W_m\}_{m=1}^\infty \) such that for every \( m \in \mathbb{N} \),

\[
\begin{align*}
(1') & \quad K_n \subseteq W_m', \\
(2') & \quad |W_m'| = |W_m| < |K_n| + \frac{1}{m}, \\
(3') & \quad W_{m+1} \subseteq W_m'.
\end{align*}
\]

Define now \( W_m = W_m' \cap V_n \). For each \( m \in \mathbb{N} \), we have \( K_n \subseteq W_m \) and \( W_m \) is open. Hence in view of Theorem 9, there exists a function \( f_{m,n} \), such that \( f_{m,n} \in C_c^{\infty}(\Omega) \), \( f_{m,n} : \Omega^d \to [0,1] \) and \( f_{m,n}|_{V_n} \equiv 1 \) and \( \text{supp} f_{m,n} \subseteq W_m \). Notice that \( f_{m,n} \leq \chi_{W_n} \) a.e., and hence \( \chi_{W_n} - f_{m,n} \leq -\chi_{W_n} \) a.e. For a fixed \( \lambda > 0 \) and any \( m \in \mathbb{N} \) by (2'), we have

\[
\{ x : \chi_{W_n}(x) - f_{m,n}(x) > \lambda \} \leq |\{ x : \chi_{W_n}(x) > \lambda \} - |W_m \setminus K_n| = |W_m| - |K_n|
\leq |W_m| - |K_n| < \frac{1}{m}.
\]

Therefore, \( \lim_{m \to \infty} f_{m,n} = \chi_{K_n} \) in measure for every \( n \in \mathbb{N} \). We can find a subsequence \( \{m_k\}_{k=1}^\infty \subset \mathbb{N} \), such that \( \lim_{k \to \infty} f_{m_{k},n} = \chi_{K_n} \) a.e. Without loss of generality, assume that \( \{f_{m_{k,n}}\}_{k=1}^\infty = \{f_{m,n}\}_{m,n=1}^\infty \). Since \( f_{m,n} \leq \chi_{W_n} \leq \chi_{V_n} \) a.e. and \( \chi_{V_n} \in E^{\Phi}(\Omega) \), then \( f_{m,n} \in E^{\Phi}(\Omega) \). For any \( \lambda > 0 \), we have \( I_{\Phi}(\lambda \chi_{V_n}) < \infty \) and

\[
\Phi(x, \lambda(f_{m,n}(x) - \chi_{K_n}(x))) \leq \Phi(x, \lambda \chi_{V_n}(x)) \text{ for a.e. } x \in \mathbb{R}^d.
\]

By the Lebesgue Dominated Convergence Theorem, \( \lim_{m \to \infty} I_{\Phi}(\lambda(f_{m,n} - \chi_{K_n})) = 0 \), so, for every \( n \in \mathbb{N} \)

\[
\lim_{m \to \infty} \| f_{m,n} - \chi_{K_n} \|_{\Phi} = 0.
\]

Now, for every \( n \in \mathbb{N} \) define \( f_n = f_{m_n,n} \), where \( m_n \) is the smallest integer such that

\[
\| f_{m_n,n} - \chi_{K_n} \|_{\Phi} < \frac{1}{n}. \tag{2}
\]

Finally, by (1) and (2),

\[
\lim_{n \to \infty} \| f_n - \chi_K \|_{\Phi} \leq \lim_{n \to \infty} \| f_n - \chi_{K_n} \|_{\Phi} + \lim_{n \to \infty} \| \chi_{K_n} - \chi_K \|_{\Phi} = 0.
\]

\[\blacksquare\]
Finally, we can show that if \(|\text{Sing } \Phi| = 0\), then functions from \(C_\infty^c(\Omega) \cap E^\Phi(\Omega)\) are dense in \(E^\Phi(\Omega)\).

**Theorem 11** Let \(\Omega \subset \mathbb{R}^d\) be open and \(\Phi\) be an MO function defined on \(\Omega\). If \(|\text{Sing } \Phi| = 0\), then \(C_\infty^c(\Omega) \cap E^\Phi(\Omega)\) is dense in \(E^\Phi(\Omega)\). In other words, \(C_\infty^c(\Omega) \cap E^\Phi(\Omega) = E^\Phi(\Omega)\).

**Proof** By Theorem 2 (5), \(E^\Phi(\Omega) \cap S(\Omega)\) is dense in \(E^\Phi(\Omega)\). Every simple function is a finite linear combination of characteristic functions of measurable sets with finite measure. Hence, by linearity and in view of Theorem 10, it suffices to show that any characteristic function \(\chi_A\) of a set of finite measure \(A\) such that \(\chi_A \in E^\Phi(\Omega)\) can be approximated by a characteristic function of a compact set.

For any such \(A\), by inner regularity of the Lebesgue measure, there exists a sequence of compact sets \(K_n \subset A\), such that \(|A \setminus K_n| < \frac{1}{n}\) for every \(n \in \mathbb{N}\). For \(\lambda > 0\), we have

\[
I_{\Phi}(\lambda(\chi_A - \chi_{K_n})) = I_{\Phi}(\lambda(\chi_{A \setminus K_n})).
\]

Clearly, \(\Phi(x, \lambda \chi_{A \setminus K_n}(x)) \leq \Phi(x, \lambda \chi_A(x))\) for a.e. \(x \in \Omega\). Since \(I_{\Phi}(\lambda \chi_A) < \infty\), so by the Lebesgue Dominated Theorem, we deduce that

\[
\lim_{n \to \infty} I_{\Phi}(\lambda(\chi_A - \chi_{K_n})) = 0,
\]

and so \(\lim_{n \to \infty} \|\chi_A - \chi_{K_n}\|_{\Phi} = 0\). \(\square\)

Since \(E^\Phi(\Omega) = L^\Phi(\Omega)\) if and only if \(\Phi\) satisfies \(\Delta_2\) (Theorem 2 (4)), the next result is an immediate consequence of Theorems 11 and 7. It is also extension of Theorem 1 in [15] and Theorem 4.5 in [8].

**Corollary 12** Let \(\Omega \subset \mathbb{R}^d\) be open and \(\Phi\) be an MO function defined on \(\Omega\) satisfying \(\Delta_2\) condition. Then, \(C_\infty^c(\Omega) \cap L^\Phi(\Omega)\) is dense in \(L^\Phi(\Omega)\) if and only if \(|\text{Sing } \Phi| = 0\).

An important class of MO functions is a class of double phase functionals consisting of \(\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+\) such that

\[
\Phi(x, r) = r^p(x) + a(x)r^r(x),
\]

where \(a, p, r\) are measurable functions on \(\Omega\), \(a(x) \geq 0\) a.e. and \(1 \leq p(x) \leq r(x) < \infty\) a.e.. Denote by \(p^+\) and \(p^-\),

\[
p^+ = \text{ess sup}_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \text{ess inf}_{x \in \Omega} p(x),
\]

and \(r^+\) and \(r^-\) analogously as above.

**Corollary 13** Let \(L^\Phi(\Omega)\) be the MO space over an open set \(\Omega \subset \mathbb{R}^d\), generated by double phase functional (3) with the assumption that the function \(a(x)\) is essentially bounded on \(\Omega\). If (i) \(r^+ < \infty\) or (ii) \(p(x) < r(x)\) for a.a. \(x \in \Omega_1\), where \(\Omega_1 = \text{supp } a\), and \(p^+ < \infty\) and \(r^+\) on \(\Omega_1 = \text{ess sup}_{x \in \Omega_1} r(x) < \infty\), then \(C_\infty^c(\Omega)\) is dense in \(L^\Phi(\Omega)\).
**Proof** Clearly, $C^\infty_c(\Omega) \subset L^\Phi(\Omega)$. By Theorem 1.8 in [10], if (i) or (ii) are satisfied then $\Phi$ satisfies $\Delta_2$ condition. Moreover in view of the assumptions on the functions $a$, $p$, $r$, $|\text{Sing } \Phi| = 0$. We finish by applying Corollary 12. 

Finally, recall that a given measurable function $1 \leq p(x) < \infty \text{ a.e. in } \Omega$, and

$$\Phi(x,t) = \frac{t^{p(x)}}{p(x)}, \quad a.a. \, x \in \Omega, \, t \geq 0,$$

the space $L^{p(x)}(\Omega) = L^\Phi(\Omega)$ is called the *variable exponent Lebesgue space* or *Nakano space* [4, 5, 7, 8, 16]. The final well-known result is also a consequence of Lemma 4 and Corollary 12 (see also Theorem 3.4.12 in [5]).

**Corollary 14** [5] Let $L^{p(x)}(\Omega)$ be the variable exponent Lebesgue space over an open set $\Omega \subset \mathbb{R}^d$. If $p^+ < \infty$ then $|\text{Sing } \Phi| = 0$. Consequently, $C^\infty_c(\Omega)$ is dense in $L^{p(x)}(\Omega)$.

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