Supersymmetric extensions of affine Toda theories

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August 18, 2018

Abstract

It is shown that all affine Toda theories admit (1,0) supersymmetric extensions. The construction is based on classical Lie algebras and supersymmetric massive sigma models. The supersymmetrized affine Toda theories have a unique, supersymmetric vacuum, their mass matrix is well defined and their energy functional is positive semi-definite.

DAMTP-1998-95

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1 Introduction

Bosonic Toda theories \cite{1, 2} provide a uniform way of constructing both conformally invariant and massive field theories, which are integrable. Since the theory of Lie algebras underlies the integrability of the bosonic models, it is natural to employ the theory of superalgebras to incorporate fermions into the bosonic models \cite{3, 4}. Whereas with this method the classical integrability of the bosonic Toda theories is maintained, most of the fermionic extended Toda theories are plagued by an indefinite signature in their kinetic term \cite{5} or by the explicit breaking of their supersymmetry \cite{5}. Some notable exceptions, which provide examples of supersymmetric, unitary and integrable field theories, are the supersymmetric extensions of Liouville theory \cite{6, 7}, of sinh- and sine-Gordon theory \cite{8, 9} and of coupled sinh- and sine-Gordon theories \cite{10}.

Toda theories are two-dimensional field theories with a flat target space and an exponential potential. Thus they constitute examples of bosonic massive sigma models. With this in mind, it was recently proposed \cite{11} to apply the framework of supersymmetric massive sigma models \cite{12} to supersymmetrize bosonic Toda theories. In particular, (1,0) supersymmetric extensions were found for all conformal Toda theories and for affine Toda theories based on the algebra $A_1^{(1)}$. Attempts to construct further supersymmetric extensions of these theories proved unsuccessful, except for affine Toda theories based on an algebra of rank one, for which a (1,1) supersymmetric extension was given. On the one hand this reconfirms the belief that there is no N=1 supersymmetric theory whose bosonic part is a Toda model based on a simple Lie algebras of rank bigger than one \cite{5}. On the other hand it points to previously overlooked (1,0) supersymmetric Toda theories,

\footnote{In chiral notation N=1 supersymmetry corresponds to (1,1) supersymmetry.}
which have the same a bosonic part as a bosonic Toda theory based on a simple Lie algebra.

It is an interesting question, whether or not these (1,0) supersymmetric extensions of Toda theories share the integrability property with their bosonic counterparts. Underlying the classical integrability of the bosonic models is a Lax pair or zero curvature conditions [13, 14], which imply the existence of an infinite number of conserved currents. Furthermore, it appears that their S-matrices are exact [15, 16], which renders these models quantum integrable. For our supersymmetric models there is no obvious way to generalize the zero-curvature condition from the bosonic models. On the contrary, in [5] it was argued that for bosonic conformal Toda theories some conserved currents of higher spin do not have an equivalent in the corresponding supersymmetric theories and it was concluded that these models are not integrable. Whereas this seems to settle the question for conformal Toda theories, it is an open question, whether or not the supersymmetric extensions of affine Toda theories are integrable.

Affine Toda theories have a unique constant solution as a ground state and apart from rank(g) fundamental particles, they also admit a set of soliton solutions in their spectrum, provided that a certain coupling constant $\beta$ is purely imaginary [17, 18]. There is also a relationship between the solitons in affine Toda theories and certain N-body integrable systems [19].

This paper presents (1,0) supersymmetric extensions of all affine Toda theories. It is sufficient to consider Toda theories related to untwisted, self-dual affine Lie algebras, i.e. those of the $A - D - E$ series, as all other cases can be related to these. The condition for a bosonic massive sigma model to admit such an extension is that its scalar potential can be written as the length of a section of a vector bundle over the sigma model manifold.
Employing this result we present fermionic extensions of bosonic affine Toda theories with the following properties:

(i) The classical vacuum is supersymmetric.

(ii) The mass matrix of the fermions is well defined at the vacuum.

(iii) Setting the fermions to zero the usual bosonic affine Toda theory is recovered.

(iv) The energy functional is positive semi-definite, i.e. the inner product of the kinetic terms is positive definite.

This construction does not render unique supersymmetric extensions; on the contrary, different ones are explicitly given for most bosonic affine Toda theories.

In section two we give the conditions for bosonic, massive sigma models to admit $(1,0)$ supersymmetric extensions. In section three we present $(1,0)$ supersymmetric affine Toda theories and in section four we give our conclusions.

2 (1,0) supersymmetric massive sigma models

In this section we revise the geometry of a class of $(1,0)$ supersymmetric massive sigma models with action

$$I = \int d^2x \left( \partial_+ \phi^i \partial_+ \phi^j g_{ij} + i \lambda^i_+ \partial_- \lambda^j_+ g_{ij} 
- \psi^a_- \partial_- \psi^b_- \delta_{ab} + m \partial_i s_a \lambda^i_+ \psi^a_- - V(\phi) \right),$$

where $\phi$ is a map from the two-dimensional Minkowski spacetime $\Sigma$ with light-cone coordinates $\{x^+, x^-\}$ into the target manifold $M$ with metric $g_{ij}$, $1 \leq i \leq \dim(M)$, $\psi^a_-$ are sections of $\phi^* E \otimes S_-$ with $E$ a real vector bundle.
over $M$, $1 \leq a \leq \dim(E)$, and $S_-$ is the bundle of right handed spinors over $\Sigma$. Furthermore, $\lambda_+$ are real chiral fermions, $m$ is a mass parameter, $V$ is the scalar potential

$$V = \frac{m^2}{4} s_a(\phi) s^a(\phi)$$

(2)

and $s^a$ are sections of the vector bundle $E$. In (1) we assumed that the sigma model has no torsion, that $M$ is flat and that $E$ is trivial, and have chosen the fibre metric of $E$ to be $\delta_{ab}$. This is the case of interest to us. The action (1) can be deduced from an off-shell (1,0) superspace formulation for massive sigma models [12] and it is invariant under the (1,0) supersymmetry transformations

$$\delta_\epsilon \phi^i = -\frac{i}{2} \epsilon_- \lambda^i_+ , \quad \delta_\epsilon \lambda^i_+ = \frac{1}{2} \epsilon_- \partial_\phi \phi^i , \quad \delta_\epsilon \psi^a = \frac{i}{4} \epsilon_- m s^a ,$$

(3)

where $\epsilon_-$ is the parameter of the transformations. Hence we conclude that bosonic massive sigma models admit a (1,0) supersymmetric extension if and only if their potential can be written as the length of the section of a vector bundle $E$ over $M$. The zeros of $V$ are the supersymmetric vacua of the theory. Whereas in the case of (1,0) supersymmetric models the vector bundle $E$ is fixed by the choice of $s$ but otherwise arbitrary, in the case of a (1,1) supersymmetric extensions $E$ has to be isomorphic to the tangent bundle of $M$. From (1) the fermion mass matrix can be read of as

$$M_{ia} = m \partial_i s_a .$$

(4)

3 Supersymmetric affine Toda theories

In this section we construct (1,0) supersymmetric extensions of all Toda theories related to affine algebras. Affine algebras can be pictorially represented by their generalized Dynkin diagrams. (For a list of the generalized Dynkin
diagrams of all affine algebras see for example \cite{20}, p. 44\). The numbering of the nodes of the generalized Dynkin diagrams which we refer to is that of E. B. Dynkin \cite{21}. For untwisted affine algebras, \( g^{(1)} \), the zeroth node represents the highest root \( \psi \) of the associated simple Lie algebra \( g \) and on removal of this node the Dynkin diagram of \( g \) is recovered with the correct Coxeter labels attached to it. For twisted algebras, \( g^{(2)} \), on removal of the zeroth node the Dynkin diagram of \( g \) is still recovered but with the wrong Coxeter labels attached to it. Self-dual untwisted affine algebras are associated to self-dual Lie algebras, which are \( A_r, D_r, E_6, E_7 \) and \( E_8 \). For all other untwisted affine algebras there exist twisted duals, i.e. affine algebras which are associated to the same Lie algebra \( B_r, C_r, F_4 \) and \( G_2 \).

Let us begin with the action of the bosonic affine Toda theories associated to the affine algebra \( g^{(\epsilon)} \), \( \epsilon = \{1, 2\} \),

\[
I = \int d^2x \left( <\partial_x \phi, \partial_x \phi> - V \right) ,
\]

where \( \phi \) is a map from \( \Sigma \) into the Cartan subalgebra \( h \) of a simple Lie algebra \( g \), \( <\cdot, \cdot> \) is a metric on \( h \) induced from the invariant metric on \( g^{(\epsilon)} \) and the potential \( V \) is

\[
V = \frac{m^2}{\beta} \left( \sum_{I=0}^{r} n_I e^{\beta \phi_I} - h \right) .
\]

The parameters \( m \) and \( \beta \) are the coupling constants of the theory, \( \phi_I = \{\phi_0, \phi_i; i = 1, 2, \ldots, r = \text{rank}(g)\} \), \( \phi_0 = \alpha_0 \cdot \phi \) and \( \phi_i = \alpha_i \cdot \phi \), where \( \{\alpha_i; i = 1, 2, \ldots, r\} \) are the simple roots and \( \alpha_0 \) is the zeroth root of the affine algebra \( g^{(\epsilon)} \). For untwisted affine algebras, \( g^{(1)} \), \( \alpha_0 = -\psi \). The integers \( n_I \) are the Coxeter labels of \( g^{(\epsilon)} \) defined through \( \sum_{I=1}^{r} n_I \alpha_I = 0 \) and \( h = \sum_{I=1}^{r} n_I \) is the Coxeter number of \( g^{(\epsilon)} \). We observe that \( n_0 = 1 \) for all affine algebras. The constant term in the scalar potential does not affect the classical theory and hence it is usually neglected. Nevertheless, we include \( h \) in \( V \) in order to render the vacuum supersymmetric.
In order to supersymmetrize the bosonic action (5) we have to write the scalar potential (6) as the sum of squares such that (i) the vacuum of $V$ at $\phi_i = 0$ is supersymmetric and (ii) the fermions have a well defined mass at the vacuum. We observe that the scalar potential only depends on the Coexter labels $n_I$ and not on the generalized Dynkin diagrams themselves. Hence for this purpose we can treat affine algebras with the same set of Coexter labels and different Dynkin diagrams, e.g. $B^{(1)}_r$ and $C^{(1)}_r$ for $r > 2$, essentially as the same, even though they give rise to distinct affine Toda theories. Furthermore, in some cases it is possible to set a subset of the fields $\phi_i$ associated to some simple roots of $g_1$ in a potential $V_1$ to zero to derive an expression for a potential $V_2$ in terms of fields $\tilde{\phi}_i$ of some algebra $g_2$. If we take for example $g_1 = F_4$ and $g_2 = G_2$ then the reduced potential of $F^{(1)}_4$ can be identified with the one of $G^{(1)}_2$ by setting $\phi_1 = \tilde{\phi}_1$, $\phi_2 = \tilde{\phi}_2$ and $\phi_3 = \phi_4 = 0$. This identification is not unique as we can also set $\phi_2 = \tilde{\phi}_2$, $\phi_4 = \tilde{\phi}_1$ and $\phi_1 = \phi_3 = 0$. We note that the truncation of a set of Coexter labels does not extend to a truncation of the full affine Toda theory. On inspection it turns out that the set of Coexter labels of any affine algebra can be derived from those of the untwisted, self-dual ones. Thus in the following we restrict our attention to these affine algebras.

Let us begin with the potential of $D^{(1)}_r$, $r > 3$, proving inductively that it can be written as the sum of squares such that the properties (i) and (ii) are satisfied. For simplicity, we set in the following the coupling constants $m$ and $\beta$ to one. Then the scalar potential of $D^{(1)}_r$ is

$$V_{D^{(1)}_r}(\phi_1, \phi_2, \ldots, \phi_r) = e^{\phi_1} + e^{\phi_{r-1}} + e^{\phi_r}$$

$$+ 2 \sum_{i=2}^{r-2} e^{\phi_i} + e^{-(\phi_1 + \phi_{r-1} + \phi_r)} \prod_{i=2}^{r-2} e^{-2\phi_i} - 2(r - 1), \quad (7)$$
which can be rewritten as

\[ V_{D_4}^{(1)}(\phi_1, \phi_2, \ldots, \phi_r) = \left( e^{-\frac{\phi_1 + \phi_{r-1} + \phi_r}{2}} \prod_{i=2}^{r-2} e^{\phi_i} - e^\phi \right)^2 + \left( e^\phi - 1 \right)^2 \]  

(8)

To complete the inductive proof we express the scalar potential of \( D_4^{(1)} \) as

\[ V_{D_4}^{(1)}(\phi_1, \phi_2, \phi_3, \phi_4) = e^{\phi_1} + 2e^{\phi_2} + e^{\phi_3} + e^{\phi_4} + e^{-(\phi_1 + 2\phi_2 + \phi_3 + \phi_4)} - 6 \]  

(9)

\[ = \left( e^{-\frac{\phi_1 + 2\phi_2 + \phi_3 + \phi_4}{2}} - e^\phi \right)^2 + \left( e^\phi - 1 \right)^2 + \left( e^\phi - 1 \right)^2 \]

\[ + 2 \left( e^\phi - 1 \right)^2 + 2 \left( e^{-\frac{2\phi_2 + \phi_3 + \phi_4}{4} - e^\phi} \right)^2 + 4 \left( e^{-\frac{2\phi_2 + \phi_4}{8} - e^\phi} \right)^2 \]

\[ + 2 \left( e^\phi - 1 \right)^2 + 4 \left( e^\phi - 1 \right)^2 + 8 \left( e^\phi - e^{-\frac{\phi_1}{8}} \right)^2 \].

The potentials of all twisted and untwisted affine algebras of the \( A, B, C \), and \( D \) series except \( A_1^{(1)} \) can be derived from the expressions (8) and (9) for the potential of \( D_4^{(1)} \). The case of \( A_1^{(1)} \) was already discussed in [11]. The potentials of all remaining affine algebras can be derived from the ones of \( E_6^{(1)}, E_7^{(1)} \) and \( E_8^{(1)} \). Let us begin with the case of \( E_6^{(1)} \):

\[ V_{E_6}^{(1)}(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = e^{\phi_1} + 2e^{\phi_2} + 3e^{\phi_3} + 2e^{\phi_4} + e^{\phi_5} + 2e^{\phi_6} \]  

(10)

\[ + e^{-(\phi_1 + 2\phi_2 + 3\phi_3 + 2\phi_4 + \phi_5 + 2\phi_6)} - 12 \]

\[ = \left( e^{-\frac{\phi_1 + 2\phi_2 + 3\phi_3 + 2\phi_4 + \phi_5 + 2\phi_6}{2}} - e^\phi \right)^2 \]

\[ + \sum_{j=1,5} \left( e^{-\frac{\phi_1 + 2\phi_2 + \phi_3 + 2\phi_4 + \phi_5 + 2\phi_6}{2}} - e^\phi \right)^2 + 8 \left( e^{\phi_4 + \phi_6} - e^{-\frac{\phi_4 + \phi_6}{4}} \right)^2 \]

\[ + 2e^{-\phi_3 + \phi_4 + \phi_5} \left( e^{\phi_1 - \phi_5} - e^{-\phi_1 - \phi_5} \right)^2 + 4 \left( e^{-\phi_1 + \phi_3 + \phi_4 + \phi_6} - e^{\phi_1 + \phi_5} \right)^2 \]

\[ + 2 \left( e^\phi - 1 \right)^2 + 2 \sum_{j=4,6} \left( e^\phi - 1 \right)^2 + 4 \left( e^\phi - e^\phi \right)^2 \].
Similarly, for $E_7^{(1)}$ we find

$$V_{E_7^{(1)}}(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7) = 2e^{\phi_1} + 3e^{\phi_2} + 4e^{\phi_3} + 3e^{\phi_4} + 2e^{\phi_5}$$

$$+e^{\phi_6} + 2e^{\phi_7} + e^{-(2\phi_1 + 3\phi_2 + 4\phi_3 + 3\phi_4 + 2\phi_5 + \phi_6 + 2\phi_7)} - 18$$

$$= \left(e^{-\frac{2\phi_1 + 3\phi_2 + 4\phi_3 + 3\phi_4 + 2\phi_5 + \phi_6 + 2\phi_7}{2}} - e^{\phi_7/2}\right)^2$$

$$+ \sum_{j=2,4} \left(e^{-\frac{2\phi_1 + 3\phi_2 + 4\phi_3 + 3\phi_4 + 2\phi_5 + \phi_6 + 2\phi_7}{8} - e^{\phi_7/2}}\right)^2$$

$$+ 2e^{2\phi_1 + 2\phi_2 + 4\phi_3 + 2\phi_4 + 2\phi_5 + \phi_6 + 2\phi_7} \left(e^{\phi_7/8} - e^\phi\right)^2$$

$$+ 2 \sum_{j=2,4} \left(e^{-\frac{2\phi_1 + 3\phi_2 + 4\phi_3 + 3\phi_4 + 2\phi_5 + \phi_6 + 2\phi_7}{8} - e^{\phi_7/2}}\right)^2$$

$$+ 4e^{2\phi_1 + 4\phi_2 + 2\phi_3 + \phi_6 + \phi_7} \left(e^{-\phi_7/8} - e^{\phi_7/8}\right)^2 + 4 \left(e^{\phi_7/2} - 1\right)^2$$

$$+ 8 \left(e^{-\frac{2\phi_1 + 3\phi_2 + 4\phi_3 + 3\phi_4 + 2\phi_5 + \phi_6 + 2\phi_7}{16} - e^{\phi_7/8}}\right)^2 + 16 \left(e^{\frac{2\phi_1 + 2\phi_2 + 4\phi_3 + 2\phi_5 + \phi_6 + 2\phi_7}{16} - e^{\phi_7/16}}\right)^2$$

$$+ 2 \left(e^{\phi_7/2} - e^{\phi_7/8}\right)^2 + \sum_{b=4,8} \left(e^{\phi_7} - 1\right)^2 + \left(e^{\phi_7/2} - e^{\phi_7/16}\right)^2$$

$$+ \sum_{b=2,4,8,16} b \left(e^{\phi_7} - 1\right)^2 + 32 \left(e^{-\phi_7/64} - e^{-\phi_7/64}\right)^2.$$
\[+4 \sum_{j=3,6} \left( e^{\phi_1 + \phi_2 + 2\phi_4 + \phi_5 + \phi_7 + \phi_8} - e^{\phi_j} \right)^2\]

\[+8e^{\phi_1 + \phi_2 + 2\phi_4 + \phi_5 + \phi_7 + \phi_8} \left( e^{\phi_1 + \phi_2} - e^\infty \right)^2\]

\[+16 \left( e^{-\phi_1 + \phi_2 + 2\phi_4 + \phi_5 + \phi_7 + \phi_8} - e^{\phi_4} \right)^2 + 4 \left( e^{\phi_4} - 1 \right)^2\]

\[+8 \left( e^{\phi_4} - 1 \right)^2 + 32 \left( e^{\phi_1 + \phi_2 + \phi_7 + \phi_8} - e^{\phi_4} \right)^2\]

\[+ \sum_{b=2,4,8,16} b \left( e^{\phi_5} - 1 \right)^2 + 2 \left( e^{\phi_4} - e^{\phi_7} \right)^2 + 2 \left( e^{\phi_2} - e^{\phi_8} \right)^2\]

\[+4 \left( e^{\phi_1 + \phi_2 + 4\phi_4 + \phi_7 + \phi_8} - e^{\phi_5} \right)^2 + \sum_{b=8,16,32} b \left( e^{\phi_1 + \phi_2 + \phi_7 + \phi_8} - 1 \right)^2\]

\[+64 \left( e^{-\phi_1 + \phi_2 + 4\phi_4 + \phi_7 + \phi_8} - e^{\phi_1 + \phi_2 + \phi_7 + \phi_8} \right)^2.\]

Let us consider as an example of our construction the untwisted affine algebra \(G_2^{(1)}\). Its scalar potential can be derived from the ones associated to any of the algebras of the \(E\) series. We choose in the following the truncation of \(E_6^{(1)}\) for which the dimension of the vector bundle \(E\) is minimal and reintroduce the coupling constants \(m\) and \(\beta\). In this case the appropriate truncation of fields is \(\phi_2 = \phi_1, \phi_3 = \phi_2\) and \(\phi_1 = \phi_4 = \phi_5 = \phi_6 = 0\). Then we find the section \(s^a\) of \(E\) for \(V_{G_2^{(1)}}\) as

\[\{s^a\} = \frac{1}{\beta} \left\{ e^{-\beta \phi_1 + 3\phi_1 - \frac{363}{2} - e^{\beta \phi_1}}, \right\}

\[\sqrt{2} \left( e^{-\beta \phi_1 + \phi_2} - 1 \right), 2 \left( e^{\beta \phi_2 + \phi_4} - e^{-\beta \phi_1 + \phi_8} \right), \sqrt{2} \left( e^{\beta \phi_2} - e^{\beta \phi_4} \right) \},\]

where \(a = \{1, 2, 3, 4\}\). The vector bundle \(E\) is four-dimensional and it is clear that we should add another four chiral fermions \(\psi^a; a = 1, 2, 3\) to the two fermions of opposite chirality \(\lambda^i; i = 1, 2\). The \((1,0)\) supersymmetric action for the affine Toda theory based on \(G_2^{(1)}\) can be found by substituting \((13)\) in \((1)\). In this case the fermion mass matrix is a \(2 \times 3\) matrix, which
takes at the vacuum, $\phi = 0$, the following form

$$M|_{\phi=0} = m \begin{pmatrix}
-\frac{3}{2} & -\frac{3}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\
-\frac{3}{2} & -\frac{3}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \quad (14)$$

This matrix has rank two and so there are two Majorana fermions constructed from the eight pairs $\{\lambda_+, \psi_-\}$ of Majorana-Weyl fermions which have non-zero mass at the supersymmetric vacuum.

4 Concluding remarks

In this paper we have constructed (1,0) supersymmetric extensions of all bosonic affine Toda theories. Our results are based on supersymmetric massive sigma models, i.e. the bosonic Toda theories are extended without reference to superalgebras. The advantage of our method is that these models are unitary and their supersymmetry is unbroken. Furthermore, the mass matrix of the fermions is well defined at the vacuum, and the usual bosonic affine Toda theories is recovered on setting the fermions to zero.

It is sufficient to consider affine Toda theories based on affine algebras of the $A - D - E$ series, since all the other cases can be deduced from these. We found that the supersymmetrization of affine Toda theories is in general not unique. On the contrary, there are as many extensions of the bosonic theories as there are ways to write their scalar potential as the sum of squares.

In this context it would be interesting to find a geometric interpretation of the vector bundle $E$, as the different extensions of affine Toda theories correspond to different choice of a section of the vector bundle $E$. Another question arises as to whether or not our supersymmetric Toda theories share the integrability property with their bosonic counterparts. The conventional
methods of proving integrability through Lax pairs or zero curvature conditions have no obvious generalization from the bosonic to the fermionic Toda theories. Whereas there are arguments why similar extensions of conformal Toda theories should not be integrable, this question remains open for the supersymmetric affine Toda theories constructed in this paper.

**Acknowledgments:** The author would like thanks G. Papadopoulos for helpful discussions and advice and the EPSRC and the German National Foundation for financial support.

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