Pointwise estimates for degenerate Kolmogorov equations with $L^p$-source term

ERICA IPOCOANA AND ANNALAURA REBUCCI

Abstract. The aim of this paper is to establish new pointwise regularity results for solutions to degenerate second-order partial differential equations with a Kolmogorov-type operator of the form

$$\mathcal{L} := \sum_{i,j=1}^{m} \partial_{x_i x_j}^2 + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t,$$

where $(x, t) \in \mathbb{R}^{N+1}$, $1 \leq m \leq N$ and the matrix $B := (b_{ij})_{i,j=1,...,N}$ has real constant entries. In particular, we show that if the modulus of $L^p$-mean oscillation of $\mathcal{L} u$ at the origin is Dini, then the origin is a Lebesgue point of continuity in $L^p$ average for the second-order derivatives $\partial_{x_i x_j}^2 u$, $i,j = 1,...,m$, and the Lie derivative $\left(\sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t\right) u$. Moreover, we are able to provide a Taylor-type expansion up to second order with an estimate of the rest in $L^p$ norm. The proof is based on decay estimates, which we achieve by contradiction, blow-up and compactness results.

1. Introduction

In this paper, we study the pointwise regularity of solutions $u$ belonging to the Sobolev–Stein space $S^p(\Omega)$ (see Sect. 2) to the following Cauchy problem

$$\begin{cases} \mathcal{L} u = f & \text{in } \Omega^-_1, \\ f \in L^p(\Omega^-_r) & \text{and } f(0) = 0, \end{cases}$$

where $\Omega^-_r = B_r \times (-r^2, 0)$ is the past cylinder defined through the open ball $B_r = \{ x \in \mathbb{R}^N : |x|_K \leq r \}$ and $| \cdot |_K$ is the semi-norm, due to the nature of operator $\mathcal{L}$, defined in (1.14).

We suppose here that $1 < p < \infty$ and that the origin $0 = (0, 0)$ is a Lebesgue point of $f$, so that we are able to define $f(0)$ if needed.

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We denote by $\mathcal{L}$ the second-order linear differential operators of Kolmogorov type of the form

$$
\mathcal{L} := \sum_{i,j=1}^{m} \partial_{x_i x_j}^2 + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t,
$$

where $(x, t) \in \mathbb{R}^{N+1}$, and $1 \leq m \leq N$.

The matrix $B := (b_{ij})_{i,j=1,...,N}$ has real constant entries.

It is natural to place operator $\mathcal{L}$ in the framework of Hörmander’s theory. More precisely, let us set

$$
X_i := \partial_{x_i}, \quad i = 1, \ldots, m, \quad Y := \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t = \langle Bx, D \rangle - \partial_t,
$$

where $\langle \cdot, \cdot \rangle$ and $D$ denote the inner product and the gradient in $\mathbb{R}^N$, respectively. Then, the operator $\mathcal{L}$ can be written as

$$
\mathcal{L} = \sum_{i=1}^{m} X_i^2 + Y.
$$

It is known that under the Hörmander’s condition (see [6])

$$
\text{rank Lie}(X_1, \ldots, X_m, Y)(x, t) = N + 1, \quad \forall (x, t) \in \mathbb{R}^{N+1},
$$

$\mathcal{L}$ is hypoelliptic, namely that every distributional solution $u$ to $\mathcal{L}u = f$ defined in some open set $\Omega \subset \mathbb{R}^{N+1}$ belongs to $C^\infty(\Omega)$, and it is a classical solution to $\mathcal{L}u = f$, whenever $f \in C^\infty(\Omega)$.

In the sequel, we will assume the following hypothesis on the Kolmogorov operator $\mathcal{L}$.

[H.1] $\mathcal{L}$ is hypoelliptic and $\delta_r$-homogeneous of degree two with respect to some dilations group $\langle \delta_r \rangle_{r>0}$ in $\mathbb{R}^{N+1}$ (see (1.6) below).

We remark that, if $\mathcal{L}$ is uniformly parabolic (i.e. $m = N$ and $B \equiv 0$), then assumption [H.1] is clearly satisfied. In fact, in this case operator $\mathcal{L}$ is simply the heat operator, which is known to be hypoelliptic. However, in this note we are mainly interested in the genuinely degenerate setting.

It is known that the natural geometry when studying operator $\mathcal{L}$ is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. More precisely, as first observed by Lanconelli and Polidoro in [8], operator $\mathcal{L}$ is invariant with respect to left translation in the group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$, where the group law is defined by

$$
(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \quad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1},
$$

and

$$
E(s) = \exp(-sB), \quad s \in \mathbb{R}.
$$
Then $\mathbb{K}$ is a non-commutative group with zero element $(0,0)$ and inverse

$$(x,t)^{-1} = (-E(-t)x,-t).$$

For a given $\zeta \in \mathbb{R}^{N+1}$, we denote by $\ell_\zeta$ the left translation on $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$ defined as follows

$$\ell_\zeta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}, \quad \ell_\zeta(z) = \zeta \circ z.$$  

Then, the operator $\mathcal{H}$ is left invariant with respect to the Lie product $\circ$, that is

$$\mathcal{L} \circ \ell_\zeta = \ell_\zeta \circ \mathcal{L} \quad \text{or, equivalently,} \quad \mathcal{L}(u(\zeta \circ z)) = (\mathcal{L}u)(\zeta \circ z),$$

for every $u$ sufficiently smooth.

We explicitly remark that, by Propositions 2.1 and 2.2 in [8], hypothesis $[\text{H.1}]$ is equivalent to assume that, for some basis on $\mathbb{R}^N$, the matrix $B$ takes the following form

$$B_0 = \begin{pmatrix} O & O & \cdots & O & O \\ B_1 & O & \cdots & O & O \\ O & B_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & B_\kappa & O \end{pmatrix} \quad (1.5)$$

where every block $B_j$ is a $m_j \times m_{j-1}$ matrix of rank $m_j$ with $j = 1, 2, \ldots, \kappa$. Moreover, the $m_j$s are positive integers such that

$$m_0 \geq m_1 \geq \cdots \geq m_\kappa \geq 1, \quad \text{and} \quad m_0 + m_1 + \cdots + m_\kappa = N.$$  

We agree to let $m_0 := m$ to have a consistent notation; moreover, $O$ denotes a block matrix whose entries are zeros. In the sequel, we shall always assume that $B$ has the canonical form (1.5).

In this case, the dilation is defined for every positive $r$ as

$$\delta_r := \text{diag}(rI_m, r^3I_{m_1}, \ldots, r^{2\kappa+1}I_{m_\kappa}, r^2),$$

where $I_k, k \in \mathbb{N}$, is the $k$-dimensional unit matrix, and the second assertion in assumption $[\text{H.1}]$ reads as follows

$$\mathcal{L}(u \circ \delta_r) = r^2 \delta_r(\mathcal{L}u), \quad \text{for every} \quad r > 0.$$  

(1.7)

It is also useful to denote by $\left(\delta^0_r\right)_{r>0}$ the family of spatial dilations defined as

$$\delta^0_r = \text{diag}(rI_m, r^3I_{m_1}, \ldots, r^{2\kappa+1}I_{m_\kappa}) \quad \text{for every} \quad r > 0.$$  

(1.8)

The integer numbers

$$Q := m_0 + 3m_1 + \cdots + (2\kappa + 1)m_\kappa, \quad \text{and} \quad Q + 2$$

(1.9)
will be named homogeneous dimension of \( \mathbb{R}^N \) with respect to \((\delta^0_r)_{r > 0}, \) and homogeneous dimension of \( \mathbb{R}^{N+1} \) with respect to \((\delta_r)_{r > 0}, \) because we have that
\[
\det \delta^0_r = r^Q \quad \text{and} \quad \det \delta_r = r^{Q+2} \quad \text{for every } r > 0.
\]

Owing to (1.6), we recall the notion of homogeneous function in a homogeneous group. We say that a function \( u \) defined on \( \mathbb{R}^{N+1} \) is homogeneous of degree \( \alpha \in \mathbb{R} \) if
\[
u(\delta_r(z)) = r^\alpha u(z) \quad \text{for every } z \in \mathbb{R}^{N+1}.
\]

According to the previous definition, it is clear that the polynomials which are homogeneous of degree two with respect to dilation (1.6) are those of degree two in the first \( m \) spatial variables and one in time. For this reason, it is natural to define the following class of polynomials, which will be greatly used in the sequel. Namely,
\[
\tilde{\mathcal{P}} = \{ P : \text{polynomials of degree less or equal to two in } x_1 \ldots x_m \\
\text{and less or equal to one in } t \}.
\]
(1.10)
\[
\mathcal{P} := \{ P \in \tilde{\mathcal{P}} : \mathcal{L} P = 0 \}.
\]
(1.11)
\[
\mathcal{P}_c := \{ P \in \tilde{\mathcal{P}} : \mathcal{L} P = c \}.
\]
(1.12)

In particular we take \( P_* \) such that \( \mathcal{L} P_* = 1 \) and set \( \mathcal{P}_c = cP_* + \mathcal{P}. \)

We next introduce a homogeneous norm of degree 1 with respect to the dilations \((\delta_r)_{r > 0}\) and a corresponding quasi-distance which is invariant with respect to the group operation in (1.3). We first rewrite the matrix \( \delta_r \) with the equivalent notation
\[
\delta_r := \text{diag}(r^{\alpha_1}, \ldots, r^{\alpha_N}, r^2),
\]
(1.13)
where \( \alpha_1, \ldots, \alpha_{m_0} = 1, \alpha_{m_0+1}, \ldots, \alpha_{m_0+m_1} = 3, \alpha_{N-m_0}, \ldots, \alpha_N = 2k + 1. \)

**Definition 1.1.** For every \((x, t) \in \mathbb{R}^{N+1}\) we set
\[
\|(x, t)\|_K = |t|^{\frac{1}{2}} + |x|, \quad |x|_K = \sum_{j=1}^{N} |x_j|^{\frac{1}{\alpha_j}}
\]
(1.14)
where the exponents \( \alpha_j, \text{for } j = 1, \ldots, N, \) were introduced in (1.13)

Note that the following pseudo-triangular inequality holds: for every bounded set \( H \subset \mathbb{R}^{N+1} \) there exists a positive constant \( c_H \) such that
\[
\|(x, t)^{-1}\|_K \leq c_H \|(x, t)\|_K, \quad \|(x, t) \circ (\xi, \tau)\|_K \leq c_H (\|(x, t)\|_K + \|(\xi, \tau)\|_K),
\]
(1.15)
for every \((x, t), (\xi, \tau) \in H. \) We then define the quasi-distance \( d_K \) by setting
\[
d_K((x, t), (\xi, \tau)) := \|(\xi, \tau)^{-1} \circ (x, t)\|_K, \quad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1},
\]
(1.16)
and the unit cylinder
\[ Q_1 = \{(x, t) \in \mathbb{R}^{N+1} \mid |x|_K < 1, \ t \in (-1, 0)\}. \]

For every \((x_0, t_0) \in \mathbb{R}^{N+1}\) and \(r > 0\), we set
\[ Q_r(x_0, t_0) := z_0 \circ \delta_r(Q_1) = \{(x, t) \in \mathbb{R}^{N+1} \mid (x, t) = (x_0, t_0) \circ \delta_r(\xi, \tau), (\xi, \tau) \in Q_1\}. \]

We also observe that the Lebesgue measure is invariant with respect to the translation group associated to \(L\), since \(\det E(s) = e^{s \text{ trace } B} = 1\). Moreover, we have
\[ \text{meas}(Q_r(x_0, t_0)) = r^{Q_1+2} \text{meas}(Q_1(x_0, t_0)), \quad \forall \ r > 0, (x_0, t_0) \in \mathbb{R}^{N+1}. \]

Finally, we recall that, under the hypothesis of hypoellipticity, Hörmander in [6] constructed the fundamental solution of \(L\) as
\[ \Gamma(z, \zeta) = \Gamma(\zeta^{-1} \circ z, 0), \quad \forall z, \zeta \in \mathbb{R}^{N+1}, z \neq \zeta, \]
where
\[ \Gamma((x, t), (0, 0)) = \begin{cases} \frac{(-N)^N}{\sqrt{2\pi}} \exp\left(-\frac{1}{4} \langle C^{-1}(t)x, x \rangle - tt r(B)\right), & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases} \]

and
\[ C(t) = \int_0^t E(s) \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & 0 \end{pmatrix} E^T(s) \, ds. \]

Note that the first condition of assumption [H.1] implies that \(C(t)\) is strictly positive for every \(t > 0\) (see [8]) and therefore \(\Gamma\) in (1.17) is well defined.

We now briefly discuss the applicative and theoretical interest in the study of operator \(L\). A simple meaningful example is the operator introduced by Kolmogorov in [7], defined for \((x, t) = (v, y, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}\) as follows
\[ \mathcal{K} := \sum_{j=1}^m \partial_{x_j}^2 - \sum_{j=1}^m x_j \partial_{x_{m+j}} - \partial_t = \Delta v - \langle v, D_y \rangle - \partial_t. \tag{1.18} \]

The operator \(\mathcal{K}\) can be written in the form (1.2) with \(\kappa = 1, m_1 = m\), and
\[ B = \begin{pmatrix} 0 & 0 \\ -I_m & 0 \end{pmatrix} \tag{1.19} \]

The operator defined in (1.18) arises in several areas of application of PDEs. In particular, in kinetic theory the density \(u\) of particles, with velocity \(v\) and position \(y\) at time \(t\), satisfies equation \(\mathcal{K}u = 0\). In this setting, the Lie group associated to Kolmogorov operator has a quite natural interpretation. In fact, the composition law (1.3) agrees with the Galilean change of variables
\[(v, y, t) \circ (v_0, y_0, t_0) = (v_0 + v, y_0 + y + t, v_0 + t), \quad (v, y, t), (v_0, y_0, t_0) \in \mathbb{R}^{2m+1}. \]
It is easy to see that $\mathcal{K}$ is invariant with respect to the above change of variables. Specifically, if $w(v, y, t) = u(v_0 + v, y_0 + y + tv_0, t_0 + t)$ and $g(v, y, t) = f(v_0 + v, y_0 + y + tv_0, t_0 + t)$, then

$$\mathcal{K}u = f \iff \mathcal{K}w = g$$

for every $(v_0, y_0, t_0) \in \mathbb{R}^{2m+1}$.

As the matrix $B$ in (1.19) is in the form (1.5), $\mathcal{K}$ is invariant with respect to the dilatation $\delta_r(v, y, t) := (rv, r^3y, r^2t)$. Note that the dilatation acts as the usual parabolic scaling with respect to the variable $v$ and $t$, as the operator is uniformly parabolic with respect to these variables. The term $r^3$ in front of $y$ is due to the fact that the velocity $v$ is the derivative of the position $y$ with respect to time $t$. For a more comprehensive description of operator $\mathcal{L}$, and of its applications, we refer to the survey article [1] by Anceschi and Polidoro and to its bibliography.

The aim of this paper is to study pointwise regularity of solutions to problem (1.1) for Kolmogorov equations with right hand side in $L^p$. This work may be seen as a generalization of [9,13], where this kind of results are obtained for elliptic and parabolic equations, respectively. However, up to our knowledge, the case of Kolmogorov type operators has not been investigated.

The main difficulty with respect to the previous literature lies in the fact that the regularity properties of the Kolmogorov equations on $\mathbb{R}^{N+1}$ depend strongly on the geometric Lie group structure introduced in (1.3). In particular, this reflects on the family of dilations we consider. Furthermore, according to (1.2), we here take into account also the case where $m < N$ and therefore $\mathcal{L}$ is strongly degenerate. We emphasize that when $m = N$ and $B \equiv O$, our result restores the one contained in [9].

We remark that several Schauder-type estimates have been proved, e.g. by Šatyro [18], Manfredini [12] in the case of dilation invariant operators, Di Francesco et al. for dilation non-invariant operators in [5] and Polidoro et al. in [17]. In particular, we note that in [17] the right hand side $f$ is Dini continuous, differently from the previous literature where $f$ was considered Hölder continuous. Moreover, we recall the works by Lunardi [11], Lorenzi [10] and Priola [16] in the framework of semigroup theory.

In [15] and then in [4], a pointwise estimate for the weak solutions to Kolmogorov equations with right hand side equal to zero is proved. In order to do so, the authors adapt the Moser iterative method to the non-Euclidean framework of the (homogeneous and non-homogeneous, respectively) Lie groups. Finally, the regularity of strong solutions to the Cauchy–Dirichlet and obstacle problem for a class of Kolmogorov-type operators was studied in [14] using a blow-up technique.

The previous results in literature were derived assuming a modulus of continuity defined on some open set $Q \subset \mathbb{R}^{N+1}$, namely,

$$\omega_f(r) := \sup_{(x, t), (\xi, \tau) \in Q} |f(x, t) - f(\xi, \tau)|, \quad (1.20)$$

where $d_K$ was defined in (1.16).

On the other hand, we here introduce a pointwise modulus of mean oscillation.
More precisely, following [13], for $p \in (1, +\infty)$, we define the following *modulus of $L^p$-mean oscillation* for the function $f$ at the origin as

$$\tilde{\omega}(f; r) := \inf_{c \in \mathbb{R}} \left( \frac{1}{|Q_r|} \int_{Q_r} |f(x, t) - c|^p \right)^{\frac{1}{p}}. \quad (1.21)$$

We now set

$$\tilde{N}(u; r) := \inf_{P \in \tilde{P}} \left( \frac{1}{r^{Q+2+2p}} \int_{Q_r} |u - P|^p \right)^{\frac{1}{p}}, \quad (1.22)$$

where $Q$ is the homogeneous dimension defined in (1.9) and $\tilde{P}$ is the class of polynomials introduced in (1.10).

Owing to (1.21), let $c_r$ be a constant which realizes the infimum of the modulus of mean oscillation, that is

$$\tilde{\omega}(f; r) = \left( \frac{1}{|Q_r|} \int_{Q_r} |f(x, t) - c_r|^p \right)^{\frac{1}{p}}. \quad (1.23)$$

If $u$ is a solution of (1.1) then

$$\hat{N}(u, f; r) = \inf_{P \in \mathcal{P}_r} \left( \frac{1}{r^{Q+2+2p}} \int_{Q_r} |u - P|^p \right)^{\frac{1}{p}}. \quad (1.24)$$

Moreover, for $0 < a < b$, we define

$$\hat{N}(u, f; a, b) = \sup_{a \leq \varrho \leq b} \hat{N}(u, f; \varrho), \quad (1.25)$$

$$\tilde{\omega}(a, b) = \sup_{a \leq \varrho \leq b} \tilde{\omega}(f; \varrho). \quad (1.26)$$

In the sequel, we will also make use of the following notation. For a given $\lambda \in (0, 1)$, we set

$$\mathcal{N}(r) = \hat{N}(u, f; \lambda r, r), \quad (1.27)$$

$$\omega(r) = \tilde{\omega}(f; \lambda^2 r, r). \quad (1.28)$$

For readers’ convenience, we eventually recall the following definition.

**Definition 1.2.** A modulus of continuity $\omega$ is said Dini if it satisfies the following integral condition

$$\int_0^1 \frac{\omega(r)}{r} dr < +\infty.$$

This paper is devoted to prove the following Theorem.

**Theorem 1.3.** Let $p \in (1, \infty)$. Then, there exist constants $\beta, r_* \in (0, 1], \lambda \in (0, 1)$ and $C > 0$, such that the following holds. If $u \in L^p(Q^{-1}_1)$ satisfies (1.1) with the associated $\tilde{\omega}$ defined in (1.21), then we have
(i) **Pointwise BMO estimate**

\[
\sup_{r \in (0, 1]} \tilde{N}(u; r) \leq C \left\{ \left( \int_{Q_i^-} |u|^p \right)^{\frac{1}{p}} + \left( \int_{Q_i^-} |f|^p \right)^{\frac{1}{p}} + \sup_{r \in (0, 1]} \tilde{\omega}(f; r) \right\}.
\]

(1.29)

(ii) **Pointwise VMO estimate**

\[
(\tilde{\omega}(f; r) \to 0 \text{ as } r \to 0^+) \implies (\tilde{N}(u; r) \to 0 \text{ as } r \to 0^+).
\]

(1.30)

(iii) **Dini continuity of \( \tilde{N}(u; \cdot) \)**

If \( \tilde{\omega}(f; \cdot) \) is Dini, then \( \tilde{N}(u; \cdot) \) is Dini. In particular, for every \( \varrho \in (0, \frac{1}{4}) \), the following holds

\[
\int_0^{4\varrho} \frac{\tilde{N}(u; r)}{r} dr \leq C \left\{ \left( \frac{4\varrho}{\lambda} \right)^\beta (\tilde{N}(u; 1) + \tilde{\omega}(f; 1)) + \int_0^{4\varrho} \frac{\tilde{\omega}(f; r)}{r} dr + \varrho^{\beta} \int_{4\varrho}^1 \frac{\tilde{\omega}(f; r)}{r^{1+\beta}} dr \right\}.
\]

where \( C \) is a constant that does not depend on \( f, u \) and \( \varrho \).

(iv) **Pointwise control on the solution**

Let \( \tilde{\omega}(f; \cdot) \) be Dini. Then, there exists a unique polynomial \( P_0 \in \mathcal{P} \), namely a solution to equation \( \mathcal{L} P_0 = 0 \), with

\[
P_0(x, t) = a + \langle b, x \rangle + \frac{1}{2} \langle cx, x \rangle + dt,
\]

where \( b \) is a vector in \( \mathbb{R}^N \) such that \( b_j = 0 \) when \( j > m \) and \( c \) is a \( N \times N \) matrix such that \( c_{ij} = 0 \) when \( i > m \lor j > m \), such that for every \( r \in (0, \frac{\varrho}{4}) \) there holds

\[
\left( \frac{1}{|Q_r|} \int_{Q_r^-} \left| \frac{u(x, t) - P_0(x, t)}{r^2} \right|^p \right)^{\frac{1}{p}} \leq C \left\{ \tilde{M}_0 \left( \frac{4r}{\lambda} \right)^\beta + \int_0^{4r} \frac{\tilde{\omega}(f; s)}{s} ds + r^{\beta} \int_{4r}^1 \frac{\tilde{\omega}(f; s)}{s^{1+\beta}} ds \right\},
\]

with

\[
\tilde{M}_0 = \int_0^1 \frac{\tilde{\omega}(f; s)}{s} ds + \left( \int_{Q_1^-} |u|^p \right)^{\frac{1}{p}} + \left( \int_{Q_1^-} |f|^p \right)^{\frac{1}{p}}.
\]

Moreover, we have

\[
|a| + |b| + |c| + |d| \leq C \tilde{M}_0.
\]
From Theorem 1.3 iv) (inequality (1.31)), it is straightforward to observe that the next result holds.

**Corollary 1.4.** If the modulus of \( L^p \)-mean oscillation of \( \mathcal{L}u \) at the origin is Dini, then the origin is a Lebesgue point of continuity in \( L^p \) average for the second-order derivatives \( \partial^2_{x_i x_j} u, i, j = 1, \ldots, m \), and the Lie derivative \( Yu \).

We observe that a simple consequence of Theorem 1.3 is that the second-order derivatives \( \partial^2_{x_i x_j} u, i, j = 1, \ldots, m \), and the Lie derivative \( Yu \) are Hölder continuous in some open set \( \Omega \subset \mathbb{R}^{N+1} \), when \( \mathcal{L}u \) is Hölder continuous with respect to the distance introduced in (1.16). Moreover, let us remark that Theorem 1.3 provides us with a Taylor-type expansion up to second order with an estimate of the rest in \( L^p \) norm. We finally emphasize that, although we consider the regularity problem for weak solutions to Kolmogorov operators in the framework of the Sobolev spaces, our procedure is basically pointwise. Indeed, we consider some \( L^p \) norm of the function \( u - P_0 \) on a cylinder of radius \( r \) and we obtain our result by letting \( r \) going to zero. Thus, this approach follows the lines of regularity theory for classical solutions rather than the ones for weak solutions, which does not seem to be usual when dealing with Kolmogorov-type operators.

The structure of the paper is the following. Some general control results are contained in Sect. 2, some from the literature and a Caccioppoli-type estimate we proved ad hoc for our problem. Finally, Sect. 3 is devoted to prove our main result Theorem 1.3. In particular, for sake of simplicity, we first derive some preliminary estimates in Sect. 3.1 in order to finally give a shorter proof of Theorem 1.3 in Sect. 3.2.

2. Preliminary results

We here list some general ultraparabolic estimates. Some of them are well known from the literature, so for their proofs we will refer to source.

First, for \( \Omega \) open set in \( \mathbb{R}^{N+1} \), \( p \in (1, +\infty) \), we define the Sobolev-Stein space

\[ S^p(\Omega) = \{ u \in L^p(\Omega) : \partial_{x_i} u, \partial^2_{x_i x_j} u, Yu \in L^p(\Omega), \ i, j = 1, \ldots, m \}. \]

If we set

\[ \| u \|_{S^p(\Omega)}^p = \| u \|_{L^p(\Omega)}^p + \sum_{i=1}^{m} \| \partial_{x_i} u \|_{L^p(\Omega)}^p + \sum_{i,j=1}^{m} \| \partial^2_{x_i x_j} u \|_{L^p(\Omega)}^p + \| Yu \|_{L^p(\Omega)}^p \]

we have the following local a priori estimates in \( S^p(\Omega) \) for solutions to \( \mathcal{L}u = f \) (see [2]).

**Theorem 2.1.** (Ultraparabolic interior \( L^p \)-estimates) Assume \([H.1]\) holds and let \( u \) be a solution to \( \mathcal{L}u = f \) in \( \Omega \), where \( \Omega \) is now a bounded open set in \( \mathbb{R}^{N+1} \). If \( \Omega_1 \subset \subset \Omega \), then we can find a constant \( c \), only depending on \( B \), \( p \), \( \Omega \) and \( \Omega_1 \), such that

\[ \| u \|_{S^p(\Omega_1)} \leq c(\| f \|_{L^p(\Omega)} + \| u \|_{L^p(\Omega)}). \]
We now state a general compactness result proved in [3].

**Theorem 2.2.** Let $\Omega$ be an open set of $\mathbb{R}^{N+1}$ and let $u \in S^p(\Omega)$ be a weak solution to $\mathcal{L} u = f$ in $\Omega$ with $f \in L^p_{\text{loc}}(\Omega)$. Then, for every $z_0 \in \Omega$ and $\rho, \sigma > 0$ such that $Q_\rho(z_0)$ is contained in $\Omega$ and $\sigma < \frac{\rho}{2c_H}$, with $c_H$ defined in (1.15), we have that if $1 < p < Q + 2$ and $p < q < p^*$ then there exists a positive constant $\tilde{C}_{p,q}$ such that

$$
\|u(\cdot \circ h) - u\|_{L^q(Q_\sigma(z_0))} \leq \tilde{C}_{p,q}(\|u\|_{L^p(Q_\rho(z_0))} + \|f\|_{L^p(Q_\rho(z_0))})\|h\|^{(Q+2)(\frac{1}{q} - \frac{1}{p^*})}
$$

where

$$
\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q+2}.
$$

As a preliminary result, we state and prove the following Caccioppoli type estimate which we obtained ad hoc for our problem.

**Lemma 2.3.** (Caccioppoli type estimate) Let $P \in \mathcal{P}_{cr}$ and let $u$ be a solution to (1.1) in $Q^-_r$. Let $p \in (1, +\infty)$ and let $\rho, r$ such that $1 \leq \rho < r$. Then, for $W := \frac{(u - P)|u - P|^{p-2}}{p-1}$, the following estimate holds:

$$
\frac{2(p-1)}{p^2} \int_{Q^-_r} |D_m W|^2 \leq \left( \frac{2}{(p-1)} \frac{c_2^2}{(r-\rho)^2} + \frac{2}{p} \frac{c_1 r^{2\alpha+1}}{r - \rho} \right) \int_{Q^-_r} W^2 + \tilde{\omega}(f; r) |Q^-_r| \left( \frac{1}{|Q^-_r|} \int_{Q^-_r} \eta^{2p'} W^2 \right)^{\frac{1}{p'}}.
$$

where $c_1, c_2$ are dimensional constants, $p'$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $D_m$ denotes the partial gradient in the first $m$ variables, that is

$$
D_m := (\partial_{x_1}, \ldots, \partial_{x_m}).
$$

**Proof.** On the past cylinder $Q^-_r$, we have

$$
- \mathcal{L} u + f = 0, \quad \text{and} \quad - \mathcal{L} P + c_r = 0,
$$

since $u$ is a solution to (1.1) and $P \in \mathcal{P}_{cr}$. We set $\phi := \eta^2 w |w|^{p-2}$, where $w := u - P$ and $\eta$ is a $C^\infty$ function with compact support, to be chosen later. Taking the difference of the two equations in (2.2) and multiplying it by $\phi$, we obtain

$$
- \int_{Q^-_r} \eta^2 w |w|^{p-2} \mathcal{L} w = - \int_{Q^-_r} \eta^2 w |w|^{p-2} (f(x) - c_r).
$$

An integration by parts shows that

$$
- \int_{Q^-_r} \eta^2 w |w|^{p-2} \mathcal{L} w = \int_{Q^-_r} \langle AD_m w, D_m(\eta^2 w |w|^{p-2}) \rangle - \int_{Q^-_r} \eta^2 w |w|^{p-2} Y(w)
$$

$$
=: I_1 + I_2,
$$

(2.4)
where $D_m$ denotes the gradient with respect to $x_1, \ldots, x_m$. We now observe that

$$D_m \phi = 2\eta D_m \eta \, w|w|^{p-2} + \eta^2 (p - 1)|w|^{p-2} D_m w$$

and therefore we can rewrite the term $I_1$ on the right hand side of (2.4) as

$$I_1 = 2 \int_{Q_r^-} \langle AD_m w, D_m \eta \rangle \eta w|w|^{p-2} + (p - 1) \int_{Q_r^-} \eta^2 |w|^{p-2} \langle AD_m w, D_m w \rangle.$$

Taking advantage of $D_m W = \frac{p}{2} |w|^{\frac{p}{p-1}} D_m w$, for $W = w|w|^{\frac{p}{p-1}}$, the previous equation rewrites as

$$I_1 = \frac{4(p - 1)}{p^2} \int_{Q_r^-} \eta^2 \langle AD_m W, D_m W \rangle + \frac{4}{p} \int_{Q_r^-} \eta W \langle AD_m W, D_m \eta \rangle. \tag{2.5}$$

We now take care of the term $I_2$ in (2.4). We first notice that

$$Y(W) = \frac{p}{2} |w|^{\frac{p}{p-1}} Y(w),$$

which, together with the divergence theorem and the identity

$$Y(W^2 \eta^2) = 2\eta W^2 Y(\eta) + 2\eta^2 W Y(W),$$

yields

$$I_2 = \frac{2}{p} \int_{Q_r^-} \eta W^2 Y(\eta). \tag{2.6}$$

Thus, combining (2.5) and (2.6), we can rewrite identity (2.3) as

$$0 = \frac{4(p - 1)}{p^2} \int_{Q_r^-} \eta^2 \langle AD_m W, D_m W \rangle + \frac{4}{p} \int_{Q_r^-} \eta W \langle AD_m W, D_m \eta \rangle$$

$$+ \frac{2}{p} \int_{Q_r^-} \eta W^2 Y(\eta) + \int_{Q_r^-} \eta^2 w|w|^{p-2} (f(x) - c_r).$$

Now, setting $\varepsilon = \frac{p-1}{2p}$ and using the estimate

$$\eta |W| \langle AD_m W, D_m \eta \rangle \leq \varepsilon \eta^2 \langle AD_m W, D_m W \rangle + \frac{W^2}{4\varepsilon} \langle AD_m \eta, D_m \eta \rangle,$$
we finally obtain
\[
\frac{2(p - 1)}{p^2} \int_{Q_r^-} \eta^2 \langle AD_m W, D_m W \rangle \\
\leq \frac{2}{(p - 1)} \int_{Q_r^-} W^2 \langle AD_m \eta, D_m \eta \rangle + \frac{2}{p} \int_{Q_r^-} W^2 \eta |Y(\eta)| \\
+ \int_{Q_r^-} |f - c_r \eta^2| W \frac{2(p-1)}{p} \\
\leq \frac{2}{(p - 1)} \int_{Q_r^-} W^2 \langle AD_m \eta, D_m \eta \rangle + \frac{2}{p} \int_{Q_r^-} W^2 \eta |Y(\eta)| \\
+ |Q_r^-| \tilde{\omega}(f; r) \left( \frac{1}{|Q_r^-|} \int_{Q_r^-} \eta^{2p' \frac{2(p-1)}{p}} |W|^{\frac{1}{p'}} \right)^{\frac{1}{p}} \\
\leq \frac{2}{(p - 1)} \int_{Q_r^-} W^2 \langle AD_m \eta, D_m \eta \rangle + \frac{2}{p} \int_{Q_r^-} W^2 \eta |Y(\eta)| \\
+ |Q_r^-| \tilde{\omega}(f; r) \left( \frac{1}{|Q_r^-|} \int_{Q_r^-} \eta^{2p' \frac{2(p-1)}{p}} |W|^{\frac{1}{p'}} \right)^{\frac{1}{p'}}. \tag{2.7}
\]

where \( p' \) is such that \( \frac{1}{p} + \frac{1}{p'} = 1 \). The thesis follows by making a suitable choice of the function \( \eta \) in (2.7). More precisely, we set

\[
\eta(x, t) = \chi \left( \| (x, 0) \|_K \right) \chi_t(t)
\]

where \( \chi \in C^\infty([0, +\infty)) \) is the cut-off function defined by

\[
\chi(s) = \begin{cases} 
0, & \text{if } s \geq r, \\
1, & \text{if } 0 \leq s \leq \varrho,
\end{cases}
|\chi'| \leq \frac{2}{r - \varrho},
\]

and \( \chi_t \in C^\infty((-\infty, 0]) \) is defined by

\[
\chi_t(s) = \begin{cases} 
0, & \text{if } s \leq -r^2, \\
1, & \text{if } -\varrho^2 \leq s \leq 0,
\end{cases}
|\chi_t'| \leq \frac{2}{r - \varrho},
\]

with \( \frac{\xi}{2} \leq \varrho < r \). We observe that

\[
|Y \eta| \leq c_1 \frac{r^{2x+1}}{r - \varrho}, \quad |\partial_{x_j} \eta| \leq \frac{c_2}{r - \varrho} \quad \text{for } j = 1, \ldots, m_0,
\]

where \( c_1 \) and \( c_2 \) are dimensional constants. Then, accordingly to (2.7), we finally obtain

\[
\frac{2(p - 1)}{p^2} \int_{Q_\varrho} |D_m W|^2 \\
\leq \frac{2}{(p - 1)} \frac{c_2^2}{(r - \varrho)^2} \int_{Q_r^-} W^2 + \frac{2}{p} c_1 \frac{r^{2x+1}}{r - \varrho} \int_{Q_r^-} W^2 \\
+ |Q_r^-| \tilde{\omega}(f; r) \left( \frac{1}{|Q_r^-|} \int_{Q_r^-} \eta^{2p' \frac{2(p-1)}{p}} |W|^{\frac{1}{p'}} \right)^{\frac{1}{p'}}
\]

and this concludes the proof. \( \square \)
3. Pointwise estimates for the Kolmogorov equation

This section is the core of the paper and it is devoted to prove our main result, Theorem 1.3. Since the proof is rather convoluted, we have decomposed it in intermediate results proved in Sect. 3.1, which will be combined in Sect. 3.2 in order to give a simpler proof of Theorem 1.3.

3.1. Preliminary estimates

The following result is a useful tool in order to prove Lemma 3.2.

**Lemma 3.1.** The following statements hold:

(i) there exists a constant $C_2 = C_2(p, Q) > 0$ s.t. for every polynomial $P \in \tilde{P}$, for any $r \geq 1$ it holds

$$\left(\frac{1}{rQ^{2+2p}} \int_{Q^{-r}_r} |P|^p \right)^{\frac{1}{p}} \leq C_2 \left(\int_{Q^{-1}_r} |P|^p \right)^{\frac{1}{p}} ;$$

(ii) there exists a constant $\tilde{C}_2 = \tilde{C}_2(p, Q) > 0$ s.t. for every polynomial $P \in \tilde{P}$, for any $r < 1$ it holds

$$\left(\int_{Q^{-1}_r} |P|^p \right)^{\frac{1}{p}} \leq \tilde{C}_2 \left(\frac{1}{rQ^{2+2p}} \int_{Q^{-r}_r} |P|^p \right)^{\frac{1}{p}}.$$

**Proof.** We only carry out the proof of assertion (i), since case (ii) is totally analogous. We start by writing the polynomial $P$ as $P(x, t) = a + (b, x) + \frac{1}{2}(cx, x) + dt$, where $b$ is a vector in $\mathbb{R}^N$ such that $b_j = 0$ when $j > m$ and $c$ is a $N \times N$ matrix such that $c_{ij} = 0$ when $i > m \lor j > m$. We moreover recall that $Q^{-r}_r = B_r \times (-r^2, 0)$, where $B_r = \{x \in \mathbb{R}^N : |x|_K \leq r\}$, with $|\cdot|_K$ as defined in (1.14). Then, owing to $\|x, t\|_K = |x|_K + |t|^{1/2}$ with in particular $|x_i| \leq r$ for $i = 1 \ldots m$ and $|t| \leq r^2$, there exists a constant $C > 0$ s.t.

$$\left(\frac{1}{rQ^{2+2p}} \int_{Q^{-r}_r} |P|^p \right)^{\frac{1}{p}} \leq C \left(\frac{|a|}{r^2} + \frac{|b|}{r} + |c| + |d| \right). \quad (3.1)$$

On the other hand, it is possible to show by contradiction that

$$|a| + |b| + |c| + |d| \leq C \left(\int_{Q^{-1}_r} |P|^p \right)^{\frac{1}{p}}.$$ \quad (3.2)

Indeed, if (3.2) is false, we have that for every constant $K > 0$ it holds

$$|a| + |b| + |c| + |d| > \frac{1}{K} \left(\int_{Q^{-1}_r} |P|^p \right)^{\frac{1}{p}}.$$

(3.3)
Therefore, for $K \to 0^+$, it follows that while the right hand side of (3.3) goes to infinity, the left hand side remains constant. This implies a contradiction, as the sum of the norms of the coefficients of $P$ would be both a constant and infinity. The thesis follows by the combination of (3.1) and (3.2), with $r \geq 1$. 

We now prove the following Lemma.

**Lemma 3.2.** (Estimates on larger cylinders) Let $u$ be solution of $\mathcal{L} u = f$ in $Q_R^-$ for $R > 2$. Then for any $\varrho \in [1, R/2]$, there exists a positive constant $C_1 = C_1(p, Q)$ s.t.

\[
\left( \frac{1}{Q^{+2+2p}_r} \int_{Q^{+2+2p}_r} |u - P_1|^p \, dx \, dt \right)^{\frac{1}{p}} \leq C_1 \int_1^{4\varrho} \frac{\hat{N}(u, f; s) + \tilde{w}(f; s)}{s} \, ds, \tag{3.4}
\]

where $P_1 \in \mathcal{P}_1$ is the polynomial realizing the infimum in definition (1.24) at level one.

**Proof.** We start working on the left hand side of (3.4). Namely, for any $\varrho \geq 1$

\[
\left( \frac{1}{Q^{+2+2p}_r} \int_{Q^{+2+2p}_r} |u - P_1|^p \right)^{\frac{1}{p}} \\
\leq \left( \frac{1}{Q^{+2+2p}_r} \int_{Q^{+2+2p}_r} |u - P_{\varrho}|^p \right)^{\frac{1}{p}} \\
+ \left( \frac{1}{Q^{+2+2p}_r} \int_{Q^{+2+2p}_r} |P_{\varrho} - P_1|^p \right)^{\frac{1}{p}} = \tilde{N}(u, f; \varrho) + I_1. \tag{3.5}
\]

where in the last line we recalled (1.24), and $P_{\varrho} \in \mathcal{P}_{c\varrho}$ is a polynomial realizing the infimum in the definition of $\tilde{N}(u, f; \cdot)$ at the level $\varrho$. We now estimate $I_1$ as follows

\[
I_1 \leq \left( \frac{1}{Q^{+2+2p}_r} \int_{Q^{+2+2p}_r} |P_r - P_1|^p \right)^{\frac{1}{p}} + \sum_{j=1}^{k} \left( \frac{1}{Q^{+2+2p}_r} \int_{Q^{+2+2p}_r} |P_{2j} - P_{2j-1}|^p \right)^{\frac{1}{p}} \\
= : I_2 + I_3, \tag{3.6}
\]

where we have written $\varrho \geq 1$ as $\varrho = 2^kr$ for an integer $k \geq 1$ and $r \in [1/2, 1)$. In order to control $I_2$ and $I_3$ we need to achieve a more general estimate. For an arbitrary $\gamma > 1$, for any $\alpha \in [1, \gamma]$ we have that

\[
\left( \frac{1}{r^{Q+2+2p}_r} \int_{Q^{+2+2p}_r} |P_{ar} - P_r|^p \right)^{\frac{1}{p}} \\
\leq \left( \frac{1}{r^{Q+2+2p}_r} \int_{Q^{+2+2p}_r} |u - P_r|^p \right)^{\frac{1}{p}} + \left( \frac{1}{r^{Q+2+2p}_r} \int_{Q^{+2+2p}_r} |u - P_{ar}|^p \right)^{\frac{1}{p}} \\
\leq \left( \frac{1}{r^{Q+2+2p}_r} \int_{Q^{+2+2p}_r} |u - P_r|^p \right)^{\frac{1}{p}} + \alpha^{\frac{Q+2+2p}{p}} \left( \frac{1}{r^{Q+2+2p}_r} \int_{Q^{+2+2p}_r} |u - P_{ar}|^p \right)^{\frac{1}{p}} \\
\leq \alpha^{\frac{Q+2+2p}{p}} \left( \hat{N}(u, f; r) + \hat{N}(u, f; \alpha r) \right) \\
\leq \gamma^{\frac{Q+2+2p}{p}} \left( \hat{N}(u, f; r) + \hat{N}(u, f; \alpha r) \right). \tag{3.7}
\]
We take care of $I_2$ choosing $\alpha r = 1$, for $r \in \left[\frac{1}{\gamma}, 1\right]$ in (3.7). Namely, applying both case (i) and (ii) from Lemma 3.1, we get

$$I_2 := \left(\frac{1}{Q^{2+2p}} \int_{Q^c} |P_r - P_1|^p\right)^{\frac{1}{p}} \leq C_2 \left(\int_{Q^c} |P_1 - P_r|^p\right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{2+2p}} \int_{Q^c} |P_1 - P_r|^p\right)^{\frac{1}{p}} \leq \gamma \frac{Q^{2+2p}}{p} (\tilde{N}(u, f; r) + \tilde{N}(u, f; 1)).$$

(3.8)

Exploiting again (3.7) together with case (i) from Lemma 3.1, we infer that for every $\rho \geq 1$, which we write as $\rho = 2^k r$ with $r \in \left[\frac{1}{\gamma}, 1\right]$, there holds

$$I_3 \leq C \sum_{j=1}^{k} \tilde{N}(u, f; 2^j r).$$

(3.9)

Now, collecting bounds (3.6), (3.8) and (3.9), we have that (3.5) reads

$$\left(\frac{1}{Q^{2+2p}} \int_{Q^c} |u - P_1|^p\right)^{\frac{1}{p}} \leq C \left(\tilde{N}(u, f; 1) + \tilde{N}(u, f; r) + \sum_{j=1}^{k} \tilde{N}(u, f; 2^j r)\right),$$

(3.10)

where $C = C(p, Q, \gamma)$ is a positive constant. We now want to estimate the right hand side of (3.10), in particular for any $\gamma > 1$ and for $\alpha \in [1, \gamma]$ it follows that

$$\tilde{N}(u, f; \alpha r) \leq \left(\frac{1}{(\alpha r)^Q} \int_{Q^c} |u - P_{\alpha r}|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{(\gamma r)^Q} \int_{Q^c} |u - P_{\gamma r}|^p\right)^{\frac{1}{p}} \left(\frac{Q^{2+2p}}{p}\right)^{\frac{1}{p}} (\gamma r)^{2p} \tilde{N}(u, f; \gamma r) + C_2 |c_{\alpha r} - c_{\gamma r}| \left(\int_{Q^c} |P_{\alpha r}|^p\right)^{\frac{1}{p}}$$

(3.11)

where in the last line we used case (i) of Lemma 3.1 and we introduced $P_{\alpha r}$ as a solution to equation $\mathcal{L} P_{\alpha r} = 1$. In particular, from (1.23), we obtain

$$|c_{\alpha r} - c_{\gamma r}| = \left(\frac{1}{|Q_{\alpha r}|} \int_{Q_{\alpha r}} |c_{\alpha r} - c_{\gamma r}|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{|Q_{\alpha r}|} \int_{Q_{\alpha r}} |f - c_{\alpha r}|^p\right)^{\frac{1}{p}} + \gamma^{\frac{Q+2}{p}} \left(\int_{Q_{\gamma r}} |f - c_{\gamma r}|^p\right)^{\frac{1}{p}} \leq \tilde{\omega}(f; \alpha r) + \gamma^{\frac{Q+2}{p}} \tilde{\omega}(f; \gamma r) \leq 2 \gamma^{\frac{Q+2}{p}} \tilde{\omega}(f; \gamma r).$$

(3.12)
Thus, combining (3.12) with (3.11) we infer that for any $\gamma > 1$ there exists a positive constant $C_{\gamma} = C_{\gamma}(p, Q, \gamma)$ s.t. for any $\alpha \in [1, \gamma]$ it holds
\[
\hat{N}(u, f; \alpha r) \leq C_{\gamma}\left(\hat{N}(u, f; \gamma r) + \tilde{\omega}(f; \gamma r)\right),
\]
\[
\tilde{\omega}(f; \alpha r) \leq C_{\gamma}\tilde{\omega}(f; \gamma r).
\]
(3.13)

Eventually, putting together (3.10) and (3.13) and choosing $\gamma = 2$ we finally obtain
\[
\left(\frac{1}{\varrho_Q^{p+2} + 2p} \int_{Q} |u - P_1|^p\right)^{1/p} \leq 3C \sum_{j=1}^{k} \left(\hat{N}(u, f; 2^{j+1}r) + \tilde{\omega}(f; 2^{j+1}r)\right)
\]
\[
\leq 6C \sum_{j=1}^{k} \frac{\hat{N}(u, f; 2^{j+1}r) + \tilde{\omega}(f; 2^{j+1}r)}{2^{j+1}}
\]
\[
\left(2^{j+2}r - 2^{j+1}r\right)
\]
\[
\leq 6C \int_{1}^{4} \frac{\hat{N}(u, f; s) + \tilde{\omega}(f; s)}{s} ds.
\]
\[
\square
\]

As a consequence, we can prove the decay estimate below.

**Proposition 3.3.** (Basic decay estimate) Given $p \in (1, +\infty)$, there exist constants $C_0 = C_0(p, Q) > 0$ and $\lambda = \lambda(p, Q), \mu = \mu(p, Q) \in (0, 1)$ such that for every function $u$ and $f$ satisfying (1.1), $\forall r \in (0, 1]$, the following estimates hold
\[
\hat{N}(u, f; \lambda^2 r, \lambda r) < \mu \hat{N}(u, f; \lambda r, r) \quad \text{or} \quad \hat{N}(u, f; \lambda^2 r, \lambda r) < C_0 \tilde{\omega}(f; \lambda^2 r, r).
\]
(3.14)

**Proof.** The proof is carried out by contradiction. Namely, if (3.14) is not true, we can find the sequences $C_k \to \infty$, $r_k \in (0, 1]$, $\lambda_k \to 0$ and $\mu_k \to 1$ such that
\[
\hat{N}(u_k, f_k; \lambda_k^2 r_k, \lambda_k r_k) \geq \mu_k \hat{N}(u_k, f_k; \lambda_k r_k, r_k)
\]
\[
\hat{N}(u_k, f_k; \lambda_k^2 r_k, \lambda_k r_k) \geq C_k \tilde{\omega}(f_k; \lambda_k^2 r_k, r_k),
\]
(3.15)
(3.16)

where $(f_k)_k$ and $(u_k)_k$ satisfy (1.1). Let us consider $\varrho_k \in [\lambda_k^2 r_k, \lambda_k r_k]$ such that, according to (1.25)
\[
\hat{N}(u_k, f_k; \lambda_k^2 r_k, \lambda_k r_k) = \hat{N}(u_k, f_k; \varrho_k) = : \varepsilon_k.
\]
(3.17)

Moreover, owing to (1.6), we define the rescaled functions
\[
u_k(x, t) = \frac{u_k(\delta \varrho_k(x, t))}{\varrho_k^2}
\]
and
\[
    w_k(x, t) = \frac{u_k(\delta_{\varrho_k}(x, t)) - P_k(\delta_{\varrho_k}(x, t))}{\varepsilon_k \varrho_k^2}
\]  
(3.18)

where \( P_k \in \mathcal{P}_{\varrho_k} \) is the homogeneous polynomial realizing the infimum at the level \( \varrho_k \).

Now we want to control \( w_k \) in order to pass to the limit. We first notice that
\[
    \inf_{P \in \mathcal{P}} \left( \int_{Q_1^-} |w_k - P|^p \right)^{\frac{1}{p}} = 1.
\]  
(3.19)

Indeed, first exploiting the definition of \( w_k \) in (3.18) and then using the change of variables \( y = \delta_{\varrho_k}(x, t) \), \( s = \varrho_k^2 t \), owing to (1.8), we infer
\[
    \inf_{P \in \mathcal{P}} \left( \int_{Q_1^-} |w_k - P|^p \right)^{\frac{1}{p}} = \inf_{P \in \mathcal{P}} \left( \int_{Q_1^-} \left| \frac{u_k(\delta_{\varrho_k}(y, s)) - P_k(\delta_{\varrho_k}(y, s)) - \varepsilon_k \varrho_k^2 P(x, t)}{\varepsilon_k \varrho_k^2} \right|^p \right)^{\frac{1}{p}} dx \, ds = \varepsilon_k \varrho_k^2 \int_{Q_1^-} \left| \frac{u_k(y, s) - P_k(y, s) - \varepsilon_k \varrho_k^2 P(\delta_{\varrho_k}(y), \frac{1}{\varrho_k}s)}{\varrho_k^2} \right|^p \right)^{\frac{1}{p}} dy \, ds.
\]

Now, since \( \mathcal{L}(P_k + \varepsilon_k \varrho_k^2 P) = \mathcal{L}P_k + \varepsilon_k \varrho_k^2 \mathcal{L}P = c_{\varrho_k} \) by (1.11) and (1.12), the identity (3.19) follows from (3.17).

In addition it holds
\[
    \hat{N}(v_k, f_k; 1) = \left( \frac{1}{\varrho_k^{Q+2+2p}} \int_{Q_{\varrho_k}} |u_k - P_k|^p \right)^{\frac{1}{p}}.
\]

We now apply Lemma 3.2 to \( v_k \), for \( s \in \left[ 1, \frac{r_k}{s \varrho_k} \right] \)
\[
    \left( \frac{1}{s \varrho_k^{Q+2+2p}} \int_{Q_{s \varrho_k}} |u_k - P_k|^p \right)^{\frac{1}{p}} = \left( \frac{1}{s \varrho_k^{Q+2+2p}} \int_{Q_{s \varrho_k}} \left| \frac{u_k(\delta_{\varrho_k}(y, s)) - P_k(\delta_{\varrho_k}(y, s))}{\varrho_k^2} \right|^p \right)^{\frac{1}{p}} \leq C_1 \int_1^{4s} \frac{\hat{N}(v_k, g_k; \tau) + \tilde{\omega}(g_k; \tau)}{\tau} d\tau \leq C_1 \int_1^{4s} \frac{\hat{N}(u_k, f_k; \tau \varrho_k) + \tilde{\omega}(f_k; \tau \varrho_k)}{\tau} d\tau,
\]  
(3.20)

where in the second line we defined \( g_k(x, t) = f_k(\delta_{\varrho_k}(x, t)) \) and in the third line we used the identities \( \hat{N}(v_k, g_k; s) = \hat{N}(u_k, f_k; s \varrho_k) \) and \( \tilde{\omega}(g_k; s) = \tilde{\omega}(f_k; s \varrho_k) \). As a
consequence, for $s \in \left[ 1, \frac{r_k}{\varrho_k} \right]$, the following holds
\[
\left( \frac{1}{s^{Q+2+2p}} \int_{Q_s^-} |w_k|^p \right)^{\frac{1}{p}} \leq \frac{C_1}{\varepsilon_k} \int_1^{4s} \frac{\hat{N}(u_k, f_k; \tau \varrho_k) + \tilde{\omega}(f_k; \tau \varrho_k)}{\tau} \, d\tau
\]
\[
\leq \frac{C_1}{\varepsilon_k} \int_1^{4s} \frac{\hat{N}(u_k, f_k; \lambda_k^2 r_k, r_k) + \tilde{\omega}(f_k; \lambda_k^2 r_k, r_k)}{\tau} \, d\tau. 
\]
(3.21)

On the other hand, combining (3.15) with (3.17), we obtain
\[
\hat{N}(u_k, f_k; \lambda_k^2 r_k, r_k) \leq \frac{\varepsilon_k}{\mu_k}
\]
and
\[
\tilde{\omega}(f_k; \lambda_k^2 r_k, r_k) \leq \frac{\varepsilon_k}{C_k}.
\]
(3.22)
These two bounds together with (3.21) yield to
\[
\left( \frac{1}{s^{Q+2+2p}} \int_{Q_s^-} |w_k|^p \right)^{\frac{1}{p}} \leq C_2 \ln 4s
\]
where $s \in \left[ 1, \frac{r}{\varrho} \right]$ and $C_2$ is a positive constant depending on $C_1$, $C_k$ and $\mu_k$.

Now, according to the dilation invariance of $L$ with respect to $\delta_r$ (see (1.7)) and (3.22), we find
\[
\left( \frac{1}{|Q_s^-|} \int_{Q_s^-} |\mathcal{L} w_k|^p \right)^{\frac{1}{p}} \leq \frac{1}{\varepsilon_k} \tilde{\omega}(f_k; s \varrho_k) \leq \frac{1}{\varepsilon_k} \tilde{\omega}(f_k; \lambda_k^2 r_k, r_k) \leq \frac{1}{C_k} \rightarrow 0
\]
(3.24)

The contradiction follows from passing to the limit. In order to do so, we need a compactness argument.

Applying Lemma 2.3 to $W_k = w_k |w_k|^\frac{p}{2} - 1$, we obtain that for every $R \in \left[ 1, \frac{r_k}{\varrho_k} \right)$
\[
\frac{2(p-1)}{p^2} \int_{Q_R^-} |D_m W_k|^2 \leq \left( \frac{2}{(p-1)} \frac{c_2^2}{(r-\varrho)^2} + \frac{2}{p} c_1 \frac{r^{2k+1}}{r-\varrho} \right)
\]
\[
\int_{Q_R^-} W_k^2 + \tilde{\omega}(f; r) |Q_R^-| \left( \frac{1}{|Q_R^-|} \int_{Q_R^-} \eta^{2p'} W_k^2 \right)^{\frac{1}{p}},
\]
for $r = \frac{r_k}{\varrho_k}$. As a consequence, for every $R \in \left( 0, \frac{r_k}{\varrho_k} \right)$, we have
\[
\| W_k \|_{S^2(Q_R^-)} \leq C_R.
\]
Thus, we can extract a non-relabelled subsequence $W_k$ such that
\[
W_k \rightharpoonup W_\infty = w_\infty |w_\infty|^\frac{p}{2} - 1 \text{ weakly in } S^2_{loc}(Q_R^-),
\]
where we denoted by \( w_\infty \) the limit of the sequence \( w_k \). Moreover, from the compact embedding provided by Theorem 2.2 it follows that

\[
W_k \rightarrow W_\infty = w_\infty |w_\infty|^{\frac{p}{2} - 1} \quad \text{in} \; L^2_{loc}(\mathcal{Q}_R^-).
\]

We now observe that

\[
\|w_k\|_{L^p(\mathcal{Q}_R^-)}^p = \|W_k\|_{L^2(\mathcal{Q}_R^-)}^2, \quad \|w_\infty\|_{L^p(\mathcal{Q}_R^-)}^p = \|W_\infty\|_{L^2(\mathcal{Q}_R^-)}^2,
\]

and therefore, we have the following convergence result for every \( \mathcal{R} \in \left(0, \frac{r_k}{\rho_k}\right) \)

\[
w_k \rightarrow w_\infty \quad \text{in} \; L^p_{loc}(\mathcal{Q}_\mathcal{R}^-).
\]

In particular, from (3.19), \( w_\infty \) satisfies

\[
\inf_{P \in \mathcal{P}} \left( \int_{\mathcal{Q}_1^-} |w_\infty - P|^p \right)^{\frac{1}{p}} = 1. \tag{3.25}
\]

Similarly, according to (3.23),

\[
\left( \frac{1}{sQ^{2+2p}} \int_{\mathcal{Q}_s^-} |w_\infty|^p \right)^{\frac{1}{p}} \leq C_2 \ln 4s.
\]

Hence, \( w_\infty \) is a function that grows quadratically in space and linearly in time up to a logarithmic correction. Moreover, in virtue of (3.24), it follows that \( w_\infty \) a.e. belongs to \( \mathcal{P} \). This contradicts (3.25) and therefore concludes the proof. \( \square \)

We now establish a sort of monotonicity result for \( N \) and \( \omega \), defined in (1.27) and (1.28), respectively.

**Proposition 3.4.** (Dini estimate) Let \( N : (0, 1] \rightarrow [0, +\infty) \), \( \omega : (0, 1] \rightarrow [0, +\infty) \) be two functions that satisfy

\[
\forall r \in (0, 1], \quad N(\lambda r) < \mu N(r) \quad \text{or} \quad N(\lambda r) < C \omega(r), \tag{3.26}
\]

and

\[
\forall r \in (0, 1], \quad \forall \alpha \in [\lambda, 1], \quad \left\{ \begin{array}{l}
N(\alpha r) \leq C (N(r) + \omega(r)), \\
\omega(\alpha r) \leq C \omega(r)
\end{array} \right. \tag{3.27}
\]

for some constants \( C > 0 \) and for \( \lambda, \mu \in (0, 1) \). Moreover, we assume that \( \omega \) is Dini. Then for every \( \varrho \in (0, \frac{1}{4}) \) and for \( \beta = \frac{\ln \mu}{\ln \lambda} \), we have

\[
\int_0^{4\varrho} \frac{N(r)}{r} dr \\
\leq C \frac{1}{\beta} \left\{ \left( \frac{4\varrho}{\lambda} \right)^\beta (N(1) + \omega(1)) + C' \left( \int_0^{4\varrho} \frac{\omega(r)}{r} dr + \varrho^\beta \int_{4\varrho}^1 \frac{\omega(r)}{r^{1+\beta}} dr \right) \right\}. \tag{3.28}
\]

where \( C' = C'((\lambda, \mu) = \frac{1}{\mu} \frac{1}{(1-\lambda)\lambda^\beta}. \)
Proof. We first prove that for all \( r \in (0, \lambda] \), we have

\[
N(r) \leq \max \left( C_1 r^\beta, \frac{1}{\mu} r^\beta \sup_{\varrho \in [r, \lambda]} \frac{\omega(\varrho)}{\varrho^\beta} \right),
\]

where \( C_1 \) is given by

\[
C_1 = C \lambda^{-\beta} (N(1) + \omega(1)).
\]

If \( r \leq \lambda \), we write it as \( r = \lambda^k r_1 \) with \( k \geq 1 \) and \( r_1 \in (\lambda, 1] \). Then, taking advantage of (3.26), we infer

\[
N(r) \leq \max \left( C \omega \left( \frac{r}{\lambda^j} \right), \mu N \left( \frac{r}{\lambda^j} \right) \right)
\]

\[
\leq \max \left( C \omega \left( \frac{r}{\lambda^j} \right), C \mu \omega \left( \frac{r}{\lambda^2} \right), \mu^2 N \left( \frac{r}{\lambda^2} \right) \right)
\]

\[
\leq \max \left( C \omega \left( \frac{r}{\lambda^j} \right), C \mu \omega \left( \frac{r}{\lambda^2} \right), C \mu^2 \omega \left( \frac{r}{\lambda^3} \right), \ldots, \right.
\]

\[
C \mu^{k-2} \omega \left( \frac{r}{\lambda^{k-j+1}} \right), \mu^k N \left( \frac{r}{\lambda^k} \right) \right).
\]

Now, if we set \( \beta = \frac{\ln \mu}{\ln \lambda} \) and \( q = \frac{r}{\lambda^{j+1}} \) for \( j = 0, \ldots, k-2 \), we deduce

\[
\mu^j \omega \left( \frac{r}{\lambda^{j+1}} \right) = e^{j \ln \mu} \omega(q) = \mu^{-1} e^{(r/q)\beta} \omega(q) = \mu^{-1} \frac{\omega(q)}{q^\beta} r^\beta.
\]

On the other hand, according to (3.27) and (3.30), we have

\[
\mu^k N \left( \frac{r}{\lambda^k} \right) \leq \mu^k C(N(1) + \omega(1)) = C_1 \mu^k \lambda^\beta \leq C_1 \mu^k r_1^\beta = C_1 \mu^k \left( \frac{r}{\lambda^k} \right)^\beta \leq C_1 r^\beta.
\]

Finally, using estimates (3.32) and (3.33) in (3.31), we get (3.29).

We now want to estimate \( \sup_{\varrho \in [r, \lambda]} \frac{\omega(\varrho)}{\varrho^\beta} \). To this end, for some \( \varrho_0 \in [r, \lambda] \), we write

\[
\sup_{\varrho \in [r, \lambda]} \frac{\omega(\varrho)}{\varrho^\beta} = \frac{\omega(\varrho_0)}{\varrho_0^\beta}
\]

\[
\leq \frac{1}{\varrho_0^\beta} \int_{\varrho_0}^{\varrho_0 + t \varrho_0} C \omega(\varrho) d\varrho
\]

\[
\leq \frac{C}{t \lambda^{1+\beta}} \int_{\varrho_0}^{\varrho_0/\lambda} \frac{\omega(\varrho)}{\varrho^{1+\beta}} d\varrho
\]

\[
\leq C_2 \int_{r}^{1} \frac{\omega(\varrho)}{\varrho^{1+\beta}} d\varrho,
\]

where in the second line we have used the monotonicity of \( \omega \) according to (3.27) and the constants \( t \) and \( C_2 \) appearing in the second and fourth line are equal to \( (1 - \lambda)/\lambda \) and \( C/((1 - \lambda)\lambda^\beta) \), respectively.
Combining the previous inequality with (3.29) and setting $C' := \frac{1}{\mu} C_2$, we obtain for any $\varrho \in (0, \frac{1}{4})$

$$
\int_0^{4\varrho} \frac{N(r)}{r} dr \leq C_1 \int_0^{4\varrho} r^{\beta-1} dr + C C' J, \tag{3.34}
$$

with

$$
J := \int_0^{4\varrho} r^{\beta-1} dr \left( \int_0^1 \frac{\omega(\tau)}{\tau^{1+\beta}} d\tau \right)^{\frac{1}{\beta}},
$$

where in the second line we have integrated by parts and in the third we have applied the dominated convergence theorem.

Inequality (3.34), together with (3.35) and the definition of $C_1$ in (3.30), yields

$$
\int_0^{4\varrho} \frac{N(r)}{r} dr \leq C \left( \frac{4\varrho}{\lambda} \right)^{\beta} \frac{1}{\beta} (N(1) + \omega(1))
$$

$$
+ C C' \left( \frac{1}{\beta} \int_0^{4\varrho} \frac{\omega(r)}{r} dr + \frac{(4\varrho)^{\beta}}{\beta} \left( \int_0^1 \frac{\omega(\tau)}{\tau^{1+\beta}} d\tau \right) \right),
$$

which concludes the proof.

Remark 3.5. We observe that hypothesis (3.27) could be substituted by (3.13), and therefore, owing to Proposition 3.3, the previous result Proposition 3.4 holds in particular for $\hat{N}$ and $\tilde{\omega}$.

Remark 3.6. We notice that:

1. the quantities $N$ and $\omega$ defined in (1.27) and (1.28) satisfy (3.26) in virtue of Proposition 3.3. Moreover, (3.13) with $\gamma = \frac{1}{\lambda}$ implies that $N$ and $\omega$ also satisfy (3.27);

2. we chose the limits of integration in order to combine effortlessly this result with the following Lemma 3.7.

We now focus on the following result, which differs from Lemma 3.2 in the choice of the polynomial and of $\varrho$. More precisely, in Lemma 3.2, we derive an estimate on large cylinders, while we here consider smaller radii.

Lemma 3.7. (Estimates on smaller cylinders) If $u$ is defined in $Q^1_{-1}$, then there exists a unique polynomial $P_0 \in \tilde{\mathcal{P}}$ such that for every $\varrho \in (0, \frac{1}{4})$, we have

$$
\left( \frac{1}{\varrho^{Q+2+2p}} \int_{Q^1_{-\varrho}} |u - P_0|^p \right)^{\frac{1}{p}} \leq C_1 \int_0^{4\varrho} \frac{\hat{N}(u, f; r) + \tilde{\omega}(f; r)}{r} dr. \tag{3.36}
$$
Proof. We suppose that \( u \) is a solution to \( \mathcal{L} u = f \) in \( Q_1^- \). Applying Lemma 3.2 to a rescaled function

\[
v(x, t) = \frac{u(\delta_r(x, t))}{r^2}
\]

it follows that for \( r \leq \frac{1}{4\gamma} \), with \( \gamma \geq 1 \)

\[
\left( \frac{1}{\gamma^{Q+2+2p}} \int_{Q_r^-} |v - P^v|^p \right)^{\frac{1}{p}} \leq C_1 \int_r^{4\gamma} \frac{\hat{N}(v, f; s) + \tilde{\omega}(f; s)}{s} ds
\]

where \( P^v \) realises the infimum in the definition of \( \hat{N}(v, f; 1) \). Now performing a change of variables with \( \varrho = \gamma r \), we infer

\[
\left( \frac{1}{\varrho^{Q+2+2p}} \int_{Q_\varrho^-} |u - P_r|^p \right)^{\frac{1}{p}} \leq C_1 \int_r^{4\varrho} \frac{\hat{N}(u, f; s) + \tilde{\omega}(f; s)}{s} ds, \tag{3.37}
\]

where we notice that \( P^v(x, t) = \frac{P_r(\delta_r(x, t))}{r^2} \) and \( \hat{N}(v, f; s) = \hat{N}(u, f; rs) \). Hence, fixing \( \varrho \in (0, 1/4) \), we may pass to the limit in (3.37) for \( r \to 0 \). Therefore up to extracting a subsequence, we can assume that \( P_r \) tends to a polynomial \( P_0 \in \tilde{P} \), namely

\[
\left( \frac{1}{\varrho^{Q+2+2p}} \int_{Q_\varrho^-} |u - P_0|^p \right)^{\frac{1}{p}} \leq C_1 \int_0^{4\varrho} \frac{\hat{N}(u, f; s) + \tilde{\omega}(f; s)}{s} ds.
\]

We now show that the Taylor-type polynomial \( P_0 \) is unique. In fact, if \( P_1 \in \tilde{P} \) is another polynomial satisfying (3.36), then for every \( \varrho \in (0, \frac{1}{4}) \) we have

\[
\left( \frac{1}{\varrho^{Q+2+2p}} \int_{Q_\varrho^-} |P_1 - P_0|^p \right)^{\frac{1}{p}} \leq 2C_1 \int_0^{4\varrho} \frac{\hat{N}(u, f; r) + \tilde{\omega}(f; r)}{r} dr.
\]

On the other hand, as \( P_1 - P_0 \in \tilde{P} \), assertion (ii) of Lemma 3.1 yields

\[
\left( \int_{Q_1^-} |P_1 - P_0|^p \right)^{\frac{1}{p}} \leq \tilde{C}_2 \left( \frac{1}{\varrho^{Q+2+2p}} \int_{Q_\varrho^-} |P|^p \right)^{\frac{1}{p}},
\]

for any \( \varrho < 1 \). Combining the previous two inequalities, infer

\[
\|P_1 - P_0\|_{L^p(Q_1^-)} \leq 2C_1 \int_0^{4\varrho} \frac{\hat{N}(u, f; r) + \tilde{\omega}(f; r)}{r} dr. \tag{3.38}
\]

As the quantity on the right hand side of (3.38) is finite and goes to 0 as \( \varrho \to 0 \), we finally obtain \( P_1 \equiv P_0 \), which concludes the proof.

We notice that, in Lemma 3.7 and the upcoming Proposition, we have \( P_0 \) belonging to the set \( \tilde{P} \). Hence, in the proof of assertion (iii) of Theorem 1.3 it is only left to show that \( P_0 \) belongs in particular to \( \hat{P} \) (i.e. \( \mathcal{L} P_0 = 0 \)) in order to prove (1.31).
**Proposition 3.8.** (Modulus of continuity of the solution up to second order) Let us assume that \( \tilde{\omega} \) is Dini continuous and let us set \( \beta = \ln \mu / \ln \lambda \). There exists a unique polynomial \( P_0 \in \tilde{P} \) and a constant \( C' = C'(C, \lambda, \mu) \) such that for every \( \varrho \in (0, \varrho^*_{4 \gamma}) \), we have

\[
\left( \frac{1}{Q^2 + 2 + 2p} \int_{Q^{-\varrho}} |u - P_0|^p \right)^{\frac{1}{p}} \leq C \frac{1}{\beta} \left( \left( \frac{4\varrho}{\lambda} \right)^{\beta} \left( \tilde{N}(u, f; 1) + \tilde{\omega}(f; 1) \right) \right) + C' \left( \int_0^{4\varrho} \frac{\tilde{\omega}(f; r)}{r} dr + \varrho \beta \int_{4\varrho}^{1} \frac{\tilde{\omega}(f; r)}{r^{1+\beta}} dr \right) \right). \tag{3.39}
\]

**Proof.** The proof simply follows from the combination of Proposition 3.4 with \( N \equiv \hat{N} \) and \( \omega \equiv \tilde{\omega} \) and Lemma 3.7, where we remark that we reabsorbed the modulus of continuity in the right hand side of (3.28). We observe that the uniqueness of \( P_0 \) follows from Lemma 3.7. \( \square \)

**Remark 3.9.** We observe that Proposition 3.8 holds, more generally, for two functions \( N \) and \( \omega \) satisfying the assumptions of Proposition 3.4.

3.2. Proof of Theorem 1.3

(i) We first observe that from definitions (1.24) and (1.22), it holds that

\[
\tilde{N}(u; r) \leq \hat{N}(u, f; r). \tag{3.40}
\]

The right hand side of (3.40) can be estimated combining (3.29) with (3.30), which yields for \( r \in (0, \lambda] \)

\[
\tilde{N}(r) \leq C \left( N(1) + \frac{1}{\mu} \sup_{\varrho \in [0, 1]} \omega(\varrho) \right),
\]

where we recall that \( N \) and \( \omega \) were defined, respectively, in (1.27) and (1.28). Moreover owing to the monotonicity-type estimate (3.13) for \( \gamma = \frac{1}{\lambda} \) and \( r \in (0, 1] \)

\[
\hat{N}(u, f; r) \leq C \left( \hat{N}(u, f; 1) + \sup_{\varrho \in [0, 1]} \tilde{\omega}(f; \varrho) \right),
\]

with \( C = C(\lambda, \mu, p, Q) \). On the other hand, from definition (1.24) we get

\[
\hat{N}(u, f; 1) \leq C \left( \|u\|_{L^p(Q^{-1})} + \|f\|_{L^p(Q^{-1})} \right).
\]

Therefore, combining the estimates above, we conclude the proof of statement (i).

(ii) Assertion (ii) follows directly from estimate (3.29).

(iii) We observe that Proposition 3.4 with \( N \equiv \hat{N} \) and \( \omega \equiv \tilde{\omega} \) yields statement (iii).
(iv) We recall that we have already proved estimate (1.31) in the case where $P_0 \in \tilde{P}$, according to Proposition 3.8 and namely to (3.39). Furthermore, we notice that the coefficients of $P_0$ are bounded by choosing $\varrho = \frac{\lambda}{4}$ in (3.39). Therefore it is only left to show that $P_0$ belongs in particular to $P$, i.e. $P_0$ satisfies equation $\mathcal{L} P_0 = 0$.

To this end, we define the function

$$u^\varepsilon(x,t) = \frac{u(\delta_{\varepsilon}(x,t)) - P_0(\delta_{\varepsilon}(x,t))}{\varepsilon^2}$$

which converges in $L^p$ to a function $v \equiv 0$ by (3.39) for $\varepsilon \to 0$. Moreover from

$$\mathcal{L}(u^\varepsilon) = \varepsilon^2 \frac{\mathcal{L}u(\delta_{\varepsilon}(x,t))}{\varepsilon^2} - \varepsilon^2 \mathcal{L}P_0(\delta_{\varepsilon}(x,t))$$

$$= f(\delta_{\varepsilon}(x,t)) - \mathcal{L}P_0(\delta_{\varepsilon}(x,t))$$

and according to (2.1), it follows that for $\varepsilon \to 0$,

$$0 = \mathcal{L}v = f(0) - \mathcal{L}P_0.$$

Since by assumption $f(0) = 0$, then we have showed that $P_0$ satisfies equation $\mathcal{L}P_0 = 0$. This concludes the proof.

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Erica Ipocoana and Annalaura Rebucci
Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Università degli Studi di Modena e Reggio Emilia
via Campi 213/b
41115 Modena
Italy
E-mail: erica.ipocoana@unipr.it

Annalaura Rebucci
E-mail: annalaura.rebucci@unipr.it

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