The partition function of gauge supersymmetric Ising model on 3D regular lattice

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The partition function of the gauge system with gauge group $\mathbb{Z}_2$ coupled with Majorana fermions is calculated on the regular 3D cubic lattice

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I. INTRODUCTION

Recently a new approach for calculation the partition function of the 2D Ising model on the regular lattice has been suggested [1]. The idea of this approach is as follows. An independent generator of Clifford algebra in the matrix representation is assigned to each vertex of the lattice. Thus the number $\mathfrak{N}$ of generators of the Clifford algebra can not be less than the number of vertexes of the lattice or the number of degrees of freedom of the partition function. Some polynomial in these generators depending on the ”temperature” parameter is defined. This polynomial is the matrix of orthogonal rotation in spinor representation in the $\mathfrak{N}$-dimensional Euclidean space. The trace of the polynomial is proportional to partition function of the 2D Ising model. This statement follows from the fact that the considering trace is represented as a sum over all closed self-intersecting contours (loops) on the regular planar lattice, and the positive weight depending on the temperature is assigned to each edge of the loops.

In this paper the outlined approach is extended to three-dimensional statistical system on on the regular cubic lattice. The sum over self-intersecting surfaces is calculated. The closed surfaces are included as well as the surfaces with boundary. Each face of the surface is assigned the positive factor $\mu^2$, each edge of the boundary is assigned the factor $\lambda\mu$. The real numbers $\lambda$ and $\mu$ depend on the temperature. Thus, the total positive weight corresponds to each closed surface. However, it turns out that the total weights corresponding to the surfaces with boundary depend on

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the boundary configurations, but not on the surface configurations with the fixed boundary.

The fact that it is impossible to retain only the closed surfaces and to remove the surfaces with boundary is the important characteristics of the considered statistical sum. This means that the considered system does not involve the 3D Ising model. Indeed, the Ising model on three-dimensional regular lattice is dual to the three-dimensional lattice gauge model with the gauge group \(Z_2\) [2]. The partition function of the latter is expressed as the sum over closed self-intersecting surfaces, and each face of the surface is assigned the positive factor depending on the temperature at that. So it is naturally to interpret the boundary of the surface as creation, propagation and annihilation of the fermion pair coupled with the gauge field. It will be shown in the subsection 2.5 that these fermions are the lattice analog of the Majorana fermions.

Thus, the calculated partition function can be interpreted as the partition function of the gauge system on the regular three-dimensional lattice with the gauge group \(Z_2\) interacting with Majorana fermions[9].

II. FORMULATION OF THE PROBLEM

A. The Dirac-Clifford algebra in the matrix representation

The generators of the Clifford algebra \(\{\xi_x\}\) satisfy the following relations

\[
\xi_x \xi_y + \xi_y \xi_x = 2 \delta_{x,y}, \quad x, y, \ldots = 1, \ldots, I = 2K, \tag{2.1}
\]

and they are assumed as hermitian matrixes of the dimension \(2^K \times 2^K\). Such dimension is minimally admissible at given number of generators. The theory of such matrixes in convenient for physicist form can be found in [3] for The concrete matrix representation of the Clifford algebra generators is no object. Only some their algebraic properties are needed. It follows from the algebra \((2.1)\) that the trace of any product of the odd number of \(\xi\)-matrix is equal to zero, and

\[
\text{tr} \, \xi_x \xi_y = 2^K \delta_{x,y}, \quad \text{tr} \, \xi_x \xi_y \xi_z \xi_v = 2^K (\delta_{x,y} \delta_{z,v} - \delta_{x,z} \delta_{y,v} + \delta_{x,v} \delta_{y,z}), \tag{2.2}
\]

and so on. It is evident that by means of permutations and usage of the equalities \(\xi_x^2 = 1\) any product of \(\xi\)-matrixes is reduced either to the number \(\pm 1\) (versus the number of permutations) or to the product of pairwise different \(\xi\)-matrixes. According to Eq. \((2.2)\) in the last case the trace of the product is equal to zero, and in the firs case it is equal to \(\pm 2^K\).
B. Description of the degrees of freedom of the system

Let’s enumerate the axes the simple three-dimensional cubic lattice by the numbers $i = 1, 2, 3$. Each vertex $v_x$ has the index composed of three natural numbers $x \equiv (m, n, l)$, and

$$m = 1, \ldots, M, \quad n = 1, \ldots, N, \quad l = 1, \ldots, L.$$  \hfill (2.3)

For simplicity the numbers $M, N, L$ are considered even. It is assumed that the numbers $(m, n, l)$ increase uniformly along the first, the second and the third axis, correspondingly. Define the base vectors of the lattice: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Thus, the vector $e_i$ is directed onto positive direction of the $i$-th axis.

Further I believe $I = 3MNL$.

Divide the totality of the Clifford algebra generators into 3 groups, each of which consisting of $MNL$ generators. Let’s designate the first, the second and the third group of the generators as $\{\alpha_x\}$, $\{\beta_x\}$ and $\{\gamma_x\}$, correspondingly. The edge connecting the adjacent vertexes $v_x$ and $v_{x+e_i}$ is designated as $l_{x,i} = l_{m,n,l;i}$. The face with vertexes $(x, x+e_j, x+e_j+e_k, x+e_k)$ is designated as $f_{x,i}$, $i = 1, 2, 3$ (here $j \neq k$ and the scalar products $e_j \cdot e_i = e_k \cdot e_i = 0$).

By definition, the matrixes $\alpha_x$, $\beta_x$ and $\gamma_x$ are related to the edges $l_{x,1}$, $l_{x,2}$ and $l_{x,3}$, correspondingly. Below also the notation

$$\xi_x = \xi_x^{(i)} = (\alpha_x, \beta_x, \gamma_x), \quad i = 1, 2, 3.$$  \hfill (2.4)

is used.

C. The definition of the partition function

Relate to each face the unitary matrix of rotation in the spinor representation: The face $f_{x,1}$ is related with matrix

$$U^{(1)}_{(m,n,l)} \equiv (\lambda + \mu \beta_x \gamma_x + e_2) (\lambda + \mu \beta_x + e_3 \gamma_x),$$  \hfill (2.5)

the face $f_{x,2}$ is related with matrix

$$U^{(2)}_{(m,n,l)} \equiv (\lambda + \mu \alpha_x \gamma_x + e_1) (\lambda + \mu \alpha_x + e_3 \gamma_x),$$  \hfill (2.6)

and the face $f_{x,3}$ is related with matrix

$$U^{(3)}_{(m,n,l)} \equiv (\lambda + \mu \alpha_x \beta_x + e_1) (\lambda + \mu \alpha_x + e_2 \beta_x).$$  \hfill (2.7)

Here

$$x = (m, n, l), \quad \lambda = \cos \frac{\psi}{2}, \quad \mu = \sin \frac{\psi}{2}. \hfill (2.8)$$
In fig. 1 a fragment of the lattice containing the vertex \( x \) and three adjacent faces \( f_{x,1}, f_{x,2}, f_{x,3} \) is represented, and also the \( \xi \)-matrixes related to the corresponding edges are designated. The arrows, curved orthogonally, lying in the faces and showing the crossing from one edge to the neighboring edge, represent the rotation matrixes corresponding to one of the round brackets in the right hand side of Eqs. (2.5), (2.6) and (2.7); the directions of the arrows indicate the ordering of \( \xi \)-matrixes. For example, the arrows (1) and (2) represent the first and the second round brackets in Eq. (2.5), correspondingly, etc. Since the round brackets in the right hand sides of Eqs. (2.5), (2.6) and (2.7) do not commute, so their order is significant. Below we need the representation of the quantities (2.5)-(2.7) in the form of the following sums

\[
U^{(1)}_{(m,n,l)} = \left( \lambda^2 + \mu^2 \beta x \gamma x + e_2 \beta x + e_3 \gamma x \right) + t \lambda \mu \left( \beta x \gamma x + e_2 + \beta x + e_3 \gamma x \right),
\]

\[
U^{(2)}_{(m,n,l)} = \left( \lambda^2 + \mu^2 \alpha x \gamma x + e_1 \alpha x + e_3 \gamma x \right) + t \lambda \mu \left( \alpha x \gamma x + e_1 + \alpha x + e_3 \gamma x \right),
\]

\[
U^{(3)}_{(m,n,l)} = \left( \lambda^2 + \mu^2 \alpha x \beta x + e_1 \alpha x + e_2 \beta x \right) + t \lambda \mu \left( \alpha x \beta x + e_1 + \alpha x + e_2 \beta x \right).
\]

In the considered theory

\[
t = 1.
\]

In the case \( t \neq 1 \) the matrixes (2.9)-(2.11) are not the rotation matrixes and therefore the used here calculation method turns out useless.
The partition function of the system is proportional to the following trace of the ordered product of rotation matrixes in spinor representation

$$Z = C \text{tr} \mathcal{U}, \quad (2.13)$$

$$\mathcal{U} = \left( \hat{\mathcal{P}} \prod_l \right) \mathcal{U}(l), \quad (2.14)$$

$$\mathcal{U}(l) = \left( \hat{\mathcal{P}} \prod_n \right) \mathcal{U}(n,l), \quad (2.15)$$

$$\mathcal{U}(n,l) = \left( \hat{\mathcal{P}} \prod_m \right) U(m,n,l) = \ldots U(m-1,n,l) U(m,n,l) U(m+1,n,l) \ldots \quad (2.16)$$

$$U_{(m,n,l)} \equiv U^{(1)}_{(m,n,l)} U^{(2)}_{(m,n,l)} U^{(3)}_{(m,n,l)}, \quad (2.17)$$

Here $C$ is some numerical constant which is not interesting here. The symbol $(\hat{\mathcal{P}} \prod_m)$ in $(2.16)$ denotes the ordered product of the matrices $(2.17)$ over all meaning of $m$ at fixed values of the numbers $(n, l)$, so that in this product the matrix $U_{(m,n,l)}$ is placed to the left of the matrix $U_{(m',n,l)}$ for $m < m'$. The ordered products in $(2.15)$ (at fixed $(l)$) and in $(2.14)$ are determined analogously.

**D. The closed surfaces with self-intersections**

Let’s consider the contribution of zeroth power in the parameter $t$ to the matrix $(2.14)$. According to Eqs. $(2.9)-(2.11)$ in this case each face gives two contributions to $\mathcal{U}$: the first contribution is proportional to the number $\lambda^2$ and the second one is proportional to the matrix $\{\mu^2 \times (\text{the ordered product of four } \xi\text{-matrixes corresponding to four edges of the face})\}$. It is obvious that under the trace operator in $(2.13)$ some summands of the matrix $\mathcal{U}$ are inessential. For example, according to $(2.2)$

$$\text{tr} C \left\{ (\lambda^2)^{3MNL-1} \mu^2 \beta_\lambda \gamma_\lambda \varepsilon\right\} = 0. \quad (2.18)$$

Only that summands of the matrix $\mathcal{U}$ give the nonzero contribution to the partition function in which each contained $\xi$-matrix is in even power. It follows from here that the considered part of partition function is represented as the sum over closed surfaces on the lattice with self-intersections. Each edge of the surface can belong to
two or four faces of the surface. In the last case the intersection or self-intersection of the surface occur in the edge. To each face $f_{x,i}$ of such surface the summand from $U^{(i)}_{(m,n,l)}, i = 1, 2, 3$, proportional to $\mu^2$ and the fourth power of $\xi$-matrix (see (2.9)-(2.11)), is assigned. The order of the $\xi$-matrix arrangement in the summand in $\mathcal{U}$ corresponding to the closed surface (not obvious connected) is determined by the formulae (2.13)-(2.17). It follows from the aforesaid that in the summand in $\mathcal{U}$ corresponding to any closed surface each $\xi$-matrix can be contained either in the power 0 (the corresponding edge does not belong to the surface) either in the power 2 (the corresponding edge belong to two faces of the surface only) or in the power 4 (the corresponding edge belong to four faces of the surface). If $S$ is the number of faces of a the closed surface, then the contribution to the partition function of the surface is equal to (at macroscopic dimensions of the lattice the total number of faces of the lattice can be considered equal to $3MNL$)

$$\Delta Z = C \left[ 2^{3MNL/2} (\lambda^2)^{3MNL} \right] (\mu/\lambda)^{2S}, \quad C > 0. \quad (2.19)$$

Up to the sign the equality (2.19) is evident. It remains to prove only that each closed surface gives the positive contribution to the partition function.

Consider the contribution to $\mathcal{U}$ from the simplest closed surface without self-intersections: the elementary cube with the faces $f_{x,1}$, $f_{x,2}$, $f_{x,3}$, $f_{x+e_2,2}$, $f_{x+e_3,3}$. Here the order of the face enumeration corresponds to the order of construction from elementary ”bricks” the corresponding summand in $\mathcal{U}$. Thus

$$\Delta \mathcal{U} = (\beta_x \gamma_{x+e_2} \beta_{x+e_3} \gamma_{x}) (\alpha_x \gamma_{x+e_1} \alpha_{x+e_3} \gamma_{x}) (\alpha_x \beta_{x+e_1} \alpha_{x+e_2} \beta_{x}) \times$$

$$\times (\beta_x \gamma_{x+e_1} \gamma_{x+e_2} \beta_{x+e_3} \gamma_{x+e_1}) (\alpha_x \gamma_{x+e_1} \beta_{x+e_2} \alpha_{x+e_3} \gamma_{x+e_2}) \times$$

$$\times (\alpha_x \beta_{x+e_1} \alpha_{x+e_2} \beta_{x+e_3} \gamma_{x+e_3}) = 1. \quad (2.20)$$

In this product each $\xi$-matrix is contained twice.

To prove the positivity of the expression (2.19) in general case let’s cut any closed surface by the plane parallel to the plane (12), intersecting the middles of the edges $l_{m,n,l};3$ with fixed $l$ and $m = 1, \ldots, M$, $n = 1, \ldots, N$. Denote this plane by $\mathfrak{P}_l$. In each plane $\mathfrak{P}_l$ there is the natural structure of the square plane lattice with the vertexes designated as $v_{m,n}$ which belong to the middles of the edges $l_{m,n,l};3$. The edges of this lattice are designated by $l_{m,n;1}$ and $l_{m,n;2}$. The edge $l_{m,n;1}$ connects the vertexes $v_{m,n}$ and $v_{m+1,n}$, and the edge $l_{m,n;2}$ connects the vertexes $v_{m,n}$ and $v_{m,n+1}$. Thus the edge $l_{m,n;1}$ belong to the face $f_{x,2}$ and the edge $l_{m,n;2}$ belong to the face $f_{x,1}$. The outlined lattice also is denoted as $\mathfrak{P}_l$. It is convenient to consider that the matrixes $\gamma_{m,n,l}$ with fixed $l$ and $m = 1, \ldots, M$, $n = 1, \ldots, N$ are related
to the vertexes $v_{m,n}$ of the lattice $\mathfrak{P}_l$.

Let’s fix $l$ and represent by thick segments that edges of the lattice $\mathfrak{P}_l$ which belong to the faces of the considered closed surface. It is evident that the totality of thick segments or edges on the lattice $\mathfrak{P}_l$ forms a closed contour with self-intersections on the cubic plane lattice, so that in one vertex $v_{m,n}$ 0, 2 or 4 thick edges can converge. It is known that the sum over such closed contours with a certain positive weight assigned to each edge is proportional to the partition function of 2D Ising model \cite{1, 4}. Let us transform the matrix factors corresponding to the faces $f_{x,1}$ and $f_{x,2}$ (belonging to the closed surface and intersecting with the plane $\mathfrak{P}_l$) as follows:

\[(\beta_x \gamma_{x+e_2} \beta_x \gamma_{x+e_3}) = (\gamma_{x+e_2} \beta_x) (\beta_x \gamma_{x+e_3}), \quad (2.21)\]

\[(\alpha_x \gamma_{x+e_1} \alpha_x \gamma_{x+e_3}) = (\gamma_{x+e_1} \alpha_x) (\alpha_x \gamma_{x+e_3}). \quad (2.22)\]

Note that the matrixes $(\gamma_{x+e_i})$, $i = 1, 2$ commute with all matrixes $\alpha_{x'}$, $\beta_{x'}$ without restrictions and also with the matrixes $(\gamma_{x'+e_j})$, $j = 1, 2$ for $l \neq l'$. But the matrixes $(\gamma_{x+e_i})$, $i = 1, 2$ and $(\gamma_{x'+e_j})$, $j = 1, 2$, for $l = l'$, generally speaking, do not commute since in the set of four matrixes $(\gamma_{x}, \gamma_{x+e_1}, \gamma_{x'}, \gamma_{x'+e_2})$ can be found coinciding matrixes. It is seen from here and from Eqs. (2.21), (2.22) that in the summand in $\mathcal{U}$ corresponding to a closed surface one can remove to the left all matrix factors $(\gamma_{x+e_2})$ and $(\gamma_{x+e_1})$ without changing the relative arrangement of the factors themselves. As a result the sign of the summand does not change. Moreover, the product of all removed to the left $\gamma$-matrixes is converted into 1. Indeed, the ordering product of the removed to the left $\gamma$-matrixes (at fixed $l$) corresponds to the ordering product of $\gamma$-matrixes in the case of 2D Ising model \cite{1}. It was proved that in the last case the considered product of $\gamma$-matrixes is equal to 1.

Analogously the $\gamma$-matrixes for all others $l$ are mutually cancelled. Thus in any summand in the matrix $\mathcal{U}$ we can cut out all $\gamma$-matrixes without changing the sign.

Further we work analogously. Let’s denote the plane cutting a closed surface through the middles of the edges $l_{m,n,1;2}$, $m = 1, \ldots, M$, $l = 1, \ldots, L$ by $\mathfrak{P}_n$. On the plane $\mathfrak{P}_n$ the natural structure of the right square plane lattice is present. The vertexes of this lattice (at fixed $n$) are designated as $v_{m,l}$. The matrixes $\beta_{m,n,l}$ are related to the vertexes $v_{m,l}$. Taking into account Eqs. (2.21) and

\[(\alpha_x \beta_{x+e_1} \alpha_x \beta_{x+e_2}) = (\beta_x \beta_{x+e_1}) (\alpha_x \alpha_{x+e_2}), \quad (2.23)\]

we can (at fixed $n$) remove from the rest of the $\xi$-matrix product (corresponding to the closed surface) all factors $(\beta_{x+e_3})$ and $(\beta_{x+e_1})$ to the left without changing the relative arrangement of the factors themselves. As above, the total sign of the
ξ-matrix product does not change, and the product of β-matrixes at each value of \( n \) is converted to 1. Further only the product of α-matrixes remains which is converted to 1 at each fixed \( m \).

Thus it is proved that the product of ξ-matrixes in each matrix summand in \( \mathcal{U} \) corresponding to any closed surface is converted to 1, and the formula (2.19) is true.

E. The surfaces with boundary

Let us consider the contributions to the partition function (2.13) which are nonzero relative to the parameter \( t \) in (2.9)-(2.11).

The elementary example of the surfaces with boundary is represented in fig. 2. The totality of curved orthogonally thick arrows forms the boundary of the surface. The surface itself is the shaded face in fig. 2 with the vertexes \( v_2, v_3, v_4, v_6 \).

![Fig. 2](image)

The contribution to the partition function of the surface with boundary represented in fig. 2 is proportional to \( t^4 \). According to the rule the contribution of this surface to \( \mathcal{U} \) is proportional to the expression

\[
\Delta_1 \mathcal{U} \sim (\beta_1 \gamma_1)(\alpha_1 \beta_1)(\alpha_1 \gamma_2 \alpha_2 \gamma_1)(\beta_2 \gamma_2)(\alpha_2 \beta_2) = -1. \tag{2.24}
\]

Here and below the proportionality coefficients are positive. In (2.24) the factors \((\beta_1 \gamma_1)\) contained in parentheses are the contributions from the curved arrow in the face with vertexes \( v_1, v_2, v_6, v_7 \). Generally, the set of factors contained in parentheses is the factor in \( \Delta \mathcal{U} \) either from one arrow in some face (only two ξ-matrixes), or the factor from the whole face (four ξ-matrixes). Each face gives no more than one factor which can be either zeroth power relative to the ξ-matrixes and proportional to \( \lambda^2 \), either the second power (represented by one curved arrow and proportional
to $t$), or the fourth power relative to $\xi$-matrixes. For example, the shaded face with vertexes $v_2, v_3, v_4, v_6$ gives the contribution of the fourth power relative to $\xi$-matrixes: $(\alpha_1 \gamma_2 \alpha_2 \gamma_1)$.

Now let’s consider the contribution to $U$ from the surface with boundary represented in fig. 3. As in the previous example, the curved orthogonally thick arrows form the boundary of the surface, and the surface itself consists of the three shaded faces. The contribution of this surface to the matrix $U$ is proportional to $t^6$ and the matrix

$$\Delta_2 U \sim (\beta_1 \gamma_1)(\alpha_1 \beta_1)(\alpha_1 \gamma_2 \alpha_2 \gamma_1)(\beta_3 \gamma_2)(\beta_2 \gamma_3) \times$$

$$\times (\alpha_2 \beta_3 \alpha_3 \beta_2)(\alpha_3 \gamma_4 \alpha_4 \gamma_3)(\beta_4 \gamma_4)(\alpha_4 \beta_4) = 1.$$  \hspace{1cm} (2.25)

![FIG. 3](image)

It is seen from the comparison of the right hand sides of the relations (2.24) and (2.25) that the factor in the summand of the partition function corresponding to a surface with boundary can be of any sign. The more detailed consideration of this problem shows that the signs of the factors of the surfaces with boundary depend on the boundary configuration but not on the surface configuration with given boundary.
The outlined situation resembles the situation in the case of Majorana fermions on three-dimensional lattice coupled with Abelian gauge field. It is shown in Appendix B that in the last case the fermion contribution to the partition function also is represented as a sum over the surfaces with boundary, and the boundary of the surface is interpreted as creation, propagation and annihilation of the fermion pair. As in the studied here model, each such surface gives some factor in one of summand of all functional integral. The sign of this factor can be either positive or negative, and the sign is determined by the boundary configuration but not by the surface configuration with given boundary. The modulo of this factor is proportional to $\epsilon^s$, where $\epsilon$ is some positive number and $s$ is the number of faces of the corresponding surface.

Stated above founds to accept the hypotheses that the studied here partition function is the partition function of the abelian gauge system with the gauge group $Z_2$ (this is evident) coupled with Majorana fermions. Just owing to this hypotheses the word ”supersymmetric” is introduced into the title of the paper.

In connection with this hypotheses it is necessary to make the following comment.

As it is known [5], it is impossible to construct on the periodic lattice a satisfactory variant of the local action of Dirac fermions which transforms into the well known Dirac action in the continual limit. In particular, the problem consists in the imposibility of construction of fermion action on the periodic lattices describing only one chiral (Weyl) fermion field and transforming into the usual action of chiral field in the continual limit. This problem is known as a ”fermion doubling” problem. It is known also that at the cost of increase the lattice dimension one can construct only one Weyl field on the lattice hypersurface of the codimension one. For example, on the 4+1-dimensional lattice one can construct the action of only one Weyl field on 4-dimensional sublattice [6]. But this construction does not seem refined and therefore prospective. Besides the problem of coupling of the such Weyl fields with the gauge field is not solved in general case beyond the framework of perturbation theory.

From here one can make the following conclusion: the approach to the study of the lattice gauge systems including fermion degrees of freedom which is based on the usage of a lattice action is incorrect in general. But another approach can be used for the study the gauge-fermion systems on the lattice. Indeed, the exponent action summed (integrated) over the degrees of freedom of the system is necessary only for finding the quantum transition amplitudes. Though in the continuous theory the method of finding of transition amplitudes with the help of functional integral of the exponent action is very effective, we see that in the lattice theories
on the such way can arise serious difficulties by reason of lack of the action itself. Fundamentally this obstacle can be overcame by means of the direct definition of transition amplitudes omitting the stage of action construction [11]. This approach seems more perspective when studying the fundamental phenomena, especially in the case of discrete space-time: it is not ruled out that exactly quantum transition amplitudes are the most elementary mathematical objects in fundamental physics. Such approach is realized in this work.

According to the aforesaid the question of the representation of the partition function (quantum vacuum-vacuum transition amplitude) \((2.13)\) in the form of exponent action summed over the degrees of freedom of the system is not correct entirely. Indeed, the considering sum over closed surfaces is nothing but the partition function of the gauge Ising model, that is summed over the system degrees of freedom exponent lattice action of the gauge field without matter with the gauge group \(Z_2\). But the sum over surfaces with boundary unlikely can be expressed in a similar manner as an integral over fermions of the corresponding exponent, though this sum must be interpreted as a Majorana fermions contribution. The last statement follows from the comparison of the deciding properties of the weights of the surfaces with boundary (the signs of the weights depend on the boundary surface shape only) in the considered model and in the case of the lattice Majorana fermions (see Appendix B). But the exact coincidence of these weights unlikely can be achieved by means of the search of the appropriate lattice fermion model because the fermion doubling problem prevents. The correct problem statement seems to be as follows: Suppose that in the model presented by the quantum transition amplitudes (but not by the action) the second kind phase transition exists, that is the long-wave or macroscopic limit takes place. Then one must study the properties of the amplitudes in the long-wave limit and compare them with the properties of amplitudes in the continuous theories. As a result the corresponding identification can be determined in the phase transition point.

Note that the free energy of the studied model has a peculiarity by the parameter \(\psi\) (see below). This means that there is the phase transition in the model. But the nature of this phase transition is not studied here.

III. THE FIRST STAGE OF DIAGONALIZATION OF THE MATRIX \(O_{x, y}\)

In the Appendix A the formulas are given which express the partition function \((2.13)\) through the eigenvalues of the orthogonal rotation matrix \(O_{x, y}\) (see \((A1)-(A4)\)). Therefore here and in the subsequent section the problem of finding the matrix \(O_{x, y}\) eigenvalues is solved and the solution is given in \((4.17)\). The reader
wishing to omit these bulky calculations can continue the reading of the paper starting from the Eqs. (4.17).

In statistical limit, when \( M, N, L \to \infty \), the problem of matrix \( \mathcal{O}_{x,y} \) diagonalization simplifies radically since the matrices \( \mathcal{O}_{x,x+z} \) and \( \omega_{x,x+z} \) become dependent only on \( z \) at relatively small distances from boundary of the lattice. This property is known as translational invariance. Therefore the diagonalization of these matrices is performed by means of Fourier transformation, i.e. by passing to quasi-momentum representation. The following complete orthonormal set of functions on the lattice is used for that purpose:

\[
|p\rangle \equiv \Psi_p(m) = \frac{1}{\sqrt{M}} e^{i\pi m}, \quad |q\rangle \equiv \Psi_q(n) = \frac{1}{\sqrt{N}} e^{i\pi n}, \quad |r\rangle \equiv \Psi_r(l) = \frac{1}{\sqrt{L}} e^{i\pi l},
\]

\[
p = -\frac{\pi (M - 2)}{M}, -\frac{\pi (M - 4)}{M}, \ldots, 0, \frac{2\pi}{M}, \ldots, \pi,
\]

\[
q = -\frac{\pi (N - 2)}{N}, -\frac{\pi (N - 4)}{N}, \ldots, 0, \frac{2\pi}{N}, \ldots, \pi,
\]

\[
r = -\frac{\pi (L - 2)}{L}, -\frac{\pi (L - 4)}{L}, \ldots, 0, \frac{2\pi}{L}, \ldots, \pi,
\]

\[
|k\rangle \equiv \Psi_k(x) = \Psi_p(m) \Psi_q(n) \Psi_r(l), \quad k = (p, q, l),
\]

\[
\sum_x \bar{\Psi}_k(x) \Psi_k'(x) = \delta_{kk'} \longleftrightarrow \sum_k \Psi_k(x) \bar{\Psi}_k(x') = \delta_{xx'}.
\] (3.1)

Let’s divide the problem into the successive series of steps. It follows from Eqs. (2.14)-(2.16) and (A2)

\[
U_{(n,l)}^{\dagger} \xi_x U_{(n,l)} = \mathcal{O}_{x,y}^{(n,l)} \xi_y, \quad (3.2)
\]

\[
U_{(l)}^{\dagger} \xi_x U_{(l)} = \mathcal{O}_{x,y}^{(l)} \xi_y, \quad \mathcal{O}^{(l)} = \ldots \mathcal{O}^{(n-1,l)} \mathcal{O}^{(n,l)} \mathcal{O}^{(n+1,l)} \ldots, \quad (3.3)
\]

\[
\mathcal{O} = \ldots \mathcal{O}^{(l-1)} \mathcal{O}^{(l)} \mathcal{O}^{(l+1)} \ldots. \quad (3.4)
\]

A. The first step

To find the matrixes \( \mathcal{O}_{x,y}^{(n,l)} \) we use the elementary equations

\[
(\lambda + \mu \xi_x \xi_y)^\dagger \xi_x (\lambda + \mu \xi_x \xi_y) = (\cos \psi) \xi_x + (\sin \psi) \xi_y, \quad (3.5)
\]
\[(\lambda + \mu \xi x \xi y)^\dagger \xi y (\lambda + \mu \xi x \xi y) = (\cos \psi) \xi y - (\sin \psi) \xi x, \quad (3.6)\]

\[(\lambda + \mu \xi x \xi y)^\dagger \xi z (\lambda + \mu \xi x \xi y) = \xi z, \quad z \neq x, y \quad (3.7)\]

which follow from (2.1) and (2.8). In (3.5)-(3.7) \( x \neq y. \)

The direct calculation with the help of Eqs. (2.16), (2.17) and (3.5)-(3.7) gives (remember that \( x = (m, n, l) \)):

\[U^\dagger_{(n,l)} \alpha_x U_{(n,l)} = \left[ (\cos^2 \psi) \alpha_x + (\cos^3 \psi \sin \psi) \beta_{x+e_1} + (\cos^2 \psi \sin \psi) \gamma_{x+e_1} + (\cos \psi \sin^2 \psi) \alpha_{x+e_1+e_3} + (-\sin^2 \psi) \beta_{x+e_1+e_3} \right] + \]

\[+ \left[ (-\cos^2 \psi \sin^2 \psi) \alpha_{x+e_1+e_2} + (\cos \psi \sin^2 \psi) \gamma_{x+e_1+e_2} \right], \quad (3.8)\]

\[U^\dagger_{(n,l)} \beta_x U_{(n,l)} = \left[ (-\sin \psi) \alpha_{x-e_1} + (\cos \psi \beta_x) \right] + \]

\[+ \left[ (-\cos^2 \psi \sin \psi) \alpha_{x+e_2} + (\cos \psi \sin \psi) \gamma_{x+e_2} \right], \quad (3.9)\]

\[U^\dagger_{(n,l)} \gamma_x U_{(n,l)} = \left[ (-\cos \psi \sin \psi) \alpha_{x-e_1} + (-\cos^2 \psi \sin^2 \psi) \beta_x + (\cos^3 \psi) \gamma_x + (-\cos^2 \psi \sin \psi) \alpha_{x+e_3} + (-\cos \psi \sin \psi) \beta_{x+e_3} \right] + \]

\[+ \left[ (\cos \psi \sin^2 \psi) \alpha_{x+e_2} + (-\sin^3 \psi) \gamma_{x+e_2} \right], \quad (3.10)\]

\[U^\dagger_{(n,l)} \alpha_{x+e_3} U_{(n,l)} = \left[ (\sin \psi) \gamma_x + (\cos \psi) \alpha_{x+e_3} \right], \quad (3.11)\]

\[U^\dagger_{(n,l)} \beta_{x+e_3} U_{(n,l)} = \left[ (\cos \psi \sin \psi) \gamma_x + (-\sin^2 \psi) \alpha_{x+e_3} + (\cos \psi) \beta_{x+e_3} \right], \quad (3.12)\]

\[U^\dagger_{(n,l)} \alpha_{x+e_2} U_{(n,l)} = \left[ (\sin \psi) \beta_x \right] + \left[ (\cos \psi) \alpha_{x+e_2} \right], \quad (3.13)\]

\[U^\dagger_{(n,l)} \gamma_{x+e_2} U_{(n,l)} = \left[ (-\cos \psi \sin \psi) \beta_x \right] + \left[ (\sin^2 \psi) \alpha_{x+e_2} + (\cos \psi) \gamma_{x+e_2} \right]. \quad (3.14)\]

All other \( \xi \)-matrixes in \( U^\dagger_{(n,l)} \ldots U_{(n,l)} \) remain unchangeable. Therefore the right-hand sides of Eqs. (3.8)-(3.14) define completely the matrixes \( C_{x,y}^{(n,l)} \). It is evident
that if the vector $\mathbf{x}$ is not too close to the boundary of the lattice, then Eqs. (3.8)–(3.14) are invariant relative to the synchronous change of the number $m$ in $\mathbf{x}$ in all summands of these equations. In the lattice size infinite limit this means the translational invariance relative to the shifts $m \rightarrow m + m'$ along the first axis. The translational invariance is used for the partial diagonalization of the matrix $O_{n,l}^{(n,l)}$ by Fourier transformation along the first axis:

$$\xi_{n,l}(p) = \sum_{m} \bar{\Psi}_p(m) \xi_{m,n,l} = \xi_{n,l}^\dagger(-p),$$

$$\begin{bmatrix} \xi_{n,l}^{(i)}(p), \xi_{n',l'}^{(j)}(p') \end{bmatrix}_+ = 2\delta_{i,j}\delta_{n,l}(n',l')\delta_{p,p'}, \quad i, j = 1, 2, 3, \quad (3.15)$$

$$\mathcal{U}_{(n,l)}^{\dagger}\xi_{n',l'}^{(i)}(p)\mathcal{U}_{(n,l)} = \sum_{j,n'',l''} \left[ \sum_{m'} O_{m,n',l',i;m+m',n'',l'',j} e^{ipm'} \right] \xi_{n'',l''}^{(j)}(p). \quad (3.16)$$

The matrix in the square brackets in the right-hand side of Eq. (3.16) is designated $[O_{n,l}^{(n,l)}(p)]_{n',l',i;n'',l'',j}$ and it is calculated with the help of Eqs. (3.8)–(3.14). The essential part of this matrix is written out below. Under the "essential part" here the part of matrix elements with $l' = l$, $(l + 1)$ and $l'' = l$, $(l + 1)$ is implied. Indeed, the linear operator $[O_{n,l}^{(n,l)}(p)]_{n',l',i;n'',l'',j}$ acts trivially for the quantities $\xi_{n,l}(p)$ with $l' \neq l$, $(l + 1)$. Therefore it is convenient to order the quantities $\xi_{n,l}(p)$ as follows:

$$\ldots, v_{n-1}(p), v_n(p), v_{n+1}(p), \ldots, \quad (3.17)$$

where $v_n(p)$ are the six-dimensional vectors of the form

$$v_n(p) = \begin{pmatrix} \alpha_{n,l}(p) \\ \beta_{n,l}(p) \\ \gamma_{n,l}(p) \\ \alpha_{n,l+1}(p) \\ \beta_{n,l+1}(p) \\ \gamma_{n,l+1}(p) \end{pmatrix} \quad (3.18)$$

In the such designations the matrixes $[O_{n,l}^{(n,l)}(p)]_{n',l''}$ take the form of block-diagonal matrices, and each block is the $6 \times 6$-matrix, the index $n''$ grows to the right and
the index $n'$ grows down.

\[
\begin{bmatrix}
\mathcal{O}^{(n,l)}(p)\end{bmatrix}_{n', n''} = \\
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
1 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & \\
0 & 0 & A(p) & B(p) & 0 & \\
0 & 0 & D & C & 0 & \\
0 & 0 & 0 & 0 & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\end{pmatrix}
\]

\begin{align*}
& n' = n, \\
& n' = n + 1
\end{align*}

(3.19)

Here $A(p)$, $B(p)$, $D$, $C$ are $6 \times 6$-matrixes the same for all matrixes $[\mathcal{O}^{(n,l)}(p)]_{n', n''}$. The matrixes $A(p)$ and $C$ are placed on main diagonal on the points $n$ and $(n + 1)$, correspondingly. According to Eqs. (3.8)-(3.14) and (3.16)

\begin{align*}
A(p) &= \\
&= \begin{pmatrix}
(cos^2 \psi) & (e^{ip} \cos^3 \psi \sin \psi) & (e^{ip} \cos^2 \psi \sin^2 \psi) & (-e^{ip} \cos \psi \sin^2 \psi) & (-e^{ip} \sin^2 \psi) & 0 \\
(-e^{-ip} \sin \psi) & (cos \psi) & 0 & 0 & 0 & 0 \\
(-e^{-ip} \cos \psi \sin \psi) & (-\cos^2 \psi \sin^2 \psi) & (cos^2 \psi) & (-\cos^2 \psi \sin \psi) & (cos \psi) & 0 \\
0 & 0 & (cos \psi \sin \psi) & (sin^2 \psi) & 0 & 0 \\
0 & 0 & (cos \psi \sin \psi) & (sin^2 \psi) & (sin \psi) & 0 \\
0 & 0 & (cos \psi \sin \psi) & (sin^2 \psi) & (cos \psi) & 0 \\
\end{pmatrix},
\end{align*}

(3.20)

\begin{align*}
B(p) &= \\
&= \begin{pmatrix}
(-e^{ip} \cos^2 \psi \sin^2 \psi) & 0 & (e^{ip} \cos \psi \sin^2 \psi) & 0 & 0 & 0 \\
-\cos^2 \psi \sin \psi & 0 & (cos \psi \sin \psi) & 0 & 0 & 0 \\
(cos \psi \sin^3 \psi) & 0 & (-\sin^3 \psi) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} = \\
&= \begin{pmatrix}
(e^{ip} \cos \psi \sin^2 \psi) & \\
(cos \psi \sin \psi) & \\
(-\sin^3 \psi) & \\
0 & \\
0 & \\
0 & \\
\end{pmatrix} \begin{pmatrix}
(-\cos \psi) & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix},
\end{align*}

(3.21)

\begin{align*}
C &= \\
&= \begin{pmatrix}
(cos \psi) & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin^2 \psi & 0 & (cos \psi) & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\end{align*}

(3.22)
\[ D = \begin{pmatrix}
0 & (\sin \psi) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & (\cos \psi \sin \psi) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}. \quad (3.23) \]

The matrix \((3.19)\) is unitary. This means that
\[ A(p) [A(p)]^\dagger + B(p) [B(p)]^\dagger = 1, \quad C C^\dagger + D D^\dagger = 1, \]
\[ A(p) D^\dagger + B(p) C^\dagger = 0. \quad (3.24) \]

The direct check shows that the matrixes \((3.20)-(3.23)\) really satisfy the Eqs. \((3.24)\).
Thus the first step (finding the matrixes \(O^{(n,l)}(p)\)) is completed.

B. The second step

To realize the second step in which the matrixes \(O^{(l)}(p)\) are found we must multiply the matrixes \(O^{(n,l)}(p)\) at fixed values of the parameters \(l\) and \(p\) according to the rule in the right-hand side of Eq. \((3.3)\). Thus
\[
\left[ \mathcal{O}^{(l)}(p) \right]_{n',n''} = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & 1 & 0 & 0 & 0 \\
\cdot & 0 & A(p) & B(p) & 0 & 0 \\
\cdot & 0 & D & C & 0 & 0 \\
\cdot & 0 & 0 & 0 & 1 & 0 \\
\cdot & 0 & 0 & 0 & 0 & 1 \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \times \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & 1 & 0 & 0 & 0 \\
\cdot & 0 & 1 & 0 & 0 & 0 \\
\cdot & 0 & 0 & 1 & 0 & 0 \\
\cdot & 0 & 0 & 0 & 1 & 0 \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \times \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & 1 & 0 & 0 & 0 \\
\cdot & 0 & 0 & 0 & A(p) & B(p) \\
\cdot & 0 & 0 & 0 & D & C \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \times \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & 1 & 0 & 0 & 0 \\
\cdot & 0 & 0 & 0 & 0 & 1 \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \quad n' = n \quad (3.25) \]
\[ \times \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & 1 & 0 & 0 & 0 \\
\cdot & 0 & 0 & 0 & A(p) & B(p) \\
\cdot & 0 & 0 & 0 & D & C \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \times \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & 1 & 0 & 0 & 0 \\
\cdot & 0 & 0 & 0 & 0 & 1 \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \quad n' = n+1 \]
From here we find by means of the direct and simple calculation

\[
\left[ O^{(l)}(p) \right]_{n, n+n'} = \begin{cases} 
0, & \text{for } n' < -1 \\
D, & \text{for } n' = -1 \\
CA(p), & \text{for } n' = 0 \\
C [B(p)]^{n'} A(p), & \text{for } n' > 0 
\end{cases}
\] (3.26)

It is seen from Eq. (3.26) that the matrix \( [O^{(l)}(p)]_{n',n'} \) depends only on the difference \((n'' - n')\), i.e. it is translation invariant (at some distance from the boundary) under the shifts along the second axis \([12]\). This fact enables to perform the further partial diagonalization of the matrix \( [O^{(l)}(p)]_{n',n''} \) by means of Fourier transformation along the second axis:

\[
\xi_l(p, q) = \sum_n \overline{\Psi}_q(n) \xi_{n,l}(p) = \xi_l^\dagger(-p, -q),
\]

\[
\left[ \xi_l^{(i)}(p, q), \xi_l^{(j)\dagger}(p', q') \right] = 2\delta_{i,j}\delta_{p,p'}\delta_{q,q'}, \quad i, j = 1, 2, 3.
\] (3.27)

Now equation (3.3) takes the form

\[
U_l^{\dagger} \xi_l^{(i)} (p, q) U_l = \sum_{j, l''} \left[ \sum_{n'} \left[ O^{(l)}(p) \right]_{n', l; n''; l, j} e^{i q n} \right] \xi_l^{(j)} (p, q).
\] (3.28)

In (3.28) we returned to the accounting of all values of indexes \( l \). Let’s introduce also the designation

\[
\sum_{n'} \left[ O^{(l)}(p) \right]_{n', l; n''; l, j} e^{i q n} \equiv \left[ O^{(l)}(p, q) \right]_{l; n''; l, j}.
\] (3.29)

Remind that the matrix (3.29) acts trivially for the elements \( \xi_l^{(j)} (p, q) \) with \( l'' \neq l, (l + 1) \). According to (3.21),

\[
[B(p)]^n = (-\kappa)^{n-1} B(p), \quad \kappa = \sin^2 \psi \left[ e^{ip} \cos^2 \psi + \sin \psi \right],
\] (3.30)

Taking into account Eqs. (3.26) and (3.30) we rewrite the matrix (3.29) in the form

\[
O^{(l)}(p, q) = \left\{ e^{-iq} D + C \left( 1 + \frac{e^{iq}}{1 + e^{iq} \kappa} B(p) \right) A(p) \right\} = \begin{pmatrix} E(p, q) & F(p, q) \\ H & G \end{pmatrix}
\] (3.31)

Here \( E(p, q), F(p, q), H, G \) are \( 3 \times 3 \) matrixes which are calculated with the help of Eqs. (3.20)-(3.23):

\[
E(p, q) = \frac{1}{1 + e^{iq} \kappa} \times
\]

\[
\begin{pmatrix} 
\cos^3 \psi \\
- \sin \psi \left[ e^{ip} \cos^2 \psi + e^{-ip} \sin \psi + e^{-ip} \cos \psi \sin \psi \sin \psi \sin \psi \right] \\
\left[ e^{ip} \cos^2 \psi \sin \psi \left[ e^{ip} \sin \psi \sin \psi \sin \psi \sin \psi - 1 \right] \\
\end{pmatrix}
\]

\[
\begin{pmatrix} 
\sin \psi \left[ e^{ip} \cos^2 \psi + e^{-ip} \sin \psi + e^{-ip} \cos \psi \sin \psi \sin \psi \sin \psi \sin \psi \right] \\
\left[ - \cos \psi \sin \psi \left[ e^{-ip} \sin \psi \sin \psi \sin \psi \sin \psi \right] \\
\left[ e^{ip} \cos \psi \cos \psi \sin \psi \left[ e^{ip} \sin \psi \sin \psi - 1 \right] \\
\end{pmatrix}
\]

\[
\begin{pmatrix} 
\cos^3 \psi \\
- \sin \psi \left[ e^{ip} \cos^2 \psi + e^{-ip} \sin \psi + e^{-ip} \cos \psi \sin \psi \sin \psi \sin \psi \sin \psi \right] \\
\left[ e^{ip} \cos^2 \psi \sin \psi \left[ e^{ip} \sin \psi \sin \psi \sin \psi \sin \psi - 1 \right] \\
\end{pmatrix}
\]

\[
\begin{pmatrix} 
\cos \psi \left[ e^{ip} \cos \psi \cos \psi \sin \psi \left[ e^{ip} \sin \psi \sin \psi \sin \psi \sin \psi \sin \psi \right] \\
\end{pmatrix}
\]
The unitarity condition of the matrix (3.31) is contained in the following equations:

\[ E(p, q) [E(p, q)]^\dagger + F(p, q) [F(p, q)]^\dagger = 1, \quad G G^\dagger + H H^\dagger = 1, \]

\[ E(p, q) H^\dagger + F(p, q) G^\dagger = 0. \]  

(3.36)

The last equalities are verified directly with the help of Eqs. (3.32)-(3.35).

Further it is necessary to return to the accounting of all values of index \( l \) and at that it is natural to order the quantities \( \xi_l(p, q) \) as follows (compare with the ordering in (3.17) and (3.18)):

\[ \ldots, w_{l-1}(p, q), w_l(p, q), w_{l+1}(p, q), \ldots, \]  

(3.37)

where \( w_l(p, q) \) are the three-dimensional vectors of the form

\[ w_l(p, q) = \begin{pmatrix} \alpha_l(p, q) \\ \beta_l(p, q) \\ \gamma_l(p, q) \end{pmatrix}. \]  

(3.38)

In this notations the matrix \([\mathcal{O}^{(l)}(p, q)]_{l', l''}\) takes the form of a block-diagonal matrix with the blocks of the size 3 × 3, and the index \( l'' \) grows to the right, the index \( l' \) grows down, so that the matrixes \( E(p, q) \) and \( G \) are placed on the main diagonal.
on the places with indexes $l$ and $(l + 1)$, correspondingly (compare with (3.19)): 

$$
[\mathcal{O}^{(l)}(p, q)]_{\nu, \nu'} = \begin{cases}
\cdots \cdots 
\cdot 1 0 0 0 0 \\
\cdot 0 1 0 0 0 \\
\cdot 0 0 E(p, q) F(p, q) 0 \\
\cdot 0 0 H G 0 \\
\cdot 0 0 0 0 1 \\
\cdots \cdots 
\end{cases}, \quad \nu = l, \quad \nu' = l+1
$$

(3.39)

The second step on finding the matrix $\mathcal{O}^{(l)}(p, q)$ is completed hereon.

C. The third step

On the third, the last step, the matrix $\mathcal{O}$ is constructed by means of the ordered product of the matrixes $\mathcal{O}^{(l)}$ according to the rule (3.4). As a result we obtain the matrix invariant relative to the translations along the third axis. The calculations are identical to that on the second step, the only difference is in the following substitutions:

$$
A(p) \rightarrow E(p, q), \quad B(p) \rightarrow F(p, q), \\
C \rightarrow G, \quad D \rightarrow H,
$$

(3.40)

so that instead of (3.26) now we have

$$
[\mathcal{O}(p, q)]_{l, l'+l} = \begin{cases}
0, & \text{for } l' < -1 \\
H, & \text{for } l' = -1 \\
G E(p, q), & \text{for } l' = 0 \\
G [F(p, q)]^{l'} E(p, q), & \text{for } l' > 0
\end{cases}.
$$

(3.41)

Using the translational invariance of the matrix (3.41) we pass to the full Fourier-components of $\xi$-matrixes:

$$
\xi(p, q, r) = \sum_l \overline{\Psi}_r(l) \xi_l(p, q) = \xi^\dagger(-p, -q, -r),
$$

$$
[\xi^{(i)}(p, q, r), \xi^{(j)}(p', q', r')]_+ = 2\delta_{i, j} \delta_{p, p'} \delta_{q, q'} \delta_{r, r'}, \quad i, j = 1, 2, 3.
$$

(3.42)

$$
\mathcal{U}^\dagger \xi(p, q, r) \mathcal{U} = \sum_{\nu'} [\mathcal{O}(p, q)]_{l, l'+l} e^{i\nu l'} \xi(p, q, r),
$$

(3.43)
\[
\sum_{l'} [\mathcal{O}(p, q)]_{l, l'+r} e^{i rl'} = \mathcal{O}(p, q, r) = \left\{ e^{-ir} H + G \left( 1 + \frac{e^{ir}}{1 + e^{ir} \sigma} F(p, q) \right) E(p, q) \right\}, \quad \sigma = \frac{(e^{ip} + e^{iq}) \cos^2 \psi \sin^2 \psi}{1 + e^{iq} \kappa}.
\]

(3.44)

Here it was used the fact that according to Eq. (3.33)
\[
[F(p, q)]^l = (-\sigma)^{l-1} F(p, q).
\]

(3.45)

The last equality is verified easily with the help of the definition (3.44) and equalities (3.36), (3.45).

Thus the problem of finding the free energy of the system is reduced to the visible problem of finding the eigenvalues of 3 \times 3-matrix \( \mathcal{O}(p, q, r) \) (3.44).

IV. THE CALCULATION OF THE EIGENVALUES OF THE MATRIX \( \mathcal{O}(p, q, r) \)

Let’s give the products of the matrixes in (3.44) in the evident form:

\[
GE(p, q) = \frac{1}{1 + e^{iq} \kappa} \times
\]

\[
\left( \begin{array}{c}
\cos^4 \psi \\
- \cos \psi \sin \psi \left[ \cos^2 \psi \sin \psi + e^{iq} \cos^2 \psi + e^{(i p + q)} \sin \psi + e^{-i q} \sin \psi \right] \\
\left( e^{-ip} \cos^2 \psi \sin \psi \left[ e^{ip} \sin \psi - 1 \right] \right)
\end{array} \right) \times \left( \begin{array}{c}
\cos \psi \sin \psi \left[ e^{ip} \cos^2 \psi + e^{-iq} + \sin^2 \psi \right] \\
\cos^4 \psi \sin \psi \left[ e^{ip} \cos^2 \psi + e^{-iq} + \sin^2 \psi \right] \\
\left( - \cos \sin \psi \left[ e^{-i \kappa} + \sin \psi \right] \right)
\end{array} \right), \quad (4.1)
\]

\[
GF(p, q) E(p, q) = \frac{1}{(1 + e^{iq} \kappa)^2} \left( \begin{array}{c}
\left( - e^{ip} \cos^2 \psi \sin^2 \psi \left[ e^{iq} \sin \psi + 1 \right] \right) \\
\cos \psi \sin^2 \psi \left[ e^{(i p + q)} \sin \psi + e^{ip} \sin^2 \psi - e^{iq} \cos^2 \psi \right] \\
\left( - \sin \psi \left[ \cos^2 \psi + e^{ip} \sin^2 \psi \left[ e^{iq} + \sin \psi \right] \right] \right)
\end{array} \right) \times \left( u_1, u_2, u_3 \right),
\]

(4.2)
Note that all columns of the matrix (4.2) are proportional to the third column of the matrix (4.1).

For completeness let’s give also the following formulae:

\[
\mathcal{O}(p, q, r) = \left\{ e^{-ir}H + G\left(1 + \frac{e^{ir}}{1 + e^{ir}\sigma}F(p, q)\right)E(p, q) \right\},
\]

\[
\sigma = \frac{(e^{ip} + e^{iq}) \cos^2 \psi \sin^2 \psi}{1 + e^{iq}\chi}, \quad \chi = \sin^2 \psi \left[e^{ip} \cos^2 \psi + \sin \psi\right],
\]

\[
\chi \equiv \left(1 + e^{iq}\chi\right)\left(1 + e^{ir}\sigma\right) = 1 + e^{iq}\sin^3 \psi + \left[e^{i(p+q)} + e^{i(p+r)} + e^{i(q+r)}\right] \cos^2 \psi \sin^2 \psi,
\]

\[
H = \begin{pmatrix} 0 & 0 & (\sin \psi) \\ 0 & 0 & (\cos \psi \sin \psi) \\ 0 & 0 & 0 \end{pmatrix}.
\]

We must solve the cubic equation

\[
\det \|\mathcal{O} - \rho E\| = 0 \quad (4.5)
\]

relative to the variable \(\rho\). This equation is rewritten as

\[
\rho^3 - (\text{tr} \mathcal{O})\rho^2 + (m_{11} + m_{22} + m_{33}) \rho - \det \mathcal{O} = 0. \quad (4.6)
\]

Here \(m_{ij}\) is the minor of the matrix element \(\mathcal{O}_{ij}\).

With the help of Eqs. (4.1)-(4.4) we find:

\[
(\text{tr} \mathcal{O}) = 1 - \frac{\eta}{\chi}, \quad \eta \equiv (1 - 3 \cos^4 \psi + \sin^6 \psi) + \sin^3 \psi \left(e^{iq} + e^{-iq}\right) = \overline{\eta}. \quad (4.7)
\]

Since the coefficients in Eq. (4.6) are the invariants which do not depend on the matrix \(\mathcal{O}\) representation, we have:

\[
(m_{11} + m_{22} + m_{33}) = (\rho_1 \rho_2 + \rho_1 \rho_3 + \rho_2 \rho_3) =
\]

\[
= (\rho_1^{-1} + \rho_2^{-1} + \rho_3^{-1}) \rho_1 \rho_2 \rho_3 = (\text{tr} \mathcal{O}^\dagger) \det \mathcal{O} = (\overline{\text{tr} \mathcal{O}}) \det \mathcal{O}. \quad (4.8)
\]

Here it was taken into account that \(\mathcal{O}^\dagger \mathcal{O} = 1\). Thus it remains to calculate the quantity \(\det \mathcal{O}\) only.

Let \(v_i, i = 1, 2, 3\) be the vector-columns, so that

\[
GE = \frac{1}{1 + e^{iq}\chi} (v_1, v_2, v_3). \quad (4.9)
\]
Then the matrix $\mathcal{O}$ is represented as follows

$$
\mathcal{O} = \left( \begin{array}{ccc}
v_1 & v_2 & w_3
\end{array} \right) - \frac{e^{ir} \sin \psi}{\cos^2 \psi (1 + e^{iq} \chi)} v_3 \times (u_1, u_2, 0),
$$

$$
w_3 = \frac{1}{\chi} v_3 + \tau, \quad \tau \equiv e^{-ir} \sin \psi \left( \begin{array}{c} 1 \\ \cos \psi \\ 0 \end{array} \right), \tag{4.10}
$$

and $(u_1, u_2, u_3)$ is the row matrix in (4.2).

Let’s add to the first two columns of the matrix $\mathcal{O}$ such columns, proportional to $w_3$, which cancel the second summand in Eq. (4.10). Thus the matrix $\mathcal{O}'$ is obtained with the same determinant:

$$
\Delta \mathcal{O} = \frac{e^{ir} \sin \psi}{\cos^2 \psi (1 + e^{iq} \chi)} w_3 \times (u_1, u_2, 0),
$$

$$
\mathcal{O}' \equiv \mathcal{O} + \Delta \mathcal{O} = \left( \begin{array}{ccc}
v_1 & v_2 & w_3
\end{array} \right) +
$$

$$
+ \frac{\sin^2 \psi}{\cos^2 \psi (1 + e^{iq} \chi)} \left( \begin{array}{c} 1 \\ \cos \psi \\ 0 \end{array} \right) \times (u_1, u_2, 0). \tag{4.11}
$$

Now let’s subtract the first row of the matrix $\mathcal{O}'$ multiplied by $(\cos \psi)$ from the second row of this matrix. As a result we obtain the matrix $\mathcal{O}''$ the determinant of which coincides with the determinant of matrix $\mathcal{O}$:

$$
\mathcal{O}'' = \left( \begin{array}{ccc}
\cos^4 \psi - e^{iq} \cos^2 \psi \sin^3 \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
-\cos \psi \cos^4 \psi + e^{iq} \cos^2 \psi \sin \psi \sin \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^2 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^4 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^2 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^4 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^2 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^4 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^2 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi \\
\cos^4 \psi \sin \psi \sin \psi \cos \psi & -e^{ip} \sin \psi \sin \psi \cos \psi & e^{ir} \sin \psi
\end{array} \right), \tag{4.12}
$$

$$
det \mathcal{O}'' = det \mathcal{O}.
Let $m''_{ij}$ be the minor of the matrix element $O''_{ij}$. We need the following minors:

\[
\begin{align*}
m''_{13} &= \cos^2 \psi \sin \psi \left( e^{-ip} + e^{-iq} \right), \\
m''_{23} &= \frac{e^{-ip} \sin^3 \psi - e^{-iq} \cos^2 \psi \sin \psi + e^{-i(p+q)} \sin^2 \psi}{\cos \psi (1 + e^{iq} \kappa)}, \\
m''_{33} &= \frac{\cos^2 \psi + e^{-ip} \sin^3 \psi + e^{-i(p+q)} \sin^2 \psi}{\cos^2 \psi (1 + e^{iq} \kappa)}. \\
\end{align*}
\]

(4.13)

To compute the determinant let us decompose $\det O''$ in the last column of the matrix $O''$. Thus we obtain:

\[
\det O = \frac{\chi}{\chi}. 
\]

(4.14)

With the help of formulae (4.7) and (4.8) we obtain:

\[
(m_{11} + m_{22} + m_{33}) = \frac{\chi - \eta}{\chi}. 
\]

(4.15)

As a result the equation (4.6) takes the form:

\[
\rho^3 - \left( 1 - \frac{\eta}{\chi} \right) \rho^2 + \left( \frac{\chi}{\chi} - \frac{\eta}{\chi} \right) \rho - \frac{\chi}{\chi} = (\rho - 1) \left( \rho^2 + \frac{\eta}{\chi} \rho + \frac{\chi}{\chi} \right) = 0. 
\]

(4.16)

Now we find easily all eigenvalues of the matrix $O$:

\[
\rho_{1,2}(p, q, r) = -\eta \pm i \sqrt{4\chi \chi - \eta^2} \quad \frac{2\chi}{2\chi}, \quad \rho_3 = 1. 
\]

(4.17)

Note that the expression $4\chi \chi - \eta^2$ is a non-negative number. Only under this condition we have

\[
|\rho_{1,2,3}(p, q, r)| = 1.
\]

The last equalities follow also from the unitarity condition of the matrix $O(p, q, r)$. Otherwise it would be

\[
|\rho_{1,2}(p, q, r)| \neq 1.
\]

The last inequalities mean the violation of the matrix $O(p, q, r)$ unitarity.
V. THE FREE ENERGY

According to (A4) and (4.17)

\[ \text{tr} \mathcal{U} = 2^{MNL/2} \prod_{p,q,r} \left( \sqrt{\rho_1} + \sqrt{\rho_1} \right) \left( \sqrt{\rho_2} + \sqrt{\rho_2} \right) = 2^{MNL/2} \prod_{p,q,r} \frac{\chi + \chi - \eta}{\sqrt{\chi}} = \]

\[ = 2^{MNL/2} \prod_{p,q,r} \left( 1 + 3 \cos^4 \psi - \sin^6 \psi \right) + 2 \nu(p, q, r) \cos^2 \psi \sin^2 \psi, \]

\[ \nu(p, q, r) \equiv [\cos(p + q) + \cos(p + r) + \cos(q + r)]. \quad (5.1) \]

The free energy is of interest. Up to inessential summand the free energy has the form

\[ \mathcal{F} = -T \ln Z = T \frac{MNL}{16\pi^3} \int_{-\pi}^{\pi} d p \int_{-\pi}^{\pi} d q \int_{-\pi}^{\pi} d r \left( \ln \chi + \ln \bar{\chi} \right) - \]

\[ -T \frac{MNL}{8\pi^3} \int_{-\pi}^{\pi} d p \int_{-\pi}^{\pi} d q \int_{-\pi}^{\pi} d r \ln \left\{ (1 + 3 \cos^4 \psi - \sin^6 \psi) + 2 \nu \cos^2 \psi \sin^2 \psi \right\}. \quad (5.2) \]

Here it was taken into account that at \( M, N, L \to \infty \) the substitution

\[ \sum_{p, q, r} \to \frac{MNL}{8\pi^3} \int_{-\pi}^{\pi} d p \int_{-\pi}^{\pi} d q \int_{-\pi}^{\pi} d r. \]

is valid.

The first summand in the right-hand side of Eq. (5.2) is equal to zero. This fact is the consequence of the slack inequalities

\[ |\sigma| \leq 1, \quad |\kappa| \leq 1, \quad (5.3) \]

so that the equalities in (5.3) take place only on the subset of the zeroth measure in the space of the variables \( \{p, q, r\} \). Indeed, according to (4.3) and (5.3) we have
the following chain of equalities:

\[
\int_{-\pi}^{\pi} d\rho \int_{-\pi}^{\pi} d\eta \int_{-\pi}^{\pi} d\xi \ln \chi = \\
= \int_{-\pi}^{\pi} d\rho \int_{-\pi}^{\pi} d\eta \int_{-\pi}^{\pi} d\xi \left\{ \ln (1 + e^{i\eta \lambda}) + \ln (1 + e^{i\xi \sigma}) \right\} = \\
= \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n} \int_{-\pi}^{\pi} d\rho \int_{-\pi}^{\pi} d\eta \int_{-\pi}^{\pi} d\xi \left\{ e^{in\eta \lambda[\lambda,\rho]} n + e^{in\xi [\lambda(\psi, \rho)] n} \right\} = 0.
\]

(5.4)

Prove the slack inequalities (5.3).

According to (3.45) \( F^2 = -\sigma F \). Let’s take the modulo of the matrix element (1, 1) of the last equality:

\[
|\sigma||_{F_{11}} = \left| \sum_i F_{1i} F_{i1} \right| \leq \sqrt{\left( \sum_i F_{1i} \overline{F}_{i1} \right) \left( \sum_j \overline{F}_{j1} F_{j1} \right)} = \\
= \sqrt{(F^\dagger F)_{11} (F F^\dagger)_{11}}.
\]

(5.5)

We have also from the unitarity condition (3.36) and the evident form of the matrix \( G \) (3.34)

\[
(F^\dagger F)_{11} = 1 - (G^\dagger G)_{11} = \cos^2 \psi \sin^2 \psi.
\]

(5.6)

By means of the direct calculation we find also that

\[
(F^\dagger F)_{11} = \frac{1 + \cos^2 \psi}{\cos^2 \psi} |F_{11}|^2.
\]

(5.7)

Combining all formulae (5.5)-(5.7) we obtain:

\[
|\sigma| \leq \sqrt{\sin^2 \psi (1 \cos^2 \psi) \leq 1}
\]

Using the evident expression (4.3) for \( \lambda \) we find:

\[
|\lambda| \leq \cos^2 \psi \sin^2 \psi + \sin^3 \psi \leq 1 \quad \text{for} \quad 0 \leq \psi \leq \pi.
\]

Thus, both inequalities (5.3) are verified, and therefore

\[
\mathcal{F} = -T \frac{M_{NL}}{8 \pi^3} \int_{-\pi}^{\pi} d\rho \int_{-\pi}^{\pi} d\eta \int_{-\pi}^{\pi} d\xi \times \\
\times \ln \left\{ (1 - \sin^2 \psi) \left[ \sin^4 \psi + 2(\nu - 1) \sin^2 \psi + 4 \right] \right\}.
\]

(5.8)
It is seen from the expression for free energy (5.8) that the earlier separation of the second axis in the previous formulae for the intermediate quantities, including the eigenvalues of the matrix $\mathcal{O}(p, q, r)$, was an artifact: the free energy of the system, which presents the physical interest, is completely symmetric relative to mutual substitutions of the all three axes.

From the definition of the parameter $\psi$ (see the second point) it is evident that this parameter plays a part of the temperature $T$. At $T \to \infty$ the angle $\psi \to 0$. This follows from the fact that at $T \to \infty$ the contribution of surfaces into the partition function (2.13) tends to zero. That corresponds to the high-temperature limit in the gauge compact lattice theories. It is also seen from the comparison of the partition function (2.13) with the corresponding quantity of the lattice gauge theory (with the gauge group $Z_2$) that the decrease of the temperature from the infinity up to zero means the surgeless increase of the angular parameter $\psi$ from zero up to $(\pi/2)$:

$$
\frac{d\psi(T)}{dT} < 0, \quad \psi(0) = \frac{\pi}{2}, \quad \psi(\infty) = 0.
$$

The answer for the important question is need: do the phase transitions exist at the temperature reduction from the infinity up to zero, and what is its kind?

Since the phase transition point coincides with the peculiarity of a free energy by the temperature, so one needs look for that values of $\psi$ at which the expression (5.8) can have a peculiarity. Just as in the case of the two-dimensional Ising model, the only possibility to have a peculiarity in the free energy consists in nullification of argument of the logarithm situated under the sign of the integral over quasi-momentum. This can occur at some values of the temperature and quasi-momenta. Then the integral over quasi-momenta near the peculiarity gives irregular contribution into free energy and other thermodynamic quantities. In the case of the two-dimensional Ising model there is the only point in the three-dimensional manifold of the totality variables ”two quasi-momenta+temperature” in which the logarithm argument vanish: $p = q = 0$ and $T = T_c$.

In the considered three-dimensional theory the situation differs qualitatively from the indicated one in two-dimensions: for $\psi = 0$ ($T = \infty$) the logarithm argument in (5.8) is positive always; for $p = q = r = 0$ the logarithm argument in (5.8) vanish only for $\psi = \pi/2$ ($T = 0$). But there is a possibility of the logarithm argument nullification in (5.8) at finite temperatures $0 < \psi < \pi/2$ and nonzero quasi-momenta. Indeed, at the quasi-momentum variations the parameter $\nu$ varies in the following range:

$$
-3 \leq \nu \leq 3.
$$
To zero the logarithm argument in (5.8), it is necessary realization of the condition
\[ x_c^2 + 2(\nu - 1)x_c + 4 = 0, \quad x \equiv \sin^2 \psi. \tag{5.11} \]

Only the real solutions on the segment \(0 \leq x_c \leq 1\) are of interest. This is possible only under the condition
\[ -3 \leq \nu \leq -\frac{3}{2}. \tag{5.12} \]

The only solution has the form
\[ x_c(\nu) = (1 - \nu) - \sqrt{(1 - \nu)^2 - 4}, \tag{5.13} \]

at that \(\sqrt{x_c(\nu)}\) is a monotonic increasing function on the segment (5.12), so that
\[ \sqrt{x_c(-3)} = \sin \psi_c = \sqrt{4 - \sqrt{12}} < 1 \tag{5.14} \]
is its minimum value, and
\[ \sqrt{x_c(-3/2)} = \sin \psi_c = 1 \tag{5.15} \]
is the maximum possible value.

The point \(\nu = -3\) is the evolved point in the space of quasi-momenta since it is obtained for two values of quasi-momenta \(p, q, r\) only. Indeed, for that the quasi-momenta must satisfy at least one of the following eight equations (all signs ”plus” and ”minus” in the right hand sides of Eqs. (5.16) are mutually independent):
\[ p + q = \pm \pi, \quad p + r = \pm \pi, \quad q + r = \pm \pi. \tag{5.16} \]

Since the system of linear equations (5.16) is non-degenerate, so for each value of the right hand side in (5.16) there is only one solution for quasi-momenta \(p, q, r\). Taking into account the fact that the quasi-momenta are defined modulo \(2\pi\) we can reduce all eight solutions of the systems (5.16) to two independent solutions:
\[ p_c^{(1)}(1) = q_c^{(1)} = r_c^{(1)} = \frac{\pi}{2}, \]
\[ p_c^{(2)}(2) = q_c^{(2)} = r_c^{(2)} = -\frac{\pi}{2}. \tag{5.17} \]

On the contrary, for all values of \(\nu\) from the half-interval
\[ -3 < \nu(p, q, r) \leq -\frac{3}{2} \tag{5.18} \]
there is a whole continuum of the quasi-momenta values for which the logarithm argument in (5.8) vanish.

The further study of the free energy properties and the properties of a phase transition in the considered model will be performed in an another work.
VI. CONCLUSION

In the present work the calculation method of the partition function of the gauge system with the gauge group $\mathbb{Z}_2$ coupled with Majorana field on the three-dimensional cubic lattice is suggested. Actually the sum over closed surfaces and the surfaces with boundary is computed. The surfaces can be with self-intersections and their weights are proportional to the factors $\mu^{2S}$, where $S$ is the number of the faces of the surface. The weights of the closed surfaces are positive always. The total sign of the surface with boundary depends essentially on the boundary configuration but not on the surface form with the given boundary. This sum is computed completely, it is represented as a threefold integral over quasi-momenta and it is the function of one parameter (the coupling constant or the temperature).

It should be noted that there is little examples of the integrable and ”rational” (that is having physical interpretation) statistical systems in three dimensions, showing a phase transition relative to the temperature. Thereupon I want pay attention to the work of A.B. Zamolodchikow [7] in which the Yang–Baxter triangle equation has been generalized to the three dimensional case where the corresponding equation is named as tetrahedron equation.

APPENDIX A

It is known [3] that the rotation matrix in spinor representation (2.14) can be expressed in the form

$$U = \exp \left( \frac{1}{4} \omega_{x,y} \gamma_x \gamma_y \right), \quad \omega_{x,y} = -\omega_{y,x},$$

(A1)

and

$$U^\dagger \gamma_x U = \Omega_{x,y} \gamma_y, \quad \Omega_{x,y} \equiv (e^{\omega})_{x,y} = \delta_{x,y} + \omega_{x,y} + \frac{1}{2!} \omega_{x,z} \omega_{z,y} + \ldots.$$  

(A2)

The trace of the matrix $U$ is expressed simply through the eigenvalues of real orthogonal matrix $\Omega_{x,y}$. Let the set of numbers

$$\left( \rho_1, \overline{\rho}_1, \rho_2, \overline{\rho}_2, \ldots, \rho_{3MNL/2}, \overline{\rho}_{3MNL/2} \right) = \{ \rho_k, \overline{\rho}_k \}, \quad k = 1, \ldots, 3MNL/2$$  

(A3)

form the complete set of eigenvalues of the matrix $\Omega_{x,y}$. Then (see Appendix A)

$$\text{tr} U = \prod_{k=1}^{3MNL/2} \left[ 2 \text{ch} \left( \frac{\ln \rho_k}{2} \right) \right] = \prod_{k=1}^{3MNL/2} \left[ 2 \cos \left( \frac{\phi_k}{2} \right) \right] = \prod_{k=1}^{3MNL/2} \left( \sqrt{\rho_k} + \sqrt{\overline{\rho}_k} \right),$$

$$\rho_k = e^{i\phi_k}.$$  

(A4)
Let us prove the formula (A4).

Let \( \{v_x^{(k)}, v_x^{(k)}\} \), \( k = 1, \ldots, 3MNL/2 \), be the complete orthonormal set of eigenvectors of the matrix \( \mathcal{O}_{x,y} \), so that the eigenvalue \( \rho_k \) \( (\overline{\rho}_k) \) corresponds to the eigenvector \( v_x^{(k)} \) \( (\overline{v}_x^{(k)}) \). Further also the designation
\[
\{v_x^{(k)}, v_x^{(k)}\} \equiv \{v_x^a\}, \quad a = 1, \ldots, 3MNL
\]
is used. We shall consider the introduced vectors as vector-columns and the upper indices \( T \) and \( \dagger \) denote the transposition and Hermitian conjugation of vectors and matrices. By definition
\[
v^{(k)} v^{(k')} = 0, \quad v^{(k)} \dagger v^{(k')} = \delta_{kk'},
\]
(\( A6 \))

The given definitions imply the following formulas:
\[
S_{xa} \equiv v_x^a \text{ or } S \equiv \left( v^{(1)}, \overline{v}^{(1)}, v^{(2)}, \overline{v}^{(2)}, \ldots \right), \quad (S^\dagger S)_{ab} = \delta_{ab},
\]
(\( A7 \))

\[
(S^\dagger \mathcal{O} S)_{ab} = \text{diag}(\rho_1, \overline{\rho}_1, \rho_2, \overline{\rho}_2, \ldots) \equiv D_{ab}.
\]
(\( A8 \))

It is shown in \( [3] \) that
\[
(S^\dagger \omega S)_{ab} = \text{diag}(\ln \rho_1, -\ln \rho_1, \ln \rho_2, -\ln \rho_2, \ldots) \equiv \Delta_{ab}.
\]
(\( A9 \))

Due to (\( A7 \)) and (\( A9 \)) we have
\[
\sum_{x, y} \frac{1}{4} \xi_x \omega_{x,y} \xi_y = \sum_{x,y} \sum_{a,b} \frac{1}{4} (\xi_x S_{xa}) \Delta_{ab} \left( S_{by}^\dagger \xi_y \right).
\]
(\( A10 \))

\( 2^{3MNL/2} \times 2^{3MNL/2} \)-matriizes
\[
c_{k}^\dagger \equiv \gamma_x v_x^{(k)}, \quad c_k \equiv \overline{v}_x^{(k)}
\]
(\( A11 \))

possess all properties of fermion creation and annihilation operators. Indeed, in consequence of (\( 2.1 \)) and (\( A6 \))
\[
[c_k, c_{k'}^\dagger]_+ = \delta_{kk'}, \quad [c_k, c_k^\dagger]_+ = [c_k^\dagger, c_k^\dagger]_+ = 0.
\]
(\( A12 \))

According to the definitions (\( A7 \)) and (\( A11 \)) we have
\[
\sum_{x} \xi_x S_{xa} = \left( c_1^\dagger, c_1, \ldots, c_{3MNL/2}^\dagger, c_{3MNL/2} \right).
\]
(\( A13 \))
With the help of Eqs. (A9), (A12) and (A13) the quantity (A10) is rewritten as

\[
\sum_{x,y} \frac{1}{4} \xi_x \omega_{x,y} \xi_y = \frac{1}{2} \sum_{k=1}^{3MNL/2} \left[ \ln \rho_k \left( c_k^\dagger c_k - c_k c_k^\dagger \right) \right] = \sum_{k=1}^{3MNL/2} \left[ \left( \ln \rho_k \right) c_k^\dagger c_k - \frac{1}{2} \ln \rho_k \right].
\]

(A14)

Equality (A4) follows immediately from (A14) since the calculation of the trace in terms of \(\xi\)-matrixes is equivalent to the calculation of trace in terms of the corresponding fermionic operators (A11).

**APPENDIX B**

Let us consider the Majorana spinors on the three-dimensional cubic lattice and their contribution into the partition function.

In the simplest case the action of the Dirac fermions on the cubic lattice has the form

\[
S_D = \frac{i}{2} \sum_x \sum_{i=1}^{3} \overline{\psi}_x \gamma^i \left( U_{x,e_i} \psi_{x+e_i} - U_{x-e_i,e_i}^\dagger \psi_{x-e_i} \right),
\]

\[
\gamma^1 = \sigma_x, \quad \gamma^2 = \sigma_y, \quad \gamma^3 = \sigma_z, \quad U_{x,e_i}^\dagger U_{x,e_i} = 1.
\]

(B1)

Here \(\sigma_i\) are the Pauli matrixes and \(U_{x,e_i}\) is the gauge field. The Fermi-fields \(\psi_x\) and \(\overline{\psi}_x\) are the elements of the Grassman algebra and all their elements are considered as a mutually independent variables. The Fermi contribution to the partition function is defined as the integral

\[
Z_D\{U\} = \prod_x \int d\overline{\psi}_x d\psi_x \exp S_D,
\]

(B2)

where \(d\overline{\psi}_x\) and \(d\psi_x\) denote the products of the differentials of the both components of the corresponding spinors.

The Majorana spinors are determined by the following system of identifications:

\[
\overline{\psi}_x = -\psi_x^T \gamma^2.
\]

(B3)

The ”electrical current” of the Majorana spinors is equal to zero identically:

\[
J^i_x = \overline{\psi}_x \gamma^i \psi_x =
\]

\[
= \begin{pmatrix} \psi_{x1}, \psi_{x2} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \psi_{x1} \\ \psi_{x2} \end{pmatrix} =
\]

\[
= \{ i(\psi_{x1}^2 - \psi_{x2}^2), -(\psi_{x1}^2 + \psi_{x2}^2), -i(\psi_{x1}\psi_{x2} + \psi_{x2}\psi_{x1}) \} \equiv \{ 0, 0, 0 \}.
\]

(B4)
The last identity in (B4) follows from the fact that all components of the field $\psi$ are odd elements of the Grassman algebra. The majorana action is obtained from the Dirac action (B1) by means of the substitutions $\bar{\psi}_x \rightarrow -\psi_x^T \gamma^2$ and division by 2:

$$S_M = -\frac{i}{4} \sum_x \sum_{i=1}^3 \psi_x^T \gamma^2 \gamma^i \left( U_{\mathbf{x}, \mathbf{e}_i} \psi_{\mathbf{x}+\mathbf{e}_i} - U_{\mathbf{x}-\mathbf{e}_i, \mathbf{e}_i}^\dagger \psi_{\mathbf{x}-\mathbf{e}_i} \right). \quad (B5)$$

The contribution of Majorana spinors to the partition function is determined by the Grassman integral

$$Z_M \{ \mathbf{U} \} = \prod_x \int d\psi_{x2} d\psi_{x1} \exp S_M. \quad (B6)$$

Since

$$\int d\psi_{x2} d\psi_{x1} (\psi_{x1} \psi_{x2})^n = \begin{cases} 1, \text{ for } n = 1 \\ 0, \text{ for } n \neq 1 \end{cases},$$

so

$$\int d\psi_x = 0,$$

$$-i \int d\psi_x \cdot (\psi_x \psi_x^T) = -i \int d\psi_{x2} d\psi_{x1} \begin{pmatrix} 0 & \psi_{x1} \psi_{x2} \\ \psi_{x2} \psi_{x1} & 0 \end{pmatrix} = \gamma^2,$$

$$\int d\psi_x \cdot (\psi_x \psi_x^T) \otimes (\psi_x \psi_x^T) = 0, \quad (B7)$$

and so on.

Draw on the lattice the system of closed broken contours without intersections and self-intersections. The elementary link of each contour is an edge $l_{\mathbf{x}, i}$ (see Section 2.2) and each vertex belong to one and only one contour. Each contour is oriented, that is the direction (arrow) is assigned to each edge of the contour, so that the continuous movement along the arrows reduce to the whole round of the contour. We shall call the edge as positive (negative) oriented if its arrow is directed in the line of the positive (negative) direction of the corresponding lattice axis. Below the particular case of the elementary closed contour based on on only one edge with both orientations at once is described.

The nonzero contribution in the integral (B6) give only those summands in the exponent expansion in the quantities

$$\left(-\frac{i}{4} \psi_x^T \gamma^2 \gamma^i U_{\mathbf{x}, \mathbf{e}_i} \psi_{\mathbf{x}+\mathbf{e}_i} \right) \quad (B8)$$
and
\[
\left( \frac{i}{4} \psi^T \gamma^2 \gamma^i U_{x-e_i, e_i} \psi_{x-e_i} \right), \tag{B9}
\]
in which these quantities are in the first power. The factor (B8) corresponds to each positively oriented edge and the factor (B9) corresponds to each negatively oriented edge [13]. The both factors (B8) and (B9) correspond to the elementary closed contour based on the edge $l_{x,i}$. Thereby, under the sign of the integral in each vertex $x$ there is the element of the Grassman algebra $(\psi_x \psi^T_x)$ in the first power giving a nonzero factor according to (B7). Note that the linkage of this rule with the contours orientation in the Majorana case is due to the interaction between fermions and gauge field. Indeed, the calculation of the integral over the gauge field in the frame of the high temperature expansion leads to the nullification of all contributions from the non-oriented closed contours (in the case of the higher gauge symmetry than $Z_2$) which do not vanish as a result of the fermion integration (B6).

Here the specific question is interesting for us. Therefore further we put $U_{x,e_i} = 1$.

From the aforesaid and with the help of Eqs. (B5), (B7), (B8) and (B9) the following rules follow for the calculation of the integral (B6):

1) Let’s draw on the lattice the system of the outlined above closed broken oriented lines and, rounding each contour successively along the arrows, relate successively to each edge $l_{x,i}$ the factor $(1/4) \gamma^i$ in the case of the positive orientation of the edge and the factor $(-1/4) \gamma^i$ in the case of the negative orientation of the edge.

2) After ending the rounding process of each contour let’s calculate the trace of the ordered product of the $\gamma$-matrixes corresponding to the contour according to the rule 1, and add the factor $(-1)$, common for the whole contour. The obtained number is called as the factor of the contour.

3) Let’s multiply the factors of all contours. The obtained number is called as the factor of the system of contours.

4) For finding the integral (B6) it is necessary to summarize the factors of all possible systems of contour.

Now let’s show the change of the sign of the contour system factors on the simplest examples. Since only the sign is interesting for us, the others positive factors are ignored.

Let’s consider the factor of the elementary closed contour based on the edge $l_{x,i}$. This contour is represented in the Fig. 4.
According to the given rules the factor of this contour is
\[ \Phi_{x, i} = (-1) \text{tr} \gamma^i (\gamma^i) = 1. \]
(B10)

One of the possible and at the same time simplest system of contours is represented in the Fig. 5. In Fig. 5 one of the mutually parallel planes containing the base vector \( e_i \) is represented. Thus, all these planes contain the identical configurations of elementary closed contours and each vertex of the lattice belong to one and only one contour. It is evident that due to (B10) the factor corresponding to this system of contours is equal to the product of units and thus it is equal to the unity.

It is supposed that in the subsequent examples of the closed contour systems almost all closed contours are elementary, so that the change of the number of the elementary contours does not affect on the total sign of the contour system. Therefore only that closed contours will be considered and represented in the figures which can effect on the total sign of the contour system factor.

Let’s consider the factor corresponding to the closed contour in the plane \((1, 2)\) which is represented in the Fig. 6.
The factor corresponding to the contour in the Fig. 6 is
\[ \Phi_{x, e_1, e_2} = (-1) \text{tr} \gamma^1 \gamma^2 (-\gamma^1) (-\gamma^2) = 1. \] (B11)

Therefore the total sign of the contour system factor corresponding to the Fig. 6 is also equal to unity.

Now let's consider the factor corresponding to the closed contour in the plane \((1, 2)\) and represented in Fig. 7.

\[ \Phi_{x, 2e_1, 2e_2} = (-1) \text{tr} \gamma^1 \gamma^1 \gamma^2 (-\gamma^1) (-\gamma^1) (-\gamma^2) (-\gamma^2) = -1. \] (B12)

From here it is seen that the total sign of the contour system factor corresponding to the Fig. 7 is negative.

Thus we see that the contour system factor, the sum of which defines the integral \((B6)\), can have either positive or negative sign. It is important that the change of the contour configurations leads, generally speaking, to the change of the factor sign.

In conclusion note that the fermion Dirac contribution into the partition function \((B2)\) is expressed also as a sum of the contour system factor. But in the Dirac case
the sign of the factor does not depend on the contour configurations. It is easy to see that in the long-wave continuous limit in a weak gauge field. In this limit the lattice action (B13) transforms into the usual Dirac action
\[ S_D = \int d^3 x \bar{\psi} \left( i \gamma^i \partial_i - e \gamma^i A_i \right) \psi. \] (B13)

The corresponding contribution into the partition function can be written in the form
\[ \det \left( i \gamma^i \partial_i - e \gamma^i A_i \right) = \text{Const} \cdot \exp \{ \text{tr} \ln \left[ 1 - e (i \gamma^i \partial_i)^{-1} \gamma^j A_j \right] \} = \]
\[ = \text{Const} \cdot \exp \left\{ -e^2 \int d^3 x \, d^3 y \, \text{tr} (i \gamma^k \partial_k)^{-1}_{x,y} \gamma^i A_i(y)(i \gamma^k \partial_k)^{-1}_{y,x} \gamma^j A_j(x) - \ldots \right\}. \] (B14)

In the last expansion in the exponent under integrals the enough smooth function \((i \gamma^k \partial_k)^{-1}_{x,y}\) is present. Since the space integrations mean the contour variations, so it is seen that the contour variations is not conjugated with the sign variation of the corresponding factor. The variations of the contour factor signs of the lattice Majorana fermions mean that in the continuous long-wave limit the contributions into the partition function from these contours are cancelled mutually. From here the impossibility of Majorana fermions-Abelian gauge field interaction in the case of continuous theory is seen. This state also follows directly from (B4).

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[8] The self-intersections are admissible only orthogonally.
[9] Possibly, it would be rather to denominate the calculated partition function as the quantum transition amplitude from vacuum to vacuum. However, I leave here for the value the term “partition function” since the signature of the used metrics is Euclidean.
[10] These arrows are the same as in fig. 1.
[11] For example, the specification of the diagram technique rules defines the elementary transition amplitudes and the rules of their superposition and construction of any amplitudes from the elementary amplitudes. Thus the knowledge of the diagram technique rules free from the necessity of the action knowledge and the study of functional integral leading to these rules.
[12] This translational invariance follows already from the form of Eq. (3.3).
[13] This statement is wrong in the case of the gauge group $Z_2$, but in the Appendix B the others gauge group are meant.