The Weyl group of type $A_1$ root systems extended by an abelian group*

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Abstract

We investigate the class of root systems $R$ obtained by extending an $A_1$-type irreducible root system by a free abelian group $G$. In this context there are two reflection groups with respect to a discrete symmetric space $T$ associated to $R$, namely, the Weyl group $W$ of $R$ and a group $U$ with a so-called presentation by conjugation. We show that the natural homomorphism $U \rightarrow W$ is an isomorphism if and only if an associated subset $T^{ab} \setminus \{0\}$ of $G_2 = G/2G$ is 2-independent, i.e. its image under the map $G_2 \rightarrow G_2 \otimes G_2$, $g \mapsto g \otimes g$ is linearly independent over the Galois field $\mathbb{F}_2$.

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1 Introduction

We consider a root system $R$ extended by an abelian group $G$, a notion that is introduced in [Yos04]. It generalizes the concepts of extended affine root systems (see [AAB+97], for instance) and affine root systems in the sense of [Sai85], both of which are generalizations of root systems of affine Kac-Moody algebras (see [MP95], for instance). The Weyl group $W$ of $R$ is not necessarily a Coxeter group, so a more general presentation is needed to capture the algebraic structure of $W$. The group $U$ is given by the so-called presentation by conjugation with respect to $R$:

$$U \cong \langle (\hat{r}_\alpha)_{\alpha \in R^\times} \mid \hat{r}_\alpha = \hat{r}_\beta \text{ if } \alpha \text{ and } \beta \text{ are linearly dependent}, \hat{r}_\alpha^2 = 1, \hat{r}_\alpha \hat{r}_\beta \hat{r}_\alpha^{-1} = \hat{r}_{\alpha(\beta)} \rangle; \text{ for } \alpha, \beta \in R \rangle.$$

There is a natural group homomorphism from $U$ onto $W$.

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Question Is $U \to W$ injective? In other words, does $W$ have the presentation by conjugation with respect to $R$?

This question has been studied for various root systems in [Kry00], [Aza99], [Aza00], [AS07], [AS08], [Hof07], [Hof08].

In this note we investigate the case that $R$ is of type $A_1$, i.e. the underlying finite root system consists of two roots. This type of root system $R$ allows for less rigidity then other types and is therefor of special interest as a prototype. We prove the following result that allows an answer to the question above using an algorithmic approach.

Suppose $R$ is a type $A_1$ root system extended by a free abelian group $G$. Then a subset $T^{ab}$ of $G_2 = G/2G$ can be associated to it in a natural way. This subset is called 2-independent, if its image under $G_2 \to G_2 \otimes G_2, g \mapsto g \otimes g$ is a linearly independent set.

Theorem The natural homomorphism $U \to W$ is an isomorphism if and only if $T^{ab} \setminus \{0\}$ is 2-independent in $G_2$.

This result provides an attractive alternative to a characterization proved in [AS08] using so-called integral collections. Our answer to the question above is more general than that in [AS08] as $G$ is not required to be finitely generated.

We expect that the idea of 2-independence that we have introduced will play an important role in understanding the question for root systems of the types $B_n$ and $C_n$.

2 Discrete symmetric spaces and their reflection groups

In this section we provide the basic terminology for the following sections. The notion of a discrete symmetric space is a special case of the symmetric spaces introduced in [Loo69]. The associated category of reflection groups is introduced in [Hof08] and more details can be found there.

Definition 2.1 (Discrete symmetric space) Let $T$ be a set with a (not necessarily associative) multiplication

$$\mu : T \times T \to T, \ (s,t) \mapsto s \cdot t.$$ 

Then the pair $(T,\mu)$ is called a discrete symmetric space if the following conditions are satisfied for all $s, t$ and $r \in T$:

(S1) $s \cdot s = s$,
(S2) $s \cdot (s \cdot t) = t$,
(S3) $r \cdot (s \cdot t) = (r \cdot s) \cdot (r \cdot t)$. 

By abuse of language, we will sometimes say that $T$ is a discrete symmetric space instead of saying that $(T, \mu)$ is a discrete symmetric space. If $s \cdot t = t$ for all $s$ and $t \in T$ then we call $\mu$ the trivial multiplication.

For the remainder of this section, let $T$ be a discrete symmetric space.

**Definition 2.2 (Reflection group)** Let $X$ be a group acting on $T$. We will denote the element in $T$ obtained by $x$ acting on $t$ by $x.t$. Let $q_X : T \rightarrow X, t \mapsto t^X$ be a function. Then $(X, q_X)$ is called a $T$-reflection group, if the following conditions are satisfied:

1. (G1) The group $X$ is generated by the set $T^X := \{ t^X \mid t \in T \}$.
2. (G2) For all $s$ and $t \in T$ we have $t^X.s = t \cdot s$.
3. (G3) For all $s$ and $t \in T$ we have $t^X.s^{X}(t^X)^{-1} = (t.s)^X$.
4. (G4) For every $t \in T$ we have $(t^X)^2 = 1$.

If we do not need to specify the map $\cdot^X$ we will also say that $X$ is a reflection group instead of saying that $(X, \cdot^X)$ is a reflection group.

**Definition 2.3 (Reflection morphism)** Let $X$ and $Y$ be $T$-reflection groups. Then a group homomorphism $\varphi : X \rightarrow Y$ is called a $T$-morphism, if $\varphi(t^X) = t^Y$ for every $t \in T$.

Let the group $U$ be given by the presentation $U := \langle (t^U)_{t \in T} \mid (t^U)^2 = 1 \text{ and } t^U.s^{U}(t^U)^{-1} = (t.s)^U \text{ for } s \text{ and } t \in T \rangle$. There is map $\cdot^U : T \rightarrow U, t \mapsto t^U$ associated to the presentation. An action of $U$ on $T$ can be defined satisfying $t^U.s = t.s$ for all $s$ and $t \in T$. With respect to this action the pair $(U, \cdot^U)$ is a $T$-reflection group. There is a unique $T$-morphism from $U$ into any other $T$-reflection group.

**Definition 2.4** The pair $(U, \cdot^U)$ is called the initial $T$-reflection group.

### 3 Type A₁ root systems extended by an abelian group

In this section we introduce the concept of a type A₁ root system extended by an abelian group $G$ in an ad hoc manner. Thus we avoid presenting the details of the definition for more general types.

Let $(G, +)$ be an abelian group. Define the multiplication

$$G \times G \rightarrow G, \ (g, h) \mapsto g \cdot h = 2g - h. \quad (1)$$
Now let $T$ be a generating subset of $G$ such that $0 \in T$ and $G \cdot T \subseteq T$. It is straightforward to verify that $T$ with the restriction of the multiplication above is a discrete symmetric space. The set $R := T \times \{1, -1\}$ is a type $A_1$ root system extended by the abelian group $G$ in the sense of [Yos04] or [Hof08].

Consider the two-element group $\mathcal{V} := \{1, -1\}$ with its action on $G$ characterized by $-1g = -g$ for all $g \in G$. Set $\mathcal{A} := G \times \mathcal{V}$. Then $\mathcal{A}$ acts on $T$ via

$$(g, v).t = 2g + vt.$$ 

The map

$$\cdot^A : T \to \mathcal{A}, \ t \mapsto t^A = (t, -1)$$

turns $\mathcal{A}$ into a $T$-reflection group.

In general, if $B$ is a group, $A$ is an abelian group, and $f : B \times B \to A$ is a cocycle, then the set $A \times B$ with the multiplication given by

$$(a, b)(a', b') = (a + a' + f(b, b'), bb')$$

defines a group denoted by $A \times f B$ which is a central extension of $B$.

The set $(G \wedge G) \times G \times \mathcal{V}$ with the multiplication

$$(l, g, v)(l', g', v') := (l + l' + g \wedge (vg'), g + vg', vv')$$

is a group. We denote it by $(G \wedge G) \wedge \times G \times \mathcal{V}$. It can equally be interpreted as the semidirect product of the Heisenberg group $(G \wedge G) \wedge \times G$ with $\mathcal{V}$ or a central extension of $\mathcal{A}$ by $G \wedge G$ with cocycle $f : \mathcal{A}^2 \to G \wedge G$, $((g, v), (g', v')) \mapsto g \wedge (vg')$.

Set

$$\cdot^W : T \to (G \wedge G) \wedge \times G \times \mathcal{V}, \ t \mapsto t^W = (0, t, -1).$$

Let $W$ be the subgroup of $(G \wedge G) \wedge \times G \times \mathcal{V}$ generated by $T^W$. Then $(W, \cdot^W)$ is a $T$-reflection group with the action of $W$ on $T$ induced by the action of $\mathcal{A}$ on $T$.

**Definition 3.1** The group $W$ is called the Weyl group of $R$. 

This definition of the Weyl group coincides with the definition of Weyl groups given in [Hof08] if $G$ is free abelian and the one given in [Aza99] if $G$ is finitely generated free abelian.

4 The abelian 2-group case

In this section we investigate the case where $G$ is an elementary abelian 2-group. So we may think of $G$ as a vector space over the Galois field $\mathbb{F}_2$ with two elements. From it immediately follows that $T$ has the trivial multiplication.
Denote by $G \otimes_{\text{sym}} G$ the subgroup of $G \otimes G$ generated by the elements of the set $\{v \otimes v \mid v \in G\}$. The group homomorphism

$$G \otimes G \to G \otimes_{\text{sym}} G$$

characterized by $g \otimes h \mapsto g \otimes h - h \otimes g$

factors through $G \wedge G$ giving a group homomorphism

$$\pi : G \wedge G \to G \otimes_{\text{sym}} G$$

characterized by $u \wedge v \mapsto u \otimes v - v \otimes u$.

If $B$ is an ordered basis of $G$ then $\{b_1 \wedge b_2 \mid b_1 < b_2 \in B\}$ is a basis of $G \wedge G$. Its image under $\pi$ is linearly independent, so $\pi$ is injective.

Define the map

$$\varphi : (G \wedge G) \times \wedge G \to G \otimes_{\text{sym}} G, \ (t,g) \mapsto (\pi(t) + g \otimes g).$$

**Theorem 4.1** The map $\varphi$ is a group isomorphism such that $\varphi(0,g) = g \otimes g$ for all $g \in G$.

**Proof.** To see that $\varphi$ is a group homomorphism let $s, t \in G \wedge G$ and $g, h \in G$. Then

$$\varphi((s,g)(t,h)) = \varphi(s + t + g \wedge h, g + h)
= \varphi(s) + \pi(t) + g \otimes h + h \otimes g + (g + h) \otimes (g + h)
= \varphi(s) + \pi(t) + g \otimes g + h \otimes h = \varphi(s,g) + \varphi(t,h).$$

It is clear that $\varphi$ is surjective, since it has a generating set in its image.

Since we are working with characteristic 2, the map

$$G \to G \vee G, \ v \mapsto v \vee v$$

is an injective group homomorphism. We denote by $G \vee_{\text{sym}} G$ the additive subgroup of $G \vee G$ generated by $\{g \vee g \mid g \in G\}$. So we have a group isomorphism

$$G \vee_{\text{sym}} G \to G.$$

Its composition with the quotient homomorphism $G \otimes_{\text{sym}} G \to G \vee_{\text{sym}} G$ yields a homomorphism

$$\sqrt{\cdot} : G \otimes_{\text{sym}} G \to G$$

satisfying $\sqrt{g \otimes g} = g$.

It vanishes on the image of $\pi$, since

$$\sqrt{\pi(g \wedge h)} = \sqrt{g \otimes h - h \otimes g} = \sqrt{g \otimes h + h \otimes g}
= \sqrt{(g + h) \otimes (g + h)} - \sqrt{g \otimes g} - \sqrt{h \otimes h}
= g + h - g - h = 0$$

for all $g$ and $h \in G$. 

To show that \( \varphi \) is injective, let \( (t,v) \in \ker(\varphi) \), so \( \pi(t) = v \otimes v \). Taking the square root on both sides yields \( 0 = v \). We conclude \( \pi(t) = 0 \). Since \( \pi \) is injective we obtain \( t = 0 \).

In this section the action of \( V \) on \( G \) is trivial, so the reflection group \( A \) is given by the direct product \( A = G \times V \). The Weyl group \( W \) is given as the subgroup of \( ((G \wedge G) \times G) \times V \) generated by the image of \( \pi \):

\[
\pi^W: T \to ((G \wedge G) \times G) \times V, \quad t \mapsto t^W = (0, t, -1).
\]

Due to the preceding theorem, the Weyl group can also be given as the subgroup of \( (G \otimes \text{sym} G) \times V \) generated by the image of \( \pi \):

\[
\pi^W: T \to (G \otimes \text{sym} G) \times V, \quad t \mapsto t^W = (t \otimes t, -1).
\]

Let \( F := F(T \setminus \{0\}) \) be the free vector space on the set \( T \setminus \{0\} \) with the embedding \( \iota: T \setminus \{0\} \to F \). The initial reflection group is given by \( U = F \times V \) with the map

\[
T \to U, \quad t \mapsto \begin{cases} \iota(t), -1 & \text{if } t \neq 0 \\ (0, -1) & \text{if } t = 0. \end{cases}
\]

**Definition 4.2** A subset \( M \) of \( G \) is called 2-dependent, if the elements of the set \( \{g \otimes g \mid g \in M\} \) are linearly dependent in \( G \otimes G \). The set \( M \) is called 2-independent if it is not 2-dependent.

**Example 4.3**

a) A linearly independent subset \( M \) of \( G \) is 2-independent, due to the homomorphism \( \sqrt{\cdot} \) used in the proof of Theorem 4.1.

b) Set \( G = (\mathbb{F}_2)^2 \). Then the set \( M \) of all nonzero vectors in \( G \) is 2-independent, since the matrices

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

are linearly independent.

c) Set \( G = (\mathbb{F}_2)^n \). Any subset \( M \) of \( G \) with cardinality \( |M| > \frac{n(n+1)}{2} \) is 2-dependent, since \( \text{dim}_{\mathbb{F}_2} (G \otimes \text{sym} G) = \frac{n(n+1)}{2} \).

**Theorem 4.4** The reflection morphism \( \mathcal{U} \to \mathcal{W} \) is injective if and only if the set \( T \setminus \{0\} \) is 2-independent in \( G \).

**Proof.** We will use the form of the Weyl group \( W \) given in (2). Suppose \( \mathcal{U} \to \mathcal{W} \) is not injective. Then there is a non-trivial element in its kernel. This element can be written as \( (\sum_{i=1}^n \iota(t_i), \sigma) \in G \otimes \text{sym} G \times V \) for distinct elements \( t_1, t_2, \ldots, t_n \in T \setminus \{0\} \) and \( \sigma \in V \). It follows that \( \sigma = 1 \) and \( \sum_{i=1}^n t_i \otimes t_i = 0 \). So \( t_1, t_2, \ldots, t_n \) are 2-dependent. This implies that \( T \setminus \{0\} \) is 2-dependent.
Conversely, suppose $T \setminus \{0\}$ is 2-dependent, say $\sum_{i=1}^{n} t_i \otimes t_i = 0$ for distinct elements $t_1, t_2, \ldots, t_n \in T \setminus \{0\}$ and $n \geq 1$. Then $\left( \sum_{i=1}^{n} t_i, 0 \right)$ is a nontrivial element in the kernel of $U \to W$. \hfill $\blacksquare$

Denote the reflection morphism $U \to W$ above by $\varphi$. Then Example 4.3 yields

**Corollary 4.5**

(i) The map $\varphi$ is injective if $T \setminus \{0\}$ is a basis of $G$.

(ii) The map $\varphi$ is not injective if $|T \setminus \{0\}| > \frac{n(n+1)}{2}$, where $n = \dim(G)$.

(iii) If $T = G$, then $\varphi$ is an isomorphism if and only if $\dim(G) \leq 2$.

5 The free abelian case

In this section let $G$ be a free abelian group. We will reduce the situation to that of the former section. More details can be found in [Hof08] Section 2, in particular in Construction 2.10.

Let $U$ be the initial $T$-reflection group and let $W$ be the Weyl group. The abelianizations $U^\text{ab}$ and $W^\text{ab}$ are $T^\text{ab}$-reflection groups, where $T^\text{ab}$ is the image of $T$ under the quotient homomorphism $G \to G_2 = G/2G$. This is a discrete symmetric space with the trivial multiplication. More precisely $U^\text{ab}$ is the initial $T^\text{ab}$-reflection group and $W^\text{ab} = (G_2 \wedge G_2) \times G_2 \times V$ is the Weyl group for the discrete symmetric space $T^\text{ab}$.

The $T$-reflection morphism $U \to W$ yields a $T^\text{ab}$-morphism $U^\text{ab} \to W^\text{ab}$ and there is a group homomorphism $\psi$ making the following diagram commute:

$$
\begin{array}{ccc}
\ker(U \to W) & \longrightarrow & U \\
\downarrow \psi & & \downarrow \psi \\
\ker(U^\text{ab} \to W^\text{ab}) & \longrightarrow & U^\text{ab} \\
\end{array}
$$

According to [Hof08] Theorem 4.16 the map $\psi$ is an isomorphism. With Theorem 4.4 we have obtained the main result of this article:

**Theorem 5.1** The $T$-reflection homomorphism $U \to W$ is an isomorphism if and only if $T^\text{ab} \setminus \{0\}$ is 2-independent in $G/2G$.

Corollary 4.5 gives more information in some specific cases. In particular, it confirms the observation made in [Hof07] and [AS08] that $U \to W$ is not always injective. If $n$ is the rank of $G$ then testing for 2-dependence involves testing for linear dependence of $|T \setminus \{0\}|$ vectors in an $n(n+1)/2$-dimensional vector space over the Galois field $\mathbb{F}_2$. This is more practical than testing for the existence of a so-called non-trivial integral collection according to [AS08] Theorem 5.16. This theorem also requires $G$ to be finitely generated, a hypothesis that we don’t require for our Theorem 5.1.
The hypotheses “free” for $G$ is only used to apply Theorem 4.16 of [Hof08]. We would be interested in understanding if it could be weakened to “torsion free”, “involution free” or even omitted completely.

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