On the condition of conversion of classical probability distribution families into quantum families

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Abstract

The purpose of the paper is to study the condition for a probability distribution family to a quantum state family. This is an (relatively) easy example of quantum version of "comparison of statistical experiments", which had turned out to supply deep insight into the foundation of classical and quantum statistics [11][13]. It turns out use of maximal quantum f-divergence is useful in characterizing the classical-quantum transformability.

1 Introduction

The purpose of the paper is to study the condition for a probability distribution family to a quantum state family. This is an (relatively) easy example of quantum version of "comparison of statistical experiments", which had turned out to supply deep insight into the foundation of classical and quantum statistics [11][13]. Consideration of such a problem nicely characterizes some known statistically important quantities, giving their new operational meaning and proving some of their properties in smart way. For example, RLD Fisher information, which is known to be the achievable lower bound to the minimum mean square error of the Gaussian shift model [5] and ‘coherent’ pure state models [3][7], is characterized as the smallest classical Fisher information to simulate the quantum statistical model locally. Also, a version of quantum relative entropy, first studied by [2], turned out to be the smallest classical entropy to generate two point quantum state family [8]. Due to this characterization, this version of quantum relative entropy had turned out to be the largest monotone relative entropy which coincide with its classical counter part in commutative case [8].

Below, we describe our setting precisely. In the paper, the dimension of Hilbert space $\mathcal{H}$ is finite. $L(\mathcal{H})$ is the set of all linear transforms on the Hilbert
space $\mathcal{H}$. A trace preserving completely positive map from a finite dimensional commutating matrices (which is interpreted as a real function over a finite set) to $\mathcal{L}(\mathcal{H})$ is called classical-to-quantum (CQ) map. We consider parameterized family of probability distributions and quantum states, where the parameter space is binary set, $\Theta := \{0, 1\}$. Hereafter, a probability density function $p$ over the finite set $\mathcal{X}$ is always identified with the finite dimensional matrix

$$\sum_{x \in \mathcal{X}} p(x) |e_x\rangle \langle e_x|,$$

where $\{|e_x\rangle\}_{x \in \mathcal{X}}$ is an orthonormal set of vectors. Our problem is to investigate the conditions for the existence of a CPTP map $\Gamma$ with

$$\Gamma (p_\theta) = \sigma_\theta, \quad \forall \theta \in \Theta,$$

(1)

where $\{p_\theta\}_{\theta \in \Theta}$ and $\{\sigma_\theta\}_{\theta \in \Theta}$ are given family of probability density functions and density operators, respectively.

## 2 $f$-Divergence

When $\{\sigma_\theta\}_{\theta \in \Theta}$ is also commutative, i.e., the condition for classical-to-classical conversion is well-studied, and the necessary and sufficient condition is characterized by $f$-divergence; Given a convex function $f$ on $[0, \infty)$, $f$-divergence between probability distributions $p_0$ and $p_1$ is

$$D_f(p_0||p_1) := \sum_{x \in \text{supp} p_1} p_1(x) f\left(\frac{p_0(x)}{p_1(x)}\right) + \left(\sum_{x \notin \text{supp} p_1} p_0(x)\right) \lim_{\lambda \to \infty} f\left(\frac{\lambda}{\lambda}\right).$$

Lemma 1 (([12]), ([13])) There is a transition probability matrix $P$ with

$$PP_\theta = q_\theta, \quad \forall \theta \in \Theta$$

exists if and only if

$$D_f(p_0||p_1) \geq D_f(q_0||q_1)$$

holds for any proper and closed convex function $f$ on $[0, \infty)$.

Motivated by the above Lemma, we study the relation between the condition ([1]) and a quantum version of $f$-divergence. Among many quantum versions of $f$-divergence, we use the following one which is defined using classical-quantum conversion problem:

$$D_f^{\max}(\sigma_0||\sigma_1) := \min \{D_f(p_0||p_1) : \{p_\theta\}_{\theta \in \Theta}, \Gamma: \text{CPTP with } [1]\}.$$  

This quantity can be written more or less explicitly, if $f$ is an operator convex function on $[0, \infty)$:

$$D_f^{\max}(\sigma_0||\sigma_1) = \text{tr} \sigma_1 f\left(\sigma_1^{-1/2} \sigma_0 \sigma_1^{-1/2}\right) + (1 - \text{tr} \bar{\sigma}_0) \lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda}$$

(2)
where, with $\pi_X$ denoting the projector onto $\text{supp} \ X$,

\[
\begin{align*}
\sigma_{0,11} & : = \pi_{\sigma_0} \sigma_0 \pi_{\sigma_1}, \\
\sigma_{0,12} & : = \pi_{\sigma_1} \sigma_0 (1 - \pi_{\sigma_1}), \\
\sigma_{0,21} & : = (1 - \pi_{\sigma_1}) \sigma_0 \pi_{\sigma_1}, \\
\sigma_{0,22} & : = (1 - \pi_{\sigma_1}) \sigma_0 (1 - \pi_{\sigma_1}), \\
\tilde{\sigma}_0 & = \sigma_{0,11} - \sigma_{0,12} (\sigma_{0,22})^{-1} \sigma_{0,21}.
\end{align*}
\]

Observe, if $\sigma_0$ is invertible,

\[
\tilde{\sigma}_0 = (\pi_{\sigma_1} \sigma_0^{-1} \pi_{\sigma_1})^{-1}.
\]

The following property of $D_f^{\max}$ will turn out to be useful.

**Lemma 2** \cite{10} (i) If a real valued two-point function $D_f^Q (\cdot | \cdot)$ of operators is monotone decreasing by application of CPTP maps,

\[
D_f^Q (\Lambda (\sigma_0) \mid \Lambda (\sigma_1)) \leq D_f^Q (\sigma_0 \mid \sigma_1)
\]

and coincide with $D_f (\cdot | \cdot)$ on commutative subalgebra, then

\[
D_f^Q (\sigma_0 \mid \sigma_1) \leq D_f^{\max} (\sigma_0 \mid \sigma_1).
\]

(ii) There is a pair $\left( \{q_0\}_{\theta \in \Theta}, \Gamma \right)$ which satisfies

\[
D_f (q_0 | q_1) = \text{tr} \sigma_1 f \left( \sigma_1^{-1/2} \tilde{\sigma}_0 \sigma_1^{-1/2} \right) + (1 - \text{tr} \tilde{\sigma}_0) \lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda}
\]

for all convex functions on $[0, \infty)$ at the same time.

In the paper we sometimes use the following family of operator convex functions

\[
f^s (\lambda) : = -\lambda^s, \ (0 < s \leq 1).
\]

If $s < 1$, $\lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda} = 0$. Thus,

\[
\begin{align*}
\lim_{s \uparrow 1} D_{f^s} (p_0 | p_1) & = - \lim_{s \uparrow 1} \sum_{x \in \text{supp} p_1} (p_1 (x))^{1-s} (p_0 (x))^s \\
& = - \sum_{x \in \text{supp} p_1} p_0 (x), \\
\lim_{s \uparrow 1} D_{f^s}^{\max} (\sigma_0 | \sigma_1) & = - \lim_{s \uparrow 1} \text{tr} \sigma_1 \left( \sigma_1^{-1/2} \tilde{\sigma}_0 \sigma_1^{-1/2} \right)^s \\
& = - \text{tr} \tilde{\sigma}_0.
\end{align*}
\]

Meantime,

\[
D_{f^s} (p_0 | p_1) = D_{f^s}^{\max} (\sigma_0 | \sigma_1) = -1.
\]

3
3 A necessary and sufficient condition

To obtain the necessary and sufficient condition for the existence of a CPTP map \( \Gamma \) with (1), we use the quantum randomization criterion\[6\][9]. Let \( \mathcal{H}_D \) be a Hilbert space, and \( \mathfrak{B} = \{ W_\theta \}_{\theta \in \Theta} \) be a pair of (bounded) operators on \( \mathcal{H}_D \).

We define

\[
D_{\mathfrak{B}} (\sigma_0 \| \sigma_1) := \max \left\{ \sum_{\theta \in \Theta} \text{tr} \ W_\theta \Lambda (\sigma_\theta) \ ; \ \Lambda: \text{CPTP from } \mathcal{L} (\mathcal{H}) \text{ to } \mathcal{L} (\mathcal{H}_D) \right\}.
\]

Then a CPTP map \( \Gamma \) with (1) exists if and only if

\[
D_{\mathfrak{B}} (p_0 \| p_1) \geq D_{\mathfrak{B}} (\sigma_0 \| \sigma_1)
\]

holds for all \( \mathfrak{B} \) and all (finite dimensional) \( \mathcal{H}_D \).

Observe

\[
D_{\mathfrak{B}} (p_0 \| p_1) = \max_{\Lambda: \text{CPTP}} \sum_{\theta \in \Theta} \sum_{x \in \mathcal{X}} p_\theta (x) \text{tr} \ W_\theta \Lambda (|e_x \rangle \langle e_x|),
\]

\[
= \max_{\rho_x: \text{a state on } \mathcal{H}_D} \sum_{\theta \in \Theta} \sum_{x} p_\theta (x) \text{tr} \ W_\theta \rho_x,
\]

\[
= \sum_{x \in \mathcal{X} \rho: \text{a state on } \mathcal{H}_D} \max_{\theta \in \Theta} \text{tr} \sum_{\theta} p_\theta (x) W_\theta,
\]

\[
= \sum_{x \in \mathcal{X}} r_{\max} \left( \sum_{\theta} p_\theta (x) W_\theta \right)
\]

\[
= \sum_{x \in \text{supp } p_1} p_1 (x) f \left( \frac{p_0 (x)}{p_1 (x)} \right) + \left( \sum_{x \notin \text{supp } p_1} p_0 (x) \right) \max_{\rho: \text{a state on } \mathcal{H}_D} \text{tr} \ W_0 \rho.
\]

where \( r_{\max} (X) \) is the largest eigenvalue of \( X \), and

\[
f (\lambda) := r_{\max} (\lambda W_0 + W_1).
\]

Since

\[
\lim_{\lambda \to \infty} \frac{f (\lambda)}{\lambda} = \lim_{\lambda \to \infty} r_{\max} \left( W_0 + \frac{1}{\lambda} W_1 \right) = r_{\max} (W_0),
\]

we have

\[
D_{\mathfrak{B}} (p_0 \| p_1) = D_f (p_0 \| p_1).
\]

Note, there are many \( D_{\mathfrak{B}} (\cdot \| \cdot) \) whose restriction equals \( D_f (\cdot \| \cdot) \). Since \( D_{\mathfrak{B}} (\cdot \| \cdot) \) is monotone decreasing by application of CPTP maps almost by definition, they are all bounded from above by \( D_f^{\max} (\cdot \| \cdot) \) :

\[
D_f^{\max} (\sigma_0 \| \sigma_1) \geq D_{\mathfrak{B}} (\sigma_0 \| \sigma_1).
\]

holds. Therefore, if

\[
D_f (p_0 \| p_1) \geq D_f^{\max} (\sigma_0 \| \sigma_1) \quad (4)
\]
holds for any closed proper convex function \( f \) on \([0, \infty)\), a CPTP map with (1) exists. Since \( D_{\text{max}}^f (\cdot || \cdot) \) is monotone decreasing by application of CPTP maps, this condition is obviously necessary. Thus:

**Theorem 3** A CPTP map \( \Gamma \) with (1) exists if and only if (4) holds for any closed proper convex function \( f \) on \([0, \infty)\).

To our regret, no closed formula of \( D_{\text{max}}^f (\cdot || \cdot) \) had been found out unless \( f \) is operator convex. Indeed, we have the following negative implication (The proof is done later):

**Proposition 4** If (2) holds for any positive operators \( \rho \) and \( \sigma \) which not necessarily with unit trace, then \( f \) has to be operator convex.

### 4 Sufficient conditions

Lemma 2 implies an upper bound to \( D_{\text{max}}^f (\sigma_0 ||\sigma_1) \). Therefore, we have the following sufficient condition.

**Corollary 5** If
\[
D_f (p_0 || p_1) \leq \text{tr} \sigma_1 f \left( \sigma_1^{-1/2} \tilde{\sigma}_0 \sigma_1^{-1/2} \right) + (1 - \text{tr} \tilde{\sigma}_0) \lim_{\lambda \to \infty} \frac{f (\lambda)}{\lambda}
\]
holds for any closed proper convex function \( f \) on \([0, \infty)\), (1) holds.

There is a sufficient condition which can be described only using operator convex functions, where the formula (2) applies.

**Lemma 6** (Lemma 5.2 of [4]) If \( f \) is a complex valued function on finitely many points \( \{ \lambda_i ; i \in I \} \subset [0, \infty) \), then for any pairwise different positive numbers \( \{ t_i ; i \in I \} \) there exist complex numbers \( \{ c_i ; i \in I \} \) such that
\[
f (\lambda_i) = \sum_{j \in I} c_j \lambda_i + t_j i \in I.
\]

**Theorem 7** If
\[
D_f (p_0 || p_1) = D_{\text{max}}^f (\sigma_0 || \sigma_1)
\]
for any operator convex function \( f \) on \([0, \infty)\), a CPTP map \( \Gamma \) with (1) exists. In fact, one only has to check identity for \((t + \lambda)^{-1} - \lambda^s\), where \( t \geq 0 \) and \( s \in (s_0, 1) \). Here \( s_0 \) is an arbitrary positive number smaller than 1.

**Proof.** Let
\[
\{ \lambda_i ; i \in I \} := \left\{ \frac{p_0 (x)}{p_1 (x)} ; x \in \text{supp} \; p_1 \right\} \cup \text{spec} \left\{ \sigma_1^{-1/2} \tilde{\sigma}_0 \sigma_1^{-1/2} \right\}.
\]
and apply Lemma 6. Suppose (3) holds for \((t + \lambda)^{-1} \), \( \forall t \geq 0 \). Then
\[
\sum_{x \in \text{supp} \; p_1} p_1 (x) f \left( \frac{p_0 (x)}{p_1 (x)} \right) = \text{tr} \sigma_1 f \left( \sigma_1^{-1/2} \tilde{\sigma}_0 \sigma_1^{-1/2} \right)
\]
holds for any convex function $f$ on $[0, \infty)$. Suppose (5) holds for $f^s(\sigma) = -\lambda^s$, $\forall s \in (s_0, 1)$. Then considering $s \uparrow 1$, by (3),

$$\sum_{x \in \text{supp} \sigma} p_0(x) = \text{tr} \tilde{\sigma}_0.$$ 

Therefore, for any $f$,

$$\left( \sum_{x \notin \text{supp} \sigma} p_0(x) \right) \lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda} = (1 - \text{tr} \tilde{\sigma}_0) \lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda}.$$

Summing up, we have (5) for any convex function $f$ on $[0, \infty)$. Then application of Theorem 3 leads to the assertion. 

5 A necessary and sufficient condition for special case

There is a case where we can give "tractable" necessary and sufficient condition. An example is the case where $\sigma_1$ is a pure state (the dimension of the Hilbert space $\mathcal{H}$ is arbitrary finite integer) Then

$$\tilde{\sigma}_0 = \sigma_1^{-1/2} \tilde{\sigma}_0 \sigma_1^{-1/2} = \gamma \sigma_1,$$

where

$$\gamma := \sigma_{0,11} - \sigma_{0,12} (\sigma_{0,22})^{-1} \sigma_{0,21}.$$ 

Therefore, by (2),

$$D_f^{\max} (\sigma_0 || \sigma_1) = f(\gamma) + (1 - \gamma) \lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda}.$$ 

Suppose (1) holds. With $\delta_{x_0}$ being a delta distribution concentrated on $x_0$, we should have

$$\Gamma (\delta_x) = \sigma_1, \forall x \in \text{supp} \sigma_1,$$

since $\sigma_1$ is rank-1 projector. Therefore,

$$\sigma_0 = \Gamma (p_0) = \sum_{x \in \text{supp} \sigma_1} p_0(x) \sigma_1 + \sum_{x \notin \text{supp} \sigma_1} p_0(x) \Gamma (\delta_x).$$ 

For this to hold for some choice of $\Gamma (\delta_x)$ ($x \notin \text{supp} \sigma_1$), it is necessary and sufficient that

$$\sigma_0 - \sum_{x \in \text{supp} \sigma_1} p_0(x) \sigma_1 \geq 0$$

holds. (Necessity is trivial. On the other hand, if this inequality holds, we only have to define

$$\Gamma (\delta_x) := \frac{1}{\sum_{x \notin \text{supp} \sigma_1} p_0(x)} \left( \sigma_0 - \sum_{x \in \text{supp} \sigma_1} p_0(x) \sigma_1 \right), \ x \notin \text{supp} \sigma_1.$$ 

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A necessary and sufficient condition of this is
\[ \sum_{x \in \text{supp } p_1} p_0 (x) \leq \gamma. \] (6)

On the other hand, consider \( f^s (\lambda) := -\lambda^s \) \((0 < s < 1)\), which is operator convex. Suppose
\[ \sum_{x \in \text{supp } p_1} p_1 (x) f^s \left( \frac{p_0 (x)}{p_1 (x)} \right) \geq f^s (\gamma) \]
holds for all \(0 < s < 1\). Then letting \( s \uparrow 1 \), we have
\[ -\sum_{x \in \text{supp } p_1} p_0 (x) \geq -\gamma, \]
which is (6), or equivalently, (1). The result above is summarized as follows.

**Proposition 8** When \( \sigma_1 \) is a pure state, then a CPTP map \( \Gamma \) with (1) exists if and only if (4) with \( f (\lambda) = -\lambda^s \) for all \( s \in (s_0, 1) \). Here \( s_0 \) is an arbitrary positive number smaller than 1.

## 6 Operator convex functions are not enough

A bad news is that (4) for all operator convex functions is not enough to show the existence of a CPTP map \( \Gamma \) with (1). A counter example is constructed by letting both \( \{ p_\theta \} \in \Theta \) and \( \{ \sigma_\theta \} \in \Theta \) be probability distributions on 3-points set \( \{1, 2, 3\} \). In addition we suppose \( p_1 \) and \( \sigma_1 \) are uniform distributions, and parameterize \( p_0 \) and \( \sigma_0 \) by
\[ \sigma_0 = (a, b, c), \quad p_0 := (a_0, b_0, c_0), \]
where \( c = 1 - a - b \), \( c_0 = 1 - a_0 - b_0 \) and
\[ a_0 < b_0 < c_0. \]

Since the uniform distribution is a fixed point, the stochastic map which sends \( p_\theta \) to \( q_\theta \) is doubly stochastic, or equivalently, a convex combination of permutations. Therefore, \( \sigma_0 \) has to be in the convex hull of six points, \((a_0, b_0, c_0)\), \((a_0, c_0, b_0)\), \((b_0, a_0, c_0)\), and so on.

**Lemma 9** (Theorem 8.1 if [4]) A continuous real valued function \( f \) on \([0, \infty)\) is operator convex if and only if
\[ f (\lambda) = f (0) + \alpha \lambda + \beta \lambda^2 + \int_{(0, \infty)} \left( \frac{\lambda}{1 + \lambda t} - \frac{\lambda}{\lambda + t} \right) d\mu (t), \]
where $\alpha$ is a real number, $\beta$ is a non-negative real number, and $\mu$ is a finite non-negative measure satisfying

$$\int_{(0, \infty)} \frac{d\mu(t)}{(1 + t)^2} < \infty.$$  

By Lemma 9, instead of all operator convex functions, we only have to check \[4\] for $\lambda^2 \frac{1}{\lambda + t}$. Let

$$g_t(a, b) := \frac{1}{a + t} + \frac{1}{b + t} + \frac{1}{1 - a - b + t} - \left\{ \frac{1}{a_0 + t} + \frac{1}{b_0 + t} + \frac{1}{c_0 + t} \right\}, \quad (t \geq 0)$$

$$g_{-1}(a, b) := a^2 + b^2 + (1 - a - b)^2 - a_0^2 - b_0^2 - c_0^2.$$  

So our purpose is to prove that the set

$$C_2 := \bigcap_{t: t \geq 0, t = -1} \{(a, b) : g_t(a, b) \leq 0\},$$

is not identical to the projection $C_1$ of the convex hull of the six points to $(a, b)$-plane. Note the set $C_2$ is convex, and contains the six points. Hence, our task is to find a point of the set $C_2$ which is not in $C_1$. Observe that the vertices of $C_1$ are

$(a_0, c_0), (b_0, c_0), (c_0, b_0), (c_0, a_0), (b_0, a_0), (a_0, b_0),$  

the maximum of $b$-coordinate of $C_1$ is $c_0$, and the edge connecting $(a_0, c_0)$ and $(b_0, c_0)$ forms the "upper bound" of $C_1$. Hence, we only have to show that there is a point in $C_2$ whose $b$-coordinate is strictly larger than $c_0$.

Observe also the line $b = 1 - 2a$ \[7\] intersects with the edge connecting $(a_0, c_0)$ and $(b_0, c_0)$ at $(a_0 + b_0, c_0)$. Thus this line intersects $g_t(a, b) = 0$ in the region above $b = c_0$. Denote the intersection point $(a_t, b_t)$. Then since the line segment connecting $(a_t, b_t)$ and $(a_0 + b_0, c_0)$ is in the set $\{(a, b) : g_t(a, b) \leq 0\}$, $C_2$ contains the line segment connecting $(a_*, b_*)$, where

$$b_* := \inf_{t: t \geq 0, t = -1} b_t.$$  

So we only have to show $b_* > c_0$.

Solving $g_t\left(\frac{1}{2}(1 - b_t), b_t\right) = 0$,

$$b_t = \frac{1}{2(3t^2 - t + e_t)} \left\{ \pm \sqrt{(24e_t - 8) t^4 + 8e_t t^3 + (9e_t^2 - 6e_t + 1) t^2 + (6e_t^2 - 2e_t) t + e_t^2} \right\},$$

where $e_t$ is defined by the identity

$$\frac{1}{t} \left( 3 - \frac{1}{t} + \frac{e_t}{t^2} \right) = \frac{1}{a_0 + t} + \frac{1}{b_0 + t} + \frac{1}{c_0 + t}.$$
Here, let \( t \to \infty \). Then \( e_t \to a_0^2 + b_0^2 + c_0^2 \) and

\[
b_t \to \frac{1}{3} \left( 1 \pm \sqrt{6 (a_0^2 + b_0^2 + c_0^2) - 2} \right).
\]

Elementally calculations show that the larger solution of the two is strictly larger than \( c_0 \). Thus, there is \( t_0 > 0 \) such that

\[
\inf_{t > t_0} b_t > \frac{1}{2} \left( c_0 + \lim_{t \to \infty} b_t \right).
\]

Therefore,

\[
b_* > \min \left\{ \frac{1}{2} \left( c_0 + \lim_{t \to \infty} b_t \right) , \inf_{t \in [0,t_0] \cup \{-1\}} b_t \right\}.
\]

Since the function \( t \to b_t \) is continuous, there is \( t_* \) such that

\[
\inf_{t \in [0,t_0] \cup \{-1\} } b_t = b_{t_*}.
\]

Since \( g_{t_*} (a, b) = 0 \) is an algebraic convex curve and passes through the two points \((a_0, c_0)\) and \((b_0, c_0)\), it cannot coincide with the line connecting these two points. Thus, \( b_{t_*} > c_0 \). Therefore, we have \( b_* > c_0 \), and \( C_1 \neq C_2 \). Thus (4) for all operator convex functions is not enough for the existence of a CPTP map \( \Gamma \) with (1).

7 Proof of Proposition 4

Suppose \( \sigma_1 > 0 \) and a CPTP map \( \Gamma \) satisfies (1). Also let

\[
M_x : = p_1 (x) \sigma_1^{-1/2} \Gamma (|e_x \rangle \langle e_x|) \sigma_1^{-1/2},
\]

\[
d : = \sigma_1^{-1/2} \sigma_0 \sigma_1^{-1/2},
\]

\[
r (x) : = \frac{p_0 (x)}{p_1 (x)}.
\]

Then

\[
\sum_{x \in \mathcal{X}} M_x = \sum_x p_1 (x) \sigma_1^{-1/2} \Gamma (|e_x \rangle \langle e_x|) \sigma_1^{-1/2}
\]

\[
= \sigma_1^{-1/2} \sigma_1 \sigma_1^{-1/2} = \sigma_1,
\]

\[
\sum_{x \in \mathcal{X}} r (x) M_x = d,
\]

\[
\text{tr} \sigma_1 M_x = p_1 (x).
\]

Also,

\[
D_f (p_0 || p_1) = \sum_{x \in \mathcal{X}} p_1 (x) f (r (x))
\]

\[
= \text{tr} \sigma_1 \sum_{x \in \mathcal{X}} f (r (x)) M_x,
\]
and
\[ \text{tr} \sigma_1 f (d) = \text{tr} \sigma_1 f \left( \sum_{x \in \mathcal{X}} r(x) M_x \right), \]

Therefore,
\[
\begin{aligned}
\min \left\{ D_f (p_0 || p_1) ; \Gamma, \{ p_\theta \} \theta \in \Theta \text{ with } (\square) \right\} &= \min \left\{ \text{tr} \sigma_1 \sum_{x \in \mathcal{X}} f (r (x)) M_x ; \sum_{x \in \mathcal{X}} M_x = \pi_x, \text{ } d = \sum_{x \in \mathcal{X}} r (x) M_x \right\} \\
&= \min \left\{ \text{tr} \sigma_1 V^\dagger f (d') V ; V : \text{ isometry from } \mathcal{H}' \text{ to } \mathcal{H}, \text{ } d = V^\dagger d' V \right\}
\end{aligned}
\]

The second identity above is by Naimark extension theorem, which states there is a Hilbert space \( \mathcal{H}' \) which is larger in dimension than \( \mathcal{H} \), and the projectors \( \{ P_x \} \), the isometry \( V \) from \( \mathcal{H}' \) to \( \mathcal{H} \) such that
\[ M_x = V^\dagger P_x V. \]

Note that \( \mathcal{H}' \), \( d' \), \( V \) is not restricted only by \( d \), and not by \( \sigma_1 \). (Here, recall we had removed the restriction of trace of \( \sigma_0 \), so that \( d \) and \( \sigma_1 \) can move freely.)

For fixed \( (\mathcal{H}', d', V) \), if the inequality
\[ \text{tr} \sigma_1 f (d) = \text{tr} \sigma_1 f (V^\dagger d' V) \leq \text{tr} \sigma_1 V^\dagger f (d') V \]
holds true for any \( \sigma_1 > 0 \), we should have
\[ f (V^\dagger d' V) \leq V^\dagger f (d') V. \]

If this holds for any \( (\mathcal{H}', d', V) \), then \( f \) has to be operator convex (See Exercise V.2.2 of \[1\], for example).

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