We find coordinates, the metric tensor, the inverse metric tensor and the Laplace-Beltrami operator for the orbit space of Hamiltonian SU(2) gauge theory on a finite, rectangular lattice, with open boundary conditions. This is done using a complete axial gauge fixing.
I. INTRODUCTION

There is no analytic method to determine the spectrum of the Hamiltonian of the gauge theory of the strong interaction. There is a long-standing conjecture that there is a gap in this spectrum between the ground-state and first-excited-state energies. One strategy to illuminate this problem is to study the physical space of configurations. These configurations are not choices of gauge field; rather they are gauge orbits. In this paper, we find coordinates on this space, eliminating gauge-fixing ambiguities. This is more difficult in three or more space-time dimensions than in two.

The space of gauge orbits is not a manifold, but an orbifold. The Hamiltonian of the gauge theory is a linear combination of the Laplace-Beltrami operator and a certain potential function on orbit space.

Determining some geometric quantities on orbit space (such as the Laplace-Beltrami operator, the Ricci curvature or the scalar curvature) require the evaluation of a trace. Such a trace does not exist without a regularization. Singer proposed zeta-function regularization for this purpose. In this paper we will regularize with a lattice. In particular, we use Kogut and Susskind’s gauge theory, defined on a spatial lattice, but with continuous time (see Reference for a derivation of the Kogut-Susskind formalism from Euclidean path-integral lattice gauge theory, with the transfer matrix).

To make our treatment of gauge fixing simple, we break with the practice of using a toroidal lattice, using instead an open rectangular lattice.

The metric tensor on the space of gauge orbits can be understood as the projection operator which vanishes on gauge transformations. Another point of view is to regard orbit space as a metric space; the same metric tensor arises naturally in such a context. This projection operator is singular by definition; it acts on functionals of the gauge field, not the physical wave functionals of gauge orbits. This is why it is desirable to find coordinates on orbit space. The metric between points of orbit space on the lattice was discussed in Reference. Several papers have partly reduced the number of orbit-space coordinates in
an axial gaug[11], but here all redundancies are completely eliminated. There are conically-singular points in the orbit-space orbifold, which arise from gauge configurations invariant under a subgroup of the gauge group[11].

Our approach to finding coordinates and the metric on lattice orbit space makes it possible to find the Riemann, Ricci and scalar curvatures, at least in principle. A lower bound on the Ricci curvature implies a gap in the spectrum of the kinetic term of the Hamiltonian[12]. The problem of calculating the curvature with the lattice regularization is more difficult than might be expected. This is because the metric tensor and inverse metric tensor need to be explicitly calculated. The lattice metric constructed in[10] was derived by taking an infimum over distances between elements of two orbits. This can then be coordinatized to give a metric tensor. Finding the inverse metric tensor is nontrivial, but possible. The inverse metric tensor is contained in the Laplace-Beltrami operator, so can be extracted once coordinates have been chosen. Determining the inverse metric tensor is the focus of this paper.

We should mention that there is another way of studying the space of configurations of 2+1-dimensional gauge theories using holomorphic coordinates which appear very useful[13]. Some results similar to those in Reference[13] were obtained in a simple formalism[13].

This paper is organized as follows. Some definitions are given in Section II. Gauss’ law and the definition of orbit space are given in Section III. In particular, we discuss how gauge-equivalent gauge configurations are eliminated. Section III A is the heart of the paper, where the last step in the gauge fixing is done. To place the ideas in context, we review the metric on orbit space in Section IV. To make the coordinates on this space explicit, we introduce Euler angles for gauge fields in Section V. In Section VI, we describe the form of the inverse metric tensor on a finite, 2-dimensional rectangular lattice for gauge group SU(2). Finally, we summarize our results and discuss some avenues for further work in Section VII.
II. PRELIMINARIES

**Definition 1.** The $D$-dimensional lattice is the graph whose set of vertices is a subset of $\mathbb{Z}^D$, and whose edges connect each vertex to its nearest neighbors.

We will work with finite rectangular lattices. An example of such a lattice with $D = 2$ is shown in Figure 1.

The vertices of the lattice are denoted by $\mathbf{x} \equiv (x_1, x_2, \ldots, x_D)$. The numbers $x_1, \ldots, x_D$ are integer multiples of the lattice spacing $a$, specifically $x_j = 0, a, 2a, \ldots, L_j$, for $j = 1, \ldots, D$. Let $\hat{1}, \ldots, \hat{D}$ be unit vectors in the positive $1-\ldots, D-$directions, respectively. We denote the edge adjacent to the two vertices $\mathbf{x}$ and $\mathbf{x} + \hat{j}$ by $(\mathbf{x}, j)$, for each $j = 1, \ldots, D$.

An element of $SU(n)$ is assigned to each edge of the lattice. The $SU(n)$ element at the edge $(\mathbf{x}, j)$ is denoted by $U_j(\mathbf{x})$. Henceforth we shall take $n = 2$, for simplicity. In the lattice-gauge literature, the vertices are called sites and the edges are called links.

**Definition 2.** A wave function is a complex-valued function of all of the variables $U_j(\mathbf{x})$ on all the edges.

**Definition 3.** Gauge state space is the Hilbert space of square-integrable wave functions, with the inner product

$$\langle \Psi | \Phi \rangle = \int \overline{\Psi(U_j(\mathbf{x}))} \Phi(U_j(\mathbf{x})) \prod_{\mathbf{x}, j} dU_j(\mathbf{x}),$$

where the integration measure on each edge is the Haar measure.

We remark that only $n = 2$ will be considered in any detail. We denote the basis vectors of $su(2)$ by $t_1$, $t_2$ and $t_3$, normalized by $\text{Tr} t_a t_b = \delta_{ab}$.

A column vector $l_j(\mathbf{x})$ of the three differential operators, $[l_j(\mathbf{x})]_1$, $[l_j(\mathbf{x})]_2$ and $[l_j(\mathbf{x})]_3$, is assigned to each edge $(\mathbf{x}, j)$ of the lattice. We call these the electric-field operators. They are defined by the commutation relations:

$$[l_j(\mathbf{x})_b, l_k(\mathbf{y})_c] = i\sqrt{2} \delta_{xy} \delta_{jk} \epsilon^{bcd} l_j(\mathbf{x})_d,$$
\[ [I_j(x)b, U_k(y)] = -\delta_{xy} \delta_{jk} t_b U_j(x), \]

with all other commutators zero.
Definition 4. The Laplace-Beltrami operator is

$$-\Delta \equiv \sum_{D} \sum_{j=1}^{n^2-1} \sum_{j=1}^{D} [l_j(x)]^2.$$ 

Its spectrum is unbounded and discrete. In 2 dimensions this operator is:

$$-\Delta = \sum_{x_1=0}^{L_1-1} \sum_{x_2=0}^{L_2-1} \sum_{b=1}^{2} \sum_{j=1}^{3} [l_j(x_1, x_2)]^2.$$ 

Definition 5. The covariant derivative of $l_j(x)$ is

$$\mathcal{D}j l_j(x) \equiv \mathcal{D}j(x) \cdot l_j(x) \equiv l_j(x) - (1 - \delta^x_j)U_j(x - j)l_j(x - j)U_j(x - j)^{-1}. \quad (1)$$

The factor $(1 - \delta^x_j)$ is needed because the lattice is finite and rectangular.

An element of the adjoint representation $\mathcal{R}_j(x)$, is assigned to $U_j(x)$ by

$$Ut_bU^{-1} \equiv \mathcal{R}t_b,$$

where the arguments denoting the edge are implicit. Notice that $\mathcal{R}_j(x)$ lies in $SO(3)$.

Hence (1) may be written

$$\mathcal{D}j l_j(x) \equiv l_j(x) - (1 - \delta^x_j)\mathcal{R}_j(x - j)l_j(x - j). \quad (2)$$

III. ORBIT SPACE

Definition 6. Gauss’ law is

$$\sum_{j}^{D} \mathcal{D}j l_j(x) = 0.$$ 

Gauss’ law is imposed at every vertex.

We denote by $\{U\}$ the collection of $U_j(x) \in SU(2)$ for all the edges $(x, j)$. The equivalence relation $\{U\} \simeq \{V\}$ between two lattice-gauge configurations $\{U\}$ and $\{V\}$ means
that there is gauge transformation \{K\}, i.e. some collection \(K(x) \in \text{SU}(2)\) at sites \(x\) such that

\[
V_j(x) = K(x + j\alpha)^{-1} U_j(x) \ K(x).
\]

We will sometimes use the obvious notation \(\{V\} = \{U\}^{\{K\}}\) for this expression.

**Definition 7.** A *gauge orbit* \(u\) is an equivalence class of lattice-gauge configurations under the equivalence relation \(\simeq\), defined above.

Gauss’ law is the statement that wave functions depend on orbits rather than gauge configurations\(^{56}\). To put coordinates on orbit space, we must first assign a unique element configuration for each equivalence class of gauge configurations. This is the procedure called *gauge fixing*.

A gauge transformation can easily be used to set the \(\text{SU}(2)\) elements on edges in the 1-direction to unity. A further gauge transformation can then used to set the \(\text{SU}(2)\) elements on the edges in the 2-direction for which \(x_1 = 0\), and so on. We have thereby fixed the gauge on a maximal tree:

\[
\begin{align*}
U_1(x_1, x_2, x_3, \ldots) &= \mathbb{I}, \\
U_2(0, x_2, x_3, \ldots) &= \mathbb{I}, \\
U_2(0, 0, x_3 \ldots) &= \mathbb{I}, \\
&\vdots
\end{align*}
\]

As this is done, we use Gauss’ law to write the electric-field operators on the fixed edges in terms of the electric-field operators on the unfixed edges. In 2 dimensions this is

\[
\begin{align*}
l_1(x_1, x_2) &= -\sum_{y_1=0}^{x_1} D_2 l_2(y_1, x_2), \\
l_2(0, x_2) &= -\sum_{y_2=0}^{x_2} \sum_{y_1=1}^{L_1} D_2 l_2(y_1, y_2).
\end{align*}
\]
The procedure is similar for $D \geq 3$.

The Laplace-Beltrami operator for $D = 2$ may now be rewritten as

$$-\Delta = \sum_{b=1}^{3} \left\{ \sum_{x_1=2}^{L_1} \sum_{x_2=0}^{L_2-1} [l_2(x_1, x_2)]^2 + \sum_{x_2=1}^{L_2-1} [l_2(1, x_2)]^2 
- \sum_{x_1=0}^{L_1-1} \sum_{x_2=0}^{L_2} \left[ \sum_{y_1=0}^{x_1} \sum_{y_2=0}^{x_2} D_2 l_2(y_1, x_2) \right]^2 
- \sum_{x_2=0}^{L_2-1} \left[ \sum_{y_2=0}^{L_2} \sum_{y_1=0}^{L_1} D_2 l_2(y_1, y_2) \right]^2 
+ [l_2(1, 0)]^2 \right\}. \tag{5}$$

The gauge fixing in (3) and (4) is not yet complete. There are three remaining conditions to solve:

$$\sum_{y_2=0}^{L_2} \sum_{y_1=1}^{L_1} D_2 l_2(y_1, y_2) = 0. \tag{6}$$

A. Fixing the Last Edge

The remaining global condition (6) can be solved by making a single element of SU(2) (at one edge) diagonal. No further gauge fixing is then possible. For $D = 2$, we chose to diagonalize $U_2(1, 0)$. As a result $R_2(1, 0)$ will also be diagonal. For this purpose, we rewrite (6) as

$$- [I - R_2(1, 0)] l_2(1, 0) = l_2(1, 1) + \sum_{y_2=2}^{L_2} D_2 l_2(1, y_2) + \sum_{y_2=0}^{L_2} \sum_{y_1=2}^{L_1} D_2 l_2(y_1, y_2) \equiv \Xi. \tag{7}$$
IV. THE METRIC

The metric distance $\rho(u, v)$ between two gauge orbits $v$ and $v$ on the lattice is given by

$$\rho(u, v)^2 = N - \frac{1}{2} \inf_{\{K\}} \sum_x \sum_{j=1}^D \left[ \text{Tr} \, K(x)V_j(x)^{-1}K(x + ja)^{-1}U_j(x) \\
+ \text{Tr} \, K(x + ja)V_j(x)K(x)^{-1}U_j(x)^{-1} \right], \quad (8)$$

where $\{U\}$ is any element of $u$ and $\{V\}$ is any element of $v$. This function of two orbits is gauge invariant. Furthermore, it is a metric.

The partition function of a Wilson lattice gauge theory in $D + 1$ dimensions, with discrete time $t$, is

$$\prod_{x, t, \mu} \int dU_\mu(x) \, e^{-S}, \quad (9)$$

where the index $\mu$ runs from 0 to $D$. The action $S$ may be split as

$$S = \frac{a^{D-2}}{a_t g_0^2} \sum_t L_{st} + \frac{a_t a^{D-4}}{g_0^2} \sum_t L_{ss},$$

where $a_t$ is the lattice spacing in the time direction, and where $L_{st}$ is the contribution of a space-time plaquette and $L_{ss}$ is the contribution of a space-space plaquette. Explicitly

$$L_{st} = \frac{N}{2} - \frac{1}{2} \sum_x \sum_{j=1}^D \left[ \text{Tr} \, U_0(x, t)U_j(x, t + a_t)^{-1}U_0(x + ja, t)^{-1}U_j(x, a) \\
+ \text{Tr} \, U_0(x + ja, t)U_j(x, t + a_t)U_0(x, t)^{-1}U_j(x, t)^{-1} \right], \quad (10)$$

and

$$L_{ss} = \frac{N}{4} - \frac{1}{4} \sum_x \sum_{j \neq k} \left[ \text{Tr} \, U_j(x, t)U_k(x + ja, t)U_j(x + \tilde{a}a, t)^{-1}U_k(x, t)^{-1} \\
+ \text{Tr} \, U_j(x, t)U_j(x + \tilde{a}a, t)U_k(x + ja, t)^{-1}U_j(x, t)^{-1} \right]. \quad (11)$$

Note that the right-hand sides of $(8)$ and $(10)$ are very similar; if we substitute for each $x$ and $j$ $U_j(x, t) \rightarrow U_j(x)$, $U_j(x, t + a_t) \rightarrow V_j(x)$, and $U_0(x, t) \rightarrow K(x)$, into the right-hand
side of (10), and take the infimum with respect to $K(x)$, we obtain the lattice metric. Thus, by an appropriate gauge fixing of the temporal gauge configuration $U_0(x, t)$, we may replace $\mathcal{L}_{st}$ by $\rho(u(t), u(t + a t))$, where $u(t)$ is the gauge orbit containing $\{U\}$ at time $t$ and $u(t + a t)$ is the gauge orbit containing $\{U\}$ at time $t + a t$. Alternatively, if we simply integrate out $U_0(x, t)$, the dominant contribution to (9) at weak coupling will come from this choice of $U_0(x, t)$.

To see that (8) is a metric, we note that for any two orbits $u$ and $v$, $\rho(u, v) = \rho(v, u) \geq 0$, with $\rho(u, v) = 0$ if and only if $u = v$. The only remaining property we need is the triangle inequality, proved in Reference [10]. As the proof is not hard, we repeat it below.

Notice that (8) is the same as

$$\rho(u, v) = \inf_{\{K\}} I(\{U\}, \{V\}^{\{K\}}) = \inf_{\{K\}} I(\{U\}^{\{K\}}, \{V\})$$

$$= \inf_{\{K\}; \{L\}} I(\{U\}^{\{K\}}, \{V\}^{\{L\}}), \quad (12)$$

where

$$I(\{U\}, \{V\})^2 = \frac{1}{2} \sum_x \sum_{j=1}^D \text{Tr} \left[ V_j(x) - U_j(x) \right]^{\dagger} \left[ V_j(x) - U_j(x) \right]. \quad (13)$$

Now for any three sets of matrices $\{U\}$ and $\{V\} \{W\}$ we have that

$$I(\{U\}, \{V\}) + I(\{V\}, \{W\}) \geq I(\{U\}, \{W\}),$$

which is a consequence of the triangle inequality of a vector space over the complex field (this is formally true by (13), even if we are not dealing with special-unitary matrices). Introducing gauge transformations $\{K\}$, $\{L\}$ and $\{M\}$, we have

$$I(\{U\}^{\{K\}}, \{V\}^{\{L\}}) + I(\{V\}^{\{L\}}, \{W\}^{\{M\}}) \geq I(\{U\}^{\{K\}}, \{W\}^{\{M\}}),$$

which implies that

$$I(\{U\}^{\{K\}}, \{V\}^{\{L\}}) + I(\{V\}^{\{L\}}, \{W\}^{\{M\}}) \geq \rho(u, w).$$
Taking the infimum of the left-hand side of this equation gives the triangle inequality
\[ \rho(u, v) + \rho(v, w) \geq \rho(u, w) . \tag{14} \]

We next show that (8) provides a Riemannian metric, except at conically-singular orbits\[^4\].

Let us substitute
\[ U_j(x) = e^{-iA_j(x) \cdot t}, \quad V_j(x) = e^{-i[A_j(x) + dA_j(x)] \cdot t} \quad \text{and} \quad K(x) = e^{d\phi(x) \cdot t} \]
into (8) and expand to second order in \( dA_j(x) \) and \( d\phi(x) \). The result is
\[ d\rho^2 = \rho(u, v)^2 = \inf_{d\phi} \sum_{x, j} \left\{ e_j(x)_{\alpha} b \, dA_j(x)^{\alpha} + \left[ -D^{\dagger}_j d\phi(x + \hat{a}) \right]_{\alpha} \right\}^2 . \]

The minimum of the sum on the right-hand side is unique. We find that \( d\phi(x) \) is
\[ d\phi(x) = \sum_{y, j} (-D^{\dagger} \cdot D)^{-1} \delta_{xy} D_j \cdot e_j A_j(y) , \]
where
\[ e_{\alpha} a t_a = -iU^{-1} \partial_{\alpha} U , \]
(this is given explicitly by
\[ e_{\alpha} a = -i \left( \frac{\mathbb{I} - e^{iA \cdot T}}{A \cdot T} \right)^a_{\alpha} , \]
in canonical coordinates, where \( T_{1,2,3} \) constitutes a basis of the adjoint representation of the Lie algebra) and where the Green’s function \((-D^{\dagger} \cdot D)^{-1} \delta_{xy}\) is uniquely determined by the boundary conditions. This variational problem has the solution\[^{10}\]
\[ d\rho^2 = G_{(x, j, \alpha)(y, k, \beta)} dA_j(x)^{\alpha} dA_k(y)^{\beta} , \tag{15} \]
where we sum over lattice edges in our summation convention and where the metric tensor is
\[ G_{(x, j, \alpha)(y, k, \beta)} = e_j(x)_{\alpha} b \left\{ \delta_{xy} \delta_{jk} \delta_{bc} - \left[ (-D^{\dagger})_{jk} \frac{1}{-D^{\dagger} \cdot D} D_k \right]_{bc} \delta_{xy} \right\} e_k(y)_{\beta}^{c} . \tag{16} \]
Notice that the quantity in curled brackets in (16) is idempotent, hence it is a projection. In fact, the metric projects out gauge transformations in inner products. To remove the zero eigenvalues, it is necessary to fix the gauge. The resulting induced metric is that on all of orbit space, except at conical singularities. The set of these singularities is of measure zero, but it is an open question whether they have consequences for the Yang-Mills spectrum.

V. EULER ANGLES

As in our previous discussion, we specialize to gauge group SU(2). We introduce Euler coordinates at each edge:

\[ U_j(x) = e^{i\alpha_j(x)\sigma_x} e^{i\beta_j(x)\sigma_y} e^{i\theta_j(x)\sigma_z}, \quad (17) \]

where \( \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices. The reason we use these coordinates (instead of the canonical coordinates of the previous section) is technical, not fundamental. It is easier to use Euler angles to perform the gauge fixing at the last edge.

In much of the discussion which follows, we denote the angles at the last edge \( \alpha_2(1,0), \beta_2(1,0) \) and \( \theta_2(1,0) \) by \( \alpha, \beta \) and \( \theta \), respectively and \( U_j(1,0) \) by \( U \).

A choice of basis vectors of \( su(2) \) is

\[ M_\gamma \sigma_a \equiv -i \partial_\gamma U, \quad (18) \]

where \( \gamma \) denotes \( \alpha, \beta \) or \( \theta \). From (17), we can find \( M_\gamma \):

\[ M = \begin{pmatrix}
\sin 2\alpha \sin 2\beta & -\cos 2\alpha \sin 2\beta & \cos 2\beta \\
\cos 2\alpha & \sin 2\alpha & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (19) \]

This result can be used to express the electric-field operators in terms of derivatives of the coordinates:
\[
\begin{pmatrix}
\ell_1 \\
\ell_2 \\
\ell_3
\end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix}
\partial_\alpha \\
\partial_\beta \\
\partial_\theta
\end{pmatrix}.
\]

Explicitly:

\[
[l_2(1,0)_1] = \frac{\sin(2\alpha)}{\sin(2\beta)} \partial_\alpha - \frac{\cos(2\alpha) \sin(2\alpha)}{\sin(2\beta)} \partial_\beta,
\]

\[
[l_2(1,0)_2] = \frac{\cos(2\alpha)}{\cos(2\beta)} \partial_\alpha + \sin(2\alpha) \partial_\beta - \sin(2\alpha) \partial_\theta,
\]

\[
[l_2(1,0)_3] = \partial_\theta.
\]

From (17) and (20), \( R_j(x_1, x_2) \) can be explicitly calculated:

\[
R_j(x_1, x_2) = \begin{pmatrix}
\cos(2\beta) & -\cos(2\alpha) \sin(2\beta) & \sin(2\alpha) \sin(2\beta) \\
\sin(2\beta) \cos(2\theta) & -\sin(2\alpha) \sin(2\theta) & -\cos(2\alpha) \sin(2\theta) \\
\sin(2\beta) \sin(2\theta) & \sin(2\alpha) \cos(2\theta) & \cos(2\alpha) \cos(2\theta)
\end{pmatrix} + \cos(2\alpha) \cos(2\beta) \sin(2\theta) - \sin(2\alpha) \cos(2\beta) \sin(2\theta),
\]

(21)

Diagonalizing \( R_2(1,0) \) by some \( g \in SU(3) \) yields the expression:

\[
g^{-1}R_2(1,0)g = \begin{pmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & \frac{1}{k}
\end{pmatrix},
\]

(22)

where

\[
k = -\frac{1}{2} \left( (1 - \cos(2\alpha - 2\theta)[1 + \cos(2\beta) - \cos(2\beta)]
\right.
\]

\[
+ \{4 + [(1 - \cos(2\alpha - 2\theta)(1 + \cos(2\beta)) - \cos(2\beta)]^2\}^{\frac{1}{2}} \right).
\]

(23)
Substitution of (22) and (23) into (7) gives two conditions, which allow the identification \( \beta = \theta = \frac{\pi}{4} \). The gauge fixing is now complete. We remove the ubiquitous factors of two in the remainder of this paper, by redefining \( \alpha \to 2\alpha \), \( \beta \to 2\beta \), and \( \theta \to 2\theta \).

Two of the derivatives on the \((1, 0)\) edge may be written in terms of the third derivative, in addition to some of the angles on other edges:

\[
\frac{\partial \theta}{\partial x} = \frac{\Xi_3}{1 - \frac{1}{k}},
\]

\[
\frac{\partial^* \beta}{\partial x} = \frac{1}{\sin \alpha} \left( \sin \alpha \frac{\Xi_3}{1 - \frac{1}{k}} + \frac{\Xi_2}{1 - \frac{1}{k}} \cos \alpha - \cos \beta \right),
\]

where \( \Xi \) is defined in (7). The asterisk on the derivative with respect to \( \beta \) means that derivatives with respect to \( \theta \) are replaced by the right-hand side of (24).

The Laplace-Beltrami operator may now be written as:

\[
-\Delta = -\Delta_1 - \Delta_2 - \Delta_3 - \Delta_4 - \Delta_5 - \Delta_6,
\]

where

\[
-\Delta_1 = \sum_{x_2=1}^{L_2} [l_2(1, x_2)]^2 + \sum_{x_1=2, x_2=0}^{L_1} [l_2(x_1, x_2)]^2,
\]

\[
-\Delta_2 = -\sum_{x_1=0}^{L_1-1} \sum_{x_2=0}^{L_2} \left[ \sum_{y_1=0}^{x_1} D_2 l_2(y_1, x_2) \right]^2,
\]

\[
-\Delta_3 = -\sum_{x_2=0}^{L_2-1} \left[ \sum_{y_2=0}^{x_2} \sum_{y_1=1}^{L_1} D_2 l_2(y_1, y_2) \right]^2,
\]

\[
-\Delta_4 = \left( \sqrt{2} \sin 2\alpha \ \partial_\alpha - \cos 2\alpha \ \partial^*_\beta - \sin 2\alpha \frac{\Xi_3}{1 - \frac{1}{k}} \right)^2,
\]

\[
-\Delta_5 = \left( \sqrt{2} \cos 2\alpha \ \partial_\alpha + \sin 2\alpha \ \partial^*_\beta - \sin 2\alpha \frac{\Xi_3}{1 - \frac{1}{k}} \right)^2,
\]

\[
-\Delta_6 = \left( \frac{\Xi_3}{1 - \frac{1}{k}} \right)^2,
\]
where the angle \( \alpha \equiv \alpha_2(1,0) \) in \( -\Delta_4, -\Delta_5 \) and \( -\Delta_6 \) is the sole remaining coordinate specifying \( U_2(1,0) \) (which is now diagonal).

A comparison of with the standard form of the Laplace-Beltrami operator: \( -\Delta \equiv -\frac{1}{\sqrt{g}} \partial_{\mu} \sqrt{g} g^{\mu\nu} \partial_{\nu} \), yields the inverse metric tensor.

### VI. THE INVERSE METRIC TENSOR

The components of the inverse metric tensor may be read off by examining (26). Fortunately, the determinant of the metric is not needed to find these components. Finding any given component (that is, \( g^{\mu\nu} \)) is done by selecting the function between two partial derivatives of the associated coordinates and multiplying by the function in front. This is because each term of the Laplace-Beltrami operator in equation (26) has the form \( -\frac{1}{h_{\mu\nu}} \partial_{\mu} h_{\mu\nu} g^{\mu\nu} \partial_{\nu} \) (the square root of the determinant of the metric \( \sqrt{g} \), automatically divides the product \( \prod_{\mu\nu} h_{\mu\nu} \)).

To illustrate how the inverse metric tensor can be extracted, we give the example of the one-edge Beltrami-Laplace operator \( -\Delta_{\text{one-edge}} = l^2 \). From the expressions (20) for the components of \( l \), we find

\[
\begin{align*}
g^{\alpha \alpha} &= \frac{1}{\sin^2 \beta}, \\
g^{\alpha \beta} &= 0, \\
g^{\alpha \theta} &= \frac{\sin \alpha (\sin (\beta - \alpha))}{\sin^2 \beta}, \\
g^{\beta \beta} &= 1, \\
g^{\beta \theta} &= \frac{\sin^2 \alpha + \sin \alpha \cos \alpha \cos \beta}{\sin \beta}, \\
g^{\theta \theta} &= \frac{\sin^2 \alpha}{\sin^2 \beta} + 1,
\end{align*}
\]

Nothing is new about this result, which is simply the inverse metric tensor of a three sphere.

Using (2), (19), and (21), the components of \( [\mathcal{D}_2 l_2(y_1, x_2)]_b \) (which are in \( -\Delta_2 \) and
–Δ_3) reduce to:

\[ [D_2 l_2(y_1, x_2)]_1 = \sin \alpha \frac{\partial_x}{\sin \beta} \partial_\alpha - \cos \alpha \partial_\beta - \frac{\sin \alpha \cos \beta}{\sin \beta} \partial_\theta \]

(27)

\[ + \left( \cos^2 \alpha - \frac{\sin \alpha \cos \beta}{\sin \beta} \right) \partial_{\alpha_2(y_1, x_2-1)} \]

\[ + \cos \alpha (\cos \beta + \sin \alpha \sin \beta) \partial_{\beta_2(y_1, x_2-1)} \]

\[ + \sin \alpha \left( \frac{\cos^2 \beta}{\sin \beta} + \cos \alpha \sin \beta - \sin \beta \right) \partial_{\theta_2(y_1, x_2-1)}, \]

\[ [D_2 l_2(y_1, x_2)]_2 \]

(28)

\[ = \frac{\cos \alpha}{\sin \beta} \partial_\alpha + \sin \alpha \partial_\beta + \sin \alpha \partial_\theta \]

\[ + \sin \alpha \cos \alpha \sin \theta - \cos^2 \alpha \cos \beta \cos \theta - \sin \alpha \cos \theta \sin \beta \partial_{\alpha_2(y_1, x_2-1)} \]

\[ + (\sin^2 \alpha \sin \theta - \sin \alpha \cos \alpha \cos \beta \cos \theta + \cos \alpha \sin \beta \cos \theta) \partial_{\beta_2(y_1, x_2-1)} \]

\[ + (\sin \alpha \cos \beta \cos \theta + \sin^2 \alpha \sin \theta + \cos \alpha \sin \theta \]

\[ - \sin \alpha \cos \beta \cos \theta - \sin \alpha \cos \alpha \cos \beta \cos \theta) \partial_{\theta_2(y_1, x_2-1)}, \]

\[ [D_2 l_2(y_1, x_2)]_3 \]

(29)

\[ = \partial_\theta - \left( \sin \alpha \sin \theta + \frac{\sin \alpha \cos \alpha \cos \theta}{\sin \beta} + \frac{\cos \alpha \cos \beta \sin \theta}{\sin \beta} \right) \partial_{\alpha_2(y_1, x_2-1)} \]

\[ + (\cos \alpha \sin \beta \sin \theta - \sin^2 \alpha \cos \theta - \sin \alpha \cos \alpha \cos \beta \sin \theta) \partial_{\beta_2(y_1, x_2-1)} \]

\[ + (\sin \alpha \cos \beta \sin \theta - \sin^2 \alpha \cos \theta - \sin \alpha \cos \alpha \cos \beta \sin \theta \]

\[ + \cos \alpha \cos \theta - \sin \alpha \cos \beta \sin \theta) \partial_{\theta_2(y_1, x_2-1)}, \]

where the edge direction and adjacent vertex are only indicated explicitly for coordinates other than \( \alpha \equiv \alpha_2(y_1, x_2), \beta \equiv \beta_2(y_1, x_2), \theta \equiv \theta_2(y_1, x_2). \)

In –Δ_2 we sum over spatial dimensions after squaring, but only in the 1-direction. These terms are merely a local term coupled with an adjoint term from the edge below. These can be constructed similarly by multiplying the associated pieces, and then summing.

The term –Δ_3 is somewhat more complicated, as the sums run in both directions. When constructing the contribution from this term for a given component of the metric
tensor, it must be noted that there will be overlap from $D_{2l_2}(y_1, y_2 + 1)$ with $D_{2l_2}(y_1, y_2)$, except on the boundary. The form of these terms is $(l(x_\gamma) - Rl(x_\gamma))(l(y_\xi) - Rl(y_\xi))$, and can also be constructed similarly to the above.

The last three terms $-\Delta_4, -\Delta_5$ and $-\Delta_6$ contain many pieces. They have many more combinations than than the above, because they each contain sums over most of the lattice. Only the third component of the vector is taken, however, so summing of components is not required in their construction.

This completely defines the inverse metric tensor for the finite rectangular lattice with $D = 2$, working over SU(2). Similar methods work for $D \geq 3$. We believe it is possible to generalize to the gauge group SU($n$). Angular coordinates are considerably more complicated for $n > 2$, however.

\section{VII. Conclusions}

To summarize, we have explicitly found the metric tensor for SU(2) on the lattice with open boundary conditions, and determined the inverse metric tensor. It is noteworthy that the gauge-fixing problem only becomes complicated when fixing the last edge.

The methodology used here can be generalized to construct results for higher-dimensional lattices and and other gauge groups. There are no marked differences for $D \geq 3$. Generalizing the results to gauge group SU($n$) is cumbersome, but the strategy is the same as for SU(2); this is under investigation.

It should be possible to study orbit-space geodesics in our coordinates. Any geodesic in the full space of lattice gauge fields is described by the real parameter $t$ through

$$U_j(x; t) = \exp i \tau(x, j) t,$$

where $\tau(x, j)$ is an arbitrary chosen element of the Lie algebra chosen for each edge $(x, j)$. The geodesics in the completely-fixed axial gauge are obtained by gauge-fixing (30) according to the prescription given in this paper.
Finally, we believe that a detailed study of the set of conically-singular points in the lattice formulation of gauge theory should be very fruitful. This set is of measure zero, but that does not mean it has no significance. Presumably the Riemann curvature diverges at such points (even on the lattice). An interesting question is whether this divergence can be regularized in a sensible and physically-meaningful way.

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