THE UNRAMIFIED COMPUTATION OF A SHIMURA INTEGRAL FOR
SL(2) × GL(2)

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Abstract. In this note, we revisit the Rankin-Selberg integral of Shimura type for generic representations of SL_2 × GL_2, constructed by Ginzburg, Rallis, and Soudry. We give a different and more “intrinsic” proof of the unramified computation. In contrast to their proof we avoid local functional equation for the general linear groups but use the Casselman-Shalika formulas for unramified Whittaker functions for SL_2 and GL_2.

1. Introduction

In [4], Ginzburg, Rallis and Soudry constructed global integrals of Shimura type, which represent the (partial) tensor product L-function for a pair of irreducible automorphic cuspidal generic representations, one of Sp_{2n}, and the other of GL_k. They presented two different constructions that are “dual” to each other: one for the case n ≥ k and one for the case n < k. The integrals for the case n = k were also constructed by Gelbart and Piatetski-Shapiro in a prior work [3]. These integrals have been used to construct explicit functorial liftings from the general linear groups to the symplectic groups (see [5, 6]).

In this note, we consider the construction of Ginzburg, Rallis and Soudry [4] in the low rank case n = 1, k = 2. To give more details about their global integral in this case, let us introduce some notations. Let F be a global field with the ring of adeles A. Let ψ be a non-trivial additive character on F\A. We denote by ω_ψ the Weil representation of ˜SL_2(A) ⋉ H(A), where H is the Heisenberg group of dimension 3, corresponding to the character ψ. Let ˜θ_Φ be a theta series associated to a Schwartz function Φ ∈ S(A). Let P = M_P × N_P and Q = M_Q × N_Q be the Siegel parabolic subgroup and the Klingen parabolic subgroup of Sp_4 respectively. The unipotent group N_Q has a structure of the Heisenberg group H; see (3). Let τ be an irreducible automorphic cuspidal representation of GL_2(A). Let ˜E(g, ˜f_{τ,s}) be a Siegel Eisenstein series on ˜Sp_4(A), associated to a smooth holomorphic ˜K-finite section

where γ_ψ is the inverse of the Weil factor attached to the character ψ regarded as a function on GL_2(A) via the pullback of γ_ψ by the determinant map, and ˜K is the preimage of the standard maximal compact subgroup K ⊂ Sp_4(A) in ˜Sp_4(A). Let π be an irreducible automorphic cuspidal ψ^{-1}-generic representation of SL_2(A). For a cusp form φ_π ∈ V_π, we consider the following integral

\[ I(φ_π, ̃θ_Φ, ̃E(·, ̃f_{τ,s})) = \int_{SL_2(F)\backslash SL_2(A)} \int_{N_Q(F)\backslash N_Q(A)} φ_π(ug) ̃θ_Φ(ug) ̃E(ut(g), ̃f_{τ,s}) dudg. \]
Here, for $g \in \text{SL}_2$, $t(g) = \text{diag}(1, g, 1)$. We remark that the function $\tilde{\theta}_4(ug)\tilde{E}(ut(g), f_{r,s})$ can be viewed as a function on $SL_2(\mathbb{A}) \times N_Q(\mathbb{A})$, and (1) is the special case when $n = 1, k = 2$ of the integrals considered in [4, Section 5].

For $h \in \tilde{\text{Sp}}_4(\mathbb{A})$, we denote

$$\tilde{f}_{W(\tau, \psi_2), s}(h) = \int_{F^\wedge} \tilde{f}_{r,s} \left( \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) h \right) \psi_2(x) dx$$

where $\psi_2(x) = \psi(2x)$. The model $W(\tau, \psi_2)$ is the Whittaker model of the representation $\tau$ with respect to the character $(1 \ x) \mapsto \psi_2(x) = \psi(2x)$. The space of $W(\tau, \psi_2)$ consists of smooth functions $W_{\phi, \psi_2}$ on $GL_2(\mathbb{A})$ of the form

$$W_{\phi, \psi_2}(g) = \int_{N_2(F) \backslash N_2(\mathbb{A})} \phi(n g)\psi_2^{-1}(n) dn, \quad \phi \in V_{\tau}, \quad N_2 = \left\{ \left( \begin{array}{cc} 1 & \bar{z} \\ 0 & 1 \end{array} \right) \right\},$$

and the group $GL_2(\mathbb{A})$ acts in the space of $W(\tau, \psi_2)$ by right translation.

The main results on the integral (1) are summarized in the following theorem.

**Theorem 1.1** (Ginzburg-Rallis-Soudry [4]).

(i) The integral $I(\varphi_\pi, \tilde{\theta}_\Phi, \tilde{E}(:, \tilde{f}_{r,s}))$ is absolutely convergent when $\text{Re}(s) \gg 0$ and can be meromorphically continued to all $s \in \mathbb{C}$. Moreover, when $\text{Re}(s) \gg 0$, the integral $I(\varphi_\pi, \tilde{\theta}_\Phi, \tilde{E}(:, \tilde{f}_{r,s}))$ unfolds to

$$(2) \quad \int_{N_2(\mathbb{A}) \backslash SL_2(\mathbb{A})} \int_{N_2(\mathbb{A})} W_{\varphi_\pi}(g) \int_{N_Q(\mathbb{A})} \omega_\psi(r g)\Phi(1) \tilde{f}_{W(\tau, \psi_2), s}(\gamma rt(g)) dr dg,$$

where $W_{\varphi_\pi} \in W(\pi, \psi^{-1})$ is the $\psi^{-1}$-Whittaker function of $\varphi_\pi$, and

$$\gamma = \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right), \quad N_0^0 = \left\{ \left( \begin{array}{cc} 1 & \bar{z} \\ 0 & 1 \end{array} \right) \right\} \subset N_Q.$$

(ii) At a finite local place $\nu$ when all data are unramified and normalized, the local integral is equal to

$$\frac{L(\pi_\nu \times \tau_\nu, s + \frac{1}{2})}{L(\tau_\nu, \text{Sym}^2, 2s + 1)}.$$

**Remark 1.2.** We refer the reader to [4, Theorem 5.1] for a proof of Theorem 1.1 (i). We point out that since we use a different isomorphism (see Remark 2.1) between the unipotent group $N_Q$ and the Heisenberg group $\mathcal{H}$ than that of [4], after the unfolding process, we get the section $\tilde{f}_{W(\tau, \psi_2), s}$ in (2) (while in [4, Theorem 5.1] the global integral unfolds to an expression involving the section $\tilde{f}_{W(\tau, \psi), s}$).

The proof of Theorem 1.1 (ii) in [4] is based on similar ideas in [14], and it involves several key ingredients. The first one is a formal identity, analogous to the one proved in [14, Section 11.4], which relates between the local integral and certain local gamma factor; see [4, Proposition 6.1]. The second one is the application of the Casselman-Shalika formula, from which one obtains the normalizing factor for certain unramified Whittaker functional. The third is the unramified computation for the local integral for $\text{Sp}_{2n} \times GL_k$ in the case $n \geq k$. Moreover, the proof of Theorem 1.1 (ii) in [4] also involves the local functional equation for $GL_n \times GL_k$ developed in [9].

The goal of this paper is to give a direct and more “intrinsic” proof of Theorem 1.1 (ii), i.e., without resorting to the local functional equation for the general linear groups. Our method is
based on the Casselman-Shalika formulas for \( \text{SL}_2 \) and \( \text{GL}_2 \); see Theorem 3.1. Henceforth until the end of the paper, we drop the reference to the local place \( \nu \) to ease notation.

**Remark 1.3.** We remark that in [10, 11] Kaplan also obtained direct unramified computations for local integrals, but for orthogonal groups.

**Remark 1.4.** We also remark that in the general case when \( n < k \), there is an additional unipotent integration involving the section in the global integral in [4] after unfolding (i.e., the global integral unfolds to a triple integral as opposed to a double integral in (2)). However, when \( n = k - 1 \) (such as the case \( n = 1, k = 2 \) we consider in this paper), this unipotent integration is absent, making the local unramified integral closer to the even rank case for which a direct unramified computation is known. Although we only consider the low rank case \( n = 1, k = 2 \) in this paper, we hope that our calculation will shed light on the techniques for direct unramified computation.

The organization of this paper is as follows. In Section 2, we introduce general notations, the Weil representation, and the \( L \)-functions that are relevant in this paper. In Section 3, we compute the local integral when all data are unramified, and prove Theorem 1.1 (ii).

## 2. Preliminaries

### 2.1. Notations.**

Let \( F \) be a non-Archimedean local field with ring \( \mathcal{O}_F \) of integers, with a fixed uniformizer \( \varpi \). We assume that the residue characteristic of \( F \) is odd, and let \( q \) be the cardinality of the residue field. Let \( \varpi \) be the valuation function on \( F \). The absolute value \( | \cdot | \) on \( F \) is normalized so that \( |\varpi| = q^{-1} \). We fix a non-trivial additive unramified character \( \psi \) of \( F \). For any \( a \in F^x \), the character \( \psi_a \) is defined by \( \psi_a(x) = \psi(ax) \). The local Hilbert symbol is denoted by \( (\cdot, \cdot)_F \).

For a positive integer \( n \), we let \( J_n \) be the \( n \times n \) matrix whose antidiagonal entries are ones and all other entries are zeros. We realize the symplectic group \( \text{Sp}_{2n}(F) \) as

\[
\text{Sp}_{2n}(F) = \left\{ g \in \text{GL}_{2n}(F) : t^g \left( -J_n \right) g = \left( -J_n \right) \right\}.
\]

Let \( \widetilde{\text{Sp}}_{2n}(F) \) be the metaplectic double cover of \( \text{Sp}_{2n}(F) \), so we have an exact sequence

\[
1 \longrightarrow \{ \pm 1 \} \longrightarrow \widetilde{\text{Sp}}_{2n}(F) \longrightarrow \text{Sp}_{2n}(F) \longrightarrow 1.
\]

As a set, we may write elements of \( \widetilde{\text{Sp}}_{2n}(F) \) as pairs \((g, \varepsilon)\) where \( g \in \text{Sp}_{2n}(F) \), \( \varepsilon \in \{ \pm 1 \} \), with group law given by

\[
(g_1, \varepsilon_1) \cdot (g_2, \varepsilon_2) = (g_1 g_2, \varepsilon_1 \varepsilon_2 \cdot c(g_1, g_2))
\]

where \( c \) is the Ranga Rao’s 2-cocycle on \( \text{Sp}_{2n}(F) \) valued in \( \{ \pm 1 \} \) described in [13]. We refer the reader to [1] for more details.

In this paper, we will only consider the small rank symplectic groups when \( n = 1 \) and \( n = 2 \). For \( n = 2 \), the Siegel parabolic subgroup of \( \text{Sp}_4(F) \) is denoted by \( P(F) \), and the Klingen parabolic subgroup is denoted by \( Q(F) \). They have the following Levi decompositions:

\[
P = M_P \rtimes N_P, M_P = \left\{ \left( \begin{smallmatrix} m & 0 \\ J_2 & m^{-1} J_2 \end{smallmatrix} \right) : m \in \text{GL}_2 \right\},
\]

\[
Q = M_Q \rtimes N_Q, M_Q = \left\{ \left( \begin{smallmatrix} a & g \\ 0 & a^{-1} \end{smallmatrix} \right) : a \in \text{GL}_1, g \in \text{SL}_2 \right\},
\]

\[
N_P = \left\{ u(x, y, z) = \left( \begin{smallmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right) \right\}.
\]

The group \( \text{Sp}_2(F) = \text{SL}_2(F) \) embeds into \( \text{Sp}_4(F) \) via

\[
t(g) := \left( \begin{smallmatrix} 1 \\ g \\ 1 \\ 1 \end{smallmatrix} \right) \in \text{Sp}_4(F), \quad g \in \text{SL}_2(F).
\]

We regard elements of \( \text{Sp}_{2n}(F) \) as elements of \( \widetilde{\text{Sp}}_{2n}(F) \) using the trivial section \( g \mapsto (g, 1) \) (the map \( g \mapsto (g, 1) \) is not a group homomorphism). If \( \alpha \) and \( \beta \) are genuine functions on \( \text{SL}_2(F) \) and \( \text{Sp}_4(F) \) respectively, the function \( g \mapsto \alpha(g) \beta(t(g)) \) is well defined on \( \text{SL}_2(F) \).
We have the following subgroups of $\text{SL}_2(F)$:

\[ B_{\text{SL}_2}(F) = \{ (\begin{smallmatrix} a & b \\ a^{-1} & 1 \end{smallmatrix}) : a \in F^\times, b \in F \}, \quad A_2(F) = \{ (\begin{smallmatrix} a & -1 \\ a^{-1} & 1 \end{smallmatrix}) : a \in F^\times \}, \quad N_2(F) = \{ (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) : b \in F \}. \]

2.2. The Weil Representation of $\widetilde{\text{SL}}_2(F) \times \mathcal{H}(F)$. Let $\mathcal{H}(F)$ be the Heisenberg group in three variables, where the multiplication is given by

\[(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1y_2 - x_2y_1).\]

The group $\text{SL}_2(F)$ acts on $\mathcal{H}(F)$ via

\[(x, y, z) \cdot g = ((x, y)g, z), \quad g \in \text{SL}_2(F),\]

where $(x, y)g$ is the usual matrix multiplication. We then form the semi-direct product $\text{SL}_2(F) \ltimes \mathcal{H}(F)$. Associated to the character $\psi$, there is a Weil representation $\omega_\psi$ of the group $\text{SL}_2(F) \ltimes \mathcal{H}(F)$, and it can be realized on the Schwartz space $\mathcal{S}(F)$. For $a \in F^\times, b \in F, (x, y, z) \in \mathcal{H}(F)$ and $\Phi \in \mathcal{S}(F)$, we have the following formulas (see [1, Sections 2.2 and 2.5]):

\[\omega_\psi \left( \begin{smallmatrix} a & b \\ a^{-1} & 1 \end{smallmatrix} \right), \varepsilon \right) \Phi(x) = \varepsilon \gamma_\psi(a)|a|^{1/2}\Phi(x),\]

\[\omega_\psi \left( \begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right), \varepsilon \right) \Phi(x) = \varepsilon \psi(bx^2)\Phi(x),\]

\[\omega_\psi \left( \begin{smallmatrix} x & y & z \\ 1 & y & 1-x \end{smallmatrix} \right), \varepsilon \right) \Phi(x) = \varepsilon \psi(z + 2xy + xy)\Phi(x + x).\]

Here, $\gamma_\psi(a)$ is the inverse of the Weil factor associated with the character $\psi$, which satisfies the following properties (see [13, Appendix]):

\[\gamma_\psi(ab) = \gamma_\psi(a) \cdot \gamma_\psi(b) \cdot (a, b)_F, \quad \gamma_\psi(b^2) = 1, \quad \gamma_\psi(ab^2) = \gamma_\psi(a), \quad \gamma_\psi(a)^4 = 1 \quad \text{for all } a, b \in F^\times.\]

The unipotent group $N_Q(F)$ is isomorphic to the Heisenberg group $\mathcal{H}(F)$ via the following map

\[\mathcal{N}_Q(F) \to \mathcal{H}(F)\]

\[\left( \begin{array}{ccc} 1 & x & y \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \mapsto (x, y, z).\]

Remark 2.1. We remark that the above isomorphism is different from the isomorphism in [4, Section 1.3], by a factor of 2 on the $y$-coordinate. The isomorphism in (3) is more natural, while the isomorphism in [4, Section 1.3] makes the unipotent matrix $u(0, y, 0) \in N_Q(F)$ act by $\psi(y)$ under the Weil representation. We remark that both the isomorphism in (3) and the isomorphism in [4, Section 1.3] are commonly used (see, for example, [7, 16, 17]).

2.3. The tensor product $L$-function for $\text{SL}_2 \times \text{GL}_2$. In this subsection we define the $L$-functions that we study in this paper.

Let $\pi$ be an irreducible unramified representation of $\text{SL}_2(F)$ and $\tau$ be an irreducible unramified representation of $\text{GL}_2(F)$. Note that the $L$-group of $\text{SL}_2$ is $\text{SO}_3(\mathbb{C})$. Let

\[t_\pi = \text{diag}(b_1, 1, b_1^{-1})\]

be the semisimple conjugacy class in $\text{SO}_3(\mathbb{C})$ attached to $\pi$. Similarly, let

\[t_\tau = \text{diag}(a_1, a_2)\]

be the semisimple conjugacy class in $\text{GL}_2(\mathbb{C})$ attached to $\tau$. Then the local tensor product $L$-function for $\pi \times \tau$ is defined by

\[L(\pi \times \tau, s) = \det(1 - t_\pi \otimes t_\tau q^{-s})^{-1}.\]

The local symmetric square $L$-function for $\tau$ is

\[L(\tau, \text{Sym}^2, s) = (1 - a_1^2q^{-s})^{-1}(1 - a_1a_2q^{-s})^{-1}(1 - a_2^2q^{-s})^{-1}.\]
3. A New Proof of the Unramified Computation

In this section we compute the local unramified integral corresponding to the global integral (2) after unfolding.

Let $\pi$ be an irreducible unramified $\psi^{-1}$-generic summand of the induced representation

$$\text{Ind}_{B_{SL_2}(F)}^{SL_2(F)}(\chi)$$

(normalized induction)

where $\chi$ is an unramified quasi-characters of $F^\times$. Let $\tau$ be an irreducible unramified generic principal series representation of $GL_2(F)$, with

$$\tau = \text{Ind}_{B_{GL_2}(F)}^{GL_2(F)}(\chi_1 \otimes \chi_2)$$

(normalized induction)

where $\chi_1, \chi_2$ are unramified quasi-characters of $F^\times$. Note that 2 is used as a subscript for the multiplicative character $\chi_2$, but $\psi_2$ is the additive character given by $x \mapsto \psi(2x)$, and we hope that the use of 2 is clear from the context.

For $a \in F^\times$, recall that $\gamma_\psi(a)$ was defined in Section 2.2. We also regard $\gamma_\psi$ as a function on $GL_2(F)$ via the pullback of $\gamma_\psi$ by the determinant map. Consider the space

$$\text{Ind}_{\tilde{P}(F)}^{Sp_4(F)}(W(\tau, \psi_2) \otimes |\det|^s \otimes \gamma_\psi^{-1})$$

(normalized induction),

consisting of smooth functions $\tilde{f}_{W(\tau, \psi_2), s}$ on the group $\tilde{Sp}_4(F)$ which takes values in $W(\tau, \psi_2)$, i.e., for any $g \in \tilde{Sp}_4(F)$, there is a function $W^0_{\tau, s} \in W(\tau, \psi_2)$ (depending on $g$) such that

$$\tilde{f}_{W(\tau, \psi_2), s}((\begin{pmatrix} m & Z \\ m^{-1} & m^* \end{pmatrix}, \varepsilon) g) = \varepsilon \gamma_\psi(|\det(m)|^{-1}|\det(m)|^s + \frac{1}{2} \tilde{W}^g_{\tau, s}(m))$$

where $m \in GL_2(F)$, $\varepsilon \in \{\pm 1\}$.

The local unramified integral corresponding to the global integral in (2) is

$$I(W^0_{\pi}, \Phi^0, \tilde{f}^0_{W(\tau, \psi_2), s}) := \int_{N_2(F) \setminus SL_2(F)} W^0_{\pi}(g) \int_{N_0^0(F)} (\omega_\psi(rg)\Phi^0)(1)\tilde{f}^0_{W(\tau, \psi_2), s}(\gamma rt(g)) dr dg.$$ 

Here $W^0_{\pi} \in W(\pi, \psi^{-1})$ is the unramified Whittaker function for $\pi$ normalized so that $W^0_{\pi}(I_2) = 1$, $\Phi^0 = 1_{O_F}$ is the characteristic function of $O_F$, and $\tilde{f}^0_{W(\tau, \psi_2), s} \in \text{Ind}_{\tilde{P}(F)}^{Sp_4(F)}(W(\tau, \psi_2) \otimes |\det|^s \otimes \gamma_\psi^{-1})$ is the normalized unramified section so that its value at the identity is exactly the normalized unramified Whittaker function $W^0_\tau$ in $W(\tau, \psi_2)$ which has value 1 at the identity. Note that for $r \in N_0^0(F)$, the function $g \mapsto (\omega_\psi(rg)\Phi^0)(1)\tilde{f}^0_{W(\tau, \psi_2), s}(\gamma rt(g))$ is well-defined on $SL_2(F)$. Also, the map $(h, \varepsilon) \mapsto (t(h), \varepsilon)$ is an embedding of $\tilde{Sp}_2(F)$ in $\tilde{Sp}_4(F)$. To see this, one can take $(g, \varepsilon) = (I_2, 1) \in \tilde{Sp}_2(F)$ in the homomorphism given by [8, (1.28)] and note that Ranga Rao’s $x$-function is trivial on the identity and that $(x(I_2), x(h))_F = 1$. Moreover, the integral $I(W^0_{\pi}, \Phi^0, \tilde{f}^0_{W(\tau, \psi_2), s})$ converges absolutely for $\text{Re}(s) \gg 0$ [4, Proposition 6.5]. We re-state Theorem 1.1 (ii) as follows.

**Theorem 3.1.** [4, Theorem 6.3] For $\text{Re}(s) \gg 0$, we have

$$I(W^0_{\pi}, \Phi^0, \tilde{f}^0_{W(\tau, \psi_2), s}) = \frac{L(\pi \times \tau, s + \frac{1}{2})}{L(\tau, \text{Sym}^2, 2s + 1)}.$$ 

The rest of this section is devoted to a proof of Theorem 3.1, which is different from [4]. Our method is based on the Casselman-Shalika formulas for $SL_2$ and $GL_2$ which we recall below.
Lemma 3.2. [2, Theorem 5.4] Let \( a \in F^\times \). Then
\[
W_\pi^0 (a \ a_{-1}) = \begin{cases} 
|a| \cdot \frac{\chi(\omega)^{\text{ord}(a)+1}-\chi(\omega)^{-\text{ord}(a)}}{\chi(\omega)-1} & \text{if ord}(a) \geq 0 \\
0 & \text{if ord}(a) < 0 
\end{cases}
\]
and
\[
W_\tau^0 (a \ a_{1}) = \begin{cases} 
|a|^{\frac{1}{2}} \cdot \frac{\chi(\omega)^{\text{ord}(a)+1}-\chi(\omega)^{\text{ord}(a)+1}}{\chi(\omega)-\chi(\omega)^{-1}} & \text{if ord}(a) \geq 0 \\
0 & \text{if ord}(a) < 0 
\end{cases}
\]

Using the Iwasawa decomposition \( SL_2(F) = N_2(F)A_2(F)SL_2(O_F) \), we have
\[
I(W_\pi^0, \Phi^0, \tilde{f}_W^0(\tau, \psi_2), s)
= \int_{F^\times} \int_{F^2} W_\pi^0(g) \int_{N_2(F) \backslash SL_2(F)} (\omega_\psi(g)\Phi^0) (1) \tilde{f}_W^0(\tau, \psi_2), s (\gamma t g) \ dr dg \\
= \int_{F^\times} \int_{F^2} W_\pi^0 (a \ a_{-1}) \int_{F^2} (\omega_\psi((x, 0, z) \ (a \ a_{-1}) \Phi^0) (1) \tilde{f}_W^0(\tau, \psi_2), s (\gamma u(x, 0, z) \ t(a)) \ dxdz |a|^{-2} d^x a \\
= \int_{F^\times} \int_{F^2} W_\pi^0 (a \ a_{-1}) \int_{F^2} (\omega_\psi((a \ a_{-1}) (xa, 0, z))\Phi^0) (1) \tilde{f}_W^0(\tau, \psi_2), s (\gamma t(a) u(xa, 0, z)) \ dxdz |a|^{-2} d^x a.
\]

We remind the reader that we have identified an element \( r = u(x, 0, z) \in N_2^0(F) \) with an element \((x, 0, z)\) in the Heisenberg group \( H(F) \). By a change of variable \( x \mapsto xa^{-1} \) and by Lemma 3.2, we get
\[
\int_{F^\times \cap O_F} \int_{F^2} W_\pi^0 (a \ a_{-1}) \int_{F^2} (\omega_\psi((a \ a_{-1}) (x, 0, z))\Phi^0) (1) \tilde{f}_W^0(\tau, \psi_2), s (\gamma t(a) u(x, 0, z)) \ dxdz |a|^{-3} d^x a.
\]

Note that
\[
(\omega_\psi((a \ a_{-1}) (x, 0, z))\Phi^0) (1) = |a|^{-1/2} \gamma_\psi(a) (\omega_\psi((x, 0, z))\Phi^0) (a).
\]
Conjugating \( t(a) \) to the left of \( \gamma \), then we have
\[
\int_{F^\times \cap O_F} \int_{F^2} W_\pi^0 (a \ a_{-1}) \int_{F^2} (\omega_\psi((x, 0, z))\Phi^0) (a) \tilde{f}_W^0(\tau, \psi_2), s (\left( \begin{array}{c} a \\ 1 \\ a_{-1} \end{array} \right) \gamma u(x, 0, z)) \ dxdz \gamma_\psi(a)|a|^{-\frac{3}{2}} d^x a.
\]

Since \( \Phi^0 \) is supported in \( O_F \), the section \( \tilde{f}_W^0(\tau, \psi_2), s \) is unramified, and by the formula
\[
(\omega_\psi((x, 0, z))\Phi^0) (a) = (\omega_\psi((0, 0, z))\Phi^0) (a + x) = \psi(z)\Phi^0(a + x),
\]
we see that the integration over \( x \) evaluates to 1. It’s convenient to use Jacquet’s style notation for the section \( \tilde{f}_W^0(\tau, \psi_2), s \), which we will use for the remainder of the paper. Note that
\[
\tilde{f}_W^0(\tau, \psi_2), s \left( I_2; \left( \begin{array}{c} a \\ 1 \\ a_{-1} \end{array} \right) \right) \gamma u(0, 0, z) = \gamma_\psi(a)^{-1}|a|^{s+3/2} \tilde{f}_W^0(\tau, \psi_2), s (\left( \begin{array}{c} a \\ 1 \end{array} \right) \gamma u(0, 0, z))
\]
since the representation \( W(\tau, \psi_2) \otimes |\det|^s \otimes \gamma_\psi^{-1} \) acts on the Levi part by
\[
((m \ m^*) \cdot 1) \mapsto \gamma_\psi(\det(m))^{-1} |\det(m)|^{s+3/2} W(\tau, \psi_2)(m).
\]
Thus
\[
I(W_\pi, \Phi^0, \tilde{\mathcal{f}}_{W(\tau, \psi^2), s})
= \int_{F^\times \cap \mathcal{O}_F} W_\pi^0(a^{-1}) \int_F (\omega_\psi((0, 0, z))\Phi^0)(a) \tilde{\mathcal{f}}_{W(\tau, \psi^2), s}(\gamma u(0, 0, z)) dz |a|^{s-1} a^\times
\]
(6)
\[= \sum_{k=0}^{\infty} \int_{\mathcal{O}_F^k} W_\pi^0(a^{-1}) \int_F (\omega_\psi((0, 0, z))\Phi^0)(a) \tilde{\mathcal{f}}_{W(\tau, \psi^2), s}(\gamma u(0, 0, z)) dz |a|^{s-1} a^\times
\]
\[= \sum_{k=0}^{\infty} \int_{\mathcal{O}_F^k} W_\pi^0(a^{-1}) J(a) |a|^{s-1} a^\times
\]
where
\[J(a) := \int_F (\omega_\psi((0, 0, z))\Phi^0)(a) \tilde{\mathcal{f}}_{W(\tau, \psi^2), s}(\gamma u(0, 0, z)) dz.
\]
By dividing the domain of \(z\) into two parts, we can write
\[J(a) = J_1(a) + J_2(a)
\]
where
\[J_1(a) := \int_{\mathcal{O}_F} \omega_\psi((0, 0, z))\Phi^0(a) \tilde{\mathcal{f}}_{W(\tau, \psi^2), s}(\gamma u(0, 0, z)) dz
\]
and
\[J_2(a) := \int_{F \setminus \mathcal{O}_F} \omega_\psi((0, 0, z))\Phi^0(a) \tilde{\mathcal{f}}_{W(\tau, \psi^2), s}(\gamma u(0, 0, z)) dz.
\]
The values of \(J_1(a)\) and \(J_2(a)\) are computed in the following two propositions.

**Proposition 3.3.** For \(a \in F^\times \cap \mathcal{O}_F\), we have
\[
J_1(a) = |a|^{\frac{1}{2}} \cdot \frac{\chi_1(\varpi)^{\text{ord}(a)+1} - \chi_2(\varpi)^{\text{ord}(a)+1}}{\chi_1(\varpi) - \chi_2(\varpi)}.
\]

**Proof.** Since both the section \(\tilde{\mathcal{f}}_{W(\tau, \psi^2), s}\) and the character \(\psi\) are unramified, we have
\[
J_1(a) = \int_{\mathcal{O}_F} (\omega_\psi((0, 0, z))\Phi^0)(a) \tilde{\mathcal{f}}_{W(\tau, \psi^2), s}(\gamma u(0, 0, z)) dz
\]
\[= \int_{\mathcal{O}_F} \psi(z) \tilde{\mathcal{f}}_{W(\tau, \psi^2), s}(\gamma u(0, 0, z)) dz
\]
\[= W_\tau^0(a^{-1})
\]
\[= |a|^{\frac{1}{2}} \cdot \frac{\chi_1(\varpi)^{\text{ord}(a)+1} - \chi_2(\varpi)^{\text{ord}(a)+1}}{\chi_1(\varpi) - \chi_2(\varpi)}
\]
where the last equality follows from Lemma 3.2.

**Proposition 3.4.** For \(a \in F^\times \cap \mathcal{O}_F\), we have
\[
J_2(a) = |a|^{\frac{1}{2}} \cdot \frac{\chi_1(\varpi)^{\text{ord}(a)} - \chi_2(\varpi)^{\text{ord}(a)}}{\chi_1(\varpi) - \chi_2(\varpi)} \cdot \chi_1(\varpi)\chi_2(\varpi) \cdot q^{-s+1/2}.
\]
Proof. To deal with the integral when \( z \in F \setminus \mathcal{O}_F \), we consider the following matrix identity

\[
\gamma u(0, 0, z) \gamma^{-1} = \begin{pmatrix}
1 & 1 \\
-z & 1 \\
1 & z-1 \\
1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
z-1 & 1 \\
1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}.
\]

Then

\[
J_2(a) = \sum_{m=1}^{\infty} \int_{\omega^{-m} \mathcal{O}_F^\times} \omega \psi((0, 0, z)) \Phi^0(a) \tilde{p}_0^0(\gamma u(0, 0, z)) \, dz
\]

\[
= \sum_{m=1}^{\infty} \int_{\omega^{-m} \mathcal{O}_F^\times} \tilde{p}_0(\gamma u(0, 0, z)) \, dz
\]

\[
= \sum_{m=1}^{\infty} \int_{\omega^{-m} \mathcal{O}_F^\times} \gamma \psi(z^{-1})^{-1} |z^{-1}|^{s+3/2} W^0_\gamma(z^{-1}) \psi(z) \, dz.
\]

Note that

\[
W^0_\gamma(z^{-1}) = W^0_\gamma(z^{-1}) \left( \frac{z^{-1}}{z^{-1}} \right) = \chi_1(z^{-1}) \chi_2(z^{-1}) W^0_\gamma(z^{-1})
\]

\[
= \begin{cases} 
\chi_1(z^{-1}) \chi_2(z^{-1}) |az|^{\frac{1}{2}}, & \text{if } \text{ord}(z) \geq -\text{ord}(a) \\
0, & \text{if } \text{ord}(z) < -\text{ord}(a)
\end{cases}
\]

where the last equality follows from Lemma 3.2. If \( \text{ord}(a) = 0 \), then for any \( z \in \omega^{-m} \mathcal{O}_F^\times \) with \( m \geq 1 \), we have \( \text{ord}(z) < -\text{ord}(a) \), and hence \( J_2(a) = 0 \). If \( \text{ord}(a) > 0 \), then

\[
J_2(a) = \sum_{m=1}^{\text{ord}(a)} \int_{\omega^{-m} \mathcal{O}_F^\times} \gamma \psi(z^{-1})^{-1} |z^{-1}|^{s+3/2} W^0_\gamma(z^{-1}) \psi(z) \, dz.
\]

Now we assume \( \text{ord}(a) > 0 \). Then

\[
J_2(a) = \sum_{m=1}^{\text{ord}(a)} \int_{\omega^{-m} \mathcal{O}_F^\times} \gamma \psi(z^{-1})^{-1} |z^{-1}|^{s+3/2} \chi_1(z^{-1}) \chi_2(z^{-1}) |az|^{\frac{1}{2}} \cdot \frac{\chi_1(\omega) \text{ord}(az)+1}{\chi_1(\omega)} - \frac{\chi_2(\omega) \text{ord}(az)+1}{\chi_2(\omega)} \psi(z) \, dz
\]

\[
= |a|^{\frac{1}{2}} \sum_{m=1}^{\text{ord}(a)} \chi_1(\omega^m) \chi_2(\omega^m) \frac{\chi_1(\omega) \text{ord}(az)+m+1}{\chi_1(\omega)} - \chi_2(\omega) \frac{\chi_2(\omega) \text{ord}(az)+m+1}{\chi_2(\omega)} \cdot q^{-m(s+1)} \int_{\omega^{-m} \mathcal{O}_F^\times} \gamma \psi(z^{-1})^{-1} \psi(z) \, dz
\]

\[
= |a|^{\frac{1}{2}} \sum_{m=1}^{\text{ord}(a)} \chi_1(\omega^m) \chi_2(\omega^m) \frac{\chi_1(\omega) \text{ord}(az)+m+1}{\chi_1(\omega)} - \chi_2(\omega) \frac{\chi_2(\omega) \text{ord}(az)+m+1}{\chi_2(\omega)} \cdot q^{-ms} \int_{\omega^{-m} \mathcal{O}_F^\times} \gamma \psi(\omega^{-m} u^{-1})^{-1} \psi(\omega^{-m} u) \, du.
\]

We remind the reader that here both \( dz \) and \( du \) are additively invariant.
In Lemma 3.5 below, we will prove that only the term corresponding to \( m = 1 \) contributes to the sum. Assuming Lemma 3.5 for the moment, then for \( \text{ord}(a) > 0 \) we have

\[
J_2(a) = |a|^{1/2} \cdot \frac{\chi_1(\varpi)^{\text{ord}(a)} - \chi_2(\varpi)^{\text{ord}(a)}}{\chi_1(\varpi) - \chi_2(\varpi)} \cdot q^{-s} \int_{\mathcal{O}_{\varpi}^\times} \gamma_\psi(\varpi u^{-1})^{-1} \psi(\varpi^{-1} u) du.
\]

It follows from the formula \( J_1(\mathbb{F}, \psi, \chi^0) = q^{-1/2} \) in the proof of [15, Lemma 1.12] that

\[
\int_{\mathcal{O}_{\varpi}^\times} \gamma_\psi(\varpi u^{-1})^{-1} \psi(\varpi^{-1} u) du = q^{-1/2}.
\]

Therefore,

\[
J_2(a) = |a|^{1/2} \cdot \frac{\chi_1(\varpi)^{\text{ord}(a)} - \chi_2(\varpi)^{\text{ord}(a)}}{\chi_1(\varpi) - \chi_2(\varpi)} \cdot \chi_1(\varpi) \chi_2(\varpi) \cdot q^{-s-1/2}.
\]

Note that the above formula also applies to \( J_2(a) = 0 \) when \( \text{ord}(a) = 0 \).

Thus, we will finish the proof of Proposition 3.4 once we take care of Lemma 3.5. □

**Lemma 3.5.** For any \( m \geq 2 \), we have

\[
\int_{\mathcal{O}_{\varpi}^\times} \gamma_\psi(\varpi^m u^{-1})^{-1} \psi(\varpi^{-m} u) du = 0.
\]

**Proof.** This follows from the proof of [15, Lemma 1.11].

For even \( m \), we take \( \chi = 1 \) in the proof of [15, Lemma 1.11], and note that \( e = 0 \) since the residue characteristic of \( F \) is odd.

Now we consider the case where \( m \) is odd (hence \( m \geq 3 \)). Note that

\[
\gamma_\psi(\varpi^m u^{-1})^{-1} = \gamma_\psi(\varpi^m u^{-1}(\varpi^{-m} u)^2)^{-1} = \gamma_\psi(\varpi^{-m} u)^{-1}.
\]

Thus

\[
(10) \quad \int_{\mathcal{O}_{\varpi}^\times} \gamma_\psi(\varpi^m u^{-1})^{-1} \psi(\varpi^{-m} u) du = \int_{\mathcal{O}_{\varpi}^\times} \gamma_\psi(\varpi^{-m} u)^{-1} \psi(\varpi^{-m} u) du.
\]

The right-hand side of (10) is equal to the integral \( J_m(\mathbb{F}, \psi, \chi) \) in [15] with \( \chi = 1 \), which is zero by the proof of [15, Lemma 1.11]. This finishes the proof of Lemma 3.5. □
Now we plug in the formulas for $J_1(a)$ and $J_2(a)$ from Proposition 3.3 and Proposition 3.4 and the formula for $W_\pi^0$ from Lemma 3.2 into (6), so we get

\begin{align}
(11) \quad & I(W_\pi^0, \Phi^0_0, \tilde{J}_{W(r, \psi_2), s}^0) \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{Z}[O_{K}^0\mathcal{O}_{K}^0]} \frac{\chi(\varpi)^{\text{ord}(a) + 1} - \chi(\varpi)^{-\text{ord}(a)}}{\chi(\varpi) - 1} \cdot \frac{\chi_1(\varpi)^{\text{ord}(a) + 1} - \chi_2(\varpi)^{\text{ord}(a) + 1}}{\chi_1(\varpi) - \chi_2(\varpi)} |a|^{s + \frac{1}{2}} d^\times a \\
&\quad + \sum_{k=1}^{\infty} \int_{\mathbb{Z}[O_{K}^0\mathcal{O}_{K}^0]} \frac{\chi(\varpi)^{k+1} - \chi(\varpi)^{-k}}{\chi(\varpi) - 1} \cdot \frac{\chi_1(\varpi)^{k+1} - \chi_2(\varpi)^{k+1}}{\chi_1(\varpi) - \chi_2(\varpi)} \cdot \chi_1(\varpi) \chi_2(\varpi) \cdot q^{-(s + \frac{1}{2})} |a|^{s + \frac{1}{2}} d^\times a \\
&= \sum_{k=0}^{\infty} \frac{\chi(\varpi)^{k+1} - \chi(\varpi)^{-k}}{\chi(\varpi) - 1} \cdot \frac{\chi_1(\varpi)^{k+1} - \chi_2(\varpi)^{k+1}}{\chi_1(\varpi) - \chi_2(\varpi)} q^{-(s + \frac{1}{2})} \\
&\quad + \sum_{k=1}^{\infty} \frac{\chi(\varpi)^{k+1} - \chi(\varpi)^{-k}}{\chi(\varpi) - 1} \cdot \frac{\chi_1(\varpi)^{k} - \chi_2(\varpi)^{k}}{\chi_1(\varpi) - \chi_2(\varpi)} \cdot \chi_1(\varpi) \chi_2(\varpi) \cdot q^{-(k+1)(s + \frac{1}{2})}.
\end{align}

Note that

\[
\frac{\chi_1(\varpi)^{k+1} - \chi_2(\varpi)^{k+1}}{\chi_1(\varpi) - \chi_2(\varpi)} = p_k(\chi_1(\varpi), \chi_2(\varpi))
\]

where $p_k(x_1, x_2)$ is the complete homogeneous symmetric polynomial of degree $k$ in two variables, whose generating function (see [12, (2.5)]) is

\[
\sum_{k=0}^{\infty} p_k(x_1, x_2)t^k = \frac{1}{(1 - x_1 t)(1 - x_2 t)}.
\]

To ease notation, we denote

\[
A := \chi(\varpi), \quad a_1 := \chi_1(\varpi), \quad a_2 := \chi_2(\varpi), \quad X := q^{-(s + \frac{1}{2})}.
\]

Then

\[
L(\pi \times \tau, s + \frac{1}{2}) = \frac{1}{(1 - Aa_1 X)(1 - Aa_2 X)(1 - a_1 X)(1 - a_2 X)(1 - A^{-1}a_1 X)(1 - A^{-1}a_2 X)},
\]

\[
L(\tau, \text{Sym}^2, 2s + 1) = \frac{1}{(1 - a_1^2 X^2)(1 - a_1 a_2 X^2)(1 - a_2 X^2)}.
\]
The first summation in (11) can be computed as follows:

\[
\sum_{k=0}^{\infty} \frac{\chi(\omega)^{k+1} - \chi(\omega)^{-k}}{\chi(\omega) - 1} \cdot \frac{\chi_1(\omega)^{k+1} - \chi_2(\omega)^{k+1}}{\chi_1(\omega) - \chi_2(\omega)} \cdot q^{-k(s+\frac{1}{4})} = \sum_{k=0}^{\infty} \frac{A}{A-1} \cdot p_k(a_1, a_2) \cdot A^k X^k - \sum_{k=0}^{\infty} \frac{1}{A-1} \cdot p_k(a_1, a_2) \cdot A^{-k} X^k
\]

\[
= \frac{A}{(A-1)(1-Aa_1 X)(1-Aa_2 X)} - \frac{1}{(A-1)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)}
\]

\[
= \frac{(A-1)(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)}{A(1-A^{-1}a_1 X)(1-A^{-1}a_2 X) - (1-Aa_1 X)(1-Aa_2 X)}
\]

\[
= \frac{A - (a_1 + a_2)X + A^{-1}a_1a_2X^2 - 1 + A(a_1 + a_2)X - A^2a_1a_2X^2}{(A-1)(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)}
\]

\[
= \frac{1 + (a_1 + a_2)X - A^2 + A^{-1}a_1a_2X^2}{(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)}.
\]

The second summation in (11) is equal to

\[
\sum_{k=0}^{\infty} \frac{\chi_1(\omega)^{k+1} - \chi_2(\omega)^{k+1}}{\chi_1(\omega) - \chi_2(\omega)} \cdot \chi(\omega) \chi_2(\omega) \cdot q^{-(k+2)(s+\frac{1}{4})} = \sum_{k=0}^{\infty} \frac{a_1a_2A^2X^2}{A-1} \cdot p_k(a_1, a_2) \cdot A^k X^k - \sum_{k=0}^{\infty} \frac{a_1a_2A^{-1}X^2}{A-1} \cdot p_k(a_1, a_2) \cdot A^{-k} X^k
\]

\[
= \frac{a_1a_2A^2X^2}{(A-1)(1-Aa_1 X)(1-Aa_2 X)} - \frac{a_1a_2A^{-1}X^2}{A(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)}
\]

\[
= \frac{a_1a_2A^3X^2(1-A^{-1}a_1 X)(1-A^{-1}a_2 X) - a_1a_2A^{-1}X^2(1-Aa_1 X)(1-Aa_2 X)}{A(A-1)(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)}
\]

\[
= \frac{1}{(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)} \cdot \left( \frac{A^2 + A + 1}{a_1a_2X^2 - a_1a_2(a_1 + a_2)X^3 - a_1^2a_2X^4} \right).
\]

Thus,

\[
I(W_{\pi, \Phi}^0, \tilde{I}^0_{\mathcal{W}(\tau, \psi_2), s}) = \frac{1}{(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)} \cdot \left( 1 + (a_1 + a_2)X - a_1a_2(a_1 + a_2)X^3 - a_1^2a_2X^4 \right)
\]

\[
= \frac{1}{(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)} \cdot \left( 1 - a_1a_2X^2 \right) \left( 1 + a_1X \right) \left( 1 + a_2X \right)
\]

\[
= \frac{1}{(1-a_1a_2X^2)(1-a_1^{-2}X^2)(1-a_2^{-2}X^2)} \frac{1}{(1-Aa_1 X)(1-Aa_2 X)(1-A^{-1}a_1 X)(1-A^{-1}a_2 X)(1-a_1X)(1-a_2X)}
\]

\[
= \frac{L(\pi \times \tau, s + \frac{1}{2})}{L(\tau, \text{Sym}^2, 2s + 1)}.
\]

This completes the proof of Theorem 3.1.

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REFERENCES

[1] R. Berndt and R. Schmidt, Elements of the representation theory of the Jacobi group, Progress in Mathematics, Vol. 163, Birkhäuser Verlag, Basel, 1998.

[2] W. Casselman and J. Shalika, The unramified principal series of $p$-adic groups. II. The Whittaker function, Compositio Math., **41**, (1980), no. 2, 207–231.

[3] S. Gelbart and I. Piatetski-Shapiro, $L$-functions for $G \times GL(n)$. In: Explicit constructions of automorphic $L$-functions. Springer Lecture Notes in Mathematics, Vol. 1254, 1987.

[4] D. Ginzburg, S. Rallis, and D. Soudry, $L$-functions for symplectic groups, Bull. Soc. Math. France, **126** (1998), no. 2, 181–244.

[5] D. Ginzburg, S. Rallis, and D. Soudry, On explicit lifts of cusp forms from $GL_n$ to classical groups, Ann. of Math. (2), **150** (1999), 807–866.

[6] D. Ginzburg, S. Rallis, and D. Soudry, The descent map from automorphic representations of $GL(n)$ to classical groups, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[7] D. Ginzburg and D. Soudry, Integrals derived from the doubling method, Int. Math. Res. Not. IMRN, (2020), no. 24, 10553–10596.

[8] D. Ginzburg and D. Soudry, Two identities relating Eisenstein series on classical groups, J. Number Theory, **221** (2021), 1–108.

[9] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika, Rankin-Selberg convolutions, Amer. J. Math., **105** (1983), no. 2, 367–464.

[10] E. Kaplan, An invariant theory approach for the unramified computation of Rankin-Selberg integrals for quasi-split $SO_{2n} \times GL_n$, J. Number Theory, **130** (2010), no. 8, 1801–1817.

[11] E. Kaplan, The unramified computation of Rankin-Selberg integrals for $SO_{2l} \times GL_n$, Israel J. Math., **191** (2012), no. 1, 137–184.

[12] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[13] R. Ranga Rao, On some explicit formulas in the theory of Weil representation, Pacific J. Math., **157** (1993), no. 2, 335–371.

[14] D. Soudry, Rankin-Selberg convolutions for $SO_{2l+1} \times GL_n$: local theory, Mem. Amer. Math. Soc., **105**, no. 500, vi+100, 1993.

[15] D. Szpruch, Computation of the local coefficients for principal series representations of the metaplectic double cover of $SL_2(F)$, J. Number Theory, **129** (2009), no. 9, 2180–2213.

[16] P. Yan, $L$-function for $Sp_4 \times GL_2$ via a non-unique model, arXiv preprint arXiv:2110.05693, 2021.

[17] Q. Zhang, A local converse theorem for $Sp_{2r}$, Math. Ann., **372** (2018), no. 1, 451–488.

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