Effective rheology of immiscible two-phase flow in porous media

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Abstract – We demonstrate through numerical simulations and a mean-field calculation that immiscible two-phase flow in a porous medium behaves effectively as a Bingham viscoplastic fluid. This leads to a generalized Darcy equation where the volumetric flow rate depends quadratically on an excess pressure difference in the range of flow rates where the capillary forces compete with the viscous forces. At higher rates, the flow is Newtonian.

The simultaneous flow of immiscible fluids in porous media[1–3] lies at the heart of a wide range of important applications ranging from oil recovery to ground water management. Within the statistical physics community, there has been considerable interest in this problem since the 1980s when the fractal structure of fluid invasion was discovered and explored [4,5]. Such invasion phenomena are transient and may be characterized by the macroscopic flow parameters, such as injected pore volumes of invading fluids change on the same time scale as those associated with the internal flow. Much less attention, however, has been offered to steady-state flow which occurs when macroscopic parameters change slowly compared to those associated with the internal flow [6–12].

The steady state has recently been studied experimentally by Tallakstad et al. [13,14] in a two-dimensional Hele-Shaw cell filled with glass beads. There are 15 evenly spaced tubes at one edge of the cell. Air and glycerol are injected at equal rates through each alternate inlet. Fluids leave at the opposing edge, which is kept open. The other two edges of the cell, perpendicular to the direction of overall flow, are closed. As the two immiscible fluids move along the Hele-Shaw cell, they form interpenetrating clusters separated by interfaces. This constitutes the steady state.

The steady state is characterized by a number of macroscopic parameters: Capillary number Ca, viscosity ratio $M$, total volumetric flow rate $Q$, non-wetting fractional flow rate $F_{nw}$, non-wetting saturation $S_{nw}$ and pressure $P$.

The capillary number is the ratio between the typical viscous pressure drop across the pores and the typical capillary force due to the interface between the fluids. Tallakstad et al. observed that the average pressure gradient $\Delta P$ throughout the system scales as a power law with the capillary number Ca as

$$\Delta P \sim Ca^\beta,$$

where $\beta = 0.54 \pm 0.08$.

More recently, Rassi et al. [15] have measured the exponent $\beta$ in eq. (1), finding it to vary from 0.3 to 0.45 depending on the saturation, in steady-state two-phase flow of water and air in a three-dimensional porous medium constructed from glass beads.

These observations have profound implications on the description of multiphase flow in porous media. An important application would be in the reservoir simulators, used in the exploitation of oil reservoirs. They are based on effective transport equations for the fluids where a linear relation between the pressure gradients and flow rates [1] is assumed so far.

In this letter we will demonstrate that, in the regime where capillary forces are comparable to the viscous forces (low Ca), the capillary effects at the interfaces between the immiscible fluids effectively create a yield threshold, making the fluids reminiscent of a Bingham viscoplastic fluid [16,17] in the porous medium — i.e., a fluid possessing a yield stress and a constant effective viscosity. This introduces a overall threshold pressure $\Delta P_c$ in the system due to the random distribution of capillary pressure thresholds. We therefore propose, and will show in the following via numerical simulations and a mean-field
calculation, that the steady-state two-phase flow in porous media in this flow regime is governed by a generalized Darcy equation

\[ Q = -C \frac{A}{L} \frac{K(S_{nw})}{\mu_{\text{eff}}(S_{nw})} \times \text{sgn}(\Delta P) \left\{ \begin{array}{ll} (|\Delta P| - \Delta P_c(S_{nw}))^2, & \text{if } |\Delta P| > \Delta P_c, \\ 0, & \text{if } |\Delta P| \leq \Delta P_c, \end{array} \right. \]  

(2)

where \( \text{sgn} \) is the sign function. Here \( A \) is the cross-section of the representative elementary volume, \( L \) is its length, \( C \) a constant with units of inverse pressure, \( K(S_{nw}) \) the effective permeability which depends on the saturation \( S_{nw} \), and \( \mu_{\text{eff}}(S_{nw}) \) is the saturation-weighted viscosity given by \( S_{nw}\mu_w + (1 - S_{nw})\mu_s \), where \( \mu_w \) and \( \mu_s \) are the viscosities of wetting and non-wetting fluids, respectively.

As a result, the correct scaling relation in general between \( \Delta P \) and \( C_a \) is not eq. (1), but

\[ (|\Delta P| - \Delta P_c) \sim C_a^\beta, \]  

(3)

with \( \beta = 1/2 \) for low \( C_a \). The relation (1) is just a special case of this relation where \( \Delta P_c \approx 0 \). We will return to this point in the discussion.

For high flow rates, i.e. at high \( C_a \), the linear Darcy equation

\[ Q = -\left( \frac{A}{L} \right) \left( \frac{K}{\mu_{\text{eff}}(S_{nw})} \right) \Delta P \]  

(4)

is recovered. Here \( |\Delta P| \gg \Delta P_c \) and \( \Delta P_c \) can be ignored. The flow of single-phase Bingham fluid through porous medium has been studied recently [18], where the system disorder, fluid rheology and inertial effects are found to make an enhancement in the hydraulic conductance at intermediate Reynolds conditions. However, the appearance of Bingham-type behavior in the flow of two immiscible Newtonian fluids in a disordered system, due to the interplay of capillary effects at the interfaces, shape of the pores and the disorder pore geometry has not been reported before elsewhere to our knowledge.

The physical existence of the global threshold pressure \( \Delta P_c \) and the quadratic dependence of \( Q \) on \((|\Delta P| - \Delta P_c)\) for small to moderate flow rates can be understood intuitively following the argument of Roux and Herrmann [17] for networks with link conductances having characteristics like a Bingham fluid: The essential ingredient is a threshold pressure for each link, distributed according to some probability density. The sum over all the thresholds over a continuous flow path throughout the entire system gives the total threshold pressure along that path and \( \Delta P_c \) corresponds to the minimum sum among all such possible paths. Now, if we raise the pressure difference across the network by a value \( \Delta P \), a number \( dN \) of tubes will cross their flow thresholds. With a reasonably smooth distribution of thresholds, we will have that \( dN \propto dP \).

The conductance of the network \( \Sigma \) will then change by an amount \( d\Sigma \), and \( d\Sigma \propto dN \propto dP \). An integration over this equation leads to eq. (2) with the appearance of \( \Delta P_c \). Moreover, at a very high value of \( P \), when all the links have crossed their individual flow thresholds, each link behaves linearly with excess pressure drop, makings \( \Sigma \) constant with \( P \) and effectively the overall flow rate becomes linear with the excess pressure drop in the high \( C_a \) regime —and hence described by the linear Darcy equation (4).

In the following, we will first present our numerical simulations. Afterwards, we will derive the effective Darcy equation (2), adapting the mean-field calculation of the conductivity of heterogeneous conductors by Kirkpatrick [19].

In our numerical studies, the porous medium is modeled by a two-dimensional network of tubes, forming a square lattice tilted by 45° with respect to the imposed pressure gradient. Two immiscible fluids, one is more wetting than the other with respect to the pore walls, flow inside the tubes. Disorder is introduced in the system by choosing the radius \( r \) of each tube randomly from a uniform distribution of random numbers in the range \([0.1, 0.4]\), where \( l \) is the length of the tubes. In order to incorporate the shape of the pores in between spherical particles (beads) that introduces the capillary effect in the system, each tube is considered hour-glass shaped so that the capillary pressure \( p_c \) at a meniscus at position \( x \) is proportional to \( 2\gamma/r/\left[1 - \cos(2\pi x/l)\right] \) where \( \gamma \) is the surface tension [2,20].

The flow is driven by setting up an external global pressure drop. The local flow rate \( q \) in a tube with a pressure difference \( \Delta p \) between the two ends of that tube follows the Washburn equation of capillary flow [2,20]

\[ q = -\left( \frac{ak}{\mu_{\text{eff}}(S_{nw})} \right) \left( \Delta p - \sum p_c \right) = -\sigma_0 \left( \Delta p - \sum p_c \right), \]  

(5)

where \( k = r^2/8 \) is the permeability for cylindrical tubes. The conical shape of the tubes leads only to an overall geometrical factor. Here \( a \) is the cross-sectional area of the tube and \( \mu_{\text{eff}}(S_{nw}) \) is the volume average of the viscosities of the two phases present inside the tube. Hence, it is a function of the saturation \( S_{nw} \) in the tube. The sum over \( p_c \) runs over all menisci within the tube. \( \sigma_0 \) is thus the tube conductivity. The set consisting of one equation (5) per tube, together with the Kirchhoff equations balancing the in and out flow at each node are then solved using Cholesky factorization combined with a conjugate gradient solver. The system is then integrated in time using an explicit Euler scheme. Inside a tube all menisci move with a speed determined by \( q \). When a meniscus reaches the end of a tube, new menisci are formed in the neighboring tubes. Further details of the model and how the menisci are moved can be found in [9,20].

The steady-state condition in the simulation is achieved in two ways. The conventional way is to use bi-periodic boundary conditions (BP), by connecting the inlet and outlet rows so that the network takes a toroidal topology [10]. It is then initialized by filling with two fluids
randomly or sequentially so that the network attains the desired saturation $S_{nw}$. As the system is closed in this boundary condition, the saturation of the network $S_{nw}$ is an independent parameter which remains constant throughout the simulation along with the total flow rate $Q$ whereas the fractional flow $F_{nw}$ fluctuates over time.

It is not possible to implement a bi-periodic boundary condition in the experiments by Tallakstad et al. [13,14], where two fluids are injected at one edge of the system through a series of alternate inlets and the opposite edge is kept open. Flow rates of the two fluids may be controlled independently there. The control parameters in this case are the total flow rate $Q$ and the fractional flow $F_{nw}$ whereas the saturation $S_{nw}$ fluctuates. Therefore, in order to have a close emulation of the experimental system, we also implement open boundary conditions (OB) in our simulation here, controlling the individual flow rates of inlet links. The simulation starts with injecting the two fluids with constant flow rates in a system completely saturated with the wetting fluid. Both drainage and imbibition therefore take place at the pore level creating new menisci. Away from the inlets, the fluids mix and a steady state is attained as in the experiment. In order to illustrate about the two different regimes as discussed before, the distribution of $\Sigma p_c$ over the system during steady-state simulations are shown in fig. 1 for (a) Ca = $1.0 \times 10^{-3}$ and (b) Ca = $4.0 \times 10^{-1}$ which correspond to $\beta = 0.5$ and 1, respectively. In (c) and (d) the same two systems as (a) and (b), respectively, are represented at the same time step, but here the links for which $|\Delta p| > \Sigma p_c$ are drawn as black. Noticeably, in the first regime $|\Delta p|$ is greater than $\Sigma p_c$ only for a fraction of links. The fraction of conducting links increases with the external pressure drop and in the second regime, in almost all the links $|\Delta p|$ have crossed $\Sigma p_c$ as seen in (d), which is very much consistent with the intuitive physical description stated before.

Simulations are performed with constant flow rate $Q$ which sets the capillary number Ca given by $Ca = \mu F_c Q / (\gamma A)$. A range of Ca from $10^{-4}$ to 1 is considered for a network of $64 \times 64$ links and an average over 10 different samples is taken for each simulation. We report results for $M = 1$ and $10^{-4}$ with $S_{nw} = 0.2$ and 0.3 for BP, and $M = 1$ with $F_{nw} = 0.3$ and 0.5 for OB. The steady state is identified from the total pressure drop $\Delta P$, which starts to fluctuate over an average value as steady state is reached. By increasing and then lowering the total flux $Q$ we also verify that it returns to the same steady state. In order to verify eq. (3), we now need to calculate $\Delta P_c$. Roux and Herrmann [17], for their Bingham system, determined $\Delta P_c$ by decreasing the external current from a large value and identifying the current paths by a search algorithm. This procedure is not feasible here as the flow patterns and menisci positions change with global flow rate and time due to fluid instabilities. Moreover, it is also not possible to follow this in the experiments as it would necessitate the knowledge of flow rates at every single pore. We therefore measure $\Delta P_c$ with a minimization procedure. A series of trial $\Delta P_c$ values are considered, for which the slope (\beta) and the least square fit errors, when fitted to eq. (3), are calculated. $\Delta P_c$ is then identified corresponding to the minimum value of the error or the best fit. This, is illustrated in fig. 2 for (a) BP and (b) OB. In table 1, the absolute values of all $\Delta P_c$s, identified by the same procedure, are listed. In fig. 3, $(|\Delta P| - \Delta P_c)$ is then plotted with Ca for (a) BP and (b) OB according to eq. (3). For different saturations ($S_{nw}$), fractional flows ($F_{nw}$) and boundary conditions, the minimum error corresponds to different values of $\Delta P_c$.
fact that one of the fluid swas percolating in their system.

In the experiments by Tallakstad et al. [13,14], β is found as 0.54 ± 0.08 with scaling relation (1) which is due to the fact that one of the fluids was percolating in their system [23], making \( \Delta P_c = 0 \).

In support of our numerical results, we now derive the generalized Darcy equation, eq. (2) in a mean-field approximation. The derivation only assumes a regular lattice with coordination number \( z \). Hence, it is not restricted to two dimensions. Equation (5) describes the instantaneous flow in a single tube. The average flow rate inside a single tube can be obtained from the time averaging of this equation under steady-state conditions. The problem then becomes equivalent to a forced overdamped oscillator, giving rise to a threshold pressure followed by a square-root singularity [24] leading to an effective flow equation for the single tube, given by

\[
q = -\sigma_0 \text{sgn}(\Delta p) \left\{ \begin{array}{ll}
\sqrt{\Delta p^2 - \Delta p_c^2}, & \text{if } |\Delta p| > \Delta p_c, \\
0, & \text{if } |\Delta p| \leq \Delta p_c,
\end{array} \right.
\]

where \( \Delta p_c \) is an effective flow threshold that depends on the shape of the tube. The effective viscosity that enters into \( \sigma_0 \) is the saturation-weighted sum of the viscosities of each liquid, where the saturation is time averaged over the tube.

The square-root singularity near \( \Delta p_c \) is caused by a saddle-node bifurcation and, hence, is a universal feature of the system [25].

The effective conductivity is thus

\[
\sigma(\Delta p) = -\frac{dq}{d(\Delta p)} = \sigma_0 \left\{ \begin{array}{ll}
\frac{|\Delta p|}{\sqrt{\Delta p^2 - \Delta p_c^2}}, & \text{if } |\Delta p| > \Delta p_c, \\
0, & \text{if } |\Delta p| \leq \Delta p_c.
\end{array} \right.
\]

In the following, we will derive eq. (2) based on the mean-field theory originally developed for calculating the conductivity of percolating systems by Kirkpatrick [19]. Our starting point is a regular lattice with coordination number \( z \). Each tube in the lattice has a non-linear conductivity \( \sigma(\Delta p) \), eq. (7). The flow thresholds of the tubes are drawn from a spatially uncorrelated probability distribution \( \pi(\Delta p_c) \).

The mean-field calculation proceeds by focusing on one tube inside the network. We then replace the rest of the network by an *equivalent homogeneous network* where all tubes have the same conductivity \( m(\Delta p) \) so that the tube we have singled out experiences the same average effect from the homogeneous network as from the original network. We then average over this last conductance and determine \( m(\Delta p) \) in a self-consistent way.

Table 1: Values of threshold pressures (\( \Delta P_c \)) measured by minimizing the least square fit errors for different parameters and boundary conditions as shown in fig. 2.

| \( S_{nw} \) | \( M \) | \( \Delta P_c \) (KPa) | \( F_{nw} \) | \( M \) | \( \Delta P_c \) (KPa) |
|---|---|---|---|---|---|
| 0.2 | 1 | 3.45 ± 0.05 | 0.3 | 1 | 6.55 ± 0.05 |
| 0.3 | 1 | 5.10 ± 0.05 | 0.5 | 1 | 6.15 ± 0.05 |
| 0.2 \( 10^{-4} \) | 3.45 ± 0.05 | 0.5 \( 10^{-4} \) | 5.10 ± 0.05 |

(a) Bi-periodic (BP). (b) Open (OB).
There is one caveat. In Kirkpatrick’s original calculation, the conductances were assumed to be linear, whereas in our case, the conductances are highly non-linear, see eq. (7). However, the end result of the calculation,

$$\left( \frac{m(\Delta p) - \sigma(\Delta p)}{\frac{\Delta p}{2}} \right) = 0,$$

is the same. This equation provides a self-consistent expression for the equivalent conductivity $m(\Delta p)$. In terms of the distribution of flow thresholds $\pi(\Delta p_c)$, eq. (8) becomes

$$\int_0^{\Delta p} dp \, \pi(p) \left( \frac{m(\Delta p) - \sigma(\Delta p)}{\frac{\Delta p}{2}} \right) = 0. \quad (9)$$

We now combine this expression with eq. (7), finding

$$\int_0^{\Delta p} dp \, \pi(p) \left( \frac{m(\Delta p) - \sigma(\Delta p)}{\frac{\Delta p}{2}} \right) \left[ \Delta p^2 - \pi^2(\Delta p) - \sigma_0(\Delta p) \right] + 1 - \Pi(\Delta p) = 0, \quad (10)$$

where $\Pi(p) = \int_0^p dp' \, \pi(p')$ is the cumulative probability to find a flow threshold less than or equal to $p$. By setting $m(\Delta p) = 0$ in eq. (10), we determine the effective flow threshold $\Delta p_c$ for the effective tubes:

$$\Pi(\Delta p_c) = \frac{1}{2}. \quad (11)$$

If we now set $\Delta p > \Delta p_c$ in eq. (10) and expand to lowest order in $m(\Delta p)$, we find

$$m(\Delta p) = \frac{4\sigma_0}{8 - z} \pi(\Delta p_c) \left( \frac{\Delta p - \Delta p_c}{\frac{\Delta p}{2}} \right)^2, \quad (12)$$

where we have used that $\Pi(\Delta p) - 1/2 = \pi(\Delta p_c)(\Delta p - \Delta p_c)$ to lowest order. We integrate eq. (12) and find

$$\bar{q} = \frac{8-2z}{8-z} \frac{\pi(\Delta p_c) \sigma_0}{\frac{\Delta p}{2}} \left( \frac{\Delta p}{\Delta p_c} \right)^2 \left[ \right], \quad (13)$$

where $\bar{q}$ is the effective flow rate, $\pi(\Delta p_c)$ is the cumulative probability to find a flow threshold less than or equal to $\Delta p_c$. There is no reason to assume strong non-linear corrections in this region.

To summarize, we have demonstrated numerically and through a mean-field calculation that steady-state immiscible two-phase flow in a porous medium behaves similar to a Bingham viscoplastic fluid. This leads to a non-linear Darcy equation where the volumetric flow rate depends quadratically on an excess pressure drop at capillary numbers at which the capillary forces compete with the viscous forces. With the subsequent increase of the flow rate, the system undergoes a crossover from the quadratic to linear regime at high capillary numbers, and the flow becomes Newtonian described by the linear Darcy equation.

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