Energy corrections due to the Non-commutative Phase-Space of the Charged Harmonic Oscillator in a constant magnetic field in 3D

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In this paper, we study the effects of non-commutative quantum mechanics in three dimensions on the energy levels of a charged harmonic oscillator in the presence of a constant magnetic field in the \( z \)-direction. The extension of this problem to three dimensions proves to be non-trivial. We obtain the first-order corrections to the energy-levels in closed form in the low energy limit of weak non-commutativity.

I. INTRODUCTION

With Heisenberg’s introduction of the uncertainty principle [1], the classical paradigm that position and momentum commute at no cost crashed. Additionally, the discussion of a charged particle in an electromagnetic field in the framework of quantum mechanics leads us to the introduction of the kinetic momentum operator, which in contrast to the canonical momentum operator, also does not commute. These two facts, emerging from the nature of quantum mechanics, evidently brings up the question, is the assumption of the commutation of the position and momentum operators among themselves an accurate assumption? Or, under which conditions is the non-commutativity of the phase-space and \( \eta_{ij} \) and \( \theta_{ij} \) are antisymmetric tensors with the properties

\[
\eta_{ij} = \begin{cases} 
\eta & \text{if } ij = 12, 23, 31 \\
-\eta & \text{if } ij = 21, 32, 13 \\
0 & \text{else}
\end{cases}
\]

and

\[
\theta_{ij} = \begin{cases} 
\theta & \text{if } ij = 12, 23, 31 \\
-\theta & \text{if } ij = 21, 32, 13 \\
0 & \text{else}
\end{cases}
\]

respectively.

Mathematically, the non-commutativity of the base manifold can be realized by application the Weyl-Moyal product [24]

\[
(f \star g)(x,p) = e^{i \frac{\hbar}{2 \alpha^2} \theta_{ij} \partial_i \partial_j} e^{i \frac{\hbar}{2 \alpha^2} \eta_{ij} \partial_i \partial_j} f(x)g(p) =
\]

\[
= f(x,p)g(x,p) + i \frac{\hbar}{2 \alpha^2} \partial_i \partial_j f(x)g(p) \bigg|_{x_i = x_j} + i \frac{\hbar}{2 \alpha^2} \partial_i \partial_j f(x)g(p) \bigg|_{p_i = p_j} + \mathcal{O}(\theta^2_{ij}) + \mathcal{O}(\eta^2_{ij}) + \mathcal{O}(\theta_{ij}\eta_{ij})
\]

So, the shift from ordinary Quantum Mechanics to non-commutative Quantum Mechanics is performed by employing the Weyl-Moyal product instead of the ordinary product. So, the non-commutative time-independent Schrödinger equation becomes

\[
H(x,p) \star \psi(x) = E \psi(x).
\]

By employing the Bopp’s shift [25], we can turn the Weyl-Moyal product again to the ordinary product by substituting \( x \) and \( p \) in the non-commutative equation by \( \hat{x} \) and \( \hat{p} \), namely

\[
H(x,p) \star \psi(x) = H(\hat{x}, \hat{p})\psi(x).
\]
Based on Harko [20], we can see that the non-commutativity parameters $\eta$ and $\theta$ can be considered as energy-dependent and that both become sufficiently small in the low energy limit. We will use this fact in order to extend the discussion of a charged particle in a harmonic oscillator with the presence of a magnetic field. The isotropic charged harmonic oscillator in a magnetic field could be solved exactly in the framework of 2D non-commutativity [27]. Let us now include the $z$-direction into the non-commutative framework. In the low energy limit, we can consider the parameters $\eta$ and $\theta$ small, and we can calculate the effect of the non-commutativity in 3D using first-order perturbation in $\eta$ and $\theta$.

In light of this, we will discuss the non-commutative charged harmonic oscillator in the presence of a constant magnetic field employing non-commutativity to all three spacial parameters. In section, we are going to discuss the non-commutative Hamiltonian of the charged particle in a parabolic potential in the presence of a constant magnetic field. Then we will expand the Hamiltonian in terms of $\theta$ and $\eta$, and calculate the energy correction due to non-commutativity in the domain of weak non-commutativity, i.e., in the low energy limit. Finally, we will compare the energy corrections with respect to the magnetic field’s magnitude, noting that the sign of the correction is a function of the magnitude of the employed magnetic field. Then we will close with some concluding remarks.

II. 3D NON-COMMUTATIVE CHARGED HARMONIC OSCILLATOR IN A CONSTANT MAGNETIC FIELD

Our starting point is the commutative Hamiltonian for the charged harmonic oscillator presence of a constant magnetic field.

$$H_0(x, p) = \frac{1}{2m} \left( \vec{p} - \frac{\eta}{\epsilon A} \right)^2 + \frac{1}{2} m \omega_z^2 \left( x^2 + y^2 + z^2 \right)$$

Without loss of generality we will choose the direction of the constant magnetic field in $z$-direction, i.e., $\vec{B} = B\hat{z}$ and $\vec{A}(\vec{x}, t) = \frac{1}{\epsilon} \left( -yB\hat{i} + xB\hat{j} \right)$ in Coulomb gauge. So, our Hamiltonian $H_0$ modifies to

$$H_0(x, p) = \frac{1}{2m} \left( \left( p_x + \frac{qB}{2c} y \right)^2 + \left( p_y - \frac{qB}{2c} x \right)^2 + p_z^2 + \frac{1}{2} m \omega_z^2 \left( x^2 + y^2 + z^2 \right) \right).$$

Expanding the Hamiltonian (10) and regrouping the terms we get

$$H_0(x, p) = \frac{1}{2m} \left( p_x^2 + p_y^2 + p_z^2 \right) - \frac{1}{2} \omega_z L_z + \frac{1}{2} m \omega_z^2 \left( x^2 + y^2 \right) + \frac{1}{2} m \omega_z^2 z^2,$$

where $L_z = xp_y - yp_x$ is the $z$-component of the angular momentum operator, $\omega_c = \frac{eb}{mc}$ the cyclotron frequency, and $\tilde{\omega}^2 = \omega^2 + \frac{\eta^2}{\omega_z^2}$ is the modified frequency of the harmonic oscillator in the $xy$-plane. From [3] we know that the Weyl-Moyal product can be turned into a normal product by substituting commutative $x$ and $p$ by the non-commutative operators $\hat{x}$ and $\hat{p}$, so let us first consider the Hamiltonian $H_0(\hat{x}, \hat{p})$.

$$H_0(\hat{x}, \hat{p}) = \frac{1}{2m} \left( \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 \right) - \frac{1}{2} \tilde{\omega}_z \hat{L}_z + \frac{1}{2} m \tilde{\omega}_z^2 \hat{z}^2$$

Let us state all non-commutative operators explicitly in the 3D version using (2) and (3) together with (4) and (5), respectively.

$$\hat{x} = \alpha x - \frac{\theta}{2\hbar} p_y + \frac{\theta}{2\hbar} p_z$$

$$\hat{y} = \alpha y - \frac{\theta}{2\hbar} p_x + \frac{\theta}{2\hbar} p_z$$

$$\hat{z} = \alpha z - \frac{\theta}{2\hbar} p_x + \frac{\theta}{2\hbar} p_y$$

$$\hat{p}_x = \alpha p_x + \eta \frac{\theta}{2\hbar} y - \eta \frac{\theta}{2\hbar} z$$

$$\hat{p}_y = \alpha p_y + \eta \frac{\theta}{2\hbar} x - \eta \frac{\theta}{2\hbar} z$$

$$\hat{p}_z = \alpha p_z + \eta \frac{\theta}{2\hbar} x - \eta \frac{\theta}{2\hbar} y$$

Based on the position and momentum operators defined in the equations (13)-(18), we can construct all other operators needed in this calculation.

As a consequence the non-commutative angular momentum operator $\hat{L}_z$ can be stated explicitly as following

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \alpha^2 L_z + \theta \left( -\hat{p}_x^2 + \hat{p}_y^2 + p_x p_\hat{p}_z + p_y p_z \right) + \frac{\eta}{2\hbar} \left( -x^2 - y^2 + xz + yz \right) + \frac{3\theta^2}{4\alpha^2 \hbar^2} \left( L_x + L_y + L_z \right).$$

Furthermore, the sum of the squares of the components of the non-commutative momentum operator $\hat{p}_x^2 + \hat{p}_\theta^2 + \hat{p}_z^2$

$$\hat{p}_x^2 + \hat{p}_\theta^2 + \hat{p}_z^2 = \alpha^2 \left( \hat{p}_x^2 + \hat{p}_\theta^2 + \hat{p}_z^2 \right) + \frac{\eta}{\hbar} \left( L_x + L_y + L_z \right) + \frac{3\theta^2}{2\alpha^2 \hbar^2} \left( x^2 - xy + y^2 - xz - yz + z^2 \right),$$

and the sum of the squares of the $x$ and $y$ components of the non-commutative squared position operator $\hat{x}^2 + \hat{y}^2$

$$\hat{x}^2 + \hat{y}^2 = \alpha^2 \left( x^2 + y^2 \right) + \frac{\theta^2}{\hbar} \left( -L_z + \left( x - y \right) p_z \right) + \frac{3\theta^2}{4\alpha^2 \hbar^2} \left( \hat{p}_x^2 + \hat{p}_\theta^2 + 2\hat{p}_x^2 - 2p_x p_z - 2p_y p_z \right),$$

(21)
and finally square of the z component of the noncommutative position operator ˘z

\[ ˘z^2 = \alpha^2 z^2 + \frac{\theta}{\hbar} (p_y - p_x) + \frac{\theta^2}{4\alpha^2 \hbar^2} (p_z - p_y)^2. \]  

(22)

Substituting (19), (22) into (12) gives the noncommutative Hamiltonian in the commutative algebra. After regrouping and summarizing all terms we get the expanded noncommutative Hamiltonian in the commutative space.

\[
H_0(\hat{x}, \hat{p}) = \alpha^2 H_0(x, p) + \frac{\eta}{\hbar} H_\eta(x, p) + \frac{\theta}{\hbar} H_\theta(x, p) + \frac{\eta\theta}{\hbar^2} H_{\eta\theta}(x, p) + \frac{\eta^2}{\hbar^2} H_{\eta^2}(x, p) + \frac{\theta^2}{\hbar^2} H_{\theta^2}(x, p)
\]

(23)

with

\[
H_\eta = -\frac{1}{2m} \left( (L_x + L_y + L_z) - \frac{1}{4} \omega \chi (-x^2 - y^2 + xz + yz) \right)
\]

\[
H_\theta = -\frac{1}{4} \omega \chi (-p_x^2 - p_y^2 + p_z p_x + p_y p_z) + \frac{1}{2} m \omega^2 (L_z + (x - y)p_z) + \frac{\omega}{8\hbar} \left( L_x + L_y + L_z \right)
\]

\[
H_{\eta\theta} = \frac{\omega}{8\alpha^2} \left( \frac{m}{\hbar} \right)^{1/4} \left( \frac{m}{\hbar} \right)^{1/4} H_{\eta\theta} \left( \sqrt{\frac{m}{\hbar}} \right)
\]

(24)

(25)

(26)

\[
H_{\eta^2} = \frac{1}{4m^2} \left( x^2 - xy + y^2 - xz + yz + z^2 \right)
\]

\[
H_{\theta^2} = \frac{1}{4\alpha^2} \left[ \frac{m}{\hbar} \omega^2 \left( p_x^2 + p_y^2 + 2p_z^2 - 2p_x p_z - 2p_y p_z \right) + \frac{1}{2} m \omega^2 (p_z - p_y)^2 \right]
\]

(27)

(28)

Obviously we return to the well known commutative case if \( \alpha = 1, \theta = \eta = 0. \)

**III. PERTURBATIVE APPROACH**

According to Harko et al. [20], the contribution of the second-order terms \( \eta^2, \theta^2, \) and \( \eta\theta \) are small compared to the terms in \( \eta \) and \( \theta \) in the low energy limit. Consequently, we can determine the effect of the noncommutativity on the binding energy by employing first-order perturbation theory.

To determine the impact of non-commutativity on the energy levels of a charged harmonic oscillator in 3D in the presence of a constant magnetic field, we first have to revisit the well-known commutative case. The Hamiltonian in the commutative case in cylindrical coordinates is then given as

\[
H_0(x, p) = \frac{\hbar^2}{2m} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) - \frac{\omega^2}{2} \hbar \frac{\partial}{\partial \varphi} + \frac{1}{2} \omega \rho \frac{\partial}{\partial \varphi} + \frac{1}{2} \left( \omega^2 + \frac{\omega^2}{\rho^2} \right) \rho^2 - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \frac{1}{2} m \omega^2 z^2,
\]

(29)

where \( x = \rho \cos \varphi, \ y = \rho \sin \varphi \) consequently \( \rho^2 = x^2 + y^2, \)

\( L_z = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}, \) and \( p_\rho^2 = -\hbar^2 \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) \right). \) With \( \omega^2 = \omega^2 + \frac{\omega^2}{\rho^2} \) we get

\[
H_0(x, p) = \frac{\hbar^2}{2m} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) - \frac{\omega^2}{2} \hbar \frac{\partial}{\partial \varphi} + \frac{1}{2} \omega \rho \frac{\partial}{\partial \varphi} + \frac{1}{2} m \omega^2 \rho^2 - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \frac{1}{2} m \omega^2 z^2.
\]

(30)

In cylindrical coordinates, the time-independent Schrödinger equation for a particle in an isotropic harmonic oscillator in the presence of a constant magnetic field can be solved by separation of variables as

\[
\psi_{n_\eta, n_\mu, n_z}(x) = \chi(\rho) e^{in\varphi} \zeta(z).
\]

(31)

After substitution into the time independent Schrödinger equation we get the eigenfunction as:

\[
\zeta(z) = \frac{1}{\sqrt{2n!}} \left( \frac{m}{\hbar} \right)^{1/4} e^{-\frac{m \omega^2}{\hbar}} H_{n_z} \left( \sqrt{\frac{m \omega}{\hbar}} z \right)
\]

(32)

\[
\chi(\rho) = A_1 \left( \frac{\sqrt{2\rho}}{\sqrt{2\rho}} \right)^{|\mu|} e^{-\frac{m \omega^2}{\hbar}} U(-n_\rho, 1 + |\mu|, \frac{m \omega^2}{2\hbar}) + A_2 \left( \frac{\sqrt{2\rho}}{\sqrt{2\rho}} \right)^{|\mu|} e^{-\frac{m \omega^2}{\hbar}} L_{n_\rho}^{|\mu|} \left( -n_\rho, \frac{m \omega^2}{2\hbar} \right)
\]

(33)

and the eigenvalue

\[
E_{n_\eta, n_\mu, n_z} = \hbar \omega (2n_\rho + |\mu| + 1) + \frac{1}{2} \hbar \omega_i \mu + \hbar \omega \left( n_z + \frac{1}{2} \right).
\]

(34)

The corrections to the binding energy for weak noncommutativity in first-order perturbation theory are then according to [23] given as

\[
\Delta E^{(1)} = \frac{\eta}{\hbar} \left( \left| n_\rho, n_\mu, n_z \right| H_\eta \left| n_\rho, n_\mu, n_z \right> + \left| n_\rho, n_\mu, n_z \right| H_\theta \left| n_\rho, n_\mu, n_z \right> \right)
\]

(35)

Due to the symmetry of the problem all following ma-
So, the only matrix elements that are non-vanishing are

\[
\begin{align*}
&= \langle n_\rho, \mu, n_z | L_x | n_\rho, \mu, n_z \rangle = \langle n_\rho, \mu, n_z | L_y | n_\rho, \mu, n_z \rangle = \\
&= \langle n_\rho, \mu, n_z | xz | n_\rho, \mu, n_z \rangle = \langle n_\rho, \mu, n_z | yz | n_\rho, \mu, n_z \rangle = \\
&= \langle n_\rho, \mu, n_z | px | n_\rho, \mu, n_z \rangle = \langle n_\rho, \mu, n_z | py | n_\rho, \mu, n_z \rangle = \\
&= \langle n_\rho, \mu, n_z | pz | n_\rho, \mu, n_z \rangle = 0
\end{align*}
\]

So, the only matrix elements that are non-vanishing are

\[
\Delta E^{(1)} = \frac{\eta}{\hbar} \left( \frac{n_\rho, \mu, n_z}{2m} - \frac{\omega_c^2 \rho^2}{4} \right) + \\
\frac{1}{\hbar} \left( \frac{n_\rho, \mu, n_z}{2m} - \frac{\omega_c^2 \rho^2}{4} \right) - \frac{1}{2} m \omega_c^2 L_z | n_\rho, \mu, n_z \rangle.
\]

(36)

With the help of \([28, 29]\), the lengthy integrals can be solved in closed form and we get for the first-order corrections in \(\eta\)

\[
\Delta E^{(1)}_{\eta} = - \frac{\eta |\mu|}{2m} - \frac{\eta \omega_c}{4m \omega} (2n_\rho + |\mu| + 1)
\]

and \(\theta\)

\[
\Delta E^{(1)}_{\theta} = - \frac{1}{2} \theta m \omega_c \left( \omega - \frac{1}{2} \omega_c f(n_\rho, |\mu|) \right)
\]

(38)

with

\[
f(n_\rho, \mu) = 2 \left( \frac{n_\rho + \mu}{\mu} \right) - 4 \mu \left( \frac{\mu + n_\rho + 2}{n_\rho - 1} \right) - \\
- \mu(1 + \mu) \left[ 2 \left( \frac{\mu + n_\rho}{n_\rho} \right) + 4 \left( \frac{\mu + n_\rho - 2}{n_\rho} \right) + \\
+ \left( \frac{\mu + n_\rho + 1}{n_\rho} \right) - \left( \frac{\mu + n_\rho + 2}{n_\rho} \right) \right].
\]

(39)

A short dimensional analysis shows that \(\eta\) has the dimension of \(mass^2 \text{Length}^{-2}\), and \(\theta\) has the dimension of \(\text{Length}^2\). So, the calculated corrections have the correct dimension of energy.

So, finally, we can summarize the results of our calculation in first-order perturbation theory. Recalling the non-commutative Hamiltonian \([23]\) we see that the unperturbed energy is

\[
E^{(0)}_{n_\rho, \mu, n_z} = \langle n_\rho, |\mu|, n_z | \alpha^2 H_0(x, p) | n_\rho, |\mu|, n_z \rangle =
\]

\[
= \alpha^2 \left[ \hbar \omega (2n_\rho + |\mu| + 1) + \frac{1}{2} \hbar \omega_c \mu + \hbar \omega \left( n_z + \frac{1}{2} \right) \right].
\]

(40)

The first order energy corrections are

\[
\Delta E^{(1)} = - \frac{\eta |\mu|}{2m} - \frac{\eta \omega_c}{4m \omega} (2n_\rho + |\mu| + 1) - \\
- \frac{1}{2} \theta m \omega_c \left( \omega - \frac{1}{2} \omega_c f(n_\rho, |\mu|) \right)
\]

with \(f(n_\rho, |\mu|)\) given in \([39]\). These results hold for the situations, where \(\eta \ll \hbar \omega_c\) and \(\theta \ll \frac{1}{m \omega_c}\).

IV. CONCLUSION

We studied the charged harmonic oscillator within a constant magnetic field in the context of the non-commutative quantum mechanics. In this line, we assumed the commutators to be up to linear order of \(\theta\) and \(\eta\) in eq.\([\text{I}]\) based on the definition given in eq.\([\text{I}]\) and \([\text{I}]\). Following the standard Schrödinger equation in non-commutative quantum mechanics, eq.\([\text{I}]\), we constructed the non-commutative Hamiltonian in the equation \([23]\). Finally, we have solved the Schrödinger equation, using the time-independent perturbation method. We obtained the corrections in the energy levels, up to the first order of the non-commutative parameters. Our final energy correction in eq.\([\text{I}]\) turns zero if the non-commutative parameters vanish.

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