Gravity on the octonion algebra

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Abstract
Gravitational interactions are treated as non-associative part of a Lagrangian of octonion fields. The Lagrangian is defined in the charge space as squared curvature with respect to the octonion fields. The applications of suggested formalism to homogeneous and isotropic space are studied.

Introduction
Generalizations of physical theories to octonions were studied in various aspects (see [1] for a review). However the main problem is that we have to speak about physical interpretations of new algebraic models having in mind possible inadequacy of the physical model to its mathematical counterpart, which is primarily the effect of non-associativity of the algebras in question [1]. Suitable matrix representations of non-associative algebras, suggested in [2, 3] was a reasonable step forward, but the price for this advancement was a peculiar multiplication law (12). As a consequence, in Feynman’s interpretation of the motion in external field, the state of a moving quantum particle, even in the absence of charges, is subject to change caused by the non-associative character of the interactions. It is reasonable to associate these interactions with the gravitational field. A holistic picture of gravitation field arises along these lines, which is the main object of this paper.

1 Non-associativity and curvature tensor
The free Dirac equations for spinor fields \( \Psi(x) \) and \( \overline{\Psi}(x) \), considered in Minkowskian space \( M_4 \) with the metric
\[
 ds^2 = \eta_{ab} dx^a dx^b = dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - dl^2 \tag{1}
\]
have the form\(^1\) \[4\]
\[
 \left( i \gamma^a \overleftarrow{\partial}_a - m \right) \Psi(x) = 0, \quad \overline{\Psi}(x) \left( i \overrightarrow{\partial}_a \gamma^a + m \right) = 0, \tag{2}
\]

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1from now on the summation over repeated indices \( a, b, c, \ldots = 0, 1, 2, 3 \) and \( \bar{a}, \bar{b}, \bar{c} = 1, 2, 3 \) is assumed and the indices are lowered and raised in \( M_4 \) by the tensors \( \eta_{ab} \) and \( \eta^{ab} \), respectively
where
\[
\overrightarrow{\partial}_a \Psi(x) = \partial \Psi/\partial x^a = \Psi_a,
\]
\[
\overrightarrow{\Psi}(x) = \Psi(x)^+ \gamma^0, \quad x \in M_4,
\]
with \(\gamma^a\) \((a = 0, 1, 2, 3)\) standing for Dirac matrices:
\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & i \sigma^a \\ -i \sigma^a & 0 \end{pmatrix}, \quad (3)
\]
and \(\sigma^a\) \((\bar{a} = 1, 2, 3)\) for Pauli matrices:
\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)
\]

The electromagnetic field \(A_a(x)\) is introduced as minimal connection in the covariant derivative \(\nabla_a = \partial_a - ieA_a(x)\) in the equations of motion. Under these conditions the Dirac equations read:
\[
\left( \gamma^a (\overrightarrow{\partial}_a - ieA_a) - m \right) \Psi(x) = 0, \quad \overrightarrow{\Psi}(x) \left( (\overrightarrow{\partial}_a + ieA_a)i\gamma^a + m \right) = 0. \quad (5)
\]

Let \(\gamma(s)\) be a contour in spacetime defined by the equations \(x_a = x_a(s)\). The vector field \(\dot{\gamma}(s)\) with the components \(X_a = dx_a/ds\) is tangent to the curve \(\gamma(s)\). A field \(\Psi(x)\) is said to be parallel transported along the curve \(\gamma(s)\), if at each its point
\[
\nabla_a \Psi(x) X^a|_{\gamma(s)} = 0.
\]

The variation of the field \(\Delta^{12}\Psi(x)\) after parallel transport along an infinitely small contour \(\gamma = \gamma(s)\), having parallelogram form \((x, x + \Delta_1 x, x + \Delta_1 x + \Delta_2 x, x + \Delta_1 x, x)\) is
\[
\Delta_{12}\Psi(x) = F_{ab}\Psi(x)\Delta S_{12}^{ab}, \quad (6)
\]
where \(F_{ab} = \partial_b A_a - \partial_a A_b\) stands for the curvature in the charge space and \(\Delta S_{12}^{ab} = \Delta_1 x^a \Delta_2 x^b - \Delta_1 x^b \Delta_2 x^a\) is the area of the parallelogram outlined by the contour.

The field \(iA_a(x)\) is a complex function. The duplication of the algebra of complex numbers leads to a new algebra. The most interesting duplication is the algebra of quaternions whose duplication yields, in particular, the algebra of octonions. In this case the field \(A_a(x)\) can be decomposed over the generators \(\Sigma^\bar{a}, \bar{a} = 1, 2, \ldots, 7\) : \(A_a(x) = A^\bar{a}_a(x)\Sigma^\bar{a}\) having the following properties
\[
\Sigma^\bar{a}\Sigma^\bar{b} = \delta^\bar{a}\bar{b} + i\epsilon^{\bar{a}\bar{b}\bar{c}}\Sigma^\bar{c}, \quad (7)
\]
where \(\epsilon^{\bar{a}\bar{b}\bar{c}}\) is completely antisymmetric symbol, whose only non-vanishing components are
\[
\epsilon^{123} = \epsilon^{145} = \epsilon^{176} = \epsilon^{246} = \epsilon^{257} = \epsilon^{347} = \epsilon^{365} = 1, \quad (8)
\]
while the values $\Sigma^\alpha$ form a non-commutative \(^7\) and non-associative \(^9\) algebra, for which:
\[
\{\Sigma^\alpha, \Sigma^\beta, \Sigma^\gamma\} = (\Sigma^\alpha \Sigma^\beta)\Sigma^\gamma - \Sigma^\alpha(\Sigma^\beta \Sigma^\gamma) = 2\varepsilon^{\alpha\beta\gamma\delta}\Sigma^\delta,
\]
(9)
taking into account that only the following coefficients of the completely anti-symmetric symbol $\varepsilon^{\alpha\beta\gamma\delta}$ do not vanish:
\[
\varepsilon^{1247} = \varepsilon^{1265} = \varepsilon^{2345} = \varepsilon^{2376} = \varepsilon^{3146} = \varepsilon^{3157} = \varepsilon^{4567} = 1.
\]
(10)

Then the variation of the field $\Delta_{12}\Psi(x)$ under a parallel transport along the infinitely small contour $\gamma = \gamma(s)$ also equals (6), but
\[
F_{ab} = \partial_b A_a - \partial_a A_b + i[A_a, A_b].
\]

Since the full matrix algebra is associative, the values $\Sigma^\alpha$ cannot be represented by matrices using conventional matrix multiplication. Although, following \(^2\), we may represent them by $4 \times 4$ matrices
\[
\Sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma^\alpha = \begin{pmatrix} 0 & -i\sigma^\alpha \\ i\sigma^\alpha & 0 \end{pmatrix},
\]
\[
\Sigma^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma^{4+\alpha} = \begin{pmatrix} 0 & -\sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix},
\]
(11)
with the following multiplication law \(^3\):
\[
\begin{pmatrix} \lambda & A \\ B & \xi \end{pmatrix} \ast \begin{pmatrix} \lambda' & A' \\ B' & \xi' \end{pmatrix} = \begin{pmatrix} \lambda\lambda' + \frac{1}{2}\text{tr}(AB') & \lambda A' + \xi' A + \frac{i}{2}[B, B'] \\ \lambda' B + \xi B' - \frac{1}{2}[A, A'] & \xi\xi' + \frac{1}{2}\text{tr}(BA') \end{pmatrix},
\]
(12)
where $A, A', B, B'$ are real $2 \times 2$ matrices and $\lambda, \lambda', \xi, \xi'$ are scalar real $2 \times 2$ matrices.

Introduce the field Lagrangian $A_a(x)$ in the Minkowskian space as the following scalar:
\[
L = -\frac{1}{16\pi e^2} F_{ab} F^{ab}.
\]
(13)

In the Riemannian space $\Omega_4$ with the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, a vector $A_\mu(x)$ being parallel transported along the contour $\gamma(s)$ an infinitely small distance $dx$ is subject to the variation \(^7\): $\delta A_\mu = \Gamma^\lambda_{\mu\nu} A_\lambda$, therefore the covariant derivative of the vector $A_\mu$ in the Riemannian space $\Omega_4$ has the form
\[
\nabla_\nu A_\mu = A_{\mu,\nu} = \partial_\nu A_\mu + \Gamma^\lambda_{\mu\nu} A_\lambda.
\]
When the vector $A_\mu$ is parallel transported along the closed contour $\gamma(s)$ in $\Omega_4$, it changes according to the following formula:
\[
\Delta A_\mu = R_{\mu\nu\lambda}^\tau A_\tau \Delta S^{\sigma\lambda},
\]
(14)
where, as above, $\Delta S^{\nu\lambda}$ is an element of the surface outlined by the contour and $R_{\mu\nu\lambda}^\tau$ is Riemann tensor:
\[
R_{\mu\nu\lambda}^\tau = \Gamma_{\mu\lambda,\nu}^{\tau} - \Gamma_{\mu\nu,\lambda}^{\tau} + \Gamma_{\nu\sigma}^{\tau} \Gamma_{\mu\lambda}^{\sigma} - \Gamma_{\sigma}^{\tau} \Gamma_{\mu\nu}^{\sigma},
\]
(15)
The curvature tensor can also be defined in tetrad representation. For that, introduce orthogonal basis tetrads $e^a_\mu(x)$ at each point of $\Omega$ defined by the conditions

$e^a_\mu(x)e^\nu_\alpha(x) = g_{\mu\nu}(x), \ e^a_\mu(x)e^b_\nu(x) = \delta_{ab}(x)$

and Ricci rotation coefficients $\gamma_{abc} = e^a_\mu e^b_\nu e^c_\lambda$. Then the tetrad representation gives the curvature tensor the following form:

$R_{abcd} = \gamma_{abc,d} - \gamma_{abd,c} + \gamma_{abf}(\gamma_{fcd} - \gamma_{fcd}) + \gamma_{afc}\gamma_{fbd} - \gamma_{afd}\gamma_{fbc}$. (16)

When a material point causally connected with certain body moves in Minkowskian space, an octonion field $A^a(x)$ gives rise to an interaction (like in classical electrodynamics), which changes the initial direction of its motion. In order to evaluate this change, exact calculations based on octonion algebra should be carried out. Due to non-associativity of octonion algebra, the direction of the motion will change as if there is an “internal curvature” in Minkowski space which we interpret as Riemannian curvature. In other words, we assume the gravitation to be induced by the non-associative part (denote it $\tilde{\text{tr}}$) as follows:

$\frac{1}{672\pi e^2} \tilde{\text{tr}}(F_{ab}F_{cd}) = \frac{1}{\kappa} R_{abcd}$, (17)

where $R_{abcd}$ is the curvature tensor of the Riemannian space in tetrad representation. Note that the equation (17) is Lorentz-covariant. It can be verified directly that Bianchi identity does not hold for the introduced curvature tensor. Therefore, in the geometrical interpretation of curvature, the Bianchi identity should be treated as additional dynamical equations when finding representations of the solution (17).

Let us find the non-associative part $\tilde{\text{tr}}(F_{ab}F_{cd})$ in case of classical fields. Since the associator is calculated on at least three fields, we exclude the terms proportional to the square of $A^a$. Since the matrices $\Sigma^{\tilde{a}}$ have zero trace, the associator of three matrices $\Sigma^{\tilde{a}}$ vanishes, as (17) can contain at least four matrices $\Sigma^{\tilde{a}}$. In the meantime

$\tilde{\text{tr}}\{\Sigma^{\tilde{a}}, \Sigma^{\tilde{b}}, \Sigma^{\tilde{c}}, \Sigma^{\tilde{d}}\} = \text{tr}\{\{\Sigma^{\tilde{a}}, \Sigma^{\tilde{b}}, \Sigma^{\tilde{c}}\}\Sigma^{\tilde{d}} - \Sigma^{\tilde{a}}\{\Sigma^{\tilde{b}}, \Sigma^{\tilde{c}}, \Sigma^{\tilde{d}}\}\} = 8\epsilon^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}}$. (18)

Therefore,

$\frac{1}{\kappa} R_{abcd} = -\frac{1}{168\pi e^2} \epsilon^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \left( A^\tilde{a}_a A^b_b - A^\tilde{a}_a A^b_b \right) \left( A^\tilde{c}_d A^\tilde{d}_a - A^\tilde{c}_d A^\tilde{d}_a \right)$. (19)

2 Uniform isotropic space

Consider a uniform isotropic space with the Friedmann metric

$ds^2 = a^2(\eta)(d\eta^2 - dl^2)$ (20)

for which the only non-vanishing components of the curvature tensor are

$R_{0\alpha 0\beta} = \frac{a'^2}{a^3} \eta_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = \frac{a'^2}{a^3} (\eta_{\beta\gamma} \eta_{\alpha\delta} - \eta_{\alpha\gamma} \eta_{\beta\delta})$. (21)
Substitute (21) into (19) and denote $\theta^4 = \kappa/16\pi\varepsilon^2$, then

$$
\frac{a'^2 - a''a}{a^4\theta^4}\eta_{\alpha\beta} = -\varepsilon\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}
\left(A^{(a)}_0 A^{(b)}_0 - A^{(a)}_0 A^{(b)}_0\right)
\left(A^{(c)}_0 A^{(d)}_0 - A^{(c)}_0 A^{(d)}_0\right) =
$$

$$
= \left(A^{(1)}_0 A^{(2)}_0 - A^{(1)}_0 A^{(2)}_0\right)
\left(A^{(4)}_0 A^{(7)}_0 - A^{(4)}_0 A^{(7)}_0\right) +
$$

$$
+ \left(A^{(4)}_0 A^{(7)}_0 - A^{(4)}_0 A^{(7)}_0\right)
\left(A^{(1)}_0 A^{(2)}_0 - A^{(1)}_0 A^{(2)}_0\right) + \ldots
$$

$$
(22)
$$

$$
\frac{a'^2}{a^4\theta^4}(\eta_{\alpha\beta} - \eta_{\alpha\gamma}\eta_{\beta\delta}) = -\varepsilon\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}
\left(A^{(a)}_0 A^{(b)}_0 - A^{(a)}_0 A^{(b)}_0\right)
\left(A^{(c)}_0 A^{(d)}_0 - A^{(c)}_0 A^{(d)}_0\right) =
$$

$$
= \left(A^{(1)}_0 A^{(2)}_0 - A^{(1)}_0 A^{(2)}_0\right)
\left(A^{(4)}_0 A^{(7)}_0 - A^{(4)}_0 A^{(7)}_0\right) +
$$

$$
+ \left(A^{(4)}_0 A^{(7)}_0 - A^{(4)}_0 A^{(7)}_0\right)
\left(A^{(1)}_0 A^{(2)}_0 - A^{(1)}_0 A^{(2)}_0\right) + \ldots,
$$

$$
(23)
$$

where the dots stand for summation over all indices (10). Denote

$$
x^2 = (a'^2 - a''a)/a^4\theta^4,
$$

$$
y^4 = a'^2/a^4\theta^4
$$

and

$$
A^{(a)}_0 = \tilde{A}^{(a)}_0 y, 
A^{(a)}_0 = \tilde{A}^{(a)}_0 x/y,
$$

then we obtain the system

$$
\begin{align*}
\varepsilon\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}
\left(A^{(a)}_0 A^{(b)}_0 - A^{(a)}_0 A^{(b)}_0\right)
\left(A^{(c)}_0 A^{(d)}_0 - A^{(c)}_0 A^{(d)}_0\right) &= -\eta_{\alpha\beta} \\
\varepsilon\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}
\left(A^{(a)}_0 A^{(b)}_0 - A^{(a)}_0 A^{(b)}_0\right)
\left(A^{(c)}_0 A^{(d)}_0 - A^{(c)}_0 A^{(d)}_0\right) &= -\eta_{\beta\gamma}\eta_{\alpha\delta} + \eta_{\alpha\gamma}\eta_{\beta\delta},
\end{align*}
$$

whose solutions $A^{(a)}_0$ have the form:

$$
A^{(a)}_0 = \frac{1}{\tilde{a}\theta} \sqrt{a'^2 - a''a/a^\prime} \cdot \text{Const}_{0},
A^{(a)}_0 = \frac{1}{\tilde{a}\theta} \sqrt{a'} \cdot \text{Const}_{0}.
$$

For the state of the matter $p = \varepsilon/3$ when $a(\eta) = C \cdot \eta$, we get the solutions:

$$
A^{(a)}_0 = \text{Const}_{0}/\eta.
$$

In case of dust-like matter $p = 0$ when $a(\eta) = C \cdot \eta^2$ we obtain:

$$
A^{(a)}_0 = \text{Const}_{0} \cdot \eta^{-3/2}.
$$
3 Concluding remarks

We have assumed that in classical case matter fields induce the field of octonions, which, in turn, interacts with the matter in such a way that in order to correctly describe the motion in terms of associative algebra a Riemannian metric needs to be introduced. As a result, from a formal perspective, there is no need to recalculate the known effects of General Relativity which might be different in the suggested model. However, the form of the equation (19) may restrict the possible geometry of physical space.

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