Lifting to Passivity for $\mathcal{H}_2$-Gain-Scheduling Synthesis with Full Block Scalings

Christian A. Rössinger and Carsten W. Scherer

Abstract: We focus on the $\mathcal{H}_2$-gain-scheduling synthesis problem for time-varying parametric scheduling blocks with scalings. Recently, we have presented a solution of this problem for $D$- and positive real scalings by guaranteeing finiteness of the $\mathcal{H}_2$-norm for the closed-loop system with suitable linear fractional plant and controller representations. In order to reduce conservatism, we extend these methods to full block scalings by designing a triangular scheduling function and by introducing a new lifting technique for gain-scheduled synthesis that enables convexification.

Keywords: Linear parameter-varying systems, Controller constraints and structure, Convex optimization

1. INTRODUCTION

The design of linear parametrically-varying (LPV) systems is widely spread over the control literature and can be roughly divided into two classes. On the one hand, parameter-dependent Lyapunov functions, as in Becker (1995), Apkarian and Adams (1997), Wu and Dong (2005), de Souza and Trofino (2006), and Sato (2011), are used for synthesis with linear matrix inequalities (LMIs) by approximating the parameter space of the scheduling variable. On the other hand, the so-called scaling approach can directly handle rational parameter dependence, as in Packard (1994), Apkarian and Gahinet (1995) for $D$-scalings, Helmersson (1998) for positive real-scalings, Scorletti and El Ghaoui (1998) for $D/G$-scalings, and Scherer (2000), Veenman and Scherer (2014) for the least conservative full block scalings. These approaches are as well of interest because of their link to distributed controller design (see Langbort et al. (2004)) and their flexibility for handling more complex scheduling blocks such as delays as considered in Rössinger and Scherer (2019).

In this work, we look at the concrete configuration in Fig. 1 which has shown to be well-suited for analysis and synthesis of LPV controllers (see Packard (1994), Apkarian and Gahinet (1995)). For an uncertain plant $G(\Delta)$ with $\Delta$ being an arbitrary fast time-varying matrix-valued parametric uncertainty, we employ constant full block scalings to synthesize a controller $K(\Delta)$ which achieves an $\mathcal{H}_2$-cost criterion imposed on $wp \rightarrow z_p$. Concrete applications of LPV design with $\mathcal{H}_2$-performance guarantees are, e.g., the control of autonomous cars and helicopters in Mustaki et al. (2019) and Guerreiro et al. (2007), respectively. Recently, Rössinger and Scherer (2019) present the first scaling solution to this problem with $D$-scalings in case that the uncertainty takes values in the unit disk or with positive-real scalings in case that the uncertainty is passive. Technically, this approach uses a convexifying transformation for controller and scaling parameters based on Masubuchi et al. (1998), Scherer et al. (1997), while suitable structured plant and controller descriptions guarantee well-posedness for the closed-loop $\mathcal{H}_2$-norm by design. However, these results heavily rely on the particular structure of $D$- and positive real scalings and cannot be easily extended to the less conservative full block scalings.

As the main contribution of this work, we present a complete solution for the $\mathcal{H}_2$-gain scheduling problem with full block scalings in terms of LMIs. For this purpose, we introduce a new design approach based on what we call lifting to passivity. This amounts to a loss-less embedding of the original synthesis problem into a passivity framework involving a suitable structural extension (or lifting) of the plant and the controller, and is the enabling factor for being able to convexify the problem through a transformation that operates on both the controller and the scaling parameters. The use of a related passivation step has been beneficial already for a completely different objective in robustness analysis and synthesis involving integral quadratic constraints in Veenman and Scherer (2013), Veenman and Scherer (2014). As a novel feature of this paper, we develop a systematic approach for using such a procedure in the context of gain-scheduled synthesis. As a further contribution, we reveal how suitable structured plant and controller representations can be exploited in our designs to render the $\mathcal{H}_2$-norm finite.

Outline. After introducing the notation used in this work, Section 2 formulates the $\mathcal{H}_2$-gain scheduling problem under investigation, while Section 3 presents the lifting

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design technique. The resulting specifically structured design problem is solved in Section 4. Finally, a short example clarifies that our results are less conservative than those in Rößler and Scherer (2019).

**Notation.** Let $\mathbb{S}^n$ denote the set of real symmetric matrices of dimension $n \times n$. For some matrices $M \in \mathbb{R}^{r \times s}$ and $P \in \mathbb{R}^{r \times r}$ we abbreviate $M^T P M$ by $(*)^T P M$ and $P + P^T$ by $P(P)$ and denote by $tr(P)$ the trace of $P$. Matrix entries that can be inferred by symmetry are indicated by *. We drop superscripts specifying partitions and dimensions of matrices if they are clear from the context. Further, $I$ and $I_m$ denote identity matrices (with $m$ specifying the dimension if not clear from the context) and $\text{col}(u_1, u_2) := \begin{pmatrix} u_1^T & u_2^T \end{pmatrix}^T$ is used for vectors. If $X, R, S$ and $A_{ij}, B_i, C_i, D$ are some suitable matrices for $i, j = 1, 2$, we abbreviate

$$\mathcal{L}(X, R, S, \begin{pmatrix} A_{ij} & B_i \\ C_i & D \end{pmatrix}) := (\ast)^T \begin{pmatrix} X & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} I & A_{11} & A_{12} \\ 0 & I & A_{21} \\ C_1 & C_2 & D \end{pmatrix}^{-1}$$

and refer to its left upper sub-block as

$$\mathcal{L}_{\text{sub}}(X, R, S, \begin{pmatrix} A_{ij} \\ C_i \end{pmatrix}) := (\ast)^T \begin{pmatrix} X & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} I & A_{11} \\ 0 & I \\ C_1 & C_2 \end{pmatrix}.$$  

### 2. PROBLEM FORMULATION

In the sequel, we introduce the $H_2$-gain scheduling problem for full block scalings.

#### 2.1 Structured plant and controller representations

For some full block time-varying uncertainty $\Delta$ taking values in some polytope, let us consider the standard LPV configuration in Fig. 1 with a $\Delta$-dependent LPV system $G(\Delta)$ and a corresponding controller $K(\Delta)$. To systematically guarantee finiteness of the closed-loop $H_2$-norm, we use specifically structured linear fractional representations (LFRs) for $G(\Delta)$, $K(\Delta)$. Let $G(\Delta)$ be structured as in

$$\begin{pmatrix} \dot{z} \\ \frac{\dot{z}}{y} \end{pmatrix} = \begin{pmatrix} A(\Delta) & B^p(\Delta) \\ C^p(\Delta) & D^p(\Delta) \end{pmatrix} \begin{pmatrix} x \\ w_p \end{pmatrix},$$

$$\begin{pmatrix} x \\ w_p \end{pmatrix} = \begin{pmatrix} A(\Delta) & B^m(\Delta) \\ C^m(\Delta) & D^m(\Delta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$  

(1)

with $D(0) = 0$, performance channel $w_p \to z_p$, control channel $u \to y$, and let us describe the controller $K(\Delta)$ by

$$\begin{pmatrix} \dot{x} \\ u \end{pmatrix} = \begin{pmatrix} A^e(\Delta) & B^e(\Delta) \\ C^e(\Delta) & D^e(\Delta) \end{pmatrix} \begin{pmatrix} x_e \\ w \end{pmatrix},$$

$$\begin{pmatrix} x_e \\ w \end{pmatrix} = \begin{pmatrix} A^e(\Delta) & B^e(\Delta) \\ C^e(\Delta) & D^e(\Delta) \end{pmatrix} \begin{pmatrix} x_e \\ y \end{pmatrix}.$$  

(2)

such that all $\Delta$-dependent operator blocks in (1), (2) are LFRs in $\Delta$. Analogous to the approach for one repeated block in Rößler and Scherer (2019), the zero block structures in (1), (2) guarantee that the performance channel in Fig. 1 has an identically vanishing direct feedthrough term. Since $w_p \to z_p$ is zero in (1), standard techniques for linear fractional transformations (LFTs) show that $G(\Delta)$ can be expressed as the LFR

$$\begin{pmatrix} \dot{z}_1 \\ \frac{\dot{z}_2}{\frac{\dot{z}_p}{p}} \end{pmatrix} = \begin{pmatrix} A_1 & A_{12} & B^p_1 \\ A_{21} & A_{22} & B^p_2 \\ C_1 & C_{21} & D^p_1 \end{pmatrix} \begin{pmatrix} x \\ w_1 \\ w_p \end{pmatrix},$$

(3)

$$\begin{pmatrix} x \\ w_1 \\ w_p \end{pmatrix} = \begin{pmatrix} A_1 & A_{12} & B^p_1 \\ A_{21} & A_{22} & B^p_2 \\ C_1 & C_{21} & D^p_1 \end{pmatrix} \begin{pmatrix} x \\ w_1 \\ w_p \end{pmatrix},$$

$$\dot{w} = \Delta \hat{z}$$

with matrices $A_{ij} \in \mathbb{R}^{r_i \times r_j}$, $B_i \in \mathbb{R}^{n_i \times r_j}$, $C_i \in \mathbb{R}^{m_i \times n_j}$, as well as with a structured uncertainty channel $\hat{w} \to \hat{z}$ for $\hat{z} := \text{col}(\hat{z}_1, \hat{z}_2)$ and $\hat{w} := \text{col}(\hat{w}_1, \hat{w}_2)$; the matrices associated to $\hat{w} \to \hat{z}$ are indicated with the symbols $\wedge$ or $-$ in (3). W.l.o.g., the LFT manipulations can be always performed such that $\Delta = \text{diag}(\Delta, \Delta)$ has a diagonal structure which is compatible with the partition of $A_{22}$. Since we only work with $\Delta$ in the sequel, we write $G(\Delta)$ for (3) and assume that $\Delta \in \Delta$ where $\Delta := \mathcal{C}(0, \infty), \mathcal{V}$ is the corresponding class of full block time-varying uncertainties for some given value set $\mathcal{V} = \text{Co}(\Delta_1, \ldots, \Delta_N)$ $\geq 0$ represented as the convex hull of finitely many real matrices $\Delta_i \in \mathbb{R}^{n_i \times n_i}$. We hence consider (3) with $\Delta \in \Delta$ as the precise mathematical description for (1).

As the zero block structure for $K(\Delta)$ in (2) resembles that in (1), the above LFT manipulations motivate to look at the following structured controller LFR

$$\begin{pmatrix} \dot{x}_e \\ \frac{\dot{x}_e}{x_e} \end{pmatrix} = \begin{pmatrix} A^e_1 & A^e_{12} & B^e_c \\ A^e_{21} & A^e_{22} & B^e_d \\ C^e_1 & C^e_{21} & D^e_c \end{pmatrix} \begin{pmatrix} x_e \\ w_{c,1} \\ w_{c,2} \end{pmatrix},$$

(4)

$$\begin{pmatrix} x_e \\ w_{c,1} \\ w_{c,2} \end{pmatrix} = \begin{pmatrix} A^e_1 & A^e_{12} & B^e_c \\ A^e_{21} & A^e_{22} & B^e_d \\ C^e_1 & C^e_{21} & D^e_c \end{pmatrix} \begin{pmatrix} x_e \\ w_{c,1} \\ w_{c,2} \end{pmatrix},$$

$$\hat{w}_c = \Delta_c(\Delta)z_c$$

with $z_c := \text{col}(\hat{z}_1, \hat{z}_2)$, $\hat{w}_c := \text{col}(\hat{w}_{c,1}, \hat{w}_{c,2})$ and the matrices $A^e_1 \in \mathbb{R}^{n_e \times n_e}$, $B^e_c \in \mathbb{R}^{n_e \times k}$, $C^e_1 \in \mathbb{R}^{m_e \times n_e}$. We refer to (4) as $K(\Delta)$ in order to display the dependence on $\Delta$. In order to have large enough flexibility in synthesis, we search for a lower block-triangular scheduling function

$$\Delta_c : \mathcal{V} \to \mathbb{R}^{r_e \times r_e} \text{ with } \Delta_c(\mathcal{V}) := \begin{pmatrix} \Delta_c^V(\mathcal{V}) & 0 \\ \Delta_c^V(\mathcal{V}) & \Delta_c^V(\mathcal{V}) \end{pmatrix}$$

(5)

of partition $r_e^1 := r_1^1 + r_2^1$. Indeed, for such a triangular $\Delta_c(\cdot)$, the controller LFR (4) still ensures the structure in (2). Note that $\Delta_c(\Delta)$ might depend in a nonlinear fashion on $\Delta \in \Delta$, while the choice of $r_e^1$, $n_e^1$ is part of the design problem. The closed-loop system for the plant (3) interconnected with (4) is then given by

$$\begin{pmatrix} \dot{x}_c \\ \frac{\dot{x}_c}{x_c} \end{pmatrix} = \begin{pmatrix} A_1 & A_{12} & B^c_1 \\ A_{21} & A_{22} & B^c_2 \\ C_1 & C_{21} & D^c \end{pmatrix} \begin{pmatrix} x_c \\ w_c \\ w_{c,1} \end{pmatrix},$$

(6)

with extended state $x_c := \text{col}(x, x_e)$, extended scheduling block $\Delta_c(V) := \begin{pmatrix} 0 & \Delta_c^V(\mathcal{V}) \end{pmatrix}$, and suitable closed-loop matrices $A_{ij}, B_i, C_i, D_i$ for $i, j = 1, 2$.

**Definition 1.** The controlled system (6) is well-posed if $I - \Delta_c(\mathcal{V})A_{22}$ is non-singular for all $V \in \mathcal{V}$. It is stable if
there exist constants $K$ and $\alpha > 0$ such that every solution of (6) is obtained for $w_p = 0$ and any $\Delta \in \Delta$ fulfills
\[ \|x_c(t)\| \leq K e^{-\alpha(t-t_0)}\|x_c(0)\| \quad \text{for all } t \geq t_0. \]

If (6) is well-posed, we can close the loop with $\Delta x_c(\Delta)$ to get
\[ (\dot{z}_c) = (\gamma_n) (\hat{w}_p) \]
where the entries with $*$ depend on $\Delta$ and $\Delta x_c(\Delta)$; note that the structured LFRs (3), (4) imply (1), (2) which lead to the desired zero block for $w_p \to z_p$ to render the $H_2$-norm finite. Hence, the $H_2$-gain-scheduling problem involves a nontrivial structural requirement.

Problem 2. For a given bound $\gamma > 0$, determine a controller $K(\Delta)$ structured as in (4)-(5) such that
\[ (\gamma_4) \text{ the controlled LFR (6) is well-posed and stable,} \]
(G1) the controlled LFR (6) is well-posed and stable, (G2) the squared $H_2$-norm of $w_p \to z_p$ for linear time-varying systems (in the stochastic setting as in Pagani and Feron (2000)) is smaller than $\gamma$ for $x_c(0) = 0$ and for all $\Delta \in \Delta$.

2.2 Analysis conditions for the original system

As well-known by the full block $S$-procedure, the conditions (G1)-(G2) are achieved if some matrix inequalities are feasible. This is formulated in the following standard analysis result from Scherer (2000) based on the class $\mathcal{P}$ of full block scalings $\mathcal{P} \subseteq \mathbb{S}^{n^2+r^2+s^2+r^2}$ satisfying
\[ (s)^T \mathcal{P} \left( \frac{\Delta x_c(V)}{I_{n+c}} \right) > 0 \quad \text{for all } V \in \mathbb{V}. \]

Theorem 3. The design goals (G1)-(G2) are reached for the structured controller $K(\Delta)$ with (4)-(5) if there exist $X_1 > 0$, $Z > 0$ with $tr(Z) < 1$ as well as $\mathcal{P} \subseteq \mathcal{P}$ such that
\[ L_{\text{sub}} \left( \frac{(-X_1) 0}{0 X_1} \right), \tilde{P}, P_2, \left( \frac{X_1}{0} \right) < 0, \]

\[ L \left( \frac{0 X_1}{0 X_1} \right), \tilde{P}, P_2, \left( \frac{X_1}{0} \right) < 0 \]
hold for the closed-loop system (6) with
\[ P_Z := \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 
\end{array} \right), \quad P_\gamma := \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 
\end{array} \right). \]

Since (8) involve two inequalities with specific outer factors and $\mathcal{P}$ is unstructured, we cannot directly eliminate or substitute the controller parameters for convexification. In the sequel, we thus introduce a novel design procedure, while, in view of Scherer (2000), we anticipate the synthesis result to be formulated with the full block scaling class
\[ \mathcal{P}_p := \left\{ P \in \mathbb{S}^{n^2+r^2} \mid (s)^T \mathcal{P} \left( \frac{I_0}{0} \right) > 0 \right\} \]
\[ \left( \begin{array}{c} x_c \\ z_c \\ \hat{z}_p \\ \hat{w}_p \\ \end{array} \right) = \left( \begin{array}{c} A_{11} \hat{z}_p \\ A_{12} \hat{z}_p \\ 0 \\ 0 \\ 
\end{array} \right) + \left( \begin{array}{c} B_{1} \hat{z}_p \\ B_{2} \hat{z}_p \\ C_{1} \hat{z}_p \\ C_{2} \hat{z}_p \\ 
\end{array} \right), \quad \left( \begin{array}{c} x_c \\ z_c \\ \hat{w}_p \\ \end{array} \right) = \left( \begin{array}{c} A_{11} \hat{z}_p \\ A_{12} \hat{z}_p \\ 0 \\ 0 \\ 
\end{array} \right) + \left( \begin{array}{c} B_{1} \hat{z}_p \\ B_{2} \hat{z}_p \\ C_{1} \hat{z}_p \\ C_{2} \hat{z}_p \\ 
\end{array} \right), \quad (13) \]

3. LIFTING DESIGN PROCEDURE

If $\hat{\mathcal{P}}$ is restricted in Theorem 3 to the class of positive real scalings $\left( \begin{array}{cc} 0 & Q \end{array} \right)$ satisfying the passivity condition related to (7), i.e. $\text{He} [\hat{\mathcal{P}} Q] > 0$ for all real $\Delta \geq 0$, the approach in Kössinger and Scherer (2019) shows that the anti-diagonal scaling block is a fundamental stumbling block for convexification by transformation. This motivates to replace the intractable inequalities (8) by a suitable, sufficient analysis condition for a certain class of positive scalings.

3.1 Lifted plant and closed-loop formulation

First, let us define a new LFR by reformulating the equations for $G(\Delta)$ in (3). Note that $\hat{w} = \Delta \hat{z}$ is equivalent to $\hat{w} = -\hat{w} + 2\Delta \hat{z}$ and thus to $\hat{w} = \Delta(\Delta) \hat{z}$ for $\Delta \in \Delta$, where
\[ \hat{w} := z := \left( \begin{array}{c} \hat{w} \\ \hat{z} \\ \end{array} \right), \quad \Delta(\Delta) := \left( \begin{array}{cc} -I_\Delta & 2V \\ 0 & I_\Delta \\ 
\end{array} \right) \quad \text{for } V \in \mathbb{V}. \]

Similarly, we can rearrange the matrices in (3) related to the uncertainty channel $\hat{w} \to \hat{z}$ to infer that (3) is true iff
\[ \left( \begin{array}{c} \dot{z}_c \\ z_c \\ \hat{w}_p \\ \end{array} \right) = \left( \begin{array}{c} A_{11} \frac{\hat{A}_{12}}{I_\Delta} \frac{B_{1}^{\prime}}{B_{2}} \frac{B_{1}}{B_{2}} \\ C_{1}^{\prime} \frac{C_{2}}{I_\Delta} \frac{D_{1}}{D_{2}} \frac{D_{1}}{D_{2}} \\ \frac{C_{1}}{I_\Delta} \frac{C_{2}}{I_\Delta} \frac{D_{1}}{D_{2}} \frac{D_{1}}{D_{2}} \\ \frac{C_{1}}{I_\Delta} \frac{C_{2}}{I_\Delta} \frac{D_{1}}{D_{2}} \frac{D_{1}}{D_{2}} \\ 
\end{array} \right) \left( \begin{array}{c} x_c \\ z_c \\ \hat{w}_p \\ \end{array} \right), \quad (14) \]

for some $V \in \mathbb{V}$ and for the relevant dimensions
\[ n := n^2 + n^r, \quad r^* := (n+1)^2 + \left( \begin{array}{c} n+1 \\ r^2 + r^2 \\ 
\end{array} \right); \]

the closed-loop matrices can be routinely expressed as
\[ \left( \begin{array}{c} A_{11} \left[ \begin{array}{c} B_{1}^{\prime} \\ B_{2}^{\prime} \\ C_{1}^{\prime} B_{1}^{\prime} \\ C_{2}^{\prime} B_{2}^{\prime} \\ C_{1}^{\prime} B_{2}^{\prime} \\ C_{2}^{\prime} B_{2}^{\prime} \\ C_{1}^{\prime} B_{2}^{\prime} \\ C_{2}^{\prime} B_{2}^{\prime} \\ 
\end{array} \right] \frac{I_{\Delta}}{I_{\Delta}} \frac{D_{1}}{D_{2}} \frac{D_{1}}{D_{2}} \frac{I_{\Delta}}{I_{\Delta}} \\ 
\end{array} \right) \left( \begin{array}{c} 0 \ I_{\Delta} \\ 0 \ I_{\Delta} \\ 0 \ I_{\Delta} \\ 0 \ I_{\Delta} \\ 
\end{array} \right), \quad (15) \]

3.2 Lifted analysis conditions with passive scaling classes

As a first observation, the scalings of $\mathcal{P}_p, \mathcal{P}_d$ in (10)-(11) already fulfill a passivity condition for the lifted block, i.e.
\[ \mathcal{P}_p = \left\{ P \in \mathbb{S}^r \mid \text{He} [P \Delta(\Delta) V] > 0 \right\} \quad \text{for all } V \in \mathbb{V}, \]
\[ \mathcal{P}_d = \left\{ P \in \mathbb{S}^r \mid \text{He} [P \Delta(\Delta) V]^T > 0 \right\} \quad \text{for all } V \in \mathbb{V}, \]

this can be seen, e.g., for $\mathcal{P}_p$ by applying a congruence transformation with the invertible $\left( \frac{I_\Delta}{I_\Delta} \right)$ to the condition $\text{He} [P \Delta(\Delta) V] > 0$ for some $P \in \mathbb{S}^n$ and $\Delta \in \Delta$. Similarly, if we replace $V$ by the lifted block $\Delta(\Delta)$, the extended block $\Delta_{x_c}(\Delta)$ from Section 2.1 becomes $\Delta_{x_c}(\Delta)$ in (15).
The following result covers step 3.4 Consequences for the original system

**Theorem 4.** Suppose there exist a structured controller $K(\Delta)$ with (4)-(5) as well as $X_{i} \succ 0$, $Z \succ 0$ with $\text{tr}(Z) < 1$, $\mathcal{P} \in \mathbf{P}$ such that the closed-loop system (14) for the lifted LFR (13) fulfills (18) with $P_z$, $P_r$ structured as in (9). Then we can construct a full block scaling $\hat{\mathcal{P}} \in \hat{\mathbf{P}}$ with (7) such that the inequalities (8) of Theorem 3 are true for the closed-loop system (6) obtained for the initial plant LFR (3) and the same controller $K(\Delta)$.

**Proof.** For some matrices $A$, $B$, $C$, $Q \in \mathbb{S}$, $R \in \mathcal{S}$ and $S$ of suitable dimension, we first observe that

$$\text{He}\left[\begin{pmatrix} A & QT \cr \odot & S \end{pmatrix} \begin{pmatrix} QT \cr \odot & R \end{pmatrix} \begin{pmatrix} \hat{A} & \hat{B} \cr \odot & \hat{C} \end{pmatrix} \right] = (\odot)^T \begin{pmatrix} 2Q & ST \cr ST & S \cr S & R \cr R & 0 \cr R & 0 \end{pmatrix} \begin{pmatrix} \hat{A} & \hat{B} \cr \odot & \hat{C} \end{pmatrix}.$$

Now, let the analysis inequalities in (18) be satisfied for some $\mathcal{P} \in \mathbf{P}$ and for the lifted LFR interconnected with a given controller $K(\Delta)$. By the definition of $\mathbf{P}$, we infer $\text{He}[\mathcal{P}\Delta_{lc}(V)] > 0$ for all $V \in \mathcal{V}$. Applying for each $V \in \mathcal{V}$ a congruence transformation with

$$\begin{pmatrix} V & 0 \\ I_v & 0 \end{pmatrix} \text{ yields } \text{He}\left[\begin{pmatrix} V & 0 \\ I_v & 0 \end{pmatrix} \mathcal{P} \begin{pmatrix} V & 0 \\ I_v & 0 \end{pmatrix}^T \right] > 0$$

for all $V \in \mathcal{V}$. Next, let us partition $\mathcal{P}$ according to the outer factors of the latter inequality as

$$\mathcal{P} = \begin{pmatrix} Q_1 & Q_{12} \cr Q_{12}^T & S \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \cr Q_{12}^T & S \end{pmatrix}^T \begin{pmatrix} V & 0 \\ S & R \end{pmatrix} \begin{pmatrix} V & 0 \\ S & R \end{pmatrix}^T \begin{pmatrix} \hat{A} & \hat{B} \cr \odot & \hat{C} \end{pmatrix}.$$

Thus $\hat{\mathcal{P}} \in \hat{\mathbf{P}}$. It is essential that the analysis inequalities (8) obtained for (3) and for the same $K(\Delta)$ are also valid for the constructed $\hat{\mathcal{P}}$ from (20). This follows by applying suitable congruence transformations to (18) along with (19); we need to omit the details for reasons of space. 

3.5 Comparison of scaling classes

Let $\hat{\mathbf{P}}_F$ be the full block scaling class used for gain-scheduling in Scherer (2000). Note that $\hat{\mathbf{P}}_F$ is a subset of $\hat{\mathbf{P}}$ from Section 2.2 and consists of all scalings $\hat{\mathcal{P}} \in \mathcal{S}^{(6+r^2+a^2)}$ satisfying in addition to (7) the constraints

$$(\odot)^T \hat{\mathcal{P}} \begin{pmatrix} I_{a+r^2} \\ 0 \end{pmatrix} = 0$$

and

$$(\odot)^T \hat{\mathcal{P}} \begin{pmatrix} 0 \\ I_{a+r^2} \end{pmatrix} > 0.$$  

We emphasize that it is not at all clear how to convexify the synthesis problem based on (8) for the class $\hat{\mathbf{P}}_F$. Still, let us briefly sketch that the choice of the specifically structured scalings in (17) causes no conservatism, i.e., if $\gamma_F$ is the optimal bound obtained for (8) with $\hat{\mathbf{P}}_F$, and $\gamma_\Omega$ denotes the one for synthesis based on (18) with the lifted LFR and $\mathbf{P}$, the relation $\gamma_\Omega \leq \gamma_F$ always holds. For this purpose, let us perform the lifting step in Section 3.1 both for the plant $G(\Delta)$ and for $K(\Delta)$. This leads to the lifted LFR $\hat{G}(\Delta)$ in (13) as well as to a lifted controller LFR $\hat{K}(\Delta)$ with a scheduling channel resembling the structure of those for $G(\Delta)$, while being scheduled by the structured $\Delta_{lc}(\Delta_{lc}(\Delta))$ with $\Delta_{lc}(\Delta)$ from

![Fig. 2. Steps of lifting technique: Build plant LFR $G(\Delta)$ in (1) and lifted plant LFR (2), design controller $K(\Delta)$ for the lifted LFR (3) and interconnect with $G(\Delta)$ in (4).](image-url)
(12). Note that the resulting LFR of $K_l(\Delta)$ can be always obtained by a structural restriction of the LFR matrices for $K(\Delta)$. By exploiting the scaling properties (21), (7) imposed for $\hat{P} \in \hat{\mathcal{P}}_F$, it is crucial to see that the original analysis inequalities (8) hold for some $\mathcal{P} \subseteq \hat{\mathcal{P}}_F$ if and only if the modified analysis inequalities (18) are satisfied for the closed-loop system obtained from interconnecting $G_l(\Delta)$ with the lifted controller LFR $K_l(\Delta)$, and for some scaling $\mathcal{P} \subseteq \mathcal{S}$ satisfying the passivity constraint

$$\text{He}[\mathcal{P} \left( \begin{array}{cc} A_l(V) & 0 \\ 0 & \Delta_l(\Delta(V)) \end{array} \right)] > 0 \quad \text{for all} \quad V \in \mathcal{V}.$$ (22)

We omit the details for reasons of space, but remark that, upon permutation, $\mathcal{P}$ in (22) equals $\hat{\mathcal{P}}$. We observe that (22) is exactly the condition that appears for the passive scalings $\mathcal{P}$ in (17) if replacing $\Delta_l(\Delta(V))$ by $\Delta_l(V)$. Since the class of all LFRs for $K_l(\Delta)$ encompasses that of all LFRs for $K_l(\Delta)$ as argued above, we infer $\gamma_l \leq \gamma_F$.

4. SYNTHESIS FOR LIFTED SYSTEM

In the following part we deal with the synthesis step $\mathfrak{3}$ in Fig. 2, i.e., we use a structured controller parameter transformation combined with a suitable scaling factorization to solve the $\mathcal{H}_\infty$-gain-scheduling problem for the lifted LFR. In the context of structured $\mathcal{H}_\infty$-design, a related factorization is established for positive definite matrices in Scherer (2014) to design triangular, time-invariant controllers, as well as for positive real matrices in Rosinger and Scherer (2019) to synthesize gain-scheduled controllers with a diagonal scheduling function of scalar parameters. Technically, we show as a novel step that the passivity with a diagonal scheduling function of scalar parameters.

Fig. 3. Optimal bounds $\gamma_{opt}$ for the lifted design (dashed red) and $D/G$-scalings (full blue) with $a \in [0.4, 1.4]$.

are satisfied after inserting for $i, j = 1, 2$ the blocks

$$\begin{bmatrix} Y_1 & I_\nu \\ I_\nu & X_1 \end{bmatrix},$$

$$(A_{ij}Y_j) \begin{bmatrix} B_{p} \\ 0 \end{bmatrix} \begin{bmatrix} X^TA_{ij} \\ X^T B_p \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix} \begin{bmatrix} K_l & I \\ 0 & D \end{bmatrix}.$$

Since $\mathcal{V} = \mathcal{C}(\Delta_1, \ldots, \Delta_N)$ and the sets $\mathcal{P}_p, \mathcal{P}_d$ can be expressed as in (10), (11), the conditions $Q \in \mathcal{P}_p, \hat{Q} \in \mathcal{P}_d$ reduce to finitely many inequalities (see Scherer (2000)):

$$(*)^TQ(\hat{\Delta}) < 0, \quad (**)^TQ(\hat{\Delta}) > 0,$$

$$+ (**)^T \hat{Q}(\hat{\Delta}) < 0 \quad \text{for} \quad i = 1, \ldots, N.$$ After applying the Schur complement to (25), we get a standard LMI test with finitely many constraints such that a direct minimization over $\gamma$ is possible. We present the proof of Theorem 5 in Appendix A. Note that our proof is constructive, i.e., if the associated LMIs are feasible, a suitable $\mathcal{H}_\infty$-controller (4)-(5) can be constructed with McMillan degree of at most $n^e$ and scheduling block size $r^e$ of at most $2r_w$, while we give an explicit formula for $\Delta_l(\cdot)$.

Remark 6. Analogously to Remark 5 and 6 in Rosinger and Scherer (2019), Theorem 5 can also handle gain-scheduling with quadratic performance and multiple objectives by properly modifying $P_m$. Also $K_{11}, K_{12}, K_{13}, L_1$ can be partially eliminated to reduce the number of variables.

5. A NUMERICAL EXAMPLE

To present a short academic example, let the matrices of the structured LFR in (3) be given as in Section 4.2 of Rosinger and Scherer (2019) with $A_{12}$ depending on some parameter $a \in [0.4, 1.4]$. Moreover, let $\delta = \text{diag}(\delta_1, I_2)$ be of size $3 \times 3$ with time-varying parametric uncertainties $\delta_1(t) \in [-0.8, 0.8], \delta_2(t) \in [-0.6, 0.6]$. Based on implementations of our algorithms in the Matlab Robust Control Toolbox, we compare in Fig. 3 the optimal bounds $\gamma_{opt}$ of the squared $\mathcal{H}_\infty$-norm for the lifted design (dashed red) obtained for the passive scaling class $\mathcal{P}$ from (17) with $D/G$-scalings (full blue). Note that $\mathcal{H}_\infty$-gain-scheduling synthesis for $D/G$-scalings with structured LFRs can be performed with the positive real scaling results from Rosinger and Scherer (2019) for the original LFR (3) by using the well-known Möbius transformation to map the uncertainty intervals for $\delta_1$ into $[0, \infty]$. To the best knowledge of the authors, there exist no alternative approaches that solve the underlying structured $\mathcal{H}_\infty$-design problem in this
generality. The results confirm that the lifted approach is less conservative than $D/G$-scalings as expected from Section 3.5. In particular, beyond the shown parameter range for $a$, the synthesis LMIs get infeasible for $D/G$-scalings if $a$ approaches 1.67, while the lifted design is feasible up to $a = 2.17$.

6. CONCLUSION AND OUTLOOK

In this work, we have introduced a new lifting technique to synthesize controllers for the $H_2$-gain-scheduling problem with full block scalings. Especially, our design framework guarantees finiteness of the closed-loop $H_2$-norm by relying on structured plant and controller LFRs, and by constructing a block-triangular scheduling function. We hope that these new methodologies offer manifold potential for refined synthesis results as the combination with parameter-dependent Lyapunov functions. A further task is the investigation of possible numerical advantages of the used scaling extension over existing approaches.

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Appendix A. PROOF OF THEOREM 5

Necessity. Let (18) be satisfied for (14), $X_1 > 0$, $Z > 0$ with $\text{tr}(Z) < 1$, and $X_2 := P \in \mathbb{P}$, i.e.

$$\mathcal{L}_{\text{sub}}\left(\left(\begin{array}{cc} X_1 & 0 \\ 0 & 0 \end{array}\right), \left(\begin{array}{c} I \\ 0 \end{array}\right), P_{22}, \left(\begin{array}{c} X_1 A_1 \\ C \end{array}\right) \right) \prec 0,$$

$$\mathcal{L}\left(\left(\begin{array}{c} I \\ 0 \end{array}\right), \left(\begin{array}{c} I \\ 0 \end{array}\right), P_{21}, \left(\begin{array}{c} X_1 A_{1j} \\ C_k \end{array}\right) \right) \prec 0.$$

(A.1)

Step 1 (Factorizations)

W.l.o.g., we assume that $n^e \geq n^s$ to factorize $X_1$ as

$$X_1 \mathcal{Y}_i = \mathcal{Z}_i \text{ with } \mathcal{Y}_i := \left(\begin{array}{c} I \\ 0 \end{array}\right), \mathcal{Z}_i := \left(\begin{array}{c} I X_{1j} \\ 0 U_j \end{array}\right).$$

(A.2)

for $i = 1$ such that $\mathcal{Y}_1$ has full column rank (see Scherer et al. (1997)).

Moreover, if we assume that $r_1^i \geq r^s$ and $r_2^i \geq r^s$, let us show that $X_2$ can be also factorized as in (A.2) such that $\mathcal{Y}_2$ has full column rank where $V_2$ and $U_2$ are lower and upper block-triangluar matrices, respectively, with respect to the partition $(r_1^1 + r_2^1) \times (r^s + r^s)$, and where $X_2, Y_2$ are partitioned as in (23) for some suitable blocks $Q_2, Q_3, Q_1$. For this purpose, let us first clarify that $X_2 \in \mathbb{P}$ is invertible with some sub-blocks of full column rank, while we use the following partitions according to $r = r^s + r_1^1 + r_2^1$:

$$X_2 = \left(\begin{array}{c} Q_3 : S_{13}^T \end{array} \begin{array}{c} S_{12}^T \\ S_{23} \end{array} \begin{array}{c} R_{11}^T \\ R_{12}^T \end{array} \end{array} \begin{array}{c} S_{21}^T \\ S_{22} \end{array} \begin{array}{c} R_{21} \\ R_{22} \end{array} \right), X_2^{-1} = \left(\begin{array}{c} Q_1 : S_{11}^T \\ S_{12} \\ S_{21}^T \\ S_{22} \end{array} \begin{array}{c} R_{11} \\ R_{12} \\ R_{21} \\ R_{22} \end{array} \right).$$

(A.3)

For the given partition of $X_2$ in (A.3), we note that $S_{13}, S_{23}$ are tall due to $r_2^j \geq r^s$ for $j = 1, 2$. Let us firstly
perturb $R_{11}, R_{21}, R_{22}$ to achieve invertibility of $R_{22}$ and
$(R_{11}, R_{12}^T)$. This allows to perturb $S_{13}, S_{23}, Q_3$ such that
\[ H := -(I_0) \left( \begin{array}{c} R_{11} \ R_{12}^T \\ R_{21} \ R_{22} \end{array} \right)^{-1} \left( \begin{array}{c} S_{11} \\ S_{21} \end{array} \right), \quad \tilde{S}_{22} := -R_{22}^{-1} S_{23} \] (A.4)
have full column rank and $Q_3 := (s^T_{13} s^T_{23})^{-1} (R_{11} R_{21})^{-1} (s^T_{13} s^T_{23})$ is invertible. In particular, this implies invertibility of $\mathcal{X}_2$. Immediately, we infer that (A.2) is true for $i = 2$ with
\[ \begin{pmatrix} X_2 \\ U_2 \end{pmatrix} := \begin{pmatrix} Q_2 \\ \tilde{S}_{22} \\ S_{23} \end{pmatrix}, \quad \begin{pmatrix} Y_2 \\ V_2 \end{pmatrix} := \begin{pmatrix} Q_1 \\ l_0 \tilde{S}_{11} \\ 0 \end{pmatrix} \] (A.5)
where $Q_2 := Q_3 - S_{13}^T R_{22}^{-1} S_{23}$ and $S_{12} := S_{13} - R_{13}^T R_{12}^{-1} S_{23}$. By the block-inversion formula, we note that $\tilde{Q}_3$ is invertible which, combined with (A.4), reveals that $S_{11} = H_0 \tilde{Q}_1$ and $\tilde{S}_{22}$ have full column rank. Thus, $V_2$ has full column rank which implies the same for $\mathcal{Y}_2$ in (A.2).

Step 2 (Proof that $Q_2, Q_3 \in \mathbb{P}_p$, $Q_1 \in \mathbb{P}_d$).
For brevity, let us omit the argument of $\Delta_1(-), \Delta_3(-)$ and $\Delta_{10}(-)$. Further, let us split $\Delta_0$ into two parts such that
\[ 0 \prec \text{He}[X_2 \Delta_0] := \text{He} \left[ \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} \right] + \text{He} \left[ \begin{pmatrix} 0 & \Delta_0 \end{pmatrix} \right] \] (A.6)
Let us perform a congruence transformation with $Y_2$ on (A.6) while using (A.2) for $i = 2$ and (A.5). This leads to
\[ 0 \prec \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{He}[Q_2 \Delta_0] + \begin{pmatrix} \Delta_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{He}[Q_3 \Delta_0] + \text{He}[Q_0 \Delta_0] \] (A.7)
\[ + \text{He} \left[ \begin{pmatrix} 0 & \Delta_0 \end{pmatrix} \right] \quad (V_2; 0) \cdot 0 \succ \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \Delta_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{He}[Q_2 \Delta_0] + \begin{pmatrix} \Delta_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{He}[Q_3 \Delta_0] + \text{He}[Q_0 \Delta_0] \] (A.8)
Since $U_2, V_2, \Delta_0$ are lower block-triangular, the diagonal entries of (A.7) just read as $Q_2, Q_3 \in \mathbb{P}_p$, $Q_1 \in \mathbb{P}_d$.

Step 3 (Derivation of synthesis inequalities (25)).
Let us use the factorizations in (A.2) to apply congruence transformations with $\mathcal{Y}_1$ to (A.1) for $i = 1, 2$. We get
\[ \begin{pmatrix} -Z^T Y_1 & 0 \\ \tilde{Z}^T A_y^T \tilde{Y}_j & \tilde{Z}^T B_y^T \tilde{Y}_j \end{pmatrix} \prec 0 \] (A.8)
\[ \begin{pmatrix} -Z^T Y_1 & 0 \\ \tilde{Z}^T A_y^T \tilde{Y}_j & \tilde{Z}^T B_y^T \tilde{Y}_j \end{pmatrix} \prec 0 \] (A.9)
By matching (A.8) to (25), the necessity part can then be finished similarly to Rössinger and Scherer (2019): By symmetry, $Z^T Y_1$ equals $X$ from (26). Further, some calculations reveal that
\[ \begin{pmatrix} \tilde{Z}^T B_y^T \tilde{Y}_j & \tilde{Z}^T A_y^T \tilde{Y}_j \end{pmatrix} \prec 0 \] (A.8)
\[ \begin{pmatrix} \tilde{Z}^T B_y^T \tilde{Y}_j & \tilde{Z}^T A_y^T \tilde{Y}_j \end{pmatrix} \prec 0 \] (A.9)
for $i, j = 1, 2$ after performing the substitution
\[ \begin{pmatrix} K_{ij} L_i \\ M_{ij} N_i \end{pmatrix} := \begin{pmatrix} X^T A_y L_i \\ \tilde{X}^T B_y \tilde{Y}_j \\ \tilde{X}^T \tilde{B}_y \tilde{Y}_j \end{pmatrix} \] (A.10)
Moreover, by exploiting the sparsity structure of the controller matrices and $U_2, V_2$, we can introduce
\[ \begin{pmatrix} K_{i1} & K_{i2} : K_{i13} L_i \\ K_{i2} R_{22} & 0 \\ K_{i3} & \tilde{K}_{i2} R_{22} = (0 \ 0 \ \tilde{Q}_2 A_{y2}^T \tilde{L}_2) \end{pmatrix} \] (A.10)