A new approach to the correlation functions of $W$-algebras

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Abstract

We propose a new approach to the study of the correlation functions of $W$-algebras. The conformal blocks (chiral correlation functions), for fixed arguments, are defined to be those linear functionals on the product of the highest weight (h.w.) representation spaces which satisfy the Ward identities. First we investigate the dimension of the chiral correlation functions in the case when there is no singular vector in any of the representations. Then we pass to the analysis of the completely degenerate representations. A special subspace of the h.w. representation spaces, introduced by Nahm, plays an important role in the considerations. The structure of these subspaces shows a deep connection with the quantum and classical Toda models and relates certain completely degenerate representations of the $WG$ algebra to representations of $G$. This is confirmed by an analysis for the Virasoro, $WA_2$ and $WBC_2$ algebra. We also relate our work to Nahms, Feigen-Fuchs’ and Watts’ results.
1 Introduction

Two dimensional conformal field theories have wide range of applications. They describe the critical behaviour of two dimensional statistical physical models since at the second order phase transition point the theory is scale and consequently conformal invariant. The conformal invariance in string theory arises as a remnant of the reparametrization invariance of the worldsheet which guarantees that the physics does not depend on the coordinate choices. The success of conformal field theories and the reason of their mathematical applications are due to the fact that they possess infinite dimensional symmetry algebras, which help to solve the models. These algebras necessarily contain the conformal or Virasoro algebra, in most cases however, they are larger. The extensions of the conformal algebra are called $W$-algebras.

The analysis of the $W$-symmetric theories, theories whose symmetry algebra is a $W$-algebra, can be divided into two steps. The first concerns the investigation of the symmetry algebra itself. It contains the presentation of the algebra, which can be done by commutation relations or by localizing it as a subalgebra in a known algebra, etc. Then the representation theory of the symmetry algebra has to be developed concentrating on the irreducibility of the representations and the singular vectors. The next step is the solution of a model with a given $W$-symmetry. This means the determination of the correlation functions, which is usually done either by solving the differential equations that arises from the decoupling equations of the singular vectors, or by using some auxiliary field technics.

In most cases the analysis of $W$-symmetric theories was focused on the symmetry algebra and their representation theory on one hand [8], or concentrated on auxiliary constructions which work well in principle only for rational theories, on the other hand [7, 10]. In the string theoretical applications, however, nonrational and rational theories are equally important. In Calabi-Yau theories the theory and consequently the symmetry algebra depend on several parameters, which not necessarily take rational values.

In this paper we concentrate on the second step, i.e. we try to analyse the correlation functions of general $W$-symmetric models. For this we give a new definition for the conformal blocks, placing emphasis on the generality and calculability.

The definition of the chiral correlation functions or conformal blocks presented here is the adaptation what was used by Felder for Kac-Moody algebras in [1]. In principle the conformal blocks are defined to be those linear functionals on the product of the highest weight (h.w.) representation spaces that respect the Ward identities. The simple idea behind this can be understood in the following way: consider a correlation function of some descendant field in the Virasoro theory

\[
\langle \Phi_1(z_1) \ldots \mathcal{L}_n \Phi_i(z_i) \ldots \Phi_N(z_N) \rangle = \oint_{C_i} \frac{dz}{2\pi i} \langle \Phi_1(z_1) \ldots (z-z_i)^{n+1} L(z) \Phi_i(z_i) \ldots \Phi_N(z_N) \rangle, \ n < 0,
\]

(1)

where all insertion points, \( \{z_i\} \), are different and none is zero or infinite. Here and from now on \( C_i \) denotes a small integration contour around \( z_i \). Now we can make a usual contour deformation. Observe that \( n < 0 \) so there is no pole at infinity and at zero (this is also true for \( n < 2 \) due to the quasi primary nature of \( L(z) \)). Consequently we obtain the following
formula:

\[ \langle \Phi_1(z_1) \ldots L_n \Phi_i(z_i) \ldots \Phi_N(z_N) \rangle = - \sum_{j:j \neq i} \oint_{C_j} \frac{dz}{2\pi i} \langle \Phi_1(z_1) \ldots (z-z_i)^{n+1} L(z) \Phi_j(z_j) \ldots \Phi_N(z_N) \rangle. \]  

(2)

Expanding \((z-z_i)^{n+1}\) around \(z_j\) we get an infinite sum, each term of it being associated with an action of the Virasoro algebra. Since we are working with h.w. representations the sum is finite and we end up with a well-defined action. Denoting the sum by \(\tilde{L}_n\) we have:

\[ \langle \Phi_1(z_1) \ldots \tilde{L}_n \Phi_i(z_i) \ldots \Phi_N(z_N) \rangle = - \sum_{j:j \neq i} \langle \Phi_1(z_1) \ldots \tilde{L}_n \Phi_j(z_j) \ldots \Phi_N(z_N) \rangle. \]  

(3)

In the singular terms of the operator product expansion we can replace \(L_{-1} \Phi_j(z_j)\) by \(\partial_z \Phi_j(z_j)\) and \(L_0 \Phi_j(z_j)\) by \(h_j \Phi_j(z_j)\) and recover the well-known Ward identities. (This relation will be used to define the conformal blocks later).

Doing the same in the WA_2 theory we realize a problem which concerns the modes \(W_{-1}\) and \(W_{-2}\): consider a correlation function of some \(W\)-descendants in a model with WA_2 symmetry:

\[ \langle \Phi_1(z_1) \ldots \mathcal{W}_n \Phi_i(z_i) \ldots \Phi_N(z_N) \rangle = \oint_{C_n} \frac{dz}{2\pi i} \langle \Phi_1(z_1) \ldots (z-z_i)^{n+2} W(z) \Phi_i(z_i) \ldots \Phi_N(z_N) \rangle, \ n < 0. \]  

(4)

Next we do the same contour deformation manipulations as before and expand the \((z-z_i)^{n+2}\) term around \(z_j\). The sum gives rise to a well-defined operator on h.w. \(W\)-primary fields which is denoted by \(\mathcal{W}_n\) with which the result is:

\[ \langle \Phi_1(z_1) \ldots \mathcal{W}_n \Phi_i(z_i) \ldots \Phi_N(z_N) \rangle = - \sum_{j:j \neq i} \langle \Phi_1(z_1) \ldots \mathcal{W}_n \Phi_j(z_j) \ldots \Phi_N(z_N) \rangle. \]  

(5)

There is an important difference compared to the Virasoro theory. Namely in the singular terms of the OPE one can only replace \(\mathcal{W}_0 \Phi_j(z_j)\) by \(w_j \Phi_j(z_j)\), but in principle one cannot relate the descendant fields \(\mathcal{W}_{-1} \Phi_j(z_j)\) and \(\mathcal{W}_{-2} \Phi_j(z_j)\) to the primary field \(\Phi_j(z_j)\). In the Virasoro theory \(L_{-1}\) has a nice interpretation as the differentiation operator however this fails in the \(W\)-case.

What to do now?

We will give up in some sense the \(L_{-1} = \partial\) relation and reformulate the technics described above in our new framework. We will see that relevant algebraic questions like the dimension of the conformal blocks, or the vanishing of the two and three point functions (the possible couplings), can be analyzed without using the \(L_{-1} = \partial\) relation.

We start with the Virasoro symmetric models in Section 1. The aim of this section is to reexplore the well-known results of the conformal symmetric models in order to be getting acquainted with the new framework. In doing so we define the space of the conformal blocks to be the space of those linear functionals on the product of the h.w. representation spaces which satisfy the Ward identities \([\mathbb{F}].\) Then the dimension of the two point functions is investigated, and it is shown that it is one if the conformal dimensions coincide and zero otherwise. In the case of the three point functions we analyse various things. The dimension of the space of the three point functions is one for arbitrary representations. If one of the representations has a singular vector than fixing one from the remaining weights and supposing non vanishing three
point functions we can determine the possible values for the last weight. We also comment on the fusion defined by Feigin and Fuchs. In the case of the four point function we show that its dimension is infinite in general. Moreover if there is a singular vector in any of the representation at level \( n \) then the dimension of the space of the conformal blocks is not greater than \( n \). After considering the general \( n \)-point functions we define the \( z \)-dependence of the conformal blocks via the Friedan-Shenker connection. Geometrically the conformal blocks are horizontal sections of a holomorphic bundle. Although this approach naturally generalizes to higher genus surfaces we restrict our attention to the sphere. In this paper we focus on the algebraic and not on the geometric aspects and on the way it can be extended to general \( W \)-algebras.

In generalising we start with the simplest \( W \)-algebra, namely with theories having the \( WA_2 \) algebra, \([20]\), as the symmetry algebra in Section 2. We define the space of the conformal blocks in an analogous manner namely by means of the Ward identities. Then the dimension of the space of the correlation functions is investigated. It is shown that a nonvanishing two point function implies that the Virasoro weights are the same and the \( W \)-weights are opposite. In the case of the three point functions even the dimension of the space of the conformal blocks is infinite in the general case. In analysing the three point functions it turns out that a "special" subspace of the h.w. representation spaces plays a very important role. This space is a factor space spanned by those negative modes that annihilate the vacuum modulo those that do not.

In the case of a general \( W \)-algebra we give the definition of the conformal blocks and analyse the dimension of the space of the correlation functions. The \( z \)-dependence of the correlation function is defined similarly to the Virasoro case.

In any of the consideration the special subspace of the h.w. representation space plays a crucial role. If one associate for a h.w. representation of \( G \) a h.w. representation of \( WG \), then the dimension of the special subspace coincides with the dimension of the underlying representation. This connection can be made more explicit with the help of the singular vectors of the representation and by exploiting the relation with the classical Toda models. This is illustrated in section 4.

The relation between the classical and quantum representations is supported by the example of the \( B_2 \) and \( C_2 \) Toda models which are given in the appendix.

## 2 The Virasoro theory

Let \( \Sigma \) be the Riemann sphere with \( N \) distinct points \( p_1, \ldots, p_N \). For the point \( p_i \) we associate an irreducible h.w. representation of the Virasoro algebra, \( V_i \), and denote the product of the representations by \( V = V_1 \otimes V_2 \otimes \ldots \otimes V_n \). We choose coordinate functions on the two covering neighbourhoods on the sphere: \( z \) and \( w = 1/z \) and write \( z(p_i) = z_i \). Introduce the space of meromorphic vector fields on the sphere with poles at \( z_i \):

\[
W^2(\Sigma \setminus \{p_i\}) = \left\{ f(z) \text{ meromorphic} \mid f(z) \frac{d}{dz} \text{ holomorphic except } z_1, \ldots, z_N \right\}. \quad (6)
\]

(We note that \( \frac{d}{dz}, z \frac{d}{dz}, z^2 \frac{d}{dz} \) are holomorphic everywhere.) In the following we use the abbreviation \( W^2 = W^2(\Sigma \setminus \{p_i\}) \). For each function \( f \in W^2 \) one can associate an action of the
Virasoro algebra on \( V \) in the following way. Make a Laurent expansion of \( f(z) \) around \( z_i \) and for the \((z - z_i)^{k+1}\) term associate the action of \( L_k \) on \( V \) and consequently the action of \( 1 \otimes \ldots \otimes L_k \otimes \ldots \otimes 1 \) on \( V \), for which we write \( L_k^{(i)} \). Since the functions of \( \mathcal{W}^2 \) are spanned by the functions of the form \((z - z_i)^n, n < 3\), we give the action explicitly for these:

\[
(z - z_i)^{-n} \to L_{-n-1}^{(i)} + \sum_{l \neq i} \tilde{L}_{-n-1}^{(l)},
\]

where we have to make a difference depending on whether \( n > 0 \) or \( n \leq 0 \). In these cases

\[
\tilde{L}_{-n-1}^{(l)} = \left\{ \begin{array}{ll}
(-1)^n \sum_{k=0}^{\infty} z_i^{-n-k} (\frac{n+k-1}{n-1}) L_{k-1}^{(l)} & n > 0 \\
(-1)^n \sum_{k=0}^{\infty} z_i^{-n-k} (\frac{n}{k}) L_{k-1}^{(l)} & n \leq 0
\end{array} \right.
\]

Although we have an infinite sum in \( \tilde{L}_{-n-1}^{(l)} \) only finitely many terms contribute when it acts on h.w. representation spaces. We note that any mode that appears in \( \tilde{L}_{-n-1}^{(l)} \) annihilates the vacuum and its index is greater than \( n - 1 \) (if \( n < -1 \)).

The conformal blocks or chiral correlation functions assign a complex number to fixed insertion points \( \{ z_i \} \) and associated representations \( V_i \), in such a way, that the Ward identity (4) holds. Reformulating this we define the space of the conformal blocks associated to vacuum and its index is greater than \( n - 1 \) (if \( n < -1 \)).

We will vary the insertion points and consider the \( z \)-dependence later, now we focus on the dimension of the space of the conformal blocks.

We start with the two point function \( \langle \varphi_1(z_1) \varphi_2(z_2) \rangle \). Consider a conformal block \( u \) acting on a general element

\[
v = L_{-n_1} \ldots L_{-n_k} | \varphi_1 \rangle \otimes L_{-m_1} \ldots L_{-m_l} | \varphi_2 \rangle, \quad n_i \geq n_{i+1}, \quad m_i \geq m_{i+1}
\]

of the product of the h.w. representation spaces. (The h.w. representation spaces are graded, the eigenvalue of the operator \( L_0 \) is the level. This grading naturally extends to \( V \): the levels sum up). Now if \( v \) contains a mode \( L_{-n} \), \( n > 1 \) then without loss of generality we have

\[
v = (L_{-n_1}^{(1)} + \tilde{L}_{-m_1}^{(2)}) v' - \tilde{L}_{-m_1}^{(2)} v',
\]

where clearly \( v' = L_{-n_2} \ldots L_{-n_k} | \varphi_1 \rangle \otimes L_{-m_1} \ldots L_{-m_l} | \varphi_2 \rangle \). Since \( u \) annihilates the first term \( u(v) = u(v'') \), where \( v'' = -\tilde{L}_{-m_1}^{(2)} v' \). Note that the terms in \( v'' \) have level smaller than the level of \( v \). This means that the value of the conformal block acting on a general element can be expressed with its values acting on vectors with smaller level. Since we are working with h.w. representation spaces the levels are bounded from below, so applying this procedure from level to level we can eliminate all the modes, which are not \( L_{-1} \). Consequently we analyse the case of the vector \( v \) of the form \( v = (L_{-1})^n | \varphi_1 \rangle \otimes (L_{-1})^m | \varphi_2 \rangle \). Up to now we have not used the
constraints coming from the everywhere homomorphic (global) transformations, which are:

\[
\begin{align*}
u \left( (L_1^{(1)} + \tilde{L}_1^{(2)})v \right) &= u \left( (L_1^{(1)} + L_1^{(2)})v \right) = 0 \\
u \left( (L_0^{(1)} + \tilde{L}_0^{(2)})v \right) &= u \left( (L_0^{(1)} + L_0^{(2)} + z_{21} L_1^{(2)})v \right) = 0 \\
u \left( (L_1^{(1)} + \tilde{L}_1^{(2)})v \right) &= u \left( (L_1^{(1)} + L_1^{(2)} + 2z_{21} L_0^{(2)} + z_{21}^2 L_1^{(2)})v \right) = 0.
\end{align*}
\]  

Two of the equations above can be used to eliminate all the \(L_{-1}\)-s. Using an appropriate combination of all the three we have one more constraint, which reads on \(v = |\varphi_1\rangle \otimes |\varphi_2\rangle\) as:

\[
u \left( (L_1^{(1)} + L_1^{(2)} + z_{21} (-L_0^{(1)} + L_0^{(2)}))v \right) = z_{21}(L_0^{(2)} - L_0^{(1)})u(v) = 0.
\]  

This means however, that fixing one of the \(L_0\) eigenvalues, say \(L_0^{(1)}|\varphi_1\rangle = h_1\), we have only the \(L_0^{(2)}|\varphi_2\rangle = h_2 = h_1\) possibility for the other. Consequently the dimension of the space of the chiral two point function is one if the conformal weights coincide and zero otherwise. 

Now we can make different interpretations. We can fix say all the three \(L_0\) eigenvalues. In this case all the \(L_{-1}\)-s can be eliminated, and no other constraints remain. This shows that the dimension of the chiral three point function is one for all possible h.w. vectors. 

In order to interpret the results in another way we can express \(L_0^{(1)}\) as:

\[
z_{21} L_0^{(1)} = L_1^{(1)} + L_1^{(2)} + L_1^{(3)} + z_{21} L_0^{(2)} + (z_{31} + z_{32}) L_0^{(3)} + z_{31} z_{32} L_{-1}^{3}
\]  

Now we can fix the eigenvalue \(L_0^{(2)} = h_2\) and \(L_0^{(3)} = h_3\) and investigate the possible weights \(L_0^{(1)}\) that can couple two them. We see that there are infinitely many possible values for \(L_0^{(1)}\), the symmetry does not restrict the possible couplings. Of course this will change drastically if we have singular vectors in the third representation space. We come back later to investigate this.

Now consider the case of the \(N\) point functions with insertion points: \(z_1, z_2, \ldots z_N\). The analysis follow the same line we used for the two and three point functions. We show that the value of the conformal block, \(u\), on the general vector

\[
v = L_{-n_1} \ldots L_{-n_k} |\varphi_1\rangle \otimes \ldots \otimes L_{-m_1} \ldots L_{-m_l} |\varphi_N\rangle,
\]  

(where \(n_i \geq n_{i+1}\), \(m_i \geq m_{i+1}\)), can be re-expressed in terms of its values on vectors at lower level, if \(v\) contains operators \(L_{-n}, n > 1\). Suppose that at the position \(i\) we have a mode
\(L_{-j-1}, j > 0\), i.e. \(v = L_{-j-1}^{(i)}v'.\) In this case, using the defining relations of the conformal blocks, we get \(u(v) = u(v'')\) where \(v'' = - \sum_{t \neq i} \tilde{L}_{-j-1}^{(t)}v'.\) Each term in \(v''\) has smaller level than the level of \(v.\) Following this procedure we can eliminate all the modes inductively in the level, which are not \(L_{-1}.\) The induction becomes complete since at the lowest levels we have vectors containing only \(L_{-1}-\)s. This shows that it is enough to define the value of the conformal blocks on the vectors containing only \(L_{-1}.\) They are not all independent however, since we have the global Ward identities:

\[
u((L_{-1}^{(i)} + \sum_{t \neq i} \tilde{L}_{-1}^{(t)})v) = u((L_{-1}^{(i)} + \sum_{t \neq i} L_{-1}^{(t)})v) = 0
\]

\[
u((L_{0}^{(i)} + \sum_{t \neq i} \tilde{L}_{0}^{(t)})v) = u((L_{0}^{(i)} + \sum_{t \neq i} (L_{0}^{(t)} + z_{it}L_{-1}^{(t)}))v) = 0
\]

(17)

\[
u((L_{1}^{(i)} + \sum_{t \neq i} \tilde{L}_{1}^{(t)})v) = u((L_{1}^{(i)} + \sum_{t \neq i} (L_{1}^{(t)} + 2z_{it}L_{0}^{(t)} + z_{it}^{2}L_{-1}^{(t)}))v) = 0.
\]

By the help of these equations we can eliminate the \(L_{-1}\) modes in three arbitrary representations, however the others remain undetermined. This shows that the dimension of the space of the \(N\)-point functions is infinite in the general case. (Sometimes we omit to write out the conformal block, \(u,\) in relations like above).

Focusing on the four point case we have to define the values of the conformal blocks on the vectors: \(|\varphi_{1}\rangle \otimes |\varphi_{2}\rangle \otimes |\varphi_{3}\rangle \otimes (L_{-1})^{n} |\varphi_{4}\rangle.\) This space is infinite dimensional in the general case. If however the representation corresponding to \(\varphi_{4}\) is degenerate, then it contains a singular vector at some level, say \(n,\) of the form \((L_{-1})^{n} + \ldots |\varphi_{4}\rangle = 0.\) This shows that it is enough to define \(u_{i} = u ((L_{-1}^{(i)})^{i}v), i < n,\) all others can be re-expressed in terms of these, i.e. the dimension of the space of the conformal blocks is \(n\) in this case, which is the level of the singular vector.

Now consider the effect of the singular vectors in the case of the three point functions. In equation (15) \(L_{0}\) becomes an \(n \times n\) matrix since \((L_{-1}^{(3)})^{n}\) can be expressed in terms of \((L_{-1}^{(3)})^{k}, k < n.\) Fixing \(L_{0}^{(2)} = h_{2}\) we can diagonalize \(L_{0}^{(1)}\) to obtain the possible nonvanishing couplings. Clearly the number of the nonzero couplings is bounded by the level of the singular vector.

We analyse the degenerate representations starting from the simplest cases. The simplest degenerate representation is the vacuum representation it contains a singular vector at level one of the form \(L_{-1} |\varphi_{3}\rangle = 0.\) Since \(L_{i}^{(3)} |\varphi_{3}\rangle = 0\) for \(i = -1, 0, 1\) then the analysis of the possible couplings reduces for the same analysis, what we have performed in the two point case, i.e. \(L_{0}^{(1)} = L_{0}^{(2)}.\)

The next simplest representation contains a singular vector at level two: \((L_{-1}^{2} - aL_{-2}) |\varphi_{3}\rangle = 0\) where \(a = \frac{4h_{1}^{2} + 2}{3} + \frac{4h_{2}^{2}}{3} + \frac{4h_{3}^{2}}{3}.\) The action of the generator \(L_{0}^{(1)}\) on the two dimensional space spanned by \(v = |\varphi_{1}\rangle \otimes |\varphi_{2}\rangle \otimes |\varphi_{3}\rangle\) and \(L_{-1}^{(1)}v\) is given by

\[
L_{0}^{(1)}v = z_{21}^{-1} \left\{ (z_{21}h_{2} + (z_{31} + z_{32})h_{3})v + z_{31}z_{32}L_{-1}^{(3)}v \right\}
\]

\[
L_{0}^{(1)}L_{-1}^{(3)}v = z_{21}^{-1} \left\{ 2h_{3}v + (z_{21}h_{2} + (z_{31} + z_{32}(h_{3} + 1))L_{-1}^{(3)}v + z_{31}z_{32}(L_{-1}^{(3)})^{2}v \right\}
\]

(18)
Now we can replace \((L^{-3}_-) v\) with \(a L^{-3}_- v\) and use
\[
L^{-3}_- = z^{-1}_2 L^{-2}_- + z^{-2}_2 L^{-3}_0 + \ldots + z^{-1}_1 L^{-1}_- + z^{-2}_1 L^{-1}_0 + \ldots,
\]
which is a consequence of the definition of the conformal blocks. All the relations are valid when they act on conformal blocks on the right. Using the global transformations we can express \(L^{-2}_-\) and \(L^{-1}_-\) in terms of the zero modes and \(L^{(3)}_1\) and \(L^{(3)}_0\), the result is the following matrix:
\[
L^{(1)}_0 = z^{-1}_2 \begin{pmatrix}
2h_3 + a \left( h_2 (z^{-1}_2 z_3 - 2 z^{-1}_2 z_3 + z^{-1}_2 z_3 + z^{-1}_2 z_3) \\
(2h_3 + a) (-2 z^{-1}_2 z_3 + 2 z^{-1}_2 z_3 + 2 z^{-1}_2 z_3 + z^{-1}_2 z_3)
\end{pmatrix}
\]
This matrix looks a bit complicated, however it turns out that its eigenvalues are \(z\)-independent. This leads to take the \(z_1 \to \infty, z_2 \to 1\) and \(z_3 \to 0\) limit. In this case the matrix takes the following simple form:
\[
L^{(1)}_0 = \begin{pmatrix}
h_2 + h_3 & -ah_2 \\
-1 & h_2 + h_3 + 1 - a
\end{pmatrix}
\]
Now if we use the standard parametrisation, \(h(r, s) = ((rt - s)^2 - (s - t)^2)/4t\), then \(h_3 = h(2, 1)\) and for \(h_2 = h(r, s)\) the eigenvalues become \(h(r + 1, s)\) and \(h(r - 1, s)\). This is very similar to the fusion of the fundamental representation of \(sl_2\).

Since the eigenvalues are \(z\)-independent we can take the limit above at the beginning. The resulting Ward identities can be described as
\[
\begin{align*}
L^{(2)}_0 &= L^{(3)}_2 - L^{(3)}_1 \\
L^{(1)}_0 &= L^{(3)}_2 - 2L^{(3)}_1 + L^{(3)}_0 \\
0 &= L^{(3)}_{n} - 2L^{(3)}_{n+1} + L^{(3)}_{n+2} & n > 2
\end{align*}
\]
These relations hold when acting on conformal blocks on the right. These are the operators used by Feigin and Fuchs in [24]. They considered a given representation \(\varphi_3\) and analysed the possible values for \(L^{(1)}_0\) and \(L^{(2)}_0\). The restriction comes from the decoupling of the singular vectors of the third representation, which can be expressed as a polynomial in \(L^{(1)}_0\) and \(L^{(2)}_0\).

The \(z\)-dependence of the correlation functions can be described in the following way: The configuration space where the arguments of the correlation function take values are \(C_N = \{z_1, \ldots, z_N\} \in \Sigma_N, z_i \neq z_j\). Now take an open neighbourhood, \(U\), of a point in the configuration space and associate with it a holomorphic dual to \(V\) as
\[
V^*(U) = \{u, : U \to V^*, \text{ holomorphic} \mid u_{z_1,\ldots,z_N}(v) \text{ holomorphic} \forall v \in V\}.
\]
The meromorphic test-functions naturally extends to \(U\), i.e. they have poles only when their argument coincides with one of the \(z_i\)-s, \(\{z_1, \ldots, z_N\} \in U\). This space is denoted by \(W^2(U)\). The space of the conformal blocks associated to the open set \(U\) and the representation \(V\) can be defined as those elements of the holomorphic dual which satisfy the Ward identities:
\[
E(U) = \{u \in V^*(U) \mid u(xv) = 0 \forall v \in V, x \in W^2(U)\}.
\]
The space of the conformal blocks can be regarded as a holomorphic vector bundle over the moduli space of the punctured sphere. The last thing we have to demand is that the operator $L_{-1}$ to be the operator of the translation on the conformal blocks. This can be achieved by defining a flat connection on the space of the conformal blocks. We define the covariant differentiation, $\nabla = \sum_i \nabla z_i \otimes dz_i$, as

$$\nabla z_i u(v) = \partial z_i u(v) - u(L_{-1}^i v) \tag{25}$$

for each $v \in V$. From the fact that $\nabla z_i (ux) = (\nabla z_i u)x + u \partial z_i x$ for $x \in \mathcal{W}^2(U)$ it follows, that $\nabla$ maps the space $E(U)$ into $\Omega^1(U) \otimes E(U)$, i.e. it preserves the space of the conformal blocks. We note that the connection is flat:

$$[\nabla z_i, \nabla z_j] = 0. \tag{26}$$

Now the conformal blocks are defined to be those sections of $E(U)$ which satisfy the horizontality condition:

$$\nabla u = 0. \tag{27}$$

These equations in the Kac-Moody case are nothing but the famous Knizhnik-Zamolodchikov equations, if we represent the Virasoro algebra via the Sugawara construction.

3 The $W_{A_2}$ theory

Now let us try to apply the techincs introduced above for the simplest nontrivial $W$-theory, namely for a theory having the $W_{A_2}$ algebra as the symmetry algebra. We define the space of the conformal blocks in analogy with the Virasoro case. Since the symmetry algebra contains the Virasoro algebra as a subalgebra the previous considerations can be adapted for this case. The only new constraint we have to take into account comes from the Ward identities of the $W(z)$ current, \cite{Ward}.

More precisely, consider the Riemann sphere, $\Sigma$, coordinatized by $z$ and $w = 1/z$ and marked with $N$ fixed points, $\{z_i\}_{i=1}^N$. For each point we associate a h.w. representation of the $W_{A_2}$ algebra and take the tensor product $V = V_1 \otimes \ldots \otimes V_i \otimes \ldots \otimes V_N$. We have the space of functions $\mathcal{W}^2$ as before, however now we define an other space of functions $\mathcal{W}^3(\Sigma \setminus \{p_1, \ldots, p_n\})$ – they correspond to the $W$-transformations – to be the space of complex valued meromorphic functions, $f(z)$, for which $f(z)dz^{-2}$ is holomorphic except the points $p_1, \ldots, p_N$. (Clearly the globally defined $W$-transformations corresponds to the functions $1, z, z^2, z^3, z^4$).

Now we associate to each function, $f(z)$ of $\mathcal{W}^3(\Sigma \setminus \{p_1, \ldots, p_n\})$ an action of the $W(z)$ field on $V$ in the following way. Make a Taylor expansion for $f(z)$ around the points $z_i$ and for the $(z - z_i)^{k+2}$ term associate the action of $W_k$ on $V_i$ thus the action of $W_n^{(l)} = 1 \otimes \ldots \otimes W_n \otimes \ldots \otimes 1$ on $V$. This map reads explicitly as:

$$(z - z_i)^{-n} \rightarrow W_n^{(l)}_{-n-2} + \sum_{k \neq i} W_k^{(l)}_{-n-2}, \tag{28}$$

where we have to make a difference depending on whether $n > 0$ or $n \leq 0$:

$$W_n^{(l)}_{-n-2} = \begin{cases} (-1)^n \sum_{k=0}^n z_{il}^{-n-k} \binom{n+k-1}{n-1} W_k^{(l)}_{-2} & n > 0 \\ (-1)^n \sum_{k=0}^n z_{il}^{-n-k} \binom{n}{k} W_k^{(l)}_{-2} & n \leq 0 \end{cases} \tag{29}$$

9
We note that the generators with tilde all annihilate the vacuum.

Similarly to the Virasoro case we define the space of conformal blocks associated to the points \( \{ z_1, \ldots, z_N \} \) and representation \( V \) as the space of linear functionals on \( V \) which is annihilated by \( W^2 \) and \( W^3 \), (\( \mathcal{W} \) for short):

\[
E(V, \{ p_i \}) = \{ u \in V^* \mid u(xv) = 0 \quad ; \forall v \in V, \ x \in \mathcal{W} \} , \tag{30}
\]

Before defining the \( z \)-dependence of the conformal blocks, we analyze the dimension of their space. In order to describe the h.w. representation spaces, spanned by the ordered monoms of the generators with negative indices, we introduce some subalgebras of the \( \mathcal{W} \) algebra. Define \( W_s \) to be the subalgebra spanned by those negative modes that annihilate the vacuum, ie. it is generated by \( W_s = \{ L_{-1}, W_{-1}, W_{-2} \} \). The other negative modes generate another subalgebra which is denoted by \( W_- \). Now a general element of the Verma module \( V \) can be written as

\[
\begin{align*}
\begin{array}{c}
\nu_i = W_i \varphi_i = W_i^{j_1} \ldots W_i^{j_k} W_i^{-m_1} \ldots W_i^{-m_l} \varphi_i \quad ; \quad W_i^{j} \in W_- , \ W_i^{-m} \in W_s \tag{31}
\end{array}
\end{align*}
\]

where in the subalgebras defined above we choose the following ordering \( j_i < j_{i+1} \) and if \( j_i = j_{i+1} \) then \( n_i \geq n_{i+1} \), (\( i \leftrightarrow j \), \( n \leftrightarrow m \)) and we introduced a compact notations for the generators: \( L = W^2 \), \( W = W^3 \). The h.w. representation spaces, as in the Virasoro case are graded by the eigenvalue of \( L_0 \).

Similarly to the Virasoro case it can be shown that the value of a conformal block on any vector is determined by its value on the vectors generated by the generators of \( W_s \). The proof is analogous to the Virasoro case: if one of the representations contains a mode from \( W_- \) then by the aid of the defining relation of the conformal blocks we can express the value of any conformal block in terms of its other values on lower level vectors only. We describe this procedure in more detail in the general \( N \)-point case.

Focusing on the two point functions the value of the conformal blocks has to be defined on the vectors

\[
v = (L_{-1})^{i_1}(W_{-2})^{i_2}(W_{-1})^{i_3} \varphi_1 \otimes (L_{-1})^{j_1}(W_{-2})^{j_2}(W_{-1})^{j_3} \varphi_2 \tag{32}
\]

Besides the constraints coming form the global conformal transformation, (\ref{13}), we have the constraints of the global \( W \)-transformations:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{align*}
&u \left( (W_{-1}^{(1)} + \tilde{W}_{-1}^{(2)})v \right) = u \left( (W_{-2}^{(1)} + W_{-2}^{(2)})v \right) = 0 \\
u \left( (W_{-1}^{(1)} + \tilde{W}_{-2}^{(2)})v \right) = u \left( (W_{-1}^{(1)} + W_{-1}^{(2)} + z_{21} W_{-2}^{(2)})v \right) = 0 \\
u \left( (W_{0}^{(1)} + \tilde{W}_{1}^{(2)})v \right) = u \left( (W_{0}^{(1)} + W_{0}^{(2)} + 2 z_{21} W_{-1}^{(2)} + z_{21}^2 W_{-2}^{(2)})v \right) = 0 \tag{33} \\
u \left( (W_{1}^{(1)} + \tilde{W}_{1}^{(2)})v \right) = u \left( (W_{1}^{(1)} + W_{1}^{(2)} + 3 z_{21} W_{-1}^{(2)} + 3 z_{21}^2 W_{-1}^{(2)} + z_{21}^3 W_{-2}^{(2)})v \right) = 0 \\
u \left( (W_{2}^{(1)} + \tilde{W}_{2}^{(2)})v \right) = u \left( (W_{2}^{(1)} + W_{2}^{(2)} + 4 z_{21} W_{1}^{(2)} + 6 z_{21}^2 W_{0}^{(2)} + 4 z_{21}^3 W_{1}^{(2)} + z_{21}^4 W_{2}^{(2)})v \right) = 0.
\end{align*}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

They correspond to the element \( (z-z_1)^n \), \( n = 0 \ldots 4 \) of \( \mathcal{W}^3 \). Clearly we have analogous but not independent equations for \( (z-z_2)^n \), \( n = 0 \ldots 4 \). From the constraints (\ref{13}) the mode \( L_{-1} \) can
be eliminated. Similarly, using the equations above the modes $W_{-1}$ and $W_{-2}$ can be removed by the aid of the combinations $W^{(1)}_1 + \bar{W}^{(2)}_1 - z_{12}(W^{(1)}_0 + \bar{W}^{(2)}_0)$, $W^{(1)}_1 + \bar{W}^{(2)}_1 - 3/2z_{12}(W^{(1)}_0 + \bar{W}^{(2)}_0)$ and $1 \leftrightarrow 2$, respectively. Considering a linear combination of the last three equations which does not contain any negative modes, we obtain

$$u \left( (W^{(1)}_2 + \bar{W}^{(2)}_2 - 2z_{12}(W^{(1)}_1 + \bar{W}^{(2)}_1) + \frac{3}{2} z_{12}^2(W^{(1)}_0 + \bar{W}^{(2)}_0) \right) v =$$

$$= u \left( (W^{(1)}_0 + W^{(2)}_0)v \right) = (w_1 + w_2)u(v) = 0,$$

(34)

where $W_0|\varphi_i⟩ = w_i|\varphi_i⟩$. This shows that the dimension of the space of the conformal blocks is one if the Virasoro weights coincide and the $W$-weights are opposite and in all other cases it is zero.

Now consider the $N$-point functions with insertion points $z_1, \ldots, z_N$. The conformal block $u$, by definition, assigns a complex number for each element of the form

$$v = W^{j_1}_{-n_1} \cdots W^{j_k}_{-n_k} |\varphi_1⟩ \otimes \cdots \otimes W^{i_1}_{-m_1} \cdots W^{i_l}_{-m_l} |\varphi_N⟩,$$

(35)

where we used the ordering introduced earlier. Now we will show that if $v$ contain modes from $W_{-n}$ than we can express the value of the conformal block with its values on vectors generated by the modes of $W_s$. Without loss of generality we suppose that $v = (W^{j}_{-n})^{(i)} v', n ≥ 0$. We use the definition of the conformal blocks, $u(xv) = 0$, for $(z - z_i)^{-n-1}$ and obtain $u(v) = -\sum_{k,k\neq i} (W^{j}_{-n})^{(i)} v' = u(v'')$. Clearly the level of $v''$ is smaller than the level of $v$. Proceeding the same way we can eliminate all the modes of $W_{-n}$ and replace them with the modes of $W_s$. Since during the procedure the level of the state in question is always decreasing and we are working with h.w. representation spaces the procedure terminates.

In order to reduce further the space of conformal blocks we use the global $W$-transformations:

$$u\left((W^{(1)}_{-2} + \sum_{j:j \neq i} \bar{W}^{(j)}_{-2})v\right) = u\left((W^{(1)}_{-2} + \sum_{j:j \neq i} W^{(j)}_{-2})v\right) = 0$$

$$u\left((W^{(1)}_{-1} + \sum_{j:j \neq i} \bar{W}^{(j)}_{-1})v\right) = u\left((W^{(1)}_{-1} + \sum_{j:j \neq i} W^{(j)}_{-1} + z_{ji} W^{(j)}_{-2})v\right) = 0$$

$$u\left((W^{(1)}_{0} + \sum_{j:j \neq i} \bar{W}^{(j)}_{0})v\right) = u\left((W^{(1)}_{0} + \sum_{j:j \neq i} W^{(j)}_{0} + 2z_{ji} W^{(j)}_{-1} + z_{ji}^2 W^{(j)}_{-2})v\right) = 0$$

$$u\left((W^{(1)}_{1} + \sum_{j:j \neq i} \bar{W}^{(j)}_{1})v\right) = u\left((W^{(1)}_{1} + \sum_{j:j \neq i} W^{(j)}_{1} + 3z_{ji} W^{(j)}_{0} +
+ 3z_{ji}^2 W^{(j)}_{-1} + z_{ji}^3 W^{(j)}_{-2})v\right) = 0$$

$$u\left((W^{(1)}_{2} + \sum_{j:j \neq i} \bar{W}^{(j)}_{2})v\right) = u\left((W^{(1)}_{2} + \sum_{j:j \neq i} W^{(j)}_{2} + 4z_{ji} W^{(j)}_{1} +
+ 6z_{ji}^2 W^{(j)}_{0} + 4z_{ji}^3 W^{(j)}_{-1} + z_{ji}^4 W^{(j)}_{-2})v\right) = 0.$$  

We determine the value of the conformal block $u$ on the vector

$$v = (L_{-1})^{i_1} (W_{-2})^{i_2} (W_{-1})^{i_3} |\varphi_1⟩ \otimes \cdots \otimes (L_{-1})^{j_1} (W_{-2})^{j_2} (W_{-1})^{j_3} |\varphi_N⟩$$

(37)

Proceeding the same way as we did in the case of the two point function, we take the difference of the $\frac{3}{2} z_{ji}$ multiple of the third equation and the fourth equation. This combination makes
it possible to eliminate the mode $W_{-2}^{(j)}$ at position $j$ without increasing the level at position $i$. 
 Similarly taking the difference of the $z_{ij}$ multiple of the third equation and the fourth equation the mode $W_{-1}^{(j)}$ can be eliminated. Using an analogous procedure the modes $W_{-1}$ and $W_{-2}$ can be eliminated at the position $i$ and $j$ simultaneously. Now taking the combination

$$W_2^{(i)} + \sum_{j:j \neq i} W_2^{(j)} - 2z_{ij}(W_1^{(i)} + \sum_{j:j \neq i} W_1^{(j)}) + z_{ij}^2(W_0^{(i)} + \sum_{j:j \neq i} W_0^{(j)})$$

we can get rid of the mode $W_{-2}^{(k)}$.

All in all this means that the value of the conformal blocks on any vector is determined if it is given on the vectors:

$$v = W_{-n_1}^{j_1} \ldots W_{-n_k}^{j_k} |\varphi_1\rangle \otimes \ldots \otimes |\varphi_i\rangle \otimes \ldots$$

$$\ldots \otimes |\varphi_j\rangle \otimes \ldots \otimes (W_{-1})^{l} |\varphi_k\rangle \otimes \ldots \otimes W_{-m_1}^{j_1} \ldots W_{-m_l}^{j_l} |\varphi_N\rangle$$

Concretely in the case of the three point function the value of the conformal block $u$ should be known on the vector $|\varphi_1\rangle \otimes |\varphi_2\rangle \otimes (W_{-1})^{l} |\varphi_3\rangle$. There is an important difference between the purely Virasoro symmetric theories and the theories having $WA_2$ symmetry. In the second case the space of the three point functions is infinite dimensional. (We note that the independence of the $|\varphi_1\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle$ three point function from $|\varphi_1\rangle \otimes |\varphi_2\rangle \otimes W_{-1} |\varphi_3\rangle$ can be seen even in the simplest minimal model).

In the three point case, as in the Virasoro theory, we can analyze the possible nonvanishing couplings. We leave the operator $W_0^{(1)}$ undetermined and eliminate the negative modes in the first two representations. Contrary to the previous case we cannot use the equation which contains $W_0^{(1)}$, so we keep the equation (38) for defining $W_0^{(1)}$. This shows that in the third representation space everything, which is generated by the elements of $W_s$ remains undetermined in the general case. If however, we have singular vectors which makes this space finite then we have a finite dimensional matrix for $W_0^{(1)}$, whose eigenvalues give the nonvanishing couplings. This matrix has a simple form if we take the limit $z_1 \to \infty$, $z_2 \to 1$ and $z_3 \to 0$. In this case the constraints, which annihilate the conformal blocks on the right, are:

$$W_0^{(2)} = -W_{-3}^{(3)} + 2W_{-2}^{(3)} - W_{-1}^{(3)}$$
$$W_0^{(1)} = -W_{-3}^{(3)} + 3W_{-2}^{(3)} - 3W_{-1}^{(3)} + W_0^{(3)}$$
$$0 = -W_{-n}^{(3)} + 3W_{-n+1}^{(3)} - 3W_{-n+2}^{(3)} + W_{-n+3}^{(3)} \quad ; \quad n > 3$$

They can be used to analyze the fusion a la Feigin and Fuchs. This was performed by Watts, see [14] for the details.

We investigate the possible couplings, similarly as we did in the Virasoro case, by starting with the simplest cases. Consider the representation space where the factorspace $V_s := W_s v / W_{-s} v$ is one dimensional. This is the vacuum representation with two singular vectors at level one which can be chosen as $L_{-1} |\varphi_3\rangle = W_{-1} |\varphi_3\rangle = 0$. We also have the descendant singular vector $W_{-2} |\varphi_3\rangle = 0$ which shows that the special subspace is one dimensional. This implies that the analysis of the three point function reduces to the analysis of the two point functions as in the Virasoro case.
In the next simplest case we have a singular vector at level one and another one at level two. This corresponds to the Toda model \[21\]. In order to describe this representation space we parametrize the central charge of the WA\(_2\) algebra as

\[ c = 2 - 24(\beta - \beta^{-1})^2. \]  

(41)

Now in this completely degenerate representation space we have a h.w. singular vector at level one

\[ \left(W_{-1} - (5/6\beta + 1/2\beta^{-1})L_{-1}\right) |u\rangle = 0. \]  

(42)

We also have a combination of the h.w. singular vector at level two and a descendant of the h.w. singular vector above, which has a very simple form

\[ \left(W_{-2} - \beta^{-1}L_{-1}^2 + 2/3\beta L_{-2}\right) |u\rangle = 0. \]  

(43)

Moreover we have the Toda equation of motion which is a descendant singular vector at level three:

\[ \left(W_{-3} - \beta^{-3}L_{-1}^3 + \beta^{-1}L_{-1}L_{-2} + (\beta/6 - 1/2\beta^{-1})L_{-3}\right) |u\rangle = 0. \]  

(44)

This shows that the special subspace in question is three dimensional.

For each pair of the h.w. representations of the sl\(_3\) algebra labelled by the highest weights \(\Lambda(n_1, n_2) = n_1\lambda_1 + n_2\lambda_2\), \(n \leftrightarrow m\), (where \(\lambda_1, \lambda_2\) denote the fundamental weights), a h.w. representation of the WA\(_2\) algebra can be associated via the following formulae:

\[ h = \frac{1}{3}(x_1^2 + x_1x_2 + x_2^2) - (\beta - \beta^{-1})^2 \quad w = \frac{1}{27}(x_1 - x_2)(2x_1 + x_2)(x_1 + 2x_2), \]  

(45)

where \(x_i = (n_i + 1)\beta - (m_i + 1)\beta^{-1} =: (n_i, m_i)\). The vacuum corresponds to the trivial representation, \(x_1 = (0, 0), x_2 = (0, 0)\), and the Toda field to the fundamental representation \(x_1 = (1, 0)\) and \(x_2 = (0, 0)\). Note that the dimension of the special subspace is the same as the dimension of the sl\(_3\) representation. (This correspondence is much more straightforward at the classical level). Now if the parameters of the second representations are \((n_1, n_2)\) then the three possible eigenvalues of the operators \(L_0^{(1)}\) and \(W_0^{(1)}\) correspond to the parameters \((n_1 + 1, n_2 - 1)\), \((n_1 - 1, n_2)\) and \((n_1, n_2 - 1)\), respectively, \[21\], (where the \(m\) values are unchanged). This is very similar to the fusion rules of the fundamental representation of sl\(_3\).

The \(z\)-dependence of the conformal blocks are defined in the same way as in the Virasoro case, the only novelty is that the meromorphic testfunctions corresponding to the \(W\)-transformations have to extended to the neighbourhood \(U\). The horizontality condition gives differential equations for the conformal blocks, which can be solved in the simplest case, for details see \[21\].

### 4 The case of general \(W\)-algebras

We now follow the previous line and generalise the results obtained to other \(W\)-algebras. Let \(\Sigma\) denote the Riemann sphere with coordinates \(z\) and \(w = 1/z\) and \(V = V_1 \otimes V_2 \otimes \ldots \otimes V_N\) the product of the highest weight representations, \(V_i\), of the chiral algebra associated to the point
The chiral algebra is supposed to be a $W$-algebra with integer spin fields only. Introduce the space of complex valued functions on the Riemann sphere, $W^j(\Sigma \setminus \{p_1, \ldots, p_n\})$, for whose elements, $f(z)$, $f(\zeta)\, d\zeta^{-j+1}$, are holomorphic except the points $p_1, \ldots, p_n$. The globally defined transformations correspond to the functions $1, \zeta, \ldots, \zeta^{2j-2}$. For each functions of this type we associate an action of the spin $j$ generator of the chiral algebra on $V$ in the following way: We Laurent expand $f(z)$ around $z_i$ and for the term $(z - z_i)^{k+j-1}$ we associate the action of $W^j_k$ on $V_i$ thus the action of $(W^j)^{(i)}_{k}$ is $1 \otimes \ldots \otimes W^j_k \otimes \ldots \otimes 1$ on $V$. Since the functions of $W^j$ are of the form $(z - z_i)^n$, $n < 2j - 1$ we give the action explicitly for them:

$$
(z - z_i)^{-n} \rightarrow W^{(i)}_{-n-j+1} \oplus \sum_{\ell \neq i} \tilde{W}^{(l)}_{-n-j+1},
$$

or depending on whether $n$ is positive or negative

$$
\tilde{W}^{(i)}_{-n-j+1} = \begin{cases} 
(-1)^n \sum_{k=0}^{\infty} z_i^{-n-k} (n_{n-1}) W^{(i)}_{k-j+1} & n > 0 \\
(-1)^n \sum_{k=0}^{\infty} z_i^{-n-k} (n_k) W^{(l)}_{k-j+1} & n \leq 0
\end{cases}.
$$

As in the previous cases the generators with tilde contain modes only which annihilate the vacuum.

Now having set the stage we define the space of the conformal blocks associated to the representation $V$ and the points $p_1, \ldots, p_N$, as the space of linear functionals on $V$ which are annihilated by $W_i$, the collection of the $W^j$-s:

$$
E(V, \{p_i\}) = \{u \in V^* | u(xv) = 0 \; ; \; \forall v \in V, \; x \in W\}.
$$

(48)

As a first step we determine the dimension of the space of the conformal blocks. We introduce a basis in the h.w. representation, $V_i$, of the $W$-algebra. By definition $V_i = U(W_-) v_i$, where $W_-$ contains the negative modes of the generating fields. We devide the negative modes of the $W$-algebra into two groups depending on whether they annihilate the vacuum or not.

$$
W_s = \{W^j_n | j > n > 0\} \; ; \; W_- = \{W^j_n | n \geq j\}.
$$

(49)

Clearly the vacuum module is $U(W_-)|0\rangle$. The elements of $W_s$ generate global transformations. The Verma modules can be described as $V_i = U(W_-)U(W_s)v_i$, where in the subalgebras the generators are ordered in such a way that the generators with smaller spins stay before the highers or in the case of the same spins the more negative preceeds the others. As before we will show that the value of the conformal block $u$ on every vector $v$ of $V$ is completely determined by its value on the vectors generated by the elements of $W_s$. This means that we have to give the value of $u$ on the tensor products, $V_s$, of the vectors, $(V_i)_s = U(W_s)v_i$. The proof is inductive in the level defined on the Verma module in the usual way: the level of a vectors in $V$ is simply the sum of its levels in $V_i$. Clearly at the lowest grades we have the elements of $V_s$. Now suppose that we have a vector $v$ not in $V_s$, without loss of generality it has the following form:

$$
v = W^{i_1}_{n_1} \ldots W^{i_r}_{n_r} v_1 \otimes \ldots \otimes W^{i_1}_{m_1} \ldots W^{i_p}_{m_p} v_i \otimes \ldots \otimes W^{k_1}_{l_1} \ldots W^{k_s}_{l_s} v_N = (W^{i_1})^{(i)}_{m_1} v'
$$

(50)
where \( m_1 \leq -i_1 \). No we use the definition of the conformal blocks \( u(xv) = 0 \) for the element \( x = (W^{(i)})_{m_1}^{i_1} + \sum_{j \neq i_1} (W^{(j)})_{m_1}^{i_1} \), which correspond to the function \((z - z_i)^{m_1 + i_1} \in W^{i_1}\) and obtain

\[
    u(v) = u \left( \sum_{j \neq i_1} (W^{(j)})_{m_1}^{i_1} v' \right) = u(v')
\]

(51)

Since the level of \( v' \) is smaller than the level of \( v \) the statement is proved.

Now we turn to the two point functions. The constraints coming from the global transformations with test functions \((z - z_1)^n \), \( n = 0, 1, \ldots, 2j - 2 \) for the spin \( j \) generators are:

\[
    u \left( (W^{(1)}_{-j+1} + W^{(2)}_{-j+1}) v \right) = u \left( (W^{(1)}_{-j+1} + W^{(2)}_{-j+1}) v \right) = 0
\]

\[
    u \left( (W^{(1)}_{-j+2} + W^{(2)}_{-j+2}) v \right) = u \left( (W^{(1)}_{-j+2} + W^{(2)}_{-j+2} + z_{21} W^{(2)}_{-j+1}) v \right) = 0
\]

\[
    \sum_{i=0}^{j-1} (z^{i-1}_i) z_{21}^{i} W^{(2)}_{-j-i} v \right) = 0
\]

(52)

Now consider the last \( j \) equations. Descarding the last one for a while the others contain the negative modes linearly independently due to the binomial nature of the coefficients. As a consequence we can find such a combinations of these equations which contain the negative mode \( W^{(2)}_{-i} \) only and nothing else. Applying this equations recursively the mode \( W^{(2)}_{-i} \) can be eliminated. Doing the same procedure for the other generators we arrive at the h.w. vector. Clearly using the \( 1 \leftrightarrow 2 \) trick, only the h.w. vectors remain in both representations spaces.

Now using also the last equation we get:

\[
    u \left( (W^{(1)}_{j-1} + W^{(2)}_{j-1} + (j - 1)z_{12} (W^{(1)}_{j-2} + W^{(2)}_{j-2}) + \ldots + (z^{j-1}_{j-1}) z_{12}^{j-1} (W^{(1)}_{0} + W^{(2)}_{0})) v \right) = 0
\]

(53)

This shows that the dimension of the space of the conformal blocks is one if the eigenvalue of the even generators are the same and the odd generators are opposite and in every other cases it is zero.

Now consider the \( N \) point functions. Clearly it is enough to take into account the value of the conformal blocks on the vectors generated by the elements of \( W_s \). The constraints from the global transformations are:

\[
    u \left( (W^{(i)}_{-j+1} + \sum_{l \neq i}^{N} W^{(l)}_{-j+1}) v \right) = u \left( (W^{(i)}_{-j+1} + \sum_{l \neq i}^{N} W^{(l)}_{-j+1}) v \right) = 0
\]

\[
    u \left( (W^{(i)}_{-j+2} + \sum_{l \neq i}^{N} W^{(l)}_{-j+2} + z_{li} W^{(l)}_{-j+1}) v \right) = 0
\]

51
\[
\begin{align*}
  u\left((W^{(i)}_0 + \sum_{l \neq i}^N \tilde{W}^{(i)}_l)v\right) &= u\left((W^{(i)}_0 + \sum_{l \neq i}^{N-1} z^{(i)}_l W^{(i)}_{-l})v\right) = 0 \\
  u\left((W^{(i)}_{j-2} + \sum_{l \neq i}^N \tilde{W}^{(i)}_{j-2})v\right) &= u\left((W^{(i)}_{j-2} + \sum_{l \neq i}^{N-2} z^{(i)}_l W^{(i)}_{-j+2-l})v\right) = 0 \\
  u\left((W^{(i)}_{j-1} + \sum_{l \neq i}^N \tilde{W}^{(i)}_{j-1})v\right) &= u\left((W^{(i)}_{j-1} + \sum_{l \neq i}^{N-1} z^{(i)}_l W^{(i)}_{-j+1-l})v\right) = 0,
\end{align*}
\]

where we did not write out explicitly the spin of the generator. Now taking the same linear combination of the equations which we saw at the two point case we can eliminate the negative modes of \(W^{(i)}_n\) without increasing the level at the position \(i\) and \(l\). Doing so reductively all negative modes at \(l\) can be eliminated. Clearly we can do the same for \(i\). Moreover with the aid of the last equation the mode \(W^{(k)}_{j+1}\) of the spin \(j\) generator at position \(k\) can be reexpressed by others. This means that the correlation function should be defined on the vectors of the form:

\[
v = W^{j_1}_{n_1} \ldots W^{j_r}_{n_r} \otimes \ldots \otimes |\nu_i\rangle \otimes \ldots \otimes W^{i_1}_{m_1} \ldots W^{i_p}_{m_p} |v_k\rangle \otimes \ldots \otimes |\nu_j\rangle \otimes W^{k_1}_{l_1} \ldots \ldots \otimes W^{k_s}_{l_s} |v_N\rangle
\]

where \(-i_s + 1 < m_s < 0\). Concretely in the three point case this means that we have to define the conformal blocks on the vectors:

\[
v = |\nu_1\rangle \otimes |\nu_2\rangle \otimes W^{k_1}_{l_1} \ldots W^{k_s}_{l_s} |\nu_3\rangle ; \quad -l_i < k_i
\]

which means that the symmetry does not fix the three point functions completely as it is fixed in the Virasoro case.

Turning to the completely degenerate representation we can express \(W^{(1)}_0\) as an operator which contain negative modes only in the third representation, moreover only the modes of \(W_s\). This shows that the number of possible couplings is bounded by the dimension of the factorspace \(V_s = W_s|\varphi_3\rangle/W_s|-\varphi_3\rangle\) which was called the special subspace by Nahm. He also obtained the same result considering the fusion of a field with finite dimensional subspace \([18]\).

One feasible way to characterise the \(W\)-algebras can be achieved by analysing their special subspaces. The simplest representation space is always the vacuum representation which contains the same number of singular vectors as the number of the generating fields of the \(W\)-algebra in question. These singular vectors are independent so they can be chosen in the following form: \(W^{j_1}_{-1}v = 0\) for all the generating primary fields. The primary nature of the fields yields that we have descendant singular vectors of the spin \(j\) generator of the form \(Ad_{L_{k-1}^j}W^{j}_{-1}v \sim W^{j}_{-k-1}v = 0\), \(k = 1, \ldots, j-2\). These show that the dimension of the special subspace is one and accordingly the analysis of the three point function reduces to the analyses of the two point function.

In the next simplest representation we have one singular vector at level two and all the others at level one. We believe that this representation characterizes the \(W\)-algebra as fundamentally as the vacuum representation. As the vacuum representation is the trivial representation in the quantum case the next simplest should be the fundamental representation at the quantum level. This is where the special subspace have the smallest nontrivial dimension and consequently the number of fields that appear in the fusion product is the smallest. That
is why we call this representation the quantum fundamental representation, since it shares this property with the fundamental representation of Lie algebras. (The importance of this representation was also observed by Hornfeck [17]).

This connection is explicit in the case of the $WG$ algebras: One possible way to quantize the Toda theories is to quantize their equation of motions, [21, 22]. In this language the special subspace has the same dimension as it has in the classical case which we analyse in the next section.

The $z$-dependence of the correlation functions can be described very similarly to the Virasoro case. Take a point in the configuration space $C_N = \{ z_1, \ldots z_N \} \in \Sigma^N$, $z_i \neq z_j$ and for its neighbourhood, $U$, associate a holomorphic dual to $V$ as

$$V^*(U) = \{ u: U \rightarrow V^*, \text{ holomorphic} : \text{ that is } u_{z_1,\ldots,z_N}(v) \text{ holomorphic } \forall v \in V \}. \quad (57)$$

Extend the meromorphic test-functions corresponding to the $W$-transformation of the spin $j$ generator to $U$ and denote this space by $W^j(U)$. The space of the conformal blocks associated to the open set $U$ and the representation $V$ are defined to be those elements of the holomorphic dual which satisfy the Ward identities:

$$E(U) = \{ u \in V^*(U) | \quad u(xv) = 0 \quad ; \quad \forall v \in V , x \in W(U) \}. \quad (58)$$

(Here $W(U)$ denotes the collection of the $W^j(U)$-s). Similarly to the Virasoro case the space of the conformal blocks can be regarded as a holomorphic vector bundle over the moduli space of the punctured sphere. The $z$-dependence is defined by the aid of a flat connection:

$$\nabla = \sum_i \nabla_{z_i} \otimes dz_i,$$

where

$$\nabla_{z_i}u(v) = \partial_{z_i}u(v) - u(L^{(i)}_{-1}v). \quad (59)$$

for each $v \in V$. Since $\nabla_{z_i}(ux) = (\nabla_{z_i}u)x + u\partial_{z_i}x$ for $x \in W(U)$ it maps the space $E(U)$ into $\Omega^1(U) \otimes E(U)$, ie. it preserves the space of the conformal blocks. Now the conformal blocks are defined to be those sections of $E(U)$ which satisfy the horizontality condition:

$$\nabla u = 0. \quad (60)$$

5 Classical Toda theory and singular vectors

The classical $W$-algebras corresponding to the principal $sl_2$ embedding were defined in [2, 3]. They are nothing but the symmetry algebras of the Toda models, the theories that arises as reductions of the Wess-Zumino-Witten-Models. Now we summarize the results we need. One imposes first class constraints on the space of the Kac-Moody currents. The $W$-algebra is the gauge invariant part of the constrained phase space, ie. it is invariant under the gauge transformations generated by the constraints. The defining relations of the $W$-algebra are given by the Dirac brackets, and can be calculated very easily, which is one consequence of the reduction procedure. More concretely the $W$-transformations can be implemented by an appropriately chosen, KM transformation, $J_{\text{imp}}(W^i)$:

$$\delta_{f_{\text{fix}}(W^i(y))} = \int dx \epsilon_i(x) \{ W^i(x), J_{f_{\text{fix}}(W^i(y))} \} = [J_{f_{\text{fix}}(W^i)}(y), J_{f_{\text{fix}}(W^i(y))}] + \partial J_{\text{imp}}(y). \quad (61)$$
The current, \( J_{\text{imp}}(W^i) \), also defines an action of the \( W \)-algebra on the group element of the WZNW model as \( \delta g(x_+) = J_{\text{imp}}(x_+)g(x_+) \), where this space is constrained as \( \partial_+g = J_{\text{fix}}g \). All information on a given classical \( W \)-algebra are contained in the matrices \( J_{\text{fix}} \) and \( J_{\text{imp}} \).

Concentrating on the Virasoro case we have for the fundamental representation the following results:

\[
J_{\text{fix}} = \begin{pmatrix} 0 & L(x) \\ 1 & 0 \end{pmatrix} = t_- + L(x)t_+ \tag{62}
\]

for the gauge fixed current and

\[
J_{\text{imp}} = \begin{pmatrix} \frac{1}{2} \epsilon_0(x) & \epsilon_+(x) \\ \epsilon_-(x) & -\frac{1}{2} \epsilon_0(x) \end{pmatrix} = \epsilon_-(x)t_- + \epsilon_0(x)\rho + \epsilon_+(x)t_+ \tag{63}
\]

for the KM implementation, where

\[
\epsilon_0(x) = \epsilon_-(x)' \quad ; \quad \epsilon_+(x) = L(x)\epsilon_-(x) - \frac{1}{2} \epsilon_-(x)'' \tag{64}
\]

Considering the classical representation space we have to solve the equation \( J_{\text{fix}}(x)g(x) = g(x)' \). Concentrating on the columns we have

\[
\begin{pmatrix} 0 & L(x) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1(x) \\ u_0(x) \end{pmatrix} = \begin{pmatrix} u_1(x)' \\ u_0(x)' \end{pmatrix} \tag{65}
\]

In detail it reads as

\[
u_1(x) = u_0(x)' \quad ; \quad u_0(x)'' - L(x)u_0(x) = 0 \tag{66}
\]

The action of the \( W \)-transformation on the representation space described above is defined as \( \delta_\epsilon g(x) = J_{\text{imp}}(x)g(x) \), ie.

\[
\delta_\epsilon u_0(x) = \epsilon(x)u_0(x)' - \frac{1}{2} \epsilon(x)'u_0(x) \tag{67}
\]

This means, that classically \( u_0 \) is a hw. primary field with weight \(-\frac{1}{2}\) and the equation (66) can be considered as a classical singular vector at level two. This has a very similar structure as the quantum singular vector of the quantum defining representation.

In the case of the \( WA_2 \) algebra starting from the fundamental representation of \( sl_3 \) the gauge fixed current reads as

\[
J_{\text{fix}} = \begin{pmatrix} 0 & L(x) & W(x) \\ 1 & 0 & L(x) \\ 0 & 1 & 0 \end{pmatrix} = t_- + L(x)t_+ + W(x)l_{++} = t_- + W^2(x)t_+ + W^3(x)l_{++} \tag{68}
\]

The current \( J_{\text{fix}} \) can be obtained by imposing the constraint \( \delta_\epsilon, \epsilon J_{\text{fix}} = [J_{\text{imp}}, J_{\text{fix}}] + J_{\text{imp}} \) for the element

\[
\epsilon L(x)t_- + \epsilon_0(x)\rho + \epsilon_+(x)t_+ + \epsilon_+(x)l_{++} + \epsilon_+(x)l_+ + \epsilon_0(x)l_0 + \epsilon_-(x)l_- + \epsilon_+(x)l_-. \tag{69}
\]
The result is
\[
\begin{align*}
\epsilon^W_+ &= \frac{1}{2}(\epsilon^W)'\, , \quad \epsilon^L_0 = -(\epsilon^L)'\, , \quad \epsilon^W_0 = \frac{1}{6}(\epsilon^W)'' - \frac{1}{3}L\epsilon^W \\
\epsilon^L_+ &= W\epsilon^W - (\epsilon^L)'' + L\epsilon^L , \quad \epsilon^W_+ = -\frac{1}{6}(\epsilon^W)''' + \frac{5}{6}L(\epsilon^W)' + \frac{1}{3}L'\epsilon^W \\
\epsilon^W_{++} &= W\epsilon^L + \frac{1}{6}(\epsilon^W)''' - \frac{2}{9}(L(\epsilon^W)'' - L'(\epsilon^W)) - \frac{1}{3}L''\epsilon^W + \frac{2}{3}L^2\epsilon^W,
\end{align*}
\]
and
\[
\begin{align*}
\delta L &= [\epsilon^L L' + 2(\epsilon^L)' L - 2(\epsilon^L)'''] + [2\epsilon^W W' + 3(\epsilon^W)' W] \\
\delta W &= [\epsilon^L W' + 3(\epsilon^L)' W_2] \\
&\quad + [\epsilon^W (-\frac{1}{6}L''' + \frac{2}{3}L(L)) + (\epsilon^W)' (-\frac{3}{4}L'' + \frac{2}{3}L^2)] \\
&\quad - \frac{5}{4}(\epsilon^W)'' L' - \frac{5}{6}(\epsilon^W)''' L + \frac{1}{6}(\epsilon^W)^{(W)}].
\end{align*}
\]

The representations are defined on the solution space of

\[
J_{fix} = \begin{pmatrix}
0 & L(x) & W(x) \\
1 & 0 & L(x) \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
u_2 \\
u_1 \\
u_0
\end{pmatrix} = \begin{pmatrix}
u_2' \\
u_1' \\
u_0'
\end{pmatrix}
\]

via \(\delta g = J_{map} g\). Explicitly we have
\[
u_1 = \nu_0' \quad ; \quad \nu_2 = \nu_0'' - Lu_0 \quad ; \quad \nu_0''' - 2L'\nu_0' + (L' + W)u_0 = 0.
\]

The transformation of the last component reads as
\[
\delta \nu_0 = \epsilon^L \nu_0' - (\epsilon^L)' \nu_0 \\
\quad + \frac{1}{6}(\epsilon^W)'' \nu_0 + \frac{1}{2}(\epsilon^W)' \nu_0' + \epsilon^W \left(\nu_0'' - \frac{2}{3}Lu_0\right).
\]

We can read off that it is a Virasoro and a \(W\)-primary field, moreover it corresponds to a completely degenerate representation, since we have two independent classical singular vectors \(\{W_{-1}, \nu_0\} = \frac{1}{2}u_0'\) and \(\{W_{-2}, \nu_0\} = \nu_0'' - \frac{2}{3}Lu_0\). Clearly they have the same structure we had in the quantum case. We expect that this structure naturally generates for other representations of the classical algebra. (See the appendix for one example). In the general case the \(\partial_+ g = J_{fix} g\) relation shows that the classical special subspace has the same dimension as the dimension of the representaion of the underlying algebra.

### 6 Conclusion

We defined the space of the conformal blocks for a general \(W\)-algebra as the space of those linear functionals on the product of the representation spaces that is annihilated by certain meromorphic \(W\)-transformations. In determining the dimension of this space we observed the importance of a special subspace of the h.w. representation spaces. This space is a factorspace spanned by those negative modes of the \(W\)-algebra that annihilate the vacuum modulo those.
that do not. Analysing the three point functions of a field – with finite dimensional special subspace – and any fixed h.w. primary field, the number of the fields that can have nonzero three point functions is bounded by the dimension of the special subspace. This number is supposed to be finite for all the soluble models. We can characterization the $W$-algebras by their special subspaces. For the simplest module, the vacuum module, the special subspace is one dimensional. We called the next simplest representation space, a representation where the dimension of the special subspace takes the smallest nontrivial values, the quantum fundamental representation since in the case of the $WG$ algebras this is the quantum analogue of the fundamental representation of $G$. This is supported by the dimension of the special subspace and by the fusion of this field, moreover can be corroborated by exploiting the relation between the classical and quantum Toda models. A further elaboration of this relationship and the analysis of the geometry of the conformal blocks is work in progress.

7 Appendix

The $WBC_2$ algebra, which is generated by the energy momentum tenzor $W_2$ and a spin four current $W_4$, is one of the simplest $W$-algebras \[13, 15, 16\]. The singular vectors in the "quantum defining representation" are very simple, see \[22\] for the details. We have a h.w. singular vector at level one

$$W_{-1}|u\rangle + \beta^1 L_{-1}|u\rangle = 0$$

(75)

Combining the h.w. singular vector of level two with one of the descendants of the h.w. singular vector above we have a singular vector at level two of the form:

$$W_{-2}|u\rangle + \beta^2 L_{-2}|u\rangle + \beta^{11} L_{-1}^2|u\rangle = 0.$$  

(76)

They have a descendant at level three which has a particularly nice form

$$W_{-3}|u\rangle + \beta^3 L_{-3}|u\rangle + \beta^{12} L_{-1} L_{-2}|u\rangle + \beta^{111} L_{-1}^3|u\rangle = 0.$$  

(77)

Among the descendants at level four we can find the quantum equation of motion of the $C_2$ Toda model:

$$W_{-4}|u\rangle + \beta^4 L_{-4}|u\rangle + \beta^{22} L_{-2}^2|u\rangle + \beta^{13} L_{-1} L_{-3}|u\rangle + \beta^{112} L_{-1}^2 L_{-2}|u\rangle + \beta^{1111} L_{-1}^4|u\rangle = 0.$$  

(78)

This shows that the dimension of the special subspace is four dimensional in this case. If we had considered the $B_2$ Toda model instead of the $C_2$ then we would have obtained analogous singular vectors at the first three levels. The singular vector at level four in the form above is missing, however we have a singular vector at level five in the form:

$$\tilde{\beta}^{14} L_{-1}^4 u - \frac{2}{(k+1)} \tilde{\beta} L_{-1} W_{-4} u + \tilde{\beta} W_{-5} u + \tilde{\beta}^{122} L_{-1} L_{-2}^2 u + \tilde{\beta}^{113} L_{-1}^2 L_{-3} u + \tilde{\beta}^{1112} L_{-1}^3 L_{-2} u + \tilde{\beta}^{11111} L_{-1}^5 u + \tilde{\beta}^{23} L_{-2} L_{-3} u + \tilde{\beta}^5 L_{-5} u = 0.$$  

(79)
It is in agreement with the fact that the dimension of the fundamental representation of $C_2$ is four and the $B_2$ is five. The fusion of these fields follows the selection rules of the four and five dimensional representations of the underlying algebra. See [23] for the details.

It is interesting to compare this quantum representations to the classical ones. In the simplest case, which corresponds to the defining representation of the $C_2$ algebra, the form of the constrained current is

$$ J_{\text{fix}} = \begin{pmatrix} 0 & 3L(x) & 0 & W(x) \\ 1 & 0 & 4L(x) & 0 \\ 0 & 1 & 0 & -3L(x) \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (80) $$

The transformation of the Toda field and consequently the classical singular vectors are:

$$ \delta u_0 = \epsilon_1 u_0' - \frac{3}{2} \epsilon_1 u_0 + \epsilon_2 (-u_0'' + \frac{41}{50} W_2 u_0' + \frac{27}{100} W_2' u_0) + \epsilon_2' \frac{1}{2} u_0'' - \frac{23}{100} W_2 u_0 - \frac{1}{5} \epsilon_2 u_0' + \frac{1}{20} \epsilon_2'' u_0. $$

The analogous investigation in the $B_2$ case gives

$$ J_{\text{fix}} = \begin{pmatrix} 0 & 4L(x) & 0 & W(x) & 0 \\ 1 & 0 & 6L(x) & 0 & -W(x) \\ 0 & 1 & 0 & -6L(x) & 0 \\ 0 & 0 & -1 & 0 & -4L(x) \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad (81) $$

for the constrained current and

$$ \delta u_0 = \epsilon_1 u_0' - 2 \epsilon_1 u_0 + \epsilon_2 (-u_0'' + \frac{16}{25} L u_0' + \frac{7}{25} L' u_0) + \epsilon_2' u_0'' - \frac{18}{25} L u_0 - \frac{3}{5} \epsilon_2 u_0' + \frac{1}{5} \epsilon_2'' u_0. $$

for the transformation of the classical Toda field.

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