EQUIVARIANT SCHRÖDINGER MAP FLOW ON TWO DIMENSIONAL HYPERBOLIC SPACE

JIAXI HUANG¹, YOUDÉ WANG²,³ AND LIFENG ZHAO¹,*

¹School of Mathematical Sciences, University of Science and Technology of China
Hefei 230026, China

² College of Mathematics and Information Sciences, Guangzhou University
Guangzhou 510006, China

³ Hua Loo-Keng Key Laboratory of Mathematics, Institute of Mathematics, AMSS
and School of Mathematical Sciences, UCAS
Beijing 100190, China

(Communicated by Joachim Krieger)

Abstract. In this article, we consider the Schrödinger flow of maps from two dimensional hyperbolic space \(\mathbb{H}^2\) to sphere \(\mathbb{S}^2\). First, we prove the local existence and uniqueness of Schrödinger flow for initial data \(u_0 \in \mathbb{H}^3\) using an approximation scheme and parallel transport introduced by McGahagan [32]. Second, using the Coulomb gauge, we reduce the study of the equivariant Schrödinger flow to that of a system of coupled Schrödinger equations with potentials. Then we prove the global existence of equivariant Schrödinger flow for small initial data \(u_0 \in \mathbb{H}^1\) by Strichartz estimates and perturbation method.

1. Introduction. Let \((\mathbb{H}^2, g)\) be the two dimensional hyperbolic space, the Schrödinger map flow is defined by the initial value problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= J(u)\tau(u), \\
u(x,0) &= u_0,
\end{aligned}
\]

where \(u(x,t) : (\mathbb{H}^2, g) \times [0, \infty) \to \mathbb{S}^2\), \(J\) is the complex structure on \(\mathbb{S}^2\), and \(\tau(u)\) is the tension field of \(u\). In the local coordinates \((x_1, x_2)\) on \(\mathbb{H}^2\) and \((y_1, y_2)\) on \(\mathbb{S}^2\), \(\tau(u)\) is given by

\[
\tau(u) = (\Delta_{\mathbb{H}^2} u^\gamma + g^{ij} \bar{\Gamma}_{\alpha \beta}^\gamma(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}) \frac{\partial}{\partial y^\gamma},
\]

where \(\Delta_{\mathbb{H}^2}\) is the Laplace-Beltrami operator on \((\mathbb{H}^2, g)\), and \(\bar{\Gamma}_{\alpha \beta}^\gamma(u)\) are the Christoffel symbols on \(\mathbb{S}^2\).

The Schrödinger flow on Euclidean spaces has been intensely studied in the last decades. The local well-posedness of Schrödinger flow was established by Sulem, Sulem and Bardos [36] for \(\mathbb{S}^2\) target, Ding and Wang [11, 12] and McGahagan...
for general Kähler manifolds. The first global well-posedness of Schrödinger flow for maps $\mathbb{R}^d \to S^2$, $d \geq 3$ with small data in the critical Besov spaces was proved by Ionescu and Kenig [16], and independently by Bejenaru [3]. This was later improved to global regularity for small data in the critical Sobolev spaces by Bejenaru, Ionescu and Kenig [4] ($d \geq 4$) and Bejenaru, Ionescu, Kenig and Tataru [5] ($d \geq 2$) for target $S^2$, and by Li [27, 28] for general compact Kähler manifolds. However, the Schrödinger map flow with large data is a much more difficult problem. When the target is $S^2$, there exists a collection of families $Q^m$ (see [8]) of finite energy stationary solutions for integer $m \geq 1$; When the target is $H^2$, there is no nontrivial equivariant stationary solutions with finite energy. Hence, Bejenaru, Ionescu, Kenig and Tataru [6, 7] proved the global well-posedness and scattering for equivariant Schrödinger flow of maps $\mathbb{R}^2 \to S^2$ with energy below the ground state and maps $\mathbb{R}^2 \to H^2$ with finite energy. When the energy of maps is larger than that of ground state, the dynamic behaviors are complicated. The asymptotic stability and blow-up for Schrödinger flow have been considered by many authors for instance [13, 14, 15, 8, 33, 34]. We refer to [18] for more open problems in this field.

It’s natural to consider geometric flows starting from curved manifolds. Because the hyperbolic spaces are symmetric and noncompact, the geometric flows from hyperbolic spaces are natural starting points. An interesting model is the heat flow from hyperbolic spaces, which is related to the Schoen-Li-Wang conjecture (see Lemm, Markovic [24]). By solving the heat flow, Li and Tam [30] gave the sufficient conditions to ensure the existence of the harmonic map between hyperbolic spaces. In recent years, there are many works concerning wave maps on hyperbolic spaces which are expected to share many similar phenomena with Schrödinger flow. D’Ancona and Zhang [10] showed the global existence of equivariant wave maps from hyperbolic spaces $\mathbb{H}^d$ for $d \geq 3$ to general targets with small data in $H^{\frac{d}{2}} \times H^{\frac{d}{2} - 1}$. The problem was also intensely studied by Lawrie, Oh and Shahshahani [20, 21, 22, 23], Li, Ma and Zhao [29] and Li [25, 26]. Since the wave maps $\mathbb{H}^2 \to \mathbb{H}^2$ or $S^2$ admit a family of equivariant harmonic maps, the stability of stationary $k$-equivariant wave maps was proved by analyzing spectral properties of the linearized operator in [20] and [21], and the soliton resolution for equivariant wave maps $\mathbb{H}^2 \to \mathbb{H}^2$ with finite energy data was proved by profile decomposition in [22]. For data without any symmetric assumption, Li, Ma and Zhao [29] proved that the small energy harmonic maps from $\mathbb{H}^2$ to $\mathbb{H}^2$ are asymptotically stable under the wave map, then Li [25, 26] further showed the asymptotic stability of large energy harmonic maps. Lawrie, Oh and Shahshahani [23] established the global well-posedness and scattering for wave maps from $\mathbb{H}^d$ for $d \geq 4$ into Riemannian manifolds of bounded geometry for small data in the critical Sobolev space. As a geometric flow, Schrödinger flow is a special case of Landau-Lifshitz flows. Li and Zhao [31] proved that the solution of Landau-Lifshitz flow $u(t, x)$ from $\mathbb{H}^2$ to $\mathbb{H}^2$ converges to some harmonic map $P(x)$ as $t \to \infty$, i.e. $\lim_{t \to \infty} \sup_{x \in \mathbb{H}^2} d_{\mathbb{H}^2}(u(t, x), P(x)) = 0$, when the Gilbert coefficient is positive.

The Schrödinger flow on $\mathbb{H}^2$ exhibits markedly different phenomena from its Euclidean counterpart. First, the most interesting feature is that there is an abundance of equivariant harmonic maps as shown in [20]. Precisely, there is a family of equivariant harmonic maps $\mathbb{H}^2 \to S^2$, indexed by a parameter that measures how far the image of each harmonic map wraps around the sphere. These maps have energies taking all values between zero and the energy of the unique co-rotational Euclidean
harmonic map, $Q_{\text{euc}}$ from $\mathbb{R}^2$ to $\mathbb{S}^2$. Second, the notable feature of the problem is the better dispersive estimates of the operator $e^{it\Delta x^2}$ than the Euclidean counterpart, see [2]. The stronger dispersion is possible due to the exponential volume growth of concentric geodesic spheres on the domain. The above features make (1) an interesting model.

In this paper, we establish the local existence and uniqueness of Schrödinger flow for large data and global existence and uniqueness of equivariant Schrödinger flow for small data. To explain the main results in more detail, we need to introduce some notations.

Let $u$ be a smooth map from $\mathbb{H}^2$ to $\mathbb{S}^2$. The pullback bundle $u^*TS^2$ is the vector bundle over $\mathbb{H}^2$ whose fiber at $x \in \mathbb{H}^2$ is the tangent space $T_{u(x)}\mathbb{S}^2$. Smooth sections of $u^*TS^2$ are maps $V : \mathbb{H}^2 \to TS^2$ so that $V(x) \in T_{u(x)}\mathbb{S}^2$ for each $x \in \mathbb{H}^2$. Let $\nabla$ denote the connections on different vector bundles which are naturally induced by Levi-Civita connections on $\mathbb{H}^2$ and $\mathbb{S}^2$. Sometimes in the context, we also use more specific notations such as $\nabla^{\mathbb{H}^2}$ and $\nabla^{\mathbb{S}^2}$ to emphasize which connection we are using. In the local coordinates $(x_1, x_2)$ on $\mathbb{H}^2$ and $(y_1, y_2)$ on $\mathbb{S}^2$, if $V = V^\alpha \frac{\partial}{\partial y^\alpha}$, then the covariant derivative on $u^*TS^2$ are given by

$$\nabla_{\frac{\partial}{\partial y^\alpha}} V = (\partial_i V^\gamma + \bar{\Gamma}_{\alpha\beta}^\gamma(u) V^\alpha \partial_i u^\beta) \frac{\partial}{\partial y^\gamma}.$$ 

Hence, we may define the intrinsic Sobolev spaces $H^k(\mathbb{H}^2; \mathbb{S}^2)$ by

$$\|u\|_{H^k(\mathbb{H}^2; \mathbb{S}^2)}^2 = \|\partial u\|_{H^{k-1}}^2 := \sum_{l=0}^{k-1} \int_{\mathbb{H}^2} |\nabla^l \partial u|^2_g \, \text{dvol}_g. \quad (2)$$

where $\text{dvol}_g$ is the volume form of $(\mathbb{H}^2, g)$. $\nabla^{l-1}$ is the $(l-1)$-th covariant derivative of tangent vectors $\partial_i$, and in the local coordinates,

$$|\nabla^{l-1} \partial u|^2_g = g^{i_1i_2} \cdots g^{i_{l-1}i_l} \left( \nabla_{\frac{\partial}{\partial x^{i_1}}} \cdots \nabla_{\frac{\partial}{\partial x^{i_{l-1}}}} \partial_i u, \nabla_{\frac{\partial}{\partial x^{i_{l-2}}}} \cdots \nabla_{\frac{\partial}{\partial x^{i_{l-1}}}} \partial_i u \right)_{u^*TS^2}.$$ 

For simplicity, denote $H^k := H^k(\mathbb{H}^2; \mathbb{S}^2)$. It is easy to check that the flow (1) admits the conserved energy

$$E(u) = \frac{1}{2} \int_{\mathbb{H}^2} |\nabla u|^2_g \, \text{dvol}_g,$$

where $|\nabla u|^2_g$, in the local coordinates, is given by

$$|\nabla u|^2_g = g^{ij} \left( \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j} \right)_{u^*TS^2}.$$ 

The first main result is the following.

**Theorem 1.1** (Local existence and uniqueness in $H^3$). *Given initial data $u_0 \in H^3$, then there exists $T > 0$ depending on $\|u_0\|_{H^3}$, such that (1) has a unique solution in $L^\infty_t([0,T]; H^3)$. Moreover, we have*

$$\sup_{t \in [0,T]} \|u(t)\|_{H^3} \leq C(\|u_0\|_{H^3}).$$

**Remark 1.** The proof of Theorem 1.1 follows closely that of [32, 35]. The local existence for Schrödinger map flow with data $u_0 \in H^3$ is proved by approximation of wave maps on hyperbolic space $(\mathbb{H}^2, g)$ and the uniqueness is given by parallel transport.
The second result concerns the equivariant Schrödinger map flow with small data \( u_0 \in H^1 \). To explain this result, we introduce the geodesic polar coordinates on \( \mathbb{H}^2 \). We consider the Minkowski space \( \mathbb{R}^{2+1} \) with the Minkowski metric \( dx_1^2 + dx_2^2 - dx_3^2 \), and the bilinear form on \( \mathbb{R}^{2+1} \times \mathbb{R}^{2+1} \),

\[
[x, y] = -x_1 y_1 - x_2 y_2 + x_3 y_3.
\]

Then hyperbolic space \( \mathbb{H}^2 \) is defined as

\[
\mathbb{H}^2 = \{ x \in \mathbb{R}^3 : [x, x] = 1, x_3 > 0 \},
\]

and the Riemannian metric \( g \) on \( \mathbb{H}^2 \) is exactly the one induced by the Minkowski metric on \( \mathbb{R}^{2+1} \).

We denote the origin of \( \mathbb{H}^2 \) by \( O = (0, 0, 1) \). And \( r : \mathbb{H}^2 \to [0, \infty) \), \( r(x) = d(x, O) \) denotes the Riemannian distance function to the origin. Then the geodesic polar coordinates on \( \mathbb{H}^2 \) are defined by

\[
\phi : \mathbb{R}_+ \times [0, 2\pi) \to \mathbb{H}^2 \subset \mathbb{R}^3, \quad \phi(r, \theta) = (\sinh r \cos \theta, \sinh r \sin \theta, \cosh r).
\]

In these coordinates, the hyperbolic metric \( g \) is given by

\[
g = dr^2 + \sinh^2 r d\theta^2,
\]

the volume element \( d\text{vol}_g \) on \( \mathbb{H}^2 \) is given by \( \sinh r dr d\theta \) and the Laplace-Beltrami operator is written as

\[
\Delta_{\mathbb{H}^2} = \partial_r^2 + \coth r \partial_r + \sinh^{-2} r \partial_\theta^2.
\]

Since \( u \) is a map \( \mathbb{H}^2 \to S^2 \subset \mathbb{R}^3 \) here, \( u \) is \( m \)-equivariant if and only if \( u \) can be written in the polar coordinates as

\[
u(r, \theta) = e^{m\theta R} \bar{u}(r).
\]

Here \( R \) is the generator of horizontal rotations, which is defined as

\[
R := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Ru = \vec{k} \times u,
\]

where \( \vec{k} = (0, 0, 1)^T \). The energy of \( m \)-equivariant maps can be expressed as

\[
E(u) = \pi \int_0^\infty \left( [\partial_r \bar{u}]^2 + \frac{m^2}{\sinh^2 r} (\bar{u}_1^2 + \bar{u}_2^2) \right) \sinh r dr.
\]

If \( m \neq 0 \), then \( E(u) < \infty \) implies \( \lim_{r \to 0} u_1(r) = \lim_{r \to 0} u_2(r) = 0 \), and then \( \lim_{r \to 0} u_3(r) = \pm 1 \) by \( |u| = 1 \). To fix matters we agree that this limit is \( 1 \) for all \( t \). However, due to the exponential decay of \( \sinh^{-1} r \) as \( r \to \infty \) in the last term in the integrand of (4), we can choose the endpoints \( \lim_{r \to \infty} \bar{u}_1(r) \), \( \lim_{r \to \infty} \bar{u}_2(r) \in [-1, 1] \) arbitrarily, which is ultimately responsible for the existence of the family of harmonic maps mentioned in the above. This stands in contrast to the corresponding problem for Schrödinger map flow on Euclidean spaces where the endpoints of \( \bar{u}_1(r), \bar{u}_2(r) \) at \( r \to \infty \) can only be zero. For any \( m \)-equivariant Schrödinger map flow of maps \( u : \mathbb{H}^2 \to S^2 \) with endpoints \( \lim_{r \to 0} u_3(r) = 1 \), \( \lim_{r \to \infty} u_3(r) = \frac{1 - \lambda^2}{1 + \lambda^2} \) for \( \lambda \in [0, \infty) \), the minimal energy is

\[
E(u) \geq E_{\text{min}} = 4m\pi \frac{\lambda^2}{1 + \lambda^2}
\]

and is attained by the 1-parameter family of harmonic maps

\[
o_m := \{ e^{m\theta R} h^m_{\alpha}(r) \mid \alpha \in [0, 2\pi) \},
\]

where

\[
h^m_{\alpha}(r) := e^{\alpha R} h_{\lambda}(r),
\]
and
\[
\lambda(r) = \left( \frac{2\lambda\tanh \frac{r}{2}}{1 + (\lambda\tanh \frac{r}{2})^2}, 0, \frac{1 - (\lambda\tanh \frac{r}{2})^2}{1 + (\lambda\tanh \frac{r}{2})^2} \right).
\]
This leads us to consider the equivariant Schrödinger flow with data in the classes
\[
\mathcal{E}_\lambda = \left\{ u : \mathbb{H}^2 \to \mathbb{S}^2 \mid E(u) < \infty, \lim_{t \to 0} u_3 = 1, \lim_{t \to \infty} u_3 = \frac{1 - \lambda^2}{1 + \lambda^2} \right\}, \quad \lambda \in [0, \infty).
\]

Next we state our second main result.

**Theorem 1.2** (Global existence for small data). There exists a sufficiently small constant \(\epsilon_0 > 0\) such that for any 1-equivariant map \(u_0\) with \(\|u_0\|_{H^1} \leq \epsilon_0\) and for any compact interval \(J \subset \mathbb{R}\), the equivariant Schrödinger map flow (1) has a unique solution \(u \in L^\infty(J; H^1)\) in the class \(\mathcal{E}_\lambda\) for \(\lambda_0\) satisfying \(\lim_{r \to 0} u_{0,3}(r) = \frac{1 - \lambda_0^2}{1 + \lambda_0^2}\), defined as the unique limit of smooth solution in \(H^1\) with initial data \(u(0) = u_0\). Moreover,
\[
\sup_{t \in J} \|u(t)\|_{H^1} \leq C\|u_0\|_{H^1},
\]
and if \(u_0 \in H^3\),
\[
\sup_{t \in J} \|u(t)\|_{H^3} \leq C(J, \|u_0\|_{H^3}).
\]

**Remark 2.** (i) The result in Theorem 1.2 can be extended to all \(m\)-equivariant cases for \(m \neq 0\).

(ii) Under the equivariant condition, the smallness assumption on \(u_0\) and the inequality (5) imply that \(\lambda_0\) must be small.

(iii) In light of the works [20, 25, 26, 29], which concerns the asymptotic stability of finite energy harmonic maps from \(\mathbb{H}^2\) to \(\mathbb{S}^2\) or \(\mathbb{H}^2\) under the wave maps evolution, one expects the asymptotic stability of harmonic maps under the Schrödinger flow. In fact, Lawrie-Lührmann-Oh-Shahshahani [19] proved the asymptotic stability of a finite energy equivariant harmonic map \(Q\) under the Schrödinger flow with respect to non-equivariant perturbations, provided \(Q\) obeys a suitable linearized stability condition.

Theorem 1.2 is of similar flavor to the result of [6] in the flat domain \(\mathbb{R}^2\). First, since we restrict ourselves to the class of equivariant Schrödinger flow, the Coulomb gauge suffices in our study. In fact, in this gauge, we can rewrite the equations for \(\partial_\nu u\) and \(\partial_\theta u\) which leads to a system of coupled Schrödinger equations on \(\mathbb{H}^2\) with potentials, i.e.
\[
\begin{cases}
(i\partial_t + \Delta_{\mathbb{H}^2} - 2\frac{\cosh r + 1}{\sinh^2 r})\psi^+ = (A_0 + 2\frac{\cosh r(A_2 - 1)}{\sinh^2 r})\psi^+ - \Im(\psi^+ \frac{\bar{\psi}_2}{\sinh r})\psi^+,
\end{cases}
\]

\[
\begin{cases}
(i\partial_t + \Delta_{\mathbb{H}^2} + 2\frac{\cosh r - 1}{\sinh^2 r})\psi^- = (A_0 - 2\frac{\cosh r(A_2 - 1)}{\sinh^2 r})\psi^- + \Im(\psi^- \frac{\bar{\psi}_2}{\sinh r})\psi^-,
\end{cases}
\]

where \(\psi^\pm : \mathbb{H}^2 \times [0, \infty) \to \mathbb{C}\) and \(A_2, A_0 : \mathbb{H}^2 \times [0, \infty) \to \mathbb{R}\) are radial functions. Moreover, \(A_2, A_0\) can be expressed in terms of \(\psi^\pm\), i.e.
\[
A_2 - 1 = \int_0^r \frac{|\psi^+|^2 - |\psi^-|^2}{4} \sinh s ds,
\]
and
\[
A_0 = -\frac{1}{2} \Re(\bar{\psi}_2 \psi^-) + \int_r^\infty \frac{\cosh s}{\sinh s} \Re(\psi^+ \psi^-) ds.
\]
Next, we consider the Cauchy problem of the system (8). In fact, we establish the Strichartz estimates for Schrödinger operators with nonnegative potentials, and prove the global existence of (8) for small data $\psi_0^\pm$ in the space $L^2(\mathbb{H}^2)$. Since our interest lies in the solutions which correspond to the geometric flow, we will show that the solutions of the system satisfy the compatibility condition. Finally, to construct the Schrödinger map flow $u$ from $\psi^\pm$, the key observation is that $\psi^+$ or $\psi^-$ contain all the information of the flow as in [6]. Hence, we can recover the flow $u(t)$ from $\psi^\pm(t)$ for small initial data $\psi^+(0), \psi^-(0) \in L^2$. Furthermore by Theorem 1.1, we show that the flow $u(t)$ is a Schrödinger map flow for data $u_0$ in $H^3$. At the same time, we obtain the Lipschitz continuity of $u(t)$ with respect to $u_0$ in $H^1$, which yields Theorem 1.2.

There are two main obstacles in the above arguments. One is the a priori higher order energy estimates for approximate wave map equations, which guarantees the uniform lifespan $T > 0$ for approximate solutions. In order to simplify the computation, the global system of coordinates related to the Iwasawa decomposition is used. Meanwhile the uniform estimates follows from a bootstrap argument. The other obstacle lies in the establishment of the well-posedness for the coupled Schrödinger system with potentials. In order to avoid the presence of the singular potential $2\frac{\cosh r + 1}{\sinh^2 r}$ in the $\psi^+$-equation in (8), we multiply $e^{it\varphi}$ to this equation, and denote $R_k\psi^+(r, \theta) = e^{ik\theta}\psi^+(r)$, for $k = 0, 1, 2 \cdots$, then the $\psi^+$-equation can be rewritten as

$$\left(i\partial_t + \Delta_{\mathbb{H}^2} - 2\frac{\cosh r - 1}{\sinh^2 r}\right)R_2\psi^+ = R_2F^+, \quad (9)$$

where $F^+$ denotes the nonlinearity of $\psi^+$-equation in (8). Note that, the potential $2\frac{\cosh r - 1}{\sinh^2 r}$ in (9) is regular. In order to establish the Strichartz estimates for (9), we prove the dispersive estimates for $e^{it(\Delta_{\mathbb{H}^2} - V)}$ with nonnegative potential $V \in e^{-\alpha t}L^\infty(\mathbb{H}^2), \alpha \geq 1$. Since the dispersive estimate for $t > 1$ has been provided by [9], we only need to establish the similar estimate for $0 < t < 1$, namely

$$\left\|e^{it(\Delta_{\mathbb{H}^2} - V)}\right\|_{L^1 \to L^\infty} \lesssim t^{-1}. \quad (10)$$

By Birman-Schwinger type resolvent expansion, the resolvent $R_V$ can be expressed as a series with respect to $R_0$, $R_V$ and $V$, then the Schrödinger propagator in (10) can be written as a series. The estimates of the dominant terms can be derived from the pointwise bounds of the free resolvent kernel in [9] and Lemma 5.4 since these terms only depend on $R_0$ and $V$. For the remainder term, we use the meromorphic continuity of resolvent $R_V$ in Lemma 5.3. For the $\psi^-$-equation, the negative regular potential can be regarded as a perturbation term of the nonlinear term, then we prove the global existence for small data by perturbation method (see [37]).

The rest of the paper is organized as follows: In Section 2 we recall the basic properties of the hyperbolic spaces and introduce some functional spaces and basic inequalities. In Section 3 we use the approximation scheme and parallel transport to prove local existence and uniqueness for Schrödinger map flow (1) in $H^3$, i.e Theorem 1.1. In the rest of the article, we consider the equivariant Schrödinger map flow with small initial data. Precisely, in Section 4, we rewrite the Schrödinger map flow in the Coulomb gauge as two coupled Schrödinger equations, i.e ($\psi^+, \psi^-$)-system. Moreover, we show that the Schrödinger map flow $u$ can be recovered from $\psi^+$ in $L^2(\mathbb{H}^2)$ with $\|\psi^+\|_{L^2(\mathbb{H}^2)}$ sufficiently small. In Section 5 we prove the Strichartz estimates and get the well-posedness of ($\psi^+, \psi^-$)-system for data $\psi_0^\pm \in L^2$. Finally, we give the proof of Theorem 1.2.
2. Preliminaries. In this section we review the geometry of two dimensional hyperbolic space and the function spaces, then state some basic inequalities.

2.1. Hyperbolic space \((\mathbb{H}^2, g)\). For latter use, we introduce a global system of coordinates on \(\mathbb{H}^2\) related to the Iwasawa decomposition (see [17]): define the diffeomorphism \(\varphi : \mathbb{R}^2 \to \mathbb{H}^2\),

\[
\varphi(x, y) = (\sinh y + e^{-y}|x|^2/2, e^{-y}x, \cosh y + e^{-y}|x|^2/2).
\]

(11)

Then the metric \(g\) and volume form \(\text{dvol}_g\) can be written as

\[
g = e^{-2y}(dx^2 + e^{2y}dy^2), \quad \text{dvol}_g = e^{-y}dx\,dy.
\]

If we fix the global orthonormal frame

\[
e_1 = e^y\partial_x, \quad e_2 = \partial_y,
\]

it’s easily shown that \([e_2, e_1] = e_1\) and

\[
\nabla e_1 e_1 = e_2, \quad \nabla e_1 e_2 = -e_1, \quad \nabla e_2 e_1 = \nabla e_2 e_2 = 0.
\]

2.2. Function spaces and basic inequalities. Here we define some relevant function spaces on \(\mathbb{H}^2\) and recall some basic inequalities. For a smooth function \(f : \mathbb{H}^2 \to \mathbb{R}\), the \(L^p(\mathbb{H}^2)\)-norm for \(1 \leq p \leq \infty\) are defined by

\[
\|f\|_{L^p(\mathbb{H}^2)} := \left( \int_{\mathbb{H}^2} |f(x)|^p \, \text{dvol}_g \right)^{1/p}, \quad \|f\|_{L^\infty(\mathbb{H}^2)} := \sup_{x \in \mathbb{H}^2} |f(x)|,
\]

(13)

and for smooth function \(f(x, t) : \mathbb{H}^2 \times I \to \mathbb{R}\), the space-time norms \(L^p(I; L^q(\mathbb{H}^2))\) are defined by

\[
\|f\|_{L^p(I; L^q(\mathbb{H}^2))} := \left( \int_I \left( \int_{\mathbb{H}^2} |f(x, t)|^q \, \text{dvol}_g \right)^{p/q} \, dt \right)^{1/p}. \]

For simplicity, denote \(\|f\|_{L^p(I; L^q(\mathbb{H}^2))} := \|f\|_{L^p(I; L^q(\mathbb{H}^2))}\). We can also define the Sobolev norm \(H^k(\mathbb{H}^2; \mathbb{R})\) by

\[
\|f\|^2_{H^k} := \sum_{l \leq k} \int_{\mathbb{H}^2} |\nabla^lf|^2 \, \text{dvol}_g,
\]

where \(\nabla^l\) is the \(l\)-th covariant derivative of \(f\). In the local coordinates \((x_1, x_2)\) on \(\mathbb{H}^2\), the components of \(\nabla f\) are given by \((\nabla f)_i = \partial_{x_i} f\), and those of \(\nabla^2 f\) are then

\[
(\nabla^2 f)_{ij} = \partial_{x_i} \partial_{x_j} f - \Gamma^k_{ij} \partial_k f,
\]

where \(\Gamma^k_{ij}\) are the Christoffel symbols of \(\mathbb{H}^2\). We then have

\[
|\nabla^lf|^2 = g^{ij_1 \cdots j_l} \cdots g^{ij_1}(\nabla f)_{i_1 \cdots i_l} \cdot (\nabla f)_{j_1 \cdots j_l}.
\]

As were shown in [23], for \(l = 0, 1, 2, \ldots\),

\[
\|f\|_{H^l} \simeq \left\| (-\Delta)^l f \right\|_{L^2},
\]

\[
\|f\|_{H^{l+1}} \simeq \| \nabla^{l+1} f \|_{L^2} \simeq \| (-\Delta)^l f \|_{L^2}.
\]

(14)

For a smooth map \(u : \mathbb{H}^2 \to S^3 \subset \mathbb{R}^3\), if \(u_i - u_i(\infty) \in H^k\) for \(i = 1, 2, 3\), then the Sobolev spaces of the map can also be defined extrinsically, whose \(H^k(\mathbb{H}^2; \mathbb{R}^3)\)-norm is defined as

\[
\|u\|_{H^k(\mathbb{H}^2; \mathbb{R}^3)} := \sum_{i=1}^3 \|u_i - u_i(\infty)\|_{H^k}.
\]

(15)
Moreover, the extrinsic Sobolev norm (15) and intrinsic Sobolev norm (2) are equivalent (see [31]) in the sense that there exist polynomials $P$ and $Q$ such that

$$
\|u\|_{H^3} \leq P(\|u\|_{H^3}), \quad \|u\|_{H^3} \leq Q(\|u\|_{H^3}).
$$

We now recall the Sobolev inequalities and diamagnetic inequality (see [31], [23]).

**Lemma 2.1.** Let $f$ be a Schwartz function on $\mathbb{H}^2$, then for $1 < p < \infty$, $p \leq q \leq \infty$, $0 < \theta < 1$, $1 < r < 2$, $r \leq l < \infty$, the following inequalities hold:

\[
\|f\|_{L^q} \lesssim \|f\|_{L^p}^{1-\theta} \|\nabla f\|_L^\theta, \quad \text{for} \quad \frac{1}{q} = \frac{1}{p} - \frac{\theta}{2},
\]

\[
\|f\|_{L^\infty} \lesssim \left\|(-\Delta)^{\frac{\alpha}{2}} f\right\|_{L^2}, \quad \text{for} \quad \alpha > 1,
\]

\[
\|\nabla f\|_{L^2} \sim \left\|(-\Delta)^{\frac{\alpha}{2}} f\right\|_{L^2},
\]

\[
\|f\|_{L^\infty} \lesssim \|f\|_{L^1}^{1/2} \|(-\Delta)^{1/2} f\|_{L^2}.
\]

**Lemma 2.2.** If $T$ is some $(r, s)$-type tensor or tensor matrix defined on $\mathbb{H}^2$, then in the distribution sense, one has

$$
\left| \nabla \left| T \right| \right| \leq \left| \nabla T \right|.
$$

In Section 4 and 5, we will consider the equivariant Schrödinger flow (1), then we will work mainly with functions of a single variable $r$. Hence, for radial functions, the Lebesgue integral and spaces are with respect to the sinh $r dr$-measure, unless otherwise specified. In fact, for a smooth radial function $f$, by (13) the $L^p$-norm of $f$ is

$$
\|f\|_{L^p} = (2\pi \int_0^\infty |f|^p \sinh r dr)^{1/p},
$$

and we define the $\dot{H}^1_e$ norm by

$$
\|f\|^2_{\dot{H}^1_e} : = \int_0^\infty |\partial_r f|^2 \sinh r dr + \int_0^\infty \left| \frac{f}{\sinh r} \right|^2 \sinh r dr,
$$

which is a natural space as it can be seen from the expression of $E(u)$. The functions in $\dot{H}^1_e$ admit the following important properties: they are continuous and have limit 0 at $r = 0$. These properties imply the Sobolev embedding immediately

$$
\|f\|_{L^\infty} \lesssim \|f\|_{\dot{H}^1_e}.
$$

For a radial function $f$ and for an integer $k$ we define

$$
R_k f(r, \theta) = e^{ik\theta} f(r).
$$

By direct computation, we have

$$
\|R_k f\|_{H^1} \sim \|f\|_{\dot{H}^1_e}, \quad \text{for} \quad k \geq 1, \quad \|R_0 f\|_{H^1} \sim \|\partial_r f\|_{L^2}.
$$

We also have the following bounds

**Lemma 2.3.** (i) If $R_0 f \in H^2$, then

$$
\|\partial_r^2 f\|_{L^2} + \left| \frac{\cosh r}{\sinh r} \partial_r f \right|_{L^2} \lesssim \|R_0 f\|_{H^2}.
$$

(ii) If $R_k f \in H^2$, for $k \geq 2$, then

$$
\|\partial_r^2 f\|_{L^2} + \left| \frac{\cosh r}{\sinh r} \partial_r f \right|_{L^2} + \left| \frac{f}{\sinh r} \right|_{L^2} \lesssim \|R_k f\|_{H^2}.
$$
\[
|\nabla^2 R_0 f|^2 = |\partial_x^2 f|^2 + \left| \frac{\cosh r}{\sinh r} \partial_r f \right|^2.
\]
Thus the bound (23) follows.

(ii) Since
\[
|\nabla^2 R_k f|^2 = |\partial_x^2 f|^2 + k^2 \left| \frac{1}{\sinh r} \partial_r f - \frac{\cosh r}{\sinh^2 r} f \right|^2 + \left| \frac{\cosh r}{\sinh r} \partial_r f - k^2 \frac{f}{\sinh^2 r} \right|^2.
\]
Then by (22) we have
\[
\left\| \frac{\cosh r}{\sinh r} \partial_r f - \frac{f}{\sinh^2 r} \right\|_{L^2} \lesssim \left\| \frac{1}{\sinh r} \partial_r f - \frac{\cosh r}{\sinh^2 r} f \right\|_{L^2} + \left\| \frac{\cosh r - 1}{\sinh r} \partial_r f + \frac{\cosh r - 1}{\sinh^2 r} f \right\|_{L^2} \lesssim \|R_k f\|_{H^2} + \|\partial_r f\|_{L^2} + \left\| \frac{f}{\sinh r} \right\|_{L^2} \lesssim \|R_k f\|_{H^2}.
\]
Hence, these imply that for \(k \geq 2\)
\[
\left\| \frac{f}{\sinh^2 r} \right\|_{L^2} \lesssim \left\| \frac{\cosh r}{\sinh r} \partial_r f - \frac{f}{\sinh^2 r} \right\|_{L^2} + \left\| \frac{\cosh r}{\sinh r} \partial_r f - \frac{k^2 f}{\sinh^2 r} \right\|_{L^2} \lesssim \|R_k f\|_{H^2}.
\]
We also obtain
\[
\left\| \frac{\cosh r}{\sinh r} \partial_r f \right\|_{L^2} \lesssim \left\| \frac{\cosh r}{\sinh r} \partial_r f - \frac{f}{\sinh^2 r} \right\|_{L^2} + \left\| \frac{f}{\sinh^2 r} \right\|_{L^2} \lesssim \|R_k f\|_{H^2}.
\]
Thus the bound (24) follows. \(\Box\)

Finally, the following estimates are frequently used for radial functions, which can be obtained by Schur’s test easily.

**Lemma 2.4.** Let \(f \in L^p\) be radial function, we have
\[
\left\| \frac{\cosh r}{\sinh^2 r} J f(s) \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 < p \leq \infty,
\]
\[
\left\| \int_0^r \frac{\cosh s}{\sinh s} f(s) ds \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 \leq p < \infty,
\]
\[
\left\| \int_r^\infty e^{-\rho} \frac{\sinh^{-1} s f(\rho) d\rho}{\sinh s} \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 \leq p < \infty,
\]
\[
\left\| \frac{1}{r^2} \int_0^r f(\rho) \rho^{p-1} d\rho \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}, \quad 1 \leq p \leq \infty.
\]

3. **Local well-posedness for Schrödinger maps.** In order to prove the local well-posedness in \(H^1\), we apply the approximation scheme and parallel transport introduced by McGahagan [32], see also [31, 35]. For any \(\delta > 0\), we introduce the wave map model equation:
\[
\begin{aligned}
\delta^2 \nabla_t \partial_t u^\delta - J \partial_t u^\delta - \tau(u^\delta) &= 0, \\
u^\delta(x, 0) &= u_0, \quad \partial_t u^\delta(x, 0) = g_0^\delta.
\end{aligned}
\]
where \(u^\delta(x, t) : \mathbb{H}^2 \times [0, T] \to \mathbb{S}^2\) and \(g_0^\delta \in T_{u_0(x)} \mathbb{S}^2\). In this section we use the global coordinates (11) and the global orthonormal frame (12). For simplicity, denote
\[
\nabla_i := \nabla_{e_i}, \quad \text{for } i = 1, 2,
\]
denote $u = u^\delta$ and $C(H^k) = C([0,T];H^k)$ in the proofs of Lemma 3.1 and Theorem 1.1. The covariant derivatives on $u^T \mathbb{S}^2$ do not commute, and their lack of commutation is measured by the curvature tensor $\tilde{R}^{S^2}$ on $\mathbb{S}^2$:

$$[\nabla_i, \nabla_j]V := \nabla_i \nabla_j V - \nabla_j \nabla_i V = \tilde{R}^{S^2}(e_i(u), e_j(u))V + \nabla_{[e_i,e_j]}V,$$

where $V(x) \in T_{u(x)}\mathbb{S}^2$. Before proving Theorem 1.1, we need the following lemma.

**Lemma 3.1.** For $\tilde{T} > 0$, there exists a constant $C > 0$ independent of $\delta$, and polynomials $P$ and $Q$ such that for any $u^\delta : [0,\tilde{T}] \times \mathbb{S}^2 \to \mathbb{S}^2$, $u^\delta \in C([0,T];H^3)$, a solution of the approximate equation, and any $1 \leq k \leq 2$, the following estimate holds for $u^\delta$:

$$\| \nabla^{k-1}\partial_t u^\delta \|_{C([0,T];L^2)} \leq CP(\|g_0^\delta\|_{H^{k+1}})Q(\|u^\delta\|_{C([0,T];H^{k+1})})$$

for some $0 < T < \tilde{T}$, depending only on the size of the solution $\|u^\delta\|_{C([0,T];H^3)}$ and on the size of the initial data $\|g_0^\delta\|_{H^1}$.

**Proof.** For $k = 1$, take the inner product of the above wave map equation with $J(u)\nabla_t \partial_t u$, then the first term will disappear by orthogonality and we get

$$\int_{\mathbb{S}^2} \langle \partial_t u, \nabla_t \partial_t u \rangle \text{dvol} = \int_{\mathbb{S}^2} \langle J\tau(u), \nabla_t \partial_t u \rangle \text{dvol}$$

for $k = 1$, take the inner product of the above wave map equation with $J(u)\nabla_t \partial_t u$, then the first term will disappear by orthogonality and we get

$$\int_{\mathbb{H}^2} \langle J\nabla_t \tau(u), \partial_t u \rangle \text{dvol} = \int_{\mathbb{H}^2} \langle J\nabla_t \tau(u), \partial_t u \rangle \text{dvol} - \int_{\mathbb{H}^2} \langle J \nabla_t \tau(u), \partial_t u \rangle \text{dvol}.$$  \hfill (29)

In the system of coordinates, $\tau(u)$ can be written as $\tau(u) = \nabla_i e_i(u) - (\nabla_i e_i)(u)$, then commute $\nabla_i$ and $\nabla_t$. Hence by integration by parts, the second term of (30) becomes

$$\int_{\mathbb{H}^2} \langle J \nabla_t \tau(u), \partial_t u \rangle \text{dvol} = \int_{\mathbb{H}^2} \langle J \nabla_t (\nabla_i e_i(u) - (\nabla_i e_i)(u)), \partial_t u \rangle \text{dvol}$$

$$= \int_{\mathbb{H}^2} \langle J \nabla_t, \nabla_t \rangle e_i(u) + J \nabla_t e_i(u) - J \nabla_t e_2 u, \partial_t u \rangle \text{dvol}$$

$$= \int_{\mathbb{H}^2} \langle J \nabla_t, \nabla_t \rangle e_i(u) + J \nabla_t e_i(u) - J \nabla_t e_2 u, \partial_t u \rangle \text{dvol}$$

Integrating (29) in time, we have

$$\frac{1}{2} \| \partial_t u(t) \|_{L^2}^2 = \frac{1}{2} \| \partial_t u(0) \|_{L^2}^2 + \int_{\mathbb{H}^2} \langle J \tau(u), \partial_t u \rangle(t) \text{dvol} - \int_{\mathbb{H}^2} \langle J \tau(u), \partial_t u \rangle(0) \text{dvol}$$

$$- \int_0^t \int_{\mathbb{H}^2} \langle J \nabla_t, \nabla_t \rangle e_i(u, \partial_t u) \text{dvol} \text{ds}.$$
\[ \int \frac{1}{2} \| \partial_t u(t) \|_{L^2}^2 + \| \tau(u(t)) \|_{L^2} \| \partial_t u(t) \|_{L^2} + \| \tau(u)(0) \|_{L^2} \| \partial_t u(0) \|_{L^2} \]
\[ + \int_0^t \| \partial_t u \|_{L^2}^2 \| D u \|_{L^\infty}^2 \, ds \]
\[ \leq C \| \partial_t u(0) \|_{L^2}^2 + C \| \tau(u)(t) \|_{L^2}^2 + C \| \tau(u)(0) \|_{L^2}^2 \]
\[ + \int_0^t \| \partial_t u \|_{L^2}^2 \| D u \|_{L^\infty}^2 \, ds, \]

which yields

\[ \| \partial_t u(t) \|_{L^2}^2 \leq \| \partial_t u(0) \|_{L^2}^2 + \| \tau(u) \|_{C([0,T];L^2)}^2 + \int_0^t \| \partial_t u \|_{L^2}^2 \| D u \|_{H^2}^2 \, ds. \]

Therefore, choosing \( T \) such that \( \| D u \|_{C([0,T];H^2)} \) small, we have by Gronwall inequality that

\[ \| \partial_t u(t) \|_{L^2}^2 \leq (\| \partial_t u(0) \|_{L^2}^2 + \| \tau(u) \|_{C([0,T];L^2)}^2)(1 + \| \partial_t u \|_{C([0,T];H^2)}^2). \]  

(31)

For \( k = 2 \), applying \( \nabla_t \) to the approximate equation (28):

\[ \delta^2 \nabla_t \nabla_t \partial_t u - J \nabla_t \partial_t u - \nabla_t \tau(u) = 0, \]

then we take the inner product of the above equation with \( J(u) \nabla_t \nabla_t \partial_t u \) and commute \( \nabla_t \) and \( \nabla_t \), we have

\[ 0 = \int_{\mathbb{H}^2} \langle J \nabla_t \partial_t u, J \nabla_t \nabla_t \partial_t u \rangle \, dvol_g - \int_{\mathbb{H}^2} \langle J \nabla_t \tau, \nabla_t \nabla_t \partial_t u \rangle \, dvol_g \]
\[ = \frac{1}{2} \frac{d}{dt} \| \nabla_t \partial_t u \|_{L^2}^2 + \int_{\mathbb{H}^2} \langle \nabla_t \partial_t u, [\nabla_t, \nabla_t] \partial_t u \rangle \, dvol_g - \frac{d}{dt} \int_{\mathbb{H}^2} \langle J \nabla_t \tau, \nabla_t \partial_t u \rangle \, dvol_g \]
\[ + \int_{\mathbb{H}^2} \langle J \nabla_t \nabla_t \partial_t u, \nabla_t \nabla_t \partial_t u \rangle \, dvol_g - \int_{\mathbb{H}^2} \langle J \nabla_t \tau, [\nabla_t, \nabla_t] \partial_t u \rangle \, dvol_g \]
\[ \triangleq \frac{1}{2} \frac{d}{dt} \| \nabla_t \partial_t u \|_{L^2}^2 + 1 - \frac{d}{dt} \int_{\mathbb{H}^2} \langle J \nabla_t \tau, \nabla_t \partial_t u \rangle \, dvol_g + II - III, \]  

(32)

which, together with integration in time and Hölder inequality, gives

\[ \frac{1}{2} \| \nabla_t \partial_t(u(t)) \|_{L^2}^2 \leq \frac{1}{2} \| \nabla_t \partial_t(u(0)) \|_{L^2}^2 + \| \nabla \tau(t) \|_{L^2} \| \nabla \partial_t(u(t)) \|_{L^2}^2 \]
\[ + \| \nabla \tau(0) \|_{L^2} \| \nabla \partial_t(u(0)) \|_{L^2}^2 + \int_0^t | - I - II + III | \, ds. \]

Thus,

\[ \| \nabla_t \partial_t(u(t)) \|_{L^2}^2 \leq \| \nabla_t \partial_t(u(0)) \|_{L^2}^2 + \| \nabla \tau(t) \|_{L^2}^2 + \| \nabla \tau(0) \|_{L^2}^2 \]
\[ + \int_0^t | - I - II + III | \, ds. \]  

(33)

Since \( II \) can be rewritten as

\[ II = \int \langle J(\nabla_t \tau, \nabla_t \nabla_t \partial_t u) + e_i(\nabla_t \nabla_t \partial_t u) - \langle J \nabla_t \tau, \nabla_t \nabla_t \partial_t u \rangle \rangle \, dvol_g \]
\[ \triangleq II_1 + II_2 + II_3. \]
Due to \( \tau(u) \) and that \( (\nabla_j e_j) u = e_2 u \), \( \Pi_2 \) becomes

\[
\Pi_2 = - \int_{\mathbb{H}^2} e_i (\nabla_i \tau, J\nabla_i \partial_t u) \, dV_g
\]

\[
= - \int_{\mathbb{H}^2} \langle \nabla_t (\nabla_j e_j) u - (\nabla_j e_j) u, J\nabla_t e_2 u \rangle \, dV_g
\]

\[
= - \int_{\mathbb{H}^2} \langle \nabla_j \nabla_t e_j u, J\nabla_t e_2 u \rangle + \langle [\nabla_t, \nabla_j] e_j u, J\nabla_t e_2 u \rangle \, dV_g.
\]

For \( \Pi_3 \), integration by parts gives

\[
\Pi_3 = - \int_{\mathbb{H}^2} \langle J [\nabla_t, \nabla_j] e_j u, \nabla_t \nabla_i \partial_t u \rangle + \langle J [\nabla_j, \nabla_j] e_j u, \nabla_i \partial_t u \rangle
\]

\[
- \langle J \nabla_t e_2 u, \nabla_i \nabla_i \partial_t u \rangle \, dV_g
\]

\[
= \int_{\mathbb{H}^2} \langle [\nabla_t, \nabla_j] e_j u, J\nabla_t e_2 u \rangle + \langle J [\nabla_i, \nabla_j] e_j u, \nabla_i \partial_t u \rangle
\]

\[
+ \langle J \nabla_t e_2 u, \nabla_i \nabla_i \partial_t u \rangle \, dV_g.
\]

Hence,

\[
\Pi = \int_{\mathbb{H}^2} \langle J [\nabla_t, \nabla_i] \tau, \nabla_i \partial_t u \rangle + \langle J [\nabla_i, \nabla_j] e_j u, \nabla_i \partial_t u \rangle \, dV_g. \tag{34}
\]

From (32), (34), Hölder inequality and (17), we have

\[
| - I - II + III | \leq \| \nabla \partial_t u \|_{L^2} \left( \| [\nabla_t, \nabla_i] \tau \|_{L^2} + \| [\nabla_i, \nabla_j] e_j u \|_{L^2} \right)
\]

\[
+ \left( \| \partial_t u \|_{H^2} + \| \nabla \partial_t u \|_{L^2} \right) \| [\nabla_t, \nabla_i] \partial_t u \|_{L^2}
\]

\[
\leq \| \nabla \partial_t u \|_{L^2} \left( \| \partial_t u \|_{L^4} \| \tau(u) \|_{L^4} \| \partial_t u \|_{L^\infty}
\right.
\]

\[
+ \| \partial_t u \|_{L^\infty} \| \partial_t u \|_{L^2} \| \partial_t u \|_{L^\infty} \| \nabla \partial_t u \|_{L^2}
\]

\[
+ \| \partial_t u \|_{L^4} \| \partial_t u \|_{L^2} \| \nabla \partial_t u \|_{L^2}
\]

\[
+ \left( \| \partial_t u \|_{H^2} + \| \nabla \partial_t u \|_{L^2} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

\[
\leq \| \nabla \partial_t u \|_{L^2} \left( \| \nabla \partial_t u \|_{L^2} + \| \partial_t u \|_{L^2} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

\[
+ \| \nabla \partial_t u \|_{L^2} \| \partial_t u \|_{L^2} \| \nabla \partial_t u \|_{H^2}
\]

\[
\leq \| \nabla \partial_t u \|_{L^2} \left( \| \partial_t u \|_{L^2} + \| \partial_t u \|_{L^2} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

\[
+ \| \partial_t u \|_{L^2} \left( \| \nabla \partial_t u \|_{L^2} + \| \partial_t u \|_{L^2} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

\[
+ \| \partial_t u \|_{L^2} \left( \| \nabla \partial_t u \|_{L^2} + \| \partial_t u \|_{L^2} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

\[
+ \| \partial_t u \|_{L^2} \left( \| \nabla \partial_t u \|_{L^2} + \| \partial_t u \|_{L^2} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

This, together with (31), implies

\[
| - I - II + III | \leq \| \nabla \partial_t u \|_{L^2} \left( \| \partial_t u(0) \|_{L^2} + \| \partial_t u \|_{C(H^2)} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

\[
+ \| \partial_t u \|_{L^2} \left( \| \partial_t u \|_{C(H^2)} \right) \| [\nabla_t, \nabla_j] e_j u \|_{H^2}
\]

Hence, combining this with (33) and Gronwall inequality, we get

\[
\| \nabla \partial_t u \|_{L^2} \leq P(\| g_0 \|_{H^1}) Q(\| u \|_{C(H^3)}).
\]

This completes the proof of the lemma. \( \square \)
Proof of Theorem 1.1. Step 1. Prove the existence of (1). We choose data $g_0^\delta$ such that $\|g_0^\delta\|_{H^1} < C$ and $\delta^2 \|g_0^\delta\|^2_{H^2} < C$. Without any restriction we make the bootstrap assumption

$$\|u\|_{C([0,T];H^3)} + \|\nabla \tau(u)\|_{C([0,T];L^2)} \leq 2C(\|u(0)\|_{H^1}, \|\nabla \tau(u(0))\|_{L^2}).$$

(35)

Define the energy functional by

$$E_1(u, \partial_t u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\delta^2}{2} \|\partial_t u\|_{L^2}^2,$$

then by (28), we have $\frac{d}{dt} E_1 = 0$. Define the second order energy functional by

$$E_2(u, \partial_t u) = \frac{1}{2} \|\nabla \partial_t u\|_{L^2}^2 + \frac{\delta^2}{2} \|\nabla \partial_t u\|_{L^2}^2,$$

by (28) we have

$$\frac{d}{dt} E_2 = \frac{1}{2} \|\nabla \partial_t u\|_{L^2}^2 + \delta^2 \int_{\mathbb{H}^2} \langle \nabla \nabla_t \partial_t u, \nabla \nabla_t u \rangle + \langle \nabla_t \nabla_t u, \nabla \nabla_t u \rangle dvol_g
= \frac{1}{2} \|\nabla \partial_t u\|_{L^2}^2 + \int_{\mathbb{H}^2} \langle J \nabla_t \partial_t u + \nabla \partial_t \tau(u), \nabla \partial_t u \rangle dvol_g$$

(36)

$$+ \int_{\mathbb{H}^2} \delta^2 \langle \nabla \nabla_t \partial_t u, \nabla \partial_t u \rangle dvol_g.$$

By integration by parts and $\langle JX, X \rangle = 0$, the second term of (36) becomes

$$\int_{\mathbb{H}^2} \langle J \nabla_t \partial_t u + \nabla \partial_t \tau(u), \nabla \partial_t u \rangle dvol_g
= \int_{\mathbb{H}^2} \langle \nabla \tau_j e_j u - \langle \nabla_j e_j u, \nabla \partial_t u \rangle dvol_g
= \int_{\mathbb{H}^2} \langle \nabla_i e_j u, \nabla \partial_t u \rangle - \langle \nabla_i e_j u, [\nabla_j, \nabla_i] e_i u \rangle - \langle \nabla_i e_j u, \nabla \delta \nabla_j e_i u \rangle dvol_g$$

(37)

and the last term of (37) becomes

$$\int_{\mathbb{H}^2} -\langle \nabla_i e_j u, \nabla \nabla_j e_i u \rangle dvol_g
= \int_{\mathbb{H}^2} -\langle \nabla_i e_j u, \nabla \nabla_j e_i u \rangle - \langle \nabla_i e_j u - \langle \nabla_i e_j u, \nabla \nabla_j e_i u \rangle \rangle
- \langle \nabla_i e_j u, \nabla \delta \nabla_j e_i u \rangle + \langle \nabla_i e_j u, \nabla \delta \nabla_j e_i u \rangle dvol_g$$

(38)

$$= -\frac{1}{2} \|\nabla \partial_t u\|_{L^2}^2 + \int_{\mathbb{H}^2} [ - \langle \nabla_i e_j u, \nabla \nabla_j e_i u \rangle + \langle \nabla_i e_j u, \nabla \partial_t \tau(u) \rangle dvol_g
- \langle \nabla_i e_j u, [\nabla_j, \nabla_i] e_i u \rangle - \langle \nabla_i e_j u, \nabla \delta \nabla_j e_i u \rangle
+ \langle \nabla_i e_j u, \nabla \delta \nabla_j e_i u \rangle dvol_g.$$

Hence, by (18), Hölder inequality, (36), (37) and (38), we have

$$\frac{d}{dt} E_2 \lesssim (\|\nabla \partial_t u\|_{L^2} + \|\nabla \delta \partial_t u\|_{L^2}) P(\|u\|_{H^3}) + \delta^2 \|\nabla \partial_t u\|_{L^2}^2 \|\partial_t u\|_{L^2} \|u\|_{H^3}.$$

Define the third order energy functional by

$$E_3 = \frac{1}{2} \|\nabla \tau(u)\|_{L^2}^2 + \frac{\delta^2}{2} \|\nabla \partial_t u\|_{L^2}^2.$$
Then integration by parts gives
\[
\frac{d}{dt}E_3 \leq \int_{\mathbb{R}^2} \delta^2 \left( |\partial_t u| |\partial u| |\nabla \partial_t u| + |\partial u|^2 |\partial_t u|^2 + |\partial_t u|^2 |\nabla^2 u| + |\partial u||\partial_t u|^2 \right) \cdot (|\nabla^2 \partial_t u| + |\nabla \partial u|) dvol_g \\
+ \int_{\mathbb{R}^2} \left[ |\partial u|^2 |\tau(u)| |\nabla \partial_t u| + |\nabla \tau(u)| (|\partial_t u| |\partial u|^2 + |\nabla \partial_t u|) \\
+ |\nabla \tau(u)| (|\partial u|^2 |\partial_t u| + |\partial u|^2 |\nabla \partial_t u|) \right] dvol_g
\]

By (17) and Hölder inequality, we have
\[
(39) \leq \delta^2 \left( |\nabla^2 \partial_t u|_{L^2} + |\nabla \partial_t u|_{L^2} \right) \left( |\partial_t u|_{L^4} |\nabla \partial_t u|_{L^4} + |\partial u|_{L^2}^2 |\partial u|_{L^2}^2 \\
+ |\partial_t u|_{L^2}^2 |\nabla u|_{L^4} + |\partial u|_{L^4}^2 |\partial u|_{H^2} \\
+ |\nabla \partial_t u|_{L^2} |\partial u|_{L^2}^2 |\partial u|_{L^2}^2 + |\nabla \partial_t u|_{L^2}^2 |\partial u|_{L^2}^2 |\partial u|_{H^2} \\
+ |\nabla \partial_t u|_{L^2} |\partial u|_{L^2}^2 |\partial u|_{H^2} \right)
\]

Similarly, we also have
\[
(40) \leq (|\nabla \partial_t u|_{L^2} + |\partial_t u|_{L^2}) P(|u|_{H^3}).
\]

Hence,
\[
\frac{d}{dt} (E_1 + E_2 + E_3) \\
\leq C \delta^2 |\partial_t u|_{H^2}^2 |\nabla \partial_t u|_{L^2}^2 P(|u|_{H^3}) + C (|\nabla \partial_t u|_{L^2} + |\partial_t u|_{L^2}) P(|u|_{H^3}).
\]

Since integration by parts yields
\[
|\nabla^2 \partial_t u|_{L^2} \leq |\nabla \tau(u)|_{L^2}^2 + |\partial u|_{L^6}^6 + |\nabla \partial_t u|_{L^2}^2 |\partial u|_{L^4}^4 + |\nabla u|_{L^2}^2,
\]

(17) gives
\[
|\nabla^2 \partial_t u|_{L^2}^2 \leq P(|u|_{H^2})^2 + |\nabla \tau(u)|_{L^2}^2.
\]

Thus, integrating (41) in time and taking the supremum over \( t \in [0, T] \), we have
\[
\delta^2 |\partial_t u|_{C(H^2)}^2 + |u|_{C(H^2)}^2 + |\nabla \tau(u)|_{C(L^2)}^2 \\
\leq \delta^2 |g(0)|_{H^2}^2 + |u(0)|_{H^2}^2 + |\nabla \tau(u(0))|_{L^2}^2 \\
+ CT \delta^2 |\partial_t u|_{C(H^2)}^2 P(|u|_{C(H^2)} + |\nabla \tau(u)|_{C(L^2)}) \\
+ CTQ (|u|_{C(H^2)} + |\nabla \tau(u)|_{C(L^2)}).
\]

Choosing \( T \) small such that \( CTP(|u|_{C([0,T];H^2)} + |\nabla \tau(u)|_{C([0,T];L^2)}) < \frac{1}{2} \), from (43) we have
\[
|u|_{C([0,T];H^2)}^2 + |\nabla \tau(u)|_{C([0,T];L^2)}^2 + \frac{\delta^2}{2} |\partial_t u|_{C(H^2)}^2 - \delta^2 |g_0|_{H^2}^2 \\
\leq |u(0)|_{H^2}^2 + |\nabla \tau(u(0))|_{L^2}^2 + CTQ (|u|_{C(H^2)} + |\nabla \tau(u)|_{C(L^2)}).
\]
If \( \| \partial_t u \|_{C^2(\mathbb{H}^2)}^2 \geq 2 \| g_0 \|_{\mathbb{H}^2}^2 \), (44) implies
\[
\| u \|_{C^2(\mathbb{H}^2)}^2 + \| \nabla u \|_{C^2(\mathbb{H}^2)}^2 \leq \| u(0) \|_{\mathbb{H}^2}^2 + \| \nabla u(0) \|_{\mathbb{H}^2}^2 + C T Q(\| u \|_{C^1(\mathbb{H}^2)} + \| \nabla u \|_{C^1(\mathbb{H}^2)}) \cdot
\]

If \( \| \partial_t u \|_{C^2(\mathbb{H}^2)}^2 < 2 \| g_0 \|_{\mathbb{H}^2}^2 \), we obtain from (43) that
\[
\| u \|_{C^2(\mathbb{H}^2)}^2 + \| \nabla u \|_{C^2(\mathbb{H}^2)}^2 \leq C + \| u(0) \|_{\mathbb{H}^2}^2 + \| \nabla u(0) \|_{\mathbb{H}^2}^2 + 3 C T Q(\| u \|_{C^1(\mathbb{H}^2)} + \| \nabla u \|_{C^1(\mathbb{H}^2)}) .
\]

Hence, by the bootstrap assumption (35), there exists \( T \) small such that
\[
\| u \|_{C^2(\mathbb{H}^2)} + \| \nabla u \|_{C^2(\mathbb{H}^2)} < \frac{3}{2} C .
\]
Therefore, by (42) we have
\[
\| u \|_{C^2(\mathbb{H}^2)} \leq C(\| u(0) \|_{\mathbb{H}^2})
\]
for some fixed \( T > 0 \) depending only on the size of data \( u(0) \). This concludes the local existence of (1) in \( \mathbb{H}^3 \).

**Step 2.** Prove the uniqueness of (1). The proof of the uniqueness is standard. Here, we only show that the uniqueness holds in \( \mathbb{H}^3 \) using the ideas of McGahagan [32] and Song-Wang [35].

Assume that \( u^{(1)}, u^{(2)} \in L^\infty([0, T], \mathbb{H}^3) \) are two solutions to the Schrödinger flow (1) with the same initial map \( u_0 \in \mathbb{H}^3 \).

By \( \mathbb{S}^2 \subset \mathbb{R}^3 \) and (1), we have for \( \lambda = 1, 2 \)
\[
\| u^{(\lambda)}(t, x) - u_0(x) \|_{L^2} \leq \| \int_0^t \partial_s u^{(\lambda)}(s, x) ds \|_{L^2} \leq C t \| \tau(u^{(\lambda)}) \|_{L^2} \leq C t .
\]

This, together with (20), implies
\[
\| u^{(\lambda)} - u_0 \|_{L^\infty} \leq C \| u^{(\lambda)} - u_0 \|_{L^2}^{1/2} \| \Delta(u^{(\lambda)} - u_0) \|_{L^2}^{1/2} \leq C t^{1/2} .
\]
From this, there exists \( T' > 0 \) such that \( \| u^{(1)} - u^{(2)} \| < \delta_0 \) for any \( (t, x) \in [0, T'] \times \mathbb{H}^2 \), and hence, there exists a unique minimizing geodesic \( \gamma_{(t, x)} : [0, 1] \to \mathbb{S}^2 \) such that \( \gamma_{(t, x)}(0) = u^{(1)}(t, x) \) and \( \gamma_{(t, x)}(1) = u^{(2)}(t, x) \). Let \( (t, x) \) vary, the family of geodesics give rise to a map \( U : [0, 1] \times [0, T'] \times \mathbb{H}^2 \to \mathbb{S}^2 \) connecting \( u^{(1)} \) and \( u^{(2)} \), where \( U(s, t, x) = \gamma_{(t, x)}(s) \). Therefore, we can define a global bundle morphism \( P : u^{(2)} \times \mathbb{T}\mathbb{S}^2 \to u^{(1)} \times \mathbb{T}\mathbb{S}^2 \) by the parallel transportation along each geodesic.

By the similar argument to [35, Section 3.3-3.6], we have
\[
\frac{1}{2} \frac{d}{dt} \| u^{(1)} - u^{(2)} \|_{L^2}^2 \leq \| P \nabla u^{(2)} - \nabla u^{(1)} \|_{L^2}^2 + C \| u^{(1)} - u^{(2)} \|_{L^2}^2 ,
\]
where the constant \( C \) depends on the \( L^\infty \)-norm of \( \nabla u^{(1)} \) and \( \nabla u^{(2)} \), and
\[
\frac{1}{2} \frac{d}{dt} \| P \nabla u^{(2)} - \nabla u^{(1)} \|_{L^2}^2 \leq C(\| u^{(1)} - u^{(2)} \|_{L^2}^2 + \| P \nabla u^{(2)} - \nabla u^{(1)} \|_{L^2}^2 \\
+ \| \nabla^2 u^{(1)} \|_{L^4}^2 + \| \nabla^2 u^{(2)} \|_{L^4}^2)^2 \| u^{(1)} - u^{(2)} \|_{L^4}^2 ) ,
\]
where the constant $C$ depends on the Riemannian curvature of $\mathbb{H}^2$ and $S^2$ and the $L^\infty$-norm of $\nabla u^{(1)}$ and $\nabla u^{(2)}$. Using the interpolation inequality (17) and Sobolev embedding (18), we obtain
\[
\|\nabla^2 u^{(1)}\|_{L^4} + \|\nabla^2 u^{(2)}\|_{L^4} + \|\nabla u^{(1)}\|_{L^\infty} + \|\nabla u^{(2)}\|_{L^\infty} \lesssim \|u^{(1)}\|_{H^3} + \|u^{(2)}\|_{H^3}.
\]
Let $d_{g^2}(u^{(1)}, u^{(2)})$ be the intrinsic distance. Since $S^2$ has bounded geometry, by (17) and the estimate in Lemma 2.2 of [35], we have
\[
\|u^{(1)} - u^{(2)}\|_{L^4} \leq C\|d_{g^2}(u^{(1)}, u^{(2)})\|_{L^4} \leq C\|d_{g^2}(u^{(1)}, u^{(2)})\|_{H^1}
\]
\[
\leq C(\|u^{(1)} - u^{(2)}\|_{L^2} + \|P\nabla u^{(2)} - \nabla u^{(1)}\|_{L^2}).
\]
The above four bounds yield
\[
\frac{1}{2} \frac{d}{dt}(\|u^{(1)} - u^{(2)}\|_{L^2}^2 + \|P\nabla u^{(2)} - \nabla u^{(1)}\|_{L^2}^2)
\]
\[
\leq C(\|u^{(1)} - u^{(2)}\|_{L^2}^2 + \|P\nabla u^{(2)} - \nabla u^{(1)}\|_{L^2}^2),
\]
where $C$ depends on the $H^2$-norm of $u^{(1)}$ and $u^{(2)}$. Since $\|u^{(1)} - u^{(2)}\|_{L^2} = \|P\nabla u^{(2)} - \nabla u^{(1)}\|_{L^2} = 0$ at initial time, we then obtain $u^{(1)} = u^{(2)}$ on $[0, T]$ by Gronwall’s inequality. By repeating the above argument, we can prove $u^{(1)} = u^{(2)}$ on the whole interval $[0, T]$ and finish the proof of the uniqueness. \qed

4. The Coulomb gauge representation of the equation. In this section, we rewrite the equivariant Schrödinger map flow (1) in the Coulomb gauge and obtain the $(\psi^+, \psi^-)$-system (8) of coupled Schrödinger equations. Conversely, we can recover the map $u$ from $\psi^+$ or $\psi^-$ at fixed time.

4.1. The Coulomb gauge. We choose $v \in u^*TS^2$ such that $v \cdot v = 1$ and define $w = u \times v$. Thus
\[
w \cdot w = 1, \quad w \cdot u = w \cdot v = 0, \quad w \times u = v, \quad v \times w = u.
\]
(45)
Since $u$ is 1-equivariant it is natural to work with 1-equivariant frame, that is
\[
v(r, \theta, t) = e^{\theta R} \bar{v}(r, t), \quad w(r, \theta, t) = e^{\theta R} \bar{w}(r, t),
\]
where $\bar{v}, \bar{w} \in u^*TS^2$ are unit vectors depending on $r$ and $t$. Let the differentiation operators $\partial_0, \partial_r, \partial_\theta$ stand for $\partial_\partial, \partial_r, \partial_\theta$, respectively. On one hand in such a frame we obtain the differentiated fields $\psi_k$ and the connection coefficients $A_k$, by
\[
\psi_k = \partial_k u \cdot v + i\partial_k u \cdot w, \quad A_k = \partial_k v \cdot w, \quad k \in \{0, 1, 2\},
\]
(46)
where $\psi_k, A_k, k = 0, 1, 2$, are radial functions. On the other hand, suppose $\psi_k$ and $A_k$ are given, the frame $(u, v, w)$ can be recovered via the system:
\[
\begin{cases}
\partial_k u = (R\psi_k)v + (\bar{3}\psi_k)w, \\
\partial_k v = -(R\psi_k)u + A_k w, \\
\partial_k w = -(\bar{3}\psi_k)u - A_k v.
\end{cases}
\]
(47)
If we introduce the covariant differentiation
\[
D_k = \partial_k + iA_k, \quad k \in \{0, 1, 2\},
\]
(48)
then it is easy to check the compatibility conditions
\[
D_k \psi_l = D_l \psi_k, \quad l, k \in \{0, 1, 2\}.
\]
(49)
Moreover, the curvature of this connection is given by
\[
D_k D_l - D_l D_k = i(\partial_k A_l - \partial_l A_k) = i\bar{3}(\psi_k \bar{\psi}_l).
\]
(50)
It is important to note that $\psi_2, A_2$ are closely related to the original map. Precisely, the definitions of $\psi_k$ and $A_k$ imply $A_2 = u_3$ and $\psi_2 = w_3 - i v_3$. Hence we obtain $|\psi_2|^2 = u_1^2 + u_2^2$, and the important conservation law

$$A_2^2 + |\psi_2|^2 = 1 \quad (51)$$

We now turn to choose the orthonormal frame $(\bar{v}, \bar{w})$ on $S^2$. For the equivariant Schrödinger map flow $u$, the Coulomb gauge $\text{div} A = 0$ can be reformulated in the polar coordinate,

$$\partial_r A_1 + \coth r A_2 + \sinh^{-2} r \partial_\theta A_2 = 0.$$  

Since $A_2 = u_3$ is radial, one can choose $A_1 = 0$, i.e. $\partial_r \bar{v} \cdot \bar{w} = 0$, which can be represented as ODE

$$\partial_r \bar{v} = - (\bar{v} \cdot \partial_r \bar{u}) \bar{u}. \quad (52)$$

The ODE (52) need to be initialized at some point. To avoid introducing a constant time-dependent potential into the equation via $A_0$, we need to choose this initialization uniformly with respect to $t$. Since we restrict the solution $u \in \mathcal{E}_\lambda$, by rotation we may restrict the data $\lim_{r \to \infty} \bar{u}(r, t) = \left( \frac{2\lambda}{1+\lambda^2}, 0, \frac{1-\lambda^2}{1+\lambda^2} \right)$, then we can fix the choice of $\bar{v}$ and $\bar{w}$ at infinity,

$$\lim_{r \to \infty} \bar{v}(r, t) = \left( \frac{1-\lambda^2}{1+\lambda^2}, 0, -\frac{2\lambda}{1+\lambda^2} \right)^\top, \quad \lim_{r \to \infty} \bar{w}(r, t) = \bar{j} = (0, 1, 0)^\top. \quad (53)$$

The existence and uniqueness of (52) with boundary condition (53) and $\|u\|_{H^1} \ll 1$ is standard. Indeed, using the Picard iteration scheme

$$\bar{v} = \sum_{i=0}^{\infty} \bar{v}^{(i)}, \quad \bar{v}^{(0)} = \bar{v}(\infty), \quad \bar{v}^{(i)}(r) = \int_r^\infty (\bar{v}^{(i-1)} \cdot \partial_r \bar{u}) \bar{u} ds. \quad (54)$$

it’s easily obtained from (54), integration by parts, Hölder and (21)

$$\|\bar{v}^{(i)}(r)\|_{C(0, \infty)} \leq \|\bar{v}^{(i-1)}(r) \cdot \partial_s (\bar{u} - \bar{k}) \bar{u} ds\|_{C(0, \infty)}$$

$$\leq \|\bar{v}^{(i-1)}(r) \cdot (\bar{u}(r) - \bar{k}) \bar{u}(r)\|_{C(0, \infty)}$$

$$+ \int_r^\infty \|\partial_s \bar{v}^{(i-1)}(r) \cdot (\bar{u} - \bar{k}) \bar{u} ds\|_{C(0, \infty)}$$

$$+ \int_r^\infty \|\bar{v}^{(i-1)}(r) \cdot (\bar{u} - \bar{k}) \partial_s \bar{u} ds\|_{C(0, \infty)}$$

$$\lesssim \|\bar{v}^{(i-1)}\|_{C(0, \infty)} \|\bar{u} - \bar{k}\|_{C(0, \infty)}$$

$$+ (\|\partial_r \bar{v}^{(i-1)}\|_{L^2} + \|\bar{v}^{(i-1)}\|_{C(0, \infty)}) \|\bar{u} - \bar{k}\|_{\sinh r} \|L^2$$

$$\lesssim (\|\partial_r \bar{v}^{(i-1)}\|_{L^2} + \|\bar{v}^{(i-1)}\|_{C(0, \infty)}) \|u\|_{H^1},$$

and

$$\|\partial_r \bar{v}^{(i)}\|_{L^2} \lesssim \|\bar{v}^{(i-1)}\|_{C(0, \infty)} \|\partial_r \bar{u}\|_{L^2} \lesssim \|\bar{v}^{(i-1)}\|_{C(0, \infty)} \|u\|_{H^1}.$$  

These give

$$\|\bar{v}^{(i)}\|_{C(0, \infty)} + \|\partial_r \bar{v}^{(i)}\|_{L^2} \lesssim (\|\bar{v}^{(i-1)}\|_{C(0, \infty)} + \|\partial_r \bar{v}^{(i-1)}\|_{L^2}) \|u\|_{H^1}.$$  

Therefore, by the above Picard iteration scheme procedure, there exists a unique solution of (52) satisfying (53). Moreover, we have

$$\|\bar{v}\|_{C(0, \infty)} + \|\partial_r \bar{v}\|_{L^2(0, \infty)} \lesssim |\bar{v}(\infty)|.$$
4.2. The Schrödinger maps system in the Coulomb gauge: Dynamic equations for $\psi^{\pm}$. In the subsection we derive the Schrödinger equations for the differentiated fields $\psi^{\pm}$.

In the geodesic polar coordinates, by (45), (46) and (48), the Schrödinger map flow (1) can be written as

$$\psi_0 = i(D_1\psi_1 + \coth r \psi_1 + \frac{1}{\sinh^2 r} D_2\psi_2).$$  \hspace{1cm} (55)

Applying the operators $D_1$ and $D_2$ to both sides of this equation, we obtain

$$D_1\psi_0 = i(D_1 D_1\psi_1 + \coth r D_1\psi_1 + \frac{1}{\sinh^2 r} D_1 D_2\psi_2)$$

$$- \psi_1 \frac{2 \cosh r}{\sinh^3 r} D_2\psi_2),$$

$$D_2\psi_0 = i(D_2 D_1\psi_1 + \coth r D_2\psi_1 + \frac{D_2 D_2\psi_2}{\sinh^2 r}).$$  \hspace{1cm} (56)

By (49), (50) and $A_1 = 0$, one can derive the equations for $\psi_1$ and $\psi_2$,

$$\begin{cases} 
(i\partial_t + \Delta)\psi_1 = (A_0 + \frac{A_2^2}{\sinh^2 r} + \frac{1}{\sinh^2 r})\psi_1 + \frac{2i \cosh r A_2}{\sinh^2 r} \psi_2 - \frac{i \Im(\psi_1 \bar{\psi}_2)}{\sinh^2 r} \\
(i\partial_t + \Delta)\psi_2 = (A_0 + \frac{A_2^2}{\sinh^2 r})\psi_2 + i \Im(\psi_1 \bar{\psi}_2)\psi_1,
\end{cases}$$

which can be further written as

$$\begin{cases} 
(i\partial_t + \Delta - \frac{2}{\sinh^2 r})\psi_1 - \frac{2i \cosh r A_2}{\sinh^2 r} \psi_2 - \frac{2i \cosh r (A_2 - 1)}{\sinh^2 r} \psi_1 - \frac{i \Im(\psi_1 \bar{\psi}_2)}{\sinh^2 r} \\
(i\partial_t + \Delta - \frac{2}{\sinh^2 r})\psi_2 + \frac{2i \cosh r A_2}{\sinh^2 r} \psi_1 + \frac{2i \cosh r (A_2 - 1)}{\sinh^2 r} \psi_2 - \frac{i \Im(\psi_1 \bar{\psi}_2)}{\sinh^2 r},
\end{cases}$$  \hspace{1cm} (57)

where $A_0$ and $A_2 - 1$ can be expressed in terms of $\psi_1$ and $\psi_2$. In fact, (49) and (50) for $k = 1$, $l = 2$ imply

$$\partial_r A_2 = \Im(\psi_1 \bar{\psi}_2), \quad \partial_r \psi_2 = i A_2 \psi_1.$$  \hspace{1cm} (58)

Since $A_2(0, t) = 1$ for all $t$, (58) gives

$$A_2 - 1 = \int_0^t \Im(\psi_1 \bar{\psi}_2)(s) ds.$$  \hspace{1cm} (59)

By (50) when $k = 0$, $l = 1$ and (55), we have

$$\partial_r A_0 = \Im(\psi_1 \bar{\psi}_0) = -\frac{1}{2 \sinh^2 r} \partial_r (\sinh^2 r |\psi_1|^2 - |\psi_2|^2),$$

which, together with (53), yields

$$A_0(t, r) = \frac{1}{2} (|\psi_1|^2 - |\psi_2|^2) + \int_{r}^{+\infty} \frac{\cosh s}{\sinh s} (|\psi_1|^2 - |\psi_2|^2) ds.$$  \hspace{1cm} (60)

Therefore the connection coefficients $A_0$ and $A_2$ depend on $\psi_1$ and $\psi_2$. 

Since the linear part of this system is not decoupled, we introduce the two new variables $\psi^+$ and $\psi^-$, defined as

$$
\psi^+ = \psi_1 + i \frac{\psi_2}{\sinh r}, \quad \psi^- = \psi_1 - i \frac{\psi_2}{\sinh r}.
$$

(61)

By (51) and (61), the system (57) can be reduced to

$$
\begin{cases}
(i\partial_t + \Delta_{\mathbb{H}^2} - 2\cosh \frac{r + 1}{\sinh^2 r}) \psi^+ = (A_0 + 2\cosh \frac{r(A_2 - 1)}{\sinh^2 r} - \Im(\psi^+ \frac{\bar{\psi}_2}{\sinh r})) \psi^+,
\end{cases}
$$

(62)

Thus the linear part of $\psi^{\pm}$ system is decoupled. By (61), the compatibility condition (49) is rewritten as

$$
\partial_r[\sinh r(\psi^+ - \psi^-)] = -A_2(\psi^+ + \psi^-),
$$

(63)

and the coefficients $A_0$ (60) and $A_2 - 1$ (59) can be expressed in terms of $\psi^{\pm}$,

$$
A_2 - 1 = \int_0^r \frac{\|\psi^+\|^2 - \|\psi^-\|^2}{4} \sinh s ds,
$$

(64)

$$
A_0 = -\frac{1}{2} \Re(\bar{\psi}^+ \psi^-) + \int_r^\infty \frac{\cosh s}{\sinh s} \Re(\bar{\psi}^+ \psi^-) ds.
$$

(65)

If define $V^{\pm}$ as the vector

$$
V^{\pm} = \partial_r u \pm \frac{u \times (\vec{k} \times u)}{\sinh r} \in u^* T\mathbb{S}^2,
$$

(66)

then $\psi^{\pm}$ is the representation of $V^{\pm}$ in the coordinate frame $(v, w)$ and the energy of $u$ can be reformulated as

$$
E(u) = \pi \int_0^\infty (|\partial_r \bar{u}|^2 + \frac{|\bar{u} \times (\vec{k} \times \bar{u})|^2}{\sinh^2 r}) \sinh r dr = \pi \int_0^\infty |V^{\pm}|^2 \sinh r + 2\partial_r \bar{u} \cdot (\bar{u} \times (\vec{k} \times \bar{u})) dr = \pi \|V^{\pm}\|_{L^2}^2 \pm 2\pi \int_0^\infty \partial_r \bar{u}_3 dr = \pi \|\psi^{\pm}\|_{L^2}^2 \pm 2\pi (\bar{u}_3(\infty) - \bar{u}_3(0)).
$$

From (62), $\|\psi^{\pm}\|_{L^2}^2$ is conserved for all time, then by the above equality and the $u_3(r)$ is fixed.

Moreover, assume that $\psi^{\pm}, \bar{\psi}^{\pm}$ are the differentiated fields corresponding to equivariant Schrödinger flows $u, \bar{u}$, respectively, and $E(u), E(\bar{u}) \ll 1$. Then the Lipschitz continuity of $\psi^{\pm}$ with respect to $u$ is valid, namely

$$
\|\psi^{\pm} - \bar{\psi}^{\pm}\|_{L^2} \lesssim \|u - \bar{u}\|_{H^1}.
$$

(67)
In fact, by (52) and the assumptions \(E(u), E(\tilde{u}) \ll 1\), it’s easy to obtain the coordinate frames \((v, \bar{w}), (\tilde{v}, \tilde{\bar{w}})\), respectively. \(\lim_{r \to 0} u(r) = \lim_{r \to 0} \tilde{u}(r) = (0, 0, 1)\), and \(u_3, \tilde{u}_3 > \frac{1}{2}\). Then from (52) and integration by parts, we have

\[
v - \tilde{v} = \int_{-\infty}^{r} -((v - \tilde{v}) \cdot \partial_s u)u + (\tilde{v} \cdot \partial_s (\tilde{u} - u))u + (\tilde{v} \cdot \partial_s \tilde{u})(\tilde{u} - u)ds
\]

and

\[
= - (v - \tilde{v}) \cdot (u - \tilde{k})u|_{r}^{\infty} + \int_{-\infty}^{r} \partial_s (v - \tilde{v}) \cdot (u - \tilde{k})u + (v - \tilde{v}) \cdot (u - \tilde{k})\partial_s u ds
\]

\[
+ \tilde{v} \cdot (\tilde{u} - u)|_{r}^{\infty} - \int_{-\infty}^{r} \partial_s \tilde{v} \cdot (\tilde{u} - u)u + \tilde{v} \cdot (\tilde{u} - u)\partial_s u - \tilde{v} \cdot \partial_s \tilde{u}(\tilde{u} - u)ds, \]

which, together with (21), implies

\[
\|v - \tilde{v}\|_{C(0, \infty)} \lesssim \|v - \tilde{v}\|_{C(0, \infty)}\|u - \tilde{u}\|_{L^\infty} + \|\partial_s (v - \tilde{v})\|_{L^2} \|\frac{u - \tilde{k}}{\sinh r}\|_{L^2}
\]

\[
+ \|v - \tilde{v}\|_{C(0, \infty)}\|u - \tilde{u}\|_{L^\infty} + \|\partial_s (v - \tilde{v})\|_{L^2} \|\frac{\tilde{u} - u}{\sinh r}\|_{L^2}
\]

\[
\lesssim \|v - \tilde{v}\|_{C(0, \infty)}\|u\|_{H^1} + \|\partial_s (v - \tilde{v})\|_{L^2} \|u\|_{H^1} + \|u - \tilde{u}\|_{H^1}. \tag{68}
\]

From (52), we also have

\[
\|\partial_s (v - \tilde{v})\|_{L^2} \lesssim \|v - \tilde{v}\|_{C(0, \infty)}\|u\|_{H^1} + \|u - \tilde{u}\|_{H^1}. \tag{69}
\]

Hence, the assumptions \(E(u), E(\tilde{u}) \ll 1\), (68) and (69) yield

\[
\|v - \tilde{v}\|_{C(0, \infty)} \lesssim \|u - \tilde{u}\|_{H^1}. \tag{70}
\]

Combining this with \(w = u \times v\) and (21), one obtain

\[
\|w - \tilde{w}\|_{C(0, \infty)} \lesssim \|u - \tilde{u}\|_{H^1}. \tag{71}
\]

Thus (66), (70) and (71) imply

\[
\left\|\psi^\pm - \tilde{\psi}^\pm\right\|_{L^2(0, \infty)} = \left\|\partial_s u \cdot (v + iw) - \partial_s \tilde{u} \cdot (\tilde{v} + i\tilde{w})\right\|_{L^2(0, \infty)}
\]

\[
+ \left\|\frac{v_3 + iw_3}{\sinh r} - \frac{\tilde{v}_3 + i\tilde{w}_3}{\sinh r}\right\|_{L^2(0, \infty)}
\]

\[
\lesssim \|u - \tilde{u}\|_{H^1} + \left\|\frac{v_3 - \tilde{v}_3}{\sinh r}\right\|_{L^2} + \left\|\frac{w_3 - \tilde{w}_3}{\sinh r}\right\|_{L^2}. \tag{72}
\]

Since \(v_3 = u_1w_2 - w_2u_1\), then (71) give

\[
\left\|\frac{v_3 - \tilde{v}_3}{\sinh r}\right\|_{L^2} = \left\|\frac{w_1u_2 - w_2u_1 - \tilde{w}_1\tilde{u} + \tilde{w}_2\tilde{u}_1}{\sinh r}\right\|_{L^2}
\]

\[
= \left\|\frac{(w_1 - \tilde{w}_1)u_2 + \tilde{w}_1(u_2 - \tilde{u} - \tilde{w}_2)u_1 - \tilde{w}_2(u_1 - \tilde{u})}{\sinh r}\right\|_{L^2}
\]

\[
\lesssim \|w - \tilde{w}\|_{L^\infty}\|u\|_{H^1} + \|\tilde{w}\|_{L^\infty}\|u - \tilde{u}\|_{H^1}
\]

\[
\lesssim \|u - \tilde{u}\|_{H^1}. \tag{73}
\]

using \(w_3 = u_1v_2 - u_2v_1\) and (70), we also obtain \(\left\|\frac{w_3 - \tilde{w}_3}{\sinh r}\right\|_{L^2} \lesssim \|u - \tilde{u}\|_{H^1}. \) Hence, the Lipschitz continuity (67) follows.

Suppose \(\psi^\pm\) satisfies the compatibility condition (63) and \(\left\|\psi^\pm\right\|_{L^2} < \infty\), we define \(A_2, \psi_1, \psi_2\) by (64) and (61), then they satisfy the relation (58). Furthermore, we
claim that $\psi_1 \in L^2$ and $\psi_2$, $A_2-1 \in H^1_0$. In fact, by (64) and (61), we have $\psi_1 \in L^2$, $A_2 \in L^{\infty}$, from (61), (58) and (25), we get $\frac{\psi_2}{\sinh r}$, $\partial_r \psi_2$, $\partial_r A_2$, and $\frac{A_2-1}{\sinh r} \in L^2$.

Finally, in order to obtain the global existence of equivariant Schrödinger flow (1), it is natural to prove global existence of the system (62) first. Since $\psi^+$ satisfies a Schrödinger equation with singular potential, we may multiply $e^{i2\theta}$ to $\psi^+$-equation, and obtain the following system with regular potential,

$$
\begin{align*}
(i\partial_t + \Delta_{g^2} - 2 & \frac{\cosh r - 1}{\sinh^2 r} )R_2 \psi^+ \\
= & [A_0 + 2 \frac{\cosh A_2 - 1}{\sinh^2 r} - \mathfrak{A}(\psi^+, \bar{\psi}_2)] R_2 \psi^+, \\
(i\partial_t + \Delta_{g^2} + 2 & \frac{\cosh r - 1}{\sinh^2 r}) \psi^- \\
= & [A_0 - 2 \frac{\cosh A_2 - 1}{\sinh^2 r} + \mathfrak{A}(\psi^-, \bar{\psi}_2)] \psi^-.
\end{align*}
$$

(72)

Then we have

**Proposition 1** (Regularity of gauge elements). If the equivariant Schrödinger map $u \in H^3$, then $R_2 \psi^+ \in H^2$ and

$$\|u\|_{H^3} \sim \|R_2 \psi^+\|_{H^2} + \|\psi^-\|_{H^2}.$$  

**Proof.** If $u \in H^3$, it’s easily obtained $\|\nabla u\|^2_{L^2} = \pi \|\psi^\pm\|^2_{L^2}$.

**Step 1.** If $u \in H^2$, by the equivariance condition, we have

$$\partial_r u = (\cosh r \frac{\partial_r}{\sinh r} - \frac{1}{\sinh^2 r})(u_1, u_2), \quad \frac{\cosh r}{\sinh r} \partial_r u_3 \in L^2.$$  

(73)

Since $A_1 = 0$, then $\nabla \frac{\partial_r}{\sinh r} v = \nabla \frac{\partial_r}{\sinh r} w = 0$, which implies

$$\partial_r \psi^\pm = \partial_r (V^\pm \cdot (v + iw)) = \partial_r V^\pm \cdot (v + iw).$$

Since the representation of $V^\pm$ (66) implies

$$\partial_r V^\pm = \partial_r (\frac{u_3 + \bar{k}}{\sinh r}) = \partial_r u + \frac{u_3}{\sinh r} \partial_r u + \frac{u_3}{\sinh^2 r} \partial_r u + \cosh r \frac{\partial_r}{\sinh r} - \frac{1}{\sinh^2 r} \partial_r u + \frac{u_3}{\sinh^2 r} \partial_r u$$

(74)

Denote $F^\pm = (74) \cdot (v + iw)$, then we have

$$\partial_r \psi^\pm = F^\pm \pm \frac{1}{\sinh r} (u_3 - 1 \psi^- + \frac{u_3}{\sinh r} (u_3 - 1 \psi^-).$$

(75)

Note that $|\frac{u_3}{\sinh r}| \leq \frac{u_3^2}{\sinh^2 r} \in L^1(dr)$. Applying $e^{J_r \frac{u_3}{\sinh r} dr}$ to both sides of (75), since (73) imply $F^\pm \in L^2$, we have $\partial_r (e^{J_r \frac{u_3}{\sinh r} dr} \psi^\pm) \in L^2$. Therefore, $e^{J_r \frac{u_3}{\sinh r} dr}$
\[ \psi^- \in \mathcal{H}^1 \] since \( u_3 \) and \( \psi^- \) are radial. Therefore \( \psi^- \in L^4 \) by (17). Hence, by (75) and \( \frac{|u_3-1|}{\sinh r} \lesssim \frac{|u_3|^2 + |u_2|^2}{\sinh r} \in L^4 \), we obtain \( \partial_r \psi^- \in L^2 \). Similarly, we have \( \partial_r \psi^+ \in L^2 \).

In order to prove \( \psi^+ \in L^2 \), we rewrite
\[
\psi^+ = \left( \frac{\partial_r u}{\sinh r} + \frac{-u_3 u + \tilde{k}}{\sinh^2 r} \right) \cdot (v + iw) \\
= \left( \frac{\partial_r}{\sinh r} - \frac{1}{\sinh^2 r} \right) u_1, \left( \frac{\partial_r}{\sinh r} - \frac{1}{\sinh^2 r} \right) u_2, \left( \frac{\partial_r}{\sinh r} - \frac{1}{\sinh^2 r} \right) u_3 \cdot (v + iw) \\
+ \left( \frac{1 - u_3}{\sinh^2 r} u_1, \frac{1 - u_3}{\sinh^2 r} u_2, \frac{u_1^2 + u_2^2}{\sinh^2 r} \right) \cdot (v + iw).
\]

Then (73) implies \( \psi^+ \in L^2 \).

Conversely, if \( R_2 \psi^+, \psi^- \in \mathcal{H}^1 \), (17) implies \( \psi^+ \in L^4 \). Then by \( \frac{|u_3-1|}{\sinh r} \lesssim \frac{|v_2|}{\sinh r} \in L^4 \) and (75), we have \( F^\pm \in L^2 \), namely, \( \partial_{\mathcal{F}r} = \left( \frac{\cosh r}{\sinh r} \partial_r + \frac{\partial_r^2}{\sinh^2 r} \right) \mid u \cdot (v + iw) \in L^2 \).

The part of \( \partial_{\mathcal{F}r} = \left( \frac{\cosh r}{\sinh r} \partial_r + \frac{\partial_r^2}{\sinh^2 r} \right) \mid u \) in the normal direction is \( -|\psi_1|^2 \pm \frac{|v_2|}{\sinh r} \in L^2 \) by \( \psi^+ \in L^4 \).

**Step 2.** If \( u \in \mathbb{H}^3 \), by (14) and (16), we obtain \( \nabla(-\Delta) u_i \in L^2 \) for \( i = 1, 2, 3 \). Then by equivariance condition, we get
\[
\partial_r \left( \partial_r + \cosh \frac{r}{\sinh r} \partial_r - \frac{4}{\sinh^2 r} \right)(u_1, u_2), \partial_r \left( \partial_r + \cosh \frac{r}{\sinh r} \partial_r \right) u_3 \in L^2, (76)
\]
and
\[
\frac{1}{\sinh r} \left( \partial_r + \cosh \frac{r}{\sinh r} \partial_r - \frac{4}{\sinh^2 r} \right)(u_1, u_2) \in L^2. (77)
\]

In order to prove \( R_2 \psi^+, \psi^- \in \mathcal{H}^2 \), it suffices to prove
\[
(\partial_{\mathcal{F}r} + \cosh \frac{r}{\sinh r} \partial_r - \frac{4}{\sinh^2 r}) \psi^+, (\partial_{\mathcal{F}r} + \cosh \frac{r}{\sinh r} \partial_r) \psi^- \in L^2.
\]

First, by (75), we have
\[
\partial_{\mathcal{F}r} \psi^\pm = \partial_r \left( \left( \partial_{\mathcal{F}r} = \left( \frac{\partial_r}{\sinh r} - \cosh \frac{r}{\sinh r} \right) u_1, \left( \frac{\partial_r}{\sinh r} - \cosh \frac{r}{\sinh r} \right) u_2, \left( \frac{\partial_r}{\sinh r} - \cosh \frac{r}{\sinh r} \right) u_3 \right) \cdot (v + iw) \\
\pm \left( \frac{u_3 - 1}{\sinh r} \right) \partial_r \psi^- \\
- \left( \frac{u_3 - 1}{\sinh r} \right) \partial_r \left( \frac{\cosh r - 1}{\sinh r} \psi_2 \right) \pm \left( \frac{u_3 - 1}{\sinh r} \right) \psi^- \\
= \partial_r A_2 \left( \frac{\cosh r - 1}{\sinh r} \psi_2 \right) \pm \left( \frac{u_3 - 1}{\sinh r} \right) \partial_r \psi^- \\
\pm \left( \frac{u_3^2 + u_2^2}{\sinh^2 r} \right) \partial_r \psi^- \cdot (v + iw). (78)
\]

Since \( R_2 \psi^+, \psi^- \in \mathcal{H}^1 \), (17) implies \( \psi^\pm \in L^4 \cap L^6 \). Then by \( |u_3 - 1| \lesssim |v_2|^2 \) and (59), the terms (78) and (79) are in \( L^2 \). From (76), we also have \( \partial_r F^\pm \in L^2 \).

Hence, \( \partial_r (e^{\int_{-\infty}^r e_2} \cdot R_2 \psi^-) \in L^2 \), which further gives \( e^{\int_{-\infty}^r e_2} \cdot \partial_r \psi^- \in L^4 \), and so \( \partial_r \psi^- \in L^4 \). Since \( \frac{|u_3-1|}{\sinh r} \in L^4 \), we obtain \( \partial_r \psi^- \in L^2 \), therefore, we also get \( \partial_r \psi^+ \in L^2 \).
Next, we estimate the term
\[
\frac{\cosh r}{\sinh r} \partial_r \psi = \frac{\cosh r}{\sinh r} (F - \frac{\cosh r}{\sinh r} - \frac{\cosh r - 1}{\sinh r} (u_3 - 1) - \frac{u_3 - 1}{\sinh r} \psi - 1). \tag{80}
\]
By (77), the first two components of \( \frac{\cosh r}{\sinh r} F \) are in \( L^2 \). The third component can be written as
\[
\frac{\cosh r}{\sinh r} (\partial_{rr} + \frac{\partial_r}{\sinh r}) u_3 \cdot (v + iw)
= \frac{i (\cosh r - 1)}{\sinh r} \psi_2 (\partial_{rr} + \frac{\partial_r}{\sinh r}) u_3 + \frac{i \psi_2}{\sinh r} (\partial_{rr} + \frac{\partial_r}{\sinh r}) u_3. \tag{81}
\]
By (73) and (76), we have \( \Delta u_3 \in L^4 \). Hence, the right hand side of (81) is in \( L^2 \).

For \( R_2 \psi^+ \), we denote \( G := \partial_r^3 + \frac{\cosh r - 1}{\sinh r} \partial_r^2 + \frac{\cosh r - 2}{\sinh r} \partial_r + \frac{3}{\sinh^2 r} \). It suffices to estimate
\[
(\partial_{rr} + \frac{\cosh r + 1}{\sinh r} \partial_r - \frac{4}{\sinh^2 r}) \psi^+
= \left( G u_1, \ G u_2, \ G + \frac{3 - \cosh r}{\sinh^2 r} \partial_r - \frac{3}{\sinh^3 r} u_3 \right) \tag{82}
\]
By (73) and (77), the first two components of (82)
\[
G u_i = \partial_r \Delta u_i - \frac{1}{\sinh r} \Delta u_i + (\frac{\cosh r - 1}{\sinh r} \partial_r^2 + \frac{4 \cosh r - 1}{\sinh^2 r} \partial_r) u_i \in L^2, \text{ for } i = 1, 2.
\]
For the third component, since \( \partial_r \Delta u_3 \in L^2 \), it suffices to estimate
\[
\frac{\cosh r}{\sinh r} \partial_r^2 u_3 \cdot (v + iw) = \frac{\cosh r - 1}{\sinh r} \psi_2 \partial_r^2 u_3 + \frac{1}{2} (\psi^+ - \psi^-) \partial_r^2 u_3.
\]
By (73) and (77), \( |\psi_2| \leq 1 \), we get \( \frac{\cosh r - 1}{\sinh r} \psi_2 \partial_r^2 u_3 \in L^2 \). By (21) and (18), we have \( \| \psi^+ \|_{L^\infty} \lesssim \| \psi^+ \|_{H^2} \) and \( \| \psi^- \|_{L^\infty} \lesssim \| \psi^- \|_{H^2} \), which implies \( \frac{1}{2} (\psi^+ - \psi^-) \partial_r^2 u_3 \in L^2 \).

Therefore, (82) are in \( L^2 \). The other terms are also easily obtained by Sobolev embedding and \( A_2 + |\psi_2|^2 = 1 \). Thus, \( R_2 \psi^+ \in H^2 \) is obtained.

Conversely, if \( R_2 \psi^+ \), \( \psi^- \in H^2 \), then (78), (79) and (80) imply \( \partial_r \Delta u \cdot (v + iw), \frac{1}{\sinh \Delta u} \cdot (v + iw) \in L^2 \). Since \( \nabla \psi^+, \psi^- \in L^4 \), one also easily obtain the part of \( \partial_r \Delta u \) and \( \frac{1}{\sinh \Delta u} \) in the normal direction are in \( L^2 \). Therefore (76) and (77) are obtained.

\[\square\]

4.3. Recovering the map \( u \) from \( \psi^+ \) with \( \| \psi^+ \|_{L^2} \ll 1 \). Here we will keep track of \( \psi^+ \in L^2 \), since it contains all the information about the map (see [6], [8]). Indeed, by (58), we have the system of (\( A_2, \psi_2 \))
\[
\begin{align*}
\partial_r A_2 &= \Imi (\psi^+ \bar{\psi}_2) - \frac{|\psi_2|^2}{\sinh r}, \\
\partial_r \psi_2 &= i A_2 \psi^+ + A_2 \frac{\psi_2}{\sinh r}, \\
\lim_{r \to \infty} (A_2(r), \psi_2(r)) &= (1 - \frac{\lambda^2}{1 + \lambda^2}, \frac{2i \lambda}{1 + \lambda^2}), \text{for } \lambda \in [0, \infty).
\end{align*}
\tag{83}
\]
The boundary condition in (83) is from the choice of \( \langle \hat{\psi}(\infty), \hat{\psi}(\infty) \rangle \) (53). Given \( \psi^+ \in L^2 \) with \( \|\psi^+\|_{L^2} < \epsilon_0 \) and boundary data \( |A_2(\infty) - 1| + |\psi_2(\infty)| < \epsilon_0 \), we reconstruct \( A_2 - 1, \psi_2 \in H^1_\infty \) by above system (83). Then by the system (47) with (53), map \( u \) can be recovered.

**Lemma 4.1.** Let \( \psi^+ \in L^2 \), such that \( \|\psi^+\|_{L^2} < \epsilon_0 \), and the boundary condition \( (A_2(\infty), \psi_2(\infty)) \) satisfying \( |A_2(\infty) - 1| + |\psi_2(\infty)| < \epsilon_0 \), the system (83) has a unique solution \( (A_2, \psi_2) \) satisfying \( \psi_2; A_2 - 1 \in H^1_\infty \), and

\[
\|\psi_2\|_{H^1_\infty} + \|A_2 - 1\|_{H^1_\infty} + \left\| \frac{A_2 - 1}{\sinh r} \right\|_{L^1(\infty)} \lesssim \|\psi^+\|_{L^2} + |\psi_2(\infty)|. 
\] (84)

Moreover, we have the following properties:

(i) If \( \psi^+ \in L^p \), with \( 1 < p < \infty \), then \( \psi^- = \frac{\psi_2}{\sinh r}, A_2 \frac{1}{\sinh r} \in L^p \) and

\[
\|\psi^-\|_{L^p} + \left\| \frac{\psi_2}{\sinh r} \right\|_{L^p} + \left\| \frac{A_2 - 1}{\sinh r} \right\|_{L^p} \lesssim \|\psi^+\|_{L^p} + |\psi_2(\infty)|. 
\] (85)

(ii) If \( (\tilde{A}_2, \tilde{\psi}_2) \) is another solution to (83) corresponding to \( \tilde{\psi}^+ \), then

\[
\|\psi^- \! - \! \tilde{\psi}^-\|_{L^2} + \left\| \psi_2 - \tilde{\psi}_2 \right\|_{H^1_\infty} + \|A_2 - \tilde{A}_2\|_{H^1_\infty} \lesssim \|\psi^- - \tilde{\psi}^-\|_{L^2}. 
\] (86)

(iii) If \( R_2 \psi^+ \in H^s \), then \( \psi^- \in H^s \) for \( s \in \{1, 2\} \).

**Proof.** We consider the ODE system (83) with

\[
\|\psi^+\|_{L^2} < \epsilon_0, \quad |A_2(\infty) - 1| + |\psi_2(\infty)| < \epsilon_0.
\]

The system and boundary condition imply the conservation law

\[
A^2_2 + |\psi_2|^2 = 1, 
\] (87)

which will be used frequently in the proof. Define \( \psi^- = \psi^+ - 2i \frac{\psi_2}{\sinh r}, \psi_1 = \psi^+ - i \frac{\psi_2}{\sinh r} \), then we get \( \partial_r A_2 = \frac{1}{4} \sinh r (|\psi^+|^2 - |\psi^-|^2) \) from (83). Integrating from infinity yields

\[
A_2 - A_2(\infty) = \frac{1}{4} \int_{r}^{\infty} (|\psi^-|^2 - |\psi^+|^2) \sinh sds.
\]

Thus we have \( A_2 > A_2(\infty) - \frac{1}{4} \|\psi^+\|_{L^2}^2 > \frac{1}{2} \).

To prove the existence, we consider the \( \psi_2 \)-equation in

\[
X = \{ \psi_2 \in H^1_{r}(0, \infty) : \|\psi_2\|_{H^1_{r}(0, \infty)} \leq 4C\epsilon_0 \}.
\]

Rewrite the \( \psi_2 \)-equation as

\[
\partial_r \psi_2 - A_2(\infty) \frac{\psi_2}{\sinh r} = iA_2 \psi^+ + (A_2 - A_2(\infty)) \frac{\psi_2}{\sinh r}.
\]

Multiply by \( e^{A_2(\infty) \int_{r}^{\infty} \sinh^{-1} sds} \) on both sides and integrating from infinity we obtain

\[
\psi_2(r) = e^{-A_2(\infty) \int_{r}^{\infty} \sinh^{-1} sds} \psi_2(\infty)
\]

\[
+ \int_{\infty}^{r} e^{A_2(\infty) \int_{\rho}^{\infty} \sinh^{-1} sds} (iA_2 \psi^+ + (A_2 - A_2(\infty)) \frac{\psi_2}{\sinh \rho}) d\rho.
\]
Define the map $T : \dot{H}^1_e(0, \infty) \to \dot{H}^1_e(0, \infty)$ by
\[
T(\psi_2)(r) = e^{-A_2(\infty)} \int_0^r \sinh^{-1} s ds \psi_2(\infty) + \int_0^\infty e^{A_2(\infty)} \int_0^r \sinh^{-1} s ds (i A_2 \psi^+ + (A_2 - A_2(\infty)) \frac{\psi_2}{\sinh \rho}) d\rho.
\]

Now it suffices to show that $T$ is a contraction map in $X$. Indeed, $A_2(\infty) > \frac{1}{2}$, (26) and (21) lead to
\[
\|T\psi_2\|_{\dot{H}^1_e} \leq C|\psi_2(\infty)| + \|i A_2 \psi^+ + (A_2 - A_2(\infty)) \frac{\psi_2}{\sinh r}\|_{L^2} \leq C|\psi_2(\infty)| + C\left(\|\psi^+\|_{L^2} + \left\|\frac{|\psi_2|^2 - |\psi_2(\infty)|^2}{A_2 + A_2(\infty)} \frac{\psi_2}{\sinh r}\right\|_{L^2}\right) \leq 2Ce_0 + C\|\psi_2\|^3_{\dot{H}^1_e} \leq 4Ce_0.
\]
And the map $T$ is Lipschitz with a small Lipschitz constant,
\[
\|T(\psi_2) - T(\tilde{\psi}_2)\|_{\dot{H}^1_e} = \left\|\int_0^\infty e^{A_2(\infty)} \int_0^r \sinh^{-1} s ds i A_2 \psi^+ + (A_2 - A_2(\infty)) \frac{\psi_2}{\sinh r} d\rho\right\|_{\dot{H}^1_e} \leq C\left(\|\psi^+\|_{L^2} + \left\|\frac{\psi_2}{A_2 + A_2(\infty)} \frac{\psi_2 - \tilde{\psi}_2}{\sinh r}\right\|_{L^2}\right) \leq 100C^2 e_0^2 \left\|\psi_2 - \tilde{\psi}_2\right\|_{\dot{H}^1_e}.
\]
Therefore there exists a unique solution $\psi_2 \in \dot{H}^1_e$ and the $A_2$ is obtained by $A_2(r) = \sqrt{1 - |\psi_2|^2}$.

Next we obtain the bound for (84). Let $G = \frac{\psi_2}{1 + A_2}$, then the system (83) gives
\[
\frac{d}{dr} |G|^2 - \frac{2}{\sinh r} |G|^2 = -2\Re\left(\frac{\psi^+}{1 + A_2} G\right),
\]
which implies
\[
\left|\frac{d}{dr} |G| - \frac{|G|}{\sinh r}\right| \leq \frac{|\psi^+|}{1 + A_2}.
\]
Therefore
\[
|G|(r) \leq e^{\int_r^\infty \sinh^{-1} s ds} |G(\infty)| + \int_r^\infty e^{\int_0^r \sinh^{-1} s ds} \frac{|\psi^+|}{1 + A_2(\rho)} d\rho.
\]
Since $|A_2| \leq 1$, we get
\begin{equation}
|\psi_2|^2(r) \lesssim e^{r \cosh^{-1} s} ds \langle \psi_2(\infty) \rangle + \int_r^\infty e^{r \cosh^{-1} s} ds \left| \frac{\psi^+}{1 + A_2} \right| d\rho. \tag{88}
\end{equation}

By (26) and $A_2 > \frac{1}{2}$, (88) gives $\| \psi_2 \|_{L^2(\Omega)} \lesssim \| \psi^+ \|_{L^2(\Omega)} + \| \psi_2(\infty) \|$. The bounds for $\| \partial_r \psi_2 \|_{L^2}$ and $\| \partial_r A_2 \|_{L^2}$ follow directly from (83). The bounds for $\| \frac{A_2 - 1}{\sinh^2 \theta} \|_{L^2}$ and $\| \frac{A_2 - 1}{\sinh^2 \theta} \|_{L^1(\Omega)}$ are obtained by (87).

Now we prove the additional properties (i)-(iii). First, we have the bound (85). If $\psi^+ \in L^p$, (26) and (88) imply $\| \psi_2 \|_{L^p(\Omega)} \lesssim \| \psi^+ \|_{L^p(\Omega)} + \| \psi_2(\infty) \|$, then the $L^p$-bound for $\psi^-$ and $\frac{A_2 - 1}{\sinh^2 \theta}$ are obtained immediately by the definition of $\psi^-$ and (87).

Second, we get the Lipschitz continuity (86). For notational convenience we denote
\begin{equation}
\delta \psi^+ = \psi^+ - \tilde{\psi}^+, \quad \delta \psi_2 = \psi_2 - \tilde{\psi}_2, \quad \delta A_2 = A_2 - \tilde{A}_2.
\end{equation}
Without any restriction in generality, we can make the assumption $\| \delta \psi^+ \|_{L^2(\Omega)} \ll 1$. By (83) and (87), we derive the equations
\begin{equation}
\begin{aligned}
\partial_r \delta \psi_2 &= \frac{\delta \psi_2}{\sinh \theta} + i \psi^+ \frac{A_2 - 1}{\sinh \theta} \delta \psi_2 + \frac{A_2 - 1}{\sinh \theta} \delta A_2 + \frac{\tilde{\psi}_2}{\sinh \theta} \delta A_2 + i A_2 \delta \psi^+, \\
\partial_r \delta A_2 &= \frac{\delta A_2}{\sinh r} \sinh r + \Re(\psi^+ \frac{\delta \psi_2}{\sinh \theta}) + \frac{2(\tilde{A}_2 - 1)}{\sinh \theta} \delta \psi_2 + \frac{2(\tilde{A}_2 - 1)}{\sinh \theta} \delta A_2 + 2 \Im(\psi^+ \frac{\delta \psi_2}{\sinh \theta}) + \frac{(\delta A_2)^2}{\sinh \theta}.
\end{aligned}
\end{equation}

Since $\| \delta \psi^+ \|_{L^2(\Omega)} \ll 1$ and $(\frac{\delta A_2}{\sinh \theta})^2$ is a high order term, $\Im(\frac{\delta \psi^+}{\sinh \theta}) + \frac{(\delta A_2)^2}{\sinh \theta}$ and $i A_2 \delta \psi^+$ can be regarded as error terms. Let $X = (\Re \delta \psi^+, \Im \delta \psi^+, \delta A_2)^T$, we have
\begin{equation}
\partial_r X = \frac{1}{\sinh \theta} LX + BX + E, \tag{89}
\end{equation}
where $L = \text{diag} \{1, 1, 2\}$,\[ B = \begin{pmatrix} \frac{A_2 - 1}{\sinh \theta} & 0 & -\Im \psi^+ + \frac{\Re \psi_2}{\sinh \theta} - \frac{\Re \tilde{\psi}_2}{\sinh \theta} \\ 0 & \frac{A_2 - 1}{\sinh \theta} & \Im \psi^+ + \frac{\Re \psi_2}{\sinh \theta} + \frac{\Re \tilde{\psi}_2}{\sinh \theta} \end{pmatrix}, \quad E = \begin{pmatrix} \Re(i A_2 \delta \psi^+) \\ \Im(i A_2 \delta \psi^+) \\ \frac{2(\tilde{A}_2 - 1)}{\sinh \theta} \Im(\delta \psi^+ \tilde{\psi}_2) + (\frac{\delta A_2}{\sinh \theta})^2 \end{pmatrix}.
\]

Since (89) can be rewritten as
\begin{equation}
X = \int_r^\infty \text{diag}(e^{r \cosh^{-1} s} ds, e^{r \cosh^{-1} s} ds, e^{2 r \cosh^{-1} s} ds)(BX + E)d\rho.
\end{equation}
By the above expression of $X$, (26) and (84), we have
\begin{equation}
\frac{X}{\sinh \theta} \lesssim \| B \|_{L^2} \left\| \frac{X}{\sinh \theta} \right\|_{L^2} + \left\| \frac{X}{\sinh \theta} \right\|_{L^2}^2 + \left\| \frac{X}{\sinh \theta} \right\|_{L^2}^2, \tag{88}
\end{equation}
and
\begin{equation}
\frac{X}{\sinh \theta} \lesssim \| B \|_{L^2} \| X \|_{L^\infty} + \left\| \frac{X}{\sinh \theta} \right\|_{L^2}^2 + \left\| \frac{X}{\sinh \theta} \right\|_{L^2}^2.
\end{equation}
Therefore, we have
\begin{equation}
\| X \|_{L^\infty} \lesssim \| \frac{X}{\sinh \theta} \|_{L^2} + \| \frac{X}{\sinh \theta} \|_{L^2}^2.
\end{equation}
Using the method of continuity, we get
\[ \|X\|_{L^\infty} + \left\| \frac{X}{\sinh r} \right\|_{L^2} \lesssim \|\delta\psi^+\|_{L^2}. \]

Furthermore, by (89) we have \( \|\partial_r X\|_{L^2} \lesssim \|\delta\psi^+\|_{L^2} \). Hence, the Lipschitz continuity (86) is obtained.

Finally we prove (iii). If \( s = 1 \), by (83), we have

\[ \partial_r \psi^- = \partial_r \psi^+ - 2i\partial_r \frac{\psi_2}{\sinh r} \]
\[ = \partial_r \psi^+ + 2A_2 \frac{\psi^+}{\sinh r} - 2iA_2 - \frac{1}{\sinh r} \frac{\psi_2}{\sinh r} + 2i \frac{\cosh r - 1}{\sinh r} \psi_2 \]

then by (84), (85) and (17), we obtain

\[ \|\partial_r \psi^-\|_{L^2} \lesssim \|\partial_r \psi^+\|_{L^2} + \left\| \frac{\psi^+}{\sinh r} \right\|_{L^2} + \left\| \frac{\psi_2}{\sinh r} \right\|_{L^2} + \left\| \frac{\psi_2}{\sinh r} \right\|_{L^2} \]
\[ \lesssim \|R_2 \psi^+\|_{H^1} + \|\psi^+\|_{L^4} \]
\[ \lesssim \|R_2 \psi^+\|_{H^1} \left( 1 + \|\psi^+\|_{L^2} \right). \]

If \( s = 2 \), by (83) and (87), we have

\[ \partial_{rr} \psi^- = \partial_{rr} \psi^+ + (3(\psi^+ \frac{\psi_2}{\sinh r}) - \frac{\psi_2}{\sinh r})(2\psi^+ - 2i \frac{\psi_2}{\sinh r}) \]
\[ + 2A_2 \frac{\cosh r}{\sinh r} \partial_r \psi^+ - 2A_2 \frac{\psi^+}{\sinh^2 r} \]
\[ + 2A_2 \frac{1 - \cosh r}{\sinh r} \partial_r \psi^+ - 2A_2 \frac{\cosh r - 1}{\sinh^2 r} \psi^+ \]
\[ + (iA_2 \psi^+ + A_2 \frac{\psi_2}{\sinh r})(-2i \frac{1}{A_2 + 1} \frac{\psi_2}{\sinh^2 r} \]
\[ + 4i \frac{1}{A_2 + 1} \frac{\psi_2}{\sinh^3 r} \frac{\psi_2}{\sinh r} + 4i \frac{\cosh r - 1}{\sinh^2 r} \frac{\psi_2}{\sinh r} \]
\[ - 2i \frac{\cosh r - 1}{\sinh^2 r} \frac{\psi_2}{\sinh r}. \]

by Sobolev embedding \( H^2 \hookrightarrow L^6 \), we get \( \partial_{rr} \psi^- \in L^2 \). Similarly, \( \frac{\cosh r}{\sinh r} \partial_r \psi^- \in L^2 \). Hence, \( \psi^- \in H^2 \).

**Proposition 2.** Given \( \psi^+ \in L^2 \) with \( \|\psi^+\|_{L^2} < \epsilon_0 \) and \( |A_2(\infty) - 1| + |\psi_2(\infty)| < \epsilon_0 \), then there is a unique map \( u : \mathbb{H}^2 \rightarrow \mathbb{S}^2 \) with the property that \( \psi^+ \) is the representation of \( \mathcal{V}^+ \) relative to a Coulomb gauge satisfying (66) with \( E(u) = \pi \|\psi^+\|_{L^2}^2 + 2\pi(1 - A_2(\infty)) \). Moreover, the map \( \psi^+ \rightarrow u \) is Lipschitz continuous in the following sense:

\[ \|u - \tilde{u}\|_{H^1} \lesssim \left\| \psi^+ - \tilde{\psi}^+ \right\|_{L^2}. \]  

**Proof.** Given \( \|\psi^+\|_{L^2} < \epsilon_0 \), by Lemma 4.1, there is a unique solution \( (A_2, \psi_2) \). Let \( \psi_1 = \psi^+ - i \frac{\psi_2}{\sinh r} \). Now we solve the system of \( U = (\tilde{u}, \tilde{v}, \tilde{w})^\top \), that is
Since $\psi$ and from which we obtain

By (84), we have

\[ \| \psi \|_{L^1(dr)} + \| \psi \|_{H^1_r} \lesssim \| \psi^+ \|_{L^2} + |\psi_2(\infty)|, \]

which allow us to construct solutions with data at $r = \infty$ by using the iteration scheme

\[ U = \sum_i U_i, \quad U_0 = U(\infty), \quad U_i(r) = \int^r_\infty M(s)U_{i-1}(s)ds. \]

Let $X = \{ U \in C(0, \infty) : \partial_r U \in L^2(0, \infty), \lim_{r \to \infty} U(r) \text{ exists} \}$. We run the iteration scheme in $X$. For $U_{i-1} \in X$, we have

\[ U_i(r) = \int^r_\infty M(s)U_{i-1}(s)ds = \int^r_\infty M_1U_{i-1} + \partial_r M_2U_{i-1}ds = \int^r_\infty M_1U_{i-1}ds + M_2(r)U_{i-1}(r) - \int^r_\infty M_2\partial_r U_{i-1}ds, \]

from which we obtain

\[ \| U_i \|_{C(0, \infty)} \leq (\| M_1 \|_{L^1(dr)} + \| M_2 \|_{C(0, \infty)}) \| U_{i-1} \|_{C(0, \infty)} + \left\| \frac{M_2}{\sinh r} \right\|_{L^2} \| \partial_r U_{i-1} \|_{L^2} \]

\[ \lesssim (\| \psi^+ \|_{L^2} + |\psi_2(\infty)|)(\| U_{i-1} \|_{C(0, \infty)} + \| \partial_r U_{i-1} \|_{L^2}), \]

and

\[ \| \partial_r U_i \|_{L^2} = \| M_1U_{i-1} + \partial_r M_2U_{i-1} \|_{L^2} \leq (\| M_1 \|_{L^2} + \| \partial_r M_2 \|_{L^2}) \| U_{i-1} \|_{C(0, \infty)} \]

\[ \lesssim (\| \psi^+ \|_{L^2} + |\psi_2(\infty)|) \| U_{i-1} \|_{C(0, \infty)}. \]

Therefore,

\[ \| U_i \|_{C(0, \infty)} + \| \partial_r U_i \|_{L^2} \lesssim (\| \psi^+ \|_{L^2} + |\psi_2(\infty)|)^i. \]

Then by $\| \psi^+ \|_{L^2} + |\psi_2(\infty)| \lesssim \epsilon_0$, we can use the iteration scheme to construct a solution $U$. As a byproduct,

\[ \| U - U_0 \|_{C(0, \infty)} + \| \partial_r U \|_{L^2(0, \infty)} \lesssim \| \psi^+ \|_{L^2} + |\psi_2(\infty)|. \]
The uniqueness of (91) is obtained by conservation law, that is, apply \((\bar{u}, \bar{v}, \bar{w})\) to both side of (91), we have \(\frac{1}{2} \partial_t (|\bar{u}|^2 + |\bar{v}|^2 + |\bar{w}|^2) = 0\).

Since \(\partial_t \bar{u} \cdot \bar{v} = -\partial_t \bar{v} \cdot \bar{u}\) by (91), we have \(\bar{u} \cdot \bar{v}(r) = \text{const}\), which together with \(\lim_{r \to \infty} U(r) = U(\infty)\) yields \(\bar{u} \cdot \bar{v}(r) = \bar{u} \cdot \bar{v}(\infty) = 0\). Similarly, we also have \(\bar{u} \cdot \bar{w} = \bar{v} \cdot \bar{w} = 0\) and \(|\bar{v}| = |\bar{w}| = 1\). Thus \(U\) satisfies the orthonormality condition.

From the system (91), we know that \(\bar{u}_3\) and \(q = \bar{w}_3 - i\bar{v}_3\) solve the system

\[
\begin{cases}
\partial_t q = \bar{u}_3 \psi_1, \\
\partial_t \bar{u}_3 = \Im (\psi_1 q).
\end{cases}
\]

with boundary condition \((\bar{u}_3, q)(\infty) = (\frac{1-\lambda^2}{1+\lambda^2}, \frac{2i\lambda}{1+\lambda^2})\). By uniqueness, \(A_2 = \bar{u}_3, \psi_2 = q\).

Then, we construct the system of \((u, v, w)\) by equivariant setup, that is, apply \((\bar{u}, \bar{v}, \bar{w})\) by \(e^{iBR}\). From \(\psi_2 = \bar{w}_3 - i\bar{v}_3\) and the orthonormality condition, (47) is satisfied for \(k = 2\).

Finally, proving the Lipschitz continuous (90). Given \(\psi^+, \tilde{\psi}^+ \in L^2\) with \(\|\psi^+\|_{L^2}, \|\tilde{\psi}^+\|_{L^2} < \epsilon_0\), we construct \(U\) and \(\tilde{U}\) as above. From (91) it follows that

\[
\partial_t (U - \tilde{U}) = (M_1 - \tilde{M}_1 + \partial_r (M_2 - \tilde{M}_2))U + (\tilde{M}_1 - \partial_r \tilde{M}_2)(U - \tilde{U}).
\]

Since, by (86), \(\psi_1 = \psi^+ - i\frac{\bar{w}_3}{\sinh \tilde{r}}, (87)\) and (21) we have

\[
\|\psi_1 - \tilde{\psi}_1\|_{L^2} \lesssim \|\psi^+ - \tilde{\psi}^+\|_{L^2} + \|\frac{\bar{w}_2 - \tilde{\psi}_2}{\sinh \tilde{r}}\|_{L^2} \lesssim \|\psi^+ - \tilde{\psi}^+\|_{L^2},
\]

and

\[
\|\psi_2 - \tilde{\psi}_2\|_{L^\infty} \lesssim \|\psi_2 - \tilde{\psi}_2\|_{L^2} \lesssim \|\psi^+ - \tilde{\psi}^+\|_{L^2}.
\]

Then from (92) and (93) we obtain

\[
\|U - \tilde{U}\|_{L^\infty} \lesssim \|
\begin{align*}
(M_1 - \tilde{M}_1)_{L^1(dr)} + \|M_2 - \tilde{M}_2\|_{L^\infty})\|U\|_{L^\infty} & + \frac{M_2 - \tilde{M}_2}{\sinh \tilde{r}} \|\partial_r U\|_{L^2} \\
\left(\|M_1\|_{L^1(dr)} + \|M_2\|_{L^\infty}\right)\|U - \tilde{U}\|_{L^\infty} & + \frac{\tilde{M}_2}{\sinh \tilde{r}} \|L^2\| + \|\partial_r (U - \tilde{U})\|_{L^2} \\
\|\psi^+ - \tilde{\psi}^+\|_{L^2} & + (\|\tilde{\psi}^+\|_{L^2} + |\psi_2(\infty)|)\|U - \tilde{U}\|_{L^\infty} + \|\partial_r (U - \tilde{U})\|_{L^2}
\end{align*}
\]

and

\[
\|\partial_r (U - \tilde{U})\|_{L^2} \lesssim \|
\begin{align*}
(M_1 - \tilde{M}_1)_{L^2} + \|\partial_r (M_2 - \tilde{M}_2)\|_{L^2})\|U\|_{L^\infty} & + \|	ilde{M}_1\|_{L^2} + \|\partial_r \tilde{M}_2\|_{L^2})\|U - \tilde{U}\|_{L^\infty} \\
\|\psi^+ - \tilde{\psi}^+\|_{L^2} & + (\|\tilde{\psi}^+\|_{L^2} + |\psi_2(\infty)|)\|U - \tilde{U}\|_{L^\infty}
\end{align*}
\]

Using \(\|\tilde{\psi}^+\|_{L^2} + |\psi_2(\infty)| \lesssim \epsilon_0\), it follows that

\[
\|U - \tilde{U}\|_{C(0, \infty)} + \|\partial_r (U - \tilde{U})\|_{L^2(0, \infty)} \lesssim \|\psi^+ - \tilde{\psi}^+\|_{L^2}.
\]
which implies $\| \partial_r (u - \bar{u}) \|_{L^2(0, \infty)} \lesssim \| \psi^+ - \bar{\psi}^+ \|_{L^2}$. Since $u_1 = v_2w_3 - v_3w_2$, $\tilde{u}_1 = \tilde{v}_2\tilde{w}_3 - \tilde{v}_3\tilde{w}_2$, $\psi_2 = w_3 - iv_3$, $\tilde{\psi}_2 = \tilde{w}_3 - iv_3$, by (94), we have

$$
\| u_1 - \tilde{u}_1 \|_{L^2} \leq \left\| \frac{1}{\sinh r} \left[ (v_2 - \tilde{v}_2)\tilde{w}_3 + v_2(w_3 - \tilde{w}_3) \right] \right\|_{L^2} \\
+ \left\| \frac{1}{\sinh r} \left[ (v_3 - \tilde{v}_3)w_2 + \tilde{v}_3(w_2 - \tilde{w}_2) \right] \right\|_{L^2} \\
\lesssim \left\| U - \tilde{U} \right\|_{L^\infty} \left\| \frac{\tilde{\psi}_2}{\sinh r} \right\|_{L^2} + \| U \|_{L^\infty} \left\| \frac{\psi_2 - \tilde{\psi}_2}{\sinh r} \right\|_{L^2} \\
\lesssim \| \psi^+ - \bar{\psi}^+ \|_{L^2}.
$$

A similar argument shows that $\| \frac{u_1 - \tilde{u}_1}{\sinh r} \|_{L^2} + \| \frac{u_2 - \tilde{u}_2}{\sinh r} \|_{L^2} \lesssim \| \psi^+ - \bar{\psi}^+ \|_{L^2}$. Therefore, (90) follows.

5. The Cauchy problem. In this section we are concerned with the $(\psi^+, \psi^-)$-system (72). For consistency in writing we choose to use $R_+ = R_2$ and $R_- = R_0$, then we rewrite (72) as

$$
\begin{align*}
(i\partial_t + \Delta_{\mathbb{H}^2} - & 2\cosh r - \frac{1}{\sinh^2 r})R_+ \psi^+ \\
= & \left[ A_0 + 2\cosh r(A_2 - 1) - 3(\psi^+ \bar{\psi}_2 \sinh r) \right] R_+ \psi^+, \\
(i\partial_t + \Delta_{\mathbb{H}^2} + & 2\cosh r - \frac{1}{\sinh^2 r})R_- \psi^- \\
= & \left[ A_0 - 2\cosh r(A_2 - 1) - 3(\psi^- \bar{\psi}_2 \sinh r) \right] R_- \psi^-.
\end{align*}
$$

(95)

with data $\psi^+(t_0) = \psi^+_0$. Where $A_0$, $A_2$, $\psi_2$ are given by (65), (64), (61). Since the system (95) arised from the Schrödinger map flow (1), we will show that $(\psi^+, \psi^-)$ satisfies the compatibility condition (63).

For simplicity of notations, we denote $\| f^\pm \| = \| f^+ \| + \| f^- \|$, and denote the nonlinearities by

$$
F^+(\psi^+) = \left[ A_0 + 2\cosh r(A_2 - 1) - 3(\psi^+ \bar{\psi}_2 \sinh r) \right] R_+ \psi^+, \\
F^-(\psi^-) = \left[ A_0 - 2\cosh r(A_2 - 1) + 3(\psi^- \bar{\psi}_2 \sinh r) \right] R_- \psi^-.
$$

For the Schrödinger operators in (95), we have the following property:

**Proposition 3.** The Schrödinger operators $-\Delta_{\mathbb{H}^2} \pm \frac{2\cosh r - 1}{\sinh^2 r}$ have continuous spectrum $[\frac{1}{4}, \infty)$, with no embedded eigenvalues in the range $(\frac{1}{4}, \infty)$. Moreover, the operator $-\Delta_{\mathbb{H}^2} - 2\cosh r - \frac{1}{\sinh^2 r}$ admits discrete spectrum $0$.

**Proof.** Since $\frac{\cosh r - 1}{\sinh^2 r} \lesssim e^{-r}$, by Theorem 1 in [9] we easily obtain the first property. Next, for the operator $-\Delta_{\mathbb{H}^2} - 2\cosh r - \frac{1}{\sinh^2 r}$, it is easy to check that the function $\frac{\cosh r - 1}{\sinh^2 r} \in L^2(\mathbb{H}^2)$ is an eigenfunction of $(-\Delta_{\mathbb{H}^2} - 2\cosh r - \frac{1}{\sinh^2 r})\phi = \mu \phi$. 

with eigenvalue $\mu = 0$.

5.1. **Strichartz estimates.** To understand the well-posedness of (95), we need to obtain the Strichartz estimates. The $R_+\psi^+$-equation in (95) is a nonlinear Schrödinger equation with nonnegative and exponential decay potential. More generally, we consider the Schrödinger equation

$$
\begin{cases}
(i\partial_t + \Delta_{\mathbb{H}^2} - V)u = F, \\
u(0) = f,
\end{cases}
$$

(96)

where $V \in e^{-\alpha r}L^\infty(\mathbb{H}^2; \mathbb{R})$ for $\alpha \geq 1$ is a nonnegative potential. In this section we always denote potential $V$ as (96). For simplicity, we denote $p' = \frac{p}{p-1}$ for $p \in [1, \infty]$.

Then we obtain the following Strichartz estimates.

**Theorem 5.1 (Strichartz estimates).** Let $(p, q), (\tilde{p}', \tilde{q}')$ be admissible pairs, $I \subset \mathbb{R}$ be open interval,

(i) If $f \in L^2(\mathbb{H}^2)$, then

$$
\left\| e^{it(\Delta_{\mathbb{H}^2} - V)}f \right\|_{L^p_I L^q} \lesssim \| f \|_{L^2},
$$

(ii) If $F \in L^{\tilde{p}}_I L^{\tilde{q}}$, then

$$
\left\| \int_0^t e^{i(t-s)(\Delta_{\mathbb{H}^2} - V)}F(s)ds \right\|_{L^p_I L^q} \lesssim \| F \|_{L^{\tilde{p}}_I L^{\tilde{q}}}. 
$$

Based on a standard theory, the above results are obtained by the following dispersive estimates immediately.

**Proposition 4 (Dispersive estimates).** Assume $V \in e^{-\alpha r}L^\infty(\mathbb{H}^2; \mathbb{R})$, $\alpha \geq 1$, is a nonnegative potential, then

$$
||e^{it(\Delta_{\mathbb{H}^2} - V)}||_{L^1 \to L^\infty} \leq \begin{cases} 
C|t|^{-\frac{1}{2}}, & \text{if } 0 < |t| < 1, \\
C|t|^{-\frac{3}{2}}, & \text{if } |t| \geq 1.
\end{cases}
$$

(97)

By standard convention the resolvent of Laplacian $-\Delta_{\mathbb{H}^2}$ on $\mathbb{H}^2$ is written as $R_0(s) = (-\Delta_{\mathbb{H}^2} - s(1-s))^{-1}$ with $\Re s > \frac{1}{2}$ corresponding to the resolvent set $s(1-s) \in \mathbb{C} - [\frac{1}{4}, \infty)$. The kernel of $R_0(s)$ is

$$
R_0(s; z, w) = Q_{s-1}^0(\cosh r),
$$

(98)

where $Q_{s-1}^0$ is Legendre function, $r := d(z, w)$. The continuous part of the spectral resolution is given by (see [9])

$$
d\Pi(\lambda) = 2i\lambda[R_0(\frac{1}{2} + i\lambda) - R_0(\frac{1}{2} - i\lambda)]d\lambda
$$

$$
= -4\lambda \Re R_0(\frac{1}{2} + i\lambda)d\lambda.
$$
Then one can use the spectral resolution to write (see [2])
\[
e^{it\Delta}f(z) = \frac{e^{it}}{2\pi i} \int_0^{+\infty} e^{it\lambda^2} f(w) d\Pi(\lambda; z, w) d\lambda
\]
\[
= \frac{e^{it}}{2\pi i} \int_0^{+\infty} \int_{\mathbb{R}^2} e^{it\lambda^2} \lambda \sin \lambda s \sqrt{\cosh s - \cosh r} f(w) dw d\lambda.
\]
Similarly, from [9], the resolvent of \(-\Delta_{\mathbb{H}^2} + V\) for potential \(V\) defined as above is given by 
\[
R_V(s) = (-\Delta_{\mathbb{H}^2} + V - s(1 - s))^{-1}
\]
and the continuous component of spectral resolution is given by
\[
d\Pi_V(\lambda) = -4\lambda \Im R_V\left(\frac{1}{2} + i\lambda\right) d\lambda,
\]
then the kernel of Schrödinger propagator can be written as
\[
e^{it(\Delta_{\mathbb{H}^2} - V)}f(z) = \frac{e^{it}}{2\pi i} \int_0^{+\infty} e^{it\lambda^2} \Im R_V\left(\frac{1}{2} + i\lambda; z, w\right) f(w) dw d\lambda.
\]
By Birman-Schwinger type resolvent expansion for all frequencies:
\[
R_V(s) = R_0(s) + R_0(s)[-VR_0(s)] + [R_0(s)VR_V(s)[VR_0(s)],
\]
we get
\[
e^{it(\Delta_{\mathbb{H}^2} - V)}f(z)
= \frac{e^{it}}{2\pi i} \int_0^{+\infty} \int_{\mathbb{R}^2} e^{it\lambda^2} \Im R_0\left(\frac{1}{2} + i\lambda; z, w\right) f(w) dw d\lambda
\]
\[
+ \frac{e^{it}}{2\pi i} \int_0^{+\infty} \int_{\mathbb{R}^2} e^{it\lambda^2} \Im R_0\left(\frac{1}{2} + i\lambda|VR_0\left(\frac{1}{2} + i\lambda\right)\right) f(w) dw d\lambda
\]
\[
+ \frac{e^{it}}{2\pi i} \int_0^{+\infty} \int_{\mathbb{R}^2} e^{it\lambda^2} \Im [R_0\left(\frac{1}{2} + i\lambda\right)VR_V\left(\frac{1}{2} + i\lambda\right)
\]
\[
\cdot [VR_0\left(\frac{1}{2} + i\lambda\right)] f(w) dw d\lambda.
\]
Before proving Proposition 4, we recall the pointwise bounds on the resolvent kernel from [9]. These bounds will be crucial for the dispersive estimates.

**Lemma 5.2.** For the free resolvent kernel the pointwise bounds are valid for \(0 \leq \lambda \leq 1, \ r \in (0, \infty)\)
\[
|R_0\left(\frac{1}{2} + i\lambda; z, w\right)| \leq \begin{cases}
C|\log r|, & r \leq 1, \\
\frac{C}{\lambda} \frac{e^{-\frac{1}{2} r}}{r}, & r > 1,
\end{cases}
\]
\[
|\partial_\lambda R_0\left(\frac{1}{2} + i\lambda; z, w\right)| \leq \begin{cases}
C|\log r|, & r \leq 1, \\
\frac{C}{\lambda} \frac{e^{-(1-\epsilon) r}}{r}, & r > 1,
\end{cases}
\]
and for \(\lambda \geq 1, \ r \in (0, \infty)\)
\[
|R_0\left(\frac{1}{2} + i\lambda; z, w\right)| \leq \begin{cases}
C|\log r|, & \lambda r \leq 1, \\
\frac{C}{\lambda^2} \frac{e^{-\frac{1}{2} r}}{r}, & \lambda r > 1,
\end{cases}
\]
\[
|\partial_\lambda R_0\left(\frac{1}{2} + i\lambda; z, w\right)| \leq \begin{cases}
C|\log r|, & \lambda r \leq 1, \\
\frac{C}{\lambda^2} \frac{e^{-(1-\epsilon) r}}{r}, & \lambda r > 1,
\end{cases}
\]
derwhere \(\epsilon := d(z, w)\).

We also recall the meromorphic continuation from [9].
Lemma 5.3 (Meromorphic continuation). For $V \in e^{-\alpha r}L^\infty(\mathbb{H}^2)$ with $\alpha > 0$, the resolvent $R_V(s)$ admits a meromorphic continuation to the half-plane $\Re s > \frac{1}{2} - \delta$ as a bounded operator

$$R_V(s) : e^{-\delta r}L^2(\mathbb{H}^2) \to e^{\delta r}L^2(\mathbb{H}^2)$$

for $\delta < \frac{\alpha}{2}$. And there exists a constant $M_V$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq M_V$,

$$\|e^{-\frac{r^2}{2}}\partial^2_{\lambda} R_V(\frac{1}{2} + i\lambda)e^{-\frac{r^2}{2}}\|_{L^2 \to L^2} \leq C_{q,\alpha}|\lambda|^{-1}.$$  

If the $R_V(\frac{1}{2} + i\lambda)$ has no pole at $\lambda = 0$, one can extend the estimate through $\lambda = 0$ to give

$$\|e^{-\frac{r^2}{2}}\partial^2_{\lambda} R_V(\frac{1}{2} + i\lambda)e^{-\frac{r^2}{2}}\|_{L^2 \to L^2} \leq C_{q,\alpha}(\lambda)^{-1}.$$  

In order to prove Proposition 4, we also need the following lemma.

Lemma 5.4.

$$\left| \int_r^\infty \frac{e^{r^2/4}((s+a)^2/r+a)}{\sqrt{\cosh s - \cosh r}} ds \right| \lesssim \begin{cases} \sqrt{t} \sqrt{\frac{r+a}{\sinh r}}, & r \geq \frac{\sqrt{t}}{2}, \\ \sqrt{t}(1 + \frac{a}{r}), & r < \frac{\sqrt{t}}{2}. \end{cases}$$  

(102)

Proof. The proof roughly follows the approach in [2]. Before proving the lemma, we recall two useful estimates, that is,

$$\frac{1}{\sqrt{\cosh s - \cosh \rho}} \leq \frac{c}{(s-\rho)\sqrt{\cosh \rho}} \leq \frac{c}{s-\rho},$$  

(103)

for $s > \rho \geq 0$,

and

$$\frac{1}{\sqrt{\cosh s - \cosh \rho}} \leq \frac{c}{\sqrt{(s-\rho)\sinh \rho}},$$  

(104)

for $s > \rho > 0$.

Case 1. $r \geq \frac{\sqrt{t}}{2}$. Let $s = \frac{r^2}{4(r+a)^2} + r$, then

$$\text{LHS}(102) = \left| \int_0^\infty \frac{e^{r^2/4}(s^2/r+a^2)(\pi t/r+a) + a)}{\sqrt{\cosh(s/r+a) - \cosh r}} \frac{t}{r+a} \frac{d\tau}{1} \right|$$

$$\lesssim \sqrt{t} \int_0^\infty \frac{e^{r^2/4(r+a)^2 + \frac{\pi t}{r+a}}}{\sqrt{\cosh(s/r+a) - \cosh r}} \frac{r+a}{r+a} d\tau.$$  

(105)

Denote $\Phi(\tau) := \frac{\pi t}{4(r+a)^2} + \frac{\pi t}{2(r+a)^2} - \frac{r+a}{\sqrt{\sinh(t/r+a)}}$ and $\alpha(\tau) := \frac{r+a}{\sqrt{\sinh(t/r+a)}}(\cosh(t/r+a) - \cosh r)^{-1/2}$, (105) can be written as

$$\sqrt{t} \sqrt{\frac{r+a}{\sinh r}} \int_0^\infty \alpha(\tau) e^{\Phi(\tau)} \Phi'(\tau) d\tau = \sqrt{t} \sqrt{\frac{r+a}{\sinh r}} \int_0^1 \int_1^\infty \left| \frac{r+a}{\sqrt{\sinh(t/r+a)}}(I_1 + I_2) \right|.$$  

(106)

Now it suffices to prove the boundedness of $I_1$ and $I_2$. For $I_1$, by (104) and $r \geq \frac{\sqrt{t}}{2}$ we have

$$I_1 \lesssim \int_0^1 \frac{\pi t}{2(r+a)^2} + \frac{1}{2} d\tau \lesssim \int_0^1 \frac{1}{\sqrt{\tau}} d\tau \lesssim 1.$$
For $I_2$, integration by parts and $\alpha(1) \lesssim 1, \alpha'(\tau) < 0$ give

$$I_2 = \int_1^\infty \partial_\tau e^{i\Phi(\tau)} \alpha(\tau) d\tau \lesssim \sup_{\tau \geq 1} |\alpha(\tau)| + \int_1^\infty |\alpha'(\tau)| d\tau \lesssim 1.$$ 

Therefore (102) follows from (106) in the region $r \geq \sqrt[3]{\frac{\tau}{2}}$.

**Case 2.** $r < \sqrt[3]{\frac{\tau}{2}}$. Let us split the left hand side of (102) into three parts:

$$\text{LHS}(102) = \left| \int_r^{2r} + \int_{2r}^{\sqrt{\tau}} + \int_{\sqrt{\tau}}^\infty \right| J_1 + J_2 + J_3 |.$$ 

For $J_1$, we assume $r > 0$, otherwise $J_1 = 0$ immediately, then (104) and $r < \frac{t}{2}$ give

$$J_1 \lesssim \int_{r+a}^{2r+a} \frac{u}{\sqrt{\cosh(u-a) - \cosh r}} du \lesssim \int_{r+a}^{2r+a} \frac{u}{(u-a-r) \sinh r} du$$

$$\lesssim (2r+a) \sqrt{\frac{r}{\sinh r}} \lesssim \sqrt{t} + a.$$

For $J_2$, by (103) we have

$$J_2 \lesssim \int_{2r+a}^{\sqrt{\tau}+a} \frac{u}{\sqrt{\cosh(u-a) - \cosh r}} du \lesssim \int_{2r+a}^{\sqrt{\tau}+a} \frac{u}{u-a-r} du$$

$$= \int_{2r+a}^{\sqrt{\tau}+a} \left(1 + \frac{a+r}{u-a-r} du \lesssim \sqrt{t} + a \sqrt{t}.$$ 

For $J_3$, let $s = \sqrt{\tau}$, we get

$$J_3 = \int_1^\infty e^{i(\tau + a^2 + 2\sqrt{\tau})} (\sqrt{\tau} + a) \sqrt{\cosh(\sqrt{\tau}) - \cosh r} d\tau = 2te^{\frac{a^2}{4\sqrt{\tau}}} \int_1^\infty e^{i\left(\frac{a^2}{2\sqrt{\tau}} + \frac{a}{\sqrt{\tau}}\right)} \left(\frac{\tau}{2} + \frac{a}{2\sqrt{\tau}}\right) d\tau.$$ 

Denote $\psi(\tau) := \frac{\tau}{4} + \frac{a}{2\sqrt{\tau}}$ and $\beta(\tau) := [\cosh(\sqrt{\tau}) - \cosh r]^{-\frac{1}{2}}$, then $\beta'(<) < 0$ implies

$$|J_3| = 2t \left| \int_1^\infty e^{i\psi(\tau)} \psi'(\tau) \beta(\tau) d\tau \right| \lesssim t |\beta(1)| + \int_1^\infty |\beta'(\tau)| d\tau$$

$$\lesssim t |\beta(1)| \lesssim \sqrt{t}.$$ 

Therefore, (102) follows in the region $r < \sqrt[3]{\frac{\tau}{2}}$. \qed

**Proof of Proposition 4.** The estimate for $|t| \geq 1$ in (97) has been proved in [9], we only prove the case $0 < |t| < 1$ here. In order to estimate $e^{it(\Delta u^2 - V)} f$ for $f \in L^1$, it suffices to bound (99)-(101) respectively. (99) is indeed $e^{it\Delta_0 u^2} f$, which can be estimated in [1]. To estimate (100), we rewrite it by (98) as

$$\int_0^\infty e^{-i\lambda \hat{\Psi}} \int_{z_0, z_1} V(z_1) \int_{r_0}^\infty \int_{r_1}^\infty \frac{\sin \lambda(s + s')}{\sqrt{\cosh s - \cosh r_1 \sqrt{\cosh s'} - \cosh r_0}} ds' ds f(z_0) d\omega_0 dz_1 d\omega_1$$

$$= t \frac{1}{2} \int_{z_0, z_1} V(z_1) \int_{r_0}^\infty \int_{r_1}^\infty \frac{e^{i\frac{(z + z')^2}{4t} (s + s')}}{\sqrt{\cosh s - \cosh r_1 \sqrt{\cosh s'} - \cosh r_0}} ds' ds f(z_0) d\omega_0 dz_1,$$

(107)
where \( r_0 = d(z_0, z_1) \), \( r_1 = d(z_1, z) \). By Lemma 5.4, we get

\[
\int_{r_0}^{\infty} \int_{r_1}^{\infty} e^{i(s+s')/2} (s+s') \sqrt{\cosh s - \cosh s - \cosh r_0} ds' ds \quad (108)
\]

It suffices to estimate the integrals in the right hand side. For \( r_0 > 0 \), we have from (104)

\[
\int_{r_0}^{\infty} \frac{1}{\sqrt{\cosh s - \cosh r_0}} ds \\
\lesssim \int_{r_0}^{r_0+1} \frac{1}{\sqrt{(s-r_0) \sinh r_0}} ds + \int_{r_0+1}^{\infty} e^{-\frac{s}{2}} ds \lesssim \frac{1}{\sqrt{\sinh r_0}}, \quad (109)
\]

which implies the following estimate immediately

\[
\int_{r_0}^{\infty} \frac{1}{\sqrt{\cosh s - \cosh r_0}} ds \\
\lesssim \int_{r_0}^{\infty} \frac{1}{\sqrt{s-r_0 + \sqrt{r_0}}} ds \\
\lesssim \int_{0}^{\infty} \frac{1}{\sqrt{\cosh(u + r_0) - \cosh r_0}} ds + \int_{r_0}^{\infty} \frac{1}{\sqrt{\cosh s - \cosh r_0}} ds \\
\lesssim \int_{0}^{\infty} \frac{1}{\sqrt{\cosh u - 1}} \frac{1}{\cosh r_0} du + \frac{1}{\sqrt{\sinh r_0}} \\
\lesssim \frac{1}{\sqrt{\cosh r_0}} + \frac{1}{\sqrt{\sinh r_0}}.
\]

Similarly,

\[\int_{r_0}^{\infty} \frac{s}{\sqrt{\cosh s - \cosh r_0}} ds = \int_{r_0}^{\infty} \frac{s-r_0}{\sqrt{\cosh s - \cosh r_0}} ds + \int_{r_0}^{\infty} \frac{r_0}{\sqrt{\cosh s - \cosh r_0}} ds \lesssim \frac{1}{\sqrt{\cosh r_0}} + \frac{r_0}{\sqrt{\sinh r_0}}.
\]

In conclusion, we obtained

\[
(108) \lesssim t^{1/2} \left[ \frac{1}{\sqrt{\sinh r_0}} \left( \frac{1}{\sinh r_1} \right) \geq \varphi (r_1) + \frac{1}{\sqrt{\varphi}} (r_1) \right] \\
+ \frac{1}{\sqrt{\cosh r_0}} \frac{1}{\sqrt{\sinh r_1}} \geq \varphi (r_1) + \frac{1}{\sqrt{\cosh r_0}} \frac{1}{\sqrt{\sinh r_1}} \geq \varphi (r_1)
\]

\[
(107) \lesssim t^{-1} \int_{z_0, z_1} V(z_1) \left[ \left( \frac{1}{\sqrt{\sinh r_0}} \right) \left( \frac{1}{\sinh r_1} + \frac{1}{\sqrt{\cosh r_0}} \frac{1}{\sqrt{\sinh r_1}} \right) \right] \geq \varphi (r_1)
\]
Finally, we estimate the (101). By duality, it suffices to prove for any \(\|h\|_{L^1} = 1\),
\[
\left( \int_0^\infty e^{it\lambda^2} \int_{\mathbb{H}^2} \Im[R_0 V][V R_0] f(w) dw d\lambda, h \right) \lesssim |t|^{-1} \|f\|_{L^1}.
\] (110)

Denote
\[
A(f)(z) = \int V(z) R_0(-\frac{1}{2} + i\lambda; z_0) f(z_0) dz_0,
\]
\[
B(f)(z) = \int V(z) \partial_\lambda R_0(-\frac{1}{2} + i\lambda; z_0) f(z_0) dz_0.
\]

Then by integration by parts and Lemma 5.4, we have
\[
(110) = \int_0^\infty e^{it\lambda^2} \int_{\mathbb{H}^2} \Im[R_0 V][V R_0] f(w) dw d\lambda \lesssim |t|^{-1} \|f\|_{L^1}.
\]
Similarly, we have
\[ \left\| e^{\frac{t}{2}}z B(f)(z) \right\|_{L^\infty} \lesssim (\lambda)^{-\frac{1}{2}} \| f \|_{L^1}. \]

Therefore,
\[ (101) \lesssim \sup_{\| h \|_{L^1} = 1} |t|^{-1} \int_0^\infty (\lambda)^{-\frac{1}{2}} \| f \|_{L^1} \| h \|_{L^1} \, d\lambda \lesssim |t|^{-1} \| f \|_{L^1}. \]

Thus Proposition 4 follows. \( \square \)

5.2. The Cauchy theory. Here we consider the Cauchy problem for (95). The local well-posedness of (95) is implied by Strichartz estimates in Theorem 5.1. Then for small initial data, since the operator \(-\Delta_{\mathbb{H}^2} - 2\cosh r^{-1}\) has discrete spectrum (see Proposition 3), we use perturbation method (see [37]) to prove global well-posedness.

For simplicity, we denote the potential in (95) as
\[ V = 2 \frac{\cosh r - 1}{\sinh^2 r}. \]

**Theorem 5.5.** Consider the problem (95) with data \( \psi^\pm_0 \in L^2(\mathbb{H}^2) \), where \( A_0, A_2, \psi_2 \) are given by (65), (64), (61). Then there exists a unique maximal-lifespan solution pair \((\psi^+, \psi^-) : I \times \mathbb{R}^2 \rightarrow \mathbb{C} \times \mathbb{C} \) with \( t_0 \in I \) and \( \psi^\pm(t_0) = \psi^\pm_0 \) with the following additional properties:

(i) \( I \) is open.

(ii) For every \( M > 0 \), time interval \( I \), and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( \psi^\pm \) is a solution satisfying \( \| \psi^\pm \|_{L^4_t L^4(\mathbb{H}^2)} \leq M \) and \( \| \psi^0_\pm - \tilde{\psi}^0_\pm \|_{L^2(\mathbb{H}^2)} \leq \delta \), then there exists a solution \( \tilde{\psi}^\pm \in L^2_t L^4 \cap L^\infty_t L^2 \) with initial data \( \tilde{\psi}^\pm(0) = \tilde{\psi}^\pm_0 \) such that
\[ \| \psi^\pm_\pm \|_{L^1_t L^4(\mathbb{H}^2)} \leq \epsilon, \] and \( \| \psi^\pm - \tilde{\psi}^\pm \|_{L^2(\mathbb{H}^2)} \leq \epsilon, \forall t \in I. \)

(iii) There exists \( \epsilon_0 > 0 \) such that \( \| \psi^0_\pm \|_{L^2(\mathbb{H}^2)} \leq \epsilon_0 \), then for any compact interval \( J \subset \mathbb{R}, \) (95) has a unique global solution \( \psi^\pm(t) \in L^4(J; L^4(\mathbb{H}^2)) \cap C(J; L^2(\mathbb{H}^2)) \), moreover,
\[ \| \psi^\pm \|_{L^4_t L^4(\mathbb{H}^2)} \leq C |J| \epsilon_0, \]

and
\[ \| \psi^\pm \|_{C(J; L^2(\mathbb{H}^2))} \leq C \epsilon_0, \]
where \( C \) depends on \( \| \psi^\pm_0 \|_{L^2} \) and \( \| V \|_{L^2}. \)

(iv) Assume that \( R^\pm \psi^0_\pm \in H^s, \) for \( s \in \{1, 2\} \). If \( \| \psi^\pm \|_{L^4_t L^4(\mathbb{H}^2)} \leq M \), then the solution \( \psi^\pm \) satisfies
\[ \| R^\pm \psi^\pm \|_{H^s} \lesssim M \| R^\pm \psi^0_\pm \|_{H^s} + \| \psi^\pm \|_{L^4_t L^4(\mathbb{H}^2)}, \quad \forall t \in I, \quad (111) \]
and it has Lipschitz dependence with respect to the initial data.

**Proof.** Consider the system (95) in the space
\[ X = \left\{ (\psi^+, \psi^-) \in C([0, T]; L^2) \cap L^4_t L^4 : \| \psi^\pm \|_{C([0, T]; L^2)} \leq 2 \| \psi^0_\pm \|_{L^2}, \right. \]
\[ \left. \| \psi^\pm \|_{L^4_t L^4} \leq \epsilon \right\}. \]

Given the formulas for \( A_0, A_2 \) and \( \psi_2 \) by (65), (64) and (61), using Lemma 2.4, we obtain
\[ \| A_0 \|_{L^2} + \left\| \frac{\cosh r(A_2 - 1)}{\sinh^2 r} \right\|_{L^2} \lesssim \| \psi^\pm \|_{L^4}, \quad \left\| \frac{\psi_2}{\sinh r} \right\|_{L^4} \lesssim \| \psi^\pm \|_{L^4}. \quad (112) \]
In a similar argument, we also have
\[
\|F^\pm(\psi^\pm) - F^\pm(\tilde{\psi}^\pm)\|_{L^4_x} \lesssim \|\psi^\pm - \tilde{\psi}^\pm\|_{L^4_x} (\|\psi^\pm\|_{L^4_x}^2 + \|\tilde{\psi}^\pm\|_{L^4_x}^2).
\] (113)

Since \( V \in L^2(\mathbb{H}^2) \) is independent of \( t \), the \( V \psi^- \) in \( \psi^- \)-equation can be regarded as a nonlinear term. Based on a standard argument, by Strichartz estimate in Theorem 5.1, (112) and (113), the system has a unique solution with \( \psi^\pm(t_0) = \psi_0^\pm \) on some maximal life-time open interval \( I \) with \( t_0 \in I \). Hence (i) follows. (ii) is standard in light of (113).

Next, we prove (iii). Let \( u^\pm : I \times R^+ \rightarrow \mathbb{C} \) be an approximate solution to system (95) in the sense that
\[
\begin{aligned}
(i\partial_t + \Delta - V)u^+ &= F^+(u^+), \\
(i\partial_t + \Delta)u^- &= F^-(u^-), \\
u^+(0) &= e^{it\theta} \psi_0^+, \quad u^-(0) = \psi_0^-.
\end{aligned}
\] (114)

Based on standard fixed point argument, by the Strichartz estimates for Schrödinger operators \(-\Delta_{\mathbb{H}^2}\) and \(-\Delta_{\mathbb{H}^2} + V\), there exists \( \epsilon_0 > 0 \) such that if \( \|\psi_0^\pm\|_{L^2} \leq \epsilon_0 \), then (114) has a unique global solution \( u^\pm \in C(\mathbb{R}; L^2) \cap L^4 L^4 \), moreover, \( \|u^\pm\|_{L^\infty L^2 \cap L^4 L^4(\mathbb{R} \times \mathbb{H}^2)} \leq C\epsilon_0 \).

Now we show using a perturbative argument that (95) is global well-posed for \( \|\psi_0^\pm\|_{L^2} < \epsilon_0 \). First we show that for \( T \) sufficiently small depending only on \( \|\psi_0^\pm\|_{L^2} \) and \( V \), the solution \((\psi^+, \psi^-)\) to (95) on \([0,T]\) satisfies an a priori estimate
\[
\|\psi^\pm\|_{L^\infty L^2 \cap L^4 L^4(0,T)} \leq 8C\epsilon_0.
\] (115)

In fact, by Duhamel formula, Strichartz estimates and \( \|u^\pm\|_{L^\infty L^2 \cap L^4 L^4} \leq C\epsilon_0 \), we have
\[
\begin{aligned}
\|e^{i(t-s)(\Delta - V)} u^+(0)\|_{L^4_x} &\leq \|u^+\|_{L^4_x} + \left\| \int_0^t e^{i(t-s)(\Delta - V)} F^+(u^+) ds \right\|_{L^4_x} \\
&\leq C\epsilon_0 + C(C\epsilon_0)^3 \\
&\leq 2C\epsilon_0,
\end{aligned}
\] (116)

and
\[
\begin{aligned}
\|e^{i(t-s)\Delta} u^-(s)\|_{L^4_x} &\leq \|u^-\|_{L^4_x} + \left\| \int_0^t e^{i(t-s)\Delta} F^-(u^-) ds \right\|_{L^4_x} \\
&\leq 2C\epsilon_0.
\end{aligned}
\] (117)

Since \((\psi^+, \psi^-)\) satisfies (95) and \( u^+(0) = e^{it\theta} \psi_0^+, \quad u^-(0) = \psi_0^- \), applying Duhamel formula, (116) and (117), we obtain
\[
\|\psi^\pm\|_{L^4_x} \leq \left\| e^{i(t-s)(\Delta - V)} u^+(0) \right\|_{L^4_x} + C \|\psi^\pm\|_{L^4_x}^3 \\
\leq 2C\epsilon_0 + C \|\psi^\pm\|_{L^4_x}^3,
\] (118)

and
\[
\begin{aligned}
\|\psi^-\|_{L^4_x} &\leq \|e^{i(t-s)\Delta} u^-(s)\|_{L^4_x} + C \|\psi^\pm\|_{L^4_x}^3 + CT^{1/2} \|V\|_{L^2} \|\psi^-\|_{L^4_x} \\
&\leq 2C\epsilon_0 + C \|\psi^\pm\|_{L^4_x}^3 + CT^{1/2} \|V\|_{L^2} \|\psi^-\|_{L^4_x}.
\end{aligned}
\]
Choosing $T$ sufficiently small such that $CT^{1/2}\|V\|_{L^2} < \frac{1}{3}$, it follows that
\[
\|\psi^\pm\|_{L^2_tL^4_x} \leq 3C\epsilon_0 + \frac{3}{2}C\|\psi^\pm\|_{L^4_xL^3_t}^3. \tag{119}
\]

Combining (118) and (119), we have
\[
\|\psi^\pm\|_{L^2_tL^4_x} \leq 5C\epsilon_0 + 3C\|\psi^\pm\|_{L^4_xL^3_t}^3.
\]

Then by continuity argument, we get
\[
\|\psi^\pm\|_{L^2_tL^4_x} \leq 7C\epsilon_0,
\]

which, together with Strichartz estimates, gives
\[
\|\psi^\pm\|_{L^\infty_tL^2_x} \leq \left\| e^{it(\Delta - V)} e^{i20\varphi}\psi^+_0 \right\|_{L^2_x} + \left\| e^{it\Delta} \psi^-_0 \right\|_{L^2_x} + 2C\|\psi^\pm\|_{L^4_xL^3_t}^3 + C\|V\|_{L^2_xL^2_t} \|\psi^-\|_{L^4_xL^3_t} 
\leq 2C\epsilon_0 + 2C(7C\epsilon_0)^3 + \frac{7}{3}C\epsilon_0 < 8C\epsilon_0.
\]

Therefore (115) is obtained.

Since the system (95) admits energy conservation $\|\psi^\pm\|_{L^2} = \|\psi^+_0\|_{L^2}$ and $T$ depends only on $\|\psi^+_0\|_{L^2}$ and $V = 2\text{cosh}r \frac{1}{\sinh^2 r}$, we obtain from (i) that $(\psi^+, \psi^-)$ is a global solution with
\[
\|\psi^\pm\|_{L^4_tL^4_x} \leq C|J|\epsilon_0,
\]

and
\[
\|\psi^\pm\|_{C(J;L^2)} \leq C\epsilon_0,
\]

for any compact interval $J \subset \mathbb{R}$, where $C$ depends on $\|\psi^+_0\|_{L^2}$ and $\|V\|_{L^2}$.

Finally, we prove the additional regularity (iv). Applying $(-\Delta)^{\frac{s}{2}}$ for $s = 1, 2$ to both sides of the system (95), we obtain
\[
\begin{cases}
(i\partial_t + \Delta)(-\Delta)^{\frac{s}{2}} R_+ \psi^+ = (-\Delta)^{\frac{s}{2}} F^+ + (-\Delta)^{\frac{s}{2}} (VR_+ \psi^+), \\
(i\partial_t + \Delta)(-\Delta)^{\frac{s}{2}} R_- \psi^- = (-\Delta)^{\frac{s}{2}} F^- - (-\Delta)^{\frac{s}{2}} (VR_- \psi^-).
\end{cases}
\]

The nonlinearities $F^\pm$ can be written as
\[
F^\pm = \left( -\frac{1}{2} |R_\pm \psi^\pm|^2 \right. + \int_r^\infty \frac{\cosh s}{\sinh s} \Re(\psi^+ \bar{\psi}^-) ds) R_\pm \psi^\pm \\
\pm \frac{\cosh r}{2\sinh^2 r} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \sinh sds R_\pm \psi^\pm \\
\pm F_2^\pm = F_1^\pm + F_2^\pm.
\]

Let $\varphi(r) \in C^\infty_c$ be a bump function with $0 \leq \varphi \leq 1$, $\varphi|_{B_{1}(0)} = 1$ and $\varphi|_{B_{2}(0)} = 0$, then $F_2^\pm$ can be rewritten as
\[
F_2^\pm = \varphi F_2^\pm + (1 - \varphi) F_2^\pm \triangleq I^\pm + II^\pm.
\]

Since $\|\psi^\pm\|_{L^2_tL^4_x} \leq M$, Strichartz estimates imply $\|\psi^\pm\|_{L^4_tL^8_x} \lesssim 1$. Then we split the interval into $I = \bigcup I_j$ such that $\|\psi^\pm\|_{L^4_xL^8_t} < \epsilon$, $\|(-\Delta)^{1/2} \psi^+_0\|_{L^2} \|\psi^\pm\|_{L^4_tL^8_x} \lesssim 1$, and
Then we have could use Littlewood-Paley decomposition to deal with $I_4$. Hence, (120), (121) and (123) imply
\[
\left\| (-\Delta)^{\frac{3}{2}} R_{\pm} \psi^\pm \right\|_{L^6} \lesssim \left\| (-\Delta)^{\frac{3}{2}} F^\pm \left( (-\Delta)^{\frac{3}{2}} (VR_{\pm} \psi^\pm) \right) \right\|_{L^6}.
\]
To obtain (111), it suffices to estimate the second term of the right hand side of (120). Define
\[
A = \left\| \psi^\pm \right\|_{L^6_j L^6} , \quad B = \left\| \partial_r \psi^\pm \right\|_{L^6_j L^6} + \left\| \psi^\pm \right\|_{L^6_j L^6} , \quad C = \left\| \partial_r^2 \psi^\pm \right\|_{L^6_j L^6} + \left\| \psi^\pm \right\|_{L^6_j L^6} ,
\]
For $s = 1$, from (19) we easily get
\[
\left\| (-\Delta)^{\frac{3}{2}} (F_1^\pm \pm VR_{\pm} \psi^\pm) \right\|_{L^6_j L^2} \lesssim B(A^2 + \left\| V \right\|_{L^{3/2} L^3}) + A^3 + \left\| \partial_r V \right\|_{L^{3/2} L^3} A.
\]
Since the operator $\frac{1}{r} \int_0^r s ds$ keeps the two dimensional frequency localization, one could use Littlewood-Paley decomposition to deal with $I^\pm$. To estimate $I^\pm$, we claim that for $f$ radial, the following estimate holds
\[
\left\| (-\Delta_R)^{\frac{1}{2}} \frac{1}{r^2} \int_0^r f s ds \right\|_{L^p(R^2)} \lesssim \left\| (-\Delta_R)^{\frac{1}{2}} f \right\|_{L^p(R^2)} , \quad \text{for } p \geq 2.
\]
Then we have
\[
\left\| (-\Delta)^{\frac{3}{2}} I^\pm \right\|_{L^6_j L^2} \lesssim \left\| \partial_r (\varphi \frac{\cosh r}{\sinh^2 r} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \sinh s ds) R_{\pm} \psi^\pm \right\|_{L^6_j L^2} + \left\| \varphi \frac{\cosh r}{\sinh^2 r} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \sinh s ds \right\|_{L^{3/2} L^3} \cdot B
\]
\[
\lesssim \left\| \partial_r (\varphi \frac{r^2 \cosh r}{\sinh^2 r} \frac{1}{r^2} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \sinh s s \varphi(\frac{r}{2}) s ds \right\|_{L^{3/2} L^3} \cdot A + \left\| (-\Delta_R)^{1/2} \left( \frac{1}{r^2} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \sinh s s \varphi(\frac{r}{2}) s ds \right) \right\|_{L^{3/2} L^3} \cdot A + BA^2
\]
\[
\lesssim A^3 + BA^2.
\]
Hence, (120), (121) and (123) imply
\[
\left\| (-\Delta)^{1/2} R_{\pm} \psi^\pm \right\|_{L^6_j L^2} \lesssim \left\| (-\Delta)^{1/2} R_{\pm} \psi^0 \right\|_{L^2} + \left\| \psi^\pm \right\|_{L^6_j L^6}.
\]
We repeat the above procedure for $I_{j+1}$ to obtain the similar estimate in $I_{j+1}$. Thus, (111) valid for $s = 1$. 

For $s = 2$, similarly, we also easily have
\[
\left\| \left( -\Delta \right) (F^\pm_1 \pm II^\pm \pm VR^\pm_0 \psi^\pm) \right\|_{L^2_1 L^2_2} \\
\lesssim C(A^2 + \|V\|_{L^{3/2}_2 L^3}^3) + B(A^2 + \|\partial_s V\|_{L^{3/2}_2 L^3}) \\
+ B^2 A + A^3 + \|\Delta V\|_{L^{3/2}_2 L^3} A,
\]
where $I^\pm$ can be rewritten as
\[
(-\Delta) I^\pm \\
= (-\Delta) (\varphi(r) r^2 \cosh r R_\pm \psi^\pm \cdot \frac{1}{r^2} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \varphi(\frac{s}{2}) \sinh s s ds)
\]
\[
= (-\Delta) (\varphi(r) r^2 \cosh r R_\pm \psi^\pm) \cdot \varphi(\frac{s}{2}) \frac{1}{r^2} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \varphi(\frac{s}{2}) \sinh s s ds
\]
\[
+ \varphi(r) r^2 \cosh r R_\pm \psi^\pm \cdot (-\Delta) (\frac{1}{r^2} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \varphi(\frac{s}{2}) \sinh s s ds)
\]
\[
- 2\partial_r \varphi(r) r^2 \cosh r R_\pm \psi^\pm \cdot \partial_r (\varphi(\frac{s}{2}) \frac{1}{r^2} \int_0^r (|\psi^+|^2 - |\psi^-|^2) \varphi(\frac{s}{2}) \sinh s s ds)
\]
\[
\lesssim I^+_1 + I^+_2 + I^+_3.
\]

By (27) and (122) we have
\[
\left\| I^+_1 + I^+_2 \right\|_{L^2_1 L^2_2} \lesssim C A^2 + B A^2 + A^3 + B^2 A.
\]
(125)

Meanwhile, (122) implies
\[
\left\| I^+_2 \right\|_{L^2_1 L^2_2} \lesssim A \left\| \Delta (|\psi^+|^2 - |\psi^-|^2) \varphi(\frac{s}{2}) \sinh r \right\|_{L^{3/2}_1 L^3}
\]
\[
\lesssim C A^2 + B A^2 + B^2 A + A^3.
\]
(126)

Thus, (120), (124), (125) and (126) give
\[
\left\| \left( -\Delta \right) R_\pm \psi^\pm \right\|_{L^2_1 L^2_2 \cap L^6_1 L^6} \\
\lesssim \left\| \left( -\Delta \right) R_\pm \psi^\pm_0 \right\|_{L^2_2} + \left\| \left( -\Delta \right)^{1/2} R_\pm \psi^\pm_0 \right\|_{L^2_2} + \|\psi^\pm\|_{L^3_1 L^6}.
\]

Hence, (111) follows for $s = 2$.

To complete the proof of (iv), it remains to prove (122). Denote $B_r = B_r(0)$ and $m_k(r) = \varphi(2^{-k} r) - \varphi(2^{-k+1} r)$. Since $f$ is radial, we have
\[
\frac{1}{r^2} \int_0^r f s ds = C \frac{1}{m(B_r)} \int_{R^2} f \cdot 1_{B_r} (y) dy = \frac{1}{m(B_r)} f * 1_{B_r}(0).
\]

Then for $P_k f = F^{-1}(m_k(\xi) \widehat{f}(\xi))$, we have
\[
\int_{R^2} \frac{1}{m(B_{|x|})} (P_k f * 1_{B_{|x|}})(0) e^{-ix\eta} dx
\]
\[
= \int_{R^2} \frac{1}{m(B_{|x|})} \int \widehat{P_k f}(\xi) \widehat{1_{B_{|x|}}}(\xi) d\xi e^{-ix\eta} dx
\]
\[= \int \hat{P}_k f(\xi) \overline{1_{B_1(|x|)}} e^{-ix\eta} d\xi\]

which implies

\[\frac{1}{r^2} \int_0^r P_k f \, s \, ds = P_{\leq k} \left( \frac{1}{r^2} \int_0^r P_k f \, s \, ds \right). \tag{127}\]

Hence, by Littlewood-Paley decomposition and (127) we have

\[
\| \frac{1}{r^2} \int_0^r P_k f \, s \, ds \|_{L^p} \lesssim \left\| \sum_j \left( \sum_k 2^{sj} P_j \left( \frac{1}{r^2} \int_0^r P_k f \, s \, ds \right) \right)^2 \right\|_{L^p}^{1/2}. \tag{128}
\]

From (27), we get

\[
(128) \lesssim \left\| \sum_j \left( \sum_k 2^{sj} \mathbf{1}_{\leq 0(j-k)} \left( 2^{sk} P_k f \right) \right)^2 \right\|_{L^p}^{1/2} \lesssim \left\| \{2^{sk} \mathbf{1}_{\leq 0(k)} \ast |2^{sk} P_k f| \}^2 \right\|_{L^p} \lesssim \left\| \{2^{sk} P_k f \}^2 \right\|_{L^p} \lesssim \left\| (-\Delta_{R^2})^{s/2} f \right\|_{L^p}. \tag{129}
\]

Thus, (122) follows. \qed

The above theorem is only concerned with the general solutions of (95). Since the system of \((\psi^+, \psi^-)\) is derived from the Schrödinger map flow (1), if we want to reconstruct the map \(u\) by \(\psi^\pm\), the solution \(\psi^\pm\) of (95) must satisfy the compatibility condition (63).

**Theorem 5.6.** If \(\psi^\pm_0 \in L^2\) with \(\|\psi^\pm_0\|_{L^2} \leq \epsilon_0\) satisfies the compatibility condition, then \(\psi^\pm(t)\) satisfies the compatibility condition for any \(t \in I\). If, in addition, \(R_{\pm} \psi^\pm_0 \in H^2\), then (49) and (50) are satisfied.

**Proof.** Given \(\psi_1 = \frac{\psi^++\psi^-}{2}, \quad \psi_2 = \frac{\psi^+-\psi^-}{2i}, \quad A_1 = 0\). To prove the compatibility condition (63), it suffices to show that \(\partial_1 \psi_2 = D_2 \psi_1\) is preserved for \(t \in I\). For this we need to derive the equation for

\[F = D_2 \psi_1 - D_1 \psi_2.\]

Before deriving the equation for \(F\), we give some identities from (95). First, (64) gives

\[\partial_1 A_2 - \partial_2 A_1 = \Im(\psi_1 \overline{\psi}_2), \tag{129}\]
Second, the system of \((\psi^+, \psi^-)\) (95) and (64) imply that
\[
\partial_0 A_2 - \partial_2 A_0 = \frac{1}{2} \int_0^T \left( \Re(\partial_t \psi^+ \bar{\psi}^+) - \Re(\partial_t \psi^- \bar{\psi}^-) \right) \sinh s ds \\
= -\frac{1}{2} \int_0^T \partial_s (3(\partial_s \psi^+ \bar{\psi}^+) \sinh s) - \partial_s (3(\partial_s \psi^- \bar{\psi}^-) \sinh s) ds \\
= 3(\psi \bar{\psi}_2) + \Re(F\bar{\psi}_1),
\]
where \(\psi_0\) is given by (55). Third, (65) implies
\[
\partial_1 A_0 - \partial_0 A_1 = \Re(\psi_1 \bar{\psi}_0) - \Re(\frac{F\bar{\psi}_2}{\sinh^2 r}).
\]
Finally, we obtain the following two equations from (95) by algebraic computation and \(A_2^2 + |\psi|^2 = 1\),
\[
D_0 \psi_1 = i[D_1(D_1 + \coth r)\psi_1 + \frac{D_2 D_2 \psi_1}{\sinh^2 r} - 2\frac{\cosh r A_2}{\sinh^2 r} + \frac{\coth r}{\sinh r} F, F = \Re(F\bar{\psi}_1)\psi_1 \\
D_0 \psi_2 = i[D_1(D_1 + \coth r)D_1 \psi_2 + \frac{D_2 D_2 \psi_2}{\sinh^2 r} + 2\frac{\cosh r}{\sinh r} F - i\Re(\psi_1 \bar{\psi}_2)\psi_1],
\]
which, combined with (56), yields
\[
D_1 \psi_0 - D_0 \psi_1 = \frac{-i}{\sinh r} D_2 F, \quad D_2 \psi_0 - D_0 \psi_2 = i(D_1 - \coth r) F.
\]
Applying the operator \(D_0\) to \(F\), by (129)-(132), we have
\[
D_0 F = D_2 D_2 \psi_1 - D_1 D_2 \psi_0 + i\Re(\psi_0 \bar{\psi}_2)\psi_1 + i\Re(F\bar{\psi}_1)\psi_1 \\
- i\Re(\frac{F \bar{\psi}_2}{\sinh^2 r}) \psi_2, \\
= D_2 D_1 \psi_0 + D_2 \left( \frac{i}{\sinh^2 r} D_2 F \right) - D_1(D_2 \psi_0 - i(D_1 - \coth r) F) \\
+ i\Re(\psi_0 \bar{\psi}_2)\psi_1 - i\Re(F \bar{\psi}_1)\psi_1 - i\Re\left( \frac{F \bar{\psi}_2}{\sinh^2 r} \right) \psi_2 \\
= D_2 D_1 \psi_0 - D_1 D_2 \psi_0 + i\Re(\psi_0 \bar{\psi}_2)\psi_1 + i\Re(F \bar{\psi}_1)\psi_1 - \frac{i A_2^2}{\sinh^2 r} F \\
+ i\partial_r (\partial_r - \coth r) F + i\Re(F \bar{\psi}_1)\psi_1 - i\Re\left( \frac{F \bar{\psi}_2}{\sinh^2 r} \right) \psi_2, \\
= -\frac{i A_2^2}{\sinh^2 r} F + i\partial_r (\partial_r - \coth r) F + i\Re(F \bar{\psi}_1)\psi_1 - i\Re\left( \frac{F \bar{\psi}_2}{\sinh^2 r} \right) \psi_2.
\]
So we get the equation for \(F\):
\[
(i\partial_t + \partial_r^2 - \coth r \partial_r) F = \left( A_0 + \frac{A_2^2}{\sinh^2 r} + \partial_r (\coth r) \right) F - \Re(F \bar{\psi}_1)\psi_1 + \Re\left( \frac{F \bar{\psi}_2}{\sinh^2 r} \right) \psi_2,
\]
namely
\[
\left( i\partial_t + \Delta - \frac{1}{\sinh^2 r} \right) \frac{F}{\sinh r} \\
= A_0 \frac{F}{\sinh r} + \frac{A_2^2 - 1}{\sinh^2 r} \frac{F}{\sinh r} - \Re\left( \frac{F}{\sinh r} \bar{\psi}_1 \right) \psi_1 + \Re\left( \frac{F}{\sinh r} \bar{\psi}_2 \right) \psi_2.
\]
If \( R^\pm \psi^\pm \in H^1 \), we can write
\[
\frac{F}{\sinh r} = \frac{1}{\sinh r} (i A_2 \psi_1 - \partial_r \psi_2)
\]
\[
= \frac{i}{2} \left[ (A_2 + 1) \frac{\psi^+}{\sinh r} + \frac{A_2 - 1}{\sinh r} \psi^- + \partial_r (\psi^+ - \psi^-) + \cosh r - 1 \left( \frac{\psi^+}{\sinh r} + \frac{\psi^-}{\sinh r} \right) \right],
\]
Due to the boundedness of \( A_2 \) and \( \frac{\cosh r - 1}{\sinh r} \), we get \( \frac{F}{\sinh r} \in L^2 \).
If \( R^\pm \psi^\pm \in H^2 \), we using \( A_2^2 + |\psi_2|^2 = 1 \) and Sobolev embedding, yields \( \frac{F}{\sinh r} \in H^1 \) by the representation
\[
-2i \frac{F}{\sinh r} = (A_2 + 1) \frac{\psi^+}{\sinh r} + \frac{A_2 - 1}{\sinh r} \psi^- + \frac{\cosh r - 1}{\sinh r} (\partial_r \psi^+ - \partial_r \psi^-)
\]
\[
+ \frac{\cosh r - 1}{\sinh r} (\psi^+ - \psi^-) + \frac{\cosh r - 1}{\sinh r} \partial_r (\psi^- - \psi^+),
\]
Then we begin to prove \( \psi^\pm(t) \) satisfies the compatibility condition. Let \( P_\epsilon \) for \( \epsilon > 0 \) be the smoothing operator defined by the Fourier multiplier \( \lambda \to e^{-\epsilon^2 \lambda^2} \). Denote \( N \) is the nonlinearity of (133). Applying \( P_\epsilon \) to both sides of (133), we obtain
\[
(i \partial_t + \Delta_{\mathbb{H}^2}) P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) = P_\epsilon (e^{i\theta} N).
\]
Since \( P_\epsilon (e^{i\theta} \frac{F}{\sinh r}), \partial_r P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \) and \( \frac{1}{\sinh r} P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \in L^2 \), it follows that
\[
\partial_r P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \cdot \sinh r, \quad P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \to 0, \quad \text{as } r \to 0,
\]
and
\[
\partial_r P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \sinh^{1/2} r, \quad P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \sinh^{1/2} r \to 0, \quad \text{as } r \to \infty.
\]
Hence, by integration by parts and (18), we get
\[
\partial_t \left\| P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \right\|_{L^2}^2 = 2 \Re \int_0^\pi (i \partial_r P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \cdot \overline{P_\epsilon (e^{i\theta} \frac{F}{\sinh r})} \sinh r) \, d\theta
\]
\[
-2 \int \Re (i P_\epsilon (e^{i\theta} N) P_\epsilon (e^{i\theta} \frac{F}{\sinh r})) \, d\text{vol}_g
\]
\[
\leq 2 \left\| P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \right\|_{L^2} \left\| P_\epsilon (e^{i\theta} N) \right\|_{L^2}
\]
\[
\leq 2 \left\| P_\epsilon (e^{i\theta} \frac{F}{\sinh r}) \right\|_{L^2}^2 \left\| \psi^\pm \right\|_{H^2}^2.
\]
which further gives
\[
\left\| P_t(e^{i\theta} \frac{F}{\sinh r}) \right\|_{L^2}^2 (t) \leq \left\| P_t(e^{i\theta} \frac{F}{\sinh r}) \right\|_{L^2}^2 (0) + 2 \int_0^t \left\| \frac{F}{\sinh r} \right\|_{L^2}^2 \| \psi^\pm \|_{H^2}^2 ds
\]

Then let \( \epsilon \to 0 \), we obtain
\[
\left\| \frac{F}{\sinh r} \right\|_{L^2}^2 (t) \leq \left\| \frac{F}{\sinh r} \right\|_{L^2}^2 (0) + 2 \int_0^t \left\| \frac{F}{\sinh r} \right\|_{L^2}^2 \| \psi^\pm \|_{H^2}^2 ds.
\]

By Gronwall inequality and \( F(0) = 0 \), we get \( F(t) = 0 \) for all \( t \in I \).

In general, if \( \psi^\pm_0 \in \mathcal{L}^2 \) with \( \| \psi^\pm_0 \|_\mathcal{L}^2 \leq \epsilon_0 \), there exists \( R^+ \psi^\pm_{0,n} \in \mathcal{H}^2 \) such that
\[
\| \psi^\pm_0 - \psi^\pm_{0,n} \|_\mathcal{L}^2 \leq \frac{\epsilon_0}{n}. \]

By Lemma 4.1, there exist compatible pair \( R^\pm \psi^\pm_{0,n} \in \mathcal{H}^2 \) and \( \| \psi^\pm_0 - \psi^\pm_{0,n} \|_\mathcal{L}^2 \leq \frac{\epsilon_0}{n} \). Moreover, the solutions \( \psi^\pm_n \) with initial data \( \psi^\pm_{0,n} \) satisfy compatibility condition. Then the compatibility condition for \( \psi^\pm_n \) can be written as
\[
\psi^\pm_n - \psi^\pm = \int_r^\infty \frac{A_2 (\psi^\pm_n + \psi^-)}{\sinh s} + \frac{\cosh s}{\sinh s} (\psi^\pm_n - \psi^-) ds.
\]

Hence, by Theorem 5.5 (iv), Lemma 2.4 and the expression of \( A_2 \), we have
\[
\left\| \psi^+ - \psi^- - \int_r^\infty \frac{A_2 (\psi^+ + \psi^-)}{\sinh s} + \frac{\cosh s}{\sinh s} (\psi^+ - \psi^-) ds \right\|_{L^2} \\
\leq \left\| (\psi^+ - \psi^-) - (\psi^+_n - \psi^-_n) \right\|_{L^2} \\
+ \left\| \int_r^\infty \frac{A_2 (\psi^+ + \psi^-)}{\sinh s} - A_{2,n} (\psi^+_n + \psi^-_n) + \frac{\cosh s}{\sinh s} [(\psi^+ - \psi^-) - (\psi^+_n - \psi^-_n)] ds \right\|_{L^2} \\
\lesssim \| \psi^\pm - \psi^\pm_n \|_\mathcal{L}^2 + \| A_2 - A_{2,n} \|_{L^\infty} \| \psi^\pm \|_{L^2} + \| A_{2,n} \|_{L^\infty} \| \psi^\pm - \psi^\pm_n \|_{L^2} \\
\lesssim \frac{1}{n} + \left\| \int_0^T |\psi^+|^2 - |\psi^-|^2 - |\psi^+_n|^2 + |\psi^-_n|^2 ds \right\|_{L^\infty} + (1 + \| \psi^\pm_n \|_{L^2}^2) \frac{1}{n} \\
\lesssim \frac{1}{n},
\]

which completes the proof of Theorem 5.6.
Using Theorem 1.1 again, the solution $u(t)$ can be continued past time $T$, and in fact for the above interval $J$, we have
\[ \|u\|_{L^\infty(J; H^3)} \lesssim C(|J|, \|u_0\|_{H^3}). \]
Thus the bound (7) follows.

Next, we continue to prove the global existence of the equivariant Schrödinger flow (1) with small initial data $u_0 \in H^3$. For such initial data, there exist $u_{0,n} \in H^3$ such that $\|u_0 - u_{0,n}\|_{H^1} < \frac{1}{n}$. Then by the above proof, there exists a unique solution $u_n \in L^\infty(J; H^3)$ with $u_n(0) = u_{0,n}$ satisfying
\[ \|u_n\|_{L^\infty(J; H^3)} \leq C(|J|, \|u_{0,n}\|_{H^3}), \]
and
\[ \|u_n\|_{L^\infty(J; H^1)} \leq C\|u_{0,n}\|_{H^1}. \]
By (67), Theorem 5.5(ii), and Proposition 2, we obtain the Lipschitz continuity of $u(t)$ with respect to $u_0$ in $H^1$, i.e.
\[ \|u - u_n\|_{H^1} \lesssim \|u_0 - u_{0,n}\|_{H^1}. \]
Thus we get the solution $u \in L^\infty(J; H^3)$ defined as the unique limit of solutions $u_n$ satisfying
\[ \|u\|_{L^\infty(J; H^1)} \leq C\|u_0\|_{H^1}. \]
Hence, the bound (6) follows. This completes the proof of Theorem 1.2. \qed

Acknowledgments. J. Huang thanks Dr. Ze Li for helpful discussions.

REFERENCES

[1] J.-P. Anker and V. Pierfelice, Nonlinear Schrödinger equation on real hyperbolic spaces, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), 1853–1869.
[2] V. Banica, The Nonlinear Schrödinger equation on hyperbolic space, Comm. Partial Differential Equations, 32 (2007), 1643–1677.
[3] I. Bejenaru, Global results for Schrödinger maps in dimensions $n \geq 3$, Comm. Partial Differential Equations, 33 (2008), 451–477.
[4] I. Bejenaru, A. D. Ionescu and C. E. Kenig, Global existence and uniqueness of Schrödinger maps in dimensions $d \geq 4$, Adv. Math., 215 (2007), 263–291.
[5] I. Bejenaru, A. D. Ionescu, C. E. Kenig and D. Tataru, Global Schrödinger maps in dimensions $d \geq 2$: Small data in the critical Sobolev spaces, Ann. of Math., 173 (2011), 1443–1506.
[6] I. Bejenaru, A. D. Ionescu, C. E. Kenig and D. Tataru, Equivariant Schrödinger maps in two spatial dimensions, Duke Math. J., 162 (2013), 1967–2025.
[7] I. Bejenaru, A. D. Ionescu, C. E. Kenig and D. Tataru, Equivariant Schrödinger maps in two spatial dimensions: The $\mathbb{H}^2$ target, Kyoto J. Math., 56 (2016), 283–323.
[8] I. Bejenaru and D. Tataru, Near soliton evolution for equivariant Schrödinger maps in two spatial dimensions, Mem. Amer. Math. Soc., 228 (2014), no. 1069.
[9] D. Borthwick and J. L. Marzuola, Dispersion estimates for scalar and matrix Schrödinger operators on $\mathbb{H}^{n+1}$, Math. Phys. Anal. Geom., 18 (2015), 26 pp.
[10] P. D’Ancona and Q. Zhang, Global existence of small equivariant wave maps on rotationally symmetric manifolds, Int. Math. Res. Not. IMRN, (2016), no. 4, 978–1025.
[11] W. Ding and Y. Wang, Schrödinger flow of maps into symplectic manifolds, Sci. China Ser. A, 41 (1998), 746–755.
[12] W. Ding and Y. Wang, Local Schrödinger flow into Kähler manifolds, Sci. China Ser. A, 44 (2001), 1446–1464.
[13] S. Gustafson, K. Kang and T.-P. Tsai, Schrödinger flow near harmonic maps, Comm. Pure Appl. Math., 60 (2007), 463–499.
[14] S. Gustafson, K. Kang and T.-P. Tsai, Asymptotic stability of harmonic maps under the Schrödinger flow, Duke Math. J., 145 (2008), 537–583.
[15] S. Gustafson, K. Nakanishi and T.-P. Tsai, Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on $\mathbb{R}^2$, Comm. Math. Phys., 300 (2010), 205–242.

[16] A. D. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps. II. Global well-posedness in dimensions $d \geq 3$, Comm. Math. Phys., 271 (2007), 523–559.

[17] A. D. Ionescu and G. Staffilani, Semilinear Schrödinger flows on hyperbolic spaces: Scattering $H^1$, Math. Ann., 345 (2009), 133–158.

[18] H. Koch, D. Tataru and M. Visan, Dispersive Equations and Nonlinear Waves, Vol. 45, Oberwolfach Seminars, Birkhäuser/Springer, Basel, 2014.

[19] A. Lawrie, J. Lührmann, S.-J. Oh and S. Shahshahani, Asymptotic stability of harmonic maps on the hyperbolic plane under the Schrödinger maps evolution, preprint, arXiv:1909.06899.

[20] A. Lawrie, S.-J. Oh and S. Shahshahani, Stability of stationary equivariant wave maps from the hyperbolic plane, Amer. J. Math., 139 (2017), 1085–1147.

[21] A. Lawrie, S.-J. Oh and S. Shahshahani, Gap eigenvalues and asymptotic dynamics of geometric wave equations on hyperbolic space, J. Funct. Anal., 271 (2016), 3111–3161.

[22] A. Lawrie, S.-J. Oh and S. Shahshahani, Equivariant wave maps on the hyperbolic plane with large energy, Math. Res. Lett., 24 (2017), 449–479.

[23] A. Lawrie, S.-J. Oh and S. Shahshahani, The Cauchy problem for wave maps on hyperbolic space in dimensions $d \geq 4$, Int. Math. Res. Not. IMRN, (2018), no. 7, 1954–2051.

[24] Z. Li, Asymptotic stability of large energy harmonic maps under the wave map from 2D hyperbolic spaces to 2D hyperbolic spaces, preprint, arXiv:1707.01362.

[25] Z. Li, Endpoint Strichartz estimates for magnetic wave equations on two dimensional hyperbolic spaces, emph{Differential Integral Equations}, 32, (2019), 369–408.

[26] Z. Li, Global Schrödinger map flows to Kähler manifolds with small data in critical Sobolev spaces: Energy critical case, preprint, arXiv:1811.10924.

[27] Z. Li, Global Schrödinger map flows to Kähler manifolds with small data in critical Sobolev spaces: High dimensions, preprint, arXiv:1903.05551.

[28] Z. Li, X. Ma and L. Zhao, Asymptotic stability of harmonic maps between 2D hyperbolic spaces under the wave map equation. II. Small energy case, Dyn. Partial Differ. Equ., 15 (2018), 283–336.

[29] P. Li and L.-F. Tam, The heat equation and harmonic maps of complete manifolds, Invent. Math., 105 (1991), 1–46.

[30] Z. Li and L. Zhao, Convergence to harmonic maps for the Landau-Lifshitz flows between two dimensional hyperbolic spaces, Discrete Contin. Dyn. Syst., 39 (2019), 607–638.

[31] H. McGahagan, An approximation scheme for Schrödinger maps, Comm. Partial Differential Equations, 32 (2007), 375–400.

[32] F. Merle, P. Raphael and I. Rodnianski, Blowup dynamics for smooth data equivariant solutions to the critical Schrödinger map problem, Invent. Math., 193 (2013), 249–365.

[33] G. Perelman, Blow up dynamics for equivariant critical Schrödinger maps, Comm. Math. Phys., 330 (2014), 69–105.

[34] C. Song and Y. Wang, Uniqueness of Schrödinger flow on manifolds, Comm. Anal. Geom., 26 (2018), 217–235.

[35] P.-L. Sulem, C. Sulem and C. Bardos, On the continuous limit for a system of classical spins, Comm. Math. Phys., 107 (1986), 431–454.

[36] T. Tao, M. Visan and X. Zhang, The nonlinear Schrödinger equation with combined power-type nonlinearities, Comm. Partial Differential Equations, 32 (2007), 1281–1343.

Received August 2019; revised December 2019.

E-mail address: jiaxih@mail.ustc.edu.cn
E-mail address: wyd@math.ac.cn
E-mail address: zhaolf@ustc.edu.cn