Hydrodynamic Approach to the Evolution of Cosmic Structures II: Study of \( N \)-body Simulations at \( z = 0 \)

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ABSTRACT
We present a series of cosmological \( N \)-body simulations which make use of the hydrodynamic approach to the evolution of structures (Domínguez 2000). This approach addresses explicitly the existence of a finite spatial resolution and the dynamical effect of subresolution degrees of freedom. We adapt this method to cosmological simulations of the standard ΛCDM structure formation scenario and study the effects induced at redshift \( z = 0 \) by this novel approach on the large–scale clustering patterns as well as (individual) dark matter halos.

Comparing these simulations to usual \( N \)-body simulations, we find that (i) the new (hydrodynamic) model entails a proliferation of low–mass halos, and (ii) dark matter halos have a higher degree of rotational support. These results agree with the theoretical expectation about the qualitative behaviour of the ”correction terms” introduced by the hydrodynamic approach: these terms act as a drain of inflow kinetic energy and a source of vorticity by the small–scale tidal torques and shear stresses.

Key words: gravitation – methods: numerical – methods: \( N \)-body simulations – galaxies: formation – cosmology: theory

1 INTRODUCTION
Gravitational instability is commonly accepted as the basic mechanism for structure formation on large scales. Combined with the CDM model it leads to the picture of hierarchical clustering with wide support from deep galaxy and cluster observations. During the recent phase of cosmic evolution groups and clusters of galaxies condense from large scale density enhancements, and they grow by accretion and merger processes of the environmental cosmic matter.

But despite the fact that the currently favoured ΛCDM model has proven to be remarkably successful on large scales (cf. WMAP results, Spergel et al. 2003, Spergel et al. 2006), recent high–resolution \( N \)-body simulations still seem to be in contradiction with observation on sub–galactic scales. There is, for instance, the problem with the steep central densities of galactic halos as the highest resolution simulations favor a cusp with a logarithmic inner slope for the density profile of approximately \(-1.2 \) (Diemand, Moore & Stadel 2005; Fukushige, Kawai & Makino 2004; Power et al. 2003), whereas high resolution observations of low surface brightness galaxies are best fit by halos with a core of constant density (Simon et al. 2005; de Block & Bosma 2002; Swaters et al. 2003). A further problem relates to the overabundance of small–sized (satellite) halos; there are many more subhaloes predicted by cosmological simulations than actually observed in nearby galaxies (e.g., Moore et al. 1998, Klypin et al. 1999, Gottl¨ ober et al. 2003). The lack of observational evidence for these satellites has led to the suggestion that they are completely (or almost completely) dark, with strongly suppressed star formation due to the removal of gas from the small protogalaxies by the ionising radiation from the first stars and quasars (Bullock et al. 2000; Tully et al. 2002; Somerville 2002; Hoef et al. 2005). Other authors suggest that perhaps low mass satellites never formed in the predicted numbers in the first place, indicating problems with the ΛCDM model in general, which is replaced with Warm Dark Matter instead (Knebe et al. 2002; Bode, Ostriker & Turok 2001; Colín et al. 2000). Suggested solutions also include the introduction of self-interactions into collisionless \( N \)-body simulations (e.g. Spergel & Steinhardt 2000; Bento et al. 2000), and non-standard modifications to an otherwise unperturbed CDM power spectrum (e.g. bumpy power spectra: Little, Knebe & Islam 2003; tilted CDM: Bullock 2001c). Recent results from (strong) lensing statistics though suggest that the predicted excess of substructure is in fact required to reconcile some observations with theory (Dahle et al. 2003, Dalal & Kochanek 2002), but this con-
The discovery of the mismatch between observations and simulations is a result of the increase in the resolution of N-body simulations over the last years. This has emphasized the importance of the influence of subresolution scales on the simulated dynamics, at least when it comes to halo properties. The purpose of this work is to study the hydrodynamic-like formulation of the formation of cosmological structure proposed recently by Domínguez (2000), dubbed SSE (small-size expansion). The SSE addresses explicitly the existence of a finite spatial resolution and the dynamical effect of subresolution degrees of freedom. Although developed independently, the SSE approach is close in spirit to the Large–Eddy Simulation of turbulent flow though developed independently, the SSE approach is close in spirit to the Large–Eddy Simulation of turbulent flow. It therefore only appears natural to implement these correction terms into an (adaptive) mesh N-body code where the density is treated in a coarse–grained fashion, too: mesh–based Poisson solvers frequently used for cosmological simulations smooth the particle distribution onto a grid and hence deal with a coarse–grained density field when solving for the potential via Poisson’s equation. For this purpose we will adapt the open source N–body code MLAPM (Multi–Level Adaptive Particle–Mesh)\(^\ast\). The particles of the N-body simulations presented in this study are effectively hydrodynamical Lagrangian particles which move under the action not only of the mesh–computed gravitational force, but also of the additional, correction terms modeling sub-grid degrees of freedom in the context of the SSE.

The rest of the work is structured as follows: in Sec. 2 we describe the theoretical background of the SSE and provide the link to mesh–based N-body codes. In Sec. 3 we present the simulation of several models (standard $\Lambda$CDM model, AWDM model, and two runs incorporating the SSE corrections). In Sec. 4 we perform a comparative analysis of the four runs at redshift $z = 0$ from two complementary points of view: properties of the mass density and velocity fields, and properties of halos. Finally, Sec. 5 includes a discussion of the results and the conclusions.

\section{The Hydrodynamic Approach}

We deal with a system of nonrelativistic, identical point particles which (i) are supposed to interact with each other via gravity only, (ii) look homogeneously distributed on sufficiently large scales, so that the evolution at such scales corresponds to an expanding Friedmann-Lemaître cosmological background, and (iii) deviations to homogeneity are relevant only on scales small enough that a Newtonian approximation is valid to follow their evolution. Let $a(t)$ then denote the expansion factor of the Friedmann-Lemaître cosmological background, $H(t) = \dot{a}/a$ the associated Hubble function, and $\varrho_0(t)$ the homogeneous (background) density on large scales. $x_\alpha$ is the comoving spatial coordinate of the $\alpha$-th particle, $u_\alpha = ax_\alpha$ its peculiar velocity, and $m$ its mass. In terms of these variables the evolution is described by the following set of equations (Peebles 1980) ($\nabla_\alpha$ denotes a partial derivative with respect to $x_\alpha$):

\begin{align}
\dot{x}_\alpha &= \frac{1}{a} u_\alpha, \\
\dot{u}_\alpha &= w_\alpha - H u_\alpha, \\
\nabla_\alpha \cdot w_\alpha &= -4\pi G a m \sum_{\beta \neq \alpha} \delta^3(x_\alpha - x_\beta) - \varrho_0, \\
\nabla_\alpha \times w_\alpha &= 0,
\end{align}

where $w_\alpha$ is the peculiar gravitational acceleration acting on the $\alpha$-th particle. Finally, Eqs. (1) must be subjected to periodic boundary conditions in order to yield a Newtonian description consistent with the Friedmann-Lemaître solution on large scales (Buchert & Eilers 1997).

If we now assume that the actual measure of the density field in an N-body code depends on a smoothing window $W(z)$, the microscopic field $\varrho_{\text{mic}}$ relates to the measured (coarse–grained) field $\varrho$ in the following way:

\begin{align}
\varrho_{\text{mic}}(x, t) &= \frac{m}{a(t)^3} \sum_{\alpha} \delta^3(x - x_\alpha(t)), \\
\varrho(x, t; L) &= \int \frac{d^3y}{L^3} W \left( \frac{|x - y|}{L} \right) \varrho_{\text{mic}}(y, t).
\end{align}

The physical interpretation of the field $\varrho(x; L)$ follows straightforwardly from the properties of the smoothing window: it is proportional to the number of particles contained within the coarsening cell of size $\approx L$ centered at $x$. A microscopic peculiar–momentum density field and the corresponding coarse–grained field can be defined in the same way:

\begin{align}
\mathbf{j}_{\text{mic}}(x, t) &= \frac{m}{a(t)^3} \sum_{\alpha} u_\alpha(t) \delta^3(x - x_\alpha(t)), \\
\mathbf{j}(x, t; L) &= \int \frac{d^3y}{L^3} W \left( \frac{|x - y|}{L} \right) \mathbf{j}_{\text{mic}}(y, t).
\end{align}

One can introduce peculiar velocity fields $\mathbf{u}_{\text{mic}}$ and $\mathbf{u}$ by definition as $\mathbf{j} = \varrho \mathbf{u}$ and similarly for $\mathbf{u}_{\text{mic}}$. The physical meaning of $\mathbf{u}(x; L)$ is also simple: it is the center-of-mass peculiar velocity of the subsystem defined by the particles inside the coarsening cell of size $\approx L$ centered at $x$. Notice that $\mathbf{u}$ is not obtained by coarse-graining $\mathbf{u}_{\text{mic}}$: from a dynamical point of view, it is more natural to coarse-grain the momentum rather than the velocity, since the former is an additive quantity for a system of particles. Finally, one can define peculiar gravitational acceleration fields $\mathbf{w}_{\text{mic}}$ and $\mathbf{w}$

\footnotesize
\begin{flushleft}
\textit{Available at http://www.aip.de/People/AKnebe/MLAPM}\end{flushleft}
through coarse-graining of the force:
\[
\varrho_{\text{mic}} w_{\text{mic}}(x, t) = \frac{m}{a(t)} \sum_a w_a(t) \delta^{(3)}(x - x_a(t)), \quad (4a)
\]
\[
\varrho w(x, t; L) = \int \frac{dy}{L^3} W \left( \frac{x - y}{L} \right) \varrho_{\text{mic}} w_{\text{mic}}(y, t). \quad (4b)
\]

The field \( w(x) \) has the physical meaning of the center-of-mass peculiar gravitational acceleration of the subsystem defined by the coarsening cell at \( x \).

From these definitions and Eqs. (1), it is straightforward to derive the evolution equations obeyed by the coarse-grained fields \( \varrho \) and \( u \) (from now on, \( \partial / \partial t \) is taken at constant \( x \) and \( L \), and \( \nabla \) means partial derivative with respect to \( x \)):
\[
\frac{\partial \varrho}{\partial t} + 3 H \varrho = - \frac{1}{a} \nabla \cdot (\varrho u), \quad (5a)
\]
\[
\frac{\partial (\varrho u)}{\partial t} + 4 H \varrho u = \varrho w - \frac{1}{a} \nabla \cdot (\varrho u u + \Pi), \quad (5b)
\]
where a new second-rank tensor field has been defined (dyadic notation):
\[
\Pi(x, t; L) = \int \frac{dy}{L^3} W \left( \frac{x - y}{L} \right) \varrho_{\text{mic}}(y, t) \left[ u_{\text{mic}}(y, t) - u(x, t; L) \right] \left[ u_{\text{mic}}(y, t) - u(x, t; L) \right]. \quad (6)
\]
The field \( \Pi(x) \) is due to the velocity dispersion, i.e., to the fact that the particles in the coarsening cell have in general a velocity different from that of the center of mass. The physical meaning of Eqs. (5) is simple: they are just balance equations, stating mass conservation and momentum conservation, respectively. The term \( \varrho w \) codifies the gravitational interaction between the coarsening cells and does not satisfy, in general, Poisson’s equation or the curl–free condition. The term \( \nabla \cdot \Pi \) represents a kinetic pressure force due to the exchange of particles between neighboring coarsening cells (just like for an ideal gas) and it has the same physical origin as the convective term \( \nabla \cdot (\varrho u u) \), i.e., a nonlinear mode-mode coupling of the velocity field. The difference is that the convective term couples only modes on scales \( \gtrsim L \), while the velocity dispersion term codifies the dynamical effect of the coupling of the modes on scales \( \gtrsim L \) with the modes on scales \( \lesssim L \). Eqs. (5) are exact as one changes the smoothing length, the fields \( \varrho, u, w, \Pi \) change in such a way that the form of the equations remains the same (for example, upon increase of the smoothing length, part of the dynamical effect described by \( \nabla \cdot (\varrho u u) \) is shifted towards \( \nabla \cdot \Pi \)). This property is reflected in that the equations are not an autonomous system for \( \varrho \) and \( u \). In fact, they are just the first ones of an infinite hierarchy, as can be checked by computing the evolution equations for the fields \( \Pi \) and \( \Pi \).

To obtain a useful set of equations, it is necessary to truncate this hierarchy by looking for a functional dependence of \( \varrho \) and \( \Pi \) on \( \varrho \) and \( u \).

The peculiarities of the problem at hand (collisionless matter in the non-stationary state of structure formation) prevent the usual truncation of the hierarchy leading to the Euler or Navier-Stokes equations, respectively (see, e.g., Chapman & Cowling 1991). The small-size expansion (SSE) is a specific truncation for this problem (Domínguez 2000, 2002; Buchert & Domínguez 2005), that starts from the physical assumption that the coupling to the small-scales is weak (this can be argued on the basis that, in a hierarchical scenario, the smaller scales “virialize” earlier and thus “decouple” from the evolution of the larger scales). Then the fields \( \Pi \) and \( u \) are derived as a formal expansion in \( L \): Keeping terms up to order \( L^2 \), Eqs. (5) become (\( \partial_i = \partial / \partial x_i \), summation over the repeated index \( i \) is understood)
\[
\frac{\partial \varrho}{\partial t} + 3 H \varrho = - \frac{1}{a} \nabla \cdot (\varrho u), \quad (7a)
\]
\[
\frac{\partial (\varrho u)}{\partial t} + 4 H \varrho u = \varrho w^{\text{mf}} - \frac{1}{a} \nabla \cdot (\varrho uu + \Pi), \quad (7b)
\]
\[
\nabla \cdot w^{\text{mf}} = -4 \pi G a (\varrho - \varrho_b), \quad (7c)
\]
\[
\nabla \times w^{\text{mf}} = 0, \quad (7d)
\]
with the additional acceleration
\[
C = \frac{B L^2}{a} \left[ (\nabla \varrho \cdot \nabla) w^{\text{mf}} - \frac{1}{a} \nabla \cdot [\varrho (\partial_i u)(\partial_i u)] \right]. \quad (8)
\]
The constant \( B \) is determined by the smoothing window \( W(z) \),
\[
B = \frac{1}{3} \int dz z^2 W(z) = \frac{4 \pi}{3} \int_0^{+\infty} dz z^4 W(z). \quad (9)
\]
To order \( L^3 \), Eqs. (7) reduce to the “dust (pressureless) approximation” for cosmological structure formation (Sahni & Cole 1995): \( w^{\text{mf}} \) represents the mean-field gravity created by the monopole moment of the matter distribution in the coarsening cells, i.e., the total mass, and the coupling to the small-scales is neglected altogether. To order \( L^5 \) there are two kinds of corrections: the term proportional to the tidal tensor \( \nabla w^{\text{mf}} \) (stemming from a term \( w - w^{\text{mf}} \) in Eqs. (5)) models the gravitational force of the higher-order multipoles moments, i.e., the coupling to the subresolution configurational degrees of freedom; the term proportional to \( (\partial_i u)(\partial_i u \) (stemming from \( \Pi \) in Eqs. (5)) models the effect of velocity dispersion, that is, the coupling to the subresolution kinetic degrees of freedom.

The expected dynamical effect of these new terms with respect to the “dust evolution” has been studied theoretically (Domínguez 2000, 2002; Buchert & Domínguez 2005). There is evidence that, assuming a locally plane-parallel collapse, these terms mimic the “adhesion model” (Gurbatov, Saichev & Shandarin 1989, Sahni & Cole 1995), in which recently collapsed regions stabilize — or more generally speaking, the term due to velocity dispersion tends to reduce the inflow velocity in collapsing regions and favors the formation of gravitationally bound systems. It has also been shown that the correction terms act as a source of vorticity via small-scale tidal forces and shear stresses. The “dust model” lacks a source of vorticity \( \omega = \nabla \times u \), and the initially present one is damped by the cosmological expansion in the linear regime. Thus, in that respect the corrections to the “dust model” can be particularly important. Taking the curl of Eq. (7b) we obtain
\[
\frac{\partial \omega}{\partial t} = - H \omega + \frac{1}{a} \nabla \times (u \times \omega) + \nabla \times C, \quad (10)
\]
where the term \( \nabla \times C \) is a source of vorticity, i.e., it does not vanish in general even if \( \omega = 0 \), as has been confirmed perturbatively by Domínguez (2002). We comment further on the relationship between vorticity and angular momentum.
in App. A, where also some results concerning the conservation of energy, momentum, and angular momentum are derived.

The dynamical evolution predicted by Eqs. (7) can be implemented without much difficulties in a particle–mesh (PM) code of $N$-body simulation. Mass conservation, Eq. (7a), is automatically satisfied by the code. The acceleration $\mathbf{w}^\text{mf}$ given by Eqs. (7c) and (7d) agrees with the value returned by the Poisson solver on a grid, and the grid constant sets naturally the resolution $L$. In principle, one only needs to take care of Eq. (7b), which can be re-written in Lagrangian coordinates as

$$
\mathbf{u} = \mathbf{w}^\text{mf} - H \mathbf{u} + \mathbf{C}.
$$

(11)

The computation of $\mathbf{C}$, given by Eq. (8), on the grid of the Poisson solver is a highly non-trivial but manageable task. Thus, Eq. (11) — together with $\mathbf{x} = \mathbf{u}/a$, see Eq. (1a) — determines the motion of Lagrangian fluid elements, which are sampled by the particles of the simulation. If we set $B = 0$ (\Rightarrow $C = 0$) in Eq. (11) we recover the equations of motions that are being integrated in a standard PM code for the update of particle velocities and positions during the course of the simulation.

3 THE $N$-BODY SIMULATIONS

3.1 The Setup

The $N$-body simulations presented in this study were carried out using a version of the open source adaptive mesh refinement code {MLAPM} (Knebe, Green & Binney 2001). This code reaches high force resolution by refining high-density regions with an automated refinement algorithm. These adaptive meshes are recursive: refined regions can also be refined, each subsequent refinement having cells that are half the size of the cells in the previous level. This creates a hierarchy of refinement meshes of different resolutions covering regions of interest. The refinement is done cell by cell (individual cells can be refined or de-refined) and meshes are not constrained to have a rectangular (or any other) shape. The criterion for (de-)refining a cell is simply the number of particles within that cell and a detailed study of the appropriate choice for this number can be found elsewhere (Knebe et al. 2001). The code also uses multiple time steps on different refinement levels where the time step for each level is two times smaller than the step on the previous level. The latest version of {MLAPM} also includes an adaptive time stepping that adjusts the actual time step after every major step to restrict particle movement across a cell to a particular fraction of the cell spacing, hence, fine tuning accuracy and computational time.

As outlined above, the only necessary modification required to model the "Hydrodynamic APProximation" (or "HAPPy" here afterwards) is to account for the correction term $\text{C}$ in Eq. (11) when updating the particle velocities\textsuperscript{†}. {MLAPM} has therefore been modified to not only calculate the density field on its hierarchy of nested refinement grids but also the velocity field. The $\nabla$-operator and spatial derivatives, respectively, have been approximated by finite-differences using the two nearest neighbors (in each dimension) to the cell for which the correction term is being calculated. Cells close to a refinement boundary for which not enough surrounding nodes are present, obtain their correction values interpolated downwards from the next coarser grid level. The assignment of mass and momentum on the grid is done with a triangular–shaped–cloud window (Hockney & Eastwood 1988),

$$
W(z) = \begin{cases} 
\frac{4}{3} - z^2 & \text{for } |z| \leq 0.5, \\
\frac{1}{3} (2 - |z|)^2 & \text{for } 0.5 < |z| \leq 1.5, \\
0 & \text{otherwise},
\end{cases}
$$

(12)

for which $B = 1/4$ according to Eq. (9). We remark that, due to the dynamical (de)refinement procedure of the {MLAPM} code, the resolution is space–dependent in a discrete manner, while Eqs. (7) were derived under the assumption of a spatially homogeneous length $L$. The SSE can be generalized to the case of a smoothly varying $L(x)$ (Domínguez, unpublished; Domínguez 2002 contains the generalization to a time–dependent $L$), but we have neglected this additional complication because the fraction of particles which are in regions where $L$ jumps is less than 1% during the run. Finally, we also note that this numerical method of integrating the hydrodynamic equations is different from, albeit similar to, the Smoothed Particle Hydrodynamics method (Gingold & Monaghan 1977; Lucy 1977) frequently used in cosmological simulations involving baryonic matter.

We ran four CDM simulations with cosmological parameters in agreement with the so-called concordance model, i.e. $\Omega_0 = 1/3$, $\lambda_0 = 2/3$, $\sigma_8 = 0.88$, $h = 2/3$:

- one standard $\Lambda$CDM model,
- one AWDM model ($m_{\text{WDM}} = 0.5 \text{keV}$),
- two $\Lambda$CDM models with $B=1/4$ and $B=1$, respectively.

Even though $B$ is actually determined by the smoothing window, we also considered a four times larger value as if it were a free parameter of the model. This model is to be understood as an "academic toy model" where we hope to gain better insight into the effects of the correction term on cosmological structure formation.

All simulations consist of $N = 128^3$ particles in a box of side length $25h^{-1}\text{Mpc}$ (the mass of a simulation particle is $m_{\text{part}} \approx 7 \times 10^8 h^{-1}\text{M}_{\odot}$), and they were started at redshift $z = 35$. The two $\Lambda$CDM models with $B \neq 0$ are also dubbed $\Lambda$CDMhappi\textsuperscript{1} ($B=1/4$) and $\Lambda$CDMhappi\textsuperscript{2} ($B=1$). We chose to also run an AWDM model to allow for a more complete comparison of the new HAPPy models to other, alternative cosmologies. A more elaborate study of the AWDM model and Warm Dark Matter can be found in Knebe et al. (2002).

The force resolution in {MLAPM} is determined by the finest refinement level reached throughout the run. While all four models applied exactly the same refinement criterion (six particles per cell), the $\Lambda$CDMhappi\textsuperscript{2} run only invoked five refinement levels whereas all other runs used seven levels at redshift $z = 0$. In terms of force resolution this translates to $10h^{-1}\text{kipc}$ spatial resolution for $\Lambda$CDMhappi\textsuperscript{2} (corresponding to an estimated maximum density $\sim 5 \times 10^4 \rho_8$) and $2.5h^{-1}\text{kipc}$ for all the other models (maximum density $\sim 3 \times 10^6 \rho_8$). This difference can be ascribed to the smoothening effect of the correction terms in the dynamical equations that are more effective for higher values of $B$.

\textsuperscript{†} The modifications are part of {MLAPM} v1.4 (and all later versions) and can be switched on using the {	exttt{DHAPPI}} upon compile time.
3.2 Accuracy of the code

A useful check of the accuracy of a simulation is provided by the global invariants of the dynamical system, which we derive and discuss in App. A.

The dynamical evolution conserves the quantity $\alpha \sum u_a$ (if the force and density interpolation schemes are identical). It was found that departures from the initially vanishing value satisfy the bound

$$\frac{1}{N} \sum_{a=1}^{N} u_a \leq 3 \times 10^{-3} u_{\text{mean}}, \quad u_{\text{mean}} = \frac{1}{N} \sum_{a=1}^{N} |u_a|, \quad (13)$$

where at redshift $z = 0$ the average particle velocity was $u_{\text{mean}} \approx 300 \, \text{km s}^{-1}$ in all four models.

In standard $N$–body codes it is common practice to check constancy of the invariant $\mathcal{I}$ (see Eq. (A5)) following from the Layzer–Irvine equation (e.g., Knebe et al. 2001). This is not the case for the two HAPPI models, since the correction term $C$ is a source or drain of energy as discussed in the App. A. We though chose to plot in Fig. 1 the dimensionless quantity

$$\tilde{I}(t) = \mathcal{I}(t) - \mathcal{I}(t_1) \frac{1}{a U_{\text{mf}}(t)}, \quad (14)$$

as a function of cosmic expansion factor $a$, where $U_{\text{mf}}$ is the mean–field potential energy defined in Eqs. (A1). For the $\Lambda$CDM and $\Lambda$WDM models, the Layzer–Irvine equation holds and predicts $\tilde{I} = 0$. Departures from this result are ascribed to both integration/truncation errors introduced by the code and the fact that the particle shape is neither constant in time nor space due to the adaptive nature of MLAPM (cf. Knebe et al. 2001). $\Lambda$CDMhappi1 performs rather similar to the standard $\Lambda$CDM model, indicating that the effect of the correction term $C$ in the evolution of $\mathcal{I}$ is small compared to the numerical errors. $\Lambda$CDMhappi2, on the other hand, departs noticeably from the other models and $\tilde{I}$ changes sign at around a redshift of $z \approx 3$, which lets us expect to find larger differences between $\Lambda$CDMhappi2 and $\Lambda$CDM.

### Table 1. Fraction of HAPPI particles in the subset of low-density particles, and in the subset of high-density particles, respectively.

| redshift | low-density | high-density |
|----------|-------------|--------------|
| $z = 0$  | 10%         | 6%           |
| $z = 0.5$| 14%         | 9%           |
| $z = 1$  | 16%         | 12%          |
| $z = 4$  | 14%         | 23%          |

3.3 The Importance of the HAPPI correction

In order to test the importance of the correction term Eq. (8) we calculated the ratio of the mean–field acceleration (i.e. $F = |w_{\text{mf}}|$) and the additional acceleration ($C = |C|$) for each individual particle as a function of the local density at various redshifts. The result for the $\Lambda$CDMhappi2 model, for which the effect of $C$ is the largest, can be viewed in Fig. 2. This figure indicates that the effect of the newly added terms is generally rather small especially at late times. At redshift $z = 0$ the fraction of all particles with $F/C < 1$ ("HAPPI particles") is a mere 9% while it increases to 15% at $z = 4$. In order to examine a possible trend with density, we consider at each redshift two subsets of particles according to whether the density is above or below the virial overdensity $\Delta_{\text{vir}}$ (cf. Eq. (17) in Section 4.2) and hence particles of the high-density subset either already belong to virialized structures or will be part of them at a later time. Table 1 gives the fraction of HAPPI particles in each subset.

This raises the question about the exact locations as well as the "dynamics" of HAPPI particles. A visual inspection shows that, at high redshift, they are preferentially located within the filaments flowing towards halos. At later times though, they can be found either in regions of strong dynamical activity (e.g., mergers, the outskirts of halos and infalling to halos), or at the very centres of relaxed systems. The smaller spatial force resolution of the $\Lambda$CDMhappi2 mentioned earlier can hence be linked to the influence of the HAPPI particles at the very centres of halos.

4 ANALYSIS OF THE MATTER DISTRIBUTION

4.1 Large Scale Structure and Global Properties

A visual impression of the density field of the particle distribution at redshift $z = 0$ for all four models is shown in Fig. 3, where the local density at each particle position was determined by smoothing the distribution onto a regular grid (256$^3$ nodes) and interpolating the density on the mesh back to the particle positions. Not surprisingly, the $\Lambda$WDM model appears less clumpy and far smoother than the $\Lambda$CDM simulation. However, there appears to be more small scale structure in both the HAPPI runs, or at least the smaller objects are more contrasted. We can readily relate this phenomenon to the influence of $B$ and larger $B$ values give higher contrasts (remember that the fiducial $\Lambda$CDM model is nothing else than a HAPPI model with $B=0$). But despite the more grainy appearance of the HAPPI runs in Fig. 3, the power on small scales is reduced compared to the power of the $\Lambda$CDM model. This can be verified in Fig. 4, where we plot the dark matter power spectrum of density fluctuations for all four
models at redshifts $z = 3$ (lower curves) and $z = 0$ (upper curves). Especially ΛCDMhappi2 falls behind ΛCDM even though it appears to be marginally more evolved at higher redshift. We will though see that these two results, the "graininess" of the ΛCDMhappi2 model and the lack of small scale power, do not exclude each other. The absence of small scale power on scales below $\sim 1–2\ h^{-1}\ Mpc$ reflects the fact that the halos corresponding to those scales (i.e. halos of mass $> 10^{11} h^{-1}\ M_\odot$) are internally less concentrated than their ΛCDM counterparts.

4.1.1 Beyond the two–point estimators

Two–point estimators like the power spectrum are sensitive only to the amplitude of the modes of the fluctuating field $\varphi(x) - \varphi_b$. The information concerning the relative phases is contained in the higher–order correlations. In the literature there have been several meaningful quantities proposed depending on higher–order correlations with a more or less transparent physical interpretation. In this work, we have employed the scalar Minkowski functionals (MFs) (see, e.g., Mecke & Stoyan 2000; Domínguez 2001), which allow a quantification of the morphological aspects appreciated by visual inspections. Given a density field $\varphi(x)$ and a density threshold $\rho$, one constructs the isodensity surface $S = \{x|\varphi(x) = \rho\}$ (with the convention that the region $\varphi > \rho$ is taken as the interior of $S$). The four MFs $V_\nu,\ \nu = 0, 1, 2, 3$, can be defined as surface integrals over $S$ and have the following geometrical meaning (up to a conventional constant prefactor):

$$
\begin{align*}
V_0 &\propto \text{volume enclosed by the isodensity surface } S \\
V_1 &\propto \text{total area of } S \\
V_2/V_1 &\propto \text{mean curvature of } S, \text{ averaged over } S
\end{align*}
$$

Figure 2. The correction term in comparison to the mean–field force acting on a single particle as a function of density. The four panels are for all particles in the simulation at redshifts $z = 0$ (upper left), $z = 0.5$ (upper right), $z = 1.0$ (lower left), and $z = 4.0$ (lower right). The vertical line indicates the virial overdensity at the respective redshift. The number in each quadrant lists the total number of particles in this regime of the plot.
$V_3/V_1 \propto \text{Gaussian curvature of } S$, averaged over $S$

$V_0$ is in fact proportional to the probability that $g(x) > \bar{\rho}$ (assuming spatial ergodicity of the realization). The ratio $V_1/V_0$ is a measure of how compact the volume enclosed by $S$ is packed. $V_2/V_1$ is a measure of the convexity of the surface $S$, while $V_3$ is proportional to the Euler characteristic or genus of the body defined by $S$:

$V_3 \propto \text{number of disconnected objects} + \text{number of holes} - \text{number of tunnels}$.

There seems to be a close relation between the threshold value at which $V_3$ vanishes and the percolation threshold of the volume enclosed by $S$ (Mecke & Wagner 1991; Neher 2003) — as a matter of fact, the use of percolation analysis is not rare in the analysis of cosmological structures (e.g. Yess & Shandarin 1996). As an example of how the MFs are to be interpreted, in App. B we discuss the case that $g(x)$ is derived from a realization of a Poissonian distribution of points.

Starting from the positions of the particles given by the simulation, a density field $g(x)$ is generated on a grid by smoothing with the window Eq. (12). We generated 20 dif-
different realizations of the field by displacing randomly the grid and we present the MFs averaged over these realizations. The MFs are functions of the grid constant and the density threshold. We observe that the measurement of the MFs in an $N$-body simulation is affected by two kinds of errors (which are more pronounced for higher orders $\nu$ of the MF $V_\nu$):

(i) **Finite–volume effects**: the tails of the probability distribution of the density field cannot be probed correctly in a finite volume, so that the measured MFs are noisy and take discrete values (due to the grid) as the threshold approaches the extremal values probed in a given simulation. This effect can be reduced by increasing the number of realizations.

(ii) **Finite–mass effects**: at threshold values corresponding to $\lesssim 1$-10 particles per grid node, the MFs detect that the density field is actually derived from a distribution of point particles.

In order to emphasize the differences between the four simulations and to facilitate the physical interpretation, we plotted as a function of $\hat{\rho}$ the relative difference between a measure in one model and the corresponding one in the reference $\Lambda$CDM model (denoted by ‘ref’):

$$\frac{V_\nu}{V_\nu^{\text{ref}}} - 1 \quad (\nu = 0, 3),$$

$$\frac{V_\nu}{V_{\nu-1}} \times \left(\frac{V_{\nu-1}}{V_\nu}\right)^{\text{ref}} - 1 \quad (\nu = 1, 2).$$

The plots are shown in Fig. 5 for a grid of $128^3$ nodes. As a general feature, the trend of the $\Lambda$WDM model with respect to the reference $\Lambda$CDM model is always opposite to the trend of the two HAPPI runs, with the deviations of $\Lambda$CDMhappi2 being larger than those of $\Lambda$CDMhappi1. Only at very high density thresholds, $\Lambda$CDMhappi2 seems to follow an opposite trend to $\Lambda$CDMhappi1: this is because the maximum density in the $\Lambda$CDMhappi2 run is smaller than in the $\Lambda$CDMhappi1 or $\Lambda$CDM runs. This (physical) effect has been already noticed concerning the final force resolution of the runs: it is due to the larger effect of the additional acceleration $\mathbf{C}$ and can be directly linked to the location of "HAPPI particles" at redshift $z = 0$ at the center of relaxed structures, see Secs. 3.1 and 3.3.
The results can be summarized as follows:

- The dependence $V_0(\rho)$ quantifies the visual impression that low-density regions ($\rho \lesssim 10\rho_0$) are clearly less likely in the WDM model, while overdense regions ($10 \lesssim \rho/\rho_0 \lesssim 10^3$) are more abundant in the HAPPI models. As remarked, higher density areas are rarer in $\Lambda$CDMhappi2 and more frequent in $\Lambda$CDMhappi1 compared to $\Lambda$CDM.
- From the fact that $V_2, V_3 > 0$ (not shown in the plot) in the whole range of threshold values, one infers that, observed at the resolution of $128^3$ nodes, the matter distribution consists mainly of “clusters” (i.e., disconnected objects which are convex on average). According to the plot of $(V_2/V_3)(\rho)$, these clusters in the HAPPI runs tend to be rounder than in the $\Lambda$CDM run, while they tend to be more cigar-shaped (filament–like) in the AWDM model. From the plot of $V_3(\rho)$ one deduces that the AWDM model has less clusters at all threshold values; the HAPPI runs have more clusters, except for $\Lambda$CDMhappi2 at very high densities because this particular model does not reach as high densities as the other ones.
- Finally, the plot of $(V_1/V_0)(\rho)$ shows that matter is more compactly packed in the AWDM model, and less compactly in the HAPPI runs. This is likely due to the different cluster abundances just discussed.

We have repeated the analysis at different spatial resolutions ($16^3, 32^3, 64^3$ and $256^3$ nodes) for the Minkowski functionals. The quantitative differences between models decrease as the grid becomes coarser, but the same conclusions hold roughly in a qualitative manner.

Summarizing, the AWDM run has less voids and more filamentary–like structures than the reference $\Lambda$CDM model, while the HAPPI runs have comparatively more mass concentrated in small, roundish clusters.

### 4.1.2 The velocity field

Motivated by the theoretical discussion, we have also addressed the distribution of the (comoving) vorticity, $\omega = \nabla \times \mathbf{u}$, and the (comoving) divergence, $\theta = \nabla \cdot \mathbf{u}$, of the peculiar velocity field $\mathbf{u}(x)$. Using the positions and velocities of the particles, we compute the fields $\omega(x)$ and $\theta(x)$ on a regular grid (see Sec. 3). The cumulative probabilities that $|\omega|^2 > \omega^2$ and that $\theta^2 > \hat{\theta}^2$ are given by the first Minkowski functional $V_0$, which we compute as explained previously.

Fig. 6 shows $P/\rho^{\text{crit}} - 1$, where $P$ is the cumulative probability measured in a simulation, and $\rho^{\text{crit}}$ is the corresponding probability in the reference $\Lambda$CDM simulation. The $\Lambda$CDMhappi2 model shows a clear tendency to have a larger vorticity and a lower divergence (in absolute value) than the $\Lambda$CDM model. The vorticity in the $\Lambda$CDMhappi1 model exhibits a similar tendency, but the differences are quantitatively smaller. In order to minimize finite–mass effects, we considered only grid nodes for which the value of the smoothed density field corresponds to more than 10 particles per node. If we restrict the measurement of the probability distributions to higher densities, the quantitative differences tend to be smaller.

![Figure 6. Cumulative probabilities $P(|\omega|^2 > \omega^2), P(\theta^2 > \hat{\theta}^2)$ relative to the probabilities in the $\Lambda$CDM model at redshift $z = 0$ at a resolution of $128^3$ grid nodes. The solid symbols represent the (relative) cumulative probability of the vorticity, the joined open symbols correspond to the divergence. The threshold $(\omega^2, \hat{\theta}^2)$ is given in units of $H_0^2$.](image)

### 4.2 Dark Matter Halos

The analysis in the following Subsection is primarily based upon gravitational bound objects which were identified using MLAPM’s Halo Finder (MHF) (Gill, Knebe & Gibson 2004). This newly developed halo finder uses the adaptive grid structure invoked by the $N$-body code MLAPM and reorganizes it into a tree. The centres of the grids at the end leaves of a branch of the tree serve as (potential) halo centres and all gravitationally bound particles about these centres are being collected. For a more elaborate discussion of this halo finder we refer the reader to Gill et al. (2004). We only like to note at this point that MHF is essentially parameter free and naturally finds halos with exactly the same accuracy as the simulation. Besides of the growth of the halo mass function and the abundance evolution of gravitationally bound objects, respectively, we confine our analysis to objects identified at redshift $z = 0$. The investigation of the evolutionary history and hierarchical growth of structures will be published elsewhere.

#### 4.2.1 Global Properties

In Fig. 7 we show the evolution of the cumulative mass function $n(> M)$ of dark matter halos, i.e., the number of DM halos with mass larger than $M$. The AWDM model clearly has less low–mass objects as already pointed out by other authors (Knebe et al. 2002, Bode et al. 2001, Colín et al. 2000). All three CDM models though perfectly agree at a redshift $z = 5$, but there is a clear trend for the HAPPI models to give rise to more small mass halos, in agreement with the conclusion derived in the previous Section. This can be verified in Fig. 8, where we plot the number density evolution of objects more massive than $M > 10^{12} h^{-1} M_\odot$ (> 20 particles). In $\Lambda$CDMhappi2 there are roughly 50% more halos above our mass cut than in the reference $\Lambda$CDM run.

In order to study the clustering properties of these halos we estimated the two–point correlation function for low– and high–mass objects, respectively, as
Figure 7. Cumulative mass function \( n(M) \) of DM halos.

Figure 8. Number density evolution of objects more massive than \( M > 10^{10} h^{-1} M_{\odot} \).

Figure 9. Two-point correlation function of objects more massive (upper panel) and less massive (lower panel) than \( 10^{12} h^{-1} M_{\odot} \).

\[ \xi_{\text{gal}}(r) = -1 + \frac{V_{\text{gal}}}{N_{\text{gal}}} \sum_{\alpha=1}^{N_{\text{gal}}} \frac{n_{\alpha}(r; \Delta r)}{v(r; \Delta r)} \quad (15) \]

where \( N_{\text{gal}} \) is the total number of objects in the simulation volume \( V \), and \( n_{\alpha}(r; \Delta r) \) is the total number of objects in a spherical shell of radius \( r \) and thickness \( \Delta r \) (and volume \( v(r; \Delta r) \)), centered at the \( \alpha \)-th object. The result (along with Poisson error bars based upon the number of pairs in each bin) is shown in Fig. 9. The low-mass objects show very similar clustering patterns, with a higher amplitude of \( \xi_{\text{gal}} \) for \( \Lambda \text{WDM} \) and a marginally decreased correlation for the \( \Lambda \text{CDMhappi2} \) model. The situation though is difficult to judge at the high-mass end as we have too few pairs per bin to make conclusive statements. The errors bars are larger than the differences amongst the models and hence have been omitted. It seems, however, that the clustering pattern of high mass objects is similar in all models and does not show differences when including the HAPPI correction term.

The most striking (and interesting) difference between standard \( \Lambda \text{CDM} \) and the two HAPPI models, however, emerges when we turn to the spin parameter distribution. The spin parameter \( \lambda \) was calculated using the definition given by Bullock et al. (2001a),

\[ \lambda = \frac{|\mathbf{L}|}{\sqrt{2M_{\text{vir}} v_{\text{vir}} r_{\text{vir}}}} \quad (16) \]

where \( \mathbf{L} \) is the angular momentum of the halo with respect to its center of mass, \( r_{\text{vir}} \) is the virial radius of the halo, \( M_{\text{vir}} \) is the virial mass (mass enclosed within the virial radius), and \( v_{\text{vir}} = \sqrt{GM_{\text{vir}}/r_{\text{vir}}} \) is the circular velocity at the virial radius. The virial radius and mass are determined by the condition

\[ M_{\text{vir}} = \frac{4\pi}{3} \rho_{\text{vir}} v_{\text{vir}}^3 \quad (17) \]

where \( \rho_{\text{vir}} = \Delta_{\text{vir}} \rho_{0}(z = 0) \) is a fiducial density with \( \Delta_{\text{vir}} \approx 340 \) (at redshift \( z = 0 \)) based on the dissipational spherical top-hat collapse model for the cosmological parameters of the \( \Lambda \text{CDM} \) model. The probability distribution, \( P(\lambda) \), of the spin parameter was fitted to a log-normal distribution (e.g. Frenk et al. 1988; Warren et al. 1992; Cole & Lacey 1996; Maller, Dekel & Somerville 2002; Gardner 2001),

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The results are presented in Fig. 10 and in Table 2, showing that a larger value of $B$ entails a larger spin parameter. In the lower panel of Fig. 10 — where we plot the cumulative distribution of the spin parameter — we clearly see that for a given $\lambda$ the probability to find halos with a smaller $\lambda$ is greatly lowered in $\Lambda$CDMhappi2 — or in other words, it is more likely to find halos with larger spin parameters in $\Lambda$CDMhappi2. As we will see later on (cf. Fig. 13 in Section 4.2.2), there is a mass dependence of this result: lower mass halos tend to dominate the signal seen in Fig. 10. When plotting the mass–weighted spin parameter distribution (not shown) the peaks for all four models approach each other. However, even then the HAPPI runs still show a distinct tail out to larger $\lambda$ values.

The HAPPI correction term also affects the concentration of dark matter halos. We define the concentration $c_{1/5}$ as the ratio of the virial radius $r_{\text{vir}}$ and the radius of the sphere that contains 1/5 of the virial mass (i.e. $r_{1/5}$ is defined via $M(< r_{1/5}) = M_{\text{vir}}/5$):

$$c_{1/5} = \frac{r_{\text{vir}}}{r_{1/5}}.$$  

(19)

In view of the definition (17), it follows that the average density within a radius $r_{1/5}$ is given by $(1/5)c_{1/5}^3\rho_{\text{vir}}$. Fig. 11 plots the cumulative probability distribution of the concentration of halos. We observe an obvious trend for an over-abundance of low–concentration halos in the WDM and $\Lambda$CDMhappi2 models. However, the opposite actually holds for $\Lambda$CDMhappi1, where there appear to be of order 10% more concentrated halos. The relative lack of power on scales $\lesssim 1h^{-1}$ Mpc noted in Fig. 4 for WDM and $\Lambda$CDMhappi2 is related to the relatively lower concentration (and increased smoothness for WDM) of the halos observed in these models (one must bear in mind that the halos have a virial radius $\lesssim 800h^{-1}$ kpc, see Fig. 16).

Although not shown, we confirm that the scaling of the concentration $c_{1/5}$, defined by Eq. (19), with mass $M_{\text{vir}}$ follows the same relation as the one proposed by Bullock et al. (2001b) for the ”NFW concentration” of the halo, $c_{\text{NFW}} = r_{\text{vir}}/r_s$ (see Eq. (21) for the definition of $r_s$).

### 4.2.2 Cross Correlations

While the last Section dealt with the distribution of halo properties, we now compare these properties across the models, i.e. how do these properties change in a given halo when moving from one model to another?

---

**Figure 10.** Upper plot: Measured spin parameter distribution for all four models at redshift $z = 0$, and the corresponding fit to Eq. (18). Lower plot: Measured cumulative distribution of the spin parameter.

**Figure 11.** Cumulative distribution of the concentration parameter $c_{1/5}$.

**Table 2.** Parameters derived from fitting the spin parameter distributions $P(\lambda)$ to Eq. (18).

| model       | $\lambda_0$ | $\sigma_0$ |
|-------------|-------------|------------|
| $\Lambda$CDM | 0.0287      | 0.4485     |
| $\Lambda$CDMhappi1 | 0.0333  | 0.4611     |
| $\Lambda$CDMhappi2 | 0.0596  | 0.4914     |
| AWDM        | 0.0259      | 0.4001     |
In order to find corresponding halos across the four models, we compare their individual particle content. We start with a halo in the ΛCDM model, whose particles are tagged, and locate the corresponding halo in the other three models as the one that shares the largest number of tagged particles. In the Figs. (12–14) we always plot the value of the property under investigation in the ΛCDM model, divided by the value of said property in the other model for all "cross-identified" halos. These "scatter-plots" are always accompanied by histograms, where we average the ratios presented in the respective figure in nine bins across the actual mass range. The percentages of counterparts amount to practically 100% for the HAPPI models while there are only 40% cross-identified halos in the ΛWDM model. This number though increases to about 90% when we consider halos containing more than 200 particles (i.e., $M > \sim 10^{11} h^{-1} M_\odot$), and hence we set this as a lower limit in the cross-correlation plots.

We start with the most obvious halo attribute, namely the halo mass itself. Fig. 12 shows that there is a very tight correlation for the masses of individual halos, especially for ΛCDM and ΛCDMhappi1. The scatter about the 1:1 relation (the flat line of value 1) marginally increases from ~11% (averaged 1σ-value) for ΛCDMhappi1 to ~18% for ΛCDMhappi2. At the high-mass end, the ΛCDMhappi2 halos tend to have a slightly larger mass. The most pronounced differences can be found for ΛWDM though: at the low-mass end the halos in ΛWDM appear to be less massive than their ΛCDM counterparts.

In the previous Section we showed that the HAPPI correction term lead to an increase in angular momentum by investigating the spin parameter distributions. But can we be sure that the observed rise in $\lambda$ as defined by Eq. (16) is not related to a possible decrease of virial radius $r_{\text{vir}}$ and/or virial velocity $v_{\text{vir}}$? To clarify this uncertainty we show in Fig. 13 the cross correlation of total specific angular momentum,

$$J = \frac{|L|}{M_{\text{vir}}}. \quad (20)$$

In view of the already mentioned minimal scatter in the mass of cross-identified halos between the ΛCDM model and the HAPPI models, Fig. 13 confirms the previous result that one effect of the term $C$ in Eq. (7b) is to inject angular momentum to halos. We also note that there is a mass dependence in this trend: the differences in angular momentum are on average larger for lower mass objects, this being particularly noticeable for the ΛCDMhappi2 and ΛWDM models; this "break" roughly happens at around $10^{12} h^{-1} M_\odot$.

We close this Section with an investigation of the cross-correlation of the concentration parameter $c_1/5$ in Fig. 14. The results are consistent with Fig. 11: ΛCDMhappi1 halos are more concentrated than their ΛCDM counterparts, while the excess of low-concentration halos for ΛCDMhappi2 is due to high-mass halos. The mass trend already noted in Fig. 13 can also be acknowledged in this figure.

Finally, we mention as a general property that the dispersion in the scatter plots increases as the halo mass diminishes. We attribute most of this scatter to differences in the halo’s formation history, but a detailed study as function of redshift is required which we will postpone to a later paper.

**Figure 12.** Ratio of halo masses for cross-identified objects. The virial mass $M_{\text{vir}}$ is given in units of $h^{-1} M_\odot$. The histograms represent the mean ratio in the respective bin.

Moreover, numerical effects could also contribute to some extent.

4.2.3 A Closer View of Individual Halos

A visual representation of the two most massive halos in all four models is given in Fig. 15. This figure nicely demonstrates the result regarding the lower concentrations in (high mass) ΛCDMhappi2 halos: the second most massive halo does not even show a distinct centre in ΛCDMhappi2 and appears more "puffy" than in any of the other models.

The question now arises whether the density profiles of the HAPPI models can still be fitted by the universal density profile advocated by Navarro, Frenk & White (NFW, 1997)

$$\rho(r) = \frac{\rho_s r_s^2}{r(r_s + r)^2}. \quad (21)$$

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Fig. 16 now shows $\rho(r)$ and the corresponding best fits to NFW profiles for a selection of halos covering the mass range from the most massive one (upper left) to rather light halos (lower right) containing a mere 300 particles. For (nearly) all $\Lambda$CDMhappi2 halos we observe a relative flattening in the central regions.

In order to gauge the quality of the fits (for the 16 presented sample profiles) we calculate the $\chi^2$ value defined as

$$
\chi^2 = \frac{1}{N_{\text{bins}}} \sum_{i=1}^{N_{\text{bins}}} \left( \frac{\rho_i - \rho_{\text{NFW}}(r_i)}{\rho_{\text{NFW}}(r_i)} \right)^2,
$$

where $\rho_i$ are the binned density profiles derived from the simulation data and $\rho_{\text{NFW}}$ the best-fit NFW profiles. This analysis then indicates that all four models are equally well fit by Eq. (21) with $\chi^2$ varying in the range $\chi^2 \sim 0.02-0.05$ depending on the weighing scheme applied for each individual bin. This entails that the dark matter halos of the HAPPI runs still exhibit the rather infamous "cusp" at the center.

We remind the reader again that the force resolution throughout the runs varies. Whereas $\Lambda$CDM, $\Lambda$CDMhappi1, and $\Lambda$WDM reached $2.5h^{-1}$ kpc resolution, $\Lambda$CDMhappi2 reliably resolves structures only on scales larger than $10h^{-1}$ kpc. Moreover, the resolution can also change from halo to halo due to the adaptive mesh nature of both the halo finder and the $N$-body code: not all halo centres lie on the finest grid level reached in the simulation. However, we plot profiles starting from the distance $r_{\text{min}}$ that corresponds to a sphere containing at least 10 particles (and hence $r_{\text{min}}$ can be actually smaller than the nominal resolution of the simulation).

For the same set of halos we present in Fig. 17 the rotation curves out to half the respective virial radius. The rotational velocity $v_{\text{circ}}(r)$ is defined as
There are a number of interesting observations to discuss now. We find that in nearly every halo the ΛCDMhappi1 rotation curve rises to higher values than any of the other models. While the maximum is still at comparable distances in ΛCDM and ΛCDMhappi1, the latter shows a steeper inner increase and a subsequent faster decline to nearly the same level in the “outer” parts. Moreover, the ΛCDMhappi1 rotation curves are always slightly above the corresponding ΛCDM curves. Quite the opposite is true for ΛCDMhappi2. Here we find that in most of the cases the circular rotation values at a given radius are substantially smaller than in ΛCDM. However, this difference becomes less prominent in lower mass systems and the flat part of the rotation curve nearly reaches the same level as ΛCDM. This discrepancy in the trends between ΛCDMhappi1 and ΛCDMhappi2 for high-mass halos is likely related to the also opposite trends concerning the concentration, Figs. 11 and 14, and the small-scale power, Fig. 4.

5 DISCUSSION & CONCLUSIONS

We have presented a series of cosmological N-body simulations which made use of the hydrodynamic approach to the evolution of structures (Domínguez 2000). This approach is novel in that it deals with the mass density and velocity fields with explicit account of the coarse-grained nature intrinsic to any approach of solving, for instance, Poisson’s equation via Monte Carlo sampling of phase-space. This N-body approach unavoidably introduces finite resolution effects and there have been systematic studies of the consequences in the context of cosmological structure formation (Kuhlman, Melott & Shandarin 1996; Splinter et al. 1998; Moore et al. 1998; Knebe et al. 2000; Power et al. 2003). N-body simulations invariably neglect the dynamical effect of sub-resolution degrees of freedom altogether. For the first time we have run simulations including a physical model of the coupling to the neglected scales. N-body codes are usually viewed as integrators of the Vlasov–Poisson system of equations. However, we have argued how grid-based N-body codes such as MLAPM can be reinterpreted to integrate hydrodynamic-like equations for the mass density and velocity fields.

The additional, correction term introduced in the hydrodynamic approach is proportional to a “coupling constant” B which depends on the smoothing window used to calculate the coarse-grained fields. It is found to be B = 1/4 for the triangular-shaped-cloud window used throughout the N-body code MLAPM. In order to get a better understanding of the effects of the correction term onto the evolution of cosmic structures we also performed a simulation with a higher value B = 1 — this later model is not physically motivated but rather serves as an "academic toy model" for comparison. The standard ΛCDM simulation can be understood as another HAPPI run with the value B = 0. In order to allow for a better comparison with other feasible alternatives to the concordance ΛCDM model as well as to better gauge the influence of the correction term, we also simulated the evolution of the same structures in a ΛWDM universe.

In this work we concentrated on the comparison of the four simulations at redshift z = 0. We analyzed the resulting structures in two complementary manners: global properties of the mass density and velocity fields, on the one hand, and specific properties of DM halos, on the other hand. We find appreciable differences between the B ≠ 0 runs and the reference (B = 0) ΛCDM run, even though the force due to the correction terms are for most particles one or even two orders of magnitude smaller than the total force (cf. Fig. 2). Most remarkably, the correction term favors the proliferation of low-mass halos, giving the mass distribution a more "grainy" aspect, as well as the gain of angular momentum specially by low-mass halos, which also shows up in a ve-
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Figure 16. Density profiles for halos at redshift $z = 0$. The number printed into each halo panel is the mass of the halo in units of $h^{-1} M_\odot$. The vertical lines to the right indicate the respective virial radius while the two vertical lines to the left indicate the spatial resolution of the ΛCDM, ΛCDMhappi1, ΛWDM (dashed) and ΛCDMhappi2 (dotted) run, respectively.

The relatively small quantitative differences between the $B = 1/4$ and $B = 0$ runs evidenced in the properties that we have measured suggest that the $B = 1/4$ correction term could be considered a small perturbation to the $B = 0$ evolution. By contrast, the results of the $B = 1$ run indicate that the correction term should not be treated as a perturbation in this case.

Our results agree with the theoretical expectation for the qualitative behavior of the correction term, which models the effect of small-scale tidal torques and shear stresses (Domínguez 2000, 2002; Buchert & Domínguez 2005). We observed that the correction term is dominant preferentially in walls at high redshifts, and later on in filaments, regions of mass accretion onto halos, halo centers as well as in regions of particular dynamical activity (e.g., mergers), that is, regions of large gradients in the fields, in concordance with the form of the correction term (8). The term is expected to act as a drain of kinetic energy in collapsing regions: this can explain the formation of small clusters of particles, which would otherwise fly by each other — instead, they can be gravitationally confined by a potential well that is lower than in the $B = 0$ model. This could explain the low mass halo proliferation for $B = 1/4$ or $B = 1$, as well as the observed tendency of halos to attain a slightly more concentrated con-

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figuration in the case of $B = 1/4$, when the correction term can be considered a small perturbation. If $B = 1$, on the other hand, the loss of kinetic energy is apparently so important that, in some cases of halos in regions of high dynamical activity, dynamical relaxation and coalescence are slowed down noticeably, leading to a multiple-core structure. These not completely relaxed halos would then have a lower concentration and a lower mass than their LCDM counterparts, similarly to the simulation results.

We further confirmed explicitly that the correction term acts as a source of vorticity. This relates directly to the gain in angular momentum of halos, which tends to increase with growing value of $B$, especially at the low-mass end of the halo distribution.

Finally, we remark that our findings agree qualitatively with conclusions following from a comparative study of identical initial conditions evolved at different resolutions. We have run a series of test simulations where we either switched on the HAPPI correction term or increased the actual force resolution; both methods lead to comparable results that are in qualitative agreement with the conclusions presented here. For a more quantitative analysis we though refer the reader to a future paper in preparation where we will investigate the relationship between HAPPI simulations and higher-resolution ones in more detail. The proliferation of small halos with increasing resolution has also been reported by other authors in and around (massive) halos (Klypin et al. 1999, Moore et al. 1998) as well as in voids (Gottlöber et al. 2003). Concerning the generation of angular momentum though, the relevance for the formation of realistic disk galaxies has yet to be determined but there are clear indications that this task requires good mass and force resolution (Governato et al. 2004). In conclusion, the HAPPI implementation seems indeed to be qualitatively consistent with what one expects from higher resolution simulations and hence may provide a framework for a better understanding of resolution effects in $N$-body simulations. However, further work is required to substantiate this possibility.

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REFERENCES

Bento M.C., et al. Phys. Rev. D 62, (2000)
Bode, P., Ostriker, J. P., Turok, N., ApJ 539, 93 (2001)
Buchert T., Domínguez A., A&A 438, 443 (2005)
Buchert T., Ehlers J., A&A 320, 1 (1997)
Bullock J.S., Dekel A., Kolatt T.S, Kravtsov A.V., Klypin A.A, Colín P., Avila-Reese V., Valenzuela O., ApJ 588, 25 (2003)
Dalal N., Kochanek C., ApJ 572, 25 (2002)
Dominguez A., Progress in Astronomy 19, 32 (Suppl.) (2001), astro-ph/0102197
Evans N.W., Witt H.J., MRAS 345, 1351 (2003)
Frenkel C.S., White S.D.M., Davis M., Efstathiou G., ApJ 327, 507 (1988)
Fukugishi T., Kawai A., Makino J., ApJ 606, 625 (2004)
Gardner J., ApJ 557, 616 (2001)
Gill S.P.D., Knebe A., Gibson B.K., MRAS 351, 399 (2004)
Gingold R.A., Monaghan J.J., MRAS 181, 375 (1977)
Gottlöber S., Kokkonen P., Gottlöber S., Hoffman Y., MRAS 344, 715 (2003)
Governato F., Mihos J.C., Bird B., Quinn T., Stadel J., Lake G., ApJ 607, 688 (2004)
Gurbatov S.N., Saichev A.I., Shandarin S.F., MRAS 236, 385 (1989)
Hoeknes R.W., Eastwood J.W., Computer Simulations Using Particles, Bristol: Adam Hilger (1988)
Hoefnagels J., Yese G., Gottlöber S., Springel V., astro-ph/0501304
Klypin A.A., Kravtsov A.V., Valenzuela O., Prada F., ApJ 522, 82 (1999)
Knebe A., Devriendt J.E.G., Mahmood A., Silk J., MRAS 325, 845 (2001)
Knebe A., Green A., Binney J., MRAS 328, 813 (2002)
Knebe A., Kravtsov A.V., Gottlöber S., Klypin A.A., MRAS 317, 630 (2000)
Kuhlen B., Melott A.L., Shandarin S.F., ApJ 470, L41 (1996)
Little B., Knebe A., Islam R.R., MRAS 341, 617 (2003)
Lucy L.B., ApJ 82, 1013 (1977)
Maller A.H., Dekel A., Somerville R.S., MRAS 329, 423 (2002)
Maccio K., Willman B., J. Stat. Phys. 64, 843 (1991)
Maccio K., Stoyan D. (Eds.), Statistical Physics and Spatial Statistics, Springer–Verlag (2000)
Moore B., Governato F., Quinn T., Stadel J., Lake G. ApJ 499, L5 (1998)
Navarro J., Frenk C.S., White S.D.M., ApJ 490, 493 (1997)
Nehe R., Stochastische Geometrie und Perkolation, Thesis work, University of Munich (2003)
Pope S.B., Turbulent Flows, Cambridge UP (2000)
Power C., Navarro J.F., Jenkins A., Frenk C.S., White S.D.M., Springel V., Stadel J., Quinn T., MRAS 338, 14 (2003)
Sahni V., Coles, P., Phys. Rep. 262, 1 (1995)
Sand, D. J., Treu, T., Smith, G. P., Ellis, R. S., ApJ 604, 88 (2004)
Schechter & Wambsganss ApJ 580, 685 (2002)
Simon J.D., Bolatto A.D., Leroy, A., Blitz L., Gates E.L., ApJ 621, 757 (2005)
Somerville, R. ApJ 572, 23 (2002)
Spergel D.N., Steinhardt P.J., Phys. Rev. Lett. 84, 3760 (2000)
Spergel, D. N., Verde, L., Peiris, H. V., Komatsu, E., Nolta, M. R., Bennett, C. L., Halpern, M., Hinshaw, G., Jarosik, N., Kogut, A., Limon, M., Meyer, S. S., Page, L., Tucker, G. S., Weiland, J. L., Wollack, E., Wright, E. L., ApJ Suppl. 148, 175 (2003)
Spergel, D. N. et al., 2006, submitted to ApJ, astro-ph/0603494
Splinter R.J., Melott A.L., Shandarin S.F., Suto Y., ApJ 497, 38 (1998)
Swaters R. A., Madore B. F., van den Bosch F. C., Balcells M., ApJ 583, 732 (2003)
Tully, B., Somerville, R., Trentham, N., Verheijen, M., ApJ 569, 573 (2002)
Warren M.S., Quinn P.J., Salmon J.K., Zurek W.H., ApJ 399, 405 (1992)
Yess C., Shandarin S.F., ApJ 465, 2 (1996)

APPENDIX A: CONSERVATION OF ENERGY, MOMENTUM AND ANGULAR MOMENTUM

The coupling to the small–scales modelled by C in Eqs. (7) injects energy into (or drains energy from) the resolved spatial scales. Given the simulation box $V_{\text{box}}$ with periodic boundary conditions, we can define the total peculiar kinetic energy and the total peculiar mean–field potential energy as follows:

$$ K = \frac{1}{2} \rho a^3 \int_{V_{\text{box}}} \, dx \, \rho \mathbf{u}^2, \quad (A1) $$

$$ \mathbf{r}_{\text{mf}} = \frac{1}{2 \rho_b} \int_{V_{\text{box}}} \, dxdy \, \left[ \rho(x) - \rho_b \right] \left[ \rho(y) - \rho_b \right] S(x-y), \quad (A2) $$

where the time–independent, symmetric kernel $S(x)$ is the solution of the problem

$$ \nabla^2 S = 4\pi G (a^3 \rho_b) \delta_{\text{Dirac}}(x) \quad (x \in V_{\text{box}}). $$

The mean–field gravitational acceleration is given by

$$ \mathbf{w}_{\text{mf}}(x) = -\frac{1}{\rho_b a^2} \nabla \int_{V_{\text{box}}} \, dy \, \left[ \rho(y) - \rho_b \right] S(x-y). \quad (A3) $$

Then, it is easy to show from Eqs. (7) that the total peculiar energy $\mathcal{H} = K + U_{\text{mf}}$ satisfies the evolution equation

$$ \frac{d\mathcal{H}}{dt} = -H(2K + U_{\text{mf}}) + a^3 \int_{V_{\text{box}}} \, dx \, \rho \mathbf{u} \cdot \mathbf{C}. \quad (A4) $$

This is a generalization of the Layzer–Irwin equation. Due to the correction term, the condition of “mean–field virialization”, $2K + U_{\text{mf}} = 0$, does not imply a time–independent $\mathcal{H}$. The quantity

$$ I = a \mathcal{H} + \int da \, K \quad (A5) $$

is conserved by the original Layzer–Irwin equation but is not constant according to the generalized equation (A4).

Concerning momentum and angular momentum, Eqs. (7) do not violate global conservation. Let $V(t)$ denote a time–dependent volume defined by the condition that the
mass enclosed is constant, i.e., a Lagrangian domain. The peculiar momentum of the domain,
\[ \mathcal{P}_V = a^3 \int_{V(t)} dx \, \rho u, \tag{A6} \]
verifies the evolution equation
\[ \frac{d\mathcal{P}_V}{dt} = -H \mathcal{P}_V + a^3 \int_{V(t)} dx \, \rho (\mathbf{w}^{mf} + \mathbf{C}). \tag{A7} \]
The correction term can be written as the divergence of a tensor (Buchert & Domínguez 2005)
\[ \rho \mathbf{C}_i = B L^2 \partial_i \left[ \rho (\partial_j \mathbf{w}^{mf}_j) + 2\pi \gamma G a^2 \partial_{ij} - \frac{1}{a} \rho (\partial_k u_i) (\partial_k u_j) \right], \]
so that its contribution in Eq. (A7) is a surface integral over the border of \( V(t) \). In particular, when \( V(t) = V_{\text{box}} \), this surface integral vanishes by periodic boundary conditions and, since the contribution by \( \mathbf{w}^{mf} \) also vanishes in this case, Eq. (A7) states that \( a \mathcal{P}_{V_{\text{box}}} \) is a constant of motion.

In the same manner, one defines the angular momentum of the domain \( V(t) \) with respect to its center of mass \( \mathbf{X}_{\text{cm}}(t) \),
\[ \mathcal{L}_V = a^4 \int_{V(t)} dx \, (\mathbf{x} - \mathbf{X}_{\text{cm}}) \times \rho \mathbf{u}. \tag{A8} \]
The evolution equation for this quantity is
\[ \frac{d\mathcal{L}_V}{dt} = a^4 \int_{V(t)} dx \, (\mathbf{x} - \mathbf{X}_{\text{cm}}) \times \rho (\mathbf{w}^{mf} + \mathbf{C}). \tag{A9} \]
The contribution by \( \mathbf{C} \) can be written again as a surface integral over the border of \( V(t) \). Thus, when \( V(t) = V_{\text{box}} \), Eq. (A9) predicts that \( \mathcal{L}_{V_{\text{box}}} \) is also a constant of motion.

As discussed in Sec. 2, the correction \( \mathbf{C} \) is a source of vorticity in the otherwise curl–free flow of the "dust model". Eq. (A9) shows that the correction also affects the evolution of the angular momentum of a domain. In this case, however, the effect may not be so noticeable, since already at the level of the "dust model" there are tidal torques by the mean–field gravity \( \mathbf{w}^{mf} \). Moreover, since the contribution by \( \mathbf{C} \) is a surface integral, it may be expected to be less relevant for a larger domain \( V(t) \). Actually, we can rewrite the definition (A8) by inserting the identity \( 2(\mathbf{x} - \mathbf{X}_{\text{cm}}) = \nabla |\mathbf{x} - \mathbf{X}_{\text{cm}}|^2 \) as
\[ \mathcal{L}_V = \frac{1}{2} a^4 \int_{\partial V} d\mathbf{S} \times \rho \mathbf{u} |\mathbf{x} - \mathbf{X}_{\text{cm}}|^2 - \frac{1}{2} a^4 \int_V dx \, |\mathbf{x} - \mathbf{X}_{\text{cm}}|^2 [\rho \mathbf{\omega} - (\nabla \rho) \times \mathbf{u}], \tag{A10} \]
and we see that vorticity is but one contribution to the angular momentum of a Lagrangian domain.

APPENDIX B: MINKOWSKI FUNCTIONALS OF A POISSON DISTRIBUTION

As an illustration of the dependence of the MFs on the threshold, Fig. B1 shows the MFs of a realization of a Poisson distribution of points: 128\(^3\) particles were distributed randomly in a cubical box, and the density field \( \rho(\mathbf{x}) \) was obtained by smoothing with the window Eq. (12) in a cubic grid of 16\(^3\) nodes. The plots are symmetric about the mean value of the density (512 particles per node) and span a width \( \approx \) rms density (\( = \sqrt{512} \)) along the threshold axis.

- The volume \( V_0(\hat{\rho}) \) decreases monotonically as the threshold is increased and the high–density regions (\( \rho > \hat{\rho} \)) shrink.
- The area \( V_1(\hat{\rho}) \) first increases as the low–density regions (\( \rho < \hat{\rho} \)) expand and, after reaching a maximum, it decreases as the high–density regions (\( \rho > \hat{\rho} \)) shrink.
- The average mean curvature \( V_2(\hat{\rho}) \) increases monotonously from a negative value (\( \mathcal{S} \) is concave towards the shrinking high–density region) to a positive value (\( \mathcal{S} \) is convex towards the shrinking high–density region).
- Finally, the genus \( V_3(\hat{\rho}) \) is positive when \( \mathcal{S} \) looks bubble–like: there are many unconnected expanding low–density regions ("holes") at small \( \hat{\rho} \), and many unconnected shrinking high–density regions ("clusters") at large \( \hat{\rho} \). \( V_3 \) is negative when \( \mathcal{S} \) is predominantly saddle–shaped: one observes many intertwined high– and low–density regions ("tunnels") at intermediate \( \hat{\rho} \).