DESCENT POLYNOMIALS FOR $k$ BUBBLE-SORTABLE PERMUTATIONS OF TYPE B

MATTHEW HYATT

Abstract. Motivated by the work of Chung, Claesson, Dukes, and Graham in [5], we define a natural type B analog of the classic bubble sort, and use it to define a type B analog of the maximum drop statistic. We enumerate (by explicit, recursive, and generating function formulas) signed permutations with $r$ type B descents and type B maximum drop at most $k$. We also find a connection between these signed permutations and certain 2-colored juggling sequences.

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1. Introduction

Let $S_n$ denote the group of permutations on the set $[n] := \{1, 2, \ldots, n\}$. An adjacent transposition $\tau_i \in S_n$, is the permutation $(i, i+1)$ written in cycle notation. Given any permutation $\pi \in S_n$ written in one-line notation, i.e. $\pi = \pi(1), \pi(2), \ldots, \pi(n)$, multiplying $\pi$ by $\tau_i$ on the right has the effect of switching $\pi(i)$ and $\pi(i+1)$. For example if

$$\pi = 3, 4, 1, 5, 2,$$

then $\pi \tau_2 = 3, 1, 4, 5, 2$.

Any permutation can be expressed as a product of adjacent transpositions, so the set $\{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$ generates $S_n$. Moreover, $S_n$ is a Coxeter group and is also called the type A Coxeter group (see also [2]). Given $\pi \in S_n$, $\pi = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_k}$ and $k$ is minimal among all such expressions, then $k$ is called the Coxeter length of $\pi$ and we write $l(\pi) = k$.

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The (right) descent set of $\pi \in S_n$, denoted $\text{DES}_A(\pi)$, is defined by

$$\text{DES}_A(\pi) := \{i \in [n-1] : l(\pi \tau_i) < l(\pi)\}.$$ 

It is well known that the descent set is also given by

$$\text{DES}_A(\pi) = \{i \in [n-1] : \pi(i) > \pi(i+1)\}.$$ 

The type A descent number of a permutation $\pi \in S_n$, denoted $\text{des}_A(\pi)$, is the cardinality of the descent set of $\pi$, i.e.

$$\text{des}_A(\pi) := |\{i \in [n-1] : \pi(i) > \pi(i+1)\}|.$$ 

For example if

$$\pi = 4, 1, 3, 5, 2,$$

then $\text{DES}_A(\pi) = \{1, 4\}$, and $\text{des}_A(\pi) = 2$.

The classic bubble sort, which we will denote by $\text{bubble}_A$, can be described completely in terms of generators (adjacent transpositions) and descents. Basically, one reads a permutation from left to right, and applies an adjacent transposition at each step where there is a descent. More precisely, for $i \in [n-1]$ define a map $s_i : S_n \to S_n$ by $s_i(\pi) = \pi \tau_i$ if $i \in \text{DES}_A(\pi)$, and $s_i(\pi) = \pi$ otherwise. Then

$$\text{bubble}_A(\pi) = s_{n-1} \circ s_{n-2} \circ \cdots \circ s_1(\pi)$$

For example if $\pi = 3, 1, 4, 5, 2$ then we can find $\text{bubble}_A(\pi)$ in 4 steps

- Step 1: $3, 1, 4, 5, 2 \mapsto 1, 3, 4, 5, 2$
- Step 2: $1, 3, 4, 5, 2 \mapsto 1, 3, 4, 5, 2$
- Step 3: $1, 3, 4, 5, 2 \mapsto 1, 3, 4, 5, 2$
- Step 4: $1, 3, 4, 5, 2 \mapsto 1, 3, 4, 2, 5$

thus $\text{bubble}_A(\pi) = 1, 3, 4, 2, 5$.

There is a natural way to extend this sorting algorithm to other Coxeter groups, in particular the type B Coxeter group (see also [1], [3], [10]). We use the convention that the type B Coxeter group $B_n$ consists of signed permutations (see also [2]). A signed permutation $\pi$ is a bijection on the integers $[-n, n]$, with the condition that $\pi(-i) = -\pi(i)$. This forces $\pi(0) = 0$, so $\pi$ is determined by its image on $[n]$, and we write signed permutations in one-line notation as $\pi = \pi(1), \pi(2), \ldots, \pi(n)$. Given $\pi \in B_n$, we use $|\pi|$ to denote the permutation $|\pi| := |\pi(1)|, |\pi(2)|, \ldots, |\pi(n)|$. Note that $|\pi| \in S_n$, and $S_n$ is a subgroup of $B_n$.

To realize $B_n$ as a Coxeter group, we need a generating set. Let $\tau_0$ denote the signed permutation $\tau_0 := -1, 2, 3, \ldots, n$. Given any $\pi \in B_n$, $\pi \tau_0$ is obtained from $\pi$ by simply changing the sign of $\pi(1)$. Then $B_n$ is generated by the set $\{\tau_0, \tau_1, \ldots, \tau_{n-1}\}$, where each $\tau_i$ for $i \in [n-1]$ is the adjacent transposition as defined above in the type A case. Define the Coxeter length in the same way as above, which one uses to define the (right) descent set of $\pi \in B_n$, denoted $\text{DES}_B(\pi)$, by

$$\text{DES}_B(\pi) := \{i \in [0, n-1] : l(\pi \tau_i) < l(\pi)\}.$$
And it turns out that
\[ \text{DES}_B(\pi) = \{ i \in [0, n-1] : \pi(i) > \pi(i+1) \}, \]
recalling that \( \pi(0) = 0 \). The \textit{type B descent number} of \( \pi \in B_n \), denoted \( \text{des}_B(\pi) \), is the cardinality of the descent set of \( \pi \), i.e.
\[ \text{des}_B(\pi) := | \{ i \in [0, n-1] : \pi(i) > \pi(i+1) \} |. \]
For example if \( \pi = -3, 4, -1, -5, 2 \), then \( \text{DES}_B(\pi) = \{ 0, 2, 3 \} \), and \( \text{des}_B(\pi) = 3 \).

Since the definition of the classic bubble sort, \( \text{bubble}_A \), was given in terms of generators and descents, one can extend the definition in a very natural way to other Coxeter groups. This paper treats the case of extending the definition to the type B Coxeter group.

**Definition 1.1.** For \( i \in [0, n-1] \) define a map \( s_i : B_n \to B_n \) by \( s_i(\pi) = \pi \tau_i \) if \( i \in \text{DES}_B(\pi) \), and \( s_i(\pi) = \pi \) otherwise. We define the \textit{type B bubble sort}, denoted \( \text{bubble}_B \), by
\[ \text{bubble}_B(\pi) = s_{n-1} \circ s_{n-2} \circ \cdots \circ s_0(\pi). \]

As before, one can think of this as reading a signed permutation from left to right, and applying a generator at each step where there is a descent.

For example if \( \pi = -3, 1, 4, -5, 2 \), then we find \( \text{bubble}_B(\pi) \) in 5 steps
\[
\begin{array}{c|c}
\text{Step 0} & -3, 1, 4, -5, 2 \\
\text{Step 1} & 3, 1, 4, -5, 2 \\
\text{Step 2} & 1, 3, 4, -5, 2 \\
\text{Step 3} & 1, 3, 4, -5, 2 \\
\text{Step 4} & 1, 3, -5, 4, 2 \\
\end{array}
\]
thus \( \text{bubble}_B(\pi) = 1, 3, -5, 2, 4 \).

**Remark 1.2.** Note that we can extend this definition in the obvious way to words \( \pi \) over \( \mathbb{Z} \), provided that \( |\pi| \) is a permutation on some subset of positive integers.

By successively applying \( \text{bubble}_A \) to a permutation, or \( \text{bubble}_B \) to a signed permutation, one eventually obtains the identity permutation \( \text{id} \). Define the \textit{type A bubble sort complexity} of \( \pi \in S_n \) by
\[ \text{bsc}_A(\pi) := \min \{ k : \text{bubble}_A^k(\pi) = \text{id} \}. \]
Define the \textit{type B bubble sort complexity} of \( \pi \in B_n \) by
\[ \text{bsc}_B(\pi) := \min \{ k : \text{bubble}_B^k(\pi) = \text{id} \}. \]
Given \( \pi \in S_n \), one can quickly compute type A bubble sort complexity using a statistic called the maximum drop, which we denote by \( \text{maxdrop}_A(\pi) \).
In [5], Chung, Claesson, Dukes, and Graham give the distribution of descents in $S_n$ with respect to maximum drop. Much of the work in this paper is inspired by, and relies on, results from their work. Define
\begin{equation}
\text{maxdrop}_A(\pi) := \max\{i - \pi(i) : 1 \leq i \leq n\}.
\end{equation}
By induction, one can prove that for all $\pi \in S_n$ we have
\begin{equation}
\text{bsc}_A(\pi) = \text{maxdrop}_A(\pi).
\end{equation}

Our motivation is to find a type B analog of the maximum drop statistic.

**Definition 1.3.** For $\pi \in B_n$, we say $\pi$ has a drop at position $i$ if $\pi(i) < i$. If $\pi$ has a drop at position $i$, then we say the drop size at position $i$ is $\min\{i - \pi(i), i\}$. The type B maximum drop of $\pi$, denoted $\text{maxdrop}_B(\pi)$, is the maximum of all drop sizes occurring in $\pi$. In other words
\begin{equation}
\text{maxdrop}_B(\pi) := \max\left\{\max\{i - \pi(i) : \pi(i) > 0\}, \max\{i : \pi(i) < 0\}\right\}.
\end{equation}
For example $\text{maxdrop}_B(-3, 4, -1, -5, 2) = 4$ since there is a drop of size 4 at position 4. Note there are drops of size 3 at positions 3 and 5, and a drop of size 1 at position 1.

In Proposition 2.3, we show that type B maximum drop has the analogous property to (8), that is for all $\pi \in B_n$ we have
\begin{equation}
\text{bsc}_B(\pi) = \text{maxdrop}_B(\pi).
\end{equation}

Since our work is motivated by the results appearing in [5], we state some of their results below. First define
\[ A_{n,k} := \{\pi \in S_n : \text{maxdrop}_A(\pi) \leq k\}, \]
and
\[ a_{n,k}(r) := |\{\pi \in A_{n,k} : \text{des}(\pi) = r\}|, \]
then define a $k$-maxdrop$_A$-restricted descent polynomial
\[ A_{n,k}(x) := \sum_{\pi \in A_{n,k}} x^{\text{des}_A(\pi)} = \sum_{r \geq 0} a_{n,k}(r) x^r. \]
Note that for $k \geq n - 1$, $A_{n,k} = S_n$ and $A_{n,k}(x)$ becomes the type A Eulerian polynomial, denoted $A_n(x)$, defined by
\[ A_n(x) = \sum_{\pi \in S_n} x^{\text{des}_A(\pi)} = \sum_{k \geq 0} \binom{n}{k} x^k. \]
The coefficient $\binom{n}{k}$ is called an Eulerian number. It is the number of permutation in $S_n$ with exactly $k$ descents, and is given explicitly by
\[ \binom{n}{k} = \sum_{i=0}^{k} (-1)^i \binom{n + 1}{i} (k + 1 - i)^n. \]
The $k$-maxdrop$_A$-restricted descent polynomials $A_{n,k}$ satisfy the following recurrence, which is equivalent to a generating function formula.

**Theorem 1.4** ([5, Theorem 1]). For $n \geq 0$,

$$A_{n+k+1,k}(x) = \sum_{i=1}^{k+1} \binom{k+1}{i}(x-1)^{i-1}A_{n+k+1-i,k}(x),$$

with initial conditions $A_{i,k}(x) = A_i(x)$ for $0 \leq i \leq k$.

Consequently,

$$\sum_{n \geq 0} A_{n,k}(x)z^n = \left(1 + \sum_{t=1}^{k} \left(A_t(x) - \sum_{i=1}^{t} \binom{k+1}{i}(x-1)^{i-1}A_{t-i}(x)\right)z^t\right)^{1 - \sum_{i=1}^{k+1} \binom{k+1}{i}z^i(x-1)^{i-1}}.$$

The following theorem gives an explicit formula for $a_{n,k}(r)$.

**Theorem 1.5** ([5, Theorem 2]). We have $A_{n,k}(x) = \sum_{d} \beta_k((k+1)d)x^d$, where

$$\sum_j \beta_k(j)u^j = P_k(u) \left(1 - \frac{u^{k+1}}{1-u}\right)^{n-k},$$

with

$$P_k(u) = \sum_{j=0}^{k} A_{k-j}(u^{k+1})(u^{k+1} - 1)^j \sum_{i=j}^{k} \binom{i}{j}u^{-i}.$$

In other words, the coefficients $a_{n,k}(r)$ of the polynomial $A_{n,k}(x)$, have the remarkable property that they are given by every $(k+1)^{st}$ coefficient in the polynomial $P_k(u)(1 + u + u^2 + \cdots + u^k)^{n-k}$.

For example setting $n = 4$ and $k = 2$ we have

$$P_2(u)(1 + u + u^2 + \cdots + u^k)^{4-2} = (1 + u + 2u^2 + u^3 + u^4)(1 + u + u^2)^2$$

$$= 1 + 3u + 7u^2 + 10u^3 + 12u^4 + 10u^5 + 7u^6 + 3u^7 + u^8.$$ So the coefficients of $A_{4,2}(x)$ are given by every third coefficient in the above polynomial, that is

$$A_{4,2}(x) = 1 + 10x + 7x^2.$$

We provide analogous enumerations for signed permutations in $B_n$ with $r$ descents and maximum drop size less than or equal to $k$. First we define

$$\mathcal{B}_{n,k} := \{\pi \in B_n : \text{maxdrop}_B(\pi) \leq k\},$$

and

$$b_{n,k}(r) := |\{\pi \in \mathcal{B}_{n,k} : \text{des}_B(\pi) = r\}|,$$

then define a $k$-maxdrop$_B$-restricted descent polynomial

$$B_{n,k}(x) := \sum_{\pi \in \mathcal{B}_{n,k}} x^{\text{des}_B(\pi)} = \sum_{r \geq 0} b_{n,k}(r)x^r.$$
Note that for \( k \geq n \), \( B_{n,k} = B_n \) and \( B_{n,k}(x) \) becomes the type B Eulerian polynomial, denoted \( B_n(x) \), defined by

\[
B_n(x) = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)} = \sum_{k=0}^{n-1} \binom{n}{k} B_k x^k.
\]

The coefficient \( \binom{n}{k} B \) is called a type B Eulerian number. It is the number of signed permutations in \( B_n \) with exactly \( k \) type B descents, and is given explicitly by

\[
\binom{n}{k} B = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (2k + 1 - 2i)^n.
\]

It turns out that the \( k \)-maxdrop\(_B\)-restricted descent polynomials \( B_{n,k}(x) \), satisfy the same recurrence as their type A counterparts (compare with Theorem 1.4).

**Theorem 1.6.** For \( n \geq 0 \),

\[
B_{n+k+1,k}(x) = \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} B_{n+k+1-i,k}(x),
\]

with initial conditions \( B_{i,k}(x) = B_i(x) \) for \( 0 \leq i \leq k \).

Consequently,

\[
\sum_{n=0}^{\infty} B_{n,k}(x) z^n = \frac{1 + \sum_{t=1}^{k} (B_t(x) - \sum_{i=1}^{t} \binom{k+1}{i} (x-1)^{i-1} B_{t-i}(x)) z^t}{1 - \sum_{i=1}^{k+1} \binom{k+1}{i} z^i (x-1)^{i-1}}.
\]

Similar to the polynomial \( P_k(u) \) defined above (see Theorem 1.5), we define

\[
Q_k(u) = \sum_{j=0}^{k} B_{k-j}(u^{k+1})(u^{k+1}-1)^j \sum_{i=j}^{k} \binom{i}{j} u^{-i}.
\]

It also turns out that the coefficients \( b_{n,k}(r) \) of the polynomial \( B_{n,k}(x) \), have the analogous remarkable property that they are given by every \((k+1)^{st}\) coefficient in the expression

\[
Q_k(u)(1 + u + u^2 + \ldots + u^k)^{n-k}.
\]

For example setting \( n = 4 \) and \( k = 2 \) we have

\[
Q_2(u)(1 + u + u^2)^{4-2} = (1 + 4u + 6u^2 + 6u^3 + 4u^4 + 2u^5 + u^6)(1 + u + u^2)^2
\]

\[
= 1 + 6u + 17u^2 + 32u^3 + 43u^4 + 44u^5 + 35u^6 + 22u^7 + 11u^8 + 4u^9 + u^{10}.
\]

The coefficients of \( B_{4,2}(x) \) are in fact given by every third coefficient above, that is

\[
B_{4,2}(x) = 1 + 32x + 35x^2 + 4x^3.
\]

We state this precisely this in the following theorem.
Theorem 1.7. We have $B_{n,k}(x) = \sum_d \gamma_k((k+1)d)x^d$, where

$$\sum_j \gamma_k(j)u^j = Q_k(u) \left( \frac{1 - u^{k+1}}{1 - u} \right)^{n-k},$$

with

$$Q_k(u) = \sum_{j=0}^k B_{k-j}(u^{k+1})(u^{k+1} - 1)^j \sum_{i=j}^k \binom{i}{j} u^{-i}.$$

While our work here is motivated by the work of Chung, Claesson, Dukes, and Graham in [5], that paper was motivated by its connection to a paper of Chung and Graham [6] that deals with juggling sequences. We explain this connection in section 4 and show how our work can also be interpreted in terms of certain juggling sequences.

2. The type B maxdrop statistic

We introduced the type B maxdrop statistic in Definition 1.3 and claimed that it is equal to our type B bubble sort complexity (see (9)). Here we prove this claim. The first step is to derive an alternate description of the type B bubble sort.

Definition 2.1. Given any signed permutation $\pi = \pi(1), \pi(2), \ldots, \pi(n)$, let $\sigma := |\pi(1)|, |\pi(2)|, \ldots, |\pi(n)| \in B_n$. There exists a unique index $j$ such that $\sigma(j) \geq \sigma(i)$ for $i \in [n]$. Split $\pi$ into blocks $L$ and $R$ to the left and right of $\pi(j)$, i.e. $\pi = L\pi(j)R$. Define an operator $O$ which acts recursively by

$$O(L\pi(j)R) = O(LR|\pi(j)|),$$

where $O(\emptyset) = \emptyset$.

Proposition 2.2. For any signed permutation $\pi$, we have $O(\pi) = \text{bubble}_B(\pi)$. Consequently, if $\pi = L\pi(j)R$ as in Definition 2.1 then

$$\text{bubble}_B(L\pi(j)R) = \text{bubble}_B(LR|\pi(j)|).$$

Proof. Let $\pi \in B_n$, and induct on $n$. The base case is obvious.

Now let $n \geq 2$ and assume $O(\alpha) = \text{bubble}_B(\alpha)$ whenever $\alpha$ is a word over $Z$ with length less than $n$ (assume $|\alpha|$ is a permutation on some subset of positive integers, see also Remark 1.2). Let $\pi \in B_n$ and let $\pi = L\pi(j)R$ as in Definition 2.1. Then

$$O(\pi) = O(LR|\pi(j)|) = \text{bubble}_B(LR|\pi(j)|).$$

If $j = 1$, then $L = \emptyset$ and

$$\text{bubble}_B(\pi) = \text{bubble}_B(\pi(1)R) = R|\pi(1)| = O(\pi).$$

This is because $\text{bubble}_B$ will make $\pi(1)$ positive (if it’s not already) in the 0th step. Then $|\pi(1)| > \pi(i)$ for $2 \leq i \leq n$ and each successive step in $\text{bubble}_B$ will apply an adjacent transposition, moving $|\pi(1)|$ to the last position.
If $j > 1$, then $\pi(j)$ must be positive, and it is also the largest letter in $\pi$. Also, $L$ will be a word of length less than $n$. It is clear then that

$$\text{bubble}_B(\pi) = \text{bubble}_B(L\pi(j)R) = \text{bubble}_B(L)R|\pi(j)| = O(L)R|\pi(j)| = O(\pi).$$

□

Next we use this alternate description of $\text{bubble}_B$ to show that $\maxdrop_B$ has the desired property of being equal to $\text{bsc}_B$.

**Proposition 2.3.** For all signed permutations $\pi$ other than $\text{id}$ we have

(11) \[ \maxdrop_B(\text{bubble}_B(\pi)) = \maxdrop_B(\pi) - 1. \]

For all signed permutations $\pi$ we have

(12) \[ \maxdrop_B(\pi) = \text{bsc}_B(\pi). \]

**Proof.** First note that (12) follows easily from (11), since $\text{id}$ is the only signed permutation whose type B maximum drop is equal to zero. To prove (11), induct on the length of $\pi$. The base case $n = 1$ is trivial. Now assume $\pi \in B_n$ and

(13) \[ \maxdrop_B(\text{bubble}_B(\pi)) = \maxdrop_B(\pi) - 1 \]

for all signed permutations $\sigma$ of length less than $n$. As in Proposition 2.2, let

$$\text{bubble}_B(\pi) = \text{bubble}_B(L\pi(j)R) = \text{bubble}_B(L)R|\pi(j)|.$$

First we show that $\maxdrop_B(\text{bubble}_B(\pi)) \leq \maxdrop_B(\pi) - 1$. Using (13) on $\text{bubble}_B(L)$ and the fact that $R$ has been shifted to the left, we see that the largest drop size of $\text{bubble}_B(\pi)$ among letters from $L$ or $R$, is exactly one less than the largest drop size of $\pi$ among letters from $L$ or $R$. It remains to check that the drop size at position $n$ in $\text{bubble}_B(\pi)$ is no bigger than $\maxdrop_B(\pi) - 1$. Indeed, the description of $\text{bubble}_B$ given in Proposition 2.2 implies that the $n - |\pi(j)|$ letters in $\pi$ which have absolute value larger than $|\pi(j)|$, must be negative and cannot appear in position 1. We obtain a lower bound on $\maxdrop_B(\pi)$ by assuming these negative letters must occur as far to the left as possible in $\pi$, so

(14) \[ \maxdrop_B(\pi) \geq n - |\pi(j)| + 1. \]

But the drop size at position $n$ in $\text{bubble}_B(\pi)$ is precisely $n - |\pi(j)|$, which is less than or equal to $\maxdrop_B(\pi) - 1$ as desired.

Next we show that $\maxdrop_B(\text{bubble}_B(\pi)) \geq \maxdrop_B(\pi) - 1$. If the maximum drop size of $\pi$ does not occur at position $j$, then the maximum drop size is contributed by a letter from $L$ or $R$. Again using (13) on $\text{bubble}_B(L)$ and the fact that $R$ has been shifted to the left, this implies $\maxdrop_B(\text{bubble}_B(\pi)) = \maxdrop_B(\pi) - 1$. If $\pi$ does have a maximum drop size at position $j$, then we claim that $\maxdrop_B(\pi) = 1$, in which case we are done. Indeed suppose $\pi$ has a drop of maximum size at position $j$ and $2 \leq j \leq n$. Then $|\pi(j)| = \pi(j)$ and

$$\maxdrop_B(\pi) = j - \pi(j).$$
But then we have a contradiction since
\[
\text{maxdrop}_B(\pi) = j - \pi_j \leq n - |\pi(j)| < \text{maxdrop}_B(\pi),
\]
where the last inequality follows from (13). On the other hand, if \( j = 1 \) and \( \pi \) has a maximum drop at position \( j \), then \( \text{maxdrop}_B(\pi) = 1 \).
\[\square\]

3. Type B maxdrop-restricted descent polynomials

In this section we provide the proofs of Theorem 1.6 and Theorem 1.7. In particular, we prove Theorem 1.6 by modifying the proof of Theorem 1.4 found in [5]. We then explain how Theorem 1.7 is a consequence of Theorem 1.6 and Theorem 1.5 from [5].

Suppose we have a finite set \( C = \{c_1, c_2, \ldots, c_n\} \subset \mathbb{N} \) with \( c_1 < c_2 < \cdots < c_n \), and a permutation \( \pi \) on \( C \). The standardization of \( \pi \), denoted \( \text{st}(\pi) \), is the permutation on \( S_n \) obtained from \( \pi \) by replacing \( c_i \) with \( i \). For example, \( \text{st}(4,5,2,9,7) = 2,3,1,5,4 \). We will call \( \pi \) a signed permutation on the set \( C \), if \( \pi \) is a word over \( \mathbb{Z} \) and \( |\pi| \) is permutation on \( C \). We define the signed standardization of a signed permutation, denoted \( \text{sst}(\pi) \), by
\[
\text{sst}(\pi)(i) = \begin{cases} 
\text{st}(|\pi|)(i) & \text{if } \pi(i) > 0 \\
-\text{st}(|\pi|)(i) & \text{if } \pi(i) < 0 
\end{cases}
\]

In other words, \( \text{sst}(\pi) \) is the standardization of \( |\pi| = |\pi(1)|, |\pi(2)|, \ldots, |\pi(n)| \), but then put the minus signs back in the same positions. For example \( \text{sst}(-5,4,-2,9,7) = -3,2,-1,5,4 \). If the set \( C \) is fixed, then the inverse of \( \text{sst} \), denoted \( \text{sst}_C^{-1} \), is well-defined. For example if \( C = \{2,4,5,7,9\} \) and \( \pi = -3,2,-1,5,4 \), then \( \text{sst}_C^{-1}(\pi) = -5,4,-2,9,7 \). If we use [3] to define the type B descent set for any word over \( \mathbb{Z} \), then it is clear that \( \text{sst} \) and \( \text{sst}_C^{-1} \) preserve type B descent set. For example \( \text{DES}_B(-5,4,-2,9,7) = \{0,2,4\} = \text{DES}_B(-3,2,-1,5,4) \).

For any \( S \subseteq [0,n-1] \), define
\[
B_{n,k}(S) := \{ \pi \in B_{n,k} : \text{DES}_B(\pi) \supseteq S \},
\]
\[
t_n(S) := \max\{i \in \mathbb{N} : [n-i,n-1] \subseteq S \},
\]
with \( t_n(S) = 0 \) if \( n-1 \notin S \). Given any \( \pi \in B_{n,k}(S) \) where \( k < n \), define a map \( f \) by
\[
f(\pi) = (\alpha, X), \quad \text{where}
\]
\[
\alpha = \text{sst}(\pi(1), \pi(2), \ldots, \pi(n-i-1)), \quad X = \{\pi(n-i), \pi(n-i+1), \ldots, \pi(n)\}
\]
with \( i = t_n(S) \). Note that since \( k < n \), \( \pi(n) \) must be positive. Moreover, since \( [n-i,n-1] \subseteq S \subseteq \text{DES}_B(\pi) \), we know that \( X \) is a set of positive integers.

For example, let \( S = \{0,2,6,7\} \) and choose \( \pi = -6,2,-1,-3,8,7,5,4 \in B_{8,5}(S) \). Note that \( \text{DES}_B(\pi) = \{0,2,3,5,6,7\} \supset S \), and \( \text{maxdrop}_B(\pi) = 4 \leq k = 5 \). Since \( i = t_8(S) = 2 \), we have \( f(\pi) = (\alpha, X) \) where \( \alpha = \text{sst}(-6,2,-1,-3,8) = -4,2,-1,-3,5 \) and \( X = \{4,5,7\} \).
Lemma 3.1. For $0 \leq k \leq n - 1$ we have
\[ f : B_{n,k}(S) \to B_{n-i-1,k}(S \cap [0, n-i-2]) \times \binom{[n-k,n]}{i+1} \]
where $i = t_n(S)$, and $\binom{X}{m}$ denotes the set of $m$-element subsets of $X$.

Proof. Let $\pi \in B_{n,k}(S)$, and let $f(\pi) = (\alpha, X)$. Let $i = t_n(S)$, thus $[n-i,n-1] \subseteq \text{DES}_B(\pi)$, and
\[ \pi(n-i) > \pi(n-i+1) > \cdots > \pi(n) \]
Since maxdrop$_B(\pi) \leq k < n$, we must have that $\pi(n)$ is a positive integer, and $\pi(n) \geq n - k$. Therefore $X$ is an $(i+1)$-element subset of $[n-k,n]$.

Clearly $\alpha \in B_{n-i-1}$, and we also need to show that maxdrop$_B(\alpha) \leq k$. Let $j$ be an index such that $1 \leq j \leq n-i-1$. If $\pi(j)$ is negative, then $\pi$ has a drop of size $j$ at position $j$. After applying signed standardization to $\alpha$, we see that $\alpha(j)$ is also negative, so $\alpha$ has a drop of size $j$ at position $j$. Since the drop size is unchanged, it must be less than or equal to $k$.

If $0 < \pi(j) < \pi(n)$, then $\alpha(j) = \pi(j)$ and again the drop size is unchanged.

If $\pi(j) > \pi(n)$, then $\alpha(j)$ will be a smaller than $\pi(j)$, but no smaller than $\pi(n)$. Since $\pi \in B_{n,k}$, we have $\alpha(j) \geq \pi(n) \geq n-k$. The corresponding drop size in $\alpha$ must satisfy
\[ j - \alpha(j) \leq n - (n-k) = k. \]
In all three cases, we have maxdrop$_B(\alpha) \leq k$.

Since signed standardization preserves type B descent sets, the type B descent set of $\alpha$ still contains the part of $S$ that is between 0 and $n-i-2$, i.e. $\alpha \in B_{n-i-1,k}(S \cap [0, n-i-2])$.

\[ \square \]

Our goal is to show that $f$ is a bijection. To accomplish this, we describe the inverse map of $f$. Let $\pi \in B_{n-i-1,k}(T)$ where $T \subseteq [0, n-i-2]$. Let $X = \{x_1, \ldots, x_{i+1}\}$ be an $(i+1)$-element subset of $[n-k,n]$, where $x_1 \leq x_2 \leq \cdots \leq x_{i+1}$. Define a map $g$ (which we will show is the inverse of $f$) by
\[ g(\pi, X) = \text{sst}^{-1}_C(\pi) \ast (x_{i+1}, x_i, \ldots, x_1) \]
where $C = [n] \setminus X$, and $\ast$ denotes concatenation.

For example, let $T = \{2\}$ and let $\pi = -3, 1, -4, 2, 5 \in B_{5,4}(T)$. Note that maxdrop$_B(\pi) = 3 \leq k = 4$ and $\text{DES}_B(\pi) = \{0,2\} \supset T$. Chose $i = 2$, then $5 = n-i-1$ implies that $n = 8$, and let $X = \{4,5,7\} \in [n-k,n] = [4,8]$. Then $C = [n] \setminus X = \{1,2,3,6,8\}$, and
\[ g(\pi, X) = \text{sst}^{-1}_C(-3, 1, -4, 2, 5) \ast (7, 5, 4) = -3, 1, -6, 2, 8, 7, 5, 4 \in B_{8,4}(T \cup [6,7]). \]

Lemma 3.2. For $0 \leq k \leq n - 1$ we have
\[ g : B_{n-i-1,k}(T) \times \binom{[n-k,n]}{i+1} \to B_{n,k}(T \cup [n-i,n-1]). \]
Proof. Let $\alpha = g(\pi, X)$. Clearly $\alpha \in B_n$, so next we will show that $\max\text{drop}_B(\alpha) \leq k$. Let $j$ be an index such that $1 \leq j \leq n-i-1$. If $\alpha(j)$ is negative, then it must be that $\pi(j)$ is also negative. Thus the drop size at position $j$ of $\alpha$, is the same as the drop size at position $j$ of $\pi$, hence less than or equal to $k$. If $\alpha(j) > 0$, then $\alpha(j) \geq \pi(j)$. If there is a drop at position $j$ of alpha, then the drop size satisfies

$$j - \alpha(j) \leq j - \pi(j) \leq k.$$ 

Now let $j$ be an index such that $n-i \leq j \leq n$. Since $\alpha(n-i) > \alpha(n-i+1) > \cdots > \alpha(n)$, the largest drop occurs at position $n$. But $\alpha(n) \in [n-k,n]$, so

$$n - \alpha(n) \leq n - (n-k) = k,$$

and $\alpha \in B_{n,k}$.

Since $\text{sst}_{\alpha}^{-1}$ preserves type B descent sets, and since $\alpha(n-i) > \alpha(n-i+1) > \cdots > \alpha(n)$, it follows that $\text{des}_B(\alpha) \supset (T \cup [n-i,n-1])$, hence $\alpha \in B_{n,k}(T \cup [n-i,n-1])$ as desired.

\[\square\]

**Lemma 3.3.** The map $f$ is a bijection, and its inverse is the map $g$.

**Proof.** Let

$$(\pi, X) \in B_{n-i-1,k}(T) \times \binom{[n-k,n]}{i+1}$$

where $T \subseteq [n-i-2]$ and $X = \{x_1 \leq \cdots \leq x_{i+1}\}$. Let $\alpha = g(\pi, X) \in B_{n,k}(T \cup [n-i,n-1])$, so

$$\alpha = \text{sst}_{\pi}^{-1}(\pi) \ast (x_{i+1}, x_i, \ldots, x_1)$$

where $C = [n] \setminus X$. To find $f(\alpha)$, we first compute

$$t_n(T \cup [n-i,n-1]) = \max\{j \in \mathbb{N} : [n-j,n-1] \subseteq (T \cup [n-i,n-1]) \} = i.$$ 

So

$$f(g(\pi, X)) = f(\alpha) = (\text{sst}(\text{sst}_{\pi}^{-1}(\pi)), \{x_{i+1}, x_i, \ldots, x_1\}) = (\pi, X).$$

Now given $\pi \in B_{n,k}(S)$, let $f(\pi) = (\alpha, X)$ where

$$(\alpha, X) = (\text{sst}(\pi(1), \pi(2), \ldots, \pi(n-i-1)), \{\pi(n-i), \pi(n-i+1), \ldots, \pi(n)\}),$$

where $i = t_n(S)$. Since this means that $[n-i,n-1] \subseteq S \subseteq \text{DES}_B(\pi)$, we know that

$$\pi(n) < \pi(n-1) < \cdots < \pi(n-i+1).$$

Thus

$$g(f(\pi)) = g(\alpha, X)$$

$$= \text{sst}_{[n]\setminus X}^{-1}(\text{sst}(\pi(1), \pi(2), \ldots, \pi(n-i-1))) \ast \pi(n-i), \pi(n-i+1), \ldots, \pi(n)$$

$$= (\pi(1), \pi(2), \ldots, \pi(n-i-1) \ast \pi(n-i), \pi(n-i+1), \ldots, \pi(n)$$

$$= \pi.$$ 

\[\square\]
Corollary 3.4. Let \( b_{n,k}(S) := |B_{n,k}(S)| \). For \( 0 \leq k \leq n - 1 \) we have

\[
b_{n,k}(S) = b_{n-i-1,k}(S \cap [0, n - i - 2]) \binom{k + 1}{i + 1},
\]

where \( i = t_n(S) \).

Theorem 3.5. For \( n \geq 0 \),

\[
B_{n,k}(x) = \sum_{j=1}^{k+1} \binom{k + 1}{j} (x - 1)^{j-1} B_{n-j,k}(x),
\]

with initial conditions \( B_{j,k}(x) = B_j(x) \) (the type B Eulerian polynomial) for \( 0 \leq j \leq k \). By replacing \( n \) by \( n + k + 1 \), one obtains the recurrence in Theorem 1.6.

Proof. First we express \( B_{n,k}(x + 1) \) in terms of \( b_{n,k} \) as follows:

\[
B_{n,k}(x + 1) = \sum_{\pi \in B_{n,k}} (x + 1)^{\text{des}_{B}(\pi)}
\]

\[
= \sum_{\pi \in B_{n,k}} \sum_{j=0}^{\text{des}_{B}(\pi)} \binom{\text{des}_{B}(\pi)}{j} x^j
\]

\[
= \sum_{\pi \in B_{n,k}} \sum_{S \subseteq \text{DES}_{B}(\pi)} x^{|S|} \sum_{\pi \in B_{n,k}(S)} 1
\]

\[
= \sum_{S \subseteq [0,n-1]} b_{n,k}(S) x^{|S|}
\]

Now take Corollary 3.4, multiply both sides by \( x^{|S|} \) and sum over all \( S \subseteq [0, n - 1] \):

\[
B_{n,k}(x + 1) = \sum_{S \subseteq [0,n-1]} x^{|S|} b_{n-t_n(S)-1,k}(S \cap [0, n - t_n(S) - 2]) \binom{k + 1}{t_n(S) + 1}
\]

\[
= \sum_{i \geq 0} \sum_{S \subseteq [0,n-1]} \sum_{t_n(S) = i} x^{|S|} b_{n-i-1,k}(S \cap [0, n - i - 2]) \binom{k + 1}{i + 1}
\]

\[
= \sum_{i \geq 0} \binom{k + 1}{i + 1} x^i \sum_{S \subseteq [0,n-1]} \sum_{t_n(S) = i} x^{|S|-i} b_{n-i-1,k}(S \cap [0, n - i - 2])
\]

Recall that if \( t_n(S) = i \), then \( S \supseteq [n - i, n - 1] \) and \( n - i - 1 \notin S \). Therefore each such \( S \) can be expressed as \( S = T \cup [n-i, n-1] \) for some \( T \subseteq [0, n-i-2] \).
Thus
\[ B_{n,k}(x+1) = \sum_{i \geq 0} \binom{k+1}{i+1} x^i \sum_{T \subseteq [0,n-i-2]} x^{|T|} b_{n-i-1,k}(T) \]
\[ = \sum_{i \geq 0} \binom{k+1}{i+1} x^i B_{n-i-1,k}(x+1) \]
\[ = \sum_{i \geq 1} \binom{k+1}{i} x^{i-1} B_{n-i,k}(x+1) \]

Now we simply replace \( x \) by \( x-1 \), and note that \( \binom{k+1}{i} = 0 \) for \( i > k+1 \). \( \square \)

The following corollary also appears in Theorem 1.6.

**Corollary 3.6.** We have
\[ \sum_{n \geq 0} B_{n,k}(x)z^n = \frac{1 + \sum_{t=1}^{k} \left( B_{t}(x) - \sum_{i=1}^{t} \binom{k+1}{i} (x-1)^{i-1} B_{t-i}(x) \right) z^i}{1 - \sum_{i=1}^{k+1} \binom{k+1}{i} z^i (x-1)^{i-1}}. \]

**Proof.** Let \( I = \sum_{n \geq 0} B_{n,k}(x)z^n \),
then
\[ I = \sum_{n=0}^{k} B_{n}(x)z^n + \sum_{n \geq k+1} B_{n,k}(x)z^n \]
\[ = \sum_{n=0}^{k} B_{n}(x)z^n + \sum_{n \geq k+1} \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} z^i \sum_{n \geq k+1} B_{n-i,k}(x)z^{n-i} \]
\[ = \sum_{n=0}^{k} B_{n}(x)z^n + \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} z^i \sum_{n \geq k+1} B_{n-i,k}(x)z^{n-i} \]
\[ = \sum_{n=0}^{k} B_{n}(x)z^n + \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} z^i \left( I - \sum_{j=0}^{k-i} B_{j}(x)z^{j} \right) \].

Next we bring the term containing \( I \) over to the left. If we set
\[ D = 1 - \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} z^i, \]
then
\[
ID = \sum_{n=0}^{k} B_n(x)z^n - \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} z^i \sum_{j=0}^{k-i} B_j(x)z^j
\]
\[
= 1 + \sum_{n=1}^{k} B_n(x)z^n - \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} B_{i-1}(x)z^i
\]
\[
= 1 + \sum_{n=1}^{k} B_n(x)z^n - \sum_{i=1}^{k} \sum_{t=1}^{i} \binom{k+1}{i} (x-1)^{i-1} B_{t-1}(x)z^t
\]
\[
= 1 + \sum_{t=1}^{k} \left( B_t(x) - \sum_{i=1}^{t} \binom{k+1}{i} (x-1)^{i-1} B_{t-i}(x) \right) z^t.
\]

And now divide by D. □

In comparing the recurrence found in our Theorem 1.4 with the recurrence from Theorem 1.4 from [5], we see that \( B_{n,k}(x) \) satisfies the same recurrence as \( A_{n,k}(x) \), but with different initial conditions. In [5], the authors use Theorem 1.4 to obtain an explicit formula for \( A_{n,k}(x) \) (see Theorem 1.5), but after what they describe as a “fierce battle” with these polynomials. Fortunately for us, a careful check of their proof of Theorem 1.5 shows that it depends only on the recurrence and the initial conditions. To be more specific, we re-state Theorem 1.5 in generality as the following Lemma.

**Lemma 3.7** ([5, Theorem 2]). Fix \( k \in \mathbb{N} \). For \( 0 \leq i \leq k \), let \( \{ C_{i,k}(x) \} \) be any collection of polynomials. For \( n \geq 0 \), define \( C_{n+k+1,k}(x) \) recursively by

\[
C_{n+k+1,k}(x) := \sum_{i=1}^{k+1} \binom{k+1}{i} (x-1)^{i-1} C_{n+k+1-i,k}(x).
\]

Then
\[
C_{n,k}(x) = \sum_{d \geq 0} \beta_k((k+1)d)x^d,
\]
where
\[
\sum_{j \geq 0} \beta_k(j)u^j = O_k(u) \left( \frac{1-u^{k+1}}{1-u} \right)^{n-k},
\]
with
\[
O_k(u) = \sum_{j=0}^{k} C_{k-j,k}(u^{k+1})(u^{k+1} - 1)^j \sum_{i=j}^{k} \binom{i}{j} u^{-i}.
\]
**Proof of Theorem 1.7.** From Theorem 1.6, $B_{n,k}(x)$ satisfies the recurrence (16) in Lemma 3.7, with initial conditions $B_{i,k}(x) = B_i(x)$ for $0 \leq i \leq k$. The result follows immediately from this.

The polynomial $P_k(u)$ appearing in Theorem 1.5, defined by

$$P_k(u) = \sum_{j=0}^{k} A_{k-j}(u^{k+1})(u^{k+1} - 1)^j \sum_{i=j}^{k} \binom{i}{j} u^{-i},$$

has symmetric and unimodal coefficients (Theorem 4). Recall that any sequence $a_0, a_1, \ldots, a_n$ is called symmetric if $a_i = a_{n-i}$ for $0 \leq i \leq n$. A sequence is called unimodal if there exists an index $0 \leq j \leq n$ such that $a_0 \leq a_1 \leq \cdots \leq a_j$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n$. We also say that a polynomial has a certain property, such as symmetric or unimodal, if its sequence of coefficients has that property. The proof that $P_k(u)$ is symmetric and unimodal uses the well known generating function for the type A Eulerian polynomials

$$\sum_{n \geq 0} A_n(u) \frac{z^n}{n!} = \frac{1 - u}{e^{u-1}z - u},$$

which goes back to Euler (see [9]).

We remark here that the expression $Q_k(u)$ appearing in Theorem 1.7, defined by

$$Q_k(u) = \sum_{j=0}^{k} B_{k-j}(u^{k+1})(u^{k+1} - 1)^j \sum_{i=j}^{k} \binom{i}{j} u^{-i},$$

does not have symmetric coefficients. However, using MAPLE we have checked that for $k \leq 7$, $Q_k(u)$ is a unimodal polynomial. We list the sequence of coefficients, starting with the constant term, for the first few $Q_k(u)$ here:

| $k$ | coefficients of $Q_k(u)$ |
|-----|-------------------------|
| 0   | 1                       |
| 1   | 1, 2, 1                 |
| 2   | 1, 4, 6, 6, 4, 2, 1     |
| 3   | 1, 8, 12, 18, 23, 32, 32, 28, 23, 8, 4, 2, 1 |
| 4   | 1, 16, 24, 36, 54, 76, 176, 200, 220, 230, 230, 176, 152, 124, 98, 76, 16, 8, 4, 2, 1 |

**Conjecture 3.8.** For $k \geq 0$, $Q_k(u)$ is a unimodal polynomial.

A sequence $a_0, a_1, \ldots, a_n$ is called log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$. For a sequence with positive coefficients, log-concavity implies unimodality. It is natural to ask if $Q_k(u)$ has this stronger property of log-concavity. However, we have checked that for $3 \leq k \leq 7$, $Q_k(u)$ is not log-concave. For example in $Q_3(u)$ we have $23^2 < 18 \cdot 32$, and $8^2 < 23 \cdot 4$. 
4. Juggling sequences

Since maxdrop$^A_A(\pi) = \text{bsc}_A(\pi)$ for all permutations $\pi \in S_n$, we see that $A_{n,k}$ is the set of all $k$-bubble sortable permutations in $S_n$ (i.e. $\text{bubble}_A^k(\pi) = \text{id}$ for all $\pi \in A_{n,k}$). In [5], the authors also show that there is a bijective correspondence between $A_{n,k}$, and certain juggling sequences of period $n$. In this section, we show that there is a bijection between $B_{n,k}$ and a natural $2$-colored analog of the aforementioned juggling sequences.

Formally, an $n$-periodic juggling sequence, $T = (t_1, t_2, \ldots, t_n)$, is a sequence of $n$ nonnegative integers such that the values $t_i + i \mod n$ are all distinct. One can interpret each $t_i$ as corresponding to a ball being thrown at time $i$ which remains in the air for $t_i$ time units. If $t_i = 0$, then no ball is thrown at time $i$. We also think of the pattern defined by $T$ as being repeated indefinitely, so that the juggling sequence is an infinite sequence of period $n$. The condition that the values $t_i + i \mod n$ are distinct guarantees that no two balls will land at the same time. Furthermore, the average of the $t_i$’s, which is an integer, is the number of balls being juggled (see also [3], [4], [5], [6], [7]).

Next we define the state of a juggling sequence (see also [4], [6]). Suppose the juggler, who has been repeating the $n$-periodic juggling pattern $T$ infinitely many times in the past, stops juggling after the ball corresponding to $t_n$ is thrown. The balls that are in the air will land at various times, and we record the landing schedule as a binary sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_h)$. If $\sigma_i = 1$, then a ball landed $i$ time units after the last ball was thrown. If $\sigma_i = 0$, then no ball landed at that time. So $\sigma_i = 1$ if and only if there is some $j \in \{1, 2, \ldots, n\}$ and some $d > 0$ such that $t_j + j = i + dn$. Since the tail of the sequence $\sigma$ must be all zeros, we let $\tilde{\sigma}$ be a finite sequence where the last entry is the last 1 appearing in $\sigma$. The state of a juggling sequence is the sequence $\tilde{\sigma}$ determined by the landing schedule. There is a particular state which is called the ground state. If $T$ is an $n$-periodic juggling sequence with $k$ balls being juggled, we say that $T$ has the ground state if its state is $\sigma = (k \times \underbrace{1, 1, \ldots, 1})$.

**Theorem 4.1** ([5]). There is a bijection, $\phi$, between permutations in $A_{n,k}$ and juggling sequences which are $n$-periodic, juggle $k$ balls, and have the ground state. Given $\pi \in A_{n,k}$, this map is defined by

$$\phi(\pi) = (t_1, t_2, \ldots, t_n), \text{ where } t_i = k - i + \pi(i).$$

**Definition 4.2.** We now define 2-colored (or signed) juggling sequences. Suppose the juggler may initially select $k$ balls to juggle, and there are two colors to choose from for each ball. We encode this as a signed juggling sequence $T = (t_1, t_2, \ldots, t_n)$ where each $t_i$ is now an integer. The sign of $t_i$ is the color of the ball being thrown at $t_i$, and it remains in the air for $|t_i|$ time units. We still do not allow more than one ball to land at the same time, so we require that the values $|t_i| + i \mod n$ are all distinct. Furthermore,
we do not allow the juggler to change colors once the juggling has started. Therefore, if $T$ has period $n$, we also require that

$$\text{sgn}(t_i) = \text{sgn}(t_j), \quad \text{where } 1 \leq i \leq n \text{ and } j = i + |t_i| \mod n.$$  

(17)

We will also define $|T| = (|t_1|, |t_2|, \ldots, |t_n|)$. Thus $|T|$ is a (1-colored) juggling sequence as described above. We define the state of a 2-colored juggling sequence $T$ to be the state of $|T|$, and we say $T$ juggles $k$ balls if $|T|$ juggles $k$ balls.

Now consider the special case when $T = (t_1, t_2, \ldots, t_n)$ is a 2-colored (or 1-colored) juggling sequence, where $T$ has the ground state, juggles $k$ balls, and $k \leq n$. The fact that $T$ has the ground state implies that $t_1, t_2, \ldots, t_k$ are all nonzero. Informally, we define the landing permutation of $T$, which we will denote by $\tau$, to be the order in which the first $k$ balls thrown will land if we stop juggling at time $n$. More precisely, for $t_i \neq 0$ we define the landing time of $i$, denoted $\text{ltime}(i)$, recursively by

$$\text{ltime}(i) := \begin{cases} 
  i + |t_i| & \text{if } i + |t_i| > n \\
  \text{ltime}(i + |t_i|) & \text{if } i + |t_i| \leq n
\end{cases}.$$  

Then

$$\tau := \text{ltime}(1), \text{ltime}(2), \ldots, \text{ltime}(k).$$

For example, consider the 1-colored juggling sequence $T = (4, 6, 3, 0, 2, 3, 3)$, which juggles 3 balls, has period 7, and has the ground state. We can visualize $T$ with the following diagram:

![Diagram of juggling sequence]

The landing permutation associated to $T$ is $\tau = 3, 1, 2$. This is because for each $i = 1, 2, 3$, if we follow the path of the ball thrown at time $i$, it (eventually) lands at time $n + \tau(i)$.

**Theorem 4.3.** For $k \leq n$, let $J_{n,k}$ be the set of 2-colored juggling sequences, where $T \in J_{n,k}$ if $T$ is $n(k!)$-periodic, juggles $k$ balls, has the ground state, and $|T|$ is $n$-periodic. There is a bijection $\psi : B_{n,k} \to J_{n,k}$.
Proof. First we describe $\psi$, so let $\pi \in B_{n,k}$. Recall that $|\pi| = |\pi(1)|, \ldots, |\pi(n)| \in S_n$. Moreover, we have $|\pi| \in A_{n,k}$ since all drop sizes decrease or stay the same if we remove the signs. Define a 1-colored juggling sequence $S = (s_1, \ldots, s_{n(k!)})$ using the map $\phi$ from Theorem 4.1:

$$S = k! \times \underbrace{\phi(|\pi|) \ast \cdots \ast \phi(|\pi|)}_{\text{k! times}},$$

where $\ast$ denotes concatenation. Note that theorem 4.1 guarantees that $S$ is a juggling sequence which juggles $k$ balls and has the ground state. We construct $\psi(\pi) = T = (t_1, \ldots, t_{n(k!)})$ by assigning signs to each $s_i$ for $1 \leq i \leq n$, so $|T| = S$. First we set

$$\text{(18)} \quad \text{sgn}(t_i) = \text{sgn}(\pi(i)) \text{ for } 1 \leq i \leq k.$$

We then assign signs recursively by

$$\text{(19)} \quad \text{sgn}(t_{i+|t_i|}) = \text{sgn}(t_i) \text{ for } k + 1 \leq i + |t_i| \leq n(k!).$$

Since $|T| = S$, it is clear (using Theorem 4.1) that $T \in J_{n,k}$ if we can show that $T$ satisfies (17). Since $T$ has the ground state, a ball thrown at time $i$ where $1 \leq i \leq k$, lands no sooner than time $k + 1$, thus (18) is well defined. Since no two balls land at the same time, the assignments from (19) are also well defined. Also, the fact that $T$ has the ground state guarantees that each nonzero $s_i$ is assigned a sign. It remains to show that

$$\text{(20)} \quad \text{sgn}(t_i) = \text{sgn}(t_j), \text{ where } 1 \leq i \leq n \text{ and } j = i + |t_i| > n.$$

Again, the fact that $T$ has the ground state ensures that $j \leq n + k$, therefore (20) will be satisfied if we can show that the landing permutation associated to $T$ is the identity permutation. Indeed, let $\tau \in S_k$ be the landing permutation associated to $\phi(|\pi|)$, and let $\tau'$ be the landing permutation associated to $T$. Since $|T|$ is just $k!$ copies of $\phi(|\pi|)$, it follows that

$$\tau'(i) = \tau^{(k!)}(i) = i.$$

Thus $\tau' = \text{id}$ as desired, and $\psi$ is well defined.

For example, consider $\pi = 4, -2, 1, 3 \in B_{4,2}$. Then $\phi(|\pi|) = (5, 2, 0, 1)$, and

$$S = (5, 2, 0, 1, 5, 2, 0, 1).$$

Since only the first $k$ letters of $\pi$ may be negative, we essentially use the first $k$ letters of $\pi$ to determine the colors of the balls being juggled. Thus

$$\psi(\pi) = (+5, -2, 0, -1, -5, +2, 0, +1).$$

We can visualize this juggling sequence with the following diagram:
The inverse map $\psi^{-1}$ is straightforward. Given $T \in J_{n,k}$, let

$$\sigma = \phi^{-1}(|t_1|, \ldots, |t_n|) \in \mathcal{A}_{n,k}.$$ 

Then

$$\psi^{-1}(T) = \text{sgn}(t_1)\sigma(1), \ldots, \text{sgn}(t_k)\sigma(k), \sigma(k+1), \ldots, \sigma(n).$$

Since only the first $k$ letters of $\psi^{-1}(T)$ may be negative, it follows that $\psi^{-1}(T) \in B_{n,k}$. Moreover, it is clear that $\psi^{-1}$ is in fact the inverse of $\psi$.

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Department of Mathematics, University of California, San Diego, La Jolla, CA 92093
E-mail address: mdhyatt@math.ucsd.edu