On integration of multidimensional generalizations of classical $C$- and $S$-integrable nonlinear partial differential equations

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Abstract

We develop a new integration technique allowing one to construct a rich manifold of particular solutions to multidimensional generalizations of classical $C$- and $S$-integrable Partial Differential Equations (PDEs). Generalizations of (1+1)-dimensional $C$-integrable and (2+1)-dimensional $S$-integrable N-wave equations are derived among examples. Examples of multidimensional second order PDEs are represented as well.

1 Introduction

After the original work [1], the integrability technique was intensively developing. At present time, it covers large class of nonlinear Partial Differential Equations (PDEs) applicable in different branches of physics and mathematics. One should mention a few most popular integration methods, such as (a) the linearization by some direct substitution, for instance, by the Hopf-Cole substitution [2] and by its multidimensional generalization [3] (appropriate nonlinear PDEs are referred to as $C$-integrable PDEs, [4, 5, 6, 7, 8, 9]); (b) the method of characteristics [10] and its matrix generalization [11, 12] ($Ch$-integrable PDEs); (c) the inverse spectral transform method [1, 13, 14], the dressing method [15, 16, 17, 18, 19] and Sato approach [20] ($S$-integrable PDEs). $S$-integrable nonlinear PDEs are most applicable in physics. We recall a few types of these equations: the soliton equations in (1+1)-dimensions, such as the Korteweg-de Vries (KdV) [1, 21] and the Nonlinear Shrödinger (NLS) [22] equations; the soliton (2+1)-dimensional equations, such as the Kadomtsev-Petviashvili (KP) [23] and the Davey-Stewartson (DS) [24] equations; the self-dual type PDEs having instanton solutions, like the Self-dual Yang-Mills equation (SDYM); PDEs associated with commuting vector fields [25, 26, 27, 28, 29, 30, 31]. Nevertheless, the class of completely integrable nonlinear PDEs is very restrictive. Thus, extensions of the integrability technique on new types of nonlinear PDEs is an actual problem.

In this paper we suggest a new version of the dressing method allowing one to construct a rich manifold of particular solutions to new class of nonlinear PDEs in any dimension. The novelty of this class of PDEs is that integrability technique does not generate commuting flows to them in usual sense, unlike all methods mentioned above, where any nonlinear PDE appears together with the commuting hierarchy of nonlinear PDEs. We show that our algorithm may provide arbitrary functions of $m-1$ independent variables in the solution space to $m$-dimensional PDEs, which suggests us to consider these PDEs as candidates for complete integrable PDEs. However we do not represent rigorous justification of complete integrability.

To anticipate, we give simple examples of nonlinear PDEs for the matrix field $V$, derived in this paper.
1. The system of first order $D$-dimensional PDEs,

\[ \sum_{m=1}^{D} (V_{t_m} + V C^{(m)} V) B^{(m)} = 0, \]  

where $B^{(m)}$ and $C^{(m)}$ are some constant matrices. This is multidimensional generalization of $C$-integrable (1+1)-dimensional nonlinear $N$-wave equation.

Simple example of this equation corresponds to $D = 2$, $t_1 = x$, $t_2 = y$, $B^{(1)}_2 = B^{(2)}_1 = 0$,

\[ V = \begin{pmatrix} u & q \\ p & v \end{pmatrix}. \]

Then eq.(1) reads

\[ \begin{align*}
&u_x + u^2 C^{(1)}_1 + pq C^{(1)}_2 = 0, \\
&p_x + p(uC^{(1)}_1 + vC^{(1)}_2) = 0, \\
&v_y + v^2 C^{(2)}_2 + pq C^{(2)}_1 = 0, \\
&q_y + q(uC^{(2)}_1 + vC^{(2)}_2) = 0,
\end{align*} \]

which reduces to the Liouville equation

\[ f_{xy} = C^{(1)}_2 C^{(2)}_1 \exp(2f), \]

if $C^{(1)}_1 = C^{(2)}_2 = 0$, $q = p = e^f$. Here and below, $B^{(m)}_{\alpha}$ and $C^{(m)}_{\alpha}$ mean the $\alpha$th diagonal elements of the diagonal matrices $B^{(m)}$ and $C^{(m)}$ respectively.

2. The first order $D_1 D_2$-dimensional PDEs

\[ \sum_{m_1=1}^{D_1} \sum_{m_2=1}^{D_2} L^{(m_1)} (V_{t_{m_1 m_2}} + V C^{(m_1 m_2)} V) R^{(m_2)} = 0, \]

where $L^{(m_1)}$, $C^{(m_1 m_2)}$ and $R^{(m_2)}$ are some constant matrices. The simple example corresponds to $D_1 = D_2 = 2$, $L^{(m_1)} = R^{(m_2)} = 0$ if $\alpha \neq m_1$ and $\beta \neq m_2$ respectively, $t_{11} = x$, $t_{12} = y$, $t_{21} = z$, $t_{22} = t$. Let $V$ be given by eq.(2), then eq.(5) yields:

\[ \begin{align*}
&u_x + u^2 C^{(11)}_1 + pq C^{(11)}_2 = 0, \\
&p_x + p(uC^{(21)}_1 + vC^{(21)}_2) = 0, \\
&v_y + v^2 C^{(22)}_2 + pq C^{(22)}_1 = 0, \\
&q_y + q(uC^{(12)}_1 + vC^{(12)}_2) = 0,
\end{align*} \]

which reduces to the following four-dimensional generalization of Liouville equation

\[ g_{xy} = C^{(11)}_2 C^{(12)}_1 e^{f+g}, \quad f_{zt} = C^{(22)}_1 C^{(21)}_2 e^{f+g} \]

if $C^{(11)}_1 = C^{(22)}_2 = C^{(12)}_2 = C^{(21)}_1 = 0$, $p = \exp f$, $q = \exp g$. 

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In particular, eq. (5) reduces to the following $D_0(D_0 - 1)/2$-dimensional PDE, $D_1 = D_2 = D_0$

$$\sum_{m_2=1}^{D_0} \sum_{m_1=1}^{D_0} \left[ i \left( L^{(m_1)} V_{\tau_{m_1 m_2}} L^{(m_2)} - L^{(m_2)} V_{\tau_{m_1 m_2}} L^{(m_1)} \right) + \right.$$\n$$L^{(m_1)} V C^{(m_1 m_2)} V L^{(m_2)} - L^{(m_2)} V C^{(m_1 m_2)} V L^{(m_1)} \right] = 0,$$

where $C^{(m_1 m_2)} = -C^{(m_2 m_1)}$; $C^{(m_1 m_2)}$ and $L^{(m_1)}$ are constant diagonal matrices and $+$ means hermitian conjugate. This equation has a physical meaning describing interaction of $n_0(n_0-1)/2$ waves if $V$ is $n_0 \times n_0$ matrix. This is multidimensional generalization of $S$-integrable $(2+1)$-dimensional nonlinear $N$-wave equation.

3. The system of second order $D$-dimensional PDEs,

$$\sum_{m,n=1}^{D} \left( V_{t_m t_n} + (V C^{(n)} V)_{t_m} + V C^{(m)} V_{t_n} + V C^{(m)} V C^{(n)} V \right) B^{(mn)} = 0, \quad (9)$$

where $B^{(mn)}$ are constant diagonal matrices. The scalar version of this equation reads

$$\sum_{m,n=1}^{D} V_{t_m t_n} B^{(mn)} + V \sum_{m=1}^{D} V_{t_m} \hat{C}^{(m)} + V^2 \hat{C} = 0, \quad (10)$$

$$\hat{C}^{(m)} = 2 \sum_{n=1}^{D} C^{(n)} B^{(mn)} + \sum_{m=1}^{D} C^{(m)} B^{(mn)}, \quad \hat{C} = \sum_{m,n=1}^{D} C^{(m)} C^{(n)} B^{(mn)}.$$

Note that not all constant coefficients may be arbitrary in the above nonlinear PDEs. Constructing particular solutions, we will reveal some relations among coefficients.

Particular examples of nonlinear PDEs, such as eqs. (3,6,8,10), will not be considered in this paper. Instead of this, we concentrate on the dressing algorithm allowing one to derive general equations, such as eqs. (1,3,9), and study the richness of available solution space for them.

4. The structure of this paper is the following. We will derive generalization of the classical $C$-integrable first and second order nonlinear PDEs in Sec.2 with eqs. (1) and (9) as particular examples. Richness of the solution space to eq. (1) will be discussed briefly and explicite particular solutions to this equation with $D = 3$ will be given. Generalization of the classical $S$-integrable nonlinear PDEs will be consider in Sec.3 with eqs. (5) and (8) as particular examples. Richness of the solution space to eq. (5) will be discussed briefly and explicite particular solutions to this equation with $D_1 = D_2 = 2$ will be given. Conclusions will be represented in Sec.4.

2 Generalization of $C$-integrable nonlinear PDEs

2.1 Starting equations

Our algorithm is based on the following integral equation

$$P(\mu) \ast \chi(\mu, \lambda; t) = W(\mu; t) \ast \chi(\mu, \lambda; t) + W(\lambda; t) \equiv W(\mu; t) \ast \left( \chi(\mu, \lambda; t) + I_1(\mu, \lambda) \right) \quad (11)$$
where \( P(\mu), \chi(\mu, \lambda; t), W(\lambda; t) \) are \( n_0 \times n_0 \) matrix functions of arguments, \( t_i \) are independent variables of nonlinear PDEs, \( t = (t_1, \ldots, t_D) \), \( \text{rank}(P) = n_0, \lambda, \mu, \nu \) are complex parameters. Here \(*\) means the integral operator defined for any two functions \( f(\mu) \) and \( g(\mu) \) as follows:

\[
f(\mu) * g(\mu) = \int f(\mu)g(\mu)d\Omega_1(\mu),
\]

and \( \Omega_1(\mu) \) is some measure. We also introduce unit \( \mathcal{I}_1(\lambda, \mu) \) and inverse \( f^{-1}(\lambda, \mu) \) operators as follows:

\[
\begin{align*}
  f(\lambda, \nu) * \mathcal{I}_1(\nu, \mu) &= \mathcal{I}_1(\lambda, \nu) * f(\nu, \mu) = f(\lambda, \mu), \\
  f(\lambda, \nu) * f^{-1}(\nu, \mu) &= f^{-1}(\lambda, \nu) * f(\nu, \mu) = \mathcal{I}_1(\lambda, \mu).
\end{align*}
\]

We introduce parameters \( t_i \) through the function \( \chi(\lambda, \mu; t) \), which is defined as a solution to the following system of linear equations:

\[
\chi_{t_m}(\lambda, \mu; t) = A^{(m)}(\lambda, \nu) \chi(\nu, \mu; t) + \tilde{A}^{(m)}(\lambda)P(\nu) \chi(\nu, \mu; t) + A^{(m)}(\lambda, \mu), \quad m = 1, \ldots, D,
\]

where \( A^{(m)}(\lambda, \nu) \) and \( \tilde{A}^{(m)}(\lambda) \) are \( n_0 \times n_0 \) matrix functions of arguments. An important requirement to eq. (11) is that it must be uniquely solvable with respect to \( W(\lambda; t) \), i.e. operator \(*(\chi(\mu, \lambda; t) + \mathcal{I}_1(\mu, \lambda))\) must be invertible.

Matrices \( A^{(m)} \) and \( \tilde{A}^{(m)} \) may not be arbitrary. They have to provide compatibility of the system (15). We will show that there are two different methods which provide this compatibility. The first one (Sec.2.2) yields classical \( \mathcal{C} \)-integrable nonlinear PDEs, linearizable by the multidimensional version of the Hopf-Cole transformation [2], while the second method (Sec.2.3) yields a new type of nonlinear PDEs whose complete integrability is not clarified yet. However, our algorithm supplies, at least, a rich manifold of particular solutions to these PDEs.

The following theorem is valid for both cases.

**Theorem 2.1.** Matrix function \( W(\lambda; t) \), obtained as a solution to the integral equation (11) with \( \chi \) defined by eq. (15) satisfies the following system of compatible linear equations:

\[
\begin{align*}
  E^{(m)}(\lambda; t) := W_{t_m}(\lambda; t) + V^{(m)}(t)W(\lambda; t) + (W(\mu; t) - P(\mu)) * A^{(m)}(\mu, \lambda) &= 0, \\
  V^{(m)}(t) &= (W(\mu; t) - P(\mu)) * A^{(m)}(\mu), \quad m = 1, \ldots, D.
\end{align*}
\]

**Proof:** To derive eq. (16), we differentiate eq. (11) with respect to \( t_m \). Then, in view of eq. (15), one gets the following equation:

\[
E^{(m)}(\mu; t) * (\chi(\mu, \lambda; t) + \mathcal{I}_1(\mu, \lambda)) = 0
\]

where \( E^{(m)} \) is defined in eq. (16). Since operator \(*(\chi(\mu, \lambda; t) + \mathcal{I}_1(\mu, \lambda))\) is invertible, eq. (18) yields: \( E^{(m)}(\mu; t) = 0 \), which coincides with eq. (16). \(\square\)

**Remark:** Following the classical integrability theory, we refer to eq. (16) as the linear equation for the function \( W(\lambda; t) \). However this is not correct, because functions \( V^{(m)}(t) \) are defined in terms of \( W(\lambda; t) \) by eq. (17). Thus, strictly speaking, eq. (16) is a nonlinear equation for \( W(\lambda; t) \).

System (16) is overdetermined system of compatible linear equations with potentials \( V^{(m)}(t) \) in analogy with the classical integrability theory. In the classical theory, nonlinear PDEs for potentials \( V^{(m)} \) may be obtained as compatibility conditions for the appropriate overdetermined linear system. However, this approach does not work in our case because of the last term in eqs. (16). Instead of this, we suggest a different method of derivation of nonlinear PDEs, see Secs. 2.2 and 2.3.

Now we analyze two methods that provide the compatibility of system (15) and derive nonlinear PDEs associated with each of them.
2.2 First method: classical $C$-integrable nonlinear PDEs

In this subsection we write the compatibility condition of eqs.\ref{15} as follows:

\[
\left( A^{(m)}(\lambda, \nu) + \bar{A}^{(m)}(\lambda)P(\nu) \right) \ast \chi_n(\nu, \mu) = \left( A^{(n)}(\lambda, \nu) + \bar{A}^{(n)}(\lambda)P(\nu) \right) \ast \chi_m(\nu, \mu), \quad \forall n, m. (19)
\]

Substituting eqs.\ref{15} for derivatives of $\chi$ into eq.\ref{19} we obtain the following equation:

\[
\begin{align*}
(L^{(m)} \ast L^{(n)} - L^{(n)} \ast L^{(m)}) \ast \chi + L^{(m)} \ast A^{(n)} - L^{(n)} \ast A^{(m)} &= 0, \quad (20) \\
L^{(m)}(\lambda, \mu) &= A^{(m)}(\lambda, \mu) + \bar{A}^{(m)}(\lambda)P(\mu), \quad .
\end{align*}
\]

Let eq.\ref{20} be satisfied for any function $\chi(\lambda, \mu; t)$ (which is a solution to the system \ref{15}). Then eq.\ref{20} is equivalent to two following equations relating matrix functions $A^{(m)}$, $\bar{A}^{(m)}$ and $P$:

\[
\begin{align*}
L^{(m)} \ast A^{(n)} - L^{(n)} \ast A^{(m)} &= 0, \quad (21) \\
L^{(m)} \ast L^{(n)} - L^{(n)} \ast L^{(m)} &= 0 \quad \text{eq.\ref{21}} \\
\left( L^{(m)}(\lambda, \nu) \ast \bar{A}^{(n)}(\nu) - L^{(n)}(\lambda, \nu) \ast \bar{A}^{(m)}(\nu) \right) P(\mu) &= 0 \quad \text{eq.\ref{22}}.
\end{align*}
\]

Since rank($P$) = $n_0$, eq.\ref{22} is equivalent to the following one:

\[
\begin{align*}
L^{(m)} \ast \bar{A}^{(n)} - L^{(n)} \ast \bar{A}^{(m)} &= 0 \quad \Rightarrow \\
A^{(m)} \ast \bar{A}^{(n)} - A^{(n)} \ast \bar{A}^{(m)} &= \bar{A}^{(n)}P \ast \bar{A}^{(m)} - \bar{A}^{(m)}P \ast \bar{A}^{(n)}, \quad n, m = 1, \ldots, D. \quad \text{eq.\ref{23}}
\end{align*}
\]

Eqs.\ref{21} and \ref{23} represent two constraints on the functions $A^{(m)}(\lambda, \mu)$ and $\bar{A}^{(m)}(\lambda)$.

Now we have everything for derivation of nonlinear PDEs for the fields $V^{(m)}(t)$. For this purpose, let us consider the following combination of eqs.\ref{16}:

\[
E^{(m)}(\lambda; t) \ast \bar{A}^{(n)}(\lambda) - E^{(n)}(\lambda; t) \ast \bar{A}^{(m)}(\lambda), \quad (24)
\]

which yields, in view of eq.\ref{23}:

\[
V^{(n)}_{tm}(t) - V^{(n)}_{tn}(t) + V^{(m)}(t)V^{(n)}(t) - V^{(n)}(t)V^{(m)}(t) = 0. \quad (25)
\]

This equation is known to be linearizable by the Hopf-Cole transformation [2] [3]:

\[
E^{(n)}_H := \Psi^{(n)} = \Psi(\lambda) \frac{V^{(n)}(t)}{
\}

where $\Psi(t)$ is an arbitrary $n_0 \times n_0$ matrix function of all variables $t_i$. The presence of an arbitrary function $\Psi(t)$ is associated with the fact that the system of nonlinear PDEs \ref{25} is not complete. One needs one more equation relating $V^{(n)}$, $n = 1, \ldots, D$.

To derive this additional equation we introduce either additional relations among $A^{(m)}$ and $\bar{A}^{(m)}$ in our algorithm based on eq.\ref{11} or an additional linear PDE for $\Psi$ in the classical algorithm based on eq.\ref{20} [3].

For instance, let

\[
\sum_{m=1}^{D} A^{(m)}(\lambda, \nu) \ast \bar{A}^{(n)}(\nu)B^{(m)} = - \sum_{m=1}^{D} \bar{A}^{(m)}(\lambda)P(\nu) \ast \bar{A}^{(n)}(\nu)B^{(m)} \quad (27)
\]
where \( B^{(m)} \) are \( n_0 \times n_0 \) arbitrary constant matrices. Then both the combination of eqs.(16), \( \sum_{m=1}^{D}(E^{(n)})B^{(m)} \), and the appropriate combination of eqs.(26), \( \Psi^{-1}\sum_{m=1}^{D}(F^{(n)}_{H})t_mB^{(m)} \), yield the same nonlinear equation for \( V^{(n)} \):

\[
\sum_{m=1}^{D}\left(V_{tm}^{(n)}+V^{(n)}V^{(m)}\right)B^{(m)}=0, \quad n = 1, \ldots, D. \tag{29}
\]

This \( N \)-wave type equation is supplemented by constraints \([25],[3]\).

In particular, introducing reduction \( V^{(m)}(t) = V(t)C^{(m)} \) (where \( C^{(m)} \) are \( n_0 \times n_0 \) constant matrices) we reduce the system \([25]\) into the following one:

\[
V_{tm}C^{(n)}-V_{tn}C^{(m)}+VC^{(m)}VC^{(n)}-VC^{(n)}VC^{(m)}=0, \quad \forall n,m, \tag{30}
\]

which is a \((1+1)\)-dimensional hierarchy of commuting \( C \)-integrable \( N \)-wave equations.

Higher order nonlinear PDEs may be obtained in a similar way introducing appropriate equations instead of eq.\([27]\) and/or eq.\([28]\).

\( C \)-integrable PDEs will not be considered in this paper.

### 2.3 Second method: new class of nonlinear PDEs

In this section we represent another way to provide the compatibility of eqs.\([15]\). As a result we obtain a new class of nonlinear PDEs together with the rich manifold of particular solutions. In particular, there are solutions in the form of rational functions of exponents.

Let us use the following representation of \( A^{(m)}(\lambda, \mu) \) and \( \tilde{A}^{(m)}(\lambda) \):

\[
A^{(m)}(\lambda, \mu) = \alpha^{(m)}(\lambda, \nu) \ast \beta^{(m)}(\nu, \mu), \quad \tilde{A}^{(m)}(\lambda) = \alpha^{(m)}(\lambda, \nu) \ast \tilde{\beta}^{(m)}(\nu), \tag{31}
\]

where \( \alpha^{(m)}(\lambda, \nu) \), \( \beta^{(m)}(\nu, \mu) \) and \( \tilde{\beta}^{(m)}(\nu) \) are \( n_0 \times n_0 \) matrix functions of arguments. Here we introduce one more integral operator \( \ast \) defined for any two functions \( f(\mu) \) and \( g(\mu) \) as follows:

\[
f(\mu) \ast g(\mu) \equiv \int f(\mu)g(\mu)d\Omega_{2}(\mu), \tag{32}
\]

where \( \Omega_{2}(\mu) \) is some measure. We also introduce unit \( \mathcal{I}_{2}(\lambda, \mu) \) and inverse \( f^{-1}(\lambda, \mu) \) operators as follows:

\[
f(\lambda, \nu) \ast \mathcal{I}_{2}(\nu, \mu) = \mathcal{I}_{2}(\lambda, \nu) \ast f(\nu, \mu) = f(\lambda, \mu), \tag{33}
\]

\[
f(\lambda, \nu) \ast f^{-1}(\nu, \mu) = f^{-1}(\lambda, \nu) \ast f(\nu, \mu) = \mathcal{I}_{2}(\lambda, \mu).
\]

Now let us write eq.\([15]\) using representation \([31]\) in the following form:

\[
\chi_{tm}(\lambda, \mu; t) = \alpha^{(m)}(\lambda, \nu) \ast \xi^{(m)}(\nu, \mu; t). \tag{34}
\]

Here

\[
\xi^{(m)}(\lambda, \mu; t) = \left(\beta^{(m)}(\lambda, \nu) + \tilde{\beta}^{(m)}(\lambda)\right)P(\nu) \ast \chi(\nu, \mu; t) + \beta^{(m)}(\lambda, \mu), \quad m = 1, \ldots, D. \tag{35}
\]
It is obvious that the compatibility condition of the system (15) is equivalent to the compatibility condition of the system (34) which reads

\[ \alpha^{(m)}(\lambda, \nu) \star \xi_{m}^{(m)}(\nu, \mu; t) = \alpha^{(n)}(\lambda, \nu) \star \xi_{m}^{(n)}(\nu, \mu; t), \quad \forall \, n, m \]  

instead of eq.(19). To satisfy this condition we assume the following relations between \( \xi \) and \( \xi^{(1)} \):

\[ \xi^{(m)}(\lambda, \mu; t) = \eta^{(m)}(\lambda, \nu) \star \xi^{(1)}(\nu, \mu; t), \quad m > 1, \quad \eta^{(1)}(\lambda, \mu) = \mathcal{I}_2(\lambda, \mu), \]  

where \( \eta^{(m)}(\lambda, \mu) \) are some \( n_0 \times n_0 \) matrix functions, which will be specified below. We also define \( t \)-dependence of \( \xi^{(1)} \) as follows:

\[ \xi^{(1)}_{m}(\lambda, \mu; t) = T^{(m)}(\lambda, \nu) \star \xi^{(1)}(\nu, \mu; t), \]  

where \( T^{(m)}(\lambda, \nu) \) are some \( n_0 \times n_0 \) matrix functions. Substituting eqs.(37) and (38) into eq.(36) we obtain the following representation for \( \alpha^{(m)}, m > 1 \):

\[ \alpha^{(m)}(\lambda, \mu) = \alpha^{(1)}(\lambda, \nu) \star T^{(m)}(\nu, \tilde{\nu}) \star (T^{(1)})^{-1}(\tilde{\nu}, \mu) \star (\eta^{(m)})^{-1}(\mu, \tilde{\mu}). \]  

In turn, the compatibility condition of eqs.(38) requires

\[ T^{(m)} \star T^{(n)} - T^{(n)} \star T^{(m)} = 0, \quad \Rightarrow \]  

\[ T^{(m)}(\lambda, \mu) = \left( T(\lambda, \nu) T^{(m)}(\nu) \right) \star T^{-1}(\nu, \mu), \]  

\[ [\tau^{(m)}(\nu), \tau^{(n)}(\nu)] = 0, \quad \forall \, n, m, \]  

where \( T(\lambda, \mu) \) and \( \tau^{(m)}(\mu) \) are some \( n_0 \times n_0 \) matrix functions of arguments. Thus compatibility condition of the system (34) generates eqs. (37)-(41).

Now, integrating eq.(38) with \( m = 1 \) we obtain:

\[ \xi^{(1)}(\lambda, \mu; t) = \left( T(\lambda, \nu)e^{\sum_{i=1}^{D} \tau^{(1)}(\nu_{i}) t_{i}} \right) \star T^{-1}(\nu, \tilde{\nu}) \star \xi_{0}(\tilde{\nu}, \mu). \]  

Finally, integrating eq.(41) with \( m = 1 \) we derive the following explicite formula for \( \chi(\lambda, \mu; t) \):

\[ \chi(\lambda, \mu; t) = \alpha^{(1)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star \xi^{(1)}(\tilde{\nu}, \mu; t) + \chi_{0}(\lambda, \mu), \]  

where \( \chi_{0}(\lambda, \mu) \) is \( n_0 \times n_0 \) matrix integration constant.

It is convenient to rewrite expressions for \( A^{(m)} \) and \( \tilde{A}^{(m)} \) (see eqs.(31)) using eq.(39) as follows:

\[ A^{(m)}(\lambda, \mu) = \alpha^{(1)}(\lambda, \nu) \star \Gamma^{(m)}(\nu, \mu), \]  

\[ \tilde{A}^{(m)}(\lambda) = \alpha^{(1)}(\lambda, \nu) \star \tilde{\Gamma}^{(m)}(\nu), \]  

\[ \Gamma^{(m)}(\lambda, \mu) = T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star (\eta^{(m)})^{-1}(\tilde{\nu}, \tilde{\mu}) \star \beta^{(m)}(\tilde{\mu}, \mu), \]  

\[ \tilde{\Gamma}^{(m)}(\lambda) = T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star (\eta^{(m)})^{-1}(\tilde{\nu}, \tilde{\mu}) \star \tilde{\beta}^{(m)}(\tilde{\mu}). \]  

Since \( \eta^{(1)}(\lambda, \mu) = \mathcal{I}_2(\lambda, \mu) \), one has \( \Gamma^{(1)} = \beta^{(1)} \) and \( \tilde{\Gamma}^{(1)} = \tilde{\beta}^{(1)} \). Representations (44) will be used in the rest of Sec.2.3.
2.3.1 Internal constraints for $\Gamma^{(m)}$, $\tilde{\Gamma}^{(m)}$, $T$ and $\tau^{(m)}$

Remember that definition of $\xi^{(m)}$ in terms of $\chi$, see eq. (35), must be consistent with eq. (43). This requirement generates a set of constraints on $\Gamma^{(m)}$, $\tilde{\Gamma}^{(m)}$, $T$ and $\tau^{(m)}$. To derive these constraints, we apply operator $(\beta^{(m)} + \tilde{\beta}^{(m)} P)^*$ to eq. (43) from the left. One gets

$$\xi^{(m)} - \beta^{(m)} = (\beta^{(m)} + \tilde{\beta}^{(m)} P)^* \alpha^{(1)} (T^{(1)})^{-1} \xi^{(1)} + (\beta^{(m)} + \tilde{\beta}^{(m)} P)^* \chi_0. \quad (45)$$

Substituting eq. (37) for $\xi^{(m)}$ one has to obtain an identity valid for any $\xi^{(1)}$. This requirement yields, first of all, the following equation relating $\beta^{(m)}$, $\tilde{\beta}^{(m)}$ and $\chi_0$ (the first constraint on the functions $\Gamma^{(m)}$ and $\tilde{\Gamma}^{(m)}$):

$$\left(\beta^{(m)} + \tilde{\beta}^{(m)} P\right)^* \chi_0 + \beta^{(m)} = 0 \Rightarrow \left(\Gamma^{(m)}(\lambda, \nu) + \tilde{\Gamma}^{(m)}(\nu) P(\nu)\right)^* \chi_0(\nu, \mu) + \Gamma^{(m)}(\nu, \mu) = 0, \quad (47)$$

and the following definition of $\eta^{(m)}$:

$$\eta^{(m)} = \left(\beta^{(m)} + \tilde{\beta}^{(m)} P\right)^* \alpha^{(1)} (T^{(1)})^{-1}, \quad m = 1, \ldots, D. \quad (48)$$

Finally, eq. (48) written in terms of $\Gamma^{(m)}$ and $\tilde{\Gamma}^{(m)}$ yields:

$$T^{(m)}(\lambda, \mu) = T^{(1)}(\lambda, \nu) \left(\Gamma^{(m)}(\nu, \tilde{\nu}) + \tilde{\Gamma}^{(m)}(\nu) P(\tilde{\nu})\right)^* \alpha^{(1)}(\tilde{\nu}, \tilde{\mu}) (T^{(1)})^{-1}(\tilde{\mu}, \mu), \quad (49)$$

where $T^{(m)}$ are represented by eq. (11) in terms of $T$ and $\tau^{(m)}$. This is the second constraint imposed on the matrix functions $\Gamma^{(m)}$, $\tilde{\Gamma}^{(m)}$, $T$ and $\tau^{(m)}$.

Constraints obtained in this section are produced, generally speaking, by system (15) and its compatibility condition. For this reason we refer to them as the internal constraints. In contrast, the external constraints will be introduced "by hand" in order to derive the nonlinear PDEs, see Theorems 2.2 and 2.3.

2.3.2 First order nonlinear PDEs for the functions $V^{(m)}(t)$, $m = 1, \ldots, D$

**Theorem 2.2.** In addition to eqs. (11) and (15), we impose the following external constraint for the matrix functions $A^{(m)}(\lambda, \mu)$ and $\tilde{A}^{(m)}(\lambda)$:

$$\sum_{m=1}^{D} A^{(m)}(\lambda, \nu) \star \tilde{A}^{(m)}(\nu) B^{(mn)} = \sum_{m=1}^{D} \tilde{A}^{(m)}(\lambda) P^{(mn)}, \quad n = 1, \ldots, D, \quad (50)$$

where $B^{(mn)}$ and $P^{(mn)}$ are some $n_0 \times n_0$ constant matrices. Then $n_0 \times n_0$ matrix functions $V^{(m)}(t)$, $m = 1, \ldots, D$, are solutions to the following system of nonlinear PDEs:

$$\sum_{m=1}^{D} \left[ (V_{tn}^{(n)} + V^{(m)}V^{(n)}) B^{(mn)} + V^{(m)} \left( A^{(n)} B^{(mn)} + P^{(mn)} \right) \right] = 0, \quad n = 1, \ldots, D, \quad (51)$$

$A^{(n)} = P(\lambda) \star \tilde{A}^{(n)}(\lambda)$. 

8
Proof: Applying operator $\star \tilde{A}^{(m)}$ to the eq.(16) from the right one gets the following equation:

$$E^{(mn)}(t) := V^{(n)}_m(t) + V^{(m)}(t) \left( V^{(n)}(t) + A^{(n)} \right) + U^{(mn)}(t) = 0, \tag{52}$$

which introduces a new set of fields $U^{(mn)}$,

$$U^{(mn)}(t) = \left( W(\mu; t) - P(\mu) \right) * A^{(m)}(\mu, \lambda) * \tilde{A}^{(n)}(\lambda). \tag{53}$$

Due to the relation (50), we may eliminate these fields using a proper combinations of eqs.(52). Namely, combination $\sum_{m=1}^{D} E^{(mn)}(t) B^{(mn)}$ results in the system (51). □

Reduction 1. Let

$$P^{(mn)} = -A^n B^{(mn)}. \tag{54}$$

Then eq.(51) reduces as follows:

$$\sum_{m=1}^{D} \left( V^{(n)}_m + V^{(m)} V^{(n)} \right) B^{(mn)} = 0 \tag{55}$$

Note that eq.(55) coincides with the linearizable eq.(29) if $B^{(mn)} = B^{(m)}$. However, eq.(29) is supplemented by constraints (25), which are not valid for eq.(55) in general. Constraint (50) reads in this case:

$$\sum_{m=1}^{D} \left( A^{(m)}(\lambda, \nu) + \tilde{A}^{(m)}(\lambda) P(\nu) \right) * \tilde{A}^{(n)}(\nu) B^{(mn)} = 0, \quad n = 1, \ldots, D. \tag{56}$$

Reduction 2. Let

$$\tilde{\Gamma}^{(m)}(\lambda) = \tilde{\Gamma}(\lambda) C^{(m)}, \quad C^{(1)} \equiv I_{n_0}, \quad C^{(n)} B^{(mn)} = B^{(m)}, \quad m, n = 1, \ldots, D, \tag{57}$$

in addition to reduction (54). Here $C^{(m)}$ and $B^{(m)}$ are some $n_0 \times n_0$ constant matrices and $\tilde{\Gamma}(\lambda)$ is $n_0 \times n_0$ matrix function. As a consequence, we obtain

$$\tilde{A}^{(m)}(\lambda) = \tilde{A}(\lambda) C^{(m)}, \quad \tilde{A}(\lambda) = \alpha^{(1)}(\lambda, \mu) * \tilde{\Gamma}(\mu), \quad V^{(m)}(t) = V(t) C^{(m)}, \quad \tilde{\Gamma}(\lambda) = \tilde{\Gamma}^{(1)}(\lambda). \tag{58}$$

Then the system of $D$ equations (55) reduces to the following single PDE:

$$\sum_{m=1}^{D} \left( V_{tm} + VC^{(m)} V \right) B^{(m)} = 0. \tag{59}$$

This equation is written in the Introduction, see eq.(11), and may be referred to as a multidimensional generalization of $(1+1)$-dimensional $C$-integrable $N$-wave equation (30). Reduction (57) changes internal constraints (47) and (49) as follows:

$$\left( \Gamma^{(m)}(\lambda, \nu) + \tilde{\Gamma}(\lambda) C^{(m)} P(\nu) \right) * \chi_0(\nu, \mu) + \Gamma^{(m)}(\lambda, \mu) = 0, \tag{60}$$

$$T^{(m)}(\lambda, \mu) = T^{(1)}(\lambda, \nu) * \left( \Gamma^{(m)}(\nu, \tilde{\nu}) + \tilde{\Gamma} C^{(m)}(\nu) P(\tilde{\nu}) \right) * \alpha^{(1)}(\tilde{\nu}, \tilde{\mu}) * (T^{(1)})^{-1}(\tilde{\mu}, \mu), \tag{61}$$

$m = 1, \ldots, D$.

In turn, external constraint (56) reduces to the following single equation:

$$\sum_{m=1}^{D} \left( A^{(m)}(\lambda, \nu) + \tilde{A}(\lambda) C^{(m)} P(\nu) \right) * \tilde{A}(\nu) B^{(m)} = 0. \tag{62}$$
2.3.3 Second order nonlinear PDEs for the functions \( V^{(m)}, m = 1, \ldots, D \)

**Theorem 2.3.** In addition to eqs. (1115), we impose the following external constraint for the matrix functions \( A^{(m)}(\lambda, \mu) \) and \( \tilde{A}^{(m)}(\lambda) \) (instead of constraint (50)):

\[
\begin{align*}
\sum_{m,n=1}^{D} A^{(m)}(\lambda, \nu) & \ast A^{(n)}(\nu, \mu) \ast \tilde{A}^{(l)}(\mu) B^{(ml)} = 0, \quad \text{eq. (63)} \\
\sum_{m,n}^{D} A^{(m)}(\lambda, \mu) \ast \tilde{A}^{(n)}(\mu) P^{(ml)} & + \sum_{m=1}^{D} \tilde{A}^{(m)}(\lambda) P^{(ml)},
\end{align*}
\]

where \( B^{(ml)}, P^{(ml)}, \) and \( P^{(ml)} \) are some constant \( n_0 \times n_0 \) matrices. Then \( n_0 \times n_0 \) matrix functions \( V^{(m)}(t) \) are solutions to the following system of nonlinear PDEs:

\[
\begin{align*}
\sum_{m,n=1}^{D} \left[ V^{(l)}_{t_{lm}} + (V^{(n)} V^{(l)})_{t_{nm}} + V^{(m)} V^{(l)}_{t_{nm}} + V^{(m)} V^{(n)} V^{(l)} \right] B^{(ml)} & + \\
\sum_{m,n=1}^{D} (V^{(n)} + V^{(m)} V^{(n)}) A^{(ml)}_1 + \sum_{m=1}^{D} V^{(m)} A^{(ml)}_2 & = 0, \quad l = 1, \ldots, D, \\
A^{(ml)}_1 = A^{(l)} B^{(ml)} + P^{(ml)}, \quad A^{(ml)}_2 = \sum_{n=1}^{D} A^{(n)} P^{(ml)} - P^{(ml)}.
\end{align*}
\]

**Proof:** Applying operator \( \ast \tilde{A}^{(n)} \) to the eq. (16) from the right one gets eq. (52), which introduces a new set of fields \( U^{(mn)} \) defined by eq. (53). However, we may not eliminate these fields from the system (52) because constraint (50) is not valid in this theorem. Instead of this, we derive nonlinear PDEs for these fields as follows. Applying operator \( \ast A^{(n)} \ast \tilde{A}^{(l)} \) to the eq. (16) from the right one gets

\[
E^{(ml)}(t) := U^{(ml)}_t(t) + V^{(m)}(t) U^{(nl)}(t) + U^{(mn)}(t) = 0, \quad \text{eq. (65)}
\]

where one more set of matrix fields appears:

\[
U^{(mn)}(t) = (W(\lambda; t) - P(\lambda)) \ast A^{(m)}(\lambda, \nu) \ast A^{(n)}(\nu, \mu) \ast \tilde{A}^{(l)}(\mu).
\]

Due to the constraint (63), these fields may be eliminated in a proper combination of equations (65), namely, \( \sum_{m,n=1}^{D} E^{(ml)} B^{(ml)} \). Then, substituting \( U^{(mn)} \) from eq. (52) one ends up with eq. (64). □

Emphasize that nonlinear equations (51) and (64) do not represent commuting flows since we assume different external constraints (50) and (63) for matrix functions \( A^{(m)}(\lambda, \mu) \) and \( \tilde{A}^{(m)}(\lambda) \) deriving these equations. These two constraints are not compatible in general.

**Reduction 1.** Let

\[
P^{(ml)} = -A^{(l)} B^{(ml)}, \quad \text{eq. (67)}
\]

\[
P^{(ml)} = \sum_{n=1}^{D} A^{(n)} P^{(ml)}.
\]
Then eq. (64) reads
\[
\sum_{m,n=1}^{D} \left[ V^{(l)}_{t_m} + (V^{(n)}V^{(l)})_{t_m} + V^{(m)}V^{(l)}_{t_n} + V^{(m)}V^{(n)}V^{(l)} \right] B^{(mnl)} = 0, \quad l = 1, \ldots, D. \tag{68}
\]

Constraint (63) reduces to the following one:
\[
\sum_{m,n=1}^{D} \left( A^{(m)}(\lambda, \nu) * A^{(n)}(\nu, \mu) + A^{(m)}(\lambda, \nu) * \tilde{A}^{(n)}(\nu) P(\mu) + \tilde{A}(\lambda) P(\nu) * \tilde{A}(\nu) P(\mu) \right) * \tilde{A}(\mu) B^{(mnl)} = 0. \tag{69}
\]

Reduction 2. Along with reduction (67) we consider reduction (57,58) with
\[
C^{(l)}B^{(mnl)} = B^{(mn)},
\]
where \(B^{(mn)}\) are some constant matrices. System (68) reduces to the following single PDE:
\[
\sum_{m,n=1}^{D} \left[ V_{t_m} + (V C^{(n)}V)_{t_m} + V C^{(m)}V_{t_n} + V C^{(m)}V C^{(n)}V \right] B^{(mn)} = 0. \tag{70}
\]

This equation is written in the Introduction, see eq.(9). Internal constraints (60) and (61) remain valid for this case as well. External constraint (69) reduces to the following single equation:
\[
\sum_{m,n=1}^{D} \left( A^{(m)}(\lambda, \nu) * A^{(n)}(\nu, \mu) + A^{(m)}(\lambda, \nu) * \tilde{A}(\nu) C^{(n)} P(\mu) + \tilde{A}(\lambda) C^{(m)} P(\nu) * \tilde{A}(\nu) C^{(n)} P(\mu) \right) * \tilde{A}(\mu) B^{(mn)} = 0. \tag{71}
\]

### 2.3.4 Solutions to the first order nonlinear PDE (59)

The problem of richness of the available solution space will be considered for the nonlinear PDE (59). We show that solution space may be full provided that all constraints (60), (61) and (62) may be resolved keeping proper arbitrariness of functions \(\tau^{(m)}(\nu)\). Examples of particular solutions will be considered as well.

**On the dimensionality of the available solution space.** We estimate the dimensionality of solution space for small \(\chi\). In this case formula (11) yields \(W(\lambda; t) \approx P(\nu) * \chi(\nu; \lambda, t)\) and formula (17) gives us
\[
V(t) \approx \left( P(\nu) * \chi(\nu, \lambda, t) - P(\lambda) \right) * \tilde{A}(\lambda). \tag{72}
\]

By construction, if all \(\tau^{(m)}(\nu) (m = 1, \ldots, D)\) are arbitrary functions, this expression preserves the following arbitrary \(n_0 \times n_0\) matrix function of all \(D\) variables:
\[
F(t) = P(\nu) \alpha^{(1)} * (T^{(1)})^{-1} * \xi^{(1)} * \tilde{A} \equiv \int g(\nu)e^{\sum_{i=1}^{D} \tau^{(i)}(\nu) t_i} g_2(\nu) d\Omega_2(\nu), \tag{73}
\]
\[
g_1(\nu) = P(\nu) \alpha^{(1)}(\nu, \mu) * (T^{(1)})^{-1} (\mu, \lambda) * T(\lambda, \nu), \quad g_2(\nu) = T^{-1}(\nu, \nu) * \xi_0(\nu, \mu) * \tilde{A}(\mu)
\]
However, dimensionality of this function reduces due to the presence of constraints (60), (61) and (62) which impose relations among \( \tau^{(m)}(\nu) \). An important question is whether the dimensionality of the function (74) may be equal to \( D - 1 \), which is necessary for fullness of the solution space. At first glance, we may expect the positive answer. In fact, eq. (60) may be satisfied using special structures of \( \Gamma^{(m)} \), \( P \) and \( \chi_0 \), as it is done in the example below, see eqs. (89). Next, eq. (61) relates \( \tau^{(m)} \) with \( \Gamma^{(m)} \) and \( \hat{\Gamma}^{(m)} \) which, in general, keeps arbitrariness of all \( \tau^{(m)}(\nu) \) and consequently does not restrict dimensionality of the above written arbitrary function. Finally, eq. (62) must be considered as a single relation among \( \tau^{(m)}(\nu) \), \( m = 1, \ldots, D \), reducing the dimensionality of the function (74) from \( D \) to \( D - 1 \), which means the full dimensionality of the solution space. Thus, we may expect new completely integrable nonlinear PDEs in the derived class of equations.

We have outlined a rough analysis of the solution space dimensionality. More detailed analysis must be carried out for particular equations and remains beyond the scope of this paper.

**Construction of explicite solutions.** Now we derive a family of particular solutions in the form of rational functions of exponents. Solitons and kinks are the most famous representatives of this family. To derive such solutions, we take

\[
\begin{align*}
\Omega_1(\lambda) &= \sum_{i=1}^{M} \delta(\lambda - a_i)d\lambda, \\
\Omega_2(\lambda) &= \sum_{i=1}^{N} \delta(\lambda - b_i)d\lambda, \\
T(\lambda, \mu) &= I_2(\lambda, \mu), \\
\mathcal{I}_1 &\rightarrow I_{Mn_0}, \\
\mathcal{I}_2 &\rightarrow I_{Nn_0}.
\end{align*}
\]

Then all integral equations reduce to the algebraic ones. We use notations

\[
\begin{align*}
\hat{W} &= \begin{bmatrix} W(a_1) & \cdots & W(a_M) \end{bmatrix}, \\
\hat{\chi} &= \begin{bmatrix} \chi(a_1, a_1) & \cdots & \chi(a_1, a_M) \\
& \cdots & \cdots \\
\chi(a_M, a_1) & \cdots & \chi(a_M, a_M) \end{bmatrix}, \\
\hat{\chi}_0 &= \begin{bmatrix} \chi_0(a_1, a_1) & \cdots & \chi_0(a_1, a_M) \\
& \cdots & \cdots \\
\chi_0(a_M, a_1) & \cdots & \chi_0(a_M, a_M) \end{bmatrix}, \\
\hat{\Gamma}^{(m)} &= \begin{bmatrix} \Gamma^{(m)}(b_1, a_1) & \cdots & \Gamma^{(m)}(b_1, a_M) \\
& \cdots & \cdots \\
\Gamma^{(m)}(b_M, a_1) & \cdots & \Gamma^{(m)}(b_M, a_M) \end{bmatrix}, \\
\hat{\xi}_0 &= \begin{bmatrix} \xi_0(b_1, a_1) & \cdots & \xi_0(b_1, a_M) \\
& \cdots & \cdots \\
\xi_0(b_M, a_1) & \cdots & \xi_0(b_M, a_M) \end{bmatrix}, \\
\tau^{(m)} &= \text{diag}(\tau^{(m)}(b_1), \ldots, \tau^{(m)}(b_M)).
\end{align*}
\]

Solution \( V \) is given by eq. (17) together with reduction (57, 58) as follows:

\[
V = (\hat{W} - \hat{P})\hat{\chi}
\]

where \( \hat{W} \) is solution to eq. (11):

\[
\hat{W} = \hat{P} \hat{\chi}(\hat{\chi} + I_{Mn_0})^{-1}
\]
Substituting eq. (78) into eq. (77) we obtain

\[ V = \hat{P} \left( \hat{\chi} + (I_{M_n})^{-1} - I_{M_n} \right) \hat{\alpha}^{(1)} \hat{\Gamma}. \]  

(79)

Since \( \chi \) is given by eq. (43), one has

\[ \hat{\chi} = \hat{\alpha}^{(1)}(\hat{\tau}^{(1)})^{-1} e^{\sum_{i=1}^{D} \hat{\tau}_{i} t_i} \hat{\Gamma}_0 + \hat{\chi}_0, \]

(80)

where we substitute eq. (42) for \( \hat{\xi}^{(1)} \).

Matrices \( \hat{\Gamma}^{(m)}, \hat{\Gamma} \) and \( \hat{\tau}^{(m)} \) must satisfy constraints (60), (61) and (62), which read in our case, \( m = 1, \ldots, D \):

\[ \left( \hat{\Gamma}^{(m)} + \hat{\Gamma} C^{(m)} \hat{P} \right) \chi_0 + \hat{\Gamma}^{(m)} = 0, \]

(81)

\[ \hat{\tau}^{(m)} = \hat{\tau}^{(1)} \left( \hat{\Gamma}^{(m)} + \hat{\Gamma} C^{(m)} \hat{P} \right) \hat{\alpha}^{(1)}(\hat{\tau}^{(1)})^{-1}, \]

(82)

\[ \sum_{m=1}^{D} \hat{\alpha}^{(1)}(\hat{\Gamma}^{(m)} + \hat{\Gamma} C^{(m)} \hat{P}) \hat{\alpha}^{(1)} \hat{\Gamma} B^{(m)} = 0. \]

(83)

Analysis of eqs. (81) points on two different types of solutions to them. First type is associated with \( \det(\hat{\chi}_0 + I_{M_n}) \neq 0 \). Then eqs. (81) may be solved for \( \Gamma^{(m)} \) and one can show that multidimensional PDE (59) may be splitted into a set of independent compatible Ordinary Differential Equations (ODEs). We will not consider this case. Second type is associated with \( \det(\hat{\chi}_0 + I_{M_n}) = 0 \) and leads to truly multidimensional solutions to eq. (59). Namely this case is considered hereafter.

Looking for the particular solutions to eq. (81) we decompose it into two equations:

\[ \hat{P} \hat{\chi}_0 = 0, \quad \hat{\Gamma}^{(m)}(\hat{\chi}_0 + I_{M_n}) = 0, \]

(84)

which means that the rows of \( \hat{P} \) and \( \hat{\Gamma}^{(m)} \) are orthogonal to the columns of \( \hat{\chi}_0 \) and \( \hat{\chi}_0 + I_{M_n} \) respectively.

Note that eq. (82) with \( m = 1 \) reads:

\[ \hat{\tau}^{(1)} = \left( \hat{\Gamma}^{(1)} + \hat{\Gamma} \hat{P} \right) \hat{\alpha}^{(1)}. \]

(85)

Since, \( \hat{\tau}^{(1)} \) must be invertable, we require \( M > N \). Then eq. (83) may be simplified removing \( \hat{\alpha}^{(1)} \) as a left factor in this equation:

\[ \sum_{m=1}^{D} (\hat{\Gamma}^{(m)} + \hat{\Gamma} C^{(m)} \hat{P}) \hat{\alpha}^{(1)} \hat{\Gamma} B^{(m)} = 0. \]

(86)

**Simple example of solution.** We consider the three-dimensional nonlinear PDE (59), i.e. \( D = 3 \):

\[ \sum_{m=1}^{3} (V t_m + V C^{(m)} V) B^{(m)} = 0. \]

(87)
Let \( n_0 = 2, M = 5, N = 2, \)

\[
B^{(1)} = C^{(1)} = I_2, \quad B^{(i)} = \text{diag}(b_1^{(i)}, b_2^{(i)}), \quad C^{(i)} = \text{diag}(c_1^{(i)}, c_2^{(i)}), \quad i = 2, 3. \tag{88}
\]

In order to satisfy eqs. (82), (84) and (86) we take the following matrices \( \hat{\chi}_0, \hat{\chi}, \hat{\Gamma}^{(m)}, \hat{\Gamma}: \)

\[
\hat{\chi}_0 = \begin{bmatrix}
-I_2 & Z_{2,8} \\
F_{2,2} & F_{2,8} \\
Z_{0,2} & Z_{6,8}
\end{bmatrix}, \quad F_{2,2} = Z, \quad F_{2,8} = [J_4 \ Z \ J_5 \ Z], \tag{89}
\]

\[
\hat{\alpha}^{(1)} = \begin{bmatrix}
I_4 \\
I_4 \\
J_0
\end{bmatrix}, \quad \hat{\xi}_0 = (\alpha^{(1)})^T
\]

\[
\hat{\chi} = [Z \ Z \ Z \ J_1 \ Z],
\]

\[
\hat{\Gamma}^{(1)} = \begin{bmatrix}
J_2 & Z & Z & Z & Z \\
Z & Z & Z & Z & Z
\end{bmatrix}, \quad \hat{\Gamma}^{(2)} = \hat{\Gamma}^{(3)} = Z_{4,10},
\]

\[
\hat{\Gamma} \equiv \hat{\Gamma}^{(1)} = \begin{bmatrix}
Z \\
J_3
\end{bmatrix}, \quad \hat{\tau}^{(1)} = \text{diag}(1, 2, 3, 4), \quad \hat{\tau}^{(2)} = \text{diag}(0, 0, 3c_2^{(2)}, 4c_1^{(2)}), \quad \hat{\tau}^{(3)} = \text{diag}(0, 0, 3c_2^{(3)}, 4c_1^{(3)}).
\]

In addition, we obtain expressions for \( c_i^{(3)}, i = 1, 2: \)

\[
c_i^{(3)} = -\frac{1 + b_i^{(2)}c_i^{(2)}}{b_i^{(3)}}, \quad i = 1, 2. \tag{90}
\]

Here \( Z_{i,j} \) and \( Z \) are \( i \times j \) and \( 2 \times 2 \) zero matrices respectively,

\[
J_0 = [I_2 \ Z], \quad J_1 = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad J_2 = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}, \quad J_3 = \begin{bmatrix}
0 & 3 \\
4 & 0
\end{bmatrix}, \tag{91}
\]

\[
J_4 = \begin{bmatrix}
8p_3 + p_2(6 - p_3) - 4p_4 - p_1(12 - p_4) \\
\frac{2(6p_1 - 3p_2 - 2p_3 + p_4)}{(-12 + 3p_2 - p_4)(-2p_3 + p_4)} \\
\frac{12(6p_1 - 3p_2 - 2p_3 + p_4)}{12(6p_1 - 3p_2 - 2p_3 + p_4)}
\end{bmatrix}
\]

\[
J_5 = \begin{bmatrix}
0 & 1 + \frac{4(3p_1 - p_3 - 6)}{12 - 3p_2 + p_4} \\
-1 + \frac{p_2}{4} - \frac{p_4}{12} & 0
\end{bmatrix}, \tag{92}
\]

Diagonal elements of matrices \( B^{(i)}, i = 2, 3, \) and \( C^{(2)} \) remain arbitrary. Substituting eqs. (89) into eq. (79, 80) one obtains \( V \) as a rational expression of exponents:

\[
V(t) = \frac{1}{D} \begin{bmatrix}
-4(e^{n_2}p_2 + p_4) \\
f_{12}e^{n_1} \\
-f_{21}e^{n_2} \\
-3(e^{n_3}p_3 + p_4)
\end{bmatrix}, \tag{93}
\]

\[
D = e^{n+m}p_1 + e^{n_2}p_2 + e^{n_3}p_3 + p_4,
\]

\[
f_{12} = -3\frac{(-12 + 3p_2 - p_4)(p_2p_3 - p_1p_4)}{4(-12p_1 + 6p_2 + 4p_3 - 2p_4)}, \quad f_{21} = 16\frac{12p_1 - 6p_2 - 4p_3 + 2p_4}{-12 + 3p_2 - p_4}.
\]
Here
\[ \eta_1 = 4t_1 + 4c_1^{(2)}t_2 - \frac{4}{b_1^{(3)}}(1 + b_1^{(2)}c_1^{(2)})t_3, \]
\[ \eta_1 = 3t_1 + 3c_2^{(2)}t_2 - \frac{3}{b_2^{(3)}}(1 + b_2^{(2)}c_2^{(2)})t_3. \]

We see that all elements of \( V \) are kinks.

Relations (90) show that not all matrix coefficients in eq.(87) are arbitrary. They are related by the following equation:
\[ C^{(3)}B^{(3)} + C^{(2)}B^{(2)} + I_2 = 0. \] (95)

Thus we have constructed particular solution to the three-dimensional nonlinear PDE (87) with diagonal matrices \( C^{(i)}, B^{(i)} \) related by eq.(95).

3 Multidimensional generalization of \( S \)-integrable PDEs

3.1 Starting equations

Algorithm developed in this section is based on the same equation (11) with different function \( \chi(t) \), which is defined by the following system of equations
\[ \chi_{tm}(\lambda, \mu; t) = \left( A^{(m)}(\lambda, \nu) + \tilde{A}^{(m)}(\lambda)P(\nu) \right) * \chi(\nu, \mu; t) - \chi(\lambda, \nu; t) * A^{(m)}(\nu, \mu), \]
\[ m = 1, \ldots, D \] (96)

instead of system (15). Here, again, \( A^{(m)}(\lambda, \nu) \) and \( \tilde{A}^{(m)}(\lambda) \) are \( n_0 \times n_0 \) matrix functions of arguments.

Matrices \( A^{(m)} \) and \( \tilde{A}^{(m)} \) have to provide compatibility of system (96). Similar to eq.(15), there are two different methods that provide this compatibility. The first one yields the classical \( S \)-integrable nonlinear PDEs, Sec.3.2, while the second method yields a new type of nonlinear PDEs whose complete integrability is not clarified yet, Sec.3.3. However, our algorithm supplies, at least, a rich manifold of particular solutions to these PDEs.

3.2 First method: classical \( S \)-integrable (2+1)-dimensional \( N \)-wave equation

Consider the compatibility condition of eqs.(96) in the following form:
\[ \left( A^{(m)}(\lambda, \nu) + \tilde{A}^{(m)}(\lambda)P(\nu) \right) * \chi_{tn}(\nu, \mu; t) - \chi_{tn}(\lambda, \nu; t) * A^{(m)}(\nu, \mu) = \]
\[ \left( A^{(n)}(\lambda, \nu) + \tilde{A}^{(n)}(\lambda)P(\nu) \right) * \chi_{tm}(\nu, \mu; t) - \chi_{tm}(\lambda, \nu; t) * A^{(n)}(\nu, \mu), \quad \forall n, m \] (97)

Substituting eq.(96) for derivatives of \( \chi \) we reduce eq.(97) to the following one:
\[ (L^{(m)} * L^{(n)} - L^{(n)} * L^{(m)}) * \chi - \chi * (A^{(m)} * A^{(n)} - A^{(n)} * A^{(m)}) = 0, \]
\[ L^{(m)}(\lambda, \mu) = A^{(m)}(\lambda, \mu) + \tilde{A}^{(m)}(\lambda)P(\mu). \]
Since eq. (98) must be valid for any function \( \chi(t) \) (which is a solution to the system (96)), it is equivalent to two following equations relating matrix functions \( A^{(m)} \), \( A^{(m)} \) and \( P \):

\[
A^{(m)}(\lambda, \nu) * A^{(n)}(\nu, \mu) - A^{(n)}(\lambda, \nu) * A^{(m)}(\nu, \mu) = 0, 
\]

\[
L^{(m)} * L^{(n)} - L^{(n)} * L^{(m)} = 0 \quad \text{eq. (99)}
\]

\[
L^{(m)}(\lambda, \nu) * \tilde{A}(\nu)P(\mu) - L^{(n)}(\lambda, \nu) * \tilde{A}(\nu)P(\mu) = 0
\]

\[
\tilde{A}(\lambda)P(\nu) * A^{(m)}(\nu, \mu) - \tilde{A}(\lambda)P(\nu) * A^{(n)}(\nu, \mu).
\]

In order to satisfy eq. (100) we require the following representation of \( \tilde{A}^{(m)}(\lambda) \):

\[
\tilde{A}^{(m)}(\lambda) = \tilde{A}(\lambda)B^{(m)}, \quad [B^{(m)}, B^{(n)}] = 0,
\]

where \( \tilde{A}(\lambda) \) and \( B^{(m)} \) are \( n_0 \times n_0 \) matrix function and constant matrix respectively. Then eq. (100) is equivalent to the following system:

\[
L^{(m)}(\lambda, \nu) * \tilde{A}(\nu)B^{(n)} - L^{(n)}(\lambda, \nu) * \tilde{A}(\nu)B^{(m)} = 0,
\]

\[
B^{(n)}P(\nu) * A^{(m)}(\nu, \mu) - B^{(m)}P(\nu) * A^{(n)}(\nu, \mu) = 0.
\]

Eqs. (99, 102, 103) represent three constraints for matrices \( A^{(m)} \) and \( \tilde{A} \).

**Theorem 3.1.** Let the matrix function \( W(\lambda; t) \) be obtained as a solution to the integral equation (11) with \( \chi \) defined by eqs. (96) supplemented with constraints (99, 102, 103). Then

1. Function \( W(\lambda; t) \) satisfies the following system of compatible linear equations

\[
E^{(nm)}(\mu; t) := B^{(n)}(W_{tm}(\mu; t) + V(t)B^{(m)}W(\mu; t) + W(\nu; t) * A^{(m)}(\nu, \mu)) -
\]

\[
B^{(m)}(W_{tn}(\mu; t) + V(t)B^{(n)}W(\mu; t) + W(\nu; t) * A^{(n)}(\nu, \mu)) = 0,
\]

\[
V(t) = (W(\nu; t) - P(\nu)) * \tilde{A}(\nu), \quad \text{eq. (105)}
\]

\[
n, m = 1, \ldots, D.
\]

2. Matrix field \( V(t) \), given by eq. (105), satisfies the following \( S \)-integrable \( N \)-wave equation:

\[
\sum_{\text{perm}(n, m, l)} \left( B^{(n)}(V_{tn} + VB^{(m)}V) - B^{(m)}(V_{tn} + VB^{(n)}V) \right)B^{(l)} = 0,
\]

where \( \text{perm}(n, m, l) \) means clockwise circle permutations.

**Proof:** 1. To derive eq. (104), we differentiate eq. (11) with respect to \( t_m \). Then, in view of eq. (96), one gets the following equation:

\[
E^{(m)}(\mu; t) := P(\lambda) * A^{(m)}(\lambda, \nu) * \chi(\nu, \mu; t) = E^{(m)}(\nu; t) * \left( \chi(\nu, \mu) + \mathcal{I}(\nu, \mu) \right),
\]

\[
E^{(m)}(\mu; t) = W_{tm}(\mu; t) + V^{(m)}(t)W(\mu; t) + W(\nu; t) * A^{(m)}(\nu, \mu).
\]

Due to the constraint (103), LHS in eqs. (107) may be removed using the following combination of these equations:

\[
B^{(n)}E^{(m)} - B^{(m)}E^{(n)} \Rightarrow
\]

\[
(\text{eqs. (107)}) (B^{(n)}E^{(m)}(\nu; t) - B^{(m)}E^{(n)}(\nu; t)) * (\chi(\nu, \mu; t) + \mathcal{I}(\nu, \mu)) = 0.
\]
Since operator \( \ast (\chi (\nu, \mu) + \mathcal{I}_1 (\nu, \mu)) \) is invertible, one gets
\[
\mathcal{E}^{(nm)} := B^{(n)} E^{(m)} - B^{(m)} E^{(n)} = 0, \tag{110}
\]
which coincides with eq. (104).

2. In order to derive nonlinear PDE (106) we consider the following combination of eqs. (104):
\[
\sum_{\text{perm}(n,m,l)} E^{(nm)} \ast \bar{A} B^{(l)},
\tag{111}
\]
which yields eq. (106) in view of constraint (102). \(\blacksquare\)

S-integrable PDEs will not be considered in this paper.

### 3.3 Second method: new class of nonlinear PDEs

We will use double indexes hereafter in this section, i.e.
\[
V^{(m)} \equiv V^{(m_1m_2)}, \quad t_m \equiv t_{m_1m_2}, \quad \sum_{m_1} D^{(m)} \equiv \sum_{m_2=1}^{D_2} \sum_{m_1=1}^{D_1} f^{(m_1m_2)}, \quad \forall f, \quad D \equiv (D_1, D_2),
\tag{112}
\]
and notation \( f^{(1)} \equiv f^{(11)}, \forall f \), for the sake of brevity. For instance, \( \tau^{(1)} \equiv \tau^{(11)}, \eta^{(1)} \equiv \eta^{(11)} \) and so on. Similar to Sec 2.3 we will use representations (51) for the matrix functions \( A^{(m)}(\lambda, \mu) \) and \( \bar{A}^{(m)}(\lambda) \) and write eq. (96) in the following form:
\[
\chi_{t_m}(\lambda, \mu; t) = \alpha^{(m)}(\lambda, \nu) \ast \xi^{(m)}(\nu, \mu) - \bar{\xi}^{(m)}(\lambda, \nu) \ast \beta^{(m)}(\nu, \mu),
\tag{113}
\]
\[
\xi^{(m)}(\lambda, \mu) = \left( \beta^{(m)}(\lambda, \nu) + \bar{\beta}^{(m)}(\lambda) P(\nu) \right) \ast \chi(\nu, \mu; t),
\tag{114}
\]
\[
\bar{\xi}^{(m)}(\lambda, \mu) = \chi(\lambda, \nu) \ast \alpha^{(m)}(\nu, \mu), \quad m_i = 1, \ldots, D, \quad i = 1, 2.
\tag{115}
\]

Then the compatibility condition for system (96) is equivalent to the compatibility condition for system (113) which reads
\[
\alpha^{(m)} \ast \xi_{t_n}^{(m)} = \bar{\xi}_{t_n}^{(m)} \ast \beta^{(m)} = \alpha^{(n)} \ast \xi_{t_m}^{(n)} - \bar{\xi}_{t_m}^{(n)} \ast \beta^{(n)}, \quad \forall n, m.
\tag{116}
\]

To satisfy this condition we, first, assume that \( \xi^{(m)} \) and \( \bar{\xi}^{(m)} \) are expressed in terms of \( \xi^{(1)} \) and \( \bar{\xi}^{(1)} \) as follows:
\[
\xi^{(m)}(\lambda, \mu; t) = \eta^{(m)}(\lambda, \nu) \ast \xi^{(1)}(\nu, \mu), \quad m_1 + m_2 > 2, \quad \eta^{(1)}(\lambda, \nu) = \mathcal{I}_2(\lambda, \nu),
\tag{117}
\]
\[
\bar{\xi}^{(m)}(\lambda, \mu; t) = \bar{\eta}^{(m)}(\lambda, \nu) \ast \bar{\xi}^{(1)}(\nu, \mu), \quad m_1 + m_2 > 2, \quad \bar{\eta}^{(1)}(\nu, \mu) = \mathcal{I}_2(\lambda, \nu),
\tag{118}
\]
where \( \eta^{(m)}(\nu, \mu) \) and \( \bar{\eta}^{(m)}(\nu, \mu) \) are some \( n_0 \times n_0 \) matrix functions which will be specified below. Second, we define \( \xi_{t_1}^{(1)} \) and \( \bar{\xi}_{t_1}^{(1)} \) in terms of \( \xi^{(1)} \) and \( \bar{\xi}^{(1)} \) as follows:
\[
\xi_{t_1}^{(1)}(\lambda, \mu; t) = T^{(m)}(\lambda, \nu) \ast \xi^{(1)}(\nu, \mu; t),
\tag{119}
\]
\[
\bar{\xi}_{t_1}^{(1)}(\lambda, \mu; t) = \bar{T}^{(m)}(\nu, \mu) \ast \bar{\xi}^{(1)}(\lambda, \nu; t),
\tag{120}
\]
where \( T^{(m)}(\lambda, \nu) \) and \( \bar{T}^{(m)}(\lambda, \nu) \) are \( n_0 \times n_0 \) matrix functions. At last, substitute eqs. (117)(120) into eq. (116). Since resulting equation must be valid for any possible \( \xi^{(1)} \) and \( \bar{\xi}^{(1)} \), we get the following expressions for \( \alpha^{(m)} \) and \( \beta^{(m)} \):
\[
\alpha^{(m)}(\lambda, \mu) = \left( \alpha^{(1)}(\lambda, \nu) \ast T^{(m)}(\nu, \nu) \ast (T^{(1)})^{-1}(\nu, \nu) \right) \ast (\eta^{(m)})^{-1}(\nu, \mu),
\tag{121}
\]
\[
\beta^{(m)}(\lambda, \mu) = (\bar{\eta}^{(m)})^{-1}(\lambda, \nu) \ast (T^{(1)})^{-1}(\nu, \nu) \ast T^{(m)}(\nu, \nu) \ast \bar{\beta}^{(1)}(\nu, \mu).
\tag{122}
\]
In turn, compatibility of eqs. (119) and (120) requires
\[ T^{(m)} \star T^{(n)} - T^{(n)} \star T^{(m)} = 0 \Rightarrow T^{(m)}(\lambda, \mu) = (T(\lambda, \nu) \tau^{(m)}(\nu)) \star T^{-1}(\nu, \mu), \quad (123) \]
\[ \bar{T}^{(m)} \star \bar{T}^{(n)} - \bar{T}^{(n)} \star \bar{T}^{(m)} = 0 \Rightarrow \bar{T}^{(m)}(\lambda, \mu) = \bar{T}^{-1}(\lambda, \nu) \star (\tau^{(m)}(\nu) T^{-1}(\nu, \mu)), \]
\[ [\tau^{(m)}(\nu), \tau^{(n)}(\nu)] = 0, \quad [\bar{\tau}^{(m)}(\nu), \bar{\tau}^{(n)}(\nu)] = 0, \]
where \( T(\lambda, \nu), \bar{T}(\lambda, \nu), \tau^{(m)}(\nu) \) and \( \bar{\tau}^{(m)}(\nu) \) are \( n_0 \times n_0 \) matrix functions (compare with eq. 111). Thus, compatibility condition of eqs. (113) generates eqs. 117-123).

Now we may integrate eqs. (119) and (120), obtaining the following expressions for \( \xi^{(1)} \) and \( \bar{\xi}^{(1)} \):
\[ \xi^{(1)}(\lambda, \mu; t) = \left( T(\lambda, \nu) e^{\sum_{i=1}^{D} \tau^{(i)}(\nu) t_i} \right) \star T^{-1}(\nu, \bar{\nu}) \star \xi_0(\bar{\nu}, \mu), \quad (124) \]
\[ \bar{\xi}^{(1)}(\lambda, \mu; t) = \bar{\xi}_0(\lambda, \nu) \star \bar{T}^{-1}(\nu, \bar{\nu}) \star \left( e^{\sum_{i=1}^{D} \bar{\tau}^{(i)}(\bar{\nu}) t_i} \bar{T}(\bar{\nu}, \mu) \right). \quad (126) \]

Here \( \xi_0(\bar{\nu}, \mu) \) and \( \bar{\xi}_0(\lambda, \nu) \) are \( n_0 \times n_0 \) matrix functions. Finally, integrating eq. (113) with \( m_1 = m_2 = 1 \), one obtains
\[ \chi(\lambda, \mu; t) = \alpha^{(1)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \bar{\nu}) \star \xi^{(1)}(\bar{\nu}, \mu) - \bar{\xi}^{(1)}(\lambda, \nu) \star (\bar{T}^{(1)})^{-1}(\nu, \bar{\nu}) \star \beta^{(1)}(\bar{\nu}, \mu). \quad (127) \]

It is convenient to rewrite eqs. 31 using eqs. 121 and 122 as follows:
\[ A^{(m)}(\lambda, \mu) = \alpha^{(1)}(\lambda, \nu) \star \gamma^{(m)}(\nu, \bar{\nu}) \star \beta^{(1)}(\bar{\nu}, \mu), \quad (128) \]
\[ \bar{A}^{(m)}(\lambda) = \alpha^{(1)}(\lambda, \nu) \star \bar{\Gamma}^{(m)}(\nu), \quad \forall m, \quad (129) \]
where
\[ \gamma^{(m)}(\lambda, \mu) = T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \bar{\nu})(\eta^{(m)})^{-1}(\bar{\nu}, \bar{\mu}) \star \]
\[ (\bar{\eta}^{(m)})^{-1}(\bar{\mu}, \mu) \star (T^{(1)})^{-1}(\bar{\mu}, \bar{\lambda}) \star T^{(m)}(\bar{\lambda}, \mu), \quad (130) \]
\[ \bar{\Gamma}^{(m)}(\lambda) = T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \bar{\nu}) \star (\eta^{(m)})^{-1}(\bar{\nu}, \bar{\mu}) \star \bar{\beta}^{(m)}(\bar{\mu}). \quad (131) \]
Representations (128,129) will be used in the rest of Sec 3.3.

### 3.3.1 Internal constraints for \( \alpha^{(1)}, \beta^{(1)}, \bar{\Gamma}^{(m)}, T, \bar{T}, \tau^{(m)} \) and \( \bar{\tau}^{(m)} \)

Note that the definitions of \( \xi^{(m)} \) and \( \bar{\xi}^{(m)} \) in terms of \( \chi \), i.e. eqs. (114) and (115), must be consistent with eq. (127). This requirement generates some constraints for \( \alpha^{(1)}, \beta^{(1)}, \bar{\Gamma}^{(m)}, T, \bar{T}, \tau^{(m)} \) and \( \bar{\tau}^{(m)} \). To derive constraints associated with definition (114), we apply operator \((\bar{\beta}^{(m)} + \beta^{(m)}) P)\) to the eq. (127) from the left obtaining the following equation:
\[ \xi^{(m)}(\lambda, \mu; t) = (\beta^{(m)}(\lambda, \nu) + \bar{\beta}^{(m)}(\lambda) P(\nu)) \star \alpha^{(1)}(\nu, \bar{\nu}) \star (T^{(1)})^{-1}(\bar{\nu}, \bar{\mu}) \star \xi^{(1)}(\bar{\mu}, \mu) - (132) \]
\[ (\beta^{(m)}(\lambda, \nu) + \bar{\beta}^{(m)}(\lambda) P(\nu)) \star \xi^{(1)}(\nu, \bar{\nu}) \star (T^{(1)})^{-1}(\bar{\nu}, \bar{\mu}) \star \beta^{(1)}(\bar{\mu}, \mu). \]

Substitute eq. (114) for \( \xi^{(m)} \) and require that resulting equation is identity for any \( \xi^{(1)} \). Then eq. (132) becomes decomposed into two following constraints:
\[ (\beta^{(m)}(\lambda, \nu) + \bar{\beta}^{(m)}(\lambda) P(\nu)) \star \xi_0(\nu, \mu) = 0, \quad (133) \]
\[ \eta^{(m)}(\lambda, \mu) = (\beta^{(m)}(\lambda, \nu) + \bar{\beta}^{(m)}(\lambda) P(\nu)) \star \alpha^{(1)}(\nu, \bar{\nu}) \star (T^{(1)})^{-1}(\bar{\nu}, \mu), \quad (134) \]
\[ m_i = 1, \ldots, D_t, \quad i = 1, 2. \]
Similarly, to derive constraints associated with definition (115), we apply operator \(*\alpha^{(m)}\) to the eq. (127) from the right. One obtains

\[
\tilde{\xi}^{(m)} = \alpha^{(1)} \ast (T^{(1)})^{-1} \ast \xi^{(1)} \ast \alpha^{(m)} - \tilde{\xi}^{(1)} \ast (T^{(1)})^{-1} \ast \beta^{(1)} \ast \alpha^{(m)}.
\] (135)

Substitute eq. (118) for \(\tilde{\xi}^{(m)}\) and require that resulting equation is identity for any \(\tilde{\xi}^{(1)}\). Then eq. (135) becomes decomposed into two following constraints:

\[
\xi_{0}(\lambda, \nu) \ast \alpha^{(1)}(\nu, \mu) = 0,
\] (136)

\[
\eta^{(m)}(\lambda, \mu) = - (\tilde{T}^{(1)})^{-1}(\lambda, \nu) \ast \beta^{(1)}(\nu, \tilde{\nu}) \ast \alpha^{(m)}(\tilde{\nu}, \mu).
\] (137)

Eqs. (133) and (136) represent two constraints for \(\alpha^{(1)}, \beta^{(m)}\) and \(\tilde{\beta}^{(m)}\). Eqs. (134) and (137) with \(m_{1} + m_{2} > 2\) may be considered as definitions of \(\eta^{(m)}\) and \(\tilde{\eta}^{(m)}\). However, since \(\eta^{(1)}(\lambda, \mu) = \tilde{\eta}^{(1)}(\lambda, \mu) = \mathcal{L}_{2}(\lambda, \mu),\) eq. (137) with \(m_{1} = m_{2} = 1\) yields:

\[
\tilde{T}^{(1)}(\lambda, \mu) = - \beta^{(1)}(\lambda, \nu) \ast \alpha^{(1)}(\nu, \mu),
\] (138)

In turn, eq. (134) with \(m_{1} = m_{2} = 1\) in view of eq. (138) yields

\[
T^{(1)}(\lambda, \mu) = - \tilde{T}^{(1)}(\lambda, \mu) + \tilde{\beta}^{(1)}(\lambda)P(\nu) \ast \alpha^{(1)}(\nu, \mu),
\] (139)

Both eqs. (138) and (139) may be treated as constraints for \(T^{(1)}, \tilde{T}^{(1)}, \alpha^{(1)}\) and \(\beta^{(1)}\).

Now we simplify eq. (137), \(m_{1} + m_{2} > 2\), replacing \(\beta^{(1)} \ast \alpha^{(1)}\) in accordance with eq. (138). One gets

\[
\tilde{\eta}^{(m)} \ast \eta^{(m)} = T^{(m)} \ast (T^{(1)})^{-1} \Rightarrow \quad \text{eq. (130)}
\] (140)

\[
\gamma^{(m)}(\lambda, \mu) = (\tilde{T}^{(1)})^{-1}(\lambda, \nu) \ast T^{(m)}(\nu, \mu), \quad m_{1} + m_{2} > 2,
\] (141)

which is the definition of \(\gamma^{(m)}\). Deriving eq. (141) we assume invertibility of the operator \(*T^{(m)}, \forall m\).

Constraints (133) and (134) may be simplified multiplying them by \(\tilde{\eta}^{(m)}\) from the left, using definitions of \(\alpha^{(m)}\) (121), \(\beta^{(m)}\) (122), \(\tilde{\Gamma}^{(m)}\) (131) and eliminating \(\tilde{\eta}^{(m)} \ast \eta^{(m)}\) with eq. (140). We obtain in result:

\[
\left( (\tilde{T}^{(1)})^{-1}(\lambda, \nu) \ast \tilde{T}^{(m)}(\nu, \tilde{\nu}) \ast \beta^{(1)}(\tilde{\nu}, \tilde{\mu}) + \tilde{\Gamma}^{(m)}(\lambda)P(\tilde{\mu}) \right) \ast \xi_{0}(\tilde{\mu}, \mu) = 0, \quad \forall m,
\] (142)

\[
\tilde{\eta}^{(m)} \ast \tilde{\beta}^{(m)}P \ast \alpha^{(1)} = T^{(m)} + \tilde{T}^{(m)} \Rightarrow \quad \text{eq. (131)}
\] (143)

\[
\tilde{\Gamma}^{(m)}(\lambda)P(\nu) \ast \alpha^{(1)}(\nu, \mu) = T^{(m)}(\lambda, \mu) + \tilde{T}^{(m)}(\lambda, \mu), \quad m_{1} + m_{2} > 2.
\] (144)

One has to take into account that \(T^{(m)}, \tilde{T}^{(m)}\) are represented by eqs. (123) in terms of \(T(\lambda, \mu), \tilde{T}(\lambda, \mu), \tau^{(m)}(\lambda, \mu)\) and \(\tilde{\tau}^{(m)}(\lambda, \mu)\). Thus, we have obtained a set of constraints for the functions \(\alpha^{(1)}, \beta^{(1)}, \tilde{\Gamma}^{(m)}, T, \tilde{T}, \tau^{(m)}\) and \(\tilde{\tau}^{(m)}\); eqs. (136) (138) (139) (142) (144).

Note that, similar to Sec. 2.3.1, all these constraints are generated by system (96) and its compatibility condition. For this reason we refer to them as the internal constraints in order to defer them from so-called external constraints which will be introduced "by hand" for the purpose of derivation of the nonlinear PDEs, see Theorems 3.3, 3.4, 3.5.
3.3.2 System of compatible linear equations for $W(\lambda; t)$.

**Theorem 3.3.** Let matrices $A^{(m)}(\lambda, \mu)$ satisfy the following external constraint:

$$\sum_{m_1=1}^{D_1} L^{(m_1)} P(\lambda) \ast A^{(m)}(\lambda, \mu) = S^{(m_2)} P(\lambda), \quad (145)$$

where $L^{(m_1)}$ and $S^{(m_2)}$ are some $n_0 \times n_0$ constant matrices. Then matrix function $W(\lambda; t)$ obtained as a solution to the integral equation (11) with $\chi$ defined by eq.(96) is a solution to the following system of compatible linear equations

$$E^{(m_2)}(\lambda; t) := \sum_{m_1=1}^{D_1} L^{(m_1)} (W_{tm}(\lambda; t) + V^{(m)}(t)W(\lambda; t) + W(\mu; t) \ast A^{(m)}(\mu, \lambda; t)) = S^{(m_2)} W(\lambda; t), \quad (146)$$

where $V^{(m)}$ is given by eq.(17).

**Proof:** To derive eq.(146), we differentiate eq.(11) with respect to $t_m$. Then, in view of eq.(96), one gets the following integral equation:

$$E^{(m)}(\mu; t) := P(\nu) \ast A^{(m)}(\nu, \lambda) \ast \chi(\lambda, \mu; t) = \tilde{E}^{(m)}(\nu; t) \ast (\chi(\nu, \mu; t) + \mathcal{I}_1(\nu, \mu)), \quad (147)$$

$$\tilde{E}^{(m)}(\mu; t) = W_{tm}(\mu; t) + V^{(m)}(t)W(\mu; t) + W(\nu; t) \ast A^{(m)}(\nu, \mu).$$

Consider the following combination of eqs.(147): $\sum_{m_1=1}^{D_1} L^{(m_1)} E^m$. Then, using constraint (145), one gets in result:

$$\sum_{m_1=1}^{D_1} L^{(m_1)} E^m := \sum_{m_1=1}^{D_1} L^{(m_1)} \tilde{E}^{(m)}(\nu; t) \ast (\chi(\nu, \mu; t) + \mathcal{I}_1(\nu, \mu)) = S^{(m_2)} W(\nu; t) \ast (\chi(\nu, \mu; t) + \mathcal{I}_1(\nu, \mu)). \quad (148)$$

Since operator $\ast (\chi(\nu, \mu; t) + \mathcal{I}_1(\nu, \mu))$ is invertable, eq.(148) is equivalent to eq.(146). ■

**Remark:** Similar to eq.(16), eq.(146) is, strictly speaking, nonlinear equation for $W(\lambda; t)$ since $V^{(m)}(t)$ are defined by eq.(17) in terms of $W(\lambda; t)$.

System (146) is an analogy of the overdetermined system of linear equations in the classical inverse spectral transform method. According to this method, nonlinear PDEs for potentials of overdetermined linear system appear as compatibility conditions for this system. However, nonlinear PDEs may not be obtained by this method in our case because of the last term in the LHS of eq.(146). Instead of this, we represent another algorithm of derivation of the nonlinear PDEs in Secs.3.3.3 and 3.3.4.

3.3.3 First order nonlinear PDEs for the fields $V^{(m)}(t)$, $m_i = 1, \ldots, D_i$, $i = 1, 2$.

**Theorem 3.4.** In addition to eqs.(1196) and external constraint (145), we impose one more external constraint:

$$\sum_{m_2=1}^{D_2} A^{(m)}(\lambda, \nu) \ast \tilde{A}^{(m)}(\nu) R^{(m_2n)} = \sum_{j=1}^{D} \tilde{A}^{(j)}(\lambda) P^{(m_1jn)}, \quad n_i = 1, \ldots, D_i, \quad i = 1, 2, \quad (149)$$
where \( R^{(m_2n)} \) and \( P^{(m_1jn)} \) are some \( n_0 \times n_0 \) constant matrices. Then \( n_0 \times n_0 \) matrix functions \( V^{(m)}(t) \) are solutions to the following system of nonlinear PDEs:

\[
\begin{align*}
\sum_{m=1}^{D} L^{(m_1)} \left( V^{(n)}_{t_m} + V^{(m)} V^{(n)} + V^{(m)} A^{(n)} \right) R^{(m_2n)} + \sum_{m_1=1}^{D_1} L^{(m_1)} \sum_{j=1}^{D} V^{(j)} P^{(m_1jn)} &= (150) \\
\sum_{m_2=1}^{D_2} S^{(m_2)} V^{(n)} R^{(m_2n)}, \quad A^{(n)} = P(\lambda, \nu) * \tilde{A}^{(n)}(\nu), \quad n_i = 1, \ldots, D_i, \quad i = 1, 2.
\end{align*}
\]

**Proof:** Applying operator \(*\tilde{A}^{(n)}\) to eq.(146) from the right one gets the following equation

\[
E^{(m_2n)}(t) = E^{(m_2)}(\lambda; t) * \tilde{A}^{(n)}(\lambda) :=
\sum_{m_1=1}^{m_1} L^{(m_1)} \left( V^{(n)}_{t_m} + V^{(m)} V^{(n)} + V^{(m)} A^{(n)} + U^{(mn)} \right) = \sum_{m_2=1}^{m_2} S^{(m_2)} V^{(n)},
\]

which introduces a new set of fields \( U^{(mn)} \), see eq.(5). Due to the constraint (149), we may eliminate these fields using a proper combinations of eqs.(151). Namely, combinations \( \sum_{m_2=1}^{D_2} E^{(m_2n)} R^{(m_2n)} \) results in the system (150). ■

**Reduction 1** The derived equation (150) admits the following reduction for its coefficients:

\[
P^{(m_1jn)} = -A^{(n)} R^{(j_2n)} \delta_{j_1 m_1}, \quad S^{(m_2)} = 0.
\]

Then eq.(150) reads:

\[
\sum_{m=1}^{D} L^{(m_1)} \left( V^{(n)}_{t_m} + V^{(m)} V^{(n)} \right) R^{(m_2n)} = 0,
\]

which is represented in the introduction, see eq.(5). This reduction does not effect constraint (145) while constraint (149) reads:

\[
\sum_{m_2=1}^{D_2} \left( A^{(m)}(\lambda, \nu) + \tilde{A}^{(m)}(\lambda) P(\nu) \right) * \tilde{A}^{(n)}(\nu) R^{(m_2n)} = 0, \quad n_i = 1, \ldots, D_i, \quad i = 1, 2.
\]

**Reduction 2** Along with reduction (152), we consider reduction (57,58) with

\[
C^{(n)} R^{(m_2n)} = R^{(m_2)}.
\]

where \( C^{(m)} \) and \( R^{(m_2)} \) are some \( n_0 \times n_0 \) constant matrices. Then the system (153) reduces to the single PDE:

\[
\sum_{m=1}^{D} L^{(m_1)} \left( V^{(m)} + VC^{(m)} V \right) R^{(m_2)} = 0,
\]

which is written in the Introduction, see eq.(5). Reduction (155) does not effect constraint (145), while constraint (154) reduces to the following single equation

\[
\sum_{m_2=1}^{D_2} \left( A^{(m)}(\lambda, \nu) + \tilde{A}(\lambda) C^{(m)} P(\nu) \right) * \tilde{A}(\nu) R^{(m_2)} = 0.
\]
If, in addition, $C^{(m_1m_2)}$, $L^{(m_1)}$ and $R^{(m_2)}$ are diagonal,

$$C^{(m_1m_2)} = -C^{(m_2m_1)}, \quad R^{(m_2)} \equiv L^{(m_2)}, \quad V_{m_1m_2} = -V_{m_2m_1}, \quad D_1 = D_2 = D_0,$$  \hspace{1cm} (158)

then nonlinear equation (156) reduces to the following equation

$$\sum_{m_1, m_2 = 1 \atop m_2 > m_1}^{D_0} \left( L^{(m_1)} V_{t_{m_1m_2}} L^{(m_2)} - L^{(m_2)} V_{t_{m_1m_2}} L^{(m_1)} + L^{(m_1)} VC^{(m_1m_2)} VL^{(m_2)} - L^{(m_2)} VC^{(m_1m_2)} VL^{(m_1)} \right) = 0,$$  \hspace{1cm} (159)

which is a multidimensional generalization of the classical (2+1)-dimensional $S$-integrable $N$-wave equation (106).

Eq.(159) admits reduction

$$t_{m_1m_2} = -i\tau_{m_1m_2}, \quad V = -V^+, \quad m_2 > m_1,$$  \hspace{1cm} (160)

see eq.(5), which is important for physical applications.

### 3.3.4 Second order nonlinear PDEs for $V^{(m)}(t), \ m_i = 1, \ldots, D_i, \ i = 1, 2$

**Theorem 3.5.** In addition to eqs.(11) and external constraint (145) we impose one more external constraint on the matrix functions $A^{(m)}(\lambda, \nu)$ and $\tilde{A}^{(m)}(\lambda)$ (instead of constraint (149)):

$$\sum_{n=1}^{D} \sum_{m_2=1}^{D_2} A^{(m)}(\lambda, \nu) * A^{(n)}(\nu, \mu) * \tilde{A}^{(l)}(\mu) R^{(m_2nl)} =$$

$$\sum_{n,p=1}^{D} A^{(n)}(\lambda, \mu) * \tilde{A}^{(p)}(\mu) P^{(m_1npl)} + \sum_{n=1}^{D} \tilde{A}^{(n)}(\lambda) P^{(m_1nl)},$$

where $R^{(m_2nl)}$, $P^{(m_1npl)}$, and $P^{(m_1nl)}$ are some $n_0 \times n_0$ constant matrices. Let $S^{(m_2)} = 0$ for the sake of simplicity. Then $n_0 \times n_0$ matrix functions $V^{(m)}(t)$ are solutions to the following system of nonlinear PDEs:

$$\sum_{m,n=1}^{D} L^{(m_1)} \left( U^{(nl)}_{t_{m}} + V^{(m)} U^{(nl)} + V^{(m)} A^{(nl)} \right) R^{(m_2nl)} +$$

$$\sum_{m_1=1}^{D_1} L^{(m_1)} \left( \sum_{n,p=1}^{D} U^{(np)} P^{(m_1npl)} + \sum_{n=1}^{D} V^{(n)} P^{(m_1nl)} \right) = 0,$$

$$A^{(nl)} = P(\lambda) * A^{(n)}(\lambda, \mu) * \tilde{A}^{(l)}(\mu), \quad l_i = 1, \ldots, D_i, \quad i = 1, 2,$$

where fields $U^{(mn)}$ are related with $V^{(m)}$ due to eq.(151):

$$\sum_{m_1=1}^{D_1} L^{(m_1)} \left( V^{(n)}_{t_{m_1}} + V^{(m)} V^{(n)} + V^{(m)} A^{(n)} + U^{(mn)} \right) = 0,$$  \hspace{1cm} (163)

$$A^{(n)} = P(\lambda) * \tilde{A}^{(n)}(\lambda).$$

**Proof:** First of all, applying operator $* \tilde{A}^{(n)}$ to eq.(146) from the right one gets equation (163), which introduces a new set of fields $U^{(mn)}(t)$, see eq.(53). This step is equivalent to the
first step in derivation of eq.  (150). However, we may not eliminate these fields from the system (163) because constraint (149) is not valid in this Theorem. Instead of this, we derive nonlinear equations for fields \( U^{(mn)}(t) \) applying operator \(*A(n)\*\tilde{A}(l)\) to eq. (164) from the right. One gets

\[
E^{(mnl)}(t) = E^{(m2)}(\lambda; t) * A(n)(\lambda, \mu) * \tilde{A}(l)(\mu) :=
\]

\[
\sum_{m1=1}^{D1} L^{(m1)} \left( U^{(nt)} + V^{(m)} U^{(nl)} + V^{(m)} A^{(nl)} + U^{(mnl)} \right) = 0,
\]

where the fields \( U^{(mnl)} \) are defined by eq. (166). Due to the constraint (161), these fields may be eliminated taking a proper combination of equations (164), namely, \( \sum_{m2=1}^{D2} E^{(m2nl)} R^{(m2nl)} \). In result, one obtains eq. (162).

Reduction 1. Let

\[
P^{(m1np)} = \delta_{m1n1} \tilde{P}^{(n2p)}, \quad P^{(m2nl)} = \delta_{m1n1} \tilde{P}^{(n2l)},
\]

\[
\tilde{P}^{(n2l)} = \sum_{p=1}^{D} (A(p) \tilde{P}^{(n2p)} - A(p) R^{(n2p)}).
\]

Then eq. (162) reduces to the following one:

\[
\sum_{m,n=1}^{D} L^{(m1)} \left( \left( U^{(nl)} + V^{(m)} U^{(nl)} \right) R^{(m2nl)} + \left( V^{(n)} + V^{(m)} V^{(n)} \right) \tilde{P}^{(m2nl)} \right) = 0,
\]

\( l_i = 1, \ldots, D_i, \quad i = 1, 2. \)

Constraint (145) remains the same (with \( S^{(m2)} = 0 \)) while constraint (161) reads

\[
\sum_{m=1}^{D1} \sum_{n2=1}^{D2} \left( A^{(m)}(\lambda, \nu) + \tilde{A}^{(m)}(\lambda) P(\nu) \right) * \left( A^{(n)}(\nu, \mu) * \tilde{A}^{(l)}(\mu) R^{(m2nl)} - \tilde{A}^{(n)}(\tilde{\nu}) \tilde{P}^{(m2nl)} \right),
\]

\( m_i = 1, \ldots, D_i, \quad l_i = 1, \ldots, D_i, \quad i = 1, 2. \)

Reduction 2. Along with reduction (165) we consider reduction (57,58) together with the following conditions

\[
U^{(ml)}(t) = U^{(m)}(t) C^{(l)}, \quad C^{(l)} R^{(m2nl)} = R^{(m2n)}, \quad C^{(n)} \tilde{P}^{(m2nl)} = P^{(m2n)}.
\]

Then the system (166) reduces to the following single PDE:

\[
\sum_{m,n=1}^{D} L^{(m1)} \left( \left( U^{(nt)} + V C^{(m)} U^{(n)} \right) R^{(m2n)} + \left( V^{(nl)} + V C^{(m)} V^{(n)} \right) P^{(m2n)} \right) = 0
\]

This reduction does not effect constraint (145), while constraint (167) reduces to the following single equation:

\[
\sum_{m=1}^{D2} \sum_{n=1}^{D} \left( A^{(m)}(\lambda, \nu) + \tilde{A}(\lambda) C^{(m)} P(\nu) \right) * \left( A^{(n)}(\nu, \mu) * \tilde{A}(\mu) R^{(m2n)} - \tilde{A}(\nu) P^{(m2n)} \right).
\]

Emphasize that systems (150) and (162,163) do not represent commuting flows since constraints (149) and (161) are not compatible in general.
3.3.5 Solutions to the first-order equation (156)

On the dimensionality of the available solution space. The important question is whether the derived nonlinear DPEs are completely integrable. In other words, regarding eq.(156), is it possible to introduce a proper number of arbitrary functions of \( D_1D_2 - 1 \) variables in the solution space.

In order to clarify this problem, first of all, let us consider small \( \chi \). Then, using equation (11), we approximate \( V \) by eq.(73). If \( \tau^{(m)}(\nu) \) and \( \tau^{(m)}(\nu) \) are arbitrary functions of arguments, then the above expression for \( V \) involves two arbitrary functions of \( D_1D_2 \) variables \( t_m, m_i = 1, \ldots, D_i, i = 1, 2 \):

\[
F_1(t) = P * \alpha^{(1)} * (T^{(1)})^{-1} * \xi^{(1)} * \alpha^{(1)} * \tilde{\Gamma} = \int g_{11}(\nu)e^{\sum_{i=1}^{D_2} \tau^{(i)}t_i}g_{12}(\nu)d\Omega_2(\nu), \\
F_2(t) = P * \alpha^{(1)} * \tilde{\xi}^{(1)} * (\tilde{T}^{(1)})^{-1} * \beta^{(1)} * \alpha^{(1)} * \tilde{\Gamma} = \int g_{21}(\nu)e^{\sum_{i=1}^{D_2} \tau^{(i)}t_i}g_{22}(\nu)d\Omega_2(\nu),
\]

where

\[
g_{11} = P * \alpha^{(1)} * (T^{(1)})^{-1} * T, \quad g_{12} = T^{-1} * \xi_0 * \alpha^{(1)} * \tilde{\Gamma}, \\
g_{21} = P * \alpha^{(1)} * \tilde{\xi}_0 * \tilde{T}^{-1}, \quad g_{22} = \tilde{T} * (\tilde{T}^{(1)})^{-1} * \beta^{(1)} * \alpha^{(1)} * \tilde{\Gamma}.
\]

However, not all \( \tau^{(m)}(\nu) \) and \( \tau^{(m)}(\nu) \) are arbitrary since we have to resolve constraints \( 136, 138, 142, 144, 145, 157 \). Constraints \( 136, 142 \) may be satisfied using a special structures of \( \alpha^{(1)}, \beta^{(1)}, P, \xi_0, \xi_0 \) (see example of explicite solution, eqs.181), which does not reduce the dimensionality of the solution space. Constraints \( 138 \) may be considered as constraints for \( \alpha^{(1)} \) and \( \beta^{(1)} \). Constraints \( 139, 144 \) relate \( T^{(m)} \) with \( T^{(m)} \) and \( \Gamma \), so that, generally speaking, only one of functions \( 171 \) remains arbitrary. Two remaining constraints \( 145, 157 \) introduce \( D_2 \) and \( D_1 \) relations among parameters \( \tau^{(m)} \), \( m_i = 1, \ldots, D_i, i = 1, 2 \), which, in general, reduces significantly the dimensionality of the solution space. However, this depends on the particular choice of the coefficients \( L^{(mj_i)}, R^{(i\mu_i)} \) and \( C^{(m)} \).

This is a preliminary analysis which suggests us to look for examples of completely integrable PDEs in the derived class of new nonlinear PDEs.

Construction of explicite solutions. We construct explicite solutions in the form of rational functions of exponents for the first order PDE (156) with \( d\Omega_i(\nu) \), \( i = 1, 2 \), given by eqs.(75). Along with notations (76) we use the following ones:

\[
\hat{\xi}_0 = \left[ \begin{array}{ccc} \xi_0(a_1, b_1) & \cdots & \xi_0(a_1, b_N) \\ \cdots & \cdots & \cdots \\ \xi_0(a_M, b_1) & \cdots & \xi_0(a_M, b_N) \end{array} \right], \quad \hat{\tau}^{(m)} = \text{diag}(\tau^{(m)}(b_1), \ldots, \tau^{(m)}(b_N)), \\
T(\lambda, \mu) = \tilde{T}(\lambda, \mu) = I_2(\lambda, \mu).
\]

Field \( V(t) \) is represented by eq.(79) where function \( \chi \) is given by eq.(127). In turn, the functions \( \xi^{(1)} \) and \( \tilde{\xi}^{(1)} \) are given by eqs.(125) and (126). In result, using notations (76, 173), we obtain the following formula for \( \hat{\chi} \):

\[
\hat{\chi} = \hat{\alpha}^{(1)}(\hat{\tau}^{(1)})^{-1}e^{\sum_{i=1}^{D_2} \hat{\tau}^{(i)}t_i}\hat{\xi}^{(0)} + \hat{\xi}^{(0)}e^{\sum_{i=1}^{D_2} \hat{\tau}^{(i)}t_i}(\hat{\tau}^{(1)})^{-1}\hat{\beta}^{(1)}.
\]

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Finally, constraints (136, 138, 139, 142, 144, 145, 157) must be satisfied, which read in view of notations (76, 173) as follows, for instance, for \( \hat{m} \):

\[
\hat{m}_0 \alpha^{(1)} = 0, \\
\hat{m}_1 = -\hat{\beta}^{(1)} \hat{\alpha}^{(1)}, \\
\hat{C}(m) \hat{P} \hat{\alpha}^{(1)} = \hat{z}^{(m)} + \hat{\xi}^{(m)} \\
(\hat{z}^{(1)})^{-1} \hat{z}^{(m)} \hat{\beta}^{(1)} + \hat{\Gamma} C(m) \hat{P} \hat{\xi}_0 = 0, \\
\sum_{m_1=1}^{D_1} L^{(m_1)} \hat{P}_a^{(1)} (\hat{z}^{(1)})^{-1} \hat{z}^{(m)} \hat{\beta}^{(1)} = 0, \\
\sum_{m_2=1}^{D_2} \hat{a}^{(1)} ((\hat{z}^{(1)})^{-1} \hat{z}^{(m)} \hat{\beta}^{(1)} + \hat{\Gamma} C(m) \hat{P}) \hat{a}^{(1)} \hat{G}^R(m_2) = 0.
\]

Here we combine equations (139) and (144) into the single equation (177). Let us transform some of eqs. (175-180). For instance, considering only particular solutions to eq. (178) we assume

\[
\hat{\beta}^{(1)} \hat{G}^\xi_0 = \hat{P} \hat{G}^\xi_0 = 0.
\]

In other words, the rows of \( \hat{\beta}^{(1)} \) and \( \hat{P} \) are orthogonal to the columns of \( \hat{G}^\xi_0 \). Similarly, eq. (175) means that the rows of \( \hat{G}^\xi_0 \) are orthogonal to the columns of \( \hat{\alpha}^{(1)} \).

In turn, constraint (180) may be transformed into the following one using eq. (176).

\[
\sum_{m_2=1}^{D_2} \hat{a}^{(1)} ((\hat{z}^{(1)})^{-1} \hat{z}^{(m)} \hat{\beta}^{(1)} + \hat{\Gamma} C(m) \hat{P}) \hat{a}^{(1)} \hat{G}^R(m_2) = 0
\]

Since \( \hat{z}^{(1)} \) is invertible, one requires \( M > N \), which is evident due to eq. (176). Then we may rewrite eq. (182) without factor \( \hat{\alpha}^{(1)} \) as follows:

\[
\sum_{m_2=1}^{D_2} (\hat{z}^{(m)} - \hat{\Gamma} C(m) \hat{P} \hat{a}^{(1)}) \hat{G}^R(m_2) = 0.
\]

All in all, the following constraints must be satisfied: (175, 177, 179, 181, 183).

**Simple examples of explicite solutions.** We obtain a particular solution to the four-dimensional nonlinear PDE (156), i.e. \( D_1 = D_2 = 2 \). Remember that we use double indices so that, for instance, \( \hat{\beta}^{(m)} = \hat{\beta}^{(m_1 m_2)} \). Let \( M = 5, N = 2, n_0 = 2, L^{(1)} = R^{(1)} = I, L^{(2)} = \text{diag}(l_1, l_2), R^{(2)} = \text{diag}(r_1, r_2), C^{(n_1 n_2)} = \text{diag}(c_1^{(n_1 n_2)}, c_2^{(n_1 n_2)}), c_1^{(11)} = c_2^{(11)} = 1. \) To satisfy eqs. (175) and (181) we use the following structures of matrices:

\[
\hat{\beta}^{(11)} = \begin{bmatrix} Z & Z & I & Z & J_1 \\ Z & Z & Z & J_1 & I \end{bmatrix}, \\
\hat{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 & 2 & 1 & 2 \end{bmatrix}, \\
\hat{\Gamma}^{(11)} = \begin{bmatrix} J_0 \\ Z \end{bmatrix}, \\
\hat{\xi}_0 = \begin{bmatrix} J_0 & Z & Z & Z & Z \\ Z & -J_2 & Z & Z & Z \end{bmatrix}, \\
\hat{\xi}_0 = \begin{bmatrix} K_2 & Z \\ Z & J_0 \\ Z & Z \\ Z & Z \\ Z & Z \end{bmatrix}, \\
\hat{\alpha}^{(11)} = \begin{bmatrix} Z_4 \\ K_{6,4} \end{bmatrix},
\]

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where \( Z \) and \( I \) are \( 2 \times 2 \) zero and identity matrices respectively, \( Z_4 \) is \( 4 \times 4 \) zero matrix, \( K_{6,4} \) is \( 6 \times 4 \) constant matrix which will be defined below,

\[
J_0 = \text{diag}(1, -1), \quad J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} s_3 & 1 \\ s_4 & s_5 \end{bmatrix} \tag{185}
\]

We take \( \hat{\tau}^{(11)} = \text{diag}(1, 2, 3, 4) \).

To satisfy eqs. (176) and (177) \((m_1 = m_2 = 1)\) we take

\[
K_{6,4} = \begin{bmatrix}
\frac{1}{5}(2s_1 - 3) & s_2 & -1 & \frac{4}{4} \\
-1 & 2 & \frac{3}{3} & \frac{8}{8} \\
\frac{2}{5}(s_1 + 1) & s_2 & -1 & \frac{3}{3} \\
s_1 & 2 & \frac{-2}{2} & \frac{4}{4} \\
-s_1 & -s_2 & -1 & \frac{3}{3} \\
-\frac{2}{5}(s_1 + 1) & -s_2 & 1 & \frac{3}{3}
\end{bmatrix}
\tag{186}
\]

\( \tau^{(11)} = \text{diag} \left( \frac{2}{5}(3s_1 - 2, 3s_2, -3, -4) \right) \).

To satisfy eqs. (179) and (183), \( m_1 + m_2 > 2 \), we take matrices \( \hat{\Gamma}^{(n_1n_2)} \) in the following form:

\[
\hat{\Gamma}^{(12)} = \begin{bmatrix} J_3 \\ Z \end{bmatrix}, \quad \hat{\Gamma}^{(21)} = \begin{bmatrix} J_4 \\ Z \end{bmatrix}, \quad \hat{\Gamma}^{(22)} = \begin{bmatrix} J_5 \\ Z \end{bmatrix},
\tag{187}
\]

\[
J_3 = \text{diag} \left( \frac{1}{r_1} \left( 1 + c_1^{(21)}l_1 + c_1^{(22)}l_1r_1 \right), \frac{1}{r_2} \left( 1 + c_2^{(21)}l_2 + c_2^{(22)}l_2r_2 \right) \right),
\]

\[
J_4 = \text{diag}(c_1^{(21)}, -c_1^{(21)}), \quad J_5 = \text{diag}(c_1^{(22)}, -c_2^{(22)}).
\]

Elements of matrix \( C^{(12)} \) must be defined as follows:

\[
c_i^{(12)} = -\frac{1}{r_i} \left( 1 + l_i(c_i^{(21)} + c_i^{(22)}r_i) \right), \quad i = 1, 2,
\tag{188}
\]
and $\hat{\tau}^{(n_1, n_2)}$, $\hat{\tau}^{(n_1, n_2)}$ are following:

\[
\hat{\tau}^{(12)} = \text{diag} \left( \frac{4 - 6 s_1}{5 r_1}, -\frac{3 s_2}{r_2}, \hat{\tau}_3^{(12)}, \hat{\tau}_4^{(12)} \right), \tag{189}
\]

\[
\hat{\tau}^{(21)} = \text{diag} \left( \frac{1}{5} c_1^{(21)} (6 s_1 + 1) + \frac{1}{l_1}, c_2^{(21)} (2 + 3 s_2) + \frac{2}{l_2}, \hat{\tau}_3^{(21)}, \hat{\tau}_4^{(21)} \right), \tag{189}
\]

\[
\hat{\tau}^{(22)} = \text{diag} \left( -\frac{5 + c_1^{(21)} (1 + 6 s_1) l_1}{5 l_1 r_1}, -\frac{2 + c_2^{(21)} (2 + 3 s_2) l_2}{l_2 r_2}, \hat{\tau}_3^{(22)}, \hat{\tau}_4^{(22)} \right), \tag{189}
\]

\[
\hat{\tau}^{(12)} = \text{diag} \left( -\frac{1}{5 r_1} (5 + (1 + 6 s_1) l_1 (c_1^{(21)} + c_1^{(22)} r_1)), \right.
\]

\[
-\frac{1}{r_2} (2 + (2 + 3 s_2) l_2 (c_2^{(21)} + c_2^{(22)} r_2)), -\hat{\tau}_3^{(12)}, -\hat{\tau}_4^{(12)} \right), \tag{189}
\]

\[
\hat{\tau}^{(21)} = -\text{diag} \left( \frac{1}{l_1}, \frac{2}{l_2}, \tau_3^{(21)}, \tau_4^{(21)} \right), \tag{189}
\]

\[
\hat{\tau}^{(22)} = \text{diag} \left( \frac{1}{5 l_1 r_1} (5 + (1 + 6 s_1) l_1 (c_1^{(21)} + c_1^{(22)} r_1)), \right.
\]

\[
\frac{1}{l_2 r_2} (2 + (2 + 3 s_2) l_2 (c_2^{(21)} + c_2^{(22)} r_2)), -\hat{\tau}_3^{(22)}, -\hat{\tau}_4^{(22)} \right), \tag{189}
\]

Introduce positive parameters $p_i$, $i = 1, 2, 3, 4$, by the following formulas:

\[
s_2 = \frac{p_3}{18 s_1 - 12}, \quad s_3 = -\frac{2}{5 p_3} (3 s_1 - 2) p_1, \tag{190}
\]

\[
s_4 = \frac{1}{5 p_3} (p_3 p_4 - p_1 p_2), \quad s_5 = \frac{p_2}{6 s_1 - 4},
\]

so that solution $V(t)$ reads

\[
V(t) = \frac{1}{D} \begin{bmatrix}
    f_{11}(p_2 e^{\hat{\eta}_2} + p_3 e^{\hat{\eta}_3 + \hat{\eta}_2}) & f_{12} e^{\hat{\eta}_3} \\
    f_{21} e^{\hat{\eta}_1 + \hat{\eta}_2 - \hat{\eta}_3} & f_{22}(p_1 e^{\hat{\eta}_1} + p_3 e^{\hat{\eta}_3 + \hat{\eta}_2})
\end{bmatrix}, \tag{191}
\]

\[
D = p_1 e^{\hat{\eta}_1} + p_2 e^{\hat{\eta}_2} + p_3 e^{\hat{\eta}_3 + \hat{\eta}_2} + p_4,
\]

\[
f_{11} = -\frac{1}{5} (1 + 6 s_1), \quad f_{12} = \frac{p_3 (1 + 6 s_1)}{2 (3 s_1 - 2)},
\]

\[
f_{21} = \frac{1}{5 p_3} (p_1 p_2 - p_3 p_4) (12 s_1 + p_3 - 8), \quad f_{22} = -\frac{12 s_1 + p_3 - 8}{2 (3 s_1 - 2)}.
\]

Here

\[
\hat{\eta}_1 = \eta_1 - \eta_3, \quad \hat{\eta}_2 = \eta_2 - \eta_3, \quad \hat{\eta}_3 = \eta_4 - \eta_3, \quad \eta_i = \sum_{m_1, m_2 = 1}^{2} a^{(i)}_{m_1 m_2} t^{m_1 m_2}, \quad i = 1, 2, 3, 4, \tag{192}
\]

\[
a^{(1)}_{11} = \frac{1 + 6 s_1}{5}, \quad a^{(1)}_{12} = \frac{1}{2 r_2} \left( 4 + \frac{p_3}{3 s_1 - 2} \right), \tag{193}
\]

\[
a^{(1)}_{21} = \frac{1}{5 l_2} \left( 10 + l_2 c_1^{(21)} (1 + 6 s_1) \right),
\]

\[
a^{(1)}_{22} = \frac{1}{5} \left( 5 + l_1 c_1^{(21)} - 10 c_2^{(21)} + c_1^{(22)} - 10 c_2^{(22)} + \frac{5 (c_2^{(21)} + r_2 c_2^{(22)}) p_3}{r_2 (4 - 6 s_1)} + \frac{6 c_1^{(21)} + r_1 c_1^{(22)}}{r_1} \right),
\]

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\[ a_{11}^{(2)} = 2 + \frac{p_3}{6s_1 - 4}, \]
\[ a_{12}^{(2)} = \frac{1}{5} \left( \frac{1 + l_1 c_1^{(21)} + (1 + 6s_1)}{r_1} \right) + \frac{l_1 c_1^{(22)} (1 + 6s_1) - 5l_2 (c_2^{(21)} + r_2 c_2^{(22)}) (-8 + p_3 + 12s_1)}{2r_2 (3s_1 - 2)}, \]
\[ a_{21}^{(2)} = \frac{2}{l_2} + \frac{c_2^{(21)}}{2 + \frac{p_3}{6s_1 - 4}}, \quad a_{22}^{(2)} = \frac{5 + l_1 c_1^{(21)} (1 + 6s_1)}{5l_1 r_1} + \frac{c_2^{(21)}}{2r_2} \left( -\frac{2}{r_2} + \frac{p_3}{4r_2 - 6r_2 s_1} \right), \]
\[ a_{11}^{(3)} = \frac{11}{5} + \frac{6s_1}{5} + \frac{p_3}{6s_1 - 4}, \quad a_{12}^{(3)} = -\frac{l_2 (c_2^{(21)} + r_2 c_2^{(22)}) (-8 + p_3 + 12s_1)}{2r_2 (3s_1 - 2)}, \]
\[ a_{21}^{(3)} = \frac{2}{l_2} + \frac{c_1^{(21)} (1 + 6s_1)}{5} + c_2^{(21)} \left( 2 + \frac{p_3}{6s_1 - 4} \right), \]
\[ a_{22}^{(3)} = \frac{-5l_1 r_1 c_2^{(21)} (-8 + p_3 + 12s_1) + 2r_2 (3s_1 - 2) (5 + l_1 c_1^{(21)} + r_1 c_1^{(22)}) (1 + 6s_1)}{10l_1 r_1 r_2 (3s_1 - 2)}, \]
\[ a_{11}^{(4)} = \frac{6 (1 + s_1)}{5}, \quad a_{12}^{(4)} = \frac{1}{5r_1} \left( 5 + l_1 c_1^{(21)} + r_1 c_1^{(22)} (1 + 6s_1) \right) + \frac{p_3 - l_2 (c_2^{(21)} + r_2 c_2^{(22)}) (-8 + p_3 + 12s_1)}{2r_2 (3s_1 - 2)}, \]
\[ a_{21}^{(4)} = \frac{1}{l_1} + \frac{c_1^{(21)} (1 + 6s_1)}{5}, \quad a_{22}^{(4)} = \frac{2}{l_2 r_2}. \]

Note that not all constant matrices may be arbitrary diagonal matrices in eq. (156). In fact, eqs. (188) mean the following relation

\[ C^{(12)} R^{(2)} + L^{(2)} (C^{(21)} + C^{(22)} R^{(2)}) + I_2 = 0 \]

Since all \( p_i \) are positive, solution \( V(t) \) (191) has no singularities unless \( \sum_{m_1, m_2=1}^{2} |t_{m_1 m_2}| \rightarrow \infty \). However, offdiagonal elements of \( V \) tend to infinity in some directions in the space of parameters \( t_{m_1 m_2} \). Thus, \( V \) is not bounded solution. Now we derive a simple example of the bounded soliton-kink solution. For this purpose we take

\[ s_3 = s_5 = 0, \quad s_4 = -1 \]

instead of eqs. (190). Then one gets the following formula for \( V \):

\[ V(t) = \left[ \begin{array}{cc} f_{11} & f_{12} \\ d + 5e^{-\eta_{12} + \eta_{21}} & de^{-\eta_{12} + 5e^{\eta_{21}}} \\ f_{21} & f_{22} \\ de^{-\eta_{21} + 5e^{\eta_{12}}} & d + 5e^{\eta_{12} + \eta_{21}} \end{array} \right], \]
\[ f_{11} = 6 \frac{(3s_1 - 2)(6s_1 + 1)s_2}{(3s_1 - 2)(6s_1 + 1)s_2}, \quad f_{12} = -3(6s_1 + 1)s_2, \]
\[ f_{21} = -2(3s_1 - 2)(3s_2 + 2), \quad f_{22} = 6(3s_1 - 2)(3s_2 + 2)s_2, \]
\[ d = -6(3s_1 - 2)s_2, \]
where \( \eta_{n1,n2} \) are the linear functions of \( t_{n1,n2} \):

\[
\eta_{12} = \sum_{n1,n2=1}^{2} a_{n1,n2} t_{n1,n2}, \quad \eta_{21} = \sum_{n1,n2=1}^{2} b_{n1,n2} t_{n1,n2}, \quad (200)
\]

\[
a_{11} = \frac{6}{5} (s_1 + 1), \quad a_{12} = -\frac{1}{r_1 r_2}(r_1 (2 + (3s_2 + 2)l_2 (c_2^{(21)} + c_2^{(22)})) + r_2 (a_{11} - 2)),
\]

\[
a_{21} = \frac{c_1^{(21)}}{5} (6s_1 + 1) + \frac{1}{r_1 r_2}(r_1 (2 + (3s_2 + 2)l_2 (c_2^{(21)} + c_2^{(22)})) - r_2 (a_{21} l_2 + 2)),
\]

\[
b_{11} = 3s_2 + 1, \quad b_{12} = -\frac{1}{5r_1 r_2}(r_2 (5 + (6s_1 + 1)l_1 (c_1^{(21)} + c_1^{(22)})) + 5r_1 (b_{11} - 1)),
\]

\[
b_{21} = c_2^{(21)} (3s_2 + 2) - \frac{1}{l_1} + \frac{2}{l_2}, \quad b_{22} = \frac{1}{5l_1 r_1 r_2}(r_2 (5 + (6s_1 + 1)l_1 (c_1^{(21)} + c_1^{(22)})) - 5r_1 (b_{21} l_1 + 1)).
\]

If \( d > 0 \), then the diagonal elements of matrix \((199)\) are kinks, while offdiagonal elements tend to infinity in some directions in the space of parameters \( t_{m1,m2} \). In order to obtain bounded solution we require

\[
\eta_{12} = a \eta_{21} \Rightarrow a_{ij} = ab_{ij}, \quad a > 0.
\]

For the sake of simplicity, we solve eqs.\((201)\) for the particular choice of the arbitrary parameters:

\[
r_1 = 2, \quad r_2 = 3, \quad l_1 = 4, \quad l_2 = 5, \quad c_1^{(21)} = 6.
\]

One has

\[
s_1 = \frac{741 + 366 c_2^{(22)} - 12 c_1^{(22)} (7 + 24 c_2^{(22)})}{8 (-559 + 108 c_2^{(22)} + 36 c_1^{(22)} (2 + c_2^{(22)}))},
\]

\[
s_2 = \left( - (14521 - 181090 c_2^{(22)} + 43200 (c_2^{(22)})^2 + 144 (c_1^{(22)})^2 (-1 + 20 c_2^{(22)}) + 40 c_1^{(22)} (-20 + 9 c_2^{(22)} + 360 (c_2^{(22)})^2) \right) \times \left( 40 (-1 + 15 c_2^{(22)})(-559 + 108 c_2^{(22)} + 36 c_1^{(22)} (2 + c_2^{(22)})) \right)^{-1},
\]

\[
a = \frac{246 - 3690 c_2^{(22)}}{-233 + 2160 c_2^{(22)} + 36 c_1^{(22)} (-1 + 20 c_2^{(22)})},
\]

\[
c_2^{(21)} = \frac{25 - 36 (3 + c_1^{(22)}) c_2^{(22)}}{-89 + 12 c_2^{(22)}}.
\]

In addition, one has to provide positivity of \( a = p_1 > 0 \) in eq.\((199)\) and positivity of \( d = p_2 > 0 \). This requirement yields constraint for \( c_i^{(22)}, i = 1, 2: \)

\[
c_1^{(22)} = (1240p_1 - p_1^3 (725 + 144p_2) - p_1^2 (525 + 248p_2) \pm p_1 (1 + p_1) \sqrt{25 (-2 + p_1)^2 - 20 p_1 p_2}) \left(48 p_1^2 (-5 + p_1 (5 + p_2)) \right)^{-1},
\]

\[
c_2^{(22)} = \left( - 1200p_1 + 5 p_1^2 (-263 + 48p_2) + 5 p_1^3 (497 + 100p_2) \pm 3p_1 (1 + p_1) \sqrt{25 (-2 + p_1)^2 - 20 p_1 p_2} \right) \times \left( 60 (10 p_1 - 5 p_1^3 - p_1^2 (15 + 2p_2) \pm p_1 (1 + p_1) \sqrt{25 (-2 + p_1)^2 - 20 p_1 p_2}) \right)^{-1}.
\]
In particular, if \( p_1 = 1/2 \) and \( p_2 = 1 \), one has
\[
s_1 = \frac{1 \pm \sqrt{185}}{24}, \quad s_2 = \frac{15 \pm \sqrt{185}}{30}, \quad c_2^{(21)} = \frac{3(-145 \pm 123\sqrt{185})}{2080},
\]
\[
\begin{align*}
c_1^{(22)} &= -\frac{2545 \pm 3\sqrt{185}}{192}, \\
c_2^{(22)} &= \frac{70 \pm 93\sqrt{185}}{780}.
\end{align*}
\]

Now expression for \( V \), eq.(199) reads
\[
V(t) = -\frac{5 \pm \sqrt{185}}{20(1 + 5e^{(a+1)\eta_{21}})} \begin{pmatrix} 13 \pm \sqrt{185} \\ 2(35 \pm \sqrt{185}) \\ 10(1 + 5e^{(a+1)\eta_{21}}) \end{pmatrix}, \quad \eta_{21} = \sum_{n_1,n_2=1}^{\mathbf{2}} b_{n_1n_2} t_{n_1n_2}. \tag{209}
\]

Appropriate expressions for \( b_{n_1n_2} \) are following
\[
\begin{align*}
b_{11} &= \frac{25 \pm \sqrt{185}}{10}, \quad b_{12} = \frac{295 \pm 61\sqrt{185}}{30}, \\
b_{21} &= \frac{3 \left(9 \pm 2\sqrt{185}\right)}{10}, \quad b_{22} = -\frac{214 \pm 43\sqrt{185}}{60}.
\end{align*}
\] We see that diagonal elements of \( V \) represent kinks, while off-diagonal elements represent non-symmetrically shaped solitons.

Note that solution (209) causes some restrictions on the coefficients of eq.(156), which are eqs.(206) and (207).

4 Conclusions

We represent a new algorithm allowing one to construct a rich variety of particular solutions to a new class of nonlinear PDEs of any order in any dimensions. These equations can be considered as multidimensional generalizations of well known C- and S-integrable equations. We show that the solution space may be rich enough to provide complete integrability of some of these equations. However, the problem of complete integrability requires further study.

The suggested algorithm allows evident generalizations. For instance, let us generalize constraint (50) as follows:
\[
\sum_{m,n=1}^{D} A^{(m)} \ast \tilde{A}^{(n)} B^{(mpn)} = \sum_{m=1}^{D} \tilde{A}^{(m)} P^{(mp)}, \quad p = 1, \ldots, D, \quad j = 1, \ldots, m_0, \tag{211}
\]
where \( B^{(mpn)} \) and \( P^{(mn)} \) are some constant \( n_0 \times n_0 \) matrices. Then one can show that \( n_0 \times n_0 \) matrix functions \( V^{(m)}(t) \) are solutions to the following system of nonlinear PDEs:
\[
\sum_{m,n=1}^{D} \left[ \left( V^{(n)}_m + (V^{(m)}V^{(n)} + V^{(m)}A^{(n)}) \right) B^{(mpn)} \right] + \sum_{m=1}^{D} V^{(m)} P^{(mp)} = 0, \tag{212}
\]
\[
p = 1, \ldots, D.
\]

Remark that the physical application of some of the derived PDEs is obvious. For instance, the multidimensional \( N \)-wave equation (8) appears in multiple-scale analysis of any physical dispersion system. However, physical applications must be considered in more details.

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