DERIVED EQUIVALENCES OF CALABI-YAU FIBRATIONS AND MIRROR SYMMETRY

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Abstract. Given a CY fibration in the sense of Bridgeland-Maciocia, fibered by abelian varieties or $K3$ surfaces, we describe their categorical derived equivalent fibrations. We study its consequences from the point of view of Mirror symmetry relating the result with SYZ mirror conjecture.

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1. INTRODUCTION

One of the main motivations to the study of algebraic varieties via their categories of coherent sheaves is its relation with string theory. The conformal field theory associated to a variety in string theory contains a lot of information packaged in a categorical way rather than in directly geometric terms. A second motivation to study varieties via their sheaves is that this approach is expected to generalize more easily to non-commutative varieties and categorical methods enable one to obtain a truer description of certain varieties than current geometric
techniques allow. For example many equivalences relating the derived categories of pairs of varieties are now known to exist.

Let $X$ be a smooth complex projective variety over a field $k$ with structure sheaf $\mathcal{O}_X$ and $D^b(X)$ the bounded derived category of coherent sheaves. It is an interesting question how much information about $X$ is contained in $D^b(X)$. Certain invariants of $X$ can be shown to depend only on $D^b(X)$. Because of the uniqueness of Serre functors, the dimension of $X$ and whether it is Calabi-Yau or not, can be read off from $D^b(X)$.

A theorem of Bondal and Orlov [BO] says that if the canonical sheaf $K_X$ or its inverse is ample, $X$ can be absolutely reconstructed from $D^b(X)$.

A Calabi-Yau threefold is a smooth compact connected threefold with vanishing first Betti number and trivial canonical class:

$$3 \bigwedge \Omega_X \cong K_X \cong \mathcal{O}_X.$$

Let $X, X'$ be two Calabi-Yau three-folds such that $D^b(X) \cong D^b(X')$ then certain invariants of $X$ and $X'$ should coincide, for example, the Hodge numbers of the two spaces often coincide. In general it is an interesting problem to know which invariants are preserved under equivalence.

**Conjecture.** Let $X, Y$ be two CY-3 folds fibred over the same base $B$ by elliptic curves, $K3$ surfaces or abelian varieties such that the derived categories of the fibres are equivalent $D^b(X_b) \cong D^b(Y_b) \ \forall b \in B$, then $D^b(X) \cong D^b(Y)$.

According to [HLS], the conjecture is true if there exists certain homological finite complex over the fibre product of $X$ and $Y$. We prove that the conjecture is true under certain conditions.

All the schemes considered in this paper are of finite type over an algebraically closed field and all the sheaves are coherent.

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2. **Calabi-Yau fibrations**

We are interested in derived categories of Calabi-Yau manifolds, and in particular of CY-fibrations (in connection with Mirror symmetry).
Let $X, Y$ be two smooth complex projective varieties. The Fourier-Mukai tranform $FMT$ by an object $P \in D^b(X \times Y)$ is the exact functor:

$$
\phi^P : D^b(X) \rightarrow D^b(Y)
$$

where $\pi_1$ and $\pi_2$ are the projections maps over the first and second component respectively.

If $\phi_P$ is an equivalence, then $P$ must satisfy a partial Calabi-Yau condition:

(2) \[ P \otimes \pi_1^* K_X \otimes \pi_2^* K_Y^{-1} \cong P. \]

The set of all objects $P$ satisfying condition (2) is the subcategory of invertible elements in $D^b(X \times Y)$ or the subcategory of elements inducing an equivalence of derived categories.

**Definition 2.1.** Two Calabi-Yau manifolds $X$ and $Y$ are derived equivalent if there exists a Fourier-Mukai equivalence $\phi_P : D^b(X) \rightarrow D^b(Y)$ defined by an object $P \in D^b(X \times Y)$ satisfying a Calabi-Yau condition.

In particular $X$ and $Y$ are non-singular projective varieties of the same dimension, and $P$ is a sheaf on $Y \times X$, flat over $Y$ satisfying that $P_y \otimes K_X = P_y$ for every point $y \in Y$, and for any pair of distinct points $y_1, y_2$ of $Y$, and any integer $i$, $Ext^i_x(P_{y_1}, P_{y_2}) = 0$, where $K_X$ is the canonical bundle of $X$.

We adopt the definition of Calabi-Yau fibration of Bridgeland and Maciocia in [BM].

**Definition 2.2.** Let $\pi : X \rightarrow S$ be a Calabi-Yau fibration, that is, a connected morphism of non-singular projective varieties whose general fibre has trivial canonical bundle and such that $K_C \cdot C = 0$ for any curve $C$ contained in a fibre of $\pi$. Suppose that $\pi$ has relative dimension one or two, so that the generic fiber is an abelian surface, a $K3$ surface or an elliptic curve.

Since the total space and the base are projective varieties the fibration morphism is automatically proper. We will assume in addition that the morphism is equidimensional and consequently it is flat.

**Conjecture 2.3.** Let $X, Y$ be two Calabi-Yau 3-folds fibred over the same base $B$ by elliptic curves, $K3$ surfaces or abelian surfaces such that the derived categories of the fibres are equivalent $D^b(X_b) \cong D^b(Y_b)$ for all $b \in B$ then $D^b(X) \cong D^b(Y)$. 
A relative integral functor is an equivalence if and only if its restriction to every fibre is an equivalence. As a consequence of Theorem 3.3.6 of [HLS] it follows the following result:

**Proposition 2.4.** If there exists an object $K \in D^b(X \times_S Y)$ of finite homological dimension over both $X$ and $Y$, then the integral functor $\phi^X_\ast : D^b(X) \to D^b(Y)$ is an equivalence if and only if $\phi_s : D^b(X_s) \to D^b(Y_s)$ is an equivalence for every closed point $s \in S$.

In general the problem of finding a universal object $K$ is closely related with the problem of finding fine components of the relative moduli space $\mathcal{M}^b(X/S)$ of stable sheaves on the fibration of the right dimension, (see Theorem 1.2 of [BM]).

Let $\pi : X \to S$ be a Calabi-Yau 3-fold fibered by K3 surfaces, and $Y$ is a component of the relative moduli space $\mathcal{M}^b(X/S)$ of stable sheaves on the fibration $\pi$. Points of $Y$ represents stable sheaves supported on the fibres of $\pi$, and there is a natural map $\hat{\pi} : Y \to S$, sending a sheaf supported on $\pi^{-1}(s)$ to the corresponding point $s \in S$. If $Y$ is of the same dimension that $X$ and it is fine, there is a universal sheaf on $Y \times X$, then $Y$ is another non-singular projective variety and the morphism $\hat{\pi} : Y \to S$ is another Calabi-Yau fibration (Mukai dual of $\pi$). The universal sheaf on $Y \times X$ is taken as kernel to construct the integral functor $\phi : D^b(X) \to D^b(Y)$.

In general, if we are not considering Calabi Yau’s fibered, the most general result for derived categories of Calabi Yau’s is due to T. Bridgeland.

**Theorem 2.5.** (Bridgeland) If $X$ is a projective threefold with terminal singularities and $Y_1 \xrightarrow{f_1} X_1$, $Y_2 \xrightarrow{f_2} X_2$ are crepant resolutions, then there is an equivalence of derived categories of coherent sheaves $D(Y_1) \to D(Y_2)$.

In particular two birational Calabi-Yau threefolds have equivalent derived categories, and therefore they have the same Hodge numbers. But not all Calabi-Yau threefold’s that are derived equivalent are birational.

2.1. Abelian fibrations. Let $\Gamma \cong \mathbb{Z}^{2d}$ be a lattice in a real vector space $U$ of dimension $2d$ or complex vector space of dimension $d$, and let $\Gamma^\ast \subset U^\ast$ be the dual lattice. The complex torus $(U/\Gamma)$ is an abelian variety $A$ of dimension $d$ over $\mathbb{Z}$ if it is algebraic, that is, it has embedding to the projective space. The dual abelian variety $\hat{A}$ is given by the dual torus $(U^\ast/\Gamma^\ast)$. There is a unique line bundle $P$ on the product $A \times \hat{A}$ such that for any point $\alpha \in \hat{A}$ the restriction $P_\alpha$ on
Given a family of abelian varieties over the unit disc $\Delta \subset \mathbb{C} = \{ t \in \mathbb{C} | |t| < 1 \}$, such as the fiber consists of the product of an abelian variety with its dual $A \times A^\vee$, there is a natural polarization defined, by considering the product $\pi_1^* \mathcal{P}_A \otimes \pi_2^* \mathcal{P}_{\hat{A}}$, where $\mathcal{P}_A$ and $\mathcal{P}_{\hat{A}}$ are the respective Poincaré bundles over $A \times \hat{A}$ and $\hat{A} \times A$. Due to a result of Deligne (see Theorem 1.11 of [Del]), one can extend this polarization to the all family. It is of special interest when the family has the structure of a scheme, these are the abelian schemes that can be seen as schemes is groups.

**Definition 2.6.** Let $S$ be a noetherian scheme. A group scheme $\pi : X \to S$ is called an abelian scheme if $\pi$ is smooth and proper, has a global section and the geometric fibers of $\pi$ are connected.

For abelian schemes Mukai has introduced the Fourier-Mukai transform in the early 80s as a duality among sheaves on abelian varieties. In general we will allow singular fibers.

Let us start with an abelian fibration $\pi : X \to B$ which is Calabi-Yau, that is, a projective smooth morphism whose fibers are abelian varieties admitting a global polarization, that is, a very ample line bundle $\mathcal{L}$ on it, in particular this means that there is a global embedding $(X/B) \hookrightarrow \mathbb{P}^N$. We denote by $X_b$, the fiber of $\pi$ over $b \in B$.

We are interested in defining a dual fibration $X^\vee/B$ in such a way that over the smooth locus, the fibers correspond to the dual abelian varieties of the original fibration, that is, if $\mathcal{P}$ is the Poincaré sheaf, then $\forall s \in S$, $\mathcal{P}_b$ is the Poincaré bundle over $X_b \times X_b^\vee$. The corresponding derived equivalence of the fibres $X_b$ and $X_b^\vee$ is given by the Poincaré bundle over the product $X_b \times X_b^\vee$. Therefore the dual fibration $\hat{\pi} : X^\vee \to B$ admits also a global polarization. We assume that the singular fibres are irreducible, reduced and have no multiplicity. We note that this is the generic singularity, all the other types of singularities that can appear are degenerations of this one.

**Lemma 2.7.** Given $X$ and $Y$ abelian varieties, such that $D^b(X) \cong D^b(X^\vee)$ and $Y \cong Y^\vee$, then $D^b(X \times Y) \cong D^b(X^\vee \times Y^\vee)$.

**Proof.** Since $D^b(X) \cong D^b(X^\vee)$, there is an isometric isomorphism between $X$ and $X^\vee$. Now by assumption $Y \cong Y^\vee$ and then there is also an isometric isomorphism between $X \times X^\vee$ and $Y \times Y^\vee$ and therefore by Theorem 2.19 of [Or2], there is an equivalence $D^b(X) \cong D^b(Y)$ and consequently $D^b(X \times Y) \cong D^b(X^\vee \times Y^\vee)$. \qed
Proposition 2.8. The derived categories of the two fibrations $\pi : X \to B$ and $\hat{\pi} : X^\vee \to B$ are fiberwise equivalent if and only if the derived categories of the two fibrations are equivalent.

Proof. First we fix our attention on the smooth locus. Let $\Sigma(\pi) \hookrightarrow B$ be the discriminant locus of $\pi$, that is, the closed subvariety in the parameter space $B$ corresponding to the singular fibers.

\[ X^\vee \supset X^\vee - \pi^{-1}(\Sigma(\pi)) \hookrightarrow \mathbb{P}^N \]

(3)

We can take then the Zariski closure of $X^\vee - \pi^{-1}(\Sigma(\pi))$ in $\mathbb{P}^N$. For each $b \in B - \Sigma(\pi)$, the corresponding derived equivalence of the fibers $X_b$ and $X_b^\vee$ is given by the Poincare bundle $P_b$ over the product $X_b \times X_b^\vee$. Moreover, its first Chern class $c_1(P_b)$ lives in $H^{1,1}(X_b \times X_b^\vee, \mathbb{Z}) \cap H^2(X_b \times X_b^\vee, \mathbb{Z})$. The monodromy group is defined by the action of the fundamental group of the complement of the discriminant locus $\pi_1(B - \Sigma(\pi))$ on the cohomology $H^*(X_b \times X_b^\vee, \mathbb{Z})$ of a fixed non-singular fiber, and since the class of the polarization is invariant by the monodromy, by Deligne theorem we can extend the class of the Poincare bundle to the non-singular fibers, it is the relative Poincare sheaf of the fibred product of the two families over the base $B$ and we will call it $E$. Then the Fourier-Mukai transform given by $E$ is an equivalence over the smooth locus.

We now need to deal with the singular fibers. The singular fibre is a degeneration of an abelian variety $X_n$ of the form $((\mathbb{C}^*)^k \times A_k$ where $A_k$ is an abelian variety of dimension $k$. In this case, since $((\mathbb{C}^*)^k$ is a Stein manifold, it follows by Cartan’s Theorem B that if $F$ is a coherent sheaf on $((\mathbb{C}^*)^k$ then $H^i((\mathbb{C}^*)^k, F) = 0$ for $i > 0$. Then $E|_{((\mathbb{C}^*)^k} \cong \mathcal{O}_{(\mathbb{C}^*)^k}$ and then by Lemma [2,7] and applying the Fourier-Mukai transform of kernel $E$, it follows the fiberwise equivalence in this case and therefore by [HLS], the equivalence of the two dual fibrations. \end{proof}

Now, let $\rho : A \to B$ be an abelian fibration, admitting a global polarization and we don’t make further assumptions on the fibers.

Definition 2.9. We define the dual fibration $\hat{\rho} : \hat{A} \to B$ as the moduli stack representing the relative Poincaré sheaf $E$.

Theorem 2.10. The integral functor $\phi^A_{\mathcal{E}} : D^b(A) \to D^b(\hat{A})$ is an equivalence if and only if $\phi_b : D^b(A_b) \to D^b(\hat{A}_b)$ is an equivalence for every closed point $b \in B$. 


Proof. Let $\sigma : B' \to B$ be a covering of the base, that is, locally (around a point $y \in B'$ and $x = \sigma(y)$, $\sigma$ is simply the function $\{z \in \mathbb{C} : |z| < 1\} \to \{z \in \mathbb{C} : |z| < 1\}$ given by $z \to z^k$, where $k$ is the multiplicity of $\sigma$ at $y$). Moreover, we can assume that $\sigma : B' \to B$ is a Galois covering with finite Galois group $G$, just we observe that any normal extension of fields admits a Galois extension. If $\mathcal{M}(B), \mathcal{M}(B')$ are the fields of meromorphic functions on $B$ and $B'$ respectively, $\sigma^* : \mathcal{M}(B) \to \mathcal{M}(B')$ is a Galois field extension of degree $k$, with Galois group $G$ (acting by pull-back on $\mathcal{M}(B')$).

Now we take the fibred product of $A$ and $B'$ over the base $B$, that is:

$$A' = A \times_B B' \to A \quad \text{and} \quad \pi : B' \to B$$

The advantage now is that the fibration $A' \to B'$ admits a section $s$ given by the identity in $B'$, and therefore there is a relative Poincaré sheaf $\mathcal{E}$ as in the proof of Proposition 3.5, that restricted on smooth fibers $A'_b \times \hat{A}'_b$, where $\hat{A}'_b$ is the corresponding dual abelian variety, is just the Poincare bundle. We are taking as dual fibration of $A'/B'$, the relative moduli space. If $\mathcal{J}$ is the relative moduli functor, of semistable sheaves of the fibers containing line bundles of degree 0 on smooth fibres, $\mathcal{J} : \text{VAR} \to \text{SETS}$. Over the smooth locus, $\mathcal{J}$ is represented by the relative Jacobian $\text{Pic}^0(A'/B')$, which is the dual fibration $\hat{A}'/B'$.

Due to the presence of singular fibers, the corresponding coarse moduli space is not a fine moduli space, but the stack maybe a FM partner, using the relative Poincaré sheaf $\mathcal{E}$ as kernel of the FM transform, that is, the relative Poincaré sheaf $\mathcal{E}$ is the family representing an element of $\mathcal{J}(\text{Pic}^0(A'/B'))$ such that for each variety $S$ and each $\mathcal{F} \in \mathcal{J}(S)$ there exists a unique morphism $f : S \to M$ satisfying that $\mathcal{F} \cong f^*\mathcal{E}$. Therefore $\mathcal{E}$ induces a natural transformation $\Phi : \mathcal{J} \to \text{Hom}(-, \hat{A}'/B')$ giving a stack structure $((\hat{A}'/B'), \Phi)$. It is universal in the sense that for every other variety $N$ and every natural transformation

$$\psi : \text{Hom}(-, N) \to \text{Hom}(-, M),$$

the following diagram commutes:
It follows by [HLS] that there is an equivalence of categories
\[ D^b(A'/B') \cong D^b(\hat{A}'/B'), \]
where \( \hat{\rho} : \hat{A}' \to B' \) is the dual abelian fibration. Now the Galois group \( G \) acts on bundles on the fibres, and there is an equivalence between the respective invariant subcategories \( (D^b(A'/B'))^G \cong (D^b(\hat{A}'/B'))^G \), therefore by the fundamental theorem of Galois theory, there is an equivalence between the original categories \( D^b(A/B) \cong D^b(\hat{A}/B) \). □

2.1.1. \textit{The twisted fibration.} More generally, let us consider now the twisted fibration obtained by considering for each \( b \in B \) an automorphism \( f_b \) of the fiber \( X_b \), that is, a biholomorphic map of the abelian variety which preserves the polarization class. Then we get a derived equivalent abelian variety \( Y_b \) (not necessarily the dual one).

\textbf{Proposition 2.11.} \textit{The abelian variety} \( Y_b \) \textit{image of} \( X_b \) \textit{under automorphism} \( f_b \) \textit{is derived equivalent to the original} \( X_b \).

\textit{Proof.} First, observe that the corresponding dual varieties \( \hat{X}_b \) and \( \hat{Y}_b \) as defined in [2.1] are isomorphic, therefore, there is an isomorphism which is an isometry
\[ \mathcal{J} : X_b \times \hat{X}_b \simeq Y_b \times \hat{Y}_b, \]
where \( \hat{X}_b \) and \( \hat{Y}_b \) are the corresponding dual varieties. Therefore, due to Theorem 2.19 of Orlov ([Or2]), it follows the result. □

By continuity, for every \( t \in B \) there is an open neighborhood \( t \in U \subset B \) where for all \( b \in U \), \( f \) defines an isomorphism \( f_U : X_U \times \hat{X}_U \simeq Y_U \times \hat{Y}_U \). If \( g \) is another isomorphism on another open set \( V \subset B \), there is a compatibility condition \( f_U \circ g_V^{-1} = g_V \circ f_U^{-1} \) on the intersection \( U \cap V \). The fibration \( \tilde{\pi} : Y \to B \) we get in this way, is a twisted fibration of the original fibration \( \pi : X \to B \).

\textbf{Lemma 2.12.} \textit{Given an abelian variety} \( A \) \textit{whose underlying complex tori is defined by a lattice of rank} \( d \), \textit{the group of automorphisms of the lattice is the linear group} \( PSL(d, \mathbb{Z}) \).

\textit{Proof.} It follows easily that if we apply a linear map \( T \in PSL(d, \mathbb{Z}) \) to the lattice that defines the abelian variety \( A \), the corresponding
abelian variety is isomorphic to the original one, and therefore \( T \) is an automorphism of the lattice. Reciprocally, if two abelian varieties are isomorphic, the corresponding lattices are related by linear transformation up to a scalar, and therefore by an element in \( PSL(d, \mathbb{Z}) \).

The abelian variety \( A \) has the structure of an algebraic group finitely generated as an abelian group. Let \( e \) be the identity point of this group. It can be checked that any endomorphism of \( A \) that sends the point \( e \) to itself is an endomorphism of the algebraic group. Such endomorphisms form a ring which contains \( \mathbb{Z} \) as a subring and for a generic abelian variety coincides with it. However the ring of \( e \)-preserving endomorphisms of \( A \) can be bigger than \( \mathbb{Z} \), for example in the case of complex multiplication. Then the ring of automorphisms of the lattice tensored by the rational numbers contains a field of degree \( 2d \) which is a quadratic extension over a totally real field. Then the group of automorphisms coincides with the units of the ring of integers.

For example if the abelian variety is \( A = E_\tau \) with \( E_\tau \) an elliptic curve with complex multiplication, that is, \( \tau \) is a root of a quadratic polynomial with integral coefficients.

**Proposition 2.13.** The derived categories of the two fibrations \( \pi : X \to B \) and \( \tilde{\pi} : Y \to B \) are fiberwise equivalent if and only if the derived categories of the two fibrations are equivalent.

**Proof.** If the fibrations are derived equivalent from the definition of \( Y/B \), it follows that there is fiberwise equivalence.

Now consider for every \( t \in B - \Sigma(p) \), the product \( X_t \times Y_t \) of the corresponding abelian variety with its derived equivalent given by an automorphism \( f \). Consider a sheaf \( \mathcal{L} \in D^b(X \times Y) \) such that for every \( \beta \in Y_t \), \( \mathcal{L}|_{\beta} \) on \( X_t \times \{ \beta \} \) is an element of \( Pic^0(X_t) \), that is, if \( \mathcal{J} \) is the relative moduli functor, \( \mathcal{L} \in \mathcal{J}(Pic^0(X/B)) \).

The family \( \{ \mathcal{L}_t : t \in B \} \) over the non singular locus satisfies a Calabi-Yau condition, since the generic fiber has trivial canonical bundle, and since the total fibration is Calabi-Yau, the extended object \( \overline{\mathcal{L}} \) over the non singular locus is invertible and determines a Fourier-Mukai transform which depends on \( f \). In particular we can take as dual of \( X_t \) the same fibration, in this case \( f \in Aut(X/B) \) and \( \mathcal{L} = \Gamma_f, \mathcal{L} \) where \( \Gamma_f \) is the graph of the isomorphism \( f \). Then the functor

\[
\phi^F : D(X) \to D(Y) \\
F \mapsto f_*(F \otimes \pi_1^*(\overline{\mathcal{L}}))
\]

defines a FM transform. \(\square\)
Remark 2.14. Observe that the case of the dual fibration corresponds to take the identity as $f$, and in this case $\mathcal{P}$ is the Poincaré sheaf and satisfies a universal property. For each variety $S$ and each $\mathcal{L} \in \mathcal{J}(S)$ there exists a unique morphism $f : S \to \text{Pic}^0(X/B)$ satisfying $\mathcal{L} \cong f^*(\mathcal{P})$.

As a consequence of the theorem 2.10 and proposition 2.13, given two Calabi-Yau’s 3-folds $X, Y$ fibred by abelian surfaces over the same base $S$, such that the derived categories of the fibres are equivalent,

$$D^b(X_s) \cong D^b(Y_s) \; \forall s \in S, \; D^b(X) \cong D^b(Y).$$

A more general problem, is that of Calabi-Yau’s fibered by complex tori (not necessarily algebraic), and this is also interesting from the point of view of mirror symmetry, and it is according to SYZ mirror prediction.

R. Donagi and T. Pantev in [DP] start with an elliptic fibration (admitting a section) and this has many associated genus one fibrations without sections codified by the Tate-Shafarevich group (twisted versions of the original fibration coming from replacing the derived category of sheaves on a space with the derived category of sheaves on a gerbe over the space). The twist of a fibration $X/B$ is given by an automorphism $f \in \text{Aut}(X/B)$, and the respective relative Jacobians are isomorphic $\text{Pic}^0(X/B) \cong \text{Pic}^0(X_f/B)$. Let $X$ be an elliptic fibration and let $\alpha$ be a class in its Tate-Shafarevich group. Under rather strong assumptions on $X$ (including smoothness and integrality of the fibers) they are able to understand the relation between the Tate-Shafarevich group of $X$ and the Brauer group of an elliptic fibration $X_{\alpha}$ obtained from $X$ by twisting by the class $\alpha$. Given another class $\beta$ in the Tate-Shafarevich group (compatible with $\alpha$ in the sense specified in [DP]), it is possible to construct a gerbe over $X_{\alpha}$ corresponding to the image class of $\beta$ in $\text{Br}(X_{\alpha})$, denoted by $\beta X_{\alpha}$. Interchanging the role of $\alpha$ and $\beta$ one gets another gerbe $\alpha X_{\beta}$ over a fibration $X_{\beta}$ locally isomorphic to $X_{\alpha}$. They conjecture that for $\pi : X \to B$ an elliptic fibration with $X$ and $B$ smooth, $\alpha$ and $\beta$ a pair of compatible classes (in the sense of [DP]) there is an equivalence of derived categories of coherent sheaves of weights 1 and -1

$$D^b(\beta X_{\alpha}, 1) \sim D^b(\beta X_{\alpha}, -1).$$

However, in dimension greater than 2 they can only prove the conjecture for smooth fibrations (smooth fibers).

Remark 2.15. Due to Theorem 5.11 of [Bas], if $\mathfrak{X}$ is a gerbe on a twisted version of $X$, and $\mathfrak{Y}$ the dual gerbe to $\mathfrak{X}$, then there is an
equivalence of categories,

\[ D^b_c(\mathcal{Y}, -1) \cong D^b_c(\mathcal{X}, -1). \]

SYZ mirror conjecture. (A. Stromminge, S.T. Yau, E. Zaslov). Mirror dual Calabi-Yau manifolds should be fibred over the same base in such a way that generic fibers are dual tori and each fiber of any of these two fibrations is a Lagrangian submanifold. In particular, each of these fibrations admits a canonical section that is an \(n\)-cycle having intersection number 1 with the fiber cycle.

We can take as a base, the moduli space of flat \(SU(n)\)-connections

\[ S = \text{Hom}(\pi_1(\Sigma), SU(n))/SU(n), \]

as in the geometric Langland program, and then, the Lagland dual of a torus \(F_p\) over a connection \(v \in S\) correspond to the dual torus \(F_p^\vee\) sitting on the dual connection, that is, the mirror brane of a torus is the dual torus, and therefore mirror symmetry in this case is T-duality.

3. \(K3\) fibrations and Mirror symmetry.

Mirror symmetry is part of a new branch String Theory of Theoretical Physics. String Theory is the first theory which has no mathematical contradictions and which unify the four main forces, gravity, weak and strong interaction and electromagnetism. It is based on conformal field theories. In string theory it was suggested that we live in 10 dimensional world which is a product of three dimensional real world, time and CY manifolds of complex dimension three. String Theory predicted that to each CY manifold one can find a mirror CY such that both of them describe the same physics. This prediction had a deep impact on Algebraic Geometry. There are two main mathematical conjectures in Mirror Symmetry, Kontsevich homological mirror symmetry conjecture and the conjecture of Strominger, Yau and Zaslow, which predicts the structure of the CY manifolds and how to get the mirror of a given CY manifold.

From a topological point of view, complexes of coherent sheaves are D-branes of the B-model (twist of a \(N = 2\) SCFT), and morphisms between the objects of \(D^b(X)\) are identified with the states of the topological string and composition of morphisms is computed by correlators of the B-model. Originally, a \(D\)-brane in string theory is by definition an embedding of a manifold, variety or cycle in the space-time that serves as a boundary condition for open-strings moving in the space-time.

Our main aim in this context is to obtain interpretations and geometric criteria for the equivalence between the derived categories of
Calabi-Yau manifolds, in particular in the case of Calabi-Yau $n$-folds fibered by abelian varieties or $K3$ surfaces.

**Definition 3.1.** A $K3$ surface is a compact complex projective 2-dimensional smooth variety with trivial canonical bundle and such that its first Betti number $b_1 = 0$ vanishes.

Let $X$ be a $K3$ surface simply connected, its first homology group $H_2(X, \mathbb{Z})$ is a torsion free group of rank 22. Let $e \in H^1(X \otimes \mathbb{R}) \cap H^2(X, \mathbb{Z})$ be the class of an ample divisor. Then $(X, e)$ is a polarized $K3$ surface. The degree of the polarization is an integer $2d$, such that the scalar product $<e, e> = 2d = 2rs$ where $d, r, s$ are any positive integers and their greatest common divisor $(r, s)$ is 1.

Consider the fine moduli space $M(r, e, s)$ parametrizing $e$-stable sheaves $E$ on $X$ such that $rk(E) = r$, $c_1(E) = l$ and $\chi(E) = r + s$. Due to a result of Mukai (see [Muk]), if $\hat{X} := M(r, e, s)$ is non-empty, it is a $K3$ surface as well, and is the dual surface to $X$.

**Theorem 3.2.** (Derived Torelli) The following conditions are equivalent:

1. Mukai’s duality is an involution $\hat{X} = X$.
2. There exists an equivalence $D^b(X) \cong D^b(\hat{X})$ induced by the universal family on $X \times \hat{X}$.
3. $\hat{X}$ has a polarization $e'$, such that $<e', e'> = 2d$.
4. There exists a Hodge isometry $f : \hat{H}(X, \mathbb{Z}) \to \hat{H}(\hat{X}, \mathbb{Z})$.

Let us now assume that the base is of dimension 1, even more let $B$ be an algebraic curve $C$ of genus $g$ and let $\pi : Y \to C$ be a three dimensional projective non-singular variety such that for each $t \in C$, $\pi^{-1}(t) = X_t$ is a $K3$ surface. We must have at least 3 singular fibers. From Hironaka’s theorem on the resolution of singularities, we may assume that the singular fibers are normal crossing divisors.

Suppose that on $Y$ we have a polarization class $H$ such that restricted to the fiber $H|_{X_t} = e$. Let $m$ be the number of points on $C$ for which the local monodromy operator is of infinite order. Then the number of singular fibers of $\pi$ is less or equal to $2g - 2 + m$.

Since every $K3$ surface is Kähler by Theorem 2 of [To], the Mumford semi-stable reduction theorem applies in this case. So we may assume that the fibers of the map $\pi$ are given locally by $z_1^{k_1} \cdots z_n^{k_n} = t$, where $k_j$ is either 0 or 1.

If $X$ is an elliptic fibration with integral fibers, then the relative Picard functor is representable and the compactified relative Jacobian $Y$ is derived equivalent to the original one.
3.1. **The dual fibration.** Now, we want to construct, given a fibration $p : X \to B$ with a polarization, its dual fibration, that should play the role of mirror fibration.

We can assume that the fibration is Calabi-Yau. Since the singular fibers are normal crossing divisors, by the theorem of U. Person and H. Pinkham [PP], there exists a birational map $\varphi : X \to X'$ where $X'$ has trivial canonical bundle and it is an isomorphism over the smooth locus such that the following diagram is commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
B & & B
\end{array}
$$

The idea is to replace each fiber by its derived categorically equivalent one. The problem is that singular fibers can appear for such a fibration. A natural idea is as in the case of abelian fibrations, to contruct the dual fibration away from the singularities, and then trying to extend it over the singular locus.

**Theorem 3.3.** Given a $K3$ non singular fibration $p : X \to B$ with a fixed polarization $l$, there exists a dual fibration which is derived equivalent to the original one and corresponds to a projective component of the relative moduli space $\mathcal{M}^l(X/B)$.

**Proof.** Let $\Sigma(p) \hookrightarrow B$ be the discriminant locus of $p$, that is, the closed subvariety in the parameter space $B$ corresponding to the singular fibers.

Consider for every $t \in B - \Sigma(p)$, the product $X_t \times \check{X}_t$ of the corresponding $K3$ surface with its Mukai dual. In order to have such duality we assume there exists a torsion free semistable bundle on $X_t$. This condition satisfies, for example, according to Morrison [Mo], if the rank of the Neron Severi group $NS(X_t)$ is bigger than 12. Then we consider the universal family $\mathcal{P}_t$ over the product $X_t \times \check{X}_t$. Proceeding as in Proposition 2.7 of section 2.1, extending the family $\mathcal{P} := \{\mathcal{P}_t : t \in B\}$ over the non singular locus by Deligne theorem, we get a dual fibration $(X/B)^\vee$ to the original one. The fibration constructed thus far, is a connected component of the relative moduli space $\mathcal{M}^l(X/B)$ of stable sheaves on $p$ with respect to the polarization, (Prop. 3.4. of [BM]). By Mumford semi-stable reduction theorem, the singular fibers have sufficiently high codimension, and then by Bridgeland-Maciocia theorem on removability of singularities (see [BM]), both fibrations are derived equivalent, and the equivalence is given by the extended universal object $\mathcal{P}$.

$\square$
Corollary 3.4. There exists at least one fine component of the relative moduli space or equivalently a sheaf $E$ on a non singular fiber with fixed Mukai vector.

A closed point of a relative moduli space corresponds to a sheaf $E$ on a fibre (not to a sheaf on the whole fibration). Let $X_s$ be a $K3$ surface or an abelian surface. The tangent space at that point to the moduli space of sheaves $M(X/S)$ on the fibration, can be identified with

$$T_M(E) \cong Ext^1_S(E, E).$$

If $Ext^2_S(E, E) = 0$, then $M$ is smooth at $E$. There are bounds (Corollary 4.5.2 of [HL]),

$$\text{ext}^1(E, E) \geq \dim E \geq \text{ext}^1(E, E) - \text{ext}^2(E, E).$$

In general to construct such components $Y$ of the relative moduli space, we assume that there exists a divisor $L$ on $X$ and integer numbers $r, s > 0$, such that there exists a sheaf $E$ on a non singular fiber $X_t$ which is stable with respect to $H_t$ and $s = ch_2(E) + r$. The component $Y(E)$ containing the class of the sheaf $E$ is a fine projective moduli space and the fibration $q : Y \rightarrow B$ is equidimensional. Thus there is a universal family on the product $Y \times Y(E)$ that gives the equivalence of the derived categories of both fibrations over $B$.

3.1.1. Categorical equivalences of the $K3$ fibration and Mirror symmetry. Now we are going to proceed in the inverse way, by considering components $Y$ of the relative moduli space $M^e(X/B)$ of stable sheaves of fixed numerical invariants $r, s$ (hence fixed Hilbert polynomial) on the fibers of $p$ which are stable with respect to a fixed polarization $e$. In order to avoid parametrizations, we can look at the component of the relative moduli space of sheaves containing line bundles in degree 0.

Proposition 3.5. Every fine projective component $Y$ of the relative moduli space $M^e(X/B)$ of stable sheaves with respect to a fixed polarization $e$ is derived equivalent to the original Calabi-Yau fibration $(X/B)$ and therefore are derived equivalent between them. Reciprocally, any projective variety derived equivalent to the original fibration is a component of the relative moduli space.

Proof. By Corollary 3.4 we can consider components $Y$ of the relative moduli space $M^e(X/B)$ of stable sheaves on the fibers of the CY-fibration $(X/B)$, stable with respect to the polarization $e$. It is a fine moduli space, so there is a universal sheaf $P$ over the product
$X \times Y$. Bridgeland and Maciocia proved in [BM] that $Y$ is a nonsingular projective variety, $\hat{p}: Y \to B$ is a $K3$ fibration and the integral functor $\mathcal{D}(Y) \to \mathcal{D}(X)$ with kernel $\mathcal{P}$ is an equivalence of derived categories, that is, a Fourier-Mukai transform. It is Calabi-Yau because one has $\mathcal{D}(X) \cong \mathcal{D}(Y)$.

Now, we start with an equivalence $\mathcal{D}(Y) \cong \mathcal{D}(X)$, then by a result of Orlov [Or1], it is given by an object $\mathcal{E} \in \mathcal{D}(X \times_B Y)$ which satisfies a Calabi-Yau condition and thus by Theorem 3.3 this defines a fine component of the relative moduli space. All the equivalences of the original fibration are obtained in this way. □

Now the next question is what is the connection with the Mirror Symmetry program, that is, the component $Y$ can be interpreted as a mirror to the original fibration. This is related with the problem of all CY manifolds giving rise to the same derived category and for a description of all the equivalences of it. Let $\mathcal{C}$ be the set of all sheaves on non singular fibers giving rise to fine projective components of the relative moduli space. By Proposition 3.5 and Corollary 3.4 $\mathcal{C}$ is in bijection with the set of derived equivalences of the original fibration $X/B$.

Homological mirror symmetry states that there should be an equivalence of categories behind mirror duality, one category being the derived category of coherent sheaves on a Calabi-Yau manifold $X$ and the other one being the Fukaya category of the mirror manifold $X$. As a consequence, two Calabi-Yau manifolds that have the same mirror have equivalent derived categories of coherent sheaves.

For two sheaves $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{C}$, the corresponding components $Y(\mathcal{E})$ and $Y(\mathcal{F})$ of the relative moduli, have the same mirror, that is, the original fibration $X/B$, and this is according to homological mirror symmetry conjecture.

3.2. K3 fibred Calabi-Yau folds. Finally we give some specific examples of Calabi-Yau fibrations by $K3$ surfaces. One example of the $K3$ fibered Calabi-Yau threefold $X$ is obtained by resolving singularities of the degree 8 hypersurface $\tilde{X} \subset \mathbb{P}_{1,1,2,2,2}$.

The Kähler cone of $X$ is generated by positive linear combinations of the linear system $H = 2l + e$ and $l$ where $e$ is an exceptional divisor coming from blowing-up a curve of singularities and the linear system $l$ is a pencil of $K3$ surfaces.

3.3. Elliptically fibred Calabi-Yau 3-fold. For elliptic Calabi Yau’s, the problem is reduced to study derived categories of elliptic curves. Any complex torus can be expressed as the quotient of $\mathbb{C}$ by a lattice
of the form $\Lambda_\tau = \{ \mathbb{Z} + \mathbb{Z} \tau \}$ with $\tau \in \mathbb{H}$ an element of the upper-half plane. We denote the complex torus given by the lattice $\Lambda_\tau$ by $E_\tau$. The complex torus is called an elliptic curve if it is an algebraic curve, in other words if it has a distinguished point.

**Theorem 3.6.** (Torelli) For smooth elliptic curves $C, C'$

$$D^b(C) \cong D^b(C') \text{ iff } C \cong C'.$$

**Proposition 3.7.** Let $X/B$ and $Y/B$ be two fibrations over $B$ such that for all $b \in B$, the corresponding Teichmüller parameters $\tau, \tau'$ of $X_b$ and $Y_b$ are related by a linear transformation $A \in SL(2, \mathbb{Z})$, then the two fibrations are derived equivalent.

**Proof.** Two lattices $\Lambda_{\tau_1}$ and $\Lambda_{\tau_2}$ determine the same complex torus, when there exists a biholomorphic map from $E_{\tau_1}$ to $E_{\tau_2}$; $\Lambda_{\tau_1}$ and $\Lambda_{\tau_2}$ give holomorphic tori if and only if $\tau_1 = A \tau_2$ for some element $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2, \mathbb{Z})$ with its usual action on $\mathbb{H}$. Then the corresponding elliptic curves are isomorphic and by Torelli theorem they are derived equivalent. \hfill \Box

The second modulus of a Calabi-Yau space is the kähler class $[w] \in H^2(E, \mathbb{C})$ that we can parametrize with $t \in \mathbb{H}$ as $\int_E w = 2\pi i t$. Mirror symmetry for elliptic curves is simply the interchange of $\tau$ and $t$.

**The STU model.** If $X \to S$ is a $K3$ surface fibred over a curve $S$, the generic fibre is an elliptic curve and $S$ is a projective line. The fibred $K3$ surfaces of the STU model are themselves elliptically fibred. It is a particular non-singular projective CY–3-fold $X$ equipped with a fibration $X \to \mathbb{P}^1$, $X_\xi = \pi^{-1}(\xi)$. Except for 528 points $\xi \in \mathbb{P}^1$, the fibers are non-singular elliptically fibered $K3$ surfaces. The 528 singular fibers $X_\xi$ have exactly 1 ordinary double point singularity each one. In this particular example of fibred CY, we know what is the dual fibration by theorem 3.3 which turns to be also the mirror fibration. For every $t \in \mathbb{P}^1$, $X_t$ admits a fibration $f : X_t \to B$ by elliptic curves. Then $\forall b \in B$, $X_{t,b}$ and $Y_{t,b}$ are related by linear transformation or mirror transformation (according to 3.7).

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