A linear seesaw model with $A_4$-modular flavor and local $U(1)_{B-L}$ symmetries

Takaaki Nomura$^a$ and Hiroshi Okada$^{b,c}$

$^a$School of Physics, KIAS, Seoul 02455, Republic of Korea
$^b$Asia Pacific Center for Theoretical Physics (APCTP) — Headquarters San 31, Hyoja-dong, Nam-gu, Pohang 790-784, Republic of Korea
$^c$Department of Physics, Pohang University of Science and Technology, Pohang 37673, Republic of Korea

E-mail: nomura@kias.re.kr, hiroshi.okada@apctp.org

Received June 29, 2022
Revised August 7, 2022
Accepted August 28, 2022
Published September 15, 2022

Abstract. We discuss a linear seesaw model with local $U(1)_{B-L}$ and modular $A_4$ symmetries. The neutrino mass matrix for linear seesaw mechanism is realized by $U(1)_{B-L}$ charge assignment and the nature of modular $A_4$ symmetry. We formulate neutrino mass and carry out numerical $\chi^2$ square analysis showing some predictions for observables in neutrino sector.

Keywords: neutrino theory, neutrino properties

ArXiv ePrint: 2007.04801
1 Introduction

The understanding of flavor structure is one of the important issues in particle physics since we do not have any symmetry to control flavor in the standard model (SM). Thus introduction of a flavor symmetry is typical strategy in constructing a model of physics beyond the SM. In particular, we need to understand structure of the neutrino sector, since only the two mass differences and three mixings are experimentally confirmed in basis of diagonal charged-lepton sector. Heavier neutral fermions are frequently introduced in order to explain the sector, which is in favor of a gauged $U(1)_{B-L}$ symmetry due to cancellation of chiral anomaly. Even though the most famous neutrino model with the $U(1)_{B-L}$ model is canonical seesaw introducing three right-handed neutrinos, but it typically requires right-handed neutrino mass scale of grand unified theory: $M_{\nu_R} \sim 10^{15}$ GeV. On the other hand linear seesaw mechanism is known as one of the TeV scale scenarios, and it can be realized by introducing left-handed neutral fermions under the $U(1)_{B-L}$ symmetry in addition to the right handed neutrinos. Thus, we can test several phenomenologies with our current experiments. Since these models typically require more free parameters than the other three sectors in the SM fermion, flavor symmetries are also introduced in these models frequently in order to reduce the parameters and get predictions.

One of the interesting approach is application of modular flavor symmetries proposed by [1, 2] to describe flavor structures. In this framework, a coupling can be transformed under a non-trivial representation of a non-Abelian discrete group and we can realize flavor structure without many scalar fields such as flavons. Then some typical groups are found to be available in basis of the modular group $A_4$ [2–23], $S_3$ [24–27], $S_4$ [28–34], $A_5$ [33, 35, 36], larger groups [37], multiple modular symmetries [38], and double covering of $A_4$ [39] and $S_4$ [40, 41] in which masses, mixing, and CP phases for quark and/or lepton are predicted.\(^1\)

Furthermore, a systematic approach to understand the origin of CP transformations has been discussed in ref. [50], and CP violation in models with modular symmetry is also discussed in refs. [51, 52], and a possible correction from Kähler potential is also discussed in ref. [53]. In particular, it is interesting to apply a modular symmetry in constructing a new physics model for neutrino mass generation in which we would obtain prediction for signals of new physics correlated with observables in neutrino sector under the linear seesaw model with local $U(1)_{B-L}$ and modular symmetries. Moreover nature of modular symmetry can be used to realize linear seesaw mechanism in addition to constraining flavor structure.

\(^1\) Some reviews are useful to understand the non-Abelian group and its applications to flavor structure [42–49].
We assign modular weight $k_i$ to these fields as summarized in table 1. Only lepton doublet $L_i$ and right-handed sterile neutrino $N^c_i$ are chosen to be $A_4$ triplet.

Then we impose superpotential should be invariant under these symmetries where each term has vanishing modular weight; properties of modular symmetry are referred to appendix. Here we denote each of vacuum expectation value (VEV) to be $\langle H_{1,2}^{I} \rangle \equiv [0, v_1^{I}/\sqrt{2}]^T$, and $\langle \varphi^{(1)} \rangle \equiv v_{\varphi^{(1)}} / \sqrt{2}$, $\varphi^{(1)}$ plays a role in inducing $(H_2 H_1') \varphi$, $(H_1 H_2') \varphi'$ terms in order to avoid massless CP-odd scalar from Higgs doublets. The mass scale of the SM singlet scalars are taken to be much higher than electroweak scale and we obtain well-known two Higgs doublet potential after they develop VEVs. Also $Z'$ boson from $U(1)_{B-L}$ gets mass by singlet scalar

2A linear seesaw model with modular $A_4$ and global symmetry is found in ref. [23] providing different flavor structure of neutrino mass and predictions from ours.

| $Q_i$ | $u_i^c$ | $d_i^c$ | $L_i$ | $\ell_i^c$ | $N^c_i$ | $S_i$ | $H_1$ | $H_2$ | $\varphi$ | $H_1'$ | $H_2'$ | $\varphi'$ |
|-------|---------|---------|-------|-----------|--------|------|-------|-------|----------|--------|--------|---------|
| SU(3)$_C$ | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| SU(2)$_L$ | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 2 |
| U(1)$_Y$ | $\frac{1}{6}$ | $-\frac{2}{3}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| U(1)$_{B-L}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $-1$ | 1 | 1 | 0 | 0 | 1 | $-1$ | 0 | $-1$ | 1 |
| $A_4$ | 1 | 1 | 3 | 1, $1''$, $1'$ | 3 | 1, $1'$, $1''$ | 1 | 1 | 1 | 1 | 1 |
| $-k_I$ | 0 | 0 | 0 | $-1$ | $-1$ | $-1$ | $-1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1. Content of supersymmetric fields for implementation of linear seesaw mechanism and their charge assignments under $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times A_4 \times k_I$ where $k_I$ is the number of modular weight and the lower index $i(= 1, 2, 3)$ represents family.

In this study, we construct a linear seesaw model with local $U(1)_{B-L}$ and modular $A_4$ symmetry.\(^2\) In our scenario desired mass matrix for linear seesaw mechanism \([54–56]\) can be realized by $U(1)_{B-L}$ charge assignment and the nature of modular $A_4$ symmetry. We then formulate neutrino mass matrix under the symmetry and carry out numerical analysis searching for parameters fitting neutrino measurements. Our numerical $\chi^2$ square analysis shows some predictions for observables in neutrino sector.

This paper is organized as follows. In section 2 we introduce our model and formulate neutrino mass from linear seesaw mechanism with modular $A_4$ symmetry. In section 3 we carry out numerical analysis and show correlations between observables in the neutrino sector, and conclude our results in section 4.

2 Model

In this section we briefly discuss the model for linear seesaw mechanism introducing $B - L$ local Abelian symmetry $U(1)_{B-L}$ and modular $A_4$ symmetry in supersymmetric framework, where we construct the model as minimum assignment as possible. In the model, we introduce three families of right(left)-handed $SU(2)$ singlet superfields $N^c(S)$ with $-1(0)$ charge under the $U(1)_{B-L}$ gauge symmetry, and two isospin singlet superfields $\varphi, \varphi'$ with $(-1,+1)$ charges under the same $U(1)$ symmetry. Furthermore, four Higgs doublet $H_1, H_1', H_2, H_2'$ are introduced where $H_2$ has charge 1 under $U(1)_{B-L}$ while $H_1$ has no $B - L$ charge to induce the masses of SM fermions from the Yukawa Lagrangian after the spontaneous symmetry breaking as in the SM. $(H_1', H_2')$ have opposite charges to $(H_1, H_2)$ under $U(1)_Y$ and $U(1)_{B-L}$. We assign modular weight $k_I$ to these fields as summarized in table 1. Only lepton doublet $L_i$ and right-handed sterile neutrino $N^c_i$ are chosen to be $A_4$ triplet.
VEV, and we just assume the mass and gauge coupling satisfy current experimental constraints. In this paper, we omit the details of the scalar/gauge sector and focus on the neutrino sector.

Using the particle contents and symmetries mentioned in table 1, the renormalizable superpotential for leptons — including charged leptons and neutral leptons — is written as,

$$-\mathcal{W}_{\text{lepton}} = \mathcal{W}_{M_L} + \mathcal{W}_L + \mathcal{L}_{M_D} + \mathcal{W}_{M_D},$$

(2.1)

where $\mathcal{W}_{M_L}$ is superpotential inducing charged lepton masses, $\mathcal{W}_{M_D}$ is for Dirac neutrino mass term connecting active light neutrinos $\nu$ and $N^c$, $\mathcal{W}_L$ is for mixing term between two types of sterile neutrinos $N^c$ and $S$, and $\mathcal{W}_{M_D}$ is for mass term connecting $\nu$ and $S$. The Majorana mass terms for the sterile neutrinos $\tilde{N}^c$ and $S$ are absent; the former one is forbidden by $U(1)_{B-L}$ symmetry and the latter one cannot be constructed due to the nature of modular $A_4$ symmetry since $A_4$ singlets have to have 4 modular weights at least. In this paper we focus on neutrino mass matrix which is obtained from the superpotential and we do not consider effect of super partners as well as soft-SUSY breaking terms assuming mass scale of super partners are much larger than TeV scale.

**Charged lepton mass matrix.** In this model, leptons constitute $A_4$ triplet as $L \equiv [L_e, L_\mu, L_\tau]^T$ and singlets $\ell^c \equiv [e^c, \mu^c, \tau^c]$. Similar to $L$, modular couplings are also defined by $Y_3^{(2)} = [y_1, y_2, y_3]^T$ under $A_4$ triplet. The superpotential to give the charged-lepton mass matrix is given by

$$\mathcal{W}_{M_L} = Y_3^{(2)} \otimes \ell^c \otimes L \otimes H_1$$

that is explicitly written in terms of three free parameters, requiring invariance under $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times A_4 \times k_I$, as follows:

$$a_\ell \ell^c(y_1 L_e + y_2 L_\tau + y_3 L_\mu)H_1 + b_\ell \mu^c(y_3 L_\tau + y_1 L_\mu + y_2 L_e)H_1 + c_\ell \tau^c(y_2 L_\mu + y_1 L_\tau + y_3 L_e)H_1.$$  

(2.2)

Then the mass matrix for charged-lepton in basis of $[e, \mu, \tau]$ is given by

$$(M_\ell)^{RL} = \frac{v_1}{\sqrt{2}} \begin{pmatrix} a_\ell & 0 & 0 \\ 0 & b_\ell & 0 \\ 0 & 0 & c_\ell \end{pmatrix} \begin{pmatrix} y_1 & y_3 & y_2 \\ y_2 & y_1 & y_3 \\ y_3 & y_2 & y_1 \end{pmatrix}.$$  

(2.3)

The charged-lepton mass eigenstates are found by diagonalizing $\text{diag}[m_e, m_\mu, m_\tau] = V_{L_{\ell},R_{\ell}}^\dagger M_\ell V_{R_{\ell}}$, where $V_{L_{\ell},R_{\ell}}$ are unitary matrices. In our numerical analysis, we will determine the free parameters $a_\ell, b_\ell, c_\ell$ so as to fit the three charged-lepton mass eigenstates after giving all the numerical values, by applying the relations:

$$\text{Tr}[M_\ell M_\ell^\dagger] = |m_e|^2 + |m_\mu|^2 + |m_\tau|^2, \quad \text{Det}[M_\ell M_\ell^\dagger] = |m_e|^2 |m_\mu|^2 |m_\tau|^2,$$

$$\text{(Tr}[M_\ell M_\ell^\dagger])^2 - \text{Tr}[(M_\ell M_\ell^\dagger)^2] = 2(|m_e|^2 |m_\mu|^2 + |m_\mu|^2 |m_\tau|^2 + |m_e|^2 |m_\tau|^2).$$  

(2.4)

**Neutral fermion mass matrix.** The superpotential to give the neutral mass matrices are given by

$$\mathcal{W}_{M_D} + \mathcal{W}_{M_D'} + \mathcal{W}_L = Y_3^{(2)} \otimes N^c \otimes L \otimes H_1 + Y_3^{(2)} \otimes L \otimes S \otimes H_2 + Y_3^{(2)} \otimes N^c \otimes S \otimes \varphi.$$
The first term is explicitly written in terms of two parameters that is given by

\[ W_{M_D} = \left[ \frac{\alpha_1}{3} [y_1 (2N_1^c L_e - N_3^c L_\mu - N_2^c L_\tau) + y_2 (2N_2^c L_\mu - N_3^c L_e - N_1^c L_\tau)] + \frac{\alpha_2}{2} (y_3 (N_2^c L_\tau - N_3^c L_\mu) + y_2 (N_3^c L_e - N_1^c L_\tau) + y_3 (N_1^c L_\mu - N_2^c L_e)] \right] H_1. \quad (2.5) \]

After the spontaneous symmetry breaking, we obtain the mass matrix

\[ (m_D)^{N^c\nu} = \frac{v_1\alpha_1}{\sqrt{2}} \left[ \frac{1}{3} \begin{pmatrix} 2y_1 & -y_3 & -y_2 \\ -y_3 & 2y_2 & -y_1 \\ -y_2 & -y_1 & 2y_3 \end{pmatrix} - \frac{\bar{\alpha}_2}{2} \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \right] \equiv \frac{v_1\alpha_1}{\sqrt{2}} (\tilde{m}_D)^{N^c\nu}, \quad (2.6) \]

where \( \bar{\alpha}_2 \equiv \alpha_2/\alpha_1. \)

The second term is written in terms of two parameters that is given by

\[ W_{M'_{D_2}} = \beta_1 [y_1 L_e + y_2 L_\tau + y_3 L_\mu] H_2 S_1 + \beta_2 [y_3 L_\mu + y_1 L_\tau + y_3 L_e] H_2 S_2 + \beta_3 [y_3 L_\tau + y_1 L_\mu + y_2 L_e] H_2 S_L. \quad (2.7) \]

After the spontaneous symmetry breaking, we obtain the mass matrix

\[ (m'_{D_2})^{\nu S} = \frac{v_2\beta_1}{\sqrt{2}} \begin{pmatrix} y_1 & y_3 & y_2 \\ y_3 & y_2 & y_1 \\ y_2 & y_1 & y_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{\beta}_2 & 0 \\ 0 & 0 & \tilde{\beta}_3 \end{pmatrix} \equiv \frac{v_2\beta_1}{\sqrt{2}} (\tilde{m}'_{D_2})^{\nu S}, \quad (2.8) \]

where \( \tilde{\beta}_2 \equiv \beta_2/\beta_1 \) and \( \tilde{\beta}_3 \equiv \beta_3/\beta_1. \)

The third term is written in terms of two parameters that is given by

\[ L_M = \gamma_1 S_1 [y_1 N_1^c + y_2 N_3^c + y_3 N_2^c] \varphi + \gamma_2 S_2 [y_2 N_3^c + y_1 N_2^c + y_3 N_1^c] \varphi + \gamma_3 S_3 [y_3 N_3^c + y_2 N_1^c + y_2 N_3^c] \varphi. \quad (2.9) \]

After the spontaneous symmetry breaking, we obtain the mass matrix

\[ (M)^{S N^c} = \frac{v_\varphi\gamma_1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{\gamma}_2 & 0 \\ 0 & 0 & \tilde{\gamma}_3 \end{pmatrix} \begin{pmatrix} y_1 & y_3 & y_2 \\ y_3 & y_2 & y_1 \\ y_2 & y_1 & y_3 \end{pmatrix} \equiv \frac{v_\varphi\gamma_1}{\sqrt{2}} (\tilde{M})^{S N^c}, \quad (2.10) \]

where \( \tilde{\gamma}_2 \equiv \gamma_2/\gamma_1 \) and \( \tilde{\gamma}_3 \equiv \gamma_3/\gamma_1. \)

In basis of \([\nu, N^c, S]^T, \) the neutral fermion mass matrix is given by

\[ M_N = \begin{pmatrix} 0 & m_D & m'_D \\ m_D & 0 & M' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \alpha_1 v_1 \tilde{m}_D & \beta_1 v_2 \tilde{m}'_D \\ \alpha_1 v_1 \tilde{m}'_D & 0 & \gamma_1 \nu \varphi \tilde{M}' \end{pmatrix}. \quad (2.11) \]

Then, block diagonalizing the above matrix, the active neutrino mass matrix is given by

\[ m_\nu = m'_D (M'^*)^{-1} m'_D + [m'_D (M'^*)^{-1} m_T D]^T \frac{\alpha_1 \beta_1 v_1 v_2}{\sqrt{2} \gamma_1 \nu \varphi} \left( \tilde{m}'_D (M'^*)^{-1} \tilde{m}'_D + \tilde{m}'_D (M'^*)^{-1} \tilde{m}'_D \right)^T \]

\[ = \kappa \tilde{m}_\nu, \quad (2.12) \]
where \( \kappa \equiv \frac{\alpha_2 |v_{12}| v_{32}}{\sqrt{2} v_{13} v_{32}} \) and we have assumed hierarchy for scale of mass matrices as \( m_D, m'_D \ll M \).

Note that such hierarchy of mass matrix can be realized by choosing \( v_{12} \ll v_{13} \).

The neutrino mass eigenstate is found by diagonalizing the mass matrix, \( D_\nu = \kappa D'_\nu = \bar{U}_\nu m_D U_\nu = \kappa U_\nu \bar{m}_L U_\nu \), where \( U_\nu \) is a unitary matrix. Then, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is given by \( U \equiv V^L \bar{U}_\nu \). Then \( |\kappa| \) is determined by

\[
\text{(NO)}: \quad |\kappa|^2 = \frac{|\Delta m^2_{\text{atm}}|}{D_{\nu_3}^2 - D_{\nu_1}^2}, \quad \text{(IO)}: \quad |\kappa|^2 = \frac{|\Delta m^2_{\text{atm}}|}{D_{\nu_2}^2 - D_{\nu_3}^2},
\]

(2.13)

where \( \Delta m^2_{\text{atm}} \) is atmospheric neutrino mass difference squared and NO and IO stand for normal and inverted ordering respectively. Subsequently, the solar mass difference squared can be written in terms of \( |\kappa| \) as follows:

\[
\Delta m^2_{\text{sol}} = |\kappa|^2 (\bar{D}_{\nu_2}^2 - \bar{D}_{\nu_1}^2),
\]

(2.14)

which can be compared to the observed value. Note that \( \kappa \) is complex in general but its phase is not physical one and we just take the phase to be zero. In our model, the observed PMNS matrix is parametrized by three mixing angle \( \theta_{ij} (i, j = 1, 2, 3; i < j) \), one CP violating Dirac phase \( \delta_{CP} \), and two Majorana phases \( \{ \alpha_{21}, \alpha_{31} \} \) as follows:

\[
U = \begin{pmatrix}
  c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta_{CP}} \\
  -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_{CP}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{CP}} & s_{23} c_{13} \\
  s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta_{CP}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta_{CP}} & c_{23} c_{13}
\end{pmatrix}
\]

(2.15)

where \( c_{ij} \) and \( s_{ij} \) stand for \( \cos \theta_{ij} \) and \( \sin \theta_{ij} \) respectively. Then, these mixings are given in terms of the components of \( U \) as follows:

\[
\sin^2 \theta_{13} = |U_{e3}|^2, \quad \sin^2 \theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2}, \quad \sin^2 \theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}.
\]

(2.16)

Also we compute the Jarlskog invariant \( J_{CP} \) that is derived from PMNS matrix elements as follows:

\[
J_{CP} = \text{Im}[U_{e1} U_{\mu 2} U^*_{e2} U^*_{\mu 1}] = s_{23} c_{23} s_{12} c_{13} s_{13} c_{13} \sin \delta_{CP}.
\]

(2.17)

The above equation determines the angle of Dirac phase; \( \delta_{CP} \) for \( 0 \leq \cos \delta_{CP} \) and \( \pi - \delta_{CP} \) for \( \cos \delta_{CP} \leq 0 \). \( \delta_{CP} \) is found to be

\[
\cos \delta_{CP} = -\frac{|U_{\tau 1}|^2 - s_{23}^2 s_{23}^2 - c_{12}^2 c_{23}^2 s_{13}^2}{2 c_{12} s_{12} c_{23} s_{23} s_{13}}.
\]

(2.18)

Similar to the Dirac phase, Majorana phases are estimated in terms of following relations:

\[
\text{Im}[U^*_{e1} U_{e2}] = c_{12} s_{12} c_{13} s_{13} \sin \left( \frac{\alpha_{21}}{2} \right), \quad \text{Im}[U^*_{e1} U_{e3}] = c_{12} s_{12} c_{13} c_{13} \sin \left( \frac{\alpha_{31}}{2} - \delta_{CP} \right).
\]

(2.19)

The above equations determine the angles of Majorana phases; \( \alpha_{21}/2 \) for \( 0 \leq \cos(\alpha_{21}/2) \), \( \pi - \alpha_{21}/2 \) for \( \cos(\alpha_{21}/2) \leq 0 \), \( \alpha_{31}/2 - \delta_{CP} \) for \( 0 \leq \cos(\alpha_{31}/2 - \delta_{CP}) \), \( \pi - (\alpha_{31}/2 - \delta_{CP}) \) for \( \cos(\alpha_{31}/2 - \delta_{CP}) \leq 0 \). \( \cos(\alpha_{21}/2) \) and \( \cos(\alpha_{31}/2 - \delta_{CP}) \) are respectively given as follows:

\[
\text{Re} [U^*_{e1} U_{e2}] = c_{12} s_{12} c_{13} \cos \left( \frac{\alpha_{21}}{2} \right), \quad \text{Re} [U^*_{e1} U_{e3}] = c_{12} s_{12} c_{13} \cos \left( \frac{\alpha_{31}}{2} - \delta_{CP} \right).
\]

(2.20)

In addition, the effective mass for the neutrinoless double beta decay is written by

\[
\langle m_{ee} \rangle = |\kappa| \bar{D}_{\nu_e} \cos^2 \theta_{12} \cos^2 \theta_{13} + \bar{D}_{\nu_2} \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i \alpha_{21}} + \bar{D}_{\nu_3} \sin^2 \theta_{13} e^{i (\alpha_{31} - 2 \delta_{CP})},
\]

(2.21)

where its value could be measured by KamLAND-Zen in future [59].

\(^3\)In some models, hierarchy of mass matrices is realized dynamically [57, 58].
Non-unitarity. Here, let us briefly discuss non-unitarity matrix $U'_{\text{PMNS}}$. This is typically parametrized by the form

$$U'_{\text{PMNS}} \equiv \left(1 - \frac{1}{2} F F^\dagger \right) U_{\text{PMNS}},$$

(2.22)

where $F \equiv (M^*)^{-1} m_D^T$ is a hermitian matrix, and $U'_{\text{PMNS}}$ represents the deviation from the unitarity. The global constraints are found via several experimental results such as the SM $W$ boson mass $M_W$, the effective Weinberg angle $\theta_W$, several ratios of $Z$ boson fermionic decays, invisible decay of $Z$, electroweak universality, measured Cabibbo-Kobayashi-Maskawa, and lepton flavor violations [62]. The result is then given by [63]

$$|F F^\dagger| \leq \begin{bmatrix} 2.5 \times 10^{-3} & 2.4 \times 10^{-5} & 2.7 \times 10^{-3} \\ 2.4 \times 10^{-5} & 4.0 \times 10^{-4} & 1.2 \times 10^{-3} \\ 2.7 \times 10^{-3} & 1.2 \times 10^{-3} & 5.6 \times 10^{-3} \end{bmatrix}.$$  

(2.23)

In our case, $F \equiv (M^*)^{-1} m_D^T = \frac{v_\tau}{v_\chi} (M^*)^{-1} m_D^T$. Since we suppose to be $M \gg m_D$ (coming from $v_\tau \gg v_1$) that is naturally realized by the difference of breaking scale. Therefore, $v_\tau$ is B-L breaking scale which is chosen to be higher than TeV scale, while $v_1$ is electroweak scale whose order is 0.1 TeV. Taking $v_\tau \sim 10^2$ TeV we obtain $(v_1/v_\tau)^2 \approx 10^{-6}$, and we find $|F F^\dagger| \leq 10^{-6}$ that is totally safe for the above bounds of the non-unitarity.

3 Numerical analysis

In this section, we carry out numerical analysis searching for parameters satisfying neutrino data, and show our predictions.

In our numerical $\chi$ square analysis referring to [60], we scan free parameters in following ranges

$$|\text{Re}[\tau]| \in [0, 0.5], \quad |\text{Im}[\tau]| \in [0.5, 2], \quad v_\tau \in [10^3, 10^5] \text{ GeV},$$

$$\{|ar{\delta}_2|, |ar{\delta}_3|, |	ilde{\beta}_2|, |	ilde{\beta}_3|, |	ilde{\gamma}_1|, |	ilde{\gamma}_3|\} \in [10^{-5}, 100],$$

(3.1)

where couplings are taken to be complex values, perturbative limit is implicitly taken into account, and these inputs are taken so that non-unitarity constraints are satisfied, employing the four reliable experimental data: $\Delta m^2_{solar}, \sin^2 \theta_{13}, \sin^2 \theta_{23}, \sin^2 \theta_{12}$. Here, $\Delta m^2_{\text{atm}}$ is considered as the input parameter from the experimental result. Notice here that we consider CP phases $\delta_{\text{CP}}, \alpha_{21}, \alpha_{31}$ as predictive values. The parameters $\{a_e, b_\ell, c_\ell\}$ are fixed to reproduce the observed charged lepton masses where we numerically solve the conditions in eq. (2.4).

Observable in neutrino sector. As a result of numerical analysis, we find allowed parameter sets satisfying neutrino data for NO. For IO case it is more difficult to find allowed parameter sets. Since we found few points of allowed points for IO, we will show benchmark points later instead of figuring out. In figure 1, we show the allowed region of $\tau$ in fundamental region, where each of colors corresponds to the range of $\sqrt{\Delta \chi^2}$ value such that blue: $\sqrt{\Delta \chi^2} \leq 1$, green: $1 < \sqrt{\Delta \chi^2} \leq 2$, yellow: $2 < \sqrt{\Delta \chi^2} \leq 3$, and red: $3 < \sqrt{\Delta \chi^2} \leq 5$. In our model, there are not any localized regions on this space.

Figure 2 show allowed regions for several output parameters; the left-top one is the relation between two Majorana phases, the right-top one is the relation between Dirac CP phase and the neutrinoless double beta decay, the right-bottom one is the relation between
Figure 1. The region of modulus $\tau$ in fundamental region satisfying the neutrino oscillation data, where each of colors corresponds to the range of $\sqrt{\chi^2}$ value such that blue: $\sqrt{\chi^2} \leq 1$, green: $1 < \sqrt{\chi^2} \leq 2$, yellow: $2 < \sqrt{\chi^2} \leq 3$, and red: $3 < \sqrt{\chi^2} \leq 5$.

Figure 2. Left-top figure is the relation between two Majorana phases, the right-top one is the relation between Dirac CP phase and the neutrinoless double beta decay, the right-bottom one is the relation between sum of neutrino masses and the neutrinoless double beta decay, and the left-bottom one is the relation between the lightest neutrino mass and the neutrinoless double beta decay. Here, the color legend is the same as the one in figure 1.
Figure 3. Correlation among pseudo Dirac sterile neutrino masses where the color legend is the same as the one in figure 1.

sum of neutrino masses and the neutrinoless double beta decay, and the left-bottom one is the relation between the lightest neutrino mass and the neutrinoless double beta decay. The color legend is the same as the one in figure 1. Any values are allowed for phases, but there is a correlation between Majorana phases. \( m_{ee} \) is allowed in the range of \([0 - 0.022]\) eV within 5\( \sigma \). \( \sum m_i \) is allowed in the range of \([0.058 - 0.088]\) eV within 5\( \sigma \). \( m_1 \) is allowed in the range of \([0.0 - 0.017]\) eV within 5\( \sigma \).

Sterile neutrino mass. In our model, sterile neutrinos are pseudo Dirac fermions whose masses are dominantly given by \( M_{N_L^R S_L} \). In figure 3, we also show the mass eigenvalues where \( M_1 \leq M_2 \leq M_3 \) and the color legend is the same as the one in figure 1. Some parameter regions could be tested at the LHC experiments, since sterile neutrino can be produced through \( Z' \) boson. In addition, \( N_i \) can be produced through mixing with active neutrinos, \( \theta_N \nu \simeq \sqrt{m_D M^{-1}} \), through the process \( pp \to W \to \ell N_i \). Detailed analysis of collider signature is beyond the scope of this work and will be given elsewhere.

Benchmark points for IO. We could not find allowed region within \( \sqrt{\Delta \chi^2} = 3 \) in case of IO. In addition, number of allowed parameter points tends to be much smaller than the case of NO when we carry out parameter scan. Thus, we demonstrate two benchmark points (Bps) in table 2 and table 3. One is the point that minimizes the \( \chi^2 \) square in table 2, the second one is the point that is nearby the best fit of \( \delta_{CP} \sim 286^\circ \) in table 3.

4 Summary and discussion

We have constructed a linear seesaw model with local \( U(1)_{B-L} \) and modular \( A_4 \) symmetry. Majorana mass terms of sterile neutrinos are forbidden by \( U(1)_{B-L} \) charge conservation and the nature of modular \( A_4 \) symmetry, and we can realize mass matrix for linear seesaw mechanism. The Yukawa couplings for leptons are written by modular form which restricts the flavor structure of corresponding interactions.

After formulating neutrino mass matrix, we have carried out \( \chi^2 \) square numerical analysis searching for parameters satisfying neutrino data as well as unitarity bounds. We have shown predicted observables such as Dirac CP phase, sum of neutrino masses, effective mass for neutrinoless double beta decay and sterile neutrino mass hierarchy. Then some characteristic relations have been found for these observables.
\[ \tau \rightarrow \gamma \tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where } a, b, c, d \in \mathbb{Z} \quad \text{and } in [\imath] > 0. \quad (A.1) \]

This is isomorphic to \( \text{PSL}(2,\mathbb{Z}) = \text{SL}(2,\mathbb{Z})/\{I, -I\} \) transformation. Then modular transformation is generated by two transformations \( S \) and \( T \) defined as follows;

\[ S: \tau \rightarrow -\frac{1}{\tau}, \quad T: \tau \rightarrow \tau + 1, \quad (A.2) \]

and they satisfy the following algebraic relations,

\[ S^2 = I, \quad (ST)^3 = I. \quad (A.3) \]
Here we introduce the series of groups $\Gamma(N)$ ($N = 1, 2, 3, \ldots$) which are defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

(A.4)

and we define $\tilde{\Gamma}(2) = \Gamma(2)/\{I, -I\}$ for $N = 2$. Since the element $-I$ does not belong to $\Gamma(N)$ for $N > 2$ case, we have $\tilde{\Gamma}(N) = \Gamma(N)$, that are infinite normal subgroup of $\Gamma$ known as principal congruence subgroups. We thus obtain finite modular groups as the quotient groups defined by $\Gamma_N = \tilde{\Gamma}/\tilde{\Gamma}(N)$. For these finite groups $\Gamma_N$, $T^N = I$ is imposed, and the groups $\Gamma_N$ with $N = 2, 3, 4, 5$ are isomorphic to $S_3$, $A_4$, $S_4$ and $A_5$, respectively [1].

Modular forms of level $N$ are holomorphic functions $f(\tau)$ which are transformed under the action of $\Gamma(N)$ given by

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma(N),$$

(A.5)

where $k$ is the so-called as the modular weight.

Here we discuss the modular supersymmetric theory, considering the $A_4$ ($N = 3$) modular group. Under the modular transformation in eq. (A.1), a field $\phi^{(I)}$ is also transformed as

$$\phi^{(I)} \rightarrow (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)},$$

(A.6)

where $-k_I$ is the modular weight and $\rho^{(I)}(\gamma)$ denotes an unitary representation matrix of $\gamma \in \Gamma(2)$ ($A_4$ representation). Thus superpotential can be invariant if sum of modular weight

Table 3. Numerical BP of our input parameters and observables taken such that $\delta_{CP}$ is closest to the best fit value $286^\circ$.

| Parameter | Value |
|-----------|-------|
| $\tau$    | $-0.0907742 + 1.68693I$ |
| $[\hat{\beta}_2, \hat{\beta}_3]$ | $[-2.21395 - 0.000424618I, -0.000456818 + 0.812065I]$ |
| $[\hat{\gamma}_2, \hat{\gamma}_3]$ | $[0.0653089 - 0.161376I, -2.97741 + 7.35402I]$ |
| $\hat{\alpha}_2$ | $0.402348 + 0.00316048I$ |
| $[a_\ell, b_\ell, c_\ell]$ | $[0.0594865, 0.00028681, 0.600903]$ |
| $[v_\nu, |\kappa|]/\text{GeV}$ | $[2800.52, 2.91 \times 10^{-12}]$ |
| $\Delta m^2_{\text{atm}}$ | $2.52 \times 10^{-3}\text{eV}^2$ |
| $\Delta m^2_{\text{sol}}$ | $7.12 \times 10^{-5}\text{eV}^2$ |
| $\sin \theta_{12}$ | $0.564$ |
| $\sin \theta_{23}$ | $0.779$ |
| $\sin \theta_{13}$ | $0.155$ |
| $[\delta_{CP}, \alpha_{21}, \alpha_{31}]$ | $[301.638^\circ, 277.244^\circ, 257.24^\circ]$ |
| $\sum m_i$ | $102\text{meV}$ |
| $\langle m_{\nu e} \rangle$ | $38.3\text{meV}$ |
| $\sqrt{\Delta \chi^2}$ | $5.38$ |
from fields and modular form in corresponding term is zero (also invariant under $A_4$ and gauge symmetry).

The kinetic terms and quadratic terms of scalar fields can be written by

\[
\sum_I \frac{|\partial_I \phi^I|^2}{(-i\tau + i\bar{\tau})k_I}, \quad \sum_I \frac{|\phi^I|^2}{(-i\tau + i\bar{\tau})k_I},
\]

which is invariant under the modular transformation and overall factor is eventually absorbed by a field redefinition consistently. Therefore the superpotential associated with these terms should be invariant under the modular symmetry.

The basis of modular forms with weight 2, $Y = (y_1, y_2, y_3)$, transforming as a triplet of $A_4$ is written in terms of the Dedekind eta-function $\eta(\tau)$ and its derivative [2]:

\[
y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right),
\]

\[
y_2(\tau) = \frac{-i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right),
\]

\[
y_3(\tau) = \frac{-i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right).
\]

Notice here that any singlet couplings under $A_4$ start from $-k = 4$ constructed from the modular forms with $-k = 2$ while it is absent if $-k = 2$.

References

[1] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, Finite Modular Groups and Lepton Mixing, Nucl. Phys. B 858 (2012) 437 [arXiv:1112.1340] [SPIRE].

[2] F. Feruglio, Are neutrino masses modular forms?, in From My Vast Repertoire...: Guido Altarelli’s Legacy, A. Levy, S. Forte and G. Ridolfi eds., pp. 227–266 (2019) [DOI] [arXiv:1706.08749] [SPIRE].

[3] J.C. Criado and F. Feruglio, Modular Invariance Faces Precision Neutrino Data, SciPost Phys. 5 (2018) 042 [arXiv:1807.01125] [SPIRE].

[4] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, Modular $A_4$ invariance and neutrino mixing, JHEP 11 (2018) 196 [arXiv:1808.03012] [SPIRE].

[5] H. Okada and M. Tanimoto, CP violation of quarks in $A_4$ modular invariance, Phys. Lett. B 791 (2019) 54 [arXiv:1812.09677] [SPIRE].

[6] T. Nomura and H. Okada, A modular $A_4$ symmetric model of dark matter and neutrino, Phys. Lett. B 797 (2019) 134799 [arXiv:1904.03937] [SPIRE].

[7] H. Okada and M. Tanimoto, Towards unification of quark and lepton flavors in $A_4$ modular invariance, Eur. Phys. J. C 81 (2021) 52 [arXiv:1905.13421] [SPIRE].

[8] F.J. de Anda, S.F. King and E. Perdomo, SU(5) grand unified theory with $A_4$ modular symmetry, Phys. Rev. D 101 (2020) 015028 [arXiv:1812.05620] [SPIRE].

[9] P.P. Novichkov, S.T. Petcov and M. Tanimoto, Trimaximal Neutrino Mixing from Modular $A_4$ Invariance with Residual Symmetries, Phys. Lett. B 793 (2019) 247 [arXiv:1812.11289] [SPIRE].

[10] T. Nomura and H. Okada, A two loop induced neutrino mass model with modular $A_4$ symmetry, Nucl. Phys. B 966 (2021) 115372 [arXiv:1906.03927] [SPIRE].
[11] G.-J. Ding, S.F. King and X.-G. Liu, *Modular A4 symmetry models of neutrinos and charged leptons*, *JHEP* **09** (2019) 074 [arXiv:1907.11714] [INSPIRE].

[12] H. Okada and Y. Orikasa, *A radiative seesaw model in modular A4 symmetry*, arXiv:1907.13520 [INSPIRE].

[13] T. Nomura, H. Okada and O. Popov, *A modular A4 symmetric scotogenic model*, *Phys. Lett. B* **803** (2020) 135294 [arXiv:1908.07457] [INSPIRE].

[14] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, *A lepton flavor model and modulus stabilization from S4 modular symmetry*, *Phys. Rev. D* **100** (2019) 115045 [Erratum ibid. **101** (2020) 039904] [arXiv:1909.05139] [INSPIRE].

[15] H. Okada and Y. Orikasa, *A modular mass matrix model from neutrino mass, mixing and leptogenesis with linear seesaw symmetry*, *Phys. Rev. D* **101** (2020) 114935 [arXiv:1910.07869] [INSPIRE].

[16] G.-J. Ding, S.F. King, X.-G. Liu and J.-N. Lu, *Modular S4 and A4 symmetries and their fixed points: new predictive examples of lepton mixing*, *JHEP* **12** (2019) 030 [arXiv:1910.03460] [INSPIRE].

[17] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, *New A4 lepton flavor model from S4 modular symmetry, JHEP* **02** (2020) 097 [arXiv:1907.09141] [INSPIRE].
[31] S.F. King and Y.-L. Zhou, *Trimaximal TM\(_4\) mixing with two modular S\(_4\) groups*, *Phys. Rev. D* **101** (2020) 015001 [arXiv:1908.02770] [inSPIRE].

[32] H. Okada and Y. Orikasa, *Neutrino mass model with a modular S\(_4\) symmetry*, *arXiv:1908.08409* [inSPIRE].

[33] J.C. Criado, F. Feruglio and S.J.D. King, *Modular Invariant Models of Lepton Masses at Levels 4 and 5*, *JHEP* **02** (2020) 001 [arXiv:1908.11867] [inSPIRE].

[34] X. Wang and S. Zhou, *The minimal seesaw model with a modular S\(_4\) symmetry*, *JHEP* **05** (2020) 017 [arXiv:1910.09473] [inSPIRE].

[35] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Modular A\(_5\) symmetry for flavour model building*, *JHEP* **04** (2019) 174 [arXiv:1812.02158] [inSPIRE].

[36] G.-J. Ding, S.F. King and X.-G. Liu, *Neutrino mass and mixing with A\(_5\) modular symmetry*, *Phys. Rev. D* **100** (2019) 115005 [arXiv:1903.12588] [inSPIRE].

[37] A. Baur, H.P. Nilles, A. Trautner and P.K.S. Vaudrevange, *Unified Models of Neutrinos, Flavour and CP-violation*, *Nuclear Physics B* **947** (2020) 114737 [arXiv:1908.00805] [inSPIRE].

[38] A. Baur, H.P. Nilles, A. Trautner and P.K.S. Vaudrevange, *A String Theory of Flavor and CP-violation*, *Phys. J. C* **76** (2013) 056201 [arXiv:1301.1340] [inSPIRE].

[39] S.F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, *Neutrino Mass and Mixing: from Theory to Experiment*, *New J. Phys.* **16** (2014) 045018 [arXiv:1402.4271] [inSPIRE].

[40] S.F. King, *Unified Models of Neutrinos, Flavour and CP-violation*, *Prog. Part. Nucl. Phys.* **94** (2017) 217 [arXiv:1701.04413] [inSPIRE].

[41] S.T. Petcov, *Discrete Flavour Symmetries, Neutrino Mixing and Leptonic CP-violation*, *Eur. Phys. J. C* **78** (2018) 709 [arXiv:1711.10806] [inSPIRE].

[42] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T.H. Tatsuishi and H. Uchida, *CP violation in modular invariant flavor models*, *Phys. Rev. D* **101** (2020) 055046 [arXiv:1910.11553] [inSPIRE].
[52] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, Generalised CP Symmetry in Modular-Invariant Models of Flavour, JHEP 07 (2019) 165 [arXiv:1905.11970] [inSPIRE].

[53] M.-C. Chen, S. Ramos-Sánchez and M. Ratz, A note on the predictions of models with modular flavor symmetries, Phys. Lett. B 801 (2020) 135153 [arXiv:1909.06910] [inSPIRE].

[54] D. Wyler and L. Wolfenstein, Massless Neutrinos in Left-Right Symmetric Models, Nucl. Phys. B 218 (1983) 205 [inSPIRE].

[55] E.K. Akhmedov, M. Lindner, E. Schnapka and J.W.F. Valle, Left-right symmetry breaking in NJL approach, Phys. Lett. B 368 (1996) 270 [hep-ph/9507275] [inSPIRE].

[56] E.K. Akhmedov, M. Lindner, E. Schnapka and J.W.F. Valle, Dynamical left-right symmetry breaking, Phys. Rev. D 53 (1996) 2752 [hep-ph/9509255] [inSPIRE].

[57] A. Das, T. Nomura, H. Okada and S. Roy, Generation of a radiative neutrino mass in the linear seesaw framework, charged lepton flavor violation, and dark matter, Phys. Rev. D 96 (2017) 075001 [arXiv:1704.02078] [inSPIRE].

[58] W. Wang and Z.-L. Han, Radiative linear seesaw model, dark matter, and U(1)B−L, Phys. Rev. D 92 (2015) 095001 [arXiv:1508.00706] [inSPIRE].

[59] KamLAND-Zen collaboration, Search for Majorana Neutrinos near the Inverted Mass Hierarchy Region with KamLAND-Zen, Phys. Rev. Lett. 117 (2016) 082503 [Addendum ibid. 117 (2016) 109903] [arXiv:1605.02889] [inSPIRE].

[60] I. Esteban, M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz and A. Zhou, The fate of hints: updated global analysis of three-flavor neutrino oscillations, JHEP 09 (2020) 178 [arXiv:2007.14792] [inSPIRE].

[61] I. Esteban, M.C. Gonzalez-Garcia, A. Hernandez-Cabezudo, M. Maltoni and T. Schwetz, NuFIT 4.1, http://www.nu-fit.org/ (2019).

[62] E. Fernandez-Martinez, J. Hernandez-Garcia and J. Lopez-Pavon, Global constraints on heavy neutrino mixing, JHEP 08 (2016) 033 [arXiv:1605.08774] [inSPIRE].

[63] N.R. Agostinho, G.C. Branco, P.M.F. Pereira, M.N. Rebelo and J.I. Silva-Marcos, Can one have significant deviations from leptonic 3 × 3 unitarity in the framework of type-I seesaw mechanism?, Eur. Phys. J. C 78 (2018) 895 [arXiv:1711.06229] [inSPIRE].

[64] T. Kobayashi and H. Otsuka, Challenge for spontaneous CP violation in Type IIB orientifolds with fluxes, Phys. Rev. D 102 (2020) 026004 [arXiv:2004.04518] [inSPIRE].

[65] R. Srivastava, C.A. Ternes, M. Tórtola and J.W.F. Valle, Testing a lepton quarticity flavor theory of neutrino oscillations with the DUNE experiment, Phys. Lett. B 778 (2018) 459 [arXiv:1711.10318] [inSPIRE].