Microscopic states of Kerr black holes from boundary-bulk correspondence

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(Dated: June 13, 2019)

It was claimed by the author that black holes can be considered as topological insulators. They both have boundary modes and those boundary modes can be described by BF theory. In this paper, we analyze the boundary modes of black holes with the method developed for topological insulator. Firstly the BTZ black hole is considered, and the results are compatible with the previous works. Then we generalize those results to Kerr black hole. Some new results are obtained: dimensionless right- and left-temperature can be defined and have well behavior both in Schwarzschild limit \( a \to 0 \) and extremal limit \( a \to M \). Also a central charge \( c = 12M^2G \) is associated with arbitrary Kerr black hole. We also identify the microstates of Kerr black hole with the quantum states of the boundary scalar field. From this identification we can count the number of microstates of Kerr black hole and give the entropy approximately.
I. INTRODUCTION

It has been well accepted that black holes behave like thermodynamics objects. Due to Bekenstein and Hawking, the black holes have temperature and entropy

$$T_H = \frac{\hbar \kappa}{2\pi}, \quad S = \frac{A}{4G\hbar},$$

where $\kappa$ is the surface gravity, and $A$ the area of the horizon. Understanding those properties is a fundamental challenge of quantum gravity, especially identifying the black hole microstates which account for the entropy.

Since this is a difficult problem, it is helpful to map this problem to a solved one. In the previous works, the author claim that the black holes can be considered as kind of topological insulators. For BTZ black hole in three dimensional spacetime this claim is tested in Ref. The boundary modes on the horizon of BTZ black hole can be described by two chiral massless scalar fields with opposite chirality. This is the same as topological insulator in three dimensional spacetime (also called quantum spin Hall states). From those chiral scalar fields one can construct the $W_{1+\infty}$ algebra which contain the near horizon symmetry algebra of BTZ black hole as subalgebra. The $W_{1+\infty}$ algebra was used to classify the BTZ black holes and give the ‘W-hairs’ of black holes, which maybe essential to solve the information paradox. The microstates of BTZ black hole are identified as the quantum states of those chiral scalar fields.

For higher dimensional black holes, such as Kerr black hole in four dimension, the boundary modes can be described by effective BF theory, which is also the same as higher dimensional topological insulators. From the BF theory, one can construct a free scalar field theory. An essential property of topological insulators is boundary-bulk correspondence, which relates the topological structure of the bulk state to the presence of gapless boundary modes. From those boundary modes one can get some key properties of the bulk states. Just like the BTZ black hole, we also identify the quantum states of the scalar field with the microstates of black hole for Kerr black hole.

In this paper, we start from the boundary BF theory description of boundary modes of black holes. Firstly we analyze the BTZ black hole and relate the results with the previous works. Then we apply the same method to Kerr black holes. For Kerr black holes, we get some new results:

- We can define dimensionless left- and right-temperature $T_{L/R} = \frac{r \pm a}{2\pi MG}$ for all Kerr black holes (including Schwarzschild black hole), which coincide with the temperatures in Kerr/CFT correspondence in the extremal limit and with the Unruh temperature in the Schwarzschild limit.
- For all Kerr black holes, if a dual conformal field theory (CFT) exist, one can associate a central charge $c = 12M^2G$. It coincides with that in the extremal case for Kerr/CFT correspondence.
- We identify the microscopic states for Kerr black hole with the quantum states of the boundary scalar field and count the number of those states which can give the Bekenstein-Hawking area law approximately. For extremal Kerr black hole, one can get the entropy exactly.

The paper is organized as follows. In section II, we analyze the BTZ black hole. In section III, the Kerr black hole is analyzed with the same method. Section IV is the conclusion. In the following we set $G = 1$. 

II. THE BTZ BLACK HOLE

In this section, we consider the boundary modes on the horizon of BTZ black hole from the boundary BF theory. To quantize the boundary BF theory, we mainly follow the method in Ref. [14, 15].

The metric of BTZ black hole is

\[ ds^2 = -N^2 dv^2 + 2dvdr + r^2(d\varphi + N\varphi dv)^2, \]

where \( N^2 = -8M + \frac{4}{L^2} + \frac{4\hat{J}^2}{r^2}, N\varphi = -\frac{4J}{r}. \)

Choose the following Newman-Penrose null co-triads

\[ l = -\frac{1}{2}N^2dv + dr, \quad n = -dv, \quad m = rN\varphi dv + rd\varphi, \]

then the corresponding spin connection which is used later is \( A_2 = \alpha m - \kappa n \) with \( \alpha = N\varphi, \kappa = r/L^2 - r(N\varphi)^2. \)

For later use, we define new coordinate \( dv' = dv/\gamma \) with the coefficient \( \gamma \) will be determined later. Then on the horizon \( \Delta : r = r_+, \)

\[ m = (rN\varphi dv + rd\varphi)|_{\Delta} = -\frac{\gamma r}{L} dv' + r_+ d\varphi. \]

A. The canonical formula

On the horizon \( r = r_+ \), we choose \((x^0, x^1) = (v', \varphi)\). The boundary BF theory on the horizon \( \Delta \) is given by

\[ S = \int_{\Delta} BF = \int_{\Delta} BdA, \]

with the constraints

\[ dB = \frac{1}{8\pi} m, \quad dA = 0, \]

where \( A \) is the non-rotating component of the spin connection \( A_2 \).

The canonical form is

\[ S = \int_{\Delta} d\varphi d'\varphi B(\partial_0 A_1 - \partial_1 A_0) = \int_{\Delta} d\varphi d'\varphi (B\partial_0 A_1 + A_0\partial_1 B). \]

Choosing the gauge \( A_0 = 0 \) gives \( A_1 = \partial_1 \phi \), so

\[ S = -\int_{\Delta} d\varphi d'\varphi \partial_1 B \partial_0 \phi. \]

Since the horizon is a null hypersurface, the metric is degenerate. To make things easier, we assume the effective metric of the horizon as

\[ d\tilde{s}^2 = -dv'^2 + r_+^2 d\varphi'^2, \]

then

\[ \sqrt{-g} = r_+, \quad \epsilon^{01} = \frac{1}{r_+}. \]
The action can be rewritten as

\[ S = -\int_\Delta d^2x \sqrt{-g} \frac{1}{\sqrt{-g}} \partial_1 B \partial_0 \phi = \int_\Delta d^2x \sqrt{-g} \pi \dot{\phi}, \]  

(11)

where \( \pi = -\frac{1}{\sqrt{-g}} \partial_1 B \) is the canonical momentum. It is easy to show that the Hamiltonian is zero. To describe a relativistic dynamics, one can add a Hamiltonian to get \[ S' = \int_\Delta d^2x \sqrt{-g} (\pi \dot{\phi} - \mathcal{H}(\pi, \phi)) = \int_\Delta d^2x \sqrt{-g} (\pi \dot{\phi} - \frac{1}{2m_0} \pi^2 - \frac{m_0}{2} g^{ij} \partial_i \phi \partial_j \phi), \]  

(12)

where \( m_0 \) is a free parameter. Other Hamiltonian can also be chosen, but the above one is the simplest \[. \]  

The equations of motion are

\[ \pi = m_0 \dot{\phi}, \quad \dot{\pi} = m_0 \Delta \phi, \]  

(13)

where \( \Delta \) is the Laplace-Beltrami operator.

The above equation can be recast into a duality relation

\[ \epsilon_{\mu \nu} \partial_\nu B = m_0 \partial_\mu \phi. \]  

(14)

The action (12) is just the action for massless scalar field

\[ S' = \frac{m_0}{2} \int d^2x \sqrt{-g} \mu^\nu \partial_\nu \phi \partial_\nu \phi. \]  

(15)

The Hamiltonian can be given by

\[ H = \frac{m_0}{2} \int d\phi \sqrt{-g} ((\partial_0 \phi)^2 + (\frac{\partial_1 \phi}{r_+})^2). \]  

(16)

We can also define the angular momentum

\[ J = m_0 \int d\phi \sqrt{-g} (\partial_0 \phi \partial_1 \phi) \]  

(17)

### B. Quantization

Next we quantize the massless scalar field theory (12). Expanding the fields with the solution of motion gives

\[ \phi(v', \varphi) = \phi_0 + p_\varphi v' + p_v \varphi + \sqrt{\frac{1}{m_0 A}} \sum_{n \neq 0} \sqrt{\frac{1}{2\omega_n}} [a_n e^{-i(\omega_n v' - k_n \varphi)} + a_n^* e^{i(\omega_n v' - k_n \varphi)}], \]  

\[ B(v', \varphi) = -m_0 (B_0 + \frac{p_\varphi}{r_+} v' + r_+ p_v \varphi) + \sqrt{\frac{4m_0}{r_+ A}} \sum_{n \neq 0} \sqrt{\frac{1}{(2\omega_n)^3}} k_n [a_n e^{-i(\omega_n v' - k_n \varphi)} + a_n^* e^{i(\omega_n v' - k_n \varphi)}], \]  

(18)

where \( \omega_n = \frac{|n|}{r_+}, k_n = n \) and \( A = 2\pi r_+ \) is the length of the circle. It is straight to show that the above expression satisfy the dual relation (14).

The quantum operators satisfy the commutative relation

\[ [\hat{\phi}(v', \varphi), \hat{\pi}(v', \varphi')] = i\delta(\varphi - \varphi'), \]  

(19)
which gives
\[
[\hat{\phi}_0, m_0 \hat{p}_v] = \frac{i}{A}, \quad [\hat{a}_n, \hat{a}^+_m] = \delta_{n,m}.
\] (20)

We can also consider \( B \) and \( \partial_1 \phi \) as two canonical variables, and lead to a further commutation relation
\[
[\hat{B}_0, m_0 \frac{\hat{p}_\phi}{r_+}] = -\frac{i}{A}.
\] (21)

Since the zero mode \( \phi_0, B_0 \) are constants on the cylinder, the canonical momentum are quantized with
\[
m_0 p_v = \frac{n_1}{A}, \quad m_0 \frac{p_\phi}{r_+} = \frac{n_2}{A} \quad n_1, n_2 \in \mathbb{Z}.
\] (22)

For BTZ black hole, the \( B \) field satisfy the constraint (6), or with the component
\[
\partial_0 B = -\frac{r_- \gamma}{8\pi L}, \quad \partial_1 B = \frac{r_+}{8\pi}.
\] (23)

Submitting the expression (18) gives
\[
p_v = -\frac{1}{8\pi m_0}, \quad p_\phi = \frac{\gamma r_+ r_-}{8\pi L m_0}.
\] (24)

Combing with the Eqn.(22) give the quantization condition
\[
r_+ = 4n_1, \quad r_- = 4n_2 \frac{L}{\gamma r_+}, \quad n_1, n_2 \in \mathbb{N}.
\] (25)

The dimensionless right- and left-temperature are defined as
\[
T_{R/L} = \frac{r_+ \pm r_-}{2\pi L}.
\] (26)

To fix the coefficient \( \gamma \) we make the following assumption:
\[
T_{R/L} \propto p_v + \frac{p_\phi}{r_+},
\] (27)

which gives \( \gamma = \frac{L}{r_+} \). So
\[
r_+ = 4n_1, \quad r_- = 4n_2, \quad n_1, n_2 \in \mathbb{N}.
\] (28)

From the above expression, the entropy \( S_+, S_- \) of the outer and inner horizon of black hole are also quantized according to
\[
S_+ = \frac{2\pi r_+}{4} = 2\pi n_1, \quad S_- = \frac{2\pi r_-}{4} = 2\pi n_2, \quad n_1, n_2 \in \mathbb{N}.
\] (29)

The Hamiltonian operator and angular momentum operator are given by
\[
\hat{H} = \pi m_0 r_+ (p_v^2 + \frac{p_\phi^2}{r_+^2}) + \sum_{n \neq 0} \frac{|n|}{r_+} \hat{a}_n^+ \hat{a}_n,
\]
\[
\hat{J} = 2\pi m_0 r_+ \hat{p}_v \hat{p}_\phi + \sum_{n \neq 0} n \hat{a}_n^+ \hat{a}_n,
\] (30)

where we omit the zero-point energy.
C. Connection with the chiral boson theory

We have two descriptions for the boundary modes of BTZ black hole. In the previous works \cite{7, 19}, it was shown that the boundary mode on the horizon of BTZ black hole can be described by two compact chiral massless scalar field $\Psi_1(\varphi + \frac{v}{L})$ and $\Psi_2(\varphi - \frac{v}{L})$. In the previous section, we show that the boundary modes can be described by the massless scalar field $\phi(v', \varphi)$.

Those compact chiral scalar fields have expansions

$$
\Psi_1(\varphi + \frac{v}{L}) = \Psi_{10} - \alpha_0(\varphi + \frac{v}{L}) + i \sum_{n \neq 0} \frac{\alpha_n}{n} e^{i n (\varphi + \frac{v}{L})}, \quad \alpha_n^* = \alpha_{-n},
$$

$$
\Psi_2(\varphi - \frac{v}{L}) = \Psi_{20} - \alpha_0^+(\varphi - \frac{v}{L}) + i \sum_{n \neq 0} \frac{\alpha_n^+}{n} e^{i n (\varphi - \frac{v}{L})}, \quad (\alpha_n^+)^* = \alpha_n^+, \quad (31)
$$

where

$$
\alpha_0 = -\pi T_L, \quad \alpha_0^+ = \pi T_R. \quad (32)
$$

The relation between those two descriptions is very simple,

$$
\phi(v', \varphi) = - (\Psi_1(\varphi + \frac{v}{L}) + \Psi_2(\varphi - \frac{v}{L})). \quad (33)
$$

In components, it reads

$$
\phi_0 = - (\Psi_{10} + \Psi_{20}), \quad p_v = - \frac{\alpha_0^+ - \alpha_0}{r_+}, \quad p_\varphi = \alpha_0^+ + \alpha_0, \quad (34)
$$

and

$$
a_n = \begin{cases} 
- i \alpha_n^+ \sqrt{4 \pi m_0/n}, & n > 0 \\
- i \alpha_n \sqrt{-4 \pi m_0/n}, & n < 0 
\end{cases}, \quad a_n^+ = \begin{cases} 
 i \alpha_n^+ \sqrt{4 \pi m_0/n}, & n > 0 \\
 i \alpha_n \sqrt{-4 \pi m_0/n}, & n < 0 
\end{cases}, \quad (35)
$$

The quantum operators satisfy the commutation relation

$$
[\hat{a}_n, \hat{a}_m^+] = \delta_{n,m}, \quad [\hat{a}_n^+, \hat{a}_m^+] = \frac{n}{2k} \delta_{n+m,0} = - [\hat{a}_n, \hat{a}_m], \quad (36)
$$

and fix the free parameter $m_0 = \frac{k}{2\pi} = \frac{L}{8\pi}$. Thus the dimensionless temperatures are

$$
T_{R/L} = - \frac{r_+}{2\pi} (p_v \mp \frac{p_\varphi}{r_+}). \quad (37)
$$

It is easy to show that they satisfy the condition

$$
\gamma \pi (T_L + T_R) = 1. \quad (38)
$$

The Hamiltonian operator and angular momentum operator \cite{30} can be represented by the operators $\{\hat{a}_n^\pm\}$ as

$$
\hat{H} = \frac{1}{r_+} (\hat{L}_0^+ + \hat{L}_0^-), \quad \hat{J} = \hat{L}_0^+ - \hat{L}_0^-, \quad (39)
$$

where the operators $\hat{L}_0^\pm$ are defined by

$$
\hat{L}_0^\pm = k (\hat{a}_0^\pm)^2 + 2k \sum_{n=1}^{\infty} \hat{a}_n^\pm \hat{a}_n^\pm, \quad \hat{a}_n^\pm \equiv \hat{a}_{-n}, \quad (40)
$$
and :: is normal ordering.

The scalar field $\phi(v', \varphi)$ can be considered as collectives of harmonic oscillators, and a general quantum state can be represented as $|p_v, p_\varphi; \{n_k\}] > \text{where } p_v, p_\varphi \text{ are zero mode parts, and } \{n_k\} \text{ are oscillating parts. As was shown in Ref.} [6, 7, 8, 9, 10, 11], \text{the BTZ black hole corresponds to the zero mode part, thus}$

$$< p_v, p_\varphi; \{0\}|\hat{J}|p_v, p_\varphi; \{0\} > = J, \quad < p_v, p_\varphi; \{0\}|\hat{H}|p_v, p_\varphi; \{0\} > = \gamma M = ML/r_+.$$  (41)

On the other hand, the microstates of BTZ black hole can be represented by $|0, 0; \{n_k\}] >$, and satisfy

$$\frac{1}{c} < 0, 0; \{n_k\}|\hat{J}|0, 0; \{n_k\} > = J, \quad \frac{1}{c} < 0, 0; \{n_k\}|\hat{H}|0, 0; \{n_k\} > = ML/r_+,$$  (42)

where $c = 3L/2G$ is the Brown-Henneaux central charge. It can be written as

$$\sum n_k = cJ, \quad \sum |n_k| = cML, \quad n_k \in \mathbb{Z}/\{0\}. \quad (43)$$

Different sequences $\{n_k\}$ correspond to different microstate of the BTZ black hole with fixed $(M, J)$. For non-extremal black holes, the total number of the microstates for BTZ black hole with parameters $(M, J)$ can be calculated through Hardy-Ramanujan formula, and gives

$$N(M, J) \approx \frac{1}{c(ML + J)c(ML - J)} \exp(2\pi \sqrt{cML + J/12} + 2\pi \sqrt{cML - J/12}). \quad (44)$$

So the entropy of BTZ black hole is given by

$$S = \ln N(M, J) = \frac{2\pi r_+}{4G} - 2\ln \frac{r_+^2}{G^2} + \cdots, \quad (45)$$

which is just the Bekenstein-Hawking entropy with some low order corrections.

For extremal BTZ black hole $J = ML$, the entropy is given by

$$S = \ln N(M) = \ln\left(\frac{1}{cML} \exp(2\pi \sqrt{cML/6})\right) = \frac{2\pi r_+}{4G} - 2\ln \frac{r_+}{G} + \cdots,$$  (46)

which is the same as the results in “horizon fluff” proposal [21].

### III. KERR BLACK HOLE

In this section we analyse the Kerr black hole with the same method. The metric of Kerr black hole can be written as

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dv^2 + 2dvdr - 2a\sin^2 \theta dv d\varphi - \frac{4aMr \sin^2 \theta}{\rho^2} dv d\varphi + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\varphi^2, \quad (47)$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta^2 = r^2 - 2Mr + a^2$, $\Sigma^2 = (r^2 + a^2)/\rho^2 + 2a^2 Mr \sin^2 \theta$. The horizon is localized at $r = r_+$.

A suitable null co-tetrad $(l, n, m, m)$ can be chosen as

$$l = -\frac{\Delta}{2(r^2 + a^2)} dv + \frac{\rho^2}{r^2 + a^2} dr + \frac{\Delta a \sin^2 \theta}{\rho^2} dv d\varphi,$$

$$n = \frac{r^2 + a^2}{\rho^2} (-dv + a \sin^2 \theta d\varphi), \quad (48)$$

$$m = -\frac{a \sin \theta}{\sqrt{2} \rho} dv + \frac{(r^2 + a^2) \sin \theta}{\sqrt{2} \rho} d\varphi + \frac{i}{\sqrt{2}} d\varphi.$$
where $\dot{\rho} = r + ia \cos \theta$. It is easy to show that area element of the horizon $\Delta$ is
\[
-Im \wedge \bar{m} = a \sin \theta dv \wedge d\theta + (r_+^2 + a^2) \sin \theta d\theta \wedge d\varphi = a \gamma \sin \theta dv' \wedge d\theta + r_0^2 \sin \theta d\theta \wedge d\varphi,
\]
with new coordinate $dv' = dv / \gamma$ and $r_0^2 = r_+^2 + a^2$.

A. The canonical formula

On the horizon $\Delta : r = r_+$, we choose $(x^0, x^1, x^2) = (v', \theta, \varphi)$. The boundary BF theory on the horizon $\Delta$ is
\[
S = \int_{\Delta} BF = \int_{\Delta} BdA,
\]
with the constraints
\[
 dB = -i \frac{1}{8\pi G} m \wedge \bar{m}, \quad dA = 0.
\]
Assume the effective metric of the horizon is
\[
\hat{ds}^2 = -dv'^2 + r_+^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]
then
\[
\epsilon_{012} = \sqrt{-g} = r_+^2 \sin \theta, \quad \epsilon^{012} = \epsilon^{12} = \frac{1}{r_+^2 \sin \theta}.
\]
Following the same method as in BTZ black hole case, we choose the gauge $B_0 = A_0 = 0$ to gives $A_i = \partial_i \phi$, and the action becomes
\[
S = -\int_{\Delta} d^3x (\partial_2 B_1 - \partial_1 B_2) \partial_0 \phi = \int_{\Delta} d^3x \sqrt{-g} \pi \dot{\phi},
\]
where $\pi = -\epsilon^{ij} \partial_i B_j$ is the canonical momentum. We add a Hamiltonian to get
\[
S' = \int_{\Delta} d^3x \sqrt{-g} (\pi \partial_0 \phi - \frac{1}{2m_0} \pi^2 - \frac{m_0}{2} g^{ij} \partial_i \phi \partial_j \phi) = \frac{m_0}{2} \int_{\Delta} d^3x \sqrt{-g} \gamma^{\mu
u} \partial_\mu \phi \partial_\nu \phi,
\]
where $m_0$ is a mass parameter to adjust the mismatch of dimensions between boundary and bulk.

The equations of motion can be recast into a duality relation
\[
\epsilon^{\mu \nu \rho} \partial_\nu B_\rho = m_0 \partial^\mu \phi.
\]
The Hamiltonian can be written as
\[
H = \frac{m_0}{2} \int_{\Sigma} d^2x \sqrt{-g} (\partial_0 \phi \partial_0 \phi + g^{ij} \partial_i \phi \partial_j \phi),
\]
where we omit the zero-point energy.

The angular momentum can be defined by
\[
J = m_0 \int_{\Sigma} d^2x \sqrt{-g} (\partial_0 \phi \partial_2 \phi).
\]
B. Quantization

Next we quantize the massless scalar field theory (55). Expanding the fields with the solution of motion gives

\[ \phi(v', \theta, \varphi) = \phi_0 + p_v v' + p_\varphi \ln(\cot \frac{\theta}{2}) + p_\varphi \varphi + \sqrt{\frac{1}{m_0 A}} \sum_{l \neq 0} \sum_{m \neq 0} \sqrt{\frac{1}{2 \omega_i}} a_{l,m} e^{-i \omega_i v'} Y_l^m(\theta, \varphi) + a_{l,m}^+ e^{i \omega_i v'} (Y_l^m)^*(\theta, \varphi), \]

\[ B_1(v', \theta, \varphi) = m_0 (B_{10} r_+ + \frac{p_\varphi}{\sin \theta}) v' + \frac{1}{2} r_+^2 \sin \theta p_\varphi \varphi - 2 \sqrt{\frac{1}{m_0 A}} \sum_{l \neq 0} \sum_{m \neq 0} \sqrt{\frac{1}{2 \omega_i}} \frac{m}{\sin \theta} a_{l,m} e^{-i \omega_i v'} Y_l^m(\theta, \varphi) + a_{l,m}^+ e^{i \omega_i v'} (Y_l^m)^*(\theta, \varphi), \]

\[ B_2(v', \theta, \varphi) = m_0 (B_{20} r_+ + p_\theta v' v_\theta - \frac{1}{2} r_+^2 \sin \theta p_\varphi \varphi - 2i \sqrt{\frac{1}{m_0 A}} \sum_{l \neq 0} \sum_{m \neq 0} \sqrt{\frac{1}{2 \omega_i}} \frac{1}{(2 \omega_i)^3} \sin \theta \frac{\partial}{\partial \theta} [a_{l,m} e^{-i \omega_i v'} Y_l^m(\theta, \varphi) - a_{l,m}^+ e^{i \omega_i v'} (Y_l^m)^*(\theta, \varphi)], \]

where \( \omega_i^2 = \frac{l(l+1)}{r_+^2}, \) \( Y_l^m(\theta, \varphi) \) are spherical harmonics and \( A = 4 \pi r_+^2. \) It is straight to show that the above expression satisfy the dual relation (56).

The commutative relation between \( \hat{\phi} \) and \( \hat{\pi} \) is

\[ [\hat{\phi}(v', \vec{x}), e^{ij} \partial_i \hat{\phi}_j(v', \vec{y})] = -i \delta(\vec{x} - \vec{y}), \]

(60)

gives

\[ [\hat{\phi}_0, m_0 \hat{p}_v] = \frac{i}{A}, \]

\[ [\hat{a}_{l,m}, \hat{a}_{l',m'}^+] = \delta_{l,l'} \delta_{m,m'}. \]

We can also consider \( B_i \) and \( e^{ij} \partial_j \phi \) as two canonical variables, and lead to two further commutation relations

\[ [\hat{B}_{10}, m_0 r_+ + \frac{\hat{p}_\varphi}{r_+^2 \sin \theta}] = \frac{i}{A}, \]

\[ [\hat{B}_{20}, m_0 r_+ + \frac{\hat{p}_\theta}{r_+^2}] = -\frac{i}{A}. \]

(62)

For Kerr black hole, the \( B \) field satisfy the constraint (51), or with the component

\[ \partial_0 B_1 = \frac{a r_0^2}{8 \pi r_+^2}, \]

\[ \partial_1 B_2 - \partial_2 B_1 = \frac{r_0^2 \sin \theta}{8 \pi}, \]

\[ \partial_0 B_2 = 0, \]

(63)

which gives

\[ p_v = -\frac{r_0^2}{8 \pi m_0 r_+}, \]

\[ p_\varphi = \frac{a \sin^2 \theta \gamma}{8 \pi m_0}, \]

\[ p_\theta = 0. \]

(64)

To get rid the trigonometric functions we define \( \hat{p}_\varphi = \frac{a \gamma}{8 \pi m_0} \) and can be considered as quantum operator of \( p_\varphi. \)

Then the commutation relations gives

\[ [\hat{\phi}_0, -\frac{4 \pi r_0^2}{8 \pi}] = i, \]

\[ [\hat{B}_{10}, \frac{\hat{a} r_+ \gamma}{2}] = i. \]

(65)

Since the zero mode \( \phi_0, B_{00} \) are constant on the horizon, the canonical momentum are quantized with

\[ \frac{4 \pi r_0^2}{8 \pi} = n_1, \]

\[ \frac{a r_+ \gamma}{2} = n_2, \]

\[ n_1, n_2 \in N. \]

(66)

The Hamiltonian operator and the angular momentum operator can be written as

\[ \hat{H} = \frac{m_0}{2} A (\hat{p}_v^2 + \frac{|\hat{p}_\varphi|^2}{r_+^2}) + \sum_{l \neq 0} \sum_{m = -l}^{m = l} \frac{\sqrt{l(l+1)}}{r_+} \hat{a}_{l,m} \hat{a}_{l,m}^+, \]

\[ \hat{J} = m_0 A \hat{p}_v |\hat{p}_\varphi| + \sum_{l \neq 0} \sum_{m = -l}^{m = l} m \hat{a}_{l,m}^+ \hat{a}_{l,m}, \]

(67)
To fix the parameters $\gamma, m_0$, we define the dimensionless left- and right-temperature as

$$T_{L/R} = -\frac{r_+}{2\pi}(p_v \pm |p_\varphi|\frac{r_+}{r_+}) = \frac{r_0^2}{16\pi^2 m_0 r_+} (1 \pm \frac{a\gamma r_+}{r_0^2}).$$

(68)

In the extremal limit $a \rightarrow M$, $T_R \rightarrow 0$. A natural choice should be

$$\gamma = \frac{r_0^2}{r_+^2}.$$

(69)

We assume that they also satisfy the condition (38), that is

$$\gamma \pi (T_L + T_R) = 1 \Rightarrow m_0 = \frac{r_0^4}{8\pi r_+^3} = \frac{M^2}{2\pi r_+} \sim M.$$

(70)

The dimensionless temperatures are given by

$$T_{L/R} = \frac{r_+}{4\pi M} (1 \pm \frac{a}{r_+}).$$

(71)

Notice that in the Schwarzschild case $a = 0$, those temperatures reduce to dimensionless Unruh temperature $T = \frac{1}{2\pi}$. In the extremal limit $a \rightarrow M$, they coincide with the temperatures in Kerr/CFT correspondence $T_{L/R} = \frac{1}{4\pi r_+} (r_+ \pm r_-)$.

If we assume a dual CFT exist for arbitrary Kerr black hole, the central charge $c$ can be obtained through the Cardy formula

$$S_{\text{Cardy}} = \frac{\pi^2 c}{3}(T_R + T_L) = S_{\text{BH}} = 2\pi Mr_+, \quad (72)$$

which gives $c = 12M^2$.

The quantized condition (66) gives

$$S = 2\pi n_1, \quad J = n_2, \quad n_1, n_2 \in N,$$

(73)

that is, the entropy and the angular momentum of Kerr black hole are both quantized.

The scalar field $\phi(v', \theta, \varphi)$ can be considered as collectives of harmonic oscillators, and a general state can be represented as $|p_v, p_\varphi; \{n_{l,m}\} >$ where $p_v, p_\varphi$ are zero mode part, and $\{n_{l,m}\}$ are oscillator part. Just like the BTZ black hole, it is easy to show that the Kerr black hole with parameters $(M, J)$ corresponds to the zero mode part, thus

$$< p_v, p_\varphi; \{0\}|\hat{J}|p_v, p_\varphi; \{0\} > = \frac{J}{\gamma} = \frac{ar_+}{2}, \quad < p_v, p_\varphi; \{0\}|\hat{H}|p_v, p_\varphi; \{0\} > = \frac{M}{2}.$$

(74)

One may confuse with the $\frac{M}{2}$ in expression (74). Actually this can be explained by associating the Hamiltonian with the heat context (enthalpy) [22].

On the other hand, we can define the microscopic states of Kerr black hole as states represented by $|0, 0; \{n_{l,m}\} >$. We require

$$\frac{1}{c} < 0, 0; \{n_{l,m}\}|\hat{J}|0, 0; \{n_{l,m}\} > = \frac{J}{\gamma} = \frac{ar_+}{2}, \quad \frac{1}{c} < 0, 0; \{n_{l,m}\}|\hat{H}|0, 0; \{n_{l,m}\} > = \frac{M}{2},$$

(75)

where $c = 12M^2$ is the central charge. Different sequence $\{n_{l,m}\}$ corresponds to different microstate of the Kerr black hole with same $(M, J)$. 

The constraints (75) equivalent to
\[ \sum \sqrt{l(l+1)} = \frac{cMr}{2}, \quad \sum m = \frac{car}{2}. \] (76)

The calculation of the Kerr black hole entropy then transforms into a mathematical problem: count the number of all different sequences \( \{l, m\} \) that satisfy the constraints (76). The exact number of states is difficult to obtain, so we make some approximates. First note \( \sqrt{l(l+1)} \sim l \geq |m| \). So from the first equation we can get
\[ \sum |m| \leq \frac{cMr}{2}, \] (77)

The number of the different sequences \( \{m\} \) is given by (see the appendix)
\[ N(m) \approx \frac{1}{M^2r_+(M+a)} \exp(2\pi Mr_+). \] (78)

Notice that every sequence \( \{m\} \) can correspond to at least one sequence \( \{l\} \), so the number of sequence \( \{l, m\} \) is about
\[ N(l, m) \sim N(m) \approx \exp(2\pi Mr_+). \] (79)

The entropy of Kerr black hole is given by the logarithm of the number of microstates \( \{n_{l,m}\} \),
\[ S = \ln N(l, m) \sim 2\pi Mr_+ + \cdots, \] (80)
which is just the Bekenstein-Hawking entropy with some low order corrections.

For extremal Kerr black hole \( a = M \), then all \( l = m \), so the entropy is
\[ S = \ln N(l, m) = \ln N(m) = \ln \left( \frac{1}{car} \exp \left( 2\pi \sqrt{\frac{car}{12}} \right) \right) = 2\pi M^2 - 2 \ln(M^2), \] (81)
which coincide the result of Ref. [23] for log-corrections.

IV. CONCLUSION

In this paper, we analyze the boundary modes of black holes with the method developed for topological insulator. Firstly the BTZ black hole is considered, and the results are compatible with the previous works. Then we generalize those results to Kerr black holes. Some new results are obtained: dimensionless right- and left-temperature can be defined and have well behavior both in Schwarzschild limit \( a \to 0 \) and extremal limit \( a \to M \). Also a central charge \( c = 12M^2 \) is associated with arbitrary Kerr black hole. We also identify the microstates of Kerr black hole with the quantum states of the boundary scalar field. From this identification one can count the number of microstates of Kerr black hole. The calculation of the Kerr black hole entropy transforms into a mathematical problem: count the number of all different sequences \( \{l, m\} \) that satisfy some constraints. After making some approximates, we get the area law.

Due to the compactness of the scalar field, one can get interesting results that the entropy of the black hole are quantized with an equally spaced spectrum both for BTZ black hole (29) and Kerr black hole (73). It is consistent with the early works [24, 25].
Based on the “horizon fluff” proposal \[20, 26, 27\], the microstates of the extremal Kerr black hole are identified in Ref.\[23\]. The number of those microstates gives the Bekenstein-Hawking area law. This approach rely on the $U(1)$ Kac-Moody algebra and Virasoro algebra, which has close relation with Kerr/CFT correspondence. In our approach, we identify the microstates of Kerr black hole with the quantum states of the massless scalar field, and the algebra is just the commutation algebra for harmonic oscillator. Can one get the Virasoro algebra from our approach is under investigate.

ACKNOWLEDGMENTS

This work is supported by Nanhu Scholars Program for Young Scholars of XYNU.

Appendix A: The calculation of $N(m)$

In this appendix, we show how to calculate the $N(m)$. The sequence $\{m\}$ should satisfy the constraints

$$\sum |m| \leq \frac{cMr_+}{2}, \quad \sum m = \frac{cMr_+}{2}. \quad (A1)$$

First we consider $\sum |m| = \frac{cMr_+}{2}$. Separate the sequence $\{m\}$ into positive part $\{m_+\}$ with all $m > 0$, and negative part $\{m_-\}$ with all $m < 0$. Then the constraints becomes

$$\sum |m| = \sum m_+ - \sum m_- = \frac{cMr_+}{2}, \quad \sum m = \sum m_+ + \sum m_- = \frac{cMr_+}{2}. \quad (A2)$$

Then

$$\sum m_+ = \frac{cr_+(M+a)}{4}, \quad \sum m_- = -\frac{cr_+(M-a)}{4}. \quad (A3)$$

Due to the Hardy-Ramanujan formula, the number of different sequence $\{m\}$ is

$$N_0(m) \approx \frac{1}{cr_+(M + a)} \exp(2\pi \sqrt{\frac{cr_+(M + a)}{24}}) \frac{1}{cr_+(M - a)} \exp(2\pi \sqrt{\frac{cr_+(M - a)}{24}}) \approx \frac{1}{M^4r_+^2(M^2 - a^2)} \exp(2\pi Mr_+), \quad (A4)$$

where we omit all numerical factor in front of the exponential function for simplification.

Next we consider $\sum |m| = \frac{cMr_+}{2} - 1$. With the same method we can get

$$N_1(m) \approx N_0(m) \exp(-\frac{1}{\sqrt{cr_+(M + a)}} - \frac{1}{\sqrt{cr_+(M - a)}}) = N_0q, \quad (A5)$$

where $q = \exp(-\frac{1}{\sqrt{cr_+(M + a)}} - \frac{1}{\sqrt{cr_+(M - a)}}) \approx 1 - \frac{1}{\sqrt{cr_+(M + a)}} - \frac{1}{\sqrt{cr_+(M - a)}}$.

For general $\sum |m| = \frac{cMr_+}{2} - n$, we can get

$$N_n(m) \approx N_{n-1}(m)q. \quad (A6)$$

The largest $n$ is $n_{\text{max}} = \frac{cr_+(M-a)}{2}$. The sequence $\{N_n(m)\}$ is approximate a geometric series, and the sum of this sequence gives the $N(m)$, thus

$$N(m) \approx \frac{N_0(m)(1 - q^{n_{\text{max}}})}{1 - q} \approx n_{\text{max}} N_0(m) = \frac{1}{M^2r_+(M + a)} \exp(2\pi Mr_+). \quad (A7)$$
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