A NOTE ON SOME CLASSICAL RESULTS OF GROMOV-LAWSON

MOSTAFA ESFAHANI ZADEH

Abstract. In this short note we show how the higher index theory can be used to prove results concerning the non-existence of complete riemannian metric with uniformly positive scalar curvature at infinity. By improving some classical results due to M. Gromov and B. Lawson we show the efficiency of these methods in dealing with such non-existence theorems.

1. Introduction

Let \((M, g)\) be an oriented complete non-compact manifold partitioned by a compact hypersurface \(N\) into two parts \(M_+\) and \(M_-\) with \(M_+ \cap M_- = \emptyset\) and \(\overline{M}_+ \cap \overline{M}_- = N\). We assume that the positive unite normal to \(N\) points out from \(M_-\) to \(M_+\). Let \(W\) be a Clifford bundle on \(M\) which is at the same time a Hilbert \(A\)-module bundle. We assume that this bundle is equipped with a connection which is compatible to the Clifford action of \(TM\) and denote the corresponding \(A\)-linear Dirac type operator by \(D\). As an example let \(M\) be a spin manifold with spin bundle \(S\) and \(V\) be a Hilbert \(A\)-module bundle on \(M\) with a hermitian connection. Then the spin Dirac operator twisted by \(V\) is an example of such operator which acts on the smooth sections of \(W = S \otimes V\).

Let \(U = (D - i)(D + i)^{-1}\) be the Cayley transform of \(D\) which is a \(A\)-linear bounded operator on \(H = L^2(M, W)\).

Let \(\phi_+\) be a smooth function on \(M\) which coincide with the characteristic function of \(M_+\) outside a compact set and put \(\phi_- := 1 - \phi_+\). It turns out that the operators \(U_+ = \phi_+ + \phi_- U\) is \(A\)-Fredholm in the sense of Fomenko-Mischenko. The Fomenko-Mischenko index \(\text{ind}(U_+)\) does not depend on \(\phi_+\) but on the cobordism class of the partitioning manifold \(N\). This index is denoted by \(\text{ind}(D, N)\). A basic property of this index is the following c.f. [2] theorem 2.4:

\[
\text{If } M \text{ is spin and } W = S \otimes V, \text{ where } V \text{ is a flat Hilbert } A\text{-module bundle then } \text{ind}(D, N) = 0 \in K_0(A) \text{ provided that the scalar curvature of } g \text{ is uniformly positive.}
\]

The Clifford action of \(i\) provides a \(\mathbb{Z}_2\) grading for \(W|_N\) and makes \(W|_N\) a graded Clifford bundle on \(N\). Let \(D_N\) denote the associated Dirac type operator which acts on smooth sections of \(W|_N\). It is a \(A\)-linear elliptic operator and has the Fomenko-Mischenko index \(\text{ind}(D, N) \in K_0(A)\). The following equality generalizing a result due to J. Roe and N. Higson [10] [6] is proved in [2]

\[
\text{ind } D_N = \text{ind } (D, N).
\]

This equality has been used in [2] to prove the following

\[
\text{If a complete spin manifold } (M, g) \text{ is partitioned by an enlargeable hypersurface } N \text{ and if there is a smooth map } \phi : M \to N \text{ whose restriction to } N \text{ is of non-zero degree, then the scalar curvature of } g \text{ can not be uniformly positive.}
\]

In this short note we improve this result by showing that under the same conditions the scalar curvature of \(g\) can not be uniformly positive even outside a compact subset of \(M\). We will use this stronger result to improve some classical results due to M. Gromov and B. Lawson. The section [2] deals with some analytical aspect of regular operators on Hilbert Modules. Here we construct the wave operator for the spin Dirac operators twisted by flat Hilbert module bundles and prove its unit speed property. In the forthcoming section we use these results to prove the vanishing theorem [3] and its implication in improving some classical results due to M. Gromov and B. Lawson.

Here we give the definition of the enlargeability as it is introduced by Gromov and Lawson in [3].

Definition: Let \(N\) be a closed oriented manifold of dimension \(n\) with a fixed riemannian metric \(g\). The

2000 Mathematics Subject Classification. 58J22 (19K56 46L80 53C21 53C27).

Key words and phrases. Higher index theory, enlargeability, Dirac operators.
manifold $N$ is enlargeable if for each real number $\epsilon > 0$ there is a riemannian spin cover $(\tilde{N}, \tilde{g})$, with lifted metric, and a smooth map $f : \tilde{N} \to S^n$ such that: the function $f$ is constant outside a compact subset $K$ of $\tilde{N}$; the degree of $f$ is non-zero; and the map $f : (\tilde{N}, \tilde{g}) \to (S^n, g_0)$ is $\epsilon$-contracting, where $g_0$ is the standard metric on $S^n$. Being $\epsilon$-contracting means that $|f(x)| \leq \epsilon$ for each $x \in N$, where $T_x f : T_x \tilde{N} \to T_{f(x)} S^n$. The manifold $N$ is said to be area-enlargeable if the function $f$ is $\epsilon$-area contracting. This means $|\Lambda^2 T_x f\| \leq \epsilon$ for each $x \in \tilde{N}$, where $\Lambda^2 T_x f : \Lambda^2 T_x \tilde{N} \to \Lambda^2 T_{f(x)} S^n$. An enlargeable manifold do not admit riemannian metrics with positive scalar curvature. The relevance of this theorem will be clear by noticing that enlargeability depend only on the homotopy type of $M$ not on differential structures used in its definition while the existence of a metric with positive scalar curvature does depend in general on the underlying differential structure, c.f. [8].

Acknowledgment: The author would like to thanks J. Roe and T. Schick for helpful hints.

2. Functional calculus of the Dirac operator

Let $H$ be a Hilbert $A$-module with $A$ a $C^*$-algebra and let $T$ be an $A$-linear map which is defined on a dense subspace $\text{Dom}(T)$ of $H$. The graph of $T$ is the following subset of $H \oplus H$

$$\text{graph}(T) := \{(u, T(u)) | u \in \text{Dom}(T)\}$$

The closure of this graph with respect to norm topology in $H \oplus H$ turns out to be the graph of an operator $\tilde{T}$ which is called the closure of $T$. The domain $\text{Dom}(\tilde{T})$ of this closure is a closed $A$-subspace of $H$. The adjoint of $T$ is the closed operator $T^*$ such that $(T(u, v)) = (u, T^*v)$ for $u \in \text{Dom}(T)$ and $v \in \text{Dom}(T^*)$. Let $\text{Dom}(T) = \text{Dom}(T^*)$. Following [7] $T$ is said to be self-adjoint if $T^* = T$ and it is said to be normal if

$$\langle Tu, Tv \rangle = \langle T^*u, T^*v \rangle,$$

for $u, v \in \text{Dom}(T)$.

The operator $T$ is called regular if there is a bounded adjointable operator $P \in \mathcal{L}_A(H \oplus H, H)$ with $\text{Im} P = \text{Dom} \tilde{T}$ (c.f proposition 5 of [7]). Regularity and self adjointness of the operator $T$ make it possible to associate to each continuous (not necessarily bounded) function $f$ on $\text{spec}(T)$ a closed $A$-linear operator $f(T)$ on $H$. This correspondence define a continuous functional calculus for $T$. This construction has been worked out in [11]. Here we follow the more geometric approach of [7]. The key observation of [7] is that the transformation $T \to Q(T) = T(1 + T^* T)^{-1/2}$ provides a $*$-preserving bijection between the set of all normal regular $A$-linear operator $\mathcal{R}_A(H)$ and the following set

$$\mathcal{V}(H) = \{Q \in \text{End}_A(H) ||Q|| \leq 1 \text{ and } \text{Im}(1 - Q^* Q) \text{ and } \text{Im}(1 - QQ^*) \text{ are dense}\}.$$  

Let $D$ denote the open unit disc in $\mathbb{C}$. Each function $g \in C_0(\mathbb{C})$ determines a function $\tilde{g} \in C_0(D)$ by the following relation

$$\tilde{g}(z(1 + |z|^2)^{-1/2}) = g(z)(1 + |g(z)|^2)^{-1/2}.$$  

Clearly $||\tilde{g}|| \leq 1$, so the bounded operator $\tilde{g}(Q(T))$ belongs again to $\mathcal{V}_A(H)$ and corresponds to a unique operator in $\mathcal{R}_A(H)$ which is defined to be $g(T)$. From this construction it is clear that if $g = g'$ on a closed subset containing the spectrum $\text{spec}(T)$ then $g(T) = g'(T)$. It is clear also that the corresponding $g \to g(T)$ provides a $C^*$-representation $\phi : C_0(X) \to \text{End}_A(H)$.

The regularity of $T$ implies $C_0(X)(T)H = H$ (see corollary 14 and theorem 15 of [7]), so given any $u \in H$ there is $g \in C_0(X)$ and $v \in H$ with $u = g(T)v$. If $h$ is a bounded continuous function on $X$ then $hg \in C_0(X)$ and one defines $h(T)u := (hg)(T)v$. It is easy to verify that $h(T)$ is well defined and that it is bounded with $||h(T)|| \leq ||h||$, where $||h|| = \sup ||h(\lambda)||, \lambda \in \text{spec}(T)$.

For an unbounded continuous function $h$, the set

$$\mathcal{C}(h) := \{g \in C_0(X)| hg \in C_0(X)\}$$

is a $*$-subalgebra of $C_0(X)$. The above argument can be used to define the unbounded operator $h(T)$ with domain $\phi(\mathcal{C}(h))(H)$. It is clear that $\text{Dom} f(T) \subset \text{Dom} g(T)$ if $|g| \leq |f|$ on $\text{spec}(T)$. As an example if $g(z) = z$ then it turns out that $g(T) = T$. The following theorem summarizes some properties of this functional calculus which are relevant to our purposes. These properties are straightforward consequences of the above discussion.
Theorem 2.1. Let $T$ be a densely defined regular normal $A$-linear operator on Hilbert $A$-module $\mathbb{H}$ and let $X \subset \mathbb{C}$ be a closed subset containing the spectrum $\text{spec}(T)$. Then to each continuous function $f \in C(X)$ one can correspond the regular normal operator $f(T)$ on $\mathbb{H}$ satisfying $f(T)^* = f(T^*)$ such that

1. for $f, g$ in $C(X)$ the operator $(f + g)(T)$ is the closure of $f(T) + g(T)$, $(fg)(T)$ is the closure of $f(T)g(T)$ and $f \circ g(T) = f(g(T))$.
2. If $T$ is bounded then $f \to f(T)$ coincides with the functional calculus in the $C^*$-algebra of bounded operators on $\mathbb{H}$.
3. Let $\{f_k\}_k$ be a sequence of continuous functions on $X$ which is dominated by a continuous function $F$, i.e. $|f_k(z)| \leq |F(z)|$ for all $z \in X$. If $f_k \to f$ uniformly on compact subsets of $X$ then $f_k(T)(u) \to f(T)(u)$ for $u \in \text{Dom}(T)$. If $f_k$'s are uniformly bounded and the convergence to $f$ is uniform then $f_k(T) \to f(T)$ in norm topology.

As a special case of this theorem, if $T$ is self adjoint then $f(t, T) = e^{itT}$, for $t \in \mathbb{R}$, is a one parameter family of unitary operators on $\mathbb{H}$ satisfying $e^{i(t+s)T} = e^{itT}e^{isT}$. Moreover $Te^{itT} = e^{itT}T$ which implies that $e^{itT}$ provides a unitary bijection between $\text{Dom}(T)$ and $\text{Im}(T)$. By the third part of the previous theorem for $u \in \text{Dom}(T)$ we have

$$
\left( \frac{d}{dt} \right)_{t=0} e^{itT}u = \lim_{t \to 0} \left( \frac{e^{itz} - 1}{t} \right)(T)(u) = iT(u).
$$

So, for a self adjoint regular operator $T$ the $t$-parameterized family of unitary operators $e^{itT}$ satisfies the wave equation and the initial condition

\begin{align}
(\frac{d}{dt} - iT)e^{itT}(u) &= 0 \quad \text{for } u \in \text{Dom}(T), \\
\lim_{t \to 0} e^{itT}u &= u.
\end{align}

The wave equation (2.1) implies the following vanishing result for $u \in \text{Dom}(T)$

$$
\frac{d}{dt}\langle e^{itT}(u), e^{itT}(u) \rangle = \langle iT e^{itT}(u), e^{itT}(u) \rangle + \langle e^{itT}(u), iTe^{itT}(u) \rangle = 0.
$$

This conservation law and initial condition (2.2) prove the uniqueness of heat operator with properties (2.1) and (2.2). If $f$ is a smooth function in the Schwartz space $S(\mathbb{R})$ then the Fourier transform $\hat{f}$ is in $S(\mathbb{R})$ and

$$
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{isx} \, ds.
$$

By applying the above theorem we get the following formula

$$
f(T)(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{isT}(u) \, ds, \quad \text{for } u \in \text{Dom}(T).
$$

The Hilbert module that we shall study in the sequel is the space of the $L^2$-sections of Hilbert module bundles over complete manifolds. Let $(M, g)$ be a complete riemannian manifold and let $W$ be a Clifford Hilbert $A$-modules bundle over $M$, where $A$ is a complex $C^*$-algebra. For $\sigma$ and $\eta$ two compactly supported smooth sections of $W$ put

$$
\langle \sigma, \eta \rangle = \int_M \langle \sigma(x), \eta(x) \rangle \, d\mu_g(x) \in A.
$$

It is easy to show that $|\sigma| = \|\langle \sigma, \sigma \rangle\|^{1/2}$ is a norm on $C_c(M, W)$. The completion of $C_c^\infty(M, W)$ with respect to this norm is the Hilbert $A$-module $\mathbb{H} = L^2(M, W)$. Let $D$ be an $A$-linear Dirac type operator acting on the compactly supported smooth sections of $W$ which form a dense sub-space of $\mathbb{H}$. We recall that $D$ is formally self adjoint, i.e. $\langle D\sigma, \eta \rangle = \langle \sigma, D\eta \rangle$ for $\sigma$ and $\eta$ as in above. Moreover $D$ is a regular operator (see e.g. [2] lemma 2.1), so one can apply the theorem 2.1 to $D$ and define the bounded operator $f(D)$ on $L^2(M, W)$ for each bounded continuous function $f$ on $\mathbb{R}$. In particular we
can define the wave operator $e^{itD}$. In the following lemma we describe a context in which the wave operator has finite propagation speed.

**Lemma 2.2.** Let $W = S \otimes V$ be the spin bundle $S$ twisted by the flat Hilbert $A$-module bundle $V$. The wave operator $e^{itD}$ has unite propagation speed.

**proof** To prove the assertion we give an another construction for the wave operator which satisfies the unite propagation speed. Then the uniqueness of the wave operator implies the desired assertion. In this proof we denote by $V_0$ the fiber of $V$ which is a Hilbert $A$-module. Let $\{U_\alpha, \phi_\alpha^V \otimes \phi_\alpha^A\}$ be a trivializing atlas for $M$ such that $W|_{U_\alpha} \simeq U_\alpha \times \mathbb{C} V_0$. Since $V$ is flat, we can assume that the transition functions $\phi_{\alpha\beta}^V : U_\alpha \cap U_\beta \to \text{End}_A(V_0)$ are locally constant. Let $D'$ denote the spin Dirac operator. Then $D = D' \otimes I_{V_0}$ on smooth sections $\xi = \sum_\alpha \xi^\alpha$ of $W$ where $\xi^\alpha$ is supported in $U_\alpha$ and $\xi^\alpha = s(x) \otimes v$ for a fixed $v$ in $V_0$. From the unite propagation formula for $D'$, for $|t|$ sufficiently small the section $\xi^\alpha_t(x) := e^{itD'} s(x) \otimes v$ is supported in $U_\alpha$ too. Moreover it is the unique solution of the following wave equation with the given initial condition $\xi$.

\[
\frac{d}{dt} - iD' \otimes I_{V_0})\xi_t^\alpha(x) = 0.
\]

Since $\phi_{\alpha\beta}^V$ is constant, the transition of solution $\xi_t^\alpha$ to another chart is the solution of the wave equation in that chart with transitioned initial condition. Therefore these local solutions of local wave equations actually paste together to define a global solution $\xi_t$ of the wave equation for $D$ for sufficiently small values of $t$. We can use $\xi_t$ as he initial condition and repeat the above procedure to define the solution of the wave equation beyond $t$. This way we get a solution which is defined for $t \in \mathbb{R}$. We define $e^{itD} \xi_0$ to be $\xi_t$. From this construction it is clear that the wave operator $e^{itD}$ has unite propagation speed. \hfill $\square$

For next uses we rewrite the relation (2.2) with the Dirac operator $D$

\[
f(D)(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{isD}(u) \, ds , \quad \text{for } u \in \text{Dom}(D).
\]

3. Vanishing theorem and its implications

As mentioned in above the higher index $\text{ind}(D,N)$ vanishes if the scalar curvature of the underlying riemannian metric $g$ is uniformly positive. In fact this index vanishes even if the scalar curvature is uniformly positive outside a compact subset of $M$. More precisely we prove the following theorem

**Theorem 3.1.** With above notation if the scalar curvature of $g$ is uniformly positive at infinity and if $W = S \otimes V$, where $V$ is a flat Hilbert $A$-module bundle, then $\text{ind}(D,N) = 0 \in K_0(A)$.

**proof** Let $U_0$ and $U_1$ be disjoint open subsets of $M$ such that the closure of $U_0$ is compact and $M = \bar{U}_0 \cup U_1$. Moreover we assume that the scalar curvature $\kappa$ of $g$ is uniformly positive in $U_1(2r)$, e.g $\kappa > 4\kappa_0$. Here $U_1(2r)$ consists of all point suited within distance $2r \geq 0$ from $U_1$ and $r$ is a sufficiently big number that will be determined in below. By multiplying the metric $g$ with a sufficiently small positive number we can and will assume that the constant $\kappa_0$ is arbitrary big. Let $\phi_0, \phi_1$ and $\phi_r$ be respectively the characteristic functions of $\bar{U}_0$, $U_1$ and $U_1(r)$ in $M$. If $\phi$ is a function on $M$ which is locally constant outside a compact subset then $([D + i]^{-1}, \phi)$ is compact, c.f. [2] lemma 2.2]. Therefore

\[
U_+ = \text{Id} - 2i\phi_+ \sum_{i,j=0}^1 \phi_i(D + i)^{-1}\phi_j
\]

\[
\sim \text{Id} - 2i\phi_+ \phi_1(D + i)^{-1}\phi_r
\]

The function $(x + 1)^{-1}$ can be uniformly approximated by compactly supported smooth functions, and hence by smooth functions with compactly supported Fourier transform. Let $h$ be such a function whose Fourier transform $\hat{h}$ is supported in $[-r, r]$ (the $r$ at the beginning of the proof is determined here). If $h$ is sufficiently close to $(x + i)^{-1}$ in sup-norm, then $\text{Id} - 2i\phi_+ \phi_1 h(D) \phi_r$ being close to
\[ -2i\phi_+\phi_1 (D+i)^{-1}\phi_r \text{ in operator norm (c.f. theorem 2.1), is an } A\text{-Fredholm operator with the same index. Therefore we need to prove the vanishing of the index of the following operator} \]

\[ (3.1) \quad \text{Id} - 2i\phi_+\phi_1 h(D)\phi_r. \]

Let \( \sigma \) be a smooth section of \( W = S \otimes V \) supported in \( U_1(r) \). Since \( V \) is flat the following generalized Lichnerowicz formula holds with respect to the \( A \)-valued \( L^2 \)-inner product, c.f. [11], page 199

\[ D^2 = \nabla^*\nabla + \frac{\kappa}{4} \]

which implies

\[
\langle D\sigma, D\sigma \rangle = \langle D^2\sigma, \sigma \rangle
\]

\[ = \langle \nabla\sigma, \nabla\sigma \rangle + \left( \frac{\kappa}{4} \right) \langle \sigma, \sigma \rangle \]

\[ \geq \kappa_0 \|\sigma\|^2 \]

In the last inequality we have used the fact that for \( a \) and \( b \) in a \( C^* \)-algebra, \( a+b \geq 0 \) and \( \|a+b\| \geq |b| \) provided that \( a \) and \( b \) are positive and self adjoint. Consider the restriction of the Dirac operator \( D \) to the Hilbert \( A \)-module \( \mathcal{H} := L^2(U_1(2r), W) \) and denote it by \( D_{2r} \). This is an unbounded operator acting on smooth sections compactly supported in \( U_1(2r) \). This operator is symmetric and satisfies the above positivity condition. In fact it has a self-adjoint regular extension to \( \mathcal{H} \) as we are going to show. We recall here the notation of the proof of the lemma 2.2. We assume the trivializing charts \( \{ U_\alpha,\phi_\alpha^\alpha \} \) and \( \{ U_\alpha,\phi_\alpha^\beta \} \) for vector bundle \( S \) and \( V \) over \( M \) such that the transition functions \( \phi_{\alpha\beta} = \phi_\alpha^\gamma \circ \phi_\beta^\gamma \) defined from \( U_\alpha \cap U_\beta \) into \( \text{End}_A(V_0) \) are constant. Since the twisting bundle \( V \) is flat, the Hilbert \( A \)-module \( \mathcal{H} \) is generated by elements \( s \otimes v \) where \( s \) is a smooth section of the spin bundle \( S \to U_1(2r) \) supported in one of \( U_\alpha \)'s and \( v \) is a constant element of \( V_0 \) (i.e. a flat section of \( V_{U_\alpha} \)).

On these sections the operator \( D_{2r} \) takes the form \( D' \otimes \text{Id} \), where \( D' \) denotes the spin Dirac operator acting on the smooth sections of \( S \) which are compactly supported in \( U_1(2r) \). With this domain, \( D' \) is a symmetric operator on \( L^2(U_1(2r), S) \) satisfying the following positivity relation in \( \mathbb{R} \)

\[ \langle D'(s), D'(s) \rangle \geq \kappa_0 \| s \|^2 . \]

The Friedrichs’ extension theorem provides a self adjoint extension \( \bar{D}' \) of \( D' \) to \( L^2(U_1(2r), S) \) satisfying still the above positivity condition. We recall two fact from the construction of the Friedrichs’ extension. If \( s \) is compactly supported in \( U_\alpha \) then \( \bar{D}'s \) is compactly supported in \( U_\alpha \) too. Moreover if \( s \) belongs to \( \text{Dom}(\bar{D}') \) then \( \phi s \in \text{Dom}(\bar{D}') \) for a smooth compactly supported function \( \phi \). Now define the operator \( D_{2r} \), as follows. The domain consists of sum of sections \( s \otimes v \) where \( s \) is supported in \( U_\alpha \) and belongs to \( \text{Dom}(\bar{D}') \) and \( v \) is an element of \( V \). On these sections we define \( D_{2r}(s \otimes v) = \bar{D}'s \otimes v \).

This is a self adjoint operator on \( \mathcal{H} \) satisfying the following relation

\[ \langle D_{2r}\sigma, D_{2r}\sigma \rangle \geq \kappa_0 \|\sigma\|^2 ; \quad \text{for } \sigma \in \text{Dom}(\bar{D}_{2r}) \subset \mathcal{H} \]

Since \( \bar{D}' \) is self adjoint \( \text{Im}(\bar{D}' + i) = L^2(U_1(2r), S) \), c.f. [11] page 257, so the above definition shows that \( \text{Im}(\bar{D}_{2r} + i) = \mathcal{H} \). Therefore \( D_{2r} \), as an operator on \( \mathcal{H} \) is self-adjoint and regular. Consequently we can apply the functional calculus of the previous section to define the bounded operator \( h(D_{2r}) \) and \( e^{itD_{2r}} \) on \( L^2(U_1(2r), W) \). The point is that the spectrum of \( D_{2r} \) is outside of the interval \( (-\kappa_0,\kappa_0) \).

Since \( h \) goes to zero at infinity, if \( \kappa_0 \) is sufficiently big then \( \| h(D_{2r}) \| \) is arbitrary small. Let \( \sigma \) be a smooth section of \( W \) supported in \( U_r \). The smooth sections \( e^{itD_{2r}}\sigma \) and \( e^{itD'}\sigma \) both satisfy the same wave equation with the same initial condition provided \( t \) be smaller than \( r \). Here we have used the unit speed propagation property of the theorem 2.2. The uniqueness of the wave operator implies their equality for \( 0 \leq t \leq r \). Now using the relation 2.4 we conclude the equality \( \phi_1 h(D)\phi_r = \phi_1 h(D_{2r})\phi_r \) which implies the invertibility of the operator (3.1) and the vanishing of its index in \( K_0(A) \). □

Now let \( M \) be a spin manifold and \( N \) be an enlargeable partitioning hypersurface of \( M \). Moreover assume that there is a smooth map \( \phi : M \to N \) such that its restriction to \( N \) is of non-zero degree. Under this condition there is a flat Hilbert \( A \)-module bundle \( V \) on \( N \) with the following properties (see [4,5] and more explicitly [2] theorem 3.1) :

(1) the index of the spin Dirac operator of \( N \) twisted by \( V \) is a non-zero element of \( K_0(A) \),
the index of the spin Dirac operator of \( N \) twisted by \( \phi_N^* V \) is equal to the index of the spin Dirac operator multiplied by \( \deg \phi_N \).

Now by applying the formula (2) to the Clifford bundle \( W := S \otimes \phi^* V \) we conclude the non-vanishing of \( \text{ind}(D, N) \). This fact along the above theorem show that the scalar curvature of \( g \) cannot be uniformly positive outside a compact subset of \( M \). So we get the following theorem.

**Theorem 3.2.** Let \( (M,g) \) be a complete riemannian spin manifold and let \( N \subset M \) be an area-enlargeable partitioning hypersurface. If there is a smooth map \( \phi : M \to N \) such that its restriction to \( N \) is of non-zero degree then the scalar curvature of \( g \) cannot be uniformly positive outside a compact subset of \( M \).

The following corollary is a direct consequence of this theorem.

**Theorem 3.3.** Let \((M,g)\) be a non-compact orientable complete spin \( n \)-manifold. Let \( N \) be a \((n-1)\)-dimensional sub-manifold of \( M \) which is area-enlargeable. Let \( M - N = M_+ \sqcup M_- \) and, say, \( M_+ \) is not compact. If there is a map \( \phi : \overline{M}_+ \to N \) such that its restriction to \( N \) has non-zero degree, then the scalar curvature of \( g \) cannot be uniformly positive outside a compact subset of \( \overline{M}_+ \).

**proof** By deforming the riemannian metric \( g \) in a compact collar neighborhood \( N \times [0,1) \) in \( \overline{M}_+ \) we can and will assume that \( g \) takes the product form \( gn + dt^2 \) where \( gn \) is a riemannian metric on \( N \). Similarly we can deform \( \phi \) in the same collar neighborhood and assume its restriction to \( N \times [0,1) \) is independent of \( t \). Let \( M^+ t \) denote the (non-complete) riemannian manifold \( M_+ \) with reversed orientation. The riemannian metric \( g|_{M^+_t} \) extends naturally (by reflection) to a complete riemannian metric \( g' \) on the manifold \( M_+ \sqcup M_{-t} \). The scalar curvature of \( g' \) is uniformly positive at infinity provided that the scalar curvature of \( g \) is uniformly positive at infinity. Moreover the map \( \phi \) extend (by reflection) to a smooth map \( \tilde{\phi} := \phi \sqcup \phi \) from \( M_+ \sqcup M_{-t} \) onto \( N \) with \( \deg \tilde{\phi}|_N \neq 0 \). Now we can apply the above theorem to deduce that the scalar curvature of \( g \) can not be uniformly positive outside a compact.

The set \( M_+ \) satisfying the above properties is called a bad-end for \( M \). M. Gromov and B. Lawson proved the assertion of the above theorem under the additional condition that the Ricci curvature of \( g \) be bounded from below on \( M_+ \), c.f [3] Theorem 7.46]. Therefore the above theorem is a considerable improvement of their result. Moreover, the above theorem improves the following theorem of Gromov-Lawson, c.f. [3] theorem 7.44] (for if \( N_0 \) is area-enlargeable then \( N_0 \times S^1 \) is area-enlargeable too).

Let \((M,g)\) be a connected complete riemannian manifold which contain a compact hypersurface \( N \) such that: \( N \) is diffeomorphic to \( N_0 \times S^1 \) where \( N_0 \) is enlargeable, \( \pi_1(N_0) \to \pi_1(M) \) is injective and there is a non-compact component \( M_+ \) of \( M - N \) and a map \( \overline{M}_+ \to N \) such that its restriction to \( N \) has non-zero degree. If one of the following two conditions is satisfied then the scalar curvature of \( g \) cannot be uniformly positive on \( M \)

1. The map \( \overline{M}_+ \to N \) is bounded,
2. \( N_0 \) has no transversal of finite area.

Here a transversal to \( N_0 \) is a properly embedded 2-dimensional sub manifold of \( M \) which is transversal to \( N_0 \) with non-zero intersection number. By comparing this theorem with the theorem [3,3] it is clear that we have relaxed the strong condition on the topology of \( N \) (it has not to be of the product form \( N_0 \times S^1 \)) and we need no condition on the fundamental groups in our theorem. Moreover we have relax the boundedness condition of the map \( \overline{M}_+ \to N \). In addition we have the stronger result that the scalar curvature cannot be positive even at infinity. Of course we have paid a price for all these: we have assumed \( M \) be a spin manifold while in the Gromov-Lawson theorem the spin condition is implicit in the enlargeability condition on \( N \).

**References**

[1] Saad Baaj. Calcul pseudo-différentiel et produits croisés de C*-algèbres. I. C. R. Acad. Sci. Paris Sér. I Math., 307(11):581–586, 1988.

[2] Mostafa Esfahani Zadeh. Index theory and partitioning by enlargeable manifolds. to appear in the Journal of Noncommutative Geometry.
[3] Mikhael Gromov and H. Blaine Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.

[4] B. Hanke and T. Schick. Enlargeability and index theory. *J. Differential Geom.*, 74(2):293–320, 2006.

[5] Bernhard Hanke and Thomas Schick. Enlargeability and index theory: infinite covers. *K-Theory*, 38(1):23–33, 2007.

[6] Nigel Higson. A note on the cobordism invariance of the index. *Topology*, 30(3):439–443, 1991.

[7] Dan Kucerovsky. Functional calculus and representations of \( C_0(C) \) on a Hilbert module. *Q. J. Math.*, 53(4):467–477, 2002.

[8] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[9] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York, 1972.

[10] John Roe. Partitioning noncompact manifolds and the dual Toeplitz problem. In *Operator algebras and applications, Vol. 1*, volume 135 of *London Math. Soc. Lecture Note Ser.*, pages 187–228. Cambridge Univ. Press, Cambridge, 1988.

[11] Jonathan Rosenberg. \( C^* \)-algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (58):197–212 (1984), 1983.

Mostafa Esfahani Zadeh.
Mathematisches Institute, Georg-August-Universität, Göttingen, Germany and
Institute for Advanced Studies in Basic Sciences(IASBS), Zanjan-Iran

*E-mail address*: zadeh@uni-math.gwdg.de