Possible Counterexample of the Riemann Hypothesis

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Abstract Under the assumption that the Riemann hypothesis is true, von Koch deduced the improved asymptotic formula \( \theta(x) = x + O(\sqrt{x} \times \log^2 x) \), where \( \theta(x) \) is the Chebyshev function. A precise version of this was given by Schoenfeld: He found under the assumption that the Riemann hypothesis is true that \( \theta(x) < x + \frac{1}{8\pi} \times \sqrt{x} \times \log^2 x \) for every \( x \geq 599 \). On the contrary, we prove if there exists some real number \( x \geq 2 \) such that \( \theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x \), then the Riemann hypothesis should be false. In this way, we show that under the assumption that the Riemann hypothesis is true, then \( \theta(x) < x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x \).

Keywords Riemann hypothesis · Nicolas inequality · Chebyshev function · prime numbers

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1 Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part \( \frac{1}{2} \) [2]. The Riemann hypothesis belongs to the David Hilbert’s list of 23 unsolved problems [2]. Besides, it is one of the Clay Mathematics Institute’s Millennium Prize Problems [2]. This problem has remained unsolved for many years [2]. In mathematics, the Chebyshev function \( \theta(x) \) is given by

\[
\theta(x) = \sum_{p \leq x} \log p
\]
where \( p \leq x \) means all the prime numbers \( p \) that are less than or equal to \( x \). Say Nicolas\((p_n)\) holds provided

\[
\prod_{q \leq p_n} \frac{q}{q-1} > e^\gamma \times \log(p_n).
\]

The constant \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant, \( \log \) is the natural logarithm, and \( p_n \) is the \( n \)th prime number. The importance of this property is:

**Theorem 1.1** [7], [8]. Nicolas\((p_n)\) holds for all prime numbers \( p_n > 2 \) if and only if the Riemann hypothesis is true.

We know the following properties for the Chebyshev function:

**Theorem 1.2** [11]. If the Riemann hypothesis holds, then

\[
\theta(x) = x + O(\sqrt{x} \times \log^2 x)
\]

for all \( x \geq 10^8 \).

**Theorem 1.3** [9]. For \( 2 \leq x \leq 10^8 \)

\[
\theta(x) < x.
\]

We also know that

**Theorem 1.4** [10]. If the Riemann hypothesis holds, then

\[
\left( \frac{e^{-\gamma}}{\log x} \times \prod_{q \leq p_n} \frac{q}{q-1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}
\]

for all numbers \( x \geq 13.1 \).

Let’s define \( H = \gamma - B \) such that \( B \approx 0.2614972128 \) is the Meissel-Mertens constant [6]. We know from the constant \( H \), the following formula:

**Theorem 1.5** [3].

\[
\sum_q \left( \log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = \gamma - B = H.
\]

For \( x \geq 2 \), the function \( u(x) \) is defined as follows

\[
u(x) = \sum_{q > x} \left( \log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right).
\]

We use the following theorems:

**Theorem 1.6** [5]. For \( x > -1 \):

\[
\frac{x}{x+1} \leq \log(1+x).
\]
Theorem 1.7 [4]. For $x \geq 1$:

$$\log\left(1 + \frac{1}{x}\right) < \frac{1}{x + 0.4}.$$  

Let’s define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B\right).$$

Definition 1.8 We define another function:

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B\right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if Nicolas($p$) holds, where $p$ is the greatest prime number such that $p \leq x$. In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.

2 Results

Theorem 2.1 The Riemann hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.

Proof In the paper [8] is defined the function:

$$f(x) = e^\gamma \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q - 1}{q}.$$  

We know that $f(x)$ is lesser than 1 when Nicolas($p$) holds, where $p$ is the greatest prime number such that $2 < p \leq x$. In the same paper, we found that

$$\log f(x) = U(x) + u(x)$$

where $U(x) = -\varpi(x)$ [8]. When $f(x)$ is lesser than 1, then $\log f(x) < 0$. Consequently, we obtain that

$$-\varpi(x) + u(x) < 0$$

which is the same as $\varpi(x) > u(x)$. Therefore, this is a consequence of the theorem 1.1.

Theorem 2.2 If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > 1$$

for all numbers $x \geq 13.1$.  

Proof Under the assumption that the Riemann hypothesis is true, then we would have
\[
\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times \log x \times \left( 1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right)
\]
after of distributing the terms based on the theorem 1.4 for all numbers \(x \geq 13.1\). If we apply the logarithm to the both sides of the previous inequality, then we obtain that
\[
\sum_{q \leq x} \log \left( \frac{q}{q-1} \right) < \gamma + \log \log x + \log \left( 1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right).
\]
That would be equivalent to
\[
\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]
where we know that
\[
\log \left( 1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right) < \frac{1}{8 \times \pi \times \sqrt{x} + 0.4} = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]
according to theorem 1.7 since \(\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} \geq 1\) for all numbers \(x \geq 13.1\). We use the theorem 1.5 to show that
\[
\sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right) = H - u(x)
\]
and \(\gamma = H + B\). So,
\[
H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]
which is the same as
\[
H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.
\]
We eliminate the value of \(H\) and thus,
\[
-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]
which is equal to
\[
u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).
\]
We know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$

Suppose that $\theta(x) = \varepsilon \times x$ for some constant $\varepsilon > 1$. Then,

$$\log \log \theta(x) - \log \log x = \log \log(\varepsilon \times x) - \log \log x$$
$$= \log(\log x + \log \varepsilon) - \log \log x$$
$$= \log \log x + \log(1 + \frac{\log \varepsilon}{\log x}) - \log \log x$$
$$= \log(1 + \frac{\log \varepsilon}{\log x}).$$

In addition, we know that

$$\log(1 + \frac{\log \varepsilon}{\log x}) \geq \frac{\log \varepsilon}{\log \theta(x)}$$

using the theorem 1.6 since $\log \log \theta(x) > -1$ when $\varepsilon > 1$. Certainly, we will have that

$$\log(1 + \frac{\log \varepsilon}{\log x}) \geq \frac{\log \varepsilon}{\log \theta(x)}.$$ 

Thus,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \varepsilon}{\log \theta(x)}.$$

If we add the following value of $\log \log \theta(x)$ to the both sides of the inequality, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \log \log \theta(x) > \frac{\log \varepsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)}$$
$$= \frac{\log \varepsilon + \log x}{\log \theta(x)}$$
$$= \log \theta(x)$$
$$= 1.$$ 

We know this inequality is satisfied when $0 < \varepsilon \leq 1$ since we would obtain that $\frac{\log \varepsilon}{\log \theta(x)} \geq 1$. Therefore, the proof is done.
Theorem 2.3 If there exists some real number \( x \geq 2 \) such that

\[
\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x,
\]

then the Riemann hypothesis is false.

Proof If the Riemann hypothesis holds, then

\[
\theta(x) = x + O(\sqrt{x} \times \log^{2} x)
\]

for all \( x \geq 10^8 \) from the theorem 1.2. Now, suppose there is a real number \( x \geq 2 \) such that \( \theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x \). That would be equivalent to

\[
\log \theta(x) > \log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x)
\]

and so,

\[
\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x)}
\]

for all numbers \( x \geq 10^8 \). Hence,

\[
\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x)}
\]

If the Riemann hypothesis holds, then

\[
3 \times \log x + 5 \quad \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x)} > 1
\]

for those values of \( x \) that complies with

\[
\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x
\]

due to the theorem 2.2. By contraposition, if there exists some number \( y \geq 10^{8} \) such that for all \( x \geq y \) the inequality

\[
3 \times \log x + 5 \quad \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x)} \leq 1
\]

is satisfied, then the Riemann hypothesis should be false. Let’s define the function

\[
u(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^{2} x)} - 1.
\]

The Riemann hypothesis is false when there exists some number \( y \geq 10^{8} \) such that for all \( x \geq y \) the inequality \( \nu(x) \leq 0 \) is always satisfied. We ignore when \( 2 \leq x \leq 10^{8} \) since \( \theta(x) < x \) according to the theorem 1.3. We know that the function \( \nu(x) \) is monotonically decreasing for every number \( x \geq 10^{8} \). The derivative of \( \nu(x) \) is negative for all \( x \geq 10^{9} \). Indeed, a function \( \nu(x) \) of a real variable \( x \) is monotonically decreasing
in some interval if the derivative of \( v(x) \) is lesser than zero and the function \( v(x) \) is continuous over that interval [1]. It is enough to find a value of \( y \geq 10^8 \) such that \( v(y) \leq 0 \) since for all \( x \geq y \) we would have that \( v(x) \leq v(y) \leq 0 \), because of \( v(x) \) is monotonically decreasing. We found the value \( y = 10^8 \) complies with \( v(y) \leq 0 \).

\[ \theta(x) < x + \frac{1}{\log \log x} \times \sqrt{x} \times \log^2 x. \]

**Theorem 2.4** Under the assumption that the Riemann hypothesis is true, then

\[ \theta(x) < x + \frac{1}{\log \log x} \times \sqrt{x} \times \log^2 x. \]

**Proof** This is a direct consequence of the theorem 2.3.

### Appendix

We found the derivative of \( v(x) \) in the web site https://www.wolframalpha.com/input. Besides, we determine the sign of the function \( v(x) \) using the tool \textit{gp} from the web site https://pari.math.u-bordeaux.fr. In the project \textit{PARI/GP}, the method \textit{sign}(\( F(X) \)) returns \(-1\) when the function \( F(X) \) is negative in the value of \( X \).

We checked that is negative for \( X = 10^8 \) with a real precision of 1000016 significant digits when \( F(X) = v(x) \). We also checked that is still negative for \( X = 100000! \), where \( (\ldots)! \) means the factorial function.

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