Dirac Branes, Characteristic Currents and Anomaly Cancellations in 5-Branes

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The aim of this note is to discuss, in a somewhat informal language, the cancellation of anomalies (in topologically trivial space-time) for 5-branes using as "building blocks": i) a generalization to $p$-branes of the Dirac strings of monopoles (Dirac branes) and a refinement of this idea involving a geometric regularization of Dirac branes, leading to the formalism of "characteristic currents" ii) the PST formalism. As an example of the potentiality of the developed framework we discuss in some detail the anomaly cancellation in the D=10 effective theory of heterotic string and 5-brane coupled to supergravity, where the anomaly inflow is automatically generated. Some remarks are also made on a similar approach to the problem of anomaly cancellation in the effective theory of M5-brane coupled to D=11 supergravity, developed in collaboration with M.Tonin, where however still as open problem remains a Dirac anomaly.

1. Currents

P-branes, Dirac strings and their generalizations found a natural mathematical formulation in terms of (de Rham) currents: a $p$-current in a $d$-dimensional space $M$ is a "$p$-form whose coefficients are distributions" or, more precisely, a linear functional on the space of smooth $(d-p)$-forms with compact support, continuous in the sense of distributions [1]. As an example let us consider $M = \mathbb{R}^4, \gamma$ a curve, which one might identify as the worldline of a charged particle, parametrized in terms of the proper time $\tau$. There is a 3-current naturally associated to $\gamma$:

$$j(x) = j_{\mu\nu\rho}(x)dx^\mu \wedge dx^\nu \wedge dx^\rho \quad (1)$$

$$j_{\mu\nu\rho}(x) = \epsilon_{\mu\nu\rho\sigma} \int_{\tau} d\tau \delta^{(4)}(x - x(\tau))d\tau.$$  

This example also justifies the term "current" because, with the physical identification alluded above, $j$ is really the (Hodge dual of the) electromagnetic current generated by a charged particle of unit charge. Obviously, in particular, all smooth $p$-forms are $p$-currents.

Moving from to $p$-forms to $p$-currents we gain a very useful extension of Poincaré duality. We recall that Poincaré duality for forms [2] is a map from the homology class of a $k$-cycle ("closed $k$-surface") $c$ into the cohomology class of a $d-k$ closed form $\alpha_c$ such that for any closed $k$-form $\beta$ of compact support

$$\int_c \beta = \int_M \alpha_c \wedge \beta. \quad (2)$$

In the space of currents there is a natural map from $k$-chains ("$k$-surfaces") $\Sigma_k$ to $d-k$ currents, $PD(\Sigma_k)$, generalizing Poincaré duality in the sense that

$$\int_{\Sigma_k} \beta = \int_M PD(\Sigma_k) \wedge \beta, \quad (3)$$

for every smooth $k$-form $\beta$ of compact support.

It is clear that, loosely speaking, the coefficients of $PD(\Sigma_k)$ are $\delta$-functions with support on $\Sigma_k$ as in the previous example where $j = PD(\gamma)$. In particular, the image by $PD$ of a closed surface is a closed current and

$$PD(\partial \Sigma_k) = dPD(\Sigma_k), \quad (4)$$

where $\partial$ denotes the boundary operator. $k$-currents which are linear combinations with in-
integer coefficients of $PD$ of $k$-chains are called integral currents. If well defined, an integral like

$$\int PD(\Sigma_k) \wedge PD(\Sigma_{d-k})$$

is an integer, the intersection number of $\Sigma_k$ and $\Sigma_{d-k}$.

2. Dirac branes

Let us show how one uses currents to describe Dirac strings or their generalizations. Suppose one has an equation of motion for an invariant curvature $H$, e.g. the field strength of a monopole,

$$dH = j,$$

where $j = PD(\Sigma)$ for some worldvolume $\Sigma$ of a closed brane. A generalization of Poincaré lemma then proves that there exists a surface $S$ such that $\Sigma = \partial S$, so that by

$$j = dC, \quad C = PD(S).$$

The surface $S$ is defined modulo a boundary, i.e. we can replace $S$ by $S' = S + \partial V$, then accordingly we replace $C$ by

$$C' = PD(S') = C + PD(\partial V) = C + dPD(V).$$

It is clear that if $H$ is the field strength of a monopole and $j$ the generated magnetic current in $d = 3 + 1$, then $C$ may be identified as the worldvolume of its Dirac string. Hence in general if $j$ is a $(d-p)$-current describing the worldvolume of a $(p-1)$-brane, we say that the corresponding $(d-p-1)$-current $C$ describes the worldvolume of a $p$-Dirac brane. In terms of this $p$-Dirac brane, one can solve (3), in a topologically trivial spacetime, by

$$H = dA + C,$$

where $A$ is a $d - p - 2$ gauge form, and must be interpreted in a distributional sense, i.e. as a $d - p - 2$-current. The "curvature" (2) is invariant under the change of Dirac brane:

$$A \to A - PD(V), \quad C \to C + dPD(V),$$

where $V$ is a $p + 2$ surface.

3. Characteristic currents

Actually we need a refinement of the notion of current because in supergravity (SUGRA) one has to consider what naively can be interpreted as the restriction of an integer current to its support, which is clearly mathematically ill defined. As an example consider, in $d = 4$, a surface current $j = PD(\Sigma)$ where $\Sigma$ is a 2-surface parametrized by $\tau \equiv \{\tau_1, \tau_2\}$, which admits a local representation analogous to (4):

$$j(x) = j_{\mu\nu}(x)dx^\mu \wedge dx^\nu$$

and identifying the bundle $N(X)$ with $N(X)$.

By abuse of language we still refer to such $j^\epsilon$ as a characteristic current. One can easily verify that in a flat space-time a characteristic current is obtained by replacing the $\delta$-function appearing in the definition of $j = PD(X)$ by a gaussian:

$$\delta(x - x(\tau)) \to e^{-\frac{(x-x(\tau))^2}{(\pi\epsilon)^{d/2}}}$$

and identifying the bundle $N$ with $N(X)$. For characteristic currents we have

$$j^\epsilon \wedge j^\epsilon \to_{\epsilon \to 0} \chi_k(N) \wedge PD(X).$$
We remark that by means of the regularization one can extend the notion of characteristic current also to non-closed submanifolds and via covariantization to non flat spaces-time. Furthermore since this regularization corresponds to a convolution, it commutes with the exterior differential $d$.

4. PST formalism

The last ingredient we need is the Pasti-Sorokin-Tonin (PST) formalism \cite{6} to deal with duality conditions, e.g. for invariant curvatures of the rank 2 and 6 Ramond-Ramond fields in type IIA SUGRA 10:

$$H_3 = *H_7,$$

or self-duality conditions, e.g. for the invariant curvature of the 2-form on the worldvolume of the M 5-brane in SUGRA 11:

$$h_3 = *h_3.$$ (17)

The PST approach allows to write, in a topologically trivial space time, an action which is local, although non-polynomial, diffeomorphism invariant and which permits to derive duality or self-duality conditions as equations of motion. Let us sketch the basic idea, for details see \cite{6}. (To simplify the notation from now on the wedge product symbol $\wedge$ is understood and some sign depending on the dimension $d$ and the rank of the form is omitted, see \cite{6}).

One introduces a 0-form $a$ and defines the 1-form

$$v(x) = \frac{da(x)}{|da||(x)},$$ (18)

where $|da|^2 = *(da * da)$. Let us assume that we are dealing with a model involving a pair of forms with invariant curvatures of rank $k$ and $d-k$, denoted by $F^1$ and $F^2$, satisfying a duality condition

$$F^1 = *F^2, \quad F^\alpha = dA^\alpha + C^\alpha,$$ (19)

and whose dynamics is determined by a "Maxwell" like action. Defining

$$f^\alpha = i_v(F^\alpha - *F^\beta \epsilon^{\alpha\beta}),$$ (20)

where $i_v$ denotes the contraction with the vector field corresponding to $v$, one can verify that the action

$$S_{PST}(A^\alpha, C^\alpha) = \frac{1}{4} \int (F^\alpha * F^\alpha + f^\alpha * f^\alpha) + \frac{1}{2} \int (C^1 dA^2 - dA^1 C^2)$$ (21)

gives the duality condition as equation of motion, besides the "Maxwell" equations. The analogous action for the self-dual case $F = *F$ is obtained identifying $A^1 = A^2, C^1 = C^2$ and it is given by

$$S_{PST} = \frac{1}{2} \int (F * F + f * f) + \int C dA.$$ (22)

The basic reason why the PST formalism works correctly is the existence of a (PST)-symmetry

$$\delta A^\alpha = -\frac{\varphi}{|da|} f^\alpha, \quad \delta a = \varphi,$$ (23)

allowing to choose the 0-form $a$ arbitrarily, so that it does not propagate unphysical degrees of freedom, in addition to a symmetry

$$\delta A^\alpha = \Phi^\alpha da$$ (24)

allowing to reduce the second order equations of motions for the gauge fields to the first order duality equation. $\varphi$ and $\Phi^\alpha$ are transformation parameters. Together with the invariance under Dirac brane changes, the PST symmetries fix the action completely in the self-dual case, leaving the freedom to add a term

$$\frac{1}{2} \int C^1 C^2$$ (25)
in the dual-case.

Furthermore if the invariant curvatures have the more general structure

$$F^\alpha = dA^\alpha + C^\alpha + L^\alpha,$$ (26)

where $L^\alpha$ are fields not transforming under Dirac brane changes, then the above symmetries identify the invariant action as

$$S_{PST}(A^\alpha, C^\alpha + L^\alpha) + \frac{1}{2} \int (C^1 L^2 - L^1 C^2).$$ (27)

Finally let us make a remark on the nature of the coupling $C$-$A$ in PST.
To get an intuitive idea consider the case where \( C_2 = j_2 = 0 \) then, a "magnetic coupling" \( C \cdot A \) in the action would have the structure

\[
\frac{1}{2} \int F \star F = \frac{1}{2} \int (dA + C) \star (dA + C)
\]

(28)

whereas on "electric coupling" would be

\[
\int dAC = \int AdC = \int Aj.
\]

(29)

In the PST approach instead we have

\[
\frac{1}{4} \int F \star F + \frac{1}{2} \int dAC
\]

(30)

hence it is a kind of "dyonic" \( C \cdot A \) coupling with "\( \frac{1}{2} \) magnetic coupling" and "\( \frac{1}{2} \) electric coupling".

5. Dirac branes approach to \( p \)-brane systems

To every (elementary) closed \( p \)-brane we associate a \( d - (p + 1) \) current \( j_{d-p+1} \) describing the worldvolume \( \Sigma_{p+1} \) of the brane, via \( PD: j_{d-p+1} = PD(\Sigma_{p+1}) \). Since \( j \equiv j_{d-p+1} \) is closed we can associated a \((p+1)\)-Dirac brane via:

\[
\Sigma_{p+1} = \partial \Sigma_{p+2}, \quad PD(\Sigma_{p+2}) = C_{d-(p+2)} \equiv C
\]

(31)

so that \( j = dC \).

The \( p \)-brane interacts with a gauge \((p+1)\)-form either electrically, and then no Dirac brane is needed, or magnetically, and then in our formalism we have an associated Dirac brane. In particular if a \( p \)-brane is electrically coupled, its dual \((d-(p+1))\)-brane is magnetically coupled. In the PST formalism, due to the "dyonic coupling" Dirac branes are always needed.

One introduces a manifestly Lorentz-invariant action describing the interaction between gauge forms, \( p \)-branes and \((p+1)\)-Dirac branes via the PST formalism (or the related Schwinger \([8]\) or Zwanziger \([8]\) formalisms, see \([8]\)). This procedure requires as consistency condition the indepdendence of the choice of the Dirac branes. This put strong restrictions, such as Dirac quantization conditions, which in turn might generate new physics like spin-statistics transmutation for dyons in \( d = 4 \) \([8]\). This transmutation has been discussed at quantum field theory level in \([10]\), combining a formalism involving Mandelstam strings, developed in \([11]\), with Schwinger or PST techniques).

6. Anomaly cancellation in 5-branes

Let us apply the formalism sketched above to analyze the anomaly cancellations in 5-branes.

We discuss in particular the system SUGRA 10+ heterotic 5-brane + heterotic string \([12]\). The gauge \( p \)-forms of SUGRA are denoted by \( B_2 \) and \( B_6 \), the current associated to the 5-brane by \( J_4 = PD(\Sigma_6) \) and its Dirac brane by \( C_3 \) and the current associated to the heterotic string by \( J_8 = PD(\Sigma_2) \) and its Dirac brane by \( C_7 \). We denote the \( SO(1,9) \) curvature by \( R \), the \( SO(32) \otimes SU(2) \) curvature by \( F \otimes G \) and the curvature of the normal \( SO(4) \) bundle \( N \) of the 5-brane by \( T \). In terms of these curvatures one defines

\[
X_8 = \frac{1}{192(2\pi)^2}(tr R^4 + \frac{1}{4}(tr R^2)^2 - tr R^2 tr F^2 + 8 tr F^4)
\]

(32)

\[
X_4 = \frac{1}{48(2\pi)^2}(tr R^2 - tr F^2)
\]

(33)

\[
Y_4 = \frac{1}{48(2\pi)^2}(tr R^2 - 2 tr T^2 - 24 tr G^2)
\]

(34)

\[
\chi_4 = \frac{1}{8(2\pi)^2} \epsilon_{a_1 a_2 a_3 a_4} T^{a_1 a_2} T^{a_3 a_4}
\]

(35)

where \( \chi_4 \) is the Euler form of \( N \). The invariant polynomials are then given by \( X_8 X_4 \) (SUGRA), \( X_8 + (X_4 + \chi_4)Y_4 \) (5-brane), \( X_4 \) (string), a factor \( 2\pi \) being omitted. With the standard convention, e.g.

\[
X_8 = dX_7, \quad \delta X_7 = dX_6^1
\]

(36)

where \( \delta \) denotes the gauge variation, the anomaly polynomials obtained via transgression are given by \( X_8^2 X_4 \) (SUGRA), \( X_6^1 + (X_4^1 + \chi_4^1)Y_4^1 \) (5-brane) \( X_2^1 J_6 \) (string).

The cancellation of anomalies is due to Green-Schwarz \([13]\) plus inflow \([14]\) mechanisms. In the standard treatment the cancelling terms in the lagrangian are \( B_2 X_8 \) (SUGRA), \( (B_6 + B_2 Y_4) J_4 \) (5-brane), \( B_2 J_8 \) (string) with the gauge transformations:

\[
\delta B_2 = -X_2^1, \quad \delta B_6 = X_6^1.
\]

(37)
but with an additional contribution on the 5-brane given by
\[
(\delta B_2)_{\Sigma_6} = -\chi^4 \delta \chi_4 
\]  
(38)
to cancel \(\chi^4 \delta \chi_4\); this last transformation corresponds to the inflow.

However, a clear-cut definition strictly localized on the 5-brane appears somewhat cumbersome, in fact if the additional transformation \(\chi^4 \delta \chi_4\) is added with a characteristic function of \(\Sigma_6\), then from the point of view of distribution theory it is irrelevant, because \(\Sigma_6\) is of measure zero in \(\mathbb{R}^{10}\) and the characteristic function is bounded, on the other hand if it is added with a \(\delta\)-function on \(\Sigma_6\), then it modifies the equations of motion. A solution is obtained via a smoothing operation, mathematically using Thom forms \([1]\), but then this contribution is no more localized strictly on the 5-brane.

The Dirac brane approach automatically solves this problem \([3]\). In fact, we start from the equations for the invariant curvatures
\[
dH_4 = X_4 + 4_1 
\]
(39)
\[
dH_7 = X_8 + Y_4 J_4 + J_8 
\]
(40)
\[
H_3 = *H_7 
\]
(41)
and we write \(X_4 = dX_3\), \(X_8 = dX_7\), \(Y_4 = dY_3\), where \(X_3\), \(X_7\), \(Y_3\) are the corresponding Chern-Simons forms, and
\[
h_2 = i_\nu (H_3 - *H_7), \quad h_6 = i_\nu (H_7 - *H_3). 
\]
(42)
According to the rules outlined in previous section we pose
\[
H_3 = dB_2 + X_3 + C_3 
\]
(43)
\[
H_7 = dB_6 + X_7 + Y_3 J_4 + C_7, 
\]
where \(X_3\) and \(X_7 + Y_3 J_4\) play the role of \(\mathcal{L}_o\) in the equation \([26]\). In quadratic approximation for \(h_2\) and \(h_6\), but a Born-Infeld action can be easily arranged \([13]\), the relevant part of the lagrangian of the coupled system is given by:
\[
\frac{1}{2} (H_3 * H_3 + H_7 * H_7 + h_2 * h_2 + h_6 * h_6) 
\]
\[
+ \frac{1}{3} (X_7 + Y_3 J_4 + C_7) dB_2 + \frac{1}{2} (X_3 + C_3) dB_6 
\]
\[
+ \frac{1}{2} (X_7 + Y_3 J_4) C_3 + \frac{1}{2} X_3 C_7 + \frac{1}{2} C_3 C_7. 
\]
(44)

The gauge transformations are given by
\[
\delta B_2 = -X_2^4, \quad \delta B_6 = -Y_6^4 - Y_2^4 J_4, 
\]
(45)
completely free of ambiguities. The corresponding contribution to the anomaly is given by,
\[
- \int [\frac{1}{2} (X_2^4 X_8 + X_6^4 X_4) + X_2^4 J_8 
\]
\[
+ (X_6^4 + \frac{1}{2} (X_2^4 Y_4 + Y_2^4 X_4) + Y_2^4 J_4) J_4 
\]
(46)
which is equal, up to trivial cocycles, to
\[
- \int [X_6^4 X_4 + X_2^4 J_8 + (X_6^4 + X_2^4 Y_4 + Y_2^4 J_4) J_4]. 
\]
(47)
The last term in \(\delta B_2\) is naively ill-defined, but using the theory of characteristic currents developed above (i.e. replacing \(J_4 J_4\) by \(\lim_{r \to 0} J_4 J_4\)) gives the desired term \(Y_2^4 \chi_4 J_4 = \chi_4^3 J_4\) + trivial cocycle. Hence we obtain the cancellation of anomalies with the inflow contribution strictly localized on the 5-brane.

7. Some remarks on M5

An action for the coupled system SUGRA 11 + M5-brane was first proposed in \([1]\), but it fails to exhibit Dirac brane independence and does not consider the anomaly cancellation. Anomaly cancellation was achieved in \([17]\) but via smoothing of the M5-brane worldvolume, using a regular Thom form, and this might cause some problem if one considers the coupling also to the M2-brane, due to a violation of Dirac quantization for fluxes, introduced by the non-locality involved in the construction. A local mechanism for anomaly cancellation was proposed in \([13]\), but it needs reducible connections in the \(SO(5)\)-normal bundle of the 5-brane and the complete action for the coupled system is not discussed; a residual dependence of a 5-vector field on M5 remains in this approach.

We reconsider this problem in collaboration with M. Tonin \([13]\) using the formalism of Dirac branes and characteristic currents. We analyze the complete coupled system and we achieve a complete cancellation of gauge and gravitational
anomalies, in local form. However, as in [18], it remains a dependence of the direction along which the worldvolume of the Dirac brane, \( \Sigma_7 \), is attached to the worldvolume, \( \Sigma_6 = \partial \Sigma_7 \), of the M5-brane. This dependence however, completely disappears in the system of NS-5-brane coupled to the II A SUGRA 10, treated with our formalism, a priori corresponding to a dimensional reduction. For details of our analysis of the M5-brane see [19], here we simply sketch how the formalism of characteristic currents helps in the game. One of the new features appearing in the treatment of this system is the Chern-Simons term \( A_3 dA_3 \), where \( A_3 \) is the gauge 3-form of SUGRA 11. When a coupling with an M5-brane is considered, it is clear how to modify \( dA_3 \): simply adding the Poincaré dual of the worldvolume associated to the 6-Dirac brane, \( C_4 = PD(\Sigma_7) \), but it is less clear how to modify \( A_3|_{\Sigma_7} \) since \( C_4 \) is not a closed form. However, using the theory of characteristic currents, replacing \( C_4 \) by \( C_4|_{\Sigma_7} = \chi_4 \), we have a natural candidate, because the pull-back of \( \chi_4 \) to \( \Sigma_6 \) is given by \( \chi_4 \), the Euler form of the normal bundle of the Dirac 6-brane. \( \chi_4 \) is a closed form, hence locally \( \chi_4 = d\chi_3 \), and \( \chi_3 \) is the natural candidate we are looking for.

Finally we remark a suggestive feature of the formalism: a priori it is consistent with the modified Dirac quantization condition \([20]\) for the invariant curvature \( H_4 = dA_3 + C_4 \). In fact \( H_4|_{\Sigma_6} \) involves a term \( C_4|_{\Sigma_6} \), naively ill-defined. Using the characteristic currents we replace it by

\[
\lim_{\epsilon \to 0} C_4^\epsilon|_{\Sigma_6} = \frac{\chi_4}{2}|_{\Sigma_6} \tag{48}
\]

where \( \chi_4 \) is the Euler characteristic of the \( SO(4) \)-bundle of the Dirac brane. Hence for a 4-cycle \( Z_4 \) in \( \Sigma_6 \) we have

\[
\int_{Z_4} H_4 = \int_{Z_4} \frac{\chi_4}{2} \in \mathbb{Z} \tag{49}
\]

so that the cohomology class of \( H_4 \) is a priori half-integer valued, as discussed in \([20]\).

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