Riemann-Roch, Stability and New Non-Abelian Zeta Functions for Number Fields

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In this paper, we introduce a geometrically stylized arithmetic cohomology for number fields. Based on such a cohomology, we define and study new yet genuine non-abelian zeta functions for number fields, using an intersection stability.

1. A New Cohomology

(1.1) Arithmetic Cohomology Groups

Let $F$ be a number field with discriminant $\Delta_F$. Denote its (normalized) absolute values by $S_F$, and write $S_F = S_{\text{fin}} \cup S_{\infty}$, where $S_{\infty}$ denotes the collection of all archimedean valuations. For simplicity, we use $v$ (resp. $\sigma$) to denote elements in $S_{\text{fin}}$ (resp. $S_{\infty}$).

Denote by $A = A_F$ the ring of adeles of $F$, by $\text{GL}_r(A)$ the rank $r$ general linear group over $A$, and write $A := A_{\text{fin}} \oplus A_{\infty}$ and $\text{GL}_r(A) := \text{GL}_r(A_{\text{fin}}) \times \text{GL}_r(A_{\infty})$ according to their finite and infinite parts.

For any $g = (g_{\text{fin}} : g_{\infty}) = (g_v : g_{\sigma}) \in \text{GL}_r(A)$, define the injective morphism $i(g) := i(g_{\infty}) : F^r \to A^r$ by $(f) \mapsto (f; g_{\sigma} \cdot f)$. Let $F^r(g) := \text{Im}(i(g))$ and set

$$A^r(g) := \{ (a_v : a_{\sigma}) \in A^r : g_v(a_v) \in \mathcal{O}_v, \forall v; \text{ and } \exists f \in F^r \text{ s.t. } g_v(f) \in \mathcal{O}_v, \forall v \text{ and } (f; a_{\sigma}) = i(g_{\infty})(f) \}.$$

Then we have the following 9-diagram with exact columns and rows:

$$
\begin{array}{cccccc}
0 & \downarrow & 0 & \downarrow & 0 & \downarrow \\
0 \to & A^r(g) \cap F^r(g) & \to & A^r(g) & \to & A^r(g) / A^r(g) \cap F^r(g) \to 0 \\
0 & \downarrow & A^r & \downarrow & A^r / F^r(g) & \to 0 \\
0 \to & F^r(g) & \to & A^r & \to & A^r / A^r(g) + F^r(g) \to 0 \\
0 & \downarrow & 0 & \downarrow & 0 \\
\end{array}
$$

Motivated by this and Weil's adelic cohomology theory for divisors over algebraic curves, (see e.g., [24] and [29]), we introduce the following

Definition. For any $g \in \text{GL}_r(A)$, define its 0-th and 1-st arithmetic cohomology groups by

$$H^0(A_F, g) := A^r(g) \cap F^r(g), \quad \text{and} \quad H^1(A_F, g) := A^r / A^r(g) + F^r(g).$$

Theorem. (Serre Duality=Pontrjagin Duality) As locally compact groups,

$$H^1(A_F, g) \simeq H^0(A_{F, k_F} \otimes g^{-1}).$$

Here $k_F$ denotes an idelic dualizing element of $F$, and $\otimes$ the Pontrjagin dual. In particular, $H^0$ is discrete and $H^1$ is compact.

Remark. For $v \in S_{\text{fin}}$, denote by $\partial_v$ the local different of $F_v$, the $v$-completion of $F$ at $v$, and by $\mathcal{O}_v$ the valuation ring with $\pi_v$ a local parameter. Then $\partial_v =: \pi_v^{\text{ord}_v(\partial_v)} \cdot \mathcal{O}_v$. We call $\kappa_F := (\partial_v^{\text{ord}_v(\partial_v)} : 1) \in I_F := \text{GL}_1(A)$ an idelic dualizing element of $F$.  

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Proof. As usual, introduce a basic character $\chi$ on $A$ by $\chi := (\chi^r; \chi^{z(r)})$ where $\chi^z := \lambda \circ \text{Tr}_{g}$ with $\lambda : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, and $\lambda_{\sigma} := \lambda_{\infty} \circ \text{Tr}_{g}$ with $\lambda_{\infty} : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Then the pairing

$$(x, y) \mapsto e^{2\pi i x \langle \sigma, y \rangle}$$

induces natural isomorphisms $\hat{A}^r \simeq A^r$ (as locally compact groups) and $(F^r)^{\perp} \simeq F^r$ (as discrete subgroups). With this, a direct local calculation shows that $(A^r(g))^{\perp} = A^r(\kappa_{g} \otimes g^{-1})$ and $(F^r(g))^{\perp} \simeq F^r(\kappa_{g} \otimes g^{-1})$. This completes the proof since

$$(A^r(g) \cap F^r(g))^{\perp} = (A^r(g))^{\perp} + (F^r(g))^{\perp}.$$

(1.2) Arithmetic Counts

Motivated by the Pontrjagin duality and the fact that the dimension of a vector space is equal to the dimension of its dual, one basic principal we adopt in counting locally compact groups is the following:

Counting Axiom. If $\#_{\mathbb{g}}$ counts a certain class of locally compact groups $G$, then $\#_{\mathbb{g}}(G) = \#_{\mathbb{g}}(\hat{G})$.

Practically, our counts of arithmetic cohomology groups are based on the Fourier inverse formula, or more accurately, the Plancherel formula in Fourier analysis over locally compact groups. (See e.g. [9].)

While any reasonable test function on $A^r$ would do, as a continuation of a more classical mathematics and also for simplicity, we set $f := \prod_{v} f_{v} \cdot \prod_{v} f_{\sigma}$. Here $f_{v}$ is the characteristic function of $\mathcal{O}_{v}^{c}$; $f_{\sigma}(x_{\sigma}) := e^{-\pi \lvert x_{\sigma} \rvert^2}/2$ if $\sigma$ is real; and $f_{\tau}(x_{\sigma}) := e^{-\pi \lvert x_{\sigma} \rvert^2}$ if $\sigma$ is complex. Moreover, we take the following normalization for the Haar measure $dx$, which we call standard, on $A$: locally for $v$, $dx$ is the measure for which $\mathcal{O}_{v}$ gets the measure $N(\partial_{v})^{-1/2}$, while for $\sigma$ real (resp. complex), $dx$ is the ordinary Lebesgue measure (resp. twice the ordinary Lebesgue measure).

Definition. (1) The arithmetic counts of the 0-th and the 1-st arithmetic cohomology groups for $g \in \text{GL}_{r}(A)$ are defined to be

$$\#_{\mathbb{g}}(H^{0}(A_{F}, g)) := \#_{\mathbb{g}}(H^{0}(A_{F}, g); f, dx) := \int_{H^{0}(A_{F}, g)} |f(x)|^{2} dx;$$

$$\#_{\mathbb{g}}(H^{1}(A_{F}, g)) := \#_{\mathbb{g}}(H^{1}(A_{F}, g); \hat{f}, d\xi) := \int_{H^{1}(A_{F}, g)} |\hat{f}(\xi)|^{2} d\xi.$$

Here $dx$ denotes (the restriction of) the standard Haar measure on $A$, $d\xi$ (the induced quotient measure from) the dual measure (with respect to $\chi$), and $\hat{f}$ the corresponding Fourier transform of $f$;

(2) The 0-th and the 1-st arithmetic cohomologies of $g \in \text{GL}_{r}(A)$ are defined to be

$$h^{0}(A_{F}, g) := \log \left( \#_{\mathbb{g}}(H^{0}(A_{F}, g)) \right)$$

and

$$h^{1}(A_{F}, g) := \log \left( \#_{\mathbb{g}}(H^{1}(A_{F}, g)) \right).$$

(1.3) Serre Duality and Riemann-Roch

For the arithmetic cohomologies just introduced, we have the following

Theorem. (1) (Serre Duality) $h^{1}(A_{F}, g) = h^{0}(A_{F}, \kappa_{F} \otimes g^{-1})$;

(2) (Riemann-Roch Theorem)

$$h^{0}(A_{F}, g) - h^{1}(A_{F}, g) = \deg(g) - \frac{r}{2} \cdot \log |A_{F}|.$$
and Tate’s Riemann-Roch theorem ([28, Thm. 4.2.1] and/or [14, XIV, §6]), i.e., the Poisson summation formula, by the fact that \( \left( H^0(A_F, g) \right)^\perp = H^0(A_F, \kappa_F \otimes g^{-1}) \). This then completes the proof.

Often, for our own convenience, we also write \( e^{\eta(A_F, g)} = \#_{ga}(H^i(A_F, g)) \) simply as \( H^i_{ga}(F, g) \), \( i = 1, 2 \).

With this, the above additive version may be rewritten as

**Theorem’.** (1) (Serre Duality) \( H^1_{ga}(F, g) = H^0_{ga}(F, \kappa_F \otimes g^{-1}) \);

(2) (Riemann-Roch Theorem) \( H^0_{ga}(F, g) = H^1_{ga}(F, g) \cdot N(g) \cdot N(\kappa_F)^{-2}, \) where as usual \( N(g) \) denotes \( e^{\deg(g)} \).

**Remarks.** (1) Our work here is motivated by the works of Weil, Tate, van der Geer-Schoof, and Li, as well as the works of Lang, Arakelov, Szpiro, Moreno, Neukirch, Deninger, Connes, and Borisov. For details, please see the references below, in particular [31]. Also, it would be extremely interesting if one could relate the work here with that of Connes [5] and Deninger [7, 8].

(2) One may apply the discussion in this paper to wider classes of (multiplicative) characters and test functions. We leave this to the reader. (See e.g., [28], [29] and [18].)

2. New Non-Abelian Zeta Functions

(2.1) Intersection Stability

For a metrized vector sheaf \((E, \rho)\) on \( \text{Spec}(O_F) \), define its associated \( \mu \)-invariant by

\[
\mu(E, \rho) := \frac{\deg_A(E, \rho)}{\text{rank}(E)},
\]

where \( O_F \) denotes the ring of integers of a number field \( F \) and \( \deg_A \) the Arakelov degree of \((E, \rho)\). (See e.g. [16].) By definition, a proper sub metrized vector sheaf \((E_1, \rho_1)\) of \((E, \rho)\) consists of a proper sub vector sheaf \( E_1 \) of \( E \) such that \( \rho_1 \) is induced from the restriction of \( \rho \) via the injection \( E_1 \hookrightarrow E \).

**Definition.** A metrized vector sheaf \((E, \rho)\) is called stable (resp. semi-stable) if for all proper sub metrized vector sheaf \((E_1, \rho_1)\) of \((E, \rho)\),

\[
\mu(E_1, \rho_1) < \mu(E, \rho) \quad (\text{resp.} \quad \mu(E_1, \rho_1) \leq \mu(E, \rho)).
\]

**Remarks.** (1) Despite the fact that we define it independently, the intersection stability in arithmetic, motivated by Mumford’s work [20] in geometry, was first introduced by Stuhler in [26,27], see also [11,12,19 and 4]. Standard facts concerning Harder-Narasimhan filtrations and Jordan-Hölder graded metrized vector sheaves hold in this setting as well. For details, see e.g., [4,19 and 31].

(2) The intersection stability plays a key role in our work on non-abelian class field theory for Riemann surfaces in [30]. Motivated by this, as a fundamental problem, we ask whether a Narasimhan-Seshadri type correspondence holds in arithmetic in [31].

For \( g = (g_v; g_s) \in \text{GL}_r(A) \), introduce a torsion-free \( O_F \)-module

\[
H^0(A_F, g)_{\text{fin}} := H^0(\text{Spec}(O_F), g) := \{ f \in F^m : g_v f \in O_v, \forall v \}
\]

in \( F^r \). Denote the associated vector sheaf on \( \text{Spec}(O_F) \) by \( \mathcal{E}(g) \), that is,

\[
\mathcal{E}(g) := H^0(\text{Spec}(O_F), g).
\]

Moreover, note that \( F^r \), via completion, is densely embedded in \( A^r_\infty \). Thus to introduce metrics on \( \mathcal{E}(g) \) is the same as to assign metrics on the determinants, i.e., on the top exterior products, of the associated data.
(See e.g., [16, Chap. V].) Hence without loss of generality, we may assume that \( r = 1 \). In this case, the metric on \( \mathcal{E}(g) \) associated to \( g \) is defined to be the one such that for the rational section \( 1 \in F, \)
\[
\|1\|_\sigma := \|g_\sigma\|_\sigma := |g_\sigma|\text{N}_r:=[F_r:Q_r].
\]
Denote such a metric on \( \mathcal{E}(g) \) by \( \rho(g) \) for \( g \in GL_r(A) \).

As such, we obtain a canonical map \( (\mathcal{E}(\cdot), \rho(\cdot)) : GL_r(A) \to \Omega_{\text{Spec}(O_F), r} \) by assigning \( g \) to \( (\mathcal{E}(g), \rho(g)) \), where \( \Omega_{\text{Spec}(O_F), r} \) denotes the collection of all metrized vector sheaves of rank \( r \) over \( \text{Spec}(O_F) \). Clearly, \( (\mathcal{E}(\cdot), \rho(\cdot)) \) factors through the quotient group \( GL_r(F) \backslash GL_r(A) \) where \( GL_r(F) \) is embedded diagonally in \( GL_r(A) \). Denote this resulting map by \( (\mathcal{E}(\cdot), \rho(\cdot)) \) too by an abuse of notation.

Denote by \( \mathcal{M}_{F,r}(d) \) the subset of \( \Omega_{\text{Spec}(O_F), r} \) consisting of semi-stable metrized vector sheaves of (Arakelov) degree \( d \). Since for a fixed, the semi-stability condition is a bounded and closed one, with respect to the natural topology, \( \mathcal{M}_{F,r}(d) \) is compact. (See e.g. [11, 26 and 27].)

Denote by \( \mathcal{M}_{A,r}(d) \subset GL_r(F) \backslash GL_r(A) \) the inverse image of \( \mathcal{M}_{F,r}(d) \) with respect to \( (\mathcal{E}(\cdot), \rho(\cdot)) \), and denote the corresponding map by
\[
\Pi_{F,r}(d) : \mathcal{M}_{A,r}(d) \to \mathcal{M}_{F,r}(d)
\]
which we call the (algebraic) moment map. As a subquotient of \( GL_r(A) \), \( \mathcal{M}_{A,r}(d) \) admits a natural topology, the induced one. Moreover, by a general result due to Borel [2], which in our case is more or less obvious, the fibers of \( \Pi_{F,r}(d) \) are all compact. Thus in particular, \( \mathcal{M}_{A,r}(d) \), which we call the moduli space of semi-stable adelic bundles of rank \( r \) and degree \( d \), is compact. In particular, as a subquotient of \( GL_r(A) \), \( \mathcal{M}_{A,r}(d) \) carries a natural measure induced from the standard one on \( GL_r(A) \), which we call the Tamagawa measure, and denote by \( d\mu_{A,r}(d) \). For the same reason, there is also a natural measure on \( \mathcal{M}_{F,r}(d) \), which we call the hyperbolic measure, and denote it by \( d\mu_{F,r}(d) \).

Clearly, the total volumes of \( \mathcal{M}_{A,r}(d) \) (resp. \( \mathcal{M}_{F,r}(d) \)) with respect to \( d\mu_{A,r}(d) \) (resp. \( d\mu_{F,r}(d) \)) are different important non-commutative invariants for number fields. Note that according to what we call the Bombieri-Vaaler trick [1], i.e., by multiplying \( g \) with \((1; e^r)\) where \( t_{sr} := N_r \cdot t \) with \( t \in \mathbb{R} \), we obtain a natural isomorphism between \( \mathcal{M}_{A,r}(d) \) and \( \mathcal{M}_{A,r}(d-n\cdot t) \) where \( n := |F : Q| \). (Even though it is an open problem that semi-stability is closed under tensor operation [4], the case here in which one is of rank 1 is rather obvious.) Consequently, the above volumes are independent of degrees \( d \). Denote them by \( W_F(r) \) and \( w_F(r) \) respectively.

**2.2 Functional Equation: A Formal Calculation**

Let \( F \) be a number field with discriminant \( \Delta_F \). Denote by \( \mathcal{M}_{A,r} \) the moduli space of semi-stable adelic bundles of rank \( r \), that is, \( \mathcal{M}_{A,r} := \bigcup_{N \in \mathbb{R}_+} \mathcal{M}_{A,r}[N] \) where \( \mathcal{M}_{A,r}[N] := \mathcal{M}_{A,r}(\log N) \). By using the Bombieri-Vaaler trick in (2.1), as topological spaces, \( \mathcal{M}_{A,r} \simeq \mathcal{M}_{A,r}[\|\Delta_F\|^2] \times \mathbb{R}_+ \). Hence we obtain a natural measure \( d\mu \) on \( \mathcal{M}_{A,r} \) from the Tamagawa measures on \( \mathcal{M}_{A,r}[N] \) and \( \frac{dt}{t} \) on \( \mathbb{R}_+ \).

For any \( E \in \mathcal{M}_{A,r} \), define \( H^i_{ga}(F, E) := H^i_{ga}(F, g) = e^{j(i)(\mathcal{A}_F, g)} \) for any \( g \in GL_r(A) \) such that \( E = [g] \). Since for any \( a \in GL_r(F) \), \( H^i_{ga}(F, a \cdot g) = H^i_{ga}(F, g) \) \( H^i_{ga}(F, E) \) is well-defined for \( i = 0, 1 \).

With respect to fixed real constants \( A, B, C, \alpha \) and \( \beta \), introduce the formal integration \( Z_{F,r; A, B, C; \alpha, \beta}(s) \) as follows:
\[
Z_{F,r; A, B, C; \alpha, \beta}(s) := (\|\Delta_F\|^{\frac{s}{2}})^s \int_{E \in \mathcal{M}_{A,r}} \left(H_{ga}^0(F, E)^A \cdot N(E)^{B+C} - N(E)^{\alpha s + \beta}\right) d\mu(E).
\]
Then formally,
\[
Z_{F,r; A, B, C; \alpha, \beta}(s) = I(s) - II(s) + III(s),
\]
where
\[
I(s) := (\|\Delta_F\|^{\frac{s}{2}})^s \int_{E \in \mathcal{M}_{A,r}, N(E) \leq \|\Delta_F\|^2} \left(H_{ga}^0(F, E)^A \cdot N(E)^{B+C} - N(E)^{\alpha s + \beta}\right) d\mu(E); \\
II(s) := (\|\Delta_F\|^{\frac{s}{2}})^s \int_{E \in \mathcal{M}_{A,r}, N(E) \geq \|\Delta_F\|^2} N(E)^{\alpha s + \beta} d\mu(E); \\
III(s) := (\|\Delta_F\|^{\frac{s}{2}})^s \int_{E \in \mathcal{M}_{A,r}, N(E) \geq \|\Delta_F\|^2} H_{ga}^0(F, E)^A \cdot N(E)^{B+C} d\mu(E).
\]
By Theorem 1.3, i.e., the multiplicative Serre duality and Riemann-Roch theorem, we have

\[ III(s) = (|\Delta_F|^{-\frac{B}{2}})^{-s} \int_{E \in \mathcal{M}_{A,F,r}, N(E) \leq |\Delta_F|^\frac{1}{2}} H_{g\alpha}^0(F,E)^A \cdot N(E)^B(-s-\frac{A+2C}{B}+\alpha) d\mu(E), \]

by the fact that \( N(E_1 \otimes E_2^\vee) = N(E_1)^{\text{rank}(E_2)} \cdot N(E_2)^{-\text{rank}(E_1)} \). Hence, formally,

\[ Z_{F,r,A,B,C;\alpha,\beta}(s) = I(s) + I(-s - \frac{A+2C}{B}) - II(s) + IV(s), \]

where

\[ IV(s) := (|\Delta_F|^{-\frac{B}{2}})^{-s-\frac{A+2C}{B}} \int_{E \in \mathcal{M}_{A,F,r}, N(E) \leq |\Delta_F|^\frac{1}{2}} N(E)^\alpha(-s-\frac{A+2C}{B}+\beta) d\mu(E). \]

Moreover, by definition,

\[ -II(s) = - \int_{E \in \mathcal{M}_{A,F,r}, N(E) \geq |\Delta_F|^\frac{1}{2}} N(E)^\alpha+\beta d\mu(E) = - \int_{\mathcal{M}_{A,F,r},|\Delta_F|^\frac{1}{2}} d\mu(E) \cdot \int_1^\infty t^{\alpha+\beta} dt \]

\[ = - W_F(r) \cdot t^{\alpha+\beta} \bigg|_1^\infty = W_F(r) \cdot \frac{1}{\alpha s + \beta}, \]

provided that \( \alpha s + \beta < 0 \).

Similarly,

\[ IV(s) = \int_{\mathcal{M}_{A,F,r},|\Delta_F|^\frac{1}{2}} d\mu(E) \cdot \int_0^1 t^{\alpha(-s-\frac{A+2C}{B}+\beta)} dt = W_F(r) \cdot \frac{1}{\alpha(-s-\frac{A+2C}{B}+\beta)}, \]

provided that \( \alpha(-s-\frac{A+2C}{B}) + \beta > 0 \).

Therefore, formally,

\[ Z_{F,r,A,B,C;\alpha,\beta}(s) = I(s) + I(-s - \frac{A+2C}{B}) + W_F(r) \cdot \left( \frac{1}{\alpha s + \beta} + \frac{1}{\alpha(-s-\frac{A+2C}{B}) + \beta} \right). \]

As a direct consequence, we have the following

**Functional Equation.** With the same notation as above, formally,

\[ Z_{F,r,A,B,C;\alpha,\beta}(s) = Z_{F,r,A,B,C;\alpha,\beta}(-s - \frac{A+2C}{B}). \]

(2.3) Non-Abelian Zeta Functions for Number Fields

To justify the arguments in (2.2), we consider convergences of two types.

**Type 1.** Convergence for II(s) and IV(s), where

\[ II(s) = (|\Delta_F|^{-\frac{B}{2}})^{-s} \int_{E \in \mathcal{M}_{A,F,r}, N(E) \geq |\Delta_F|^\frac{1}{2}} N(E)^\alpha d\mu(E); \]

\[ IV(s) = (|\Delta_F|^{-\frac{B}{2}})^{-s-\frac{A+2C}{B}} \int_{E \in \mathcal{M}_{A,F,r}, N(E) \leq |\Delta_F|^\frac{1}{2}} N(E)^\alpha(-s-\frac{A+2C}{B}+\beta) d\mu(E). \]

From the calculation in (2.2), when \( \text{Re}(\alpha \cdot s + \beta) < 0 \) and \( \text{Re}(\alpha \cdot (-s-\frac{A+2C}{B}) + \beta) > 0 \), being holomorphic functions,

\[ II(s) = -W_F(r) \cdot \frac{1}{\alpha s + \beta} \quad \text{and} \quad IV(s) = W_F(r) \cdot \frac{1}{\alpha(-s-\frac{A+2C}{B}) + \beta}. \]
Type 2. Convergence for $I(s)$ and $I(-s - \frac{A+2C}{B})$ where

$$I(s) = (|\Delta_F|^{-\frac{s}{2}})^s \int_{E \in \mathcal{M}_{A,F,r}} \left( H^0_{ga}(F,E)^A \cdot N(E)^{B_s+C} - N(E)^{\alpha s+\beta} \right) d\mu(E).$$

By the discussion above about $II(s)$, unless $B = \alpha$, $I(s)$ and $I(-s - \frac{A+2C}{B})$, and hence $Z_{F,r;A,B,C;\alpha,\beta}(s)$ cannot be meromorphically extended as a meromorphic function to the whole $s$-plane. (See also the discussion below.) Thus, we introduce the following

**Compatibility Conditions:** $\alpha = B$ and $\beta = C$.

With this, by a change of variables, we also assume that $B = 1$ and $C = 0$ so as to obtain the following integration:

$$Z_{F,r;A}(s) := Z_{F,r;A_{-1,0};-1,0}(s) := (|\Delta_F|^s)^s \int_{E \in \mathcal{M}_{A,F,r}} \left( H^0_{ga}(F,E)^A - 1 \right) \cdot N(E)^{-s} d\mu(E).$$

Furthermore, for any $g \in \text{GL}_r(\mathbb{A})$, from the definition and the fact that $H^0_{ga}(A_F,g)$ is discrete, by writing down each term precisely,

$$H^0_{ga}(F,g) = 1 + \sum_{\alpha \in H^0(\text{Spec} \mathbb{O}_F,\mathcal{E}(g))\setminus \{0\}} \exp \left( -\pi \sum_{\sigma : \mathbb{R}} |g_\sigma \cdot \alpha|^2 - 2\pi \sum_{\sigma : \mathbb{C}} |g_\sigma \cdot \alpha|^2 \right) =: 1 + H^0_{ga}(F,g).$$

In this expression, the first term is simply the constant function 1 on the moduli space, while each term in the second decays exponentially. With this, by a standard argument about convergence of an integration of theta series for higher rank lattices in reduction theory, see e.g., [25, Chap. III, Lect. 15], and the fact that 1 in the first term $H^0_{ga}(F,E)^A$ cancels with the second term 1 in the combination $H^0_{ga}(F,E)^A - 1$, we conclude that $II(s)$ and $II(-s - A)$ are all holomorphic functions, provided $A > 0$. All in all, we have proved the following

**Main Theorem.** For any strictly positive real number $A$,

$$Z_{F,r;A}(s) := (|\Delta_F|^s)^s \int_{E \in \mathcal{M}_{A,F,r}} \left( H^0_{ga}(F,E)^A - 1 \right) \cdot N(E)^{-s} d\mu(E)$$

is holomorphic when $\text{Re}(s) > A$. Moreover,

1. $Z_{F,r;A}(s)$ admits a meromorphic continuation to the whole complex $s$ plane which only has simple poles at $s = 0$ and $s = A$ with the same residue $W_F(r)$, i.e., the Tamagawa volume of the moduli space $\mathcal{M}_{A,F,r}[|\Delta_F|];$
2. $Z_{F,r;A}(s)$ satisfies the functional equation

$$Z_{F,r;A}(s) = Z_{F,r;A}(A - s).$$

**Main Definition.** The function $Z_{F,r}(s) := Z_{F,r;1}(s)$ is called the rank $r$ non-abelian zeta function of $F$.

**Remarks.** The latest definition may be justified by Iwasawa’s ICM talk at MIT. (See e.g., [13 and/or 31].) As they stand, our non-abelian zeta functions expose non-abelian aspect of number fields. For details, see e.g., [31].

(2) One may simply use moduli spaces $\mathcal{M}_{F,r}(d)$ to introduce new non-commutative zeta functions for number fields as well.

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