Analytic QCD – a short review

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I. INTRODUCTION

Perturbative QCD calculations involve coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ which has Landau singularities on the space-like semiaxis $0 \leq Q^2 \leq \Lambda^2$ ($q^2 \equiv -Q^2$). These lead to Landau singularities for the evaluated space-like observables $D(Q^2)$ at low $Q^2 \lesssim \Lambda^2$. The existence of such singularities is in contradiction with the general principles of the local quantum field theories [1]. Further, lattice simulations [2] confirm that such singularities are not present in $a(Q^2)$.

An analytized coupling $A_1(Q^2)$, which agrees with the perturbative $a(Q^2)$ at $Q^2 \rightarrow \infty$ and is analytic in the Euclidean part of the $Q^2$-plane ($Q^2 \epsilon C, Q^2 \leq 0$), addresses this problem, and has been constructed by Shirkov and Solovtsov about ten years ago [3].

Several other analytic QCD (anQCD) models for $A_1(Q^2)$ can be constructed, possibly satisfying certain additional constraints at low and/or at high $Q^2$.

Another problem is the analytization of higher power terms $a^n \rightarrow A_n$ in the truncated perturbation series (TPS) for $D(Q^2)$. Also here, several possibilities appear.

Application of the Operator Product Expansion (OPE) approach, in the ITEP sense, to inclusive space-like observables appears to make sense only in a restricted class of such anQCD models.

This is a short and incomplete review of the activity in the area; relatively large space is given to the work of the review’s authors. For an earlier and more extensive review, see e. g. Ref. [4].

Section III contains general aspects of analytization of the Euclidean coupling $a(Q^2) \rightarrow A_1(Q^2)$, and the definition of the time-like (Minkowskian) coupling $A_0(s)$. Further, in Sec. II we review the minimal analytization (MA) procedure developed by Shirkov and Solovtsov, and a variant thereof developed by Nesterenko [5]. In Sec. III we present various approaches of going beyond the MA procedure, i.e., various models for $A_1(s)$, and thus for $A_1(Q^2)$ and $A_2(Q^2)$ in MA model are presented [10, 11], and an alternative approach which is applicable to any model of analytic $A_1(Q^2)$ is outlined [13]. In Sec. VI methods of evaluations of space-like and of inclusive time-like observables in models with analytic $A_1(Q^2)$ are described, and some numerical results are presented for semihadronic $\tau$ decay rate ratio $r_\tau$, Adler function $d_{11}(Q^2)$ and Bjorken polarized sum rule (BjPSR) $d_0(Q^2)$ 10, 11, 12, 13, 14, 15. In Sec. VII two sets of models are presented [17, 18] whose analytic couplings $A_1(Q^2)$ preserve the OPE-ITEP philosophy, i.e., at high $Q^2$ they fulfill: $|A_1(Q^2) - a(Q^2)| < (\Lambda^2/Q^2)^k$ for any $k \epsilon N$. Section VIII contains a summary of the presented themes.

II. ANALYTIZATION $a(Q^2) \rightarrow A_1(Q^2)$

In perturbative QCD (pQCD), the beta function is written as a truncated perturbation series (TPS) of coupling $a$. Therefore, the renormalization group equation (RGE) for $a(Q^2)$ has the form

$$\frac{\partial a(\ln Q^2, \beta_2, \ldots)}{\partial \ln Q^2} = -\sum_{j=2}^{j_{\text{max}}} \beta_{j-2} a^j(\ln Q^2, \beta_2, \ldots). \quad (1)$$
The first two coefficients \( [\beta_0 = (1/4)(11 - 2n_f/3), \beta_1 = (1/16)(102 - 38n_f/3)] \) are scheme-independent in mass-independent schemes. The other coefficients \((\beta_2, \beta_3, \ldots)\) characterize the renormalization scheme (RSch). The solution of perturbative RGE \( (1) \) can be written in the form

\[
a(Q^2) = \sum_{k=1}^{\infty} K_{k} \frac{(\ln L)^{k}}{L^{k}},
\]

where \( L = \ln(Q^2/\Lambda^2) \) and \( K_{k} \) are constants depending on \( \beta_{j}'s \). In \( \overline{\text{MS}} \): \( \Lambda = \overline{\Lambda} \sim 10^{-1} \text{GeV} \).

The pQCD coupling \( a(Q^2) \) is nonanalytic on \(-\infty < Q^2 < \Lambda^2 \). Application of the Cauchy theorem gives the dispersion relation

\[
a(Q^2) = \frac{1}{\pi} \int_{\sigma=-\Lambda^2}^{\infty} \frac{d\rho_{1}^{(\text{pt})}(\sigma)}{(\sigma + Q^2)}, \quad (\eta \to 0),
\]

where \( \rho_{1}^{(\text{pt})}(\sigma) \) is the (pQCD) discontinuity function of \( a \) along the cut axis in the \( Q^2 \)-plane: \( \rho_{1}^{(\text{pt})}(\sigma) = \text{Im}a(-\sigma - i\epsilon) \). The MA procedure of Shirkov and Solovtsov \( [3] \) removes the pQCD contribution of the unphysical cut \( 0 < -\sigma < \Lambda^2 \), keeping the discontinuity elsewhere unchanged ("minimal analytization" of \( a \))

\[
A_{1}^{(\text{MA})}(Q^2) = a(Q^2),
\]

In general:

\[
A_{1}(Q^2) = a(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\rho_{1}(\sigma)}{(\sigma + Q^2)},
\]

where \( \rho_{1}(\sigma) = \text{Im}A_{1}(-\sigma - i\epsilon) \). Relation \( [5] \) defines an analytic coupling in the entire Euclidean complex \( Q^2 \)-plane, i.e., excluding the time-like semiaxis \(-s = Q^2 \leq 0 \).

On this semi-axis, it is convenient to define the time-like (Minkowskian) coupling \( A_1(s) \) \( [12, 13, 14] \)

\[
A_1(s) = \frac{i}{2\pi} \int_{-s+i\epsilon}^{s-i\epsilon} \frac{d\sigma'}{\sigma'} A_1(\sigma').
\]

The following relations hold between \( A_1, A_1 \) and \( \rho_1 \):

\[
A_1(s) = \frac{1}{\pi} \int_{s}^{\infty} \frac{d\sigma}{\sigma} \rho_1(\sigma),
\]

\[
A_1(Q^2) = Q^2 \int_{0}^{\infty} \frac{ds A_1(s)}{(s + Q^2)^2},
\]

\[
d\ln A_1(\sigma) = -\frac{1}{\pi} \rho_1(\sigma).
\]

The MA is equivalent to the minimal analytization of the TPS form of the \( \beta(\alpha) = \partial a(Q^2)/\partial \ln Q^2 \) function \( [19] \)

\[
\frac{\partial A_{1}^{(\text{MA})}(\ln Q^2; \beta_2, \ldots)}{\partial \ln Q^2} = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\rho_{\beta}^{(\text{pt})}(\sigma)}{(\sigma + Q^2)},
\]

where \( \rho_{\beta}^{(\text{pt})}(\sigma) = \text{Im}a(\sigma) \).

\[
\beta(a) = -\sum_{j=2}^{\infty} \beta_{j-2} a^{j} \ln(Q^2; \beta_2, \ldots).
\]

The MA couplings \( A_1(Q^2) \) and \( A_1(s) \) are finite in the IR (with the value \( 1/\beta_0 \) at \( Q^2 = 0 \), or \( s = 0 \)) and show strong stability under the increase of the loop-level \( n_m = j_{\text{max}} - 1 \) (see Figs. \( \text{I} \) \( \text{II} \)), and under the change of the renormalization scale (RSc) and scheme (RSch). Another similar pQCD-approach is to anal-

\[
\text{FIG. 1: Left: one-loop MA } a_{\text{MK}}(Q) = \pi A_1(Q^2) \text{ and its one-loop perturbative counterpart } a_{\text{p}}(Q^2) \text{ in } \overline{\text{MS}}, \text{ for } n_f = 3 \text{ and } \Lambda = \overline{\Lambda} = 0.2 \text{ and } 0.4 \text{ GeV. Right: stability of the MA } a_{\text{MK}}(Q) = \pi A_1(Q^2) \text{ under the loop-level increase. Both figures from: Shirkov and Solovtsov, 1997} [3].
\]

\[
\text{FIG. 2: The MA time-like and space-like couplings } A_1(s^{1/2}) \text{ and } A_1(Q) \text{ at 1-loop, 2-loop (3-loop) level; in } \overline{\text{MS}} \text{ for } n_f = 3 \text{ and } \Lambda = 0.35 \text{ GeV} [3] \text{ and } A_1 \text{ in figure are } \pi A_1 \text{ and } \pi A_1 \text{ in our normalization convention}. \text{Figure from: Shirkov and Solovtsov, 2006} [16].
\]

This leads to an IR-divergent analytic (MA) coupling, \( A_1(Q^2) \sim (\Lambda^2/Q^2)(\ln(\Lambda^2/Q^2))^{-1} \) when \( Q^2 \to 0 \). At one-loop:

\[
A_1(Q^2) = a(Q^2) = \frac{(Q^2/\Lambda^2) - 1}{\beta_0 (Q^2/\Lambda^2) \ln(Q^2/\Lambda^2)}.
\]

Also this coupling has improved stability under the loop-level change, and under the RSc and RSch changes (see
one-loop level, for various observables, were performed in Ref. [21], and they agree with the experimental results within the experimental uncertainties and the theoretical uncertainties of the one-loop approximation.

III. BEYOND THE MA

The idea to make the QCD coupling IR finite phenomenologically is an old one, by the substitution \( \ln(Q^2/\Lambda^2) \rightarrow \ln[(Q^2+4m_g^2)/\Lambda^2] \) where \( m_g \) is an effective gluon mass, cf. Refs. [22, 23, 24].

On the other hand, the analytic MA, or \( \overline{\text{MA}} \), couplings can be modified at low energies, bringing in additional parameter(s) such that there is a possibility to reproduce better a wide set of low energy QCD experimental data. Among the recent proposed analytic couplings are:

1. Synthetic coupling proposed by Alekseev [6]:

\[
\alpha_{\text{syn}}(Q^2) = \alpha^{(\overline{\text{MA}})}(Q^2) + \frac{\pi}{\beta_0} \left[ \frac{cL^2}{Q^2} - \frac{d}{Q^2 + m_g^2} \right], \tag{13}
\]

where the three new parameters \( c, d \) and gluon mass \( m_g \) were determined by requiring \( \alpha_{\text{syn}}(Q^2) - \alpha_{\text{pt}}(Q^2) \sim (\Lambda^2/Q^2)^3 \) (for the convergence of the gluon condensate) and by the string condition \( V(r) \sim \sigma r (r \rightarrow \infty) \) with \( \sigma \approx 0.42\text{GeV}^2 \). This coupling is IR-divergent.

2. The coupling by Sriwastawa et al. [5]:

\[
\frac{1}{\alpha_{\text{SPPW}}^{(1)}(Q^2)} = \frac{1}{\alpha_{\text{SPPW}}^{(1)}(\Lambda^2)} + \frac{\beta_0}{\pi} \int_0^\infty \frac{(z-1)z^p}{(\sigma-z+i\epsilon)(\sigma+1)(1+z^p)} \, d\sigma, \tag{14}
\]

where \( z = Q^2/\Lambda^2 \) and \( 0 < p < 1 \). This formula coincides with Nesterenko’s (one-loop) \( \overline{\text{MA}} \) coupling when \( p = 1 \).

3. An IR-finite coupling proposed by Webber [8]:

\[
\alpha_{\text{IR}}^{(1)}(Q^2) = \frac{\pi}{\beta_0} \left[ \frac{1}{\ln z} + \frac{1}{1-z} + \frac{1+c}{z+c} \right]^p, \tag{15}
\]

where \( z = Q^2/\Lambda^2 \) and specific values are chosen for parameters \( b = 1/4, c = 4, \) and \( p = 4 \); \( \alpha_{\text{IR}}^{(1)}(0) \sim \pi^2/(2\beta_0) \).

4. “Massive” \( \overline{\text{MA}} \) or MA couplings \( \tilde{A}_1(Q^2) \) and \( \tilde{A}_1(s) \) proposed by Nesterenko and Papavassiliou [9]:

\[
\tilde{A}_1^{(m)}(s) = \Theta(s-4m^2)\tilde{A}_1(s),
\]

\[
\tilde{A}_1^{(m)}(Q^2) = \frac{Q^2}{Q^2+4m^2} \int_{4m^2}^{\infty} \frac{\rho_1(\sigma)}{\sigma+Q^2} \frac{d\sigma}{\sigma}, \tag{16}
\]

where \( m \sim \overline{\Lambda} \); and \( \rho_1(\sigma) = \rho_1(\overline{r})(\sigma) \) in the MA case. In this case: \( \tilde{A}_1^{(m)}(0) = \tilde{A}_1^{(m)}(0) = 0 \). The mass \( m \) is some kind of threshold, and can be expected to be \( \sim m_T \).

5. Two specific models of IR-finite analytic coupling (10, 11): on the time-like axis \( s \equiv -Q^2 > 0 \), the perturbative discontinuity function \( \rho_1(s) \), or equivalently \( \tilde{A}_1^{(MA)}(s) \), was modified in the in the IR regime \( s \sim \overline{\Lambda}^2 \).

A first possibility (model ‘M1’):

\[
\tilde{A}_1^{(M1)}(s) = c_f M_1^2 \delta(s - M_1^2) + k_0 \Theta(s - M_0^2 - s) + \Theta(s - M_0^2) \tilde{A}_1^{(MA)}(s),
\]

where \( c_f, k_0, c_r = M_r^2/\Lambda^2, c_0 = M_0^2/\Lambda^2 \) are four dimensionless parameters of the model, all \( \sim 1 \). One of them \( (k_0) \) can be eliminated by requiring the (approximate) merging of M1 with MA at large \( Q^2 \):

\[
|\tilde{A}_1^{(M1)}(Q^2) - \tilde{A}_1^{(MA)}(Q^2)| \sim (\overline{\Lambda}^2/Q^2)^2.
\]

The Euclidean \( A_1^{(M1)}(Q^2) \) is

\[
A_1^{(M1)}(Q^2) = A_1^{(MA)}(Q^2) + \Delta A_1^{(M1)}(Q^2),
\]

\[
\Delta A_1^{(M1)}(Q^2) = -\frac{1}{\pi} \int_{s=0}^{M_1^2} \frac{d\sigma \rho_1^{(pt)}(\sigma)}{(\sigma + Q^2)} + c_{f} M_1^2 Q^2 \left( \frac{Q^2 + M_r^2}{Q^2 + M_r^2} \right)^2,
\]

\[
-\frac{d}{\pi} \frac{M_0^2}{(Q^2 + M_0^2)}, \tag{17}
\]
where the constant \( d_f \) is
\[
d_f \equiv -k_0 + \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma} \rho_k^{(pt)}(\sigma).
\]

Another, simpler, possibility is (model 'M2'):
\[
\begin{align}
a_1^{(M1)}(s) &= \mathcal{A}_1^{(MA)}(s) + c_v \Theta(\Lambda_p^2 - s), \\
A_1^{(M1)}(Q^2) &= A_1^{(MA)}(Q^2) + \frac{c_v}{Q^2} \frac{\Lambda_p^2}{(Q^2 + \Lambda_p^2)},
\end{align}
\]
where \( c_v \) and \( c_p = \frac{\Lambda_p^2}{\Lambda_p^2 + \Lambda^2} \) are the model parameters.

6. Those anQCD models which respect the OPE-ITEP condition are presented in Sec. VII.

IV. ANALYTIZATION OF HIGHER POWERS

\( a^k \mapsto A_k \)

In MA model, the construction is \[ \mathbf{3} \quad \mathbf{12} \quad \mathbf{13} \quad \mathbf{14} \]
(MSSSh: Milton, Solovtsov, Solovtsova, Shirkov):
\[
a^k(Q^2) \mapsto A_k^{(MA)}(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \rho_k^{(pt)}(\sigma),
\]
where \( k = 1, 2, \ldots; \rho_k^{(pt)}(\sigma) = \text{Im}[a^k(-\sigma - i\epsilon)]; \) and \( a \) is given, e.g., by Eq. (2). In other words, “minimal analyticization” (MA) is applied to each power \( a^k \).

As a consequence, in MA we have \[ 19 \]
\[
\begin{align}
\frac{\partial A_1^{(MA)}(\mu^2)}{\partial \ln \mu^2} &= -3\beta_0 A_2^{(MA)}(\mu^2) - \beta_1 A_3^{(MA)}(\mu^2) - \cdots, \\
\frac{\partial^2 A_1^{(MA)}(\mu^2)}{\partial (\ln \mu^2)^2} &= 2\beta_0^2 A_3^{(MA)} + 5\beta_0 \beta_1 A_4^{(MA)} + \cdots,
\end{align}
\]

and analogously for \( \partial A_1/\partial \beta_3 \), etc. In our approach, the basic space-like quantities are \( A_1(\mu^2) \) of a given anQCD model (e.g., MA, M1, M2) and its logarithmic derivatives
\[
\tilde{A}_n(\mu^2) \equiv \frac{(-1)^{n-1}}{\beta_0^{n-1}(n-1)!} \frac{\partial^{n-1} A_1(\mu^2)}{\partial (\ln \mu^2)^{n-1}}, \quad (n = 1, 2, \ldots),
\]
whose pQCD analogs are
\[
\tilde{a}_n(\mu^2) \equiv \frac{(-1)^{n-1}}{\beta_0^{n-1}(n-1)!} \frac{\partial^{n-1} a(\mu^2)}{\partial (\ln \mu^2)^{n-1}}, \quad (n = 1, 2, \ldots).
\]

At loop-level three \( (n_m = 3) \), where we include in RGE \[ 11 \]
term with \( j_{\text{max}} = 4 \) (thus \( \beta_2 \)), relations \[ 21 \]
are
\[
\tilde{A}_2(\mu^2) = A_2(\mu^2) + \frac{\beta_1}{\beta_0} \tilde{A}_3(\mu^2), \quad \tilde{A}_3(\mu^2) = \tilde{A}_3(\mu^2),
\]

implying
\[
A_2(\mu^2) = \tilde{A}_2(\mu^2) - \frac{\beta_1}{\beta_0} \tilde{A}_3(\mu^2), \quad A_3(\mu^2) = \tilde{A}_3(\mu^2).
\]

The RSc \( (\beta_2) \) dependence is obtained from the truncated Eqs. \[ 22 \] and \[ 21 \]
\[
\frac{\partial \tilde{A}_1(\mu^2; \beta_2)}{\partial \beta_2} \approx \frac{1}{2\beta_0^3} \frac{\partial^2 \tilde{A}_1(\mu^2; \beta_2)}{\partial (\ln \mu^2)^2},
\]

where \( (j = 1, 2, \ldots) \) and \( \tilde{A}_1 = A_1 \).

At loop-level four \( (n_m = 4) \), where we include in RGE \[ 11 \]
term with \( j_{\text{max}} = 5 \) (thus \( \beta_3 \)), relations analogous to \[ 20 \]-\[ 27 \] can be found \[ 11 \].

It turns out that there is a clear hierarchy in magnitudes \( |A_1(Q^2)| > |A_2(Q^2)| > |A_3(Q^2)| > \cdots \) at all \( Q^2 \), in all or most of the anQCD models (cf. Fig. 3 for MA, M1, M2; and Fig. 9 in Sec. VII for another model).

![Fig. 5: A1 and A2 for various models (M1, M2 and MA) with specific model parameters: c0 = 2.54, cν = 0.45, c_f = 1.08 for M1; c_v = 0.1, c_p = 3.4 for M2; n_f = 3, \( \Lambda_p(n_f = 3) = 0.4 \) GeV in all three models. The upper three curves are \( A_1 \); the lower three are \( 3 \times A_2 \). All couplings are in v-scheme (see Subsec. VIIA). A2 is constructed with our approach. Figure from: Ref. 11.](image)
We recall that the perturbation series of a space-like observable $D(Q^2)$ ($Q^2 = -q^2 > 0$) can be written as

$$D(Q^2)_{pt} = a + d_1 a^2 + d_2 a^3 + \cdots , \quad (28)$$

where the second form (29) is the reorganization of the perturbative power expansion (28) into a perturbation expansion in terms of $\bar{a}_n$'s (24) (note: $\bar{a}_1 \equiv a$). The basic analytization rule we adopt is the replacement

$$\bar{a}_n \rightarrow \tilde{A}_n \quad (n = 1, 2, \ldots) , \quad (30)$$

term-by-term in expansion (29), and this is equivalent to the analytization rule $a^n \rightarrow A_n$ term-by-term in expansion (28). However, in principle, other analytization procedures could be adopted, e.g. $a^n \rightarrow A_1^n$, or $a^n \rightarrow A_1 A_{n-1}$, etc. The described analytization $a^n \rightarrow A_n$ reduces to the MSSSh analytization in the case of the MA model (i.e., in the case of $A_1 = A_1^{(MA)}$), because the aforementioned RGE-type relations hold also in the MA case.

Let's denote by $D^{(nm)}(Q^2)$ the TPS of (28) with terms up to (and including) the term $\sim a^{nm}$, and by $\partial D^{(nm)}(Q^2)/\partial \mu^2 \sim A_{n+1}$ the corresponding truncated analytic series (TAS) obtained from the previous one by the term-by-term analytization $a^n \rightarrow A_n$. The evolution of $A_{n+1}(Q^2)$ under the changes of the RSch was truncated in such a way that $\partial D^{(nm)}(Q^2)/\partial \beta_j \sim A_{n+1}$ (where $j \geq 2$). Further, our definition of $A_k$'s ($k \geq 2$) via Eqs. (21) [cf. Eqs. (20)] involves truncated series which, however, still ensure the “correct” RScI-dependence $\partial D^{(mn)}(Q^2)/\partial \mu^2 \sim A_{n+1}$. This is all in close analogy with the pQCD results for TPS's: $\partial D^{(mn)}(Q^2)/\partial \beta_j \sim a^{nm+1}$, and $\partial D^{(mn)}(Q^2)/\partial \mu^2 \sim a^{nm+1}$. In conjunction with the mentioned hierarchy depicted in Fig. 3 this means that the evaluated TAS will have increasingly weaker RScI and RScI dependence when the number of TAS terms increases, at all values of $Q^2$.

On the other hand, if the analytization of powers were performed by another rule, for example, by the simple rule $a^n \rightarrow A_1^n$, the above RScI:RScI dependence of the TAS would not be valid anymore. An increasingly weaker RScI:RScI dependence of TAS (when the number of TAS terms is increased) would not be guaranteed anymore.

V. CALCULATION OF $A_\nu$ FOR $\nu$ NONINTEGER

Analytization of noninteger powers $a^\nu$ or $a^\nu \ln a$, is needed in calculations of pion electromagnetic form factor, and in some resummed expressions for Green functions or observables, calculated within an anQCD model.

In the mentioned approach, use is made of the Laplace transformation $f_L$ of function $f$,

$$f(z) \mapsto (f)_L(t) : f(z) = \int_0^\infty dt e^{-zt} (f)_L(t) ,$$

where $z \equiv \ln(Q^2/\Lambda^2)$. Using notations (21) and (23), it can be shown

$$\langle \bar{a}_n \rangle_L(t) = \frac{\nu_{n-1}}{\beta_0^{-1} \Gamma(\nu)} (a)_L(t) , \quad (31)$$

$$\langle \tilde{A}_n \rangle_L(t) = \frac{\nu_{n-1}}{\beta_0^{-1} \Gamma(\nu)} (A_1)_L(t) . \quad (32)$$

Therefore, it is natural to define for any real $\nu$ the following Laplace transforms:

$$\langle \bar{a}_\nu \rangle_L(t) = \frac{\nu_{\nu-1}}{\beta_0^{-1} \Gamma(\nu)} (a)_L(t) ; \quad (33)$$

$$\langle \tilde{A}_\nu \rangle_L(t) = \frac{\nu_{\nu-1}}{\beta_0^{-1} \Gamma(\nu)} (A_1)_L(t) . \quad (34)$$

In MA model, at one-loop level, $(a)_L(t)$ and $(A_1)_L(t)$ are known

$$a(z) = \frac{1}{\beta_0 z} \Rightarrow (a)_L(t) = \frac{1}{\beta_0} . \quad (35)$$

$$A_1(z) = \frac{1}{\beta_0} \left( \frac{1}{z} - \frac{1}{e^z - 1} \right) \Rightarrow (A_1)_L(t) = \frac{1}{\beta_0} \left( 1 - \sum_{k=1}^{\infty} \delta(t-k) \right) . \quad (36)$$

Since at one-loop $\tilde{A}_\nu = A_\nu$, it follows in one-loop MA model

$$A_\nu(z) = \int_0^\infty dt e^{-zt} \beta_0^{-1} \Gamma(\nu) \left( 1 - \sum_{k=1}^{\infty} \delta(t-k) \right) . \quad (37)$$

Similarly, since

$$a^\nu(z) \ln a(z) = \frac{d}{d\nu} a^\nu(z) ,$$

it can be defined

$$\left[ \frac{d}{d\nu} a^\nu(z) \right]_{MA} \equiv \frac{d}{d\nu} A_\nu(z) . \quad (38)$$

To calculate higher (two-)loop level $A_\nu(z)$ in MA model, the authors of Ref. [15] expressed the two-loop $a^{(2)}(z)$ in terms of one-loop powers $a_1^{(1)}(z)$ = $z$ in $a_1(Z) \ln(a_1(Z))$ and then followed the above procedure.
VI. EVALUATION METHODS FOR OBSERVABLES

In pQCD, the most frequent method of evaluation of the leading-twist part of a space-like physical quantity is the evaluation of the available (RG-improved) truncated perturbation series (TPS) in powers of perturbative coupling $a$. Within the anQCD models, an analogous method is the aforementioned replacement $a^n \rightarrow A_n$ in the TPS (where $A_n$ are constructed in Sec. [IV]), and the evaluation thereof. More specifically, consider an observable $D(Q^2)$ depending on a single space-like physical scale $Q^2 (\equiv -q^2) > 0$. Its usual perturbation series has the form [23], where $a = a(\mu^2; \beta_2, \beta_3, \ldots)$, with $\mu^2 \sim Q^2$.

For each TPS $D(Q^2)^{(N)}$ of order $N$, in the minimal anQCD (MA) model, the authors MSSSh [12, 13, 14] introduced the aforementioned replacement $a^n \rightarrow A_n^{(MA)}$:

$$D(Q^2)^{(N)}_{\text{an(MSSSh)}} = A_1^{(MA)} + d_1 A_2^{(MA)} + \cdots + d_{N-1} A_N^{(MA)}.$$  \hspace{1cm} (39)

This method of evaluation (via $a^n \rightarrow A_n$) was extended to any anQCD model in [9, 11] (cf. Sec. IV). Further, in the case of inclusive space-like observables, the evaluation was extended to the resummation of the large-$\beta_0$ terms:

A. Large-$\beta_0$-motivated expansion of observables

We summarize the presentation of Ref. [11]. We work in the RScl's where each $\beta_k$ ($k \geq 2$) is a polynomial in $n_f$ of order $k$; in other words, it is a polynomial in $\beta_0$:

$$\beta_k = \sum_{j=0}^{k} b_{kj} \beta_0^j, \quad k = 2, 3, \ldots$$ \hspace{1cm} (40)

The MS scheme belongs to this class of schemes. In such schemes, the coefficients $d_n$ of expansion have the following specific form in terms of $\beta_0$:

$$D(Q^2)_{\text{pt}} = a + (c_{11} \beta_0 + c_{10} a^2)
+ (c_{22} \beta_0^2 + c_{21} \beta_0 + c_{20} + c_{2, -1} \beta_0^{-1}) a^3 + \cdots.$$ \hspace{1cm} (41)

We can construct a separation of this series into a sum of two RScl-independent terms – the leading-$\beta_0$ (L-$\beta_0$), and beyond-the-leading-$\beta_0$ (BL-$\beta_0$) \begin{equation}
D_{\text{pt}} = D_{\text{pt}}^{(L-\beta_0)} + D_{\text{pt}}^{(BL-\beta_0)}, \end{equation}

where

$$D_{\text{pt}}^{(L-\beta_0)} = a + a^2 \left[ \beta_0 c_{11} \right] + a^3 \left[ \beta_0^2 c_{22} + \beta_1 c_{11} \right]
+ a^4 \left[ \beta_0^3 c_{33} + \frac{5}{2} \beta_0 \beta_1 c_{22} + \beta_2 c_{11} \right] + O(\beta_0^5 a^5).$$ \hspace{1cm} (43)

Expression (43) is not the standard leading-$\beta_0$ contribution, since it contains also terms with $\beta_j^\prime$ ($j \geq 1$), but only in a minimal way to ensure that the expression contains all the leading-$\beta_0$ terms and at the same time remains RScl-independent. It can be shown that, for inclusive observables, all the coefficients in this L-$\beta_0$ contribution can be obtained, and can be expressed in the integral form

$$D_{\text{pt}}^{(L-\beta_0)}(Q^2)_{\text{pt}} = \int_0^\infty \frac{dt}{t} F_{\text{pt}}^\beta(t) a(t e^C Q^2),$$ \hspace{1cm} (44)

where $F_{\text{pt}}^\beta(t)$ is the (Euclidean) L-$\beta_0$ -characteristic function. In MS scheme, $\Lambda = \overline{\Lambda}$ which corresponds here to $C = \overline{C} = -5/3$. No RScl $\mu$ appears in (44). Expression (44) is referred to in the literature sometimes as dressed gluon approximation.

The BL-$\beta_0$ contribution is usually known only to $\sim a^3$ or $\sim a^4$. For it, we can use an arbitrary RScl $\mu \equiv Q^2 e^C \sim Q^2$. Further, the powers $a^k$ can be reexpressed in terms of $\tilde{a}_n (\mu^2)$ [21]:

$$a^2 = \tilde{a}_2 - (\beta_1 / \beta_0) \tilde{a}_3 + \cdots, \quad a^3 = \tilde{a}_3 + \cdots.$$ \hspace{1cm} (45)

Therefore,

$$D(Q^2)_{\text{pt}} = D_{\text{pt}}^{(L-\beta_0)}(Q^2)_{\text{pt}}
+ \tilde{t}_2 \tilde{a}_2(Q^2 e^C) + \tilde{t}_3 \tilde{a}_3(Q^2 e^C) + \tilde{t}_4 \tilde{a}_4(Q^2 e^C).$$ \hspace{1cm} (46)

where $\tilde{t}_2 = c_{10}$ is scheme-independent, and coefficients $\tilde{t}_3$ and $\tilde{t}_4$ have a scheme dependence (depend on $\beta_2, \beta_3$ – i.e., on $b_{2j}$ and $b_{3j}$). We note that expression (46) is not really a pure TPS, because its L-$\beta_0$ contribution is not truncated. An observable-dependent scheme (D-scheme) can be chosen such that $\tilde{t}_3 = \tilde{t}_4 = 0$. For the Adler function $D = d_v$, such a scheme will be called v-scheme. The analytization of the obtained $D(Q^2)_{\text{pt}}$ is performed by the substitution $\alpha_n \rightarrow \tilde{\alpha}_n$, Eq. (39), leading to the truncated analytic series (TAS)

$$D(Q^2) = D(Q^2)_{\text{TAS}} + O(\beta_0^3 A_3),$$ \hspace{1cm} (47)

$$D(Q^2)_{\text{TAS}} = \int_0^\infty \frac{dt}{t} F_{\text{pt}}^\beta(t) A_1(t e^C Q^2)
+ \tilde{c}_{10} \tilde{A}_2(Q^2 e^C) + \tilde{t}_3 \tilde{A}_3(Q^2 e^C) + \tilde{t}_4 \tilde{A}_4(Q^2 e^C).$$ \hspace{1cm} (48)

In the D-scheme, the last two terms disappear. Eq. (48) is a method that one can use to evaluate any inclusive space-like QCD observable in any anQCD model. As argued in Sec. [IV], the scale and scheme dependence of the TAS is very suppressed

$$\frac{\partial D(Q^2)_{\text{TAS}}}{\partial X} \sim \beta_0^3 A_5 \sim \beta_0^4 A_5 \quad (X = \ln \mu^2, \beta_j).$$ \hspace{1cm} (49)

If the BL-$\beta_0$ perturbative contribution is known exactly only up to (and including) $\sim a^3$, then no $\tilde{t}_4$ term appears in Eq. (48) and the precision in Eqs. (47) and (49) is diminished: $O(\beta_0^3 A_3) \rightarrow O(\beta_0^2 A_4)$. It is interesting to note that the Taylor expansion of $A_1(t e^C Q^2)$ in $D^{(L-\beta_0)}(Q^2)_{\text{an}}$ in [45] around a chosen RScl
TABLE I: Various order contributions to observables within PT, and MSSSh (=APT) methods [14, 15].

| Process | Method | 1st order | 2nd | 3rd |
|---------|--------|-----------|-----|-----|
| GLS     | PT     | 65.1%     | 24.4% | 10.5% |
| \((Q \sim 1.76\text{GeV})\) | APT | 75.7% | 20.7% | 3.6% |
| \(r_\tau\) | PT | 54.7% | 29.5% | 15.8% |
| \((M_\tau = 1.78\text{GeV})\) | APT | 87.9% | 11.0% | 1.1% |

\[\ln(\mu^2)\] reveals just the aforementioned \(a^n \rightarrow A_n\) analytization of the large-\(\beta_0\) part [43], in any aNQCD:

\[
D^{(L,\beta_0)}_{an} = \int_0^\infty \frac{dt}{t} F_D^\xi(t) A_1(t e^C Q^2) = A_1 + A_2 [\beta_0 c_{11} + \beta_1 c_{11}] + A_3 [\beta_0^2 c_{22} + \beta_1 c_{11}] + A_4 \left[ \beta_0^3 c_{33} + \frac{5}{2} \beta_0 \beta_1 c_{22} + \beta_2 c_{11} \right] + O(\beta_0^4 A_3),
\]

where \(A_k = A_k(\mu^2; \beta_2, \beta_3, \ldots)\). In other words, at the leading-\(\beta_0\) level, the natural analytization \(a^n \rightarrow A_n\) \((\equiv \bar{a}_n \rightarrow \bar{A}_n)\) in the corresponding perturbation series. This thus represents yet another motivation for the analytization \(a^n \rightarrow A_n\) \((\equiv \bar{a}_n \rightarrow \bar{A}_n)\) postulated in Sec. LV of all the available perturbation terms in \(D\). For the first motivation, based on the systematic weakening of the RScl & RSch dependence of the truncated analytized \(D\), see the end of Sec. LV.

B. Applications in phenomenology

Evaluations in MA model, with the MSSSh-approach \(a^n \rightarrow A_n^{(MA)}\) [12, 13, 14], are usually performed in MS scheme. The only free parameter is \(\lambda = (\overline{\lambda})\). Fitting the experimental data for \(\Upsilon\)-decay, \(Z \rightarrow \) hadrons, \(e^+e^- \rightarrow \) hadrons, to the MSSSh approach for MA at the two- or three-loop level, they obtained \(\Lambda_{n_f=5} \approx 0.26-0.30\) GeV, corresponding to: \(\Lambda_{n_f=3} \approx 0.40-0.44\) GeV, and \(\pi A_1^{(MA)}(M_Z^2) \approx 0.124\), which is above the pQCD world-average value \(\alpha_s(M_Z^2) \approx 0.119 \pm 0.001\). The apparent convergence of the MSSSh nonpower truncated series is also remarkable – see Table II.

In Refs. [10, 11], the aforementioned TAS evaluation method [48] in aNQCD models MA [41], M1 [17] and M2 [19] was applied to the inclusive observables Bjorken-polarized sum rule (BjPSR) \(d_0(Q^2)\), Adler function \(d_1(Q^2)\) and semihadronic \(\tau\) decay ratio \(r_\tau\). The exact values of coefficients \(d_1\) and \(d_2\) are known for space-like observables BjPSR \(d_0(Q^2)\) [28] and (massless) Adler function \(d_1(Q^2)\) [29, 30]. (The exact coefficient \(d_3\) of \(d_0\) has been recently obtained [31], but was not included in the analysis of Ref. [11] that we present here; rather, an estimated value of \(d_3\) was used.) In the \(v\)-scheme, the evaluated massless \(d_v(Q^2)\) is

\[
d_v(Q^2)_{(TAS)} = \int_0^\infty \frac{dt}{t} F_v^E(t) A_1(t e^C Q^2; \beta_2(x), \beta_3(x)) + \frac{1}{12} \bar{A}_2(e^C Q^2),
\]

while BjPSR \(d_0(Q^2)_{(TAS)}\) has one more term \(\bar{A}_3(e^C Q^2)\). The difference between the (massless) true \(d_2(Q^2)\) \((x = v, b)\) and \(d_2(Q^2)_{(TAS)}\) is \(O(\bar{A}_3)\). The semihadronic \(\tau\) decay ratio \(r_\tau\) is, on the other hand, a time-like quantity, but can be expressed as a contour integral involving the Adler function \(d_v\):

\[
r_\tau(\Delta S = 0, m_q = 0) = \frac{2}{\pi} \int_0^{\pi} ds \left( 1 - \frac{s}{m^2} \right)^2 \left( 1 + 2 \frac{s}{m^2} \right) \text{Im} \Pi(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \left( 1 + e^{i\phi} \right)^3 \left( 1 - e^{i\phi} \right) d_v(Q^2 = m^2_e e^{i\phi}).
\]

This implies for the leading-\(\beta_0\) term of \(r_\tau\)

\[
r_\tau(\Delta S = 0, m_q = 0)_{(L,\beta_0)}^{(BL,\beta_0)} = \int_0^\infty \frac{dt}{t} F_v^M(t) \bar{A}_1(t e^C m_\tau^2),
\]

where \(\bar{A}_1\) is the time-like coupling appearing in Eqs. [39-41], and superscript \(M\) in the characteristic function indicates that it is Minkowskian (time-like). The latter was obtained by Neubert (second entry of Refs. [25]). The beyond-the-leading-\(\beta_0\) (BL/\(\beta_0\)) contribution is the contour integral

\[
r_\tau(\Delta S = 0, m_q = 0)_{(BL,\beta_0)} = \frac{1}{24\pi} \int_{-\pi}^{\pi} d\phi \left( 1 + e^{i\phi} \right)^3 \left( 1 - e^{i\phi} \right) \bar{A}_2(e^C m_\tau^2 e^{i\phi}).
\]

The parameters of aNQCD models M1 [17] and M2 [19] were then determined [11] by fitting the evaluated observables to the experimental central values \(r_\tau(\Delta S = 0, m_q = 0) = 0.204 \pm 0.005\) (for M1 and M2), and to \(d_0(Q^2 = 1\text{GeV}^2) = 0.17 = 0.11\) and \(d_0(Q^2 = 2 \text{GeV}^2) = 0.16 \pm 0.11\) (for M1). For M1 we obtained: \(c_f = 1.08\), \(c_\tau = 0.45\), \(c_0 = 2.94\). For M2 we obtained: \(c_v = 0.1\) and \(c_e = 3.4\).

The numerical results were then obtained [11]. In models MA, M1 and M2 they are given for \(r_\tau\) in Table II for
Adler function $d_{v}(Q^{2})$ in Fig. 6 and for BjPSR $d_{b}(Q^{2})$ (in M1 and M2) in Figs. 7 and 8 (Table II and Figs. 6, 7, 8 are taken from Ref. 11). All results were calculated in the $v$-scheme. For details, we refer to Ref. 11.

Analytic QCD models have been used also in the physics of mesons [35, 34], in calculating various meson masses by summing two contributions: that of the confining part and that of the (one-loop) perturbative part of the Bethe-Salpeter potential. In Refs. 35, the (one-loop) MA coupling [3] was used to calculate/predict the masses; in Refs. 34, the experimental mass spectrum was used to extract the approximate values of the (analytic) coupling $A_1(Q^2)$ at low $Q^2$. In this formalism, the current quark masses were replaced by the constituent quark masses, accounting in this way approximately for the quark self-energy effects. The results by the authors of Ref. 34 indicate that $A_1(Q^2)$ remains finite (and becomes possibly zero) when $Q^2 \to 0$.

VII. ANALYTIC QCD AND ITEP-OPE PHILOSOPHY

In general, the deviations of analytic $A_1(Q^2)$ from the perturbative coupling $a_{pt}(Q^2)$ at high $Q^2 \gg \Lambda^2$ are power terms

$$|\delta A_1(Q^2)| \equiv |A_1(Q^2) - a_{pt}(Q^2)| \sim \left( \frac{\Lambda^2}{Q^2} \right)^k \quad (Q^2 \gg \Lambda^2),$$

where $k$ is a given positive integer. Such a coupling introduces in the evaluation (of the leading-twist) of inclusive space-like observables $D(Q^2)$, already at the leading-$\beta_0$ level, an UV contribution $\delta D^{(UV)}(Q^2)$ which behaves like a power term [18]

$$\delta D^{(UV)}(Q^2) \sim \left( \frac{\Lambda^2}{Q^2} \right)^{\min(k,n)} \quad \text{if } k \neq n,$$

where $n \in \mathcal{N}$ is the position of the leading IR renormalon of the observable $D(Q^2)$; if $k = n$, then the left-hand side of Eq. (54) changes to $(\Lambda^2/Q^2)^n \ln(\Lambda^2/Q^2)$ [18]. Such nonperturbative contributions coming from the UV sector contradict the ITEP Operator Product Expansion (OPE) philosophy (the latter saying that such terms can come only from the IR sector) [35].

Two specific sets of models of anQCD have been introduced in the literature so far such that they do not contradict the ITEP-OPE:

(A) a model set based on a modification of the $\beta(a)$ function [17]:

(B) a model set obtained by a direct construction [18].

A. Set of models A

This is the set of models constructed in Refs. 17. The TPS $\beta(a)$ used in pQCD is

$$\frac{\partial a}{\partial \ln Q^2} = \beta^{(N)}(a) = -\beta_0 a^2 \left( 1 + \sum_{j=1}^{N} c_j a^j \right).$$

FIG. 6: Adler function as predicted by pQCD, and by our approach in several anQCD models: MA, M1, M2. The full quantity is depicted, with the contribution of massive quarks included. The experimental values are from [32]. Figure from: Ref. 11.

FIG. 7: Bjorken polarized sum rule (BjPSR) $d_{b}(Q^{2})$ in model M1, in various RSch’s and at various RScl’s. The vertical lines represent experimental data, with errorbars in general covering the entire depicted range of values.

FIG. 8: As in the Fig. 6 but this time for model M2. Both figures from: Ref. 11.
This was then modified, \( \beta^{(N)}(a) \to \tilde{\beta}^{(N)}(a) \), by fulfilling three main conditions:

1.) \( \tilde{\beta}^{(N)}(a) \) has the same expansion in powers of \( a \) as \( \beta^{(N)}(a) \);

2.) \( \tilde{\beta}^{(N)}(a) \sim -\zeta a^p \) with \( \zeta > 0 \) and \( p \leq 1 \), for \( a \gg 1 \), in order to ensure the absence of Landau singularities;

3.) \( \tilde{\beta}^{(N)}(a) \) is analytic function at \( a = 0 \), in order to ensure \( |a(Q^2) - a_{pt}(Q^2)| < (\Lambda^2/Q^2)^k \) for any \( k > 0 \) at large \( Q^2 \) (thus respecting the ITEP-OPE approach).

This modification was performed by the substitution \( a \to u(a) \equiv a/(1+\eta a) \), \( \eta > 0 \) being a parameter, and

\[
\tilde{\beta}^{(N)}(a) = -\beta_0 \left[ \kappa(a-u(a)) + \sum_{j=0}^{N} \tilde{c}_j u(a)^{j+2} \right], \tag{56}
\]

and \( \tilde{c}_j \) are adjusted so that the first condition is fulfilled

\[
\tilde{c}_0 = 1-\eta \kappa, \quad \tilde{c}_1 = c_1 + 2\eta - \eta^2 \kappa, \quad \text{etc.}
\]

This procedure results in an analytic coupling \( a(Q^2) \), with \( p = 1 \) and \( \zeta = \beta_0 \kappa \), and with two positive adjusting parameters \( \kappa \) and \( \eta \). The QCD parameter \( \Lambda \) was taken the same as in the pQCD. Evaluation of observables was carried out in terms of power expansion, with the replacement \( a_{pt}^n \to a^n \). Further, the couplings in this set are IR infinite: \( a(Q^2) \sim 1/(Q^2)^{\beta_0 \kappa} \to \infty \) when \( Q^2 \to 0 \). These new \( a(Q^2) \)'s are analytic (\( a \equiv A_1 \)). The RScl and RSch sensitivity of the modified TPS's of space-like observables turned out to be reduced. The author of Refs. [17] chose \( \kappa = 1/\beta_0 \); by fitting the predicted values of the static interquark potential to lattice results, he obtained \( \eta \approx 4.1 \).

### B. Set of models B

This is the set of models for \( A_1 \) constructed in Ref. [18]. A class of IR-finite analytic couplings which respect the ITEP-OPE philosophy can be constructed directly. The proposed class of couplings has three parameters \((\eta, h_1, h_2)\). In the intermediate energy region \((Q \sim 1 \text{ GeV})\), the proposed coupling has low loop-level and renormalization scheme dependence. We outline here the construction. We recall expansion (2) for the perturbative coupling \( a(Q^2) \), where \( L = \log Q^2/\Lambda^2 \) and \( K_{k\ell} \) are functions of the \( \beta \)-function coefficients. This expansion (sum) is in practice usually truncated in the index \( k \) \((k \leq k_m)\). The proposed coupling is obtained by modifying (the nonanalytic) \( L \)'s to analytic quantities \( L_0 \) and \( L_1 \) that fall faster than any inverse power of \( Q^2 \) at large \( Q^2 \), and by adding to the truncated sum another quantity with such properties:

\[
A^{(k_m)}_1(Q^2) = \sum_{k=1}^{k_m} \sum_{\ell=0}^{k-1} K_{\ell\ell} \left( \frac{\log L_1}{L_0^\ell} \right) + e^{-\eta \sqrt{x}} f(x), \tag{57}
\]

where \( x = Q^2/\Lambda^2 \). The second term is only relevant in the IR region, and the first term (double sum) plays, in the UV region, the role of the perturbative coupling. \( L_0 \) and \( L_1 \) are analytic and chosen aiming at a low \( k_m \)-dependence in the IR region.

\[
\frac{1}{L_i} = \frac{1}{L} + \frac{e^{\nu_1(1-\sqrt{x})}}{1-x} g_i(x), \quad \nu_1 > 0, \quad i = 0, 1. \tag{58}
\]

Functions \( g_i(x) \) are chosen in simple meromorphic form

\[
g_0(x) = \frac{2x}{(1+\nu_0)+x(1-\nu_0)}, \quad 0 < \nu_0 < 1; \tag{59}
g_1(x) = \frac{de^{-\nu_1}+x(d+1-de^{-\nu_1})}{d+x}, \quad d > 0, \tag{60}
\]

with the constants fixed at typical values \( \nu_0 = 1/2 \) and \( \nu_1 = d = 2 \). The additional exponential term in \( 57 \) is chosen in a similar meromorphic form

\[
e^{-\eta \sqrt{x}} f(x) = h_1 + h_2 \frac{1}{(1+x/2)^2} e^{-\eta \sqrt{x}}, \tag{61}
\]

Results for \( A_1, A_2 \) and \( A_3 \), for specific typical values of parameters \( \eta, h_1 \) and \( h_2 \), are shown in Fig. 9. Couplings \( A_2 \) and \( A_3 \) are constructed via \( A_2 \) and \( A_3 \), according to the procedure described in Sec. [18] Eqs. (26).

A general remark: if \( A_1(Q^2) \) differs from the perturbative \( a(Q^2) \) by less than any negative power of \( Q^2 \) at large \( Q^2 \) \((\gg \Lambda^2)\), then the same is true for the difference between any \( \tilde{A}_k(Q^2) \) and \( \tilde{A}_k(Q^2) \) \((k = 2, 3, \ldots)\).

### VIII. SUMMARY

Various analytic (anQCD) models, i.e., analytic couplings \( A_1(Q^2) \), were reviewed, including some of those beyond the minimal analytization (MA) procedure.

Analytization of the higher powers \( a^n \to A_n \) was considered; an RGE-motivated approach, which is applicable to any model of analytic \( A_1 \), was described. Analytization of noninteger powers \( a^n \) in MA model was outlined.
Evaluation methods for space-like and time-like observables in anQCD models were reviewed. A large-$\beta_0$-motivated expansion of space-like inclusive observables is proposed, with the resummed leading-$\beta_0$ part; on its basis, an evaluation of such observables in anQCD models is proposed: truncated analytic series (TAS). Several evaluated observables in various anQCD models were compared to the experimental data. We recall that evaluated expressions for space-like observables in anQCD respect the physical analyticity requirement even at low energy, in contrast to those in perturbative QCD (pQCD).

Finally, specific classes of analytic couplings $\mathcal{A}_1(Q^2)$ which preserve the OPE-ITEP philosophy were discussed, i.e., at high $Q^2$ they approach the pQCD coupling faster than any inverse power of $Q^2$. Such analytic couplings should eventually enable us to use the OPE approach in anQCD models.

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