Asymptotic profile of a two-dimensional chemotaxis–Navier–Stokes system with singular sensitivity and logistic source

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Abstract

The chemotaxis–Navier–Stokes system

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \triangle n - \chi \nabla \cdot \left( \frac{n}{c} \nabla c \right) + n(r - \mu n), \\
    c_t + u \cdot \nabla c &= \triangle c - nc, \\
    u_t + (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \phi, \\
    \nabla \cdot u &= 0,
\end{align*}
\]

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is considered in a bounded smooth domain \( \Omega \subset \mathbb{R}^2 \), where \( \phi \in W^{1,\infty}(\Omega) \), \( \chi > 0 \), \( r \in \mathbb{R} \) and \( \mu > 0 \) are given parameters. It is shown that there exists a value \( \mu^*(\Omega, \chi, r) \geq 0 \) such that whenever \( \mu > \mu^*(\Omega, \chi, r) \), the global-in-time classical solution to the system is uniformly bounded with respect to \( x \in \Omega \). Moreover, for the case \( r > 0 \), \((n, c, |\nabla c|, u)\) converges to \((\frac{\chi}{r}, 0, 0, 0)\) in \( L^\infty(\Omega) \times L^\infty(\Omega) \times L^p(\Omega) \times L^\infty(\Omega) \) for any \( p > 1 \) exponentially as \( t \to \infty \), while in the case \( r = 0 \), \((n, c, |\nabla c|, u)\) converges to \((0, 0, 0, 0)\) in \((L^\infty(\Omega))^4\) algebraically. To the best of our knowledge, these results provide the first precise information on the asymptotic profile of solutions.

**Keywords:** Chemotaxis, Navier–Stokes, singular sensitivity, asymptotic profile.

**AMS Classification** (2010): 35K55, 35Q30, 35B40, 35Q92, 76D05, 92C17.

# 1 Introduction

In their seminal work ([14, 15]), using cross-diffusive parabolic PDE systems, Keller and Segel studied the phenomenon of the emergence of spatial structures in biological systems through chemically induced processes. In particular, they looked at situations where components of the biological system were able to actively secrete a chemoattractive signal which then directed the movements of components in the system; or where, instead of having the ability to produce such signals, components of the system simply followed and consumed a chemical nutrient produced externally. A prototypical example of the former is the dictyostelium discoideum colony, while that of the latter is an E. coli population. The movement thus induced by either a chemical signal or nutrient is called chemotaxis, and the corresponding model of the latter type is sometimes called a chemotaxis–consumption model. Often, such chemotactic movements take place in a fluid environment, and experimental findings and analytical studies have revealed the remarkable effects of chemotaxis–fluid interaction on the overall behavior of the respective chemotaxis systems, such as the prevention of blow-up and improvement of efficiency of mixing ([8, 16, 17, 29, 37]).

In this paper, we are concerned with the chemotaxis–consumption system coupled with
the incompressible Navier–Stokes equations

\[
\begin{cases}
  n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot \left( \frac{n}{c} \nabla c \right) + f(n), & x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, t > 0, \\
  u_t + (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, t > 0, \\
  \nabla \cdot u = 0, & x \in \Omega, t > 0,
\end{cases}
\]

(1.1)

describing the biological population density \( n \), the chemical signal concentration \( c \), the incompressible fluid velocity \( u \) and the associated pressure \( P \) of the fluid flow in the physical domain \( \Omega \subset \mathbb{R}^N \). It is assumed that \( n \) and \( c \) diffuse randomly as well as are transported by the fluid, with a bouyancy effect on \( n \) through the presence of a given gravitational potential \( \phi \). Further, it is assumed that the chemotactic stimulus is perceived in accordance with the Weber–Fechner Law ([34, 40, 52]) which states that subjective sensation is proportional to the logarithm of the stimulus intensity, in other words, the population \( n \) partially direct their movement toward increasing concentrations of the chemical nutrient \( c \) that they consume with the logarithmic sensitivity. In addition, on the considered time scales of cell migration, we allow for population growth to take place, through the term \( f(n) = rn - \mu n^2 \) with the effective growth rate \( r \in \mathbb{R} \) and strength of the overcrowding effect \( \mu > 0 \); we note that \( r = 0 \) is allowed and has indeed been argued for in certain models ([12, 16]).

The system (1.1) appears to generate interesting, non-trivial dynamics. However, to the best of our knowledge, no analytical result is available yet which rigorously describes the qualitative behavior of such solutions. This may be due to the circumstance that (1.1) joins two subsystems which are far from being fully understood even when decoupled from each other. Indeed, (1.1) contains the Navier–Stokes equations which themselves do not admit a complete existence and regularity theory ([32]).

At the same time, by setting \( u \equiv 0 \) in (1.1), we arrive at the following chemotaxis–consumption model

\[
\begin{cases}
  n_t = \Delta n - \chi \nabla \cdot \left( \frac{n}{c} \nabla c \right), \\
  c_t = \Delta c - nc,
\end{cases}
\]

(1.2)

where population growth has been ignored, which was introduced by Keller and Segel ([15]) to describe the collective behaviour of the bacteria E. coli set in one end of a capillary.
tube featuring a gradient of nutrient concentration observed in the celebrated experiment of Adler ([11]). Later, this model was also employed to describe the dynamical interactions between vascular endothelial cells and vascular endothelial growth factor (VEGF) during the initiation of tumor angiogenesis (see [9, 26]). It has already been demonstrated that the logarithmic sensitivity featured in (1.2) renders a significant degree of complexity in the system; in particular, it plays an indispensable role in generating wave-like solutions without any type of cell kinetics ([12, 14, 32, 33, 40]), which is a prominent feature in the Fisher equation ([18]).

In comparison with (1.2), the related chemotaxis system

\[
\begin{align*}
  n_t &= \Delta n - \chi \nabla \cdot (\frac{n}{c} \nabla c) + f(n), \\
  c_t &= \Delta c - c + n,
\end{align*}
\]  

(1.3)

where the chemical signal \( c \) is actively secreted by the bacteria rather than consumed (see [2, 12]), has been more extensively studied. It is observed that the chemical signal production mechanism in the \( c \)-equation inhibits the tendency of \( c \) to take on small values, and thereby the singularity in the sensitivity function is mitigated. Accordingly, for such higher dimensional systems with reasonably smooth but arbitrarily large data, the global existence of bounded smooth solutions can be achieved. Indeed, global existence and boundedness of classical solutions to (1.3) without source terms is guaranteed if \( \chi \in (0, \sqrt{\frac{2}{N}}) \) ([11, 45]), or if \( N = 2, \chi \in (0, \chi_0) \) with some \( \chi_0 > 1.015 \) ([22]), while certain generalized solutions have been constructed for general \( \chi > 0 \) in the two-dimensional radially symmetric case ([35, 45]). Moreover, without any symmetry hypothesis, Winkler and Lankeit established the global solvability of generalized solutions for the cases \( \chi < \infty, N = 2; \chi < \sqrt{8}, N = 3; \) and \( \chi < \frac{N}{N-2}, N \geq 4 \) ([24]).

Furthermore, in accordance with known results for the classical Keller–Segel chemotaxis model (see [21, 44, 46] for example), the presence of the logistic source term \( f(n) = n(r - \mu n) \) in (1.3) can inhibit the tendency toward explosions of cells at least under some restrictions on certain parameters. Indeed, it is known that (1.3) with \( N = 2 \) possesses a global classical solution \((n, c)\) for any \( r \in \mathbb{R}, \chi, \mu > 0, \) and \((n, c)\) is globally bounded if \( r > \frac{\chi^2}{4} \) for \( 0 < \chi \leq 2 \) or \( r > \chi - 1 \) for \( \chi > 2 \) ([53]). Moreover, \((n, c)\) exponentially converges to \((\frac{r}{\mu}, \frac{r}{\mu})\) in \( L^\infty(\Omega) \)
provided that \( \mu > 0 \) is sufficiently large (\([56]\)). As for the higher dimensional cases (\( N \geq 2 \)),
the global very weak solution of (1.3) with \( f(n) = rn - \mu n^k \) is constructed when \( k, \chi \) and \( r \) fulfill a certain condition. In addition, when \( N = 2 \) or 3, this solution is global bounded provided \( \frac{r}{\mu} \) and the initial data \( \|n_0\|_{L^2}, \|\nabla c_0\|_{L^4} \) are suitably small (\([55]\)).

In contrast to (1.3), system (1.2) is more challenging due to the combination of the consumption of \( c \) with the singular chemotaxis sensitivity of \( n \). Intuitively, the absorption mechanism in the \( c \)-equation of (1.2), which induces the preference for small values of \( c \), considerably intensifies the destabilizing potential of singular sensitivity in the \( n \)-equation. Up to now, it seems that only limited results on global classical solvability in the spatial two-dimensional case are available. In fact, only recently have certain global generalized solutions to (1.2) been constructed for general initial data in \([20, 47, 50]\), whereas with respect to global classical solvability, it has only been shown for some small initial data (see \([41, 48]\)). In particular, Winkler (\([48]\)) showed that the global classical solutions to (1.2) in bounded convex two-dimensional domains exist and converge to the homogeneous steady state under an essentially explicit smallness condition on \( n_0 \) in \( L \log L(\Omega) \) and \( \nabla \ln c_0 \) in \( L^2(\Omega) \).

We would, however, like to note that numerous variants of (1.2), such as those involving nonlinear diffusion, logistic-type cell kinetics and saturating signal production (\([10, 13, 19, 23, 25, 28, 38, 54]\)), have been studied. For example, the authors of \([54]\) proved that the particular version of (1.2) by adding \( f(n) = rn - \mu n^k \) \( (r > 0, \mu > 0, k > 1) \) into the \( n \)-equation admits a global classical solution \((n, c)\) in the bounded domain \( \Omega \subset \mathbb{R}^N \) if \( k > 1 + \frac{N}{2} \), and in the two-dimensional setting, \((n, c, \frac{\nabla c}{c}) \rightarrow \left( \left( \frac{r}{\mu} \right)^{\frac{1}{k-1}}, 0, 0 \right) \) for sufficiently large \( \mu \). In particular, it is shown in the recent paper \([19]\) that (1.2) with logistic source \( f(n) = rn - \mu n^2 \) \( (r \in \mathbb{R}, \mu > 0) \) possesses a unique global classical solution if \( 0 < \chi < \sqrt{\frac{2}{N}}, \mu > \frac{N-2}{2N} \), and a globally bounded solution only in one dimension for any \( \chi > 0, \mu > 0 \). Also, the author of \([39]\) showed that if \( \mu > \mu_0 \) with some \( \mu_0 = \mu_0(\Omega, \chi) > 0 \) then the corresponding classical solution is globally bounded, and \((n, c, \frac{\nabla c}{c}) \rightarrow \left( \frac{r}{\mu}, \lambda, 0 \right) \) with \( \lambda \in [0, \frac{1}{|\Omega|} \int_{\Omega} c_0] \) in \((L^\infty(\Omega))^3 \) as \( t \rightarrow \infty \). Of course, this leaves open the possibility of blow-up of solutions when \( \mu \) is positive but small. Anyhow, it have been shown in \([49]\) that when \( \mu > 0 \) is suitably small, the strongly destabilizing
action of chemotactic cross–diffusion may lead to the occurrence of solutions which attain possibly finite but arbitrarily large values.

Coming back to our chemotaxis–consumption–fluid model (1.1), as we have already pointed out, very little seems to be known regarding the qualitative behavior of solutions ([4, 5, 6]). In fact, we are aware of one result only which is concerned with the asymptotic behavior and eventual regularity of solutions to the Stokes–variant of (1.1). Namely, it is shown in [4] that for small initial mass \( \int_{\Omega} n_0 \), the corresponding system upon neglect of \( u \cdot \nabla u \) and \( f(n) \) in (1.1) possesses at least one global generalized solutions, which will become smooth after some waiting time and stabilize toward the steady state \( \left( \frac{1}{|\Omega|} \int_{\Omega} n_0, 0, 0 \right) \) with respect to the topology of \((L^{\infty}(\Omega))^3\). Since the presence of the fluid interaction does not have any regularizing effect on the large time behavior, it is expected that instead of the small restriction on the initial data, the quadratic degradation may have a substantial regularizing effect on the dynamic behavior of solutions to (1.1).

The goal of the present work is to give the asymptotic profile in time of solutions to (1.1) in the two-dimensional case. In order to state our main results, we shall impose on (1.1) the boundary conditions

\[
\nabla n \cdot \nu = \nabla c \cdot \nu = 0 \quad \text{and} \quad u = 0 \quad \text{for} \quad x \in \partial \Omega,
\]

(1.4)

and initial conditions

\[
n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), \quad u(x, 0) = u_0(x) \quad \text{for} \quad x \in \Omega.
\]

(1.5)

Throughout this paper, it is assumed that

\[
\begin{cases}
  n_0 \in C^0(\bar{\Omega}), \quad n_0 \geq 0 \text{ and } n_0 \not\equiv 0 \text{ in } \Omega, \\
  c_0 \in W^{1,\infty}(\Omega), \quad c_0 > 0 \text{ in } \bar{\Omega} \quad \text{as well as} \\
  u_0 \in D(A^\beta) \text{ for all } \beta \in \left(\frac{1}{2}, 1\right)
\end{cases}
\]

(1.6)

with \( A \) denoting the Stokes operator \( A = -\mathcal{P}\Delta \) with domain \( D(A) := W^{2,2}(\Omega; \mathbb{R}^2) \cap W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega) \), where \( L^2_\sigma(\Omega) := \{ \varphi \in L^2(\Omega; \mathbb{R}^2) | \nabla \cdot \varphi = 0 \} \) and \( \mathcal{P} \) stands for the Helmholtz projection of \( L^2(\Omega) \) onto \( L^2_\sigma(\Omega) \).
Within this framework, by straightforward adaptation of arguments in [19] with only some necessary modifications, one can see that the problem (1.1), (1.4), (1.5) admits a global classical solution \((n, c, u, P)\) whenever \(\chi \in (0, 1), r \in \mathbb{R}\) and \(\mu > 0\), which is unique up to addition of constants in the pressure variable \(P\), and satisfies \(n > 0, c > 0\) in \(\Omega \times [0, \infty)\).

The first of our main results is concerned with the global boundedness of the solution as well as its asymptotic behavior.

**Theorem 1.1.** Let \(f(n) = rn - \mu n^2\), \(r \in \mathbb{R}\), \(\mu > 0\) and \(\phi \in W^{1,\infty}(\Omega)\), and suppose that \((n_0, c_0, u_0)\) satisfy (1.6). If \((n, c, u, P)\) denotes the corresponding global classical solution to (1.1), (1.4), (1.5), then there exists a value \(\mu_0 := \mu_0(\Omega, \chi, r) \geq 0\) with \(\mu_0(\Omega, \chi, 0) = 0\) such that whenever \(\mu > \mu_0\), \((n, c, u)\) is global bounded,

\[
\|n(\cdot, t) - \frac{r+r_+}{\mu}\|_{L^\infty(\Omega)} \to 0, \quad \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0
\]

and when \(r > 0\),

\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} \to 0
\]

as \(t \to \infty\).

As indicated in the above discussion, we need to introduce new ideas to show how the regularizing effect of the quadratic degradation in the chemotaxis–fluid model (1.1) can counterbalance the strongly destabilizing action of chemotactic cross–diffusion caused by the combination of the consumption of \(c\) with the singular chemotaxis sensitivity of \(n\). Specifically, we develop the conditional energy functional method in [48] to show the global boundedness of solutions in the case of \(r > 0\), in which the key point is to verify that

\[
\mathcal{F}(n, w) := \int_\Omega H(n) + \frac{\chi}{2} \int_\Omega |\nabla w|^2, \quad w := -\ln\left(\frac{c}{\|c_0\|_{L^\infty(\Omega)}}\right)
\]

with \(H(s) := s \ln \frac{\mu s}{er} + \frac{r}{\mu}\) constitutes an energy functional in the sense that \(\mathcal{F}(n, w)\) is non-increasing in time whenever \(\mu\) is appropriately large relative to \(r\) (see Lemma 3.2). Indeed, from (3.13), one can obtain the global bound of \(\int_\Omega n|\ln n|dx\) and \(\int_\Omega |\nabla w|^2dx\), which then serves as a starting point to derive the uniform bound of \(\|n(\cdot, t)\|_{L^\infty(\Omega)}\) via the Neumann heat semigroup estimates. Furthermore, by making appropriate use of the dissipative information
expressed in (3.13), we can establish the convergence result asserted in Theorem 1.1. It is noted that compared to that of the case \( r > 0, \mu > 0 \), the proof of Theorem 1.1 in the case of \( r \leq 0, \mu > 0 \) involves a more delicate analysis. In fact, unlike in the case \( r = \mu = 0 \) or \( r > 0, \mu > 0 \), with \( r \leq 0, \mu > 0 \) seems to lack the favorable structure that facilitates such conditional energy-type inequalities. Taking full advantage of the decay information on \( n \) in \( L^1 \)-norm expressed in (2.2), our approach toward Theorem 1.1 is to construct the quantity

\[
F(n, w) := \int_\Omega n(\ln n + a) + \frac{\chi}{2} \int_\Omega |\nabla w|^2
\]

with parameter \( a > 0 \) determined below (see (3.49)). Unlike in the case of \( r > 0 \), \( F(n, w) \) does not enjoy monotonicity property, it however satisfies a favorable non-homogeneous differential inequality (3.55) in the sense that it can provide us a priori information on solution such as the global bound of \( \int_\Omega n|\ln n|dx \) and \( \int_\Omega |\nabla w|^2dx \) (see Lemma 3.4), as well as \( \lim_{t \to \infty} \int_\Omega |\nabla w(\cdot, t)|^2 = 0 \) (see (3.61)).

As an important step to understand the model (1.1) more comprehensively, we shall consider the convergence rate of its classical solutions in the form of the following result:

**Theorem 1.2.** Let the assumptions of Theorem 1.1 hold and \( r > 0 \). Then one can find \( \mu_*(\chi, \Omega, r) > 0 \) such that if \( \mu > \mu_*(\chi, \Omega, r) \), the classical solution of (1.1), (1.4), (1.5) presented in Theorem 1.1 satisfies

\[
\|n(\cdot, t) - \frac{r}{\mu} \|_{L^\infty(\Omega)} \to 0, \quad \|c(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0
\]

as well as

\[
\|\frac{\nabla c}{c}(\cdot, t)\|_{L^p(\Omega)} \to 0
\]

for all \( p > 1 \) exponentially as \( t \to \infty \).

This implies that suitably large \( \mu \) relative to \( r \) enforces asymptotic stability of the corresponding constant equilibria of (1.1); however, the optimal lower bound on \( \frac{\mu}{r} \) seems yet lacking. The main ingredient of the our approach toward Theorem 1.2 involves a so-called self-map-type reasoning. More precisely, making use of the convergence properties of \((n, \frac{\nabla c}{c})\)
asserted in Theorem 1.1, we prove by a self-map-type reasoning that whenever $\mu$ is suitably large compared with $r$,

$$(n(\cdot,t) - \frac{r}{\mu} c(\cdot,t), u) \longrightarrow (0,0,0) \quad \text{and} \quad \frac{\nabla c}{c}(\cdot,t) \longrightarrow 0$$

in $(L^\infty(\Omega))^3$ and $L^6(\Omega)$ exponentially as $t \rightarrow \infty$, respectively (see Lemma 4.2).

As aforementioned, the limit case $r = 0$ becomes relevant in several applications. In this limiting situation, the total cell population can readily be seen to decay in the large time limit (cf. Lemma 2.3 below). As a consequence, we can obtain the decay properties of solutions, namely that the decay on $n$ in $L^1$ actually occurs in $L^\infty$, and also for $c$. More precisely, our result reads as follows:

**Theorem 1.3.** Let the assumptions of Theorem 1.1 hold and $r = 0$. Then the classical solution of (1.1), (1.4), (1.5) from Theorem 1.1 satisfies

$$(n, c, \frac{\nabla c}{c}, u) \longrightarrow (0,0,0,0)$$

in $(L^\infty(\Omega))^4$ algebraically as $t \rightarrow \infty$.

The result indicates that structure generating dynamics in the spatially two-dimensional version of (1.1), (1.4), (1.5), if at all, occur on intermediate time scales rather than in the sense of a stable large-time pattern formation process. Apparently, it leaves open the questions whether the more colorful large time behavior can appear in the three-dimensional version of (1.1).

Our approach toward Theorem 1.3 uses an alternative method, which, at its core, is based on the argument that the $L^\infty$-norm of $n$ can be controlled from above by appropriate multiples of $\frac{1}{t+1}$. This results from a suitable variation-of-constants representation of $n$, by which and in view of the decay information on $|\nabla w|$ in $L^\infty(\Omega)$, the $L^1$ decay information on $u$ from (2.2) can be turned into the $L^\infty$-norm of $n$ (see Lemma 4.3). As a consequence, by comparison argument, we have a pointwise upper estimate for $w$ as well as a lower estimate for $v$ (see Lemma 4.4). Using $L^p - L^q$ estimates for the Neumann heat semigroup $(e^{t\Delta})_{t>0}$, we then successively show that $\|\nabla w\|_{L^\infty}$ and $\|n\|_{L^\infty(\Omega)}$ can be controlled by appropriate multiples of $\frac{1}{t+1}$ from above and below, respectively (see Lemma 4.5). These a priori estimates allow
us to get the pointwise lower estimate for \(w\) as well as the upper estimate for \(c\), which complement the lower bound for \(c\) previously obtained, and thereby prove that \(c\) actually decays algebraically.

2 Preliminaries

In this section, we begin by recalling the important \(L^p - L^q\) estimates for the Neumann heat semigroup \((e^{t\Delta})_{t>0}\) on bounded domains (see [7, 13]).

**Lemma 2.1.** Let \((e^{t\Delta})_{t>0}\) denote the Neumann heat semigroup in the bounded domain \(\Omega \subset \mathbb{R}^n\) and \(\lambda_1 > 0\) denote the first nonzero eigenvalue of \(-\Delta\) in \(\Omega\) under the Neumann boundary condition. Then there exists \(c_i > 0\) (\(i = 1, 2, 3\)) such that for all \(t > 0\),

(i) If \(1 \leq q \leq p \leq \infty\), then for all \(\omega \in L^q(\Omega)\),

\[
\|\nabla e^{t\Delta} \omega\|_{L^p(\Omega)} \leq c_1 \left(1 + t^{-\frac{1}{2} - \frac{n}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}\right) e^{-\lambda_1 t} \|\omega\|_{L^q(\Omega)};
\]

(ii) If \(2 \leq q \leq p < \infty\), then for all \(\omega \in W^{1,q}(\Omega)\),

\[
\|\nabla e^{t\Delta} \omega\|_{L^p(\Omega)} \leq c_2 \left(1 + t^{-\frac{n}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}\right) e^{-\lambda_1 t} \|\nabla \omega\|_{L^q(\Omega)};
\]

(iii) If \(1 \leq q \leq p \leq \infty\) or \(1 < q < \infty\) and \(p = \infty\), then for all \(\omega \in (L^q(\Omega))^n\),

\[
\|e^{t\Delta} \nabla \cdot \omega\|_{L^p(\Omega)} \leq c_3 \left(1 + t^{-\frac{1}{2} - \frac{n}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}\right) e^{-\lambda_1 t} \|\omega\|_{L^q(\Omega)}.
\]

**Lemma 2.2.** (Gagliardo–Nirenberg inequality) Let \(\Omega \subset \mathbb{R}^2\) be a bounded Lipschitz domain. Then i) there is \(K_1 > 0\) such that

\[
\|\nabla \varphi\|_{L^4(\Omega)}^4 \leq K_1 \|\Delta \varphi\|_{L^2(\Omega)}^2 \|\nabla \varphi\|_{L^2(\Omega)}^2
\]

for all \(\varphi \in W^{2,2}(\Omega)\) fulfilling \(\frac{\partial \varphi}{\partial n}|_{\partial \Omega} = 0\);

ii) there is \(K_2 > 0\) such that

\[
\|\varphi\|_{L^3(\Omega)}^3 \leq K_2 \|\varphi\|_{W^{1,2}(\Omega)}^2 \|\varphi\|_{L^1(\Omega)}
\]

for all \(\varphi \in W^{1,2}(\Omega)\).
In order to derive some essential estimates, it would be more convenient to deal with a nonsingular chemotaxis term of the form $\nabla \cdot (n \nabla w)$ instead of $\nabla \cdot (\frac{n}{c} \nabla c)$ in (1.1). To this end, we employ the following transformation as in \[19, 23, 47\]:

$$w := -\ln\left(\frac{c}{\|c_0\|_{L^\infty(\Omega)}}\right),$$

whereupon $0 \leq w \in C^0(\bar{\Omega} \times (0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$, and the problem (1.1), (1.4), (1.5) transforms to

$$\begin{cases}
n_t + u \cdot \nabla n = \Delta n + \chi \nabla \cdot (n \nabla w) + n(r - \mu n), & x \in \Omega, t > 0, \\
w_t + u \cdot \nabla w = \Delta w - |\nabla w|^2 + n, & x \in \Omega, t > 0, \\
u_t + (u \cdot \nabla)u = \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, t > 0, \\
\nabla \cdot u = 0, & x \in \Omega, t > 0, \\
\nabla n \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial \Omega, t > 0, \\
n(x, 0) = n_0(x), & w(x, 0) = -\ln\left(\frac{c_0(x)}{\|c_0\|_{L^\infty(\Omega)}}\right), & u = u_0(x), & x \in \Omega.
\end{cases}$$

(2.1)

Let us first recall some basic but important information about $(n, w)$ due to the presence of the quadratic degradation term in the first equation of (2.1).

**Lemma 2.3.** The classical solution $(n, w, u, P)$ of (2.1) satisfies

(i) \(\limsup_{t \to \infty} \|n(\cdot, t)\|_{L^1(\Omega)} \leq \frac{\|\Omega\| r_+}{\mu}\);

(ii) \(\int_{t_0}^t \|n(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq \frac{r_+}{\mu} \int_{t_0}^t \|n(\cdot, s)\|_{L^1(\Omega)} ds + \frac{1}{\mu} \|n(\cdot, t_0)\|_{L^1(\Omega)} \) for all \(t > t_0\);

(iii) \(\int_\Omega \int_\Omega |\nabla w|^2 dx ds \leq \int_\Omega w(x, t_0) dx + \int_{t_0}^t \|n(\cdot, s)\|_{L^1(\Omega)} ds \) for all \(t > t_0\).

In particular, if \(r \leq 0\), then

\[\|n(\cdot, t)\|_{L^1(\Omega)} \leq \frac{\|\Omega\|}{\mu(t + \gamma)} \] for all \(t > t_0\)

(2.2)

with \(\gamma = \frac{\|\Omega\|}{\mu \int_\Omega n_0(x) dx}\).

**Proof.** Integrating the first equation in (2.1) and using the Cauchy–Schwarz inequality, we get

\[
\frac{d}{dt} \int_\Omega n = r \int_\Omega n - \mu \int_\Omega n^2 \leq r_+ \int_\Omega n - \frac{\mu}{\|\Omega\|} \left( \int_\Omega n \right)^2
\]

(2.3)
which yields (i) readily. By the time integration of (2.3) over \((t_0, t)\), we get (ii) immediately. In addition, from the second equation in (2.1), \(\nabla \cdot u = 0\) and \(u = 0\) on \(\partial \Omega\), it follows that
\[
\frac{d}{dt} \int_{\Omega} w = - \int_{\Omega} |\nabla w|^2 + \int_{\Omega} n, \tag{2.4}
\]
and thus establishes (iii).

When \(r \leq 0\), it follows from (2.3) that
\[
\frac{d}{dt} \int_{\Omega} n \leq - \frac{\mu}{|\Omega|} (\int_{\Omega} n)^2 \tag{2.5}
\]
which then yields (2.2) by the time integration.

In order to make use of the spatio-temporal properties provided by Lemma 2.3(ii) to estimate the ultimate bound of \(\int_{\Omega} |\nabla u|^2\), we shall utilize the following elementary lemma (see Lemma 3.4 of [51]):

**Lemma 2.4.** Let \(t_0 \geq 0, T \in (t_0, \infty], a > 0\) and \(b > 0\), and suppose that the nonnegative function \(h \in L^1_{loc}(\mathbb{R})\) satisfies \(\int_{t}^{t+1} h(s)ds \leq b\) for all \(t \in [t_0, T]\). If \(y \in C^0([t_0, T]) \cap C^1([t_0, T])\) has the property that
\[
y'(t) + ay(t) \leq h(t) \quad \text{for all } t \in (t_0, T),
\]
then
\[
y(t) \leq e^{-a(t-t_0)}y(t_0) + \frac{b}{1 - e^{-a}} \text{ for all } t \in [t_0, T].
\]

With Lemmas 2.3 and 2.4 at hand, we can employ the standard energy inequality associated with the fluid evolution system in (2.1) to derive some boundedness results for \(u\).

**Lemma 2.5.** For the global classical solution \((n, w, u)\) of (2.1), we have

i) if \(r > 0\), then
\[
\limsup_{t \to \infty} \|u(\cdot, t)\|_{L^2(\Omega)}^2 \leq \frac{3(1 + r)|\Omega|}{\mu} \frac{\|\nabla \phi\|_{L^\infty(\Omega)}^2 r}{C_p(1 - e^{-\frac{C_p}{T}})} \tag{2.6}
\]
as well as
\[
\limsup_{t \to \infty} \int_{t}^{t+1} \|\nabla u(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq \frac{5(1 + r)|\Omega|}{\mu} \frac{\|\nabla \phi\|_{L^\infty(\Omega)}^2 r}{C_p(1 - e^{-\frac{C_p}{T}})} \tag{2.7}
\]
with Poincaré constant $C_p > 0$.

ii) if $r \leq 0$, then

$$\int_\Omega |u(\cdot, t)|^2 \leq \|u(\cdot, t_0)\|_{L^2(\Omega)}^2 e^{-\frac{C_p}{2}(t-t_0)} + \frac{2|\Omega|}{\mu^2} \|\nabla \phi\|_{L^\infty(\Omega)}^2 \frac{1}{C_p(1-e^{-\frac{C_p}{2}}) t_0 + \gamma}$$

for all $t > t_0$ (2.8)

as well as

$$\int_t^{t+1} \|\nabla u(\cdot, s)\|_{L^2(\Omega)}^2 ds$$

$$\leq \|u(\cdot, t_0)\|_{L^2(\Omega)}^2 e^{-\frac{C_p}{2}(t-t_0)} + \frac{4|\Omega|}{\mu^2} \|\nabla \phi\|_{L^\infty(\Omega)}^2 \frac{1}{C_p(1-e^{-\frac{C_p}{2}}) t_0 + \gamma}$$

for all $t > t_0$. (2.9)

Proof. i) According to the Poincaré inequality, one can find some constant $C_p > 0$ such that

$$C_p \int_\Omega |u|^2 \leq \int_\Omega |\nabla u|^2.$$

Testing the third equation in (2.1) by $u$ and using the Hölder inequality, we obtain

$$\frac{d}{dt} \int_\Omega |u|^2 + C_p \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \leq 2 \int_\Omega \nabla \phi \cdot u$$

$$\leq 2 \|\nabla \phi\|_{L^\infty(\Omega)} \|n\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$$

$$\leq C_p \frac{2}{2} \|u\|_{L^2(\Omega)}^2 + \frac{2}{C_p} \|\nabla \phi\|_{L^\infty(\Omega)} \|n\|_{L^2(\Omega)}^2,$$

due to $u|_{\partial \Omega} = 0$ and $\nabla \cdot u = 0$.

Writing $h(t) = \frac{2}{C_p} \|\nabla \phi\|_{L^\infty(\Omega)} \|n(\cdot, t)\|_{L^2(\Omega)}^2$, we see that $y(t) : = \int_\Omega |u(\cdot, t)|^2$ satisfies

$$y'(t) + \frac{C_p}{2} y(t) + \int_\Omega |\nabla u(\cdot, t)|^2 \leq h(t) \quad \text{for all } t > 0.$$  (2.10)

In view of Lemma 2.3 (i) and (ii), we know that

$$\limsup_{t \to \infty} \int_t^{t+1} h(s) ds \leq \frac{2}{C_p} \|\nabla \phi\|_{L^\infty(\Omega)}^2 \frac{(1 + r)|\Omega|}{\mu} \frac{r}{\mu}.$$  (2.11)

An application of Lemma 2.4 thus shows that there exists positive $t_0 > 0$ such that

$$\int_\Omega |u(\cdot, t)|^2 \leq \|u(\cdot, t_0)\|_{L^2(\Omega)}^2 e^{-\frac{C_p}{2}(t-t_0)} + \frac{3(1 + r)|\Omega|}{\mu} \|\nabla \phi\|_{L^\infty(\Omega)}^2 \frac{r}{C_p(1-e^{-\frac{C_p}{2}}) \mu}$$

for all $t > t_0$ and thereby verifies (2.6). Thereafter, again thanks to (2.11), an integration of (2.10) in time yields (2.7).
ii) In view of (2.2), we have
\[
\int_t^{t+1} h(s) \, ds \leq \frac{2}{C_p} \| \nabla \phi \|_{L^\infty(\Omega)}^2 \frac{|\Omega|}{\mu t + \gamma},
\]  
(2.12)
whereupon Lemma 2.4 guarantees that
\[
\int_{\Omega} |u(\cdot,t)|^2 \leq \| u(\cdot,t_0) \|_{L^2(\Omega)}^2 e^{-\frac{2C_p}{\mu^2}(t-t_0)} + \frac{2|\Omega|}{\mu^2} \frac{\| \nabla \phi \|_{L^\infty(\Omega)}^2}{C_p(1-e^{-\frac{C_p}{\mu^2}})} 1^{t_0 + \gamma}
\]  
for all \( t > t_0 \).
This precisely warrants (2.8), and thereby in turn yields (2.9) after integrating (2.10) over \((t, t + 1)\) and once more employing (2.12).

Now by a further testing procedure, we can turn the above information into the estimate of \( \| \nabla u(\cdot,t) \|_{L^2(\Omega)} \), particularly its decay in the case of \( r = 0 \), on the basis of an interpolation argument, which is inspired by an approach illustrated in section 3.2 of [36].

**Lemma 2.6.** For the global classical solution \((n, w, u, P)\) of (2.1), we have

i) if \( r > 0 \), then there exists \( \mu_1 := \mu_1(\Omega, r) > 0 \) such that for all \( \mu > \mu_1 \),
\[
\limsup_{t \to \infty} \| \nabla u(\cdot,t) \|_{L^2(\Omega)} \leq \frac{1}{17K_1|\Omega|}
\]  
(2.13)

ii) if \( r \leq 0 \), then for any \( \mu > 0 \),
\[
\lim_{t \to \infty} \| \nabla u(\cdot,t) \|_{L^2(\Omega)} = 0.
\]  
(2.14)

**Proof.** Applying the Helmholtz projector \( \mathcal{P} \) to the third equation in (2.1), multiplying the resulting identity \( u_t + Au = -\mathcal{P}[(u \cdot \nabla)u] + \mathcal{P}[n \nabla \phi] \) by \( Au \), and using the Gagliardo–Nirenberg inequality, we can find \( C_1 > 0 \) such that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |Au|^2 = -\int_{\Omega} \mathcal{P}[(u \cdot \nabla)u] \cdot Au + \int_{\Omega} \mathcal{P}[n \nabla \phi] \cdot Au
\]  
\[\leq \frac{1}{2} \int_{\Omega} |Au|^2 + \int_{\Omega} |(u \cdot \nabla)u|^2 + \| \nabla \phi \|_{L^\infty(\Omega)}^2 \int_{\Omega} n^2
\]  
\[\leq \frac{1}{2} \int_{\Omega} |Au|^2 + \| u \|_{L^2(\Omega)}^2 \| \nabla u \|_{L^2(\Omega)}^2 + \| \nabla \phi \|_{L^\infty(\Omega)}^2 \int_{\Omega} n^2
\]  
\[\leq \frac{1}{2} \int_{\Omega} |Au|^2 + C_1|Au| \| u \|_{L^2(\Omega)} \| \nabla u \|_{L^2(\Omega)}^2 + \| \nabla \phi \|_{L^\infty(\Omega)}^2 \int_{\Omega} n^2
\]  
\[\leq \int_{\Omega} |Au|^2 + \frac{C_1^2}{2} \| u \|_{L^2(\Omega)}^4 \| \nabla u \|_{L^2(\Omega)}^4 + \| \nabla \phi \|_{L^\infty(\Omega)}^2 \int_{\Omega} n^2.
\]
which entails \( y(t) := \int_{\Omega} |\nabla u(\cdot, t)|^2 \) satisfies

\[
y'(t) \leq h_1(t)y(t) + h_2(t) \quad \text{for all } t > 0 \tag{2.15}
\]

with \( h_1(t) = C_1^2 \|u(\cdot, t)\|_{L^2(\Omega)}^2 \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 \) and \( h_2(t) = 2\|\nabla \phi\|_{L^\infty(\Omega)}^2 \|n(\cdot, t)\|_{L^2(\Omega)}^2 \).

i) In order to prepare the integration of (2.15), we may use Lemma 2.5 i) to find some \( t_0 > 0 \) such that

\[
\|u(\cdot, t)\|_{L^2(\Omega)}^2 \leq C_2 := \frac{3(1 + r)|\Omega|}{\mu} \frac{\|\nabla \phi\|_{L^\infty(\Omega)}^2 \cdot r}{C_p(1 - e^{-\frac{C_p}{t_0}})^2} \mu
\]

and

\[
\int_{t-1}^{t} \|\nabla u(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq 2C_2
\]

for all \( t > t_0 + 1 \).

Hence for any \( t > t_0 + 1 \), we can find \( t_* = t_*(t) \in [t-1, t) \) such that

\[
\|\nabla u(\cdot, t_*)\|_{L^2(\Omega)}^2 \leq 2C_2, \tag{2.16}
\]

and then integrating (2.15) over \((t_*, t)\) yields

\[
y(t) \leq y(t_*) e^{\int_{t_*}^{t} h_1(\sigma) d\sigma} + \int_{t_*}^{t} e^{\int_{t_*}^{s} h_1(\sigma) d\sigma} h_2(s) ds \\
\leq (2 + C_p)C_2 e^{2C_1^2C_2^2 t_0 + \gamma}
\]

and thereby verifies (2.13).

ii) For any \( t_0 > 1 \) and \( t > t_0 + 2 \), we use Lemma 2.5 ii) to pick \( t_* = t_*(t) \in [t-1, t) \)

fulfilling

\[
\|\nabla u(\cdot, t_*)\|_{L^2(\Omega)}^2 = \int_{t-1}^{t} \|\nabla u(\cdot, s)\|_{L^2(\Omega)}^2 ds \\
\leq \|u(\cdot, t_0)\|_{L^2(\Omega)}^2 e^{-\frac{C_p}{t_0}(t-1-t_0)} + \frac{4|\Omega|}{\mu^2} \frac{\|\nabla \phi\|_{L^\infty(\Omega)}^2}{C_p(1 - e^{-\frac{C_p}{t_0}})^2} t_0 + \gamma,
\]

as well as

\[
\int_{t-1}^{t} h_1(\sigma) d\sigma \leq C_1^2 \max_{t-1 \leq s \leq t} \|u(\cdot, s)\|_{L^2(\Omega)}^2 \int_{t-1}^{t} \|\nabla u(\cdot, s)\|_{L^2(\Omega)}^2 ds \\
\leq C_1^2 (\|u(\cdot, t_0)\|_{L^2(\Omega)}^2 e^{-\frac{C_p}{t_0}(t-1-t_0)} + \frac{4|\Omega|}{\mu^2} \frac{\|\nabla \phi\|_{L^\infty(\Omega)}^2}{C_p(1 - e^{-\frac{C_p}{t_0}})^2} t_0 + \gamma)^2.
\]
In addition, by (2.12) we also have
\[ \int_{t-1}^{t} h_2(\sigma) d\sigma = 2 \| \nabla \phi \|_{L^\infty(\Omega)}^2 \int_{t-1}^{t} \| n(\cdot, s) \|_{L^2(\Omega)}^2 ds \leq 2 \| \nabla \phi \|_{L^\infty(\Omega)}^2 |\Omega| \frac{1}{\mu^2} t - 1 + \gamma. \]

Therefore combining the above inequalities, (2.13) implies that
\[ y(t) \leq y(t_*) e^{\int_{t-1}^{t} h_1(\sigma) d\sigma} + e^{\int_{t-1}^{t} h_1(\sigma) d\sigma} \int_{t-1}^{t} h_2(s) ds \]
and thus (2.14) holds readily.

3 Global boundedness of solutions

In this section, we show that the classical solution of the problem (2.1) is globally bounded in the cases of $r > 0$ and $r \leq 0$, respectively.

3.1 The case $r > 0$

In this subsection, we derive the global boundedness of solutions to (2.1) whenever $\mu$ is suitably large compared with $r$. As in [48], the main idea is to examine the behavior of the functional
\[ F(n, w) := \int_{\Omega} H(n) + \frac{\chi}{2} \int_{\Omega} |\nabla w|^2 \]  
where $H(s) := s \ln \frac{es}{e^r} + \frac{s}{\mu}$, along trajectories of the boundary value problem (2.1).

The following elementary property of $H(n)$ will be used in the sequel.

**Lemma 3.1.** For all nonnegative function $n \in C(\bar{\Omega})$, $H(n) \geq 0$.

**Proof.** It is easy to verify that $H(\frac{e^r}{\mu}) = 0$, $H'(\frac{e^r}{\mu}) = 0$ and $H''(s) = \frac{1}{s} \geq 0$, which implies $H(n) \geq 0$ for all $n \geq 0$.

Now we can describe the evolution of $F(n, w)$ along the trajectories of (2.1) by the standard testing procedure.

**Lemma 3.2.** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $(n, w, u)$ be the global classical solution of (2.1) with $r > 0, \mu > 0$. Then there exists $t_* > 0$ such that
\[ \frac{d}{dt} F(n, w) \leq 0 \text{ for all } t \geq t_* \]  
(3.2)
whenever $\mu > \mu_2(\Omega, \chi, r) := \max\{\mu_1, \frac{K_1(36 + 16\chi)}{\chi} |r|\}$.

**Proof.** Multiplying the first equation in (2.1) by $H(n)$ and integrating by parts, we get

$$
\frac{d}{dt} \int_{\Omega} H(n) = \int_{\Omega} H'(n)(\Delta n + \chi \nabla \cdot (n \nabla w) + rn - \mu n^2 - u \cdot \nabla n)
= -\int_{\Omega} \frac{|\nabla n|^2}{n} - \chi \int_{\Omega} \nabla n \cdot \nabla w + \int_{\Omega} (\ln n - \ln \frac{r}{\mu})(rn - \mu n^2)
\leq -\int_{\Omega} \frac{|\nabla n|^2}{n} - \chi \int_{\Omega} \nabla n \cdot \nabla w
$$

(3.3)
due to $(\ln n - \ln \frac{r}{\mu})(rn - \mu n^2) \leq 0$, $\nabla \cdot u = 0$ and $u = 0$ on $\partial \Omega$.

On the other hand, testing the second equation in (2.1) by $-\Delta w$, using $\nabla \cdot u = 0$ and $u = 0$ on $\partial \Omega$ again, we can obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w|^2 + \frac{1}{2} \int_{\Omega} |\Delta w|^2 = \int_{\Omega} |\nabla w|^2 \Delta w + \int_{\Omega} \nabla n \cdot \nabla w + \int_{\Omega} (u \cdot \nabla w) \Delta w
\leq \frac{1}{2} \int_{\Omega} |\Delta w|^2 + \frac{1}{2} \int_{\Omega} |\nabla w|^4 + \int_{\Omega} |\nabla n| \cdot |\nabla w| + \int_{\Omega} (u \cdot \nabla w) \Delta w
= \frac{1}{2} \int_{\Omega} |\Delta w|^2 + \frac{1}{2} \int_{\Omega} |\nabla w|^4 + \int_{\Omega} |\nabla n| \cdot |\nabla w| - \int_{\Omega} \nabla w \cdot (\nabla u \cdot \nabla w).
$$

Furthermore, by Lemma 2.2 i) and the Cauchy–Schwarz inequality, we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w|^2 + \frac{1}{2} \int_{\Omega} |\nabla w|^4 \leq \frac{K_1}{2} \|\nabla w\|^2_{L^2(\Omega)} \int_{\Omega} |\Delta w|^2 + \int_{\Omega} |\nabla n| \cdot |\nabla w| + \int_{\Omega} |\nabla u| \cdot |\nabla w|^2
\leq (\frac{K_1}{2} \|\nabla w\|^2_{L^2(\Omega)} + K_1 |\Omega|^\frac{1}{2} \|\nabla u\|_{L^2(\Omega)}) \int_{\Omega} |\Delta w|^2 + \int_{\Omega} |\nabla n| \cdot |\nabla w|
$$

and thus

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w|^2 + \frac{1}{2} (1 - K_1 \|\nabla w\|^2_{L^2(\Omega)} - 2K_1 |\Omega|^\frac{1}{2} \|\nabla u\|_{L^2(\Omega)}) \int_{\Omega} |\Delta w|^2 \leq \int_{\Omega} |\nabla n| \cdot |\nabla w|. \quad (3.4)
$$

Since $2\mathcal{F}(n, w) \geq \chi \|\nabla w\|^2_{L^2(\Omega)}$ due to $H(n) \geq 0$, combining (3.4) with (3.3) yields

$$
\frac{d}{dt} \mathcal{F}(n, w) + \int_{\Omega} \frac{|\nabla n|^2}{n} + (\frac{\chi}{2} - K_1 \mathcal{F}(n, w) - 2\chi K_1 |\Omega|^\frac{1}{2} \|\nabla u\|_{L^2(\Omega)}) \int_{\Omega} |\Delta w|^2 \leq 0 \text{ for } t > 0. \quad (3.5)
$$

On the other hand, from (2.13), it is possible to pick some $t_0 > 0$ such that

$$
16K_1 |\Omega|^\frac{1}{2} \|\nabla u(\cdot, t)\|_{L^2(\Omega)} < 1 \text{ for all } t > t_0
$$

whenever $\mu > \mu_1$, and thereby

$$
\frac{d}{dt} \mathcal{F}(n, w) + \int_{\Omega} \frac{|\nabla n|^2}{n} + (\frac{3\chi}{8} - K_1 \mathcal{F}(n, w)) \int_{\Omega} |\Delta w|^2 \leq 0 \text{ for } t > t_0. \quad (3.6)
$$
In what follows, we shall show that there exists \( t_\ast > t_0 \) such that \( 4K_1\mathcal{F}(n, w)(t_\ast) < \chi \).

Firstly by Lemma 2.3(i), there exists \( t_1 > t_0 \) such that for all \( t > t_1 \)

\[
\|n(\cdot, t)\|_{L^1(\Omega)} \leq \frac{3|\Omega|}{2\mu} r,
\]

which along with Lemma 2.3(iii) yields

\[
\int_{t_1}^{t_2} \int_\Omega |\nabla w|^2 \, \leq \int_\Omega w(\cdot, t_1) + \int_{t_1}^{t_2} \|n(\cdot, s)\|_{L^1(\Omega)} \, ds \\
\leq \int_\Omega w_0(x) + \int_{t_1}^{t_2} \|n(\cdot, s)\|_{L^1(\Omega)} \, ds + \frac{3|\Omega|}{2\mu}(t_2 - t_1).
\]

Similarly invoking Lemma 2.3(i) and (ii), we find that

\[
\int_{t_1}^{t_2} \|n(\cdot, s)\|_{L^2(\Omega)}^2 \, ds \leq \frac{3|\Omega|}{2}(\frac{r}{\mu})^2(t_2 - t_1) + \frac{1}{\mu} \|n(\cdot, t_1)\|_{L^1(\Omega)}.
\]

Hence there exists \( t_\ast > t_1 \) suitably large such that whenever \( t_2 \geq t_\ast \),

\[
\int_{t_1}^{t_2} \int_\Omega |\nabla w|^2 \, \leq \frac{2|\Omega|}{\mu}(t_2 - t_1)
\]

and

\[
\int_{t_1}^{t_2} \|n(\cdot, s)\|_{L^2(\Omega)}^2 \, ds \leq 2|\Omega|(\frac{r}{\mu})^2(t_2 - t_1).
\]

Let

\[
S_1 := \{ t \in [t_1, t_2] \mid \int_\Omega |\nabla w(\cdot, t)|^2 \geq \frac{8|\Omega| r}{\mu} \}
\]

and

\[
S_2 := \{ t \in [t_1, t_2] \mid \|n(\cdot, t)\|_{L^2(\Omega)}^2 \geq 8|\Omega|(\frac{r}{\mu})^2 \}.
\]

Then

\[
|S_1| \leq \frac{|t_2 - t_1|}{4}, \quad |S_2| \leq \frac{|t_2 - t_1|}{4}.
\]

In order to estimate the size of \( S_1 \) and \( S_2 \), we recall (3.8) to get

\[
\frac{8|\Omega| r}{\mu} |S_1| \leq \int_{t_1}^{t_2} \int_\Omega |\nabla w|^2 \, \leq \frac{2|\Omega|}{\mu}(t_2 - t_1)
\]

and thus \( |S_1| \leq \frac{|t_2 - t_1|}{4} \) is valid. Similarly, one can verify that \( |S_2| \leq \frac{|t_2 - t_1|}{4} \).

As (3.10) warrants that

\[
|(t_1, t_2) \setminus (S_1 \cup S_2)| \geq \frac{|t_2 - t_1|}{2},
\]

(3.7)
one can conclude that there exists $t^* \in (t_1, t_2)$ such that

$$\|n(\cdot, t^*)\|_{L^2(\Omega)}^2 < 8|\Omega| \left(\frac{r}{\mu}\right)^2$$

(3.11)

and

$$\int_\Omega |\nabla w(\cdot, t^*)|^2 < \frac{8|\Omega|r}{\mu}.$$  

(3.12)

Applying $\xi \ln \frac{\xi}{\eta} \leq \eta \xi^2 + \ln \frac{1}{\eta} \cdot \xi$ for all $\xi > 0, \eta > 0, \sigma > 0$ (see Lemma 5.5 of [48]), and combining (3.7) with (3.11), we then arrive at

$$\int_\Omega H(n(\cdot, t^*)) \leq \frac{\mu}{r} \int_\Omega n^2(\cdot, t^*) - \int_\Omega n(\cdot, t^*) + \frac{r}{\mu} |\Omega|$$

$$\leq \frac{9|\Omega|r}{\mu}.$$  

Thereupon from (3.12) and the definition of $F(n, w)$, it follows that

$$F(n, w)(t^*) < (9 + 4\chi)|\Omega| \frac{r}{\mu},$$

which entails that $4K_1F(n, w)(t^*) < \chi$ provided $\mu > \frac{K_1(36+16\chi)|\Omega|r}{\chi}$.

As an immediate consequence of (3.5), we have

$$\frac{d}{dt}F(n, w) + \int_\Omega \frac{\nabla n^2}{n} + \frac{\chi}{8} \int_\Omega |\nabla w|^2 \leq 0 \text{ for all } t > t^*$$

(3.13)

when $\mu > \mu_2(\Omega, \chi, r)$, and thus end the proof of this lemma.

Additionally from (3.13), one can also conclude that

**Corollary 3.1.** Under the conditions of Lemma 3.2, we have

$$F(n, w)(t) + \int_{t_*}^t \int_\Omega \frac{\nabla n^2}{n} + \frac{\chi}{8} \int_{t_*}^t \int_\Omega |\nabla w|^2 \leq (9 + 4\chi)|\Omega| \frac{r}{\mu} \text{ for all } t > t_*.$$  

(3.14)

Next by a further testing procedure, we can turn the above information into the uniform-in-time boundedness of $\|n(\cdot, t)\|_{L^2(\Omega)}$ and $\|\nabla w(\cdot, t)\|_{L^4(\Omega)}$ if $\mu$ is appropriately large compared with $r$, which will serve as the foundation for the proof of the global boundedness of $\|n(\cdot, t)\|_{L^\infty(\Omega)}$ and $\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)}$.

**Lemma 3.3.** Under the assumptions in Lemma 3.2, there exists $C > 0$ such that

$$\|n(\cdot, t)\|_{L^2(\Omega)} + \|\nabla w(\cdot, t)\|_{L^4(\Omega)} \leq C \text{ for all } t \geq t_*$$

(3.15)

provided $\mu > \frac{182K_2|\Omega|r}{\chi^2}$.  

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Proof. Multiplying the first equation in (2.1) by \( n \) and integrating the result over \( \Omega \), we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega n^2 = - \int_\Omega |\nabla n|^2 - \chi \int_\Omega n \nabla n \nabla w + r \int_\Omega n^2 - \mu \int_\Omega n^3
\]
\[
\leq - \frac{1}{2} \int_\Omega |\nabla n|^2 + \frac{1}{2} \int_\Omega n^2 |\nabla w|^2 + r \int_\Omega n^2 - \mu \int_\Omega n^3.
\]
(3.16)

On the other hand, by the second equation in (2.1) and the identity \( \nabla w \cdot \nabla \Delta w = \frac{1}{2} \Delta |\nabla w|^2 - |D^2 w|^2 \), we obtain
\[
\frac{d}{dt} \int_\Omega |\nabla w|^4 = 2 \int_\Omega |\nabla w|^2 \Delta |\nabla w|^2 - 4 \int_\Omega |\nabla w|^2 |D^2 w|^2 - 4 \int_\Omega |\nabla w|^2 \nabla w \cdot \nabla |\nabla w|^2
\]
\[
+ 4 \int_\Omega |\nabla w|^2 \nabla n \cdot \nabla w - 4 \int_\Omega |\nabla w|^2 \nabla w \cdot (u \cdot \nabla w)
\]
\[
= -2 \int_\Omega \Delta |\nabla w|^2 - 4 \int_\Omega |\nabla w|^2 |D^2 w|^2 - 4 \int_\Omega |\nabla w|^2 \nabla w \cdot \nabla |\nabla w|^2 - 4 \int_\Omega n |\nabla w|^2 \Delta w
\]
\[
- 4 \int_\Omega n |\nabla w|^2 \cdot \nabla w + 2 \int_{\partial \Omega} |\nabla w|^2 \frac{\partial |\nabla w|^2}{\partial \nu} - 4 \int_\Omega |\nabla w|^2 \nabla w \cdot (\nabla u \cdot \nabla w)
\]
(3.17)
due to \( \nabla \cdot u = 0 \) and \( u = 0 \) on \( \partial \Omega \).

According to
\[
\frac{\partial |\nabla w|^2}{\partial \nu} \leq c_1 |\nabla w|^2 \quad \text{on} \quad \partial \Omega \quad \text{for some} \quad c_1 > 0
\]
and
\[
|||\nabla w|||_{L^2(\partial \Omega)} \leq \eta |||\nabla |\nabla w|||_{L^2(\Omega)} + c_2(\eta) |||\nabla w|||_{L^1(\Omega)} \quad \text{for any} \quad \eta \in (0, \frac{5}{4})
\]
(see Lemma 4.2 of \[30\] and Remark 52.9 in \[31\]), one can conclude that
\[
2 \int_{\partial \Omega} |\nabla w|^2 \frac{\partial |\nabla w|^2}{\partial \nu} \leq \frac{1}{4} \int_\Omega |\nabla |\nabla w||^2 + c_3(\int_\Omega |\nabla w|^2)^2
\]
(3.18)
for some \( c_3 > 0 \).

For the other integrals on the right-side of (3.17), we use the Young inequality to estimate
\[
- 4 \int_\Omega |\nabla w|^2 \nabla w \cdot \nabla |\nabla w|^2 \leq \frac{1}{3} \int_\Omega |\nabla |\nabla w||^2 + 12 \int_\Omega |\nabla w|^6
\]
(3.19)
\[
- 4 \int_\Omega n |\nabla w|^2 \cdot \nabla w \leq \frac{1}{3} \int_\Omega |\nabla |\nabla w||^2 + 12 \int_\Omega n^2 |\nabla w|^2
\]
(3.20)
as well as
\[
- 4 \int_\Omega n |\nabla w|^2 \Delta w \leq \frac{1}{6} \int_\Omega |\nabla w|^2 |\Delta w|^2 + 24 \int_\Omega n^2 |\nabla w|^2
\]
\[
\leq \frac{1}{3} \int_\Omega |\nabla w|^2 |D^2 w|^2 + 24 \int_\Omega n^2 |\nabla w|^2
\]
(3.21)
due to $|\Delta w|^2 \leq 2|D^2w|^2$ on $\Omega$.

Substituting (3.18)–(3.21) into (3.17), we readily get

$$\frac{d}{dt}\int_{\Omega} |\nabla w|^4 + \frac{13}{12}\int_{\Omega} |\nabla |\nabla w|^2|^2 + \frac{11}{3}\int_{\Omega} |\nabla w|^2|D^2w|^2 \leq 12 \int_{\Omega} |\nabla w|^6 + 36 \int_{\Omega} n^2|\nabla w|^2 + c_3\left(\int_{\Omega} |\nabla w|^2\right)^2 + 4 \int_{\Omega} |\nabla w|^4|\nabla u|$$

and thus

$$\frac{d}{dt}\int_{\Omega} |\nabla w|^4 + 2 \int_{\Omega} |\nabla |\nabla w|^2|^2 \leq 12 \int_{\Omega} |\nabla w|^6 + 36 \int_{\Omega} n^2|\nabla w|^2 + c_3\left(\int_{\Omega} |\nabla w|^2\right)^2 + 4 \int_{\Omega} |\nabla w|^4|\nabla u|$$

(3.22)
due to the fact $|\nabla |\nabla w|^2|^2 \leq 4|\nabla w|^2|D^2w|^2$ on $\Omega$.

Therefore combining (3.16) with (3.22) leads to

$$\frac{d}{dt}\left(\int_{\Omega} n^2 + \int_{\Omega} |\nabla w|^4\right) + 2 \int_{\Omega} |\nabla |\nabla w|^2|^2 + \int_{\Omega} |\nabla n|^2 \leq 12 \int_{\Omega} |\nabla w|^6 + 37 \int_{\Omega} n^2|\nabla w|^2 + c_3\left(\int_{\Omega} |\nabla w|^2\right)^2 + 2r \int_{\Omega} n^2 - 2\mu \int_{\Omega} n^3 + 4 \int_{\Omega} |\nabla w|^4|\nabla u|$$

$$\leq 13 \int_{\Omega} |\nabla w|^6 + 37^2 \int_{\Omega} n^3 + c_3\left(\int_{\Omega} |\nabla w|^2\right)^2 + 2r \int_{\Omega} n^2 - 2\mu \int_{\Omega} n^3 + 4 \int_{\Omega} |\nabla w|^4|\nabla u|.$$ 

(3.23)

Furthermore by Lemma 2.2 (ii), we get $\|\varphi\|^3_{L^3} \leq K_2\|\nabla \varphi\|^2_{L^2}\|\varphi\|_{L^1} + c_4\|\varphi\|^3_{L^3}$, and thus

$$\int_{\Omega} |\nabla w|^6 \leq K_2\left(\int_{\Omega} |\nabla |\nabla w|^2|^2\right)\left(\int_{\Omega} |\nabla w|^2\right) + c_4\left(\int_{\Omega} |\nabla w|^2\right)^3.$$

Upon inserting this into (3.23) and (3.14), we obtain

$$\frac{d}{dt}\left(\int_{\Omega} n^2 + \int_{\Omega} |\nabla w|^4\right) + (2 - 13K_2) \int_{\Omega} |\nabla w|^2 \int_{\Omega} |\nabla |\nabla w|^2|^2 + \int_{\Omega} |\nabla n|^2 + \int_{\Omega} n^2 + \int_{\Omega} |\nabla w|^4$$

$$\leq 37^2 \int_{\Omega} n^3 + (2r + 1) \int_{\Omega} n^2 - 2\mu \int_{\Omega} n^3 + \int_{\Omega} |\nabla w|^4 + 4 \int_{\Omega} |\nabla w|^4|\nabla u| + c_5,$$

which, along with

$$\int_{\Omega} |\nabla w|^4 \leq \frac{1}{7} \int_{\Omega} |\nabla |\nabla w|^2|^2 + c_6$$

and

$$4 \int_{\Omega} |\nabla w|^4|\nabla u| \leq 4\|\nabla w|^2\|_{L^6(\Omega)}\|\nabla u\|_{L^\frac{3}{2}(\Omega)}$$

$$\leq \frac{13}{56} \int_{\Omega} |\nabla |\nabla w|^2|^2 + c_7$$
by the Gagliardo–Nirenberg inequality and (2.13), implies that

\[
\frac{d}{dt} \left( \int_{\Omega} n^2 + \int_{\Omega} |\nabla w|^4 \right) + \frac{13}{8} - 13K_2 \int_{\Omega} |\nabla w|^2 \int_{\Omega} |\nabla|\nabla w|^2| + \int_{\Omega} |\nabla n|^2 + \int_{\Omega} n^2 + \int_{\Omega} |\nabla w|^4 \\
\leq 37^2 \int_{\Omega} n^3 + (2r + 1) \int_{\Omega} n^2 - 2\mu \int_{\Omega} n^3 + c_8.
\]

(3.24)

Finally according to an extended variant (3.3), (3.7) and (3.14), one can infer that

\[
37^2 \int_{\Omega} n^3 \leq c_9 \left( \int_{\Omega} |\nabla n|^2 \right) \left( \int_{\Omega} n |\ln n| \right) + c_9 \left( \int_{\Omega} n \right)^3 + c_9
\]

\[
\leq \frac{1}{2} \int_{\Omega} |\nabla n|^2 + c_{10}.
\]

Hence, in conjunction with (3.14), (3.23) yields \(c_{11} > 0\) such that for all \(t > t^*\)

\[
\frac{d}{dt} \int_{\Omega} (n^2 + |\nabla w|^4) + \int_{\Omega} (n^2 + |\nabla w|^4) + \frac{13}{8} - 13K_2 \int_{\Omega} |\nabla w|^2 \int_{\Omega} |\nabla|\nabla w|^2| \leq c_{11}.
\]

(3.25)

Now in view of (3.14), we can see that \(\int_{\Omega} |\nabla w|^2 \leq r \left( \frac{18}{\chi} + \frac{8}{\chi^2} \right)|\Omega|\) for all \(t > t^*\) and thus

\[
\frac{13}{7} - 13K_2 \int_{\Omega} |\nabla w|^2 \geq 0
\]

provided that \(\frac{r}{\mu} < \frac{\chi^2}{18^2K_2|\Omega|}\), which guarantees that

\[
\frac{d}{dt} \int_{\Omega} (n^2 + |\nabla w|^4) + \int_{\Omega} (n^2 + |\nabla w|^4) \leq c_{11}
\]

(3.26)

for all \(t > t^*\) and thereby (3.15) is valid.

We are now ready to prove Theorem 1.1 in the case of \(r > 0\).

Proof of Theorem 1.1 in the case of \(r > 0\). From the above lemmas, it follows that there exists \(C > 0\) such that

\[
\|n(\cdot,t)\|_{L^2(\Omega)} + \|\nabla w(\cdot,t)\|_{L^4(\Omega)} + \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \leq C
\]

(3.27)

whenever \(\mu > \mu_0(\chi,\Omega, r) := \max\{\mu_2(\chi, \Omega, r), \frac{18^2K_2|\Omega|}{\chi^2r}\}\). Thereupon, by the argument in e.g. Lemma 4.4 of [4], we can readily prove that \(\|n(\cdot,t)\|_{L^\infty(\Omega)}, \|\nabla w(\cdot,t)\|_{L^\infty(\Omega)}\) and \(\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}\) with some \(\alpha \in (\frac{1}{2}, 1)\) are globally bounded; we refer the reader to the proof of Lemma 4.4 in [4], Lemma 3.12 and Lemma 3.11 in [36] for the details.

Based on the global boundedness of solutions, we are able to derive the convergence result claimed in Theorem 1.1, namely,

\[
\lim_{t \to \infty} \|n(\cdot,t) - \frac{r}{\mu}\|_{L^\infty(\Omega)} = 0,
\]

(3.28)
\[
\lim_{t \to \infty} \|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} = 0,
\] (3.29)
\[
\lim_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} = 0
\] (3.30)
as well as
\[
\lim_{t \to \infty} \inf_{x \in \Omega} w(x, t) = \infty.
\] (3.31)

In fact, due to
\[
\int_0^\infty \int_{\Omega} \frac{|\nabla n|^2}{n} + \int_0^\infty \int_{\Omega} |\Delta w|^2 \leq C
\] established in (3.14), we can show (3.29), (3.30) and
\[
\lim_{t \to \infty} \|n(\cdot, t) - \overline{n}(t)\|_{L^\infty(\Omega)} = 0
\] (3.32)
with \(\overline{n}(t) = \frac{1}{|\Omega|} \int_{\Omega} n(\cdot, t)\) by the arguments in Proposition 4.15 of [4]. Therefore it suffices to show that
\[
\lim_{t \to \infty} |\overline{n}(t) - \frac{r}{\mu}| = 0. \tag{3.33}
\]
To this end, we adapt the idea of [27] and give the details of the proof for the convenience of readers.

Integrating the first equation in (2.1) on the spatial variable over \(\Omega\), we obtain
\[
\overline{n}_t = \lambda \overline{n} - \frac{\mu}{|\Omega|} \int_{\Omega} n^2 = \lambda \overline{n} - \mu \overline{n}^2 - \frac{\mu}{|\Omega|} \int_{\Omega} (n - \overline{n})^2.
\]
Putting \(a(t) := \frac{\mu}{|\Omega|} \int_{\Omega} (n(\cdot, t) - \overline{n})^2\), the above equation then becomes
\[
\overline{n}_t = \mu (\overline{n} - \frac{\lambda}{\mu}) - a(t) \tag{3.34}
\]
Thereupon multiplying (3.34) by \(\overline{n} - \frac{\lambda}{\mu}\), we get
\[
\frac{d}{dt} (\overline{n} - \frac{\lambda}{\mu})^2 + 2\mu \overline{n} (\overline{n} - \frac{\lambda}{\mu})^2 = -2a(t)(\overline{n} - \frac{\lambda}{\mu})
\]
and then
\[
2\mu \int_1^\infty \overline{n} (\overline{n} - \frac{\lambda}{\mu})^2 \leq (\overline{n}(1) - \frac{\lambda}{\mu})^2 + 2 \sup_{t \geq 1} |\overline{n}(t) - \frac{\lambda}{\mu}| \int_1^\infty a(t). \tag{3.35}
\]
In addition, invoking the Poincaré–Wintinger inequality
\[
\int_{\Omega} |\varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi(y) dy|^2 \leq C_p \int_{\Omega} |\varphi|^2 \int_{\Omega} \frac{|
abla \varphi|^2}{|\varphi|} \quad \text{for all } \varphi \in W^{1,2}(\Omega)
\]
for some $C_p > 0$, one can find
\[ \int_1^\infty a(s) ds \leq C_p \sup_{t \geq 1} \|n(t)\|_{L^1(\Omega)} \int_1^\infty \int_{\Omega} \frac{\nabla n(s)}{n(s)} ds \leq C \] (3.36)
due to (3.14) and Lemma 2.5(i). Hence combining (3.36) with (3.35) yields
\[ \int_1^\infty \pi (\pi - \frac{\lambda}{\mu})^2 \leq C. \] (3.37)

On the other hand,
\[ \frac{d}{dt} \pi (\pi - \frac{\lambda}{\mu})^2 = \pi \left( (\pi - \frac{\lambda}{\mu})^2 + 2\pi (\pi - \frac{\lambda}{\mu}) \right), \]
which along with $|\pi_t| \leq \lambda \pi + \frac{\mu}{|\Omega|} \int_{\Omega} n^2 \leq C$ implies that
\[ \left| \frac{d}{dt} (\pi - \frac{\lambda}{\mu}) \right| \leq C. \] (3.38)

Therefore by Lemma 6.3 of [27], (3.38) and (3.37) show that
\[ \lim_{t \to \infty} \pi (\pi - \frac{\lambda}{\mu}) = 0. \] (3.39)

From (3.32), it follows that there exists $t_1 > t_*$ such that $\|n(\cdot, t) - \pi(t)\|_{L^\infty(\Omega)} \leq \frac{\lambda}{2\mu}$ for all $t > t_1$, and thus
\[
\pi(t) = \frac{\lambda}{\mu} - \mu \frac{n^2}{|\Omega|} \int_{\Omega} n - \pi \nabla \pi \nabla \pi \geq \mu \frac{n^2}{|\Omega|} \int_{\Omega} n - \pi \sup_{t > t_1} \|n(\cdot, t) - \pi(t)\|_{L^\infty(\Omega)} \geq \mu \frac{n}{2\mu} - \pi. \] (3.40)

On the other hand, noticing that the solution $y(t)$ of the ODE
\[ y'(t) = \mu \overline{y} \left( \frac{\lambda}{2\mu} - \overline{y} \right), \quad y(t_1) > 0 \]
satisfies $\lim_{t \to \infty} y(t) = \frac{r}{2\mu}$, by the comparison principle, (3.40) implies that there exists $t_2 > t_1$ such that for all $t \geq t_2$,
\[ \pi(t) \geq \frac{\lambda}{4\mu}. \]

This together with (3.39) yields (3.33).

Finally, in view of (3.28), one can find $t_3 > 1$ such that $n(x, t) \geq \frac{r}{2\mu}$ for all $x \in \Omega$ and $t \geq t_3$, and thereby $w(x, t)$ satisfies
\[ w_t \geq \nabla w - |\nabla w|^2 + \frac{r}{2\mu} - u \cdot \nabla \]
for \( t \geq t_3 \). Hence if \( y(t) \) denotes the solution of ODE: 
\[
y'(t) = \frac{r}{2\mu}, \quad y(t_3) = \min_{x \in \Omega} w(\cdot, t_3)
\]
then
\[
w(x, t) \geq \frac{r}{2\mu}(t - t_3)
\]
by means of a straightforward parabolic comparison which warrants that (3.31) holds and thereby completes the proof.

### 3.2 The case \( r \leq 0 \)

In this subsection, we show the global boundedness of solutions to (1.1), (1.4), (1.5) in the case \( r \leq 0, \mu > 0 \). As mentioned in the introduction, due to the structure of (1.1) with \( r \leq 0, \mu > 0 \), it is difficult to find a decreasing energy functional compared with the situation when \( r > 0, \mu > 0 \) considered in the previous subsection or when \( r = \mu = 0 \) considered in [48]. Indeed, the energy-type functional \( \mathcal{F}(n, w) \) in (3.1) of [48] decreases along a solution in \( \Omega \times (t_0, \infty) \) if \( \mathcal{F}(n(\cdot, t_0), w(\cdot, t_0)) \) is suitably small, namely
\[
\frac{d}{dt} \mathcal{F}(n, w) \leq 0 \quad \text{for all} \quad t \geq t_0.
\]
The main idea underlying our approach is to make use of the quadratic degradation in the first equation of (1.1) which should enforce some suitable regularity properties. More precisely, on the basis of (2.2), we can show that the quantity of form
\[
\mathcal{F}(n, w) := \int_{\Omega} n(\ln n + a)dx + \frac{\chi}{2} \int_{\Omega} |\nabla w|^2 dx,
\]
with parameter \( a > 0 \) determined below (see (3.49)), satisfies a certain of differential inequality. Although unlike the case of \( r > 0 \) in which it enjoys the monotonicity property, \( \mathcal{F}(n, w) \) also provides us the global boundedness of \( \int_{\Omega} n|\ln n|dx \) and \( \int_{\Omega} |\nabla w|^2dx \). This is encapsulated in the following lemma.

**Lemma 3.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain and \((n, w, u)\) be the global classical solution (2.1) with \( r \leq 0, \mu > 0 \). Then there exists \( t_* > 0 \) such that for all \( t > t_* \)
\[
\int_{\Omega} |\nabla w(\cdot, t)|^2 \leq \frac{1}{4K_1}
\]
(3.43)
as well as
\[ \int_{\Omega} n|\ln n| \leq C \] (3.44)
for some \( C > 0 \).

**Proof.** We test the first equation in (2.1) against \( \ln n + a + 1 \), and integrate by parts to see that
\[
\frac{d}{dt} \int_{\Omega} n(\ln n + a) \geq - \int_{\Omega} \frac{|\nabla n|^2}{n} - \chi \int_{\Omega} n \nabla \cdot w + \int_{\Omega} (n(r - \mu n) - u \cdot \nabla n)(\ln n + a + 1) \\
\leq - \int_{\Omega} \frac{|\nabla n|^2}{n} - \chi \int_{\Omega} n \nabla \cdot w + \int_{\Omega} n(r - \mu n)(\ln n + a). \tag{3.45}
\]
due to \( r \leq 0 \) and \( \nabla \cdot u = 0 \).

On the other hand, recalling (3.4) and (2.14), it is possible to fix \( t_0 > 0 \) such that for all \( t \geq t_0 \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w|^2 + \frac{1}{4} \int_{\Omega} |\Delta w|^2 + \frac{1}{4}(\frac{3}{4} - 2K_1\|\nabla w\|_{L^2(\Omega)}^2) \int_{\Omega} |\Delta w|^2 \leq \int_{\Omega} \nabla u \cdot \nabla w. \tag{3.46}
\]
From Lemma 2.2(i), there exists a constant \( K_3 > 0 \) such that
\[
8K_3\|\nabla w\|_{L^2(\Omega)}^2 \leq \|\Delta w\|_{L^2(\Omega)}^2. \tag{3.47}
\]
Hence combining (3.46) with (3.45), we get
\[
\frac{d}{dt} F(n, w) + \int_{\Omega} \frac{|\nabla n|^2}{n} + \frac{\chi}{4} \int_{\Omega} |\Delta w|^2 + K_3 \int_{\Omega} n(\ln n + a) \\
+ \frac{\chi}{4}(\frac{3}{4} - 2K_1\|\nabla w\|_{L^2(\Omega)}^2) \int_{\Omega} |\Delta w|^2 \leq \int_{\Omega} n(K_3 - \mu n)(\ln n + a) + r \int_{\Omega} n(\ln n + a) \quad \text{for} \quad t \geq t_0. \tag{3.48}
\]
Now for any fixed \( \varepsilon < \min\{\frac{\chi}{24K_1}, \frac{\chi}{42K_2}\} \), we pick \( a > 1 \) sufficiently large such that
\[
e^{-a} < \frac{K_3}{\mu}, \quad (1 - r)|\Omega| \max_{0 < n \leq e^{-a}} |n \ln n| < \varepsilon \min\{K_3, 1\}, \tag{3.49}
\]
due to \( \lim_{n \to 0} n \ln n = 0 \) and \( n \ln n < 0 \) for all \( n \in (0, 1) \), and thereby fix \( t_1 > \max \{1, t_0\} \) fulfilling
\[
\frac{a|\Omega|}{\mu(t_1 + \gamma)} < \frac{\varepsilon}{4}, \quad \frac{|\Omega|}{\mu^2(t_1 + \gamma)} < \frac{\varepsilon}{16}, \quad \frac{(a + (\ln K + \varepsilon))|\Omega|}{\mu(t_1 + \gamma)} + \frac{2\varepsilon(\frac{|\Omega|}{\mu} + \|n_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)})}{t_1} < \frac{\varepsilon}{4}. \tag{3.50}
\]

Let \( t_2 = t_1 + t_1^2 \),
\[
S_1 \triangleq \{ t \in [t_1, t_2] \mid \int_\Omega |\nabla w(\cdot, t)|^2 \geq \frac{\varepsilon}{2\chi} \}
\]
and
\[
S_2 \triangleq \{ t \in [t_1, t_2] \mid \|n(\cdot, t)\|_{L^2(\Omega)}^2 \geq \frac{\varepsilon}{4} \}.
\]
Then
\[
|S_1| \leq \frac{|t_2 - t_1|}{4}, \quad |S_2| \leq \frac{|t_2 - t_1|}{4}. \tag{3.51}
\]

By Lemma 2.3(iii), (2.2) and the second equation in (2.1), we obtain that
\[
\int_{t_1}^{t_2} \int_\Omega |\nabla w|^2 \leq \int_{t_1}^{t_2} \int_\Omega n + \int_\Omega w(\cdot, t_1)
= \int_{t_1}^{t_2} \int_\Omega n + \int_\Omega w_0 + \int_0^{t_1} \int_\Omega n
\leq \frac{|\Omega|}{\mu(t_1 + \gamma)}(t_2 - t_1) + \int_\Omega w_0 + t_1 \int_\Omega n_0.
\]
Furthermore, by (3.50)
\[
\int_{t_1}^{t_2} \int_\Omega |\nabla w|^2 dx ds \leq \frac{|\Omega|}{\mu(t_1 + \gamma)} + \frac{t_1\|n_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)}}{t_2 - t_1} + \frac{\|n_0\|_{L^2(\Omega)}^2}{t_1}((t_2 - t_1)
\leq \frac{\varepsilon}{8\chi}(t_2 - t_1).
\]
On the other hand, by the definition of \( S_1 \), we see that
\[
\frac{\varepsilon}{2\chi}|S_1| \leq \int_{t_1}^{t_2} \int_\Omega |\nabla w|^2.
\]
and thereby \( |S_1| \leq \frac{|t_2 - t_1|}{4} \).

In addition, by (2.2) and (3.50), we get
\[
\int_{t_1}^{t_2} \int_\Omega n^2 \leq \frac{1}{\mu} \int_\Omega n(\cdot, t_1) \leq \frac{|\Omega|}{\mu^2(t_1 + \gamma)} < \frac{\varepsilon}{16},
\]
27
which implies that $|S_2| \leq \frac{|t_2-t_1|}{4}$.

Therefore from (3.51), it follows that $|(t_1, t_2) \setminus (S_1 \cup S_2)| \geq \frac{|t_2-t_1|}{2}$, and thereby there exists $t_* \in (t_1, t_2)$ such that

$$\|n(\cdot, t_*)\|_{L^2(\Omega)}^2 < \frac{\varepsilon}{4}$$

(3.52)

and

$$\int_\Omega |\nabla w(\cdot, t_*)|^2 < \frac{\varepsilon}{2\chi} < \frac{1}{6K_1}. \quad (3.53)$$

By (3.53), we can see that the set

$$S \triangleq \{ t \in (t_*, \infty) | K_1 \int_\Omega |\nabla w(\cdot, t)|^2 < \frac{1}{4} \}$$

is not empty and hence $T_S = \sup S$ is a well-defined element of $(t_*, \infty]$. In fact, we claim that $T_S = \infty$. To this end, supposing on the contrary that $T_S < \infty$, we then have $K_1 \int_\Omega |\nabla w(\cdot, t)|^2 < \frac{1}{4}$ for all $t \in [t_*, T_S)$, but

$$K_1 \int_\Omega |\nabla w(\cdot, T_S)|^2 = \frac{1}{4}. \quad (3.54)$$

Hence from (3.48) and (3.47), it follows that for all $t \in [t_*, T_S)$,

$$\frac{d}{dt} F(n, w) + \int_\Omega |\nabla n|^2 + \frac{\chi}{4} \int_\Omega |\Delta n|^2 + K_3 \int_\Omega n \ln n + a + \frac{K_3 \chi}{2} \int_\Omega |\nabla w|^2$$

$$\leq \int_\Omega n(K_3 - \mu n)(\ln n + a) + r \int_\Omega n \ln n$$

$$\leq \int_{e^{-a} < n \leq \frac{K_3}{\mu}} n(K_3 - \mu n)(\ln n + a) + \int_{0 < n \leq e^{-a}} n \ln n$$

$$\leq a K_3 \int_\Omega n + K_3 \int_{e^{-a} < n \leq \frac{K_3}{\mu}} n \ln n - r |\Omega| \max_{0 < n \leq e^{-a}} |\ln n|$$

$$\leq K_3 (a + (\ln \frac{K_3}{\mu})_+) \int_\Omega n + \varepsilon K_3$$

$$\leq \frac{(a + (\ln \frac{K_3}{\mu})_+) |\Omega|}{\mu(t_1 + \gamma)} + \varepsilon K_3,$$

where we have made use of $t_* \geq t_1$, the decay estimate (2.2) and (3.50), and thus

$$F(n, w)(T_*) + \int_{t_*}^{T_S} e^{-K_3(T_* - \sigma)} \left( \int_\Omega |\nabla n|^2 \frac{n}{\sigma} + \frac{\chi}{4} \int_\Omega |\Delta w(\cdot, \sigma)|^2 d\sigma \right) d\sigma$$

$$\leq F(n, w)(t_*) + \frac{(a + (\ln \frac{K_3}{\mu})_+) |\Omega|}{\mu(t_1 + \gamma)} + \varepsilon,$$
which implies that

\[
\frac{\chi}{2} \int_{\Omega} |\nabla w(\cdot, T_s)|^2 \leq \mathcal{F}(n, w)(t_*) + \frac{(a + (\ln \frac{K_3}{\mu})_+)|\Omega|}{\mu(t_1 + \gamma)} - \int_{\Omega} n(\ln n + a)(\cdot, T_s) + \varepsilon
\]

\[
\leq \int_{\Omega} n(\ln n + a)(\cdot, t_*) + \frac{\chi}{2} \int_{\Omega} |\nabla w|^2(\cdot, t_*) + \varepsilon
\]

\[
+ \frac{(a + (\ln \frac{K_3}{\mu})_+)|\Omega|}{\mu(t_1 + \gamma)} - \int_{\Omega} n(\ln n + a)(\cdot, T_s)
\]

\[
\leq \int_{\Omega} (n^2 + an)(\cdot, t_*) + \frac{\chi}{2} \int_{\Omega} |\nabla w|^2(\cdot, t_*) + \varepsilon
\]

\[
+ \frac{(a + (\ln \frac{K_3}{\mu})_+)|\Omega|}{\mu(t_1 + \gamma)} - \int_{\Omega} n(\ln n + a)(\cdot, T_s),
\]

(3.56)

due to \( n \geq \ln n \) for all \( n > 0 \).

In addition, by (3.50), we see that

\[
\int_{\Omega} n(\ln n + a)(\cdot, T_s) \geq \int_{0 < n \leq e^{-a}} n(\ln n + a)(\cdot, T_s)
\]

\[
\geq \int_{0 < n \leq e^{-a}} n \ln n(\cdot, T_s)
\]

\[
\geq - |\Omega| \max_{0 < n \leq e^{-a}} |n \ln n|
\]

\[
\geq - \varepsilon.
\]

(3.57)

Upon inserting (3.57) into (3.56), we see that

\[
\frac{\chi}{2} \int_{\Omega} |\nabla w(\cdot, T_s)|^2 \leq \int_{\Omega} (n^2 + an)(\cdot, t_*) + \frac{\chi}{2} \int_{\Omega} |\nabla w|^2(\cdot, t_*)
\]

\[
+ \frac{(a + (\ln \frac{K_3}{\mu})_+)|\Omega|}{\mu(t_1 + \gamma)} + 2\varepsilon,
\]

(3.58)

which along with (3.52), (3.53), (2.2) and (3.50), establishes that

\[
\frac{\chi}{2} \int_{\Omega} |\nabla w(\cdot, T_s)|^2 \leq \frac{5\varepsilon}{2} + a \int_{\Omega} n(\cdot, t_*) + \frac{(a + (\ln \frac{K_3}{\mu})_+)|\Omega|}{\mu(t_1 + \gamma)}
\]

\[
< 3\varepsilon
\]

\[
\leq \frac{\chi}{8K_1}.
\]

(3.59)

This contradicts (3.54) and thereby \( T_s = \infty \), which means that the differential inequality (3.55) is actually valid for all \( t > t_* \).
Now revisiting the proof of (3.59), upon integration in time over \((t_*, t)\), we have
\[
\frac{X}{2} \int_{\Omega} |\nabla w(\cdot, t)|^2 \leq 3\varepsilon \quad \text{for all} \quad t > t_*
\]
which implies that (3.43) is valid by the choice of \(\varepsilon\), as well as
\[
\int_{\Omega} n \ln n(\cdot, t) \leq C_1 \quad \text{for all} \quad t > t_*
\] (3.60)
for some \(C_1 > 0\).

Since \(\xi \ln \xi \geq -\frac{1}{\varepsilon}\) for all \(\xi > 0\),
\[
\int_{\Omega} |\ln n(\cdot, t)| = \int_{\Omega} n \ln n(\cdot, t) - 2 \int_{0<n<1} n \ln n(\cdot, t)
\leq \int_{\Omega} n \ln n(\cdot, t) + \frac{2|\Omega|}{e},
\]
which along with (3.60) readily implies that (3.44) is actually valid with \(C = C_1 + \frac{2|\Omega|}{e}\).

Furthermore, from (3.55), one can also conclude that:

**Corollary 3.2.** Under the conditions of Lemma 3.4, we have
\[
\lim_{t \to \infty} \int_t^{t+1} \int_{\Omega} \left(\frac{|\nabla n|^2}{n} + |\Delta w|^2\right) = 0, \quad \lim_{t \to \infty} \int_{\Omega} |\nabla w(\cdot, t)|^2 = 0. \quad (3.61)
\]

**Proof.** On the basis of the decay estimate (2.2) and revisiting the argument in the proof of Lemma 3.4, one can conclude that for any fixed \(\varepsilon \in (0, \frac{X}{42K_1})\), there exists \(t_\varepsilon > 1\) such that
\[
\int_{\Omega} |\nabla w(\cdot, t)|^2 + \int_{t_\varepsilon}^t e^{-K_3(t-\sigma)} \left(\int_{\Omega} \frac{|\nabla n|^2}{n}(\cdot, \sigma) + \frac{X}{8} \int_{\Omega} |\Delta w(\cdot, \sigma)|^2\right) d\sigma \leq \varepsilon
\]
for all \(t > t_\varepsilon\). Furthermore, it follows from the above inequality that
\[
\int_{t-1}^t \left(\int_{\Omega} \frac{|\nabla n|^2}{n}(\cdot, \sigma) + \frac{X}{8} \int_{\Omega} |\Delta w(\cdot, \sigma)|^2\right) d\sigma \leq \varepsilon e^{K_3}
\]
for any \(t > t_\varepsilon + 1\), which implies that (3.61) is indeed valid.

At this point, we can prove Theorem 1.1 in the case of \(r \leq 0\).

**Proof of Theorem 1.1 in the case \(r \leq 0\).** We can repeat the argument in the proof of Theorem 1.1 in the case \(r > 0\). In fact, in view of (3.43) and (3.44), (3.45) is also valid for \(r \leq 0, \mu > 0\), and thereby the global boundedness of solutions can be proven. In addition, similar to the case of \(r > 0\), we can show
\[
\lim_{t \to \infty} \|n(\cdot, t)\|_{L^\infty(\Omega)} = 0, \quad (3.62)
\]

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\[
\lim_{t \to \infty} \| \nabla w(\cdot, t) \|_{L^\infty(\Omega)} = 0 \quad (3.63)
\]
as well as
\[
\lim_{t \to \infty} \| u(\cdot, t) \|_{L^\infty(\Omega)} = 0.
\quad (3.64)
\]
For the sake of completeness we shall only recount the main steps and refer to the mentioned sources for more details. Invoking standard parabolic regularity theory (see the proofs of Lemma 4.5 and Lemma 4.9 of [48] for details), one can see that there exist \( \theta \in (0, 1) \) and \( \alpha \in (\frac{1}{2}, 1) \) and \( c_1 > 0 \) such that for all \( t > 1 \)
\[
\| n \|_{C^{\theta, \theta}(\Omega \times [t, t+1])} + \| \nabla w(\cdot, t) \|_{C^\alpha(\Omega)} + \| A^\alpha u(\cdot, t) \|_{L^2(\Omega)} \leq c_1.
\quad (3.65)
\]
If (3.62) were false, then there would be \( c_2 > 0 \), \( (t_k)_{k \in \mathbb{N}} \) and \( (x_k)_{k \in \mathbb{N}} \subseteq \Omega \) such that \( t_k \to \infty \) as \( k \to \infty \), and \( n(x_k, t_k) > c_2 \) for all \( k \in \mathbb{N} \), which, along with the uniform continuity of \( n \) in \( \overline{\Omega} \times [t, t+1] \) as shown by (3.65), entails that one can find \( r > 0 \) such that \( B(x_k, r) \subseteq \Omega \) for all \( k \in \mathbb{N} \) and
\[
\frac{c_2}{2} \quad \text{for all} \quad x \in B(x_k, r).
\]
This shows
\[
\int_{\Omega} n(\cdot, t_k) \geq \int_{B(x_k, r)} n(\cdot, t_k) \geq \frac{c_2}{2} \pi r^2
\]
which contradicts (2.2) and thus proves (3.62). Similarly, on the basis of (3.61) and (3.65), (3.63) can be proved. Finally, (3.64) results from (2.14), (3.65) and a simple interpolation, and thereby completes the proof.

4 Asymptotic profile of solutions

It is observed that in the case \( r < 0 \), solutions to (1.1), (1.4), (1.5) enjoy the exponential decay property due to the exponential decay of \( \| n(\cdot, t) \|_{L^1(\Omega)} \). Therefore the present paper focuses on the asymptotic profile of (1.1), (1.4), (1.5) in the cases \( r > 0 \) and \( r = 0 \), namely, we will give the proofs of Theorems 1.2 and 1.3 respectively.
4.1 The case $r > 0$

Making use of the convergence properties of $(n, \frac{\|\nabla c\|}{c})$ asserted in Theorem 1.1, we apply $L^p - L^q$ estimates for the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ to show $(n, c, u) \rightarrow (\frac{r}{\mu}, 0, 0)$ in $L^\infty(\Omega)$ and $\frac{\|\nabla c\|}{c} \rightarrow 0$ in $L^p(\Omega)$ at some exponential rate as $t \rightarrow \infty$, respectively, whenever $\mu$ is suitably large compared with $r$. To this end, we first make an observation which will be used in the proof of the subsequent lemma:

**Lemma 4.1.** For any $\alpha \in (0, \min\{\lambda_1, r\})$, for $I := \int_0^\infty (1 + \sigma^{-\frac{2}{3}} + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1 - \alpha)^2} d\sigma$, and $c_i > 0$ $(i=1,3)$ as given by Lemma 2.1, there exist $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that

$$4c_1 \varepsilon_2 < 1, \quad 4c_1 |\Omega| \frac{\lambda}{\varepsilon_1} \leq \varepsilon_2, \quad 8\chi c_3 I \varepsilon_2 < 1,$$

$$8\mu \varepsilon_1 < r - \alpha. \quad (4.1)$$

**Lemma 4.2.** Let $(n, w, u)$ be the global bounded solution of (2.1). If $\mu > 32\chi c_3 |\Omega|^{\frac{1}{2}} I^2 r$, one can find constants $C_i > 0$, $i = 1, 2, 3$, $\alpha \in (0, \min\{\lambda_1, r\})$ and $\beta < \alpha$ such that

$$\|n(\cdot, t) - \frac{r}{\mu}\|_{L^\infty(\Omega)} \leq C_1 e^{-\alpha t}, \quad (4.3)$$

$$\|\nabla w(\cdot, t)\|_{L^6(\Omega)} \leq C_2 e^{-\alpha t} \quad (4.4)$$

as well as

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3 e^{-\beta t} \quad (4.5)$$

for all $t \geq 1$.

**Proof.** Let $N(x,t) = n(x,t) - \frac{r}{\mu}, \varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be given by Lemma 4.1. Then from (3.28), (3.29) and (3.30), there exists $t_0 > 1$ suitably large such that for $t \geq t_0$

$$\|N(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{\varepsilon_1}{8}, \quad (c_2 + 1)\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{\varepsilon_2}{8} \quad (4.6)$$

and

$$8c_1 \|u(\cdot, t)\|_{L^\infty(\Omega)} \int_0^\infty (1 + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1 - \alpha)^2} d\sigma \leq 1. \quad (4.7)$$
Now we consider

\[ T \triangleq \sup \left\{ \tilde{T} \in (t_0, \infty) \left| \begin{array}{l}
\| N(\cdot, t) \|_{L^\infty(\Omega)} \leq \varepsilon_1 e^{-\alpha(t-t_0)} \quad \text{for all } t \in [t_0, \tilde{T}], \\
\| \nabla w(\cdot, t) \|_{L^6(\Omega)} \leq \varepsilon_2 e^{-\alpha(t-t_0)} \quad \text{for all } t \in [t_0, \tilde{T}].
\end{array} \right. \right\} \quad (4.8) \]

By (4.6), \( T \) is well-defined. In what follows, we shall demonstrate that \( T = \infty \).

To this end, we first invoke the variation-of-constants representation of \( w \):

\[
w(\cdot, t) = e^{(t-t_0)\Delta} w(\cdot, t_0) - \int_{t_0}^{t} e^{(t-s)\Delta} |\nabla w(\cdot, s)|^2 ds + \int_{t_0}^{t} e^{(t-s)\Delta} N(\cdot, s) ds - \int_{t_0}^{t} e^{(t-s)\Delta} (u \cdot \nabla) (\cdot, s) ds + \frac{r}{\mu} (t - t_0),
\]

and use Lemma 2.1(i), (ii) to estimate

\[
\| \nabla w(\cdot, t) \|_{L^6(\Omega)}
\leq \| e^{(t-t_0)\Delta} w(\cdot, t_0) \|_{L^6(\Omega)} + \int_{t_0}^{t} \| e^{(t-s)\Delta} |\nabla w(\cdot, s)|^2 \|_{L^6(\Omega)} ds
\]

\[
+ \int_{t_0}^{t} \| e^{(t-s)\Delta} N(\cdot, s) \|_{L^6(\Omega)} ds + \int_{t_0}^{t} \| e^{(t-s)\Delta} (u \cdot \nabla) (\cdot, s) \|_{L^6(\Omega)} ds
\leq 2c_2 e^{-\lambda_1(t-t_0)} \| \nabla w(\cdot, t_0) \|_{L^6(\Omega)} + c_1 \int_{t_0}^{t} (1 + (t - s)^{-\frac{3}{4}}) e^{-\lambda_1(t-s)} \| \nabla w(\cdot, s) \|_{L^6(\Omega)}^2 ds
\]

\[
+ c_1 |\Omega|^\frac{1}{4} \int_{t_0}^{t} (1 + (t - s)^{-\frac{1}{4}}) e^{-\lambda_1(t-s)} \| N(\cdot, s) \|_{L^\infty(\Omega)} ds
\]

\[
+ c_1 \int_{t_0}^{t} (1 + (t - s)^{-\frac{1}{4}}) e^{-\lambda_1(t-s)} \| u(\cdot, s) \|_{L^\infty(\Omega)} \| \nabla w(\cdot, s) \|_{L^6(\Omega)} ds
\]

\[
:= I_1 + I_2 + I_3 \quad (4.10)
\]

for all \( t_0 < t < T \).

Now we estimate the terms \( I_i \ (i = 1, 2, 3) \) respectively. Firstly, from (4.6), we have \( I_1 \leq \frac{\varepsilon_2}{4} e^{-\lambda_1(t-t_0)} \). By the definition of \( T \) and (4.1), we can see that

\[
I_2 \leq c_1 \varepsilon_2^2 \int_{t_0}^{t} (1 + (t - s)^{-\frac{3}{4}}) e^{-\lambda_1(t-s)} e^{-2\alpha(s-t_0)} ds
\]

\[
\leq c_1 \varepsilon_2^2 \int_{t_0}^{t} (1 + (t - s)^{-\frac{3}{4}}) e^{-\lambda_1(t-s)} e^{-\alpha(s-t_0)} ds
\]

\[
\leq c_1 \varepsilon_2^2 \int_{t_0}^{\infty} (1 + \sigma^{-\frac{3}{4}}) e^{-(\lambda_1-\alpha)\sigma} d\sigma \cdot e^{-\alpha(t-t_0)}
\]

\[
\leq \frac{\varepsilon_2}{4} e^{-\alpha(t-t_0)}.
\]
By the definition of $T$, (4.7) and (4.11) again, we also have

$$I_3 \leq (c_1|\Omega|^2 \varepsilon_1 + c_1 \sup_{t \geq t_0} \|u(\cdot, t)\|_{L^\infty(\Omega)} \varepsilon_2) \int_{t_0}^t (1 + (t - s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} e^{-\alpha(s-t_0)} ds$$

$$= (c_1|\Omega|^2 \varepsilon_1 + c_1 \sup_{t \geq t_0} \|u(\cdot, t)\|_{L^\infty(\Omega)} \varepsilon_2) \int_{t_0}^t (1 + (t - s)^{-\frac{1}{2}}) e^{-(\lambda_1 - \alpha)(t-s)} e^{-\alpha(t-t_0)} ds$$

$$\leq (c_1|\Omega|^2 \varepsilon_1 + c_1 \sup_{t \geq t_0} \|u(\cdot, t)\|_{L^\infty(\Omega)} \varepsilon_2) \int_0^\infty (1 + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1 - \alpha)\sigma} d\sigma \cdot e^{-\alpha(t-t_0)}$$

$$\leq \frac{3\varepsilon_2}{8} e^{-\alpha(t-t_0)}.$$

Substituting these estimates into (4.10), we get

$$\|\nabla w(\cdot, t)\|_{L^6(\Omega)} \leq \frac{7\varepsilon_2}{8} e^{-\alpha(t-t_0)} < \varepsilon_2 e^{-\alpha(t-t_0)} \quad \text{for all} \quad t \in [t_0, T). \quad (4.11)$$

On the other hand, since

$$N_t = \Delta N + \chi \nabla \cdot (n \nabla w) - rN - \mu N^2 - u \cdot \nabla N,$$

the variation-of-constants representation of $N$ yields

$$N(\cdot, t) = e^{(t-t_0)(\Delta-r)} N(\cdot, t_0) + \chi \int_{t_0}^t e^{(t-s)(\Delta-r)} \nabla \cdot (n \nabla w)(\cdot, s) ds - \mu \int_{t_0}^t e^{(t-s)(\Delta-r)} N^2(\cdot, s) ds$$

$$\quad - \int_{t_0}^t e^{(t-s)(\Delta-r)} (u \cdot \nabla N)(\cdot, s) ds.$$

Then by $\nabla \cdot u = 0$ we can see that

$$\|N(\cdot, t)\|_{L^\infty(\Omega)}$$

$$\leq \|e^{(t-t_0)(\Delta-r)} N(\cdot, t_0)\|_{L^\infty(\Omega)} + \mu \int_{t_0}^t \left\|e^{(t-s)(\Delta-r)} N^2(\cdot, s)\right\|_{L^\infty(\Omega)} ds$$

$$+ \int_{t_0}^t \left\|e^{(t-s)(\Delta-r)} \nabla \cdot (uN)(\cdot, s)\right\|_{L^\infty(\Omega)} ds + \chi \int_{t_0}^t \left\|e^{(t-s)(\Delta-r)} \nabla \cdot (n \nabla w)(\cdot, s)\right\|_{L^\infty(\Omega)} ds$$

$$:= J_1 + J_2 + J_3 + J_4.$$

Here the maximum principle together with (4.6) ensures that

$$J_1 \leq e^{-r(t-t_0)} \|N(\cdot, t_0)\|_{L^\infty(\Omega)} \leq \frac{\varepsilon_1}{8} e^{-\alpha(t-t_0)}.$$
By the definition of $T$ and comparison principle, we infer that

\[ J_2 \leq \mu \int_{t_0}^{t} e^{-r(t-s)} \| e^{(t-s)A} N^2(\cdot, s) \|_{L^\infty(\Omega)} ds \]
\[ \leq \mu \int_{t_0}^{t} e^{-r(t-s)} \| N(\cdot, s) \|_{L^\infty(\Omega)}^2 ds \]
\[ \leq \mu \varepsilon_1^2 \int_{t_0}^{t} e^{-r(t-s)} e^{-2\alpha(s-t_0)} ds \]
\[ \leq \mu \varepsilon_1^2 \int_{t_0}^{t} e^{-(r-\alpha)(t-s)} ds \cdot e^{-\alpha(t-t_0)} \]
\[ \leq \frac{\mu \varepsilon_1^2}{r-\alpha} e^{-\alpha(t-t_0)} \]
\[ \leq \frac{\varepsilon_1}{8} e^{-\alpha(t-t_0)} \]

due to (4.2) and $\alpha < r$. Similarly by (4.7), we have

\[ J_3 \leq c_1 \sup_{t \geq t_0} \| u(\cdot, t) \|_{L^\infty(\Omega)} \int_{t_0}^{t} (1 + (t-s)^{-\frac{3}{2}}) e^{-(\lambda_1+r)(t-s)} \| N(\cdot, s) \|_{L^\infty(\Omega)} ds \]
\[ \leq c_1 \sup_{t \geq t_0} \| u(\cdot, t) \|_{L^\infty(\Omega)} \varepsilon_1 \int_{t_0}^{t} (1 + (t-s)^{-\frac{3}{2}}) e^{-(\lambda_1+r)(t-s)} e^{-\alpha(s-t_0)} ds \]
\[ \leq c_1 \sup_{t \geq t_0} \| u(\cdot, t) \|_{L^\infty(\Omega)} \varepsilon_1 \int_{t_0}^{\infty} (1 + \sigma^{-\frac{3}{2}}) e^{-(\lambda_1-\alpha)\sigma} d\sigma \cdot e^{-\alpha(t-t_0)} \]
\[ \leq \frac{\varepsilon_1}{8} e^{-\alpha(t-t_0)}. \]

As for the term $J_4$, we recall (4.1), (4.8) and apply Lemma 2.1(iii) to get

\[ J_4 \leq \chi c_3 \int_{t_0}^{t} (1 + (t-s)^{-\frac{3}{2}}) e^{-(\lambda_1+r)(t-s)} \| \langle n \nabla w \rangle(\cdot, s) \|_{L^p(\Omega)} ds \]
\[ \leq \chi c_3 \varepsilon_2 \int_{t_0}^{t} (1 + (t-s)^{-\frac{3}{2}}) e^{-(\lambda_1+r)(t-s)} \left( \frac{r}{\mu} + \varepsilon_1 e^{-\alpha(s-t_0)} \right) e^{-\alpha(s-t_0)} ds \]
\[ \leq \chi c_3 \varepsilon_2 \left( \frac{r}{\mu} + \varepsilon_1 \right) \int_{t_0}^{\infty} (1 + \sigma^{-\frac{3}{2}}) e^{-(\lambda_1+r-\alpha)\sigma} d\sigma \cdot e^{-\alpha(t-t_0)} \]
\[ \leq \frac{\varepsilon_1}{8} e^{-\alpha(t-t_0)} + \chi c_3 \frac{r}{\mu} I \varepsilon_1 e^{-\alpha(t-t_0)} \]
\[ \leq \frac{\varepsilon_1}{8} e^{-\alpha(t-t_0)} \]

provided that

\[ \chi c_3 \varepsilon_2 \frac{r}{\mu} I < \frac{\varepsilon_1}{8}. \]  (4.12)

Therefore, letting $\varepsilon_2 = 4c_1|\Omega|^{\frac{3}{2}} I \varepsilon_1$ in Lemma 4.1, (4.12) can be warranted provided $\mu > 32\chi c_3 c_1|\Omega|^{\frac{3}{2}} I^2 r$, and thereby

\[ \| N(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{5\varepsilon_1}{8} e^{-\alpha(t-t_0)} \] for all $t \in [t_0, T)$. 

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This along with (4.11) readily shows that $T$ cannot be finite. In combination with the decay property (4.3), a straightforward interpolation argument can be employed to prove (4.5).

**Proof of Theorem 1.2.** According to (3.41) and $w = -\ln(\frac{c(t)}{\|c\|_{L^\infty(\Omega)}})$, we have $c(x, t) \leq \|c_0\|_{L^\infty(\Omega)} e^{-\frac{1}{2\mu}(t-t_3)}$ for all $t \geq t_3$. On the other hand, if $\mu_*(\chi, \Omega, r) := \max\{\mu_0, 32\chi c_1|\Omega|^{\frac{1}{2}}f^2r\}$, then as an immediate consequence of Theorem 1.1 and Lemma 4.3, $n(\cdot, t) \to \frac{\tau}{\mu}$ and $\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \to 0$ in $L^\infty(\Omega)$ and $L^6(\Omega)$, respectively, at an exponential rate when $\mu > \mu_*(\chi, \Omega, r)$. Moreover, with the help of the uniform boundedness of $\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)}$ with respect to $t > 0$, one can show that $\|\nabla c(\cdot, t)\|_{L^p(\Omega)}$ for any $p > 1$ exponentially by the interpolation argument. The proof of this theorem is thus complete.

**4.2 The case $r = 0$**

The proof of Theorem 1.3 proceeds on an alternative reasoning. To this end, making use of the decay information on $|\nabla w|$ in $L^\infty(\Omega)$ in (3.61) and the quadratic degradation in the $n-$equation, we first turn the decay property of $\|n(\cdot, t)\|_{L^1(\Omega)}$ from (2.2) into an upper bound estimate of $\|n(\cdot, t)\|_{L^\infty(\Omega)}$.

**Lemma 4.3.** Let $(n, w, u)$ be the global bounded solution of (2.1) obtained in Theorem 1.1 with $r = 0, \mu > 0$. Then one can find constant $C > 0$ such that

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{C}{t + 1} \text{ for all } t > 0. \quad (4.13)$$

**Proof.** According to the known smoothing properties of the Neumann heat semigroup $(e^{\tau\Delta})_{t>0}$ on $\Omega \subset \mathbb{R}^n$ (see [43]), one can pick $c_1 > 0$ and $c_2 > 0$ such that for all $0 < \tau \leq 1$,

$$\|e^{\tau\Delta}\varphi\|_{L^\infty(\Omega)} \leq c_1 \tau^{\frac{n}{2}} \|\varphi\|_{L^1(\Omega)} \text{ for all } \varphi \in L^1(\Omega) \quad (4.14)$$

and

$$\|e^{\tau\Delta}\nabla \cdot \varphi\|_{L^\infty(\Omega)} \leq c_2 \tau^{-\frac{n}{2}} \|\nabla \varphi\|_{L^p(\Omega)} \text{ for all } \varphi \in C^1(\Omega; \mathbb{R}^n). \quad (4.15)$$

By (3.63) and (3.64), there exists $t_0 > 3$ such that

$$24c_2(\chi\|\nabla w(\cdot, t)\|_{L^3(\Omega)} + \|u(\cdot, t)\|_{L^2(\Omega)}) \leq 1 \text{ for all } t > t_0 - 1. \quad (4.16)$$
Now in order to prove the lemma, it is sufficient to derive a bound, independent of $T \in (t_0, \infty)$, for $M(T) \triangleq \sup_{t_0-1 < t < T} \{t\|n(\cdot, t)\|_{L^\infty(\Omega)}\}$.

By the variation-of-constants representation of $n$, we have

$$
\begin{aligned}
n(\cdot, t) &= e^{\Delta} n(\cdot, t-1) + \chi \int_{t-1}^t e^{(t-s)\Delta} \nabla \cdot (n \nabla w)(\cdot, s) ds - \int_{t-1}^t e^{(t-s)\Delta} (u \cdot \nabla n)(\cdot, s) ds \\
&\quad - \mu \int_{t-1}^t e^{(t-s)\Delta} n^2(\cdot, s) ds.
\end{aligned}
$$

(4.17)

Since $e^{(t-s)\Delta}$ is nonnegative in $\Omega$ for all $0 < s < t$ due to the maximum principle, it follows from the nonnegativity of $n$ that for all $t \in (t_0, T)$

$$
\begin{aligned}
\|n(\cdot, t)\|_{L^\infty(\Omega)} &\leq \|e^{\Delta} n(\cdot, t-1)\|_{L^\infty(\Omega)} + \chi \int_{t-1}^t \|e^{(t-s)\Delta} \nabla \cdot (n \nabla w)(\cdot, s)\|_{L^\infty(\Omega)} ds + \int_{t-1}^t \|e^{(t-s)\Delta} (u \cdot \nabla n)(\cdot, s)\|_{L^\infty(\Omega)} ds \\
&\leq c_1 \|n(\cdot, t-1)\|_{L^1(\Omega)} + c_2 \chi \int_{t-1}^t (t-s)^{-\frac{5}{2}} \|n \nabla w(\cdot, s)\|_{L^1(\Omega)} ds + c_2 \int_{t-1}^t (t-s)^{-\frac{5}{2}} \|u(\cdot, s)\|_{L^1(\Omega)} ds \\
&\leq \frac{c_1 |\Omega|}{\mu(t-1+\gamma)} + \frac{6c_2}{t-1} \left(\chi \max_{t_0-1 < s < T} \|\nabla w(\cdot, s)\|_{L^1(\Omega)} + \|u(\cdot, s)\|_{L^2(\Omega)}\right) \cdot M(T) \\
&\quad + \frac{1}{4(t-1)} M(T).
\end{aligned}
$$

Hence,

$$
M(T) \leq \frac{4c_1 |\Omega|}{\mu} + \frac{2}{t_0-1 < s < T} \{s\|n(\cdot, s)\|_{L^\infty(\Omega)}\},
$$

which readily yields (4.13) since $T > t_0$ is arbitrary, and thus ends the proof.

In light of Lemma 4.3, we can derive a pointwise estimate $c(x, t)$ from below.

Lemma 4.4. Let $(n, w, u)$ be the global classical solution of (2.1) obtained in Theorem 1.1 with $r = 0, \mu > 0$. Then there exists $\kappa > 0$ fulfilling

$$
c(x, t) \geq \inf_{x \in \Omega} \frac{c_0(x)}{(t+1)^\kappa}.
$$

(4.18)

Proof. By the second equation of (2.1) and Lemma 4.3 we can see that

$$
w_t \leq \Delta w - |\nabla w|^2 + \frac{c_1}{t+1} - u \cdot \nabla w
$$

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with some $c_1 > 0$ for all $t > 0$. Let $y \in C^1([0, \infty))$ denote the solution of the initial-value problem $y'(t) = \frac{c_1}{t+1}$, $y(0) = \|w_0\|_{L^\infty(\Omega)}$, then from the comparison principle, we infer that

$$w(x, t) \leq \|w_0\|_{L^\infty(\Omega)} + c_1 \ln(t+1) \quad \text{for all } t > 0,$$

which along with $w = -\ln(\frac{c_1}{\|w_0\|_{L^\infty(\Omega)}})$, yields (4.18) with $\kappa = c_1$.

Now utilizing the decay information on $|\nabla w|$ in $L^\infty(\Omega)$ in (3.61) again, and thanks to the precise information on the decay of $\|n(\cdot, t)\|_{L^\infty(\Omega)}$ in Lemma 4.3, we can obtain the desired estimate for $\|n(\cdot, t)\|_{L^\infty(\Omega)}$ from below as well as the upper estimate for $\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)}$.

**Lemma 4.5.** Let $(n, w, u)$ be the solution of (2.1) obtained in Theorem 1.1 with $r = 0, \mu > 0$. Then one can find $C_1 > 0$ and $C_2 > 0$ fulfilling

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} \geq \frac{1}{|\Omega|} \|n(\cdot, t)\|_{L^1(\Omega)} \geq \frac{C_1}{t+1} \quad \text{for all } t > 0$$

as well as

$$\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{C_2}{t+1} \quad \text{for all } t > 0.$$

**Proof.** We first adapt the method in Lemma 4.3 to derive the precise decay rate of $\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)}$. By (3.63) and (3.64), one can choose some $t_0 > 2$ such that

$$4c_1 \int_0^\infty (1 + \sigma^{-\frac{3}{2}})e^{-\lambda_1 \sigma} d\sigma (\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)}) \leq 1 \quad \text{for all } t > \frac{t_0}{2},$$

and then let $M(T) \triangleq \sup_{\frac{t_0}{2} < s < T} \{s \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)}\}$ for all $T > t_0$.

By the variation-of-constants representation of $w$, we have

$$w(\cdot, t) = e^{\frac{t}{2} \Delta} w(\cdot, \frac{t}{2}) - \int_{\frac{t}{2}}^t e^{(t-s)\Delta} |\nabla w|^2(\cdot, s) ds + \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (n - u \cdot \nabla w)(\cdot, s) ds$$

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for all $t_0 < t < T$. We then show that

$$
\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)}
\leq \|\nabla e^{\frac{t}{2}\Delta}w(\cdot, \frac{t}{2})\|_{L^\infty(\Omega)} + \int_{\frac{t}{2}}^{t} \|\nabla e^{(t-s)\Delta} |\nabla w|^2\|_{L^\infty(\Omega)} + \int_{\frac{t}{2}}^{t} \|\nabla e^{(t-s)\Delta} n\|_{L^\infty(\Omega)}
+ \int_{\frac{t}{2}}^{t} \|\nabla e^{(t-s)\Delta} (u \cdot \nabla w)\|_{L^\infty(\Omega)}
\leq c_1 (1 + t^{\frac{1}{2}}) e^{-\frac{\lambda_1 t}{2}} \|w(\cdot, \frac{t}{2})\|_{L^\infty(\Omega)} + c_1 \int_{\frac{t}{2}}^{t} (1 + (t - s)^{-\frac{1}{2}}) e^{-\lambda_1 (t-s)} \|n(\cdot, s)\|_{L^\infty(\Omega)}
+ c_1 \int_{\frac{t}{2}}^{t} (1 + (t - s)^{-\frac{1}{2}}) e^{-\lambda_1 (t-s)} \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)}
\leq c_1 (1 + t^{\frac{1}{2}}) e^{-\frac{\lambda_1 t}{2}} (\|w_0\|_{L^\infty(\Omega)} + c_2 \ln(t + 1)) + \frac{2c_1c_2}{t} \int_{0}^{\infty} (1 + \sigma^{\frac{1}{2}}) e^{-\lambda_1 \sigma} d\sigma
\leq c_1 (1 + t^{\frac{1}{2}}) e^{-\frac{\lambda_1 t}{2}} (\|w_0\|_{L^\infty(\Omega)} + c_2 \ln(t + 1)) + \frac{2c_1c_2}{t} \int_{0}^{\infty} (1 + \sigma^{\frac{1}{2}}) e^{-\lambda_1 \sigma} d\sigma
+ \frac{1}{2t} M(T)
$$

by using Lemma 2.1(i), (4.19), (4.13) and (4.22). This along with the definition of $M(T)$ yields

$$
M(T) \leq 2 \sup_{\mathcal{A}} \{ s \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)} \} + c_3
$$

with some constant $c_3 > 0$ as $\lim_{t \to \infty} t \ln(t + 1) e^{-\lambda_1 t} = 0$. Hence, upon the definition of $M(T)$, we arrive at (4.21) with an evident choice of $C_2$.

Continuing with the proof, we claim that there exists $c_4 > 0$ such that

$$
\|n(\cdot, t)\|_{L^\infty(\Omega)} \geq \frac{1}{|\Omega|} \|n(\cdot, t)\|_{L^1(\Omega)} \geq \frac{c_4}{t + 1} \quad \text{for all } t > 0.
$$

(4.23)

Indeed, from the $n$–equation of (2.1) with $r = 0$ and Young’s inequality, it follows that

$$
\frac{d}{dt} \int_{\Omega} \ln n = \int_{\Omega} \frac{\|\nabla n\|^2}{n^2} + \chi \int_{\Omega} \frac{1}{n} \nabla \cdot (n \nabla w) - \mu \int_{\Omega} n
\geq -\frac{\chi^2}{4} \int_{\Omega} \|\nabla w\|^2 - \mu \int_{\Omega} n.
$$

Inserting (2.2) and (4.21) into the above inequality yields

$$
\frac{d}{dt} \int_{\Omega} \ln n \geq -\frac{\chi^2}{4} \frac{C_2^2 |\Omega|}{(t + 1)^2} - \frac{|\Omega|}{t + \gamma}
$$
and thus
\[ \int_{\Omega} \ln n(\cdot, t) \geq -|\Omega| \ln(t + \gamma) - c_5 \quad \text{for all } t > 1 \quad (4.24) \]
with some \( c_5 > 0 \). On the other hand, by the Jensen inequality, we have
\[ |\Omega| \ln(\int_{\Omega} n(\cdot, t)) - |\Omega| \ln |\Omega| = |\Omega| \ln\{ \frac{1}{|\Omega|} \int_{\Omega} n(\cdot, t) \} \geq \int_{\Omega} \ln n(\cdot, t). \]
This inequality together with (4.24) readily leads to (4.20).

With the above lemmas at hand, we can now complete the proof of Theorem 1.3. 

**Proof of Theorem 1.3.** By \( w = -\ln( \frac{c}{\|c_0\|_{L^\infty(\Omega)}} ) \), Lemma 4.3 and Lemma 4.5, one can see that \((n, |\nabla c|) \to (0, 0)\) in \( L^\infty(\Omega) \) algebraically as \( t \to \infty \). Hence it suffices to show the decay property of \( c(x, t) \). In view of the \( w \)-equation in (2.1), (4.23), (4.21) and \( \nabla \cdot u = 0 \), we can pick \( c_1 > 0, c_2 > 0 \) and \( c_3 > 0 \) such that
\[
\frac{d}{dt} \int_{\Omega} w = \int_{\Omega} n - \int_{\Omega} |\nabla w|^2 - \int_{\Omega} u \cdot \nabla w \\
\geq \frac{c_1 |\Omega|}{t + 1} - \frac{c_2 |\Omega|}{(t + 1)^2},
\]
and hence
\[
\int_{\Omega} w(\cdot, t) \geq c_1 |\Omega| \ln(t + 1) - c_3,
\]
which entails that for any \( t > 0 \) there exists \( x_0(t) \in \Omega \) such that
\[
w(x_0(t), t) \geq c_1 \ln(t + 1) - \frac{c_3}{|\Omega|}.
\]
Since for each \( \varphi \in W^{1,p}(\Omega) \) with \( p > 2 \), there exists \( c_4 > 0 \) such that
\[
|\varphi(x) - \varphi(y)| \leq c_4 |x - y|^{1-\frac{2}{p}} \| \nabla \varphi \|_{L^p(\Omega)} \quad \text{for all } x, y \in \Omega,
\]
we therefore obtain from (4.21) that
\[
w(x, t) \geq w(x_0(t), t) - |x - x_0(t)| \| \nabla w(\cdot, t) \|_{L^\infty(\Omega)} \\
\geq c_1 \ln(t + 1) - \frac{c_3}{|\Omega|} - c_4 \text{diam}(\Omega),
\]
and thereby
\[
c(x, t) \leq \frac{c_5}{(t + 1)^{c_1}} \quad \text{for } x \in \Omega, t > 0
\]
with some \( c_5 > 0 \). This together with (4.18) shows that \( c(x, t) \) actually converges to 0 in \( L^\infty(\Omega) \) algebraically as \( t \to \infty \), and thus ends the proof of Theorem 1.3.
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