VOLUMES OF 5-FREE HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. Let $M$ be a closed, orientable hyperbolic 3-manifold. We show that if $\pi_1(M)$ is 5-free then $\text{vol}(M) > 3.77$. As an application, we show that if $\text{vol}(M) \leq 3.77$ then $\dim H_1(M; \mathbb{F}_2) \leq 10$.

1. Introduction

It is a consequence of the Mostow Rigidity Theorem that a finite-volume hyperbolic 3-manifold is determined up to isometry by its topological type. Hence any geometrically defined invariant of such a manifold, such as its volume, may be regarded as a topological invariant. The theme of such papers as [13], [7], [6], [15], [3], [11], [18], [4], [16], [17], which will be pursued further in this paper, is to develop explicit quantitative relationships between the volume $\text{vol}(M)$ of a closed, orientable hyperbolic 3-manifold $M$ and topologically defined numerical invariants of $M$.

To illustrate the theme of “quantitative Mostow rigidity,” we shall first discuss the relationship between the volume and the homology of a closed, orientable hyperbolic 3-manifold $M$. It has long been known that for any prime $p$, the dimension of $H_1(M; \mathbb{F}_p)$ (where $\mathbb{F}_p$ denotes the field of order $p$) is linearly bounded in terms of $\text{vol}(M)$. According to [5, Proposition 2.2], which builds on results proved in [13], [1] and [10], it was shown that $\dim H_1(M; \mathbb{F}_p) \leq 334.08 \cdot \text{vol}(M)$ for every prime $p$. For small values of $\text{vol}(M)$, these results were improved by a couple of orders of magnitude in [3], [16], and [17]. Theorem 1.1 of [3] asserts that if $\text{vol}(M) \leq 1.22$ then $\dim H_1(M; \mathbb{F}_p) \leq 2$ for $p \neq 2, 7$, while $\dim H_1(M; \mathbb{F}_p) \leq 3$ if $p$ is 2 or 7; this result is sharp, as we have $\dim H_1(M; \mathbb{F}_p) = 2$ when $M$ is the Weeks manifold. Theorem 1.2 of [16] asserts that if $\text{vol}(M) \leq 3.08$ then $\dim H_1(M; \mathbb{F}_2) \leq 5$. Theorem 1.7 of [17] asserts that if $\text{vol}(M) \leq 3.44$ then $\dim H_1(M; \mathbb{F}_2) \leq 7$.

Theorem 15.4 of this paper, which is a consequence of our main result, asserts that if $\text{vol}(M) \leq 3.77$ then $\dim H_1(M; \mathbb{F}_2) \leq 10$. While this is similar in flavor to the results mentioned above, proving it requires a crucially new idea, which we hope will be a step toward the discovery of a general result of this kind.

In the context of volumes of hyperbolic 3-manifolds, the bound 3.77 which appears in Theorem 15.4 is qualitatively different from the bounds that appear in the results mentioned above. The set $\mathcal{V}$ of all finite volumes of orientable hyperbolic 3-manifolds is a well-ordered set in the ordering inherited from the standard ordering of the real numbers. Each element of $\mathcal{V}$ is represented by only finitely many hyperbolic manifolds (up to isometry). The ordinal type of $\mathcal{V}$ is $\omega$, so that there is a unique order-preserving bijection $\alpha \mapsto V_\alpha$ from the set of...
VOLUMES OF 5-FREE HYPERBOLIC 3-MANIFOLDS

For every non-limit ordinal \( \alpha \), the number \( V_{\alpha} \) is realized as the volume of a closed orientable hyperbolic 3-manifold. It follows from the main theorem of [2] that \( V_{\omega^2} \) is equal to \( V_{\text{oct}} = 3.66 \ldots \), the volume of a regular ideal hyperbolic octahedron; it is realized as the volume of the complement of the Whitehead link, an orientable hyperbolic 3-manifold. Thus the set \( \mathcal{V} \cap (0, 3.77] \) has ordinal type at least \( \omega^2 \), whereas the set \( \mathcal{V} \cap (0, 3.44] \) has ordinal type only \( m\omega + n \) for some integers \( m, n \geq 0 \). (The ordinal types of the sets of known elements of \( \mathcal{V} \) that are at most 3.44 and 3.77 respectively are \( 36\omega + 1 \) and \( \omega^2 + 8\omega + 24 \).) In this sense, Theorem 15.4 applies to far “more” hyperbolic manifolds than the other results mentioned above.

While Theorem 15.4, and the corresponding results involving the bounds 3.08 and 3.44, are vast improvements over previously known results, there is no reason to think that they are sharp. Indeed, the smallest known closed orientable 3-manifold \( M \) with \( \dim H_1(M; \mathbb{F}_2) \geq 4 \) has volume 6.35 \ldots .

The main theorem of this paper, Theorem 14.6, from which Theorem 15.4 is derived, is similar in nature to the results in earlier papers that underlie the results mentioned above about manifolds of volume at most 3.08 or 3.44. Recall that the rank of a group \( A \) is defined to be the minimal cardinality of a generating set for \( A \). A group \( \Gamma \) is said to be \( k \)-free, where \( k \) is a given non-negative integer, if every subgroup of \( \Gamma \) having rank at most \( k \) is a free group. According to [4, Corollary 9.3], which is an extension of [7, Theorem 6.1], if a closed, orientable hyperbolic 3-manifold \( M \) has a 3-free fundamental group then \( \text{vol} M > 3.08 \). According to [17, Theorem 1.6], if \( \pi_1(M) \) is 4-free then \( \text{vol} M > 3.44 \). (A minor correction to the proof of the latter theorem was given in [12].) Theorem 14.6 of the present paper asserts that if \( \pi_1(M) \) is 5-free then \( \text{vol} M > 3.77 \).

In both this paper and its predecessors, the transition between results relating \( k \)-freeness to volume and those relating volume to homology involves deep geometric and topological considerations, which are embodied in the results of [22], [8], [7], [6], [4], [16], and [11]. In the case of the present paper, the relevant arguments occupy Section 15, in which several of the papers listed above are quoted. In this connection, we would like to call the reader’s attention to Proposition 15.2, which is deduced almost formally from the results in [11] but was overlooked when that paper was written.

To put the discussion of our proof of Theorem 14.6 in context, we shall begin by reviewing some material from our earlier paper [19], in which the proofs of the volume estimates mentioned above for manifolds with 3-free and 4-free fundamental groups are put in a general setting. If \( p \) is a point of a closed, orientable hyperbolic 3-manifold \( M \), each non-trivial element of \( \pi_1(M, p) \) lies in a unique maximal cyclic subgroup. The main theorem of [19] asserts that if \( \pi_1(M) \) is \( k \)-free for a given \( k > 0 \), and if \( C_1, \ldots, C_m \) are maximal cyclic subgroups of \( \pi_1(M, p) \), each of which contains at least one non-trivial element represented by a loop of length less that \( \log(2k-1) \) based at \( p \), then the subgroup of \( \pi_1(M, p) \) generated by \( C_1, \ldots, C_m \) has rank at most \( k - 3 \). In the introduction to [19] the explicit geometric information provided by this theorem in the cases \( k = 3 \) and \( k = 4 \) is discussed, and it is
pointed out that this is exactly the information that was used in [7] and [17] to establish the strict lower volume bounds of 3.08 and 3.44 respectively.

The proof of the main theorem of [19], which builds on ideas developed in [14], [7], [17], and [20], involves writing \( M \) as a quotient \( H^3/\Gamma \), where \( \Gamma \cong \pi_1(M) \) is a discrete, cocompact, \( k \)-free group of orientation-preserving isometries of \( H^3 \), and studying a family \( Z_\lambda(\Gamma) \) of open subsets of \( H^3 \), where \( \lambda \) is a positive number. The family is indexed by the set \( C \) of all maximal cyclic subgroups of \( \Gamma \), and the element of the family indexed by any \( C \in C \) is the set \( Z_\lambda(C) \) consisting of all points \( P \in H^3 \) such that \( dist(P, x \cdot P) < \lambda \) for some non-trivial element \( x \) of \( C \). (Throughout the paper we will use “dist” to denote the distance function in a metric space, wherever this is possible to do without creating ambiguity.) If, for \( \lambda = \log(2k-1) \), the family \( Z_\lambda(\Gamma) \) does not cover \( H^3 \), it is easily shown that no non-trivial element of \( \pi_1(M, p) \) is represented by a loop of length less than \( \log(2k-1) \) based at \( p \); this is obviously stronger than the conclusion of the main theorem of [19]. When \( Z_\lambda(\Gamma) \) does cover \( H^3 \), the arguments of [19] are based on a consideration of the nerve of the covering (in the sense that is standard in algebraic topology; cf. 3.1 below). The nerve is a simplicial complex whose vertices are canonically in bijective correspondence with elements of \( C \). In [19], we defined the rank of any simplex \( \sigma \) of the nerve to be the rank of the subgroup of \( \Gamma \) generated by the elements of \( C \) that correspond to the vertices of \( \sigma \). We defined the internal rank of a simplex to be the maximum of the ranks of its faces.

A key step in the proof of the main theorem of [19] is the proof of the following assertion:

**1.0.1.** If \( \Gamma \) is a discrete, cocompact, \( k \)-free group of orientation-preserving isometries of \( H^3 \) such that \( Z_{\log(2k-1)}(\Gamma) \) covers \( H^3 \), then the nerve of \( Z_{\log(2k-1)}(\Gamma) \) contains a simplex that has internal rank at most \( k - 3 \) and has a non-contractible link.

Although the assertion 1.0.1 was not made explicit in [19], its proof and its role in the proof of the main theorem of [19] are easily extracted from the statements and proofs of Lemma 3.3 and Theorem 5.2 of that paper. The proof of 1.0.1 makes essential use of the so-called \( \log(2k-1) \) theorem, a result first proved under restrictive hypotheses in [7] and generalized in [3], which asserts that if orientation-preserving isometries \( x_1, \ldots, x_k \) of \( H^3 \) generate a free discrete group of rank \( k \), then \( \max_{1 \leq i \leq k} dist(P, x_i \cdot P) \geq \log(2k-1) \) for every \( P \in H^3 \). The proof of 1.0.1 combines the \( \log(2k-1) \) theorem with topological, group-theoretical and combinatorial arguments that build on ideas from [14], [7], [17], and [20]. The passage from 1.0.1 to the main theorem of [19] is largely an application of the Borsuk Nerve Theorem (cf. Propositions 3.3 and 3.4 below).

For \( k = 5 \), the main theorem of [19] is not strong enough to give a new volume estimate. Our volume estimate for manifolds with 5-free fundamental groups depend on a central result, Theorem 5.2 of the present paper, which is strictly stronger than the assertion 1.0.1; the implication is formalized in Subsection 5.3 below. The statement of Theorem 5.2 of this paper involves the notion of the “height” of a simplex (see Definition and Remarks 4.12), a quantity which is a priori greater than or equal to the internal rank. The theorem asserts that if \( \Gamma \) is a discrete, cocompact, \( k \)-free group of orientation-preserving isometries of \( H^3 \)
such that \(Z_{\log(2k-1)}(\Gamma)\) covers \(H^3\), then the nerve of \(Z_{\log(2k-1)}(\Gamma)\) contains a simplex that has height at most \(k - 3\) and has a non-contractible link. The definition of height blends algebraic and geometric information about elements of the discrete group \(\Gamma\), and therefore makes possible the application, in the proof of Theorem 5.2, of a stronger version of the \(\log(2k-1)\) theorem, also proved in a special case in [7] and generalized in [3] (cf. Theorem 4.2 below), in which the conclusion \(\max_{1 \leq i \leq k} \text{dist}(P, x_i \cdot P) \geq \log(2k-1)\) is replaced by the stronger inequality \(\sum_{i=1}^k 1/(1 + \exp(\text{dist}(P, x_i \cdot P))) \leq 1/2\).

The actual definition of height, which appears below as Definition 4.12, is subtle and involved, and depends on a good deal of machinery, the development of which occupies much of Sections 2—4. This includes the purely group-theoretical material in Section 2 on the structure of \(k\)-free groups, which we believe may be of independent interest: it provides a natural closure operation on the class of subgroups of a \(k\)-free group which have local rank less than \(k\). (See 2.1 for the definition of local rank, and see Subsection 2.11 and Proposition 2.13 for the definition and properties of the closure operation.) The general material on nerves that we need is reviewed and systematized in Section 3.

In analogy to the use of the assertion 1.0.1 to prove the main theorem of [19], we use Theorem 5.2 of the present paper to prove a result, Proposition 7.2, which contains geometric information that is strong enough to establish our improved lower bound of 3.77 for the volume of \(M\) when \(\pi_1(M)\) is 5-free.

In order to explain the part of the statement of Proposition 7.2 which is relevant to the volume estimate, we anticipate here some definitions that are given in a more systematic setting in Subsection 6.2 below. If \(p\) is a point of the closed, orientable hyperbolic 3-manifold \(M\), we denote by \(s_1(p)\) the minimum length of a homotopically non-trivial loop based at \(p\).

If there is only one maximal cyclic subgroup \(C\) of \(\pi_1(M, p)\) containing a non-trivial element represented by a loop of length \(s_1(p)\), we denote by \(s_2(p)\) the minimum length of a loop \(\alpha\) with \([\alpha] \notin C\;\); if there is more than one such subgroup, we set \(s_2(p) = s_1(p)\).

The relevant part of Proposition 7.2 states that if the closed, orientable hyperbolic 3-manifold \(M\) has a 5-free fundamental group, and if \(\lambda\) is a positive real number with \(\lambda \leq \log 9\), then either (a) \(M\) contains a point \(p\) with \(s_2(p) \geq \lambda\); or (b) \(\lambda > \log 5\), and there exists a point \(p_1 \in M\) with \(s_1(p_1) > \log((e^\lambda + 7)/(e^\lambda - 5))\); or (c) there is a point \(p \in M\) such that \(1/(1 + e^{s_1(p)}) + 1/(1 + e^{s_2(p)}) \leq 1/2 - 2/(1 + e^\lambda)\).

The rather involved arguments needed to deduce Proposition 7.2 from Theorem 5.2 are given in Section 7, following the elementary observations and establishment of conventions that largely occupy Section 6.

Section 8 is a brief section devoted to the proof of a result, Proposition 8.3, which is a variant of Proposition 7.2 and is deduced from Proposition 7.2. Proposition 8.3 allows one to strengthen the lower bound on volume to 3.77 from 3.75, which one would obtain from a more direct application of Proposition 7.2.

Sections 9—14 provide the transition from Proposition 8.3 to the volume estimate given by Theorem 14.6. The results of Section 10 give lower bounds for the volumes of certain
subsets of a hyperbolic 3-manifold in terms of the geometry of the manifold; some of these are reviewed from [17], but a number of them constitute improvements on the results that appeared there. Section 9 is an observation about hyperbolic triangles that is used in Section 10. The results of Section 11 concerning volumes, diameters, and Margulis numbers are strict improvements over the corresponding results in Section 10 of [17]. The results of Section 12 are stronger and more general than those established by the corresponding arguments in [17] (which are contained in the last step of the proof of [17, Lemma 13.4]).

In Section 13, the results of Sections 10—12 are combined with Proposition 8.3 to prove Proposition 13.2, which for each integer \( k \geq 5 \) provides sufficient conditions, stated entirely in terms of ranges of analytically defined functions, for a given number \( V_0 \) to be a lower bound for the volumes of all closed, orientable hyperbolic 3-manifolds with \( k \)-free fundamental group. In Section 14 we prove Theorem 14.6 by verifying that these conditions hold when \( V_0 = 3.77 \) and \( k = 5 \). As in [17], this requires resorting to a brute-force partitioning method because we do not have analytical techniques for handling the functions involved. Preliminary calculations suggest that for \( k = 6 \), Proposition 13.2 is likely to yield a lower bound of about 3.85; we will not give details, because we hope that a variant of the methods presented here will yield a stronger result for \( k \geq 6 \) than the one given by Proposition 13.2.

We think of the paper as being divided into three chapters. The first chapter, which comprises Sections 2—5, is centered around the study of groups (abstract groups in Section 2 and discrete groups of orientation-preserving isometries of \( \mathbb{H}^3 \) in Sections 3—5). The second chapter, which consists of Sections 6—8, is concerned with hyperbolic 3-manifolds, and the third chapter, Sections 9—15, is more specifically about volumes.

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2. The structure of \( k \)-free groups

Definitions and Remarks 2.1. If \( \Gamma \) is a group, we will write \( A \leq \Gamma \) to mean that \( A \) is a subgroup of \( \Gamma \).

We shall say that elements \( x_1, \ldots, x_m \) of a group \( \Gamma \) are independent if the subgroup \( \langle x_1, \ldots, x_m \rangle \) of \( \Gamma \) is free on its generators \( x_1, \ldots, x_m \).

A group is said to be locally free if each of its finitely generated subgroups is free.

As we mentioned in the introduction, the rank of a group \( A \), denoted \( \text{rank } A \), is defined to be the minimal cardinality of a generating set for \( A \); and a group \( \Gamma \) is said to be \( k \)-free, where \( k \) is a given non-negative integer, if every subgroup of \( \Gamma \) having rank at most \( k \) is a free group.

Every group is 0-free. If \( k \) and \( k' \) are integers with \( 0 \leq k' \leq k \), every \( k \)-free group is \( k' \)-free. We define the index of freedom of a group \( \Gamma \), denoted \( \text{iof}(\Gamma) \), to be the supremum of all integers \( k \) such that \( \Gamma \) is \( k \)-free. Thus we have \( \text{iof}(\Gamma) = \infty \) if and only if \( \Gamma \) is locally free,
i.e. every finitely generated subgroup of $\Gamma$ is free. If $\Gamma$ is not locally free then $\text{iof}(\Gamma)$ is the largest integer $k$ such that $\Gamma$ is $k$-free.

If $S$ is a subset of a group $\Gamma$, the internal rank of $S$, denoted $\text{IR}(S)$, is defined to be the maximum of $\text{rank}(T)$, where $T$ ranges over all subsets of $S$. (This is a slight change from the terminology in [19], where the internal rank is defined for a set of maximum cyclic groups of an ICC-group (see 4.3), rather than a set of elements of a group.)

Let $A$ be a group. If there is a non-negative integer $r$ with the property that every finitely generated subgroup of $A$ is contained in a subgroup of $A$ having rank $r$, then the smallest integer with this property will be called the local rank of $A$. If no such integer $r$ exists, we say that the local rank of $A$ is infinite.

Note that the local rank of a finitely generated group is equal to its rank. Note also that if $k$ is a positive integer and $\Gamma$ is a $k$-free group, then every subgroup of $\Gamma$ having local rank at most $k$ is a locally free group.

**Proposition 2.2.** Let $A$ be a group of local rank $r < \infty$, and let $\langle t \rangle$ be an infinite cyclic group. Then the free product $A \star \langle t \rangle$ has local rank $r + 1$.

**Proof.** If $A$ is finitely generated, so that $r = \text{rank} A$, then $A \star \langle t \rangle$ is finitely generated, and $\text{localrank}(A \star \langle t \rangle) = \text{rank}(A \star \langle t \rangle) = r + 1$ by Grushko’s Theorem.

For the proof in the general case, first note that if $B$ is any finitely generated subgroup of $A \star \langle t \rangle$, we have $B \leq C_0 \star \langle t \rangle$ for some finitely generated subgroup $C_0$ of $A$. Since $\text{localrank}(A) = r$, there is a finitely generated subgroup $C_1$ of $A$ with $\text{rank} C_1 = r$ and $C_0 \leq C_1$. Then $B \leq C_1 \star \langle t \rangle$, and $\text{rank}(C_1 \star \langle t \rangle) = r + 1$. Hence $\text{localrank}(A \star \langle t \rangle) \leq r + 1$.

Now assume that the latter inequality is strict, and consider an arbitrary finitely generated subgroup $D$ of $A$. Since we have assumed that $\text{localrank}(A \star \langle t \rangle) \leq r$, there is a finitely generated subgroup $E$ of $A \star \langle t \rangle$ such that $\text{rank} E \leq r$ and $D \star \langle t \rangle \leq E$. According to the Kurosh Subgroup Theorem, we may identify $E$ with a free product $G_1 \star \cdots \star G_m \star F$, where $F$ is a free group of finite rank and $G_1, \ldots, G_m$ are precisely the non-trivial subgroups that arise as intersections of $E$ with conjugates of $A$. If we regard $D$ as a subgroup of $D \star \langle t \rangle$, we have $D \leq E \cap A$; hence $D \leq G_i$ for some $i$, and after re-indexing we may assume that $D \leq G_1$. On the other hand, since $t \in E$, the subgroup $E$ of $A \star \langle t \rangle$ cannot be generated by its intersections with conjugates of $A$; hence $F$ has strictly positive rank. By Grushko’s Theorem we have $r \geq \text{rank} E = \text{rank} G_1 + \cdots + \text{rank} G_m + \text{rank} F \geq \text{rank} G_1 + \text{rank} F > \text{rank} G_1$, so that $\text{rank} G_1 \leq r - 1$. As $D$ was an arbitrary finitely generated subgroup of $A$, and $D \leq G_1$, this shows that $\text{localrank}(A) \leq r - 1$, a contradiction. \qed

**Notation and Remark 2.3.** Let $A$ be a subgroup of a group $\Gamma$, and let $x$ be an element of $\Gamma$. If $\langle t \rangle$ is the standard infinite cyclic multiplicative group, we will denote by $\iota_{A,x}$ the homomorphism from the free product $A \star \langle t \rangle$ to $\Gamma$ defined by taking $\iota_{A,x}|A$ to be the inclusion, and setting $\iota_{A,x}(t) = x$.

Note that if $B$ is any subgroup of $A$ then $B \star \langle t \rangle$ is canonically identified with a subgroup of $A \star \langle t \rangle$, and that under this identification we have $\iota_{B,x} = \iota_{A,x}|(B \star \langle t \rangle)$.
Definition 2.4. A subgroup $A$ of a group $\Gamma$ will be said to be closable if $\text{localrank}(A) < \text{iof}(A)$. Note that if $A$ is closable then in particular we have $\text{localrank}(A) < \infty$.

Proposition 2.5. Let $A$ be a closable subgroup of a group $\Gamma$. Then for any $x \in \Gamma$, we have $\text{localrank}(\langle A, x \rangle) > \text{localrank}(A)$ if and only if the homomorphism $\iota_{A,x} : A \star \langle t \rangle \to \Gamma$ is injective.

Proof. Set $r = \text{localrank}(A)$. By hypothesis we have $r < \text{iof}(\Gamma)$.

If $\iota_{A,x}$ is injective then $\langle A, x \rangle$ is isomorphic to $A \star \langle t \rangle$, where $\langle t \rangle$ is infinite cyclic, so that Proposition 2.2 gives $\text{localrank}(\langle A, x \rangle) = 1 + r > r$.

To prove the converse, let us first consider the case in which $A$ is finitely generated. In this case, we have rank $A = r$. Suppose that that rank $\langle A, x \rangle > r$. Since $r < \text{iof}(\Gamma)$, the rank-$r$ group $A$ is free. Fix a basis $(u_1, \ldots, u_r)$ for $A$. Then $(u_1, \ldots, u_r, x)$ is a generating set for $\langle A, x \rangle$; in particular the rank of $\langle A, x \rangle$ is at most $r + 1$. Since this rank has been assumed to be strictly greater than $r$, it must be exactly $r + 1$. Now since $r + 1 < \text{iof}(\Gamma)$, the rank-$(r + 1)$ group $\langle A, x \rangle$ is free. But a generating set for a finite-rank free group, whose cardinality is equal to the rank of the group, must be a basis (see [24, vol. 2, p. 59]). It follows that the generating set $(u_1, \ldots, u_r, x)$ for $\langle A, x \rangle$ must be a basis for $\langle A, x \rangle$. Now since $(u_1, \ldots, u_r)$ and $(u_1, \ldots, u_r, x)$ are bases for $A$ and $\langle A, x \rangle$ respectively, $\iota_{A,x}$ is injective.

For the proof in the general case, we assume that $\text{localrank}(\langle A, x \rangle) > r$. We must show that for an arbitrary element $z$ of $A \star \langle t \rangle$ we have $\iota_{A,x}(z) \neq 1$. The hypothesis $\text{localrank}(\langle A, x \rangle) > r$ gives a finitely generated subgroup $B$ of $\langle A, x \rangle$ such that every finitely generated subgroup of $\langle A, x \rangle$ containing $B$ has rank at least $r + 1$. Since $\langle B, z \rangle \leq \langle A, x \rangle$ is finitely generated, we may choose a finitely generated subgroup $C$ of $A$ so that $\langle B, z \rangle \leq \langle C, x \rangle$. Since $\text{localrank}(A) = r$, there is a finitely generated subgroup $D$ of $A$ with $C \leq D$ and $\text{rank } D \leq r$. We have $B \leq \langle C, x \rangle \leq \langle D, x \rangle$, and our choice of $B$ therefore guarantees that $\text{rank } \langle D, x \rangle \geq r + 1$.

Hence $\text{rank } \langle D, x \rangle > \text{rank } D$, and $\text{rank } D \leq r < \text{iof}(\Gamma)$. Since we have already proved the assertion in the case of a finitely generated subgroup, we may apply this special case with $D$ playing the role of $A$ to deduce that $\iota_{D,x} : D \star \langle t \rangle \to \Gamma$ is injective. But according to 2.3 we have $\iota_{D,x} = \iota_{A,x}|(D \star \langle t \rangle)$. As $z \in C \leq D$, we have $\iota_{A,x}(z) = \iota_{D,x}(z) \neq 1$, as required. $\square$

Proposition 2.6. Let $k$ be a positive integer, let $\Gamma$ be a $k$-free group, let $S$ be a finite subset of $\Gamma$, and set $m = \min(k, \text{IR}(S))$. Then $S$ contains $m$ independent elements.

Proof. We may assume $m > 0$, as otherwise the assertion is trivial. Since $0 < m \leq \text{IR}(S)$, there exist an integer $n > 0$ and elements $x_1, \ldots, x_n$ of $S$ such that $\text{rank } \langle x_1, \ldots, x_n \rangle \geq m$. Set $A_0 = \{1\}$, and for $s = 1, \ldots, n$ set $A_s = \langle x_1, \ldots, x_s \rangle$. We have rank $A_0 = 0$ and rank $A_n \geq m$, and for each $s$ with $0 \leq s < n$ we have rank $A_{s+1} \leq 1 + \text{rank } A_s$. Hence for each $j \in \{0, \ldots, m\}$ there is an index $s_j$ such that rank $A_{s_j} = j$, and rank $A_s < j$ for every $s$ with $0 \leq s < s_j$. Note that $0 = s_0 < \ldots < s_m$.

Consider an arbitrary index $j \in \{1, \ldots, m\}$. We have rank $A_{s_j-1} < j \leq m$. Since $m \leq k$, and $\Gamma$ is $k$-free, it follows that rank $A_{s_j-1} < \text{iof}(\Gamma)$, i.e. $A_{s_j-1}$ is closable. Since rank $\langle A_{s_j-1}, x_{s_j} \rangle =$
rank $A_{s_j} = j > \text{rank } A_{s_j-1}$, it now follows from Proposition 2.5 that the homomorphism $\iota_{A_{s_j-1}, x_{s_j}} : A_{s_j-1} \ast \langle t \rangle \to \Gamma$ is injective. But we have $s_{j-1} \leq s_j - 1$ and hence $A_{s_{j-1}} \leq A_{s_j-1}$, so that 2.3 gives $\iota_{A_{s_{j-1}}, x_{s_{j-1}}} = \iota_{A_{s_j-1}, x_{s_j}}(A_{s_j-1} \ast \langle t \rangle)$. Thus in particular $\iota_{A_{s_{j-1}}, x_{s_{j-1}}}$ is injective. Since this is true for $j = 1, \ldots, m$, it follows that $x_{s_1}, \ldots, x_{s_m}$ are independent. 

Proposition 2.7. Let $A$ and $B$ be closable subgroups of a group $\Gamma$, such that $B \leq A$. Suppose that $x$ is an element of $\Gamma$ such that localrank$(\langle B, x \rangle) \leq$ localrank$(B)$. Then localrank$(\langle A, x \rangle) \leq$ localrank$(A)$.

Proof. Assume that localrank$(\langle A, x \rangle) >$ localrank$(A)$. Then according to Proposition 2.5, \( \iota_{A, x} : A \ast \langle t \rangle \to \Gamma \) is injective. But according to 2.3 we have $\iota_{B, x} = \iota_{A, x}(B \ast \langle t \rangle)$. Hence $\iota_{B, x}$ is injective. According to Proposition 2.5, this implies that localrank$(\langle B, x \rangle) >$ localrank$(B)$, a contradiction to the hypothesis.

Definition and Remark 2.8. Let $A$ be a closable subgroup of a group $\Gamma$. We define an $A$-admissible sequence in $\Gamma$ to be a finite sequence of elements $x_1, \ldots, x_m$ of $\Gamma$, where $m$ is a positive integer, such that the following chain of inequalities holds:

\[
\text{localrank}(A) \geq \text{localrank}(\langle A, x_1 \rangle) \geq \text{localrank}(\langle A, x_1, x_2 \rangle) \geq \cdots \geq \text{localrank}(\langle A, x_1, \ldots, x_m \rangle).
\]

Note that if $x_1, \ldots, x_m$ is an $A$-admissible sequence in $\Gamma$, then for any $s$ with $1 \leq s \leq m$, the sequence $x_1, \ldots, x_s$ is also $A$-admissible.

Lemma 2.9. Let $A$ and $B$ be closable subgroups of a group $\Gamma$ such that $B \leq A$. Then every $B$-admissible sequence in $\Gamma$ is also $A$-admissible.

Proof. Let $x_1, \ldots, x_m$ be a $B$-admissible sequence. We prove by induction on the length $m \geq 1$ of $x_1, \ldots, x_m$ that it is $A$-admissible. If $m = 1$, the definition of $B$-admissibility gives localrank$(\langle B, x_1 \rangle) \leq$ localrank$(B)$. Since $A$ and $B$ are closable, we may apply Proposition 2.7, with $x_1$ playing the role of $x$, to deduce that localrank$(\langle A, x_1 \rangle) \leq$ localrank$(A)$; this says that the one-term sequence $x_1$ is $A$-admissible. Now suppose that $m > 1$ and that the result is true for admissible sequences of length $m - 1$. Let us set $A_0 = A$ and $B_0 = B$, and for $1 \leq i \leq m$ let us set $A_i = \langle A, x_1, \ldots, x_i \rangle$ and $B_i = \langle B, x_1, \ldots, x_i \rangle$. Then $B_i \leq A_i$ for $i = 0, \ldots, m$. The $B$-admissibility of $x_1, \ldots, x_m$ gives

\[
\text{localrank}(B_0) \geq \text{localrank}(B_1) \cdots \geq \text{localrank}(B_m).
\]

In particular we have localrank$(B_0) \geq$ localrank$(B_1) \geq \cdots \geq$ localrank$(B_m - 1)$, so that $x_1, \ldots, x_{m-1}$ is $B$-admissible, and is therefore $A$-admissible by the induction hypothesis; that is, we have localrank$(A_0) \geq$ localrank$(A_1) \geq \cdots \geq$ localrank$(A_{m-1})$. In view of the closability of $A_0 = A$ and $B_0 = B$, we have localrank$(A_{m-1}) \leq$ localrank$(A_0) <$ iof$(\Gamma)$ and localrank$(B_{m-1}) \leq$ localrank$(B_0) <$ iof$(\Gamma)$. Hence $A_{m-1}$ and $B_{m-1}$ are closable. Since (2.9.1) also gives localrank$(\langle B_{m-1}, x_m \rangle) =$ localrank$(B_m) \leq$ localrank$(B_{m-1})$, we may apply Proposition 2.7, with $A_{m-1}$, $B_{m-1}$ and $x_m$ playing the respective roles of $A$, $B$ and $x$, to deduce that localrank$(A_m) =$ localrank$(\langle A_{m-1}, x_m \rangle) \leq$ localrank$(A_{m-1})$. Thus we have localrank$(A_0) \geq$ localrank$(A_1) \geq \cdots \geq$ localrank$(A_m)$, i.e. $x_1, \ldots, x_m$ is $A$-admissible. 

□
Lemma 2.10. Let $A$ be a closable subgroup of a group $\Gamma$, and let $p$ be a positive integer. For $j = 1, \ldots, p$ let $x_1^{(j)}, \ldots, x_{m_j}^{(j)}$ be an $A$-admissible sequence in $\Gamma$. Then the sequence
\[ x_1^{(1)}, \ldots, x_{m_1}^{(1)}, x_1^{(2)}, \ldots, x_{m_2}^{(2)}, \ldots, x_1^{(p)}, \ldots, x_{m_p}^{(p)} \]
of elements of $\Gamma$ is $A$-admissible.

Proof. It is enough to give the proof in the case $p = 2$, as the general case then follows by induction. Thus it suffices to prove that if $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are an $A$-admissible sequences in $\Gamma$, then the sequence $x_1, \ldots, x_m, y_1, \ldots, y_n$ is $A$-admissible. Let us set $A' = \langle A, x_1, \ldots, x_m \rangle$. The $A$-admissibility of $x_1, \ldots, x_m$, and the closability of $A$, imply that $\text{localrank}(A') \leq \text{localrank}(A) < \text{iof}(\Gamma)$. Hence $A'$ is closable. Since in addition we have $A \leq A'$, and $y_1, \ldots, y_n$ is $A$-admissible, it follows from Lemma 2.9 that $y_1, \ldots, y_n$ is $A'$-admissible. But in view of the definition of admissibility, the $A$-admissibility of $x_1, \ldots, x_m$ and the $A'$-admissibility of $y_1, \ldots, y_n$ immediately imply that $x_1, \ldots, x_m, y_1, \ldots, y_n$ is $A$-admissible. 

Notation and Remarks 2.11. If $A$ is a closable subgroup of a group $\Gamma$, we will denote by $c(A)$ the set of all elements $x \in \Gamma$ with the property that there exists an $A$-admissible sequence $x_1, \ldots, x_m$ with $x_m = x$.

We observed in 2.8 that if $x_1, \ldots, x_m$ is an $A$-admissible sequence then the sequence $x_1, \ldots, x_s$ is also $A$-admissible for any $s$ with $1 \leq s \leq m$. It follows that if $x_1, \ldots, x_m$ is an $A$-admissible sequence, then we have $x_1, \ldots, x_m \in c(A)$.

Note that, a priori, $c(A)$ is only a subset of $\Gamma$. However, Proposition 2.13 below will assert, among other things, that it is a subgroup of $\Gamma$; it will be seen from the proof of Proposition 2.13 that all the asserted properties of $c(A)$ depend on the closability of $A$. It is for this reason that we have defined $c(A)$ only when $A$ is closable.

Lemma 2.12. Let $A$ be a closable subgroup of a group $\Gamma$, and let $S$ be a finite subset of $c(A)$. Then there is an $A$-admissible sequence $y_1, \ldots, y_q$ such that $S \subset \{y_1, \ldots, y_q\} \subset c(A)$, and we have $\text{localrank}(\langle A, y_1, \ldots, y_q \rangle) \leq \text{localrank}(A)$. In particular, $\langle A, y_1, \ldots, y_q \rangle$ is a closable subgroup of $\Gamma$.

Proof. Write $S = \{x^{(1)}, \ldots, x^{(p)}\}$ for $B$. For each $j \in \{1, \ldots, p\}$, since $x^{(j)} \in c(A)$, there is an $A$-admissible sequence $x_1^{(j)}, \ldots, x_{m_j}^{(j)}$ with $x_j^{(j)} = x_{m_j}^{(j)}$. We observed in 2.11 that $x_j^{(j)} \in c(A)$ for $i = 1, \ldots, m_j$.

According to Lemma 2.10, the sequence
\[(2.12.1)\]
\[ x_1^{(1)}, \ldots, x_{m_1}^{(1)}, x_1^{(2)}, \ldots, x_{m_2}^{(2)}, \ldots, x_1^{(p)}, \ldots, x_{m_p}^{(p)} \]
is $A$-admissible. If we set
\[ C = \langle A, x_1^{(1)}, \ldots, x_{m_1}^{(1)}, x_1^{(2)}, \ldots, x_{m_2}^{(2)}, \ldots, x_1^{(p)}, \ldots, x_{m_p}^{(p)} \rangle, \]
the definition of an $A$-admissible sequence then implies that $\text{localrank}(C) \leq \text{localrank}(A) = r$. Thus if we define $y_1, \ldots, y_q$ to be the sequence (2.12.1),

\[ y_1, \ldots, y_q \in c(A), \]

then $\langle A, y_1, \ldots, y_q \rangle$ is a closable subgroup of $\Gamma$. \hfill $\square$
we have $\text{localrank}(\langle A, y_1, \ldots, y_q \rangle) \leq \text{localrank}(A)$. Since $A$ is closable, we have

$$\text{localrank}(\langle A, y_1, \ldots, y_q \rangle) \leq \text{localrank}(A) < \text{iof}(\Gamma),$$

so that $\langle A, y_1, \ldots, y_q \rangle$ is closable.

\[\square\]

**Proposition 2.13.** Let $A$ be a closable subgroup of a group $\Gamma$. Then:

1. $\mathfrak{c}(A)$ is a subgroup of $\Gamma$, and $\text{localrank}(\mathfrak{c}(A)) \leq \text{localrank}(A)$; in particular $\mathfrak{c}(A)$ is closable;
2. $A \leq \mathfrak{c}(A)$;
3. for any subgroup $B$ of $A$ with $\text{localrank}(B) < k$, we have $\mathfrak{c}(B) \leq \mathfrak{c}(A)$; and
4. $\mathfrak{c}(\mathfrak{c}(A))$ (which is defined since $\mathfrak{c}(A)$ is closable by Assertion (1)) is equal to $\mathfrak{c}(A)$.

**Proof.** Set $r = \text{localrank}(A)$. Since $A$ is closable, we have $r < \text{iof}(\Gamma)$.

We first show that the set $\mathfrak{c}(A)$ contains $A$. If $x$ is any element of $A$, the one-term sequence $x$ is admissible since $\langle A, x \rangle = A$. Hence $x \in \mathfrak{c}(A)$, and the inclusion $A \subseteq \mathfrak{c}(A)$ is established.

Next we prove that $\mathfrak{c}(A)$ is a subgroup of $\Gamma$. Suppose that $x$ and $y$ are elements of $\mathfrak{c}(A)$. By definition this means that there are $A$-admissible sequences $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ such that $x_m = x$ and $y_n = y$. According to Lemma 2.10, the sequence $x_1, \ldots, x_m, y_1, \ldots, y_n$ is admissible. If we set $A' = \langle A, x_1, \ldots, x_m, y_1, \ldots, y_n \rangle$, we have $xy = x_my_n \in A'$; thus $\langle A', xy \rangle = A'$, and in particular $\text{localrank}(\langle A', xy \rangle) = \text{localrank}(A')$. In view of the definition of $A$-admissibility, it now follows that $x_1, \ldots, x_m, y_1, \ldots, y_n, xy$ is admissible. Hence $xy \in \mathfrak{c}(A)$.

Since we have shown $A \subseteq \mathfrak{c}(A)$, we have $1 \in \mathfrak{c}(A)$. If $x$ is any element of $\mathfrak{c}(A)$, we may choose an $A$-admissible sequence $x_1, \ldots, x_m$ with $x_m = x$. Since $\langle A, x_1, \ldots, x_{m-1}, x_m^{-1} \rangle = \langle A, x_1, \ldots, x_{m-1}, x_m \rangle$, the sequence $x_1, \ldots, x_{m-1}, x_m^{-1}$ is also $A$-admissible, and hence $x^{-1} = x_m^{-1} \in \mathfrak{c}(A)$. This completes the proof that $\mathfrak{c}(A)$ is a subgroup of $\Gamma$.

Now that $\mathfrak{c}(A)$ has been shown to be a subgroup, the inclusion $A \subseteq \mathfrak{c}(A)$, which was proved above, may be rewritten in the form $A \leq \mathfrak{c}(A)$. This proves Assertion (2).

To complete the proof of Assertion (1), it remains to prove that $\text{localrank}(\mathfrak{c}(A)) \leq r = \text{localrank}(A)$. (This will imply in particular that $\text{localrank}(\mathfrak{c}(A)) < \text{iof}(\Gamma)$, so that $\mathfrak{c}(A)$ is closable.) Suppose that $B$ is a finitely generated subgroup of $\mathfrak{c}(A)$, and fix a finite generating set $S$ for $B$. According to Lemma 2.12, there is an $A$-admissible sequence $y_1, \ldots, y_q$ such that $S \subseteq \{ y_1, \ldots, y_q \} \subseteq \mathfrak{c}(A)$, and the group $C := \langle A, y_1, \ldots, y_q \rangle$ has local rank at most $r$. Since $S \subseteq \{ y_1, \ldots, y_q \} \subseteq C$, we have $B \subseteq C$. Hence $B$ is contained in some finitely generated subgroup $B'$ of $C$ with rank $B' \leq r$.

Since $\{ y_1, \ldots, y_q \} \subseteq \mathfrak{c}(A)$, and since $A \leq \mathfrak{c}(A)$ by Assertion (2), we have $C \leq \mathfrak{c}(A)$. Hence $B'$ is in particular a finitely generated subgroup of $\mathfrak{c}(A)$ which contains $B$ and has rank at most $r$. As $B$ was an arbitrary finitely generated subgroup of $\mathfrak{c}(A)$, this shows that $\text{localrank}(\mathfrak{c}(A)) \leq r$, and the proof of Assertion (1) is complete.
To prove Assertion (3), consider an arbitrary subgroup $B$ of $A$ with $\text{localrank}(B) < k$, and an arbitrary element $x$ of $c(B)$. The definition of $c(A)$ gives a $B$-admissible sequence $x_1, \ldots, x_m$ with $x_m = x$. According to Lemma 2.9, $x_1, \ldots, x_m$ is also $A$-admissible. Hence $x \in c(A)$.

To prove Assertion (4), first note that since $c(A)$ is a closable subgroup of $\Gamma$ by Assertion (1), we may apply Assertion (2), with $c(A)$ playing the role of $A$, to deduce that $c(A) \leq c(c(A))$.

The main step in the proof of the reverse inclusion will be the proof of the following fact:

**2.13.1.** If $z$ is an element of $\Gamma$ such that $\text{localrank}(\langle c(A), z \rangle) \leq \text{localrank}(c(A))$, then $z \in c(A)$.

To prove 2.13.1, we first invoke Lemma 2.5, letting $z$ and $c(A)$ play the respective roles of $A$; again, this is permissible since $c(A)$ is a closable subgroup of $\Gamma$. According to Lemma 2.5, the assumption $\text{localrank}(\langle c(A), z \rangle) \leq \text{localrank}(c(A))$ implies that the homomorphism $\iota_{c(A),z} : c(A) \ast \langle t \rangle \to \Gamma$ is non-injective, where $\langle t \rangle$ denotes an infinite cyclic group. Choose a non-trivial element $w$ of the kernel of $\iota_{c(A),z}$. Let us write $w = u_1 t^{e_1} \cdots u_p t^{e_p}$, where $u_1, \ldots, u_p$ are elements of $c(A)$ and $e_1, \ldots, e_p$ are integers. (It is unimportant for this argument whether some of the $u_i$ are trivial or some of the $e_i$ are zero.) Set $U = \{u_1, \ldots, u_p\}$. According to Lemma 2.12, applied with $U$ playing the role of $S$, there is an $A$-admissible sequence $v_1, \ldots, v_q$ such that $U \subset \{v_1, \ldots, v_q\} \subset c(A)$, and the group $C := \langle A, v_1, \ldots, v_q \rangle$ has local rank at most $r$. In particular we have $\text{localrank}(C) < \text{iof}(\Gamma)$, so that $C$ is a closable subgroup of $\Gamma$. Since $U \subset \{v_1, \ldots, v_q\} \subset C$, we have $w \in C \ast \langle t \rangle$. Since $\{v_1, \ldots, v_q\} \subset c(A)$, and since Assertions (1) and (2) give that $c(A)$ is a subgroup of $\Gamma$ containing $A$, we have $C \leq c(A)$.

According to an observation made in 2.3, we have $\iota_{C,z} = \iota_{c(A),z} | (C \ast \langle t \rangle)$. Since $C$ contains the non-trivial element $w$ of the kernel of $\iota_{c(A),z}$, the homomorphism $\iota_{C,z}$ is non-injective. It therefore follows from Lemma 2.5 (applied with $C$ playing the role of the closable subgroup $A$ in that lemma) that $\text{localrank}(\langle C, z \rangle) \leq \text{localrank}(C)$, i.e.

\begin{equation}
\text{localrank}(\langle A, v_1, \ldots, v_q, z \rangle) \leq \text{localrank}(\langle A, v_1, \ldots, v_q \rangle).
\end{equation}

The inequality (2.13.2), together with the $A$-admissibility of $v_1, \ldots, v_q$, implies that the sequence $v_1, \ldots, v_q, z$ is $A$-admissible. Hence $z \in c(A)$, and 2.13.1 is proved.

Now to complete the proof of (4), suppose that $x \in c(c(A))$ is given. Then there is a $c(A)$-admissible sequence $x_1, \ldots, x_m$ with $x = x_m$. By induction on $i = 1, \ldots, m$ we will show that $x_i \in c(A)$. From the $c(A)$-admissibility of $x_1, \ldots, x_m$ it follows that $\text{localrank}(\langle c(A), x_i \rangle) \leq \text{localrank}(c(A))$; applying 2.13.1 with $z = x_1$, we deduce that $x_1 \in c(A)$. Now suppose that we are given an index $i$ with $1 < i \leq m$, and that $x_1, \ldots, x_{i-1} \in c(A)$. Then $\langle c(A), x_1, \ldots, x_{i-1} \rangle = c(A)$, and the $c(A)$-admissibility of $x_1, \ldots, x_m$ implies that

\[ \text{localrank}(\langle c(A), x_i \rangle) = \text{localrank}(\langle c(A), x_1, \ldots, x_i \rangle) \leq \text{localrank}(\langle c(A), x_1, \ldots, x_{i-1} \rangle) = \text{localrank}(c(A)); \]

applying 2.13.1 with $z = x_i$, we deduce that $x_i \in c(A)$. This completes the induction. In particular we have $x = x_m \in c(A)$. This shows that $c(c(A)) \leq c(A)$, and the proof of (4) is complete. \(\square\)
Note that Proposition 2.13 may be regarded as meaning that the operation $A \mapsto c(A)$ is a “closure operation” on the class of all closable subgroups of a given group, analogous, for example, to the operation that assigns to each subfield of a field $K$ its relative algebraic closure in $K$. (One difference is that whereas the class of subfields of a field $K$ includes $K$ itself, the class of closable subgroups of a group $\Gamma$ does not in general include $\Gamma$.)

**Proposition 2.14.** Let $A$ be a closable subgroup of a group $\Gamma$. Let $x_1, \ldots, x_m$ be independent elements of $c(A)$, and let $y$ be an element of $\Gamma$ with $y \notin c(A)$. Then $x_1, \ldots, x_m, y$ are independent.

**Proof.** Set $C = c(A)$, so that $C$ is a closable subgroup of $\Gamma$ by Assertion (1) of Proposition 2.13. According to Assertion (4) of Proposition 2.13, we have $c(C) = C$. By hypothesis we have $y \notin C$, i.e. $y \notin c(C)$; hence the one-term sequence $y$ cannot be $C$-admissible. We therefore have $\text{localrank}(\langle C, y \rangle) > \text{localrank}(C)$. By Proposition 2.5, this implies that the homomorphism $\iota_{C,y} : C \ast \langle t \rangle \to \Gamma$ is injective. Now set $B = \langle x_1, \ldots, x_m \rangle$ since the $x_i$ lie in $C$ by hypothesis, we have $B \leq C$. In view of an observation made in 2.3, we have $\iota_{B,y} = \iota_{C,y} | B \ast \langle t \rangle$; the injectivity of $\iota_{C,y}$ therefore implies that $\iota_{B,y}$ is injective. The independence of $x_1, \ldots, x_m$ and the injectivity of $\iota_{B,y}$ imply that $x_1, \ldots, x_m, y$ are independent. \hfill \Box

**Proposition 2.15.** Let $A$ be a closable subgroup of a group $\Gamma$. Let $B$ be a non-trivial subgroup of $A$. Then the normalizer of $B$ in $\Gamma$ is contained in $c(A)$.

**Proof.** Let $x$ be any element of the normalizer of $B$. In order to show that $x \in c(A)$, it suffices to show that $x$ is a one-term $A$-admissible sequence. This is equivalent to showing that $\text{localrank}(\langle A, x \rangle) \leq \text{localrank}(A)$, which by Proposition 2.5 is equivalent to showing that the homomorphism $\iota_{A,x} : A \ast \langle t \rangle \to \Gamma$ is non-injective.

Assume to the contrary that $\iota_{A,x} : A \ast \langle t \rangle \to \Gamma$ is injective. As we observed in 2.3, we have $\iota_{B,x} = \iota_{A,x}|(B \ast \langle t \rangle)$, and hence $\iota_{B,x}$ is injective. Hence $\iota_{B,x}$ may be regarded as an isomorphism of $B \ast \langle t \rangle$ onto $\langle B, x \rangle$, which restricts to the identity map on $B$. Since $x$ normalizes $B$, the subgroup $B$ of $\langle B, x \rangle$ is normal; hence $B$ is a normal subgroup of $B \ast \langle t \rangle$. But this is false, because $B$ is non-trivial by hypothesis, and for any non-trivial element $y$ of $B$, the element $tyt^{-1}$ of $B \ast \langle t \rangle$ is defined by a reduced word of length 3, and hence is not an element of $B$. \hfill \Box

**Proposition 2.16.** Let $A$ be a closable subgroup of a group $\Gamma$. Then for every element $u$ of $\Gamma$, the subgroup $uAu^{-1}$ of $\Gamma$ is closable, and $c(uAu^{-1}) = uc(A)u^{-1}$.

**Proof.** First note that since $A$ is closable, we have $\text{localrank}(uAu^{-1}) = \text{localrank}(A) < \text{iof}(\Gamma)$, so that $uAu^{-1}$ is closable. Now suppose that $x \in c(A)$ is given, and fix an $A$-admissible sequence $x_1, \ldots, x_m$ with $x_m = x$. For $1 \leq i \leq m$, set $A_i = \langle A, x_1, \ldots, x_i \rangle$; thus $uAu^{-1} = \langle uAu^{-1}, ux_1u^{-1}, \ldots, ux_mu^{-1} \rangle$. For $1 < i \leq m$ we have $\text{localrank}(uAu^{-1}) = (\text{localrank}(A_i) \leq \text{localrank}(uAu^{-1}) = \text{localrank}(uAu^{-1})u^{-1})$. This means that $ux_1u^{-1}, \ldots, ux_mu^{-1}$ is a $uAu^{-1}$-admissible sequence, so that $uxu^{-1} = ux_mu^{-1} \in c(uAu^{-1})$. Thus we have shown that $uc(A)u^{-1} \leq c(uAu^{-1})$. As this holds for every $u \in \Gamma$ and every closable subgroup $A$ of
Γ, we may replace u by \( u^{-1} \) and A by \( uAu^{-1} \) to deduce that \( u^{-1}c(uAu^{-1})u \leq c(A) \), i.e. \( c(uAu^{-1}) \leq uc(A)u^{-1} \).

\[ \square \]

3. Nerves

We begin this section by reviewing some general conventions from [19].

3.1. Except when we specify otherwise, the term simplicial complex will be understood in the geometric sense; that is, a simplicial complex is a set \( L \) of pairwise-disjoint finite-dimensional open simplices in a (possibly infinite-dimensional) real vector space, with the property that any face of any simplex in \( L \) is itself in \( L \). The geometric realization of an abstract simplicial complex is a simplicial complex in this sense, and every simplicial complex is simplicially isomorphic to the geometric realization of an abstract simplicial complex. The simplicial complexes referred to in this paper are not assumed to be locally finite. If \( L \) is a simplicial complex, the union of its simplices will be denoted by \( |L| \). The set \(|L|\) will always be understood to be endowed with the weakest topology which induces the standard topology on the closure of each simplex.

Let \( L \) be a simplicial complex. We say that a subset \( W \) of \(|L|\) is saturated if \( W \) is a union of (open) simplices of \( L \).

An indexed family \( \mathcal{F} = (U_i)_{i \in I} \) of nonempty (open) subsets of a topological space \( X \) is said to cover \( X \) (or to be an open covering of \( X \)) if \( X = \bigcup_{i \in I} U_i \). (Here the index set \( I \) can be any set whatsoever.) We define the abstract nerve of a covering \( \mathcal{F} = (U_i)_{i \in I} \) of \( X \) to be the abstract simplicial complex that is well defined up to canonical simplicial isomorphism as follows. The vertex set \( V \) of the complex is a bijective copy of the index set \( I \), equipped with a specific bijection \( i \mapsto v_i \) from \( I \) to \( V \). A simplex \( \sigma \) is a set \( \{v_{i_0}, \ldots, v_{i_d}\} \), with \( d \geq 0 \) and \( i_0, \ldots, i_d \in I \), such that \( U_{i_0} \cap \cdots \cap U_{i_d} \neq \emptyset \). The nerve of a covering is defined to be the geometric realization of its abstract nerve. The nerve of a covering \( \mathcal{F} \) will be denoted \( K_\mathcal{F} \).

Note that as in [19], the definition of a covering and its nerve given here, unlike the most classical definition, allows the possibility that the covering \( \mathcal{F} \) is “non-faithfully indexed” in the sense that there exist distinct \( i, j \in I \) for which \( U_i = U_j \). This affects the definition of the nerve of \( \mathcal{F} \), and will be needed for the proof of Proposition 3.4 below, in which the covering \( \mathcal G \) may be “non-faithfully indexed.” The proof of Proposition 3.4 depends on a version of the Borsuk Nerve Theorem, which is proved in [19] and is paraphrased below as Prop. 3.3, and applies to coverings that are not necessarily “faithfully indexed.”

**Notation and Remarks 3.2.** Let \( \mathcal{F} = (U_i)_{i \in I} \) be an open covering of a topological space \( X \). For every simplex \( \sigma \) of \( K_\mathcal{F} \), we will denote by \( \mathcal{I}_\sigma^\mathcal{F} \) (or by \( \mathcal{I}_\sigma \) when the covering \( \mathcal{F} \) is understood) the set of all indices \( i \in I \) such that \( v_i \) is a vertex of \( \sigma \). (In [19, Definitions 2.9], the object denoted here by \( \mathcal{I}_\sigma \) was defined in a special case, and was denoted by \( I_\sigma \).) We will denote by \( \mathcal{U}_\sigma^\mathcal{F} \) (or by \( \mathcal{U}_\sigma \) when the covering \( \mathcal{F} \) is understood) the set \( \bigcap_{i \in \mathcal{I}_\sigma} U_i \).

Thus the definition of the nerve \( K_\mathcal{F} \) implies that \( \mathcal{U}_\sigma^\mathcal{F} \neq \emptyset \) for every simplex \( \sigma \) of \( K_\mathcal{F} \).

Note that if \( \tau \) is a face of a simplex \( \sigma \) of \( K_\mathcal{F} \), we have \( \mathcal{U}_\tau \subset \mathcal{U}_\sigma \).
If an index \( i \) belongs to \( \mathcal{I}_\sigma \), then by definition we have \( \mathcal{U}_\sigma \subset U_i \). We denote by \( \mathcal{F}_\sigma^\mathcal{F} \) (or by \( \mathcal{F}_\sigma \) when the covering \( \mathcal{F} \) is understood) the set of all indices \( i \in I \) such that \( i \notin \mathcal{I}_\sigma \) but \( \mathcal{U}_\sigma \cap U_i \neq \emptyset \).

Here is our version of the Borsuk Nerve Theorem:

**Proposition 3.3.** Let \( \mathcal{F} = (U_i)_{i \in I} \) be an open covering of a paracompact space \( X \). Suppose that for every simplex \( \sigma \) of \( K_\mathcal{F} \), the set \( \mathcal{U}_\sigma \) is contractible. Then \( |K_\mathcal{F}| \) is homotopy-equivalent to \( X \).

**Proof.** This is a paraphrase of Proposition 2.7 of [19]. \( \square \)

Various special cases of the following result were implicit in the proofs of [17, Lemma 5.7], [20, Lemma 3.8], and [19, Lemma 3.3].

**Proposition 3.4.** Let \( \mathcal{F} = (U_i)_{i \in I} \) be an open covering of a paracompact space \( X \). Suppose that for every simplex \( \sigma \) of \( K_\mathcal{F} \), the set \( \mathcal{U}_\sigma \) is contractible. Let \( \sigma \) be a simplex of \( |K_\mathcal{F}| \). Then the set \( \mathcal{U}_{\sigma} \cap \bigcup_{i \in \mathcal{I}_{\sigma}} U_i \) is homotopy-equivalent to \( \text{link}_{K_\mathcal{F}} \sigma \).

**Proof.** Set \( \mathcal{U} = \mathcal{U}_{\sigma}^\mathcal{F} \). Set \( \mathcal{W} = \mathcal{U} \cap \bigcup_{i \in \mathcal{I}_{\sigma}} U_i \).

Set \( V_i = U_i \cap \mathcal{U} \) for each \( i \in \mathcal{I}_\sigma \), and \( \mathcal{G} = (V_i)_{i \in \mathcal{I}_\sigma} \). We have \( \bigcup_{i \in \mathcal{I}_{\sigma}} V_i = \bigcup_{i \in \mathcal{I}_{\sigma}} (U_i \cap \mathcal{U}) = (\bigcup_{i \in \mathcal{I}_{\sigma}} U_i) \cap \mathcal{U} = \mathcal{W} \). Hence \( \mathcal{G} \) is a cover for \( \mathcal{W} \).

According to 3.1, the nerve \( K_{\mathcal{G}} \) comes equipped with a bijection from the index set \( \mathcal{I}_\sigma \) of the covering \( \mathcal{G} \) to the vertex set \( K_{\mathcal{G}}^{(0)} \) of \( K_{\mathcal{G}} \). We will denote this bijection by \( i \mapsto w_i \).

Since \( \mathcal{I}_\sigma \subset I \), and since \( i \mapsto v_i \) is a bijection from the index set \( I \) of \( \mathcal{F} \) to the vertex set \( K_{\mathcal{F}}^{(0)} \) of \( K_{\mathcal{F}} \), there is a well-defined injection \( f^{(0)} : K_{\mathcal{G}}^{(0)} \to K_{\mathcal{F}}^{(0)} \) given by \( f^{(0)}(w_i) = v_i \).

If \( \tau \) is any simplex of \( K_{\mathcal{G}} \), we have \( \bigcap_{i \in \mathcal{I}_{\tau}} V_i \neq \emptyset \) by the definition of the nerve \( K_{\mathcal{G}} \). But \( \bigcap_{i \in \mathcal{I}_{\tau}} V_i = \bigcap_{i \in \mathcal{I}_{\tau}} (U_i \cap \mathcal{U}) \subset \bigcap_{i \in \mathcal{I}_{\tau}} U_i \), and hence \( \bigcap_{i \in \mathcal{I}_{\tau}} U_i \neq \emptyset \); in view of the definition of the nerve \( K_{\mathcal{F}} \), it follows that \( \{v_i : i \in \mathcal{I}_{\tau}\} \), the image under \( f^{(0)} \) of the vertex set \( \{w_i : i \in \mathcal{I}_{\tau}\} \) of \( \tau \), is the vertex set of a simplex of \( K_{\mathcal{F}} \). Since this is the case for every simplex \( \tau \) of \( K_{\mathcal{G}} \), the map \( f^{(0)} \) extends to a simplicial map \( f : K_{\mathcal{G}} \to K_{\mathcal{F}} \). Since \( f^{(0)} \) is injective, \( f \) is also injective.

We now claim:

**3.4.1.** For every simplex \( \tau \) of \( K_{\mathcal{G}} \), we have \( f(\tau) \subset \text{link}_{K_{\mathcal{F}}} \sigma \), and the set \( \mathcal{U}_{\tau}^\mathcal{G} \) is contractible.

To prove 3.4.1, first note that by the definition of the nerve \( K_{\mathcal{G}} \) we have \( \bigcap_{i \in \mathcal{I}_{\tau}} V_i \neq \emptyset \). But \( \bigcap_{i \in \mathcal{I}_{\tau}} V_i = \bigcap_{i \in \mathcal{I}_{\tau}} (U_i \cap \mathcal{U}) = (\bigcap_{i \in \mathcal{I}_{\tau}} U_i) \cap \mathcal{U} = (\bigcap_{i \in \mathcal{I}_{\tau}} U_i) \cap (\bigcap_{i \in \mathcal{I}_{\sigma}} U_i) = \bigcap_{i \in \mathcal{I}_{\tau} \cup \mathcal{I}_{\sigma}} U_i \). Hence \( \bigcap_{i \in \mathcal{I}_{\tau} \cup \mathcal{I}_{\sigma}} U_i \neq \emptyset \), which by the definition of \( K_{\mathcal{G}} \) means that \( \mathcal{F}_{\tau} \cup \mathcal{F}_{\sigma} = \mathcal{F}_{\phi} \) for some simplex \( \phi \) of \( K_{\mathcal{F}} \). But according to 3.2, the index set \( \mathcal{I}_{\sigma} \subset I \) of \( \mathcal{G} \) is disjoint from \( \mathcal{I}_{\sigma} \); in particular, \( \mathcal{F}_{\tau} \cap \mathcal{F}_{\sigma} = \emptyset \). Thus \( \mathcal{F}_{\phi} \) is the disjoint union of \( \mathcal{F}_{\tau} \) and \( \mathcal{F}_{\sigma} \), so that the vertex set \( \{v_i : i \in \mathcal{I}_{\phi}\} \) of \( \phi \) is the disjoint union of the vertex sets \( \{v_i : i \in \mathcal{I}_{\tau}\} \) and \( \{v_i : i \in \mathcal{I}_{\sigma}\} \) of \( f(\tau) \) and \( \sigma \). This shows that \( f(\tau) \subset \text{link}_{K_{\mathcal{F}}} \sigma \).
On the other hand, the equality \( \bigcap_{i \in \mathcal{I}} V_i = \bigcap_{i \in \mathcal{I} \cup \mathcal{J}_\psi} U_i \) may be written as \( \mathcal{U}_\psi^Q = \bigcap_{i \in \mathcal{J}_\psi} U_i = \mathcal{U}_\psi^F \); since the hypothesis of the proposition implies that \( \bigcap_{i \in \mathcal{J}_\psi} U_i \) is contractible, we have shown that \( \mathcal{U}_\psi^Q \) is contractible. Thus (3.4.1) is proved.

Next, we claim:

\[
3.4.2 \quad f(K_G) = \text{link}_{K_x}(\sigma).
\]

To prove (3.4.2), first note that the inclusion \( f(K_G) \subset \text{link}_{K_x}(\sigma) \) is immediate from the first assertion of 3.4.1. To establish the reverse inclusion, consider an arbitrary simplex \( \psi \) of \( \text{link}_{K_x}(\sigma) \). There is a simplex \( \phi \) of \( K_x \) whose domain is \( (0, 1/e^u) \) and \( \phi \in \mathcal{J}_\psi \). Hence \( \mathcal{J}_\psi \cap \mathcal{J}_\phi = \emptyset \) and \( \mathcal{J}_\psi \cup \mathcal{J}_\phi = \mathcal{J}_\phi \). By the definition of \( K_x \) we have \( \emptyset \neq \bigcap_{i \in \mathcal{J}_\phi} U_i = \bigcap_{i \in \mathcal{J}_\psi} U_i \cap \bigcap_{i \in \mathcal{J}_\phi} U_i = \mathcal{U} \cap \bigcap_{i \in \mathcal{J}_\psi} U_i \), i.e.

\[
3.4.3 \quad \bigcap_{i \in \mathcal{J}_\psi} (U_i \cap \mathcal{U}) \neq \emptyset.
\]

In particular, it follows from (3.4.3) that if \( i \) is any index in \( \mathcal{J}_\psi \), we have \( U_i \cap \mathcal{U} \neq \emptyset \); on the other hand, since \( \mathcal{J}_\phi \cap \mathcal{J}_\psi = \emptyset \), we have \( i \notin \mathcal{J}_\phi \). By definition we therefore have \( i \in \mathcal{J}_\phi \).

This shows that \( \mathcal{J}_\psi \subset \mathcal{J}_\phi \). We may now rewrite (3.4.3) in the form \( \bigcap_{i \in \mathcal{J}_\phi} V_i \neq \emptyset \). By the definition of \( K_G \) it follows that \( \mathcal{J}_\psi = \mathcal{J}_\tau \) for some simplex \( \tau \) of \( K_G \). Hence \( f \) maps the vertex set \( \{ w_i : i \in \mathcal{J}_\tau \} \) of \( \tau \) onto the vertex set \( \{ v_i : i \in \mathcal{J}_\psi \} \) of \( \psi \), so that \( f(\tau) = \psi \). This establishes the inclusion \( \text{link}_{K_x}(\sigma) \subset f(K_G) \), and completes the proof of (3.4.2).

It follows from (3.4.2) that the injective simplicial map \( f \) may be regarded as a simplicial isomorphism between \( K_G \) and \( \text{link}_{K_x}(\sigma) \). But by the second assertion of 3.4.1, the covering \( \mathcal{G} \) of \( \mathcal{W} \) has the property that for every simplex \( \tau \) of \( K_G \), the set \( \mathcal{U}_\psi^Q \) is contractible. It therefore follows from Proposition 3.3 that \( K_G \) is homotopy-equivalent to \( \mathcal{W} \). Hence \( \text{link}_{K_x}(\sigma) \) is also homotopy-equivalent to \( \mathcal{W} \), as asserted by the present proposition. \( \square \)

4. Heights

**Notation 4.1.** We shall denote by \( \text{Isom}_+ (\mathbb{H}^3) \) the group of all orientation-preserving isometries of \( \mathbb{H}^3 \). For each \( x \in \text{Isom}_+ (\mathbb{H}^3) \) and for each point \( P \in \mathbb{H}^3 \), we shall write \( d(x, P) = \text{dist}(P, x \cdot P) \).

We define a function \( Q \) on the non-negative real numbers by \( Q(u) = 1/(1 + e^u) \).

Note that \( Q \) is strictly monotone decreasing and has range \((0, 1/2]\). The inverse function \( Q^{-1} \), whose domain is \((0, 1/2]\), is given by \( Q^{-1}(x) = \log(1/x - 1) \).

If \( P \) is a point of \( \mathbb{H}^3 \) and \( x \) is an isometry of \( \mathbb{H}^3 \), we set \( Q(x, P) = Q(d(x, P)) \).

The following result is fundamental for the arguments in this paper.

**Theorem 4.2.** Suppose that \( p \) is a positive integer and that \( x_1, \ldots, x_p \) are independent elements of \( \text{Isom}_+ (\mathbb{H}^3) \) which generate a discrete group. Then for every point \( P \in \mathbb{H}^3 \) we have

\[
Q(x_1, P) + \cdots + Q(x_p, P) \leq \frac{1}{2}.
\]
Proof. For \( p = 1 \) the assertion is trivial. For \( p \geq 2 \) it is a paraphrase of [3, Theorem 4.1]. (The latter result is an extension of [7, Theorem 6.1] obtained by combining the results of [7] with those of [1], [10], [25], and [26].) □

4.3. As in [19], we will say that a group has the \textit{infinite cyclic centralizer property}, or is an \textit{ICC-group}, if the centralizer of every non-trivial element of \( \Gamma \) is infinite cyclic. If \( \Gamma \) is an ICC-group then \( \Gamma \) is torsion-free, and every non-trivial element \( x \) of \( \Gamma \) belongs to a unique maximal cyclic subgroup of \( \Gamma \), namely the centralizer of \( x \) in \( \Gamma \). Hence two elements of \( \Gamma \) commute if and only if they lie in the same maximal cyclic subgroup. A purely loxodromic, discrete subgroup of Isom_+(H^3) is an ICC-group.

4.4. This subsection is in large part devoted to review of material from [17], [20] and [19]. One minor difference between the conventions that we give here and those used in the cited papers is that we use the notation \( d(x, P) \) that was introduced in 4.1. A couple of other minor differences will be pointed out parenthetically as they arise in this subsection; they involve relaxing some assumptions on a discrete group that were built into the machinery developed in [19]. The flexibility obtained by doing this will allow us to streamline the statements of many lemmas proved in this and subsequent sections, by avoiding irrelevant hypotheses.

By a \textit{hyperbolic cylinder} in \( \text{H}^3 \) we mean a set of the form \( Z = \{ P \in \text{H}^3 : \text{dist}(P, l) < R \} \), where \( l \) is a line in \( \text{H}^3 \) and \( R \) a positive number; the line \( l \) and the number \( R \), which are uniquely determined by the set \( Z \), will be respectively called the \textit{axis} and \textit{radius} of the cylinder. Note that any hyperbolic cylinder is a convex subset of \( \text{H}^3 \).

Let \( \Gamma \) be a discrete, purely loxodromic subgroup of Isom_+(H^3). (In the corresponding discussion in [19, Section 2] it was assumed that \( \Gamma \) is cocompact. While we have not built this condition into the general machinery described in this subsection, it is a crucial hypothesis for [19, Proposition 2.13], which will be quoted in the proof of Theorem 5.2 of this paper.) As we observed in 4.3, \( \Gamma \) is an ICC-group. Given a real number \( \lambda > 0 \), we define \( C_\lambda(\Gamma) \) to be the set of maximal cyclic subgroups \( C \) of \( \Gamma \) such that a (loxodromic) generator of \( C \) has translation length less than \( \lambda \).

For every non-trivial element \( x \) of \( \Gamma \), we set \( Z_\lambda(x) := \{ P \in \text{H}^3 : d(x, P) < \lambda \} \), and for each \( C \in C_\lambda(\Gamma) \), set \( Z_\lambda(C) = \bigcup_{1 \neq x \in C} Z_\lambda(x) \). For each \( C \in C_\lambda(\Gamma) \), we have \( Z_\lambda(C) = Z_\lambda(x) \) for some element \( x \neq 1 \) of \( C \); furthermore, \( Z_\lambda(C) \) is a hyperbolic cylinder whose axis is the common translation axis of all non-trivial elements of \( C \). For any \( \lambda > 0 \) we set \( Z_\lambda(\Gamma) = (Z_\lambda(C))_{C \in C_\lambda(\Gamma)} \).

In this context, we will denote by \( \mathcal{X}_\lambda(\Gamma) \) the set of all points \( P \in \text{H}^3 \) such that \( d(x, P) < \lambda \) for some non-trivial element \( x \) of \( \Gamma \). (In the corresponding discussion in [19, Section 2] it was assumed that \( \mathcal{X}_\lambda(\Gamma) = \text{H}^3 \). While we have not built this condition into the general machinery described in this subsection, it will appear crucially in the hypothesis of Theorem 5.2 of this paper, where it is needed for an application of the Borsuk Nerve Theorem (Proposition 3.3).)

It follows from the definitions that \( \mathcal{X}_\lambda(\Gamma) = \bigcup_{C \in C_\lambda(\Gamma)} Z_\lambda(C) \), so that \( Z_\lambda(\Gamma) \) is a covering of \( \mathcal{X}_\lambda(\Gamma) \), in the sense discussed in 3.1. It should be borne in mind that the index set for this
covering is $C_\lambda(\Gamma)$, a collection of maximal cyclic groups of $\Gamma$. The nerve $K_{\mathcal{Z}_\lambda(\Gamma)}$ is well defined by 3.1.

Recall from 3.1 that the nerve $K_{\mathcal{Z}_\lambda(\Gamma)}$ comes equipped with a bijection $C \mapsto v_C$ from $C_\lambda(\Gamma)$, the index set of the covering $\mathcal{Z}_\lambda(\Gamma)$ of $\mathcal{X}_\lambda(\Gamma)$, to the vertex set $K_{\mathcal{Z}_\lambda(\Gamma)}^0$ of $K_{\mathcal{Z}_\lambda(\Gamma)}$. We will denote the inverse bijection by $v \mapsto C_v$.

According to the conventions of 3.2, for any simplex $\sigma$ of $K_{\mathcal{Z}_\lambda(\Gamma)}$, the set $S_\sigma = S_{\mathcal{Z}_\lambda(\Gamma)}^\sigma$ consists of all maximal cyclic subgroups $C \in C_\lambda(\Gamma)$ such that $v_C$ is a vertex of $\sigma$. Equivalently, $S_\sigma$ may be described as the set of all indices $C_v$, where $v$ ranges over the vertices of $\sigma$.

For each simplex $\sigma$ we will set $\Theta(\sigma) = \langle S_\sigma \rangle$, so that $\Theta(\sigma)$ is a finitely generated subgroup of $\Gamma$. Note that if $S$ is a set of the form $\{z_v : v \text{ a vertex of } \sigma\}$, where $z_v$ is a generator of $C_v$ for each vertex $v$ of $\sigma$, then $\Theta(\sigma) = \langle S \rangle$. If $S$ is such a set, we also set $\text{IR}(\sigma) = \text{IR}(S)$, where $\text{IR}(S)$ is defined as in 2.1; this definition is clearly independent of the choice of the generators $z_v$.

Note that these definitions imply that $\text{IR}(\sigma) = \max\{\text{rank } \Theta(\tau) : \tau \leq \sigma\}$ for every simplex $\sigma$. (This shows that the definition of $\text{IR}(\sigma)$ given here is equivalent to the definition given in [19, Definitions 2.8].)

In particular we have

\begin{equation}
\text{rank } \Theta(\sigma) \leq \text{IR}(\sigma).
\end{equation}

If $W$ is a saturated subset of $K_{\mathcal{Z}_\lambda(\Gamma)}$, we define $\Theta(W)$ to be the subgroup of $\Gamma$ generated by all the subgroups $\Theta(\sigma)$ where $\sigma$ ranges over the simplices contained in $W$. (The definitions of $\Theta(\sigma)$ and $\Theta(W)$ given here are equivalent to the definitions given in [19], although the latter are framed in terms of the notion of a “labeled complex,” which will not be used in this paper.)

We will say that vertices $v_1, \ldots, v_p$ of $K_{\mathcal{Z}_\lambda(\Gamma)}$ are independent if the elements $z_1, \ldots, z_p$ are independent (see 2.1), where $z_i$ is a generator of $C_{v_i}$ for $i = 1, \ldots, p$.

Note that if $\tau$ is a face of a simplex $\sigma$ of $K_{\mathcal{Z}_\lambda(\Gamma)}$, we have $\mathcal{I}_\tau \subset \mathcal{I}_\sigma$ and therefore $\Theta(\tau) \leq \Theta(\sigma)$.

Since $C \mapsto v_C$ is a bijection, we may define an action of $\Gamma$ on $K_{\mathcal{Z}_\lambda(\Gamma)}^0$ by setting $x \cdot v_C = v_{xCx^{-1}}$ for each $x \in \Gamma$ and each $C \in C_\lambda(\Gamma)$. It is pointed out in [19, Subsection 2.12] that this action extends to a simplicial action on $K_{\mathcal{Z}_\lambda(\Gamma)}$, which, as in [19], will be referred to as the canonical action of $\Gamma$ on $K_{\mathcal{Z}_\lambda(\Gamma)}$. The definitions imply that, under the canonical action, we have

\begin{equation}
C_{x \cdot v} = xC_vx^{-1} \text{ for every } x \in \Gamma \text{ and every } v \in K_{\mathcal{Z}_\lambda(\Gamma)}^0.
\end{equation}

It follows that for every $x \in \Gamma$ and for every saturated set $W \subset K_{\mathcal{Z}_\lambda(\Gamma)}$, we have $\Theta(x \cdot W) = x\Theta(W)x^{-1}$.

Note also that for every $C \in C_\lambda(\Gamma)$ and every $x \in \Gamma$, we have

\begin{equation}
Z_\lambda(xCx^{-1}) = x \cdot Z_\lambda(C).
\end{equation}
The conventions of \ref{assump} give
\begin{equation}
\mathcal{U}_\sigma = \mathcal{U}_\sigma^{K_{Z_\lambda(\Gamma)}} = \bigcap_{C \in \mathcal{C}_\lambda} Z_\lambda(C)
\end{equation}
for each simplex \(\sigma\) of \(K_{Z_\lambda(\Gamma)}\).

We have observed that the hyperbolic cylinder \(Z_\lambda(C)\) is convex for each \(C \in \mathcal{C}_\lambda(\Gamma)\); hence for each simplex \(\sigma\) of \(K_{Z_\lambda(\Gamma)}\), the non-empty set \(\mathcal{U}_\sigma\) is an intersection of convex sets, and is therefore convex. In particular, \(\mathcal{U}_\sigma\) is contractible for each simplex \(\sigma\) of \(K_{Z_\lambda(\Gamma)}\). Hence by Proposition \ref{prop:independence}, \(K_{Z_\lambda(\Gamma)}\) is homotopy-equivalent to \((\Gamma)\). (These observations are related to the proof of \cite[Lemma 3.3]{19}.)

**Lemma 4.5.** Let \(\Gamma\) be a discrete purely loxodromic subgroup of \(\text{Isom}_+(\mathbb{H}^3)\), let \(\lambda\) be a positive real number, and let \(\sigma\) be any simplex of \(K_{Z_\lambda(\Gamma)}\). Let \(p\) be a non-negative integer, and suppose that \(\sigma\) has \(p\) independent vertices (in the sense of \ref{def:independence}). Then for every point \(P \in \mathcal{U}_\sigma\), there is a \(p\)-tuple \((x_1, \ldots, x_p)\) of independent elements of \(\Theta(\sigma)\) such that \(d(x_i, P) < \lambda\) for each \(i\) with \(1 \leq i \leq p\).

**Proof.** If \(p = 0\) the assertion is vacuously true. Now assume that \(p \geq 1\), and let \(v_1, \ldots, v_p\) be independent vertices of \(\sigma\). For \(i = 1, \ldots, p\) choose a generator \(z_i\) of \(C_{v_i}\). The independence of \(v_1, \ldots, v_p\) says that \(z_1, \ldots, z_p\) are independent elements of \(\Gamma\). Since \(P \in \mathcal{U}_\sigma \subset Z_\lambda(C_{v_i})\), there is a non-trivial element \(x_i\) of \(C_{v_i}\) such that \(d(x_i, P) < \lambda\). Since \(z_1, \ldots, z_p\) are independent, and \(x_i\) is a non-zero power of \(z_i\) for each \(i\), the elements \(x_1, \ldots, x_p\) are also independent. Furthermore, since the \(v_i\) are vertices of \(\sigma\), the definition of \(\Theta(\sigma)\) implies that \(C_{v_i} \leq \Theta(\sigma)\) for \(i = 1, \ldots, p\), so that in particular we have \(x_1, \ldots, x_p \in \Theta(\sigma)\). \(\Box\)

The following result has considerable overlap with \cite[Lemma 2.5]{20}, but we find it valuable to give a self-contained proof from the point of view of this paper.

**Proposition 4.6.** Let \(k\) be a positive integer, let \(\Gamma\) be a purely loxodromic, discrete, \(k\)-free subgroup of \(\text{Isom}_+(\mathbb{H}^3)\), and let \(\lambda\) be a real number with \(0 < \lambda \leq \log(2k - 1)\). Let \(\sigma\) be any simplex of \(K_{Z_\lambda(\Gamma)}\), and set \(r = \text{IR}(\sigma)\). Then \(r < k\), and \(\sigma\) has \(r\) independent vertices (in the sense of \ref{def:independence}). In particular we have \(\text{rank} \Theta(\sigma) < k\).

**Proof.** For each vertex \(v\) of \(\sigma\) choose a generator \(z_v\) of \(C_v\). As in \ref{def:independence}, let \(S\) denote the set consisting of all the elements \(z_v\), where \(v\) ranges over the vertices of \(\sigma\). According to the definitions (see \ref{def:independence}) we have \(r = \text{IR}(S)\). Hence if we set \(m = \min(k, r) > 0\), it follows from Proposition \ref{prop:independence} that \(S\) contains \(m\) independent elements. Again by the definition, this means that \(\sigma\) has \(m\) independent vertices. Hence by Lemma \ref{lem:independence}, if we fix a point \(P \in \mathcal{U}_\sigma\), there are independent elements \(x_1, \ldots, x_m\) of \(\Theta(\sigma) \leq \Gamma\) such that \(d(x_i, P) < \lambda\) for \(i = 1, \ldots, m\). In particular we have \(d(x_i, P) < \log(2k - 1)\), and hence \(Q(x_i, P) = Q(d(x_i, P)) > Q(\log(2k - 1)) = 1/(2k)\). But according to Theorem \ref{thm:volume}, we have \(Q(x_1, P) + \cdots + Q(x_m, P) \leq 1/2\). It now follows that \(m/(2k) < 1/2\), i.e. \(m < k\). In view of the definition of \(m\), this means that \(r < k\) and that \(m = r\); thus \(\sigma\) has \(r\) independent vertices. The inequality \(\text{rank} \Theta(\sigma) < k\) follows from the inequality \(r < k\), since we have \(\text{rank} \Theta(\sigma) \leq r\) by \ref{eq:rank}. \(\Box\)
Definition 4.7. A positive real number $\lambda$ will be said to be compatible with a group $\Gamma$ if there exists an integer $k \geq 2$ such that $\Gamma$ is $k$-free and $\lambda \leq \log(2k - 1)$.

Proposition 4.8. Let $\Gamma$ be a purely loxodromic, discrete subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and let $\lambda$ be a real number which is compatible with $\Gamma$. Then for every simplex $\sigma$ of $K_{\mathbb{Z}_\lambda(\Gamma)}$, the subgroup $\Theta(\sigma)$ of $\Gamma$ is closable (see 2.4).

Proof. By the definition of compatibility, there is an integer $k \geq 2$ such that $\Gamma$ is $k$-free and $\lambda \leq \log(2k - 1)$. For any simplex $\sigma$ of $K_{\mathbb{Z}_\lambda(\Gamma)}$, it then follows from Proposition 4.6 that $\text{rank} \Theta(\sigma) < k$. But since $\Gamma$ is $k$-free, the definition of the index of freedom (2.1) gives that $k \leq \text{iof}(\Gamma)$. Thus we have $\text{localrank}(\Theta(\sigma)) = \text{rank} \Theta(\sigma) < \text{iof}(\Gamma)$, which by definition means that $\Theta(\sigma)$ is a closable subgroup of $\Gamma$. \(\square\)

Remark and Notation 4.9. Let $k$ be a positive integer, let $\Gamma$ be a purely loxodromic, discrete, $k$-free subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and let $\lambda$ be a real number with $0 < \lambda \leq \log(2k - 1)$. Then in particular $\lambda$ is compatible with $\Gamma$, so that for every simplex $\sigma$ of $K_{\mathbb{Z}_\lambda(\Gamma)}$, the subgroup $\Theta(\sigma)$ of $\Gamma$ is closable by Proposition 4.8, and hence $c(\Theta(\sigma))$ is a well-defined subgroup of $\Gamma$ by 2.11 and Proposition 2.13. For every simplex $\sigma$ of $K_{\mathbb{Z}_\lambda(\Gamma)}$, we will denote by $\mathcal{H}(k, \Gamma, \lambda)(\sigma)$ the set of all strictly positive integers $h$ such that for every $P \in U_\sigma$, at least one of the following conditions holds:

(i) there exist an integer $m \geq 0$ and independent elements $x_1, \ldots, x_m$ of $c(\Theta(\sigma))$ such that

\[
Q(x_1, P) + \cdots + Q(x_m, P) > \frac{1}{2} - (k - h)Q(\lambda);
\]

or

(ii) there exist an integer $m \geq 0$, independent elements $x_1, \ldots, x_m$ of $c(\Theta(\sigma))$, and an element $y$ of $\Gamma - c(\Theta(\sigma))$, such that

\[
Q(x_1, P) + \cdots + Q(x_m, P) + Q(y, P) > \frac{1}{2} - (k - h - 1)Q(\lambda).
\]

We will abbreviate $\mathcal{H}(k, \Gamma, \lambda)(\sigma)$ by writing $\mathcal{H}(\sigma)$ when $k$, $\Gamma$ and $\lambda$ are understood.

Lemma 4.10. Let $k$ be a positive integer, let $\Gamma$ be a purely loxodromic, discrete, $k$-free subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and let $\lambda$ be a real number with $0 < \lambda \leq \log(2k - 1)$. Then for every simplex $\sigma$ of $K_{\mathbb{Z}_\lambda(\Gamma)}$, we have $\text{IR}(\sigma) \in \mathcal{H}(k, \Gamma, \lambda)(\sigma)$ and $k \notin \mathcal{H}(k, \Gamma, \lambda)(\sigma).

Proof. To prove the first assertion, set $r = \text{IR}(\sigma)$. According to Proposition 4.6, $\sigma$ has $r$ independent vertices $v_1, \ldots, v_r$. Hence if $P$ is an arbitrary point of $U_\sigma$, it follows from Lemma 4.5 that there are independent elements $x_1, \ldots, x_r$ of $\Theta(\sigma)$ such that $d(x_i, P) < \lambda$. Thus if we set $\alpha = Q(\lambda)$, we have $Q(x_i, P) = Q(d(x_i, P)) > \alpha$ for each $i$. On the other hand, since $\lambda \leq \log(2k - 1)$, we have $\alpha \geq 1/(2k)$, so that $1/2 - k\alpha \leq 0$. Hence

\[
\frac{1}{2} - (k - r)\alpha = \left(\frac{1}{2} - k\alpha\right) + r\alpha \leq r\alpha < Q(x_1, P) + \cdots + Q(x_r, P).
\]

Since $x_i \in \Theta(\sigma)$, and $\Theta(\sigma) \leq c(\Theta(\sigma))$ by Proposition 2.13, this shows that Condition (i) of 4.9 holds with $m = h = r$. Since $P$ was an arbitrary point of $U_\sigma$, this shows that $r \in \mathcal{H}(k, \Gamma, \lambda)(\sigma)$, and the first assertion of the lemma is established.
To prove the second assertion, assume that $k \in \mathcal{S}_{(k, \Gamma, \lambda)}(\sigma)$, and choose a point $P \in \mathcal{U}_\alpha \subset \mathbf{H}^3$. Then one of the conditions (i) or (ii) of 4.9 holds with $h = k$. We define an integer $p$ and a $p$-tuple $(z_1, \ldots, z_p)$ of elements of $\Gamma$ as follows: if (i) holds we set $p = m$ and $z_i = x_i$ for each $i$; and if (ii) holds we set $p = m + 1$, $z_i = x_i$ for $1 \leq i \leq m$, and $z_{m+1} = y$. In either case we have $Q(z_1, P) + \cdots + Q(z_p, P) > 1/2$. But the elements $z_1, \ldots, z_p$, which belong to the discrete group $\Gamma \leq \text{Isom}_+(\mathbf{H}^3)$, are independent; this independence assertion is immediate if (i) holds, and if (ii) holds it follows from Proposition 2.14. Hence the inequality $Q(z_1, P) + \cdots + Q(z_p, P) > 1/2$ contradicts Theorem 4.2. This proves the second assertion.

\[ \Box \]

**Proposition 4.11.** Let $k$ be a positive integer, let $\Gamma$ be a purely loxodromic, discrete, $k$-free subgroup of $\text{Isom}_+(\mathbf{H}^3)$, and let $\lambda$ be a real number with $0 < \lambda \leq \log(2k - 1)$. Then for every simplex $\sigma$ of $K_{Z_\lambda(\Gamma)}$, there is a unique integer $h_0 > 0$ such that $\mathcal{S}_{(k, \Gamma, \lambda)}(\sigma) = \{1, \ldots, h_0\}$. Furthermore, we have $\text{IR}(\sigma) \leq h_0 < k$.

**Proof.** Set $r = \text{IR}(\sigma)$ and $\mathcal{H} = \mathcal{H}_{(k, \Gamma, \lambda)}(\sigma)$. It is immediate from the definition of $\mathcal{H}$ that if $h$ is any element of $\mathcal{H}$, then for any integer $h'$ with $0 < h' \leq h$ we have $h' \in \mathcal{H}$. Since Lemma 4.10 gives that $k \notin \mathcal{H}$, it now follows that $k$ is a strict upper bound for $\mathcal{H}$. Since Lemma 4.10 also gives that $r \in \mathcal{H}$, we have $\mathcal{H} \neq \emptyset$. Hence $\mathcal{H} = \{1, \ldots, h_0\}$ for some positive integer $h_0 < k$. The uniqueness of $h_0$ is trivial. Again using that $r \in \mathcal{H}$, we deduce that $h_0 \geq r$. \[ \Box \]

**Definition and Remarks 4.12.** Let $k$ be a positive integer, let $\Gamma$ be a purely loxodromic, discrete, $k$-free subgroup of $\text{Isom}_+(\mathbf{H}^3)$, and let $\lambda$ be a real number with $0 < \lambda \leq \log(2k - 1)$. For every simplex $\sigma$ of $K_{Z_\lambda(\Gamma)}$, we define the $(k, \Gamma, \lambda)$-height of $\sigma$, denoted by $\text{height}_{(k, \Gamma, \lambda)}(\sigma)$, to be the integer $h_0$ given by Proposition 4.11. If $k$, $\Gamma$ and $\lambda$ are understood, we shall refer to $\text{height}_{(k, \Gamma, \lambda)}(\sigma)$ simply as the height of $\sigma$, and denote it by $\text{height}(\sigma)$.

Note that according to Proposition 4.11 we have $\text{IR}(\sigma) \leq \text{height}(\sigma) < k$.

It should be borne in mind that the $(k, \Gamma, \lambda)$-height of a simplex of $K_{Z_\lambda(\Gamma)}$ is defined only if $\Gamma$ is $k$-free and $\lambda \leq \log(2k - 1)$.

**Proposition 4.13.** Let $k$ be a positive integer, let $\Gamma$ be a purely loxodromic, discrete, $k$-free subgroup of $\text{Isom}_+(\mathbf{H}^3)$, and let $\lambda$ be a real number with $0 < \lambda \leq \log(2k - 1)$ (so that in particular $\lambda$ is compatible with $\Gamma$). Suppose that $\sigma$ is a simplex of $K_{Z_\lambda(\Gamma)}$, and that $\tau$ is a face of $\sigma$. Then we have $\text{height}_{(k, \Gamma, \lambda)}(\tau) \leq \text{height}_{(k, \Gamma, \lambda)}(\sigma)$. Furthermore, if $\text{height}_{(k, \Gamma, \lambda)}(\tau) = \text{height}_{(k, \Gamma, \lambda)}(\sigma)$, then $\Theta(\sigma) \leq c(\Theta(\tau))$. (Here, since $\Theta(\tau)$ is closable according to Proposition 4.8, $c(\Theta(\tau))$ is a well-defined subgroup of $\Gamma$ by 2.11 and Proposition 2.13.)

**Proof.** Let us set $\alpha = Q(\lambda)$.

Since $\tau$ is a face of $\sigma$, it follows from an observation made in 3.2 that

\[ \text{(4.13.1)} \quad \mathcal{U}_\sigma \subset \mathcal{U}_\tau, \]

and it follows from an observation made in 4.4 that

\[ \text{(4.13.2)} \quad \Theta(\tau) \leq \Theta(\sigma). \]
According to Proposition 4.8, $\Theta(\sigma)$ and $\Theta(\tau)$ are closable subgroups of $\Gamma$, so that $c(\Theta(\tau))$ and $c(\Theta(\sigma))$ are well-defined subgroups of $\Gamma$ by 2.11 and Proposition 2.13. From 4.13.2 and Assertion (3) of Proposition 2.13, it follows that

\[(4.13.3) \quad c(\Theta(\tau)) \leq c(\Theta(\sigma)).\]

According to Proposition 4.11 and Definition 4.12, the first assertion of the present proposition, that $\text{height}(\tau) \leq \text{height}(\sigma)$, is equivalent to the assertion that $\mathcal{S}(\tau) = \mathcal{S}_{(k,\Gamma,\lambda)}(\tau)$ is contained in $\mathcal{S}(\sigma) = \mathcal{S}_{(k,\Gamma,\lambda)}(\sigma)$. Let $h \in \mathcal{S}(\tau)$ be given. In order to show that $h \in \mathcal{S}(\sigma)$, we must consider an arbitrary point $P \in \mathcal{U}_\sigma$, and show that at least one of the conditions (i), (ii) of 4.9 holds.

Since $P \in \mathcal{U}_\sigma$, it follows from 4.13.1 that $P \in \mathcal{U}_\tau$. Since $h \in \mathcal{S}(\tau)$, it then follows that at least one of the conditions (i), (ii) of 4.9 holds with $\tau$ in place of $\sigma$. If (i) holds with $\tau$ in place of $\sigma$, i.e. if there exist an integer $m \geq 0$ and independent elements $x_1, \ldots, x_m$ of $c(\Theta(\tau))$ such that (4.9.1) holds, then by (4.13.3) we have $x_1, \ldots, x_m \in c(\Theta(\sigma))$. This means that (i) holds for the simplex $\sigma$ (and with the given choices of $h$ and $P$).

Now suppose that Condition (ii) holds with $\tau$ in place of $\sigma$. Thus there exist an integer $m \geq 0$ and independent elements $x_1, \ldots, x_m$ of $c(\Theta(\tau))$, and an element $y$ of $\Gamma - c(\Theta(\tau))$, such that (4.9.2) holds. It follows from Proposition 2.14, applied with $A = \Theta(\tau)$, that $x_1, \ldots, x_m, y$ are independent.

Consider the subcase in which $y \notin c(\Theta(\sigma))$. Since (4.9.2) holds, and since $x_1, \ldots, x_m \in c(\Theta(\sigma))$ and $y \notin c(\Theta(\sigma))$, Condition (ii) holds for the simplex $\sigma$.

Now consider the subcase in which $y \in c(\Theta(\sigma))$. By (4.9.2) we have

\[Q(x_1, P) + \cdots + Q(x_m, P) + Q(y, P) > \frac{1}{2} - (k - h - 1)\alpha > \frac{1}{2} - (k - h)\alpha.\]

This means that (4.9.1) holds with the independent $(m + 1)$-tuple $(x_1, \ldots, x_m, y)$ of elements of $\Theta(\sigma)$ in place of $(x_1, \ldots, x_m)$. Hence in this subcase, Condition (i) of 4.9 holds for the simplex $\sigma$. This completes the proof of the first assertion of the proposition.

To prove the second assertion, we will assume that $\Theta(\sigma) \not\leq c(\Theta(\tau))$, and prove that $\text{height}(\tau) < \text{height}(\sigma)$. To prove the latter inequality, we will set $h_0 = \text{height}(\tau)$, and prove that $h_0 + 1 \in \mathcal{S}(\sigma)$. Thus we must consider an arbitrary point $P \in \mathcal{U}_\sigma$, and show that at least one of the conditions (i), (ii) of 4.9 holds with this choice of $P$, with the given choice of $\sigma$, and with $h_0 + 1$ playing the role of $h$. Since $P \in \mathcal{U}_\sigma$, it follows from 4.13.1 that $P \in \mathcal{U}_\tau$.

By definition $\Theta(\sigma)$ is generated by the cyclic groups $C_v$, where $v$ ranges over the vertices of $\sigma$. Hence the assumption $\Theta(\sigma) \not\leq c(\Theta(\tau))$ implies that there is a vertex $v_0$ of $\sigma$ such that $C_{v_0} \not\leq c(\Theta(\tau))$. We fix a generator $z$ of $C_{v_0}$. Then $z \notin c(\Theta(\tau))$.

Since $v_0$ is a vertex of $\sigma$, we have $\mathcal{U}_\sigma \subset Z(C_{v_0})$. In particular we have $P \in Z(C_{v_0})$; by definition this means that there is a non-trivial element $w$ of $C_{v_0}$ such that $d(w, P) < \lambda$. Hence

\[(4.13.4) \quad Q(w, P) > \alpha.\]
We may write \( w = z^e \) for some non-zero integer \( e \).

Since \( h_0 = \text{height}(\tau) \in \mathcal{H}(\tau) \), at least one of the conditions (i), (ii) of 4.9 holds with \( \tau \) and \( h_0 \) playing the respective roles of \( \sigma \) and \( h \), and with the given choice of \( P \). Hence there exist an integer \( m \geq 0 \) and independent elements \( x_1, \ldots, x_m \) of \( c(\Theta(\tau)) \) such that either (a) (4.9.1) holds with \( h = h_0 \), or (b) there is an element \( y \) of \( \Gamma - c(\Theta(\tau)) \) such that (4.9.2) holds with \( h = h_0 \). Now since \( x_1, \ldots, x_m \) are independent elements of \( c(\Theta(\tau)) \), and since \( z \notin c(\Theta(\tau)) \), it follows from Proposition 2.14 that \( x_1, \ldots, x_m, z \) are independent. Since \( w = z^e \) and \( e \neq 0 \), the elements \( x_1, \ldots, x_m, w \) are independent. Note also that since \( x_1, \ldots, x_m \in c(\Theta(\tau)) \), it follows from 4.13.3 that \( x_1, \ldots, x_m \in c(\Theta(\sigma)) \); and that since \( w \in C_{v_0} \leq \Theta(\sigma) \), it follows from Assertion (2) of Proposition 2.13 that \( w \in c(\Theta(\sigma)) \). Thus the independent elements \( x_1, \ldots, x_m, w \) all lie in \( c(\Theta(\sigma)) \).

Consider Case (a), in which (4.9.1) holds with \( h = h_0 \). From (4.9.1), with \( h = h_0 \), and from (4.13.4), we find

\[
Q(x_1, P) + \cdots + Q(x_m, P) + Q(w, P) > \frac{1}{2} - (k - h_0)\alpha + \alpha = \frac{1}{2} - (k - (h_0 + 1)),
\]

which says that (4.9.1) holds when \( h \) and \( m \) are replaced by \( h_0 + 1 \) and \( m + 1 \) respectively, and the \( m \)-tuple \( (x_1, \ldots, x_m) \) is replaced by the \( (m + 1) \)-tuple \( (x_1, \ldots, x_m, w) \) of independent elements of \( c(\Theta(\sigma)) \). This shows that in this case Condition (i) of 4.9 holds with the given choices of \( P \) and \( \sigma \), and with \( h_0 + 1 \) playing the role of \( h \).

We now turn to Case (b), in which there is an element \( y \) of \( \Gamma - c(\Theta(\tau)) \) such that (4.9.2) holds with \( h = h_0 \). Here there are two subcases. Consider first the subcase in which \( y \notin c(\Theta(\sigma)) \).

From (4.9.2), with \( h = h_0 \), and from (4.13.4), we find

\[
Q(x_1, P) + \cdots + Q(x_m, P) + Q(w, P) + Q(y, P) = (Q(x_1, P) + \cdots + Q(x_m, P) + Q(y, P)) + Q(w, P) > \frac{1}{2} - (k - h_0 - 1)\alpha + \alpha = \frac{1}{2} - (k - h_0 - 2)\alpha,
\]

which says that (4.9.2) holds when the \( m \)-tuple \( (x_1, \ldots, x_m) \) is replaced by the \( (m + 1) \)-tuple \( (x_1, \ldots, x_m, w) \) of independent elements of \( \Theta(\sigma) \) and \( h \) is replaced by \( h_0 + 1 \), while \( y \in \Gamma - c(\Theta(\sigma)) \) is given as above. This shows that, in this subcase, Condition (ii) of 4.9 holds with the given choices of \( P \) and \( \sigma \), and with \( h_0 + 1 \) playing the role of \( h \).

There remains the subcase in which \( y \in c(\Theta(\sigma)) \). In this subcase we observe that since \( x_1, \ldots, x_m \) are independent elements of \( c(\Theta(\tau)) \), and since \( y \notin c(\Theta(\tau)) \), it follows from Proposition 2.14 that the elements \( x_1, \ldots, x_m, y \) of \( c(\Theta(\sigma)) \) are independent. From (4.9.2), with \( h = h_0 \), we find

\[
Q(x_1, P) + \cdots + Q(x_m, P) + Q(y, P) > \frac{1}{2} - (k - h_0 - 1)\alpha.
\]

Thus (4.9.1) holds with \( h_0 + 1 \) and \( m + 1 \) playing the respective roles of the quantities denoted \( h \) and \( m \) in (4.9.1), and with the \( (m + 1) \)-tuple \( (x_1, \ldots, x_m, y) \) of independent elements of \( c(\Theta(\sigma)) \) playing the role of the \( m \)-tuple \( (x_1, \ldots, x_m) \). Hence, in this final subcase, Condition (i) of 4.9 holds with the given choices of \( P \) and \( \sigma \), and with \( h_0 + 1 \) playing the role of \( h \). □
The first assertion of Proposition 4.13 immediately implies:

**Corollary 4.14.** Let \( k \) be a positive integer, let \( \Gamma \) be a purely loxodromic, discrete, \( k \)-free subgroup of \( \text{Isom}_+ (\mathbb{H}^3) \), and let \( \lambda \) be a real number with \( 0 < \lambda \leq \log(2k - 1) \). Let \( r \) be a positive integer, and let \( K' \) denote the set of all simplices of \( K_{Z, \lambda}(\Gamma) \) that have height at most \( r \). Then \( K' \) is a subcomplex of \( K_{Z, \lambda}(\Gamma) \). □

**Proposition 4.15.** Let \( k \) be a positive integer, let \( \Gamma \) be a purely loxodromic, discrete, \( k \)-free subgroup of \( \text{Isom}_+ (\mathbb{H}^3) \), and let \( \lambda \) be a real number with \( 0 < \lambda \leq \log(2k - 1) \) (so that in particular \( \lambda \) is compatible with \( \Gamma \)). Suppose that \( W \) is a connected, saturated subset of \( K_{Z, \lambda}(\Gamma) \), with the property that all simplices contained in \( W \) have the same \((k, \Gamma, \lambda)\)-height. Let \( \sigma_0 \) be any simplex contained in \( W \), and let \( H \) denote the normalizer of \( \Theta(W) \) in \( \Gamma \). Then \( H \leq c(\Theta(\sigma_0)) \). (Here, since \( \Theta(\sigma_0) \) is closable according to Proposition 4.8, \( c(\Theta(\sigma_0)) \) is a well-defined subgroup of \( \Gamma \) by 2.11 and Proposition 2.13.) Furthermore, \( H \) is locally free.

**Proof.** We will first prove that

\[
(4.15.1) \quad \Theta(W) \leq c(\Theta(\sigma_0)).
\]

By definition (see 4.4) the subgroup \( \Theta(W) \) of \( \Gamma \) is generated by the subgroups \( \Theta(\sigma) \), where \( \sigma \) ranges over the simplices contained in \( W \). Hence in order to prove (4.15.1) it suffices to show that for each simplex \( \sigma \subset W \), we have \( \Theta(\sigma) \leq c(\Theta(\sigma_0)) \).

Let a simplex \( \sigma \subset W \) be given. If \( \sigma = \sigma_0 \), Assertion (2) of Proposition 2.13 gives \( \Theta(\sigma) \leq c(\Theta(\sigma_0)) \). Now suppose that \( \sigma \neq \sigma_0 \). Since \( W \) is connected, there exist a positive integer \( n \) and a finite sequence \( \sigma_1, \ldots, \sigma_n \) of simplices contained in \( W \) such that \( \sigma_n = \sigma \), and for each \( i \) with \( 0 < i \leq n \), either \( \sigma_i \) is a face of \( \sigma_{i-1} \) or \( \sigma_{i-1} \) is a face of \( \sigma_i \). We will prove by induction on \( i \) for \( 0 \leq i \leq n \), that \( \Theta(\sigma_i) \leq c(\Theta(\sigma_0)); \) the case \( i = n \) will give the required conclusion. The base case \( i = 0 \) again follows from Assertion (2) of Proposition 2.13.

Now suppose that \( i \) is given with \( 0 < i \leq n \), and that \( \Theta(\sigma_{i-1}) \leq c(\Theta(\sigma_0)) \). By construction, either \( \sigma_i \) is a face of \( \sigma_{i-1} \), or \( \sigma_{i-1} \) is a face of \( \sigma_i \). If \( \sigma_i \) is a face of \( \sigma_{i-1} \), we have \( \Theta(\sigma_i) \leq \Theta(\sigma_{i-1}) \). Since \( \Theta(\sigma_{i-1}) \leq c(\Theta(\sigma_0)) \), we have \( \Theta(\sigma_i) \leq c(\Theta(\sigma_0)) \) in this case. Now consider the case in which \( \sigma_{i-1} \) is a face of \( \sigma_i \). Since \( \sigma_{i-1} \) and \( \sigma_i \) are both contained in \( W \), the hypothesis of the proposition implies that \( \text{height}(\sigma_{i-1}) = \text{height}(\sigma_i) \). It therefore follows from the second assertion of Proposition 4.13 that \( \Theta(\sigma_i) \leq c(\Theta(\sigma_{i-1})) \) (where \( c(\Theta(\sigma_{i-1})) \) is defined because \( \Theta(\sigma_{i-1}) \) is closable by Proposition 4.8). But since \( \Theta(\sigma_{i-1}) \leq c(\Theta(\sigma_0)) \), Assertions (3) and (4) of Proposition 2.13 give \( c(\Theta(\sigma_{i-1})) \leq c(\Theta(\sigma_0)) = c(\Theta(\sigma_0)) \). Hence \( \Theta(\sigma_i) \leq c(\Theta(\sigma_0)) \). This completes the proof of (4.15.1).

According to Assertion (1) of Proposition 2.13, the subgroup \( c(\Theta(\sigma_0)) \) of \( \Gamma \) is closable; and according to (4.15.1), we have \( \Theta(W) \leq c(\Theta(\sigma_0)) \). Applying Proposition 2.15 with \( c(\Theta(\sigma_0)) \) and \( \Theta(W) \) playing the respective roles of \( A \) and \( B \), we deduce that the normalizer \( H \) of \( \Theta(W) \) is contained in \( c(\Theta(\sigma_0)) \); in view of Assertion (4) of Proposition 2.13, this means that \( H \leq c(\Theta(\sigma_0)) \). This is the first assertion of the present proposition.

To prove the second assertion, note that since \( \Gamma \) is \( k \)-free and \( \lambda \leq \log(2k - 1) \), it follows from Proposition 4.6 that \( \text{rank} \Theta(\sigma_0) < k \). Assertion (1) of Proposition 2.13 gives
localrank(\(c(\Theta(\sigma_0))\)) \leq \text{localrank}(\Theta(\sigma_0)) = \text{rank}(\Theta(\sigma_0)) < k$; again using that \(\Gamma\) is \(k\)-free, we deduce that \(c(\Theta(\sigma_0))\) is locally free. Hence its subgroup \(H\) is also locally free. □

5. A central result

**Lemma 5.1.** Let \(k\) be a positive integer, let \(\Gamma\) be a purely loxodromic, discrete, \(k\)-free subgroup of \(\text{Isom}_+(\mathbb{H}^3)\), and let \(\lambda\) be a real number with \(0 < \lambda \leq \log(2k - 1)\) (so that in particular \(\lambda\) is compatible with \(\Gamma\)). Set \(K = K_{\lambda,\Gamma}\), and let \(K\) be equipped with the canonical action described in 4.4. Then:

1. for every \(P \in \mathbb{H}^3\) and for all elements \(x, z \in \Gamma\), we have \(d(zxz^{-1}, P) = d(x, z^{-1} \cdot P)\) and \(Q(zxz^{-1}, P) = Q(x, z^{-1} \cdot P)\); and

2. for every simplex \(\sigma\) of \(K\) and for every \(z \in \Gamma\), we have (a) \(U_{z,\sigma} = z \cdot U_\sigma\); (b) \(c(\Theta(z \cdot \sigma)) = zc(\Theta(\sigma))z^{-1}\) (where \(c(\Theta(z \cdot \sigma))\) and \(c(\Theta(\sigma))\) are well-defined subgroups of \(\Gamma\) because \(\Theta(z \cdot \sigma)\) and \(\Theta(\sigma)\) are closable by Proposition 4.8); and (c) height\(_{\Gamma,\lambda}(z \cdot \sigma)\) = height\(_{(k,\Gamma,\lambda)}(z \cdot \sigma)\).

**Proof.** We have \(d(zxz^{-1}, P) = \text{dist}(zxz^{-1} \cdot P, P) = \text{dist}(xz^{-1} \cdot P, z^{-1} \cdot P) = d(x, z^{-1} \cdot P)\). Hence \(Q(zxz^{-1}, P) = Q(d(zxz^{-1}, P)) = Q(d(x, z^{-1} \cdot P)) = Q(x, z^{-1} \cdot P)\). This proves (1).

Now note that for each vertex \(v\) of \(\sigma\) we have \(C_{zv} = zC_vz^{-1}\) by (4.4.2). But by (4.4.3) we have \(Z_\lambda(zC_vz^{-1}) = z \cdot Z_\lambda(C_v)\). Thus we have \(Z_\lambda(C_{zv}) = z \cdot Z_\lambda(C_v)\). Hence, letting \(v\) range over the vertices of \(\sigma\), we have \(U_{z,\sigma} = \bigcap_v Z_\lambda(C_{zv}) = \bigcap_v z \cdot Z_\lambda(C_v) = z \cdot \bigcap_v Z_\lambda(C_v) = z \cdot U_\sigma\). This proves (2a). Next note that, again letting \(v\) range over the vertices of \(\sigma\), we have \(z\Theta(z)z^{-1} = z(\bigcup_v C_v)z^{-1} = (\bigcup_v zC_vz^{-1}) = (\bigcup_v C_{zv}) = \Theta(z \cdot \sigma)\). Hence \(z\Theta(z)z^{-1} = z(\bigcup_v C_v)z^{-1} = (\bigcup_v C_{zv}) = \Theta(z \cdot \sigma)\). Since \(\Theta(z \cdot \sigma)\) is closable by Proposition 4.8, it follows that \(c(z\Theta(z)z^{-1})\) is defined and equal to \(c(\Theta(z \cdot \sigma))\). By Proposition 2.16 we have \(c(z\Theta(z)z^{-1}) = zc(\Theta(\sigma))z^{-1}\); Assertion (2b) now follows.

In order to prove (2c), let us consider an arbitrary element \(h \in \mathfrak{H}(\sigma)\). Let \(P\) be an arbitrary point of \(U_{z,\sigma}\). In view of (2a), we have \(z^{-1} \cdot P \in U_\sigma\). Since \(h \in \mathfrak{H}(\sigma)\), one of the conditions (i), (ii) of 4.9 holds with \(z^{-1} \cdot P\) in place of \(P\). Thus there exist independent elements \(x_1, \ldots, x_m\) of \(c(\Theta(\sigma))\) such that either (i) the inequality (4.9.1) is satisfied with \(z^{-1} \cdot P\) in place of \(P\), or (ii) for some \(y \in \Gamma - c(\Theta(\sigma))\), the inequality (4.9.2) is satisfied with \(z^{-1} \cdot P\) in place of \(P\). It follows from (2b) that the independent elements \(zx_1z^{-1}, \ldots, zx_mz^{-1}\) belong to \(c(\Theta(z \cdot \sigma))\); and that when (ii) holds we have \(zyz^{-1} \notin c(\Theta(z \cdot \sigma))\). Furthermore, by Assertion (1) we have \(Q(zxz^{-1}, P) = Q(x_i, z^{-1} \cdot P)\) for \(i = 1, \ldots, m\); and if (ii) holds so that the element \(y\) is defined, then \(Q(zyz^{-1}, P) = Q(y, z^{-1} \cdot P)\). It follows that when (i) holds, (4.9.1) is satisfied with the given choice of \(P\), and with \(zx_iz^{-1}\) in place of \(x_i\) for \(i = 1, \ldots, m\); and that when (ii) holds, (4.9.2) is satisfied with the given choice of \(P\), with \(zx_iz^{-1}\) in place of \(x_i\) for \(i = 1, \ldots, m\), and with \(zyz^{-1}\) in place of \(y\). This shows that one of the alternatives of 4.9 holds for the arbitrary point \(P \in U_{z,\sigma}\) and the given \(h\), and hence that \(h \in \mathfrak{H}(z \cdot \sigma)\). Thus we have \(\mathfrak{H}(\sigma) \subset \mathfrak{H}(z \cdot \sigma)\). In view of Proposition 4.11 and Definition 4.12, it follows that height\(_{\sigma}\) \leq height\(_{(z \cdot \sigma)}\). As this holds for every \(z \in \Gamma\), we may replace \(z\) by \(z^{-1}\) and \(\sigma\) by \(z \cdot \sigma\) to obtain height\(_{(z \cdot \sigma)}\) \leq \text{height}(z^{-1} \cdot z \cdot \sigma) = \text{height}(\sigma)\). This proves (2c). □
Theorem 5.2. Let \( k \) be a positive integer. Let \( \Gamma \) be a discrete subgroup of \( \text{Isom}_+ (H^3) \) which is cocompact and \( k \)-free (and therefore purely loxodromic). Let \( \lambda \) be a real number with \( 0 < \lambda \leq \log (2k-1) \). Assume that, in the notation of 4.4, we have \( \mathfrak{X}_\lambda (\Gamma) = H^3 \). Set \( K = K_{Z_\lambda (\Gamma)} \) (again in the notation of 4.4). Then there is a simplex \( \sigma \) of \( K \) such that (1) \( \text{height}_{(k, \Gamma, \lambda)} (\sigma) \leq k - 3 \) and (2) \( \text{link}_K \sigma \) is non-contractible.

Remark 5.2.1. Since the height of a simplex is always strictly positive (see Proposition 4.11 and Definition 4.12), it is a formal consequence of Theorem 5.2 that if \( \Gamma \leq \text{Isom}_+ (H^3) \) is discrete, cocompact, and \( k \)-free, and if \( \mathfrak{X}_\lambda (\Gamma) = H^3 \) for some \( \lambda \leq \log (2k-1) \), then \( k \geq 4 \). Equivalently, if \( \Gamma \) is \( k \)-free for a given \( k \leq 3 \), then for any \( \lambda \leq \log (2k-1) \) we have \( \mathfrak{X}_\lambda (\Gamma) \neq H^3 \). This fact is not new; it is trivial for \( k = 1 \), since we then have \( \lambda \leq \log 1 = 0 \) and \( \mathfrak{X}_\lambda (\Gamma) = \emptyset \); and for \( k = 2 \) and \( k = 3 \) it is equivalent (via the self-contained and elementary material in 6.2 below) to the known facts, included in [3, Corollary 4.2] and [4, Corollary 9.3], that an orientable hyperbolic 3-manifold with 2-free or 3-free fundamental group contains a hyperbolic ball of radius \( (\log 3)/2 \) or \( (\log 5)/2 \) respectively.

Proof of Theorem 5.2. As we pointed out in 4.4, \( U_\sigma \) is convex and hence contractible for each simplex \( \sigma \) of \( K_{Z_\lambda (\Gamma)} \), and Proposition 3.3 therefore implies that \( K_{Z_\lambda (\Gamma)} \) is homotopy-equivalent to \( \mathfrak{X}_\lambda (\Gamma) \). Since the hypothesis gives \( \mathfrak{X}_\lambda (\Gamma) = H^3 \), it follows that \( |K| \) is contractible.

Since \( \Gamma \) is cocompact, it follows from [19, Proposition 2.13] that \( K \) is finite-dimensional.

Let \( K' \) denote the set of simplices \( \sigma \in K \) such that \( \text{height}(\sigma) = \text{height}_{(k, \Gamma, \lambda)} (\sigma) \leq k - 3 \). According to Corollary 4.14, \( K' \) is a subcomplex of \( K \).

In order to establish the conclusion of the theorem, it suffices to show that there is a simplex \( \sigma \in K' \) such that link\(_K \)\( \sigma \) is non-contractible.

Assume that this is false, i.e. that link\(_K \)\( \sigma \) is contractible for every \( \sigma \in K' \). But according to [19, Proposition 3.2], if \( K' \) is a subcomplex of a finite-dimensional simplicial complex \( K \), and \( K' \) is a subcomplex of \( K \) such that link\(_K \)\( \sigma \) is contractible for every simplex \( \sigma \in K' \), then the inclusion map of the saturated set \( |K| - |K'| \) into \( |K| \) is a homotopy equivalence.

In the present situation, since \( |K| \) is contractible, it follows that the saturated set \( |K| - |K'| \) is also contractible.

According to the definition of \( K' \), a simplex is contained in \( |K| - |K'| \) if and only if its height is at least \( k - 2 \). On the other hand, according to Proposition 4.11 and Definition 4.12, we have \( \text{height}(\sigma) < k \) for every simplex \( \sigma \) of \( K \). Hence the saturated subset \( |K| - |K'| \) is the union of simplices of \( K \) of height equal to \( k - 2 \) or \( k - 1 \). For \( h = k - 1, k - 2 \), we let \( X_h \) denote the union of simplices of height equal to \( h \), so that \( |K| - |K'| \) is the set-theoretical disjoint union of \( X_{k-2} \) and \( X_{k-1} \).

For each \( h \in \{ k - 2, k - 1 \} \), let \( \mathcal{W}_h \) denote the set of connected components of \( X_h \). We will say that elements \( W_{k-2} \) of \( \mathcal{W}_{k-2} \) and \( W_{k-1} \) of \( \mathcal{W}_{k-1} \) are adjacent if there are simplices \( \sigma_{k-2} \subset W_{k-2} \) and \( \sigma_{k-1} \subset W_{k-1} \) such that either \( \sigma_{k-2} < \sigma_{k-1} \) or \( \sigma_{k-1} < \sigma_{k-2} \).

We construct an abstract bipartite graph \( \mathcal{G} \) as follows: The vertex set \( Y \) of \( \mathcal{G} \) is a disjoint union \( Y_{k-2} \cup Y_{k-1} \), where for \( h = k - 2, k - 1 \) the set \( Y_h \) is a bijective copy of \( \mathcal{W}_h \) and is
equipped with a bijection from $\mathcal{W}_h$ to $Y_h$, which we denote $W \mapsto s_W$. An edge of $\mathcal{G}$ is defined to be an unordered pair of the form $\{s_{W_{k-2}}, s_{W_{k-1}}\}$, where $W_h$ is an element of $\mathcal{W}_h$ for $h = k - 2, k - 1$, and $W_{k-2}$ and $W_{k-1}$ are adjacent.

Let $T$ denote the geometric realization of $\mathcal{G}$ (regarded as a simplicial complex of dimension at most 1). Because $|K| - |K'|$ is the disjoint union of the saturated subsets $X_{k-2}$ and $X_{k-1}$ of $|K|$, it follows from [17, Lemma 5.12] that $|T|$ is a homotopy-retract of $|K| - |K'|$. Since $|K| - |K'|$ is contractible, it follows that $T$ is a tree.

We consider the canonical action of $\Gamma$ on $K$ that was discussed in 4.4. According to Assertion (2c) of Lemma 5.1, we have $\text{height}(z \cdot \sigma) = \text{height}(\sigma)$ for every $\sigma \in K$ and every $z \in \Gamma$. Consequently, $X_{k-2}$ and $X_{k-1}$ are invariant under the action of $\Gamma$.

Since the canonical action of $\Gamma$ on $K$ is simplicial, it defines a continuous action of $\Gamma$ on $|K|$. Hence for $h = k - 2, k - 1$, the restricted action of $\Gamma$ on $X_h$ gives rise to an action on $\mathcal{W}_h$. The simplicial nature of the action on $K$ also implies that if $W_{k-1} \in \mathcal{W}_{k-1}$ and $W_{k-2} \in \mathcal{W}_{k-2}$ are adjacent, then $x \cdot W_{k-1}$ and $x \cdot W_{k-2}$ are adjacent for every $x \in \Gamma$. Hence the action of $\Gamma$ on $Y$ defined by $x \cdot s_W = s_{x \cdot W}$ extends to a simplicial action of $\Gamma$ on $\mathcal{G}$, which gives rise to an action on $T$.

Note that since $X_{k-2}$ and $X_{k-1}$ are $\Gamma$-invariant, the sets $Y_{k-2}, Y_{k-1} \subset \mathcal{G}$ are also $\Gamma$-invariant. Since each edge of $\mathcal{G}$ has one vertex in $Y_{k-2}$ and one in $Y_{k-1}$, the action of $\Gamma$ on $T$ has no inversions.

If $s$ is an arbitrary vertex of $\mathcal{G}$, we may write $s = s_W$ for some $W \in \mathcal{W}_h$, where $h \in \{k - 2, k - 1\}$. If $x$ is any element of the vertex stabilizer $\Gamma_s \leq \Gamma$, it follows from the definition of the action of $\Gamma$ on $\mathcal{G}$ that $x \cdot W = W$. On the other hand, it was pointed out in 4.4 that $\Theta(x \cdot W) = x\Theta(W)x^{-1}$. Hence $\Theta(W) = x\Theta(W)x^{-1}$. This shows that $\Gamma_s$ is contained in the normalizer of $\Theta(W)$ in $\Gamma$. But since $W \in \mathcal{W}_h$ is by definition a component of $X_h$, it is a connected saturated subset of $K$, and each simplex contained in $W$ has height $h$. It therefore follows from Proposition 4.15 that the normalizer of $\Theta(W)$ in $\Gamma$ is locally free. In particular, $\Gamma_s$ is locally free.

We have shown that $\Gamma$ acts simplicially, without inversions, on the tree $T$, and that the stabilizer of every vertex in $\Gamma$ is locally free. But since $\Gamma$ is cocompact, it is isomorphic to the fundamental group of a compact, orientable hyperbolic 3-manifold. According to [17, Lemma 5.13], the fundamental group of a closed, orientable, aspherical 3-manifold cannot act simplicially, without inversions, on a tree in such a way that each vertex stabilizer is locally free. This contradiction completes the proof. □

**5.3.** We observed in 4.12 that if $k$ is a positive integer, if $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is purely loxodromic, discrete and $k$-free for a given $k > 0$, and if $0 < \lambda \leq \log(2k - 1)$, then for every simplex $\sigma$ of $K_Z(\Gamma)$ we have $\text{IR}(\sigma) \leq \text{height}(\sigma) < k$. It therefore follows from Theorem 5.2 that, under the hypotheses of the theorem, there is a simplex $\sigma$ of $K$ such that $\text{IR}(\sigma) \leq k - 3$ and $\text{link}_K\sigma$ is non-contractible. This fact was stated as Assertion 1.0.1 in the introduction, where its implicit role in [19] was discussed.
6. Elementary Quantitative Geometry of Hyperbolic Manifolds

**Notation and Remarks 6.1.** If $p$ is a point of a metric space, and $r$ is a real number, we will denote by $\text{nbhd}_r(p)$ the set of all $x \in X$ such that $\text{dist}(x, p) < r$. Thus $\text{nbhd}_r(p)$ is a neighborhood of $p$ if $r > 0$, and is empty if $r \leq 0$.

We will often use the following consequence of the triangle inequality: if $p$ and $p'$ are points of a metric space, and $r$ and $r'$ are real numbers with $r + r' > \text{dist}(p, p')$, then $\text{nbhd}_r(p) \cap \text{nbhd}_{r'}(p') = \emptyset$.

If $X$ is a subset of a hyperbolic manifold $M$, one can make $X$ into a metric space by defining the distance between two points of $X$ to be their distance in $M$; this is the *extrinsic distance function*. If $X$ is compact and non-empty, the extrinsic distance function gives rise to the *extrinsic diameter* of $U$, which is the maximum of the set of all distances in $M$ between points of $X$. On the other hand, if $X$ is connected and open, the *intrinsic distance* between points $p$ and $p'$ of $X$ is the infimum of all lengths of paths in $U$ between $p$ and $p'$. This gives rise to the notion of an *intrinsic isometry* between open connected subsets $X$ and $X'$ of hyperbolic manifolds $M$ and $M'$: it is a diffeomorphism between $X$ and $X'$ that preserves lengths of paths.

A *hyperbolic ball* in a hyperbolic 3-manifold $M$ is a subset of $M$ which is intrinsically isometric to a ball $N \subset \mathbb{H}^3$; its *radius* is the radius of $N$, and its *center* is the unique point that is mapped to the center of $N$ by an intrinsic isometry.

A *tube* in a hyperbolic 3-manifold $M$ is a set $T \subset M$ which is intrinsically isometric to a quotient $Z/\langle \tau \rangle$, where $Z \subset \mathbb{H}^3$ is a hyperbolic cylinder (see 4.4), and $\tau$ is a loxodromic transformation whose axis is the axis $l$ of $Z$. The *radius* of $T$ is the radius of $Z$, and its *core* is the unique simple closed geodesic that is mapped to $l/\langle \tau \rangle$ by an intrinsic isometry from $T$ to $Z/\langle \tau \rangle$.

The frontier of a subset $A$ of a topological space $X$, defined as $\overline{A} \cap X - A$, will be denoted by $\text{Fr}_X A$, or simply by $\text{Fr} A$ when the space $X$ is understood.

**Definition, Notation and Remarks 6.2.** If $p$ is a point of a closed, orientable hyperbolic 3-manifold $M$, we denote by $s_1(p) > 0$ the minimum length of a homotopically non-trivial loop based at $p$. We shall say that $p$ is *$\alpha$-thin* for a given $\alpha > 0$ if $s_1(p) < \alpha$, and that $p$ is *$\alpha$-thick* if $s_1(p) \geq \alpha$. The point $p$ is $\alpha$-thick if and only if it is the center of a hyperbolic ball of radius $\alpha/2$ in $M$.

We denote by $M_{\text{thin}}(\alpha)$ the set of all $\alpha$-thin points of $M$; thus $M_{\text{thick}}(\alpha) := M - M_{\text{thin}}(\alpha)$ is the set of all $\alpha$-thick points of $M$. Note that if $M$ is written as $\mathbb{H}^3/\Gamma$ where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete, torsion-free and cocompact, and therefore purely loxodromic, and if $q: \mathbb{H}^3 \to M$ denotes the quotient map, then in the notation of 4.4 we have $q^{-1}(M_{\text{thin}}(\alpha)) = \mathfrak{X}_\alpha(\Gamma)$ for every $\alpha > 0$. In particular we have $\mathfrak{X}_\alpha(\Gamma) = \mathbb{H}^3$ if and only if $M_{\text{thin}}(\alpha) = M$, i.e. if and only if $M_{\text{thick}}(\alpha) = \emptyset$.

Given a point $p$ of the closed, orientable hyperbolic 3-manifold $M$, we will say that a maximal cyclic subgroup $C$ of $\pi_1(M, p)$ is *short* if some non-trivial element of $C$ is represented by a
loop of length \( s_1(p) \) based at \( p \). It follows from 4.3 that \( \pi_1(M) \) is an ICC-group. Thus every element of \( \pi_1(M, p) \) lies in a unique maximal cyclic subgroup, and hence it follows from the definitions that \( \pi_1(M, p) \) has at least one short maximal cyclic subgroup.

We also define a positive real number \( s_2(p) \) as follows: if \( \pi_1(M, p) \) has only one short maximal cyclic subgroup, denoted \( C_0 \), we define \( s_2(p) \) to be the minimum length of a loop defining an element of \( \pi_1(M, p) - C_0 \). If there are two or more short maximal cyclic subgroups, we set \( s_2(p) = s_1(p) \).

Thus in all cases, if \( C \) is any short maximal cyclic subgroup, then \( s_2(p) \) is the minimum length of a loop defining an element of \( \pi_1(M, p) - C \).

If \( M \) is given as a quotient \( \mathbb{H}^3/\Gamma \), where \( \Gamma \leq \text{Isom}^+_0(\mathbb{H}^3) \) is discrete, cocompact and torsion-free, and if \( P \in \mathbb{H}^3 \) is a point that projects to \( p \in M \) under the quotient map, we have \( s_1(p) = \min_{x \neq z} d(x, P) \). Furthermore, if \( z \) is a non-trivial element of \( \Gamma \) with \( d(z, P) = s_1(p) \), and if \( C \) denotes the maximal cyclic group of the ICC-group \( \Gamma \) containing \( z \), then \( s_2(p) = \min_{x \in \Gamma - C} d(x, P) \).

Notice also that we have \( s_2(p) \geq s_1(p) \) for every \( p \in M \).

6.3. In [17] (see Definition 1.3 of that paper), a point \( p \) of a closed, orientable hyperbolic 3-manifold \( M \) is defined to be \( \lambda \)-semithick, where \( \lambda \) is a given positive number, if any two loops of length less than \( \lambda \) based at \( p \) define elements of \( \pi_1(M, p) \) which commute. Since \( \pi_1(M, p) \) is an ICC-group, this is equivalent to saying that any two homotopically non-trivial loops of length less than \( \lambda \) based at \( p \) define elements in the same maximal cyclic subgroup of \( \pi_1(M, p) \).

It is an immediate consequence of this definition, together with the definition of \( s_2(p) \) given in 6.2, that \( p \) is \( \lambda \)-semithick if and only if \( s_2(p) \geq \lambda \).

In [17, 3.8], a set \( \mathcal{G}_M \) is defined to be the set of all points \( p \in M \) for which the elements of \( \pi_1(M, p) \) generated by all elements represented by loops of length \( s_1(p) \) is a cyclic group. (The quantity that we denote by \( s_1(p) \) in this paper is denoted by \( \ell_p \) in [17].) In terms of the definition given in 6.2, this means that \( p \in \mathcal{G}_M \) if and only if \( \pi_1(M, p) \) has only one short maximal cyclic subgroup. Furthermore, in [17, 3.8], when \( p \in \mathcal{G}_M \), the unique maximal cyclic subgroup of \( \pi_1(M, p) \) containing at least one non-trivial element represented by a loop of length less than \( s_1(p) \) is denoted by \( C_p \), and \( s_M(p) \) denotes the smallest length of a loop based at \( p \) that does not represent an element of \( C_p \); thus from the point of view of the present paper, \( C_p \) is the unique short maximal cyclic subgroup of \( \pi_1(M, p) \) when \( p \in \mathcal{G}_M \), and \( s_M \) is simply the restriction of \( s_2 \) to \( \mathcal{G}_M \). Note also that, according to our definitions, we have \( s_2(p) = s_1(p) \) for any \( p \in M - \mathcal{G}_M \).

These comparisons of the conventions of the present paper to those of [17] will be useful in the proofs of Proposition 10.3 and Lemmas 10.8 and 10.12 below.

Lemma 6.4. Let \( M \) be a closed, orientable 3-manifold.
(a) For every point \( p \in M \) we have \( s_1(p) + s_2(p) = \min_{(\alpha, \beta)}(\text{length}(\alpha) + \text{length}(\beta)) \), where \((\alpha, \beta)\) ranges over all pairs of loops based at \( p \) such that \([\alpha]\) and \([\beta]\) do not commute in \( \pi_1(M, p) \).

(b) For any two points \( p, p' \in M \) we have \(|s_1(p) - s_1(p')| \leq 2 \text{dist}(p, p')\) and \(|(s_1(p) + s_2(p)) - (s_1(p') + s_2(p'))| \leq 4 \text{dist}(p, p')\).

(c) The functions \( s_1 \) and \( s_2 \) are continuous on \( M \).

**Proof.** First note that by the definition of \( s_1(p) \), we may fix a loop \( \alpha_0 \) based at \( p \) such that \( 1 \neq [\alpha_0] \in \pi_1(M, p) \) and \( \text{length}(\alpha_0) = s_1(p) \); and according to the discussion in 6.2, if \( C \) denotes the maximal cyclic subgroup of the ICC-group \( \pi_1(M, p) \) containing \([\alpha_0]\), we may fix a loop \( \beta_0 \) based at \( p \) such that \([\beta_0] \notin C \) and \( \text{length}(\beta_0) = s_2(p) \). Since \([\beta_0] \notin C \), the elements \([\alpha_0]\) and \([\beta_0]\) do not commute. We have \( \text{length}(\alpha_0) + \text{length}(\beta_0) = s_1(p) + s_2(p) \).

To prove (a), it remains only to prove that if \( \alpha \) and \( \beta \) are arbitrary loops based at \( p \) such that \([\alpha]\) and \([\beta]\) do not commute in \( \pi_1(M, p) \), we have \( \text{length}(\alpha) + \text{length}(\beta) \geq s_1(p) + s_2(p) \).

Since \([\alpha]\) and \([\beta]\) do not commute, they cannot both lie in \( C \). Hence by symmetry we may assume that \([\beta] \notin C \). The discussion in 6.2 then shows that \( \text{length}(\beta) \geq s_2(p) \). Furthermore, since \([\alpha]\) is in particular non-trivial, the definition of \( s_1(p) \) gives \( \text{length}(\alpha) \geq s_1(p) \). Hence \( \text{length}(\alpha) + \text{length}(\beta) \geq s_1(p) + s_2(p) \) as required, and (a) is proved.

To prove (b), set \( d = \text{dist}(p, p') \), and let \( \gamma \) denote a geodesic path from \( p \) to \( p' \) with \( \text{length}(\gamma) = d \). If \( \alpha \) is a homotopically non-trivial loop based at \( p \), then \( \alpha' := \overline{\gamma} \ast \alpha \ast \gamma \) is a homotopically non-trivial loop based at \( p' \), and \( \text{length}(\alpha') = 2d + \text{length}(\alpha) \). Since \( s_1(p) \) and \( s_1(p') \) are by definition the minimum lengths of homotopically non-trivial loops based at \( p \) and \( p' \) respectively, it follows that

\[
(6.4.1) \quad s_1(p') \leq s_1(p) + 2d.
\]

Likewise, if \( \alpha \) and \( \beta \) are loops based at \( p \) such that \([\alpha]\) and \([\beta]\) do not commute, then \( \alpha' := \overline{\gamma} \ast \alpha \ast \gamma \) and \( \beta' := \overline{\gamma} \ast \beta \ast \gamma \) are loops based at \( p' \) such that \([\alpha']\) and \([\beta']\) do not commute, and \( \text{length}(\alpha') + \text{length}(\beta') = 4d + \text{length}(\alpha) + \text{length}(\beta) \). Since, by (a), we have \( s_1(p) + s_2(p) = \min_{(\alpha, \beta)}(\text{length}(\alpha) + \text{length}(\beta)) \), where \((\alpha, \beta)\) ranges over all pairs of loops based at \( p \) such that \([\alpha]\) and \([\beta]\) do not commute, and \( s_1(p') + s_2(p') = \min_{(\alpha', \beta')}(\text{length}(\alpha') + \text{length}(\beta')) \), where \((\alpha', \beta')\) ranges over all pairs of loops based at \( p' \) such that \([\alpha']\) and \([\beta']\) do not commute, it follows that

\[
(6.4.2) \quad s_1(p') + s_2(p') \leq s_1(p) + s_2(p) + 4d.
\]

Interchanging the roles of \( p \) and \( p' \) in (6.4.1) and in (6.4.2), we obtain

\[
(6.4.3) \quad s_1(p) \leq s_1(p') + 2d
\]

and

\[
(6.4.4) \quad s_1(p) + s_2(p) \leq s_1(p') + s_2(p') + 4d.
\]

Now (6.4.1) and (6.4.3) imply the inequality \(|s_1(p) - s_1(p')| \leq 2d\), while (6.4.2) and (6.4.4) imply \(|(s_1(p) + s_2(p)) - (s_1(p') + s_2(p'))| \leq 4d\). Thus (b) is proved.
To prove (c), note that it follows from (b) that the functions \( s_1 \) and \( s_1 + s_2 \) are continuous on \( M \); hence \( s_2 \) is also continuous.

**Remark and Notation 6.5.** Let \( M \) be a closed, orientable hyperbolic 3-manifold. Since by Lemma 6.4 the function \( s_2 \) is continuous on the compact space \( M \), this function takes a greatest and a least value on \( M \), which we will denote by \( \lambda_M \) and \( \mu_M \) respectively.

**Definition 6.6.** Recall that a Margulis number for an orientable hyperbolic 3-manifold \( M \) is defined to be a positive number \( \mu \) such that, for every point \( p \in M \) and for any two loops \( \alpha \) and \( \beta \) of based at \( p \) and having length less than \( \mu \), the elements \([\alpha]\) and \([\beta]\) of \( \pi_1(M,p) \) commute.

**Proposition 6.7.** Let \( M \) be a closed, orientable hyperbolic 3-manifold. Then the interval \([\mu_M, \lambda_M]\) is the range of the function \( s_2 \) on \( M \), and the interval \((0, \mu_M]\) is the set of all Margulis numbers for \( M \).

**Proof.** Since, by Lemma 6.4, \( s_2 \) is a continuous function on the compact connected space \( M \), its range is a closed interval. It follows from the definitions of \( \mu_M \) and \( \lambda_M \) that they are respectively the left-hand and right-hand endpoints of this interval. This proves the first assertion.

To prove the second assertion, first consider an arbitrary number \( \mu \) with \( 0 < \mu \leq \mu_M \). Let \( p \) be an arbitrary point of \( M \), and let \( \alpha \) and \( \beta \) be loops based at \( p \) and having length less than \( \mu \). Then their lengths are less than \( \mu_M \), which is in turn at most \( s_2(p) \) by definition. According to 6.2, we may choose a short maximal cyclic subgroup \( C \) of \( \pi_1(M,p) \), and \( s_2(p) \) is the minimal length of any loop representing an element of \( \pi_1(M,p) - C \). Since \( \alpha \) and \( \beta \) have length less than \( s_2(p) \), they represent elements of \( C \), and therefore commute. This shows that \( \mu \) is a Margulis number for \( M \).

Now consider an arbitrary number \( \nu > \mu_M \). By the definition of \( \mu_M \) we may choose a point \( p \in M \) with \( s_2(p) = \mu_M \). According to 6.2, we may choose a short maximal cyclic subgroup \( C \) of \( \pi_1(M,p) \), and there are loops \( \alpha \) and \( \beta \) based at \( p \), having respective lengths \( s_1(p) \) and \( s_2(p) \), and respectively representing elements of \( C \) and \( \pi_1(M,p) - C \). Since \( C \) is a maximal cyclic subgroup of the ICC-group \( \pi_1(M,p) \), the elements \( \alpha \) and \( \beta \) do not commute. But their lengths are both bounded above by \( s_2(p) = \mu_M < \nu \). This shows that \( \nu \) is not a Margulis number for \( M \), and the proof of the second assertion is complete.

**Proposition 6.8.** Let \( M \) be a closed, orientable hyperbolic 3-manifold, and suppose that \( \mu \) is a Margulis number for \( M \). Then every component of \( M_{\text{thin}}(\mu) \) is an open solid torus. Hence \( M_{\text{thick}}(\mu) \) is connected (and in particular non-empty).

**Proof.** We use the notation of 4.4. Write \( M = \mathbb{H}^3/\Gamma \), where \( \Gamma \leq \text{Isom}_+(\mathbb{H}^3) \) is discrete, torsion-free and cocompact. If \( C_1 \) and \( C_2 \) are distinct elements of \( \mathcal{C}_N(\Gamma) \) and \( P \) is a point of \( Z_\mu(C_1) \cap Z_\mu(C_2) \), then for \( i = 1, 2 \) the definition of \( Z_\mu(C_i) \) gives a non-trivial element \( x_i \) of \( C_i \) such that \( d(x_i, P) < \mu \). Since the \( C_i \) are distinct maximal cyclic subgroups of the ICC-group \( \Gamma \), the elements \( x_1 \) and \( x_2 \) do not commute. It follows that if \( p \in M \) denotes the image
of \( P \) under the quotient map, there are loops of length less than \( \mu \) based at \( p \) representing non-commuting elements of \( \pi_1(M,p) \); this contradicts the definition of a Margulis number. Hence the family \( (Z_\mu(C))_{C \in C_\mu(\Gamma)} \) is pairwise disjoint, and the sets in this family are therefore the components of \( \mathcal{X}_\mu(\Gamma) = \bigcup_{C \in C_\mu(\Gamma)} Z_\mu(C) \). It follows that each component of \( M_{\text{thin}}(\mu) \) is the quotient of \( Z_\mu(C) \), for some \( C \in C_\mu(\Gamma) \), by its stabilizer, which is \( C \). The assertion follows.

The following result will be used at a couple of points in this paper:

**Proposition 6.9.** Let \( k \) and \( m \) be integers with \( k \geq 2 \) and \( 0 \leq m \leq k \). Suppose that \( M \) is a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is \( k \)-free, and let \( \mu \) be a Margulis number for \( M \). Let \( p \) be a point of \( M \), let \( u_1, \ldots, u_m \) be independent elements of \( \pi_1(M,p) \) represented by loops \( \alpha_1, \ldots, \alpha_m \) based at \( p \), and let \( d_j \) denote the length of \( \alpha_j \). Then there is a \( \mu \)-thick point \( p' \in M \) such that \( \rho := \text{dist}(p,p') \) satisfies

\[
(k-m)Q(2\rho) + \sum_{j=1}^{m} Q(d_j) \leq \frac{1}{2}.
\]

**Proof.** This is a paraphrase of [15, Corollary 6.2]. □

**Reformulation 6.10.** For applications of Lemma 6.9, it will be convenient to define, for every positive integer \( n \), a function \( \xi_n \) on the interval \((0, 1/2)\), by setting

\[
\xi_n(u) = \frac{1}{2} Q^{-1} \left( \frac{1}{n} \left( \frac{1}{2} - u \right) \right)
\]

whenever \( 0 < u < 1/2 \). Since \( Q \) is strictly monotone decreasing and has range \((0, 1/2)\) (see 4.1), the function \( \xi_n \) is well defined, and is strictly monotone increasing. In terms of these definitions, we may reformulate Proposition 6.9 (at least in the case \( k > m \)) as follows: if \( k \) and \( m \) are integers with \( k \geq 2 \) and \( 0 \leq m < k \), if \( M \) is a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is \( k \)-free, if \( \mu \) is a Margulis number for \( M \), if \( p \) is a point of \( M \), if \( u_1, \ldots, u_m \) are independent elements of \( \pi_1(M,p) \) represented by loops \( \alpha_1, \ldots, \alpha_m \) based at \( p \), and if \( d_j \) denotes the length of \( \alpha_j \), then \( \sum_{j=1}^{m} Q(d_j) < 1/2 \) (so that \( \xi_{k-m}(\sum_{j=1}^{m} Q(d_j)) \) is defined), and there is a \( \mu \)-thick point \( p' \in M \) such that \( \text{dist}(p,p') \geq \xi_{k-m}(\sum_{j=1}^{m} Q(d_j)) \).

**Lemma 6.11.** Let \( k > 2 \) be an integer, let \( M \) be a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is \( k \)-free, and let \( \mu \) be a Margulis number for \( M \). Then for every point \( p \in M \) we have \( Q(s_1(p) + Q(s_2(p))) < 1/2 \) (so that \( \xi_{k-2}(Q(s_1(p) + Q(s_2(p))) \) is defined), and there is a \( \mu \)-thick point \( p' \in M \) such that \( \text{dist}(p,p') \geq \xi_{k-2}(Q(s_1(p) + Q(s_2(p)))) \).

**Proof.** By 6.2 we may choose a short maximal cyclic subgroup \( C \) of \( \pi_1(M,p) \), and there are elements \( u_1 \in C - \{1\} \) and \( u_2 \in \pi_1(M,p) - C \) that are represented by loops of respective lengths \( s_1(p) \) and \( s_2(p) \). Since \( \pi_1(M) \) is an ICC-group, the elements \( u_1 \) and \( u_2 \) do not commute; and since the \( k \)-free group \( \pi_1(M) \) is in particular 2-free, \( u_1 \) and \( u_2 \) are independent. The conclusion now follows from the case \( m = 2 \) of 6.10. □
7. Geometry of hyperbolic 3-manifolds with 5-free fundamental group, I

**Notation 7.1.** For $\lambda > \log 5$, we set

$$f_1(\lambda) = \log \left( \frac{1 + 6Q(\lambda)}{1 - 6Q(\lambda)} \right) = \log \left( \frac{e^\lambda + 7}{e^\lambda - 5} \right).$$

(Here $Q(\lambda)$ is defined by 4.1.) Note that $f_1$ is monotone decreasing on its domain.

This section will be devoted to the proof of the following result:

**Proposition 7.2.** Let $M$ be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 5-free, and let $\lambda$ be a positive real number with $\lambda \leq \log 9$. Then at least one of the following alternatives holds:

(i) $M$ contains a point $p$ with $s_2(p) \geq \lambda$;
(ii) we have $\lambda > \log 5$ (so that $f_1(\lambda)$ is defined); and there exist a point $p_1 \in M$ with $s_1(p_1) > f_1(\lambda)$, and independent elements $u_1, u_2, u_3$ of $\pi_1(M, p_1)$, represented by loops whose respective lengths $d_1, d_2, d_3$ satisfy $Q(d_1) + Q(d_2) + Q(d_3) = 1/2 - Q(\lambda)$; or
(iii) there is a point $p \in M$ such that $Q(s_1(p)) + Q(s_2(p)) \leq 1/2 - 2Q(\lambda)$.

**Remark 7.3.** If Alternative (ii) of Proposition 7.2 holds, then for any arbitrary Margulis number $\mu$ for $M$, and any $k \geq 5$ such that $\pi_1(M)$ is $k$-free, Proposition 6.9 gives a $\mu$-thick point $p_2 \in M$ such that $\rho := \text{dist}(p_1, p_2)$ satisfies $(k - 3)Q(2\rho) + \sum_{j=1}^{3} Q(d_j) \leq 1/2$. Hence $(k - 3)Q(2\rho) + (1/2 - Q(\lambda)) \leq 1/2$, i.e. $(k - 3)Q(2\rho) \leq Q(\lambda)$. Recalling that $Q(2\rho) = 1/(1 + \exp(2\rho))$, we deduce that the point $p_2$ satisfies

$$\text{dist}(p_1, p_2) \geq \frac{1}{2} \log \left( \frac{k - 3}{Q(\lambda) - 1} \right).$$

Somewhat surprisingly, although the existence of a point $p_2$ satisfying (7.3.1) seems geometrically significant, it is not used in the rest of this paper because it turns out that it would contribute little to our final volume estimates. In fact, the part of Alternative (ii) of Proposition 7.2 involving the independent elements $u_1, u_2$ and $u_3$ is not quoted, and in particular it has no counterpart in Lemma 8.1 or Proposition 8.3, which provide the beginning of the transition between Proposition 7.2 and the volume estimates to be given later in the paper. It is possible that the existence of elements $u_1, u_2$ and $u_3$ having the properties stated in Alternative (ii) will be useful in a later paper.

Subsections 7.4—7.11 below are preparation for the proof of Proposition 7.2. The material in Subsections 7.4—7.10 is relatively elementary in that it does not involve Theorem 5.2; it is based on the material in Sections 2—4 and 6, and a number of the lemmas involve the hypothesis of compatibility (see 4.7) to make application of Proposition 4.8 possible. The crucial application of Theorem 5.2 occurs in the proof of Lemma 7.11.

**Lemma 7.4.** Let $\Gamma$ be a discrete subgroup of $\text{Isom}_+(\mathbb{H}^3)$, let $n$ be a positive integer, and let $C$ be a positive constant. Suppose that $X$ is a subset of the Cartesian power $\Gamma^n$, and let $\mathcal{R}$
denote the set of all points \( P \in \mathbb{H}^3 \) such that the inequality \( Q(x_1, P) + \cdots + Q(x_n, P) \leq C \) holds for every \((x_1, \ldots, x_n) \in X\). Then for every point \( P \in \partial \mathbb{H}^3 \), there is an element \((x_1, \ldots, x_n) \in X\) such that \( Q(x_1, P) + \cdots + Q(x_n, P) = C \).

Proof. Since \( P \in \partial \mathbb{H}^3 \), there is a sequence of points \( P^{(1)}, P^{(2)}, \ldots \) in \( \mathbb{H}^3 - \mathcal{R} \) converging to \( P \). For each \( j \geq 1 \), since \( P^{(j)} \notin \mathcal{R} \), there is an element \((x_1^{(j)}, \ldots, x_n^{(j)}) \) of \( X \) such that

\[
Q(x_1^{(j)}, P^{(j)}) + \cdots + Q(x_n^{(j)}, P^{(j)}) > C.
\]

After passing to a subsequence, we may assume that for every index \( i \in \{1, \ldots, n\} \), the sequence \( x_1^{(1)}, x_1^{(2)}, \ldots \) is bounded or tends to \( \infty \). Let \( S \subset \{1, \ldots, n\} \) denote the set of indices for which \( x_1^{(1)}, x_1^{(2)}, \ldots \) is bounded. Since \( \Gamma \) is discrete, we may assume, after again passing to a subsequence, that the sequence \( x_1^{(1)}, x_1^{(2)}, \ldots \) is constant for each \( i \in S \).

Consider an arbitrary index \( i \in S \). Since \( P^{(j)} \to P \) as \( j \to \infty \), we have

\[
d(x_1^{(1)}, P^{(j)}) \to d(x_1^{(1)}, P);
\]

hence the quantity \( Q(x_1^{(1)}, P^{(j)}) = Q(d(x_1^{(1)}, P^{(j)})) \) tends to \( Q(x_1, P) = Q(d(x_1^{(1)}, P^{(j)})) \) as \( j \to \infty \). Since the sequence \( x_1^{(1)}, x_2^{(2)}, \ldots \) is constant, we may rewrite this as

\[
Q(x_1^{(j)}, P^{(j)}) \to Q(x_1^{(1)}, P) \quad \text{as} \quad j \to \infty \quad \text{for each} \quad i \in S.
\]

On the other hand, for any index \( i \in S' := \{1, \ldots, n\} - S \), since \( x_i^{(j)} \) tends to infinity in \( \text{Isom}_+(\mathbb{H}^3) \) as \( j \to \infty \), and since the sequence \( P^{(1)}, P^{(2)}, \ldots \) is convergent and therefore bounded, we have \( d(x_i^{(1)}, P^{(j)}) \to \infty \) as \( j \to \infty \). Hence

\[
Q(x_i^{(1)}, P^{(j)}) \to 0 \quad \text{as} \quad j \to \infty \quad \text{for each} \quad i \in S'.
\]

In view of (7.4.2) and (7.4.3), we may take limits in (7.4.1) as \( j \to \infty \) to obtain

\[
\sum_{i \in S} Q(x_i^{(j)}, P) \geq C.
\]

Combining this with the inequality \( \sum_{1 \leq i \leq n} Q(x_i^{(1)}, P) \leq C \), which holds because \( P \in \mathcal{R} \), we obtain

\[
C \leq \sum_{i \in S} Q(x_i^{(1)}, P) \leq \sum_{1 \leq i \leq n} Q(x_i^{(1)}, P) \leq C,
\]

which implies that \( Q(x_1^{(1)}, P) + \cdots Q(x_n^{(1)}, P) = C \), and thus gives the conclusion of the lemma. (Incidentally, it also follows that \( S = \{1, \ldots, n\} \).)

Notation 7.5. Let \( \Gamma \) be a discrete, purely loxodromic subgroup of \( \text{Isom}_+(\mathbb{H}^3) \), and let \( \lambda \) be a positive number which is compatible (4.7) with \( \Gamma \). Let a simplex \( \sigma \) of \( K_{Z, \lambda}(\Gamma) \) be given (see 4.4). According to Proposition 4.8, \( \Theta(\sigma) \) is a closable subgroup of \( \Gamma \), so that \( c(\Theta(\sigma)) \) is a well-defined subgroup of \( \Gamma \) by 2.11 and Proposition 2.13.

We will denote by \( \mathcal{J}(\sigma) \) (or simply by \( \mathcal{J} \)) the set of all points \( P \in \mathcal{U}_\sigma \) for which both the following conditions hold:

- for every pair of non-commuting elements \( x_1, x_2 \in c(\Theta(\sigma)) \) we have

\[
Q(x_1, P) + Q(x_2, P) \leq \frac{1}{2} - 2Q(\lambda);
\]
and

- for every pair of non-commuting elements $x_1, x_2 \in \mathbf{c}(\Theta(\sigma))$, and every element $y$ of $\Gamma - \mathbf{c}(\Theta(\sigma))$, we have

$$(7.5.2) \quad Q(x_1, P) + Q(x_2, P) + Q(y, P) \leq \frac{1}{2} - Q(\lambda).$$

We will denote by $J^0_{(\Gamma, \lambda)}(\sigma)$ (or simply by $J^0(\sigma)$ when the choices of $\Gamma$ and $\lambda$ are understood) the set of all points $P \in J(\sigma)$ for which the following condition holds:

- there exist non-commuting elements $x_1, x_2 \in \mathbf{c}(\Theta(\sigma))$, and an element $y$ of $\Gamma - \mathbf{c}(\Theta(\sigma))$, for which the inequality $(7.5.2)$ is an equality.

We will denote by $L_{(\Gamma, \lambda)}(\sigma)$ (or simply by $L(\sigma)$ when the choices of $\Gamma$ and $\lambda$ are understood) the set of all points $P \in U_\sigma$ such that, for every $y \in \Gamma - \mathbf{c}(\Theta(\sigma))$, we have $d(y, P) \geq \lambda$.

We will denote by $L^0_{(\Gamma, \lambda)}(\sigma)$ (or simply by $L^0(\sigma)$ when the choices of $\Gamma$ and $\lambda$ are understood) the set of all points $P \in U_\sigma$ such that, for every $y \in \Gamma$ with $d(y, P) < \lambda$, we have $y \in C_v$ for some vertex $v$ of $\sigma$.

It should be borne in mind that $J_{(\Gamma, \lambda)}(\sigma)$, $J^0_{(\Gamma, \lambda)}(\sigma)$ and $L_{(\Gamma, \lambda)}(\sigma)$ are defined only under the hypothesis that $\lambda$ is compatible with $\Gamma$.

**Remark 7.6.** Let $\Gamma$ be a discrete, purely loxodromic subgroup of $\text{Isom}_+\left(\mathbb{H}^3\right)$, let $\lambda$ be a positive number which is compatible with $\Gamma$, and let $\sigma$ be a simplex of $K_{\mathbb{Z}_\lambda(\Gamma)}$. For each vertex $v$ of $\sigma$, the definition of $\Theta(\sigma)$ and assertion (2) of Proposition 2.13 give $C_v \leq \Theta(\sigma) \leq \mathbf{c}(\Theta(\sigma))$. It therefore follows from the definitions of $L(\sigma)$ and $L^0(\sigma)$ given in 7.5 that $L^0(\sigma) \subset L(\sigma)$.

**Lemma 7.7.** Let $\Gamma$ be a discrete, purely loxodromic subgroup of $\text{Isom}_+\left(\mathbb{H}^3\right)$, and let $\lambda$ be a positive real number which is compatible with $\Gamma$. Then for every simplex $\sigma$ of $K_{\mathbb{Z}_\lambda(\Gamma)}$, we have $\text{Fr}_{U_\sigma} J(\sigma) \subset J^0(\sigma) \cup (J(\sigma) \cap L(\sigma))$.

**Proof.** Let $R_1$ denote the set of all points $P \in \mathbb{H}^3$ such that $(7.5.1)$ holds for every pair of non-commuting elements $x_1, x_2 \in \mathbf{c}(\Theta(\sigma))$, and let $R_2$ denote the set of all points $P \in \mathbb{H}^3$ such that $(7.5.2)$ holds for every pair of non-commuting elements $x_1, x_2 \in \mathbf{c}(\Theta(\sigma))$, and every element $y$ of $\Gamma - \mathbf{c}(\Theta(\sigma))$. Then by definition we have $J(\sigma) = R_1 \cap R_2 \cap U_\sigma$. Since $U_\sigma$ is open, we have $\text{Fr}_{U_\sigma} J(\sigma) \subset \text{Fr}_{\mathbb{H}^3}(R_1 \cap R_2) \subset \left(\left(\text{Fr}_{\mathbb{H}^3} R_1\right) \cup \left(\text{Fr}_{\mathbb{H}^3} R_2\right)\right) \cap (J(\sigma) \cap L(\sigma))$. But since $J(\sigma)$ is by definition closed in the subspace topology of $U_\sigma$, we also have $\text{Fr}_{U_\sigma} J(\sigma) \subset J(\sigma)$. Hence $\text{Fr}_{U_\sigma} J(\sigma) \subset (J(\sigma) \cap \left(\text{Fr}_{\mathbb{H}^3} R_1\right)) \cup (J(\sigma) \cap \left(\text{Fr}_{\mathbb{H}^3} R_2\right))$. We will complete the proof by establishing the inclusions (1) $J(\sigma) \cap \left(\text{Fr}_{\mathbb{H}^3} R_2\right) \subset J^0(\sigma)$ and (2) $J(\sigma) \cap \left(\text{Fr}_{\mathbb{H}^3} R_1\right) \subset J(\sigma) \cap L(\sigma)$.

To this end, we first apply Lemma 7.4 taking $n = 3$, taking $C = 1/2 - Q(\lambda)$, and taking $X \subset \Gamma^3$ to be the set of all triples of the form $(x_1, x_2, y)$ where $x_1$ and $x_2$ are non-commuting elements of $\mathbf{c}(\Theta(\sigma))$, and $y$ is an element of $\Gamma - \mathbf{c}(\Theta(\sigma))$. With this choice of $X$, the set $R$ defined in the statement of Lemma 7.4 is equal to $R_2$. Hence Lemma 7.4 implies that if $P$ is any point of $\text{Fr}_{\mathbb{H}^3} R_2$, we have $Q(x_1, P) + Q(x_2, P) + Q(y, P) = 1/2 - Q(\lambda)$ for some non-commuting elements $x_1$ and $x_2$ of $\mathbf{c}(\Theta(\sigma))$ and some element $y$ of $\Gamma - \mathbf{c}(\Theta(\sigma))$. If we
assume in addition that $P \in \mathcal{J}(\sigma)$, then by the definitions given in 7.5 we have $P \in \mathcal{J}^0(\sigma)$; this establishes the inclusion (1).

Next, we apply Lemma 7.4 taking $n = 2$, taking $C = 1/2 - 2Q(\lambda)$, and taking $X \subset \Gamma^2$ to be the set of all pairs of the form $(x_1, x_2)$ where $x_1$ and $x_2$ are non-commuting elements of $c(\Theta(\sigma))$. With this choice of $X$, the set $\mathcal{R}$ defined in the statement of Lemma 7.4 is equal to $\mathcal{R}_1$. Hence if $P$ is any point of $\text{Fr}_{\mathcal{H}} \mathcal{R}_1$, Lemma 7.4 provides non-commuting elements $x_{1,0}$ and $x_{2,0}$ of $c(\Theta(\sigma))$ such that $Q(x_{1,0}, P) + Q(x_{2,0}, P) = 1/2 - 2Q(\lambda)$. Now assume in addition that $P \in \mathcal{J}(\sigma)$, and consider an arbitrary element $y$ of $\Gamma - c(\Theta(\sigma))$. Since $P \in \mathcal{J}(\sigma)$, we have $Q(x_{1,0}, P) + Q(x_{2,0}, P) + Q(y, P) \leq 1/2 - Q(\lambda)$. This inequality, together with the equality $Q(x_{1,0}, P) + Q(x_{2,0}, P) = 1/2 - 2Q(\lambda)$, gives $Q(y, P) \leq Q(\lambda)$. As the latter inequality holds for an arbitrary element $y$ of $\Gamma - c(\Theta(\sigma))$, we have $P \in \mathcal{L}(\sigma)$ according to the definition given in 7.5. This establishes the inclusion (2). \hfill \Box

**Lemma 7.8.** Let $\Gamma$ be a discrete, purely loxodromic subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and let $\lambda$ be a positive number which is compatible with $\Gamma$. Suppose that $\sigma$ is a simplex of $K_{\mathbb{Z}_3}(\Gamma)$ such that $\mathcal{J}(\sigma)$ and $\mathcal{L}(\sigma)$ are both non-empty. Then either $\mathcal{J}^0(\sigma)$ or $\mathcal{J}(\sigma) \cap \mathcal{L}(\sigma)$ is non-empty.

**Proof.** Set $\mathcal{J} = \mathcal{J}(\sigma)$ and $\mathcal{L} = \mathcal{L}(\sigma)$. Since $\emptyset \neq \mathcal{J} \subset \mathcal{U}_\sigma$, and since $\mathcal{U}_\sigma$ is convex by 4.4 and hence connected, we have either $\text{Fr}_{\mathcal{U}_\sigma} \mathcal{J} \neq \emptyset$ or $\mathcal{J} = \mathcal{U}_\sigma$. If $\text{Fr}_{\mathcal{U}_\sigma} \mathcal{J}(\sigma) \neq \emptyset$, it follows from Lemma 7.7 that either $\mathcal{J}^0(\sigma)$ or $\mathcal{J}(\sigma) \cap \mathcal{L}(\sigma)$ is non-empty. If $\mathcal{J} = \mathcal{U}_\sigma$, then since $\emptyset \neq \mathcal{L} \subset \mathcal{U}_\sigma$, we in particular have $\mathcal{J} \cap \mathcal{L} \neq \emptyset$. \hfill \Box

**Lemma 7.9.** Let $\Gamma$ be a discrete, purely loxodromic, 5-free subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and let $\lambda$ be a positive number which is compatible with $\Gamma$. Let $\sigma$ be a simplex of $K_{\mathbb{Z}_3}(\Gamma)$ with $\dim \sigma > 0$.

(a) If $P$ is a point of $\mathcal{J}(\sigma)$, then for every non-trivial element $z$ of $\Gamma$, we have $Q(z, P) < 1/2 - 3Q(\lambda)$.

(b) If $P$ is a point of $\mathcal{J}(\sigma) \cap \mathcal{L}(\sigma)$, then for any two non-commuting elements $u_1$ and $u_2$ of $\Gamma$, we have $Q(u_1, P) + Q(u_2, P) \leq 1/2 - 2Q(\lambda)$.

**Proof.** Set $\alpha = Q(\lambda)$.

We begin by considering an arbitrary point $P$ of $\mathcal{U}_\sigma$. Since $\dim \sigma > 0$, we may choose two distinct vertices $v_1, v_2$ of $\sigma$. For $i = 1, 2$, since $v_i$ is a vertex of $\sigma$, we have $\mathcal{U}_\sigma \subset Z(C_{v_i})$. In particular we have $P \in Z(C_{v_i})$; by definition this means that there is a non-trivial element $w_i$ of $C_{v_i}$ such that $d(w_i, P) < \lambda$. Hence $Q(w_i, P) > \alpha$ for $i = 1, 2$.

Note that since $v_1 \neq v_2$, and the assignment $v \mapsto C_v$ is bijective by 4.4, we have $C_{v_1} \neq C_{v_2}$. Thus $w_1$ and $w_2$ are non-trivial elements of distinct maximal cyclic subgroups of the ICC-group $\Gamma$, and therefore do not commute. Note also that by the definition of $\Theta(\sigma)$, and Assertion (2) of Proposition 2.13, we have $C_{v_i} \leq \Theta(\sigma) \leq c(\Theta(\sigma))$ for $i = 1, 2$. In particular we have $w_1, w_2 \in c(\Theta(\sigma))$.

In order to prove Assertion (a), we specialize to the situation in which $P \in \mathcal{J}(\sigma) \subset \mathcal{U}_\sigma$. Let $z$ be a non-trivial element of $\Gamma$. First consider the case in which $z \notin c(\Theta(\sigma))$. Since $w_1$ and $w_2$ are non-commuting elements of $c(\Theta(\sigma))$, and since $P \in \mathcal{J}(\sigma)$, the inequality (7.5.2) holds
with \( w_1 \) and \( w_2 \) playing the roles of \( x_1 \) and \( x_2 \), and with \( z \) playing the role of \( y \); that is, we have \( Q(w_1, P) + Q(w_2, P) + Q(z, P) \leq 1/2 - \alpha \). Since \( Q(w_i, P) > \alpha \) for \( i = 1, 2 \), it follows that \( Q(z, P) < 1/2 - 3\alpha \) in this case.

Next consider the case in which \( z \in c(\Theta(\sigma)) \). Let \( C_0 \) denote the maximal cyclic subgroup of the ICC-group \( \Gamma \) containing \( z \) (see 4.3). Since \( C_{v_1} \) and \( C_{v_2} \) are distinct maximal cyclic subgroups of \( \Gamma \), they cannot both coincide with \( C_0 \); hence we may fix an index \( t \in \{1, 2\} \) such that \( C_{v_t} \neq C_0 \). It follows (see 4.3) that the elements \( z \) and \( w_t \) of \( c(\Theta(\sigma)) \) do not commute. Since \( P \in \mathcal{F}(\sigma) \), the inequality (7.5.1) holds with \( z \) and \( w_t \) playing the roles of \( x_1 \) and \( x_2 \); that is, we have \( Q(z, P) + Q(w_t, P) \leq 1/2 - 2\alpha \). Since \( Q(w_t, P) > \alpha \), it follows that \( Q(z, P) < 1/2 - 3\alpha \) in this case as well. Thus (a) is proved.

To prove Assertion (b), we specialize to the situation in which \( P \in \mathcal{F}(\sigma) \cap \mathcal{L}(\sigma) \subset U_\sigma \). Let \( u_1 \) and \( u_2 \) be any non-commuting elements of \( \Gamma \). First consider the case in which \( u_1 \) and \( u_2 \) both lie in \( c(\Theta(\sigma)) \). Since \( P \in \mathcal{F}(\sigma) \), the inequality (7.5.1) then holds with \( u_i \) playing the role of \( x_i \) for \( i = 1, 2 \); that is, we have \( Q(u_1, P) + Q(u_2, P) \leq 1/2 - 2\alpha \), as required.

Next consider the case in which exactly one of the \( u_i \) lies in \( c(\Theta(\sigma)) \); by symmetry we may assume that \( u_1 \in c(\Theta(\sigma)) \) and that \( u_2 \notin c(\Theta(\sigma)) \). Let \( C_1 \) denote the maximal cyclic subgroup of \( \Gamma \) containing \( u_1 \). Since \( C_{v_1} \) and \( C_{v_2} \) are distinct maximal cyclic subgroups of \( \Gamma \), they cannot both coincide with \( C_1 \); hence we may fix an index \( s \in \{1, 2\} \) such that \( C_{v_s} \neq C_1 \). Since by definition we have \( w_s \in C_{v_s} \), it follows (see 4.3) that the elements \( u_1 \) and \( w_s \) of \( c(\Theta(\sigma)) \) do not commute. Since \( P \in \mathcal{F}(\sigma) \), the inequality (7.5.1) holds with \( u_1 \) and \( w_s \) playing the roles of \( x_1 \) and \( x_2 \); that is, we have \( Q(u_1, P) + Q(w_s, P) \leq 1/2 - 2\alpha \). On the other hand, since \( P \in \mathcal{L}(\sigma) \) and \( u_2 \notin c(\Theta(\sigma)) \), the definition of \( \mathcal{L}(\sigma) \) gives \( Q(u_2, P) \leq \alpha \). Since \( Q(w_s, P) > \alpha \), it follows that \( Q(u_1, P) + Q(u_2, P) \leq Q(u_1, P) + Q(w_s, P) \leq 1/2 - 2\alpha \) in this case.

There remains the case in which neither \( u_1 \) nor \( u_2 \) lies in \( c(\Theta(\sigma)) \). In this case, since \( P \in \mathcal{L}(\sigma) \), we have \( Q(u_i, P) \leq \alpha \) for \( i = 1, 2 \). Since \( Q(w_i, P) > \alpha \), it follows that \( Q(u_i, P) < Q(w_i, P) \) for \( i = 1, 2 \). But since \( w_1 \) and \( w_2 \) are non-commuting elements of \( c(\Theta(\sigma)) \), the inequality (7.5.1) holds with \( u_1 \) and \( w_2 \) playing the roles of \( x_1 \) and \( x_2 \); that is, we have \( Q(w_1, P) + Q(w_2, P) \leq 1/2 - 2\alpha \). Hence \( Q(u_1, P) + Q(u_2, P) < Q(w_1, P) + Q(w_2, P) \leq 1/2 - 2\alpha \) in this case.

\[ \Box \]

**Lemma 7.10.** Let \( \Gamma \) be a discrete, purely loxodromic subgroup of \( \text{Isom}_+(\mathbf{H}^3) \), and let \( \lambda \) be a positive number which is compatible with \( \Gamma \). Set \( M = \mathbf{H}^3/\Gamma \). Let \( \sigma \) be a simplex of \( K_{\mathbf{Z}_\lambda(\Gamma)} \).

(a) If \( \dim \sigma = 0 \) and \( \mathcal{L}^0(\sigma) \neq \emptyset \), then \( M \) contains a point \( p \) with \( s_2(p) \geq \lambda \).
(b) If \( \dim \sigma > 0 \) and \( \mathcal{F}^0(\sigma) \neq \emptyset \), then \( \lambda > \log 5 \) (so that \( f_1(\lambda) \) is defined); furthermore, there exist a point \( p_1 \in M \) with \( s_1(p_1) > f_1(\lambda) \), and independent elements \( u_1, u_2, u_3 \) of \( \pi_1(M, p_1) \), represented by loops whose respective lengths \( d_1, d_2, d_3 \) satisfy \( Q(d_1) + Q(d_2) + Q(d_3) = 1/2 - Q(\lambda) \).
(c) If \( J(\sigma) \cap L(\sigma) \neq \emptyset \), then either (i) \( \lambda > \log 3 \) (so that \( f_2(\lambda) \) is defined) and \( f_2(\lambda) \) is a Margulis number for \( M \), or (ii) there is a point \( p \in M \) such that \( Q(\mathfrak{s}_1(p)) + Q(\mathfrak{s}_2(p)) \leq 1/2 - 2Q(\lambda) \).

**Proof.** Let \( q : H^3 \to M \) denote the quotient map. Set \( \alpha = Q(\lambda) \).

To prove Assertion (a), choose a point \( P \in L^0(\sigma) \), and set \( p = q(P) \). Since \( \sigma \) is 0-dimensional, it has a unique vertex \( v_0 \). According to the definition of \( L^0(\sigma) \), for every \( y \in \Gamma \) with \( d(y, P) < \lambda \), we have \( y \in C_{v_0} \). On the other hand, the definition of \( L^0(\sigma) \) also implies that \( P \in U_\sigma \), and since \( v_0 \) is the unique vertex of \( \sigma \), it follows from (4.4.4) that \( U_\sigma = Z_\lambda(C_{v_0}) \). Hence \( P \in Z_\lambda(C_{v_0}) \), which means that for some non-trivial element \( w \) of \( C_{v_0} \) we have \( d(w, P) < \lambda \). It follows that if we set \( d_0 = \mathfrak{s}_1(p) = \min_{1 \neq y \in \Gamma} d(y, P) \) (see 6.2), then \( C_{v_0} \) contains all elements \( y \) of \( \Gamma \) with \( d(y, P) = d_0 \). It therefore follows from the definition of \( \mathfrak{s}_2(p) \) given in 6.2 that \( \mathfrak{s}_2(p) = d(z, P) \) for some \( z \in \Gamma - C_{v_0} \); hence \( \mathfrak{s}_2(p) \geq \lambda \), and (a) is proved.

To prove (b), choose a point \( P_1 \in J^0(\sigma) \), and set \( p = q(P_1) \). Since \( \dim \sigma > 0 \), and since the definition of \( J^0(\sigma) \) (7.5) implies that we have in particular \( P_1 \in J(\sigma) \), it follows from Assertion (a) of Lemma 7.9 that for every non-trivial element \( z \) of \( \Gamma \) we have \( Q(z, P_1) < 1/2 - 3\alpha \). In particular, since \( Q(z, P_1) > 0 \), we have \( 1/6 > \alpha = Q(\lambda) = 1/(1 + e^\lambda) \), so that \( \lambda > \log 5 \). Furthermore, since \( Q(z, P_1) = Q(d(z, P_1)) = 1/(1 + \exp(d(z, P_1))) \), the inequality \( Q(z, P_1) < 1/2 - 3\alpha \) gives \( d(z, P_1) > (1 + 6\alpha)/(1 - 6\alpha) = f_1(\lambda) \) for every \( z \in \Gamma - \{1\} \). By 6.2 we have \( \mathfrak{s}_1(p_1) = \min_{1 \neq z \in \Gamma} d(z, P_1) \), and hence \( \mathfrak{s}_1(p_1) > f_1(\lambda) \).

The definition of \( J^0(\sigma) \) also implies that there exist non-commuting elements \( x_1, x_2 \in c(\Theta(\sigma)) \), and an element \( y_0 \) of \( \Gamma - c(\Theta(\sigma)) \), for which the inequality (7.5.2) is an equality when we set \( y = y_0 \); that is, we have \( Q(x_1, P_1) + Q(x_2, P_1) + Q(y_0, P_1) = 1/2 - \alpha \). Since \( \Gamma \) is in particular 2-free, the non-commuting elements \( x_1, x_2 \) of \( c(\Theta(\sigma)) \) are independent. Since \( y_0 \notin c(\Theta(\sigma)) \), it follows from Proposition 2.14 that the three elements \( x_1, x_2, y_0 \) of \( \Gamma \) are independent. Hence there are independent elements \( u_1, u_2, u_3 \) of \( \pi_1(M, p_1) \), represented by loops whose respective lengths \( d_1, d_2, d_3 \) satisfy \( Q(d_1) + Q(d_2) + Q(d_3) = 1/2 - \alpha \). This completes the proof of (b).

To prove (c), we choose a point \( P \in J(\sigma) \cap L(\sigma) \). According to Assertion (b) of Lemma 7.9, the inequality \( Q(x_1, R) + Q(x_2, R) \leq 1/2 - 2\alpha \) holds for any two non-commuting elements \( x_1, x_2 \in \Gamma \). We set \( p = q(P) \).

According to the discussion given in 6.2, we may choose a short maximal cyclic subgroup \( C \) of the ICC-group \( \Gamma \), and \( C \) contains a non-trivial element \( z_1 \) such that \( d(z_1, P) = \mathfrak{s}_1(p) \); hence \( Q(z_1, P) = Q(\mathfrak{s}_1(p)) \). It also follows from the discussion in 6.2 that we may choose an element \( z_2 \) of \( \Gamma - C \) such that \( d(z_2, P) = \mathfrak{s}_2(p) \); hence \( Q(z_2, P) = Q(\mathfrak{s}_2(p)) \). Since \( C \) is a maximal cyclic subgroup of the ICC-group \( \Gamma \), and since \( z_1 \) is a non-trivial element of \( C \) and \( z_2 \in \Gamma - C \), the elements \( z_1 \) and \( z_2 \) do not commute. Hence \( Q(x_1, R) + Q(x_2, R) \leq 1/2 - 2\alpha \), i.e. \( Q(\mathfrak{s}_1(p)) + Q(\mathfrak{s}_2(p)) \leq 1/2 - 2\alpha \). This establishes (c). \( \square \)

**Lemma 7.11.** Let \( \Gamma \) be a discrete, purely loxodromic, 5-free subgroup of \( \text{Isom}_+(H^3) \), and let \( \lambda \) be a real number with \( 0 < \lambda \leq \log 9 \) (so that in particular \( \lambda \) is compatible with \( \Gamma \)). Suppose
that the manifold $\mathbb{H}^3/\Gamma$ has no $\lambda$-thick point (see 6.2). Then there is a simplex $\sigma$ of $K_{\mathcal{Z}}(\Gamma)$ such that $\mathcal{J}(\sigma)$ and $\mathcal{L}^0(\sigma)$ are both non-empty.

**Proof.** Set $M = \mathbb{H}^3/\Gamma$, and set $\mathcal{Z} = \mathcal{Z}_\lambda(\Gamma)$. By hypothesis we have $M_{\text{thick}}(\lambda) = \emptyset$; as we observed in 6.2, this is equivalent to saying that $\mathcal{X}_\lambda(\Gamma) = \mathbb{H}^3$. Since in addition $\Gamma$ is 5-free and $\lambda \leq \log 9$, we may apply Theorem 5.2, with $k = 5$, to obtain a simplex $\sigma$ of $K := K_{\mathcal{Z}}$ such that ${\text{height}}_{(5, \Gamma, \lambda)}(\sigma) \leq 2$ and $\text{link}_K \sigma$ is non-contractible. We shall complete the proof by showing that $\mathcal{J}(\sigma)$ and $\mathcal{L}^0(\sigma)$ are both non-empty.

Assume that $\mathcal{L}^0(\sigma) = \emptyset$. Consider an arbitrary point $P \in \mathcal{U}_\sigma$. Since $P \notin \mathcal{L}^0(\sigma)$, it follows from the definition of $\mathcal{L}^0(\sigma)$ (see 7.5) that there is an element $y \in \Gamma$ with $d(y, P) < \lambda$, but $y \notin C_v$ for any vertex $v$ of $\sigma$. In particular $y \neq 1$. Hence $y$ belongs to a unique maximal cyclic subgroup $C_0$ of the ICC-group $\Gamma$, and $C_0$ is not equal to $C_v$ for any vertex $v$ of $\sigma$. Since the assignment $v \mapsto C_v$ is the inverse of the bijection $C \rightarrow v_C$ (see 4.4), this means that $C_0$ is not a vertex of $\sigma$, so that in the notation of 3.2 we have $C_0 \notin \mathcal{F}_\sigma^2$. But we have $P \in Z_\lambda(C_0)$ since $d(y, P) < \lambda$; in particular $\mathcal{U}_\sigma \cap Z_\lambda(C_0) \neq \emptyset$.

It follows that, in the notation of 3.2, we have $C_0 \in \mathcal{F}_\sigma^2 = \mathcal{F}_\sigma$. Thus $P \in Z_\lambda(C_0) \subset \bigcup_{C \in \mathcal{F}_\sigma} Z_\lambda(C)$. Since $P$ was an arbitrary point of $\mathcal{U}_\sigma$, this shows that $\mathcal{U}_\sigma \subset \bigcup_{C \in \mathcal{F}_\sigma} Z_\lambda(C)$. Hence the set $\mathcal{U}_\sigma$ is equal to $\mathcal{U}_\sigma \cap \bigcup_{C \in \mathcal{F}_\sigma} Z_\lambda(C)$, which by Proposition 3.4 is homotopy-equivalent to $\text{link}_K \sigma$. But $\mathcal{U}_\sigma$ is convex (see 4.4) and hence contractible, whereas $\text{link}_K \sigma$ is non-contractible by our choice of $\sigma$. This contradiction shows that $\mathcal{L}^0(\sigma) \neq \emptyset$.

Now assume that $\mathcal{J}(\sigma) = \emptyset$. Consider an arbitrary point $P \in \mathcal{U}_\sigma$. Since $P \notin \mathcal{J}(\sigma)$, it follows from the definition of $\mathcal{J}(\sigma)$ (see 7.5) that either (i) there exist non-commuting elements $x_1, x_2 \in \mathcal{c}(\Theta(\sigma))$ such that (7.5.1) fails to hold, or (ii) there exist non-commuting elements $x_1, x_2 \in \mathcal{c}(\Theta(\sigma))$, and an element $y$ of $\Gamma - \mathcal{c}(\Theta(\sigma))$, such that (7.5.2) fails to hold. In Case (i), since (7.5.1) does not hold, we have

$$Q(x_1, P) + Q(x_2, P) > \frac{1}{2} - 2Q(\lambda).$$

(7.11.1)

In Case (ii), since (7.5.2) does not hold, we have

$$Q(x_1, P) + Q(x_2, P) + Q(y, P) > \frac{1}{2} - Q(\lambda).$$

(7.11.2)

In both cases, since $\Gamma$ is in particular 2-free, the non-commuting elements $x_1, x_2 \in \mathcal{c}(\Theta(\sigma))$ are independent. Note also that (7.11.1) is the inequality (4.9.1) with $k = 5$, $h = 3$ and $m = 2$, while (7.11.2) is the inequality (4.9.2) with the same values of $k$, $h$, and $m$. Since $P$ was an arbitrary point of $\mathcal{U}_\sigma$, it now follows from the definition of $\mathcal{F}_1(\sigma)$ given in 4.9 that $3 \in \mathcal{F}_1(\sigma)$, which according to Definition 4.12 means that $\text{height}(\sigma) \geq 3$. This contradicts our choice of $\sigma$, and we therefore have $\mathcal{J}(\sigma) \neq \emptyset$. \qed

**Proof of Proposition 7.2.** If $M$ contains a $\lambda$-thick point $p$, so that $s_1(p) \geq \lambda$, then according to 6.2 we have $s_2(p) \geq s_1(p) \geq \lambda$, so that Alternative (i) of the conclusion holds. For the rest of the proof we will assume that $M$ contains no $\lambda$-thick point. Let us write $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete and 5-free. According to Lemma 7.11 (which applies because $\Gamma$ is
5-free and \( \lambda \leq \log 9 \), we may now fix a simplex \( \sigma \) of \( K_{\mathcal{L}(\Gamma)} \) such that \( \mathcal{J}(\sigma) \) and \( \mathcal{L}^0(\sigma) \) are both non-empty. Since Remark 7.6 gives \( \mathcal{L}^0(\sigma) \subset \mathcal{L}(\sigma) \), we in particular have \( \mathcal{L}(\sigma) \neq \emptyset \).

If \( \dim \sigma = 0 \), then since \( \mathcal{L}^0(\sigma) \neq \emptyset \), it follows from Assertion (a) of Lemma 7.10 that \( M \) contains a point \( p \) with \( s_2(p) \geq \lambda \). This is Alternative (i) of the conclusion of the present proposition.

Now suppose that \( \dim \sigma > 0 \). Since \( \mathcal{J}(\sigma) \) and \( \mathcal{L}(\sigma) \) are both non-empty, it follows from Lemma 7.8 that either \( \mathcal{J}^0(\sigma) \) or \( \mathcal{J}(\sigma) \cap \mathcal{L}(\sigma) \) is non-empty. If \( \mathcal{J}^0(\sigma) \neq \emptyset \), then according to Assertion (b) of Lemma 7.10, we have \( \lambda > \log 5 \), and there exist a point \( p_1 \in M \) with \( s_1(p_1) > f_1(\lambda) \), and independent elements \( u_1, u_2, u_3 \) of \( \pi_1(M, p_1) \), represented by loops whose respective lengths \( d_1, d_2, d_3 \) satisfy \( Q(d_1) + Q(d_2) + Q(d_3) = 1/2 - Q(\lambda) \). This is Alternative (ii) of the present proposition. If \( \mathcal{J}(\sigma) \cap \mathcal{L}(\sigma) \neq \emptyset \), then by Assertion (c) of Lemma 7.10, there is a point \( p \in M \) such that \( Q(s_1(p)) + Q(s_2(p)) \leq 1/2 - 2Q(\lambda) \). This is Alternative (iii) of the present proposition. \( \Box \)

8. Geometry of hyperbolic 3-manifolds with 5-free fundamental group, II

In this brief section we prove a result, Proposition 8.3, which is an almost formal consequence of Proposition 7.2, but is stated in a convenient form for applications to volume estimates in the later sections.

Recall that in Subsection 6.5, the quantity \( \lambda_M \) was defined for any closed, orientable, hyperbolic 3-manifold \( M \).

**Lemma 8.1.** Let \( M \) be a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is 5-free. Then at least one of the following alternatives holds:

(i) \( \lambda_M \geq \log 9 \); or  
(ii) we have \( \lambda_M > \log 5 \) (so that \( f_1(\lambda_M) \) is defined), and there is a point \( p_1 \in M \) such that \( s_1(p_1) \geq f_1(\lambda_M) \); or  
(iii) there is a point \( p \in M \) such that \( Q(s_1(p)) + Q(s_2(p)) \leq 1/2 - 2Q(\lambda_M) \).

**Proof.** We will assume that Alternative (i) does not hold, and show that one of the alternatives (ii), (iii) must hold. Since (i) does not hold, we have \( \lambda_M < \log 9 \), and we may therefore choose a sequence \( (\lambda^{(i)})_{i \geq 1} \) of real numbers such that \( \lambda_M < \lambda^{(i)} \leq \log 9 \) for every \( i \), and \( \lambda^{(i)} \to \lambda_M \). For each \( i \) we apply Proposition 7.2, taking \( \lambda = \lambda^{(i)} \) and \( \mu = \mu_M \). The hypotheses of Proposition 7.2 hold because \( \pi_1(M) \) is 5-free and \( \lambda^{(i)} \leq \log 9 \). Hence, for each \( i \), one of the alternatives (i)—(iii) of Proposition 7.2 must hold with \( \lambda = \lambda^{(i)} \). If, for some \( i \), Alternative (i) of the conclusion of Proposition 7.2 holds with \( \lambda = \lambda^{(i)} \), there is a point \( p \) of \( M \) with \( s_2(p) \geq s_2(\lambda^{(i)}) > \lambda_M \); this contradicts the definition of \( \lambda_M \). Hence, for each \( i \), one of the alternatives (ii), (iii) of Proposition 7.2 must hold with \( \lambda = \lambda^{(i)} \). After passing to a subsequence we may assume that there is a single one of the alternatives (ii), (iii) of Proposition 7.2 which holds, with \( \lambda = \lambda^{(i)} \), for each \( i \geq 1 \). If Alternative (ii) of Proposition 7.2 holds for each \( i \), then \( \lambda_M \geq \log 5 \), and for each \( i \) the manifold \( M \) contains a point \( p_1^{(i)} \) such that \( s_1(p_1^{(i)}) > f_1(\lambda^{(i)}) \). Since \( M \) is compact, we may assume after passing to
a subsequence that the sequence \((p_1^{(i)})_{i \geq 1}\) converges to a point \(p_1 \in M\). If \(\lambda_M = \log 5\) then \(f_1(\lambda^{(i)}) \to \infty\) and hence \(s_1(p^{(i)}) \to \infty\), a contradiction since \(s_1\) is continuous by Lemma 6.4. Hence \(\lambda_M > \log 5\). Again using the continuity of \(s_1\), we find that \(s_1(p_1) \geq f_1(\lambda_M)\). This gives Alternative (ii) of the present lemma. If Alternative (iii) of Proposition 7.2 holds for each \(i\), we have a sequence \((p^{(i)})_{i \geq 1}\) of points in \(M\) such that \(Q(s_1(p^{(i)})) + Q(s_2(p^{(i)})) \leq 1/2 - 2Q(\lambda^{(i)})\) for each \(i\). We may assume after passing to a subsequence that \((p^{(i)})\) converges to a point \(p \in M\). Since \(s_1\) and \(s_2\) are continuous by Lemma 6.4, and \(Q\) is continuous, we have \(Q(s_1(p)) + Q(s_2(p)) \leq 1/2 - 2Q(\lambda_M)\). This gives Alternative (iii) of the present lemma. □

**Notation 8.2.** For \(\lambda > \log 3\), we set

\[
f_2(\lambda) = \log \left( \frac{3 + 4Q(\lambda)}{1 - 4Q(\lambda)} \right).
\]

**Proposition 8.3.** Let \(M\) be a closed, orientable hyperbolic 3-manifold such that \(\pi_1(M)\) is 5-free. Let \(\lambda^+\) be a real number with \(\log 3 < \lambda^+ \leq \log 9\). Then at least one of the following alternatives holds:

(i) there is a point \(p \in M\) with \(s_2(p) = \lambda^+\);
(ii) there is a point \(p_0 \in M\) with \(\log 5 < s_2(p_0) \leq \lambda^+\) and \(\max_{p \in M} s_1(p) \geq f_1(s_2(p_0))\);
(iii) \(\min(\lambda^+, f_2(\lambda^+))\) is a Margulis number for \(M\); or
(iv) there is a point \(p_0 \in M\) such that \(Q(s_1(p_0)) + Q(s_2(p_0)) = 1/2 - 2Q(\lambda^+)\) and \(s_2(p_0) < \lambda^+\).

**Proof.** According to Proposition 6.7, the range of the function \(s_2\) is the interval \([\mu_M, \lambda_M]\). Hence if \(\mu_M \leq \lambda^+ \leq \lambda_M\), then Alternative (i) of the present lemma holds. If \(\lambda^+ < \mu_M\), then in particular we have \(\min(\lambda^+, f_2(\lambda^+)) < \mu_M\), which by Proposition 6.7 implies that \(\min(\lambda^+, f_2(\lambda^+))\) is a Margulis number for \(M\); this is Alternative (iv) of the present lemma. For the rest of the proof we will assume that \(\lambda^+ > \lambda_M\).

The hypotheses of the present lemma imply those of Lemma 8.1. Hence one of the alternatives of the conclusion of Lemma 8.1 must hold. Alternative (i) of Lemma 8.1 would give \(\lambda_M \geq \log 9 \geq \lambda^+\), a contradiction. Hence one of the alternatives (ii) or (iii) of Lemma 8.1 must hold.

Suppose that Alternative (ii) of Lemma 8.1 holds (so that in particular \(\lambda_M > \log 5\)), and choose \(p_1 \in M\) such that \(s_1(p_1) \geq f_1(\lambda_M)\). We now have \(\log 5 < \lambda_M < \lambda^+\). But we have \(\max_{p \in M} s_1(p) \geq s_1(p_1) \geq f_1(\lambda_M) = f_1(s_2(p_0))\). Thus Alternative (ii) of the present lemma holds.

Now suppose that Alternative (iii) of Lemma 8.1 holds. This means that the set

\[
X = \{p \in M : Q(s_1(p)) + Q(s_2(p)) \leq 1/2 - 2Q(\lambda_M)\}
\]

is non-empty. Since \(M\) is connected, either \(\text{Fr}_M X \neq \emptyset\), or \(X = M\). If \(\text{Fr}_M X \neq \emptyset\), we select a point \(p_0 \in \text{Fr}_M X\). Since \(s_1\) and \(s_2\) are continuous on \(M\) by Lemma 6.4, and \(Q\) is continuous on the positive real line, we have \(Q(s_1(p_0)) + Q(s_2(p_0)) = 1/2 - 2Q(\lambda_M)\). Furthermore, since \(\lambda^+ > \lambda_M = \max_{p \in M} s_2(p)\), we have \(s_2(p_0) < \lambda^+\); thus Alternative (v) of the present lemma holds. Finally, if \(X = M\), we have \(Q(s_1(p)) + Q(s_2(p)) \leq 1/2 - 2Q(\lambda_M)\).
for every \( p \in M \). Since \( Q \) is monotone decreasing, and since \( s_1(p) \leq s_2(p) \) for every \( p \in M \) by 6.2, we have \( Q(s_2(p)) \leq 1/4 - Q(\lambda_M) \) for every \( p \in M \); since \( \lambda^+ > \lambda_M \), we have \( Q(s_2(p)) \leq 1/4 - Q(\lambda^+) \) But the definitions of \( Q \) and \( f_2 \) imply that \( 1/4 - Q(\lambda^+) = Q(f_2(\lambda^+)) \); hence \( s_2(p) \geq f_2(\lambda^+) \geq \min(\lambda^+, f_2(\lambda^+)) \) for every \( p \in M \). It then follows from Proposition 6.7 that \( \min(\lambda^+, f_2(\lambda^+)) \) is a Margulis number for \( M \), which is Alternative (iv) of the present proposition. \( \square \)

9. AN OBSERVATION ABOUT HYPERBOLIC TRIANGLES

The lemma established in this brief section will be needed in the proof of Lemma 10.8 to give an upper bound for an angle of a hyperbolic triangle, given certain constraints on the sides of the triangle.

**Notation and Remarks 9.1.** We define a function \( \omega \) on \((0, \infty)^3\) by

\[
\omega(x, y, z) = \frac{(\cosh x)(\cosh y) - \cosh z}{(\sinh x)(\sinh y)}.
\]

We denote by \( \mathcal{M} \) the subset of \( \mathbb{R}^3 \) consisting of all points \((x, y, z)\) such that \( x, y \) and \( z \) are all positive, and satisfy \( x + y \geq z \), \( x + z \geq y \) and \( y + z \geq x \). For any \((x, y, z) \in \mathcal{M} \) we have \(|\omega(x, y, z)| \leq 1\), and we define a function \( \Omega \) on \( \mathcal{M} \) by setting \( \Omega(x, y, z) = \arccos(\omega(x, y, z)) \in [0, \pi] \) whenever \((x, y, z) \in \mathcal{M} \).

If a hyperbolic triangle has sides of lengths \( x, y \) and \( z \), then we have \((x, y, z) \in \mathcal{M} \), and according to the hyperbolic law of cosines, \( \Omega(x, y, z) \) is the angle between the sides of lengths \( x \) and \( y \).

We shall extend \( \Omega \) to a function \( \overline{\Omega} \) on \((0, \infty)^3\) by setting

\[
\overline{\Omega}(x, y, z) = \arccos(\min(\max(\omega(x, y, z), -1), 1)) \in [0, \pi]
\]

for all \( x, y, z > 0 \).

Note that if \( x > 0 \) and \( y > z > 0 \), the definition of \( \omega \) implies that \( \omega(x, y, z) > 0 \). Hence:

\[(9.1.1) \quad \overline{\Omega}(x, y, z) < \frac{\pi}{2} \text{ whenever } x > 0 \text{ and } y > z > 0.\]

We define a function \( \theta \) on \((0, \infty)^2\) by setting

\[(9.1.2) \quad \theta(C, x) = \overline{\Omega}\left(\text{arccosh}\left(\frac{\cosh x}{\cosh C}\right), x, C\right)\]

if \( x > C \), and \( \theta(C, x) = \pi/2 \) if \( x \leq C \). (It is easily checked that \( \theta \) is continuous, although this fact will not be used.)

We observe that if \( x > C \), it follows from (9.1.1) that the right-hand side of (9.1.2) is less than \( \pi/2 \). Hence for all positive numbers \( x \) and \( C \), we have \( 0 \leq \theta(C, x) \leq \pi/2 \).

If \( C \) is a positive number, and if \( a_{1,0}, a_{1,1}, a_{2,0} \) and \( a_{2,1} \) are numbers such that \( 0 < a_{1,0} \leq a_{1,1} \) and \( 0 < a_{2,0} \leq a_{2,1} \), we set

\[
A_1(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) = \max_{(i,j) \in \{0,1\} \times \{0,1\}} \overline{\Omega}(a_{1,i}, a_{2,j}, C),
\]
\[ A_2(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) = \max_{(m,i) \in \{1,2\} \times \{0,1\}} \theta(C, a_{m,i}), \]

and
\[ A(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) = \max(A_1(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}), A_2(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1})). \]

**Lemma 9.2.** Suppose that for \( m = 1, 2 \), we are given positive numbers \( a_{m,0}, a_{m,1} \) with \( a_{m,0} \leq a_{m,1} \). Let a positive number \( C \) be given, and suppose that \( (x_1^{(0)}, x_2^{(0)}) \) is a point of \([a_{1,0}, a_{1,1}] \times [a_{2,0}, a_{2,1}]\) such that \( (x_1^{(0)}, x_2^{(0)}, C) \in \mathcal{M} \). Then we have \( \Omega(x_1, x_2, C) \leq A(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) \).

**Proof.** Set \( R = [a_{1,0}, a_{1,1}] \times [a_{2,0}, a_{2,1}] \). The function \( f \) defined on \((0, \infty)^2 \supset R\) by \( f(x_1, x_2) = \omega(x_1, x_2, C) \) is continuous and therefore takes a least value on \( R \); we choose a point \((\eta_1, \eta_2) \in R\) where this least value is achieved.

Since \((x_1^{(0)}, x_2^{(0)}, C) \in \mathcal{M}\), we have \( f(x_1^{(0)}, x_2^{(0)}) = \omega(x_1^{(0)}, x_2^{(0)}, C) \leq 1 \); since \((x_1^{(0)}, x_2^{(0)}) \in R\), it follows that \( \omega(\eta_1, \eta_2) \leq 1 \). This means that \( \omega(\eta_1, \eta_2, C) \leq 1 \), which with the definition of \( \overline{\Omega} \) gives \( \overline{\Omega}(\eta_1, \eta_2, C) = \arccos(\max(\omega(\eta_1, \eta_2, C), -1)), \) i.e.
\[
(9.2.1) \quad \overline{\Omega}(\eta_1, \eta_2, C) = \arccos(\max(f(\eta_1, \eta_2), -1)).
\]

On the other hand, since \((x_1^{(0)}, x_2^{(0)}, C) \in \mathcal{M}\), we have \( \omega(x_1^{(0)}, x_2^{(0)}, C) \geq -1 \); and since \((x_1^{(0)}, x_2^{(0)}) \in R\), we have \( \omega(x_1^{(0)}, x_2^{(0)}, C) \geq f(\eta_1, \eta_2) \). Hence \( \omega(x_1^{(0)}, x_2^{(0)}, C) \geq \max(f(\eta_1, \eta_2), -1) \).

With (9.2.1) and the definition of \( \Omega \), this gives
\[
(9.2.2) \quad \Omega(x_1^{(0)}, x_2^{(0)}, C) \leq \overline{\Omega}(\eta_1, \eta_2, C).
\]

We have \( f(x_1, x_2) = (\coth x_1)(\coth x_2) - (\cosh C)(\cosech x_1)(\cosech x_2) \) for all \( x_1, x_2 > 0 \). Hence the partial derivative of \( f \) with respect to the first variable is
\[ f_1'(x_1, x_2) = -(\cosech^2 x_1)(\coth x_2) + (\cosh C)(\cosech x_1)(\coth x_2)(\cosech x_2), \]

which vanishes precisely when \( \cosh x_2 = (\cosh C)(\cosh x_1) \). Likewise, we have \( f_2'(x_1, x_2) = 0 \) precisely when \( \cosh x_1 = (\cosh C)(\cosh x_2) \). It follows that the two partial derivatives can vanish simultaneously only if \( \cosh C = 1 \), which is not the case since \( C > 0 \). Hence \( f \) has no critical points in \( R \), and \((\eta_1, \eta_2)\) must be a boundary point of \( R \).

If \((\eta_1, \eta_2)\) is a corner point of the rectangle \( R \), we have \((\eta_1, \eta_2) = (a_{1,i}, a_{2,j})\) for some \((i,j) \in \{0,1\} \times \{0,1\}\). Using (9.2.2) and the definitions of \( A_1(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) \) and \( A(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) \), we find that
\[
\Omega(x_1^{(0)}, x_2^{(0)}, C) \leq \overline{\Omega}(\eta_1, \eta_2, C) = \overline{\Omega}(a_{1,i}, a_{2,j}, C) \leq A_1(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) \leq A(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}).
\]

There remains the case in which \((\eta_1, \eta_2)\) is an interior point of a side of \( R \). In this case we will show that \( \overline{\Omega}(\eta_1, \eta_2, C) \leq A_2 \leq A \), which with (9.2.2) implies the conclusion. By the symmetry in the definitions of \( f \), \( \Omega \) and \( A_2 \), we may assume that the side containing \((\eta_1, \eta_2)\) has the form \([a_{1,0}, a_{1,1}] \times (a_{2,i})\) for some \( i \in \{0,1\} \). Thus we have \( \eta_2 = a_{2,i} \) and \( f_1'(\eta_1, a_{2,i}) = 0 \), so that \( \cosh a_{2,i} = \cosh \eta_2 = (\cosh C)(\cosh \eta_1) \). Thus we have \( \cosh a_{2,i} > \cosh C \), and
\[ \eta_1 = \text{arccosh}((\cosh a_2, i)/(\cosh C)). \] According to the definition of the function \( \theta \), it follows that in this case we have
\[ \overline{\Omega}(\eta_1, \eta_2, C) = \theta(C, a_2, i) \leq A_2 \]
as required. \( \square \)

10. **Elementary facts about volumes**

This section, and to some extent the next two, concern ways of using quantitative geometric information about a closed, orientable hyperbolic 3-manifold \( M \) to obtain lower bounds for the volumes of certain subsets of \( M \).

In Sections 13, these methods will be used to make the transition from Proposition 8.3 to a sufficient condition for a number to be a lower bound for \( \text{vol} \, M \) when \( \pi_1(M) \) is \( k \)-free for a given \( k \geq 5 \); this is in turn used, in Section 14, to give an explicit lower volume bound when \( \pi_1(M) \) is 5-free.

In Subsections 10.1–10.9, we obtain lower bounds for the volume of \( \text{nbhd}_{a_2}(p)(p) \), where \( p \) is a point of \( M \); these lower bounds depend on \( s_2(p) \) and on the minimal length of a loop based at \( p \) representing the generator of a short max cyclic subgroup of \( \pi_1(M, p) \) (see 6.2). Lemma 10.12 is similar in nature, but deal more specifically with the case in which \( p \) lies on a suitably short closed geodesic. In Subsection 10.14 we obtain lower bounds for the volume of a certain metric neighborhood of a point \( p \in M \) in terms of \( s_1(p) \), given that there is a point of \( M \) which is suitably distant from \( p \). Subsections 10.15 and 10.16 are devoted to establishing a lower bound for the complement of a certain metric neighborhood of a point \( p \) of \( M \), given that there is a \( \mu \)-thick point—where \( \mu \) is a Margulis number for \( M \)—which is suitably distant from \( p \). In Subsections 10.17–10.19, some of these results are combined in ways that will prove useful in Section 13.

A significant part of this section consists of review of material from [17], but much of this material will be reorganized here.

**Notation 10.1.** We define a strictly increasing function \( B(x) = \pi(\sinh(2x) - 2x) \) for \( x > 0 \). Geometrically, \( B(x) \) gives the volume of a ball in \( \mathbb{H}^3 \) of radius \( x \).

10.2. We shall review some material from Subsections 6.1 and 6.5 of [17]. If \( N \) is an open ball in \( \mathbb{H}^3 \), then for each point \( \zeta \) in the sphere \( \partial \overline{N} \), we will denote by \( \eta_{N, \zeta} \) the ray originating at the center of \( N \) and passing through \( \zeta \). For each point \( \zeta \in \partial \overline{N} \) and each number \( w \geq 0 \) we shall denote by \( \Pi_N(\zeta, w) \) the plane which meets \( \eta_{N, \zeta} \) perpendicularly at a distance \( w \) from the center of \( N \), and by \( H_N(\zeta, w) \) the closed half-space which is bounded by \( \Pi_N(\zeta, w) \) and has unbounded intersection with \( \eta_{N, \zeta} \). For any given \( w > 0 \) and for an arbitrary point \( \zeta \in \partial \overline{N} \), we denote by \( K_N(\zeta, w) \) the “cap”: \( N \cap H(\zeta, w) \). When the choice of the ball \( N \) is understood, we will write \( \eta_K, \Pi(\zeta, w), H(\zeta, w) \) and \( K(\zeta, w) \) in place of \( \eta_{N, \zeta}, \Pi_N(\zeta, w), H_N(\zeta, w) \) and \( K_N(\zeta, w) \) respectively.

Now let \( R \) be a positive real number, and let \( N \) denote an open ball of radius \( R \) in \( \mathbb{H}^3 \). We set \( \kappa(R, w) = \text{vol} \, K(\zeta, w) \) for an arbitrary point \( \zeta \in \partial \overline{N} \). We endow \( \partial \overline{N} \) with the spherical
metric in which the distance between two points $\zeta, \zeta' \in S$ is the angle between $\eta_{\zeta}$ and $\eta_{\zeta'}$, and we denote by $\iota(R, w, w', \alpha)$ and $\sigma(R, w, w', \alpha)$ the respective volumes of $K(\zeta, w) \cap K(\zeta', w')$ and $K(\zeta, w) \cup K(\zeta', w')$, where $w, w' > 0$ and $\alpha \in [0, \pi]$ are given, and $\zeta$ and $\zeta'$ are points in $S$ whose spherical distance is $\alpha$. Then $\kappa$ is a well-defined function of two positive real variables, monotone increasing in its first argument and monotone decreasing in its second; while $\iota$ and $\sigma$ are well-defined functions of three positive real variables and a fourth variable whose values are restricted to $[0, \pi]$. (If $w \geq R > 0$, then $\kappa(R, w)$ vanishes, as does $\iota(R, w, w', \alpha)$ for any $w' > 0$ and any $\alpha \in [0, \pi]$.)

Note that we have

$$\sigma(R, w, w', \alpha) = \kappa(R, w) + \kappa(R, w') - \iota(R, w, w', \alpha) \tag{10.2.1}$$

for any positive numbers $R, w, w'$ and any $\alpha \in [0, \pi]$. Analytic expressions for the functions $\kappa$ and $\iota$ (and hence for $\sigma$) are given in [17, Section 14].

According to [17, Proposition 6.7], $\sigma$ is monotone decreasing in its third argument and monotone increasing in its fourth argument. Furthermore, the first paragraph of the proof of [17, Proposition 6.7], with union signs replaced by intersection signs, shows that $\iota$ is monotone decreasing in its third argument. Since $\sigma$ and $\iota$ are symmetric in their second and third arguments (cf. [17, 6.5]), they are also monotone decreasing in their second argument. In view of (10.2.1), the fact that $\sigma$ is monotone increasing in its fourth argument may be interpreted as meaning that $\iota$ is monotone decreasing in its fourth argument.

Here is the first of several results that we will extract from Sections 6 and 7 of [17] and apply later in this paper:

**Proposition 10.3.** Let $M$ be a closed, orientable hyperbolic 3-manifold, and write $M = H^3/\Gamma$ where $\Gamma \leq \text{Isom}(H^3)$ is discrete and torsion-free. Let $\alpha$ be a positive number, and let $p$ be a point of $M$ with $s_2(p) \geq \alpha$. Let $P$ be a point of $H^3$ which projects to $p$ under the quotient map, let $j : \pi_1(M, p) \to \Gamma$ denote the isomorphism determined by the compatible base points $P \in H^3$ and $p \in M$, let $x$ denote the image under $j$ of a generator of a short maximal cyclic subgroup (6.2) of $\pi_1(M, p)$. Let $N \subset H^3$ denote the open ball of radius $\alpha/2$ centered at $P$. For each integer $n \neq 0$, set $d_n = d(x^n, P)$ (in the notation of 4.1). Let $\zeta_n$ denote the point of intersection of $\partial N$ with the ray originating at $P$ and passing through $x^n \cdot P$. Then in the notation of 10.2 we have

$$\text{vol nbhd}_{\alpha/2}(p) = B(\alpha/2) - \text{vol} \left( \bigcup_{0 \neq n \in \mathbb{Z}} K\left(\zeta_n, \frac{d_n}{2}\right) \right). \tag{10.3.1}$$

**Proof.** First consider the case in which $\pi_1(M, p)$ has only one short maximal cyclic subgroup. As we pointed out in 6.3, this says, in the notation of [17], that $p \in \mathcal{S}_M$, and we then have $s_2(p) = s_2(p) \geq \alpha$, where $s_M$ is defined as in [17]; furthermore, the short maximal cyclic subgroup of $\pi_1(M, p)$ is denoted $C_p$ in the notation of [17]. In this case, the statement of the present proposition is precisely that of [17, Proposition 6.2], with $p$, $P$ and $\alpha$ playing the respective roles of $P$, $\tilde{P}$ and $\lambda$ in the latter result.
In the case where $\pi_1(M, p)$ has more than one short maximal cyclic subgroup, we have by definition (see 6.2) that $\mathfrak{s}_2(p) = \mathfrak{s}_1(p)$; hence $\mathfrak{s}_1(p) \geq \alpha$, so that $p$ is the center of a hyperbolic ball of radius $\alpha/2$. Hence $\text{vol nbhd}_{\alpha/2}(p) = B(\alpha/2)$. But in this case, for each integer $n \neq 0$, the definition of $\mathfrak{s}_1(p)$ gives $d_n \geq \mathfrak{s}_1(p) \geq \alpha$ so that $K(\zeta_n, d_n/2) = \emptyset$. Hence the left-hand side of (10.3.1) is also equal to $B(\alpha/2)$.

**Corollary 10.4.** Let $M$ be a closed, orientable hyperbolic 3-manifold, let $\alpha$ be a positive number, and let $p$ be a point of $M$ with $\mathfrak{s}_2(p) \geq \alpha$. Suppose that $C$ is a short maximal cyclic subgroup of $\pi_1(M, p)$, and let $l$ denote the length of a closed geodesic in $M$ representing the conjugacy class in $\pi_1(M)$ of a generator of $C$. Then

$$\text{vol nbhd}_{\alpha/2}(p) \geq B(\alpha/2) - 2\left\lfloor \frac{n}{l} \right\rfloor \cdot \kappa\left(\frac{\alpha}{2}, \frac{\mathfrak{s}_1(p)}{2}\right).$$

**Proof.** Let $P$ be a point of $H^3$ which projects to $p$ under the quotient map, let $j : \pi_1(M, p) \to \Gamma$ denote the isomorphism determined by the compatible base points $P \in H^3$ and $p \in M$, and let $x$ denote the image under $j$ of a generator of $C$. Let $N$ denote the ball of radius $\alpha/2$ centered at $P$, and for each integer $n \neq 0$ define the quantity $d_n$, and the point $\zeta_n \in \partial N$, as in Proposition 10.3. For each $n \in \mathbb{Z}$ we have $d_n \geq |n|l$; hence $K(\zeta_n, d_n/2) = \emptyset$ for $|n| > \lfloor \alpha/l \rfloor$. We therefore have

$$\text{vol}\left(\bigcup_{0 \neq n \in \mathbb{Z}} K\left(\zeta_n, \frac{d_n}{2}\right)\right) = \text{vol}\left(\bigcup_{0 < |n| \leq \lfloor \alpha/l \rfloor} K\left(\zeta_n, \frac{d_n}{2}\right)\right) \leq \sum_{0 < |n| \leq \lfloor \alpha/l \rfloor} \text{vol} K\left(\zeta_n, \frac{d_n}{2}\right) = \sum_{0 < |n| \leq \lfloor \alpha/l \rfloor} \kappa\left(\frac{\alpha}{2}, \frac{d_n}{2}\right).$$

But the definition of $\mathfrak{s}_1(p)$ implies that $d_n \geq \mathfrak{s}_1(p)$ for each $n$, and since $\kappa$ is monotone decreasing in its second argument, we have $\kappa(\alpha/2, d_n/2) \leq \kappa(\alpha/2, \mathfrak{s}_1(p)/2)$ for each $n$. Hence

$$\text{vol}\left(\bigcup_{0 \neq n \in \mathbb{Z}} K\left(\zeta_n, \frac{d_n}{2}\right)\right) \leq 2\left\lfloor \frac{n}{l} \right\rfloor \cdot \kappa\left(\frac{\alpha}{2}, \frac{\mathfrak{s}_1(p)}{2}\right),$$

which, with the equality in the conclusion of Proposition 10.3, gives the conclusion of the present corollary. \hfill \Box

**10.5.** We shall review the definitions and basic properties of some more functions that are introduced in [17].

As in [17, Subsection 7.1], we define, for each integer $n \geq 1$, a function $\Phi_n$ on the domain $\{(\delta, D) : 0 < \delta \leq D\} \subset \mathbb{R}^2$ by

$$\Phi_n(\delta, D) = \arccosh\left(\cosh(n\delta) + \frac{(\cosh(n\delta) - 1)(\cosh D - \cosh \delta)}{\cosh \delta + 1}\right).$$

Note that $\Phi_n$ is monotonically increasing in its second argument.

The significance of the function $\Phi_n$ arises from Lemma 7.3 of [17], which asserts (in the notation of 4.1 above) that if $P$ is a point of $H^3$, and $x$ is a loxodromic isometry of $H^3$ whose
translation length is bounded below by a given positive number $\delta$ then for every positive integer $n$ we have

$$d(x^n, P) \geq \Phi_n(\delta, d(x, P)).$$

(The statement of Lemma 7.3 given in [17] unfortunately contained a typographical error: the inequality (10.5.1) was reversed. The form given here matches both the proof of Lemma 7.3 given in [17], and the applications given there.)

We will also use Lemma 7.4 of [17], which asserts that if $n$ is a positive integer, and if $\delta$ and $D$ are real numbers with $0 < \delta \leq D$, then

$$n\delta \leq \Phi_n(\delta, D) \leq nD.$$  

10.6. As in [17], Subsection 7.5, we define a function $\Psi$ on the domain $\{(x, y) : 0 < y \leq 2x\} \subset \mathbb{R}^2$, with values in $[0, \pi/2]$, by

$$\Psi(x, y) = \arccos((\coth x)(\coth y - \cosech y)).$$

Thus in the notation of 9.1, we have $\Psi(x, y) = \Omega(y, x, x)$. Hence if an isosceles hyperbolic triangle has base $y$ and has its other two sides equal to $x$, we have $y \leq 2x$ (i.e. $(y, x, x) \in \mathbb{R}$ in the notation of 9.1), and the base angles of the triangle are equal to $\Psi(x, y)$.

It is pointed out in [17], Subsection 7.5] that $\Psi$ is monotone increasing in its first argument and monotone decreasing in its second.

As in [17], Subsection 6.3], we denote by $\Theta$ the real-valued function with domain $\{(w, R) : 0 < w < R\} \subset \mathbb{R}^2$ which is defined by

$$\Theta(w, R) = \arccos\left(\frac{\tanh w}{\tanh R}\right)$$

and takes values in $(0, \pi/2)$. Note that $\Theta$ is monotonically decreasing in its first argument and monotonically increasing in its second argument.

**Notation 10.7.** We will need one function that is not defined in [17]. Recall from 9.1 that $A(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1})$ is defined whenever $0 \leq a_{1,0} \leq a_{1,1}$ and $0 \leq a_{2,0} \leq a_{2,1}$. If $D$ and $\delta$ are numbers with $0 \leq \delta \leq D$, then for $n = 2, 3$ we have $n\delta \leq \Phi_n(\delta, D)$ by (10.5.2). Hence we may define a function $\Lambda$ on $\{(\delta, D) : 0 \leq \delta \leq D\} \subset \mathbb{R}^2$ by

$$\Lambda(\delta, D) = A(D, 2\delta, \Phi_2(\delta, D), 3\delta, \Phi_3(\delta, D)).$$

**Lemma 10.8.** Let $p$ be a point of a closed, orientable hyperbolic 3-manifold $M$, and let $C$ be a short maximal cyclic subgroup (see 6.2) of $\pi_1(M, p)$. Let $\delta$ and $\alpha$ be constants with $0 < \alpha < 4\delta$. Assume that

- $s_2(p) \geq \alpha$, and that
- the conjugacy class of a generator of $C$ is represented by a closed geodesic in $M$ having length at least $\delta$. 


Let $D$ denote the minimal length of a loop based at $p$ that represents a generator of $C \leq \pi_1(M, p)$. Then $\delta \leq D$, so that $T_n := \Phi_n(\delta, D)$ is defined for every $n \geq 1$. Furthermore, we have

$$\text{vol} \text{nbhd}_{\alpha/2}(p) \geq B \left( \frac{\alpha}{2} \right) - 2\sigma \left( \frac{\alpha}{2}, D, T_2, \Psi(D, T_2) \right) - 2\kappa \left( \frac{\alpha}{2}, T_3, \frac{\sigma(D, T_2)}{\psi(D, T_2)} \right) + 2t \left( \frac{\alpha}{2}, T_2, T_3, \Lambda(\delta, D) \right),$$

(where $\Psi(D, T_2)$ is defined because $T_2 \leq 2D$ by 10.5.1). If in addition we have $D < T_3 < \alpha$, so that, in particular, the quantities $\Theta(D/2, \alpha/2)$ and $\Theta(T_2, \alpha/2)$ are defined (see 10.5), and if

$$\cos \left( \Theta \left( \frac{D}{2}, \frac{\alpha}{2} \right) - \Theta \left( \frac{T_3}{2}, \frac{\alpha}{2} \right) \right) < \frac{\cosh D \cosh T_3 - \cosh 2D}{\sinh D \sinh T_3},$$

then

$$\text{vol} \text{nbhd}_{\alpha/2}(p) \geq B \left( \frac{\alpha}{2} \right) - 2\sigma \left( \frac{\alpha}{2}, D, T_2, \Psi(D, T_2) \right).$$

**Remark 10.8.1.** The quantity that is denoted $D$ in the statement of Lemma 10.8 is not necessarily the same as $s_1(p)$. Whereas $D$ is the minimal length of a loop based at $p$ that represents a generator $x$ of $C \leq \pi_1(M, p)$, the definitions in 6.2 imply that $s_1(p)$ is the minimal length of a loop based at $p$ that represents any non-trivial power of $x$. It is a standard observation that these need not be equal: for example, if we write $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$, if $P$ is a point of $\mathbb{H}^3$ which projects to $p$ under the quotient map, if $j : \pi_1(M, p) \to \Gamma$ denotes the isomorphism determined by the compatible base points $P \in \mathbb{H}^3$ and $p \in M$, and if $j(x)$ is a loxodromic isometry having very small translation length and twist angle very close to $\pi$, then the minimal length of a loop representing $x^2$ will be less than the minimal length of a loop representing $x$.

Because $D$ and $s_1(p)$ may be distinct, one cannot rule out the possibility that $D > s_2(p)$. (This is related to the issue addressed in Lemma 14.5 below.)

**Proof of Lemma 10.8.** Let $P$ be a point of $\mathbb{H}^3$ which projects to $p$ under the quotient map, let $j : \pi_1(M, p) \to \Gamma$ denote the isomorphism determined by the compatible base points $P \in \mathbb{H}^3$ and $p \in M$, and let $x$ denote the image under $j$ of a generator of $C$. Let $N$ denote the ball of radius $\lambda/2$ centered at $P$, and for each integer $n \neq 0$ define the quantity $d_n$, and the point $\zeta_n \in \partial N$, as in Proposition 10.3. Note that for any $n \neq 0$ we have $d_{-n} = d_n$. The hypothesis of the present lemma implies that $x$ has translation length at least $\delta$, so that in particular $D \geq \delta$, and thus each $T_n$ is defined. According to (10.5.1) we have

$$d_n = d_{|n|} \geq T_{|n|}$$

for each $n \neq 0$. Let us set $K_n = K(\zeta_n, d_n/2)$ for each integer $n \neq 0$. For $0 \neq n \in \mathbb{Z}$ we have $d_n \geq |n|\delta$; since $\lambda < 4\delta$, it follows that

$$K_n = \emptyset \text{ for } |n| > 3.$$

Let us set $L_n = K(\zeta_n, T_{|n|}/2)$ for each $n \neq 0$. It follows from (10.8.1) that

$$K_n \subset L_n \text{ for each } n \neq 0.$$
Combining Proposition 10.3 with (10.8.2) and (10.8.3), we obtain

\[
\text{vol nbhd}_{\lambda/2}(p) = B\left(\frac{\lambda}{2}\right) - \text{vol}\left(\bigcup_{0 \neq n \in \mathbb{Z}} K_n\right)
\]
\[
= B\left(\frac{\lambda}{2}\right) - \text{vol}\left(\bigcup_{0 < |n| \leq 3} K_n\right)
\]
(10.8.4)
\[
\geq B\left(\frac{\lambda}{2}\right) - \text{vol}\left(\bigcup_{0 < |n| \leq 3} L_n\right)
\]
\[
\geq B\left(\frac{\lambda}{2}\right) - (\text{vol}(L_1 \cup L_2 \cup L_3) + \text{vol}(L_{-1} \cup L_{-2} \cup L_{-3})).
\]

Let \(\varepsilon \in \{-1, 1\}\) be given. Consider the triangle with vertices \(P, x^\varepsilon \cdot P\) and \(x^{2\varepsilon} \cdot P\). The side joining \(P\) to \(x^{2\varepsilon} \cdot P\) has length \(d_2\), and each of the other sides has length \(D\). It therefore follows from 10.6 that the angle of the triangle at \(P\) is \(\Psi(D, d_2)\). Hence \(\Psi(D, d_2)\) is the spherical distance between \(\zeta_\varepsilon\) and \(\zeta_{2\varepsilon}\). The definition of the function \(\sigma\) given in 10.2 then implies that \(\text{vol}(L_\varepsilon \cup L_{2\varepsilon}) = \sigma(\lambda/2, D, T_2/2, \Psi(D, d_2))\). But we have \(d_2 \geq T_2\) by (10.8.1), and since the function \(\Psi\) is monotone decreasing in its second argument by 10.6, we have \(\Psi(D, d_2) \leq \Psi(D, T_2)\). Since, according to [17, Proposition 6.7] (cf. 10.2), the function \(\sigma\) is monotone increasing in its fourth argument, we deduce that

\[
\text{vol}(L_\varepsilon \cup L_{2\varepsilon}) \leq \sigma\left(\frac{\lambda}{2}, \frac{D}{2}, \frac{T_2}{2}, \Psi(D, T_2)\right).
\]
(10.8.5)

By definition we have \(\text{vol}L_{3\varepsilon} = \kappa(\lambda/2, T_3/2)\), so that

\[
\text{vol}(L_\varepsilon \cup L_{2\varepsilon} \cup L_{3\varepsilon}) = \text{vol}(L_\varepsilon \cup L_{2\varepsilon}) + \kappa\left(\frac{\lambda}{2}, \frac{T_3}{2}\right) - \text{vol}(L_\varepsilon \cup L_{2\varepsilon}) \cap L_{3\varepsilon}).
\]
(10.8.6)

Now consider the triangle with vertices \(P, x^{3\varepsilon} \cdot P\) and \(x^{2\varepsilon} \cdot P\); the sides opposite these three vertices have respective lengths \(D, d_2\) and \(d_3\). It therefore follows from 9.1 that we have \((d_2, d_3, D) \in \mathfrak{M}\), and that the angle of the triangle at \(P\) is \(\Omega(d_2, d_3, D)\). Hence \(\Omega(d_2, d_3, D)\) is the spherical distance between \(\zeta_{2\varepsilon}\) and \(\zeta_{3\varepsilon}\). In view of the definition of \(\iota\) (see 10.2), it follows that

\[
\text{vol}(L_\varepsilon \cup L_{2\varepsilon} \cap L_{3\varepsilon}) \geq \text{vol}(L_{2\varepsilon} \cap L_{3\varepsilon}) = \iota\left(\frac{\lambda}{2}, \frac{T_2}{2}, \frac{T_3}{2}, \Omega(d_2, d_3, D)\right).
\]
(10.8.7)

From (10.8.5), (10.8.6) and (10.8.7) it follows that

\[
\text{vol}(L_\varepsilon \cup L_{2\varepsilon} \cup L_{3\varepsilon}) \leq \sigma\left(\frac{\lambda}{2}, \frac{D}{2}, \frac{T_2}{2}, \Psi(D, T_2)\right) + \kappa\left(\frac{\lambda}{2}, \frac{T_3}{2}\right) - \iota\left(\frac{\lambda}{2}, \frac{T_2}{2}, \frac{T_3}{2}, \Omega(d_2, d_3, D)\right).
\]
(10.8.8)

Now by 10.5.2 we have \(n\delta \leq d_n \leq T_n\) for \(n = 2, 3\). We may therefore apply Lemma 9.2, letting \(D, 2\delta, T_2, 3\delta\) and \(T_3\) play the respective roles of \(C, a_{1,0}, a_{1,1}, a_{2,0}\) and \(a_{2,1}\), and taking \(x_1^{(0)} = d_2\) and \(x_2^{(0)} = d_3\), to deduce that \(\Omega(d_2, d_3, D) \leq A(C, 2\delta, T_2, 3\delta, T_3) = A(C, 2\delta, \Phi_2(\delta, D), 3\delta, \Phi_3(\delta, D))\). In view of the definition given in 10.7, this means that
\[ \Omega(d_2, d_3, D) \leq \Lambda(\delta, D). \] But \( \iota \) is monotone decreasing in its fourth argument; indeed, it was pointed out in 10.2 that this is included in [17, Proposition 6.7]. It now follows that \( \iota(\lambda/2, T_2/2, T_3/2, \Omega(d_2, d_3, D)) \geq \iota(\lambda/2, T_2/2, T_3/2, \Lambda(\delta, D)) \). Combining this inequality with (10.8.8), we deduce that

\[
\text{vol}(L_2 \cup L_2' \cup L_3) \leq \sigma \left( \frac{\lambda}{2}, \frac{D}{2} \right) + \kappa \left( \frac{T_2}{2}, \frac{T_3}{2} \right) + \iota \left( \frac{\lambda}{2}, \frac{T_2}{2}, \frac{T_3}{2}, \Lambda(\delta, D) \right).
\]

We may now combine (10.8.4) with the cases \( \varepsilon = 1 \) and \( \varepsilon = -1 \) of (10.8.9) to obtain

\[
\text{vol}_{\text{nbhd}}(\alpha/2)(p) \geq B \left( \frac{\lambda}{2} \right) - 2 \left( \sigma \left( \frac{\lambda}{2}, \frac{D}{2}, \frac{T_2}{2}, \Psi(D, T_2) \right) + \kappa \left( \frac{T_2}{2}, \frac{T_3}{2} \right) - \iota \left( \frac{\lambda}{2}, \frac{T_2}{2}, \frac{T_3}{2}, \Lambda(\delta, D) \right) \right),
\]

which proves the first assertion of the lemma.

To prove the second assertion, we first consider the case in which \( \pi_1(M, p) \) has only one short maximal cyclic subgroup. As we pointed out in 6.3, this says, in the notation of [17], that \( p \in G_M \), and we then have \( s_M(p) = s_2(p) \geq \alpha \), where \( s_M \) is defined as in [17]; furthermore, the short maximal cyclic subgroup of \( \pi_1(M, p) \) is denoted \( C_p \) in the notation of [17]. Furthermore, in this case the quantity denoted by \( D \) in the statement of the present proposition is the minimal length of a loop based at \( p \) that represents a generator of the unique maximal cyclic subgroup of \( \pi_1(M, p) \); in the notation of [17], this quantity is denoted by \( D_M(p) \). The statement of the second assertion of the present proposition is then seen to be precisely that of the second assertion of [17, Lemma 7.6], with \( p \) and \( \alpha \) playing the respective roles of \( P \) and \( \Lambda \) in the latter result.

In the case where \( \pi_1(M, p) \) has more than one short maximal cyclic subgroup, we have by definition (see 6.2) that \( s_2(p) = s_1(p) \); hence \( s_1(p) \geq \alpha \), so that \( p \) is the center of a hyperbolic ball of radius \( \alpha/2 \). Hence \( \text{vol}_{\text{nbhd}}(\alpha/2)(p) = B(\alpha/2) \). This is enough to imply the second assertion in this case, and the proof of the lemma is thus complete.

(\text{It may be noted that the argument that has been used to deduce the second assertion of the present lemma from the second assertion of [17, Lemma 7.6] would also allow us to deduce a weaker version of the first assertion of the present lemma from the first assertion of [17, Lemma 7.6]. The stronger version of the first assertion of the present lemma was made possible by the use of the material in Section 9 above.})

\textbf{Reformulation 10.9.} For applications of Lemma 10.8, it will be convenient to define subsets \( \mathfrak{W}, \mathfrak{W}' \) and \( \mathfrak{W}'' \) of \( \mathbb{R}^3 \), and functions \( W \) and \( V_{\text{st}}^\text{near} \) with domain \( \mathfrak{W} \), as follows. We set \( \mathfrak{W} = \{ (\alpha, \delta, D) \in \mathbb{R}^3 : 0 < \delta \leq D \text{ and } \alpha > 0 \} \). Note that if \( (\alpha, \delta, D) \in \mathfrak{W} \), then \( \Phi_n(\delta, D) \) is defined for every \( n \geq 1 \). Furthermore, we have \( \Phi(\delta, D) \leq 2D \) by 10.5.2, so that \( \Psi(D, \Phi(\delta, D)) \) is defined. For each \( (\alpha, \delta, D) \in \mathfrak{W} \) we set

\[
W(\alpha, \delta, D) = B(\alpha/2) - 2\sigma(\alpha/2, D/2, \Phi(\delta, D)/2, \Psi(D, \Phi(\delta, D))).
\]

Next we define \( \mathfrak{W}' = \{ (\alpha, \delta, D) \in \mathfrak{W} : D < \Phi_3(\delta, D) < \alpha \} \), and observe that by 10.6, the quantities \( \Theta(D/2, \alpha/2) \) and \( \Theta(\Phi_3(\delta, D)/2, \alpha/2) \) are defined for every \( (\alpha, \delta, D) \in \mathfrak{W}' \). We
define $\mathcal{W}'$ to be the set of all $(\alpha, \delta, D) \in \mathcal{W}$ such that
\begin{equation}
\qquad \qquad \cos \left( \frac{\alpha}{2}, \frac{\alpha}{2} \right) - \Theta\left( \frac{\Phi_3(\delta, D)}{2}, \frac{\Phi_3(\delta, D)}{2} \right) \leq \cosh D \cosh \Phi_3(\delta, D) - \cosh 2D \frac{\sinh \Phi_3(\delta, D)}{\sinh D}.
\end{equation}

We then define the function $V_{\text{ST}}^\text{near}$ on $\mathcal{W}$ by setting
\[ V_{\text{ST}}^\text{near}(\alpha, \delta, D) = W(\alpha, \delta, D) \]
if $(\alpha, \delta, D) \in \mathcal{W}'$, and
\[ V_{\text{ST}}^\text{near}(\alpha, \delta, D) = W(\alpha, \delta, D) - 2\kappa\left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D)}{2} \right) + 2\ell\left( \frac{\alpha}{2}, \frac{\Phi_2(\delta, D)}{2}, \frac{\Phi_3(\delta, D)}{2}, \Lambda(\delta, D) \right) \]
if $(\alpha, \delta, D) \in \mathcal{W} - \mathcal{W}'$.

In terms of these definitions, we may reformulate Lemma 10.8 as follows. Let $p$ be a point of a closed, orientable hyperbolic 3-manifold $M$, and let $C$ be a short maximal cyclic subgroup of $\pi_1(M, p)$. Let $\delta$ and $\alpha$ be constants with $0 < \alpha < 4\delta$. Assume that $s_2(p) \geq \alpha$, and that the conjugacy class of a generator of $C$ is represented by a closed geodesic in $M$ having length at least $\delta$. Let $D$ denote the minimal length of a loop based at $p$ that represents a generator of $C$. Then $(\alpha, \delta, D) \in \mathcal{W}$, and $\text{vol nbhd}_{\alpha/2}(p) \geq V_{\text{ST}}^\text{near}(\alpha, \delta, D)$. (The superscript “near” and the subscript “ST” are meant to indicate that $V_{\text{ST}}^\text{near}$ gives a lower bound for the volume of a suitable neighborhood of a point which is “$\alpha$-semi-thick” in the sense mentioned in 6.3.)

**Remark 10.10.** For any $(\alpha, \delta, D) \in \mathcal{W}$, it follows from the geometric definitions of $\kappa$ and $\ell$ (see 10.2) that
\[ \kappa\left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D)}{2} \right) \geq \ell\left( \frac{\alpha}{2}, \frac{\Phi_2(\delta, D)}{2}, \frac{\Phi_3(\delta, D)}{2}, \Lambda(\delta, D) \right). \]

In view of the definition of $V_{\text{ST}}^\text{near}$, it then follows that for any $(\alpha, \delta, D) \in \mathcal{W}$ we have
\begin{equation}
V_{\text{ST}}^\text{near}(\alpha, \delta, D) \geq W(\alpha, \delta, D) - 2\kappa\left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D)}{2} \right) + 2\ell\left( \frac{\alpha}{2}, \frac{\Phi_2(\delta, D)}{2}, \frac{\Phi_3(\delta, D)}{2}, \Lambda(\delta, D) \right).
\end{equation}

We shall extract one more result, Lemma 10.12 below, from [17, Section 7].

**Notation 10.11.** We define a function $V_{SG}^\text{near}$ on the quadrant $(0, \infty)^2 \subset \mathbb{R}^2$ by $V_{SG}^\text{near}(\lambda, l) = B(\lambda/2) - 2\kappa(\lambda/2, l/2)$.

(The superscript “near” and the subscript “SG” are meant to indicate that $V_{SG}^\text{near}$ gives a lower bound for the volume of a suitable neighborhood of a point which lies on a short geodesic in a hyperbolic manifold; this is the content of Lemma 10.12 below.)

**Lemma 10.12.** Let $M$ be a closed, orientable hyperbolic 3-manifold, and let $\mu$ be a Margulis number for $M$. Suppose that $c$ is a closed geodesic in $M$ of length $l < \mu$, and let $p$ be any point of $c$. Then $s_1(p) = l$, and
\[ \text{vol nbhd}_{s_2(p)/2}(p) = V_{SG}^\text{near}(s_2(p), l). \]
Proof. Since $c$ is a closed geodesic of length $l$, there is an element $g$ of $\pi_1(M,p)$, generating a maximal cyclic subgroup $C$, such that for every non-zero integer $n$, the shortest loop based at $p$ and representing $g^n$ has length $nl$. For any $h \in \pi_1(M) - C$, the elements $g$ and $h$ do not commute; since $l < \mu$, and $\mu$ is a Margulis number, it follows that a loop representing $h$ must have length at least $\mu > l$. This shows that $s_1(p) = l$, which is the first assertion of the lemma.

According to [17, Proposition 13.1], we have $p \in C_M$ and

\begin{equation}
\text{vol nbhd}_{s_2(p)/2}(p) = B(s_2(p)/2) - 2\kappa(s_2(p)/2, l/2),
\end{equation}

where $C_M$ is the set, and $s_M$ the function with domain $C_M$, whose definitions were reviewed in 6.3. We observed in 6.3 that $s_M$ is simply the restriction of $s_2$ to $C_M$. Thus (10.12.1) becomes $\text{vol nbhd}_{s_2(p)/2}(p) = B(s_2(p)/2) - 2\kappa(s_2(p)/2, l/2)$, which by the definition of $V_{SG}^{\text{near}}$ is equivalent to the second assertion of the present lemma. \hfill $\square$

The following result will be needed in Section 14.

Lemma 10.13 (cf. [12]). The function $V_{SG}^{\text{near}}$ is monotonically increasing (in the weak sense) in each of its arguments.

Proof. Suppose that numbers $\lambda_1$, $\lambda_2$ and $l$ are given, with $l > 0$ and $0 < \lambda_1 \leq \lambda_2$. Fix a line $L \subset H^3$ and a point $P \in L$, and let $Y$ denote the closed connected subset of $H^3$ bounded by the two planes that are perpendicular to $L$ and have distance $l/2$ from $P$. Then for $i = 1, 2$, if $N_i$ denotes the ball of radius $\lambda_i/2$ centered at $P$, we have $\text{vol}(Y \cap N_i) = V_{SG}^{\text{near}}(\lambda_i, l)$. Since $\lambda_1 \leq \lambda_2$, we have $N_1 \subset N_2$ and therefore $Y \cap N_1 \subset Y \cap N_2$; hence $V_{SG}^{\text{near}}(\lambda_1, l) \leq V_{SG}^{\text{near}}(\lambda_2, l)$. This shows that $V_{SG}^{\text{near}}$ is monotonically increasing in its first argument. It is monotonically increasing in its second argument because $\kappa$ monotonically decreasing in its second argument (see 10.2). \hfill $\square$

Notation, Review and Remarks 10.14. As in [9] and [17], for any $n \geq 2$ and any $R > 0$ we shall denote by $h_n(R)$ the distance from the barycenter to a vertex of a regular hyperbolic $n$-simplex $G_{n,R}$ with sides of length $2R$. For any $n \geq 2$ the function $h_n(R)$ is strictly monotone increasing. Formulae for $h_2(R)$ and $h_3(R)$ are given in [17, Subsection 9.1].

For $R > 0$ we define a function $\text{density}(R)$ (denoted $d_3(R)$ in [9] and $d(R)$ in [17]) by

\[\text{density}(R) = (3\beta(R) - \pi)(\sinh((2R) - 2R)/\tau(r)),\]

where the functions

\[\beta(R) = \arccsc(\text{sech}(2R) + 2) \quad \text{and} \quad \tau(R) = 3 \int_{\beta(R)}^{\arccsc3} \text{arcsech}((\sec t) - 2) \, dt\]

respectively give the dihedral angle and the volume of $G_{n,R}$.

Suppose that $p$ is a point of a hyperbolic 3-manifold $M$ and that $R$ is a positive number. It is pointed out in [17, Subsection 9.2], as a consequence of the results on sphere packing proved in [9], that and if there is a radius-$R$ hyperbolic ball in $M$ with center $p$ (see 6.1)
then vol nbhd\(_{h_3(R)}(p) \geq B(R)/\text{density}(R)\). It is also observed in [17, Subsection 9.4] that we have \(B(h_3(R)) \geq B(R)/\text{density}(R)\) for any \(R > 0\).

A stronger version of the lower bound for vol nbhd\(_{h_3(R)}(p)\) mentioned above is established in [17]. For \(R > 0\) and \(\rho > h_3(R)\), one defines

\[
\phi(R, \rho) = \arcsin\left(\frac{\sqrt{\cosh^2 \rho - \cosh^2 R}}{\sinh \rho \cosh R}\right) - \arcsin\left(\frac{\sqrt{\cosh^2 h_3(R) - \cosh^2 R}}{\sinh h_3(R) \cosh R}\right),
\]

and

\[
V_{\text{B" or}}(R, \rho) = \left(1 - \cos \phi(R, \rho)\right) B(h_3(R)) + \left(1 + \cos \phi(R, \rho)\right) \frac{B(R)}{d(R)}.
\]

Note that since \(B(h_3(R)) \geq B(R)/\text{density}(R)\), we have \(V_{\text{B" or}}(R, \rho) \geq B(R)/\text{density}(R)\).

According to [17, Remark 9.6], the function \(V_{\text{B" or}}\) is monotone increasing in its second argument. Furthermore, [17, Proposition 9.7] asserts that if \(p\) is a point of a hyperbolic 3-manifold \(M\), if \(R\) and \(\rho\) satisfy \(\rho > h_3(R)\), if \(p\) is the center of a hyperbolic ball of radius \(R\), and if there is a point of \(M\) whose distance from \(p\) is \(\rho\), then vol nbhd\(_{h_3(R)}(p)\) \(\geq V_{\text{B" or}}(R, \rho)\).

**Notation and Remark 10.15.** We define a function \(V^{\text{far}}\) on \((0, \infty)^3\) by

\[
V^{\text{far}}(\rho, R, \mu) = \begin{cases} 
B(\max(0, \rho - R)) & \text{if } \rho - R < \mu/2 \\
B(\mu/2) & \text{if } \mu/2 \leq \rho - R < h_3(\mu/2) \\
V_{\text{B" or}}(\mu/2, \rho) + B(\min(\mu/2, (\rho - R - h_3(\mu/2))/2)) & \text{if } h_3(\mu/2) \leq \rho - R < 3h_3(\mu/2) \\
V_{\text{B" or}}(\mu/2, \rho) + B(\mu/2)/\text{density}(\mu/2) & \text{if } \rho - R \geq 3h_3(\mu/2).
\end{cases}
\]

Note that \(V^{\text{far}}\) is monotone decreasing (in the weak sense) in its second argument, since we have \(V_{\text{B" or}}(\mu/2, \rho) \geq B(\mu/2)/\text{density}(\mu/2) \geq B(\mu/2)\) whenever \(\rho > \mu/2\) (see 10.14). Furthermore, the latter fact, together with the fact that \(V_{\text{B" or}}\) is monotone increasing in its second argument by [17, Remark 9.6] (cf. 10.14), implies that \(V^{\text{far}}\) is (weakly) monotone increasing in its first argument.

The notation \(V^{\text{far}}\) is meant to suggest that this function provides a lower bound for the volume of the complement of a suitable neighborhood of a point in a hyperbolic manifold under certain hypotheses. More precisely, we have the following result, the proof of which will incorporate ideas from the proofs of Lemmas 8.3 and 11.3 of [17].

**Lemma 10.16.** Let \(M\) be a hyperbolic 3-manifold, let \(\mu\) be a Margulis number for \(M\), let \(\rho\) and \(R\) be positive numbers. Let \(p_0 \in M\) and \(p_1 \in M_{\text{thick}}(\mu)\) be points such that \(\text{dist}(p_0, p_1) \geq \rho\). Suppose that \(R \geq \delta_2(p_0)/2\). Then

\[
\text{vol}(M - \text{nbhd}_R(p_0)) \geq V^{\text{far}}(\rho, R, \mu).
\]

**Proof.** We set \(\rho_1 = \text{dist}(p_0, p_1) \geq \rho\).

If \(\rho \leq R\) we have \(V^{\text{far}}(\rho, R, \mu) = 0\), and the result is therefore trivial in this case. For the rest of the proof we will assume \(\rho > R\).
Set \( s = \min(\rho - R, h_3(\mu/2)) > 0 \). Since \( s + R \leq \rho \leq \rho_1 = \text{dist}(p_0, p_1) \), it follows from the triangle inequality (see 6.1) that \( \text{nbhd}_s(p_1) \cap \text{nbhd}_R(p_0) = \emptyset \), i.e.

\[
(10.16.1) \quad M - \text{nbhd}_R(p_0) \supset \text{nbhd}_s(p_1)
\]

The hypothesis \( p_1 \in M_{\text{thick}}(\mu) \) means that \( s_1(p_1) \geq \mu \). Hence there is a hyperbolic ball in \( M \) having radius \( \mu/2 \) and center \( p_1 \) (see 6.2). In particular, if we set \( s' = \min(\rho - R, \mu/2) > 0 \), so that \( 0 < s' \leq s \), then there is a hyperbolic ball in \( M \) having radius \( s' \) and center \( p_1 \). Hence \( \text{vol}\text{nbhd}_s(p_1) \geq \text{vol}\text{nbhd}_{s'}(p_1) = B(s') = B(\min(\rho - R, \mu/2)) \). But it follows from (10.16.1) that \( \text{vol}(M - \text{nbhd}_R(p_0)) \geq \text{vol}\text{nbhd}_s(p_1) \), and therefore

\[
(10.16.2) \quad \text{vol}(M - \text{nbhd}_R(p_0)) \geq B(\min(\rho - R, \mu/2)).
\]

In the case where \( \rho - R < h_3(\mu/2) \), the left-hand side of (10.16.2) is equal to \( V^{\text{fat}}(\rho, R, \mu) \), and hence the conclusion of the lemma is true in this case.

The remainder of the proof will be devoted to the case in which \( \rho - R \geq h_3(\mu/2) \). In this case we have \( s = h_3(\mu/2) \). We set \( u = (\rho - s - R)/2 \geq 0 \) and \( \lambda = s_2(p_0) \). By hypothesis we have \( R \geq \lambda/2 \).

According to 6.2, we may choose a short maximal cyclic subgroup \( C \) of \( \pi_1(M, p_0) \); furthermore, some non-trivial element of \( C \) is represented by a loop \( \alpha \) of length \( s_1(p_0) \), while some element of \( \pi_1(M, p_0) - C \) is represented by a loop \( \beta \) of length \( \lambda \). Since \( \pi_1(M) \) is an ICC-group, the elements \( [\alpha] \) and \( [\beta] \) of \( \pi_1(M, p_0) \) do not commute. If we set \( K = \alpha([0, 1]) \cup \beta([0, 1]) \), it follows that the image of the inclusion homomorphism \( \pi_1(K) \to \pi_1(M) \) is non-abelian. But by Proposition 6.8, each component of \( M_{\text{thick}}(\mu) := M - M_{\text{thick}}(\mu) \) has an abelian fundamental group. Hence \( K \not\subset M_{\text{thick}}(\mu) \). We may therefore choose a point \( p_2 \in K \cap M_{\text{thick}}(\mu) \).

Let us define a continuous function \( F \) on \( M \) by \( F(p) = \text{dist}(p_0, p) \). Since the respective lengths of \( \alpha \) and \( \beta \) are \( s_1(p_0) \leq \lambda \) and \( \lambda \), we have \( F(K) \subset [0, \lambda/2] \). In particular \( F(p_2) \leq \lambda/2 \leq R < \rho \leq \rho_1 \). On the other hand, the definition of \( \rho_1 \) gives \( F(p_1) = \rho_1 \). Since \( p_1 \) and \( p_2 \) both lie in the set \( M_{\text{thick}}(\mu) \), which is connected by Proposition 6.8, we have \( F(M_{\text{thick}}(\mu)) \supset [F(p_2), p_1] \supset [R, \rho_1] \). Since the definition of \( u \) implies that \( R \leq R + u < \rho \leq \rho_1 \), there is a point \( p^* \in M_{\text{thick}}(\mu) \) such that \( \text{dist}(p_0, p^*) = F(p^*) = R + u \). It then follows from the triangle inequality (see 6.1) that \( \text{nbhd}_R(p_0) \cap \text{nbhd}_u(p^*) = \emptyset \), i.e.

\[
(10.16.3) \quad M - \text{nbhd}_R(p_0) \supset \text{nbhd}_u(p^*).
\]

On the other hand, we have \( \text{dist}(p_1, p^*) \geq \text{dist}(p_0, p_1) - \text{dist}(p_0, p^*) = \rho_1 - (R + u) \geq \rho - (R + u) \), and the definition of \( u \) implies that \( \rho - (R + u) = u + s \). Thus \( \text{dist}(p_1, p^*) \geq u + s \), and the triangle inequality gives

\[
(10.16.4) \quad \text{nbhd}_s(p_1) \cap \text{nbhd}_u(p^*) = \emptyset.
\]

From (10.16.1), (10.16.3) and (10.16.4), it follows that

\[
(10.16.5) \quad \text{vol}(M - \text{nbhd}_R(p_0)) \geq \text{vol}(\text{nbhd}_s(p_1)) + \text{vol}(\text{nbhd}_u(p^*)).
\]

We have observed that \( p_1 \) is the center of a hyperbolic ball in \( M \) having radius \( \mu/2 \). Since the point \( p_0 \) lies at a distance \( \rho_1 \) from \( p_1 \), and since \( s = h_3(\mu/2) \) in the present case, it follows
from [17, Proposition 9.7], which was reviewed in 10.14, that \( \text{vol nbhd}_s(p_1) \geq V_{\text{B"or}}(\mu/2, \rho_1) \); as \( V_{\text{B"or}} \) is monotone increasing in its second argument by [17, Remark 9.6], which was also reviewed in 10.14, we obtain
\[
(10.16.6) \quad \text{vol nbhd}_s(p_1) \geq V_{\text{B"or}}(\mu/2, \rho).
\]
Since \( p^* \in M_{\text{thick}}(\mu) \), we have \( s_1(p^*) \geq \mu \), so that \( p^* \) is the center of a hyperbolic ball of radius \( \mu/2 \); if we set \( u' = \min(u, \mu/2) > 0 \), it follows that \( p^* \) is the center of a hyperbolic ball of radius \( u' \), and hence
\[
(10.16.7) \quad \text{vol nbhd}_{u'}(p^*) \geq \text{vol nbhd}_{u'}(p^*) = B(u').
\]
From (10.16.5), (10.16.6) and (10.16.7), we obtain
\[
(10.16.8) \quad \text{vol}(M - \text{nbhd}_R(p_0)) \geq V_{\text{B"or}}(\mu/2, \rho) + B(u').
\]
In the subcase where \( h_3(\mu/2) \leq \rho - R < 3h_3(\mu/2) \), the right-hand side of (10.16.8) is equal to 
\( V_{\text{B"or}}(\mu/2, \rho) + B(\min(\rho - h_3(\mu/2) - R)/2, \mu/2)) = V_{\text{far}}(\rho, R, \mu) \) according to the definitions, and the conclusion of the lemma follows in this subcase.

Finally, consider the subcase in which \( \rho - R \geq 3h_3(\mu/2) = 3s \). We then have \( u = (\rho - s - R)/2 \geq s \). Hence \( \text{nbhd}_{u}(p^*) \supset \text{nbhd}_{s}(p^*) \). But since \( p^* \) is the center of a hyperbolic ball of radius \( \mu/2 \), it follows from the discussion in 10.14 that \( \text{vol nbhd}_{s}(p^*) = \text{vol nbhd}_{h_3(\mu/2)}(p^*) \geq B(\mu/2)/\text{density}(\mu/2) \). Hence
\[
(10.16.9) \quad \text{vol nbhd}_{u}(p^*) \geq B(\mu/2)/\text{density}(\mu/2).
\]
In this subcase, (10.16.5), (10.16.6) and (10.16.9) give
\[
\text{vol}(M - \text{nbhd}_R(p_0)) \geq V_{\text{B"or}}(\mu/2, \rho) + B(\mu/2)/\text{density}(\mu/2) = V_{\text{far}}(\rho, R, \mu),
\]
where the last equality is simply the definition of \( V_{\text{far}}(\rho, R, \mu) \) in the subcase \( \rho - R \geq 3h_3(\mu/2) \).

\begin{lemma}
Let \( k > 2 \) be an integer, and let \( M \) be a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is \( k \)-free. Let \( \mu \) be a Margulis number for \( M \). Let \( p \) be a point of \( M \), set \( \lambda = s_2(p) \), and let \( \alpha \) be a number with \( 0 < \alpha \leq \lambda \). Let \( \delta \) be a number with \( \delta > \alpha/4 \), and suppose that every closed geodesic in \( M \) has length strictly greater than \( \delta \). Then there is a number \( D \), with \( D \geq \delta \), such that \( Q(\alpha) + Q(D) < 1/2 \) (so that \( \xi_{k-2}(Q(\lambda) + Q(D)) \) is defined), and
\[
\text{vol } M \geq V_{\text{ST}}^\text{near}(\alpha, \delta, D) + V_{\text{far}}(\xi_{k-2}(Q(\lambda) + Q(D)), \lambda/2, \mu).
\]
\end{lemma}

\textbf{Proof.} By 6.2 we may fix a short maximal cyclic subgroup \( C \) of \( \pi_1(M, p) \). We choose a generator \( g \) of \( C \), and we define \( D \) to be the minimal length of a loop based at \( p \) that represents \( g \). The hypotheses of the present proposition imply that \( \alpha < 4\delta \), and that the conjugacy class of \( g \) is represented by a closed geodesic in \( M \) having length greater than \( \delta \). Hence it follows from 10.9 that \( (\alpha, \delta, D) \in \mathfrak{W} \) (i.e. \( D \geq \delta \)), and that \( \text{vol nbhd}_{\alpha/2}(p) \geq V_{\text{ST}}^\text{near}(\alpha, \delta, D) \).

Next, note that according to 6.11 we have \( Q(\lambda) + Q(s_1(p)) < 1/2 \) (so that \( \xi_{k-2}(Q(\lambda) + Q(s_1(p))) \) is defined), and there is a \( \mu \)-thick point \( p' \in M \) such that \( \text{dist}(p, p') \geq \xi_{k-2}(Q(\lambda) + Q(s_1(p))) \).
The definition of $s_1(p)$ implies that $D \geq s_1(p)$; since $Q$ is strictly monotone decreasing, and $\xi_{k-2}$ is strictly monotone increasing on $(0,1/2)$, it follows that $Q(\lambda) + Q(D) < 1/2$ (so that $\xi_{k-2}(Q(\lambda) + Q(D))$ is defined), and $\text{dist}(p,p') \geq \xi_{k-2}(Q(\lambda) + Q(D))$. Since $\lambda = s_2(p)$ and $p' \in M_{\text{thick}}(\mu)$, we may apply \ref{lem:vol-bound} with $p$, $p'$, $\xi_{k-2}(Q(\lambda) + Q(D))$ and $\lambda/2$ playing the respective roles of $p_0$, $p_1$, $\rho$ and $R$, to deduce that $\text{vol}(M - \text{nbhd}_{\lambda/2}(p)) \geq V^{\text{far}}(\xi_{k-2}(Q(\lambda) + Q(D)), \lambda/2, \mu)$. Since $\lambda \geq \alpha$, it now follows that

$$\text{vol } M = \text{vol } \text{nbhd}_{\lambda/2}(p) + \text{vol } (M - \text{nbhd}_{\lambda/2}(p)) \geq \text{vol}(\text{nbhd}_{\alpha/2}(p)) + \text{vol } (M - \text{nbhd}_{\lambda/2}(p)) \geq V^{\text{near}}(\alpha, \delta, D) + V^{\text{far}}(\xi_{k-2}(Q(\lambda) + Q(D)), \lambda/2, \mu).$$

$\square$

**Notation and Remarks \ref{lem:vol-bound}**. We define a function $f_3$ on $(0, \infty)$ by

$$f_3(x) = Q^{-1}(1/2 - Q(x)) = \log \left( \frac{e^x + 3}{e^x - 1} \right).$$

Since, by \ref{lem:Q-properties}, $Q$ has domain $(0, \infty)$ and range $(0, 1/2)$ and is strictly monotone decreasing, $f_3$ is well defined and is also strictly monotone decreasing.

Now let $k > 2$ be an integer, and let $l$, $h$ and $\mu$ be positive real numbers. Since the function $Q$ is strictly monotone decreasing on $(0, \infty)$, we have $Q(f_3(l) + h) + Q(l) < Q(f_3(l)) + Q(l) = 1/2$, and hence $\xi_{k-2}(Q(f_3(l) + h) + Q(l))$ is a well-defined positive number. We set

$$\psi_k(h, l, \mu) = V^{\text{near}}_{\text{SG}}(f_3(l) + h, l) + V^{\text{far}}(\xi_{k-2}(Q(f_3(l) + h) + Q(l)), (f_3(l) + h)/2, \mu)$$

(where the functions $V^{\text{near}}_{\text{SG}}$ and $V^{\text{far}}$ are defined by \ref{def:volumes} and \ref{def:vol-far}).

**Lemma \ref{lem:vol-bound}**. Let $k > 2$ be an integer, and let $M$ be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is $k$-free. Let $\mu$ be a Margulis number for $M$. Let $l$ be a positive real number which is the length of some closed geodesic in $M$, and suppose that $l \leq \log 3$. Then there is a positive real number $h$ such that $\text{vol } M \geq \psi_k(h, l, \mu)$.

**Proof.** Since $\pi_1(M)$ is in particular 2-free, $\log 3$ is a Margulis number for $M$ by [3, Corollary 4.2]. Let us fix a closed geodesic $c$ of length $l$, and a point $p$ of $c$. We set $x = s_2(p)$. We apply Lemma \ref{lem:vol-bound}, letting $\log 3$ play the role of the Margulis number that is denoted $\mu$ in that lemma. This gives $s_1(p) = l$ and $\text{vol } \text{nbhd}_{x/2}(p) = V^{\text{near}}_{\text{SG}}(x, l)$.

Next, note that according to \ref{lem:volumes} we have $Q(x) + Q(l) = Q(s_2(p)) + Q(s_1(p)) < 1/2$ (so that $\xi_{k-2}(Q(x) + Q(l))$ is defined), and there is a $\mu$-thick point $p' \in M$ such that $\text{dist}(p,p') \geq \xi_{k-2}(Q(x) + Q(l))$. Since $x = s_2(p)$, we may apply Lemma \ref{lem:vol-bound} with $p$, $p'$, $\xi_{k-2}(Q(x) + Q(l))$ and $x/2$ playing the respective roles of $p_0$, $p_1$, $\rho$ and $R$, to deduce that $\text{vol}(M - \text{nbhd}_{x/2}(p)) \geq V^{\text{far}}(\xi_{k-2}(Q(x) + Q(l)), x/2, \mu)$. It now follows that

$$\text{vol } M = \text{vol } \text{nbhd}_{x/2}(p) + \text{vol } (M - \text{nbhd}_{x/2}(p)) \geq V^{\text{near}}_{\text{SG}}(x, l) + V^{\text{far}}(\xi_{k-2}(Q(x) + Q(l)), x/2, \mu).$$

(10.19.1)

Now since $Q(x) + Q(l) < 1/2$, we have $Q(x) < 1/2 - Q(l) = Q(f_3(l))$; since $Q$ is strictly monotone decreasing on $(0, \infty)$, this gives $x > f_3(l)$. We may therefore write $x = f_3(l) + h$
for some $h > 0$. The definition of $\psi$ now gives $\psi_k(h,l,\mu) = V^\text{near}_{SG}(x,l) + V^\text{far}(x,Q(l),x/2,\mu)$, which with (10.19.1) gives $\text{vol} M \geq \psi_k(h,l,\mu).$ \hfill $\square$

11. Volumes, diameters, and Margulis numbers

**Notation and Remarks 11.1.** For any positive numbers $R$ and $\rho$ we set

$$\tilde{V}_\text{Bör}(R,\rho) = \begin{cases} V_\text{Bör}(R,\rho) & \text{if } \rho > h_3(R) \\ B(R)/\text{density}(R) & \text{if } \rho \leq h_3(R). \end{cases}$$

Note that the strict monotonicity of $V_\text{Bör}$ in its second argument (cf. 10.14), together with the inequality $V_\text{Bör}(R,\rho) \geq B(R)/\text{density}(R)$ (cf. 10.14), implies that $\tilde{V}_\text{Bör}$ is monotone increasing, in the weak sense, in its second argument.

**Proposition 11.2.** Let $p$ be a point of a hyperbolic 3-manifold $M$, and let $R$ and $\rho$ be positive numbers. Suppose that $R \leq s_1(p)/2$, and that there is a point of $M$ whose distance from $p$ is at least $\rho$. Then $\text{vol nbhd}_{h_3(R)}(p) \geq \tilde{V}_\text{Bör}(R,\rho)$.

**Proof.** Since $R \leq s_1(p)/2$, there is a hyperbolic ball in $M$ having radius $R$ and center $p$ (see 6.2). If $\rho \leq h_3(R)$, the conclusion now follows from the inequality $\text{vol nbhd}_{h_3(R)}(p) \geq B(R)/\text{density}(R)$, which was reviewed in 10.14. Now suppose that $\rho < h_3(R)$. Since $M$ is connected, and some point of $M$ has distance at least $\rho$ from $p$, there is a point of $M$ whose distance from $p$ is exactly $\rho$. Hence [17, Proposition 9.7], which was also reviewed in 10.14, gives $\text{vol nbhd}_{h_3(R)}(p) \geq V_\text{Bör}(R,\rho)$, which in this case is equivalent to the conclusion of the present proposition. \hfill $\square$

**Proposition 11.3.** Let $M$ be a closed, orientable hyperbolic 3-manifold. Let $\mu$ be a Margulis number for $M$ (so that $M_\text{thick}(\mu) \neq \emptyset$ by Proposition 6.8). Let $\Delta$ denote the extrinsic diameter (see 6.1) of $M_\text{thick}(\mu)$, and assume that $\Delta \geq \mu$. Let $R$ be a positive number such that $R \leq \max_{p \in M}(s_1(p)/2)$. Then $\text{vol } M \geq \tilde{V}_\text{Bör}(R,\Delta/2) + \min(B(\mu/2),2B(\max(0,\Delta/2 - h_3(R))))$

(\text{where the function } B \text{ is defined as in 10.1 and the functions } h_3 \text{ is defined as in 10.14}).

**Proof.** According to the hypothesis, we may fix a point $p_0 \in M$ such that $s_1(p_0) \geq 2R$.

According to the definition of $\Delta$, we may fix points $p_1, p_2 \in M_\text{thick}(\mu)$ such that $\text{dist}(p_1,p_2) = \Delta$. We may suppose the $p_i$ to be indexed so that $\text{dist}(p_0,p_1) \geq \text{dist}(p_0,p_2)$. Since $\Delta \leq \text{dist}(p_0,p_1) + \text{dist}(p_0,p_2)$, it follows that $\text{dist}(p_0,p_1) \geq \Delta/2$. Now since $s_1(p_0) \geq 2R$, it follows from Proposition 11.2 that $N := \text{nbhd}_{h_3(R)}(p_0)$ has volume at least $\tilde{V}_\text{Bör}(R,\Delta/2)$.

For $i = 1, 2$, set $d_i = \max(\text{dist}(p_i,p_0),h_3(R))$. Since $d_i \geq h_3(R)$ we have $e_i := d_i - h_3(R) \geq 0$. Since $\text{dist}(p_0,p_1) \geq \text{dist}(p_0,p_2)$, we have $d_1 \geq d_2$ and hence $e_1 \geq e_2$. On the other hand, we have $\text{dist}(p_1,p_2) \leq \text{dist}(p_1,p_0) + \text{dist}(p_0,p_2) \leq d_1 + d_2$.

For $i = 1, 2$, set $U_i = \text{nbhd}_{e_i}(p_i)$ (so that in particular $U_i = \emptyset$ if $e_i = 0$). We claim that

$$U_i \cap N = \emptyset$$

(11.3.1)
for \(i = 1, 2\). To prove (11.3.1) for a given \(i\), first note that if \(e_i = 0\), the assertion is immediate since \(U_i = \emptyset\). Now suppose that \(e_i > 0\). Then we have \(d_i > h_3(R)\), which in view of the definition of \(d_i\) means that \(\text{dist}(p_i, p_0) > h_3(R)\) and that \(d_i = \text{dist}(p_i, p_0)\).

Hence \(\text{dist}(p_i, p_0) = e_i + h_3(R)\), which by the triangle inequality (see 6.1) implies that \(\text{nbhd}_{e_i}(p_i) \cap \text{nbhd}_{h_3(R)}(p_0) = \emptyset\), which establishes 11.3.1.

Consider the case in which \(e_1 \geq \mu/2\). In this case we have \(W := \text{nbhd}_{\mu/2}(p_1) \subset U_1\); in view of 11.3.1, it follows that \(W \cap N = \emptyset\). Hence \(\text{vol } M \geq \text{vol}(W) + \text{vol}(N)\). We have seen that \(\text{vol } N \geq \widetilde{\text{V_{Bor}}} (R, \Delta/2)\). Since \(p_1 \in M_{\text{thick}}(\mu)\), the set \(W\) is intrinsically isometric to a ball of radius \(\mu/2\) in \(\mathbb{H}^3\). Hence \(\text{vol } W = B(\mu/2)\). Thus in this case we have \(\text{vol } M \geq \widetilde{\text{V_{Bor}}} (R, \Delta/2) + B(\mu/2)\), which implies the conclusion of the proposition.

The rest of the proof will be devoted to the case in which \(e_1 < \mu/2\). Since \(e_1 \geq e_2\), we have \(e_2 < \mu/2\); and since \(\mu \leq \Delta\) by hypothesis, it follows that \(e_1 + e_2 < \mu \leq \Delta = \text{dist}(p_1, p_2)\). Hence by the triangle inequality (see 6.1), we have \(\text{nbhd}_{e_1}(p_1) \cap \text{nbhd}_{e_2}(p_2) = \emptyset\), i.e.

(11.3.2) \[ U_1 \cap U_2 = \emptyset. \]

According to (11.3.1) and (11.3.2), the sets \(U_1, U_2\) and \(N\) are pairwise disjoint. Hence \(\text{vol } M \geq \text{vol}(U_1) + \text{vol}(U_2) + \text{vol}(N)\). We have seen that \(\text{vol } N \geq \widetilde{\text{V_{Bor}}} (R, \Delta/2)\). For \(i = 1, 2\), since \(p_i \in M_{\text{thick}}(\mu)\), and since \(e_i < \mu/2\), the set \(U_i\) is intrinsically isometric to a ball in \(\mathbb{H}^3\) of radius \(e_i\), and hence \(\text{vol } U_i = B(e_i)\). Thus we obtain \(\text{vol } M \geq \widetilde{\text{V_{Bor}}} (R, \Delta/2) + B(e_1) + B(e_2)\).

But the function \(B(x) = \pi(\sinh(2x) - 2x)\) is convex for \(x \geq 0\), so that \((B(e_1) + B(e_2))/2 \geq B((e_1 + e_2)/2)\). We therefore have \(\text{vol } M \geq \widetilde{\text{V_{Bor}}} (R, \Delta/2) + 2B((e_1 + e_2)/2)\). But we have \(e_1 + e_2 = d_1 + d_2 - 2h_3(R) \geq \text{dist}(p_1, p_0) + \text{dist}(p_2, p_0) - 2h_3(R) \geq \text{dist}(p_1, p_2) - 2h_3(R) = \Delta - 2h_3(R)\). As \(e_1\) and \(e_2\) are non-negative, we have \((e_1 + e_2)/2 \geq \max(0, \Delta/2 - h_3(R))\). Since \(B(x)\) is also monotone increasing for \(x \geq 0\), it now follows that \(\text{vol } M \geq 2B(\max(0, \Delta/2 - h_3(R)))\), which implies the conclusion of the proposition in this case.

**Reformulation 11.4.** For applications of Proposition 11.3, it will be convenient to define a function \(V_{D\cdot R}\) on \((0, \infty)^3\) by \(V_{D\cdot R}(R, x, \mu) = \widetilde{\text{V_{Bor}}} (R, x/2) + \min(B(\mu/2), 2B(\max(0, x/2 - h_3(R))))\). In terms of this definition, we may reformulate Proposition 11.3 as follows: if \(M\) is a closed, orientable hyperbolic 3-manifold, if \(\mu\) is a Margulis number for \(M\), if \(\Delta\) denotes the extrinsic diameter of \(M_{\text{thick}}(\mu)\), if \(\Delta \geq \mu\), and if \(R\) is a positive number such that \(R \leq \max_{p \in M} (\theta_1(p)/2)\), then \(\text{vol } M \geq V_{D\cdot R}(R, \Delta, \mu)\). (The subscript \(D\cdot R\) stands for “diameter-radius,” as this function allows one to estimate the volume of a hyperbolic 3-manifold in terms of the diameter of its thick part and the radius of a certain hyperbolic ball in the manifold (cf. 6.2).)

Proposition 11.3 (or its reformulation 11.4) will be applied in conjunction with the following result, which is stated in terms of the function \(Q\) defined in 4.1:

**Proposition 11.5.** Let \(k > 2\) be an integer and let \(M\) be a closed, orientable hyperbolic 3-manifold such that \(\pi_1(M)\) is \(k\)-free. Let \(\mu_0\) be a Margulis number for \(M\), and let \(\Delta\) denote the extrinsic diameter (see 6.1) of \(M_{\text{thick}}(\mu_0)\) in \(M\). Let \(\mu\) be a positive real number such that
\[ Q(\mu) + Q(\mu_0) \geq 1/2 \text{ and } 2Q(\mu) + (k - 2)Q(2\Delta) \geq 1/2. \text{ Then } \mu \text{ is itself a Margulis number for } M. \]

**Proof.** This is a paraphrase of [17, Corollary 10.3], using the function \( Q \). The quantities denoted here by \( \mu_0 \) and \( \mu \) are denoted by \( \mu \) and \( \lambda \) respectively in [17, Corollary 10.3]. \( \square \)

**Notation 11.6.** For each integer \( k > 2 \), we define a function \( g_k \) on \((\log 3, \infty)\) by

\[
g_k(x) = \frac{1}{2} Q^{-1}\left(\frac{1/2 - 2Q(x)}{k - 2}\right).
\]

The function \( g_k \) is well defined for each \( k > 2 \) since \( 0 < Q(x) < 1/4 \) whenever \( x > \log 3 \).

**Proposition 11.7.** Let \( k > 2 \) be an integer, and let \( V, R, \) and \( \mu \) be positive real numbers. Suppose that \( \mu > \log 3 \), that \( V_{D-R}(R, g_k(\mu), f_3(\mu)) \geq V \), and that \( g_k(\mu) \geq f_3(\mu) \) (where \( f_3(\mu) \) is defined by 10.18). Let \( M \) be a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is \( k \)-free and \( \max_{p\in M}(s_1(p)/2) \geq R \). Then either \( \text{vol } M \geq V \), or \( \mu \) is a Margulis number for \( M \).

**Remark 11.8.** Proposition 11.7 gives strictly stronger information than [Lemma 10.4] of [17] even in the special case addressed by the latter lemma. Let \( M \) be a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is 4-free. Since \( \pi_1(M) \) is in particular 3-free, [4, Corollary 9.3] (a generalization of a result proved in [7]), we have \( \max_{p\in M}(s_1(p)/2) \geq (\log 5)/2 \). We apply Proposition 11.7, taking \( V = 3.48 \), \( R = (\log 5)/2 \) and \( \mu = 1.128 \). This gives \( g_4(\mu) = 2.80579 \ldots \) and \( f_3(\mu) = 1.06950 \ldots \), so that \( g_4(\mu) > f_3(\mu) \), and \( V_{D-R}(R, g_4(\mu), f_3(\mu)) = 3.484 \ldots \). Thus either \( \text{vol } M > 3.48 \), or 1.128 is a Margulis number for \( M \). By contrast, [Lemma 10.4] of [17] asserts only that either \( \text{vol } M > 3.468 \) or 1.119 is a Margulis number for \( M \).

**Proof of Proposition 11.7.** Set \( \mu_0 = f_3(\mu) \) and \( G = g_k(\mu) \). According to the definitions of \( f_3 \) and \( g_k \), we have \( Q(\mu) + Q(\mu_0) = 1/2 \) and \( 2Q(\mu) + (k - 2)Q(2G) = 1/2 \).

Since \( \mu > \log 3 \), we have \( \mu_0 < \log 3 \). But since \( \pi_1(M) \) is in particular 2-free, it follows from [3, Corollary 4.2] that \( \log 3 \) is a Margulis number for \( M \); hence \( \mu_0 \) is a Margulis number.

We shall denote by \( \Delta \) the extrinsic diameter of \( M_{\text{thick}}(\mu_0) \).

Consider the case in which \( \Delta \geq G \). Since \( G \geq \mu_0 \) by hypothesis, we have \( \Delta \geq \mu_0 \). Since \( \max_{p\in M}(s_1(p)/2) \geq R \) by the hypothesis of the present lemma, we may apply 11.4, with the Margulis number \( \mu_0 \) playing the role of \( \mu \) in that proposition, to deduce that \( \text{vol } M \geq V_{D-R}(R, \Delta, \mu_0) \).

The hypothesis gives \( V_{D-R}(R, G, \mu_0) \geq V \). On the other hand, since \( V_{B\ddot{u}r} \) is monotone increasing, in the weak sense, in its second argument (see 11.1), and \( B \) is (strictly) monotone increasing, the function \( V_{D-R} \) is monotone increasing, in the weak sense, in its second argument. Since \( \Delta \geq G \), we now have \( \text{vol } M \geq V_{D-R}(R, \Delta, \mu_0) \geq V_{D-R}(R, G, \mu_0) \geq V \), and the first alternative of the conclusion of the lemma is true in this case.

There remains the case in which \( \Delta < G \). Since \( Q \) is monotone decreasing, in this case we have \( 2Q(\mu) + (k - 2)Q(2\Delta) > 2Q(\mu) + (k - 2)Q(2G) = 1/2 \). Since \( Q(\mu) + Q(\mu_0) = 1/2 \),
it now follows from Proposition 11.5 that \( \mu \) is a Margulis number for \( M \), and the second alternative of the conclusion holds.

\[ \square \]

**Notation 11.9.** For any integer \( k > 2 \), we define an interval \( I_k \subset \mathbb{R} \) as follows: if \( k < 7 \) we set

\[ I_k = \left( \log 3, \log \left( \frac{20}{7-k} - 1 \right) \right), \]

and if \( k \geq 7 \) we set \( I_k = (\log 3, \infty) \). It follows from the definitions of the functions \( g_k \) and \( Q \) that \( I_k \) is the set of all real numbers \( x > \log 3 \) such that \( g_k(x) > \log 3 \).

In terms of the intervals \( I_k \), we have the following corollary to Proposition 11.7:

**Corollary 11.10.** Let \( k > 2 \) be an integer, and let \( V, R, \) and \( \mu \) be positive real numbers. Suppose that \( \mu \in I_k \) and that \( V_{D,R}(R, g_k(\mu), f_3(\mu)) \geq V \). Let \( M \) be a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is \( k \)-free and \( \max_{p \in M}(s_1(p)/2) \geq R \). Then either \( \text{vol} \, M \geq V \), or \( \mu \) is a Margulis number for \( M \).

**Proof.** It suffices to show that the hypotheses of Proposition 11.7 hold with the given choices of \( k, V, R, \mu \) and \( M \). Since \( \mu \in I_k \), we have \( g_k(\mu) > \log 3 \) by 11.9. But, again since \( \mu \in I_k \), we have \( \mu > \log 3 \), and hence \( f_3(\mu) < \log 3 \). It now follows that \( g_k(\mu) > f_3(\mu) \), which is one of the hypotheses of Proposition 11.7. The other hypotheses of Proposition 11.7 are included in those of the present corollary. \( \square \)

### 12. Very short geodesics

**Lemma 12.1.** Let \( k \geq 3 \) be an integer, and let \( \delta \) be a positive real number less than \( \min(0.7, \log(k-1)/2) \). Set

\[ C = \frac{4}{\cosh(\delta/2)e^{\delta}(e^\delta + 3)}. \]

Suppose that \( M \) is a closed, orientable hyperbolic 3-manifold such that \( \pi_1(M) \) is \( k \)-free, and that \( M \) contains a closed geodesic of length at most \( \delta \). Let \( \mu \) be a Margulis number for \( M \). Then

\[ \text{vol} \, M \geq \pi C - \frac{\pi \delta}{2} + B\left( \min\left( \frac{\mu}{2}, \frac{\log(k-1)}{2} - \frac{\delta}{2} \right) \right), \]

where the function \( B(x) \) is defined as in 10.1.

**Proof.** We fix a closed geodesic \( c \) in \( M \) of some length \( l \leq \delta \). We denote by \( R \) the tube radius of \( c \); by definition this means that \( R = 0 \) if \( c \) is not simple, and otherwise \( R \) is the maximum radius of a tube having core \( c \) (see 6.1). Since \( \pi_1(M) \) is in particular 2-free, it follows from [3, Corollary 4.2] that \( \log 3 \) is a strong Margulis number for \( M \) in the sense defined in [7]; according to [7, Proposition 10.1], the fact that \( \log 3 \) is a strong Margulis number implies that

\[ \cosh 2R \geq \frac{e^{2l} + 2e^l + 5}{(\cosh(l/2))(e^l - 1)(e^l + 3)} > \frac{8}{(\cosh(l/2))(e^l - 1)(e^l + 3)}. \]
(The fact that (12.1.1) holds when \( \pi_1(M) \) is 2-free was used in the proof of [17, Lemma 13.4], but the argument given there was incomplete.) The mean value theorem gives \( e^l - 1 = le^{l_0} \) for some \( l_0 \in (0, l) \). Since \( l \leq \delta \), it follows that \( e^l - 1 < le^\delta \), and (12.1.1) then yields

\[
(12.1.2) \quad \cosh 2R > \frac{8}{(\cosh(\delta/2))(le^\delta)(e^\delta + 3)} = \frac{2C}{l}.
\]

Since \( \delta \leq 0.7 \), the definition of \( C \) gives \( C \geq 0.373 \ldots \). In particular we have \( 2C > 0.7 > \delta \geq l \). Hence there is a unique strictly positive number \( R_0 \) such that \( \cosh 2R_0 = 2C/l \). It follows from (12.1.2) that \( R_0 < R \). In particular \( R > 0 \), so that \( c \) is simple, and there is a tube \( T \) having core \( c \) and radius \( R_0 \). We have

\[
\text{vol } T = \pi l \sinh^2 R_0 = \pi l \left( \frac{1}{2} \cosh 2R_0 - \frac{1}{2} \right) = \pi l \left( \frac{C}{l} - \frac{1}{2} \right) \geq \pi C - \frac{\pi \delta}{2}.
\]

Now fix a point \( p \) of the simple closed geodesic \( c \). Since the extrinsic diameter (see 6.1) of \( c \) is at most \( l/2 \), and every point of \( T \) is at a distance at most \( R_0 \) from a point of \( c \), we have \( T \subset \text{nbhd}_{R_0+l/2}(p) \). Hence

\[
(12.1.3) \quad \text{vol } \text{nbhd}_{R_0+l/2}(p) \geq \text{vol } T \geq \pi C - \frac{\pi \delta}{2}.
\]

An orientation of \( c \) defines a non-trivial element \( u \) of the torsion-free group \( \pi_1(M,p) \), which is represented by a loop of length \( l \), and we may regard \( \{u\} \) as a one-element independent set. Applying Proposition 6.9 with \( m = 1 \), we obtain a \( \mu \)-thick point \( p' \in M \) such that \( \rho := \text{dist}(p,p') \) satisfies \( (k - 1)Q(2\rho) + Q(l) \leq 1/2 \), i.e.

\[
\frac{k - 1}{1 + e^{2\rho}} + \frac{1}{1 + e^l} \leq \frac{1}{2}.
\]

Solving the latter inequality for \( e^{2\rho} \), we obtain

\[
e^{2\rho} \geq \frac{(2k - 3)e^l + (2k - 1)}{e^l - 1} = (2k - 3) + \frac{4k - 4}{e^l - 1} > \frac{4k - 4}{e^l - 1} \geq \frac{4k - 4}{le^\delta},
\]

so that

\[
(12.1.4) \quad \rho > \frac{1}{2} \log(4k - 4) + \frac{1}{2} \log(1/l) - \delta/2.
\]

In view of the definition of \( R_0 \), we have \( e^{2R_0} \leq 2 \cosh(2R_0) = 4C/l \), and hence

\[
(12.1.5) \quad R_0 \leq \frac{1}{2} \log(4C) + \frac{1}{2} \log(1/l).
\]

Combining (12.1.4) and (12.1.5), we find

\[
(12.1.6) \quad \rho - (R_0 + l/2) > \frac{1}{2} \log(4k - 4) - \frac{1}{2} \log(4C) - \delta/2 - l/2.
\]
But we have \( l \leq \delta \), and the definition of \( C \) immediately implies \( C < 1 \). Hence (12.1.6) gives
\[
\rho - (R_0 + l/2) > \log(k - 1)/2 - \delta, \quad \text{i.e.} \quad \rho > (R_0 + l/2) + \left( \frac{\log(k - 1)}{2} - \delta \right).
\]
By hypothesis we have \((\log(k - 1))/2 - \delta > 0\). In view of (12.1.7) and the triangle inequality
(cf. (see 6.1), we obtain \( \text{nbhd}_{R_0+l/2}(p) \cap \text{nbhd}_{\log(k-1)/2-\delta}(p') = \emptyset \), which implies
\[
(12.1.8) \quad \text{vol} M \geq \text{vol}(\text{nbhd}_{R_0+l/2}(p)) + \text{vol}(\text{nbhd}_{\log(k-1)/2-\delta}(p')).
\]
Since \( p' \in M_{\text{thick}}(\mu) \), i.e. \( s_1(p') \geq \mu \), there is a hyperbolic ball in \( M \) having radius \( \mu/2 \) and
center \( p' \) (see 6.2). In particular, \( p' \) is the center of a ball of radius \( s := \min(\mu/2, \log(k - 1)/2 - \delta) \). We have \( \text{nbhd}_{\log(k-1)/2-\delta}(p') \supset \text{nbhd}_{s}(p') \), and hence \( \text{vol}\text{nbhd}_{\log(k-1)/2-\delta}(p') \geq \text{vol}\text{nbhd}_{s}(p') = B(s) \); that is,
\[
(12.1.9) \quad \text{vol}\text{nbhd}_{\log(k-1)/2-\delta}(p') \geq B\left( \min\left( \frac{\mu}{2}, \frac{\log(k-1)}{2} - \delta \right) \right).
\]
The conclusion of the lemma follows from (12.1.8), (12.1.3) and (12.1.9). \( \square \)

Reformulation 12.2. For applications of Lemma 12.1, it will be convenient to define a function \( V_{\text{VSG}} \) on a subset of \( \mathbb{Z} \times \mathbb{R}^2 \) as follows. The domain of \( V_{\text{VSG}} \) consists of all triples \((k, \delta, \mu) \in \mathbb{Z} \times \mathbb{R}^2 \) such that \( k > 2 \), \( \mu > 0 \), and \( 0 < \delta < \log(k - 1)/2 \). For each such triple \((k, \delta, \mu) \) we set
\[
V_{\text{VSG}}(k, \delta, \mu) = \frac{4\pi}{\cosh(\delta/2)e^{\delta}(e^{\delta} + 3)} - \frac{\pi\delta}{2} + B\left( \min\left( \frac{\mu}{2}, \frac{\log(k-1)}{2} - \delta \right) \right).
\]
In terms of these definitions, we may reformulate Lemma 12.1 as follows. Let \( k \geq 3 \) be an integer, let \( \delta \) be a positive real number less than \( \min(0.7, \log(k - 1)/2) \), let \( M \) be a closed, orientable hyperbolic 3-manifold which has \( k \)-free fundamental group and contains a closed geodesic of length at most \( \delta \), and let \( \mu \) be a Margulis number for \( M \). Then \( \text{vol} M \geq V_{\text{VSG}}(k, \delta, \mu) \).

(The subscript VSG stands for “very short geodesic.”)

13. A numerical criterion for lower bounds

The main result of this section, Proposition 13.2, provides a sufficient condition for a given constant to be a lower bound for the volumes of all closed, orientable hyperbolic 3-manifolds with \( k \)-free fundamental group, where \( k \geq 5 \) is a given integer. The statement and proof involve a number of functions and sets that were defined earlier in the paper: \( Q \) (Subsection 4.1); \( \xi_n \) for \( n > 0 \) (6.10); \( f_1 \) (7.1); \( f_2 \) (8.2); \( B \) (10.1); \( \kappa \) (10.2); \( V_{\text{near}} \) (10.9); \( h_3 \) and density (10.14); \( V_{\text{far}} \) (10.15); \( f_3 \), and \( \psi_k \) for \( k > 2 \) (10.18); \( V_{\text{D-R}} \) (11.4); \( g_k \) for \( k > 2 \) (11.6); \( I_k \) for \( k > 2 \) (11.9); and \( V_{\text{VSG}} \) (12.2).

Proposition 13.2 will also involve two functions that were not mentioned in earlier sections.
Notation and Remark 13.1. For each number $\lambda > \log 3$, we set
\[
H_\lambda = Q^{-1}\left(\frac{1}{4} - Q(\lambda)\right) = \log\left(\frac{3e\lambda + 7}{e\lambda - 3}\right).
\]

If we are given a number $\lambda$ with $\lambda > \log 7$, we have $H_\lambda < \lambda$, and for any $y \in [H_\lambda, \lambda]$ we have
\[
1/2 > 1/2 - 2Q(\lambda) - Q(y) \geq 1/2 - 2Q(\lambda) - Q(H_\lambda) = 1/4 - Q(\lambda) > 0.
\]
Hence we may define a positive-valued function $F_\lambda$ on the domain $[H_\lambda, \lambda]$ by
\[
F_\lambda(y) = Q^{-1}\left(\frac{1}{2} - 2Q(\lambda) - Q(y)\right).
\]

Note that since $Q$ is monotone decreasing on its domain, $F_\lambda$ is monotone decreasing on $[H_\lambda, \lambda]$ for any $\lambda > \log 7$.

Proposition 13.2. Let $k \geq 5$ be an integer, and let $V_0 > 0$ be a real number. Suppose that $\lambda^-$ and $\lambda^+$ are constants, with $\log 7 < \lambda^- \leq \lambda^+ \leq \log 9$; that $\delta_0$ and $\delta_1$ are positive constants with $\max(\delta_0, \lambda^+/4, f_3(\lambda^-)) < \delta_1 < \log 3$; that $\mu^*$ is a constant lying in the interval $I_k$, so that $g_k(\mu^*)$ is defined by 11.9; and that $\mathcal{M}$ is a function defined on $[\lambda^-, \lambda^+]$, and taking values in the interval $I_k$, so that $g_k(\mathcal{M}(\lambda))$ is defined whenever $\lambda \in [\lambda^-, \lambda^+]$. Assume that the following conditions hold.

1a) $V_{D-R}(([\log 5]/2), g_k(\mu^*), f_3(\mu^*)) > V_0$.

1b) For each $\lambda \in [\lambda^-, \lambda^+]$, there is an $R \in (0, f_1(\lambda)/2]$ such that $V_{D-R}(R, g_k(\mathcal{M}(\lambda)), f_3(\mathcal{M}(\lambda))) > V_0$.

2) $0 < \delta_0 < \min(0.7, \log(k - 1)/2)$ (so that $V_{VSG}(k, \delta_0, \mu^*)$ is defined by 12.2), and $V_{VSG}(k, \delta_0, \mu^*) > V_0$.

3) For every $l$ with $\delta_0 < l \leq \delta_1$, and for every $h > 0$, we have $\psi_k(h, l, \mu^*) > V_0$.

Note that for any $D \geq \delta_1$, and any $\lambda \geq \lambda^-$, we have in particular $D > f_3(\lambda^-) \geq f_3(\lambda)$, which, in view of the definition of $f_3$ and the fact that $Q$ is monotone decreasing, implies that $Q(\lambda) + Q(D) < 1/2$; since $\xi_{k-2}$ has domain $(0, 1/2)$ by 6.10, it follows that $\xi_{k-2}(Q(\lambda) + Q(D))$ is defined. Assume that the following two conditions hold.

4a) For every $D \geq \delta_1$ we have
\[
V_{ST}^{\text{near}}(\lambda^+, \delta_1, D) + V_{\text{far}}(\xi_{k-2}(Q(\lambda^+) + Q(D)), \lambda^+/2, \mu^*) > V_0.
\]

4b) For every $\lambda \in [\lambda^-, \lambda^+]$, there is a number $\alpha \in (0, \lambda]$, such that for every $D \geq \delta_1$ we have
\[
V_{ST}^{\text{near}}(\alpha, \delta_1, D) + V_{\text{far}}(\xi_{k-2}(Q(\lambda) + Q(D)), \lambda/2, \mathcal{M}(\lambda)) > V_0.
\]

Set $r = (\min(f_1(\lambda^-), \lambda^+, f_2(\lambda^+))/2$, and assume:

5) $B(r)/\text{density}(r) > V_0$.

Finally, note that since $\lambda^+ \geq \lambda^- > \log 7$, it follows from 13.1 that the quantity $H_{\lambda^+}$ is defined and is less than $\lambda^+$, and $F_{\lambda^+}$ is defined on $[H_{\lambda^+}, \lambda^+]$. Furthermore, we have $0 < 2Q(\lambda^+)/(k - 2) \leq (2/3)Q(\lambda^+) < 1/3 < 1/2$, so that $Q^{-1}(2Q(\lambda^+)/(k - 2))$ is well defined. Set $E = Q^{-1}(2Q(\lambda^+)/(k - 2))/2$, and assume that the following condition holds.
(6) For any \( y \in [H_{\lambda^+}, \lambda^+] \), we have

\[
B\left(\frac{y}{2}\right) - 6\kappa\left(\frac{y}{2} \cdot \frac{F_{x^+}(y)}{2}\right) + V^{\text{far}}\left(E, \frac{y}{2}, \mu^*\right) > V_0.
\]

Then for every closed, orientable hyperbolic 3-manifold \( M \) such that \( \pi_1(M) \) is \( k \)-free, we have \( \text{vol} M \geq V_0 \).

Proof. Since \( k \geq 5 > 3 \), the group \( \pi_1(M) \) is in particular 3-free. Hence according to [4, Corollary 9.3] (a generalization of a result proved in [7]), we have \( \max_{p \in M} r_1(p)/2 \geq (\log 5)/2 \). Furthermore, we have \( \mu^* \in I_k \) by hypothesis, and according to Condition (1a) of the hypothesis, we have \( V_{\text{D-R}}((\log 5)/2, g_k(\mu^*), f_3(\mu^*)) > V_0 \). It therefore follows from Corollary 11.10, applied with \( R = (\log 5)/2, \mu = \mu^*, \) and \( V = V_0 \), that either \( \mu^* \) is a Margulis number for \( M \), or \( \text{vol} M \geq V_0 \). Hence we may assume that \( \mu^* \) is a Margulis number for \( M \); this assumption will be used without mention for the rest of the proof.

Let \( l_0 \) denote the minimal length of a closed geodesic in \( M \). We claim that if \( l_0 \leq \delta_1 \) then \( \text{vol} M > V_0 \). To prove this, first consider the case in which \( l_0 \leq \delta_0 \). Since \( \delta_0 < \min(0.7, \log (k-1)/2) \) by Condition (2), it follows from 12.2 that \( \text{vol} M \geq V_{\text{VSG}}(k, \delta, \mu^*) \). Since Condition (2) also gives \( V_{\text{VSG}}(k, \delta, \mu^*) > V_0 \), the inequality \( \text{vol} M > V_0 \) is established in this case.

Now consider the case in which \( \delta_0 < l_0 \leq \delta_1 \). By hypothesis we have \( \log 3 > \delta_1 \geq l_0 \). By definition \( l_0 \) is the length of some closed geodesic in \( M \). Since \( \pi_1(M) \) is \( k \)-free, it now follows from Lemma 10.19 that there is a positive number \( h \) such that \( \text{vol} M \geq \psi(h, l, \mu) \). But according to Condition (3) of the present proposition, the inequalities \( \delta_0 < l_0 \leq \delta_1 \) imply that \( \psi(h, l_0, \mu^*) > V_0 \), and hence \( \text{vol} M > V_0 \) in this case as well. This completes the proof that, if \( l_0 \leq \delta_1 \), then \( \text{vol} M > V_0 \). We may therefore assume, for the rest of the proof, that \( l_0 > \delta_1 \).

We now apply Proposition 8.3. Since \( \pi_1(M) \) is \( k \)-free, and \( k \geq 5 \), one of the alternatives (i)—(iv) of that lemma must hold, with \( \lambda^+ \) given by the hypothesis of the present proposition.

Consider the case in which Alternative (i) holds. We fix a point \( p \in M \) with \( s_2(p) = \lambda^+ \), and we apply Lemma 10.17, with \( \mu^* \) and \( \delta_1 \) playing the respective roles of \( \mu \) and \( \delta \) in that lemma, and with \( \lambda^+ \) playing the roles of both \( \lambda \) and \( \alpha \). We have \( l_0 > \delta_1 \), so that every closed geodesic in \( M \) has length greater than \( \delta_1 \); and according to the hypotheses of the present proposition we have \( \delta_1 > \lambda^+/4 \). It therefore follows from Lemma 10.17 that there is a number \( D \geq \delta_1 \) such that

\[
\text{vol} M \geq V_{\text{ST}}^{\text{near}}(\lambda^+, \delta_1, D) + V^{\text{far}}(\xi_{k-2}(Q(\lambda^+)) + Q(D)), \lambda^+/2, \mu^*).
\]

(The inequality \( Q(\lambda^+) + Q(D) < 1/2 \), which forms part of the conclusion of Lemma 10.17, is redundant in the present context, since it was observed in the statement of the present proposition that \( Q(\lambda) + Q(D) < 1/2 \) for any \( D \geq \delta_1 \) and any \( \lambda \geq \lambda^- \).)

According to Condition (4a) of the present proposition, the right-hand side of (13.2.1) is greater than \( V_0 \), and hence the required conclusion \( \text{vol} M > V_0 \) holds in this case.
Next we consider the case in which Alternative (ii) of Proposition 8.3 holds. We fix a point \( p_0 \in M \) such that \( \log 5 < s_2(p_0) \leq \lambda^+ \) and \( \max_{p \in M} \alpha_1(p) \geq f_1(s_2(p_0)) \). We set \( \lambda = s_2(p_0) \).

We first consider the subcase \( \log 5 < \lambda < \lambda^- \). Since \( f_1 \) is strictly monotone decreasing on \((\log 5, \infty)\), we have \( \max_{p \in M} \alpha_1(p) \geq f_1(\lambda) > f_1(\lambda^-) \); in particular there is a point \( p \in M \) such that \( s_1(p)/2 > r \), where \( r \) is defined as in the statement of the present proposition. Hence \( p \) is the center of a ball of radius \( r \) in \( M \) (see 6.2). By the results reviewed in 10.14, it follows that \( \text{vol} \ M \geq B(r) / \text{density}(r) \). According to Condition (5) of the hypothesis, the right-hand side of the latter inequality is greater than \( V_0 \).

We now turn to the subcase in which \( \lambda^- \leq s_2(p_0) \leq \lambda^+ \). We set \( \mu^\dagger = \mathcal{M}(\lambda) \). Since by hypothesis \( \mathcal{M} \) takes its values in \( I_k \), we have \( \mu^\dagger \in I_k \). According to Condition (1b) of the hypothesis, we may fix an \( R \in (0, f_1(\lambda)/2] \) such that \( V_{D,R}(R, g_k(\mu^\dagger), f_3(\mu^\dagger)) > V_0 \). We have \( \max_{p \in M} \alpha_1(p)/2 > f_1(s_2(p_0)/2) = f_1(\lambda/2) \geq R \). Thus all the hypotheses of Corollary 11.10 hold with \( V_0 \) and \( \mu^\dagger \) playing the respective roles of \( V \) and \( \mu \), with \( k \) given by the hypothesis of the present proposition, and with the choice of \( R \) made above. It therefore follows from Corollary 11.10 that either \( \mu^\dagger \) is a Margulis number for \( M \), or \( \text{vol} \ M \geq V_0 \). Hence we may assume, for the argument in the present case, that \( \mu^\dagger \) is a Margulis number for \( M \).

Now fix an \( \alpha \in (0, \lambda] \) having the property stated in Condition (4b) of the hypothesis, where \( \lambda \) is chosen as above.

We apply Lemma 10.17, with \( \lambda \) and \( \alpha \) chosen as above, and with \( p_0, \delta_1, \) and \( \mu^\dagger \) playing the respective roles of \( p, \delta, \) and \( \mu \) in that lemma. Since \( l_0 > \delta_1 \), every closed geodesic in \( M \) has length strictly greater than \( \delta \). Furthermore, according to the hypotheses of the present proposition, we have \( \delta_1 > \lambda^+ / 4 \geq \lambda / 4 \). Since in addition we have \( \alpha \leq \lambda \), it now follows from Lemma 10.17 that there is a number \( D \geq \delta_1 \) such that

\[
\text{vol} \ M \geq V_{ST}^{\text{near}}(\alpha, \delta_1, D) + V_{\text{far}}(\xi_{k-2}(Q(\lambda) + Q(D)), \lambda/2, \mu^\dagger).
\]

(The inequality \( Q(\lambda) + Q(D) < 1/2 \), which forms part of the conclusion of Lemma 10.17, is redundant in the present context, since it was observed in the statement of the present proposition that \( Q(\lambda) + Q(D) < 1/2 \) for any \( D \geq \delta_1 \) and any \( \lambda \geq \lambda^- \).)

According to the inequality in Condition (4b), the right-hand side of (13.2.2) is greater than \( V_0 \), and hence we have \( \text{vol} \ M > V_0 \) in this subcase.

In the case in which Alternative (iii) of Proposition 8.3 holds, so that \( \min(\lambda^+, f_2(\lambda^+)) \) is a Margulis number for \( M \), it follows from Proposition 6.8 that \( M_{\text{thick}}(\min(\lambda^+, f_2(\lambda^+))) \neq \emptyset \); that is, there is a point \( p \in M \) such that \( s_1(p) \geq \min(\lambda^+, f_2(\lambda^+)) \). In particular we have \( s_1(p)/2 \geq r \), where \( r \) is defined as in the statement of the present proposition. Hence \( p \) is the center of a ball of radius \( r \) in \( M \) (see 6.2). By the results reviewed in 10.14, it follows that \( \text{vol} \ M \geq B(r)/\text{density}(r) \). According to Condition (5) of the hypothesis, the right-hand side of the latter inequality is greater than \( V_0 \).

There remains the case in which Alternative (iv) of Proposition 8.3 holds. We fix a point \( p_0 \in M \) such that \( Q(s_1(p_0)) + Q(s_2(p_0)) = 1/2 - 2Q(\lambda^+) \) and \( s_2(p_0) < \lambda^+ \). We set \( x = s_1(p_0) \) and \( y = s_2(p_0) \), so that \( 0 < x \leq y < \lambda^+ \) and \( Q(x) + Q(y) = 1/2 - 2Q(\lambda^+) \). Since \( Q \) is
monotone decreasing, we have \( Q(y) \leq Q(x) \) and therefore \( Q(y) \leq 1/4 - Q(\lambda^+) = Q(H_{\lambda^+}) \) (see 13.1); hence \( y \geq H_{\lambda^+} \), so that \( y \in [H_{\lambda^+}, \lambda^+] \). Furthermore, since \( Q(x) + Q(y) = 1/2 - 2Q(\lambda^+) \), 13.1 gives \( x = F_{\lambda^+}(y) \).

Let \( C \) be a short maximal cyclic subgroup of \( \pi_1(M, p_0) \), and let \( g \) be a generator of \( C \). If \( l \) denotes the length of a closed geodesic representing the conjugacy class of \( g \), then the definition of \( l_0 \) implies that \( l \geq l_0 \); since \( l_0 > \delta_1 \), we have \( l > \delta_1 \). According to the hypothesis of the present proposition, we have \( \delta_1 > \lambda^+/4 > y/4 \). Hence \( l > y/4 \), so that \( |y/l| \leq 3 \). Applying Corollary 10.4, we obtain

\[
\text{vol nbhd}_{y/2}(p_0) \geq B \left( \frac{y}{2} \right) - 6\kappa \left( \frac{y}{2}, \frac{x}{2} \right) = B \left( \frac{y}{2} \right) - 6\kappa \left( \frac{y}{2}, \frac{F_{\lambda^+}(y)}{2} \right).
\]

Now we apply Lemma 6.11, letting \( p_0 \) play the role of \( p \) in that lemma. This gives a point \( p_1 \in M \) such that \( \text{dist}(p_0, p_1) \geq \xi_{k-2}(Q(x) + Q(y)) = \xi_{k-2}(1/2 - 2Q(\lambda^+)) \). The definition of \( \xi_{k-2} \) (see 6.10) gives \( \xi_{k-2}(1/2 - 2Q(\lambda^+)) = Q^{-1}(2Q(\lambda^+)/(k-2))/2 = E \). Since \( y = s_2(p_0) \), we may apply Proposition 10.16, taking \( \rho = E \) and \( R = y/2 \), to obtain

\[
\text{vol}(M - \text{nbhd}_{y/2}(p_0)) \geq V_{\text{far}}(E, \frac{y}{2}, \mu^*).
\]

Hence

\[
\text{vol } M = \text{vol}(\text{nbhd}_{y/2}(p_0)) + \text{vol}(M - \text{nbhd}_{y/2}(p_0)) \\
\geq B \left( \frac{y}{2} \right) - 6\kappa \left( \frac{y}{2}, \frac{F_{\lambda^+}(y)}{2} \right) + V_{\text{far}}(E, \frac{y}{2}, \mu^*).
\]

According to Condition (6) of the present proposition, the right-hand side of this last inequality is greater than \( V_0 \).

\section*{14. A numerical estimate for the 5-free case}

**Notation and Remarks 14.1.** We will need estimates for the functions \( \omega, \Omega \) and \( \theta \) defined in 9.1.

If \( x^-, x^+, y^-, y^+ \) and \( z^+ \) are positive numbers with \( x^- \leq x^+ \) and \( y^- \leq y^+ \), we set

\[
\omega^-(x^-, x^+, y^-, y^+, z^+) = \text{cosech}(\text{cosh} x^+)(\text{cosech} y^+),
\]

Comparing this definition with the definition of \( \omega \) given in 9.1, we find that \( \omega(x, y, z) \geq \omega^-(x^-, x^+, y^-, y^+, z^+) \) whenever \( x^- \leq x \leq x^+ \), \( y^- \leq y \leq y^+ \), and \( 0 < z \leq z^+ \). Hence if we define a function \( \Omega^+ \) on the same domain as \( \omega^- \) by setting

\[
\Omega^+(x^-, x^+, y^-, y^+, z^+) = \arccos(\min(\max(\omega^-(x^-, x^+, y^-, y^+, z^+), -1), 1)) \in [0, \pi]
\]

for all \( x^-, x^+, y^-, y^+, z^+ > 0 \), then we have

\[
\Omega(x, y, z) \leq \Omega^+(x^-, x^+, y^-, y^+, z^+)
\]

whenever \( 0 < x^- \leq x \leq x^+ \), \( 0 < y^- \leq y \leq y^+ \), and \( 0 < z \leq z^+ \).
If $C^-, C^+, x^-$ and $x^+$ are numbers satisfying $0 < C^- \leq C^+$ and $0 < x^- \leq x^+$, we set

\begin{equation}
\theta^+(C^-, C^+, x^-, x^+) = \overline{\Omega}^+(\arccosh(\cosh x^-/\cosh C^+), \arccosh(\cosh x^+/(\cosh C^-), x^-, x^+, C^+))
\end{equation}

if $x^- > C^+$, and setting $\theta^+(C^-, C^+, x^-, x^+) = \pi/2$ if $x^- \leq C^+$.

We then have

\begin{equation}
\theta(C, x) \leq \theta^+(C^-, C^+, x^-, x^+)
\end{equation}

whenever $C^- \leq C \leq C^+$ and $x^- \leq x \leq x^+$. Indeed, to establish (14.1.3) in the case where $x^- > C^+$, we note that in this case we have $x > C$, so that $\theta(C, x)$ and $\theta^+(x^-, x^+, y^-, y^+, z^+)$ are defined by (9.1.2) and (14.1.2) in this case. We have $\arccosh((\cosh x^-)/(\cosh C^+)) \leq \arccosh((\cosh x^-)/(\cosh C^-))$ and (14.1.3) is then simply the instance of (14.1.1) in which $\arccosh((\cosh x^-)/(\cosh C^+))$, $\arccosh((\cosh x^-)/(\cosh C^-))$, $x^−$, $x^+$, $C$ and $C^+$ play the respective roles of $x^−$, $x^+$, $y^−$, $y^+$, $z$ and $z^+$. In the case where $x^- \leq C^+$, we have $\theta^+(C^-, C^+, x^-, x^+) = \pi/2$, and (14.1.3) follows from the observation made in 9.1 that $\theta(C, x) \leq \pi/2$ for all positive $C$ and $x$.

If $\underline{a} = (C^-, C^+, a_{i,0}^+, a_{i,0}^-, a_{i,1}^+, a_{i,1}^-, a_{2,0}^+, a_{2,0}^-, a_{2,1}^+, a_{2,1}^-)$ is a decouple of positive numbers such that $C^- \leq C^+$ and $a_{i,0}^- \leq a_{i,0}^+ \leq a_{i,1}^- \leq a_{i,1}^+$ for $i = 1, 2$, we set

\begin{align*}
A^+_1(\underline{a}) &= \max_{(i,j) \in \{0,1\} \times \{0,1\}} \Omega^+(a_{i,j}^−, a_{i,j}^+, a_{j,i}^-), \\
A^+_2(\underline{a}) &= \max_{(m,i) \in \{1,2\} \times \{0,1\}} \theta^+(C^-, C^+, a_{m,i}^−, a_{m,i}^+)
\end{align*}

and

\begin{equation}
A^+(\underline{a}) = \max(A^+_1(\underline{a}), A^+_2(\underline{a})).
\end{equation}

Now given any $C \in [C^-, C^+]$, and given $a_{(m,i)} \in [a_{(m,i)}^−, a_{(m,i)}^+]$ for each $(m, i) \in \{1, 2\} \times \{0,1\}$, it follows from (14.1.1) and (14.1.3), and from the definitions of $A_1$, $A_2$ and $A$ given in 9.1, that $A_t(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) \leq A^+_t(\underline{a})$ for $t = 1, 2$, and hence

\begin{equation}
A(C, a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}) \leq A^+(\underline{a}).
\end{equation}

If $\delta$, $D^-$ and $D^+$ are numbers satisfying $0 < \delta \leq D^- \leq D^+$, we set

\begin{equation}
A^+(\delta, D^-, D^+) = A^+(D^-, D^+, 2\delta, 2\delta, \Phi_2(\delta, D^-), \Phi_2(\delta, D^+), 3\delta, 3\delta, \Phi_3(\delta, D^-), \Phi_3(\delta, D^+)).
\end{equation}

Given any $D \in [D^-, D^+]$, since $\Phi_2$ and $\Phi_3$ are monotone increasing in their second argument (cf. 10.5), we have $\Phi_n(\delta, D) \in [\Phi_n(\delta, D^-), \Phi_n(\delta, D^+)]$ for $n = 2, 3$. Hence (14.1.4), with the definition of $\Lambda^+$ given above and the definition of $\Lambda$ given in 10.7, implies:

\begin{equation}
\Lambda(\delta, D) \leq \Lambda^+(\delta, D^-, D^+) \text{ whenever } D \in [D^-, D^+].
\end{equation}

**Notation and Remarks 14.2.** We set $\mathfrak{M}_0 = \{(\alpha, \delta, D^-, D^+) \in \mathbb{R}^4 : 0 < \delta \leq D^- \leq D^+ \text{ and } \alpha > 0\}$. Note that if $(\alpha, \delta, D^-, D^+) \in \mathfrak{M}_0$, then $\Phi_n(\delta, D^-)$ is defined for every $n \geq 1$. Furthermore, by 10.5.2 we have $\Phi_2(\delta, D^-) \leq 2D^- \leq 2D^+$, so that $\Psi(D^+, \Phi_2(\delta, D^-))$ is defined (see 10.6). For each $(\alpha, \delta, D^-, D^+) \in \mathfrak{M}_0$ we set

\begin{equation}
W^-(\alpha, \delta, D^-, D^+) = B(\alpha/2) - 2\sigma(\alpha/2, D^-/2, \Phi_2(\delta, D^-)/2, \Psi(D^+, \Phi_2(\delta, D^-))).
\end{equation}
Next we define $\mathcal{W}_0' = \{(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0 : D^- < \Phi_3(\delta, D^-) < \Phi_3(\delta, D^+) < \alpha\}$, and observe that by 10.6, the quantities $\Theta(D^+/2, \alpha/2)$ and $\Theta(\Phi_3(\delta, D^-)/2, \alpha/2)$ are defined for every $(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0'$. We define $\mathcal{W}_0^{''}$ to be the set of all points $(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0'$ such that

$$
(14.2.1) \quad \cos \left( \Theta \left( \frac{D^+}{2}, \frac{\alpha}{2} \right) - \Theta \left( \frac{\Phi_3(\delta, D^-)}{2}, \frac{\alpha}{2} \right) \right) < \frac{\cosh D^- \cosh \Phi_3(\delta, D^-) - \cosh 2D^+}{\sinh D^+ \sinh \Phi_3(\delta, D^+)}.
$$

We then define the function $V_{ST}^-$ on $\mathcal{W}_0'$ by setting

$$
V_{ST}^-(\alpha, \delta, D^-, D^+) = W^-(\alpha, \delta, D^-, D^+)
$$

if $(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0'^{''}$, and setting

$$
(14.2.2) \quad V_{ST}^-(\alpha, \delta, D^-, D^+) = W^-(\alpha, \delta, D^-, D^+) - 2\kappa \left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D^-)}{2} \right) + 2\iota \left( \frac{\alpha}{2}, \frac{\Phi_2(\delta, D^+)}{2}, \frac{\Phi_3(\delta, D^+)}{2}, \Lambda^+(\delta, D^-, D^+) \right)
$$

if $(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0^{''} - \mathcal{W}_0'^{''}$.

**Remarks 14.3.** Note that for any $(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0'$ and any $D \in [D^-, D^+]$, we have $(\alpha, \delta, D) \in \mathcal{W}$. It follows from the fact that $\Phi_2$ is increasing in its second argument (see 10.5) that if $(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0'$, then $(\alpha, \delta, D) \in \mathcal{W}^\prime$ for any $D \in [D^-, D^+]$.

If $(\alpha, \delta, D^-, D^+) \in \mathcal{W}_0'^{''}$, then $(\alpha, \delta, D) \in \mathcal{W}^{''}$ for any $D \in [D^-, D^+]$. To see this, since we have already shown that $(\alpha, \delta, D) \in \mathcal{W}$, it suffices to observe that the left- and right-hand sides of (10.9.1) are respectively bounded above and below by the left- and right-hand sides of (14.2.1); this observation in turn follows from the fact that $\Phi_3$ is increasing in its second argument (see 10.5), and the fact that $\Theta$ is decreasing in its first argument (see 10.6).

**Lemma 14.4.** Let $\alpha$, $\delta$, $D^-$ and $D^+$ be positive numbers such that $\delta \leq D^- \leq D^+$. Then for every $D$ with $D^- \leq D \leq D^+$, we have $V_{ST}^{near}(\alpha, \delta, D) \geq V_{ST}^-(\alpha, \delta, D^-, D^+)$.

**Proof.** First note that since $\sigma$ is decreasing in its second and third arguments and increasing in its fourth argument (see 10.2), while $\Phi_2$ and $\Phi_3$ are increasing in their second argument (see 10.5) and $\Psi$ is increasing in its first argument and decreasing in its second (see 10.6), it follows from the definition of $W$ given in 10.9 and the definition of $W^-$ given above that

$$
(14.4.1) \quad W(\alpha, \delta, D) \geq W^-(\alpha, \delta, D^-, D^+).
$$

Likewise, since $\kappa$ is decreasing in its second argument (10.2), we find, again using that $\Phi_3$ is increasing in its second argument, that

$$
(14.4.2) \quad \kappa \left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D)}{2} \right) \leq \kappa \left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D^-)}{2} \right).
$$

Next note that, according to (14.1.5), we have $\Lambda(\delta, D) \leq \Lambda^+(\delta, D^-, D^+)$. Using that $\iota$ is decreasing in its second, third and fourth arguments (see 10.2), and again using that $\Phi_2$ and
\[ \Phi_3 \] are increasing in their second argument, we deduce that
\[ (14.4.3) \]
\[ \iota \left( \frac{\alpha}{2}, \frac{\Phi_2(\delta, D)}{2}, \frac{\Phi_3(\delta, D)}{2}, \Lambda(\delta, D) \right) \geq \iota \left( \frac{\alpha}{2}, \frac{\Phi_2(\delta, D^+)}{2}, \frac{\Phi_3(\delta, D^+)}{2}, \Lambda^+(\delta, D^-, D^+) \right). \]

We now distinguish two cases. The first is the case in which \((\alpha, \delta, D, D^-) \in \mathcal{W}_0^\delta\). It then follows from Remark 14.3 that \((\alpha, \delta, D) \in \mathcal{W}^\delta\). Hence, according to the definitions, we have \(V_{ST}^-(\alpha, \delta, D^-) = W^- (\alpha, \delta, D^-)\) and \(V_{ST}^{\text{near}}(\alpha, \delta, D) = W(\alpha, \delta, D)\) in this case, and the conclusion of the lemma follows from (14.4.1).

Now consider the case in which \((\alpha, \delta, D, D^-) \notin \mathcal{W}_0^\delta\), so that \(V_{ST}^- (\alpha, \delta, D^-, D^+)\) is given by (14.2.2). According to Remark 10.10, \(V_{ST}^{\text{near}}(\alpha, \delta, D)\) satisfies (10.10.1). But it follows from (14.4.1), (14.4.2) and (14.4.3) that the right-hand side of (10.10.1) is bounded below by the right-hand side of (14.2.2). Hence the conclusion of the lemma is true in this case as well.

**Lemma 14.5.** Let \(\lambda^-, \lambda^+\) and \(\delta\) be positive numbers such that \(\delta < \lambda^- \leq \lambda^+\). Then for every \(\alpha \in [\lambda^-, \lambda^+]\) and every \(D \in [\lambda^+, \infty)\), we have
\[ V_{ST}^{\text{near}}(\alpha, \delta, D) \geq B \left( \frac{\lambda^-}{2} \right) - 2\kappa \left( \frac{\lambda^+}{2}, \frac{\Phi_2(\delta, \lambda^+)}{2} \right) - 2\kappa \left( \frac{\lambda^+}{2}, \frac{\Phi_3(\delta, \lambda^+)}{2} \right). \]

**Proof.** It follows from Remark 10.10 and the definition of the function \(W\) that
\[ (14.5.1) \]
\[ V_{ST}^{\text{near}}(\alpha, \delta, D) \geq W(\alpha, \delta, D) - 2\kappa \left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D)}{2} \right) \]
\[ = B(\alpha/2) - 2\sigma \left( \frac{\alpha}{2}, \frac{D}{2}, \frac{\Phi_2(\delta, D)}{2}, \Psi(D, \Phi_2(\delta, D)) \right) - 2\kappa \left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D)}{2} \right). \]

Since \(D/2 \geq \lambda^+/2 \geq \alpha/2\), in the notation of 10.2 we have \(K_N(\zeta, D/2) = \emptyset\), where \(N \subset \mathbb{H}^3\) is an open ball of radius \(\alpha/2\) and \(\zeta\) is a point of \(\partial N\). The definitions in 10.2 then give \(\sigma(\alpha/2, D/2, \Phi_2(\delta, D)/2, \Psi(D, \Phi_2(\delta, D))) = \kappa(\alpha/2, \Phi_2(\delta, D)/2)\). Thus (14.5.1) becomes
\[ (14.5.2) \]
\[ V_{ST}^{\text{near}}(\alpha, \delta, D) \geq B(\alpha/2) - 2\kappa \left( \frac{\alpha}{2}, \frac{\Phi_2(\delta, D)}{2} \right) - 2\kappa \left( \frac{\alpha}{2}, \frac{\Phi_3(\delta, D)}{2} \right). \]

Since \(\alpha \geq \lambda^-\), we have \(B(\alpha/2) \geq B(\lambda^-/2)\). On the other hand, according to 10.2, \(\kappa\) is increasing in its first argument and decreasing in its second, while according to 10.5, \(\Phi_2\) and \(\Phi_3\) are increasing in their second argument; since \(\alpha \leq \lambda^+ \leq D\), it follows that for \(n = 2, 3\) we have \(\kappa(\alpha/2, \Phi_n(\delta, D)/2) \leq \kappa(\lambda^+/2, \Phi_n(\delta, \lambda^+)/2)\). The conclusion of the lemma therefore follows from (14.5.2).

**Theorem 14.6.** Let \(M\) be a closed, orientable hyperbolic 3-manifold such that \(\pi_1(M)\) is 5-free. Then \(\text{vol} M > 3.77\).

**Proof.** We apply Proposition 13.2, taking \(k = 5\), \(V_0 = 3.77001\), \(\lambda^- = \log 7.64\), \(\lambda^+ = \log 7.935\), \(\delta_0 = 0.033\), \(\delta_1 = 0.545\), and \(\mu^* = 1.1319\). In order to define the function \(\mathcal{M}\), we first define
a 51-tuple \((u_0, \ldots, u_{50})\) to be
\((7.64, 7.67, 7.7, 7.72, 7.73, 7.75, 7.76, 7.77, 7.78, 7.785, 7.79, 7.795, 7.8, 7.805, 7.81, 7.815, 7.82, 7.825, 7.83, 7.8325, 7.835, 7.8375, 7.84, 7.8425, 7.845, 7.8475, 7.85, 7.8525, 7.855, 7.857, 7.86, 7.8625, 7.865, 7.8675, 7.87, 7.874, 7.878, 7.88, 7.883, 7.885, 7.89, 7.895, 7.9, 7.905, 7.9075, 7.91, 7.9125, 7.915, 7.92, 7.9275, 7.935)\).

Set \(\lambda_i = \log u_i\), for \(i = 0, \ldots, 50\), so that \(\lambda^- = \lambda_0 < \cdots < \lambda_{50} = \lambda^+\). Now define a 50-tuple \((\mu_1, \ldots, \mu_{50})\) to be
\[
(1.156, 1.156, 1.152, 1.15, 1.147, 1.1457, 1.1445, 1.1434, 1.1428, 1.1423, 1.1415, 1.1415, 1.141, 1.1405, 1.1402, 1.1398, 1.13951.1391, 1.1388, 1.1388, 1.1385, 1.1385, 1.1381, 1.1381, 1.1378, 1.1375, 1.1375, 1.1373, 1.1373, 1.137, 1.137, 1.1367, 1.1367, 1.13635, 1.13635, 1.1362, 1.1361, 1.136, 1.1357, 1.1355, 1.1353, 1.1351, 1.1349, 1.13489, 1.1338, 1.1338, 1.1338, 1.1338)\).

We then take \(\mathcal{M}\) to be the step function defined on \([\lambda^-, \lambda^+]\) by setting \(\mathcal{M}(\lambda^-) = \mu_1\), and \(\mathcal{M}(\lambda) = \mu_i\) whenever \(1 \leq i \leq 50\) and \(\lambda_{i-1} < \lambda \leq \lambda_i\). (The partition of the interval \([\lambda^-, \lambda^+]\) defined by the \(\lambda_i\) is somewhat finer than is needed to define the step function \(\mathcal{M}\), but this partition will be useful later in the proof.)

Our choices of \(\lambda^-\) and \(\lambda^+\) give \(\log 7 < \lambda^- \leq \lambda^+ < \log 9\). We have \(f_3(\lambda^-) = 0.4715\ldots\), which with our choices of \(\delta_0\), \(\delta_1\) and \(\lambda^+\) gives \((\delta_0, \lambda^+/4, f_3(\lambda^-)) < \delta_1 < \log 3\). According to 11.9 we have \(I_5 = (\log 3, \log 9)\), so that \(\mu^* \in I_5\); furthermore, the definition of \(\mathcal{M}\) shows that it takes its values in \([1.1338, 1.156] \subseteq I_5\). It remains to verify Conditions (1a)–(6) of Proposition 13.2.

By direct calculation we find that \(g_5(\mu^*) = 2.7431\ldots\), and that \(V_{D,R}(((\log 5)/2), g_5(\mu^*), f_3(\mu^*)) = 3.7717\ldots > V_0\). This is Condition (1a).

If \(\lambda \in [\lambda^-, \lambda^+]\) is given, there is an index \(i \in \{1, \ldots, 50\}\) with \(\lambda_{i-1} \leq \lambda \leq \lambda_i\) and \(\mathcal{M}(\lambda) = \mu_i\). Let us set \(R = f_1(\lambda_i)/2\). Since \(f_1\) is monotone decreasing on its domain (see 7.1, we have \(R \in (0, f_1(\lambda)/2)\)). We have \(V_{D,R}(R, g_5(\mathcal{M}(\lambda)), f_3(\mathcal{M}(\lambda))) = V_{D,R}(f_1(\lambda_i)/2, g_5(\mu_i), f_3(\mu_i))\). For \(i = 1, \ldots, 50\), we verify directly that \(V_{D,R}(f_1(\lambda_i)/2, g_5(\mu_i), f_3(\mu_i)) > V_0\). (The smallest value of \(V_{D,R}(f_1(\lambda_i)/2, g_5(\mu_i), f_3(\mu_i))\) is 3.77017\ldots\), and is achieved when \(i = 9\).) This establishes Condition (1b).

Our choice of \(\delta_0\) guarantees that \(0 < \delta_0 < \min(0.7, (\log 4)/2)\) (so that \(V_{VSG}(5, \delta_0, \mu^*)\) is defined), and we have \(V_{VSG}(5, \delta_0, \mu^*) = V_{VSG}(5, 0.033, 1.319) = 3.7715\ldots > V_0\). This gives Condition (2).

To verify Condition (3), we note that, for every \(l\) with \(\delta_0 < l \leq \delta_1\), and for every \(h > 0\), the definition of \(\psi_5\) gives
\[
\psi_5(h, l, \mu^*) = V_{SG}^{\text{far}}(f_3(l) + h, l) + V_{SG}^{\text{far}}\left(\xi_3(Q(f_3(l) + h) + Q(l)), \frac{f_3(l) + h}{2}, \mu^*\right).
\]
Recall from 4.1 and 10.18 that the functions $Q$ and $f_3$ are strictly decreasing. Hence, given any $h^-, h^+, l^-, l^+$ with $0 \leq h^- < h^+$ and $\delta_0 \leq l^- < l^+$, we have $Q(f_3(l^-) + h^+) + Q(l^+) < Q(f_3(l^-)) + Q(l^-) = 1/2$, so that $\xi_3(Q(f_3(l^-) + h^+) + Q(l^+))$ is defined (see 6.10). We set

$$
\psi^-(h^-, h^+, l^-, l^+) = V_{SG}^{\text{near}}(f_3(l^+) + h^-, l^-) + V_{SG}^{\text{far}}\left(\xi_3(Q(f_3(l^-) + h^+) + Q(l^+)), \frac{f_3(l^-) + h^+}{2}, \mu^*\right).
$$

Now recall that $\xi_3$ is strictly increasing (see 6.10), and that $V_{SG}^{\text{far}}$ is increasing in its first argument and decreasing in its second (10.15). Recall also, from Lemma 10.13, that the function $V_{SG}^{\text{near}}$ is monotonically increasing in both its arguments. Combining these monotonicity properties with the monotonicity properties of $Q$ and $f_3$ already mentioned, we deduce that $\psi^-(h^-, h^+, l^-, l^+)$ is a lower bound for $\psi_5(h, l, \mu^*)$ whenever $h^- < h \leq h^+$ and $l^- \leq l \leq l^+$. (The strict inequality $h^- < h$ is important in the case $h^- = 0$, since $\psi_5$ is not defined when its first argument is zero.) Hence the inequality $\psi_k(h, l, \mu^*) > V_0$ holds whenever $(h, l)$ lies in a rectangle $(h^-, h^+) \times [l^-, l^+] \subset (0, \infty) \times [\delta_0, \delta_1]$ with $\psi^-(h^-, h^+, l^-, l^+) > V_0$.

For $0 \leq i \leq 2000$, we set $h_i = 0.54i/2000$, so that $0 < h_0 < \cdots < h_{2000} = 0.54$. We also define numbers $l_0, \ldots, l_{83}$ with $0.03 = l_0 < \cdots < l_{83} = \delta_1 = 0.545$, by stipulating that $l_j - l_{j-1}$ is equal to 0.005 for $1 \leq j \leq 6$; to 0.01 for $7 \leq j \leq 14$; to 0.02 for $15 \leq j \leq 27$; to 0.01 for $28 \leq j \leq 35$; to 0.005 for $36 \leq j \leq 42$; to 0.002 for $43 \leq j \leq 52$; to 0.001 for $53 \leq j \leq 57$; to 0.0005 for $58 \leq j \leq 63$; and to 0.0001 for $64 \leq j \leq 83$. Then the rectangle $(0, 0.54] \times [0.03, \delta_1]$, which contains $(0, 0.54] \times [\delta_0, \delta_1]$, is the union of the subrectangles $(h_{i-1}, h_i] \times [l_{j-1}, l_j]$ for $(i, j) \in \{1, \ldots, 2000\} \times \{1, \ldots, 83\}$. For each $(i, j)$ in the latter set, it is verified by direct computation that $\psi^-(h_{i-1}, h_i, l_{j-1}, l_j) > V_0$; the smallest value is 3.77029\ldots, which occurs at $i = 679$ and $j = 63$. This shows that the inequality in Condition (3) holds when $\delta_0 \leq l \leq \delta_1$ and $0 < h \leq 0.54$.

To establish the inequality when $\delta_0 \leq l \leq \delta_1$ and $h > 0.54$, we note that in this case, for any subinterval $[l^-, l^+]$ of $[\delta_0, \delta_1]$ containing $l$, the definition of $\psi_5$ gives

$$
\psi_5(h, l, \mu^*) \geq V_{SG}^{\text{near}}(f_3(l) + h, l).
$$

Since $V_{SG}^{\text{near}}$ is increasing in both its arguments by Lemma 10.13, and since $f_3$ is monotone decreasing, it follows that

$$
\psi_5(h, l, \mu^*) \geq V_{SG}^{\text{near}}(f_3(l^+) + 0.54, l^-).
$$

It therefore suffices to show that every $l \in [\delta_0, \delta_1]$ lies in a subinterval $[l^-, l^+]$ of $[\delta_0, \delta_1]$ such that

$$
V_{SG}^{\text{near}}(f_3(l^+) + 0.54, l^-) > V_0.
$$

But if we set $l_j^+ = \delta_0 + .001j$ for $j = 0, \ldots, 512$, so that $\delta_0 = l_0^+ < \cdots < l_{512}^+ = \delta_1$, then direct calculation shows that

$$
V_{SG}^{\text{near}}(f_3(l_j^+) + 0.54, l_{j-1}^-) > V_0
$$

for $j = 1, \ldots, 512$. (The smallest value of the left-hand side is 3.82\ldots, achieved when $j = 512$.) This completes the verification of Condition (3). (The last step in the argument is similar to the argument used in [12].)
As a preliminary to verifying Conditions (4a) and (4b), we observe that for any \( \alpha \in [\lambda^-, \lambda^+] \) and any \( D > \lambda^+ \), we may apply Lemma 14.5, letting \( \delta_1 \) play the role of \( \delta \), to obtain

\[
V_{ST}^{\text{near}}(\lambda_{i-1}, \delta_1, D) \geq B\left(\frac{\lambda^-}{2}\right) - 2\kappa\left(\frac{\lambda^+}{2}, \Phi_2 \left(\frac{\lambda^-}{2}, \frac{\lambda^+}{2}\right)\right) - 2\kappa\left(\frac{\lambda^+}{2}, \Phi_3 \left(\frac{\lambda^-}{2}, \frac{\lambda^+}{2}\right)\right) = 5.06\ldots.
\]

In particular:

**14.6.1.** For every \( \alpha \in [\lambda^-, \lambda^+] \) and every \( D > \lambda^+ \), we have \( V_{ST}^{\text{near}}(\alpha, \delta_1, D) > V_0 \).

We define numbers \( D_0, \ldots, D_{654} \) by stipulating that \( D_0 = \delta_0 \), and that \( D_j - D_{j-1} \) is equal to 0.00005 for \( 1 \leq j \leq 600 \); to 0.001 for \( 601 \leq j \leq 610 \); to 0.005 for \( 611 \leq j \leq 641 \); to 0.04 for \( 642 \leq j \leq 653 \); and to the quantity \( \lambda^+ - 1.22 = 0.85\ldots \) for \( j = 654 \). Thus we have \( \delta_0 = D_0 < \cdots < D_{654} = \lambda^+ \).

To verify Condition (4a), we first note that for any \( D > \lambda^+ \), the first term of the left-hand side of the inequality in (4a) is already greater than \( V_0 \) according to 14.6.1. It therefore suffices to verify the inequality in (4a) in the case where \( \delta_1 \leq D \leq \lambda^+ \). For such a value of \( D \), we have \( D \in [D_{j-1}, D_j] \) for some \( j \in \{1, \ldots, 654\} \). It now follows from Lemma 14.4, applied with \( \lambda^+ \) playing the role of \( \alpha \), that

\[
V_{ST}^{\text{near}}(\lambda^+, \delta_1, D) \geq V_{ST}^{\text{near}}(\lambda^+, \delta_1, D_{j-1}, D_j).
\]

On the other hand, since \( V^{\text{far}} \) is increasing in its first argument and decreasing in its second by 10.15, and \( \xi_3 \) is increasing and \( Q \) is decreasing (see 6.10 and 4.1), we have

\[
V_{ST}^{\text{near}}(\lambda^+, \delta_1, D) + V^{\text{far}} \left(\xi_3(Q(\lambda^+) + Q(D)), \frac{\lambda^+}{2}, \mu^+\right) \geq V^{\text{far}} \left(\xi_3(Q(\lambda^+) + Q(D_j)), \frac{\lambda^+}{2}, \mu^+\right)
\]

It follows from (14.6.2) and (14.6.3) that

\[
V_{ST}^{\text{near}}(\lambda^+, \delta_1, D) + V^{\text{far}} \left(\xi_3(Q(\lambda^+) + Q(D)), \frac{\lambda^+}{2}, \mu^+\right)
\geq V_{ST}^{\text{near}}(\lambda^+, \delta_1, D_{j-1}, D_j) + V^{\text{far}} \left(\xi_3(Q(\lambda^+) + Q(D_j)), \frac{\lambda^+}{2}, \mu^+\right).
\]

Hence, in order to establish the inequality in (4a), it suffices to check that

\[
V_{ST}^{\text{near}}(\lambda^+, \delta_1, D_{j-1}, D_j) + V^{\text{far}} \left(\xi_3(Q(\lambda^+) + Q(D_j)), \frac{\lambda^+}{2}, \mu^+\right) > V_0
\]

for each \( j \in \{1, \ldots, 654\} \). This is verified by 654 numerical calculations. The smallest value of the left-hand side of (14.6.4) is 3.7700697\ldots, achieved when \( j = 191 \). This completes the verification of Condition (4a).

To verify Condition (4b), we first note that if any \( \lambda \in [\lambda^-, \lambda^+] \) is given, we have \( \lambda \in [\lambda_{i-1}, \lambda_i] \) for some \( i \in \{1, \ldots, 50\} \), and that we then have \( \mathcal{M}(\lambda) = \mu_i \). We will show that the conclusion of (4b) holds with \( \lambda_{i-1} \in (0, \lambda] \) playing the role of \( \alpha \). Thus it suffices to show that for each \( i \in \{1, \ldots, 50\} \), for each \( \lambda \in [\lambda_{i-1}, \lambda_i] \) and for each \( D \geq \delta_1 \), we have

\[
V_{ST}^{\text{near}}(\lambda_{i-1}, \delta_1, D) + V^{\text{far}} \left(\xi_3(Q(\lambda) + Q(D)), \lambda/2, \mu_i\right) > V_0.
\]
For \( D > \lambda^+ \), the first term of the left-hand side of (14.6.5) is already greater than \( V_0 \) according to 14.6.1. It therefore suffices to verify (14.6.5) in the case where \( \delta_1 \leq D \leq \lambda^+ \). We then have \( D \in [D_{j-1}, D_j] \) for some \( j \in \{1, \ldots, 654\} \). It now follows from Lemma 14.4 that

\[
14.6.6 \quad V_{ST}^{\text{near}}(\lambda_{i-1}, \delta_1, D) \geq V_{ST}^{\text{far}}(\lambda_{i-1}, \delta_1, D_{j-1}, D_j).
\]

On the other hand, using the same monotonicity properties that were recalled in the proof of (4a), together with the fact that \( V^{\text{far}} \) is decreasing in its second argument (see 10.15), we find that

\[
14.6.7 \quad V^{\text{far}}\left(\xi_3(Q(\lambda) + Q(D)), \frac{\lambda_i}{2}, \mu_i\right) \geq V^{\text{far}}\left(\xi_3(Q(\lambda_i) + Q(D_j)), \frac{\lambda_i}{2}, \mu_i\right).
\]

It follows from (14.6.6) and (14.6.7) that

\[
V_{ST}^{\text{near}}(\lambda_{i-1}, \delta_1, D) + V^{\text{far}}\left(\xi_3(Q(\lambda) + Q(D)), \frac{\lambda}{2}, \mu_i\right) \geq V_{ST}^{\text{far}}(\lambda_{i-1}, \delta_1, D_{j-1}, D_j) + V^{\text{far}}\left(\xi_3(Q(\lambda_i) + Q(D_j)), \frac{\lambda_i}{2}, \mu_i\right).
\]

Hence, in order to establish (14.6.5), it suffices to check that

\[
14.6.8 \quad V_{ST}^{\text{near}}(\lambda_{i-1}, \delta_1, D_{j-1}, D_j) + V^{\text{far}}(\xi_3(Q(\lambda_i) + Q(D_j)), \lambda_i/2, \mu_i) > V_0
\]

for each \((i, j) \in \{1, \ldots, 50\} \times \{1, \ldots, 654\} \). This is verified by \(50 \times 654\) numerical calculations. The smallest value of the left-hand side of (14.6.8) is 3.770205\ldots, achieved when \( i = 19 \) and \( j = 329 \). This completes the verification of Condition (4b).

We have \( f_1(\lambda^-) = 1.7129\ldots, \lambda^+ = 2.0712\ldots, \) and \( f_2(\lambda^+) = 1.8313\ldots \). Hence, in the notation of Proposition 13.2, we have \( r = (\min(f_1(\lambda^-), \lambda^+, f_2(\lambda^+))/2 = f_1(\lambda^-)/2, \) and we find that \( B(r)/\text{density}(r) = 3.7764\ldots > V_0 \). This establishes Condition (5).

Finally, to verify Condition (6), let \( E \) be defined as in Proposition 13.2, with \( k = 5 \) and \( \lambda^+ = \log 7.935 \); we have \( E = 1.2589\ldots \). We have \( H_{\lambda^+} = H_{\log 7.935} = 1.8313\ldots \). Recall that the function \( \kappa \) is monotone increasing in its first argument and monotone decreasing in its second (10.2); that \( V^{\text{far}} \) is (weakly) monotone decreasing in its second argument (10.15); and that the function \( F_k \) defined in 13.1 is monotone decreasing.

Hence for every subinterval \([a, b]\) of \([H_{\lambda^+}, \lambda^+]\), and every \( \lambda \in [a, b] \), we have

\[
B\left(\frac{y}{2}\right) - 6\kappa\left(\frac{y}{2}, \frac{F_{\lambda^+}(y)}{2}\right) + V^{\text{far}}\left(E, \frac{y}{2}, \mu^*\right) \geq L(a, b),
\]

where

\[
L(a, b) = B\left(\frac{a}{2}\right) - 6\kappa\left(\frac{b}{2}, \frac{F_{\lambda^+}(b)}{2}\right) + V^{\text{far}}\left(E, \frac{b}{2}, \mu^*\right).
\]

To establish (6), it therefore suffices to exhibit \([H_{\lambda^+}, \lambda^+]\) as a union of intervals \([a, b]\) for which \( L(a, b) > 3.77 \). We write \([H_{\lambda^+}, \lambda^+] = [H_{\lambda^+}, 1.9] \cup [1.9, 2.0] \cup [2.0, \lambda^+] \), and we have \( L(H_{\lambda^+}, 1.9) = 3.86\ldots, L(1.9, 2.0) = 3.98\ldots, \) and \( L(2.0, \lambda^+) = 4.38\ldots \).
Volume and homology

Let us recall some standard definitions in 3-manifold theory. By a surface in a closed 3-manifold $M$ we will mean a connected 2-dimensional submanifold $S$ of $M$ which is tame, i.e. is smooth with respect to some smooth structure on $M$. (Some authors use the word “embedded” to emphasize that $S$ is a submanifold rather than a 2-manifold equipped with an immersion in $M$.) An orientable 3-manifold $M$ is said to be irreducible if $M$ is connected and every surface in $M$ which is homeomorphic to $S^2$ is the boundary of a 3-ball in $M$. An incompressible surface in an irreducible, closed, orientable 3-manifold $M$ is an orientable surface $S$ which is not a 2-sphere, but has the property that the inclusion homomorphism $\pi_1(S) \to \pi_1(M)$ is injective. If $M$ is hyperbolic, then $M$ is irreducible and every incompressible surface in $M$ has genus at least 2.

Following [11], we will say that $M$ is $(g, h)$-small, where $g$ and $h$ are given positive integers, if every incompressible surface in $M$ has genus at least $h$, and every separating incompressible surface in $M$ has genus at least $g$.

**Lemma 15.1.** Let $g$ be a positive integer, and $M$ be a closed, orientable hyperbolic 3-manifold that contains an incompressible (embedded) surface of genus $g$. Then there exist an integer $g' \geq 2$ and an incompressible surface $T \subset M$ of genus $g'$, such that either

(i) $g' \leq 2g - 2$, the surface $T$ separates $M$, and $M$ is $(g', g'/2 + 1)$-small, or

(ii) $g' \leq g$, the surface $T$ does not separate $M$, and $M$ is $(2g' - 1, g')$-small.

**Proof.** Let $g_0 \leq g$ denote the smallest positive integer which occurs as the genus of an incompressible surface in $M$. Since $M$ is hyperbolic we have $g_0 \geq 2$. Consider first the case in which $M$ contains a separating incompressible surface of genus $g_0$; we choose such a surface and denote it by $T_0$. According to our choice of $g_0$, every incompressible surface in $M$ has genus at least $g_0$; and since $g_0 \geq 2$ we have $g_0/2 + 1 \leq g_0$. Thus $M$ is $(g_0, g_0/2 + 1)$-small in this case. Furthermore, since $2 \leq g_0 \leq g$, we have in particular that $g_0 \leq 2g - 2$. Thus, in this case, Alternative (i) of the conclusion holds with $T = T_0$ and $g' = g_0$.

Now consider the case in which $M$ contains no separating incompressible surface of genus $g_0$. According to our choice of $g_0$, there is an incompressible surface $T_1 \subset M$ whose genus is $g_0$, and in this case $T_1$ must be non-separating.

We distinguish two subcases. The first is the subcase in which every separating incompressible surface in $M$ has genus at least $2g_0 - 1$. Our choice of $g_0$ guarantees that every incompressible surface in $M$ has genus at least $g_0$. Thus, according to the definition, $M$ is $(2g_0 - 1, g_0)$-small in this subcase, and Alternative (ii) of the conclusion holds with $T = T_1$ and $g' = g_0$.

There remains the subcase in which $M$ contains an incompressible surface whose genus is strictly less than $2g_0 - 1$. Let $g_2$ denote the smallest positive integer which occurs as the genus of a separating incompressible surface in $M$; then $g_2 \leq 2g_0 - 2$ and $g_2 \geq 2$ by hyperbolicity. Let us fix a separating incompressible surface $T_2$ of genus $g_2$. According to our choice of $g_0$, every incompressible surface in $M$ has genus at least $g_0 \geq g_2/2 + 1$; and according to
our choice of \( g_2 \), every separating incompressible surface in \( M \) has genus at least \( g_2 \). Thus, according to the definition, \( M \) is \((g_2, g_2/2 + 1)\)-small in this case. Since in addition we have \( g_2 \leq 2g_0 - 2 \leq 2g - 2 \), Alternative (i) of the conclusion holds with \( T = T_2 \) and \( g' = g_2 \). \( \square \)

We now review some more definitions from [11]. The Euler characteristic of a finitely triangulable space \( Y \) will be denoted \( \chi(Y) \), and we will set \( \overline{\chi}(Y) = -\chi(Y) \). If \( S \) is an incompressible surface in a closed, irreducible 3-manifold \( M \), we denote by \( M \setminus S \) the manifold with boundary obtained by splitting \( M \) along \( S \). Each component of the manifold \( M \setminus \) is irreducible and boundary-irreducible in the sense of [21]. Furthermore, each component of \( M \setminus \) is strongly atoral in the sense that its fundamental group has no rank-2 free abelian subgroup.

Any compact, orientable 3-manifold \( K \) which is irreducible and boundary-irreducible has a well-defined relative characteristic submanifold \( \Sigma_K \) in the sense of [23] and [22]. (In the notation of [22], \((\Sigma_K, \Sigma_K \cap \partial K)\) is the characteristic pair of \((K, \partial K)\).) If \( B \) is a compact, orientable 3-manifold whose components are irreducible and boundary-irreducible, we denote by \( \Sigma_B \subset B \) the union of the submanifolds \( \Sigma_K \), where \( K \) ranges over the components of \( B \). In the case where the components of \( B \) are strongly atoral, component \( C \) of \( \Sigma_B \) may be given the structure of an \( I \)-bundle over a compact 2-manifold with boundary in such a way that \( C \cap \partial B \) is the associated \( \partial I \)-bundle. We denote by \( \k(\Sigma) \) the union of all components of \( B - \Sigma_B \) that have (strictly) negative Euler characteristic.

Let \( B \) be a compact, orientable 3-manifold whose components are irreducible, boundary-irreducible, and strongly atoral. To say that \( B \) is acylindrical means that \( \Sigma_B = \emptyset \); this is equivalent to saying that \( \k(\Sigma) = B \).

**Proposition 15.2.** Let \( g \) be a positive integer, and \( M \) be a closed, orientable hyperbolic 3-manifold that contains an incompressible surface of genus \( g \). Suppose that the Heegaard genus of \( M \) is strictly greater than \( 2g + 1 \). Then there exist an integer \( g' \geq 2 \) and an incompressible surface \( S \subset M \) of genus \( g' \), such that either

(i) \( g' \leq 2g - 1 \), the surface \( S \) separates \( M \), and \( M \setminus S \) has an acylindrical component; or

(ii) \( g' \leq 2g - 1 \), the surface \( S \) separates \( M \), and for each component \( B \) of \( M \setminus S \) we have \( \k(B) \neq \emptyset \); or

(iii) \( g' \leq g \), the surface \( S \) does not separate \( M \), and \( \overline{\chi}(\k(M \setminus S)) \geq 2g' - 2 \).

**Proof.** Let \( G \geq 2g + 2 \) denote the Heegaard genus of \( M \).

The hypothesis of the present proposition include those of Lemma 15.1. Hence there exist an integer \( g' \) and an incompressible surface \( T \subset M \) of genus \( g' \), such that one of the alternatives (i) or (ii) of the conclusion of Lemma 15.1 holds.

Consider first the case in which Alternative (i) of Lemma 15.1 holds. Thus \( g' \leq 2g - 2 \), the surface \( T \) separates \( M \), and \( M \) is \((g', g'/2 + 1)\)-small. We have \( G \geq 2g + 2 \geq g' + 4 \). We now apply Theorem 5.1 of [11], with \( g' \) playing the role of \( g \) in that theorem. The theorem asserts that if \( M \) is a closed, orientable 3-manifold containing a separating incompressible surface of some genus \( g' \), and if \( M \) has Heegaard genus at least \( g' + 4 \) and is \((g', g'/2 + 1)\)-small, then
$M$ contains a separating incompressible surface $S$ of genus $g$ such that either $M \setminus S$ has at least one acylindrical component, or $\text{kish}(B) \neq \emptyset$ for each component $B$ of $M \setminus S$. Thus in this case, one of the alternatives (i), (ii) of the present proposition holds.

Now consider the case in which Alternative (ii) of Lemma 15.1 holds. Thus $g' \leq g$, the surface $T$ does not separate $M$, and $M$ is $(2g' - 1, g')$-small. We have $G \geq 2g + 2 \geq 2g' + 2$. In this case we set $S = T$, and we apply Theorem 3.1 of [11], with $g'$ playing the role of $g$ in that theorem. The theorem asserts that if $M$ is a closed, orientable, hyperbolic 3-manifold containing a non-separating incompressible surface $S$ of genus $g'$, and if $\chi(\text{kish}(M \setminus S)) < 2g' - 2$, and if $M$ is $(2g' - 1, g')$-small, then the Heegaard genus $G$ of $M$ is at most $2g' + 1$. Since in the present situation we have $G \geq 2g' + 2$, we must have $\chi(\text{kish}(M \setminus S)) \geq 2g' - 2$. Thus in this case, Alternative (iii) of the present proposition holds.

**Proposition 15.3.** Let $M$ be a closed, orientable hyperbolic 3-manifold, and let $g$ be a positive integer. Suppose that $M$ contains an incompressible surface of genus $g$, and that the Heegaard genus of $M$ is strictly greater than $2g + 1$. Then $\text{vol } M > 6.45$.

*Proof.* Since $g$ is the genus of some incompressible surface in the closed, orientable hyperbolic 3-manifold $M$, we have $g \geq 2$. Thus the hypothesis implies that the Heegaard genus of $M$ is at least 6. According to [11, Theorem 6.1], a closed, orientable hyperbolic 3-manifold which contains an incompressible surface of genus 2, and has Heegaard genus at least 6, must have volume greater than 6.45. Thus the conclusion of the proposition is true if $M$ contains an incompressible surface of genus 2. For the rest of the proof, we will assume that $M$ contains no such surface.

The hypotheses of the present proposition are the same as those of Proposition 15.2. Hence we may fix an integer $g' \geq 2$ and an incompressible surface $S \subset M$ of genus $g'$, such that one of the alternatives (i)—(iii) of the conclusion of Proposition 15.2 holds. Since we have assumed that $M$ contains no incompressible surface of genus 2, we have $g' \geq 3$.

We claim:

$\chi(\text{kish}(M \setminus S)) \geq 2.$  \hspace{1cm} (15.3.1)

To prove (15.3.1), we first consider the case in which Alternative (i) of Proposition 15.2 holds. In this case we may label the components of $M \setminus S$ as $B_1$ and $B_2$ in such a way that $B_1$ is acylindrical. Then $\text{kish}(B_1) = B_1$; and since by definition every component of $\text{kish}(B_2)$ has negative Euler characteristic, we have $\chi(\text{kish}(B_2)) \geq 0$. Hence $\chi(\text{kish}(M \setminus S)) = \chi(\text{kish}(B_1)) + \chi(\text{kish}(B_2)) \geq \chi(\text{kish}(B_1)) = \chi(B_1) = \chi(S)/2 = g' - 1$. Since $g' \geq 3$, (15.3.1), is established in this case.

In the case where Alternative (ii) of Proposition 15.2 holds, let $B_1$ and $B_2$ denote the components of $M \setminus S$. For $i = 1, 2$, since $\text{kish}(B_i) \neq \emptyset$, and since by definition every component of $\text{kish}(B_i)$ has negative Euler characteristic, we have $\chi(\text{kish}(B_i)) \geq 1$; hence $\chi(\text{kish}(M \setminus S)) = \chi(\text{kish}(B_1)) + \chi(\text{kish}(B_2)) \geq 2$, and (15.3.1) holds. In the case where Alternative (iii) of Proposition 15.2 holds, we have $\chi(\text{kish}(M \setminus S)) = \chi(\text{kish}(B_1)) + \chi(\text{kish}(B_2)) \geq 2g' - 2 \geq 2$; thus (15.3.1) is established in all cases.
Let $V_{oct} = 3.66 \ldots$ denote the volume of a regular ideal hyperbolic octahedron. According to [6, Theorem 9.1], whenever $S$ is an incompressible surface in a closed, orientable hyperbolic 3-manifold $M$, we have $\text{vol} M \geq V_{oct} \chi(kish(M \setminus S))$. In the present situation, (15.3.1) then gives $\text{vol} M \geq 2V_{oct} > 6.45$.

**Theorem 15.4.** Let $M$ be a closed, orientable hyperbolic 3-manifold with $\text{vol} M \leq 3.77$. Then $\dim H_1(M; \mathbb{F}_2) \leq 10$.

**Proof.** Assume that $\dim H_1(M; \mathbb{F}_2) \geq 11$. We apply [16, Proposition 8.1], which asserts that if $k \geq 3$ is an integer, and $M$ is a closed orientable hyperbolic 3-manifold with $\dim H_1(M; \mathbb{F}_2) \geq \max(3k - 4, 6)$, then either $\pi_1(M)$ is $k$-free, or $M$ contains an incompressible surface of genus at most $k - 1$. (The actual statement of the result quoted is slightly stronger than this.) Taking $k = 5$, we deduce that either $\pi_1(M)$ is 5-free, or $M$ contains an incompressible surface of genus at most 4. If $\pi_1(M)$ is 5-free, then Theorem 14.6 above gives $\text{vol} M > 3.77$, a contradiction to the hypothesis. Now suppose that $M$ contains an incompressible surface of some genus $k \leq 4$. We have $\dim H_1(M; \mathbb{F}_2) \geq 11 \geq 9 \geq 2k + 1$. In particular, the Heegaard genus of $M$ is strictly greater than $2k + 1$. Proposition 15.3 now gives $\text{vol} M > 6.45$, and again the hypothesis is contradicted. □

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