LOCAL COHOMOLOGY AND $D$-AFFINITY
IN POSITIVE CHARACTERISTIC

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1. Introduction

Let $k$ be a field. Consider the polynomial ring

$$R = k \left[ \begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{array} \right]$$

and $I \subset R$ the ideal generated by the three $2 \times 2$ minors

$$f_1 := \begin{vmatrix} X_{12} & X_{13} \\ X_{22} & X_{23} \end{vmatrix}, \quad f_2 := \begin{vmatrix} X_{13} & X_{11} \\ X_{23} & X_{21} \end{vmatrix} \quad \text{and} \quad f_3 := \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix}.$$

If $k$ is a field of positive characteristic, the local cohomology modules $H^j_I(R)$ vanish for $j > 2$ (see Chapitre III, Proposition (4.1) in [2]). However, if $k$ is a field of characteristic zero, $H^3_I(R)$ is non-vanishing (see Proposition 2.4 of this paper or Remark 3.13 in [1]).

Consider the Grassmann variety $X = \text{Gr}(2, V)$ of 2-dimensional vector subspaces of a 5-dimensional vector space $V$ over $k$. Let us take a two dimensional subspace $W$ of $V$. Then the singularity $R/I$ appears in the Schubert variety $Y \subset X$ of 2-dimensional subspaces $E$ such that $\dim(E \cap W) \geq 1$. Therefore $H^3_Y(\mathcal{O}_X)$ does not vanish in characteristic zero while it does vanish in positive characteristic.

In this paper we show how this difference in the vanishing of local cohomology translates into a non-vanishing first cohomology group for the $D_X$-module $H^3_Y(\mathcal{O}_X)$ in positive characteristic.

Previous work of Haastert ([3]) showed the Beilinson-Bernstein equivalence ([1]) to hold for projective spaces and the flag manifold of $SL_3$ in positive characteristic. However, as we show in this paper, $D$-affinity breaks down for the flag manifold of $SL_5$ in all positive characteristics. The Beilinson-Bernstein equivalence, therefore, does not carry over to flag manifolds in positive characteristic.

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2. Local cohomology

Keep the notation from §1. A topological proof of the following proposition is given in §5.

**Proposition 2.1.** \(H^3_I(R)\) does not vanish in characteristic zero.

**Corollary 2.2.** \(H^3_Y(\mathcal{O}_X)\) does not vanish in characteristic zero.

The local to global spectral sequence

\[ H^p(X, H^q_Y(\mathcal{O}_X)) \Rightarrow H^{p+q}(X, \mathcal{O}_X) \]

and \(\mathcal{D}\)-affinity in characteristic zero (II) implies

\[ H^3_Y(X, \mathcal{O}_X) = \Gamma(X, H^3_Y(\mathcal{O}_X)) \neq 0. \]

On the other hand, if \(k\) is a field of positive characteristic, \(H^q_Y(\mathcal{O}_X) = 0\) if \(q \neq 2\), since \(Y\) is a codimension two Cohen Macaulay subvariety of the smooth variety \(X\) ([2], Chapitre III, Proposition (4.1)). This gives a totally different degeneration of the local to global spectral sequence. In the positive characteristic case we get

\[ H^p(X, H^2_Y(\mathcal{O}_X)) \cong H^{p+2}_Y(X, \mathcal{O}_X). \]

We will prove that \(H^3_Y(X, \mathcal{O}_X) \neq 0\) even if \(k\) is a field of positive characteristic. This will give the desired non-vanishing

\[ H^1(X, H^3_Y(\mathcal{O}_X)) \neq 0 \]

in positive characteristic.

3. Lifting to \(\mathbb{Z}\)

To deduce the non-vanishing of \(H^3_Y(X, \mathcal{O}_X)\) in positive characteristic, we need to compute the local cohomology over \(\mathbb{Z}\) and proceed by base change. Flag manifolds and their Schubert varieties admit flat lifts to \(\mathbb{Z}\)-schemes. In this section \(X_{\mathbb{Z}}\) and \(Y_{\mathbb{Z}}\) will denote flat lifts of a flag manifold \(X\) and a Schubert variety \(Y \subset X\) respectively.

The local Grothendieck-Cousin complex (cf. [3], §8) of the structure sheaf \(\mathcal{O}_{X_{\mathbb{Z}}}\)

\[ (3.1) \quad \mathcal{H}^0_{X_{\mathbb{Z}}/X_i}(\mathcal{O}_{X_{\mathbb{Z}}}) \to \mathcal{H}^1_{X_{\mathbb{Z}}/X_1}(\mathcal{O}_{X_{\mathbb{Z}}}) \to \cdots, \]

where \(X_i\) denotes the union of Schubert schemes of codimension \(i\), is a resolution of \(\mathcal{O}_{X_{\mathbb{Z}}}\), since \(\mathcal{O}_{X_{\mathbb{Z}}}\) is Cohen Macaulay, \(\text{codim } X_i \geq i\) and \(X_i \setminus X_{i+1} \to X\) are affine morphisms for all \(i\) (see [3], Theorem 10.9). The sheaves in this resolution decompose into direct sums

\[ \mathcal{H}^i_{X_{\mathbb{Z}}/X_{i+1}}(\mathcal{O}_{X_{\mathbb{Z}}}) = \bigoplus_{\text{codim}(C) = i} \mathcal{H}^i_C(\mathcal{O}_{X_{\mathbb{Z}}}) \]
of local cohomology sheaves $\mathcal{H}_C^i(\mathcal{O}_{X_Z})$ with support in Bruhat cells $C$ of codimension $i$. The degeneration of the local to global spectral sequence gives

$$H^p_{Y_Z}(X_Z, \mathcal{H}_C^c(\mathcal{O}_{X_Z})) = H^{p+c}_{C\cap Y_Z}(X_Z, \mathcal{O}_{X_Z}),$$

since $\mathcal{H}_C^c(\mathcal{O}_{X_Z}) = 0$ if $i \neq c = \text{codim}(C)$.

Since the scheme $X_i \setminus X_{i+1}$ is affine it follows that $H^p_C(X_Z, \mathcal{O}_{X_Z}) = 0$ if $p \neq \text{codim}(C)$ (cf. [5], Theorem 10.9). This shows that the resolution (3.1) is acyclic for the functor $\Gamma_{Y_Z}$ and

$$\Gamma_{Y_Z}(X_Z, \mathcal{H}_C^c(\mathcal{O}_{X_Z})) = \begin{cases} 0 & \text{if } C \not\subset Y_Z \\ H^c_C(X_Z, \mathcal{O}_{X_Z}) & \text{if } C \subset Y_Z, \end{cases}$$

where $c = \text{codim}(C)$. Applying $\Gamma_{Y_Z}(X_Z, -)$ to (3.1) we get the complex

$$M^c_Y : H^c_{C_Y}(X_Z, \mathcal{O}_{X_Z}) \to \bigoplus_{\text{codim}(C) = c+1, C \subset Y_Z} H^{c+1}_C(X_Z, \mathcal{O}_{X_Z}) \to \cdots,$$

where $c$ is the codimension of $Y_Z$, $C_Y$ is the open Bruhat cell in $Y_Z$ and $H^c_{C_Y}(X_Z, \mathcal{O}_{X_Z})$ sits in degree $c$. Notice that $H^i(M^c_Y) = H^i_{Y_Z}(X_Z, \mathcal{O}_{X_Z})$ and that $M^c_Y$ is a complex of free abelian groups. In fact the individual entries $H^c_C(X_Z, \mathcal{O}_{X_Z})$ are direct sums of weight spaces, which are finitely generated free abelian groups (cf. [5], Theorem 13.4). By weight spaces we mean eigenspaces for a fixed $\mathbb{Z}$-split torus $T$. The differentials in $M^c_Y$, being $T$-equivariant, the complex $M^c_Y$ is a direct sum of complexes of finitely generated free abelian groups. Since $H^i(M^c_Y) = H^i_{Y_Z}(X_Z, \mathcal{O}_{X_Z})$, one obtains the following lemma.

**Lemma 3.1.** Every local cohomology group $H^i_{Y_Z}(X_Z, \mathcal{O}_{X_Z})$ is a direct sum of finitely generated abelian groups. In the codimension $c$ of $Y_Z$ in $X_Z$, $H^c_{Y_Z}(X_Z, \mathcal{O}_{X_Z})$ is a free abelian group.

### 4. The Counterexample

For a field $k$, let us set $X_k = X_Z \otimes k$ and $Y_k = Y_Z \otimes k$. Then one has $H^0_{Y_k}(X_k, \mathcal{O}_{X_k}) = H^0_{Y_Z}(X_Z, \mathcal{O}_{X_Z} \otimes k)$. Since $\mathcal{O}_{X_Z}$ is flat over $\mathbb{Z}$, one has a spectral sequence

$$\text{Tor}^\mathbb{Z}_{-p}(H^0_{Y_Z}(X_Z, \mathcal{O}_{X_Z}), k) \Rightarrow H^{p+q}_{Y_k}(X_k, \mathcal{O}_{X_k}).$$

This shows that the natural homomorphism

$$H^i_{Y_Z}(X_Z, \mathcal{O}_{X_Z}) \otimes k \to H^i_{Y_k}(X_k, \mathcal{O}_{X_k})$$

is an injection, and it is an isomorphism if the field $k$ is flat over $\mathbb{Z}$.

In our example (cf. [3]), one has

$$H^3_{Y_Z}(X_Z, \mathcal{O}_{X_Z}) \otimes \mathbb{C} \cong H^3_{Y_Z}(X_Z, \mathcal{O}_{X_Z}) \neq 0.$$

By Lemma 3.1, the cohomology $H^3_{Y_Z}(X_Z, \mathcal{O}_{X_Z})$ must contain $\mathbb{Z}$ as a direct summand. Therefore the injection

$$H^3_{Y_Z}(X_Z, \mathcal{O}_{X_Z}) \otimes k \to H^3_{Y_k}(X_k, \mathcal{O}_{X_k})$$
shows that $H^3_{Y_k}(X_k, \mathcal{O}_{X_k})$ is non-vanishing for any field $k$ of positive characteristic. Since $H^3_{Y_k}(X_k, \mathcal{O}_{X_k}) \cong H^1(X_k, \mathcal{H}^2_{Y_k}(\mathcal{O}_{X_k}))$, one obtains the following result.

**Proposition 4.1.** $H^1(X_k, \mathcal{H}^2_{Y_k}(\mathcal{O}_{X_k})) \neq 0$ if $k$ is of positive characteristic.

5. **Proof of non-vanishing of $H^3_I(R)$**

In this section, we shall give a topological proof of Proposition 2.1. We may assume that the base field is the complex number field $\mathbb{C}$.

The local cohomologies $H^*_I(R)$ are the cohomology groups of the complex

$$R \rightarrow R[f_1^{-1}] \oplus R[f_2^{-1}] \oplus R[f_3^{-1}]$$

$$\rightarrow R[(f_1f_2)^{-1}] \oplus R[(f_2f_3)^{-1}] \oplus R[(f_1f_3)^{-1}] \rightarrow R[(f_1f_2f_3)^{-1}].$$

Hence one has

$$H^3_I(R) = \frac{R[(f_1f_2f_3)^{-1}]}{R[(f_1f_2)^{-1}] + R[(f_2f_3)^{-1}] + R[(f_1f_3)^{-1}]}.$$ 

In order to prove the non-vanishing of $H^3_I(R)$, it is enough to show

$$\frac{1}{f_1f_2f_3} \notin R[(f_1f_2)^{-1}] + R[(f_2f_3)^{-1}] + R[(f_1f_3)^{-1}].$$ (5.2)

Consider the 6-cycle

$$\gamma = \left\{ \left( \begin{array}{ccc} -t_2u + t_3v & u & -t_1\bar{v} \\ -t_2v - t_3u & v & t_1\bar{u} \end{array} \right) \mid |t_1| = |t_2| = |t_3| = 1, |u|^2 + |v|^2 = 1 \right\}$$

$$= \left\{ k \left( \begin{array}{ccc} -t_2 & 1 & 0 \\ -t_3 & 0 & t_1 \end{array} \right) \mid |t_1| = |t_2| = |t_3| = 1, k = \left( \begin{array}{cc} u & -\bar{v} \\ v & \bar{u} \end{array} \right) \in SU(2) \right\}$$

in $X \setminus (f_1f_2f_3)^{-1}(0)$, where $X = \text{Spec}(R) \cong \mathbb{C}^6$. Then on $\gamma$ one has $f_1 = t_1$, $f_2 = t_1t_2$ and $f_3 = t_3$.

Set $\omega = \bigwedge dX_{ij}$. Then one has $\omega = t_1dt_1dt_2dt_3\theta$ on $\gamma$, where $\theta$ is a non-zero invariant form on $SU(2)$. Therefore one has

$$\int_{\gamma} \frac{\omega}{f_1f_2f_3} = \int_{\gamma} \frac{dt_1dt_2dt_3\theta}{t_1t_2t_3} \neq 0.$$ 

Hence, in order to show (5.2), it is enough to prove that

$$\int_{\gamma} \varphi \omega = 0$$ (5.3)
for any $\varphi \in R[(f_1f_2)^{-1}] + R[(f_2f_3)^{-1}] + R[(f_1f_3)^{-1}]$.

For $\varphi \in R[(f_1f_2)^{-1}]$, the equation (5.3) holds because we can shrink the cycle $\gamma$ by $|t_3| = \lambda$ from $\lambda = 1$ to $\lambda = 0$. For $\varphi \in R[(f_1f_3)^{-1}]$, the equation (5.3) holds because we can shrink the cycle $\gamma$ by $|t_2| = \lambda$ from $\lambda = 1$ to $\lambda = 0$.

Let us show (5.3) for $\varphi \in R[(f_2f_3)^{-1}]$. Let us deform the cycle $\gamma$ by $\gamma_\lambda = \left\{ k \begin{pmatrix} (1 - \lambda) & 0 & -\lambda t_1 t_2 t_3^{-1} \\ -t_3 & 0 & t_1 \end{pmatrix}; |t_1| = |t_2| = |t_3| = 1, k \in SU(2) \right\}$. Note that the values of $f_1$, $f_2$ and $f_3$ do not change under this deformation. Hence $\gamma_\lambda$ is a cycle in $X \setminus (f_1f_2f_3)^{-1}(0)$. One has

$\gamma_1 = \left\{ k \begin{pmatrix} 0 & 1 & -t_1 t_2 t_3^{-1} \\ -t_3 & 0 & t_1 \end{pmatrix}; |t_1| = |t_2| = |t_3| = 1, k \in SU(2) \right\}$

$= \left\{ k \begin{pmatrix} 0 & 1 & -t_2 \\ -t_3 & 0 & t_1 \end{pmatrix}; |t_1| = |t_2| = |t_3| = 1, k \in SU(2) \right\}$.

In the last coordinates of $\gamma_1$, one has $f_2 = t_2 t_3$ and $f_3 = t_3$. Hence, for $\varphi \in R[(f_2f_3)^{-1}]$,

$\int_{\gamma} \varphi \omega = \int_{\gamma_1} \varphi \omega$

vanishes because we can shrink the cycle $\gamma_1$ by $|t_1| = \lambda$ from $\lambda = 1$ to $\lambda = 0$.

Remark. Although we do not give a proof here, $H^3_J(R)$ is isomorphic to $H^3_J(R)$ as a D-module. Here $J$ is the defining ideal of the origin.

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