Matroids And Greedy Algorithms
A Deeper Justification of Using Greedy Approach To Find A Maximal set of a Matroid

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Abstract -- Greedy algorithms are used in solving a diverse set of problems in small computation time. However, for solving problems using greedy approach, it must be proved that the greedy strategy applies. The greedy approach relies on selection of optimal choice at a local level reducing the problem to a single sub problem, which actually leads to a globally optimal solution. Finding a maximal set from the independent set of a matroid M(S, I) also uses greedy approach and justification is also provided in standard literature (e.g. Introduction to Algorithms by Cormen et al.). However, the justification does not clearly explain the equivalence of using greedy algorithm and contraction of M by the selected element. This paper thus attempts to give a lucid explanation of the f...
III. VALIDITY OF MAXWEIGHT

The section gives the formal proof that is provided in support of the above algorithm in standard literature. We make the following observations which will be used.

1. If \( \{x\} \notin I \), then \( x \notin A \) for all \( A \in I \)

   To prove these, suppose otherwise, i.e. \( \{x\} \notin S \) and \( A \in I \) such that \( x \in A \)

   \[ \Rightarrow \{x\} \notin A \]

   \[ \Rightarrow \{x\} \in I \] by hereditary property.

   \[ \Rightarrow \text{Contradiction} \]

   Hence the theorem is proved.

2. If \( M (S, I) \) is a weighted matroid with \( S \) sorted into monotonically decreasing order by weight, then if \( x \) is the first element such that \( \{x\} \notin I \) (if such an element exists), there exists an optimal subst A of S such that \( x \notin A \) (the optimal refers to maximum weight independent subset).

   This is typically known as optimal substructure property.

   Proof: Let \( B \) be a nonempty optimal set. If \( x \in B \), the theorem is true.

   If \( x \notin B \), let \( A = \{x\} \).

   Till \( |A| < |B| \) we can add some \( y \in B \) such that \( A = A \cup \{y\} \).

   So at a point \( |A| = |B| \), such that \( A \) and \( B \) have \( |A|-1 \) same elements such that \( x \in A, x \notin B, z \in B, z \notin A \) for some \( w (z) \leq w (x) \).

   Because \( z \in B \Rightarrow w (z) \leq w (x) \) as \( x \) is heaviest independent element of \( S \).

   \[ \Rightarrow A = B \cup \{z\} \cup \{x\} \wedge w (A) = w (B) + w (x) \]

   \[ \Rightarrow w (A) = w (B) + w (x) - w (z) \geq w (s) \]

   \[ \Rightarrow w (A) \text{ is optimal.} \]

   Hence the theorem is proved.

3. Matroids exhibited optimal substructure property. If we select maximum weight element \( x \in S \) such that \( \{x\} \in I \), their remaining problem is to find an optimal subset of matroid \( M' (S', I') \) such that \( I'. \ S' = \{y : x \notin S \text{ and } (x,y) \in I \} \) is independent of \( M \).

   The basis for verification of MAXWEIGHT is that the element passed over by MAXWEIGHT can never be included in any independent subset (by 1). Thus, after selecting \( x \), the problem is reduced to applying the same algorithm on contraction of \( M \) by \( x \) because \( B \) is independent is \( M' \) if and only if \( B \cup \{x\} \) is independent is \( M \).

   This is an overview of the basic understanding of applying a greedy algorithm for solving this problem. However, this explanation does not clearly indicate the equivalence of MAXWEIGHT and contraction of \( M \). Specifically, the objective of section IV of the paper is to prove that at each iteration of MAXWEIGHT, further iteration is equivalent to applying MAXWEIGHT on the contraction of \( M \) by \( x \) where \( x \) is the element selected in the latest loop iteration. For example, it disproves the assumption that there may be some \( y \in S \) such that \( \{x,y\} \notin I \) for which \( A \cup \{y\} \notin I \) and thus, \( y \) should have been selected but the algorithm would not select it. This is not usually explained in standard literature and is the core motivation of this producing this paper.

IV. EQUIVALENCE OF MAXWEIGHT AND CONTRACTION OF MATROID

We will now try to prove that, at every iteration, the element selected is an element of contraction of \( M \) by previous element and that every element of \( S \) of current reduced matroid is considered.

Proof: At every step \( A \) is selected if \( A \cup \{x\} \notin I \). Let the loop run \( N \) time and let \( x_k \) denote element selected at the \( k \)th iteration such that optimal set formed finally is \( A = \{x_1 \ldots x_N\} \).

Now \( x_n \) is selected if \( A \cup \{x_n\} \notin I \), where \( A = \{x_1, x_2, \ldots, x_{n-1}\} \).

Assuming that till (k-1)th iteration, all elements selected were part of corresponding contraction, i.e. \( \{x_1, x_2\} \in I_k \) for \( i = 2 \).

At \( k \)th iteration, \( A = \{x_1, x_2, \ldots, x_k\} \notin I \), \( \Rightarrow A_2 = A \cup \{x_1\} \notin I \) (by definition of contraction) \( \Rightarrow A_3 = A_2 \cup \{x_2\} \notin I_3 \) and so on up to \( \Rightarrow A_k = A_k \cup \{x_k\} \notin I_k \) \( \Rightarrow \{x_1, x_2, \ldots, x_k\} \notin I_k \).

Therefore we can rewrite the algorithm as MAXWEIGHT \((M, w)\)

1. \( A = A_1 \)

2. Sort \( S \) in monotonically decreasing order

3. For every element \( x \in S \)

   - If \( x \notin I \)
   - \( A = A \cup \{x\} \)
   - \( \text{prev } = x \)
   - \( M(S, I) = M'(S', I') \) where \( M'(S', I') \) is contraction of \( M(S, I) \)

4. Return \( A \)

The results of both the algorithms will be same.

V. CONCLUSION

Thus both forms of the algorithm are equivalent. The running time of both the algorithm is \( O(nlg n + n f(n)) \) where \( f(n) \) is the asymptotic time taken for test, be it \( A \cup \{x\} \notin I \) or \( \{x\} \notin I \).

Thus, if in any problem, the computation of test \( \{x\} \notin I \), takes lesser time, the algorithm claimed in the paper might give better result in terms of the running time. Also the paper gives
a clear explanation of the validity of using greedy approach in finding a maximum weight maximal independent set of a matroid. Thus, further scope of research may lie towards finding the test which takes lesser time to check independence of the set containing element being considered at every loop iteration in this greedy approach.

VI. REFERENCES

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