A special modulus of continuity and the $K$–functional
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Abstract

We consider the questions connected with the approximation of a real continuous $1$-periodic functions and give a new proof of the equivalence of the special Boman–Shapiro modulus of continuity with Peetre's $K$–functional. We also prove Jackson’s inequality for the approximation by trigonometric polynomials.

keywords: Modulus of continuity, $K$–functional, Jackson’s theorem

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1. Introduction

Denote by $C(T), T = \mathbb{R}/\mathbb{Z}$ the space of real continuous $1$–periodic functions $f$ with the uniform norm

$$
\|f\| = \sup_{u \in T}|f(u)|.
$$

The derivative operator is denoted by the symbol $D$, and the space of functions $f$ with $D^2 f \in C(T)$ will be denoted by $C^2(T)$.

Let $L(T)$ be the space of measurable, integrable functions with norm

$$
\|f\|_1 = \int_T |f(u)| \, du
$$

and let $T_{n-1}$ be the set of real trigonometric $1$–periodic polynomials $\tau$ of degree at most $n-1$:

$$
\tau(t) := \sum_{j=-n+1}^{n-1} \alpha_j \exp(2\pi ij t), \quad \alpha_j = \bar{\alpha}_{-j}.
$$

For $f \in C(T)$, we denote by $E_{n-1}(f)$ the value of the best approximation of $f$ by real trigonometric polynomials of degree at most $n-1$

$$
E_{n-1}(f) := \inf_{\tau \in T_{n-1}} \|f - \tau\|.
$$

We will use the convolution of periodic functions $f$ with positive functions $g$, with finite support. In this case, the convolution can be understood in the following sense:

$$
(f * g)(t) := \int_\mathbb{R} f(u) g(t-u) \, du.
$$

We denote by $\chi_{h}^k$, $k = 1, 2, \ldots$ the convolution powers of the normalized characteristic function of the interval $(-h/2, h/2)$, $h > 0$:

$$
\chi_{h}^k := \chi_{h}^{k-1} * \chi_h, \quad \chi_h(t) := \begin{cases} 
\frac{1}{h} & t \in (-h/2, h/2), \\
0 & t \notin (-h/2, h/2).
\end{cases}
$$
In particular, 
\[ \chi^k_h(t) = \begin{cases} 
\frac{1}{h} \left(1 - \frac{|t|}{h}\right), & t \in (-h, h), \\
0, & t \notin (-h, h). 
\end{cases} \]

The functions \( \chi^k_h \) are the cardinal \( B- \) splines with support \([-kh/2, kh/2] \) and \( \|\chi^k_h\|_1 = 1 \).

We will use the following moduli of continuity (see [9, 11, 2])
\[ W_2(f, \chi^k_h) := \|f - f \ast \chi^k_h\|, \]
\[ W^*_2(f, \chi^k_h) := \sup_{0 < u \leq h} W_2(f, \chi^k_u). \]

They are special cases of the Boman–Shapiro moduli of continuity (see [11, 3, 4]).

This paper is the continuation of [1]. The main result of [1] is the following Jackson inequality for the uniform approximation of continuous \( 1- \) periodic functions by trigonometric polynomials:

**Let** \( f \) **be a continuous** \( 1- \) **periodic function and** \( n \in \mathbb{N} \), \( h = \alpha/(2n) \), \( \alpha > 2/\pi \). **Then the following inequality holds**
\[ E_{n-1}(f) \leq \left(\sec \frac{1}{\alpha} + \tan \frac{1}{\alpha}\right) W_2(f, \chi_h). \tag{1.1} \]

**The estimate is exact for** \( \alpha = 1, 3, . . . \).

In [1] the following sharp Bernstein–Nikolsky–Stechkin inequality for \( \tau \in T_n \) was also obtained:

**Let** \( \tau \) **be a real trigonometric** \( 1- \) **periodic polynomial of degree at most** \( n - 1 \) **for** \( n \in \mathbb{N} \), **and suppose** \( h \in (0, 1/n] \). **Then**
\[ \|D^2 \tau\| \leq (2\pi n)^2 W_2(c_n, \chi_h)^{-1} W_2(\tau, \chi_h), \quad c_n(t) := \cos(2\pi nt). \tag{1.2} \]

The Jackson inequality (1.1) and the Bernstein–Nikolsky–Stechkin estimate (1.2) allowed us to prove the equivalence of a special modulus of continuity and the second Peetre’s \( K- \) functional [1]:

**Let** \( h \in (0, 1] \). **Then**
\[ 1/4K_2(f, h/(4\sqrt{6})) \leq W_2(f, \chi_h) \leq 4K_2(f, h/(4\sqrt{6})). \tag{1.3} \]

The equivalence of moduli of this type and the \( K- \) functional is known (see [7] and [14]). Here we give a new form of this equivalence with the calculation of the constants. We present a new simple proof of the estimates of the type (1.3) with better constants (Theorem 1). Theorem 1 and a new construction in the proof of Theorem 1 are the main results of the present paper. Further, we introduce a generalized \( K- \) functional which is related to the new approach to the direct theorems of approximation theory [9, 1] and give the analogue of Theorem 1 for it (Theorem 2). We also give a proof of the estimates of the type (1.1) which hold for \( \alpha > 0 \) and better than (1.1) for \( \alpha < 0.778 \) (Theorem 3).
2. Some auxiliary results

2.1. The classical moduli of continuity and the special moduli of continuity.

In this paper, we consider the modulus of continuity of the second order (modulus of smoothness). The classic definition of the modulus of smoothness is the following [6]:

$$\omega_2(f, h) := \sup_{0 < u \leq h} \sup_{t \in \mathbb{T}} |f(t + u) - 2f(t) + f(t - u)| = \sup_{0 < u \leq h} \|\Delta^2_u f\|.$$ 

In [9, 1, 2] the importance of the following moduli of continuity was indicated

$$W_2(f, \chi_h^k) = \|f - f \ast \chi_h^k\|, \quad W_2^*(f, \chi_h^k) = \sup_{0 < u \leq h} \|f - f \ast \chi_u^k\|.$$ 

It is obvious that $W_2(f, \chi_h^k) \leq W_2^*(f, \chi_h^k)$. Note that

$$f(t) - (f \ast \chi_h^k)(t) = f(t) - \int_{kh/2}^{kh/2} f(t - u) \chi_h^k(u) \, du$$

$$= f(t) - \int_0^{kh/2} (f(t + u) + f(t - u)) \chi_h^k(u) \, du = - \int_0^{kh/2} \Delta^2_u f(t) \chi_h^k(u) \, du,$$

and hence

$$\|f - f \ast \chi_h^k\| = \int_0^{kh/2} \Delta^2_u f(\cdot) \chi_h^k(u) \, du.$$ 

We give some simple properties of the modulus $W_2$.

**Lemma 1.** Let $f, D^2 g \in C(\mathbb{T})$, $k, l \in \mathbb{N}$, $h > 0$. Then for $W_2$, the following inequalities hold

$$W_2(f, \chi_h^k) \leq W_2(f - g, \chi_h^k) + W_2(g, \chi_h^k), \quad (2.4)$$

$$W_2(f, \chi_h^k) \leq 1/2 \omega_2(f, kh/2), \quad (2.5)$$

$$W_2(f, \chi_h^k) \leq 2\|f\|, \quad (2.6)$$

$$W_2(f, \chi_h^k) \leq W_2(f, \chi_h^l) + W_2(f, \chi_h^{l-k}), \quad \text{in particular} \quad (2.7)$$

$$W_2(f, \chi_h^k) \leq k W_2(f, \chi_h^1), \quad (2.8)$$

$$W_2(g, \chi_h^k) \leq c_k(h)\|D^2 g\|, \quad c_k(h) = \frac{2h^2}{(k + 2)!} \sum_{j=0}^{[k/2]} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^{k+2}, \quad (2.9)$$

Specifically, for $k = 1, 2$ (2.9) gives

$$W_2(g, \chi_h) \leq \frac{h^2}{24}\|D^2 g\|, \quad (2.10)$$

$$W_2(g, \chi_h^2) \leq \frac{h^2}{12}\|D^2 g\|. \quad (2.11)$$

**Proof.** The inequalities (2.4), (2.5) follow directly from the definition of a special modulus of continuity. We have

$$W_2(f, \chi_h^k) = \|(f - g) - (f - g) \ast \chi_h^k + g - g \ast \chi_h^k\| \leq W_2(f - g, \chi_h^k) + W_2(g, \chi_h^k)$$
and

\[ W_2(f, \chi^k_h) = \| \int_0^{kh/2} \Delta^2_u f(\cdot) \chi^k_h(u) \, du \| \leq 1/2 \omega_2(f, kh/2). \]

To prove (2.6) it is enough to use the semi-additive property of the norm, the following property of convolution: \( \| f * \chi^k_h \| \leq \| f \| * \| \chi^k_h \| \) and the equality \( \| \chi^k_h \|_1 = 1 \). To prove (2.7), we write \( f - f * \chi^k_h \) in the form

\[ f - f * \chi^k_h = (f - f * \chi^j_h) + (f * \chi^j_h - f * \chi^k_h), \]

and similarly to the proof of (2.6), obtain

\[ W_2(f, \chi^k_h) \leq W_2(f, \chi^j_h) + W_2(f, \chi^{[k-j]}_h). \]

To prove (2.9) we will use the representation (see [5], p. 245)

\[ \chi^k_h(u - kh/2) = \frac{1}{h(k-1)!} \sum_{j \geq 0, u/h - j > 0} (-1)^j \binom{k}{j} (u/h - j)^{k-1}, \quad 0 < u < kh. \]

Using inequality \( \| u^{-2} \Delta^2_u g(\cdot) \| \leq \| D^2 g \| \), we obtain

\[
\begin{align*}
W_2(g, \chi^k_h) &\leq \| D^2 g \| \int_0^{kh/2} u^2 \chi^k_h(u) \, du = \| D^2 g \| \int_0^{kh/2} (kh/2 - u)^2 \chi^k_h(kh/2 - u) \, du \\
&= \frac{\| D^2 g \|}{h^{k}(k-1)!} \sum_{j=0}^{[k/2]} (-1)^j \binom{k}{j} \int_{kh}^{kh/2} (u - kh/2)^2 (u - jh)^{k-1} \, du \\
&= \left( \frac{2h^2}{(k+2)!} \sum_{j=0}^{[k/2]} (-1)^j \binom{k}{j} (k/2 - j)^{k+2} \right) \| D^2 g \|. \quad \square
\end{align*}
\]

**Remark 1.** Lemma 1 holds for \( W^*_2 \) with the same proofs.

### 2.2. The second modulus of continuity and the \( K \)–functional

For \( f \in C(\mathbb{T}) \) define the second \( K \)–functional as follows:

\[ K_2(f, h) := \inf_{g \in C^2(\mathbb{T})} \{ \| f - g \| + h^2 \| D^2 g \| \}. \]

The second \( K \)–functional characterizes the values of the best approximation \( f \in C(\mathbb{T}) \) by smooth functions \( g \in C^2(\mathbb{T}) \), with a control on the norm \( g \). Note that \( K_2 \) has the following properties:

\[ K_2(f, h) \leq \max\{1, h^2/\delta^2\} K_2(f, \delta), \quad h, \delta > 0. \]

Indeed, for \( \delta > h \), we have \( \max\{1, h^2/\delta^2\} = 1 \) and

\[ K_2(f, h) \leq K_2(f, \delta). \quad (2.12) \]

If \( \delta < h \), then \( \max\{1, h^2/\delta^2\} = h^2/\delta^2 \) and

\[ h^2/\delta^2 K_2(f, \delta) = \inf_{g \in C^2} \{ h^2/\delta^2 \| f - g \| + h^2 \| D^2 g \| \} \geq K_2(f, h). \]
Thus, when \( \delta < h \), we have

\[
\frac{K_2(f, h)}{h^2} \leq \frac{K_2(f, \delta)}{\delta^2}.
\] (2.13)

In other words, the function \( K_2(f, h) \) is a monotonically increasing function of the argument \( h > 0 \), and the function \( K_2(f, h)/h^2 \) is a monotonically decreasing function for \( h > 0 \).

The following lemma is well known. The idea of using an intermediate approximation of a smooth function belongs to Steklov [13, 12] and Favard [8]. To present a standard approach and compare it with our approach, we give the lemma with the proof. Of special interest is the constant in the first inequality in (2.14).

**Lemma 2.** For \( f \in C(T) \) and \( h > 0 \), we have

\[
\frac{2}{3} K_2(f, h/2) \leq \frac{1}{2} \omega_2(f, h) \leq 2 K_2(f, h/2).
\] (2.14)

**Proof.** To prove the right inequality, we will use the well-known properties of the second modulus of continuity

\[
\omega_2(f_1 + f_2, h) \leq \omega_2(f_1, h) + \omega_2(f_2, h), \quad \omega_2(f, h) \leq 4\|f\|, \quad \omega_2(g, h) \leq h^2\|D^2 g\|.
\]

We have

\[
\omega_2(f, h) \leq \inf_{g \in C^2} \left\{ \omega_2(f - g, h) + \omega_2(g, h) \right\}
= \inf_{g \in C^2} \left\{ 4\|f - g\| + h^2\|D^2 g\| \right\} = 4 K_2(f, h/2).
\]

If \( g = f \star \chi_h^2 \), then by using the identity \( D^2(f \star \chi_h^2) = h^{-2} \Delta_h^2 f \) we obtain the left inequality

\[
K_2(f, h/2) \leq \|f - f \star \chi_h^2\| + \frac{h^2}{4} \|D^2(f \star \chi_h^2)\|
\leq \frac{1}{2} \omega_2(f, h) + \frac{1}{4} \omega_2(f, h) = \frac{3}{4} \omega_2(f, h).
\]

\[\square\]

### 3. The special modulus of continuity and the \( K \)– functional.

In this and the following sections, we restrict ourselves to statements about the special moduli of continuity \( W_2(f, \chi_h^k), W_2^*(f, \chi_h^k) \) for \( k = 1, 2 \).

The main result of this paper is the following theorem and the new construction \[8, 17\], which was used in the proof.

**Theorem 1.** Let \( f \in C(T) \) and \( h > 0 \). Then

\[
\frac{2}{3} K_2(f, h/(2\sqrt{6})) \leq W_2^*(f, \chi_h^2) \leq 2 K_2(f, h/(2\sqrt{6})), \quad (3.15)
\]

\[
\frac{2}{5} K_2(f, h/(4\sqrt{3})) \leq W_2^*(f, \chi_h) \leq 2 K_2(f, h/(4\sqrt{3})). \quad (3.16)
\]
Proof. Firstly, we prove (3.15). Let \( f \in C(T) \), \( g \in C^2(T) \), \( h > 0 \). The proof of the right inequality is standard. By (2.14), (2.6), (2.11) we have:

\[
W_2^*(f, \chi_h^2) \leq \inf_{g \in C^2} \{ W_2^*(f - g, \chi_h^2) + W_2^*(g, \chi_h^2) \} \\
\leq \inf_{g \in C^2} \{ 2\|f - g\| + \frac{h^2}{12}\|D^2g\| \} = 2K_2(f, h/(2\sqrt{6})).
\]

Consider the inverse estimate. Let

\[
g(t) = \frac{12}{h^2} \int_0^h (f * \chi_h^2)(t) u^2 \chi_h^2(u) \, du. \tag{3.17}
\]

From the definition of the second \( K \)-functional we obtain

\[
K_2(f, h/(2\sqrt{6})) \leq \|f - g\| + \frac{h^2}{24}\|D^2g\|.
\]

We can compute

\[
\|D^2g\| = \frac{12}{h^2} W_2(f, \chi_h^2) \leq \frac{12}{h^2} W_2^*(f, \chi_h^2). \tag{3.18}
\]

Indeed, since \( D^2(f * \chi_h^2) = u^{-2}\Delta u^2 f \), we have

\[
D^2 g(t) = \frac{12}{h^2} \int_0^h D^2((f * \chi_h^2)(t)) u^2 \chi_h^2(u) \, du \\
= \frac{12}{h^2} \int_0^h \Delta u^2 f(t) \chi_h^2(u) \, du = -\frac{12}{h^2} \left( f(t) - (f * \chi_h^2)(t) \right),
\]

which implies (3.18). Further, from

\[
12/h^2 \int_0^h u^2 \chi_h^2(u) \, du = 1 \quad \text{(see the proof of (2.9) and (2.11))}
\]

we have

\[
|f(t) - g(t)| = \left| \frac{12}{h^2} \int_0^h (f(t) - (f * \chi_h^2)(t)) u^2 \chi_h^2(u) \, du \right| \leq W_2^*(f, \chi_h^2) \frac{12}{h^2} \int_0^h u^2 \chi_h^2(u) \, du = W_2^*(f, \chi_h^2). \tag{3.19}
\]

From (3.18) and (3.19) we obtain

\[
K_2(f, h/(2\sqrt{6})) \leq W_2^*(f, \chi_h^2) + \frac{1}{2} W_2^*(f, \chi_h^2).
\]

We proceed with the proof of (3.16). The right inequality of (3.16) can be proved as in the previous case.

\[
W_2^*(f, \chi_h) \leq \inf_{g \in C^2} \{ W_2^*(f - g, \chi_h) + W_2^*(g, \chi_h) \} \\
\leq \inf_{g \in C^2} \{ 2\|f - g\| + \frac{h^2}{24}\|D^2g\| \} = 2K_2(f, h/(4\sqrt{3})).
\]

In the proof of the inverse estimate we will use two auxiliary functions, \( g_1(t) \) and \( g_2(t) \).

\[
g_1(t) = \frac{24}{h^2} \int_0^{h/2} (f * \chi_u)(t) u^2 \chi_h(u) \, du,
\]

\[
g_2(x) = \frac{24}{h^2} \int_0^{h/2} (f * \chi_u^2)(t) u^2 \chi_h(u) \, du.
\]
From the definition of the second $K$– functional it follows that
\[ K_2(f, h/(4\sqrt{3})) \leq \|f - g_1\| + \|g_1 - g_2\| + \frac{h^2}{48} \|D^2 g_2\|. \] (3. 20)

Similarly, (3. 18) and (3. 19) yield
\[ \|f - g_1\| \leq W_2^*(f, \chi_h), \] (3. 21)
\[ \|D^2 g_2\| \leq \frac{24}{h^2} W_2^*(f, \chi_h). \] (3. 22)

It remains to estimate the norm $g_1 - g_2$. We have
\[ |g_1(t) - g_2(t)| = \left| \frac{24}{h^2} \int_0^{h/2} (\left((f * \chi_u)(t) - (f * \chi_u^2)(t)\right) u^2 \chi_h(u) \, du \right|. \]

By using the inequalities $\|f * g\| \leq \|f\| \|g\|_1$ and $W_2^*(f, \chi_{h/2}) \leq W_2^*(f, \chi_h)$ and (2. 10), we obtain
\[ \|g_1 - g_2\| \leq W_2^*(f, \chi_h) \left( \frac{24}{h^2} \int_0^{h/2} u^2 \chi_h(u) \, du = W_2^*(f, \chi_h). \right. \]

By (3. 20), (3. 21), (3. 22) and the previous inequality, we get
\[ K_2(f, h/(4\sqrt{3})) \leq 2W_2^*(f, \chi_h) + \frac{1}{2} W_2^*(f, \chi_h). \] \[ \square \]

Theorem 1 and Lemma 2 yield the following fact.

**Corollary 1.** Let $h > 0$. Then for $h_1 = h/(2\sqrt{6})$, $h_2 = h/(4\sqrt{3})$
\[ \frac{1}{3} W_2^*(f, \chi_{h_1}^2) \leq \frac{2}{3} K_2(f, h_1) \leq \frac{1}{2} \omega_2(f, 2h_1) \leq 2K_2(f, h_1) \leq 3W_2^*(f, \chi_h). \]

and
\[ \frac{1}{3} W_2^*(f, \chi_h) \leq \frac{2}{3} K_2(f, h_2) \leq \frac{1}{2} \omega_2(f, 2h_2) \leq 2K_2(f, h_2) \leq 5W_2^*(f, \chi_h). \]

Moreover, Theorem 1 and the inequality (2. 7) of Lemma 1 allow us to compare the moduli $W_2^*(f, \chi_h)$ and $W_2^*(f, \chi_{h_1}^2)$:

**Corollary 2.** For $h > 0$
\[ W_2^*(f, \chi_{h_1}^2) \leq 2W_2^*(f, \chi_h) \leq 6W_2^*(f, \chi_h). \] (3. 23)
where the left inequality cannot be improved for $h = 1/(2n)$, $n = 1, 2, \ldots$. 

**Proof.** The left inequality in (3. 23) is (2. 7) for $k = 2$, $l = 1$. The right inequality follows from (3. 16) and (2. 12):
\[ W_2^*(f, \chi_h) \leq 2 K_2(f, h/(4\sqrt{3})) \leq 2 K_2(f, h/(2\sqrt{6})) \leq 3 W_2^*(f, \chi_h). \]

The left inequality of Corollary 2 is exact for $h = 1/(2n)$, $n = 1, 2, \ldots$. To show this, fix $n \in \mathbb{N}$ and $h = 1/(2n)$. Consider the following construction, which was used in [11]. Let
\[ \varepsilon_j(t) := \text{sign} \cos(2\pi nt) \star \chi_h^j(t), \quad j = 0, 1, 2, \ldots, \quad \chi_h^0 = \delta, \quad f \star \delta = f, \]
\[ \phi := \sum_{j=0}^{\infty} \varepsilon_j. \]
It was proved in [2] that $$\| \varepsilon_j \| = E_{n-1}(\chi^j) \leq (2/\pi)^{j-1}$$ and the series converges uniformly. We have

$$\| \phi - \phi * \chi^2_h \| = \| \sum_{j=0}^{\infty} \varepsilon_j - \sum_{j=1}^{\infty} \varepsilon_j \| = \| \varepsilon_0 + \varepsilon_1 \| = 2,$$

$$\| \phi - \phi * \chi_h \| = \| \sum_{j=0}^{\infty} \varepsilon_j - \sum_{j=1}^{\infty} \varepsilon_j \| = \| \varepsilon_0 \| = 1. \square$$

Now we show that Theorem 1 implies the estimates (1.3), obtained in [1] by another method.

**Corollary 3.** For $$h > 0$$

$$\frac{2}{5} K_2(f, h/(4\sqrt{6})) \leq W^*_2(f, \chi^h) \leq 4 K_2(f, h/(4\sqrt{6})).$$

**Proof.** The fact that $$K_2(h)$$ is increasing plus the left inequality (3.16) imply

$$\frac{2}{5} K_2(f, h/(4\sqrt{6})) \leq \frac{2}{5} K_2(f, h/(4\sqrt{3})) \leq W^*_2(f, \chi^h).$$

The property (2.13) and the right inequality in (3.16) give

$$W^*_2(f, \chi^h) \leq 2 K_2(f, h/(4\sqrt{3})) \leq 4 K_2(f, h/(4\sqrt{6})). \square$$

Note that the constant in the lower estimate (3.16) is worse than the constant in the appropriate lower estimate (3.15). The estimate (3.15) can be improved if we consider the generalized $$\tilde{K}_2$$-functional, which is a sharper characteristic than $$K_2$$. Its definition is motivated by the proof of (3.16).

Let

$$\tilde{K}_2(f, h_1, h_2) = \inf_{g_1 \in C, g_2 \in C^2} \{\| f - g_1 \| + h_1 \| D(g_1 - g_2) \| + h_2^2 \| D^2 g_2 \| \}.$$ 

Clearly, one can take $$g_2 \equiv g_1$$ and obtain

$$\tilde{K}_2(f, h_1, h_2) \leq K_2(f, h_2).$$

**Theorem 2.** Let $$f \in C(\mathbb{T})$$ and $$h > 0$$. Then

$$\frac{4}{9} \tilde{K}_2 \left( f, \frac{h}{8}, \frac{h}{4\sqrt{3}} \right) \leq W^*_2(f, h) \leq 2 \tilde{K}_2 \left( f, \frac{h}{8}, \frac{h}{4\sqrt{3}} \right).$$

**Proof.** Let $$f, Dg_1, D^2 g_2 \in C(\mathbb{T})$$ and $$h > 0$$. By (2.4), (2.6), (2.10) and the inequality

$$W^*_2(g_1 - g_2, h) \leq \| \int_0^{h/2} \Delta^1_u \Delta^1_u (g_1 - g_2)(u) \chi_h(u) du \| \leq \| D(g_1 - g_2) \| \frac{2}{h} \int_0^{h/2} u du = \frac{h}{4} \| D(g_1 - g_2) \|$$

we obtain

$$W^*_2(f, h) \leq W^*_2(f - g_1, h) + W^*_2(g_1 - g_2, h) + W^*_2(g_2, h) \leq 2 \tilde{K}_2 \left( f, \frac{h}{8}, \frac{h}{4\sqrt{3}} \right).$$
Consider the inverse estimate. By the definition of \( \tilde{K}_2(f, h_1, h_2) \), we have
\[
\tilde{K}_2 \left( f, \frac{h}{8}, \frac{h}{4\sqrt{3}} \right) \leq \| f - g_1 \| + \frac{h}{8} \| D(g_1 - g_2) \| + \frac{h^2}{48} \| D^2 g_2 \| , \tag{3. 24}
\]
where
\[
g_1(t) = \frac{24}{h^2} \int_0^{h/2} (f * \chi(u))(t) u^2 \chi(u) du,
\]
\[
g_2(t) = \frac{24}{h^2} \int_0^{h/2} (f * \chi^2(u))(t) u^2 \chi(u) du.
\]
We can estimate the norm of \( D(g_1 - g_2) \). From the inequality
\[
\| D(f * \chi - f * \chi^2) \| \leq 2u^{-1}W_2^*(f, \chi)
\]
we conclude that
\[
\| D(g_1 - g_2) \| \leq \frac{48}{h^2} W_2^*(f, \chi) \int_0^{h/2} u \chi(u) du = \frac{6}{h} W_2^*(f, \chi).
\]
Therefore, by (3. 24), (3. 21), (3. 22) and the previous inequality, we deduce
\[
\tilde{K}_2 \left( f, \frac{h}{8}, \frac{h}{4\sqrt{3}} \right) \leq W_2^*(f, \chi) + \frac{3}{4} W_2^*(f, \chi) + \frac{1}{2} W_2^*(f, \chi) = \frac{9}{4} W_2^*(f, \chi). \quad \Box
\]

4. Jackson’s theorem with the special modulus of continuity

To prove Jackson’s theorem, we will use a method that dates back to Steklov[12, 13] and Favard[8]. A new element here is our intermediate function \( g \).

**Theorem 3.** Let \( f \in C(T) \), \( \alpha > 0 \), \( c(\alpha) = 1 + 3/(2\alpha^2) \). Then for \( n \in \mathbb{N} \)
\[
E_{n-1}(f) \leq c(\alpha) W_2^*(f, \chi_{\alpha/(2n)}), \quad (4. 25)
\]
\[
E_{n-1}(f) \leq 2c(\alpha) W_2^*(f, \chi_{\alpha/(2n)}). \quad (4. 26)
\]

**Proof.** Firstly, we prove the inequality (4. 25). As an intermediate approximation we will use
\[
g(t) = \frac{12}{h^2} \int_0^h (f * \chi^2(u))(t) u^2 \chi^2(u) du, \quad h > 0.
\]
We can write the function \( f \) as
\[
f = f - g + g,
\]
and obtain
\[
E_{n-1}(f) \leq \| f - g \| + E_{n-1}(g).
\]
We have (see [3, 19])
\[
\| f - g \| \leq W_2^*(f, \chi^2).
\]
By using the Favard inequality
\[
E_{n-1}(g) \leq \frac{1}{32n^2} \| D^2 g \| \quad (4. 27)
\]
and (3.18), we see that
\[ E_{n-1}(g) \leq \frac{1}{32n^2} \|D^2g\| \leq \frac{3}{8(nh)^2} W_2^*(f, \chi_h^2). \]

By choosing \( h = \alpha/(2n) \), \( \alpha > 0 \), \( n \in \mathbb{N} \) we finally obtain
\[ E_{n-1}(f) \leq \left( 1 + \frac{3}{2\alpha^2} \right) W_2^*(f, \chi_{\alpha/(2n)}^2). \]

The inequality (4.26) follows from Corollary 2 and the inequality (4.25).

The estimates of Theorem 3 are valid for \( \alpha > 0 \). The inequality (4.26) complements the inequality (1.1), which is valid for \( \alpha > 2/\pi \). Note that the constant in (4.26) is better than in (1.1) for \( \alpha \leq 0.778 \).

For the classical modulus of smoothness \( \omega_2 \) the standard Steklov–Favard approach (see for example [9, Theorem 9.2]) gives
\[ E_{n-1}(f) \leq \left( \frac{1}{2} + \frac{1}{8\alpha^2} \right) \omega_2^2(f, \alpha/(2n)), \quad \alpha > 0. \quad (4.28) \]

The inequality (4.28) is sharp for \( \alpha = 1/(2j) \), \( j = 1, 2, \ldots \) (see [10]). The estimates (4.28) and \( \omega_2(f, h) \leq h^2 \|D^2f\| \) imply the weak form of the Favard inequality,
\[ E_{n-1}(f) \leq \frac{1}{32n^2} (1 + 4\alpha^2) \|D^2f\|. \]

The value \( 4\alpha^2 \) is a measure of the distance between the sharp difference estimate (4.28) and the sharp differential estimate (4.27).

By the inequalities
\[ W_2^*(f, \chi_h) \leq \frac{h^2}{24} \|D^2f\|, \quad W_2^*(f, \chi_h^2) \leq \frac{h^2}{12} \|D^2f\|, \]

and (4.25), (4.26) we have
\[ E_{n-1}(f) \leq \frac{1}{32n^2} \left( 1 + \frac{2\alpha^2}{3} \right) \|D^2f\|. \]

Probably, by using sharp estimates in Theorem 3, for small \( \alpha \), one can reduce the \( \alpha^2 \)-coefficient \( 2/3 \) and obtain the better correspondence between differential and difference estimates in the direct theorems of approximation theory.

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