Metastable spin textures and Nambu-Goldstone modes of a ferromagnetic spin-1 Bose-Einstein condensate confined in a ring trap

Masaya Kunimi
Department of Engineering Science, University of Electro-Communications, Tokyo 182-8585, Japan
(Dated: August 20, 2014)

We investigate the metastability of a ferromagnetic spin-1 Bose-Einstein condensate confined in a quasi-one-dimensional rotating ring trap by solving the spin-1 Gross-Pitaevskii equation. We find analytical solutions that exhibit spin textures. By performing linear stability analysis, it is shown that the solutions can become metastable states. We also find that the number of Nambu-Goldstone modes changes at a certain rotation velocity without changing the continuous symmetry of the order parameter.

PACS numbers: 67.85.Fg, 03.75.Mn, 03.75.Lm, 14.80.Va

I. INTRODUCTION

Owing to recent developments in the experimental techniques associated with cold atomic gases, Bose-Einstein condensates (BECs) confined in multiply connected geometries have been realized [1,13]. Such systems are suitable for investigating the fundamental properties of superfluidity. In fact, many interesting features of superfluidity have already been observed, such as persistent current [1,4], phase slip and vortex nucleations [5,8], hysteresis [10], and a current-phase relationship [12].

Although the above experiments, except Ref. [4], were performed using scalar BECs, BECs with internal degrees of freedom (spinor BECs [14,15]) confined in simply connected geometries have also been realized experimentally [16,17]. In spinor BECs, there exist various topological defects [18,28], spin textures [29,32], and Nambu-Goldstone modes (NGMs) [33–38] due to their spontaneous symmetry breaking.

The previous theoretical works regarding multi-component systems (two-component Bose gases [39,43] and spinor BECs [44,45]) in a ring trap concern the stability of persistent currents. Other important properties, including metastability under external rotation, have also been thoroughly investigated theoretically [46,47] and experimentally [5,8,10] for scalar BECs. However, a complete understanding of the metastability of spinor BECs in multiply connected systems under external rotation has not yet been established.

In this paper, we investigate the properties of spin-1 BECs in a rotating ring trap within the framework of the mean-field approach. We present analytical solutions of the spin-1 Gross-Pitaevskii equation (GPE) [33,34] under a twisted periodic boundary condition. This solution exhibits spin textures. By performing linear stability analysis, we show that these solutions can become metastable states. Furthermore, we determine the NGMs and find the change in the number of NGMs for a given rotational velocity without changing the continuous symmetry of the order parameter. This change in the number of NGMs is called a type-I-type-II transition [48].

II. MODEL

We consider $N$ spin-1 bosons confined in a rotating ring trap. Within the mean-field approximation, the system can be described by a three-component order parameter (a condensate wave function), $\Psi(r) \equiv [\Psi_1(r), \Psi_0(r), \Psi_{-1}(r)]^T$, where $T$ denotes the transpose. For simplicity, we treat the system as a quasi-one-dimensional torus. We assume that the spatial dependence of the order parameter is given by $\Psi(x) \equiv \Psi(x)/\sqrt{S}$, where $S$ is the cross-section of the torus and $x$ represents the coordinate [49]. The energy functional of the system is given by

$$E = \int_{-L/2}^{+L/2} dx \left[ \frac{\hbar^2}{2M} \frac{\partial^2 \Psi_m(x)}{\partial x^2} - \frac{c_0}{2} n(x)^2 + \frac{c_1}{2} F(x)^2 \right]$$

where $M$ is the mass of the boson, $n(x)$ the magnetic sublevels and can take the values of $1$, $0$, and $-1$, $L$ the length of the torus, $n_m(x) \equiv \sum_m |\Psi_m(x)|^2$ is the particle density, $F(x) \equiv \sum_{m,n} \Psi_m(x) (f_{mn} \Psi_n(x)$ is the magnetization density vector, $(f_{\nu})_{mn}(\nu = x,y,z)$ are the spin-1 matrices, $c_0 \equiv 4\pi\hbar^2(a_0 + 2a_2)/(3MS)$ and $c_1 \equiv 4\pi\hbar^2(a_2 - a_0)/(3MS)$ are the spin-independent and spin-dependent interaction strengths, and $a_0$ and $a_2$ are the s-wave scattering lengths of the spin-0 and -2 channels, respectively. In this paper, we consider the case in which $c_0 > 0$, $c_1 < 0$, and $c_0 \gg |c_1|$. This corresponds to $^{87}$Rb.

The time-independent GPE for spin-1 bosons [33,34] can be derived using the functional derivative of the energy functional $\delta[E - \mu N]/\delta \Psi_m(x) = 0$,

$$L_{\pm} \Psi_{\pm}(x) + c_1 [F_\ast(x) \Psi_\ast(x) + F_\ast(x) \Psi_{-\ast}(x)]/\sqrt{2} = 0,$$

$$L_0 \Psi_0(x) + c_1 [F_\ast(x) \Psi_\ast(x) + F_\ast(x) \Psi_{-\ast}(x)]/\sqrt{2} = 0,$$

$$L_m \equiv -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} - \mu + c_0 n(x) + mc F_\ast(x),$$
where \( F_\pm(x) \equiv F_x(x) \pm iF_y(x) \) and \( \mu \) is the chemical potential of the system, which is determined by the total number of particles \( N = \int_{-L/2}^{+L/2} dx \sum_m |\Psi_m(x)|^2 \).

The effects of the rotation are described by imposing the twisted periodic boundary condition \( \Psi_m(x + L) = \Psi_m(x) e^{iMvL/\hbar}, \) where \( -\nu \) is the rotational velocity of the ring. This corresponds to the boundary condition in the rotating frame \( [27] \).

We note that the twisted boundary condition is invariant under the transformation \( v \rightarrow v + lv_0, \) where \( v_0 \equiv 2\pi \hbar/(ML) \) and \( l \in \mathbb{Z} \). Throughout this paper, we use the parameters \( (L, c_1) = (96\xi_0, -0.005\xi_0) \) to demonstrate our results, where \( \xi_0 \equiv \hbar/Mc_0v_0 \) is the healing length. These values are close to those used in previous experiments \( [6,52] \).

Our results presented below are valid for other parameter regions as long as \( c_1 < 0 \).

### III. RESULTS

First, we present trivial plane-wave solutions of the GPE. It can be shown that ferromagnetic plane-wave (FPW) states, \( (\Psi_F(x) = \sqrt{n_0} e^{iM(v + W_F)x/\hbar}[1,0,0]^T, W_F \in \mathbb{Z}) \), and polar plane-wave (PPW) states, \( (\Psi_P(x) = \sqrt{n_0} e^{iM(v + W_P)x/\hbar}[0,1,0]^T, W_P \in \mathbb{Z}) \), are the stationary solutions of the GPE, where \( n_0 \equiv N/L \) is the mean-particle density. The velocity dependences of these states are shown in Fig. 1 \([53] \).

We find that the ground state is the ferromagnetic state. The stability of these states can be determined by the Bogoliubov excitation spectra \([14] \).

The low-lying excitation of the ferromagnetic state is a magnon mode as long as \( c_0 \gg |c_1| \) and its expression is given by \( \epsilon_F^M = -Mv_0|v + W_Fv_0| + Mv_0^2/2 \). The low-lying magnon mode becomes negative, namely, Landau instability occurs for \( |v + W_Fv_0| > v_0/2 \). The low-lying excitations of the polar state are also magnon modes (doubly degenerated) and their expressions are given by \( \epsilon_P^M = -Mv_0|v + W_Pv_0| + (Mv_0^2/2)\sqrt{1 + 4c_1n_0/(Mv_0^2)} \). Dynamical instability \( (\text{Im}(\epsilon_F^M) \neq 0) \) occurs in the polar state for \( L/\xi_0 > \pi \sqrt{c_0}/c_1 \).

Next, we present a non-trivial solution of the GPE, which is expressed using the following ansatz:

\[
\Psi_m(x) = \sqrt{n_0} e^{iMvtx/\hbar} e^{iMW_mv_0x/\hbar} \phi_m
\]

Here, \( W_m \in \mathbb{Z} \) is the winding number of the component \( m \) and \( \phi_m \) is a complex constant. We assume that all components of \( \phi_m \) are non-zero. According to the \( U(1) \times SO(3) \) symmetry of the system and the requirement that the chemical potential be real valued, we can choose \( \phi_1 \) and \( \phi_0 \) to be real and positive and \( \phi_{-1} \) to be real without loss of generality. \( \{\phi_m\} \) satisfy \( \sum_m \phi_m^2 = 1 \) due to the total particle number condition. When the winding numbers \( \{W_m\} \) satisfy the relations

\[
W = W_0 - W_1 = W_{-1} - W_0 \neq 0,
\]

the condensate wave function \( \phi \) becomes a solution of the GPE. The expressions of \( \{\phi_m\} \) are given explicitly by

\[
\phi_{\pm 1} = \left| \frac{W}{2} \pm \left( \frac{v}{v_0} + W_0 \right) \right| \sqrt{\frac{M_z/N}{2W(v/v_0 + W_0)}},
\]

\[
\phi_0 = \sqrt{1 - \frac{M_z/N}{W(v/v_0 + W_0)}} \left[ \frac{W^2}{4} + \left( \frac{v}{v_0} + W_0 \right)^2 \right],
\]

\[
\frac{M_z}{N} = 2 \left[ \frac{v}{v_0} + W_0 \right] \left[ 1 - \frac{g(v,W,W_0)}{|c_1|n_0} \right],
\]

where \( M_z \equiv \int_{-L/2}^{+L/2} dx F_z(x) = LF_z \) is the magnetization of the \( z \)-component \( (F_z \) does not depend on \( x) \), \( \phi_{-1} \) is positive due to Eq. (11), and \( g(v,W,W_0) \) is a function defined for convenience as

\[
g(v,W,W_0) = M_y^2 \left[ \frac{W^2}{4} - \left( \frac{v}{v_0} + W_0 \right)^2 \right].
\]

The parameter regions where the solution exists are given by

\[
\left| \frac{v}{v_0} + W_0 \right| \leq \left| \frac{W}{2} \right|, \quad \left( \frac{v}{v_0} + W_0 \right)^2 \geq \frac{W^2}{4} + \frac{|c_1|n_0}{M_y^2}.
\]

The explicit expressions of the physical quantities such as the chemical potential, total energy, total momentum.
of the rotating frame, and local magnetizations become
\[ \mu = (c_0 + c_1)n_0 + \frac{1}{2}M(v + W_0v_0)^2 + g(v, W, W_0), \]
\[ \frac{E}{N} = \mu - \frac{1}{2}(c_0 + c_1)n_0 - \frac{g(v, W, W_0)^2}{2|c_1|n_0}, \]
\[ P = -\frac{i}{2} \int_{-L/2}^{+L/2} dx \sum_m \left[ \Psi_m(x) \frac{d}{dx} \Psi_m(x) - \text{c.c.} \right] \]
\[ = NMv_0 \left( \frac{v}{v_0} + W_0 - \frac{W M_z}{N} \right), \]
\[ F_x(x) = \sqrt{2}n_0\phi_0(\phi_1 + \phi_{-1}) \cos \left( \frac{2\pi W L}{x} \right), \]
\[ F_y(x) = \sqrt{2}n_0\phi_0(\phi_1 + \phi_{-1}) \sin \left( \frac{2\pi W L}{x} \right). \]

From the expressions for the magnetization density (12) and (13), this solution (we call it the three-component plane-wave (TCPW) state) represents the spin texture. The spin rotates |W| times around the ring. This texture is similar to the polar core vortex (see Fig. 2 in Ref. [57]). We note that similar solutions for infinite systems were reported in Refs. [31, 58, 59].

The velocity dependences of the total energy for the TCPW states are shown by the solid red lines in Fig. 1. At \( v = -W_0v_0 \pm v_0/2 \), the TCPW states with |W| = 1 branches appear (see Eq. (11)). These points are the same as the points at which Landau instability occurs in the FPW branches due to the magnon mode, indicating that the instability of the magnon mode in the FPW state triggers the TCPW states, namely, the transition from the green line to the red line in Fig. 1. For |W| ≥ 3 branches, the TCPW branches continuously connect the PPW branches. These bifurcations occur when the right-hand side of the second inequality in Eq. (11) becomes positive.

To investigate the stability of the TCPW states, we solve the Bogoliubov equation for spin-1 systems [54, 55, 56]. The Bogoliubov equation can be derived by substituting \( \Psi(x, t) = e^{-i\mu t/\hbar} \left[ \Psi(x) + \tilde{u}_j(x)e^{-i\epsilon_j t/\hbar} - \tilde{v}_j(x)e^{i\epsilon_j t/\hbar} \right] \) into the time-dependent GPE and neglecting the higher order terms \( \tilde{u}_j(x) \equiv [u_{1,j}(x), u_{0,j}(x), u_{-1,j}(x)]^T \) and \( \tilde{v}_j(x) \equiv [v_{1,j}(x), v_{0,j}(x), v_{-1,j}(x)]^T \), where \( \epsilon_j \) is the excited energy of the j-th excited state. The velocity dependence of the excited energy is shown in Fig. 2.

From the results of the numerical diagonalization of the 6 × 6 Bogoliubov matrix [14], we find that the branch for |W| = 1 is a metastable state because the energy is higher than the FPW branches and all excitation energies \( \epsilon_i \) are real and positive. We also find that Landau or dynamical instability occurs in the branches for |W| > 1 (data are not shown).

We now discuss the NGMs of the TCPW states. The Hamiltonian of the system has \( U(1) \times SO(3) \) internal symmetry [60]; however, the TCPW state breaks this symmetry. Therefore, it can be expected that the system has a number of NGMs. The generators of these symmetries \{Q_i\} are given by \( f_x, f_y, f_z \), and \( I \), where \( I \) is the 3 × 3 unit matrix. According to Ref. [48], the zero-energy eigenstates of the Bogoliubov equation (zero-mode) originating from the spontaneous symmetry breaking are given by \( \Psi = [\Psi(x), \Psi^*(x)]^T \). In the present case, we can analytically obtain four zero-modes: \( \Psi_B \equiv [\Psi(x), \Psi^*(x)]^T \), \( \Psi_z \equiv [f_x \Psi(x), f_y \Psi^*(x)]^T \), \( \Psi_y \equiv [f_y \Psi(x), f_x \Psi^*(x)]^T \), and \( \Psi_z \equiv [f_x \Psi(x), f_y \Psi^*(x)]^T \), where the global phase transformation and the spin rotation around the \( x \), \( y \), and \( z \)-axes, respectively. From these zero-modes, we can construct the NGMs.

According to recent developments of the theory of NGMs for non-relativistic systems [61–63], the NGMs can be classified as two different types: type-I (unpaired) and type-II (paired). The type-I NGMs are given by \( [Q_i, \Psi(x), \Psi^*(x)]^T \). On the other hand, the type-II NGMs are given by a linear combination of two zero-modes, \( [Q_i, \Psi(x), Q^* \Psi^*(x)]^T \) and \( [Q_i, \Psi(x), Q^* \Psi^*(x)]^T \). The pair is formed when the commutator \( [Q_i, Q_k] \) has a non-zero expectation value. This is analogous to the canonical commutation relation of the coordinate and the momentum operators for the quantum mechanics: \( \{ \hat{x}, \hat{p} \} = i\hbar \). Two zero-modes describe one degree of freedom when the canonical pair is formed.

We can obtain the total number of NGMs and determine which zero-modes form the canonical pair in the following way [61, 63]. We calculate the Watanabe-Brauner (WB) matrix \( \rho \), which is defined as

\[ \rho_{ij} = -\frac{i}{\hbar} \int_{-L/2}^{+L/2} dx \sum_{m,n} \Psi_m^*(x)[Q_i, Q_j]_{mn} \Psi_n(x). \] (17)

Each component of the matrix is proportional to the expectation value of the commutator \( [Q_i, Q_j] \). The number of type-II NGMs \( n_{II} \) is given by \( n_{II} = \text{rank}(\rho)/2 \). The to-
tal number of NGMs $n_{\text{NGM}}$ is given by $n_{\text{NGM}} = n_{\text{BG}} - n_{\text{HI}},$ where $n_{\text{BG}}$ is the number of broken generators. In the present case, $\rho$ reduces to

$$
\rho = \frac{1}{L} \begin{bmatrix}
0 & +M_z & 0 & 0 \\
-M_z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
$$

(18)

where we used the fact that the magnetizations of the $x$ and $y$ components are zero, i.e., $M_x = \int_{-L/2}^{+L/2} dx F_x(x) = 0$ and $M_y = \int_{-L/2}^{+L/2} dx F_y(x) = 0$ (see Eqs. (13) and (16)). The total number of NGMs $n_{\text{NGM}}$ is thus given by

$$
n_{\text{NGM}} = n_{\text{BG}} - n_{\text{HI}} = 4 - \frac{1}{2} \text{rank}(\rho) = \begin{cases} 
3 & \text{for } M_z \neq 0 \\
4 & \text{for } M_z = 0
\end{cases},
$$

(19)

where $n_{\text{BG}} = 4$ in the present case. When $M_z \neq 0$, a $2 \times 2$ block of the WB matrix exists and $n_{\text{HI}} = \text{rank}(\rho)/2 = 1$ due to the commutation relation $[f_z, f_y] = i f_z$. Therefore, one type-II NGM exists, which is given by a linear combination of $x_z$ and $x_y$, in addition to two type-I NGMs, which are given by $x_B$ and $x_z$. On the other hand, when $M_z = 0$, the four zero-modes, namely, $x_B$, $x_x$, $x_y$, and $x_z$, are type-I NGMs because $n_{\text{HI}} = \text{rank}(\rho)/2 = 2$.

The above results imply that the number of NGMs changes at the point where $M_z = 0$ without changing the broken continuous symmetries. This is called the type-I-type-II transition [48]. From Eqs. (9) and (10), the transition point is given by $v = -W_0 v_0$ [64]. In fact, we can see this transition for the $(W, W_0) = (-1, 0)$ branch in Fig. 2 the first excited energy $\epsilon_1$ becomes zero at $v = 0$. This means that the additional zero-mode exists at $v = 0$. We note that similar behavior was found in the unstable parameter region in the H-phase of the spin-3 spinor BEC [48]. In our system, the type-I-type-II transition can occur in the metastable parameter regions such as the $|W| = 1$ branches.

Finally, we discuss the applicability of the above results to actual experiments. The TCPW states have different winding numbers, namely, different angular momenta for each component. Such states can be prepared through two-photon Raman transitions with circularly polarized Laguerre-Gaussian and standard Gaussian beams [23]. The type-I-type-II transition can be indirectly observed by utilizing the vanishment of the first excited energy, because the first excited state, which is a magnetic excitation, is converted to the zero-mode at $v = -W_0 v_0$, as described above. The magnon excited energy and its dispersion relation have been observed in a recent experiment using a magnon contrast interferometry technique [38]. By applying this technique to ring trap experiments, the type-I-type-II transition can be observed.

IV. SUMMARY

We investigated the metastability of the spin textures and excitations of ferromagnetic spin-1 BECs confined in a rotating ring trap using mean-field theory. We found analytical solutions of the GPE (TCPW solutions) that exhibit spin textures analogous to polar-core vortices. By numerically solving the Bogoliubov equation, it was shown how the TCPW states can become metastable states. On the basis of these analytical solutions, we determined the number of NGMs by using the WB matrix [61, 63]. We found that the number of type-II NGMs changes at $v = -W_0 v_0$ without changing the continuous symmetry of the order parameter.

In future work we will study the stability of the TCPW states in the presence of an external potential that breaks translational symmetry. An additional goal is to perform a many-body calculation for spinor Bose gases in a ring trap.

ACKNOWLEDGMENTS

We thank D. A. Takahashi, H. Saito, and Y. Kato for fruitful discussions. This work was supported by a Grant-in-Aid for Scientific Research on Innovative Areas “Fluctuation & Structure” (No. 25103007) from the Ministry of Education, Culture, Sports, Science, and Technology of Japan.
[9] C. Ryu, K. C. Henderson, and M. G. Boshier, New J. Phys. 16, 013046 (2014).
[10] S. Eckel, J. G. Lee, F. Jendrzejewski, N. Murray, C. W. Clark, C. J. Lobb, W. D. Phillips, M. Edwards, and G. K. Campbell, Nature 506, 200 (2014).
[11] F. Jendrzejewski, S. Eckel, N. Murray, C. Lanier, M. Edwards, C. J. Lobb, and G. K. Campbell, Phys. Rev. Lett. 113, 045305 (2014).
[12] S. Eckel, F. Jendrzejewski, A. Kumar, C. J. Lobb, and G. K. Campbell, arXiv:1406.1095 (2014).
[13] L. Cormann, L. Chomaz, T. Bienaimé, R. Desbuquois, C. Weintenberg, S. Nascimbène, J. Dalibard, and J. Beugnon, arXiv:1406.4073 (2014).
[14] Y. Kawaguchi and M. Ueda, Phys. Rep. 520, 253 (2012).
[15] D. M. Stamper-Kurn and M. Ueda, Rev. Mod. Phys. 85, 1191 (2013).
[16] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle, Phys. Rev. Lett. 80, 2027 (1998).
[17] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle, Nature 396, 345 (1998).
[18] U. Leonhardt and G. E. Volovik, JETP. Lett. 72, 411 (1999).
[19] U. Al Khawaja and H. Stoof, Nature (London) 411, 918 (2001).
[20] A. E. Leanhardt, Y. Shin, D. Kipelinson, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 90, 140403 (2003).
[21] J. Ruostekoski and J. R. Anglin, Phys. Rev. Lett. 91, 190402 (2003).
[22] Y. Kawaguchi and M. Ueda, Phys. Rev. Lett. 100, 180403 (2008).
[23] M. Kobayashi, Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. 103, 115301 (2009).
[24] L. S. Leslie, A. Hansen, K. C. Wright, B. M. Deutsch, and N. P. Bigelow, Phys. Rev. Lett. 103, 250401 (2009).
[25] K. C. Wright, L. S. Leslie, A. Hansen, and N. P. Bigelow, Phys. Rev. Lett. 102, 030405 (2009).
[26] J. Y. Choi, W. J. Kwon, and Y. I. Shin, Phys. Rev. Lett. 108, 035301 (2012).
[27] J. Y. Choi, W. J. Kwon, M. Lee, H. Jeong, K. An, and Y. I. Shin, New. J. Phys. 14, 053013 (2012).
[28] M. W. Ray, E. Ruokokoski, S. Kandel, M. Möttönen, and D. S. Hall, Nature 505, 657 (2014).
[29] L. E. Sadler, J. M. Highie, S. R. Leslie, M. Vengalattore, and D. M. Stamper-Kurn, Nature 434, 312 (2006).
[30] M. Vengalattore, S. R. Leslie, J. Guzman, and D. M. Stamper-Kurn, Phys. Rev. Lett. 100, 174043 (2008).
[31] R. W. Cherng, V. Gritsev, D. M. Stamper-Kurn, and E. Demler, Phys. Rev. Lett. 100, 180404 (2008).
[32] M. Vengalattore, J. Guzman, S. R. Leslie, F. Serwane, and D. M. Stamper-Kurn, Phys. Rev. A 81, 053612 (2010).
[33] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. 67, 1822 (1998).
[34] T. L. Ho, Phys. Rev. Lett. 81, 742 (1998).
[35] K. Murata, H. Saito, and M. Ueda, Phys. Rev. A 75, 013607 (2007).
[36] S. Uchino, M. Kobayashi, and M. Ueda, Phys. Rev. A 81, 063632 (2010).
[37] L. M. Symes, D. Baillie, and P. B. Blakie, Phys. Rev. A 89, 053628 (2014).
[38] G. E. Marti, A. MacRae, R. Olf, S. Lourette, F. Fang, and D. M. Stamper-Kurn, arXiv:1404.5631 (2014).
[39] J. Smynakis, S. Bargi, G. M. Kavoulakis, M. Magiropoulos, K. Kärkkäinen, and S. M. Reimann, Phys. Rev. Lett. 103, 100404 (2009).
[40] S. Bargi, F. Malet, G. M. Kavoulakis, and S. M. Reimann, Phys. Rev. A 82, 043631 (2010).
[41] K. Anoshkin, Z. Wu, and E. Zaremba, Phys. Rev. A 88, 013609 (2013).
[42] J. Smynakis, M. Magiropoulos, G. M. Kavoulakis, and A. D. Jackson, Phys. Rev. A 87, 013603 (2013).
[43] Z. Wu and E. Zaremba, Phys. Rev. A 88, 063640 (2013).
[44] H. Mäkelä and E. Lundh, Phys. Rev. A 88, 033622 (2013).
[45] A. I. Yakimenko, K. O. Isaieva, S. I. Vilchinskii, and M. Weyrauch, Phys. Rev. A, 88, 051602(R) (2013).
[46] E. J. Mueller, Phys. Rev. A 66, 063603 (2002).
[47] R. Kanamoto, L. D. Carr, and M. Ueda, Phys. Rev. A 79, 063616 (2009).
[48] D. A. Takahashi and M. Nitta, arXiv:1404.7696 (2014).
[49] The relation between $x$ and the azimuthal angle $\theta$ is given by $x = R\theta$, where $R$ is the radius of the ring. We assume that $R \gg \sqrt{x}$.
[50] E. H. Lieb, R. Seiringer, and J. Yngvason, Phys. Rev. B 66, 134529 (2002).
[51] This boundary condition can be derived as follows: First, we consider the laboratory frame; in this frame, the periodic boundary condition is imposed on the condensate wave function $\Psi_{Lab}(x_L, t)$ because there is a requirement that $\Psi_{Lab}(x_L, t)$ be single valued, where $x_L$ denotes the coordinate of the laboratory frame. The condensate wave function in the rotating frame with velocity $-v$ is given by $\Psi(x, t) = e^{i(Mx^2/2 + Mv^2t)/\hbar} \Psi_{Lab}(x_L, t)$, where $x_L = x_L + vt$. Therefore, $\Psi(x, t)$ satisfies the twisted periodic boundary condition.
[52] E. G. M. van Kempen, S. J. J. M. F. Kokkelmans, J. M. Heinitz, and B. J. Verhaar, Phys. Rev. Lett. 88, 093201 (2002).
[53] Although other types of solution such as soliton states [54], counterflow states [55], and half-quantum vortex states [56] can appear in Fig. 1, we do not consider them in this paper.
[54] Z. H. Zhang, C. Zhang, S. J. Yang, and S. Feng, J. Phys. B: At. Mol. Opt. Phys. 45, 215302 (2012).
[55] K. Fujimoto and M. Tsubota, Phys. Rev. A 85, 033642 (2012).
[56] S. Hoshi and H. Saito, Phys. Rev. A 78, 033618 (2008).
[57] T. Isoshima, K. Machida, and T. Ohmi, J. Phys. Soc. Jpn. 70, 1604 (2001).
[58] A. S. Rodrigues, P. G. Kevrekidis, R. Carretero-González, D. J. Frantzeskakis, P. Schmelcher, T. J. Alexander, and Yu. S. Kivshar, Phys. Rev. A, 79, 043603 (2009).
[59] R. S. Tasgal and Y. B. Band, Phys. Rev. A 87, 023626 (2013).
[60] The TCPW states spontaneously break the translational symmetry because they exhibit spin textures. However, an NGM originating from the translational symmetry does not exist because the translation of the TCPW can be written as $\Psi(x + x_0) = e^{iM(x + x_0\omega_0)/\hbar} e^{-iMv\omega_0 t/2\hbar} \Psi(x)$, where $x_0$ is a constant. This means that the translation can be represented by a combination of the $U(1)$ gauge transformation and the spin rotation around z-axis.
[61] H. Watanabe and T. Brauner, Phys. Rev. D 84, 125013 (2011).
[62] H. Watanabe and H. Murayama, Phys. Rev. Lett. 108, 251602 (2012).

[63] Y. Hidaka, Phys. Rev. Lett. 110, 091601 (2013).

[64] Although $M_z/N = 0$ also holds at $(v/v_0 + W_0)^2 = W^2/4 - |c_1|n_0/Mv_0^3$, we do not consider such a case, because this point is a transition point from a TCPW state to a PPW state.