Semi-classical spectral estimates for Schrödinger operators at a critical level. Case of a degenerate maximum of the potential.

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Abstract

We study the semi-classical trace formula at a critical energy level for a Schrödinger operator on $\mathbb{R}^n$. We assume here that the potential has a totally degenerate critical point associated to a local maximum. The main result, which establishes the contribution of the associated equilibrium in the trace formula, is valid for all time in a compact subset of $\mathbb{R}$ and includes the singularity in $t = 0$. For these new contributions the asymptotic expansion involves the logarithm of the parameter $\hbar$. Depending on an explicit arithmetic condition on the dimension and the order of the critical point, this logarithmic contribution can appear in the leading term.

Key words: Semi-classical analysis; Trace formula; Degenerate oscillatory integrals; Schrödinger operators.

1 Introduction.

Let us consider $P_\hbar$ a self-adjoint $\hbar$-pseudodifferential operator, or more generally $\hbar$-admissible (see [18]), acting on a dense subset of $L^2(\mathbb{R}^n)$. A classical and accessible problem is to study the asymptotic behavior, as $\hbar$ tends to 0,
of the spectral distributions:

\[
\gamma(E, h, \varphi) = \sum_{|\lambda_j(h) - E| \leq \varepsilon} \varphi\left(\frac{\lambda_j(h) - E}{h}\right),
\]

(1)

where the \(\lambda_j(h)\) are the eigenvalues of \(P_h\), \(E\) is an energy level of the principal symbol of \(P_h\) and \(\varphi\) a function. Here we suppose that the spectrum is discrete in \([E - \varepsilon, E + \varepsilon]\), a sufficient condition for this is given below. If \(p_0\) is the principal symbol of \(P_h\) we recall that an energy \(E\) is regular when \(\nabla p_0(x, \xi) \neq 0\) on the energy surface:

\[
\Sigma_E = \{(x, \xi) \in T^*\mathbb{R}^n \mid p_0(x, \xi) = E\},
\]

(2)

and critical when it is not regular.

A classical result is the existence of a link between the asymptotics of (1), as \(h\) tends to 0, and the closed trajectories of the Hamiltonian flow of \(p_0\) on the energy surface \(\Sigma_E\), i.e.:

\[
\lim_{h \to 0} \gamma(E, h, \varphi) \equiv \{(t, x, \xi) \in \mathbb{R} \times \Sigma_E / \Phi_t(x, \xi) = (x, \xi)\},
\]

where \(\Phi_t = \exp(tH_{p_0})\) and \(H_{p_0} = \partial_\xi p_0.\partial_x - \partial_x p_0.\partial_\xi\). This duality between spectrum and periodic orbits exists in a lot of various settings such as in the Selberg trace formula or for the trace of the wave operator on compact manifolds [10]. In the semi-classical setting this correspondence was initially pointed out in the physicist literature: Gutzwiller [12], Balian&Bloch [1].

For a rigorous mathematical approach, and when \(E\) is a regular energy, a non-exhaustive list of references is Helffer&Robert [13], Brummelhuis&Uribe [3], Paul&Uribe [16], and more recently Combescure et al. [8], Petkov&Popov [17].

Equilibriums are suspected to give special contributions in both sides of the trace formula. When \(E\) is no more a regular value, the asymptotic behavior of Eq. (1) depends on the nature of the singularities of \(p\) on \(\Sigma_E\) which is too complicated to be treated in general position. The case of a non-degenerate critical energy for \(p_0\), that is such that the critical-set \(C(p_0) = \{(x, \xi) \in T^*\mathbb{R}^n \mid dp_0(x, \xi) = 0\}\) is a compact \(C^\infty\) manifold with a Hessian \(d^2p_0\) transversely non-degenerate along this manifold, has been investigated first by Brummelhuis et al. in [2]. They treated this question for quite general operators but for some ”small times”, i.e. it was assumed that 0 was the only period of the linearized flow in \(\text{supp}(\dot{\varphi})\). Later, Khuat-Duy in [15] has obtained the contributions of the non-zero periods of the linearized flow for \(\text{supp}(\dot{\varphi})\) compact, but for Schrödinger operators with symbol \(\xi^2 + V(x)\) and a non-degenerate potential \(V\). Our contribution to this subject was to generalize his result for some more general operators, always with \(\dot{\varphi}\) of compact support.
and under some geometrical assumptions on the flow (see [4]). In [7] we have studied the case of a Schrödinger operator near a degenerate minimum of the potential and the objective of the present work is to investigate the situation near a degenerate maximum which leads to a totally different asymptotic problem.

After a reformulation, via the theory of Fourier integral operators, the spectral distribution of Eq. (1) can be expressed in terms of oscillatory integrals whose phases are related to the classical flow of \( p_0 \). Moreover, the asymptotic behavior as \( h \) tends to 0 of these oscillatory integrals is related to closed orbits of this flow. When \( (x_0, \xi_0) \) is a critical point of \( p_0 \), and hence an equilibrium of the flow, it is well known that the relation:

\[
\mathfrak{F}_t = \text{Ker}(d_{x,\xi} \Phi_t(x_0, \xi_0) - \text{Id}) \neq \{0\}, \tag{3}
\]

leads to the study of degenerate oscillatory integrals. In the present work we consider the case of a Schrödinger operator:

\[
P_h = -h^2 \Delta + V(x), \tag{4}
\]

but a generalization to an \( h \)-admissible operator (in the sense of [18]) of principal symbol \( \xi^2 + V(x) \) is outlined in the last section. In particular, we will consider the case of a potential \( V \) with a single and degenerate critical point \( x_0 \) attached to a local maximum. A typical example is the top of a polynomial double well. An immediate consequence is that the symbol admits a unique critical point \( (x_0, 0) \) on the energy surface \( \{\xi^2 + V(x) = V(x_0)\} \) and that the linearized flow at this point is given by the flow of the free Laplacian. A fortiori, Eq. (3) is automatically satisfied with:

\[
\mathfrak{F}_t = \{(\delta u, \delta v) \in T_{x_0,\xi_0} T^* \mathbb{R}^n / \delta v = 0 \} \simeq \mathbb{R}^n; \quad t \neq 0, \\
\mathfrak{F}_0 = T_{x_0,\xi_0} (T^* \mathbb{R}^n) \simeq \mathbb{R}^{2n}.
\]

In particular, the stationary phase method cannot be applied at all in a microlocal neighborhood of \( t = 0 \).

The core of the proof lies in establishing suitable local normal forms for the local phase functions of a Fourier integral operator approximating the propagator in the semi-classical regime and in a generalization of the stationary phase formula for these normal forms. This generalization, based on an analytic representation of the associated class of oscillatory integrals, is more complicated than in the case of a local minimum but however allows to compute, invariantly, the leading term of the related local trace formula.
2 Hypotheses and main result.

Let \( p(x, \xi) = \xi^2 + V(x) \) where the potential \( V \) is smooth on \( \mathbb{R}^n \) and real valued. To this symbol is attached the \( h \)-differential operator \( P_h = -h^2 \Delta + V(x) \) and by a classical result \( P_h \) is essentially self-adjoint, for \( h \) small enough, if \( V \) is bounded from below. Moreover, if \( E \) is an energy level of \( p \) satisfying :

\((H_1)\) There exists \( \varepsilon_0 > 0 \) such that \( p^{-1}([E - \varepsilon_0, E + \varepsilon_0]) \) is compact,

then, by Theorem 3.13 of [18] the spectrum \( \sigma(P_h) \cap [E - \varepsilon, E + \varepsilon] \) is discrete and consists in a sequence \( \lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_j(h) \) of eigenvalues of finite multiplicities, if \( \varepsilon \) and \( h \) are small enough. For example, \((H_1)\) is certainly satisfied if \( V \) goes to infinity at infinity and, more generally, this is true when \( E < \lim \inf V < \infty \).

We want to study the asymptotic behavior of the spectral distribution :

\[
\gamma(E_c, h, \varphi) = \sum_{\lambda_j(h) \in [E_c - \varepsilon, E_c + \varepsilon]} \varphi\left(\frac{\lambda_j(h) - E_c}{h}\right). \tag{5}
\]

We use the subscript \( E_c \) to recall that this energy level is critical. To avoid any problem of convergence we impose the condition :

\((H_2)\) We have \( \hat{\varphi} \in C_0^\infty(\mathbb{R}) \) with a sufficiently small support near the origin.

**Remark 1** This extra condition on the size of the support is simply here to avoid contributions of non-trivial closed orbits and can be easily relaxed. An explicit characterization of \( \text{supp}(\hat{\varphi}) \) is given in Lemma 8.

To simplify notations we write \( z = (x, \xi) \) for any point of the phase space and let be \( \Sigma_{EC} = p^{-1}(\{E_c\}) \). In the next condition degenerate means that the second derivative at the critical point \( x_0 \) is zero. We impose now the type of singularity of the potential :

\((H_3)\) On \( \Sigma_{EC} \) the symbol \( p \) has a unique critical point \( z_0 = (x_0, 0) \). This critical point is degenerate and associated to a local maximum of the potential \( V \) of the form :

\[
V(x) = E_c + V_{2k}(x) + O(||x - x_0||^{2k+1}), \tag{6}
\]

where \( V_{2k} \) is homogeneous of degree \( 2k \) w.r.t. \( (x - x_0) \). Also, \( k \geq 2 \) and \( V_{2k} \) is definite negative.

**Remark 2** Since all previous derivatives are 0 in \( x_0 \), the function \( V_{2k} \) does not depend on the choice of local coordinates near \( x_0 \). Contrary to the case exposed in [7], \( z_0 \) is not isolated on \( \Sigma_{E_c} \) and might eventually interfere with non-trivial closed trajectories. A generalization to more general maximums,
e.g. to a sum of such homogeneous terms with different degrees, is possible but to simplify we only consider the homogeneous case (H3).

In this work, we are particularly interested in the contribution of the fixed point $z_0$. To understand the new phenomenon it suffices to study the localized problem:

$$\gamma_{z_0}(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int e^{i(tE_c + \hat{\varphi}(t))} \psi^w(x, hD_x) \exp(-\frac{it}{h}P_h) \Theta(P_h) dt. \quad (7)$$

Here $\Theta$ is a function of localization near the critical energy surface $\Sigma_{E_c}$, $\psi \in C_0^\infty(T^*\mathbb{R}^n)$ is micro-locally supported near $z_0$ and $\psi^w(x, hD_x)$ the associated operator obtained by $h$-Weyl quantization. Rigorous justifications are given in section 3 for the introduction of $\Theta(P_h)$ and in section 4 for $\psi^w(x, hD_x)$. In [7] it was proven that:

**Theorem 3** Under (H1), if $x_0$ is a local minimum of the potential, homogeneous as in (H3), then for all $\varphi$ with $\hat{\varphi} \in C_0^\infty(\mathbb{R})$ we have:

$$\gamma_{z_0}(E_c, h, \varphi) \sim h^{-n+\frac{\delta}{2} + \frac{\mu}{2}} \sum_{j,l \in \mathbb{N}^2} h^{\frac{\delta}{2} + \frac{\mu}{2}} \Lambda_{j,l}(\varphi),$$

where the $\Lambda_{j,l}$ are some computable distributions. The leading coefficient is:

$$\Lambda_{0,0}(\varphi) = \frac{S(S^{n-1})}{(2\pi)^n} \int_{S^{n-1}} |V_{2k}(\eta)|^{-\frac{\mu}{2}} d\eta \int_{\mathbb{R}^+ \times \mathbb{R}^+} \varphi(u^2 + v^{2k})u^{n-1}v^{n-1}dudv, \quad (8)$$

where $S(S^{n-1})$ is the surface of the unit-sphere of $\mathbb{R}^n$.

Clearly, this result shows that the spectral estimates are related to a purely local problem, namely to estimate the asymptotic behavior of fiber integrals:

$$\int_{T^*\mathbb{R}^n} \varphi\left(\frac{\xi^2 + V_{2k}(x)}{h}\right) dxd\xi, \quad h \to 0^+. \quad (9)$$

This result can be interpreted as a scaling of the trace of $-\Delta + V_{2k}$, cf. the first term of the trace formula. Evidently, such an interpretation cannot hold in our setting since the trace does not exists if $V_{2k}$ is a negative definite function.

The main result of the present work is:

**Theorem 4** Under hypotheses (H1) to (H3) we have:

$$\gamma_{z_0}(E_c, h, \varphi) \sim \sum_{m=0,1} h^{-n+\frac{\delta}{2} + \frac{\mu}{2}} \sum_{j,l \in \mathbb{N}^2} h^{\frac{\delta}{2} + \frac{\mu}{2}} \log(h)^m \Lambda_{j,l,m}(\varphi),$$

where the $\Lambda_{j,l,m}$ are in $S'(\mathbb{R})$.

As concerns the leading term of the expansion, when $\frac{n(k+1)}{2k} \notin \mathbb{N}$, the first
non-zero coefficient of this local trace formula is given by:

\[ h^{-n+\frac{n}{2}+\frac{n}{2k}} \langle T_{n,k}, \varphi \rangle \frac{S(S^{n-1})}{(2\pi)^n} \int_{S^{n-1}} |V_{2k}(\eta)|^{-\frac{n}{2k}} d\eta. \]  

(10)

The distributions \( T_{n,k} \) are given by:

\[ \langle T_{n,k}, \varphi \rangle = \int_{\mathbb{R}} (C_{n,k}^+ |t|^n_{\frac{n}{2k}+1} + C_{n,k}^- |t|_{\frac{n}{2k}-1}) \varphi(t) dt, \]  

if \( n \) is odd. \hfill (11)

\[ \langle T_{n,k}, \varphi \rangle = C_{n,k}^- \int_{\mathbb{R}} |t|_{\frac{n}{2k}-1} \varphi(t) dt, \]  

if \( n \) is even. \hfill (12)

But if \( \frac{n(k+1)}{2k} \in \mathbb{N} \) and \( n \) is odd then the top-order term is:

\[ C_{n,k} \log(h) h^{-n+\frac{n}{2}+\frac{n}{2k}} \frac{S(S^{n-1})}{(2\pi)^n} \int_{S^{n-1}} |V_{2k}(\eta)|^{-\frac{n}{2k}} d\eta \int_{\mathbb{R}} |t|^{n}_{\frac{n}{2k}-1} \varphi(t) dt. \]  

(13)

Finally, if \( \frac{n(k+1)}{2k} \in \mathbb{N} \) and \( n \) is even, \( C_{n,k}^+ = C_{n,k}^- \) and we have:

\[ C_{n,k}^\pm h^{-n+\frac{n}{2}+\frac{n}{2k}} \frac{1}{(2\pi)^n} \int_{S^{n-1}} |V_{2k}(\eta)|^{-\frac{n}{2k}} d\eta \int_{\mathbb{R}} |t|^{n}_{\frac{n}{2k}-1} \varphi(t) dt. \]  

(14)

In all expressions above \( C_{n,k} \) are non-zero universal constants depending only on \( n \) and \( k \).

Explicit formulas for the numbers \( C_{n,k}^\pm \) are given in section 6. The arithmetical condition on \( k \) and \( n \) might be surprising at the first look. But this condition becomes clear when the oscillatory integrals of our spectral problem are analytically reformulated in section 6. Also, viewing the top order coefficient of the trace as a tempered distribution on the Schwartz function \( \varphi \), we obtain that the singular support is only located at the origin.

3 Oscillatory representation of the spectral functions.

The construction below is more or less classical and will be sketchy. For a more detailed exposition the reader can consult [2], [4] or [15].

Let be \( \varphi \in \mathcal{S}(\mathbb{R}) \) with \( \hat{\varphi} \in C^\infty_0(\mathbb{R}) \), we recall that:

\[ \gamma(E, h, \varphi) = \sum_{\lambda_j(h) \in I_c} \varphi\left(\frac{\lambda_j(h) - E}{h}\right), \]
where \( I_\varepsilon = [E_c - \varepsilon, E_c + \varepsilon] \), with \( 0 < \varepsilon < \varepsilon_0 \), and \( p^{-1}(I_\varepsilon) \) compact in \( T^*\mathbb{R}^n \). We localize near the critical energy level \( E_c \) by inserting a cut-off function \( \Theta \in C^\infty_0([E_c - \varepsilon, E_c + \varepsilon]) \), such that \( \Theta = 1 \) in a neighborhood of \( E_c \) and \( 0 \leq \Theta \leq 1 \) on \( \mathbb{R} \). The corresponding decomposition is:

\[
\gamma(E_c, h, \varphi) = \gamma_1(E_c, h, \varphi) + \gamma_2(E_c, h, \varphi),
\]

with:

\[
\gamma_1(E_c, h, \varphi) = \sum_{\lambda_j(h) \in I_\varepsilon} (1 - \Theta)(\lambda_j(h))\varphi\left(\frac{\lambda_j(h) - E_c}{h}\right),
\]

\[
\gamma_2(E_c, h, \varphi) = \sum_{\lambda_j(h) \in I_\varepsilon} \Theta(\lambda_j(h))\varphi\left(\frac{\lambda_j(h) - E_c}{h}\right).
\]

Since \( \varphi \in \mathcal{S}(\mathbb{R}) \) a classical estimate, see e.g. Lemma 1 of [5], is:

\[
\gamma_1(E_c, h, \varphi) = O(h^\infty), \quad as \quad h \to 0.
\]

By inversion of the Fourier transform we have:

\[
\Theta(P_h)\varphi\left(\frac{P_h - E_c}{h}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\frac{P_h}{h}} \varphi(t)\exp\left(-it\frac{P_h}{h}\right)\Theta(P_h)dt.
\]

The trace of the left hand-side is precisely \( \gamma_2(E_c, h, \varphi) \) and Eq. (17) gives:

\[
\gamma(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{it\frac{P_h}{h}} \varphi(t)\exp\left(-it\frac{P_h}{h}\right)\Theta(P_h)dt + O(h^\infty).
\]

If \( U_h(t) = \exp(-\frac{it}{h}P_h) \) is the evolution operator, we can approximate \( U_h(t)\Theta(P_h) \) by a Fourier integral-operator depending on a parameter \( h \). If \( \Lambda \) is the Lagrangian manifold associated to the flow of \( p \):

\[
\Lambda = \{(t, \tau, x, \xi, y, \eta) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \tau = p(x, \xi), \ (x, \xi) = \Phi_t(y, \eta)\},
\]

a classical and general result, see e.g. Duistermaat [9], is:

**Theorem 5** The operator \( U_h(t)\Theta(P_h) \) is an \( h \)-FIO associated to \( \Lambda \). For each \( N \in \mathbb{N} \) there exists \( U_h^{(N)}(t) \) with integral kernel in Hörmander’s class \( I(\mathbb{R}^{2n+1}, \Lambda) \) and \( R_h^{(N)}(t) \) bounded, with a \( L^2 \)-norm uniformly bounded for \( 0 < h \leq 1 \) and \( t \) in a compact subset of \( \mathbb{R} \), such that:

\[
U_h(t)\Theta(P_h) = U_h^{(N)}(t) + h^N R_h^{(N)}(t).
\]

The remainder associated to \( R_h^{(N)}(t) \) is controlled by the classical trick:
Corollary 6 Let $\Theta_1 \in C^\infty_0(\mathbb{R})$, with $\Theta_1 = 1$ on $\text{supp}(\Theta)$ and $\text{supp}(\Theta_1) \subset I_\varepsilon$, then $\forall N \in \mathbb{N}$:

$$\text{Tr}(\Theta(P_h)\varphi(\frac{P_h - E_c}{\hbar})) = \frac{1}{2\pi} \text{Tr} \int_\mathbb{R} \hat{\varphi}(t)e^{\frac{i}{\hbar(E_c - \langle x, \xi \rangle)}U^{(N)}_{\Theta,h}(t)\Theta_1(P_h)}dt + O(h^{N-n}).$$

The proof is easy by cyclicity of the trace (see [5] or [18]).

For the particular case of a Schrödinger operator the BKW ansatz shows that the integral kernel of $U^{(N)}_{\Theta,h}(t)$ can be recursively constructed as:

$$K^{(N)}_h(t, x, y) = \frac{1}{(2\pi\hbar)^n} \int_\mathbb{R^n} b^{(N)}_h(t, x, y, \xi)e^{\frac{i}{\hbar}(S(t,x,\xi)-\langle y, \xi \rangle)}d\xi,$$

where $S$ satisfies the Hamilton-Jacobi equation:

$$\left\{ \begin{array}{l}
\partial_t S(t, x, \xi) + p(x, \partial_x S(t, x, \xi)) = 0, \\
S(0, x, \xi) = \langle x, \xi \rangle.
\end{array} \right.$$}

In particular we obtain that:

$$\{(t, \partial_t S(t, x, \eta), x, \partial_x S(t, x, \eta), \partial_\eta S(t, x, \eta), -\eta) \} \subset \Lambda_{\text{flow}},$$

and that the function $S$ is a generating function of the flow, i.e.:

$$\Phi_t(\partial_\eta S(t, x, \eta), \eta) = (x, \partial_x S(t, x, \eta)).$$

(19)

Inserting this approximation in Eq. (18) we find that, modulo an error $O(h^{N-n})$, the trace $\gamma(E_c, \hbar, \varphi)$ can be written for all $N \in \mathbb{N}$ as:

$$\gamma(E_c, \hbar, \varphi) = \sum_{j < N} \frac{\hbar^j}{(2\pi\hbar)^n} \int_{\mathbb{R}^n \times T^*\mathbb{R}^n} e^{\frac{i}{\hbar}(S(t,x,\xi)-\langle x, \xi \rangle)+tE_c}a_j(t, x, \xi)\hat{\varphi}(t)dtdxd\xi,$$

(20)

where $a_j(t, x, \eta) = b_j(t, x, x, \eta)$ is the evaluation of $b_j$ on the diagonal $\{x = y\}$.

Remark 7 By a theorem of Helffer&Robert, see [18] Theorem 3.11 and Remark 3.14, $\Theta(P_h)$ is $h$-admissible with a symbol supported in $p^{-1}(I_\varepsilon)$. This allows to consider only oscillatory-integrals with compact support. □
Critical points of the phase function of (20) are given by the equations:

\[
\begin{aligned}
E_c &= -\partial_t S(t, x, \xi), \\
x &= \partial_\xi S(t, x, \xi), \\
\xi &= \partial_x S(t, x, \xi),
\end{aligned}
\]

where the right hand side defines a closed trajectory of the flow inside \(\Sigma_{E_c}\).

Since we are mainly interested in the contribution of the critical point, we choose a function \(\psi \in C^\infty_0(T^*\mathbb{R}^n)\), with \(\psi = 1\) near \(z_0\), hence:

\[
\gamma_2(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_\mathbb{R} e^{iE_c/h} \hat{\varphi}(t) \psi^w(x, hD_x) \exp(-\frac{i}{h} t P_h) \Theta(P_h) dt
\]

Under the additional hypothesis of having a clean flow, the asymptotics of the second term is given by the semi-classical trace formula on a regular level. We also observe that the contribution of the first term, which is precisely the distribution \(\gamma_{z_0}(E_c, h, \varphi)\) of Theorem 4, is micro-local. Hence this allows to introduce local coordinates near \(z_0\). As pointed out in Remark 2, \(\{z_0\}\) is not a connected component of the energy surface \(\Sigma_{E_c}\) and elements:

\[
\{(T, z), T \neq 0, \ z \in \Sigma_{E_c} \cap \text{supp}(\psi), \ \Phi_T(z) = z\},
\]

(21)

could contribute in the asymptotic expansion. In fact, the only contribution arises from the set \(\{(t, z_0), \ t \in \text{supp}(\hat{\varphi})\}\) if the support of \(\hat{\varphi}\) is small enough.

**Lemma 8** There exists a \(T > 0\), depending only on \(V\), such that \(\Phi_t(z) \neq z\) for all \(z \in \Sigma_{E_c} \setminus \{z_0\}\) and all \(t \in [-T, 0] \cup [0, T]\).

**Proof.** If \(H_p\) is our hamiltonian vector field and \(z = (x, \xi)\) we have:

\[
||H_p(z_1) - H_p(z_2)||^2 = 4||\xi_1 - \xi_2||^2 + ||\nabla_x V(x_1) - \nabla_x V(x_2)||^2.
\]

Since our potential is smooth, when \(z_1\) and \(z_2\) are in in the energy surface \(\Sigma_{E_c}\), which is compact by assumption, there exists \(M > 0\) such that:

\[
||H_p(z_1) - H_p(z_2)|| \leq M||z_1 - z_2||.
\]

The main result of [19] shows that any periodic trajectory inside \(\Sigma_{E_c}\) has a period \(p\) such that \(p \geq \frac{2\pi}{M}\). 

\[\blacksquare\]
Remark 9 The result of Lemma 8 can perhaps be improved for any $T > 0$ by replacing $\Sigma_{E_i}$ by $U_T \cap \Sigma_{E_i}$, where $U_T$ is a restricted neighborhood of $z_0$. This is suggested by the fact that such a result is proven in [15] for non-degenerate potentials. Anyhow, via Lemma 8 we can understand the contribution in $t = 0$ of elements $\{t, z_0\}$ in the trace formula if $\dot{\varphi} \in C_0^\infty([-p, p])$. \hfill \Box

Now, we restrict our attention to the singular contribution generated by the critical point. Since $z_0$ is an equilibrium of the flow we obtain that:

$$d_{x,\xi} \Phi_t(z_0) = \exp(tH - \Delta), \; \forall t. \quad (22)$$

The computation of this linear operator is easy and gives:

$$d_{x,\xi} \Phi_t(z_0)(u, v) = (u + 2tv, v), \; \forall (u, v) \in T_{z_0}T^*\mathbb{R}^n. \quad (23)$$

From classical mechanics we know that the singularities, in the sense of the Morse theory, of the function $S(t, x, \xi) - \langle x, \xi \rangle$ are supported in the set:

$$\mathfrak{F}_t = \text{Ker}(d_{x,\xi} \Phi_t(z_0) - \text{Id}),$$

see e.g. Lemma 9 of [4]. As mentioned in the introduction we obtain:

$$\mathfrak{F}_0 = T_{z_0}(T^*\mathbb{R}^n) \cong \mathbb{R}^{2n},$$

$$\mathfrak{F}_t = \{(u, v) \in T_{z_0}(T^*\mathbb{R}^n) / v = 0\} \cong \mathbb{R}^n, \; t \neq 0.$$

To simplify notations, and until further notice, all derivatives will be taken with respect to the initial conditions $(x, \xi)$. The next non-zero terms of the Taylor expansion of the flow are computed via the technical result:

Lemma 10 Let be $z_0$ an equilibrium of the $C^\infty$ vector field $X$ and $\Phi_t$ the flow of $X$. Then for all $m \in \mathbb{N}^*$, there exists a polynomial map $P_m$, vector valued and of degree at most $m$, such that:

$$d^m \Phi_t(z_0)(z^m) = d \Phi_t(z_0) \int_0^t d \Phi_{-s}(z_0)P_m(d \Phi_s(z_0))(z), ..., d^{m-1} \Phi_s(z_0)(z^{m-1}))ds.$$ 

For a proof we refer to [4] or [5]. Here we shall use that our vector field is:

$$H_p = 2\xi \frac{\partial}{\partial x} - \partial_x V(x) \frac{\partial}{\partial \xi}.$$ 

We identify the linearized flow in $z_0$ with a matrix multiplication operator:

$$d \Phi_t(z_0) = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}.$$
Clearly, with the hypothesis \((H_3)\) we obtain the polynomials:

\[ P_j = 0, \quad \forall j \in \{2, \ldots, 2k - 2\}, \]

\[ P_{2k-1}(Y_1, \ldots, Y_{2k-2}) = \begin{pmatrix} 0 \\ d^{2k-1} \nabla_x V(x_0)(Y_1^{2k-1}) \end{pmatrix} \neq 0. \]

Where the notation \(Y^l_i\) stands for \((Y_1, \ldots, Y_1)\) \(l\)-times. Inserting the definition of \(d\Phi_z(0)\) and integration from 0 to \(t\) yields:

\[ d^{2k-1} \Phi_t(z_0)((x, \xi)^{2k-1}) = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} \int_0^t \begin{pmatrix} 2sd^{2k-1} \nabla_x V(x_0)((x + 2s\xi)^{2k-1}) \\ -d^{2k-1} \nabla_x V(x_0)((x + 2s\xi)^{2k-1}) \end{pmatrix} ds. \]

Terms of higher degree can be obtained similarly by successive integrations. If we assume that \(z_0 = 0\) the jet of order \(2k - 1\) of the flow is:

\[ \Phi_t(z) = d\Phi_t(0)(z) + \frac{1}{(2k-1)!} d^{2k-1} \Phi_t(0)(z^{2k-1}) + O(||z||^{2k}), \quad (24) \]

and can be computed explicitly with a given \(V_{2k}\).

5 Normal forms of the phase function.

Since the contribution we study is local, cf. the introduction of \(\psi\) in Eq. (7), we work with some local coordinates \((x, \xi)\) near the critical point \(z_0\). With these coordinates we identify locally \(T^*\mathbb{R}^n \cap V(z_0)\) with an open of \(\mathbb{R}^{2n}\). We define:

\[ \Psi(t, z) = \Psi(t, x, \xi) = S(t, x, \xi) - \langle x, \xi \rangle + tE_c, \quad z = (x, \xi) \in \mathbb{R}^{2n}. \quad (25) \]

We start by a more precise description of our phase function.

**Lemma 11** Near \(z_0\), here supposed to be 0 to simplify, we have:

\[ \Psi(t, x, \xi) = -t||\xi||^2 + S_{2k}(t, x, \xi) + R_{2k+1}(t, x, \xi), \quad (26) \]

where \(S_{2k}\) is homogeneous of degree \(2k\) w.r.t. \((x, \xi)\) and is uniquely determined by \(V_{2k}\). Moreover, \(R_{2k+1}(t, x, \xi) = O(||(x, \xi)||^{2k+1})\), uniformly for \(t\) in a compact subset of \(\mathbb{R}\).

**Proof.** With the particular structure of the flow in \(z_0\), cf. Eq. (24), we search our local generating function as:

\[ S(t, x, \xi) = -tE_c + S_2(t, x, \xi) + S_{2k}(t, x, \xi) + O(||(x, \xi)||^{2k+1}), \]
where the \( S_j \) are time dependant and homogeneous of degree \( j \) w.r.t. \((x, \xi)\). Starting from the implicit relation \( \Phi_t(\partial_\xi S(t, x, \xi), \xi) = (x, \partial_x S(t, x, \xi)) \) and with Eq. (24), we obtain that:

\[
S_2(t, x, \xi) = \langle x, \xi \rangle - t||\xi||^2.
\]

Now we compute the term \( S_2k \). To do so, we retain only terms homogeneous of degree \( 2k - 1 \) and we get:

\[
d\Phi_t(0)((\partial_\xi S, 0)) + \frac{1}{(2k-1)!}d^{2k-1}\Phi_t(0)((\partial_\xi S_2, \xi)^{2k-1}) = (0, \partial_x S_{2k}).
\]

If \( J \) is the matrix of the usual symplectic form \( \sigma \) on \( \mathbb{R}^{2n} \), we have:

\[
J\nabla S_{2k}(t, x, \xi) = \frac{1}{(2k-1)!}d^{2k-1}\Phi_t(0)((x - 2t\xi, \xi)^{2k-1}).
\]

By homogeneity and with Eq. (24) we obtain:

\[
S_{2k}(t, x, \xi) = \frac{1}{(2k)!}\sigma((x, \xi), d^{2k-1}\Phi_t(0)((x - 2t\xi, \xi)^{2k-1})).
\]

This gives the result since \( d^{2k-1}\Phi_t(0) \) is well determined by \( V_{2k} \).

Fortunately, we will not have to compute the remainder explicitly because of some homogenous considerations. To prepare the construction of our normal forms, we study carefully \( S_{2k} \).

**Corollary 12** The function \( S_{2k} \) satisfies:

\[
S_{2k}(t, x, \xi) = -tV_{2k}(x) + t^2 \langle \xi, \nabla_x V_{2k}(x) \rangle + \sum_{j,l=1}^n \xi_j \xi_l g_{jl}(t, x, \xi),
\]

where the functions \( g_{jl} \) are smooth and vanish in \( x = 0 \).

**Proof.** If \( f \) is homogeneous of degree \( 2k > 2 \) w.r.t. \((x, \xi)\) we can write:

\[
f(t, x, \xi) = f_1(t, x) + \langle \xi, f_2(t, x) \rangle + \sum_{j,l=1}^n \xi_j \xi_l f_{jl}(t, x).
\]

It remains to compute the function \( f_1 \) and the vector field \( f_2 \). We have:

\[
S_{2k}(t, x, \xi) = \frac{1}{(2k)!}\sigma((x, \xi), d\Phi_t(0)) \int_0^t \left( 2s d^{2k-1}\nabla_x V(0)((x + 2(s - t)\xi)^{2k-1}) - d^{2k-1}\nabla_x V(0)((x + 2(s - t)\xi)^{2k-1}) \right) ds.
\]

Clearly, the term of degree homogeneous of degree \( 2k \) w.r.t. \( x \) is given by:

\[
- \frac{1}{(2k)!} \int_0^t \langle x, d^{2k-1}\nabla_x V(0)(x^{2k-1}) \rangle ds = -\frac{t}{2k} \langle x, \nabla_x V_{2k}(x) \rangle = -tV_{2k}(x).
\]
Where the last result holds by homogeneity. As concerns the linear term w.r.t. \( \xi \), by combinatoric and linear operations, this one can be written:

\[
\frac{t^2}{(2k)!} \left( (2k - 1) \langle x, d^{2k-1} \nabla_x V(0)((x^{2k-2}, \xi)) \rangle + \langle \xi, d^{2k-1} \nabla_x V(0)(x^{2k-1}) \rangle \right)
\]

\[
= \frac{t^2}{2k} \left( (2k - 1) \langle \nabla_x V_{2k}(x), \xi \rangle + \langle \xi, \nabla_x V_{2k}(x) \rangle \right) = t^2 \langle \xi, \nabla_x V_{2k}(x) \rangle.
\]

This complete the proof. 

These two terms can be also derived by a heuristic method. Starting from the Hamilton-Jacobi equation we obtain

\[
S(0, x, \xi) = \langle x, \xi \rangle, \quad \partial_t S(0, x, \xi) = -p(x, \xi)
\]

and also:

\[
\partial^2_{t,t} S(t, x, \xi) = -\langle \partial_\xi p(x, \partial_x S(t, x, \xi)), \partial^2_{t,x} S(t, x, \xi) \rangle.
\]

But, for our flow, in \( t = 0 \) we have simply:

\[
\partial^2_{t,t} S(0, x, \xi) = 2 \langle \xi, \partial_x V(x) \rangle.
\]

Hence the Taylor expansion in \( t = 0 \) provides a good result with few calculations. Unfortunately, this approach gives no information about the degree, w.r.t. \((x, \xi)\), of the remainders \( O(t^d) \) for each \( d \geq 3 \).

We have now enough material to establish the normal of our phase function.

**Lemma 13** In a micro-local neighborhood of \( z = z_0 \), there exists local coordinates \( \chi \) such that:

\[
\Psi(t, z) \simeq -\chi_0 (\chi_1^2 - \chi_2^{2k}).
\]

**Proof.** We can here assume that \( z_0 \) is the origin. We proceed in two steps. First we want to eliminate terms of high degree. Starting form Eq. (27) we define:

\[
E(t, x, \xi) = \sum_{j,l} \xi_j \xi_l g_{j,l}(t, x, \xi).
\]

Since \( S(t, x, \xi) - \langle x, \xi \rangle = O(t) \), we have \( \Psi(t, z) = O(t) \). Hence, all terms of the expansion are \( O(t) \) and this allows to write \( E = t \tilde{E} \). Similarly, we write \( R_{2k+1} = t \tilde{R}_{2k+1} \) in Eq. (26). To obtain a blow-up of the singularity, we use polar coordinates \( x = r \theta, \xi = q \eta, \theta, \eta \in \mathbb{S}^{n-1}(\mathbb{R}), q, r \in \mathbb{R}_+ \). This induces naturally a jacobian \( r^{n-1}q^{n-1} \). By construction, there exists a function \( F \) vanishing in \((q, r) = (0, 0)\) such that:

\[
\tilde{E}(t, r \theta, q \eta) = q^2 F(t, r, \theta, q, \eta).
\]

With Lemma 11 and Corollary 12, near \( z_0 \) the phase \( \Psi(t, z) \) can be written:

\[
-t \left( q^2 + r^{2k} V_{2k}(\theta) - t q r^{2k-1} \langle \eta, \nabla V_{2k}(\theta) \rangle + q^2 F(t, r, \theta, q, \eta) + \tilde{R}_{2k+1}(t, r \theta, q \eta) \right).
\]
Thanks to the Taylor formula, the remainder $\tilde{R}_{2k+1}$ can be written as:

$$\tilde{R}_{2k+1}(t, r, \theta, q, \eta) = q^2 R_1(t, r, \theta, q, \eta) + r^{2k} R_2(t, r, \theta, q, \eta),$$

where $R_1$ vanishes in $r = 0$ and $R_2$ vanishes in $q = 0$. We obtain:

$$\Psi(t, z) \simeq -t(q^2 \alpha_1(t, r, \theta, q, \eta) - r^{2k} \alpha_2(t, r, \theta, q, \eta)) + t^2 q r^{2k-1} \langle \eta, \nabla V_{2k}(\theta) \rangle,$$

where we have defined:

$$\alpha_1(t, r, \theta, q, \eta) = (1 + R_1 + F)(t, r, \theta, q, \eta),$$

$$\alpha_2(t, r, \theta, q, \eta) = |V_{2k}(\theta)| + R_2(t, r, \theta, q, \eta).$$

Since $|V_{2k}(\theta)| > 0$ on $S^{n-1}$, we can eliminate $\alpha_1$ and $\alpha_2$ by a local change of coordinates:

$$(q \alpha_1^\frac{1}{2}, r \alpha_2^\frac{1}{2k^2}) \rightarrow (Q, R),$$

near $(q, r) = (0, 0)$. This is acceptable since the corresponding Jacobian is:

$$|J(Q, R)|(0, 0) = |V_{2k}(\theta)|^\frac{1}{2k} \neq 0.$$

In these local coordinates, and still using $(r, q)$ instead of $(R, Q)$, we obtain:

$$\Psi(t, z) \simeq -t(q^2 - r^{2k}) + t^2 q r^{2k-1} \varepsilon(t, r, \theta, q, \eta),$$

where:

$$\varepsilon(t, r, \theta, q, \eta) = \langle \eta, \nabla V_{2k}(\theta) \rangle (\alpha_1^{\frac{1}{2}} \alpha_2^{\frac{1}{2k}})(t, r, \theta, q, \eta).$$

In a second time, we eliminate the nonlinear term in $t$. To do so, we write:

$$-t(q^2 - t q r^{2k-1} \varepsilon) = -t(q - \frac{t}{2} r^{2k-1} \varepsilon)^2 + \frac{1}{4} t^3 r^{4k-2} \varepsilon^2.$$

Now, we can factor out the last term if we use:

$$\alpha_3(t, r, \theta, q, \eta) = (1 - \frac{t^2}{4} r^{4k-2} \varepsilon^2(t, r, \theta, q, \eta)).$$

Then the change of variables $r \rightarrow \alpha_3^{\frac{1}{2k-2}}r$ gives:

$$\Psi(t, z) \simeq -t((q - \frac{t}{2} r^{2k-1} \varepsilon(t, r, \theta, q, \eta))^2 - r^{2k}),$$

where:

$$\varepsilon(t, r, \theta, q, \eta) = \alpha_3^{\frac{1}{2k-2}} \varepsilon(t, \alpha_3^{\frac{1}{2k-2}} r, \theta, q, \eta).$$

Finally, if we define:

$$(\chi_0, \chi_2, \chi_3, ..., \chi_{2n})(t, r, \theta, q, \eta) = (t, r, \theta, \eta),$$

$$\chi_1(t, r, \theta, q, \eta) = q - \frac{t}{2} r^{2k-1} \varepsilon(t, r, \theta, q, \eta),$$

(32)
the phase is $-\chi_0(\chi_1^2 - \chi_2^{2k})$ which is the desired result. 

If we use these local coordinates we obtain a simpler problem:

$$
\int e^{-i\frac{\chi_0}{h}(\chi_1^2 - \chi_2^{2k})} A(\chi_0, \chi_1, \chi_2) d\chi_0 d\chi_1 d\chi_2,
$$

(34)

where the amplitude $A$ is obtained via pullback and integration, i.e.:

$$
A(\chi_0, \chi_1, \chi_2) = \int \chi^*(a|J\chi|) d\chi_4 ... d\chi_{2n}.
$$

(35)

Now, we make several geometrical comments on this construction:

**Remark 14** The phase $-\chi_0(\chi_1^2 - \chi_2^{2k})$ gives a correct description of the singularities of the phase since the critical set is:

$$
\mathcal{C}(-\chi_0(\chi_1^2 - \chi_2^{2k})) = \{(\chi_0, 0, 0)\} \cup \{(0, \chi_1, \chi_1)\}.
$$

(36)

Moreover, this phase is non-degenerate w.r.t. $\chi_1$ when $\chi_0 \neq 0$. In order to express the main coefficient of the trace formula, we notice that $\chi_2 = 0 \Leftrightarrow x = x_0$ and $\chi_1(t, 0, \theta, q, \eta) = 0 \Leftrightarrow \xi = 0$, locally near $(x_0, \xi_0)$. Also, because of the introduction of the polar coordinates our amplitude satisfies:

$$
A(\chi_0, \chi_1, \chi_2) = O(\chi_l^{n-1}), \ l = 1, 2.
$$

Finally, each diffeomorphism used has Jacobian 1 in $z_0$ excepted the correction w.r.t. $\alpha_2$, cf. Eq. (31), which induces by pullback of the measure $r^{n-1}dr$ a multiplication by:

$$
|V_{2k}(\theta)|^{-\frac{m}{\pi}}.
$$

(37)

These facts will be used in the next section to obtain an invariant formulation. □

6 Proof of the main result.

Let be $2k$ the even integer attached to our potential. We state 2 technical results which allow to compute the asymptotic expansion of our oscillatory integrals. The first one proves the existence of a total asymptotic expansion for the family of oscillatory integrals attached to our normal forms. The second result, which is the hardest part of this work, computes the main coefficients of this expansion w.r.t. $h$ in the trace formula.

**Lemma 15** For $a \in C_0^\infty(\mathbb{R} \times [0, \infty]^2)$, the oscillatory integrals:

$$
J(\lambda) = \int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+} e^{-i\lambda\chi_0(\chi_1^2 - \chi_2^{2k})} A(\chi_0, \chi_1, \chi_2) d\chi_0 d\chi_1 d\chi_2,
$$

(38)
admit, as $\lambda \to \infty$, the asymptotic expansion:

$$J(\lambda) \sim \sum_{j=0}^{\infty} \lambda^{-\frac{j+1}{2k}} C_j(A) + \sum_{j=0}^{\infty} \lambda^{-j+1} \log(\lambda) D_j(A), \quad (39)$$

where $C_j$ and $D_j$ are universal (computable) distributions.

**Remark 16** The result stated obviously also holds for integration on $\mathbb{R} \times \mathbb{R}^2$ if we split up the domain of integration and use the symmetry w.r.t. $\chi_1$ and $\chi_2$ of the phase. Also, note that terms with a logarithm of the parameter only occur when $(j/2k) \in \mathbb{N}^*$. □

To compute the leading term of the trace formula we need a particular and explicit result which explains the effect of the dimension $n$ in our spectral problem. For $A \in C_0^\infty(\mathbb{R} \times \mathbb{R}^2_+)$ we define:

$$I(\lambda) = \int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+} e^{-i\lambda(r^2-q^2)} A(t,r,q) r^{n-1} q^{n-1} dt dr dq. \quad (40)$$

We note $t_\pm = \max(\pm,0)$ and $\hat{A}$ the partial Fourier transform of $A$ w.r.t. $t$.

**Lemma 17** If $n(k+1)/2k$ is not an integer then, as $\lambda \to \infty$, we have:

$$I(\lambda) = C_0(A) \lambda^{-\frac{n(k+1)}{2k}} + \mathcal{O}(\log(\lambda) \lambda^{-\frac{n(k+1)+1}{2k}}). \quad (41)$$

The distributional coefficients are given by:

$$C_0(A) = C_{n,k}^{-} \int_{\mathbb{R}} t_{-\frac{n+1}{2k}-1}^{-} \hat{A}(t,0,0) dt, \text{ if } n \text{ is even},$$

$$C_0(A) = \int_{\mathbb{R}} (C_{n,k}^{-} t_{-\frac{n+1}{2k}-1}^{-} + C_{n,k}^{+} t_{\frac{n+1}{2k}-1}^{+}) \hat{A}(t,0,0) dt, \text{ if } n \text{ is odd}.$$

But if $n(k+1)/2k \in \mathbb{N}^*$ and $n$ is odd, then:

$$I(\lambda) = -C_0(A) \lambda^{-\frac{n-1}{2k}} \log(\lambda) + \mathcal{O}(\lambda^{-\frac{n(k+1)+1}{2k}}), \quad (42)$$

with:

$$C_0(A) = C_{n,k} \int_{\mathbb{R}} |t|^{\frac{n+1}{2k}-1} \hat{A}(t,0,0) dt.$$

Finally, if $n(k+1)/2k \in \mathbb{N}^*$ and $n$ is even the asymptotic is given by Eq.(41) with:

$$C_0(A) = \int_{\mathbb{R}} (C_{n,k}^{-} t_{\frac{n+1}{2k}-1}^{-} + C_{n,k}^{+} t_{\frac{n+1}{2k}-1}^{+}) \hat{A}(t,0,0) dt.$$

**Remark 18** We will not detail all the coefficients of the asymptotic expansion because these are given by lengthy formulae and it is, a priori, not possible
to express invariantly their contributions to the trace formula. Anyhow, these can be explicitly computed with the procedure below. Also if \( \frac{n(k+1)+1}{2k} \notin \mathbb{N} \) the remainder of Eq. (41) can be optimized to \( O(\lambda^{-\frac{n(k+1)+1}{2k}}) \). □

Before entering in the proof we would like to add a comment suggested by an interesting remark of D.Barlet. The Bernstein-Sato polynomial of our phase \( \chi_0(\chi_1^2 - \chi_2^{2k}) \) can be explicitly computed as:

\[
\mathfrak{B}(z) = (z + 1)^3 \prod_{l=0, l \neq k+1}^{2k-2} (z + \frac{l + k - 1}{2k}).
\] (43)

In particular, we could expect to obtain terms \( \log(\lambda)^2 \) in the expansion since \( 1/\mathfrak{B} \) has a triple pole in \( z = 1 \). But we will see that if the amplitude is smooth there is no such contribution.

**Proof of Lemma 15.** To attain our objective we can restrict the proof of the lemma to an amplitude \( A(t, r, q) = a(t)B(r, q) \). This is justified by a standard density argument in \( C_0^\infty \) and the fact that the coefficients obtained below are linear continuous functionals. By integration w.r.t. \( t \) we obtain:

\[
J(\lambda) = \int_{\mathbb{R}^2} \hat{a}(\lambda(r^2 - q^{2k}))B(r, q)drdq.
\]

This shows that the asymptotic is supported in the set \( r = q^k \) since \( \hat{a} \) decreases faster than any polynomial at infinities. Although the new problem looks simple it is still too complicated to obtain an explicit solution. First we split our integral as:

\[
J(\lambda) = J_+(\lambda) + J_-(\lambda),
\] (44)

with:

\[
J_+(\lambda) = \int_{0 \leq q^k \leq r < \infty} \hat{a}(\lambda(r^2 - q^{2k}))B(r, q)drdq,
\]

\[
J_-(\lambda) = \int_{0 \leq r \leq q^k \leq \infty} \hat{a}(\lambda(r^2 - q^{2k}))B(r, q)drdq.
\]

Now we define the Melin transforms of \( \hat{a} \):

\[
M_+(z) = \int_0^\infty t^{z-1}\hat{a}(t)dt,
\] (45)

\[
M_-(z) = \int_0^\infty t^{z-1}\hat{a}(-t)dt.
\] (46)
By Melin inversion formula we obtain that :

\[ J_+ (\lambda) = \frac{1}{2i\pi} \int_{c+i\mathbb{R}} M_+ (z) \int_{0 \leq q^k \leq r < \infty} (\lambda (r^2 - q^{2k}))^{-z} B(r, q) dr dq dz. \]  

(47)

Here \( 0 < c < (2k)^{-1} \) so that the Melin inversion makes sense. In order to desingularize the remaining part of the phase we introduce the new coordinates \( r = sq^k, \ s > 0 \) and \( q > 0 \). We accordingly obtain that :

\[ J_+ (\lambda) = \frac{1}{2i\pi} \int_{c+i\mathbb{R}} M_+ (z) \lambda^{-z} \int_{s=1}^{\infty} \int_{q=0}^{\infty} (s^2 - 1)^{-z} q^{-2kz} B(sq^k, q) ds dq dz, \]  

(48)

and :

\[ J_-(\lambda) = \frac{1}{2i\pi} \int_{c+i\mathbb{R}} M_-(z) \lambda^{-z} \int_{s=0}^{\infty} \int_{q=0}^{\infty} |s^2 - 1|^{-z} q^{-2kz} B(sq^k, q) ds dq dz. \]  

(49)

To control the remainders, and also to justify the changes of path below, we recall the following classical result :

**Lemma 19** If \( a \in \mathcal{S}(\mathbb{R}) \) then we have :

\[ \forall c > 0 : M_\pm (c + iy) \in \mathcal{S}(\mathbb{R}), \]  

(50)

a fortiori \( M_\pm (c + iy) \in L^1(\mathbb{R}, dy). \)

Our integrals can be expanded via Cauchy’s residue method by pushing of the complex path of integration to the right. The associated distributional factors are analytic (cf. Lemma 20 below) and if we choose \( d > c \) conveniently we obtain :

\[ J_+ (\lambda) = \sum_{c < z_i < d} \text{Res}(z_i) + R(d, \lambda), \]

where the remainder satisfies :

\[ |R(d, \lambda)| \leq C(B) \lambda^{-d} \|M_+(d + iy)\|_{L^1(\mathbb{R}, dy)} = O(\lambda^{-d}). \]  

(51)

Here, for each \( d \) the constant \( C(B) \) involves the \( L^1 \) norm of a finite number of derivatives of \( B \). This will indeed lead to an asymptotic expansion.

The resulting asymptotics are related to poles of meromorphic distributions \( z \mapsto M_\pm (z) \lambda^{-z} (s^2 - 1)^{-z} q^{-2kz} \) and it remains now to extend analytically these distributions.

**Lemma 20** The family of distributions on \( C_0^\infty(\mathbb{R}^2) : z \mapsto |s^2 - 1|^{-z} q^{-2kz}, \) initially defined in the domain \( \Re(z) < 1/2k, \) is meromorphic on \( \mathbb{C} \) with poles
located at the rational points:

\[ z_{j,k} = \frac{j}{2k}, \quad j \in \mathbb{N}^*. \]  

These poles are of order 2 if \( z_{j,k} \in \mathbb{N}^* \) and of order 1 otherwise.

**Proof.** We first observe that for \( s \geq 0 \) we can write:

\[ |s^2 - 1|^{-z} = (s + 1)^{-z}|s - 1|^{-z}. \]

Since we will integrate on \( s \geq 0 \) the first term defines clearly an entire distribution. Now we remark that for \( s \geq 1 \):

\[
\frac{\partial^{2k+1}}{\partial s \partial q^{2k}}(s - 1)^{1-z}(q^{2k})^{-z} = b_0(z)(s - 1)^{-z}(q^{2k})^{-z},
\]

\[ b_0(z) = (1 - z) \prod_{j=1}^{2k} (j - 2k). \]

Hence for \( \Re(z) > 0 \) we can write:

\[
\int_{s=1}^{\infty} \int_{q=0}^{\infty} (s^2 - 1)^{-z} q^{-2kz} f(s, q) ds dq = \frac{-1}{b_0(z)} \int_{s=1}^{\infty} \int_{q=0}^{\infty} (s - 1)^{l-1} q^{2l-1} \frac{\partial^{2k+1}}{\partial s \partial q^{2k}}(1 + s)^{-z} f(s, q) ds dq. \]  

(53)

Now, the r.h.s. is meromorphic in \( \Re(z) < 1 + 1/2k \) and we can iterate to get the analytic continuation in \( \Re(z) < l + 1/2k, \ l \in \mathbb{N} \) arbitrary. The poles, and their order, can be read off the rational functions of \( z \):

\[ \Re_l(z) = \prod_{m=0}^{l-1} \frac{1}{b_0(z - m)}. \]  

(54)

Finally, a similar construction holds if we integrate w.r.t. \( s \in [0, 1] \).

With the residue method and classical estimates for the remainder (cf. Lemma 19) the last lemma proves the existence of a total asymptotic expansion for the integrals \( J(\lambda) \). This ends the proof of Lemma 15.

**Proof of Lemma 17.** With some slight modifications, we can apply the results above to our initial problem. Taking Remark 14 into account, we have to compute asymptotics of oscillatory integrals with amplitudes:

\[ B(r, q) = b(r, q)r^{n-1}q^{n-1}. \]  

(55)
By substitution, we have to study the poles of:

\[ g_+(z) = M_+(z) \int_{s=1}^{\infty} \int_{q=0}^{\infty} (s^2 - 1)^{-z} q^{-2kz} b(sq^k, q) s^{n-1} q^{n(k+1)-1} dsdq, \]  
\[ g_-(z) = M_-(z) \int_{s=0}^{\infty} \int_{q=0}^{\infty} |s^2 - 1|^{-z} q^{-2kz} b(sq^k, q) s^{n-1} q^{n(k+1)-1} dsdq. \]  
(56)

Here \( b(sq^k, q) \) is in general no more of compact support but all expressions and manipulations will be legal because terms depending on \( z \) will decrease faster and faster when we will shift the path of integrations. To avoid unnecessary calculations and discussions below, we remark that we can commute the polynomial weights w.r.t. \( q \) via the relation:

\[ \frac{\partial^{2k+1}}{\partial s \partial q^{2k}} (s - 1)^{1-z} q^{2k(1-z)} q^{n(k+1)-1} = b(z)(s - 1)^{-z} q^{-2kz} q^{n(k+1)-1}, \]

\[ b(z) = (z - 1) \prod_{j=1}^{2k} (j - 2kz + n(k + 1) - 1). \]

More generally, with the differential operator:

\[ D = \frac{\partial^{2k+1}}{\partial s \partial q^{2k}}, \]

after \( l \)-iterations one obtains:

\[ D^l(s - 1)^{1-z} q^{2k(1-z)} q^{n(k+1)-1} = \mathfrak{B}_l(z)(s - 1)^{-z} q^{-2kz} q^{n(k+1)-1}, \]

\[ \mathfrak{B}_l(z) = \prod_{i=0}^{l-1} b(z - i), \quad l \in \mathbb{N}^*. \]

A priori, this shows that there is poles of order 2 at positive integers and simple poles at rational points:

\[ z_{p,j,n,k} = p + \frac{j + n(k + 1) - 1}{2k} \notin \mathbb{N}, \quad p \in \mathbb{N}, \quad j \in [1, \ldots, 2k]. \]  
(58)

For example, the analytic extension in the half-plane \( \Re(z) < l, \quad l \in \mathbb{N}^* \), of the complex function appearing in the first integral is explicitly given by:

\[ \frac{(-1)^l}{\mathfrak{B}_l(z)} M_+(z) q^{-z} \int_{s=1}^{\infty} \int_{q=0}^{\infty} q^{2k(l-z)+n(k+1)-1} D^l(1 + s)^{-z} b(sq^k, q) s^{n-1} dsdq. \]  
(59)

A similar relation for the term involving \( M_-(z) \) is:

\[ D|1 - s|^{1-z} q^{n(k+1)-1-2k(1-z)} = -b(z)|1 - s|^{-z} q^{n(k+1)-1-2kz}. \]
This change of sign, due to the modulus, will be important below because of some symmetries. After \( t \)-iterations we have:

\[
\frac{1}{\mathcal{B}_l(z)} M_-(z) \lambda^{-z} \int_{s=0}^{\infty} \int_{q=0}^{\infty} q^{2k(t-z)+n(k+1)-1} D^t(1 + s)^{-z} b(sq^k, q)s^{n-1} ds dq. \tag{60}
\]

**Remark 21** The smallest double root of \( b \), and a fortiori of each \( \mathcal{B}_l \), is greater than \( z_{\min} = \frac{n(k+1)}{2k} \). In fact we will see below that there is no poles, and a fortiori no contributions, before this value. This insures that all integrals involved in the asymptotic expansion are absolutely convergent. □

A carefully examination of the integral w.r.t \( q \) shows that all coefficients are zero until we reach the pole:

\[
z_{\min} = \frac{k+1}{2k} = \frac{n}{2} + \frac{n}{2k}. \tag{61}
\]

This is justified by the fact that if \( \alpha > 0, \beta \in \mathbb{N}^* \) and \( \alpha + 1 > \beta \) we obtain:

\[
\int_0^\infty \partial_x^\beta (x^\alpha f(x)) dx = [\partial_x^{\beta-1}(x^\alpha f(x))]_{x=0}^\infty = 0, \forall f \in C_0^\infty(\mathbb{R}).
\]

Since poles located at integers are of order 2 the attached residuum are computed via the elementary formula:

\[
\lim_{z \to p} \frac{\partial}{\partial z} \left( (z-p)^2 \mathfrak{g}_\pm(z) \lambda^{-z} \right), \; p \in \mathbb{N}, \tag{62}
\]

where \( \mathfrak{g}_\pm \) are defined by Eqs. (56,57). For \( h \) holomorphic and \( \lambda > 0 \) we have:

\[
\frac{\partial}{\partial z}(h(z)\lambda^{-z}) = \frac{\partial h}{\partial z}(z)\lambda^{-z} - \log(\lambda)\lambda^{-z} h(z), \tag{63}
\]

and we can apply this to \( h(z) = (z-p)^2 \mathfrak{g}_\pm(z) \) near \( z = p \). Hence, a generic double pole located at \( z = p \) leads to a contribution:

\[
(-1)^p \log(\lambda)\lambda^{-p} \lim_{z \to p} \frac{(z-p)^2}{\mathcal{B}_p(z)} M_+(z) \int_{s=1}^{\infty} \int_{q=0}^{\infty} q^{n(k+1)-1} D^p(s+1)^{-p} b(sq^k, q)s^{n-1} ds dq. \tag{64}
\]

Since \( M_+(p) \) is well defined and:

\[
\lim_{z \to p} \frac{(z-p)^2}{\mathcal{B}_p(z)} = c_p \in \mathbb{Q}, \tag{65}
\]

our coefficients can be explicitly determined by the computation of the integrals. Also, cf. Remark 21, the \( c_p \) are zero until \( p \geq z_{\min} \). By integrations by
parts and up to a factorial number, the inner integral of Eq. (64) can also be written:

\[- \int_{s=1}^{\infty} \frac{\partial^p}{\partial s^p} (s+1)^{-p} s^{n-1} \left( \frac{\partial^{2kp-1}}{\partial q^{2k-1}} b(sq^k, q) q^{n(k+1)-1} \right) ds \]

\[= \left( \frac{\partial^{p-1}}{\partial s^{p-1}} (s+1)^{-p} s^{n-1} \left( \frac{\partial^{2kp-1}}{\partial q^{2k-1}} b(sq^k, q) q^{n(k+1)-1} \right) \right)_{q=0} |_{s=1} \]

But for all \(g\) smooth bounded, with bounded derivatives, and \(p\) large enough we have:

\[\int_{0}^{\infty} \frac{\partial^p}{\partial s^p} \left( (1+s)^{-p} s^{n-1} g(s) \right) ds = 0. \quad (66)\]

In particular, this is the case for all our coefficients since we have:

\[p \geq \frac{n}{2} + \frac{n(k+1)}{2k} > n.\]

This trick shows that coefficients obtained by integration w.r.t. \(s\) on \([0,1]\) and \([1, \infty]\) can be identified up to a sign and we can save some computations.

**Computation of the leading term.**

Contrary to the case of non-degenerate critical points, the evaluation of the leading term is somehow technical and some computations will be left to the reader. According to the analysis above, we distinguish out the case where the first non-zero residue is attached to a simple or a double pole.

**Case of \(z_{min}\) simple pole.**

Taking Remark 21 into account the first coefficient is given by:

\[\lim_{z \to z_{min}} (-1)^l \lambda^{-z} \left( z - z_{min} \right) \frac{M_+(z)}{B_l(z)} \]

\[\int_{s=1}^{\infty} \int_{q=0}^{\infty} (s-1)^{l-z} q^{2k(l-z)+n(k+1)-1} D^l (s+1)^{-z} b(sq^k, q) s^{n-1} ds dq, \]

with \(l \in \mathbb{N}^*\) such that \(l > z_{min}\) (any such \(l\) is acceptable). With this choice we can take the limit under the integral to obtain:

\[c_{l,k,n} \lambda^{-\frac{k+1}{2k}} \int_{s=1}^{\infty} \int_{q=0}^{\infty} (s-1)^{l-n - \frac{k+1}{2k}} q^{2k(l-z)} D^l (s+1)^{-n - \frac{k+1}{2k}} b(sq^k, q) s^{n-1} ds dq, \quad (67)\]

\[c_{l,k,n} = (-1)^l M_+ \left( \frac{n}{2k} \right) \lim_{z \to z_{min}} \left( z - z_{min} \right) \frac{B_l(z)}{B_l(z)}. \quad (68)\]
By integrations by parts, the integral w.r.t. $q$ of Eq. (67) is given by:

$$
\int_0^\infty q^{2kl-1} \frac{\partial^{2kl}}{\partial q^{2kl}} b(sq^k, q) dq = (2kl - 1)! b(0, 0).
$$

Hence we obtain the asymptotic relation for the positive part of our integral:

$$
I_+(\lambda) = c_1 b(0, 0) \lambda^{-\frac{k+1}{2k}} \int_1^\infty (s - 1)^{l-n\frac{k+1}{2k}} \frac{\partial^{l}}{\partial s^l} ((1 + s)^{-n\frac{k+1}{2k}} s^{n-1}) ds + R_1(\lambda),
$$

$$
c_1 = (2kl - 1)! c_{l,k,n}.
$$

As concerns the convergence of the integral w.r.t. $s$, by construction the singularity in $s = 1$ is controlled. For the behavior at infinity we remark that the degree w.r.t. $s$ is:

$$
l - n\frac{k+1}{2k} - n\frac{k+1}{2k} + n - 1 - l = -n\frac{k+1}{k} + n - 1 < -1, \quad (69)
$$

so that the integral is absolutely convergent. To compute explicitly the values of our integrals we can choose $l = n$ since $n - z_{\text{min}} > 0$. First, by induction on $n$ and for $\frac{n}{2} < \Re(\alpha) < n + 1$ we obtain that:

$$
E(n, \alpha) = \int_1^\infty (s - 1)^{n-\alpha} \frac{\partial^n}{\partial s^n} ((1 + s)^{-\alpha} s^{n-1}) ds = 0, \text{ if } n \text{ is even.}
$$

Next, for $n$ odd, and always with $\frac{n}{2} < \Re(\alpha) < n + 1$, we have:

$$
E(n, \alpha) = \prod_{j=1}^{n-1} (-2j - 1)2^{\frac{n+j-2}{2}} \frac{\Gamma(n + 1 - \alpha)\Gamma(-n + 2\alpha)}{\Gamma(\frac{1-n}{2} + \alpha)}.
$$

Hence for $\alpha = n(k + 1)/2k$ and $n$ odd, we obtain:

$$
\int_1^\infty (s - 1)^{n-n\frac{k+1}{2k}} \frac{\partial^n}{\partial s^n} ((1 + s)^{-n\frac{k+1}{2k}} s^{n-1}) ds = c_{n,k} \frac{\Gamma(1 + n\frac{k+1}{2k})\Gamma(\frac{n}{k})}{\Gamma(\frac{k+n}{2k})}, \quad (70)
$$

$$
c_{n,k} = \prod_{j=1}^{n-1} (-2j - 1)2^{\frac{n+j-2}{2}} \frac{n-j}{k}.
$$

For any integer $l > n(k + 1)/2k$, similar computations show that:

$$
I_-(\lambda) = c_2 b(0, 0) \lambda^{-n\frac{k+1}{2k}} \int_0^1 |s - 1|^l-n\frac{k+1}{2k} \frac{\partial^l}{\partial s^l} ((1 + s)^{-n\frac{k+1}{2k}} s^{n-1}) ds + R_2(\lambda),
$$

$$
c_2 = (2kl - 1)! c_{p,M} \frac{(n(k + 1)}{2k}).
$$
Once more the choice of $l = n$ is admissible and we define:

$$a_{n,k} = \int_0^1 |s - 1|^{n-k-1} \frac{\partial^n}{\partial s^n}(1 + s)^{-n^{k+1} \frac{1}{2k} s^{n-1})ds.$$ 

since these integrals seem to have no formulation by mean of elementary functions, unless by mean of hypergeometric functions. These numbers $a_{n,k}$ are finite and non-zero in general position.

From the analysis of the poles above we know that the remainders $R_1$ and $R_2$ are of order $O(\frac{n(k+1)+1}{2k})$ if the next pole is not an integer, respectively $O(\log(\lambda)\lambda^{-n(k+1)+1})$ if this is an integer (cf. Remark 18). By summation we obtain the leading term of the asymptotic expansion with a precise remainder.

**Case of $z_{\text{min}}$ pole of order 2.**

Starting from Eq. (63) we see that the associated coefficients are given by:

$$-\log(\lambda)\lambda^{-z_{\text{min}}} \left((z - z_{\text{min}})^2 g(z)M_{\pm}(z)\right)|_{z = z_{\text{min}}}.$$ (72)

But a great part of this limit was precisely computed above. Since $z_{\text{min}}$ is by assumption an integer we will obtain some particular values. By induction on $p > 1$, and assuming recursively that $n < 2p$, we have:

$$\int_1^\infty \frac{\partial^p}{\partial s^p}((s + 1)^{-p}s^{n-1})ds = -\frac{1}{2p} \prod_{j=0}^{p-1}(n - 2j).$$ (73)

If $n$ is odd this coefficient is not zero and we get the result. But if $n$ is even we obtain that the associated contribution vanishes and there is no logarithm in the leading term. To obtain the top order coefficient we must compute the coefficient obtained by derivation of our meromorphic distributions.

Starting from Eqs. (59,60), by Leibnitz rule, we have 3 possibilities: derivation of the rational function, of the Melin transform or of the analytic integrals in $(q, s)$. Since the integral w.r.t. $s$ vanishes in $z = z_{\text{min}}$, the 2 first terms do not contribute. Similarly, the derivative of $q^{-2kz}$ can be discarded. Hence, the only contribution comes from derivation of the distribution w.r.t. $s$ and we have to use the modified constants:

$$\tilde{a}_{p,k}^+ = -\int_1^\infty \log(s^2 - 1) \frac{\partial^p}{\partial s^p}((s + 1)^{-p}s^{n-1})ds,$$ (74)

$$\tilde{a}_{p,k}^- = -\int_0^1 \log(s^2 + 1) \frac{\partial^p}{\partial s^p}((s + 1)^{-p}s^{n-1})ds.$$ (75)
respectively for $I_+(\lambda)$ and $I_-(\lambda)$. By similar considerations as above, these integrals are absolutely convergent for any $p \geq z_{\text{min}}$.

**The other coefficients.**
To obtain a complete overview of the asymptotic expansion we show also how to compute the coefficients attached to logarithmic distributions. For $p \in \mathbb{N}^*$, $p \geq z_{\text{min}}$, by Leibnitz rule, these derivatives are equal to:

$$\partial_z M_\pm(p) \lim_{z \to p} (z-p)^2 g_\pm(z) + M_\pm(p) \lim_{z \to p} (\partial_z(z-p)^2 g_\pm(z)).$$

Hence the asymptotic involves also the distributional terms:

$$\int_0^\infty \log(t) t^{p-1} \hat{a}(\pm t) dt, \ p \in \mathbb{N}^*,$$

and, by derivation of the meromorphic distributions, terms:

$$\int_0^\infty \int_{s=1}^{\infty} \int_{q=0}^{\infty} \log(q) q^a D^p (1+s)^{-p} b(sq^k, q) s^{n-1} ds dq,$$

$$\int_{s=1}^{\infty} \int_{q=0}^{\infty} \log(s^2 - 1) q^a D^p (1+s)^{-p} b(sq^k, q) s^{n-1} ds dq.$$
principal symbol of $\Theta(P_h)$ is $\Theta(p)$ with $\Theta(p(z_0)) = 1$. Hence, at the critical point, the term homogeneous of degree 0 w.r.t. $h$ of the amplitude of our FIO is given by $a(t, z_0) = \hat{\varphi}(t)$ (cf. section 3). Substituting Eq. (76) in all integral formulas for the leading terms of the asymptotic expansion the Fourier inversion formula yields:

$$M_+ (\hat{\alpha}(t, z_0))(n \frac{k+1}{2k}) = \int_0^\infty \varphi(t) t^n \frac{k+1}{2k} dt,$$

$$M_- (\hat{\alpha}(t, z_0))(n \frac{k+1}{2k}) = \int_0^\infty \varphi(-t) t^n \frac{k+1}{2k} dt.$$

Setting $\lambda = h^{-1}$, so that $\log(h) = - \log(\lambda)$, dividing by $(2\pi h)^n$ we obtain, via Lemma 17, the results stated in Theorem 4.

**Extension.**

We show here shortly how to extend the result of Theorem 4 to the case of an $h$-admissible operator $P_h$ of symbol $p_h \sim \sum h^j p_j$ whose principal symbol is $p_0 = \xi^2 + V(x)$ with a non-vanishing subprincipal symbol $p_1$. In this case the Fourier integral operator approximating the propagator has the amplitude:

$$\tilde{\alpha}(t, z) = a(t, z) \exp(i \int_0^t p_1(\Phi_s(z)) ds),$$

see Duistermaat [9] concerning the solution of the first transport equation. Since $z_0$ is an equilibrium we have simply $p_1(\Phi_s(z_0)) = p_1(z_0)$ and hence:

$$\tilde{\alpha}(t, z_0) = \hat{\varphi}(t) e^{it p_1(z_0)}. \quad (77)$$

If the subprincipal symbol vanishes at the critical point, which is the case in a lot of practical situations, the top order coefficient of the trace formula remains the same. Finally, when $p_1(z_0) \neq 0$ by Fourier inversion formula we replace $\varphi(t)$ by $\varphi(t + p_1(z_0))$ in all integral formulae of Theorem 4.

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