Hawking temperature for various kinds of black holes from Heisenberg uncertainty principle

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Abstract

Hawking temperature is computed for a large class of black holes (with spherical, toroidal and hyperboloidal topologies) using only laws of classical physics plus the "classical" Heisenberg Uncertainty Principle. This principle is shown to be fully sufficient to get the result, and there is no need to this scope of a Generalized Uncertainty Principle.

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1 Introduction

The Hawking temperature has been heuristically derived for various kinds of black holes by using several different versions of the uncertainty principle. For the Schwarzschild black hole, the "classical" Heisenberg principle alone seems to be sufficient to obtain the Hawking formula (see, for example, Ref. [1]), even if some small adjustments of the numerical constants are sometime used [2]. Recently, in Ref. [3], the Hawking temperature for the (Anti) de Sitter black hole has been computed with a generalized version of the uncertainty principle. In this derivation some numerical constants are chosen in a somehow arbitrary way, and a particular version of the generalized uncertainty principle (GUP), namely

$$\Delta x \Delta p \geq \hbar \left( 1 + \beta^2 \frac{\Delta x^2}{l_p^2} \right),$$

(1.1)

(where $\beta$ is an arbitrary constant and $l_p$ is the Planck length) is claimed to be necessary to get the exact form of the Hawking temperature in the (Anti) de Sitter case. From this claim, the authors speculate on some connection between the fact that the Anti de Sitter black hole thermodynamics admits a holographic description in terms of a dual conformal quantum field theory, and the fact that the GUP finds its natural collocation in string theory and in non commutative geometry.

The aim of the present paper is, on the contrary, to show that the local Hawking temperature of a quite large class of black holes (Schwarzschild, Reissner-Nordström, (Anti) de Sitter, with the various topologies, spherical, toroidal and hyperboloidal) can be obtained directly from the "classical" Heisenberg principle casted in the form $\Delta E \Delta x \simeq \hbar c/2$ and from the effective (newtonian)
potential approximating the metric, without need of any form of Generalized Uncertainty Principle. The present derivation does not contain any adjustable parameter and, although heuristic, gives us a final result in very good agreement, even numerical, with the exact formulae worked out from the quantum field theory on curved space-times. This result in turn can be seen as a further evidence of the purely kinematic nature of the Hawking effect \cite{5} and of its independence from the dynamical Einstein equations.

2 Effective potential from the metric

We consider here space-times with a metric that locally has the form

\[ ds^2 = -F(r)c^2 dt^2 + F(r)^{-1} dr^2 + r^2 d\Omega_k^2 = g_{\mu\nu} dx^\mu dx^\nu \]

(2.1)

where

\[ F(r) = k - \frac{2GM}{c^2 r} + \frac{GQ^2}{c^4 r^2} + \lambda r^2 \]

(2.2)

and the time-like coordinate is chosen as \( x^0 = ct \). The parameters \( M \) (mass), \( Q \) (electric charge), \( \lambda \) (cosmological constant, up to a factor) are real and continuous. \( \lambda < 0 \) corresponds to a de Sitter space-time, \( \lambda > 0 \) to an Anti de Sitter space-time. The discrete parameter \( k \) takes the values 1, 0, \(-1\) and \( d\Omega_k^2 \) is the metric on a two dimensional surface \( \Sigma_k \) of constant Gaussian curvature \( k \).

In local coordinates \((\theta, \phi)\) on \( \Sigma_k \) we have

\[
d\Omega_k^2 = \begin{cases} 
  d\theta^2 + \sin^2 \theta \, d\phi^2, & k = 1, \text{ spherical,} \\
  d\theta^2 + \theta^2 \, d\phi^2, & k = 0, \text{ toroidal,} \\
  d\theta^2 + \sinh^2 \theta \, d\phi^2, & k = -1, \text{ hyperboloidal.}
\end{cases}
\]

(2.3)

The vector \( \partial/\partial t \) is a Killing vector, time-like for \( F > 0 \) and space-like for \( F < 0 \). Incidentally, the metric (2.2) solves the Einstein-Maxwell equations with a cosmological constant \( \Lambda = -3\lambda \).

However, we shall see that this fact does not enter in the derivation of the Hawking temperature. An expression of the effective potential may be obtained considering the equation of motion for a neutral particle of negligible mass in a gravitational field \( g_{\mu\nu} \), given by the geodesic

\[ \frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \]

(2.4)

where, as usual, \( \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \).

If we suppose that the particle is moving slowly, in a stationary and weak gravitational field, then we can follow well known steps, described for example in Ref. \cite{6}, and we easily arrive to define the effective (newtonian) potential

\[ V = -\frac{c^2}{2} h_{00} + C, \]

(2.5)

where \( C \) is a constant, and where (since the field is weak) the metric is supposed to be close to the flat metric

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

(2.6)
with $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu} \equiv (-k, 1, 1, 1)$, with $k = 1, 0, -1$.
In case $V \to 0$ and $h_{00} \to 0$ for $r \to \infty$, then $C = 0$.
We can now express $V$ in terms of the metric (2.2). In fact from (2.1) we have

$$g_{00} = -F(r)$$

(2.7)

and from (2.6), when $|h_{00}| \ll 1$, we can write

$$g_{00} = \eta_{00} + h_{00} = -k + h_{00}.$$  

(2.8)

Therefore using (2.5) we have

$$V = \frac{c^2}{2} (F - k) + C.$$  

(2.9)

which is the effective potential generated by the metric (2.1).

3 Horizons

In this section we consider some general properties of the (Killing) horizons of the metric (2.1)-(2.2).
As usual the horizons are located at the positive zeros of the function $F(r)$. They are coordinate singularities on null hyper-surfaces. The vector $\partial/\partial t$ is a globally defined Killing vector, time-like in the regions $F > 0$, space-like in the regions $F < 0$ and null on the hyper-surfaces with $F = 0$.

The regions with $F > 0$ are therefore static, and the hyper-surfaces $F = 0$ are Killing horizons $^1$.

We are interested here in describing some behaviours of the zeros of $F(r)$ for various values of the parameters $M, Q, \lambda$, and especially what are the regions where the term $h_{00}$ becomes small, as required by the approximation used to obtain the effective potential $V$.

In units where $G = c = 1$, the function $F(r)$ can be rewritten as

$$F(r) = k - \frac{2M}{r} + \frac{Q^2}{r^2} + \lambda r^2.$$  

(3.1)

We restrict our analysis to the case $M > 0$, on the grounds of physical plausibility.

For example, for $k = 1$ and $\lambda > 0$ (Anti de Sitter space-time) a straightforward analysis reveals that for $\Delta := M^2 - Q^2 \leq 0$ there are no zeros of $F(r)$, therefore we do not have horizons. For $\Delta := M^2 - Q^2 > 0$ we can still have no horizons, or two coinciding horizons, or even two different positive horizons. It is easy to show that it is impossible to have one single horizon (because $Q^2 > 0$). The situation, for a couple of cases, is illustrated in Fig. 1, where is plotted the function $(F - 1)$, which coincides with $2V$ when $|F - 1| \simeq 0$.

In the diagrams, the horizons are located at the intersection point(s) between the graph of $F - 1$ and the straight line $y = -1$. The regions where $V$ can be defined, and where $2V \simeq (F - 1)$, are located in the neighborhood of the (second) intersection point between the graph of $F - 1$ and the axis $y = 0$. When $\lambda \to 0^+$ (small cosmological constant) and $Q^2 \to 0$ (small electric charge of the hole), the zero going to the Schwarzschild horizon, $R = 2M$, is the outer one (see also Ref. [3]).

$^1$For a complete analysis of the causal structure and the Penrose diagrams of these zeros/horizons we refer the reader to Ref. [3], for the case $k = 1$, and to Ref. [4], for the cases $k = 0, k = -1$.
For $k = 1$ and $\lambda < 0$ (de Sitter space-time) the situation is a bit more complicated. The analysis reveals that there is always one negative non physical zero. As regard the positive zeros of $F(r)$ (i.e. those originating the horizons), we can have just one zero, 2 coincident zeros and another positive zero, 3 different zeros, one zero and other two coincident zeros, one zero only. The second and third of these circumstances are summarized in the diagrams of Fig. 2, where again the function $(F - 1)$ is plotted. As before, $V$ is defined in the regions where $|F - 1| \simeq 0$. The third case, the one with three positive zeros, is probably the most representative from the physical point of view, with $M \gg |Q| \gg |\lambda|$. In this case, the zero tending to the Schwarzschild value, $R \to 2M$, when $\lambda \to 0^{-}$ and $Q^2 \to 0$, is the "middle" one.

![Diagram](image1.png)

Figure 1: For $k = 1, \lambda > 0$, the cases of two coincident horizons and two different horizons.

Analog diagrams can be drawn for the situations $k = 0$ ($\lambda > 0, \lambda < 0$), and $k = -1$ ($\lambda > 0, \lambda < 0$). From them we can visualize, as before, the regions where the effective potential $V$ can be defined. A very useful parametrization for the black hole space-times, instead of the pair $(M, Q)$, is the pair $(R_h, Q)$, where $R_h$ is the value of $r$ at the (outer)(or middle) horizon. The mass is then given (from the eq. $F(R_h) = 0$) in terms of $(R_h, Q)$ as

$$M = \frac{R_h}{2} \left( \lambda R_h^2 + k + \frac{Q^2}{R_h^2} \right). \quad (3.2)$$

As we see, the relation (3.2) yields the usual Schwarzschild relation (namely $R_h = 2M$) for $\lambda \to 0$, $Q \to 0$, $k = 1$.

![Diagram](image2.png)

Figure 2: For $k = 1, \lambda < 0$, the cases of 2 coincident plus 1 horizons, and of 3 different horizons.

4
4 Hawking temperature from Heisenberg uncertainty principle

In order to get the Hawking formula, consider the following gedanken experiment. Suppose to have a gravitational field described by the metric \( g_{\mu\nu} \) and a region where \( |F - k| \approx 0 \). There the effective potential \( V \) can be defined, and the eq. (2.9) holds. Imagine to have a neutral (uncharged) particle of rest mass \( m \) which falls radially in such a field. Then the newtonian potential energy of the particle, due to gravity, is, classically,

\[
U = mV = \frac{1}{2} mc^2 (F - k). \tag{4.1}
\]

Let us now suppose that this expression of the classical potential energy holds in any region, also where \( |F - k| \) differs strongly from zero. This means to extrapolate the validity of eq. (4.1), which is a weak field result, to a strong field situation. If the particle falls for a small radial displacement \( \Delta r \), then it feels a variation in potential energy equal to

\[
\Delta U = \frac{1}{2} mc^2 F'(r) \Delta r. \tag{4.2}
\]

During the shift \( \Delta r \), the particle acquires a kinetic energy \( \Delta K \) which equals the lost potential energy \( \Delta U \). Suppose that \( \Delta K \) is sufficient to create a particle-antiparticle pair from the quantum vacuum,

\[
\Delta K = 2mc^2. \tag{4.3}
\]

Then we can compute the \( \Delta r \) needed for this process

\[
\Delta U = \Delta K = 2mc^2 \quad \Rightarrow \quad F'(r) \Delta r = 4, \tag{4.4}
\]

that is

\[
\Delta r = \frac{4}{F'(r)} \tag{4.5}
\]

This process, repeated many times, creates a gas of particles in the region where it takes place. Since the particles are confined in a space slice of thickness \( \Delta r \), each of them has an uncertainty in (kinetic) energy equal to

\[
\Delta E = \frac{\hbar c}{2\Delta r} = \frac{\hbar c}{8} F'(r). \tag{4.6}
\]

Suppose now to interpret the quantum uncertainty in the (kinetic) energy of these particles as due to thermal agitation, which means that we can write, again using simply the classical Maxwell-Boltzmann statistics,

\[
\Delta E \sim \frac{3}{2} k_B T \tag{4.7}
\]

where \( T \) is the temperature of this gas of particles. Therefore

\[
\frac{3}{2} k_B T = \frac{\hbar c}{8} F'(r) \tag{4.8}
\]
or

\[ T = \frac{\hbar c}{12 k_B} F'(r). \] (4.9)

This is the temperature of the particle gas confined in the space slice \( \Delta r \) around the value \( r \) of the radial coordinate.

Imagine that the whole process just described takes place close to the external edge of the (event) horizon, i.e. for \( r \simeq R_h \). By the hypothesis assumed, the expression (4.1) for the effective potential energy \( U \) maintains its validity even close to the outer (or middle) horizon, i.e. for \( r \simeq R_h \). In other words, we extrapolate the validity of the expression (4.1) for \( U \) even to the region \( r \simeq R_h \). This will be sufficient to get the Hawking temperature of the hole with a very good agreement with the QFT formula.

In fact, from eqs. (4.9) and (2.2) we can write for \( r = R_h \)

\[ T = \frac{\hbar c}{12 k_B} \left( \frac{2GM}{c^2 R_h^2} - \frac{2GQ^2}{c^4 R_h^3} + 2\lambda R_h \right) \] (4.10)

and, inserting \( M \) from the mass formula (3.2) with the constants restored

\[ \frac{2GM}{c^2} = kR_h + \frac{GQ^2}{c^4 R_h^3} + \lambda R_h^3, \] (4.11)

we have finally

\[ T = \frac{\hbar c}{12 k_B} \left( \frac{k}{R_h} - \frac{GQ^2}{c^4 R_h^3} + 3\lambda R_h \right). \] (4.12)

This is the temperature of the particle gas thickened on the surface of the (event) horizon, that is the temperature of the hole, as seen by an observer set far in the quasi-flat region of the space-time. In particular, for a Schwarzschild black hole we can write

\[ T = \frac{\hbar c}{12 k_B R_h} = \frac{\hbar c^3}{24 k_B GM}. \] (4.13)

It is remarkable that the exact formula for the Hawking temperature, namely

\[ T = \frac{\hbar c}{4\pi k_B} \left( \frac{k}{R_h} - \frac{GQ^2}{c^4 R_h^3} + 3\lambda R_h \right), \] (4.14)

worked out with the apparatus of QFT in curved space-time (see Ref. [4]), differs from the (4.12) only for the factor \( 4\pi \simeq 12 \). This is especially remarkable if we think that for the derivation of (4.12) we used only laws of classical physics plus the Heisenberg principle.

\(^2\)We know from the previous sections that the effective potential \( V \) (because of the hypothesis \( |h_{\text{ho}}| \ll 1 \)) can be defined only in regions where \( |F - k| \simeq 0 \). In such regions we have \( V = (F - k)c^2/2 \) (eq. 2.9). On the contrary, in the proximity of an (event) horizon, \( r \simeq R_h \), we have by definition \( F \simeq 0 \), that is \( (F - k) \simeq k \). Therefore, in such regions, the effective potential could not, rigorously speaking, be defined.
5 Comment and conclusion

We note that this derivation can be used for the temperature of the horizons of acoustic black holes and, due to its generality, wherever a metric admitting an effective (newtonian) potential is present. So, in this paper we have shown that it is possible to compute the Hawking temperature for a large class of black holes (Schwarzschild, Reissner-Nordström, (Anti) de Sitter) in a unified way, using always only the Heisenberg uncertainty principle, plus laws from classical physics. The Hawking temperature does not seem, also in this analysis, to have any link with the dynamical Einstein equations. Moreover, no generalized uncertainty principle has been used, in any form. As a byproduct we can therefore say that the fact that (A)dS black hole thermodynamics admits a holographic description in terms of dual conformal QFT, whereas Schwarzschild black hole does not, does not seem to rely or depend on the diversity of the uncertainty principle(s) used to derive such thermodynamics.

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