Ecalle’s paralogarithmic resurgence monomials
and effective synthesis

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Abstract

Paralogarithms constitute a family of special functions, which are some
generalizations of hyperlogarithms. They have been introduced by Jean
Ecalle in the context of the classification of complex analytic dynamical
systems with irregular singularities, to solve the so-called “synthesis
problem” in an effective and very general way.

We describe the formalism of resurgence monomials, introduce the paralogarithmic family and present the effective synthesis with paralogarithmic monomials, of analytic vector fields having a saddle-node singularity, following Ecalle.

1 Introduction

Resurgent functions and alien calculus were introduced by Jean Ecalle more
than 40 years ago for classifying singularities of dynamical systems with ana-
lytic data. Divergent series appearing in the solutions of these complex analytic
dynamical systems at irregular singularities, when expressed in a suitable vari-
able \( z \sim \infty \) can quite systematically be treated by the Borel–Laplace summation
formalism (see the next section). The Borel transform of these series have iso-
lated singularities and display properties of “self–reproduction” at their singular
points (11, 12), which are at the origin of the very name resurgence; families of
so–called alien operators were defined by Ecalle to characterize this resurgent
behaviour.

Alien calculus has made it possible to solve difficult problems of moduli in
an effective manner, which was the ultimate goal; many consequences for local
dynamics can then easily be derived thanks to the precise and effective descrip-
tion of the moduli spaces obtained with the help of alien derivations (12, 13).

Beyond the original field of applications, the phenomenon of resurgence had
been described in the early 80’s for perturbation series in quantum mechanics
and quantum field theory; more recently, there has been an explosion of works
in theoretical Physics involved with resurgence (see e.g. 20, 8, 6, 3 and notably
1 and the references therein) – and significant connections with “wall–crossing”
(19, 21).
The classification of analytic saddle–nodes, with a complete description of the corresponding moduli spaces, had been a major breakthrough in the understanding of the irregular singularities of complex analytic dynamical systems. It has been obtained around 1980, independently by J. Ecalle, with resurgent functions, and by J. Martinet and J.–P. Ramis (22), who pioneered a “cohomological approach” to study the Stokes phenomenon.

Saddle–node is a traditional terminology to designate singularities of differential equations which display in the real phase–space a set of trajectories with a saddle behaviour on one side and a node behaviour on the other. The formal classification of these dynamical systems and their generalizations, in 2 complex dimensions, had been known for a long time and there were partial results regarding the transformations that normalize these germs of vector fields.

Such formal normalizations correspond to analytic functions in the critical variable, but only in sectorial domains stemming from the singular point under study: the formal series involved are generically divergent. This divergence was characterized by Martinet–Ramis by a delicate study of the differences of these analytic sectorial functions on overlapping sectors and of the corresponding Stokes phenomenon; in their approach, some non–abelian first cohomology space characterizes the moduli.

For the simplest formal normal form, a point in the moduli space amounts to the data of a pair of Stokes diffeomorphisms \( S_+ \), \( S_- \) in the respective singular directions (cf below) \( \mathbb{R}_\rightarrow \) and \( \mathbb{R}_\leftarrow \); the diffeo. \( S_- \) being a translation by a constant and \( S_+ \) being any analytic tangent to identity germ of diffeo. To solve the inverse problem, notably, Martinet–Ramis’s scheme performed the synthesis in a non–effective way of a vector field \( X \) eventually yielding the given pair \( S_+, S_- \) first in the \( C^\infty \) category and then using the crucial Newlander–Nirenberg integrability theorem to obtain analyticity (22).

On his side, Ecalle solved the problem by the use of alien calculus and had in particular presented a solution to the inverse problem (“synthesis”), for moduli of germs of tangent to identity diffeomorphisms in dimension 1, early on in vol. 2 of [11], yet in a non constructive way; more recently, however, he had described in [14] a family of new special functions, the so–called paralogarithmic resurgence monomials, that can be used in an effective way to synthesize in full generality analytic objects which have prescribed moduli expressed in the resurgent formalism.

The aim of this text is to present these beautiful objects, describe in the framework of Ecalle’s theories this systematic and constructive solution to the synthesis problem, focusing on the example of the saddle–node singularity – and restraining to the simplest formal normal form.

We indicate now how the present text is organized. To make the article self–contained, we begin by giving a brief introduction to resurgent functions and alien derivations. Next, we present all the necessary background on mould calculus which is required to formulate the scheme for the resurgent treatment of the synthesis problem and to solve it, using notably the crucial apparatus of arborification–coarborification.

We recall after that the results pertaining to the resurgence properties of the
saddle-node ("the direct problem") and then move to the "inverse problem", with a presentation of J. Ecalle's way to solve it. We introduce Ecalle's paralogarithmic resurgence monomials and give schemes of proofs for their properties, to eventually reach a complete and explicit solution of the synthesis problem.

Remark 1 No claim of originality is made concerning the main results contained in the present article, which can be viewed as an invitation to read Ecalle’s foundational texts, in particular [14]. Resurgent functions, alien calculus, moulds, arborification, the problematics of resurgence monomials, the resurgence treatment of the problem of synthesis, paralogarithmic monomials themselves and the schemes to prove their essential properties are all due to Ecalle.

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2 Resurgent functions and alien differential calculus

2.1 Basic facts about resurgent functions

We briefly recall basic definitions and properties of resurgent functions, adapted to the context of saddle–node singularities which we shall study in detail below.

Most resurgent functions of natural origin (say, formal expansions appearing in the general solution of some functional equation or expansions in some critical parameter in Physics) appear as Borel transforms with respect to some critical variable $z \sim \infty$, where the Borel transform $B$ of a formal series $\tilde{\varphi}(z) = \sum_{n \geq 0} c_n z^{-(n+1)}$ is:

$$B(\tilde{\varphi}) := \tilde{\varphi}(z) \rightarrow \sum_{n \geq 0} c_n \frac{z^n}{n!}$$

As a rule, the presence of exponentials of the type $e^{az}$ ($a \in \mathbb{C}$) in formal solutions at an irregular singularity of a given dynamical system is concomitant with the Gevrey–1 type of divergence of the formal series implied.
Definition 1 A formal series $\sum_{n \geq 0} a_n t^{n+1}$ is called Gevrey–1 iff $\exists C, M > 0$

such that $|a_n| \leq CM^n n!$, $\forall n \geq 0$

Any series $\tilde{\varphi}(z) = \sum_{n \geq 0} c_n z^{-(n+1)}$ which is Gevrey–1 has a Borel transform $\hat{\varphi}(\zeta) = \sum_{n \geq 0} c_n \frac{\zeta^n}{n!}$, which belongs to $\mathbb{C}\{\zeta\}$ and it is a general fact that the $\hat{\varphi}(\zeta)$ corresponding to formal series involved in the solutions at the singularity under study can be analytically continued, with isolated singularities $\omega$ in the $\zeta$–plane. There are operators with good algebraic properties which make it possible to characterize the singular behaviour at the points $\omega$, notably the so–called alien derivations $\Delta_\omega$ defined below.

If $\hat{\varphi}(\zeta)$ has at most exponential growth in the direction $\mathbb{R}_>$ (meaning that $\exists H, K > 0$ such that $\forall \zeta \in \mathbb{R}_>, |\hat{\varphi}(\zeta)| \leq He^{K\zeta}$), we can perform a Laplace transform of this function and get a function:

$$\varphi(z) := L(\hat{\varphi})(z) = \int_0^\infty e^{-\zeta z} \hat{\varphi}(\zeta) d\zeta$$

This function $\varphi(z)$ is analytic in the half plane $\text{Re}(z) > K$; more generally, for any direction $d_\theta = e^{i\theta} \mathbb{R}_>$ from the origin in the $\zeta$–plane, a Laplace integration in the direction $d_\theta$

$$L_\theta(\hat{\varphi})(z) = \int_0^{e^{i\theta}} e^{-\zeta z} \hat{\varphi}(\zeta) d\zeta$$

yields a function $\varphi(z)$ which is analytic in some half plane bisected by the conjugate direction $d := e^{-i\theta} \mathbb{R}_\infty^\infty$ and this scheme is licit whenever $\hat{\varphi}(\zeta)$ can be analytically continued with exponential growth at $\infty$ along $d_\theta$.

The Borel transform $B$ is a morphism of differential algebras, where the product in the $\zeta$ plane is the following convolution (and the derivation wrt this convolution is multiplication by $-\zeta$):

$$\hat{\varphi} * \hat{\psi}(\zeta) = \int_0^\zeta \hat{\varphi}(s) \hat{\psi}(\zeta - s) ds \quad (\zeta \sim 0)$$

Resurgent functions appear in practice in 3 “models” :

1. The formal model, involving formal series in $z^{-1}$
2. The geometric model(s), involving holomorphic functions in sector(s)
3. The Borel plane, involving functions with isolated singularities and tame behaviour at $\infty$

In fact, we shall very soon need to consider germs, defined on the Riemann surface of the logarithm $\mathbb{C}_\infty$, which are a priori also singular at the origin; these 3 models and the morphisms that relate them will be accordingly enriched ($[12, 13, 14]$).

Let us focus on the Borel plane and consider a function $\psi(\zeta)$ which is defined in particular for any $\zeta$ in a universal cover $\tilde{D}_*$ of a pointed disk $D_* :=
\( D(\omega, \rho) - \{\omega\} \) centered at a given complex number \( \omega \), which is thus a priori a singular point of \( \psi(\zeta) \); the singularity of \( \psi(\zeta) \) at \( \omega \) can be characterized by the class of \( \psi(\zeta) \), mod. analytic germs at \( \omega \), for \( \zeta \) on \( \tilde{D}_* \).

We now proceed to define resurgent functions, at a level of generality which is sufficient for the present text (we notably refer to [12] for more general cases).

**Definition 2** A function, defined at the origin of \( \mathbb{C}_\infty \) (meaning: for any \( \zeta \) in a universal cover \( \tilde{D}_* \) of a pointed disk \( D_* := D(0, \rho) - \{0\} \)), is called endlessly continuable iff it can be analytically continued, with isolated singularities. The set of germs of functions which are endlessly continuable is denoted by \( \mathcal{P} \).

The set \( \mathcal{P} \) has a natural structure of vector space; it encodes any type of ramified singularity at the origin of \( \mathbb{C} \) and, by discarding the functions which are regular at the origin of \( \mathbb{C} \) we arrive at the following:

**Definition 3** Let \( \mathcal{R} = \mathcal{P} / \mathcal{P}_1 \) the quotient of \( \mathcal{P} \) by the subspace \( \mathcal{P}_1 \) of regular germs (\( \mathcal{P}_1 = \mathcal{P} \cap \mathbb{C}\{\zeta\} \)).

An element of \( \mathcal{R} \) will be denoted by \( \varphi^\nabla \) (or simply \( \varphi \) for short, keeping in mind that \( \varphi \) designates a class of functions) and any representative of such an element will be called a major of \( \varphi^\nabla \) and denoted by \( \hat{\varphi} \). For any major \( \hat{\varphi} \) of a given element \( \varphi^\nabla \) in \( \mathcal{R} \), the function \( \hat{\varphi} \) defined for \( \zeta \sim 0 \) by

\[
\hat{\varphi}(\zeta) = (1 - R)\hat{\varphi}(\zeta) = \hat{\varphi}(\zeta) - \hat{\varphi}(\zeta e^{-2\pi i}) \quad (R \text{ is the rotation of } -2\pi, \text{ on } \mathbb{C}_\infty)
\]

does not depend on the choice of the major; it is called the minor of \( \varphi^\nabla \).

**Remark 2** One thing can be highlighted from the start: as defined above, resurgent functions are not functions – they are classes of functions – and *stricto sensu* there is nothing resurgent in them, either. It is *in the applications* that the phenomenon of resurgence will be displayed and the vast majority of divergent formal series of natural origin, appearing in solutions to dynamical systems at singularities, or in various physical theories, show some form of resurgent behaviour: at the singularities in the Borel plane of a given solution, some other solution resurges, in a shape that depends upon the problem under study.

**Example 1** To any analytic regular germ \( \varphi(\zeta) \) at the origin of \( \mathbb{C} \) we can associate a germ on \( \mathbb{C}_\infty \) which defines a logarithmic singularity, namely \( \varphi^\nabla \) defined by the major:

\[
\hat{\varphi}(\zeta) = \frac{1}{2\pi i} \log(\zeta)\varphi(\zeta)
\]

The minor of \( \varphi^\nabla \) is the germ \( \varphi(\zeta) \); by abuse of language, in this case it is harmless to speak there of “the resurgent function \( \hat{\varphi}(\zeta) \)” (or simply the function \( \varphi(\zeta) \))

Ecalle’s formalism of majors and minors makes it possible to deal with any isolated singularity in the Borel plane and to eventually define alien operators.
for very general spaces of resurgent functions, but for our purposes in the present
text, we shall mostly work with resurgent functions with analytic continuations
which have simple singularities, according to the following:

**Definition 4** We say that the analytic continuation along a path \( \gamma \) from the
origin to \( \omega \in \mathbb{C} \) of a function \( \hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\} \) has a simple singularity at the point
\( \omega \) if and only if there exist 2 analytic functions \( a(\zeta) \), \( b(\zeta) \) at 0 and a constant
\( \alpha_{\omega} \) such that

\[
\hat{\varphi}(\omega + \zeta) = \frac{\alpha_{\omega}}{2\pi i \zeta} + \frac{1}{2\pi i}(\log \zeta)a(\zeta) + b(\zeta)
\]

The singularity at \( \omega \) is thus characterized by the data \( (\alpha_{\omega}, a(\zeta)) \) and the minor
is \( a(\zeta) \)

For such functions, the minor is equal to the germ \( a(\zeta) \) and the singularity,
up to the simple pole \( \frac{\alpha_{\omega}}{2\pi i \zeta} \), is characterized by the minor.

**Remark 3** In the present text, the resurgent functions we deal with have m i-
ners which are uniform at the origin; in that case, it is harmless to keep the
same letter \( \zeta \) for germs at the origin of \( \mathbb{C} \) or \( \mathbb{C}_\infty \).

There is a convolution product, defined at the level of majors, that boils
down, for logarithmic singularities, to the formula given above; we shall not
need the general convolution product, for the saddle–node case we will focus on
later and refer to \[12, 13, 14\] for this material but point out that the simple pole
singularity defined by \( \hat{\varphi} = \frac{1}{2\pi i} \) is a unit for this product and will be denoted by
\( \delta \) (in the Borel plane it is a unit for the convolution, thus acts as a Dirac at the
origin for the Laplace transform of distributions or hyperfunctions).

### 2.2 Alien operators; systems of alien derivations

Let us consider an analytic function \( \hat{\varphi} \) of the variable \( \zeta \), which can be analytically
continued on a sector bisected by the positive real axis, with isolated singularities
at points \( \omega_i \) on \( \mathbb{R}_{>0} : \)

\[
0 < \omega_1 < \omega_2 < \ldots < \omega_n < \ldots
\]

The hypothesis on \( \hat{\varphi} \) we just stated precisely means here that \( \hat{\varphi}(\zeta) \) can be
analytically continued along any path \( \gamma \) starting from 0 and avoiding any point
in the set \( \Omega = \{\omega_1, \omega_2 \ldots\} \). We are going to define the alien operators for
such functions, referring for more general situations to the foundational texts
\[10, 12, 13\] and also to \[20, 25\].

For any \( n \in \mathbb{N}_+ \), let us denote by \( \hat{\varphi}_{\omega_n}^+ \) the function which is the analytic
continuation of the germ \( \hat{\varphi} \) to the point \( \omega_n \), by following the positive real axis
from below; \( \hat{\varphi}_{\omega_n}^+(\xi) \) is in particular defined for \( \xi \) belonging to a universal cover
of a pointed disk \( \mathbb{D}_n := \mathbb{D}(\omega_n, \rho) \setminus \{\omega_n\} \)centered at \( \omega_n \).
We can define an operator $\Delta^+_{\omega_n}$ which “measures” the singular behaviour of the function $\hat{\varphi}$ at $\omega_n$, by:

$$\Delta^+_{\omega_n}(\hat{\varphi})(\zeta) := (\hat{\varphi}^+_{\omega_n}(\zeta + \omega_n))^V$$

where $\zeta$ is positive and close to 0 (we thus extract the singularity of $\hat{\varphi}^+_{\omega_n}$ at the singular point $\omega_n$ after translation of the variable in order to obtain a singularity at the origin). For a resurgent function $\varphi^V$ whose minor $\hat{\varphi}$ is regular at the origin, with isolated singularities at the $\omega_i$ on $\mathbb{R}_{>0}$, $\Delta^+_{\omega_n}$ is defined by the same formula.

These operators satisfy a simple commutation relation with the ordinary derivation $\partial$, which in the Borel plane is the multiplication by $-\zeta$ (of majors and minors), namely:

$$[\Delta^+_{\omega}, \partial] = -\omega \Delta^+_{\omega} \quad (\forall \omega \in \Omega)$$

The $\Delta^+_{\omega_n}$ are not derivations of the algebra of resurgent functions, but they satisfy the following property:

$$\Delta^+_{\omega}(\hat{\varphi} \ast \hat{\psi}) = \sum_{\omega' + \omega'' = \omega} \Delta^+_{\omega'}(\hat{\varphi}) \Delta^+_{\omega''}(\hat{\psi})$$

This corresponds to the fact that the $\Delta^+_{\omega_n}$ are the homogeneous components of the Stokes automorphism, here in the direction $d = \mathbb{R}_+$, (with respect to the grading by the $e^{\omega_n z}$ for alien operators acting on the relevant space of formal expansions: we are sketchy here and refer to [26, 25] for precise formulations) when we deal with resurgent functions with exponential growth at $\infty$.

The collection of the $\Delta^+_{\omega_n}$ can be used as building blocks to construct other operators $\Gamma_{\omega_n}$ in these graded spaces of linear endomorphisms, by formulas of the following type:

$$\Gamma_{\omega_n} = \sum_{\omega_1 + \ldots + \omega_r = \omega_n} M^{(\omega_1,\ldots,\omega_r)}(\Delta^+_{\omega_1} \ldots \Delta^+_{\omega_r})$$

Such a collection $M^{(\omega_1,\ldots,\omega_r)}$ is called a mould and the collection of the $\Delta^+_{(\omega_1,\ldots,\omega_r)}$, where $\Delta^+_{(\omega_1,\ldots,\omega_r)} := \Delta^+_{\omega_1} \ldots \Delta^+_{\omega_r}$, a comould; basic facts about moulds and comoulds are recalled below. When the coefficients satisfy suitable symmetry properties, the $\Gamma_{\omega_n}$ will be derivations of the algebras of resurgent functions on which they act: these are the alien derivations and there are several systems of them, depending on the family of coefficients $M^{(\omega_1,\ldots,\omega_r)}$. When dealing with resurgent functions which have minors with isolated singularities $\omega$ over a given direction $d$ which is not necessarily $\mathbb{R}_{>0}$, alien operators $\Delta^+_\omega$, $\Delta_\omega$ are defined by the same process (in case of minors which are not uniform at the origin, alien operators shall have to be indexed by points $\omega$ on $\mathbb{C}_{\infty}$, we refer to [12, 13]).

The standard system of alien derivations $\Delta^+_{\omega_{\text{stan}}}$ are defined by $M^{(\omega_1,\ldots,\omega_r)} := \gamma^{(\omega)}_n$ and are simply denoted by $\Delta_\omega$; they correspond to the homogeneous components of the “directional derivation” which is the logarithm of the Stokes
diffeomorphism in the direction \( d \) under consideration – which we suppose here to be the positive real axis.

There are many other systems of alien derivations, due to Ecalle ([14]), which are particularly relevant to the synthesis problem; however, as we explain below, we have chosen to work in the present text with the operators \( \Delta_+^\omega \) and we refer to the appendix for brief explanations on the other families of alien derivations, notably the so-called organic ones \( \Delta_\omega^{\text{org}} \) and to [14] for detailed descriptions.

By pullback under the formal Borel transform, we get alien operators and in particular derivations – acting on algebras of formal series, denoted by the same symbols \( \Delta_+^\omega, \Delta_\omega. \)

Finally we can define, on sets of formal expansions involving at the same time formal series in \( z^{-1} \) and exponential factors of the type \( e^{\omega z} \) (so-called transseries, see e.g. [14, 1]), the “exponential bearing” operators \( \dot{\Delta}_+^\omega := e^{-\omega z} \Delta_+^\omega \) and \( \dot{\Delta}_\omega := e^{-\omega z} \Delta_\omega \) which commute with the ordinary one \( \partial = \frac{d}{dz}. \)

3 The resurgent study of the saddle–node

3.1 Analysis

Saddle–nodes which have the simplest normal form correspond to non linear differential equations in the resurgent variable \( z \) of the following type

\[
(E) \quad \partial_z y(z) = y(z) + b(z^{-1}, y) \quad (z \sim \infty, b \text{analytic})
\]

They have as formal normal form the elementary equation:

\[
(E^{\text{nor}}) \quad \partial_z y(z) = y(z) \quad (z \sim \infty)
\]

The general solution of \( E^{\text{nor}} \) being \( y(z) = u e^{z} \) (\( u \in \mathbb{C} \)), when \( \frac{1}{z} = x \sim 0 \), the real phase space indeed displays a saddle–type behaviour on the right and node–type one on the left.

Let \( y(z, u) \in \mathbb{C}[[z^{-1}, u e^z]] \) be the formal integral – namely the general formal solution of \( (E) \); the resurgence properties of the saddle–node are summed up in the following theorem:

**Theorem 1** (Ecalle [13]). The formal integral \( y(z, u) = \sum_{n \geq 0} u^n e^{nz} y_n(z) \) is resurgent in \( z \): the formal series \( y_n \) are Gevrey–1, their Borel transforms have simple singularities (simple poles + logarithmic singularities) at the non zero integers and they have exponential growth in any non–real direction. We have moreover, for any given family of alien derivations the bridge equation:

\[
\dot{\Delta}_n y(z, u) = A_n y(z, u) \quad n \in \Omega = \{-1\} \cup \mathbb{N}^*
\]

Where \( A_n = A_n u^{n+1} \frac{\partial}{\partial u} \) \((A_n \in \mathbb{C})\) is a derivation in the \( u \) variable; the equality that casts a bridge between alien and ordinary calculus has to be understood component-wise:

\[
\Delta_n(y_m) = (m-n)A_n y_{m-n} \quad \text{if } 1 \leq n \leq m-1 \quad \text{and } 0 \text{ else}
\]

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The family of numbers $(A_n)_{n \in \Omega}$ constitutes a complete set of analytic invariants: two equations that are analytically conjugate to the same normal form $E^{nor}$ have the same family of invariants and conversely, two equations which share the same family are analytically conjugate.

Once the resurgent character of the series involved is proved (beside e.g. [12], see also [25] and the extensive calculations in [26]), the bridge equation is easy to obtain with alien derivations $\Delta_n$, by formal considerations involving the Leibniz rule and homogenity properties satisfied by the $\Delta_n$ ([12] [13] [26] [25]).

Then, one can derive a version of the bridge equation for the action of the operators $\Delta^+_n$: we get ordinary differential operators $A^+_n$ in the $u$ variable which are no longer derivations, yet each $A^+_n$ shares with the corresponding $A_n$ the same property of homogeneity (of order $n$).

Moreover, as the $A^+_n$ represent the homogeneous components of the Stokes automorphism, their growth estimates in $n$, as linear operators, are exponential ($\lim \sup \frac{\log(|A^+_n|)}{n} < +\infty$) where the $A_n$ associated to the standard alien derivations generically display a superexponential growth ([13] [14], see also the appendix).

That property of the $A^+_n$ make them very convenient to formulate the inverse problem.

3.2 Synthesis

Let us now tackle the inverse problem within the framework of resurgence, for the saddle–node singularity – and considering the simplest normal form, as above. For this purpose, the data are a given family of ordinary differential operators $(A^+_n)_{n \in \Omega = (-1) \cup \mathbb{N}^0}$, which play the role of the ones to be obtained, through the bridge equation, by the action of a family of alien operators $\Delta_n^+$ on the formal solution of an analytic vector field that we wish to construct.

We will moreover suppose that $\mathbb{R}^*_+$ is the only singular ray, which amounts to supposing that $A_{-1} = 0$; this simplifying assumption will enable us to focus on the main difficulties, but the scheme works in the general case (see the last section).

Our aim will thus be to build an analytic vector field $X$ by a conjugation to the normal form $X^{nor}$, through a change of variables that can be chosen ([22] [20] [25]) as a fibered transformation $\theta : (z, u) \rightarrow (z, \varphi(z, u))$, which will have the required moduli; the transformation $\theta$ will encode all the information.

In fact, the main object which we shall construct will be an endomorphism $\Theta$ of a space of formal series, which is equivalent to $\theta$; indeed, there is a biunivoque correspondance between tangent to identity formal diffeomorphisms $\theta$ and tangent to identity automorphisms $\Theta$ of spaces of formal series ([12] [13] [14] [17]):

$$\theta \mapsto \Theta \quad \text{with} \quad \Theta(f) := f \circ \theta$$

All the following calculations will thus be performed at the operatorial level and we are going to build $\Theta$ by taking as input a given family $(A^+_n)_{n \geq 1}$ as above;
for this, we write an Ansatz and a priori express $\Theta$ as:

$$\Theta = \sum_{\mathbf{n} = (n_1, \ldots, n_r), n_i \in \mathbb{N}_0} \mathcal{L}^n(z) \mathcal{A}^+_n \ldots \mathcal{A}^+_1$$

The $\mathcal{L}^n(z)$ designate resurgent functions of the variable $z$, indexed by sequences $\mathbf{n} = (n_1, \ldots, n_r)$ of nonnegative integers.

If the $\mathcal{L}^n(z)$ have nondecreasing valuations in $n$, where $n = ||(n_1, \ldots, n_r)|| := n_1 + \ldots + n_r$, and taking into account the homogeneity properties of the $\mathcal{A}_n$, the formal sum above will make sense in the space of operators acting on $\mathbb{C}[[z^{-1}, u]]$ and indeed yield a tangent to identity linear endomorphism. Now, such an object will solve our synthesis problem if the following conditions are satisfied:

1. The linear endomorphism $\Theta$ must be an automorphism of the algebra of formal series

2. $X = \Theta X \text{nor} \Theta^{-1}$ must be analytic

3. The moduli obtained by the action of the $\Delta^+_n$ on $X$ are the $\mathcal{A}^+_n$ we started from

The first point will follow from the symmetry properties of the mould $\mathcal{L}^*$: if it is symmetric, then $\Theta$ will indeed be a morphism of algebras, due to the cosymmetry of the comould $\mathcal{A}^*_n$ (invertibility of $\Theta$ is immediate, when $\mathcal{L}^\omega = 1$, as $\mathcal{A}_{\emptyset}^* = \text{Id}$).

Of course, if we had decided to take as input the $\mathcal{A}_n$ associated to some family of alien derivations $\Delta_n$ instead of the $\mathcal{A}^+_n$, the comould $\mathcal{A}^*$ would have been symmetric and we would have required the symmetry of the mould to get the good algebraic properties we need.

The commutation relation $[\Delta^+_n, \Theta] = -\Theta \Delta^+_n$, which expresses the bridge equation in operatorial form, will be automatically satisfied if the endomorphism $\Theta$ is built by using a mould $\mathcal{L}^*$ with $\mathcal{L}^\omega = (-1)^r \mathcal{G}^\omega$, where $\mathcal{G}^*$ is a “$\Delta$-friendly” family of resurgent functions, in the following sense:

$$\Delta_{n_0} \mathcal{G}^{n_1, \ldots, n_r}(z) = \mathcal{G}^{n_2, \ldots, n_r}(z) \quad (\text{if } n_0 = n_1)$$
$$= 0 \quad (\text{if } n_0 \neq n_1)$$

Moreover, all alien derivations will, by the very definition of $X$ act trivially on it, which will entail the analyticity of $X$. But for this scheme to work, we of course first have to prove that $\Theta$ defines an endomorphism of the relevant space of resurgent functions – and this is the crux.

There are several technical difficulties to overcome, notably the following one: as the data $\mathcal{A}^+_n$ involve composing ordinary differential operators $\mathcal{A}^+_n$, even if the family of operators $\mathcal{A}^+_n$ display an exponential growth, we will only get by brute force estimates of the type $r!K^{n_1 + \ldots + n_r}$ for the compositions $\mathcal{A}^+_1 \ldots \mathcal{A}^+_n$. The spurious $r!$ factor will eventually prevent us to get normal convergence of the sums defining $\theta$ in spaces of resurgent functions.
Thus, a clever reorganization of the expansion expressing $\Theta$ is required and this will be provided by Ecalle’s arbomould/coarmould formalism (13, 17, 16), explained in the section on moulds below. The point is that we must “reorder” the mould–comould sums and consider expansions over all forests $F$ decorated by nonnegative integers involving armoulds $\mathcal{L}^< := (\mathcal{L}^F)$ and coarmoulds $(\mathcal{A}_F^-)$:

$$\Theta = \sum_{F} \mathcal{L}^F \mathcal{A}_F^-$$

To sum up, the punchline amounts to finding an armould $(\mathcal{L}^F)$ which is built out of “$\Delta$-friendly” resurgent monomials, with exponential bounds, for suitable semi–norms defining the topology of the relevant topological vector space of resurgent functions.

That requirement is highly non trivial and it was only in the late 90’s that such good families of (ar)moulds were found by Ecalle: the so–called perilogarithms, among which the family of paralogarithms stands out.

### 3.3 Formal synthesis with hyperlogarithmic moulds

The hyperlogarithmic mould $\mathcal{V}^\bullet$ satisfies exponential growth estimates, in spaces of resurgent functions with tame behaviour at infinity (see [13] and, for the case of the saddle–node, the detailed calculations in [26]).

These properties have in particular been used by Lappo-Danilevsky for his beautiful partial solution of the Riemann–Hilbert problem with hyperlogarithms (cf e.g. the Bourbaki lecture by Beauville [2]). It is worth recalling here that Lappo-Danilevsky’s scheme to give explicit solutions to the inverse problem for linear systems involves a clever use of the inverse function theorem, that works when the (monodromy) data are close to the identity. Ecalle’s scheme with paralogarithmic resurgence monomials, explained below, applies to any (Stokes) data, thanks to the presence of a parameter $c$ which we can adjust.

The mould $\mathcal{V}^\bullet$, and some generalizations of it, is most useful for the analysis of the resurgence properties of very general dynamical systems and we refer to the foundational [12, 13] for far-reaching results obtained with it for vector fields in any complex dimension.

Already in the early 80’s, J. Ecalle had introduced a “$\Delta$–friendly” companion mould $\mathcal{U}^\bullet$ (described below) to the mould $\mathcal{V}^\bullet$ and implemented it to tackle the synthesis problem, but he has shown that the objects thus synthesised with the help of this hyperlogarithmic $\mathcal{U}^\bullet$ mould were in fact formal series that are generically divergent.

We refer to [14] section 4 for a careful analysis of this divergence and of the related specific resurgence properties, in particular about the necessity to introduce some parameters in the process, because of the “multivaluedness” of the relations between the “invariants” and the “coinvariants” (which are by definition the coefficients of the dynamical systems synthesized, here by the use of hyperlogarithmic resurgence monomials).
4 Elements of mould calculus

4.1 Moulds and comoulds

**Definition 5** A mould $M^*$ is a family of elements $M^\omega$ of some commutative algebra $A$ over $\mathbb{C}$, indexed by the set $\Omega^*$ of sequences $\omega = (\omega_1, \ldots, \omega_r)$ of elements of some set $\Omega$, also denoted as words $\omega_1 \ldots \omega_r$ with letters in the alphabet $\Omega$.

A mould can alternatively be seen as a linear function from the free vector space spanned by sequences/words to the algebra $A$; this point of view can be quite fruitful (see e.g. [9, 10]), although we shall stick here to Ecalle’s definitions and notations.

In the applications, moulds are very often associated with elements $B_\omega$ of some bialgebra, to constitute formal sums; more precisely, we shall deal with expressions of the following type:

$$\Phi = \sum M^\omega B_\omega = \sum_{r \geq 0} \sum_{\omega = \omega_1 \ldots \omega_r} M^\omega B_\omega$$

The family of the $B_\omega$ is called a comould and such an expansion a mould–comould contraction; when a relevant condition for $\Omega$ of “local finiteness” is satisfied, such sums make sense on the completed free associative algebra spanned by the $B_\omega$, under an hypothesis that the components of the mould and comould have reasonable homogeneity properties.

If the $B_\omega$ represent products of derivations of $A$ ($B_\omega = B_{\omega_r} \ldots B_{\omega_1}$, with the $B_{\omega_i}$ satisfying Leibniz identity), such an expansion $F$ will itself be an automorphism if the coefficients satisfy corresponding symmetry properties, hence the following:

**Definition 6** A mould is called symmetral iff $M^\emptyset = 1$ and, for every pair of non–empty words $\omega', \omega''$, we have

$$M^\omega M^{\omega'} = \sum_{\omega \in \text{sh}(\omega', \omega'')} M^\omega$$

Where $\text{sh}(\omega', \omega'')$ designates the set of words $\omega$ that can be obtained by shuffling $\omega'$ and $\omega''$, counting multiplicities; thus, for distinct elements $\alpha, \beta, \gamma$ we have

$$M^{\alpha\beta} M^\gamma = M^{\alpha\beta\gamma} + M^{\alpha\gamma\beta} + M^{\gamma\alpha\beta}$$

and

$$M^{\alpha\beta} M^\beta = 2M^{\alpha\beta\beta} + M^{\beta\alpha\beta}$$

**Definition 7** A mould is called alternal iff $M^\emptyset = 0$ and, for every pair of non–empty words $\omega', \omega''$, we have

$$0 = \sum_{\omega \in \text{sh}(\omega', \omega'')} M^\omega$$
As we shall deal with mould–comould contractions where the comould is made with shall also need the following:

**Definition 8** A mould is called symmetrel iff $M^\emptyset = 1$ and, for every pair of non-empty words $\omega', \omega''$, we have

$$M^{\omega'} M^{\omega''} = \sum_{\omega \in \text{csh}(\omega', \omega'')} M^\omega$$

In this definition, ordinary shuffles are replaced by contracting shuffles csh: we shuffle 2 words $\omega', \omega''$ and then also perform all the possible contractions $\omega'_i, \omega''_{i+1} \rightarrow \omega'_i + \omega''_{i+1}$ of 2 or more consecutive elements of $\omega'$ and $\omega''$ respectively; thus:

$$M^{\alpha\beta} M^\gamma = M^{\alpha\beta\gamma} + M^{\alpha\gamma\beta} + M^{\gamma\alpha\beta} + M^{\alpha(\beta+\gamma)} + M^{(\alpha+\gamma,\beta)}$$

**Definition 9** A mould is called alternel iff $M^\emptyset = 0$ and, for every pair of non-empty words $\omega', \omega''$, we have

$$0 = \sum_{\omega \in \text{csh}(\omega', \omega'')} M^\omega$$

**Definition 10** A comould $B_\omega$ is a family of differential operators $B_\omega$ (or more generally of some cocommutative bialgebra) indexed by the set $\Omega^*$ of sequences $\omega = (\omega_1, \ldots, \omega_r)$ of elements of some set $\Omega$.

Most of the times in the applications, the components $B_\omega$ will have a certain degree of homogeneity $N_\omega$, as linear operators acting on formal series (thus, in one dimension, $B_\omega(x^n) = \beta_\omega x^{n+N_\omega}$, with $\beta_\omega \in \mathbb{C}$ and $N_\omega \in \mathbb{N}$).

### 4.2 Operations

There are 2 very natural operations, to be performed on mould/comould contractions:

1. We can change the family of coefficients, keeping the set of letters $B_\omega$ fixed; this will give us the product $\times$ of moulds.

2. We can change the set of letters $B_\omega$, through the action of some mould $M^\bullet$, this way

$$B_\bullet \rightarrow C_\bullet \quad \text{with} \quad C_{\omega_0} = \sum_{\|\omega\|=\omega_0} M^\omega B_\omega$$

and this will give the composition $\circ$ of moulds.
Definition 11 Let $M^\bullet$ and $N^\bullet$ be 2 moulds

1. The product $P^\bullet = M^\bullet \times N^\bullet$ is defined by

$$P^{(\omega_1, \ldots, \omega_r)} = \sum_{i=0}^{r} M^{\omega_1 \ldots \omega_i} N^{\omega_{i+1} \ldots \omega_r}$$

2. The composition $Q^\bullet = M^\bullet \circ N^\bullet$ is defined by

$$Q^{(\omega_1, \ldots, \omega_r)} = \sum M((\omega^1 \ldots \omega^s)) N^{\omega_1} \ldots N^{\omega_s}$$

Where the summation is over all partitions of $\omega$ by non-empty words $\omega = \omega_1 \ldots \omega^s$.

Both operations are associative but they are noncommutative as soon as $\Omega$ has more than one element. The mould $1^\bullet$, which takes the value 1 for the empty sequence and 0 else, is the inverse for the multiplication; the mould $I^\bullet$ which takes the value 1 for sequences with a single letter and 0 else is a composition inverse on the set of moulds $M^\bullet$ with $M^\emptyset = 0$.

4.3 Armouse and coarmoulds

The arborification/coarborification process appears in the context of mould–comould contractions:

$$\sum M^\omega B^\omega$$

When one composes ordinary differential operators in $\mathbb{R}^N$ or $\mathbb{C}^N$, trees naturally come in as combinatorial tools, as one can already see in one dimension: let us consider functions $\varphi$ of one complex variable $x$, denote by $\partial = \frac{d}{dx}$ and take a family of operators $B_i$ which are derivations $B_i = b_i(x) \partial$; for any $C^\infty$ function $\varphi(x)$, the action of the composition of $B_2$ and $B_1$ on $\varphi$ yields 2 terms:

$$B_2 B_1 (\varphi)(x) = b_2(x)(\partial b_1(x)) \partial(\varphi)(x) + b_2(x) b_1(x) \partial^2(\varphi)(x)$$

If we leave aside the observable $\varphi$ and work at the level of operators, $B_2 B_1$ is the sum of one operator $B'$ which is the output of the “action of $B_2$ on $B_1$”, the resulting operator the acting on observables and a second one $B''$ which corresponds to $B_2$ and $B_1$ each acting each on the observables. Pictorially:

$$B' : B_2 \uparrow^{B_1} B'' : B_1 \rightarrow B_2$$

When composing 3 operators, $P = B_3 B_2 B_1$ can be represented by the sum of 6 “forests”:

$$B_3 \uparrow^{B_2} B_4 \uparrow^{B_3} B_5 \uparrow^{B_2} B_6 \uparrow^{B_3} B_7 \uparrow^{B_2} B_8 \uparrow^{B_3} B_9 \uparrow^{B_2} B_{10} \uparrow^{B_3} B_{11} \uparrow^{B_2} B_{12} \uparrow^{B_3} B_{13} \uparrow^{B_2} B_{14} \uparrow^{B_3} B_{15} \uparrow^{B_2} B_{16} \uparrow^{B_3} B_{17} \uparrow^{B_2} B_{18} \uparrow^{B_3} B_{19} \uparrow^{B_2} B_{20} \uparrow^{B_3} B_{21} \uparrow^{B_2} B_{22} \uparrow^{B_3} B_{23}$$
This observation was first made by Cayley in the middle of the 19th century (18) and it has been used in a number of works since then, and in particular been implemented in a Hopf–algebraic formalism by Grossman and Larson (cf e.g. 16 17 18 for related combinatorial constructions).

Ecalle has developed a much richer and structured algebraic formalism by considering operations and symmetries involved both for the families of composition of differential operators and the families of coefficients with which they are matched, in mould/comould expansions. We now state the definitions and properties we shall need, referring for further information and proofs to the foundational texts by Ecalle 13 14 15 and also to 16 17 23 24 (in particular, all the required terminology and facts regarding the combinatorics on trees and forest are explained in detail in 17).

**Definition 12**

An arborescent mould (armould, for short) $M^<$ is a family of elements of some commutative algebra $A$, indexed by forests which are decorated by elements of a given set $\Omega$. The mould $M^<$ is called separative iff it is multiplicative on the monoid of forests:

$$M^{F'F''} = M^{F'}M^{F''} \quad \text{(and } M^\emptyset = 1_A)$$

An arborescent comould (coarmould, for short) $B^<$ is a family of ordinary differential operators (or elements of some cocommutative bialgebra), indexed by forests which are decorated by elements of a given set $\Omega$. The comould $B^<$ is called coseparative iff it satisfies for every forest $F$:

$$B_F(fg) = \sum_{F=F'F''} B_{F'}(f)B_{F''}(g) \quad \text{(and } B^\emptyset = \text{Id})$$

We now introduce the process of arborification/coarborification, which was first presented out of the blue by Ecalle in 13, through which we reorder mould/comould contractions to express them as sums over decorated forests – thus called armould/coarmould contractions. For this purpose, we shall need the following definition, to go from totally ordered sequences of elements of a set $\Omega$ to $\Omega$–decorated forests.

**Definition 13**

For any forest $F$ decorated by elements $\omega_i$ and any totally ordered sequence $\omega' = (\omega'_1, \ldots, \omega'_r)$, we denote $\omega' \ll F$ iff there is a bijection from the set of summits of $F$ to the set of indices of $\omega'$, which is order preserving (wrt to the arborescent order of $F$ and the total order of the sequence, respectively) and transports the decorations:

$$\forall j \in \{1, \ldots, r\} \quad \omega'_j = \omega_i \quad \text{where } j = \sigma(i)$$
For any forest $F$ decorated by elements $\omega_i$ and any totally ordered sequence $\omega' = (\omega'_1, \ldots, \omega'_s)$, we denote $\omega' \prec F$ iff there is a surjection $\pi$ from the set of summits of $F$ to the set of indices of $\omega'$, which preserves the strict order and “projects” the decorations:

$$\forall j \in \{1, \ldots, s\} \quad \omega'_j = \sum_{\pi(i)=j} \omega_i$$

**Definition 14**

The arborified of a given mould $M^\bullet$ is the armould $M^<$ defined by

$$M^F := \sum_{\omega',\text{such that } \omega' \prec F} M^{\omega'}$$

The contracted arborified of a given mould $M^\bullet$ is the armould $M^<^<$ defined by

$$M^F := \sum_{\omega',\text{such that } \omega' \prec F} M^{\omega'}$$

The arborified (resp. contracted arborified) of any symmetric (resp. symmetrical) mould is separative ([13, 16]). Moreover, the process of arborification is a morphism of algebras, the product for arborescent moulds being the one introduced by Ecalle in 1992 — and corresponding to the coproduct of Connes-Kreimer’s Hopf algebra, when armoulds are interpreted as characters ([16, 17]).

**Example 2**

For $F = \begin{array}{c}
\beta \\
\gamma
\end{array}$ we have, for the arborified and the contracted arborified respectively:

$$M^F = M^\beta + M^\gamma$$

$$M^F = M^\beta + M^\gamma + 2M^\beta$$

**Definition 15**

A coarmould $B_<$ is called a coarborified of a given mould $B^\bullet$, iff

$$B_\omega = \sum_{\omega \prec F} B_F$$

A coarmould $B_<$ is called a contracted coarborified of a given mould $B^\bullet$, iff

$$B_\omega = \sum_{\omega \prec F} B_F$$
On the comould side, thus, as the process of coarborification involves a decomposition, no uniqueness of the coarborified is to be expected.

There is however a recursive process, due to Ecalle, producing a coarmould out of any given family of ordinary differential operators \( (B_\omega) \) with reasonable homogeneity properties, yielding a coseparative comould \( B_< \) whenever the inputs \( B_\omega \) are all derivations or all satisfy \( B_\omega(fg) = \sum_{\omega=\omega',\omega''} B_{\omega'}(f)B_{\omega''}(g) \); this coarmould is such that in the former case \( B_< \) is a coarborified of the symmetric mould defined by \( B_\omega \), and in the latter a coarborified of the symmetrel mould defined by the same formula.

The key property satisfied by the homogeneous coarborification is that the coarmould it produces satisfies exponential growth estimates, as soon as its building blocks \( B_\omega \) do (e.g., in one dimension, \( B_\omega(x^n) = \beta_\omega x^{n+\mathcal{N}_\omega}, \) with \( |\beta_{\mathcal{N}_\omega}| \leq H_{\mathcal{N}_\omega} \), for some \( H > 0 \)). This was stated, at the level of operators, with very concise arguments to justify it, in the foundational article [13] (see also section 11 of [14]). Later, a proof of this crucial property was given, for the functions obtained by the action of these operators on the coordinates, by F. Menous in [23, 24] and implemented again in [16, 17]; we refer to all these references for the precise definitions, statements and proofs.

For any symmetrel/cosymmetrel (resp. symmetrel/cosymmetrel) mould contraction \( \Phi = \sum M^\omega B_\omega \), we have thus a systematic way to reorganize it as an armould/coarmould contraction, using the homogeneous coarborified and the arborified (resp. contracting arborified) of the involved mould by

\[
\Phi = \sum_F M^F B_F
\]

## 5 Resurgence monomials

### 5.1 The problematics of resurgence monomials

As the alien operators \( \Delta^\omega \) and the alien derivations \( \Delta_\omega \) are powerful and efficient tools for characterizing the singularities of the resurgent functions and exploring their behaviour on all the sheets of their Riemann surfaces, it is most natural to search for families of resurgent functions which are in some sense dual to these alien operators.

More precisely, rather than looking for families \( (M^\omega(z))_{\omega \in \Omega} \), as the process of analytic continuation which underlies the very definition of the alien operators entails non-commutativity, it is more sensible to look for families \( (M^{(\omega_1,\ldots,\omega_r)}(z)) \) of resurgent functions indexed by sequences of elements of a given discrete set \( \Omega \) of complex numbers.

We thus wish to define moulds with values in spaces of resurgent functions, which shall be as simple as possible and numerous enough to expand any given resurgent function as an infinite sum of the \( M^{(\omega_1,\ldots,\omega_r)}(z) \); moreover, we expect these moulds to have good symmetry properties – symmetrel or symmetrel, depending on the type of problems we wish to treat, see [13].
Beyond that, we need to have moulds that behave simply under the action of the ordinary multiplication (which correspond to multiplication by $-\zeta$ in the Borel plane), and the alien operators. In fact, there is a tension between these last 2 requirements: as explained in [12], only trivial functions will at the same time behave very simply with regard to both the ordinary and the alien derivations; correspondingly, we shall have to relax our exigencies and look for distinct families of resurgence monomials:

1. Monomials with very simple properties wrt $\partial$, at the cost of satisfying a bit more complicated properties wrt the $\Delta_\omega$’s

2. Monomials with very simple properties wrt the $\Delta_\omega$’s, at the cost of satisfying a bit more complicated properties wrt $\partial$

Monomials of the first type, called $\partial$–friendly, are particularly well suited for the analysis of the resurgence properties: namely proving the resurgence properties and deriving the bridge equation. Those of the second type, called $\Delta$–friendly, are the ones which will have to be used for synthesis problems.

5.2 Hyperlogarithms

The most natural family of resurgence monomials is built with hyperlogarithms ([10, 12, 13, 26]). We define a mould $\mathcal{V}^\omega$, by recurrence on the length of the sequence (with $\mathcal{V}^\omega = 1$), by the following formula:

$$(\partial + \|\omega\|)\mathcal{V}^\omega(z) = -z^{-1}\mathcal{V}^{\omega_1,\ldots,\omega_{r-1}}(z)$$

We recall that $\|\omega_1,\ldots,\omega_r\| := \omega_1 + \ldots + \omega_r$

In the Borel plane, this relation becomes (for minors):

$$(-\zeta + \|\omega\|)\mathcal{V}^{(\omega_1,\ldots,\omega_r)}(\zeta) = -\int_0^{\zeta} \mathcal{V}^{(\omega_1,\ldots,\omega_{r-1})}(s)ds$$

The mould $\mathcal{V}^\omega$ takes its values in the space of so–called simple resurgent functions: at each of its singular point (for any given sequence, there is a finite number of them, by construction), the singularity is logarithmic plus a simple pole; moreover, $\mathcal{V}^\omega$ satisfies the shuffle relations:

$$\mathcal{V}^{\omega'}(z)\mathcal{V}^{\omega''}(z) = \sum_{\omega \in \text{sh}(\omega',\omega'')} \mathcal{V}^{\omega}(z)$$

By their very definition, the $\mathcal{V}^\omega(z)$ are “$\partial$–friendly”: their behaviour under the ordinary derivation $\partial$ is most simple. With respect to the action of alien derivations, it is more involved and we have the following formulas for the action of the standard alien derivations $\Delta_\omega$, implying complex coefficients $V^\omega$ ([10, 12, 14]):

$$\Delta_\omega \mathcal{V}^\omega(z) = \sum_{\omega_1 + \ldots + \omega_1 = \omega_0} V^{\omega_1,\ldots,\omega_1,\omega_0+1,\ldots,\omega_r}(z)$$
\[ \Delta_{\omega_0} V_\omega(z) = 0 \quad \text{if } \omega_1 + \ldots + \omega_i \neq \omega_0, \forall i \in \{1, \ldots, r\} \]

We then make use of the family of coefficients \( V_\omega \) (which are built with multiple zeta values, see \([10], [26]\)) to construct the “hyperlogarithmic \( \Delta \)-friendly mould \( \mathcal{U} \).”

If we collect the \( V_\omega \)'s for all words \( \omega \), we indeed get a mould, which is \( \mathbb{C} \)-valued; this mould is alternal, it has an inverse \( U^* \) for the operation of composition \( \circ \) and we then define \( \mathcal{U} \) by:

\[ \mathcal{U}^* = \mathcal{V} \circ U^* \]

The mould \( \mathcal{U}^* \) (beside \([10]\), detailed calculations can be found in \([26]\)) is a symmetrical mould with values in the space of simple resurgent function and it satisfies:

\[ \Delta_{\omega_0} U_{\omega_1, \ldots, \omega_r}(z) = U_{\omega_1, \ldots, \omega_r}(z) \quad (\text{if } \omega_0 = \omega_1) \]
\[ = 0 \quad (\text{if } \omega_0 \neq \omega_1) \]

For short, these relations expressing the behaviour of the \( U_{\omega_1, \ldots, \omega_r} \) under the action of the alien derivations, and similar ones for other families of operators, will be called “orthogonality relations”. It is important to keep in mind that they depend upon the particular set of alien operators we use to express them: here, it is the family of standard alien derivations \( \Delta_{\omega} \) but we can also use other families: e.g. the organic alien derivations \( \Delta_{\omega, \text{org}} \), or the \( \Delta_{\omega, +} \) and in this last case, the corresponding monomials will be denoted by \( \pm U \).

With regards to the ordinary derivation \( \partial \), the monomials \( U \) satisfy the following resurgence relations:

\[ (\partial z + \|\omega\|) U_\omega(z) = \sum_{\omega_1 + \ldots + \omega_i = \omega_0} U_{\omega_1, \ldots, \omega_i}(z) U_{\omega_{i+1}, \ldots, \omega_r} z^{-1} \]

where the \( U_\omega \)'s are complex constants.

The \( \Delta \)-friendly mould \( \mathcal{U} \) will for short be called hyperlogarithmic, as it stems from the hyperlogarithmic \( \partial \)-friendly mould \( \mathcal{V} \).

**Remark 4** The reason for the divergence of the objects which are synthesized with the use of the “hyperlogarithmic \( \mathcal{U} \)” can in fact be traced to the operation of taking an inverse for the composition of moulds, in the process of the construction of the mould \( \mathcal{U} \) from the mould \( \mathcal{V} \). Indeed, both the product \( \times \) and the composition \( \circ \) respect the property of exponential growth of moulds and so does the operation of inversion with respect to the product, but the calculation of the composition inverse is a very costly operation (already at the level of ordinary moulds, and also for arborescent ones) and that very fact eventually entails divergence of the constructions involving the hyperlogarithmic \( \mathcal{U} \) mould for the synthesis problem.
6 Paralogarithmic resurgent monomials and synthesis

6.1 Paralogarithms

We shall now present, following Ecalle, a family of resurgence monomials that possess all the required properties, notably the crucial one regarding exponential growth. These monomials are built from the hyperlogarithmic ones; they require a parameter — but as we have mentioned above, some parameters shall be expected in this process — and the striking fact is that one is enough and that this family (conveniently generalized, as explained in the next section) will be truly universal: it will enable to solve all inverse problems, for dynamical systems displaying resurgence.

We have described in the previous section the “standard procedure” to get the \( \Delta \)-friendly family of monomials \( U \), from the hyperlogarithmic one \( V \). In fact, there is a compact formula, already given in [10], that expresses the monomials \( U \) as a multiple integral:

\[
+U_{\omega_1 \ldots \omega_r}(z) = \int_0^\infty \cdots \int_0^\infty \frac{\exp(-\omega_1 y_1 - \cdots - \omega_r y_r)}{(y_r - y_{r-1}) \cdots (y_2 - y_1)(y_1 - z)} dy_1 \cdots dy_r
\]

The integration is performed bypassing every \( y_i \) from under \( \text{Re} \) (or rather performing all integrations on distinct directions \( \theta_i \) that are slightly under the positive real axis); we have of course a corresponding formula for the monomials \( -U_{\omega_1 \ldots \omega_r}(z) \). This formula is valid in any of the 3 models of the resurgent functions involved, but the term \( \frac{1}{y_1 - z} \) has to be understood accordingly ([14], section 10):

1. in the formal plane as the series \( \sum_{n=0}^{\infty} z^{-n-1} y_1^n \)
2. in the geometric models as a function of the complex variable \( z \)
3. in the Borel plane as the function \( \exp(-y_1 \zeta) \)

Remark 5 The geometric models depend by definition upon the choice of a direction \( d \) bisecting a sector of opening \( \pi \) on which the given function is a priori defined. In particular, if we express \( +U_{\omega_1 \ldots \omega_r}(z) \) in a sector bisected by a direction \( d_+ \) (resp. \( d_- \)) slightly under (resp. above) the positive real axis, we shall keep a tag for this; we can keep the notation adopted in [14], section 10 and denote these sectorial incarnations of \( +U_{\omega_1 \ldots \omega_r}(z) \) by \( \eta_+ +U_{\omega_1 \ldots \omega_r}(z) \), where \( \eta = + \) or \( - \) respectively.

To introduce the paralogarithmic resurgent monomials, we reconsider now, here for points \( \omega_i \) on the positive real axis, the integral formulas expressing the (exponential–bearing) hyperlogarithmic resurgent monomials \( +Ue \), written in the following way:
Accordingly, Laplace transform:

\[ g(y) = \int_0^\infty \frac{1}{\exp(-y_1) \cdots \exp(-y_r)} \frac{1}{\exp(-y z) (y_r - y_{r-1}) \cdots (y_2 - y_1)(y_1 - z)} dy_1 \cdots dy_r \]

The idea is to start from this formula and modify the exponential integration kernel in such a way as to preserve the asymptotics and the symmetry properties, in order to ensure that the new functions we build shall indeed be resurgent and orthogonal to the corresponding alien operators but also that they possess the crucial property of exponential growth.

For that purpose, we introduce now a nonnegative real parameter \( c \), and consider the following function:

\[ g_{c,\omega}(y) := \exp(-\omega y - \omega c y^2 y^{-1}) \]

Along with \( g_{c,\omega}(y) \), we shall also have to deal with the function \( f_{c,\omega} \), which is its Laplace transform:

\[ f_{c,\omega}(x) = \int_0^\infty e^{-xy} g_{c,\omega}(y) dy \]

Accordingly, \( g_{c,\omega} \) is the Borel transform of \( f_{c,\omega} \):

\[ g_{c,\omega}(y) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{xy} f_{c,\omega}(x) dx \]

**Definition 16** The paralogarithmic resurgent monomials are defined by:

\[ \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) := \int_0^\infty \cdots \int_0^\infty g_{c,\omega_1}(y_1) \cdots g_{c,\omega_r}(y_r) \frac{1}{y_1 - y_{r-1}} \cdots (y_2 - y_1)(y_1 - z) dy_1 \cdots dy_r \]

Alongside the exponential–bearing \( \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) \) we shall of course have to consider the monomials \( \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) \), which satisfy:

\[ U_c e^{\omega_1, \cdots, \omega_r}(z) = \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) e^{\|\omega_1, \cdots, \omega_r\| z} \]

In fact, for the proofs of the main properties, it will be more convenient to work with the auxiliary functions \( \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) \) (introduced in [14]), related to \( \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) \) in the following way:

\[ U_{c} e^{\omega_1, \cdots, \omega_r}(z) = \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) e^{\|\omega_1, \cdots, \omega_r\| z + c^2 \|\omega_1, \cdots, \omega_r\| z^{-1}} \]

Our working formula, with the same integration rule, will thus be:

\[ \hat{U}_c^{\omega_1, \cdots, \omega_r}(z) = \int_0^\infty \cdots \int_0^\infty \frac{g_{c,\omega_1}(y_1) \cdots g_{c,\omega_r}(y_r)}{(y_1 - y_{r-1}) \cdots (y_2 - y_1)(y_1 - z)} dy_1 \cdots dy_r \]

**Remark 6** For \( c = 0 \), of course, the defining formula of the \( U_c(z) \) boils down to hyperlogarithms but the asymptotics properties completely change as soon as \( c > 0 \), which we will implicitly assume in what follows, when working with the \( U_c(z) \) monomials.
6.2 Schemes for the proofs

As was pointed out in the introduction, the schemes for the proofs of the properties of paralogarithmic resurgence monomials are due to Ecalle and they are clearly indicated in the article [14]. That reference, however, contains a wealth of material on the topic of synthesis but also on other related important topics, which makes it not easily accessible to the readers who have no prior familiarity with mould calculus (this may explain why these beautiful objects had not yet been digested by the dynamical systems community, or beyond); we urge the reader to focus on sections 4, 5, 10, 11 of that reference (cf in particular subsection 11.3).

To work on the proofs, we have the choice between 2 expressions: the defining one with integration over the $y$ variables or another formula, deduced from it through the Borel transform, namely:

$$U_{c,\omega_1,\ldots,\omega_r}(z) = \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \sigma_+(\hat{x}_1) \sigma_-(\hat{x}_2) \cdots \sigma_-(\hat{x}_r) f_{c,\omega_1}(x_1) \cdots f_{c,\omega_r}(x_r) e^{\|\omega\|z} dx_1 \cdots dx_r$$

In this expression, $\sigma$ designates Heaviside’s step function $1_{\mathbb{R}_+}$ and, for any sequence $x_1,\ldots,x_r$ and index $i \in \{1,\ldots,r\}$, $\hat{x}_i := \sum_{i \leq j \leq r} x_j$.

For the proofs of the required properties of the monomials, we can take advantage of the 2 multiple integrals; in particular, as explained in section 10 of [14], both expressions can be used to obtain the crucial exponential growth of the arborified moulds.

1) The resurgent character of the $U_{c,\omega_1,\ldots,\omega_r}(z)$ is checked using the “$y$-integrals”, by recurrence on $r$.

2) The orthogonality of the $^+U_{c,\omega_1,\ldots,\omega_r}(z)$ to the $\Delta^+_{\omega}$ is a direct consequence of the fact that the $^+U_{c,\omega_1,\ldots,\omega_r}(z)$ have the same asymptotics than the hyperlogarithmic $^+U_{\omega_1,\ldots,\omega_r}(z)$, because for any $c$, $\log(g_{c,\omega}(y)) \sim -y$, when $y \sim +\infty$ on $\mathbb{R}_+$

3) The symmetricality of the mould $^+U^*_c(z)$ can be checked on the $y$-integrals

4) To prove the exponential growth property, we can use the $x$-integrals. For every positive $\omega$ we have:

$$g_{c,\omega}(y) \leq \exp(-2\omega c) \quad (\forall y > 0)$$

and for complex values of $y$ we have a saddle–point at $y = c$ (see also the estimates for the integrals in section 4 of [5] and those in appendix C of [19]). The lack of such a property for $c = 0$ of course corresponds to the inadequacy of the hyperlogarithmic $\mathcal{U}$ for this scheme to yield convergent series as output.

As the product $\sigma_+(\hat{x}_1) \sigma_-(\hat{x}_2) \cdots \sigma_-(\hat{x}_r)$ is obviously always 0 or 1, we indeed get an exponential growth estimate:

$$|U_{c,\omega_1,\ldots,\omega_r}(z)| \leq K^{\omega_1+\ldots+\omega_r}$$
which is valid uniformly for any \( z \) in a sector of opening \( < \pi \), centered on any direction except \( \mathbb{R}_> \): we indeed work in geometric models, perform Laplace transforms on directions \( d \not\in \mathbb{R}_> \) and use the saddle–point method to obtain the inequality – with a constant \( K \) explicitly depending on the parameter \( c \) in the following way:

\[
K \leq K_0 e^{-\alpha c}
\]

with \( K_0 \) independent of \( c \), and \( \alpha > 0 \).

Now, as we have seen, the ultimate estimate that we shall need is one of the same type for the contracted arboreified of the mould \( +Ua \) (the ones for \( +U \) will follow easily: they differ by a factor which is an exponential in \( z^{-1} \), which corresponds in the Borel plane to a convolution by a constant of resurgence).

This is the crucial point and it is settled by the fact that we have for the (contracted) arboreified \( Ua^- \) an integral formula of exactly the same shape, but the sums \( \sum_{i \leq j \leq r} x_j \) are replaced by sums \( \sum_{i < j} x_j \) where \( \leq \) designates the arborescent order, for any given forest decorated by the \( x_i \).

This point is essential: without such an lemma, there is no chance of getting any estimate better than \( r!C x_1 + ... + x_r \) for a given forest.

Here lies the true “miracle” of the arboreification technique: it yields closed formulas for the arboreified of “moulds of practical interest” ([13, 14]) and it is thanks to these closed formulas that the required exponential growth can be checked. We thus give the crucial:

**Lemma 1** For any forest \( F \), with decorations \( x_i \), we have for \( Ua^F_c(z) \) the following expression:

\[
\int_{-i\infty}^{+i\infty} ... \int_{-i\infty}^{+i\infty} \sigma_+(\hat{x}_1)\sigma_-(\hat{x}_2) ... \sigma_-(\hat{x}_r)f_{c,\omega_1}(x_1) ... f_{c,\omega_r}(x_r)e^{\|\omega\|z}dx_1 ... dx_r
\]

where for any \( i \), \( \hat{x}_i := \sum_{i < j} x_j \), with \( \leq \) designating the arborescent order.

Of course, with this lemma, the exponential growth is obtained for \( Ua^F_c(z) \) as for \( Ua^-_c(z) \).

### 6.3 The paralogarithmic synthesis of analytic saddle–nodes

We collect now all the tools at our disposal to implement the synthesis of saddle–nodes with paralogarithmic resurgence monomials. For the sake of simplicity, we still consider families of given invariants with \( \mathbb{R}_> \) as only singular direction in the Borel plane but we stress that the method works in the general case.

We thus take as data any family \( \{A^+_n = A_n u^{n+1} \frac{d}{du}\}_{n \in \mathbb{N}_>} \) of exponential growth, which we view as a family of invariants obtained through the bridge equation, by using the operators \( \Delta^+_n \). We consider the comould \( A^+: \)

\[
A^+_{n_1, ..., n_r} = A^+_{n_r} ... A^+_{n_1}
\]

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Plugging in the paralogarithmic monomials $^+\mathcal{U}e_c$, we get a formal operator

$$\Theta_c := \sum_{(n_1, \ldots, n_r)} (-1)^r \mathcal{U}e_{n_1, \ldots, n_r}^+ \mathcal{A}_{n_1, \ldots, n_r}^+$$

As the mould $\mathcal{U}e_c^*$ is symmetrel, the operator $\Theta_c$ is an automorphism of the algebra of formal series in $(z, u)$ that yields, when applied to the variables $z, u$ themselves, a fibered formal normalization transformation:

$$\theta_c := (z, u) \mapsto (z, \varphi_c(z, u))$$

The operator $\Theta_c$ can be re-expressed as a sum over all forests $F$ decorated by positive integers

$$\Theta_c := \sum_F (-1)^r \mathcal{U}e_c^F \mathcal{A}_F^+$$

Finally, we consider the derivation $X_c$, acting a priori on spaces of formal series, defined by:

$$X_c = \Theta_c X^{\text{nor}} \Theta_c^{-1}$$

By what we have just seen, $X_c$ defines an operator that acts on spaces of holomorphic functions in the $z$ variable, on sectors at $\infty$, with uniform bounds: for any ray $d$ except the positive real axis, $X_c$ induces an endomorphism of the space of holomorphic functions, on this sector, because we have exponential growth estimates both for the armould ($^+\mathcal{U}e_c^F$) and the coarmould ($\mathcal{A}_F^+$), hence normal convergence of the sums expressing $\Theta_c$ in the relevant spaces of holomorphic functions in sectors, with uniform exponential bounds at $\infty$.

Moreover, the Stokes phenomenon for $X_c$ in the critical direction $\mathbb{R}_>$ is trivial because by its very construction all the alien derivatives of the components of $X_c$ at the positive integers vanish (at the operatorial level, we have $[X_c, \Delta_n] = 0, \forall n \in \mathbb{N}_>$)! We can now conclude that $X_c$ is analytic.

On the other hand, the commutation relation which we just used to define $X_c$ is no other than the expression of the bridge equation at the level of operators; thus $X_c$ has the prescribed family of analytic invariants $(\mathcal{A}_n^+)_{n \in \mathbb{N}_>}$.

We are done: when expressed in the original variables $x, y$ near the origin, $X_c$ is a vector field with analytic coefficients; we have achieved the synthesis of an analytic vector field with saddle–node singularity that has the prescribed set of invariants.

**Remark 7** We insist that the scheme that we have just described for the saddle–node case will work in full generality for the effective synthesis of any dynamical system displaying resurgence features ([14], sections 4, 10 and 11). Having at our disposal a family of good resurgence monomials, it can, in the same way, be plugged in mould–comould expansions to treat any inverse problem, hence this synthesis process can be called canonical – see the last section. We refer to the examples given in [14], for various classes of dynamical systems with discrete or continuous time (see also [24] for normalization results of some non-linear q–difference equations within the (ar)mould formalism, the synthesis
scheme also has a vocation to apply to these): all that is needed is a precise
resurgent analysis yielding some form of the bridge equation, the shape of which
will depend upon the class of functional equation under study.

Remark 8 Writing for example \( X_c = \Theta_c(X^{\text{nor}}\Theta_c^{-1}) \) the analytic object \( X_c \)
thus appears as the product of 2 divergent–resurgent objects, in the space of
linear endomorphisms acting on formal series in 2 variables.

7 Other examples and outlook

7.1 More general families of paralogarithmic monomials

We have presented in the previous section paralogarithmic resurgence monomials \( (U^{(\omega_1,\ldots,\omega_r)}_c) \) indexed by sequences of positive integers; to deal with more
general indexing sets \( \Omega \), we shall need the following:

Definition 17 For any complex number \( \omega \), and any positive parameter \( c \) we
define

\[
g_{c,\omega}(y) := \exp(\omega y - c^2 \omega y^{-1})
\]

With this enhanced definition (14), these \( U^{(\omega_1,\ldots,\omega_r)}_c(z) \) will enable us to deal
with synthesis problems involving any discrete set \( \Omega \) corresponding to singulari-
ties in the Borel plane, by following the same scheme, for dynamical systems
that have the simplest normal form (14).

The paralogarithmic resurgence monomials \( (U^{(\omega_1,\ldots,\omega_r)}_c) \) moreover satisfy 2 im-
portant properties:

Proposition 1

1. The \( U^{(\omega_1,\ldots,\omega_r)}_c(z) \) induce real functions on the real line
2. The \( U^{(\omega_1,\ldots,\omega_r)}_c(z) \) obey the following homogeneity relations, in the multipli-
cative models (geometric or formal), they are invariant under dilations:

\[
(y, c, \omega_i) \rightarrow (ly, lc, l^{-1} \omega_i) \quad \forall l > 0
\]

The first point of the proposition is an immediate consequence of the defini-
tion of the \( U^{(\omega_1,\ldots,\omega_r)}_c(z) \); however elementary, it is quite important if we want to
synthesize dynamical systems with real coefficients, when we plug the monomials
in expansions involving \( \mathbb{A}^+_1 \) with real coefficients.

The second point is easy to check and is notably relevant when one wants
to explore the dependance in \( c \) of the dynamical systems synthesized with these
monomials.
In order to deal with more general normal forms which involve some ramified powers $z^\sigma$, we need to introduce ramified paralogarithms built with the functions:

$$g_{c,\sigma}(y) := \exp(\omega y - c^2\omega y^{-1} + \sigma \log(y))$$

Finally, in order to synthesize general dynamical systems involving several critical times, we must have recourse to “polycritical ramified paralogarithmic monomials”; these objects are also defined in [14], they are outside the scope of the present text.

Remark 9 The simplifying assumption we had made (namely $A_{-1} = 0$) in presenting for the simplest normal form of saddle–nodes the general scheme of synthesis with paralogarithmic resurgence monomials had the effect that any $c > 0$ is sufficient to ensure convergence of the armoulds/coarmould expansions; without this assumption, the scheme works and yields analytic objects when the parameter $c$ is large enough (see [14], sections 10-11).

7.2 The linear case

Resurgent functions, alien and mould calculus were introduced and developed to solve problems of moduli for non–linear dynamical systems, yet these concepts can prove quite useful already in linear problems – with less technical complications. For the synthesis problem notably, paralogarithmic resurgence monomials entail a systematic and effective procedure to build analytic linear dynamical systems with arbitrary Stokes data: they are thus relevant to tackle the Riemann–Hilbert problem.

If we consider an analytic or meromorphic linear dynamical system $X$ in any complex dimension $\nu$, with a normal form $X = \sum_{i=1}^{\nu} \lambda_i x_i \partial x_i$ (for simplicity, we restrain to systems with trivial formal monodromy), the fundamental matrix solution of $X$ involves Gevrey–1 and generically divergent components, which are resurgent with singularities in the Borel plane at points $\omega_{ij} := \lambda_i - \lambda_j$, $1 \leq i, j \leq \nu$, $i \neq j$ ([12], [14]).

As we are dealing with linear equations – and thus a finite number of singularities, growth conditions for the coefficients produced by the bridge equation is of course not a concern and we can take as input for the solution with resurgence monomials of the synthesis problem, any family $(A_{\omega_{ij}})$ corresponding to the action of the $\Delta_{\omega_{ij}}$ or whatever system of alien derivations $\Delta_{\omega_{ij}}$ – in particular the standard ones.

The process yields an analytic linear dynamical system $X_c$ with prescribed bridge equation (or equivalently Stokes) data, depending on a parameter $c$, which must be chosen large enough, depending on the values of the data $(A_{\omega_{ij}})$ for the mould expansions to converge in the relevant spaces of analytic functions.

The arborification/coarborification process is not needed for linear systems: the mould/comould expansions with paralogarithmic resurgence monomials already converge (for suitably large $c$), but the corresponding expansions with
hyperlogarithmic resurgence monomials diverge, except when the data \((A_{\omega_{ij}})\) are small, or in otherwise particular situations.

The dependence on the parameter \(c\) can be exploited, with the help of the many formulas given in section 12 of [14], to tackle questions of deformations for the linear dynamic systems thus synthesized.

**Remark 10** It is quite striking that, in order to solve Riemann–Hilbert problems, Cecotti–Vafa ([5]) and then Gaiotto–Moore–Neitzke ([19]) used multiple integrals quite similar to the one defining paralogarithmic resurgence monomials, but without a parameter.

### 7.3 Antipodal symmetry

A very intriguing feature of the use of paralogarithmic resurgence monomials for the effective solution of inverse problems is that, alongside the sought analytic objects in \(z \sim \infty\) that they enable to build, they also yield other analytic objects, but in \(z \sim 0\). This fact, which is totally absent when using hyperlogarithms, of course has its origin in the simultaneous presence of \(z\) and of \(z^{-1}\) in the mould formulas defining the paralogarithms and there are in fact compact formulas given in section 11.4 of [14], relating the fundamental objects \(\Theta_{c}(z)\) and \(\Theta_{c}(c^{2}z^{-1})\).

That feature is reminiscent of the phenomenon of “UV/IR mixing” for a number of perturbative series in Physics; on the mathematical side, anyway, it raises many interesting questions (e.g. about possible overlapping of the analytic continuations of the 2 objects, from \(\infty\) and 0 respectively).

### 8 Appendix

#### 8.1 Alien derivations and related moulds

Systems of alien operators \(\Gamma_{\eta}\), notably of alien derivations \((\Delta_{\eta})\), acting on spaces of resurgent functions which have minors with isolated singularities \(\eta_{1}, \eta_{2}, \ldots\) (we keep here the notations of [14]: the \(\eta\)’s designate the successive singular points and the \(\omega\)’s the increments: \(\omega_{i} = \eta_{i} - \eta_{i-1}\)) on a given direction \(d\) in the Borel plane, can be defined either:

1. by averaging various analytic continuations following the direction \(d\) and circumventing the points \(\eta_{i}\) by the right or by the left (with a tag \(\varepsilon_{i} = +\) or \(-\), resp.), involving families of weights \(d\left(\begin{array}{c}
\varepsilon_{1} \\
\omega_{1} \\
\vdots \\
\varepsilon_{r} \\
\omega_{r}
\end{array}\right)\)

2. or, as mentioned in section 2 above, by relating them to the lateral operators \(\Delta_{+}^{\eta}\) (or \(\Delta_{-}^{\eta}\)), with so–called “transition moulds” denoted by (led*) and (red*), respectively.
We refer to \([10, 12, 13, 14]\) for the former point of view (see also \([2, 6]\)) and focus here on the latter and recall the formula:

\[
\Gamma_{\eta_n} = \sum_{\omega_1 + \ldots + \omega_r = \omega_n} \text{led}(\omega_1, \ldots, \omega_r) \Delta_{\omega_1}^+ \ldots \Delta_{\omega_r}^+
\]

**Proposition 2** Let and \(\text{red}^*\) and \(\text{led}^*\) be the 2 “lateral moulds” defining a family of alien operators; then we have \((14)\):

The operators \(\Gamma_{\eta}\) are derivations \(\Leftrightarrow\) \(\text{red}^*\) is an alternel mould \(\Leftrightarrow\) \(\text{led}^*\) is an alternel mould

**Definition 18** The organic alien derivations \(\text{dom}\) (there is another related system, called \(\text{don}\), which also satisfies exponential bounds at the arborified level; we refer to \((14)\)) can be defined by the explicit lateral alternel moulds:

\[
\text{redom}^{(\omega_1, \ldots, \omega_r)} := (-1)^r \frac{1}{2} \frac{\omega_1 + \omega_r}{\omega_1 + \ldots + \omega_r} = - \text{ledom}^{(\omega_1, \ldots, \omega_r)}
\]

The crucial property satisfied by the organic system of alien derivation is contained in the following:

**Proposition 3** The contracted arborified of the moulds \(\text{redom}^*, \text{ledom}^*\) have exponential bounds: \(\exists H, K > 0\) such that, for any forest \(F\) with \(r\) summits

\[|\text{redom}^F| \leq H^r\text{ and } |\text{ledom}^F| \leq K^r\]

It is this proposition that entails that the ordinary differential operators produced by expressing the bridge equation with the organic alien derivations have exponential growth estimates; as mentioned above, the ones corresponding to the standard system of alien derivations do not (this is related to the fact that the latter are the homogeneous components of the directional derivation which is the infinitesimal generator of a Stokes diffeomorphism and such a derivation is generically divergent \([11, 12, 25]\)), which make them less convenient for the synthesis problem. Beside the organic system of derivations, they are many other ones, with interesting properties and beautiful analytic or combinatorial constructions; we refer to \([14]\) for detailed information on this.

### 8.2 Resurgence monomials and systems of alien derivations

We have given in section 6 the multiple integrals that define the paralogarithmic resurgence monomials which are orthogonal to the operators \(\Delta_{\omega}^+\) and worked with them, as in our presentation we have chosen to take as input the operators obtained through the action of the \(\Delta_{\omega}^+\).

It is however possible to express the monomials which are orthogonal to the alien derivations of any given system, by an average of similar integrals
formulas. Indeed, by considering the same integrands as in section 6.1, we can decide to integrate on the direction $\mathbb{R}_>$ successively in the variables $y_1, \ldots, y_r$ by prescribing that, when integrating over $y_i$, we give a $\varepsilon_i = + \tag{resp. $\varepsilon_i = -$} if we bypass the point $y_{i+1}$ from below (resp. from above); correspondingly, we shall obtain an integral that depends on the sequence $\varepsilon$ of the signs $\varepsilon_i$.

Now, if, for a given sequence $\omega_1, \ldots, \omega_r$ we perform an average of these integrals, with the family $\delta$ of coefficients 

$$d \left( \begin{array}{c} \varepsilon_1 \\ \omega_1 \\ \vdots \\ \varepsilon_r \\ \omega_r \end{array} \right)$$

used to define the corresponding alien derivation (\[14\], 2.3 and 6.3), we will automatically get a family $\left( \delta \mathcal{U}^{\omega}(z) \right)$ of monomials which are orthogonal to them, namely $\Delta$–friendly for this particular set of derivations.

For any given family of averaging weights defining good (wrt to growth property of the ordinary operators produced by the corresponding bridge equation) alien derivations, the passage from the (arborified of the) $^+ \mathcal{U}(z)$ to the (arborified of the) $\delta \mathcal{U}(z)$ can be expressed by a product by a scalar (ar)mould with exponential bounds and thus, if we are able to prove that the $^+ \mathcal{U}^{\omega_1, \ldots, \omega_r}(z)$ satisfy exponential growth estimates, then we shall have for free the same property for monomials $\mathcal{U}$ that correspond to a good family of alien derivations, e. g. the organic ones.

It is most valuable in the proofs, if we wish to formulate the synthesis problem when we take as input bridge equation operators obtained through the action of good alien derivations: we can still focus on the monomials $^+ \mathcal{U}^{\omega_1, \ldots, \omega_r}(z)$ and the mould combinatorics takes care of the rest.

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