On the construction and the cardinality of finite $\sigma$-fields

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Abstract. In this note, we first discuss some properties of generated $\sigma$-fields and a simple approach to the construction of finite $\sigma$-fields. It is shown that the $\sigma$-field generated by a finite class of $\sigma$-distinct sets which are also atoms, is the same as the one generated by the partition induced by them. The range of the cardinality of such a generated $\sigma$-field is explicitly obtained. Some typical examples and their complete forms are discussed. We discuss also a simple algorithm to find the exact cardinality of some particular finite $\sigma$-fields. Finally, an application of our results to statistics, with regard to independence of events, is pointed out.

Keywords. Finite $\sigma$-fields, induced partition of sets, cardinality, independence of events.

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1. Introduction

The role of $\sigma$-fields in probability and statistics is well known. A $\sigma$-field generated by a class of subsets of a given space $\Omega$, also known as the generated $\sigma$-field, is defined to be the intersection of all $\sigma$-fields containing that class. However, a constructive approach to obtain the $\sigma$-field generated by finite number of subsets has not been well studied in the literature. Indeed, no easy constructive method is available. It is a challenging task to construct a $\sigma$-field generated by, say, four arbitrary subsets and to know its exact cardinality. Given a finite class of $\sigma$-distinct sets (see Definition 2.2) which are also atoms, we discuss a simple approach to obtain the generated $\sigma$-field based on the partition induced by the given class and then look at the $\sigma$-field generated by this partition. In this direction, we first obtain the cardinality of the induced partition by a finite class of such sets. Using this result, we obtain the cardinality of the generated $\sigma$-field. When all members of the induced partition are non-empty, the cardinality of the generated $\sigma$-field is $2^n$, a known result in the literature (see Ash and Doleans-Dade (2000), p 457).

Let $\mathbb{R}$ denote the set of real numbers. Then the cardinality of $\mathbb{R}$ (also called cardinality of the continuum) is given by $2^{\aleph_0}$, where $\aleph_0$ is the cardinality of $\mathbb{N}$.

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the set of natural numbers. One of the fascinating results in measure theory is that there is no $\sigma$-field whose cardinality is countably infinite. In other words, the cardinality of a $\sigma$-field can be either finite or equal to $2^{\aleph_0}$ (uncountable) (Billingsley (1995), p. 34). Our focus is on the cardinality of a $\sigma$-field, generated by a finite class of $n$ sets, and to show that it assumes only particular values within a fixed range. To the best of our knowledge, this problem has been addressed only for some special cases (Ash and Doleans-Dade (2000), p. 11). Several typical examples are discussed to bring out the nature of the finitely generated $\sigma$-fields. An algorithm to find the exact cardinality of some specific $\sigma$-fields of interest is also presented. Finally, we discuss an application to statistics with regard to the independence of some events and establish some new results in this direction.

2. Range for Cardinality of a finite $\sigma$-field

Let us begin with a simple example. Let $\Omega$ be the given space, and consider two distinct subsets $B$ and $C$ of $\Omega$ such that $B \cup C \neq \Omega$ and $B \cap C \neq \phi$. We henceforth suppress the intersection symbol $\cap$, unless the context demands it. For instance, $A \cap B$ will be denoted by $AB$.

It is well known that the $\sigma$-field generated by the class $\{B, C\}$, denoted by $\sigma(B, C)$, is obtained by the usual operations of complementations, unions and intersections, as

$$\sigma(B, C) = \{\phi, \Omega, B, C, B^c, C^c, B \cup C, B \cup C^c, B^c \cup C, B^c \cup C^c, BC, BC^c, B^cC, B^cC^c, (BC) \cup (B^cC), (BC) \cup (B^cC^c)\}.$$  

Let $\mathcal{P}_{B,C} = \{BC, B^cC, BC^c, B^cC^c\}$ denote the partition (see Definition 2.2) of $\Omega$ induced by $B$ and $C$. The elements of the $\sigma$-field generated by a partition consist of the empty set $\phi$, the sets in the partition and all possible (finite) unions of them. The resulting class is evidently closed under complementation. Hence, the $\sigma$-field generated by $\mathcal{P}_{B,C}$ is given by

$$\sigma(\mathcal{P}_{B,C}) = \sigma(BC, B^cC, BC^c, B^cC^c)$$

$$= \{\phi, \Omega, BC, B^cC, BC^c, B^cC^c, (BC) \cup (B^cC),$$

$$BC \cup (BC^c), (BC) \cup (B^cC^c), (BC) \cup (BC^c),$$

$$(BC) \cup (B^cC) \cup (B^cC^c),$$

$$(BC) \cup (B^cC^c), (BC) \cup (B^cC) \cup (B^cC^c),$$

$$(BC) \cup (BC) \cup (B^cC^c), (BC) \cup (BC^c) \cup (B^cC^c),$$

$$(BC) \cup (BC^c), B^c, C, B, (BC) \cup (B^cC)^c,$$

$$(B^cC) \cup (BC^c), B^c, C^c, B \cup C, B^c \cup C, C^c \cup B, B^c \cup C^c\}.$$  

(2.1)
as some unions in (2.1) admit simple forms which can be obtained using De Morgan’s laws. It is interesting to note that \( \sigma(B, C) = \sigma(\mathcal{P}_{B,C}) \). Note that \( |\sigma(B, C)| = 16 \).

Consider next the case when \( BC = \phi \). In this case,

\[
\mathcal{P}_{B,C} = \{BC^c, B^cC, B^cC^c\} = \{B, C, B^cC^c\};
\]

\[
\sigma(\mathcal{P}_{B,C}) = \{\phi, \Omega, B, C, B^cC^c, B \cup C, B \cup (B^cC^c), C \cup (B^cC^c)\}
\]

\[
= \{\phi, \Omega, B, C, B^cC^c, B \cup C, C^c, B^c\} = \sigma(B, C)
\]

and \( |\sigma(B, C)| = 8 \) here. Thus, when \( BC = \phi \) also, we have \( \sigma(B, C) = \sigma(\mathcal{P}_{B,C}) \).

Consider next the cardinality of \( \mathcal{P}_{B,C} \), denoted by \( |\mathcal{P}_{B,C}| \), where \( B \) and \( C \) are arbitrary subsets of \( \Omega \). Two cases arise: (i) \( B \cup C = \Omega \) and (ii) \( B \cup C \neq \Omega \).

Case (i): If \( BC = \phi \), then \( \mathcal{P}_{B,C} = \{B, C\} \). If \( BC \neq \phi \), then \( \mathcal{P}_{B,C} = \{BC, B^cC, B^cC^c\} \) or \( \{B, B^cC\} \) (if \( B \subset C \)) or \( \{C, BC^c\} \) (if \( C \subset B \)). Hence, in this case, \( |\mathcal{P}_{B,C}| \in \{2, 3\} \).

Case (ii): If \( BC = \phi \), then \( \mathcal{P}_{B,C} = \{B, C, B^cC^c\} \). If \( BC \neq \phi \), then \( \mathcal{P}_{B,C} = \{BC, B^cC, BC^c, B^cC^c\} \) or \( \{B, B^cC, B^cC^c\} \) (if \( B \subset C \)) or \( \{C, BC^c, B^cC^c\} \) (if \( C \subset B \)). Hence, in this case, \( |\mathcal{P}_{B,C}| \in \{3, 4\} \).

Thus, from cases (i) and (ii), we have \( |\mathcal{P}_{B,C}| \in \{2, 3, 4\} \) for arbitrary subsets \( B \) and \( C \).

It is of interest to know the cardinality of the partition based on \( n \) arbitrary sets. To answer the question, we first introduce the following formal definitions.

**Definition 2.2.** We call a class \( A = \{A_1, A_2, \ldots, A_n\} \) of sets \( \sigma \)-distinct if no set in \( A \) can be obtained from other sets by an operation of union or intersection or complementation.

Let \( A^1 \equiv A \) and \( A^0 \equiv A^c \) henceforth. A formal definition of the partition induced by \( n \) sets is the following.

**Definition 2.3.** Let \( A = \{A_1, A_2, \ldots, A_n\} \) be a \( \sigma \)-distinct class of subsets of \( \Omega \). Then the finest partition induced by \( A \), denoted by \( \mathcal{P}_{A_1,A_2,\ldots,A_n} \), is the collection of sets of the form

\[
\mathcal{P}_{A_1,A_2,\ldots,A_n} = \left\{ \bigcap_{i=1}^{n} A_i^{\epsilon_i} \left| \epsilon_i \in \{0, 1\}, 0 \leq \sum_{i=1}^{n} \epsilon_i \leq n \right. \right\} = \mathcal{P}_A,
\]

where the empty sets are excluded.
Remark 2.5. Note that $\mathcal{P}_{A_1,A_2,A_1 \cup A_2} = \mathcal{P}_{A_1,A_2, A_2} = \mathcal{P}_{A_1,A_2,A_1^c} = \mathcal{P}_{A_1,A_2}$. That is, $\mathcal{P}_{A_1,A_2,B} = \mathcal{P}_{A_1,A_2}$ for $B \in \sigma(A_1,A_2)$. Hence, we consider, without loss of generality, only the $\sigma$-distinct sets $A_1$ and $A_2$.

The following definition is well known.

Definition 2.6. Let $\mathcal{A}$ be a collection of sets. A set $A \in \mathcal{A}$ is called an atom of $\mathcal{A}$ if $B \subseteq A$, and $B \in \mathcal{A}$, imply $B = A$; that is, no proper subset of $A$ belongs to $\mathcal{A}$.

We start with a simple fact.

Lemma 2.7. If $\{A_1, A_2, \ldots, A_n\}$ themselves form a partition of $\Omega$, then $\mathcal{P}_{A_1,A_2,\ldots,A_n} = \{A_1, A_2, \ldots, A_n\}$ and $|\mathcal{P}_{A_1,A_2,\ldots,A_n}| = n$.

Proof. Since $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ forms a partition of $\Omega$, $A_1^c \ldots A_n^c = \emptyset$ and all intersections of $A_j$'s order 2 to $n$ are empty. This implies that all the sets in $\mathcal{P}_{A_1,A_2,\ldots,A_n}$ with $2 \leq \sum_{i=1}^{n} \epsilon_i \leq n$ are empty. Consider next the sets of the form $A_1^{\epsilon_1} \ldots A_n^{\epsilon_n}$ with $\sum_{i=1}^{n} \epsilon_i = 1$. Then, for example, $A_1 A_2^c \ldots A_n^c = A_1 \cap (\bigcup_{i=2}^{n} A_i)^c = A_1 \neq \emptyset$, since $\bigcup_{i=1}^{n} A_i = \Omega$. Thus, $\mathcal{P}_{A_1,A_2,\ldots,A_n} = \{A_1, A_2, \ldots, A_n\}$ and hence $|\mathcal{P}_{A_1,A_2,\ldots,A_n}| = n$. \hfill \blacksquare

The following example clearly shows the nature of the partition and its cardinality.

Example 2.8. Consider the class $\{A, B, C, D\}$ of 4 subsets of $\Omega$. Then the partition $\mathcal{P}_{A,B,C,D}$ induced by them is

$$
\mathcal{P}_{A,B,C,D} = \{ABCD, A^cBCD, AB^cCD, ABC^cD, ABCD^c, A^cB^cCD, A^cBC^cD, A^cBCD^c, AB^cC^cD, AB^cCD^c, ABC^cD^c, A^cB^cC^cD, A^cB^cCD, A^cBC^cD^c, AB^cC^cD^c, A^cB^cC^cD^c\}.
$$

If $A, B, C$ and $D$ themselves form a partition of $\Omega$, then $\mathcal{P}_{A,B,C,D} = \{A, B, C, D\}$ and $|\mathcal{P}_{A,B,C,D}| = 4$. If $A \cup B \cup C \cup D \neq \Omega$, there may be two cases as follows.

(a) If $ABCD \neq \emptyset$ and all other elements of $\mathcal{P}_{A,B,C,D}$ are also non-empty, then $|\mathcal{P}_{A,B,C,D}| = 2^4 = 16$.

(b) If $ABCD = \emptyset$, there can be the following sub cases:
(i) If only $ABCD = \phi$, then $|\mathcal{P}_{A,B,C,D}| = 15$ and
\[
\mathcal{P}_{A,B,C,D} = \{A^cB^cC^cD, AB^eC^eD, ABC^eC^eD, ABC^eD, A^cB^cC^eD, A^cB^cD^e, \\
AB^eC^eD, AB^eC^eD, AB^eD^e, A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, AB^eC^eD, AB^eC^eD^e, \\
A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, A^cB^cC^eD^e \}.
\]

(ii) If only $BCD = \phi$, along with the implied case (i) (not mentioned later), then $|\mathcal{P}_{A,B,C,D}| = 14$ and
\[
\mathcal{P}_{A,B,C,D} = \{AB^eC^eD, ABC^eD, ABC^eC^eD, A^cB^cD^e, AB^eC^eD, AB^eC^eD^e, \\
AB^eC^eD, AB^eC^eD^e, A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, AB^eC^eD, AB^eC^eD^e \}.
\]

(iii) If only $BCD = ACD = \phi$, then $|\mathcal{P}_{A,B,C,D}| = 13$ and
\[
\mathcal{P}_{A,B,C,D} = \{ABC^eD, ABC^eC^eD, A^cB^cD^e, AB^eC^eD, AB^eC^eD^e, \\
A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, A^cB^cC^eD^e, AB^eC^eD, AB^eC^eD^e \}.
\]

(iv) If only $BCD = ACD = AB = \emptyset$, then $|\mathcal{P}_{A,B,C,D}| = 12$ and
\[
\mathcal{P}_{A,B,C,D} = \{ABCD^e, A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, AB^eC^eD, AB^eC^eD^e, \\
A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, A^cB^cC^eD^e \}.
\]

(v) If only $BC = AC = DBD = \emptyset$, then $|\mathcal{P}_{A,B,C,D}| = 11$ and
\[
\mathcal{P}_{A,B,C,D} = \{A^cB^cD^e, A^cB^cD^e, A^cB^cC^eD, AB^eC^eD, AB^eC^eD^e, \\
A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, A^cB^cC^eD^e \}.
\]

(vi) If only $AB = AC = CD = \emptyset$, then $|\mathcal{P}_{A,B,C,D}| = 10$ and
\[
\mathcal{P}_{A,B,C,D} = \{A^cB^cD^e, A^cB^cD^e, A^cB^cC^eD, AB^eC^eD, AB^eC^eD^e, \\
A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, A^cB^cC^eD^e \}.
\]

(vii) If only $ABC = CD = BD = \emptyset$, then $|\mathcal{P}_{A,B,C,D}| = 9$ and
\[
\mathcal{P}_{A,B,C,D} = \{A^cB^cD^e, AB^eC^eD, ABC^eD^e, A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, \\
A^cB^cD^e, AB^eC^eD^e, A^cB^cD^e \}.
\]

(viii) If only $CD = BD = BC = \emptyset$, then $|\mathcal{P}_{A,B,C,D}| = 8$ and
\[
\mathcal{P}_{A,B,C,D} = \{AB^eC^eD, AB^eC^eD^e, ABC^eC^eD, A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, \\
AB^eC^eD, A^cB^cC^eD^e \}.
\]

(ix) If only $CD = BD = BC = AD = \emptyset$, then $|\mathcal{P}_{A,B,C,D}| = 7$ and
\[
\mathcal{P}_{A,B,C,D} = \{AB^eC^eD^e, ABC^eD^e, A^cB^cD^e, A^cB^cC^eD, A^cB^cD^e, \\
AB^eC^eD^e, A^cB^cC^eD^e \}.
\]

(x) If only $CD = BD = BC = AD = AC = \emptyset$, then $|\mathcal{P}_{A,B,C,D}| = 6$ and
\[
\mathcal{P}_{A,B,C,D} = \{ABC^eD^e, A^cB^cC^eD, A^cB^cD^e, A^cB^cC^eD, AB^eC^eD^e, A^cB^cC^eD^e \}.
\]
We next look at the cardinality of $P_{A,B,C,D}$ in the above cases, where the partitions are different, though their cardinality is the same.

Proof. Let $\sigma$ be the partition induced by $A$, $B$, $C$, and $D$. Then $|P_{A,B,C,D}| = 5$ and

$$P_{A,B,C,D} = \{A^cB^cC^cD, A^cB^cC^cD^c, A^cB^cC^CD, AB^cC^cD, A^cB^cC^cD\}.$$

Hence, we observe that $|P_{A,B,C,D}| \in \{4, 5, 6, \ldots, 16\}$.

Remark 2.9. Two or more different partitions $P_{A_1,A_2,\ldots,A_n}$ may have the same cardinality. For example, if $A = \{A,B,C,D\}$ such that $A \cup B \cup C \cup D \neq \Omega$, then $|P_{A,B,C,D}| = 9$ corresponds to the following distinct cases:

(i) If only $AB = CD = \phi$, along with the implied empty intersections, then

$$P_{A,B,C,D} = \{A^cB^cC^cD, A^cB^cC^cD^c, A^cB^cC^cD, AB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD^c, A^cB^cC^cD^c, A^cB^cC^cD^c\}.$$  

(ii) If only $AB = AC = AD = \phi$, along with the implied empty intersections, then

$$P_{A,B,C,D} = \{A^cB^cCD, A^cB^cCD^c, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD^c\}.$$  

(iii) If only $AB = AC = BCD = \phi$, along with the implied empty intersections, then

$$P_{A,B,C,D} = \{A^cB^cCD, A^cB^cCD^c, AB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD, A^cB^cC^cD^c\}.$$  

In the above cases, the partitions are different, though their cardinality is the same.

We next look at the cardinality of $P_{A_1,A_2,\ldots,A_n}$ induced by $\{A_1, A_2, \ldots, A_n\}$. It looks difficult to argue for the arbitrary sets. However, we have the following result for a fairly large and reasonable class of sets.

Theorem 2.10. Let $A = \{A_1, A_2, \ldots, A_n\}, n \geq 3$, be a class of $\sigma$-distinct sets of $\Omega$, where each $A_i$ is an atom of $A$, and $P_{A_1,A_2,\ldots,A_n}$ be the partition induced by $A_1, A_2, \ldots, A_n$. Then $|P_{A_1,A_2,\ldots,A_n}| \in \{n, n+1, \ldots, 2^n\}$.

Proof. Let

$$Q_j = \left\{ \bigcap_{i=1}^n A_i^{\epsilon_i} \bigg| \epsilon_i \in \{0,1\}, 1 \leq i \leq n, \sum_{i=1}^n \epsilon_i = j \right\},$$

for $0 \leq j \leq n$. Then $P_{A_1,A_2,\ldots,A_n}$ can be represented as

$$P_{A_1,A_2,\ldots,A_n} = \bigcup_{j=0}^n Q_j.$$
where only non-empty elements of $Q_j$'s are considered. This is because some $Q_j$'s may contain empty sets. Note that $Q_j$ consists of at most $\binom{n}{j}$ distinct non-empty sets corresponding to a selection of $j$ of the $n$ $\epsilon_i$'s as 1 and the rest as 0. So $|Q_j| \leq \binom{n}{j}$, and hence

\begin{equation}
|P_{A_1, A_2, \ldots, A_n}| = \left| \bigcup_{j=0}^{n} Q_j \right| \leq \sum_{j=0}^{n} |Q_j| \leq \sum_{j=0}^{n} \binom{n}{j} = 2^n.
\end{equation}

Note that the elements of $Q_j$, for $1 \leq j \leq n-1$ are all disjoint. Indeed, if $D_1 \in Q_i$ and $D_2 \in Q_j$, for some $j \neq i$, then $D_1 D_2 = \phi$. Therefore, an element $E \in Q_j$ being empty or non-empty does not affect the nature (emptiness or non-emptiness) of another element $D$ coming from $Q_i$ or $Q_j$ for all $i \neq j$, in general. Consider next the four exhaustive cases.

Case (i): $\bigcup_{i=1}^{n} A_i \neq \Omega$, $\bigcap_{i=1}^{n} A_i \neq \phi$. In this case, our claim is

\begin{equation}
|P_{A_1, A_2, \ldots, A_n}| \in \{n + 2, n + 3, \ldots, 2^n\}.
\end{equation}

Here, both $Q_0$ and $Q_n$ contain non-empty sets, so that $|Q_0| = |Q_n| = 1$. Suppose now $A_i$'s are such that all elements of $Q_1$ to $Q_{n-2}$ are empty. We show that none of the elements of $Q_{n-1}$ is empty. Suppose an element of $Q_{n-1}$, say, $A_{3}^{c} A_{2} \ldots A_{n}$ = $\phi$. Note first that $\bigcup_{0 \leq \sum_{j=3}^{n} \epsilon_j \leq n-2} A_{3}^{c} \ldots A_{n}^{c} = \Omega$, since the LHS is the union of sets in $P_{A_3 \ldots A_n}$. Hence,

\begin{equation}
(A_3 \ldots A_n)^c = \bigcup_{0 \leq \sum_{j=3}^{n} \epsilon_j \leq n-3} A_{3}^{c} \ldots A_{n}^{c}.
\end{equation}

Now,

\[
A_1^{c} A_2 = A_1^{c} A_2 A_3 \ldots A_n + A_1^{c} A_2 (A_3 \ldots A_n)^c \\
= A_1^{c} A_2 \bigcap \left( \bigcup_{0 \leq \sum_{j=3}^{n} \epsilon_j \leq n-3} A_{3}^{c} \ldots A_{n}^{c} \right) \quad \text{(using (2.15))} \\
= \bigcup_{0 \leq \sum_{j=3}^{n} \epsilon_j \leq n-3} A_{1}^{c} A_2 A_{3}^{c} \ldots A_{n}^{c} \\
= \phi,
\]

since each element $A_{1}^{c} A_2 A_{3}^{c} \ldots A_{n}^{c}$, with $1 \leq \sum_{j=3}^{n} \epsilon_j \leq n - 3$ belongs to one of $Q_1$ to $Q_{n-2}$. So $A_2 \subset A_1$, a contradiction, since each $A_i$ is an atom of $\mathcal{A}$. Also, in this case, $|P_{A_1, A_2, \ldots, A_n}| = 2 + \binom{n}{n-1} = n + 2$. 


Let now $2 \leq k \leq n - 1$. Consider the case where all elements of $Q_j$ for $1 \leq j \leq k-1$ are non-empty, only $r$ elements of $Q_k$ are empty, and all elements of $Q_l$ for $k+1 \leq l \leq n-1$ are empty. In this case, $|P_{A_1, A_2, \ldots, A_n}| = 2 + \sum_{j=1}^{k-1} \binom{n}{j} + r$.

where $1 \leq r \leq \binom{n}{k}$. If $k = n - 1$ and $r = \binom{n}{n-1} = n$, then $|P_{A_1, A_2, \ldots, A_n}| = 2 + \sum_{j=1}^{n-2} \binom{n}{j} + n = 2^n$. That is, when all elements of $Q_0$ to $Q_n$ are non-empty, we get $|P_{A_1, A_2, \ldots, A_n}| = 2^n$. Thus, $|P_{A_1, A_2, \ldots, A_n}| \in \{n + 2, n + 3, \ldots, 2^n\}$.

Case (ii): $\bigcup_{i=1}^{n} A_i = \Omega$, $\bigcap_{i=1}^{n} A_i \neq \phi$. In this case, we claim

$$|P_{A_1, A_2, \ldots, A_n}| \in \{n + 1, n + 2, \ldots, 2^n - 1\}.$$ (2.16)

Here $Q_0 = \{\phi\}$ and $Q_n \neq \{\phi\}$ and so $|Q_n| = 1$. The argument for the minimum value of $|P_{A_1, A_2, \ldots, A_n}|$ is the same as that in Case (i). Also, $|P_{A_1, A_2, \ldots, A_n}|$ is exactly 1 less than that in Case (i), and so $|P_{A_1, A_2, \ldots, A_n}| \in \{n + 1, n + 2, \ldots, 2^n - 1\}$.

Case (iii): $\bigcup_{i=1}^{n} A_i \neq \Omega$, $\bigcap_{i=1}^{n} A_i = \phi$. In this case also, we claim

$$|P_{A_1, A_2, \ldots, A_n}| \in \{n + 1, n + 2, \ldots, 2^n - 1\}.$$ (2.17)

Here $Q_0 \neq \{\phi\}$ and $|Q_0| = 1$, while $Q_n = \{\phi\}$. Suppose $A_i$’s are such that all elements of $Q_2$ to $Q_{n-1}$ are empty. Let now an element of $Q_1$, say, $A_1 A_2 \ldots A_n = \phi$. Then $A_1 = \bigcup_{0 \leq \sum_{i=2}^{n} c_i \leq n-1} (A_1 A_2^c \ldots A_n^c) = \phi$, by assumptions, which is a contradiction. Thus, all elements of $Q_1$ are non-empty, so that $|P_{A_1, A_2, \ldots, A_n}| = 1 + \binom{n}{1} = n + 1$ in this case. The rest of the arguments are similar to those in Case (i). Hence, $|P_{A_1, A_2, \ldots, A_n}| \in \{n + 1, n + 2, \ldots, 2^n - 1\}$.

Case(iv): $\bigcup_{i=1}^{n} A_i = \Omega$, $\bigcap_{i=1}^{n} A_i = \phi$. Our claim in this case is

$$|P_{A_1, A_2, \ldots, A_n}| \in \{n, n + 1, \ldots, 2^n - 2\}.$$ (2.18)

Here, $Q_0 = Q_n = \{\phi\}$. Observe first that it is not necessary that $A_1, A_2, \ldots, A_n$ themselves form a partition of $\Omega$. The argument for the minimum value of $|P_{A_1, A_2, \ldots, A_n}|$ is the same as that in Case (i). Also, $|P_{A_1, A_2, \ldots, A_n}|$ is exactly 1 less than that in Case (iii), and so $|P_{A_1, A_2, \ldots, A_n}| \in \{n, n + 1, \ldots, 2^n - 2\}$.

Thus, from all the above cases, $|P_{A_1, A_2, \ldots, A_n}| \in \{n, n + 1, \ldots, 2^n\}$. ■
Remark 2.19. If \( R_j = \bigcup_i B_i \), where \( B_i \in Q_j \), for \( 0 \leq j \leq n \), then \( \Omega = \bigcup_{j=0}^n R_j \).

Note that \( R = \{R_0, \ldots, R_n\} \) forms a partition of \( \Omega \).

Our main interest is on the construction and the cardinality of finite \( \sigma \)-fields. Our approach is via the partition induced by the generating class \( A = \{A_1, A_2, \ldots, A_n\} \). Note first that \( \sigma(\mathcal{P}_{A_1, A_2, \ldots, A_n}) \) is obtained by including the empty set and taking all the sets in \( \mathcal{P}_{A_1, A_2, \ldots, A_n} \), and all possible unions taken two at a time, three at a time, and so on till the union of all the sets in \( \mathcal{P}_{A_1, A_2, \ldots, A_n} \). The following result justifies our approach.

**Theorem 2.20.** Let \( A = \{A_1, A_2, \ldots, A_n\} \) be a class of \( n \) subsets of \( \Omega \) that satisfies the conditions of Theorem 2.10. Then

\[
(2.21) \quad (i) \quad \sigma(A) = \sigma(\mathcal{P}_A) \\
(ii) \quad |\sigma(A)| = 2^{|\mathcal{P}_A|}.
\]

**Proof.** Let \( \epsilon_i \in \{0, 1\} \) for \( 1 \leq i \leq n \). Since \( A_i \in A \), we have \( A_i^{\epsilon_i} \in \sigma(A) \), for \( 1 \leq i \leq n \) and hence \( A_1^{\epsilon_1} \ldots A_n^{\epsilon_n} \in \sigma(A) \). Thus, \( \mathcal{P}_A \subseteq \sigma(A) \) and hence \( \sigma(\mathcal{P}_A) \subseteq \sigma(A) \).

Conversely, for \( 1 \leq i \leq n \),

\[
A_i = A_i \cap \left( \bigcup_{\epsilon_j: j \neq i} A_1^{\epsilon_1} \ldots A_{i-1}^{\epsilon_{i-1}} A_{i+1}^{\epsilon_{i+1}} \ldots A_n^{\epsilon_n} \right)
\]

\[
= \bigcup_{\epsilon_j: j \neq i} \left( A_1^{\epsilon_1} \ldots A_{i-1}^{\epsilon_{i-1}} A_i A_{i+1}^{\epsilon_{i+1}} \ldots A_n^{\epsilon_n} \right) \in \sigma(\mathcal{P}_A),
\]

since each \( A_1^{\epsilon_1} \ldots A_{i-1}^{\epsilon_{i-1}} A_i A_{i+1}^{\epsilon_{i+1}} \ldots A_n^{\epsilon_n} \in \mathcal{P}_A \). Thus, \( \{A_1, A_2, \ldots, A_n\} = A \subseteq \mathcal{P}_A \Rightarrow \sigma(A) \subseteq \mathcal{P}_A \). Thus, \( \sigma(A) = \sigma(\mathcal{P}_A) \).

Let now \( |\mathcal{P}_{A_1, A_2, \ldots, A_n}| = k \). Since \( \sigma(\mathcal{P}_{A_1, A_2, \ldots, A_n}) \) is the \( \sigma \)-field obtained by taking all possible unions of the sets in \( \mathcal{P}_{A_1, A_2, \ldots, A_n} \), we have

\[
|\sigma(\mathcal{P}_{A_1, A_2, \ldots, A_n})| = 1 + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k} = 2^k,
\]

where the unity is added for the empty set. Hence, \( \sigma(\mathcal{P}_A) = 2^{|\mathcal{P}_A|} \).

It is known that the cardinality of a finite \( \sigma \)-field is of the form \( 2^m \) for some \( m \in \mathbb{N} \) (see, for example, Rosenthal (2006), p. 24). The following corollary, which follows from Theorems 2.10 and 2.20 gives the explicit range for \( m \).
Corollary 2.22. Let \( \mathcal{F} \) be a finitely generated \( \sigma \)-field of subsets of \( \Omega \). Then \(|\mathcal{F}| = 2^m\), for some \( n \leq m \leq 2^n \) and \( n \geq 1 \).

Suppose \( \mathcal{F} = \sigma(\mathcal{B}) \) for some class \( \mathcal{B} \) of subsets of \( \Omega \). Let \( \mathcal{A} \) be the largest \( \sigma \)-distinct subclass of atoms (of \( \mathcal{B} \)). Then \( m = |\mathcal{P}_A| \).

Remark 2.23. (i) Suppose \( A_1, A_2, \ldots, A_n \) themselves form a partition of \( \Omega \). Then, from Lemma 2.7 and Theorem 2.20, \(|P| = 2^m\) for some \( n \leq m \leq 2^n \) and \( n \geq 1 \).

(ii) Suppose \( \bigcup_{i=1}^{n} A_i \neq \Omega \) and \( \bigcap_{i=1}^{n} A_i \neq \phi \). Also, if all elements of \( \mathcal{Q}_j \) for \( 1 \leq j \leq n-1 \) are non-empty, then \(|\mathcal{P}_{A_1,A_2,\ldots,A_n}| = 2^n\) (Case (i) of the proof of Theorem 2.20) and hence, \(|\sigma(A_1, A_2, \ldots, A_n)| = 2^{2n}\), a known result (Ash and Doleans-Dade (2000), p 457.)

Example 2.24. (i) Let \( \mathcal{A} = \{A\} \). Then \( \mathcal{P}_A = \{A, A^c\} \) and

\[ |\mathcal{P}_A| = 2, \sigma(A) = \sigma(\mathcal{P}_A) = \{\phi, \Omega, A, A^c\}, |\sigma(A)| = |\sigma(\mathcal{P}_A)| = 4. \]

(ii) Let \( \mathcal{A} = \{A, B, C\} \) be a collection of three \( \sigma \)-distinct subsets such that \( A \cup B \cup C \neq \Omega \). Then

\[ \mathcal{P}_{A,B,C} = \{ABC, A^cBC, AB^cC, ABC^c, A^cB^cC, AB^cC^c, A^cB^cC^c \}. \]

The different possible values of \(|\mathcal{P}_{A,B,C}|\) are listed in the following cases:

(i) If \( ABC \neq \phi \), then \( \mathcal{P}_{A,B,C} = \{ABC, A^cBC, AB^cC, ABC^c, A^cB^cC, AB^cC^c, A^cB^cC^c \} \). Also, \(|\mathcal{P}_{A,B,C}| = 8 \) and \(|\sigma(A, B, C)| = |\sigma(\mathcal{P}_{A,B,C})| = 2^8 = 256. \)

(ii) If only \( ABC = \phi \), then \( \mathcal{P}_{A,B,C} = \{A^cBC, AB^cC, ABC^c, A^cB^cC, AB^cC^c, A^cB^cC^c \} \). Also, \(|\mathcal{P}_{A,B,C}| = 7 \) and \(|\sigma(A, B, C)| = |\sigma(\mathcal{P}_{A,B,C})| = 2^7 = 128. \)

(iii) If \( AB = \phi \), then \( \mathcal{P}_{A,B,C} = \{A^cBC, AB^cC, A^cB^cC, A^cB^cC^c \} \). Also, \(|\mathcal{P}_{A,B,C}| = 6 \) and \(|\sigma(A, B, C)| = |\sigma(\mathcal{P}_{A,B,C})| = 2^6 = 64. \)

(iv) If \( AB = AC \neq \phi \), then \( \mathcal{P}_{A,B,C} = \{A^cBC, AB^cC, A^cB^cC, AB^cC^c, A^cB^cC^c \} \). Also, \(|\mathcal{P}_{A,B,C}| = 5 \) and \(|\sigma(A, B, C)| = |\sigma(\mathcal{P}_{A,B,C})| = 2^5 = 32. \)

(v) If \( AB = AC = BC \neq \phi \), then \( \mathcal{P}_{A,B,C} = \{A^cB^cC, AB^cC^c, A^cB^cC^c \} \). Also, \(|\mathcal{P}_{A,B,C}| = 4 \) and \(|\sigma(A, B, C)| = |\sigma(\mathcal{P}_{A,B,C})| = 2^4 = 16. \)

Also, in this case,

\[ \sigma(A, B, C) = \sigma(\mathcal{P}_{A,B,C}) = \{\phi, \Omega, A, B, C, A^c, B^c, C^c, A \cup B, A \cup C, B \cup C, A \cup B \cup C, A^c B^c C, A^c B^c C^c, B^c C^c \}. \]
Thus, we see that $|\sigma(A, B, C)| \in \{16, 32, 64, 128, 256\}$. This result easily follows also from (2.14) and (2.17) and Theorem 2.20.

3. Exact cardinality of some finite $\sigma$-fields

Let $\pi = \{1, 2, \ldots, n\}$. Given a class $A = \{A_1, A_2, \ldots, A_n\}$ of $\sigma$-distinct sets, we call henceforth $A_i A_j, i \neq j$, 2-factor intersections, $A_i A_j A_k, i \neq j \neq k$, 3-factor intersections, and so on. Clearly, there is only one $n$-factor intersection, namely $A_1 A_2 \ldots A_n$. We denote $A_{i_1} A_{i_2} \ldots A_{i_k} = A_S$, where $S = \{i_1, i_2, \ldots, i_k\}$. We now discuss a simple algorithm to find the exact cardinality of some special $\sigma$-fields:

Let $\bigcup_i^n A_i \neq \emptyset$, so that $|Q_0| = 1$. Note that an element $A_1 A_2^2 \ldots A_n^c \in Q_1$ is empty if $A_1 \subset (\bigcup_{j \neq 1} A_j)$. Also, let $s_1$ denote the number of $A_i$’s such that $A_i \subset \bigcup_{j=1, j \neq i}^n A_j$. Then, $n - s_1 = |Q_1|$, the cardinality of $Q_1$.

Let now $3 \leq l \leq n$. Suppose none of the $j$-factor intersections, for $2 \leq j \leq l - 1$, is empty and only $k$ of the $l$-factor intersections, say, $A_{L_1}, \ldots, A_{L_k}$ are empty for some $L_i \subseteq \pi$ with $|L_i| = l, 1 \leq i \leq k$. Define

$$S_{L_i} = \{B \mid L_i \subseteq B \subseteq \pi\}.$$  

We call $S_{L_i}$ the set of indices of implied empty intersections of $L_i$, as $A_B = \emptyset, \forall B \in S_{L_i}$. Then

$$|\mathcal{P}_{A_1, A_2, \ldots, A_n}| = 2^n - \left| \bigcup_{i=1}^k S_{L_i} \right| - s_1,$$

if none of the higher order intersections $A_M$ is empty, where $l < |M| \leq n - 1$ and $M \notin \bigcup_{i=1}^k S_{L_i}$.

If there exist $M_1, \ldots, M_r$ such that $l < |M_i| \leq n - 1$ and $M_i \notin \bigcup_{i=1}^k S_{L_i}$, then

$$|\mathcal{P}_{A_1, A_2, \ldots, A_n}| = 2^n - \left| \Big( \bigcup_{i=1}^k S_{L_i} \big) \bigcup \bigcup_{j=1}^r S_{M_j} \right| - s_1.$$

Indeed, it suffices to consider only those $M_j$’s such that

$$M_j \notin \bigcup_{i=1}^k S_{L_i} \bigcup \bigcup_{r=1}^{j-1} S_{M_r},$$

for $2 \leq j \leq n - 1$. 
We next consider some particular cases of interest under the case $s_1 = 0$.

(i) When none of the intersections of $A_j$’s of order two or more is empty,
$$|P_{A_1, A_2, \ldots, A_n}| = 2^n.\]

(ii) When only the $(n - 1)$-factor intersections, say $q_{n-1}$ in number, are empty, then
$$|P_{A_1, A_2, \ldots, A_n}| = 2^n - q_{n-1} - 1,$$
since $A_1 \ldots A_n$ is the only implied empty intersection.

(iii) When all the 2-factor intersections $A_i A_j = \phi$, then all the collections $Q_2$ to $Q_n$ are empty and in this case all the elements of $Q_1$ are non-empty. Thus,
$$|P_{A_1, A_2, \ldots, A_n}| = n + 1,$$
since $\bigcup_{i=1}^{n} A_i \neq \Omega$ which implies $Q_0$ is nonempty.

Two typical examples follow.

**Example 3.3.** Let $\Omega = \{5, 6, \ldots, 20\}$ and consider the sets $A_1 = \{8, 10, 14, 16, 17\}$, $A_2 = \{6, 7, 18\}$, $A_3 = \{7, 8, 9, 14, 16, 19\}$ and $A_4 = \{9, 10, 11, 14, 20\}$. Then
$$\bigcup_{i=1}^{4} A_i \neq \Omega,$$ and only $A_1 A_2 = A_2 A_4 = \phi$. Hence, $L_1 = \{1, 2\}, L_2 = \{2, 4\}$ and

(i) $S_{L_1} = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$,
(ii) $S_{L_2} = \{\{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$.

Thus, $|S_{L_1} \cup S_{L_2}| = 6$. Also, here $s_1 = 0$. Using (3.1), we get $|P_{A_1, A_2, A_3, A_4}| = 2^4 - 6 = 10$ and by Theorem 2.20 $|\sigma(A_1, A_2, A_3, A_4)| = 2^{10} = 1024$.

**Example 3.4.** Consider the case where $\Omega = \{5, 6, \ldots, 15\}$, $A_1 = \{8, 9, 10\}$, $A_2 = \{7, 11, 12, 15\}$,

$A_3 = \{7, 11, 12, 13\}$ and $A_4 = \{8, 10, 13, 15\}$. Then $\bigcup_{i=1}^{4} A_i \neq \Omega$ and $A_1 A_2 = A_1 A_3 = \phi$ are the only 2-factor intersections so that $L_1 = \{1, 2\}, L_2 = \{1, 3\}$ and

(i) $S_{L_1} = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$,
(ii) $S_{L_2} = \{\{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.

Also, $A_2 A_3 A_4 = \phi$ so that $M = \{2, 3, 4\} \not\subseteq S_{L_1} \cup S_{L_2}$ and $S_{M} = \{\{2, 3, 4\}, \{1, 2, 3, 4\}\}$.

Therefore, $|S_{L_1} \cup S_{L_2} \cup S_{M}| = 7$. Note also that $A_2 \subseteq \bigcup_{j \neq 2} A_j$, $A_3 \subseteq \bigcup_{j \neq 3} A_j$ and
$A_4 \subset \bigcup_{j \neq 4} A_j$, so that $s_1 = 3$. Using (3.2), $|\mathcal{P}_{A_1,A_2,A_3,A_4}| = 2^4 - 7 - 3 = 6$ and, by Theorem 2.20 $|\sigma(A_1, A_2, A_3, A_4)| = 2^6 = 64$.

Finally, we look at the question of cardinality $\sigma(\mathcal{B}, D)$, where $D \notin \mathcal{B}$ and $\mathcal{B} = \sigma(\mathcal{A})$ is a $\sigma$-field. Here, we need to talk about $\mathcal{P}\{\mathcal{B}, D\}$, where the members are not $\sigma$-distinct.

**Definition 3.5.** Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an arbitrary class of $n$ subsets of $\Omega$. Then the partition $\mathcal{P}_A^*$ induced by $\mathcal{A}$ is defined as the subclass of $\mathcal{P}_A$, defined in (2.1), where each element of $\mathcal{P}_A^*$ is an atom. That is, the sets in $\mathcal{P}_A$ which can be obtained as the union of other sets, are removed from $\mathcal{P}_A$ to obtain $\mathcal{P}_A^*$.

For simplicity and continuity, we denote $\mathcal{P}_A^*$ by $\mathcal{P}_A$ only.

**Theorem 3.6.** Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a class of $\sigma$-distinct sets, and $\mathcal{B} = \sigma(\mathcal{A})$ be the generated $\sigma$-field. Let $D \subset \Omega$ and $D \notin \mathcal{B}$. Then

\[
|\sigma(\mathcal{B}, D)| = 2^{|\mathcal{P}_A^*,D|}.
\]

**Proof.** Let $\mathcal{P}_{A,D}$ be the finest partition induced by the class $\{\mathcal{A}, D\}$. Then

\[
\mathcal{P}_{A,D} = \left\{ A_1^{\epsilon_1} \ldots A_n^{\epsilon_n} D^{\epsilon_{n+1}} \mid \epsilon_j \in \{0, 1\}, 0 \leq \sum_{j=1}^{n+1} \epsilon_j \leq n + 1 \right\}
\]

\[
= \left\{ A_1^{\epsilon_1} \ldots A_n^{\epsilon_n} D \mid \epsilon_j \in \{0, 1\}, 0 \leq \sum_{j=1}^{n} \epsilon_j \leq n \right\}
\]

\[
\bigcup \left\{ A_1^{\epsilon_1} \ldots A_n^{\epsilon_n} D^c \mid \epsilon_j \in \{0, 1\}, 0 \leq \sum_{j=1}^{n} \epsilon_j \leq n \right\}.
\]

When none of the elements of $\mathcal{P}_{A,D}$ is empty,

\[
|\mathcal{P}_{A,D}| = 2|\mathcal{P}_A|.
\]

Note also that $\mathcal{P}_{\sigma(\mathcal{A})} = \mathcal{P}_A$, since $\sigma(\mathcal{A})$ is the collection of all possible unions of the sets in $\mathcal{P}_A$. Hence,

\[
\mathcal{P}_{\sigma(\mathcal{A}),D} = \mathcal{P}_{A,D}
\]

for $D \notin \sigma(\mathcal{A})$. Observe also that the sets in $\{\mathcal{A}, D\}$ are $\sigma$-distinct. Now

\[
\sigma(\mathcal{B}, D) = \sigma(\mathcal{P}_{B,D}) = \sigma(\mathcal{P}_{\sigma(\mathcal{A}),D}) = \sigma(\mathcal{P}_{A,D}).
\]

Hence, from Theorem 2.20

\[
|\sigma(\mathcal{B}, D)| = 2^{|\mathcal{P}_{A,D}|},
\]

which proves the result. \[\blacksquare\]
Remark 3.10. (i) When none of the elements of $\mathcal{P}_{A,D}$ is empty, we have from (3.7) and (3.8),

\begin{equation}
|\sigma(B, D)| = 2^{2^{|\mathcal{P}_{A,D}|}} = 2^{2^{|\mathcal{P}_{A}|}} = (2^{|\mathcal{P}_{A}|})^2 = |B|^2.
\end{equation}

(ii) It follows from (3.9) that

\begin{equation}
\sigma(\sigma(A), D) = \sigma(\mathcal{P}_{\sigma(A), D}) = \sigma(\mathcal{P}_{A,D}) = \sigma(A, D),
\end{equation}

an expected result.

Example 3.13. Consider the simplest case, where $\mathcal{B} = \sigma(A) = \{\phi, \Omega, A, A^c\}$. Let $D \notin \mathcal{B}$ be such $AD \neq \phi$ and $A \cup D \neq \Omega$. Then $\{B, D\} = \{\phi, \Omega, A, A^c, D\}$ and $\mathcal{P}_{A,D} = \{AD, A^cD, AD^c, A^cD^c\} = \mathcal{P}_{A,D}$. Note here all the elements of $\mathcal{P}_{A,D}$ are non-empty. Hence, $|\mathcal{P}_{B,D}| = |\mathcal{P}_{A,D}| = 2|\mathcal{P}_A| = 4$ and $|\sigma(B, D)| = |B|^2 = 16$.

Example 3.14. Consider the case when $A_1$ and $A_2$ are two $\sigma$-distinct subsets of $\Omega$ such that $A_1 \cup A_2 \neq \Omega$ and $A_1 \cap A_2 = \phi$. Let $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \sigma(\mathcal{A})$. Here $\mathcal{P}_{\mathcal{A}} = \{A_1^cA_2, A_1A_2^c, A_1^cA_2^c\}$ and $\mathcal{B} = \{\phi, \Omega, A_1, A_2, A_1^c, A_2^c, A_1 \cup A_2, A_1^c \cap A_2^c\}$. Let $D \notin \mathcal{B}$ be a non-empty subset of $\Omega$, so that $\{B, D\} = \{\phi, \Omega, A_1, A_2, A_1^c, A_2^c, A_1 \cup A_2, A_1^c \cap A_2^c, D\}$ and $\{A, D\} = \{A_1, A_2, D\}$. Then $\mathcal{P}_{B,D} = \{A_1^cA_2D, A_1A_2^cD, A_1^cA_2^cD, A_1^cA_2D^c, A_1A_2^cD^c, A_1^cA_2^cD^c\} = \mathcal{P}_{A,D}$. Assume all elements of $\mathcal{P}_{A,D}$ to be non-empty. Then $|\mathcal{P}_{B,D}| = |\mathcal{P}_{A,D}| = 2|\mathcal{P}_A| = 6$. Hence, $|\sigma(B, D)| = 2^{2^{|\mathcal{P}_{B,D}|}} = 2^{2^{|\mathcal{P}_A|}} = |\sigma(A)|^2 = |B|^2 = 64$.

4. An Application to Independence of Events

Let us consider three events $A, B$ and $C$ such that $A$ is independent of $B$ (denoted by $A \perp B$) and $A$ is independent of $C$ (denoted by $A \perp C$). It is known that these assumptions neither imply $A \perp (B \cup C)$ nor $A \perp (BC)$, unless $B$ and $C$ are disjoint (see Whittaker (1990) p. 27). However, we have the following lemma.

Lemma 4.1. Let $A \perp B$ and $A \perp C$. Then $A \perp (B \cup C)$ iff $A \perp (BC)$.

Proof. Consider

\[ P[A \cap (B \cup C)] = P[(AB) \cup (AC)] = P(AB) + P(AC) - P(ABC) = P(A)[P(B) + P(C)] - P(ABC) \]

(by assumption)

so that

\begin{equation}
P[A \cap (B \cup C)] - P(A)[P(B \cup C)] = P(A)[P(BC)] - P(ABC).
\end{equation}

Hence, $A \perp (B \cup C)$ (that is, L.H.S. of (4.2) = 0) iff $A \perp (BC)$ (that is, R.H.S. of (4.2) = 0).
Lemma 4.3. If $A \perp B$, $A \perp C$ and $A \perp (B \cup C)$, then $A \perp \mathcal{P}_{B,C}$.

Proof. We need to show that under the assumptions, $A \perp E$, for every $E \in \{BC, B^cC, BC^c, B^cC^c\} = \mathcal{P}_{B,C}$. Using Lemma 4.1 and the assumptions, we have $A \perp (BC)$. Since,

$$P[A \cap (B^cC)] = P(AC) - P(ABC) = P(A)P(C) - P(A)P(BC) = P(A)[P(C) - P(BC)] = P(A)P(B^cC)$$

shows that $A \perp (B^c \cap C)$. Similarly, $A \perp (BC^c)$, by symmetry. Further, $A$ is independent of $B \cup C$ implies $A$ is independent of $(B \cup C)^c = B^cC^c$. Hence, $A \perp \mathcal{P}_{B,C}$.

The following stronger form of independence of three events is used especially in graphical models (see Whittaker (1990) p. 27).

Definition 4.4. $A \perp [B, C]$ if $A$ is independent of $E$, for every $E \in \mathcal{P}_{B,C}$.

As another application Theorem 2.20, we have the following result which states that $A \perp [B, C]$ implies the independence of $A \perp E$, where $E \in \sigma(B, C)$ and $4 \leq |\sigma(B, C)| \leq 16$.

Theorem 4.5. The following statements are equivalent.

(i) $A \perp [B, C]$.
(ii) $A \perp B$, $A \perp C$ and $A \perp (B \cup C)$.
(iii) $A \perp \sigma(B, C)$.

Proof. Assume (i) holds. Since the elements of $\mathcal{P}_{B,C} = \{BC, B^cC, BC^c, B^cC^c\}$ are disjoint, $A$ is independent of all possible unions of elements in $\mathcal{P}_{B,C}$. Note that $B = BC \cup BC^c, C = BC \cup B^cC$ and $B \cup C = BC \cup B^cC \cup BC^c$. So, $A \perp B$, $A \perp C$ and $A \perp (B \cup C)$. Hence, (i) $\implies$ (ii). Assume now (ii) holds. Then by Lemma 4.3, $A \perp \mathcal{P}_{B,C}$. Since $\sigma(\mathcal{P}_{B,C})$ consists the collection of all possible unions of sets in $\mathcal{P}_{B,C}$, we have $A \perp \sigma(\mathcal{P}_{B,C}) = \sigma(B, C)$, by Theorem 2.20. So, (ii) $\implies$ (iii). Consider now statements (i) and (iii). Since $\mathcal{P}_{B,C} \subseteq \sigma(B, C)$, $A \perp \sigma(B, C)$ implies $A \perp \mathcal{P}_{B,C}$ and thus $A \perp [B, C]$, by Definition 4.4. Hence, (iii) $\implies$ (i). Thus, statements (i), (ii) and (iii) are equivalent.
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