HIGHER LAMINATIONS AND AFFINE BUILDINGS

IAN LE

Abstract. We give a Thurston-like definition for laminations on higher Teichmüller spaces associated to a surface \( S \) and a semi-simple group \( G \) for \( G = SL_m \) and \( PGL_m \). The case \( G = SL_2 \) or \( PGL_2 \) corresponds to the classical theory of laminations. Our construction involves positive configurations of points in the affine building. We show that these laminations are parameterized by the tropical points of the spaces \( X_{G,S} \) and \( A_{G,S} \) of Fock and Goncharov. Finally, we explain how these laminations give a compactification of higher Teichmüller spaces.

Contents

1. Introduction 1
2. Background 4
3. The tropical points of higher Teichmüller space 5
4. Background on affine buildings 6
5. Laminations for the \( \mathcal{A} \) space 7
5.1. Outline of the definition of \( \mathcal{A} \)-laminations 7
5.2. Canonical Coordinates on Configurations in the Flag Variety and Affine Grassmannian 10
5.3. Positive Configurations 11
5.4. Virtual Positive Configurations 15
5.5. Definition of laminations for the \( \mathcal{A} \) space of a disc with marked points 16
5.6. Laminations for the \( \mathcal{A} \)-space of a general surface 19
6. Laminations for the \( \mathcal{X} \) space 21
6.1. Laminations for the \( \mathcal{X} \)-space of a disc 21
6.2. Laminations for the \( \mathcal{X} \)-space of a general surface 24
6.3. Laminations on a closed surface 25
7. Comparison with other works 28
References 30

1. Introduction

The study of higher Teichmüller theory was initiated in the 90’s by Hitchin [H], who studied the space of representations of \( \pi_1 (S) \), the fundamental group of a surface \( S \), in a split-real group \( G(\mathbb{R}) \). He showed that one of the components of this space was contractible, and behaved much like classical Teichmüller space, which is obtained in the special case where \( G = SL_2 \). Hitchin’s approach was analytic, and involved the study of Higgs bundles on Riemann surfaces.

More recently, Fock and Goncharov discovered a completely different approach to higher Teichmüller theory [FG1], which is more algebraic, combinatorial and explicit. For a surface \( S \) with boundary and possibly marked points on the boundary, they look at \( G(\mathbb{R}) \)-local systems
on the surface with some extra data of framing on the boundary components (with some modification the theory extends to closed surfaces). The main ingredient is Lusztig’s theory of total positivity, which allows them to define a pair of moduli spaces \( \mathcal{X}_{G,S} \) and \( \mathcal{A}_{G,S} \), each with an atlas of coordinate charts such that all transition functions involve only addition, multiplication and division. It thus makes sense to take the positive \( \mathbb{R}_{>0} \) points of these moduli spaces. With some work, Fock and Goncharov show that taking the positive points gives an algebro-geometric description of the component of representations studied by Hitchin. The theory gives an explicit parameterization of positive representations, and it becomes manifest that the space of representations is contractible. Their theory has many interesting features: there is a close connection to cluster algebras; their moduli spaces can be quantized; the moduli spaces come in dual pairs which are a manifestation of Langlands duality.

The theory has geometric consequences as well. For example, Fock and Goncharov show that the corresponding representations are discrete and faithful (this was also shown by Labourie and Guichard for \( G = \text{SL}_m \) using different methods). Another surprising consequence of their approach is that they can completely recover Thurston’s theory of laminations. In the 80’s, Thurston invented the theory of laminations in a completely different context—the geometry and topology of two- and three-dimensional manifolds.

Fock and Goncharov show that Thurston’s space of measured laminations arises by taking the tropical points of the \( \text{PGL}_2 \) (or \( \text{SL}_2 \)) moduli space. Moreover, by analogy, they define higher laminations as the tropical points of higher Teichmüller space. For the case of \( \text{SL}_2 \), they show how their definition of laminations is equivalent to the definition by Thurston. However, for groups of higher rank, a more concrete definition in the spirit of Thurston has remained elusive. Such a definition would confirm that the definition of Fock and Goncharov is the correct one, and clarify the bridge between ideas from geometric topology and the study of \( G \)-bundles on surfaces.

In this paper, we give a definition of higher laminations for the space of framed \( G(\mathbb{R}) \)-local systems on a surface \( S \) and show that it coincides with the tropical points of higher Teichmüller space. We will show in a future paper that we can extend much of what was done in [FG3] for \( \text{SL}_2 \) laminations—the construction of functions corresponding to laminations, and duality pairings—to the case of general \( G \); moreover, the construction of these functions should lead to analogues of length functions. Another reason our definition is the right one is that it involves affine buildings, and is in agreement with work of Morgan, Shalen, Alessandrini, and Parreau [MS], [A], [P]. Our main results are as follows:

**Theorem 1.1.** Let \( S \) be a (hyperbolic) surface with marked points, and let \( C \) be its cyclic set at \( \infty \). Associated to any tropical point of \( \mathcal{A}_{G,S}(\mathbb{Z}^t) \) there is an \( \mathcal{A} \)-lamination: a \( \pi_1(S) \)-equivariant virtual positive configuration of points in the affine building of \( G \) parameterized by \( C \). This virtual positive configuration is unique up to equivalence.

Analogously, associated to any tropical point of \( \mathcal{X}_{G,S}(\mathbb{Z}^t) \) there is an \( \mathcal{X} \)-lamination: a family of positive configurations of points in the affine building parameterized by every finite set of the set \( C \), compatible under restriction from one finite set to another, and equipped with an action of \( \pi_1(S) \) on these families of configurations. This family of positive configurations is unique up to equivalence.

The space of projectived laminations provides a spherical boundary for the corresponding higher Teichmüller spaces \( \mathcal{A}_{G,S}(\mathbb{R}_{>0}) \) and \( \mathcal{X}_{G,S}(\mathbb{R}_{>0}) \).

We expect our theory of laminations on higher Teichmüller spaces to have lots of applications, and we list a few here.

1. The general philosophy of cluster algebras and the duality conjectures of Fock and Goncharov lead us to expect that the cluster complex associated to \( \mathcal{A}_{G,S} \) embeds inside
the space of laminations for the space $X_{G^\vee, S}$, where $G^\vee$ is the Langlands dual of $G$. This gives a parameterization of all cluster variables and all clusters. This is of interest because higher Teichmüller spaces give many examples of cluster algebras of rather general (and sometimes mysterious) type and also include many well-studied cases (for example, most finite mutation type cluster algebras as well as the elliptic $E_7$ and $E_8$ algebras).

(2) In fact, one expects more: laminations should parameterize atomic/canonical bases for the cluster algebras. These have not been constructed, but are of interest in physics, where they correspond with line operators [GMN1]. Higher laminations are also related to the spectral networks defined in [GMN2].

(3) A better understanding of the structure of the cluster algebra on the space $\mathcal{A}_{G, S}$ should in particular lead to a better understanding of its symmetries. These symmetries are expected to be a higher analogue of the mapping class group. One of our motivations for a geometric definition of laminations was to obtain some understanding of the higher mapping class group. It is possible that higher laminations can be used as a tool to study of the higher mapping class group and its dynamics in the way that laminations are used to study the mapping class group.

(4) One hopes to construct length functions associated to laminations on higher Teichmüller space. One approach might be to define them by studying asymptotics of canonical functions on higher Teichmüller space. This could lead to a better understanding of the geometry of representation varieties.

(5) The study of positive configurations of points inside the affine building for $G$ is related to counting tensor product multiplicities for representations of $G^\vee$. Positive configurations of points inside the affine building also seem to be in bijection with Satake fibers [K].

(6) Positive configurations of points in the affine building are new objects in the geometry of the affine building. They are fairly rigid, they can be parameterized, and they are of a tropical nature, unlike general configurations of points in the affine building. There should be a duality pairing between positive configurations for Langlands dual groups which encapsulates rich geometric structure.

Let us summarize the contents of this paper. In section 2, we begin by reviewing the constructions of higher Teichmüller spaces in [FG1]. We will adapt some of the exposition from their introduction. In section 3 we discuss some generalities on tropical points of positive varieties, spelling out some ideas that are implicit in [FG1]. In section 4 we review the definition and basic properties of the affine Grassmanian and affine buildings. In section 5 we first outline a conceptual description of virtual positive configurations of points in the affine building, the central object by which we define higher laminations. We then explain the construction of positive configurations and give a precise definition of virtual positive configurations of points in the affine Grassmanian and the affine building. This leads up to Theorem 5.11, the central result of this paper, describing $\mathcal{A}$-laminations on the disc. From here we deduce how to define $\mathcal{A}$-laminations on any surface. In section 6, we give the analogous result for $X$-laminations, and extend the definition to closed surfaces. Finally, in section 7, we describe an application of the theory: a spherical compactification of higher Teichmüller space as a closed ball such that the action of the (higher) mapping class group extends to the boundary.

Acknowledgments I would like to thank Vladimir Fock for his generosity in clarifying the ideas of [FG1] and [FG2], and also pointing out that the study of higher laminations was an interesting problem. I thank Francois Labourie for helpful conversations and encouragement. My ideas are very indebted to Joel Kamnitzer’s work, and he helped me understand the relationship of his work to my own. My thinking about cluster algebras was very influenced by Lauren Williams,
Greg Musiker and David Speyer. Finally, my advisor David Nadler has continually been a valuable sounding board for my ideas, and I am grateful for his support and encouragement.

2. BACKGROUND

Let $S$ be a compact oriented surface, with or without boundary, and possibly with a finite number of marked points on each boundary component. We will refer to this whole set of data—the surface and the marked points on the boundary—by $S$. We will always take $S$ to be hyperbolic, meaning it either has negative Euler characteristic, or contains enough marked points on the boundary (in other words, we can give it the structure of a hyperbolic surface such that the boundary components that do not contain marked points are cusps, and all the marked points are also cusps).

Let $G$ be a semi-simple algebraic group. When $G$ is adjoint, i.e., has trivial center, we can define a higher Teichmuller space $X_{G,S}$; while for $G$ simply-connected, we can define the higher Teichmuller space $A_{G,S}$. They will be the space of local systems of $S$ with structure group $G$ with some extra structure on the boundaries. Alternatively, these spaces describe homomorphisms of $\pi_1(S)$ into $G$ modulo conjugation plus some extra data.

When $S$ does has at least one hole, the spaces $X_{G,S}$ and $A_{G,S}$ have a distinguished collection of coordinate systems, equivariant under the action of the mapping class group of $S$. All the transition functions are subtraction-free, and give a positive atlas on the corresponding moduli space. This positive atlas gives the spaces $X_{G,S}$ and $A_{G,S}$ the structure of a positive variety.

If $X$ is a positive variety (for example, $X = A_{G,S}$ or $A_{G,S}$), we can take points of $X$ with values in any semifield, i.e., in any set equipped with operations of addition, multiplication and division, such that these operations satisfy their usual properties (the most important being distributivity). For us, the important examples of semifields will be the positive real numbers $\mathbb{R}_{>0}$; any tropical semifield; and the semifield which interpolates between these two: the field of formal Laurent series over $\mathbb{R}$ with positive leading coefficient, which we denote $K_{>0}$. The tropical semifields $\mathbb{Z}^t, \mathbb{Q}^t, \mathbb{R}^t$ are obtained from $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ by replacing the operations of multiplication, division and addition by the operations of addition, subtraction and taking the maximum, respectively.

Taking $\mathbb{R}_{>0}$-points of the spaces $X_{G,S}$ and $A_{G,S}$ allows us to recover higher Teichmuller spaces. The proof of this takes up the bulk of [FG1]. This space consists of the real points of $X_{G,S}$ and $A_{G,S}$ whose coordinates in one, and hence in any, of the constructed coordinate charts are positive.

The existence of these extraordinary positive co-ordinate charts depends on G. Lusztig’s theory of positivity in semi-simple Lie groups [Lu], [Lu2], and is a reflection of the cluster algebra structure of the ring of functions on these spaces.

We now review the definition of the spaces $X_{G,S}$ and $A_{G,S}$. Let $B$ be a Borel subgroup, a maximal solvable subgroup of $G$. Then $B = G/B$ is the flag variety. Let $U := [B, B]$ be a maximal unipotent subgroup in $G$.

Let $L$ be a $G$-local system on $S$. For any space $X$ on equipped with a $G$-action, we can form the associated bundle $L_X$. For $X = G/B$ we get the associated flag bundle $L_B$, and for $X = G/U$, we get the associated principal flag bundle $L_A$.

A framed $G$-local system on $S$ is a pair $(L, \beta)$, where $L$ is a $G$-local system on $S$, and $\beta$ a flat section of the restriction of $L_B$ to the punctured boundary of $S$.

**Definition 2.1.** The space $X_{G,S}$ is the moduli space of framed $G$-local systems on $S$.

Now suppose $G$ is simply-connected. The maximal length element $w_0$ of the Weyl group of $G$ has a natural lift to $G$, denoted $\overline{w}_0$. Let $s_G := \overline{w}_0^2$. It turns out that $s_G$ is in the center of
We will consider the positive semifield of formal Laurent series over \( A \) and describe higher laminations. Tropical points. The main task of this paper will be to use this construction to give a concrete system on \( S \).

A twisted \( G \)-local system is a representation \( \pi_1(S) \) in \( G \) such that \( \sigma_S \) maps to \( s_G \). Such a representation gives a local system on \( T'S \).

Let \( G \) be a simply-connected split semi-simple algebraic group. Let \( \mathcal{L} \) be a twisted \( G \)-local system on \( S \). Such a twisted local system gives an associated principal affine bundle \( \mathcal{L}_A \) on the punctured tangent bundle \( T'S \). We may restrict this bundle to the boundary of \( S \).

**Definition 2.2.** A decorated \( G \)-local system on \( S \) consists of \(( \mathcal{L}, \alpha )\), where \( \mathcal{L} \) is a twisted local system on \( S \) and \( \alpha \) is a flat section of \( \mathcal{L}_A \) restricted to the boundary.

The space \( \mathcal{A}_{G,S} \) is the moduli space of decorated \( G \)-local systems on \( S \).

Note that in the case where \( s_G = e \), a decorated local system is just a local system on \( S \) along with a flat section of \( \mathcal{L}_A \) restricted to the boundary.

The positive co-ordinate systems on \( \mathcal{A}_{G,S} \) and \( \mathcal{A}_{G,S} \) arise by rationally identifying them with spaces of configurations of flags. We will only outline this part of the story. Let \( S \) be a hyperbolic surface, and that \( c \) is a central extension of \( \pi_1(S) \). The quotient of \( \pi_1(S) \) by the central subgroup \( 2\mathbb{Z} \subset \mathbb{Z} \), gives \( \pi_1(S) \) which is a central extension of \( \pi_1(S) \) by \( \mathbb{Z}/2\mathbb{Z} \). Let \( \sigma_S \) denote the non-trivial element of the center.

A twisted \( G \)-local system is a representation \( \pi_1(S) \) in \( G \) such that \( \sigma_S \) maps to \( s_G \). Such a representation gives a local system on \( T'S \).

\[ \beta(\gamma \cdot c) = \rho(\gamma) \cdot c \]

for all points \( c \in C \).

**Theorem 2.3.** [FG1] The space \( \mathcal{A}_{G,S} \) has a positive atlas that comes from identifying a framed local system with a \( \pi_1(S) \)-equivariant positive configuration of flags parameterized by \( C \).

The space \( \mathcal{A}_{G,S} \) has a positive atlas that comes from identifying decorated a local system with a \( \pi_1(S) \)-equivariant twisted positive cyclic configuration of principal affine flags parameterized by \( C \).

In particular, when \( S \) is a disk with marked points on the boundary we simply get moduli spaces of cyclically ordered configurations of points in the flag variety \( B := G/B \) and twisted cyclic configurations of points of the principal affine variety \( A := G/U \). For more details, see [FG1]

3. The tropical points of higher Teichmuller space

We have constructed higher Teichmuller space. We will now give one construction of its tropical points. The main task of this paper will be to use this construction to give a concrete description of higher laminations.

We now give a simple way to construct tropical points of any positive variety. Let \( K \) be the field of formal Laurent series over \( \mathbb{R}, \mathbb{R}((t)) \). This ring has a natural valuation \( \text{val} : K \to \mathbb{Z} \).

We will consider the positive semifield \( K_{>0} \) which consists of those Laurent series with positive
leading coefficient. Let $X$ be any positive variety, in other words a variety with an atlas of charts such that all transition functions involve only multiplication, division and addition (for example $X_{G,S}$ or $A_{G,S}$). We may then consider the $K_{>0}$ points of these varieties.

Let $x \in X(K_{>0})$. Then there is a corresponding tropical point $x^t$ of the space $X(Z^t)$. This point $x^t$ is characterized by the property that if $f$ is one of the positive coordinates of a positive chart, then

$$f(x^t) = -\text{val}(f(x)).$$

In other words, we specify the tropical coordinates of $x^t$ in each chart as being negative of the valuation of the coordinate of $x$. To see that $x^t$ is well-defined, we only need to check that under a change of coordinate charts, the functions $-\text{val}(f(x))$ transform tropically. However, this is clear, because all transition functions between coordinate charts involve only multiplication, division and addition, and we have

$$-\text{val}(xy) = -\text{val}(x) - \text{val}(y),$$

$$-\text{val}(x/y) = -\text{val}(x) + \text{val}(y),$$

$$-\text{val}(x + y) = \max\{-\text{val}(x), -\text{val}(y)\}.$$  

The last equality holds whenever both $x$ and $y$ have positive leading coefficient.

In other words, the negative valuations of $K_{>0}$-points of a positive variety automatically satisfy the tropical relations. Thus we get a map $-\text{val} : X(K_{>0}) \to X(Z^t)$. The map is surjective because we may specify the valuations of coordinates of a point of $X(K_{>0})$ in any particular chart as we wish. Thus, in order to understand tropical points of $X_{G,S}$ or $A_{G,S}$, we must analyze the fibers of this map and see what the points in one fiber have in common, or isolate what invariant information is contained in the tropical functions.

We will find a satisfactory answer in the case where $G = SL_m$. Many of the steps will have clear generalizations to general groups (for example, the definition of laminations in terms of affine buildings); we will note those steps which do not extend as straightforwardly. We hope to treat in a future paper the case of a general semi-simple Lie group, for which we believe the best approach would be to explicitly construct cluster coordinates on Teichmuller spaces associated to these groups. These coordinates would be analogous to the “canonical coordinates” of Fock and Goncharov on Teichmuller spaces for $SL_m$. Henceforth we will be concerned primarily with the case $G = SL_m$ or $G = PGL_m$.

If instead of considering $K$, we consider the ring of Laurent series over $t^\lambda$ where $\lambda$ is allowed to vary in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, we get obtain different types of laminations with coefficients in $Z^t$, $\mathbb{Q}^t$ or $\mathbb{R}^t$.

4. Background on affine buildings

In this section we follow some of the treatment of [FKK], which in many ways was inspiration for our work. Let $G$ be a simple, simply-connected complex algebraic group and let $G^\vee$ be its Langlands dual group. Let us define the real affine Grassmanian. (The affine Grassmanian is an ind-scheme, or an inductive limit of schemes, and it can be defined over any ring, but that will not concern us here.) Let $O = \mathbb{R}[\lbrack t\rbrack]$ be the ring of formal power series over $\mathbb{R}$ and let $K = \mathbb{R}(\lbrack t\rbrack)$ be its function field. Then

$$\text{Gr} = \text{Gr}(G) = G(K)/G(O)$$

is the real affine Grassmannian for $G$. It can be viewed as a direct limit of real varieties of increasing dimension.

The affine Grassmannian $\text{Gr}$ is also a coweight-valued metric space: the double cosets

$$G(O) \backslash G(K)/G(O)$$


are in bijection with the cone $\Lambda_+$ of dominant coweights of $G$. Recall that the coweight lattice $\Lambda$ is defined as $\text{Hom}(G_m, T)$. The coweight lattice contains dominant coweights, those coweights lying in the dominant cone.

Given any dominant coweight $\mu$ of $G$, there is an associated point $t^\mu$ in the (real) affine Grassmannian. Any two points $p$ and $q$ of the affine Grassmannian can be translated to $t^0$ and $t^\mu$, respectively, for some unique dominant coweight $\mu$. Under this circumstance, we will write

$$d(p, q) = \mu$$

and say that the distance from $p$ to $q$ is $\mu$. Note that this distance function is not symmetric; one can easily check that

$$d(p, q) = -w_0d(q, p)$$

where $w_0$ is the longest element of the Weyl group of $G$ (recall that the Weyl group acts on both the weight space $\Lambda^*$ and its dual $\Lambda$). However, there is a partial order on $\Lambda$ defined by $\lambda > \mu$ if $\lambda - \mu$ is dominant. Then the distance function satisfies a version of the triangle inequality.

The action of $G(\mathcal{K})$ on the real affine Grassmannian preserves this distance function.

The points of the affine Grassmannian $\text{Gr}$ are also a subset of the vertices of an associated simplicial complex called the affine building $\Delta = \Delta(G) \mathbb{H}$ whose type is the extended Dynkin type of $G$. The simplices of this affine building correspond to parahoric subgroups of the affine Kac-Moody group $\tilde{G}$. In the case where $G = \text{SL}_m$ or $\text{PGL}_n$, the set of vertices of the affine building is precisely given by the points of the affine Grassmannian $\text{Gr}$. For a detailed description of affine buildings from this perspective, see [GL].

An affine building $\Delta$ satisfies the following axioms:

1: The building $\Delta$ is a non-disjoint union of apartments, each of which is a copy of the Weyl alcove simplicial complex of $G$.

2: Any two simplices of $\Delta$ of any dimension are both contained in at least one apartment $\Sigma$.

3: Given two apartments $\Sigma$ and $\Sigma'$ and two simplices $\alpha, \alpha' \in \Sigma \cap \Sigma'$, there is an isomorphism $f : \Sigma \to \Sigma'$ that fixes $\alpha$ and $\alpha'$ pointwise.

The axioms imply that the vertices of $\Delta$, denoted $\text{Gr}'$, are canonically colored by the vertices of the extended Dynkin diagram $\tilde{I} = I \cup \{0\}$ of $G^\vee$, or equivalently the vertices of the standard Weyl alcove $\delta$ of $G^\vee$. Moreover, every maximal simplex of $\Delta$ is a copy of $\delta$; it has exactly one vertex of each color. The affine Grassmannian consists of those vertices colored by 0 and by minuscule nodes of the Dynkin diagram of $G^\vee$. For $\text{SL}_m$ or $\text{PGL}_m$, all roots are minuscule.

5. Laminations for the $A$ space

5.1. Outline of the definition of $A$-laminations. We will first consider the space of integral laminations $A_{G,S}(\mathbb{Z})$. For the purpose of orientation for the next few sections, in which we give the construction of positive virtual configurations in the building, we first give a conceptual outline. All the constructions here can be extended without difficulty to laminations rational or real lamination spaces ($A_{G,S}(\mathbb{Q})$ or $A_{G,S}(\mathbb{R})$). In the following, we will assume that $G = \text{SL}_m$.

We will begin by describing laminations on a disc with $n$ marked points. The arguments in [FG1] show that understanding laminations on a surface reduces, via cutting and gluing and $\pi_1$-equivariance, to the case of laminations on a disc with 2, 3 or 4 marked points. We will build up to the following definition:

**Definition 5.1.** A $G$-lamination on a disc with $n$ marked points is a virtual positive configuration of $n$ points in the affine building for $G$ up to equivalence.
A configuration of $n$ points in the building is a set of $n$ points of the affine building for $G$. We will study configurations up to equivalence. We will define equivalence inductively. Let $p_1, \ldots, p_n$ and $p'_1, \ldots, p'_n$ be two configurations of points of the affine building. If they are to be equivalent, we first require that the pairwise distances $d(p_i, p_j)$ and $d(p'_i, p'_j)$ are equal. Define a perimeter of a configuration $p_1, \ldots, p_n$ to be a union of the some choice of geodesics between each $p_i$ and $p_{i+1}$, where indices are taken cyclically. We can then choose a corresponding perimeter for $p'_1, \ldots, p'_n$. Then choose some geodesic between two points $a$ and $b$ in the perimeter. Suppose $a$ is on the geodesic between $p_i$ and $p_{i+1}$ and $b$ on the geodesic between $p_j$ and $p_{j+1}$. Take the corresponding points $a'$ and $b'$ on the perimeter of $p'_1, \ldots, p'_n$. Then we make a “cut” to form the configurations

$$a, p_1 + 1, \ldots, p_j, b$$

and

$$b, p_j + 1, \ldots, p_i, a$$

and the corresponding configurations

$$a', p'_1 + 1, \ldots, p'_j, b'$$

and

$$b', p'_j + 1, \ldots, p'_i, a'.$$

Then $p_1, \ldots, p_n$ and $p'_1, \ldots, p'_n$ are equivalent if and only if $a, p_{i+1}, \ldots, p_j, b$ is equivalent to $a', p'_{i+1}, \ldots, p'_{j}, b'$ and $b, p_{j+1}, \ldots, p_i, a$ is equivalent to $b', p'_{j+1}, \ldots, p'_i, a'$. Using cuts, we can reduce to the case of triangles with miniscule side lengths. Finally, we say that two such triangles are equivalent if their side lengths coincide. It turns out that our definition of equivalence, when restricted to positive configurations, does not depend on the sequence of cuts. From now on, when we discuss configurations in the affine building, we will be tacitly considering them up to equivalence.

(It is likely that positivity allows us to strengthen our notion of equivalence. For example, we believe that the equivalence of two positive configurations $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$ implies that their convex hulls are isometric, where the convex hull of $x_1, \ldots, x_n$ is the smallest geodesically closed subset containing $x_1, \ldots, x_n$. It is even possible that there is an isometry of the entire affine building which carries $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$. It may be useful in applications to have a somewhat stronger notion of equivalence, though for this paper, the one given above is sufficient.)

In this section we will try to give the ideas behind the adjectives “virtual” and “positive.” We will later give description of a family of maps from configurations of points in $G/U((t))$ to configurations of points in the affine flag variety and also to configurations of points in the affine Grassmanian. There will in general be many such maps, and they will depend on choices.

We will then describe positive configurations in $G/U((t))$. This carries over to the notion of positive configurations in both the affine flag variety and the affine Grassmanian. Thus we construct positive configurations of points in the affine Grassmanian and the affine building from positive configurations for points in $G/U((t))$ via our previous construction. The points of the affine Grassmanian can then be considered as points in the building, and we can carry over the notion of a positive configuration of points there as well. Most of the work will initially involve positive configurations in the affine Grassmanian, while later, when we consider the notion of equivalence, we will use positive configurations in the affine building. Positivity will allow us to deduce certain properties of these configurations of points in the affine building, for example the fact that tropical coordinates completely determine the configuration up to equivalence. The idea is that positivity ensures certain genericity properties of our configuration. Moreover,
we will see that once we pass to configurations of points in the affine building, many of the choices we made will become irrelevant.

**Note 5.2.** Fock and Goncharov define of positive configurations of points in the flag variety. Their definition should extend to a definition of positive configurations of points in the affine flag variety. It would be interesting to compare our notion of positive configurations with theirs.

In order to define virtual configurations, we need to understand the action of $\Lambda^+_{\mathfrak{a}}$ (the dominant coweights) on the space of positive configurations of $n$ points in the affine building up to equivalence. It turns out that positive configurations in the affine building come equipped automatically with an action of $\Lambda^+_{\mathfrak{a}}$. (We will later see that there is an action of $\Lambda^+_{\mathfrak{a}}$ for every marked point and every hole on the surface $S$. Furthermore, this will extend to an action of the entire coweight lattice $\Lambda$ on the space of virtual configurations. On a surface with holes, this will extend to an action of an affine Weyl group at each puncture on the space of laminations.)

Let us describe the action of the monoid $\Lambda^+_{\mathfrak{a}}$ on the space of positive configurations of $n$ points in the building. There is one factor $\Lambda^+_{\mathfrak{a}}$ acting on each of the $n$ points in the configuration. Let $p_1, p_2, \ldots, p_n$ be a configuration of $n$ points. $\lambda \in \Lambda^+_{\mathfrak{a}}$ acts on the point $p_1$ by moving $p_1$ to another point $p'_1$ of the building which is distance $\lambda$ away, i.e. such that

$$d(p_1, p'_1) = \lambda.$$ 

There are of course many choices for $p'_1$; however, there is a family of most generic choices such that we are moving $p_1$ a distance $\lambda$ away from the other points in the configuration. It turns out all these choices are equivalent in the sense that the configurations $p'_1, p_2, \ldots, p_n$ will be equivalent for all such choices of $p'_1$.

A virtual positive configuration of points in the affine building is a set of $n$ pairs $(\lambda_i, p_i)$, $i = 1, 2, \ldots, n$, where $\lambda_i$ are all coweights and the $p_i$ form a positive configuration. Let $(\lambda_1, p_1)$ and $(\nu_1, p_1)$ be two virtual configurations. If all the $\lambda_i$ and $\nu_i$ are positive coweights then we may allow $(\lambda_1, \ldots, \lambda_n)$ to act on $(p_1, \ldots, p_n)$ and $(\nu_1, \ldots, \nu_n)$ to act on $(\nu_1, \ldots, \nu_n)$. Suppose the resulting configurations are $(\mu'_1, \ldots, \mu'_n)$ and $(\lambda'_1, \ldots, \lambda'_n)$. We will say that $(\mu'_1, \ldots, \mu'_n)$ realizes the virtual configuration $(\lambda_1, \ldots, \lambda_n)$.

Then we will say that $(\lambda_1, \ldots, \lambda_n)$ and $(\nu_1, \ldots, \nu_n)$ are equivalent virtual configurations if and only if $(\mu'_1, \ldots, \mu'_n)$ and $(\lambda'_1, \ldots, \lambda'_n)$ are equivalent as configurations. More generally, two configurations $(\lambda_1, \ldots, \lambda_n)$ and $(\nu_1, \ldots, \nu_n)$ are equivalent if there exists $\nu_i$ such that $(\lambda_1 + \nu_1)$ and $(\nu_1, \ldots, \nu_n)$ are equivalent virtual configurations of points in the affine building, then for any $n$ dominant coweights $\lambda_1, \ldots, \lambda_n$, if we allow the $\lambda_i$ to act on $p_i$ and $q_i$, the resulting positive configurations will still be equivalent.

Finally, let us note that $n$ marked points on a disc come with the natural cyclic ordering. Virtual positive configurations will attach a virtual point of the affine building to each of these $n$ marked points. In other words, virtual positive configurations will have a natural cyclicity: $(p_1, \lambda_1), \ldots, (p_n, \lambda_n)$ is a positive virtual configuration if and only if every cyclic shift is a positive virtual configuration.

This concludes the outline of the definition of virtual positive configurations. Let us make these definitions more precise.
The Grassmanian is given by \( G \) functions where points in the affine Grassmanian for \( SL \) be thought of as a finitely generated, rank \( m \) generators for this submodule, then strictly less than \( m \) functions face Grassmanian, thought of as \( O \) edge depend on two of the flags. We can call such functions and division) [FG1].

by a positive rational transformation (a transformation involving only addition, multiplication and division) [FG1].

Given a cyclic configuration of \( n \) flags, imagine the flags sitting at the vertices of an \( n \)-gon, triangulate the \( n \)-gon. Then taking the edge and face functions on the edges and faces of this triangulation, we get a set of functions on a cyclic configuration of flags. These functions form a coordinate chart, and different triangulations yield different functions that are related to these by a positive rational transformation (a transformation involving only addition, multiplication and division) [FG1].

We will now analogously define the triple distance functions \( f^1_{ijk} \) on configuration of three points in the affine Grassmanian for \( SL_m \). The functions \( f^1_{ijk} \) are essentially the same as the functions \( H_{ijk} \), which were defined in a slightly different way in [K]. Recall that the affine Grassmanian is given by \( G(\mathcal{K})/G(O) \). For \( G = SL_m \), a point in the affine Grassmanian can be thought of as a finitely generated, rank \( m \) \( O \)-submodule of \( \mathcal{K}^m \) such that if \( v_1, \ldots, v_m \) are generators for this submodule, then

\[
v_1 \wedge \cdots \wedge v_m = e_1 \wedge \cdots \wedge e_m
\]

where \( e_1, \ldots, e_m \) is the standard basis of \( \mathcal{K}^m \). Let \( x_1, x_2, x_3 \) be three points in the affine Grassmanian, thought of as \( O \)-submodules of \( \mathcal{K}^n \). We will consider

\[
- \text{val}(\det(u_1, \ldots, u_i, v_1, \ldots, v_j, w_1, \ldots, w_k))
\]

as \( u_1, \ldots, u_i \) range over elements of the \( O \)-submodule \( x_1 \), \( v_1, \ldots, v_j \) range over elements of the \( O \)-submodule \( x_2 \), and \( w_1, \ldots, w_k \) range over elements of the \( O \)-submodule \( x_3 \). Define \( f^1_{ijk}(x_1, x_2, x_3) \) as the largest value of

\[
- \text{val}(\det(u_1, \ldots, u_i, v_1, \ldots, v_j, w_1, \ldots, w_k))
\]

as all the vectors \( u_1, \ldots, u_i, v_1, \ldots, v_j, w_1, \ldots, w_k \) range over elements of the \( O \)-submodules \( x_1, x_2, x_3 \).

There is a more invariant way to define \( f^1_{ijk} \). Lift \( x_1, x_2, x_3 \) to elements \( g_1, g_2, g_3 \) of \( G(\mathcal{K}) \), then project to three flags \( F_1, F_2, F_3 \in G/U(\mathcal{K}) \). In some sense, we are lifting from \( G(\mathcal{K})/G(O) \) to \( G/U(\mathcal{K}) \). Then define \( f^1_{ijk} \) to be the maximum of \( - \text{val}(f_{ijk}(F_1, F_2, F_3)) \) over the different possible lifts from \( G(\mathcal{K})/G(O) \) to \( G/U(\mathcal{K}) \).
Note 5.4. It is not hard to check that the edge functions recover the distance between two points in the building in this way. More precisely, \( f_{ij}^{kl}(x_1, x_2, x_3) \) is given by \( \omega_j \cdot d(x_1, x_2) = \omega_i \cdot d(x_2, x_1) \).

5.3. Positive Configurations. We now move on to our central construction and define positive configurations of \( n \) points in the affine Grassmanian, and thus positive configurations of \( n \) points in the affine building. For this section, let \( S \) be the disc with \( n \) marked points on the boundary. The associated higher Teichmüller space \( A_{G,S} \) consists of configurations of \( n \) principal flags. Recall that \( \mathcal{K}_{>0} \) is the ring of positive Laurent series, or Laurent series with positive leading term. We will consider the set \( A_{G,S}(\mathcal{K}_{>0}) \). This set consists of configurations of \( n \) principal flags \( F_1, F_2, \ldots, F_n \) in \( \mathcal{K}^m \). For each flag \( F_i \), choose a lift to \( G(K) = SL_m(K) \). Call this lift \( g_i \). The columns of \( g_i \) will be \( v_{i1}, \ldots, v_{im} \), where \( v_{i1} \wedge \cdots \wedge v_{ik} \) for \( k = 1, 2, \ldots, m-1 \) will be the successive subspaces (with volume form) in the flag.

One naive guess would be to associated to the flag \( F_i \) the \( \mathcal{O} \)-submodule of \( \mathcal{K}^m \) spanned by \( v_{i1}, \ldots, v_{im} \), which would then give us an element of \( \text{Gr}(G) \). This of course is not well-defined, as it would depend on our choice of lift \( g_i \). However, it turns out that there a fix. For each flag \( F_i \) choose a lift \( v_{i1}, \ldots, v_{im} \). Suppose we have some other lift \( v'_{i1}, \ldots, v'_{im} \).

**Lemma 5.5.** For some large enough coweights \( \lambda_i = (\lambda_{i1}, \ldots, \lambda_{im}) \), the vectors

\[ t^{-\lambda_{i1}}v_{i1}, t^{-\lambda_{i2}}v_{i2}, \ldots, t^{-\lambda_{im}}v_{im} \]

generate the same \( \mathcal{O} \)-module as

\[ t^{-\lambda'_{i1}}v'_{i1}, t^{-\lambda'_{i2}}v'_{i2}, \ldots, t^{-\lambda'_{im}}v'_{im}. \]

(Recall that a coweight for \( SL_m \) consists of an \( m \)-tuple of integers that sum to 0. A “large” coweight is one where the this \( m \)-tuple is decreasing, and the gaps between the integers are large.)

**Proof.** This is fairly straightforward. We show this statement separately for each flag. Different lifts from \( G/U^-(K) \) to \( G(K) \) differ by some element in \( U^- \). In other words, \( v_{i1}, \ldots, v_{im} \) and \( v'_{i1}, \ldots, v'_{im} \) are related by some lower triangular matrix \( u \) with entries in \( K \). If the entries of \( u \) and \( u^{-1} \) have entries such that all their valuations are greater than \( -C \) (where \( C > 0 \)), then if we choose \( \lambda_i \) large enough that its gaps are all greater than \( C \), then the vectors

\[ t^{-\lambda_{i1}}v_{i1}, t^{-\lambda_{i2}}v_{i2}, \ldots, t^{-\lambda_{im}}v_{im} \]

and the vectors

\[ t^{-\lambda'_{i1}}v'_{i1}, t^{-\lambda'_{i2}}v'_{i2}, \ldots, t^{-\lambda'_{im}}v'_{im} \]

generate the same \( \mathcal{O} \)-module.

In more invariant terms, for such a \( \lambda_i \), conjugating \( u \) by \( t^\lambda \) will give us an element of \( G(\mathcal{O}) \). This more invariant argument works for groups other than \( SL_m \).

The motivation for this construction can be explained as follows. A configuration of \( n \) principal flags is a configuration of \( n \) points of \( G/U^- \) up to a diagonal action of \( G \). Here by convention \( U^- \) is the group of unipotent lower triangular matrices. But on this space there is a right action of \( T \) on each principal flag. On the space of positive configurations \( A_{G,S,\mathbb{R}_{>0}} \), there is an action of \( T(\mathbb{R}_{>0})^n \). This action can be thought of as changing the horocycle (for \( SL_2 \), this is exactly changing the horocycle at a cusp of the Riemann surface). Analogously, on \( A_{G,S,\mathcal{K}_{>0}} \) there is an action of \( T(\mathcal{K}_{>0})^n \), with one copy of \( T(\mathcal{K}_{>0}) \) acting at each point. Inside \( T(\mathcal{K}_{>0}) \) we have the elements \( t^\lambda \) for \( \lambda \in \Lambda \), the coweight lattice. We will take the convention that the action of \( \lambda \) will be by multiplication by \( t^{-w_0(\lambda_i)} \) on the right, which takes the flag

\[ v_{i1}, \ldots, v_{im} \]
to the flag

\[ t^{-\lambda_i}v_{i1}, \ldots, t^{-\lambda_i}v_{im}. \]

We will see that this action will become the action of \( \Lambda^n \) on the space of positive configurations of \( n \) points in the building.

Thus although there is no sensible map from configurations in \( G/U(K) \) to configurations in \( G(K)/G(O) \), we can define a map up to some choice of lifts \( v_{i1}, \ldots, v_{im} \) and some choice of “large enough” coweights \( \lambda_i \). Our strategy will then be to assign to \( F_i \) the \( O \)-submodule of \( K^m \) spanned by

\[ t^{-\lambda_i}v_{i1}, \ldots, t^{-\lambda_i}v_{im}. \]

Our task then becomes defining a notion of “large enough.” This can be thought of as a reverse procedure to the lifting from \( G(K)/G(O) \) to \( G/U(K) \) that we used to define the functions \( f_{ijk}^t \).

Now given the configurations of \( n \) principal flags \( F_1, \ldots, F_n \) in \( K^m \), choose lifts \( v_{i1}, \ldots, v_{im} \) and choose a triangulation of the \( n \)-gon. We will say that the lifts \( v_{i1}, \ldots, v_{im} \) are good if for any triangle of flags \( F_p, F_q, F_r \),

\[ f_{ijk}^t(x_p, x_q, x_r) = -\text{val}(\det(v_{p1}, \ldots, v_{pi}, v_{qi}, \ldots, v_{qj}, v_{rl}, \ldots, v_{rk})) \]

where \( x_p \) is the \( O \)-module spanned by \( v_{p1}, v_{p2}, \ldots, v_{pm} \) and similarly for \( x_q \) and \( x_r \).

In other words, the lifts \( v_{i1}, \ldots, v_{im} \) are good if

\[ -\text{val}(f_{ijk}(F_p, F_q, F_r)) = f_{ijk}^t(x_p, x_q, x_r) \]

or if they realize the maximum of minus the valuation of all the \( f_{ijk} \) simultaneously. This is of course quite unlikely to be the case. But just as before, we are saved by the action of \( \Lambda \):

**Lemma 5.6.** We can choose \( \lambda_i \) large enough such that for each triangle of flags \( F_p, F_q, F_r \) in our triangulation of the \( n \)-gon and every \( i, j, k \) with \( i + j + k = m \),

\[ f_{ijk}^t(x_p, x_q, x_r) = -\text{val}(\det(t^{-\lambda_p}v_{p1}, \ldots, t^{-\lambda_p}v_{pi}, t^{-\lambda_q}v_{q1}, \ldots, t^{-\lambda_q}v_{qj}, t^{-\lambda_r}v_{rl}, \ldots, t^{-\lambda_r}v_{rk})) \]

where now \( x_p \) is the \( O \)-module spanned by

\[ t^{-\lambda_p}v_{p1}, \ldots, t^{-\lambda_p}v_{pm} \]

and similarly for \( x_q \) and \( x_r \). In other words, we act upon the flags \( F_i \), so that the vectors

\[ t^{-\lambda_p}v_{p1}, \ldots, t^{-\lambda_p}v_{pm} \]

are a good lift of \( F_i \cdot \lambda_i \) and

\[ -\text{val}(f_{ijk}(F_p \cdot \lambda_p, F_q \cdot \lambda_q, F_r \cdot \lambda_r)) = f_{ijk}^t(x_p, x_q, x_r) \]

**Proof.** The different spans \( x_p, x_q, x_r \) vary as we change the \( \lambda_i \). Consider the different possible values for minus the valuation of the determinant of some subset of \( i \) vectors among

\[ t^{-\lambda_p}v_{p1}, \ldots, t^{-\lambda_p}v_{pm}, \]

some subset of \( j \) vectors among

\[ t^{-\lambda_q}v_{q1}, \ldots, t^{-\lambda_q}v_{qm}, \]

and some subset of \( k \) vectors among

\[ t^{-\lambda_r}v_{r1}, \ldots, t^{-\lambda_r}v_{rm}. \]
Observe that for any choice of $\lambda$, $f'_{ijk}(x_p, x_q, x_r)$ is the maximum of all these values, as the determinant of any $i$ vectors in $x_p$, $j$ vectors in $x_q$, and $k$ vectors in $x_r$ is a linear combination of the determinants considered above. However, as the $\lambda_i$ get large, the valuation of
\[
\det(t^{-\lambda_{i_1} v_{p_1}} t^{-\lambda_{i_2} v_{p_2}} \cdots, t^{-\lambda_{i_j} v_{p_j}}, t^{-\lambda_{r_1} v_{q_1}}, \cdots, t^{-\lambda_{r_k} v_{q_k}})
\]
get negative the fastest. Thus for large enough $\lambda_i$, we have our claim.

\[\square\]

We observe here that if $F_i$ is a positive configuration of flags in $G/U(t)$, then $F_i \cdot \lambda_i$ will also be a positive configuration of flags for any choice of $\lambda_i$. Now we can define positive configurations in the affine Grassmanian. Given a positive configuration of flags $F_1, F_2, \ldots, F_n$ coming from a point in $A_{G,S}K_{>0}$, we choose some lifts $v_{i_1}, \ldots, v_{i_m}$ of the $F_i$ and choose a triangulation of the $n$-gon. Then taking $\lambda_i$ large enough to give us good lifts as in Lemma 5.6, we can then obtain the points $x_1, x_2, \ldots, x_n \in G(K)/G(O)$. We will call configurations of points in the affine Grassmanian that arise in this way positive configurations of points.

We now must analyze how this construction depends on various choices. The choice of lifts $v_{i_1}, \ldots, v_{i_m}$ only affects the choice of $\lambda_i$. Note that if $t^{-\lambda_{i_1}} v_{i_1}, \ldots, t^{-\lambda_{i_m}} v_{i_m}$ is a good lift of $F_i \cdot \lambda_i$, then replacing the $\lambda_i$ with any set of larger coweights will still give us a good lift.

Let us analyze the dependence on the choice of lifts.

**Lemma 5.7.** If we have two different sets of lifts $v_{i_1}, \ldots, v_{i_m}$ and $v'_{i_1}, \ldots, v'_{i_m}$, and some $\lambda_i$ such that $t^{-\lambda_{i_1}} v_{i_1}, \ldots, t^{-\lambda_{i_m}} v_{i_m}$ and $t^{-\lambda_{i_1}} v'_{i_1}, \ldots, t^{-\lambda_{i_m}} v'_{i_m}$ are both good lifts of $F_i \cdot \lambda_i$, then in fact both sets of vectors span the same $O$-modules.

**Proof.** The two sets of vectors
\[
t^{-\lambda_{i_1}} v_{i_1}, \ldots, t^{-\lambda_{i_m}} v_{i_m}
\]
and
\[
t^{-\lambda_{i_1}} v'_{i_1}, \ldots, t^{-\lambda_{i_m}} v'_{i_m}
\]
differ by lower triangular matrices $U_i$ which takes the former to the latter. The entries of these matrices must be in $O$, otherwise the sets of vectors could not both be good lifts.

For example, suppose $j < k$ and $t^{-\lambda_{i_j}} v'_{i_k} = t^{-\lambda_{i_k}} v_{i_j} + a \cdot t^{-\lambda_{i_j}} v_{i_j}$, where $a$ has negative valuation. Then if $t^{-\lambda_{i_1}} v_{i_1}, \ldots, t^{-\lambda_{i_m}} v_{i_m}$ is a good lift, then replacing any occurrence of $t^{-\lambda_{i_j}} v'_{i_k}$ by $t^{-\lambda_{i_k}} v'_{i_j}$ in some determinant expression will result in a smaller valuation, so that $t^{-\lambda_{i_1}} v'_{i_1}, \ldots, t^{-\lambda_{i_m}} v'_{i_m}$ won’t be a good lift. \[\square\]

We now analyze the dependence on the triangulation. We have the following:

**Lemma 5.8.** If $F_i \cdot \lambda_i$ comes from a positive configuration and has good lifts
\[
t^{-\lambda_{i_1}} v_{i_1}, \ldots, t^{-\lambda_{i_m}} v_{i_m}
\]
for some triangulation of the $n$-gon, this lift remains good for any other triangulation. In other words, if we change the triangulation of the $n$-gon, we do not need to change the $\lambda_i$.

**Proof.** The general case is equivalent to the case where $\lambda_i = 0$. Every change of triangulation comes from a sequence of flips; thus is suffices to consider the case where we have just four points $F_i$, $i = 1, 2, 3, 4$. We will consider this simpler case. Assume that $v_{i_1}, \ldots, v_{i_m}$ are a good lift of $F_i$ in the triangulation with triangles 123, 134. We want to show that they remain good lifts for the triangles 124, 234. Let the span of $v_{i_1}, \ldots, v_{i_m}$ be $x_i$.

In other words, we know that
\[
f'_{ijk}(x_p, x_q, x_r) = -\text{val}(\det(v_{p_1}, \ldots, v_{p_i}, v_{q_1}, \ldots, v_{q_j}, v_{r_1}, \ldots, v_{r_k}))
\]
for \((p, q, r) = (1, 2, 3)\) and \((1, 3, 4)\), and want to conclude this for \((p, q, r) = (1, 2, 4)\) and \((2, 3, 4)\).

Recall that \(f'_{ijk}(x_p, x_q, x_r)\) is defined by taking the maximum of minus the valuation of the determinant of some subset of \(i\) vectors in \(x_p\), some subset of \(j\) vectors in \(x_q\), and some subset of \(k\) vectors in \(x_r\). Equivalently, it comes from taking some \(i\)-dimensional subspace of the \(\mathbb{R}\) span of \(v_{p1}, \ldots, v_{pm}\), some \(j\)-dimensional subspace of \(\mathbb{R}\) span of \(v_{q1}, \ldots, v_{qm}\), and some \(k\)-dimensional subspace of the \(\mathbb{R}\) span of \(v_{r1}, \ldots, v_{rm}\). Among all these subspaces, the set of them that achieve the maximum of minus the valuation of the determinant is an open set.

Now consider the space of flags in the \(\mathbb{R}\) span of \(v_{q1}, \ldots, v_{qm}\). Let \(y_i\) be such flags in the span of \(v_{q1}, \ldots, v_{qm}\), and let them be represented by \(w_{i1}, \ldots, w_{im}\), where the \(r\)-dimensional subspace in \(y_i\) is the span of \(w_{i1}, w_{i2}, \ldots, w_{ir}\). Then there is an open subset of choices for the \(y_i\) such that

\[
f'_{ijk}(x_p, x_q, x_r) = -\text{val}(\det(w_{p1}, \ldots, w_{pi}, \ldots, w_{qj}, w_{r1}, \ldots, w_{rk})).
\]

Thus there is some choice of \(y_i\) such that the above equality holds for all \(i, j, k\) as well as for all triples \((p, q, r)\). Now let

\[
f'_{ijk}(p, q, r) = \det(w_{p1}, \ldots, w_{pi}, \ldots, w_{qj}, w_{r1}, \ldots, w_{rk}).
\]

By [FG1], we know that all the functions

\[
f'_{ijk}(1, 2, 4), f'_{ijk}(2, 3, 4)
\]

can be expressed in terms of the functions

\[
f'_{ijk}(1, 2, 3), f'_{ijk}(1, 3, 4)
\]

using only addition, multiplication and division. This gives us that

\[-\text{val}(f'_{ijk}(1, 2, 4)), -\text{val}(f'_{ijk}(2, 3, 4))\]

are less than or equal to some tropical expression in

\[-\text{val}(f'_{ijk}(1, 2, 3)), -\text{val}(f'_{ijk}(1, 3, 4)).
\]

However, by positivity, we know that

\[-\text{val}(f_{ijk}(F_1, F_2, F_4)), -\text{val}(f_{ijk}(F_2, F_3, F_4))\]

are equal to (and are not just less than or equal to) the same tropical expressions in

\[-\text{val}(f_{ijk}(F_1, F_2, F_3)), -\text{val}(f_{ijk}(F_1, F_3, F_4)).
\]

Thus by the maximality of

\[-\text{val}(f'_{ijk}(1, 2, 4)), -\text{val}(f'_{ijk}(2, 3, 4)),
\]

we must have

\[-\text{val}(f'_{ijk}(1, 2, 4)) = -\text{val}(f_{ijk}(F_1, F_2, F_4))
\]

and

\[-\text{val}(f'_{ijk}(2, 3, 4)) = -\text{val}(f_{ijk}(F_2, F_3, F_4)).
\]

Therefore

\[
f_{ijk}(x_p, x_q, x_r) = -\text{val}(\det(v_{p1}, \ldots, v_{pi}, \ldots, v_{qj}, v_{r1}, \ldots, v_{rk})))
\]

for \((p, q, r) = (1, 2, 4)\) and \((2, 3, 4)\), as \(-\text{val}(f_{ijk}(1, 2, 4))\) and \(-\text{val}(f_{ijk}(2, 3, 4))\) are equal to the left hand side and \(-\text{val}(f_{ijk}(F_1, F_2, F_4))\) and \(-\text{val}(f_{ijk}(F_2, F_3, F_4))\) are equal to the right hand side. \(\square\)
Thus the map from configurations in \( G/U(\mathcal{K}) \) to configurations in \( G(\mathcal{K})/G(\mathcal{O}) \) does not depend on a choice of triangulation. Of course, the map does depend on choosing the \( \lambda_i \); we get a different map for each choice of \( \lambda_i \). Thus it is better to think that for every point in \( A_{G,S}(\mathcal{K}_{>0}) \), we have a family of positive configurations in \( G(\mathcal{K})/G(\mathcal{O}) \), parameterized by all choices of \( \lambda_i \) that are “large enough” for some lifts of the flags. It turns out that there exist lifts \( g_i \) of \( F_i \) such that \( \lambda_i \) are large enough for some lifts if and only if they are large enough for the lifts \( g_i \).

5.4. Virtual Positive Configurations. To give a better framework for thinking of these families, let us define virtual positive configurations. A virtual positive configuration of \( n \) points in the affine Grassmanian consists of \( n \) pairs \( (x_1, \lambda_1), \ldots, (x_n, \lambda_n) \) where \( x_1, \ldots, x_n \) is a positive configuration of points in the affine Grassmanian, and the \( \lambda_i \) are coweights. Starting with a positive configuration of \( n \) flags \( F_1, \ldots, F_n \), let \( v_{i1}, \ldots, v_{im} \) be some lifts of these flags. Then suppose that for a choice of \( \lambda_1, \ldots, \lambda_m \), \( t^{-\lambda_1i}v_{i1}, \ldots, t^{-\lambda_mi}v_{im} \) is a good lift of \( F_i \cdot \lambda_i \). Let \( x_i \) be the \( \mathcal{O} \)-module spanned by \( t^{-\lambda_1i}v_{i1}, \ldots, t^{-\lambda_mi}v_{im} \). Then we can associate to \( F_1, \ldots, F_n \) the virtual positive configuration

\[
(x_1, -\lambda_1), \ldots, (x_n, -\lambda_n)
\]

If instead we had chosen a different set \( \lambda_i' \) instead of \( \lambda_i \) (for example, it is clear that we could take any set of \( \lambda_i' \) that are all greater than \( \lambda_i \)), we would have ended up with a different configuration \( (x_1', -\lambda_1'), \ldots, (x_n', -\lambda_n') \). Thus we get a family of virtual positive configurations \( (x_1, -\lambda_1), \ldots, (x_n, -\lambda_n) \) as the set of \( \lambda_i \) vary over some subset of \( \Lambda^n \). All we know about this subset is that it is closed under addition by any element of the monoid \( \Lambda^n_+ \). We will later see that there is a smallest possible value for the \( \lambda_i \) such that all other values can be obtained by adding some element of the monoid \( \Lambda^n_+ \). Finally, observe that by Lemma 5.7 the \( x_i \) only depend on the \( \lambda_i \) and not the lifts \( v_{i1}, \ldots, v_{im} \).

Definition 5.9. A family of virtual positive configurations is a set of virtual positive configurations \( (x_1, -\lambda_1), \ldots, (x_n, -\lambda_n) \) (the \( x_i \) vary with choices of \( \lambda_i \)), which comes from some positive configuration of flags \( F_i \) in \( G/U(\mathcal{K}) \).

Thus we have a well-defined map from a positive configuration in \( G/U(\mathcal{K}) \) to a family of virtual configurations in \( G(\mathcal{K})/G(\mathcal{O}) \). One can check as an exercise (and though we do not logically need this it is useful psychologically), that knowing only such a family of virtual configurations in \( G(\mathcal{K})/G(\mathcal{O}) \), we can reconstruct the original positive configuration in \( G/U(\mathcal{K}) \) up to the action of \( T(\mathbb{R}) \) on each flag. Contrast this with the fact that any particular virtual configuration can come from many different positive configurations in \( G/U(\mathcal{K}) \).

We can now define an action of \( \Lambda^n \) on such families (which is not to be confused with the action of \( \Lambda^n \) among the different virtual positive configurations in a family of virtual positive configurations). Note that if \( (x_1, -\lambda_1), \ldots, (x_n, -\lambda_n) \) is a virtual positive configuration, then so is \( (x_1, -\lambda_1 + \mu_1), \ldots, (x_n, -\lambda_n + \mu_n) \) for any coweights \( \mu_1, \ldots, \mu_n \). Whereas \( (x_1, -\lambda_1), \ldots, (x_n, -\lambda_n) \) came from a family of virtual positive configurations associated to \( F_1, \ldots, F_n \), the new configuration comes from a family of virtual positive configuration of flags \( F_1 \cdot \mu_1, \ldots, F_n \cdot \mu_n \). We define an action of \( \Lambda^n \) on families of virtual positive configurations in this way: it takes the family associated to \( F_1, \ldots, F_n \) to the one associated to \( F_1 \cdot \mu_1, \ldots, F_n \cdot \mu_n \), and in terms of virtual positive configurations can be thought of as taking each \( (x_1, -\lambda_1), \ldots, (x_n, -\lambda_n) \) in one family to \( (x_1, -\lambda_1 + \mu_1), \ldots, (x_n, -\lambda_n + \mu_n) \) in the other.

We will say that a virtual positive configuration \( (x_1, -\lambda_1), \ldots, (x_n, -\lambda_n) \) is realized by an actual configuration \( x_1', \ldots, x_n' \) if \( (x_1, -\lambda_1), \ldots, (x_n, -\lambda_n) \) and \( (x_1', 0), \ldots, (x_n', 0) \) are in the same family of virtual positive configurations.
Let us now make some observations. We have a positive configuration of \(n\) flags \(F_1, \ldots, F_n\), and let \((x_1, -\lambda_1), \ldots, (x_n, -\lambda_n)\) be any of the associated virtual configurations. By definition, the positive configuration \(x_1, \ldots, x_n\) is defined by good lifts, so that we know that

\[
-\text{val}(f_{ijk}(F_p \cdot \lambda_p, F_q \cdot \lambda_q, F_r \cdot \lambda_r)) = f_{ijk}^t(x_p, x_q, x_r)
\]

for all \(1 \leq p, q, r \leq n\) and \(0 \leq i, j, k \leq m - 1\) with \(i + j + k = m\). It is also clear from direct calculation that

\[
f_{ijk}(F_p \cdot \lambda_p, F_q \cdot \lambda_q, F_r \cdot \lambda_r) = f_{ijk}(F_p, F_q, F_r) \cdot t^{-\lambda_p \cdot \omega_i - \lambda_q \cdot \omega_j - \lambda_r \cdot \omega_k}
\]

where \(\omega_i, \omega_j, \omega_k\) are the fundamental weights, so that we can conclude that

\[
f_{ijk}^t(x_p, x_q, x_r) = -\text{val}(f_{ijk}(F_p, F_q, F_r)) + \lambda_p \cdot \omega_i + \lambda_q \cdot \omega_j + \lambda_r \cdot \omega_k.
\]

We extend the definition of \(f_{ijk}^t\) to virtual positive configurations of points in the affine Grassmanian as follows. We define

\[
f_{ijk}^t((x_p, \lambda_p), (x_q, \lambda_q), (x_r, \lambda_r)) = f_{ijk}^t(x_p, x_q, x_r) + \lambda_p \cdot \omega_i + \lambda_q \cdot \omega_j + \lambda_r \cdot \omega_k.
\]

Then if the positive configuration of flags \(F_i\) is associated with a family of virtual positive configuration of points which includes the virtual positive configuration \((x_i, \lambda_i)\), then we have

\[
f_{ijk}^t((x_p, \lambda_p), (x_q, \lambda_q), (x_r, \lambda_r)) = -\text{val}(f_{ijk}(F_p, F_q, F_r)).
\]

**5.5. Definition of laminations for the \(\mathcal{A}\) space of a disc with marked points.** We are now very close to getting a complete definition of higher laminations. We have defined the functions \(f_{ijk}^t\) on virtual positive configurations in the affine Grassmanian. Moreover, because these virtual positive configurations come from positive configurations of flags the functions we have:

**Proposition 5.10.** Suppose we have a virtual positive configuration of points in the affine Grassmanian \((x_i, \lambda_i)\). Then all the functions \(f_{ijk}^t((x_p, \lambda_p), (x_q, \lambda_q), (x_r, \lambda_r))\) for different \(i, j, k, p, q, r\) satisfy the tropical relations satisfied by tropical points of \(\mathcal{A}_{G,S}\).

**Proof.** The functions \(f_{ijk}(F_p, F_q, F_r)\) satisfy some relations defining \(\mathcal{A}_{G,S}\). These relations involve only addition, multiplication, and division. Moreover, because we have a positive configuration,

\[
f_{ijk}(F_p, F_q, F_r) \in \mathcal{K}_{>0}.
\]

Therefore the negative valuations of these functions must satisfy the tropicalizations of the corresponding relations. Because

\[
f_{ijk}^t((x_p, \lambda_p), (x_q, \lambda_q), (x_r, \lambda_r)) = \text{val}(f_{ijk}(F_p, F_q, F_r))
\]

the \(f_{ijk}^t((x_p, \lambda_p), (x_q, \lambda_q), (x_r, \lambda_r))\) satisfy these relations as well.

\( \square \)

We can thus associate to any tropical point of \(\mathcal{A}_{G,S}\) some virtual positive configuration of points in the affine Grassmanian. The problem is that there are many associated virtual positive configurations: not only do they come in families, but there are also many (an infinite dimensional space) of such families.

Let us now define virtual positive configurations of points in the affine building. Let \((x_1, -\lambda_1), \ldots, (x_n, -\lambda_n)\) be a virtual positive configurations in the affine Grassmanian, and suppose the points \(x_1, \ldots, x_n\) map to points \(p_1, \ldots, p_n\) in the affine building. In this situation, we will say that

\[
(p_1, -\lambda_1), \ldots, (p_n, -\lambda_n)
\]
is the corresponding virtual positive configuration of points in the affine building. The reason we
want to view these configurations inside the affine building rather than the affine Grassmanian is
that the affine building is where it is most natural to define equivalence between configurations
of points.

As before, equivalence of two configurations of points in the affine building was defined by
a recursive cutting procedure, and miniscule triangles were declared equivalent if they were
isometric. Let us now define an equivalence relation on virtual positive configurations of points
in the affine building.

Let \((p_i, \lambda_i)\) and \((q_i, \mu_i)\) be two virtual configurations. Suppose all the \(\lambda_i\) and \(\mu_i\) are positive
coweights then both \((p_i, \lambda_i)\) and \((q_i, \mu_i)\) come in a family containing the actual configurations
\((p'_i, 0)\) and \((q'_i, 0)\) We will say that the positive configuration \(p'_1, \ldots, p'_n\) realizes the virtual con-
figuration \((p_i, \lambda_i)\). Then we will say that \((p_i, \lambda_i)\) and \((q_i, \mu_i)\) are equivalent virtual configurations
if and only if \(p'_1, \ldots, p'_n\) and \(q'_1, \ldots, q'_n\) are equivalent as configurations.

More generally, two configurations \((p_i, \lambda_i)\) and \((q_i, \mu_i)\) are equivalent if there exists \(\nu_i\) such
that \((p_i, \lambda_i + \nu_i)\) and \((q_i, \mu_i + \nu_i)\) are equivalent. For large enough \(\nu_i\), \(\lambda_i + \nu_i\) and \(\mu_i + \nu_i\) will be
dominant, so that \((p_i, \lambda_i + \nu_i)\) and \((q_i, \mu_i + \nu_i)\) can be realized by some configurations \(p'_1, \ldots, p'_n\)
and \(q'_1, \ldots, q'_n\). In other words, if \((p_i, \lambda_i)\) and \((q_i, \mu_i)\) are equivalent, they are in the same families
as \((p'_i, -\nu_i)\) and \((q'_i, -\nu_i)\), respectively, where \(p'_1, \ldots, p'_n\) and \(q'_1, \ldots, q'_n\) are equivalent positive
configurations. From this definition it is clear that virtual positive configurations coming from
the same family are equivalent.

**Theorem 5.11.** There is a bijection between tropical points of \(\mathcal{A}_{G,S}(\mathbb{Z}^t)\) and virtual positive
configurations of points in the affine building up to equivalence. Given a virtual positive con-
figuration \((p_i, \lambda_i)\) of points in the affine building, it comes from a virtual positive configuration
of points \((x_i, \lambda_i)\) in the affine Grassmanian, which in turn comes from a positive configuration
of flags \(F_i \in G/U^-(K)\). Given a triangulation of the \(n\)-gon, we associate to each triangle
\((p_a, \lambda_a), (p_b, \lambda_b), (p_c, \lambda_c)\) the functions

\[
f^t_{ijk}((p_a, \lambda_a), (p_b, \lambda_b), (p_c, \lambda_c)) := f^t_{ijk}(x_a, \lambda_a, x_b, \lambda_b, x_c, \lambda_c) = \text{val}(f_{ijk}(F_a, F_b, F_c)).
\]

These functions satisfy the tropical relations, and therefore give a well-defined point of
\(\mathcal{A}_{G,S}(\mathbb{Z}^t)\). The values of these functions completely determine the virtual positive configu-
trations of points in the affine building up to equivalence, and vice versa.

**Proof.** Most of this theorem has already been proved. We need only to show that two virtual
positive configurations of points in the affine building have the same tropical coordinates \(f^t_{ijk}\)
if and only if they are equivalent.

Recall that any two virtual positive configurations that lie in the same family have the same
coordinates. Moreover, it is clear from the definitions that virtual positive configurations in the
same family are equivalent. Also note that the properties of equivalence and having the same
coordinates are stable under the action of \(\Lambda\): two virtual positive configurations of points in
the affine building \((p_i, \lambda_i)\) and \((q_i, \mu_i)\) have the same coordinates if and only if for any choice of
coweights \(\lambda_i\), \((p_i, \lambda_i + \nu_i)\) and \((q_i, \mu_i + \nu_i)\) have the same coordinates. Thus it suffices to work
with positive configurations \(p_i\) and \(q_i\) and show that they are equivalent if and only if they have
the same (tropical) coordinates.

Let us fix notation. Let \(p_i\) and \(q_i\) be two positive configurations of points in the affine
building. Let them come from positive configurations \(p_i\) and \(q_i\) in the affine Grassmanian
which in turn come from positive configurations \(F_i, F'_i\) of flags in \(G/U^-(K)\). We will need the
following lemma, which we shall use repeatedly:
Lemma 5.12. Given any positive configuration \( p_1, \ldots, p_n \) in the affine building, let \( y \) be a point in the geodesic between \( p_a \) and \( p_{a+1} \). Then \( p_1, \ldots, p_a, y, p_{a+1}, \ldots, p_n \) is a positive configuration of \( n + 1 \) points in the affine building.

Proof. We can actually do this on the level of flags. The idea is to construct a flag \( F \) such that \( F_1, \ldots, F_a, F, F_{a+1}, \ldots, F_n \) is a positive configuration of flags that maps down to \( p_1, \ldots, p_a, y, p_{a+1}, \ldots, p_n \) in the affine building.

Let us denote by

\[
 f_{i,m-i}(F_a, F_{a+1})
\]

the edge function corresponding to

\[
 \det(v_1, \ldots, v_i, w_1, \ldots, w_{m-i})
\]

where the sequence of vectors \( v_1, \ldots, v_m \) gives the flag \( F_a \) and the sequence of vectors \( w_1, \ldots, w_m \) gives the flag \( F_{a+1} \). Let \( f^i_{i,m-i} \) be the corresponding tropical function given by the negative valuation of \( f_{i,m-i} \). Recall that

\[
 f_{i,m-i}(p_a, p_{a+1}) = d(p_a, p_{a+1}) \cdot \omega_{m-i}.
\]

Now we want to construct a flag \( F \) that will map down to the point \( y \) in the affine building. Note that

\[
 d(p_a, y) + d(y, p_{a+1}) = d(p_a, p_{a+1}).
\]

Let

\[
 d_i = d(p_a, y) \cdot \omega_{m-i}.
\]

We will construct \( F \) by stipulating that

\[
 f_{i,j,0}(F_a, F, F_{a+1}) = t^{d_i},
\]

\[
 f_{i,j,k}(F_a, F, F_{a+1}) = f_{j,k}(F_a, F_{a+1})t^{-d_j},
\]

\[
 f_{i,j,k}(F_a, F, F_{a+1}) = f_{i+j,k}(F_a, F_{a+1})t^{-d_{i+j}+d_{i}}.
\]

One sees that by construction, \( F_a, F, F_{a+1} \) is a postive configuration in \( G/U^-(K) \). Moreover, it maps to an actual configuration of points in the affine Grassmanian, not merely a virtual one. Consider the triple of flags

\[
 F_a \cdot -d(y, p_a), F, F_{a+1} \cdot -d(y, p_{a+1}).
\]

If we show that this gives an actual configuration, then because \( d(y, p_a) \) and \( d(y, p_{a+1}) \) are positive coweights, \( F_a, F, F_{a+1} \) will also be an actual configuration. But one can calculate that the functions

\[
 f'_{i,j,k}(x_a, x, x_{a+1})
\]

all have valuation 0. This means that that we may choose the flags to be generated by vectors of valuation 0 (an explicit formula is given in section 9 of [FG1]). Let the \( \mathcal{O} \)-module spanned by these vectors give the corresponding points \( x_a, x, x_{a+1} \) in the affine Grassmanian. Then \( f'_{i,j,k}(x_a, x, x_{a+1}) \) can’t be any larger than 0, so

\[
 F_a \cdot -d(y, p_a), F, F_{a+1} \cdot -d(y, p_{a+1})
\]

is a good lift of \( x_a, x, x_{a+1} \). Thus \( x_a, x, x_{a+1} \), and correspondingly \( p_a, y, p_{a+1} \) is a positive configuration of points. By Lemma 5.8 we may glue to see that

\[
p_1, \ldots, p_a, y, p_{a+1}, \ldots, p_n
\]

is a positive configuration of points in the building.

\(\square\)
We now return to a proof the theorem. Suppose we have two positive configurations of points $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ that under some (and hence any) triangulation of an $n$-gon have the same coordinates. We wish to show that they are equivalent.

First observe that the distance between any two points $p_i$ and $p_j$ is the same as the distance between the corresponding points $q_i$ and $q_j$ by virtue of the coordinates of the configurations being the same. In particular, the distance between $p_i$ and $p_{i+1}$ is the same as the distance between $q_i$ and $q_{i+1}$. Then choosing any geodesic between $p_i$ and $p_{i+1}$, we can choose the corresponding geodesic between $q_i$ and $q_{i+1}$, and if we like, we may extend the configurations $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ by adding points along the geodesics. Assume we had done this to begin with, and that the result was $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$.

Now take any two points $p_i$ and $p_j$. Then take any geodesic between them, and take the corresponding geodesic between $q_i$ and $q_j$. Then $p_i, p_{i+1}, \ldots, p_j$ and $p_j, p_{j+1}, \ldots, p_i$ are both positive configurations of points. Moreover, their coordinates are completely determined by the configuration $p_1, p_2, \ldots, p_n$. Therefore, the corresponding configurations $q_i, q_{i+1}, \ldots, q_j$ and $q_j, q_{j+1}, \ldots, q_i$ are both positive and have the same respective coordinates.

Thus if two configurations have the same coordinates, taking corresponding cuts gives configurations that still have the same coordinates. Continuing this process, we get that $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ are equivalent, because the coordinates on miniscule triangles completely determine their side lengths.

Now let us show the converse. Suppose that $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ don’t have the same coordinates. If the pairwise distances between each $p_i$ and $p_j$ and the corresponding $q_i$ and $q_j$ aren’t the same, then clearly they can’t be equivalent. If they are the same, then let us make some cut of the configuration choosing any $p_i$ and $p_j$ and cutting to create two configurations $p_i, p_{i+1}, \ldots, p_j$ and $p_j, p_{j+1}, \ldots, p_i$. Then either $p_i, p_{i+1}, \ldots, p_j$ and $q_i, q_{i+1}, \ldots, q_j$ have different coordinates or $p_j, p_{j+1}, \ldots, p_i$ and $q_j, q_{j+1}, \ldots, q_i$ have different coordinates. This is because if both pairs had the same coordinates, then gluing together would give that the larger configurations had the same coordinates. Then with the smaller configurations, we may again add in points on the perimeter if necessary and make more cuts. Eventually, we will be left with non-equivalent miniscule triangles.

Finally, we are able to show that our notion of virtual configuration of points is well defined.

The next lemma is now quite simple:

**Lemma 5.13.** If $p_i$ and $q_i$ are equivalent positive configurations of $n$ points in the affine building, then for any $n$ dominant coweights $\lambda_1, \lambda_2, \ldots, \lambda_n$, if we allow the $\lambda_i$ to act on $p_i$ and $q_i$, the resulting positive configurations will still be equivalent.

**Proof.** The action of the $\lambda_i$ on the configurations $p_i$ and $q_i$ changes the co-ordinates in an explicit way (see the end of section 5.4). We have just shown that configurations are equivalent if and only if they have the same coordinates.

**Note 5.14.** It is an interesting question to determine when a virtual positive configuration it is equivalent to an actual configuration. The duality conjectures imply that actual configurations on an $n$-sided polygon parameterize invariants of $n$-fold tensor products of representations of $G$. We will discuss this in a future paper.

5.6. Laminations for the $A$-space of a general surface. Our goal now is to define laminations on a general surface $S$, possibly with marked points. The basic idea is that any surface can be glued together from triangles. Once we know laminations on a triangle and how to glue
together laminations along an edge, we know how to construct laminations on a general surface. There is one subtlety, which we will discuss below.

We have defined laminations on $S$, where $S$ is a disc with marked points. We will sometimes refer to such $S$ as a "polygon." Proposition $\ref{prop:lam}$ shows that coordinates on laminations on a polygon satisfy the tropical relations, while Theorem $\ref{th:lam}$ shows that these coordinates determine the lamination completely. Thus for $S$ a disc with marked points, we have an identification between tropical points of $A_{G,S}$ and $G$-laminations on $S$.

It is clear that our construction is compatible with cutting and gluing of polygons: laminations are completely determined by their coordinates, and the coordinates are constructed locally with respect to a triangulation, with coordinates associated to the edges and triangles of a triangulation. We now wish to extend this to surfaces. We will first analyze the case of polygons more closely.

Suppose we have two laminations

$$(p_1, \lambda_1), (p_2, \lambda_2), \ldots, (p_n, \lambda_n)$$

and

$$(q_1, \mu_1), (q_2, \mu_2), \ldots, (q_l, \mu_l)$$
on an $n$-gon and an $l$-gon. We may glue laminations on these polygons (in a unique way) by gluing the edge $(p_1, \lambda_1)$ to the edge $(q_1, \mu_1)$ if these two configurations of two points in the affine building are equivalent. If they are, by possibly increasing $\lambda_1$ and $\lambda_2$ so that they are larger than $\mu_2$ and $\mu_1$, we get a configuration of two points $(p_1', \lambda_1'), (p_2', \lambda_2')$ such that $(p_1', \lambda_1'), (p_2', \lambda_2'), (p_3, \lambda_3), \ldots, (p_n, \lambda_n)$
is equivalent to

$$(p_1, \lambda_1), (p_2, \lambda_2), \ldots, (p_n, \lambda_n).$$

Additionally, we may move the $(q_i, \mu_i)$, $i \neq 1, 2$, to obtain the configuration

$$(p_2', \lambda_2'), (p_1', \lambda_1), (q_3', \mu_3), \ldots, (q_l', \mu_l)$$

which is equivalent to

$$(q_1, \mu_1), (q_2, \mu_2), \ldots, (q_l, \mu_l).$$

Then the we will get a lamination

$$(p_2', \lambda_2'), (p_3, \lambda_3), \ldots, (p_n, \lambda_n), (p_1', \lambda_1'), (q_3', \mu_3), \ldots, (q_l', \mu_l)$$
on an $n + l - 2$-gon.

The result of this discussion is that if we are gluing triangles, we may have to replace one of the virtual points of our configuration, $(p, \lambda)$, with another point, $(p', \lambda')$, that gives an equivalent configuration. The reason is that if a vertex belongs to several different triangles, the value of $\lambda$ at this vertex in each triangle may be different, and we have to choose a $\lambda$ large enough for all the triangles containing this vertex simultaneously.

Lemma $\ref{lem:lambda}$ gave us a way to explicitly find such a $\lambda$: Start with a configuration of $n$ flags in $G(\mathcal{K})/U(\mathcal{K})$, and choose some triangulation of the $n$-gon. Then if a vertex belongs to $r$ triangles, the virtual point at the vertex in each of these triangles is $(p_i, \lambda_i)$, $i = 1, 2, \ldots, r$. By choosing $\lambda$ larger than all the $\lambda_i$, we may replace all these $(p_i, \lambda_i)$ by a single virtual point $(p, \lambda)$. This choice of $\lambda$ then works regardless of the triangulation. Thus we know that we just have to choose $\lambda$ larger than the value necessary in each of the triangles to which a vertex belongs. It turns out that in a general surface the same holds, though this is not obvious a priori.

For a general surface $S$ with marked points, we consider the cyclic set at $\infty$ formed by all the lifts of these marked points to the universal cover of $S$. This set $C$ is infinite, and comes with a free action of $\pi_1(S)$. Then we may define
**Definition 5.15.** A $G$-lamination on a surface $S$ with marked points is a virtual positive cyclic configuration of points in the affine building parameterized by the set $C$, equipped with an equivariant action of $\pi_1(S)$.

For more on the definition of this cyclic set, see the introduction of [FG1]. An equivariant action of $\pi_1(S)$ means that for any $\gamma \in \pi_1(S)$, pulling back the configuration by the map $\gamma$ gives an equivalent configuration.

Almost everything that holds for a finite virtual positive configuration carries over to the infinite case. We shall say that for an infinite set of points, two positive configurations parameterized by this cyclic infinite set are equivalent if for every finite subset, the configurations are equivalent.

One might worry that because we have an infinite number of points, there is not suitable choice of a large enough $\lambda$ at a given vertex. But if we start with a triangulation of a surface and lift this to the universal cover, although there are an infinite number of triangles at each vertex, there are only a finite number of values $\lambda_i$ that we need to choose $\lambda$ to be larger than. Thus we have

**Theorem 5.16.** Let $S$ be a (hyperbolic) surface with marked points, and let $C$ be its cyclic set at $\infty$. Associated to any tropical point of $\mathcal{A}_{G,S}(\mathbb{Z})$ there is a $\pi_1(S)$-equivariant virtual positive configuration of points in the affine building of $G$ parameterized by $C$. This configuration is unique up to equivalence.

The tropical coordinates on this lamination come from a triangulation of the surface $S$. We lift this triangulation to a triangulation of the disc with the cyclic set $C$ at the boundary. On each triangle, we have a virtual positive configuration $(p_i, \lambda_i)$ of points in the affine building, which comes from a virtual positive configuration of points $(x_i, \lambda_i)$. We shall say that for an infinite set of points, two positive configurations parameterized by this cyclic infinite set are equivalent if for every finite subset, the configurations are equivalent.

Let $S$ be a (hyperbolic) surface with marked points, and let $C$ be its cyclic set at $\infty$. Associated to any tropical point of $\mathcal{A}_{G,S}(\mathbb{Z})$ there is a $\pi_1(S)$-equivariant virtual positive configuration of points in the affine building of $G$ parameterized by $C$. This configuration is unique up to equivalence.

The tropical coordinates on this lamination come from a triangulation of the surface $S$. We lift this triangulation to a triangulation of the disc with the cyclic set $C$ at the boundary. On each triangle, we have a virtual positive configuration $(p_i, \lambda_i)$ of points in the affine building, which comes from a virtual positive configuration of points $(x_i, \lambda_i)$. We shall say that for an infinite set of points, two positive configurations parameterized by this cyclic infinite set are equivalent if for every finite subset, the configurations are equivalent.

Laminations for the $\mathcal{X}$ space

6.1. **Laminations for the $\mathcal{X}$-space of a disc.** We will now treat the dual case of laminations for the $\mathcal{X}$ space. For the $\mathcal{X}$ space, we work with the group $G = PGL_n$. Let $S$ be a polygon. Recall that any positive configuration $p_1, p_2, \ldots, p_n$ of points in the affine building comes in a family of equivalent virtual configurations $(q_1, \lambda_1), (q_2, \lambda_2), \ldots, (q_n, \lambda_n)$, where the $q_i$ vary with the $\lambda_i$, and $(p_1, 0), (p_2, 0), \ldots, (p_n, 0)$ are among these equivalent virtual configurations. We will now define a similar concept.

Start with a positive configuration of $n$ (principal) flags $F_1, \ldots, F_n$, $F_i \in GL_n/U(K)$. Note that here we no longer require that the flags have determinant 1. As before, we can choose $v_{i1}, \ldots, v_{im}$ for some lifts of these flags to $GL_n(K)$, then choose $\lambda_i$, such that

$$t^{-\lambda_{i1}} v_{i1}, \ldots, t^{-\lambda_{im}} v_{im}$$

is a good lift of $F_i \cdot \lambda_i$. Here the $\lambda_i$ are coweights for $GL_n$.

Let $x_i$ be the $O$-module spanned by $t^{-\lambda_{i1}} v_{i1}, \ldots, t^{\lambda_{im}} v_{im}$. We view $x_i$ inside the affine Grassmanian for $PGL_n$ by considering this $O$-module up to scaling by any element of $K$. We also only will care about the $\lambda_i$ up to its image in the coweight lattice for $PGL_n$ (as opposed to $GL_n$).
Then we can associate to $F_1, \ldots, F_n$ the virtual positive configuration $(x_1, -\lambda_1), \ldots, (x_n, -\lambda_n)$, where the $x_i$ form a positive configuration inside the affine Grassmanian for $PGL_n$, and we retain the notation $\lambda_i$ for the image in the coweight lattice of $PGL_n$. We can then map to the affine building to get a virtual positive configuration

$$(q_1, \lambda_1), (q_2, \lambda_2), \ldots, (q_n, \lambda_n).$$

Different choices of $\lambda_i$ give different configurations $q_i$, so we get a family of virtual positive configurations $(q_1, \lambda_1), \ldots, (q_n, \lambda_n)$ as the $\lambda_i$ vary. We can further forget the data of the $\lambda_i$ and just get a family of configurations $q_1, \ldots, q_n$.

**Definition 6.1.** A family of positive configurations in the affine building is a family arising from the above construction.

Whereas before we were interested in virtual positive configurations, we now are interested in these families. Note that we lose some degrees of freedom in passing from the former to the latter because we forget the data of the $\lambda_i$. It is as if we were considering equivalence classes of virtual positive configurations modulo the action of $\Lambda$. These families can be thought of as a configuration $p_1, \ldots, p_n$ with the data of a cone inside the building attached to each $p_i$. We can take as representatives of this family any $p'_1, \ldots, p'_n$, where $p'_i$ is in the cone corresponding to $p_i$.

We will consider these families of positive configurations up to equivalence. We define two families of configurations to be equivalent if they contain some configurations $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ which are equivalent in the sense we defined previously.

We now recall the functions on the $X$ space, and also describe their tropicalization. The $X$ space describes configurations of points in $G/B$. The functions on the $X$ space come from a triangulation, and there are functions attached to each face and each interior (non-boundary) edge of the triangulation. For each face of a triangulation of the $n$-gon, we have a set of functions $g_{ijk}$, where $i + j + k = m$ and $i, j, k > 0$. Suppose that some triangle has flags $B_1, B_2, B_3$ at its vertices, where $B_i \subset G/B$. Then

$$g_{ijk}(B_1, B_2, B_3) = \frac{f_{i-1,j+1,k}(F_1, F_2, F_3) \cdot f_{i,j-1,k+1}(F_1, F_2, F_3) \cdot f_{i+1,j,k-1}(F_1, F_2, F_3)}{f_{i,j,k+1}(F_1, F_2, F_3) \cdot f_{i,j+1,k-1}(F_1, F_2, F_3) \cdot f_{i,j-1,k+1}(F_1, F_2, F_3)}$$

where $F_1, F_2, F_3$ are any lifts of $B_1, B_2, B_3$ from $G/B$ to $GL_n/U^-$. Moreover, for any edge of the triangulation, we can look at the two triangles it belongs to, and construct a set of functions $g_{ij}$, for $i + j = m$ and $i, j > 0$. If the four flags $B_1, B_2, B_3, B_4$ form two triangles $B_1, B_2, B_3$ and $B_3, B_4, B_1$ which share the edge $B_1, B_3$, we have the functions

$$g_{ij}(B_1, B_2, B_3, B_4) = \frac{f_{i-1,j-1}(F_1, F_2, F_3) \cdot f_{j-1,i-1}(F_3, F_4, F_1)}{f_{i,j-1}(F_1, F_2, F_3) \cdot f_{j,i-1}(F_3, F_4, F_1)}$$

where $F_1, F_2, F_3, F_4$ are any lifts of $B_1, B_2, B_3, B_4$ from $G/B$ to $GL_n/U^-$. It is shown in [CG1] that these functions are independent of the choice of these lifts for both the face and edge functions. We will also define the tropicalization of these functions. For a family of positive configurations, choose some representative configuration $p_1, p_2, \ldots, p_n$. Then if a triangle has $p_a, p_b, p_c$ at its vertices, where the $p_i$ are points of the building, then we define

$$g_{ijk}^t(p_a, p_b, p_c) := g_{ijk}(x_a, x_b, x_c) = -\text{val}(g_{ijk}(F_a, F_b, F_c))$$

for $i + j + k = m$ and $i, j, k > 0$. Here, the configuration of points in the building $p_a, p_b, p_c$ comes from the configuration of points $x_a, x_b, x_c$ in the affine Grassmannian of $PGL_n$ and the
configuration of flags $F_a, F_b, F_c$ respectively. Similarly, for a positive configuration of four points in the building $p_a, p_b, p_c, p_d$, we define

$$g_{ij}^t(p_a, p_b, p_c, p_d) := g_{ij}^t(x_a, x_b, x_c, x_d) = -\val(g_{ij}(F_a, F_b, F_c, F_d)).$$

for $i + j = m$ and $i, j > 0$. We will show in the proof of the next theorem that the choice of configuration $p_i$ within the family of configurations does not affect the values of these functions. The following theorem is similar to the $A$ case:

**Theorem 6.2.** Let $S$ be a surface with $n$ marked points. There is a bijection between tropical points of $X_{G, S}(Z^t)$ and families of positive configurations of points in the affine building up to equivalence. Given a family of positive configurations, take one configuration $p_1, p_2, \ldots, p_n$ of points in the affine building. It comes from a virtual positive configuration of points $(x_i, \lambda_i)$ in the affine Grassmanian, which in turn comes from a positive configuration of flags $F_i \in GL_n/U^-(K)$. Given a triangulation of the $n$-gon, we associate to each triangle $p_a, p_b, p_c$ the functions $g_{ijk}(p_a, p_b, p_c)$, and to each edge $p_a, p_c$ bordering two triangles $p_a, p_b, p_c$ and $p_b, p_c, p_d$, we associate the functions $g_{ij}^t(p_a, p_b, p_c, p_d)$.

These functions satisfy the tropical relations, and therefore give a well-defined point of $X_{G, S}(Z^t)$. The values of these functions completely determine the family of positive configurations of points in the affine building up to equivalence, and vice versa.

**Proof.** First we must show that the functions $g_{ij}^t$ and $g_{ij}^t$ do not depend on the choice of configuration $p_1, p_2, \ldots, p_n$ in a family of positive configurations. However this is not difficult. The functions are in the end defined in terms of the configurations of flags that they come from. The different possible configurations of points in the affine building (or affine Grassmanian) are related to each other by the action of $\Lambda^n$. In terms of flags, the action of $\Lambda^n$ comes via particular elements of $T(K)$. However, as we mentioned above, the right action of the torus $T(K)$ on $GL_n/U^-(K)$ does not affect the values of these functions. This is evident by construction; more explanation can be found in section 9.3 of [FG1]. Thus the action of $\Lambda^n$ does not change the values of the functions $g_{ijk}$ and $g_{ij}$ on the flags $F_i$ or the corresponding tropical functions $g_{ijk}^t$ and $g_{ij}^t$ on the points of the building $p_i$.

Now we construct a map from $X_{G, S}(K_{>0})$ to the space of families of positive configurations of $n$ points in the affine building for $G$. We can choose some triangulation, and then choose arbitrarily the values of $g_{ijk}$ and $g_{ij}$ for each triangle and edge. We choose the values of these functions to lie in $K_{>0}$, such that they have prescribed valuations. This is always possible. This gives a positive configuration of points in $G/B(K)$. The results of [FG1] show that one can lift this to a positive (twisted) configuration of points in $G/U^-(K)$ (this is not explicitly stated, but comes from examining equation 5.2 on page 73, theorem 7.3 on page 96, and equation 8.9 on page 119). On this configuration of flags, the functions $f_{ijk}$ are defined, and they give the functions $g_{ijk}$ and $g_{ij}$ as described above.

Then we can map to a family of configurations and choose one representative configuration $p_1, \ldots, p_n$. The values of $g_{ijk}^t$ and $g_{ij}^t$ will be negative the valuations of the values of $g_{ijk}$ and $g_{ij}$ by construction; choosing some other representative configuration is the same as choosing a virtual positive configuration $(p_1, \lambda_1), (p_2, \lambda_2), \ldots, (p_n, \lambda_n)$. On this virtual configuration, the functions $f_{ijk}^t$ are defined, and then

$$g_{ijk}^t(p_a, p_b, p_c) = (f_{i-1, j+1, k}^t + f_{i, j-1, k+1}^t + f_{i+1, j-1, k+1}^t - f_{i+1, j-1, k}^t - f_{i, j+1, k+1}^t - f_{i-1, j, k+1}^t) (p_a, p_b, p_c)$$

and

$$g_{ij}(p_a, p_b, p_c, p_d) = f_{i-1, 1, j}^t (p_a, p_b, p_c) + f_{j-1, 1, i}^t (p_c, p_d, p_a) - f_{i, 1, j}^t (p_a, p_b, p_c) - f_{j, 1, i}^t (p_c, p_d, p_a).$$
One easily checks that the resulting functions are independent of the $\lambda_i$, and therefore they really are functions of the family of positive configurations containing $p_1, \ldots, p_n$, and not just of the configuration itself.

Thus we have a surjective map from $X_{G,S}(K > 0)$ to families of positive configurations of points in the affine building. As explained in section 2, the co-ordinate transformations under changes of triangulation behave tropically, and we get a well-defined point of $X_{G,S}(\mathbb{Z}^t)$.

Finally, we need to show that different points of $X_{G,S}(K > 0)$ which have the same valuations give equivalent families of configurations of points. In other words, we need to show the values of $g^t_{ijk}$ and $g^t_{ij}$ completely determine a family of configurations. Recall that the functions $f^t_{ijk}$ completely determined a virtual configuration of points in $F_1$ (and therefore the functions $f_{ijk}$) up to the action of $T(K)$. Therefore the tropicalizations $g^t_{ijk}$ and $g^t_{ij}$ completely determine the functions $f^t_{ijk}$—and therefore the (possibly virtual) configuration of points $(p_i, \lambda_i)$—up to the action of $\Lambda^n$. Thus they completely determine the family of configurations of points.

We now discuss a bit the idea behind $X$-laminations. In our presentation, the definition of the functions $g_{ijk}$ and $g_{ij}$, tropical or not, depended on some choice, of either a configuration within a family or a principal flag (an element of $G/U$) dominating a regular flag (an element of $G/B$). Note that the choice of a configuration or dominating flag is analogous to a choice of horocycles at the marked points. We construct the functions using these horocycles, show that they are independent of the choice of horocycles, and then forget the horocycles. This means that we lose some degrees of freedom in going from $A_{G,S}$ to $X_{G,S}$. For example, on a triangle, the space $X_{G,S}$ doesn’t have co-ordinates corresponding to the boundary edges. On the other hand, the gluings between triangles are more interesting: for the space $A_{G,S}$ we can only glue two triangles if they have the same edge functions, whereas for the space $X_{G,S}$ any two triangles have an interesting space of gluings based on the co-ordinates assigned to the edges. The different possible gluings are related to each other by shearing. This is one important feature that distinguishes the spaces $A_{G,S}$ and $X_{G,S}$. This will become even more important in the next section.

6.2. Laminations for the $\mathcal{X}$-space of a general surface. We now define $\mathcal{X}$-laminations on a general open surface $S$, possibly with marked points. Again, because any open surface can be glued together from triangles, knowing laminations on a triangle and how they can glue together to get laminations on a quadrilateral, will allow us to construct laminations on a general surface.

We have defined laminations on $S$, where $S$ is a “polygon.” The previous section gave an identification between tropical points of $X_{G,S}$ and $G$-laminations on $S$.

It is clear that our construction is compatible with cutting and gluing of polygons. In the $A$ case, we took some care to show that in gluing, we didn’t have to increase the values of $\lambda_i$ indefinitely. However, because here we are interested in families of positive configurations, we do not have this concern.

For a general surface $S$ with marked points, recall the cyclic set at $\infty$ formed by all the lifts of these marked points to the universal cover of $S$. This set $C$ is infinite, and comes with a free action of $\pi_1(S)$. Then we may define

**Definition 6.3.** A $G$-lamination for the $\mathcal{X}$-space of a surface $S$ with marked points is the data of a family of positive cyclic configurations of points in the affine building parameterized by every finite set of the set $C$, compatible under restriction from one finite set to another, and equipped with an action of $\pi_1(S)$ on these families of configurations.
We should understand the action of $\pi_1(S)$ as follows. If some element $\gamma$ of the fundamental group moves the set finite subset $S \subset C$ to the finite subset $S' \subset C$, then family of configurations on $S$ is equivalent to the family of configurations on $S'$.

Almost everything that holds for families of finite positive configurations carries over to the infinite case. We shall say that two $G$ laminations for the $X$ space are equivalent if for any finite subset of $C$, the two families of positive configurations parameterized by this finite set are equivalent.

**Theorem 6.4.** Let $S$ be a (hyperbolic) surface with marked points, and let $C$ be its cyclic set at $\infty$. Associated to any tropical point of $X_G,S(\mathbb{Z})$ there is a $G$-lamination as defined above. This $G$-lamination is unique up to equivalence.

The tropical coordinates on this lamination come from a triangulation of the surface $S$. We lift this triangulation to a triangulation of the disc with the cyclic set $C$ at the boundary. On each triangle or quadrilateral, we have a family of positive configurations among which we can take a representative positive configuration $p_i$ of points in the affine building, which comes from a positive configuration of points $x_i$ in the affine Grassmanian, which in turn comes from a positive configuration of flags $F_i \in GL_n/U^-(K)$. The tropical functions are

$$g^t_{ijk}(p_a,p_b,p_c) := g^t_{ijk}(x_a,x_b,x_c) = -\text{val}(g_{ijk}(F_a,F_b,F_c))$$

and

$$g^d_{ij}(p_a,p_b,p_c,p_d) := g^d_{ij}(x_a,x_b,x_c,x_d) = -\text{val}(g_{ij}(F_a,F_b,F_c,F_d))$$

These functions satisfy the tropical relations, and therefore completely determine lamination.

We give a word of caution. We cannot talk about the family of positive configurations of points parameterized by the entire set $C$. This is because of the way gluing works. Suppose we had a family of positive configurations of four points. Let $p_1,p_2,p_3,p_4$ be one configuration in this family. Then it restricts on the triangle 123 to the family containing the configuration $p_1,p_2,p_3$. However, not every configuration in the family containing $p_1,p_2,p_3$ comes from a configuration in the family containing $p_1,p_2,p_3,p_4$. Thus in gluing together configurations, we may need to replace a configuration by another, larger configuration in the same family in order to glue.

Thus the problem that we avoided for the $A$-space turns out to be important here: as we take larger and larger subsets of $C$, the actual configurations in our family of positive configurations may get larger and larger, so that in the limiting case we don’t actually have a configuration. Thus we can only choose from a family of positive configurations a representative one for any finite subset of $C$.

Moreover, the more robust gluing allowed in the case of laminations on the $X$ space allows us to have interesting monodromy around a hole in our surface. This is a reflection of the fact that edge co-ordinates allow for a kind of “shearing.” Let $x$ be a point of $C$, the boundary at infinity. Then if $\gamma \in \pi_1(S)$ preserves $x$, it may not preserve the points of the affine building attached to the point $x$; it may move $p_x$ to another point $p'_x$, where $p_x$ and $p'_x$ are related by the action of $A$ at the point $x$.

**6.3. Laminations on a closed surface.** Instead of using the cyclic set $F_\infty$ at infinity formed by all the cusps and marked points of our surface $S$, we could have instead used the larger cyclic set $G_\infty = F_\infty \cup G'_\infty$, where $G'_\infty$ consists of preimages on the boundary at infinity of all endpoints of non-boundary geodesics on $S$. We can then define laminations for a closed surface $S$. 
Definition 6.5. A $G$-lamination for a possibly closed surface $S$ is the data of a family of positive cyclic configurations of points in the affine building parameterized by every finite set of the set $\mathcal{G}_\infty$, compatible under restriction from one finite set to another, and equipped with an action of $\pi_1(S)$ on these families of configurations. These families of positive cyclic configurations are considered up to equivalence.

This definition is equivalent to our old one for surfaces with boundary. To understand this better in the case of closed surfaces, we use the cutting and gluing properties of higher Teichmüller spaces. We follow here the treatment given in sections 6.9, 7.6-7.9 of [FG1] and give tropical versions of those arguments.

Let $S$ be a surface with or without boundary, and suppose we have a point $x$ of $\mathcal{X}_{G,S}(K>0)$. Then along any any closed (oriented) curve $\gamma$ on $S$ we may calculate the monodromy $\rho(\gamma)$ of the local system around that closed curve.

By the results of [FG1], we know that $\rho(\gamma)$ will lie in $G(K>0)$. A result of Lusztig ([Lu2, Theorem 5.6]) gives us that

Theorem 6.6. Let $g \in G(K>0)$. Then there exists a unique split maximal torus of $G$ containing $g$. In particular, $g$ is regular and semi-simple.

We can conjugate $\rho(\gamma)$ to $H(K>0)$, and then take valuations to get an element of the coweight lattice $\Lambda$. However the conjugation of $\rho(\gamma)$ into $H(K>0)$ is only well-defined up an action of the Weyl group $S_n$. Thus we actually get a well-defined element of the dominant coweights $\Lambda^+$ which we will denote $d(\gamma)$. Let $l$ be the lamination corresponding to $x$. Then $d(\gamma)$ should be viewed as the “length” of the lamination associated to our along the loop $\gamma$. This analogy will be explored further in the next section.

We can state a version of theorem 7.6 of [FG1] for the semi-field $K>0$. Let $S$ be a surface, with or without boundary, with $\chi(S) < 0$. Let $\gamma$ be a non-trivial loop on $S$, not homotopic to a boundary component of $S$. Denote by $S'$ the surface obtained by cutting $S$ along $\gamma$. We assume that $S'$ is connected. It has two boundary components, $\gamma_+$ and $\gamma_-$, whose orientations are induced by the one of $S'$. The surface $S'$ has one or two components, each of them of negative Euler characteristic. Denote by

$$\mathcal{X}_{G,S'}(\gamma_+, \gamma_-)(K>0)$$

the subspace of $\mathcal{X}_{G,S}(K>0)$ given by the following condition: the monodromies along $\gamma_+$ and $\gamma_-$ are inverse.

Theorem 6.7. Let $S$ be a surface with $\chi(S) < 0$, and $S'$ is obtained by cutting along a loop $\gamma$, as above. Then the restriction from $S$ to $S'$ provides us a principal $H(K>0)$-bundle

$$\mathcal{X}_{G,S}(K>0) \longrightarrow \mathcal{X}_{G,S'}(\gamma_+, \gamma_-)(K>0).$$

As a set, the space $\mathcal{X}_{G,S}(K>0)$ is simply $K_{>0}^{-\chi(S)\dim G}$.

Thus we have

1. We may restrict $G(K>0)$-bundles on $S$ to obtain $G(K>0)$-bundles on $S'$.
2. The image of the restriction map consists of $G(K>0)$-bundles on $S'$ satisfying the constraint that the monodromies around the oriented loops $\gamma_+$ and $\gamma_-$ are opposite.
3. Given a $G(K>0)$-bundle on $S'$ satisfying this monodromy constraint on the boundaries, one can glue it to a $G(K>0)$-bundle on $S'$, and the group $H(K>0)$ acts simply transitively on the set of the gluings.

The proof of the above theorem relies on an analysis of configurations of flags parameterized by the cyclic set at infinity. In particular, it relies on analysis of the relationship between the
cyclic sets at infinity of $S$ and $S'$. The proof is no different in the $\mathbb{R}_{>0}$ and $K_{>0}$ cases. For $S$ a closed or an open surface, the tropical points of $\mathcal{X}_{G,S}(\mathbb{Z}^t)$ arise as equivalence classes of points of $X_{G,S}(K_{>0})$. Points of $X_{G,S}(K_{>0})$ correspond to configurations of points in $G/B(K)$ parameterized by $\mathcal{G}_\infty$ and equipped with an action of $\pi_1(S)$. Points of $X_{G,S}(\mathbb{Z}^t)$ consist of equivalence classes of families of configurations of points in the affine building for $G$ parameterized by (finite subsets) $\mathcal{G}_\infty$ and equipped with an action of $\pi_1(S)$.

We can now state the tropical version of theorem 7.6 of [FG1]. Let $S$ be a surface, with or without boundary, with $\chi(S) < 0$. Let $\gamma$ be a non-trivial loop on $S$, not omotopic to a boundary component of $S$. Denote by $S'$ the surface obtained by cutting $S$ along $\gamma$. We assume that $S'$ is connected. It has two boundary components, $\gamma_+$ and $\gamma_-$, whose orientations are induced by the one of $S'$. We obtain a lamination on $S'$, induced by the inclusion of cyclic sets

$$\mathcal{G}_\infty(S') \subset \mathcal{G}_\infty(S).$$

The surface $S'$ has one or two components, each of them of negative Euler characteristic. Denote by

$$\mathcal{X}_{G,S}^+(\gamma_+,\gamma_-)(\mathbb{Z}^t)$$

the subspace of $\mathcal{X}_{G,S}^+(\mathbb{Z}^t)$ given by the following condition: The lengths of the lamination along $\gamma_+$ and $\gamma_-$ are inverse:

$$d(\gamma_+) = -w_0d(\gamma_-).$$

Because the (semi-simple part of the) monodromy around a boundary component is given by a monomial map, this subspace of laminations will be a linear subspace of the space of laminations $\mathcal{X}_{G,S}^+(\mathbb{Z}^t)$ of codimension $\dim H$.

**Theorem 6.8.** Let $S$ be a surface with $\chi(S) < 0$, and $S'$ is obtained by cutting along a loop $\gamma$, as above. Then the restriction from $S$ to $S'$ provides us a principal $H(\mathbb{Z}^t)$-bundle

$$\mathcal{X}_{G,S}(\mathbb{Z}^t) \twoheadrightarrow \mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{Z}^t).$$

As a set, the space $\mathcal{X}_{G,S}(\mathbb{Z}^t)$ is is simply $\mathbb{Z}^{-\chi(S)\dim G}$.

Thus we have

1. We may restrict laminations on $S$ to obtain laminations on $S'$.
2. The image of the restriction map consists of laminations $S'$ satisfying the constraint that the lengths of the laminations around the oriented loops $\gamma_+$ and $\gamma_-$ are opposite.
3. Given a lamination on $S'$ satisfying this length constraint on the boundaries, one can glue it to a lamination on $S'$, and the group $H(\mathbb{Z}^t)$ acts simply transitively on the set of the gluings.

The spaces $\mathcal{X}_{G,S}(\mathbb{R}_{>0})$ and $\mathcal{X}_{G,S}(\mathbb{Z}^t)$ are naturally only $H(\mathbb{R}_{>0})$- and $H(\mathbb{Z}^t)$-bundles over $\mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{R}_{>0})$ and $\mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{Z}^t)$, respectively. It is convenient to (non-canonically) trivialize these bundles.

Choose a hyperbolic structure on $S$ so that it’s universal is identified with the hyperbolic plane $\mathcal{H}$ and its cyclic subset at infinity is a subset of $\partial \mathcal{H}$. Let $c_1$ and $c_2$ be the two endpoints of $\gamma$, the lift of the geodesic $\gamma$ to the universal cover of $S$. Copies of the universal cover of $S$ can naturally be glued together in way respecting the hyperbolic structure to form the universal cover of $S$, see [FG1]. The result is that on one side of $\gamma$ we have a copy of the universal cover of $S'$ with boundary $\gamma_-$ and on the other we have a copy of the universal cover of $S'$ with boundary $\gamma_+$. Then we may choose two points $c_-$ and $c_+$ in the boundary at infinity of each of these respective copies of the universal cover of $S'$, so that $c_1$, $c_+$, $c_2$ and $c_-$ occur in cyclic order. Then the edge coordinates (respectively the tropicalization of these coordinates)
on the edge $c_1c_2$ of the quadrilateral $c_1c_2c_1c_2$ give a trivialization of the $H(\mathbb{R}_{>0})$- (respectively $H(\mathbb{Z}'_1)$-)torsors.

Alternatively, we may choose some geodesic $\gamma'$ that intersects $\gamma$ transversely (for example, we may take the geodesic used to trivialize the torso of twist parameters for the Fenchel-Nielsen coordinates corresponding to a pants decomposition). Then we may take as a basepoint for the $H(\mathbb{R}_{>0})$-torso the $G(\mathbb{R})$-bundle with the smallest monodromy around $\gamma'$. By smallest, we mean that the coefficients of the characteristic polynomial of $\rho(\gamma')$ are simultaneously minimized. Such a point exists and is unique because these functions are convex on $H(\mathbb{R}_{>0})$ (in fact, they are positive Laurent polynomials).

Analogously, we take as basepoint for the $H(\mathbb{Z}'_1)$-torso the unique $G$-lamination for which $d(\gamma') \in \Lambda^+$ is minimized.

Choosing a basepoint for the $H(\mathbb{R}_{>0})$- (respectively $H(\mathbb{Z}'_1)$-)torsors in one of the two ways above, we then have a product decompositions:

$$\mathcal{X}_{G,S}(\mathbb{R}_{>0}) = \mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{R}_{>0}) \times H(\mathbb{R}_{>0})$$

$$\mathcal{X}_{G,S}(\mathbb{Z}'_1) = \mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{Z}'_1) \times H(\mathbb{Z}'_1)$$

The functions which parameterize the factor $H(\mathbb{Z}'_1)$ are a tropicalization of the functions that parameterize the factor $H(\mathbb{R}_{>0})$. Finally, we have a theorem analogous to those in previous sections:

**Theorem 6.9.** Let $S$ be a closed (hyperbolic) surface, and let $C$ be its cyclic set at $\infty$. Associated to any tropical point of $\mathcal{X}_{G,S}(\mathbb{Z}'_1)$ (= $L_{G,S}(\mathbb{Z}'_1)$) there is a $G$-lamination as defined above. This $G$-lamination is unique up to equivalence.

Choosing a curve $\gamma$ and a trivialization of the $H(\mathbb{Z}'_1)$-torso, we obtain tropical coordinates on this lamination coming from the tropical coordinates on the factors $\mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{Z}'_1)$ and $H(\mathbb{Z}'_1)$. These tropical functions completely determine the lamination.

In fact, for any triangle or quadrilateral in the cyclic set $G_\infty$ we obtain a set of tropical functions on $\mathcal{X}_{G,S}(\mathbb{Z}'_1)$. When these functions are related by flips, they satisfy the usual the tropical relations.

On the other hand, while the different coordinate systems on the factors $\mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{R}_{>0})$ and $H(\mathbb{R}_{>0})$ (respectively $\mathcal{X}_{G,S}(\gamma_+,\gamma_-)(\mathbb{Z}'_1)$ and $H(\mathbb{Z}'_1)$) are related by positive (respectively tropical) transition functions, transformations between coordinate systems for different choices of $\gamma$ are more complicated, and are not usually rational (respectively piecewise-linear). This is one advantage of having a coordinate-free description of laminations as equivalence classes of $G(\mathcal{K})$-bundles or as $\pi_1(S)$-equivariant families of configurations of points in the affine building.

7. **Comparison with other works**

One application of our definition of laminations is that projectived $G$-laminations give a spherical compactification of higher Teichmüller space. This will give a Thurston-type compactification of higher Teichmüller space. We will explain this below, and compare this compactification with those found in the works of Alessandrini, Parreau, and, in the case of $G = SL_2$, the work of Morgan and Shalen. We will need to understand length functions on higher Teichmüller spaces, and study their degenerations to the boundary.

We first review how to construct this compactification. Much of this was explained in [FG4]. For any positive space $\mathcal{X}$ (throughout this section, $\mathcal{X}$ will be either $A_{G,S}$ or $\mathcal{X}_{G,S}$; we will assume for simplicity, except where noted, that $S$ is a surface with boundary), we may form its tropicalization by taking the points of $\mathcal{X}$ with values in the semifields $\mathbb{A}^t$, for $\mathbb{A} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. 
In the later two cases, we have an action of $A^*$ (the multiplicative group of positive elements) on the tropical points $X(A^*)$. One can define this in each chart of the positive atlas and show that the action is compatible with changes of co-ordinate chart. Then we have:

**Definition 7.1.** Let $X$ be a positive space. Let $A$ be either $\mathbb{Q}$ or $\mathbb{R}$. The projectivisation $\mathbb{P}X(A^*)$ of the tropical $A$-points of $X$ is

$$\mathbb{P}X(A^*) := \left( X(A^*) - \{0\} \right)/A^*.$$

Observe that $\mathbb{P}X(\mathbb{R}^t)$ is a sphere, and the transition maps between co-ordinate charts on this sphere are tropical maps, and hence piecewise-linear maps. The set $\mathbb{P}X(\mathbb{Q}^t)$ is an everywhere dense subset of $\mathbb{P}X(\mathbb{R}^t)$.

This sphere lives at the boundary of higher Teichmüller space $X(\mathbb{R}_{>0})$, and gives us a logarithmic compactification as in [A], [P], [MS]. Let the dimension of $X$ be $d$. Then in any co-ordinate chart $H_\alpha$, taking logarithms of the co-ordinates gives an identification of $X(\mathbb{R}_{>0})$ with $\mathbb{R}^d$. Then the compactification we seek is simply the radial compactification of $\mathbb{R}^d$. Any point in the spherical boundary of $\mathbb{R}^d$ corresponds to some relative growth rates of the coordinates in the chart $H_\alpha$.

One of the main theorems of [FG1] tells us that transition functions between co-ordinate charts are given by positive rational functions (in fact, they are expected to be positive Laurent polynomials). Because of this, the growth rates in any chart $H_\alpha$ completely determine the growth rates in any other chart $H_\beta$. A point $(x_1, x_2, \ldots, x_d) \in H_\alpha(\mathbb{R}^t)$ of the boundary corresponds to the limit of the points

$$(e^{sx_1}, e^{sx_2}, \ldots, e^{sx_d})$$

as $s \to \infty$. Suppose we have another co-ordinate chart $H_\beta$ with $\phi_{\alpha\beta} : H_\alpha \to H_\beta$ the transition map between co-ordinate charts. Let

$$(y_1, y_2, \ldots, y_d) = \phi_{\alpha\beta}(x_1, x_2, \ldots, x_d)$$

be the tropicalization of $\phi_{\alpha\beta}$ applied to $(x_1, x_2, \ldots, x_d)$. Then

$$\phi_{\alpha\beta}(e^{sx_1}, e^{sx_2}, \ldots, e^{sx_d})$$

is asymptotic to

$$(e^{sy_1}, e^{sy_2}, \ldots, e^{sy_d})$$

as $s \to \infty$.

Thus the radial logarithmic compactifications in different co-ordinate charts transform tropically, and this compactification is naturally identified with $\mathbb{P}X(\mathbb{R}^t)$.

Now recall that we had a map from $X(K_{>0})$ to the space of laminations $X(\mathbb{Z}^t)$. Now let

$$x = (x_1, x_2, \ldots, x_d)$$

be a point of $X(K_{>0})$. Suppose that the $x_i$ are in fact convergent power series in $K_{>0} = \mathbb{R}((t))_{>0}$. Then for small enough $t$, the $x_i$ are positive when evaluated at $t$, and we may view $x$ as a path in $X(\mathbb{R}_{>0})$. The growth rate of $x_i$ as $t$ goes to 0 is $\text{val}(x_i)$. Thus given any lamination $l \in X(\mathbb{R}^t)$ which is non-zero, we can construct a path in higher Teichmüller space that approaches the projectivization of this lamination in the boundary. Laminations which are related by the action of $\mathbb{A}^*$ approach the same point on the boundary at different speeds. Laminations measure growth rates of paths in $X(\mathbb{R}_{>0})$, while projectivized laminations measure the relative growth rates of the co-ordinates. In summary, we have mapped out the relationship between valuations, growth rates and tropical points.

To summarize: projectivized $G$-laminations give a spherical boundary for higher Teichmüller space. Points in the boundary parameterize relative growth rates of paths in higher Teichmüller...
space that go to infinity. We note that like Thurston’s compactification of Teichmuller space, our compactification has a natural action of the (higher) mapping class group (the higher mapping class group is defined as the symmetries of the cluster algebra underlying the higher Teichmuller space). This turns out to be tautological from the definition of the higher mapping class group.

Note 7.2. In the case of $S$ closed, we form the compactification by taking logarithms of the natural functions on $X_{G,S}(\gamma_{+},\gamma_{-})(\mathbb{R}_{>0})$ and $H(\mathbb{R}_{>0})$ and then forming the radial compactification. This gives a spherical compactification of $X_{G,S}(\mathbb{R}_{>0})$ by $\mathbb{P}\mathcal{X}(\mathbb{R}^{1})$. For every point in the boundary, we can find a corresponding point of $X_{G,S}(K_{>0})$ that represents a path approaching this point in the boundary.

We now compare this compactification with the ones given by [A] and [P]. The construction outlined above, due mostly to Fock and Goncharov, works in the context of positive spaces. On the other hand, the constructions of Alessandrini and Parreau work in greater generality (for example, Parreau works in the context of representation varieties of finitely generated groups, while Alessandrini works in the context of compactifications of general algebraic varieties). We are for this reason able to avoid some technical arguments that they use. As we understand it, to a boundary point in their compactification, they associate some $\pi_{1}$ action on an affine building; however, the association is fairly non-constructive, it is one-to-many (each point in the boundary may be associated to many $\pi_{1}$ actions on different affine buildings), and it is difficult to pin down the invariant properties of the different possible answers.

Our contribution is to identify what kinds of configurations in the affine building can occur and define an equivalence relation on configurations coming from the same boundary point. Moreover, affine buildings are large and infinite objects; we are able to give finite invariant subsets of the building that completely capture the lamination. For example, for any triangulation of the surface, we can lift the triangulation to the universal cover. Attached to each triangle is a configuration of points in the affine building. Take the convex hull of these points. The union of these convex hulls over all the triangles in our triangulation gives a subset of the affine building which (in the case of $\mathcal{A}$-laminations) is finite up to the action of $\pi_{1}$. (One can do something similar in the case of $\mathcal{X}$-laminations.)

Finally, we will show that our Thurston-type compactification surjects onto the compactification in [P], although we conjecture that the compactifications are in fact the same. The compactification found in [A] and [P] is very similar to ours, except that instead of radially compactifying for the cluster co-ordinate systems, they use a different set of functions. For each path $\gamma$ on $S$, they consider the different coefficients of the characteristic polynomial of the monodromy around $\gamma$. Let $f$ be any such function.

Recall that for a point in higher Teichmuller space, the monodromy of $\gamma$ lies in $G(\mathbb{R}_{>0})$. One can easily check that the coefficients of the characteristic polynomial of a matrix $g \in G(\mathbb{R}_{>0})$ are given in terms of a positive expression in the generalized minors of this matrix. Hence, the function $f$ is given by positive rational functions of the cluster co-ordinates. Because the function $f$ is positive, it can be tropicalized to give the function $f^{t}$. This gives the c-length functions of [P]. Moreover, the expression of $f$ as a positive Laurent polynomial means that the growth rates of cluster co-ordinates in any chart completely determine the growth rates of $f$. From this, we get first that compactifying by growth rates of cluster co-ordinates is at least as refined as compactifying by coefficients of the characteristic polynomial around all loops in $S$.

References

[A] D. Alessandrini. Tropicalization of group representations, preprint on arXiv:math.GT/0703608v3.
[FG1] V.V. Fock, A.B. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. Publ. Math. IHES, n. 103 (2006) 1-212. math.AG/0311149.
[FG2] V.V. Fock, A.B. Goncharov. Cluster ensembles, quantization and the dilogarithm. Ann. Sci. Ecole Norm. Sup. vol 42, (2009) 865-929. math.AG/0311245.
[FG3] V.V. Fock, A.B. Goncharov. Dual Teichmüller and lamination spaces. Handbook of Teichmüller theory. Vol. I, 647684, IRMA Lect. Math. Theor. Phys., 11, Eur. Math. Soc., Zurich, 2007. math.DG/0510312.
[FG4] V.V. Fock, A.B. Goncharov. Cluster X-varieties at infinity. math.AG/1104.0407
[FKK] B. Fontaine, J. Kamnitzer, G. Kuperberg. Buildings, spiders, and geometric Satake. math.QA/1103.3519
[GL] S. Gaussent and P. Littelmann. LS galleries, the path model, and MV cycles, Duke Math. J. 127 (2005), no. 1, 3588. arXiv:math/0307122.
[GMN1] D. Gaitto, G. Moore, A. Neitzke. Spectral networks. arXiv:1204.4824
[GMN2] D. Gaitto, G. Moore, A. Neitzke. Spectral networks. arXiv:1006.0146.
[H] N.J. Hitchin. Lie groups and Teichmüller space., Topology 31, (1992), no. 3, 449473.
[K] J. Kamnitzer. Hives and the fibres of the convolution morphism, Selecta Math. N.S. 13 no. 3 (2007), 483-496.
[L] F. Labourie. Anosov Flows, Surface Groups and Curves in Projective Space. Inventiones Mathematicae 165 no. 1, 51–114 (2006).
[Lu] G. Lusztig. Total positivity and canonical bases, in “Algebraic groups and Lie groups” ed. G.I.Lehrer, Cambridge U.Press 1997, 281-295.
[Lu2] G. Lusztig. Total positivity in reductive groups, Lie theory and geometry, Progr. Math., 123, Birkhauser Boston, Boston, MA, (1994), 531-568.
[M] D. Maclagan and B. Sturmfels. Introduction to Tropical Geometry. [http://homepages.warwick.ac.uk/staff/D.J.Maclagan/papers/TropicalBook.pdf]
[MS] J. Morgan, P. Shalen. Valuations, Trees, and Degenerations of Hyperbolic Structures. The Annals of Mathematics, Second Series, Vol. 120, No. 3 (Nov., 1984), pp. 401-476.
[P] A. Parreau. Compactification d’espaces de représentations de groupes de type fini. Mathematische Zeitschrift (2011). DOI: 10.1007/s00209-011-0921-8.
[R] Mark Ronan. Lectures on buildings, University of Chicago Press, 2009, Updated and revised.

Department of Mathematics, Northwestern University, Evanston, IL 60208-2370
E-mail address: i-le@math.northwestern.edu