TOWARD A GEOMETRIC CONSTRUCTION OF FAKE PROJECTIVE PLANES

JONGHAE KEUM

Abstract. We give a criterion for a projective surface to become a quotient of a fake projective plane. We also give a detailed information on the elliptic fibration of a (2,3)-elliptic surface that is the minimal resolution of a quotient of a fake projective plane. As a consequence, we give a classification of $\mathbb{Q}$-homology projective planes with cusps only.

It is known that a compact complex surface with the same Betti numbers as the complex projective plane $\mathbb{P}^2$ is projective (see e.g. [BHPV]). Such a surface is called a fake projective plane if it is not isomorphic to $\mathbb{P}^2$.

Let $X$ be a fake projective plane. Then its canonical bundle is ample. So a fake projective plane is exactly a surface of general type with $p_g(X) = 0$ and $c_1(X)^2 = 3c_2(X) = 9$. By [Au] and [Y], its universal cover is the unit 2-ball $B \subset \mathbb{C}^2$ and hence its fundamental group $\pi_1(X)$ is infinite. More precisely, $\pi_1(X)$ is exactly a discrete torsion-free cocompact subgroup $\Pi$ of $PU(2,1)$ having minimal Betti numbers and finite abelianization. By Mostow’s rigidity theorem [Mos], such a ball quotient is strongly rigid, i.e., $\Pi$ determines a fake projective plane up to holomorphic or anti-holomorphic isomorphism. By [KK], no fake projective plane can be anti-holomorphic to itself. Thus the moduli space of fake projective planes consists of a finite number of points, and the number is the double of the number of distinct fundamental groups $\Pi$. By Hirzebruch’s proportionality principle [Hir], $\Pi$ has covolume 1 in $PU(2,1)$. Furthermore, Klingler [Kl] proved that the discrete torsion-free cocompact subgroups of $PU(2,1)$ having minimal Betti numbers are arithmetic (see also [Ye]).

With these information, Prasad andYeung [PY] carried out a classification of fundamental groups of fake projective planes. They describe the algebraic group $\widetilde{G}(k)$ containing a discrete torsion-free cocompact arithmetic subgroup $\Pi$ having minimal Betti numbers and finite abelianization as follows. There is a pair $(k,l)$ of number fields, $k$ is totally real, $l$ a totally complex quadratic extension of $k$. There is a central simple algebra $D$ of degree 3 with center $l$ and an involution $\iota$ of the second kind on $D$ such that $k = l^\iota$. The algebraic group $\widetilde{G}$ is defined over $k$ such that

$$\widetilde{G}(k) \cong \{z \in D|\iota(z)z = 1\}/\{t \in l|\iota(t)t = 1\}.$$ 

There is one Archimedean place $\nu_0$ of $k$ so that $\widetilde{G}(k_{\nu_0}) \cong PU(2,1)$ and $\widetilde{G}(k_\nu)$ is compact for all other Archimedean places $\nu$. The data $(k,l,D,\nu_0)$ determines $\widetilde{G}$ up.

2000 Mathematics Subject Classification. 14J29; 14J27.

Key words and phrases. fake projective plane; $\mathbb{Q}$-homology projective plane; surface of general type; properly elliptic surface.

Research supported by Basic Science Research Program through the National Research Foundation(NRF) of Korea funded by the Ministry of Education, Science and Technology (2007-C00002).
to \(k\)-isomorphism. Using Prasad’s volume formula \([P]\), they were able to eliminate most \((k, l, D, \nu)\), making a short list of possibilities where \(\Pi\)'s might occur, which yields a short list of maximal arithmetic subgroups \(\tilde{\Gamma}\) which might contain a \(\Pi\). If \(\Pi\) is contained, up to conjugacy, in a unique \(\Gamma\), then the group \(\Pi\) or the fake projective plane \(B/\Pi\) is said to belong to the class corresponding to the conjugacy class of \(\Gamma\). If \(\Pi\) is contained in two non-conjugate maximal arithmetic subgroups, then \(\Pi\) or \(B/\Pi\) is said to form a class of its own. They exhibited 28 non-empty classes \([PY]\), Addendum). It turns out that the index of such a \(\Pi\) in a \(\Gamma\) is 1, 3, 9, or 21, and all such \(\Pi\)'s in the same \(\Gamma\) have the same index.

Then Cartwright and Steger \([CS]\) have carried out a computer-based but very complicated group-theoretic computation, showing that there are exactly 28 non-empty classes, where 25 of them correspond to conjugacy classes of maximal arithmetic subgroups and each of the remaining 3 to a \(\Pi\) contained in two non-conjugate maximal arithmetic subgroups. This yields a complete list of fundamental groups of fake projective planes: the moduli space consists of exactly 100 points, corresponding to 50 pairs of complex conjugate fake projective planes.

It is easy to see that the automorphism group \(\text{Aut}(X)\) of a fake projective plane \(X\) can be given by

\[
\text{Aut}(X) \cong N(\pi_1(X))/\pi_1(X),
\]

where \(N(\pi_1(X))\) is the normalizer of \(\pi_1(X)\) in a suitable \(\tilde{\Gamma}\).

**Theorem 0.1.** \([PY],[CS],[CS2]\) For a fake projective plane \(X\),

\[
\text{Aut}(X) = \{1\}, \ C_3, \ C_2^3, \ 7 : 3,
\]

where \(C_n\) denotes the cyclic group of order \(n\), and \(7 : 3\) the unique non-abelian group of order 21.

According to \([CS],[CS2]\), 68 of the 100 fake projective planes admit a nontrivial group of automorphisms.

Let \((X, G)\) be a pair of a fake projective plane \(X\) and a non-trivial group \(G\) of automorphisms. In \([K08]\), all possible structures of the quotient surface \(X/G\) and its minimal resolution were classified:

**Theorem 0.2.** \([K08]\)

1. If \(G = C_3\), then \(X/G\) is a \(\mathbb{Q}\)-homology projective plane with 3 singular points of type \(\frac{1}{3}(1, 2)\) and its minimal resolution is a minimal surface of general type with \(p_g = 0\) and \(K^2 = 3\).
2. If \(G = C_2^3\), then \(X/G\) is a \(\mathbb{Q}\)-homology projective plane with 4 singular points of type \(\frac{1}{4}(1, 2)\) and its minimal resolution is a minimal surface of general type with \(p_g = 0\) and \(K^2 = 1\).
3. If \(G = C_7\), then \(X/G\) is a \(\mathbb{Q}\)-homology projective plane with 3 singular points of type \(\frac{1}{4}(1, 5)\) and its minimal resolution is a \((2, 3)\)-, \((2, 4)\)-, or \((3, 3)\)-elliptic surface.
4. If \(G = 7 : 3\), then \(X/G\) is a \(\mathbb{Q}\)-homology projective plane with 4 singular points, 3 of type \(\frac{1}{3}(1, 2)\) and one of type \(\frac{1}{4}(1, 5)\), and its minimal resolution is a \((2, 3)\)-, \((2, 4)\)-, or \((3, 3)\)-elliptic surface.

Here a \(\mathbb{Q}\)-homology projective plane is a normal projective surface with the same Betti numbers as \(\mathbb{P}^2\). A fake projective plane is a nonsingular \(\mathbb{Q}\)-homology projective plane, hence every quotient is again a \(\mathbb{Q}\)-homology projective plane. An
(a, b)-elliptic surface is a relatively minimal elliptic surface over \( \mathbb{P}^1 \) with two multiple fibres of multiplicity \( a \) and \( b \) respectively. It has Kodaira dimension 1 if and only if \( a \geq 2, b \geq 2, a + b \geq 5 \). It is an Enriques surface iff \( a = b = 2 \), and it is rational iff \( a = 1 \) or \( b = 1 \). An \((a, b)\)-elliptic surface has \( p_g = q = 0 \), and by \([D]\) its fundamental group is the cyclic group of order the greatest common divisor of \( a \) and \( b \).

**Remark 0.3.** (1) Since \( X/G \) has rational singularities only, \( X/G \) and its minimal resolution have the same fundamental group. Let \( \bar{\Gamma} \) be the maximal arithmetic subgroup of \( \text{PU}(2, 1) \) containing \( \pi_1(X) \). There is a subgroup \( \tilde{G} \subset \bar{\Gamma} \) such that \( \pi_1(X) \) is normal in \( \tilde{G} \) and \( G = \tilde{G}/\pi_1(X) \). Thus,

\[ X/G \cong B/\tilde{G}. \]

It is well known (cf. \([\text{Arr}]\)) that

\[ \pi_1(B/\tilde{G}) \cong \tilde{G}/H, \]

where \( H \) is the minimal normal subgroup of \( \tilde{G} \) containing all elements acting non-freely on the 2-ball \( B \). In our situation, it can be shown that \( H \) is generated by torsion elements of \( \tilde{G} \), and Cartwright and Steger have computed, along with their computation of the fundamental groups, the quotient group \( \tilde{G}/H \) for each pair \((X, G)\).

- **[CS]** If \( G = C_3 \), then
  \[ \pi_1(X/G) \cong \{1\}, C_2, C_3, C_4, C_6, C_7, C_{13}, C_{14}, C_2^2, C_2 \times C_4, S_3, D_8 \text{ or } Q_8, \]
  where \( S_3 \) is the symmetric group of order 6, and \( D_8 \) and \( Q_8 \) are the dihedral and quaternion groups of order 8.

- **[CS2]** If \( G = C_3^2 \) or \( C_7 \) or \( 7 : 3 \), then
  \[ \pi_1(X/G) \cong \{1\} \text{ or } C_2. \]

  This eliminates the possibility of \((3, 3)\)-elliptic surfaces in Theorem 0.2, as \((3, 3)\)-elliptic surfaces have \( \pi_1 = C_3 \).

(2) It is interesting to consider all ball quotients which are covered irregularly by a fake projective plane. Indeed, Cartwright and Steger have considered all subgroups \( \tilde{G} \subset PU(2, 1) \) such that \( \pi_1(X) \subset \tilde{G} \subset \bar{\Gamma} \) for some maximal arithmetic subgroup \( \bar{\Gamma} \) and some fake projective plane \( X \), where \( \pi_1(X) \) is not necessarily normal in \( \tilde{G} \). It turns out \([CS2]\) that, if \( \pi_1(X) \) is not normal in \( \tilde{G} \), then there is another fake projective plane \( X' \) such that \( \pi_1(X') \) is normal in \( \tilde{G} \), hence \( B/\tilde{G} \cong X'/G' \) where \( G' = \tilde{G}/\pi_1(X') \). Thus such a general subgroup \( \tilde{G} \) does not produce a new surface.

It is a major step toward a geometric construction of a fake projective plane to construct a \( \mathbb{Q} \)-homology projective plane satisfying one of the descriptions (1)-(4) from Theorem 0.2. Suppose that one has such a \( \mathbb{Q} \)-homology projective plane. Then, can one construct a fake projective plane by taking a suitable cover? In other words, does the description (1)-(4) from Theorem 0.2 characterize the quotients of fake projective planes? The answer is affirmative in all cases.

**Theorem 0.4.** Let \( Z \) be a \( \mathbb{Q} \)-homology projective plane satisfying one of the descriptions (1)-(4) from Theorem 0.2.
(1) If $Z$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 3$, then there is a $C_3$-cover $X \to Z$ branched at the three singular points of $Z$ such that $X$ is a fake projective plane.

(2) If $Z$ is a $\mathbb{Q}$-homology projective plane with 4 singular points of type $\frac{1}{4}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 1$, then there is a $C_4$-cover $Y \to Z$ branched at three of the four singular points of $Z$ and a $C_3$-cover $X \to Y$ branched at the three singular points on $Y$, the pre-image of the remaining singularity on $Z$, such that $X$ is a fake projective plane.

(3) If $Z$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{3}(1,5)$ and its minimal resolution is a $(2,3)$- or $(2,4)$-elliptic surface, then there is a $C_7$-cover $X \to Z$ branched at the three singular points of $Z$ such that $X$ is a fake projective plane.

(4) If $Z$ is a $\mathbb{Q}$-homology projective plane with 4 singular points, 3 of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{5}(1,5)$, and its minimal resolution is a $(2,3)$- or $(2,4)$-elliptic surface, then there is a $C_3$-cover $Y \to Z$ branched at the three singular points of type $\frac{1}{3}(1,2)$ and a $C_7$-cover $X \to Y$ branched at the three singular points, the pre-image of the singularity on $Z$ of type $\frac{1}{3}(1,5)$, such that $X$ is a fake projective plane.

In the case (4), we give a detailed information on the types of singular fibres of the elliptic fibration on the minimal resolution of $Z$.

**Theorem 0.5.** Let $Z$ be a $\mathbb{Q}$-homology projective plane with 4 singular points, 3 of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{5}(1,5)$. Assume that its minimal resolution $\tilde{Z}$ is a $(2,3)$-elliptic surface. Then the following hold true.

1. The triple cover $Y$ of $Z$ branched at the three singular points of type $\frac{1}{3}(1,2)$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{5}(1,5)$. The minimal resolution $\tilde{Y}$ of $Y$ is a $(2,3)$-elliptic surface, and every fibre of the elliptic fibration on $\tilde{Z}$ does not split in $\tilde{Y}$.
2. The elliptic fibration on $\tilde{Z}$ has 4 singular fibres of type $\mu_1I_3 + \mu_2I_3 + \mu_3I_3 + \mu_4I_3$, where $\mu_i$ is the multiplicity of the fibre.
3. The elliptic fibration on $\tilde{Y}$ has 4 singular fibres of type $\mu_9I_9 + \mu_1I_1 + \mu_2I_1 + \mu_3I_1$.

The case where $\tilde{Z}$ is a $(2,4)$-elliptic surface was treated in [K10]. The assertions (2) and (3) of Theorem 0.5 were given without proof in Corollary 4.12 and 1.4 of [K08].

As a consequence of Theorem 0.4 and the result of Cartwright and Steger ([CS], [CS2]), we give a classification of $\mathbb{Q}$-homology projective planes with cusps, i.e., singularities of type $\frac{1}{3}(1,2)$, only.

**Theorem 0.6.** Let $Z$ be a $\mathbb{Q}$-homology projective plane with cusps only. Then $Z$ is isomorphic to one of the following:

1. $X/C_3$, where $X$ is a fake projective plane with an order 3 automorphism;
2. $X/C_3^2$, where $X$ is a fake projective plane with $\text{Aut}(X) = C_3^2$;
3. $\mathbb{P}^2/(\sigma)$, where $\sigma$ is the order 3 automorphism given by $\sigma(x, y, z) = (x, \omega y, \omega^2 z)$;
\(\mathbb{P}^2/\langle \sigma, \tau \rangle\), where \(\sigma\) and \(\tau\) are the commuting order 3 automorphisms given by

\[\sigma(x, y, z) = (x, \omega y, \omega^2 z), \quad \tau(x, y, z) = (z, ax, a^{-1}y),\]

where \(a\) is a non-zero constant and \(\omega = \exp(\frac{2\pi i}{\sqrt{-1}})\).

**Remark 0.7.** In differential topology, they use two notions “exotic \(\mathbb{P}^2\)” and “fake \(\mathbb{P}^2\).” An exotic \(\mathbb{P}^2\) is a simply connected symplectic 4-manifold homeomorphic to, but not diffeomorphic to \(\mathbb{P}^2\). The existence of such a 4-manifold is not known yet. It does not exist in complex category.

**Notation**

- \(K_Y\): the canonical class of \(Y\)
- \(b_i(Y)\): the \(i\)-th Betti number of \(Y\)
- \(e(Y)\): the topological Euler number of \(Y\)
- \(q(X) := \dim H^1(X, \mathcal{O}_X)\), the irregularity of a surface \(X\)
- \(p_g(X) := \dim H^2(X, \mathcal{O}_X)\), the geometric genus of a surface \(X\)

**1. Preliminaries**

First, we recall the topological and holomorphic Lefschetz fixed point formulas.

**Topological Lefschetz Fixed Point Formula.** Let \(M\) be a topological manifold of dimension \(m\) admitting a homeomorphism \(\sigma\). Then the Euler number of the fixed locus \(M^\sigma\) of \(\sigma\) is equal to the alternating sum of the trace of \(\sigma^*\) acting on \(H^j(M, \mathbb{Z})\), i.e.,

\[e(M^\sigma) = \sum_{j=0}^{m} (-1)^j Tr\sigma^*|H^j(M, \mathbb{Z}).\]

**Holomorphic Lefschetz Fixed Point Formula.** (AS3, p. 567) Let \(M\) be a complex manifold of dimension 2 admitting an automorphism \(\sigma\). Let \(p_1, \ldots, p_l\) be the isolated fixed points of \(\sigma\) and \(R_1, \ldots, R_k\) be the 1-dimensional components of the fixed locus \(S^\sigma\). Then

\[\sum_{j=0}^{2} (-1)^j Tr\sigma^*|H^1(M, \mathcal{O}_M) = \sum_{j=1}^{l} \frac{1}{\det(I - d\sigma)|T_{p_j}|} + \sum_{j=1}^{k} \left\{ \frac{1 - g(R_j)}{1 - \xi_j} - \frac{\xi_j R_j^2}{(1 - \xi_j)^2} \right\},\]

where \(T_{p_j}\) is the tangent space at \(p_j\), \(g(R_j)\) is the genus of \(R_j\) and \(\xi_j\) is the eigenvalue of the differential \(d\sigma\) acting on the normal bundle of \(R_j\) in \(M\).

Assume further that \(\sigma\) is of finite and prime order \(p\). Then

\[\frac{1}{p-1} \sum_{i=1}^{p-1} \sum_{j=0}^{2} (-1)^j Tr\sigma^*|H^1(M, \mathcal{O}_M) = \sum_{i=1}^{p-1} a_i r_i + \sum_{j=1}^{k} \left\{ \frac{1 - g(R_j)}{2} + \frac{(p + 1)R_j^2}{12} \right\},\]

where \(r_i\) is the number of isolated fixed points of \(\sigma\) of type \(\frac{1}{p}(1, i)\), and

\[a_i = \frac{1}{p-1} \sum_{j=1}^{p-1} \frac{1}{(1 - \zeta^j)(1 - \xi^j)} \]

with \(\zeta = \exp(\frac{2\pi i \sqrt{-1}}{p})\), e.g., \(a_1 = \frac{5-p}{12}, \ a_2 = \frac{11-p}{24}, \) etc.
For a complex manifold $M$ of dimension $2$ with $K_M^2 = 3c_2(M) = 9$, it is known that
\[ p_g(M) = q(M) \leq 2. \]
Indeed, such a surface $M$ has $\chi(O_M) = 1$, $p_g(M) = q(M)$, and is a ball-quotient or $\mathbb{P}^2$. Since $c_2(M) = 3$, $M$ cannot be fibred over a curve of genus $\geq 2$. Thus by Castelnuovo-de Franchis theorem, $p_g(M) \geq 2q(M) - 3$, which implies $p_g(M) = q(M) \leq 3$. The case of $p_g(M) = q(M) = 3$ was eliminated by the classification result of Hacon and Pardini [HP] (see also [Pi] and [CCM]).

**Proposition 1.1.** Let $M$ be a complex manifold $M$ of dimension $2$ with $K_M^2 = 3c_2(M) = 9$. Then, the following hold true.

1. If $M$ admits an order $7$ automorphism $\sigma$ with isolated fixed points only, then $p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = p_g(M) = q(M)$, and $M/\langle \sigma \rangle$ has either $3$ singular points of type $\frac{1}{7}(1,5)$ or $2$ singular points of type $\frac{1}{7}(1,2)$ and $1$ singular point of type $\frac{1}{6}(1,6)$.
2. If $M$ has $p_g(M) = q(M) = 1$ and admits an order $3$ automorphism $\sigma$ with isolated fixed points only, then
   a) $p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = 0$, and $M/\langle \sigma \rangle$ has $6$ singular points of type $\frac{1}{3}(1,1)$; or
   b) $p_g(M/\langle \sigma \rangle) = 1$, $q(M/\langle \sigma \rangle) = 0$, and $M/\langle \sigma \rangle$ has $3$ singular points of type $\frac{1}{3}(1,1)$ and $6$ singular points of type $\frac{1}{3}(1,2)$; or
   c) $p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = 1$, and $M/\langle \sigma \rangle$ has $3$ singular points of type $\frac{1}{3}(1,2)$.

**Proof.** Note that $M$ cannot admit an automorphism of finite order acting freely, because $\chi(O_M) = 1$ not divisible by any integer $\geq 2$.

1. By Hodge decomposition theorem,
\[ Tr\sigma^*|H^1(M,\mathbb{Z}) = Tr\sigma^*|H^1(M,\mathbb{C}) = Tr\sigma^*|(H^{0,1}(M) \oplus H^{1,0}(M)). \]

Note that this number is an integer. Let $\zeta = \exp(\frac{2\pi i}{7})$.

Assume that $p_g(M) = q(M) = 2$. Let $\zeta^i$ and $\zeta^j$ be the eigenvalues of $\sigma^*$ acting on $H^{0,1}(M)$. Then
\[ Tr\sigma^*|H^1(M,\mathbb{Z}) = \zeta^i + \zeta^j + \bar{\zeta}^i + \bar{\zeta}^j, \]
and this is an integer iff $\zeta^i = \zeta^j = 1$. This implies that $Tr\sigma^*|H^{0,1}(M) = 2$ and $q(M/\langle \sigma \rangle) = q(M) = 2$. By the Topological Lefschetz Fixed Point Formula, $e(M^\sigma) = -6 + Tr\sigma^*|H^2(M,\mathbb{Z})$, so $6 < Tr\sigma^*|H^2(M,\mathbb{Z})$. Since
\[ \text{rank } H^2(M,\mathbb{Z}) = 1 + 4q(M) = 9, \]
it follows that $Tr\sigma^*|H^2(M,\mathbb{Z}) = 9$ and $e(M^\sigma) = 3$. In particular, $Tr\sigma^*|H^{0,2}(M) = 2$ and $p_g(M/\langle \sigma \rangle) = p_g(M) = 2$. By the Holomorphic Lefschetz Fixed Point Formula,
\[ 1 = \frac{1}{6}r_1 + \frac{1}{6}(r_2 + r_4) + \frac{1}{3}(r_3 + r_5) + \frac{2}{3}r_6, \]
where $r_i$ is the number of isolated fixed points of $\sigma$ of type $\frac{1}{6}(1, i)$. Since
\[ \sum r_i = e(M^\sigma) = 3, \]
we have two solutions:
\[ r_3 + r_5 = 3, r_1 = r_2 = r_4 = r_6 = 0; \quad r_2 + r_4 = 2, r_6 = 1, r_1 = r_3 = r_5 = 0. \]
Assume that $p_g(M) = q(M) = 1$. By the same argument, $Tr\sigma^*|H^{0,1}(M) = 1$,
$Tr\sigma^*|H^2(M, \mathbb{Z}) = 5$, $e(M^\sigma) = 3$ and $Tr\sigma^*|H^{0,2}(M) = 1$.
Assume that $p_g(M) = q(M) = 0$. Then $Tr\sigma^*|H^{0,1}(M) = Tr\sigma^*|H^{0,2}(M) = 0$,
$Tr\sigma^*|H^2(M, \mathbb{Z}) = 1$ and $e(M^\sigma) = 3$.

First note that $p_g(M/\langle \sigma \rangle) \leq 1$ and $q(M/\langle \sigma \rangle) \leq 1$.
Let $\zeta^i$ and $\bar{\zeta}^j$ be the eigenvalues of $\sigma^*$ acting on $H^{0,1}(M)$ and $H^{0,2}(M)$, respectively,
where $\zeta = \exp(\frac{2\pi i}{3})$.

Assume that $p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = 0$. Then $\zeta^i \neq 1$ and $\bar{\zeta}^j \neq 1$.

$$Tr\sigma^*|H^1(M, \mathbb{Z}) = Tr\sigma^*|(H^{0,1}(M) \oplus H^{1,0}(M)) = \zeta^i + \bar{\zeta}^j = -1,$$
$$Tr\sigma^*|(H^{0,2}(M) \oplus H^{2,0}(M)) = \zeta^j + \bar{\zeta}^i = -1.$$

The latter implies that $Tr\sigma^*|H^{1,1}(M)$ is an integer, hence $Tr\sigma^*|H^{1,1}(M) = 3$.
Then by the Topological Lefschetz Fixed Point Formula, $e(M^\sigma) = 6$. By the Holomorphic Lefschetz Fixed Point Formula,
$$1 = \frac{1}{6}r_1 + \frac{1}{3}r_2,$$
where $r_i$ is the number of isolated fixed points of $\sigma$ of type $\frac{1}{3}(1, i)$. Since $r_1 + r_2 = e(M^\sigma) = 6$, we have a unique solution: $r_1 = 6$, $r_2 = 0$. This gives (a).

Assume that $p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = 1$. Then $\zeta^i \neq 1$ and $\bar{\zeta}^j = 1$.

$$Tr\sigma^*|H^1(M, \mathbb{Z}) = Tr\sigma^*|(H^{0,1}(M) \oplus H^{1,0}(M)) = \zeta^i + \bar{\zeta}^i = -1,$$
$$Tr\sigma^*|(H^{0,2}(M) \oplus H^{2,0}(M)) = 1 + 1 = 2.$$

The latter implies that $Tr\sigma^*|H^{1,1}(M)$ is an integer, hence $Tr\sigma^*|H^{1,1}(M) = 3$.
Then by the Topological Lefschetz Fixed Point Formula, $e(M^\sigma) = 9$. By the Holomorphic Lefschetz Fixed Point Formula,
$$\frac{1}{2}\{(1 - \zeta^i + 1) + (1 - \zeta^j + 1)\} = \frac{5}{2} = \frac{1}{6}r_1 + \frac{1}{3}r_2.$$

Since $r_1 + r_2 = 9$, we have a unique solution: $r_1 = 3$, $r_2 = 6$. This gives (b).

Assume that $p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = 1$. Then $\zeta^i = \zeta^j = 1$.

$$Tr\sigma^*|(H^{0,1}(M) \oplus H^{1,0}(M)) = Tr\sigma^*|(H^{0,2}(M) \oplus H^{2,0}(M)) = 2,$$
$$Tr\sigma^*|H^{1,1}(M) = 3$$
and $e(M^\sigma) = 3$. By the Holomorphic Lefschetz Fixed Point Formula,
$$1 = \frac{1}{6}r_1 + \frac{1}{3}r_2.$$

Since $r_1 + r_2 = 3$, we have a unique solution: $r_1 = 0$, $r_2 = 3$. This gives (c).

Assume that $p_g(M/\langle \sigma \rangle) = 0$ and $q(M/\langle \sigma \rangle) = 1$. Then $\zeta^i = 1$ and $\bar{\zeta}^j \neq 1$.

$$Tr\sigma^*|(H^{0,1}(M) \oplus H^{1,0}(M)) = 2,$$
$$Tr\sigma^*|(H^{0,2}(M) \oplus H^{2,0}(M)) = \zeta^i + \bar{\zeta}^j = -1,$$
$$Tr\sigma^*|H^{1,1}(M) = 3$$
and $e(M^\sigma) = 0$. Thus $\sigma$ acts freely, a contradiction. \hfill \Box

**Proposition 1.2.** Let $M$ be an abelian surface. Assume that it admits an order 3 automorphism $\sigma$ such that $p_g(M/\langle \sigma \rangle) = 0$. Then $b_2(M/\langle \sigma \rangle) = 4$ or 2.
Proof. First note that \( p_g(M) = 1 \) and \( \text{rank} H^{1,1}(M) = 4 \). Let \( \zeta = \exp(2\pi i / 3) \).
Let \( \zeta^k \) be the eigenvalue of \( \sigma^* \) acting on \( H^{0,2}(M) \). Since \( p_g(M/\langle \sigma \rangle) = 0 \), we have \( \zeta^k \neq 1 \), hence
\[
Tr\sigma^*(H^{0,2}(M) \oplus H^{2,0}(M)) = \zeta^k + \zeta^3 = -1.
\]
It implies that \( Tr\sigma^*|H^{1,1}(M) \) is an integer, hence is equal to 4, 1 or \(-2\). The last possibility can be ruled out, as there is a \( \sigma \)-invariant ample divisor yielding a \( \sigma^* \)-invariant vector in \( H^{1,1}(M) \). Finally note that \( b_2(M/\langle \sigma \rangle) = \text{rank} H^{1,1}(M)^\sigma \). \( \square \)

Remark 1.3. If in addition, \( q(M/\langle \sigma \rangle) = 0 \), then either

1. \( r_2 = 0, \ r_1 = \sum R^2_j = 9, \ b_2(M/\langle \sigma \rangle) = 4; \) or
2. \( r_2 = 3, \ r_1 = \sum R^2_j = 3, \ b_2(M/\langle \sigma \rangle) = 2. \)

Here \( r_i \) is the number of isolated fixed points of type \( \frac{1}{i}(1,i) \), and \( \cup R_j \) is the \( i \)-dimensional fixed locus of \( \sigma \).

Proposition 1.4. Let \( M \) be a surface of general type with \( p_g(M) = q(M) = 2 \). Assume that it admits an order 3 automorphism \( \sigma \) with isolated fixed points only such that \( p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = 0 \). Let \( \tilde{\alpha} : M/\langle \sigma \rangle \to \text{Alb}(M)/\langle \sigma \rangle \) be the map induced by the Albanese map \( \alpha : M \to \text{Alb}(M) \). Then \( \tilde{\alpha} \) cannot factor through a surjective map \( M/\langle \sigma \rangle \to N \) to a normal projective surface \( N \) with Picard number 1.

Proof. Suppose that \( \tilde{\alpha} \) factors through a surjective map \( M/\langle \sigma \rangle \to N \) to a normal projective surface \( N \) with Picard number 1, i.e.,
\[
\tilde{\alpha} : M/\langle \sigma \rangle \to N \to \text{Alb}(M)/\langle \sigma \rangle.
\]
Let \( b : N \to \text{Alb}(M)/\langle \sigma \rangle \) be the second map. Since a normal projective surface with Picard number 1 cannot be fibred over any curve, the map \( b \) is surjective. Since \( p_g(M/\langle \sigma \rangle) = q(M/\langle \sigma \rangle) = 0 \), we have
\[
p_g(N) = q(N) = 0 \quad \text{and} \quad p_g(\text{Alb}(M)/\langle \sigma \rangle) = q(\text{Alb}(M)/\langle \sigma \rangle) = 0.
\]
Since \( \text{Alb}(M)/\langle \sigma \rangle \) has quotient singularities only, its minimal resolution has \( p_g = q = 0 \), hence
\[
\text{Pic}(\text{Alb}(M)/\langle \sigma \rangle) \otimes \mathbb{Q} \cong H^2(\text{Alb}(M)/\langle \sigma \rangle, \mathbb{Q}).
\]
By Proposition 1.2, \( \text{Alb}(M)/\langle \sigma \rangle \) has Picard number 4 or 2. This is a contradiction, as a normal projective surface with Picard number 1 cannot be mapped surjectively onto a surface with Picard number \( \geq 2 \). \( \square \)

Let \( S \) be a normal projective surface with quotient singularities and
\[
f : S' \to S
\]
be a minimal resolution of \( S \). It is well-known that quotient singularities are log-terminal singularities. Thus one can write the adjunction formula,
\[
K_{S'} \equiv \sum_{\text{num}} f^*K_S - \sum_{p \in \text{Sing}(S)} D_p,
\]
where \( D_p = \sum(a_jA_j) \) is an effective \( \mathbb{Q} \)-divisor with \( 0 \leq a_j < 1 \) supported on \( f^{-1}(p) = \cup A_j \) for each singular point \( p \). It implies that
\[
K^2_{S'} = K^2_S - \sum_p D^2_p = K^2_{S'} + \sum_p D_pK_{S'}.
\]
The coefficients of the \( \mathbb{Q} \)-divisor \( D_p \) can be obtained by solving the equations

\[
D_p A_j = -K_{\mathbb{P}^2} A_j = 2 + A_j^2
\]
given by the adjunction formula for each exceptional curve \( A_j \subset f^{-1}(p) \).

2. The Proof of Theorem 0.4

2.1. The case: \( Z \) has 3 singular points of type \( \frac{1}{3} (1, 2) \). Let \( p_1, p_2, p_3 \) be the three singular points of \( Z \) of type \( \frac{1}{3} (1, 2) \), and \( \tilde{Z} \to Z \) be the minimal resolution.

Lemma 2.1. There is a \( C_3 \)-cover \( X \to Z \) branched at the three singular points of \( Z \).

Proof. We use a lattice theoretic argument. Consider the cohomology lattice

\[
H^2(\tilde{Z}, \mathbb{Z})_{\text{free}} := H^2(\tilde{Z}, \mathbb{Z})/(\text{torsion})
\]

which is unimodular of signature \((1, 6)\) under intersection pairing. Since \( Z \) is a \( \mathbb{Q} \)-homology projective plane, \( p_g(\tilde{Z}) = q(\tilde{Z}) = 0 \) and hence \( \text{Pic}(\tilde{Z}) = H^2(\tilde{Z}, \mathbb{Z}) \).

Let \( R_i \subset H^2(\tilde{Z}, \mathbb{Z})_{\text{free}} \) be the sublattice spanned by the numerical classes of the components of \( f^{-1}(p_i) \). Consider the sublattice \( R_1 \oplus R_2 \oplus R_3 \). Its discriminant group is 3-elementary of length 3, and its orthogonal complement is of rank 1. It follows that there is a divisor class \( L \in \text{Pic}(\tilde{Z}) \) such that

\[
3L = B + \tau
\]

for some torsion divisor \( \tau \), where \( B \) is an integral divisor supported on the six \((-2)\)-curves contracted to the points \( p_1, p_2, p_3 \) by the map \( \tilde{Z} \to Z \). Here all coefficients of \( B \) are greater than 0 and less than 3.

If \( \tau = 0 \), \( L \) gives a \( C_3 \)-cover of \( \tilde{Z} \) branched along \( B \), hence yielding a \( C_3 \)-cover \( X \to Z \) branched at the three points \( p_1, p_2, p_3 \). Clearly, \( X \) is a nonsingular surface.

If \( \tau \neq 0 \), let \( m \) denote the order of \( \tau \). Write \( m = 3^i m' \) with \( m' \) not divisible by 3. By considering \( 3(m'L) = m'B + m' \tau \), and by putting \( B' = m'B \) (modulo 3), \( r' = m' \tau \), we may assume that \( \tau \) has order \( 3^i \). The torsion bundle \( \tau \) gives an unramified \( C_3 \)-cover

\[
p : V \to \tilde{Z}.
\]

Let \( g \) be the corresponding automorphism of \( V \). Pulling \( 3L = B + \tau \) back to \( V \), we have

\[
3p^*L = p^*B.
\]

Obviously, \( g \) can be linearized on the line bundle \( p^*L \), hence gives an automorphism of order \( 3^i \) of the total space of \( p^*L \). Let \( V' \to V \) be the \( C_3 \)-cover given by \( p^*L \).

We regard \( V' \) as a subvariety of the total space of \( p^*L \). Since \( g \) leaves invariant the set of local defining equations for \( V' \), \( g \) restricts to an automorphism of \( V' \) of order \( 3^i \). Thus we have a \( C_3 \)-cover

\[
V'/\langle g \rangle \to \tilde{Z}.
\]

This yields a \( C_3 \)-cover \( X \to Z \) branched at the three points \( p_1, p_2, p_3 \). Clearly, \( X \) is a nonsingular surface. \( \square \)

Since \( Z \) has only rational double points, the adjunction formula gives \( K_{\tilde{Z}}^2 = K_{\tilde{Z}}^2 = 3. \) Hence \( K_X^2 = 3K_{\tilde{Z}}^2 = 9. \) The smooth part \( Z^0 \) of \( Z \) has Euler number \( e(Z^0) = e(\tilde{Z}) - 9 = 0 \), so \( e(X) = 3e(Z^0) + 3 = 3. \) This shows that \( X \) is a ball quotient with \( p_g(X) = q(X). \) It is known that such a surface has \( p_g(X) = q(X) \leq 2. \)
2.2. The case: $Z$ has 4 singular points of type $\frac{1}{3}(1,2)$. Let $p_1, p_2, p_3, p_4$ be the four singular points of $Z$, and $f : Z \to Z$ the minimal resolution.

**Lemma 2.2.** If there is a $C_3$-cover $Y \to Z$ branched at three of the four singular points of $Z$, then the minimal resolution $\tilde{Y}$ of $Y$ has $K_Y^2 = 3$, $e(\tilde{Y}) = 9$ and $p_g(\tilde{Y}) = q(\tilde{Y}) = 0$.

**Proof.** We may assume that the three points are $p_1, p_2, p_3$. Note that $Y$ has 3 singular points of type $\frac{1}{3}(1,2)$, the pre-image of $p_4$. Let $\tilde{Y} \to Y$ be the minimal resolution. It is easy to see that $K_Y^2 = 3$, $e(\tilde{Y}) = 9$ and $p_g(\tilde{Y}) = q(\tilde{Y}) = 0$.

Suppose that $p_g(\tilde{Y}) = q(\tilde{Y}) = 1$. Consider the Albanese fibration $Y \to Alb(\tilde{Y})$. It induces a fibration $Y \to Alb(\tilde{Y})$. Let $\sigma$ be the order 3 automorphism of $Y$ corresponding to the $C_3$-cover $Y \to Z$. It induces a fibration $\phi : \tilde{Z} \to Alb(\tilde{Y})/\langle \sigma \rangle$. Since $q(Z) = 0$, we have $Alb(\tilde{Y})/\langle \sigma \rangle \cong \mathbb{P}^1$. The eight $(-2)$-curves of $\tilde{Z}$ are contained in a union of fibres of $\phi$. It follows that $\tilde{Z}$ has Picard number $\geq 8 + 2 = 10$, a contradiction.

Suppose that $p_g(\tilde{Y}) = q(\tilde{Y}) = 2$. Consider the Albanese map $a : \tilde{Y} \to Alb(\tilde{Y})$. It contracts the six $(-2)$-curves of $\tilde{Y}$, hence the induced map $\tilde{a} : \tilde{Y}/\langle \sigma \rangle \to Alb(\tilde{Y})/\langle \sigma \rangle$ factors through a surjective map $\tilde{Y}/\langle \sigma \rangle \to Z$, where $\sigma$ is the order 3 automorphism of $Y$ corresponding to the $C_3$-cover $Y \to Z$. Since $Z$ has Picard number 1 and $p_g(Z) = q(Z) = 0$, Proposition 1.4 gives a contradiction.

The possibility of $p_g(\tilde{Y}) = q(\tilde{Y}) \geq 3$ can be ruled out by considering a $C_3$-cover $X \to Y$ branched at the three singular points of $Y$. See the paragraph below Lemma 2.3.

**Lemma 2.3.** There is a $C_3$-cover $Y \to Z$ branched at three of the four singular points of $Z$, and a $C_3$-cover $X \to Y$ branched at the three singular points of $Y$.

**Proof.** The existence of two $C_3$-covers can be proved by a lattice theoretic argument. Note that $\text{Pic}(\tilde{Z}) = H^2(\tilde{Z}, \mathbb{Z})$. We know that $H^2(\tilde{Z}, \mathbb{Z})_{\text{free}}$ is a unimodular lattice of signature $(1,8)$ under intersection pairing. Let $R_1 \subset H^2(\tilde{Z}, \mathbb{Z})_{\text{free}}$ be the sublattice spanned by the numerical classes of the components of $f^{-1}(p_i)$. Consider the sublattice $R_1 \oplus R_2 \oplus R_3 \oplus R_4$. Its discriminant group is 3-elementary of length 4, and its orthogonal complement is of rank 1. It follows that there are two divisor classes $L_1, L_2 \in \text{Pic}(\tilde{Z})$ such that

$$3L_1 = B_1 + \tau_1, \quad 3L_2 = B_2 + \tau_2$$

for some torsion divisors $\tau_i$, where $B_i$ is an integral divisor supported on the six $(-2)$-curves lying over three of the four points $p_1, p_2, p_3, p_4$. We may assume that $B_i$ is supported on $\cup_{j \neq i} f^{-1}(p_j)$ and all coefficients of $B_i$ are greater than 0 and less than 3.
By the same argument as in Lemma 2.1 we can take a $C_3$-cover $Y \to Z$ branched at the three points $p_2$, $p_3$, $p_4$. Then $Y$ has 3 singular points of type $\frac{1}{2}(1, 2)$, the preimage of $p_1$. This can be done by using the line bundle $L_1$ if $\tau_1 = 0$. Otherwise, we first take an unramified cover $p : V \to \tilde{Z}$ corresponding to $\tau_1$ and then lift the covering automorphism $g$ to the $C_3$-cover $V' \to V$ given by $p^*L_1$, then take the quotient $V'/\langle g \rangle$.

Let $\psi : \tilde{Y} \to \tilde{Z}$ be the $C_3$-cover corresponding to the $C_3$-cover $Y \to Z$, composed with a normalization. Then $\tilde{Y}$ is a normal surface and there is a surjection $f : \tilde{Y} \to Y$. Now

$$3f_*(\psi^*L_2) = f_*(\psi^*B_2) + f_*(\psi^*\tau_2)$$

and $f_*(\psi^*B_2)$ is an integral divisor supported on the exceptional locus of $\tilde{Y} \to Y$ with coefficients greater than 0 and less than 3. Now by the same argument as in Lemma 2.1 there is a $C_3$-cover $X \to Y$ with $X$ nonsingular. \hfill \square

It is easy to see that $K_X^2 = 9$, $e(X) = 3$ and $p_g(X) = q(X)$. Such a surface has $p_g(X) = q(X) \leq 2$. (See the paragraph before Proposition 1.1) It implies that $p_g(Y) = q(Y) \leq 2$, which completes the proof of Lemma 2.2.

By Lemma 2.2 $p_g(Y) = q(Y) = 0$, so $Y$ has Picard number 1 and has three singular points of type $\frac{1}{5}(1, 2)$. Then by the previous subsection, $p_g(X) = q(X) = 0$.

2.3. The case: $Z$ has 3 singular points of type $\frac{1}{7}(1, 5)$. Let $p_1, p_2, p_3$ be the three singular points of $Z$ of type $\frac{1}{7}(1, 5)$. Then there is a $C_7$-cover $X \to Z$ branched at the three points. In the case of $\pi_1(Z) = \{1\}$, this was proved in [K08], p922. In our general situation, we consider the lattice $\operatorname{Pic}(\tilde{Z})/(\text{torsion})$, where $\tilde{Z} \to Z$ is the minimal resolution. Then by the same lattice theoretic argument as in [K08], there is a divisor class $L \in \operatorname{Pic}(\tilde{Z}) = H^2(\tilde{Z}, \mathbb{Z})$ such that $7L = B + \tau$ for some torsion divisor $\tau$, where $B$ is an integral divisor supported on the exceptional curves of the map $\tilde{Z} \to Z$. Here all coefficients of $B$ are not equal to 0 modulo 7. If $\tilde{Z}$ is a $(2, 4)$-elliptic surface and if $\tau \neq 0$, then $2\tau = 0$. By considering $7(2L) = 2B$, and by putting $L' = 2L$ and $B' = 2B$, we get $7L' = B'$. This implies the existence of a $C_7$-cover $X \to Z$ branched at the three points $p_1, p_2, p_3$. Then $X$ is a nonsingular surface.

Note that $K_{\tilde{Z}}^2 = 0$. So by the adjunction formula, $K_{\tilde{Z}}^2 = \frac{1}{7}$. It is easy to see that $K_X^2 = 9$, $e(X) = 3$ and $p_g(X) = q(X)$. Such a surface has $p_g(X) = q(X) \leq 2$. (See the paragraph before Proposition 1.1) Now by Proposition 1.1 $p_g(X) = q(X) = 0$.

2.4. The case: $Z$ has 3 singular points of type $\frac{1}{4}(1, 2)$ and one of type $\frac{1}{7}(1, 5)$. Let $\tilde{Z} \to Z$ be the minimal resolution. Then $\tilde{Z}$ is a $(2, 3)$- or $(2, 4)$-elliptic surface. Let

$$\phi : \tilde{Z} \to \mathbb{P}^1$$

be the elliptic fibration. Let $Z' \to Z$ be the minimal resolution of the singular point of type $\frac{1}{7}(1, 5)$. Then $\phi : \tilde{Z} \to \mathbb{P}^1$ induces an elliptic fibration

$$\phi' : Z' \to \mathbb{P}^1.$$

Lemma 2.4. (1) There is a $C_3$-cover $Y \to Z$ branched at the three points of type $\frac{1}{2}(1, 2)$. The cover $Y$ has 3 singular points of type $\frac{1}{4}(1, 5)$.

(2) The minimal resolution $\hat{Y}$ of $Y$ is a $(2, 3)$- or $(2, 4)$-elliptic surface. Its multiplicities are the same as those of $\hat{Z}$. Furthermore, every fibre of $\hat{Z}$ does not split in $\hat{Y}$. 
The elliptic fibration $\varphi$ so cannot have any more branch points. The minimal resolution $\tilde{Z}$ of $\bar{Z}$ by $E$. Note that every $(-\mu)$ contained in a union of fibres, only in one of the following three cases. Here $\bar{Z}$ contains nine curves whose dual diagram is

$$(-2) - (-2) - (-2) - (-2) - (-2) - (-2) - (-2) - (-3).$$

Note that every $(-2)$-curve on $\bar{Z}$ is contained in a fiber. The eight $(-2)$-curves are contained in a union of fibres, only in one of the following three cases. Here $\mu$ or $\mu_i$ is the multiplicity of the fibre.

(a) $IV^* + \mu I_3$,  
(b) $IV^* + IV$,  
(c) $\mu_1 I_3 + \mu_2 I_3 + \mu_3 I_3 + \mu_4 I_3$.

In the first two cases, the $(-3)$-curve must intersects with multiplicity 2 the central component of the $IV^*$-fibre. Thus, the image in $Z'$ of the $IV^*$-fibre contains the 3 singular points of $Z'$, so it does not split in $\bar{Y}$. This means that the base point of the $IV^*$-fibre is another branch point of the base change map $\gamma$, a contradiction. In the last case, we also get at least 3 branch points of $\gamma$, a contradiction. Therefore, every fibre of $Z'$ does not split in $\bar{Y}$. In particular, the multiplicity of a fibre in $\bar{Y}$ is the same as that of the corresponding fibre in $\bar{Z}$. Thus $\bar{Y}$ is an elliptic surface over $\mathbb{P}^1$ having 2 multiple fibres with multiplicity 2 and 3, resp. Since $K^2\bar{Z} = 0$ and $Z'$ has only rational double points, the adjunction formula gives $K^2_{\bar{Y}} = K^2_{\bar{Z}} = 0$. Hence $K^2_{\bar{Y}} = 3K^2_{\bar{Z}} = 0$. In particular, $\bar{Y}$ is minimal. The smooth part $Z'$ has Euler number $e(Z) = e(\bar{Z}) - 9 = 3$, so $e(\bar{Y}) = 3e(Z) + 3 = 12$. This shows that $\bar{Y}$ is a $(2,3)$-elliptic surface.

Now by the previous subsection, there is a $C_7$-cover $X \to Y$ branched at the three singular points such that $X$ is a fake projective plane.

3. Proof of Theorem 0.5

(1) was proved in Lemma 2.3.
(2) As we have seen in the proof of Lemma 2.3, the eight (−2)-curves on \( \tilde{Z} \) are contained in a union of fibres, only in one of the following three cases. Here \( \mu \) or \( \mu_i \) is the multiplicity of the fibre.

\[
(a) IV^* + \mu I_3, \quad (b) IV^* + IV, \quad (c) \mu_1 I_3 + \mu_2 I_3 + \mu_3 I_3 + \mu_4 I_3.
\]

Recall that every fibre in \( \tilde{Z} \) does not split in \( \tilde{Y} \), and the (−3)-curve in \( \tilde{Z} \) is a 6-section. We will eliminate the first two cases. Let \( Z' \to Z \) be the minimal resolution of the singular point of type \( \frac{1}{3}(1, 5) \).

Case (a): \( IV^* + \mu I_3 \). In this case, the surface \( \tilde{Z} \) has a fibre of type \( \mu' I_1 \). Since the (−3)-curve in \( \tilde{Z} \) is a 6-section, it intersects with multiplicity 2 the central component of the \( IV^* \)-fibre. Thus both the \( \mu I_3 \)-fibre and the \( \mu' I_1 \)-fibre are disjoint from the branch of the \( C_3 \)-cover \( \tilde{Y} \to Z' \). It is easy to see that these two fibres will give a \( I_3 \)-fibre and a \( \mu' I_3 \)-fibre in \( \tilde{Y} \), so \( \tilde{Y} \) has Picard number \( \geq 12 \), a contradiction.

Case (b): \( IV^* + IV \). This case can be eliminated in a similar way as above. The \( IV \)-fibre on \( \tilde{Z} \) does not contain any of the (−2)-curves contracted by the map \( \tilde{Z} \to Z' \). But there is no unramified connected triple cover of an \( IV \)-fibre.

(3) If the image in \( Z' \) of the \( \mu I_3 \)-fibre contains a singular point of \( Z' \), then it will give a \( \mu I_1 \)-fibre in \( \tilde{Y} \). If it does not, then it will give a \( \mu I_3 \)-fibre in \( \tilde{Y} \).

4. \( \mathbb{Q} \)-HOMOLOGY PROJECTIVE PLANES WITH CUSPS

In this section we will prove Theorem 0.6.

Let \( Z \) be a \( \mathbb{Q} \)-homology projective plane with cusps, i.e., singularities of type \( \frac{1}{3}(1, 2) \), only. Let \( \tilde{Z} \to Z \) be the minimal resolution.

Let \( k \) be the number of cusps on \( Z \). A \( \mathbb{Q} \)-homology projective plane with quotient singularities can have at most 5 singular points, and the case with the maximum possible number of quotient singularities was classified in [HK]. According to this classification, there is no \( \mathbb{Q} \)-homology projective plane with 5 cusps. Thus we have \( k \leq 4 \). It is easy to see that \( K_2^2 = K_2^2 = 9 - 2k \). Since \( K_2^2 > 0 \), \( K_2 \) is not numerically trivial. By Lemma 3.3 of [HK], the product of the orders of local abelianized fundamental groups and \( K_2^2 \) is a positive square number. In our situation, the product is \( 3^k(9-2k) \), and this number is a square only if \( k = 4 \) or 3.

Since \( K_2 \) is not numerically trivial, either \( K_2 \) or \( -K_2 \) is ample.

Assume that \( K_2 \) is ample. Then \( K_2 \) is nef, hence \( \tilde{Z} \) is a minimal surface of general type. By Theorem 0.4, \( Z \) is the quotient of a fake projective plane by a group of order 9 if \( k = 4 \), by order 3 if \( k = 3 \).

Assume that \( -K_2 \) is ample. Then \( Z \) is a log del Pezzo surface of Picard number 1 with 4 or 3 cusps. Assume that \( Z \) has 3 cusps. By a similar argument as in Section 2, there is a \( C_3 \)-cover \( \mathbb{P}^2 \to Z \) branched at the 3 cusps. It is easy to see that the covering automorphism is a conjugate of the order 3 automorphism

\[
\sigma: (x, y, z) \mapsto (x, \omega y, \omega^2 z).
\]

Assume that \( Z \) has 4 cusps. By a similar argument as in Section 2, there is a \( C_3^2 \)-cover \( \mathbb{P}^2 \to Z \) branched at the 4 cusps, the composition of two \( C_3 \)-covers. It is easy to see that the Galois group is a conjugate of \( \langle \sigma, \tau \rangle \), where \( \sigma \) and \( \tau \) are the commuting order 3 automorphisms given by

\[
\sigma(x, y, z) = (x, \omega y, \omega^2 z), \quad \tau(x, y, z) = (z, ax, a^{-1}y),
\]
where \( a \) is a non-zero constant and \( \omega = \exp\left(\frac{2\pi i}{3}\right) \).

**Remark 4.1.** (1) In the case (1) and (2), the fundamental group \( \pi_1(Z) \) is given by the list of Cartwright and Steger. See Remark 0.3.

(2) One can construct a log del Pezzo surface of Picard number 1 with 4 or 3 cusps in many ways other than taking a global quotient. One different way is to consider a rational elliptic surface \( V \) with 4 singular fibres of type \( I_3 \). Such an elliptic surface can be constructed by blowing up \( \mathbb{P}^2 \) at the 9 base points of the Hesse pencil. Every section is a \((-1)\)-curve. Contracting a section, we get a nonsingular rational surface \( W \) with eight \((-2)\)-curves forming a diagram of type \( 4A_2 \). Contracting these eight \((-2)\)-curves, we get a log del Pezzo surface of Picard number 1 with 4 cusps. On \( W \), we contract a string of two rational curves forming a diagram \((-1)\)-\((-2)\) to get a nonsingular rational surface with six \((-2)\)-curves forming a diagram of type \( 3A_2 \). Contracting these six \((-2)\)-curves, we get a log del Pezzo surface of Picard number 1 with 3 cusps.

**References**

[Arm] M. A. Armstrong, *The fundamental group of the orbit space of a discontinuous group*, Proc. Camb. Phil. Soc. 64 (1968), 299-301.

[AS3] M. F. Atiyah, and I. M. Singer, *The index of elliptic operators, III*, Ann. of Math. 87 (1968), 546-604.

[Au] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, C. R. Acad. Sci. Paris Ser. A-B 283 (1976), no. 3, A119–A121.

[BHPV] W. Barth, K. Hulek, Ch. Peters, A. Van de Ven, *Compact Complex Surfaces*, second ed. Springer 2004.

[CS] D. Cartwright, T. Steger, *Enumeration of the 50 fake projective planes*, C. R. Acad. Sci. Paris, Ser. I 348 (2010), 11-13.

[CS2] D. Cartwright, T. Steger, *private communication*.

[CCM] F. Catanese, C. Ciliberto, M. Mendes Lopes, *On the classification of irregular surfaces of general type with nonbirational bicanonical map*, Trans. Amer. Math. Soc. 350 (1998), no. 1, 275-308.

[D] I. Dolgachev, *Algebraic surfaces with \( q = p_g = 0 \)*, C.I.M.E. Algebraic surfaces, pp 97-215, Liguori Editori, Napoli 1981.

[HP] C. D. Hacon, R. Pardini, *Surfaces with \( p_g = q = 3 \)*, Trans. Amer. Math. Soc. 354 (2002), no. 7, 2631-2638.

[Hir] F. Hirzebruch, *Automorphe Formen und der Satz von Riemann-Roch* in: 1958 Symposium International de Topologia Algebraica, UNESCO, pp.129-144.

[HK] D. Hwang and J. Keum, *The maximum number of singular points on rational homology projective planes*, [arXiv:0801.3921] to appear in J. Algebraic Geom.

[K06] J. Keum, *A fake projective plane with an order 7 automorphism*, Topology 45 (2006), 919-927.

[K08] J. Keum, *Quotients of fake projective planes*, Geom. Top. 12 (2008), 2497-2515.

[K10] J. Keum, *A fake projective plane constructed from an elliptic surface with multiplicities (2,4)*, preprint (2010).

[KK] V. S. Kharlamov, V. M. Kulikov, *On real structures on rigid surfaces*, Izv. Russ. Akad. Nauk. Ser. Mat. 66, no. 1, (2002), 133-152; Izv. Math. 66, no. 1, (2002), 133-150.

[Kl] B. Klingler, *Sur la rigidité de certains groupes fondamentaux, l’arithméticité des réseaux hyperboliques complexes, et les “faux plans projectifs”*, Invent. Math. 153 (2003), 105-143.

[Mos] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Annals Math. Studies 78, Princeton Univ. Press, Princeton, N.J.; Univ. Tokyo Press, Tokyo 1973.

[Pi] G.P. Pirola, *Surfaces with \( p_g = q = 3 \)*, Manuscripta Math. 108 (2002), no. 2, 163–170.

[P] G. Prasad, *Volumes of S-arithmetic quotients of semi-simple groups*, Inst. Hautes Études Sci. Publ. Math. 69 (1989), 91-117.

[PY] G. Prasad, and S.-K. Yeung, *Fake projective planes*, Invent. Math. 168 (2007), 321-370; Addendum, Invent. Math. 182 (2010), 213-227.
S.-T. Yau, *Calabi’s conjecture and some new results in algebraic geometry*, Proc. Nat. Ac. Sc. USA 74 (1977), 1798-1799.

S.-K. Yeung, *Integrality and arithmeticity of cocompact lattices corresponding to certain complex two-ball quotients of Picard number one*, Asian J. Math. 8 (2004), 107-130; Erratum, Asian J. Math. 13 (2009), 283-286.

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea

E-mail address: jhkeum@kias.re.kr