Textures of Spin-Orbit Coupled $F=2$ Spinor Bose Einstein Condensates

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Abstract. Motivated by recent studies of Bose Einstein condensates (BECs) under synthetic gauge fields, we study the textures of $F=2$ BECs with a Rashba-like spin-orbit coupling (SOC). By analyzing the SOC term, we demonstrate that the SOC favors a helical modulation of order parameters (OPs). In addition, we find the stable textures obtained by the numerical minimization of the Gross-Pitaevskii energy functional and interpret them as a one- or two-dimensional helical modulation. In the parameter region where the polar OP is favored, the emergence of the skyrmion of the uniaxial polar OP and its energetics relative to one-dimensional helical modulation of biaxial polar OP are discussed in detail. Through the comparison of the lattice texture in the cyclic favored parameter region and uniaxial polar skyrmion, we show the instability of the lattice of the uniaxial polar skyrmion.

1. Introduction

Since the recent success of generating gauge fields in neutral cold atoms [1], much attention has been paid to spin-orbit coupled Bose Einstein condensates (BECs). The synthetic gauge field technique enables to realize the quantum Hall effect and the novel textures of order parameter (OP) in cold atoms.

The synthetic gauge field technique is based on the Berry phase in systems with $N$-fold degenerate levels. The vector potential is generated by the adiabatic motion of the degenerate eigenstate of the laser-atom interaction [1, 2]. These eigenstates play a role of the internal degree of freedom, that is, pseudospins. Different from a “natural” gauge field, one can adjust the form of the pseudospin state and the vector potential by tuning the laser configuration. The non-Abelian gauge field couples the angular momentum of the pseudospin with the orbital one. That coupling generates the nontrivial textures, which can not be compensated by the $U(N)$ gauge transformation. In the $F=1$ and $F=1/2$ BECs, the possible textures are the plane and standing wave, which are stable in the parameter region favoring the ferromagnetic (FM) and polar phase [3]. On the other hand, the textures of the $F=2$ BEC have more variety because of the existence of the two kinds of polar phases, the biaxial polar (BP) and uniaxial polar (UP) phase, and the cyclic (CY) phase.

In this paper, we present how textural structures can emerge in $F=2$ spinor BECs. In particular, we focus on the skyrmion texture of the UP OP as the simplest case of the two-dimensional textures. Their energetics relative to the one-dimensional texture of the BP state is discussed in detail.
2. Helical modulation

In this section, we introduce our analytical argument which is the basis for our interpretation of the numerical calculation. We consider a zero temperature $F=2$ spinor BEC with a SOC. The single particle Hamiltonian including the Rashba-like SOC term is described as

\[
H_0 = \int d^2r \bar{\Psi}(r) \left( i \hbar \nabla - \frac{\hbar^2}{2m} \hat{M}_S \cdot \nabla \right) \Psi(r), \quad \hat{M}_S = \begin{bmatrix}
0 & e_- & 0 & 0 & 0 \\
e_+ & 0 & \beta e_- & 0 & 0 \\
0 & \beta e_+ & 0 & \beta e_- & 0 \\
0 & 0 & \beta e_+ & 0 & e_- \\
0 & 0 & 0 & e_+ & 0
\end{bmatrix},
\]

where, $\hat{M}_S$ consists of the kinetic energy and the trap potential term, and $\Psi(x) = \begin{bmatrix} \psi_2, \psi_1, \psi_0, \psi_{-1}, \psi_{-2} \end{bmatrix}^T$ is the OP vector in a $F=2$ BEC, $h_0 = -\nabla^2/2m + V_{\text{pot}}(r)$ consists of the kinetic energy and the trap potential term, and $e_{\pm} = \hat{z} \pm i\hat{y}$. The SOC term is generalized by the parameter $\beta$. The SO introduced by hexapod setup [2] and the precise Rashba-type SOC correspond to $\beta = 1$ and $\beta = \sqrt{6}/2$, respectively. By using the simplified OP $\Psi(r) = R[k \cdot r, \hat{n}_R] \Psi_1(r_0)$, one can obtain the SOC energy density:

\[
h_{\text{SOC}}(k, \hat{n}_R, \Psi_1) = -\frac{\hbar^2}{16m} \bar{\Psi}_1^T \hat{M}'_{\text{SO}} \cdot k \Psi_1
\]

where, $\hat{M}'_{\text{SO}} = \begin{bmatrix}
4A\hat{n}_R & -i\omega^{-1}\hat{A}_{\perp} & 0 & i\omega^3\hat{C}_{\perp} & 0 \\
2i\omega\hat{A}_{\perp} & 4\hat{n}_R & 0 & 0 & 2i\omega^{-3}\hat{C}_{\perp} \\
\sqrt{6}\omega^2\hat{C}_{\perp} & i\omega\hat{B}_{\perp} & 0 & i\omega^{-1}\hat{B}_{\perp} & \sqrt{6}\omega^{-2}\hat{C}_{\perp} \\
2i\omega^3\hat{C}_{\perp} & 0 & 0 & 4\hat{n}_R & 2i\omega^{-1}\hat{A}_{\perp} \\
0 & i\omega^{-3}\hat{C}_{\perp} & 0 & i\omega^{-1}\hat{A}_{\perp} & 4A\hat{n}_R
\end{bmatrix}, \quad \omega = \exp[ik \cdot r]
\]

where, $\hat{R}(\theta, \hat{n})$ denotes the rotation matrix with the angle $\theta$ about $\hat{n}$ in the pseudospin space, $A = (1 + \sqrt{6}/2), B = \sqrt{3}/2(3 + \sqrt{6}/2), C = (3 - \sqrt{6}/2), \Psi_1 = \hat{R}(\pi/3, \hat{e}_{111}) \Psi_1, \hat{e}_{111} = \hat{z} + i\hat{y} + \hat{z}$, and $\hat{n}_R = \hat{n}_R \times \hat{z}$. The unit vectors $\hat{n}_R$ and $\hat{k}$ are the rotation axis and the modulation vector in the $xy$-plane, respectively. $\Psi_1$ is an arbitrary OP vector at a certain point $r_0$.

The diagonal elements of $\hat{M}'_{\text{SO}}$ in Eq. (2) are found to be energetically dominant, compared to the off-diagonal elements which have the oscillatory factor $\omega^n (n \neq 0)$. The dominant elements are proportional to $k \cdot \hat{n}_R$. Therefore, the SOC favors the helical modulation where the rotation axis $\hat{n}_R$ corresponds to the modulation vector $\hat{k}$. In addition, this observation is independent of the details of SOC term $\beta$ and the configuration of OP $\Psi_1$.

The same arguments are also applicable to spin-orbit coupled spinor BECs with pseudospin $F$. Actually, in terms of the helical modulation, the plain (standing) wave texture of $F=1$ BECs [3] is interpreted as the one-dimensional helical modulation of the FM (polar) OP.

3. Numerical results

In this section, we present the numerical results by minimizing the full Gross-Pitaevskii (GP) energy functional $H_0 + H_{\text{int}}$. We note that our results in this section are obtained by setting $\beta = 1$ (hexapod setup) although they are independent of $\beta$ qualitatively. Here, we assume that the atoms in pseudospin states interact through the most symmetric interaction:

\[
H_{\text{int}} = \frac{1}{2} \int d^2r \left[ c_0 n^2 + c_1 \mathbf{S} \cdot \mathbf{S} + c_2 |A_{20}|^2 \right],
\]

where $n = \bar{\Psi}^\dagger \Psi$, $\mathbf{S} = \bar{\Psi}^\dagger \sigma \Psi$, and $A_{20} = (2\psi_2 \psi_{-2} - 2\psi_1 \psi_{-1} + \psi_0 \psi_0) / \sqrt{5}$ are the particle density, the spin density, and the singlet pair amplitude, respectively [4, 5]. The purpose of using this assumption is to concentrate on the symmetry breaking due to the SOC term. In the absence of the SOC, $H_{\text{int}}$ leads to OP of four magnetic ground states: FM, BP, UP, and CY OPs [4].
The OP configurations in the spin space have their favorable rotation axis \( \hat{n}_R \) to reduce the energy by the Rashba-like SOC. For example, the OPs of the FM and BP state have a single favorable axis. That axis of FM OP corresponds to the magnetization vector and that of BP OP is the direction shown in figure 1. Therefore, the FM and BP OPs form the helical texture, which consists of the one-dimensional helical modulation. Actually, in our numerical calculation the resulting OPs can be simplified to \( \Phi(r) = \hat{R}[k, \hat{y}]\hat{\Psi}_{FM(BP)} \) with \( \hat{\Psi}_{FM} = \hat{R}^{-1}(\pi/3, \hat{e}_{111})[1, 0, 0, 0, 0]^T \) and \( \hat{\Psi}_{BP} = \hat{R}^{-1}(\pi/3, \hat{e}_{111})[1, 0, 0, 0, 1]^T \). These textures correspond to the plane and standing wave textures of \( F=1 \) BECs [3].

In contrast, the favorable axis of UP OP \( \hat{\Psi}_{UP} = [0, 0, 1, 0, 0]^T \) shown in figure 1(b) is arbitrary in the \( xy \)-plane. Thus the favorable OP in the UP state reduces to \( \hat{\Psi}(r) = \hat{R}[k \cdot \hat{r}, k] \hat{\Psi}_{UP} \). Due to the nonuniqueness of \( k \) the radial helical modulation \( k = k\hat{r} \) becomes the most favorable modulation and the texture results in the skyrmion of UP OP shown in figure 1(c). The UP-skyrmion is the simplest case of the two-dimensional texture arising from the Rashba-like SOC.

Here, we argue the energetics of the BP-helical and UP-skyrmion texture in the parameter region \( c_1/c_0 > 0 \) and \( c_2/c_0 < 0 \), where the polar OP is favored. The relative energy of them as a function of the spin-orbit coupling constant \( \kappa \) is shown in figure 2. The stable region of the UP-skyrmion is \( 1/\kappa \gtrsim R_{TF} \) where \( R_{TF} \) is the Thomas-Fermi radius. These results are interpreted as follows. In the center of the skyrmion \( U_0 \) in figure 1(c), from which the multiple modulation vectors start, the integration of the contribution of all the helical modulation reduces more energy than the helical texture of the BP OP. With distance from the center \( U_0 \), the OPs have the unique rotation axis \( \hat{n}_R \parallel \hat{r} \), and hence, their helical modulations correspond to one-dimensional modulation. In addition, the skyrmion yields the non-helical modulation, which has the rotation axis \( \hat{n}_R \parallel \hat{z} \) and modulation vector \( k \parallel \hat{z} \times \hat{r} \). This modulation does not reduce the SOC energy in (1). In the region \( 1/\kappa \gtrsim R_{TF} \), the contribution of the center of the skyrmion is dominant relatively, so that the UP-skyrmion is more stable than BP-helical texture. Except for the small system size, the SOC in the UP state may stabilize the two-dimensional lattice of UP-skyrmion with a period \( R = \pi/2k \). However, the lattice is unstable in our calculation because the radius of the unit cell is too large.

In the parameter region \( c_1/c_0 > 0 \) and \( c_2/c_0 > 0 \), where the CY OP is favored, lattice textures are possible to be solution of GP equation as the absolute minimum energy [6, 7]. One of them is the UP/CY lattice shown in figure 3. As shown in figure 3, UP-skyrmion
Figure 3. (Color online) The CY/UP lattice texture: (a) The order parameter, and (b) singlet trio amplitude $A_{30}(r)$. The parameters are set to be $\kappa R_{TF} = 10.1$ and $c_1/c_0 = c_2/c_0 = 0.2$.

texture realize near the center of the unit cell, $U_0$. However, on the path from the center $U_0$ to the edge $P_6$ in figure 3(a), with the continuous change to the CY OP, the OP rotates in the pseudospin space by $\theta_1 = \arccos(1/\sqrt{3})$ implying that this unit cell is smaller than that of UP-skyrmion. Figure 3(b) shows that the cyclic order characterized with the singlet trio amplitude $|A_{30}|^2/n^3 = 2$ forms the honeycomb network structure. The singlet trio amplitude is defined as $A_{30} = 3\sqrt{6}/2(\psi_1^2\psi_{-2} + \psi_{-1}^2\psi_2) + \psi_0(\psi_0^2 - 3\psi_1\psi_{-1} - 6\psi_2\psi_{-2})$. As shown in figure 3(a), these networks are connected by the helical modulation, and also reduce GP energy $H_0 + H_{\text{int}}$ by the SOC. For these reasons, the lattice texture shown in figure 3 becomes stable.

In this parameter region, the Rashba-like SOC also forms the 1/3-vortex lattice texture by the networks of the helical modulation in the parameter region $c_2 \gg c_1$. More detailed arguments of the emergence and energetics of lattice texture in the CY favorable region are given in [7].

4. Summary
In this paper, we have studied the textures of $F=2$ spinor BEC with a spin-orbit coupling (SOC). In particular, we demonstrate the detailed discussion of the energetics of the textures in the parameter region where the polar order parameter (OP) is favored. The helical modulation of the OP has been introduced by analyzing the SOC term of Gross-Pitaevskii (GP) energy functional. We obtain the stable textures by minimizing the full GP energy functional numerically. We find the following remarks. The OP of the ferromagnetic and biaxial polar state consists of the one-dimensional helical modulation texture and that of uniaxial polar (UP) state yields the skyrmion texture. The former texture originates from the single rotation axis of OP in the pseudospin space and latter is from the nonuniqueness of the favorable axis. In the parameter region where the polar state is favorable, the skyrmion is stable in the small size of the system within $1/\kappa \lesssim R_{TF}$. The UP skyrmion can not form the lattice structure because of the large size of the unit cells and the non-helical modulation encircles them. The lattice textures which have small size and are encircled by helical modulation realize in the cyclic stable region.

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