An Almost Perfect Lattice Action for infrared QCD

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A block-spin transformation on the dual lattice leads us to an almost perfect lattice action for monopoles and strings in QCD. The perfect operator for a static quark potential is fixed when we compare the above action with the perfect action obtained analytically after infinite-step block-spin transformations in a simple case. The continuum rotational invariance is restored and the physical value of the string tension is reproduced fairly well. Gauge independence of the abelian and the monopole scenario is discussed.

1 Introduction

Low-energy effective theory of QCD is important for analytical understanding of hadron physics. One of approaches is to search for relevant dynamical variables and to construct an effective theory for these variables.

From this point of view, the idea proposed by ’t Hooft is very promising. The confinement of quarks can be explained as the dual Meissner effect which is due to condensation of these monopoles after an abelian projection.

Many numerical results support the dual superconductor picture of confinement in the Maximal abelian (MA) gauge in the framework of lattice QCD. We expect hence, after integrating out all degrees of freedom other than the monopoles, an effective theory described by the monopoles works well in the IR region of QCD.

The effective monopole action on the MA projection of SU(2) lattice QCD was first obtained by Shiba and Suzuki using an inverse Monte-Carlo method. See also Ref. 1.

The purpose of this talk is to review briefly the results in our recent publications.

2 An (almost) perfect monopole action

The method to derive the monopole action is the following.

1 We generate link fields \( \{ U(s, \mu) \} \) using the simple Wilson action for SU(2) and SU(3) QCD. We consider 24\(^4\) and 48\(^4\) hyper-cubic lattice for \( \beta = 2.0 \sim 2.8 \) (SU(2)) and for \( \beta = 5.6 \sim 6.4 \) (SU(3)).

\(^{a}\)This talk is based on our recent works.
Next we perform the abelian projection in the MA gauge to separate abelian link variables \( \{ u(s, \mu) \} \).

Monopole currents \( k_\mu(s) \) can be defined from abelian link variables following DeGrand and Toussaint\(^\text{12}\).

We determine the set of couplings \( G_i \) from the monopole current ensemble \( \{ k_\mu(s) \} \) with the aid of an inverse Monte-Carlo method first developed by Swendsen\(^\text{10}\) and extended to closed monopole currents by Shiba and Suzuki\(^\text{9}\). Here the monopole action can be written as a linear combination of some operators \( S[k] = \sum_i G_i S_i[k] \).

Practically, we have to restrict the number of interaction terms. The form of actions adopted here is 27 quadratic interactions and 4-point and 6-point interactions.

We perform a block-spin transformation in terms of the monopole currents on the dual lattice to investigate the renormalization flow in the IR region. We adopt \( n = 1, 2, 3, 4, 6, 8 \) extended monopole currents as an \( n \) blocked operator\(^\text{13}\):

\[
K_\mu(s^{(n)}) = \sum_{i,j,t=0}^{n-1} k_\mu(ns^{(n)}) + (n - 1)\hat{\mu} + i\hat{\nu} + j\hat{\rho} + l\hat{\sigma})
\]  

(1)

The renormalized lattice spacing is \( b = na(\beta) \) and the continuum limit is taken as the limit \( n \to \infty \) for a fixed physical length \( b \).

The physical length \( b = na(\beta) \) is taken in unit of the physical string tension \( \sqrt{\sigma_{\text{phys}}} \).

### 3 Numerical results

1. The couplings are fixed clearly. We see the scaling \( S[k_\mu, n, a(\beta)] \to S[k_\mu, b = na(\beta)] \) for fixed \( b = na(\beta) \) looks good. The continuum limit is taken as \( a \to 0 \), \( n \to \infty \) for \( b = n \cdot a \) fixed.

2. The four- and six-point interactions become negligibly small for IR region. Two-point interactions are relatively dominant for large \( b \) region.

3. We see the direction dependence of the current action from the data. For example, two nearest-neighbor interactions \( G_2 \) and \( G_3 \) are quite different for small \( b \) region.
4. The $SU(3)$ case has three types of monopole currents $\{k^a(\mu)(s), a = 1 \sim 3\}$ with one constraint $\sum_a k^a(\mu)(s) = 0$. But the behaviors of the effective action are similar to those of the $SU(2)$ case especially for large $b$ region.

4 A perfect operator for a physical quantities

In QCD, the string tension from the static potential is an important physical quantity. A naive abelian Wilson loop operator on the coarse lattice is not good, because the cut-off effect is of order of the lattice spacing of the coarse lattice. We should use an improved operator on the coarse lattice in order to get the correct values of the physical observables. An operator giving a cut-off
independent value on RT is called **perfect operator**.

4.1 The method

The abelian monopole action $S[k]$ which is obtained numerically is well approximated by quadratic interactions alone. We can perform the analytic block spin transformation along the flow projected on the quadratic coupling constant plane. When we define an operator on the fine $a$ lattice, we can find a perfect operator along the projected flow in the $a \to 0$ limit for fixed $b$. We adopt the perfect operator on the projected space as an approximation of the correct operator for the action $S[k]$ on the coarse $b$ lattice.

4.2 The operator for the abelian static potential

First let us consider the following abelian gauge theory of the generalized Villain form on a fine lattice with a very small lattice distance:

$$S[\theta, n] = \frac{1}{4\pi^2} \sum_{s, s', \mu > \nu} \left( \partial_{\mu} \theta_{\nu}(s) + 2\pi n_{\mu\nu}(s) \right) \left( \Delta D_0(s - s') \partial_{\mu} \theta_{\nu}(s') + 2\pi n_{\mu\nu}(s') \right),$$

where $\theta_\mu(s)$ is a compact abelian gauge field and integer-valued tensor $n_{\mu\nu}(s)$ stands for Dirac string. Both of variables are defined on the original lattice. Since we are considering a fine lattice near to the continuum limit, we assume the direction symmetry of $D_0$. In this model, it is natural to use an abelian Wilson loop $W(C) = \exp \left\{ i \sum_{C} (\theta_\mu(s), J_\mu(s)) \right\}$ for particles with fundamental abelian charge, where $J_\mu(s)$ is an abelian integer-charged electric current. The expectation value of $W(C)$ is written as

$$\langle W(C) \rangle = \left\langle \exp \left\{ i \sum_{s, \mu} J_\mu(s) \theta_\mu(s) \right\} \right\rangle = Z[J]/Z[0],$$

$$Z[J] \equiv \int_{-\pi}^{\pi} \prod_{s, \mu} d\theta_\mu(s) \sum_{n_{\mu\nu}(s) = -\infty}^{+\infty} \exp \left\{ -S[\theta, n] + i \sum_{s, \mu} J_\mu(s) \theta_\mu(s) \right\},$$

When use is made of BKT transformation, the area law term from the partition function is equivalent to that from the following monopole expression

$$\langle W_m(C) \rangle = \frac{1}{Z} \sum_{k_\mu(s) = -\infty}^{+\infty} \exp \left\{ - \sum_{s, s', \mu} k_\mu(s) D_0(s - s') k_\mu(s') + 2\pi i \sum_{s, \mu} N_\mu(s) k_\mu(s) \right\},$$

4
\[ N_{\mu}(s, S_J) = \sum_{s'} \Delta^{-1}_L(s - s') \frac{1}{2} \epsilon_{\mu\alpha\beta\gamma} \partial_{\alpha} S_{\beta\gamma}^{J}(s' + \hat{\mu}), \]  \hfill (4)

where \( S_{\beta\gamma}^{J}(s' + \hat{\mu}) \) is a plaquette variable satisfying \( \partial'_{\beta} S_{\beta\gamma}^{J}(s) = J_{\gamma}(s) \).

### 4.3 The perfect operator for the static potential on the coarse lattice

We perform a block spin transformation (4) of the monopole currents. Let us start from (4). The cutoff effect of the operator (4) is \( O(a) \) by definition. The \( \delta \)-function renormalization group transformation (4) can be done analytically. Taking the continuum limit \( a \to 0, n \to \infty \) (with \( b = na \) is fixed) finally, we obtain the expectation value of the operator on the coarse lattice with spacing \( b = na \):

\[
\langle W(C) \rangle = \frac{1}{Z} \langle W_m(C) \rangle_{cl} \exp \left\{ \pi^2 \sum_{s(n), s(n)'} B_{\mu}(s(n)) D_{\mu\nu}(s(n) - s(n)') B_{\nu}(s(n)') \right\}
\times \sum_{\partial_\mu K_{\mu}=0} \exp \left\{ -S[K_{\mu}(s^{(n)})] + 2\pi i \sum_{s(n)} B_{\mu}(s(n)) D_{\mu\nu}(s(n) - s(n)') K_{\nu}(s(n)') \right\},
\hfill (5)
\]

where \( B_{\mu}(s^{(n)}) \) is the renormalized electric source term. \( S[K_{\mu}(s^{(n)})] \) denotes the effective action defined on the coarse lattice:

\[
S[K_{\mu}(s^{(n)})] = \sum_{s(n), s(n)'} \sum_{\mu, \nu} K_{\mu}(s^{(n)}) D_{\mu\nu}(s^{(n)} - s^{(n)'} \nu) K_{\nu}(s^{(n)'}). \hfill (6)
\]

Here \( \langle W_m(C) \rangle_{cl} \) is defined by

\[
\langle W_m(C) \rangle_{cl} = \exp \left\{ -\pi^2 \int_{-\infty}^{\infty} d^4x d^4y \sum_{\mu} N_{\mu}(x) D_{\mu\nu}(x) N_{\nu}(y) \right\}, \hfill (7)
\]

Since we take the continuum limit analytically, the operator (5) does not have any cutoff effect. For details, see Ref. 1.

Performing BKT transformation on the coarse lattice, we can get the loop operator for the static potential in the framework of the string model:

\[
\langle W_m(C) \rangle = \sum_{\sigma_{\mu\nu}(s)=-\infty}^{\infty} \exp \left\{ -\pi^2 \sum_{s, s', \mu, \nu} \sigma_{\mu\nu}(s) \partial_{\mu} \partial'_{\nu} D_{\mu\nu}^{-1} \Delta^{-2}_\nu(s - s') \sigma_{\nu\beta}(s') -2\pi^2 \sum_{s, s', \mu, \nu} \sigma_{\mu\nu}(s) \partial_{\mu} \Delta^{-1}_\nu(s - s') B_{\nu}(s') \right\} \times \langle W_m(C) \rangle_{cl}. \hfill (8)
\]
4.4 Parameter fitting

In order to compare our analysis with the inverse Monte-Carlo method, we expand $D_0(s - s')$ in the monopole action (4) as

$$\alpha \delta_{s,s'} + \beta \Delta_1(s - s') + \gamma \Delta_2(s - s'),$$

where $\alpha$, $\beta$ and $\gamma$ are free parameters. Then by matching the set of numerically obtained coupling constants of the monopole action $\{G_i(b)\}$ with $D_{\mu\nu}(s^{(n)} - s^{(n)'}))$ we fix the free parameters as shown in Ref. 2.

5 Analytic evaluation of physical quantities

5.1 The string tension

Let us evaluate the string tension using the perfect operator (8). The plaquette variable $S_{\alpha\beta}$ in Eq.(4) for the static potential $V(Ib, 0, 0)$ is expressed by

$$S_{\alpha\beta}(z) = \delta_{\alpha 1} \delta_{\beta 4} \delta(z_2) \delta(z_3) \theta(z_1) \theta(Tb - z_1) \theta(z_4) \theta(Tb - z_4).$$

(9)

The string model is in the strong coupling region for large $b$. Therefore, we evaluate Eq.(8) by the strong coupling expansion. As shown in Ref. 4, the (classical) string tension coming from Eq. (7) is dominant and it becomes in SU(2)

$$\sigma_{cl} = \frac{\pi \kappa}{2} \ln \frac{m_1}{m_2},$$

(10)

Using the optimal values $\kappa$, $m_1$ and $m_2$, we get the string tension $\sqrt{\sigma_{cl}/\sigma_{phys}}$ as shown in Table [1].

5.2 On the continuum rotational invariance

We here comment on the continuum rotational invariance of the quark-antiquark static potential. We get the static potential $V(Ib, Ib, 0)$ in SU(2) can be written as

$$V(Ib, Ib, 0) = \frac{\sqrt{2} \pi \kappa Ib}{2} \ln \frac{m_1}{m_2}.$$
The potentials from the classical part take only the linear form and the rotational invariance is recovered completely even for the nearest $I = 1$ sites.

6 Gauge independence

If the color confinement mechanism is due to the condensation of the monopoles, it should be gauge independent, since the monopole condensation in some special gauge means only abelian charge confinement which is quite different from color confinement.

Recently Ogilvie has developed a character as well as a strong coupling expansions for Abelian projection and found that gauge fixing is unnecessary, i.e., Abelian projection yields string tensions of the underlying non-Abelian theory even without gauge fixing. Hence at least gauge independence of abelian dominance seems to be realized. In the following, we show numerical analyses of gauge dependence for a general class of gauge between MA gauge and no-gauge fixing. Then we next prove if abelian dominance is gauge independent, gauge independence of monopole dominance is derived.

6.1 Numerical analyses in a general gauge

We consider the Langevin equation with stochastic gauge fixing term:

$$U_\mu(x, \tau + \Delta \tau) = \omega^\dag(x, \tau) \exp(i f^a_\mu t^a) U_\mu(x, \tau) \omega(x + \Delta X, \tau),$$

$$f^a_\mu = -\frac{\partial S}{\partial A^a_\mu} \Delta \tau + \eta^a_\mu(x, \tau) \sqrt{\Delta \tau},$$

$$\omega(x, \tau) = \exp(i \beta \Delta a(x, \tau) t^a \Delta \tau/2N_c \alpha).$$

We set $\Delta(x, \tau) = i[\sigma_3, X(x, \tau)]$, where $X$ is the operator to be diagonalized in MA gauge. $\alpha = 0 (\alpha = \infty)$ corresponds to the MA gauge fixing (no gauge fixing).

We have performed numerical simulations on $8^3 \times 12$ and $16^3 \times 24$ lattices with improved Iwasaki action. See Ref. for details. We get the results shown in Fig. 3. Abelian and monopole dominances are seen for a wide range of $\alpha$.

6.2 Gauge independence of monopole dominance

Assume gauge independence of abelian dominance is realized. Then we expect the existence of an abelian effective action. We express the abelian action in terms of the Villain form $Z = \int_{\mathbb{R}} D\theta \sum_{n \in \mathbb{Z}} e^{-F[d\theta + 2\pi n]}$. The general Villain
action can be expressed as follows:

\[
e^{-\left(d\theta + 2\pi n, D(d\theta + 2\pi n)\right)} - F\left[d\theta + 2\pi n\right] = e^{-F'[\left[-i\delta/\delta B\right]e^{-\left(d\theta + 2\pi n, D(d\theta + 2\pi n)\right)}]} \bigg|_{B=0}
\]

where \([D, d] = [D, \delta] = 0\) are satisfied in the large \(\beta\) scaling region.

The abelian Wilson loop \(e^{i(\theta, J)}\) is estimated with this action, where \(J\) is the electric current. When we use the BKT transformation, we get an action in terms of monopole currents:

\[
Z(J) = e^{-F'[\left[-i\delta/\delta B\right]e^{-\left(d\theta + 2\pi n, D(d\theta + 2\pi n)\right)}]} \sum_{k \in \mathbb{Z}, dk = 0} e^{-\frac{i}{4}(\delta B, (\Delta D)^{-1}\delta B)} \times e^{\frac{i}{4}(iB, 4\pi\delta\Delta^{-1}k + i(\Delta D)^{-1} dJ)} e^{-2\pi i(\delta\Delta^{-1}k, S)} e^{-\frac{i}{4}(J, (\Delta D)^{-1}J)} \bigg|_{B=0}
\]

1) Electric-electric current \(J - J\) interactions (with no monopole \(k\)) come from the exchange of regular photons and have no line singularity leading to a linear potential. Hence the linear potential of abelian Wilson loops is due to the monopole contribution alone. Monopole dominance is proved from abelian dominance. 2) The linear potential comes only from \(\exp(2\pi i(\delta\Delta^{-1}k, S))\). The surface independence of the static potential is assured due to the 4-d linking number.

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