POINT PROCESSES AND
THE INFINITE SYMMETRIC GROUP
PART III: FERMION POINT PROCESSES

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ABSTRACT. In Part I (G. Olshanski) and Part II (A. Borodin) we developed an
approach to certain probability distributions on the Thoma simplex. The latter has
infinite dimension and is a kind of dual object for the infinite symmetric group.
Our approach is based on studying the correlation functions of certain related point
stochastic processes.

In the present paper we consider the so-called tail point processes which describe
the limit behavior of the Thoma parameters (coordinates on the Thoma simplex)
with large numbers. The tail processes turn out to be stationary processes on the
real line. Their correlation functions have determinantal form with a kernel which
generalizes the well-known sine kernel arising in random matrix theory. Our second
result is a law of large numbers for the Thoma parameters. We also produce Sturm–
Liouville operators commuting with the Whittaker kernel introduced in Part II and
with the generalized sine kernel.

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The present note is a continuation of [O] (Part I) and [B1] (Part II). Our aim
here is to draw some conclusions from the computations of Part II.

In Part I we started the study of a family \( \{ P_{zz'} \} \) of probability Borel measures
living on an infinite-dimensional simplex \( \Omega \), the Thoma simplex. Recall that this
family consists of two parts: the principal series and the complementary series, each
of which is indexed by two real parameters. For the principal series, \( z \) is an arbitrary
complex number distinct from \( 0, \pm 1, \pm 2, \ldots \), and \( z' = \bar{z} \). For the complementary
series, \( z \) and \( z' \) are real numbers which are both contained in a unit interval with

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integer ends. When $z$ is a noninteger real number and $z' = z$, the corresponding measure belongs to the intersection of the both series.

Note that $z - z'$ ranges over the imaginary axis plus the open interval $(-1, 1)$ — a picture that immediately evokes the principal and complementary series for $SL(2, \mathbb{R})$.

The measures $P_{zz'}$ originated from the work [KOV]: they govern the decomposition of certain reducible representations which seem to be right analogs of the regular representation for the infinite symmetric group. It is known that, excepting the symmetry relation $P_{zz'} = P_{z'z}$, the measures $P_{zz'}$ are pairwise disjoint.

Recall that the points of the Thoma simplex $\Omega$ are the double sequences $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0), \beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0)$ such that $\sum (\alpha_j + \beta_j) \leq 1$. The basic idea of Part I was to interpret the measures $P_{zz'}$ as point processes (denoted as $\mathcal{P}_{zz'}$) on the punctured interval $I = [-1, 1] \setminus \{0\}$, the random configuration being of the form $(-\beta_1, -\beta_2, \ldots, \alpha_2, \alpha_1)$, where only nonzero $\alpha_j$'s and $\beta_j$'s are considered and the points are accumulated near 0. A method of calculating the correlation functions $\rho^{zz'}_n$, $n = 1, 2, \ldots$, of the processes $\mathcal{P}_{zz'}$ was developed. In Part I, we calculated the first correlation function $\rho^{zz'}_1$, and in Part II — the higher correlation functions $\rho^{zz'}_n$. More advanced results were obtained for the process $\mathcal{P}^{zz'}_+$, the restriction of $\mathcal{P}_{zz'}$ to $(0, 1] \subset I$. The process $\mathcal{P}^{zz'}_+$ reflects the behavior of the Thoma parameters $\alpha_j$; the study of the $\beta_j$'s is reduced to that of the $\alpha_j$'s simply by change of the sign for $z$ and $z'$.

The initial definition of the processes $\mathcal{P}_{zz'}$ is rather indirect and we know no explicit probabilistic mechanism generating them. From the beginning it was unclear what known processes they resemble. Now, the knowledge of the correlation functions makes it possible to conclude that $\mathcal{P}_{zz'}$ (or at least certain derived processes) are similar to the point processes arising in the scaling limit of certain random matrix ensembles. The basic common feature is that the correlation functions are given by determinantal formulas involving a kernel; such processes are called fermion point processes after [Ma1, Ma2, DV]. Note that determinantal correlation functions also appear in certain models of mathematical physics [KBI].

The results of Part II lead to interesting kernels: the Whittaker kernel (see sections 1–2), the sin/sh kernel, the sh/sh kernels and their degenerations (see sections 3–4). The Whittaker kernel seems to be a new example; the sin/sh kernel already appeared in works of mathematical physicists, see [BCM, MCIN]. We think that the connection of our problem with the random matrix theory is interesting and promising.

The note is organized as follows. In section 1, we briefly review some general facts about the fermion processes. In section 2, we discuss the “lifting” of the process $\mathcal{P}^{zz'}_+$, which leads to the Whittaker kernel. As an application, we calculate the mean value for $\sum \alpha_i$ and $\sum \beta_i$. In section 3, we introduce the “tail process” for $\mathcal{P}^{zz'}_+$; it turns out to be a stationary fermion process on $\mathbb{R}$ depending on $z, z'$. In this way, we get a two–parametric family of kernels generalizing the sine kernel; they are discussed in section 4. In section 5, we prove that

$$\lim_{j \to \infty} \alpha^{1/j}_j = \lim_{j \to \infty} \beta^{1/j}_j = e^{-1/C},$$

with probability 1, where $C > 0$ is a certain (explicitly determined) constant depending on $z, z'$. Roughly speaking, this means that the Thoma parameters decay
with the rate of a geometric progression. In section 6, we produce the Sturm–
Liouville differential operators which commute with the integral operators given
by the Whittaker kernel and the stationary kernels of section 4. In section 7, we
compare the processes \( P_{zz}^+ \) with the Poisson–Dirichlet processes \( \mathcal{PD}(t) \) [Ki].

1. The fermion point processes. Let \( \mathcal{P} \) be a point process on a space \( X \) and
\( \rho_n(x_1, \ldots, x_n) \) denote its correlation functions relative to a reference measure \( \mu \) on
\( X, \ n = 1, 2, \ldots \). By a test set \( A \subset X \) we mean a Borel set such that the first
correlation function \( \rho_1 \) is integrable on \( A \); this means that \( A \) intersects the random
configuration at a finite number of points with probability 1.

In many concrete situations, it turns out that the functions \( \rho_n \) are given by a
determinantal formula,

\[ \rho_n(x_1, \ldots, x_n) = \det [K(x_a, x_b)]_{a,b=1}^n, \]

where \( K(x, y) \) is a kernel on \( X \times X \) not depending on \( n \).

The processes with determinantal correlation functions seem to be important
enough to deserve a special name. Though key examples of such processes were al-
ready considered in the 1st edition (1967) of Mehta’s book on random matrices [Me]
and some earlier papers, the first (to our knowledge) general discussion appeared
in Macchi’s paper [Ma1]. In her works and in the book [DVJ] these processes are
called the fermion point processes, and we shall adopt this terminology.

Let us list some general properties of the fermion processes (see [Ma1, Ma2] and
[DVJ], Example 5.4(c) and Exercises 5.4.7–9).

If the reference measure \( \mu \) is replaced by an equivalent one,
\( \mu \mapsto f\mu \), where \( f \) is
a strictly positive function, then the kernel must be divided by
\( \sqrt{f(x)f(y)} \).

Note that a determinantal correlation function vanishes when some of the argu-
ments coincide; this means that the points of the random configuration have some
repulsion properties.

Given a kernel \( K(x, y) \), the following conditions ensure the existence of a fermio
process, see [Ma1]:

(*) The functions \( \rho_n \) as defined above are nonnegative.

(**) For a test set \( A \subset X \), let \( K_A \) denote the integral operator whose kernel
\( K_A(x, y) \) is defined as the restriction of \( K(x, y) \) to \( A \times A \). It is required that
the norm of \( K_A \) in \( L^2(A, \mu) \) be strictly less than 1 for any test set \( A \).

In many cases the kernel \( K(x, y) \) turns out to be symmetric, so that \( K_A \) is a
nonnegative self–adjoint operator in the Hilbert space \( L^2(A, \mu) \). But there also exist
interesting examples of non symmetric kernels (see, e.g., [B2]). Even for symmetric
kernels, a direct verification of the above sufficient conditions can be difficult.\(^1\) But
\( K(x, y) \) often arises as a limit of kernels which certainly satisfy (*) and (**) . In
such a situation we can conclude at least that \( K(x, y) \) satisfies (*) and \( \| K_A \| \leq 1 \),
which is a weak form of (**).

Let us denote by \( \pi_n^{(A)}(x_1, \ldots, x_n) \) the finite–dimensional distribution functions
of \( \mathcal{P} \). Here \( n = 1, 2, \ldots, A \subset X \) is a test set and, by definition,

\[ \pi_n^{(A)}(x_1, \ldots, x_n) \mu(dx_1) \ldots \mu(dx_n) = \text{Prob} \{ \text{exactly } n \text{ points in } A, \]

one point located in each of the infinitesimal regions \( dx_i \}. \]

\(^1\)It greatly simplifies when \( K(x, y) \) is a translation invariant kernel, say, on the real axis, see
section 4 below.
By the well–known inclusion–exclusion principle, the $\pi$ functions can be expressed through the $\rho$ functions as follows

$$
\pi^{(A)}_n(x_1, \ldots, x_n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{A^k} \rho_{n+k}(x_1, \ldots, x_n, y_1, \ldots, y_k) \mu(dy_1) \cdots \mu(dy_k),
$$

see [DVJ], section 5.4. For the fermion processes this relation takes the following form

$$
\pi^{(A)}_n(x_1, \ldots, x_n) = \text{Det}(1 - K_A) \det[L_A(x_a, x_b)],
$$

where $K_A$ is the integral operator whose kernel $K_A(x, y)$ is obtained by restricting the kernel $K(x, y)$ to $A \times A$, $\text{Det}$ is the Fredholm determinant $\text{Det}(1 - \lambda K_A)$ evaluated at $\lambda = 1$, and $L_A(x, y)$ is the kernel of the operator $L_A := K_A(1 - K_A)^{-1}$. In particular, the probability $\pi^{(A)}_0$ that $A$ is empty of points of the random configuration equals $\text{Det}(1 - K_A)$.

For a fermion process on the line, there is a relation between the Fredholm determinant and the probability distribution of the spacings, see, e.g., [TW4], first formula after (5.35). For other relations involving the Fredholm determinant, see [Me, TW1].

Now let us give examples of the kernels $K(x, y)$ originated from concrete problems.

The most known is the sine kernel

$$
K(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)}, \quad x, y \in \mathbb{R}.
$$

It appears in scaling limit of various random matrix ensembles “in the bulk of the spectrum”, see [Me, NW1, TW1]. Since the kernel is translation invariant, the corresponding process is stationary. The sine kernel can be included into a more general family of translation invariant kernel, see section 4 below.\(^2\)

Taking scaling limit “at the edge of the spectrum” leads to other kernels: the Airy kernel

$$
K(x, y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x - y}, \quad x, y \in \mathbb{R}
$$

(where $Ai(\cdot)$ is the Airy function) and the Bessel kernel

$$
K(x, y) = \frac{J_\alpha(x^{\frac{1}{2}})y^{\frac{1}{2}}J'_\alpha(y^{\frac{1}{2}}) - J_\alpha(y^{\frac{1}{2}})x^{\frac{1}{2}}J'_\alpha(x^{\frac{1}{2}})}{2(x - y)}, \quad x, y > 0
$$

(where $J_\alpha(\cdot)$ is the Bessel function of order $\alpha > -1$), see [F, NW2, TW2, TW3].

The results of Part II, section 3, lead to one more kernel expressed through special functions, the Whittaker kernel

$$
K(x, y) = \text{const} \cdot (xy)^{-\frac{1}{2}} \frac{W_{\kappa, \mu}(x)W_{\kappa-1, \mu}(y) - W_{\kappa, \mu}(y)W_{\kappa-1, \mu}(x)}{x - y}, \quad x, y > 0
$$

\(^2\)About the meaning of the factor $\pi$, see the first comment to Proposition 4.2.
(where $W_{\kappa,\mu}(\cdot)$ is the Whittaker function, a version of the confluent hypergeometric function, see [E1]). See also section 2. We did not encounter this kernel in literature, maybe this is a new example.

Note that for the process with the sine kernel the points are accumulated near $\pm \infty$; for the Airy kernel — near $-\infty$; for the Bessel kernel — near $+\infty$; and for the Whittaker kernel — near 0.

An important problem is studying the Fredholm determinant $\text{Det}(1 - \lambda K_A)$ (the test set $A$ being an interval or a finite union of intervals), in particular, its asymptotics as an end of an interval tends to an accumulation point. See [TW1–5].

2. **Lifting and the Whittaker kernel.** Given a point process $Q$ on $(0, 1]$ and a probability distribution $\sigma$ on $(0, +\infty)$, we can construct a new point process $\tilde{Q}$, living on $(0, +\infty)$, as follows. We take the random configuration $\xi$ corresponding to $Q$ and multiply it by an independent scale factor $s$ distributed according to $\sigma$.

Choose as $\sigma$ the gamma distribution with density

$$\frac{\sigma(ds)}{ds} = s^{t-1}e^{-s}\Gamma(t), \quad t > 0.$$ 

Following Part II, section 3, we shall call $\tilde{Q}$ the **lifting** of $Q$ with parameter $t$.

Let us denote by $P_{zz'}^+$ the restriction of the process $P_{zz'}$ to $(0, 1] \subset I$. The process $P_{zz'}^+$, governs the random behavior of the Thoma parameters $\alpha_1, \alpha_2, \ldots$ and neglects the parameters $\beta_1, \beta_2, \ldots$. On the contrary, to focus on the beta part of the Thoma parameters it suffices to replace $z, z'$ by $-z, -z'$.

In Part II, Theorem 3.3.4, we proved the following result:

**Theorem 2.1.** Application of lifting with parameter $t = zz'$ to the process $Q = P_{zz'}^+$ gives a fermion process on $(0, +\infty)$. Its kernel $K(x, y)$ is the Whittaker kernel

$$\frac{(xy)^{-1/2}}{\Gamma(z)\Gamma(z')} \cdot \frac{W_{zz'+1, zz'}(x)W_{zz'+1, zz'}(y) - W_{zz'+1, zz'-1}(x)W_{zz'-1, zz'}(y)}{x - y}.$$ 

See section 6.1 in [E1] for the definition of the Whittaker function $W_{\kappa,\mu}$.

**Comments.**

1) Comparison with Theorems 2.2.1 and 2.4.1 from Part II shows that lifting greatly simplifies the structure of the expressions for the correlation functions. For instance, the dimension of integrals involved in the description of the unlifted correlation functions $\rho_n^{(zz')}$, grows with $n$ while for the lifted correlation functions $\tilde{\rho}_n^{(zz')}$ we need only a one–dimensional integral implicit in the definition of the Whittaker function.

2) By Proposition 3.1.1 (Part II), the passage from $\rho^{(zz')}$ to $\tilde{\rho}_n^{(zz')}$ is effected by the integral transform $L^t$ defined in Proposition 3.2.1. This transform is readily reduced to one–dimensional Laplace transform and so, in principle, can be inverted via the Laplace inversion formula. This implies that the lifted process retains the whole information about the initial process.

3) Let $(\alpha_1 > \alpha_2 > \ldots)$ be the random configuration of the process $P_{zz'}^+$ and $(\tilde{\alpha}_1 > \tilde{\alpha}_2 > \ldots)$ be the similar object for the lifted process. The distribution functions of $\alpha_1$ and $\tilde{\alpha}_1$ are also related by the transform $L^t$ and the same is true for joint distributions of any finite number of coordinates.


4) Note that the distribution of $\tilde{\alpha}_1$ is given by the Fredholm determinant:

$$\text{Prob}\{\tilde{\alpha}_1 < \tau\} = \text{Det}(1 - K_{(\tau,+\infty)}), \quad \tau > 0,$$

where $K_{(\tau,+\infty)}(x,y)$ is the restriction of the Whittaker kernel to $(\tau,+\infty)$.

5) The transform $\mathcal{L}^t$ has a simple meaning in the language of moments. Its application, say, to a one-dimensional distribution results in multiplying the $m$th moment by $(t)^m$, $m = 1,2,\ldots$. Thus, if we calculate numerically a few moments of the random variable $\tilde{\alpha}_1$ then we immediately get the corresponding moments of $\alpha_1$.

6) The fact that application of lifting can simplify a point process is also demonstrated on the example of the Poisson–Dirichlet process whose lifting is simply a Poisson process (Part II, Proposition 3.1.2). Note that from this result one can very easily get Griffiths’ formulas [G1, G2] for the mean values of the random coordinates $x_1 > x_2 > \ldots$ distributed according to the Poisson–Dirichlet law.

As a simple application of Theorem 2.1 we shall prove the following result.

**Proposition 2.2.** Consider the probability space $(\Omega, P_{zz'})$ where $\Omega$ is the Thoma simplex and $P_{zz'}$ is one of the measures of the principal or complementary series (see Part I). Let us view the Thoma parameters as random variables defined on this probability space and let the symbol $\mathbb{E}$ mean expectation. Then we have

$$\mathbb{E}\left(\sum_{i=1}^{\infty} \alpha_i\right) = \frac{\sin \pi z \cdot \sin \pi z'}{\pi \sin \pi (z - z')} \left[ \frac{(z-z')}{2zz'} + \psi(-z') - \psi(-z) \right], \quad (2.1)$$

where $\psi(a) = \Gamma'(a)/\Gamma(a)$.

Similarly,

$$\mathbb{E}\left(\sum_{i=1}^{\infty} \beta_i\right) = \frac{\sin \pi z \cdot \sin \pi z'}{\pi \sin \pi (z - z')} \left[ \frac{(z'-z)}{2zz'} + \psi(-z) - \psi(-z') \right]. \quad (2.2)$$

**Proof.** By Proposition 4.6 of Part I, the symmetry map $(\alpha,\beta) \mapsto (\beta,\alpha)$ of the Thoma simplex takes the measure $P_{zz'}$ to the measure $P_{-z,-z'}$. On the other hand, the right-hand sides of the formulas (2.1), (2.2) differ exactly by change of sign in $z, z'$. So, the both formulas are equivalent, and it suffices to check one of them, say, (2.1).

Let $\rho_1(x)$ denote the density function of the process $P_{zz'}$. By the very definition of the density function (see Part I, §4),

$$\mathbb{E}\left(\sum_{i=1}^{\infty} \alpha_i\right) = \int_0^1 x \rho_1(x) dx. \quad (2.3)$$

Since we know various expressions for the density function (see Part I, Theorem 5.10, Theorem 5.12; Part II, Corollary 2.4.2), we could try to employ one of them to calculate the integral (2.3) explicitly. However, this does not seem to be easy, so we have preferred to use a roundabout way – reduction to the lifted process.
Let $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots$ stand for the “lifted” random variables $\alpha_1, \alpha_2, \ldots$ and $\tilde{\rho}_1$ be the density function of the lifted process (see section 3.1 in Part II). Similarly to (2.3) we have
\[
E \left( \sum_{i=1}^{\infty} \tilde{\alpha}_i \right) = \int_0^\infty x \tilde{\rho}_1(x) dx. \tag{2.4}
\]

On the other hand, it follows from the definition of lifting (see also Comment 5 above) that
\[
\int_0^1 x \rho_1(x) dx = \frac{1}{t} \int_0^\infty x \tilde{\rho}_1(x) dx,
\]
where, as usual, $t = zz'$. Let $K(x,y)$ be the Whittaker kernel (Theorem 2.1). Then
\[
\tilde{\rho}_1(x) = K(x,x), \quad x > 0.
\]

Let us abbreviate
\[
\kappa = \frac{z + z' + 1}{2}, \quad \mu = \frac{z - z'}{2}, \tag{2.5}
\]
and assume $x > 0$. Then, applying the l'Hospital rule, we get
\[
x \tilde{\rho}_1(x) = \frac{1}{\Gamma(z)\Gamma(z')} (W'_{\kappa,\mu}(x)W_{\kappa-1,\mu}(x) - W_{\kappa,\mu}(x)W'_{\kappa-1,\mu}(x)). \tag{2.6}
\]

Let us employ the relation
\[
W'_{\kappa,\mu} = \left( -\frac{1}{2} + \frac{\kappa}{x} \right) W_{\kappa,\mu} + \frac{(\frac{1}{2} - \kappa + \mu)(\frac{1}{2} - \kappa - \mu)}{x} W_{\kappa-1,\mu},
\]
which follows from formulas 6.9 (2) and 6.6 (1) in [E1]. Then we get from (2.5) and (2.6)
\[
\frac{1}{t} x \tilde{\rho}_1(x) = \frac{W_{\kappa,\mu}(x)W_{\kappa-1,\mu}(x) + zz'W^2_{\kappa-1,\mu}(x) - (z - 1)(z' - 1)W_{\kappa,\mu}(x)W_{\kappa-2,\mu}(x)}{zz' \cdot \Gamma(z)\Gamma(z') \cdot x}.
\]

The integral of the above expression over $(0, +\infty)$ can be found by making use of the table integrals
\[
\int_0^\infty x^{-1} W_{\kappa_1,\mu}(x)W_{\kappa_2,\mu}(x) dx = \frac{\pi}{(\kappa_1 - \kappa_2) \sin(2\pi \mu)} \times \left[ \frac{1}{\Gamma\left(\frac{1}{2} - \kappa_1 + \mu\right)\Gamma\left(\frac{1}{2} - \kappa_2 - \mu\right)} - \frac{1}{\Gamma\left(\frac{1}{2} - \kappa_1 - \mu\right)\Gamma\left(\frac{1}{2} - \kappa_2 + \mu\right)} \right]
\]
and
\[
\int_0^\infty x^{-1} W^2_{\kappa,\mu}(x) dx = \frac{\pi}{\sin 2\pi \mu} \frac{\psi\left(\frac{1}{2} - \kappa + \mu\right) - \psi\left(\frac{1}{2} - \kappa - \mu\right)}{\Gamma\left(\frac{1}{2} - \kappa + \mu\right)\Gamma\left(\frac{1}{2} - \kappa - \mu\right)},
\]
which can be found in [PBM], section 2.19.23, formulas 3 and 4. Note that these two integrals are convergent provided that $|\Re \mu| < \frac{1}{2}$, which indeed holds in our case, because of the restrictions imposed on the parameters $z, z'$, see §2 in Part I.

Finally, we also need the relation
\[
\psi(a + 1) = \psi(a) + \frac{1}{a},
\]
see [E1], 1.7 (8).

Then, after elementary calculations we get the desired formula (2.1).
Remark 2.3. Assume that \( z' = -z \); according to the assumptions on the parameters \( z, z' \), this means that \( z = -z' \) is purely imaginary. Then the measure \( P_{zz'} \) is stable under the symmetry map transposing the \( \alpha \)'s and the \( \beta \)'s, so that the expressions (2.1) and (2.2) must be equal to 1/2. One can check that this is indeed the case by making use of the relation

\[
\psi(a) - \psi(-a) = -\pi \cotg(\pi a) - \frac{1}{a},
\]

which can be found in [E1], 1.7 (11).

Further, we know that \( \sum (\alpha_i + \beta_i) = 1 \) almost surely with respect to \( P_{zz'} \) (Part I, Theorem 6.1). It follows that the sum of the expressions (2.1) and (2.2) must be identically equal to 1. Again, this can be readily verified by making use of the above relation.

Remark 2.4. Note that for certain special values of the parameters \( z, z' \) the Whittaker kernel degenerates to the Christoffel–Darboux kernel for the Laguerre polynomials.

Specifically, let \( L^2_{N} \) stand for the \( N \)th Laguerre polynomial with the weight function \( x^2 e^{-x} \) on \( \mathbb{R}_+ \), where \( 2\mu > -1 \); the normalization is that of [E1]. We have

\[
x^{-\frac{1}{2}} W_{\mu+N+\frac{1}{2},\mu}(x) = (-1)^N N! x^\mu e^{-\frac{x}{2}} L^2_{N}(x),
\]

see, e.g. [E1], 6.9 (36).

Assume \( N - 1 < z, z' < N \), where \( N = 1, 2, \ldots \), and let \( z' \) tends to \( N \) while \( z \) remains fixed. Denote \( 2\mu = z - N \); then \( -1 < 2\mu < 0 \). Using the above formula we readily get that the limit of the Whittaker kernel as \( z' \to N \) is equal to

\[
\frac{N!}{\Gamma(N+2\mu)} (xy)^\mu e^{\frac{x+y}{2}} L^2_{N-1}(x) L^2_{N}(y) - L^2_{N-1}(y) L^2_{N}(x)
\]

\[
x = y \sum_{i=0}^{N-1} \frac{L^2_{i}(x) L^2_{\mu}(y)}{\int (L^2_{i}(x))^2 x e^{-x} dx}.
\]

The latter expression coincides with the kernel of the projection in the Hilbert space \( L^2(R_+, dx) \) on the linear span of the functions \( x^{i+\mu} e^{-\frac{x}{2}} \), where \( i = 0, \ldots, N-1 \); this is exactly the kernel associated with the “\( N \)-point Laguerre polynomial ensemble”, see [FK], [Br], [NW1].

Finally, note that the restriction \( \mu < 0 \), which comes from the assumption \( N-1 < z, z' < N \), is inessential, because there exists a natural “degenerate series” of the measures \( P_{zz'} \) with the parameters \( z' = N, z > N - 1 \).

3. The tail process. Let \( Q \) be a point process on \( (0, 1] \) or on \( (0, +\infty) \) and \( \rho_1(x) \) be its first correlation function. We assume that \( \rho_1 \) is integrable on the right of any \( \varepsilon > 0 \) and nonintegrable on \( (0, \varepsilon) \), so that the points are accumulated to 0. Consider the mapping

\[
(0, 1] \to [0, +\infty), \quad x \mapsto \xi := \int_x^1 \rho_1(y) dy,
\]
and let \( \hat{Q} \) be the image of the process \( Q \) (or rather of its restriction to \((0, 1]\), in case \( Q \) is defined on the whole ray) under that mapping. Then \( \hat{Q} \) is a point process on \([0, +\infty)\) and its first correlation measure coincides with Lebesgue measure.

Further, for any \( \tau \geq 0 \), let \( \hat{Q}_\tau \) be the process on \([-\tau, +\infty)\) obtained from \( \hat{Q} \) by the shift \( \xi \mapsto \xi - \tau \). We let \( \tau \rightarrow +\infty \) and assume that there exists a point process \( \hat{Q}_\infty \) on the whole axis \( \mathbb{R} \) such that the limit

\[
\lim_{\tau \rightarrow +\infty} \hat{Q}_\tau = \hat{Q}_\infty
\]

exists in a reasonable sense. Then we shall say that \( \hat{Q}_\infty \) is the *tail process* for \( Q \).

Of course, the exact meaning of the limit above has to be precised. We shall be content with the following type of convergence: for any \( n \), the \( n \)th correlation function of \( \hat{Q}_\tau \) tends, as \( \tau \rightarrow +\infty \), to the \( n \)th correlation function of \( \hat{Q}_\infty \), uniformly on compact sets in \( \mathbb{R} \). Perhaps, the definition can be elaborated. But anyway, the idea is clear: we restrict the initial process to a small interval \((0, \varepsilon)\), next rescale it to make the density function constant, and then look at the asymptotics as \( \varepsilon \rightarrow 0 \).

In the examples considered below the correlation functions of \( Q \) can be represented in the form

\[
\rho_n(x_1, \ldots, x_n) = \frac{C^n}{x_1 \cdots x_n} (f_n(x_1, \ldots, x_n) + o(1)), \quad x_1, \ldots, x_n > 0,
\]

where \( C > 0 \) is a constant not depending on \( n \), \( f_n(x_1, \ldots, x_n) \) is a continuous homogeneous function,

\[
f_n(rx_1, \ldots, rx_n) = f_n(x_1, \ldots, x_n) \quad r > 0,
\]

and the rest term, denoted as \( o(1) \), tends to zero as \( \max\{x_1, \ldots, x_n\} \rightarrow 0 \). Note that the function \( f_1 \) should be a constant, and we choose \( C \) in such a way that \( f_1(\cdot) \equiv 1 \).

In such a situation, we make a change of variables \( x_i \mapsto \xi_i \), where \( x_i = e^{-\xi_i/C} \). In the new variables, the correlation functions take the form

\[
\rho'_n(\xi_1, \ldots, \xi_n) = g_n(\xi_1, \ldots, \xi_n) + o(1), \quad \xi_1, \ldots, \xi_n \in \mathbb{R},
\]

where the rest term \( o(1) \) tends to zero as \( \min\{\xi_1, \ldots, \xi_n\} \rightarrow +\infty \) and

\[
g_n(\xi_1, \ldots, \xi_n) := f_n(e^{-\xi_1/C}, \ldots, e^{-\xi_n/C})
\]

is a translation invariant function. Consequently, the desired “tail” correlation functions have the form

\[
\hat{\rho}(\xi_1, \ldots, \xi_n) = g_n(\xi_1, \ldots, \xi_n).
\]

As illustration, examine first the Poisson–Dirichlet process.
Proposition 3.1. The tail process for the Poisson–Dirichlet process $PD(t)$ is the Poisson process on $\mathbb{R}$ with constant density 1.

Proof. Recall that the correlation functions of $PD(t)$ are given by Watterson’s formula
\[
\rho_n(x_1, \ldots, x_n) = \frac{t^n(1 - x_1 - \cdots - x_n)^{t-1}}{x_1 \cdots x_n},
\]
see [W] and Part I, Corollary 7.4. In particular, the first correlation function is
\[
\rho_1(x) = \frac{t(1 - x)^{t-1}}{x}.
\]
These correlation functions fit into the above scheme with $C = t$ and all the functions $f_n$ identically equal to 1. It follows that the “tail” correlation functions $\hat{\rho}_n$ are identically equal to 1, which corresponds to the standard Poisson process. \qed

As in section 2 above, let $P_{zz'}^+$ denote the restriction of the process $P_{zz'}$ to $(0, 1] \subset I$.

Theorem 3.2. Take as $Q$ the process $P_{zz'}^+$ or its lifting with parameter $t = zz'$. In both cases the tail process $\tilde{Q}$ is a fermion process on $(0, +\infty)$ with the same translation invariant kernel $K(\xi, \eta)$, which has the following form.

- For the principal series, when $z' = \bar{z}$ and $z$ is not real,
  \[
  K(\xi, \eta) = \frac{B \sin A(\xi - \eta)}{\text{sh} B(\xi - \eta)},
  \]
  where
  \[
  B = \frac{\pi \sin \pi(z - z')}{2(z - z') \sin \pi z \cdot \sin \pi z'} > 0, \quad A = \pm i(z - z')B.
  \]

- For the supplementary series, when $m < z, z' < m + 1$ for a certain $m \in \mathbb{Z}$ and $z \neq z'$,
  \[
  K(\xi, \eta) = \frac{B \text{sh} A(\xi - \eta)}{\text{sh} B(\xi - \eta)},
  \]
  where $B$ is given by the same formula and $A = \pm (z - z')B$.

- Finally, on the intersection of the both series, when $z = z' \in \mathbb{R} \setminus \mathbb{Z}$, the kernel is given by the limit expression
  \[
  K(\xi, \eta) = \frac{B(\xi - \eta)}{\text{sh} B(\xi - \eta)},
  \]
  where
  \[
  B = \frac{\pi^2}{2 \sin^2 \pi z}.
  \]

Proof. For the lifted process, the behavior of the correlation functions near zero is given by the asymptotics of the Whittaker kernel as described in Part II, Theorem 4.1.1. For the process $P_{zz'}^+$, itself this requires knowledge of the asymptotics of the multivariate Lauricella functions of type $B$; the final result is described in Part II, Theorem 4.3.1. According to these theorems, the correlation functions, both in
lifted and non lifted case, fit into the above scheme with the same constant \( C \) and the same functions \( f_n \).

Specifically, we have

\[
C = \begin{cases} 
\frac{(z - z') \sin \pi z \cdot \sin \pi z'}{\pi \sin \pi(z - z')}, & z' \neq z \\
\frac{\sin^2 \pi z}{\pi^2}, & z' = z \in \mathbb{R} \setminus \mathbb{Z}
\end{cases}
\]

and

\[
f_n(x_1, \ldots, x_n) = \det[K'(x_i, x_j)],
\]

where

\[
K'(x, y) = \begin{cases} 
\frac{1}{z - z'} \cdot \frac{(x/y)^{z - z'} - (x/y)^{z' - z}}{(x/y)^{z - z'} - (x/y)^{z' - z}}, & z' \neq z \\
\frac{\ln x - \ln y}{(x/y)^{z/2} - (x/y)^{-z/2}}, & z = z' \in \mathbb{R} \setminus \mathbb{Z}
\end{cases}
\]

Passing from the functions \( f_n(x_1, \ldots, x_n) \) to the functions \( g_n(\xi_1, \ldots, \xi_n) \) as described above we get the expressions indicated in the statement of the theorem. \( \square \)

**Remark 3.3.** Let us notice two remarkable features of the expression for the kernel in Theorem 3.2.

- First, the kernel does not change under the transform \((z, z') \mapsto (-z, -z')\). This implies that the tail properties of the \( \alpha_j \)'s are the same as that of the \( \beta_j \)'s.
- Second, the kernel does not change under the shift \((z, z') \mapsto (z + 1, z' + 1)\). This means a quite surprising periodicity of the tail process with respect to the parameters \( z, z' \). One can ask whether this phenomenon is somehow related to degeneration of \( P_{zz'} \) at integer values [KOV].

**4. The sin/sh and sh/sh kernels.** Here we shall examine in more detail the kernels that appeared in Theorem 3.2. These are the stationary kernels of the form

\[
K(x, y) = \frac{B \sin A(x-y)}{A \sin B(x-y)}
\]

where \( B \) is real and \( A \) is either real or pure imaginary (here and below we use the letters \( x, y \) instead of \( \xi, \eta \)). There are two main types and two limit types:

1) The “sin/sh kernel”,

\[
K(x, y) = \frac{B \sin A(x-y)}{A \sin B(x-y)}, \quad B > 0, \quad A \in \mathbb{R}, \quad A \neq 0.
\]

2) The “sh/sh kernel”,

\[
K(x, y) = \frac{B \sinh A(x-y)}{A \sinh B(x-y)}, \quad B > 0, \quad A \in \mathbb{R}, \quad A \neq 0.
\]

3) The limit case \( B = 0 \):

\[
K(x, y) = \frac{\sinh A(x-y)}{A(x-y)}, \quad A \in \mathbb{R}, \quad A \neq 0.
\]

4) The limit case \( A = 0 \):

\[
K(x, y) = \frac{B(x-y)}{\sinh B(x-y)}, \quad B > 0.
\]

In all the cases we normalized the kernels so that \( K(x, x) \equiv 1 \); by a change of variable, \( x \mapsto \text{const} \cdot x \), we could replace 1 by an arbitrary constant.
Proposition 4.1. Let \( K(x,y) = k(x - y) \) be a translation invariant kernel on \((\mathbb{R}, dx)\) and assume that \( k(\cdot) \) is the inverse Fourier transform of an integrable function \( \hat{k}(\cdot) \),

\[
    k(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixy} \hat{k}(y) dy,
\]

such that \( 0 \leq \hat{k}(y) \leq 1 \) for all \( y \in \mathbb{R} \) and \( \hat{k}(y) < 1 \) when \( |y| \) is large enough.

Then the conditions (*) and (**) stated in section 1 are satisfied, so that the kernel corresponds to a fermion process on \( \mathbb{R} \).

Proof. Since \( \hat{k} \) is nonnegative, the function \( k \) is Hermitian–symmetric and nonnegative definite, whence the condition (*) is satisfied.

Let \( K \) denote the integral operator in \( L^2(\mathbb{R}, dx) \) with the kernel \( K(x,y) \). The image of \( K \) under the Fourier transform is the operator of multiplication by the function \( \hat{k} \). This implies that \( 0 \leq K \leq 1 \).

It remains to check that \( K_A \) is strictly less than 1 for any test set \( A \subset \mathbb{R} \), i.e., for any bounded \( A \); without loss of generality one can assume that \( A \) is a finite interval \([a,b] \). The function \( k \) being continuous, the operator \( K_A \) is a compact Hermitian operator. Assume \( K_A \phi = \phi \) for a function \( \phi \in L^2(\mathbb{R}) \). Then \( \phi \) is concentrated on \( A \) and \( K \phi = \phi \). Taking the Fourier transform we see that the Fourier image \( \hat{\phi} \) is concentrated on the region where \( \hat{k}(\cdot) = 1 \). By the assumption, this region is bounded, so that both \( \phi \) and \( \hat{\phi} \) are compactly supported, which implies \( \phi \equiv 0 \). This means that \( K_A < 1 \), which completes the proof. \( \square \)

Let \( k(x) = B \sin Ax/\text{sh} Bx \) or \( k(x) = B \text{sh} Ax/\text{sh} Bx \), where in the latter case we assume \( |A| < B \) (otherwise \( k(x) \) certainly does not have the required form). Then the Fourier transform \( \hat{k}(y) \) is given by the formula

\[
    \hat{k}(y) = \frac{\pi \text{sh}(\pi A/B)}{A[\text{ch}(\pi A/B) + \text{ch}(\pi y/B)]},
\]

or

\[
    \hat{k}(y) = \frac{\pi \sin(\pi A/B)}{A[\cos(\pi A/B) + \text{ch}(\pi y/B)]},
\]

respectively. These formulas (which are related by analytic continuation with respect to the parameter \( A \)) can be found, e.g., in [E2], section 1.9, formula 14.

Proposition 4.2. The above four stationary kernels generate a fermion point process if the parameters \( A,B \) satisfy the following conditions, respectively.

1) The “\( \sin / \text{sh} \) kernel”:

\[
    \text{th} \frac{\pi |A|}{2B} \leq \frac{|A|}{\pi}.
\]

2) The “\( \text{sh} / \text{sh} \) kernel”:

\[
    0 \leq \text{tg} \frac{\pi |A|}{2B} \leq \frac{|A|}{\pi}, \quad \frac{|A|}{B} < 1.
\]

3) The limit case \( B = 0 \):

\[
    A \geq \pi.
\]
4) The limit case \( A = 0 \):
\[
B \geq \frac{\pi^2}{2}.
\]

Proof. The above expressions for \( \hat{k}(y) \) correspond to the first two cases. For the remaining two cases we get \( \hat{k}(y) \) by an obvious limit transition. We must find conditions on \( A, B \) under which \( \hat{k}(y) \) satisfies the two inequalities of Proposition 4.1. This is done by an elementary calculation. □

Comments. 1) In our scheme, the famous sine kernel corresponds to the degenerate case \( B = 0 \) and the minimal allowed value \( A = \pi \).
2) The “\( \sin/\sinh \)” kernel with \( A = \pi \) and arbitrary \( B \) appeared in the papers [BCM, MCIN].
3) For the principal series, when \( z = a + ib, \ z' = a - ib \), where \( a, b \) are real and \( b \) is nonzero, the kernel of the tail process is of type “\( \sin/\sinh \)” with the parameters
\[
A = \frac{\pi \sin \pi (z - z')}{2 \sin \pi z \cdot \sin \pi z'}, \quad B = \frac{\pi \sin \pi (z - z')}{2(z - z') \sin \pi z \cdot \sin \pi z'}.
\]
The inequality imposed on \( A, B \) becomes evident in terms of \( a, b \):
\[
\frac{\text{ch}(2\pi b) - \cos(2\pi a)}{\text{ch}(2\pi b) + 1} \leq 1.
\]
Note that we do not get all the allowed couples \((A, B)\).
4) For the complementary series, when \( z \) and \( z' \) are real such that \( m < z, z' < m + 1 \) for a certain \( m \in \mathbb{Z} \) and \( z \neq z' \), the tail kernel is of type “\( \sinh/\sinh \)” with the parameters
\[
A = \frac{\pi \sin \pi (z - z')}{2 \sin \pi z \cdot \sin \pi z'}, \quad B = \frac{\pi \sin \pi (z - z')}{2(z - z') \sin \pi z \cdot \sin \pi z'}.
\]
The first inequality imposed on \( A, B \) turns into the evident one:
\[
0 \leq \frac{2 \sin \pi z \cdot \sin \pi z'}{\cos \pi (z - z')} = \frac{\cos \pi (z - z') - \cos \pi (z + z')}{\cos \pi (z - z') + 1} \leq 1.
\]
The second inequality is also evident, because
\[
\frac{|A|}{B} = |z - z'| < 1.
\]
Again, we do not get all the allowed couples \((A, B)\).
5) For the intersection of the both series, when \( z = z' \in \mathbb{R} \setminus \mathbb{Z} \), the kernel is of limit type with the parameters
\[
A = 0, \quad B = \frac{\pi^2}{2 \sin^2 \pi z},
\]
and the inequality on \( B \) takes the form
\[
\frac{\pi^2}{2 \sin^2 \pi z} \geq \frac{\pi^2}{2}.
\]
As \( \sin^2 \pi z \) ranges over \((0, 1]\), we get here all the allowed values of the parameter \( B \).
6) Consider the principal series \((z = a + ib, \ z' = a - ib)\) and let \(|b| \to \infty\); then, irrespective to the behavior of \( a \), the corresponding tail kernel tends to the sine kernel \( \sin \pi (\xi - \eta) / \pi (\xi - \eta) \).
5. Rate of decay of the Thoma parameters.

**Theorem 5.1.** Let $P_{zz'}$ be an arbitrary measure of principal or complementary series. Then, for almost every point $\omega = (\alpha, \beta) \in \Omega$, with respect to the measure $P_{zz'}$, there exist the limits

$$\lim_{j \to \infty} (\alpha_j)^{1/j} = \lim_{j \to \infty} (\beta_j)^{1/j} = e^{-C^{-1}},$$

where $C$ is the same as in the proof of Theorem 3.2, i.e.,

$$C = \begin{cases} 
\frac{(z-z') \sin \pi z \cdot \sin \pi z'}{\pi \sin \pi (z-z')} , & z' \neq z \\
\frac{\sin^2 \pi z}{\pi^2} , & z' = z \in \mathbb{R} \setminus \mathbb{Z}
\end{cases}$$

**Proof.** Step 1 (change of a variable). We shall prove that the first limit exists and equals $e^{-C^{-1}}$. Then this will also imply the claim concerning the second limit, because, on the one hand, $\alpha$ and $\beta$ change places as $z$ and $z'$ are multiplied by $-1$, and, on the other hand, this does not affect the value of $C$.

So, we shall examine the point process $P_{zz'}$ on $(0, 1]$. It is convenient to pass from $(0, 1]$ to $[0, +\infty)$ via the map $x \mapsto \xi = -\ln x$. Let $x_1 \geq x_2 \geq \ldots$ be the random configuration of the process $P_{zz'}$ and $\xi_1 \leq \xi_2 \leq \ldots$ be its image under this map. (Actually, we know that the inequalities are strict (see Theorem 2.5.1 of Part II) but here this is unessential.)

We have

$$\lim_{j \to \infty} (x_j)^{1/j} = e^{-C^{-1}} \iff \lim_{j \to \infty} \frac{\xi_j}{j} = C^{-1}$$

(here we mean limits with probability 1).

Step 2 (reduction to $N_\tau$). Let $\xi_1 \leq \xi_2 \leq \ldots$ be the random configuration for a point process on $[0, +\infty)$ and $N_\tau$ denote the number of points in $[0, \tau]$, where $\tau > 0$ is arbitrary. Then the following equivalence holds ($C > 0$ is a constant):

$$\lim_{j \to \infty} \frac{\xi_j}{j} = C^{-1} \iff \lim_{\tau \to +\infty} \frac{N_\tau}{\tau} = C,$$

limits with probability 1.

Actually, in this claim, randomness is unessential; we shall prove it for a fixed (nonrandom) configuration.

Assume that $\xi_j/j$ tends to $C^{-1}$ with a certain $C > 0$. Then $\xi_j \to +\infty$. Further, for any $\tau > 0$, there exists a unique $j$ such that $\xi_j \leq \tau < \xi_{j+1}$. Then $N_\tau = j$ and

$$\frac{j}{\xi_{j+1}} < \frac{N_\tau}{\tau} \leq \frac{j}{\xi_j}.$$

As $\tau \to +\infty$, we have $j \to +\infty$, whence, by the assumption, the both bounds tend to $C$. Consequently, $N_\tau/\tau$ tends to $C$, too.

Conversely, assume that $N_\tau/\tau$ tends to a certain $C > 0$. Then it follows that $\xi_j \to +\infty$ as $j \to \infty$. Further, fix an arbitrary $\varepsilon > 0$ and remark that

$$N_{\xi_j - \varepsilon} < j \leq N_{\xi_j}.$$
This implies
\[
\frac{N_{\xi - \varepsilon}}{\xi} < \frac{j}{\xi} \leq \frac{N_{\xi}}{\xi}.
\]
As \( j \to \infty \), we have \( \xi_j \to +\infty \), whence, by the assumption, the both bounds tend to \( C \). It follows that \( j/\xi_j \) tends to \( C \), too, so that \( \xi_j/j \) tends to \( C^{-1} \).

**Step 3 (use of correlation functions).** Consider a point process on \([0, +\infty)\). As above, by \( N_\tau \) we denote the number of points in \([0, \tau]\). Let \( \rho_1 \) and \( \rho_2 \) be the first and the second correlation functions.

**Lemma 5.2.** Assume that \( \rho_1 \) and \( \rho_2 \) satisfy the following conditions, where \( \varepsilon \) and \( C \) are certain constants \((0 < \varepsilon < 1 \text{ and } C > 0)\) and \( \tau \to +\infty \):

\[
\int_0^\tau \rho_1(\xi)d\xi = C\tau + O(\tau^{1-\varepsilon})
\]

\[
\int_0^\tau \int_0^\tau \rho_2(\xi, \eta)d\xi d\eta = C^2\tau^2 + O(\tau^{2-\varepsilon}).
\]

Then \( N_\tau/\tau \to C \).

**Proof of the lemma.** We shall adapt Kingman’s argument in [Ki], section 4.2. By the definition of the correlation measures,

\[
\int_0^\tau \rho_1(\xi)d\xi = \mathbb{E}(N_\tau)
\]

\[
\int_0^\tau \int_0^\tau \rho_2(\xi, \eta)d\xi d\eta = \mathbb{E}(N_\tau(N_\tau - 1)).
\]

By the assumptions of the lemma, this implies

\[
\mathbb{E}(N_\tau) = C\tau + O(\tau^{1-\varepsilon})
\]

\[
\mathbb{E}(N_\tau^2) = \mathbb{E}(N_\tau(N_\tau - 1) + N_\tau) = C^2\tau^2 + O(\tau^{2-\varepsilon}).
\]

It follows

\[
\mathbb{E} \left( \left( \frac{N_\tau}{\tau} - C \right)^2 \right) = \frac{\mathbb{E}(N_\tau^2)}{\tau^2} - 2C \frac{\mathbb{E}(N_\tau)}{\tau} + C^2 = O(\tau^{-\varepsilon}).
\]

By the Chebyshev inequality, for any \( \delta > 0 \),

\[
\text{Prob} \left( \left| \frac{N_\tau}{\tau} - C \right| \geq \delta \right) \leq \frac{\text{const}}{\delta^2\tau^\varepsilon},
\]

where the constant does not depend on \( \tau \). Taking \( \tau = k^{2/\varepsilon} \), where \( k = 1, 2, \ldots \), we conclude that the series

\[
\sum_{k=1}^\infty \text{Prob} \left( \left| \frac{N_{k^{2/\varepsilon}}}{k^{2/\varepsilon} - C} \right| \geq \delta \right).
\]
converges for any $\delta > 0$. By the Borel–Cantelli lemma,

$$\lim_{k \to \infty} \frac{N_{k^{2/\varepsilon}}}{k^{2/\varepsilon}} = C$$

with probability 1.

Finally, for an arbitrary $\tau > 0$, define a natural $k$ from the relation

$$k^{2/\varepsilon} \leq \tau < (k + 1)^{2/\varepsilon}.$$

Then

$$N_{k^{2/\varepsilon}} \leq N_{\tau} \leq N_{(k+1)^{2/\varepsilon}}.$$

Since

$$\frac{(k + 1)^{2/\varepsilon}}{k^{2/\varepsilon}} \sim 1, \quad k \gg 1,$$

we get

$$\lim_{\tau \to +\infty} \frac{N_{\tau}}{\tau} = C$$

with probability 1. □

Step 4 (reduction to the lifted process). Remark that multiplication of a sequence $(x_1 > x_2 > \ldots)$ by a positive factor does not affect on the limit behavior of $x_j^{1/j}$. By the very definition of the lifting, it follows that in the claim of the theorem, we may replace our process by its lifting. The only purpose of this reduction is that below we may employ Theorem 4.1.1 (Part II) for the lifted process instead of the parallel Theorem 4.3.1 whose proof is more difficult.

Step 5 (estimation of correlation functions). Consider the lifting of the process $P_{z z'}^+$ and then make change of a variable $x \mapsto \xi = -\ln x$. Denote by $\rho_n(\xi_1, \ldots, \xi_n)$ the correlation functions of the resulting process. By the above discussion it remains to check that $\rho_1$ and $\rho_2$ obey the assumptions of Lemma 5.2.

Theorem 4.1.1 of Part II implies that our correlation functions can be written in the form

$$\rho_n(\xi_1, \ldots, \xi_n) = C^n \det[K(\xi_i, \xi_j)] + r_n(\xi_1, \ldots, \xi_n), \quad \xi_1, \ldots, \xi_n > 0,$$

where the constant $C$ is the same as in the statement of the theorem,

$$K(\xi, \eta) = k(\xi - \eta),$$

$$k(\zeta) = \begin{cases} 
\text{sh} \left( \frac{z - z'}{2} \zeta \right), & z' \neq z \\
\frac{\zeta}{\text{sh} \left( \frac{1}{2} \zeta \right)}, & z = z' \in \mathbb{R} \setminus \mathbb{Z},
\end{cases}$$

and the rest term admits the estimate

$$r_n(\xi_1, \ldots, \xi_n) = O(e^{-\delta \min\{|\xi_1|, \ldots, |\xi_n|\}})$$
with a certain \( \delta > 0 \) (this \( \delta \) is equal to 1 for the principal series with \( z' \neq z \); to \( 1 - |z - z'| \) for the complementary series with \( z \neq z' \); and can be any number strictly less than 1, for the intersection of the both series).

Note that \( k(0) = 1 \) and the function \( k(\zeta) \) is an even square integrable function on the whole real axis.

We have
\[
\int_0^\tau r_1(\xi)d\xi = O(1), \quad \int_0^\tau \int_0^\tau r_2(\xi, \eta)d\xi d\eta = O(\tau).
\]

Using this we get
\[
\int_0^\tau \rho_1(\xi)d\xi = C \int_0^\tau k(0)d\xi + O(1) = C\tau + O(1),
\]
\[
\int_0^\tau \int_0^\tau \rho_2(\xi, \eta)d\xi d\eta = C^2 \int_0^\tau \int_0^\tau \frac{1}{k(\xi - \eta)} d\xi d\eta + O(\tau)
\]
\[
= C^2 \tau^2 - C^2 \int_0^\tau \int_0^\tau k^2(\xi - \eta)d\xi d\eta + O(\tau).
\]

Finally,
\[
\int_0^\tau \int_0^\tau k^2(\xi - \eta)d\xi d\eta = 2 \int_0^\tau d\xi \int_0^\tau k^2(\zeta)d\zeta = O(\tau),
\]

because \( k(\zeta) \) is square integrable.

Thus, we have verified the assumptions of Lemma 5.2, which completes the proof. \( \Box \)

**Remark 5.3.** The same argument can be applied to the Poisson–Dirichlet process \( \mathcal{P}D(t) \). Here the final step 5 is much easier, because of a simpler structure of the correlation functions. We get in this way that for \( \mathcal{P}D(t) \), \((x_j)^{1/j}\) tends to \( e^{-t} \), the result originally obtained (for \( t = 1 \)) in [VS] by a quite different way. We can also use lifting, as suggested on step 4, which provides a quick reduction to the law of large numbers for the Poisson process, see [Ki], section 4.2.

Thus, both for \( \mathcal{P}^{zz'} \) and \( \mathcal{P}D(t) \), the rate of decay of the \( x_j \)'s is of the same type.

**Remark 5.4.** Let \( \Omega' \) denote the subset of points \( \omega = (\alpha, \beta) \in \Omega \) such that the limit \( F(\omega) := \lim(\alpha_j)^{1/j} \) exists. Clearly, this is a Borel subset. According to Theorem 5.1, \( \Omega' \) is of full measure with respect to any \( P_{zz'} \) and the function \( F \) takes constant values almost everywhere, with the constant depending on \( z, z' \) in a nontrivial way. This agrees with the fact that the measure \( P_{zz'} \) are pairwise disjoint. However, this does not provide an alternative proof, because a single function is not sufficient to separate points in the two–dimensional space of the parameters.

6. **Associated Sturm–Liouville operators.** It is well known\(^3\) that the integral operator with the sine kernel, restricted to an arbitrary finite interval, commutes with a certain Sturm–Liouville differential operator
\[
Df = (pf')' + qf.
\]

\(^3\)One of the authors (G. O.) is grateful to F. Alberto Grünbaum for drawing his attention to this fact.
Specifically, take the interval \([-\tau, \tau]\) with \(\tau > 0\); then

\[
p(x) = x^2 - \tau^2, \quad q(x) = \pi^2 x^2.
\]

Gaudin’s proof of this fact, sketched in [Me], section 5.3, uses a trick but the claim can also be verified by brute force.

Similar results also hold for the Airy kernel and the Bessel kernel, and they turn out to be useful in the study of the corresponding Fredholm determinants, see [TW2, TW3].

Now we shall produce analogous differential operators for the kernels discussed in section 4 and for the Whittaker kernel.

**Proposition 6.1.** The integral operator with the \(\sin / \text{sh}\) kernel or the \(\text{sh} / \text{sh}\) kernel, restricted to the interval \([-\tau, \tau]\), \(\tau > 0\), commutes with the Sturm–Liouville operator \(Df = (pf')' + qf\), where

\[
p(x) = \frac{\text{sh}^2(Bx) - \text{sh}^2(B\tau)}{B^2}, \quad q(x) = \frac{(B^2 \pm A^2) \text{sh}^2(Bx)}{B^2}
\]

and the plus sign is taken for the \(\sin / \text{sh}\) kernel while the minus sign is taken for the \(\text{sh} / \text{sh}\) kernel; the same is true in the limit cases \(A = 0\) or \(B = 0\).

**Proposition 6.2.** The integral operator with the Whittaker kernel restricted to the semi–infinite interval \((\tau, +\infty)\) with arbitrary \(\tau > 0\), commutes with the Sturm–Liouville operator \(Df = (pf')' + qf\), where

\[
p(x) = x(x-\tau), \quad q(x) = -\frac{((a - x/2)^2 - t)(x - \tau)}{x}, \quad a = (z + z')/2, \quad t = zz'.
\]

**Proof of the propositions.** One checks by a direct computation that the kernel satisfies the relation \(D_x K(x, y) = D_y K(x, y)\) which implies the desired claim. □

7. **Comparison with Poisson–Dirichlet.** Like the processes \(P_{zz'}\), the Poisson–Dirichlet processes are closely related to harmonic analysis on the infinite symmetric group. The both families of processes play a similar role but on different levels. Namely, the Poisson–Dirichlet processes describe the decomposition on ergodic components for certain measures. Those measures live on a compactification of the infinite symmetric group and are employed in the construction of the generalized regular representations, see [KOV]. The processes \(P_{zz'}\) correspond to the dual level, as they govern the decomposition of that representations.

Our study makes it possible to compare the both families of processes.

The correlation functions of \(PD(t)\) are given by simple formulas, those of \(P_{zz'}\) look much more complicated. This is already seen for the first correlation functions.

Both \(PD(t)\) and \(P_{zz'}^+\) are simplified after lifting. But the lifting of the former is a (non stationary) Poisson process while the lifting of the latter is a less elementary object — the fermion process with the Whittaker kernel.

A similar conclusion can be made about the corresponding tail processes. These are the standard Poisson process and a stationary fermion process, respectively. This means, for instance, that the asymptotic probability distribution of the ratio \(\alpha_j/\alpha_{j+1}\) (as \(j \to \infty\)) looks quite differently.
Thus, the processes $P_{zz'}$ seem to be much more sophisticated objects than the Poisson–Dirichlet processes.

On the other hand, as is shown in section 5 above, there is a rough characteristic with respect to which $PD(t)$ and $P_{zz'}^+$ behave similarly: in the both cases, the rate of decay of the $\alpha_j$’s is that of a geometric progression.

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