ABSTRACT. Over an algebraically closed field $K$ with any characteristic, on an $N$-dimensional smooth projective $K$-variety $P$ equipped with $c \geq N/2$ very ample line bundles $L_1, \ldots, L_c$, we study the General Debarre Ampleness Conjecture, which expects that for all large degrees $d_1, \ldots, d_c \geq d \gg 1$, for generic $c$ hypersurfaces $H_1 \in |L_1^{\otimes d_1}|, \ldots, H_c \in |L_c^{\otimes d_c}|$, the complete intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle $\Omega_X$.

First, we introduce a notion of formal matrices and a dividing device to produce negatively twisted symmetric differential forms, which extend the previous constructions of Brotbek and the author. Next, we adapt the moving coefficients method (MCM), and we establish that, if $L_1, \ldots, L_c$ are almost proportional to each other, then the above conjecture holds true. Our method is effective: for instance, in the simple case $L_1 = \cdots = L_c$, we provide an explicit lower degree bound $d = N^{N^2}$.

1. Introduction

Smooth projective varieties having ample cotangent bundle suit well with the phenomenon/philosophy that ‘geometry governs arithmetic’, in the sense that, on one hand, over the complex number field $\mathbb{C}$, none of them contain any entire curve, on the other hand, over a number field $K$, each of them is expected to possess only finitely many $K$-rational points (Lang’s conjecture). For instance in the one-dimensional case, the first property is due to the Uniformization Theorem and the Liouville’s Theorem, while the second assertion is the famous Mordell Conjecture/Faltings’s Theorem.

For a long time, few such varieties were known, even though they were expected to be reasonably abundant. In this aspect, Debarre conjectured in [4] that the intersection of $c \geq N/2$ generic hypersurfaces of large degrees in $\mathbb{P}_\mathbb{C}^N$ should have ample cotangent bundle.

By introducing the moving coefficients method (MCM) and the product coup, the Debarre Ampleness Conjecture was first established in [5], with an additional effective lower degree bound.

**Theorem 1.1** ([5]). The cotangent bundle $\Omega_X$ of the complete intersection $X := H_1 \cap \cdots \cap H_c \subset \mathbb{P}_\mathbb{C}^N$ of $c \geq N/2$ generic hypersurfaces $H_1, \ldots, H_c$ with degrees $d_1, \ldots, d_c \geq N^{N^2}$ is ample.

The proof there extends the approach of [2], by adding four major ingredients as follows.

(1) Generalizations of Brotbek’s symmetric differential forms [2, Lemma 4.5] by means of a geometric approach, and also by a scheme-theoretic approach.
(2) Make use of ‘hidden’ symmetric differential forms constructed over any intersection of Fermat-type hypersurfaces with coordinate hyperplanes:

\[ H_1 \cap \cdots \cap H_c \cap \{ z_{v_1} = \cdots = z_{v_\eta} = 0 \} \quad (\forall \eta = 1, \ldots, N-1; 0 \leq v_1 < \cdots < v_\eta \leq N). \]

(3) ‘Flexible’ hypersurfaces designed by MCM, which produce many more negatively twisted symmetric differential forms than pure Fermat-type ones.

(4) The product coup, which produces ample examples of all large degrees \( d_1, \ldots, d_c \).

Recently, Brotbek and Darondeau [3] provided another approach to the Debarre Ampleness Conjecture, by means of new constructions and deep theorems in algebraic geometry. As mentioned in [3, p. 2], it is tempting to extend the Debarre Ampleness Conjecture from projective spaces to projective varieties, equipped with several very ample line bundles.

**General Debarre Ampleness Conjecture.** For any smooth projective \( \mathbb{K} \)-variety \( \mathbf{P} \) of dimension \( N \geq 1 \), for any positive integer \( c \geq N/2 \), for any very ample line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_c \) over \( \mathbf{P} \), there exists some lower bound:

\[ d = d(\mathbf{P}, \mathcal{L}_1, \ldots, \mathcal{L}_c) \gg 1 \]

such that, for all large degrees \( d_1, \ldots, d_c \geq d \), for \( c \) generic hypersurfaces:

\[ H_1 \in |\mathcal{L}_1^{\otimes d_1}|, \ldots, H_c \in |\mathcal{L}_c^{\otimes d_c}|, \]

the complete intersection \( X := H_1 \cap \cdots \cap H_c \) has ample cotangent bundle \( \Omega_X \).

Sharing the same flavor as [5, p. 6, Conjecture 1.5], this general conjecture attracts our interest. To this aim, we develop further our previous method in [5], and generalize several results.

We work over an algebraically closed field \( \mathbb{K} \) with any characteristic. First of all, by adapting the techniques in [5], we can confirm the General Debarre Ampleness Conjecture in the case \( \mathcal{L}_1 = \cdots = \mathcal{L}_c = \mathcal{L} \).

**Theorem 1.2.** Let \( \mathbf{P} \) be an \( N \)-dimensional smooth projective \( \mathbb{K} \)-variety, equipped with a very ample line bundle \( \mathcal{L} \). For any positive integer \( c \geq N/2 \), for all large degrees \( d_1, \ldots, d_c \geq N^2 \), for \( c \) generic hypersurfaces \( H_1 \in |\mathcal{L}_1^{\otimes d_1}|, \ldots, H_c \in |\mathcal{L}_c^{\otimes d_c}| \), the complete intersection \( X := H_1 \cap \cdots \cap H_c \) has ample cotangent bundle \( \Omega_X \).

In fact, we will prove a stronger result, in the case that the \( c \) rays:

\[ \mathbb{R}_+ \cdot [\mathcal{L}_1], \ldots, \mathbb{R}_+ \cdot [\mathcal{L}_c] \subset \text{Ample Cone of } \mathbf{P} \]

have small pairwise angles. More rigorously, we introduce the

**Definition 1.3.** Let \( \mathbf{P} \) be an \( N \)-dimensional projective variety, and let \( \mathcal{L}, \mathcal{I} \) be two ample line bundles on \( \mathbf{P} \). Then \( \mathcal{I} \) is said to be almost proportional to \( \mathcal{L} \), if there exist two elements \( \alpha \in \mathbb{R}_+ \cdot [\mathcal{I}] \) and \( \beta \in \mathbb{R}_+ \cdot [\mathcal{L}] \) such that \( \beta < \alpha < (1 + \epsilon_0)\beta \), i.e. both \( \alpha - \beta, (1 + \epsilon_0)\beta - \alpha \) lie in the ample cone of \( \mathbf{P} \), where \( \epsilon_0 := 3/(N^2/2 - 1) \).

The value \( \epsilon_0 \) is due to the effective degree estimates of MCM, see Proposition 6.1.

**Theorem 1.4.** Let \( \mathbf{P} \) be an \( N \)-dimensional smooth projective \( \mathbb{K} \)-variety, equipped with a very ample line bundle \( \mathcal{L} \). For any integers \( c, r \geq 0 \) with \( 2c + r \geq N \), for any \( c + r \) ample line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_{c+r} \) which are almost proportional to \( \mathcal{L} \), there exists some integer:

\[ d = d(\mathcal{L}_1, \ldots, \mathcal{L}_{c+r}, \mathcal{L}) \gg 1 \]

such that, for all large integers \(d_1, \ldots, d_c, d_{c+1}, \ldots, d_{c+r} \geq d\), for generic \(c+r\) hypersurfaces:

\[
H_1 \in |L_1 \otimes d_1|, \ldots, H_{c+r} \in |L_{c+r} \otimes d_{c+r}|,
\]

the cotangent bundle \(\Omega_V\) of the intersection of the first \(c\) hypersurfaces \(V := H_1 \cap \cdots \cap H_c\) restricted to the intersection of all the \(c+r\) hypersurfaces \(X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}\) is ample.

We will see that in our proof, the lower degree bound \(d = d(L_1, \ldots, L_{c+r}, L)\) is effective. In particular, when \(r = 0\) and all \(L_1 = \cdots = L_c = L\) coincide, we will obtain the effective degree bound \(N^{N^2}\) of Theorem 1.2. See Subsection 6.8 for the details.

This paper is organized as follows. In Section 2, we outline the general strategy for the Debarre Ampleness Conjecture, which serves as a guiding principle of our approach. Next, in Section 3, we introduce a notion of formal matrices, and use their determinants to produce symmetric differential forms. Then, for the purpose of making negative twist, we play a dividing trick in Section 4, and thus generalize the aforementioned ingredient (1). Consequently, we are able to generalize (2) in Section 5. Thus, by adapting the ingredients (3), (4) as well, we establish Theorem 1.4 in Section 6, by means of the moving coefficients method developed in [5]. Lastly, we fulfill some technical details in Section 7.

It is worth to mention that, by means of formal matrices, we can also construct higher order jet differential forms. Therefore, we can also apply MCM to study the ampleness of certain jet subbundle of hypersurfaces in \(\mathbb{P}_C^N\), notably when \(N = 3\). We will discuss this in our coming paper.

Acknowledgments. I would like to thank Damian Brotbek and Lionel Darondeau for inspiring discussions. Also, I thank my thesis advisor Joël Merker for valuable suggestions and remarks.

2. General Strategy

It seems that, up to date, there has been only one strategy to settle the Debarre Ampleness Conjecture. To be precise, for fixed degrees \(d_1, \ldots, d_c\) of hypersurfaces, the strategy is firstly to choose a certain subfamily of \(c\) hypersurfaces, and then secondly to construct sufficiently many negatively twisted symmetric differential forms over the corresponding subfamily of intersections, and lastly to narrow their base locus up to discrete points over a generic intersection. Thus, there exists one desirable ample example in this subfamily, which suffices to conclude the generic ampleness of the whole family thanks to a theorem of Grothendieck.

Following this central idea, the first result [1] was obtained in the case \(c = N - 2\) for complex surfaces \(X = H_1 \cap \cdots \cap H_c \subset \mathbb{P}_C^N\), by employing a method related to Kobayashi hyperbolicity problems, in which the existence/quantity of negatively twisted symmetric differential forms was guaranteed/measured by the holomorphic Morse inequality. Such an approach would fail in the higher dimensional case, simply because one could not control the base locus of the implicitly given symmetric forms.

To find an alternative approach, the key breakthrough happened when Brotbek constructed explicit negatively twisted symmetric differential forms [2, Lemma 4.5] by a cohomological approach, for the subfamily of pure Fermat-type hypersurfaces of the same degree \(d + \epsilon\) defined by:

\[
F_i = \sum_{j=0}^{N} A_j^i \omega_j^{d_j} \quad (i = 1, \ldots, c),
\]
where \( d, \epsilon \geq 1 \), and where all coefficients \( A^j_i \) are some homogeneous polynomials with \( \deg A^j_i = \epsilon \geq 1 \). Then, in the case \( 4c \geq 3N - 2 \), Brotbek showed that over a generic intersection \( X \), the obtained symmetric forms have discrete base locus, and hence he established the conjectured ampleness.

However, when \( 4c < 3N - 2 \), this approach would not work, because the obtained symmetric differential forms keep positive dimensional base locus, for instance in the limiting case \( 2c = N \), there is only one obtained symmetric form, whereas \( \dim \mathbb{P}(\Omega_X) = N - 1 \gg 1 \).

To overcome this difficulty, the author [5] introduced the moving coefficients method (MCM), the cornerstone of which is a generalization of Brotbek’s symmetric differential forms for general Fermat-type hypersurfaces defined by:

\[
F_i = \sum_{j=0}^{N} A^j_i z_j^{\lambda_j} \quad (i = 1 \cdots c),
\]

where \( \lambda_0, \ldots, \lambda_N \geq 1 \) and where all polynomial coefficients \( A^j_i \) satisfy \( \deg A^j_i + \lambda_j = \deg F_i \). Then, by employing the other major ingredients (2), (3), (4) mentioned before, the Debarre Ampleness Conjecture finally turned into Theorem 1.1.

Recently, Brotbek and Darondeau [3] discovered a new way to construct negatively twisted symmetric differential forms for a certain subfamily of hypersurfaces, using pullbacks of some Plücker-embedding like morphisms, and they successfully controlled the base loci by means of deep theorems in algebraic geometry. Their approach together with the product coup gives another proof of the Debarre Ampleness Conjecture. Also, it is expected to achieve an effective lower bound on hypersurface degrees, which would ameliorate the preceding bound \( N^{N/2} \) of Theorem 1.1.

### 3. Formal Matrices Produce Symmetric Differential Forms

Aiming at the General Debarre Ampleness Conjecture, and following the general strategy above, we would like to first construct negatively twisted symmetric differential forms. Recalling the determinantal structure of Brotbek’s symmetric differential forms [2, Lemma 4.5], in fact, we can take any formal matrices for construction, regardless of negative twist at the moment.

Take an arbitrary scheme \( \mathcal{P} \). For any positive integers \( 1 \leq n \leq e \), for any \( e \) line bundles \( \mathcal{S}_1, \ldots, \mathcal{S}_e \) over \( \mathcal{P} \), we construct an \( (e + n) \times (e + n) \) formal matrix \( K \) such that, for \( p = 1 \cdots e \) its \( p \)-th row consists of global sections \( F^1_p, \ldots, F^{e+n}_p \in H^0(\mathcal{P}, \mathcal{S}_p) \), and for \( q = 1 \cdots n \) its \( (e + q) \)-th row is the formal differential — to be defined — of the \( q \)-th row:

\[
K := \begin{pmatrix}
F^1_1 & \cdots & F^{e+n}_1 \\
\vdots & \ddots & \vdots \\
F^1_e & \cdots & F^{e+n}_e \\
dF^1_1 & \cdots & dF^{e+n}_1 \\
\vdots & \ddots & \vdots \\
dF^1_n & \cdots & dF^{e+n}_n
\end{pmatrix}
\]

(2)

We will see later that the determinant of \( K \) produces a twisted symmetric differential form on \( \mathcal{P} \). First of all, we define the above formal differential entries \( dF^j_i \) in a natural way.

**Definition 3.1.** Let \( \mathcal{S} \) be a line bundle over \( \mathcal{P} \), with a global section \( S \). For any Zariski open set \( U \subset \mathcal{P} \) with a trivialization \( \mathcal{S}\big|_U = \mathcal{O}_U \cdot s \) (\( s \in H^0(U, \mathcal{S}) \) is invertible), denote \( S/s \) for the unique
\( \bar{s} \in \mathcal{O}_p(U) \) such that \( S = \bar{s} \cdot s \). Also, define the formal differential \( dS \) in the local coordinate \((U, s)\) by:

\[
dS(U, s) := d(S/s) \cdot s \in H^0(U, \Omega^1_p \otimes \mathcal{I}),
\]

where ‘d’ stands for the usual differential.

Let us check that the above definition works well with the usual Leibniz’s rule. Indeed, let \( \mathcal{I}_1, \mathcal{I}_2 \) be two line bundles over \( P \), with any two global sections \( S_1, S_2 \) respectively. For any Zariski open set \( U \subset P \) with trivializations \( \mathcal{I}_1|_U = \mathcal{O}_U \cdot s_1 \) and \( \mathcal{I}_2|_U = \mathcal{O}_U \cdot s_2 \), we may compute:

\[
d(S_1 \otimes S_2)(U, s_1 \otimes s_2) = d(S_1/s_1) \cdot S_2/s_2 \cdot s_1 \otimes s_2 + d(S_2/s_2) \cdot S_1/s_1 \cdot s_1 \otimes s_2
\]

[Dropping the tensor symbol ‘\( \otimes \)’ and coordinates \((U, s_1, s_2)\), we abbreviate the above identity as:

\[
d(S_1 \cdot S_2) = dS_1 \cdot S_2 + dS_2 \cdot S_1.
\]

Now, let us compute the determinant of (2) in local coordinates. For any Zariski open set \( U \subset P \) with trivializations \( \mathcal{I}_1|_U = \mathcal{O}_U \cdot s_1, \ldots, \mathcal{I}_e|_U = \mathcal{O}_U \cdot s_e \), writing \( f^j_i := F^j_i/s_i \in \mathcal{O}_p(U) \), we may factor:

\[
K(U, s_1, \ldots, s_e) := \begin{pmatrix}
f^1_1 \cdot s_1 & \cdots & f^1_e \cdot s_1 \\
\vdots & \ddots & \vdots \\
f^e_1 \cdot s_e & \cdots & f^e_e \cdot s_e \\
d f^1_1 \cdot s_1 & \cdots & d f^e_1 \cdot s_n \\
\vdots & \cdots & \vdots \\
d f^1_n \cdot s_n & \cdots & d f^e_n \cdot s_n
\end{pmatrix} = \begin{pmatrix}
s_1 & \cdots & s_e \\
\vdots & \ddots & \vdots \\
s_1 & \cdots & s_e \\
d f^1_1 \cdot s_1 & \cdots & d f^e_1 \cdot s_n \\
\vdots & \cdots & \vdots \\
d f^1_n \cdot s_n & \cdots & d f^e_n \cdot s_n
\end{pmatrix}
\]

Denoting the last two matrices by \( T^U_{s_1, \ldots, s_e} \) and \( (K)^U_{s_1, \ldots, s_e} \), we obtain:

\[
\det K(U, s_1, \ldots, s_e) = \det T^U_{s_1, s_e} \cdot \det (K)^U_{s_1, s_e}
\]

\[
= s_1 \cdots s_e \cdot s_1 \cdots s_n \cdot \det (K)^U_{s_1, \ldots, s_e} \in H^0(U, \text{Sym}^n \Omega^1_p \otimes \mathcal{I}(\bigvee)),
\]

where for shortness we denote:

\[
\mathcal{I}(\bigvee) := (\otimes_{p=1}^e \mathcal{I}_p) \otimes (\otimes_{q=1}^n \mathcal{I}_q).
\]

**Proposition 3.2.** The local definition:

\[
\det K|_U := \det K(U, s_1, \ldots, s_e) \in H^0(U, \text{Sym}^n \Omega^1_p \otimes \mathcal{I}(\bigvee))
\]

does not depend on the particular choices of invertible sections \( s_1, \ldots, s_e \) over \( U \).

**Proof.** Assume that \( s'_1, \ldots, s'_e \) are any other invertible sections of \( \mathcal{I}_1|_U, \ldots, \mathcal{I}_e|_U \). Abbreviating the \( p \)-th row of the formal matrix \( K \) by \( F^p \), we may compute:

\[
F^p/s'_p = s_p/s'_p \cdot F^p/s_p
\]

[Leibniz's rule]

\[
d(F^p/s'_p) = d(s_p/s'_p) \cdot F^p/s_p + s_p/s'_p \cdot d(F^p/s_p)
\]
Thus we receive the transition identity:

\[ (K)_{s'_1, \ldots, s'_e}^{U} = T_{s'_1, \ldots, s'_e}^{s_1, \ldots, s_e} \cdot (K)_{s_1, \ldots, s_e}^{U}, \]

where \( T_{s'_1, \ldots, s'_e}^{s_1, \ldots, s_e} \) is an \((e+n) \times (e+n)\) lower triangular matrix with the diagonal entries \( s_1/s'_1, \ldots, s_e/s'_e, s_1/s'_1, \ldots, s_n/s'_n \) in the exact order. Taking determinant on both sides of (6) thus yields:

\[
\det (K)_{s'_1, \ldots, s'_e}^{U} = \det T_{s'_1, \ldots, s'_e}^{s_1, \ldots, s_e} \cdot \det (K)_{s_1, \ldots, s_e}^{U} \]

\[= (\det T_{s'_1, \ldots, s'_e}^{s_1, \ldots, s_e})^{-1} \cdot (\det T_{s_1, \ldots, s_e}^{s_1, \ldots, s_e}) \cdot \det (K)_{s_1, \ldots, s_e}^{U} \]

Multiplying by \( \det T_{s'_1, \ldots, s'_e}^{U} \) on both sides, we conclude the proof. \( \square \)

Consequently, we receive

**Proposition 3.3.** The determinant of the formal matrix (2) is globally well defined:

\[ \det K \in H^0 \left( P, \text{Sym}^n \Omega^1_P \otimes \mathcal{S}(\bigvee) \right). \]

To grasp the essence of the above arguments, we provide another wholly formal ‘Smart Proof’. Suppose that we do not know the meaning of formal differential \( dF \), for any global section \( F \) of a line bundle \( \mathcal{S} \) over \( P \). Nevertheless, we still try to compute the determinant of the formal matrix (2).

First of all, we would like to extract some useful information out of the ‘mysterious’ \( dF \). A priori, we may assume that the formal differential satisfies the Leibniz’s rule in a certain sense, and also that when \( \mathcal{S} = \mathcal{O}_P \) it coincides with the usual differential \( d \). Thus, starting with any local section \( z \) of \( \mathcal{S} \), we would have:

\[ F = z \cdot F/z, \]

\[ dF = dz \cdot F/z + z \cdot (F/z), \]

that is:

\[
\begin{pmatrix}
F \\
dF
\end{pmatrix} =
\begin{pmatrix}
z & 0 \\
\star & z
\end{pmatrix}
\begin{pmatrix}
F/z \\
(d(F/z))
\end{pmatrix},
\]

where \( \star = dz \) is meaningless/negligible in our coming computations. Indeed, all we need is that the above underlined \( 2 \times 2 \) formal matrix is lower triangular, with meaningful diagonal.

Back to our formal proof, we abbreviate every row of \( K \) as \( F_1, \ldots, F_e, dF_1, \ldots, dF_n \), and for convenience we write:

\[ K = (F_1, \ldots, F_e, dF_1, \ldots, dF_n)^T. \]

Over any Zariski open set \( U \subset P \) with invertible sections \( z_1, \ldots, z_e \) of \( \mathcal{S}_1, \ldots, \mathcal{S}_e \) respectively, using identity (7), we can dehomogenize \( K \) with respect to \( z_1, \ldots, z_e \) by:

\[
K = T_{z_1, \ldots, z_e} \cdot \begin{pmatrix}
(F_1/z_1, \ldots, F_e/z_e, d(F_1/z_1), \ldots, d(F_n/z_n)
\end{pmatrix}^T,
\]

where \( T_{z_1, \ldots, z_e} \) is a lower triangular \((e+n) \times (e+n)\) formal matrix with diagonal entries \( z_1, \ldots, z_e, z_1, \ldots, z_n \) in the exact order. Now, it is desirable to notice that, on the right-hand-side of (8), the matrix \( (K)_{z_1, \ldots, z_e} \) and the diagonal of the formal matrix \( T_{z_1, \ldots, z_e} \) are well-defined, thus all the ‘mysterious
differentials’ of the matrix $K$ appear only in the strict lower-left part of $T_{z_1,\ldots,z_e}$, which would immediately disappear after taking determinant on both sides of (8):

$$
\det K_{\lfloor_U} = \det T_{z_1,\ldots,z_e} \cdot \det (K)_{z_1,\ldots,z_e} = z_1 \cdots z_e z_1 \cdots z_0 \cdot \det (K)_{z_1,\ldots,z_e} \in H^0\left(U, \Sym^n \Omega^1_P \otimes \mathcal{J}(\vee)\right).
$$

Bien sûr, it is independent of the choices of $z_1,\ldots,z_e$, since the left-hand-side — a formal determinant — is. □

**Remark 3.4.** The formal differential $d$ is much the same as the usual differential $d$, in the sense that both of them can be defined locally, and both of them obey the Leibniz’s rule. These two facts constitute the essence of Proposition 3.3.

Next, we consider $e$ sections:

$$
F_i = \sum_{j=0}^{e+n} F_i^j \in H^0\left(P, \mathcal{J}_i\right) \quad (i = 1\ldots e),
$$

(9)

each $F_i$ being the sum of $e + n + 1$ global sections of the same line bundle $\mathcal{J}_i$. Let $V$ be the intersection of the zero loci of the first $n$ sections:

$$
V := \{F_1 = 0\} \cap \cdots \cap \{F_n = 0\} \subset P,
$$

and let $X$ be the intersection of the zero loci of all the $e \geq n$ sections:

$$
X := \{F_1 = 0\} \cap \cdots \cap \{F_e = 0\} \subset V \subset P.
$$

Let $K$ be the $(e + n) \times (e + n + 1)$ formal matrix whose $e + n$ rows copy the $e + n + 1$ terms of $F_1,\ldots,F_e,dF_1,\ldots,dF_n$ in the exact order:

$$
K := \begin{pmatrix}
F_0^1 & \cdots & F_{e+n}^1 \\
\vdots & \ddots & \vdots \\
F_0^e & \cdots & F_{e+n}^e \\
\vdots & \ddots & \vdots \\
dF_0^1 & \cdots & dF_{e+n}^1 \\
\vdots & \ddots & \vdots \\
dF_0^n & \cdots & dF_{e+n}^n
\end{pmatrix}.
$$

Also, for $j = 0\cdots e + n$, let $\widehat{K}_j$ denote the submatrix of $K$ obtained by omitting the $(j+1)$-th column.

Since the restricted cotangent sheaf $\Omega^1_V|_X$ is formally defined by the $e + n$ equations:

$$
F_1 = 0,\ldots,F_e = 0, dF_1 = 0,\ldots,dF_n = 0,
$$

i.e. the sum of all $e + n + 1$ columns of $K$ vanishes, by Observation 3.6 below, we may receive

**Proposition 3.5.** For all $j = 0\cdots e + n$, the $e + n + 1$ sections:

$$
\psi_j = (-1)^j \det \widehat{K}_j \in H^0\left(P, \Sym^n \Omega^1_P \otimes \mathcal{J}(\vee)\right),
$$

when restricted to $X$, give one and the same section:

$$
\psi \in H^0\left(X, \Sym^n \Omega^1_V \otimes \mathcal{J}(\vee)\right).
$$
Observation 3.6. In a commutative ring $R$, for all positive integers $N \geq 1$, let $A^0, A^1, \ldots, A^N \in R^N$ be $N + 1$ column vectors satisfying:

$$A^0 + A^1 + \cdots + A^N = 0.$$ 

Then for all $0 \leq j_1, j_2 \leq N$, there hold the identities:

$$(-1)^{j_1} \det (A^0, \ldots, \hat{A}^{j_1}, \ldots, A^N) = (-1)^{j_2} \det (A^0, \ldots, \hat{A}^{j_2}, \ldots, A^N).$$

Proof of Proposition 3.5. Using the same notation as in (4), for $j = 0 \cdots e + n$, we obtain an $(e + n) \times (e + n)$ matrix $(\hat{K})_{s_1, \ldots, s_e}$. We also define:

$$K(U, s_1, \ldots, s_e) := T_{s_1, \ldots, s_e}^U \cdot (K)_{s_1, \ldots, s_e}^U,$$

where the $(e + n) \times (e + n + 1)$ matrix $(K)_{s_1, \ldots, s_e}^U$ satisfies that, for $j = 0 \cdots e + n$, the matrix $(\hat{K})_{s_1, \ldots, s_e}$ is obtained by omitting the $(j + 1)$-th column of $(K)_{s_1, \ldots, s_e}^U$.

We may view all entries of $(K)_{s_1, \ldots, s_e}^U$ as sections in $H^n(U \cap X, \text{Sym}^* \Omega^1_V)$, where:

$$\text{Sym}^* \Omega^1_V := \bigoplus_{k=0}^{\infty} \text{Sym}^k \Omega^1_V.$$

Thus the sum of all columns of $(K)_{s_1, \ldots, s_e}^U$ vanishes, and hence Observation 3.6 yields:

$$(-1)^{j_1} \det (\hat{K})_{s_1, \ldots, s_e}^U = (-1)^{j_2} \det (\hat{K})_{s_1, \ldots, s_e}^U \in H^0(U \cap X, \text{Sym}^n \Omega^1_V) \quad (j_1, j_2 = 0 \cdots e + n).$$

By multiplication of $\det T_{s_1, \ldots, s_e}^U$ on both sides, we conclude the proof. \qed

Remember that our goal is to construct negatively twisted symmetric differential forms. One idea, foreshadowed by the constructions in [2, 5], is to find some $e + n + 1$ line bundles $\mathcal{T}_0, \ldots, \mathcal{T}_{e+n}$ with respective global sections $t_0, \ldots, t_{e+n}$ having empty base locus, such that the line bundle:

$$\mathcal{J}(\mathcal{V}) \otimes \mathcal{T}_0^{-1} \otimes \cdots \otimes \mathcal{T}_{e+n}^{-1} =: \mathcal{J}(\mathcal{V}') < 0,$$

is negative, and such that:

$$\hat{\omega}_j := \frac{\psi_j}{t_0 \cdots t_{e+n}} = \frac{(-1)^j}{t_0 \cdots t_{e+n}} \det \hat{K}_j \in H^0(D(t_j), \text{Sym}^n \Omega^1_V \otimes \mathcal{J}(\mathcal{V}')) \quad (j = 0 \cdots e + n) \quad (10)$$

have no poles. Then, these $e + n + 1$ sections, restricted to $X$, would glue together to make a global negatively twisted symmetric differential form:

$$\omega \in H^0(X, \text{Sym}^n \Omega^1_V \otimes \mathcal{J}(\mathcal{V}')).$$

For the purpose of (10), we may require that every $t_0, \ldots, t_{e+n}$ subsequently `divides' the corresponding column of $K$ in the exact order. With some additional effort, we shall make this idea rigorous in our central applications.

4. A Dividing Trick

Let $\mathcal{L}$ be a line bundle over $P$ such that it has $N + 1$ global sections $\zeta_0, \ldots, \zeta_N$ having empty common base locus. Let $c \geq 1, r \geq 0$ be two integers with $2c + r \geq N$ and $c + r < N$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_{c+r}$ be $c + r$ auxiliary line bundles to be determined. Now, we consider $c + r$ Fermat-kind sections having the same shape as (1):

$$F_i = \sum_{j=0}^N A_j^i \zeta^j \in \mathcal{A}_i \otimes \mathcal{L}_i \otimes \mathcal{L}^i = \mathcal{A}_i \otimes \mathcal{L}^i \quad (i = 1 \cdots c + r), \quad (11)$$
where \( e_j, \lambda_j, d_i \geq 1 \) are integers satisfying \( e_j + \lambda_j = d_i \), and where every \( A^j_i \) is some global section of the line bundle \( \mathcal{A}_i \otimes \mathcal{L}^j \).

For the first \( c \) equations of (11), a formal differentiation yields:

\[
dF_i = \sum_{j=0}^{N} \frac{d(A^j_i \xi_j)}{\zeta_j(\xi_j \frac{\partial A^j_i}{\partial \zeta_j} + \lambda_j A^j_i \frac{d\zeta_j}{\zeta_j})} = B_i^j, \quad (i = 1 \ldots c).
\] (12)

Now, we construct the \((c + r + c) \times (N + 1)\) matrix \( M \), whose first \( c + r \) rows consist of all \((N + 1)\) terms in the expressions (11) of \( F_1, \ldots, F_{c+r} \) in the exact order, and whose last \( c \) rows consist of all \((N + 1)\) terms in the expressions (12) of \( dF_1, \ldots, dF_c \) in the exact order:

\[
M := \begin{pmatrix}
A^0_1 \xi_0 & \cdots & A^N_1 \xi_N \\
A^0_2 \xi_0 & \cdots & A^N_2 \xi_N \\
\vdots & & \vdots \\
A^0_{c+r} \xi_0 & \cdots & A^N_{c+r} \xi_N \\
A^0_1 \xi_0 & \cdots & A^N_1 \xi_N \\
\vdots & & \vdots \\
A^0_1 \xi_0 & \cdots & A^N_1 \xi_N \\
\end{pmatrix} = \begin{pmatrix}
A^0_1 \xi_0 & \cdots & A^N_1 \xi_N \\
A^0_2 \xi_0 & \cdots & A^N_2 \xi_N \\
\vdots & & \vdots \\
A^0_{c+r} \xi_0 & \cdots & A^N_{c+r} \xi_N \\
B^0_1 \xi_0 & \cdots & B^N_1 \xi_N \\
\vdots & & \vdots \\
B^0_1 \xi_0 & \cdots & B^N_1 \xi_N \\
\end{pmatrix}, (13)
\]

Denote \( n := N - c - r \), observe that \( 1 \leq n \leq c \). For every \( 1 \leq j_1 < \cdots < j_n \leq c \), denote by \( M_{j_1, \ldots, j_n} \) the \((c + r + n) \times (N + 1)\) submatrix of \( M \) consisting of the first upper \( c + r \) rows and the selected rows \( c + r + j_1, \ldots, c + r + j_n \). Also, for \( j = 0 \cdots N \), denote by \( M_{j_1, \ldots, j_n} \) the submatrix of \( M_{j_1, \ldots, j_n} \) obtained by omitting the \((j + 1)\)-th column.

Let \( V \subset \mathbb{P} \) be the subvariety defined by the first \( c \) sections \( F_1, \ldots, F_c \), and let \( X \subset \mathbb{P} \) be the subvariety defined by all the \( c + r \) sections \( F_1, \ldots, F_{c+r} \). Now, applying Proposition 3.5, denoting:

\[
\mathcal{A}^{1 \ldots c+r}_{j_1, \ldots, j_n} \ := \ 
\mathcal{A}_{j_1} \otimes \cdots \otimes \mathcal{A}_{j_n} \otimes \mathcal{A}_c \otimes \cdots \otimes \mathcal{A}_c,
\]

we receive

**Proposition 4.1.** For every \( 1 \leq j_1 < \cdots < j_n \leq c \), for all \( j = 0 \cdots N \), the \( N + 1 \) sections:

\[
\psi_{j_1, \ldots, j_n} \ := \ (-1)^j \det M_{j_1, \ldots, j_n} \in H^0 \left( \mathbb{P}, \mathcal{A}^{1 \ldots c+r}_{j_1, \ldots, j_n} \right),
\]

when restricted to \( X \), give one and the same symmetric differential form:

\[
\psi_{j_1, \ldots, j_n} \in H^0 \left( X, \mathcal{A}^{1 \ldots c+r}_{j_1, \ldots, j_n} \otimes \mathcal{L}^{\nu}_{j_1, \ldots, j_n} \right),
\]

with the twisted degree:

\[
\nu_{j_1, \ldots, j_n} = \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n} d_i,
\] (14)
Observe in (13) that the $N + 1$ columns of $M$ are subsequently divisible by $\zeta_0^{\lambda_0-1}, \ldots, \zeta_N^{\lambda_N-1}$. Dividing out these factors, we receive the formal matrix:

$$
C := \begin{pmatrix} 
A_0^0 \zeta_0 & \cdots & A_0^N \zeta_N \\
\vdots & \ddots & \vdots \\
A_{c+r}^0 \zeta_0 & \cdots & A_{c+r}^N \zeta_N \\
B_1^0 & \cdots & B_1^N \\
\vdots & \ddots & \vdots \\
B_c^0 & \cdots & B_c^N 
\end{pmatrix}.
$$

By mimicking the notation of the submatrices $M_{j_1, \ldots, j_n}, \tilde{M}_{j_1, \ldots, j_n; j}$ of $M$, we analogously define the submatrices $C_{j_1, \ldots, j_n}, \tilde{C}_{j_1, \ldots, j_n; j}$ of $C$.

Now, we interpret Proposition 4.1 in terms of the matrix $C$, starting by the formal computation:

$$
(-1)^j \det \tilde{M}_{j_1, \ldots, j_n; j} = (-1)^j \zeta_0^{\lambda_0-1} \cdots \zeta_j^{\lambda_j-1} \cdots \zeta_N^{\lambda_N-1} \det \tilde{C}_{j_1, \ldots, j_n; j} = \frac{(-1)^j}{\zeta_j^{\lambda_j-1}} \det \tilde{C}_{j_1, \ldots, j_n; j} \cdot \zeta_0^{\lambda_0-1} \cdots \zeta_{j-1}^{\lambda_{j-1}} \zeta_{j+1}^{\lambda_j-1} \cdots \zeta_N^{\lambda_N-1}.
$$

Dividing by $\zeta_0^{\lambda_0-1} \cdots \zeta_N^{\lambda_N-1}$ on both sides above, we receive the following $N + 1$ ‘coinciding’ forms:

$$
\frac{(-1)^j \det \tilde{M}_{j_1, \ldots, j_n; j}}{\zeta_0^{\lambda_0-1} \cdots \zeta_N^{\lambda_N-1}} = \frac{(-1)^j}{\zeta_j^{\lambda_j-1}} \det \tilde{C}_{j_1, \ldots, j_n; j} \quad (j = 0 \ldots N), \tag{15}
$$

This is the aforementioned dividing trick.

**Proposition 4.2.** For all $j = 0 \cdots N$, the formal symmetric differential forms:

$$
\tilde{o}_{j_1, \ldots, j_n} = \frac{(-1)^j \det \tilde{M}_{j_1, \ldots, j_n; j}}{\zeta_0^{\lambda_0-1} \cdots \zeta_N^{\lambda_N-1}}
$$

are well-defined sections in:

$$
H^0 \left( \mathcal{D}(\zeta_j), \text{Sym}^n \Omega^1_\mathcal{P} \otimes \mathcal{O}^{1, \ldots, c+r}_{j_1, \ldots, j_n} \otimes \mathcal{L}^\gamma_{j_1, \ldots, j_n} \right),
$$

with the twisted degree:

$$
\gamma'_{j_1, \ldots, j_n} := \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n} d_{j_q} - \sum_{k=0}^{N} (\lambda_k - 1).
$$

Moreover, when restricted to $X$, they glue together to make a global section:

$$
\omega_{j_1, \ldots, j_n} \in H^0 \left( X, \text{Sym}^n \Omega^1_\mathcal{V} \otimes \mathcal{O}^{1, \ldots, c+r}_{j_1, \ldots, j_n} \otimes \mathcal{L}^{\gamma'}_{j_1, \ldots, j_n} \right).
$$

While the formal identity (15) transparently shows the essence of this proposition, it is not yet a proof by itself, since both sides are to be defined. Indeed, to bypass the potential trouble of divisibility, the rigorous proof below is much more involved than one would first expect.
Proof. Without loss of generality, we only prove the case \( j_1 = 1, \ldots, j_n = n \), and we will often drop the indices \( j_1, \ldots, j_n \) since no confusion could occur. Here is a sketch of the proof.

Step 1. Over each Zariski open set \( U \subset D(\xi_j) \) with trivializations \( \mathcal{A}^1|_U = \mathcal{O}_U \cdot a_1, \ldots, \mathcal{A}^r|_U = \mathcal{O}_U \cdot a_{r+1}, \) we compute the expression of \( \omega_j := \omega_{j_1, \ldots, j_n} \) in coordinates \((U, a_1, \ldots, a_{r+1}, \xi_j)\).

Step 2. We show that the obtained symmetric form \( \omega_j|_U \) is independent of the choices of trivializations \( a_1, \ldots, a_{r+1} \), whence we conclude the first claim.

Step 3. For any distinct indices \( 0 \leq \ell_1, \ell_2 \leq N \), over any Zariski open set \( U \subset D(\xi_{\ell_1}) \cap D(\xi_{\ell_2}) \) with trivializations \( \mathcal{A}^1|_U = \mathcal{O}_U \cdot a_1, \ldots, \mathcal{A}^r|_U = \mathcal{O}_U \cdot a_{r+1}, \) we show that:

\[
\omega_{\ell_1} = \omega_{\ell_2} \in H^0 \left( U, \text{Sym}^n \Omega^1_U \otimes \mathcal{A}^{1-c+r} \otimes \mathcal{L}^{\varphi|_{j_1,\ldots,j_n}} \right)
\]

by computations in coordinates. Thus we conclude the second claim.

Proof of Step 1. Recalling (3), by trivializations:

\[
\mathcal{A}^j \otimes \mathcal{L}^d|_U = \mathcal{O}_U \cdot a_j \xi^d_j \quad (i = 1 \cdots c+r),
\]

the formal matrix \( K := M_{j_1,\ldots,j_n} \) has coordinates:

\[
K = T^j_{a_1,\ldots,a_{c+r}} \cdot \left( F_1/s_1, \ldots, F_{c+r}/s_{c+r}, d(F_1/s_1), \ldots, d(F_n/s_n) \right)^T
\]

where \( T^j_{a_1,\ldots,a_{c+r}} \) is an \( N \times N \) diagonal matrix with the diagonal \( s_1, \ldots, s_{c+r}, s_1, \ldots, s_n \), and where like (8) we abbreviate the first \( c + r \) rows of \( K \) by \( F_1, \ldots, F_{c+r} \). Further computation yields:

\[
F_i/s_i := (A_0^0 \xi^d_0, \ldots, A_i^N \xi^d_i) / s_i = \left( A_i^0 / A_i^0 \cdot (\xi_0/\xi_i)^{d_0}, \ldots, A_i^N / A_i^0 \cdot (\xi_N/\xi_i)^{d_N} \right) \quad (i = 0 \cdots c+r),
\]

where \( a_i := a_i \cdot \xi^d_i \). ‘Dividing’ every column of \((K)_{a_1,\ldots,a_{c+r}}\) subsequently by \((\xi_0/\xi_i)^{d_0-1}, \ldots, (\xi_N/\xi_i)^{d_N-1}\), we obtain an \( N \times (N + 1) \) matrix \((C)_{a_1,\ldots,a_{c+r}}\). For every \( \ell = 0 \cdots N \), we denote by \((\widetilde{C})_{a_1,\ldots,a_{c+r}} \) the submatrix of \((C)_{a_1,\ldots,a_{c+r}}\) obtained by deleting its \((\ell+1)\)-th column. Now, formula (4) yields:

\[
\det \widetilde{M}_{j_1,\ldots,j_n} = s_1 \cdots s_{c+r} s_1 \cdots s_n (\xi_0/\xi_i)^{d_0-1} \cdots (\xi_N/\xi_i)^{d_N-1} \det (\widetilde{C})_{a_1,\ldots,a_{c+r}}.
\]

Thus, in coordinates \((U, a_1, \ldots, a_{c+r}, \xi_j)\), we obtain/define:

\[
\omega_j = \frac{(-1)^j \det \widetilde{M}_{j_1,\ldots,j_n}}{\xi_0^{d_0-1} \cdots \xi_N^{d_N-1}} := \frac{(-1)^j s_1 \cdots s_{c+r} s_1 \cdots s_n}{\xi_0^{d_0-1} \cdots \xi_N^{d_N-1}} \cdot \det (\widetilde{C})_{a_1,\ldots,a_{c+r}}
\]

\[
\quad = (-1)^j a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n \cdot \xi^d_{j_1,\ldots,j_n} \cdot \det (\widetilde{C})_{a_1,\ldots,a_{c+r}}
\]

\[
\in H^0 \left( U, \text{Sym}^n \Omega^1_U \otimes \mathcal{A}^{1-c+r} \otimes \mathcal{L}^{\varphi|_{j_1,\ldots,j_n}} \right).
\]

Proof of Step 2. We only need to show that:

\[
a_1 \cdots a_{c+r} a_1 \cdots a_n \cdot \det (\widetilde{C})_{a_1,\ldots,a_{c+r}} \in H^0 \left( U, \text{Sym}^n \Omega^1_U \otimes \mathcal{A}^{1-c+r} \otimes \mathcal{L}^{\varphi|_{j_1,\ldots,j_n}} \right)
\]

is independent of the choices of \( a_1, \ldots, a_{c+r} \).
Let $\overline{a}_1, \ldots, \overline{a}_{c+r}$ be any other choices of invertible sections of $\mathscr{A}|_U$, $\mathscr{A}_{c+r}|_U$. Accordingly, we obtain the matrices $(C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$, $(\overline{C})_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$, and we denote $\overline{a}_i^j := \overline{a}_i \cdot \xi_j^i$. Then, for $i = 1 \cdot c + r$, the $i$-th row of the matrix $(C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$ is:

$$[\,a_i^j/\overline{a}_i^j = a_i/\overline{a}_i, \quad (A_i^j/\overline{a}_i^j \cdot (\xi_0^j/\xi_j), \ldots, A_i^j/\overline{a}_i^j \cdot (\xi_N^j/\xi_j)) = a_i/\overline{a}_i \cdot (A_i^0/\alpha_i^0 \cdot (\xi_0/\xi_j), \ldots, A_i^N/\alpha_i^N \cdot (\xi_N/\xi_j))].$$

the $i$-th row of the matrix $(C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$

Also, for $i = 1 \cdot n$, $k = 0 \cdot N$, using:

$$d(A_i^k/\overline{a}_i^k) = d(A_i^k/\alpha_i^k \cdot a_i/\overline{a}_i) = a_i/\overline{a}_i \cdot d(A_i^k/\alpha_i^k) + A_i^k/\alpha_i^k \cdot d(a_i/\overline{a}_i),$$

we see that the $(c + r + i, k + 1)$-th entry of $(C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$ satisfies:

$$(\xi_k/\xi_j) \cdot d(A_i^k/\overline{a}_i^k) + \lambda_k(A_i^k/\overline{a}_i^k) \cdot d(\xi_k/\xi_j)$$

$$= a_i/\overline{a}_i \cdot (\xi_k/\xi_j) \cdot d(A_i^k/\alpha_i^k) + \lambda_k(A_i^k/\alpha_i^k) \cdot d(\xi_k/\xi_j) + d(a_i/\overline{a}_i) \cdot A_i^k/\alpha_i^k \cdot (\xi_k/\xi_j).$$

Hence we receive the transition identity:

$$(C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j = T_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j \cdot (C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j,$$

where $T_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$ is a lower triangular matrix with the product of the diagonal:

$$\det T_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j = a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n.$$ 

In particular, we have:

$$(\overline{C})_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j = T_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j \cdot (\overline{C})_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j,$$

hence, by taking determinant on both sides above, we obtain:

$$\overline{a}_1 \cdots \overline{a}_{c+r} \cdot \overline{a}_1 \cdots \overline{a}_n \cdot \det (\overline{C})_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j = a_1 \cdots a_{c+r} a_1 \cdots a_n \cdot \det (\overline{C})_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j,$$

which is our desired identity.

**Proof of Step 3.** First of all, we recall the famous

**Cramer’s Rule.** In a commutative ring $R$, for all positive integers $N \geq 1$, let $A^0, A^1, \ldots, A^N \in R^N$ be $N + 1$ column vectors, and suppose that $z_0, z_1, \ldots, z_N \in R$ satisfy:

$$A^0 z_0 + A^1 z_1 + \cdots + A^N z_N = 0.$$

Then for all indices $0 \leq \ell_1, \ell_2 \leq N$, there hold the identities:

$$(-1)^{\ell_1} \det (A^0, A^{\ell_1}, \ldots, A^N) z_{\ell_2} = (-1)^{\ell_2} \det (A^0, A^{\ell_2}, \ldots, A^N) z_{\ell_1}.$$

In the rest of the proof, we shall view all entries of the matrices $(K)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$, $(C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$ as elements in the ring $H^0(U \cap X, \text{Sym}^a \Omega^1_U)$. Note that the sum of all columns of $(K)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$ vanishes:

$$C_0 \left(\frac{\xi_0}{\xi_j}\right)^{k_0-1} + \cdots + C_N \left(\frac{\xi_N}{\xi_j}\right)^{k_0-1} = 0,$$

where we denote the $(i + 1)$-th row of $(C)_{\overline{a}_1, \ldots, \overline{a}_{c+r}}^j$ by $C_i$. Applying Cramer’s rule, we receive:

$$(-1)^{\ell_1} \det (\overline{C})_1 \left(\frac{\xi_0}{\xi_j}\right)^{k_0-1} = (-1)^{\ell_2} \det (\overline{C})_2 \left(\frac{\xi_0}{\xi_j}\right)^{k_1-1} \in H^0(U \cap X, \text{Sym}^a \Omega^1_U) \quad (0 \leq \ell_1, \ell_2 \leq N).$$
Lastly, we can check the desired identity (16) by the following computation:

\[
\widehat{\omega}_{\ell_1} = (-1)^{\ell_1} a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n \cdot \xi_{\ell_1}^{q_{1/n}} \cdot \det (\widehat{\mathcal{C}}_{\ell_1})_{a_1 \cdots a_{c+r}} \quad \text{[use (19)]}
\]

\[
\text{[use (21) for } j = \ell_1]\]

\[
= (-1)^{\ell_2} a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n \cdot \xi_{\ell_1}^{q_{1/n}} \cdot \det (\widehat{\mathcal{C}}_{\ell_2})_{a_1 \cdots a_{c+r}} \cdot \left(\xi_{\ell_1}/\xi_{\ell_2}\right)^{\pm 1}
\]

\[
\text{[use Proposition 4.3 below]}
\]

\[
= (-1)^{\ell_2} a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n \cdot \xi_{\ell_1}^{q_{1/n}} \cdot \left(\frac{\xi_{\ell_2}}{\xi_{\ell_1}}\right)^{\varphi(\ell_2)} \cdot \det (\widehat{\mathcal{C}}_{\ell_2})_{a_1 \cdots a_{c+r}} \cdot \left(\frac{\xi_{\ell_1}}{\xi_{\ell_2}}\right)^{\pm 1}
\]

\[
[\varphi(\ell_2) = \varphi_{j_1, \ldots, j_n} + \lambda_{\ell_2} - 1]
\]

\[
= (-1)^{\ell_2} a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n \cdot \xi_{\ell_2}^{q_{1/n}} \cdot \det (\widehat{\mathcal{C}}_{\ell_2})_{a_1 \cdots a_{c+r}}
\]

\[
[\text{use (19)]} = \widehat{\omega}_{\ell_2}
\]

Thus we finish the proof.

An essential ingredient in the above proof is to compare the same determinant in different trivializations $\xi_{\ell_1}, \xi_{\ell_2}$. Now we give general transition formulas.

**Proposition 4.3.** For all $0 \leq j, \ell_1, \ell_2 \leq N$, for any Zariski open set $U \subset \mathcal{D}(\xi_{\ell_1}) \cap \mathcal{D}(\xi_{\ell_2})$ with trivializations $\mathcal{A}_{\ell_1}^U = \mathcal{O}_U \cdot a_1, \ldots, \mathcal{A}_{\ell_2}^U = \mathcal{O}_U \cdot a_{c+r}$, there hold the transition formulas:

\[
\det (\widehat{\mathcal{C}}_{j})_{a_1 \cdots a_{c+r}} = \left(\frac{\xi_{\ell_1}}{\xi_{\ell_2}}\right)^{\varphi(j)} \cdot \det (\widehat{\mathcal{C}}_{\ell_1})_{a_1 \cdots a_{c+r}} \in H^0 \left(U, \text{Sym}^n \Omega^1_{\mathcal{P}}\right),
\]

(22)

with $\varphi(j) = \varphi_{j_1, \ldots, j_n} + \lambda - 1$.

**Proof.** Our idea is to expand the two determinants and to compare each pair of corresponding terms. Without loss of generality, we may assume $j = 0$.

For $i = 1 \cdots N, k = 1 \cdots N$, we denote the $(i, k)$-th entry of $(\widehat{\mathcal{C}}_{\ell_1})_{a_1 \cdots a_{c+r}}$ (resp. $(\widehat{\mathcal{C}}_{\ell_2})_{a_1 \cdots a_{c+r}}$) by $c_{i,j}^1$ (resp. $c_{i,j}^2$). First of all, we recall all the entries:

\[
c_{p,k}^0 := \frac{A_p^k}{a_p} \cdot \frac{\xi_k}{\xi_{\ell_2}}, \quad c_{c+r+q,k}^0 := d \left(\frac{A_q}{a_q} \cdot \frac{\xi_k}{\xi_{\ell_2}}\right) \cdot \xi_k + \lambda_k \cdot \frac{A_q}{a_q} \cdot d \left(\frac{\xi_k}{\xi_{\ell_0}}\right) \quad (\ell_1, \ell_2, p = 1 \cdots c+r, q = 1 \cdots n, k = 1 \cdots N).
\]

By much the same reasoning as in (20), we can obtain the transition formulas:

\[
c_{c+r+q,k}^1 = c_{c+r+q,k}^2 \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{\varepsilon_{k}^{q+1}},
\]

\[
c_{c+r+q,k}^1 = c_{c+r+q,k}^2 \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{\varepsilon_{k}^{q+1}} + c_{c+r+q,k}^2 \cdot \left(\xi_{\ell_2} + \lambda_k\right) \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{\varepsilon_{k}^{q}} d \left(\xi_{\ell_2}/\xi_{\ell_1}\right).
\]

Recalling that $\varepsilon_{p} + \lambda_k = d_p$, we thus rewrite the above identities as:

\[
c_{c+r+q,k}^1 = c_{c+r+q,k}^2 \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{d_p-(\varepsilon_{k}^{q+1})},
\]

\[
c_{c+r+q,k}^1 = c_{c+r+q,k}^2 \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{d_p-(\varepsilon_{k}^{q+1})} + c_{c+r+q,k}^2 \cdot \left(\xi_{\ell_2} + \lambda_k\right) \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{d_p-\varepsilon_{k}^{q}} d \left(\xi_{\ell_2}/\xi_{\ell_1}\right).
\]

(23)

Now, comparing (23) with the desired formula (22), we may anticipate that, the underlined terms would bring some trouble, since no terms $d \left(\xi_{\ell_2}/\xi_{\ell_1}\right)$ appear on the right-hand-side of (22). Nevertheless, we can overcome this difficulty firstly by observing:

\[
\begin{vmatrix}
\varepsilon_{c+r+q,k_1}^1 & \varepsilon_{c+r+q,k_2}^1 \\
\varepsilon_{c+r+q,k_1}^1 & \varepsilon_{c+r+q,k_2}^1 \\
\end{vmatrix}
= \begin{vmatrix}
\varepsilon_{c+r+q,k_1}^2 & \varepsilon_{c+r+q,k_2}^2 \\
\varepsilon_{c+r+q,k_1}^2 & \varepsilon_{c+r+q,k_2}^2 \\
\end{vmatrix} \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{d_p-(\varepsilon_{k_1}^{q+1})} \cdot \left(\xi_{\ell_2}/\xi_{\ell_1}\right)^{d_p-(\varepsilon_{k_2}^{q+1})} \quad (q = 1 \cdots n; k_1, k_2 = 1 \cdots N).
\]

(24)
and secondly by using a tricky Laplace expansion of the determinant:

$$\det(\widehat{C}_0)_{a_1, \ldots, a_{c+r}}^{\xi_{\ell_1}} = \sum \text{Sign}(\pm) \cdot \prod_{q=1}^{n} \left| \begin{array}{ccc}
    c^1_{q,k^1_1} & c^1_{q,k^1_2} \\
    c^2_{c+r+q,k^1_1} & c^2_{c+r+q,k^1_2}
  \end{array} \right| \cdot \prod_{p=n+1}^{c+r} c^2_{p,k_p}, \quad (25)
$$

where the sum runs through all choices of $N = 2n + (c + r - n)$ indices $k^1_1 < k^1_2, \ldots, k^1_n < k^2_n, k_{n+1}, \ldots, k_{c+r}$ such that their union is exactly $\{1, \ldots, N\}$, and where \text{Sign}(\pm) is either $1$ or $-1$ uniquely determined by the choices of indices. Now, using (24), we see that each term in (25) is equal to:

$$\text{Sign}(\pm) \cdot \prod_{q=1}^{n} \left| \begin{array}{ccc}
    c^2_{q,k^1_1} & c^2_{q,k^1_2} \\
    c^2_{c+r+q,k^1_1} & c^2_{c+r+q,k^1_2}
  \end{array} \right| \cdot \prod_{p=n+1}^{c+r} c^2_{p,k_p}
$$

multiplied by $(\zeta_{\ell_2}/\zeta_{\ell_1})^\circ$, where:

$$\diamond := \sum_{q=1}^{n} \left[ d_q - (\lambda^1_{k^1_q} - 1) + d_q - (\lambda^2_{k^2_q} - 1) \right] + \sum_{p=n+1}^{c+r} (d_q - (\lambda^1_{k^1_p} - 1))
$$

$$= \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n} d_q - \sum_{k=1}^{N} (\lambda_k - 1)
$$

$$= \varnothing + \lambda_0 - 1.
$$

Thus (25) factors as:

$$\det(\widehat{C}_0)_{a_1, \ldots, a_{c+r}}^{\xi_{\ell_1}} = (\zeta_{\ell_2}/\zeta_{\ell_1})^\circ \cdot \sum \text{Sign}(\pm) \cdot \prod_{q=1}^{n} \left| \begin{array}{ccc}
    c^2_{q,k^1_1} & c^2_{q,k^1_2} \\
    c^2_{c+r+q,k^1_1} & c^2_{c+r+q,k^1_2}
  \end{array} \right| \cdot \prod_{p=n+1}^{c+r} c^2_{p,k_p}
$$

[use Laplace expansion again] \quad = (\zeta_{\ell_2}/\zeta_{\ell_1})^\circ \cdot \det(\widehat{C}_0)_{a_1, \ldots, a_{c+r}},
$$

whence we conclude the proof. \qed

5. ‘Hidden’ Symmetric Differential Forms

Comparing the two approaches in [5, Section 6], the scheme-theoretic one has the advantage in further generalizations, while the geometric one is superior in discovering the ‘hidden’ symmetric differential forms [5, Proposition 6.12]. Skipping the thinking process, we present the corresponding generalizations of these symmetric forms as follows.

We assume that $\lambda_0, \ldots, \lambda_N \geq 2$ in this section. For any $\eta = 1 \cdots n - 1$, for any indices $0 \leq v_1 < \cdots < v_\eta \leq N$ and $1 \leq j_1 < \cdots < j_{N-\eta} \leq c$, write $\{0, \ldots, N\} \setminus \{v_1, \ldots, v_\eta\}$ in the ascending order $r_0 < r_1 < \cdots < r_{N-\eta}$, and then denote by $v_1, \ldots, v_\eta M_{j_1, \ldots, j_{N-\eta}}$ the $(N-\eta) \times (N-\eta+1)$ submatrix of $M$ determined by the first $c + r$ rows and the selected rows $c + r + j_1, \ldots, c + r + j_{N-\eta}$ as well as the $(N-\eta+1)$ columns $r_0 + 1, \ldots, r_{N-\eta} + 1$. Next, for every index $j \in \{0, \ldots, N\} \setminus \{v_1, \ldots, v_\eta\}$, let $v_1, \ldots, v_\eta \overline{M}_{j_{1, \ldots, j_{N-\eta}}} j$ denote the submatrix of $v_1, \ldots, v_\eta M_{j_1, \ldots, j_{N-\eta}}$ obtained by deleting the column which is originally contained in the $(j+1)$-th column of $M$. Lastly, denote by $v_1, \ldots, v_\eta P \subset \mathcal{P}$ the subvariety defined by sections $\zeta_{v_1}, \ldots, \zeta_{v_\eta}$ (‘vanishing coordinates’), and denote $v_1, \ldots, v_\eta X := X \cap v_1, \ldots, v_\eta P$. Setting:

$$\mathcal{A}^{1, \ldots, r}_{j_1, \ldots, j_{N-\eta}} := \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r \otimes \mathcal{A}_{j_1} \otimes \cdots \otimes \mathcal{A}_{N-\eta},
$$

by much the same reasoning as in Proposition 4.1, we have
Proposition 5.1. For all \( j = 0 \cdots N - \eta \), the following \( N + 1 - \eta \) sections:

\[
v_{1, \ldots, v_{\eta}} H_{1, \ldots, j_{-\eta}} r_j : = (-1)^j \det \left( v_{1, \ldots, v_{\eta}} M_{1, \ldots, j_{-\eta}} r_j \right)
\]

\[\in H^0 \left( v_{1, \ldots, v_{\eta}} P, \text{Sym}^{n-\eta} \Omega_p^{1} \otimes \mathcal{A}^{1, \ldots, c+r} \otimes L^0 \right),\]

when restricted to \( v_{1, \ldots, v_{\eta}} X \), give one and the same symmetric differential form:

\[v_{1, \ldots, v_{\eta}} \omega_{1, \ldots, j_{-\eta}} \in H^0 \left( v_{1, \ldots, v_{\eta}} X, \text{Sym}^{n-\eta} \Omega_p^{1} \otimes \mathcal{A}^{1, \ldots, c+r} \otimes L^{0} \right),\]

with the twisted degree:

\[
\omega_{1, \ldots, j_{-\eta}} = \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n-\eta} d_{jq}. \quad \square
\]

Moreover, playing the dividing trick again, we obtain an analogue of Proposition 4.2.

Proposition 5.2. For all \( j = 0 \cdots N - \eta \), the formal symmetric differential forms:

\[
v_{1, \ldots, v_{\eta}} \omega_{1, \ldots, j_{-\eta}} : = (-1)^j \det \left( v_{1, \ldots, v_{\eta}} M_{1, \ldots, j_{-\eta}} r_j \right)
\]

are well-defined sections in:

\[H^0 \left( D(\xi_j) \cap v_{1, \ldots, v_{\eta}} P, \text{Sym}^{n-\eta} \Omega_p^{1} \otimes \mathcal{A}^{1, \ldots, c+r} \otimes L^{0} \right),\]

with the twisted degree:

\[
v_{1, \ldots, v_{\eta}} \omega_{1, \ldots, j_{-\eta}} : = \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n-\eta} d_{jq} - \sum_{k=0}^{N} (\lambda_k - 1) + \sum_{\mu=1}^{\eta} (\lambda_{\mu} - 1).
\]

Moreover, when restricted to \( v_{1, \ldots, v_{\eta}} X \), they glue together to make a global section:

\[v_{1, \ldots, v_{\eta}} \omega_{1, \ldots, j_{-\eta}} \in H^0 \left( v_{1, \ldots, v_{\eta}} X, \text{Sym}^{n-\eta} \Omega_p^{1} \otimes \mathcal{A}^{1, \ldots, c+r} \otimes L^{0} \right). \quad \square
\]

6. Applications of MCM

6.1. Motivation. Recall [5, Section 7] that the moving coefficients method is devised to produce as many negatively twisted symmetric differential forms as possible, by manipulating the determinantal structure of the constructed symmetric differential forms. Since Propositions 4.2, 5.2 exactly share the same determinantal shape, it is possible to adapt MCM for the aim of Theorem 1.2, which coincides with Theorem 1.1 in the case that \( P = \mathbb{P}^N_{\mathbb{K}}, L = \mathcal{O}_{\mathbb{P}^N_{\mathbb{K}}}(1) \). Indeed, by introducing \( c \) auxiliary line bundles \( \mathcal{A}_1, \ldots, \mathcal{A}_c \approx \text{trivial line bundle} \), we can even treat the case of \( c \) ample line bundles \( L + \mathcal{A}_1, \ldots, L + \mathcal{A}_c \approx L \), and eventually we will obtain Theorem 1.4.

6.2. Adaptation. Let \( P \) be a smooth projective \( \mathbb{K} \)-variety of dimension \( N \), equipped with a very ample line bundle \( L \). By Bertini’s theorem, we may choose \( N + 1 \) simple normal crossing global sections \( \xi_0, \ldots, \xi_N \) of \( L \), and we shall view them as the ‘homogeneous coordinates’ of \( P \). Thus, we may ‘identify’ \( (P, L) \) with \( (\mathbb{P}^N_{\mathbb{K}}, \mathcal{O}_{\mathbb{P}^N_{\mathbb{K}}}(1)) \) in the sense that locally they have the same coordinates \([\xi_0 : \cdots : \xi_N] \approx [\xi_0 : \cdots : \xi_N]\), and therefore we can generalize local computations of the later one to the former one, like what we perform in Section 4. This treatment is also visible in [3].
Let $c \geq 1$, $r \geq 0$ be two integers with $2c + r \geq N$ and $c + r < N$, and let $\mathcal{A}_1, \ldots, \mathcal{A}_{c+r}$ be $c + r$ auxiliary line bundles to be determined. Now, we start to adapt the machinery of MCM. First of all, introduce the following $c + r$ ‘flexible’ sections which copy the major ingredient (3):

$$
F_i = \sum_{j=0}^{N} A_j^i \xi_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0 \leq c < \cdots < j \leq N \atop j \neq l \text{ for some } 0 \leq d \leq l} M_{i,0}^{j_0 \cdots j_1 \cdots j_l \cdots j_{c+r}} \xi_{j_0}^{\mu_{i,0}} \cdots \xi_{j_l}^{\mu_{i,l}} \cdots \xi_{j_{c+r}}^{\mu_{i,c+r}} 
$$

\begin{equation}
\in H^0 \left( \mathbb{P}, \mathcal{A}_i \otimes \mathcal{L}^d \right) \quad (i = 1 \cdots c + r),
\end{equation}

where all coefficients $A_j^i, M_j^{i}\cdot \cdot \cdot$ are some global sections of $\mathcal{A}_i \otimes \mathcal{L}$ for some fixed integers $i \geq 1$, and where all positive integers $\mu_j^l, d$ are to be chosen by a certain Algorithm, which is designed to make all the symmetric differential forms obtained later have negative twist. For better comprehension, we will make the Algorithm clear in Subsection 6.4 below, and for the time being we roughly summarize it as:

$$
1 \leq \max \{ \epsilon_i \} = 1 \cdots c + r \leq \mu_{c+r+1,0} \leq \cdots \leq \mu_{c+r+1,c+r+1} \leq \cdots \leq \mu_{N,0} \leq \cdots \leq \mu_{N,N} \leq d. \quad (27)
$$

Let $V \subset \mathbb{P}$ be the subvariety defined by the first $c$ sections $F_1, \ldots, F_c$, and let $X \subset \mathbb{P}$ be the subvariety defined by all the $c + r$ sections $F_1, \ldots, F_{c+r}$. A priori, we require all the line bundles $\mathcal{A}_i \otimes \mathcal{L}_i$ to be very ample, so that for generic choices of parameters:

$$
A_j^i, M_j^{i}\cdot \cdot \cdot \in H^0 \left( \mathbb{P}, \mathcal{A}_i \otimes \mathcal{L}_i \right) \quad (i = 1 \cdots c + r),
$$

both intersections $V, X$ are smooth complete (the proof is much the same as that of Bertini’s Theorem, see Subsection 7.2).

### 6.3. Manipulations

Now, we apply MCM to construct a series of negatively twisted symmetric differential forms. For shortness, we will refer to [5, Section 7] for skipped details, in which the canonical setting $\{\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1), (z_0, \ldots, z_N)\}$ there plays the same role as that of $\{\mathbb{P}, \mathcal{L}, (\xi_0, \ldots, \xi_N)\}$ in our treatment here.

To begin with, we rewrite each section $F_i$ in (26) as (cf. [5, p. 43, (104)]):

$$
F_i = \sum_{j=0}^{N} \left( A_j^i \xi_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0 \leq c < \cdots < j \leq N \atop j \neq l \text{ for some } 0 \leq d \leq l} M_{i,0}^{j_0 \cdots j_1 \cdots j_l \cdots j_{c+r}} \xi_{j_0}^{\mu_{i,0}} \cdots \xi_{j_l}^{\mu_{i,l}} \cdots \xi_{j_{c+r}}^{\mu_{i,c+r}} \right) + \sum_{k=0}^{N} M_{i,0}^{j_0 \cdots j_k \cdots j_{c+r}} \xi_{j_0}^{\mu_{i,0}} \cdots \xi_{j_k}^{\mu_{i,k}} \cdots \xi_{j_{c+r}}^{\mu_{i,c+r}} 
$$

\begin{equation}
\text{for each } k = 0 \cdots N, \text{ we view it as one section }
\end{equation}

Thus, we view each $F_i$ as the sum of $2N + 2 = \sum_{j=0}^{N} 1 + \sum_{k=0}^{N} 1$ sections of the same line bundle, as indicate above.

Next, we construct a $(c + r + c) \times (2N + 2)$ formal matrix $M$ such that, for every $i = 1 \cdots c + r$, $j = 1 \cdots c$, its $i$-th row copies the $2N + 2$ sections in the sum of $F_i$ in the exact order, and its $(c + r + j)$-th row is the formal differential of the $j$-th row.

Write the $2N + 2$ columns of $M$ as:

$$
M = \begin{pmatrix}
A_0 & \cdots & A_N & B_0 & \cdots & B_N
\end{pmatrix}.
$$

\begin{equation}
(29)
\end{equation}

For every $\nu = 0 \cdots N$, we construct the matrix:

$$
K^{\nu} := \begin{pmatrix}
A_0 & \cdots & \tilde{A}_\nu & \cdots & A_N & A_\nu + \sum_{j=0}^{N} B_j
\end{pmatrix}.
$$

\begin{equation}
(30)
\end{equation}
where the last column is understood to appear in the ‘omitted’ column. Also, for every $\tau = 0 \cdots N-1$ and every $\rho = \tau + 1 \cdots N$, we construct the matrix:

$$K^{\tau,\rho} := \left( A_0 + B_0 | \cdots | A_\tau + B_\tau | A_{\tau+1} | \cdots | \widehat{A}_\rho | \cdots | A_N | A_\rho + \sum_{j=\tau+1}^{N} B_j \right). \quad (31)$$

Now, fix a positive integer $\heartsuit \geq 1$ such that:

$$\mathcal{A}_i \otimes \mathcal{L}^{\heartsuit} < 0 \quad (i = 1 \cdots c+r). \quad (32)$$

Recalling the rough Algorithm (27), observe in (28), (30) that the $N + 1$ columns of $K^\tau$ are subsequently divisible by:

$$\xi_0^{d-\delta_N}, \cdots, \xi_{\tau}^{d-\delta_N}, \cdots, \xi_N^{d-\delta_N}, \xi_V^{\mu(N,0)},$$

where $\delta_N := (N-1)\mu_{N-1,N-1}$. Thus, applying Proposition 4.2, for every $1 \leq j_1 < \cdots < j_n \leq c$, we obtain a global symmetric differential form:

$$\phi_{j_1,\cdots,j_n}^\nu \in H^0(X, \text{Sym}^n \Omega_V \otimes \mathcal{A}_{j_1,\cdots,j_n} \otimes \mathcal{L}^{\nu}_{j_1,\cdots,j_n}) \quad (\nu = 0 \cdots N),$$

with negative twist:

$$\nabla_{j_1,\cdots,j_n}^\nu = \sum_{p=1}^{c+r} (d + \epsilon_p) + \sum_{q=1}^{n} (d + \epsilon_{i_q}) - \sum_{j=0, j \neq v}^{N} (d - \delta_N - 1) - (\mu_{N,0} - 1)$$

$$= -\mu_{N,0} + N \delta_N + \sum_{p=1}^{c+r} \epsilon_p + \sum_{q=1}^{n} \epsilon_{i_q} + N + 1 \quad (33)$$

[by Algorithm (27)] \quad \leq -N \heartsuit.

Similarly, observe that the $N + 1$ columns of $K^{\tau,\rho}$ are subsequently divisible by:

$$\xi_0^{d-N\mu_{N,0}}, \cdots, \xi_{\tau}^{d-N\mu_{N,\tau}}, \cdots, \xi_{\tau+1}^{d-N\mu_{N,\tau+1}}, \cdots, \xi_N^{d-N\mu_{N,N}}, \xi_V^{\mu(N,\tau+1)},$$

thus by Proposition 4.2 we obtain:

$$\psi_{j_1,\cdots,j_n}^{\tau,\rho} \in H^0(X, \text{Sym}^n \Omega_V \otimes \mathcal{A}_{j_1,\cdots,j_n} \otimes \mathcal{L}^{\tau,\rho}_{j_1,\cdots,j_n}) \quad (\tau = 0 \cdots N-1, \rho = \tau+1 \cdots N),$$

with negative twist:

$$\nabla_{j_1,\cdots,j_n}^{\tau,\rho} = \sum_{p=1}^{c+r} (d + \epsilon_p) + \sum_{q=1}^{n} (d + \epsilon_{i_q}) - \sum_{k=0}^{\tau} (d - N \mu_{N,k} - 1) - \sum_{j=\tau+1, j \neq p}^{N} (d - \delta_N - 1) - (\mu_{N,\tau+1} - 1)$$

$$= -\mu_{N,\tau+1} + \sum_{k=0}^{\tau} N \mu_{N,k} + (N - \tau - 1) \delta_N + \sum_{p=1}^{c+r} \epsilon_p + \sum_{q=1}^{n} \epsilon_{i_q} + N + 1 \quad (34)$$

[by Algorithm (27)] \quad \leq -N \heartsuit.

Recalling the notation in Section 5, for any $\eta = 1 \cdots n - 1$, for any ‘vanishing’ indices $0 \leq \nu_1 < \cdots < \nu_\eta \leq N$, by applying Proposition 5.2, we can construct a series of negatively twisted symmetric differential forms over the ‘coordinates vanishing part’:

$$\omega_\ell \in \Gamma(\nu_1,\cdots,\nu_\eta, X, \text{Sym}^{n-\eta} \Omega_V \otimes \text{negative twist}) \quad (\ell = 1,2,\ldots).$$
The procedure is much the same as before. First, we rewrite each section $F_i$ in (26) as:

$$F_i = \sum_{j=0}^{N-\eta} A_i^j r_j + \sum_{j=N-\eta+1}^{N-\eta} \sum_{0\leq j_i < \ldots < j_n \leq N-\eta} \sum_{k=0}^{l} M_j^{N-n,\eta,\rho_k} \xi_{r_{j_i}} \ldots \xi_{r_{j_n}} \xi_{r_k} + \text{negligible terms},$$

so that $F_j$ has the same structure as (26), in the sense of replacing:

$$N \leftrightarrow N - \eta, \quad \{0, \ldots, N\} \leftrightarrow \{r_0, \ldots, r_{N-\eta}\}.$$

Thus, we can repeat the above manipulations. For shortness, we skip all details (cf. [5, Subsection 7.3]) and only state the results.

For every $1 \leq j_1 < \cdots < j_{N-\eta} \leq c$, for every $\nu = 0 \cdots N - \eta$, we obtain a symmetric differential form:

$$\phi_{j_1, \ldots, j_{N-\eta}}^{\nu} \in \Gamma(v_1, \ldots, v_{N}, X, \text{Sym}^{N-\eta} \Omega_V \otimes \mathcal{A}^{1, \ldots, c+r} \otimes \mathcal{L}^{1, \ldots, N-\eta, \nu^{\eta}_{j_1, \ldots, j_{N-\eta}}}),$$

with negative twist (set $\delta_{N-\eta} := (N - \eta - 1) \mu_{N-\eta-1,N-\eta-1}$):

$$v_1, \ldots, v_{\nu}^{\nu} \otimes^{\nu}_{j_1, \ldots, j_{N-\eta}} = -\mu_{N-\eta,0} + (N - \eta) \delta_{N-\eta} + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{N-\eta} \epsilon_{\ell} + (N - \eta) + 1 \leq -(N - \eta).$$

(35)

Also, for every $\tau = 0 \cdots N - \eta - 1$ and every $\rho = \tau + 1 \cdots N - \eta$, we obtain:

$$\phi_{j_1, \ldots, j_{N-\eta}}^{\tau, \rho} \in \Gamma(v_1, \ldots, v_{N}, X, \text{Sym}^{N-\eta} \Omega_V \otimes \mathcal{A}^{1, \ldots, c+r} \otimes \mathcal{L}^{1, \ldots, N-\eta, \nu^{\eta}_{j_1, \ldots, j_{N-\eta}}}),$$

with negative twist:

$$v_1, \ldots, v_{\nu}^{\nu} \otimes^{\nu}_{j_1, \ldots, j_{N-\eta}} = -\mu_{N-\eta, \tau+1} + \sum_{k=0}^{\tau} (N - \eta) \mu_{N-\eta,k} + (N - \eta - \tau - 1) \delta_{N-\eta} + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{N-\eta} \epsilon_{\ell} + (N - \eta) + 1 \leq -(N - \eta).$$

(36)

6.4. A Natural Algorithm. We will construct $\mu^{lk}$ in a lexicographic order with respect to indices $(l, k)$, for $l = c + r + 1 \cdots N$, $k = 0 \cdots l$, together with positive integers $\delta_l$.

For simplicity, we start by setting:

$$\delta_{c+r+1} \geq \max \{\epsilon_1, \ldots, \epsilon_{c+r}\}. \quad (37)$$

For every $l = c + r + 1 \cdots N$, in this step, we begin with choosing $\mu_{l,0}$ that satisfies:

$$[\text{see (35), (33)}] \quad \mu_{l,0} \geq l \delta_l + l \delta_{c+r+1} + l + 1 + l \heartsuit, \quad (38)$$

then inductively we choose $\mu_{l,k}$ satisfying:

$$[\text{see (36), (34)}] \quad \mu_{l,k} \geq \sum_{j=0}^{k-1} l \mu_{l,j} + (l-k) \delta_l + l \delta_{c+r+1} + l + 1 + l \heartsuit \quad (k = 1 \cdots l). \quad (39)$$

If $l < N$, we end this step by setting:

$$\delta_{l+1} := l \mu_{l,l} \quad (40)$$

as the starting point for the next step $l + 1$. At the end $l = N$, we require that:

$$d \geq (N + 1) \mu_{N,N}$$

be large enough.
6.5. **Controlling the base loci.** We will provide some technical preparations in Section 7.

By adapting the arguments in [5, Section 9], we can show that, for generic choices of parameters \(A^*, \, M^*, \), firstly, the:

\[
\text{Base Locus of } \left\{ \phi_{j_1,...,j_{\eta-1}}^{\tau}, \psi_{j_1,...,j_{\eta-1}}^{\tau,p} \right\}_{\eta \leq j_1 < \cdots < j_{\eta} < c} =: \text{BS}
\]

is discrete/empty over the ‘coordinates nonvanishing part’ \(\{\zeta_0 \cdots \zeta_N \neq 0\}\), and secondly, for each \(1 \leq \eta \leq n-1\), for every \(0 \leq v_1 < \cdots < v_\eta \leq N\), the:

\[
\text{Base Locus of } \left\{ \phi_{v_1,...,v_\eta}^{\tau,j_1,...,j_{\eta-1}}, \psi_{v_1,...,v_\eta}^{\tau,j_1,...,j_{\eta-1}} \right\}_{1 \leq j_1 < \cdots < j_{\eta} < c} =: v_1,...,v_\eta \text{BS}
\]

is discrete/empty over the corresponding ‘coordinates nonvanishing part’ \(\{\zeta_{v_0} \cdots \zeta_{v_N} \neq 0\}\).

For the sake of completeness, we sketch the proof in Subsection 7.3 below.

6.6. **Effective degree estimates.** In the Algorithm above, we first set \(\nu = 2, \epsilon_1 = \cdots = \epsilon_{s+l} = 1\), and next we demand all inequalities (37) – (40) to be exactly equalities. Thus we receive the estimate (cf. [5, Section 11]):

\[
(N + 1) \mu_{N,N} < N^{N^2/2} - 1 := d_0 \quad (\forall N \geq 3).\]

Now, recall the value \(e_0 = 3/d_0\) in Definition 1.3. In fact, the motivation is the following

**Proposition 6.1.** Let \(L, \mathcal{I}\) be two ample line bundles on \(P\). Then \(\mathcal{I}\) is almost proportional to \(L\) if and only if there exist some positive integers \(d \geq d_0, s, l \geq 1\), such that \(\mathcal{I}^s = \mathcal{A} \otimes L^{l} \otimes L^{l \cdot d}\), where the line bundle \(\mathcal{A}\) satisfies that \(\mathcal{A} \otimes L^l\) is very ample and that \(\mathcal{A} \otimes L^{-l} < 0\) is negative.

**Proof.** “\(\Rightarrow\)” We can take \(\alpha = s \cdot [\mathcal{I}]\) and \(\beta = l \cdot d \cdot [L]\), so that \(\alpha - \beta = [\mathcal{A} \otimes L^l] > 0\), and that \((1 + e_0)\beta - \alpha \geq (1 + 3/d)\beta - \alpha = [\mathcal{A} \otimes L^{-l} - 1] > 0\).

“\(\Leftarrow\)” Since \(Q_+\) is dense in \(\mathbb{R}_+\), we may assume that \(\alpha \in \mathbb{Q}_+ \cdot [\mathcal{I}]\) and \(\beta \in \mathbb{Q}_+ \cdot [L]\). Next, we can choose a sufficiently divisible integer \(m \geq 0\) such that \(m \cdot \alpha = s_0 \cdot [\mathcal{A}] \) and \(m \cdot \beta = l_0 \cdot [L]\) for some positive integers \(s_0, l_0 > 0\). Set the line bundle \(A_0 := \mathcal{I}^{s_0} \otimes (L^{l_0} \otimes L^{l_0 \cdot d_0})^{-1}\), hence \(\mathcal{I}^{s_0} = \mathcal{A} \otimes L^{l_0} \otimes L^{l_0 \cdot d_0}\).

Now, using \(\beta \leq \alpha < (1 + \epsilon)\beta\), we receive:

\[
0 < m \cdot (\alpha - \beta) = m \cdot \alpha - m \cdot \beta = s_0 \cdot [\mathcal{I}] - l_0 \cdot [L] = [\mathcal{I}^{s_0} \otimes L^{-l_0 \cdot d_0}] = [\mathcal{A} \otimes L^{-l_0 \cdot d_0}],
\]

\[
0 > m \cdot (\alpha - (1 + e_0)\beta) = m \cdot \alpha - (1 + 3/d_0) \cdot m \cdot \beta = s_0 \cdot [\mathcal{I}] - (1 + 3/d_0) \cdot l_0 \cdot [L] = [\mathcal{A} \otimes L^{-l_0 \cdot d_0}].
\]

The first line above implies that \(\mathcal{A} \otimes L^{l_0 \cdot d_0}\) is very ample for some positive integer \(m' > 0\). Thus we can set \(s := s_0 m', l := l_0 m', \mathcal{A} := \mathcal{A}_0^{m'}\), then \(\mathcal{I}^s = \mathcal{A} \otimes L^{l} \otimes L^{l \cdot d_0}\) satisfies that \(\mathcal{A} \otimes L^l\) is very ample and that \(\mathcal{A} \otimes L^{-l} < 0\) is negative. \(\square\)

**Remark 6.2.** In the above proof, we see that the second assertion holds for \(d = d_0\). In fact, it holds for any positive integer \(d' \leq d\), since we have:

\[
L^{s(1+d')} = (\mathcal{A} \otimes L^{l(1+d)})^{1+d'} = \mathcal{A}^{1+d'} \otimes L^{l(1+d)} \otimes (L^{l(1+d)})^{d'},
\]

where:

\[
\mathcal{A}^{1+d'} \otimes L^{l(1+d)} = \left( \mathcal{A} \otimes L^{l} \right)^{1+d'} \otimes L^{l(d-d')} \text{ is very ample and where:}
\]

\[
\mathcal{A}^{1+d'} \otimes L^{-2l(1+d')} = \left( \mathcal{A} \otimes L^{-2l} \right)^{1+d'} \otimes L^{-2l(1+d')} < 0.
\]

Thus the second assertion holds not only for \((d, s, l)\) but also for \((d', s(1+d'), l(1+d))\).
6.7. Proof of Theorem 1.4. Summarizing the above Subsections 6.2 – 6.6, we can obtain

**Theorem 6.3.** Let $P$ be a smooth projective variety of dimension $N$, and let $L$ be a very ample line bundle over $P$. For any integers $c, r \geq 0$ with $2c + r \geq N$, for any integer $d \geq d_0$, for any $c + r$ line bundles $\mathcal{A}_i (i = 1 \cdots c + r)$ such that $\mathcal{A}_i \otimes L$ are very ample and that $\mathcal{A}_i \otimes L^{-2} < 0$, setting:

$$L_i = \mathcal{A}_i \otimes L \otimes L^d \quad (i = 1 \cdots c + r),$$

then, for generic $c + r$ hypersurfaces:

$$H_1 \in |L_1|, \ldots, H_{c+r} \in |L_{c+r}|,$$

the cotangent bundle $\Omega_V$ of the intersection of the first $c$ hypersurfaces $V = H_1 \cap \cdots \cap H_c$ restricted to the intersection of all the $c + r$ hypersurfaces $X = H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}$ is ample.

Denote the projectivization of the cotangent bundle $\Omega_P$ of $P$ by:

$$\mathbb{P}(\Omega_P) := \text{Proj}(\oplus_{k \geq 0} \text{Sym}^k \Omega_P),$$

and denote the associated Serre line bundle by $\mathcal{O}_{\mathbb{P}(\Omega_P)}(1)$. For any integers $a, b \geq 0$, for any $a + b$ global sections $F_1, \ldots, F_a, F_{a+1}, \ldots, F_{a+b}$ of arbitrary $a + b$ line bundles over $P$, denote by:

$$F_{a+1}, \ldots, F_{a+b} \mathbb{P}F_1, \ldots, F_a \subset \mathbb{P}(\Omega_P),$$

the unique subscheme defined by equations $F_1, \ldots, F_{a+b}, dF_1, \ldots, dF_a$. Thus, we reformulate the above theorem as:

**Theorem 6.3’.** For generic $c + r$ sections:

$$F_1 \in H^0(P, L_1), \ldots, F_{c+r} \in H^0(P, L_{c+r}),$$

the Serre line bundle $\mathcal{O}_{\mathbb{P}(\Omega_P)}(1)$ is ample over the subvariety $F_{a+1} \cdots F_{a+b} \mathbb{P}F_1 \cdots F_a$.

Proof of Theorem 6.3. We may assume that $N \geq 3$ and $c + r < N$, otherwise there is nothing to prove. Set $n = N - c - r$, observe that $1 \leq n \leq c$. Since ampleness is a Zariski open condition in family (Grothendieck), we only need to provide one ample example $H_1, \ldots, H_{c+r}$. In fact, we will construct $c + r$ sections $F_1, \ldots, F_{c+r}$ of the MCM shape (26) to conclude the proof.

Step 1. Since $d \geq d_0$, by the effective degree estimates in preceding subsection, we can construct integers $\{\mu^{l,k}\}$ that satisfy the Algorithm in Subsection 6.4. Now, the structure of (26) is fixed, and we will choose some appropriate coefficients $A^*_i, M^*_i$ for $i = 1 \cdots c + r$.

Step 2. For generic choices of parameters $A^*_i, M^*_i$, both $X, V$ are smooth complete, and moreover, for all $1 \leq \eta \leq n = N - c - r$, for all indices $0 \leq v_1 < \cdots < v_\eta \leq N$, the further intersection varieties $v_1, \ldots, v_\eta X$ are all smooth complete. The reasoning is much the same as in Bertini’s Theorem. For the sake of completeness, we provide a proof in Subsection 7.2 below.

Step 3. For generic choices of parameters $A^*_i, M^*_i$, all the constructed negatively twisted symmetric differential forms have discrete based loci outside ‘coordinates vanishing part’, see Subsection 6.5 for details. This is the core of the moving coefficients method.

Step 4. Choose any generic parameters $A^*_i, M^*_i$ that satisfy the properties in the above two steps. We claim that the corresponding sections $F_1, \ldots, F_{c+r}$ constitute one ample example.

Proof of the claim. Abbreviate $P := F_{a+1}, \ldots, F_{a+b} F_1, \ldots, F_a$ and $v_1, \ldots, v_\eta P := F_{a+1}, \ldots, F_{a+b} \mathcal{O}_{\mathbb{P}(\Omega_P)}(n - \eta) \mathbb{P}F_1 \cdots F_a$. Let $\pi: \mathbb{P}(\Omega_P) \rightarrow P$ be the canonical projection. Note that all the obtained symmetric differential forms in Step 3 can be viewed as sections (when $\eta = 0$, we agree $v_1, \ldots, v_\eta P = P$):

$$v_1, \ldots, v_\eta \omega \in H^0(v_1, \ldots, v_\eta P, \mathcal{O}_{\mathbb{P}(\Omega_P)}(n - \eta) \otimes \pi^* v_1, \ldots, v_\eta L),$$

(43)
where we always use $\ast$ to denote auxiliary integers, and where all $v_1, \ldots, v_q \mathcal{L}_v < 0$ are some negative line bundles. Choose an ample $\mathbb{Q}$-divisor $\mathcal{F} > 0$ over $\mathbb{P}$ such that all $v_1, \ldots, v_q \mathcal{L}_v / (n - \eta) + \mathcal{F} < 0$ are still negative. Then we claim that $\mathcal{N} := \mathcal{O}_{\mathbb{P}(\mathbb{P}_p)}(1) \otimes \pi^* \mathcal{F}^{-1}$ is nef over $\mathbb{P}$.

Indeed, for any irreducible curve $C \subset \mathbb{P}$, if $C$ lies in at least $n$ ‘coordinate hyperplanes’ defined by $\zeta_{v_1}, \ldots, \zeta_{v_q}$, then by Step 2 we see that $C$ must contract to a point by $\pi$, thus $\mathcal{N}|_C \cong \mathcal{O}_{\mathbb{P}(\mathbb{P}_p)}(1)|_C$ is not only nef but ample. Assume now that $C$ lies in at best $\eta < n$ ‘coordinate hyperplanes’ defined by $\zeta_{v_1}, \ldots, \zeta_{v_q}$ ($\eta$ could be zero). Since the base locus of all sections in $(43)$ is discrete over the ‘coordinates nonvanishing part’ $\{\zeta_{v_1} \cdots \zeta_{v_q} \neq 0\}$, and $C \cap \{\zeta_{v_1} \cdots \zeta_{v_q} \neq 0\}$ is one-dimensional, we can find some $v_1, \ldots, v_q \omega_{\ast}$ such that $v_1, \ldots, v_q \omega_{\ast}|_C \neq 0$. Thus the intersection number $C \cdot \mathcal{O}_{\mathbb{P}(\mathbb{P}_p)}((n - \eta) \otimes \pi^* \mathcal{L}_v)$ is $\geq 0$. Since $v_1, \ldots, v_q \mathcal{L}_v / (n - \eta) + \mathcal{F} < 0$, we immediately conclude that $C \cdot \mathcal{N} > 0$.

Lastly, since $\mathcal{F} > 0$ over $\mathbb{P}$, there exists some large integer $m > 1$ such that $\mathcal{P} \coloneqq \mathcal{O}_{\mathbb{P}(\mathbb{P}_p)}(1) \otimes \pi^* \mathcal{L}^m > 0$ is positive over $\mathbb{P}(\mathbb{P}_p)$. In particular, it is also positive over $\mathbb{P}$. Since ‘nef+ample=ample’, we have $m \mathcal{N} + \mathcal{P} > 0$ over $\mathbb{P}$, that is $\mathcal{O}_{\mathbb{P}(\mathbb{P}_p)}(1)|_C > 0$.

Thus we conclude the proof. □

Finally, using the product coup, we obtain

**Proof of Theorem 1.4.** For every $i = 1 \cdots c + r$, since $\mathcal{L}_i$ is almost proportional to $\mathcal{L}$, by Proposition 6.1, there exist some positive integers $s_1, l_1 > 1, d_1 > d_0$ such that $\mathcal{L}_i^{s_1} = \mathcal{A}_i \otimes \mathcal{L}_i^{l_1} \otimes \mathcal{L}_i^{d_1}$, where the line bundle $\mathcal{A}_i$ satisfies that $\mathcal{A}_i \otimes \mathcal{L}_i$ is very ample and that $\mathcal{A}_i \otimes \mathcal{L}_i^{-2l_1} < 0$ is negative. In order to apply Theorem 6.3’, first of all, we need an

**Observation 6.4.** There exist some positive integers $\tilde{s}_1, \ldots, \tilde{s}_{c+r}, l \geq 1$ and $d \geq d_0$ such that:

$$
\mathcal{L}_i^{\tilde{s}_1} = \mathcal{A}_i \otimes \mathcal{L} \otimes \mathcal{L}_i^0, \quad \mathcal{L}_i^{\tilde{s}_1+1} = \mathcal{B}_i \otimes \mathcal{L} \otimes \mathcal{L}_i^d \quad (i = 1 \cdots c + r),
$$

where $\tilde{\mathcal{L}} := \mathcal{L}^l$ is very ample, and where $\mathcal{A}_i \otimes \tilde{\mathcal{L}}, \mathcal{B}_i \otimes \tilde{\mathcal{L}}$ are very ample, and where $\mathcal{A}_i \otimes \mathcal{L}_i^{-2}, \mathcal{B}_i \otimes \mathcal{L}_i^{-2}$ are negative.

**Proof.** First, by Remark 6.2, we may assume that $d_1 = \cdots = d_{c+r} = d > d_0$.

Next, we may assume that $l_1 = \cdots = l_{c+r} = l$. Otherwise, we can choose a positive integer $l$ which is divisible by $l_1, \ldots, l_{c+r}$, then we receive/rewrite:

$$
\mathcal{L}_i^{s_1 l_1} = (\mathcal{A}_i \otimes \mathcal{L}_i^{l_1} \otimes \mathcal{L}_i^{d_1})^{l_1} = \mathcal{A}_i^{l_1} \otimes \mathcal{L}^l \otimes \mathcal{L}_i^{d_1} \quad (i = 1 \cdots c + r),
$$

while $\mathcal{A}_i^{l_1} \otimes \mathcal{L}^l$ remains very ample and also $\mathcal{A}_i^{l_1} \otimes \mathcal{L}_i^{-2l_1} = (\mathcal{A}_i \otimes \mathcal{L}_i^{-2l_1})^{l_1} < 0$.

Lastly, we can choose one large integer $m \gg 1$ such that, for all $i = 1 \cdots c + r$, not only $\mathcal{L}_i \otimes (\mathcal{A}_i \otimes \mathcal{L}_i)^m$ are very ample, but also $\mathcal{L}_i \otimes (\mathcal{A}_i \otimes \mathcal{L}_i^{-2l_1})^m < 0$ are negative. Thus, the following data:

$$
l := m l, \quad \tilde{s}_i := m s_i, \quad \tilde{\mathcal{A}}_i := m \mathcal{A}_i, \quad \tilde{\mathcal{B}}_i := \mathcal{L}_i \otimes \mathcal{A}_i^m \quad (i = 1 \cdots c + r)
$$

satisfy the claimed observation. □

Now, we can set:

$$
d = d(\mathcal{L}_1, \ldots, \mathcal{L}_{c+r}, \mathcal{L}) = \max_{1 \leq i \leq c+r} \lfloor \tilde{s}_i (\tilde{s}_i - 1) \rfloor.
$$

(44)

For any integers $d_1, \ldots, d_{c+r} \geq d$, all of them can be written as:

$$
d_i = p_i \tilde{s}_i + q_i (\tilde{s}_i + 1) \quad (i = 1 \cdots c + r)
$$

for some integers $p_i, q_i \geq 0$. Let every:

$$
F_i := f_1^i \cdots f_{p_i}^i \tilde{f}_1^i \cdots \tilde{f}_{q_i}^i \in \mathcal{H}^0(\mathbb{P}, \mathcal{L}_i^d)
$$

for some integers $p_i, q_i \geq 0$. Let every:
be a product of some sections:
\[ f^i_1, \ldots, f^i_p \in H^0(\mathbf{P}, \mathcal{L}^i_1), \quad f^i_{p+1}, \ldots, f^i_{p+q_i} \in H^0(\mathbf{P}, \mathcal{L}^i_{c+1}) \]
to be chosen, then the product coup reveals the decomposition:
\[ F_{c+1} \cdots F_{c+r} = \bigcup_{k=0}^{c} \bigcup_{j} \bigcup_{l=1}^{c-k} \bigcup_{l=1}^{c-k} \mathcal{L}^i_{k} \bigcup_{l=1}^{c-k} \mathcal{L}^i_{i+1} \cdots \mathcal{L}^i_{c+1} \bigcup_{l=1}^{c-k} \mathcal{L}^i_{r+1} \cdots \mathcal{L}^i_{c+1} \mathcal{L}^i_{f_1} \cdots \mathcal{L}^i_{f_k}. \]

Now, applying Theorem 6.3, for generic choices of \( \{ f^*_i \} \), the Serre line bundle \( \mathcal{O}_\mathbf{P}(\zeta)(1) \) is ample on every subscheme \( f^*_1 \cdots f^*_k \mathcal{L}^i_{f_1} \cdots \mathcal{L}^i_{f_k} \cdots \mathcal{L}^i_{c+1} \), and therefore is also ample on their union \( F_{c+1} \cdots F_{c+r} \). Since ampleness is a generic property in family, we conclude the proof.

6.8. Effective lower degree bound \( N^{N^2} \) of Theorem 1.2. Now, we provide an effective degree estimate of Theorem 1.4 in the case \( \mathcal{L} = \cdots = \mathcal{L}_{c+r} = \mathcal{L} \).

When \( N = 1, 2 \), Theorem 1.4 holds trivially for \( d = N^{N^2} \). When \( N \geq 3 \), denote the trivial line bundle on \( \mathbf{P} \) by \( 0_\mathbf{P} \). Note that in Observation 6.4 we can take \( \tilde{s}_1 = \cdots = \tilde{s}_{c+r} = d_0 + 1, \ell = 1 \), so that:
\[
\mathcal{L}^{d_0+1} = 0_\mathbf{P} \otimes \mathcal{L} \otimes \mathcal{L}^{d_0}, \quad \mathcal{L}^{d_0+2} = \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}^{d_0}
\]
satisfy the requirements. Thus by (44) we can set:
\[
d = d(\mathcal{L}) = \max_{1 \leq i \leq c+r} (\tilde{s}_i - 1) = (d_0 + 1) = (N^{N^2} - 1) N^{N^2} < N^{N^2}.
\]
In particular, when \( r = 0 \), we recover Theorem 1.2.

7. Some Technical Details

7.1. Surjectivity of evaluation maps. Recalling the notation in Definition 3.1, at every closed point \( z \in \mathbf{P} \), for every tangent vector \( \xi \in T_\mathbf{P}_z \), whenever we can choose any local trivialization \( (U, s) \) of the line bundle \( \mathcal{S} \) at \( z \) and then evaluate \( S, dS \) at \( (z, \xi) \) by:
\[
S(z, U, s) := S/s(z) \in \mathbb{K},
\]
\[
dS(z, U, s) := d(S/s)(z, \xi) \in \mathbb{K}.
\]
If \( (U, s') \) is another local trivialization of \( \mathcal{S} \), then we have the transition formula:
\[
\begin{pmatrix}
S \\
\frac{dS}{dS}
\end{pmatrix}(z, \xi)(U, s) = \begin{pmatrix}
\frac{s'}{s} & 0 \\
\frac{d}{dS}(\frac{s'}{s}) & \frac{s'}{s}
\end{pmatrix} \begin{pmatrix}
S \\
\frac{dS}{dS}
\end{pmatrix}(z, \xi)(U, s')^{-1}
\]
(45)

Thanks to the above identity, in assertions which do not depend on the particular choice of \( (U, s) \), we can just write \( S(z), dS(z, \xi) \) by dropping \( (U, s) \).

**Proposition 7.1.** Let \( \mathcal{S} \) be a very-ample line bundle over a smooth \( \mathbb{K} \)-variety \( \mathbf{P} \). Then one has:

(i) at every closed point \( z \in \mathbf{P} \), for any nonzero tangent vector \( 0 \neq \xi \in T_\mathbf{P}_z \), the evaluation map:
\[
\begin{pmatrix}
\frac{V_z}{d_z(\xi)}
\end{pmatrix} : H^0(\mathbf{P}, \mathcal{S}) \longrightarrow \mathbb{K}^2
\]
\[
S \longmapsto (S(z), dS(z, \xi))^T
\]
is surjective;

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(ii) at every closed point $z \in P$, for any $N = \dim P$ linearly independent tangent vectors $\xi_1, \ldots, \xi_N \in T_P(z),$ the evaluation map:

$$\begin{pmatrix}
  v_z \\
  d_z(\xi_1) \\
  \vdots \\
  d_z(\xi_N)
\end{pmatrix} : H^0(P, \mathcal{S}) \rightarrow \mathbb{K}^{N+1}
$$

is surjective.

Proof. We have the following three elementary observations.

1. By transition formula (45), property (i) is independent of the choice of local trivialization $(U, s)$ of $\mathcal{S}$ near $z$, so it makes sense.

2. In any fixed local trivialization $(U, s)$ of $\mathcal{S}$ near $z$, by basic linear algebra, properties (i), (ii) are equivalent to each other.

3. Property (i) is the usual property of ‘very-ampleness’.

Thus we may conclude the proof by the reasoning ‘very-ampleness’ $\implies$ (i) $\iff$ (ii). \hfill \Box

Proposition 7.2. Let $\mathcal{S}$ be a very ample line bundle over a smooth $\mathbb{K}$-variety $P$, and let $\mathbb{A}$ be any line bundle over $P$ with a nonzero section $A \neq 0$. Then, at every closed point $z \in D(A) \subset P$, for any nonzero tangent vector $0 \neq \xi \in T_P(z)$, the evaluation map:

$$\begin{pmatrix} A \cdot v_z \\
  d_z(A \cdot \eta)(\xi) \end{pmatrix} : H^0(P, \mathcal{S}) \rightarrow \mathbb{K}^2$$

is surjective.

Proof. It is a direct consequence of the formula:

$$\begin{pmatrix} A \cdot v_z \\
  d_z(A \cdot \eta)(\xi) \end{pmatrix} = \begin{pmatrix} A(z) & 0 \\
  dA(z, \xi) & A(z) \end{pmatrix} \cdot \begin{pmatrix} v_z \\
  d_z(\xi) \end{pmatrix}$$

[Leibniz’s rule]

and of the preceding proposition. \hfill \Box

7.2. Bertini-type assertions. Recalling (26) and that for all $i = 1 \cdots c + r$ the line bundles $\mathcal{A}_i \otimes L^i$ are very ample, we now fulfill the step 2 in the proof of Theorem 6.3. We start with

**Observation 1.** ‘Smooth complete’ is a Zariski open condition in family.

**Observation 2.** We only need to prove that, for generic choices of $A_1^*, M_{1^*}$, the hypersurface $H_1 = \{F_1 = 0\} \subset P$ is smooth complete.

Proof. Indeed, replacing $P$ by $H_1$, we can repeat the same argument to choose $A_2^*, M_{2^*}$, and so on. Thus we know that there exists at least one choice of parameters $A_2^*, M_{2^*}$ such that $X, V$ are both smooth complete. Immediately, by Observation 1 above, it holds for generic choices of parameters.

Next, to show that generically $v_1, \ldots, v_q \cdot X$ is smooth complete, we can start with $v_1, \ldots, v_q \cdot P$ instead of $P$, and use the same reasoning to conclude the proof. \hfill \Box
Observation 3. We can first set all $M^*_i = 0$, and then thanks to the following proposition, we can find some appropriate $A_i$ such that $H_i$ is smooth complete. Thus we finish the proof of step 2.

**Proposition 7.3.** Let $P$ be a smooth $\mathbb{K}$-variety of dimension $N$, and let $\mathcal{A}$, $\mathcal{B}$ be two line bundles over $P$. Assume that $\mathcal{A}$ is very ample, and that $\mathcal{B}$ has $N + 1$ global sections $B_0, \ldots, B_N$ having empty common base locus. Then, for generic choices of parameters $A_0, \ldots, A_N \in H^0(P, \mathcal{A})$, the section:

$$F = \sum_{j=0}^{N} A_j B_j \in H^0(P, \mathcal{A} \otimes \mathcal{B})$$

defines a smooth complete subvariety.

**Proof.** Denoting $\mathcal{M} := \oplus_{j=0}^{N} H^0(P, \mathcal{A})$, then $\mathbb{P}(\mathcal{M})$ stands for the projective parameter space of $t = (A_0, \ldots, A_N)$. Now we introduce the universal subvariety:

$$S := \{([t], z) : F_i(z) = 0, dF_i(z, \xi) = 0, \forall \xi \in T_{P_t}|_{z}\} \subset \mathbb{P}(\mathcal{M}) \times P$$

consisting of singular points. We claim that $\dim S < \dim \mathbb{P}(\mathcal{M})$. It suffices to show that, for every closed point $z \in P$, the fibre $S_z \subset \mathbb{P}(\mathcal{M}) \times \{z\}$ is smooth over $z$ satisfies that $\text{codim} S_z > \dim P$.

Indeed, choose $N$ linearly independent tangent vector $\xi_1, \ldots, \xi_N$ at point $z$, and then consider the formal $\mathbb{K}$-linear map:

$$\text{ev} : \mathcal{M} \rightarrow \mathbb{K}^{N+1}$$

$$t \mapsto (F_1(z), dF_1(z, \xi_1), \ldots, dF_1(z, \xi_N))^T.$$ 

By Proposition 7.2, $\text{ev}$ is surjective. Note that $S_z \subset \mathbb{P}(\mathcal{M})$ consists of points $[t] \in \mathbb{P}(\mathcal{M})$ such that $\text{ev}(t) = 0$. Thus we see that:

$$\text{codim} S_z = \text{codim} \{[0] \subset \mathbb{K}^{N+1}\} = N + 1 > N = \dim P. \quad \square$$

We will see in the proof of Proposition 7.6 that the *Core Lemma* of MCM plays the same role as that of the above underlined codimension equality/estimate.

7.3. **Emptiness of the base loci.** Recalling (41), (42), in order to characterize the base loci $\text{BS, BS, \ldots} \subset \mathbb{P}(\Omega_P)$, we introduce the following subvarieties (cf. [5, p. 62, (148))):

$$\mathcal{M}^a \subset \text{Mat}_{b \times 2(a+1)}(\mathbb{K}) \quad (\forall 2 \leq a \leq b)$$

consisting of all $b \times 2(a+1)$ matrices $(\alpha_0 | \alpha_1 | \cdots | \alpha_a | \beta_0 | \beta_1 | \cdots | \beta_a)$ such that:

(i) the sum of all $2a + 2$ columns is zero:

$$\alpha_0 + \alpha_1 + \cdots + \alpha_a + \beta_0 + \beta_1 + \cdots + \beta_a = 0; \quad (46)$$

(ii) for every $\nu = 0 \cdots a$, there holds the rank inequality:

$$\text{rank}_{\mathbb{K}} \{\alpha_0, \ldots, \alpha_\nu, \ldots, \alpha_a, \alpha_\nu + (\beta_0 + \beta_1 + \cdots + \beta_a)\} \leq a - 1; \quad (47)$$

(iii) for every $\tau = 0 \cdots a - 1$, for every $\rho = \tau + 1 \cdots a$, there holds:

$$\text{rank}_{\mathbb{K}} \{\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \alpha_\rho, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_a)\} \leq a - 1. \quad (48)$$
From now on, we only consider the closed points in each scheme. For instance, we shall regard:

\[ P(\Omega_p) = \{ (z, [\xi]) : \forall z \in P, \xi \in T_{\mathcal{P}}[z] \}. \]

By the same reasoning as in [5, Proposition 9.3], we get:

**Proposition 7.4.** For generic choices of parameters \( A^*, M^* \), a point:

\[ (z, [\xi]) \in P(\Omega_p) \setminus \{ \zeta_0 \cdots \zeta_N \neq 0 \} \]

lies in \( BS \) if and only if:

\[ \text{[recall (29)]} \quad M(z, \xi) \in \mathcal{M}^N_{2c+r}. \]

Now, we introduce the engine of MCM (slightly different from the original [5, Lemma 9.5]):

**Core Lemma.** For all positive integers \( 2 \leq a \leq b \), there hold the codimension estimates:

\[ \text{codim} \mathcal{M}^a_b \geq a + b - 1. \]

‘Naive proof’. First of all, the equation (46) eliminates the first variable column:

\[ \alpha_0 = - (\alpha_1 + \cdots + \alpha_a + \beta_0 + \beta_1 + \cdots + \beta_a), \]

so it contributes codimension value \( b \). Next, denoting \( S_i := \sum_{j=i}^a \beta_j \) for \( i = 0 \cdots a \), we may rewrite the restriction (47) as:

\[ \text{rank}_{\mathbb{C}} [\alpha_0, \ldots, \alpha_i, \ldots, \alpha_a, \alpha_v + S_0] \leq a - 1 \quad (v = 0 \cdots a). \]

By (46), the sum of all columns above vanishes, hence we can drop the first column and state it equivalently as:

\[ \text{rank}_{\mathbb{C}} [\alpha_1, \ldots, \alpha_i, \ldots, \alpha_a, \alpha_v + S_0] \leq a - 1 \quad (v = 0 \cdots a). \] (49)

Similarly, we can reformulate (48) equivalently as:

\[ \text{rank}_{\mathbb{C}} [\alpha_1 + S_1 - S_2, \ldots, \alpha_t + S_{t+1} - S_{t+1}, \ldots, \alpha_p, \ldots, \alpha_a, \alpha_r + S_{r+1}] \leq a - 1 \quad (\tau = 0 \cdots a - 1, \rho = \tau + 1 \cdots a). \] (50)

Observe in (49), (50) that the variable columns \( S_0, \ldots, S_a \) have distinct status, and moreover that for \( i = 1 \cdots a \), subsequently, each variable \( S_i \) satisfies nontrivial new equations involving only the former variables \( \alpha_*, S_1, \ldots, S_{i-1} \). Thus, the restrictions (49), (50) should contribute at least \( a + 1 \) codimension value. Summarizing, we should have:

\[ \text{codim} \mathcal{M}^a_b \geq b + (a + 1) \geq a + b - 1. \]

**Remark 7.5.** However, a rigorous proof (cf. [5, Subsection 10.6]) is much more demanding and delicate, because of the unexpected algebraic complexity behind (cf. [5, Subsection 10.7]).

Thereby, we can exclude positive-dimensional base locus in Proposition 7.4.

**Proposition 7.6.** For generic choices of parameters \( A^*, M^* \), the base locus over the ‘coordinates nonvanishing part’:

\[ BS \setminus \{ \zeta_0 \cdots \zeta_N \neq 0 \} \]

is discrete or empty.
Proof. The proof goes much the same as that of Proposition 7.3, in which the underlined codimension estimate is replaced by:

\[
\text{codim } \mathcal{M}^{N}_{2c+r} \geq N + 2c + r - 1 \quad \text{[by the Core Lemma]}
\]

\[
\text{[use } 2c + r \geq N \text{]} \geq N + N - 1
\]

\[
\text{[exercise]} = \dim \mathbb{P}(\Omega_p)
\]

\[
\text{[⊗]} = \dim (\mathbb{P}(\Omega_p) \setminus \{ζ_0 \cdots ζ_N \neq 0\}).
\]

For the remaining details, we refer the reader to [5, Propositions 9.6, 9.7]. □

This is exactly the first emptiness assertion on the base loci in Subsection 6.5. By much the same reasoning, we can also establish the second one there (cf. [5, Proposition 9.11]).

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