Invertibility of spectral x-ray data with pileup–two dimension-two spectrum case

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Abstract—In the Alvarez-Macovski\cite{1} method, the line integrals of the x-ray basis set coefficients are computed from measurements with multiple spectra. An important question is whether the transformation from measurements to line integrals is invertible. This paper presents a proof that for a system with two spectra and a photon counting detector, pileup does not affect the invertibility of the system. If the system is invertible with no pileup, it will remain invertible with pileup although the reduced Jacobian may lead to increased noise.

Key Words: Jacobian, photon counting, dead time, pileup, spectral x-ray, dual energy, energy selective

I. INTRODUCTION

In the Alvarez-Macovski\cite{1} method, the line integrals of the x-ray basis set coefficients are computed from measurements with multiple spectra. The introduction of photon counting detectors into medical x-ray imaging\cite{2} gives the possibility of providing the spectral data by pulse height analysis (PHA). These detectors, however, have multiple defects\cite{3} that affect the information they provide. In some cases, the defects have been found to produce sharply increased noise in the estimates of the line integrals at specific object attenuation\cite{4}. The increased noise may indicate potential non-invertibility of the transformation between the spectral measurements and the line integrals. Therefore, it is important to develop mathematical descriptions of the invertibility of the transformation.

This paper is a step towards this mathematical description. It presents a proof that measurements with two spectra and a photon counting detector with pileup do not affect the invertibility of the system. If the system is invertible with no pileup, it will remain invertible with pileup although the reduced Jacobian may lead to increased noise. An example of the system analyzed is measurements with a photon counting detector of the transmitted flux with two different x-ray tube voltages. Note that the results of this paper are not applicable to pulse height analysis (PHA) since pileup changes the effective spectra of the bins\cite{5}. My recent paper\cite{4} gives an example of a three bin PHA system that becomes non-invertible for high pileup.

This paper addresses only the invertibility with deterministic, non-noisy measurements although it describes the conditioning of the system, which also affects noise.

II. THE ALVAREZ-MACOVSKI METHOD

For biological materials we can approximate the x-ray attenuation coefficient $\mu(r, E)$ accurately with a two function basis set\cite{5}

$$\mu(r, E) = a_1(r)f_1(E) + a_2(r)f_2(E). \quad (1)$$

In this equation, $a_i(r)$ are the basis set coefficients and $f_i(E)$ are the basis functions, $i = 1\ldots2$. As implied by the notation, the coefficients $a_i(r)$ are functions only of the position $r$ within the object and the functions $f_i(E)$ depend only on the x-ray energy $E$. If there is a high atomic number contrast agent, we need to extend the basis set to higher dimensions.

Neglecting scatter, the expected value of the number of transmitted photons $\lambda_k$ with an effective measurement spectrum $S_k(E)$ is

$$\lambda_k = \int S_k(E)e^{-\int L \mu(r, E) dr} dE \quad (2)$$

where the line integral in the exponent is on a line $L$ from the x-ray source to the detector.

Using the decomposition, Eq. \cite{1} the line integral in Eq. \cite{2} is

$$\int L \mu(r, E) dr = A_1 f_1(E) + A_2 f_2(E) \quad (3)$$

where $A_i = \int a_i(r) dr$, $i = 1\ldots2$ are the line integrals of the basis set coefficients. Summarizing the $A_i$ as the components of the $A$-vector, $A$, and the basis functions at energy $E$ as a vector $f(E) = [f_1(E), f_2(E)]$, we can write the line integral as the inner product of $A$ and $f(E)$,

$$\int L \mu(r, E) dr = A \cdot f(E). \quad (4)$$

The measurements are summarized by a vector, $N$, whose components are the expected photon counts with each effective spectrum. Since the body transmission is exponential in $A$, we can approximately linearize the measurements by taking logarithms. The results is the log measurement vector $L = -\log(N/N_0)$, where $N_0$ is the expected value of the measurements with no object in the beam and the division means that corresponding members of the vectors are divided.

Equations \cite{2} define a relationship between $A$ and the expected value measurement vector, $L(A)$. The invertibility of this transformation is the subject of this paper.

III. INVERTIBILITY WITHOUT PILEUP—SIMPLE CASES

Before discussing the invertibility in general, two simple but important cases will be discussed.

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The transformation is invertible if the determinant of $M$ is not equal to zero:

$$\begin{vmatrix} f_1(E_1) & f_2(E_1) \\ f_1(E_2) & f_2(E_2) \end{vmatrix} \neq 0$$

That is if

$$\frac{f_1(E_1)}{f_2(E_1)} \neq \frac{f_1(E_2)}{f_2(E_2)}$$

The bottom graph of Fig. 1 shows the ratio of the Compton scattering/photoelectric cross-sections basis set functions versus energy in the medical diagnostic region. Note that the ratio is monotonically decreasing. Thus, the condition in equation (6) will be true if the two energies are different. The $M$ matrix for any other basis set that spans the space of attenuation coefficients will be $M' = TM$ where $T$ is an invertible matrix so it has a nonzero determinant. Since these are square matrices, $\det(M') = \det(T)\det(M)$ and if $\det(M) \neq 0$ then $\det(M') \neq 0$ so this result is true for any valid basis set.

### A. Delta function spectra

The first case is two monoenergetic spectra with energies $E_1$ and $E_2$. In this case, $L_i = A \cdot f(E_i)$, $i = 1, 2$ and the transformation can be written $L(A) = MA$ where $M$ is a matrix with coefficients

$$M = \begin{bmatrix} f_1(E_1) & f_2(E_1) \\ f_1(E_2) & f_2(E_2) \end{bmatrix}$$

The derivative of $\lambda$ with respect to $t$ is always negative

$$\langle \lambda \rangle' = - \int \mu(E)S_1(E)\exp[-\mu(E)t]dE < 0$$

because the integrand is always greater than zero. Since $\lambda(t)$ is monotonically decreasing, it can be inverted to compute $t$.

### IV. INVERTIBILITY FOR THE GENERAL CASE WITH NO PILEUP

For the general case, the following theorem is useful (Fulks 1978[7] page 284):

Let $F$ be a continuously differentiable mapping defined on an open region $D$ in $E2$, with range $R$ in $E2$, and lets its Jacobian be never zero in $D$. Suppose further that $C$ is a simple closed curve that, together with its interior (recall the Jordan curve theorem), lies in $D$, and that $F$ is one-to-one on $C$. Then the image $T$ of $C$ is a simple closed curve that, together with its interior, lies in $R$. Furthermore, $F$ is one-to-one on the closed region consisting of $C$ and its interior, so that the inverse transformation can be defined on the closed region consisting of $T$ and its interior.

The first quadrant will be used as the region with the closed curve $C$ consisting of segments along the positive axes and a circle joining the ends of the segments, as shown in Figure 2. This is a region of theoretical and practical importance because a basis set consisting of the attenuation coefficients of the calibration materials is usually used. Since only positive equivalent thicknesses of the calibration materials can be used, this region must contain all the measured values.

In the theorem, the Jacobian is the determinant of the matrix of all the partial derivatives of the transformation.
The remaining part of the theorem requires us to show that \( J(A) \) is non-zero inside \( C \). This must be tested with individual spectra. The following sections show that, for the system studied, pileup does not affect this condition. If the Jacobian is non-zero without pileup, it will also be non-zero with pileup.

V. EXPECTED NUMBER OF PHOTONS RECORDED WITH PILEUP

The response time of a photon counting detector is modeled using the dead time, \( \tau \), which is defined to be the minimum time between two photons that are recorded as separate events[8]. The dead time is an abstraction that combines the contributions to the response time of all the physical effects in the detector. In this model, the detector is assumed to start in a “live” state. With the arrival of a photon, the detector enters a separate state where it does not count additional photons. The non-paralyzable model will be used where the time in the separate state is assumed to be fixed and independent of the arrival of any other photons during the dead time. There is a second model commonly used, called paralyzable, where the arrival of photons extends the time in the non-counting state. Both models give similar recorded counts at low interaction rates but give different results at high rates where the probability of multiple interactions during the dead time becomes significant.

Measurements by Taguchi et al.[9] indicate that for the detectors they studied the non-paralyzable model is more accurate at higher count rates. It also leads to simpler analytical results[10].

In a previous paper[5], I used the central limit theorem of renewal processes[11] to show that, for recording times much greater than the mean inter-event time, the expected value of recorded counts with pileup approaches

\[
\langle N_{rec} \rangle = \frac{\lambda}{1 + \eta}.
\]

In this equation, \( \lambda \) is the expected number of photons incident on the detector during the measurement time and \( \eta \) is the expected number of photons arriving during the dead time \( \tau \). If \( \rho \) is the average rate of photon arrivals then \( \eta = \rho \tau \). If \( T_{rec} \) is the measurement time then \( \rho = \lambda / T_{rec} \). Defining \( b = \tau / T_{rec} \), \( \eta = b \lambda \) and the expected recorded counts are

\[
\langle N_{rec} \rangle = \frac{\lambda}{1 + b \lambda}.
\]

VI. INVERTIBILITY OF TWO BASIS FUNCTIONS-TWO SPECTRA CASE WITH PILEUP

In this section I give a proof that with photon counting detector measurements of the total number of transmitted photons with two spectra, pileup does not affect invertibility. An example would be making two sequential measurements of an object using an x-ray tube with different voltages. Note that this does not prove that any two spectrum measurement with pileup is invertible.
For example with two bin photon counting with PHA, pileup causes the recorded spectrum to change so the assumptions of this section would not be met.

The proof is analogous to the proof for the measurements with no pileup described in the previous Sec. [II].

A. Invertibility with pileup on the contour C

First, I will show that if the transformation is invertible on the path in the first quadrant shown in Fig. 2 without pileup it is also invertible with pileup.

The proof for invertibility on the circular segment joining the segments on the axes is also applicable with pileup since, for large thicknesses, the count rate is very low so the pileup parameter \( \eta \) is essentially equal to zero and the pileup counts are the same as those without pileup.

Next we need to show that the data are invertible on the paths from the origin along the coordinates axes. The equation for the expected value of the recorded number of photons with pileup is

\[
\langle N_{rec} \rangle = \frac{\lambda}{1 + b\lambda}. \tag{12}
\]

Differentiating this equation along the axes

\[
\langle N_{rec} \rangle' = \frac{\lambda'}{1 + b\lambda} - \frac{b(\lambda')\lambda}{(1 + b\lambda)^2}
\]

where a prime denotes a derivative with respect to object thickness. Factoring the equation

\[
\langle N_{rec} \rangle' = \frac{\lambda'}{(1 + b\lambda)} \left[ 1 - \frac{b\lambda}{1 + b\lambda} \right] = \frac{\lambda'}{(1 + b\lambda)^2}. \tag{13}
\]

From Eq. 7 for a single material \( \lambda' \) is always less than zero. Since the denominator of Eq. 13 is always positive, the derivative of the recorded counts with pileup with respect to zero so the transformation of the recorded counts with pileup is invertible along the coordinate axes.

This shows that if the transformation without pileup is invertible on \( C \) then the transformation with pileup is also invertible on the curve.

B. Jacobian inside \( C \)

The log measurement with pileup is

\[
L_{rec,i} = -\log \left( \frac{N_{rec,i}}{N_{rec,i,0}} \right)
\]

and the Jacobian matrix with pileup has elements

\[
M_{rec,ij} = -\frac{\partial L_{rec,i}}{\partial A_j} = -\frac{1}{N_{rec,i}} \frac{\partial N_{rec,i}}{\partial A_j}. \tag{14}
\]

From Eq. 11

\[
N_{rec,i} = \frac{\lambda_i}{1 + b\lambda_i}
\]

and from Eq. 13

\[
\frac{\partial N_{rec,i}}{\partial A_j} = \frac{\lambda_i}{(1 + b\lambda_i)^2}.
\]

Substituting in Eq. 14

\[
M_{rec,ij} = \frac{1}{1 + b\lambda_i} \left[ \frac{\lambda'_i}{\lambda_i} \right] \tag{15}
\]

From the definition in Eq. 8 the term in brackets in Eq. 15 is the element of the Jacobian matrix without pileup.

Therefore,

\[
M_{rec,ij} = \frac{M_{0,ij}}{1 + b\lambda_i}
\]

Since for the dual energy case the matrices are \( 2 \times 2 \), the determinant is

\[
J_{rec} = M_{rec,11}M_{rec,22} - M_{rec,21}M_{rec,12}
= \frac{M_{0,11}M_{0,22} - M_{0,21}M_{0,12}}{(1 + b\lambda_1)(1 + b\lambda_2)}.
\]

where \( J_0 \) is the Jacobian without pileup. Since the terms in the denominator are always positive, the zero values of Jacobian determinant with pileup, if any, will occur at the same points as the Jacobian without pileup.

VII. Discussion

The data acquisition model used is unrealistic—most systems with photon counting detectors would also use PHA. Nevertheless, it provides an example of invertibility with pileup.

As discussed in my previous paper[5], due to the exponential probability distribution of x-ray photon inter-arrival times, pulse pileup is a fundamental effect. It will always be present in photon counting systems no matter how small the response time. Pileup has accurate analytical models and its analysis may lead to an understanding of invertibility with the other detector defects such as charge trapping and sharing, polarization, and incomplete photon energy deposition due to Compton scattering and K radiation escape[3].

VIII. Conclusion

A proof is given that pileup does not affect the invertibility of a two spectrum, photon counting data with pileup system.

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