Well-posedness of the vector advection equations by stochastic perturbation.

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Abstract

A linear stochastic vector advection equation is considered. The equation may model a passive magnetic field in a random fluid. The driving velocity field is integrable to a certain power and the noise is infinite dimensional. We prove that, thanks to the noise, the equation is well posed in a suitable sense, opposite to what may happen without noise.

Keywords: Stochastic vector advection equations, Cauchy problem, multiplicative noise, nonregular coefficients, regularization by noise, stochastic flows, infinite dimensional noise

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1 Introduction

Consider the linear stochastic vector advection equation in the unknown random field $B : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$

$$
\begin{align*}
dB + (v \cdot \nabla B - B \cdot \nabla v) \, dt + \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla B - B \cdot \nabla \sigma_k) \circ dW^k_t = 0
\end{align*}
$$

where $v : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma_k : \mathbb{R}^3 \to \mathbb{R}^3$, $k \in \mathbb{N}$, are given divergence free vector fields and $(W^k)_{k \in \mathbb{N}}$ is a family of independent real-valued Brownian motions on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The stochastic integration is to be understood in the Stratonovich sense. This equation may model a passive vector field $B$, like a magnetic field, in a turbulent fluid with a non-trivial average component $v$ and random component $\sum_{k=1}^{\infty} \sigma_k dW^k_t$. The general structure of the noise assumed here is inspired to the theory of diffusion of passive scalars and vector fields in turbulent fluids, see for instance [3] and is also motivated by the recent proposal for a variational principle approach to fluid

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mechanics, see [11] (although the equations in [11] are always nonlinear, with random \( v \) influenced by \( B \), hence more difficult than those studied here). Particular cases of this equation have been considered before in [7], [6]; see also [2] and references therein; but the generality assumed here is important from the physical viewpoint and the proofs are new. We impose below some simplifying assumptions on the vector fields \( \sigma_k \).

We aim at studying existence and uniqueness, under low regularity assumption on \( v \). More precisely, we assume that

\[
v \in L^{\infty} ([0, T], L^p(\mathbb{R}^3)) \quad \text{for some } p > 3.
\]

Under this condition, existence and uniqueness is not a classical result: indeed, in the deterministic case (all \( \sigma_k = 0 \)), it is not true, as the examples in [7], [6] show. Thus the result of existence and uniqueness is due to the random perturbation. The same question was considered in [7] under an Hölder condition on \( v \) and a partial result is given in [6] for \( v \) having suitable integrability. But in both cases the noise was easier. The main novelties of the present work with respect to [7], [6] are: i) the approach, based on the new concept of quasi-regular solution, recently introduced in [5] for transport type equations, approach which allows one to prove certain properties in a much easier way; ii) the noise is much more general and in line with the physical and geometrical literature, [3], [11]; iii) the proof for \( v \) with only integrability properties (instead of Hölder continuity) is here complete, w.r.t. [6] which gave only general arguments, also due to the more synthetic approach used here.

The uniqueness result has to be understood in a slightly more restrictive way than in other works: indeed it requires that solutions, which are compared, both are defined on a probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) fully described by the noise, as it is the classical Wiener space (see below precise statements).

About condition (2), let us add some historical remarks. Our source of inspiration is the paper by [9], where they proved existence and uniqueness of strong solutions to the SDE (the equation of characteristics for (1))

\[
X_{s,t}(x) = x + \int_{s}^{t} v(r, X_{s,r}(x)) \, dr + W_t - W_s.
\]

This kind condition also was considered in [1], [4] and [13] to study scalar problems like linear transport equations and linear continuity equations. In fluid mechanics, in the viscous case of Navier-Stokes equations, when such condition holds for a weak solution, then such solution is unique and more regular (it is a particular case of the so called Ladyženskaja-Prodi-Serrin condition). Here the framework is of course different: \( v \) is given, not the unknown, and the equation is inviscid; hence a true comparison is not possible. We only stress some parallelism between these theories.
2 Preliminaries and assumptions

2.1 Assumptions on the noise

We first assume \[ \sum_{k=1}^\infty |\sigma_k(x)|^2 < \infty \] for every \( x \in \mathbb{R}^3 \), so that a matrix-valued function \( Q(x, y) \in \mathbb{R}^{3 \times 3} \), \( x, y \in \mathbb{R}^3 \), is well defined by

\[
Q^{\alpha\beta}(x, y) := \sum_{k=1}^\infty \sigma_k^\alpha(x) \sigma_k^\beta(y)
\]

(we write \( Q^{\alpha\beta}(x, y) \), \( \alpha, \beta = 1, 2, 3 \) for its components and similarly for \( \sigma_k^{\alpha}(x) \)). Concerning the vector fields \( \sigma_k \), we assume they are twice differentiable and divergence free (although the latter assumption is not explicitly used, it is only coherent with the other fields of the equation). We assume that the matrix-function \( Q(x, y) \) is twice continuously differentiable in \((x, y)\), bounded with bounded first and second derivatives. Finally, for reasons which will appear below, we assume that

\[
Q(x, x) \geq \nu \text{Id}_{\mathbb{R}^3} \tag{4}
\]

for some \( \nu > 0 \), uniformly in \( x \in \mathbb{R}^3 \).

Remark 1 In the literature it often assumed that there exists a matrix-valued function \( Q(x) \in \mathbb{R}^{3 \times 3}, x \in \mathbb{R}^3 \), such that

\[
Q(x, y) = Q(x - y)
\]

(this is equivalent to assume that the Gaussian random field \( \sum_{k=1}^\infty \sigma_k(x) W^k_t \) is space homogeneous). The value \( Q(0) \) plays a special role and is often assumed to be a non-degenerate matrix, for simplicity

\[
Q(0) = \text{Id}
\]

the identity matrix in \( \mathbb{R}^3 \). We do not impose these additional conditions but only (4) which corresponds to the non degeneracy of \( Q(0) \).

2.2 Itô formulation

To continue, it is convenient to introduce the notation of the Lie bracket between vector fields

\[
[A, B] = A \cdot \nabla B - B \cdot \nabla A
\]

which is also equal to the Lie derivative \( \mathcal{L}_A B \) and also, for divergence free fields, to \( \text{curl} (A \wedge B) \). In Stratonovich form equation (1) then reads

\[
dB + [v, B] \, dt + \sum_{k=1}^\infty [\sigma_k, B] \circ dW_t^k = 0.
\]
Its Itô formulation is
\[
\begin{align*}
  dB + [v, B] \ dt + \sum_{k=1}^{\infty} [\sigma_k, B] \ dW^k_t = \frac{1}{2} \sum_k [\sigma_k, [\sigma_k, B]] \ dt.
\end{align*}
\]

Before we justify this claim we have to clarify that we wrote the Stratonovich formulation above in a formal way, for the purpose of a better physical understanding, but at the rigorous level we shall always use its Itô formulation. For this reason, we do not provide a rigorous proof of the equivalence of the two formulations but only a formal argument. Then, in the next section, we give a rigorous definition of solution of the Itô equation only.

Let us show that (1) leads to (5). Recall that Stratonovich integral differs from Itô integral by 1/2 mutual variation:
\[
X \circ dW = X dW + \frac{1}{2} d \langle X, W \rangle;
\]
where, in the case of interest to us when \(X\) is vector valued and \(W\) is real valued, by \(\langle X, W \rangle\) we mean the vector of components \(\langle X_\alpha, W \rangle\). Then
\[
[\sigma_k, B] \circ dW^k_t = [\sigma_k, B] dW^k_t + d \langle [\sigma_k, B], W^k_t \rangle t.
\]

Now
\[
d \langle [\sigma_k, B], W^k \rangle t = (\sigma_k \cdot \nabla) d \langle B, W^k \rangle t - d \langle B, W^k \rangle t \cdot \nabla \sigma_k.
\]

From the equation for \(dB\) and the property that the mutual variations between \(W^k\) and \(BV\) functions or stochastic integrals with respect to \(W^j\) for \(j \neq k\) are zero (and \(d \langle W^k, W^k \rangle_t = dt\)) we get
\[
d \langle B, W^k \rangle t = d \left\langle \int_0^t [\sigma_k, B_s] \ dW^k_s, W^k \right\rangle_t = [\sigma_k, B_t] dt.
\]

Therefore we deduce (formally speaking) (5).

We have introduced the second order differential operator, acting on smooth vector fields \(B\), defined as
\[
\mathcal{L}B(x) := \frac{1}{2} \sum_k [\sigma_k, [\sigma_k, B]](x).
\]

### 2.3 The differential operator \(\mathcal{L}\)

**Proposition 2** Assume \(Q\) to be twice continuously differentiable, bounded with bounded first and second derivatives, and
\[
Q(x, x) \geq \nu I_{\mathbb{R}^3}
\]
for some \(\nu > 0\), uniformly in \(x \in \mathbb{R}^3\). Then \(\mathcal{L}\) is uniformly elliptic. In particular, there exists \(C > 0\) such that
\[
- \int_{\mathbb{R}^3} \mathcal{L}B(x) \cdot B(x) \ dx \geq \nu \int_{\mathbb{R}^3} |DB(x)|^2 \ dx - C \int_{\mathbb{R}^3} |B(x)|^2 \ dx
\]
for all $B \in W^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$. Moreover it has the form

\[
(LB)^\alpha (x) = \sum_{i,j=1}^3 a_{ij}(x) \partial_i \partial_j B^\alpha (x)
+ \sum_{i,\beta=1}^3 b_i^{\alpha \beta}(x) \partial_i B^\beta (x) + \sum_{\beta=1}^3 c^{\alpha \beta}(x) B^\beta (x)
\]

where $a_{ij}$ is twice continuously differentiable, bounded with bounded first and second derivatives, $b_i^{\alpha \beta}$ is continuously differentiable, bounded with bounded first derivatives and $c^{\alpha \beta}$ is bounded continuous.

We prepare the proof by the explicit computation of $[\sigma_k, [\sigma_k, B]]$. We have

\[
[\sigma_k, [\sigma_k, B]] = (\sigma_k \cdot \nabla) [\sigma_k, B] - ([\sigma_k, B] \cdot \nabla) \sigma_k
= (\sigma_k \cdot \nabla) (\sigma_k \cdot \nabla) B_t - (\sigma_k \cdot \nabla) (B_t \cdot \nabla) \sigma_k
- ((\sigma_k \cdot \nabla) B_t \cdot \nabla) \sigma_k + ((B_t \cdot \nabla) \sigma_k \cdot \nabla) \sigma_k.
\]

All terms can be expressed by means of $Q$, after the following remarks. The function $Q(x, y)$ is defined on $\mathbb{R}^3 \times \mathbb{R}^3$ with values in matrices $\mathbb{R}^{3 \times 3}$. When we differentiate $Q_{\alpha \beta}(x, y)$ with respect to the first set of components, we write

\[
\left( \partial^{(1)}_i Q_{\alpha \beta} \right)(x, y) = \lim_{\epsilon \to 0} \frac{Q_{\alpha \beta}(x + \epsilon e_i, y) - Q_{\alpha \beta}(x, y)}{\epsilon}
\]

while when we differentiate $Q_{\alpha \beta}(x, y)$ with respect to the second set of components, we write

\[
\left( \partial^{(2)}_i Q_{\alpha \beta} \right)(x, y).
\]

We have

\[
\left( \partial^{(1)}_i Q_{\alpha \beta} \right)(x, y) = \partial_{x_i} \left( Q_{\alpha \beta}(x, y) \right) = \sum_{k=1}^{\infty} (\partial_i \sigma_i^\alpha (x)) \sigma_k^\beta (y)
\]

\[
\left( \partial^{(2)}_i Q_{\alpha \beta} \right)(x, y) = \partial_{y_i} \left( Q_{\alpha \beta}(x, y) \right) = \sum_{k=1}^{\infty} \sigma_i^\alpha (x) \left( \partial_i \sigma_i^\beta \right)(y).
\]

Hence, when we evaluate at $y = x$,

\[
\sum_{k=1}^{\infty} (\partial_i \sigma_i^\alpha (x)) \sigma_k^\beta (x) = \left( \partial^{(1)}_i Q_{\alpha \beta} \right)(x, x)
\]

\[
\sum_{k=1}^{\infty} \sigma_i^\alpha (x) \left( \partial_i \sigma_i^\beta \right)(x) = \left( \partial^{(2)}_i Q_{\alpha \beta} \right)(x, x).
\]
Similarly,

\[
\left( \partial_j^{(1)} \partial_i^{(1)} Q^{\alpha \beta} \right) (x, y) = \partial_x \partial_x \left( Q^{\alpha \beta} (x, y) \right) = \sum_{k=1}^{\infty} (\partial_j \partial_i \sigma_k^\alpha) (x) \sigma_k^\beta (y)
\]

whence, at \( y = x \),

\[
\sum_{k=1}^{\infty} (\partial_j \partial_i \sigma_k^\alpha) (x) \sigma_k^\beta (x) = \left( \partial_j^{(1)} \partial_i^{(1)} Q^{\alpha \beta} \right) (x, x)
\]

and so on for the other second derivatives. Let us denote by \([\sigma_k, [\sigma_k, B]]^{(\alpha)}\) the \(\alpha\)-component of the vector \([\sigma_k, [\sigma_k, B]]\).

**Lemma 3**

\[
\sum_k [\sigma_k, [\sigma_k, B]]^{(\alpha)} (x)
\]

\[
= \sum_{i,j=1}^{3} Q^{ij} (x, x) \partial_i \partial_j B^\alpha (x)
\]

\[
+ \sum_{i=1}^{3} \sum_{\gamma=1}^{3} \partial_i^{(2)} Q^{\gamma i} (x, x) \partial_i B^\alpha (x) - \sum_{i,\beta=1}^{3} 2 \left( \partial_i^{(2)} Q^{\alpha \beta} \right) (x, x) \partial_i B^\beta (x)
\]

\[
+ \sum_{\beta,\gamma=1}^{3} \partial_{\beta}^{(1)} \partial_{\gamma}^{(2)} Q^{\gamma \alpha} (x, x) B^\beta (x) - \sum_{\gamma,\beta=1}^{3} \left( \partial_{\gamma}^{(2)} \partial_{\beta}^{(2)} Q^{\gamma \alpha} \right) (x, x) B^\beta (x).
\]

Therefore the operator \( \mathcal{L} \) has coefficients given by

\[
a_{ij} (x) = \frac{1}{2} Q^{ij} (x, x)
\]

\[
b_{i}^{\alpha \beta} (x) = \frac{1}{2} \sum_{\gamma=1}^{3} \partial_{\gamma}^{(2)} Q^{\gamma i} (x, x) \delta_{\alpha \beta} - \left( \partial_{\beta}^{(2)} Q^{\alpha \gamma} \right) (x, x)
\]

\[
c^{\alpha \beta} (x) = \frac{1}{2} \sum_{\gamma=1}^{3} \partial_{\gamma}^{(1)} \partial_{\beta}^{(2)} Q^{\gamma \alpha} (x, x) - \frac{1}{2} \sum_{\gamma=1}^{3} \left( \partial_{\gamma}^{(2)} \partial_{\beta}^{(2)} Q^{\gamma \alpha} \right) (x, x).
\]
Proof.

\[
\sum_k (\sigma_k \cdot \nabla) (\sigma_k \cdot \nabla) B^i_k - \sum_k (\sigma_k \cdot \nabla) (L_k \cdot \nabla) \sigma^i_k + \sum_k ((\sigma_k \cdot \nabla) L_k \cdot \nabla) \sigma^i_k + \sum_k ((L_k \cdot \nabla) \sigma_k \cdot \nabla) \sigma^i_k
\]

\[
= \sum_k \sum_{\alpha \beta} (\sigma_k^\alpha \partial_\alpha (\sigma_k^\beta \partial_\beta B^i_k) - \sigma_k^\alpha \partial_\alpha (B^i_k \partial_\beta \sigma^\beta_k) - \sigma_k^\alpha \partial_\alpha B^i_k \partial_\beta \sigma^\beta_k + B^i_k \partial_\alpha \sigma_k^\beta \partial_\beta \sigma^\beta_k)
\]

\[
= \sum_k \sum_{\alpha \beta} (\sigma_k^\alpha \sigma_k^\beta \partial_\alpha \partial_\beta B^i_k + \sigma_k^\alpha \partial_\alpha \sigma_k^\beta \partial_\beta B^i_k - \sigma_k^\alpha B^i_k \partial_\alpha \partial_\beta \sigma^\beta_k)
\]

\[
+ \sum_k \sum_{\alpha \beta} (-\sigma_k^\alpha \partial_\alpha B^i_k \partial_\beta \sigma^\beta_k - \sigma_k^\alpha \partial_\alpha \sigma_k^\beta \partial_\beta \sigma^\beta_k + B^i_k \partial_\alpha \sigma_k^\beta \partial_\beta \sigma^\beta_k)
\]

\[
= \sum_{\alpha \beta} Q^{\alpha \beta} (x, x) \partial_\alpha \partial_\beta B^i_k + (\partial^{(2)} \alpha \beta) (x, x) \partial_\beta B^i_k - (\partial^{(2)} \alpha \beta Q^{\alpha \beta}) (x, x) B^i_k
\]

\[
+ \sum_{\alpha \beta} (-2 \partial^{(2)} \alpha \beta Q^{\alpha \beta} \partial_\alpha B^i_k + \partial^{(1)} \alpha \beta \partial_\beta Q^{\alpha \beta} B^i_k)
\]

The result of the lemma is just a rewriting of this expression. ■

We can now prove the proposition.

Proof. Denoting by \( R_0 \) a remainder that we shall handle below, we have

\[
- \int_{\mathbb{R}^3} \mathcal{L} B (x) \cdot B (x) \, dx = - \sum_i \int_{\mathbb{R}^3} \sum_{\alpha \beta} Q^{\alpha \beta} (x, x) \partial_\alpha \partial_\beta B^i (x) B^i (x) \, dx + R_0
\]

\[
= \sum_i \int_{\mathbb{R}^3} \sum_{\alpha \beta} Q^{\alpha \beta} (x, x) \partial_\beta B^i (x) \partial_\alpha B^i (x) \, dx
\]

\[
+ \sum_i \int_{\mathbb{R}^3} \sum_{\alpha \beta} \partial_\alpha Q^{\alpha \beta} (x, x) \partial_\beta B^i (x) B^i (x) \, dx + R_0
\]

\[
\geq \nu \sum_i \int_{\mathbb{R}^3} |\nabla B^i (x)|^2 \, dx - \sum_{i \alpha \beta} \int_{\mathbb{R}^3} \left| \partial_\alpha Q^{\alpha \beta} (x, x) \right| |\partial_\beta B^i (x)| |B^i (x)| \, dx - |R_0|
\]

\[
= \nu \int_{\mathbb{R}^3} |DB (x)|^2 \, dx - R_1 - |R_0|
\]

with \( R_1 \) defined by the identity. The estimates on \(|R_0|\) are similar to the estimate on \( R_1 \), so we limit ourselves to this one. We have

\[
R_1 \leq C_1 \sum_{i \alpha \beta} \int_{\mathbb{R}^3} |\partial_\beta B^i (x)| |B^i (x)| \, dx
\]

because we have assumed that \( Q \) has bounded derivatives,

\[
\leq C_2 \int_{\mathbb{R}^3} |DB (x)| |B (x)| \, dx \leq \frac{\nu}{4} \int_{\mathbb{R}^3} |DB (x)|^2 \, dx + C_3 \int_{\mathbb{R}^3} |B (x)|^2 \, dx.
\]
Here we have denoted by $C_i > 0$ some constants, possibly depending on $\nu$ and other factors, but not on $B$. ■

### 2.4 Stochastic exponentials

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be the filtered probability space introduced above, with the sequence $\{W^k_t\}_{k \in \mathbb{N}}$ of independent Brownian motions. Let $\mathcal{G}_t$ be the associated filtration:

$$\mathcal{G}_t = \sigma \{ B^k_s; s \in [0,t], k \in \mathbb{N} \}.$$  

Let $\mathcal{G}'_t$ be the completed filtration. For some $T > 0$, let

$$\mathcal{H} = L^2(\Omega, \mathcal{G}_T, P)$$

$$F = \cup_{n \in \mathbb{N}} L^2(0,T; \mathbb{R}^n)$$

$$\mathcal{D} = \{ e_f(T); f \in F \}$$

where, for $n \in \mathbb{N}$, $f \in L^2(0,T; \mathbb{R}^n)$, with components $f_1, ..., f_n$, we set

$$e_f(t) = \exp \left( \sum_{k=1}^{n} \int_{0}^{t} f_k(s) \, dW^k_s - \frac{1}{2} \sum_{k=1}^{n} \int_{0}^{t} |f_k(s)|^2 \, ds \right)$$

for $t \in [0,T]$. From Itô formula

$$d e_f(t) = \sum_{k=1}^{n} f_k(t) \, e_f(t) \, dW^k_t.$$

The following result is known, see the argument in [14]:

**Lemma 4** $\mathcal{D}$ is dense in $\mathcal{H}$.

### 2.5 Interpolation inequalities

**Lemma 5** If $f, h \in W^{1,2}(\mathbb{R}^3)$ and $g \in L^p(\mathbb{R}^3)$ for some $p > 3$, then

$$\int_{\mathbb{R}^3} f(x) g(x) \partial_i h(x) \, dx \leq C_{\epsilon} \| g \|_{L^p(\mathbb{R}^3)} \| f \|_{W^{1,2}(\mathbb{R}^3)} \| h \|_{W^{1,2}(\mathbb{R}^3)}$$

where $C > 0$ is a constant independent of $f, g, h$ and for every $\epsilon > 0$ there is a constant $C_{\epsilon} > 0$ such that

$$\int_{\mathbb{R}^3} f(x) g(x) \partial_i h(x) \, dx \leq \epsilon \| h \|^2_{W^{1,2}(\mathbb{R}^3)} + \epsilon \| f \|^2_{W^{1,2}(\mathbb{R}^3)} + C_{\epsilon} \| g \|^2_{L^p(\mathbb{R}^3)} \| f \|^2_{L^2(\mathbb{R}^3)}.$$
Proof. Step 1.

\[ \int_{\mathbb{R}^3} |f|^2 |g|^2 \, dx \leq \|f\|_{W^{1,2}(\mathbb{R}^3)}^2 \left( C \|g\|_{L^p(\mathbb{R}^3)}^2 \|f\|_{L^2(\mathbb{R}^3)}^{2-\frac{6}{p}} \right). \]

Indeed,

\[ \int_{\mathbb{R}^3} |f|^2 |g|^2 \, dx \leq \left( \int_{\mathbb{R}^3} |g|^p \, dx \right)^{\frac{2}{p}} \left( \int_{\mathbb{R}^3} |f|^{\frac{2p}{p-2}} \, dx \right)^{\frac{p-2}{p}} = \|g\|_{L^p(\mathbb{R}^3)}^2 \|f\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^3)}^{2p-2}. \]

Recall now that, by Sobolev embedding theorem,

\[ \|f\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^3)} \leq C \|f\|_{W^{\frac{3}{2},2}(\mathbb{R}^3)} \]

(because \( W^{s,2} \subset L^r \) for \( \frac{1}{r} = \frac{1}{2} - \frac{s}{3} \), namely \( s = \frac{3}{2} - \frac{3(p-2)}{2p} = \frac{3p-3(p-2)}{2p} = \frac{3}{p} \)) and the interpolation inequality gives, for \( \alpha \in (0,1) \),

\[ \|f\|_{W^{\alpha,2}(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)}^{1-\alpha} \|f\|_{W^{1,2}(\mathbb{R}^3)}^\alpha \]

hence

\[ \|f\|_{W^{\frac{3}{2},2}(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{p}} \|f\|_{W^{1,2}(\mathbb{R}^3)}^{\frac{3}{p}}. \]

Summarizing,

\[ \int_{\mathbb{R}^3} |f|^2 |g|^2 \, dx \leq \|f\|_{W^{1,2}(\mathbb{R}^3)}^2 \left( C \|g\|_{L^p(\mathbb{R}^3)}^2 \|f\|_{L^2(\mathbb{R}^3)}^{2-\frac{6}{p}} \right). \]

Step 2. Obviously

\[ \int_{\mathbb{R}^3} f(x) g(x) \partial_i h(x) \, dx \leq C \int_{\mathbb{R}^3} |f(x)||g(x)||\partial_i h(x)| \, dx \leq C \|h\|_{W^{1,2}(\mathbb{R}^3)} \left( \int_{\mathbb{R}^3} |f|^2 |g|^2 \, dx \right)^{1/2}. \]

Hence from the inequality of Step 1,

\[ \int_{\mathbb{R}^3} f(x) g(x) \partial_i h(x) \, dx \leq C \|h\|_{W^{1,2}(\mathbb{R}^3)} \|f\|_{W^{1,2}(\mathbb{R}^3)} \|g\|_{L^p(\mathbb{R}^3)}. \]

Step 3. Again

\[ \int_{\mathbb{R}^3} f(x) g(x) \partial_i h(x) \, dx \leq \epsilon \|h\|_{W^{1,2}(\mathbb{R}^3)}^2 + C \epsilon \int_{\mathbb{R}^3} |f|^2 |g|^2 \, dx. \]

\[ \leq \epsilon \|h\|_{W^{1,2}(\mathbb{R}^3)}^2 + C \epsilon \|f\|_{W^{1,2}(\mathbb{R}^3)}^6 \left( C \|g\|_{L^p(\mathbb{R}^3)}^2 \|f\|_{L^2(\mathbb{R}^3)}^{2-\frac{6}{p}} \right). \]
By Young inequality $ab \leq \delta a^r + C_b b^{r'}, \frac{1}{r} + \frac{1}{r'} = 1$, we have
\[
\|f\|_{W^{1,2}(\mathbb{R}^3)}^p \left( \|g\|_{L^p(\mathbb{R}^3)}^2 \|f\|_{L^2(\mathbb{R}^3)}^{2-r} \right) \\
\leq \delta \|f\|_{W^{1,2}(\mathbb{R}^3)}^2 + C\delta \|g\|_{L^p(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3)}^2
\]

where was used with $r = \frac{6}{3}$, hence $r' = \frac{p}{p-3}$. By a proper choice of $\epsilon$ and $\delta$, we get the last inequality of the lemma.

### 2.6 Definition of weak solutions

We present now the setting and a suitable definition of quasiregular weak solutions to equation (1), adapted to treat the problem of well-posedness. Throughout the paper we assume that the vector field $v$ satisfies
\[
v \in L^\infty \left([0, T], L^p(\mathbb{R}^3; \mathbb{R}^3)\right) \quad \text{for some } p > 3, \quad (6)
\]
and
\[
\text{div } v(t, x) = 0. \quad (7)
\]
Moreover, the initial condition is taken to be
\[
B_0 \in L^4(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \quad \text{div } B_0 = 0.
\]

The next definition tells us in which sense a stochastic process is a quasiregular weak solution of (1).

**Definition 6** A stochastic process $B : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$, $B \in L^2 \left(\Omega \times [0, T], L^2_{\text{loc}}(\mathbb{R}^3)\right)$ is called a quasiregular weak solution of the Cauchy problem (1) when:

i) for any $\varphi \in C^\infty_c(\mathbb{R}^3, \mathbb{R}^3)$, the real valued process $\int B(t, x) \varphi(x) dx$ has a continuous modification which is an $\mathcal{F}_t$-semimartingale, and for all $t \in [0, T]$ ii) for any $\phi \in C^\infty_c(\mathbb{R}^3, \mathbb{R}^3)$, we have $\mathbb{P}$-almost surely

\[
\int B(t, x) \phi(x) \ dx - \int_0^t \int [v, \phi](s, x) B(s, x) \ dx dt \\
\quad - \sum_{k=1}^\infty \int_0^t \left( \int [\sigma_k, \phi](s, x) B(s, x) \ dx \right) dW_s^k \\
= \int B_0(x) \phi(x) \ dx + \frac{1}{2} \int_0^t \sum_k [\sigma_k, [\sigma_k, \phi]](s, x) B(s, x) \ dx dt \quad (8)
\]
For each function \( f \in L^2(0, T; \mathbb{R}^n) \), with components \( f_1, \ldots, f_n \), the deterministic function \( V(t, x) := \mathbb{E}[B(t, x) e_f(t)] \) is a measurable bounded function, which belongs to \( L^2([0, T]; H^1(\mathbb{R}^3)) \cap C([0, T]; L^2(\mathbb{R}^3)) \) and satisfies the parabolic equation

\[
\partial_t V + [v + h, V] = \mathcal{L}V
\]
in the weak sense, where \( h(t, x) := \sum_{k=1}^n f_k(t) \sigma_k(x) \).

\[
\text{div} B = 0, \text{ in the sense of distributions .}
\]

### 3 Existence

Assume \( v \) and \( B_0 \) smooth and let \( B \) be a smooth solution (it exists by stochastic flows, see the next remarks). In this section we prove a priori estimates which depend only on the norms \( \|v\|_{L^\infty(0, T; L^p(\mathbb{R}^3))} \) and \( \|B_0\|_{L^4} \). By a classical procedure we shall deduce the existence of a quasi-regular weak solution: given our non-smooth data \( v \) and \( B_0 \), taken a family of standard symmetric mollifiers \( \{\rho_\varepsilon\}_\varepsilon \) we define the family of regularized coefficients \( v_\varepsilon(t, x) = [v(t, \cdot) * \rho_\varepsilon(\cdot)](x) \) and initial conditions \( B_\varepsilon^0(x) = [B_0(\cdot) * \rho_\varepsilon(\cdot)](x) \); we prove the estimates for the corresponding solutions \( B_\varepsilon(t, x) \) and extract a subsequence which converges weakly; then we pass to the limit just by weak convergence since the equation is linear and the coefficients \( v_\varepsilon(t, x) \) converge strongly.

Let us remark about the existence of a smooth solution \( B_\varepsilon(t, x) \). By the classical results of [10], the equation

\[
dX_t^\varepsilon = v_\varepsilon(t, X_t^\varepsilon) \, dt + \sum_k \sigma_k(X_t^\varepsilon) \circ dW_t^k, \quad X_0 = x
\]
generates a stochastic flow of smooth diffeomorphisms \( \Phi_t^\varepsilon(x) \), with inverse \( \Psi_t^\varepsilon(x) \). Then formula

\[
B_\varepsilon(t, x) = (D\Phi_t^\varepsilon)(\Psi_t^\varepsilon(x))B_0^\varepsilon(\Psi_t^\varepsilon(x)) \quad (10)
\]
gives us a smooth solution.
3.1 Estimates on $B^\varepsilon$ in $L^2 ([0, T] \times \Omega, L^2_{loc} (\mathbb{R}^3))$

In the first part of this section we denote $B^\varepsilon$ and $v^\varepsilon$ by $B$ and $v$, to simplify notations. The constants in the estimates are independent of $\varepsilon$. The strategy used in this section is inspired to [1].

We formally work on the Stratonovich formulation but the same computations can be done in a rigorous, although blind, way on the Itô form. Thus we start, componentwise, from

$$dB^\alpha + [v, B]^\alpha dt + \sum_{k=1}^{\infty} [\sigma_k, B]^\alpha \circ dW^k_t = 0.$$  

We have, for any $\alpha, \beta = 1, \ldots, 3$,

$$d \left( B^\alpha B^\beta \right) + [v, B]^\alpha B^\beta dt + \sum_{k=1}^{\infty} [\sigma_k, B]^\alpha B^\beta \circ dW^k_t$$

$$+ B^\alpha [v, B]^\beta dt + \sum_{k=1}^{\infty} B^\alpha \sigma_k^\beta \circ dW^k_t = 0.$$

Notice that

$$[A, B]^\alpha B^\beta + [A, B]^\beta B^\alpha$$

$$= (A \cdot \nabla B^\alpha) B^\beta - (B \cdot \nabla A^\alpha) B^\beta + \left( A \cdot \nabla B^\beta \right) B^\alpha - \left( B \cdot \nabla A^\beta \right) B^\alpha$$

$$= (A \cdot \nabla) \left( B^\alpha B^\beta \right) - (B \cdot \nabla A^\alpha) B^\beta - \left( B \cdot \nabla A^\beta \right) B^\alpha$$

$$= (A \cdot \nabla) \left( B^\alpha B^\beta \right) - \left( B^\beta B \cdot \nabla \right) A^\alpha - (B^\alpha B \cdot \nabla) A^\beta$$

hence

$$d \left( B^\alpha B^\beta \right) + (v \cdot \nabla) \left( B^\alpha B^\beta \right) dt + \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla) \left( B^\alpha B^\beta \right) \circ dW^k_t$$

$$- \left( (B^\beta B \cdot \nabla) v^\alpha + (B^\alpha B \cdot \nabla) v^\beta \right) dt - \sum_{k=1}^{\infty} \left( (B^\beta B \cdot \nabla) \sigma_k^\alpha + (B^\alpha B \cdot \nabla) \sigma_k^\beta \right) \circ dW^k_t = 0.$$

We go now to Itô form; the corrections are, on the LHS:

$$\frac{1}{2} \sum_{k=1}^{\infty} d \left( (\sigma_k \cdot \nabla) \left( B^\alpha B^\beta \right), W^k \right)_t = \frac{1}{2} \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla) d \left( B^\alpha B^\beta, W^k \right)_t$$

$$= -\frac{1}{2} \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla) (\sigma_k \cdot \nabla) \left( B^\alpha B^\beta \right) dt$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla) \left( (B^\beta B \cdot \nabla) \sigma_k^\alpha + (B^\alpha B \cdot \nabla) \sigma_k^\beta \right) dt$$
and
\[-\frac{1}{2} \sum_{k=1}^{\infty} d \left< \left( B^\beta B \cdot \nabla \right) \sigma_k^\alpha + (B^\alpha B \cdot \nabla) \sigma_k^\beta, W^k \right>_t \]
\[-\frac{1}{2} \sum_{k=1}^{\infty} \sum_{i=1}^{3} d \left< B^\beta B^i, W^k \right>_t \partial_i \sigma_k^\alpha - \frac{1}{2} \sum_{k=1}^{\infty} \sum_{i=1}^{3} d \left< B^\alpha B^i, W^k \right>_t \partial_i \sigma_k^\beta \]
\[= \frac{1}{2} \sum_{k=1}^{\infty} \sum_{i=1}^{3} \left( \sigma_k \cdot \nabla \right) \left( B^\beta B^i \right) \partial_i \sigma_k^\alpha dt - \frac{1}{2} \sum_{k=1}^{\infty} \sum_{i=1}^{3} \left( \left( B^\beta B \cdot \nabla \right) \sigma_k^\alpha + (B^i B \cdot \nabla) \sigma_k^\beta \right) \partial_i \sigma_k^\alpha dt \]
\[+ \frac{1}{2} \sum_{k=1}^{\infty} \sum_{i=1}^{3} \left( \sigma_k \cdot \nabla \right) \left( B^\alpha B^i \right) \partial_i \sigma_k^\beta dt - \frac{1}{2} \sum_{k=1}^{\infty} \sum_{i=1}^{3} \left( \left( B^i B \cdot \nabla \right) \sigma_k^\alpha + (B^\alpha B \cdot \nabla) \sigma_k^\beta \right) \partial_i \sigma_k^\beta dt. \]

We may summarize this large amount of terms in the form
\[-\frac{1}{2} \sum_{i,j=1}^{3} Q^{ij}(x, x) \partial_i \partial_j \left( B^\alpha B^\beta \right) dt \]
\[+ \sum_{i=1}^{3} \theta^{\alpha,\beta} \cdot \nabla \left( B^\alpha B^\beta \right) dt + \sum_{i=1}^{3} \theta_i^\alpha \cdot \nabla \left( B^\beta B^i \right) dt + \sum_{i=1}^{3} \theta_i^\beta \cdot \nabla \left( B^\alpha B^i \right) dt \]
\[+ \sum_{i=1}^{3} \eta_i^\alpha B^\beta B^i dt + \sum_{i=1}^{3} \eta_i^\beta B^\alpha B^i dt + \sum_{i,j=1}^{3} \eta_{i,j}^\alpha B^i B^j dt \]

where we have also used the fact that
\[-\frac{1}{2} \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla) (\sigma_k \cdot \nabla) \left( B^\alpha B^\beta \right) \]
\[= -\frac{1}{2} \sum_{i,j=1}^{3} Q^{ij}(x, x) \partial_i \partial_j \left( B^\alpha B^\beta \right) + \sum_{i=1}^{3} \theta^{\alpha,\beta} \cdot \nabla \left( B^\alpha B^\beta \right). \]

Summarizing, we get
\[d \left( B^\alpha B^\beta \right) + (v \cdot \nabla) \left( B^\alpha B^\beta \right) dt + \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla) \left( B^\alpha B^\beta \right) dW_t^k \]
\[- \left( \left( B^\beta B \cdot \nabla \right) v^\alpha + (B^\alpha B \cdot \nabla) v^\beta \right) dt - \sum_{k=1}^{\infty} \left( \left( B^\beta B \cdot \nabla \right) \sigma_k^\alpha + (B^\alpha B \cdot \nabla) \sigma_k^\beta \right) dW_t^k \]
\[ \frac{1}{2} \sum_{i,j=1}^{3} Q^{ij}(x, x) \partial_i \partial_j \left( B^\alpha B^\beta \right) dt \]

\[ - \sum_{i=1}^{3} \theta^{\alpha,\beta} \cdot \nabla \left( B^\alpha B^\beta \right) dt - \sum_{i=1}^{3} \theta_i^\alpha \cdot \nabla \left( B^\beta B^i \right) dt - \sum_{i=1}^{3} \theta_i^\beta \nabla \left( B^\alpha B^i \right) dt \]

\[ - \sum_{i=1}^{3} \eta_j^\alpha B^\beta dt - \sum_{i=1}^{3} \eta_j^\beta B^\alpha dt - \sum_{i,j=1}^{3} \eta_i^{\alpha,\beta} B^i B^j dt. \]

Then, taking expectation, we deduce

\[ \frac{\partial}{\partial t} \mathbb{E} \left[ B^\alpha B^\beta \right] + (v \cdot \nabla) \mathbb{E} \left[ B^\alpha B^\beta \right] - \left( \mathbb{E} \left[ B^\beta B \right] \cdot \nabla \right) v^\alpha - \left( \mathbb{E} \left[ B^\alpha B \right] \cdot \nabla \right) v^\beta \]

\[ = \frac{1}{2} \sum_{i,j=1}^{3} Q^{ij}(x, x) \partial_i \partial_j \mathbb{E} \left[ B^\alpha B^\beta \right] \]

\[ - \sum_{i=1}^{3} \theta^{\alpha,\beta} \cdot \nabla \mathbb{E} \left[ B^\alpha B^\beta \right] - \sum_{i=1}^{3} \theta_i^\alpha \cdot \nabla \mathbb{E} \left[ B^\beta B^i \right] - \sum_{i=1}^{3} \theta_i^\beta \nabla \mathbb{E} \left[ B^\alpha B^i \right] \]

\[ - \sum_{i=1}^{3} \eta_j^\alpha \mathbb{E} \left[ B^\beta B^i \right] - \sum_{i=1}^{3} \eta_j^\beta \mathbb{E} \left[ B^\alpha B^i \right] - \sum_{i,j=1}^{3} \eta_i^{\alpha,\beta} \mathbb{E} \left[ B^i B^j \right]. \]

Define \( u_{\alpha \beta}(t, x) := \mathbb{E} \left[ B^\alpha (t, x) B^\beta (t, x) \right] \). We have the system of parabolic equations

\[ \frac{\partial u_{\alpha \beta}}{\partial t} + (v \cdot \nabla) u_{\alpha \beta} - \sum_{i=1}^{3} u_{\beta i} \partial_i v^\alpha - \sum_{i=1}^{3} u_{\alpha i} \partial_i v^\beta \]

\[ = \frac{1}{2} \sum_{i,j=1}^{3} Q^{ij}(x, x) \partial_i \partial_j u_{\alpha \beta} + M^{\alpha \beta}(u) \]

where \( u \) is the matrix-function \( (u_{ij})_{ij} \) and \( M^{\alpha \beta}(u) \) is a first order differential operator with bounded continuous coefficients.

Then

\[ \frac{1}{2} \frac{d}{dt} \int u_{\alpha \beta}^2(t, x) \, dx - \frac{1}{2} \sum_{i,j=1}^{3} \int Q^{ij}(x, x) \partial_i \partial_j u_{\alpha \beta} u_{\alpha \beta} \, dx \]

\[ = \sum_{i=1}^{3} \int u_{\alpha \beta} u_{\beta i} \partial_i v^\alpha \, dx + \sum_{i=1}^{3} \int u_{\alpha \beta} u_{\alpha i} \partial_i v^\beta \, dx + \int M^{\alpha \beta}(u) u_{\alpha \beta} \, dx. \]
Similarly to Proposition 2 we have

\[-\frac{1}{2} \sum_{i,j=1}^{3} Q^{ij}(x, x) \partial_i \partial_j u_{\alpha \beta} u_{\alpha \beta} dx \geq \nu \int |\nabla u_{\alpha \beta}|^2 dx - C \int |u_{\alpha \beta}|^2 dx\]

for some constants \( \nu > 0, \) \( C \geq 0, \) and similarly, for every \( \epsilon > 0, \)

\[\int M^{\alpha \beta}(u) u_{\alpha \beta} dx \leq \epsilon \sum_{\alpha, \beta = 1}^{3} \int |\nabla u_{\alpha \beta}|^2 dx + C \epsilon \sum_{\alpha, \beta = 1}^{3} \int u_{\alpha \beta}^2 dx\]

hence

\[\frac{1}{2} \frac{d}{dt} \sum_{\alpha, \beta = 1}^{3} \int u_{\alpha \beta}^2 (t, x) dx + \nu \int |\nabla u_{\alpha \beta}|^2 dx - \epsilon \sum_{\alpha, \beta = 1}^{3} \int |\nabla u_{\alpha \beta}|^2 dx \leq C \epsilon \sum_{\alpha, \beta = 1}^{3} \int u_{\alpha \beta}^2 dx + \sum_{i=1}^{3} \int u_{\alpha \beta} u_{\beta i} \partial_i v^\alpha dx + \sum_{i=1}^{3} \int u_{\alpha \beta} u_{\alpha i} \partial_i v^\beta dx.\]

We deduce, with a proper choice of \( \epsilon > 0, \)

\[\frac{1}{2} \frac{d}{dt} \sum_{\alpha, \beta = 1}^{3} \int u_{\alpha \beta}^2 dx + \frac{\nu}{2} \sum_{\alpha, \beta = 1}^{3} \int |\nabla u_{\alpha \beta}|^2 dx \leq C \sum_{\alpha, \beta = 1}^{3} \int u_{\alpha \beta}^2 dx + \sum_{i, \alpha, \beta = 1}^{3} \int u_{\alpha \beta} u_{\beta i} \partial_i v^\alpha dx + \sum_{i=1}^{3} \int u_{\alpha \beta} u_{\alpha i} \partial_i v^\beta dx.\]

Let us treat only the term \( \int u_{\alpha \beta} u_{\beta i} \partial_i v^\alpha dx, \) the others being equal. We have

\[\int u_{\alpha \beta} u_{\beta i} \partial_i v^\alpha dx = - \int v^\alpha \partial_i (u_{\alpha \beta} u_{\beta i}) dx.\]

From Lemma 5 (for instance with \( f = u_{\alpha \beta}, g = v^\alpha, h = u_{\beta i} \)), we deduce

\[\int u_{\alpha \beta} u_{\beta i} \partial_i v^\alpha dx \leq \epsilon \|u\|_{W^{1,2}(\mathbb{R}^3)}^2 + C \epsilon \|v\|_{L^p(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}^2.\]

Choosing \( \epsilon > 0 \) small enough, we get

\[\frac{1}{2} \frac{d}{dt} \sum_{\alpha, \beta = 1}^{3} \int u_{\alpha \beta}^2 dx + \frac{\nu}{4} \sum_{\alpha, \beta = 1}^{3} \int |\nabla u_{\alpha \beta}|^2 dx \leq C \sum_{\alpha, \beta = 1}^{3} \int u_{\alpha \beta}^2 dx\]

where \( C \) depends also on \( \|v\|_{L^\infty(0,T;L^p(\mathbb{R}^3))}.\)
For the sake of clarity, let us restore now the notations $B^\varepsilon$ and $v^\varepsilon$, different from $B$ and $v$. We continue to denote by $C > 0$ a constant which depends only on $\|v\|_{L^\infty(0,T;L^p(\mathbb{R}^3))}$ and not on $\varepsilon$. The previous identity, by Gronwall lemma and some other elementary computations, gives us

$$
\sup_{t \in [0,T]} \left[ \int \left( \mathbb{E} \left[ |B^\varepsilon(t,x)|^2 \right] \right)^2 dx \right] \leq C \int \left( \mathbb{E} \left[ |B_0^\varepsilon(x)|^2 \right] \right)^2 dx
$$

$$
= C \int |B_0^\varepsilon(x)|^4 dx \leq C \int |B_0(x)|^4 dx
$$

(here we see the need to assume $B_0 \in L^4(\mathbb{R}^3, \mathbb{R}^3)$). Therefore, given any $R > 0$, we have

$$
\int_0^T \int_{B_R} \mathbb{E} \left[ |B^\varepsilon(t,x)|^2 \right] dx \leq TR \left( \int_0^T \int_{B_R} \left( \mathbb{E} \left[ |B^\varepsilon(t,x)|^2 \right] \right)^2 dx \right)^{\frac{1}{2}} \leq CRT
$$

(11)

where the constant $C > 0$ depends only on $\|v\|_{L^\infty(0,T;L^p(\mathbb{R}^3))}$ and $\int |B_0(x)|^4 dx$.

### 3.2 Estimates on $E[B^\varepsilon e_f]$ 

Since $B^\varepsilon$ is regular, by Itô calculus we can prove that $V^\varepsilon(t,x) := \mathbb{E}[B^\varepsilon(t,x)e_f(t)]$ satisfies equation (11), namely (we often omit again in the first part of the section the superscript $\varepsilon$)

$$
\partial_t V + (v + h) \cdot \nabla V = \mathcal{L}V + V \cdot \nabla (v + h).
$$

Hence, with $V_0^\varepsilon(x) := \mathbb{E}[B_0^\varepsilon(x)e_f(0)] = B_0^\varepsilon(x)$,

$$
\int_{\mathbb{R}^3} |V(t,x)|^2 dx - \int_0^t \int_{\mathbb{R}^3} V \cdot \mathcal{L}V dx ds
$$

$$
= \int_{\mathbb{R}^3} |V_0(x)|^2 dx + \sum_{i,j=1}^3 \int_0^t \int_{\mathbb{R}^3} V^i \partial_i (v + h)^j V^j dx ds
$$

$$
= \int_{\mathbb{R}^3} |B_0(x)|^2 dx - \sum_{i,j=1}^3 \int_0^t \int_{\mathbb{R}^3} (v + h)^j \partial_i (V^i V^j) dx ds.
$$

We observe that

$$
\sum_{i,j=1}^3 \int_0^t \int_{\mathbb{R}^3} h^j \partial_i (V^i V^j) dx ds.
$$

$$
\leq C \int_0^t |f| \int_{\mathbb{R}^3} |V(t,x)|^2 dx ds.
$$
From Proposition \[2\] and Lemma \[5\] (for instance with \( f = V^i, \ g = v^j, \ h = V^j \)), we get

\[
\int_{\mathbb{R}^3} V^2(t, x) \, dx + \nu \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla V(s, x)|^2 \, dx \, ds \\
\leq \int_{\mathbb{R}^3} V_0^2(x) \, dx + \int_{0}^{t} (|f| + C) \int_{\mathbb{R}^3} |V(t, x)|^2 \, dx \, ds \\
+ \epsilon \int_{0}^{t} \|V\|^2_{W^{1,2}(\mathbb{R}^3)} \, ds + C_{\epsilon} \sup_{t \in [0,T]} \|v(t)\|_{L^p(\mathbb{R}^3)}^{2p} \int_{0}^{t} \|V\|^2_{L^2(\mathbb{R}^3)} \, ds.
\]

When \( \epsilon \) is small enough, by Gronwall lemma and the inequality itself we deduce (here we restore the superscript \( \varepsilon \))

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^3} V^\varepsilon(t, x)^2 \, dx + \int_{0}^{T} \|V^\varepsilon(s, \cdot\|)^2_{W^{1,2}(\mathbb{R}^3)} \, ds \leq C \tag{12}
\]

where again (given \( h \)) the constant \( C > 0 \) depends only on \( \|v\|_{L^\infty(0,T;L^p(\mathbb{R}^3))} \) and \( \int |B_0(x)|^4 \, dx \).

### 3.3 Passage to the limit

From the bound \[(11)\] and a diagonal procedure, we may construct a sequence \( B^\varepsilon_n \) which converges weakly to a progressively measurable process \( B \) in \( L^2([0,T] \times B(0,B) \times \Omega, \mathbb{R}^3) \) for every \( R > 0 \). Since \( B^\varepsilon \) is a solution of \[(1)\], it is also a weak solution. The equation is linear and thus, over compact support test functions, we may pass to the limit by means of the previous weak convergence property; we apply the classical argument of \[15\] Sect. II, Chapter 3], see also \[8\] Theorem 15. This proves the existence of a weak solutions to \[(1)\].

From \[(12)\] there exists a subsequence \( \varepsilon_n \), which can be extracted from the subsequence used in the previous step, such that \( V^\varepsilon_n(t, x) \) converges weakly star to the function \( V(t, x) = \mathbb{E}[B(t, x) e_j] \) in \( C([0,T]; L^2(\mathbb{R}^3, \mathbb{R}^3)) \) and such that \( \nabla V^\varepsilon_n(t, x) \) converges weakly to \( \nabla V(t, x) \) in \( L^2([0,T] \times \mathbb{R}^3; \mathbb{R}^3) \). This allows us to conclude that \( V \in L^2([0,T]; H^1(\mathbb{R}^3, \mathbb{R}^3)) \cap C([0,T]; L^2(\mathbb{R}^3, \mathbb{R}^3)) \) and, again thanks to the linearity of the equations, to show that \( V \) solves the PDE \[(9)\]. Therefore, also the second condition of Definition \[6\] is satisfied, which proves that \( B \) a quasiregular weak solution.

### 3.4 Extra regularity in the case of finite dimensional noise

Consider the special case when \( \sigma_k(x) = e_k \) for \( k = 1, 2, 3 \), where \( e_1, e_2, e_3 \) is a basis of \( \mathbb{R}^3 \) and \( \sigma_k(x) = 0 \) for \( k \geq 4 \). The equation of characteristics, for the regularized field \( v^\varepsilon \), is simply

\[
dX^t_i = v^\varepsilon(t, X^t) \, dt + dW^t_i, \quad X^0 = x
\]

where \( W_t = (W^t_1, W^t_2, W^t_3) \). Recall we have the representation formula

\[
B^\varepsilon(t, x) = (D\Phi^\varepsilon_t)(\Psi^\varepsilon_t(x))B^0_0(\Psi^\varepsilon_t(x)) \tag{13}
\]
in terms of the (regularized) initial condition and the direct and inverse flows \( \Phi^\varepsilon_t \) and \( \Psi^\varepsilon_t \) associated to this equation. By Lemma 5 of [4] we have that for every \( p \geq 1 \), there exists \( C_{p,T} > 0 \) such that

\[
\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^3} \mathbb{E}[|D\Phi^\varepsilon_t(x)|^p] \leq C_{p,T}, \quad \text{uniformly in } \varepsilon > 0.
\]

(14)

Since \( \Phi^\varepsilon_t(\Psi^\varepsilon_t) = \text{Id} \) we have

\[
(D\Phi^\varepsilon_t)(\Psi^\varepsilon_t) = (D\Psi^\varepsilon_t)^{-1}.
\]

We observe that \( (D\Psi^\varepsilon_t)^{-1} \) is equal to

\[
\frac{1}{\det(D\Phi^\varepsilon_t)} \text{Cof}(D\Phi^\varepsilon_t)^T
\]

where \( \text{Cof} \) denoted the cofactor matrix of \( D\Phi^\varepsilon_t \). By the solenoidal hypothesis on \( \nu \) we have

\[
\text{Det}(D\Phi^\varepsilon_t) = 1
\]

and by inequality [14] we deduce that \( \text{Cof}(D\Phi^\varepsilon_t)^T \in L^\infty ([0, T] \times \mathbb{R}^3, L^2(\Omega)) \). Then, under the assumption that \( B_0 \) is bounded, \( B^\varepsilon(t,x) \) is uniformly bounded in \( L^2(\Omega \times [0,T] \times \mathbb{R}^3) \) and in \( L^\infty (\mathbb{R}^3 \times [0,T], L^2(\Omega)) \). Arguing as above on weakly star converging subsequences, this allows to prove the existence of a quasiregular solution with the additional property

\[
B \in L^\infty (\mathbb{R}^3 \times [0,T], L^2(\Omega))
\]

4 Uniqueness

As above, let \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a filtered probability space with a sequence \( \{W^k_t\}_{k \in \mathbb{N}} \) of independent Brownian motions. Let \( \mathcal{G}_t \) be the filtration associated to the Brownian motions, \( \mathcal{G}_t = \sigma \{B^k_s; s \in [0,t], k \in \mathbb{N} \} \), and let \( \mathcal{G}_t \) be the completed filtration.

**Theorem 7** Let \( B^i, i = 1, 2 \), be two quasi-regular weak solutions of equation [1] with the same initial condition \( B_0 \). Assume that \( \int B^i(t,x)\varphi(x)dx \) is \( \mathcal{G}_t \)-adapted, for both \( i = 1, 2 \), for every \( \varphi \in C^\infty_c (\mathbb{R}^3, \mathbb{R}^3) \). Then \( B^1 = B^2 \).

**Proof.** Step 0. Set of solutions. Remark that the set of quasiregular weak solutions is a linear subspace of \( L^2 (\Omega \times [0,T] \times \mathbb{R}^3) \), because the stochastic advection equation is linear, and the regularity conditions is a linear constraint. Therefore, it is enough to show that a quasiregular weak solution \( B \) with initial condition \( B_0 = 0 \) vanishes identically.

Step 1. \( V = 0 \). Let \( V(t,x) = \mathbb{E}[B(t,x)e_f(t)] \), with \( f \in L^2([0,T], \mathbb{R}^n) \cap L^\infty([0,T], \mathbb{R}^n) \). If we prove that \( V = 0 \), for arbitrary \( f \), by Lemma [3] we deduce \( B = 0 \). The function \( V \) satisfies

\[
\partial_t V + [v + h, V] = \mathcal{L}V
\]
with initial condition $V_0 = 0$, where $h(t, x) := \sum_{k=1}^n f_k(t) \sigma_k(x)$. It is thus sufficient to prove that a solution $V$ (in weak sense) of class $L^2([0, T]; H^1(\mathbb{R}^3)) \cap C([0, T]; L^2(\mathbb{R}^3))$ of this equation, such that $V_0 = 0$, is identically equal to zero. Let us see that this is a classical result of the variational theory of evolution equations.

Let $\mathcal{V} \subset \mathcal{H} = \mathcal{V}'$ be the Gelfand triple defined by

$$
\mathcal{H} = L^2_\sigma(\mathbb{R}^3, \mathbb{R}^3), \\
\mathcal{V} = H^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)
$$

where the subscript $\sigma$ denotes the fact that we take these vector fields with divergence equal to zero. The norm $|.|_\mathcal{H}$ and scalar product $\langle . , . \rangle_\mathcal{H}$ are the usual ones, and the norm $||.||_\mathcal{V}$ in $\mathcal{V}$ is defined by

$$
||f||^2_\mathcal{V} = \sum_{i=1}^3 \int_{\mathbb{R}^3} |\nabla f^i(x)|^2 \, dx + \int_{\mathbb{R}^3} |f(x)|^2 \, dx.
$$

Let $a : [0, T] \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be the bilinear form defined on smooth fields $f, g$ as

$$
a(t, f, g) = -\int_{\mathbb{R}^3} \mathcal{L} f(x) \cdot g(x) \, dx + \int_{\mathbb{R}^3} [v + h, f](x) \cdot g(x) \, dx
$$

and extended to $\mathcal{V} \times \mathcal{V}$ by one integration by parts of the second order term in $\mathcal{L}$; moreover, since $v$ is not differentiable, we have to interpret also one term in $\int_{\mathbb{R}^3} [v + h, f](x) \cdot g(x) \, dx$ by integration by parts. More precisely,

$$
a(t, f, g) = \sum_{i,j,\alpha=1}^3 \int_{\mathbb{R}^3} a_{ij}^\alpha(x) \partial_j f^\alpha(x) \partial_i g^\alpha(x) \, dx + \sum_{i,j,\alpha=1}^3 \int_{\mathbb{R}^3} g^\alpha(x) \partial_j f^\alpha(x) \partial_i a_{ij}^\alpha(x) \, dx
$$

$$
- \sum_{i,\alpha,\beta=1}^3 \int_{\mathbb{R}^3} b_i^{\alpha\beta}(x) \partial_i f^\beta(x) g^\alpha(x) \, dx - \sum_{\alpha,\beta=1}^3 \int_{\mathbb{R}^3} c^{\alpha\beta}(x) f^\beta(x) g^\alpha(x) \, dx
$$

$$
+ \sum_{\alpha,\beta=1}^3 \int_{\mathbb{R}^3} (v^\alpha(t, x) + h^\alpha(t, x)) \partial_\alpha f^\beta(x) g^\beta(x) \, dx
$$

$$
+ \sum_{\alpha,\beta=1}^3 \int_{\mathbb{R}^3} (v^\beta(t, x) + h^\beta(t, x)) \partial_\alpha (f^\alpha(x) g^\beta(x)) \, dx
$$

where we recall that $\partial_i a_{ij}$ is bounded continuous. Then the weak form of equation $\partial_t V + [v + h, V] = \mathcal{L} V$, with $V_0 = 0$, is equivalent to

$$
\langle V(t), \phi \rangle_\mathcal{H} + \int_0^t a(s, V(s), \phi) \, ds = 0.
$$

(15)
for all $\phi \in \mathcal{V}$. Uniqueness for equations (9) and (15) are equivalent, in the class $V \in L^2(0, T; \mathcal{V}) \cap C([0, T]; \mathcal{H})$. It is known, see [12], that uniqueness (and existence) in this class holds when $a$ is measurable in the three variables, continuous and coercive in the last two variables, namely

$$|a(t, f, g)| \leq C \|f\|_V \|g\|_V$$

(16)

$$a(t, f, f) \geq \nu \|f\|_V^2 - \lambda |f|^2_H$$

(17)

for some constants $C, \lambda \geq 0, \nu > 0$, for a.e. $t$ and all $f, g \in \mathcal{V}$. Let us prove these two properties. It is sufficient to check them on the subset of smooth compact support divergence free fields $f, g$.

Let us prove (16). The first four terms in the explicit expression for $a(t, f, g)$ can be bounded above by $C \|f\|_V \|g\|_V$ because $a_{ij}, \partial_i a_{ij}, b^{\alpha \beta i}, c^{\alpha \beta}$ are bounded. The difficult terms are the last two. Again, since $h$ is bounded, the terms

$$\sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} h^\alpha(t, x) \partial_\alpha f^{\beta}(x) \partial_\beta(x) \, dx + \sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} h^\beta(t, x) \partial_\alpha \left( f^\alpha(x) g^\beta(x) \right) \, dx$$

can be bounded above by $C \|f\|_V \|g\|_V$. It remains to bound

$$\sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} v^\alpha(t, x) \partial_\alpha f^{\beta}(x) \partial_\beta(x) \, dx + \sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} v^\beta(t, x) \partial_\alpha \left( f^\alpha(x) g^\beta(x) \right) \, dx.$$

But here we use repeatedly the first claim of Lemma 5 and bound also these terms with $C \|f\|_V \|g\|_V$. We have proved (16).

Finally, let us show property (17). From Proposition 2 the part of $a(t, f, f)$ related to

$$- \int_{\mathbb{R}^3} \mathcal{L} f(x) \cdot f(x) \, dx$$

is bounded below by

$$\nu \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, dx - C \int_{\mathbb{R}^3} |f(x)|^2 \, dx.$$

The remaining terms, namely

$$\sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} (v^\alpha(t, x) + h^\alpha(t, x)) \partial_\alpha f^{\beta}(x) f^{\beta}(x) \, dx$$

(18)

$$+ \sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} (v^\beta(t, x) + h^\beta(t, x)) \partial_\alpha \left( f^\alpha(x) f^{\beta}(x) \right) \, dx$$

(19)

are bounded above in absolute value by

$$\frac{\nu}{2} \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, dx + C \int_{\mathbb{R}^3} |f(x)|^2 \, dx$$

20
because of Lemma 5 with a suitable choice of $\epsilon > 0$. This implies $a(t, f, f) \geq \frac{\epsilon}{2} \|f\|_V^2 - C |f|_{H_2}$.

**Step 2. Conclusion.** Until now we have proved that, for every $f \in L^2([0, T], \mathbb{R}^n) \cap L^\infty([0, T], \mathbb{R}^n)$, the function $(t, x) \mapsto \mathbb{E}[B(t, x)e_f(t)]$ is the zero element of the space $L^2(0, T; V) \cap C([0, T]; \mathcal{H})$. We have to deduce that $B = 0$.

Being $(t, x) \mapsto \mathbb{E}[B(t, x)e_f(t)]$ the zero element of $C([0, T]; \mathcal{H})$, we know that for every $t \in [0, T]$ we have

$$
\int_{\mathbb{R}^3} \mathbb{E}[B(t, x)e_f(t)]g(x) \, dx = 0
$$

for all $g \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$; and this holds true for all $e_f \in \mathcal{D}$. By linearity of the integral and the expected value we also have that

$$
\int_{\mathbb{R}^3} \mathbb{E}[B(t, x)Y]g(x) \, dx = 0 \quad (20)
$$

for every random variable $Y$ which can be written as a linear combination of a finite number of $e_f(t)$ and by density also the restriction $f \in L^\infty([0, T], \mathbb{R}^n)$ can be removed. Since by Lemma 4 the span generated by $e_f(t)$ is dense in $L^2(\Omega, \mathcal{G}_t)$, (20) holds for any $Y \in L^2(\Omega, \mathcal{G}_t)$. Namely, we have

$$
\mathbb{E} \left[ \int_{\mathbb{R}^3} B(t, x)g(x) \, dx \right] Y = 0
$$

for every $Y \in L^2(\Omega, \mathcal{G}_t)$. Since, by assumption, $\int_{\mathbb{R}^3} B(t, x)g(x) \, dx$ is $\mathcal{G}_t$-measurable, we deduce

$$
\int_{\mathbb{R}^3} B(t, x)g(x) \, dx = 0.
$$

This holds true for every $g \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$, hence $B(t, \cdot) = 0$. □

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