ITERATION OF STRONGLY $\kappa^+$-CC FORCING POSETS

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1. Introduction

One of the basic results in iterated forcing states that a finite support iteration of ccc forcing is ccc. It is natural to look for extensions of this result: the most natural setting for generalisations is to let $\kappa$ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$, and consider $\kappa$-support iterations in which each iterand is $\kappa$-closed and $\kappa^+$-cc. It is known that (even for the case where $\kappa = \aleph_1$ and CH holds) such iterations do not in general have $\kappa^+$-cc \cite{5}, so we will need to strengthen the closure and chain condition hypotheses on the iterands.

Shelah \cite{4} proved that if we strengthen the chain condition assumption a lot and the closure assumption a little then we get a useful iteration theorem. More precisely, let $\kappa = \kappa^{<\kappa}$ and say that a poset $P$ is *regressively $\kappa^+$-cc* if it enjoys the following property: for every sequence $(p_i)_{i<\kappa^+}$ of conditions in $P$ there exist a club set $E \subseteq \kappa^+$ and a regressive function $f$ on $E \cap \text{Cof}(\kappa)$ such that $f(\alpha) = f(\beta)$ implies $p_\alpha$ is compatible with $p_\beta$. This looks technical, but can be motivated by the observation that if $P$ was proved to be $\kappa^+$-cc by the standard $\Delta$-system and amalgamation arguments then the proof very likely shows that $P$ is regressively $\kappa^+$-cc. Shelah’s iteration theorem states that a $\kappa$-support iteration with $\kappa$-closed, well met, and regressively $\kappa^+$-cc iterands is regressively $\kappa^+$-cc. Here a poset is *well met* if any pair of compatible conditions has a greatest lower bound (glb): Shelah \cite{4} showed that in general this technical condition can not be removed.

We will prove an iteration theorem where the chain condition hypothesis is strengthened in a different direction. Motivation for this work includes some results by Mekler \cite{2} where the ccc is proved using elementary submodels, and the more recent surge of interest (initiated by Mitchell’s work on $I_{[\omega_2]}$ \cite{3}) in the notion of strong properness.

In Section 2 we give some background on forcing posets, elementary submodels and generic conditions. Section 3 contains the statement and proof of our main theorem. Finally Section 4 discusses some generalisations.

2. Background

For the rest of this paper we fix an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$. We make the convention that when we write “$N \prec H_\theta$” we mean “$N \prec (H_\theta, \in ,<_\theta)$” where $<_\theta$ is a wellordering of $H_\theta$. The structure $(H_\theta, \in ,<_\theta)$ has definable Skolem functions, so that if $N,N' \prec H_\theta$ then $N \cap N' \prec H_\theta$. When $N \prec H_\theta$ we write $\bar{N}$ for the transitive collapse of $N$, $\rho_N : N \simeq \bar{N}$ for the transitive collapsing map, and $\pi_N : \bar{N} \simeq N$ for its inverse.

**Definition 1.** Let $Q$ be a forcing poset and let $M \prec H_\theta$. A model $M$ is *$\kappa$-good for $Q$* if and only if $\kappa, Q \in M$, $|M| = \kappa$ and $^{<\kappa}M \subseteq M$. 

Remark 1. If $Q \in H_\theta$, then the set of $M$ which are $\kappa$-good for $Q$ is stationary in $P_{\kappa^+}H_\theta$.

When $M$ is $\kappa$-good for $Q$ and $G$ is $Q$-generic over $V$, we will study the subset $G \cap M$ of $Q \cap M$. In a mild abuse of notation we sometimes write $\hat{G}$ for the subset $\rho_M[G \cap M]$ of the poset $Q$. We write $M[G]$ for the set of elements of form $\hat{\tau}$ where $\tau$ is a $Q$-name in $M$.

Definition 2. Let $M$ be $\kappa$-good for $Q$. Then:

1. A condition $q \in Q$ is $(M, Q)$-generic iff $q$ forces that $\hat{G}$ is $Q$-generic over $M$, and strongly $(M, Q)$-generic iff it forces that $\hat{G}$ is $Q$-generic over $V$.
2. If $q \in Q$ and $r \in Q \cap M$, then $r$ is a strong properness residue of $q$ (for $M$) iff for every $s \in Q \cap M$ with $s \leq r$, $q$ is compatible with $s$. We write spr to abbreviate strong properness residue.

Assume that $M$ is $\kappa$-good for $Q$. The following facts are standard:

- If $G$ is $Q$-generic over $V$, then $M[G] \prec H_\theta[G] = H_\theta[V[G]]$. If in addition $Q$ is $\kappa$-closed then $V[G] = H_\theta[V[G]]$. If in addition $Q$ is $\kappa$-closed then $V[G] = H_\theta[V[G]]$. If in addition $Q$ is $\kappa$-closed then $V[G] = H_\theta[V[G]]$.
- A condition $q$ is $(M, Q)$-generic iff $q$ forces that $M[\hat{G}] \cap V = M$. In this case $q$ forces that $\pi_M$ can be lifted to an elementary embedding $\pi_M : M[\hat{G}] \rightarrow H_\theta[G]$.
- A condition $q$ is strongly $(M, Q)$-generic iff the set of conditions in $Q$ which have a spr for $M$ is dense below $q$.
- The poset $Q$ is $\kappa^+$-cc iff every condition in $Q$ is $(M, Q)$-generic.

Definition 3. A forcing poset $Q$ is strongly $\kappa^+$-cc if and only if for all large $\theta$, for every $M \prec H_\theta$ which is $\kappa$-good for $Q$, every condition in $Q$ is strongly $(M, Q)$-generic. Equivalently, densely many conditions have a spr for $M$, and this implies that in fact all conditions have a spr for $M$.

3. An iteration theorem

Theorem 1. Let $\kappa$ be uncountable with $\kappa^{<\kappa} = \kappa$. Let $P$ be an iteration with $< \kappa$-supports such that each iterand $Q_\alpha$ is forced at stage $\alpha$ to have the following properties:

1. $Q_\alpha$ is strongly $\kappa^+$-cc.
2. $Q_\alpha$ is well met.
3. Every directed subset of $Q_\alpha$ of size less than $\kappa$ has a glb.

Then $P$ is strongly $\kappa^+$-cc.

Depending on the exact way one defines “directed” in condition [3], condition [3] may be read to subsume condition [2].

Before proving the theorem, we digress briefly to illustrate the difficulties and motivate the main idea. Consider the case of an iteration $P_2 = Q_0 * Q_1$ of length two, where $Q_0$ is strongly $\kappa^+$-cc and forces that $Q_1$ is strongly $\kappa^+$-cc. Let $M$ be $\kappa$-good for $P_2$, and let $(q_0, q_1)$ be an arbitrary condition for which we aim to construct a spr. If $r_0$ is a spr for $q_0$ and $M$, while $\hat{r}_1$ names a spr for $q_1$ and $M[\hat{G}_0]$, then we are not warranted in claiming that $(r_0, \hat{r}_1)$ is a spr for $(q_0, q_1)$. The issue is that while $\hat{r}_1$ names something which is the denotation of a term in $M$, there is no reason to think $\hat{r}_1$ itself is in $M$. In this simple case we can cope by first extending $q_0$ to some $q_0'$, which determines the identity of some term $r_1'$ which denotes a spr for $\hat{q}_1$, ...
and then choosing \( r_0^* \) which is a spr for \( q_0^* \): this clearly becomes problematic for an iteration of infinite length. We will deal with this kind of problem by building a spr on every relevant coordinate simultaneously. This is similar to the approach taken by [1], but without a need for side conditions.

**Remark 2.** It is easy to see that condition \( 3 \) is preserved by iteration with \( < \kappa \)-supports, so that \( \mathbb{P} \) satisfies it. To be explicit, if \( D \) is a directed subset of \( \mathbb{P} \) with \( |D| < \kappa \) then we construct a glb \( p \) for \( D \) inductively. We build \( p \) so that \( \text{supp}(p) = \bigcup_{t \in D} \text{supp}(t) \): at stage \( i \) we have that \( p \upharpoonright i \) is a glb for \( \{ t \upharpoonright i : t \in D \} \), observe that \( p \upharpoonright i \) forces \( \{ t(i) : t \in D \} \) to be directed, and choose \( p(i) \) to name a glb for this set.

**Proof of Main Claim:** We let \( s \) be a glb \( \leq \kappa \), \( q \) be arbitrary. By Remark 2, \( \text{supp}(q) = \bigcup_n \text{supp}(p_n) \) and \( q \upharpoonright \alpha \) forces that \( q(\alpha) \) is the glb of the sequence \( (p_n(\alpha)) \).

We may choose \( p_{i+1} \) because (using Remark 2) \( \mathbb{P} \) is \( \kappa \)-closed. At the end we set \( H = \bigcup_n H_n \). By Remark 2, the sequence \( (p_n) \) has a glb \( q \).

We record some information:

1. By construction \( H < H_\theta \), \( |H| < \kappa \) and \( p, M \in H \).
2. By Remark 2, \( \text{supp}(q) = \bigcup_n \text{supp}(p_n) \) and \( q \upharpoonright \alpha \) forces that \( q(\alpha) \) is the glb of the sequence \( (p_n(\alpha)) \).
3. If \( g = \{ x \in \mathbb{P} \cap H : \exists i \leq \kappa \\} \), then \( g \) is a filter on \( \mathbb{P} \cap H \) which meets every dense open set in \( H \).
4. By definition, \( q \) is the glb of \( g \). We claim that \( g = \{ x \in \mathbb{P} \cap H : q \leq x \} \). Clearly if \( x \in g \) then \( q \leq x \), and if \( x \notin g \) then by genericity there is \( n \) such that \( p_n \perp x \) and so \( q \not\leq x \).
5. We claim that the support of \( q \) is \( H \cap \gamma \). By construction \( \text{supp}(p_n) \subseteq H_n \cap \gamma \) for all \( n \), and so \( \text{supp}(q) \subseteq H \cap \gamma \); conversely if \( \alpha \in H \cap \gamma \) then by genericity there is \( n \) such that \( \alpha \in \text{supp}(p_n) \).

The set \( g \cap M \) is a directed subset of \( \mathbb{P} \) and \( |g \cap M| \leq |H| < \kappa \), so \( g \cap M \) has an glb \( r \). Since \( \kappa M \subseteq M \), \( g \cap M \in M \) and so by elementarity \( r \in M \).

**Main Claim:** \( r \) is a spr for the condition \( q \) and the model \( M \).

**Proof of Main Claim:** We let \( s \leq r \) with \( s \in M \) and build inductively a condition \( q^* \) such that \( q^* \) is a common refinement of \( s \) and \( q \). The induction is easy except at coordinates \( \alpha \in \text{supp}(s) \cap \text{supp}(q) \), so fix such an \( \alpha \). The support of \( s \) is contained in \( M \), and the support of \( q \) is contained in \( H \), so \( \alpha \in H \cap M \cap \gamma \). Note that \( s \leq r \) and by induction \( q^* \upharpoonright \alpha \leq s \upharpoonright \alpha \), so that \( q^* \upharpoonright \alpha \models s(\alpha) \leq r(\alpha) \).

For each \( i < \omega \), define a set \( D_i \subseteq \mathbb{P} \) as follows: \( D_i \) is the set of \( t \in \mathbb{P} \) such that either \( t \perp p_i \), or \( t \leq p_i \) and there is \( \dot{r} \in M \) such that \( t \upharpoonright \alpha \) forces \( t(\alpha) \leq \dot{r} \), and
is a spr for \( p_i(\alpha) \) and \( M[G_\alpha]^\omega \). Since \( \alpha, p_i, M \in H \) we have by elementarity that \( D_i \in H \).

We claim that \( D_i \) is dense. Let \( t_0 \in \mathbb{P} \) be arbitrary. If \( t_0 \) is incompatible with \( p_i \) then \( t_0 \in D_i \), otherwise we find \( t_1 \leq t_0, p_i \). Extending \( t_1 \) to \( \alpha \) if necessary, we may assume that \( t_1 \upharpoonright \alpha \) determines some \( \dot{r} \in M \) which denotes a spr for \( t_1(\alpha) \); now \( t_1 \upharpoonright \alpha \) forces that \( \dot{r} \) and \( t_1(\alpha) \) are compatible so extending \( t_1 \) at coordinate \( \alpha \) we obtain a condition \( t_2 \leq t_1 \) such that \( t_2 \upharpoonright \alpha \) forces \( t_2(\alpha) \leq \dot{r} \). Since \( t_2 \leq t_1 \leq p_i \) we have \( t_2 \upharpoonright \alpha \vdash t_1(\alpha) \leq p_i(\alpha) \), so \( t_2 \upharpoonright \alpha \) forces that \( \dot{r} \) is a spr for \( p_i(\alpha) \).

By the construction of the sequence \( (p_i) \), we find \( j \) such that \( p_j \in D_i \). From the definitions \( p_j \leq p_i \) (that is \( j \geq i \)), and \( p_j \upharpoonright \alpha \) forces \( p_j(\alpha) \leq \dot{r} \) and \( \dot{r} \) is a spr for \( p_j(\alpha) \) for some \( \dot{r} \in M \). As \( p_j, p_i, \alpha, M \in H \) we may assume by elementarity that \( \dot{r} \in M \cap H \). Now if we let \( r^* \) be the condition in \( \mathbb{P} \) that has \( \dot{r} \) at coordinate \( \alpha \) and is otherwise trivial, \( p_j \leq r^* \in M \cap H \) so that \( r^* \in g \cap M \).

So \( r \leq r^* \), and since \( q^* \upharpoonright \alpha \leq s \upharpoonright \alpha \leq r \upharpoonright \alpha \) we have \( q^* \upharpoonright \alpha \vdash r(\alpha) \leq r^*(\alpha) = \dot{r} \).

Since also \( q^* \upharpoonright \alpha \leq p_j \upharpoonright \alpha \), \( q^* \upharpoonright \alpha \) forces that \( \dot{r} \) is a spr for \( p_j(\alpha) \). Since \( q^* \upharpoonright \alpha \vdash s(\alpha) \leq r(\alpha) \leq \dot{r} \), \( q^* \upharpoonright \alpha \) forces that \( s(\alpha) \) is compatible with \( p_j(\alpha) \).

Now we force below \( q^* \upharpoonright \alpha \) to obtain a generic object \( G_\alpha \), and work in \( V[G_\alpha] \) to compute a lower bound for the decreasing sequence \( (s(\alpha) \upharpoonright p_j(\alpha)) \). Let \( q^*(\alpha) \) name a lower bound, then \( q^* \upharpoonright \alpha \) forces that \( q^*(\alpha) \) is a lower bound for the sequence \( (p_j(\alpha)) \), and (since \( q^* \upharpoonright \alpha \leq g \upharpoonright \alpha \)) also that \( q(\alpha) \) is the glb for the sequence \( (p_j(\alpha)) \). Hence \( q^* \upharpoonright \alpha \) forces that \( q^*(\alpha) \leq q(\alpha) \). Hence \( q^* \upharpoonright \alpha \vdash q^*(\alpha) \leq q(\alpha), s(\alpha) \) as required.

\[ \square \]

4. Further results

With more work we can weaken the closure hypotheses on the iterands as follows: it is enough to assume that each iterand \( \mathbb{Q}_\alpha \) is forced to be \( \kappa \)-strategically closed, to be countably closed, and to satisfy the strengthened form of countable strategic closure in which move \( \omega \) is required to be a glb for the moves played at finite stages.

The iteration theorem can also be generalised in other directions. For example let \( S \subseteq \kappa^+ \cap \text{Cof}(\kappa) \) be stationary, and define a poset to be \( S \)-strongly \( \kappa^+ \)-cc if sprs exist for \( \kappa \)-good models \( M \) with \( M \cap \kappa^+ \in S \). Then \( S \)-strongly \( \kappa^+ \)-cc forcing posets preserve the stationarity of \( S \), and an iteration of \( S \)-strongly \( \kappa^+ \)-cc posets with appropriate closure properties is \( S \)-strongly \( \kappa^+ \)-cc. To prove the generalisation to \( S \)-strongly \( \kappa^+ \)-cc posets, simply restrict throughout to \( M \) such that \( M \cap \kappa^+ \in S \).

We briefly sketch the proof of the generalisation weakening the closure hypothesis on the iterands.

We can construct \( p_i \) and \( q \) as in the proof of Theorem [1] from the weaker hypotheses. \( p_{i+1} \) can be constructed using \( \kappa \)-strategic closure. If \( \sigma \) is a strategy for player II to produce descending chains of length \( \omega \) with a glb, and taking \( \sigma \in H_0 \), one can use the fact that \( p_{i+1} \) meets all dense open sets in \( H \) to find a play \( (u_n)_{n<\omega} \) by \( \sigma \) so that \( p_{i+1} \leq u_{2i+1} \leq u_{2i} \leq p_i \). This ensures that \( (p_i)_{i<\omega} \) has a glb.

The final argument in the proof of Theorem [1] obtaining a lower bound for the sequence \( (s(\alpha) \upharpoonright p_i(\alpha)) \), goes through with countable closure.

The only other use of closure in the proof is in defining \( r \), a glb for \( g \cap M \). We prove that this can be done with the weakened assumptions.

The support of \( r \) is \( M \cap H \cap \gamma \). We work by induction on \( \alpha \in M \cap H \cap \gamma \) to define \( r(\alpha) \), assuming that \( r \upharpoonright \alpha \) has been defined and is a glb for \( (g \cap M) \upharpoonright \alpha \). Passing to the transitive collapse \( H \) of \( H \), we have that \( \bar{g} = \rho_H[\bar{g}] \) is generic over
\(\bar{H}\) for \(\rho_H(\mathbb{P})\). So \(\bar{g} \upharpoonright \alpha\) is generic for \(\rho_H(\mathbb{P} \upharpoonright \alpha)\) over \(\bar{H}\), and \(\bar{g}(\alpha)\) is generic for \(\bar{Q}_\alpha = \rho_H(\bar{Q}_\alpha[\bar{g} \upharpoonright \alpha])\) over \(H[\bar{g} \upharpoonright \alpha]\).

By the strong chain condition, \(\bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \upharpoonright \alpha]\) is generic over \(\rho_H(M)[\bar{g} \upharpoonright \alpha]\).

Using the strategic closure of the \(\alpha^{th}\) iterand it follows that for each \(i\), there is a lower bound \(\dot{w}_i \in \bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \upharpoonright \alpha]\) for \(\bar{g}(\alpha) \cap \rho_H(H_i)[\bar{g} \upharpoonright \alpha]\). Let \(\tau_\alpha \in H \cap M\) be a strategy for player II to produce descending chains of length \(\omega\) with a glb in \(\bar{Q}_\alpha\). Using the genericity of \(\bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \upharpoonright \alpha]\) one can pick \(\dot{w}_i\) to be part of a play by \(\rho_H(\bar{\tau}_\alpha)[\bar{g} \upharpoonright \alpha]\). Let \(\dot{w}_i\) name \(w_i\). Note that by genericity the fact that the conditions \(\dot{w}_i\) are part of a play by \(\rho_H(\bar{\tau}_\alpha)\) is forced by conditions in \(\bar{g}(\alpha) \cap \rho_H(H_i)[\bar{g} \upharpoonright \alpha]\). Then \(r \upharpoonright \alpha\), being a lower bound for \((g \cap M) \upharpoonright \alpha\), forces that the conditions \(\pi_H(\dot{w}_i)\) are part of a play according to \(\bar{\tau}_\alpha\), and therefore \((\pi_H(\dot{w}_i))_{i<\omega}\) has a glb. Let \(\dot{r}(\alpha)\) name this glb. One can check that then \(r \upharpoonright \alpha + 1\) is a glb for \(g \upharpoonright \alpha + 1\).

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