ULAM–HYERS–RASSIAS STABILITY OF A NONLINEAR
STOCHASTIC ITO–VOLterra INTEGRAL EQUATION

NGO PHUOC NGUYEN NGOC AND NGUYEN VAN VINH

(Communicated by Marek T. Malinowski)

Abstract. In this paper, by using the classical Banach contraction principle, we investigate and
establish the stability in the sense of Ulam-Hyers and in the sense of Ulam-Hyers-Rassias for the
following stochastic integral equation

\[ X_t = \xi_t + \int_0^t A(t,s,X_s)ds + \int_0^t B(t,s,X_s)dW_s, \]

where \( \int_0^t B(t,s,X_s)dW_s \) is Ito integral.

1. Introduction

The study of stability problems for various functional equations originated from a
famous talk given by Ulam in 1940. In the talk, Ulam discussed a problem concerning
the stability of homomorphisms (see [24] and [25]). More precisely, he proposed the
following problem:

Given a group \( G_1 \), a metric group \((G_2,d)\) and a positive number \( \varepsilon \), does there
exist a \( \delta > 0 \) such that if a function \( f : G_1 \rightarrow G_2 \) satisfies the following inequality

\[ d(f(xy),f(x)f(y)) < \delta, \]

for all \( x,y \in G_1 \), then there exists a homomorphism \( T : G_1 \rightarrow G_2 \) such that:

\[ d(f(x),T(x)) < \varepsilon, \]

for all \( x \in G_1 \)?

When this problem has a solution, we say that the homomorphisms from \( G_1 \) to \( G_2 \)
are stable, or that the equation defining group homomorphisms is stable (in the sense of
Ulam).

In 1941, D.H. Hyers (see [8]) gave a partial solution of Ulam’s problem under the
assumption that \( G_1 \) and \( G_2 \) are Banach spaces. In 1950, T. Aoki (see [2]) studied the
stability problem for additive mappings by using unbounded Cauchy differences (see

Mathematics subject classification (2010): Primary 60H20, 34K20; Secondary 47H10.

Keywords and phrases: Ulam-Hyers-Rassias stability, Stochastic integral equations, Banach’s fixed
point theorem.
also [16]). In 1978, Th.M. Rassias (see [22]) studied a similar problem. The stability considered in [22] is often called the Ulam-Hyers-Rassias stability.

In [21], V. Radu introduced a simple and nice proof for the Hyers-Ulam stability of the Cauchy additive functional equation. Using the idea of V. Radu, S.M. Jung proved in [11] the Hyers-Ulam-Rassias stability of some Volterra integral equations defined on a finite interval. After that, in [5], L.P. Castro and D.A. Ramos investigated the stability of Volterra integral equation of second kind for not only the finite case but also the infinite case. A simple proof of Jung’s problem was later given in [23] by using some Gronwall lemmas.

In the references, at the end of this paper, we have listed other papers dealing with the stability of functional equations.

For a large amount of information on the stability of functional equations, the reader is invited to consult the books [7], [9] and [12] (see also the papers [1], [4], [26], and others). Especially, in [4], the authors presented some recent developments in Ulam’s type stability.

We point out that a fixed point method was used to investigate the stability of several functional equations. Works along these lines are achieved by L. Cădariu, V. Radu (see [6] and [21]), and H.A. Kenary et al ([14]). Fixed point methods were also used to study the stability of differential equations (see [13], [17], and other related papers).

Recently, N.P.N Ngoc ([18]) and X. Zhao ([27]) established the stability to stochastic differential equations on finite intervals. In this paper, we first introduce the notion of Hyers-Ulam-Rassias stability to the stochastic Ito-Volterra integral equation and then prove that kind of equations on not only finite but also infinite intervals has the Ulam-Hyers-Rassias stability.

2. Definitions and Preliminaries

Fix a probability space \((\Omega, \mathcal{F}, P)\). Let \(\| \cdot \|_2 = (E|\cdot|^2)^{1/2}\) be a norm of the space \(L_2(\Omega, P)\). Let \(W_t\) be a Brownian motion defined in \((\Omega, \mathcal{F}, P)\) and let \(\{\mathcal{F}_t, t \in I\}\) be the natural filtration associated to \(W_t\), where \(I \subset \mathbb{R} (I = [0, T] \) or \(I = [0, \infty)\)).

Denote by \(L^2_{ad}(I, \Omega)\) the space of stochastic processes \(f(t, \omega)\) such that each \(f(t, \omega)\) is adapted to the filtration \(\{\mathcal{F}_t\}\) and \(E (\int_I |f(t)|^2 dt) < \infty\).

Let \(A(t, s, x)\) and \(B(t, s, x)\) be measurable functions of \(s, t \in S\) and \(x \in \mathbb{R}\), where \(S = \{(s, t) \in I^2 : 0 \leq s \leq t\}\). Consider the stochastic integral equation of Volterra second type:

\[
X_t = \xi_t + \int_0^t A(t, s, X_s)ds + \int_0^t B(t, s, X_s)dW_s, t \in I,
\]

where \(\xi_t\) is a \(\mathcal{F}_t\)-adapted process.

About the existence and uniqueness of solution of Equation (1), we refer to [10] and [20] for more detail.

In the following definitions, we introduce the stability in the sense Ulam-Hyers and Ulam-Hyers-Rassias of the stochastic integral equation.
DEFINITION 1. Equation (1) is said to have the Ulam-Hyers stability with respect to \( \varepsilon \) if there exists a constant \( M_\varepsilon > 0 \) such that for each solution \( X_t \in L^2_{ad}(I, \Omega) \) of the following inequation
\[
\|X_t - \xi_t - \int_0^t A(t,s,X_s)ds - \int_0^t B(t,s,X_s)dW_s\|_2 \leq \varepsilon, \forall t \in I,
\]
there exists a solution \( U_t \in L^2_{ad}(I, \Omega) \) of Equation (1) such that:
\[
\|X_t - U_t\|_2 \leq M_\varepsilon \varepsilon, \forall t \in I,
\]
where \( M_\varepsilon \) is a constant that does not depend on \( X_t \).

DEFINITION 2. Equation (1) is said to have the Ulam-Hyers-Rassias stability with respect to \( u(t) \) if there exists a constant \( M_u > 0 \) such that for each solution \( X_t \in L^2_{ad}(I, \Omega) \) of the following inequation
\[
\|X_t - \xi_t - \int_0^t A(t,s,X_s)ds - \int_0^t B(t,s,X_s)dW_s\|_2 \leq u(t), \forall t \in I,
\]
there exists a solution \( U_t \in L^2_{ad}(I, \Omega) \) of the equation (1) such that:
\[
\|X_t - U_t\|_2 \leq M_u u(t), \forall t \in I,
\]
where \( M_u \) is a constant that does not depend on \( X_t \).

In order to show that Equation (1) is stable in the sense of Ulam-Hyers and Ulam-Hyers-Rassias, we shall need some definitions and remarks in [20].

DEFINITION 3. ([20]) Let \( C_u \) denote the space of all processes in \( L^2_{ad}(I, \Omega) \) that satisfy the following condition
\[
\|x(t)\|_2 \leq K u(t), \forall t \in I,
\]
where \( u(t) > 0 \) is a given continuous function and \( K \) is some positive constant.

REMARK 1. It is well known that \( C_u \) is a Banach space when a norm \( \| \cdot \|_{C_u} \) is defined by
\[
\|x\|_{C_u} = \sup_{t \in I} \left\{ \frac{\|x(t, \omega)\|_2}{u(t)} \right\}.
\]

DEFINITION 4. ([20]) If \( u(t) = 1, \forall t \in I \) in Definition 3, we shall denote the corresponding \( C_u \) by \( C_b \).

DEFINITION 5. ([20]) Let \( C_{1,u} \) denote the space of all processes \( x(t,s; \omega) \) in \( C_1 = \{x(t,s; \omega) : \|x\| := \sup_{(s,t) \in S} \|x(t,s; \omega)\|_2 < \infty \} \) such that
\[
\|x(t,s)\|_2 \leq K u(t) u(s), \forall (s,t) \in S,
\]
for some constant \( K > 0 \) and bounded positive continuous function \( u(t) \).
DEFINITION 6. ([20]) If \( u(t) = 1, \forall t \in I \) in Definition 5, we shall denote the corresponding \( C_{1,u} \) by \( C_{1,b} \).

REMARK 2. It is known that \( C_{1,u} \) is a Banach space with the norm \( \| \cdot \|_{C_{1,u}} \) defined by

\[
\| x \|_{C_{1,u}} = \sup_{(s,t) \in S} \left\{ \frac{\| x(t,s) \|_2}{u(t)u(s)} \right\}.
\]

We define the integral operators \( \Lambda_1, \Lambda_2 \) as follows:

\[
(\Lambda_1 x)(t, \omega) = \int_0^t x(t,s; \omega) ds,
\]

\[
(\Lambda_2 x)(t, \omega) = \int_0^t x(t,s; \omega) dW_s.
\]

REMARK 3. According to [20], with suitable conditions, \( (C_{1,u}, C_u) \) is admissible with respect to both \( \Lambda_1 \) and \( \Lambda_2 \). It means that \( \Lambda_1(C_{1,u}) \subset C_u \) and \( \Lambda_2(C_{1,u}) \subset C_u \). In this case, there are constants \( K_1 \) and \( K_2 \) such that:

\[
\begin{cases}
\| \Lambda_1 x \|_{C_u} \leq K_1 \| x \|_{C_{1,u}}, \\
\| \Lambda_2 x \|_{C_u} \leq K_2 \| x \|_{C_{1,u}}.
\end{cases}
\]

We now introduce Banach’s fixed point theory. This theorem will play an important role in proving our main theorems.

THEOREM 1. ([3]) (Banach’s fixed point theorem) Suppose \((X,d)\) is a complete metric space and \( T : X \to X \) is a contraction (for some \( \lambda \in [0,1) \)), \( d(T(x), T(y)) \leq \lambda d(x,y) \) for all \( x,y \in X \). Also suppose that \( u \in X, \delta > 0, \) and \( d(u, T(u)) \leq \delta \). Then there exists a unique \( p \in X \) such that \( p = T(p) \). Moreover,

\[
d(u, p) \leq \frac{\delta}{1 - \lambda}.
\]

In the rest of the paper, we shall use the following operator

\[
\Lambda(X_t) = \xi_t + \int_0^t A(t,s,X_s)ds + \int_0^t B(t,s,X_s)dW_s.
\]

3. Ulam-Hyers-Rassias stability on a finite interval

In this section, we show that Equation (1) on the finite interval \( I = [0,T] \), under some conditions given in [10], has Ulam-Hyers-Rassias property. Furthermore, this equation also has a unique solution.
THEOREM 2. (Ulam-Hyers stability) We suppose that the following assumptions are satisfied:

a) $\xi_t \in C_b$;

b) \[ \begin{cases} |A(t,s,X_s)| \leq K(1+|X_s|), \forall 0 \leq s \leq t < T, a.s; \\
|B(t,s,X_s)| \leq K(1+|X_s|), \forall 0 \leq s \leq t < T, a.s; \end{cases} \]

c) \[ \begin{cases} |A(t,s,X_s) - A(t,s,Y_s)| \leq \alpha_1 |X_s - Y_s|, \forall 0 \leq s \leq t < T, a.s; \\
|B(t,s,X_s) - B(t,s,Y_s)| \leq \alpha_2 |X_s - Y_s|, \forall 0 \leq s \leq t < T, a.s; \end{cases} \]

d) $(\alpha_1 T + \alpha_2 \sqrt{T}) < 1$.

Then:

i) Equation (1) has a unique solution belonging to the space $C_b$.

ii) Equation (1) has the Ulam-Hyers stability.

Proof. For all $X_t \in C_b$, using the triangle inequality, the estimation $\| \int_0^t \cdot \|_2 ds \leq \int_0^t \| \| ds$ and Ito isometry, we get

$$
\| \Lambda(X_t) \|_2 \leq \| \xi_t \|_2 + \| \int_0^t A(t,s,X_s)ds \|_2 + \| \int_0^t B(t,s,X_s)dW_s \|_2
$$

\[ \leq \| \xi_t \|_2 + \int_0^t \| A(t,s,X_s) \|_2 ds + \sqrt{\int_0^t \| B(t,s,X_s) \|_2^2 ds}
\]

\[ \leq \| \xi_t \|_2 + \int_0^t K(1+\| X_s \|_2) ds + \sqrt{\int_0^t K^2 (1+\| X_s \|_2^2) ds}
\]

\[ \leq \| \xi_t \|_{C_b} + K(T + \sqrt{T})(1 + \| X_t \|_{C_b}),
\]

which implies that $\| \Lambda(X_t) \|_{C_b} \leq \| \xi_t \|_{C_b} + K(T + \sqrt{T})(1 + \| X_t \|_{C_b})$. Hence, $\Lambda(C_b) \subset C_b$.

Furthermore, we have

$$
\| \Lambda(X_t) - \Lambda(Y_t) \|_2 \leq \| \int_0^t A(t,s,X_s) - A(t,s,Y_s)ds \|_2 + \| \int_0^t B(t,s,X_s) - B(t,s,Y_s)dW_s \|_2
$$

\[ \leq \int_0^t \| A(t,s,X_s) - A(t,s,Y_s) \|_2 ds + \sqrt{\int_0^t \| B(t,s,X_s) - B(t,s,Y_s) \|_2^2 ds}
\]

\[ \leq \int_0^t \alpha_1 \| X_s - Y_s \|_2 ds + \sqrt{\int_0^t \alpha_2^2 \| X_s - Y_s \|_2^2 ds}
\]

\[ \leq (\alpha_1 T + \alpha_2 \sqrt{T}) \| X_t - Y_t \|_{C_b},
\]

which implies that $\| \Lambda(X_t) - \Lambda(X_t) \|_{C_b} \leq (\alpha_1 T + \alpha_2 \sqrt{T}) \| X_t - Y_t \|_{C_b}$. By assumption d), the mapping $\Lambda$ is strictly contractive. Thus, by the Banach’s fixed point principle, Equation (1) has a unique solution $U_t \in C_b$.

Let $X_t \in C_b$ be a solution of Inequation (2). It means that $\| X_t - \Lambda(X_t) \|_2 \leq \varepsilon, \forall t \in [0,T]$, from which we get $\| X_t - \Lambda(X_t) \|_{C_b} \leq \varepsilon$. By the estimate (4) in Theorem 1, we obtain

$$
\| X_t - U_t \|_{C_b} \leq \frac{\varepsilon}{1 - M_1}, \quad (5)
$$
where $M_1 = \alpha_1 T + \alpha_2 \sqrt{T}$. On the other hand, we have

$$\|X_t - U_t\|_2 \leq \|X_t - U_t\|_{C_b}, \forall t \in [0, T]. \quad (6)$$

Thus, $\|X_t - U_t\|_2 \leq \frac{\varepsilon}{1 - M_1}$, which implies that Equation (1) is stable in the sense of Ulam-Hyers and completes the proof.

**Theorem 3.** (Ulam-Hyers-Rassias stability) We suppose that the following assumptions are satisfied:

a) $\xi_t \in L^2_{ad}([0, T], \Omega)$;

b) $\left\{ \begin{array}{l}
|A(t, s, X_s)| \leq K(1 + |X_s|), \forall 0 \leq s \leq t, a.s; \\
|B(t, s, X_s)| \leq K(1 + |X_s|), \forall 0 \leq s \leq t, a.s; 
\end{array} \right.$

c) $\left\{ \begin{array}{l}
|A(t, s, X_s) - A(t, s, Y_s)| \leq \alpha_1 |X_s - Y_s|, \forall 0 \leq s \leq t, a.s; \\
|B(t, s, X_s) - B(t, s, Y_s)| \leq \alpha_2 |X_s - Y_s|, \forall 0 \leq s \leq t, a.s; 
\end{array} \right.$

d) The function $u(t)$ is positive and there exists a constant $N_u > 0$ such that

$$\int_0^t u^2(s)ds \leq N_u u^2(t), \forall t \in [0, T];$$

e) $\sqrt{2(T\alpha_1^2 + \alpha_2^2)N_u} < 1$.

Then:

i) Equation (1) has a unique solution belonging to the space $L^2_{ad}([0, T], \Omega)$.

ii) Equation (1) has the Ulam-Hyers-Rassias stability with respect to $u(t)$.

**Proof.**

For all $X_t, Y_t \in L^2_{ad}([0, T], \Omega)$, we set

$$d_u(X_t, Y_t) = \sup_{t \in [0, T]} \frac{\|X_t - Y_t\|_2}{u(t)} < \infty.$$  

Notice that $\Lambda(L^2_{ad}([0, T], \Omega)) \subset L^2_{ad}([0, T], \Omega)$ and $(L^2_{ad}([0, T], \Omega), d_u)$ is a complete metric space.

We assert that $\Lambda$ is strictly contractive on $L^2_{ad}([0, T], \Omega)$. Given any $X_t, Y_t \in L^2_{ad}([0, T], \Omega)$, let $M_{X_t, Y_t} \in [0, \infty)$ be an arbitrary constant such that $d_u(X_t, Y_t) \leq M_{X_t, Y_t}$, from which we deduce that

$$\|X_t - Y_t\|_2 \leq M_{X_t, Y_t} u(t), \forall t \in [0, T]. \quad (7)$$
Using the inequality \( \| x + y \|_2^2 \leq 2 (\| x \|_2^2 + \| y \|_2^2) \), Schwarz inequality and Ito isometry, we have the following estimates:

\[
\| \Lambda(X_t) - \Lambda(Y_t) \|_2^2 \leq 2 \left( \| \int_0^t A(t, s, X_s) - A(t, s, Y_s) \|_2^2 ds + \| \int_0^t B(t, s, X_s) - B(t, s, Y_s) dW_s \|_2^2 \right)
\leq 2 \left( T \int_0^t \| A(t, s, X_s) - A(t, s, Y_s) \|_2^2 ds + \int_0^t \| B(t, s, X_s) - B(t, s, Y_s) \|_2^2 ds \right)
\leq 2 \left( T \int_0^t \alpha_1^2 \| X_s - Y_s \|_2^2 ds + \int_0^t \alpha_2^2 \| X_s - Y_s \|_2^2 ds \right)
\leq 2 (T \alpha_1^2 + \alpha_2^2) \int_0^t \| X_s - Y_s \|_2^2 ds.
\]

Therefore,

\[
\| \Lambda(X_t) - \Lambda(Y_t) \|_2^2 \leq 2 (T \alpha_1^2 + \alpha_2^2) \int_0^t M^2_{X_t, Y_t} u(s)^2 ds
\leq 2 (T \alpha_1^2 + \alpha_2^2) M_u^2 u(t).
\]

Hence,

\[
\| \Lambda(X_t) - \Lambda(Y_t) \|_2 \leq M_2 M_{X_t, Y_t} u(t),
\]

where \( M_2 = \sqrt{2 (T \alpha_1^2 + \alpha_2^2) N_u} \). It implies that \( d_u(\Lambda(X_t), \Lambda(Y_t)) \leq M_2 M_{X_t, Y_t} \). We may conclude that \( d_u(\Lambda(X_t), \Lambda(Y_t)) \leq M_2 d_u(X_t, Y_t) \) for any \( X_t, Y_t \in L^2_{ad}([0, T], \Omega) \). By assumption e), the mapping \( \Lambda \) is strictly contractive on the metric space \( L^2_{ad}([0, T], \Omega), d_u \). Thus, by the Banach’s fixed point principle, Equation (1) has a unique solution.

Let \( X_t \) be a solution of Inequation (3) and let \( U_t \) be the solution of Equation (1). From \( \| X_t - \Lambda(X_t) \|_2 \leq u(t), \forall t \in [0, T] \), we get \( d_u(X_t, \Lambda(X_t)) \leq 1 \). By the triangle inequality, we have

\[
d_u(X_t, U_t) \leq d_u(X_t, \Lambda(X_t)) + d_u(\Lambda(X_t), U_t)
\leq d_u(X_t, \Lambda(X_t)) + d_u(\Lambda(X_t), \Lambda(U_t))
\leq 1 + M_2 d_u(X_t, U_t),
\]

which implies that

\[
d_u(X_t, U_t) \leq \frac{1}{1 - M_2}.
\]

Hence,

\[
\| X_t - U_t \|_2 \leq M_u u(t),
\]

where \( M_u = \frac{1}{1 - M_2} \). It means that Equation (1) has the Ulam-Hyers-Rassias stability. The proof of the theorem thus is complete.
4. Ulam-Hyers-Rassias stability on an infinite interval

In this section, we investigate the stability of Equation (1) on the infinite interval \( I = [0, \infty) \) making use of some results given in the paper [20]. In the first two theorems, we use the triangle inequality, the estimation \( \| \int_0^t ds \|_2 \leq \int_0^t \cdot \| ds \) and Ito isometry in order to evaluate the \( L_2 \)-norm of \( \Lambda(X_t) - \Lambda(Y_t) \). In the last two theorems, by using Remark 3, we quickly obtain estimations for \( \| \Lambda(X_t) - \Lambda(Y_t) \|_{C_b} \) and \( \| \Lambda(X_t) - \Lambda(Y_t) \|_{C_w} \).

**THEOREM 4. (Ulam-Hyers stability)** We suppose that the following assumptions are satisfied:

a) \( \sup_{t \geq 0} \int_0^t (\gamma(t,s) + \gamma^2(t,s)) ds < \infty \);

b) \( \xi_t \in C_b \);

c) \[
\begin{align*}
|A(t,s,X_s)| &\leq \gamma(t,s)|X_s|, \forall 0 \leq s \leq t, a.s; \\
|B(t,s,X_s)| &\leq \gamma(t,s)|X_s|, \forall 0 \leq s \leq t, a.s; 
\end{align*}
\]

d) \[
\begin{align*}
|A(t,s,X_s) - A(t,s,Y_s)| &\leq \alpha_1 \gamma(t,s)|X_s - Y_s|, \forall 0 \leq s \leq t, a.s; \\
|B(t,s,X_s) - B(t,s,Y_s)| &\leq \alpha_2 \gamma(t,s)|X_s - Y_s|, \forall 0 \leq s \leq t, a.s; 
\end{align*}
\]

e) \( \sup_{t \geq 0} \left( \alpha_1 \int_0^t \gamma(t,s) ds + \alpha_2 \sqrt{\int_0^t \gamma^2(t,s) ds} \right) < 1 \).

Then:

i) Equation (1) has a unique solution belonging to the space \( C_b \).

ii) Equation (1) has the Ulam-Hyers stability.

**Proof.**

For all \( X_t \in C_b \), we have

\[
\| \Lambda(X_t) \|_2 \leq \| \xi_t \|_2 + \int_0^t |A(t,s,X_s)| ds \|_2 + \int_0^t B(t,s,X_s)dW_s \|_2 \\
\leq \| \xi_t \|_2 + \int_0^t |A(t,s,X_s)| ds \|_2 + \sqrt{\int_0^t |B(t,s,X_s)|^2 ds} \\
\leq \| \xi_t \|_2 + \int_0^t \gamma(t,s) \|X_s\|_2 ds + \sqrt{\int_0^t \gamma^2(t,s) \|X_s\|_2 ds} \\
\leq \| \xi_t \|_{C_b} + \|X_s\|_{C_b} \sup_{t \geq 0} \left( \int_0^t \gamma(t,s) ds + \sqrt{\int_0^t \gamma^2(t,s) ds} \right). 
\]

Hence,

\[
\| \Lambda(X_t) \|_{C_b} \leq \| \xi_t \|_{C_b} + \|X_s\|_{C_b} \sup_{t \geq 0} \left( \int_0^t \gamma(t,s) ds + \sqrt{\int_0^t \gamma^2(t,s) ds} \right),
\]

which implies that \( \Lambda(C_b) \subset C_b \).
As in Theorem 2, we have

$$\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \leq$$

$$\leq \int_0^t \|A(t,s,X_s) - A(t,s,Y_s)\|_2 ds + \sqrt{\int_0^t \|B(t,s,X_s) - B(t,s,Y_s)\|_2^2 ds}$$

$$\leq \int_0^t \alpha_1 \gamma(t,s)\|X_s - Y_s\|_2 ds + \sqrt{\int_0^t \alpha_2^2 \gamma^2(t,s)\|X_s - Y_s\|_2^2 ds}$$

$$\leq \sup_{t \geq 0} \left( \alpha_1 \int_0^t \gamma(t,s) ds + \alpha_2 \sqrt{\int_0^t \gamma^2(t,s) ds} \right) \|X_s - Y_s\|_{C_b}.$$  

Hence,

$$\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_b} \leq \sup_{t \geq 0} \left( \alpha_1 \int_0^t \gamma(t,s) ds + \alpha_2 \sqrt{\int_0^t \gamma^2(t,s) ds} \right) \|X_s - Y_s\|_{C_b}.$$  

By assumption e), $\Lambda$ is a contraction. Therefore, there exists unique solution $U(t) \in C_b$ of Equation (1) such that $\Lambda(U_t) = U_t$, $t \geq 0$.

We assume that $X_t$ is a solution of Inequation (2). We have $\|X_t - \Lambda(X_t)\|_2 \leq \varepsilon$, $\forall t \geq 0$, which implies that $\|X_t - \Lambda(X_t)\|_{C_b} \leq \varepsilon$. By the estimate (4) in Theorem 1, we obtain

$$\|X_t - U_t\|_{C_b} \leq \frac{\varepsilon}{1 - M_3}, \quad (12)$$

where $M_3 = \sup_{t \geq 0} \left( \alpha_1 \int_0^t \gamma(t,s) ds + \alpha_2 \sqrt{\int_0^t \gamma^2(t,s) ds} \right)$. Hence, $\|X_t - U_t\|_2 \leq \frac{\varepsilon}{1 - M_3}$ for all $t \geq 0$, which shows that the stochastic integral equation (1) is stable in the sense of Ulam-Hyers and completes the proof.

\textbf{THEOREM 5. (Ulam-Hyers-Rassias stability)} We suppose that the following assumptions are satisfied:

a) $u(t) > 0$ is a continuous function and $\sup_{t \geq 0} \int_0^t (u(s) + u^2(s)) ds < \infty$;

b) $\xi_t \in C_u$;

c) $\begin{cases} |A(t,s,X_s)| \leq u(t) |z(t,\omega) + \gamma(t,s)| X_s|, \forall 0 \leq s \leq t < \infty, a.s; \\
|B(t,s,X_s)| \leq u(t) |z(t,\omega) + \gamma(t,s)| X_s|, \forall 0 \leq s \leq t < \infty, a.s; \end{cases}$

for $0 \leq s \leq t < \infty$, where $z(s,\omega)$ is a second order stochastic process in $C_u$ and $\gamma(t,s)$ is a bounded continuous function defined for $0 \leq s \leq t$.

d) $\begin{cases} |A(t,s,X_s) - A(t,s,Y_s)| \leq \alpha_1 u(t) |X_s - Y_s|, \forall 0 \leq s \leq t < \infty, a.s; \\
|B(t,s,X_s) - B(t,s,Y_s)| \leq \alpha_2 u(t) |X_s - Y_s|, \forall 0 \leq s \leq t < \infty, a.s; \end{cases}$

e) $\sup_{t \geq 0} \left( \alpha_1 \int_0^t u(s) ds + \alpha_2 \sqrt{\int_0^t u^2(s) ds} \right) < 1.$

Then:

i) Equation (1) has a unique solution belonging to the space $C_u$.

ii) Equation (1) has the Ulam-Hyers-Rassias stability with respect to $u(t)$. 
\textbf{Proof.} According to [20], \((C_1,u,C_u)\) is admissible with respect to both the operators \(\Lambda_1\) and \(\Lambda_2\). Condition c) implies that \(A(t,s,X_s)\) and \(B(t,s,X_s)\) are in \(C_{1,u}\) whenever \(X_t \in C_u\). Therefore, \(\Lambda(C_u) \subset C_u\).

We show that if \(X_t, Y_t \in C_u\) then \((A(t,s,X_s) - A(t,s,Y_s))\) and \((B(t,s,X_s) - B(t,s,Y_s))\) belong to \(C_{1,u}\).

From the condition d), we get

\[
\|A(t,s,X_s) - A(t,s,Y_s)\|_2 \leq \alpha_1 u(t)\|X_s - Y_s\|_2 = \alpha_1 \|X_s - Y_s\|_2
\]

Thus, \(A(t,s,X_s) - A(t,s,Y_s) \in C_{1,u}\).

Similarly, we have \(B(t,s,X_s) - B(t,s,Y_s) \in C_{1,u}\).

Hence,

\[
\begin{align*}
\int_0^t A(t,s,X_s) - A(t,s,Y_s) \, ds & \in C_u, \\
\int_0^t B(t,s,X_s) - B(t,s,Y_s) \, dW_s & \in C_u.
\end{align*}
\]

As in Theorem 4, we have the following estimates:

\[
\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \leq \int_0^t \|A(t,s,X_s) - A(t,s,Y_s)\|_2 \, ds + \sqrt{\int_0^t \|B(t,s,X_s) - B(t,s,Y_s)\|^2 \, ds}
\]

\[
\leq \int_0^t \alpha_1 u(t)\|X_s - Y_s\|_2 \, ds + \sqrt{\int_0^t \alpha_2 u^2(t)\|X_s - Y_s\|^2 \, ds}
\]

\[
\leq \alpha_1 u(t) \int_0^t \|X_s - Y_s\|_2 \, ds + \alpha_2 u(t) \sqrt{\int_0^t \|X_s - Y_s\|^2 \, ds}.
\]

Therefore,

\[
\frac{\|\Lambda(X_t) - \Lambda(Y_t)\|_2}{u(t)} \leq \frac{\alpha_1 \int_0^t \|X_s - Y_s\|_2 \, ds}{u(s)} + \alpha_2 \sqrt{\int_0^t \|X_s - Y_s\|^2 \, ds}
\]

\[
\leq \alpha_1 \int_0^t \frac{\|X_s - Y_s\|_2}{u(s)} \, ds + \alpha_2 \sqrt{\int_0^t \frac{\|X_s - Y_s\|^2}{u^2(s)} \, ds}
\]

\[
\leq \sup_{t \geq 0} \left( \alpha_1 \int_0^t u(s) \, ds + \alpha_2 \sqrt{\int_0^t u^2(s) \, ds} \right) \|X_s - Y_s\|_{C_u},
\]

from which we deduce that

\[
\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_u} \leq \sup_{t \geq 0} \left( \alpha_1 \int_0^t u(s) \, ds + \alpha_2 \sqrt{\int_0^t u^2(s) \, ds} \right) \|X_s - Y_s\|_{C_u}.
\]
By assumption e), the mapping $\Lambda$ is strictly contractive. Thus, by the Banach’s fixed point principle, there exists a unique solution (say) $U_t$ in $C_u$ of Equation (1).

Let $X_t \in C_u$ be a solution of Inequation (3). We have

$$\|X_t - \Lambda(X_t)\|_2 \leq u(t),$$

from which, we deduce the following inequality $\|X_t - \Lambda(X_t)\|_{C_u} \leq 1$.

By the triangle inequality, we get:

$$\|X_t - U_t\|_{C_u} \leq \|X_t - \Lambda(X_t)\|_{C_u} + \|\Lambda(X_t) - \Lambda(U_t)\|_{C_u} \leq 1 + M_4 \|X_t - Y_t\|_{C_u},$$

where $M_4 = \sup_{t \geq 0} \left( \alpha_1 \int_0^t u(s) ds + \alpha_2 \sqrt{\int_0^t u^2(s) ds} \right)$. Therefore,

$$\|X_t - U_t\|_{C_u} \leq \frac{1}{1 - M_4}. \quad (14)$$

Thus, $\|X_t - U_t\|_2 \leq \frac{1}{1 - M_4} u(t), \forall t \geq 0$, which implies that Equation (1) has the Ulam-Hyers-Rasiasst stability with respect to $u(t)$. This ends the proof.

**Remark 4.** Theorem 2 is a consequence of Theorem 5.

In the next two theorems, we keep the assumptions in Theorem 4 and Theorem 5. Remark 3 will be used to evaluate $\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_b}$ and $\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_u}$.

**Theorem 6.** (Ulam-Hyers stability) Suppose that the assumptions a), b), c) and d) in Theorem 4 together with the following assumption are satisfied:

e) $(K_1 \alpha_1 + K_2 \alpha_2) \sup_{0 \leq s \leq t < \infty} \gamma(t, s) < 1$, where $K_1, K_2$ are the constants in Remark 3.

Then:

i) Equation (1) has a unique solution belonging to the space $C_b$.

ii) Equation (1) has the Ulam-Hyers stability.

**Proof.**
With $X_t \in C_b$, we get the following estimates:

$$\left\| \int_0^t A(t, s, X_s) ds \right\|_2 \leq \int_0^t \|A(t, s, X_s)\|_2 ds \leq \int_0^t \gamma(t, s) \|X_s\|_2 ds \leq \|X_t\|_{C_b} \sup_{t \geq 0} \int_0^t \gamma(t, s) ds < \infty,$$
and
\[ \| \int_0^t B(t,s,X_s) dW_s \|_2^2 = \int_0^t \| B(t,s,X_s) \|_2^2 ds \]
\[ \leq \int_0^t \gamma^2(t,s) \| X_s \|_2^2 ds \]
\[ \leq \| X_0 \|_{C_b}^2 \sup_{t \geq 0} \int_0^t \gamma^2(t,s) ds < \infty. \]

Hence, \( \int_0^t A(t,s,X_s) ds \in C_b, \int_0^t B(t,s,X_s) dW_s \in C_b. \)

As in Theorem 4, we have \( \Lambda(C_b) \in C_b. \)

From
\[ |\Lambda(X_t) - \Lambda(Y_t)| \leq | \int_0^t A(t,s,X_s) - A(t,s,Y_s) ds | \]
\[ + | \int_0^t B(t,s,X_s) - B(t,s,Y_s) dW_s |, \]
we get that
\[ \| \Lambda(X_t) - \Lambda(Y_t) \|_{C_b} \leq \]
\[ \leq \| \int_0^t A(t,s,X_s) - A(t,s,Y_s) ds \|_{C_b} + \| \int_0^t B(t,s,X_s) - B(t,s,Y_s) dW_s \|_{C_b} \]
\[ \leq K_1 \| A(t,s,X_s) - A(t,s,Y_s) \|_{C_{1,b}} + K_2 \| B(t,s,X_s) - B(t,s,Y_s) \|_{C_{1,b}}. \]

We also have
\[ | A(t,s,X_s) - A(t,s,Y_s) | \leq \alpha_1 \gamma(t,s) \| X_s - Y_s \|, \forall 0 \leq s \leq t, \]
then
\[ \| A(t,s,X_s) - A(t,s,Y_s) \|_{C_{1,b}} \leq \alpha_1 \sup_{0 \leq s \leq t < \infty} \gamma(t,s) \| X_s - Y_s \|_{C_b}. \]

Similarly, we obtain
\[ \| B(t,s,X_s) - B(t,s,Y_s) \|_{C_{1,b}} \leq \alpha_2 \sup_{0 \leq s \leq t < \infty} \gamma(t,s) \| X_s - Y_s \|_{C_b}. \]

Therefore,
\[ \| \Lambda(X_t) - \Lambda(Y_t) \|_{C_b} \leq (K_1 \alpha_1 + K_2 \alpha_2) \sup_{0 \leq s \leq t < \infty} \gamma(t,s) \| X_s - Y_s \|_{C_b}. \] (15)

According to Theorem 1, with \( U_t \) is the solution of Equation (1) and \( X_t \) is a solution of Inequation (3), we have the following estimate \( \| X_t - U_t \|_2 \leq \frac{\varepsilon}{1 - M_5}, \) where \( M_5 = (K_1 \alpha_1 + K_2 \alpha_2) \sup_{0 \leq s \leq t < \infty} \gamma(t,s), \) which implies that Equation (1) has the Ulam-Hyers stability. This completes the proof.
THEOREM 7. *(Ulam-Hyers-Rassias stability)* Suppose that the assumptions a), b), c) and d) in Theorem 5 together with the following assumption are satisfied:
e) $K_1 \alpha_1 + K_2 \alpha_2 < 1$, where $K_1, K_2$ are the constants in Remark 3.

Then:
i) Equation (1) has a unique solution belonging to the space $C_u$.

ii) Equation (1) has the Ulam-Hyers-Rassias stability with respect to $u(t)$.

**Proof.** We have

$$||\Lambda(X_t) - \Lambda(Y_t)||_{C_u} \leq \frac{||\int_0^t A(t,s,X_s) - A(t,s,Y_s) ds||_{C_u} + ||\int_0^t B(t,s,X_s) - B(t,s,Y_s) dW_s||_{C_u}}{u(t)u(s)} \leq K_1||A(t,s,X_s) - A(t,s,Y_s)||_{C_{1,u}} + K_2||B(t,s,X_s) - B(t,s,Y_s)||_{C_{1,u}}.$$ 

Thus,

$$\frac{||A(t,s,X_s) - A(t,s,Y_s)||_{2}}{u(t)u(s)} \leq \alpha_1 \frac{||X_s - Y_s||_{2}}{u(s)} \leq \alpha_1 ||X_s - Y_s||_{C_u}.$$ 

Therefore,

$$||A(t,s,X_s) - A(t,s,Y_s)||_{C_{1,u}} \leq \alpha_1 ||X_t - Y_t||_{C_u}.$$ 

Similarly, we have

$$||B(t,s,X_s) - B(t,s,Y_s)||_{C_{1,u}} \leq \alpha_2 ||X_t - Y_t||_{C_u}.$$ 

We get the following estimate

$$||\Lambda(X_t) - \Lambda(Y_t)||_{C_u} \leq (K_1 \alpha_1 + K_2 \alpha_2)||X_t - Y_t||_{C_u}. \quad (16)$$

By assumption e), the mapping $\Lambda$ is strictly contractive. Thus, according to the Banach’s fixed point principle, Equation (1) has a unique solution $U_t \in C_u$.

Using the estimate $||X_t - \Lambda(X_t)||_{C_u} \leq 1$ and the triangle inequality, we get that

$$||X_t - U_t||_{C_u} \leq \frac{1}{1 - M_6}, \quad (17)$$

where $X_t$ is a solution of Inequality (3) and $M_6 = K_1 \alpha_1 + K_2 \alpha_2$.

Thus, $||X_t - U_t||_2 \leq \frac{1}{1 - M_6} u(t)$, which implies that Equation (1) has the Ulam-Hyers-Rasiass stability with respect to $u(t)$.

5. Examples

In this section, we consider Section 3 with the case $T = 1$. Remark that $u(t) = t, \ t \in [0, 1]$, is a function satisfying the condition d) in Theorem 3 with $N_u = \frac{1}{3}$.

Consider the following stochastic integral equation

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s, \quad (18)$$

where $\mu$ and $\sigma$ are constants. Here, $\xi$ and the functions $A, B$ are given by

$$\xi = X_0, \ A(t,s,x) = \mu x, \ B(t,s,x) = \sigma x,$$
The functions $A$ and $B$ satisfy Lipschitz condition in $x$ with Lipschitz constants $\mu$ and $\sigma$, respectively. In the case $\mu + \sigma < 1$, all the hypotheses of Theorem 2 are satisfied. Hence, Equation (18) has Ulam-Hyers stability and its solution is a geometry Brownian motion given by

$$X_t = X_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).$$

We continue considering the Langevin equation (see Example 10.1.1. in [15])

$$X_t = X_0 - \int_0^t \alpha X_s ds + \int_0^t \beta dW_s,$$

where $\alpha, \beta$ are constants.

In the case $T = 1$ and $u(t) = t$, the condition $e'$ in Theorem 3 is equivalent to $\alpha_1^2 + \alpha_2^2 < \frac{3}{2}$. It is evident that the functions $A = -\alpha x$ and $B = \beta$ satisfy Lipschitz condition in $x$ with Lipschitz constant $|\alpha|$. Hence, with $|\alpha| < \sqrt{\frac{3}{2}}$, all the assumptions of Theorem 3 are satisfied. Thus, Equation (19) has Ulam-Hyers-Rassias stability with respect to $u(t) = t$ and its solution is an Ornstein-Uhlenbeck process given by

$$X_t = e^{-\alpha t} x_0 + \beta \int_0^t e^{-\alpha (t-s)} dW_s.$$

Acknowledgement. The authors express their sincere gratitude to the editors and anonymous referees for the careful reading of the original manuscript and useful comments which have led to a significant improvement to our original manuscript.

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(Received June 7, 2018)

Ngo Phuoc Nguyen Ngoc
Hue university of Sciences
Department of Mathematics
77 Nguyen Hue street, Hue city, Vietnam
e-mail: ngochvn@gmail.com

Nguyen Van Vinh
Hue University of Education
Department of Mathematics
32 Le Loi street, Hue city, Vietnam
e-mail: vinhnguyen0109@gmail.com