A GENERALIZATION OF HERMITE–HADAMARD’S INEQUALITY

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Abstract. In literature the Hermite–Hadamard inequality was eligible for many reasons, one of the most surprising and interesting that the Hermite–Hadamard inequality combine the midpoint and trapezoid formulae in an inequality. In this work, a Hermite-Hadamard like inequality that combines the composite trapezoid and composite midpoint formulae is proved. So that, the classical Hermite–Hadamard inequality becomes a special case of the presented result. Some Ostrowski’s type inequalities for convex functions are proved as well.

1. Introduction

Let $f : [a, b] \rightarrow \mathbb{R}$, be a twice differentiable mapping such that $f''(x)$ exists on $(a, b)$ and $\|f''\|_{\infty} = \sup_{x \in (a,b)} \|f''(x)\| < \infty$. Then the midpoint inequality is known as:

$$\left| \int_{a}^{b} f(x) \, dx - (b-a) f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{24} \|f''\|_{\infty},$$

(1.1)

and, the trapezoid inequality

$$\left| \int_{a}^{b} f(x) \, dx - (b-a) \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^3}{12} \|f''\|_{\infty},$$

(1.2)

also hold. Therefore, the integral $\int_{a}^{b} f(x) \, dx$ can be approximated in terms of the midpoint and the trapezoidal rules, respectively such as:

$$\int_{a}^{b} f(x) \, dx \approx (b-a) f \left( \frac{a+b}{2} \right),$$

and

$$\int_{a}^{b} f(x) \, dx \approx (b-a) \frac{f(a) + f(b)}{2},$$

which are combined in a useful and famous relationship, known as the Hermite-Hadamard’s inequality. That is,

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2},$$

(1.3)

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which hold for all convex functions $f$ defined on a real interval $[a, b]$.

The real beginning was (almost) in the last twenty-five years, where, in 1992 Dragomir [10] published his article about (1.3). The main result in [10] was

**Theorem 1.** Let $f : [a, b]$ is convex function one can define the following mapping on $[0, 1]$ such as:

$$H(t) = \frac{1}{b-a} \int_{a}^{b} f\left(tx + (1-t) \frac{a+b}{2}\right) dx,$$

then,

1. $H$ is convex and monotonic non-decreasing on $[0, 1]$.
2. One has the bounds for $H$

$$\sup_{t \in [0, 1]} H(t) = \frac{1}{b-a} \int_{a}^{b} f(x) dx = H(1),$$

and

$$\inf_{t \in [0, 1]} H(t) = f\left(\frac{a+b}{2}\right) = H(0).$$

A few years after 1992, many authors have took (a real) attention to the Hermite–Hadamard inequality and sequence of several works under various assumptions for the function involved such as bounded variation, convex, differentiable functions whose $n$-derivative(s) belong to $L_p(a, b)$; ($1 \leq p \leq \infty$), Lipschitz, monotonic, ... etc, have been published. For a comprehensive list of results and excellent bibliography we recommend the interested to refer to [4], [5] and [14].

In 1997, Yang and Hong [15], continued on Dragomir result [11] and they proved the following theorem:

**Theorem 2.** Suppose that $f : [a, b] \rightarrow \mathbb{R}$, is convex and the mapping $F : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$F(t) = \frac{1}{b-a} \int_{a}^{b} \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}u\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}u\right)\right] du,$$

then,

1. The mapping $F$ is convex and monotonic nondecreasing on $[0, 1]$.
2. We have the bounds

$$\inf_{t \in [0, 1]} F(t, s) = \frac{1}{b-a} \int_{a}^{b} f(x) dx = F(0),$$

$$\sup_{t \in [0, 1]} F(t) = \frac{f(a) + f(b)}{2} = F(1).$$

For other closely related results see [11], [2], [6–9] and [14–13].
In terms of composite numerical integration, we recall that the Composite Midpoint rule \((\text{(3), p.202})\)

\[
(1.4) \quad \int_a^b f(x) \, dx - 2h \sum_{j=0}^{n/2} f(x_{2j}) = \frac{(b-a)}{6} h^2 f''(\mu),
\]

for some \(\mu \in (a,b)\), where \(f \in C^2[a,b]\), \(n\) even, \(h = \frac{b-a}{n+2}\) and \(x_j = a + (j+1)h\), for each \(j = -1, 0, \cdots, n+1\).

And, the Composite Trapezoid rule \((\text{(3), p.203})\)

\[
(1.5) \quad \int_a^b f(x) \, dx - \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] = \frac{(b-a)}{12} h^2 f''(\mu),
\]

for some \(\mu \in (a,b)\), where \(f \in C^2[a,b]\), \(h = \frac{b-a}{n}\) and \(x_j = a + jh\), for each \(j = 0, 1, \cdots, n\).

The main purpose of this work, is to combine the composite Trapezoid and composite Midpoint formulae in an inequality that is similar to the classical Hermite–Hadamard inequality \((\text{(1.3)})\) for convex functions defined on a real interval \([a, b]\). In this way, we establish a conventional generalization of \((\text{(1.3)})\) which is in turn most useful and have a very constructional form.

2. A Generalization of Hermite–Hadamard’s Inequality

**Theorem 3.** Let \(f : [a, b] \to \mathbb{R}\) be a convex function on \([a, b]\), then the double inequality

\[
(2.1) \quad h \sum_{k=1}^{n} f \left( \frac{x_{k-1} + x_k}{2} \right) \leq \int_a^b f(t) \, dt \leq \frac{h}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right],
\]

holds, where \(x_k = a + k \frac{b-a}{n}\), \(k = 0, 1, 2, \cdots, n\); with \(h = \frac{b-a}{n}, n \in \mathbb{N}\). The constant '1' in the left-hand side and '1/2' in the right-hand side are the best possible for all \(n \in \mathbb{N}\). If \(f\) is concave then the inequality is reversed.

**Proof.** Since \(f\) is convex on \([a, b]\), then \(f\) so is on each subinterval \([x_j-1, x_j]\), \(j = 1, \cdots, n\), then for all \(t \in [0,1]\), we have

\[
(2.2) \quad f(tx_{j-1} + (1-t)x_j) \leq tf(x_{j-1}) + (1-t)f(x_j).
\]

Integrating \((2.2)\) with respect to \(t\) on \([0,1]\) we get

\[
(2.3) \quad \int_0^1 f(tx_{j-1} + (1-t)x_j) \, dt \leq \frac{f(x_{j-1}) + f(x_j)}{2}.
\]

Substituting \(u = tx_{j-1} + (1-t)x_j\), in the left hand side of \((2.3)\), we get

\[
\int_{x_{j-1}}^{x_j} f(u) \, du \leq \frac{x_j - x_{j-1}}{2} \left( f(x_{j-1}) + f(x_j) \right).
\]
Taking the sum over $j$ from 1 to $n$, we get
\[
\sum_{j=1}^{n} \int_{x_{j-1}}^{x_{j}} f(u) \, du = \int_{a}^{b} f(u) \, du \\
\leq \sum_{j=1}^{n} \frac{x_{j} - x_{j-1}}{2} (f(x_{j-1}) + f(x_{j})) \\
\leq \frac{1}{2} \max_{j} \{x_{j} - x_{j-1}\} \cdot \sum_{j=1}^{n} (f(x_{j-1}) + f(x_{j})) \\
= \frac{h}{2} \left[ f(x_{0}) + f(x_{1}) + \sum_{j=2}^{n-1} \{f(x_{j-1}) + f(x_{j})\} + f(x_{n-1}) + f(x_{n}) \right] \\
(2.4) = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_{j}) + f(b) \right].
\]

On the other hand, again since $f$ is convex on $I_x$, then for $t \in [0, 1]$, we have
\[
f \left( \frac{x_{j-1} + x_{j}}{2} \right) = f \left( \frac{tx_{j} + (1-t)x_{j-1}}{2} + \frac{(1-t)x_{j} + tx_{j-1}}{2} \right) \\
\leq \frac{1}{2} \left[ f(tx_{j} + (1-t)x_{j-1}) + f((1-t)x_{j} + tx_{j-1}) \right].
\]

Integrating inequality (2.5) with respect to $t$ on $[0, 1]$ we get
\[
f \left( \frac{x_{j-1} + x_{j}}{2} \right) \leq \frac{1}{2} \int_{0}^{1} [f(tx_{j} + (1-t)x_{j-1}) + f((1-t)x_{j} + tx_{j-1})] \, dt \\
(2.6) = \frac{1}{2} \int_{0}^{1} f(tx_{j} + (1-t)x_{j-1}) \, dt + \frac{1}{2} \int_{0}^{1} f((1-t)x_{j} + tx_{j-1}) \, dt.
\]

By putting $1-t = s$ in the second integral on the right-hand side of (2.6), we have
\[
f \left( \frac{x_{j-1} + x_{j}}{2} \right) \leq \frac{1}{2} \int_{0}^{1} f(tx_{j} + (1-t)x_{j-1}) \, dt + \frac{1}{2} \int_{0}^{1} f((1-t)x_{j} + tx_{j-1}) \, dt \\
(2.7) = \int_{0}^{1} f(tx_{j} + (1-t)x_{j-1}) \, dt.
\]

Substituting $u = tx_{j} + (1-t)x_{j-1}$, in the left hand side of (2.7), and then taking the sum over $j$ from 1 to $n$, we get
\[
h \sum_{k=1}^{n} f \left( \frac{x_{k-1} + x_{k}}{2} \right) \leq \int_{a}^{b} f(t) \, dt. \\
(2.8)
\]

From (2.4) and (2.8), we get the desired inequality (2.1).
To prove the sharpness let (2.1) hold with another constants $C_1, C_2 > 0$, which gives

$$C_1 \cdot h \sum_{k=1}^{n} f \left( \frac{x_{k-1} + x_k}{2} \right) \leq \int_{a}^{b} f(t) \, dt \leq C_2 \cdot h \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right].$$

(2.9)

Let $f : [a, b] \to \mathbb{R}$ be the identity map $f(x) = x$, then the right-hand side of (2.9) reduces to

$$\frac{b^2 - a^2}{2} \leq C_2 \cdot h \left[ a + 2 \sum_{k=1}^{n-1} x_k + b \right] = C_2 \cdot h \left[ a + 2 \sum_{k=1}^{n-1} \left( a + \frac{b-a}{n} \right) + b \right] = C_2 \cdot h \left[ a + 2 \sum_{k=1}^{n-1} a + 2 \frac{b-a}{n} \sum_{k=1}^{n-1} k + b \right] = C_2 \cdot \frac{b-a}{n} \left[ a + 2 (n-1) a + 2 \frac{b-a}{n} \cdot \frac{n (n-1)}{2} + b \right] = C_2 \cdot (b-a) (a+b).$$

It follows that $\frac{1}{2} \leq C_2$, i.e., $\frac{1}{2}$ is the best possible constant in the right-hand side of (2.1).

For the left-hand side, we have

$$\frac{b^2 - a^2}{2} \geq C_1 \cdot h \sum_{k=1}^{n} \frac{x_{k-1} + x_k}{2} = C_1 \cdot h \sum_{k=1}^{n} \left\{ a + (2k-1) \frac{b-a}{2n} \right\} = C_1 \cdot \frac{b-a}{n} \left[ na + \frac{b-a}{2n} \left( 2 \cdot \frac{n (n+1)}{2} - n \right) \right] = C_1 \cdot \frac{b^2 - a^2}{2},$$

which means that $1 \geq C_1$, and thus $1$ is the best possible constant in the left-hand side of (2.1). Thus the proof of Theorem 3 is completely finished. \(\square\)

**Remark 1.** In Theorem 3, if we take $n = 1$, then we refer to the original Hermite–Hadamard inequality (1.3).

As application, next we give a direct refinements of Hermite-Hadamard’s type inequalities for convex functions defined on a real interval $[a, b]$, according to the number of division ‘n’ (e.g. $n = 1, 2, 3, 4$) in Theorem 3.

**Corollary 1.** In Theorem 3 we have
(1) If \( n = 1 \), then
\[
(b - a) f \left( \frac{a + b}{2} \right) \leq \int_a^b f(t) \, dt \leq (b - a) \frac{f(a) + f(b)}{2}.
\]

(2) If \( n = 2 \), then
\[
\frac{(b - a)}{2} \left[ f \left( \frac{3a + b}{4} \right) + f \left( \frac{a + 3b}{4} \right) \right]
\leq \int_a^b f(t) \, dt
\leq \frac{(b - a)}{4} \left[ f(a) + 2f \left( \frac{a + b}{2} \right) + f(b) \right].
\]

(3) If \( n = 3 \), then
\[
\frac{(b - a)}{3} \left[ f \left( \frac{5a + b}{6} \right) + f \left( \frac{a + 2b}{2} \right) + f \left( \frac{a + 5b}{6} \right) \right]
\leq \int_a^b f(t) \, dt
\leq \frac{(b - a)}{6} \left[ f(a) + 2f \left( \frac{a + b}{3} \right) + 2f \left( \frac{a + 2b}{3} \right) + f(b) \right].
\]

(4) If \( n = 4 \), then
\[
\frac{(b - a)}{4} \left[ f \left( \frac{7a + b}{8} \right) + f \left( \frac{5a + 3b}{8} \right) + f \left( \frac{3a + 5b}{8} \right) + f \left( \frac{a + 7b}{8} \right) \right]
\leq \int_a^b f(t) \, dt
\leq \frac{(b - a)}{12} \left[ f(a) + 2f \left( \frac{3a + b}{4} \right) + 2f \left( \frac{a + b}{2} \right) + 2f \left( \frac{a + 3b}{4} \right) + f(b) \right].
\]

Now, let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\). Define the mappings \( H_j, F_j : [0, 1] \to \mathbb{R} \), given by
\[
(2.10) \quad H_j(t) = \frac{1}{h} \int_{x_{j-1}}^{x_j} f \left( tu + (1 - t) \frac{x_{j-1} + x_j}{2} \right) \, du, \quad u \in [x_{j-1}, x_j],
\]
and
\[
(2.11) \quad F_j(t) = \frac{1}{h} \int_{x_{j-1}}^{x_j} \left[ f \left( \frac{1 + t}{2} x_{j-1} + \frac{1 - t}{2} u \right) + f \left( \frac{1 + t}{2} x_j + \frac{1 - t}{2} u \right) \right] \, du, \quad u \in [x_{j-1}, x_j].
\]

Applying Theorems 1 and 2 for \( f : [x_{j-1}, x_j] \to \mathbb{R}, j = 1, 1, \ldots, n \). Then the following statements hold:
(1) \( H_j(t) \) and \( F_j(t) \) are convex for all \( t \in [0, 1] \) and \( u \in [x_{j-1}, x_j] \).
(2) \( H_j(t) \) and \( F_j(t) \) are monotonic nondecreasing for all \( t \in [0, 1] \) and \( u \in [x_{j-1}, x_j] \).
(3) We have the following bounds for $H_j(t)$

\[
\frac{1}{h} \int_{x_j-1}^{x_j} f(u) \, du = H_j(1),
\]

and

\[
f\left(\frac{x_{k-1} + x_k}{2}\right) = H_j(0).
\]

and the following bounds for $F_j(t)$

\[
\frac{f(x_{j-1}) + f(x_j)}{2} = F_j(1),
\]

and

\[
\frac{1}{h} \int_{x_{j-1}}^{x_j} f(u) \, du = F_j(0).
\]

Hence, we may establish two related mappings for the inequality (2.1).

**Proposition 1.** Let $f$ be as in Theorem 3, define the mappings $H, F : [0, 1] \to \mathbb{R}$, given by

\[
H(t) = \sum_{j=1}^{n} H_j(t) \quad \text{and} \quad F(t) = \sum_{j=1}^{n} F_j(t),
\]

where $H_j(t)$ and $F_j(t)$ are defined in (2.10) and (2.11), respectively; then the following statements hold:

1. $H(t)$ and $F(t)$ are convex for all $t \in [0, 1]$ and $u \in [a, b]$.
2. $H(t)$ and $F(t)$ are monotonic nondecreasing for all $t \in [0, 1]$ and $u \in [a, b]$.
3. We have the following bounds for $H(t)$

\[
\sup_{t \in [0, 1]} H(t) = \frac{1}{h} \int_{a}^{b} f(u) \, du = H(1),
\]

and

\[
\inf_{t \in [0, 1]} H(t) = \frac{1}{h} \sum_{k=1}^{n} f \left(\frac{x_{k-1} + x_k}{2}\right) = H(0).
\]

and the following bounds for $F(t)$

\[
\sup_{t \in [0, 1]} F(t) = \frac{1}{2} \left[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b)\right] = F(1),
\]

and

\[
\inf_{t \in [0, 1]} F(t) = \frac{1}{h} \int_{a}^{b} f(u) \, du = F(0).
\]

**Proof.** Taking the sum over $j$ from 1 to $n$, in (2.12)–(2.15) we get the required results, and we shall omit the details. □

**Remark 2.** The inequality (2.7) may written in a convenient way as follows:

\[
\sum_{k=1}^{n} f \left(\frac{x_{k-1} + x_k}{2}\right) - \sum_{k=1}^{n-1} f(x_k) \leq \frac{1}{h} \int_{a}^{b} f(t) \, dt - \sum_{k=1}^{n-1} f(x_k) \leq \frac{f(a) + f(b)}{2}
\]

which is of Ostrowski’s type.
Some sharp Ostrowski’s type inequalities for convex functions defined on a real interval \([a, b]\), are proposed in the next theorems.

**Theorem 4.** Let \(f : [a, b] \rightarrow \mathbb{R}_+\) be a positive convex function on \([a, b]\), then the inequality

\[
\int_a^b \int_0^1 f(x)\,dx - (b - a)\,f(y) \leq \frac{h}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right],
\]

holds for all \(y \in [a, b]\). where, \(x_k = a + k\frac{b-a}{n}, k = 0, 1, 2, \ldots, n\); with \(h = \frac{b-a}{n},\ n \in \mathbb{N}\). The constant \(\frac{h}{2}\) in the right-hand side is the best possible, in the sense that it cannot be replaced by a smaller one for all \(n \in \mathbb{N}\). If \(f\) is concave then the inequality is reversed.

**Proof.** Fix \(y \in [x_{j-1}, x_j], j = 1, \ldots, n\). Since \(f\) is convex on \([a, b]\), then \(f\) so is on each subinterval \([x_{j-1}, x_j]\), in particular on \([x_{j-1}, y]\), then for all \(t \in [0, 1]\), we have

\[
f(tx_{j-1} + (1 - t) y) \leq tf(x_{j-1}) + (1 - t) f(y), \quad j = 1, \ldots, n.
\]

Integrating (2.22) with respect to \(t\) on \([0, 1]\) we get

\[
\int_0^1 f(tx_{j-1} + (1 - t) y)\,dt \leq \frac{f(x_{j-1}) + f(y)}{2}.
\]

Substituting \(u = tx_{j-1} + (1 - t) y\), in the left hand side of (2.23), we get

\[
\int_{x_{j-1}}^y f(u)\,du \leq \frac{y - x_{j-1}}{2} \left(f(x_{j-1}) + f(y)\right).
\]

Now, we do similarly for the interval \([y, x_j]\), we therefore have

\[
f(ty + (1 - t) x_j) \leq tf(y) + (1 - t) f(x_j), \quad j = 1, \ldots, n.
\]

Integrating (2.25) with respect to \(t\) on \([0, 1]\) we get

\[
\int_0^1 f(ty + (1 - t) x_j)\,dt \leq \frac{f(y) + f(x_j)}{2}.
\]

Substituting \(u = ty + (1 - t) x_j\), in the left hand side of (2.26), we get

\[
\int_{y}^{x_j} f(u)\,du \leq \frac{x_j - y}{2} \left(f(y) + f(x_j)\right).
\]

Adding the inequalities (2.24) and (2.27), we get

\[
\int_{x_{j-1}}^{y} f(u)\,du + \int_{y}^{x_j} f(u)\,du = \int_{x_{j-1}}^{x_j} f(u)\,du
\]

\[
\leq \frac{y - x_{j-1}}{2} \left(f(x_{j-1}) + f(y)\right) + \frac{x_j - y}{2} \left(f(y) + f(x_j)\right)
\]

\[
\leq \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_{j-1}) + \frac{x_j - x_{j-1}}{2} f(y)
\]

\[
\leq \frac{x_j - x_{j-1}}{2} \left(f(x_{j-1}) + f(x_{j-1})\right) + hf(y).
\]
Taking the sum over \( j \) from 1 to \( n \), we get

\[
\sum_{j=1}^{n} \int_{x_{j-1}}^{x_{j}} f(u) \, du = \int_{a}^{b} f(u) \, du \\
= \sum_{j=1}^{n} \frac{x_{j} - x_{j-1}}{2} \{ f(x_{j-1}) + f(x_{j}) \} + \sum_{j=1}^{n} hf(y) \\
\leq \frac{1}{2} \max_{j=1}^{n} \{ f(x_{j}) - f(x_{j-1}) \} \cdot \sum_{j=1}^{n} \{ f(x_{j-1}) + f(x_{j}) \} + (b - a) f(y) \\
= \frac{h}{2} \left[ f(x_{0}) + f(x_{1}) + \sum_{j=2}^{n-1} \{ f(x_{j-1}) + f(x_{j}) \} + f(x_{n-1}) + f(x_{n}) \right] + (b - a) f(y) \\
= \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_{j}) + f(b) \right] + (b - a) f(y),
\]

which gives that

\[
\int_{a}^{b} f(u) \, du - (b - a) f(y) \leq \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_{j}) + f(b) \right]
\]

for all \( y \in [x_{j-1}, x_{j}] \subseteq [a, b] \) for all \( j = 1, 2, \ldots, n \), which gives the desired result (2.21).

To prove the sharpness let (2.21) hold with another constants \( C > 0 \), which gives

\[
\int_{a}^{b} f(x) \, dx - (b - a) f(y) \leq C \cdot \frac{h}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_{k}) + f(b) \right].
\]

Let \( f : [0, 1] \rightarrow \mathbb{R} \) be the identity map \( f(x) = x \), then the right-hand side of (2.29) reduces to

\[
\frac{1}{2} - y \leq \frac{C}{n} \left[ \sum_{k=1}^{n-1} x_{k} + \frac{1}{2} n(n - 1) \right] \\
= \frac{C}{n} \left[ \frac{1}{2} n \frac{n(n - 1)}{2} + \frac{1}{2} \right] \\
= C.
\]

Choose \( y = 0 \), it follows that \( \frac{1}{2} \leq C \), i.e., \( \frac{1}{2} \) is the best possible constant in the right-hand side of (2.21).

\[ \square \]

**Theorem 5.** Under the assumptions of Theorem 4, we have

\[
(2.30) \quad \int_{a}^{b} f(x) \, dx - (b - a) f(y) \\
\leq \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_{j}}{2} \right| \right] \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_{j}) + f(b) \right],
\]

for all \( y \in [a, b] \). The constant \( \frac{1}{2} \) in the right-hand side is the best possible for all \( n \in \mathbb{N} \). If \( f \) is concave then the inequality is reversed.
Proof. Repeating the steps of the proof of Theorem 4 therefore by (2.28)

\[
\int_{x_{j-1}}^{x_j} f(u) du \leq \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_j) + h f(y) \\
\leq \max \left\{ \frac{y - x_{j-1}}{2}, \frac{x_j - y}{2} \right\} \cdot (f(x_{j-1}) + f(x_j)) + h f(y) \\
\leq \left[ \frac{x_j - x_{j-1}}{2} + \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot (f(x_{j-1}) + f(x_j)) + h f(y)
\]

(2.31)

Taking the sum over \( j \) from 1 to \( n \), we get

\[
\int_a^b f(u) du \\
= \sum_{j=1}^{n} \left[ \frac{x_j - x_{j-1}}{2} + \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \{ f(x_{j-1}) + f(x_j) \} + \sum_{j=1}^{n} h f(y) \\
\leq \max_{1 \leq j \leq n} \left[ \frac{x_j - x_{j-1}}{2} + \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \sum_{j=1}^{n} (f(x_{j-1}) + f(x_j)) + (b - a) f(y) \\
\leq \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \\
\times \left[ f(x_0) + f(x_1) + \sum_{j=2}^{n-1} \{ f(x_{j-1}) + f(x_j) \} + f(x_{n-1}) + f(x_n) \right] + (b - a) f(y) \\
= \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] + (b - a) f(y),
\]

which gives that

\[
\int_a^b f(u) du - (b - a) f(y) \\
\leq \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right]
\]

for all \( y \in [x_{j-1}, x_j] \subseteq [a, b] \) for all \( j = 1, 2, \cdots, n \), which gives the desired result (2.30). The proof of sharpness goes likewise the proof of the sharpness of Theorem 4 and we shall omit the details. \( \square \)
Corollary 2. Let $\alpha_i \geq 0$, $\forall \ i = 0, 1, 2, \ldots, n$, be positive real numbers such that $\sum_{i=0}^{n} \alpha_i = 1$, then under the assumptions of Theorem 3 we have

\begin{equation}
\int_{a}^{b} f(x) \, dx - (b-a) f \left( \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i x_i \right) \leq \left[ \frac{h}{2} + \max_{1 \leq j \leq n} \left| \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i x_i - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],
\end{equation}

for all $y \in [a, b]$. The constant $\frac{1}{2}$ in the right-hand side is the best possible. If $f$ is concave then the inequality is reversed.

Theorem 6. Under the assumptions of Theorem 3 we have

\begin{equation}
\frac{1}{b-a} \int_{a}^{b} f(u) \, du - \frac{1}{n} \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \leq \frac{1}{n} \sum_{j=1}^{n-1} f(x_j) \leq \frac{f(a) + f(b)}{2n}
\end{equation}

for all $j = 1, 2, \ldots, n$. The constant $\frac{1}{2}$ in the right-hand side is the best possible. If $f$ is concave then the inequality is reversed.

Proof. Repeating the steps of the proof of Theorem 3, if we choose $y = \frac{x_{j-1} + x_j}{2}$, then we get

\begin{equation}
\int_{x_{j-1}}^{x_j} f(u) \, du \leq \frac{x_j - x_{j-1}}{2} \cdot (f(x_{j-1}) + f(x_j)) + h f \left( \frac{x_{j-1} + x_j}{2} \right)
\end{equation}

Taking the sum over $j$ from 1 to $n$, we get

\begin{align*}
\int_{a}^{b} f(u) \, du & \leq \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{2} \cdot \{f(x_{j-1}) + f(x_j)\} + \sum_{j=1}^{n} h f \left( \frac{x_{j-1} + x_j}{2} \right) \\
& \leq \frac{h}{2} \sum_{j=1}^{n} \{f(x_{j-1}) + f(x_j)\} + h \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right),
\end{align*}

which gives that

\begin{equation}
\int_{a}^{b} f(u) \, du - h \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \leq \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],
\end{equation}

for all $j = 1, 2, \ldots, n$, which gives the desired result (2.33). The proof of sharpness goes likewise the proof of the sharpness of Theorem 4 and we shall omit the details.

Theorem 7. Let $I \subset \mathbb{R}$ be an open interval and $a, b \in I, a < b$. Let $f : I \to \mathbb{R}_+$ be an increasing convex function on $[a, b]$, then the inequality

\begin{equation}
\int_{a}^{b} f(t) \, dt - \frac{(b-a)}{2} f(y) \geq \frac{h}{2} \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \geq 0,
\end{equation}

for all $y \in [a, b]$. The constant $\frac{1}{2}$ in the right-hand side is the best possible. If $f$ is concave then the inequality is reversed.
is valid for all \( y \in [a, b] \subset I \). The constant \( \frac{1}{2} \) in the right-hand side is the best possible, in the sense that it cannot be replaced by a greater one. If \( f \) is concave then the inequality is reversed.

**Proof.** Let \( y \in [x_{j-1}, x_j] \) be an arbitrary point such that \( x_{j-1} < y - p \leq y \leq y + p < x_j \) for all \( j = 1, 2, \ldots, n \) with \( p > 0 \).

It is well known that \( f \) is convex on \( I \) iff

\[
 f(y) \leq \frac{1}{2p} \int_{y-p}^{y+p} f(t) \, dt
\]

for every subinterval \( [y-p, y+p] \subset [a, b] \subset I \) for some \( p > 0 \). But since \( f \) increases on \( [a, b] \), we also have

\[
 f(y) \leq \frac{1}{h} \int_{y-p}^{y+p} f(t) \, dt \leq \frac{1}{h} \int_{x_{j-1}}^{x_j} f(t) \, dt
\]

Choosing \( p \geq \frac{h}{2} \) (this choice is available since it is true for every subinterval in \( I \)), therefore from the last inequality we get

\[
 f(y) \leq \frac{1}{h} \int_{y-p}^{y+p} f(t) \, dt \leq \frac{1}{h} \int_{x_{j-1}}^{x_j} f(t) \, dt
\]

Again by convexity we have

\[
 hf \left( \frac{x_{j-1} + x_j}{2} \right) \leq \int_{x_{j-1}}^{x_j} f(t) \, dt
\]

Adding the last two inequalities, we get

\[
 hf \left( y \right) + hf \left( \frac{x_{j-1} + x_j}{2} \right) \leq 2 \int_{x_{j-1}}^{x_j} f(t) \, dt
\]

or we write

\[
 hf \left( \frac{x_{j-1} + x_j}{2} \right) \leq 2 \int_{x_{j-1}}^{x_j} f(t) \, dt - hf \left( y \right)
\]

Taking the sum over \( j \) from 1 to \( n \), we get

\[
 h \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \leq 2 \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(t) \, dt - \sum_{j=1}^{n} hf \left( y \right)
\]

hence,

\[
 \int_{a}^{b} f(t) \, dt - \frac{b-a}{2} f(y) \geq \frac{h}{2} \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \geq 0,
\]

holds by positivity of \( f \) and this proves our assertion.

To prove the sharpness let (2.34) holds with another constant \( C > 0 \), which gives

\[
 (2.35) \quad \int_{a}^{b} f(t) \, dt - \frac{b-a}{2} f(y) \geq C \cdot h \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \geq 0.
\]
Let \( f : [0, 1] \to \mathbb{R}_+ \) be the identity map \( f(x) = x \), then the right-hand side of (2.35) reduces to
\[
\frac{1}{2} - \frac{1}{2} y \geq C \cdot \frac{1}{n} \sum_{j=1}^{n} \frac{2j - 1}{2n} \\
= C \cdot \frac{1}{n} \left[ \sum_{j=1}^{n} \frac{j}{n} - \sum_{j=1}^{n} \frac{1}{2n} \right] \\
= C \cdot \frac{1}{n} \left[ \frac{1}{n} \cdot \frac{n(n + 1)}{2} - \frac{1}{2n} \cdot n \right] \\
= \frac{1}{2} C.
\]
Choose \( y = \frac{1}{2} \), it follows that \( \frac{1}{2} \geq \frac{1}{2} C \) which means that \( \frac{1}{2} \geq C \), i.e., \( \frac{1}{2} \) is the best possible constant in the right-hand side of (2.34).

\[\square\]

**Corollary 3.** Let \( \alpha_i \geq 0, \forall \ i = 0, 1, 2, \ldots, n, \) be positive real numbers such that \( \sum_{i=0}^{n} \alpha_i = 1 \), then under the assumptions of Theorem 7, we have
\[
\int_{a}^{b} f(t) \, dt - \frac{(b - a)}{2} f \left( \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i x_i \right) \geq \frac{h}{2} \sum_{j=1}^{n} f \left( \frac{x_{j-1} + x_j}{2} \right) \geq 0. 
\]  
(2.36)

The constant \( \frac{1}{2} \) in the right-hand side is the best possible. If \( f \) is concave then the inequality is reversed.

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