An occupation time formula for semimartingales in $\mathbb{R}^N$

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Abstract

Inspired by coarea formula in geometric measure theory, an occupation time formula for continuous semimartingales in $\mathbb{R}^N$ is proven. The occupation measure of a semimartingale, for $N \geq 2$, is singular with respect to Lebesgue measure but it has a bounded density “transversal” to a foliation, under proper assumptions. In the particular case of the foliation given locally by the distance function from a manifold, the transversal density is related to a geometric local time of the semimartingale at the manifolds of the foliation.

1 Introduction

The occupation time formula for real valued continuous semimartingales reads

$$\int_0^T f(X_t) \, d\langle X \rangle_t = \int_{\mathbb{R}} f(a) \, L^a_T(X) \, da$$

for all positive Borel functions $f : \mathbb{R} \to \mathbb{R}$, where $L^a_T(X)$ is the local time at $a$ of $X$ over $[0,T]$. This formula is related to the correction term of Itô formula and explains its extensions beyond $C^2$ functions. The aim of this paper is to discuss a possible extension to semimartingales in $\mathbb{R}^N$, $N \geq 2$.

Let $X$ be a continuous semimartingale in $\mathbb{R}^N$. When $\langle X^j, X^k \rangle_t$ is differentiable a.s. in $(t,\omega)$ for all $j,k \in 1,\ldots,N$, we introduce the matrix valued process $g_t$ defined as $g_{t}^{jk} = d\langle X^j, X^k \rangle_t/dt$. In Section 2 we prove a multidimensional extension of occupation time formula inspired by coarea formula in geometric measure theory. A particular but relevant case of Theorem 10 is the following statement.

Theorem 1 Assume that $cI_n \leq g_t \leq CI_N$ a.s. in $(t,\omega)$ for some constants $C > c > 0$. Let $\phi \in C^2(\mathbb{R}^N)$ be a function such that $\inf_{x \in A} |\nabla \phi(x)| > 0$ on an open set $A \subseteq \mathbb{R}^N$. Then there exists a random bounded compact support non-negative function $L_{t,A,\phi}$ and random probability measures $Q_{t,A}(a,\cdot), Q_{t,A,\phi}^1(a,\cdot)$ concentrated on $\Gamma_a = \{x \in A : \phi(x) = a\}$ for

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\( a.e. \ a \in \mathbb{R}, \) such that
\[
\int_0^T 1_{X_t \in A} f(X_t) \, d\langle X^i \rangle_t = \int_{\mathbb{R}} \left( \int_{\Gamma_a} f(x) Q^i_{A,\phi}(a, \, dx) \right) L^a_{i,A,\phi} da. \tag{1}
\]

We omit to denote the dependence of \( L^a_{i,A,\phi} \) also on \( T \) and \( X \) since they will always be a priori given. The formula extends to \( \int_0^T f(X_t) \, d\langle X^i, X^j \rangle_t \) by polarization.

Localization on a set \( A \) is important in applications, since the non-degeneracy conditions may be too severe on the full space. In section 2.2 we give more general conditions of non-degeneracy of \( X \) and \( \phi \) than those assumed in Theorem 1, and accordingly, Theorem 10 is more general; this additional generality has applications to certain singular problems, as described in Sections 3 and 4.

As we said, formula (1) has the flavor of coarea formula in geometric measure theory. However, the classical coarea formula disintegrates Lebesgue measure along a foliation and the measures \( Q(a, dx) \) on the leaves are the \((N-1)\)-dimensional Hausdorff measures. Here we disintegrate a random measure \( \mu_i \), defined by
\[
\mu_i(f) = \int_0^T f(X_t) \, d\langle X^i \rangle_t
\]
which is singular with respect to Lebesgue measure, for \( N \geq 2 \). Thus the measures \( Q^i_{A,\phi}(a, dx) \) do not have good regularity properties.

The key point is, on the contrary, that the measure in the “transversal” direction to the leaves, \( L^a_{i,A,\phi} \), has a density with respect to Lebesgue measure. This fact is false if we replace \( X \) by a smooth function: the occupation measure of a smooth deterministic function may concentrate at some point, and the transversal density \( L^a \) is lost. Thus it is the regularity of fluctuations of a semimartingale with good non-trivial quadratic variation that produces the densities \( L^a_{i,A,\phi} \), similarly to the existence of the local time in dimension 1. Formula (1) captures a regularity property of the occupation measures \( \mu_i \) of certain semimartingales. Other regularity properties of occupation measure are reviewed for instance in [6]. The regularity of occupation measure is also a topic of interest in harmonic analysis, see for instance [15].

By formula (1), the integral \( \int_0^T |f(X_t)| \, d\langle X^i \rangle_t \) is finite for functions \( f \) that are singular along an \((N-1)\)-dimensional manifold \( \Gamma \), with a certain degree of integrability of the singularity. In Section 3 devoted to examples and applications, we call this class of functions \( L^1_{1\mathrm{loc}}(\Gamma^\perp) \) and we prove a general result of integrability.

**Theorem 2** Let \( X \) be a continuous semimartingale in \( \mathbb{R}^N \) such that \( cI_n \leq g_t \leq CI_N \) a.s. in \((t, \omega)\) for some constants \( C > c > 0 \). Let \( \Gamma \) be an \((N-1)\)-dimensional orientable manifold of class \( C^2 \), closed and without boundary, embedded in \( \mathbb{R}^N \). If \( f \in L^1_{1\mathrm{loc}}(\Gamma^\perp) \) then \( P \left( \int_0^T |f(X_t)| \, dt < \infty \right) = 1. \)
This result is, for some applications, more precise than the results offered by other approaches, in which the integrability degree of \( f \) is related to the dimension \( N \), see Remark 19.

A natural question is whether \( L_{i,A,\phi}^a \) is equal to the local time of some 1-dimensional semimartingale. When this holds true, the process \( a \mapsto L_{i,A,\phi}^a \) has a càdlàg modification. Another question is whether \( L_{i,A,\phi}^a \) is a sort of local time of \( X \) at the leave \( \Gamma_a \). We answer these two questions in a very particular case: when \( \phi \) is locally the distance from a given manifold \( \Gamma \) and \( X \) is a Brownian semimartingale, namely when \( X \) is continuous and \( g = I_N \).

In Section 4 we denote the identical \( L_{i,A,\phi}^a \) by \( L_{A,\phi}^a \), we prove its local representation as the (symmetric) local time of an 1-dimensional semimartingale.

**Theorem 3** Let \( X \) be a Brownian semimartingale in \( \mathbb{R}^N \). Let \( \Gamma \) be an \((N-1)\)-dimensional orientable manifold of class \( C^2 \), closed and without boundary, embedded in \( \mathbb{R}^N \). Then, for the distance function \( d(\cdot, \Gamma) \), there exists a neighborhood \( V \) of \( \Gamma \) and an extension \( \phi \) outside \( V \) such that the assumptions of Theorem 11 are satisfied with \( A = V \), and additionally, we have that for each fixed \( \omega \in \Omega \) there exists \( \varepsilon_1(\omega) > 0 \) such that

\[
L_{A,\phi}^a = L_{\Gamma}^a (\phi (X))
\]

for a.e. \( a < \varepsilon_1(\omega) \), and they are both null if \( a < 0 \). In particular, on the random interval \((-\infty, \varepsilon_1] \) the process \( (\omega, a) \mapsto L_{A,\phi}^a(\omega) \) is the modification of a càdlàg process.

In Section 5 we define the random variable \( L_{\Gamma}^{\Gamma_a} (X) \) and we call it the geometric local time of \( X \) at \( \Gamma_a \) on \([0, T] \). In the case \( \Gamma \) has an uniform neighborhood in which the distance its regular (see Section 4.1.1), we prove an additional representation of \( L_{A,\phi}^a \) as the geometrical local time at the leave \( \Gamma_a \).

**Theorem 4** Under the hypotheses of Theorem 3 if there exists \( \varepsilon > 0 \) such that \( \mathcal{V} = \{ x \in \mathbb{R}^N : d(x, \Gamma) < \varepsilon \} \), then we have

\[
L_{A,\phi}^a = L_{\Gamma}^{\Gamma_a} (X)
\]

for a.e. \( a \in [0, \varepsilon_0) \), where \( \Gamma_a = \{ x \in \mathcal{V} : d(x, \Gamma) = a \} \).

Then we compare \( L_{\Gamma}^{\Gamma_a} (X) \) with the similar but different local time on graphs defined by Peskir \cite{10, 11}. The research reported here has been strongly influenced by it. For the purpose of a generalized Itô formula for \( u (X_t) \) where \( u : \mathbb{R}^N \to \mathbb{R} \) is smooth except on a graph, the notion of \cite{10, 11} is very convenient. We have tried to apply that notion by local graph charts to our set-up, in order to avoid new definitions, but this turns out to be not easy and thus we prefer to develop the definition of \( L_{\Gamma}^{\Gamma_a} (X) \) from scratch in part inspired by \cite{13}. The definition given here is conceptually similar to \cite{11} but it has two differences which may be of interest: i) it treat \((N-1)\)-dimensional manifolds which are not necessarily global graphs, ii) it is intrinsic of the manifold and does not depend on the coordinate system.
2 The occupation time formula

2.1 Disintegration of random measures

Given a finite Borel measure \( \mu \) on \( \mathbb{R}^N \) and a Borel function \( \phi : \mathbb{R}^N \to \mathbb{R} \), set

\[
\nu = \phi_\# \mu
\]

the push-forward of \( \mu \) under \( \phi \) \((\nu = \mu \circ \phi^{-1})\). Then there exists a family of probability measures \((\mu_a)_{a \in \mathbb{R}}\) on \( \mathbb{R}^N \), uniquely determined for \( \nu \)-a.e. \( a \in \mathbb{R} \), called conditional probabilities of \( \mu \) w.r.t. \( \nu \), such that:

i) for every Borel set \( E \), \( a \mapsto \mu_a(E) \) is measurable

ii)

\[
\mu(f) = \int_{\mathbb{R}} \left( \int_{\Gamma_a} f(x) \mu_a(dx) \right) \nu(da)
\]

for all positive Borel functions \( f : \mathbb{R}^N \to \mathbb{R} \), where \( \Gamma_a = \{ x \in \mathbb{R}^N : \phi(x) = a \} \)

iii) \( \mu_a(\Gamma_a) = 1 \), for \( \nu \),a.e. \( a \in \mathbb{R} \).

This is a consequence of Rohlin disintegration theorem (see [14], [2] for a recent version and references therein).

Let \( \{ \mu^\omega; \text{a.e. } \omega \in \Omega \} \) be a random finite Borel measure on \( \mathbb{R}^N \), on a probability space \((\Omega, \mathcal{F}, P)\) that is universally measurable (see [14]); for instance, Polish spaces are universally measurable.

Proposition 5 The push-forward \( \nu^\omega = \phi_\# \mu^\omega \) is a random finite Borel measure on \( \mathbb{R} \); moreover the family of probability measures \( \{ \mu^\omega_a; \nu \text{-a.e. } a \in \mathbb{R}, P \text{-a.e. } \omega \in \Omega \} \) has the properties that for every Borel set \( E \), \((a, \omega) \mapsto \mu^\omega_a(E) \) is measurable on \((\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F})\); and for \( P \)-a.e. \( \omega \in \Omega \), properties 2 and 3 above hold true.

Proof. The proof that \( \nu^\omega \) is a random measure is a simple exercise. Then we could apply the previous result of Rohlin \( \omega \)-wise and construct the family \( \mu^\omega_a \); properties ii) and iii) above would be obvious but not property i), the joint measurability of \((a, \omega) \mapsto \mu^\omega_a \). To overcome this problem, let us construct a jointly measurable \( \mu^\omega_a \) by another procedure and then deduce the other properties. We define \( M = \mu^\omega \otimes P \) on the product space \( \mathbb{R}^N \times \Omega \), we consider the map \( p : \mathbb{R}^N \times \Omega \mapsto \mathbb{R} \times \Omega \) defined as \( p(x, \omega) = (\phi(x), \omega) = (\phi \otimes Id)(x, \omega) \), and we can apply the Rohlin disintegration theorem with respect to the partition \( \{ \Gamma_a \times \{ \omega \}, a \in \mathbb{R}, \omega \in \Omega \} \). We obtain a unique random family of measures \( \{ \tilde{\mu}^\omega_a \} \) concentrated on the sets of the partition such that

\[
\int_{\mathbb{R}^N \times \Omega} f(x, \omega)dM = \int_{\mathbb{R} \times \Omega} \left( \int_{\Gamma_a \times \{ \omega \}} f(x, \zeta)d\tilde{\mu}^\omega_a \right) dp_{\#}M
\]
for all positive measurable functions \( f : \mathbb{R}^N \times \Omega \to \mathbb{R} \). Now we define the family \( \mu^\omega_a \) as

\[
\mu^\omega_a(B) := \tilde{\mu}^\omega_a(B \times \Omega) = \tilde{\mu}^\omega_a(B \times \{\omega\})
\]

for each \( \omega \in \Omega \), \( a \in \mathbb{R} \) and \( B \in \mathcal{B}(\mathbb{R}^N) \). We have to prove that it satisfies properties i) and ii); for \( P \)-a.e. \( \omega \in \Omega \) property iii) above is trivial. We have that the function \((a, \omega) \mapsto \tilde{\mu}^\omega_a(E)\) is measurable for each set \( E \in \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{F} \), and in particular for the sets like \( B \times \Omega \), for each \( B \in \mathcal{B}(\mathbb{R}^N) \): so it is jointly measurable also \((a, \omega) \mapsto \mu^\omega_a(B)\). Moreover,

\[
p_a M = (\phi \otimes \text{Id})_2(\mu^\omega \otimes P) = \nu^\omega \otimes P
\]

so

\[
\int_{\mathbb{R}^N \times \Omega} f(x, \omega) d\mu^\omega \otimes P = \int_{\mathbb{R} \times \Omega} \left( \int_{\Gamma_a} f(x, \omega) d\mu^\omega_a \right) d\nu^\omega \otimes P
\]

that is the same of

\[
E \left[ \int_{\mathbb{R}^N \times \Omega} f(x, \omega) d\mu^\omega - \int_{\mathbb{R} \times \Omega} \left( \int_{\Gamma_a} f(x, \omega) d\mu^\omega_a \right) d\nu^\omega \right] = 0.
\]

We can define the event \( \Omega_1 = \{ \omega \in \Omega : \int f(x, \omega) d\mu^\omega > \int \left( \int_{\Gamma_a} f(x, \omega) d\mu^\omega_a \right) d\nu^\omega \} \) and the random function \( f_1(\omega, \cdot) := f(\omega, \cdot) \) if \( \omega \in \Omega_1 \) and zero otherwise. Then from the previous equation we have

\[
E \left[ \int_{\mathbb{R}^N \times \Omega} f_1(x, \omega) d\mu^\omega - \int_{\mathbb{R} \times \Omega} \left( \int_{\Gamma_a} f_1(x, \omega) d\mu^\omega_a \right) d\nu^\omega \right] = 0
\]

hence \( P \)-a.e.

\[
\int_{\mathbb{R}^N \times \Omega} f_1(x, \omega) d\mu^\omega - \int_{\mathbb{R} \times \Omega} \left( \int_{\Gamma_a} f_1(x, \omega) d\mu^\omega_a \right) d\nu^\omega = 0
\]

so \( \Omega_1 \) is negligible. In the same way we can prove that \( \Omega_2 := \{ \omega \in \Omega : \int f(x, \omega) d\mu^\omega < \int \left( \int_{\Gamma_a} f(x, \omega) d\mu^\omega_a \right) d\nu^\omega \} \) is negligible too. We obtained that for \( P \)-a.e. \( \omega \in \Omega \) it must be

\[
\int f(x, \omega) d\mu^\omega = \int \left( \int_{\Gamma_a} f(x, \omega) d\mu^\omega_a \right) d\nu^\omega.
\]

In particular we have proved property ii) above, extended to all random test functions. \( \square \)
2.2 Non-degeneracy conditions

**Definition 6** Given the continuous semimartingale $X$ in $\mathbb{R}^N$, consider the random positive measure $\eta_X (dt)$ defined by

$$
\int_0^T \varphi (t) \eta_X (dt) = \sum_{i=1}^N \int_0^T \varphi (t) d\langle X^i \rangle_t + \sum_{i,j=1}^N \int_0^T \varphi (t) d\langle X^i + X^j \rangle_t
$$

for all $\varphi \in C([0,T])$. Let $A$ be a Borel set of $\mathbb{R}^N$. Let $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a Borel function. We say that $\phi (X)$ controls $X$ in quadratic variation on the set $A$ if

i) $\phi (X)$ is a continuous semimartingale

ii) there is a random constant $C_A > 0$ such that

$$
\int_0^T 1_{X_t \in A} |\varphi (t)| \eta_X (dt) \leq C_A \int_0^T 1_{X_t \in A} |\varphi (t)| d\langle \phi (X) \rangle_t
$$

for all $\varphi \in C([0,T])$.

The condition that $\phi (X)$ controls $X$ in quadratic variation will be the main assumption of the multidimensional occupation time formula of the next section. Now we want to give a very general sufficient condition for it, that we use several times in the paper.

By $\text{Lip}_{\text{loc}} (A)$ of an open set $A$ we denote the set of locally Lipschitz continuous functions on $A$ and we recall that such functions are differentiable almost everywhere. Moreover, we shall say that $\langle X \rangle_t$ is Lipschitz continuous if, for each $j,k = 1, \ldots, N$, there is an adapted process with bounded paths $g^{jk}_t$ such that, with probability one,

$$
\langle X^j, X^k \rangle_t = \int_0^t g^{jk}_s ds.
$$

If $X$ solves a differential equation of the form $sX_t = b(t, X_t) dt + \sigma (t, X_t) dW_t$ then $g_t = \sigma \sigma^T$ and certain assumptions we impose on $g_t$ correspond to usual non-degeneracy assumptions on $\sigma$.

**Definition 7** Let $X$ be a continuous semimartingale in $\mathbb{R}^N$ and $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a Borel function such that $\phi (X)$ is a continuous semimartingale. Assume that $\langle X \rangle_t$ is Lipschitz continuous and set

$$
g^{jk}_t = d \langle X^j, X^k \rangle_t / dt
$$

such that

$$
d\langle \phi (X) \rangle_t = \sum_{i,j=1}^N \partial_j \phi (X_t) \partial_k \phi (X_t) g^{jk}_t dt.
$$
Let $A$ be an open set in $\mathbb{R}^N$ such that $\phi \in \text{Lip}_{loc}(A)$ and let $D_\phi \subset A$ be a set of full measure where $\phi$ is differentiable; assume that, for $P$-a.e. $\omega \in \Omega$, for a.e. $t \in [0,T]$ the property $X_t(\omega) \in A$ implies $X_t(\omega) \in D_\phi$. We say that $\langle \phi(X) \rangle$ is non-degenerate on $A$ if

$$\inf_{t \in [0,T]: X_t \in D_\phi} \sum_{i,j=1}^N \frac{\partial_j \phi(X_t) \partial_k \phi(X_t) g_{ij}^{jk}}{\sum_{j,k=1}^N \partial_j \phi(X_t) \partial_k \phi(X_t) g_{jk}^{jk}} g_t^{ij} > 0$$

with probability one. Notice that this sum is always a non-negative quantity.

**Proposition 8** Let $X$ be a continuous semimartingale in $\mathbb{R}^N$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ be a Borel function, such that $\langle \phi(X) \rangle$ is non-degenerate on an open set $A$ as described in Definition 7. Then $\phi(X)$ controls $X$ in quadratic variation on $A$.

**Proof.** We have only to check condition (ii) of Definition 7. We treat separately each component of $\eta_X$ and restrict our proof to a component of the form $d \langle X^i + X^i' \rangle_t$, the others being similar. We have

$$\int_0^T 1_{X_t \in A} |\varphi(t)| \, d \langle X^i + X^i' \rangle_t$$

$$= \int_0^T 1_{X_t \in A} |\varphi(t)| \frac{\sum_{j,k=1}^N \partial_j \phi(X_t) \partial_k \phi(X_t) g_{ij}^{jk}}{\sum_{j,k=1}^N \partial_j \phi(X_t) \partial_k \phi(X_t) g_{jk}^{jk}} \left( g_t^{ii} + g_t^{i'i'} + 2g_t^{ii'} \right) \, dt$$

$$\leq C \int_0^T 1_{X_t \in A} |\varphi(t)| \left| \sum_{j,k=1}^N \partial_j \phi(X_t) \partial_k \phi(X_t) g_{ij}^{jk} \right| \, dt$$

for a suitable random constant $C > 0$ (here we use that $g_t^{ii}$ are bounded and the non-degeneracy assumption for $\langle \phi(X) \rangle$ on $A$)

$$= C \int_0^T 1_{X_t \in A} |\varphi(t)| \sum_{j,k=1}^N \partial_j \phi(X_t) \partial_k \phi(X_t) g_{ij}^{jk} \, dt$$

(we have used the non-negativity of $\sum_{j,k=1}^N \partial_j \phi(X_t) \partial_k \phi(X_t) g_{ij}^{jk}$)

$$= C \int_0^T 1_{X_t \in A} |\varphi(t)| \, d \langle \phi(X) \rangle_t$$

and the proof is complete. □

**Corollary 9** If $c I_N \leq g_t \leq C I_N$ a.s. in $(t, \omega)$, for some constants $C \geq c > 0$, $\phi \in C^2(\mathbb{R}^N)$, and $\inf_{x \in A} |\nabla \phi(x)| > 0$ on an open set $A \subset \mathbb{R}^N$, then $\phi(X)$ controls $X$ in quadratic variation on $A$. 

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Proof. We have that
\[
\inf_{t \in [0,T]: X_t \in A} \sum_{i,j=1}^{N} \partial_j \phi (X_t) \partial_k \phi (X_t) g^i_k = \inf_{t \in [0,T]: X_t \in A} |\nabla \phi (X_t)|^2
\]
\[
= \inf_{t \in [0,T]: X_t \in A} |\nabla \phi_A (X_t)|^2 \geq \min_{x \in A} |\nabla \phi_A (x)| > 0.
\]
Hence the hypotheses of non-degeneracy of Definition 7 are satisfied taking \( D_\phi = A \) and using the Itô formula.

2.3 Local occupation time formula

Theorem 10 Let \( X \) be a continuous semimartingale in \( \mathbb{R}^N \), \( \phi : \mathbb{R}^N \to \mathbb{R} \) be a Borel function, \( A \) be an open set of \( \mathbb{R}^N \). Assume that \( \phi (X) \) controls \( X \) in quadratic variation on \( A \). Then there exists a random bounded compact support non-negative function \( L^a_{i,A,\phi} \) and random probability measures \( Q^i_{A,\phi} (a,dx) \), such that \( Q^i_{A,\phi} (a,\cdot) \) is concentrated on \( \Gamma_a = \{ x \in A : \phi (x) = a \} \) for a.e. \( a \in \mathbb{R} \), and that
\[
\int_0^T 1_{X_t \in A_i} f (X_t) d \langle X^i \rangle_t = \int_\mathbb{R} \left( \int_{\Gamma_a} f (x) Q^i_{A,\phi} (a,dx) \right) L^a_{i,A,\phi} da.
\]
Moreover we have
\[
L^a_{i,A,\phi} = \lim_{\varepsilon \to 0} \frac{1}{2 \varepsilon} \int_0^T 1_{X_t \in A_i (a-\varepsilon, a+\varepsilon)} (\phi (X_t)) d \langle X^i \rangle_t
\]
for a.e. \( a \). Similar results hold for \( \int_0^T f (X_t) d \langle X^i + X^j \rangle_t \).

Proof. Consider the random Borel measure \( \mu^i_A \) on \( \mathbb{R}^N \) defined as
\[
\mu^i_A (f) = \int_0^T 1_{X_t \in A_i} f (X_t) d \langle X^i \rangle_t
\]
for all \( f \in C_b (\mathbb{R}^N) \). We have
\[
\mu^i_A (A^c) = \int_0^T 1_{X_t \in A_i \forall t \notin A} d \langle X^i \rangle_t = 0
\]
namely \( \mu^i_A \) is concentrated on \( A \). From Proposition 5 there exists a family of probability measures \( Q^i_{A,\phi} (a,\cdot) \) concentrated on \( \Gamma_a \), for \( \nu \)-a.e. \( a \in \mathbb{R} \), and a Borel measure \( \nu^i_{A,\phi} \) on \( \mathbb{R} \), such that
\[
\mu^i_A (f) = \int_\mathbb{R} \left( \int_{\Gamma_a} f (x) Q^i_{A,\phi} (a,dx) \right) \nu^i_{A,\phi} (da)
\]
for all positive Borel functions $f : \mathbb{R}^N \to \mathbb{R}$. We want to prove that $\nu^i_{A,\phi}$ has a bounded density with respect to Lebesgue measure.

If we choose $f$ of the form $f(x) = \theta(\phi(x))$ with a positive Borel function $\theta : \mathbb{R} \to \mathbb{R}$, we get
\[
\int_0^T 1_{X_t \in A}\theta(\phi(X_t)) d\langle X^i \rangle_t = \int_{\mathbb{R}} \theta(a) \nu^i_{A,\phi}(da).
\]
Thus, let us consider the random linear functional
\[
F_{i,A}(\theta) = \int_0^T 1_{X_t \in A}\theta(\phi(X_t)) d\langle X^i \rangle_t.
\]
We have, by the main assumption,
\[
|F_{i,A}(\theta)| \leq \int_0^T 1_{X_t \in A}|\theta(\phi(X_t))| d\langle X^i \rangle_t
\leq C_A \int_0^T 1_{X_t \in A}|\theta(\phi(X_t))| d\langle \phi(X) \rangle_t \leq C_A \int_0^T |\theta(\phi(X_t))| d\langle \phi(X) \rangle_t.
\]
By the occupation time formula for $\phi(X)$,
\[
= C_A \int_{\mathbb{R}} |\theta(a)| L^a_T(\phi(X)) da
\]
where $L^a_T(\phi(X))$ is the local time at $a$ of the continuous semimartingale $\phi(X)$ on $[0,T]$. This local time, as a function of $a$, is, with probability one, càdlàg and bounded support. Hence
\[
\leq C'_A \int_{\mathbb{R}} |\theta(a)| da
\]
for a new random constant $C'_A > 0$. The functional $F_{i,A}$ is thus ($\omega$-wise) bounded continuous on $L^1(\mathbb{R})$, and it is non-negative on non-negative functions, and thus there exists a bounded non-negative function $L^{a}_{i,A,\phi}$ such that
\[
F_{i,A}(\theta) = \int_{\mathbb{R}} \theta(a) L^{a}_{i,A,\phi} da.
\]
This proves the first claim of the theorem.

If we use $\theta = 1_{(a-\varepsilon,a+\varepsilon)}$ as a test function, with a given $a$, we obtain that
\[
\frac{1}{2\varepsilon} \int_0^T 1_{X_t \in A1_{(a-\varepsilon,a+\varepsilon)}}(\phi(X_t)) d\langle X^i \rangle_t = \frac{1}{2\varepsilon} \int_{a-\varepsilon}^{a+\varepsilon} L^{a}_{i,A,\phi} da'.
\]
Thanks to the Lebesgue theorem, we get
\[
L^{a}_{i,A,\phi} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T 1_{X_t \in A1_{(a-\varepsilon,a+\varepsilon)}}(\phi(X_t)) d\langle X^i \rangle_t
\]
for a.e. $a$. The proof is complete. □
Proof of Theorem 1. It readily follows from Corollary 9 and Theorem 10. □

3 Examples and applications

3.1 Singular sets of the functions φ

The difficulty to apply Theorem 1 on the full space $\mathbb{R}^N$ is in the fact that the non-degeneracy assumption (replaced by $|\nabla \phi(x)| > 0$) is quite restrictive. However, in some cases the singular set is polar for the process and the theory applies using Theorem 10. Let us see this global corollary.

The set $\{x \in \mathbb{R}^N : |\nabla \phi(x)| = 0\}$ will be called the singular set of $\phi$ and denoted by $S_\phi$. Recall that $g_{ij}^k := d\langle X^j, X^k \rangle_t/dt$.

Corollary 11 Let $X$ be a continuous semimartingale in $\mathbb{R}^N$ such that $cI_n \leq g_t \leq CI_N$ a.s. in $(t,\omega)$ for some constants $C > c > 0$. Let $\phi \in C^2(\mathbb{R}^N)$ be a function such that the singular set $S_\phi$ is polar for $X$. Then the results of Theorem 10 hold.

Proof. The quadratic variation $\langle X \rangle_t$ is obviously Lipschitz continuous. We have only to check that $\langle \phi(X) \rangle$ is non-degenerate on the full $\mathbb{R}^N$. We have

$$\sum_{i,j=1}^{N} \partial_i \phi(X_t) \partial_j \phi(X_t) g_{ij}^k \geq c |\nabla \phi(X_t)|^2.$$ 

Given a.s. $\omega$, the function $t \mapsto |\nabla \phi(X_t(\omega))|^2$ is continuous (composition of continuous functions) and different from zero at each point, since $X_t(\omega)$ does not touch the polar set $S_\phi$. Thus, on $[0,T]$, the function $t \mapsto |\nabla \phi(X_t(\omega))|^2$ is strictly positive.

The assumptions of Theorem 10 hold true and thus the result holds. □

Example 12 If $X$ is a Brownian motion in $\mathbb{R}^N$ with $X_0 = x_0$ and $S_\phi$ is given by a finite number of points, with $\nabla \phi(x_0) \neq 0$, then the assumptions of Corollary 11 are satisfied. An example is

$$\phi(x) = |x|^2$$

when $x_0 \neq 0$. If $x_0 = 0$, we have to localize as in Theorem 1 just by taking $A = \mathbb{R}^N \setminus \{x : |x| \leq \varepsilon\}$, for some $\varepsilon > 0$.

Example 13 If $X$ is a Brownian semimartingale in $\mathbb{R}^N$ and the singular set $S_\phi$ is empty, then the assumptions of Corollary 11 are satisfied. For example this happens for

$$\phi(x) = x^N - g(x^1, ..., x^{N-1})$$

where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a $C^2$ function. Indeed,

$$|\nabla \phi(x)|^2 = |\nabla g(x^1, ..., x^{N-1})|^2 + 1 > 0$$
everywhere. Thus, in the case of a Brownian semimartingale, we may apply the theory. This is related to [11] (which is much more general). The manifolds $\Gamma_a$ are translations of the graph of $g$.

3.2 Integration of functions $f$ with singularities

The question treated in this section is when, for $i = 1, \ldots, N$,

$$P\left( \int_0^T |f(X_t)| d\langle X^i \rangle_t < \infty \right) = 1$$

for functions $f$ which are not bounded. Let us distinguish two cases:

i) we already have a foliation $\Gamma_a := \{ \phi(x) = a \}, a \in \mathbb{R}$, and $f$ is not bounded only in the transversal direction to the foliation

ii) we have only one manifold $\Gamma$ and a function $f$ which is unbounded only in the neighborhood of $\Gamma$.

In the first case we only need to apply the formula; in the second case we have to construct a suitable foliation.

Concerning case (i), we give two examples: Example 14 is elementary and global, Example 16 is its general version.

**Example 14** Let $X$ be a continuous semimartingale in $\mathbb{R}^N$. Assume $X$ has Lipschitz continuous quadratic variation with

$$P\left( \inf_{t \in [0,T]} \phi_t^{NN} > 0 \right) = 1.$$  

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function of the form

$$f(x) = f_1(x)f_2(x_N)$$

$x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, where $f_1 \in C(\mathbb{R}^N)$ and $f_2 : \mathbb{R} \to \mathbb{R}$ is a locally integrable function. Then (2) holds. To prove this fact we use $\phi : \mathbb{R}^N \to \mathbb{R}$ defined as $\phi(x) = x_N$: one has

$$\sum_{i,j=1}^N \partial_j \phi (X_t) \partial_k \phi (X_t) g_{ij} = g_t^{NN}$$

and thus the assumptions of Theorem 14 hold with $A = \mathbb{R}^N$. Therefore

$$\int_0^T |f(X_t)| d\langle X^i \rangle_t \leq C_1 \int_0^T |f_2(X_t^N)| d\langle X^i \rangle_t$$

$$= C_1 \int_{\mathbb{R}} \left( \int_{\Gamma_a} |f_2(x_N)| Q^i_{A,\phi} (a, dx) \right) \mathcal{L}^a_{i,A,\phi} da$$

11
where, denoted by $K$ and $B_i$ random compact sets containing respectively the image of the curve $X$ and the support of $\mathcal{L}_{i,A,\phi}^0$, we have set $C_1 = \sup_{K} |f_1|$, $C_2 = \sup_{B_i} \mathcal{L}_{i,A,\phi}^0$.

The following simple application of the occupation time formula will be used below. For instance, in the case when $\langle X^i \rangle_t = t$ it implies that $X_t \notin A \cap \phi^{-1}(N)$ for a.e. $t \in [0, T]$ with probability 1.

**Lemma 15** Under the assumptions and with the notations of Theorem 10, with probability one,

$$
\int_0^T 1_{A \cap \phi^{-1}(E)} (X_t) \, d \langle X^i \rangle_t = 0
$$

for every Borel set $E \subset \mathbb{R}$ of zero Lebesgue measure.

**Proof.** By the local occupation time formula of Theorem 10 we have

$$
\int_0^T 1_{A \cap \phi^{-1}(E)} (X_t) \, d \langle X^i \rangle_t = \int_\mathbb{R} \left( \int_{\phi^{-1}(a)} 1_{A \cap \phi^{-1}(E)} (x) Q_{A,\phi}^i (a, dx) \right) \mathcal{L}_{i,A,\phi}^0 \, da
$$

$$
\leq \int_\mathbb{R} 1_E (a) \mathcal{L}_{i,A,\phi}^0 \, da = 0.
$$

If $X$ is a Brownian semimartingale in $\mathbb{R}^N$ and we apply Lemma 15 we obtain

$$
\int_0^T 1_{A \cap \phi^{-1}(E)} (X_t) \, dt = 0
$$

for every Borel set $E \subset \mathbb{R}$ of zero Lebesgue measure. Due to this, the assumption in Example 14 that $f_2 : \mathbb{R} \to \mathbb{R}$ is a locally integrable function may be replaced by the assumption $f_2 \in L^1_{\text{loc}} (\mathbb{R})$; in other words, the result does not change if we modify $f_2$ on a zero-measure set or in the case when $f_2$ is not even defined on a zero-measure set. Indeed if $f_2$ is not defined on the zero-measure set $E$, then $f(x) = 1_A f_1 (x) f_2 (\phi (x)) + 1_A g(x)$ is not defined on the set $A \cap \phi^{-1}(E)$. But, with probability one, $X_t \in (A \cap \phi^{-1}(E))^c$ for a.e. $t \in [0, T]$ and thus $\int_0^T |f(X_t)| \, dt$ is well defined. This is the integral $\int_0^T |f(X_t)| \, d \langle X^i \rangle_t$ examined by Example 14.

**Example 16** Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function of the form

$$
f(x) = 1_A f_1 (x) f_2 (\phi (x)) + 1_A g(x)
$$

where $A$ is an open set, $f_1 \in C(\overline{A})$, $f_2 \in L^1_{\text{loc}} (\mathbb{R})$, $g \in C(A^c)$, $\phi : \mathbb{R}^N \to \mathbb{R}$ is such that $\phi(X)$ controls $X$ in quadratic variation on $A$. Let $X$ have a Lipschitz continuous quadratic
variation and assume that \( \langle \phi(X) \rangle \) is non-degenerate on \( A \). Then (2) holds. Indeed, from Theorem 10,

\[
\int_0^T |f(X_t)| d\langle X^i \rangle_t
= \int_0^T 1_{X_t \notin A} |f(X_t)| d\langle X^i \rangle_t + \int_\Gamma (\int_{\Gamma_a} |f(x)| Q_{A,\phi}^i (a, dx)) L_{i,A,\phi}^a da
\leq C_1 \langle X^i \rangle_T + C_2 \int_{B_i,A} |f_2(a)| L_{i,A,\phi}^a da
\]

where, denoted by \( K \) and \( B_{i,A} \) random compact sets containing respectively the image of the curve \( X \) and the support of \( L_{i,A,\phi}^a \), we have set \( C_1 = \sup_{K \cap A^c} |g|, C_2 = \sup_{K \cap \Gamma} |f_1| \).

We conclude as in the previous example.

Let us see now examples of case (ii) above, namely when we have a function \( f \) which is singular only along an \((N-1)\)-dimensional manifold \( \Gamma \). The problem here is to construct a suitable function \( \phi \). Let us see first a case which relates to [11].

**Example 17** Let us continue Example 13. We assume that \( X \) is a Brownian semimartingale and \( f: \mathbb{R}^N \setminus \Gamma \to \mathbb{R} \) is a continuous function, where \( \Gamma \) is the graph of a \( C^2 \) function \( g: \mathbb{R}^{N-1} \to \mathbb{R} \). Consider the function

\[
\phi(x) = x^N - g(x^1, ..., x^{N-1})
\]

the associated sets \( \Gamma_a \) and the numbers \( M_a^\phi(|f|) = \max_{x \in \Gamma_a} |f(x)| \) for every \( a \neq 0 \). If

\[
\int_{-1}^1 M_a^\phi(|f|) da < \infty
\]

then (2) holds. To prove this claim it is sufficient to apply the result of Example 16, with

\[
A = \{ x : -1 < \phi(x) < 1 \}, \quad g = f |A^c, \quad f_2(a) = M_a^\phi(|f|)
\]

\[
f_1(x) = \begin{cases} f(x)/f_2(\phi(x)) \quad \text{if} \quad f_2(\phi(x)) > 0 \\ 0 \quad \text{if} \quad f_2(\phi(x)) = 0 \end{cases}
\]

In the proof of Theorem 2 we will use some geometric results that we will develop in Section 4.2. It is presented here because of its conceptual unity with the previous. Let \( \Gamma \) be a manifold in \( \mathbb{R}^N \) and \( f: \mathbb{R}^N \setminus \Gamma \to \mathbb{R} \) be a measurable function. For every \( a, R > 0 \) define

\[
M_{a,R}(|f|) := \sup_{x \in \Gamma_{a,R}^d} |f(x)|
\]

where \( \Gamma_{a,R}^d = \{ x \in B(0, R) : d(x, \Gamma) = a \} \) and \( B(0, R) \) is a ball.
Definition 18. We say that \( f : \mathbb{R}^N \setminus \Gamma \rightarrow \mathbb{R} \) is in \( L^1_{loc} (\Gamma^\perp) \) if for every \( R > 0 \)
\[
\int_0^1 M_{t,R}(|f|) \, dt < \infty
\]
and \( f \) is bounded on every compact set in \( \mathbb{R}^N \setminus \Gamma \).

Proof of Theorem 2. Let \( \mathcal{U} \) and \( \delta_\Gamma \) be given by Proposition 24. Let \( \mathcal{V} \), with \( \overline{\mathcal{V}} \subset \mathcal{U} \), and \( \phi \in C^2 (\mathbb{R}^N) \) (extension of \( \delta_\Gamma \)) be given by Corollary 27. We have that \( \phi (X) \) controls \( X \) in quadratic variation on \( \mathcal{V} \) and in particular on \( A := \mathcal{V} \cap \{ x : -1 < d(x, \Gamma) < 1 \} \). From Theorem 1 we have
\[
\int_0^T |f (X_t)| \, d \langle X^i \rangle_t = \int_0^T 1_{X_t \notin A} |f (X_t)| \, d \langle X^i \rangle_t + \int_{\mathbb{R}^+} \left( \int_{\Gamma_a} |f (x)| Q^i_{A,\phi} (a, dx) \right) L^a_{i,A,\phi} da.
\]
For each fixed \( \omega \in \Omega \) the trajectory of the process \( X \) remains inside a compact ball \( B(0, R_\omega) \) and
\[
\int_0^T 1_{X_t \notin A} |f (X_t)| \, d \langle X^i \rangle_t (\omega) \leq \int_0^T 1_{A^c \cap B(0, R_\omega)} |f (X_t)| \, d \langle X^i \rangle_t (\omega) < \infty
\]
because \( A^c \cap B(0, R_\omega) \) is a compact set in \( \mathbb{R}^N \setminus \Gamma \). Moreover
\[
\int_{\mathbb{R}^+} \left( \int_{\Gamma_a} |f (x)| Q^i_{A,\phi} (a, dx) \right) L^a_{i,A,\phi} da(\omega) = \int_{\mathbb{R}^+} \left( \int_{\Gamma_a} |f (x)| Q^i_{A,\phi} (a, dx) \right) L^a_{i,A,\phi} da(\omega)
\]
using that for each \( a > 0 \) we have \( \Gamma_a \cup \Gamma_{-a} = \Gamma^d_{a,R} \). So it is
\[
\leq \int_{\mathbb{R}^+} \left( \int_{\Gamma^d_{a,R}} M_{a,R}(|f|) Q^i_{A,\phi} (a, dx) \right) L^a_{i,A,\phi} da(\omega) = \int_{\mathbb{R}^+} (M_{a,R}(|f|)) L^a_{i,A,\phi} da(\omega) < \infty
\]
because \( L^a_{i,A,\phi} \) is bounded. \( \square \)

Remark 19. In the case of a Brownian motion \( B \), using its explicit Gaussian density one can show that \( P \left( \int_0^T |f (B_t)| \, dt < \infty \right) = 1 \) is true for functions \( f \) of class \( L^q_{loc} (\mathbb{R}^N) \) with \( q > \frac{N}{2} \lor 1 \) (see for instance [9]). In a sense, the previous theorem gives us a more precise result, valid for all Brownian semimartingales and when the singularity set of \( f \) is of a special type.
3.3 SDEs with singular coefficients

The idea of the following example is taken from Cerný-Engelbert \[3\], where a similar case is treated in dimension one.

The problem from which the example arises is to construct an example of non-existence for an SDE in \(\mathbb{R}^N\) of the form

\[
dX_t = b(X_t) \, dt + dW_t, \quad X_0 = x_0
\]

outside the present classes of \(b\)'s where existence is known, in order to test the sharpness of such classes. We refer to the very general result of \[9\] which states that strong (local) existence (and pathwise uniqueness) is known when \(b \in L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)\) for some \(p > N \vee 2\). Also the result of existence of weak solutions of \[1\] for distributional drift, when particularized to distributions realized by functions, gives the same class. Thus it looks optimal, even for weak existence.

The function

\[
b(x) = C |x|^{-2} x
\]

is of class \(L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)\) only for \(p < N\), thus it is outside the boundary of the previous theory. We prove that, in the particular case \(C = -\frac{1}{2}\) and \(x_0 = 0\), no weak solution exists.

First, by weak solution \((X, W)\) on \([0, T]\) (on a local random time interval the argument is similar) we mean that there is a filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}_t, P)\), an \(\mathcal{F}_t\)-Brownian motion \(W\) in \(\mathbb{R}^d\), an \(\mathcal{F}_t\)-adapted continuous process \((X_t)_{t \geq 0}\) in \(\mathbb{R}^d\), such that

\[
\int_0^T |b(X_t)| \, dt < \infty
\]

and, a.s.,

\[
X_t = \int_0^t b(X_s) \, ds + W_t.
\]

Hence \(X\) is a continuous semimartingale, with quadratic covariation \(\langle X^i, X^j \rangle_t = \delta_{ij} t\) between its components. Take (to be more precise than above)

\[
b(x) = \begin{cases} 
0 & \text{if } x = 0 \\
-\frac{1}{2|x|^2} x & \text{if } x \neq 0
\end{cases}
\]

We shall write \(b(x) = -\frac{1_{x \neq 0}}{2|x|^2} x\).

**Proposition 20** The SDE with this vector field \(b\) and \(x_0 = 0\) does not have any weak solution \((X, W)\).
Proof. Assume, by contradiction, that \((X,W)\) is a weak solution. By Itô formula,
\[
d|X_t|^2 = -1_{X_t \neq 0} dt + 2X_t \cdot dW_t + dt = 1_{X_t = 0} dt + 2X_t \cdot dW_t.
\]
From Theorem 1 with \(A = \mathbb{R}^N\) and \(\phi(x) = x_1\), we get
\[
\int_0^T 1_{X_t = 0} dt = \int_{\mathbb{R}} \left( \int_{\Gamma_a} 1_{\{0\}}(x) Q_{A,\phi}(a,dx) \right) \mathcal{L}^a_{A,\phi} da \leq \int_{\mathbb{R}} \eta(a) \mathcal{L}^a_{A,\phi} da = 0
\]
where \(\eta(0) = Q_{A,\phi}(0,\{0\}) \leq 1\), \(\eta(a) = 0\) for \(a \neq 0\) and we omitted the (identical) dependence by \(i \in 1, \ldots, N\). Hence
\[
|X_t|^2 = \int_0^t 2X_s \cdot dW_s.
\]
Therefore \(|X_t|^2\) is a positive local martingale, null at \(t = 0\). This implies that \(|X_t|^2 \equiv 0\) hence \(X_t \equiv 0\). But this contradicts the fact that \(\langle X^i, X^j \rangle_t = \delta_{ij} t\). □

Remark 21 The property \(\int_0^T 1_{X_t = 0} dt = 0\), where we have used our multidimensional occupation time formula, can be proved also in other ways. The point of this example is not to show a striking application where the occupation time formula is strictly necessary but an example where it can be used to prove something useful and non-trivial in a line.

4 Embedding of a manifold \(\Gamma\) in a foliation

In some examples we have a process \(X\) in \(\mathbb{R}^N\), a function \(f: \mathbb{R}^N \to \mathbb{R}\) for which we want to consider \(\int_0^T f(X_t) d\langle X^i \rangle_t\), and an \((N-1)\)–dimensional manifold \(\Gamma\) in \(\mathbb{R}^N\) where \(f\) is singular (see Section 3).

In this section we pose the problem to embed \(\Gamma\) in a foliation \(\{\Gamma_a; a \in \mathbb{R}\}\) given by level sets of some function \(\phi: \mathbb{R}^N \to \mathbb{R}\), satisfying the hypotheses of Theorem 10. In order to solve this problem, we propose to use as \(\phi\) a smooth extension of the signed distance function. This requires that \(\Gamma\) is an orientable manifold with some other simple properties. The signed distance function then exists of class \(C^2\) in a neighborhood of \(\Gamma\) and we may extend it on the full space, still of class \(C^2\), equal to zero outside a larger neighborhood.

This construction is however insufficient to prove Theorem 3, which is the second aim of this section. An extension equal to zero does not work because it affects the sets \(\phi^{-1}(a)\) for small \(a\) (to be precise, Proposition 32 part (iii) would fail and it plays a basic role in the proof of Theorem 3, see identity (5)). We then choose to work with the distance function \(d(\cdot, \Gamma)\), suitably extended outside a neighborhood \(\mathcal{V}\) of \(\Gamma\). Since it is not smooth on \(\Gamma\), we have to overcome some difficulties. At the end we develop the necessary geometric preliminaries to prove Theorem 3 and later on Theorem 4 in Section 5.
4.1 Construction of $\phi$ given a manifold $\Gamma$ of class $C^2$

Given the manifold $\Gamma$, there are several functions $\phi$ such that $\{\phi = 0\} = \Gamma$. Basic requisite for us is that $\phi(X)$ is a continuous semimartingale. This is clearly achieved if $\phi$ is of class $C^2$ but there are interesting cases in which, in order to have other properties of $\phi$, we have to give up with a full $C^2$-regularity.

The second requisite on $\phi$ is a form on non-degeneracy in order to have that $\phi(X)$ controls $X$ in quadratic variation on a set of interest $A$ (typically a neighborhood of $\Gamma$). The key for non-degeneracy is that $\nabla \phi$ should not vanish (not too much) in $A$. Along with the requirement $\{\phi = 0\} = \Gamma$, this means that we have to look for non-trivial functions $\phi$.

Finally, in Section 4.2, we shall see the advantage of having further properties, related to the eikonal equation

$$|\nabla \phi(x)| = 1$$

(aimed to hold at least locally around $\Gamma$). Thus we pose in this section the following question: given a manifold $\Gamma$, construct a function $\phi : \mathbb{R}^N \to \mathbb{R}$ such that

- $\{\phi = 0\} = \Gamma$,
- $\phi(X)$ is a continuous semimartingale,
- $|\nabla \phi| = 1$ in a neighborhood of $\Gamma$.

Natural candidates are the distance function, $x \mapsto d(x, \Gamma)$ and the signed distance function, when defined. The advantage of the distance function is that it is globally and elementary defined in full generality on $\Gamma$, but when $x$ crosses $\Gamma$ it is not differentiable (it is also non-differentiable far from $\Gamma$, but this is less relevant). The drawback of the signed distance function is first of all the difficulty to define it, but then it has the advantage of some smoothness also around $\Gamma$. But let us first mention a case when the signed distance has an obvious definition.

**Example 22** Let $D$ be a non-empty open set in $\mathbb{R}^N$ with non-empty complementary set $D^c$. Let $\Gamma$ be the boundary of $D$. We call signed distance function from $\Gamma$ the function

$$\delta_\Gamma(x) = \begin{cases} 
    d(x, \Gamma) & \text{if } x \in D \\
    -d(x, \Gamma) & \text{if } x \in D^c 
\end{cases}.$$

If $\Gamma$ is piecewise smooth, then $\delta_\Gamma$ is differentiable a.e. and, where it is differentiable, it satisfies $|\nabla \delta_\Gamma(x)| = 1$. If $\Gamma$ is of class $C^2$, then there is a neighborhood $\mathcal{U}$ of $\Gamma$ where $\delta_\Gamma$ is of class $C^2$, and the neighborhood can be taken of the form

$$\mathcal{U}_\varepsilon = \{x \in \mathbb{R}^N : d(x, \Gamma) < \varepsilon\}$$

for some $\varepsilon > 0$ if $D$ is bounded. A discussion of these and other properties can be found in [7].
Inspired by these properties, let us axiomatize some properties of a signed distance function from a general manifold, so that it will be more clear what we use in each general statement. For the definition of embedded manifold see [8].

**Notation 23** If $\Gamma$ is an $(N - 1)$-dimensional orientable manifold of class $C^2$, closed and without boundary and embedded in $\mathbb{R}^N$, then we call it a leaf-manifold.

**Proposition 24** Let $\Gamma$ be a leaf-manifold. Then there exist an open neighborhood $U$ of $\Gamma$ and a function $\delta_\Gamma : U \rightarrow \mathbb{R}$ such that:

i) $\delta_\Gamma \in C^2 (U)$

ii) $|\delta_\Gamma (x)| = d (x, \Gamma)$ for all $x \in U$.

This (not unique) function is called signed distance function $\delta_\Gamma$ on the open neighborhood $U$ of $\Gamma$.

Proof in Appendix A.

**Lemma 25** Following Proposition 24 assume that the manifold $\Gamma$ has a signed distance function $\delta_\Gamma$ on $U$. Then one has:

iii) on every connected component of $U \setminus \Gamma$, the function $\delta_\Gamma (\cdot)$ is either identically equal to $d (\cdot, \Gamma)$ or to $-d (\cdot, \Gamma)$

iv) $d (x, \Gamma)$ is of class $C^2$ on $U \setminus \Gamma$

v) each $x \in U$ has a unique point $P_\Gamma (x) \in \Gamma$ of minimal distance.

vi) $|\nabla \delta_\Gamma (x)| = 1$ for all $x \in U$

Proof in Appendix A.

**Remark 26** In Proposition 24 we imposed that $\Gamma$ has no boundary since it would be incompatible with the required properties just in the simplest case of $\Gamma$ equal to the closed $(N - 1)$-dimensional disk. Indeed in general if $U \setminus \Gamma$ has only a single connected component, by property (iii) we would have that $\delta_\Gamma$ is equal to $d (\cdot, \Gamma)$ (or to $-d (\cdot, \Gamma)$) in the whole $U \setminus \Gamma$, so by continuity it is true also in $U$, and this is a contradiction with its $C^2$ regularity.

The first aim of this subsection was to construct a function $\phi$, out of a manifold $\Gamma$, in order to apply Theorem 1. If $\Gamma$ is a leaf manifold, we have solved this problem. Indeed, let $U$ and $\delta_\Gamma$ be given by Proposition 24. Let $V$ be a neighborhood of $\Gamma$ such that $V \subset U$. There exists a $C^2$ function $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\phi = \delta_\Gamma$ on $V$ and $\phi = 0$ outside $U$. Take $A = V$. Then, by Lemma 25 part (vi) we have $\inf_{x \in A} |\nabla \phi (x)| > 0$. All the assumptions of Theorem 1 hold true. To summarize:
Corollary 27 Let $X$ be a continuous semimartingale in $\mathbb{R}^N$ such that $cI_n \leq g_t \leq CI_N$ a.s. in $(t, \omega)$ for some constants $C > c > 0$. Let $\Gamma$ be leaf-manifold. Let $\mathcal{U}$ be given by Proposition 24 and let $\mathcal{V}$ be an open neighborhood of $\Gamma$ such that $\overline{\mathcal{V}} \subset \mathcal{U}$. Let $\phi \in C^2(\mathbb{R}^N)$ be an extension of $\delta_\Gamma$ from $\overline{\mathcal{V}}$. Then the assumptions of Theorem 1 are satisfied with $A = \mathcal{V}$.

4.1.1 Uniform neighborhoods $\mathcal{U}_\varepsilon(\Gamma)$

We investigate a slightly more restrictive condition since it will turn out to be relevant in Section 5, and useful in some of the next proofs.

Notation 28 Given $\varepsilon > 0$ and a set $S$, we denote by $\mathcal{U}_\varepsilon(S)$ the open neighborhood of $S$ of radius $\varepsilon$: the set of all $x \in \mathbb{R}^N$ such that $d(x, S) < \varepsilon$.

Remark 29 When $\mathcal{U} = \mathcal{U}_\varepsilon(\Gamma)$ in Proposition 24, we may choose $\mathcal{V} = \mathcal{U}_\varepsilon(\Gamma)$ with $\varepsilon_1 < \varepsilon$, in Corollary 27. We shall always make this choice, below.

Not all $C^2$ orientable manifolds $\Gamma$ have a “uniform” neighborhood of the form $\mathcal{U}_\varepsilon(\Gamma)$ satisfying the conditions of Proposition 24. For instance, in $\mathbb{R}^2$, the graph of the function $y = \sin x^2$ has a neighborhood $\mathcal{U}$ as in Proposition 24 but not a neighborhood of the form $\mathcal{U}_\varepsilon(\Gamma)$ ($\mathcal{U}$ has to shrink at infinity). Anyways if $\Gamma$ is compact, we can always take a neighborhood $\mathcal{V}$ of $\Gamma$ such that it is bounded and $\overline{\mathcal{V}} \subset \mathcal{U}$; hence we can define $\varepsilon_0 := \max_{x \in \overline{\mathcal{V}}} d(x, \Gamma)$ and restrict the signed distance on the set $\mathcal{U}_{\varepsilon_0}(\Gamma) \subset \mathcal{U}$.

We discuss now a general class of manifolds which fulfill Proposition 24 with $\mathcal{U} = \mathcal{U}_\varepsilon(\Gamma)$. A subset $S$ of $\mathbb{R}^N$ is called proximally smooth, or with positive reach, if exists $\varepsilon > 0$ such that for each $x \in \mathcal{U}_\varepsilon(S)$ (defined above) there exists a unique minimizer of the distance function from $x$ to $S$ (see also [4]). This number $\varepsilon$ is called the reach of the set $S$.

We remind here the Mises theorem (see [16]): it states that for each closed set $F \subset \mathbb{R}^N$ the one-sided directional derivatives $D_v d(x, F) := \lim_{\varepsilon \to 0^+} \frac{d(x+\varepsilon v, F) - d(x, F)}{\varepsilon}$ are well defined for all $x \in \mathbb{R}^N \setminus F, v \in \mathbb{R}^N$. In particular if we call $P_F(x)$ the set of metric projections of $x \not\in F$ on $F$, we have that $\forall v \in \mathbb{R}^N$,

$$D_v d(x, F) = \inf \left\{ \frac{v \cdot (x-y)}{|x-y|}, y \in P_F(x) \right\}.$$  \hfill (3)

Using (3) we obtain that the distance from a set $S$ with positive reach is of class $C^1$ in $\mathcal{U}_\varepsilon(S) \setminus S$ (indeed the function $P_F$ is continuous). But in the case of $S$ also being a leaf-manifold, we can prove $C^2$ regularity.

Proposition 30 Let $\Gamma$ be a leaf-manifold with positive reach $\varepsilon$. Then it is possible to define a signed distance on the open neighborhood $\mathcal{U}_\varepsilon(\Gamma)$.

Proof in Appendix A.

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The proximally smooth sets were introduced in 1959 in the seminal paper [5] by Federer, who also proved many of their most relevant properties. A proximally smooth set must be closed, and the class contains the convex sets, as well as those sets which can be defined locally by means of finitely many equations $f(x) = 0$ and inequalities $f(x) \leq 0$ using real valued continuously differentiable functions $f$ whose gradients are Lipschitz continuous and satisfy a certain independence condition (Thm 4.12, [5]).

**Theorem 31 (Federer)** Suppose $f_1, \ldots, f_m$ are continuously differentiable real valued functions on an open subset of $\mathbb{R}^N$, $\nabla f_1, \ldots, \nabla f_m$ are Lipschitz continuous for each $0 \leq k \leq m$, and

$$A = \bigcap_{i=1}^k \{x : f_i(x) = 0\} \cap \bigcap_{i=k+1}^m \{x : f_i(x) \leq 0\}.$$  

If $\forall a \in A$, we take $J = \{i : f_i(a) = 0\}$, and there do not exist real numbers $t_i$, corresponding to $i \in J$, such that $t_i \neq 0$ for some $i \in J$, $t_i \geq 0$ whenever $i \in J$, $i > k$, and

$$\sum_{i \in J} t_i \nabla f_i(a) = 0$$

then $A$ has positive reach.

**4.1.2 Good extension of $d(\cdot, \Gamma)$**

As we said at the beginning of the section, the extension of $\delta_\Gamma$ equal to zero used before Corollary [27] does not allow us to prove Theorem [3]. Thus we extend the distance function, since this extension will have better properties.

**Proposition 32** Let $\Gamma$ be a leaf-manifold. Let $U$ be given by Proposition [24] and let $V$ be an open neighborhood of $\Gamma$ such that $V \subset U$. Then there exists a function $\phi : \mathbb{R}^N \to \mathbb{R}$, such that

i) $\phi(x) = d(\cdot, \Gamma)$ on $V$

ii) the process $\phi(X_t)$ is a continuous semimartingale

iii) for each compact ball $B \subset \mathbb{R}^N$ there exists $\varepsilon_1 > 0$ such that for each $\varepsilon < \varepsilon_1$ we have $\{x \in B : \phi(x) < \varepsilon\} = \{x \in B : d(x, \Gamma) < \varepsilon\}$.

Such a function $\phi$ will be called good extension of $d(\cdot, \Gamma)$.

Proof in Appendix B.

**Lemma 33** Let $X$ be a continuous semimartingale in $\mathbb{R}^N$, $\phi$ a good extension of $d(\cdot, \Gamma)$ and $V$ an open neighborhood of $\Gamma$ satisfying Proposition [32]. Then one has:
iv) for each \( t \) such that \( X_t \in V \) we have that
\[
\langle d\phi(X) \rangle_t = \sum_{i,j=1}^N \partial_i \delta_{\Gamma}(X_t) \partial_j \delta_{\Gamma}(X_t) \langle X^i, X^j \rangle_t.
\]
Moreover if \( X \) is a Brownian semimartingale, \( d\langle \phi(X) \rangle_t = dt \).

v) If \( X \) is a Brownian semimartingale, then \( \phi \) controls \( X \) in quadratic variation on \( V \).

Proof in Appendix B.

4.2 The density \( \mathcal{L}^a_{A,\phi} \) when \( X \) is a Brownian semimartingale and \( \phi = \delta_{\Gamma} \)

Here we will restrict ourselves to Brownian semimartingales, since they mix well with the properties of \( d(\cdot, \Gamma) \). Thus \( \mathcal{L}^a_{i,A,\phi} \) is independent of \( i \) and will be denoted by \( \mathcal{L}^a_{A,\phi} \). With this choice we may identify \( \mathcal{L}^a_{i,A,\phi} \) as the the local time of the 1D semimartingale \( \phi(X) \) and deduce the existence of a càdlàg version. Denote by \( \tilde{L}^a_T(Y) \) the càdlàg modification of the local time at \( a \) on \([0, T]\) of a continuous semimartingale \( Y \) (see [12], Theorem 1.7, Chapter VI).

**Definition 34** Let \( Y \) be a continuous semimartingale in \( \mathbb{R} \) and \( a \in \mathbb{R} \), then we define the symmetrical local time of \( Y \) in \( a \) as
\[
\tilde{L}^a_T(Y) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T 1_{(a-\varepsilon, a+\varepsilon)}(Y_t) d\langle Y \rangle_t.
\]

For each fixed \( \omega \in \Omega \), it coincides with \( L^a_T(Y)(\omega) \) except if \( a \) is a point of discontinuity for it, and in that case
\[
\tilde{L}^a_T(Y)(\omega) = \frac{L^a_T(Y)(\omega) + L^{a-}_T(Y)(\omega)}{2}
\]
where \( L^{a-}_T(Y)(\omega) \) is the left limit of the local time in \( a \) (see [12], Chapter VI). In general \( \tilde{L}^a_T(Y) \) is a modification of \( L^a_T(Y) \), and whenever the local time is continuous they coincide.

**Theorem 3** Let \( X \) be a Brownian semimartingale in \( \mathbb{R}^N \). Let \( \Gamma \) be a leaf-manifold and \( \mathcal{U} \) be an open neighborhood of \( \Gamma \) as in Proposition [24]. Let \( \phi : \mathbb{R}^N \to \mathbb{R} \) be a good extension of \( d(\cdot, \Gamma) \) and \( V \subset \mathcal{U} \) be an open neighborhood of \( \Gamma \) satisfying Proposition [22]. Then the assumptions of Theorem [10] are satisfied with \( A = V \). Moreover for each fixed \( \omega \in \Omega \) there exists \( \varepsilon_1(\omega) > 0 \) such that
\[
\mathcal{L}^a_{A,\phi} = \tilde{L}^a_T(\phi(X))
\]
for a.e. \( a < \varepsilon_1(\omega) \), and they are both null if \( a < 0 \).

In particular, on the random interval \((-\infty, \varepsilon_1] \) the process \( (\omega, a) \mapsto \mathcal{L}^a_{A,\phi}(\omega) \) is the modification of a càdlàg process.
Proof. Using Lemma 33 part (v), we have that the assumptions of Theorem 10 are satisfied. Let us prove (4). Using part (iv) of the same Corollary we have
\[ d \langle \phi (X) \rangle_t = dt \]
because \( X \) is a Brownian semimartingale, and \( A = \mathcal{V} \). Hence for a.e. \( a \) (we use the formula for \( L^a_{A,\phi} \) given by Theorem 10)
\[
L^a_{A,\phi} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T 1_{X_t \in A_1(a-\varepsilon,a+\varepsilon)}(\phi(X_t))dt
\]
for each fixed \( \omega \in \Omega \) the trajectory of \( X_t, t \in [0,T] \) remains inside a compact ball \( B(\omega) \); then we can use Proposition 32 part (iii) and obtain that there exists \( \varepsilon_1(\omega) > 0 \) such that for each \( \varepsilon < \varepsilon_1(\omega) \) we have \( \{ x \in B : \phi(x) < \varepsilon_1 \} = \{ x \in B : d(x,\Gamma) < \varepsilon_1 \} \). In particular if \( a < \varepsilon_1(\omega) \) and \( \varepsilon < |a - \varepsilon_1(\omega)| \) then
\[
1_{X_t \in A_1(a-\varepsilon,a+\varepsilon)}(\phi(X_t)) = 1_{(a-\varepsilon,a+\varepsilon)}(\phi(X_t))
\]
hence, for a.e. \( a < \varepsilon_1(\omega) \),
\[
L^0_{A,\phi} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T 1_{(a-\varepsilon,a+\varepsilon)}(\phi(X_t))d \langle \phi (X) \rangle_t = \tilde{L}_{T}^0 (\phi (X)),
\]
that is a modification of the càdlàg process \( L_{T}^0 (\phi (X)) \). The proof is complete. \( \square \)

Corollary 35 Under the hypotheses of the previous theorem, if there exists \( \varepsilon > 0 \) such that \( \mathcal{U} = \mathcal{U}_{\varepsilon}(\Gamma) \), we can take \( \varepsilon_1(\omega) = \varepsilon_2 < \frac{\varepsilon}{2} \) for each \( \omega \in \Omega \). Hence we have that \( L^a_{A,\phi} \) is the modification of a càdlàg process for a.e. \( a < \varepsilon_2 \).

Proof. Repeat the proof of Proposition 32 part (iii), taking \( B = \mathbb{R}^N \) and then the proof of the previous theorem. \( \square \)

4.3 Manifolds with singularities

Until now in this section we have solved (in two ways, namely with the signed distance and the distance function) the problem of the construction of a suitable function \( \phi \), given a \( C^2 \) manifold \( \Gamma \) (with suitable additional properties). To show that, potentially, the theory developed in this paper may adapt to manifolds with singularities, we give here two examples of construction of \( \phi \) when the set \( \Gamma \) is less regular: first a manifold with some Lipschitz point, then the transversal union of smooth manifolds. Writing general statements in such cases turns out to be particularly annoying; thus we limit ourselves to show some particular examples, that the reader will easily conceptualize.

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Example 36 Let $D \subset \mathbb{R}^2$ be the square

$$D = [0, 1]^2 = \{ x = (x_1, x_2) : x_1 \in [0, 1], x_2 \in [0, 1] \}.$$ 

Let $\Gamma = \partial D$. It is a piecewise smooth manifold. The function $\delta_\Gamma$ defined in Example 22 is smooth except on the following sets:

$$\{ x_1 \in [0, 1], x_2 = x_1 \}, \quad \{ x_1 \in [0, 1], x_2 = 1 - x_1 \}$$

where it is continuous with side derivatives. We do not have the properties of Proposition 24 but still we may check directly the properties of Definition 7, for instance in the case when $X$ is a Brownian motion. Indeed we have that

$$|x_1| < |x_2| \Leftrightarrow (x_1^+ < x_2^+) \lor ((-x_1)^+ < (x_2)^+).$$

Then we consider

$$x_1^+ - (x_1^+ - x_2^+)^+ = \begin{cases} x_1^+, & \text{if } x_1^+ < x_2^+ \\ x_2^+, & \text{if } x_1^+ \geq x_2^+ \end{cases}$$

and

$$(-x_1)^+ - ((-x_1)^+ - (x_2)^+) = \begin{cases} (-x_1)^+, & \text{if } (-x_1)^+ < (x_2)^+ \\ (x_2)^+, & \text{if } (-x_1)^+ \geq (x_2)^+ \end{cases}$$

So we obtain

$$\phi(x) = x_1^+ - (x_1^+ - x_2^+)^+ - ((-x_1)^+ - ((-x_1)^+ - (x_2)^+)$$

In particular we can use Itô-Tanaka theorem on the single functions that compose $\phi$: the process $\phi(X)$ is a semimartingale with quadratic variation equal a.s. to

$$\sum_{ij} \partial_i \phi(X_t) \partial_j \phi(X_t) d\langle X^i, X^j \rangle_t = dt.$$ 

Thus we could apply Proposition 8 and then Theorem 10.

Example 37 Let $\Gamma \subset \mathbb{R}^2$ be the union of the two lines:

$$\Gamma = \{ x : x_2 = x_1 \} \cup \{ x : x_2 = -x_1 \}.$$ 

We introduce the sets

$$D_1 = \{ x : x_1 > 0, -x_1 < x_2 < x_1 \}, \quad D_2 = \{ x : x_2 > 0, -x_2 < x_1 < x_2 \},$$

$$D_3 = -D_1, \quad D_4 = -D_2.$$ 

We set

$$\phi(x) = \begin{cases} d(x, \Gamma) & \text{if } x \in D_1 \cup D_3 \\ -d(x, \Gamma) & \text{if } x \in D_2 \cup D_4 \end{cases}$$
and φ(x) = 0 on Γ. It preserves some properties of the signed distance function. It is Lipschitz continuous everywhere, but it is not differentiable on the axes

\{x : x_1 = 0\} \cup \{x : x_2 = 0\}.

If X is a Brownian motion, Definition 7 applies and thus Proposition 8 and Theorem 10 hold. Indeed we consider that

\[
x_1^+ - (x_1^+ - x_2^+) = \begin{cases} x_1^+, & \text{if } x_1^+ < x_2^+ \\ x_2^+, & \text{if } x_1^+ \geq x_2^+ \end{cases}
\]

Then we define

\[
\psi(x) = -\left|((-x_1)^+, (-x_2)^+)\right|
\]

and so

\[
\phi(x) = x_1^+ - (x_1^+ - x_2^+) + \psi(x).
\]

We can apply again the Itô-Tanaka theorem to all the single functions (remind that the modulus is convex) and get the same result as in the previous example.

5 Local times with respect to a codimension-1 manifold

In this section we shall introduce the notion of local time at an (N − 1)-dimensional manifold Γ, on [0, T], of a continuous semimartingale X in \(\mathbb{R}^N\); it will be denoted by \(L_T^\Gamma(X)\). Then we collect here its relation with \(\mathcal{L}_{A,\phi}^\alpha\) and, in the special case that Γ is globally a graph, with the notion used in [11].

We assume that X is a continuous semimartingale in \(\mathbb{R}^N\), defined on a filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}_t, P)\).

**Theorem 38** Let Γ be a leaf-manifold. Let \(U\) be an open neighborhood of Γ satisfying Proposition 24. Then the limit

\[
L_T^\Gamma(X) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \sum_{i,j=1}^{N} \int_0^T 1_{[0,\varepsilon)}(d(X_t, \Gamma)) \partial_i \delta_T(X_t) \partial_j \delta_T(X_t) d\langle X^i, X^j \rangle_t
\]

is well defined and a.s. exists. It will be called geometric local time of X at Γ on [0, T]. If X is a Brownian semimartingale, then

\[
L_T^\Gamma(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T 1_{[0,\varepsilon)}(d(X_t, \Gamma)) dt.
\]
Remark 39  Let $\psi$ be an arbitrary $C^2(\mathbb{R}^N)$ extension of the signed distance $\delta_\Gamma$. Then to be rigorous we should write $\partial_i \psi$ instead of $\partial_i \delta_\Gamma$, because $\delta_\Gamma$ is not defined outside $\mathcal{U}$. But the limit does not depend on the choice of the extension: indeed for each fixed $\omega \in \Omega$ the trajectory of the process $X$ remains inside a compact ball $B$, and so we may take an $\varepsilon_0(\omega) > 0$ such that $\mathcal{U} \cap B \supseteq \mathcal{U}_{\varepsilon_0}(\Gamma) \cap B$. Hence for each $\varepsilon < \varepsilon_0(\omega)$ the limit depends only on $\delta_\Gamma$.

Proof. Let $\phi$ be the good extension of $d(\cdot, \Gamma)$ defined in Proposition 32, by Lévy characterization of local times (see [12], Corollary 1.9, Chapter VI) there exists a.s.

$$I := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T 1_{(0, \varepsilon)}(\phi(X_t)) \, d \langle \phi(X) \rangle_t.$$  

The first claim is proved because for each fixed $\omega \in \Omega$ we have $L_{T_\Gamma}(X) = \frac{1}{2}I$. Indeed the trajectory $X_t(\omega)$, for each $t \in [0, T]$ remains inside a compact ball $B(\omega)$; then we can use Proposition 32, part (iii) and obtain that there exists $\varepsilon_1(\omega) > 0$ such that for each $\varepsilon < \varepsilon_1(\omega)$ we have $\{x \in B : \phi(x) < \varepsilon_1\} = \{x \in B : d(x, \Gamma) < \varepsilon_1\}$; and in particular for each $\varepsilon < \varepsilon_1$ we have $1_{(0, \varepsilon)}(\phi(X_t)) = 1_{(0, \varepsilon)}(d(X_t, \Gamma))$. Moreover, following Lemma 33, part (iv), for each $t$ such that $X_t \in \mathcal{V}$ we have that

$$\langle \phi(X) \rangle_t = \sum_{i,j=1}^N \partial_i \delta_\Gamma(X_t) \partial_j \delta_\Gamma(X_t) \, d \langle X^i, X^j \rangle_t$$

and this is true whenever $d(X_t, \Gamma) < \varepsilon_1$. In particular, thanks to the same Lemma, if $X$ is a Brownian semimartingale we obtain also the second claim. □

Remark 40  The formula given above which defines $L_{T_\Gamma}(X)$ in the general case may look strange at first sight. However, it is the natural one if we think to the particular case of a hyperplane $\Gamma$. In that case, the natural definition would be the classical local time (which includes the time-change due to the quadratic variation) of the projection of $X$ along the normal to $\Gamma$. This is the formula above, as we also show below in Proposition 43.

Here we will suppose that $X$ is a Brownian semimartingale and we will also assume that there exists an $\varepsilon > 0$ such that $\mathcal{U} = \mathcal{U}_\varepsilon(\Gamma)$: this is because we need that for each $a < \varepsilon$ the level sets $\Gamma_a = \{x : d(x, \Gamma) = a\}$ are leaves-manifolds.

The following geometric lemma is about the relation between $d(\cdot, \Gamma)$ and $d(\cdot, \Gamma_a)$.

Lemma 41  Let be $a > 0$ and $\varepsilon > 0$, then the following properties are equivalent:

1) $d(x, \Gamma_a) \in [0, \varepsilon)$
2) $d(x, \Gamma) \in (a - \varepsilon, a + \varepsilon)$.

Proof in Appendix A.
Corollary 42 If $\varepsilon_0 > a > 0$ and $\varepsilon < |a - \varepsilon_0|$, then the following properties are equivalent:

i) $x \in U_{\varepsilon_0}(\Gamma)$, $d(x, \Gamma_a) \in [0, \varepsilon)$

ii) $x \in U_{\varepsilon_0}(\Gamma)$, $d(x, \Gamma) \in (a - \varepsilon, a + \varepsilon)$.

Proof. We apply the previous Lemma considering that for each $x \in U_{\varepsilon_0}(\Gamma)$, both the neighborhoods $U_{\varepsilon}(\Gamma_a)$ and $\{x : d(x, \Gamma) \in (a - \varepsilon, a + \varepsilon)\}$ are subsets of $U_{\varepsilon_0}(\Gamma)$. □

Theorem 4 Let $X$ be a Brownian semimartingale in $\mathbb{R}^N$. Let $\Gamma$ be a leaf-manifold and $U$ be an open neighborhood of $\Gamma$ as in Proposition 24. Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be a good extension of $d(\cdot, \Gamma)$ and $V \subset U$ be an open neighborhood of $\Gamma$ satisfying Proposition 32. Then the assumptions of Theorem 10 are satisfied with $A = V$. Moreover suppose there exists $\varepsilon_0 > 0$ such that $U = U_{\varepsilon_0}(\Gamma)$ so that we can take also $V = U_{\varepsilon_1}(\Gamma)$ for a fixed $\varepsilon_1 \in (0, \varepsilon_0)$. Then we have

$$L_{A,\phi}^a = L_{\Gamma,a}^T(X)$$

for a.e. $a \in [0, \varepsilon_0]$, where $\Gamma_a = \{x \in V : d(x, \Gamma) = a\}$.

Proof. The set $V = U_{\varepsilon_1}(\Gamma)$ has a tubular neighborhood structure (see [8]) that we can define following Proposition 30, indeed the regularity of $\delta_\Gamma$ implies that the manifold has positive reach. Hence for each $a < \varepsilon_1$, the set $\Gamma_a$ is a leaf-manifold. Thus we can apply Theorem 38 and define

$$L_{\Gamma,a}^T(X) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T 1_{[0,\varepsilon]}(d(X_t, \Gamma_a))dt.$$ 

From Corollary 42 for each $a \in (0, \varepsilon_1)$ and very small $\varepsilon$ we have

$$1_{[0,\varepsilon]}(d(x, \Gamma_a)) = 1_{x \in A} 1_{(a-\varepsilon, a+\varepsilon)}(d(x, \Gamma)) = 1_{x \in A} 1_{(a-\varepsilon, a+\varepsilon)}(\phi(x)),$$

because $A = V$. Hence

$$L_{\Gamma,a}^T(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T 1_{x \in A} 1_{(a-\varepsilon, a+\varepsilon)}(\phi(x))dt = L_{A,\phi}^a.$$ 

□

5.1 The graph local time

The geometric local time introduced above is intrinsic, in the sense that it is independent of the coordinate system of $\mathbb{R}^N$: indeed, it is defined only in terms of $\Gamma$, $X$, and the function $\delta_\Gamma$. In this section we compare the geometric local times with the graph local times defined below and we show in particular that they are different; moreover, $L_{\Gamma,\text{graph}}^{\Gamma,\text{graph}}$ may change value if we change the coordinate system used to describe $\Gamma$ as a graph. We explain this by the simple example of Proposition 43 below.
Let \( g : \mathbb{R}^{N-1} \to \mathbb{R} \) be a \( C^2 \) function and let \( \Gamma \) be its graph:

\[
\Gamma = \left\{ (x^1, \ldots, x^N) \in \mathbb{R}^N : x^N = g(x^1, \ldots, x^{N-1}) \right\}.
\]

Following [11], let us define the graph local time of \( X \) at \( \Gamma \) as

\[
L_{\Gamma, \text{graph}}^T(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^{\varepsilon} 1_{[0,\varepsilon)}(|Y_t|) \, d\langle Y \rangle_t
\]

where

\[
Y_t = X^N_t - g(X^1_t, \ldots, X^{N-1}_t).
\]

Given \( v \in \mathbb{R}^N \), by \( v \cdot X \) we mean the process \( \sum_{i=1}^N v_i X_i \).

**Proposition 43** In \( \mathbb{R}^N \), let \( \Gamma \) be the \((N-1)\)-dimensional subspace orthogonal to a given unitary vector \( v \). Then

\[
L_{\Gamma}^T(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^{\varepsilon} 1_{[0,\varepsilon)}(d(X_t, \Gamma)) \, d\langle v \cdot X \rangle_t
\]

Given a system of coordinates in \( \mathbb{R}^N \) (we write \( x = (x^1, \ldots, x^N) \)), let \( a = (a^1, \ldots, a^{N-1}) \in \mathbb{R}^N \) be the vector such that \( \Gamma \) is defined by the equation \( x^N = \sum_{i=1}^{N-1} a^i x^i \). Then

\[
L_{\Gamma, \text{graph}}^T(X) = \sqrt{1 + |a|^2} L_{\Gamma}^T(X).
\]

**Proof.** We choose the system of coordinates in \( \mathbb{R}^N \) and the vector \( a \in \mathbb{R}^{N-1} \) as in the statement. One can introduce a global signed distance function \( \delta_{\Gamma} \), such that

\[
\nabla \delta_{\Gamma}(x) = v \quad \text{for all } x \in \mathbb{R}^N.
\]

Hence

\[
L_{\Gamma}^T(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \sum_{i,j=1}^N \int_0^{\varepsilon} 1_{[0,\varepsilon)}(d(X_t, \Gamma)) v^i v^j d\langle X^i, X^j \rangle_t
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^{\varepsilon} 1_{[0,\varepsilon)}(d(X_t, \Gamma)) \, d\langle v \cdot X \rangle_t.
\]

The first relation is proved.

We also have, using the coordinates,

\[
v \cdot X = \frac{1}{\sqrt{1 + |a|^2}} (X^N - a \cdot X) = \frac{1}{\sqrt{1 + |a|^2}} Y_t
\]

\[\text{This name is given here only to distinguish the notion from the geometric local time given above.}\]
where we have written $X = (X, X^N)$. Hence

$$L_T^\Gamma (X) = \frac{1}{1 + |a|^2} \lim_{\varepsilon \to 0} \frac{1}{2 \varepsilon} \int_0^T 1_{[0, \varepsilon)} (d(X_t, \Gamma)) d \langle Y \rangle_t.$$

Finally, it is a simple exercise to check that $d(X_t, \Gamma) = r$ if and only if $|Y_t| = r\sqrt{1 + |a|^2}$, and thus $d(X_t, \Gamma) \in [0, \varepsilon)$ if and only if $|Y_t| < \varepsilon\sqrt{1 + |a|^2}$.

The proof is complete. □

When $g$ is not linear, at present we can only guess that a suitable localization argument could lead to the identity

$$L_T^\Gamma (X) = \frac{1}{\sqrt{1 + |a|^2}} \lim_{\varepsilon \to 0} \frac{1}{2 \varepsilon} \int_0^T 1_{[0, \varepsilon\sqrt{1 + |a|^2})} (|Y_t|) d \langle Y \rangle_t = \frac{1}{\sqrt{1 + |a|^2}} L_T^\Gamma,\text{graph} (X).$$

but the proof is not trivial and will not be discussed further here.

Appendix

A) Some proofs of geometry

Proof of Proposition[24]. For each point $x \in \Gamma$ there exists $\varepsilon_0 > 0$ and a diffeomorphism $\psi$ from the ball $B(x, \varepsilon_0)$ into $B(0, 1) \subset \mathbb{R}^N$, such that the restriction of $\psi$ on $\Gamma \cap B(x, \varepsilon_0)$ is a diffeomorphism onto the $(N - 1)$-dimensional disc. Then there exists $\varepsilon_1 \in (0, \frac{\varepsilon_0}{2})$ such that $\Gamma \cap B(x, \varepsilon_1)$ because of its compactness has a tubular neighborhood of width equal to $\varepsilon_1$, on which the distance from the manifold coincides with the distance on the normal bundle (see [8]). Moreover we can use the orientation of the normal bundle to locally define a function $\delta_T^\Gamma$ satisfying (i) and (ii) w.r.t. $\Gamma \cap B(x, \varepsilon_0)$. If we restrict the previous tubular neighborhood around the submanifold $\Gamma \cap B(x, \varepsilon_1)$, and we call it $U_x$, we have also that $d(\cdot, \Gamma \cap B(x, \varepsilon_1)) = d(\cdot, \Gamma)$ on $U_x$. Moreover we can define $\mathcal{U} = \bigcup_{x \in \Gamma} U_x$ and we obtain a global neighborhood of $\Gamma$. Then thanks to its orientability we have that all the local signed distances have a compatible signature: for each $x, y \in \Gamma$, $y \in U_x \cap U_z$ we have $\delta_T^\Gamma (y) = \delta_T^\Gamma (z)$. For each $y \in \mathcal{U}$ exists $x \in \Gamma$ such that $y \in U_x$ and we can define $\delta_T^\Gamma (y) = \delta_T^\Gamma (y)$; it satisfies both (i) and (ii). □
Proof of Lemma 25. The functions $d(\cdot, \Gamma)$ and $\delta_\Gamma(\cdot)$ are both continuous and different from zero in the open set $\mathcal{U}\setminus \Gamma$. The ratio $\frac{\delta_\Gamma(x)}{d(x, \Gamma)}$ is a continuous well defined function on $\mathcal{U}\setminus \Gamma$, equal to $\pm 1$, hence it is constant on each connected component of $\mathcal{U}\setminus \Gamma$. Property (iii) is proved. Property (iv) is an easy consequence of (i) and (iii). Property (v) is true by contradiction: if exists $x \in U$ such that $P_F(x) \supseteq \{y_1, y_2\}$ with $y_1 \neq y_2$, then $\frac{x-y_1}{|x-y_1|} \neq \frac{x-y_2}{|x-y_2|}$.

Then if we substitute $v_1 = \frac{x-y_1}{|x-y_1|}$ in (4) we have that $1 = \frac{x-y_1}{|x-y_1|} \cdot \frac{x-y_1}{|x-y_1|} < \frac{x-y_1}{|x-y_1|} \cdot \frac{x-y_2}{|x-y_2|}$ by Cauchy-Schwartz inequality, hence $D_{v_1}d(x, F) = v_1 \cdot \frac{x-y}{|x-y|}$. Vice versa if we substitute $v_2 = \frac{x-y_2}{|x-y_2|}$ in (4) we obtain $D_{v_2}d(x, F) = v_2 \cdot \frac{x-y}{|x-y|}$. This is incompatible with the differentiability of $\delta_\Gamma$ in $U$. About property (vi), if $x \in U\setminus \Gamma$, thanks to (iii) we have

$$\left| \nabla \delta_\Gamma(x) \right| = \left| \nabla d(\cdot, \Gamma)(x) \right| = \max_{|v|=1} |D_v d(x, F)| = 1$$

where we substituted $v = P_F(x)$ in (3) and $P_F(\cdot)$ is well defined thanks to (v). If otherwise $x \in \Gamma$, we have $\left| \nabla \delta_\Gamma(x) \right| = 1$ by continuity thanks to (i). □

Proof of Proposition 30. We have that $\mathcal{U}_\varepsilon(\Gamma)$ is the union of all the normal segments $N_x$, $x \in \Gamma$, of lengths $\varepsilon$ on both sides of $\Gamma$, and this union must be disjoint. Indeed by contradiction let $x_1, x_2 \in \Gamma$, $x_1 \neq x_2$, be such that there exists $y \in N_{x_1} \cap N_{x_2}$, and suppose that $d(y, x_1) \geq d(y, x_2)$. Then the continuous function $d(\cdot, x_1) - d(\cdot, x_2)$ is positive in $y$ and negative in $x_1$: there exists a point $z \in N_{x_1}$ such that $d(z, x_1) - d(z, x_2) = 0$. This is incompatible with the positive reach property. Hence $\mathcal{U}_\varepsilon(\Gamma)$ is a global tubular neighborhood of $\Gamma$ (see [8] for the definition): the distance from the manifold coincides on that neighborhood with the distance on the normal bundle and we can use the orientation to locally define a function $\delta_\Gamma$ satisfying properties (i) and (ii) of the Proposition 24. □

Proof of Lemma 41. The case $a = 0$ is trivial: let us suppose $a \neq 0$. Let be $x \in \mathbb{R}^N$ and $y \in P_{\Gamma_a}(x)$ (the set of metric projections of $x$ on $\Gamma_a$), so that we have $d(x, \Gamma_a) = d(x, y)$. Then define $z \in P_\Gamma(y)$ and we have $d(y, z) = a$. Then

$$d(x, \Gamma) \leq d(x, z) \leq d(x, y) + d(y, z) = a + d(x, y)$$

and if $d(x, \Gamma_a) < \varepsilon$, we have $d(x, y) < \varepsilon$ and $d(x, \Gamma) \leq a + \varepsilon$. Moreover define $z_1 \in P_\Gamma(x)$, such that $d(x, \Gamma) = d(x, z_1)$. Then

$$d(x, \Gamma) = d(x, z_1) \geq d(y, z_1) - d(y, x) = a - d(x, y)$$

and if $d(x, \Gamma_a) < \varepsilon$, we have $d(x, y) < \varepsilon$ and $d(x, \Gamma) \geq a - \varepsilon$. We proved

$$a - \varepsilon \leq d(x, \Gamma_a) \leq a + \varepsilon.$$
Vice versa define \( y_1 \) as the intersection of the segment linking \( x \) and \( z_1 \). So we obtain 
\[
d(x, z_1) = d(x, y_1) + d(y_1, z_1) \quad \text{but} \quad d(x, y_1) \geq d(x, \Gamma_a) \quad \text{and} \quad d(y_1, z_1) \geq a, \quad \text{so}
\]
\[
d(x, \Gamma) = d(x, z_1) \geq d(x, \Gamma_a) + a.
\]
In particular from (ii) we have \( 0 \leq d(x, \Gamma) \leq a + \varepsilon \) and so \( d(x, \Gamma_a) \leq \varepsilon \). \( \square \)

**B) The good extension construction**

**Proof of Proposition 32.** Let \( \theta : \mathbb{R}^N \to \mathbb{R} \) be a cut-off function of class \( C^2 \) such that 
\[
\theta(x) = 1 \quad \text{if} \quad x \in \mathcal{V} \quad \text{and} \quad \theta(x) = 0 \quad \text{if} \quad x \in \mathcal{U}^c.
\]
Thanks to Lemma 25 the distance \( d(\cdot, \Gamma) \) is regular in \( \mathcal{U} \setminus \Gamma \), so we can regularize it on all \( \mathbb{R}^N \setminus \Gamma \) without changing its value inside \( \mathcal{V} \). We call \( \tilde{d} \) this mollified distance. Then we take, for all \( x \in \mathbb{R}^N \)
\[
\phi(x) := d_1(x) + d_2(x)
\]
where \( d_1(x) := (1 - \theta(x)) \tilde{d}(x), \quad d_2(x) := \theta(x)d(x, \Gamma) \). We have that \( d_1(X) \) is a continuous 
semimartingale because it is the composition of \( X \) with a \( C^2 \) function. Moreover if we define 
\( f(x) = \theta(x)\delta_\Gamma(x) \) for all \( x \in \mathcal{U} \) and \( f(x) = 0 \) when \( x \in \mathcal{U}^c \), we obtain that \( f \) is a function of class 
\( C^2(\mathbb{R}^N) \), and \( d_2(x) = |f(x)|; \) so, thanks to the Itô-Tanaka formula, also 
\( d_2(X) \) is a continuous semimartingale, and \( \phi(X) \) too.

To verify the third property, we remind that by compactness there exists an \( \varepsilon_0 > 0 \) such that 
\( \mathcal{V} \cap B \supseteq \mathcal{U}_{\varepsilon_0}(\Gamma) \cap B \), so if we had taken mollifiers that do not change too much 
the value of \( d(\cdot, \Gamma) \), there exists \( \varepsilon_1 < \varepsilon_0 \) such that if \( \phi(x) < \varepsilon < \varepsilon_1 \) then 
\( d(x, \Gamma) < \varepsilon_0 \), and \( x \in \mathcal{U}_{\varepsilon_0}(\Gamma) \). In particular if \( x \in B \) then \( x \in \mathcal{V} \cap B \) and \( \phi(x) = d(x, \Gamma) \). Vice versa 
if \( d(x, \Gamma) < \varepsilon < \varepsilon_1 \) then again \( d(x, \Gamma) < \varepsilon_0 \) and we obtain like above that if \( x \in B \) then 
\( \phi(x) = d(x, \Gamma) \). \( \square \)

**Proof of Lemma 33.** Let \( \psi \) be an arbitrary \( C^2(\mathbb{R}^N) \) extension of \( \delta_\Gamma \) from \( \mathcal{V} \). If \( X_t \in \mathcal{V}, \) 
hence \( \phi(X_t) = |\delta_\Gamma(X_t)| = |\psi(X_t)| \), and we can apply Itô-Tanaka formula to obtain 
\[
d\langle |\psi(X)| \rangle_t = \sum_{i,j=1}^N \partial_i \psi(X_t) \partial_j \psi(X_t) d\langle M^i, M^j \rangle_t
\]
\[
= \sum_{i,j=1}^N \partial_i \delta_\Gamma(X_t) \partial_j \delta_\Gamma(X_t) d\langle X^i, X^j \rangle_t,
\]
where \( M \) was the local martingale part of \( X \). Then, if \( X \) is a Brownian semimartingale, 
we have 
\[
\sum_{i,j=1}^N \partial_i \delta_\Gamma(X_t) \partial_j \delta_\Gamma(X_t) d\langle X^i, X^j \rangle_t = |\nabla \delta_\Gamma(X_t)|^2 dt = dt
\]
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by Lemma \[\text{Lemma 25}\]. The proof of property (iv) is complete.

Moreover, if $X$ is a Brownian semimartingale, then $\phi$ satisfies the hypotheses of Definition \[\text{Definition 7}\] with $A = \mathcal{V}$ and $D_{\phi} = \mathcal{V} \setminus \Gamma$. Hence thanks to Proposition \[\text{Proposition 8}\], it controls $X$ in quadratic variation on $\mathcal{V}$. Indeed $d\langle X \rangle_t = dt$ is trivially Lipshitz continuous, $\phi$ is locally 1-Lipshitz in $\mathcal{V}$ because it is equal to $d(\cdot, \Gamma)$ and we can use \[\text{Equation (3)}\] to estimate its partial derivatives. The property that

$$\{\omega \in \Omega : X_t(\omega) \in \Gamma\} = \{\omega \in \Omega : \phi(X_t(\omega)) = 0\}$$

implies that for a.e. $t \in [0, T]$ we have

$$P\{\omega \in \Omega : X_t(\omega) \in \Gamma\} = P\{\omega \in \Omega : \phi(X_t(\omega)) = 0\} = 0.$$ 

Otherwise we would have that on a not negligible event, there would exists $\Theta(\omega) \subset [0, T]$ with Lebesgue measure $\lambda > 0$, such that $\forall t \in \Theta$ it would be $\phi(X_t) = 0$. And this would contradict the Occupation time formula (see \[\text{[12]}\], Chapter VI, Corollary 1.6)

$$\int_\Theta d\langle \phi(X) \rangle_s \leq \int_0^T \mathbf{1}_{\{0\}}(\phi(X_s))d\langle \phi(X) \rangle_s = \int_{-\infty}^{+\infty} \mathbf{1}_{\{0\}}(a)da = 0.$$ 

Indeed $\forall t \in \Theta$, $X_t \in \Gamma \subset \mathcal{V}$ and then we can use part (iv) of this lemma to obtain $d\langle \phi(X) \rangle_t = dt$, hence

$$\int_\Theta d\langle \phi(X) \rangle_s = \int_\Theta ds = \lambda > 0.$$ 

Additionally $\phi(X)$ is trivially non-degenerate (following Definition \[\text{Definition 7}\]). \[\square\]

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