HOT TOPICS IN COLD GASES

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ABSTRACT. Since the first experimental realization of Bose-Einstein condensation in cold atomic gases in 1995 there has been a surge of activity in this field. Ingenious experiments have allowed us to probe matter close to zero temperature and reveal some of the fascinating effects quantum mechanics has bestowed on nature. It is a challenge for mathematical physicists to understand these various phenomena from first principles, that is, starting from the underlying many-body Schrödinger equation. Recent progress in this direction concerns mainly equilibrium properties of dilute, cold quantum gases. We shall explain some of the results in this article, and describe the mathematics involved in understanding these phenomena. Topics include the ground state energy and the free energy at positive temperature, the effect of interparticle interaction on the critical temperature for Bose-Einstein condensation, as well as the occurrence of superfluidity and quantized vortices in rapidly rotating gases.

1. Introduction

Bose-Einstein Condensation (BEC) was first experimentally realized in cold atomic gases in 1995 [2, 10]. In these experiments, a large number of (bosonic) atoms is confined to a trap and cooled to very low temperatures. Below a certain critical temperature condensation of a large fraction of particles into the same one-particle state occurs.

These Bose-Einstein condensates display various interesting quantum phenomena, like superfluidity and the appearance of quantized vortices in rotating traps, effective lower dimensional behavior in strongly elongated traps, etc. We refer to the review articles [9, 5, 8, 15] for an overview of the state-of-the-art of this subject and a list of references to the original literature.

BEC was predicted by Einstein in 1924 [13] from considerations of the non-interacting Bose gas, extending the work of Bose [7] to massive particles. The presence of particle interactions represents a major difficulty for a rigorous derivation of this phenomenon, however, as we shall discuss below.

1.1. The Bose Gas: A Quantum Many-Body Problem. The quantum-mechanical description of the Bose gas is given in terms of its Hamiltonian.
For a gas of $N$ bosons confined to a region $\Lambda \in \mathbb{R}^3$, and interacting via a repulsive pair-interaction potential $v$, it is given by
\[
H = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leq i < j \leq N} v(\vec{x}_i - \vec{x}_j)
\] (1)

The kinetic energy of the particles is described by the Laplacian $\Delta = \nabla^2$, and we choose Dirichlet boundary conditions on $\partial \Lambda$ for concreteness. Other boundary conditions could be used as well. The subscript $i$ stands for the action in the $i$th particle coordinate $\vec{x}_i \in \mathbb{R}^3$. Units are chosen such that $\hbar = 2m = 1$, with $m$ the mass of the particles.

The Hamiltonian $H$ acts on the Hilbert space of permutation-symmetric wave functions $\Psi \in \bigotimes^N L^2(\mathbb{R}^3)$, as appropriate for bosons; i.e., square-integrable functions of $N$ variables $\vec{x}_i \in \mathbb{R}^3$ satisfying $\psi(\vec{x}_1, \ldots, \vec{x}_N) = \psi(\vec{x}_{\pi(1)}, \ldots, \vec{x}_{\pi(N)})$ for any permutation $\pi$ of $(1, 2, \ldots, N)$.

In the following, the interaction $v$ will be assumed to be radial and non-negative. Moreover, it is sufficiently short range as to have a finite scattering length, which means that it is integrable outside some compact set. No other regularity assumptions will be made. In particular, $v$ is allowed to have a hard core, which reduces the domain of definition of $H$ to those functions $\Psi$ that vanish whenever the distance between a pair of particle coordinates is smaller than the hard-sphere radius. A particular example of an interaction potential to keep in mind are pure hard spheres where, formally, $v(\vec{x}) = \infty$ for $|\vec{x}| \leq a$, and $v(\vec{x}) = 0$ for $|\vec{x}| > a$.

The present setup can be easily generalized to describe inhomogeneous systems in a trap. One simply adds trap potential
\[
\sum_{i=1}^{N} V(\vec{x}_i)
\]

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\]
to $H$, where $V$ is a real-valued, locally bounded function with $\lim_{|\vec{x}| \to \infty} V(\vec{x}) = \infty$. The latter condition guarantees that the particles are confined to the trap, even in case $\Lambda = \mathbb{R}^3$.

Similarly, rotating systems can be described adding the term
\[
\sum_{i=1}^{N} \vec{\Omega} \cdot \vec{L}_i
\]

to the Hamiltonian $H$, with $\vec{\Omega} \in \mathbb{R}^3$ being the angular velocity and $\vec{L} = -i\vec{x} \wedge \nabla$ the angular momentum operator. This term results from a transformation to the rotating frame of reference.
1.2. Quantities of Interest. In the following, we shall distinguish two types of questions that can be asked concerning the behavior of Bose gases described by the Hamiltonian (1) above.

- **Thermodynamic quantities**, like the ground state energy per unit volume, or the free energy density at positive temperature. Here one considers homogeneous systems and is interested in the thermodynamic limit \( N \to \infty, \Lambda \to \mathbb{R}^3 \) with the particle density \( \varrho = N/|\Lambda| \) fixed.

  Of particular interest is the notion of Bose-Einstein condensation, which concerns off-diagonal long-range order in the one-particle density matrix \( \langle a^\dagger(x)a(y) \rangle \), and is expected to occur below a critical temperature.

- **Behavior of trapped systems** in the ground state. One observes interesting quantum phenomena, like effective one-dimensional behavior in strongly elongated traps, vortices in rotating systems, a bosonic analogue of the fractional quantum Hall effect in rapidly rotating gases, etc.

  Of particular relevance is the **Gross-Pitaevskii** scaling, where the ratio of the scattering length \( a \) to the diameter of the trap is \( O(N^{-1}) \).

We shall discuss our current knowledge about answers to these questions, as far as mathematical physics is concerned, in the following sections.

2. Homogeneous Systems in the Thermodynamic Limit

2.1. The Ground State Energy of Homogeneous Bose Gases. Consider first the case of a homogeneous system in the absence of a trapping potential or rotation. The ground state energy density in the thermodynamic limit is given by

\[
e(\varrho) = \lim_{\Lambda \to \mathbb{R}^3, N/|\Lambda| \to \varrho} \frac{1}{|\Lambda|} \inf \text{spec } H
\]

(2)

with \( H \) as in (1). The existence of this thermodynamic limit is well understood for appropriate sequences of domains \( \Lambda \) approaching \( \mathbb{R}^3 \). See, e.g., Ruelle’s book [34].

We will be particularly interested in the limit of low density, when the gas is dilute in the sense that \( a^3 \varrho \ll 1 \), where \( a \) denotes the scattering length of the interaction potential \( v \). It is defined as

\[
4\pi a = \inf \left\{ \int_{\mathbb{R}^3} \left( |\nabla \phi(|\vec{x}|)|^2 + \frac{1}{2}v(\vec{x})\phi(|\vec{x}|)^2 \right) d\vec{x} : \phi \geq 0, \lim_{r \to \infty} \phi(r) = 1 \right\}.
\]

(3)

For bosons at low density, one expects that

\[
e(\varrho) \approx 4\pi a \varrho^2.
\]

(4)
This formula is suggested by considering the ground state energy of two bosons in a large region Λ, which is $8\pi a/|\Lambda|$, as can be easily deduced from (3). Multiplying this by the number of pairs of bosons, $N(N-1)/2$, one arrives at (4). That this simple heuristics is correct is far from obvious, however. It fails for two-dimensional systems, for instance [35, 29].

The investigation of the ground state energy density $e(\varrho)$ goes back to Bogoliubov [6] in the 40s, and Lee, Huang and Yang in the 50s [19]. Dyson [11] computed a rigorous upper bound that shows the correct leading order asymptotics (4) for hard spheres, but his lower bound was 14 times too small. His upper bound was later generalized to arbitrary repulsive interaction potentials in [24]. The correct lower bound was proved only in 1998 by Lieb and Yngvason [28]. We formulate this result as a theorem.

\textbf{Theorem 1 (Bosons at $T = 0$).} As $\varrho \to 0$,

\begin{equation}
\boxed{e(\varrho) = 4\pi a\varrho^2 + o(\varrho^2)}
\end{equation}

Note that if one treats the interaction energy as a perturbation of the kinetic energy, naive perturbation theory would yield $\frac{1}{2} \int v$ instead of $4\pi a$. This is always too big, as (3) shows, and would even be infinite for hard spheres. In fact, the result (5) is non-perturbative in the sense that the scattering length $a$ contains terms to arbitrary high order in the interaction potential $v$.

It remains an open problem to establish the leading order correction to (4), which is expected to be given by the Lee-Huang-Yang formula [19]

\begin{equation}
e(\varrho) \approx 4\pi a\varrho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{a^3 \varrho}\right).
\end{equation}

Recent progress in this direction was made in [17] and [27], where it was shown that (6) holds for certain density-dependent and appropriately scaled interaction potentials. The general question remains open, however.

\textbf{2.2. Homogeneous Bose Gas at Positive Temperature.} At positive temperature $T > 0$, the appropriate quantity to consider is the free energy density, which is defined as

\begin{equation}
f(\varrho, T) = -T \lim_{\Lambda \to \mathbb{R}^3, N/|\Lambda| \to \varrho/|\Lambda|} \frac{1}{\ln \text{Tr} \exp(-H/T)}.
\end{equation}

For non-interacting bosons (i.e., $v \equiv 0$), it can be calculated explicitly. We denote it by $f_0(\varrho, T)$. It is given in terms of a Legendre transform as

\begin{equation}
f_0(\varrho, T) = \sup_{\mu < 0} \left[ \mu \varrho + \frac{T}{(2\pi)^3} \int_{\mathbb{R}^3} d\vec{p} \ln (1 - \exp(-\langle \vec{p}^2 - \mu \rangle/T)) \right].
\end{equation}

Note that it has the scaling property

\[ f_0(\varrho, T) = \varrho^{5/3} f_0(1, T \varrho^{-2/3}) \]
which follows from the fact that the only length scales in the problem are the mean particle spacing $\varrho^{-1/3}$ and the thermal wavelength $T^{-1/2}$, and hence $f_0$ depends, up to a prefactor, only on their ratio.

From (8) it is easy to see that $f_0$ is not an analytic function of $\varrho$ (or $T$), and hence even a non-interacting Bose gas shows a phase transition. This transition is known as Bose-Einstein condensation, and occurs at a critical density

$$\varrho_c(T) = \zeta(\frac{3}{2}) \left( \frac{T}{4\pi} \right)^{3/2},$$

where $\zeta$ denotes the Riemann zeta-function. In fact, $\partial f_0(\varrho, T)/\partial \varrho = 0$ for $\varrho \geq \varrho_c$. For $\varrho > \varrho_c(T)$, $\varrho - \varrho_c(T)$ is interpreted as the density of the Bose-Einstein condensate.

For interacting gases, there are now three length scales to consider: the interaction range $a$, the mean particle distance $\varrho^{-1/3}$, and the thermal wavelength $T^{-1/2}$. For dilute systems, one considers the case $a \ll \varrho^{-1/3} \sim T^{-1/2}$.

In this regime, the free energy turns out to be the given by the following expression.

**Theorem 2 (Bosons at $T > 0$).** For $a^3 \varrho \ll 1$ we have

$$f(\varrho, T) = f_0(\varrho, T) + 4\pi a \left( 2\varrho^2 - [\varrho - \varrho_c(T)]^2_+ \right) + o(\varrho^2) \quad (9)$$

where $[t]_+ = \max\{t, 0\}$ denotes the positive part.

The lower bound in (9) was proved in [38]. An upper bound for smooth interacting potentials of rapid decay was later obtained in [40], the more general case being still open.

The error term in (9) is uniform in $T/\varrho^{2/3}$ for bounded $T/\varrho^{2/3}$, corresponding to the quantum regime. For $T/\varrho^{2/3} \to \infty$ one obtains a classical gas, whereas for $T/\varrho^{2/3} \to 0$ the system approaches the ground state.

Note that for $\varrho < \varrho_c(T)$, the leading order correction compared to the ideal Bose gas is $8\pi a \varrho^2$ instead of the $4\pi a \varrho^2$ at zero temperature. The additional factor 2 is a result of the symmetry requirements of the wave functions and can be interpreted as an exchange term; this symmetrization applies only to particles outside the condensate, however, and this explains the subtraction of the square of the condensate density in (9). We also remark that without restricting to symmetric functions, the leading order correction compared with an ideal gas would be $4\pi a \varrho^2$ at any $T > 0$, just like at $T = 0$.

The proof of Theorem 2 is long and technical and hence can not be reproduced here. One of the key issues to understand is a certain separation of energy scales in the two terms on the right side of (9). In momentum space, these are
• large momenta $|\vec{p}| \sim 1/a$ responsible for scattering of two particles at a distance $\sim a$ of each other.
• low momenta $|\vec{p}| \sim T^{1/2} \ll 1/a$, responsible for the thermal distribution distribution of the particles’ kinetic energy.
• Bose-Einstein condensation at momentum $\vec{p} = 0$.

2.3. Critical Temperature for BEC. As discussed above, the ideal, non-interacting Bose gas displays a phase transition above a critical density. Equivalently, BEC in the ideal gas occurs below the critical temperature

$$T_c(\rho) = \frac{4\pi}{\zeta(3/2)^{2/3}} \rho^{2/3}.$$ 

A useful characterization of BEC, applicable also for interacting systems, is in terms of the one-particle density matrix of the system. This density matrix is defined as

$$\gamma = N \frac{1}{\text{Tr} e^{-H/T}} \text{Tr}(N^{-1}) e^{-H/T}$$

where $\text{Tr} N^{-1}$ stands for the partial trace over $N - 1$ particle coordinates. Hence $\gamma$ is an operator on the one-particle space $L^2(\mathbb{R}^3)$. Obviously $\gamma \geq 0$ and $\text{Tr} \gamma = N$, by definition. BEC is characterized by the fact that, in the thermodynamic limit, the integral kernel $\gamma(\vec{x}, \vec{y})$ of $\gamma$ does not vanish as $|\vec{x} - \vec{y}| \to \infty$. This is also referred to as off-diagonal long range order. For non-interacting bosons, one can show that

$$\gamma(\vec{x}, \vec{y}) = [\rho - \rho_c(T)]_+ + \sum_{n \geq 0} \frac{e^{\bar{\mu} n / T}}{(4\pi n / T)^{3/2}} e^{-T|x-y|^2/(4n)}$$

in the thermodynamic limit, with $[\bar{t}]_+ = \max\{\bar{t}, 0\}$ denoting the positive part, and $\bar{\mu} \leq 0$ the $\mu$ where the maximum in (8) is achieved. Hence the kernel $\gamma(\vec{x}, \vec{y})$ has the following characteristics:

• For $T < T_c(\rho)$, $\gamma(\vec{x}, \vec{y})$ does not decay. In fact, $\lim_{|\vec{x} - \vec{y}| \to \infty} \gamma(\vec{x}, \vec{y}) = \rho - \rho_c(T)$, the condensate density.
• For $T > T_c(\rho)$, $\gamma(\vec{x}, \vec{y})$ decays exponentially, like $e^{-\sqrt{-\bar{\mu}}|\vec{x} - \vec{y}|}$
• For $T = T_c(\rho)$, $\gamma(\vec{x}, \vec{y})$ decays algebraically. In fact, $\gamma(\vec{x}, \vec{y}) \sim |\vec{x} - \vec{y}|^{-1}$ in this case.

These features are expected to hold also for interacting Bose gases, although with a different value of the critical temperature $T_c(\rho)$. It is still an open problem to prove the existence of BEC for interacting gases, however. The only known case where BEC has been proved is the hard-core lattice gas at half filling [12], which is equivalent to the XY spin model.[31]

Although there is no proof that $T_c \neq 0$ in the interacting case, an upper bound can be derived rigorously [39].
Theorem 3 (Upper bound on $T_c$). For small $a^3 \rho$ and some $c > 0$,

$$\frac{T_c - T_c^{(0)}}{T_c^{(0)}} \leq c \sqrt{a \rho^{1/3}}$$

where $T_c^{(0)} = \frac{4\pi}{\zeta(3/2)^{2/3}} \rho^{2/3}$ is the critical temperature for the ideal Bose gas.

More precisely, it is shown in [39] that $\gamma(\vec{x}, \vec{y})$ decays exponentially if $T > T_c^{(0)}(1 + c \sqrt{a \rho^{1/3}})$. The proof uses a well-known Feynman-Kac representation of the partition function in terms of integrals over paths and sums over cycles in permutations [16].

There seems to be still no consensus in the physics literature concerning the correct power of the exponent of $a \rho^{1/3}$ in the shift in critical temperature, or even the sign of $c$. Recent numerical simulations suggest that the shift should be linear in $a \rho^{1/3}$, with a positive $c$. This expected behavior of $T_c(\rho)$, as well as the upper bound of Theorem [3] are sketched in Figure 1.

![Figure 1](image_url)

**Figure 1.** The red line shows the rigorous upper bound on the critical temperature for BEC. The dashed line corresponds to the expected behavior based on numerical simulations.

3. Trapped Bose Gases

In the previous chapter we considered homogeneous Bose gases in the thermodynamic limit. Recent experiments with cold atoms consider inhomogeneous gases in traps, however. That is, one can take $\Lambda$ to be the whole of $\mathbb{R}^3$, but adds a trap potential $\sum_{i=1}^N V(\vec{x}_i)$ to the Hamiltonian [11]. A typical example, which describes the experimental situation rather well, is
a harmonic oscillator potential \( V(\vec{x}) = \omega^2|\vec{x}|^2 \), with \( \omega > 0 \) the trap frequency. More generally, the trapping frequencies in the three directions can be different, of course.

A characteristic feature of these trapped gases is their response to rotation. One observes the appearance of quantized vortices \([30, 14]\), whose number increases with the rotation speed \(|\vec{\Omega}|\).

Even a rotating Bose gas can be described in a time-independent way, by going to the rotating reference frame. The only effect on the Hamiltonian is to add the term \( \sum_{i=1}^{N} \vec{\Omega} \cdot \vec{L}_i \), as discussed in the Introduction. To ensure stability of the system, the trap potential \( V \) has to increase fast enough at infinity to compensate for the centrifugal force in the rotating system. More precisely, we have to assume that

\[
\lim_{|\vec{x}| \to \infty} \left( V(\vec{x}) - \frac{1}{4}|\vec{\Omega} \wedge \vec{x}|^2 \right) = +\infty.
\]

3.1. The Gross-Pitaevskii Equation. The previous considerations suggest that dilute Bose gases close to zero temperature should be well described by the Gross-Pitaevskii (GP) energy functional \([18, 33]\)

\[
E^{\text{GP}}[\phi] = \left\langle \phi \left| -\Delta + V(\vec{x}) - \vec{\Omega} \cdot \vec{L} \right| \phi \right\rangle + 4\pi Na \int_{\mathbb{R}^3} |\phi(\vec{x})|^4 d\vec{x}.
\]  

\[
(11)
\]

Its ground state energy is

\[
E^{\text{GP}}(Na, \vec{\Omega}) = \inf_{\|\phi\|_2=1} E^{\text{GP}}[\phi],
\]

and any minimizer satisfies the GP equation

\[
-\Delta \phi(\vec{x}) + V(\vec{x})\phi(\vec{x}) - \vec{\Omega} \cdot \vec{L} \phi(\vec{x}) + 8\pi Na|\phi(\vec{x})|^2 \phi(\vec{x}) = \mu \phi(\vec{x}).
\]

For a minimizer, \( N|\phi(\vec{x})|^2 \) is interpreted as the particle density of the system. Hence the last term in (11) is the natural generalization of the expression \( 4\pi a_0^2 \) to inhomogeneous systems.

For \( \vec{\Omega} \neq 0 \) and axially symmetric \( V(\vec{x}) \), the rotation symmetry can be broken due to the appearance of quantized vortices. More precisely, it was shown in \([36, 37]\) that for all trap potentials \( V(\vec{x}) \) that grow faster than quadratically at infinity, there exists a \( g_{\vec{\Omega}} \) such that for all \( Na > g_{\vec{\Omega}} \) the GP minimizers necessarily are not axially symmetric. In particular, there are many (in fact, uncountably many) GP minimizers! The symmetry breaking is due to the appearance of quantized vortices which can not be arranged in a symmetric way. Many interesting results have been obtained concerning the nature and distribution of these vortices in GP minimizers. We refer to \([1]\) and references therein.
3.2. **Ground State Energy of Dilute Trapped Gases.** In typical experiments on cold atomic gases, \( N \gg 1, \ a \ll 1 \) (the length scale of the trapping potential \( V \)), but \( Na = O(1) \). To get to this dilute regime, one writes

\[
v(\vec{x}) = \frac{1}{a^2} w(\vec{x}/a)
\]

with \( w \) having scattering length 1. It is easy to see that \( v(\vec{x}) \) then has scattering length \( a \). The scattering length thus becomes a parameter, and we can write

\[
\inf \text{spec } H = E_0(N,a,\vec{\Omega}).
\]

We note that the scaling (12) is of course equivalent to a rescaling of the trap potential \( V \) while keeping \( v \) fixed. This latter procedure may seem physically more natural (as the trap potential is easier to adjust experimentally than the interaction potential) but we find it more convenient to fix \( V \) instead and scale \( v \) as in (12) instead. Our procedure corresponds to measuring all lengths in the system in units of the length scale of the trap potential.

For dilute systems, one expects that \( E_0(N,a,\vec{\Omega}) \approx NE_{\text{GP}}(Na,\vec{\Omega}) \). The proof of this fact was given in [23].

**Theorem 4 (Ground State Energy of Trapped Gases).** For fixed \( g \geq 0 \) and \( \vec{\Omega} \in \mathbb{R}^3 \),

\[
\lim_{N \to \infty} \frac{E_0(N,g/N,\vec{\Omega})}{N} = E_{\text{GP}}(g,\vec{\Omega})
\]

This theorem was previously proved in [24] for the case \( \vec{\Omega} = 0 \). The main difficulty in the generalization to rotating systems comes from the fact that the permutation symmetry of the wave functions now becomes essential. While for non-rotating systems it is well known that the ground state for bosons coincides with the ground state without symmetry restrictions (as the latter is unique and positive, hence must be symmetric), this fact fails to hold for rotating systems. In fact one can show that (13) fails to hold, in general, if the left side is replaced by the absolute ground state energy of \( H \) (viewed as an operator on \( L^2(\mathbb{R}^{3N}) \), without symmetry restrictions). [37]

3.3. **BEC for Rotating Trapped Gases.** In the previous subsection it was argued that the ground state energy of the GP functional (11) is a good approximation to the ground state energy of \( H \) for dilute gases. For the corresponding ground state \( \Psi_0(\vec{x}_1,\ldots,\vec{x}_N) \), one would also expect that its one-particle reduced density matrix satisfies

\[
\gamma_0 \equiv \text{Tr}^{(N-1)}|\Psi_0\rangle\langle\Psi_0| \approx |\phi\rangle\langle\phi|
\]

with \( \phi \) a minimizer of the GP functional. (For convenience, the normalization of \( \gamma_0 \) has been chosen differently here than we did previously in (10).) While this is indeed true in the non-rotating case [22], the rotating case is more complicated because of the non-uniqueness of the GP minimizers \( \phi \). The best one can hope for is to replace the right side of (14) by a convex
combination of rank-one projections onto GP minimizers. This is indeed the content of Theorem 5 below, which was proved in [23].

To state the following results precisely, it is necessary to introduce the concept of an approximate ground state. We will call a sequence of $N$-particle density matrices (positive trace class operators on the $N$-particle space with trace equal to one) an approximate ground state if their energy equals the ground state energy to leading order in $N$. Then we define the set $\Gamma$ as the set of limit points of one-particle density matrices of such approximate ground states. More precisely,

$$\Gamma = \left\{ \gamma : \exists \text{ sequence } \gamma_N, \lim_{N \to \infty} \frac{1}{N} \text{Tr} H \gamma_N = E^{\text{GP}}(g, \vec{\Omega}), \lim_{N \to \infty} \gamma_N^{(1)} = \gamma \right\}$$

where $\gamma_N^{(1)} = \text{Tr}^{(N-1)} \gamma_N$ denotes the one-particle density matrix of $\gamma_N$.

**Theorem 5 (BEC for Dilute Trapped Gases).** The set $\Gamma$ in (13) has the following properties.

(i) $\Gamma \subset \mathcal{J}_1$ is compact and convex.

(ii) The extreme points $\Gamma_{\text{ext}} \subset \Gamma$ are given by GP minimizers, i.e., $\Gamma_{\text{ext}} = \{ |\phi\rangle\langle \phi| : E^{\text{GP}}[\phi] = E^{\text{GP}}(g, \vec{\Omega}) \}$. 

(iii) For every $\gamma \in \Gamma$ there exists a positive (regular Borel) measure $d\mu_\gamma$, supported in $\Gamma_{\text{ext}}$ with $\int_{\Gamma_{\text{ext}}} d\mu_\gamma(\phi) = 1$, such that

$$\gamma = \int_{\Gamma_{\text{ext}}} d\mu_\gamma(\phi) |\phi\rangle\langle \phi|.$$  

Eq. (16) is the natural generalization of (14) to the case of multiple GP minimizers. It says that the one-particle density matrix of any approximate ground state is close (in trace class norm) to the convex combination of projections onto GP minimizers.

Theorem 5 represents also a proof of the spontaneous breaking of the rotation symmetry in rotating Bose gases. An infinitesimal perturbation, e.g., of the trap potential $V$, leads to a unique GP minimizer and hence to 100% condensation, since the set $\Gamma$ consists of only one element in this case. The quantized vortices are visible in the GP minimizer; they are a typical feature of superfluids. Theorem 5 can therefore also be interpreted as a proof of the superfluid behavior of rotating Bose gases.[25]

### 3.4. Rapid Rotation

Consider now the special case of a harmonic trapping potential

$$V(\vec{x}) = \frac{1}{4} |\vec{x}|^2.$$  

As discussed above, $H$ is bounded below only for $|\vec{\Omega}| \leq 1$. The results in the previous subsections are valid for fixed $|\vec{\Omega}| < 1$. The question we would like to address in this final section is what happens as $|\vec{\Omega}| \to 1$? Denoting
\( \vec{e}_\Omega = \vec{\Omega}/|\vec{\Omega}| \) the unit vector in the direction of \( \vec{\Omega} \), we can write

\[-\Delta + \frac{1}{4}|\vec{x}|^2 - \vec{\Omega} \cdot \vec{L} = \left( -i\nabla - \frac{1}{2}\vec{e}_\Omega \wedge \vec{x} \right)^2 + \frac{1}{4}|\vec{e}_\Omega \cdot \vec{x}|^2 + (\vec{e}_\Omega - \vec{\Omega}) \cdot \vec{L}. \tag{17} \]

The operator \( h \) has eigenvalues \( \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots \), each of which is infinitely degenerate.

For low energies it makes sense to restrict the allowed wave functions to the kernel of \( h - \frac{3}{2} \). This kernel is given by the Bargmann space \([3]\)

\[ \{ f(z)e^{-|z|^2/4}, \ f : \mathbb{C} \to \mathbb{C} \text{ analytic} \} \subset L^2(\mathbb{R}^3) \tag{18} \]

where we identify the complex variable \( z \) with the plane perpendicular to \( \vec{\Omega} \). In particular, \( |\vec{x}|^2 = |z|^2 + |\vec{e}_\Omega \cdot \vec{x}|^2 \). Since the Gaussian factor is fixed, it is convenient to absorb it into the measure and think of the Hilbert space as a space of analytic functions only. It can easily be checked that the angular momentum operator \( \vec{e}_\Omega \cdot \vec{L} \) acts on \( f \) as \( z\partial_z \). In particular, its eigenfunctions are \( z^n \), with eigenvalues \( n \in \{0, 1, 2, \ldots \} \).

We note that if one interprets \( \vec{e}_\Omega \) as a homogeneous magnetic field, the Bargmann space \([18]\) corresponds to the lowest Landau level in the perpendicular direction, multiplied by a fixed Gaussian in the longitudinal direction.

Now that we have identified the Bargmann space \([18]\) as the appropriate one-particle Hilbert space for rapidly rotating bosons, we have to come up with an effective Hamiltonian describing this system. The only term left in the one-particle energy \([17]\), expect for a trivial factor \( \frac{3}{2} \), is the angular momentum term \( (\vec{e}_\Omega - \vec{\Omega}) \cdot \vec{L} \). If the range of the interaction potential is much shorter than the “magnetic length” \( 1 \), it makes sense to approximate the interaction potential by a \( \delta \)-function, which becomes a bounded operator when projected to the Bargmann space. Writing the prefactor of the \( \delta \)-function as \( 8\pi a \), in accordance with previous considerations, we arrive at the effective Hamiltonian

\[
H^{LLL} := (1 - |\vec{\Omega}|) \sum_{i=1}^{N} z_i \partial_{z_i} + 8\pi a \sum_{1 \leq i < j \leq N} \delta_{ij} . \tag{19}
\]

It acts on the space of permutation-symmetric analytic functions \( f(z_1, \ldots, z_N) \) which are square-integrable with respect to the measure \( \prod_{i=1}^{N} e^{-|z_i|^2/4}dz_i \). We denote this space by \( \mathcal{B}^{\otimes N} \). The operator \( \delta_{12} : \mathcal{B}^{\otimes 2} \to \mathcal{B}^{\otimes 2} \) acts as

\[
(\delta_{12} f) (z_1, z_2) = \frac{1}{(2\pi)^{3/2}} f \left( \frac{1}{2}(z_1 + z_2), \frac{1}{2}(z_1 + z_2) \right)
\]

which takes analytic functions into analytic functions. It is obtained by projecting \( \delta(\vec{x}_1 - \vec{x}_2) \) onto \( \mathcal{B}^{\otimes 2} \).

Concerning the effective Hamiltonian \([19]\), the following questions arise naturally:
(1) Can one derive $H^\text{LLL}$ rigorously from the full Hamiltonian $H$ as $|\vec{\Omega}| \to 1$ and $a \to 0$? Such a rigorous derivation was indeed achieved in [21], where it was shown that if $|\vec{\Omega}| \to 1$ for fixed $N$ and fixed $a/(1 - |\vec{\Omega}|)$ (i.e., also $a \to 0$), then the ratio of the ground state energy of $H$, minus the trivial term $\frac{3}{2}N$, to the ground state energy of $H^\text{LLL}$ goes to 1. Similarly, one can show that also eigenfunctions converge in the same limit. Uniformity in the particle number $N$ is still an open problem, however.

(2) What are the properties of $H^\text{LLL}$, in particular concerning its spectrum and corresponding eigenfunctions? Certain features of $H^\text{LLL}$ are expected to show some similarities to the fractional quantum Hall effect which occurs in fermionic systems.

Concerning the latter question, let us first note that the two terms $\mathcal{L}_N$ and $\Delta_N$ in $H^\text{LLL}$ commute:

$$H^\text{LLL} = (1 - |\vec{\Omega}|) \sum_{i=1}^{N} z_i \partial_{z_i} + 8\pi a \sum_{1 \leq i < j \leq N} \delta_{ij}.$$ 

Hence the ground state energy $E^\text{LLL}_0(N,a,\vec{\Omega}) = \inf \text{spec } H^\text{LLL}$ is obtained from the joint spectrum of these two operators. Of particular relevance is the yrast curve, which is defined as the lowest eigenvalue of $\Delta_N$ in the sector of total angular momentum $L$:

$$\Delta_N(L) = \inf \text{spec } \Delta_N |_{\mathcal{L}_N=L}.$$ 

It is explicitly known for $L \leq N$ [4, 32, 20] (see also [21] for a simple proof)

$$\Delta_N(L) = (2\pi)^{-3/2} \left\{ \begin{array}{ll} \frac{4}{3}N(N-1) & \text{for } L \in \{0,1\} \\ \frac{2}{3}N \left(N-1 - \frac{1}{2}L\right) & \text{for } 2 \leq L \leq N. \end{array} \right.$$ 

Moreover, $\Delta_N(L) = 0$ for $L \geq N(N-1)$. The eigenfunctions for $L = N(N-1)$ corresponding to the eigenvalue 0 of $\Delta_N$ is the bosonic Laughlin wave function

$$\prod_{1 \leq i < j \leq N} (z_i - z_j)^2.$$ 

Little is known about $\Delta_N(L)$ for $N < L < N(N-1)$, except for numerical simulations for small particle number. The only rigorous result concerns the limit $N \gg 1$ and $L \ll N^2$ where one can show that the Gross-Pitaevskii approximation is exact [26]. I.e., in this regime the convex hull of $\Delta_N(L)$ is given by

$$\inf \left\{ \frac{1}{2} \langle f \otimes f | \delta_{12} | f \otimes f \rangle : f \in B, \|f\|^2 = N, \langle f | z \partial_z | f \rangle = L \right\}.$$ 

The qualitative behavior of $\Delta_N(L)$ is sketched in Figure [2].
Figure 2. (taken from [21]) A sketch of the joint spectrum of $\mathcal{L}_N$ and $\Delta_N$. The dotted line is the yrast curve, its convex hull is in red. The bold dots correspond to the possible ground states of $H^{LLL}$ as one varies $(1 - |\vec{\Omega}|)/a$. The yellow area on the left shows the validity regime of the Gross-Pitaevskii equation. For $L \geq N(N - 1)$, the interaction energy is zero.

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