C*-algebras of Toeplitz type associated with algebraic number fields

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Abstract We associate with the ring $R$ of algebraic integers in a number field a C*-algebra $\mathcal{T}[R]$. It is an extension of the ring C*-algebra $\mathfrak{A}[R]$ studied previously by the first named author in collaboration with X. Li. In contrast to $\mathfrak{A}[R]$, it is functorial under homomorphisms of rings. It can also be defined using the left regular representation of the $ax + b$-semigroup $R \rtimes R^\times$ on $\ell^2(R \rtimes R^\times)$. The algebra $\mathcal{T}[R]$ carries a natural one-parameter automorphism group $(\sigma_t)_{t \in \mathbb{R}}$. We determine its KMS-structure.

The technical difficulties that we encounter are due to the presence of the class group in the case where $R$ is not a principal ideal domain. In that case, for a fixed large inverse temperature, the simplex of KMS-states splits over the class group. The “partition functions” are partial Dedekind $\zeta$-functions. We prove a result characterizing the asymptotic behavior of quotients of such partial $\zeta$-functions, which we then use to show uniqueness of the $\beta$-KMS state for each inverse temperature $\beta \in (1, 2]$.

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1 Introduction

Let \( R \) be the ring of algebraic integers in a number field \( K \), let \( R^\times = R \setminus \{0\} \) be its multiplicative semigroup and \( R \rtimes R^\times \) its \( ax + b \)-semigroup. In the present paper we study the C*-algebra generated by the left regular representation of the semigroup \( R \rtimes R^\times \) on the Hilbert space \( \ell^2(R \rtimes R^\times) \), and its KMS-structure for a natural one-parameter automorphism group. In the first part of the paper we analyze the structural properties of the C*-algebra. We show that it can be described as a universal C*-algebra defined by generators and relations. Since the left regular C*-algebra of a semigroup is often called its Toeplitz algebra we denote this universal algebra by \( \mathfrak{A}[R] \). The relations are closely related to those characterizing the ring C*-algebra \( A \). This corresponds to the fact that \( \mathfrak{A}[R] \) is generated by the natural representation of \( R \rtimes R^\times \) on \( \ell^2(R) \) rather than on \( \ell^2(R \rtimes R^\times) \). Recall that the generators for \( \mathfrak{A}[R] \) are unitaries \( u^x \), \( x \in R \) and isometries \( s_a \), \( a \in R^\times \) satisfying the following relations:

(a) The \( u^x \) and the \( s_a \) define representations of the additive group \( R \) and of the multiplicative semigroup \( R^\times \), respectively, (i.e. \( u^x u^y = u^{x+y} \) and \( s_a s_b = s_{ab} \)) and moreover we require the relation \( s_a u^x = u^{ax} s_a \) for all \( x \in R \), \( a \in R^\times \) (i.e. the \( u^x \) and \( s_a \) together give a representation of the \( ax + b \)-semigroup \( R \rtimes R^\times \)).

(b) For each \( a \in R^\times \) one has \( \sum_{x \in R/aR} u^x s_a s_a^* u^{-x} = 1 \).

This algebra was shown to be purely infinite and simple in [4,5]. It has different representations in terms of crossed products for actions on spaces of finite or infinite adeles for \( K \), [6].

Now, to obtain a presentation of \( \Sigma[R] \) we essentially have to relax, in this presentation of \( \mathfrak{A}[R] \), condition (b) to the weaker condition \( \sum_{x \in R/aR} u^x s_a s_a^* u^{-x} \leq 1 \). This modification in the relations is sufficient to characterize the algebra \( \Sigma[R] \) in the case where \( R \) is a principal ideal domain. We are however especially interested precisely in the situation where this is not the case, i.e. where the number field \( K \) has non-trivial class group. To treat this case adequately we have to impose certain conditions on the range projections of the isometries \( s_a \). The most efficient way to formalize these conditions is to use projections associated with ideals in \( R \) as additional generators and to describe their relations. We mention that a description of the C*-algebra generated by the left regular representation of a cancellative semigroup by analogous generators and relations had been discussed before also by Li, [15], Appendix A2, see also [20], Chapter 4 for a specific example.

An important role in our analysis of \( \Sigma[R] \) is played by a canonical maximal commutative subalgebra. Its Gelfand spectrum \( Y_R \) can be understood as a completion, for a natural metric, of the disjoint union \( \bigsqcup I R/I \) over all non-zero ideals \( I \) of \( R \). It contains the profinite completion \( \hat{R} \) of \( R \) (which is the spectrum of the analogous commutative subalgebra of \( \mathfrak{A}[R] \)). It is important to note that the algebra \( \Sigma[R] \) is functorial for homomorphisms between rings while \( \mathfrak{A}[R] \) is not. This is reflected in the striking fact that the construction \( R \mapsto Y_R \) is contravariant under ring homomorphisms rather than covariant as one might expect. An inclusion of rings \( R \subset S \) induces a surjective map \( Y_S \to Y_R \). The same holds for the locally compact version of \( Y_R \) (corresponding to a natural stabilization of \( \Sigma[R] \)) which plays the role of the locally compact space of finite adeles.
Especially important for us is a natural one-parameter group \((\sigma_t)_{t \in \mathbb{R}}\) of automorphisms of \(\mathfrak{T}[R]\). It is closely related to Bost–Connes systems [1] and to Dedekind \(\zeta\)-functions. In special cases it had been considered before in [4, 13].

The Toeplitz algebra for the semigroup \(\mathbb{N} \times \mathbb{N}^\times\)—which is very closely related to the Toeplitz algebra \(\mathfrak{T}[\mathbb{Z}]\) for the ring \(\mathbb{Z}\) in the sense of the present paper—has been analyzed in [13]. In particular it was found in that paper that the canonical one-parameter automorphism group on this algebra has an intriguing KMS-structure. There is a phase transition at \(\beta = 2\) with a spontaneous symmetry breaking. In the range \(1 \leq \beta \leq 2\) there is a unique KMS-state while for \(\beta > 2\) there is a family of KMS-states labeled by the probability measures on the circle and with partition function the Riemann \(\zeta\)-function.

It turns out that, for our Toeplitz algebra, the KMS-structure is similar, but quite a bit more intricate. We show in Theorem 6.7 that for \(\beta\) in the range \(1 \leq \beta \leq 2\) (with \(\beta = 1\) playing a special role) there is a unique KMS state. The essential new feature which is also the source of the main technical difficulties in this paper is the presence of the class group, in the case where \(R\) is not a principal ideal domain. Our proof for the uniqueness of the KMS-state requires a delicate estimate of the asymptotics of partial Dedekind \(\zeta\)-functions for different ideal classes, see Theorem 6.6. This theorem seems to be new and of independent interest. We include the proof in the appendix.

For \(\beta > 2\) we obtain a splitting of the KMS states over the class group \(\Gamma\) for the number field \(K\). The KMS states for each \(\beta\) in this range are labeled by the elements \(\gamma \in \Gamma\), but moreover also by traces on a crossed product \(C(\mathbb{T}^n) \rtimes R^\times\) (n being the degree of our field extension) by an action (which depends on \(\gamma\)) of the group \(R^\times\) of units of \(R\). For a precise statement see Theorem 7.3. The partition functions are the partial Dedekind \(\zeta\)-functions \(\zeta_\gamma\) associated with the ideal classes \(\gamma\) for \(K\).

In Sect. 8 we determine the ground states. We find a situation which is similar to the one for the KMS states in the range \(\beta > 2\). The ground states are labeled by the states of a certain subalgebra of \(\mathfrak{T}[R]\).

We mention that our methods also immediately yield the KMS-structure of the much simpler, but in the case of a non-trivial class group still interesting, C*-dynamical system that one obtains from the Toeplitz algebra of the multiplicative semigroup \(R^\times\) [i.e. the C*-algebra generated by the left regular action of this semigroup on \(\ell^2(R^\times)\)] with the analogous one-parameter automorphism group, see Remark 7.5.

When we restrict to the case of a trivial class group, all our arguments become very simple indeed and can be used to get a simpler approach to the results in [13].

The presentation of \(\mathfrak{T}[R]\) in terms of generators and relations and the functoriality from Sect. 3 had been obtained and announced by the first named author before the present paper took shape. These two results have since been generalized to more general semigroups by Li [16]. The first named author is indebted to Peter Schneider for very helpful comments.

After this paper was circulated, Neshveyev [11] informed us that, using the crossed product description of \(\mathfrak{T}[R]\) in Sect. 5 and the methods developed, the KMS-structure on \((\mathfrak{T}[R], (\sigma_t))\) could be linked to that of a Bost–Connes system. The KMS-structure of this Bost–Connes system in turn was determined in [10]. Together, this would give a basis for an alternative approach to our results on KMS-states in Sects. 6 and 7.

We include a brief list of notations at the end of the appendix.
2 The Toeplitz algebra for the $ax+b$-semigroup over $R$

Let $R$ be the ring of algebraic integers in the number field $K$. The $ax+b$-semigroup for $R$ is the semidirect product $R \rtimes R^\times$ of the additive group $R$ and the multiplicative semigroup $R^\times = R \setminus \{0\}$ of $R$. We can define the Toeplitz algebra for the semigroup $R \rtimes R^\times$ as the $C^*$-algebra generated by the left regular representation of $R \rtimes R^\times$ on $\ell^2(R \rtimes R^\times)$. We set out to describe this $C^*$-algebra abstractly as a $C^*$-algebra given by generators and relations.

**Definition 2.1** We define the $C^*$-algebra $T[R]$ as the universal $C^*$-algebra generated by elements $u^x, x \in R, s_a, a \in R^\times, e_I, I$ a non-zero ideal in $R$, with the following relations

Ta: The $u^x$ are unitary and satisfy $u^x u^y = u^{x+y}$, the $s_a$ are isometries and satisfy $s_a s_b = s_{ab}$. Moreover we require the relation $s_a u^x = u^{ax} s_a$ for all $x \in R, a \in R^\times$.

Tb: The $e_I$ are projections and satisfy $e_I \cap e_J = e_I e_J, e_R = 1$.

Tc: We have $s_a e_I s_a^* = e_{aI}$.

Td: For $x \in I$ one has $u^x e_I = e_I u^x$, for $x \notin I$ one has $e_I u^x e_I = 0$.

The first condition Ta simply means that the $u^x$ and $s_a$ define a representation of the semigroup $R \rtimes R^\times$. We will see below that $T[R]$ is actually isomorphic to the Toeplitz algebra for the $ax+b$-semigroup $R \rtimes R^\times$, see Corollary 4.16.

In the following, ideals in $R$ will always be understood to be non-zero.

**Remark 2.2** In the case where $R$ is a principal ideal domain, the axioms can be reduced considerably. In fact, in that case, the projections $e_I$ are not needed to describe $T[R]$ by generators and relations (they are all of the form $s_a s_a^*$) and conditions Tb, Tc and Td can be replaced by the single very simple condition

$$\sum_{x \in R/\mathbb{R}} u^x s_a s_a^* u^{-x} \leq 1$$

Note that this inequality is a consequence of Tc and Td. In fact, by Tc one has $e_a R = s_a s_a^*$ and by Td the projections $u^x e_a R u^{-x}, x \in R/a R$ are pairwise orthogonal.

**Remark 2.3** (a) The elements $s_a$ and $s_b^*$ commute if and only if $a$ and $b$ are relatively prime (i.e. $a R + b R = R$). (Proof: $s_b^* s_a = s_a s_b^*$ iff $s_b s_b^* s_a s_a^* = s_b s_a s_b^* s_a^*$ iff $e_{aR} e_{bR} = e_{abR}$. Then use the fact, established below using explicit representations of $\Xi[R]$, that $e_I = e_J \Rightarrow I = J$)

(b) From condition Td it follows that $e_I u^x e_J = 0$ if $x \notin I + J$ and that $e_I u^x e_J = u^{x_1} e_{I \cap J} u^{x_2}$ if there are $x_1 \in I$ and $x_2 \in J$ such that $x = x_1 + x_2$.

Let us derive a few more consequences from the axioms Ta–Td. From the projections $e_I$ we can form associated projections. For each ideal $I$ in $R$ set

$$f_I = \sum_{x \in R/\mathbb{R}} u^x e_I u^{-x}$$

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(note that $u^x e_I u^{-x}$ is well defined for $x \in R/I$, since $u^{x+i} e_I u^{-x-i} = u^x e_I u^{-x}$ for $i \in I$, and that the $u^x e_I u^{-x}$ are pairwise orthogonal for different $x \in R/I$). For each prime ideal $P$ and $n \in \mathbb{N}$ set

$$\varepsilon_{P^n} = f_{P^{n-1}} - f_{P^n}$$

Moreover, for an ideal $I = P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}$ with $P_1, P_2, \ldots, P_n$ distinct primes, set

$$\varepsilon_I = \varepsilon_{P_1^{k_1}} \varepsilon_{P_2^{k_2}} \cdots \varepsilon_{P_n^{k_n}}$$

**Lemma 2.4** The $e_I, f_I, \varepsilon_I$ have the following properties:

(a) For any two ideals $I$ and $J$ in $R$ one has

$$e_I f_J = \sum_{x \in I/(I \cap J)} u^x e_{I \cap J} u^{-x} f_I f_J = \sum_{x \in (I + J)/(I \cap J)} u^x e_{I \cap J} u^{-x}$$

(b) If $I$ and $J$ are relatively prime, then $f_I f_J = f_{IJ}$. If $I \subset J$, then $f_I f_J = f_I$.

(c) If $I$ and $J$ are relatively prime, then $\varepsilon_I \varepsilon_J = \varepsilon_{IJ}$. If $I$ and $J$ have a common prime divisor but occurring with different multiplicities, then $\varepsilon_I \varepsilon_J = 0$.

(d) The family of projections $\{e_I, f_I, \varepsilon_I \mid I$ an ideal in $R\}$ is commutative.

(e) $u^x f_I u^{-x} = f_I$ and $u^x \varepsilon_I u^{-x} = \varepsilon_I$ for all $x \in R$.

**Proof** (a) Obvious from Remark 2.3.

(b) is a special case of the formula under (a).

(c) follows from the definition together with the fact that $\varepsilon_{P^n} \varepsilon_{P^m} = 0$ for a prime ideal $P$ and $n \neq m$.

(d) It follows from (a) that the $e_I$ and $f_I$ form a commutative family. However the $\varepsilon_I$ are defined as products of differences of certain $f_J$.

(e) follows directly from the definition.  

\[\square\]

### 3 Functoriality of $\mathcal{S}[R]$ for injective homomorphisms of rings

We assume that we have an inclusion $R \subset S$ of rings of algebraic integers. We are going to show that this induces an (injective) homomorphism $\kappa : \mathcal{S}[R] \rightarrow \mathcal{S}[S]$. Denote by $s_a, u^x, e_I$ the generators of $\mathcal{S}[R]$ and by $\bar{s}_a, \bar{u}^x, \bar{e}_I$ the generators of $\mathcal{S}[S]$.

The homomorphism $\kappa$ will map $s_a$ to $\bar{s}_a$, $u^x$ to $\bar{u}^x$ and it is clear that this respects the relations $Ta$. With an ideal $I$ in $R$ we associate the ideal $IS$ in $S$ and we define $\kappa(e_I) = \bar{e}_{IS}$. It is then clear that relation $Tc$ is also respected. The fact that $Tb$ and $Td$ are respected follows from the following elementary (and well-known) lemma.

**Lemma 3.1** In the situation above one has for ideals $I, J$ in $R$:

(a) $IS \cap R = I$

(b) $IS \cap JS = (I \cap J)S$
Proof Both statements can be proved in an elementary way using the unique decomposition of $I$ and $J$ into prime ideals in $R$, cf. [19], p. 45 and p. 52, Exercise 1. The statements also follow from the fact that $S$ is a flat (even projective) module over $R$, see [2], Chap. I §2.6 Prop. 6 and Corollary.

Summarizing, we obtain

**Proposition 3.2** Let $R$ and $S$ be the rings of algebraic integers in the number fields $K$ and $L$, respectively. Then any injective homomorphism $\alpha : R \to S$ induces naturally a homomorphism $\hat{T}[R] \to \hat{T}[S]$.

It follows from Theorem 4.13 below that this homomorphism is also injective.

4 The canonical commutative subalgebra

We denote by $\tilde{D}$ the C*-subalgebra of $\hat{T}[R]$ generated by all projections of the form $u^x e_I u^{-x}$, $x \in R$, $I$ a non-zero ideal in $R$. It follows from Remark 2.3 (b) that this algebra is commutative. In fact, the elements $e_I^x := u^x e_I u^{-x}$ satisfy

\[
e_I^x e_I^y = \begin{cases} 0 & \text{if } (x + I) \cap (y + J) = \emptyset \\ e_{I \cap J}^z & \text{if } z \in (x + I) \cap (y + J) \end{cases}
\]

and thus linearly span a dense $\ast$-subalgebra of $\tilde{D}$. The algebra $\tilde{D}$ also obviously contains the elements of the form $e_I$ defined above.

**Lemma 4.1** (a) If $d \in \tilde{D}$, then $s_a d s_a^*$ and $u^x d u^{-x}$ are in $\tilde{D}$ for all $a \in R^\times$, $x \in R$. (b) The set of linear combinations of elements of the form $s_a^* d u^x s_b$ with $a, b \in R^\times$, $x \in R$, $d \in \tilde{D}$ is a dense $\ast$-subalgebra in $\hat{T}[R]$.

**Proof** (a) This follows from the definition and conditions Ta–Td. (b) The set of elements of the form $s_a^* d u^x s_b$ contains the generators and, by (a), is invariant under adjoints and multiplication from the left or from the right by elements $s_c, s_c^*, u^y, e_I$ for $c \in R^\times$, $y \in R$, $I$ an ideal in $R$ (the invariance under multiplication by $s_c^*$ on the right follows from the identity $s_b s_c^* = s_c^* s_b s_c^* = s_c^* s_b s_c^*$).

Let $P$ be a prime ideal in $R$. We denote by $\tilde{D}_P$ the C*-subalgebra of $\tilde{D}$ generated by all projections of the form $u^x e_{P^n} u^{-x}$ with $x \in R$, $n = 0, 1, 2, \ldots$. The $e_{P^n}$ define pairwise orthogonal projections in $\tilde{D}_P$. We define projections in $\tilde{D}_P$ by

\[
\delta_{P^n}^x = u^x e_{P^n} u^{-x} = u^x e_{P^n} u^{-x} e_{P^{n+1}}, \quad x \in R / P^n
\]

They are pairwise orthogonal since $e_{P^n}$ and $e_{P^m}$ are orthogonal for $n \neq m$ and since the $u^x e_{P^n} u^{-x}$ are pairwise orthogonal. In our definition we allow for $n = 0$ so that $\delta_{P^0}^x = e_P$. Note that, by Lemma 4.6 below, the $\delta_{P^0}^x = e_P$ are all non-zero.
Lemma 4.2 One has

\[ \delta_{p_n}^0 = e_{p_n} - \sum_{x \in P_n / P_{n+1}} u^x e_{p_{n+1}} u^{-x} \quad \text{and} \quad \varepsilon_{p_{n+1}} = \sum_{x \in \mathbb{R} / P_n} \delta_{p_n}^x \]

Proof

\[ \delta_{p_n}^0 = e_{p_n} \varepsilon_{p_{n+1}} = e_{p_n} \left( \sum_{x \in P_{n-1} / P_n} u^x e_{p_n} u^{-x} - \sum_{y \in P_{n} / P_{n+1}} u^y e_{p_{n+1}} u^{-y} \right) \]

\[ = e_{p_n} - \sum_{y \in P_{n} / P_{n+1}} u^y e_{p_{n+1}} u^{-y} \]

The second identity is obvious from either formula for the \( \delta_{p_n}^x \).

Lemma 4.3 The algebra \( \varepsilon_{p_n} \bar{D}_P \) is finite-dimensional and isomorphic to \( \mathbb{C}(\mathbb{R} / P_{n-1}) \). For each \( \{x\} \) in \( \mathbb{C}(\mathbb{R} / P_{n-1}) \), the projection \( \delta_{p_{n-1}}^x = u^x e_{p_{n-1}} e_{p_n} u^{-x} \), \( x \in \mathbb{R} / P_{n-1} \) is minimal in this algebra and corresponds to the characteristic function of \( \{x\} \). The isomorphism \( \varepsilon_{p_n} \bar{D}_P \cong \mathbb{C}(\mathbb{R} / P_{n-1}) \) is compatible with the natural action of the additive group \( \mathbb{R} \) on these two algebras.

For each \( k \leq n \) we have

\[ \varepsilon_{p_{n+1}} e_{p_k} = \sum_{x \in P_k / P_n} \delta_{p_n}^x \]

Let \( G_k \) denote the (finite-dimensional) \( \mathbb{C}^* \)-subalgebra of \( \bar{D}_P \) generated by the projections \( 1, u^x e_{p_i} u^{-x}, i = 1, \ldots, k, x \in \mathbb{R} / P_i \). The map

\[ G_k \ni z \mapsto (z \varepsilon_{p}, z \varepsilon_{p_2}, \ldots, z \varepsilon_{p_{k+1}}) \]

defines an isomorphism \( G_k \rightarrow \bigoplus_{n \leq k+1} \varepsilon_{p_n} \bar{D}_P \).

Proof For \( k \geq n \), since \( e_{p_k} \leq e_{p_n} \), we have \( e_{p_k} \varepsilon_{p_n} = 0 \) and, since \( u^x \) commutes with \( \varepsilon_{p_n} \), also \( u^x e_{p_k} u^{-x} \varepsilon_{p_n} = 0 \) for such \( k \).

On the other hand if \( k \leq n-1 \), then \( e_{p_k} u^x e_{p_{n-1}} = 0 \) for \( x \notin P_k \) and \( e_{p_k} u^x e_{p_{n-1}} = u^x e_{p_{n-1}} \) for \( x \in P_k \). Applying this to the product \( e_{p_k} \delta_{p_n}^x = e_{p_k} u^x e_{p_n} u^{-x} \varepsilon_{p_{n+1}} \) we see that this expression vanishes for \( x \notin P_k \) and equals \( \delta_{p_n}^x \) for \( x \in P_k \).

The last assertion then is an immediate consequence.

Denote by \( D_P \) the ideal in \( \bar{D}_P \) generated by the \( \varepsilon_{p_n} \). Lemma 4.3 shows that \( D_P \cong \bigoplus \mathbb{C}(\mathbb{R} / P^n) \). Since the union of the subalgebras \( G_k \) is dense in \( \bar{D}_P \), the last statement in this lemma also shows that \( D_P \) is an essential ideal in \( \bar{D}_P \).

Let \( \iota : \mathbb{C}(\mathbb{R} / P^n) \rightarrow \mathbb{C}(\mathbb{R} / P^m) \) denote the homomorphism induced by the quotient map \( \mathbb{R} / P^m \rightarrow \mathbb{R} / P^n \) for \( m > n \).
Lemma 4.4 The C*-algebra $\overline{D}_p$ is isomorphic to the subalgebra of the infinite product

$$\prod_{n=0}^{\infty} C(R/P^n)$$

given by the “Cauchy sequences” $(d_n)$ (by this we mean that for each $\varepsilon > 0$ there is $N > 0$ such that $\|\iota(d_n) - d_m\| < \varepsilon$ for all $n, m$ such that $N \leq n \leq m$).

Proof The map $\overline{D}_p \ni z \mapsto (\varepsilon P^n z) \in \prod_{n=0}^{\infty} C(R/P^n)$ is injective since $D_p$ is essential. Each element of the form $u^x e_{p^n} u^{-x}$ is mapped, according to Lemma 4.3, to the sequence $(0, \ldots, 0, \delta^x_{p^k}, \iota(\delta^x_{p^k}), \iota^2(\delta^x_{p^k}), \ldots)$

Thus the images of these projections generate, together with the images of the $\delta^x_{p^k}$, the algebra of all “Cauchy-sequences”. □

Lemma 4.5 There is an exact sequence

$$0 \rightarrow D_p \rightarrow \overline{D}_p \rightarrow C \rightarrow 0$$

where the Gelfand spectrum $\text{Spec} D_p$ equals $\bigsqcup R/P^n$ and the Gelfand spectrum of the C*-algebra $C$ is the $P$-adic completion $R_p = \varprojlim_n R/P^n$ of $R$.

Proof The isomorphism $D_p \cong \bigoplus C(R/P^n)$ shows that $\text{Spec} D_p = \bigsqcup R/P^n$. In the quotient $\overline{C} = \overline{D}_p/D_p$ the images $(u^x e_{p^n} u^{-x})$ of the projections $u^x e_{p^n} u^{-x}$ satisfy the relation

$$\tilde{e}_{p^n} = \sum_{x \in P^n/P^m} (u^x e_{p^n} u^{-x})$$

for $m \geq n$ (see Lemma 4.2). Since $C$ is generated by these images, $C = \varprojlim_n C(R/P^n)$ and this proves the second claim. □

Now let $P_1, P_2, \ldots$ be an enumeration of the prime ideals in $R$ (say ordered by increasing norm $|R/P_i|$) and, for each $n$, let $\mathcal{I}_n$ be the set of ideals of the form $I = P_{1}^{k_1} P_{2}^{k_2} \cdots P_{n}^{k_n}$ with all $k_i \geq 0$. We write $D_n = D_{P_1} D_{P_2} \cdots D_{P_n}$ and $\overline{D}_n = \overline{D}_{P_1} \overline{D}_{P_2} \cdots \overline{D}_{P_n}$. The $\overline{D}_{P_n}$ all commute and $\overline{D}$ obviously is the inductive limit of the $\overline{D}_n$.

We now use a natural representation of $\mathfrak{T}[R]$ on the following Hilbert space $H_R$:

$$H_R = \bigoplus_{I \text{ ideal in } R} \ell^2(R/I)$$

Note that $H_R$ is isomorphic to the infinite tensor product $\bigotimes P H_P$ where the tensor product is taken over all prime ideals $P$ in $R$ and $H_R = \bigoplus \ell^2(R/P^n)$ with “vacuum vector” the standard unit vector in the one-dimensional space $\ell^2(R/R)$.

$\mathfrak{T}[R]$ acts on $H_R$ in the following way:
The units $u^x$, $x \in R$ act componentwise on $\ell^2(R/I)$ in the natural way.

- The isometries $s_a$ act through the composition: $\ell^2(R/I) \cong \ell^2(aR/aI) \leftrightarrow \ell^2(R/aI)$.
- The projection $e_J$ is represented by the orthogonal projection onto the subspace $H = \bigoplus_{I \subset J} \ell^2(J/I)$ of $H$.

It is easy to check that this assignment respects the relations between the generators and thus defines a representation $\mu$ of $\mathcal{Z}[R]$. One has

**Lemma 4.6** Let $I = P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}$ with all $k_i \geq 1$, and $x_1, x_2, \ldots, x_n \in R$. Then $\mu(\delta_p)$ acts on the subspace $\ell^2(P/I)$ of $H_R$ as the orthogonal projection onto the subspace $\ell^2(P/I)$. Thus $\mu(\delta_p)$ acts on this subspace as the orthogonal projection onto the one-dimensional subspace $\ell^2(I/I)$ and $\mu(\delta_p)$ acts as the orthogonal projection onto the one-dimensional subspace $\ell^2((I + z)/I)$ where $z$ is the unique element in $\bigcap_i (P_i^{k_i} + x_i)/I$.

Proof. This follows from the definition of $\mu(e_{P_i^{k_i}})$ and the fact that $e_{P_i^{k_i} + 1} = 1$ on $\ell^2(R/I)$ (recall that, by definition $e_{P_i^{k_i} + 1} = e_{P_i^{k_i}} e_{P_i^{k_i} + 1}$).

**Lemma 4.7** For an ideal $I = P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}$ in $\mathcal{I}_n$ and $x \in R/I$ consider the projection

$$
\delta_x = u^x e_{I} e_{P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}} u^{-x}, \quad x \in R/I
$$

These projections are non-zero according to Lemma 4.6. (Note also that our definition of $\delta_x$ depends on the fact that we consider $I$ as an element of $\mathcal{I}_n$!) Then

$$
\delta_x = u^x \sum_{x \in R/I} e_I e_{P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}} u^{-x} = e_I e_{P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}}
$$

Proof. The identity $\delta_x = u^x e_{I} e_{P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}}$ follows from the equations $e_I = e_{P_1^{k_1}} e_{P_2^{k_2}} \cdots e_{P_n^{k_n}}$ and $e_I e_{P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n}} = e_{P_1^{k_1} + 1} e_{P_2^{k_2 + 1}} \cdots e_{P_n^{k_n + 1}}$ (see Lemma 2.4). The second identity follows from the first one in combination with the corresponding identity in Lemma 4.2.

We have now shown that $\mathcal{D}_n = \mathcal{D}_{P_1^{k_1}} \mathcal{D}_{P_2^{k_2}} \cdots \mathcal{D}_{P_n^{k_n}}$ is isomorphic to the tensor product $\bigotimes_{1 \leq i < j \leq n} \mathcal{D}_{P_i} \mathcal{D}_{P_j}$ with minimal projections the $\delta_x$, $x \in \mathcal{I}_n$. Thus $\mathcal{D}_n \cong \bigoplus_{I \in \mathcal{I}_n} \mathcal{C}(R/I)$ and the spectrum of $\mathcal{D}_n$ is $\bigcup_{I \in \mathcal{I}_n} R/I$ (this is the cartesian product of the spectra $\bigcup_{k \geq 0} R/P_i^{k}$ of the $\mathcal{D}_{P_i}$).

**Corollary 4.8** $\mathcal{D}_n$ is an essential ideal in $\mathcal{D}_n$. $\mathcal{D}_n$ is isomorphic to $\mathcal{D}_{P_1^{k_1}} \mathcal{D}_{P_2^{k_2}} \cdots \mathcal{D}_{P_n^{k_n}}$ and $\mathcal{D}$ is isomorphic to the infinite tensor product $\bigotimes_{P} \mathcal{D}_{P}$. 

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Proof Consider the surjective homomorphism

$$\varphi : \bar{D}_P \otimes \bar{D}_P \cdots \otimes \bar{D}_P \to \bar{D}_{P_1} \bar{D}_{P_2} \cdots \bar{D}_{P_n} = \bar{D}_n$$

which exists by the universal property of the tensor product. The restriction of \( \varphi \) to the essential (see the comment after Lemma 4.3) ideal \( \bar{D}_{P_1} \bar{D}_{P_2} \cdots \otimes \bar{D}_{P_n} \) is an isomorphism. Thus \( \varphi \) is an isomorphism.

From Lemma 4.5 it follows that, for each prime ideal \( P \), we have \( \text{Spec } \bar{D}_P = \text{Spec } D_p \cup \text{Spec } C = \bigsqcup_n R/P^n \sqcup R_p \).

For an ideal \( I \) in \( R \) and \( x \in R/I \), we consider the projection \( e^x_I = u^x e_I u^{-x} \). Since, by Remark 2.3(b), \( e^x_I e^y_J \) is either zero or equal to \( e^{x+y}_{I \cap J} \) for \( z \in (x + I) \cap (y + J) \), the set of projections \( \{ e^x_I \mid I \text{ ideal in } R, x \in R/I \} \) is multiplicatively closed.

In particular the set of projections \( \{ e^n_{P_n} \mid n = 0, 1, 2, \ldots, x \in R/P^n \} \) in \( \bar{D}_P \) is multiplicatively closed and a sequence \( (\varphi_k) \) of characters of \( \bar{D}_P \) converges to a character \( \varphi \) if and only if \( \varphi_k(e^n_{P_n}) \to \varphi(e^n_{P_n}) \) for each \( x \) and \( n \). To describe this topology in terms of a metric we use the norm of an ideal. For an ideal \( I \) in \( R \) we denote by \( N(I) = |R/I| \) the number of elements in \( R/I \). The topology on \( \text{Spec } \bar{D}_P \) is described by the metric \( d_\alpha \) defined for any choice of \( \alpha > 1 \) by

$$d_\alpha(\varphi, \psi) = \sum_{n \geq 0, x \in R/P^n} N(P^n)^{-\alpha}|(\varphi - \psi)(e^n_{P_n})|$$

The topology on the first component \( \bigsqcup R/P^n \) of \( \text{Spec } \bar{D}_P \) is the discrete topology. The elements in this component are the characters \( \eta^n_{P_n} \) uniquely defined by the condition

$$\eta^n_{P_n}(\delta^n_{P_n}) = 1$$

The topology on the second component \( R_P \) of \( \text{Spec } \bar{D}_P \) is the usual ultrametric topology and finally a sequence \( \eta^n_{P_{n_k}} \) converges to an element \( \eta_z \) in the second component determined by \( z \in R_P \) if and only if \( n_k \to \infty \) and there is \( N > 0 \) such that (using a self-explanatory notation for the image of \( z \) in the quotient) \( z/P^{n_k} = x_{n_k} \) for \( k \geq N \).

Now, since \( \bar{D} \cong \bigotimes_p \bar{D}_P \), every character \( \varphi \) of \( \bar{D} \) is of the form \( \varphi = \bigotimes_p \varphi_p \) with each \( \varphi_p \) either of the form \( \eta^n_{P_n} \) for \( n \in \mathbb{N} \), \( x \in R/P^n \) or \( \eta_z \) with \( z \in R_P \).

Again, the set \( \{ e^x_I \mid I \text{ ideal in } R, x \in R/I \} \) of projections in \( \bar{D} \) is multiplicatively closed and generates \( \bar{D} \). Thus a sequence \( (\varphi_n) \) of characters converges to \( \varphi \) if and only if \( \varphi_n(e^x_I) \to \varphi(e^x_I) \) for each ideal \( I \) and \( x \in R/I \). This topology is described by the metric \( d_\alpha \) defined for any choice of \( \alpha > 1 \) by

$$d_\alpha(\varphi, \psi) = \sum_{I \text{ ideal in } R, x \in R/I} N(I)^{-\alpha}|(\varphi - \psi)(e^x_I)|$$
We consider special elements $\eta_I^x$ labeled by $\bigsqcup I R/I$. For $I = P_{1}^{k_1} P_{2}^{k_2} \cdots P_{n}^{k_n}$ and $x \in R/I$, $\eta_I^x$ is defined as

$$\eta_I^x = \bigotimes_{i=1}^{\infty} \eta_{p_i^{k_i}}^{x_i}$$

Here, $k_i$ is defined to be 0 if $P_i$ does not occur in the prime ideal decomposition of $I$ and $x_i = x / p_i^{k_i}$.

**Proposition 4.9** The subset $\bigsqcup I R/I$ is dense in Spec $\mathcal{D}$. Thus Spec $\mathcal{D}$ is the completion of $\bigsqcup I R/I$ for the metric $d_\alpha$ described above.

**Proof** It is clear from the discussion above that the set of elements of the form $\bigotimes_i \eta_{p_i^{k_i}}^{x_i}$ is dense. We show that each element $\eta = \bigotimes_i \eta_{p_i^{k_i}}^{x_i}$ can be approximated by the $\eta_I^x$. In fact, if $I = P_{1}^{k_1} P_{2}^{k_2} \cdots P_{n}^{k_n}$ and $x \in R$ is such that $x / p_i^{k_i} = x_i$, $i = 1, \ldots, n$, then $\eta(e_J^x) = \eta_I^x(e_J^x)$ for each ideal $J$ that contains only $P_1, \ldots, P_n$ in its prime ideal decomposition and for any $y \in R/J$.

**Remark 4.10** This description of Spec $\mathcal{D}$ also clarifies the wrong-way functoriality in $R$ of the construction. Assume that we have field extensions $Q \subset K \subset L$ and corresponding inclusions $\mathbb{Z} \subset R \subset S$ of the rings of algebraic integers. Denote by $Y_R$ and $Y_S$ the spectra of the corresponding commutative subalgebras $\mathcal{D}_R$ and $\mathcal{D}_S$ in $\mathbb{Z}[R]$ and $\mathbb{Z}[S]$, respectively. Thus $Y_R$ and $Y_S$ are completions of the metric spaces $\bigsqcup I R/I$ and $\bigsqcup J S/J$, respectively.

With every character $\eta_J^x \in \bigsqcup J S/J$ we can associate a character $(\eta_J^x)' \in$ Spec $\mathcal{D}_R$ by defining $(\eta_J^x)'(e_M) = \eta_J^x(e_M^S)$ for an ideal $M$ in $R$ and $y \in R/M$. The map $\eta_J^x \rightarrow (\eta_J^x)'$ is obviously contractive (up to a constant $n^\alpha$ with $n = [L:K]$) for the metrics $d_\alpha$ and thus extends to a continuous map Spec $\mathcal{D}_S \rightarrow$ Spec $\mathcal{D}_R$. It is surjective, since the dense subset $\bigsqcup I R/I$ of Spec $\mathcal{D}_R$ has a natural lift to Spec $\mathcal{D}_S$. In fact, one immediately checks that $(\eta_{1S}^x)' = \eta_I^x$ for an ideal $I \subset R$ and $x \in R/I$.

**Lemma 4.11** Let $a \in R^\times$ such that $aR = QL$ with $L \in \mathcal{I}_n$ and $Q$ relatively prime to each of the $P_1, \ldots, P_n$. Then, for $I \in \mathcal{I}_n$,

$$s_a \delta_{I,n}^{0} s_a^* = e_Q \delta_{L,1,n}^{0}$$

In particular, if $aI = bJ$ for two ideals $I, J$ in $\mathcal{I}_n$ and $a, b \in R$, then $s_a \delta_{I,n}^{0} s_a^* = s_b \delta_{J,n}^{0} s_b^*$.

**Proof** Using induction, it suffices to show that, for $aR = Q P_1^{k_1}$ with $Q$ relatively prime to $P_1$,

$$s_a \delta_{P_1^{k_1},n}^{0} s_a^* = e_Q \delta_{P_1^{k_1+1},n}^{0}$$
This follows from the following computation

\[
\delta_0^a p_{1}^{0 \ast} \delta_0^a = a \left( e_{p_{1}^{0}} - \sum_{x \in P_{1}^{0} / P_{1}^{0} + 1} u^x e_{p_{1}^{0} + 1} u^{-x} \right) s_{a} = e_{a} p_{1}^{0} - \sum_{x \in P_{1}^{0} / P_{1}^{0} + 1} u^{a x} e_{a} p_{1}^{0} u^{-a x}
\]

\[
= e_{Q} \left( e_{p_{1}^{0} + k_{1}} - \sum_{x \in P_{1}^{0} / P_{1}^{0} + 1} u^{a x} e_{p_{1}^{0} + k_{1} + 1} u^{-a x} \right)
\]

For the last equality we use the fact that \( u^{a x} \) commutes with \( e_{Q} \) and that \( Q \) is relatively prime to \( P_{1} \). \( \square \)

The dual \( \hat{K}^\times \) of the multiplicative group \( K^\times \) acts by automorphisms \( \alpha_{\chi}, \chi \in \hat{K}^\times \) defined by

\[
\alpha_{\chi}(s_{a}) = \chi(a) s_{a} \quad \alpha_{\chi}(u^{x}) = u^{x} \quad \alpha_{\chi}(e_{I}) = e_{I}
\]

By Lemma 4.1 (b) the fixed point algebra \( B \) is the subalgebra of \( \Sigma[R] \) generated by all \( u^{x}, x \in R \) and \( e_{I}, I \) an ideal in \( R \). Integration over \( K^\times \) gives a faithful conditional expectation \( \Sigma[R] \rightarrow B \). On \( B \) the dual \( \hat{R} \) of the additive group \( R \) acts by automorphisms \( \beta_{\chi} \) given by

\[
\beta_{\chi}(u^{x}) = \chi(x) u^{x} \quad \beta_{\chi}(e_{I}) = e_{I}
\]

The fixed point algebra for this action is \( \hat{D} \). Again, integration over the compact group \( \hat{R} \) defines a faithful conditional expectation \( B \rightarrow \hat{D} \).

Composing these two expectations we obtain the faithful conditional expectation \( E : \Sigma[R] \rightarrow \hat{D} \) which we will use now. Note that, for a typical element \( z = s_{a}^{n} d u^{x} s_{b} \), \( E(z) = 0 \), unless \( a = b, x = 0 \) in which case \( E(z) = s_{a}^{n} d s_{a} \).

**Lemma 4.12** Let \( z = d + \sum_{i=1}^{m} s_{a_{i}}^{n} d_{i} u^{x_{i}} s_{b_{i}} \) be an element of \( \Sigma[R] \) [cf. Lemma 4.1(b)] such that for each \( i, a_{i} \neq b_{i} \) or \( x_{i} \neq 0 \) and such that \( d, d_{i} \in D_{n} \) for some \( n \). Let \( n \) be large enough so that also the principal ideals \( a_{i} R, b_{i} R \) are in \( T_{n} \) for all \( i \). Let \( \varepsilon > 0 \).

(a) There is a minimal projection \( \delta \) in \( D_{n} \) such that \( \| d \delta \| = \| \delta d \delta \| \geq \| d \| - \varepsilon \).

(b) There is \( k > 0 \) and a minimal projection \( \delta' \in D_{n+k}, \delta' \leq \delta \) such that \( \delta' s_{a_{i}}^{n} d_{i} u^{x_{i}} s_{b_{i}} \delta' = 0 \) for all \( i \).

(c) For the projection \( \delta' \) in (b) one has \( \| \delta' z \delta' \| \geq \| E(z) \| - \varepsilon \)

**Proof** (a) simply expresses the fact that \( D_{n} \) is essential.
(b) Let $\delta = \delta^y_{1,n}$. $I \in \mathcal{I}_n$, $y \in R/I$. Using Lemma 4.11 we may then choose

$$\delta' = u^y \delta^0_{1,n} \delta_{p_{n+1}}^0 \cdots \delta_{p_{n+k}}^0 u^{-y}$$

such that for each $i$ the projections $s_{a_i} \delta_{a_i}^i$ and $u^{x_i} s_{b_i} \delta_{b_i}^i u^{-x_i}$ are orthogonal. This projection $\delta'$ will have the required properties.

(c) follows immediately from (a) together with (b) using the fact that $E(z) = d$ and that $d \delta$ is just a multiple of $\delta$ ($\delta$ is a minimal projection in $D_n$).

**Theorem 4.13** Let $\alpha : \mathfrak{T}[R] \to A$ be a *-homomorphism into any C*-algebra $A$. The following are equivalent:

(a) $\alpha$ is injective.
(b) $\alpha$ is injective on $\hat{D}$.
(c) $\alpha$ is injective on $D_n$ for each $n$.

**Proof** (c) implies (b) since $D_n$ is essential in $\hat{D}_n$ for each $n$.

(b) $\Rightarrow$ (a): Let $h$ be a positive element in $\mathfrak{T}[R]$ with $\alpha(h) = 0$ and $z$ a linear combination as in 4.12 such that $\|h - z\| < \varepsilon$. Let $\delta'$ be a projection as in 4.12 (b) such that $\|\delta' z \delta\| \geq \|E(z)\| - \varepsilon$ and such that $\delta' z \delta'$ is a multiple of $\delta'$. If $\alpha(h) = 0$, then $\|\alpha(\delta' z \delta')\| < \varepsilon$ and thus also $\|\delta' z \delta'\| < \varepsilon$. It follows that $\|E(h)\| < 2\varepsilon$. Since this holds for each $\varepsilon$, $E(h) = 0$ and, since $E$ is faithful, $h = 0$.

From this technical theorem we can derive the following important Corollaries 4.14.4.16 and 4.17.

**Corollary 4.14** The representation $\mu$ of $\mathfrak{T}[R]$ on $\bigoplus_I \ell^2(R/I)$ is an isomorphism.

**Proof** The restriction of $\mu$ to each $D_n$ is injective by Lemma 4.6.

Let $\mathfrak{T} \subset \mathcal{L}(\ell^2(R \times R^\times))$ denote the C*-algebra defined by the left regular representation of the semigroup $R \times R^\times$ (cf. Sect. 2). Given an ideal $I$ in $R$ we can define a projection $e'_I$ in $\mathcal{L}(\ell^2(R \times R^\times))$ as the orthogonal projection on the subspace $\ell^2(\mathfrak{T} \times I \times I^\times) \subset \ell^2(R \times R^\times)$. Denote by $u^x$ and $s_a^x$ the operators defined by the left action of $R$ and $R^\times$ on $\ell^2(R \times R^\times)$. Then it is easy to check that the $u^x$, $s_a^x$ and $e'_I$ satisfy the relations defining $\mathfrak{T}[R]$.

**Lemma 4.15** (a) Every ideal $I$ in $R$ can be written in the form $\frac{a}{b} R \cap R$ with $a, b \in R^\times$.
(b) If $I = \frac{a}{b} R \cap R$, then $e_I = s_a^* s_a s_b$ and, similarly, $e'_I = s_b^* s_a^* s_b$.

**Proof** (a) Let $Q$ and $M$ be ideals such that $I, Q, M$ are relatively prime and such that $IQ$, $QM$ are principal, say $IQ = aR$, $QM = bR$. Then $bI = IQM = aR \cap bR$.

(b) One has $bI = aR \cap bR$ and therefore $s_b e_I s_b^* = s_a s_b s_b^*$. Since this uses only the relations defining $\mathfrak{T}[R]$, it also holds in $\mathfrak{T}$.

This lemma shows that $e'_I \in \mathfrak{T}$, hence we obtain a natural homomorphism $\mathfrak{T}[R] \to \mathfrak{T}$ by assigning $s_a \mapsto s_a^*$, $u^x \mapsto u^x$, $e_I \mapsto e'_I$.

**Corollary 4.16** The natural map $\mathfrak{T}[R] \to \mathfrak{T}$ is an isomorphism.
The map is obviously surjective. To prove injectivity, suppose $I \in \mathcal{I}_n$ and $x \in R$ are given. Let $Q \not\in \{P_1, P_2, \ldots, P_n\}$ be a prime ideal in the ideal class $[1]^{-1}$ and let $a$ be a generator of the principal ideal $IQ$ (for the existence of such a $Q$ see for instance [17], chapter 7, §2, Corollary 7). Since the exponents of $\{P_1, P_2, \ldots, P_n\}$ in the prime factorization of $aR$ are identical to those of $I$, the image of the projection $\delta_{I,n}^a$ fixes the canonical basis vector $\xi_{(x,a)} \in \ell^2(R \rtimes R^\times)$, so it does not vanish. Hence the natural map is injective on $D_n$ for each $n$ and therefore injective by Theorem 4.13.

Denote, as above, by $Y_R$ the spectrum of $\tilde{D}$. The semigroup $R \rtimes R^\times$ acts on $Y_R$ in a natural way and this action corresponds to the canonical action of $R \rtimes R^\times$ on $\tilde{D}$ by conjugation by the $u^x$ and the $s_d$. We use the definition of a semigroup crossed product as in [12], Sect. 2, [15], Appendix A1.

**Corollary 4.17** The algebra $\mathcal{S}[R]$ is canonically isomorphic to the semigroup crossed product $(\mathcal{C}(Y_R) \rtimes R) \rtimes R^\times$.

**Proof** The generators $e_1^0$ of $\mathcal{C}(Y_R)$ together with the canonical generators of $R \rtimes R^\times$ in $(\mathcal{C}(Y_R) \rtimes R) \rtimes R^\times$ satisfy the relations defining $\mathcal{S}[R]$, hence they determine a surjective homomorphism $\mathcal{S}[R] \to (\mathcal{C}(Y_R) \rtimes R) \rtimes R^\times$. This homomorphism is injective on $\tilde{D}$ and therefore injective by 4.13. □

5 An alternative description of Spec $\tilde{D}$ and the dilation of $\mathcal{S}[R]$ to a crossed product by $K \rtimes K^*$

We will give a parametrization of the spectrum of $\tilde{D}$, along the lines of that obtained for the case $R = \mathbb{Z}$ in [9,13], and use it to realize $\mathcal{S}[R]$ as a full corner in a crossed product. Let $\mathbb{A}_f$ denote the ring of finite adeles over $K$ and let $\hat{R}$ be the compact open subring of (finite) integral adeles; their multiplicative groups are the finite ideles $\mathbb{A}_f^*$ and the integral ideles $\hat{R}^*$, respectively.

For each integral adele $a$ and each prime ideal $P$, let $\epsilon_P(a)$ be the smallest nonnegative integer $n$ such that the canonical projection of $a$ in $R/P^n$ is nonzero, and put $\epsilon_P(a) = \infty$ if $a$ projects onto 0 in $R/P^n$ for every $n$. If $a$ is a finite adele, then there exists $d \in R$ such that $da \in \hat{R}$, and we let $\epsilon_P(a) = \epsilon_P(da) - \epsilon_P(d)$. This does not depend on $d$. The group $\hat{R}^*$ acts by multiplication on $\mathbb{A}_f$ and the corresponding orbit space $\mathbb{A}_f/\hat{R}^*$ factors as a restricted infinite product over the set $P$ of prime ideals in $R$

$$\mathbb{A}_f/\hat{R}^* \cong \left\{ (\epsilon_P)_{P \in \mathcal{P}} : \epsilon_P \in \mathbb{Z} \cup \{\infty\} \text{ and } \epsilon_P \geq 0 \text{ for almost all } P \in \mathcal{P} \right\}$$

under the map $a \mapsto (\epsilon_P(a))_{P \in \mathcal{P}}$. This product can be viewed as a space of fractional superideals. The usual fractional ideals of $K$ appear as the elements $a\hat{R}^* \subseteq \mathbb{A}_f/\hat{R}^*$ such that $\epsilon_P(a) \in \mathbb{Z}$ for every $P$ and $\epsilon_P(a) = 0$ for all but finitely many $P$. The zero divisors in $\mathbb{A}_f/\hat{R}^*$ correspond to sequences for which $\epsilon_P(a) = \infty$ for some $P$. The elements in $\hat{R}^*/\hat{R}^*$ correspond to superideals with nonnegative exponent sequences and are analogous to the usual supernatural numbers (see e.g. wikipedia), in fact indistinguishable from them as a space—the distinction will only arise when we consider the

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multiplicative action of $K^*$ on additive classes. For each $a \in \mathbb{A}_f$ the additive subgroup $a\hat{R}$ of $\mathbb{A}_f$ is invariant under the multiplicative action of $\hat{R}^*$.

We will say that two pairs $(r, a)$ and $(s, b)$ in $\mathbb{A}_f \times \mathbb{A}_f$ are equivalent if $b \in a\hat{R}^*$ and $s - r \in a\hat{R}$ and we will denote by $\omega_{r,a}$ the equivalence class of $(r, a)$. Since equivalence classes are compact the quotient

$$\Omega_{\mathbb{A}_f} := \{ \omega_{r,a} | a \in \mathbb{A}_f, r \in \mathbb{A}_f \}$$

is a locally compact Hausdorff space, whose elements are pairs $(r, a)$ with $a \in \mathbb{A}_f / \hat{R}^*$ and $r \in \mathbb{A}_f / a\hat{R}$.

When $a$ and $b$ are superideals such that $\epsilon_P(a) \leq \epsilon_P(b)$ for every $P$ we write $a \leq b$. In this case $b\hat{R} \subseteq a\hat{R}$ and there is an obvious homomorphism reduction modulo $a$ of $\mathbb{A}_f / b\hat{R}$ to $\mathbb{A}_f / a\hat{R}$; we will write $r(a)$ for the image of $r \in \mathbb{A}_f / b\hat{R}$. When $I$ and $J$ are ideals of $R$ viewed as elements of $\mathbb{A}_f / \hat{R}^*$, then $I \leq J$ means $J \subseteq I$ and the reduction defined above is the usual reduction of ideal classes $R/I \rightarrow R/J$.

There is a natural action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$ given by

$$(m, k)\omega_{r,a} = \omega_{m+kr, ka}.$$  (1)

The additive action $(m, 0)\omega_{r,a} = \omega_{m+r,a}$ is by straightforward addition of classes in $\mathbb{A}_f / a\hat{R}$ and the multiplicative action $k(\hat{a}\hat{R}^*) = (ka)\hat{R}^*$ on the second component is also straightforward, but the multiplicative action on the first component requires the homomorphism $\times k : \mathbb{A}_f / a\hat{R} \rightarrow k\mathbb{A}_f / ka\hat{R} = \mathbb{A}_f / ka\hat{R}$.

Since the set $\hat{R}$ of integral adeles is a compact open subset of $\mathbb{A}_f$, the subset

$$\Omega_{\hat{R}} := \{ \omega_{r,a} | r \in \hat{R}, a \in \hat{R} \}$$

consisting of integral elements is compact open in $\Omega_{\mathbb{A}_f}$ and is invariant under the action of $R \rtimes R^\times$.

**Proposition 5.1** The projection $\mathbb{1}_{\Omega_{\hat{R}}}$ is full in $C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^*$ and there is a canonical isomorphism

$$C(\Omega_{\hat{R}}) \rtimes R \rtimes R^\times \cong \mathbb{1}_{\Omega_{\hat{R}}} \left( C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^* \right) \mathbb{1}_{\Omega_{\hat{R}}}.$$ 

**Proof** Clearly $(R^\times)^{-1}(R \rtimes R^\times) = K \rtimes K^*$, so $R \rtimes R^\times$ is an Ore semigroup, and $\bigcup_{k \in R^\times} (0, k)^{-1}\Omega_{\hat{R}}$ is dense in $\Omega_{\mathbb{A}_f}$ because for every element $\omega_{r,a} \in \Omega_{\mathbb{A}_f}$ there exist $k \in R^\times$ such that $kr \in \hat{R}$ and $ka \in \hat{R} / ka\hat{R}$. By Laca [8, Theorem 2.1] the action of $K \rtimes K^*$ on $C_0(\Omega_{\mathbb{A}_f})$ is the minimal automorphic dilation (see [8]) of the semigroup action of $R \rtimes R^\times$ on $C(\Omega_{\hat{R}})$. The fullness of $\mathbb{1}_{\Omega_{\hat{R}}}$ and the isomorphism to the corner then follow by Laca [8, Theorem 2.4].

**Proposition 5.2** Let $v_{m,k}$ with $(m, k) \in R \rtimes R^\times$ be the semigroup of isometries in $C(\Omega_{\hat{R}}) \rtimes R \rtimes R^\times$ implementing the action of $R \rtimes R^\times$. For each ideal $I$ in $R$ let $E_I$ be the characteristic function of the set $\{ \omega_{s,b} \in \Omega_{\hat{R}} | b \geq I, s(b) \in I \}$. Then the
maps \( u^i \mapsto v_{x,1} \), \( s_k \mapsto v_{0,k} \), and \( e_I \mapsto E_I \) extend to an isomorphism of \( \mathfrak{S}(R) \) onto \( \hat{\mathcal{C}}(\Omega_{\hat{R}}) \times R \times R^\times \).

**Proof** The set \( \{w, b \in \Omega_{\hat{R}} : b \geq I, s(b) \in I\} \) is closed open because it is defined via finitely many conditions \([(\varepsilon_P(b) \geq \varepsilon_P(I)) \text{ on the prime factors of } I \text{ and } s = 0 \text{ mod } I\}] \) each of which determines a closed open set; thus \( E_I \) is continuous. The relations \( \Gamma_a \) are satisfied because \((m, k) \mapsto v_{m,k} \) is an isometric representation of \( R \times R^\times \), and \( \Gamma_b \) holds because \( b \geq I \) and \( b \geq J \) if and only if \( b \geq I \cap J \), and \( s(I) \in I \) and \( s(J) \in J \) if and only if \( s(I \cap J) \in I \cap J \). Computing with \( m = 0 \) in Eq. (1) shows that multiplication by \( k \in R^\times \) maps the support of \( E_I \) onto the support of \( E_{kI} \), hence relation \( \Gamma_c \) holds. Similarly, setting \( k = 1 \) in Eq. (1) shows that addition of \( m \) maps the support of \( E_I \) onto itself if \( m \in I \), and onto a set disjoint from it if \( m \notin I \), showing that relation \( \Gamma_d \) holds. This gives a homomorphism \( h : \mathfrak{S}(R) \to \hat{\mathcal{C}}(\Omega_{\hat{R}}) \times R \times R^\times \).

To show that \( h \) is surjective it suffices to prove that the functions \( E_I^x := v_{x,1}E_Iv_{-x,1} \) separate points in \( \Omega_{\hat{R}} \). So let \( \omega_{r,a} \) and \( \omega_{s,b} \) be two distinct points in \( \Omega_{\hat{R}} \). If \( a \neq b \), we may assume there exists a prime ideal \( Q \) such that \( \varepsilon_Q(a) < \varepsilon_Q(b) \) (otherwise reverse the roles of \( a \) and \( b \)). If we now let \( I = Q^{\varepsilon_Q(b)} \), then \( E_I^{x(I)} \) takes on the value \( 1 \) at \( \omega_{s,b} \) but vanishes at \( \omega_{r,a} \). If \( a = b \) as superideals, since the points \( \omega_{r,a} \) and \( \omega_{s,b} \) are distinct, there exists an ideal \( I \leq a \) for which \( r(I) = s(I) \), in which case the function \( E_I^{x(I)} \) does the separation.

Next we show that this homomorphism is injective on \( D_n \) for each \( n \). Fix \( n \), let \( I \) be an ideal whose prime factors are all in \( \{P_1, P_2, \ldots, P_n\} \) and choose a class \( x \in R/I \).

Choose \( a \in I \) such that \( \varepsilon_{P_j}(a) = \varepsilon_{P_j}(I) \) for \( j = 1, 2, \ldots, n \) (if \( I \) is principal, a generator will do; otherwise adjust with a prime ideal \( Q \notin \{P_1, P_2, \ldots, P_n\} \) such that \( IQ = I \).)

Also choose \( r \in \hat{R}/a\hat{R} \) such that \( r(I) = x \) in \( R/I \). Then \( E_I^{x(I)}(\omega_{r,a}) = 1 \), but \( E_I^{x(I)}(\omega_{r,a}) = 0 \) for each \( j \), proving that \( h(\mathfrak{d}_{I,n}^x) \neq 0 \). Hence \( h \) is injective on \( D_n \) and the result follows by Theorem 4.13.

As a byproduct we see that \( \Omega_{\hat{R}} \) is an ‘adelic’ realization of the spectrum of \( \hat{D} \).

**Corollary 5.3** We view each nonzero ideal \( I \) of \( R \) as an element \( a(I) \) of \( \hat{R}/\hat{R}^\times \) and, similarly, we view each \( x \in R/I \) as a class \( r(x, I) \) in \( \hat{R}/a\hat{R} \) of \( \hat{R}/R/I \). Then the map \( \eta_I^{x} \mapsto \omega_{r(x, I),a(I)} \) defined for \( (x, I) \in \bigsqcup_x R/I \) extends to a homeomorphism of the spectrum \( Y_{\hat{R}} \) of \( \hat{D} \) onto \( \Omega_{\hat{R}} \).

**Proof** From the proof of Proposition 5.2, we know that the isomorphism \( h \) maps the projection \( e_I^{x} \) onto the projection \( E_I^{x} \), giving an isomorphism of \( \hat{D} \) to \( \hat{\mathcal{C}}(\Omega_{\hat{R}}) \). To conclude that the homeomorphism \( \hat{h}^{-1} : Y_{\hat{R}} \to \Omega_{\hat{R}} \) induced by this isomorphism maps \( \eta_I^{x} \) to \( \omega_{r,a} \) as stated, it suffices to evaluate

\[
\eta_I^{x}(e_I^{y}) = \begin{cases} 1 & \text{if } I \subset J \text{ and } x = y(\text{mod } J) \\ 0 & \text{otherwise,} \end{cases}
\]

and

\[
E_I^{y}(\omega_{r(x, I),a(I)}) = \begin{cases} 1 & \text{if } J \leq a(I) \text{ and } r(x, I) = y(\text{mod } J) \\ 0 & \text{otherwise} \end{cases}
\]
for every ideal \( J \) in \( R \) and \( y \in R/J \), and to observe that the two results coincide because \( r(x, I) = x \mod I \).

\[ \square \]

### 6 KMS-states for \( \beta \leq 2 \)

Recall that for a non-zero ideal \( I \) in \( R \) we denote by \( N(I) \) the norm of \( I \), i.e. the number \( N(I) = |R/I| \) of elements in \( R/I \). For \( a \in R^\times \) we also write \( N(a) = N(aR) \). The norm is multiplicative, [19].

Using the norm one defines a natural one-parameter automorphism group \( (\sigma_t)_{t \in \mathbb{R}} \) on \( \mathfrak{S}[R] \), given on the generators by

\[
\sigma_t(u^x) = u^x \quad \sigma_t(e_I) = e_I \quad \sigma_t(s_a) = N(a)^t s_a
\]

(this assignment manifestly respects the relations between the generators and thus induces an automorphism). Recall that a \( \beta \)-KMS state with respect to a one parameter automorphism group \( (\sigma_t)_{t \in \mathbb{R}} \) is a state \( \varphi \) which satisfies \( \varphi(xy) = \varphi(x\sigma_t(y)) \) for a dense set of analytic vectors \( x, y \) and for the natural extension of \( (\sigma_t) \) to complex parameters on analytic vectors, [3]. Here KMS-states on \( \mathfrak{S}[R] \) are always understood as KMS with respect to the one-parameter automorphism group \( \sigma \) defined above, in which case the \( \beta \)-KMS condition for a state \( \varphi \) on \( \mathfrak{S}[R] \) translates to

\[
\varphi(u^x z) = \varphi(z u^x) \quad \varphi(e_I z) = \varphi(z e_I) \quad \varphi(s_a z) = N(a)^{-\beta} \varphi(z s_a)
\]

for a set of analytic vectors \( z \) with dense linear span and for the standard generators \( u^x, e_I, s_a \) of \( \mathfrak{S}[R] \). We will usually choose \( z \) to be a product of the form \( s_b^* d u^x s_a \) with \( d \in D \). We will use in the following the notation from Sect. 4.

**Proposition 6.1** \( \varphi \) is a \( \beta \)-KMS state on \( \mathfrak{S}[R] \) for \( \beta < 1 \).

**Proof** Given \( a \in R^\times \) and \( x \in R/aR \), denote by \( e^x_a \) the projection \( u^x e_{aR} u^{-x} \). If we apply a \( \beta \)-KMS state \( \varphi \) to the inequality

\[
\sum_{x \in R/aR} e^x_a = \sum_{x \in R/aR} u^x s_a s^*_a u^{-x} \leq 1
\]

using that \( \varphi(e^x_a) = N(a)^{-\beta} \) by (2) and that \( |R/aR| = N(a) \), we obtain \( N(a) N(a)^{-\beta} \leq 1 \), which implies \( \beta \geq 1 \).

**Lemma 6.2** Let \( \varphi \) be a \( \beta \)-KMS state for \( \beta > 1 \) on \( \mathfrak{S}[R] \) and let \( \pi_\varphi \) be the associated GNS-representation on \( H_\varphi \). Then \( \pi_\varphi(D_n) H_\varphi \) is dense in \( H_\varphi \), for each \( n \).

**Proof** Fix \( n \in \mathbb{N} \) and let \( J \) be an ideal in \( \mathcal{I}_n \). Since the class group for the field of fractions \( K \) is finite, there is \( k \in \mathbb{N} \) such that \( J^k = aR \) with \( a \in R^\times \). We have \( N(J^k) = N(a) \).

The subspace \( L = \pi_\varphi(D_n) H_\varphi \) is invariant under all \( u^x, x \in R \). It is also invariant under all \( s_c, s^*_c, c \in R^\times \). The reason is that if \( cR = QS \) with \( S \in \mathcal{I}_n \) and \( Q \) relatively
prime to $P_1, P_2, \ldots, P_n$, then, according to Lemma 4.11, for every $I \in \mathcal{I}_n$ we have $s_c \delta_I^s s_a^* e_u^{u^x} \leq \delta^c_I \in \mathcal{D}_n$.

Denote by $E$ the orthogonal projection onto $L^\perp$. Then $1 - E$ is the strong limit of $\pi_\varphi(h^{1/n})$ where $h$ is a strictly positive element in $\mathcal{D}_n$. Therefore, $\varphi_E$ defined by $\varphi_E(z) = (E \pi_\varphi(z) \xi_\varphi) \xi_\varphi$, for the cyclic vector $\xi_\varphi$ in the GNS-construction, is a $\beta$-KMS functional (consider the limit $n \to \infty$ of the expression $\varphi((1 - h^{1/n})x \sigma_\beta(y)) = \varphi(y(1 - h^{1/n})x)$ using the fact that $E$ commutes with $y$). Consider the restriction $\rho$ of $\pi_\varphi$ to $L^\perp$. Then $\rho(\mathcal{D}_n) = 0$, whence $\rho(1 - f_{Jk}) = 0$.

It follows that

$$\rho(1) = \sum_{x \in R/K} \rho\left(u^x s_a s_a^* u^{-x}\right).$$

This implies that $\varphi_E(1) = N(J^k)\varphi_E(s_a s_a^*) = N(a)\varphi_E(s_a s_a^*)$. On the other hand, the fact that $\varphi_E$ is $\beta$-KMS implies that $\varphi_E(1) = \varphi_E(s_a s_a^*) = N(a)\varphi_E(s_a s_a^*)$. Since $\beta > 1$, it follows that $\varphi_E(1) = 0$ and hence $E = 0$. \hfill $\Box$

**Lemma 6.3** Let $\varphi$ be a $\beta$-KMS state for $1 \leq \beta \leq 2$ on $\mathcal{S}[R]$ and let $I$ be a fixed ideal in $R$. Then $\varphi(\delta_{I,n}^0)$ tends to 0 for $n \to \infty$.

**Proof** Consider again the GNS-representation $\pi_\varphi$ on $H_\varphi$. Let $\tilde{\delta}_I$ denote the limit, in the strong operator topology, of the decreasing sequence $(\pi_\varphi(\delta_{I,n}^0))$ as $n \to \infty$. By Lemma 4.7, the projections $\pi_\varphi(u^x s_a) \tilde{\delta}_I \pi_\varphi(s_a u^{-x})$ are pairwise orthogonal for $a \in R^x/R^x, x \in R/a R$. If we let $\tilde{\varphi}$ denote the vector state extension of $\varphi$ to $\tilde{\mathcal{L}}(H_\varphi)$ we have

$$\sum_{a \in R^x/R^x, x \in R/a R} \tilde{\varphi}(u^x s_a \delta_I s_a^* u^{-x}) \leq \varphi(1) = 1.$$

However, since $\tilde{\varphi}$ is normal we have $\lim_{n \to \infty} \varphi(\delta_{I,n}^0 P_{I_1}P_{I_2} \ldots P_{I_n}) = \varphi(\tilde{\delta}_I)$ and since $\varphi$ is $\beta$-KMS, it follows that $\varphi(u^x s_a \delta_I s_a^* u^{-x}) = N(a)^{-\beta} \varphi(\tilde{\delta}_I)$.

Thus

$$\tilde{\varphi}(\tilde{\delta}_I) \sum_{a \in R^x/R^x} N(a) N(a)^{-\beta} \leq 1,$$

The series on the left hand side represents the partial Dedekind $\xi$-function, corresponding to the trivial ideal class, at $\beta - 1$. Thus, by Neukirch [19], Theorem 5.9, it diverges for $\beta - 1 \leq 1$. Therefore the inequality above implies $\tilde{\varphi}(\tilde{\delta}_I) = 0$. \hfill $\Box$

**Lemma 6.4** Let $\varphi$ be a $\beta$-KMS state on $\mathcal{S}[R]$ and let $I, J \in \mathcal{I}_n$ be two ideals in $R$ in the same ideal class. Then, for any $x, y \in R$,

$$\varphi(\delta_{I,n}^x) = N(I)^{-\beta} N(J)^\beta \varphi(\delta_{J,n}^y)$$
and

$$\varphi(\varepsilon_{IP_1P_2\ldots P_n}) = N(I)^{1-\beta}N(J)^{\beta-1}\varphi(\varepsilon_{IP_1P_2\ldots P_n}).$$

**Proof** If $aI = bJ$ then $N(a)N(b)^{-1} = N(I)^{-1}N(J)$ and, by Lemma 4.11, $s_b\delta_{I,n}^0 s_a^* = s_b\delta_{I,n}^0 s_b^*$. Therefore

$$\varphi(\delta_{I,n}^0) = \varphi(s_b\delta_{I,n}^0 s_a^*) = N(a)\beta \varphi(s_b\delta_{I,n}^0 s_b^*)$$

and

$$\varphi(\delta_{I,n}^0) = N(I)^{-\beta}N(J)^{\beta}\varphi(\delta_{J,n}^0).$$

Summing over $x \in R/I$ and $y \in R/J$ gives the second statement. □

**Lemma 6.5** Let $\varphi$ be a $\beta$-KMS state for $1 \leq \beta \leq 2$ on $\mathfrak{T}[R]$. Let $\mathfrak{T}$ be the canonical subalgebra of $\mathfrak{T}[R]$ generated by all projections $u^*e_1u^{-x}$ and let $d \in \mathfrak{T}$. Then $\varphi(s_a^*du^ys_b)$ is zero except if $a = b$ and $y = 0$ (in which case the argument $s_a^*du^ys_b$ is also an element in $\mathfrak{T}$).

**Proof** Suppose first $\beta = 1$ and let $\varphi$ be a 1-KMS state; then for each $c \in R^\times$ and $x \in R/cR$ we get $\varphi(e_c^x) = \varphi(u^xsc_a^*u^{-x}) = N(c)^{-1}$, whence $\sum_{x \in R/cR} \varphi(e_c^x) = 1$ and

$$\varphi(z) = \varphi\left(\left(\sum_{x \in R/cR} e_c^x\right)z\right) = \varphi\left(\sum_{x \in R/cR} e_c^xze_c^x\right)$$

for each $z \in \mathfrak{T}$ [R]. Again, let $z = s_a^*du^ys_b$; then

$$e_c^xze_c^x = e_c^xsc_a^*du^ys_b^xe_c^x = s_a^*e_{bc}^{ax}e_{bc}^{by}du^ys_b$$

where the product $e_{bc}^{ax}e_{bc}^{by}$ is nonzero if and only if $(ax + acR) \cap (bx + y + bcR) \neq \emptyset$, which implies

$$(a - b)x \equiv y \mod cR. \quad (4)$$

Suppose $z \notin \mathfrak{T}$, then either $a \neq b$ or else $a = b$ and $y \neq 0$. In the first case choose $c \in R^\times$ with $cR$ relatively prime to $(a - b)R$; then there is a unique $x \in R/cR$ for which (4) holds. In the second case, choose $c \in R^\times$ with $cR$ relatively prime to $y$; then (4) has no solutions in $x$. Thus for $z \notin \mathfrak{T}$ there is at most one $x \in R/(cR)$ such that $\varphi(e_c^xze_c^x) \neq 0$, and from Eq. (3) we obtain

$$|\varphi(z)| \leq N(c)^{-1}\|z\|.$$

Since $N(c)$ can be chosen arbitrarily large this shows that $\varphi(z) = 0$. This proves that $\varphi(z) \neq 0$ only if $a = b$ and $y = 0$ in the case $\beta = 1$. 

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Suppose now that $1 < \beta \leq 2$. It follows from Lemma 6.2 and from the normality of the vector state extension of $\varphi$ to $L(H_R)$ that for all $z \in \Sigma[R]$ and $n \in \mathbb{N}$, we have

$$\varphi(z) = \sum_{I \in I_s, x \in R/I} \varphi(\delta^x_{I,n} z \delta^x_{I,n}).$$

(5)

Working this out for $z = s_a^* du^y s_b$, where we may assume that $n$ is so large that $aR, bR \in I_n$, we find

$$\delta^x_{I,n} s_a^* du^y s_b \delta^x_{I,n} = s_a^* s_{aI,n} du^y s_{bI,n} s_b = s_a^* s_{aI,n} \delta_{bI,n} du^y s_b$$

and this expression (call it $z^x_I$) does not vanish only if $aI = bI$, i.e. if $a = gb$ for a unit $g \in R^*$. We have to consider only that case.

Assume first that $g = 1$. Then $z^x_I \neq 0$ only if $bx \equiv by + y \mod aI$, that is, only if $y \in bI$. For a fixed $y \neq 0$ this last condition is satisfied only for the finitely many ideals $I$ in $R$ such that $bI$ divides $yR$. Thus, if $y \neq 0$, in the sum (5) there are at most a fixed finite number (independent of $n$) of non-zero terms and each individual term is bounded by $\varphi(\delta^0_{I,n}) \|z\|$, which is arbitrarily small for large $n$ by Lemma 6.3, whence $\varphi(z) = 0$.

Assume now that $g \neq 1$. Then $z^x_I \neq 0$ only if $x$ satisfies $(g - 1)bx \equiv y \mod bI$.

Let $D := \gcd(I, (g - 1)R)$. If $y \not\equiv bD$, then there is no such $x$. Assume thus $y \in bD$, and write $y = by'$ with $y' \in D$. The nonzero terms in the sum (5) may only arise from $y$ and $I$ such that $(g - 1)x \equiv y' \mod I$. Notice that multiplication by $(g - 1)$, viewed as a map $R/I \to R/I$ is $N(D)$-to-one. Since $N(g - 1) \geq N(D)$, $N(g - 1)$ is a uniform bound on the number of solutions $x$ of the equation $(g - 1)x \equiv y' \mod I$.

Thus, for each ideal $I$ there are at most $N(g - 1)$ classes $x + I$ in $R/I$ such that $z^x_I \neq 0$. Choosing a reference ideal $J_\gamma$ for each ideal class $\gamma$ and using Lemma 6.4, we can transform (5) into an estimate

$$|\varphi(z)| \leq \sum_{\gamma} \sum_{I \in I_s \cap \gamma} N(g - 1)\gamma^{N(I) - \beta} N(I, \gamma)^\beta \varphi(\delta^0_{J_\gamma, n}) \|z\|.

Since the series for all the partial Dedekind $\zeta$-functions converge for $\beta > 1$ and since each $\varphi(\delta^0_{J_\gamma, n}) \to 0$ as $n \to \infty$ by Lemma 6.3, we conclude that $\varphi(z) = 0$ unless $a = b$ and $y = 0$ also for $1 < \beta \leq 2$, completing the proof.

As a consequence of this lemma, in order to know $\varphi$, it suffices to know its values on $D$. Moreover, for $1 < \beta \leq 2$ it suffices to know $\varphi$ on $D_n$ for all $n$, by Lemma 6.2.

**Theorem 6.6** Let $0 < \sigma \leq 1$. For each ideal class $\gamma$ and for each $n \in \mathbb{N}$ we set $\zeta^{(n)}_\gamma(\sigma) = \sum_{I \in I_n \cap \gamma} N(I)^{-\sigma}$. Then for any two ideal classes $\gamma_1, \gamma_2$ the quotient

$$\frac{\zeta^{(n)}_{\gamma_1}(\sigma)}{\zeta^{(n)}_{\gamma_2}(\sigma)}$$

tends to 1 as $n \to \infty$.

We postpone the proof and give it in the appendix.

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Theorem 6.7 For each $\beta$ with $1 \leq \beta \leq 2$ there is exactly one $\beta$-KMS state $\varphi_\beta$ on $\mathfrak{F}[R]$, it factors through the canonical conditional expectation $E : \mathfrak{F}[R] \to \mathfrak{D}$ and it is determined by the values
\[ \varphi_\beta(e_I^x) = N(I)^{-\beta} \quad \text{with } I \text{ an ideal in } R \text{ and } x \in R/I. \] (6)

For $\beta = 1$ the state $\varphi_\beta$ factors through the natural quotient map $\mathfrak{F}[R] \to \mathfrak{A}[R]$.

Proof Suppose $\varphi$ is a $\beta$-KMS state. Lemma 6.5 implies that $\varphi$ factors through the conditional expectation $E : \mathfrak{F}[R] \to \mathfrak{D}$ for $1 \leq \beta \leq 2$.

The next step is to show that (6) holds. Since the linear combinations of the projections $e_I^x := u^x e_I u^{-x}$ are dense in $\mathfrak{D}$ and since $\varphi(u^x e_I u^{-x}) = \varphi(e_I)$, this will yield the uniqueness assertion. The argument for $\beta = 1$ is easier and we do it first.

Assume first $\beta = 1$ and let $\varphi$ be a 1-KMS state. If $I$ is any (non-zero) ideal in $R$, then
\[ 1 = \varphi(1) \geq \varphi \left( \sum_{x \in R/I} e_I^x \right) = N(I) \varphi(e_I) \]
and if $aR \subset I$ and $y \in R/I$, then
\[ \varphi(e_I^y) = \varphi(e_I) \geq \sum_{x \in I/aR} \varphi(e_a^x) = (N(a)N(I)^{-1})N(a)^{-1} = N(I)^{-1} \]

For the identity on the right hand side note that $\varphi(e_a^x) = N(a)^{-1}$ and that $|I/aR| = N(a)N(I)^{-1}$. We conclude that $\varphi(e_I^y) = N(I)^{-1}$, so (6) holds for $\beta = 1$.

It is obvious that such a state $\varphi$ satisfies $\varphi(f_P) = 1$ and hence vanishes on the projections of the form $e_P$ that generate the kernel of the quotient map $q : \mathfrak{F}[R] \to \mathfrak{A}[R]$ as an ideal, so $\varphi$ factors through this quotient. It is now easy to prove existence of a 1-KMS state. From Sect. 4 of [5], and the fact that $q$ intertwines the canonical conditional expectations on $\mathfrak{F}[R]$ and on $\mathfrak{A}[R]$, we know that the image of $\mathfrak{D}$ in $\mathfrak{A}[R]$ under $q$ is naturally isomorphic to $C(\hat{R})$ ($\hat{R}$ being the profinite completion of $R$). If we let $\lambda_1$ be the state of $C(\hat{R})$ given by normalized Haar measure on $\hat{R}$, an easy computation shows that $\varphi_1 := \lambda_1 \circ E \circ q$ satisfies the 1-KMS condition from (2). This finishes the proof in the case $\beta = 1$.

Assume now $1 < \beta \leq 2$ and let $\varphi$ be a $\beta$-KMS state. Using Lemma 4.7, for the particular element $e_I$ of $\mathfrak{D}_n$, with $I \in \mathcal{I}_n$, and working in the GNS representation $\pi_\varphi$ of $\varphi$ we obtain the formula
\[ \pi_\varphi(e_I) = \sum_{J \in \mathcal{I}_n, J \subset I, x \in I/J} \pi_\varphi(\delta_{J,n}^x), \] (7)

with strong operator convergence by Lemma 6.2. We know from Lemma 6.4 that for $J, L \in \mathcal{I}_n$ in the same ideal class we have
\[ \varphi(\delta_{L,n}^x) = N(L)^{-\beta} N(J)^{\beta} \varphi(\delta_{J,n}^x) \]
Thus, for an ideal class $\gamma$ in the ideal class group $\Gamma$ for $R$, the expression

$$\alpha^{(n)}_\gamma = N(L)^\beta \varphi(\delta_{L,n}^\gamma)$$

does not depend on the choice of an ideal $L \in \gamma \cap I_n$. Using the vector state extension $\bar{\varphi}$ of $\varphi$ to handle the infinite sum we obtain

$$1 = \varphi(1) = \sum_{I \in I_n, x \in R/I} \varphi(\delta_{I,n}^\gamma) = \sum_{I \in I_n} N(I) \varphi(\delta_{I,n}^0) = \sum_{\gamma \in \Gamma} \alpha^{(n)}_{\gamma} \zeta^{(n)}_{\gamma}(\beta - 1) \quad (8)$$

On the other hand, using (7) and computing with $\bar{\varphi}$ again, we see that

$$\varphi(e_I) = \sum_{J \subset I, x \in I/J} \varphi(\delta_{J,n}^x) = \sum_{J \subset I} \alpha^{(n)}_{IJ} N(I)^{-1} N(J)^{-\beta} = \sum_{\gamma \in \Gamma} \alpha^{(n)}_{I\gamma} \zeta^{(n)}_{I\gamma}(\beta - 1).$$

Dividing by the right hand side of Eq. (8) above and using Theorem 6.6, we see that this last expression converges to $N(I)^{-\beta}$ as $n \to \infty$, proving that (6) holds when $1 < \beta \leq 2$.

Let us now prove existence in this case. Since $D_n$ is essential in $\bar{D}_n$ there is a natural embedding $D_n \hookrightarrow \ell^\infty(\text{Spec } D_n)$ and we know that $\text{Spec } D_n = \bigcup_{I \in I_n} R/I$. The minimal projections in $D_n$ are the $\delta_{I,n}^x$, $I \in I_n$, $x \in R/I$. Thus any $d$ in $\bar{D}_n$ is represented by an $\ell^\infty$-function $(I, x) \mapsto \lambda^x_I(d)$ uniquely defined by $d I \delta_{I,n}^x = \lambda^x_I(d) \delta_{I,n}^x$ on $\text{Spec } D_n$. Notice that for $d \in D_n$ one actually has $d = \sum \lambda^x_I(d) \delta_{I,n}^x$.

We define a state $\varphi_n$ on $\bar{D}_n$ by

$$\varphi_n(d) = \frac{\sum_{I \in I_n, x \in R/I} \lambda^x_I(d) N(I)^{-\beta}}{\sum_{I \in I_n} N(I)^{1-\beta}}.$$

One obviously has $\varphi_n(u^x d u^{-x}) = \varphi_n(d)$. Since $\lambda^x_I(s_a d s_a^*) = \lambda^x_I(d)$ if $I = aJ$ and $x = ay$, and $\lambda^x_I(s_a d s_a^*) = 0$ otherwise, Lemma 4.11 implies that $\varphi_n(s_a d s_a^*) = N(a)^{-\beta} \varphi_n(d)$ for $a \in R^\times$ with $aR \in I_n$. Since we also have $\varphi_{n+1}|_{\bar{D}_n} = \varphi_n$, the sequence determines a state $\varphi_\infty$ on $\bar{D} = \bigcup_n \bar{D}_n$. We define a state $\varphi_\beta$ on $\mathfrak{I}[R]$ by $\varphi_\beta = \varphi_\infty \circ \mathfrak{I}$ where $E : \mathfrak{I}[R] \to \bar{D}$ is the canonical conditional expectation. Then for an element $z = s_a^x d u^y s_b$ one has

$$\varphi_\beta(s_c z) = N(c)^{-\beta} \varphi_\beta(z s_c) \quad \varphi_\beta(u^y z) = \varphi_\beta(z u^y) \quad \varphi_\beta(d' z) = \varphi_\beta(z d')$$

for $c \in R^\times$, $y \in R$, $d' \in \bar{D}$. This suffices to show that $\varphi_\beta$ is $\beta$-KMS. □

Remark 6.8 Note that the above construction of the state $\varphi_\infty$ of $\bar{D}$ and thus of $\varphi_\beta$ carries through for all $\beta > 1$. Note also that the state $\varphi_\infty$ of $\bar{D}$ is the infinite tensor

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The product state \( \varphi_\infty = \bigotimes_P \varphi_P \) over all prime ideals \( P \) in \( R \) of the states \( \varphi_P \) defined on \( \mathcal{D}_P \) by

\[
\varphi_P(d) = \frac{\sum_{n \geq 0, x \in \mathcal{R}/P^n} \lambda_n^x(d) N(P)^{-n\beta}}{\sum_{n \geq 0} N(P)^{n(1-\beta)}}
\]

for \( d = \sum \lambda_n^x \delta_{P^n} \). For \( \beta = 1 \) one takes \( \varphi_P \) to be induced from normalized Haar measure on the \( P \)-adic completion \( \mathcal{R}_P \).

7 KMS-states for \( \beta > 2 \)

The basis for the study of KMS-states in this range is the natural representation \( \mu \) of \( T[\mathcal{R}] \) on the Hilbert space

\[
H_R = \bigoplus_{I \text{ ideal in } \mathcal{R}} \ell^2(R/I)
\]

which has been used already in Sect. 4.

Let \( \mathcal{E}_I \) denote the projection onto the subspace \( \ell^2(R/I) \) of \( H \) and define the operator \( \Delta \) on \( H \) by \( \Delta = \sum_{I} N(I)^{-1} \mathcal{E}_I \). Then \( \Delta \) commutes with \( \mu(u^x) \) for every \( x \in \mathcal{R} \) and \( \Delta \mu(s_a) = N(a)^{-1} \mu(s_a) \Delta \), hence the dynamics \( \sigma \) is implemented spatially by the unitary group \( t \mapsto \Delta^{it} \). Since \( \text{tr}(\Delta^\beta) = \zeta(\beta - 1) \), the operator \( \Delta^\beta \) is of trace class for \( \beta > 2 \) and

\[
\varphi(z) = \text{Tr} \left( \mu(z) \Delta^\beta \right) / \zeta(\beta - 1)
\]

defines a \( \beta \)-KMS state \( \varphi \) for each \( \beta > 2 \). Denote by \( \Gamma \) the ideal class group of our number field \( K \). The Hilbert space \( H \) splits canonically into a sum of invariant subspaces \( H = \bigoplus_{\gamma \in \Gamma} H_\gamma \), where \( H_\gamma = \bigoplus_{I \in \gamma} \ell^2(R/I) \). Denoting by \( \mu_\gamma \) and \( \Delta_\gamma \) the restrictions of \( \mu \) and \( \Delta \) to \( H_\gamma \) we obtain a decomposition of \( \varphi \) as a convex linear combination:

\[
\varphi = \sum_{\gamma \in \Gamma} \frac{\zeta_\gamma(\beta - 1)}{\xi(\beta - 1)} \varphi_\gamma
\]
in which the \( \beta \)-KMS state \( \varphi_\gamma \) associated to the class \( \gamma \) is defined by

\[
\varphi_\gamma(z) = \text{Tr} \left( \mu(z) \gamma \Delta_\gamma^\beta \right) / \text{Tr} \left( \Delta_\gamma^\beta \right),
\]

where \( \text{Tr} \left( \Delta_\gamma^\beta \right) \) is the corresponding partial zeta function \( \xi_\gamma(\beta - 1) \). We will see below that this family of \( \beta \)-KMS states on \( T[\mathcal{R}] \) parametrized by \( \Gamma \) consists of different states.

To obtain the most general KMS-state, we have to consider a more general family of representations of \( T[\mathcal{R}] \). We fix temporarily a class \( \gamma \) in the class group \( \Gamma \) and we choose a reference ideal \( J = J_\gamma \) in this class.
Let \( \tau \) be a tracial state on the C*-algebra \( C^*(J \rtimes R^*) \), where the semidirect product is taken with respect to the multiplicative action of the group of invertible elements (units) \( R^* \) on the additive group \( J \). Note that these traces form a Choquet simplex [21]. By Neshveyev [18, Corollary 5] the extreme points can be parametrized by pairs in which the first component is an ergodic \( R^* \)-invariant probability measure \( \mu \) on the compact dual group \( \hat{J} \) on which the isotropy is a constant group \( \mu \)-a.e., and the second component is a character of that isotropy group.

Denote by \((H_J, \pi_J, \xi_J)\) the GNS-construction for \( \tau \), with \( H_J = L^2(C^*(J \rtimes R^*), \tau) \). If \( I \) is another integral ideal in the class \( \gamma \), then there is \( a \in K^\times \) such that \( I = aJ \). Multiplication by \( a \) induces an isomorphism \( J \rightarrow I \) which commutes with the action of \( R^* \), hence \((j, g) \mapsto (aj, g)\) induces an isomorphism of groups \( J \rtimes R^* \cong I \rtimes R^* \) and of C*-algebras \( C^*(J \rtimes R^*) \cong C^*(I \rtimes R^*) \). The trace \( \tau_a \) on \( C^*(I \rtimes R^*) \) obtained from \( \tau \) via this isomorphism is given by \( \tau_a(\delta_{(x,g)}) = \tau(\delta_{(a^{-1}x,g)}) \) where \( \delta_{(x,g)} \) runs through the canonical generators of \( C^*(I \rtimes R^*) \). If \( aJ = bJ \), then \( ab^{-1} = g \in R^* \) and for every \( \delta_{(j,g')} \) in \( C^*(I \rtimes R^*) \) we have

\[
\tau_b(\delta_{(j,g')} ) = \tau(\delta_{(b^{-1}j,g')}) = \tau_a(\delta(0,g)\delta_{(j,g')}\delta_{(0,g)}) = \tau_a(\delta_{(j,g')}),
\]

so \( \tau_a \) does not depend on the choice of such an \( a \), and we denote it simply as \( \tau_I \). From the isomorphism \( J \rtimes R^* \cong I \rtimes R^* \) we obtain an isomorphism \( H_J \rightarrow H_I \) intertwining the representations \( \pi_J \) and \( \pi_I \), in which the cyclic vector \( \xi_J \in H_J \) is mapped to the corresponding cyclic vector \( \xi_I \in H_I \).

The representation \( \pi_I \) of \( C^*(I \rtimes R^*) \) can be induced to a natural representation (which we also denote \( \pi_I \)) of \( C^*(R \rtimes R^*) \) on

\[
\ell^2(R/I, H_I) \cong \{ f : R \rightarrow H_I \mid f(x+y) = \pi_I(u^x)(f(y)), \ x \in I, \}
\]

**Lemma 7.1** The direct sum representation \( \pi_{\tau} := \bigoplus_{I \in \gamma} \pi_I \) of \( C^*(R \rtimes R^*) \) on the Hilbert space

\[
H_{\tau} = \bigoplus_{I \in \gamma} \ell^2(R/I, H_I)
\]

extends to a representation of \( \mathcal{S}[R] \) on the same Hilbert space.

**Proof** To simplify the notation let \( U^x := \pi_{\tau}(u^x) \) for \( x \in R \) and \( S_g := \pi_{\tau}(s_g) \) for \( g \in R^* \). We may view the cyclic vector \( \xi_I \in H_I \) as a vector in \( \ell^2(R/I, H_I) \) (supported on the trivial class) which is cyclic for the action of \( C^*(R \rtimes R^*) \) on \( \ell^2(R/I, H_I) \).

Next we define \( S_a \) for \( a \in R^\times \). By Marcelo and Machiel [14, Lemma 1.11] there exists a multiplicative cross section of the quotient \( R^\times \rightarrow R^\times/R^* \) and thus we have a homomorphism \( a \mapsto \tilde{a} \) of \( R^\times \) into itself such that for each \( a \in R^\times \) there exists a unique \( g \in R^* \) with \( a = \tilde{a}g \).

First we define \( S_a \) for \( a \) in the range of the cross section by

\[
S_a(u^x s_w \xi_I) := u^{\tilde{a}x} s_w \xi_{\tilde{a}I}
\]

for \( x \in R \) and \( w \in R^* \). Since

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because $\tau_I$ and $\tau_{\alpha I}$ are traces satisfying $\tau_{\alpha I}(u^\alpha s_w) = \tau_I(u^\alpha s_w)$, the map $S_{\alpha}$ is isometric on a dense set and thus extends uniquely by linearity and continuity to an isometry $S_{\alpha}$ of $l^2(R/I, H_I)$ into $l^2(R/\alpha I, H_{\alpha I})$. For general $a \in R^\times$ we simply write $a = \tilde{a}g$ and we let $S_a := S_{\tilde{a}}S_g$.

For an ideal $L \subset R$, we view $l^2(L/I, H_I)$ as the obvious subspace of $l^2(R/I, H_I)$ and we define $E_L$ to be the orthogonal projection onto $\bigoplus_{I \in \mathcal{I}, L \subset I} l^2(L/I, H_I)$.

It is easy to verify that $S$ is a representation of the semigroup $R^\times$ by isometries and that $E$ is a family of projections representing the lattice of ideals of $R$, such that $U$, $S$, and $E$ satisfy the relations defining $\mathfrak{T}[R]$. Hence there is a representation $\mu_\tau$ of $\mathfrak{T}[R]$ such that $\mu_\tau(u^x) = U^x$, $\mu_\tau(s_a) = S_a$ and $\mu_\tau(e_I) = E_I$. □

We can use the representation $\mu_\tau$ to define a $\beta$-KMS state as follows. Let $\mathcal{E}_I$ denote the orthogonal projection onto the subspace $l^2(R/I, H_I)$ and define a positive operator $\Delta$ on $H_r$ by $\Delta = \sum N(I)^{-1}\mathcal{E}_I$. Since $\Delta$ commutes with $U^x$ and with $E_I$, and since $\Delta S_a = N(a)S_a \Delta$, the unitary group $t \mapsto \Delta^{it}$ implements the dynamics, just as in our initial example, but when $H_{J_r}$ is not finite dimensional, the operator $\Delta^\beta$ is not of trace class. Nevertheless, we have $\sum_{I \in \mathcal{Y}, x \in R/I}(\Delta^\beta U^x \xi_I \mid U^x \xi_I) = \Delta_\tau(\beta - 1)$, and setting

$$\varphi_{\gamma, \tau}(z) = \frac{\sum_{I \in \mathcal{Y}, x \in R/I}(\mu_\tau(z)\Delta^\beta U^x \xi_I \mid U^x \xi_I)}{\sum_{I \in \mathcal{Y}, x \in R/I}(\Delta^\beta U^x \xi_I \mid U^x \xi_I)}$$

(9)

yields a $\beta$-KMS state for each $\beta > 2$, by (2).

As before, let $P_1, P_2, \ldots$ be an enumeration of the prime ideals in $R$. When $\rho$ is a given representation of $\mathfrak{T}[R]$, for each ideal $I$ in $R$, let $\tilde{\xi}_I$ denote the strong operator limit of the decreasing sequence of projections $\rho(\epsilon_IP_1P_2\ldots P_n)$. The $\tilde{\xi}_I$ form a family of pairwise orthogonal projections. Similarly, let $\tilde{\delta}_I$ be the strong limit of the decreasing family of projections $\rho(\delta^0_{I,n})$, as in the proof of Lemma 6.3, and let $\tilde{\delta}_I := \rho(u^x)\tilde{\delta}_I\rho(u^{-x})$. If $I$ and $L$ are ideals in $R$, then
\[ \tilde{\delta}_I^x \rho(e_L) = \begin{cases} \tilde{\delta}_I^x & \text{if } I \subset L \text{ and } x \in L/I \\ 0 & \text{otherwise,} \end{cases} \] (10)

To see why, observe that as soon as \( n \) is large enough that \( \{P_1, P_2, \ldots, P_n\} \) contains all the prime factors of \( I \) and \( L, \tilde{\delta}_{I,n}^0 e_L = \delta_{I,n}^0 \) if \( I \subset L \) and \( x \in L/I \), and is 0 otherwise, from the description of \( D_n \) in Lemma 4.7.

**Lemma 7.2** Let \( \mu_\tau \) be the representation constructed in Lemma 7.1 from a trace \( \tau \) on \( C^*(J_\gamma \rtimes R^*) \), and let \( U^x := \mu_\tau(u^x) \) and \( S_\alpha := \mu_\tau(s_\alpha) \). Suppose I is an ideal in \( R \) and \( x \in R \);

(i) if \( I \in \gamma \), then \( U^x \tilde{\delta}_I U^{-x} = U^x E_I E_I U^{-x} \), the projection onto \( U^x H_I \);

(ii) if \( I \notin \gamma \), then \( U^x \tilde{\delta}_I U^{-x} = 0 \); and

(iii) the trace \( \tau \) is retrieved from \( \varphi_{\gamma,\tau} \) by conditioning to \( \tilde{\delta}_{J_\gamma} \):

\[ \tau(u^x s_g) = N(J_\gamma)^\beta \zeta_\gamma(\beta - 1) \varphi_{\gamma,\tau}(\tilde{\delta}_{J_\gamma} U^x S_g \tilde{\delta}_{J_\gamma}) \quad x \in J_\gamma, \ g \in R^*. \]

**Proof** For part (i), notice that when \( I \in \gamma \), then \( \tilde{E}_I = \tilde{\delta}_I := \lim_{n \to \infty} \mu_\tau(\varepsilon_{I P_1 P_2\ldots}) \), then multiply by \( E_I \) and translate with \( x \in R/I \).

For part (ii), recall that if \( I \) and \( I' \) are different ideals, then the projections \( \tilde{\delta}_I^x \) and \( \tilde{\delta}_{I'}^x \) are mutually orthogonal. Since the Hilbert space \( H_\tau = \bigoplus_{I \in \gamma} \ell^2(R/I, H_I) \) is generated by the ranges of the projections \( \tilde{\delta}_I \) with \( I \in \gamma \) and \( x \in R/I \), it follows that \( \tilde{\delta}_{I'}^x = 0 \) whenever \( I' \notin \gamma \).

Finally, notice that \( H_{J_\gamma} \) viewed as a subspace of \( H_\tau \) is invariant for the action of \( C^*(J_\gamma \rtimes R^*) \) and, by construction, the restriction of \( \mu_\tau \) to \( C^*(J_\gamma \rtimes R^*) \) and to this subspace is the GNS representation of \( \tau \), with cyclic vector \( \xi_{J_\gamma} \). Since \( \tilde{\delta}_{J_\gamma} = E_{J_\gamma} \tilde{E}_{J_\gamma} \) is the projection onto \( H_{J_\gamma} \), the sum in Eq. (9) has only one term, giving the identity in part (iii). \( \square \)

It turns out that to parametrize the \( \beta \)-KMS states in the region \( \beta > 2 \) all we need to do is combine states constructed from different ideal classes.

**Theorem 7.3** Suppose \( \beta > 2 \) and choose a fixed reference ideal \( J_\gamma \in \gamma \) for each \( \gamma \) in the class group \( \Gamma \) of \( K \). For each tracial state \( \tau \) of \( \bigoplus_{\gamma} C^*(J_\gamma \rtimes R^*) \) write \( \tau = c_\gamma \tau_\gamma \) as a convex linear combination of traces on the components and define \( \varphi_\tau := \sum_{\gamma} c_\gamma \varphi_{\gamma,\tau_\gamma} \) using Eq. (9). Then the map \( \tau \mapsto \varphi_\tau \) is a continuous affine isomorphism of the Choquet simplex of tracial states of \( \bigoplus_{\gamma} C^*(J_\gamma \rtimes R^*) \) onto the simplex of \( \beta \)-KMS states for \( \mathbb{F}[R] \).

Going in the opposite direction, the \( \gamma \)-component of the trace \( \tau \) corresponding to a given \( \beta \)-KMS state \( \varphi \) is obtained by conditioning (the vector state extension of) \( \varphi \) to \( \tilde{\delta}_{J_\gamma} \):

\[ c_\gamma \tau_{\varphi,\gamma}(u^x s_g) := N(J_\gamma)^\beta \zeta_\gamma(\beta - 1) \bar{\varphi}(\tilde{\delta}_{J_\gamma} U^x S_g \tilde{\delta}_{J_\gamma}), \]

where \( c_\gamma := N(J_\gamma)^\beta \zeta_\gamma(\beta - 1) \bar{\varphi}(\tilde{\delta}_{J_\gamma}) \).

[Springer]
Proof} Since $\varphi_{\gamma, \tau}$ is a $\beta$-KMS state for each $\gamma$, so is $\varphi_{\tau} = \sum_{\gamma} c_{\gamma} \varphi_{\gamma, \tau}$. To see that $\tau$ is obtained from $\varphi_{\tau}$ by conditioning to $\delta_{I_{\gamma}}$, assume $c_{\gamma} \neq 0$ (otherwise skip $\gamma$). Then Lemma 7.2(iii) implies that

$$c_{\gamma} \varphi_{\gamma}(u^x s_g) = N(J_{\gamma})^\beta \zeta_{\gamma}(\beta - 1)\varphi_{\gamma}(\delta_{I_{\gamma}} U^x S_g \delta_{I_{\gamma}})$$

which is equal to $N(J_{\gamma})^\beta \zeta_{\gamma}(\beta - 1)\varphi_{\tau}(\delta_{I_{\gamma}} U^x S_g \delta_{I_{\gamma}})$ by Lemma 7.2(ii). This proves that the map $\tau \mapsto \varphi_{\tau}$ is injective. Next we show it is surjective.

Suppose $\varphi$ is a $\beta$-KMS state and let $\mathfrak{S}[R]$ be represented on $H_\varphi$ in the GNS-construction for $\varphi$. As usual, we denote by $\tilde{\varphi}$ the vector state extension of $\varphi$ to $\mathcal{L}(H_\varphi)$, and we also write $\pi_\varphi(u^x) = U^x$, $\pi_\varphi(s_a) = S_a$ for simplicity of notation.

We show next that $\bigoplus_I \tilde{\mathfrak{S}}_I(H_\varphi) = H_\varphi$. Using Lemma 6.2 we obtain, as soon as $I$ is in the semigroup $\mathcal{I}_n$ generated by $P_1, P_2, \ldots, P_n$,

$$\varphi(1) = \sum_{I \in \mathcal{I}_n} \varphi(\varepsilon I P_2 \ldots P_n) = \sum_{\gamma \in \Gamma} \sum_{I \in \mathcal{I}_n \cap \gamma} \varphi(\varepsilon I P_2 \ldots P_n)$$

where, according to Lemma 6.4

$$\varphi(\varepsilon I P_2 \ldots P_n) = N(J_{\gamma})^{\beta - 1} N(I)^{1 - \beta} \varphi(\varepsilon J_{\gamma} P_2 \ldots P_n)$$

for $I \in \mathcal{I}_n \cap \gamma$. In the limit $n \to \infty$ this gives

$$\varphi(1) = \sum_{\gamma \in \Gamma} N(J_{\gamma})^{\beta - 1} \zeta_{\gamma}(\beta - 1)\tilde{\varphi}(\tilde{\epsilon}_{I_{\gamma}}) = \sum_{\gamma \in \Gamma} N(J_{\gamma})^{\beta} \zeta_{\gamma}(\beta - 1)\tilde{\varphi}(\tilde{\delta}_{I_{\gamma}})$$

where $\zeta_{\gamma}$ is the partial $\zeta$-function $\zeta_{\gamma}(t) = \sum_{i \in \gamma} N(I)^{-t}$, which converges for $t > 1$.

Let $F$ denote the orthogonal complement of $\bigoplus_I \tilde{\mathfrak{S}}_I(H_\varphi)$ and $\psi$ the restriction of $\tilde{\varphi}$ to $\pi_\varphi(\mathfrak{S}[R])|_F$. Since $\psi$ is again a $\beta$-KMS functional (see [3], 5.3.4 and 5.3.29) and since the $F \tilde{\delta}_{I_{\gamma}} F = 0$, the above identity applied to $\psi$ shows that $\psi(F) = 0$ and thus $F = 0$, proving $\bigoplus_I \tilde{\mathfrak{S}}_I(H_\varphi) = H_\varphi$.

Since $\sum_{\gamma \in \Gamma} \sum_{I \in \gamma} \sum_{x \in R/I} \tilde{\delta}_I^x = \sum_{\gamma \in \Gamma} \sum_{I \in \gamma} \tilde{\epsilon}_I = 1$ in the GNS representation of $\varphi$ and since $\tilde{\delta}_I^x$ is in the centralizer of $\tilde{\varphi}$, we have

$$\tilde{\varphi}(\cdot) = \tilde{\varphi} \left( \sum_{\gamma, I, x} \tilde{\delta}_I^x \right) = \sum_{\gamma} \sum_{I \in \gamma} \sum_{x \in R/I} \tilde{\varphi}(\tilde{\delta}_I^x \cdot \tilde{\delta}_I^x).$$ \hspace{1cm} (11)

The projection $\tilde{\delta}_I$ commutes with $U^x S_g$ for $x \in I$ and $g \in R^*$, hence the canonical map $u^x s_g \mapsto \tilde{\delta}_I U^x S_g \tilde{\delta}_I$ determines a homomorphism of $C^*(I \rtimes R^*)$ to the corner $\tilde{\delta}_I \mathfrak{S}[R] \tilde{\delta}_I$. This homomorphism is surjective because

$$\tilde{\delta}_I S_a d U^x S_b \tilde{\delta}_I = S_a^x \tilde{\delta}_a \tilde{\delta}_b d U^y S_b$$
is nonzero only if \( aI = bI \) and \( y \in aI = bI \), i.e. only if \( b = ga \) for some \( g \in R^* \) and \( y' = y/a \in I \), in which case the whole expression reduces to \( \delta_I z_d U^y S^*_a S_b = z_d \delta_I U^y S_g \), where \( z_d \) is a scalar.

Next we show how to recover each \( \tilde{\phi}(\tilde{\delta}^x_I \cdot \tilde{\delta}^x_I) \) from \( \tilde{\phi}(\tilde{\delta}^x_I \cdot \tilde{\delta}^x_J \gamma) \). For each \( I \in \gamma \) there exist \( a_I \) and \( b_I \) in \( R^\times \) such that \( (a_I/b_I)J_\gamma = I \). By Lemma 4.11

\[
\tilde{\delta}^x_I := U^x \delta_I U^{-x} = U^x S^*_a S_b, \tilde{\delta}_I J_\gamma S^*_a S_b U^{-x}.
\]

Using the KMS-condition we obtain

\[
\tilde{\phi}(\tilde{\delta}^x_I \cdot \tilde{\delta}^x_I) = \tilde{\phi}(U^x S^*_a S_b a_I S^*_a S_b a_I S_b U^{-x} \cdot U^x S^*_a S_b a_I S^*_a S_b U^{-x}) = N(a_I/b_I)^{-\beta} \tilde{\phi}(\tilde{\delta}_I J_\gamma S^*_a S_b U^{-x} \cdot U^x S^*_a S_b a_I S^*_a S_b U^{-x}) = N(1)^{-\beta} N(J_\gamma)^{\beta} \tilde{\phi}(\tilde{\delta}_I J_\gamma S^*_a S_b U^{-x} \cdot U^x S^*_a S_b a_I S^*_a S_b U^{-x}).
\]

Notice that the choice of \( a_I \) and \( b_I \) does not affect the result because of Lemma 8.6 and because for \( g \in R^* \) the unitary \( S_g \) commutes with \( \tilde{\delta}_J \gamma \) and centralizes \( \phi \). Hence every \( \beta \)-KMS state for \( \beta > 2 \) is determined by the collection of conditional functionals \( \{ \tilde{\phi}(\tilde{\delta}_J \gamma \cdot \tilde{\delta}_J \gamma) \mid \gamma \in \Gamma \} \).

The state \( \phi \) gives rise to traces as follows. First let \( c_\gamma := N(J_\gamma)^{\beta} \xi_\gamma (\beta - 1) \tilde{\phi}(\tilde{\delta}_J \gamma) \) and recall that \( \sum_{\gamma} c_\gamma = 1 \) from above. When \( c_\gamma \neq 0 \), set

\[
c_\gamma \tau_{\gamma, \phi}(u^x S_g) := N(J_\gamma)^{\beta} \xi_\gamma (\beta - 1) \tilde{\phi}(\tilde{\delta}_J \gamma U^x S_g \tilde{\delta}_J \gamma),
\]

which defines a tracial state \( \tau_{\gamma, \phi} \) on \( C^*(J_\gamma \times R^*) \), by the KMS condition. This shows that the given \( \beta \)-KMS state \( \phi \) arises as \( \varphi_\tau \) from the trace \( \tau := \sum_{\gamma} c_\gamma \tau_{\gamma, \phi} \) that it determines on \( \bigoplus_{\gamma \in \Gamma} C^*(J_\gamma \times R^*) \), proving the surjectivity of the map \( \tau \mapsto \varphi_\tau \).

The map \( \varphi \mapsto \tau \) is clearly affine and continuous in the weak*-topology, and since the spaces of traces and of \( \beta \)-KMS states are compact Hausdorff, the map is a homeomorphism.

\[\Box\]

**Remark 7.4** (1) Our parameter space of traces is obviously not canonical because it depends on the arbitrary choice of representative ideals \( J_\gamma \) in each class. However, the traces are determined up to canonical isomorphisms of the underlying C*-algebras, as discussed at the beginning of the section.

(2) The \( \beta \)-KMS states can be evaluated explicitly on products of the form \( s_a^* e^x_j u^y \cdot S_b \); since these have dense linear span, this characterizes \( \varphi_\tau \). Assume first \( \tau \) is supported on a single ideal class \( \gamma \in \Gamma \). By (9) we may assume \( a^{-1} b = g \in R^* \), for otherwise \( \varphi_\tau(s_a^* e^x_j u^y \cdot S_b) = 0 \). Then

\[
\varphi_{\gamma, \tau}(s_a^* e^x_j u^y \cdot S_b) = \frac{1}{\xi_\gamma (\beta - 1)} \sum_{I \in \gamma} \sum_{x \in R/I} (S^*_a E^x_j U^y S_b U^x U^x \xi_1 | U^x \xi_1) = \frac{1}{\xi_\gamma (\beta - 1)} \sum_{I \in \gamma} \sum_{x \in R/I} (U^{-x} S^*_a E^x_j U^y S_b U^x \Delta^x \xi_1 | \xi_1).
\]
The nontrivial contributions come from terms with
\[ z - ax \notin J, \]
\[ y + a(g - 1)x \in aI \text{ and} \]
a\( I \subset J \).

Thus, recalling that \( \xi_{aI} \) is the cyclic vector for the GNS representation of \( \tau_I \) (the notation is from the construction leading up to Lemma 7.1), the sum reduces to
\[
\varphi_{\gamma, \tau}(s_a^* e_j^* u^y S_b) = \frac{1}{\zeta_\gamma(\beta - 1)} \sum_{l \in I, aI \subset J} \sum_{x \in P_l} N(I)^{-\beta} \tau_{aI}(u^{y + ax(g - 1)} S_g)
\]
where \( P_l := \{ x \in R/I \mid ax - z \in J/I, y + a(g - 1) \in I \} \).

If we now start with a trace \( \tau = \sum_{\gamma \in \Gamma} c_\gamma \tau_\gamma \), then the values of the corresponding \( \beta \)-KMS state are given by
\[
\varphi(s_a^* e_j^* u^y S_b) = \sum_{\gamma \in \Gamma} \sum_{l \in I, aI \subset J} \sum_{x \in P_l} c_\gamma N(I)^{-\beta} \tau_{\gamma, aI}(u^{y + a(g - 1)} S_g).
\]

(3) The \( \infty \)-KMS states are, by definition, the weak-* limits as \( \beta \to \infty \) of \( \beta \)-KMS states, and they too can be computed explicitly, by taking limits in the above formula. Notice that
\[
\frac{N(I)^{-\beta}}{\zeta_\gamma(\beta - 1)} \to 0 \text{ as } \beta \to \infty,
\]
except when \( I \) is norm-minimizing in its class, in which case the limit is \( k_\gamma^{-1} \) (with \( k_\gamma \) the number of norm-minimizing ideals in the class \( \gamma \)). Thus, \( \infty \)-KMS states are still indexed by traces \( \tau = \sum_{\gamma \in \Gamma} c_\gamma \tau_\gamma \) of \( \bigoplus_{\gamma} C^*(J_\gamma \rtimes R^\times) \), and are given by
\[
\varphi(s_a^* e_j^* u^y S_b) = \sum_{\gamma \in \Gamma} \sum_{l \in I, aI \subset J} \sum_{x \in P_l} c_\gamma k_\gamma^{-1} \tau_{\gamma, aI}(u^{y + a(g - 1)} S_g).
\]

where the sum is now over the subset \( \gamma \) of norm-minimizing ideals in \( \gamma \).

**Remark 7.5** As a much simpler “toy model” for the dynamical system \((\Sigma[R], (\sigma_t))\) we can also consider the Toeplitz algebra \(\Sigma[R^\times]\) associated with the multiplicative semigroup \(R^\times\) of \(R\), i.e. the C*-algebra generated by the left regular representation of this semigroup. It is generated by isometries \(s_a, a \in R^\times\) and carries an analogous one-parameter automorphism group \((\sigma^t_a)^\times\) defined by \(\sigma^t_a(s_a) = N(a)^t S_a\). Since \(R^\times\) is a split extension of \(R^\times/R^\ast\) by \(R^\ast\), [14, Lemma 1.11], we see that \(\Sigma[R^\times]\) is the tensor...
product of $C^*(R^\times)$ and the Toeplitz algebra $\mathfrak{T}[R^\times/R^\times]$ for the semigroup $R^\times/R^\times$ of principal integral ideals. In the case where $R$ is a principal ideal domain, $\mathfrak{T}[R^\times]$ is then simply an infinite tensor product of the ordinary Toeplitz algebras (i.e. universal $C^*$-algebras generated by a single isometry) generated by the isometries associated to the primes in $R$, and of $C^*(R^\times)$. In this case the situation is nearly trivial. An easy exercise shows that the KMS-states for each $\beta > 0$ are labeled by the states of $C^*(R^\times)$.

However, in the case of a non-trivial $C^*$-dynamical system, essentially, because there is an ‘interaction’ between the classes. The methods and results of the last two sections (including Theorem 5.6) immediately lead to a determination of its KMS structure. One finds that for $\beta = 0$ there is a family of 0-KMS states ($\sigma$-invariant traces) indexed by the $\sigma$-invariant states on $C^*(K^\times)$ (such a state has to factor through the quotient of $\mathfrak{T}[R]$ where each of the generators $s_a$ becomes unitary—this quotient is exactly $C^*(K^\times)$). For each $\beta$ in the range $0 < \beta \leq 1$ the $\beta$-KMS states correspond exactly to the states of $C^*(R^\times)$ (there is a unique $\beta$-KMS state on $\mathfrak{T}[R^\times/R^\times]$ which can be combined with an arbitrary state on the tensor factor $C^*(R^\times)$). For each $\beta$ in the range $1 < \beta < \infty$ the simplex of KMS states splits in addition over the class group $\Gamma$. Thus the KMS states in that range are labeled by the states of $C^*(R^\times \times \hat{\Gamma})$.

We note that it is known that the class group $\Gamma$ for $K$ is determined already by the semigroup $R^\times$. In fact $\Gamma$ coincides with the semigroup class group defined by the ideals in this semigroup (i.e. the subsets invariant under multiplication by all elements), cf. [7, section 2.10].

8 Ground states

Recall that a state $\varphi$ on a $C^*$-dynamical system $(\mathcal{B}, (\sigma_t)_{t \in \mathbb{R}})$ is a ground state if and only if the function

$$z \mapsto \varphi(w \sigma_z(w'))$$

is bounded on the upper half plane on a set of analytic vectors $w, w' \in \mathcal{B}$ with dense linear span.

**Proposition 8.1** Let $\varphi$ be a state of $\mathfrak{T}[R]$. Then the following are equivalent:

1. $\varphi$ is a ground state;
2. for all $d \in \mathfrak{D}, a, b \in R^\times, x \in R$ and $w \in \mathfrak{T}[R]$ we have $\varphi(sw^*du^xs_b) = 0$, whenever $N(a) > N(b)$;
3. for $a, b \in R^\times, x \in R$, we have $\varphi(s_bu^xas_a^*s_bu^{-x}s_b) = 0$, whenever $N(a) > N(b)$ (note that the expression under $\varphi$ depends on $x$ only via its image in $R/aR$);

**Proof** $\varphi$ is a ground state if and only if the function

$$z \mapsto \varphi(w \sigma_z(w'))$$

is bounded on the upper half plane on a set of analytic vectors $w, w' \in \mathfrak{T}[R]$ with dense linear span. We may choose $w'$ of the form $s^*_a du^x s_b$. 

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We have
\[ \varphi(w \sigma_z(s_a^*du^x s_b)) = N(b/a)^i z \varphi(w (s_a^*du^x s_b)), \]
This function is bounded on the upper half plane if and only if it vanishes when \( N(b/a) < 1 \). This proves that (1) and (2) are equivalent.

By the Cauchy–Schwarz inequality (2) is equivalent to the fact that
\[ \varphi((s_a^*du^x s_b)^*(s_a^*du^x s_b)) = 0 \]
for all \( a, b, x, d \). However \( (s_a^*du^x s_b)^*(s_a^*du^x s_b) \leq \|d\|^2 s_b^*u^x s_d s_a^*u^{-x} s_b \). This shows that (2) and (3) are equivalent. \( \square \)

We will see that the ground states on \( \mathcal{T}[R] \) are supported on projections corresponding to what we call “norm-minimizing ideals”. We say that an ideal \( I \) in \( R \) is norm-minimizing if for any other ideal \( J \) in the same ideal class we have \( N(I) \leq N(J) \).

The use of norm-minimizing ideals was suggested by work in preparation by Laca–van Frankenhuijsen.

Recall from Lemma 4.15 (a) that every ideal \( I \) in \( R \) can be written in the form \( a R \cap R \) with \( a, b \in R^\times \).

**Lemma 8.2**
(i) If a product \( J = IL \) is norm-minimizing, then so are \( I \) and \( L \).
(ii) The prime ideals that are norm-minimizing generate the ideal class group.
(iii) If \( I = \frac{a}{b} R \cap R \) is norm-minimizing, then \( N(a) \leq N(b) \).

**Proof** The proof of part (i) is obvious and (ii) follows easily from (i). To prove (iii) observe that for each \( I = \frac{a}{b} R \cap R \), the integral ideal \( I' = \frac{b}{a} I \) is in the same class and \( N(I') = N(b)N(a)^{-1}N(I) \). Thus, if \( I \) is norm minimizing, necessarily \( N(b)N(a)^{-1} \geq 1 \). \( \square \)

**Lemma 8.3** Let \( \varphi \) be a ground state of \( \mathcal{T}[R] \). Then \( \varphi(e_P^+e_I^+) = 0 \) for each ideal \( I \) in \( R \) which is not norm-minimizing and for each \( x \in R/I \).

**Proof** If \( I \) and \( J \) are two ideals in the same ideal class, then there exist integers \( a \) and \( b \) in \( R^\times \) such that \( b I = a J \), so \( e_I = s_b^*s_a e_J s_a^*s_b \). Assuming that \( J \) is norm-minimizing but \( I \) is not, then \( N(a) > N(b) \) by Lemma 8.2(iii), so we may use part (2) of Proposition 8.1 on the product \( (u^x s_b^*s_a e_J) (s_a^*s_b u^{-x}) = e_I^x \) to finish the proof. \( \square \)

In particular, the above proposition implies that \( \varphi(e_P^+) = 0 \) for each prime ideal \( P \) which is not norm-minimizing and for each \( x \in R/P \). Thus \( \varphi(e_P) = \varphi(1 - f_P) = 1 \) for such ideals. To take advantage of this feature, we will order the prime ideals in \( R \) in such a way that \( P_1, \ldots, P_k \) are norm-minimizing while all the other prime ideals \( P_{k+1}, P_{k+2}, \ldots \) are not. By Lemma 8.2(ii) the (finite) set \( \mathcal{I}_k \) of norm-minimizing ideals in the semigroup \( \mathcal{I}_k \) generated by the \( P_1, \ldots, P_k \) is in fact the finite set of all norm-minimizing ideals of \( R \). The projection \( \mathcal{e}_{\mathcal{I}_k} := \sum_{I \in \mathcal{I}_k} \mathcal{e}_I P_1 \ldots P_k \) corresponding to the norm-minimizing ideals will be the key to our characterization of ground states.
Lemma 8.4 Let \( \varphi \) be a ground state of \( \Xi[R] \) and assume \( n > k \) so that \( P_1, \ldots, P_k \) are norm-minimizing while \( P_{k+1}, P_{k+2}, \ldots, P_n \) are not. If \( \varepsilon_{\mathcal{T}_k} := \sum_{I \in \mathcal{T}_k} \varepsilon_{I 1 \cdots P_k} \), then \( \varphi(\varepsilon_{\mathcal{T}_k} \varepsilon_{P_{k+1} P_{k+2} \cdots P_n}) = 1 \).

Proof Recall the minimal projections \( \delta_{I,n}^{\varepsilon} \in \mathcal{D}_n \), for \( I \in \mathcal{I}_n, x \in R/I \), introduced in Sect. 4.

Since \( \varepsilon_{I P_1 P_2 \cdots P_n} = \sum_{x \in R/I} \delta_{I,n}^{\varepsilon} \) for each \( I \in \mathcal{I}_n \), we have

\[
\varepsilon_{\mathcal{T}_k} \varepsilon_{P_{k+1} P_{k+2} \cdots P_n} = \sum_{I \in \mathcal{T}_k, x \in R/I} \delta_{I,n}^{\varepsilon},
\]

which is a projection with finite support in \( \text{Spec} \bar{D}_n \). In view of Lemma 8.2(i), the complement of the support is covered by the supports of the \( e_j^x \) with \( J \in \mathcal{I}_n \setminus \mathcal{I}_k \) and \( x \in R/J \). By Lemma 8.3 we conclude that

\[
\varphi \left( 1 - \sum_{I \in \mathcal{T}_k, x \in R/I} \delta_{I,n}^{\varepsilon} \right) \leq \sum_{J \in \mathcal{I}_n \setminus \mathcal{I}_k, x \in R/J} \varphi(e_j^x) = 0,
\]

finishing the proof. \( \square \)

We will now consider \( \Xi[R] \) in its universal representation. Thus let \( S \) be the state space of \( \Xi[R] \) and let \( \pi_S = \bigoplus_{f \in S} \pi_f \) be its universal representation on the Hilbert space \( H_S = \bigoplus_{f \in S} H_f \). We will from now on assume that \( \Xi[R] \) is represented via \( \pi_S \) and we will omit the \( \pi_S \) from our notation.

If \( \varphi \) is a state of \( \Xi[R] \), we denote by \( \tilde{\varphi} \) its unique normal extension to the von Neumann algebra \( \Xi[R]^\prime \prime \) generated by \( \Xi[R] \).

We write \( \delta_I, \delta_I^\varepsilon, \varepsilon_I \) for the strong limits, as \( n \to \infty \), of the monotonously decreasing sequences of projections \( \delta_{I,n}, \delta_{I,n}^\varepsilon \) and \( \varepsilon_{I P_1 P_2 \cdots P_n} \), respectively (recall that \( \delta_{I,n} := \delta_{I,n}^0 \)).

In the representation \( \mu \) used in Sect. 4, the projection \( \tilde{\delta}_I^x \) is represented by the projection onto the one-dimensional subspace of \( \ell^2(R/I) \) corresponding to \( x \in R/I \). It is therefore non-zero.

We also consider the projection \( E \) defined as the strong limit of the sequence of projections \( \varepsilon_{\mathcal{T}_k} \varepsilon_{P_{k+1} P_{k+2} \cdots P_n} \). Equation (12) immediately gives the formula

\[
E = \sum_{I \in \mathcal{T}_k, x \in R/I} \tilde{\delta}_I^x.
\]

Proposition 8.5 A state \( \varphi \) of \( \Xi[R] \) is a ground state if and only if \( \tilde{\varphi}(E) = 1 \).

Proof If \( \varphi \) is a ground state, then \( \tilde{\varphi}(E) = 1 \) follows immediately from Lemma 8.4 because \( \tilde{\varphi} \) is normal.

If, conversely, \( \tilde{\varphi}(E) = 1 \), then \( \varphi(w) = \tilde{\varphi}(w) = \tilde{\varphi}(E w E) \) for each \( w \in \Xi[R] \). In order to show that condition (3) in Proposition 8.1 is satisfied, i.e. that \( \tilde{\varphi}(E s_{b}^{\varepsilon} u^{x} s_{a}^{\varepsilon} u^{-x} s_{b} E) = 0 \) whenever \( N(a) > N(b) \), it suffices to show that \( \delta_{I}^{y} s_{b}^{\varepsilon} u^{x} s_{a}^{\varepsilon} u^{-x} s_{b} \delta_{I}^{y} = 0 \) for all \( I \in \mathcal{T}_k \) and \( y \in R/I \), whenever \( N(a) > N(b) \).
This amounts to showing that \( \delta^Y_{bI} s_a s^*_a \delta^Y_{bI} = 0 \) whenever \( N(a) > N(b) \). However, by Eq. (10), this last expression can be non-zero only if \( bI \subset aR \). This inclusion implies that \( I \subset \frac{a}{b} R \cap R \), i.e. that the ideal \( \frac{a}{b} R \cap R \) divides \( I \). Since \( I \) is norm-minimizing, \( \frac{a}{b} R \cap R \) then has to be norm-minimizing, too, and \( N(a) \leq N(b) \) by Lemma 8.2(iii).

Lemma 8.6 Let \( I, J \in \mathcal{I}_R \) and let \( a, b, a', b' \in R \) such that \( aI = bJ \) and \( a'J = b'J \). Then there is \( g \in R^* \) such that \( s^*_a s_{a'} = s^*_g s_{b'} s_a \). The operators \( s^*_a s_{b'} \delta_I \) and \( s^*_b s_a \delta_I \) are partial isometries with support \( \tilde{\delta}_I \) and range \( \tilde{\delta}_J \). If \( I, J, L \) are three ideals in \( \mathcal{I}_R \) and \( aI = bJ = cL \), then \( s^*_c s_b \delta_J s^*_b s_a \delta_I = s^*_c s_a \delta_I \).

Proof We have \((a/b)I = J = (a'/b')I\) whence \( a'/b' = ga/b \) for some \( g \in R^* \). Thus \( s_{a'} = a'b' \) and \( s_g s_{a'b'} = s_{a'b'} s_{b'} \). Multiplying this from the left by \( s^*_a s^*_{a'} \), gives the first assertion (note that \( s_g \) and \( s^*_g \) commute with \( s_a, s_{a'} \)). The second assertion then follows from Lemma 4.11. Finally, \( s^*_c s_b s^*_b s_a \delta_I = s^*_c s_a \delta_I \) from Eq. (10) and the fact that \( aI \subset bR \).

Proposition 8.7 The corner \( M = E \mathfrak{X}[R]E \) is a \( C^* \)-algebra isomorphic to

\[ \bigoplus_{\gamma \in \Gamma} M_{k_\gamma N(J_\gamma)}(C^*(J_\gamma) \rtimes R^*) \]

Here \( \Gamma \) denotes the class group, \( k_\gamma = |\mathcal{I}_R \cap \gamma| \) and \( J_\gamma \) is any fixed ideal in \( \mathcal{I}_R \cap \gamma \) (they are all isomorphic).

Proof We use the partition of \( E \) as a sum of the projections \( \tilde{\delta}_I^x \), \( I \in \mathcal{I}_R \), \( x \in R/I \).

If \( I_1, I_2 \in \mathcal{I}_R \) are two ideals which are not in the same ideal class and \( w \) is an element of \( \mathfrak{X}[R] \) of the form \( w = s^*_b e_L u^y s_a \) with \( a, b \in R^x, y \in R \) then

\[ \tilde{\delta}^{x_1}_{I_1} w \tilde{\delta}^{x_2}_{I_2} = s^*_b s_{b'} \delta_{I_1} e_L u^y s_{a_2} s_{a_1} = 0 \]

because \( \tilde{\delta}^{t_1}_{I_1} \delta^{t_2}_{I_2} = 0 \) for two different ideals \( L_1, L_2 \) independently of the choice of \( t_1, t_2 \). Thus \( \tilde{\delta}^{x_1}_{I_1} \mathfrak{X}[R] \delta^{x_2}_{I_2} = 0 \). If we write

\[ E_\gamma = \sum_{I \in \mathcal{I}_R \cap \gamma, x \in R/I} \tilde{\delta}^x_I \]

then \( E = \sum_{\gamma} E_\gamma \) and \( E_{\gamma_1} \mathfrak{X}[R] E_{\gamma_2} = 0 \) whenever \( \gamma_1 \neq \gamma_2 \).

If \( I, J \in \mathcal{I}_R \) are two ideals in the same ideal class \( \gamma \), we can choose, according to Lemma 8.6, a partial isometry \( c_{IJ} \) of the form \( c_{IJ} = s^*_a s_b \delta_I \) with support \( \tilde{\delta}_I \) and range \( \tilde{\delta}_J \). This element is well determined up to multiplication by a unitary \( s_{\gamma}, g \in R^* \).

By fixing a reference ideal \( J_\gamma \) in the class \( \gamma \) and choosing first the \( c_{IJ_\gamma} \) and then putting \( c_{IJ} = c_{IJ_\gamma} c^{*}_{IJ_\gamma} \), we may assume that the \( c_{IJ} \) have the property that \( c_{IJ} = c^*_{IJ} \) and \( c_{IJ} c_{IJ} = c_{IJ} \) for \( I, J, L \in \mathcal{I}_R \cap \gamma \) (i.e. they are matrix units). They generate a matrix algebra isomorphic to \( M_{k_\gamma} (\mathbb{C}) \). Setting

\[ c^{xy}_{IJ} = u^x c_{IJ} u^{-y} \]
we obtain a system of matrix units for the larger index set \{(I, x) \mid I \in I_k \cap \gamma, x \in R/I\}. This system generates a matrix algebra isomorphic to \(M_{k, N(J_\gamma)}(\mathbb{C})\) (note that \(N(I) = N(J_\gamma)\) for all \(I \in I_k \cap \gamma\)).

Consider again an element \(w\) of \(\mathfrak{S}[R]\) of the form \(w = s^*_b u^y s_a\) with \(a, b \in R^\times, y \in R\). Then \(\tilde{\delta}_{J_\gamma} w \tilde{\delta}_{J_\gamma}\) is non-zero only if \(bJ_\gamma = aJ_\gamma\), \(y \in aJ_\gamma\) and \(L \supset aJ_\gamma\). In that case we get

\[
\tilde{\delta}_{J_\gamma} w \tilde{\delta}_{J_\gamma} = w^{y/b} s^*_b s_a \tilde{\delta}_{J_\gamma} = w^{y/b} s_g \tilde{\delta}_{J_\gamma}
\]

for a suitable \(g \in R^\times\). This shows that \(\tilde{\delta}_{J_\gamma} \mathfrak{S}[R] \tilde{\delta}_{J_\gamma}\) is isomorphic to the subalgebra \(\mathcal{E}\) of \(\mathfrak{S}[R]\) generated by the \(s_g\), \(g \in R^\times\) and the \(u^x\), \(x \in J_\gamma\). On the other hand the representation of \(\mathfrak{S}[R]\) constructed in Sect. 7 shows that the surjective map \(C^*(J_\gamma) \rtimes R^\times \to \mathcal{E}\) from the crossed product is an isomorphism. Therefore \(\tilde{\delta}_{J_\gamma} \mathfrak{S}[R] \tilde{\delta}_{J_\gamma}\) is isomorphic to the crossed product \(C^*(J_\gamma) \rtimes R^\times\).

Finally, the map that sends a matrix \((w_1^{x_1 y_1} w_2^{x_2 y_2})\) in \(M_{k, N(J_\gamma)}(\mathcal{E})\) to

\[
\sum \zeta_{I_1 I_2}^{x_1 y_1} w_1^{x_1 y_2} w_2^{0 x_2} \gamma_{J_\gamma, I_1 I_2}
\]

defines an isomorphism \(M_{k, N(J_\gamma)}(\mathcal{E}) \to E_\gamma \mathfrak{S}[R] E_\gamma\). \(\square\)

**Theorem 8.8** The ground states of \(\mathfrak{S}[R]\) are exactly the states of the form \(\varphi(w) = \psi(E w E)\) where \(\psi\) is an arbitrary state of \(E \mathfrak{S}[R] E \cong \bigoplus M_{k, N(J_\gamma)}(C^*(J_\gamma \rtimes R^\times))\).

**Proof** This is immediate from Propositions 8.5 and 8.7. \(\square\)

**Appendix A: Asymptotics for partial \(\zeta\)-functions**

As above let \(R\) be the ring of algebraic integers in a number field \(K\). Also let \(P_1, P_2, \ldots\) be an enumeration of the prime ideals in \(R\) such that \(N(P_i) \leq N(P_{i+1})\) for all \(i \geq 1\) and let \(I_n\) be the semigroup generated by \(P_1, P_2, \ldots, P_n\). For each \(\gamma\) in the class group \(\Gamma\) of \(K\) and each \(0 < \sigma \leq 1\) set

\[
\zeta_\gamma^{(n)}(\sigma) = \sum_{I \in I_n \cap \gamma} N(I)^{-\sigma}
\]

Recall the statement of Theorem 6.6: Let \(0 < \sigma \leq 1\). Then for any two ideal classes \(\gamma_1, \gamma_2\) we have

\[
\lim_{n \to \infty} \frac{\zeta_{\gamma_1}^{(n)}(\sigma)}{\zeta_{\gamma_2}^{(n)}(\sigma)} = 1.
\]

**Proof of Theorem 6.6** Let \(\psi_\gamma : \Gamma \to [0, 1]\) denote the characteristic function of the one-point set \(\{\gamma\}\). For every character \(\chi\) of the abelian group \(\Gamma\) let \(a_\gamma(\chi) = |\Gamma|^{-1} \chi(\gamma)\) so that

\[
\psi_\gamma = \sum_{\chi \in \Gamma} a_\gamma(\chi) \chi
\]
In the following we also consider $\chi$ and $\psi_\gamma$ as functions on the set of non-zero integral ideals. We have

$$
\zeta_\gamma^{(n)}(\sigma) = \sum_{I \in \mathcal{I}_n} \psi_\gamma(I) N(I)^{-\sigma} = \sum_{\chi \in \hat{\Gamma}} \left( a_\gamma(\chi) \sum_{I \in \mathcal{I}_n} \chi(I) N(I)^{-\sigma} \right)
$$

$$
= \sum_{\chi \in \hat{\Gamma}} \left( a_\gamma(\chi) \prod_{i=1}^n \left( 1 - \chi(P_i) N(P_i)^{-\sigma} \right)^{-1} \right)
$$

(13)

In order to study the asymptotics of $\prod_{i=1}^n \left( 1 - \chi(P_i) N(P_i)^{-\sigma} \right)^{-1}$ for $n \to \infty$ we consider

$$
f_n(\chi, \sigma) = \log \prod_{i=1}^n \left( 1 - \chi(P_i) N(P_i)^{-\sigma} \right)^{-1}
$$

$$
:= \sum_{\nu=1}^\infty \frac{1}{\nu} \sum_{i=1}^n \chi(P_i^{\nu}) N(P_i)^{-\nu \sigma}
$$

(14)

Up to finitely many terms the first sum is bounded by a constant which is independent of $n$:

$$
\left| \sum_{\nu > 1/\sigma} \frac{1}{\nu} \sum_{i=1}^n \chi(P_i^{\nu}) N(P_i)^{-\nu \sigma} \right| \leq \sum_{\nu > 1/\sigma} \frac{1}{\nu} \sum_{i=1}^\infty N(P_i)^{-\nu \sigma}
$$

$$
= \sum_{i=1}^\infty N(P_i)^{-\sigma \lfloor 1/\sigma \rfloor} \sum_{\nu = \nu + \lfloor 1/\sigma \rfloor}^\infty N(P_i)^{-\nu \sigma}
$$

$$
\leq \sum_{i=1}^\infty N(P_i)^{-\sigma \lfloor 1/\sigma \rfloor} \frac{N(P_i)^{-\sigma}}{1 - N(P_i)^{-\sigma}}
$$

$$
\leq \frac{1}{1 - 2^{-\sigma}} \sum_{i=1}^\infty N(P_i)^{-\sigma (1 + \lfloor 1/\sigma \rfloor)}
$$

$$
< \frac{1}{1 - 2^{-\sigma}} \zeta_K(\sigma (1 + \lfloor 1/\sigma \rfloor)) < \infty.
$$

Therefore

$$
f_n(\chi, \sigma) = \sum_{1 \leq \nu \leq 1/\sigma} \frac{1}{\nu} \sum_{i=1}^n \chi(P_i^{\nu}) N(P_i)^{-\nu \sigma} + O(1)
$$

(15)

where the $O$-constant depends on $\sigma$ but not on $n$ or $\chi$. 
Let us now fix some $1 \leq \nu \leq 1/\sigma$. The values of $\chi$ are $h$th roots of unity where $h = |\Gamma|$ is the class number. We get

$$
\sum_{i=1}^{n} \chi(P_i^\nu)N(P_i)^{-\nu\sigma} = \sum_{\zeta^h=1} \zeta^\nu \sum_{\gamma \in \chi^{-1}(\zeta)} \omega^{(\nu\sigma)}_{\gamma}(n).
$$

(16)

Here for $\kappa \in \mathbb{R}$, $\gamma \in \Gamma$ and $n \geq 1$ we have set:

$$
\omega^{(\kappa)}_{\gamma}(n) = \sum_{i=1}^{n} N(P_i)^{-\kappa}.
$$

**Lemma A.1.** Fix some $0 \leq \kappa \leq 1$ and write $\omega_{\gamma}(n) = \omega^{(\kappa)}_{\gamma}(n)$. Set

$$
\omega(n) = \frac{1}{h} \sum_{i=1}^{n} N(P_i)^{-\kappa}.
$$

Then $\omega(n) \to \infty$ as $n \to \infty$ and for arbitrary $\gamma \in \Gamma$ we have $\lim_{n \to \infty} \omega_{\gamma}(n)/\omega(n) = 1$.

The proof of the lemma is given below. For $1 \leq \nu \leq 1/\sigma$ we have $0 < \kappa = \nu\sigma \leq 1$. Using (16) and the lemma, we get for $n \to \infty$:

$$
\frac{1}{\omega(n)} \sum_{i=1}^{n} \chi(P_i^\nu)N(P_i)^{-\nu\sigma} \to \sum_{\zeta^h=1} \zeta^\nu |\chi^{-1}(\zeta)|.
$$

We have the identities

$$
\sum_{\zeta^h=1} \zeta^\nu |\chi^{-1}(\zeta)| = |\text{Ker}(\chi)| \sum_{\zeta \in \text{Im} \chi} \zeta^\nu = \begin{cases} h & \text{if } |\text{Im} \chi| \mid \nu \\ 0 & \text{if } |\text{Im} \chi| \nmid \nu \end{cases}
$$

Therefore, using (15) we get

$$
\lim_{n \to \infty} \frac{1}{\omega(n)} f_n(\chi, \sigma) = \alpha(\chi) := h \sum_{1 \leq \nu \leq 1/\sigma, \mid \text{Im} \chi \mid \mid \nu} \frac{1}{\nu} \geq 0
$$

(17)

Note that if $\chi$ is not the trivial character $1$, then $\alpha(\chi) < \alpha(1)$. Let

$$
L_n(\chi, \sigma) := \prod_{i=1}^{n} (1 - \chi(P_i)N(P_i)^{-\sigma})^{-1} = \exp f_n(\chi, \sigma)
$$

(18)
From (13) we get

$$\zeta_{\gamma}^{(n)}(\sigma) = \sum_{\chi} a_{\gamma}(\chi) L_n(\chi, \sigma)$$

Because of (17) and (18) one knows that for \( n \to \infty \)

$$0 < L_n(1, \sigma) = \prod_{i=1}^{n} (1 - N(P_i)^{-\sigma})^{-1} \to \infty$$

Also

$$\left| \frac{L_n(\chi, \sigma)}{L_n(1, \sigma)} \right| = \exp \Re (f_n(\chi, \sigma) - f_n(1, \sigma))$$

Now assume that \( \chi \neq 1 \). Since \( \omega(n) \to \infty \) and

$$\lim_{n \to \infty} \frac{1}{\omega(n)} (f_n(\chi, \sigma) - f_n(1, \sigma)) = \alpha(\chi) - \alpha(1) < 0$$

by (17), we find that

$$\lim_{n \to \infty} \Re (f_n(\chi, \sigma) - f_n(1, \sigma)) = -\infty$$

and thus

$$\lim_{n \to \infty} \frac{L_n(\chi, \sigma)}{L_n(1, \sigma)} = 0 \quad \text{for} \quad \chi \neq 1$$

This gives:

$$\lim_{n \to \infty} \frac{\zeta_{\gamma}^{(n)}(\sigma)}{L_n(1, \sigma)} = a_{\gamma}(1) = \frac{1}{h}$$

and hence

$$\lim_{n \to \infty} \frac{\zeta_{\gamma}^{(n)}(\sigma)}{\zeta_{\eta}^{(n)}(\sigma)} = 1$$

for any two ideal classes \( \gamma \) and \( \eta \).

It remains to prove lemma A.1. For this we need a version of the prime number theorem for prime ideals in a given ideal class with a simple remainder term. For \( x \geq 0 \) let \( \pi_K(\gamma, x) \) denote the number of prime ideals \( P \) in \( \gamma \) with \( N(P) \leq x \). Using the relation

$$\text{li}(x) = \frac{x}{\log x} + O \left( \frac{x}{(\log x)^2} \right) \quad \text{for} \quad x \to \infty$$

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the corollary after lemma 7.6 in chap. 7, § 2 of [17] implies the following asymptotics:

$$
\pi_K(\gamma, x) = \frac{1}{h} \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)
$$

(19)

For \( x \geq 0 \) and \( \kappa \leq 1 \) let us write:

$$
\Omega_\gamma(x) = \Omega^{(\kappa)}_\gamma(x) = \sum_{N(P) \leq x, P \in \gamma} N(P)^{-\kappa}
$$

and

$$
\Omega(x) = \Omega^{(\kappa)}(x) = \frac{1}{h} \sum_{N(P) \leq x} N(P)^{-\kappa}.
$$

We now use the following version of summation by parts: Consider a function \( f \) on the integers \( \nu \geq 1 \) and a \( C^1 \)-function \( g \) on \([1, \infty)\). For \( x \geq 1 \) we set \( M_f(x) = \sum_{\nu \leq x} f(\nu) \). Then we have

$$
\sum_{\nu \leq x} f(\nu)g(\nu) = M_f(x)g(x) - \int_1^x M_f(t)g'(t) \, dt.
$$

Setting \( f(\nu) = \left| \{ P \mid P \in \gamma \text{ and } N(P) = \nu \} \right| \) and \( g(x) = x^{-\kappa} \) we have

$$
\Omega_\gamma(x) = \sum_{\nu \leq x} f(\nu)g(\nu) \quad \text{and} \quad M_f(x) = \pi_K(\gamma, x).
$$

Hence using (19) we get for \( x \to \infty \):

$$
\Omega_\gamma(x) = \pi_K(\gamma, x)x^{-\kappa} + \kappa \int_2^x \pi_K(\gamma, t)t^{-\kappa} \, dt \\
= \frac{1}{h} \frac{x^{1-\kappa}}{\log x} + \frac{\kappa}{h} \int_2^x \frac{t^{-\kappa}}{\log t} \, dt + O\left(\frac{x^{1-\kappa}}{(\log x)^2}\right) + O\left(\int_2^x \frac{t^{-\kappa}}{(\log t)^2} \, dt\right).
$$

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For $\kappa < 1$ we have:

$$
\int_e^x \frac{t^{-\kappa}}{(\log t)^2} \, dt = \int_e^{\sqrt{x}} \frac{t^{-\kappa}}{(\log t)^2} \, dt + \int_{\sqrt{x}}^x \frac{t^{-\kappa}}{(\log t)^2} \, dt
$$

\[
\leq \int_e^{\sqrt{x}} t^{-\kappa} \, dt + \frac{1}{(\log \sqrt{x})^2} \int_{\sqrt{x}}^x t^{-\kappa} \, dt
\]

\[
= O \left( \frac{x^{1-\kappa}}{(\log x)^2} \right).
\]

Hence we get for $\kappa < 1$:

$$
\Omega_{\gamma}(x) = \frac{1}{h} x^{1-\kappa} + \frac{\kappa}{h} \int_2^x \frac{t^{-\kappa}}{\log t} \, dt + O \left( \frac{x^{1-\kappa}}{(\log x)^2} \right). \quad (20)
$$

For the case $\kappa = 1$ note that

$$
\int_2^x \frac{t^{-1}}{(\log t)^2} \, dt = \frac{1}{\log 2} - \frac{1}{\log x} = O(1)
$$

and

$$
\int_2^x \frac{t^{-1}}{\log t} \, dt = \log \log x + O(1).
$$

Thus for $\kappa = 1$ we get

$$
\Omega_{\gamma}(x) = \frac{1}{h} \log \log x + O(1). \quad (21)
$$

Relations (20) and (21) also hold for $\Omega(x)$ instead of $\Omega_{\gamma}(x)$ since the right hand sides do not depend on $\gamma$ and $\Omega(x) = h^{-1} \sum_{\gamma \in \Gamma} \Omega_{\gamma}(x)$. It follows that for $\kappa \leq 1$ we have $\Omega_{\gamma}(x) \sim \Omega(x)$. It remains to show that for $n \to \infty$ we have $\omega_{\gamma}(n) \sim \omega(n)$ as well. For a given prime number $p$ there are at most $(K : \mathbb{Q})$ different prime ideals $P$ in $R$ with $P \mid p$. It follows that for every $v \geq 1$ the equation $N(P) = v$ has at most $(K : \mathbb{Q})$ solutions in primes $P$ of $R$. Since $\overline{N}(P_i) \leq N(P_{i+1})$ for all $i$ we therefore get:

$$
\omega_{\gamma}(n) = \Omega_{\gamma}(N(P_n)) + O(N(P_n)^{-\kappa})
$$

$$
= \Omega_{\gamma}(N(P_n)) + O(1) \quad \text{since } \kappa \geq 0
$$
and analogously
\[ \omega(n) = \Omega(N(P_n)) + O(1). \]
This implies the result. \( \square \)

Appendix B: List of notations

For \( u^x, s_a, e_I \) see Definition 2.1. The projections \( f_I, \varepsilon_I \) are introduced before Lemma 2.4. The commutative subalgebra \( \tilde{D} \) is introduced at the beginning of Sect. 4. \( \mathbb{I}_n, \mathbb{D}_n, \bar{\mathbb{D}}_n \) are introduced after Lemma 4.5. The representation \( \mu \) of \( \Sigma[R] \) is defined before Lemma 4.6. The minimal projections \( \delta_{\tilde{I}, n} \) in \( \mathbb{D}_n \) are introduced in Lemma 4.7. \( e_I^x \) is defined after 4.8. For \( Y_R \) see Remark 4.10. The notation \( \tilde{\Sigma} \) is introduced after Corollary 4.14. The automorphism \( \sigma_t \) is defined at the beginning of Sect. 6. \( R^* \) denotes the group of units (invertible elements) in \( R \).

References

1. Bost, J.-B., Connes, A.: Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. Sel. Math. (N.S.) 1(3), 411–457 (1995)
2. Bourbaki, N.: Commutative algebra. In: Elements of Mathematics (Berlin), chapters 1–7. Springer, Berlin (1989, translated from the French, reprint of the 1972 edition)
3. Bratteli, O., Robinson D.W.: Operator algebras and quantum statistical mechanics. 2. Texts and monographs in physics. In: Equilibrium States. Models in Quantum Statistical Mechanics, 2nd edn. Springer, Berlin (1997)
4. Cuntz, J.: \( C^* \)-algebras associated with the \( ax + b \)-semigroup over \( \mathbb{N} \). In: Cortiñas, G. et al. (eds) K-theory and noncommutative geometry. Proceedings of the ICM.: satellite conference, Valladolid, Spain, August 31–September 6, 2006. European Mathematical Society (EMS), Zürich. Series of Congress Reports, 201–215 (2008)
5. Cuntz, J., Li, X.: The regular \( C^* \)-algebra of an integral domain. Clay Math. Proc. 10, 149–170 (2010)
6. Cuntz, J., Li, X.: \( C^* \)-algebras associated with integral domains and crossed products by actions on adele spaces. J. Noncommut. Geom. 5(1), 1–37 (2011)
7. Geroldinger, A., Halter-Koch, F.: Non-unique factorizations. In: Algebraic, Combinatorial and Analytic Theory. Pure and Applied Mathematics (Boca Raton), vol. 278. Chapman & Hall/CRC, Boca Raton (2006)
8. Laca, M.: From endomorphisms to automorphisms and back: dilations and full corners. J. London Math. Soc. (2) 61(3), 893–904 (2000)
9. Laca, M., Neshveyev, S.: Type \( III_1 \) equilibrium states of the Toeplitz algebra of the affine semigroup over the natural numbers. J. Funct. Anal. 261(2), 169–187 (2011)
10. Laca, M., Larsen, N., Neshveyev, S.: On Bost–Connes systems for number fields. J. Number Theory 129, 325–338 (2009)
11. Laca, M., Neshveyev, S., Trifkovic, M.: Bost–Connes systems. In: Hecke Algebras and Induction (preprint, arXiv:1010.4766)
12. Laca, M., Raeburn, I.: Semigroup crossed products and the Toeplitz algebras of nonabelian groups. J. Funct. Anal. 139(2), 415–440 (1996)
13. Laca, M., Raeburn, I.: Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers. Adv. Math. 225(2), 643–688 (2010)
14. Laca, M., van Frankenhuijsen, M.: Phase transitions on Hecke \( C^* \)-algebras and class-field theory over \( \mathbb{Q} \). J. Reine Angew. Math. 595, 25–53 (2006)
15. Li, X.: Ring \( C^* \)-algebras. Math. Ann. 348(4), 859–898 (2010)
16. Li, X.: Semigroup \( C^* \)-algebras and amenability of semigroups (preprint, arXiv:1105.5539, 2011)
17. Narkiewicz, W.: Elementary and Analytic Theory of Algebraic Numbers. Springer, Berlin (1990)
18. Neshveyev, S.: Traces on crossed products (preprint, 2010)
19. Neukirch, J.: Algebraic number theory. In: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322. Springer, Berlin [Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by Harder G (1999)]
20. Peebles, J.: The Toeplitz $C^*$-algebra of the semigroup of principal ideals in a number field. MSc. Thesis, University of Victoria. https://dspace.library.uvic.ca:8443/handle/1828/2380 (2005)
21. Thoma, E.: Über unitäre Darstellungen abzählbarer, diskreter. Math. Ann. 153, 111–138 (1964)