Semiclassical Approach to Black Hole Evaporation

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Black hole evaporation may lead to massive or massless remnants, or naked singularities. This paper investigates this process in the context of two quite different two dimensional black hole models. The first is the original CGHS model, the second is another two dimensional dilaton-gravity model, but with properties much closer to physics in the real, four dimensional, world. Numerical simulations are performed of the formation and subsequent evaporation of black holes and the results are found to agree qualitatively with the exactly solved modified CGHS models, namely that the semiclassical approximation breaks down just before a naked singularity appears.
1. Introduction

Ever since Hawking’s discovery [1] of the thermal character of radiation from black holes, much effort has been expended trying to understand the implications of this for a theory of quantum gravity. Conservative suggestions have been that uncharged black holes evolve into massive states somehow made stable by quantum gravity corrections, or into massless remnants such as the cornucopions, in which information is stored in a semi-infinite horn geometry. Both these possibilities allow unitarity to be preserved [2]. A radical suggestion was made by Hawking to describe the physics of the endpoint of black hole evaporation. It was suggested that pure quantum states must evolve into mixed states, leading to a breakdown in predictability. This is an inevitable consequence of the formation of a naked singularity when the event horizon ceases to exist as the black hole evaporates away to leave empty space behind. This of course violates the Cosmic Censorship hypothesis of Penrose [4] and leaves us unable to predict the future even at the classical level in half of spacetime.

Recently much progress has been made in developing toy models of black hole evaporation in two dimensions. This was initiated by the work of CGHS [5]. The quantum backreaction is included at the one loop level by integrating out the $N$ matter fields, inducing a Polyakov-type term [3] in the effective action. The large $N$ limit is then taken allowing the one loop effects of the gravitational degrees of freedom to be ignored. Static solutions of the resulting equations have been obtained in [7–9]. More recently analytic results for dynamical shock wave solutions have been obtained in modified models based on a Liouville type field theory [10–12]. There it was found that a naked singularity appears as the black hole continues evaporating to negative mass.

This paper begins by reviewing the results of CGHS and then numerical results are presented for shock wave solutions in the original CGHS model. They are found to be in qualitative agreement with the analytic results of the modified models, namely that the semiclassical approximation breaks down just before a naked singularity appears.

A model is then introduced which is obtained by the dimensional reduction of the four dimensional Einstein action. See [13] for a discussion of a general class of two dimensional dilaton-gravity theories. This model is then coupled to matter generally covariant only in two dimensions allowing the backreaction to be included in the same way as the CGHS model. The black hole solutions are of course essentially the same as their four dimensional analogs, so it is hoped that the study of this model will lead to greater insights into physics in the real world. In particular the Hawking temperature is $T = 1/8\pi M$. 
This model seems even more difficult than the CGHS model to treat analytically, so once again numerical methods are used to study both the static solutions and dynamical shock waves. The finite mass static solutions exhibit singular horizons, reminiscent of the solutions of the CGHS model. Again shock wave solutions evolve until a naked singularity appears, at which point the approximation breaks down, and fluctuations in the gravitational degrees of freedom cannot be neglected. At the very least this means observers may see high energy quantum gravity effects at the endpoint of black hole evaporation, but the question of what really happens is unresolved.

2. Review of CGHS model

CGHS begin with the classical action for dilaton gravity in two spacetime dimensions

\begin{equation}
S_{\text{cl}} = \frac{1}{\pi} \int d^2\sigma \ e^{-2\phi} \left[ -2\partial_+ \partial_- \rho + 4\partial_+ \phi \partial_- \phi - \lambda^2 e^{2\rho} \right] - \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i ,
\end{equation}

where \( \phi, \rho \) and \( f \) are the dilaton, Liouville and matter fields respectively.

Static solutions of this action representing two dimensional black holes are

\begin{equation}
e^{-2\phi} = e^{-2\rho} = \frac{M}{\lambda} - \lambda^2 x^+ x^- .
\end{equation}

Here \( M \) corresponds to the ADM mass of the solution. The case \( M = 0 \) corresponds to a coordinate transformation of the linear dilaton vacuum solution

\begin{equation}
\rho = 0 \quad \text{and} \quad \phi = -\lambda \sigma ,
\end{equation}

where \( \sigma = \frac{1}{2}(\sigma^+ - \sigma^-) \) and

\begin{equation}
e^{\lambda \sigma^+} = \lambda x^+, \quad e^{-\lambda \sigma^-} = -\lambda x^- .
\end{equation}

Shock waves corresponding to a black hole with mass \( M \) being formed by collapse take the form

\begin{equation}
e^{-2\phi} = e^{-2\rho} = -\lambda^2 x^+ x^- - \frac{M}{\lambda x_0^+} (x^+ - x_0^+) \theta(x^+ - x_0^+) .
\end{equation}

The key point of is that in the one loop approximation quantum corrections to the classical action may be computed simply by integrating out the fluctuations in the matter fields. Taking the large \( N \) limit, where \( N \) is the number of matter fields means the
contributions of the gravitational degrees of freedom may be ignored. The resulting action is

\[ S_{\text{eff}} = S_{\text{cl}} + \frac{1}{\pi} \int d^2 \sigma \frac{N}{12} \partial_+ \rho \partial_- \rho . \]  

(2.6)

The equations of motion for the case \( f_i = 0 \) can be written in the convenient form

\[ \partial_+ \partial_- \phi = \frac{(1 - \frac{N}{24} e^{2\phi})}{(1 - \frac{N}{12} e^{2\phi})} \left( 2 \partial_+ \rho \partial_- \phi + \frac{\lambda^2}{2} e^{2\rho} \right) \]  

\[ \partial_+ \partial_- \rho = \frac{1}{(1 - \frac{N}{24} e^{2\phi})} \partial_+ \partial_- \phi . \]  

(2.7)

It should be noted (2.7) become degenerate when \( \phi = \phi_{cr} = -\frac{1}{2} \log(N/12) \), where in general a curvature singularity will appear. However, one expects the large \( N \) approximation to break down before this singularity is reached. This may be seen by examining the matrix \( K \) of the dilaton-Liouville kinetic term \( \partial_+ \Phi K(\phi) \partial_- \Phi \) where \( \Phi \) denotes the 2-vector \((\rho, \phi)\). The determinant of this matrix is

\[ \det K = e^{-2\phi}(e^{-2\phi} - \frac{N}{12}) . \]  

(2.8)

This should be \( O(N^2) \) if the one loop contributions of the dilaton and Liouville fields are to be ignored, so one sees the large \( N \) approximation fails when \( \phi_{cr} - \phi \lesssim O(1) \). The quantity \( \frac{N}{12} e^{2\phi} \) is therefore a useful measure of the coupling of quantum gravity. When it becomes of order one, quantum gravity effects become strong.

The following formula for the effective mass of a shock wave solution will be useful in the next section

\[ M_{\text{eff}} = \frac{1}{4\lambda} (1 - \frac{N}{12} e^{2\phi})^\frac{3}{2} e^{-2\phi} R . \]  

(2.9)

This agrees precisely with the Bondi mass of a shock solution along the infall line \( x^+ = x_0^+ \), even in regions of strong coupling. Also in the limit \( x^- \to -\infty \) this agrees with the total initial energy of the matter shock wave. After formation, i.e. for \( x^+ > x_0^+ \), \( M_{\text{eff}} \) will decrease as \( x^- \) increases corresponding to energy loss as the black hole evaporates. Therefore an observer moving along a line of constant \( \phi \) will see all the mass of the black hole radiated away if \( R \to 0 \) as \( x^+ \) increases.
3. Numerical Results For Original CGHS Model

It may readily be seen from the form of (2.7) that for fixed $x^+$ they reduce to ordinary differential equations for the variables $\partial_+ \rho$ and $\partial_+ \phi$. These may then be integrated from large negative $x^-$ in towards the singularity at $\frac{N}{12} e^{2\phi} = 1$. With $\partial_+ \rho$ and $\partial_+ \phi$ now known the solution may be evolved a step in the $x^+$ direction. This is of course a standard method in the solution of hyperbolic differential equations known as the method of characteristics. Here the characteristic lines are simply the $x^+$ and $x^-$ directions. Details of the precise numerical method used are given in the appendix.

The boundary conditions imposed are that $\rho$ and $\phi$ match onto the linear dilaton vacuum along the line $x^+ = x^+_0$. In order to specify a unique solution boundary conditions must also be imposed at past null infinity. In practice, the conditions are that $\partial_+ \phi$ and $\partial_+ \rho$ correspond to a finite mass classical black hole solution for $x^+ > x^+_0$ and at some large negative value of $x^-$ where the quantum effects are always negligible. Note that $\lambda$ may be scaled out of the problem so in the following $\lambda = 1$.

To check that the numerical algorithm was functioning correctly the classical shock solution was evolved and the position of the apparent horizon $\partial_+ \phi = 0$ was plotted in comparison to the exact value $x^- = -M$. This is shown in fig. 1 where $M = 50$. Excellent agreement is obtained.

The typical result of evolving the semiclassical equations for a large $N$ evaporating solution is shown in fig. 2. The parameters $M = 50$ and $N = 480$ are thought to be indicative of the generic case. The apparent horizon is seen to recede as the singularity approaches and then crosses it. The numerical integration breaks down near the singularity so the path of the singularity is represented by a line of large constant $R$. This line begins to recede to larger values of $x^-$ as $x^+$ increases signalling the appearance of a timelike naked singularity at $x^- = x^-_s$. The true path of the singularity lies at slightly larger values of $x^-$ than this line and becomes tangent to $x^- = x^-_s$ at the moment when the singularity becomes naked. After this point both the apparent horizon and the singularity recede beyond the region determined by propagation of the equations. If one follows a line of constant $x^-$ close to the line $x^- = x^-_s$ the curvature will increase to some maximum value and then decrease. As this line approaches $x^-_s$ the maximum value of the curvature seems to increase without bound.

As expected the curvature on the apparent horizon seems to increase without bound as shown in fig. 3. In fig. 4 the curvature measured by an observer who travels along a line
of constant $\phi$ is shown. Equation (2.9) relates this curvature to the effective mass of the black hole. This passes through zero becoming negative as the naked singularity continues radiating away to negative mass.

These results are qualitatively the same as the results of [10–12] in that a naked singularity appears. The semiclassical equations break down just before this singularity appears, by which time the mass of the black hole is of order the Planck mass.

4. Another Model: Reduction of the Einstein Action From 4d to 2d

The starting point for this model is the four dimensional Einstein action

\[
(4) S = \frac{1}{2\pi} \int d^4\sigma \sqrt{-g} (4) R .
\]  

Restricting to spherically symmetric field configurations allows one to write the four dimensional metric as

\[
(4) ds^2 = -e^{2\rho} dx^+ dx^- + e^{-2\phi} d\Omega^2 ,
\]

where the dilaton field $\phi$ now becomes part of the four dimensional metric. The four dimensional Ricci curvature scalar is then

\[
(4) R = 2e^{2\phi} + e^{-2\rho} (8 \partial_+ \partial_- \rho + 24 \partial_+ \phi \partial_- \phi - 16 \partial_+ \partial_- \phi) .
\]

Substituting this into (4.1) then gives the two dimensional dilaton gravity action

\[
(2) S = \frac{1}{2\pi} \int d^2\sigma \sqrt{-g} e^{-2\phi} \left( (2) R + 2(\nabla \phi)^2 + 2e^{2\phi} \right) .
\]

This model has been considered at the classical level before, for example in [13].

The static solutions of this classical action are of course the familiar Schwarzschild solutions. These are most easily investigated not in conformal gauge, but in the gauge

\[
(2) ds^2 = -h(r) dt^2 + \frac{1}{h(r)} dr^2 .
\]

Solving the equations of motion in this gauge, the most general solution up to translations in $r$ obeying $h(r) \to 1$ as $r \to \infty$ is

\[
\phi(r) = \phi_0 - \log(r)
\]

\[
h(r) = 1 - \frac{2M}{r} ,
\]
as expected. These solutions are periodic in imaginary time with period $8\pi M$, so have Hawking temperature $T = 1/8\pi M$ as in the four dimensional case.

If one chooses to couple this classical dilaton gravity model to matter which is generally covariant only in two dimensions the fluctuations in the matter fields may be integrated out in the same way as the CGHS model to give the effective action

$$S_{\text{eff}} = \frac{1}{\pi} \int dx^+ dx^- \left( e^{-2\phi} \left[ -2\partial_+ \partial_- \rho + 2\partial_+ \phi \partial_- \phi \right] - \frac{1}{2} e^{2\rho} \right)

- \frac{1}{2} \sum_{i=1}^{N} \partial_+ f \partial_- f + \frac{N}{12} \partial_+ \rho \partial_- \rho \right),

$$

where conformal gauge has been used

$$^{(2)}ds^2 = -e^{2\rho} dx^+ dx^- .

$$ (4.7)

Ideally, one would like to include matter fields generally covariant in four dimensions. Integrating out the fluctuations of the matter fields in four dimensions then leads to a trace anomaly of the stress energy tensor that includes $R^2$ type terms. The effective action must then include highly nonlocal terms and the problem seems to become intractable. In the following sections the effective action (4.7) will be studied as a two dimensional toy model of black hole evaporation with properties much closer to four dimensional black hole physics than the original CGHS model.

The equations of motion that follow from (4.7) are

$$\partial_+ \partial_- \phi = \partial_+ \phi \partial_- \phi + \frac{1}{1 - \frac{N}{2\pi} e^{2\phi}} \left( \partial_+ \phi \partial_- \phi + \frac{1}{4} e^{2(\rho + \phi)} \right)

\partial_+ \partial_- \rho = \partial_+ \partial_- \phi - \partial_+ \phi \partial_- \phi

- \frac{1}{2} \left( \partial_+ f \partial_- f + \frac{N}{12} \partial_+ \rho \partial_- \rho \right),

\partial_+ \partial_- f = 0 .

(4.9)

In addition there are the constraint equations

$$0 = T_{++} = e^{-2\phi} \left( 4\partial_+ \rho \partial_+ \phi - 2\partial_+^2 \phi + 2(\partial_+ \phi)^2 \right) + \frac{1}{2} (\partial_+ f)^2

- \frac{N}{12} \left( (\partial_+ \rho)^2 - \partial_+^2 \rho + t_+(x^+) \right)

0 = T_{--} = e^{-2\phi} \left( 4\partial_- \rho \partial_- \phi - 2\partial_-^2 \phi + 2(\partial_- \phi)^2 \right) + \frac{1}{2} (\partial_- f)^2

- \frac{N}{12} \left( (\partial_- \rho)^2 - \partial_-^2 \rho + t_-(x^-) \right) .

(4.10)
where \( t_+ \) and \( t_- \) are integration functions to be fixed by the boundary conditions. They arise from the nonlocality still present in the covariant form of the effective action.

As in the CGHS model, (4.9) degenerate at a critical value of \( \phi = \phi_{cr} = -\frac{1}{2} \log(N/24) \), where in general a curvature singularity will appear. Following similar arguments as lead to (2.8) one finds the large \( N \) approximation breaks down in this region where quantum gravity effects become strong.

5. Static Solutions of the New Model

If one is interested in finite mass static solutions it is natural to choose as the radial coordinate \( \sigma = \frac{1}{2}(x^+ - x^-) \). In these coordinates the event horizon will turn out to be at \( \sigma = -\infty \). The equations of motion in the static limit become

\[
\phi'' = (\phi')^2 + \frac{1}{(1 - \frac{N}{24}e^{2\phi})}((\phi')^2 + e^{2(\rho + \phi)})
\]

\[
\rho'' = \phi'' - (\phi')^2 ,
\]

and the constraint equations when \( f_i = 0 \) become

\[
e^{-2\phi}(\rho'\phi' - \frac{1}{2}\phi'' + \frac{1}{2}(\phi')^2) - \frac{N}{48}((\rho')^2 - \rho'' + t) = 0 .
\]

The vacuum corresponds to the solution

\[
\phi = -\log(\sigma), \quad \rho = 0 .
\]

Solving the linearized equations of motion about the vacuum leads to infinite mass solutions when \( t \neq 0 \), and when \( t = 0 \) finite mass solutions with asymptotic form

\[
\phi = -\log(\sigma) + \frac{2M}{\sigma} \log(\sigma) , \quad \rho = -\frac{M}{\sigma} .
\]

The results of integrating (5.1) with boundary conditions given by (5.4) are shown in fig. 5 and fig. 6. These are very similar to the static solutions of the CGHS model \( \textit{[7–9]} \). The coupling increases as \( \sigma \) decreases before bouncing off the region of critical coupling \( \phi_{cr} = -\frac{1}{2} \log(N/24) \), and then decreases to zero coupling as \( \sigma \to -\infty \). The curvature \( R = -2e^{-2\rho}\rho'' \) rises to a maximum near this bounce region before running off to \( -\infty \) as \( \sigma \to -\infty \). As \( M \to 0 \), \( \phi_{max} \to \phi_{cr} \), and the solution closely approaches the vacuum outside a region close to the origin.
Beyond the bounce region as $\sigma \to -\infty$ the $\exp(2(\rho+\phi))$ term in (5.4) may be neglected allowing the equations to be solved analytically

$$e^{-2\phi} = -\frac{N}{12} \rho + a\sigma + b$$

$$e^{-\phi} \sqrt{e^{-2\phi} - \frac{N}{24} - \frac{N}{24} \log\left(\sqrt{e^{-2\phi} - \frac{N}{24}} + e^{-\phi}\right)} = -a\sigma + c ,$$

where $a, b$ and $c$ are constants. The asymptotic form of these solutions as $\sigma \to -\infty$ is

$$e^{-2\phi} \sim -a\sigma + \frac{N}{48} \log(-a\sigma)$$

$$ds^2 \sim e^{48a\sigma/N} \sqrt{-a\sigma} (-d\tau^2 + d\sigma^2) .$$

It is clear from the form of the metric that $\sigma = -\infty$ is an event horizon at finite proper distance and the curvature becomes singular there.

Another interesting class of static solutions are those with regular event horizons. As expected these will correspond to infinite mass black holes supported by an incoming flux of radiation. In this case the most convenient radial coordinate seems to be $r^2 = -x^+ x^-$, which ensures that the horizon is at $r = 0$. The equations of motion in these coordinates become

$$\phi'' + \frac{1}{r} \phi' = (\phi')^2 + \frac{1}{1 - \frac{N}{24} e^{2\phi}} ((\phi')^2 - e^{2(\rho+\phi)})$$

$$\rho'' + \frac{1}{r} \rho' = \phi'' + \frac{1}{r} \phi' - (\phi')^2 ,$$

and the constraint equations become

$$4\rho' \phi' - 2\phi'' + \frac{2}{r} \phi' + 2(\phi')^2 = \frac{N}{12} e^{2\phi} ((\rho')^2 - \rho'' + \frac{1}{r} \rho' + \tilde{t}) .$$

The boundary conditions for a regular horizon are therefore

$$\phi'(0) = 0, \quad \rho'(0) = 0, \quad \tilde{t} = 0, \quad \rho(0) = 0, \quad \phi(0) = \phi_h ,$$

where $\rho(0)$ has been set to zero by a scale transformation. The solutions are parameterized by the value of $\phi$ on the horizon. Only when $\phi_h < \phi_{cr}$ do the solutions approach the vacuum (5.3) as $r \to \infty$. The coupling $e^{2\phi}$ for a typical solution is shown in fig. 7.

As $\phi_h \to \phi_{cr}$ the solutions approach a limiting form as can be seen from fig. 8 where $e^{-\rho}$ is plotted for various values of $\phi_h$. These solutions match onto the vacuum solutions of (5.7) as $r \to \infty$ which are

$$e^{-\phi} = a - b \log r$$

$$\rho = \log(b/r) ,$$

as $\phi_h \to \phi_{cr}$.
where \(a\) and \(b\) are constants. The value of \(b\) for the limiting form is \(b_0 \approx 6.3\).

The evaporation of a large mass black hole may be well approximated by a succession of these static solutions with regular horizons. As the mass is radiated away \(\phi_h\) will increase towards the critical value \(\phi_{cr}\). If one expects no naked singularity to appear then this limiting form should approach zero temperature. In fact, this is not true, as may be seen by a calculation of the Hawking flux. In order that the future horizon be regular one must demand that \(t_-(x^-) = 0\). This is analogous to the condition \(\tilde{t} = 0\) for the static regular horizon solution. Transforming (5.10) to asymptotically Minkowskian coordinates \(\sigma^+\) and \(\sigma^-\) via

\[
x^+ = e^{\sigma^+/b_0}, \quad x^- = -e^{-\sigma^-/b_0},
\]
leads to

\[
t_-(\sigma^-) = \frac{N}{24}(t_-(x^-) + D_{\sigma^-}^S(x^-))\left(\frac{\partial \sigma^-}{\partial x^-}\right)^{-2}
= \frac{N}{48b_0^2},
\]
where

\[
D_{\sigma}^S(x) = \frac{\sigma'''}{\sigma'} - \frac{3}{2} \left(\frac{\sigma''}{\sigma'}\right)^2
\]
is the Schwarzian derivative and \(\sigma'\) denotes \(\partial \sigma / \partial x\). Thus the limiting form is unstable with a finite outgoing flux. This strongly suggests that a naked singularity will form. In the following section this will be confirmed by direct numerical calculation for the shockwave geometry.

6. Black Hole Evaporation

In this section the equations (4.9) will be solved numerically for incoming matter shock waves which form black holes, and then subsequently evaporate. The same algorithm is used as in Section 3.

Now the boundary conditions are that \(\rho\) and \(\phi\) match onto the vacuum (5.3) along the null line \(x^+ = 1\). Again at past null infinity (or more practically large negative \(x^-\)) the boundary condition is that \(\partial_+ \phi\) and \(\partial_+ \rho\) correspond to a classical black hole for \(x^+ > 1\). Translated into conformal gauge, this condition is

\[
\partial_+ \phi \sim -\frac{1}{x^+ - x^-} + \frac{4M}{(x^+ - x^-)^2} \left(1 - \log\left(\frac{1}{2}(x^+ - x^-)\right)\right),
\]

\[
\partial_+ \rho \sim \frac{2M}{(x^+ - x^-)^2}.
\]
The boundary condition is imposed in a region where the quantum effects are always negligible so one may be sure there is no extra incoming energy flux other than the finite energy of the initial shock wave.

From the usual adiabatic analysis the lifetime of these black holes \( t \sim M^3 \). This is observed in these numerical studies as a strong dependence of the lifetime on the mass, in contrast to the CGHS case where \( t \sim M \). Since one is interested in the endpoint of the evaporation this effectively restricts one to the study of initial masses of order the Planck mass.

In fig. 9 the paths of the singularity and the apparent horizon \( \partial_+ \phi = 0 \) are displayed. The set of parameters chosen (\( M = 3 \) and \( N = 2400 \)) are believed to represent the generic case. These collide after a finite proper time signaling the formation of a naked singularity. The path of the singularity is represented by a line of large constant \( R \). The true line of singularity lies at slightly larger \( x^- \), and becomes tangent to a line of constant \( x^- \) at the moment when the singularity becomes naked.

In fig. 10 the curvature on the apparent horizon is shown, which appears to become singular for finite \( x^+ \). The curvature decreases along a line of constant \( \phi \) as the mass of the black hole is radiated away, as shown in fig. 11.

7. Conclusions

In this paper, black hole evaporation in the context of two quite different semiclassical two dimensional dilaton gravity models has been studied. The second of these is obtained by the dimensional reduction of the four dimensional Einstein equations so is expected to closely reflect physics in four dimensions. The formation of a naked singularity appears to be a rather generic feature of these semiclassical calculations. This is in agreement with the results of the exactly solved modified models [10, 12]. However, this seems to disagree with recent results of Hawking and Stewart [14] who, in the context of the CGHS-type models, claim the apparent horizon persists, and a thunderbolt singularity forms. This kind of endpoint is similar to a stable remnant, but the singularity spreads out to infinity rather than remaining in a bounded region. This disagreement may arise because in [14] the boundary condition on a line of constant \( x^- \) is imposed in a region where quantum effects are likely to be strong. One might expect this would be equivalent to sending in some non-zero energy flux over a long period of time balancing the outgoing flux and preventing the naked singularity from appearing. In this paper the boundary condition is
imposed in a region where quantum effects are negligible so one can be certain the matter shock wave carries only a finite total energy.

In any case, the semiclassical approximation only holds until the remaining mass of the black hole is of order the Planck mass. One might hope that higher order quantum corrections will tame the naked singularity and the residual energy might then be released as some kind of gamma ray burst (a rather scaled down version of the big bang naked singularity). Due to the energy scale involved it is unlikely this could be detected unless it occurred in our close proximity. If one takes the emergence of a naked singularity seriously it is natural to ask whether naked singularities appear as nonperturbative solutions of string theory. If this is true it may make string theory unpredictable and require the introduction of a density matrix description as in [3].

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Appendix A. Numerical Algorithm

The nonlinear partial differential equations to be solved take the form

\[ \partial_+ \partial_- u = f(u, \partial_- u, \partial_+ u) \]  \hspace{1cm} (A.1)

with boundary conditions imposed along the lines \( x^+ = x^+_0 \) and \( x^- = x^-_0 \). For fixed \( x^+ \) (A.1) may be regarded as a first order ordinary differential equation for \( \partial_+ u \). A fourth order Runge-Kutta routine with adaptive stepsize in the \( x^- \) direction is used to compute \( \partial_+ u \). The efficiency of this method relies on \( u \) being known for arbitrary \( x^- \). This is accomplished using a fourth order rational function interpolation routine. The additional error introduced by this interpolation is more than compensated by the improved efficiency of the adaptive stepsize method. The derivatives \( \partial_- u \) are computed using a second order backward difference method.

Having obtained \( \partial_+ u \) as a function of \( x^- \) again a fourth order Runge-Kutta method with adaptive stepsize is used to evolve the solution a step in the \( x^+ \) direction. Repeating this procedure allows to the solution to be evolved out to large \( x^+ \).
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Figure Captions

Fig. 1. Position of the apparent horizon for a classical shockwave solution in the original CGHS model. The dashed line is the numerical result, the solid line is the exact result.

Fig. 2. Position of the apparent horizon and the singularity for the original CGHS model. The initial mass of the black hole is \( M = 50 \) and \( N = 480 \). The apparent horizon recedes and eventually crosses the singularity.

Fig. 3. Curvature on the apparent horizon for the original CGHS model, \( M = 50 \) and \( N = 480 \).

Fig. 4. Curvature along a line of constant \( \phi \) for the original CGHS model, \( M = 50 \) and \( N = 480 \).

Fig. 5. Coupling \( \frac{N}{24}e^{2\phi} \) versus the radial coordinate for static solutions of the model obtained from the 4d Einstein equations with mass \( M = 8, 10, 15 \), and \( N = 480 \).

Fig. 6. Curvature versus the radial coordinate for static solutions of the model obtained from the 4d Einstein equations with mass \( M = 8, 10, 15 \), and \( N = 480 \).

Fig. 7. The coupling \( \frac{N}{24}e^{2\phi} \) versus the radial coordinate for a regular horizon static solution of the model obtained from the 4d Einstein equations.

Fig. 8. \( e^{-\rho} \) versus the radial coordinate for a regular horizon static solution of the model obtained from the 4d Einstein equations with \( \phi_h = -2.67, -1.70, -1.52 \). Here \( \phi_{cr} = -1.50 \).

Fig. 9. Paths of the singularity and the apparent horizon for the model obtained from the 4d Einstein equations, with \( M = 3 \) and \( N = 2400 \). The apparent horizon recedes and eventually crosses the singularity.

Fig. 10. Curvature on the apparent horizon for the model obtained from the 4d Einstein equations, with \( M = 3 \) and \( N = 2400 \).

Fig. 11. Curvature along a line of constant \( \phi \) for the model obtained from the 4d Einstein equations, with \( M = 3 \) and \( N = 2400 \).