THE HASSE PRINCIPLE FOR BILINEAR SYMMETRIC FORMS OVER THE RING OF INTEGERS OF A GLOBAL FUNCTION FIELD

RONY A. BITAN

Abstract. Let $C$ be a smooth projective curve defined over the finite field $\mathbb{F}_q$ ($q$ is odd) and let $K = \mathbb{F}_q(C)$ be its function field. Removing one closed point $C_{\text{af}} = C - \{\infty\}$ results in an integral domain $O_{\{\infty\}} = \mathbb{F}_q(C_{\text{af}})$ of $K$, over which we consider a non-degenerate bilinear and symmetric form $f$ of any rank $n$. We express the set of isomorphism classes in the genus of $f$ over $O_{\{\infty\}}$ with cardinality $c(f)$ in cohomological terms and use it to present a sufficient and necessary condition imposed on $C_{\text{af}}$ only, under which $c(f) = 1$, i.e., $f$ admits the Hasse local-global principle, namely, $|\text{Pic}(C_{\text{af}})|$ is odd for any $n \neq 2$ and equal to 1 for $n = 2$. Examples are provided.

1. Introduction

Let $C$ be a smooth, projective, geometrically connected curve defined over the finite field $\mathbb{F}_q$ with $q$ odd, and let $K = \mathbb{F}_q(C)$ be its function field. For any prime $p$ of $K$, let $v_p$ be the induced discrete valuation on $K$. We remove one closed point $\infty$ from $C$, resulting in an affine curve $C_{\text{af}}$, and consider the following ring of $\{\infty\}$-integers of $K$: $O_{\{\infty\}} = \mathbb{F}_q(C_{\text{af}}) := \{a \in K \mid v_p(a) \geq 0 \quad \forall p \neq \infty\}$.

Throughout the paper, an integral form on $V \cong O_{\{\infty\}}^n$ refers to a bilinear and symmetric map $f : V \times V \to O_{\{\infty\}}$. It will be called unimodular if it is non-degenerate locally everywhere, i.e., its determinant belongs to $\mathbb{F}_q^\times$. Two integral forms $f$ and $g$ on $V$ are $O_{\{\infty\}}$-isomorphic if there exists $Q \in \text{GL}(V)$ such that $f(u, v) = g(Qu, Qv)$ for all $u, v \in V$.

The standard approach to classifying bilinear forms over a global field such as $K$, basically relies on the Hasse-Minkowski theorem which states that this classification, expressed by the first Galois cohomology set $H^1(K, \mathcal{O}_V)$ where $\mathcal{O}_V$ stands for the unitary group of $f$, is equivalent to that obtained locally everywhere, namely, by $\prod_p H^1(\hat{K}_p, (\mathcal{O}_V)_p)$ where $\hat{K}_p$ is the complete localization of $K$ at a prime $p$ and $(\mathcal{O}_V)_p$ is the geometric fiber of $\mathcal{O}_V$ there. However, if one considers the classification of integral forms, then this local-global principle fails, leading to the notion of a genus of a form. In this paper, we aim to describe geometrically the violation of this principle. We express the classification of integral unimodular forms from the same genus in cohomological terms (Proposition 4.2 below), namely as the kernel of the map sending (isomorphism classes of) principal...
bundles to their generic fibers:

\[ H^1_\text{ét}(O_{\{\infty\}}, O_V) \to H^1(K, O_V), \]

where \( O_V \) stands for the unitary group scheme of \( f \) defined over \( \text{Spec}(O_{\{\infty\}}) \).

This description leads us to assert the validity of the Hasse principle for any unimodular integral form of rank \( n \neq 2 \) if and only if \( |\text{Pic}(C^\text{af})| \) is odd and for \( n = 2 \) if and only if \( |\text{Pic}(C^\text{af})| = 1 \), i.e., if and only if \( O_{\{\infty\}} \) is a UFD (Theorem 4.6 below). More precisely, we show that under this condition imposed upon \( C^\text{af} \) only, principal \( \mu_2 \)-bundles are identified with their generic fibers and all principal \( SO_V \)-bundles are trivial, regardless of the choice of \( f \). This result can be considered as a generalization of Theorem 3.1 in [Ger1] in which the elementary case of \( O_{\{\infty\}} = \mathbb{F}_q[t] \) is treated.

Its proof was initially based on the reduction by Harder, of the unimodular theory over \( \mathbb{F}_q[t] \) to the theory of spaces over \( \mathbb{F}_q \) (see [Ger2, Theorem 7.13]).

Acknowledgements: The author thanks B. Conrad, L. Gersten, P. Gille and B. Kunyavskii for valuable discussions concerning the topics of the present article.

2. Classification over rings of integers

The geometrically connected projective curve \( C \) remains geometrically connected after removing the closed point \( \infty \), resulting in \( C^\text{af} \). In order to classify integral forms, we shall refer to the fundamental group \( \pi_1(C^\text{af}, a) \) of \( C^\text{af} \) w.r.t. some geometric base point \( a \), as defined by Grothendieck in [SGA1, V, §4 and §7]. Up to isomorphism, this group (as a topological group) does not depend on the choice of the base point (see [Mil, Ch.I, Remark 5.1]). Therefore, where one is only concerned with the group-theoretic structure of \( \pi_1(C^\text{af}, a) \), we may omit the base-point and write just \( \pi_1(C^\text{af}) \).

For any prime \( p \) of \( K \), let \( O_p \) be the discrete valuation ring of \( K \) w.r.t. to \( v_p \) and let \( K_p \) be its fraction field. Let \( \hat{K}_p \) be the completion of \( K_p \) and let \( \hat{O}_p \) be its ring of integers. Let \( k_p = \hat{O}_p/p \) be the corresponding (finite) residue field. Let \( \hat{K}_p^{\text{ur}} \) be the maximal unramified extension of \( \hat{K}_p \) and let \( \hat{O}_p^{\text{sh}} \) be its ring of integers. Given a smooth group scheme \( G_p \) defined over \( \text{Spec}(\hat{O}_p) \), the set \( H^1_\text{ét}(\hat{O}_p, G_p) \) is bijective to the Galois cohomology set \( H^1(\hat{O}_p, G_p) \), while the Galois group taken under consideration is \( \text{Gal}(\hat{O}_p^{\text{sh}}/\hat{O}_p) = \text{Gal}(\hat{K}_p/k_p) \) where \( \hat{K}_p \) stands for the algebraic closure of \( k_p \). For a smooth group scheme \( G \) defined over \( O_{\{\infty\}} \), by writing \( H^1_\text{ét}(O_{\{\infty\}}, G) \cong H^1_\text{flat}(O_{\{\infty\}}, G) \) we shall refer to the action of the aforementioned (total) fundamental group \( \pi_1(C^\text{af}) \) (see [SGA4, VIII Corollaire 2.3] for the étale, flat and Galois cohomology sets bijections in the smooth case).
3. The class group of an $\mathcal{O}_\{\infty\}$-group scheme and Nisnevich’s sequence

Let $G$ be an affine, flat and smooth group scheme defined over $\text{Spec}(\mathcal{O}_\{\infty\})$ with generic fiber $G$. For any prime $p$ of $K$, the localization $(\mathcal{O}_\{\infty\})_p$ is a base change of $\mathcal{O}_p$. Thus the bijection $\text{Spec}(\mathcal{O}_\{\infty\})_p \rightarrow \text{Spec}(\mathcal{O}_p)$ is faithfully flat (see [Liu, Theorem 3.16]) and so $G$, extended to be defined over $\text{Spec}(\hat{\mathcal{O}}_p)$ and denoted by $G_p$, is also smooth. Under these settings, we shall refer to the adelic group $G(\mathbb{A})$ and to its subgroup over the ring of $\{\infty\}$-integral ad` eles $\mathbb{A}_\{\infty\} := \hat{K}_\infty \times \prod_{p \neq \infty} \hat{\mathcal{O}}_p$.

Definition 1. The class set of $G$ is the set of double cosets $\text{Cl}_\infty(G) := G(\mathbb{A}_\{\infty\}) \backslash G(\mathbb{A}) / G(K)$. This set is finite (see [Con1, Theorem 1.3.1]). We denote its cardinality by $h_\infty(G)$ and call it the class number of $G$.

Theorem 3.1. (Ye. Nisnevich, [Nis, Theorem I.3.5]). There is an exact sequence

$$1 \rightarrow \text{Cl}_\infty(G) \rightarrow H^1_{\text{ét}}(\mathcal{O}_\{\infty\}, G) \rightarrow H^1(K, G) \times \prod_{p \neq \infty} H^1(\hat{\mathcal{O}}_p, G_p) \rightarrow$$

(see in the proof of this Theorem in the case that all $H^1(\hat{\mathcal{O}}_p, G_p)$ vanish, for the right term).

Lemma 3.2. Suppose $G$ (being affine, flat and smooth defined over $\text{Spec}(\mathcal{O}_\{\infty\})$) is connected, and that $G$ is quasi-split and simply connected. Then $H^1_{\text{ét}}(\mathcal{O}_\{\infty\}, G) = 0$.

Proof. At any prime $p$, as $\hat{\mathcal{O}}_p$ is Henselian, we have $H^i(\hat{\mathcal{O}}_p, G_p) \cong H^i(k_p, G_p)$ for $i \geq 0$ where $G_p := G_p \otimes k_p$ (see Remark 3.11(a) in [Mil, Ch. III, §3]). The right set for $i = 1$ is trivial by Lang’s Theorem (see [Ser, Ch.VI, Prop.5]). Furthermore, $H^1(K, G)$ is trivial in the simply connected case due to Harder’s result (see [Har, Satz A]), and so Nisnevich’s sequence from Theorem 3.1 obtained for $G$, shows that $\text{Cl}_\infty(G)$ is bijective to $H^1_{\text{ét}}(\mathcal{O}_\{\infty\}, G)$. But as $G$ is quasi-split and simply connected, $\text{Cl}_\infty(G)$ is trivial (see [Nis, Corollary II.5.5]), yielding the asserted result. □

4. The Hasse principle and the class number of the unitary group

Let $\mathcal{X}$ be the scheme of invertible symmetric $n \times n$-matrices with entries in $\mathcal{O}_\{\infty\}$. It is a $\text{Spec}(\mathcal{O}_\{\infty\})$-scheme, and its points correspond to (non-degenerate) $n$-dimensional integral forms, on which $\text{GL}_n$, defined over $\text{Spec}(\mathcal{O}_\{\infty\})$ acts by

$$\forall g \in \text{GL}_n, \ F \in \mathcal{X} : \ g * F = g^t F g.$$ 

This action is defined over $\mathcal{O}_\{\infty\}$ and the stabilizer of $F \in \mathcal{X}$, corresponding to an integral form $f$, is an $\mathcal{O}_\{\infty\}$-group scheme denoted by $\mathcal{O}_V$ with generic fiber denoted by $\text{O}_V := \mathcal{O}_V \otimes \mathcal{O}_\{\infty\}$. The special unitary group $\text{SU}_V$ (see [Bas, §3.1] and [Con2, Definition 1.6] for the accurate definition) is
a smooth closed subgroup with connected fibers (see [Con2, Theorem 7.1]). As \( \text{char}(K) \neq 2 \) and 2 is a unit in \( O_{\{\infty\}} \), \( O_V \cong SO_V \times \mu_2 \) (see [Con2, §1]) is smooth as well.

**Definition 2.** Two integral forms share the same genus if they are isomorphic over \( K \) and over \( \hat{\mathcal{O}}_p \) for all primes \( p \). We denote by \( \text{gen}(f) \) the set of all integral forms of the same genus as \( f \).

**Definition 3.** Given an integral form \( f \), let \( c(f) \) denote the number of \( O_{\{\infty\}} \)-isomorphism classes in \( \text{gen}(f) \). We say that the Hasse principle holds for \( f \) if \( c(f) = 1 \).

Platonov and Rapinchuk have shown in [PR, Prop. 8.4] – in the number field case – that \( c(f) \) is finite and equals the class number of its unitary group. In the following, we shall sketch briefly their proof, this time in the function field case.

We consider the above \( O_{\{\infty\}} \)-scheme \( GL_n \) (in which \( O_V \) is embedded), its subgroup \( SL_n \) and their extensions defined over \( \hat{\mathcal{O}}_p \) at any prime \( p \) (see Section 3) while referring to their adelic groups. Any element of \( O_V(\hat{\mathbb{A}}) \) can be put in \( SL_n(\hat{\mathbb{A}}) \) by multiplying by a suitable element of \( GL_n(\mathbb{A}_\infty) \). Since the \( K \)-group \( SL_n \) is split, simple and simply connected, it admits the strong approximation property whence \( SL_n(\hat{\mathbb{A}}) = SL_n(\mathbb{A}_\infty)SL_n(K) \) (see [Pra, Theorem A]). It follows that \( O_V(\hat{\mathbb{A}}) \subseteq GL_n(\mathbb{A}_\infty)GL_n(K) \). Now according to the Stabilizer Formula [PR, Theorem 8.2], \( c(f) \) is equal to the number of double cosets \( O_V(\mathbb{A}_\infty) \cdot x \cdot O_V(K) \) in the principal coset \( GL_n(\mathbb{A}_\infty)GL_n(K) \) which is \( h_\infty(O_V) \).

**Corollary 4.1.** \( c(f) = h_\infty(O_V) \).

**Proposition 4.2.** \( \text{Cl}_\infty(O_V) \cong \ker[H^1_{\text{ét}}(O_{\{\infty\}}, O_V) \to H^1(K, O_V)] \).

**Proof.** As \( O_V \) is affine, flat and smooth by the above construction, Nisnevich’s Theorem 3.1 yields the exact sequence:

\[
1 \to \text{Cl}_\infty(O_V) \to H^1_{\text{ét}}(O_{\{\infty\}}, O_V) \to H^1(K, O_V) \times \prod_{p \neq \infty} H^1(\hat{\mathcal{O}}_p, (O_V)_p) \to \prod_{p \neq \infty} H^1(\hat{K}_p, (O_V)_p) \to \prod_{p \neq \infty} H^1(\hat{K}_p, (O_V)_p). \]

Let \( W(*) \) denote the Witt ring for the ring \( * \). By Witt’s Theorem, two forms are isomorphic if and only if they belong to the same Witt class and have the same rank (see [MH, Cor. 3.3]), whence \( H^1_{\text{ét}}(\hat{\mathcal{O}}_p, (O_V)_p) \) injects into \( W(\hat{\mathcal{O}}_p) \) and \( H^1(\hat{K}_p, (O_V)_p) \) into \( W(\hat{K}_p) \). Since \( \hat{K}_p \) is complete, \( W(\hat{\mathcal{O}}_p) = W(k_p) \) injects into \( W(\hat{K}_p) \) and we obtain the following commutative diagram:

\[
\begin{array}{ccc}
H^1_{\text{ét}}(\hat{\mathcal{O}}_p, (O_V)_p) & \longrightarrow & H^1(\hat{K}_p, (O_V)_p) \\
\downarrow & & \downarrow \\
W(\hat{\mathcal{O}}_p) & \longrightarrow & W(\hat{K}_p)
\end{array}
\]
which shows that \( H^1(\hat{O}_p, (\mathcal{O}_V)_p) \) embeds into \( H^1(\hat{K}_p, (\mathcal{O}_V)_p) \) for any \( p \). On the other hand, it is clearly also a surjection. Canceling these local isomorphisms simplifies Nisnevich’s sequence to
\[
1 \to \text{Cl}_\infty(\mathcal{O}_V) \to H^1_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mathcal{O}_V) \to H^1(K, \mathcal{O}_V) \to 1,
\]
and the assertion follows from the sequence exactness. □

In particular, Proposition 4.2 yields:

**Corollary 4.3.** The Hasse principle holds for an integral unimodular form \( f \) if and only if
\[
\ker[H^1_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mathcal{O}_V) \to H^1(K, \mathcal{O}_V)] = 1.
\]

**Lemma 4.4.** The two following properties are true if and only if \(|\text{Pic}(\mathcal{C}_{\text{af}})|\) is odd:

1. \( H^1_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mu_2) \cong H^1(K, \mu_2) \) and
2. \( H^2_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mu_2) = 0 \).

**Proof.** As \( \mathcal{C}_{\text{af}} := C - \{\infty\} \) is smooth, \( H^2_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mathbb{G}_m) = \text{Br}(\mathcal{O}_{\{\infty\}}) \) classifying Azumaya \( \mathcal{O}_{\{\infty\}} \)-algebras (see [Mil § 2]). For any prime \( p \) one has \( \text{Br}(\mathcal{O}_{\{\infty\}}) \subseteq \text{Br}((\mathcal{O}_{\{\infty\}})_p) \subseteq \text{Br}(\hat{O}_p) \). Since \( \hat{O}_p \) is complete, the latter group is isomorphic to \( \text{Br}(k_p) \) (see [AG Theorem 6.5]). But \( k_p \) is a finite field; thus \( \text{Br}(k_p) \) is trivial, as well as \( H^2_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mathbb{G}_m) \). This is regardless of \(|\text{Pic}(\mathcal{C}_{\text{af}})|\) parity.

Consequently, flat cohomology applied on Kummer’s exact sequence over \( \text{Spec}(\mathcal{O}_{\{\infty\}}) \):
\[
1 \to \mu_2 \to \mathbb{G}_m \to \mathbb{G}_m \to 1 \quad (4.1)
\]
gives rise to the following long exact sequence:
\[
1 \to \mathbb{F}^\times/(\mathbb{F}^\times)^2 \to H^1_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mu_2) \to \text{Pic}(\mathcal{C}_{\text{af}}) \xrightarrow{\phi|L| \to [2]} \text{Pic}(\mathcal{C}_{\text{af}}) \to H^2_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mu_2) \to 0
\]
on which if \( \text{Pic}(\mathcal{C}_{\text{af}}) \) has an odd cardinality, i.e. has no element of order 2, \( \phi \) is an isomorphism whence the sequence is being cut by a trivial term replacing \( \phi \). This implies the validity of both asserted properties. Conversely, if \( \text{Pic}(\mathcal{C}_{\text{af}}) \) has an element of order 2, then \( \phi \) has a non-trivial kernel and co-kernel, implying that these both properties are no longer true. □

**Proposition 4.5.** Let \( f \) be a unimodular form of rank \( n \) defined over \( \mathcal{O}_{\{\infty\}} \). If \(|\text{Pic}(\mathcal{C}_{\text{af}})|\) is odd, then \( H^1_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mathcal{SO}_V) = 0 \) for \( n \neq 2 \). This is true also for \( n = 2 \) if \( \mathcal{O}_{\{\infty\}} \) is a UFD.

**Proof.** For rank 1, \( \mathcal{SO}_V \) is trivial. For rank 2, we recall that as \( \mathcal{O}_{\{\infty\}} \) is a Dedekind domain, its Picard group is isomorphic to its ideal class group. Therefore being a UFD means that \( H^1_{\text{et}}(\mathcal{O}_{\{\infty\}}, \mathbb{G}_m) = \text{Pic}(\mathcal{C}_{\text{af}}) = 1 \). The special unitary group of \( 1_2 \) represented by \( \text{diag}(1, 1) \) (with the standard basis over \( \mathcal{O}_{\{\infty\}} \)), is \( \mathcal{SO}_2 \). It is a one dimensional torus \( \text{Spec}(\mathcal{O}_{\{\infty\}})[x, y]/(x^2 + y^2 - 1) \),
splitting over $O_{\{\infty\}}$ if $-1 \in \mathbb{F}_q^2$ or over $O_{\{\infty\}}(i)$ otherwise. In the split case, $H^1_{\text{ét}}(O_{\{\infty\}}, \mathbf{SO}_2) = H^1_{\text{ét}}(O_{\{\infty\}}, \mathbf{SO}_2) = \text{Pic } (O_{\text{aff}}) = 1$. Otherwise, $\mathbf{SO}_2$ fits into the exact sequence of $O_{\{\infty\}}$-tori:

$$1 \to \mathbf{SO}_2 \to R_{O_{\{\infty\}}(i)/O_{\{\infty\}}}(\mathbb{F}_q) \to \mathbf{SO}_2 \to 1$$

(4.2)
on which the midterm is the induced torus and $\mathbb{F}_q$ is a double covering, yielding the following short exact sequence of $\text{ét}$-cohomology.

Moreover, as $\mathbf{SO}_2 := \mathbf{SO}_2 \otimes_{O_{\{\infty\}}} \mathbb{F}_q$ is connected, due to Lang’s Theorem one has:

$$H^1_{\text{ét}}(O_{\{\infty\}}, R_{O_{\{\infty\}}(i)/O_{\{\infty\}}}(\mathbb{F}_q)) = H^1_{\text{ét}}(O_{\{\infty\}}(i), \mathbb{F}_q) = \text{Pic } (O_{\text{aff}}) \oplus \text{Pic } (O_{\text{aff}}) = 1. $$

whence applying flat cohomology on the sequence (4.2) one sees that $H^1_{\text{ét}}(O_{\{\infty\}}, \mathbf{SO}_2) = 0$. Since all unitary groups associated to unimodular forms are conjugated over the algebraic closure, they all coincide in this commutative case, thus, this argument is valid for any unimodular integral form.

For rank $\geq 3$, we consider the following construction (see [Bas] §2):

Let $\mathbf{C}(f)$ be the Clifford algebra associated to $f$. It is a $\mathbb{Z}_2$-graded algebra. The linear map $v \mapsto -v$ on $V$ extends to an algebra automorphism $\alpha : \mathbf{C}(f) \to \mathbf{C}(f)$ (acting as the identity on the even part and negation on the odd part). The Clifford group associated to $(V, f)$ is

$$\mathbf{CL}(f) := \{u \in C(f)^{\times} : \alpha(u)v^2 = 1 \forall v \in V\}.$$ 

The identity map on $V$ (viewed as its inclusion in the opposite algebra of $\mathbf{C}(f)$) extends to an anti-automorphism of $\mathbf{C}(f)$ which we denote by $t$. The composition $\alpha \circ t$ mapping $v \mapsto \bar{v}$ gives rise to the norm $N : \mathbf{CL}(f) \to O_{\{\infty\}}^{\times} = \mathbb{F}_q^{\times} : v \mapsto \bar{v}$ (for $v \in V$ it is just $N(v) = \bar{v}v = -f(v, v)$). We define $\mathbf{Pin}_V(O_{\{\infty\}}) := \ker(N)$. This group admits an underlying group scheme over $\text{Spec } (O_{\{\infty\}})$ which we denote by $\mathbf{Pin}_V$. The homomorphism $\pi : \mathbf{Pin}_V \to O_V$ sending $v$ to the isometry stabilizing it is a double covering, yielding the following short exact sequence of $O_{\{\infty\}}$-group schemes:

$$1 \to \mu_2 \to \mathbf{Spin}_V \to \mathbf{SO}_V \to 1$$

where $\mathbf{Spin}_V := \pi^{-1}(\mathbf{SO}_V)$ in $\mathbf{Pin}_V$. As these schemes are smooth, étale cohomology gives rise to the long exact sequence:

$$H^1_{\text{ét}}(O_{\{\infty\}}, \mathbf{Spin}_V) \to H^1_{\text{ét}}(O_{\{\infty\}}, \mathbf{SO}_V) \to H^2_{\text{ét}}(O_{\{\infty\}}, \mu_2)$$

(4.3)
on which, as $\mathbf{Spin}_V$ is affine, smooth and connected and its generic fiber is split and simply connected, its first étale cohomology set is trivial by Lemma [A.2]. The right term is trivial due to Lemma [4.7](2). Consequently, $H^1_{\text{ét}}(O_{\{\infty\}}, \mathbf{SO}_V) = 0$, as required. □
Theorem 4.6. Let \( f \) be a unimodular form of rank \( n \) defined over \( \mathcal{O}_{\{\infty\}} \). The Hasse principle holds for \( f \) for \( n \neq 2 \) if and only if \( |\text{Pic}(C^{\text{af}})| \) is odd, and for \( n = 2 \) if and only if \( \mathcal{O}_{\{\infty\}} \) is a UFD.

Proof. As \( \mathcal{O}_V \cong \mathbf{SO}_V \times \mathbb{Z}_2 \) (see in Section 4), \( H^1(\mathcal{O}_{\{\infty\}}, \mathbb{Z}_2) \) is a direct summand in \( H^1(\mathcal{O}_{\{\infty\}}, \mathcal{O}_V) \) which due to Lemma 4.4(1) is embedded into its generic fiber if and only if \( |\text{Pic}(C^{\text{af}})| \) is odd, whence may be canceled in the classification kernel in Proposition 4.2. Otherwise, the kernel cannot be trivial. If it can be reduced, then the classification of forms in \( \text{gen}(f) \) simplifies to

\[
\ker[H^1(\mathcal{O}_{\{\infty\}}, \mathbf{SO}_V) \rightarrow H^1(K, \mathbf{SO}_V)]
\]

which is trivial in the stated case, due to Proposition 4.5. \( \square \)

Example 4.7. Let \( C \) be of genus zero. Then if \( \infty \) is rational over \( \mathbb{F}_q \), or if \( C \) has no \( \mathbb{F}_q \)-rational point, then \( \mathcal{O}_{\{\infty\}} \) is a UFD (see [Sam, Theorem 5.1]) and the Hasse principle will hold for any unimodular form over it. Thus Theorem 4.6 is a generalization of Theorem 3.1 in [Ger1] on which the elementary case of \( \mathcal{O}_{\{\infty\}} = \mathbb{F}_q[t] \) is treated.

Example 4.8. Let \( C \) be an elliptic curve and let \( C^{\text{af}} \) be the affine curve obtained by removing an \( \mathbb{F}_q \)-rational point \( \infty \) from \( C \) (such one must exist). Since \( \{\infty\} \) is an irreducible subset of co-dimension 1 in \( C \), the restriction of \( C \) to \( C^{\text{af}} \) gives rise to an exact sequence (see [Hart, Cha.II, Prop.6.5(c)]):

\[
0 \rightarrow \mathbb{Z} \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(C^{\text{af}}) \rightarrow 0
\]

on which the first map \( 1 \mapsto 1 \cdot \{\infty\} \) is injective because the degree of a curve’s divisor is well defined. As we assumed \( \infty \) is \( \mathbb{F}_q \)-rational, this sequence splits as abelian groups. The degree map on \( \text{Pic}(C) \) yields another exact sequence which again splits as abelian groups:

\[
0 \rightarrow \text{Pic}^0(C) \rightarrow \text{Pic}(C) \rightarrow \mathbb{Z} \rightarrow 0.
\]

We get an isomorphism of summands \( \text{Pic}^0(C) \cong \text{Pic}(C^{\text{af}}) \). Together with another isomorphism of abelian groups: \( C(\mathbb{F}_q) \cong \text{Pic}^0(C); P \mapsto [P] - [\infty] \) we may deduce that:

\[
C(\mathbb{F}_q) \cong \text{Pic}(C^{\text{af}}). \tag{4.4}
\]

Hence according to Theorem 4.6 any unimodular form \( f \) of rank \( \geq 3 \) defined over \( \text{Spec}(\mathcal{O}_{\{\infty\}}) \) admits the Hasse principle if and only if there is no element of order 2 in \( C(\mathbb{F}_q) \). For example, suppose \( q > 3 \) and \( \infty = (0 : 1 : 0) \in C(\mathbb{F}_q) \) is removed, so the remaining (non-singular) affine curve \( C^{\text{af}} \) is given in affine coordinates by the Weierstrass form

\[
y^2 = x^3 + ax + b \quad \text{for some } a, b \in \mathbb{F}_q.
\]

Then \( f \) admits the Hasse principle if and only if \( C^{\text{af}} \) does not have any \( \mathbb{F}_q \)-point on the \( x \)-axis.
Remark 4.9. The unimodularity condition is essential for the validity of the Hasse principle even if $O\{\infty\}$ is a UFD. For example, consider the projective line over $F_q$, from which by removing the point $\infty = (1/x)$, we get $O\{\infty\} = F_q[x]$ which is a UFD. Let $f$ and $g$ be the $O\{\infty\}$-forms represented by $F = \text{diag}((1 - x^2)^2, 1)$ and $G = \text{diag}((1 - x^2), (1 + x)^2)$, respectively. Let
\[
Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 + x \end{pmatrix} \in \text{GL}_2(O_p) \text{ \forall } p \neq (1 + x)
\]
and
\[
P = \begin{pmatrix} 0 & 1 + x \\ 1 - x & 0 \end{pmatrix} \in \text{GL}_2(O_p) \text{ \forall } p \neq (1 - x).
\]
Then $Q^f F Q = P^g F P = G$. This shows that $f$ and $g$ belong to the same genus. But they are not, however, isomorphic over $O\{\infty\}$, since mapping the eigenvalue $(1 - x^2)^2$ in $F$ to $(1 - x)^2$ or $(1 + x)^2$ in $G$ can be done only by dividing by a non-constant element, which is not allowed in $O\{\infty\} = F_q[x]$. This may happen only if the common determinant of $f$ and $g$ is not invertible in $O\{\infty\}$, unlike in the case of unimodular forms in which all eigenvalues must belong to $F_q^\times$.

Remark 4.10. The UFD condition – rather than the one of $|\text{Pic} (C^{af})|$ being odd – is essential for the validity of the Hasse principle in case of rank 2, even for unimodular forms. For example, let $C$ again be an elliptic curve defined over $F_q$ such that $-1 \in F_q^2$, and let $C^{af}$ be obtained by removing one $F_q$-rational point $\infty$. The unitary group scheme over $\text{Spec}(O\{\infty\})$ of $I_2$ is $O_2$. Consider the exact sequence of smooth $O\{\infty\}$-schemes (recall that $\text{char}(K)$ is odd):
\[
1 \to \text{SO}_2 \to O_2 \to \mu_2 \to 1.
\]
As $-1 \in F_q^2$, the one dimensional torus $\text{SO}_2$ is split, i.e., isomorphic to $\mathbb{Z}_n$ and so $H^1(\text{et},O\{\infty\},\text{SO}_2) = \text{Pic} (C^{af})$. Then due to isomorphism (4.1) we know that:
\[
C(F_q) \cong H^1(\text{et},O\{\infty\},\text{SO}_2) = \ker[H^1(\text{et},O\{\infty\},\text{SO}_2) \to H^1(K,SO_2)]
\]
as $H^1(K,SO_2)$ is trivial due to Hilbert’s Theorem 90. This kernel is embedded in
\[
\ker[H^1(\text{et},O\{\infty\},O_2) \to H^1(K,O_2)]
\]
which classifies, according to Proposition 4.12 the integral forms from the genus of $I_2$ over $O\{\infty\}$. But $C(F_q)$ is clearly not trivial in general. The geometric interpretation of this non-trivial kernel is that non-trivial principal $\text{SO}_2$-bundles, which are in this case non-trivial line bundles of $C^{af}$, become trivial after tensoring with $K$. This causes the failure of the Hasse principle.

Example 4.11. Consider the elliptic curve $C := \{ZY^2 = X^3 + 2XZ^2 + 3Z^3\}$ defined over $F_5$. Removing the closed point $\infty = (0 : 1 : 0)$, one can divide by $Z$ which results in $C^{af}$ such that
\[
O\{\infty\} = F_5(C^{af}) = F_5[x,y]/(y^2 - x^3 - 2x - 3).
Consider the group scheme $O_2$ associated to the identity form $f = 1^2$, defined over Spec $(O_{\{\infty\}})$. We have $|C(F_5)| = 7$ which, according to Remark 4.10, implies that the Hasse principle fails for $f$. The key obstruction here for finding an explicit integral form from the same genus of $1^2$ which is not isomorphic to it over $O_{\{\infty\}}$, is using the fact that $O_{\{\infty\}}$ is not a UFD in such a way that there exist distinct isomorphisms to $1^2$, defined over integer rings at distinct primes. In our case, we know that

$$y^2 = x^3 + 2x + 3 = (x + 1)^2(x - 2).$$

The matrix

$$Q = \begin{pmatrix} 1/y & 3y \\ 2/y & 2y \end{pmatrix} \in GL_2(O_p) \forall p \neq (y)$$

gives rise to the unimodular integral form $g$ represented by

$$G = Q^t Q = \begin{pmatrix} 0 & 2 \\ 2 & 3y^2 \end{pmatrix} \in GL_2(O_{\{\infty\}}).$$

Also the matrix

$$P = \begin{pmatrix} 3 \\ x+1 \\ 2(x+1)^2 \end{pmatrix} \in GL_2(O_p) \forall p \neq (x+1)$$

gives rise to

$$P^t P = \begin{pmatrix} 0 & 2 \\ 2 & 3(x^3 + 2x + 3) \end{pmatrix} = G.$$

So $g$ is from the same genus of $1^2$ but is not, however, isomorphic to it over $O_{\{\infty\}}$. The reason is that any integral form isomorphic to $1^2$ over $O_{\{\infty\}}$ has the shape

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

for some $a, b, c, d \in O_{\{\infty\}}$, but if $b^2 + d^2 = 3y^2$ as in $G$, then $b = 3y$ and $d = 2y$ (or the opposite), and therefore $ab + cd = (3a + 2c)y = 2$ which is impossible for any $a, c \in O_{\{\infty\}}$.

References

[SGA4] M. Artin, A. Grothendieck, J.-L. Verdier, Théorie des Topos et Cohomologie Étale des Schémas (SGA 4) LNM, Springer, 1972/1973.

[AG] M. Auslander, O. Goldman, The Brauer group of a commutative ring, Trans. Amer. Math. Soc. 97 (1960), 367–409.

[Bas] H. Bass, Clifford algebras and spinor norms over a commutative ring, Amer. J. Math., 96 (1974), 156–206.

[Con1] B. Conrad, Finiteness theorems for algebraic groups over function fields, Compos. Math. 148 (2012), no. 2, 555-639.

[Con2] B. Conrad, Math 252. Properties of orthogonal groups, http://math.stanford.edu/~conrad/252/Handouts/O(q).pdf

[Ger1] L. J. Gerstein, Unimodular quadratic forms over global function fields, J. Number Theory 11 (1979), 529–541.

[Ger2] L. J. Gerstein, Basic Quadratic Forms, Graduate Studies in Mathematics, Amer. Math. Soc. 90 (2008).

[EGAIV] A. Grothendieck, J. Dieudonné, Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie, Publications Mathématiques de l’I.H.E.S. 32 (1967), 5–361.
[SGA1] A. Grothendieck, M. Raynaud, *Revêtements Étales et Groupe Fondamental* (1960–1961), Lect. Notes Math. **224**, Springer, Heidelberg, 1971.

[Har] G. Harder, *Über die Galoiskohomologie halbeinfacher algebraischer Gruppen*, III, J. Reine Angew. Math. **274/275** (1975), 125–138.

[Hart] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[Liu] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2002.

[Mil] J. S. Milne, *Étale Cohomology*, Princeton University Press, Princeton, 1980.

[MH] J. Milnor, D. Husemoller, *Symmetric Bilinear Forms*, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[Nis] Y. Nisnevich, *Étale Cohomology and Arithmetic of Semisimple Groups*, PhD thesis, Harvard University, 1982.

[PR] V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, San Diego 1994.

[Pra] G. Prasad, *Strong approximation for semi-simple groups over function fields*, Ann. of Math. **105** (1977), 553–572.

[Ser] J.-P. Serre, *Algebraic Groups and Class Fields*, Springer, Berlin, 1988.

[Sam] P. Samuel, *On unique factorization domains*, Illinois J. Math. **5** (1961), 1–17.