Is Weyl unified theory wrong or incomplete?

Carlos Romero
Departamento de Física, Universidade Federal da Paraíba,
Caixa Postal 5008, 58059-970 João Pessoa, PB, Brazil and
E-mail: cromero@fisica.ufpb.br

In 1918, H. Weyl proposed a unified theory of gravity and electromagnetism based on a generalization of Riemannian geometry. In spite of its elegance and beauty, a serious objection was raised by Einstein, who argued that Weyl’s theory was not suitable as a physical theory. According to Einstein, the theory led to the prediction of a “second clock effect”, which is not observed by experiments. We briefly revisit this point and argue that a preliminary discussion on the very notion of proper time is needed in order to consider Einstein’s critical point of view.

PACS numbers: 04.20.Jb, 11.10.kk, 98.80.Cq

Keywords: (2+1)Gravity, Weyl integrable space-time theory.

I. INTRODUCTION

It has been widely recognized among historians and physicists that, in spite of its elegance and beauty, the theory formulated by H. Weyl, in 1918, in his attempt to unify gravity and electromagnetism is not suitable as a physical theory [1]. As is well known, in an appendix to Weyl’s paper, Einstein set forth a serious objection to the theory. In his critique, Einstein argued that the theory would necessarily predict the existence of the so-called “second clock effect” [2]. According to Einstein, in a space-time ruled by Weyl geometry the existence of sharp spectral lines in the presence of an electromagnetic field would not be possible since atomic clocks would depend on their past history [3]. Einstein reasoned that this predicted effect is a logical consequence of Weyl’s theory, insofar as in a Weyl space-time the length of a vector is not held constant by parallel transport, and this, in turn, would imply that the tic tac of atomic clocks, measured by some periodic physical process, should be path dependent.

In this paper, we revisit Weyl’s theory and approach Einstein’s critique from a new perspective. We argue that a preliminary discussion on the very notion of proper time is needed in order to consider Einstein’s critical point of view. Our discussion will be guided by the so-called Weyl’s Principle of Gauge Invariance, a symmetry principle that plays an essential role in the development of the theory.

The paper is organized as follows. In Section 2, we give a brief account of Weyl geometry. We then proceed to Section 3 to review Einstein’s argument and examine in more detail the assumptions implicitly made therein. In Section 4, we briefly consider recent scalar-tensor theories, which were inspired by a weaker version of Weyl’s geometry, the so-called WIST gravity theories, and show why they are not plagued by the problem of proper time. We conclude with some final remarks in Section 5.

II. A BRIEF SUMMARY OF WEYL GEOMETRY

Weyl geometry is perhaps one of the simplest generalization of Riemannian geometry, the only modification being the fact that the covariant derivative of the metric tensor g is not zero, but instead given by

\[ \nabla_\alpha g_{\beta\lambda} = A_\alpha g_{\beta\lambda}, \]  \hspace{1cm} (1)

where \( A_\alpha \) denotes the components of a one-form field \( A \) in a local coordinate basis. This weakening of the Riemannian compatibility condition is entirely equivalent to requiring that the length of a vector field may change when parallel-transported along a curve in the manifold. We shall refer to the triple \( (M, g, A) \) consisting of a differentiable manifold \( M \) endowed with both a metric \( g \) and a 1-form field \( A \) as a Weyl gauge (or, frame). Now one important discovery made by Weyl was the following. Suppose we perform the conformal transformation

\[ \bar{g} = e^f g, \]  \hspace{1cm} (2)
where \( f \) is an arbitrary scalar function defined on \( M \). Then, the Weyl compatibility condition (1) still holds provided that we let the Weyl field \( A \) transform as
\[
A = A + df.
\] (3)

In other words, the Weyl compatibility condition does not change when we go from one gauge \((M, g, A)\) to another \((M, g, A')\) by simultaneously transformations in \( g \) and \( A \).

If we assume that the Weyl connection \( \nabla \) is symmetric, a straightforward algebra shows that one can express the components of the affine connection in an arbitrary vector basis completely in terms of the components of \( g \) and \( A \):
\[
\Gamma^\alpha_{\beta\lambda} = \{^\alpha_{\beta\lambda}\} - \frac{1}{2} g^\alpha_{\mu} [g_{\mu\beta} A_\lambda + g_{\mu\lambda} A_\beta - g_{\beta\lambda} A_\mu],
\] (4)

where \( \{^\alpha_{\beta\lambda}\} \) represents the Christoffel symbols. It is not difficult to see that the connection and, consequently, the geodesic equations are invariant with respect to the transformations (2) and (3).

We now present Weyl’s second great discovery. Suppose that we are given two vector fields \( V \) and \( U \) parallel transported along a curve \( \alpha = \alpha(\lambda) \). Then, (1) leads to the following equation
\[
\frac{d}{d\lambda} g(V, U) = A \left( \frac{d}{d\lambda} \right) g(V, U).
\] (5)

where \( \frac{d}{d\lambda} \) denotes the vector tangent to \( \alpha \). If we integrate this equation along the curve \( \alpha \), starting from a point \( P_0 = \alpha(\lambda_0) \), we obtain
\[
g(V(\lambda), U(\lambda)) = g(V(\lambda_0), U(\lambda_0)) e^{\int_{\lambda_0}^{\lambda} A(\frac{d}{d\rho}) d\rho}.
\] (6)

Setting \( U = V \) and denoting by \( L(\lambda) \) the length of the vector \( V(\lambda) \) at a point \( P = \alpha(\lambda) \) of the curve, it is easy to verify that in a local coordinate system \( \{x^\alpha\} \) the equation (5) becomes
\[
\frac{dL}{d\lambda} = A_\alpha \frac{dx^\alpha}{d\lambda}.
\] (7)

Let us now consider the set of all closed curves \( \alpha : [a, b] \in R \rightarrow M \), i.e, with \( \alpha(a) = \alpha(b) \). Then, either from (6) or (7) it follows that
\[
L = L_0 e^{\int_{a}^{b} A_\alpha d\alpha},
\]

where \( L_0 \) and \( L \) denotes the values of \( L(\lambda) \) at \( a \) and \( b \), respectively. From Stokes’s theorem we can write
\[
L = L_0 e^{\int f F_{\mu\nu} d\alpha \wedge dx^\nu}.
\]

We thus see that, according to the rules of Weyl geometry, the necessary and sufficient condition for a vector to have its original length preserved after being parallel transported along any closed trajectory is that the 2-form \( F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \) vanishes, where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

Therefore Weyl realized that in his new geometry there are two kinds of curvature, a direction curvature (Richtungskrümmung) and a length curvature (Streckenkrümmung). The first is responsible for changes in the direction of parallel transplanted vectors and is given by the usual curvature tensor \( \mathcal{R}^\alpha_{\beta\mu\nu} \), while the other regulates the changes in their length, and is given by \( F_{\mu\nu} \). Weyl’s second great discovery was that the 2-form \( F \) is invariant under the gauge transformation (3). The analogy with the electromagnetic field is apparent and becomes even more so when we take into account that \( F \) satisfies the identity \( dF = 0 \).
III. THE PRINCIPLE OF GAUGE INVARIANCE AND THE FIELD EQUATIONS

Clearly, the Weyl transformations \(\mathcal{G}_1\) and \(\mathcal{G}_2\) define a whole equivalence class constituted by the set \(\{(M, g, A)\}\) of all Weyl gauges. It is then natural to expect that, as in conformal geometry, the geometrical objects of interest are those that are gauge-invariant. Surely, these invariants will be fundamental to build the action that is expected to give the field equations of the geometrical unified theory. Some basic invariants are easily found: the affine connection \(\Gamma_{\beta\alpha}^{\gamma}\), the curvature tensor \(R^\alpha_{\beta\mu\nu}\), the Ricci tensor \(R_{\mu\nu}\) and the length curvature \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). The simplest invariant scalars, in four-dimensional space-time, that can be constructed out of these are: \(\sqrt{-g}R^\alpha_{\beta\mu\nu}\), \(\sqrt{-g}R_{\alpha\beta\mu\nu}\), \(\sqrt{-g}R_{\alpha\beta}R_{\gamma\delta}\) and \(\sqrt{-g}F_{\alpha\beta}F^{\alpha\beta}\), where \(R = g^{\alpha\beta}R_{\alpha\beta}\) is the Ricci scalar calculated with the Weyl connection.

It seems evident that Weyl’s idea was to have a physical theory completely invariant with respect to change between gauges (or frames). Obviously, this was a minimal requirement of consistency of his physics with the new geometry. As we know, his choice was finally to pick up the simplest possible invariant action, namely,

\[
S = \int d^4x \sqrt{-g}[R^2 + \omega F_{\mu\nu}F^{\mu\nu}],
\]

where \(\omega\) is a constant. Variations with respect to \(A_\mu\) and \(g_{\mu\nu}\) lead, after some simplifications, to the field equations

\[
\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}F^{\mu\nu}) = \frac{3}{2} g^{\mu\nu} (RA_\nu + \partial_\nu R),
\]

\[
R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) = \omega T_{\mu\nu},
\]

where \(T_{\mu\nu} = F_{\mu\alpha}F^{\alpha\nu} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\). It is worth mentioning that when applied to the case of a static and spherically symmetric matter distribution it can be showed that Weyl’s theory correctly predicts the perihelion precession of Mercury as well as the gravitation deflection of light by a massive body. In fact, this is a consequence of the fact that all vacuum solutions of Einstein’s equations (including the Schwarzschild solution) satisfy (9) and (10) when we set \(A_\mu = 0\).

Now before we start our discussion of the Einstein’s objection to Weyl’s theory, in the next section, we would like to stress that to build his theory Weyl adopted a very strong and, at the same time, rather restrictive principle, namely, the Principle of Gauge Invariance, which asserts that all physical quantities must be invariant under the gauge transformations \(\mathcal{G}_1\) and \(\mathcal{G}_2\). This principle was strictly followed by Weyl and guided him to (8) when he had to choose an action for his theory. It should also be noted here that any invariant scalar of this geometry must necessarily be formed from both the metric \(g_{\mu\nu}\) and the Weyl gauge field \(A_\mu\). These two fields constitute an essential and intrinsically part of the geometry and neither of them can be neglected when we want to construct an invariant scalar, so they are, in this sense, inseparable, and must always appear together.

IV. EINSTEIN’S OBJECTION REVISITED

In order to examine Einstein’s objection to Weyl’s unified theory, let us first spell out two of the hypotheses on which the argument is based. They can be stated as follows:

H1) The proper time \(\Delta \tau\) measured by a clock travelling along a curve \(\alpha = \alpha(\lambda)\) is given as in general relativity, namely, by the (Riemannian) prescription

\[
\Delta \tau = \frac{1}{c} \int [g(V, V)]^{\frac{1}{2}} d\lambda = \frac{1}{c} \int [g_{\mu\nu}V^\mu V^\nu]^{\frac{1}{2}} d\lambda,
\]

where \(V\) denotes the vector tangent to the clock’s world line and \(c\) is the speed of light. This supposition is known as the clock hypothesis and clearly assumes that the proper time only depends on the instantaneous speed of the clock and on the metric field.

---

3 In conformal geometry, one basic invariant is the Weyl tensor \(W^\alpha_{\beta\mu\nu}\). In conformal gravity, this tensor is used to form the scalar \(W_{\alpha\beta\mu\nu}W^{\alpha\beta\mu\nu}\), which, then, defines the gravitation sector of the action.

4 Here we are not considering the matter action.
H2) The fundamental tic tac of clocks (in particular, atomic clocks) is to be associated with the (Riemannian) length $L = \sqrt{g(Y, Y)}$ of a certain vector $Y$. As a clock moves in space-time $T$ is parallel-transported along its worldline from a point $P_0$ to a point $P$, hence $L = L_0 e^{\frac{1}{2} \int A_\mu dx^\mu}$, $L_0$ and $L$ indicating the duration of the tic tac of the clock at $P_0$ and $P$, respectively.

Let us now have a look into these two assumptions. We start with the first hypothesis (H1). First of all, we would expect that, to be consistent with the Principle of Gauge Invariance, proper time, as a physically-relevant quantity, should be gauge invariant. It turns out, however, that there is no such invariant notion of proper time in Weyl's theory. In addition to that, the adoption of the general relativistic clock hypothesis here does not seem to be plausible, since $\Delta \tau$, as defined above, takes into account only part of the geometry, namely, the metric field, and completely ignores the other geometric field, i.e., the gauge field $A_\mu$. In the second hypothesis (H2), gauge invariance is violated: the concept of tic tac is not modelled as a gauge-invariant physical quantity and, again, the Weyl geometrical field plays no role in its determination.

To conclude this section, let us remark that, with the inexistence of an invariant notion of proper time, even the first clock effect (the "twin paradox"), which appears both in the special and general relativity, cannot be predicted in Weyl's theory.

V. THE INCOMPLETENESS OF WEYL’S THEORY

Since it does not come equipped with an appropriate notion of proper time, consistent with the requirement of gauge invariance, we are forcefully led to conclude Weyl's theory is not complete. An interesting question that now arises is whether or not one could come up with an acceptable definition of proper time $(\Delta \tau)$ in this theory. Of course the sought-after definition would have to fulfill the following requirements:

i) $\Delta \tau$ should be constructed entirely from the geometry (recall that the general relativistic proper time $(\tau)$ is proportional to length);

ii) $\Delta \tau$ should be consistent with the Principle of Gauge Invariance;

iii) $\Delta \tau$ should depend both on the metric field $g_{\mu\nu}$ and the gauge field $A_\mu$;

iv) $\Delta \tau$ should be written in the form $\Delta \tau = \int F(V, g, A) d\lambda$, with $F$, as in Finsler geometry, being a first-order homogeneous function with respect the tangent vector $V$ (This condition is necessary to guarantee invariance under reparametrization) [7];

v) Finally, it would be highly desirable, though not strictly necessary, that the new definition of $\Delta \tau$ could allow for the Weyl affine geodesic equations to be deduced from a variational principle.

To find a good definition of proper time that fulfills all the above requirements does not seem to be an easy task. First of all, because of the condition on homogeneity with respect to $V$ one has to look for second-order gauge-invariant tensors. Candidates that immediately come to mind are: $R_{\mu\nu}$ and $Rg_{\mu\nu}$. These would lead, respectively, to $\Delta \tau_1 = a_1 \int [R_{\mu\nu} V^\mu V^\nu]^{\frac{1}{2}} d\lambda$ and $\Delta \tau_2 = a_2 \int [Rg_{\mu\nu} V^\mu V^\nu]^{\frac{1}{2}} d\lambda$, $a_1$ and $a_2$ denoting dimensional constants. However, it is clear that neither $\Delta \tau_1$ nor $\Delta \tau_2$ is a good choice as they do not satisfy conditions (iv) and (vi). A rather contrived choice would be $\Delta \tau_3 = a_3 \int \frac{g_{\mu\nu} V^\mu V^\nu}{g_{\mu\nu} W^\mu W^\nu} \frac{d\lambda}{2}$, where $a_3$ designates a dimensional constant and $W$ is the gauge-invariant 1-form defined by $W = A + d(ln(R))$, with $W^\mu = g^{\mu\nu} W_\nu$. In this case, it is interesting to note that the light cone structure is gauge invariant and is determined by the metric only. However, again this choice is not consistent with conditions (iv) and (vi). Perhaps the solution of the problem of finding a satisfactory definition of proper time may lead us beyond the Weyl geometrical framework, indicating that we need a higher level of generalization, such as the one we find in Finsler geometry.

VI. FINAL REMARKS

When $A$ is an exact form, i.e., $A = d\phi$, where $\phi$ is a scalar field, then we say that we have an integrable Weyl space-time (WIST). Theories framed in this kind of geometry are not subject to Einstein’s objection since they do not predict a second clock effect. Due to this fact WIST gravity has attracted the attention of some cosmologists [10].

---

5 In order to preserve gauge invariance it is expected that $A_\mu$ should appear in $F$ only via the tensor $F_{\mu\nu}$.

6 At present, we do not know if this requirement is mathematically possible. In fact, the solution of this question will lead us to examine the so-called inverse variational problem for the case of Weyl affine geodesics [6].
Instead of the electromagnetic field, now it is the scalar field that is geometrized. It is interesting to recall here that in the case of WIST theories the definition of proper time is given by the gauge-invariant equation [11]

$$\Delta \tau = \int_a^b e^{-\frac{\phi}{2}} \left( g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{\frac{1}{2}} d\lambda.$$  \hspace{1cm} (12)

It is not difficult to verify that, with this definition, $\Delta \tau$ satisfies all the requirements listed in the previous section, with $\phi$ replacing $A_\mu$.

We do not know if it is possible to "complete" the elegant and profound theory developed by Weyl almost a century ago. Perhaps in a modified version, but still inspired in Weyl’s ideas, the essential features of the theory could be revived. As some authors have put it: "Weyl geometrical theory contains a suggestive formalism and may still have the germs of a future fruitful theory" [12].

Acknowledgements

The author would like to thank J. B. Fonseca-Neto, F. Dahia, R. G. Lima and J. B. Formiga for helpful discussions. This work was partially supported by CNPq (Brazil).

[1] H. Weyl, Sitzungseber Deutsch. Akad. Wiss. Berlin, 465 (1918). H. Weyl, *Space, Time, Matter* (Dover, New York, 1952).
[2] For a clear and pedagogical explanation of the second clock effect, see R. Penrose, *The Road to Reality*, Ch. 19 (Jonathan Cape, London, 2004).
[3] For a brief historical account of Weyl gravity theory see A. Pais, *Subtle is the Lord*, Ch. 17 (Oxford University Press, Oxford, 1982). See, also, W. Pauli, *Theory of Relativity* (Dover, New York, 1981). P. G. Bergmann, *Theory of Relativity* (Dover, New York, 1976), and L. O’Raifeartaigh and N. Straumann, Rev. Mod. Phys. **72**, 1 (2000).
[4] For a more detailed account of Weyl geometry, see F. Dahia, G.A.T. Gomez, C. Romero, J. Math. Phys. **49**, 102501 (2008). A more formal mathematical treatment is given by G. B. Folland, J. Diff. Geom. **4**, 145 (1970). For a clear and comprehensive review on Weyl geometry, see E. Scholz, math.HO [arXiv:1111.3220]. See also E. Scholz, [arXiv:1206.1559].
[5] P. D. Manheim, Found. Phys. **42**, 388 (2011).
[6] For a more detailed account of Weyl geometry, see F. Dahia, G.A.T. Gomez, C. Romero, J. Math. Phys. **49**, 102501 (2008).
[7] D. Bao, S. S. Chern and Z. Chen, *An Introduction to Riemann-Finsler geometry* (Springer, 1991).
[8] In connection with this, see C. Pfeifer, M. N. R. Wohlfarth, Phys.Rev. D **85**, 064009 (2012).
[9] In the case of auto-parallels in a Riemann-Cartan space-time, the inverse variational problem was examined by C. Maultzsch and V. Shabanov, J. Phys. A **32**, 5355 (1999).
[10] M. Novello, L.A.R. Oliveira, J.M. Salim, E. Elbas, Int. J. Mod. Phys. **D1** (1993) 641-677. J. M. Salim and S. L. Sautú, Class. Quant. Grav. **13**, 353 (1996). H. P. de Oliveira, J. M. Salim and S. L. Sautú, Class.Quant.Grav. **14**, 2833 (1997). V. Melnikov, *Classical Solutions in Multidimensional Cosmology* in Proceedings of the VIII Brazilian School of Cosmology and Gravitation II (1995), edited by M. Novello (Editions Frontières) pp. 542-560. ISBN 2-86332-192-7. K.A. Bronnikov, M.Yu. Konstantinov, V.N. Melnikov, Grav.Cosmol. **1**, 60 (1995). J. Miritzis, Class. Quantum Grav. **21**, 3043 (2004). J. Miritzis, J.Phys. Conf. Ser. **8**, 131 (2005). J.E.M. Aguilar and C. Romero, Found. Phys. **39** (2009)1205; J.E.M. Aguilar and C. Romero, Int. J. Mod. Phys. A **24**, 1505 (2009). J. Miritzis, Int. J. Mod. Phys. D **22**, 1350019 (2013). F. P. Pouls and J. M. Salim [arXiv:1305.6830]. R. Vazirian, M. R. Tannahayi and Z. A. Motahar, Adv. High Energy Physics **7**, 902396 (2015).
[11] See, for instance, T. S. Almeida, M. L. Pucheu, C. Romero and J. B. Formiga, Phys. Rev. D **89**, 064047.
[12] R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, Ch. 15, (McGraw-Hill, 1975).

---

7 Pauli refers to Weyl’s theory as "an extremely profound theory" [3].