$L^p$ BOUNDS FOR HIGHER RANK EIGENFUNCTIONS AND ASYMPTOTICS OF SPHERICAL FUNCTIONS

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Abstract. We prove almost sharp upper bounds for the $L^p$ norms of eigenfunctions of the full ring of invariant differential operators on a compact locally symmetric space. Our proof combines techniques from semiclassical analysis with harmonic theory on reductive groups, and makes use of asymptotic bounds for spherical functions which improve upon those of Duistermaat, Kolk and Varadarajan and are of independent interest.

1. Introduction

If $M$ is a compact Riemannian manifold of dimension $n$ and $\psi$ is a Laplace eigenfunction on $M$ satisfying $\Delta \psi = \lambda^2 \psi$, it is a well studied problem to investigate the asymptotic behaviour of the $L^p$ norms of $\psi$ as $\lambda \to \infty$. The fundamental upper bound for these norms was established by Hörmander [5] and Sogge [9], who prove that

$$\|\psi\|_p \ll \lambda^{\delta_0(p)} \|\psi\|_2$$

where $\delta_0(p)$ is the piecewise linear function of $1/p$ given by

$$\delta_0(p) = \begin{cases} n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{2}, & 0 \leq \frac{1}{p} \leq \frac{n-1}{2n+1}, \\ \frac{n-1}{2n+1}, & \frac{n-1}{2n+1} \leq \frac{1}{p} \leq \frac{1}{2}. \end{cases}$$

Moreover, these bounds were shown by Sogge [9] to be sharp when $M$ is the round $n$-sphere $S^n$.

It is sometimes possible to improve the upper bound in (1) by assuming that $M$ has additional symmetry, or that $\psi$ is an eigenfunction of extra differential operators which commute with $\Delta$. In the extreme case of the flat torus $T^n$, for instance, if one assumes that $\psi$ is an eigenfunction of all the translations $\{i \partial / \partial x_i\}$ then $\psi$ is a complex exponential, and so we have $\|\psi\|_p \leq C$ for all $p$ and $C$ depending only on $T^n$. A more interesting example of this phenomenon is given by Sarnak in his letter to Morawetz [6]. He proves that if $X$ is a compact locally symmetric space of dimension $n$ and rank $r$, and $\psi$ is an eigenfunction of the full ring of differential operators on $X$ with spectral parameter $t\lambda$, then

$$\|\psi\|_\infty \ll t^{(n-r)/2} \|\psi\|_2,$$

uniformly for $\lambda$ in compact subsets of $\mathfrak{a}^*_+\setminus\mathfrak{a}^*_r$ which are bounded away from the walls (notations are standard and given in section 2.1). Note that the Laplace eigenvalue of $\psi$ is roughly $t^2$, so that (3) represents an improvement in the exponent of (1).
from \((n - 1)/2\) to \((n - r)/2\). This upper bound is also sharp in the case when \(X\) is of compact type, and Sarnak states that it should be considered as the ‘convex bound’ for the sup norm of a higher rank eigenfunction.

The goal of this paper is to derive the correct convex bound for all \(L^p\) norms of an eigenfunction in higher rank, by combining basic techniques from semiclassics with a detailed analysis of spherical functions. Our main result in this direction is stated below, which in the compact case differs from the expected sharp bound only by a factor of \((\log \lambda)^{1/2}\) at the kink point. Let \(X\) be a compact locally symmetric space of dimension \(n\) and rank \(r\) which is a quotient of the globally symmetric space \(S\), and assume that \(S\) is irreducible. We then have:

**Theorem 1.** Let \(C \subset a^*_+\) be a cone in the positive dual Weyl chamber which is bounded away from the walls. If \(\psi\) is an eigenfunction of the full ring of invariant differential operators on \(X\) with spectral parameter \(\Lambda \in C\), then we have

\[
\|\psi\|_p \ll_{X,C,p} \left\{ \begin{array}{ll}
(\log \|\Lambda\|)^{1/2}\|\Lambda\|^{\delta(p)}\|\psi\|_2, & p = \frac{2(n+r)}{n-r}, \\
\|\Lambda\|^{\delta(p)}\|\psi\|_2, & p \neq \frac{2(n+r)}{n-r},
\end{array} \right.
\]

where \(\delta(p)\) is the piecewise linear function

\[
\delta(p) = \left\{ \begin{array}{ll}
n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{r}{2}, & 0 \leq \frac{1}{p} \leq \frac{n-r}{2(n+r)}, \\
\frac{n-r}{2} - \frac{1}{p}, & \frac{n-r}{2(n+r)} \leq \frac{1}{p} \leq \frac{1}{2}.
\end{array} \right.
\]

Moreover, these bounds are sharp up to the logarithmic factor in the case when \(X\) is of compact type.

It will be apparent in the course of the proof of theorem 1 that when \(X\) is the quotient of a product \(S = S_1 \times \ldots \times S_d\) of irreducible symmetric spaces, the \(L^p\) norm of an eigenfunction on \(X\) is bounded by the product of the functions \(\delta(p)\) for each irreducible factor of \(S\). Moreover, in the compact case this will again be sharp up to the logarithmic factors at the kink points. A discussion of how one might go about removing these logarithmic factors by using Sogge’s original methods is given in section 2.5.

To give an example comparing the bound produced by theorem 1 with the classical bound (1), let \(X\) be a quotient of the globally symmetric space \(SL(3, \mathbb{R})/SO(3)\). Let \(\psi\) be an eigenfunction of the full ring of invariant differential operators on \(X\), which was proven by Selberg [7] to be isomorphic to a polynomial algebra in two variables and contains \(\Delta\), and assume that the spectral parameter of \(\psi\) is restricted to a cone \(C\) as in theorem 1. The following graph compares the two exponents \(\delta_0(p)\) and \(\delta(p)\) appearing in Sogge’s bound and theorem 1.
We therefore see that by using the symmetry of $X$ in the form of its extra differential operators, we are able to significantly strengthen the bounds for $\|\psi\|_p$.

Let us take a moment to discuss the significance of the exponent $\delta(p)$ in theorem 1 and hopefully convince the reader that it is natural. Suppose that $r|n$, and let $X$ be a product of $r$ compact manifolds $X_1 \times \ldots \times X_r$ of dimension $n/r$. Let $\Delta_i$ be the Laplacian of $X_i$, and let $\psi = \psi_1 \times \ldots \times \psi_r$ be a joint eigenfunction of the Laplacians $\Delta_i$ on $X$. Let $\Delta_i \psi = \lambda_i^2 \psi$, and assume that the ratios $\lambda_i/\lambda_j$ are all bounded by some constant. By applying Sogge's bound (1) to each $\psi_i$, we may show that

$$\|\psi\|_p \ll \lambda^{r\delta_0(p)} \|\psi\|_2,$$

where $\lambda^2 = \lambda_1^2 + \ldots + \lambda_r^2$ and $\delta_0(p)$ is the function (2) with $n/r$ in place of $n$. It may easily be seen that the function $\delta(p)$ defined in (5) satisfies $\delta(p) = r\delta_0(p)$, and so theorem 1 may be summarised as saying that, from the point of view of the convex bound for $L^p$ norms of eigenfunctions, a locally symmetric space of dimension $n$ and rank $r$ whose universal cover is irreducible behaves like the product of $r$ general Riemannian manifolds of dimension $n/r$.

It would be interesting to know in which other cases this product behaviour occurs, that is when the $L^p$ bounds of theorem 1 hold for a more general compact manifold $M$ of dimension $n$ with $r$ commuting differential operators which are 'independent' in some sense. There are no nontrivial examples of this in the completely integrable case, as it was proven by Toth and Zelditch [10] that if $M$ is a quantum completely integrable manifold and all joint eigenfunctions on $M$ are uniformly bounded then $M$ is a flat torus.

In proving theorem 1, we shall in fact show that the same bounds hold for the $L^2 \to L^p$ norm of a spectral projector onto a ball of fixed radius about $\Lambda$. With this formulation, our bounds will be sharp up to the log in the case of both compact and noncompact type. The fact that this bound is sharp for individual eigenfunctions in the compact case is due to the high multiplicity of the spectrum, so that by choosing the radius of our spectral projector to be sufficiently small we know that it will always pick out exactly one eigenvalue of high multiplicity.

In both cases, the bounds of theorem 1 are realised by simple wave packets which are the higher rank analogues of the zonal functions and Gaussian beams on a general Riemannian manifold. We shall describe these packets on the globally
symmetric space $S = G/K$, their analogues on $X$ being similar. The cotangent bundle $T^*S$ of $S$ is isomorphic to the $K$-principal bundle $G \times_K p^*$, which we recall is defined to be the quotient of the trivial bundle $G \times p^*$ by the action

$$(g, v)k = (gk, \text{Ad}_{k^{-1}}v).$$

If $\lambda \in \mathfrak{a}^*$ we define $T^*_\lambda S \subset T^*S$ to be the set of points $(g, v) \in G \times_K p$ in which $v$ is in the $\text{Ad}_K$ orbit of $\lambda/||\lambda||$. Saying that $\psi \in C^\infty(S)$ is an approximate eigenfunction of the ring of invariant differential operators on $S$ with parameter $\lambda$ then implies that the microlocal support of $\psi$ is concentrated on $T^*_\lambda S$ (see section 5.4 of [8]).

Let $p \in S$ correspond to the identity coset of $K$, and let $A$ be the standard flat subspace containing $p$. Identify $T^*_A$ with $\mathfrak{a}^* \times A$, let $T^*_\lambda A$ be the subspace $\lambda/||\lambda|| \times A$, and let $L^*_\lambda \subset T^*_\lambda S$ be the orbit of $T^*_\lambda A$ under $K$. We shall see that there exists a $K$-biinvariant function $\psi \in L^2(S)$ whose microlocal support is concentrated on $L^*_\lambda$, and which saturates the $L^p$ norms on $S$ for $p$ above the kink point. The fibre of the projection map $\pi : L^*_\lambda \to S$ is isomorphic to $\text{Stab}_K(a)$, so that $\pi$ is smooth for generic $s$ and maximally singular at the origin, and correspondingly $\psi$ will be strongly peaked at this point so that we may think of $\psi$ as an analogue of the usual zonal function on a Riemannian manifold.

We shall see that $\psi$ may be taken to be the kernel of the spectral projectors $K_\lambda$ and $K_\mu$ constructed in sections 2 and 8 respectively. Note that in the case of compact type we can prove that the spherical functions $\varphi_\lambda$ also saturate the $L^p$ bounds of theorem 1 for large $p$, but we have only been able to prove that they have the microlocal structure described here in a small neighbourhood of the origin.

For $p$ below the kink point, the $L^p$ norms on $S$ are saturated by the higher rank analogue of a Gaussian beam, which is simply a wave packet concentrated on a maximal flat subspace, and whose microlocal support is concentrated on the set $\lambda/||\lambda|| \times A \subset T^*A$. These functions will be described more thoroughly in the case of compact type in section 7.3.

### 1.1. Asymptotics for Spherical Functions

In the course of proving theorem 1 we have found it necessary to develop sharp asymptotics for spherical functions of large eigenvalue on $G$, which we state here as separable theorems. First let us assume that $G$ is semisimple with no compact factors, in which case our result strengthens a theorem of Duistermaat, Kolk and Varadarajan [4]. Let $\lambda \in \mathfrak{a}_+^*$ and let $\varphi_\lambda$ denote the standard spherical function with parameter $\lambda$, normalised so that $\varphi_\lambda(e) = 1$. Fix a compact subset $C \subset \mathfrak{a}_+^*$ which is bounded away from the walls. Let $\text{Vol}_0(K)$ and $\text{Vol}_0(M)$ be the volumes of $K$ and $M$ with respect to the metric induced from minus the Killing form on $\mathfrak{k}$, and for any $w \in W$ define

$$\sigma_w(H) = -\sum_{\alpha \in \Delta^+} m(\alpha) \text{sgn}(w\alpha(H)).$$

Our result is the following:

**Theorem 2.** Let $B \subset \mathfrak{a}$ be any compact set, and $\epsilon > 0$ be arbitrary. We have the asymptotic
\( (6) \quad \varphi_{t\lambda}(\exp(H)) = (1 + O_{B,C,\epsilon}(t^{\max\{1/|\alpha(H)|\}}) \sum_{w \in W} \exp(itw\lambda(H) + i\pi\sigma_w(H)/4)t^{-(n-r)/2} \times \prod_{\alpha \in \Delta^+} \left| \frac{\langle \alpha, \lambda \rangle}{2\pi} \sinh w_0(H) \right|^{-m(\alpha)/2} \frac{\text{Vol}_0(M)}{\text{Vol}_0(K)}, \)

uniformly for \( \lambda \in C \) and \( H \in B \) satisfying \( |\alpha(H)| \geq t^{-1+\epsilon} \) for all \( \alpha \in \Delta \). We also have the upper bound

\( (7) \quad |\varphi_{t\lambda}(\exp(H))| \ll_{B,C} \prod_{\alpha \in \Delta^+} (1 + t|\alpha(H)|)^{-m(\alpha)/2} \)

uniformly for \( H \in B \).

The asymptotic (7) is the strongest possible upper bound that can be given for \( \varphi_{\lambda}(\exp(H)) \) when \( H \) is bounded and \( \lambda \) grows, at least under the regularity assumption on \( \lambda \) that we have made. In comparison, the bound of (1) gives the same constant, but requires that \( H \) be restricted to a compact equisingular set \( D \subset a \) (that is, the set of roots vanishing on every element of \( D \) is the same). In particular, it does not give the correct bound when \( H \) approaches a root hyperplane as \( t \to \infty \).

Theorem 2 will be derived from our analysis of stationary phase integrals in section 4.3.

We have an analogous result in the case of compact type, but which is weakened by the requirement that the group variable be constrained to a small ball about the origin. Let \( U \) be a compact Lie group, and \( K \) a subgroup with the property that \( (U,K) \) is a Riemannian symmetric pair. Let \( T \subset U \) be an Abelian subgroup which projects isomorphically to a maximal flat subspace of \( U/K \), let \( C \subset i\mathfrak{a}^*_+ \) be a compact set which is bounded away from the walls, and let \( \mu \) be a spherical weight of the form \( t\tilde{\mu}, \tilde{\mu} \in C \) (notations to be explained in section 7.1). If \( \varphi_{\tilde{\mu}} \) is the \( K \)-spherical function on \( U \) with parameter \( \mu \), normalised so that \( \varphi_{\tilde{\mu}}(e) = 1 \), we then have

**Theorem 3.** There exists a ball \( B \subset T \) about the origin such that

\[ |\varphi_{\tilde{\mu}}(h)| \ll_{C} \prod_{\alpha \in \Delta^+} (1 + t|\alpha(h)| - 1)^{-m(\alpha)/2}, \]

uniformly for \( h \in B \) and \( \tilde{\mu} \in C \).

Theorem 3 will be proven in section 8.2. Note that it is possible to derive the asymptotic for \( \varphi_{\mu} \) in the ball \( B \) corresponding to (6) using our methods, but we have not done so.

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2. **The Proof in the Case of Noncompact Type**

We shall begin by proving theorem 1 in the case when \( X \) is of noncompact type in sections 2 to 6. The proof in the case of compact type is formally identical, and
we shall make the technical modifications to our argument which are required to treat it in sections 7 and 8.

2.1. Notation. To introduce our notation in the noncompact case, let $G$ be a simple real Lie group with Lie algebra $\mathfrak{g}$, and let $K$ be a maximal compact subgroup. We have the usual decompositions

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k} + \mathfrak{a} + \mathfrak{q}.$$ 

We let $\Delta$ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and $\Delta^+$ the set of positive roots corresponding to a closed positive Weyl chamber $\mathfrak{a}^+$ and nilpotent subalgebra $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Define $\Phi$ and $\Phi^\vee$ to be the set of simple positive roots and coroots.

For any $\alpha \in \Delta$ we let $m(\alpha)$ denote the dimension of the corresponding root space, and for $S \subset \Delta$ we define $m(S)$ to be $\sum_{\alpha \in S} m(\alpha)$. We have $m(\Delta) = 2n - 2r$, where $n$ and $r$ are the dimension and rank of $X$ as above.

We denote the Killing form on $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle$. If $\alpha$ is a root, $H_\alpha \in \mathfrak{a}$ will be the vector dual to $\alpha$ under the Killing form, and $\mathfrak{a}^*_+ \subset \mathfrak{a}^*$ will denote the dual positive Weyl chamber. $M$ will be the subgroup of $K$ commuting with $\mathfrak{a}$, and $\mathfrak{m}$ will denote its Lie algebra. $W$ will denote the Weyl group of $\mathfrak{a}$, which we shall think of as a set of cosets of $M$ in $K$. Let $A$ and $N$ be the connected subgroups of $G$ with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. We shall write log for the inverse exponential map on $A$.

2.2. An Outline of the Proof. Let $C \subset \mathfrak{a}^*_+$ be any compact set which is bounded away from the walls, and let $\Lambda \in C$. We shall approach theorem 1 by the standard method of constructing a family of approximate spectral projectors $T_t$ onto a ball of radius 1 about $t\Lambda$, and bounding the norms of $T_t$ from $L^2$ to $L^p$. Note that all bounds we state will depend on $X$ and $C$ from now on, but will be uniform in $\Lambda \in C$.

We shall construct $T_t$ using the theory of Harish-Chandra, which will allow us to gain good control over the behaviour of the integral kernel of this operator. Choose a function $h \in S(\mathfrak{a}^*)$ of Payley-Wiener type, positive and $\geq 1$ in a ball of radius 1 about the origin. Let

$$h_t(\lambda) = \sum_{w \in W} h(w\lambda - t\Lambda),$$

and let $K^0_t$ be the $K$-biinvariant function on $S$ with Harish-Chandra transform $h_t$ (see [3] for an explanation of these notions). It is of compact support independent of $t$ and $\Lambda \in C$ by the Payley-Wiener theorem of [2]. If $T_t$ are the convolution operators on $X$ associated to $K^0_t$, it would suffice to prove bounds for $\|T_t \psi\|_p$ of the form (1) with $\|\Lambda\|$ replaced with $t$, uniformly for $\Lambda \in C$. As is common, we shall approach this by forming the adjoint square operator $T_t T_t^*$ and proving the bounds

(8) $$\|T_t T_t^* \psi\|_p \leq \left\{ \begin{array}{ll} \log t \times t^{2\delta(p)} \|\psi\|_{p'}, & p = \frac{2(n+r)}{n-r}, \\ t^{2\delta(p)} \|\psi\|_{p'}, & p \neq \frac{2(n+r)}{n-r}, \end{array} \right.$$ 

for the operator norms of $T_t T_t^*$, where $p'$ and $p$ are dual exponents.
2.3. The Case of Rank One. In the rank one case, (8) may be proven quite easily using a dyadic decomposition of the $K$-biinvariant function $K_t = K_t^0 * K_t^0$ associated to $T_1 T_1^*$ in terms of its radial support, and it is this argument which we shall extend to the case of higher rank. To briefly recall this argument, choose $\alpha \in C_0^\infty(\mathbb{R})$ to be positive, symmetric, and identically 1 in a neighbourhood of 0, and for $m \in \mathbb{Z}_{\geq 0}$ let

$$\beta_{t,m}(x) = \begin{cases} \alpha(tx), & m = 0, \\ \alpha(te^{-m}x) - \alpha(te^{-m+1}x), & m > 0. \end{cases}$$

If we fix an isomorphism between $a$ and $\mathbb{R}$ and define $K_{t,m}(k_1 a k_2) = \beta_{t,m}(\log a) K_t(k_1 a k_2)$, it may be shown that

$$\|K_{t,m} * \psi\|_\infty \ll t^{n-1} e^{-m(n-1)/2} \|\psi\|_1,$$

$$\|K_{t,m} * \psi\|_2 \ll t^{-1} e^m \|\psi\|_2.$$

By interpolating between (9) and (10) we may prove the bound

$$\|K_{t,m} * \psi\|_p \ll t^{L(\sigma,p)} \|\psi\|_{p'}, \quad 2 \leq p \leq \infty,$$

where $\sigma(m)$ is defined by $t^{-\sigma(m)} = t^{-1} e^m$ and $L(x,p)$ is a linear function of $x$ and $1/p$ which interpolates between

$$L(0,p) = (n-1)(\frac{1}{2} - \frac{1}{p}) \quad \text{and} \quad L(1,p) = 2n(\frac{1}{2} - \frac{1}{p}) - 1,$$

so that

$$2\delta(p) = \max\{L(0,p), L(1,p)\}.$$

Because $K_t$ had uniformly bounded support, we may assume that $\alpha$ was chosen so that $K_{t,m} \equiv 0$ for $m \geq \log t$. With this choice, summing $\|K_{t,m} * \psi\|_p$ over $m$ is equivalent to summing $\sigma$ over the set $[0,1] \cap \mathbb{Z}/\log t$, and the resulting upper bound for $\|K_{t,m} * \psi\|_p$ is equal to $\|\psi\|_{p'}$ times a geometric progression of length $\log t$ whose extremal terms are $t^{L(0,p)}$ and $t^{L(1,p)}$. The bounds (8) follow immediately from this.

2.4. Generalisation to Higher Rank. Our proof for higher rank groups works by applying a similar decomposition in terms of the Cartan $a^+$ co-ordinate to $K_t$. Complications arise in higher rank when we localise $K_t$ in regions very close to the walls of $a^+$, and so we shall first describe a simplified version of our method which ignores this problem, before giving the full approach at the start of section 3. If $\beta_m$ are as before, and $m$ is an $r$-tuple of non-negative integers, define

$$\beta'_{t,m} : \mathbb{R}^r_{\geq 0} \to \mathbb{R}, \quad (x_1, \ldots, x_r) \mapsto \beta_{t,m_1}(x_1) \cdots \beta_{t,m_r}(x_r).$$

Fix a linear isomorphism $L$ between $\mathbb{R}^r_{\geq 0}$ and $a^+$, and let $\beta_{t,m}$ be the function on $a^+$ obtained by transferring $\beta'_{t,m}$ under $L$ and then modifying at the walls so
that the unfolded functions on $a$ are smooth. Define the truncated kernel $K_{t,m}$ by

$$K_{t,m}(k_1ak_2) = \beta_{t,m}(\log a)K_t(k_1ak_2).$$

If we define the points $H_m \in a^+$ by

$$H_m = t^{-1} \sum_{\omega \in \Phi^+} \omega e^{m_\omega},$$

so that $H_m$ is essentially a generic point in the support of $\beta_{t,m}$, the generalisations of (9) and (10) to higher rank may be stated as follows:

\begin{align*}
\|K_{t,m} * \psi\|_\infty & \ll t^{(n-r)/2} \prod_{\alpha \in \Delta^+} \alpha(H_m)^{-m(\alpha)/2} \|\psi\|_1, \\
\|K_{t,m} * \psi\|_2 & \ll \prod_{\alpha \in \Phi} \alpha(H_m) \|\psi\|_2.
\end{align*}

We may simplify the dependence of these inequalities on $m$ by using the asymptotic

\begin{equation}
\ln \alpha(H_m) = \max_{\alpha(\omega) \neq 0} m_\omega - \log t + O(1).
\end{equation}

Define

$$L(x,p) : [0,1]^r \times [1/2,\infty] \to \mathbb{R}$$

to be the function which is linear in $1/p$ and for $p = \infty$ and 2 is given by

\begin{align*}
L(x,\infty) & = n - r - \sum_{\alpha \in \Delta^+} m(\alpha) \max_{\alpha(\omega) \neq 0} x_i/2, \\
L(x,2) & = -r + \sum_{\alpha \in \Phi} \max_{\alpha(\omega) \neq 0} x_i.
\end{align*}

Define $\sigma(m) = m/\log t$ as before, and assume that we have chosen $\beta_m$ so that $K_{t,m} \equiv 0$ unless $\sigma(m) \in [0,1]^r$. We shall show that interpolating between (12) and (13) gives the bound

\begin{equation}
\|K_{t,m} * \psi\|_p \ll t^{L(\sigma,p)} \|\psi\|_{p'}.\end{equation}

Moreover, $L(x,p)$ has the property that if we divide the unit cube $[0,1]^r$ along the hyperplanes $\{x_i = x_j : 1 \leq i < j \leq r\}$ to obtain a collection of closed simplices $\{C_i\}$, then $L(x,p)$ is piecewise linear on each $C_i$.

On summing (15) over $m$ we obtain

\begin{equation}
\|K_{t} * \psi\|_p \ll \sum_{\sigma \in [0,1]^r \cap \mathbb{Z}^r/\log t} t^{L(\sigma,p)} \|\psi\|_{p'}.
\end{equation}

By the piecewise linearity of $L(x,p)$, the constant in this bound may be thought of as a sum of generalised geometric progressions indexed by the simplices $C_i$. The common ratios of these progressions depend only on $p$, so if we let $M(p)$ be the maximum value of $L(x,p)$ and $d(p)$ the dimension of the simplex on which this maximum is attained, we may simplify (16) to

$$\|K_t * \psi\|_p \ll_p (\log t)^{d(p)} t^{M(p)} \|\psi\|_{p'}.$$
Define \( v_0 = (0, \ldots, 0) \) and \( v_1 = (1, \ldots, 1) \). We shall see that in the case of \( \Lambda \) generic, \( L(x, p) \) attains its maximum exactly at either \( v_0 \), \( v_1 \), or the line joining them. It will follow that

\[
M(p) = \max \{ L(v_0, p), L(v_1, p) \} = 2\delta(p)
\]

and that \( d(p) \) is equal to 1 at the kink point and 0 elsewhere, which gives (8).

2.5. Removal of the Logarithmic Factor. It may be possible to remove the factor of log in the theorem by combining the techniques developed here with Sogge’s original proof of (1), and we shall now sketch a way in which this might be done.

To briefly recall Sogge’s original argument, let \( M \) be a compact Riemannian manifold, and let \( T_\lambda \) be a spectral projector onto functions on \( M \) with Laplace eigenvalue near \( \lambda^2 \). We wish to prove the bound

\[
\|T_\lambda f\|_p \ll \lambda^{\delta_0(p)} \|f\|_2.
\]

One may apply a partition of unity argument to reduce to the case in which the integral kernel \( K_\lambda(x, y) \) of \( T_\lambda \) satisfies \( \text{supp} K_\lambda \subset U \times U \), where \( U \subset M \) is a small open set which we identify with a ball in \( \mathbb{R}^n \), and \( (x, y) \in \text{supp} K_\lambda \) only if the vector \( (x - y)/|x - y| \) lies in a small open set in \( S^{n-1} \). In addition, one may choose \( T_\lambda \) so that \( K_\lambda(x, y) \) is supported on the set \( \{x, y : |x - y| \in [C^{-1}\epsilon, C\epsilon]\} \) for some \( C, \epsilon > 0 \).

With this assumption, \( K_\lambda \) may be expressed in the form

\[
K_\lambda(x, y) = \lambda^{(n-1)/2} a_\lambda(x, y) e^{i\lambda \psi(x, y)},
\]

where \( a_\lambda \in C^\infty \) and has all derivatives uniformly bounded, and \( \psi \) is real and \( C^\infty \), and satisfies the Carleson-Sjölin condition (\( \psi \) will essentially be the geodesic distance from \( x \) to \( y \)). The conditions on \( \psi \) and \( \text{supp} K_\lambda \) imply that \( x \) and \( y \) may be split into two variables as

\[
x = (x', s) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad y = (y', t) \in \mathbb{R}^{n-1} \times \mathbb{R}
\]

in such a way that the restriction of the phase \( \psi \) to each pair of planes \( (\cdot, s) \times (\cdot, t) \) satisfies the nondegeneracy condition

\[
\det \left( \frac{\partial^2}{\partial x'_i \partial y'_j} \psi(x, y) \right) \neq 0
\]

on \( \text{supp} K_\lambda \).

The proof of (1) proceeds by expressing \( T_\lambda \) as an integral of ‘frozen’ operators \( T^\lambda_{s,t} \) from \( (\cdot, s) \) to \( (\cdot, t) \), defined by

\[
T^\lambda_{s,t} f(x') = \int_{\mathbb{R}^{n-1}} K_\lambda(x', s, y', t) f(y') dy'.
\]

The estimate one needs to establish for these operators is

\[
\|T^\lambda_{t,u}(T^\lambda_{s,u})^* f\|_{L^p(\mathbb{R}^{n-1})} \ll |t - u|^{-1 + (1/p - 1/p')} \lambda^{-n(n-1)/(n+1)} \|f\|_{L^p(\mathbb{R}^{n-1})}, \quad p = \frac{2(n+1)}{n+3},
\]
as when one integrates this bound over s, t and u and applies the Hardy-Littlewood-Sobolev inequality, one obtains a bound for $T_\lambda T_\lambda^*$ which implies \( [1] \). The bound is proven by interpolating between $L^2 \to L^2$ bounds, which are proven using the nondegeneracy condition \( [17] \), and $L^1 \to L^\infty$ bounds, which follow from pointwise bounds for the kernel of $T_{x,t}(T_{x,u})^*$ derived using stationary phase.

We shall now describe how this argument might be generalised to a locally symmetric space $S$, beginning with some simple geometry which we shall make use of. Given two points $x, y \in S$, let $A(x, y)$ be any flat subspace containing them. There is an isometry of $S$ which maps $A(x, y)$ to the standard flat $A$, $x$ to the origin in $A$, and $y$ into $\exp(a^+)$. After applying this isometry, there is a vector $H(x, y) \in \mathfrak{a}^+$ such that $\exp(H(x, y)) = y$, which is referred to as the composite distance from $x$ to $y$ and is uniquely determined by them. We shall say that $x$ and $y$ are relatively singular if $H(x, y)/|H(x, y)|$ is on or near (depending on the context) a wall of $\mathfrak{a}^+$, and relatively regular otherwise. Note that the flat containing $x$ and $y$ is unique when $x$ and $y$ are relatively regular.

Suppose that we can find an approximate spectral projector $T_\lambda$ whose integral kernel $K_\lambda(x, y)$ is a point pair invariant whose underlying $K$-biinvariant function $K_\lambda(x)$ is supported away from the walls of the Weyl chamber, so that $x$ and $y$ are relatively regular when $(x, y) \in \text{supp } K_\lambda$. Restrict $x$ and $y$ to a small ball $U \subset S$ as before, and perform a spherical truncation on $K_\lambda$ so that $(x, y) \in \text{supp } K_\lambda$ only if the direction of the unique flat through $x$ and $y$ lie in a small open set. Using stationary phase techniques, it should be possible to prove that $K_\lambda$ has the form

$$K_\lambda(x, y) = \|\lambda\|^{(n-r)/2} \sum_{w \in W} a(w, x, y) \exp(i \psi_{w, \lambda}(x, y)),$$

$$\psi_{\lambda}(x, y) = \lambda(H(x, y)).$$

It may also be shown that the phase $\psi$ defined here satisfies a certain higher rank analogue of the Carleson-Sjölin condition. For $x \in S$, define $Z_x$ to be the curve

$$Z_x = \{ \nabla_x \psi(x, y) : y \in S \} \subset T_x S,$$

which we may identify with the curve $Z = \text{Ad}_{K} \lambda \subset \mathfrak{p}$. The normal to $Z$ at any point is naturally identified with $\mathfrak{a}$, and the ambient curvature of $Z$ at a point $p$ is a quadratic form from $T_p Z$ to $\mathfrak{a}$. The analogue of the Carlson-Sjölin condition satisfied by $\psi$ is that this quadratic form is nondegenerate, in the sense that the associated bilinear map $T_p Z \times T_p Z \to \mathfrak{a}$ has no kernel.

We wish to bound the operator $T_\lambda$ using a freezing argument as in the general Riemannian case. The natural generalisation of the argument used there seems to be to choose a flat $A$ passing through $U$, and whose direction is contained in the set to which we have truncated $\text{supp } K_\lambda$, and to use the orthogonal projection onto $A$ to define a set of co-ordinates mapping $U$ into $\mathfrak{q} \times \mathfrak{a}$. We may then express $T_\lambda$ as an integral of operators $T_{u,v}^\lambda$, $u, v \in \mathfrak{a}$, and attempt to prove the bound

$$\|T_{u,v}^\lambda (T_{u,w}^\lambda)^* f\|_{L^{p'}(\mathfrak{q})} \ll \|v - w\|^{-1+(1/p - 1/p')} \lambda^{\frac{n(n-r)}{n+r}} \|f\|_{L^p(\mathfrak{q})}, \quad p = \frac{2(n+r)}{n+3r},$$

where
for the norms of $T^\lambda_{u,v}(T^\lambda_{u,w})^*$. The sharp form of theorem 1 would follow from this by integrating in $u, v$ and $w$ and applying the Hardy-Littlewood-Sobolev inequality.

The higher rank Carleson-Sjölin condition that we have claimed for $\psi$ should imply that its restrictions to the sets $(\cdot, u) \times (\cdot, v)$ are nondegenerate, and so the bound for the $L^2 \to L^2$ norms of $T^\lambda_{u,v}(T^\lambda_{u,w})^*$ should hold as before. The problem arises in bounding the $L^1 \to L^\infty$ norms, which is equivalent to bounding the integral kernel $K^\lambda_{u,v}(x', v, y', w)$ of $T^\lambda_{u,w}(T^\lambda_{u,w})^*$. We have seen that the kernel of $T^\lambda T^\lambda_\alpha$ is large when $x$ and $y$ are relatively singular, and we expect $K^\lambda_{u,v}$ to behave similarly.

However, it may be possible to get around this by decomposing $T^\lambda T^\lambda_\alpha$ into two parts, one whose kernel is supported on relatively singular pairs and the other on regular pairs. We may bound the first component using the techniques of this paper, as we have shown that if we require $x$ and $y$ to be relatively singular enough then the $L^p$ norms of the corresponding component of $T^\lambda T^\lambda_\alpha$ are negligible compared with the regular component. Having removed the part of $T^\lambda_{u,v}(T^\lambda_{u,w})^*$ whose kernel is large, we should then be able to apply the classical freezing argument to the remainder, as the methods for giving $L^2 \to L^2$ and $L^1 \to L^\infty$ bounds should be compatible with the spatial truncation we have applied.

3. Details of the Proof

We have shown that most of the work in proving theorem 1 lies in establishing the estimates (12) and (13) for the norms of the truncated kernels $K_{t,m}$. In this section we shall deduce slight modifications of these estimates from the pointwise bound for $K_t$ given in theorem 4 below. We shall also give a precise form of the argument sketched in section 2.4 which deduces theorem 1 from these $L^p$ estimates.

The $L^1 \to L^\infty$ norm of $K_{t,m}$ is simply $\|K_{t,m}\|_\infty$, and so the problem here is to give a good bound for $|K_t(a)|$ which is uniform in $a$. By a careful application of stationary phase we have in fact been able to prove the following sharp upper bound, which implies (12).

**Theorem 4.** We have

\[ |K_t(\exp(H))| \ll t^{n-r} \prod_{\alpha \in \Delta^+} (1 + t|\alpha(H)|)^{-m(\alpha)/2} \]

for all $H \in \mathfrak{a}$ and $\Lambda \in C$.

This is the key step in establishing theorem 1 and sections 4, 5 and 6 will be dedicated to its proof. Theorem 2 will follow immediately from the methods developed in these sections.

3.1. $L^2$ Bounds for Truncated Kernels. Estimating the $L^2 \to L^2$ norm of $K_{t,m}$ is the same as bounding the supremum of its Harish-Chandra transform. If we could show that the transform was localised near $t\Lambda$ (and in particular, away from the walls of $\mathfrak{a}^*_+$), then we could prove sharp bounds for

\[ \tilde{K}_{t,m}(\lambda) = \int_G K_{t,m}(g)\varphi_\lambda(g)dg \]

using the pointwise bounds for $K_t$ and $\varphi_\lambda$ given by theorem 4. However, when we sharply truncate $K_t$ near the walls of the Weyl chamber, the support of $\tilde{K}_{t,m}$ may
spread to the walls of the dual chamber where $\varphi_\lambda$ will be a power of $t$ larger than expected. For this reason we have found it necessary to modify our partition of unity $\beta_{t,m}$ so that it does not localise so sharply at the walls.

We shall modify our partition by defining an equivalence relation $\sim$ on $\mathbb{Z}^r \cap [0, \log t]^r$ in terms of a small parameter $\delta > 0$. If $m \in \mathbb{Z}^r \cap [0, \log t]^r$, let $|m|$ be $\max\{m_i\}$, and let $S \subseteq [1, r]$ be the set of co-ordinates at which $m_i \leq \delta |m|$. We shall say that $m \sim m'$ if $|m| = |m'|$, $m$ and $m'$ differ only in the entries in $S$, and $m'_i \leq \delta |m'|$ for $i \in S$. Let $P_t$ be the resulting set of equivalence classes.

Choose a representative $m$ from every equivalence class in $P_t$, whose entries are as large as possible, denote the class of $m$ by $S(m)$, and let $\mathcal{M}_t \subset \mathbb{Z}^r \cap [0, \log t]^r$ be the set of all these representatives. We redefine our partition of unity so that it is indexed by $m \in \mathcal{M}_t$, and so that $\beta_{t,m}$ is now the sum of the previous $\beta_{t,j}$ over $j \in S(m)$. We make the corresponding modification to $K_{t,m}$. We shall say that $m$ is regular if $S(m)$ consists of a single element, and that it is singular otherwise, and define $H_m$ as before for $m \in \mathcal{M}_t$.

With this new choice of $\beta_{t,m}$, we may prove

**Proposition 5.** $K_{t,m}$ satisfies the bound

$$\|\hat{K}_{t,m}\|_\infty \ll \prod_{\alpha \in \Phi} \alpha(H_m)$$

for all $m \in \mathcal{M}_t$.

We begin by showing that $\hat{K}_{t,m}(t\lambda)$ satisfies \(20\) when $\lambda$ is far from $\Lambda$. In particular, let $D \subset \mathfrak{a}^*$ be a small ball about $\Lambda$ which is bounded away from all root hyperplanes, and assume that $\lambda \notin D$. Let $\tilde{\beta}_{t,m}$ be the $K$-biinvariant function on $S$ corresponding to $\beta_{t,m}$, so that

$$\hat{K}_{t,m}(t\lambda) = \int_{S} \tilde{\beta}_{t,m} K_i \varphi_{t\lambda} ds.$$ 

If we substitute the definition of $\varphi_{t\lambda}$ as an integral of plane waves, and the analogous formula for $K_i$ given by lemma 6 below, this becomes

$$\hat{K}_{t,m}(t\lambda) = t^{n-r} \int_{K} \int_{B} \tilde{\beta}_{t,m}(s) g(t,k,s) \exp(it\lambda(H(s)) - it\Lambda(H(ks))) ds dk,$$ 

where $B \subset S$ is a ball containing the support of $K_i$ for all $t$, and all derivatives of $g(t,k,s)$ in $k$ and $s$ are uniformly bounded. Define $\psi(\lambda, k, s)$ to be $\lambda(H(s)) - \Lambda(H(ks))$. We shall show that the integral \(21\) is rapidly decaying by showing that the gradient of $\psi$ in $s$ is bounded from below.

Choose a large ball $D_1 \subset \mathfrak{a}^*$ centered at the origin and which contains $D$, and suppose for the moment that $\lambda \in D_1$. When $p \in S$ corresponds to the identity in $G$, we have

$$d\psi(\lambda, k, p)(v) = (\lambda - \text{Ad}_k^{-1}\Lambda)(v), \quad v \in p.$$ 

It follows that the gradient of $\psi$ in $s$ at $p$ is bounded from below, uniformly in $k \in K$ and $\lambda \in D_1 \setminus D$, and it follows from continuity that this also holds for $s$ in some small ball $B_1$ about the origin. We may assume without loss of generality that the support of $K_t$ is contained in $B_1$ for all $t$. 


We wish to prove the rapid decay of the inner integral in (21) by applying integration by parts with respect to the field $\nabla_s \psi$. We know that all derivatives of this vector field are uniformly bounded in $K$ and $\lambda \in D_1 \setminus D$, and so it only remains to show that the derivatives of $\tilde{\beta}_{t,m}$ are smaller than the power of $t$ we gain from the partial integrations.

If $\partial$ is any differential operator on $S$ of order $\alpha$ with continuous coefficients, then it follows from the definition of $\tilde{\beta}_{t,m}$ that

$$|\partial \tilde{\beta}_{t,m}| \ll |\partial| (te^{-m_i})^\alpha.$$ 

Our assumption that $\min m_i \geq \delta \max m_i$ then implies that

$$|\partial \tilde{\beta}_{t,m}| \ll |\partial| (te^{-\delta \max m_i})^\alpha.$$ 

After substituting the asymptotic $\|H_m\| \sim (te^{-\max m_i})^{-1}$, this becomes $|\partial \tilde{\beta}_{t,m}| \ll |\partial t\|H_m\|^{-\delta}$. By performing $A$ partial integrations with respect to the vector field $X = \nabla_s \psi$ on the inner integral in (21) and using $|X^A \tilde{\beta}_{t,m}| \ll t^A \|H_m\|^{-\delta A}$, we obtain

$$\hat{K}_{t,m}(t\lambda) \ll_A t^{n-r} \text{Vol}(\text{supp}(\tilde{\beta}_{t,m})) \|tH_m\|^{-\delta A}.$$ 

We have the asymptotic

(22) $$\text{Vol}(\text{supp}(\tilde{\beta}_{t,m})) \sim \prod_{\alpha \in \Phi} \alpha(H_m) \prod_{\alpha \in \Delta^+} \alpha(H_m)^{m(\alpha)},$$

and substituting this gives

$$\hat{K}_{t,m}(t\lambda) \ll_A \|tH_m\|^{-\delta A} \prod_{\alpha \in \Phi} \alpha(H_m) \prod_{\alpha \in \Delta^+} \alpha(tH_m)^{m(\alpha)}$$

$$\ll \prod_{\alpha \in \Phi} \alpha(H_m)$$

as required. If $\lambda \notin D_1$, we may replace $\lambda$, $\Lambda$ and $t$ with $\lambda/\|\lambda\|$, $\Lambda/\|\lambda\|$ and $t/\|\lambda\|$ so that $\lambda$ and $\Lambda$ are again varying in two disjoint compact sets, and apply the same argument.

Finally, the case of $\lambda \in D$ follows easily from the approach discussed at the start of the section. We have the bounds

$$|K_{t,m}| \ll t^{n-r} \prod_{\alpha \in \Delta^+} (1 + t\alpha(H_m))^{-m(\alpha)/2},$$

$$|\tilde{\beta}_{t,m}(t\lambda)| \ll \prod_{\alpha \in \Delta^+} (1 + t\alpha(H_m))^{-m(\alpha)/2},$$

and combining these with the volume asymptotic (22) gives (20).
3.2. Simplification of $L^p$ Bounds. We shall now give a precise version of the argument of section 2.4, which deduces theorem 1 from theorem 4 and proposition 5. We begin by proving that if $L(x,p)$ is defined as before, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that we have the bounds

$$
\|K_{t,m} \ast \psi\|_p \ll \delta t^{L(\sigma,p)} \|\psi\|_{p'}
$$
when $m$ is regular, and

$$
\|K_{t,m} \ast \psi\|_p \ll \epsilon t^{L(\sigma,p)+\epsilon} \|\psi\|_{p'}
$$
otherwise (recall that the definition of regularity depends on $\delta$). By interpolation, it clearly suffices to check these for $p = 2$ and $\infty$. When $p = \infty$, theorem 4 and the definition of $\beta_{t,m}$ imply that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$
\|K_{t,m}\|_\infty \ll t^{(n-r)/2+\epsilon} \prod_{\alpha \in \Delta^+} \alpha(H_m)^{-m(\alpha)/2},
$$
where we may drop the $\epsilon$ if $m$ is regular. It therefore suffices to show that if

$$
t^A(m) = t^{(n-r)/2} \prod_{\alpha \in \Delta^+} \alpha(H_m)^{-m(\alpha)/2}
$$
then we have $A(m) \leq L(\sigma, \infty) + O(1/\log t)$, and applying (14) to the above gives

$$
A(m) = (n-r)/2 - \sum_{\alpha \in \Delta^+} m(\alpha) \log \alpha(H_m)/2 \log t
$$
$$
= n-r - \sum_{\alpha \in \Delta^+} m(\alpha) \max_{\alpha(\omega) \neq 0} m_i/2 \log t + O(1/\log t)
$$
$$
= L(\sigma, \infty) + O(1/\log t)
$$
as required. Likewise, when $p = 2$ we may define $B(m)$ by

$$
t^{B(m)} = \prod_{\alpha \in \Phi} \alpha(H_m),
$$
and we wish to show that $B(m) \leq L(\sigma, 2) + O(1/\log t)$. Applying (14) again gives

$$
B(m) = -r + \sum_{\alpha \in \Phi} \max_{\alpha(\omega) \neq 0} m_i/\log t + O(1/\log t)
$$
$$
= L(\sigma, 2) + O(1/\log t),
$$
which implies (23) and (24).

Having established the role of $L(x,p)$ in bounding $K_{t,m}$, we now prove our assertion about its maximum value and the equality

$$
(25) \quad \max\{L(v_0,p), L(v_1,p)\} = 2\delta(p),
$$
where $v_0$ and $v_1$ are as before. By the piecewise linearity of $L$ on the simplices $C_i$, it suffices to understand the maximum of $L(v,p)$ when $v \in \{0,1\}^r$. We shall freely identify such $v$ with their image in $a^+$. It may be seen that $\max_{\alpha(\omega) \neq 0} v_i$ is zero exactly
when \( v \) lies in the kernel of \( \alpha \), and so if we define \( \Delta_v \) to be the set of roots vanishing on \( v \), and \( \Delta_v^+ = \Delta^+ \cap \Delta_v \) and \( \Phi_v = \Phi \cap \Delta_v^+ \), we have

\[
L(v, \infty) = (n - r)/2 + m(\Delta_v^+)/2,
\]

\[
L(v, 2) = -|\Phi_v|.
\]

We have \( m(\Delta_v^+) = n - r \) and \( |\Phi_v| = r \) when \( v = v_0 \), and \( m(\Delta_v^+) = |\Phi_v| = 0 \) when \( v = v_1 \), and it can easily be checked that this implies [25].

To prove the assertion about the maximum of \( L(x, p) \), we must prove that the linear function \( L(v, p) \) of \( 1/p \) is always strictly dominated by the two functions \( L(v_0, p) \) and \( L(v_1, p) \). If we define

\[
M(s) = \max\{m(\Delta_v^+) : |\Phi_v| = s, \quad 0 \leq s \leq r\},
\]

then for any \( v \), \( L(v, p) \) is dominated by the linear function of \( 1/p \) interpolating between the points \((0, (n - r)/2 + M(|v|)/2) \) and \((1/2, -|v|)\). Showing that this is dominated by \( L(v_0, p) \) and \( L(v_1, p) \) is therefore equivalent to showing that \( M(s) \) is strictly convex as a function of \( s \). \( \Delta_v \) is the root system of a Levi subgroup \( M_v \) in which \( \Delta_v^+ \) and \( \Phi_v \) are a system of positive and simple roots, and so the Dynkin diagram of \( \Delta_v \) is given by the diagram of \( \Delta \) restricted to \( \Phi_v \). Using this observation, together with our assumption that \( g \) is simple and Cartan’s classification of symmetric spaces, it may easily be checked that \( m(\Delta_v^+) \) obtains its maximum for fixed \( |\Phi_v| \) when \( \Phi_v \) forms a connected subset of the Dynkin diagram, and that \( M(s) \) is convex as required.

Finally, because the factor of \( t^r \) in the bounds [23] and [24] only appears when \( m \) is very close to the walls, the value of \( L(\sigma, p) \) for such \( m \) will be less than \( 2\delta(p) \) by an absolute constant. By choosing \( \epsilon \) small we may therefore assume that the contribution from these terms to the sum [16] is negligible, and so we may deduce the bound [8] from [23] and [24] by summing the geometric progressions as before.

**Remark.** We have discussed in the introduction that the \( L^p \) norms of a zonal function and Gaussian beam with parameter \( t\Lambda \) are roughly \( \max\{t^{L(v_0, p)}, 1\} \) and \( \max\{t^{L(v_1, p)}, 1\} \) respectively, so that \( v_0 \) and \( v_1 \) may be thought of as corresponding to these two regimes of concentration. In general, if \( v \in \{0, 1\}^r \) is arbitrary then we may construct an analogous wave packet \( \psi \) which is spatially concentrated on \( M_v \), and whose \( L^p \) norm is approximately \( \max\{t^{L(v, p)}, 1\} \). If \( M_v = Z_v \times M^0_v \), where \( Z_v \) and \( M^0_v \) are the centre and derived subgroup of \( M_v \), then \( \psi \) will essentially be the product of a plane wave on \( Z_v \) with a truncated spherical function on \( M^0_v \).

4. **Bounds for the Kernel \( K_t \)**

We now turn to proving theorem[1] The first step is to invert the Harish-Chandra transform, which converts the problem to one of bounding the oscillatory integral on \( K \) appearing in the following lemma.

**Lemma 6.** We have the formula

\[
K_t(a) = t^{n-r} \int_K g(t, a, k) e^{it\phi_a(k)} dk, \quad a \in A,
\]

where \( \phi_a(k) = -\Lambda(H(ka)) \), \( g \in C^\infty \), and all \( k \)-derivatives of \( g \) are uniformly bounded in \( t \) and compact subsets of \( A \).
Proof. The Harish-Chandra transform of $K_t$ is equal to

$$
\hat{K}_t(\lambda) = h_t^2(\lambda)
= \left( \sum_{w \in W} h(w\lambda - t\Lambda) \right)^2
= \sum_{w \in W} h(w\lambda - t\Lambda)^2 + \sum_{w_1 \neq w_2} h(w_1\lambda - t\Lambda)h(w_2\lambda - t\Lambda)
= \sum_{w \in W} h(w\lambda - t\Lambda)^2 + s(\lambda, t),
$$

where $s(\lambda, t)$ is a Schwartz function of $\lambda$ and all its norms in the Schwartz topology are rapidly decaying in $t$. Inverting the Harish-Chandra transform gives

$$
K_t(a) = \int_{W \setminus a^*} \hat{K}_t(\lambda)\varphi_{-\lambda}(a)|c(\lambda)|^2 d\lambda
= \int_{W \setminus a^*} \sum_{w \in W} h(w\lambda - t\Lambda)^2\varphi_{-\lambda}(a)|c(\lambda)|^2 d\lambda
+ \int_{W \setminus a^*} s(\lambda, t)\varphi_{-\lambda}(a)|c(\lambda)|^2 d\lambda.
$$

The second term is of uniform compact support in $a$ by the Paley-Wiener theorem of [2], and rapidly decaying in $t$, and so may be ignored. Substituting the expression for $\varphi_{-\lambda}$ as an integral of a plane wave over rotation by $K$, we have

$$
K_t(a) = \int_{K} \int_{a^*} h(\lambda - t\Lambda)^2 e^{(\rho - i\lambda)(H(ka))} |c(\lambda)|^2 d\lambda
= \int_{K} f(t, H(ka)) e^{-i\Lambda(H(ka))} dk,
$$

(27)

where $f(t, H)$ is given by

$$
f(t, H) = \int_{a^*} h^2(\lambda) e^{(\rho - i\lambda)(H)} |c(\lambda + t\Lambda)|^2 d\lambda.
$$

If we make the definition $g(t, a, k) = f(t, H(ka))$, it remains to give bounds for the $k$-derivatives of $f(t, H(ka))$ which are uniform in $t$ and for a varying in a fixed compact set, which would follow from similar bounds for the $H$-derivatives of $f(t, H)$. This function is equal to $e^{\rho(H)}$ times the Fourier transform on $a^*$ of $h^2(\lambda)|c(\lambda + t\Lambda)|^2$, and so we have

$$
|\partial^n_H f(t, H)| \ll_{n, B} \int_{a^*} h^2(\lambda)||\lambda||^n|c(\lambda + t\Lambda)|^2 d\lambda
$$

for $H \in B$, $B \subset a$ compact. The formula for $c(\lambda)$ (see for instance theorem 4.7.5 of [3]) implies that $|c(\lambda)|^2 \ll ||\lambda||^{-r}$, so that

$$
\int_{a^*} s(\lambda)|c(\lambda + t\Lambda)|^2 d\lambda \ll s t^{n-r}
$$

for any Schwartz function $s$. It follows that $|\partial^n_H f(t, H)| \ll_{n, C} t^{n-r}$, as required.
The oscillatory integral of lemma 6 is a slight generalisation of the one studied in [1], in order to give asymptotics for the spherical function $\varphi_\lambda$ as $\lambda$ grows along a ray. The stationary phase techniques of that paper allow one to prove the bound of theorem 4 for $H$ and $\Lambda$ fixed, or varying in compact subsets of $a^+$ and $a^*_-$ which are bounded away from the walls, but this is not strong enough for our purposes. Proving the bounds of theorem 4 for $K_t$ requires giving a description of the behaviour of $\varphi_H$ near its critical set, uniformly as $H$ varies in an arbitrary compact subset of $a^+$, which is complicated by the fact that this critical set jumps when $H$ enters or leaves root hyperplanes. We have done this in proposition 9 below, which describes the degeneration and vanishing of critical points of $\varphi_H$ accurately enough that a sharp upper bound for $K_t$ follows easily from it.

From now until the end of section 6, we shall fix $l \in K$ and bound the contribution of a small neighbourhood of $l$ to the integral (26). The bound we shall prove is stated as proposition 8, and theorem 4 follows easily from this by a partition of unity argument.

4.1. Notation. We define $\varphi_H$ and $K_t(H)$ to be $\varphi_{\exp(H)}$ and $K_t(\exp(H))$ respectively, and shall assume from now on that $H$ is confined to a fixed compact set which contains the support of $K_t(H)$ for all $t$. We shall fix $l \in K$ and define $K_{t,l}(H)$ to be the contribution of a small neighbourhood of $l$ to the integral (26), which means that we choose a small open neighbourhood $l \in B$ and a smooth cutoff function $b$ with $\text{supp}(b) \subset B$ and define

$$K_{t,l}(H) = \int_K b(k)g(t, \exp(H), k)e^{it\varphi_H(k)}dk.$$

Note that we have dropped the factor of $t^{n-r}$ for simplicity. We have also suppressed the dependence of $K_{t,l}(H)$ on $B$ and $b$, but all upper bounds we prove for $K_{t,l}(H)$ will depend on these choices in the following way: when we write $K_{t,l}(H) \ll X(t, H)$, this will mean that there is a sufficiently small neighbourhood $B$ such that $K_{t,l}(H) \ll C(b)X(t, H)$ for any $b$ with $\text{supp}(b) \subset B$. As we will only need to apply finitely many bounds of this form in the course of our proof, we may therefore ignore this dependence on the cutoff function.

The critical set $C_H$ of $\varphi_H$ was calculated in proposition 5.4 of [1] to be

$$C_H = WK_H,$$

where $K_H \subset K$ is the stabiliser of $H$. Moreover, it was shown that the critical points of $\varphi_H$ are nondegenerate in the directions transverse to $C_H$, and the Hessians at these critical points were calculated. We define $a_l \subseteq a$ to be the subspace

$$a_l = \text{Ad}_{l^{-1}} a \cap a.$$

Define $L \subset G$ to be the connected Levi subgroup with Lie algebra $l$ which fixes $a_l$ under the adjoint action. Let $K_L = K \cap L$, and decompose $l$ as $p_L + \mathfrak{t}_L$. The significance of $a_l$ for the behaviour of $\varphi_H$ is described by the following lemma.
Lemma 7.

$$a_l = \{ H \in a : l \in C_H \}.$$  

Proof. \( H \in a_l \) if and only if there exists \( H' \in a \) such that \( \text{Ad}_{l}^{-1} H' = H \). If \( \text{Ad}_{l}^{-1} H' = H \), this implies that there exists \( w \in W \) such that \( H' = wH \), so that \( \text{Ad}_{l}^{-1} wH = H \). Therefore \( l^{-1} w \in C_H \), and \( l \in C_H \). Conversely, if \( l = wH \in C_H \) for some \( H \in a \) then we have \( \text{Ad}_{l}^{-1} wH = \text{Ad}_{l}^{-1} H = H \), so \( H \in a_l \) as required.

We have \( K_L = K_H \) for \( H \in a_l \) generic, and so lemma 7 implies that we may write \( l \) as \( w l_0 \), where \( l_0 \in K_L \) and \( w \in W \). We shall decompose \( a \) as an orthogonal direct sum \( a = a_l + a' \), and write the decomposition of \( H \) as \( H_l + H' \). We let \( \Delta_l \) be the set of roots which vanish on \( a_l \), which is exactly the root system of \( L \), and let \( \Delta' = \Delta \setminus \Delta_l \) be its complement.

It may easily be seen that \( \phi_H(k) \) and \( g(t, a, k) \) are left \( \mathcal{M} \)-invariant, and from now on it will be convenient to think of them as functions on the quotient space \( R = M \setminus K \). We shall identify \( l \) with its image in \( R \). Choose a basis \( \{ Y_i \}_{i=1}^{n-r} \) for \( m^\perp \subseteq \mathfrak{k} \) which is orthonormal with respect to the Killing form, and so that for each \( 1 \leq i \leq n-r \) there is a root \( \alpha_i \in \Delta^+ \) such that \( Y_i \in g_{\alpha_i} + g_{-\alpha_i} \). Let \( X_i = \text{Ad}_{l_0}^{-1} Y_i \), and define

$$\mathfrak{s} = \text{span}\{ X_i : \alpha_i \in \Delta^l \} \subseteq \mathfrak{k}$$

so that the set \( l \exp(\mathfrak{s}) \) is transverse to \( lK_L \). We shall assume that the \( X_i \) are ordered so that \( \alpha_i \in \Delta^l \) for \( i \leq m(\Delta^l)/2 \), and \( \alpha_i \in \Delta_l \) otherwise. Let \( \bar{X}_i \) be the vector fields on \( R \) generated by right multiplication by the one-parameter subgroups \( \exp(tX_i) \), which form an orthonormal basis for \( T_R \) at the point \( l \).

We shall generally use the following notation for open sets in \( a \), \( R \), \( a \times R \), and their complexifications. Fix a large closed disk \( D \subset a \) containing the \( H \)-support of \( K_t(H) \) for all \( t \) and \( \lambda \in C \), and let \( D_t = D \cap a_t \). \( B_a \) will denote an open set in \( a \) or \( a_t \) containing \( D \) or \( D_t \), depending on the context. \( B \) and \( B_H \) will denote open neighbourhoods of \( l \) in \( R \) which are respectively independent of, and dependent on, \( H \).

Our proof of theorem 4 will involve complexifying \( a \) and \( R \). It will be convenient to have a global complexification of \( R \), which we construct by assuming that \( G \) is of adjoint type so that \( K \) occurs as the real points of a connected complex algebraic group \( K_C \). If we define \( M_C \subset K_C \) to be the subgroup fixing \( a \), then \( R_C = M_C \setminus K_C \) provides the required global complexification. \( U \) will denote an open neighbourhood of \( D \times l \) in \( a_C \times R_C \), or of \( D_t \times l \) in \( a_{t,C} \times R_C \). \( U_a \) denote the projection of \( U \) to \( a_C \) or \( a_{t,C} \), and assume that \( U_a \times l \subset U \). For \( H \in U_a \), define \( U_H = U \cap (H \times R_C) \), which we shall think of as an open subset of \( R_C \).

4.2. Framing of the Critical Points. By analysing the behaviour of \( \phi_H \) near \( l \) and using stationary phase, we will prove the following bound for \( K_{t,l}(H) \):

**Proposition 8.** We have

$$|K_{t,l}(H)| \ll_{t,A} \|tH^l\|^{-A} \prod_{\alpha \in \Delta' \cap \Delta^+} (1 + t|\alpha(H)|)^{-m(\alpha)/2}$$  

(30)
for any \( A > 0 \).

This follows easily from the following proposition, which gives the required description of \( \phi_H \) near \( l \) for varying \( H \), and lies at the heart of all of our asymptotics for spherical functions and the kernel \( K_t \).

**Proposition 9.** For all \( l \in \mathbb{R} \) there exist \( B \) and \( B_a \) as above, and \( \delta, \rho > 0 \), such that for all \( H \in B_a \) there exists an open set \( B_H \supseteq B \) with the following properties.

1. The changes from geodesic normal co-ordinates about \( l \) to \( (\Xi, Y) \) and back have all derivatives bounded uniformly in \( H \).
2. In the co-ordinates \( (\Xi, Y) \) we have

\[
\phi_H(\Xi, Y) = f(Y) + \sum_{i=1}^{N} \Lambda(wH, \alpha_i(H)) \alpha_i(\xi_l) \xi_i^2
\]

for some function \( f \in C^\infty((-\rho, \rho)^{n-r-N}) \), all of whose derivatives are bounded uniformly in \( H \).
3. For every \( H \) there exists a vector field \( X_H \) on \((-\rho, \rho)^{n-r-N} \), all of whose derivatives are bounded uniformly in \( H \) and such that

\[
|X_H f| \gg_{B, B_a} \|H\|
\]

**Remark.** We shall see in the course of the proof of proposition 9 that the sets \( B \) and \( B_a \), and all bounds on the derivatives of \( \Xi, Y, f \) and \( X_H \), may be assumed to be uniform in \( \Lambda \in \mathbb{R} \), which yields the \( \Lambda \)-uniformity of our results. This will follow from elementary book-keeping, and a simple observation which we give at the end of section 5.

Proposition 9 should be viewed as describing the way in which two different effects influence the behaviour of \( \phi_H \) at \( l \). When \( H \in a_l \), we know that \( l \) is a critical point of \( \phi_H \). Correspondingly, \( N = n - r \) in proposition 9 so that the first term in (31) is trivial, and we have a complete description of this critical point in the co-ordinates \( \Xi \) which is uniform in \( H \). After changing co-ordinates to \( \Xi \) and applying stationary phase, this gives the bound

\[
|K_{t, l}(H)| \ll \prod_{\alpha \in \Delta_l \cap \Delta^+} (1 + t|\alpha(H)|)^{-m(\alpha)/2}.
\]

When \( H \) leaves \( a_l \), \( e^{it\phi_H} \) starts to oscillate along the orbit \( lK_l \), and this lets us show that

\[
|K_{t, l}(H)| \ll_{l, A} \|H\|^{-A}
\]

for any \( A \). When \( \|H\| \gg t^{-1+\epsilon} \), this provides a bound equivalent to that of (30).

To prove proposition 9 we must give a description of \( \phi_H \) which allows us to see both the developing oscillation along \( lK_l \) and the transverse critical points at the same time, uniformly in \( H \), and this is what proposition 9 provides. Indeed, the
directions $X_i$ with $\|H\| \leq \delta \alpha_i(H_l)$ can be thought of as those for which the perturbation of $H$ off $a_l$ does not significantly alter the critical behaviour of $\phi_H$ in the direction $X_i$. As a result, an application of stationary phase in the direction $X_i$ will strengthen the upper bound for $|K_{t,l}(H)|$ by a factor of $(1 + t|\alpha(H_l)|)^{-1/2}$. The directions $X_i$ for which $\|H\| \geq \delta \alpha_i(H_l)$ are those in which the perturbation overwhelms the original curvature of $\phi_H$, but while we may no longer apply stationary phase in these directions, the factor of $\|tH\|^{-A}$ coming from the oscillatory integral along $lK_L$ makes up for this.

To formalise this argument, let $b$ be any cutoff function with support in $B$. Absorb $g(t, \exp(H), k)$ into $b$, so that we may write (28) as

$$K_{t,l}(H) = \int_R b(t, \exp(H), k) e^{it\phi_H(k)} dk$$

where all $R$-derivatives of $b$ are uniformly bounded. After changing co-ordinates to $(\Xi, Y)$ this becomes

$$K_{t,l}(H) = \int_{(-\rho, \rho)^{n-r}} b_0(t, H, Y, \Xi) \exp \left( if(Y) + it \sum_{i=1}^N w \Lambda(H_{\alpha_i}) \alpha_i(H_l) \xi^2_i \right) |J| dY d\Xi,$$

where $J$ is the Jacobian of the change of co-ordinates and has all its derivatives uniformly bounded by condition (1) of the proposition. Applying stationary phase in the $\Xi$-variables, we obtain

$$K_{t,l}(H) = \int_{(-\rho, \rho)^{n-r-N}} b_0(t, H, Y) \exp(if(Y)) dY,$$

where $b_0$ satisfies

$$|\partial^\alpha b_0| \ll_{B, \alpha} \prod_{\|H\| \leq \delta \alpha_i(H_l)} (1 + t|\alpha_i(H_l)|)^{-1/2}$$

for all $Y$-derivatives $\partial^\alpha$. Partial integration with respect to $X_H$ and the inequality

$$\prod_{\|H\| \geq \delta \alpha_i(H_l)} (1 + t|\alpha_i(H_l)|)^{-1/2} \ll_{B, A} \|tH\|^{-A}$$

for some $A$ completes the proof of proposition 8 and hence of theorem 4.

### 4.3. Proof of Theorem 2

The upper bound (7) for $\varphi_{t \lambda}$ given in theorem 4 follows immediately from the methods we have developed to bound $K_{t,l}$. To derive the asymptotic formula (6), express $\varphi_{t \lambda}$ as the integral

$$\varphi_{t \lambda}(a) = \int_R e^{(\rho + it\lambda)(H(ka))} dk, \quad a \in A,$$

and let $\varphi_{t,l}$ be the contribution of a small neighbourhood of $l$ to this integral as before. By proposition 8 it suffices to consider only the contributions $\varphi_{t,w}$ from neighbourhoods of the Weyl points in $R$. Because $a_l = a$ when $l = w$ is a Weyl point, proposition 9 gives a complete uniformisation of $\phi_H$ near $w$. Applying stationary phase then gives
\[ \phi_{t,w}(\exp(H)) = \exp(it\phi_H(w))t^{(n-r)/2}C(w,H,\lambda)(1 + O_{B,C,\epsilon}(t \max_{\alpha \in \Delta} 1/|\alpha(H)|)) \]

\[ = e^{it\phi_H(w)}t^{(n-r)/2}C(w,H,\lambda)(1 + O_{B,C,\epsilon}(t \max_{\alpha \in \Delta} 1/|\alpha(H)|)), \]

where \( C(w,H,\lambda) \) is a constant calculated in terms of \( e^{\phi(wH)} \) and the Hessian determinant of \( \phi_H \) at \( w \). This constant may be determined by comparison with the asymptotic formula in theorem 9.1 of [1], and summing over \( w \) gives [6].

5. Analysis of \( \phi_H \), Part I

We shall approach proposition 9 by first restricting \( H \) to \( a_l \), so that \( l \) is always contained in the critical point set. This simplifies matters considerably, as we only have to deal with the degenerations of the critical point rather than the way in which it vanishes, and allows us to prove a form of the proposition with much better uniformity in \( H \) which is stated as proposition 10 below. We shall consider the case of general \( H \) in section 6 by writing \( \phi_H \) as \( \phi_{H_1} + (\phi_H - \phi_{H_1}) \) and viewing \( \phi_H - \phi_{H_1} \) as a perturbation of a function which we have already described.

The Hessian \( D\phi_H \) of \( \phi_H \) at \( l \) was calculated in proposition 6.5 of [1], and with respect to the basis \( \tilde{X}_i \) of \( T_l R \) is given by the diagonal matrix

\[ (D\phi_H)_{ii} = \frac{1}{2} \Lambda(wH_{a_i})(1 - e^{-\alpha_j(H)}). \]

\( \phi_H \) is right-invariant under \( K_L \) when \( H \in a_l \), which corresponds to the fact that [32] vanishes when \( \alpha_i(a_l) = 0 \). One also sees that the critical point at \( l \) is nondegenerate transversally to \( lK_L \) for generic \( H \), but degenerates in the direction \( \tilde{X}_i \) when \( \alpha_i(H) = 0 \). These features are reflected in the following proposition, which gives us our initial uniformisation of \( \phi_H \) for \( H \in a_l \).

**Proposition 10.** Let \( U \) and \( U_H \) be complex open neighbourhoods of \( D \times l \) and \( l \) as above. For \( U \) sufficiently small, there exists a collection of analytic functions \( \{ z_i : 1 \leq i \leq n-r \} \) on \( U \) such that

1. For each \( H \in U \), \( \{ z_i \} \) restricted to \( U_H \) form a co-ordinate system about \( l \).
2. The functions \( z_i \) are real on \( a_l \times R \).
3. \( \phi_H \) has an analytic continuation to \( U \), which may be expressed in the co-ordinates \( \{ z_i \} \) as

\[ \phi_H(k) = \phi_H(l) + \sum_{i=1}^{n-r} \Lambda(wH_{a_i})\alpha_i(H)z_i^2(H,k) \]

4. For \( \alpha_i \in \Delta_i \), the value of \( z_i \) on \( lk_L \exp(s) \) for \( k_L \in K_L \) and \( s \in s \) depends only on \( k_L \).

We shall prove proposition 10 by analytically continuing \( \phi_H \) to \( U \), and applying an inductive argument to the complexified function. We shall also continue the vector fields \( \tilde{X}_i \), by thinking of the action of \( K_C \) on \( R_C \) as a biholomorphic map \( P : R_C \times K_C \rightarrow R_C \) and defining \( \tilde{X}_i \) to be the image of \( X_i \in \gamma_2 \simeq T_eK_C \) under the differential of \( P \). If we decompose (the complexification of) \( \tilde{X}_i \) into its type \((1,0)\) and \((0,1)\) parts and let \( X_i^+ \) and \( X_i^- \) be the corresponding components of \( X_i \), we
may associate vector fields \( \tilde{X}_i^\pm \) on \( R_C \) to \( X_i^\pm \) in the same way. We see that \( \tilde{X}_i^+ \) and \( \tilde{X}_i^- \) are holomorphic (resp. antiholomorphic), and that \( \tilde{X}_i = \tilde{X}_i^+ + \tilde{X}_i^- \).

Define \( C \) to be the set \( U \cap (U_a \times lK_{L,C}) \), which lies in the critical set of \( \phi_H \) for all \( H \in a_{t,C} \) by \((29)\). The inductive step of our proof may be stated as the following proposition.

**Proposition 11.** Let \( z_1, \ldots, z_{t-1}, t \leq m(\Delta_l)/2 \), be analytic functions on \( U \) satisfying the following conditions:

1. \( z_i \) are real valued on \( a_t \times R \).
2. The differentials of \( z_i|_{U_H} \) are linearly independent at \( l \) for all \( H \in U_a \).
3. The kernel of the \( z_i \) is a smooth submanifold \( Z \subset U \) which contains \( C \).
4. If \( Z_H = Z \cap U_H \), then \( T_iZ_H = \text{span}\{ \tilde{X}_i^\pm : i \geq t \} \).

Furthermore, suppose that there exists an analytic function \( \pi : U \to Z \) with the following properties:

1. \( \pi|_Z = \text{id} \).
2. \( \pi \) preserves the fibres \( U_H \) and the real submanifold \( a_t \times R \).
3. \( \ker D\pi = \text{span}\{ \tilde{X}_i^\pm : 1 \leq i \leq t-1 \} \) along \( Z \).
4. We may express \( \phi_H \) in terms of the functions \( z_i \) and \( \pi \) as

\[
\phi_H(k) = \phi_H(\pi(H,k)) + \sum_{i=1}^{t-1} \Lambda(wH_{\alpha_i})\alpha_i(H)z_i^2(H,k)
\]

Then, after shrinking \( U \) if necessary, we may find a set of \( t \) co-ordinate functions with kernel \( Z' \) and a function \( \pi : U \to Z' \) with the same properties.

Proposition \([10]\) follows from the conclusion of this inductive argument. Indeed, when \( l = m(\Delta_l)/2 \) we have \( \text{dim } Z = \text{dim } C \), so \( Z = C \) and \( \phi_H(\pi(H,k)) = \phi_H(l) \) by the right invariance of \( \phi \) under \( K_{L,C} \). This implies that the expressions \( [33] \) and \( [31] \) for \( \phi_H(k) \) agree, because we have ordered the \( \alpha_i \) so that the last \( m(\Delta_l)/2 \) terms in the first sum are zero. We may therefore establish proposition \([10]\) by extending the set of functions \( \{ z_i \} \) provided by proposition \([11]\) to a co-ordinate system about \( l \) by adding any functions which satisfy conditions (2) and (4).

**Proof.** Because \( \text{Ad}_{a_t} \) fixes \( a_t \) pointwise we have

\[
X_i = \text{Ad}_{a_t}^{-1}Y_i \in \text{span}\{ Y_i : (\alpha_i - \alpha_l)|_{a_t} = 0 \},
\]

and so \([H,X_i] = 0 \) when \( \alpha_l(H) = 0 \). If we let \( L_t = a_{t,C} \cap \ker \alpha_t \), this implies that when \( H \in L_t \) we have

\[
\begin{align*}
\tilde{X}_i\phi_H(k) &= \frac{d}{dt}\phi_H(k\exp(tX_i)) \bigg|_{t=0} \\
&= -\frac{d}{dt}\Lambda(H(k\exp(tX_i)\exp(H))) \bigg|_{t=0} \\
&= -\frac{d}{dt}\Lambda(H(k\exp(H)\exp(tX_i))) \bigg|_{t=0} \\
&= 0
\end{align*}
\]

Because \( \tilde{X}_i^-\phi \) is identically 0 due to the holomorphy of \( \phi \), this also implies that \( \tilde{X}_i^+\phi_H(k) \) vanishes when \( H \in L_t \). For \( i \geq t \), let \( V_i, V_i^+ \) and \( V_i^- \) be the analytic (resp. holomorphic and antiholomorphic) vector fields on \( Z \) defined by
\(V_{i,z} = D\pi_z\tilde{X}_i, \quad V_{i,z}^\pm = D\pi_z\tilde{X}_i^\pm, \quad \text{for } z \in Z,\)

which are nowhere vanishing by assumption (7) on \(\pi\). Moreover, assumptions (4) and (5) on \(T_lZ_H\) and \(\pi\) imply that \(V_i^\pm = \tilde{X}_i^\pm\) along \(U_a \times l\). Let \(\psi\) be the restriction of \(\phi_H(k)\) to \(Z\). By differentiating equation \((34)\) with respect to \(\tilde{X}_i^\pm\), we obtain

\[
\tilde{X}_i^+ \phi_H(k) = V_i^+ \psi(\pi(H, k)) + \sum_{i=1}^{t-1} \Lambda(wH_{\alpha_i})\alpha_i(H) z_i(H, k) \tilde{X}_i^+ z_i(H, k), \quad (k, H) \in Z.
\]

The second term on the RHS vanishes by assumption (3), and so we see that \(V_i^+ \psi(z)\) vanishes when \(z \in Z_t = Z \cap (L_t \times \mathbb{R}_C)\). Condition (2) on the functions \(z_i\) implies that \(Z_t\) is a smooth analytic submanifold of \(Z\) of codimension 1 in a neighbourhood of \(U_a \times l\) in \(U\). Because \(Z_t\) is defined by the analytic function \(\alpha_i(H)\), this implies that after possibly shrinking \(U\) we may find an analytic function \(f\) on \(Z\) such that

\[
(35) \quad V_i^+ \psi(z) = \alpha_i(H(z)) f(z).
\]

Because \(V_i^+ \psi = V_i \psi\) and \(\psi\) is real on \(Z_0 = Z \cap (a_t \times R)\), we see that \(f\) is real valued on \(Z_0\). We shall require the following about \(f\):

**Lemma 12.** The function \(f\) has the following properties:

1. \(f\) vanishes on \(C\).
2. For all \(z \in U_a \times l\), we have \(V_i^+ f \neq 0\) and \(V_i^+ f = 0\) for all \(i > t\).
3. \(\Lambda(wH_{\alpha_i}) V_i f(z)\) is real and positive for \(z \in a_t \times l\).

**Proof.** Because \(C\) is contained in both \(Z\) and the critical set of \(\phi\), it is also contained in the critical set of \(\psi\). \(f\) therefore vanishes on \(C \setminus C \cap Z_t\), and hence on \(C\) by continuity.

For \(i \geq t\) we have

\[V_i^+ f = V_i^+ V_i^+ \psi/\alpha_i(H(z)),\]

and because \(V_{i,z}^+ = \tilde{X}_{i,z}^+\) for \(z = (H, l) \in U_a \times l\) we have

\[
V_i^+ V_i^+ \psi(H, l) = \tilde{X}_i^+ \tilde{X}_i^+ \phi(H, l) = \langle X_i, D\phi_H X_i \rangle = -\frac{1}{2} \delta_H \Lambda(wH_{\alpha_i})(1 - e^{-\alpha_i(H)}),
\]

\[V_i^+ f = -\frac{1}{2} \delta_H \Lambda(wH_{\alpha_i})(1 - e^{-\alpha_i(H)})/\alpha_i(H).
\]

\(V_i^+ f\) is therefore 0 for \(i \neq t\), and when \(i = t\) it may be seen that we can ensure that \(V_i^+ f \neq 0\) by choosing \(U_a\) to be sufficiently small in the imaginary directions. The final claim about the sign of \(V_i f\) follows easily from \((36)\).

\[\square\]

Lemma 12 implies that (after possibly shrinking \(U\) again) the zero locus of \(f\) is a smooth analytic submanifold \(Z' \subset Z\) of codimension 1, which contains \(C\) and whose tangent space is everywhere transverse to \(V_i^+\). Moreover, if \(Z'_H = Z' \cap U_H\) then assertion (2) of lemma 12 and the fact that \(V_i^\pm = \tilde{X}_i^\pm\) on \(U_a \times l\) imply that
We denote $Z' \cap Z_0$ by $Z'_0$, which is a real codimension 1 submanifold of $Z_0$.

Because $V^+_t$ and $V^-_t$ are complex conjugates, they commute and we may integrate along the real parts of their span to obtain an analytic projection map $\pi_Z : Z \to Z'$. The restriction of $\pi_Z$ to $Z_0$ is just the map $Z_0 \to Z'_0$ given by integration along the real vector field $V_t$. If we define $\pi' = \pi_Z \circ \pi$, we see that $\pi' : U \to Z'$ is an analytic function which is the identity on $Z'$, preserves $U_H$ and $a_t \times R$, and has the property that for $z \in Z'$,

$$\kappa D\pi' = \text{span}\{\tilde{X}^\pm_i : 1 \leq i \leq t - 1\} \cup \{V^\pm_t\}$$

$$\pi'$$ therefore satisfies conditions (5), (6) and (7) of our inductive hypothesis with respect to $Z'$ and the vector fields $\tilde{X}^\pm_1, \ldots, \tilde{X}^\pm_t$.

It follows from (35) that the function $\Psi(z) = (\psi(z) - \psi(\pi_Z(z)))/\alpha_l(H)$ is bounded, and hence analytic, on $Z$. Moreover both $\Psi$ and $V^+_t\Psi$ vanish on $Z'$, while we have

$$(V^+_t)^2\Psi = \frac{(V^+_t)^2(\psi(z) - \psi(\pi_Z(z)))/\alpha_l(H)}{V^+_t f(z)}$$

$$\neq 0, \quad z \in U_H \times l.$$  

$\Psi$ therefore vanishes to exactly second order along $Z'$, and so has no other zeros in some small neighbourhood of that set containing $U_H \times l$. In addition, assertion (3) of lemma 12 implies that $\Lambda(wH_{a_t})\Psi$ is real and non-negative on $Z_0$. This implies that there exists an analytic function $\xi$ on $Z$ which is real on $Z_0$ and such that $\Lambda(wH_{a_t})\xi^2 = \Psi$, and so by the definition of $\Psi$ we have

$$\psi(z) = \psi(\pi_Z(z)) + \Lambda(wH_{a_t})\alpha_l(H(z))\xi(z)^2.$$  

Define the function $z_t$ on $U$ to be $\xi(\pi(z))$. Because $\pi$ mapped $a_t \times R$ to $Z_0$ and $\xi$ is real valued there, we see that $z_t$ is real on $a_t \times R$. Because $\xi|_{Z_H}$ has nonzero differential at $l$ for all $H$ we see that the collection of functions $\{z_t|_{U_H} : i \leq t\}$ again have linearly independent differentials at $l$. The kernel of $\{z_t : i \leq t\}$ is $Z'$, which is smooth and contains $C$, and we have already established condition (4) on $T_lZ'_H$, so that our new collection of functions satisfies conditions (1) through (4) of lemma 12.

It remains to establish the extension of (34) to our enlarged set of co-ordinates. Setting $z = \pi(H, K)$ in (37) and recalling the definitions $\psi = \phi|_Z$ and $\pi' = \pi_Z \circ \pi$, we may substitute (37) into the existing form of (34) to obtain

$$\phi_H(k) = \phi_H(\pi'(H, k)) + \sum_{i=1}^t \Lambda(wH_{a_i})\alpha_i(H)z^2_i(H, k).$$

This completes our inductive argument.
Remark. To make proposition 10 uniform in \( \Lambda \), introduce \( \Lambda \) as an extra complex variable and apply the argument above to an open neighbourhood of \( D \times I \times C \) in \( a_{l,c} \times R_C \times a_C^* \). It follows from this that the co-ordinate functions \( z_i \) depend analytically on \( \Lambda \), and this may be used to make all subsequent bounds uniform in \( \Lambda \) as well.

6. Analysis of \( \phi_H \), Part II

We now consider the problem of describing \( \phi_H \) for general \( H \). \( l \) is no longer a critical point of \( \phi_H \) if \( H \notin a_l \), and by quantifying this fact appropriately we will be able to prove the bound \( |K_{l,l}(H)| \ll t^{-A} \) for any \( A \) if \( ||H^l|| > t^{-1+\epsilon} \). We may therefore assume that \( H \) is very close to \( a_l \), and view \( \phi_H \) as the sum of the well understood function \( \phi_{H_l} \), and the small perturbation \( \phi_H - \phi_{H_l} \).

To describe our approach, extend the open sets \( U_a \) and \( U \) produced by proposition 10 from \( a_{l,c} \) and \( a_{l,c} \times R_C \) to \( a_C \) and \( a_C \times R_C \) by taking their products with an open ball in \( a_C \), and extend the co-ordinate functions \( z_i \) by making them independent of \( H^l \). It follows from the analyticity of \( \phi_H(k) \) that all \( R \)-derivatives of \( \phi_H - \phi_{H_l} \) are \( \ll ||H^l|| \), uniformly in \( k \) and \( H \). As a result, if a root \( \alpha_i \) satisfies \( \delta \alpha_i(H_l) > ||H^l|| \) for some small \( \delta \) then the critical point of \( \phi_H \) in the \( z_i \) co-ordinate will be preserved under perturbation by \( \phi_H - \phi_{H_l} \), while if \( \alpha_i(H_l) \) is much smaller than \( ||H^l|| \) it may be overwhelmed. Choose \( \delta > 0 \), assume that the co-ordinates \( z_i \) are ordered so that \( \delta \alpha_i(H_l) \geq ||H^l|| \) iff \( i \leq N \), and define

\[
X = (z_1, \ldots, z_N) \quad \text{and} \quad Y = (z_{N+1}, \ldots, z_{N-r}).
\]

It will follow that there exists a holomorphic function \( \Psi \) such that \( \phi_H(X, Y) \) has a nondegenerate critical point in \( X \) at \( \Psi(Y) \) for each fixed \( Y \), and by uniformising this critical point we shall be able to write \( \phi_H \) as the sum of \( \phi_H(\Psi(Y), Y) \) and a quadratic form in \( N \) new co-ordinates which vanish on the set \( Z = (\Psi(Y), Y) \).

The result of this approach is stated below as proposition 13 from which the first two points of proposition 10 will easily follow. The third point will be proven in section 6.2 after an analysis of the derivatives of \( \phi_H \) as \( H \) leaves \( a_l \).

6.1. Perturbation of \( H \). We shall assume that the co-ordinates \( z_i \) map \( U_H \) isomorphically to \( D(0, \rho)^{n-r} \) for some \( \rho > 0 \) and all \( H \in U_a \), where \( D(0, \rho) \) denotes the open disk of radius \( \rho \) in \( \mathbb{C} \), and we shall identify these two sets. With this in mind, we may state the following:

**Proposition 13.** There exist positive constants \( C_1, C_2 \) and \( C_3 \) which depend only on \( U_a \) and \( l \), such that the following is true for any choice of \( \rho < C_1 < 1, \delta < C_2 \rho^2 \), and \( H \in U_a \):

Assume that the co-ordinates \( z_i \) are ordered so that \( \alpha_i \) satisfies \( ||H^l|| \leq \delta |\alpha_i(H_l)| \) iff \( i \leq N \), and let \( X \) and \( Y \) be as above. There exists a holomorphic function

\[
\Psi : D(0, \rho)^{n-r-N} \rightarrow D(0, C_3 \delta)^N, \quad \Psi(0) = 0,
\]

a set \( Z \subset D(0, \rho)^{n-r} \) defined by

\[
Z = \{ (\Psi(Y), Y) : Y \in D(0, \rho)^{n-r-N} \},
\]
and holomorphic functions $\xi_i$, $1 \leq i \leq N$, which vanish on $Z$ such that

\begin{align}
(39) \quad \phi_H(x, y) &= \phi_H(\Psi(y), y) + \sum_{i=1}^{N} \Lambda(wH_{\alpha_i})\alpha_i(H)\xi_i^2, \\
(40) \quad \left| \frac{\partial \xi_i}{\partial z_j} - \delta_{ij} \right| &\ll \frac{\delta}{\rho^3}.
\end{align}

Moreover, if $H \in \mathfrak{a}$ the restrictions of $\Psi$ and $\xi_i$ to the submanifold $R$ are all real.

Before giving the proof of this proposition, we shall explain how it implies proposition [9] with the exception of condition (3). The holomorphy of the co-ordinates $z_i$ on $U$ implies that, after shrinking $U$, for every sufficiently small $\rho' > 0$ there exists a ball $l \in B' \subset R$ such that $B' \subset U_H = D(0, \rho')^{n-r}$ for all $H \in U_a$. Choose $\rho'$ and $\delta$ as in proposition [13], and let $\Xi = (\xi_1, \ldots, \xi_N)$ and $Y = (z_{N+1}, \ldots, z_{n-r})$ be the co-ordinates produced by the proposition. If we choose $\delta$, so that $\delta/(\rho')^3$ is small, condition (40) implies that there exist $\rho > 0$ and an open set $V_H \subset U_H$ for all $H \in U_a$ such that the co-ordinates $(\Xi, Y)$ map $V_H$ isomorphically onto $D(0, \rho)^{n-r}$, and there is a ball $l \in B$ such that $B \subset V_H$. These $\delta$, $\rho$, $B$, and the restrictions of $\Xi$ and $Y$ to $R$ will be the data satisfying the conditions of proposition [9].

To prove that the real restriction of $(\Xi, Y)$ satisfies condition (1), first note that the co-ordinates $z_i$ satisfy the same condition with respect to the complexified geodesic normal co-ordinates on compact subsets of $U$ by holomorphy. Condition (40) and our freedom to shrink $U$ therefore implies the same for the complex co-ordinates $(\Xi, Y)$, and we obtain (1) by restriction to $R$.

Condition (2) follows from (39) by setting $f(Y) = \phi_H(\Psi(Y), Y)$ once we know that all derivatives of $f$ are uniformly bounded in $Y$ and $H$. This follows from the corresponding bounds on the derivatives of $\phi_H$, and the fact that the holomorphy of $\Psi$ implies that all its derivatives are bounded after shrinking $U$.

We begin the proof of proposition [13] with a lemma which we shall need to prove that $\Psi(0) = 0$, or that $l$ remains the origin of our new co-ordinate system $(\Xi, Y)$.

**Lemma 14.** We have $\tilde{X}_i \phi_H(l) = 0$ for all $\tilde{X}_i$ such that $\alpha_i \in \Delta'$.

**Proof.** Write $l\exp(tX_i)a$ in Iwasawa co-ordinates as $n_t a_i k_t$. Define $n'_i \in \mathfrak{g}$ to be the derivative of $n_t + n_{t-1}$ at $s = 0$, and likewise for $a'_i$ and $k'_i$. It follows from the definition of $\phi_H$ that $\tilde{X}_i \phi_H(l) = -\Lambda(a'_0)$, and so we wish to calculate $a'_0$. By differentiating the equation $l\exp(tX_i)a = n_t a_i k_t$ at $t = 0$, we obtain

\begin{align*}
\text{Ad}_l X_i &= n'_0 + \text{Ad}_{n_{\alpha_i}} a'_0 + \text{Ad}_{n_{\alpha_i}} k'_0, \\
\text{Ad}_w Y_i &= n'_0 + \text{Ad}_{n_{\alpha_i}} a'_0 + \text{Ad}_{n_{\alpha_i}} k'_0, \\
\text{Ad}_{n^{-1}_0} w Y_i &= \text{Ad}_{n^{-1}_0} n'_0 + a'_0 + \text{Ad}_{w} k'_0.
\end{align*}

We therefore wish to show that the projection of $\text{Ad}_{n^{-1}_0} w Y_i$ onto $\mathfrak{a}$ is 0. $Y_i$ lies in $\mathfrak{g}_{\alpha_i} + \mathfrak{g}_{-\alpha_i}$, so that $\text{Ad}_w Y_i$ lies in $\mathfrak{g}_{w \alpha_i} + \mathfrak{g}_{-w \alpha_i}$, and we shall show that the image of these spaces under $\text{Ad}_{n^{-1}}$ is orthogonal to $\mathfrak{a}$. Write $l_0 a$ in Iwasawa co-ordinates on the group $L$ with respect to the unipotent subgroup $N_L = w^{-1} N w \cap L$ as $l_0 = n_L a_L k_L$, so that
\[ l_a = w n_L a_L k_L \]
\[ = (w n_L w^{-1})(w^{-1}a_L)w k_L. \]

By comparing this with the decomposition \( l_a = n_0 a_0 k_0 \), we see that \( n_0 = w n_L w^{-1} \in N \cap w L w^{-1} \). It follows that \( \text{Ad}_{n_0} a \in w w^{-1} \), but because \( \alpha_i \in \Delta^t \) we have \( g_{\pm w_{\alpha_i}} \perp w w^{-1} \). Therefore \( \text{Ad}_{n_0} g_{\pm w_{\alpha_i}} \) is orthogonal to \( a \) as required.

We now turn to the main part of the proof. All implied constants in the following will depend only on \( U, \alpha, l \) and \( C \). As with proposition 10, we shall proceed by induction on the number of variables. Let \( t \in \{1, \ldots, N \} \) be given, and define

\[ X = (z_1, \ldots, z_{t-1}) \]
\[ Y = (z_t, \ldots, z_{n-r}). \]

Suppose that we are able to carry out the required uniformisation of \( \phi_H \) in the \( X \) variables, i.e. that we can find a function

\[ \Psi : D(0, \rho)^{n-r-t+1} \rightarrow \prod_{i=1}^{t-1} D(0, C_3 \delta / \|H_l\|) \subseteq D(0, C_3 \delta)^{t-1}, \]
\[ \Psi(0) = 0, \]

and functions \( \xi_1, \ldots, \xi_{t-1} \) which satisfy the conditions of the proposition with respect to \( X, Y \) and \( \Psi \). We shall then show that we may do the same with respect to the co-ordinates

\[ X' = (z_1, \ldots, z_t) \]
\[ Y' = (z_{t+1}, \ldots, z_{n-r}). \]

For \( Y \in D(0, \rho)^{n-r-t+1} \), define \( f(Y) \) to be \( \phi_H(\Psi(Y), Y) \), and let \( Y = (z_t, Y') \) with \( Y' \) as above. We have

\[ \frac{\partial f}{\partial z_t}(Y) = \frac{\partial \phi_H}{\partial z_t}(\Psi(Y), Y) + \sum_{i=1}^{t-1} \frac{\partial \phi_H}{\partial z_i}(\Psi(Y), Y) \frac{\partial \Psi_i}{\partial z_t}(Y) \]
\[ = \frac{\partial \phi_H}{\partial z_t}(\Psi(Y), Y) + O(||H_l||) \]
\[ + \sum_{i=1}^{t-1} \left( \frac{\partial \phi_H}{\partial z_i}(\Psi(Y), Y) + O(||H_l||) \right) \frac{\partial \Psi_i}{\partial z_t}(Y), \]

and after substituting the expression for \( \phi_H \) given by proposition 10 this becomes

\[ \frac{\partial f}{\partial z_t}(Y) = -2\Lambda(w H_{\alpha_t}) \alpha_t(H_l) z_t + O(||H_l||) \]
\[ + \sum_{i=1}^{t-1} \left[ \alpha_i(H_l) O(\|\Psi_i(y)\|) + O(||H_l||) \right] \frac{\partial \Psi_i}{\partial z_t}(Y). \]
We have assumed that $|\Psi_t| \leq C_3 \|H_t^{\ell}\|/|\alpha_t(H_t)|$, and after replacing $\rho$ with $\rho/2$ we may combine this with the holomorphy of $\Psi$ to deduce that $|\partial \Psi_t/\partial z_t| \ll \delta/\rho$. These bounds imply

$$
\frac{\partial f}{\partial z}(Y) = -2\Lambda(w H_{\alpha_t}) \alpha_t(H_t) z_t + O(\|H_t^{\ell}\|) + \mathcal{O}(\|H_t^{\ell}\|/\|\alpha_t(H_t)\|)
$$

$$
= -2\Lambda(w H_{\alpha_t}) \alpha_t(H_t)(z_t + O(\|H_t^{\ell}\|/|\alpha_t(H_t)|))
$$

$$
= -2\Lambda(w H_{\alpha_t}) \alpha_t(H_t)(z_t + O(\delta)).
$$

Reducing $C_2$ and increasing $C_3$ if necessary, we may combine (42) with our assumption that $\delta \leq C_2 \rho^2$ to deduce that, for each fixed $Y$, $\partial f/\partial z_t$ has a simple zero in $z_t$ in the disk $D(0, C_3 \|H_t^{\ell}\|/|\alpha_t(H_t)|)$ and no others. As a result, there exists a function

$$
\psi: D(0, \rho)^{n-r-t} \rightarrow D(0, C_3 \|H_t^{\ell}\|/|\alpha_t(H_t)|)
$$

such that

$$
\frac{\partial f}{\partial z}(\psi(Y'), Y') = 0.
$$

We may show that $\psi(0) = 0$ using lemma 14. This is equivalent to showing that $\partial f/\partial z_t(0) = 0$, and setting $Y = 0$ in equation (41) and using the assumption that $\Psi(0) = 0$ gives

$$
\frac{\partial f}{\partial z_t}(0) = \frac{\partial \phi_H}{\partial z_t}(0) + \sum_{i=1}^{t-1} \frac{\partial \phi_H}{\partial z_t}(0) \frac{\partial \Psi_t}{\partial z_t}(0).
$$

Assumption (3) of proposition 10 on the final $m(\Delta_t)/2$ co-ordinates $z_i$ implies that the span of the vectors $\{\partial/\partial z_i : 1 \leq i \leq m(\Delta_t)/2\}$ at $l$ is equal to the span of $\{\tilde{X}_i : 1 \leq i \leq m(\Delta_t)/2\}$, so that lemma 14 implies that $\partial \phi_H/\partial z_i(0) = 0$ for all $i \leq t$ as required.

As a result of (43), if we define the function $\Psi'$ by

$$
\Psi': D(0, \rho)^{n-r-t} \rightarrow \prod_{i=1}^{t} D(0, C_3 \|H_t^{\ell}\|/|\alpha_t(H_t)|)
$$

$$
\Psi'(Y') = (\Psi(\psi(Y'), Y'), \psi(Y'))
$$

then one sees easily that $\phi_H(X', Y')$ has a critical point in the $X'$ variables at $X' = \Psi'(Y')$, and we shall show that $\Psi'$ plays the role of $\Psi$ with respect to the co-ordinates $(X', Y')$. Note that $\psi(0) = 0$ implies that $\Psi'(0) = 0$. To uniformise $\phi_H$ in the $X'$ co-ordinate around the set

$$
Z' = \{ (\Psi'(Y'), Y') : Y' \in D(0, \rho)^{n-r-t} \},
$$

we must show that, for each fixed $Y'$, $f(z_t, Y') - f(\psi(Y'), Y')$ has a double zero in $z_t$ at $z_t = \psi(Y')$ and no others. Applying the definition of $f$ gives

$$
f(Y) - f(\psi(Y'), Y') = \phi_H(\Psi(Y), Y) - \phi_H(\Psi'(Y'), Y')
$$

$$
= \phi_H(\Psi(Y), Y) - \phi_H(\Psi'(Y'), Y') + \mathcal{O}(\|H_t^{\ell}\|).
$$
The points \((\Psi(Y), Y)\) and \((\Psi'(Y'), Y')\) agree in the last \(n - r - t\) co-ordinates, and for \(i < t\) it follows from the definitions of \(\Psi\) and \(\Psi'\) that their \(i\)th co-ordinates are both bounded by \(C_\delta \|H^i\|/|\alpha_i(H_t)|\). The formula of proposition 10 then implies that

\[
f(Y) - f(\psi(Y'), Y') = -\Lambda(wH_{\alpha_i})\alpha_i(H_t)(z_t^2 - \psi(Y')^2) + \sum_{i=1}^{t-1} \alpha_i(H_i)O(\|H^i\|/|\alpha_i(H_t)|)^2 + O(\|H^t\|)
\]

\[
= -\Lambda(wH_{\alpha_i})\alpha_i(H_t)(z_t^2 - \psi(Y')^2) + O(\|H^t\|)
\]

\[
= -\Lambda(wH_{\alpha_i})\alpha_i(H_t)(z_t^2 + O(\delta)).
\]

Our assumption that \(\delta < C_\delta \rho^2\) implies that \(f(Y) - f(\psi(Y'), Y')\) has only two zeros with \(z_t \in D(0, \rho)\), and so both must be at \(z_t = \psi(Y')\) as required. We may therefore define the function \(\xi(Y)\) on \(D(0, \rho)^{n-r-t+1}\) by

\[
-\Lambda(wH_{\alpha_i})\alpha_i(H)\xi(Y)^2 = f(Y) - f(\psi(Y'), Y'),
\]

and we define \(\xi_t\) to be the extension of \(\xi\) to \(D(0, \rho)^{n-r}\) by making it independent of \(X\). If we substitute (45) into our inductive assumption (39), we have

\[
\phi_H(X, Y) = \phi_H(\Psi(Y), Y) - \sum_{i=1}^{t-1} \Lambda(wH_{\alpha_i})\alpha_i(H_t)\xi_t^2
\]

\[
= f(Y) - \sum_{i=1}^{t-1} \Lambda(wH_{\alpha_i})\alpha_i(H_t)\xi_t^2
\]

\[
= f(\psi(Y'), Y') - \Lambda(wH_{\alpha_i})\alpha_i(H)\xi(Y)^2 - \sum_{i=1}^{t-1} \Lambda(wH_{\alpha_i})\alpha_i(H_t)\xi_t^2
\]

\[
= \phi_H(\Psi'(Y'), Y') - \sum_{i=1}^{t} \Lambda(wH_{\alpha_i})\alpha_i(H_t)\xi_t^2,
\]

so that (39) holds for our new set of co-ordinates \(Y'\) and \(\{\xi_i\}\) with respect to \(\Psi'\).

We next show that \(\xi_t\) satisfies the derivative condition (40), which will follow from the bounds

\[
|\frac{\partial \xi}{\partial z_t} - \delta_{zt}| < \delta/\rho^3, \quad t \leq n-r.
\]

To prove this when \(i = t\), we differentiate (45) and substitute (42) to obtain

\[
-\Lambda(wH_{\alpha_i})\alpha_i(H)\xi \frac{\partial \xi}{\partial z_t} = \frac{\partial f}{\partial z_t} = -2\Lambda(wH_{\alpha_i})\alpha_i(H_t)(z_t + O(\delta))
\]

\[
\xi \frac{\partial \xi}{\partial z_t} = z_t + O(\delta).
\]
Moreover, by comparing (44) and (45) we have $\xi^2 = z^2 + O(\delta)$. By combining this with (47) we obtain the bound (46) when $|z_i| = \rho$, and the bound in the interior follows by the maximum principle.

When $i > t$, differentiating (45) gives

$$-\Lambda(wH_\alpha)\alpha_i(H)\xi(Y) \frac{\partial \xi}{\partial z_i}(Y) = \frac{\partial f}{\partial z_i}(Y) - \frac{\partial f}{\partial z_i}(\psi(Y'), Y') - \frac{\partial f}{\partial z_i} \frac{\partial \psi}{\partial z_i}(\psi(Y'), Y')$$

As before, we have

$$\frac{\partial f}{\partial z_i} = -2\Lambda(wH_\alpha)\alpha_i(H)z_i + O(||H'||),$$

so that

$$\left| \frac{\partial f}{\partial z_i}(Y) - \frac{\partial f}{\partial z_i}(\psi(Y'), Y') \right| \ll ||H'||.$$

In addition, (42) and $|\psi| \ll ||H'||/\alpha_t(H_l)$ imply that

$$\left| \frac{\partial f}{\partial z_i}(\psi(Y'), Y') \right| \ll ||H'||,$$

and as before we know that (after shrinking $\rho$) all first order derivatives of $\psi$ are bounded by $\delta/\rho$. Combining these, we have

$$\left| \Lambda(wH_\alpha)\alpha_i(H)\xi(Y) \frac{\partial \xi}{\partial z_i}(Y) \right| \ll ||H'||$$

$$\left| \frac{\partial \xi}{\partial z_i} \right| \ll \delta.$$

The required bound follows from this and (44) as before. Finally, it may easily be checked that $\Psi$ and $\xi_i$ satisfy the realness condition.

6.2. Vanishing of Critical Points. It remains to verify that the function $f(Y) = \phi_H(\Psi(Y), Y)$ determined by proposition 13 satisfies condition (3) of proposition 9. We do not need to complexify to do this, and so shall think of $(\Xi, Y)$ as real co-ordinates about $l$ on $R$. If $X$ is any vector field on $R$, then differentiating (31) with respect to $X$ along the submanifold $\{\Xi = 0\}$ gives

$$(48) \quad X\phi_H(0, Y) = \overline{X}f(Y),$$

where $\overline{X}$ is the image of $\overline{X}$ under projection to the $Y$-co-ordinate. Lower bounds for the derivative of $f$ will therefore follow from lower bounds for the derivative of $\phi_H$.

The lower bound for the derivative of $\phi_H$ that we shall use is as follows. Let the open neighbourhood $B \subset R$ of $l$ and $C_1 > 0$ be given. For $H \in B_2$, a root $\alpha \in \Delta$, will be called $H$-oscillating (with respect to $B$ and $C_1$) if there exists $X = X_\alpha + X_{-\alpha} \in \mathfrak{k}$ with $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ such that

$$(49) \quad |X\phi_H(k)| \geq C_1||X|||\alpha(H)||, \quad k \in B.$$
Proposition 15. There exist $B$ and $C_1$ such that the set of $H$-oscillating roots spans $a'$ for all $H \in B_a$.

Proof. We shall begin by finding a simple condition which implies that a given root $\alpha \in \Delta_l$ is $H$-oscillating. Let $k \in B$ and $H \in B_a$ be given, set $a = \exp(H)$, and let $X = X_\alpha + X_{-\alpha} \in \mathfrak{t}$ be associated to $a$ as above. Write $k \exp(tX)a$ in Iwasawa co-ordinates as $n_t a_t k_t$. Define $n'_t \in \mathfrak{g}$ to be the derivative of $n_t + n_t^{-1}$ at $s = 0$, and likewise for $a'$ and $k'$. It follows from the definition of $\phi_H$ that $X \phi_H(k) = -\Lambda(\alpha_0')$, and so we wish to calculate $a'_0$. By differentiating the equation $k \exp(tX)a = n_t a_t k_t$ at $t = 0$, we obtain

\[
\begin{align*}
\text{Ad}_kX &= n'_0 + \text{Ad}_{n_0}a'_0 + \text{Ad}_{n_0k_0}k'_0 \\
\text{Ad}_{n_0}^{-1}kX &= \text{Ad}_{n_0}^{-1}n'_0 + a'_0 + \text{Ad}_{a_0}k'_0 \\
\text{Ad}_{a_0k_0}^{-1}X &= \text{Ad}_{a_0}^{-1}n'_0 + a'_0 + \text{Ad}_{a_0}k'_0.
\end{align*}
\]

Let $H_\Lambda \in \mathfrak{a}$ be the vector dual to $\Lambda$ under the Killing form. As $\text{Ad}_{n_0}^{-1}n'_0 + \text{Ad}_{a_0}k'_0 \in \mathfrak{n} + \mathfrak{t}$, we have

\[
-\Lambda(a'_0) = -(H_\Lambda, \text{Ad}_{a_0k_0}^{-1}X) = -(H_\Lambda, \text{Ad}_{a_0}^{-1}X) = -(\text{Ad}_{k_0}^{-1}H_\Lambda, e^{\alpha(H)}X_\alpha + e^{-\alpha(H)}X_{-\alpha}).
\]

Let $\pi_\alpha$ be the projection from $\mathfrak{p}$ onto $\mathfrak{p}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$, and let the Iwasawa decomposition of $la$ be $n_la k_l$. We may assume without loss of generality that $X$ is the unit vector in $\mathfrak{h}_\alpha$ which maximises the value of $X \phi_H(l)$, which is equivalent to saying that

\[
-\langle X, X \rangle = (X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) = 1 \quad \text{and} \quad \pi_\alpha(\text{Ad}_{k_l}^{-1}H_\Lambda) = \|\pi_\alpha(\text{Ad}_{k_l}^{-1}H_\Lambda)(X_\alpha - X_{-\alpha})\| \in \mathfrak{p}_\alpha.
\]

Because the $\mathfrak{p}$ component of $e^{\alpha(H)}X_\alpha + e^{-\alpha(H)}X_{-\alpha}$ is $\frac{1}{2}(e^{\alpha(H)} - e^{-\alpha(H)})(X_\alpha - X_{-\alpha})$, we have

\[
-\langle X, X \rangle = -\frac{1}{2}(e^{\alpha(H)} - e^{-\alpha(H)})(\text{Ad}_{k_l}^{-1}H_\Lambda, (X_\alpha - X_{-\alpha})) \quad \text{and} \quad |\Lambda(a'_0)| \geq C|\alpha(H)|\frac{|\langle \pi_\alpha(\text{Ad}_{k_0}^{-1}H_\Lambda), \pi_\alpha(\text{Ad}_{k_l}^{-1}H_\Lambda) \rangle|}{\|\pi_\alpha(\text{Ad}_{k_l}^{-1}H_\Lambda)\|}.
\]

As a result, it suffices to prove the following statement:

There exists a ball $B$ about $l$ and $C > 0$ such that for any $H \in B_a$, there exists a subset $S(H) \subseteq \Delta_l$ spanning $a_l$, such that for all $\alpha \in S(H)$ and $k \in B$ we have

\[
\frac{|\langle \pi_\alpha(\text{Ad}_{k_0}^{-1}H_\Lambda), \pi_\alpha(\text{Ad}_{k_l}^{-1}H_\Lambda) \rangle|}{\|\pi_\alpha(\text{Ad}_{k_l}^{-1}H_\Lambda)\|} \geq C.
\]

Note that if this statement is true with $k_0 = k_l$ for all $H \in B_a$, then by continuity it will hold with $C$ replaced by $C/2$ for $k$ lying in an open neighbourhood $B_H$ of $l$.
which depends continuously on $H$. By the precompactness of $B_\alpha$, we see that we may in fact assume that is $B_H$ independent of $H$ after possibly shrinking $B_\alpha$. It therefore suffices to prove the simpler statement:

There exists $C > 0$ such that for any $H \in B_\alpha$, there exists a subset $S(H) \subseteq \Delta_l$ spanning $a_l$, such that for all $\alpha \in S(H)$ we have

$$\|\pi_\alpha(\text{Ad}_{k_l}^{-1}H_\alpha)\| \geq C.$$  

We shall assume that $k = l$ from now on, and write $la = n_0a_0k_0$. If the above statement fails for all $C > 0$, then by applying the continuity of $\text{Ad}_{k_l}^{-1}H_\alpha$ in $a_l$, the precompactness of $B_\alpha$ and our freedom to shrink this set, we find that there exists $H \in B_\alpha$ and a proper subspace $V \subset a_l^\perp$ such that $\pi_\alpha(\text{Ad}_{k_l}^{-1}H_\alpha) = 0$ for all $\alpha \in \Delta_l \setminus (\Delta_l \cap V)$, and we shall derive a contradiction from this.

Define $Q \subset L$ to be the connected Levi subgroup with Lie algebra

$$q = a + m + \bigoplus_{\alpha \in V} a_{\alpha}.$$  

It follows from the decomposition $la = n_0a_0k_0$ that $l_0a = n_0^{-1}a_0^{-1}w^{-1}k_0$. If we express $l_0a$ in Iwasawa co-ordinates on the group $L$ with respect to the unipotent subgroup $N_L = L \cap w^{-1}N w$ as $n_la_lk_L$, then by comparing the two factorisations we see that $w^{-1}k_0 = k_L \in K_L$. We therefore have $\text{Ad}_{k_0}^{-1}H_\alpha \in p_L$, and because we have assumed that $\text{Ad}_{k_0}^{-1}H_\alpha$ is orthogonal to all root spaces of $I$ which are not contained in $q$, we see that in fact $\text{Ad}_{k_0}^{-1}H_\alpha \in p_Q$. As a result, there exists $k_Q \in K_Q$ such that $\text{Ad}_{k_Q}\text{Ad}_{k_0}^{-1}H_\alpha \in a$, which implies that $k_Qk_0^{-1} \in W$ and $k_0 \in WK_Q$ by the nondegeneracy of $\Lambda$.

Because $k_L = w^{-1}k_0$ and $k_0 \in WK_Q$, we have $k_L = \bar{w}k_Q$ for some $\bar{w} \in W$ and $\bar{k}_Q \in K_Q$, and hence

$$l_0a = n_la_Lk_L = n_la_L\bar{w}k_Q$$

$$\bar{w}^{-1}l_0 = n_L^{-1}a_L^{-1}k_Qa^{-1}.$$  

If we express $a_L^{-1}k_Qa^{-1} \in Q$ in Iwasawa co-ordinates with respect to the unipotent subgroup $Q \cap \bar{w}^{-1}N_L\bar{w}$ as $n_Qa_Qk_Q'$, (50) becomes

$$\bar{w}^{-1}l_0 = n_L^{-1}n_Qa_Qk_Q'.$$  

The uniqueness of the Iwasawa factorisation implies that $\bar{w}^{-1}l_0 = k_Q'$, so that $l \in Wk_Q'$ and

$$\text{Ad}_{k_Q^{-1}}a \cap a = \text{Ad}_{k_Q^{-1}}a \cap a$$

$$\supseteq a + (a_l \cap V^\perp),$$

which contradicts the definition of $a_l$.  

\[\Box\]
To deduce condition (3) of proposition 9 from proposition 15, let $B_H$ be as in proposition 9 and let $B_1$ and $C$ satisfy the conditions of proposition 15 and assume that $B_H \subset B_1$ for all $H \in B_a$. The fact that the set of oscillating roots spans a $\mathcal{A}$ for all $H \in B_a$ implies that there exists a constant $C_2$ depending only on $G$, such that for any $H \in B_a$ and $k \in B_H$ these exists and an $H$-oscillating root $\alpha$ with $\alpha(H) \geq C_2 \| H^I \|$. If we let $\tilde{X}_H$ be the vector field produced by proposition 15 and assume it is normalised to have Killing norm 1, then we have

$$|\tilde{X}_H \phi_H(k)| \geq C_1 C_2 \| H^I \|$$

for all $H \in B_a$ and $k \in B_H$. Letting $\mathbf{X}_H$ be the vector field on $\{ \Xi = 0 \}$ as in (48), we have

$$|\mathbf{X}_H f(Y)| \geq C_1 C_2 \| H^I \|$$

for all $Y$ and $H$, and so it suffices to show that all $Y$-derivatives of $\mathbf{X}_H$ are uniformly bounded. However, condition (1) of proposition 9 implies that all derivatives of $\tilde{X}_H$ in the co-ordinates $(\Xi, Y)$ are bounded, and it follows that the same is true for $\mathbf{X}_H$.

7. The Case of Compact Type I

We now consider the case in which $X$ is a symmetric space of compact type. We assume without loss of generality that $X = S = U/K$, where $U$ is a compact simply connected Lie group and $K$ is connected. As in the noncompact case, most of the work in proving theorem 3 lies in establishing a sharp pointwise bound for the kernel of an approximate spectral projector, and the bound we shall use is exactly that of theorem 3 for the spherical function $\varphi_\mu$ on $S$. We shall prove this bound using the method of the previous sections, after first deriving an expression for $\varphi_\mu$ as a $K$-integral in the noncompact case.

7.1. Notation. Let $G/K$ be the noncompact dual of $S$, with both $G$ and $U$ analytic subgroups of the complex, simply connected group $G_C$ whose Lie algebra is the complexification $\mathfrak{g}_C$ of the Lie algebra $\mathfrak{g}$ of $G$. We write as usual

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k} + \mathfrak{a} + \mathfrak{q}$$

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p} = \mathfrak{k} + i\mathfrak{a} + i\mathfrak{q}.$$  

Let $\Delta$ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and $\Delta^+$ the set of positive roots corresponding to the Weyl chamber $\mathfrak{a}^+$ and the Lie algebra $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Extend $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Define $\mathfrak{a}_C \subset i\mathfrak{a}$ to be the set on which all of the functions $\{ e^{\alpha(H)} - 1 : \alpha \in \Delta \}$ are nonzero, otherwise known as the regular set. Define $A$ and $T$ to be the connected subgroups of $G$ and $U$ with Lie algebras $\mathfrak{a}$ and $i\mathfrak{a}$, and denote the kernel of the exponential mapping from $i\mathfrak{a}$ to $T$ by $L$. We shall identify $T$ with its image in $S$. Let $M$ be the stabiliser of $\mathfrak{a}$ in $K$ with Lie algebra $\mathfrak{m}$, and let $W$ be the Weyl group of $\mathfrak{a}$.

Define $\Lambda$ to be the set

$$\Lambda = \left\{ \mu \in \mathfrak{a}^* : \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \text{ for } \alpha \in \Delta^+ \right\}.$$
Given $\mu \in \Lambda$, let $\pi_\mu$ denote the representation of $G_C$ whose restriction to $G$ is spherical (that is, has a $K$-fixed vector) and has highest restricted weight $\mu$. It is known that every spherical representation of $G$ is of this form, see for instance chapter II, section 4 of [4]. Let $V_\mu$ be the corresponding representation space with $\pi_\mu(U)$-invariant inner product $\langle \cdot, \cdot \rangle$, and let $d(\mu)$ be the dimension of $V_\mu$. Viewing $\mu$ as a linear form on $h_C$, 0 on $h \cap k$, let $\epsilon_\mu \in V_\mu$ belong to the weight $\mu$ and let $v_\mu \in V_\mu$ be a unit vector fixed under $K$. We let $s^*$ be the automorphism of $\Lambda$ such that $\pi_\mu$ and $\pi_{s^*\mu}$ are contragredient, which is given by composing the map $\mu \mapsto -\mu$ with the long element of the Weyl group.

7.2. Approach to Proving Theorem 3. Let $C \subset a_+^*$ be a compact set as in theorem 1, and assume that $\mu$ is equal to $\tilde{\mu}^\ast$ with $\tilde{\mu} \in C$. We shall consider the two functions

$$
\varphi_\mu(g) = \langle \pi_\mu(g^{-1})v_\mu, v_\mu \rangle, \\
b_\mu(g) = \langle \pi_\mu(g^{-1})\epsilon_\mu, v_\mu \rangle, \quad g \in G_C.
$$

The first is the spherical function with spectral parameter $\mu$, and we shall see in section 7.3 that the restriction of the second function to $U$ may be thought of as a higher rank Gaussian beam. If we normalise $\epsilon_\mu$ so that $b_\mu(e) = 1$ we have

$$
\varphi_\mu(u) = \int_K b_\mu(ku)dk, \quad u \in U.
$$

We shall see in section 7.3 that there is a strong formal analogy between this expression for $\varphi_\mu$ and and the standard representation of $\varphi_\lambda$ in the noncompact case, and so one would hope to be able to prove theorem 3 by applying the techniques of the previous sections to this integral. This works well when $u$ is regular, as (51) and basic stationary phase techniques imply that if $D \subset a_r$ is compact then $\varphi_\mu$ has an expansion of the form

$$
\varphi_\mu(\exp(H)) = t^{-(n-r)/2}(1 + O_D(t^{-1})) \sum_{w \in W} A_w(\tilde{\mu}, H)e^{-w\mu(H)}, \quad H \in D,
$$

where the functions $A_w \in C^\infty(C \times a_r)$ are nowhere vanishing. However, the fact that $b_\mu$ is sharply concentrated along a flat subspace (in particular, that its absolute value has large derivatives) makes it difficult to bound $\varphi_\mu(u)$ for singular $u$ using the representation (51). We get around these difficulties by observing that the terms in the sum (52) behave much more like plane waves on $G/K$ than the function $b_\mu$, as their absolute values are not changing rapidly. As a result, we may prove theorem 3 by first averaging $b_\mu$ under the action of a small open neighbourhood of the identity in $K$ to generate a plane wave on some open set in $S$, and then expressing $\varphi_\mu$ as an average of the plane wave about a point in this set.

7.3. The Structure of Gaussian Beams. We begin by proving some results about the functions $b_\mu$ which are needed to carry out this approach, and which we shall derive by following chapter III, section 9 of [4] and applying methods from the theory of pseudodifferential operators. We have as usual
The mapping
\[(X, H, J) \rightarrow \exp X \exp H \exp J \quad (X \in \mathfrak{n}_C, H \in \mathfrak{a}_C, J \in \mathfrak{k}_C)\]
is a holomorphic diffeomorphism of of a neighbourhood of 0 in \(u_C = \mathfrak{g}_C\) onto a neighbourhood \(U^0_C\) of \(e\) in \(G_C\). We can therefore consider the map
\[H : \exp X \exp H \exp J \rightarrow X\]
of \(U^0_C\) into \(\mathfrak{a}_C\), which is an analytic continuation of the earlier map \(H\). As \(b_\mu\) is holomorphic on \(G_C\), \(N\)-invariant on the left and \(K\)-invariant on the right, we have
\[b_\mu(\exp X \exp H \exp J) = b_\mu(\exp H) = e^{-\mu(H)}b_\mu(e),\]
so normalising \(b_\mu\) by \(b_\mu(e) = 1\) we have
\[(53) \quad b_\mu(u) = e^{-\mu(H(u))}, \quad u \in U^0_C.\]
We may extend the function \(H\) to a map \(TU^0_C \rightarrow \mathfrak{a}_C/L\) using the action of \(T\), and equation \((53)\) will continue to hold on this set.

We may use the expression \((53)\) for \(b_\mu\) to prove that its restriction to \(S\) is localised sharply around \(T\), and has Gaussian decay in the transverse directions. We do this by first computing the Hessian of \(H\) transversally to \(T\), and then showing that \(b_\mu\) decays rapidly in sets which are bounded away from \(T\) by an argument involving pseudodifferential operators.

By the \(T\) covariance of \(b_\mu\), it suffices to calculate the Hessian of \(H\) at the identity. If \(\alpha \in \Delta^+\), let \(X_\alpha \in \mathfrak{g}_\alpha\) and \(X_{-\alpha} = \theta X_\alpha \in \mathfrak{g}_{-\alpha}\), and let \(V_\alpha = i(X_\alpha - X_{-\alpha})\) be a normal vector to \(T\) at \(e\). We then have

**Lemma 16.** The second derivative of \(H\) at \(e\) in the direction \(V_\alpha\) is
\[
\left. \frac{d^2}{dt^2} H(\exp(tV_\alpha)) \right|_{t=0} = H_\alpha.
\]

**Proof.** Write the second order approximation to the Iwasawa decomposition of \(\exp(tX_\alpha - tX_{-\alpha})\) in terms of unknowns \(V_1, V_2\) and \(V_3\) as
\[\exp(tX_\alpha - tX_{-\alpha}) = \exp(2tX_\alpha + t^2V_1) \exp(t^2V_2) \exp(-t(X_\alpha + X_{-\alpha}) + tV_3) + O(t^3).\]
After applying the Baker-Campbell-Hausdorff formula to the RHS we have
\[\exp(tX_\alpha - tX_{-\alpha}) = \exp(tX_\alpha - tX_{-\alpha} - t^2[X_\alpha, X_{-\alpha}] + t^2(V_1 + V_2 + V_3)) + O(t^3).\]
Equating coefficients gives
\[V_1 + V_2 + V_3 = [X_\alpha, X_{-\alpha}] \in \mathfrak{a},\]
\[V_2 = \langle X_\alpha, X_{-\alpha} \rangle H_\alpha.\]
Our normalisation of $V_\alpha$ implies that $\langle X_\alpha, X_{-\alpha} \rangle = -1/2$, from which the lemma follows.

It follows from lemma 16 that the second derivative of $-\tilde{\mu}(H(u))$ in the direction $V_\alpha$ is $-\tilde{\mu}(H_\alpha)$. This satisfies the uniform bounds $-C < -\tilde{\mu}(H_\alpha) < -c < 0$ for $\tilde{\mu} \in C$, which implies that $\|b_\mu\|_2 \gg t^{-(n-r)/4}$.

**Lemma 17.** If $D \subset S$ is any compact set which does not intersect $T$, we have

$$|b_\mu(u)| \ll_{D,A} t^{-A}, \quad u \in D.$$

**Proof.** We shall prove this using the theory of pseudodifferential operators, by comparing the action of the Laplacian and the translations $\{i\partial/\partial X : X \in ia\}$ on $b_\mu$. As we have already established that $\|b_\mu\|_2 \gg t^{-(n-r)/4}$, it suffices to prove the lemma after first rescaling $b_\mu$ to have $L^2$ norm one.

Recall the definition of the principal bundle $U \times_K ip^*$ over $S$, which is the quotient of $U \times i_\alpha p^*$ by the action $(u,v)k = (uk, Ad_k^{-1}v)$, $k \in K$.

There is a canonical isomorphism $T^*S \simeq U \times_K ip^*$ under which the symbols of the operators $\Delta$ and $i\partial/\partial X$ are

$$p_\Delta : (u,v) \mapsto -\langle v,v \rangle,$$

$$p_X : (u,v) \mapsto v(Ad_u^{-1}X).$$

The eigenvalues of $b_\mu$ under $\Delta$ and $i\partial/\partial X$ are $\langle \mu, \mu \rangle$ and $-i\mu(X)$, and by general principles of semiclassical analysis this should imply that $b_\mu$ is microlocally concentrated near the set

$$L_\mu = \{q \in T^*S : p_\Delta(q) = \langle \mu, \mu \rangle, p_X(q) = -i\mu(X) \text{ for all } X \in i_\alpha \}.$$

We shall in fact show that

$$L_\mu = \{(tm, -i\mu) : t \in T, m \in M\}.$$

This implies that $L_\mu$ projects to $T \subset S$, and so the lemma would follow by combining this with the claimed microlocal concentration phenomenon. To prove (54), let $(u,v) \in L_\mu$ and assume that we have chosen a representative with $v \in i_\alpha$. The conditions on $L_\mu$ imply that $-\langle v,v \rangle = \langle \mu, \mu \rangle$ and

$$v(Ad_u^{-1}X) = -i\mu(X), \quad X \in i_\alpha$$

$$Ad_u v(X) = -i\mu(X)$$

$$H(Ad_u v) = -i\mu.$$

By comparing norms under the positive definite form $-\langle , \rangle$ on $u$, this implies that $Ad_u v = -i\mu$. We may therefore assume that $v = -i\mu$, and as $\mu$ was generic this implies that $u \in T \cdot M$ as required.

It remains to deduce the conclusion of the lemma from (54) using a pseudodifferential operator argument. Let $B \subset S \setminus T$ be a precompact open set. It follows
from the proceeding discussion that if $X \in i\alpha$ is chosen to be the vector dual to $-i\mu$, then the symbol $P(x, \xi)$ of the operator

$$P_0 = \Delta - (i\partial/\partial X)^2$$

satisfies the estimates

$$C_1(1 + |\xi|)^2 \leq 1 + P(x, \xi) \leq C_2(1 + |\xi|)^2, \quad 0 < C_1, C_2 < \infty, \quad x \in B.$$ 

As a result, if we choose positive $C^\infty$ cutoff functions $b_1$ and $b_2$ satisfying

$$b_1(x) = 1, \quad x \in T$$

$$b_2(x) = 1, \quad x \in B$$

and define the operator $P$ by

$$P = (1 + b_1)\Delta - (i\partial/\partial X)^2,$$

then $P$ is elliptic on $S$. We also have $P(b_2\rho) = \rho$, where $\rho$ is supported in a compact set $B' \subset S \setminus (T \cup \overline{B})$. As $P$ is an elliptic differential operator it has a parametrix $E$ such that $E \cdot P = I + S$ for some smoothing operator $S$, and applying $E \cdot P$ to $b_2\rho$ gives

$$E\rho = b_2\rho + Sb_2\rho.$$ 

As $E$ is also a pseudodifferential operator it is local up to smoothing, which implies that there is a second smoothing operator $S'$ such that

$$b'_{\mu}(x) = S'b_{\mu}(x), \quad x \in B.$$ 

As $\|S'b_{\mu}\|_2 \ll_A t^{-A}$, we see that the $L^2$ norm of $b_{\mu}$ restricted to $B$ is rapidly decaying. The standard methods of bounding $L^\infty$ norms of Laplace eigenfunctions in terms of their $L^2$ norms then imply that $|b_{\mu}(x)| \ll_{D,A} t^{-A}$ uniformly on any compact set $D \subset B$, which concludes the proof.

7.4. Sharpness of Theorem in the Compact Case. We may now prove that theorem is sharp up to the logarithmic factor in the case of compact type.

The spherical function $\varphi_{\mu}$ saturates the $L^p$ bounds for $p$ above the kink point. To see this, first observe that $b_{\mu}$ is roughly constant in a ball of radius $\gg t^{-1}$ about the identity in $S$, and so the expression (51) for $\varphi_{\mu}$ implies that $|\varphi_{\mu}(s)| \gg 1$ in the same ball. Moreover, the Weyl dimension formula implies that $\|\varphi_{\mu}\|_2 \sim t^{-(n-r)/2}$. These two facts imply that $t^{(n-r)/2}\varphi_{\mu}$ has $L^2$ norm $\sim 1$, and has absolute value $\gg t^{(n-r)/2}$ on a set of measure $\gg t^{-n}$, so that $\|t^{(n-r)/2}\varphi_{\mu}\|_p \gg t^{L(n,r,p)}$ as required.

Lemmas 16 and 17 imply that the functions $t^{(n-r)/4}b_{\mu}$ saturate the bounds of theorem for $p$ below the critical point. Indeed, by lemma 17 it suffices to understand the behaviour of $b_{\mu}$ in the open neighbourhood $T \cdot U_{0.2}^\perp \cap U$ of $T$, and lemma 16 implies that $|b_{\mu}|$ is essentially the characteristic function of a ball of radius $t^{-1/2}$ around $T$ in $S$. It easily follows that the $L^p$ norm of $t^{(n-r)/4}b_{\mu}$ is approximately $t^{L(n,r,p)}$. 


8. The Case of Compact Type II

In this section we shall derive theorem 3 from the results of section 7.3, before using theorem 3 to prove theorem 4.

8.1. Construction of Plane Waves. We begin by averaging \( b_\mu \) over rotations by a small neighbourhood of the identity in \( K \) to generate the compact analogue of a plane wave. Let \( B_1, B \subset K \) be two open balls around \( e \) which satisfy \( B_1^2 \subset B \) and \( BB_1 \cap W \subset M \), and such that \( kT \subset T \cdot U^0_C \) for \( k \in BB_1 \) where \( U^0_C \) is as in section 7.3. Let \( g \in C^\infty(K) \) be a positive cutoff function which is supported in \( B \) and equal to 1 on \( B_1 \), and define the function \( \phi^0_\mu \) by

\[
\phi^0_\mu(u) = \int_K g(k) b_\mu(ku) dk, \quad u \in U.
\]

The asymptotic we require for \( \phi^0_\mu \) is as follows:

**Lemma 18.** Let \( D \) be a precompact subset of the regular points of \( T \), and define the set \( S_0 \) by

\[
S_0 = \{ k_1 h : k_1 \in B_1, h \in D \}.
\]

There then exists a function \( a \in C^\infty(\Lambda \times S_0) \), all of whose derivatives in the second factor are uniformly bounded and which satisfies \( |a(\mu, u)| \geq C > 0 \), such that

\[
(55) \quad \phi^0_\mu(u) = t^{-(n-r)/2} a(\mu, u) \mu(h)^{-1}, \quad u = k_1 h \in S_0.
\]

**Proof.** If \( u = k_1 h \in S_0 \), our assumption that \( S_0 \subset T \cdot U^0_C \) implies that

\[
\phi^0_\mu(k_1 h) = \int_K g(k) b_\mu(kk_1 h) dk
\]

\[
= \int_K g(kk_1^{-1}) e^{-t\tilde{\mu}(H(kh))} dk.
\]

To reduce this integral to one which is concentrated at a single point, define \( R = M \setminus K \) as in the noncompact case, and define the functions

\[
g(k_1, r) : B_1 \times R \to \mathbb{R}, \quad \phi_h(r) : T \times R \to \mathbb{C}
\]

by

\[
g(k_1, r) = \int_M g(mkk_1^{-1}) dm, \quad \phi_h(r) = -\tilde{\mu}(H(rh)), \quad r = Mk \in R,
\]

so that

\[
(56) \quad \phi^0_\mu(k_1 h) = \int_R g(k_1, r) e^{-t\phi_h(r)} dr.
\]

\( g(k_1, r) \) is bounded away from zero, and the support of \( g(k_1, r) \) in \( r \) is contained in the image of \( BB_1 \) in \( R \). Our assumptions on \( BB_1 \) then imply that

\[
\text{Re} \phi_h(r) \geq 0, \quad h \in D, \ r \in \text{supp} \ g(k_1, \cdot),
\]
with equality iff \( r = e \). If \( \{ Y_i : 1 \leq i \leq n - r \} \) is an orthonormal basis for \( T_e R \) as in the noncompact case, the Hessian \( D\phi_h \) of \( \phi_h \) at \( e \) may be calculated as in proposition 6.5 of [1] to be the diagonal matrix
\[
(D\phi_h)_{ii} = \tfrac{1}{2} \mu(H_{a_i})(\alpha_i^{-1}(h) - 1).
\]

Write \( g \) as \( g_0 + g_1 \), where \( g_1 \) vanishes to first order at the origin in \( r \) for all \( k_1 \) and \( g_0 \) is equal to \( g(k_1, e) \) on a small ball. Because all \( h \)-derivatives of \( \phi_h(r) - \phi_h(e) \) vanish to second order in \( r \) at the origin, and \( \exp(t\phi_h(r) - t\phi_h(e)) \) has Gaussian decay, we have the asymptotic
\[
\int_R g_1(k_1, r)e^{-t\phi_h(r)} dr = -t^{-(n-r)/2-1} a_1(\mu, k_1, h)e^{-t\phi_h(e)}
\]
\[
= -t^{-(n-r)/2-1} a_1(\mu, k_1, h)\mu(h)^{-1},
\]
where all \( k_1 \) and \( h \) derivatives of \( a_1 \) are again uniformly bounded. We may therefore assume that \( g = g_0 \) is constant in \( r \) in a small ball about \( e \). The asymptotic (55) then follows by complexifying \( R \) and deforming the contour of integration in (56) in a small ball about \( e \), so that in geodesic normal co-ordinates about \( e \) there is a smaller ball in which it is a plane on which \( D\phi_h \) is real and negative definite.

\[\square\]

**Remark.** The asymptotic (52) for \( \varphi_{\mu} \) on the regular set follows from applying the analysis we have used here, and the rapid decay of \( \beta_{\mu} \) away from \( T \), to the integral (51).

### 8.2. Proof of Theorem 3

The regular set \( a_r \) is a union of open simplices, and we choose one such simplex \( C_0 \) whose closure contains the origin. Define \( T_0 \) to be \( \exp(C_0) \), and \( \overline{C} \) and \( \overline{T} \) to be the closures of \( C_0 \) and \( T_0 \). \( \overline{T} \) is then a fundamental domain for the action of \( W \) on \( T \). The Cartan decomposition on \( U \) implies that every \( u \in U \) is equal to \( k_1 \exp(a)k_2 \) for a unique \( a \in \overline{C} \), and we define the function \( A : U \to \overline{C} \) by \( A : u \mapsto a \). If we assume that \( h \in D \cap T_0 \) in formula (55), we may rewrite it with the aid of this notation as
\[
\varphi_{\mu}^0(u) = t^{-(n-r)/2} a(\mu, u)e^{-\mu(A(u))}, \quad u \in S_0K.
\]
Note that \( \mu(A(u)) \) is purely imaginary. There is a clear similarity between \( \varphi_{\mu}^0 \) and the plane waves on \( G/K \), and this will allow us to bound \( \varphi_{\mu} \) by choosing \( h_1 \in T_0 \), expressing \( \varphi_{\mu} \) as an average of \( \varphi_{\mu}^0 \) under rotation about \( h_1 \), and applying the techniques of the previous section to the resulting integral on \( K \).

If we fix \( h_1 \in T_0 \) and normalise \( \varphi_{\mu}^0 \) by \( \varphi_{\mu}^0(h_1) = 1 \), we have the expression
\[
\varphi_{\mu}^0(h) = \int_K \varphi_{\mu}^0(h_1kh)dk, \quad h \in T
\]
\[
= \int_K a(\mu, h_1kh)e^{-t\mu(A(h_1kh))}dk,
\]
where \( a(\mu, h_1) = 1 \). We shall denote the phase function \(-\mu(A(h_1kh))\) by \( \phi_h(k) \) as before. The following proposition shows that \( \phi_h(k) \) has almost all of the properties which are needed to study the integral (57) using the techniques developed in the noncompact case.
Proposition 19. There is a ball $B$ about the origin in $i\mathfrak{a}$ such that if $h \in B$, the function $\phi_h(k)$ is left invariant under $M$ and right invariant under $K_h$, and its critical point set is equal to

$$C_h = WK_h.$$ 

If $w \in W$ and $\{X_i : 1 \leq i \leq n - r\}$ is the basis of $T_w R$ defined in section 2.1 there are nonvanishing analytic functions $F_{w, i} : B \to \mathbb{R}$, $1 \leq i \leq n - r$, such that the Hessian of $\phi_h$ at $w$ with respect to the basis $\{X_i\}$ is the diagonal matrix

$$(D\phi_h)_{ii} = F_{w, i}(h)(\alpha_i(h) - \alpha_i(h)^{-1}), \quad h \in B.$$ 

Proof. The claims about the invariance properties of $\phi_h$ are immediate. To determine the critical point set, we shall first assume that $k$ is a critical point of $\phi_h$ and show that $k = wk_h$ for some $w \in W$ and $k_h \in K_h$. Consider a vector $X \in \mathfrak{k}$ and write the Cartan decomposition of $h_1k \exp(tX)h$ as

$$h_1k \exp(tX)h = k_1(t)a(t)k_2(t).$$

We assume that $B$ is sufficiently small that $h_1k \exp(tX)h$ is regular for $h \in B$ and $t$ small, so that $k_1(t)$ and $k_2(t)$ are unique and depend smoothly on $t$. Define $k'_i(t) \in \mathfrak{g}$ to be the derivative of $k^{-1}_i(t)k_i(t + s)$ at $s = 0$, and likewise for $a'$ and $k'_2$, so that $X\phi_h(k) = -\tilde{\mu}(a')$. If we define $k_1 = k_1(0)$, $k'_1 = k'_1(0)$ etc, then differentiating (58) at $t = 0$ gives

$$\Ad^{-1}_h X = \Ad^{-1}_{k_2} k'_1 + \Ad^{-1}_{k_2} a' + k'_2$$

$$\Ad_{k_2} \Ad^{-1}_h X = \Ad^{-1}_{k_2} k'_1 + a' + \Ad_{k_2} k'_2.$$ 

As $\Ad^{-1}_{k_2} k'_1$ and $\Ad_{k_2} k'_2$ both lie in $\mathfrak{k} + i\mathfrak{q}$, we see that $a'$ is equal to the projection of $\Ad_{k_2} \Ad^{-1}_h X$ to $i\mathfrak{a}$. Our assumption that $k$ is a critical point of $\phi_h$ then implies that

$$\tilde{\mu}(\Ad_{k_2} \Ad^{-1}_h X) = 0, \quad X \in \mathfrak{k}.$$ 

As we have assumed that $h$ is contained in a small ball, the projection of $\Ad^{-1}_h \mathfrak{k}$ to $i\mathfrak{p}$ is equal to

$$i\mathfrak{p} \cap \bigoplus_{\alpha(h) \neq 0} \mathfrak{g}_{\alpha, \mathbb{C}}.$$ 

Let $H_{\tilde{\mu}} \in \mathfrak{a}$ be the vector dual to $\tilde{\mu}$. For $k$ to be critical, $\Ad_{k_2}^{-1} H_{\tilde{\mu}}$ must be orthogonal to (59) which implies that

$$\Ad_{k_2}^{-1} H_{\tilde{\mu}} \in \mathfrak{l} = \mathfrak{a} + \mathfrak{m} + \left(\mathfrak{g} \cap \bigoplus_{\alpha(h) = 0} \mathfrak{g}_{\alpha, \mathbb{C}}\right) \subseteq \mathfrak{g}.$$ 

Let $L$ be the Levi subgroup of $G$ with Lie algebra $\mathfrak{l}$, let $K_L = K_h \subset K$ be its maximal compact subgroup, and write $\mathfrak{l} = \mathfrak{p}_L + \mathfrak{t}_L$. The inclusion $\Ad_{k_2}^{-1} H_{\tilde{\mu}} \in \mathfrak{p}_L$
implies that there is an element $k_L \in K_L$ such that $\text{Ad}_{k_L} \text{Ad}_L^{-1} H_p \in a$, and so $k_2 = w k_L$ for some $w \in W$. Substituting this into (58) at $t = 0$ gives

$$
\begin{align*}
h_1 k &= k_1 a k_2 h^{-1} \\
h_1 k &= k_1 a (h^{-1})^w w k_L.
\end{align*}
$$

Because $h$ is small we know that both $h_1$ and $a(h^{-1})^w$ lie in $T_0$, so that uniqueness of the Cartan decomposition gives $k = w k_L$ as required. This shows that the critical point set of $\phi_h$ is contained in $C_h$, and the reverse inclusion follows from the right $K_h$-invariance of $\phi_h$ and the easily observed fact that $\phi_h$ is critical on $W$.

We shall prove the final claim of the proposition in the case $w = e$ by calculating the Hessian of $\phi_h$ at the identity in $K$; the case of the other Weyl points is identical. In the notation of (58), we wish to determine $a'' = a''(0)$ when $k = e$ and $X$ is of the form $X_\alpha + X_{-e_\alpha}$, where $X_{\pm \alpha} \in g_{\pm \alpha}$ are normalised so that $(X,X) = -1$. Note that the assumption $k = e$ implies that $k_1 = k_2 = e$ as well. Differentiating (58) at $t = 0$ gives

$$
\text{Ad}_h^{-1}(X_\alpha + X_{-e_\alpha}) = \text{Ad}_a^{-1} k_1' + k_2'$$

and this has the unique solution

$$
(61) \quad k_1' = \frac{\alpha(h) - \alpha(h^{-1})}{\alpha(a) - \alpha(a)^{-1}} (X_\alpha + X_{-e_\alpha}), \quad k_2' = \frac{\alpha(ah^{-1}) - \alpha(ha^{-1})}{\alpha(a) - \alpha(a)^{-1}} (X_\alpha + X_{-e_\alpha}).
$$

If we consider the second order approximation to (58) by writing

$$
k_1(t) = \exp(k_1't + k_1''t^2 + O(t^3)),
$$

and likewise for $a(t)$ and $k_2(t)$, we then have

$$
\begin{align*}
h_1 \exp(tX)h &= \exp(k_1't + k_1''t^2) a \exp(a''t^2) \exp(k_2't + k_2''t^2) + O(t^3) \\
a \exp(t\text{Ad}_h^{-1}X) &= a \exp(\text{Ad}_a^{-1} k_1't + \text{Ad}_a^{-1} k_1''t^2) \exp(a''t^2) \exp(k_2't + k_2''t^2).
\end{align*}
$$

Applying the Baker-Campbell-Hausdorff formula and equating second order terms gives

$$
\frac{1}{2} \left[ \text{Ad}_a^{-1} k_1', k_2' \right] + a'' = 0,
$$

and after substituting the values of $k_1'$ and $k_2'$ from (61) this becomes

$$
a'' = H_\alpha \frac{\alpha(ah^{-1}) - \alpha(ha^{-1})}{4(\alpha(a) - \alpha(a)^{-1})} (\alpha(h) - \alpha(h^{-1})).
$$

Because $ah^{-1} = h_1$, it follows that

$$
\frac{d^2}{dt^2} \phi_h(\exp(tX)) = -\mu(H_\alpha) \frac{\alpha(h_1) - \alpha(h_1)^{-1}}{2(\alpha(a) - \alpha(a)^{-1})} (\alpha(h) - \alpha(h^{-1})).
$$
If we choose $B$ so that $\alpha(a) = \alpha(h_1h)$ is not equal to $\pm 1$ for $h \in B$, then this expression is of the form $F(h)(\alpha(h) - \alpha(h)^{-1})$ for some nonvanishing real analytic function $F$, which completes the proof.

We may use proposition 15 to prove a structure theorem for $\phi_h$ which is equivalent to proposition 9 and hence implies theorem 3. This may be established in exactly the same way as in the noncompact case, with the exception of the argument used to prove the analogue of condition (3) of proposition 5 concerning the vanishing of critical points. Adapting this argument requires small modifications to the proof of proposition 15 which we now describe.

Let $l = w_l \in K$, $a_l$ and $a^\circ$ be as in the noncompact case, and let $H_\mu \in \mathfrak{a}$ be the dual vector to $\mu$. Let $L \subset U$ be the compact form of the Levi subgroup associated to $l$ in the noncompact case, and decompose its Lie algebra as $l = \mathfrak{ip}_L + \mathfrak{t}_L$. By differentiating the Cartan decomposition of $h_3 k \exp(tX) h$ and reasoning as in the proof of proposition 15, we are reduced to deriving a contradiction from the following statement:

There exists $h \in B$ such that if we write $h_1 lh$ as $k_1 ak_2$, there is a proper subspace $V \subset \mathfrak{a}^\circ_1$ such that $\pi_\alpha(\text{Ad}_{k_2}^\mu H_\mu) = 0$ for all $\alpha \in \Delta_l \setminus (\Delta_l \cap V)$.

To proceed from here, we follow the proof of proposition 15 with Cartan decompositions in place of Iwasawa decompositions. We again define $Q \subset L$ to be the connected subgroup whose Lie algebra $\mathfrak{q}$ has the complexification

$$\mathfrak{q}_C = \mathfrak{a}_C + \mathfrak{m}_C + \bigoplus_{\alpha \in \mathfrak{V}} \mathfrak{g}_{\alpha, C}.$$

The Cartan decomposition $h_1 lh = k_1 ak_2$ implies that $h_1^{w_l^{-1}} l_0 h = w_l^{-1} k_1 ak_2$. If we let the Cartan decomposition of $h_1^{w_l^{-1}} l_0 h \in L$ with respect to $L$ be $k_1, L a_L k_{2, L}$, then by comparing the two factorisations and using the genericity of $a$ we see that $k_2 \in W k_{2, L}$. This implies that $\text{Ad}_{k_2}^{-1} H_\mu \in \mathfrak{p}_L$, and our assumption on the projections of $\text{Ad}_{k_2}^{-1} H_\mu$ to $\mathfrak{g}_a$ allows us to strengthen this to $\text{Ad}_{k_2}^{-1} H_\mu \in \mathfrak{p}_Q$. It follows that there exists $k_Q \in K_Q$ such that $\text{Ad}_{k_Q} \text{Ad}_{k_2}^{-1} H_\mu \in \mathfrak{a}$, and because $k_2 = W k_{2, L}$ we see that $k_{2, L} = \bar{w} k_Q$ for some $\bar{w} \in W$. We then have

$$\begin{align*}
h_1^{w_l^{-1}} l_0 h & = k_1, L a_L \bar{w} k_Q \\
(w_l^{-1} k_{2, L}^{-1} w_l^{-1}) h_1 (w_l 0) & = a_L^{w_l^{-1}} k_Q h_l^{-1} \in Q.
\end{align*}$$

As the RHS of this equation is the Cartan decomposition of an element of $Q$ and $h_1$ is generic, we must have $l_0 \in WK_Q$ which contradicts the definition of $a_l$.

8.3. Conclusion. It remains to deduce theorem 1 from what we have proven. Let $b \in C^\infty(U)$ be a $K$-biinvariant cutoff function supported near the identity, and consider the function $b \varphi_{*\mu}$. As in the noncompact case, we may use our expression of $\varphi_{*\mu}$ as an integral of plane waves to show that the spherical transform of $b \varphi_{*\mu}$ is rapidly decaying away from $\mu$, in particular that for any $\delta > 0$ we have

$$\widehat{b \varphi_{*\mu}}(\nu) \ll_{\delta, A} ||\mu - \nu||^{-A}$$
if \( \mu \) and \( \nu \) satisfy \( \| \mu - \nu \| \geq \delta \| \mu \| \). In addition, the asymptotic (12) implies that the proportion of the \( L^2 \) mass of \( \varphi_{\mu}^* \) in any small ball around the origin is bounded from below as \( t \to \infty \). This implies that \( t^{n-r} b_{\varphi_{\mu}^*} (\mu) \gg 1 \), and so \( t^{n-r} b_{\varphi_{\mu}^*} \) is the kernel of an approximate spectral projector onto the spectral parameter \( \mu \). Define the function \( K_\mu \) to be

\[
K_\mu = t^{2n-2r} b_{\varphi_{\mu}^*} * b_{\varphi_{\mu}^*}.
\]

\( K_\mu \) is the integral kernel of an approximate spectral projector of adjoint-square form, and we shall prove theorem 1 by analyzing it as in the noncompact case. If we choose \( \delta > 0 \) and let \( B_\mu \subset \mathfrak{a}^* \) be the ball of radius \( \delta \| \mu \| \) about \( \mu \), then by inverting the spherical transform we have

\[
K_\mu = t^{2n-2r} \sum_{\nu \in \Lambda} d(\nu) |b_{\varphi_{\mu}^*}(\nu)|^2 \varphi_{\nu}^* + O_A(t^{-A}).
\]

If we use this to evaluate \( K_\mu(e) \), we have

\[
\sum_{\nu \in \Lambda} d(\nu) |b_{\varphi_{\mu}^*}(\nu)|^2 = \| b_{\varphi_{\mu}^*} \|^2_2
\]

\[
\ll \| \varphi_{\mu}^* \|^2_2
\]

\[
\ll t^{-(n-r)},
\]

and this implies the following pointwise upper bound for \( K_\mu \):

\[
|K_\mu(u)| \ll t^{n-r} \sup_{\nu \in \Lambda} |\varphi_{\nu}^*(u)| + O_A(t^{-A}).
\]

The required pointwise bounds for \( K_\mu \) follow by combining this with theorem 3. If \( \beta_m \) are \( K \)-biinvariant cutoff functions which are analogous to those used in the noncompact case, it remains to prove \( L^2 \to L^2 \) bounds for the truncated kernels \( \beta_m K_\mu \). We do this by first showing that the spherical transforms of \( \beta_m \hat{\varphi}_\nu \) are localised near \( \nu \) as before, which implies the same localisation for the transform of \( \beta_m K_\mu \). The result then follows from the pointwise bounds already established.

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