The 3D circulation and cyclone-anticyclone asymmetry in the shallow layers of viscous rotating fluid

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Abstract. A modified von Kármán problem that describes steady vortex flow in a rotating thin viscous fluid layer is solved. An analysis of the effect of bottom friction on the behavior of cyclonic and anticyclonic vortices at arbitrary values of the Rossby number is presented. Several anticyclonic flow patterns are examined. An approximate analytical solution obtained for steady flow is compared with numerical computations of a time-dependent problem. Experimental results on cyclonic and anticyclonic vortices in multiple-vortex quasi-turbulent flow are presented, and their interpretation based on the solution of the numerical model is given.

1. Problem statement

The statistical properties of quasi-2D turbulent flows of viscous rotating fluid created by magnetohydrodynamical forcing are recently experimentally investigated (Bokhoven et al., 2009; Ponomarev et al., 2009). The current study is devoted to theoretical description of cyclonic and anticyclonic vortices observed in such flows and dependence their properties from Reynolds and Rossby numbers (Kostrykin et al., 2011). The governing equations in terms of stream($\Psi$) and potential($\Phi$) functions can be written as follows

\[
\omega_t + [\Psi, \omega] + \nabla((f + \omega)\nabla\Phi) + w\omega_z + k \cdot \nabla w \times U_z - \nu \omega_{zz} - \nu \Delta \omega = k \cdot F_{ext},
\]

\[
\delta_t + [\Phi, \omega] - \nabla((f + \omega)\nabla\Psi) + \frac{1}{2} \Delta U^2 + w\delta_z + \nabla w \cdot U_z - \nu \delta_{zz} - \nu \Delta \delta = -g \Delta \eta,
\]

\[
w_z = -\nabla \cdot U,
\]

\[
\frac{d\eta}{dt} = w|_{z=h},
\]

\[
\omega|_{z=0} = \delta|_{z=0} = 0, \omega_z|_{z=h} = \delta_z|_{z=h} = 0,
\]

where $U = k \times \nabla \Psi + \nabla \Phi$, $\omega = \Delta \Psi$, $\delta = \Delta \Phi$, $[A, B] = A_x B_y - B_x A_y$, $\nabla = (\partial_x, \partial_y)$, $\Delta = \partial^2_{xx} + \partial^2_{yy}$. From these equations follows that the cyclone-anticyclone asymmetry is due to the third terms in equations (1)-(2), which implies that 3D-circulation is the main reason for this asymmetry.
2. Modified Kármán problem

To study the effect of bottom friction on the motion of the rotated fluid one can consider the simplified model. In the fixed cylindrical coordinate system, under assumption that the fluid motion is "solid-body" in the horizontal plane, equations (1)-(5) can be rewritten in the form of the modified Kármán problem (Hewitt & Hazel, 2007; Zandbergen & Dijkstra, 1987)

\[
\begin{align*}
F_t + H F_z & = E F_{zz} + C^2 - F^2 + p(t), \\
G_t + H G_z & = E G_{zz} - 2 F G + q, \\
2F & = -H_z
\end{align*}
\]

with boundary

\[
G|_{z=0} = s, F|_{z=0} = G_z|_{z=1} = F_z|_{z=1} = 0, H|_{z=0} = H|_{z=1} = 0
\]

and initial conditions

\[
G(z,0) = s, F(z,0) = 0, H(z,0) = 0.
\]

Here \( F = \delta/2, G = \omega/2, H = w/h, E = \nu/h^2, q = k \cdot \text{rot} F_{\text{ext}}/2, p(t) = -g \Delta \eta/2, s \) - bottom rotation speed. Analyses of the numerical solution of this model and corresponding solution of quasi-linear model will be presented in the following sections.

It is useful to take advantage of the property of self-similarity of the solution (6)-(10) from the parameter \( E \). If one substitute

\[
\tilde{t} = t E, \tilde{G} = G E, \tilde{F} = F E, \tilde{H} = H E, \tilde{q} = q E^2, \tilde{p} = p E^2, \tilde{s} = s E
\]

in the (6)-(10), we obtain equations which are not dependent on the parameter \( E \). Further we will omit tilde sign, taking in mind conversion (11).

3. Numerical results

The equations (6)-(10) were numerically solved to obtain stationary solutions for different values of \( s \geq 0 \) and \( q \). The profiles of the stationary solution for different values of \( q \) and \( s = 1, E = 0.02 \) are presented on the Fig.1.

From these experiments it follows that the flow has a quasi-2D structure for large subset of the parameter space \((q, s)\), except for the case when the external forcing is negative and of moderate (in comparison with \( s \)) value. For example, this anomalous regime is realized for \( q = -0.5 \) and is plotted on the second column from the left on the Fig.1. It is characterized by non-linear dependency of vertical velocity with height in the whole layer. It contrasts to the general case, when we have a Ekman boundary layer near bottom and a surface layer with constant values of flow vorticity, divergence and linear profile of vertical velocity.

On Fig.2 are presented a dependencies of angular velocity at the surface from \( q \) for different values of \( s \). It is seen that without rotation we have a symmetrical behaviour of angular velocity from the external forcing of opposite sign. But for large enough values of rotation frequency this symmetry is broken at some interval of negative \( q \)-values. The length of this interval grows with \( s \).

The different flow regimes can be plotted on the \((q, s)\)-plane (Fig.3). The area of quasi-2D flow is located near coordinate axis \( q \) and \( s \). In the middle we have a quasi-3D regime.
Figure 1. Stationary profiles of modified Kármán problem for different forcing magnitudes \((E = 0.02, s = 1)\). Top row - \(G - s\), bottom row - \(H\), columns from the left to the right correspond the \(q\)-values \(-0.01, -0.5, -1, 1\).

Figure 2. Dependence of angular velocity at the surface from forcing magnitude obtained from numerical solution of modified Kármán problem at \(E = 1/50\) and different values of bottom rotation speed - \(s\).

Figure 3. Flow regimes on the \((q, s)\)-plane. Logarithmic scales. I - area of quasi-2D regime, II - quasi-3D regime. Circles - numerical, lines - theoretical estimates.

4. Quasi-linear model

4.1. Statement of the problem and general solution

In the stationary case the problem (6)-(10) can be rewritten in the form

\[
\begin{align*}
K &= G + iF, \\
K_{zz} - HK_z + iK^2 &= -q - ip, \\
-2\text{Im}K &= H_z, \\
K(0) &= s, K_z(1) = 0, H(0) = H(1) = 0.
\end{align*}
\]  

(12)
Then it can be linearized near some average flow $K = G_0 + k, H = H_0 + \tilde{h}$, where

$$|k| \ll G_0, |\tilde{h}| \ll H_0$$  \hspace{1cm} (13)

to obtain equations for the disturbances

$$k_{zz} - H_0 k_z + 2iG_0 k = -q - i(p + G_0^2), -2i\text{Im} k = \tilde{h}_z,$$  \hspace{1cm} (14)

$$k(0) = s - G_0, k_z(1) = 0, h(0) = h(1) = -H_0.$$  \hspace{1cm} 

One can notice from boundary conditions (14) that the condition (13) is broken near the bottom. Nevertheless we suggest that the solution of (14) is close to the solution of the full problem to some extent. This suggestion was verified on the results of the numerical modelling.

The solution of the equations (14) can be found in the following form

$$k(z) = k_0 + k_1 e^{\lambda_1 z} + k_2 e^{\lambda_2 z},$$  \hspace{1cm} (15)

where the $\lambda_i$ satisfy characteristic equation

$$\lambda^2 - H_0 \lambda + 2iG_0 = 0.$$  \hspace{1cm} (16)

From (14) it follows that for the parameters $G_0, H_0$ are valid "dispersion" relationships

$$(s - G_0)\text{Im} \left( \frac{1}{B(0)} \right) = \frac{q}{2G_0},$$

$$(s - G_0)\text{Im} \left( \frac{1}{B(0)} \right) [1 + \frac{2}{A} (\frac{1 - \lambda_1 e^{-\lambda_1} - e^{-\lambda_1}}{\lambda_1^3} + \frac{1 - \lambda_2 e^{-\lambda_2} - e^{-\lambda_2}}{\lambda_2^3})] = -H_0,$$  \hspace{1cm} (17)

$$B(z) = 1 + \frac{1}{A} \left( \frac{1}{\lambda_1} e^{\lambda_1(z-1)} - \frac{1}{\lambda_2} e^{\lambda_2(z-1)} \right),$$

$$A = \frac{1}{\lambda_2^3} (1 - e^{-\lambda_2}) - \frac{1}{\lambda_1^3} (1 - e^{-\lambda_1}).$$

4.2. Regimes of circulation

It is convenient to study the flow regimes in dependence of parameters $G_0$ - mean angular velocity and $\beta = \frac{8G_0}{H_0}$ - parameter of two-dimensionality of the flow.

- The case of slow average rotation $G_0 \ll 1$ corresponds to the parabolic(Poiseuille) profiles $k(z)$
- In the case of fast average rotation $G_0 \gg 1$ and $\beta \gtrsim 1$ the "dispersion" relations (17) reduce to the

$$\frac{q}{2G_0} = (s - G_0)\text{Im} \frac{1}{\lambda} = H_0,$$

where $\lambda = (1 - i \text{sign}(G_0)) \sqrt{|G_0|}$. This equation has one solution at $q \geq 0, q/s^{3/2} \lessgtr -0.4$ and three solutions at $-0.4 \lessgtr q/s^{3/2} < 0$. For this regime are valid following scalings:

$$G_0 = sg_0 \left( \frac{q}{s^{3/2}} \right), H_0 = s^{1/2}h_0 \left( \frac{q}{s^{3/2}} \right)$$

and asymptotics at $q/s^{3/2} \gg 1$

$$G_0 \approx s + 0.8\text{sign}(q)q^{2/3}, H_0 \approx 0.6\text{sign}(q)q^{1/3}.$$
• In the case of fast average rotation $G_0 \gg 1$ and $\beta \ll 1$ the “dispersion” relations (17) takes the form

$$H_0 = -\frac{q}{2s} G_0 = \alpha H_0, \frac{2s^2}{q}(1 - \text{ctg}\alpha)^2 = \alpha^3.$$ 

From analyses of above equation it follows that the solution exist only at $q < 0$ and has a following scalings

$$G_0 = s_0(q), H_0 = s_0(q).$$

Comparison of the results of of quasi-linear model with numerical solution of the full model is shown on the Fig.4. One can conclude that for the quasi-2D regime the quasi-linear model well reproduces the model results.

![Figure 4](image)

Figure 4. Dependence of $G_0$ (top row), $H_0$ (bottom row) from $q$ at different values of $s$ ($s = 0$ - left column, $s = 1$ - right column) obtained from numerical solution of the full model and quasi-linear model at $E = 0.02$.

5. Comparison of theory with experimental data

We have performed an additional numerical experiments in order to simulate condition similar to the laboratory. Thus, we suggest that the external vortical force is not constant, but decaying with height as $q(z) = q_0 e^{-z_L}$. The dependence of root mean-square relative vorticity of the same sign from rotation speed obtained both from laboratory and numerical experiments are presented on the Fig. 5. We see a qualitatively and partly quantitatively agreement between the data. The most close correspondence is for the cyclones and fast-rotated anticyclones. The difference between data for slow-rotated anticyclones probably can be attributed to the strong influence of the non-stationarity and horizontal non-uniformity of the laboratory flow.
6. Concluding remarks

The theoretical analysis presented in this study has demonstrated a wide difference in behavior between cyclonic and anticyclonic vortices in rotating thin viscous fluid layers. The asymmetry is due to nonlinear bottom friction, which determines the boundary layer structure at the underlying surface. For anticyclonic vortices, the boundary layer height reaches a maximum when the Rossby number is close to unity. Simulations based on a modified von Kármán model show that the corresponding boundary layer regime is unstable and strong circulation develops in the vertical plane. This kind instability is similar to the instability arising in the case of two counter-rotating disks (Hewitt & Hazel, 2007).

Acknowledgments

We are highly indebted to the late F.V. Dolzhanskii and V.M. Ponomarev, who made significant contributions to theoretical and experimental studies of the effect of bottom friction on quasi-two-dimensional flows. This work was supported by the Russian Foundation for Basic Research, projects 11-05-01206a and 10-05-00457a and under the program of Nonlinear Waves.

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