Shallow Coulomb Gap and Weak Level-Level Correlations in a Deeply Insulating Electron System with Interactions

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(October 27, 2018)

We consider a system of two-dimensional electrons strongly localized by disorder. Interactions create a gap in the average tunneling density of states $\nu(E)$ at energies, $E$, close to the Fermi level. We derive a system of self-consistent equations for the correlators of local densities of occupied and empty states. When either the interactions are screened by the gate or the temperature, $T$, is high enough, so that the Coulomb gap is shallow, the perturbative solution of the system yields analytical expressions for the interaction-induced correction $\delta \nu(E)$ and the level-level correlator. We show that even with short-range interactions, $\delta \nu(E)$ exhibits a singular energy dependence at $T = 0$. We also demonstrate that at high $T$ this energy dependence is a universal function of the the ratio $E/T$. Regarding the level correlations, we trace how the correlator falls off as a function of spatial and energy separation between the levels. We also trace how the correlations vanish with increasing $T$. Our most noticeable observation is that for two close energies the correlator changes sign from positive (attraction) at small distances to negative (repulsion) at large distances.

PACS numbers: 73.20.Dx, 71.23.An, 71.55.Jv

I. INTRODUCTION

A distinctive difference between metallic and insulating systems lies in statistics of energy levels. It is now a common knowledge that in metallic diffusive samples, where electrons explore the entire volume in course of their motion, energy levels obey Wigner-Dyson statistics (for review, see Refs. 1-2). In the deeply insulating regime electron states are localized, and can be viewed as classical points randomly positioned in space and energy. Thus, the levels obey the Poisson statistics. There is presently no analytical theory describing how the Wigner-Dyson statistics evolve into the Poisson one as the degree of disorder grows. The most interesting critical regime, i.e. the level statistics directly at the Anderson transition, was first addressed in Refs. 3-4, which generated an ongoing stream of subsequent studies. In terms of experimentally observable quantities, the level statistics correspond to the correlation in tunnel conductances at different positions of the Fermi level.

It is also well known that electron-electron interactions affect considerably the low-energy behavior of disordered systems. In the diffusive regime, they generate a dip in the average tunneling density of states at the Fermi surface. This dip manifests itself as the zero-bias anomaly (for review, see Ref. 5). Interaction effects are even more pronounced in the insulating regime. For Coulomb interactions between the electrons, the tunneling density of states turns to zero as a power law in the vicinity of the Fermi level, as argued by Efros and Shklovskii. This phenomenon, dubbed the Coulomb gap, also became a subject of a big number of papers, which are too numerous to be cited here. The works before 1985 are reviewed in Refs. 6-7; recent theoretical activity on the classical Coulomb glass (aside from a long-standing problem of hopping transport) was focused on the shape of the density of states at zero and finite temperatures with a finite bandwidth of a bare energy distribution. Recent studies also addressed the dynamics of formation of the Coulomb glass.

Naively, one could think that the effect of interactions on the level statistics in the classical case is only to suppress the density of states, which enters as a factor in the Poisson distribution. Thus, the resulting distribution might be expected to remain Poissonian. In reality, however, even in a purely classical system interactions cause deviations from the Poisson statistics of energy levels. To be specific, let us address the following situation. Consider a narrow energy strip close to the Fermi level. Without interactions, the positions of sites with energies within the strip are completely random. The prime effect of interactions is an overall depletion of the strip (manifestation of the Coulomb gap). Less obvious is the question: to what extent, after the energies are modified by interactions, the sites within the same strip “feel” each other? In other words, how are the fluctuations in their density correlated? How does this correlation depend on the energy position of the strip? How does it vanish with increasing temperature, etc.

Getting answers to the above questions by means of numerical simulations is a much harder task than studying the shape of the Coulomb gap. In view of this, an analytical approach to the problem of level statistics in insulators with interactions is vitally important.

In this paper, we develop an analytical theory of level statistics in classical insulators with interactions. Such a theory must apparently be based on one of the existing analytical treatments of the Coulomb gap problem, which we review below. The first approach is due
to Efros. The basic underlying assumption of this approach is that at zero temperature the ground state of the system can be achieved by transpositions of electrons within singly occupied pairs of localized states (sites). Namely, for an isolated pair of sites with the energies $\varepsilon_1$ and $\varepsilon_2$, separated by the distance $\rho_{12}$, the work needed to transfer an electron from the occupied site 1 to the empty site 2 is

$$\Delta_{12} = \varepsilon_2 - \varepsilon_1 - V(\rho_{12}),$$

where $V(\rho)$ is the pair interaction potential. For the ground state, this work must be positive for each pair of sites $(1, 2)$. Efros expresses the density of states through the probability that all the conditions $\Delta_{12} > 0$ are fulfilled. The advantage of the approach is that it yields a closed equation for the density of states. The drawback is that it neglects the possibility of reducing the total energy by transpositions of electrons within the groups of three and more sites. An alternative derivation of the self-consistent equation is given in Ref. 17.

The formalism developed in Ref. 18 is also restricted to transpositions of pairs. The difference between Refs. 18 and 19 lies in the way of pair counting. In Ref. 18 the pairs are ranged according to the distance between the constituting sites $\rho_{12}$ (referred below as the pair shoulder). The theory describes the evolution of the density of states with increasing $\rho_{12}$. Consequently, the final result emerges as a solution of a “rate equation” in the limit when all the transpositions are carried out ($\rho_{12} \rightarrow \infty$). In contrast to Ref. 19 the formalism of Ref. 18 preserves automatically the total number of occupied and empty sites. By this virtue, the latter allows the generalization to finite temperatures in a natural way: if the condition $\Delta_{12} > 0$ is met for a certain pair, the transfer of the electron is not forbidden, but happens with a probability $\exp(-\Delta_{12}/T)$. Although the formalisms of Refs. 18 and 19 are different, the results they yield for the zero-temperature density of states coincide.

Another approach was put forward by Johnson and Khmelnitskii. They start from the exact expression for the partition function, expand it into the series of Feynman diagrams, which they subsequently average over disorder term by term. In the one-dimensional case they sum up the leading-order logarithmically divergent diagrams. The result of the summation reproduces the expression obtained in the framework of the approach of Ref. 19. For two- and three-dimensional cases ($D = 2, 3$), the results for the density of states $\nu(E) \propto |E|^{D-1}$ in the case of the Coulomb interactions ($V(\rho) \propto \rho^{-1}$) were reproduced in Ref. 19 by means of an $c$-expansion with $c = D - 1$.

Both approaches were designed to describe the average density of states.

Below we study the effect of interactions on the fluctuations of the density of states in disordered insulators, based on the technique developed previously in Ref. 18. In Section II we derive the general equations relating the average density of states and the level-level correlation function for an arbitrary pair interaction potential $V(\rho)$ and finite temperature. Sections III and IV are devoted to the solution of these equations for the case when this potential can be treated perturbatively. For this case we derive a general formula for the density of states (Section III) with an arbitrary potential and apply it to the two particular geometries relevant for experiment: the 2D disordered insulator located above a planar gate, and sandwiched between the two planar gates. The perturbative expression for level-level correlation function is derived and analyzed in Section IV. Concluding remarks are presented in Section V.

II. GENERAL EQUATIONS

The main subject of the study of level statistics is the level-level correlation function, defined as follows,

$$R^+(r, E_1, E_2) \equiv \langle \nu^+(E_1, R)\nu^+(E_2, R + r) \rangle_c,$$  

where the subindex $c$ stands for the irreducible part, $\langle AB \rangle_c \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle$, and $\nu^+(E, R)$ is the local density of occupied states at the space point $R$ and energy $E$,

$$\nu^+(E, R) = \sum \delta(R - R_i)\delta(\varepsilon_i - E).$$

Here we consider a system of electrons which are localized at the impurities randomly positioned at space points $R_i$, and having (single-particle) energies $\varepsilon_i$; $n_i$ are the occupation numbers, assuming the values 0 or 1. It is crucial to emphasize that the single-particle energies $\varepsilon_i$ include the interaction-induced shifts from all the surrounding sites. Note that the definition differs from the standard definition, commonly used in the diffusive regime (see e.g. Refs. 1, 2) in two respects. First, only occupied states are taken into account. The existence of the Fermi level (for which we choose $E = 0$) breaks the homogeneity in the energy space, and thus Eq. (2) depends on the two energies $E_1$ and $E_2$. Furthermore, the definition implicitly assumes that the particles are classical, i.e. the spatial extent of the electron wave function is much smaller than the average distance between the impurities. The level-level correlation function depends only on the difference $r$ of the space arguments due to the global homogeneity. Because of this, the averaging in Eq. (2) may be actually viewed as the integration over $R$.

To construct the theory, it is convenient to introduce, in addition to Eqs. (2), (3), the density of empty states,

$$\nu^-(E, R) = \sum \delta(R - R_i)\delta(\varepsilon_i - E),$$

and three more correlation functions,

$$R^-(r, E_1, E_2) \equiv \langle \nu^-(E_1, R)\nu^-(E_2, R + r) \rangle_c.$$
and
\[ S(r, E_1, E_2) \equiv \langle \nu^+(E_1, R)\nu^-(E_2, R+r) \rangle_c , \] (6)

They are related by obvious symmetries, \( R^-(r, -E_1, -E_2) = R^+(r, E_1, E_2) \), and \( \tilde{S}(r, E_1, E_2) = S(r, E_2, E_1) \). Thus, the knowledge of the functions \( R^+ \) and \( S \) is sufficient for the description of the level statistics.

Without interactions, the average density of states is constant (we denote it with \( \nu_0 \)), while the fluctuations of the local density of states are Poissonian,

\[ \langle \nu^\pm(E, R) \rangle = \nu_0 f_F(\pm E); \] (8)

\[ R^\pm(r, E_1, E_2) = \nu_0 \delta(r) \delta(E_1 - E_2) f_F(\pm E_1) f_F(\pm E_2); \]

\[ S(r, E_1, E_2) = \nu_0 \delta(r) \delta(E_1 - E_2) f_F(E_1)[1 - f_F(E_2)]. \]

Here \( f_F(E) = [\exp(E/T) + 1]^{-1} \) is the Fermi function.

**Fig. 1.** Transpositions of electrons between single and occupied localized states, illustrating the derivation of Eqs. (6) and (7). The states, characterized by location in energy and space, are shown as dashes; occupied states are marked by bullets. Processes 1 and 2 are explained in the text.

In the presence of interactions, Eqs. (6) do not actually describe the equilibrium state of the system. To find the equilibrium state, we employ the approach of Ref. [18]. Within this approach, the search for the equilibrium state is performed by transpositions of electrons within pairs of singly-occupied states (i.e. the pairs \( i,j \) with \( n_i = 1 \) and \( n_j = 0 \)). To this end, we introduce an auxiliary variable \( \rho \) (the length of the pair shoulder), and the functions \( g^\pm(E, R, \rho) \). These functions have the meaning of the local densities of occupied/empty states, which are established after the transpositions for all the pairs with the shoulders less than \( \rho \) are already carried out. For \( \rho = 0 \) the non-interacting values serve as an initial condition; the limiting case \( \rho \to \infty \) corresponds to the equilibrium state of the system. The densities of states \( \nu^\pm \) and \( g^\pm \) are related via

\[ \nu^\pm(E, R) = \lim_{\rho \to \infty} g^\pm(E, R, \rho). \] (9)

Now we derive the equation for the functions \( g^\pm \). To be specific, we consider now the two-dimensional case. The derivation is best illustrated by Fig. 1. Consider the energy strip \( (E, E + dE) \). Imagine that all the transpositions with the shoulders shorter than \( \rho \) are already performed. As we increase the shoulder from \( \rho \) to \( \rho + d\rho \), the single-particle energies change due to transpositions, and two processes take place: (i) a certain number of occupied sites leave the energy strip (Process 1 in Fig. 1); (ii) as a result of the transpositions, the empty sites with the single-particle energies within the strip \( (E + V(\rho), E + dE + V(\rho)) \) get occupied and arrive into the strip \( (E, E + dE) \) (Process 2 in Fig. 1). The net change of content of the energy strip is responsible for the change \( \delta g^+(E, R, \rho) \) of the density of states. Thus, the evolution of \( g^+ \) with \( \rho \) is described by an equation which is similar to the rate equation of the kinetic theory, with the processes (i) and (ii) corresponding to out-scattering and in-scattering terms in the collision integral, respectively. The actual form of this equation is as follows,

\[ \frac{\partial g^+(E, R, \rho)}{\partial \rho} = -\frac{\rho}{2} \int d\mathbf{n} \left\{ g^+(E, R, \rho) \right\} \]

\[ \times \left[ \int_{E-V(\rho)}^{E+V(\rho)} dE' g^-(E', R + n\rho, \rho) + \int_{E+V(\rho)}^{\infty} dE' \right] \]

\[ \times g^-(E', R + n\rho, \rho) \exp \left( \frac{-E' + V(\rho)}{T} \right) \]

\[ - g^-(E + V(\rho), R, \rho) \left[ \int_{E}^{\infty} dE' g^+(E', R + n\rho, \rho) \right. \]

\[ + \left. \int_{-\infty}^{E} dE' g^+(E', R + n\rho, \rho) \exp \left( \frac{E - E'}{T} \right) \right] \],

where \( n \) is a unit vector, and \( \int d\mathbf{n} = 2\pi \).

Let us establish the correspondence between the r.h.s. of Eq. (10) and the “scattering” processes shown in Fig. 1. The first two terms (first square bracket) in the r.h.s. describe the scattering out, i.e. departure of levels from the energy strip \( (E, E + dE) \) due to the transpositions (Process 1). Indeed, this bracket is multiplied by the factor \( g^+ \), describing the electrons which have been transferred. The bracket itself contains the function \( g^- \) as an integrand, which accounts for empty levels to which the electrons are transferred. The argument \( R + n\rho \) of this function takes into account that the transpositions occur within the pairs with the shoulder \( \rho \). Furthermore, in the integration over the energies of all available sites, it is taken into account that the transpositions with negative work \( \Delta(E, E') = E' - E - V(\rho) \) happen with the probability one, while for \( \Delta(E, E') > 0 \) the probability is given by the Boltzmann factor. The last two terms in
fluctuations of the density of states. It is important that Eq. (10) is not averaged over the position \( R \), and thus carries the information about the local fluctuations of the density of states.

Similarly, the “kinetic” equation for \( g^- \) takes the form

\[
\frac{\partial g^-(E, R, \rho)}{\partial \rho} = -\frac{\mu}{2} \int d\mathbf{n} \left\{ g^-(E, R, \rho) \right\}
\]

\[
\times \left[ \int_{-V(\rho)}^{\infty} dE' g^+(E', R + n \rho, \rho) + \int_{-\infty}^{-V(\rho)} dE' \times g^+(E', R + n \rho, \rho) \exp \left( -\frac{E' - V(\rho)}{T} \right) \right]
\]

\[
- g^+(E - V(\rho), R, \rho) \left[ \int_{-\infty}^{E} dE' g^-(E', R + n \rho, \rho) + \int_{E}^{\infty} dE' g^-(E', R + n \rho, \rho) \exp \left( -\frac{E' - E}{T} \right) \right] \right\}. \tag{11}
\]

Now we use the equations (10) and (11) to obtain the coupled equations for the average density of states and the level-level correlation functions. The analogy between Eqs. (10) and (11) and the kinetic theory of gases suggests the following course of action. We split the functions \( g^\pm \) into average (over the space) and fluctuating parts,

\[
g^\pm(E, R, \rho, \rho) = \nu^\pm(E, \rho, \rho) + \delta g^\pm(E, R, \rho), \tag{12}
\]

with \( \langle \delta g^\pm \rangle = 0 \). First, we average Eq. (10). After this procedure, the rate of change of the average value of \( g^+ \) is expressed through \( \nu^+, \nu^- \), and the correlator of the fluctuations \( \delta g^\pm \),

\[
\frac{\partial \nu^+(E)}{\partial \rho} = -\pi \lambda \int dE' \psi(E + V(\rho) - E') \nu^- (E')
\]

\[
- \nu^- (E + V(\rho)) \int dE' \psi(E' - E) \nu^+(E') + \int dE' \psi(E + V(\rho) - E') \nu^+(E') \tag{13}
\]

\[
\frac{\partial \nu^-(E)}{\partial \rho} = -\pi \lambda \int dE' \psi(E + V(\rho) - E') \nu^- (E')
\]

\[
- \nu^- (E + V(\rho)) \int dE' \psi(E' - E) \nu^+(E') + \int dE' \psi(E + V(\rho) - E') \nu^- (E') \tag{13}
\]

The equation (13) needs to be supplemented by two more equations describing the evolution of the correlation functions \( S \) and \( \rho^\pm \), defined as

\[
\rho^\pm(r, E_1, E_2) \equiv \langle \delta g^\pm(E_1, R, \rho) \delta g^\pm(E_2, R + r, \rho) \rangle. \tag{16}
\]

To derive these equations, we first subtract from Eq. (10) for \( \partial g^+(E_1, R, \rho)/\partial \rho \) its average, Eq. (13), and call the resulting equation \( A \). In the same way, we subtract from Eq. (11) for \( \partial g^+(E_2, R + r, \rho)/\partial \rho \) its average and call the resulting equation \( B \). Following the general approach of the kinetic theory of gases, we now multiply \( A \) by \( \delta g^- (E_2, R + r, \rho) \), multiply \( B \) by \( \delta g^+ (E_1, R, \rho) \), add them up and take the average, disregarding triple correlations (see below Section V). As a result, we obtain

\[
\frac{\partial S_{\rho}(r, E_1, E_2)}{\partial \rho} = -\frac{\rho}{2} \left\{ 2 \pi S_{\rho}(r, E_1, E_2) \int dE' \psi(E_1 - E' + V(\rho)) [\nu^-(E')] \right.
\]

\[
- \nu^- (E_1 + V(\rho)) \int dE' \psi(E' - E_1 + V(\rho)) \left[ R^\pm[r + n \rho], E', E_2 \right]
\]

\[
+ \nu^+(E_1) \int d\mathbf{n} \int dE' \psi(E_1 - E' + V(\rho)) R^\pm[r + n \rho], E', E_2 \right\}
\]

\[
\left. \left[ 2 \pi S_{\rho}(r, E_1, E_2) \int dE' \psi(E_1 - E' + V(\rho)) [\nu^- (E')] \right. \right.
\]

\[
- \nu^- (E_1 + V(\rho)) \int dE' \psi(E' - E_1 + V(\rho)) \left[ R^\pm[r + n \rho], E', E_2 \right]
\]

\[
+ 2 \pi S_{\rho}(r, E_1, E_2) \int dE' \psi[E_1 - E' + V(\rho)] \nu^+(E') \right. \right.
\]

\[
- 2 \pi R^\pm[r + n \rho], E_1 \] \int dE' \psi[E_2 - E' + V(\rho)] \nu^- (E')
\]

\[
+ \nu^+(E_2) \int d\mathbf{n} \int dE' \psi[E_2 - E' + V(\rho)] R^\pm[r + n \rho], E', E_1 \right\}
\]

\[
- \nu^+(E_2 - V(\rho)) \int d\mathbf{n} \int dE' \psi(E_2 - E') S_{\rho}[r + n \rho, E_1, E_2] \right\}. \tag{17}
\]
For the second equation, we ultimately multiply $A$ by $\delta q^+(E_2, \mathbf{R} + \mathbf{r}, \rho)$, multiply $B$ by $\delta q^-(E_1, \mathbf{R}, \rho)$, add up and take the average. The resulting equation reads

$$
\frac{\partial R^+(\mathbf{r}, E_1, E_2)}{\partial \rho} = -\frac{\rho}{2} \left\{ 2\pi R^+(\mathbf{r}, E_1, E_2) \int dE' \psi[E_1 - E' + V(\rho)]\nu_0^-(E') - 2\pi S\rho[r, E_2, E_1 + V(\rho)] \int dE' \psi[E' - E_1]\nu_0^+(E') \right. \\
+ \nu_0^-(E_1) \int d\mathbf{n} \int dE' \psi[E_1 - E' + V(\rho)]S\rho[\mathbf{r} + \mathbf{n}\rho, E_2, E'] \\
\left. - \nu_0^-(E_1 + V(\rho)) \int d\mathbf{n} \int dE' \psi[E_1 - E' + V(\rho)]R^+(\mathbf{r} + \mathbf{n}\rho, E_2, E'] \\
+ 2\pi R^+(\mathbf{r}, E_1, E_2) \int dE' \psi[E_2 - E' + V(\rho)]\nu_0^-(E') - 2\pi S\rho[r, E_1, E_2 + V(\rho)] \int dE' \psi[E' - E_2]\nu_0^+(E') \right. \\
+ \nu_0^+(E_2) \int d\mathbf{n} \int dE' \psi[E_2 - E' + V(\rho)]S\rho[\mathbf{r} + \mathbf{n}\rho, E_1, E'] \\
\left. - \nu_0^+(E_2 + V(\rho)) \int d\mathbf{n} \int dE' \psi[E_2 - E_2]R^+(\mathbf{r} + \mathbf{n}\rho, E_1, E') \right\}. \quad (18)
$$

Equations (13), (17), and (18) describe the evolution of the functions $\nu_\rho$, $S\rho$, and $R^+\rho$ with increasing $\rho$. Due to the electron-hole symmetry, $R^+\rho(r, E_1, E_2) = R^+\rho(r, -E_1, -E_2)$, and thus the system of the equations (13), (17), and (18) is closed. For $\rho = 0$, when no transpositions have yet been performed, the system corresponds to the equilibrium state of non-interacting localized electrons. Thus, Eqs. (8) serve as initial conditions at $\rho = 0$,

$$
\nu_0^+(E)|_{\rho=0} = \nu_0 f_F(E); \\
R^+(\mathbf{r}, E_1, E_2)|_{\rho=0} = \frac{\nu_0}{2\pi} \delta(r)\delta(E_1 - E_2)f_F(E_1)f_F(E_2); \\
S\rho(r, E_1, E_2)|_{\rho=0} = \frac{\nu_0}{2\pi} \delta(r)\delta(E_1 - E_2) \\
\times f_F(E_1)[1 - f_F(E_2)]. \quad (19)
$$

To obtain equilibrium properties of interacting system within this approach, one must solve Eqs. (13), (17), and (18). The average density of the occupied states $\nu_\rho^+(E)$ and the correlation functions $R^+\rho(r, E_1, E_2)$ and $S\rho(r, E_1, E_2)$ are then obtained as the limit $\rho \to \infty$ of the functions $\nu_\rho^+(E)$, $R^+\rho(r, E_1, E_2)$, and $S\rho(r, E_1, E_2)$, respectively.

III. DENSITY OF STATES

The cumbersome equations (13), (17), and (18) contain information about the statistical properties of the equilibrium state of the system for an arbitrary interaction potential $V$. In this and the next Sections our goal is to study $weak$ deviations of the level statistics from the Poisson distribution, and of the average density of states from its non-interacting value $\nu_0$. We solve this problem perturbatively and demonstrate that the desired deviations appear already in the first successive approximation.

To this end, we notice that the right-hand sides in Eqs. (13), (17), and (18) are effectively small due to the energy integration. Thus, in the leading order approximation it is sufficient to put the initial conditions (19) into the right-hand sides of these equations, and find the deviations from the initial conditions by integrating them over $\rho$.

For the density of states, inserting Eq. (13) into the right-hand side of Eq. (13) and performing the energy integration, we obtain the following equation for $\nu_\rho^+$,

$$
\frac{\partial \nu_\rho^+(E)}{\partial \rho} = -\pi \rho \nu_0^2 T \\
\times \left\{ \frac{\exp \frac{E+V(\rho)}{T}}{1 + \exp \frac{E}{T}} \ln \left( 1 + \exp \frac{-E-V(\rho)}{T} \right) \\
+ \frac{1}{1 + \exp \frac{E}{T}} \ln \left( 1 + \exp \frac{E+V(\rho)}{T} \right) \\
- \frac{\exp \frac{E}{T}}{1 + \exp \frac{-E-V(\rho)}{T}} \ln \left( 1 + \exp \frac{-E}{T} \right) \\
- \frac{1}{1 + \exp \frac{-E-V(\rho)}{T}} \ln \left( 1 + \exp \frac{-E-V(\rho)}{T} \right) \right\}. \quad (20)
$$

The total average density of states is found as $\nu(E) = \nu^+(E) + \nu^-(E)$, where $\nu^+(E)$ is the solution of Eq. (21) with the boundary condition (19), taken for $\rho \to \infty$. Due to the particle-hole symmetry, this becomes $\nu(E) = \nu^+(E) + \nu^+(E)$.

Although Eq. (21) can not be integrated for an arbitrary temperature, in two important limiting cases results may be obtained in a closed form.
A. \( T = 0 \)

1. General results

For zero temperature, Eq. (20) simplifies to

\[ \partial_\rho^+(E) = -\pi \rho_0^2 \left[ 2E + V(\rho) \right] \theta(-E) \theta(E + V(\rho)). \] (21)

Integrating Eq. (21), we find for the true average density of states

\[ \nu(E) = \nu_0 \left[ 1 + \pi \nu_0 |E| \rho_E^2 - \pi \nu_0 \int_0^{\rho_E} dp \rho V(\rho) \right], \] (22)

where \( \rho_E \) is the solution of the equation \( V(\rho) = |E| \).

It is remarkable that although the right-hand side of the “kinetic equation” Eq. (21) vanishes after being integrated over \( dE \), thus providing the independence of the total number of levels on the interactions, the solution (22) does not obey this relation. To realize this, we recall that for \( \rho \to 0 \) each realistic potential has a Coulomb form, \( V(\rho) = e^2/\kappa \rho \). Then it is straightforward to check the relation \( \int dE \nu(E) - \nu_0 = -\pi \nu_0^2 e^4/2\kappa^2 \), in which the right-hand side assumes a finite value due to the short-distance behavior of \( V(\rho) \). This seeming inconsistency has the following resolution. Let us introduce a cut-off distance \( \rho_{\text{min}} \) (minimal separation between the sites) which is much smaller than the characteristic decay length, \( d \), of the potential \( V(\rho) \). Then for energies \( E > e^2(\kappa \rho_{\text{min}})^{-1} \) we have \( \nu(E) = \nu_0 \). With finite \( \rho_{\text{min}} \) the conservation of the number of states, \( \int dE \nu(E) - \nu_0 = 0 \), gets restored. This restoration occurs due the fact that within a wide interval of high energies \( e^2(\kappa \rho_{\text{min}})^{-1} \ll E \ll e^2(\kappa d)^{-1} \) the deviation \( \nu(E) - \nu_0 \) exhibits a shallow maximum. While the height of this maximum vanishes with \( \rho_{\text{min}} \to 0 \), the area under the maximum remains finite.

Conceptually more interesting is the low-energy behavior of the density of states. For \( E = 0 \), we obtain \( \nu(0) = \nu_0 \left[ 1 - \pi \nu_0 \int_0^\infty dp \rho V(\rho) \right] < \nu_0 \). Thus, the average density of states at the Fermi level is suppressed by interactions. This effect is a precursor of the Coulomb gap.

The perturbative treatment is only valid in the case when the correction to the non-perturbed density of states \( \nu_0 \), given by Eq. (22), is small. Thus, the expansion parameter in the theory is \( \nu_0 d^2 V(d) \). If the potential \( V(\rho) \) falls off at large distances slower than \( \rho^{-2} \), the integral \( \int_0^\infty dp \rho V(\rho) \) diverges, manifesting that these potentials (including the important case of the Coulomb potential) are not amenable to the perturbative treatment at \( T = 0 \).

Now we apply the general result Eq. (22) to two specific realizations of short-range interaction potentials.

2. 2D system with a single gate

In the presence of a planar gate at a distance \( d \) from the 2D electron gas, the Coulomb potential is modified due to the image charge,

\[ V(\rho) = \frac{e^2}{\kappa} \left( \frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 + 4d^2}} \right), \] (23)

where \( \kappa \) is the static dielectric constant. In this case the decay length is \( d \), for larger distances \( V(\rho) \) falls off as \( \rho^{-3} \). With interaction (23) the integration in Eq. (22) can be carried out in a closed from. However, the radius \( \rho_E \) is now a solution of the fourth power algebraic equation. Then it is convenient to introduce the dimensionless energy \( \tilde{E} = 2d \kappa e^2/\epsilon^2 \) and present the result for the density of states as follows,

\[ \nu(E) - \nu_0 = -\frac{2\pi \nu_0^2 e^4}{\kappa} \frac{y(|\tilde{E}|) - 1)^2}{y^2(|\tilde{E}|) + 1}, \] (24)

where the function \( y(\tilde{E}) \) is the positive solution of the equation \( y^4 + (4/|\tilde{E}|) y^3 - 1 = 0 \). For \( E = 0 \) we obtain \( \nu(0) - \nu_0 = -2\pi \nu_0^2 e^4/\kappa \). In the vicinity of the Fermi surface the interaction-induced correction has a singularity,

\[ \nu(E) - \nu(0) = 2\pi \nu_0^2 \left( \frac{2e^2 d^2}{\kappa} \right)^{2/3} |E|^{1/3}. \] (25)

The energy dependence of the density of states calculated from Eq. (24) is shown in Fig. 2.

![Density of states](image)

FIG. 2. Density of states (24) for a 2D system with a single gate in units of \( 2\pi \nu_0^2 e^2 d/\kappa \) as a function of the dimensionless energy \( \tilde{E} \) at zero temperature.

3. 2D system between two gates

We consider now a planar electron system located between two infinite planar gates. Denote with \( d_1 \) and \( d_2 \) the distances to the upper and lower gates. For large
\( \rho \gg (d_1 + d_2) \) it can be shown that the interaction \( V(\rho) \) takes the form

\[
V(\rho) = \frac{e^2}{2 \pi \kappa \sqrt{2 \rho (d_1 + d_2)}} \cos^2 \left[ \frac{\pi (d_1 - d_2)}{2 (d_1 + d_2)} \right] \times \exp \left( -\frac{\pi \rho}{d_1 + d_2} \right) = V_0 \left( \frac{a}{\rho} \right)^{1/2} \exp(-\rho/a), \tag{26}\]

where \( a = (d_1 + d_2)/\pi \) is the interaction radius; the parameter \( V_0 \) is defined by the second equality in (26). The asymptotic expression (26) is only suitable to describe energies \( |E| \ll V_0 \). In this case, keeping the two leading terms, we write

\[
\rho \nu = a \left[ \ln \frac{V_0}{|E|} - \frac{1}{2} \ln \left( \ln \frac{V_0}{|E|} \right) \right]. \tag{27}\]

The leading order in \( V_0/|E| \) terms in the density of states are found from Eq. (23),

\[
\nu(E) - \nu_0 = \pi \nu_0^2 a^2 \left[ - \frac{\pi^{1/2}}{2} V_0 + |E| \ln \frac{V_0}{|E|} \left( \ln \frac{V_0}{|E|} - \ln \left( \ln \frac{V_0}{|E|} \right) + O(1) \right) \right]. \tag{28}\]

The singularity around the Fermi level is of the type

\[
\nu(E) - \nu(0) \propto |E| \ln^2 (V_0/|E|). \tag{29}\]

It is remarkable that the singularity survives despite the fast decay of the potential \( V(\rho) \) with distance. Moreover, since the behavior around \( E = 0 \) is determined exactly by this long-distance decay, the singularity of the form (28) is universal and characteristic of any potential which falls off exponentially at large distances. In any realistic experimental setup the sample is always surrounded by a number of gates, and may be characterized by several length scales. The long-\( \rho \) (beyond the longest scale) behavior of this potential is always exponential. Thus, we conclude that the \( |E| \ln^2 |E| \) singularity at small \( E \) is generic.

### B. High temperatures

We assume now that the temperature is higher than the potential \( V(\rho) \) for any \( \rho \). In particular, this means that the potential is finite everywhere. Expanding Eq. (24) \( \ln V(\rho)/T \) up to the second order (first-order terms cancel out) and subsequently integrating over \( \rho \), we find the high-temperature expression for the density of states,

\[
\nu(E) = \nu_0 + \frac{\nu_0^2 \mathcal{L}}{T} F(E/T), \tag{30}\]

where we have introduced the function

\[
F(x) = \frac{\pi}{2 \cosh^2 (x/2)} \left\{ \frac{1}{2} - \ln \left( 1 + e^x \right) - \ln \left( 1 + e^{-x} \right) \right\}, \tag{31}\]

and the parameter

\[
\mathcal{L} = \int_0^\infty d\rho \rho V^2(\rho). \tag{32}\]

It is straightforward to check that \( \int dx F(x) = 0 \), which automatically ensures the conservation of the total number of states. The effective upper limit in the integral (31) is \( \rho \sim d \), where \( d \) is a characteristic radius of the interaction potential. However, the integral diverges logarithmically at small \( \rho \), since at short distances the interaction is always Coulomb, \( V(\rho) = e^2/\kappa \rho \). In fact, this divergence is an artifact of the expansion of (26) with respect to \( V(\rho)/T \), which is valid only for \( \rho \gg \rho_T = e^2/\kappa T \). Thus, \( \rho_T \) serves as an ultraviolet cut-off in the integral (31).

![FIG. 3. The universal function \( F(E/T) \) (Eq. (30)).](image)

The actual form of the interaction potential affects only a numerical factor under the logarithm. So does the contribution from the short distances \( \rho \lesssim \rho_T \). The condition of validity of the high-temperature asymptotics (29) is \( T \gg V(d) \approx e^2/\kappa d \), \( d \), \( d \approx \kappa T \). The condition of validity of the interaction-induced correction should be smaller than \( \nu_0 \). The latter can be expressed as \( T \gg \nu_0 e^4/\kappa^2 \). This condition has a transparent physical interpretation: \( T \) should be bigger than the width of the Coulomb gap at zero temperature.

It is remarkable that the density of states (24) depends on energy only in combination \( E/T \), and thus the result is expressed in terms of the \textit{universal} function \( F \), shown in Fig. 3. The asymptotic expressions for \( \nu(E) \) are

\[
\nu(E) = \nu_0 - \frac{\nu_0^2 \mathcal{L}}{4T} \left( 2 \ln 2 - 1 - \frac{E^2 \ln 2}{2T^2} \right). \tag{33}\]
for $|E| \ll T$, and
\[
\nu(E) = v_0 + \frac{\pi v_0^2 L}{T} e^{-|E|/T}
\]
for $|E| \gg T$. As it is seen from Fig. 3, the maxima of \(\nu(E)\) correspond to \(E = \pm 2.2T\).

**IV. LEVEL STATISTICS**

**A. Singular part**

The level-level correlation function is defined by
\[
K(r, E_1, E_2) = \left\langle \sum_{ij} \delta(R - R_i)\delta(R - R_j)\delta(\varepsilon_i - E_1)\delta(\varepsilon_j - E_2) \right\rangle_c.
\]

If we now assume that there are no states with different energies localized at the same space point (i.e. \(R_i = R_j\) implies \(\varepsilon_i = \varepsilon_j\), this assumption effectively serving as an ultraviolet cut-off), it is evident that Eq. (36) contains a singular part,
\[
K_{sing}(r, E_1, E_2) = \nu(E_1)\delta(E_1 - E_2)\frac{\delta(r)}{2\pi r},
\]
which originates from the term with \(i = j\). On the other hand, the remaining terms are regular in \(r\), and can be obtained perturbatively from the equations (17) and (18), to which the initial conditions (1) must be inserted in right-hand sides. Below we deal in this way with the regular part \(K_{reg}\) of the level-level correlation function (36).

**B. Regular part, zero temperature**

For \(T = 0\), the correlation function \(S_{reg}(r, E_1, E_2)\) describes fluctuations of the densities of states at the opposite sides of the Fermi level, \(E_1E_2 < 0\); otherwise, it equals zero. The result for this quantity is
\[
S_{reg}(r, E_1, E_2) = -\nu_0^2 \theta(\rho^* - r).
\]

with the notation \(\rho^* = \rho_{E_1} + \rho_{E_2}\). In the derivation of (36) we have used the identity
\[
\int \frac{d\eta}{2\pi} \frac{\delta(|\eta + \nu_0\rho|)}{|\eta + \nu_0\rho|} = \frac{1}{r} \delta(r - \rho).
\]

The function \(R_{reg}^+(r, E_1, E_2)\) is responsible for the correlations of levels at the same side of the Fermi surface,
\[
R_{reg}(r, E_1, E_2) = \frac{\nu_0^2}{2} \theta(\tilde{\rho} - r),
\]
with \(\tilde{\rho} = \min[\rho_{E_1}, \rho_{E_2}]\). Eq. (36) applies for \(E_1 > 0, E_2 > 0\) as well as \(E_1 < 0, E_2 < 0\); therefore we dropped the superscript +.

The first observation to be made is that the equations (35) and (38) taken together obey the “detailed balance” relation,
\[
\int_{-\infty}^{\infty} dE_1 dE_2 K_{reg}(r, E_1, E_2)
= 2 \int_{0}^{\infty} dE_1 \int_{-\infty}^{0} dE_2 S_{reg}(r, E_1, E_2)
+ 2 \int_{0}^{\infty} dE_1 \int_{0}^{\infty} dE_2 R_{reg}(r, E_1, E_2) = 0,
\]
for any \(r\).

Obviously, \(S_{reg}\) and \(R_{reg}\) are zero without interactions. Therefore, they express interaction-induced spatial correlations between the sites. These correlations are the stronger the closer the two energies \(E_1\) and \(E_2\) are to the Fermi surface; though the amplitudes of the correlators are constant, their radii, \(\rho\) or \(\rho^*\), increase as \(E_1 \rightarrow 0\) and \(E_2 \rightarrow 0\). The sign of these correlations is negative (level repulsion) for the energies of the opposite signs. The origin of this repulsion is that the transpositions, which build up the ground state of the interacting system, involve pairs of sites across the Fermi surface. As a result, in the ground state, when the transpositions are energetically unfavorable, the occupied and empty sites tend to stay away from each other.

The positive sign of the correlator \(R\) (level attraction) can be understood as follows. Close in space occupied sites “feel” the repulsion from the same empty sites and, hence, tend to gather closer together. Note, that the phenomenon of “clustering of like sites” was identified in course of numerical simulations with \(V(\rho) \propto \rho^{-1}\) (and, thus, fully-developed Coulomb gap) in Ref. 22.
C. Regular part, finite temperatures

The first order corrections to the functions $S$ and $R^+$ can be actually calculated in a closed form for an arbitrary relation function according to Eq. (35), we obtain an expression

$$K_{reg}(r, E_1, E_2) = \frac{\nu_0^2}{2} \left\{ -2\psi|E_1 - E_2 + V(r)|f_F(E_1)f_F(-E_2) - 2\psi|E_2 - E_1 + V(r)|f_F(-E_1)f_F(E_2) \\
+ \psi(E_1 - E_2)|f_F(E_1)f_F(-E_2 - V(r)) + f_F(-E_2)f_F(E_1 - V(r))| \\
+ \psi(E_2 - E_1)|f_F(E_2)f_F(-E_1 - V(r)) + f_F(-E_1)f_F(E_2 - V(r))| \right\}. \quad (41)$$

This is a general result valid for an arbitrary relation between the energies $E_1$, $E_2$, $V(r)$, and $T$. It is straightforward to check that for $T = 0$ Eq. (11) reproduces expressions (38) and (39). Also, the detailed balance relation,

$$K_{reg}(r, E, E) = \frac{\nu_0^2 \sinh(V(r)/2T)}{\cosh^2(E/2T)} \left[ \cosh(V(r)/2T) \cosh^2(E/2T) + \sinh(V(r)/2T) \sinh^2(E/2T) \right] \cosh(E/T) + \sinh(V(r)/T). \quad (42)$$

We see that the abrupt cut-off of the correlator (38) at $T = 0$ gets smeared at finite temperature. In the vicinity of the Fermi level ($|E| \ll T$) Eq. (42) simplifies to

$$K_{reg}(r, 0, 0) = \frac{\nu_0^2}{2} \tanh\left(\frac{V(r)}{2T}\right). \quad (43)$$

The above expression illustrates the decay of the correlator with increasing either the distance $r$ or the temperature $T$. In the opposite limit ($|E| \gg T$) the generalization of (38) to the finite temperatures takes the form

$$K_{reg}(r, E, E) = \nu_0^2 \left[ \frac{\exp(V(r)/T) - 1}{\exp(|E|/T) + 2 \cosh(V(r)/T)} \right]. \quad (44)$$

which shows that at finite $r$, instead of the abrupt termination of (38) at $|E| = V(r)$, the correlator decays with departure of $E$ from the Fermi level as $\exp(-|E|/T)$.

Finally, let us consider the behavior of the correlator at large distances ($V(r) \ll T$). Expanding (42) in the first order in $V(r)/T$ yields

$$K_{reg}(r, E, E) = \frac{\nu_0^2 V(r)}{4T \cosh^2(E/2T)}, \quad (45)$$

so that [in accordance with Eq. (39)] $K_{reg}(r, E, E)$ decays as $V(r)$. The most important consequence of Eq. (42) is that at any distances and temperatures the sign of the level-level correlator at coinciding energies remains positive (level attraction). Below we will see that this is not the case for arbitrarily small mismatch $E_1 - E_2$. To illustrate this point we analyze several limiting cases for which the general expression (41) for $K_{reg}(r, E_1, E_2)$ can be simplified.

**Far tail in $r$, different energies, $V(r) \ll |E_1 - E_2|, T$.** In this limit we get

$$K_{reg}(r, E_1, E_2) = -\frac{\nu_0^2 V^2(r)}{16T^2} \exp\left(-\frac{|E_1 - E_2|}{2T}\right) \times \left\{ \frac{1}{\cosh(E_1/2T) \cosh(E_2/2T)} + \frac{1}{\cosh(E_1/2T) \cosh(E_2/2T)} \right\}. \quad (46)$$

We see that, in contrast to the case of coinciding energies, the correlation function is negative (level repulsion) and proportional to $V^2(r)$.

**Far tail in energy, $E_1 = 0$, $|E_2| \gg T, V(r)$.** In the leading order in $\exp(-|E_2|/T)$ we obtain

$$K_{reg}(r, 0, E_2) = -\frac{\nu_0^2}{2} \exp\left(\frac{V(r) - 2|E_2|}{2T}\right) \sinh^2(V(r)/2T) \cosh(V(r)/2T). \quad (47)$$

This expression indicates that for large enough energy separation the repulsion persists for arbitrary relation between $V(r)$ and $T$.

**Crossover from short- to long-distance behavior.** Consider $K_{reg}(r, E + \Delta/2, E - \Delta/2)$ with $T, \Delta \ll
We have seen already that this function (at least for $\Delta \ll T$) is positive for low distances $r$, see Eq. (43). We
are now going to show that, as a function of the separation $r$,  this function changes sign at $r = \rho_\Delta$ and becomes
negative (level repulsion) for long distances. Indeed, for
$\rho_E \ll r < \rho_\Delta$ ($\Delta < V(r) \ll |E|$) in the leading order we find

$$K_{reg}(r, E + \Delta/2, E - \Delta/2) = \nu_0^2 \exp \left( \frac{\Delta - |E|}{2T} \right) \times \left( \exp \left( \frac{V(r) - \Delta}{T} \right) - 1 \right),$$

which is positive and corresponds to the level attraction. The level-level correlation function turns to zero for $r = \rho_\Delta$. For longer distances, the leading order is proportional to $\exp(-2|E|/T)$ and becomes $(r > \rho_\Delta)$

$$K_{reg}(r, E + \Delta/2, E - \Delta/2) = -\nu_0^2 e^{-2|E|/T} \times \left( 1 + e^{-\Delta/T} \left( e^{V(r)/T} - 1 \right) \sinh \left( \frac{V(r)}{T} \right) \right),$$

which matches Eq. (43). The levels, indeed, repel each other. We emphasize once more that the transition be-
tween level attraction and level repulsion happens when $V(r) = \Delta$, which corresponds to much longer distances
than $V(r) = |E|$.

**D. Level number variance**

We now turn to the question about fluctuations of the number of levels within a certain energy strip which was
formulated in the Introduction.

For weakly disordered systems, the variance of the number of levels in the energy strip of the width $E$

deserves as quantitative measure of the deviation of the level statistics from Poissonian (see e.g. Ref. 2). Provided the strip is taken far from the band edges and contains many levels, this *level number variance* is a function of $E$ only, and does not depend on the absolute position of the strip.

Drawing an analogy with the weakly disordered case, we define the quantity $\Sigma(\Omega, E_1, E_2)$, which is the variance of the number of levels which are located within the spatial region $\Omega$ and have energies between $E_1$ and $E_2$. This level number variance can be trivially expressed through the level-level correlation function,

$$\Sigma(\Omega, E_1, E_2) = \int_{E_1}^{E_2} dE dE' \int_{\Omega} dR dR' K(|R - R'|, E, E').$$

Unlike in weakly disordered case, for many-electron system under study the level number variance depends on both $E_1$ and $E_2$, not merely on the difference $E_1 - E_2$.

Eq. (50) can be significantly simplified if we assume that the spatial domain $\Omega$ is sufficiently large, so that for
relevant energies the correlator $K$ is very small outside this domain. Then $\Sigma$ is essentially proportional to the
volume of $\Omega$. Defining the partial level number variance (referred below as LNV), $\sigma(E_1, E_2) = \Omega^{-1} \Sigma(\Omega, E_1, E_2)$, and using the decomposition of $K$ into regular and sin-
gular parts, we obtain

$$\sigma(E_1, E_2) - \nu_0(E_2 - E_1) = \sigma_{sing}(E_1, E_2) + \sigma_{reg}(E_1, E_2),$$

where $\sigma_{sing}$ and $\sigma_{reg}$ are defined as

$$\sigma_{sing}(E_1, E_2) = \int_{E_1}^{E_2} dE \left[ \nu(E) - \nu_0 \right],$$

$$\sigma_{reg}(E_1, E_2) = \int_{E_1}^{E_2} dE dE' \int_0^\infty 2\pi r dr K_{reg}(r, E, E').$$

The value $\nu_0(E_2 - E_1)$ corresponds to non-interacting electrons; $\sigma_{sing}$ and $\sigma_{reg}$ are the interaction-induced
corrections. The physical meaning of these corrections is quite different. Correction $\sigma_{sing}$ accounts for the net
depletion of the strip due to reduction of the density of states. It does not bear information about deviations from the Poissonian statistics. These deviations are de-
scribed by $\sigma_{reg}$.

Generally, both corrections $\sigma_{sing}$ and $\sigma_{reg}$ are complicated functions of $E_1, E_2, T$, and the form of the
potential. For this reason, below we consider only several instructive particular cases.

$T = 0$, symmetric interval. Suppose that the energy strip is symmetric with respect to the Fermi level:

$$E_1 = -E/2, E_2 = E/2.$$ At zero temperature the “non-

Poissonian” contribution to LNV can be evaluated by substituting Eqs. (38) and (39) into Eq. (53),

$$\sigma_{reg}(-E/2, E/2) = 2\pi \nu_0^2 \int_{E/2}^{E/2} dE_1 \int_{E/2}^{E/2} dE_2 \int_0^{\infty} dr d\theta |\tilde{\rho} - r|$$

$$- 4\pi \nu_0^2 \int_{E/2}^{E/2} dE_1 \int_{-E/2}^{E/2} dE_2 \int_0^{\infty} dr d\theta |\tilde{\rho}^* - r|. (54)$$

With the use of definitions of $\tilde{\rho}(E_1, E_2)$ and $\rho^*(E_1, E_2)$, integration over $r$ and one of the energies in Eq. (54) can be performed explicitly. Then $\sigma_{reg}(-E/2, E/2)$ takes the concise form

$$\sigma_{reg}(-E/2, E/2) = -2\pi \nu_0^2 \int_{E/2}^{E} dE' (E' - E)^2 \rho_{\nu_0}^2. (55)$$

This expression is one of the central results of the present study. Note that initial formula (54) for $\sigma_{reg}(-E/2, E/2)$. 

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contains two competing terms. The first term is positive and originates from attraction between the occupied as well as between the empty levels. The second term is negative reflecting the repulsion between the levels with different occupation numbers. Eq. (55) shows that the second term wins, i.e., the net deviation from the Poissonian statistics for \( T = 0 \) and symmetric energy interval corresponds to the repulsion. To proceed further, we choose for concreteness the potential \( V(\rho) = V_0 (\rho/a)^2 \exp(-\rho/a) \) (\( \rho \gg a \)), which pertains to the planar electron system between two gates. At short distances, \( \rho \ll a \), the potential is Coulomb (see Section 11). Consider \( E \ll V_0 \). Then the integrals in (52) and (53) are determined by distances \( \rho_E \gg a \) and can be calculated explicitly. The result for LNV reads

\[
\sigma(-E/2, E/2) = -\left[\nu_0 - \nu(0)\right]E + 2\pi \left(\ln 2 - 1/2\right) \nu_0^2 a^2 E^2 \ln \left(\frac{V_0}{E}\right),
\]

where the difference \( \nu_0 - \nu(0) \) is equal to \( \pi^{3/2} \nu_0^2 a^2 V_0^2/2 \) as follows from Eq. (27). It is seen from Eq. (27) that the leading effect of interactions on LNV comes from the depletion of the strip. The effect of level-level correlations is smaller in parameter \( E/V_0 \ll 1 \). Obviously enough, the interaction-induced correction to LNV is much smaller than \( \nu_0 E \), since \( \nu_0 V_0 a^2 \ll 1 \) is the expansion parameter of the perturbation theory.

Note that, in the leading order, \( \sigma_{reg}(-E/2, E/2) \) is proportional to \( E^2 \ln^2(V_0/E) \). However, this contribution is canceled by the corresponding part in \( \sigma_{sing}(-E/2, E/2) \). As a result, the second term in (56) contains \( \ln(V_0/E) \) in the first power.

For \( E \gg V_0 \) the interaction-induced correction to LNV is determined by the short-distance behavior of \( V(\rho) \). It depends on ultraviolet cut-off required to satisfy the sum rule \( \int \nu(E) \nu_0 dE = 0 \) (see Section 11), and is not discussed here.

**High temperatures, symmetric interval.** The above conclusion that correlations cause a negative correction to LNV does not hold as the temperature is elevated. To illustrate this, we choose the limit when the temperature is much higher than the width of the strip. In contrast to the \( T = 0 \) case considered above, we will also assume that the strip is much wider than the crossover energy \( V_0 \sim e^2/\kappa a \). As it was already mentioned, for \( T \gg V_0 \) the exact form of the long-distance fall-off of the potential appears to be not essential. More precisely, the characteristic decay radius \( a \) enters the results only under the logarithm, as we established in Section 11 while calculating the density of states [see Eq. (23)]. In this calculation, high temperature introduced a short-distance cut-off at \( \rho \sim \rho_T = e^2/\kappa T \). The major contribution to \( \nu(E) - \nu_0 \) came from the wide interval of distances \( \rho_T < \rho < a \) within which the potential is Coulomb. It appears that the same simplifications are valid for the calculation of \( \sigma_{reg}(-E/2, E/2) \). Namely, it is safe to use the expansion with respect to \( V(r) \) in the general formula (11) for the correlator \( K_{reg} \). Substituting this expansion into Eq. (13) and performing integration first over \( r_1, r_2 \) and then over \( r \) we obtain

\[
\sigma_{reg}(-E/2, E/2) = \frac{2\pi \nu_0^2 e^4}{\kappa^2 T} \ln \left(\frac{\kappa a E}{e^2}\right), \quad (57)
\]

The positive sign of \( \sigma_{reg} \) indicates that at high \( T \) correlations cause a super-Poissonian behavior of LNV. The remaining task is to compare Eq. (57) with \( \sigma_{sing} \). The expression for \( \sigma_{sing}(-E/2, E/2) \) immediately follows from Eq. (33).

\[
\sigma_{sing}(-E/2, E/2) = -\frac{2(\ln 2 - 1)e^4 V_0^2}{4\kappa^2 T} \ln \left(\frac{\kappa a T}{e^2}\right). \quad (58)
\]

Comparing (57) and (58), we see that, besides different numerical coefficients, they differ in logarithmic factors, which were assumed to be large parameters in the course of the calculation. Since \( T \) is much bigger than \( E \), the logarithmic factor in (57) dominates.

In both particular cases considered above, contribution of correlations to LNV was parametrically smaller than the contribution from the change of the density of states. It is interesting to find out whether or not the opposite relation is possible within a certain range of parameters. Below we illustrate that the opposite relation is indeed possible.

**High temperatures, asymmetric interval.** Consider now the case of strongly asymmetric (with respect to the Fermi level) energy strip, \( E - \Delta/2, E + \Delta/2 \), with \( E \gg \Delta \). The most interesting situation for this case occurs when the conditions \( V_0 \ll T \ll E \) are met. Then \( \nu(E) - \nu_0 \) is given by Eq. (14) and is small as \( \exp(-E/T) \). Hence, \( \sigma_{sing} \) is also proportional to \( \exp(-E/T) \). As a result, unlike the symmetric-strip case, the interaction-induced correction to LNV is dominated by \( \sigma_{reg} \),

\[
\sigma(E - \Delta/2, E + \Delta/2) = \sigma_{reg}(E - \Delta/2, E + \Delta/2) \approx \sigma_{reg}(E - \Delta/2, E + \Delta/2) - \nu_0 \Delta
\]

\[
= \int_{E-\Delta/2}^{E+\Delta/2} dE dE' \int_0^\infty 2\pi r dr K_{reg}(r, E, E'). \quad (59)
\]

From Eqs. (48) and (49) we see that the level-level correlation function for \( V(r) \ll E \) is also exponentially suppressed. Then the major contribution to (57) comes from the short distances where \( E \ll V(r) \). For such \( r \) we can replace \( K_{reg}(r, E_1, E_2) \) by \( K_{reg}(r, E) \) in Eq. (57). Moreover, for \( T \ll E \ll V(r) \) we can use the zero-temperature result (19) for level-level correlation function, namely \( K_{reg}(r, E, E) = \nu_0^2 \theta(\rho_E - r) \). Finally, since the center of the strip, \( E \), lies well above \( V_0 \), the parameter \( \rho_E \) is determined by the Coulomb part of the potential, i.e., \( \rho_E = e^2/\kappa E \). Combining all together, we obtain for LNV

\[
\sigma(E - \Delta/2, E + \Delta/2) = \nu_0 \Delta + \frac{\pi \nu_0^2 \Delta^2 e^4}{2E^2 \kappa^2}. \quad (60)
\]
The super-Poissonian result for LNV is not surprising, since we have already realized that like sites tend to attract each other. To estimate the ratio of non-Poissonian and Poissonian contributions to LNV, it is convenient to rewrite it as $(\Delta/E)(\nu_0 e^4/\kappa^2 E)$. The first factor is small. Less trivial is to realize that the second factor is also small. This follows from the condition $E \gg T$, which we exploited, and from the fact that in the perturbative regime the condition $T \gg \nu_0 e^4/\kappa^2$ should hold. As it was mentioned above, this condition ensures that the Coulomb gap is washed out by temperature.

V. CONCLUSIONS

In the present paper we have extended the self-consistent description of the disordered interacting system of localized electrons proposed in Ref. 18 to include the correlation of the single-particle levels (both in energy and in space). Correspondingly, our results allow to incorporate correlations in the calculation of the density of low-lying neutral excitations of the system (these excitations determine the specific heat). We obtained the analytical solution of the self-consistent equations for the case when, due to the screening by gates, interactions are weak enough to be treated perturbatively. Let us discuss the conditions when the perturbative description is valid.

In the absence of screening, the width of the Coulomb gap can be estimated by equating the low-energy asymptotics \( \nu(E) \sim \kappa^2 |E|/e^4 \) of the density of states to the bare density of states \( \nu_0 \). This yields \( \nu(E) \sim \nu_0 e^4/\kappa^2 = E_C \). A perturbative solution applies when the screening cuts off the Coulomb interaction at short enough distance, \( d \), so that \( V(d) \sim e^2/\kappa d \gg E_C \). In this case the interaction-induced correction to the density of states, \( \delta \nu(E) \), is much smaller than \( \nu_0 \) at all energies. In other words, the perturbative regime corresponds to a small enough \( \nu_0 \), which implies a broad distribution of the bare energies of sites, \( i.e. \) to the limit of a strong external disorder.

The above condition \( d \ll \kappa/\nu_0 e^2 \) can be derived from a very different consideration. Since the density of states at the Fermi level is close to \( \nu_0 \) in the perturbative regime, the combination \( R_C = \kappa/\nu_0 e^2 \) formally represents the static screening radius in two dimensions. Then the condition of validity of the perturbative description can be presented as \( d \ll R_C \), and has a transparent interpretation that screening is not important for short-range interactions.

At zero temperature and \( E = 0 \) we obtained the perturbative result \( \delta \nu(0)/\nu_0 = -\pi \nu_0 \int_0^\infty d\rho \rho V(\rho) \). Note, that the r.h.s. of this expression can be considered as a probability for an electron at the Fermi level to be transposed as the interactions are switched on. Then the fact that \( \delta \nu(0)/\nu_0 \ll 1 \) implies that the portion of sites which are involved in transpositions (\( i.e. \) the sites, \( i \), for which the transposition with another site, \( j \), is energetically favorable) is relatively small. This ensures that the deviation from the Poissonian statistics of energy levels is weak.

Obviously, the perturbative description also applies when temperature is high enough, \( T \gg E_C \).

The predictions following from the theory developed, which are easiest to test experimentally, concern the energy dependence of \( \delta \nu \). The precursor of the Coulomb gap should show up in the low-bias behavior of the differential tunneling conductance \( \delta G(U) \propto \delta \nu(U) \). For tunneling between a metallic gate and two-dimensional layer of localized electrons \( \delta G(U) \) is determined by Eq. 24 (see also Fig. 2) at \( T = 0 \), and by \( F(U)/T \), where the function \( F \) is given by (8) and plotted in Fig. 3, for high temperatures.

Let us discuss the physics left out within the self-consistent approach. Firstly, while considering transpositions within pairs of sites, we neglected the fact that, upon each transposition \( i \to j \), the single-particle energies of all other sites change. This change could trigger the “secondary” transpositions which were forbidden (at \( T = 0 \)) before the transposition \( i \to j \) took place (polaronic effect). Secondly, we were concerned only with the question whether or not a transposition within a pair is energetically favorable. The example given in Ref. 25 shows that, even when all pair conditions, \( \Delta_{12} > 0 \) [see Eq. (10)] are satisfied, the work required for simultaneous transfer of two electrons can be negative (bipolaronic effect). Note, however, that both effects are higher order in \( \nu_0 \) than the \( \nu_0^2 \)-corrections captured by the perturbative solution. For the same reason, we explicitly disregarded triple correlations of the type \( \langle \delta g_d \delta g_d \rangle \) when deriving Eqs. (17) and (18). Terms describing these correlations are also of the order of \( \nu_0^3 \) and are beyond our precision.

We now give an overview of works discussing spectral statistics in interacting systems. One avenue found in the literature is to study (mostly numerically) the statistics of many-particle levels. The early finding was that many-particle levels in generic interacting systems obey Wigner-Dyson statistics, while integrable ones are Poissonian distributed. Subsequently, statistics of many-particle states have been discussed in weakly disordered conductors, at the metal-insulator transition, and in the insulating regime, with the general conclusion that the many-particle level distribution crosses from the non-interacting one to the Poissonian one as the number of interacting particles increases (thus, the statistics of three-particle states are closer to the Poissonian form as those of two-particle states). Another opportunity, realized in Refs. 30 and 31, is to study the addition spectrum of quantum dots in the deeply insulating regime (defined as the increment of the many-particle ground state energy when electrons are consecutively added to the dot).

Our results for level-level correlations correspond to yet another experimental setup: STM tip moves along the surface of the sample and measures the tunneling conductance, proportional to the local density of states.
This is unrelated to the addition spectrum of quantum dots, since to discuss the addition spectrum, one has to account for the shift of the Fermi energy as electrons are added. What is more, it is not clear whether the really measured addition spectrum of such dots would be determined by the ground state energy rather than by energies of the single-particle intermediate states, through which an electron actually tunnels. What we discuss in this paper are also not many-particle states but rather quasiparticle excitations (corresponding to the poles of the single-particle Green’s function); this is an analog of the excitation spectrum peaks in the weakly localized regime rather than many-body states which compose each peak.

Finally, we remark that recently the issue of the Coulomb gap in quantum interacting systems became a subject of several numerical studies in connection with the quantum Hall effect and metal-insulator transition in two dimensions. The results of our paper are strictly classical and are valid in the deeply insulating regime in two dimensions. The results of our paper are also not many-particle states but rather energies of the single-particle intermediate states, through which an electron actually tunnels. What we discuss in this paper are also not many-particle states but rather quasiparticle excitations (corresponding to the poles of the single-particle Green’s function); this is an analog of the excitation spectrum peaks in the weakly localized regime rather than many-body states which compose each peak.

ACKNOWLEDGMENTS

This work was supported by the Swiss National Science Foundation (Y. M. B.). M. E. R. acknowledges the support of the NSF under grant INT-9815194. He is also grateful to the University of Geneva for hospitality. We thank The Aspen Center for Physics, where this work was started, for hospitality and support.

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