Game-theoretic discussion of quantum state estimation and cloning

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Abstract

We present a game-theoretic perspective on the problems of quantum state estimation and quantum cloning. This enables us to show why the focus on universal machines and the different measures of success, as employed in previous works, are in fact legitimate.

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The topic of *quantum games* is a new area of study within quantum information, and its potential usefulness and consequences are still being explored and understood [1, 2, 3, 4, 5]. Early on in its development, Meyer [1] discussed a connection between quantum games and quantum information processing. However the majority of the research carried out on quantum games to date, has lacked a direct connection to quantum information processing.

In this paper, we discuss two quantum games which are of fundamental interest to quantum information processing and specifically to quantum cryptography: the *quantum state estimation game* and the *quantum cloning game*. These two quantum games are no strangers to physicists, and have been investigated extensively. However there are still some significant questions waiting to be addressed. First: most work in these two areas focused on universal machines, and it was typically assumed that the initial pure state to be measured or cloned had a uniform distribution according to the corresponding unitary group. But why are these assumptions legitimate? Second: the measure of success in the cloning game was not unique, and researchers used different measures to facilitate their expositions. So why did they end up with the same answers? The purpose of this paper is to answer these two questions by means of a unifying game-theoretic scenario which describes these two games.

We start with a description of the two games concerned:

**Quantum state estimation game** Suppose there are two players, I and II, with a referee presiding over the game. Player II chooses an arbitrary pure state $|\psi\rangle \in \mathcal{H}$ where $\mathcal{H} = \mathbb{C}^d$. He then sends $|\psi\rangle^\otimes N$ to player I and $|\psi\rangle$ to the referee. Player I’s task after receiving the $N$ qubits from player II, is to perform a measurement on them. Based on the outcome, player I sends a pure state $|\phi\rangle \in \mathcal{H}$ to the referee. For example, if player I decides to use the set of POVM operators $\{M_m\}$ to do the measurement, and if he associates state $|\phi_m\rangle$ to measurement outcome $m$, the final state that he sends back to the referee will be $\sum_m \text{tr}[M_m \rho M_m^\dagger]|\phi_m\rangle\langle\phi_m|$ where $\rho = (|\psi\rangle\langle\psi|)^\otimes N$. After receiving the two qubits from player I and player II, the referee applies the SWAP-test on them: if the test says that the two states are equal, he awards a payoff of 1 to player I and $-1$ to player II. Otherwise, player I gets a payoff of $-1$ and player II a payoff of 1.

**Quantum cloning game** In this game, player II chooses a pure state $|\psi\rangle \in \mathcal{H}$ where $\mathcal{H} = \mathbb{C}^d$. He sends the state $(|\psi\rangle\langle\psi|)^\otimes N$ to player I and the state $(|\psi\rangle\langle\psi|)^\otimes M$ to the referee. After receiving the state from player II, player I designs a device that takes as input $(|\psi\rangle\langle\psi|)^\otimes N$ and outputs a state $\sigma$ such that $\sigma$ is a density operator in $\mathcal{H}^\otimes M$. He then sends $\sigma$ to the referee. Finally, the referee applies the SWAP-test on...
\[(\psi)\langle\psi|^{\otimes M} \text{ and } \sigma. \text{ If they pass the test, he awards a payoff of } 1 \text{ to player I and } -1 \text{ to player II. Otherwise he awards } -1 \text{ to player I and } 1 \text{ to player II.}

The above games are generalizations of the quantum games considered in Refs. [2, 5]. However, one fundamental difference from the games of Refs. [2, 5] is that these games involve communication via qubits. Moreover, comparing to the strategy sets discussed in Ref. [5], we are placing a severe restriction on the players’ strategy sets. This is justifiable because only one particular quantum operation is of interest. We also note that the quantum state estimation and quantum cloning games that we consider here, are equivalent to the games discussed in Ref. [7] and Ref. [8] respectively. The game-theoretic analysis which follows below, however, is new.

Before deducing any general theorems for these games, we will review some basic definitions in game theory for completeness. Further details are given in Ref. [9]. For a vector \(\vec{v} = (v_i)_{i \in N}\), where the \(k\)-th entry corresponds to the \(k\)-th player’s choice of strategy, we set \(\vec{v}_{-k}\) to be \((v_i)_{i \in N \setminus \{k\}}\) and we denote \((v_1, \ldots, v_{k-1}, v'_k, v_{k+1}, \ldots, v_N)\) by \((\vec{v}_{-k}, v'_k)\). We also define the set of so-called best replies for player \(k\) to be \(B_k(\vec{v}_{-k}) := \{\chi_k \in \Omega_k : P_k(\vec{v}_{-k}, \chi_k) \geq P_k(\vec{v}_{-k}, \chi'_k), \forall \chi'_k \in \Omega_k\}\) where \(P_k(\vec{v})\) is the payoff for the \(k\)-th player given profile \(\vec{v}\). We note that a strategy \(\chi_k \in B_k(\vec{v}_{-k})\) means that adopting \(\chi_k\) renders the optimal payoff to player \(k\) if all the other players have chosen strategies according to the strategy profile \(\vec{v}_{-k}\). Using this notion of best reply, we can easily define what a Nash equilibrium is: an operator profile \(\vec{v}\) is a Nash equilibrium if \(\chi_k \in B_k(\vec{v}_{-k})\) for all \(k\).

We are now ready to state and prove two theorems for these games:

**Theorem 1** For the quantum state estimation and cloning games,

\[
\max_{\chi \in \Omega_I} \min_{\xi \in \Omega_{II}} P(\chi, \xi) = \min_{\xi \in \Omega_{II}} \max_{\chi \in \Omega_I} P(\chi, \xi) \quad (1)
\]

where \(\Omega_I\) and \(\Omega_{II}\) are the strategic sets for player I and player II respectively and \(P(\chi, \xi)\) is the payoff for player I.

**Proof:** We use the formalism developed in Ref. [5] where the same theorem was proved. The only difference here is that we have allowed an exchange of qubits between the two players, i.e. the payoff matrix now has entries of the form:

\[
\text{tr}[R(\tilde{E} \otimes \tilde{E} \otimes I)(I \otimes \tilde{E} \otimes \tilde{E})\rho(I \otimes \tilde{E}^\dagger \otimes \tilde{E}^\dagger)(\tilde{E}^\dagger \otimes \tilde{E}^\dagger \otimes I)]
\]

\(\text{I's operators} \quad \text{II's operators} \quad \text{I's operators} \quad \text{II's operators} \quad \text{I's operators}
\]

with the summation indices omitted for clarity. We note that \(R\) corresponds to the SWAP-test by the referee. Although the overall operation on the initial state is no longer a direct product of the two
players’ strategy sets, this detail is inessential with regards the theorem for the following reason: once a basis for the vector space of operators is fixed, \( \{ \tilde{E} \} \) in this case, then all the coefficients attached to the operators commute to the front. Hence one can see that the resulting payoff function is again bilinear with respect to the vector spaces of the strategy sets of the two players. Also, since we have allowed mixed strategies for the two players, the resulting strategy sets are convex. Compactness of the strategy sets also follows from a similar discussion in Ref. [3]. We can therefore deduce the theorem by invoking the Minmax theorem of Ref. [10] because the strategy sets are compact and convex, and \( P(\chi, \xi) \) is linear and continuous in \( \chi \) and \( \xi \).

Q.E.D.

The above proof is almost identical to the classical version, but one should note that the two players’ strategy spaces no longer form a direct product of operator spaces.

The next theorem is more general than we need and it highlights the similarity of the quantum state estimation and quantum cloning games. In fact, the same proof shows that in the well-known children’s game of rock-paper-scissors, the best strategy is to adopt the three options with uniform probability.

**Theorem 2** Consider a two-player zero-sum game with pure strategy sets \( \Omega_I \) and \( \Omega_{II} \) for players I and II respectively, such that the Minmax theorem applies. Suppose that \( \Omega_{II} \) is a compact topological group with the property that for each \( e \in \Omega_I \), there is an \( e' \in \Omega_I \) such that the payoff \( P(e', f) = P(e, f'f) \) for all \( f \in \Omega_{II} \). Then the strategy \( \xi^* \) which corresponds to adopting a specific strategy with uniform probability with respect to the unique Haar measure, is a strategy at a Nash equilibrium for player II.

**Proof:** We let \( \chi^* \in B_I(\xi^*) \) where \( B_I(\xi^*) \) is the set of best replies of player I with respect to \( \xi^* \). \( \chi^* \) can be represented by the probability ensemble \( \{ Q(e)de, e \} \) where \( Q(e)de \) is the probability for choosing the pure strategy \( e \). We now define \( \bar{\chi}^* \) to be the probability ensemble \( \{ Q(e)de df', e'f \} \) where \( df' \) is a uniform probability distribution corresponding to the Haar measure of \( \Omega_{II} \). Then

\[
P(\bar{\chi}^*, \xi^*) = \int de \ df \ df' Q(e) P(e', f) \tag{2}
\]

\[
= \int de \ df \ df' Q(e) P(e, f'f) \tag{3}
\]

\[
= \int de \ df \ Q(e) P(e, f) \tag{4}
\]
Hence, \( \bar{\chi}^* \) is also in \( B_1(\xi^*) \) but with the extra property that \( P(\bar{\chi}^*, f') = P(\bar{\chi}^*, f'') \) for all \( f', f'' \in \Omega_\Pi \). Borrowing the terminology from the cloning literature, we call \( \bar{\chi}^* \) universal. Now, suppose \( \xi^* \) is not at a Nash equilibrium but \( \hat{\xi} \) is (its existence is guaranteed by the Minmax theorem). Then

\[
P(\bar{\chi}^*, \hat{\xi}) < P(\bar{\chi}^*, \xi^*).
\]

However this is impossible because \( \hat{\xi} \) is itself a probability ensemble of \( \Omega_\Pi \). The theorem therefore follows. Q.E.D.

We can also deduce the following corollary from the above theorem:

**Corollary 3** Consider a game as depicted in Theorem 2. Any universal strategy \( \bar{\chi}^* \), such that \( P(\bar{\chi}^*, \xi^*) \) is optimal and where \( \xi^* \) is the same strategy as defined in Theorem 2, is therefore at a Nash equilibrium. Q.E.D.

The quantum state estimation and quantum cloning games fall within the realm of the previous theorems. We are therefore ready to discuss the two questions we set out to answer. First: the assumption of having an initial pure state distributed with uniform probability, is legitimate because this corresponds to a strategy at Nash equilibrium for player II. Subsequently, optimizing the estimation or cloning operation with respect to this strategy gives us the value of the corresponding game. Furthermore, if the resulting operation is universal, then the operations are themselves guaranteed to be at Nash equilibrium using the above corollary. The reason why Werner obtained an identical result to Gisin and Massar, despite the fact that they adopted different measures of success, is as follows: the quantity Gisin and Massar identified for optimization (with \( \chi \in \Omega_I \)) was

\[
P(\chi, \xi^*),
\]

while that of Werner was

\[
\inf_{\xi \in \Xi} P(\chi, \xi)
\]

where \( \Xi \) is the set of strategies whereby player II can only choose one particular pure state rather than a probabilistic mixture of many. Hence using Theorem 3 and Theorem 2, we can see that the bound that they arrived at is actually the value of the cloning game. Furthermore, since Werner’s optimization was with respect to all unitary states, the operation found is guaranteed to be at Nash equilibrium by Theorem 1.
On the other hand, since the operation found by Gisin and Massar is universal, it is also at Nash equilibrium due to Theorem 2.

For completeness, we now give the Nash equilibria for the quantum state estimation and quantum cloning games:

The **quantum state estimation game** with $N$ initial qubits has a Nash equilibrium $(\chi^*, \xi^*)$ where $\chi^*$ corresponds to the following strategy:

1. Measure the initial set of qubits with the set of measurement operators $\left\{ \sum_{m,n} c_r e^{-i\psi_r (m-n)} d_{m,N/2}^{N/2}(\theta_r) d_{n,N/2}^{N/2}(\theta_r) |m\rangle \langle n| \right\}$, $1 \leq r \leq N + 1$ where the $c_r$'s are such that
   \[ \sum_r c_r e^{-i\psi_r (m-n)} d_{m,N/2}^{N/2}(\theta_r) d_{n,N/2}^{N/2}(\theta_r) = \delta_{m,n}. \] (8)
   Here $|m\rangle$ is short-hand for $|N/2, m\rangle$ with the principal axis adopted with uniform probability according to the corresponding unitary group, and with $d_{m,N/2}^{N/2}(\theta)$ being the rotation operator of a spin-$N/2$ particle \[12\].

2. Upon measurement $s$, submit to the referee the qubit $|\phi_s\rangle$ where $|\phi_s\rangle^{\otimes N} = \sum_m e^{-i\psi_s m} d_{m,N/2}^{N/2}(\theta_s) |m\rangle$.

For player II, $\xi^*$ corresponds to adopting a pure state with uniform probability with respect to the Haar measure of the unitary group. The value of the game is $\frac{N+1}{N+2}$.

The $N \mapsto M$ **quantum cloning game** in a $d$-dimensional Hilbert space has a Nash equilibrium $(\chi^*, \xi^*)$ where $\chi^*$ corresponds to the mapping \[3\]:

\[ \rho \mapsto \frac{d[N]}{d[M]} s_M (\rho \otimes 1^{\otimes (M-N)}) s_M. \] (9)

Here $s_M$ is the orthogonal projection of $\mathcal{H}^M$ onto its Bose space and $\xi^*$ corresponds to adopting a pure state with uniform probability with respect to the Haar measure of the unitary group. The value of the game is $d[N]/d[M]$ where $d[N] = \binom{d + N - 1}{N}$.

Although we have only succeeded in finding one particular strategy profile at Nash equilibrium, this is in fact sufficient as far as playing the game is concerned due to the following theorem:

**Theorem 4** In a two-player zero-sum game, let $(\chi_1, \xi_1)$ and $(\chi_2, \xi_2)$ be two equilibrium pairs. Then

1. $(\chi_1, \xi_2)$ and $(\chi_2, \xi_1)$ are also equilibrium pairs, and
2. \( P(\chi_1, \xi_1) = P(\chi_2, \xi_2) = P(\chi_1, \xi_2) = P(\chi_2, \xi_1) \).

**Proof:** The proof can be found in Ref. [13]. Q.E.D.

In contrast to general games where one should worry about multiple Nash equilibria, a strategy at equilibrium is as good as any other in a two-player zero-sum game. Therefore, finding one such equilibrium is enough.

We note that there is, in fact, more than one version of a quantum cloning game: these versions differ according to how the referee determines the payoffs. For example, the referee may perform a one-particle test or multiple-particle test on the qubits he/she receives. However it turns out that the strategies at the Nash equilibrium are equivalent [8, 14]. Unfortunately, their equivalence cannot be deduced from game-theoretic arguments since the payoff vectors \( R \) differ for these two games, and consequently they are distinct in the game-theoretic formalism. However, we can in fact deduce a bound on asymmetric cloning from the Minmax theorem by considering the one-particle-test game. In this game, the referee is going to apply the SWAP-test on \( |\psi\rangle\langle\psi| \), which is sent by player II, together with a reduced density matrix \( \sigma \in \mathcal{H} \) obtained by tracing out all but the \( k \)-th Hilbert space \( \mathcal{H} \) where \( k \) is chosen by player II. In a similar way to the above argument, one can see that Keyl and Werner have shown that the mapping in Eq. 9 is the strategy at Nash equilibrium, and the value of a \( N \mapsto M \) cloning game is \( \frac{N(d+M)+M-N}{(d+N)M} \). Hence if there exists an asymmetric cloner such that the sum of the fidelity of the \( M \) output states with respect to the original input state is greater than \( \frac{N(d+M)+M-N}{(d+N)} \), then it will violate Theorem 1. Therefore given a \( N \mapsto M \) cloner, the sum of the fidelity of the \( M \) output states with respect to the original input state is less than or equal to \( \frac{N(d+M)+M-N}{(d+N)} = N + O(1/d) \). This bound thus limits the flow of information from one system to another. We note that the above bound on asymmetric cloning is also implicit in Ref. [11]: however in that work the authors proceeded by symmetrizing the asymmetric cloning machine.

In summary we have discussed the problems of quantum state estimation and cloning using a game-theoretic perspective, and have found the corresponding Nash equilibria. We also justified the focus to date on universal machines, and the different measures of success employed. The fact that the theorems that we deduced are more general than we needed, implies that they have potential use in other adversary-type scenarios. We also note that although we have restricted the referee’s action to be physical, hence rendering some situations impossible [14], this need not be the case. In fact, the Minmax theorem holds as long as the payoff function is rendered linear with re-
spect to $\chi$ and $\xi$. We conclude by noting that it is well-known among computer scientists that bounds on classical computing can be proved by classical game-theoretic techniques [14]. So could quantum games pay back this debt by passing similar benefits back over to quantum computation? The answer awaits further investigation.

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