WAVE EQUATION AND MULTIPLIER ESTIMATES ON $ax + b$ GROUPS

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Abstract. Let $L$ be the distinguished Laplacian on certain semidirect products of $\mathbb{R}$ by $\mathbb{R}^n$ which are of $ax + b$-type. We prove pointwise estimates for the convolution kernels of spectrally localized wave operators of the form

$$e^{it\sqrt{L}}\psi(\sqrt{L}/\lambda)$$

for arbitrary time $t$ and arbitrary $\lambda > 0$, where $\psi$ is a smooth bump function supported in $[-2, 2]$ if $\lambda \leq 1$ and supported in $[1, 2]$ if $\lambda \geq 1$. As corollary, we reprove a basic multiplier estimate from [5] for this particular class of groups, and derive Sobolev estimates for solutions to the wave equation associated to $L$. There appears no dispersive effect with respect to the $L^\infty$-norms for large times in our estimates, so that it seems unlikely that non-trivial Strichartz type estimates hold.

1. Introduction

We denote by $G$ the semi-direct product $G = \mathbb{R} \ltimes \mathbb{R}^n$, endowed with the group law

$$(x, y)(x', y') = (x + x', y + e^x y').$$

This subgroup of the affine group of $\mathbb{R}^n$ is a solvable Lie group with exponential volume growth. We call $G$ an $ax + b$ group.

A basis for the Lie algebra of left-invariant vector fields is given by

$$X = \partial_x, \quad Y_1 = e^x \partial_{y_1}, \ldots, \quad Y_n = e^x \partial_{y_n}. \quad (1.1)$$

We define the distinguished left-invariant Laplacian as the second order differential operator

$$L = -X^2 - \sum_{j=1}^n Y_j^2. \quad (1.2)$$

A right-invariant Haar measure on $G$ is given by

$$dg = dx dy_1 \ldots dy_n.$$
We will use this right-invariant measure in notions such as $L^p(G)$. An operator on function spaces on $G$ is given by a right convolution kernel $k$ if

$$(1.3) \quad Tf(g') = f * k(g') = \int f(g^{-1}) k(gg') \, dg, \quad \text{for } f \in D(G).$$

The distinguished Laplacian $L$ has a self-adjoint extension in $L^2(G)$ ([3]), and thus we can use spectral calculus to define the operators

$$(1.4) \quad e^{it\sqrt{L}} m(L),$$

where the multiplier $m$ will lie in a suitable symbol class. The main purpose of this article is to prove Theorem 6.1 which states uniform (w.r. to $\lambda$ and $t$) pointwise estimates for the convolution kernels of spectrally localized multiplier operators of the form

$$(1.5) \quad e^{it\sqrt{L}} \psi(\sqrt{L}/\lambda),$$

for arbitrary time $t \in \mathbb{R}$ and arbitrary $\lambda > 0$, where $\psi$ is a bump function supported in $[-2, 2]$, if $\lambda \leq 1$, and in $[1, 2]$, if $\lambda \geq 1$. We can use these estimates to give a new proof of the basic multiplier estimate used in [5] (see Theorem 6.1 of [5]), which is based entirely on the wave equation, at least for the class of $ax + b$ groups under consideration.

As a corollary of our main theorem, we shall also deduce Sobolev estimates for solutions to the wave equation associated to $L$.

Our estimates are mainly controlled by the left-invariant Riemannian distance

$$(1.6) \quad R = R(x, y) := \text{arch}(\text{ch}(x) + \frac{1}{2} \|y\|^2 e^{-x}),$$

of a point $(x, y)$ to the identity element on $G$, where $y = (y_1, \ldots, y_n)$, $\|y\|^2 = \sum_{j=1}^n y_j^2$ and arch is understood to take $[1, \infty)$ to $[0, \infty)$.

We remark that, for $n = 2$, our main theorem could also be deduced from a transfer principle of Hebisch [6]. In that special situation, the group $G$ is of the form $AN$ where $KAN$ is the Iwasawa decomposition of the complex Lie group $SL_2(\mathbb{C})$, and Hebisch introduces a mapping from radial functions $f$ on $\mathbb{R}^3$ to functions on the group $G$, given by

$$Tf(x, y) = Ce^{-x} \frac{R}{\text{sh}(R)} f(R),$$

which preserves the $L^1$ norm and commutes with convolution and with application of the corresponding Laplacians. This allows to deduce our main Theorem 6.1 from the analogous theorem on $\mathbb{R}^3$. However, this transfer principle is somewhat misleading, since it would suggest for higher dimensions estimates different from the ones which actually hold on $G$.

We also remark that our results should extend to the distinguished Laplacians which arise from the Laplace-Beltrami operators on rank one symmetric spaces of the non-compact type by means of conjugation with the square root of the modular function (see e.g. [2]), by means of refinements of the estimates for spherical functions in [9].
However, we shall not pursue this here, since we prefer to present the completely self-contained proof which we can give for the affine group.

In Section 2, we prove a lemma which describes integration of radial functions over the affine groups. In Section 3 we derive an explicit kernel representation for the resolvent of $L$ using the theory of hypergeometric functions. It is known ([8], [4]) that the resolvent kernels can be expressed in terms of special functions, and this has been used in [3] to prove estimates for singular integrals related to $L$. We chose to derive the integral formula for the resolvent kernel from scratch, even though this task could have been done quoting tables of special functions such as [2]. We hope some readers will find benefit of our explicit calculations.

Section 4 presents a subordination argument to obtain convenient integral representations for the convolution kernels of the operators $\langle L \rangle$.

In Section 5 we prove some asymptotic formulas for the hypergeometric functions appearing in Section 3.

Section 6 assembles the results of the previous sections to prove Theorem 6.1 which states pointwise estimates for the convolution kernel of $\langle L \rangle$, and Proposition 6.3 which states $L^1(G)$ estimates for these kernels. We also reprove a multiplier theorem of [5].

In Section 8, we prove growth estimates for the wave propagator associated with $L$ using spectrally defined Sobolev norms.

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2. Integration of radial functions on the affine group

In this section we discuss integration of “radial” functions. The results will be useful in the estimation of $L^1$-norms of convolution kernels for functions of the Laplacian $L$.

Bending the notion of radial function, by radial function we mean a function of the type

$$e^{-nx/2}g(R)$$

with $R$ as in (1.6).

We briefly motivate the special form of the radial variable $R$. For $n = 1$, the affine group $G$ is a subgroup of the group of conformal automorphisms of the upper half plane via the identification of $(x, y)$ with the map $z \to e^x z + y$. This subgroup acts simply transitively on the upper half plane, thus we can naturally identify $G$ as a set with the upper half plane, identifying the neutral element with the point $i$. In particular, the hyperbolic metric on the upper half plane turns out left-invariant. The pull back of the hyperbolic distance from a point $z$ to the point $i$ (which is log $|(1 + \rho)/(1 - \rho)|$ with $\rho = |z - i|/|z + i|$) gives a natural “radial” distance from the origin in the affine group given by

$$\text{arcch}(ch x + \frac{1}{2} y^2 e^{-x}).$$
Since \( R(x, y) = R((x, y)^{-1}) \) there is no difference between a left and a right radial variable.

**Lemma 2.1.** Given a function \( g : \mathbb{R}^+ \rightarrow \mathbb{C} \), then

\[
\int_G e^{-nx/2} g(R(x, y)) \, dx \, dy = \int_0^\infty g(R)J(R) \, dR,
\]

where

\[
(2.1) \quad J(R) \sim R^n, \quad \text{if } R \leq 1,
\]

\[
(2.2) \quad J(R) \sim Re^{nR/2}, \quad \text{if } R \geq 1,
\]

and \( a \sim b \) means that each of the two numbers can be bounded by a constant times the other, the constant depending only on \( n \).

**Proof.** Define

\[
B(r) = \int_{R(x,y) \leq r} e^{-nx/2} \, dx \, dy.
\]

Then we have for all \( r \geq 0 \)

\[ J(r) = B'(r). \]

Thus we have to estimate \( B'(r) \). Observe that \( R(x,y) \leq r \) implies \( x \leq r \) and

\[ \|y\| \leq (2e^x(ch(r) - ch(x)))^{1/2}. \]

Doing the \( y \)-integration and letting \( V_n \) be the Euclidean volume of the unit ball in \( \mathbb{R}^n \), we obtain

\[
B(r) = V_n \int_{-r}^r e^{-nx/2}(2e^x(ch(r) - ch(x)))^{n/2} \, dx.
\]

Simplification and differentiation gives

\[
B'(r) = \frac{n}{2} 2^{n/2} V_n \int_{-r}^r (ch(r) - ch(x))^{n/2-1} \, sh(r) \, dx.
\]

First assume \( r \geq 1 \). We break the integral in the previous display into the sum of

\[
(2.3) \quad I_1 = sh(r) \int_{|x| < r-1} (ch(r) - ch(x))^{n/2-1} \, dx
\]

and

\[
(2.4) \quad I_2 = sh(r) \int_{r-1 \leq |x| \leq r} (ch(r) - ch(x))^{n/2-1} \, dx.
\]

In the first integral, we have \( |x| < r - 1 \) and thus

\[ chR - ch(x) \sim e^x, \quad sh(r) \sim e^r. \]

Therefore

\[ I_1 \sim re^{nr/2}. \]
Thus $I_1$ has already the correct order of magnitude which we need to show for $I_1 + I_2$. Since $I_1$ and $I_2$ are positive, we only need an upper bound for $I_2$. Since in the domain of integration of $I_2$ we still have
\[ \text{ch}(r) - \text{ch}(x) \leq 2e^{r}, \quad \text{sh}(r) \leq e^{r}, \]
we can do the same calculation as before to obtain
\[ I_2 \leq C e^{r n/2}. \]

Now assume $r < 1$. We do a similar splitting of the integral as before, now into the regions $|x| < r/2$ and $r/2 \leq |x| \leq r$. Call the corresponding integrals $I_1$ and $I_2$. In the domain $|x| < r/2$ we have
\[ \text{ch}(r) - \text{ch}(x) \sim r^2, \quad \text{sh}(r) \sim r, \quad e^{x} \sim 1. \]
Hence
\[ I_1 \sim r^{n-1} \int_0^{r/2} dx \sim r^n. \]
Similarly as before, in the domain $r/2 \leq |x| \leq r$ we have the same upper bounds as in the domain $|x| < r/2$, and thus obtain $I_2 \leq Cr^n$. This completes the proof of Lemma 2.1.

3. An explicit kernel for the resolvent of $L$

Assume that $k$ is an integrable function on $G$ such that for every compactly supported smooth function $\varphi$ we have (in the distributional sense, for this purpose we identify $G$ with $\mathbb{R}^{n+1}$)

\begin{equation}
\int \varphi(g)(L - \lambda)k(g) \, dg = \varphi(0). \tag{3.1}
\end{equation}

Then, by left invariance of $L$ and right invariance of $dg$,
\[ (L - \lambda) \int \varphi(g^{-1})k(gg') \, dg = \int \varphi(g^{-1})[(L - \lambda)k](gg') \, dg \]
\[ = \int \varphi(g'g^{-1})[(L - \lambda)k](g) \, dg = \varphi(g'). \]
Thus the resolvent operator $(L - \lambda)^{-1}$ is given by right convolution with $k$, which extends to a bounded operator on $L^2(G)$. The following lemma describes such a fundamental solution $k$.

**Lemma 3.1.** Assume $\lambda \in \mathbb{C} \setminus [0, \infty)$ and choose $\nu := i\sqrt{\lambda}$ with strictly negative real part. Then the resolvent operator $(L - \lambda)^{-1}$ is given by right convolution with the kernel $k$ defined by $(R = R(x, y)$ as in (1.6)):

\[ k(x, y) = (-1)^l \frac{2^{-1-n/2} \pi^{-n/2}}{\Gamma(1 - n/2 + l)} e^{-nx/2} \int_0^\infty D_{\text{sh}, v}^l [e^{\nu v}](\text{ch}v - \text{ch}R)^{-n/2+l} \, dv, \]
where $l$ is any integer with $-n/2 + l > -1$ and we have written $D_{sh,v}^l$ for the $l$-th power of $D_{sh} : g \to D(g/sh)$ acting in the $v$ variable.

Moreover, the kernel $k$ satisfies the estimate

$$
\int_{B_R} |k| \, dx \, dy \leq C_n (1 + |\nu|)^{\frac{n}{2}} \left[ 1 + \int_0^R e^{\Re(\nu) r} \, dr \right],
$$

where $B_R$ is the ball of radius $R$ about the origin in $G$. In particular, $k \in L^1(G)$.

Proof. Fix $\lambda$. For $n > 1$, we will show that $k$ as defined in the lemma is integrable and smooth outside the origin, satisfies $(L - \lambda)k = 0$ outside the origin, and has asymptotics

$$
k(x, y) = 2^{-2} \pi^{-(n+1)/2} \Gamma((n - 1)/2) (x^2 + \|y\|^2)^{-(n-1)/2} + O(x^2 + \|y\|^2)^{-n/2}
$$

near the origin. This will give for any compactly supported $\varphi$

$$
\int \varphi(g)(L - \lambda)k(g) \, dg = \int \eta(g) \varphi(g)(L - \lambda)k(g) \, dg,
$$

where $\eta$ is a smooth cutoff function at scale $\epsilon$, i.e., constant equal to 1 on an $\epsilon$-neighborhood around the origin and zero outside a $2\epsilon$ neighborhood, with the usual control of derivatives. By definition of the distributional derivative, the last display becomes

$$
\int (L - \lambda)(\eta \psi)(g)k(g) \, dg.
$$

If we subtract the leading order term from $k$ in this integral, then the remaining integral tends to zero as $\epsilon \to 0$. Thus we may replace $k$ by the leading order term. Also, we may disregard in the expansion of $(L - \lambda)(\eta \psi)$ by Leibniz’ rule all terms other than those taking two derivatives of $\eta$, and also may approximate the coefficient $e^x$ by 1. Thus the last display is equal to $(\Delta = -\sum \partial^2_j)$

$$
2^{-2} \pi^{-(n+1)/2} \Gamma((n - 1)/2) \int \Delta(\eta \varphi) (x^2 + |y|^2)^{-(n-1)/2} \, dx \, dy.
$$

Now standard theory in $R^n$ ([13] pp. 211, 262) gives that the last display is equal to $\varphi(0)$, which was to be proved. If $n = 1$, we use the same approach, here the asymptotic behaviour of $k$ near the origin is

$$
2^{-2} \pi^{-1} \log(x^2 + \|y\|^2) + O(1).
$$

Now we prove the properties of $k$ claimed above. Define

$$
d(x, y) := \text{ch}(R(x, y)) = \text{ch}(x) + \frac{1}{2} \|y\|^2 e^{-x}.
$$

Then the kernel $k$ is of the form

$$
k(x, y) = e^{-nx/2} f(d(x, y)),
$$

where $l$ is any integer with $-n/2 + l > -1$ and we have written $D_{sh,v}^l$ for the $l$-th power of $D_{sh} : g \to D(g/sh)$ acting in the $v$ variable.
with a function \( f \) which is smooth on \((1, \infty)\). We claim that \((L - \lambda)k = 0\) outside the origin is equivalent to \( f \) satisfying the ordinary differential equation

\[
-\frac{n^2}{4} f(d) - (n + 1)df'(d) - (d^2 - 1)f''(d) = \lambda f(d)
\]

for \( d > 1 \). To verify the claim, we observe

\[
(Xd)^2 + \sum_{j=1}^{n} (Y_j d)^2
\]

\[
= \left( \frac{1}{2} e^x - \frac{1}{2} e^{-x} - \frac{1}{2} ||y||^2 e^{-x} \right)^2 + ||y||^2 = \left( \frac{1}{2} e^x + \frac{1}{2} e^{-x} + \frac{1}{2} ||y||^2 e^{-x} \right)^2 - 1
\]

\[
= d^2 - 1
\]

and

\[
X^2 d + \sum_{j=1}^{n} Y_j^2 d - nXd
\]

\[
= \left( \frac{1}{2} e^x + \frac{1}{2} e^{-x} + \frac{1}{2} ||y||^2 e^{-x} \right) + ne^x - n(\frac{1}{2} e^x - \frac{1}{2} e^{-x} - \frac{1}{2} ||y||^2 e^{-x}) = (n+1) \left( \frac{1}{2} e^x + \frac{1}{2} e^{-x} + \frac{1}{2} ||y||^2 e^{-x} \right)
\]

\[
= (n+1)d.
\]

Therefore we have

\[
(X^2 + \sum_{j=1}^{n} Y_j^2) e^{-nx/2} f(d(x, y)) =
\]

\[
= e^{-nx/2} \left[ (Xd)^2 + (Yd)^2 \right] f''(d) + (X^2d + Y^2d)f'(d) - n(Xd)f'(d) + \frac{n^2}{4} f(d)]
\]

\[
= e^{-nx/2} \left[ \frac{n^2}{4} f(d(x, y)) + (n+1)d(x, y)f'(d(x, y)) + (d(x, y)^2 - 1)f''(d(x, y)) \right].
\]

Thus if \( f \) satisfies the ordinary differential equation (3.4) on \((1, \infty)\), then \((L - \lambda)k = 0\) outside the origin, and conversely. Equation (3.4) is a classical hypergeometric differential equation. The Riemann symbol \( \Pi \) associated to this differential equation is

\[
(3.5) \quad P \left( \begin{array}{ccc} -1 & 1 & \infty \\ 0 & \frac{n}{2} + \nu & d \\ -\frac{n-1}{2} & -\frac{n-1}{2} & -\frac{n}{2} - \nu \end{array} \right).
\]

There is a two dimensional space of solutions \( f \). However, there is only one dimensional space of solutions (those with leading asymptotics \( d^{-n/2+\nu} \) as \( d \to \infty \)), which make \( k \) as defined above integrable on the group \( G \). Of course exactly one of these solutions is normalized properly to make \( k \) a fundamental solution.

The following lemma provides an explicit solution of the differential equation (3.4) in a certain complex region of parameters \( n \) and \( \nu \).
Lemma 3.2. Assume \(-\text{Re}(n)/2 > -1\) and \(\text{Re}(\nu) - \text{Re}(n)/2 < 0\). Then the function
\[
(3.6) \quad f_0(d) = \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2} dv ,
\]
defined for \(d > 1\), satisfies the ordinary differential equation (3.4).

Proof. Under the stated assumptions on \(n\) and \(\nu\), the integral defining \(f_0\) is absolutely integrable. We first assume that \(-\text{Re}(n)/2 > 1\).

Differentiating under the integral sign gives
\[
f_0'(d) = \frac{n}{2} \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-1} dv ,
\]
\[
f_0''(d) = \frac{n}{2} (\frac{n}{2} + 1) \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-2} dv .
\]

Hence,
\[
(n + 1) f_0' = (\frac{n^2}{2} + \frac{n}{2})(-f_0 + \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-2}(\text{ch} v^2 - d\text{ch} v) dv) ,
\]
\[
(d^2 - 1) f_0'' = (\frac{n^2}{4} + \frac{n}{2})(f_0 - \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-2}(\text{ch} v^2 - 2d\text{ch} v + 1) dv) .
\]

It thus remains to prove
\[
\int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-2} \left[ -\frac{n^2}{4} \text{ch} v^2 - \frac{n}{2} d\text{ch} v + \frac{n^2}{4} + \frac{n}{2} \right] dv
\]
\[
= \lambda \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2} dv .
\]

However, by partial integrations, the right hand side is equal to
\[
= \frac{\lambda n}{2\nu} \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-1} \text{sh}(v) dv
\]
\[
= -\frac{\lambda n}{2\nu^2} \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-1} \text{ch} v dv
\]
\[
+ \frac{\lambda}{\nu^2} (\frac{n^2}{4} + \frac{n}{2}) \int_{\text{arcch}(d)}^{\infty} e^{\nu v}(\text{ch} v - d)^{-n/2-2} \text{sh}(v)^2 dv .
\]

This proves the lemma for \(-\text{Re}(n)/2 > 1\) since \(\lambda/\nu^2 = -1\) and \(\text{sh}^2(v) = \text{ch}^2(v) - 1\).

The case \(-\text{Re}(n)/2 > -1\) then follows by analytic continuation. \(\square\)

Clearly, any analytic continuation of (3.6) in the parameters \(n\) and \(\nu\) also satisfies the ordinary differential equation (3.4). The following lemma provides explicit expressions for such analytic continuations.
Lemma 3.3. Assume $-\Re(n)/2 > -1$ and $\Re(\nu) - \Re(n)/2 < 0$. Then, for each integer $l \geq -1$, we have the identity
\[
\Gamma\left(-\frac{n}{2} + 1\right)^{-1} f_0(d) = (-1)^l \Gamma\left(l - \frac{n}{2} + 1\right)^{-1} \int_{\text{arcch}}^{\infty} D_{sh,v}^l[e^{\nu v}](\text{ch} v - d)^{-n/2+l} \, dv.
\]
The right hand side provides the unique analytic continuation of the left hand side to the parameter region $-\Re(n/2) + l > -1$ and $\Re(\nu) - \Re(n/2) < 0$. The right hand side satisfies the ordinary differential equation (3.4) in this parameter region. Here we have set for $\Re(\nu) < 0$
\[
D_{sh,v}^{-1}[e^{\nu v}] = v^{-1} \text{sh}(v) e^{\nu v}.
\]
Proof. By partial integration, we have for $l \geq 0$
\[
\Gamma\left(l - \frac{n}{2} + 1\right)^{-1} \int_{\text{arcch}}^{\infty} D_{sh,v}^l[e^{\nu v}](\text{ch} v - d)^{-n/2+l} \, dv
\]
\[
= -\Gamma\left(l - \frac{n}{2} + 1\right)^{-1} \int_{\text{arcch}}^{\infty} D_{sh,v}^{l-1}[e^{\nu v}](\text{sh}(v))^{-1} D_v[(\text{ch} v - d)^{-n/2+l}] \, dv
\]
\[
= -\Gamma\left(l - \frac{n}{2}\right)^{-1} \int_{\text{arcch}}^{\infty} D_{sh,v}^{l-1}[e^{\nu v}](\text{ch} v - d)^{-n/2+l-1} \, dv
\]
By induction, this proves the identity of the lemma.

To normalize the resolvent kernel properly, we need to calculate the asymptotic behaviour of $f_0$ near 1. The reader not interested in explicit constants may skip the following lemma.

Lemma 3.4. Let $\Re(\nu) - \Re(n/2) < 0$ and assume $n$ is not an odd negative integer. For $R > 0$ and $R$ near 0 and any $\varepsilon \in (0, 1)$ we have
\[
\Gamma\left(1 - \frac{n}{2}\right)^{-1} f_0(\text{ch} R) = 2^{n/2-1}\pi^{-1/2} \Gamma((n - 1)/2) R^{1-n} + O(R^{1-\Re(n)+\varepsilon})
\]
if $n \neq 1$ and
\[
\Gamma\left(1 - \frac{n}{2}\right)^{-1} f_0(\text{ch} R) = 2^{1/2}\pi^{-1/2}|\log(R)| + O(1)
\]
in case $n = 1$. Here the left hand side is to be understood as an analytic function in the sense of Lemma 3.3.

Proof. We do the case $1 < \Re(n) < 2$. The general case follows by similar calculations or by methods of analytic continuation.

Assume $R << 1$. We split the integral
\[
f_0(\text{ch} R) = \int_{R}^{\infty} e^{\nu v} (\text{ch} v - \text{ch} R)^{-n/2} \, dv
\]
into the integrals over the intervals $[R, R^{1-\varepsilon}]$ and $[R^{1-\varepsilon}, \infty]$. Since $\text{ch} v - 1 > v^2$, the integral over the second interval is bounded by
\[
C \int_{R^{1-\varepsilon}}^{\infty} v^{-\Re(n)} \, dv \leq CR^{(1-\Re(n))(1-\varepsilon)}.
\]
Thus this integral is negligible. For $v < R^{1-\varepsilon}$ we write
\[ e^{\nu v} = 1 + O(R^{1-\varepsilon}) , \]
\[ \text{ch} v - \text{ch} R = \frac{1}{2}(v^2 - R^2)(1 + O(R^{2-\varepsilon})) . \]
Thus the first integral is
\[ 2^{n/2} \int_{R^{1-\varepsilon}}^R [v^2 - R^2]^{-n/2} dv (1 + O(R^{1-\varepsilon})) . \]
By an argument as before, we can change the domain of integration back to $[R, \infty]$ doing at most an error of order $R^{(1-n)(1-\varepsilon)}$. Thus we have to get asymptotics of the integral
\[ 2^{n/2} \int_{R}^\infty [v^2 - R^2]^{-n/2} dv \]
\[ = 2^{n/2-1}R^{1-n} \int_{1}^\infty [w - 1]^{-n/2}w^{-1/2} dw . \]
The integral in the last display can be expressed in terms of Gamma functions:
\[ \int_{1}^\infty [w - 1]^{-n/2}w^{-1/2} dw \]
\[ = \Gamma(1/2)^{-1} \int_{1}^\infty [w - 1]^{-n/2} \int_{0}^\infty t^{1/2}e^{-tw} \frac{dt}{t} dw \]
\[ = \pi^{-1/2} \int_{0}^\infty t^{1/2}e^{-t} \int_{0}^\infty r^{-n/2}e^{-tr} \frac{dt}{t} dr \]
\[ = \pi^{-1/2} \Gamma(1 - n/2) \int_{0}^\infty t^{n/2-1/2}e^{-t} \frac{dt}{t} \]
\[ = \pi^{-1/2} \Gamma(1 - n/2) \Gamma((n - 1)/2) . \]
This proves the asymptotics claimed in the lemma. \qed

With Lemma 3.4 we have completed the proof of the identities for the resolvent kernel $k$ claimed in Lemma 3.1.

It remains to prove the $L^1$ estimates for $k$ (compare with [3]). We can assume that $n$ is a positive integer. We have to estimate the integral
\[ \int_{R}^\infty D_{sh,v}[e^{\nu v}](\text{ch}(v) - \text{ch}(R))^{l-n/2} dv = H_v^1(R) + H_v^2(R) , \]
where
\[ H_v^1(R) = \int_{R}^{R+2} D_{sh,v}[e^{\nu v}](\text{ch}(v) - \text{ch}(R))^{l-n/2} dv \]
and
\[ H_v^2(R) = \int_{R+2}^\infty D_{sh,v}[e^{\nu v}](\text{ch}(v) - \text{ch}(R))^{l-n/2} dv , \]
and where we choose $l$ such that $l - n/2 \in \{-1/2, 0\}$.
It is easily seen that
\[ |H_2^\nu(R)| \lesssim |\nu|^l \int_{R+2}^\infty e^{-(1+\text{Re}(\nu))v} e^{(l-\frac{l}{2})v} dv \lesssim (1 + |\nu|)^\frac{l}{2} e^{-\frac{l}{2} R + \text{Re}(\nu) R}. \]

To estimate \( H_1^\nu(R) \), for \( R \geq 1 \) we use that in the domain of integration we have
\[ D^m \text{sh}^{-1}(v) \sim e^{-R}, \]
\[ |D^m e^{\nu v}| \sim |\nu|^m e^{\text{Re}(\nu) R}, \]
\[ \text{ch}(v) - \text{ch}(R) \sim e^R (v - R). \]

This leads to the same estimate as for \( H_2^\nu(R) \).

For \( R < 1 \) we use that in the domain of integration
\[ D^m \text{sh}^{-1}(v) \sim v^{-1-m}, \]
\[ \text{ch}(v) - \text{ch}(R) \sim v(v - R), \]
\[ |D^m e^{\nu v}| \sim |\nu|^m. \]

Thus
\[ |H_1^\nu(R)| \lesssim (1 + |\nu|^l) \int_{R}^{R+2} v^{-l-n/2}(v - R)^{-n/2+1} dv \]
\[ \lesssim (1 + |\nu|^l) R^{1-n} \int_1^\infty v^{-l-n/2}(v - 1)^{-n/2+1} dv \]
\[ \lesssim (1 + |\nu|^l) R^{1-n} \]

Put together, we find that for \( \text{Re}(\nu) < 0 \)
\[ \left| \int_R^\infty D_{\text{sh}, \nu}^l[e^{\nu v}](\text{ch} v - \text{ch} R)^{-l-n/2} dv \right| \leq C_n (1 + |\nu|)^\frac{l}{2} R^{1-n} (1 + R^{n-1}) e^{-\frac{l}{2} R + \text{Re}(\nu) R}. \]

Using Lemma 2.1, this proves the desired estimate (3.2) and completes the proof of Lemma 3.1.

4. Spectral Multipliers

Let \( \psi \in C_0(\mathbb{R}) \). Since \( \psi(L) \) is a bounded linear operator on \( L^2(G) \), by the Schwartz’ kernel theorem and left-invariance of \( L \), there exists a unique convolution kernel \( k_\psi \in \mathcal{D}'(G) \) such that \( \psi(L)\varphi = \varphi * k_\psi \), \( \varphi \in \mathcal{D}(G) \). We shall derive an integral representation for \( k_\psi \) in this section. In the sequel, we shall sometimes also use the suggestive notation \( k_\psi = \psi(L)\delta_0 \).

Since the Gauss-kernels \( g_\varepsilon(s) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{s^2}{2\varepsilon}}, \varepsilon > 0 \), form an approximation to the identity with respect to convolution, we have \( \psi_\varepsilon := \psi * g_\varepsilon \to \psi \), uniformly as \( \varepsilon \to 0 \). But, \( \psi_\varepsilon \) has an analytic continuation, given by
\[ \psi_\varepsilon(\zeta) := \frac{1}{\sqrt{2\pi\varepsilon}} \int_\mathbb{R} \psi(t) e^{-\frac{(t-\zeta)^2}{2\varepsilon}} dt, \zeta \in \mathbb{C}. \]
Therefore, \( \psi_\varepsilon(L) \) is given by the Cauchy-integral

\[
\psi_\varepsilon(L) = \frac{1}{2\pi i} \int_{\gamma_{\delta}} \psi_\varepsilon(\zeta)(L - \zeta)^{-1} d\zeta,
\]

where, for any \( \delta > 0 \), we may choose for \( \gamma_\delta \) the (clockwise) contour \( \gamma_\delta : s \mapsto (s + i\delta)^2 \), \( s \in \mathbb{R} \). By Lemma 3.1 \( \psi_\varepsilon(L) \varphi = \varphi \ast k_{\psi_\varepsilon} \), \( \varphi \in \mathcal{S} \), where the kernel \( k_{\psi_\varepsilon} \) is given, for any \( l > \frac{n}{2} - 1 \), by

\[
(4.1) \quad k_{\psi_\varepsilon}(x, y) = c_l e^{-\frac{n\pi}{2} x^2} \int_{\mathbb{R}} \psi_\varepsilon((s + i\delta)^2) F_R(s + i\delta)(s + i\delta) \, ds,
\]

where

\[
(4.2) \quad c_l := \frac{(-1)^l}{\pi i} \frac{2^{-1-\frac{n}{2}} \pi^{-\frac{n}{2}}}{\Gamma(-\frac{n}{2} + 1 + l)},
\]

and where

\[
(4.3) \quad F_R(\zeta) := \int_{\mathbb{R}} D_{\text{sh}, v}[e^{i\zeta v}](\text{ch} v - \text{ch} R)^{-\frac{n}{2} + l} \, dv,
\]

if \( \text{Im} \zeta > -\frac{n}{2} \).

The estimate (3.2) shows that the mappings \( (x, y) \mapsto e^{-\frac{n\pi}{2} x^2} F_R(x, y)(s + i\delta) \) are locally integrable on \( G \), and their integrals over compact subsets of \( G \) are of polynomial growth in \( s \), uniformly in \( 0 < \delta < 1 \).

Moreover, since

\[
\text{Re} [(t - (s + i\delta)^2)^2] - s^4 \leq C[1 + |s|^3],
\]

where \( C \) is uniform in \( 0 \leq \delta \leq 1 \), \( t \in \text{supp}(\psi) \), we have

\[
|\psi_\varepsilon((s + i\delta)^2)| \leq Ce^{-cs^4},
\]

where \( C \) and \( c > 0 \) depend on \( \varepsilon \) but not on \( \delta \).

Therefore, given \( \varphi \in \mathcal{D}(G) \), by the dominated convergence theorem the limit of

\[
\int_{\mathbb{R}} \psi_\varepsilon((s + i\delta)^2) \int_{\mathbb{R}} e^{-\frac{n\pi}{2} F_R(x, y)(s + i\delta)} \varphi(x, y) \, dx \, dy \, (s + i\delta) \, ds
\]
as \( \delta \) tends to 0 is equal to the same expression with \( \delta = 0 \). Therefore, in the sense of distributions,

\[
(4.4) \quad k_{\psi_\varepsilon}(x, y) = c_l e^{-\frac{n\pi}{2} x^2} \int_{\mathbb{R}} \psi_\varepsilon(s^2) F_R(s, y)(s) \, ds.
\]

Finally, as \( \varepsilon \to 0 \), \( \psi_\varepsilon(L) \to \psi(L) \) in the operator norm on \( L^2(G) \), which implies that \( k_{\psi_\varepsilon} \to k_\psi \) in \( \mathcal{D}'(G) \). On the other hand, \( |\psi_\varepsilon(s^2)| \leq Ce^{-\frac{n\pi}{2}} \), for \( 0 < \varepsilon < 1 \), so that, again by the dominated convergence theorem and (4.4), \( k_{\psi_\varepsilon} \to C_l e^{-\frac{n\pi}{2}} \int_{\mathbb{R}} \psi(s^2) F_R(s) \, ds \), in the sense of distributions. We have thus proved
 Proposition 4.1. Let $\psi \in C_0(\mathbb{R})$. Then, for any $l > \frac{n}{2} - 1$, the convolution kernel $k_\psi$ of $\psi(L)$ is locally integrable on $G$, and is given by

$$k_\psi(x,y) = c_l e^{-\frac{nx^2}{2}} \int_{\mathbb{R}} \psi(s^2) F_R(x,y)(s) s ds$$

with $c_l$ given by (4.2).

5. Asymptotics of $F_R(s)$

We denote by $S^\alpha$ the symbol class

$$S^\alpha := \{ b \in C^\infty(\mathbb{R}) : \| b \|_{S^\alpha,k} := \sup_s (1 + s^2)^{-\frac{\alpha+k}{2}} | b^{(k)}(s) | < \infty \text{ for all } k \in \mathbb{N} \}.$$ 

The spaces $S^\alpha$ are Fréchet-spaces, with the topology induced by the sequence of semi-norms $\| \cdot \|_{S^\alpha,k}$, $k \in \mathbb{N}$. The product of a function in $S^\alpha$ with a function in $S^\beta$ is in $S^{\alpha+\beta}$. Moreover, $S^\alpha \subset S^\beta$ if $\alpha < \beta$ and $Db \in S^{\alpha-1}$ if $b \in S^\alpha$. The following general lemma will also be useful:

Lemma 5.1. Let $b_\alpha \in S^\alpha$ and $\beta, \gamma > 0$. Then we can write for each $R > 0$

$$b_\alpha = R^\beta b_{\alpha+\beta,R} + R^{-\gamma} b_{\alpha-\gamma,R}$$

with $b_{\alpha+\beta} \in S^{\alpha+\beta}$ and $b_{\alpha-\gamma} \in S^{\alpha-\gamma}$ uniformly in $R$, i.e.,

$$\| b_{\alpha+\beta} \|_{S^{\alpha+\beta,k}} \leq C_k$$

$$\| b_{\alpha-\gamma} \|_{S^{\alpha-\gamma,k}} \leq C_k$$

with constants $C_k$ independent of $R$.

Note: We have suppressed the $R$-dependence of the symbols $b_{\alpha+\beta}$, $b_{\alpha-\gamma}$ in the notation, and we will continue to suppress any $R$ dependence of symbols $b$ throughout the rest of this paper.

Proof. Let $\chi$ be a smooth cutoff function which is constant 1 on $(-\infty, 1]$ and vanishes on $[2, \infty)$. Then we write

$$b_\alpha = R^\beta \left[ R^{-\beta} (1 - \chi(R(1 + s^2)^{1/2})) b_\alpha \right] + R^{-\gamma} \left[ R^\gamma \chi(R(1 + s^2)^{1/2}) b_\alpha \right].$$

It is easy to see by Leibniz' rule that this is the desired splitting. \qed

We wish to estimate the function

$$F_R(s) := \int_{\mathbb{R}} D_{sh,v}^{j} [e^{xsv}](chv - cR)^{-\frac{n}{2} + l} dv, \quad s \in \mathbb{R}, \quad (l > \frac{n}{2} - 1).$$

The estimates are stated in Proposition 5.2 for the case $R \geq 1$ and in Proposition 5.7 for the case $0 < R \leq 1$. 
Proposition 5.2. If $R \geq 1$, then
\begin{equation}
F_R(s) = e^{-\frac{n}{2}R} e^{iRs} b_{\frac{n}{2}-1}(s),
\end{equation}

where $b_{\frac{n}{2}-1} \in S_{\frac{n}{2}-1}$ uniformly in $R$.

This proposition will follow from the subsequent lemmas.

Lemma 5.3. For $v \geq 1$ we can write
\begin{equation}
D^l_{sh,v}[e^{isv}] = \sum_{k=0}^{l} s^k q_k(v) e^{-lv} e^{isv}
\end{equation}
with $q_k \in S^0$ for all $k$.

Proof. This is proved by induction on $l \in \mathbb{N}$, the case $l = 0$ being trivial. Assume the statement is true for some $l \in \mathbb{N}$. Then
\begin{align*}
D^{l+1}_{sh,v}[e^{isv}] &= \sum_{k=0}^{l} s^k D_v((shv)^{-1} q_k(v) e^{-lv} e^{isv}) \\
&= \sum_{k=0}^{l} s^k D_v(2\frac{1}{1-e^{-2v}} q_k(v) e^{-(l+1)v} e^{isv}).
\end{align*}

On the interval $[1, \infty)$, the function $\frac{1}{1-e^{-2v}} = \sum_{m=0}^{\infty} e^{-2mv}$ coincides with a function in $S^0$. This easily implies the statement of the lemma for $l + 1$. \hfill \Box

We can therefore decompose
\begin{align*}
F_R(s) &= \sum_{k=0}^{l} s^k \int_{R}^{\infty} q_k(v) (ch - chR)^{-\frac{n}{2}+l} e^{-lv} e^{isv} dv \\
&= \sum_{k=0}^{l} e^{iRs} e^{-\frac{\pi}{2} R} s^k \int_{0}^{\infty} q_k(R + v) [(ch(R + v) - chR)e^{-(R+v)}]^{-\frac{n}{2}+l} e^{-\frac{\pi}{2}v} e^{isv} dv.
\end{align*}

Fixing $k \in \{0, \ldots, l\}$ and writing
\begin{equation}
\gamma_R(v) := q_k(R + v) [(ch(R + v) - chR)e^{-(R+v)}]^{-\frac{n}{2}+l} e^{-\frac{\pi}{2}v},
\end{equation}
it then suffices to prove that the function
\begin{equation}
f_R(s) := \int_{0}^{\infty} \gamma_R(v)e^{isv} dv, \quad s \in \mathbb{R},
\end{equation}
lies in $S^{\frac{n}{2}-l-1}$ uniformly with respect to $R \geq 1$. 
Let \( \chi \) be a smooth cutoff function which is constant equal to 1 on \((-\infty, 1]\) and vanishes on \([2, \infty)\).

It suffices to show that

\[
(5.3) \quad f_{R,1}(s) := \int_0^\infty \chi(v)\gamma_R(v) e^{isv} dv, \quad s \in \mathbb{R},
\]

\[
(5.4) \quad f_{R,2}(s) := \int_0^\infty (1 - \chi(v))\gamma_R(v) e^{isv} dv, \quad s \in \mathbb{R},
\]

are in \(S^{\frac{n}{2} - l - 1}\) uniformly with respect to \(R \geq 1\).

The following lemma settles the question for \(f_{R,2}\).

**Lemma 5.4.** The function \((1 - \chi)\gamma_R\) is in the Schwartz class uniformly in \(R > 0\).

**Proof.** Since the function \(q_k\) is in \(S^0\), all derivatives \(D^k q_k\) are bounded. It then suffices to show that also all derivatives of

\[
(5.5) \quad [(\text{ch}(R + v) - \text{ch}R)e^{-(R+v)}]^{-\frac{n}{2}+l}
\]

are bounded on \([1, \infty)\), uniformly in \(R > 0\). However,

\[
(5.6) \quad (\text{ch}(R + v) - \text{ch}R)e^{-(R+v)} = \frac{[\text{ch}(v) - 1]}{e^v} \frac{\text{ch}R}{e^R} + \frac{\text{sh}(v) \text{sh}R}{e^v} \frac{\text{sh}R}{e^R},
\]

and this function and all its derivatives are bounded on \([1, \infty)\) uniformly in \(R > 0\). Moreover, \(5.6\) is also bounded below by some \(\varepsilon > 0\) on \([1, \infty)\) uniformly in \(R > 0\). Therefore, all derivatives of \(5.5\) are bounded, which completes the proof of the lemma. \(\square\)

It remains to show that \(f_{R,1}\) is in \(S^{\frac{n}{2} - l - 1}\). This will follow from the next two lemmas.

**Lemma 5.5.** For \(v > 0\) we can write

\[
\chi \gamma_R = g_R(v)v^{-\frac{n}{2}+l},
\]

where \(g_R\) is supported in \(v \leq 2\) and \(D^k g_R\) is bounded uniformly in \(R \geq 1\) for all \(k \in \mathbb{N}\).

**Proof.** Taylor expansion of \(\text{sh}(v)\) and \(\text{ch}(v)\) in the expression \(5.6\) gives for \(v \leq 2\)

\[
(\text{ch}(R + v) - \text{ch}R)e^{-(R+v)} = v\tilde{g}(v)
\]

for some function \(\tilde{g}\) which is bounded below by \(\varepsilon > 0\) and has all derivatives bounded above uniformly in \(R \geq 1\). This proves the lemma. \(\square\)

**Lemma 5.6.** Let \(g \in S^0\) be supported in \([-2, 2]\) and \(\alpha > -1\). Then

\[
f(s) := \int_0^\infty g(v)v^\alpha e^{ivs} dv, \quad s \in \mathbb{R},
\]

is in \(S^{-\alpha-1}\) with semi-norms \(\|f\|_{S^{-\alpha-1},k}\) controlled by the seminorms \(\|g\|_{S^0,k}\).
Proof. Since

\[ D_j f(s) := i^j \int_0^\infty v^{\alpha+j} e^{iv s} \, dv, \]

it suffices to show that

\[ |f(s)| \leq C|s|^{-\alpha-1}. \]

Assume without loss of generality that \( s > 0 \). By a change of variables, we need to show

\[ \int_0^\infty g \left( \frac{v}{s} \right) v^\alpha e^{iv} \, dv \leq C. \]

Let \( \chi \) be again a smooth cutoff function which is constant 1 on \((-\infty, 1]\) and vanishes on \([2, \infty)\). It suffices to estimate separately the terms

\[ \int_0^\infty \chi(v) g \left( \frac{v}{s} \right) v^\alpha e^{iv} \, dv, \]

\[ \int_0^\infty (1 - \chi(v)) g \left( \frac{v}{s} \right) v^\alpha e^{iv} \, dv. \]

The first term is clearly bounded. The second term, after \( N \) integrations by part, can be estimated by

\[ \int_0^{2s} \left| D^N \left[ (1 - \chi(v)) g \left( \frac{v}{s} \right) v^\alpha \right] \right| dv. \]

By Leibniz’ rule, this is dominated by a constant times

\[ \int_{2s}^{\infty} v^{\alpha-N} \, dv, \]

which is finite if \( N \) is chosen sufficiently large. \( \square \)

This completes the proof of Proposition 5.2

Proposition 5.7. Assume that \( 0 < R \leq 1 \).

(a) If \( n = 1 \), then

\[ F_R(s) = e^{i R s} R^{-\frac{\alpha}{2}} b_{\frac{1}{2}}(s), \]

where \( b_{\frac{1}{2}} \in S^{-\frac{\alpha}{2}} \) uniformly in \( R \in (0, 1] \).

(b) If \( n \geq 2 \), then

\[ F_R(s) = e^{i R s} \left[ R^{-\frac{\alpha}{2}} b_{\frac{n-1}{2}}(s) + R^{1-n} b_0(s) \right], \]

where \( b_{\frac{n-1}{2}} \in S^{\frac{\alpha}{2}-1} \) and \( b_0 \in S^0 \) uniformly in \( R \in (0, 1] \).

For the proof, we decompose \( F_R = F^1_R + F^2_R \), with

\[ F^1_R(s) := \int_R^\infty \chi(v) D_{sh,v}^l[e^{ivu}](chv - chR)^{-\frac{\alpha}{2}+l} \, dv, \]

\[ F^2_R(s) := \int_2^{\infty} (1 - \chi(v)) D_{sh,v}^l[e^{ivu}](chv - chR)^{-\frac{\alpha}{2}+l} \, dv. \]
Here, $\chi$ is a smooth cut-off function such that
$$\chi(v) = 1, \text{ if } |v| \leq 2, \text{ and } \chi(v) = 0, \text{ if } |v| \geq 4.$$ 

The function $F_R^2$ can be again estimated by means of Lemma 5.3 as in Lemma 5.4, which shows that $F_R^2$ is in $S(\mathbb{R})$, uniformly in $R$. Thus it remains to estimate $F_R^1$.

**Lemma 5.8.** For $0 < v \leq 4$, we can write
$$D_{sh,v}^l [e^{isv}] = \sum_{k=0}^{l} s^k q_k(v) v^{k-2l} e^{isv},$$
where $q_k \in S^0$.

**Proof.** This follows by induction, the case $l = 0$ being trivial. Assume the statement is true for some $l \in \mathbb{N}$. Then
$$D_{sh,v}^{l+1} [e^{isv}] = \sum_{k=0}^{l} s^k D_v((shv)^{-1} q_k(v) v^{k-2l} e^{isv}).$$
However, on the interval $[0, 4]$,
$$sh(v)^{-1} = g(v) v^{-1}$$
for some $g \in S^0$. This easily implies the statement of the lemma for $l + 1$. \hfill $\Box$

**Lemma 5.9.** For $0 \leq v \leq 4$, we have
$$\text{ch}v - \text{ch}R = \gamma(v)(v + R)(v - R)$$
for some $\gamma \in S^0$ with $\gamma(v) > \varepsilon > 0$ for $0 \leq v \leq 4$, all uniformly in $R \in (0, 1]$.

**Proof.** This follows immediately from
$$\text{ch}v - \text{ch}R = \sum_{n=1}^{\infty} \frac{(v^2)^n - (R^2)^n}{(2n)!}.$$ \hfill $\Box$

We can therefore decompose
$$F_R^1(s) = \sum_{k=0}^{l} s^k \int_{R}^{\infty} \tilde{\gamma}_{k,R}(v) v^{k-2l} (v + R)^{-\frac{n}{2} + l}(v - R)^{-\frac{n}{2} + l} e^{isu} dv$$
$$= \sum_{k=0}^{l} s^k e^{Ris} \int_{0}^{\infty} \tilde{\gamma}_{k,R}(v)(R + v)^{k-2l}(2R + v)^{-\frac{n}{2} + l}v^{-\frac{n}{2} + l} e^{isu} dv,$$
with $\tilde{\gamma}_{k,R}, \gamma_{k,R} \in S^0$ uniformly in $R \in (0, 1]$ and $\gamma_{k,R}(v) = 0$ for $v > 4$. 

\ cleared
Lemma 5.10. Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ such that $\alpha = \alpha_1 + \alpha_2 > 0$ and $-1 < \beta$. Let $\gamma \in S^0$ such that $\gamma(v) = 0$ for $v > 4$. Consider the function

$$f(s) = \int_0^\infty \gamma(v)(R + v)^{-\alpha_1}(2R + v)^{-\alpha_2}v^\beta e^{iv} dv.$$ 

Then, for every $\delta \geq 0$ with

$$\delta < \alpha \text{ and } \delta \leq \beta + 1,$$

we have that $R^{\alpha-\delta} f$ is in $S^{\delta-\beta-1}$ uniformly in $R \in (0, 1]$.

Proof. Similarly as in the proof of Lemma 5.6, it suffices to prove that

$$|f(s)| \leq C R^{-\alpha+\delta}(1 + |s|)^{\delta-\beta-1}, \quad s \in \mathbb{R},$$

with $C$ independent of $R$.

If $\delta = \beta + 1$, then $-\alpha + \beta < -1$ and we have

$$|f(s)| \lesssim R^{-\alpha+\beta+1} \int_0^\infty (1 + v)^{-\alpha}v^\beta dv,$$

which proves the desired estimate.

Now assume $\delta < \beta + 1$. Consider first the case $|s| \leq 1$. We estimate

$$\begin{align*}
(R + v)^{-\alpha_1}(2R + v)^{-\alpha_2} &\lesssim (R + v)^{-\alpha} = (R + v)^{\delta-\alpha}(R + v)^{-\delta} \\
&\lesssim R^{\delta-\alpha} v^{-\delta}.
\end{align*}$$

This gives

$$|f(s)| \lesssim \int_0^4 R^{\delta-\alpha} v^{\beta-\delta} dv,$$

which proves the desired estimate in view of $\delta < \beta + 1$.

It remains to consider $|s| > 1$. We write $f = f_1 + f_2$ with

$$f_1(s) = s^{-\beta-1} \int_0^\infty \chi(v)\gamma(v/s)(R + v/s)^{-\alpha_1}(2R + v/s)^{-\alpha_2}v^\beta e^{iv} dv,$$

$$f_2(s) = s^{-\beta-1} \int_1^\infty (1 - \chi(v))\gamma(v/s)(R + v/s)^{-\alpha_1}(2R + v/s)^{-\alpha_2}v^\beta e^{iv} dv,$$

where $\chi$ is a smooth cutoff function which is constant 1 on $(-\infty, 1]$ and vanishes on $[2, \infty)$. To estimate $f_1$, we split $(R + v/s)^{-\alpha}$ analogously to (5.11) and obtain

$$|f_1(s)| \lesssim s^{-\beta-1} \int_0^\infty |\chi(v)||s|^\delta R^{\delta-\alpha} v^{\beta-\delta} dv.$$

This proves the desired estimate for $f_1$ in view of $\delta < \beta + 1$.

To estimate $f_2$, we do $N$ times partial integration. The functions $(1 - \chi(v))\gamma(v/s)$ is in $S^0$ uniformly in $|s| > 1$, and $v^\beta$ is in $S^\beta$. With

$$|D_v^k[(R + v/s)^{-\alpha_1}(2R + v/s)^{-\alpha_2}]| \lesssim (v/s)^k(R + v/s)^{\alpha-k}v^{-k}$$

$$\lesssim (R + v/s)^{-\alpha} v^{-k} \lesssim s^k R^{\delta-\alpha} v^{-\delta-k}$$

This completes the proof.
we therefore obtain

$$|f_2(s)| \lesssim s^{\delta-\beta-1} R^{\delta-\alpha} \int_1^\infty v^{\beta-N} \, dv,$$

which proves the desired estimate for $f_2$.

We are now in a position to estimate $F_1^R$. Assume first that $n$ is even and choose $l = n/2$.

Applying Lemma 5.10 to (5.8) with $\delta = 0$ in case $k = l$ and $\delta = 1$ in case $k < l$ gives

$$F_1^R = \left[ \sum_{k=1}^{l-1} s^k e^{iRs} R^{k-l-\frac{n}{2}+1} b_{\frac{n}{2}-l} \right] + s^l e^{iRs} R^{-\frac{n}{2}} b_{\frac{n}{2}-l-1}$$

where $b_\alpha$ generally denotes a function in $S^\alpha$ (uniformly in $R \in (0, 1]$) which may be different at different places in the argument. Applying Lemma 5.1 gives

$$F_1^R = e^{iRs} [R^{-l-\frac{n}{2}+1} b_{\frac{n}{2}-l} + R^{-\frac{n}{2}} b_{\frac{n}{2}-l-1}] .$$

As $l = n/2$, we obtain the desired estimate.

Assume next that $n$ is odd and $n \geq 3$. We choose $l = (n-1)/2$. Then we apply Lemma 5.10 with $\delta = 0$ for $k = l$ and with $\delta = 1/2$ for $k < l$ and obtain

$$F_1^R = \left[ \sum_{k=1}^{l-1} s^k e^{iRs} R^{k-l-\frac{n}{2}+\frac{1}{2}} b_{\frac{n}{2}-l} \right] + s^l e^{iRs} R^{-\frac{n}{2}+\frac{1}{2}} b_{\frac{n}{2}-l-1}$$

Applying Lemma 5.1 again and using $l = (n-1)/2$ gives the desired estimate.

Finally, assume $n = 1$. We choose $l = 0$. Applying Lemma 5.10 with $\delta = 0$ gives

$$F_1^R = e^{iRs} R^{-\frac{n}{2}} b_{\frac{n}{2}-1} ,$$

which proves the desired estimate.

6. SPECTRALLY LOCALIZED ESTIMATES FOR THE WAVE PROPAGATOR

The following theorem states pointwise estimates for the convolution kernel of spectrally localized wave propagators on the $ax + b$ group.

**Theorem 6.1.** Let $t \in \mathbb{R}, \lambda > 0$, and let $\psi$ be an even bump function $\psi \in C_0^\infty(\mathbb{R})$ supported in $[-2, 2]$. If $\lambda \geq 1$, we shall in addition assume that $\psi$ vanishes on $[-1, 1]$. Then the convolution kernel of

$$m_\lambda^1(L) := \psi\left(\frac{\sqrt{L}}{\lambda}\right) \cos(t\sqrt{L})$$
is of the form
\[ k^t_\lambda(x, y) = e^{-\frac{n}{2} x} e^{-\frac{nR}{2}} [G_\lambda(R, R - t) + G_\lambda(R, R + t)] , \]
where the function \( G_\lambda \) satisfies for every \( N \in \mathbb{N} \) the following estimates:

(a) If \( R \geq 1 \), then
\[
|G_\lambda(R, \rho)| \lesssim \begin{cases} 
\lambda^{\frac{j}{2}+1} (1 + |\lambda \rho|)^{-N}, & \text{if } \lambda \geq 1, \\
\lambda^2 (1 + |\lambda \rho|)^{-N}, & \text{if } \lambda < 1.
\end{cases}
\]

(b) If \( 0 \leq R \leq 1 \), then, for \( n = 1 \),
\[
|G_\lambda(R, \rho)| \lesssim \begin{cases} 
R^{-\frac{j}{2}} \lambda^2 (1 + |\lambda \rho|)^{-N}, & \text{if } \lambda \geq 1, \\
R^{-\frac{j}{2}} \lambda^2 (1 + |\lambda \rho|)^{-N}, & \text{if } \lambda < 1,
\end{cases}
\]
and for \( n \geq 2 \),
\[
|G_\lambda(R, \rho)| \lesssim \begin{cases} 
(R^{1-n} \lambda^2 + R^{-\frac{j}{2}} \lambda^{\frac{j}{2}+1})(1 + |\lambda \rho|)^{-N}, & \text{if } \lambda \geq 1, \\
R^{1-n} \lambda^2 (1 + |\lambda \rho|)^{-N}, & \text{if } \lambda < 1,
\end{cases}
\]
where the constants in these estimates depend only on the \( C_N \)-norms of \( \psi \).

Proof. We consider first the case \( R \geq 1 \). By Proposition 4.1 and Proposition 5.2, the kernel \( k^t_\lambda \) can be written as
\[
\frac{c_\lambda}{2} e^{-\frac{n}{2} x} e^{-\frac{nR}{2}} \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) b_{\frac{j}{2}-1}(s) s^{j-1} e^{i\rho s} ds ,
\]
where \( b_{\frac{j}{2}-1} \) is in \( S^{\frac{j}{2}-1} \) uniformly in \( R \geq 1 \). Then the desired estimate follows by an application of Lemma 6.2 below with \( j = 2 \).

The case \( 0 < R \leq 1 \) is done similarly using Proposition 5.7 instead of Proposition 5.2.

Lemma 6.2. Let \( b \in S^\beta \), let \( j \geq 1 \) be an integer, and let \( \lambda > 0 \). Consider
\[
M_\lambda(\rho) := \int \psi\left(\frac{s}{\lambda}\right) b(s) s^{j-1} e^{i\rho s} ds , \quad \rho \in \mathbb{R}.
\]
Then, for every \( N \in \mathbb{N} \),
\[
|M_\lambda(\rho)| \leq C_N \begin{cases} 
\lambda^{\beta+j} (1 + |\lambda \rho|)^{-N}, & \text{if } \lambda \geq 1 \text{ and } \text{supp } \psi \subset [-2, -1] \cup [1, 2], \\
\lambda^j (1 + |\lambda \rho|)^{-N}, & \text{if } \lambda < 1 \text{ and } \text{supp } \psi \subset [-2, 2].
\end{cases}
\]
Here, the constants \( C_N \) depend only on \( N \) and the semi-norms \( ||b||_{S^{\beta,k}} \) of \( b \), and on the \( C_N \)-norm of \( \psi \).

Proof. In order to defray the notation, we shall write \( A \lesssim B \), if \( A \leq C_N B \), where \( C_N \) is an "admissible" constant in the sense described in the lemma. We may and shall assume that \( \rho \geq 0 \). We write
\[
M_\lambda(\rho) = \lambda^j \int \psi(s) b(\lambda s) s^{j-1} e^{i\lambda s} ds ,
\]
Assume $\lambda \geq 1$ and $\psi$ is supported in $[-2, -1] \cup [1, 2]$. By the symbol estimates for $b$ we have for $s$ in the support of $\psi$:

$$|D^k b(\lambda s)| \lesssim \lambda^k (1 + |\lambda s|)^{\beta - k} \lesssim \lambda^\beta \left(\frac{1}{\lambda} + |s|\right)^{\beta - k} \lesssim \lambda^\beta.$$ \hfill (6.1)

Integrating by parts $N$ times, we thus find that $|M_\lambda(\rho)| \lesssim \lambda^{\beta j} (1 + |\lambda \rho|)^{-N}$.

Next, assume $\lambda \leq 1$ and $\psi$ is supported in $[-2, 2]$. We then have for $s$ in the support of $\psi$

$$|D^k b(\lambda s)| \lesssim \lambda^k (1 + |\lambda s|)^{\beta - k} \lesssim 1.$$ \hfill (6.2)

Integrating again by parts $N$ times, we find that $|M_\lambda(\rho)| \lesssim \lambda^{\beta j} (1 + |\lambda \rho|)^{-N}$. \hfill $\square$

As a consequence of Theorem 6.1, we obtain estimates of the $L^1$-norms of the convolution kernels of $\psi(\sqrt{L_\lambda}) \cos(t \sqrt{L_\lambda}),$ and let $\varepsilon \geq 0$.

**Proposition 6.3.** Let $W^t_\lambda := k^{t/\lambda}$ denote the convolution kernel of $\psi(\sqrt{L_\lambda}) \cos(t \sqrt{L_\lambda})$, and let $\varepsilon \geq 0$.

(a) If $\lambda \geq 1$ and $\text{supp} \psi \subset [-2, -1] \cup [1, 2]$, then

$$\int_G |W^t_\lambda(x, y)| R(x, y)^\varepsilon d(x, y) \lesssim \begin{cases} \lambda^{-\varepsilon} (1 + t)^{\frac{3}{2} + \varepsilon}, & \text{if } t \leq \lambda, \\ \lambda^{\frac{\varepsilon}{2} - 1 - \varepsilon} t^{1 + \varepsilon}, & \text{if } t \geq \lambda. \end{cases}$$ \hfill (6.3)

(b) If $0 < \lambda \leq 1$ and $\text{supp} \psi \subset [-2, 2]$, then

$$\int_G |W^t_\lambda(x, y)| R(x, y)^\varepsilon d(x, y) \lesssim \lambda^{-\varepsilon} (1 + t)^{1 + \varepsilon}.$$ \hfill (6.4)

In particular, if $\lambda \geq 1$, then

$$\int_G |W^t_\lambda(x, y)| (1 + \lambda R(x, y))^\varepsilon d(x, y) \lesssim \begin{cases} (1 + t)^{\frac{3}{2} + \varepsilon}, & \text{if } n \geq 2, \\ (1 + t)^{1 + \varepsilon}, & \text{if } n = 1, \end{cases}$$ \hfill (6.5)

and, if $0 < \lambda \leq 1$, then

$$\int_G |W^t_\lambda(x, y)| (1 + \lambda R(x, y))^\varepsilon d(x, y) \lesssim (1 + t)^{1 + \varepsilon},$$ \hfill (6.6)

in each instance uniformly in $\lambda$. The constants in these estimates depend only on the $C^N$-norms of $\psi$.

**Proof.** Without loss of generality, we shall assume that $t \geq 0$. Then the dominant term in Theorem 6.1 is the one containing $G_\lambda(R, R - t)$, to which we shall therefore restrict ourselves.
We consider first $\lambda \geq 1$. By Theorem 6.1 and Lemma 2.1, we can estimate the left-hand-side of (6.1) by

$$\lambda^2 \int_0^1 (1 + |\lambda R - t|)^{-N} R^{1+\varepsilon} dR + \lambda^{\frac{n}{2}+1} \int_0^1 (1 + |\lambda R - t|)^{-N} R^{\frac{n}{2}+\varepsilon} dR$$

(6.5)$$+ \lambda^{\frac{n}{2}+1} \int_1^\infty (1 + |\lambda R - t|)^{-N} R^{1+\varepsilon} dR.$$

If $t \leq \frac{\lambda}{2}$, then we can estimate this using Lemma 6.4 below by

$$\lambda^{-\varepsilon}(1 + t)^{1+\varepsilon} + \lambda^{-\varepsilon}(1 + t)^{\frac{n}{2}+\varepsilon} + \lambda^{\frac{n}{2}+1-N}.$$

Similarly, if $\frac{\lambda}{2} \leq t \leq 2\lambda$, we estimate (6.5) by

$$\lambda^{-\varepsilon}(1 + t)^{1+\varepsilon} + \lambda^{-\varepsilon}(1 + t)^{\frac{n}{2}+\varepsilon} + \lambda^{\frac{n}{2}-1-\varepsilon}(1 + t)^{1+\varepsilon},$$

and if $2\lambda \leq t$ we estimate (6.5) by

$$(1 + t)^{-N} + \lambda^{\frac{n}{2}-1-\varepsilon}(1 + t)^{1+\varepsilon}.$$

In each case we easily verify (6.1), taking into account that for $n = 1$ the first of the three summands of (6.5) is not present.

If $0 < \lambda \leq 1$, we estimate the left-hand-side of (6.2) by

$$\lambda^2 \int_0^\infty (1 + |\lambda R - t|)^{-N} R^{1+\varepsilon} dR$$

if $n \geq 2$ and by

$$\lambda^2 \int_0^1 (1 + |\lambda R - t|)^{-N} R^{1+\varepsilon} dR + \lambda^2 \int_1^\infty (1 + |\lambda R - t|)^{-N} R^{1+\varepsilon} dR$$

if $n = 1$. In either case it is easy to verify (6.2) using Lemma 6.4. Estimates (6.3) and (6.4) follow immediately from estimates (6.1) and (6.2). \ \square

Lemma 6.4. For $\alpha \geq 0$, $t \geq 0$ and $N > \alpha + 1$, let

$$I_0 := \int_0^1 (1 + |\lambda R - t|)^{-N} R^\alpha dR,$$

$$I_\infty := \int_1^\infty (1 + |\lambda R - t|)^{-N} R^\alpha dR.$$

Then

$$I_0 \leq C \begin{cases} (1 + \lambda)^{-\alpha-1}(1 + t)^\alpha, & \text{if } t \leq 2\lambda, \\ (1 + t)^{-N}, & \text{if } t \geq 2\lambda, \end{cases}$$

(6.6)
and

\[(6.7) \quad I_\infty \leq C \begin{cases} \lambda^{-\alpha-1}(1 + \lambda)^{-N+\alpha+1}, & \text{if } t \leq \frac{\lambda}{2}, \\ \lambda^{-\alpha-1}(1 + t)^\alpha, & \text{if } t \geq \frac{\lambda}{2}. \end{cases}\]

**Proof.** We begin with \(I_0\). If \(t \geq 2\lambda\), then clearly \(I_0 \lesssim (1 + t)^{-N}\). If \(t \leq 2\lambda\), we write

\[I_0 = \lambda^{-\alpha-1} \int_{-t}^{\lambda-t} (1 + |v|)^{-N}(v + t)^\alpha dv.\]

If \(\lambda \geq 1\), one easily deduces from this representation that \(I_0 \lesssim \lambda^{-\alpha-1}(1 + t)^\alpha\). And, if \(\lambda \leq 1\), the original formula for \(I_0\) immediately implies \(I_0 \lesssim 1\), so that we obtain (6.6).

As for \(I_\infty\), if \(t \leq \frac{\lambda}{2}\), then clearly

\[I_\infty \lesssim \int_1^\infty (1 + \lambda R)^{-N} R^\alpha dR = \lambda^{-\alpha-1} \int_\lambda^\infty (1 + R)^{-N} R^{-\alpha} dR,\]

hence \(I_\infty \lesssim \lambda^{-\alpha-1}(1 + \lambda)^{-N+\alpha+1}\).

If \(t \geq \frac{\lambda}{2}\), we write

\[I_\infty = \lambda^{-\alpha-1} \int_{\lambda-t}^\infty (1 + |v|)^{-N}(v + t)^\alpha dv.\]

If \(t \leq 1\), this implies \(I_\infty \lesssim \lambda^{-\alpha-1}\), and if \(t \geq 1\), one finds that \(I_\infty \lesssim \lambda^{-\alpha-1} t^\alpha\), so that also (6.7) is verified. \(\square\)

By means of the subordination principle described e.g. in [12], we immediately obtain:

**Corollary 6.5.** (cf. [5], Theorem. 6.1)

If \(\varepsilon > 0, s_0, s_1 > \frac{n}{2} + \varepsilon\) and \(s_1 > \frac{n+1}{2} + \varepsilon\), then there exists a constant \(C\) such that, for every continuous function \(F\) supported in \([1, 2]\) and \(0 < \lambda \leq 1\),

\[\int_G |F(L_x)\delta_0(x, y)| (1 + \lambda R(x, y))^{\varepsilon} d(x, y) \leq C\|F\|_{H(s_0)},\]

while for \(\lambda \geq 1\)

\[\int_G |F(L_x)\delta_0(x, y)| (1 + \lambda R(x, y))^{\varepsilon} d(x, y) \leq C\|F\|_{H(s_1)}.\]

**Proof.** Choose an even function \(\psi \in C_0^\infty(\mathbb{R})\) such that \(\psi = 1\) on \([1, 2]\) and \(\text{supp} \, \psi \subset [-4, -\frac{1}{2}] \cup [\frac{1}{2}, 4]\). Proposition 6.3 holds for such \(\psi\) as well. Put \(f(v) := F(v^2)\). Then
\[ \|f\|_{H^{(s)}} \sim \|F\|_{H^{(s)}}, \text{ for any } s \geq 0, \text{ and } F\left(\frac{L}{\lambda^2}\right) = f\left(\frac{\sqrt{t}}{\lambda}\right) = \psi\left(\frac{\sqrt{t}}{\lambda}\right) f\left(\frac{\sqrt{t}}{\lambda}\right). \] Moreover, by the Fourier inversion formula and Fubini’s theorem, one easily obtains
\[ f\left(\frac{\sqrt{t}}{\lambda}\right) = \frac{1}{\pi} \int_0^\infty \hat{f}(t) \cos(t \frac{\sqrt{L}}{\lambda}) \, dt, \]
since \( f \) is an even function. Thus
\[ F\left(\frac{L}{\lambda^2}\right) = \frac{1}{\pi} \int_0^\infty \hat{f}(t) \psi\left(\frac{\sqrt{t}}{\lambda}\right) \cos(t \frac{\sqrt{L}}{\lambda}) \, dt, \]
which implies
\[ I_\lambda := \int_G |F\left(\frac{L}{\lambda^2}\right)\delta_0|(1 + \lambda R)^\varepsilon d(x, y) \leq \int_0^\infty |\hat{f}(t)| \left[ \int |W_t^\ell|(1 + \lambda R)^\varepsilon d(x, y) \right] dt. \]
Thus, if \( 0 < \lambda \leq 1 \), then, by (6.4),
\[ I_\lambda \lesssim \int_R |\hat{f}(t)|(1 + |t|)^{1+\varepsilon} \, dt \lesssim \left( \int_R |\hat{f}(t)(1 + |t|)^{\gamma_0}|^2 \, dt \right)^{\frac{1}{2}} = \|f\|_{H^{(\gamma_0)}} \sim \|F\|_{H^{(\gamma_0)}}. \]

For the class of groups \( G \) considered here, we have thus established a completely different approach to the basic Theorem 6.1 in [5], entirely based on the wave equation.

7. Improvements on the estimates in Theorem 6.1 for small \( R \)

The estimates in Theorem 6.1 are already good enough for \( L^1 \)-estimates, but not yet for \( L^\infty \)-estimates, since they exhibit singularities at \( R = 0 \). One knows that the singular support of the wave propagator for time \( t \) is the sphere \( R = |t| \), so that these singularities are in fact not present. We shall show in this section how to improve on our estimates when \( R \leq 1 \), which we shall assume throughout this section.
To this end we observe that by formula (4.5), we may replace $F_R(s)$ in the previous discussions by

$$
\tilde{F}_R(s) := \int_R^\infty D_{s, v}^l[\sin(sv)](chv - chR)^{-\frac{n}{2} + l}dv, \quad (s \geq 0, \quad l > \frac{n}{2} - 1).
$$

Working with $\tilde{F}_R(s)$ in place of $F_R(s)$, we can prove the following theorem.

**Theorem 7.1.** If $R \leq 1$, then the estimates in Theorem 6.1 can be improved by the following additional estimates, valid for any $N \in \mathbb{N}$ :

$$
|G_\lambda(R, \rho)| \lesssim \begin{cases} 
\lambda^{n+1}(1 + |\lambda \rho|)^{-N}, & \text{if } \lambda \geq 1, \\
\lambda^2(1 + |\lambda \rho|)^{-N}, & \text{if } \lambda < 1,
\end{cases}
$$

In order to prove this result, as in Section 5 we split $\tilde{F}_R(s) = \tilde{F}_R^1(s) + \tilde{F}_R^2(s)$, where

$$
\tilde{F}_R^1(s) := \int_R^\infty \chi(v)D_{s, v}^l[\sin(sv)](chv - chR)^{-\frac{n}{2} + l}dv.
$$

$\tilde{F}_R^2(s)$ is again a Schwartz function, uniformly for $0 \leq R \leq 1$, and its contribution to $k_\lambda^l$ can easily be seen to satisfy the estimates in Theorem 7.1.

In order to deal with $\tilde{F}_R^1(s)$, we need the following substitute for Lemma 5.8.

**Lemma 7.2.** For $0 < sv < \pi/2$ and $0 < v < 4$ we can write

$$
D_{s, v}^l[\sin(sv)] = \sum_{k=0}^l s^{2k}q_k(sv, v),
$$

where $q_k(y, v)$ has a power series expansion of the form

$$
\sum_{m,n=0}^\infty a_{mn}y^{2m+1}v^{2n}.
$$

Moreover,

$$
|D_{s}^{2l}[q_k(sv, v)]| \leq C_{j,k}(1 + s^2)^{-j/2}, \quad \text{uniformly in } v.
$$

**Proof.** We proceed by induction on $l$, the case $l = 0$ being clear. Assume that $q_k(y, v)$ is given by (7.3). Then

$$
\frac{q_k(sv, v)}{sh(v)} = sg_k(sv, v),
$$

where $g_k(y, v)$ has an expansion of the form

$$
g_k(y, v) = \sum_{m,n=0}^\infty b_{mn}y^{2m}v^{2n}.
$$
Then
\[
D_v[sg_k(sv, v)] = \sum_{m \geq 1, n \geq 0} b_{mn} 2ms^2(sv)^{2m-1}v^{2n}
+ \sum_{m \geq 0, n \geq 1} sb_{mn} 2n(sv)^{2m}v^{2n-1}
= s^2 h^1_k(sv, v) + h^2_k(sv, v),
\]
where \(h^1_k(y, v)\) and \(h^2_k(y, v)\) are of the form (7.3). This shows that (7.3) holds also for \(l + 1\) in place of \(l\).

Moreover, (7.4) is obvious for \(|s| \leq 1\), in view of (7.3). And, if \(|s| \geq 1\), it follows from
\[
D^j_s[\gamma_k(sv, v)] = v^j(D^j_y \gamma_k)(sv, v) = s^{-j}(sv)^j(D^j_y \gamma_k)(sv, v).
\]

Now, if \(\lambda R \geq 1/2\), then \(\lambda \geq 1/2\), and the estimates (7.1) follow immediately from Theorem 6.1. Let us therefore assume that \(\lambda R \leq 1/2\).

We write
\[
\tilde{F}^1_R(s) = H^1_R(s) + H^2_R(s),
\]
where
\[
H^1_R(s) := \int_{-\frac{1}{2}R}^{\frac{1}{2}R} \chi(v) D^l_{sh, v} [\sin(sv)](ch v - ch R)^{-\frac{n}{2} + l} dv,
\]
\[
H^2_R(s) := \int_{\frac{1}{2}R}^{\infty} \chi(v) D^l_{sh, v} [\sin(sv)](ch v - ch R)^{-\frac{n}{2} + l} dv.
\]

We need information on the asymptotics of these functions for \(|s| \leq 2\lambda\).

As for \(H^1_R(s)\), notice that in the integral defining \(H^1_R(s)\) we have \(|sv| \leq 1\) if \(|s| \leq 2\lambda\). Therefore, from Lemma 7.2 and Lemma 5.9 we find that, for any \(l > -\frac{n}{2} + 1\),
\[
H^1_R(s) = \sum_{k=0}^{l} s^{2k} \int_{-\frac{1}{2}R}^{\frac{1}{2}R} \chi(v) D^l_{sh, v} [\sin(sv)](ch v - ch R)^{-\frac{n}{2} + l} dv,
\]
where \(\gamma_k(sv, v)\) is supported where \(0 \leq v \leq \min(2, \frac{1}{2\lambda})\) and satisfies \(\gamma_k(-sv, v) = -\gamma_k(sv, v)\) and
\[
|D^j_s[\gamma_k(sv, v)]| \leq C_{j,k}(1 + s^2)^{-j/2}, \quad \text{for every } j \in \mathbb{N}.
\]

Choose \(l\) large enough so that \(l - \frac{n}{2} \geq 0\), and let
\[
J(\lambda) := \int_{-\frac{1}{2}R}^{\frac{1}{2}R} (v + R)^{-\frac{n}{2} + l} (v - R)^{-\frac{n}{2} + l} dv.
\]

Then clearly
\[
J(\lambda) \leq (1 + \lambda)^{-2l+n-1},
\]
and we find that, for $|s| \leq 2\lambda$,

$$H^1_R(s) = (1 + \lambda)^{-2l + n - 1} \sum_{k=0}^{l} s^{2k} b_{0,k}(s),$$

where $b_{0,k}$ is an odd function in $S^0$, uniformly in $R$ and $\lambda$. From Lemma 6.2 we therefore obtain

$$\left| \int_0^\infty \psi(\frac{s}{\lambda}) H^1_R(s) s \cos(ts) \, ds \right| = \frac{1}{2} \left| \int_\mathbb{R} \psi(\frac{s}{\lambda}) H^1_R(s) s \cos(ts) \, ds \right|$$

$$\lesssim C_N \sum_{k=0}^{l} (1 + \lambda)^{-2l + n - 1} \lambda^{2k + 2} (1 + |\lambda t|)^{-N},$$

hence

$$(7.5) \quad \left| \int_0^\infty \psi(\frac{s}{\lambda}) H^1_R(s) s \cos(ts) \, ds \right| \lesssim C_N \left\{ \begin{array}{ll} \lambda^{n+1}(1 + |\lambda t|)^{-N}, & \text{if } \lambda \geq 1, \\ \lambda^{2}(1 + |\lambda t|)^{-N}, & \text{if } \lambda < 1, \end{array} \right.$$ 

Next, we consider $H^2_R(s)$, again for $|s| \leq 2\lambda$.

Observe first that $H^2_R \equiv 0$, unless $\lambda \geq 1/4$. In the latter case, one finds that $H^2_R(s)$ behaves like $F^1_R(s)$, only with $R$ replaced by $\frac{1}{\lambda} \leq 4$. Replacing $R$ by $\frac{1}{\lambda}$ and $R \pm t$ by $\pm t$ in Theorem 6.1 (b), we therefore find that

$$\left| \int_0^\infty \psi(\frac{s}{\lambda}) H^1_{R}(s) s \cos(ts) \, ds \right|$$

satisfies estimates (7.5) too.

Noticing finally that $1 + |\lambda t| \sim 1 + |\lambda \rho|$ if $\rho = R \pm t$, since $\lambda R \leq 1/2$, the conclusion of Theorem 7.1 follows.

**Corollary 7.1.** Assume that $\lambda \geq 1$ and $t \geq 0$. Then

$$(7.6) \quad ||k^1_{\lambda}||_{\infty} \lesssim (1 + t^{-\frac{n}{2}}) \lambda^{\frac{n}{2} + 1}.$$ 

**Remark 7.2.** Notice that, for small times, this estimate agrees with the one valid for the Laplacian on Euclidean space $\mathbb{R}^{n+1}$, as is to be expected, since $L$ is elliptic. However, for large times, there appears no dispersive effect (definitely not for $n = 2$, by Hebisch’s transfer principle), so that it seems unlikely that non-trivial Strichartz-type estimates will hold for large times.

**Proof.** First we observe that $e^{-\frac{nx}{2}} e^{-\frac{nR}{2}} \leq 1$, and equality holds, if $y = 0$ and $x \leq 0$. Therefore,

$$\left| k^1_{\lambda} \right| \lesssim \sup_{R \geq 0} |G_{\lambda}(R, R-t)|.$$

If $R \geq 1$, then, by Theorem 6.1,

$$|G_{\lambda}(R, R-t)| \lesssim \lambda^{\frac{n}{2} + 1}.$$
So, assume that $R \leq 1$. Then, by Theorem 7.1,

$$|G_\lambda(R, R - t)| \lesssim \lambda^{n+1}(1 + \lambda|R - t|)^{-N}$$

for every $N \in \mathbb{N}$.

If $\lambda t \leq 1$, this implies

$$|G_\lambda(R, R - t)| \lesssim \lambda^{n+1} \leq \lambda^{\frac{n}{2} + 1} t^{-\frac{n}{2}}.$$

Assume next that $\lambda t \geq 1$.

If $R \leq t/2$, then $|R - t| \sim t$, so that (7.7) implies

$$|G_\lambda(R, R - t)| \lesssim \lambda^{n+1}(\lambda t)^{-N}$$

for every $N \in \mathbb{N}$, hence

$$|G_\lambda(R, R - t)| \lesssim \lambda^{\frac{n}{2} + 1} t^{-\frac{n}{2}}.$$  

If $R \geq t/2$, then for $n \geq 2$ Theorem 6.1 implies (7.8), since $t^{1-n} \lambda^2 \leq t^{-\frac{n}{2}} \lambda^{\frac{n}{2} + 1}$, and (7.8) is also valid for $n = 1$.  

\[ \square \]

**Remark 7.3.** The group $G$ can be considered as an Iwasawa $AN$-subgroup of the Lorentz group $S = SO(1, n + 1)$, and hence may be identified as a manifold with the symmetric space $K \setminus S$, where $K$ is a maximal compact subgroup of $S$. The spherical function $\varphi_0$ of order zero on $K \setminus S$ is comparable to $(\frac{R}{\sin(R)})^{n/2}$ in these coordinates, as one finds from Harish-Chandra’s spherical function expansion (see e.g. [7]). In view of the well-known estimates for the wave propagators in Euclidean space, a naive extrapolation of Hebisch’s transfer principle to this situation (where $S$ is not a complex semisimple Lie group, unless $n = 2$) would lead to the following “conjecture”:

$$k^t_\lambda(x, y) = e^{-\frac{\pi x}{2}} e^{-\frac{nR}{2} P^t_\lambda(R)},$$

where

(a) If $R \geq 1$, then

$$|P^t_\lambda(R)| \lesssim \begin{cases} t^{-\frac{n}{2}} \lambda^{\frac{n}{2} + 1} R^{\frac{n}{2}} (1 + \lambda R)^{-N}, & \text{if } \lambda t \geq 1, \\ \lambda^{n+1} R^{\frac{n}{2}} (1 + \lambda R)^{-N}, & \text{if } \lambda t < 1. \end{cases}$$

(b) If $0 \leq R \leq 1$, then

$$|P^t_\lambda(R)| \lesssim \begin{cases} t^{-\frac{n}{2}} \lambda^{\frac{n}{2} + 1} (1 + \lambda R)^{-N}, & \text{if } \lambda t \geq 1, \\ \lambda^{n+1} (1 + \lambda R)^{-N}, & \text{if } \lambda t < 1, \end{cases}$$

From Theorems 6.1 and 7.1 one can indeed easily verify these estimates, if $\lambda \geq 2$, say.

However, if $\lambda \leq 1$, and if we choose $R = t \geq 1$ and $\lambda t \geq 1$, the “conjecture” would predict a size of order $\lambda^{\frac{n}{2}+1}$ for $|P^t_\lambda(R)|$, whereas we find the order $\lambda^2$.  

8. Growth estimates for solutions to the wave equation in terms of spectral Sobolev norms

**Theorem 8.1.** Given a symbol $m \in S^{-\alpha}$, we define operators $T_1^t := m(\sqrt{L}) \cos(t\sqrt{L})$ and $T_2^t := m(\sqrt{L}) \frac{\sin(t\sqrt{L})}{\sqrt{L}}$, a priori on $L^2(G)$, for $t \in \mathbb{R}$. Let $1 \leq p \leq \infty$.

(a) If $\alpha > n\left|\frac{1}{p} - \frac{1}{2}\right|$, then $T_1^t$ extends from $L^p \cap L^2(G)$ to a bounded operator on $L^p(G)$, and

$$||T_1^t||_{L^p \to L^p} \leq C_p \ (1 + |t|)^{\frac{1}{p} - \frac{1}{2}}.$$

(b) If $\alpha > n\left|\frac{1}{p} - \frac{1}{2}\right| - 1$, then $T_2^t$ extends from $L^p \cap L^2(G)$ to a bounded operator on $L^p(G)$, and

$$||T_2^t||_{L^p \to L^p} \leq C_p \ (1 + |t|).$$

Note that the extension is unique, if $1 \leq p < \infty$.

**Proof.** (a) Let $\chi \in C_0^\infty(\mathbb{R})$ be an even function such that $\chi(s) = 1$ if $|s| \leq \frac{1}{2}$, and $\chi(s) = 0$, if $|s| \geq 1$. Put $\psi_0(s) := \chi(s)$, and $\psi_j(s) := \chi(2^{-j}s) - \chi(2^{-j-1}s) = \psi(2^{-j}s)$, $j = 1, \ldots, \infty$, where $\psi(s) := \chi(\frac{s}{2}) - \chi(s)$ is supported in $\{s : \frac{1}{2} \leq |s| \leq 2\}$. Then $\psi_0$ is supported in $[-2, 2]$, $\psi_j$ in $\{s : 2^{j-1} \leq |s| \leq 2^{j+1}\}$ for $j \geq 1$, and

$$\sum_{j=0}^{\infty} \psi_j(s) = 1, \quad s \in \mathbb{R}. \tag{8.1}$$

We shall restrict ourselves to the case $1 \leq p < 2$, since the case $p = 2$ is trivial and the case $p > 2$ follows from the case $p < 2$ by duality. Using (8.1), we decompose the symbol $m$ as

$$m(s) = \sum_{j=0}^{\infty} m_j(2^{-j}s),$$

where $m_0 = m\chi$ and $m_j(s) := (m\psi_j)(2^j s) = m(2^j s)\psi_j$ if $j \geq 1$. Notice that

$$||m_j||_{C^\infty} \leq C 2^{-\alpha j}, \tag{8.2}$$

where the constant $C$ depends on the semi-norms $||m||_{S^{-\alpha}, k}$ only.

Then, for every $f \in L^2(G)$,

$$T_1^t f = \sum_{j=0}^{\infty} T_j f \quad \text{in} \quad L^2(G), \tag{8.3}$$

where $T_j := m_j(\sqrt{L}) \cos((2^j t)\sqrt{L})$.

Estimating the operator norms $||T_j||_{L^1 \to L^1}$ of $T_j$ on $L^1(G)$ by means of Proposition 6.3.
and \text{(8.2)}, and interpolating these estimates with the trivial \( L^2 \)-estimate \( \| T_j \|_{L^2 \to L^2} \lesssim 2^{-\alpha j} \), we obtain the following inequalities (we assume w.l.o.g. \( t \geq 0 \)):

\[
\| T_j \|_{L^p \to L^p} \lesssim \begin{cases} 
2^{-\alpha j} (1 + 2^j t)^{\frac{1}{p} - \frac{1}{2}}, & \text{if } t \leq 1, \\
2^{-\alpha j} (2^{2j} t)^{\frac{1}{p} - \frac{1}{2}}, & \text{if } t \geq 1.
\end{cases}
\]

(8.4)

The estimate in (a) follows immediately from \text{(8.4)} by summation over all \( j \geq 0 \).

As for (b), observe first that if we replace \( m_j^{(i)}(s) = \psi(s) \cos(t \sqrt{s}) \) in Section 6 by \( \tilde{m}_j^{(i)}(s) = \psi(s) \frac{\sin(t \sqrt{s})}{\sqrt{s}} \), then the factor \( s(e^{i(R-t)s} + e^{i(R+t)s}) \) in the corresponding kernel \( k_j^{(i)}(R) \) has to be replaced by \( i(e^{i(R-t)s} - e^{i(R+t)s}) \). By Lemma 6.2 with \( j = 1 \), the estimates for the function \( \tilde{k}_j^{(i)} \) associated to \( \tilde{m}_j^{(i)} \) are therefore the same as for \( k_j^{(i)} \), except for an additional factor \( 2 \). Moreover,

\[
\sup_s | m_j \left( \frac{s}{2^j}, \sin(ts) \right) | \lesssim \begin{cases} 
2^{-\alpha j} 2^{-j}, & \text{if } j \geq 1, \\
2^{-\alpha j} (1 + t), & \text{if } j = 0.
\end{cases}
\]

Together, this implies that for \( j \geq 1 \), the operators \( \tilde{T}_j \) arising in the dyadic decomposition of \( T_2 \) satisfy the same estimates as \( T_j \), except for an additional factor \( 2^{-j} \). And, for \( j = 0 \),

\[
\| \tilde{T}_0 \|_{L^2 \to L^2} \lesssim (1 + t), \quad \| \tilde{T}_0 \|_{L^1 \to L^1} \lesssim (1 + t),
\]

hence \( \| \tilde{T}_0 \|_{L^p \to L^p} \lesssim (1 + t) \). The estimates in (b) thus follow by summing over all \( j \).

Q.E.D

Let \( u = u(t, x) = u_t(x) \) be the solution of the Cauchy problem

\[
\frac{\partial^2}{\partial t^2} u - (X^2 + \sum_{j=1}^n Y_j^2) u = 0, \quad u_0 = f, \quad \frac{\partial}{\partial t} u|_{t=0} = g.
\]

(8.5)

Then, a priori for \( f, g \in L^2(G) \), \( u_t \) is given by \( u_t = \cos(t \sqrt{L}) f + \frac{\sin(t \sqrt{L})}{\sqrt{L}} g \). If we define adapted Sobolev norms

\[
\| \varphi \|_{L^p_{\alpha}} := \| (1 + L)^{\frac{\alpha}{2}} \varphi \|_{L^p}, \quad \alpha \in \mathbb{R},
\]

we therefore immediately obtain from Theorem 8.1 the following

**Corollary 8.2.** If \( 1 \leq p < \infty \), then for \( \alpha_0 > n \left| \frac{1}{p} - \frac{1}{2} \right| \) and \( \alpha_1 > n \left| \frac{1}{p} - \frac{1}{2} \right| - 1 \)

\[
\| u_t \|_{L^p} \leq C_p ( (1 + |t|)^{\frac{\alpha_1}{2} - 1} \| f \|_{L^p_{\alpha_0}} + (1 + |t|) \| g \|_{L^p_{\alpha_1}} ).
\]

(8.6)

**Remark 8.1.** It is likely that the estimate \text{(8.6)} even holds for \( \alpha_0 = n \left| \frac{1}{p} - \frac{1}{2} \right| \) and \( \alpha_1 = n \left| \frac{1}{p} - \frac{1}{2} \right| - 1 \), if \( 1 < p < \infty \). This would be the counterpart to corresponding results by Peral \[14\] and Miyachi \[11\] in the Euclidean setting (see also \[15\] for a local variable coefficient version). The sharp result would require an introduction of a suitable Hardy respectively BMO-space on \( G \). There is strong evidence that such spaces exist on \( G \), in view of the ideas in \[21\] and \[5\], but we shall not pursue these issues here.
REFERENCES

[1] Michael Cowling, Saverio Giulini, Andrzej Hulanicki, and Giancarlo Mauceri. Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth. Studia Math., 111(2):103–121, 1994.

[2] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. Higher transcendental functions. Vol. I. Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, With a preface by Mina Rees, With a foreword by E. C. Watson, Reprint of the 1953 original.

[3] G. I. Gaudry, T. Qian, and P. Sjögren. Singular integrals associated to the Laplacian on the affine group $ax + b$. Ark. Mat., 30(2):259–281, 1992.

[4] Michael Gnewuch. Zum differenzierbaren $l^p$-Funktionalkalkül auf Lie-Gruppen mit exponentiellem Volumenwachstum. Dissertation, Kiel, 2002.

[5] W. Hebisch and T. Steger. Multipliers and singular integrals on exponential growth groups. Math. Z., 245:37–61, 2003.

[6] Waldemar Hebisch. The subalgebra of $L^1(AN)$ generated by the Laplacian. Proc. Amer. Math. Soc., 117(2):547–549, 1993.

[7] Sigurdur Helgason. Groups and geometric analysis, volume 83 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000. Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original.

[8] A. Hulanicki. On the spectrum of the Laplacian on the affine group of the real line. Studia Math., 54(3):199–204, 1975/76.

[9] Alexandru D. Ionescu. Fourier integral operators on noncompact symmetric spaces of real rank one. J. Funct. Anal., 174(2):274–300, 2000.

[10] Felix Klein. Hypergeometric Functions.

[11] Akihiko Miyachi. On some estimates for the wave equation in $L^p$ and $H^p$. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 27(2):331–354, 1980.

[12] Detlef Müller. Functional calculus on Lie groups and wave propagation. In Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), number Extra Vol. II, pages 679–689 (electronic), 1998.

[13] Edward Nelson and W. Forrest Stinespring. Representation of elliptic operators in an enveloping algebra. Amer. J. Math., 81:547–560, 1959.

[14] Juan C. Peral. $L^p$ estimates for the wave equation. J. Funct. Anal., 36(1):114–145, 1980.

[15] Andreas Seeger, Christopher D. Sogge, and Elias M. Stein. Regularity properties of Fourier integral operators. Ann. of Math. (2), 134(2):231–251, 1991.

[16] Michael E. Taylor. Partial differential equations, volume 23 of Texts in Applied Mathematics. Springer-Verlag, New York, 1996. Basic theory.

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