Duality rules for more mixed-symmetry potentials

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T- and S-duality rules among the gauge potentials in type II supergravities are studied. In particular, by following the approach of [J. J. Fernández-Melgarejo et al., arXiv:1909.01335], we determine the T- and S-duality rules for certain mixed-symmetry potentials, which couple to supersymmetric branes with tension $T \propto g_s^{-n}$ ($n \leq 4$). Although the T-duality rules are rather intricate, we find a certain redefinition of potentials which considerably simplifies the duality rules. After the redefinition, potentials are identified with components of the T-duality-covariant potentials, which have been predicted by the $E_{11}$ conjecture. Since our approach is based on U-duality, we can also determine the 11D uplifts of the mixed-symmetry potential, unlike the T-duality-covariant approach known in the literature. We also study the field strengths of the mixed-symmetry potentials.

Subject Index B11, B20

1. Introduction

Toroidally compactified 11D supergravity or type II supergravity has the U-duality symmetry [1] but this is not manifest in the standard formulation. In order to exhibit the symmetry, the standard metric, scalar fields and $p$-form gauge potentials are not enough [2]. In fact, we additionally need to introduce certain mixed-symmetry potentials, which are related to the standard potentials through a non-local relation, similar to the electric–magnetic duality. According to the $E_{11}$ conjecture [2,3], there are infinitely many mixed-symmetry potentials in each theory. By introducing an integer-valued parameter $\ell$, known as the level, the number of mixed-symmetry potentials with a fixed level $\ell$ is finite, and we can determine the full list of the mixed-symmetry potentials for each level $\ell$ (see Ref. [4,5] and references therein).

Although a list of mixed-symmetry potentials which constitutes the U-duality multiplets has been algebraically determined, their physical definitions are still obscure. In the case of the standard supergravity fields, their definitions can be fixed by the supergravity action, but the mixed-symmetry potentials do not appear in the standard supergravity action and it is not straightforward to specify their definitions. A possible way to specify their definitions is to construct the worldvolume actions for supersymmetric branes. As is well known, the Ramond–Ramond (R–R) fields couple to D-branes, and one can identify the definition of the R–R fields by looking at the Wess–Zumino (WZ) term. Similarly, mixed-symmetry potentials generally couple to certain exotic branes [6–15] and their definitions can be fixed by constructing the WZ term for exotic branes. For example, the WZ term of the Kaluza–Klein (KK) monopole has been constructed [16–19] and a precise definition of the dual graviton has been given. However, at present, worldvolume actions have been constructed for only a few exotic branes.
Then, by determining the duality rule, we can fix the convention for the mixed-symmetry potential. Type IIB:

Type IIA:

To make precise definitions of mixed-symmetry potentials, it is more straightforward to determine the T- and S-duality transformation rule. The mixed-symmetry potentials which couple to supersymmetric branes are related to the standard p-form potentials under \( T \)-/S-duality transformations. Then, by determining the duality rule, we can fix the convention for the mixed-symmetry potentials. Recently, a systematic approach to determine the duality rules has been proposed [20] and the \( T \)-/S-duality rules for the dual graviton have been determined. In this paper, we continue the analysis and obtain the \( T \)-/S-duality rules for more mixed-symmetry potentials. Concretely, we consider the duality web described in Fig. 1. There, each line (with a circled letter appended) corresponds to a \( T \)-duality that connects a type IIA brane and a type IIB brane. For example, the \( T \)-duality \( \circ \) connects the 52-brane in type IIA theory and the 52-brane in type IIB theory. Since the 52-brane and the 52-brane minimally couple to the potentials \( \mathcal{A}_{8,2} \) and \( \mathcal{A}_{7,1} \), respectively, \( T \)-duality \( \circ \) corresponds to a \( T \)-duality rule for \( \mathcal{A}_{a_1\ldots a_7,a_\gamma} \leftrightarrow \mathcal{A}_{a_1\ldots a_7,a_\gamma} \), where \( x^\gamma \) is the \( T \)-duality direction. We determine the \( T \)-duality rule for the 27 lines, \( \circ \)–\( \cdots \circ \), including non-linear terms in the duality rules.

In Fig. 1, by following the notation of Ref. [10], a \( p \)-brane in type II theory with tension

\[
T_p = \frac{g_s^{-n}}{l_s^2 (2\pi l_s)^p} \left( \frac{R_{q_1} \cdots R_{q_{c_2}}}{l_s^{c_2}} \right) \left( \frac{R_{r_1} \cdots R_{r_{c_1}}}{l_s^{c_1}} \right)^3 \quad (R_n : \text{toroidal radii}),
\]

is denoted as a \( p_n^{(c_1,\ldots,c_2)} \)-brane. In particular, the NS5-brane is denoted as \( 5_2 \equiv 5_2^{(0,\ldots,0)} \) and the fundamental string (F1) and the Dp-brane may be denoted as \( 1_0 \equiv 1_0^{(0,\ldots,0)} \) and \( p_1 \equiv p_1^{(0,\ldots,0)} \), respectively. The \( T \)-dualities \( \circ \)–\( \Theta \) and \( \circ \)–\( \Xi \) correspond to the standard \( T \)-dualities for the NS–NS fields and the R–R fields. The \( T \)-dualities \( \circ \)–\( \Theta \) were obtained in Refs. [18,20–23] and \( \Xi \) is obtained in Ref. [11]. To the author’s knowledge, other \( T \)-dualities are new.\(^1\) In our approach, type IIB fields are defined as SL(2) S-duality tensors, and the S-duality rules are simple.

The structure of this paper is as follows. In Sect. 2, we establish our convention for the (bosonic) supergravity fields. The standard fields are defined through the action, and several higher \( p \)-form potentials are introduced through the electric–magnetic duality. Additional mixed-symmetry potentials are defined in Sect. 2.4 by finding a consistent parametrization of the \( U \)-duality-covariant 1-form field \( \mathcal{A}_{\nu}^I \). In Sect. 3, we determine the \( T \)- and S-duality rules by following the approach of

\(^1\)\( T \)-duality rules for additional potentials which are not studied here are discussed in Refs. [21,22].
Refs. [20, 23]. In particular, in Sects. 3.2.1 and 3.3.1, by considering certain field redefinitions, we show that our mixed-symmetry potentials can be packaged into the $O(10, 10)$-covariant potentials $D_{M_1\ldots M_4}$, $E_{M N}^a$ and $F_{M_1\ldots M_{10}}^+$. In Sect. 4, we discuss the gauge symmetries and field strengths in each theory. Section 5 is devoted to conclusions and discussions. In Appendix A, we explain our conventions.

2. Supergravity fields
In this section, we explain our conventions for the bosonic supergravity fields.

2.1. 11D supergravity
In 11D supergravity, the bosonic fields are $\hat{g}_{ij}$ and $\hat{A}^3$, for which the Lagrangian is

$$L_{11} = \hat{\tilde{R}} - \frac{1}{2} \hat{F}_4 \wedge \hat{\tilde{F}}_4 - \frac{1}{31} \hat{A}_3 \wedge \hat{\tilde{F}}_4,$$

(2.1)

where we have defined $\hat{F}_4 \equiv d\hat{A}_3$. By introducing the dual field strength as

$$\hat{F}_7 \equiv -\hat{\tilde{F}}_4,$$

(2.2)

the equation of motion for $\hat{A}_3$ is expressed as the Bianchi identity

$$d\hat{F}_7 - \frac{1}{2} \hat{F}_4 \wedge \hat{\tilde{F}}_4 = 0.$$

(2.3)

This prompts us to introduce the 6-form potential $\hat{A}_6$ as

$$\hat{F}_7 \equiv d\hat{A}_6 + \frac{1}{2} \hat{A}_3 \wedge \hat{\tilde{F}}_4.$$

(2.4)

Although the potential $\hat{A}_6$ is not contained in the standard Lagrangian, it is necessary for manifesting the $U$-duality symmetry. For the manifest $U$-duality symmetry, in general, we need to introduce additional gauge potentials, which are generally mixed-symmetry tensors. Among these, we consider $\hat{A}_{8,1}$, $\hat{A}_{9,3}$, and $\hat{A}_{10,1,1}$ in this paper. The dual graviton $\hat{A}_{8,1}$ and the potential $\hat{A}_{10,1,1}$ respectively appear in the worldvolume action of the KK monopole [17] and the M9-brane [25–28], and their definitions are rather established. However, for the $5^{3}$-brane (which couples to $\hat{A}_{9,3}$), only the kinetic term has been constructed [29] and the WZ term including the potential $\hat{A}_{9,3}$ has not been found. Thus the definition of $\hat{A}_{9,3}$ is still unclear.

Here, instead of considering brane actions, we define the mixed-symmetry potentials by using the approach of Ref. [20]. Namely, we consider the $E_{n(n)}$ $U$-duality-covariant 1-form $A^I_1$, which appears when the 11-dimensional (11D) spacetime is compactified on an $n$-torus. It is uniquely defined (as the generalized graviphoton [20]), and by parametrizing $A^I_1$ in terms of the mixed-symmetry potentials, we can fix the convention for the mixed-symmetry potentials. The concrete parametrizations are given in Sect. 2.4. After providing the parametrizations, we can straightforwardly obtain the $T$- and $S$-duality transformation rules as explained in Ref. [20].

For convenience, below we summarize the correspondence between each gauge potential and the supersymmetric brane, which electrically couples to the potential:

| $A_3$ | $A_6$ | $A_{8,1}$ | $A_{9,3}$ | $A_{10,1,1}$ |
|-------|-------|-----------|-----------|--------------|
| M2    | M5    | $6^1$     | $5^3$     | $8^{(1,0)}$  |

(2.5)
2.2. Type IIA supergravity

In order to obtain type IIA supergravity, we consider the standard 11D–10D map,
\[
\hat{g}_{ij} \, dx^i \, dx^j = e^{-\frac{2}{3} \varphi} \, dx^m \, dx^n + e^\frac{4}{3} \varphi \, (dx^2 + \mathcal{C}_1)^2,
\]
\[
\hat{A}_3 = \mathcal{C}_3 + \mathcal{B}_2 \wedge dx^2 \quad (m, n = 0, \ldots, 9),
\] (2.6)

where \(x^2\) is the coordinate along the M-theory circle. Then, we obtain the type IIA Lagrangian
\[
L_{IIA} = e^{-2\varphi} \left( \ast R + 4 \, d\varphi \wedge \ast d\varphi - \frac{1}{2} \mathcal{H}_3 \wedge \ast \mathcal{H}_3 \right)
- \frac{1}{2} \left( \mathcal{G}_2 \wedge \ast \mathcal{G}_2 + \mathcal{G}_4 \wedge \ast \mathcal{G}_4 + \mathcal{B}_2 \wedge d\mathcal{C}_3 \wedge d\mathcal{C}_3 \right),
\] (2.7)

where we have defined
\[
\mathcal{H}_3 \equiv d\mathcal{B}_2, \quad \mathcal{G}_2 \equiv d\mathcal{C}_3 - \mathcal{H}_3 \wedge \mathcal{C}_1,
\] (2.8)

and used the identity
\[
\hat{F}_4 = \mathcal{G}_4 + \mathcal{H}_3 \wedge (dx^2 + \mathcal{C}_1).
\] (2.9)

Again, the equations of motion for \(\mathcal{B}_2\) and \(\mathcal{C}_3\) are expressed as Bianchi identities
\[
d\mathcal{H}_7 - \mathcal{G}_2 \wedge \ast \mathcal{G}_4 - \frac{1}{2} \mathcal{G}_4 \wedge \mathcal{G}_4 = 0, \quad d\mathcal{G}_6 - \mathcal{H}_3 \wedge \mathcal{G}_4 = 0,
\] (2.10)

where the dual field strengths are defined by
\[
\mathcal{H}_7 \equiv e^{-2\varphi} \ast \mathcal{H}_3, \quad \mathcal{G}_6 \equiv -\ast \mathcal{G}_4.
\] (2.11)

Then, we can introduce the dual potentials \(\mathcal{G}_5\) and \(\mathcal{B}_6\) as follows:
\[
\mathcal{G}_6 \equiv d\mathcal{G}_5 - \mathcal{H}_3 \wedge \mathcal{C}_3, \quad \mathcal{H}_7 \equiv d\mathcal{B}_6 - \mathcal{G}_6 \wedge \mathcal{C}_4 + \frac{1}{2} \mathcal{G}_4 \wedge \mathcal{C}_4 - \frac{1}{2} \mathcal{H}_3 \wedge \mathcal{C}_3 \wedge \mathcal{C}_1.
\] (2.12)

Through the electric–magnetic duality, we obtain the following 11D–10D map:
\[
\hat{A}_6 = \mathcal{B}_6 + \left( \mathcal{G}_5 - \frac{1}{2} \mathcal{H}_3 \wedge \mathcal{B}_2 \right) \wedge dx^2, \quad \hat{F}_7 = \mathcal{H}_7 + \mathcal{G}_6 \wedge (dx^2 + \mathcal{C}_1).
\] (2.13)

On the other hand, the equation of motion for \(\mathcal{C}_1\) is expressed as
\[
d\mathcal{G}_8 - \mathcal{H}_3 \wedge \mathcal{G}_6 = 0, \quad \mathcal{G}_8 \equiv \ast \mathcal{G}_2,
\] (2.14)

and we can introduce the 7-form potential \(\mathcal{C}_7\) as
\[
\mathcal{G}_8 \equiv d\mathcal{C}_7 - \mathcal{H}_3 \wedge \mathcal{C}_3.
\] (2.15)

The 11D uplift of this 8-form field strength is discussed in Sect. 4 [see Eq. (4.16)].

In 11D supergravity, we have introduced non-standard potentials \(\hat{A}_{8,i}, \hat{A}_{10,i},\) and \(\hat{A}_{9,\bar{3}}\). Here, we consider the following simple 11D–10D map for these potentials.\(^2\)

\(^2\) We do not consider overlined potentials, such as \(\overline{\mathcal{A}}_8\), which do not couple to supersymmetric branes.
The potential \( \mathcal{A}_1 \) is related to the R–R 7-form \( \mathcal{C}_7 \) introduced in Eq. (2.15) and the 9-form \( \mathcal{A}_9 \) is related to the standard R–R 9-form \( \mathcal{C}_9 \) (see Sect. 3.1).

In the type IIA case, the correspondence between the potentials and the supersymmetric branes is summarized as follows:3

| \( \mathcal{B}_2 \) | \( \mathcal{C}_1 \) | \( \mathcal{C}_3 \) | \( \mathcal{C}_5 \) | \( \mathcal{B}_6 \) | \( \mathcal{C}_7 \) | \( \mathcal{A}_{7,1} \) | \( \mathcal{A}_{8,1} \) | \( \mathcal{A}_9 \) | \( \mathcal{A}_{8,2} \) | \( \mathcal{A}_{8,3} \) | \( \mathcal{A}_{9,1,1} \) | \( \mathcal{A}_{10,1,1} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| F1 | D0 | D2 | D4 | 52 (NS) | D6 | 52 (KK/AA) | 61 (KK/AA) | D8 | 52 | 43 | 7(1,0) | 8(1,0) |

### 2.3. Type IIB supergravity

The standard SL(2) S-duality-invariant (pseudo) Lagrangian for type IIB supergravity is

\[
\mathcal{L}_{\text{IIB}} = *_E R + \frac{1}{4} F_{1a\beta} \wedge *_E F_1^{a\beta} - \frac{1}{2} m_{a\beta} F_3^a \wedge *_E F_3^\beta - \frac{1}{4} F_5 \wedge *_E F_5 + \frac{1}{4} \epsilon_{a\beta} A_4 \wedge F_3^a \wedge F_3^\beta,
\]

where \( \alpha = 1, 2 \) are indices of SL(2) doublets and \( (\epsilon_{a\beta}) = (\epsilon_{a\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The fundamental fields are \( \{g_{mn}, m_{a\beta}, A_5^a, A_3 \} \), and \( g_{mn} \) is the Einstein-frame metric, for which the Hodge star operator is denoted by \( *_E \). The scalar field \( m_{a\beta} \) is symmetric \( m_{a\beta} = m_{\beta a} \) and satisfies

\[
m^a_{\gamma} m^\gamma_{\beta} = -m^a_{\gamma} m^\gamma_{\beta} = -\delta^a_{\beta},
\]

where we have raised or lowered the SL(2) indices as \( v^a = \epsilon_{a\beta} v_\beta \) and \( v_\alpha = \nu^\beta \epsilon_{\beta a} \). The field strengths are defined by

\[
F_1^{a\beta} = m^{a\gamma} d_m^{\gamma\beta} = F_1^{(a\beta)}, \quad F_3^a = dA_5^a, \quad F_5 = dA_4 + \frac{1}{2} \epsilon_{a\beta} F_3^a \wedge A_2^\beta,
\]

which satisfy the Bianchi identities

\[
dF_1^{a\beta} + \epsilon_{a\beta} F_1^{\alpha\gamma} \wedge F_1^{\gamma\delta} = 0, \quad dF_3^a = 0, \quad dF_5 + \frac{1}{2} \epsilon_{a\beta} F_3^a \wedge F_3^\beta = 0.
\]

The self-duality relation for the 5-form field strength,

\[
F_5 = *_E F_5,
\]

should be imposed at the level of equations of motion.

As is well-known, under the self-duality relation (2.22) the equation of motion for \( A_4 \) is equivalent to the last Bianchi identity. If we additionally define the dual field strengths as4

\[
F_7^a \equiv m^a_{\beta} *_E F_3^{(\beta)} , \quad F_9^{a\beta} \equiv *_E F_1^{a\beta},
\]

---

3 We here ignore the component \( \mathcal{A}_{9,3} \), which couples to the \( S^1 \)-brane.

4 As noted in Ref. [11,30], the triplet \( F_1^{a\beta} \) has only two independent components because it satisfies \( m_{a\beta} F_1^{a\beta} = 0 \). Then, the duality (2.23) shows that the triplet \( F_9^{a\beta} \) also has only two independent components.
the equations of motion for $m_{\alpha\beta}$ and $A_{2}^{\alpha\beta}$ also can be expressed as the Bianchi identities

$$dF_{9}^{\alpha\beta} - F_{3}^{(\alpha} \wedge F_{7}^{\beta)} = 0, \quad dF_{7}^{\alpha} - F_{3}^{\alpha} \wedge F_{5} = 0. \quad (2.24)$$

This prompts us to introduce the higher potentials $A_{6}^{\alpha}$ and $A_{8}^{\alpha\beta}$ as

$$F_{7}^{\alpha} \equiv dA_{6}^{\alpha} - F_{3}^{\alpha} \wedge A_{4} + \frac{1}{3!} \epsilon_{\gamma\delta} F_{3}^{\gamma} \wedge A_{2}^{\alpha} \wedge A_{2}^{\alpha}, \quad (2.25)$$

$$F_{9}^{\alpha\beta} \equiv dA_{8}^{\alpha\beta} - F_{3}^{(\alpha} \wedge A_{6}^{\beta)} + \frac{1}{4!} \epsilon_{\gamma\delta\eta} F_{3}^{\gamma} \wedge A_{2}^{\alpha} \wedge A_{2}^{\alpha} \wedge A_{2}^{\alpha} \wedge A_{2}^{\alpha}. \quad (2.26)$$

In Refs. [30–33], a 10-form potential was also introduced by considering the supersymmetry algebra (which is also predicted by $E_{11}$ [34]), and the field strength, in our convention, is defined as

$$F_{11}^{\alpha\beta\gamma} \equiv dA_{10}^{\alpha\beta\gamma} - F_{3}^{(\alpha} \wedge A_{8}^{\beta\gamma)} + \frac{1}{5!} \epsilon_{\zeta\eta\xi} F_{3}^{\zeta} \wedge A_{2}^{\eta} \wedge A_{2}^{\alpha} \wedge A_{2}^{\alpha} \wedge A_{2}^{\alpha} \wedge A_{2}^{\alpha} = 0. \quad (2.27)$$

This satisfies the Bianchi identity (without considering the dimensionality)

$$dF_{11}^{\alpha\beta\gamma} - F_{3}^{(\alpha} \wedge F_{9}^{\beta\gamma)} = 0. \quad (2.28)$$

In this paper, we consider the following set of type IIB fields:

$$\{g_{mn}, m_{\alpha\beta}, A_{2}^{\alpha}, A_{4}, A_{6}^{\alpha}, A_{7,1}^{\alpha}, A_{8}^{\alpha}, A_{8,2}^{\alpha}, A_{10}^{\alpha\beta\gamma}, A_{9,2,1}\}, \quad (2.29)$$

which transform covariantly under SL(2) S-duality transformations. At the present stage, definitions of $A_{7,1}^{\alpha}, A_{8,2}^{\alpha}, A_{9,2,1}$, and $A_{10}^{\alpha\beta\gamma}$ are not specified. They are defined in Sect. 2.4.

In the following discussion, a redefinition of the 6-form potential

$$A_{6}^{\alpha} \equiv A_{6}^{\alpha} - A_{4} \wedge A_{2}^{\alpha} \quad (2.30)$$

makes the $T$-duality rules slightly shorter. Thus, the 6-form $A_{6}^{\alpha}$ rather than $A_{6}^{\alpha}$ is mainly used in this paper. For notational consistency, other fields may also be denoted by bold typeface:

$$A_{2}^{\alpha} \equiv A_{2}^{\alpha}, \quad A_{4} \equiv A_{4}, \quad A_{7,1} \equiv A_{7,1}, \quad A_{8}^{\alpha} \equiv A_{8}^{\alpha}, \quad A_{8,2}^{\alpha} \equiv A_{8,2}^{\alpha}, \quad A_{10}^{\alpha\beta\gamma} \equiv A_{10}^{\alpha\beta\gamma}, \quad A_{9,2,1} \equiv A_{9,2,1}. \quad (2.31)$$

The relations between the potentials and the supersymmetric branes are as follows:

| $A_{2}^{\alpha}$ | $A_{4}$ | $A_{6}^{\alpha}$ | $A_{7,1}^{\alpha}$ | $A_{8}^{\alpha\beta}$ | $A_{8,2}^{\alpha}$ | $A_{10}^{\alpha\beta\gamma}$ | $A_{9,2,1}$ |
|-----------------|---------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------|
| F1/D1           | D3      | DS5/2 (NS5)    | $\frac{5}{2}$ (KKSB) | D7/73 (NS7B)   | $\frac{5}{2}$ (NS7B) | D9/93 (NS9B) | $\frac{5}{2}$ (KKSB) |

It is noted that an SL(2) n-plet $A_{\alpha_{1}}^{\cdots\alpha_{n-1}}$ ($n \geq 2$) always couples to only two supersymmetric branes. The components which couple to supersymmetric branes are $A_{1}^{1 \cdots 1}$ and $A_{2}^{2 \cdots -2}$ as discussed in Refs. [32,35] (see also Ref. [5]).

\footnote{5 The additional 10-form potential $\tilde{A}_{10}$ was also introduced there, but here we do not consider this potential because it does not couple to supersymmetric branes.}
2.4. Parametrization of the U-duality-covariant 1-form

When 11D supergravity or type II supergravity is compactified to \( d \)-dimensions, the bosonic fields with one external index \( \mu (= 0, \ldots, d - 1) \) are packaged into the 1-form field \( A^I_\mu \), where \( I \) is the index for the so-called vector representation of the \( U \)-duality group \( E_{n(n)} \) \((n = 11 - d)\). Under the compactification, we decompose the indices in M-theory/type IIB theory as

\[
\text{M-theory: } \{i\} = \{\mu, i\} \quad (i = d, \ldots, 8, y, z),
\]
\[
\text{Type IIB theory: } \{m\} = \{\mu, m\} \quad (m = d, \ldots, 8, y).
\]

Then the vector index \( I \) in M-theory is decomposed into indices of \( \text{SL}(n) \) tensors as [2]

\[
(A^I_\mu) = \left( A^i_\mu, \frac{A_{\mu i}^{i_2 i_3}}{\sqrt{2!}}, \frac{A_{\mu i}^{i_2 i_3 i_5}}{\sqrt{3!}}, \frac{A_{\mu i}^{i_2 i_3 i_4 i_5 k}}{\sqrt{5!}}, \frac{A_{\mu i}^{i_2 i_3 i_4 i_5 k_1 k_2}}{\sqrt{813!}}, \frac{A_{\mu i}^{i_2 i_3 i_4 i_5 k_1 k_2 k_3}}{\sqrt{9!}}, \ldots \right),
\]

where only the relevant components are shown. In type IIB theory, the vector index is denoted by \( l \) and it is decomposed into indices of \( \text{SL}(n - 1) \times \text{SL}(2) \) tensors as [36]

\[
(A^I_\mu) = \left( A^m_{\mu i}, A^a_{\mu m}, \frac{A_{\mu m i}^{m a i}}{\sqrt{2!}}, \frac{A_{\mu m i}^{m a i_2 i_3}}{\sqrt{3!}}, \frac{A_{\mu m i}^{m a i_2 i_3 i_5}}{\sqrt{5!}}, \frac{A_{\mu m i}^{m a i_2 i_3 i_4 i_5 k}}{\sqrt{8!}}, \frac{A_{\mu m i}^{m a i_2 i_3 i_4 i_5 k_1 k_2}}{\sqrt{813!}}, \frac{A_{\mu m i}^{m a i_2 i_3 i_4 i_5 k_1 k_2 k_3}}{\sqrt{9!}}, \frac{A_{\mu m i}^{m a i_2 i_3 i_4 i_5 k_1 k_2 k_3 k_4}}{\sqrt{8!}}, \ldots \right).
\]

Now, we parametrize each component of the 1-form field in terms of the bosonic fields introduced in the last subsections. In fact, the 1-form has the universal form [20]

\[
A^I_\mu = \tilde{N}_\mu^I + \tilde{A}_\mu^I \tilde{N}_2^I (\text{M-theory}), \quad A^I_\mu = N^I_\mu + A^I_\mu \iota^I_N (\text{type IIB theory}),
\]

where \( \tilde{N}/N \) are the 11D/10D fields and \( \tilde{A}_\mu^I/\tilde{A}_\mu^I \) are the graviphotons, which are defined by

\[
\tilde{A}_\mu^i = g_{\mu \nu} \tilde{g}^{i \nu} \quad \left[ (g_{\mu \nu}) = (g^{\mu \nu})^{-1} \right].
\]

and it is similar for \( A^m_\mu \). Therefore, the parametrization of the 1-form field is equivalent to the parametrization of the 11D- or 10D-covariant fields \( \tilde{N} \) or \( N \).

In this paper, we parametrize the 11D tensors \( \{\tilde{N}\} = \{\tilde{N}^i, \frac{\tilde{N}^i_{i_2 i_3}}{\sqrt{2!}}, \ldots\} \) as follows:

\[
\tilde{N}^i_j = \delta^i_j, \quad (2.38)
\]
\[
\tilde{N}^i_{j i_2} = \hat{A}_{j i_2}, \quad (2.39)
\]
\[
\tilde{N}^i_{j i_2 i_3} = \hat{A}_{j i_2 i_3} - 5 \hat{A}_{[i_2 i_3} \hat{A}^{i_2 i_3]}, \quad (2.40)
\]
\[
\tilde{N}^i_{j i_2 i_3 i_4} \simeq \hat{A}_{j i_2 i_3} - 21 \hat{A}_{[i_2 i_3} \hat{A}_5 i_4], + 35 \hat{A}_{[i_2 i_3 i_4} \hat{A}_5 i_4 i_5], \quad (2.41)
\]
\[
\tilde{N}^i_{j i_2 i_3 i_4 i_5} \simeq \hat{A}_{j i_2 i_3} - 18 \hat{A}_{[i_2 i_3 i_4 i_5}, \hat{A}_5 i_3 i_4 i_5], + 3 \hat{A}_{j i_2 i_3 i_4 i_5} \hat{A}_5 i_3 i_4 i_5], \quad (2.42)
\]
\[
\tilde{N}^i_{j i_2 i_3 i_4 i_5 i_6} \simeq \hat{A}_{j i_2 i_3 i_4} - 84 \hat{A}_{j 1 i_2 i_3 i_4 i_5} \hat{A}_5 i_3 i_4 i_5], - 42 \hat{A}_{j i_2 i_3 i_4 i_5} \hat{A}_5 i_3 i_4 i_5], \quad (2.43)
\]

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Here, the overlined indices are totally antisymmetrized; e.g. \( \hat{A}_{j k_1 k_2} \hat{A}_{i_1 \ldots i_6 k_3} = \hat{A}_{j k_1 k_2} \hat{A}_{i_1 \ldots i_6 k_3} \). In addition, the equality
\[
\hat{N}_{j_1 \ldots i_p j_d \ldots j_q k_1 \ldots k_r} \simeq (\cdots)_{j_1 \ldots i_p j_d \ldots j_q k_1 \ldots k_r},
\]
(2.44)
denotes that it is valid only for the indices satisfying the restriction rule,
\[
\{i_1, \ldots, i_p\} \supset \{j_1, \ldots, j_q\} \supset \{k_1, \ldots, k_r\} \supset \cdots.
\]
(2.45)
Since the 1-form \( A_{1 \mu} \) is uniquely defined, the above parametrizations uniquely define our bosonic fields, particularly the mixed-symmetry potentials, \( \hat{A}_{k_1}, \hat{A}_{k_3} \), and \( \hat{A}_{10,11} \).

The detailed procedure of how to determine the above parametrization of \( \hat{N} \) is explained in section 2 of Ref. [20]. By considering the consistency between the M-theory and the type IIB parametrizations, the above parametrizations are uniquely determined (up to redefinitions of mixed-symmetry potentials). The same parametrization also can be obtained by constructing a matrix representation of the \( E_{11} \) generators, as discussed in Sect. 3 of Ref. [20]. In the second approach based on \( E_{11} \), the parametrizations will be completely determined without requiring the restriction rule (2.45) (see Ref. [20] for more details).

Now, let us turn to the type IIB parametrization. In type IIB theory, we parametrize the SL(2)-covariant 10D tensors \( \{N\} \) as follows:
\[
N_n^m = \delta_n^m,
\]
(2.46)
\[
N_n^{\alpha \beta} = A_{n m}^{\alpha \beta},
\]
(2.47)
\[
N_{n_1 m_1 m_2 m_3} = A_{n m_1 m_2 m_3} - \frac{1}{2} \epsilon_{\gamma \delta} A_{n m_1}^{\gamma} A_{m_2 m_3}^{\delta},
\]
(2.48)
\[
N_{n_1 m_1 \ldots m_5} = A_{n m_1 \ldots m_5}^{\alpha} + 5 A_{n_1 m_1 m_2 m_3 m_4}^{\alpha} A_{m_5}^{\beta} A_{m_2 m_3}^{\gamma} A_{m_4 m_5}^{\delta},
\]
(2.49)
\[
N_{n_1 m_1 \ldots m_6, p} \simeq A_{n m_1 \ldots m_6, p} + \epsilon_{\gamma \delta} A_{n m_1 \ldots m_6, p} A_{m_1 m_2 m_3}^{\gamma} A_{m_4 m_5 m_6}^{\delta} + 10 A_{n_1 m_1 m_2 m_3 m_4 m_5 m_6}^{\alpha \beta \gamma \delta}.
\]
(2.50)
\[
N_{n_1 m_1 \ldots m_7} = A_{n m_1 \ldots m_7}^{\alpha \beta} - 21 A_{n m_1 \ldots m_7}^{\alpha \beta} A_{m_2 m_3}^{\gamma} A_{m_4 m_5 m_6 m_7}^{\delta} - 105 A_{n_1 m_1 m_2 m_3 m_4 m_5 m_6 m_7}^{\alpha \beta \gamma \delta},
\]
(2.51)
\[
N_{n_1 m_1 \ldots m_7, p_1 p_2} \simeq A_{n m_1 \ldots m_7, p_1 p_2}^{\alpha \beta \gamma \delta} - 2 A_{n_1 m_1 \ldots m_7}^{\alpha \beta \gamma \delta} A_{n p_1}^{\gamma} A_{m_2 m_3}^{\delta} A_{m_4 m_5 m_6 m_7}^{\alpha} - 35 A_{n_1 m_1 m_2 m_3 m_4 m_5 m_6 m_7}^{\alpha \beta \gamma \delta} A_{m_7}^{\alpha},
\]
(2.52)
As discussed in Ref. [20], the two 1-forms \( A_\mu \) and \( A_\mu^l \) are the same object expressed in different bases. Indeed, they are related through a constant matrix \( S \) as

\[
A_\mu = S^{l\mu} A_\mu^l.
\]

Here, the equality

\[
N_n m_1 \ldots m_p p_1 \ldots p_q \equiv A_{nm_1 \ldots m_p p_1 \ldots p_q} + \epsilon_{\gamma \delta} A_{n m_1 \ldots m_p p_1 p_2} + 56 A_{n m_1 \ldots m_q p_1 p_2} A_{m_5 \ldots m_8} p_2 q
\]

\[
+ 168 \epsilon_{\gamma \delta} A_{n m_1 \ldots m_5 p_1 p_2} A_{m_5 m_7} A_{m_8 m_9} p_2 q
\]

\[
- 84 \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} A_{n m_1 \ldots m_5 p_1} A_{m_5 m_7} A_{m_8 m_9} p_2
\]

\[
- 84 \epsilon_{\gamma \delta} A_{n m_1 m_2 m_5} A_{m_5 m_7} A_{m_8 m_9} p_2
\]

\[
+ 210 \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} A_{n m_1} A_{m_2 m_5} A_{m_5 m_7} A_{m_8 m_9} p_2
\]

\[
- 1470 \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} \epsilon \eta A_{n m_1} A_{m_2 m_5} A_{m_5 m_7} A_{m_8 m_9} p_2
\]

\[
- 23100 \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} \epsilon \eta A_{n m_1} A_{m_2 m_5} A_{m_5 m_7} A_{m_8 m_9} p_2
\]

\[
(2.53)
\]

denotes that it is valid only for the indices satisfying the restriction rules

\[
\{m_1, \ldots, m_p\} \supset \{n_1, \ldots, n_q\} \supset \{p_1, \ldots, p_r\} \supset \cdots, \quad \alpha_1 = \cdots = \alpha_n.
\]

Now, let us comment more on the restriction rules given in Eqs. (2.45) and (2.55). As already mentioned, our parametrizations of \( \hat{N} \) and \( N \) are valid only for the restricted components. One of the motivations of this paper is to provide a firm ground for the study of the worldvolume dynamics of exotic branes. For that purpose, it will be enough to consider the restricted components, because the other components, which break the restriction rules, do not couple to any supersymmetric branes [32,35,37–39]. Components satisfying and breaking the rule are contained in different duality orbits, and they can be separated. Indeed, in our parametrizations, the restricted components of \( \hat{N} \) and \( N \) are always parametrized by the restricted components of mixed-symmetry potentials. For example, in the parametrization of \( \hat{N}_{j i_1 \ldots i_8 k_1 k_2 k_3} \) given in Eq. (2.42), as long as the rule \( \{i_1, \ldots, i_8\} \supset \{k_1, k_2, k_3\} \) is satisfied, the dual graviton \( \hat{A}_{i_1 \ldots i_8 k_3} \) appearing on the right-hand side also satisfy the rule \( \{i_1, \ldots, i_8\} \supset \{k_3\} \).\(^6\) In this sense, our parametrizations respect the restriction rule, and the \( T \)-duality rules obtained in the next section connect only the restricted components in type IIA/IIB theories. In other words, components breaking the restriction rule do not appear in our \( T \)-duality rules.

3. Duality rules

As discussed in Ref. [20], the two 1-forms \( A_\mu^l \) and \( A_\mu^l \) are the same object expressed in different bases. Indeed, they are related through a constant matrix \( S \) as

\[
A_\mu^l = S^{l\mu} A_\mu^l.
\]

\(^6\) This property can be spoiled by a redefinition; e.g. \( \hat{A}_{i_1 \ldots i_8 k_1 k_2 k_3} \rightarrow \hat{A}_{i_1 \ldots i_8 k_1 k_2 k_3} + \hat{A}_{i_1 i_2 i_3} \hat{A}_{i_4 \ldots i_8 k_1 k_2 k_3} \).
which is called the linear map \([23]\) (see also Ref. \([40]\)). In order to explain the linear map in more detail, let us further decompose the internal indices in M-theory and type IIB theory as

\[
\text{M-theory: } \{i\} = \{a, \alpha\}, \quad \text{Type IIB theory: } \{m\} = \{a, y\},
\]

where \(a = d, \ldots, 8\) and \(\alpha = y, z\). Under the decomposition, components of the 1-forms \(\{A^i_{\mu}\}\) in M-theory and \(\{A^4_{\mu}\}\) in type IIB theory are decomposed into \(\text{SL}(n - 2) \times \text{SL}(2)\) tensors. Then, the linear maps connect the \(\text{SL}(n - 2) \times \text{SL}(2)\) tensors in the two theories. In this paper, we consider the following linear maps (which are extensions of Refs. \([20, 23]\)):

\[
\begin{align*}
A^a_{\mu} & \equiv A^a_{\mu}, \quad A^{\alpha}_{\mu} \equiv A^{\alpha}_{\mu, y}, \quad A_{\mu; a_1 a_2} \equiv A_{\mu; a_1 a_2 y}, \quad A_{\mu; a a} \equiv A^\beta_{\mu; a} \epsilon_{\beta \alpha}, \quad A_{\mu; y z} \equiv A^y_{\mu}, \\
A_{\mu; a_1 \cdots a} & \equiv A_{\mu; a_1 \cdots a y}, \quad A_{\mu; a_1 \cdots a} \equiv A^\beta_{\mu; a_1 \cdots a y} \epsilon_{\beta \alpha}, \quad A_{\mu; a_1 a_2 a_1 y} \equiv A_{\mu; a_1 a_2 a_1}, \\
A_{\mu; a_1 \cdots a_6 a y} & \equiv A^8_{\mu; a_1 \cdots a_6 a y} \epsilon_{\beta \alpha}, \quad A_{\mu; a_1 \cdots a_6 (a_1, a_2)} \equiv A^8_{\mu; a_1 \cdots a_6 a y} \epsilon_{\beta_1 \alpha_1} \epsilon_{\beta_2 \alpha_2}, \\
A_{\mu; a_1 \cdots a_5 a} & \equiv A_{\mu; a_1 \cdots a_5 a} \epsilon_{\beta \alpha}, \quad A_{\mu; a_1 \cdots a_5} \equiv A^8_{\mu; a_1 \cdots a_5}, \\
A_{\mu; a_1 \cdots a_6 b_1 b_2 a y} & \equiv A^8_{\mu; a_1 \cdots a_6 b_1 b_2 a y} \epsilon_{\beta \alpha}, \\
A_{\mu; a_1 \cdots a_5 a} \equiv A_{\mu; a_1 \cdots a_5 a} \epsilon_{\beta \alpha}, \quad A_{\mu; a_1 \cdots a_5} \equiv A^8_{\mu; a_1 \cdots a_5}, \\
A_{\mu; a_1 \cdots a_7 a} & \equiv A_{\mu; a_1 \cdots a_7 a y}, \quad A_{\mu; a_1 \cdots a_7 a} \equiv A^8_{\mu; a_1 \cdots a_7 a y}, \quad A_{\mu; a_1 \cdots a_7} \equiv A^8_{\mu; a_1 \cdots a_7} \epsilon_{\beta_1 \alpha_1} \epsilon_{\beta_2 \alpha_2} \epsilon_{\beta_3 \alpha_3}.
\end{align*}
\]

These relate the M-theory fields (left-hand side) and the type IIB fields (right-hand side), and by rewriting the M-theory fields in terms of type IIA fields we obtain the \(T\)-duality rules between type IIA/IIB theories. By decomposing the indices \(\alpha, \beta\) into \(y \equiv 1\) and \(z \equiv 2\), and taking into account the restriction rule \((2.45)\), we find that there are 27 linear maps.\(^7\) They correspond to the 27 \(T\)-duality lines \([8–9]\) depicted in Fig. 1. In the following subsections, we obtain the 27 \(T\)-duality rules from the above linear maps.

Before proceeding, let us comment on the second restriction rule in type IIB theory \((2.55)\). Apparently, it looks different from the first restriction rule, but in fact both of them can be understood as consequences of the M-theory rule \((2.45)\) \([5]\). As an example, let us consider the last linear map \(A_{\mu; a_1 \cdots a_8 (a_1, a_2, a_3)} \equiv A^8_{\mu; a_1 \cdots a_8 a y} \epsilon_{\beta_1 \alpha_1} \epsilon_{\beta_2 \alpha_2} \epsilon_{\beta_3 \alpha_3}\). The M-theory rule requires

\[
\{a_1, \cdots, a_8, a_1\} \supset \{a_2\} \supset \{a_3\},
\]

and this leads to \(\alpha_1 = \alpha_2 = \alpha_3\). Then, the corresponding type IIB field \(A^8_{\mu; a_1 \cdots a_8 a y}\) needs to satisfy \(\beta_1 = \beta_2 = \beta_3\) and the second restriction rule in type IIB theory \((2.55)\) is derived.

### 3.1. Standard potentials

Let us begin with a comparison of the linear maps \([8–9]\) with the standard \(T\)-duality rules. For this purpose, we parametrize the type IIA fields and the \(S\)-duality-covariant type IIB fields by using familiar fields. We parametrize the 7-form and 9-form in type IIA theory as

\[
\mathcal{A}_7 = C_7 - \frac{1}{3} C_3 \wedge \mathcal{B}_2 \wedge \mathcal{B}_2, \quad \mathcal{A}_9 = C_9 - \frac{1}{3} C_3 \wedge \mathcal{B}_2 \wedge \mathcal{B}_2 \wedge \mathcal{B}_2,
\]

\(^7\) For example, the restriction rule \((2.45)\) for \(A_{\mu; a_1 \cdots a_9 (a_1, a_2, a_3)}\) is \(\alpha_1 = \alpha_2 = \alpha_3\) and it gives two linear maps:

\[
A_{\mu; a_1 \cdots a_9 (y, y, y)} \equiv -A^{222}_{\mu; a_1 \cdots a_9 y} \quad \text{and} \quad A_{\mu; a_1 \cdots a_9 (z, z, z)} \equiv A^{111}_{\mu; a_1 \cdots a_9 y}.
\]
These parametrizations are given such that the linear maps \( \mathfrak{A} \) in Eq. (3.3) reproduce the standard \( T \)-duality rules for NS–NS fields [41,42],

\[
(\mathfrak{A}_\mu^A) = B_{\mu y} + A_{\mu y} B_p y, \quad (\mathfrak{A}_\mu^B) = B_{\mu y} + A_{\mu y} B_p y, \quad (\mathfrak{A}_\mu^y) = B_{\mu y} + A_{\mu y} B_p y,
\]

and R–R fields [24,43,44],

\[
(\mathfrak{C}_{a_1 \cdots a_{n-1} y^y}) = \mathfrak{C}_{a_1 \cdots a_{n-1}} - \frac{(n-1) \mathfrak{C}_{a_1 \cdots a_{n-2} y^y} B_{a_{n-1} y}}{y y}, \quad (\mathfrak{C}_{a_1 \cdots a_y}) = \mathfrak{C}_{a_1 \cdots a_y} - \frac{n (n-1) \mathfrak{C}_{a_1 \cdots a_{n-1} y^y} B_{a_{n-1} y}}{y y}, \\
(\mathfrak{C}_{a_1 \cdots a_{n-1} y^y}) = \mathfrak{C}_{a_1 \cdots a_{n-1}} - \frac{(n-1) \mathfrak{C}_{a_1 \cdots a_{n-2} y^y} B_{a_{n-1} y}}{y y}, \quad (\mathfrak{C}_{a_1 \cdots a_y}) = \mathfrak{C}_{a_1 \cdots a_y} - \frac{n (n-1) \mathfrak{C}_{a_1 \cdots a_{n-1} y^y} B_{a_{n-1} y}}{y y}.
\] (3.11)

Here, the indices \( a_1, a_2, \cdots \) are 9D indices, which are orthogonal to the \( T \)-duality directions \( y \) or \( y \).

On the other hand, in the linear map (3.3), the indices are restricted to \( a_1, a_2, \cdots \) which run over the internal \( n - 2 \) dimensions [recall Eq. (3.2)]. By assuming that the \( T \)-duality rules have the 9D covariance, we have extended the linear map by replacing \( a \) with \( a \). This is always assumed in the following discussion.

Under the above parametrizations, the field strengths in type IIB supergravity become

\[
(F_{a^b}) = \begin{pmatrix} 1 & 0 \\ 0 & -C_0 \end{pmatrix} \begin{pmatrix} e^{2 \Phi} G_1 & d \Phi \\ d \Phi & -G_1 \end{pmatrix} \begin{pmatrix} 1 & -C_0 \\ 0 & 1 \end{pmatrix}, \quad (F^a_y) = \begin{pmatrix} 1 & 0 \\ -C_0 & 1 \end{pmatrix} \begin{pmatrix} H_3 \\ -G_3 \end{pmatrix},
\]

\[
F_5 = G_5, \quad (F^a_y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G_7 \\ -H_7 \end{pmatrix}, \quad (F^{a^b}_9) \simeq \begin{pmatrix} F^{11}_9 & F^{12}_9 \\ F^{21}_9 & F^{22}_9 \end{pmatrix} = \begin{pmatrix} G_9 \\ H_9 \end{pmatrix},
\] (3.12)

where we have defined

\[
H_3 \equiv dB_2, \quad G_{2p+1} \equiv dC_{2p} - H_3 \wedge C_{2p-2} \quad (C_{-2} \equiv 0), \\
H_7 \equiv dB_6 - C_4 \wedge dC_2 - \frac{1}{2} C_2 \wedge C_2 \wedge H_3 \wedge C_0 G_7,
\] (3.13) (3.14)
\[ H_9 \equiv dE_8 - B_6 \wedge dC_2 - \frac{1}{3!} H_3 \wedge C_2 \wedge C_2 \wedge C_2. \]  
\hfill (3.15)

By using the Hodge star operator * in the string frame \((\ast \omega_p = e^{\frac{\omega_p}{2}} \Phi \ast \omega_p)\), we obtain
\[ G_p = (-1)^{\frac{p(p-1)}{2}} \ast G_{10-p}, \quad \quad H_7 = e^{-2\Phi} \ast H_3, \]  
\[ H_9 = -2e^{-2\Phi} C_0 \ast d\Phi + (C_0^2 - e^{-2\Phi}) \ast G_1. \]  
\hfill (3.16)
\hfill (3.17)

The electric–magnetic duality for \(H_9\) is considered further in Sect. 3.4.

For later convenience, we also define the following 6-form, which was used in Ref. [21]:
\[ B_6 \equiv B_6 - C_4 \wedge C_2. \]  
\hfill (3.18)

We also introduce several redefinitions of the dual graviton. The type IIB dual graviton \(N'_{7,1}\) introduced in Ref. [21] is related to our \(A_{7,1}\) as
\[ N'_{m_1\cdots m_7,n} \simeq A_{m_1\cdots m_7,n} - 7 B_{[m_1\cdots m_7,B_{m_7}]n} - \frac{105}{4} C_{[m_1\cdots m_4} (B_{m_5m_6} C_{m_7]n} - 3 C_{m_5m_6} B_{m_7]n}) \]  
\[ - \frac{315}{4} C_{[m_1m_2} C_{m_3m_4} B_{m_5m_6} B_{m_7]n}. \]  
\hfill (3.19)

As we see below, this is useful in simplifying the \(T\)-duality rules (although the \(S\)-duality covariance is not manifest). In M-theory, we introduce
\[ \hat{A}_{i_1\cdots i_8,j} = \hat{A}_{i_1\cdots i_6} \hat{A}_{i_7i_8,j}, \]  
\hfill (3.20)

and define \(A_{7,1}\) as \(A_{m_1\cdots m_7,n} \equiv \hat{A}_{m_1\cdots m_7,n}\). More explicitly, we have
\[ A_{m_1\cdots m_7,n} \simeq A_{m_1\cdots m_7,n} + \frac{14}{3} B_{[m_1\cdots m_6} B_{m_7]n} \]  
\[ - 14 C_{[m_1\cdots m_5} C_{m_6m_7]n} + 70 E_{[m_1m_2m_3} E_{m_4m_5]n} E_{m_6m_7}. \]  
\hfill (3.21)

The type IIA dual graviton \(A_{7,n}\) associated with a Killing direction (i.e. isometry direction) \(n\) corresponds to \(N'(7)\) of Ref. [21], and it is also useful in simplifying the \(T\)-duality rules.

### 3.2. Potentials for solitonic five branes

In this subsection, we consider the potentials that couple to the solitonic 5-branes (i.e. 5-branes with tensions \(T \propto g_s^{-2}\)). We begin by reproducing the known \(T\)-duality rules \(\mathcal{A}_1\)–\(\mathcal{A}_6\). There, we demonstrate that redefinitions of potentials can make the \(T\)-duality rules simpler. After reproducing the known results, we obtain new \(T\)-duality rules \(\mathcal{A}_7\)–\(\mathcal{A}_9\). In Sect. 3.2.1, we find the field redefinitions, which make the \(T\)-duality rules considerably simple, and discuss the relation to the potentials studied in the context of the double field theory (DFT) [45–49].

**\(T\)-duality rule \(\mathcal{A}_1\): 5₂ (IIA) ↔ 5₂ (IIB)** By substituting our parametrizations into the linear map \(A_{\mu_1\cdots \mu_5}^{\nu_1\cdots \nu_5} = -A_{\mu_1\cdots \mu_5}^{\nu_1\cdots \nu_5}\), we find
\[ B_{a_1\cdots a_5} \equiv B_{a_1\cdots a_5} \]  
\[ B_{a_1\cdots a_5} \equiv B_{a_1\cdots a_5} + 5 C_{[a_1a_2a_3]} C_{a_4a_5}, \]  
\[ C_{a_1\cdots a_4} \equiv C_{a_1\cdots a_4} - \frac{2}{g_{yy}} \left( C_{a_1a_2a_3} \right) \left( g_{yy} \right), \]  
\hfill (3.22)

\[ \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = \mathcal{A}_5 = \mathcal{A}_6. \]

\[ \mathcal{A}_7 = \mathcal{A}_8 = \mathcal{A}_9. \]

\[ \mathcal{A}_7 = \mathcal{A}_8 = \mathcal{A}_9. \]  

---

\(^8\) In the literature (e.g. [18,21]), the last index of the dual graviton is supposed to be a particular isometry direction, and it is not written down explicitly. Accordingly, the dual graviton is treated as a 7-form.
They were obtained in Ref. [18] (see Appendix A.2 for their conventions).

They can be simplified by using the 6-form $B_6$ [21]

$$B_{a_1\ldots a_6 y} \overset{A-B}{=} B_{a_1\ldots a_6} + 5 C_{[a_1 a_2 a_3] y} \left( C_{a_4 a_5} - \frac{2 C_{a_4 y} g_{a_5 y}}{g_{yy}} \right),$$

(3.24)

$$B_{a_1\ldots a_6 y} \overset{B-A}{=} B_{a_1\ldots a_6 y} - 5 \left( C_{[a_1 a_2 a_3] y} - \frac{3 C_{[a_1 a_2] y a_3 y}}{g_{yy}} \right) C_{a_4 a_5 y}.$$  

(3.25)

On the other hand, if we employ the $S$-duality-covariant fields, we find

$$B_{a_1\ldots a_6 y} \overset{A-B}{=} D_{a_1\ldots a_6 y} + 5 A_{[a_1 a_2 a_3] y} \left( C_{a_4 a_5} - \frac{2 C_{a_4 y} g_{a_5 y}}{g_{yy}} \right)$$

$$+ \frac{5}{2} \epsilon_{y \delta} A_{[a_1 a_2} A^\delta_{a_3]} \left( C_{a_4 a_5} - \frac{6 C_{a_4 y} g_{a_5 y}}{g_{yy}} \right),$$

(3.26)

$$D_{a_1\ldots a_6 y} \overset{B-A}{=} B_{a_1\ldots a_6 y} - 5 \left( C_{[a_1 a_2 a_3] y} - \frac{2 C_{[a_1 y a_2] a_3 y}}{g_{yy}} \right) C_{a_4 a_5 y}$$

$$+ 10 C_{[a_1 a_2 y]} \left( C_{a_3} - \frac{C_{[a_1 y a_2] y}}{g_{yy}} \right) B_{a_4 a_5 y}.$$

(3.27)

Since our linear maps (3.3) have the $S$-duality covariance, the $T$-duality rules are covariant under $S$-duality. In the present example, we can uplift the $T$-duality rule (3.26) into

$$B_{a_1\ldots a_5 a_6} \overset{M-B}{=} \left[ A^\beta_{a_1\ldots a_5 y} + 5 A_{[a_1 a_2 a_3] y} \left( A^\beta_{a_4 a_5} - \frac{2 A^\beta_{a_4 y} g_{a_5 y}}{g_{yy}} \right) \right. \left. + \frac{5}{2} \epsilon_{y \delta} A_{[a_1 a_2} A^\delta_{a_3]} \left( A^\beta_{a_4 a_5} - \frac{6 A^\beta_{a_4 y} g_{a_5 y}}{g_{yy}} \right) \right] \epsilon_{\beta \alpha},$$

(3.28)

which indeed reproduces (3.26) by choosing $\alpha = y$. On the other hand, $\alpha = z$ gives

$$C_{a_1\ldots a_5 y} - 5 C_{[a_1 a_2 a_3} B_{a_4 a_5]} \overset{A-B}{=} C_{a_1\ldots a_5 y} + 5 A_{[a_1 a_2 a_3] y} \left( B_{a_4 a_5} - \frac{2 B_{a_4 y} g_{a_5 y}}{g_{yy}} \right)$$

$$+ \frac{5}{2} \epsilon_{y \delta} A_{[a_1 a_2} A^\delta_{a_3]} \left( B_{a_4 a_5} - \frac{6 B_{a_4 y} g_{a_5 y}}{g_{yy}} \right),$$

(3.29)

which corresponds to the $T$-duality rule for $R$–R potentials obtained before. Each of the $T$-duality rules obtained in this paper has this kind of $S$-dual partner.

**$T$-duality rule (II):** $5_2^1$ (IIA) $\leftrightarrow$ $5_2$ (IIB) From the linear map $A_{\mu_1 a_1\ldots a_{3\mu}, y} \overset{\theta}{=} -A_{\mu_2 a_1\ldots a_5}$, we obtain [18]

$$A_{a_1\ldots a_5 y} \overset{A-B}{=} B_{a_1\ldots a_5} - \frac{6 B_{[a_1 a_2 a_3] y} g_{a_4 y}}{g_{yy}} - 30 B_{[a_1 a_2} C_{a_3 a_4} \left( C_{a_5 a_6} - \frac{4 C_{a_5 y} g_{a_6 y}}{g_{yy}} \right)$$

$$+ \frac{20 C_{[a_1 a_2} C_{a_3 a_4} g_{a_5 y}}{g_{yy}},$$

(3.30)

$$B_{a_1\ldots a_6} \overset{B-A}{=} A_{a_1\ldots a_6 y} - 6 B_{[a_1 a_2 a_3] y} B_{a_4 a_5} - 30 C_{[a_1 a_2 a_3] y} \left( C_{a_4 a_5} - \frac{C_{[a_4 y a_5]} g_{a_6 y}}{g_{yy}} \right) B_{a_6 y}$$

$$- 10 C_{[a_1 a_2 a_3} C_{a_4 a_5] y} B_{a_6 y} + 30 C_{[a_1 a_2 y} C_{a_3 a_4]} \left( B_{a_5 a_6} - \frac{3 B_{a_5 y} g_{a_6 y}}{g_{yy}} \right).$$

(3.31)

By using the potentials $A_{7,1}$ and $B_6$, we can simplify the duality rules as [21]

$$A_{a_1\ldots a_6 y} \overset{A-B}{=} B_{a_1\ldots a_6} - \frac{2 B_{[a_1 a_2 a_3] y} g_{a_4 y}}{g_{yy}} + 5 C_{[a_1 a_2 a_3} \left( C_{a_4 a_5} - \frac{2 C_{a_4 y} g_{a_5 y}}{g_{yy}} \right),$$

(3.32)
In the following, we adopt \( S \), we obtain \( 21 \) for the S-duality counterpart is the T-duality rule (1), relating the R–R 7-form and 6-form.

In the following, we adopt \( \mathcal{B}_6, \mathcal{A}_7, \mathcal{D}_6, \mathcal{B}_6, \mathcal{N}_7 \) for the S-duality non-covariant expressions and \( \mathcal{B}_6, \mathcal{A}_7, \mathcal{D}_6, \mathcal{A}_7 \) for the S-duality covariant expressions.

**T-duality rule (3):** 5\(_2\) (IAA) ↔ 5\(_1\) (IIB) From \( A_{\mu; a_1 \cdots a_5} \equiv A_{\mu; a_1 \cdots a_y y} \), we obtain \( 21 \)

\[
B_{a_1 \cdots a_6} = B_{a_1 \cdots a_5} - 2B_{a_1 \cdots a_5} B_{a_6} + 5C_{a_1 \cdots a_4} C_{a_5 a_6} + 30 \left( C_{a_1 a_2 a_3} - \frac{3g_{a_1 a_2 a_3} g_{a_1 a_2 a_3}}{yy} \right) B_{a_4 a_5} B_{a_6},
\]

\[(3.33)\]

Instead, if we use the S-duality covariant fields, we find

\[
\mathcal{B}_{a_1 \cdots a_6} = \mathcal{B}_{a_1 \cdots a_5} - 6 \mathcal{B}_{a_1 \cdots a_5} B_{a_6} + 15 \mathcal{A}_{a_1 \cdots a_4} \left( C_{a_5 a_6} - \frac{2g_{a_5 a_6} g_{a_5 a_6}}{yy} \right) + \frac{40 \mathcal{A}_{a_1 a_2 a_3} C_{a_5 a_6} g_{a_5 a_6}}{yy} - \frac{30 \epsilon_{a_1 a_2 a_3} A_{a_5 a_6} A_{a_5 a_6} g_{a_5 a_6}}{yy},
\]

\[(3.34)\]

\[
\mathcal{D}_{a_1 \cdots a_6} = \mathcal{D}_{a_1 \cdots a_5} - 6 \mathcal{B}_{a_1 \cdots a_5} B_{a_6} + 15 \mathcal{A}_{a_1 \cdots a_4} \left( C_{a_5 a_6} - \frac{2g_{a_5 a_6} g_{a_5 a_6}}{yy} \right) + \frac{40 \mathcal{A}_{a_1 a_2 a_3} C_{a_5 a_6} g_{a_5 a_6}}{yy} + \frac{50 \epsilon_{a_1 a_2 a_3} A_{a_5 a_6} A_{a_5 a_6} g_{a_5 a_6}}{yy} - \frac{60 \mathcal{A}_{a_1 a_2 a_3} C_{a_5 a_6} g_{a_5 a_6}}{yy}.
\]

\[(3.35)\]

\[
\mathcal{N}_{a_1 \cdots a_6} = \mathcal{N}_{a_1 \cdots a_5} - 6 \mathcal{B}_{a_1 \cdots a_5} B_{a_6} + 15 \mathcal{A}_{a_1 \cdots a_4} \left( C_{a_5 a_6} - \frac{2g_{a_5 a_6} g_{a_5 a_6}}{yy} \right) B_{a_6},
\]

\[(3.36)\]

\[
\mathcal{N}_{a_1 \cdots a_6} = \mathcal{A}_{a_1 \cdots a_6} - 6 \mathcal{B}_{a_1 \cdots a_5} B_{a_6} + 15 \mathcal{A}_{a_1 \cdots a_4} \left( C_{a_5 a_6} - \frac{2g_{a_5 a_6} g_{a_5 a_6}}{yy} \right) B_{a_6},
\]

\[(3.37)\]

For the S-duality covariant fields, we find \( 20 \)

\[
B_{a_1 \cdots a_6} = \mathcal{B}_{a_1 \cdots a_5} - 15 \epsilon_{a_1 a_2 a_3} A_{a_5 a_6} \left( A_{a_4 a_6} A_{a_5 a_6} + \frac{2A_{a_4 a_6} A_{a_5 a_6} g_{a_5 a_6}}{yy} \right),
\]

\[(3.38)\]

\[
A_{a_1 \cdots a_6} = \mathcal{A}_{a_1 \cdots a_5} - 15 \epsilon_{a_1 a_2 a_3} B_{a_5 a_6} \left( C_{a_6} - \frac{g_{a_5 a_6} b_{a_5 a_6}}{yy} \right) + \frac{15 \epsilon_{a_1 a_2 a_3} C_{a_5 a_6} g_{a_5 a_6}}{yy},
\]

\[(3.39)\]

which are self-dual under S-duality.

**T-duality rule (4):** 5\(_2\) (IAA) ↔ 5\(_1\) (IIB) From the linear map \( A_{\mu; a_1 \cdots a_5} \equiv \mathcal{A}_{\mu; a_1 \cdots a_y a} \), we obtain

\[
\mathcal{A}_{a_1 \cdots a_6} = \mathcal{A}_{a_1 \cdots a_5} - \frac{1}{3} \mathcal{N}_{a_1 \cdots a_6} B_{a_6} - 4 \mathcal{B}_{a_1 \cdots a_5} B_{a_6} - \left( B_{a_1 \cdots a_6} - \frac{2B_{a_1 \cdots a_5} g_{a_5 a_6}}{yy} \right) B_{a_6} + 5 \mathcal{C}_{a_1 \cdots a_4} \left( C_{a_5 a_6} - \frac{2g_{a_5 a_6} g_{a_5 a_6}}{yy} \right) B_{a_6},
\]

\[(3.40)\]
They are partially obtained in Eq. (5.13) of Ref. [11] under the truncation $B_2 = 0 = C_2$. The full result without the truncation is obtained in Ref. [20]. The same $T$-duality map seems to be obtained in Eqs. (3.10) and (3.11) of Ref. [22], although the relation to our potentials is not clear.

On the other hand, by using the $S$-duality covariant fields, we obtain

$$\mathcal{A}_{a_1 \ldots a_6y,b} \cong \mathcal{A}_{a_1 \ldots a_6y,b} - \frac{A_{a_1 \ldots a_6y} \mathcal{B}_{by}}{g_{yy}} + 6 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1 \ldots a_5|y} \mathcal{A}^\delta_{a_6|y} + 30 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1 \ldots a_4|y} \mathcal{A}^\delta_{a_5|y} \mathcal{B}_{by}$$

$$+ 10 \left( \mathcal{B}_{a_1a_2a_3|y} + \frac{K_{a_1a_2a_3}}{g_{yy}} \right) \mathcal{A}_{a_4a_5a_6|y} + 20 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by}$$

$$+ 30 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by} - 30 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by}$$

$$+ 60 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by} + 45 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5a_6|y} \mathcal{B}_{by}$$

$$- 15 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5a_6|y} \mathcal{B}_{by} - 15 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5a_6|y} \mathcal{B}_{by}$$

$$= - \frac{A_{a_1 \ldots a_6y} \mathcal{B}_{by}}{g_{yy}} + 6 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1 \ldots a_5|y} \mathcal{A}^\delta_{a_6|y} + 30 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1 \ldots a_4|y} \mathcal{A}^\delta_{a_5|y} \mathcal{B}_{by}$$

$$- 10 \left( \mathcal{B}_{a_1a_2a_3|y} + \frac{K_{a_1a_2a_3}}{g_{yy}} \right) \mathcal{A}_{a_4a_5a_6|y} + 20 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by}$$

$$+ 30 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by} - 30 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by}$$

$$+ 60 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5|y} \mathcal{B}_{by} + 45 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5a_6|y} \mathcal{B}_{by}$$

$$- 15 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5a_6|y} \mathcal{B}_{by} - 15 \epsilon_{y\delta} \mathcal{A}^\gamma_{a_1a_2a_3|y} \mathcal{A}^\delta_{a_4a_5a_6|y} \mathcal{B}_{by}$$

$$= (3.42)$$

$$\mathcal{A}_{a_1 \ldots a_6y,b} \cong \mathcal{A}_{a_1 \ldots a_6y,b} - \mathcal{B}_{a_1 \ldots a_6} \mathcal{B}_{by} - 6 \mathcal{B}_{a_1 \ldots a_5|y} \mathcal{B}_{a_6|y} + 6 \mathcal{C}_{a_1 \ldots a_5} \mathcal{C}_{a_6|y}$$

$$- 10 \mathcal{C}_{a_1a_2a_3|by} \left( \mathcal{C}_{a_4a_5a_6} + \frac{3}{2} \mathcal{C}_{a_4a_5|y} \right)$$

$$- 15 \mathcal{C}_{a_1a_2a_3|by} \left( \mathcal{C}_{a_4} - \frac{\mathcal{C}_{a_4|y}}{g_{yy}} \right) \mathcal{B}_{a_5a_6}$$

$$+ 20 \mathcal{C}_{a_1a_2a_3} \left( \mathcal{C}_{a_4a_5|y} + \frac{\mathcal{C}_{a_4a_5|y}}{g_{yy}} \right) \mathcal{B}_{a_6|y} - 50 \mathcal{C}_{a_1a_2a_3} \mathcal{C}_{a_4|y} \mathcal{B}_{a_5a_6}$$

$$+ 15 \mathcal{C}_{a_1a_2a_3} \left( \mathcal{C}_{a_4a_5a_6} + \frac{\mathcal{C}_{a_4a_5a_6}}{g_{yy}} \right) \mathcal{B}_{a_6|y}$$

$$+ 15 \mathcal{C}_{a_1a_2a_3} \left( \mathcal{C}_{a_4a_5a_6} + \frac{\mathcal{C}_{a_4a_5a_6}}{g_{yy}} \right) \mathcal{B}_{a_6|y}$$

$$- 45 \mathcal{C}_{a_1a_2a_3} \left( \mathcal{C}_{a_4a_5a_6} + \frac{\mathcal{C}_{a_4a_5a_6}}{g_{yy}} \right) \mathcal{B}_{a_6|y}$$

$$= (3.43)$$

A short comment: In the following, we present new results. The $T$-duality rules obtained below are rather lengthy, and we determine the maps only from type IIA fields to type IIB fields. However, in Sect. 3.2.1, we find a redefinition of mixed-symmetry potentials, which transforms our potentials into the potentials $D_6, D_{7,1}$ and $D_{8,2}$. The $T$-duality rules for the new fields are very simple, and one can easily find the inverse map, if necessary.
**T-duality rule T:** $S_2^1 (IIA) \leftrightarrow S_2^1 (IIB)$ The linear map $A_{\mu; a_1 \cdots a_6, a} \overset{\oplus}{=} A_{\mu; a_1 \cdots a_6, a}$ gives

$$\mathcal{M}_{a_1 \cdots a_7, b}^{A-B} \cong A_{a_1 \cdots a_7, b} = \frac{35 A_{[a_1 a_2 a_3] b} A_{a_4 a_5 a_6} g_{y_7 y}}{g_{y y}} - \frac{105}{2} e_{y \delta} A_{[a_1 a_2 a_3] b} A_{a_4 a_5 a_6} g_{y_7 y} - \frac{105}{2} e_{y \delta} A_{[a_1 a_2 a_3] y} A_{a_4 a_5 a_6} A_{a_7} g_{y_7 y} + \frac{315}{4} e_{y \eta} A_{[a_1 a_2}] A_{a_3 a_4 a_5} A_{a_6 a_7} g_{y_7 y},$$

which is self-dual under $S$-duality.

**T-duality rule T:** $S_2^1 (IIA) \leftrightarrow S_2^1 (IIB)$ From the linear map $A_{\mu; a_1 \cdots a_6, a} \overset{\oplus}{=} A_{\mu; a_1 \cdots a_6, a}$, we obtain

$$\mathcal{M}_{a_1 \cdots a_7, b}^{A-B} \cong D_{a_1 \cdots a_7, b} - 7 A_{[a_1 \cdots a_6] y} B_{a_7} b - 42 A_{[a_1 \cdots a_5] y} B_{a_6 a_7} g_{y_7 y} - 105 e_{y \delta} C_{[a_1 \cdots a_5] y} A_{a_6 a_7} g_{y_7 y} + 140 A_{[a_1 a_2 a_3] y} A_{a_4 a_5 a_6} B_{a_7} g_{y_7 y} - 70 A_{[a_1 a_2 a_3] y} A_{a_4 a_5 a_6} B_{a_7} g_{y_7 y} - 70 e_{y \delta} A_{[a_1 a_2 a_3] y} B_{a_4 a_5} A_{a_6 a_7} g_{y_7 y} + 140 e_{y \delta} A_{[a_1 a_2 a_3] y} B_{a_4 a_5} A_{a_6 a_7} g_{y_7 y} + 105 e_{y \delta} A_{[a_1 a_2 a_3] y} B_{a_4 a_5} A_{a_6 a_7} g_{y_7 y} + 140 e_{y \delta} A_{[a_1 a_2 a_3] y} B_{a_4 a_5} A_{a_6 a_7} g_{y_7 y} - 140 e_{y \delta} A_{[a_1 a_2 a_3] y} B_{a_4 a_5} A_{a_6 a_7} g_{y_7 y} + 105 e_{y \delta} A_{[a_1 a_2 a_3] y} B_{a_4 a_5} A_{a_6 a_7} g_{y_7 y} - \frac{1575}{8} e_{y \eta} B_{[a_1 a_2 a_3 a_4] A_{a_5 a_6 a_7} g_{y_7 y}} + \frac{2825}{4} e_{y \eta} B_{[a_1 a_2 a_3 a_4] A_{a_5 a_6 a_7} g_{y_7 y}} + \frac{315}{2} e_{y \eta} B_{[a_1 a_2 a_3 a_4] A_{a_5 a_6 a_7} g_{y_7 y}}.$$  

(3.45)

Under the simplifying assumption $B_2 = C_2 = 0$, this map was obtained in Ref. [11] [the last line of Eq. (5.12)], where $\mathcal{N}^{(8)}$ corresponds to our $D_{8, by}$ (up to $B_2 = C_2 = 0$). The $S$-dual counterpart of this $T$-duality rule is obtained later in Eq. (3.69).

**T-duality rule T:** $S_2^1 (IIA) \leftrightarrow S_2^1 (IIB)$ From the linear map $A_{\mu; a_1 \cdots a_6, b} \overset{\oplus}{=} A_{\mu; a_1 \cdots a_6, b}$, we obtain

$$\mathcal{M}_{a_1 \cdots a_7, b_1 b_2}^{A-B} \cong D_{a_1 \cdots a_7, b_1 b_2} - 7 C_{[a_1 \cdots a_6] A_{a_7} b_1 b_2 y} - \frac{7}{2} e_{y \delta} C_{[a_1 \cdots a_6] A_{a_7} b_1 b_2} A_{a_6 a_7} + \frac{35}{2} C_{[a_1 a_2 a_3] b_1 b_2} \left( A_{a_4 a_5 a_6} + \frac{2 A_{a_4 a_5 a_6} g_{y_7 y}}{g_{y y}} \right) + \frac{105}{2} e_{y \delta} C_{[a_1 \cdots a_4] b_1 b_2} A_{a_5 a_6 a_7} A_{a_7} b_2 g_{y_7 y} - \frac{105}{2} e_{y \delta} C_{[a_1 a_2 a_3] b_1 b_2} A_{a_4 a_5 a_6} A_{a_7} b_2 g_{y_7 y} + 35 A_{[a_1 a_2 a_3] b_1 b_2} A_{a_4 a_5 a_6} B_{a_7} b_2 - \frac{315}{2} A_{[a_1 a_2 a_3] b_1 b_2} A_{a_4 a_5 a_6} B_{a_7} b_2 + 70 A_{[a_1 a_2 a_3] b_1 b_2} A_{a_4 a_5 a_6} B_{a_7} b_2 + \frac{105}{2} A_{[a_1 a_2 a_3] b_1 b_2} A_{a_4 a_5 a_6} B_{a_7} b_2 g_{y_7 y}.$$  

(3.46)

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Now, let us consider the field redefinition that makes the expression simpler by finding some $S$-duality-covariant field redefinitions, but we do not attempt to find such redefinitions here.

### 3.2.1. $T$-dual-manifest redefinitions

Now, let us consider the field redefinition that makes the $T$-duality rules very simple. In the case of the R–R fields in type IIA/IIB theory, the R–R polyform in the $C$-basis is defined as

$$C \equiv C_1 + C_3 + C_5 + C_7 + C_9, \quad C \equiv C_0 + C_2 + C_4 + C_6 + C_8 + C_{10}. \quad (3.47)$$

By considering a redefinition into the $A$-basis [50],

$$A = e^{-B_2 \wedge C}, \quad A = e^{-B_2 \wedge C}, \quad (3.48)$$

we find that the $T$-duality rules for the new fields are simple [51]:

$$A_{a_1 \ldots a_p} = B_{a_1 \ldots a_p}, \quad A_{a_1 \ldots a_p} = B_{a_1 \ldots a_p}. \quad (3.49)$$
This is according to the fact that the $A$-basis transforms as an $O(10, 10)$ spinor. As studied in Refs. [52–54], if we define the (real) gamma matrices $\{\Gamma^M\} = \{\Gamma^m, \Gamma_m\}$ that satisfy

$$\{\Gamma^M, \Gamma^N\} = \eta^{MN}, \quad (\eta^{MN}) = \begin{pmatrix} 0 & \delta^m_n \\ \delta^n_m & 0 \end{pmatrix}, \quad (\Gamma_m)^T = \Gamma^m, \quad (\Gamma^m)^T = \Gamma_m, \quad (3.50)$$

and also define the Clifford vacuum $|0\rangle$ as

$$\Gamma_m |0\rangle = 0, \quad (0|0) = 1, \quad \Gamma^{11}|0\rangle = |0\rangle,$$

where $(0\rangle = |0\rangle^T$ and $\Gamma^{11} \equiv (-1)^{N_F} (N_F \equiv \Gamma^m \Gamma_m)$, we find that

$$|A\rangle \equiv \sum_p \frac{1}{p!} A_{m_1 \ldots m_p} \Gamma^{m_1 \ldots m_p} |0\rangle, \quad \Gamma^{[M_1 \ldots M_p]} \equiv \Gamma^{[M_1 \ldots M_p]} \quad (3.52)$$

transforms as an $O(10, 10)$ spinor. Here, the R–R field $|A\rangle$ is defined to have a definite chirality

$$\Gamma^{11}|A\rangle = \mp |A\rangle \quad \text{(IIA/IIB)}. \quad (3.53)$$

Under the factorized $T$-duality along the $x^\gamma$-direction, it transforms as

$$|A\rangle \rightarrow |A\rangle' = (\Gamma^\gamma - \Gamma_\gamma) \Gamma^{11} |A\rangle,$$

and in terms of the components, this transformation rule gives the rules (3.49).

Similarly, the potentials which couple to the solitonic 5-branes also constitute an $O(10, 10)$-covariant potential denoted by $D_{M_1 \ldots M_4}$ [55], where the 20D indices are totally antisymmetric. This tensor can be generally decomposed into $SL(10)$ tensors:

$$D^{m_1 m_2 m_3 m_4} = \frac{1}{6!} \epsilon^{m_1 \ldots m_4 n_1 \ldots n_6} D_{n_1 \ldots n_6},$$

$$D^{m_1 m_2 m_3} = \frac{1}{7!} \epsilon^{m_1 m_2 m_3 n_1 \ldots n_7} D_{n_1 \ldots n_7, m_4} + \cdots,$$

$$D^{m_1 m_2 m_3} = \frac{1}{8!} \epsilon^{m_1 m_2 m_3 n_1 \ldots n_8} D_{n_1 \ldots n_8, m_4} + \cdots,$$

$$D^{m_1 m_2 m_3} = \frac{1}{9!} \epsilon^{m_1 m_2 m_3 n_1 \ldots n_9} D_{n_1 \ldots n_9, m_4} + \cdots,$$

$$D^{m_1 m_2 m_3} = \frac{1}{10!} \epsilon^{m_1 m_2 m_3 n_1 \ldots n_{10}} D_{n_1 \ldots n_{10}, m_4} \quad (3.55)$$

where the ellipses denote the irrelevant contribution from the potentials that do not couple to supersymmetric branes. Under the $T$-duality along the $x^\gamma$-direction, this transforms as

$$D'_{M_1 \ldots M_4} = \Lambda_{M_1} N_1 \ldots \Lambda_{M_4} N_4 D_{N_1 \ldots N_4}, \quad (\Lambda_M^N) \equiv \begin{pmatrix} 1 - e_y & e_y \\ e_y & 1 - e_y \end{pmatrix}, \quad (3.56)$$

where $e_y$ is a $10 \times 10$ matrix, $e_y = \text{diag}(0, \ldots, 0, 1_y, 0, \ldots, 0)$. By rewriting the transformation rule in terms of the component fields $D_6, D_{7,1}, D_{8,2}, D_{9,3}$ and $D_{10,4}$, we obtain

$$D_{a_1 \ldots a_6 b_1 \ldots b_9} \overset{A-B}{\rightarrow} D_{a_1 \ldots a_6 b_1 \ldots b_9 y}, \quad (n = 0, \ldots, 3),$$

$$D_{a_1 \ldots a_5 b_1 \ldots b_9 y} \overset{A-B}{\rightarrow} D_{a_1 \ldots a_5 b_1 \ldots b_9 y}, \quad (n = 0, \ldots, 4),$$

$$D_{a_1 \ldots a_5 b_1 \ldots b_9 y} \overset{A-B}{\rightarrow} D_{a_1 \ldots a_5 b_1 \ldots b_9 y}, \quad (n = 0, \ldots, 3). \quad (3.57)$$
Similarly to the case of the R–R potential $A_p$, which is related to our potentials as Eq. (3.48), it is natural to expect that $D_{6+n,n}$ are also obtained by considering a redefinition of our mixed-symmetry potentials. Indeed, if we redefine the type IIA fields as

$$D_{m_1\cdots m_6} \equiv B_{m_1\cdots m_6} - \frac{15}{2} C_{[m_1\cdots m_4] m_5 m_6} - \frac{1}{2} C_0 C_{m_1\cdots m_6},$$

$$D_{m_1\cdots m_7,n} \simeq A_{m_1\cdots m_7} + 7 B_{m_1\cdots m_6} B_{m_7} - \frac{7}{2} C_0 C_{m_1\cdots m_6} B_{m_7} - \frac{105}{4} C_{[m_1\cdots m_4] m_5 m_6} C_{m_7} B_{m_7},$$

$$D_{m_1\cdots m_8,n_1 n_2} \simeq D_{m_1\cdots m_8,n_1 n_2} - 4 C_{[m_1\cdots m_7] n_1 [m_8] n_2} + 4 C_0 C_{[m_1\cdots m_7] n_1 [m_8] n_2}$$

$$+ 28 C_{[m_1\cdots m_6] m_7 n_1 [m_8] n_2} - 28 C_0 C_{[m_1\cdots m_6] m_7 n_1 [m_8] n_2}$$

$$- 84 C_{[m_1\cdots m_5; m_6] m_7 n_1 [m_8] n_2} + 210 C_{[m_1 m_2; m_3 m_4] m_5 m_6 m_7 m_8} n_1 n_2,$$

and type IIB fields as

$$D_{m_1\cdots m_6} \equiv B_{m_1\cdots m_6} - \frac{15}{2} C_{[m_1\cdots m_4] m_5 m_6} - \frac{1}{2} C_0 C_{m_1\cdots m_6},$$

$$D_{m_1\cdots m_7,n} \simeq A_{m_1\cdots m_7} + 7 B_{m_1\cdots m_6} B_{m_7} - \frac{7}{2} C_0 C_{m_1\cdots m_6} B_{m_7} - \frac{105}{4} C_{[m_1\cdots m_4] m_5 m_6} C_{m_7} B_{m_7},$$

$$D_{m_1\cdots m_8,n_1 n_2} \simeq D_{m_1\cdots m_8,n_1 n_2} - 4 C_{[m_1\cdots m_7] n_1 [m_8] n_2} + 4 C_0 C_{[m_1\cdots m_7] n_1 [m_8] n_2}$$

$$+ 28 C_{[m_1\cdots m_6] m_7 n_1 [m_8] n_2} - 28 C_0 C_{[m_1\cdots m_6] m_7 n_1 [m_8] n_2}$$

$$- 84 C_{[m_1\cdots m_5; m_6] m_7 n_1 [m_8] n_2} + 210 C_{[m_1 m_2; m_3 m_4] m_5 m_6 m_7 m_8} n_1 n_2,$$

the complicated $T$-duality rules obtained in this subsection are surprisingly simplified as Eq. (3.57).

As a consistency check, let us express the 7-form field strength in type IIA/IIB theory by using the new 6-form $D_6$. Then, we obtain

$$\mathcal{H}_7 = d D_6 - \frac{1}{2} \left( G_6 \wedge \mathcal{C}_1 - G_4 \wedge \mathcal{C}_3 + G_2 \wedge \mathcal{C}_5 \right),$$

$$H_7 = d D_6 - \frac{1}{2} \left( G_7 C_0 - G_5 \wedge C_2 + G_3 \wedge C_4 - G_1 \wedge C_6 \right),$$

and this precisely coincides with the expression given in Ref. [55] up to conventions. This shows that our $D_{6+n,n}$ ($n = 0, 1, 2$) are precisely the same as $D_{6+n,n}$ studied there, and they can be straightforwardly extended also to $n = 3, 4$. As shown in Ref. [56], the field strength, $\mathcal{H}_7$ or $H_7$, can be regarded as a particular component of

$$H_{MNPQ} \equiv \partial^D D_{MNPO} - \frac{1}{4} \sqrt{|\mathcal{G}|} \Gamma_{MNPQ} |F|, \quad |F| \equiv \Gamma^M \partial_M |A|,$$

(3.55)
where \( [A] \equiv \langle A | C^T \equiv (|A\rangle)^T C^T \) with \( C \equiv (\Gamma^0 + \Gamma_0) \cdots (\Gamma^9 + \Gamma_9) \). The indices \( M, N, \ldots \) are raised/lowered by using \( \eta_{MN} \) and the derivative \( \partial_M \) can be understood as \( (\partial_M = (\partial_m, 0) \). Then, we can show that \( H_7 \) or \( T \) in type IIA or IIB theory is reproduced from

\[
H_7 = \frac{1}{7!} \epsilon_{m_1 \cdots m_7 n_1 n_2 n_3} H^{m_1 \cdots m_7} dx^{m_1} \wedge \cdots \wedge dx^{m_7}.
\]

Other components are also easily computed. For example, the component \( H^{a_1 a_2 \cdots n} \) associated with a Killing direction \( n \) satisfying \( n \not\in \{a_1, a_2\} \) gives the field strength of the dual graviton,

\[
t_n H_{8,n} = \frac{1}{7!} \epsilon_{m_1 \cdots m_7 a_1 a_2 n} H^{a_1 a_2 \cdots n} dx^{m_1} \wedge \cdots \wedge dx^{m_7}.
\]

In type IIA/IIB theory, this reproduces

\[
t_n H_{8,n} = dt_n D_{7,n} + \frac{1}{2} \left( t_n F_8 t_n A_1 - t_n F_6 \wedge t_n F_4 + t_n A_5 \right),
\]

\[
t_n H_{8,n} = dt_n D_{7,n} + \frac{1}{2} \left( t_n F_7 \wedge t_n A_2 - t_n F_5 \wedge t_n F_3 + t_n A_6 \right),
\]

where \( F_{p+1} \equiv dA_p \) (note that \( t_n F_1 = t_n A_0 = 0 \). The 11D uplift or the \( S \)-duality-invariant expression is given respectively in Sects. 4.1 or 4.2. We can compute the other components as well, yielding the field strengths for mixed-symmetry potentials \( D_{8,2} \) and \( D_{9,3} \).

Here it will be useful to comment on the notion of the level \( n \). If we look at, for example, the right-hand side of (3.63), terms like \( C \cdots C \cdot B \cdot B \cdot B \) appear, but \( C \cdots B \cdot B \cdot B \) never appears. This can be understood by considering the level, which has been introduced in the study of the \( E_{11} \) conjecture [57,58]. In type II theories, a potential which couples to a brane with the tension \( T \propto g_s^{-n} \) has the level \( n \) [13]. For example, \( B_2 \) has level 0 while the \( R\bar{R} \) potentials have level 1. Since the potentials \( D_{6+n,n} \) have level 2, the level on the right-hand side of Eq. (3.63) must be summed up to 2, and this is the reason why \( C \cdots B \cdot B \cdot B \cdot \) does not appear. The level is always respected in various equations, such as the parametrization of \( N \) given in Sect. 2.4, the \( T \)-duality rules, the field strengths such as Eq. (3.65) and the field redefinitions, and this helps when we find the complicated redefinitions such as Eq. (3.63).

### 3.3. Potentials for exotic branes

Here we find the \( T \)-duality rules for mixed-symmetry potentials that couple to exotic branes with tensions \( T \propto g_s^{-3} \) and \( g_s^{-4} \). Since the potentials have many indices, the \( T \)-duality rules are generally more complicated than before. Thus, we again find only the \( T \)-duality rules, each of which maps a type IIA potential to type IIB potentials, and by using those, we identify the relation between the manifestly \( T \)-duality-covariant potentials.

**\( T \)-duality rule (1):** \( 6^1 (\text{IIA}) \leftrightarrow 5^2 (\text{IIB}) \) From the linear map \( A_{\mu ; a_1 \cdots a_6 ; y ; a} \supseteq -A^{\delta}_{\mu ; a_1 \cdots a_6 ; y ; a} \), we obtain

\[
A_{a_1 \cdots a_7 , y , b} \propto \sum_{\text{perm}} \begin{pmatrix} a \& B \\ A & \mathbf{B} \end{pmatrix} E_{a_1 \cdots a_7 , y , b} - 7 A_{[a_1 \cdots a_6 ; y ; C_{a_7} ; b} - \frac{42 A_{[a_1 \cdots a_6 ; y ; C_{a_7} ; g_{a_7} ; y]} \cdot g_{a_7 ; y]}{y} + 21 \epsilon_{y \delta} D_{[a_1 \cdots a_5 ; y ; A_{a_6 ; y ; a} ; a} \cdot A^{\delta}_{a_7 ; y ; a} + 21 \epsilon_{y \delta} D_{[a_1 \cdots a_5 ; y ; A_{a_6 ; y ; a} \cdot A^{\delta}_{a_7 ; y ; a}.
\]

---

\(^9\) In terms of DFT, by supposing that \( D_{M_1 \cdots M_4} \) is a generalized tensor with weight 1, we can show \( \delta_{Y} H_{MNQ} = \sum a \cdot 2 \sum \partial_{M} \cdot V^{Y} \) under generalized diffeomorphisms. The anomalous term vanishes under the assumption that lower indices of \( H_{MNQ} \), \( H_{MPQ} \), and \( H_{QMP} \) are associated with Killing directions.
This is $S$-dual to the $T$-dual rule (3.45). Under $B_2 = C_2 = 0$, this map was obtained in Ref. [11] [the middle line of Eq. (5.12)], where $N^{(8)}$ corresponds to $E_{8,by}$ (under $B_2 = C_2 = 0$).

$T$-duality rule (II): $A_3^4$ (IIA) $\leftrightarrow$ $S_3^2$ (IIB) From the linear map $A_{\mu;\alpha_1\ldots\alpha_6\beta_1\beta_2y}$, we obtain

$$A_{\alpha_1\ldots\alpha_2\beta_1\beta_2y} = B_{\alpha_1\ldots\alpha_6\beta_1\beta_2} - 7 D_{\alpha_1\ldots\alpha_6} A_{\alpha_7\beta_1\beta_2y} - \frac{7}{2} \varepsilon_{\gamma\delta} D_{\alpha_1\ldots\alpha_5} A_{\alpha_7\beta_1\beta_2y} A_{\alpha_1\alpha_5\alpha_6\beta_1\beta_2y}$$

$$+ \frac{35}{2} D_{\alpha_1\alpha_2\beta_1\beta_2} A_{\alpha_1\alpha_2\beta_1\beta_2y} - 105 \varepsilon_{\gamma\delta} D_{\alpha_1\alpha_2\beta_1\beta_2} A_{\alpha_1\alpha_2\beta_1\beta_2y} A_{\alpha_1\alpha_2\beta_1\beta_2y} A_{\alpha_1\alpha_2\beta_1\beta_2y}$$

$$+ \frac{35}{2} \varepsilon_{\gamma\delta} A_{\alpha_1\alpha_2\beta_1\beta_2y} A_{\alpha_1\alpha_2\beta_1\beta_2y} A_{\alpha_1\alpha_2\beta_1\beta_2y}$$

$$(3.69)$$
This is S-dual to the T-dual rule (3.46).

\textbf{T-duality rule }\Psi: 6_{1}^{1}(IIA) \leftrightarrow 6_{3}^{(1,1)}(IIB)\textbf{ From the linear map }A_{\mu; a_{1} \cdots a_{7}, a} \simeq A_{\mu; a_{1} \cdots a_{7}, ay, y},\textbf{ we obtain}

\begin{align}
\mathcal{A}_{a_{1} \cdots a_{5}, b} & \simeq \mathcal{A}_{a_{1} \cdots a_{5}, hy, y} - 84 \epsilon_{\gamma \delta} \mathcal{A}_{a_{1} \cdots a_{5}, hy, y} \left( A_{\gamma a_{7} \delta}^{\nu} - \frac{2 A_{\gamma a_{7} \delta}^{|y\nu|} g_{\nu y}}{g_{\nu y}} \right) A_{a_{8} |y|}^{\delta} \\
& + 280 \epsilon_{\gamma \delta} \mathcal{A}_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{4} a_{5} a_{6}}^{|y\nu|} - \frac{A_{a_{4} a_{5} a_{6}}^{|y\nu|}}{g_{\nu y}} \right) A_{\gamma a_{7} \delta}^{\nu} A_{a_{8} |y|}^{|y\nu|} \\
& - 560 \epsilon_{\gamma \delta} \mathcal{A}_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{4} a_{5} a_{6}}^{|y\nu|} - \frac{A_{a_{4} a_{5} a_{6}}^{|y\nu|}}{g_{\nu y}} \right) A_{\gamma a_{7} \delta}^{\nu} A_{a_{8} |y|}^{|y\nu|} \\
& + 140 \epsilon_{a_{\beta} \gamma} \epsilon_{\gamma \delta} \mathcal{A}_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{1} a_{5} a_{6}}^{|y\nu|} - \frac{A_{a_{1} a_{5} a_{6}}^{|y\nu|}}{g_{\nu y}} \right) A_{\gamma a_{7} \delta}^{\nu} A_{a_{8} |y|}^{|y\nu|} \\
& + 140 \epsilon_{a_{\beta} \gamma} \epsilon_{\gamma \delta} \mathcal{A}_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{1} a_{5} a_{6}}^{|y\nu|} - \frac{A_{a_{1} a_{5} a_{6}}^{|y\nu|}}{g_{\nu y}} \right) A_{\gamma a_{7} \delta}^{\nu} A_{a_{8} |y|}^{|y\nu|} \\
\end{align}

which is self-dual under S-duality. Under \( B_{2} = C_{2} = 0 \), this map was obtained in Ref. [11] [the first line of Eq. (5.12)], where \( N^{(9)} \) may be related to \( A_{9, ny, y} \).

\textbf{T-duality rule }\Psi': 7_{3}^{(1,0)}(IIA) \leftrightarrow 6_{3}^{(1,1)}(IIB)\textbf{ From the linear map }A_{\mu; a_{1} \cdots a_{7}, a, b} \simeq A_{\mu; a_{1} \cdots a_{7}, ay, y, b},\textbf{ we obtain}

\begin{align}
\mathcal{A}_{a_{1} \cdots a_{5}, b, c} & \simeq \mathcal{A}_{a_{1} \cdots a_{5}, hy, y, c} - \frac{8 \mathcal{A}_{a_{1} \cdots a_{5}, hy, y, g_{a_{8} |y|}}}{g_{yy}} \\
& + 8 \epsilon_{\gamma \delta} A_{a_{1} \cdots a_{5}, |y\nu|} A_{a_{8} |y|}^{\delta} \\
& - 28 \epsilon_{\gamma \delta} A_{a_{1} \cdots a_{5}, |y\nu|} A_{a_{7} |y|}^{\delta} \\
& - 84 \epsilon_{\gamma \delta} A_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{7} |y|}^{\nu} - \frac{2 A_{a_{7} |y|}^{\nu}}{g_{\nu y}} \right) A_{a_{8} |y|}^{\delta} \\
& + 28 \epsilon_{\gamma \delta} A_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{7} |y|}^{\nu} - \frac{2 A_{a_{7} |y|}^{\nu}}{g_{\nu y}} \right) A_{a_{8} |y|}^{\delta} \\
& + 168 \epsilon_{a_{\beta} \gamma} \epsilon_{\gamma \delta} A_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{1} a_{5} a_{6}}^{\nu} - \frac{A_{a_{1} a_{5} a_{6}}^{\nu}}{g_{\nu y}} \right) A_{a_{7} |y|}^{\delta} \\
& + 168 \epsilon_{a_{\beta} \gamma} \epsilon_{\gamma \delta} A_{a_{1} \cdots a_{5}, |y\nu|} \left( A_{a_{1} a_{5} a_{6}}^{\nu} - \frac{A_{a_{1} a_{5} a_{6}}^{\nu}}{g_{\nu y}} \right) A_{a_{8} |y|}^{\delta} \\
\end{align}
The S-dual counterpart of this T-duality is the map $\ominus$ connecting $\mathcal{C}_9$ and $\mathcal{C}_8$.

**T-duality rule $\ominus$: $6^1$ (IIA) $\leftrightarrow 7_3$ (IIB)** From the linear map $\mathcal{A}_{\mu; a_1 \cdots a_8 y, y} = \mathcal{A}_{\mu; a_1 \cdots a_8 y}^{22}$, we obtain

$$\mathcal{A}_{a_1 \cdots a_2 y} = E_{a_1 \cdots a_2 y} - 7 \left( B_{a_1 \cdots a_6} - \frac{6 B_{a_1 \cdots a_5} y_{a_6 y}}{g_{yy}} \right) C_{a_7 y} - 35 C_{a_1 a_2 a_3} \left( C_{a_4 a_5} C_{a_6 a_7} - \frac{4 C_{a_4 a_5 y} C_{a_6 a_7 y}}{g_{yy}} \right).$$

(3.73)

The S-dual to this $T$-duality is the map $\oplus$ connecting $\mathcal{C}_9$ and $\mathcal{C}_{10}$. We can also find the inverse map as

$$E_{a_1 \cdots a_8} \equiv \mathcal{A}_{a_1 \cdots a_8 y, y} - 8 \left( C_{a_1 \cdots a_7 y} B_{a_8 y} + 56 \mathcal{A}_{a_1 \cdots a_6 y, y} B_{a_7 y} \right) \left( C_{a_8 y} - \frac{\eta_{a_1 a_2 y}}{g_{yy}} \right)$$

$$- 70 \left( C_{a_1 a_2 a_3} - \frac{3 \eta_{a_1 a_2 y} a_3 y}{g_{yy}} \right) C_{a_4 a_5 y} C_{a_6 a_7 y} B_{a_8 y}$$

$$+ 1680 B_{a_1 a_2} B_{a_3 a_4} B_{a_5 a_6} B_{a_7 a_8} \left( C_{a_8 y} - \frac{\eta_{a_1 a_2 y}}{g_{yy}} \right).$$

(3.76)

So far, we have considered the potentials which couple to exotic $(7 - p + n)_3(p - n)$-branes. Finally, let us consider the only map which is associated with branes with tension $T \propto g_s^{-4}$.

**T-duality rule $\ominus$: $8^1$ (IIA) $\leftrightarrow 9_4$ (IIB)** From the linear map $A_{\mu; a_1 \cdots a_8 y, y} = \mathcal{A}_{\mu; a_1 \cdots a_8 y}^{22}$, we find

$$\mathcal{A}_{a_1 \cdots a_2 y} = F_{a_1 \cdots a_2 y} - 9 \left( E_{a_1 \cdots a_8} - \frac{8 E_{a_1 \cdots a_7} y_{a_8 y}}{g_{yy}} \right) C_{a_9 y}$$

$$- 315 C_{a_1 a_2 a_3} \left( C_{a_4 a_5} C_{a_6 a_7} \left( C_{a_8 a_9} - \frac{6 C_{a_8 y} y_{a_9 y}}{g_{yy}} \right) \right).$$

(3.77)

The potentials $\mathcal{A}_{10,1,1}$ and $F_{10}$ have level 4 while $E_8$ has level 3, and again we can see that the levels indeed match on both sides.

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10 The correspondent of (3.73) is given in Eq. (3.3) of Ref. [11], but there seems to be a small discrepancy regarding the terms including $B^{2}$ $(C^{2})^{3}$. 

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3.3.1. \textit{T}-dual-manifest redefinitions

Having obtained the \textit{T}-duality rules, let us find the field redefinitions that map our mixed-symmetry potentials to the \textit{T}-duality-covariant potentials. As studied in Ref. [38], potentials $E_{8+n,p,n}$ ($n = 0, 1, 2$ and $p = \text{odd/even}$ in type IIA/IIB theory) that couple to the exotic $(7 - p + n)(n,p-n)$-branes constitute the \textit{T}-duality-covariant potential $E_{MN\tilde{a}}$ ($\tilde{a} = 1, \ldots, 512$). This transforms in the 87040-dimensional tensor-spinor representation of the O(10, 10) group. By using the notation of the O(10, 10) spinor given in Sect. 3.2.1, we denote it as $|E_{MN}\rangle$, which satisfies the following conditions:

$$|E_{MN}\rangle = -|E_{NM}\rangle, \quad \Gamma^N |E_{NM}\rangle = 0, \quad \Gamma^{11} |E_{MN}\rangle = \mp |E_{MN}\rangle \quad (\text{IIA/IIB}). \quad (3.78)$$

As discussed in Ref. [56], if we truncate the components which do not couple to supersymmetric branes, $|E_{MN}\rangle$ can be parametrized as

$$|E_{mn}\rangle = \sum_{p} \frac{1}{N!p!} e^{mq_1 \cdots q_p} E_{q_1 \cdots q_p, r_1 \cdots r_p} \Gamma^{r_1 \cdots r_p} \cdot |0\rangle, \quad (3.79)$$

$$|E_{m \tilde{n}}\rangle = \sum_{p} \frac{1}{N!p!} e^{mq_1 \cdots q_p} E_{q_1 \cdots q_p, r_1 \cdots r_p, \tilde{n}} \Gamma^{r_1 \cdots r_p} \cdot |0\rangle, \quad (3.80)$$

$$|E_{mn}\rangle = \sum_{p} \frac{1}{N!p!} e^{q_1 \cdots q_{10}} E_{q_1 \cdots q_{10}, r_1 \cdots r_p, mn} \Gamma^{r_1 \cdots r_p} \cdot |0\rangle. \quad (3.81)$$

The constraint $\Gamma^N |E_{NM}\rangle = 0$ is automatically satisfied under the restriction rule for the indices. Under the factorized \textit{T}-duality along the $x^\mu$-direction, it transforms as

$$|E'_{M_1 M_2}\rangle = \Lambda_{M_1}^{N_1} \Lambda_{M_2}^{N_2} \left(\Gamma^\nu - \Gamma_\nu\right) \Gamma^{11} |E_{N_1 N_2}\rangle, \quad (3.82)$$

and in terms of the components, we have

$$E_{a_1 \cdots a_{8+n}, b_1 \cdots b_p, c_1 \cdots c_n} \overset{A-B}{\simeq} E_{a_1 \cdots a_{8+n}, y_1 \cdots y_p, c_1 \cdots c_n},$$

$$E_{a_1 \cdots a_{7+n}, b_1 \cdots b_p, c_1 \cdots c_n} \overset{A-B}{\simeq} E_{a_1 \cdots a_{7+n}, y_1 \cdots y_p, c_1 \cdots c_n},$$

$$E_{a_1 \cdots a_{7+n}, b_1 \cdots b_{p-1}, c_1 \cdots c_n} \overset{A-B}{\simeq} E_{a_1 \cdots a_{7+n}, y_1 \cdots y_{p-1}, c_1 \cdots c_n},$$

$$E_{a_1 \cdots a_{8+n}, b_1 \cdots b_{p-1}, c_1 \cdots c_n} \overset{A-B}{\simeq} E_{a_1 \cdots a_{8+n}, y_1 \cdots y_{p-1}, c_1 \cdots c_n}, \quad (3.83)$$

where $n = 0, 1, 2$ and $p = 1, 3, 5, 7$. The \textit{T}-duality web for the family of potentials $E_{8+n,p,n}$, which contains our \textit{T}-dualities (1–4), can be summarized as follows:

$$\begin{array}{cccccccc}
E_{8,7} & \longleftrightarrow & E_{8,6} & \longleftrightarrow & E_{8,5} & \longleftrightarrow & E_{8,4} & \longleftrightarrow & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
E_{9,8,1} & \longleftrightarrow & E_{9,7,1} & \longleftrightarrow & E_{9,6,1} & \longleftrightarrow & E_{9,5,1} & \longleftrightarrow & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
E_{10,9,2} & \longleftrightarrow & E_{10,8,2} & \longleftrightarrow & E_{10,7,2} & \longleftrightarrow & E_{10,6,2} & \longleftrightarrow & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
E_{8,2} & \leftarrow & E_{8,1} & \leftarrow & E_8 \\
\end{array} \quad (3.84)$$
Through trial and error, we have found that the following redefinitions indeed map our mixed-symmetry potentials to the $T$-duality-covariant potentials $E_{8+n,p,n}$:

Type IIA:

$$E_{m_1\ldots m_n} \simeq \omega_{m_1\ldots m_n} - 56 \left( B_{m_1\ldots m_5} [n] + \frac{5}{6} C_{m_1\ldots m_5} [m_5] \right) C_{m_6 m_7 m_8},$$

$$E_{m_1\ldots m_n, n_1 n_2} \simeq \omega_{m_1\ldots m_n, n_1 n_2} + \frac{280}{3} B_{m_1\ldots m_5, n_1 n_2} C_{m_6 m_7 m_8} - 28 B_{m_1\ldots m_5, n_1 n_2} C_{m_6 m_7 m_8} - 28 C_{m_1\ldots m_5, n_1 n_2} C_{m_6 m_7 m_8} + 56 E_{m_1\ldots m_5, n_1 n_2} C_{m_6 m_7 m_8} + 28 C_{m_1\ldots m_5, n_1 n_2} C_{m_6 m_7 m_8},$$

$$E_{m_1\ldots m_n, n_1 n_2, n_3} \simeq \omega_{m_1\ldots m_n, n_1 n_2, n_3} - 280 B_{m_1\ldots m_5, n_1 n_2, n_3} C_{m_6 m_7 m_8} - 252 B_{m_1\ldots m_5, n_1 n_2, n_3} C_{m_6 m_7 m_8} + 21 E_{m_1\ldots m_5, n_1 n_2, n_3} C_{m_6 m_7 m_8} + 420 C_{m_1\ldots m_5, n_1 n_2, n_3} C_{m_6 m_7 m_8} - 1575 B_{m_1\ldots m_5, n_1 n_2, n_3} C_{m_6 m_7 m_8},$$

Type IIB:

$$E_{m_1\ldots m_8} = E_{m_1\ldots m_8} - 28 B_{m_1\ldots m_6, C_{m_7 m_8}} + 140 C_{m_1\ldots m_4, C_{m_5 m_6}, C_{m_7 m_8}} + \frac{35}{3} C_{m_1\ldots m_4, C_{m_5 m_6}, C_{m_7 m_8}},$$

$$E_{m_1\ldots m_8, n_1 n_2} \simeq E_{m_1\ldots m_8, n_1 n_2} + 420 B_{m_1\ldots m_6, n_1 n_2, B_{m_5 m_6, C_{m_7 m_8}}} + 56 B_{m_1\ldots m_6, B_{m_7, p_1, C_{m_8, p_2}}} - 28 B_{m_1\ldots m_6, C_{m_7 m_8, n_1 n_2}} C_{m_8, n_2} + 70 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} + 70 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} - 35 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} C_{m_8, n_2} + 650 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} B_{m_8, p_2} - 630 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} B_{m_8, p_2} - 455 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} B_{m_8, p_2} + 1995 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} B_{m_8, p_2} - 1155 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} B_{m_8, p_2} - 2310 C_{m_1\ldots m_4, C_{m_5 m_8}, C_{m_7 m_8}} B_{m_8, p_2},$$

$$E_{m_1\ldots m_8, n_1 n_2, n_3} \simeq E_{m_1\ldots m_8, n_1 n_2, n_3} - 126 A_{m_1\ldots m_5, n_1 n_2, n_3} C_{m_6 m_7 m_8, m_8} - 3 B_{m_6 m_7 m_8},$$

$$E_{m_1\ldots m_8, n_1 n_2, n_3} \simeq E_{m_1\ldots m_8, n_1 n_2, n_3} + 1260 C_{m_1\ldots m_4, n_1 n_2} B_{m_5 m_6, C_{m_7 m_8}} + 420 C_{m_1\ldots m_4, n_1 n_2} C_{m_5 m_6, m_7, p_1} C_{m_8, m_9} - 420 C_{m_1\ldots m_4, n_1 n_2} C_{m_5 m_6, m_7, p_1} C_{m_8, m_9} + 210 C_{m_1\ldots m_4, C_{m_5 m_6, m_7, p_1}} B_{m_8, p_2} - 315 C_{m_1\ldots m_4, C_{m_5 m_6, m_7, p_1}} C_{m_7 m_8} B_{m_8, p_2} - 105 C_{m_1\ldots m_4, C_{m_5 m_6, m_7, p_1}} C_{m_7 m_8} B_{m_8, p_2} + 315 C_{m_1\ldots m_4, C_{m_5 m_6, m_7, p_1}} C_{m_7 m_8} B_{m_8, p_2} + 630 C_{m_1\ldots m_4, C_{m_5 m_6, m_7, p_1}} C_{m_7 m_8} B_{m_8, p_2}.$$
Similarly, even for the potentials \( A_{10,11} \) and \( F_{10} \), if we consider the redefinitions,

\[
F_{m_1 \cdots m_{10}, b, c} = A_{m_1 \cdots m_{10}, b, c} - 120 A_{m_1 \cdots m_{10}, b, c} F_{m_5 m_9 m_10} C + 1260 C_{m_1 m_2 m_3 | p} \left( \epsilon_{m_1 m_2 m_3 | p} B_{m_5 m_9 m_10} \right) - 2520 C_{m_1 m_2 m_3 | p} \left( \epsilon_{m_1 m_2 m_3 | p} B_{m_5 m_9 m_10} \right) - 630 B_{m_1 m_2} B_{m_5 m_9} \left( \epsilon_{m_1 m_2 m_5 m_9 m_10} \right) C_{m_1 m_2 m_5 m_9 m_10}.
\]

the \( T \)-duality rule (3.77) is simplified as

\[
F_{a_1 \cdots a_9 y, y, y} \, A-B \equiv F_{a_1 \cdots a_9 y}.
\]

As is studied in Refs. [59,60], potentials that couple to the \((9-p)_{4,0}\)-branes \((p = 1, 3, 5, 7, 9, 0, 2, 4, 6, 8)\) in type IIA/IIB theory are packaged into the self-dual O(10,10) tensor \( F^{+}_{M_1 \cdots M_{10}} \). Then, the above potentials, \( F_{10,11} \) and \( F_{10} \), will be identified as the particular components of \( F^{+}_{M_1 \cdots M_{10}} \). The potential \( F^{+}_{M_1 \cdots M_{10}} \) contains the following family of potentials:

\[
\begin{align*}
\text{IIA} & \quad F_{10,9,9} \\
& \quad F_{10,7,7} \quad F_{10,5,5} \quad F_{10,3,3} \quad F_{10,1,1}
\end{align*}
\]

\[
\begin{align*}
\text{IIB} & \quad F_{10,8,8} \quad F_{10,6,6} \quad F_{10,4,4} \quad F_{10,2,2} \quad F_{10}
\end{align*}
\]

What we have explicitly confirmed is only the rightmost arrow (2), but the existence of the O(10,10)-covariant potential \( F^{+}_{M_1 \cdots M_{10}} \) suggests the validity of other maps. Namely, we can define the family of potentials \( F_{10, p, p} \) through the simple \( T \)-duality rules,

\[
F_{a_1 \cdots a_9 y, b_1 \cdots b_{p-1}, y, b_{p-1}} \, A-B \equiv F_{a_1 \cdots a_9 y, b_1 \cdots b_{p-1}} \, A-B \equiv F_{a_1 \cdots a_9 y, b_1 \cdots b_{p}, y, b_{p-1}}.
\]

for \( p = 1, 3, 5, 7, 9 \), without any non-linear correction. Of course, in order to discuss the M-theory uplifts or the \( S \)-duality rules for \( F_{10, p, p} \) \((p \geq 2)\), we need to determine how they enter into the 1-form field, \( A_{\mu}^I \) or \( A_{\mu}^I \).
3.4. S-duality rule

In this paper, the type IIB fields are defined to be S-duality covariant, and under an SL(2) transformation $\Lambda^\alpha_\beta$ the bosonic fields transform as

\[
g'_{mn} = g_{mn}, \quad m'^{\alpha\beta} = \Lambda^\alpha_\gamma \Lambda^\beta_\delta m^{\gamma\delta}, \quad A'_\alpha = \Lambda^\alpha_\beta A_\beta, \quad \Lambda' = \Lambda.
\]

In particular, under the S-duality, $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the component fields are transformed as

\[
g'_{mn} = g_{mn}, \quad C'_0 = -\frac{C_0}{(C_0)^2 + e^{-2\phi}}, \quad e^{-\Phi'} = \frac{e^{-\Phi}}{(C_0)^2 + e^{-2\phi}},
\]

\[
B'_2 = -B_2, \quad C'_2 = B_2, \quad C'_4 = C_4 - B_2 \wedge C_2,
\]

\[
C'_6 = -(B_6 - \frac{1}{3} B_2 \wedge C_2 \wedge C_2), \quad B'_6 = C_6 - \frac{1}{3} C_2 \wedge B_2 \wedge B_2,
\]

\[
C'_8 = E_8 - \frac{1}{3} B_2 \wedge C_2 \wedge C_2 \wedge C_2, \quad E'_8 = C_8 - \frac{1}{3} C_2 \wedge B_2 \wedge B_2 \wedge B_2,
\]

\[
C'_{10} = -(E_{10} - \frac{1}{3} B_2 \wedge C_2 \wedge C_2 \wedge C_2 \wedge C_2), \quad F'_{10} = C_{10} - \frac{1}{3} C_2 \wedge B_2 \wedge B_2 \wedge B_2 \wedge B_2,
\]

\[
A'_{7,1} = A_{7,1}, \quad D'_{8,2} = -E_{8,2}, \quad E'_{8,2} = D_{8,2}, \quad A'_{9,2,1} = A_{9,2,1}.
\]

It is sometimes useful to introduce the dual parametrization of $m_{\alpha\beta}$,

\[
(m_{\alpha\beta}) = e^\Phi \left( \frac{e^{-2\phi} + (C_0)^2}{C_0} \right) \equiv e^{-\tilde\phi} \begin{pmatrix} 1 & -\tilde\gamma \\ -\tilde\gamma & e^{2\tilde\phi} + \tilde\gamma^2 \end{pmatrix},
\]

which is equivalent to

\[
\tilde\gamma \equiv -\frac{C_0}{(C_0)^2 + e^{-2\phi}}, \quad e^{-\Phi'} = \frac{e^{-\Phi}}{(C_0)^2 + e^{-2\phi}}.
\]

Then, the S-duality rule becomes

\[
C'_0 = \gamma, \quad e^{\Phi'} = e^{-\tilde\phi}.
\]

The electric–magnetic duality for $H_9$ is also simplified as

\[
H_9 = e^{-2\Phi} *_E d\gamma.
\]

For the $T$-duality-covariant potentials, the S-duality transformation rules are complicated. For example, we find

\[
D'_6 = C_6 - \frac{1}{2} C_4 \wedge B_2 + \frac{\tilde\gamma}{2} \left( D_6 + \frac{1}{2} C_6 C_0 + \frac{3}{2} C_4 \wedge C_2 - \frac{1}{2} B_2 \wedge C_2^2 \right),
\]

\[
E'_8 = C_8 - C_6 \wedge B_2 + \frac{1}{2} C_4 \wedge B_2^2 + \frac{\tilde\gamma}{3} \left( C_2^2 - 2 C_4 \wedge C_2 \wedge B_2 + C_2 \wedge B_2^2 \right),
\]

\[
F'_{10} = C_{10} - C_8 \wedge B_2 + \frac{1}{2} C_6 \wedge B_2^2 - \frac{1}{8} C_4 \wedge B_2^3 - \frac{1}{30} C_2 \wedge B_2^4
\]
\[
+ \frac{\tilde\gamma}{40} \left( D_6 \wedge B_2 + \frac{1}{2} C_6 \wedge B_2 C_0 - 2 C_4^2 + \frac{9}{2} C_4 \wedge C_2 \wedge B_2 - \frac{5}{2} C_2^2 \wedge B_2^2 \right) \wedge B_2
\]
\[
- \frac{\tilde\gamma^2}{40} \left[ 2 D_6 \wedge (C_4 - B_2 \wedge C_2) + C_6 \wedge (C_4 - B_2 \wedge C_2) C_0
\]
\[
+ (C_2^2 - 2 C_4 \wedge C_2 \wedge B_2 + C_2 \wedge B_2^2) \wedge C_2 \right].
\]

The $S$-duality rules for other $T$-duality-covariant potentials can also be obtained from Eq. (3.97).
4. Field strengths and gauge transformations

In this section, we summarize the field strengths and gauge transformations studied in the literature in terms of our mixed-symmetry potentials, and make a small amount of progress.

4.1. 11D/Type IIA supergravity

In 11D supergravity, the field strengths $\hat{F}_4$ and $\hat{F}_7$ defined in Sect. 2.1 are invariant under

$$\delta \hat{A}_3 = d\hat{\lambda}_2, \quad \delta \hat{A}_6 = d\hat{\lambda}_3 - \frac{1}{2} \hat{A}_3 \wedge d\hat{\lambda}_2.$$  \hspace{2cm} (4.1)

Here, we discuss the field strengths for the mixed-symmetry potentials $\hat{A}_{8,1}$ and $\hat{A}_{10,1,1}$.

Since $\hat{A}_{8,1}$ and $\hat{A}_{10,1,1}$ are 11D uplifts of the R–R 7-form and 9-form, let us consider the 11D uplifts of the known R–R field strengths $G_8$ and $G_{10}$. The 10-form $G_{10}$ is the electric–magnetic dual to the Romans mass [61], and in order to discuss the field strength of $\hat{A}_{10,1,1}$, we need to introduce the mass deformation. Thus, let us begin by summarizing the gauge transformations and field strengths in massive type IIA supergravity. In massive type IIA supergravity, the field strength in the $A$-basis has the 0-form field strength $F_0 \equiv m$,

$$F \equiv F_0 + F_2 + \cdots + F_8 + F_{10} = dA + m, \quad A \equiv A_1 + A_3 + A_5 + A_7 + A_9.$$  \hspace{2cm} (4.2)

Accordingly, in the $C$-basis, the field strength is given by

$$G \equiv G_0 + G_2 + \cdots + G_8 + G_{10} = e^{p_2} \left[ d(e^{-p_2} \wedge \sigma) + m \right] = d\sigma - \mathcal{H}_3 \wedge \mathcal{G} + e^{p_2} m.$$  \hspace{2cm} (4.3)

The gauge transformation is given by

$$\delta \mathcal{B}_2 = d\chi_1, \quad \delta \sigma = e^{p_2} d\lambda - me^{p_2} \chi_1.$$  \hspace{2cm} (4.4)

Now, let us review the uplifts of these relations to 11D. Since the R–R 1-form is contained in the 11D metric, under gauge transformations, the 11D metric is transformed as

$$\delta \hat{g}_{ij} = -m (\chi_i \hat{g}_{iz} + \chi_j \hat{g}_{iz}),$$  \hspace{2cm} (4.5)

where the coordinate $z^2$ is also transformed as $dz^2 = -\lambda_0$. The gauge transformations for the R–R 3-form and the $B$-field are uplifted as

$$\delta \hat{A}_3 = d\hat{\lambda}_2 + m \iota_2 \hat{A}_3 \wedge \iota_2 \hat{\lambda}_2, \quad \hat{\lambda}_2 \equiv \lambda_2 + \chi_1 \wedge dx^2,$$  \hspace{2cm} (4.6)

Under these transformations, the field strength

$$\hat{F}_4 \equiv d\hat{A}_3 + \frac{m}{2} \iota_2 \hat{A}_3 \wedge \iota_2 \hat{A}_3 \equiv G_4 + \mathcal{H}_3 \wedge (dx^2 + \mathcal{G}_1),$$  \hspace{2cm} (4.7)

transforms as

$$\delta \hat{F}_4 = m \iota_2 \hat{\lambda}_2 \wedge \iota_2 \hat{F}_4.$$  \hspace{2cm} (4.8)

The non-invariance is due to $\delta(dx^2 + \mathcal{G}_1) = m \iota_2 \hat{\lambda}_2$, although $G_4$ and $\mathcal{H}_3$ are invariant. From a similar consideration, the gauge transformations for the R–R 5-, 7-, and 9-forms are also uplifted as

$$\delta \hat{A}_5 = d\hat{\lambda}_3 - \frac{1}{2} \hat{A}_3 \wedge d\hat{\lambda}_2 - m (\iota_2 \hat{\lambda}_{7,z} + \iota_2 \hat{A}_6 \wedge \iota_2 \hat{\lambda}_2),$$  \hspace{2cm} (4.9)

\begin{footnote}
[11] In our convention, the gauge transformations of $\iota_4 \hat{A}_{8,1}$ and $\iota_4 \hat{A}_{10,1,1}$ does not include the mass deformation because the R–R potentials are included there such that the mass dependence is canceled out [see (3.5)]. The R–R 6-form potential is also contained in $\hat{A}_6$ such that the mass dependence is canceled out, but the gauge transformation of the potential $\mathcal{B}_6$ gives the mass dependence of $\delta \hat{A}_6$.
\end{footnote}
We note that the 7-form field strength is not invariant similar to the 4-form, and the associated field strengths are defined by

\[
\delta (\iota_z \hat{A}_{8,z}) = \iota_z d\hat{\lambda}_{8,z} + \iota_z \hat{A}_3 \wedge \iota_z d\hat{\lambda}_{5,z} - \frac{2}{31} \iota_z \hat{A}_3 \wedge \iota_z (\hat{A}_3 \wedge d\hat{\lambda}_{2}),
\]

\[
\delta (\iota_z \hat{A}_{10,z,z}) = \iota_z d\hat{\lambda}_{9,z,z} + \iota_z \hat{A}_3 \wedge \iota_z d\hat{\lambda}_{5,z,z} - \frac{1}{2} \iota_z \hat{A}_3 \wedge \iota_z (\hat{A}_3 \wedge d\hat{\lambda}_{2}),
\]

and the projections of the 9-form and the 11-form are invariant, as is clear from

\[
G^8 = \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3
\]

\[
G^{10} = \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3 \wedge \iota_z \hat{A}_3
\]

while the projections of the 9-form and the 11-form are invariant, as is clear from

\[
\iota_z \hat{F}_{9,z} = G_8, \quad \iota_z \hat{F}_{11,z,z} = G_{10}.
\]

We note that the 7-form field strength is not invariant similar to the 4-form,

\[
\delta \hat{F}_7 = m \iota_z \hat{\lambda}_2 \wedge \iota_z \hat{F}_7,
\]

while the projections of the 9-form and the 11-form are invariant, as is clear from

\[
\iota_z \hat{F}_{9,z} = G_8, \quad \iota_z \hat{F}_{11,z,z} = G_{10}.
\]

The above gauge transformations have been discussed in Refs. [17,24,27,62] and they generate the gauge symmetry of the massive 11D supergravity [24,27,62], which reproduces the massive type IIA supergravity after the dimensional reduction.

By using the above setup, we can easily consider the 11D uplift of the electric–magnetic duality $G_8 = \ast G_2$. For this purpose, we also introduce the 2-form field strength associated with a Killing vector $k \equiv \partial_z$ as in Ref. [27] (see also Ref. [22]);

\[
\hat{F}_2 \equiv dk_1 + m |k|^2 \iota_k \hat{A}_3,
\]

where $k_1 \equiv k^i \hat{g}_{ij} dx^j$. This transforms as

\[
\delta \hat{F}_2 = m \iota_k \hat{\lambda}_2 \wedge \iota_k dk_1 = m \iota_k \hat{F}_2.
\]

In terms of the type IIA fields, we have $k_1 = e^{\frac{\varphi}{2}} (dx^2 + C_1)$, and this gives

\[
\hat{F}_2 \equiv e^{\frac{\varphi}{2}} G_2 + \frac{1}{2} e^{\frac{\varphi}{2}} d\varphi \wedge (dx^2 + C_1).
\]

Then, the electric–magnetic duality $G_8 = \ast G_2$ becomes

\[
\iota_z \hat{F}_{9,z} = \iota_z \hat{F}_{2},
\]

which shows that the dual graviton is the electric–magnetic dual to the Killing vector. In this paper, only the restricted components that couple to supersymmetric branes are considered, and the restriction corresponds to the projection $\iota_z$ in front of the field strength.\(^{12}\) We can similarly consider the 11D uplift of the electric–magnetic duality $m = G_0 = \ast G_{10}$ [27]

\[
m |k|^4 = \ast \hat{F}_{11,z,z},
\]

\(^{12}\) The full definition of the field strength $\hat{F}_{9,z}$ without the projection was proposed in Ref. [22]. It can be found by regarding the dilaton equations of motion as the Bianchi identity, and by uplifting this to 11D.
where

\[ \hat{F}_{\hat{1},x,z} \equiv \iota_z \hat{F}_{\hat{1},x,z} \wedge (dx^z + \mathcal{E}_1), \]  

(4.22)

is the gauge-invariant field strength.

Now, we consider the relation to the recent studies on mixed-symmetry potentials in DFT. If we decompose the 7-form field strength as

\[ \hat{F}_7 = \mathcal{H}_7 + \mathcal{G}_6 \wedge (dx^7 + \mathcal{E}_1), \]  

(4.23)

the 7-form field strength \( \mathcal{H}_7 \) becomes

\[ \mathcal{H}_7 = dD_6 - \frac{1}{2} (\mathcal{G}_6 \wedge \mathcal{E}_1 - \mathcal{G}_4 \wedge \mathcal{E}_3 + \mathcal{G}_2 \wedge \mathcal{E}_5 - \mathcal{G}_0 \mathcal{E}_7) + \frac{m}{2} A_7, \]  

(4.24)

which coincides with the expression given in Ref. [56], and is invariant under

\[ \delta D_6 = d\chi_5 + \frac{1}{2} (A_5 \wedge d\lambda_0 - A_3 \wedge d\lambda_2 + A_1 \wedge d\lambda_4) - m (\lambda_6 + \frac{1}{2} A_5 \wedge \chi_1). \]  

(4.25)

Here, we have parametrized

\[ \hat{\lambda}_2 = \chi_5 + \lambda_4 \wedge dx^z, \quad \hat{\lambda}_{2,4} = \lambda_7 + \lambda_6 \wedge dx^z. \]  

(4.26)

Let us also clarify the relation between the field strength of the dual graviton and the field strength \( \iota_n \mathcal{H}_{8,n} \) defined in Eq. (3.68). To this end, we assume the existence of a Killing direction denoted by \( n \) (i.e. \( \xi_n \equiv \iota_n d + d\iota_n = 0 \)) other than the M-theory circle. For simplicity, we turn off the mass parameter. Then, as before, we can easily show that the field strength

\[ \iota_n \hat{F}_{\hat{g},n} \equiv \iota_n d\hat{A}_{\hat{g},n} + \iota_n \hat{A}_{\hat{g},n} + \iota_n \hat{F}_{\hat{g},n} + \frac{1}{3!} \iota_n \hat{A}_{\hat{g},n} + \iota_n \left( \hat{A}_{\hat{g},n} + \hat{A}_{\hat{g},n} \right), \]  

(4.27)

is invariant under

\[ \delta \hat{A}_{\hat{g},n} = d\hat{\lambda}_{\hat{g},n}, \quad \delta \hat{A}_{\hat{g},n} = d\hat{\lambda}_{\hat{g},n} - \frac{1}{2} \hat{A}_{\hat{g},n} \wedge d\hat{\lambda}_{\hat{g},n}, \]  

\[ \delta \left( \iota_n \hat{A}_{\hat{g},n} \right) = \iota_n d\hat{\lambda}_{\hat{g},n} + \iota_n \hat{A}_{\hat{g},n} + \iota_n d\hat{\lambda}_{\hat{g},n} - \frac{2}{3!} \iota_n \hat{A}_{\hat{g},n} + \iota_n \left( \hat{A}_{\hat{g},n} + d\hat{\lambda}_{\hat{g},n} \right). \]  

(4.28)

Now, we consider the reduction to type IIA theory. We define the field strength of the dual graviton in type IIA theory as

\[ \iota_n \mathcal{G}_{8,n} \equiv \iota_z \iota_n \left( \hat{F}_{\hat{g},n} - \hat{F}_{\hat{g},n} \right) = \iota_z \iota_n \hat{F}_{\hat{g},n} + \iota_n \hat{G}_8 \iota_n \mathcal{E}_1, \]  

(4.29)

which is also gauge invariant because \( \iota_n \mathcal{E}_1 \) is gauge invariant due to the Killing equation, \( \xi_n \lambda_0 = \iota_n d\lambda_0 = 0 \). Then, a straightforward but slightly long computation gives

\[
\begin{align*}
\iota_n \mathcal{G}_{8,n} &= \iota_n \mathcal{H}_{7,n} + \iota_n d\mathcal{E}_3 - \iota_n \mathcal{G}_6 \wedge \mathcal{E}_3 - \frac{1}{2} \iota_n \mathcal{B}_2 \wedge \mathcal{E}_3 - \frac{2}{3} \mathcal{B}_2 \wedge \iota_n d\mathcal{E}_3 - \frac{1}{3} \iota_n d\mathcal{G}_2 \wedge \mathcal{E}_3 - \frac{5}{3} \mathcal{B}_2 \wedge \iota_n d\mathcal{E}_3 - \frac{1}{6} \iota_n \mathcal{H}_3 \wedge \mathcal{E}_3 - \frac{1}{6} \mathcal{H}_3 \wedge \iota_n d\mathcal{E}_3 + \iota_n \mathcal{G}_8 \iota_n \mathcal{E}_1 \\
&= \iota_n \mathcal{H}_{7,n} - \iota_n \mathcal{H}_7 \wedge \iota_n \mathcal{B}_2,
\end{align*}
\]  

(4.30)

where we have used Eq. (A.9). This shows that the invariant field strength \( \iota_k \mathcal{G}_{8,n} \) is the component of the untwisted tensor,

\[
\hat{H}_{M_1 M_2 M_3} \equiv \left( e^\mathcal{E}_M \right)_{M_1} \left( e^\mathcal{E}_M \right)_{M_2} \left( e^\mathcal{E}_M \right)_{M_3} \hat{H}_{N_1 N_2 N_3}, \quad \left( e^\mathcal{E}_M \right)^N_M \equiv \begin{pmatrix} \delta_m^n & \mathcal{B}_{mn} \\ 0 & \delta^m_n \end{pmatrix}.
\]  

(4.31)
Indeed, we can easily check
\begin{equation}
\iota_n \mathcal{G}_{8,n} = \frac{1}{7!} \epsilon_{m_1 \cdots m_7 a_1 a_2} \bar{F}^{a_1 a_2} \, dx^{m_1} \wedge \cdots \wedge dx^{m_7} = \iota_n \mathcal{H}^{8,n} - \iota_n \mathcal{H}^{2} \wedge \iota_n \mathcal{B}_2.
\end{equation}

Namely, the field strength \( H_{MNP} \) is similar to the R–R field strength \( F \); \( F \) is invariant under gauge transformations of the R–R potentials, but not under B-field gauge transformations, and it becomes invariant after untwisting the field strength as \( G = e^{\mathcal{B}_2} F \).

For completeness, let us also show that the gauge transformation (4.10) reproduces \( \left[ 17 \right] \)
\begin{align}
\delta (\iota_n \mathcal{A}_n^7) &= \iota_n d\lambda_{6,n} + \iota_n \mathcal{C}_3 \wedge \iota_n d\lambda_4 - \iota_n \mathcal{C}_2 \wedge \iota_n d\chi_2 - \frac{1}{3} \iota_n \mathcal{C}_3 \wedge (\iota_n \mathcal{C}_2 \wedge d\chi_1 + 2 \iota_n \mathcal{C}_2 \wedge d\lambda_2) \\
&\quad + \frac{1}{3} (\iota_n \mathcal{C}_3 \wedge \mathcal{B}_2 + \mathcal{C}_3 \wedge \iota_n \mathcal{B}_2) \wedge \iota_n d\lambda_2 + \frac{1}{3} \mathcal{C}_3 \wedge d\iota_n \mathcal{C}_2 \wedge \iota_n d\chi_1,
\end{align}
where we have parametrized
\begin{equation}
\hat{\lambda}^\alpha_{7,n} = \lambda_{7,n} + \lambda_{6,n} \wedge dx^\gamma.
\end{equation}

In terms of the \( T \)-duality-covariant tensor, we have
\begin{equation}
\delta (\iota_n D_{7,n}) = \iota_n d\lambda_{6,n} - \frac{1}{2} (\iota_n A_5 \wedge \iota_n d\lambda_2 - \iota_n A_3 \wedge \iota_n d\lambda_4 + \iota_n A_1 \wedge \iota_n d\lambda_6) - \iota_n D_6 \wedge \iota_n d\chi_1.
\end{equation}

The field strength \( \iota_n \mathcal{F}_7^{\hat{\alpha}} \) also contains the field strength of the potential \( \mathcal{A}_{9,1} \). Since we have established the 11D–10D map, it is a straightforward task to compute the field strength. The relation between \( \mathcal{A}_{9,1} \) and the potential \( E_{8,1} \) is also given, and it will not be difficult to rewrite the field strength in a manifestly \( T \)-duality-covariant form. Similarly, we can also consider the reduction of the field strength \( \iota_n \mathcal{F}_{11,n,n} \), which gives the field strengths of \( \mathcal{A}_{9,1} \) and \( \mathcal{A}_{10,1,1} \). The former can be expressed in the \( T \)-duality-covariant form by rewriting \( \mathcal{A}_{9,1} \) into \( E_{9,1,1} \). Since this is a 9-form, we need to introduce another deformation parameter associated with the non-geometric \( R \)-flux \( R^{1,1} \) (with non-vanishing component \( R^{m,n} = m \)), which is the magnetic flux of the exotic \( \gamma^{(1,0)} \)-brane (see Ref. \( [63] \)).

On the other hand, the field strength of \( \mathcal{A}_{10,1,1} \) gives the field strength of the \( T \)-duality-covariant potential \( F_{10,1,1} \), although the field strength automatically vanishes in 10D.

For the mixed-symmetry potential \( \mathcal{A}_{9,3} \), it is not straightforward to define the field strength because it is not related to the standard fields. However, since this is the 11D uplift of the type IIA potential \( D_{8,2} \), it may be useful to define the type IIA field strength by using the component \( \mathcal{H}^{m_{1,2}} \) and uplift this to 11D.

4.2. Type IIB supergravity

In type IIB supergravity, the gauge transformations are given as follows \( [30–33] \):
\begin{align}
\delta A_2^a &= d\Lambda_1^a, \\
\delta A_4 &= d\Lambda_3 - \frac{1}{21} \epsilon_{\gamma \delta} A_2^\gamma \wedge d\Lambda_1^\delta, \\
\delta A_5^a &= d\Lambda_5^a + A_2^a \wedge d\Lambda_3 - \frac{2}{31} \epsilon_{\gamma \delta} A_2^\gamma \wedge A_2^\delta \wedge d\Lambda_1^\gamma, \\
\delta A_8^{ab} &= d\Lambda_7^{ab} + A_2^{(a} \wedge d\Lambda_5^{b)} + \frac{1}{21} A_2^a \wedge A_2^b \wedge d\Lambda_3 - \frac{3}{31} \epsilon_{\gamma \delta} A_2^a \wedge A_2^b \wedge A_2^\gamma \wedge d\Lambda_1^\delta, \\
\delta A_{10}^{ab\gamma} &= d\Lambda_9^{ab\gamma} + A_2^{(a} \wedge d\Lambda_5^{b)\gamma} + \frac{1}{21} A_2^{(a} \wedge A_2^b \wedge d\Lambda_5^{\gamma)} \\
&\quad + \frac{1}{31} A_2^a \wedge A_2^b \wedge A_2^\gamma \wedge d\Lambda_3 - \frac{4}{31} \epsilon_{\delta \epsilon} A_2^a \wedge A_2^b \wedge A_2^\gamma \wedge A_2^\delta \wedge d\Lambda_1^\epsilon.
\end{align}
The gauge transformation (4.49) can also be expressed as
\[
\delta B_2 = d\tilde{\chi}_1, \quad \delta C = e^{B_2} d\lambda = d\hat{\lambda} - H_3 \wedge \hat{\lambda},
\]
(4.41)
\[
\delta B_6 = d\chi_5 + C_2 \wedge (d\lambda_3 + B_2 \wedge d\lambda_1),
\]
(4.42)
\[
\delta E_8 = d\xi_7 + C_2 \wedge d\chi_5 + \frac{1}{21} C_2 \wedge C_2 \wedge (d\lambda_3 + B_2 \wedge d\lambda_1),
\]
(4.43)
\[
\delta F_{10} = d\eta_9 + C_2 \wedge d\chi_7 + \frac{1}{21} C_2 \wedge C_2 \wedge d\chi_5 + \frac{1}{21} C_2 \wedge C_2 \wedge (d\lambda_3 + B_2 \wedge d\lambda_1),
\]
(4.44)
where the gauge parameters are parametrized as
\[
(A^{0\alpha}) = \left( \begin{array}{c} \chi^1 \\ -\lambda_1 \end{array} \right), \quad A_3 = \lambda_3, \quad (A^{2\alpha}) = \left( \begin{array}{c} \lambda_5 \\ -\chi_5 \end{array} \right),
\]
\[
\left( \begin{array}{c} A_2^{11} \\ A_2^{22} \end{array} \right) = \left( \begin{array}{c} \lambda_7 \\ \xi_7 \end{array} \right), \quad \left( \begin{array}{c} A_2^{1\parallel} \\ A_2^{2\parallel} \end{array} \right) = \left( \begin{array}{c} \lambda_9 \\ -\eta_9 \end{array} \right),
\]
(4.45)
and we have defined
\[
\lambda \equiv \lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 + \lambda_9, \quad \hat{\lambda} \equiv \hat{\lambda}_1 + \hat{\lambda}_3 + \hat{\lambda}_5 + \hat{\lambda}_7 + \hat{\lambda}_9 \equiv e^{B_2} \wedge \lambda.
\]
(4.46)
Let us also consider the field strength for the dual graviton $A_{7,1}$. Similar to the case of type IIA supergravity, by introducing the untwisted tensor
\[
\hat{H}_{M_1 M_2 M_3} \equiv \langle e^B \rangle_{M_1 N_1} \langle e^B \rangle_{M_2 N_2} \langle e^B \rangle_{M_3 N_3} \hat{H}_{N_1 N_2 N_3}, \quad \langle e^B \rangle_M^N \equiv \left( \begin{array}{c} \delta^m_n \\ 0 \end{array} \right)
\]
(4.47)
we define the field strength as
\[
\iota_n G_{8,n} \equiv \frac{1}{\sqrt{2}} \epsilon_{m_1 \cdots m_7 n} \hat{H}^{a_1 a_2 n} dx^{m_1} \wedge \cdots \wedge dx^{m_7} = \iota_n H_{8,n} - \iota_n H_7 \wedge \iota_n B_2.
\]
(4.48)
By assuming $\ell_n = 0$ for an arbitrary field, it is invariant under the gauge transformation
\[
\delta (\iota_n D_{7,n}) = \iota_n d\lambda_{6,n} + \frac{1}{2} \left( \iota_n A_6 \wedge \iota_n d\lambda_1 - \iota_n A_4 \wedge \iota_n d\lambda_3 + \iota_n A_2 \wedge \iota_n d\lambda_5 \right)
\]
\[
- \iota_n D_6 \wedge \iota_n d\chi_1,
\]
(4.49)
which is the $T$-dual counterpart of Eq. (4.35). Since the dual graviton $A_{7,n}$ is $S$-duality-invariant, it is natural to expect that this field strength is invariant under $S$-duality. Indeed, we can express the field strength in a manifestly $S$-duality-invariant form,
\[
\iota_n G_{8,n} = d\iota_n A_{7,n} - \epsilon_{\gamma \delta} \iota_n dA_\gamma^\delta \wedge \iota_n A_\delta - \frac{1}{2} \iota_n A_4 \wedge \iota_n dA_4
\]
\[
+ \frac{1}{2} \epsilon_{\gamma \delta} \iota_n (dA_4 \wedge A_2^\delta) \wedge \iota_n A_2^\delta + \epsilon_{\gamma \delta} \iota_n (A_4 \wedge dA_2^\delta) \wedge \iota_n A_2^\delta
\]
\[
- \frac{1}{16} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} A_2^\gamma \wedge dA_2^\beta \wedge \iota_n A_2^\delta + \frac{1}{32} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} \iota_n A_2^\gamma \wedge \iota_n dA_2^\beta \wedge A_2^\delta \wedge \iota_n A_2^\delta.
\]
(4.50)
The gauge transformation (4.49) can also be expressed as
\[
\delta (\iota_n A_{7,n}) = \iota_n d\lambda_{6,n} - \epsilon_{\gamma \delta} \iota_n A_\delta^\gamma \wedge \iota_n dA_\gamma - \frac{1}{2} \iota_n A_4 \wedge \iota_n dA_3
\]
\[
+ \epsilon_{\gamma \delta} \iota_n (A_4 \wedge A_2^\gamma) \wedge \iota_n dA_2^\delta - \frac{1}{4} \epsilon_{\gamma \delta} \iota_n A_4 \wedge \iota_n (A_2^\gamma \wedge dA_2^\delta)
\]
\[
- \frac{1}{16} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} A_2^\gamma \wedge dA_2^\beta \wedge \iota_n A_2^\delta + \frac{1}{32} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} A_2^\gamma \wedge \iota_n A_2^\beta \wedge A_2^\delta \wedge \iota_n dA_2^\delta.
\]
(4.51)
Here, we do not study additional potentials, but one can obtain their field strengths as follows. As mentioned in the type IIA case, the field strength of $D_{8,2}$ will be obtained by computing $\hat{H}^{m_{1}m_{2}}_{n_{1}n_{2}}$. Then, rewriting the field strength in terms of the $S$-duality covariant potentials, we can obtain the field strength of $A_{8,2}^{\alpha}$. In order to define the field strength of the potentials $E_{8+n,p,n}^{\pm}$, it is useful to find the $T$-duality-covariant expression for $|E_{MN}|$ that reproduces the field strength $H_{9}$ as the particular component. This covariant field strength contains various field strengths, and by using the relation between $E_{8+n,p,n}^{\pm}$ and our mixed-symmetry potentials we can obtain the $S$-duality-covariant expressions for field strengths.

5. Conclusions

In this paper, we have provided explicit definitions of mixed-symmetry potentials by finding their relation to the standard supergravity fields under $T$- and $S$-duality transformations. The obtained $T$-duality rules are generally very complicated, but by performing certain field redefinitions, they are considerably simplified. The redefined fields $A_{p}$, $D_{6+n,n}$, $E_{8+n,p,n}$, and $F_{10,p,p}$ are identified with certain components of the $O(10,10)$-covariant tensors $A_{\dot{a}}$, $D_{M_{1}...M_{4}}$, $E_{MN\dot{a}}$, and $F_{M_{1}...M_{10}}^{+}$. These $O(10,10)$-covariant tensors have been studied in the literature, but their relation to the standard supergravity fields has not been discussed enough. For example, the potential $E_{8}$ has been expected to be the $S$-dual of the R–R 8-form $C_{8}$, but the $S$-duality rule (3.103), including the non-linear terms, is newly determined in this paper. The $S$-duality rule for $F_{10}$ and more mixed-symmetry potentials are also newly determined. Additionally, we have also studied the field strengths of mixed-symmetry potentials. Most of the results are known in the literature (where the mixed-symmetry potentials are treated as $p$-forms), but here we have clarified the relation to the field strength $H_{MNP}$ studied in DFT. We have also provided the $S$-duality-invariant expression for the field strength of the dual graviton.

The linear map was originally studied for the generalized metric [20,23,40]. By comparing two parametrizations, duality rules for potentials that appear in the $E_{n(n)}$ generalized metric ($n \leq 8$) are determined. The linear map for the 1-form $A_{I}^{\mu}$ studied here is more efficient for finding the duality rules, but the parametrization is involved and the obtained $T$-duality rules are rather long. In the linear map for the generalized metric, the parametrization is more systematic, and by considering the linear map for the $E_{11}$ generalized metric we may find a better definition of mixed-symmetry potentials which simplify the duality rules.

As we have demonstrated, the linear map works well in finding the duality rules for mixed-symmetry potentials, and it is straightforward to consider more mixed-symmetry potentials. In addition, having clarified the definitions of mixed-symmetry potentials, it is important to consider the application to the worldvolume theories of exotic branes. It is also interesting to study the supersymmetry transformations for mixed-symmetry potentials (where Killing vectors should be involved) by extending the series of works [30–33].

Before closing this paper, let us comment on a relevant open issue, called the exotic duality [35,55,64–66], which is the electric–magnetic duality for exotic branes with co-dimensions equal to (or higher than) two. As we have already mentioned, the mixed-symmetry potentials couple to various exotic branes electrically. On the other hand, the exotic branes (with co-dimensions of two) magnetically couple to certain dual fields [66–70], such as the $\beta$-fields\(^{13}\) or the $\gamma$-fields [66,80–82],

\(^{13}\)This was originally introduced in for example Refs. [71–75] and utilized more recently in the $\beta$-supergravity [76–80].
which are associated with the non-geometric $Q$-flux [83] or the $P$-fluxes [81, 84, 85]. Thus, the exotic duality is the electric–magnetic duality between the mixed-symmetry potentials and the dual potentials. An example is given in Eq. (3.101),

$$
\frac{1}{2\eta} \epsilon^{mn_1\ldots n_q} H_{m_1\ldots n_q} = - e^{-2\phi} \sqrt{|g|} g^{mn} \partial_n \tilde{\gamma},
$$

(5.1)

where $\tilde{\gamma}$ roughly corresponds to the $\gamma$-field (as we explain below). Recently, the $T$-duality-covariant expression of the exotic duality was investigated in Refs. [106–108] but it has not yet fully succeeded. For the $T$-duality-covariant exotic duality, it will be important to establish the $T$-duality-covariant description of the dual potentials, and we make a small attempt below.

First of all, let us explain the definition of the dual fields, such as the $\beta$- and $\gamma$-fields. In the $U$-duality formulations, the supergravity fields are embedded into the generalized metric $\mathcal{M}_{IJ}$, which is defined as (see Ref. [109] and references therein)

$$
\mathcal{M}_{IJ} \equiv (\mathcal{E}^T)_{I}^{\ K} \delta_{KL} \mathcal{E}^{L, J},
$$

(5.2)

where the generalized vielbein $\mathcal{E}^I_J$ is the matrix representation of an $E_{n(n)}$ element in the vector representation. The identity matrix $\delta_{IJ}$ is invariant under the maximal compact subgroup, and the generalized vielbein $\mathcal{E}^I_J$ is generally parametrized by the Borel subalgebra. For example, in the type IIB theory, the Borel subalgebra is generated by

$$
\left\{ K^m_n \ (m \leq n), \ 2 \ R_{12}, \ R_{12}^{m_1 m_2}, \ R_{12}^{m_1 m_2 m_3}, \ R_{12}^{m_1 m_2 m_3 m_4}, \ R_{12}^{m_1 m_2 m_3 m_4 m_5}, \ldots \right\},
$$

(5.3)

and the generalized vielbein can be parametrized as [36]

$$
\mathcal{E} = e^{\sum h_n K^m_n} \epsilon^{2} \Phi R_{12} e^{-\gamma} R_{12} e^{\frac{1}{2} A^a_{m_1 m_2} R_{12}^{m_1 m_2}} e^{\frac{1}{4} A^{m_1 m_2} R_{12}^{m_1 m_2}} e^{\frac{1}{6} A^{m_1 m_2 m_3} R_{12}^{m_1 m_2 m_3}} e^{\frac{1}{7} A^{m_1 m_2 m_3 m_4} R_{12}^{m_1 m_2 m_3 m_4}} \ldots,
$$

where the standard vielbein corresponds to $(\epsilon^a_m) = e^{-h^T}$. On the other hand, we can also consider the negative Borel subalgebra, spanned by [70]

$$
\left\{ K^m_n \ (m \geq n), \ 2 \ R_{12}, \ -R_{11}, \ R_{11}^{a_1 a_2}, \ R_{11}^{a_1 a_2 a_3}, \ R_{11}^{a_1 a_2 a_3 a_4}, \ R_{11}^{a_1 a_2 a_3 a_4 a_5}, \ldots \right\},
$$

(5.4)

and introduce the dual parametrization as [15]

$$
\tilde{\mathcal{E}} = e^{\sum h_n K^m_n} \epsilon^{2} \Phi R_{12} e^{-\gamma} R_{12} e^{-\frac{1}{2} A^a_{m_1 m_2} R_{12}^{m_1 m_2}} e^{-\frac{1}{4} A^{m_1 m_2} R_{12}^{m_1 m_2}} e^{-\frac{1}{6} A^{m_1 m_2 m_3} R_{12}^{m_1 m_2 m_3}} e^{-\frac{1}{7} A^{m_1 m_2 m_3 m_4} R_{12}^{m_1 m_2 m_3 m_4}} \ldots.
$$

The dual vielbein is similarly defined by $(\tilde{\epsilon}^a_m) = e^{-h^T}$ and the dual metric is $\tilde{g}_{mn} = (\tilde{\mathcal{E}}^T \tilde{\mathcal{E}})_{mn}$. Then, by comparing the two parametrizations of the generalized metric,

$$
(\mathcal{E}^T \mathcal{E})_{IJ} = \mathcal{M}_{IJ} = (\tilde{\mathcal{E}}^T \tilde{\mathcal{E}})_{IJ},
$$

(5.5)

we can obtain the dual fields as a local redefinitions of the standard fields [70]. For example, if we consider only the NS–NS fields, the relation (5.5) is simplified as

$$
\begin{pmatrix}
\tilde{g}_{mn} - B_{mp} g^{pq} B_{qn} & -B_{mp} \tilde{g}^{pn} \\
g^{mp} B_{pn} & \tilde{g}^{pq}
\end{pmatrix}
= \begin{pmatrix}
\tilde{g}_{mn} - \beta_{mp} \tilde{g}^{pn} & 0 \\
-\beta^{mp} \tilde{g}_{pq} & \tilde{g}^{pq} - \beta^{mp} \tilde{g}^{pn}
\end{pmatrix},
$$

14 See Refs. [86–105] for an incomplete list of references utilizing non-geometric fluxes.

15 Our dual fields have opposite sign compared to those introduced in Ref. [23].
where \( g_{mn} \equiv e^{\Phi/2} g_{mn} \) is the standard string-frame metric and \( \tilde{g}_{mn} \equiv e^{\Phi/2} \tilde{g}_{mn} \) is the dual-string-frame metric. These are precisely the relations studied in the \( \beta \)-supergravity \([76–80]\). On the other hand, if we only keep the metric \( g_{mn} \) and \( m_{\alpha \beta} \), the relation (5.5) reduces to

\[
\gamma = \gamma,
\]

and this shows that the \( \gamma \)-field is similar to the \( \tilde{\gamma} \) appearing in Eq. (3.101). In general, without any truncations, these relations receive non-linear corrections.

Secondly, let us explain the \( T \)-duality rules for the dual fields. By following the discussion of Ref. [23], the \( T \)-duality rules for the dual fields are determined in the same manner as the standard potentials. To this end, we parametrize the dual fields in the same manner as (3.6)–(3.9), for example,

\[
(\tilde{A}_{\mu}^{m_{1}m_{2}}) \equiv \left( \begin{array}{c} \beta^{m_{1}m_{2}} \\ \gamma^{m_{1}m_{2}} \end{array} \right), \quad \tilde{A}^{m_{1}\ldots m_{4}} \equiv \gamma^{m_{1}m_{4}} - 3 \gamma^{[m_{1}m_{2}m_{3}m_{4}]}, \ldots .
\]

Then, we find that the gamma fields \( \gamma^{m_{1}\ldots m_{p}} \) follow the same \( T \)-duality rules as those of the R–R field \( C_{m_{1}\ldots m_{p}} \), although the positions of the indices are opposite:16

\[
\gamma^{a_{1}\ldots a_{n-1}1} = \gamma^{a_{1}\ldots a_{n-1}} - \frac{(n-1) \gamma^{[a_{1}\ldots a_{n-2}1]} \tilde{g}^{a_{n-1}1}}{g^{a_{n}1}}, \\
\gamma^{a_{1}\ldots a_{n}} = \gamma^{a_{1}\ldots a_{n}} - n \gamma^{a_{1}\ldots a_{n-1}} \tilde{g}^{a_{n}1} - n \frac{(n-1) \gamma^{[a_{1}\ldots a_{n-2}1]} \tilde{g}^{a_{n-1}1}}{g^{a_{n}1}}.
\]

The dual fields in the NS–NS sector also transform as

\[
\tilde{g}^{ab} = \tilde{g}^{ab} - \tilde{g}^{av} \tilde{g}^{bv} - \beta^{ab} \tilde{g}^{bv}, \\
\tilde{g}^{ay} = - \beta^{ay}, \\
\tilde{g}^{vy} = \frac{1}{g^{v1}},
\]

\[
\beta^{ab} = \beta^{ab} - \frac{\tilde{g}^{av} \tilde{g}^{bv} - \tilde{g}^{bv} \tilde{g}^{av}}{g^{v1}}, \\
\beta^{ay} = - \frac{\tilde{g}^{av}}{g^{v1}}, \\
e^{2\Phi} = e^{2\Phi}.
\]

Then, we find that

\[
\tilde{\mathcal{H}}_{MN} \equiv \left( \begin{array}{cc} \tilde{g}_{mn} & \tilde{g}_{mp} \beta^{pn} \\ -\beta^{mp} & \tilde{g}_{mn} - \beta^{mp} \tilde{g}_{pq} \beta^{qn} \end{array} \right), \quad e^{-2\Phi} \equiv e^{-2\Phi} \sqrt{|g|},
\]

transform covariantly under \( T \)-duality, which reduces to Eq. (5.6) when only the NS–NS fields are present. Similarly, we can show that the \( \gamma \)-field also transforms covariantly under \( T \)-duality. For this purpose, we define the dual field \( \alpha \) associated with \( A = e^{-\beta v} \wedge \Phi \) (or \( A = e^{-B_{2}} \wedge C \)) as

\[
\alpha \equiv e^{-\beta v} \wedge \gamma \quad (\beta \equiv \frac{1}{n!} \beta^{mn} \partial_{m} \wedge \partial_{n}),
\]

where \( \alpha \equiv \sum_{p} \frac{1}{p!} \alpha^{m_{1}\ldots m_{p}} \partial_{m_{1}} \wedge \cdots \wedge \partial_{m_{p}} \) and \( \gamma \equiv \sum_{p} \frac{1}{p!} \gamma^{m_{1}\ldots m_{p}} \partial_{m_{1}} \wedge \cdots \wedge \partial_{m_{p}} \) are poly-vectors and \( \wedge \) is the wedge product for poly-vectors. This \( \alpha^{m_{1}\ldots m_{p}} \) transforms as

\[
\alpha^{a_{1}\ldots a_{p}} = \alpha^{a_{1}\ldots a_{p}}, \\
\alpha^{a_{1}\ldots a_{p-1}1} = \alpha^{a_{1}\ldots a_{p-1}1},
\]

under the \( T \)-duality along the \( v^{i} \)-direction. In other words,

\[
|\alpha| \equiv \sum_{p} \frac{1}{p!} \alpha^{m_{1}\ldots m_{p}} \Gamma_{m_{1}\ldots m_{p}}[0],
\]

16 The 11D uplifts also have the same form as the standard potentials.
transforms as an O(10, 10) spinor,\textsuperscript{17} where we have defined a new vacuum annihilated by $\Gamma^m$,

$$|\tilde{0}\rangle \equiv C |0\rangle = \Gamma^{0\cdots9} |0\rangle, \quad \tilde{\Gamma}^M \equiv \Gamma^{11} \Gamma^M. \quad (5.15)$$

Other dual fields, such as $\beta^{m_1\cdots m_6}$ \textsuperscript{[66,70]}, can also be embedded into T-duality tensors.

Finally, let us consider the exotic duality, in particular (5.1). The left-hand side is the field strength of the potential $E_8$. Since $E_8$ is a component of $|E_{MN}\rangle$, the field strength will be also defined covariantly. The field strength has been discussed in Ref. \[56\], although the explicit form has not yet been determined;

$$|K^M\rangle \sim \partial_N |E^{MN}\rangle + \cdots. \quad (5.16)$$

On the other hand, the right-hand side contains $d\tilde{\gamma}$, which is roughly equal to the $P$-flux $P_1 \equiv d\gamma$, as we have seen in Eq. (5.7). The $P$-flux may be also defined T-duality covariantly,

$$|P_M\rangle \equiv \partial_M |\gamma\rangle + \cdots, \quad (5.17)$$

and the exotic duality will be a covariant relation connecting $|K^M\rangle$ and $|P_M\rangle$. The non-trivial point is that although the field strength $|K^M\rangle$ is defined in the standard parametrization, the $P$-flux $|P_M\rangle$ is defined in the dual parametrization, and it is not easy to find the relation. In order to find the covariant expression for the exotic duality, it may be useful to consider the supergravity action for the dual fields. As was (partially) worked out in Ref. \[70\], by substituting the dual parametrization into the action of the $U$-duality-covariant supergravity, known as the exceptional field theory \[110–113\] (which is based on DFT and earlier works \[2,3,109,114–118\]), we obtain the action for the dual fields

$$\mathcal{L} = \tilde{R} - \frac{1}{2} d\tilde{\phi} \wedge \tilde{\star} d\tilde{\phi} - \frac{1}{2} e^{-\tilde{\phi}} d\gamma \wedge \tilde{\star} d\gamma + \cdots. \quad (5.18)$$

Then, the equations of motion for the dual fields $\chi^{m_1\cdots m_p}$ are precisely the exotic duality, as discussed in Ref. \[66\]. Thus, in order to find the $T$-duality-covariant exotic duality, it will be useful to find the $T$-duality-covariant action for the dual fields, by defining the dual fields as $T$-duality-covariant tensors. From the covariant action, we obtain the $T$-duality-covariant equations of motion for the dual fields, and they will correspond to the exotic duality.

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\textsuperscript{17} To be more precise, it is a spinor density. The O(10, 10) spinor $|A\rangle$ has weight 1/2 and the weight can be removed by considering $e^\delta |A\rangle$. On the other hand, $|\alpha\rangle$ has weight $-1/2$ and $e^{-\delta} |\alpha\rangle$ is weightless.
Appendix A. Notations

A.1. Differential forms and mixed-symmetry potentials

We employ the following convention for differential forms:

\[(\ast \alpha_p)_{m_1 \ldots m_{10-p}} = \frac{1}{(10-p)!} \epsilon^{n_1 \ldots n_p}_{m_1 \ldots m_{10-p}} \alpha_{n_1 \ldots n_p},\]

\[(\ast (d x^{m_1} \wedge \ldots \wedge d x^{m_p})) = \frac{1}{(10-p)!} \epsilon^{m_1 \ldots m_p}_{n_1 \ldots n_{10-p}} d x^{n_1} \wedge \ldots \wedge d x^{n_{10-p}},\]

\[(\iota_v \alpha_p) = \frac{1}{(p-1)!} v^i \alpha_{nm_1 \ldots m_p-1} d x^{m_1} \wedge \ldots \wedge d x^{m_p-1},\] \hspace{1cm} (A.1)

where

\[\epsilon_{m_1 \ldots m_{10}} \equiv \sqrt{|g|} \epsilon_{m_1 \ldots m_{10}}, \quad \epsilon_{m_1 \ldots m_{10}} \equiv \frac{1}{\sqrt{|g|}} \epsilon_{m_1 \ldots m_{10}}, \quad \epsilon_{0 \ldots 9} = 1, \quad \epsilon_{0 \ldots 9} = -1.\] \hspace{1cm} (A.2)

The symmetrization and antisymmetrization are normalized as

\[A_{(m_1 \ldots m_n)} \equiv \frac{1}{n!} (A_{m_1 \ldots m_n} + \cdots), \quad A_{[m_1 \ldots m_n]} \equiv \frac{1}{n!} (A_{m_1 \ldots m_n} \pm \cdots).\] \hspace{1cm} (A.3)

Indices separated by “|” are not (anti-)symmetrized. For example,

\[3 \hat{A}_{i_1[k_1 k_2]} \hat{A}_{j_2[i_1]} = \hat{A}_{i_1 k_1 k_2} \hat{A}_{j_2[i_1]} + \hat{A}_{i_2 k_1 k_2} \hat{A}_{j_1[i_2]} + \hat{A}_{i_3 k_1 k_2} \hat{A}_{j_1[i_3]}.\] \hspace{1cm} (A.4)

When two groups of indices are antisymmetrized, we have used overlines. For example,

\[3 \hat{A}_{[i_1[k_1 k_2]} \hat{A}_{i_2 \ldots i_6]k_3} = \hat{A}_{[i_1[k_1 k_2]} \hat{A}_{i_2 \ldots i_6]k_3} + \hat{A}_{[i_1[k_2 k_1]} \hat{A}_{i_2 \ldots i_6]k_1} + \hat{A}_{[i_1[k_1 k_2]} \hat{A}_{i_2 \ldots i_6]k_2}.\] \hspace{1cm} (A.5)

In this paper, we consider only the components of the mixed-symmetry potentials that satisfy the restriction rules \((2.45)\) or \((2.55)\). For convenience, by using the equality \(\sim\) we have expressed various equations without making the restriction rule manifest. However, we can always convert the equality \(\sim\) into the exact equality \(=\) by making the restriction rule manifest. For example, let us consider Eq. \((3.59)\):

\[D_{m_1 \ldots m_7, n} \sim \mathcal{A}_{m_1 \ldots m_7, n} + 7 D_{[m_1 \ldots m_6| \mathcal{B}_{m_7]|n} - \frac{1}{2} \mathcal{C}_{m_1 \ldots m_7| \mathcal{C}_n} - \frac{21}{2} \mathcal{C}_{[m_1 \ldots m_5| \mathcal{C}_{m_6 m_7]|n}
+ 70 \mathcal{C}_{[m_1 m_2 m_3| \mathcal{C}_{m_4 m_5 m_6]| \mathcal{B}_{m_6 m_7}].\] \hspace{1cm} (A.6)

In this example, the restriction rule is \(\{m_1, \ldots, m_7\} \ni n\), and this is automatically satisfied by choosing \(m_7 = n\). We then obtain

\[D_{m_1 \ldots m_6, n} = \mathcal{A}_{m_1 \ldots m_6, n} - 6 D_{[m_1 \ldots m_5| n| \mathcal{B}_{m_6]|n} - \frac{1}{2} \mathcal{C}_{m_1 \ldots m_6| n} - \frac{15}{2} \mathcal{C}_{[m_1 \ldots m_4| n| \mathcal{C}_{m_5 m_6]|n}
+ 20 \mathcal{C}_{[m_1 m_2 m_3| \mathcal{C}_{m_4 m_5 m_6]| \mathcal{B}_{m_6}|n} + 30 \mathcal{C}_{[m_1 m_2| n| \mathcal{C}_{m_3 m_4 m_6}| \mathcal{B}_{m_5 m_6}].\] \hspace{1cm} (A.7)

In general, this makes the expressions longer, and that is the reason why we are using \(\sim\).

In order to simplify expressions, it is also useful to use the notation of the differential form. For example, several relations for the \(T\)-duality-covariant potentials become

\[D_6 = \mathcal{B}_6 - \frac{1}{2} \mathcal{C}_5 \wedge \mathcal{C}_1,\] \hspace{1cm} (A.8)

\[\iota_n D_{7, n} = \iota_n \mathcal{D}_{7, n} - \iota_n D_6 \wedge \iota_n \mathcal{B}_2 - \frac{1}{2} \left( \iota_n \mathcal{C}_7 \iota_n \mathcal{C}_1 + \iota_n \mathcal{C}_5 \wedge \iota_n \mathcal{C}_3 \right)
+ \frac{1}{2} \left( \mathcal{B}_2 \wedge \iota_n \mathcal{C}_3 - \mathcal{C}_3 \wedge \iota_n \mathcal{B}_2 \right) \wedge \iota_n \mathcal{C}_3,\] \hspace{1cm} (A.9)

\[\iota_n E_{8, n} = \iota_n \mathcal{E}_{8, n} + \iota_n \mathcal{B}_6 \wedge \iota_n \mathcal{C}_3 - \frac{1}{2} \iota_n \mathcal{C}_5 \wedge \left( \mathcal{C}_3 \iota_n \mathcal{C}_1 - \iota_n \mathcal{C}_3 \wedge \mathcal{C}_1 \right),\] \hspace{1cm} (A.10)
A.2. Supergravity fields

Our gauge potentials are related to those used in Refs. [11,17–19] as follows. In type IIA theory, their fields (left) and our fields (right) are related as

\[ B = \tilde{B}_2, \quad C^{(p)} = \tilde{C}_p, \quad B^{(6)} = -\tilde{B}_6, \quad N^{(7)} = \tilde{\omega}_8, n, \quad N^{(8)} = -\tilde{\omega}_8, n, \]  

(A.16)

where \( n \) represents a Killing direction. In type IIB theory, the relations are summarized as

\[ B = B_2, \quad C^{(0)} = -C_0, \quad C^{(2)} = -C_2, \quad C^{(4)} = -A_4, \quad C^{(6)} = -\left(C_6 - \frac{1}{4} B_2 \wedge B_2 \wedge C_2\right), \]
\[ C^{(8)} = -(C_8 - \frac{1}{2} C_2 \wedge B_2 \wedge B_2), \quad B^{(6)} = -(B_6 - \frac{1}{4} C_2 \wedge C_2 \wedge B_2), \quad \tilde{C}^{(8)} = E_8. \]

(A.17)

Although the full \( T \)-duality rule for the dual graviton \( N^{(7)} \) has not been obtained there, by comparing the gauge transformation (4.51) with Eq. (B.4) of Ref. [18], we find

\[ N^{(7)} = \iota_n A_{7,n} - \frac{1}{4} \epsilon_{\gamma \delta} A_2^{\gamma} \wedge \iota_n A_2^{\delta}. \]

(A.18)

In addition, \( N^{(8)} \) and \( \tilde{N}^{(8)} \) correspond to our \( D_{8,2} \) and \( E_{8,2} \) at least under \( B_2 = 0 \) and \( C_2 = 0 \). We have not identified the precise relation between their \( N^{(9)} \) and our \( A^{9,2,1} \).

Our 11D fields \( \tilde{A}_3, \tilde{A}_6 \) and \( \tilde{A}_8 \) are the same as those used in Refs. [17,27,28], where \( \tilde{A}_8 \) is denoted as \( \tilde{N}^{(8)} \). The 9-form \( \iota_n \tilde{C}^{(10)} \) used in Refs. [27,28] can be defined as

\[ \iota_n \tilde{C}^{(10)} = \iota_n \tilde{A}_{10,n} + \frac{1}{4} \tilde{A}_3 \wedge \iota_n \tilde{A}_3 \wedge \iota_n \tilde{A}_3 \wedge \iota_n \tilde{A}_3. \]

(A.19)

Let us also identify the relation between our type IIB fields and those used in Refs. [30–33]. For this purpose, it is useful to perform a redefinition,

\[ \tilde{A}_2^{\alpha} = A_2^{\alpha}, \quad \tilde{A}_4^{\alpha} = A_4^{\alpha}, \quad \tilde{A}_6^{\alpha} = A_6^{\alpha} - \frac{1}{2} A_4^{\alpha} \wedge A_2^{\alpha}, \]
\[ \tilde{A}_8^{\alpha \beta} = A_8^{\alpha \beta} - \frac{1}{4} A_6^{(\alpha} \wedge A_2^{\beta)}, \quad \tilde{A}_8^{\alpha \beta \gamma} = A_8^{\alpha \beta \gamma} - \frac{1}{2} A_8^{(\alpha \beta} \wedge A_2^{\gamma)}. \]

(A.20)
which makes the field strengths have the schematic form $F \sim d\tilde{A} + \sum F \wedge \tilde{A}$,

$$
\begin{align*}
F_3^\alpha &= d\tilde{A}_3^\alpha, \\
F_5 &= d\tilde{A}_4 + \frac{1}{2} \epsilon_{\alpha\beta} F_3^\alpha \wedge \tilde{A}_2^\beta, \\
F_7^\alpha &= d\tilde{A}_6^\alpha + \frac{1}{3} F_5 \wedge \tilde{A}_2^\alpha - \frac{2}{3} F_3^\alpha \wedge \tilde{A}_4, \\
F_9^{\alpha\beta} &= d\tilde{A}_8^{\alpha\beta} + \frac{1}{4} F_7^\alpha \wedge \tilde{A}_2^\beta - \frac{3}{4} F_3^{(\alpha} \wedge \tilde{A}_6^{\beta)}, \\
F_{11}^{\alpha\beta\gamma} &= d\tilde{A}_{10}^{\alpha\beta\gamma} + \frac{1}{3} F_9^{(\alpha} \wedge \tilde{A}_2^{\beta)} - \frac{4}{3} F_3^{(\alpha} \wedge \tilde{A}_8^{\beta\gamma)} = 0.
\end{align*}
$$

(A.21)

The gauge transformation can also be expressed as $\delta\tilde{\Lambda} \sim d\tilde{\Lambda} + \sum F \wedge \tilde{\Lambda}$,

$$
\begin{align*}
\delta\tilde{A}_3^\alpha &= d\tilde{\Lambda}_{3}^\alpha, \\
\delta\tilde{A}_4 &= d\tilde{\Lambda}_{3} + \frac{1}{2} \epsilon_{\alpha\beta} F_3^\alpha \wedge \tilde{\Lambda}_1^\beta, \\
\delta\tilde{A}_6^\alpha &= d\tilde{\Lambda}_{5}^\alpha + \frac{1}{3} F_5 \wedge \tilde{\Lambda}_1^\alpha - \frac{2}{3} F_3^\alpha \wedge \tilde{\Lambda}_3, \\
\delta\tilde{A}_8^{\alpha\beta} &= d\tilde{\Lambda}_{7}^{\alpha\beta} - \frac{3}{4} F_3^{(\alpha} \wedge \tilde{\Lambda}_5^{\beta)} + \frac{1}{4} F_7^{(\alpha} \wedge \tilde{\Lambda}_1^{\beta)}, \\
\delta\tilde{A}_{10}^{\alpha\beta\gamma} &= d\tilde{\Lambda}_{9}^{\alpha\beta\gamma} + \frac{1}{3} F_9^{(\alpha} \wedge \tilde{\Lambda}_2^{\beta)} - \frac{4}{3} F_3^{(\alpha} \wedge \tilde{\Lambda}_8^{\beta\gamma)}.
\end{align*}
$$

(A.22)

by considering a field-dependent redefinitions of gauge parameters:

$$
\begin{align*}
\tilde{\Lambda}_1^\alpha &= \Lambda_1^\alpha, \\
\tilde{\Lambda}_3 &= \Lambda_{3} - \frac{1}{21} \epsilon_{\gamma\delta} \tilde{A}_2^\gamma \wedge \Lambda_1^\delta, \\
\tilde{\Lambda}_5^\alpha &= \Lambda_5^\alpha + \frac{2}{3} \tilde{A}_2^\alpha \wedge \Lambda_3 - \frac{4}{3} \tilde{A}_4 \wedge \Lambda_1^\alpha - \frac{1}{31} \epsilon_{\gamma\delta} \tilde{A}_2^\gamma \wedge \tilde{A}_2^\delta \wedge \Lambda_1^\beta, \\
\tilde{\Lambda}_7^{\alpha\beta} &= \Lambda_7^{\alpha\beta} + \frac{3}{4} \tilde{A}_2^{(\alpha} \wedge \Lambda_5^{\beta)} + \frac{1}{4} \tilde{A}_2^{(\alpha} \wedge \tilde{A}_2^{\beta)} \wedge \Lambda_3 \\
&\quad - \frac{1}{4} \tilde{A}_2^{(\alpha} \wedge \Lambda_1^{\beta)} - \frac{1}{41} \epsilon_{\gamma\delta} \tilde{A}_2^\gamma \wedge \tilde{A}_2^\delta \wedge \tilde{A}_2^\beta \wedge \Lambda_1^\alpha, \\
\tilde{\Lambda}_9^{\alpha\beta\gamma} &= \Lambda_9^{\alpha\beta\gamma} + \frac{4}{5} \tilde{A}_2^{(\alpha} \wedge \Lambda_7^{\beta)} + \frac{3}{10} \tilde{A}_2^{(\alpha} \wedge \tilde{A}_2^{\beta)} \wedge \Lambda_5^{\gamma)} + \frac{1}{15} \tilde{A}_2^{(\alpha} \wedge \tilde{A}_2^{\beta} \wedge \tilde{A}_2^{\gamma)} \wedge \Lambda_3 \\
&\quad - \frac{1}{5} \tilde{A}_8^{(\alpha\beta} \wedge \Lambda_1^{\gamma)} + \frac{1}{5} \tilde{A}_2^{(\alpha} \wedge \tilde{A}_6^{\beta)} \wedge \Lambda_1^{\gamma)} - \frac{1}{51} \epsilon_{\delta\epsilon} \tilde{A}_2^\delta \wedge \tilde{A}_2^\epsilon \wedge \tilde{A}_2^\beta \wedge \tilde{A}_2^\gamma \wedge \Lambda_1^\alpha. 
\end{align*}
$$

(A.23)

Then, the tilde potentials and the field strengths are related to the fields used in Refs. [30–33] (appearing on the right-hand sides) as follows:

$$
\begin{align*}
\tilde{A}_2 &= A_2, \\
\tilde{A}_4 &= -4 A_4, \\
\tilde{A}_6^\alpha &= -A_6^\alpha, \\
\tilde{A}_8^{\alpha\beta} &= -4 A_8^{\alpha\beta}, \\
\tilde{A}_{10}^{\alpha\beta\gamma} &= 12 A_{10}^{\alpha\beta\gamma}. \\
F_3^\alpha &= F_3^\alpha, \\
F_5 &= -4 F_5, \\
F_7^\alpha &= -F_7^\alpha, \\
F_9^{\alpha\beta} &= -4 F_9^{\alpha\beta}, \\
F_{11}^{\alpha\beta\gamma} &= 12 F_{11}^{\alpha\beta\gamma}.
\end{align*}
$$

(A.24)

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