Modal approach to the controllability problem of distributed parameter systems with damping

Alexander Zuyev

Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine

Abstract

This paper is devoted to the controllability analysis of a class of linear control systems in a Hilbert space. It is proposed to use the minimum energy controls of a reduced lumped parameter system for solving the infinite dimensional steering problem approximately. Sufficient conditions of the approximate controllability are formulated for a modal representation of a flexible structure with small damping.

1 Introduction

The problems of spectral, approximate, exact, and null controllability of distributed parameter systems have been intensively studied over the last few decades [1, 2, 3]. On the one hand, the question of the approximate controllability of a linear time-invariant system on a Hilbert space can be formulated in terms of an invariant subspace of the corresponding adjoint semigroup [4], [2, p. 56]. On the other hand, the problem of effective control design remains challenging for wide classes of mechanical systems (see, e.g. [5, 7, 8, 9, 11], and references therein). The goal of this work is to propose a constructive control strategy, based on a reduced model, and to justify that this approach can be used to solve the approximate controllability problem for infinite dimensional systems.

2 Problem statement

This paper addresses the problem of approximate controllability of a linear differential equation

\[
\dot{x} = Ax + Bu, \quad x \in H, \quad u \in \mathbb{R}^m,
\]

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where $H$ is a Hilbert space, $A : D(A) \to H$ is a closed densely defined operator, and $B : \mathbb{R}^m \to H$ is a continuous operator. We assume that $A$ generates a strongly continuous semigroup of operators $\{e^{tA}\}_{t \geq 0}$ on $H$. Hence, for any $x^0 \in H$ and $u \in L^2(0, \tau)$, the mild solution of (1) corresponding to the initial condition $x|_{t=0} = x^0$ and control $u = u(t)$ can be written as follows:

$$x(t; x^0, u) = e^{tA}x^0 + \int_0^t e^{(t-s)A}Bu(s) \, ds, \quad 0 \leq t \leq \tau. \quad (2)$$

Note that that system (1) is approximately controllable in time $\tau > 0$ if (cf. [2]), given $x^0, x^1 \in H$ and $\varepsilon > 0$, there exists $u \in L^2(0, \tau)$ such that $\|x(\tau; x^0, u) - x^1\| < \varepsilon$. In order to study the approximate controllability of system (1), we use the following result.

**Proposition 1.** [10] Let $\{Q_N\}_{N=1}^\infty$ be a family of bounded linear operators on $H$ satisfying the following conditions:

1) \[ \lim_{N \to \infty} \|Q_N x\| = 0, \quad \text{for all } x \in H; \quad (3) \]

2) the operators $e^{tA}$ and $Q_N$ commute;

3) for each $x^0, x^1 \in H$, $N \geq 1$, there is a control $u^N_{x^0,x^1} \in L^\infty(0, \tau)$ such that

\[
(I - Q_N)\left(x(\tau; x^0, u^N_{x^0,x^1}) - x^1\right) = 0, \quad (4)
\]

\[
\lim_{N \to \infty} \left(\|Q_N B\| \cdot \|u^N_{x^0,x^1}\|_{L^2(0,\tau)}\right) = 0. \quad (5)
\]

Then system (1) is approximately controllable in time $\tau$, and the above family of functions $u = u^N_{x^0,x^1}(t)$, $0 \leq t \leq \tau$, can be used to solve the approximate controllability problem.

For a possible application of this proposition, we assume that each operator $P_N = I - Q_N$ is a finite dimensional projection. Let $\dim(\text{Im} P_N) = d_N$. For given $x^0, x^1 \in H$, we introduce vectors

\[
\tilde{x}^0_N = P_N x^0, \quad \tilde{x}^1_N = P_N x^1, \quad \tilde{x}_N = P_N x, \quad (\tilde{x}^0_N, \tilde{x}^1_N, \tilde{x}_N \in \text{Im} P_N),
\]

and operators $\tilde{A}_N = P_N A$, $\tilde{B}_N = P_N B$. Then condition (1) implies that $u^N_{\tilde{x}^0,x^1}(t)$ should solve the following control problem:

\[
\dot{\tilde{x}}_N = \tilde{A}_N \tilde{x}_N + \tilde{B}_N u, \quad t \in [0, \tau], \quad (6)
\]

\[
\tilde{x}_N|_{t=0} = \tilde{x}^0_N, \quad \tilde{x}_N|_{t=\tau} = \tilde{x}^1_N.
\]

Here we have used the assumption that $P_N$ and $A$ commute as well as the property $P_N = P_N^2$ of a projection. To satisfy condition (5), it is natural to look for a control $u = u^N_{\tilde{x}^0,x^1}(t)$ that minimizes the functional

\[
J = \int_0^\tau (Qu, u) \, dt \to \min \quad (7)
\]
with some symmetric positive definite $m \times m$-matrix $Q$. As control system (6) evolves on a real $d_N$-dimensional vector space $\text{Im} P_N$, we may treat (6) as a system on $\mathbb{R}^{d_N}$ without lack of generality. By applying the Pontryagin maximum principle, we get the optimal control for problem (6)-(7):

$$\tilde{u}(t) = Q^{-1} \tilde{B}_N e^{\tau-t} \tilde{A}_N^t \nu, \quad \nu = \left( \int_0^\tau e^{s \tilde{A}_N} \tilde{B}_N Q^{-1} \tilde{B}_N^t e^{s \tilde{A}_N} ds \right)^{-1} (x_N^1 - e^{\tau \tilde{A}_N} x_0^0),$$

where the prime stands for the transpose. Proposition 1 implies that the proof of the approximate controllability can be reduced to the checking conditions (3) and (5) with a family of smooth controls $u_{N,x_1}^N = \tilde{u}(t)$ given by (8). The main contribution of this paper is the application of such a scheme for a class of systems (1) representing the oscillations of a flexible structure with damping.

### 3 Flexible system with damping

Consider a particular case of system (1) as follows

$$\dot{x} = Ax + Bu, \quad x = (x_1, x_2, ...) \in \ell^2, \quad u \in \mathbb{R},$$

where $\|x\|_{\ell^2} = \left( \sum_{n=1}^\infty x_n^2 \right)^{1/2}$. We assume that the operator $A : D(A) \to \ell^2$ in (9) is given by its block-diagonal matrix:

$$A = \text{diag}(A_0, A_1, A_2, ...), \quad A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_n = \begin{pmatrix} 0 & \omega_n \\ -\omega_n & -2\kappa \end{pmatrix}, \quad n = 1, 2, ...,$$

and $B = (0, 1, 0, b_1, 0, b_2, ...) \in \ell^2$. Control system (9) is a linear model of a rotating flexible beam attached to a rigid body. The components of $x$ play the role of modal coordinates, and the control $u$ is the angular acceleration of the body. Coefficients $\omega_n$ and $b_n$ are, respectively, the modal frequency and the control coefficient corresponding to the $n$-th mode of oscillations of the beam. The coefficient $\kappa > 0$ represents the viscous damping in the beam. The procedure of deriving the equations of motion with modal coordinates is described in the paper [6] for a rotating rigid body with flexible beams.

The main result of this paper is as follows.

**Proposition 2.** Assume that $b_n \neq 0$ and $\omega_n > 0$ for all $n = 1, 2, ...$. Then there exists a $\tau > 0$ such that system (9) is approximately controllable in time $\tau$ provided that

$$\sum_{i,j=1, i \neq j}^\infty \frac{1}{(\omega_i - \omega_j)^2} < \infty$$

and that the damping coefficient $\kappa$ is small enough.
Proof. Let us introduce the family of operators \( P_N : \ell^2 \to \ell^2 \) as follows:

\[
P_N \begin{pmatrix} \xi_0 \\ \eta_0 \\ \vdots \\ \xi_N \\ \eta_N \\ \xi_{N+1} \\ \eta_{N+1} \\ \vdots \\ \xi_0 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \eta_0 \\ \vdots \\ \xi_N \\ \eta_N \\ 0 \\ 0 \\ \vdots \\ \xi_0 \\ \eta_0 \end{pmatrix},
\]

and \( Q_N = I - P_N, \ N = 1, 2, \ldots \). Then condition 1) of Proposition 1 holds. To check condition 2), we compute the semigroup \( \{e^{tA}\} \) generated by \( A \):

\[
e^{tA} = \text{diag} \left( e^{tA_0}, e^{tA_1}, e^{tA_2}, \ldots \right),
\]

\[
e^{tA_0} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad e^{tA_n} = e^{-\kappa t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]

The assertion of Proposition 2 follows from Proposition 1 by exploiting the construction of \( L^2 \)-minimal controls \( u_{x_0}^N = \tilde{u}(t) \) in (8).

4 Conclusions

This work extends the result of [10] for the case of a flexible system with damping. As it was shown earlier in [10], condition (10) is satisfied for the Euler-Bernoulli beam without damping. Hence, condition (10) is sufficient for the approximate controllability in both conservative (\( \kappa = 0 \)) and dissipative (small \( \kappa > 0 \)) cases under our assumptions. An open question is whether it is possible to relax restrictions (10) in order to justify the relevance of controls (8) under a weaker assumption on the distribution of the modal frequencies \( \{\omega_n\} \).

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