Newtonian Limit of Maxwell Fluid Flows

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Abstract

In this paper, we revise Maxwell’s constitutive relation and formulate a system of first-order partial differential equations with two parameters for compressible viscoelastic fluid flows. The system is shown to possess a nice conservation–dissipation (relaxation) structure and therefore is symmetrizable hyperbolic. Moreover, for smooth flows we rigorously verify that the revised Maxwell’s constitutive relations are compatible with Newton’s law of viscosity.

1. Introduction

Maxwell fluids are among macromolecular or polymeric fluids. A large number of experiments indicate that polymeric fluids exhibit elastic as well as viscous properties [1]. Thus, they are quite different from small molecular fluids. The latter have viscosity as the main feature, are satisfactorily characterized by Newton’s law of viscosity

$$
\tau = -\nu \left[ \nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] - \kappa \nabla \cdot v I,
$$

(1.1)

and are also called Newtonian fluids. Here \( \tau = \tau(x, t) \) is the stress tensor of the fluid at space-time \((x, t)\), \( \nu \) is the shear viscosity, \( \kappa \) is the bulk viscosity, \( v = v(x, t) \) is the velocity, \( \nabla \) is the gradient operator with respect to the space variable \( x = (x_1, x_2, x_3) \), the superscript T stands for the transpose operator, and \( I \) denotes the unit matrix of order 3. Combining Newton’s law of viscosity with the conservation laws of mass, momentum and energy, one gets the classical Navier–Stokes equations.

To account for the elastic properties of polymeric fluids, Maxwell combined Newton’s law of viscosity with Hooke’s law of elasticity and proposed the following constitutive relation [9]

$$
\varepsilon \tau + \tau = -\nu \left[ \nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] - \kappa \nabla \cdot v I.
$$

(1.2)
Here $\varepsilon$ is the ratio of the viscosity and the elastic modulus. A Maxwell fluid is that obeying the constitutive relation (1.2). This relation reflects that the stress tensor responds to the fluid motion in a delayed, instead of instant, fashion. It has motivated many more realistic and nonlinear constitutive relations, including the well-known upper-convected Maxwell (UCM) and Oldroyd-B models [3].

In this paper, we revise Maxwell’s constitutive relation (1.2), combine the conservation laws and formulate the following partial differential equations (PDEs) for compressible viscoelastic fluid flows:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho v) &= 0, \\
\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v + p I) + \frac{1}{\varepsilon_1} \nabla \cdot \tau_1 + \frac{1}{\varepsilon_2} \nabla \tau_2 &= 0, \\
\partial_t \tau_1 + \frac{1}{\varepsilon_1} \left[ \nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] &= -\frac{\tau_1}{\nu \varepsilon_1^2}, \\
\partial_t \tau_2 + \frac{1}{\varepsilon_2} \nabla \cdot v &= -\frac{\tau_2}{\kappa \varepsilon_2^2}.
\end{align*}
\] (1.3)

Here $\rho$ is the density of the fluid, $\otimes$ denotes the tensorial product, $p = p(\rho)$ is the hydrostatic pressure, $\tau_1$ is a tensor of order two, $\varepsilon_1$ and $\varepsilon_2$ are two positive parameters, and $\tau_2$ is a scalar. This is a system of first-order partial differential equations, with domain

\[ G := \{ (\rho, \rho v, \tau_1, \tau_2) : \rho > 0 \}. \]

In (1.3) there are 14 equations (for three-dimensional problems). Note that the \([\cdots]\)-term in the $\tau_1$-equation is symmetric and traceless. It is easy to see that $\tau_1$ is symmetric and traceless if it is so initially. When $\tau_1$ is symmetric and traceless, the number of independent equations in (1.3) reduces to $n = 10$. Throughout this paper, we assume that $\tau_1$ is symmetric and traceless.

We will show that the first-order system (1.3) satisfies the entropy dissipation condition proposed in [16]. This particularly implies that the system is symmetrizable hyperbolic. Moreover, we will show that the revised Maxwell’s constitutive relations (the $\tau_1$-, $\tau_2$-equations) in (1.3) are compatible with Newton’s law of viscosity (1.1) for small $\varepsilon_1$ and $\varepsilon_2$. To see this, we rewrite the two $\tau$-equations in (1.3) as follows and iterate them once to obtain

\[
\begin{align*}
\tau_1 &= -\varepsilon_1 v \left[ \nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] - \varepsilon_1^2 v \partial_t \tau_1, \\
&= -\varepsilon_1 v \left[ \nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] + O(\varepsilon_1^3), \\
\tau_2 &= -\varepsilon_2 \kappa \nabla \cdot v - \varepsilon_2^2 \kappa \partial_t \tau_2, \\
&= -\varepsilon_2 \kappa \nabla \cdot v + O(\varepsilon_2^3).
\end{align*}
\]

Substituting the truncations into the momentum equation in (1.3), we obtain the classical isentropic Navier–Stokes equations. In this sense, Newton’s law (1.1) is recovered.