On the differentiability of solutions to singularly perturbed SPDEs

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Abstract

We consider semilinear stochastic evolution equations on Hilbert spaces with multiplicative Wiener noise and linear drift term of the type $A + \varepsilon G$, with $A$ and $G$ maximal monotone operators and $\varepsilon$ a “small” parameter, and study the differentiability of mild solutions with respect to $\varepsilon$. The operator $G$ can be a singular perturbation of $A$, in the sense that its domain can be strictly contained in the domain of $A$.

1 Introduction

Let us consider a stochastic evolution equation of the type

$$du_\varepsilon + (A + \varepsilon G)u_\varepsilon \, dt = f(u_\varepsilon) \, dt + B(u_\varepsilon) \, dW, \quad u_\varepsilon(0) = u_0,$$

posed on a Hilbert space $H$, where $A$ and $G$ are linear maximal monotone operators on $H$ such that the closure of $A + \varepsilon G$ (denoted by the same symbol for simplicity), with $\varepsilon$ a positive parameter, is also maximal monotone for sufficiently small values of $\varepsilon$. Moreover, $W$ is a cylindrical Wiener process, and the random time-dependent coefficients $f$, $B$ satisfy suitable measurability and Lipschitz continuity conditions. Then (1.1) admits a unique mild solution $u_\varepsilon$ with continuous paths for every positive $\varepsilon$ close to or equal to zero. Denoting the mild solution to (1.1) with $\varepsilon = 0$ by $u$, if $A + \varepsilon G$ converges to $A$ in the strong resolvent sense as $\varepsilon \to 0$, then $u_\varepsilon$ converges to $u$ in a rather strong topology (implying, in particular, the uniform convergence in probability on compact time intervals).

Our main interest is about cases where the domain of $G$ is included in the domain of $A$ and $A$ is “hyperbolic” (in the terminology of Kato, cf. [6], i.e. the semigroup $S_A$ generated by $-A$ is not holomorphic). The aim is then to study the differentiability of arbitrary order of the map $\varepsilon \mapsto u_\varepsilon$, under suitable assumptions on the interplay between the semigroup $S_A$ and $G$ and regularity assumptions on the initial datum and the coefficients of (1.1). This allows to construct series expansions in $\varepsilon$ of functionals of $u_\varepsilon$ “centered” around the corresponding functionals of $u$, appealing to arguments based on Taylor’s formula, composition of (Fréchet) differentiable maps, and composition of formal series. Such asymptotic expansions have been obtained in [1] in the simpler case where $f$ and $B$ are random time-dependent maps that do not explicitly depend on the unknown. The results of [1] have been obtained under the assumption that the semigroups generated by $-A$ and $-G$ commute, which implies that $S_{A+\varepsilon G} = S_AS_{\varepsilon G}$ (with obvious meaning of the notation), and using a Taylor-type formula for $S_{\varepsilon G}$ (see [3, Proposition 1.1.6]). Here we proceed in a completely different way, which essentially consists in differentiating (1.1) with respect to the parameter $\varepsilon$. This method seems both clearer and more powerful, at least in the sense that, even assuming that $S_{A+\varepsilon G} = S_AS_{\varepsilon G}$, the technique of expanding $S_{\varepsilon G}$ in a series of Taylor type does not appear to be helpful in the case of equations with drift and diffusion coefficients depending on the unknown. It would not be difficult, in our approach, to allow $u_0$, 

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f and B to depend on \( \varepsilon \) too, at the cost of little more than heavier notation. We also do not assume that \( A \) and \( G \) are (negative) generators of commuting semigroups, even though our alternative weaker hypotheses are admittedly still quite strong and most likely not easy to check. On the other hand, the differentiability of \( \varepsilon \mapsto u_\varepsilon \), even in the linear deterministic setting where both \( f \) and \( B \) are zero, is difficult to obtain already at order two (cf. [2] pp. 506-ff). It seems hence unlikely that asymptotic expansions as \( \varepsilon \to 0 \) can be obtained without rather heavy assumptions.

We remark that, even though we discuss in this paper only the differentiability of \( u_\varepsilon \) with respect to \( \varepsilon \), all results about series expansions of functionals of \( u_\varepsilon \) around the corresponding functionals of \( u \) obtained in [1] immediately extend to the more general situation considered here. In particular, one can extend the results on the parabolic regularization of a suitably extended Musiela's SPDE obtained in op. cit., where the volatilities and the drift term are random time-dependent maps that do not depend on the forward curve, to the "full" Musiela SPDE where the volatilities, hence the drift term, may indeed depend on the forward curve.

The content of the remaining text is organized as follows: in section 2 after stating the main assumptions on \( A \) and \( G \), we obtain some auxiliary results of analytic nature on the behavior of the semigroup with negative generator \( A \) and \( A + \varepsilon G \) on the domain of powers of \( G \). Moreover, we recall results on existence, uniqueness, and continuous dependence on the coefficients for stochastic evolution equations on Hilbert spaces. In section 3 we formally differentiate (1.1) with respect to \( \varepsilon \) and establish the existence and uniqueness of solutions to the stochastic evolution equations thus obtained, as well as their continuous dependence on the parameter \( \varepsilon \). In the final sections 4 and 5 we prove the differentiability of the map \( \varepsilon \mapsto u_\varepsilon \), showing that its derivative of order \( k \) coincides with the mild solution to the equations obtained by formally differentiating equation (1.1) \( k \) times with respect to \( \varepsilon \).

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2 Preliminaries and auxiliary results

Let us begin with some conventions and notations. All random quantities will be defined on a fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a filtration \((\mathcal{F}_t)_{t \in [0,T]}\) satisfying the so-called usual conditions. A fixed cylindrical Wiener process on a separable Hilbert space \( U \) will be denoted by \( W \). A Hilbert space \( H \), with scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), will be fixed from now on. Let \( E \) be a Banach space. The (quasi-)normed space of adapted, continuous \( E \)-valued processes \( X \) such that

\[
\| X \|_{C^p(E)} := \| X \|_{L^p([0,T];E)} < \infty,
\]

with \( p > 0 \), is denoted by \( C^p(E) \), or just by \( C^p \) if \( E = H \). This is a Banach spaces for \( p \geq 1 \), and a quasi-Banach space for \( p < 1 \). The Banach space of multilinear maps from \( H^k \) to \( E \), with \( k \) a positive integer, will be denoted by \( \mathcal{L}_k(H;E) \). If \( E \) is a Hilbert space, we shall write \( \mathcal{L}^2(U;E) \) to mean the Hilbert space of Hilbert-Schmidt operators from \( U \) to \( E \).

Let \( A \) and \( G \) be linear maximal monotone operators on \( H \). We shall denote the strongly continuous semigroup of contractions on \( H \) generated by \(-A\) by \( S_A \) (and analogously for any other maximal monotone operator). We recall that \( G \) is a closed operator, hence so is every power \( G^k \), with \( k \) a positive integer, and the domain \( D(G^k) \) of \( G^k \), endowed with the scalar product

\[
\langle \phi, \psi \rangle_{D(G^k)} := \langle \phi, \psi \rangle + \langle G\phi, G\psi \rangle + \cdots + \langle G^k\phi, G^k\psi \rangle,
\]

is a Hilbert space.

The following assumption will be in force throughout.

Assumption 2.1. There exists \( \varepsilon_0 \) such that the closure of \( A + \varepsilon G \) is maximal monotone for every \( \varepsilon \in [0,\varepsilon_0] \). Moreover, there exist an integer \( m \geq 1 \) and constants \( \alpha_1, \ldots, \alpha_{m+1} \) such that, for every \( k \in \{1, \ldots, m+1\} \),

\[
\| G^k S_A(t) \phi \| \leq e^{\alpha_k t} \| G^k \phi \| \quad \forall \phi \in D(G^k), \ t \geq 0.
\]
The assumption is satisfied, for instance, if $S_A$ and $S_G$ commute. Moreover, it immediately implies that there exists a constant $\alpha$ such that
\[
\|S_A(t)\phi\|_{D(G^\infty)} \leq e^{\alpha t} \|\phi\|_{D(G^\infty)} \quad \forall \phi \in D(G^\infty), \; t \geq 0.
\]
Note that $A + \varepsilon G$ is closable because it is monotone, so the assumption about it is that, for every $\varepsilon \in [0, \varepsilon_0]$, there exists $\lambda > 0$ such that the image of $\lambda + A + \varepsilon G$ is dense in $H$. We shall abuse notation in the following writing $A + \varepsilon G$ to mean its closure, and we shall assume, for simplicity, that $\varepsilon_0 = 1$.

### 2.1 Analytic preliminaries

Assumption (2.1) implies analogous estimates for $S_{A+\varepsilon G}$.

**Proposition 2.2.** One has, for every $k \in \{1, \ldots, m + 1\}$,
\[
\|G^k S_{A+\varepsilon G}(t)\phi\| \leq e^{\alpha t} \|G^k \phi\| \quad \forall \phi \in D(G^k), \; t \geq 0.
\]

**Proof.** Let $k \in \{1, \ldots, m + 1\}$ and $\phi \in D(G^k)$. One has, by the Trotter product formula (see, e.g., [5, p. 227]),
\[
S_{A+\varepsilon G}(t)\phi = \lim_{n \to \infty} (S_A(t/n)S_{G}(t/n))^n \phi, \tag{2.1}
\]
where, for any natural $n \geq 1$,
\[
\|G^k S_A(t/n)S_{\varepsilon G}(t/n)\phi\| \leq e^{\alpha t/n} \|G^k S_{\varepsilon G}(t/n)\phi\|
\leq e^{\alpha t/n} \|S_{\varepsilon G}(t/n)G^k \phi\|
\leq e^{\alpha t/n} \|G^k \phi\|.
\]

Let us show that if
\[
\|G^k (S_A(t/n)S_{\varepsilon G}(t/n))^{j} \phi\| \leq e^{\alpha t/n} \|G^k \phi\|
\]
(which has just been proved with $j = 1$) holds for a natural number $j < n$, then the same inequality holds with $j$ replaced by $j + 1$. In fact,
\[
\|G^k (S_A(t/n)S_{\varepsilon G}(t/n))^{j+1} \phi\|
\leq e^{\alpha t/n} \|G^k (S_A(t/n)S_{\varepsilon G}(t/n))^{j} \phi\|
\leq e^{\alpha t/n} \|G^k \phi\|.
\]

In particular, one has
\[
\|G^k (S_A(t/n)S_{\varepsilon G}(t/n))^{n} \phi\| \leq e^{\alpha t} \|G^k \phi\|,
\]
which implies that there exists $\zeta \in H$ such that, passing to a subsequence if necessary,
\[
G^k (S_A(t/n)S_{\varepsilon G}(t/n))^{n} \phi \rightharpoonup \zeta
\]
weakly in $H$. The closedness of $G^k$ (in the strong-weak sense) and (2.1) imply that $\zeta = G^k S_{A+\varepsilon G}(t)\phi$, and weak lower semicontinuity of the norm yields
\[
\|G^k S_{A+\varepsilon G}(t)\phi\| \leq e^{\alpha t} \|G^k \phi\|.
\]

Let us introduce the resolvents of $A$ and $A + \varepsilon G$ defined, for $\lambda > 0$, as
\[
R_{\lambda} := (\lambda + A)^{-1}, \quad R_{\lambda}(\varepsilon) := (\lambda + A + \varepsilon G)^{-1},
\]
respectively. By $R_{\lambda}(0)$ we shall mean $R_{\lambda}$. 




Proposition 2.3. For any $\varepsilon \in [0,1]$ and $\lambda > \alpha_1$, $R_{\lambda}(\varepsilon + h)x$ converges to $R_{\lambda}(\varepsilon)x$ as $h \to 0$ for all $x \in H$.

Proof. Let $\phi \in D(G)$. One has, by assumption, $\|GS_{\lambda}(t)\phi\| \leq e^{\alpha_1 t}\|G\phi\|$, which implies, recalling that $G$ is closed, that, for every $\lambda > \alpha_1$,

$$GR_{\lambda}\phi = \int_0^{\infty} e^{-\lambda t}GS_{\lambda}(t)\phi \, dt,$$

hence

$$\|GR_{\lambda}\phi\| \leq \int_0^{\infty} e^{-\lambda t}\|GS_{\lambda}(t)\phi\| \, dt \leq \frac{1}{\lambda - \alpha_1}\|G\phi\|.$$

The second resolvent identity yields

$$R_{\lambda}(\varepsilon)\phi - R_{\lambda}\phi = \varepsilon R_{\lambda}(\varepsilon)GR_{\lambda}\phi,$$

where

$$\|R_{\lambda}(\varepsilon)GR_{\lambda}\phi\| \leq \|GR_{\lambda}\phi\| \leq \frac{1}{\lambda - \alpha_1}\|G\phi\|.$$

Therefore $R_{\lambda}(\varepsilon) - R_{\lambda}$ converges pointwise to zero on $D(G)$, which is dense in $H$, and is bounded in $\mathcal{L}(H)$ uniformly with respect to $\varepsilon \in [0,1]$. From this it follows that $R_{\lambda}(\varepsilon)$ converges to $R_{\lambda}$ pointwise on $H$. Taking Proposition 2.2 into account, the proof that $R_{\lambda}(\varepsilon + h)$ converges to $R_{\lambda}(\varepsilon)$ pointwise in $H$ as $h \to 0$ is completely analogous.

\[\Box\]

Proposition 2.4. The restrictions of the semigroups $S_{A}$ and $S_{A+\varepsilon G}$ on $D(G^k)$ are strongly continuous quasi-contraction semigroups thereon for every $k \in \{1, \ldots, m+1\}$.

Proof. The family of operators $S_{A}$ and $S_{A+\varepsilon G}$ are quasi-contractive endomorphisms of $D(G^k)$ for every $k \in \{1, \ldots, m+1\}$ by Assumption 2.1 and Proposition 2.2, respectively. In the following we write $S$ to mean either $S_{A}$ or $S_{A+\varepsilon G}$, as the argument is identical. Let us first prove weak continuity, i.e. that $\phi(S(t)h) \to \phi(h)$ for every $\phi \in D(G^k)'$, the dual of $D(G^k)$. By the Riesz representation theorem, for every $\phi \in D(G^k)'$ there exists $\ell_{\phi} \in D(G^k)$ such that

$$\phi: h \mapsto \langle \ell_{\phi}, h \rangle_{D(G^k)} = \langle \ell_{\phi}, h \rangle + \langle G\ell_{\phi}, Gh \rangle + \cdots + \langle G^{k-1}\ell_{\phi}, G^k h \rangle.$$

(2.2)

Let $j \in \{1, \ldots, k\}$ and $h \in D(G^j)$ be fixed. Thanks to Assumption 2.1 and Proposition 2.2, $G^jS(t)h$ is bounded in $H$, hence there exists $\zeta \in H$ and a sequence $(t_n)$ converging to zero such that $G^jS(t_n)h$ converges weakly to $\zeta$ in $H$ as $n \to \infty$. Since $G^j$ is closed, hence also strongly-weakly closed, one has $\zeta = G^j h$. By uniqueness of the limit we conclude that $G^jS(t)h$ converges weakly to $G^j h$ in $H$ as $t \to 0$. This immediately implies, in view of (2.2), that $S(t)h \to h$ weakly in $D(G^k)$ as $t \to 0$. The proof is concluded recalling that weak continuity for semigroups implies strong continuity (see, e.g., [5, p. 40]).

The convergence of resolvent of Proposition 2.3 continues to hold also on $D(G^m)$, as we now show.

Proposition 2.5. Let $\varepsilon \in [0,1]$ and $k \in \{1, \ldots, m\}$. The resolvent $R_{\lambda}(\varepsilon + h)$ converges to $R_{\lambda}(\varepsilon)$ pointwise on $D(G^k)$ as $h \to 0$ for every $\lambda > \alpha_k \vee \alpha_{k+1}$.

Proof. The proof is analogous to that of Proposition 2.3 for any $\phi \in D(G^{k+1})$ one infers, by Assumption 2.1 that

$$\|G^k R_{\lambda}\phi\| \leq \frac{1}{\lambda - \alpha_k}\|G^k \phi\|.$$

The second resolvent identity implies

$$G^k R_{\lambda}(\varepsilon)\phi - G^k R_{\lambda}\phi = \varepsilon G^k R_{\lambda}(\varepsilon)GR_{\lambda}\phi,$$
where, by Proposition 2.2,
\[ \|G^k R_\lambda(\varepsilon) GR_\lambda \phi\| \leq \frac{1}{\lambda - \alpha_k} \|G^{k+1} R_\lambda \phi\| \leq \frac{1}{(\lambda - \alpha_k)(\lambda - \alpha_{k+1})} \|G^{k+1} \phi\|. \]
This implies that \( R_\lambda(\varepsilon) - R_\lambda \) converges pointwise to zero on \( D(G^{k+1}) \), which is dense in \( D(G^k) \), and is bounded in \( L^p(D(G^k)) \) uniformly with respect to \( \varepsilon \in [0, 1] \). Therefore \( R_\lambda(\varepsilon) \) converges to \( R_\lambda \) pointwise on \( D(G^k) \) as \( \varepsilon \to 0 \). Using Proposition 2.2 instead of Assumption 2.1 allows one to establish, by the same argument, that \( R_\lambda(\varepsilon + h) \) converges to \( R_\lambda(\varepsilon) \) pointwise in \( D(G^k) \) as \( h \to 0 \) for every \( \varepsilon \in [0, 1] \).

2.2 Existence, uniqueness and convergence of mild solutions

We assume throughout that the maps
\[ f : \Omega \times [0, T] \times H \to H, \quad B : \Omega \times [0, T] \times H \to L^2(U; H) \]
are progressively measurable (in the sense that they are measurable with respect to the product \( \mathcal{B} \otimes \mathcal{B}(H) \), where \( \mathcal{B} \) stands for the \( \sigma \)-algebra of progressively measurable subsets of \( \Omega \times [0, T] \)) and they satisfy the Lipschitz continuity condition
\[ \|f(\omega, t, x) - f(\omega, t, y)\| + \|B(\omega, t, x) - B(\omega, t, y)\|_{L^2(U; H)} \lesssim \|x - y\|, \]
for all \((\omega, t) \in \Omega \times [0, T]\), with constant independent of the \( \omega \) and \( t \).

It is well known that if, for a \( p > 0 \), \( u_0 \in L^p(\Omega, \mathcal{F}_0; H) \) and there exists \( a \in H \) such that
\[ f(a) \in L^p(\Omega; L^1(0, T; H)), \quad B(a) \in L^p(\Omega; L^2(0, T; \mathcal{L}^2(U; H))), \]
then the stochastic evolution equation on \( H \)
\[ du + Au \, dt = f(u) \, dt + B(u) \, dW, \quad u(0) = u_0, \]
Admits a unique mild solution \( u \in C^p \) and the solution map \( u_0 \mapsto u \) is Lipschitz continuous from \( L^p(\Omega; H) \) to \( C^p \) (see, e.g., [1] 1).

An important tool for the developments in the following sections is a more precise continuous dependence result for the mild solution to (2.4) on its data. In particular, let us consider the family of equations indexed by \( n \in \mathbb{N} \)
\[ du_n + A_n u_n \, dt = f_n(u_n) \, dt + B_n(u_n) \, dW, \quad u_n(0) = u_{0n}, \]
where \( A_n \) is a linear maximal monotone operator on \( H \), \( u_{0n} \in L^0(\Omega, \mathcal{F}_0; H) \), and the maps
\[ f_n : \Omega \times [0, T] \times H \to H, \quad B_n : \Omega \times [0, T] \times H \to \mathcal{L}^2(U; H) \]
satisfy the same measurability conditions satisfied by \( f \) and \( B \), respectively, and are Lipschitz continuous in their third argument, uniformly over \( \Omega \times [0, T] \) as well as with respect to \( n \), i.e. there exist positive constants \( N_1 \) and \( N_2 \), not depending on \( \omega \), \( t \) and \( n \), such that
\[ \|f_n(\omega, t, x) - f_n(\omega, t, y)\| \leq N_1 \|x - y\|, \]
\[ \|B_n(\omega, t, x) - B_n(\omega, t, y)\|_{\mathcal{L}^2(U; H)} \leq N_2 \|x - y\| \]
for all \((\omega, t, n) \in \Omega \times [0, T] \times \mathbb{N} \). Therefore, for any \( p > 0 \), if \( f_n \) and \( B_n \) satisfy condition (2.3) and \( u_{0n} \in L^p(\Omega, \mathcal{F}_0; H) \), there exists a unique mild solution \( u_n \in C^p \) to (2.5).

**Theorem 2.6.** Let \( p > 0 \) Assume that \( f, B \) as well as \( f_n \) and \( B_n \) satisfy condition (2.3) for every \( n \in \mathbb{N} \). Furthermore, assume that, as \( n \to \infty \), \( A_n \to A \) in the strong resolvent sense, \( f_n(\omega, t, \cdot) \) and \( B_n(\omega, t, \cdot) \) converge to \( f(\omega, t, \cdot) \) and \( B(\omega, t, \cdot) \) pointwise in \( H \) and \( \mathcal{L}^2(U; H) \), respectively, for all \((\omega, t) \in \Omega \times [0, T] \), and \( u_{0n} \to u_0 \) in \( L^p(\Omega, \mathcal{F}_0; H) \). Then \( u_n \to u \) in \( C^p \).
The proof is entirely similar to that of \cite{10} Theorem 2.4 and hence omitted (cf. also \cite{8} where, however, \( A_t \) as well as \( A \) have to be negative generators of holomorphic semigroups).

We are now in the position to state and prove the well-posedness and convergence results for (1.1) needed in the following sections. To this purpose, we shall say that \( f \) and \( B \) satisfy condition \( H(m,p) \), \( m \in \mathbb{N}, p > 0 \), if

\[
\begin{align*}
  f : \Omega \times [0,T] \times H & \rightarrow D(G^m), \\
  B : \Omega \times [0,T] \times H & \rightarrow \mathcal{L}^2(U; D(G^m))
\end{align*}
\]

are progressively measurable and they fulfill the Lipschitz continuity condition

\[
\| f(\omega,t,x) - f(\omega,t,y) \|_{D(G^m)} + \| B(\omega,t,x) - B(\omega,t,y) \|_{\mathcal{L}^2(U; D(G^m))} \lesssim \| x - y \|, \quad (2.6)
\]

with implicit constant independent of \( \omega \) and \( t \), and there exists \( a \in H \) such that (2.3) holds with \( H \) replaced by \( D(G^m) \).

**Proposition 2.7.** Let \( p > 0 \) and \( m \in \mathbb{N} \). Assume that \( f \) and \( B \) satisfy assumption \( H(m,p) \) and \( u_0 \in L^p(\Omega, \mathcal{F}_0; D(G^m)) \). Then equation (1.1) admits a unique mild solution \( u_\varepsilon \in \mathcal{C}^p(D(G^m)) \) for every \( \varepsilon \in [0,1] \). Moreover, \( u_{\varepsilon+h} \) converges to \( u_\varepsilon \) in \( \mathcal{C}^p(D(G^m)) \) as \( h \to 0 \) for every \( \varepsilon \in [0,1] \).

**Proof.** Well-posedness is a consequence of Proposition 2.4 upon observing that (2.6) holds also when the norm on the right-hand side is replaced by the norm of \( D(G^m) \). The convergence of \( u_{\varepsilon+h} \) to \( u_\varepsilon \) in \( \mathcal{C}^p(D(G^m)) \) as \( h \to 0 \) follows by Proposition 2.5 and Theorem 2.6.

\[\square\]

### 3 Equations for formal derivatives

The purpose of this section is to establish well-posedness in appropriate spaces of the linear stochastic evolution equations obtained by formally differentiating (1.1) with respect to the parameter \( \varepsilon \). The formal \( n \)-th derivative of \( \delta \mapsto u_\delta \) at the point \( \varepsilon \) will be denoted by \( u_\varepsilon^n \). Derivatives of \( f \) and \( B \) are always meant with respect to their third argument. First order formal differentiation yields

\[
du_\varepsilon^1 + (A + \varepsilon G)u_\varepsilon^1 \, dt + Gu_\varepsilon = f(u_\varepsilon)u_\varepsilon^1 \, dt + B'(u_\varepsilon)u_\varepsilon^1 \, dW, \quad u_\varepsilon^1(0) = 0. \quad (3.1)
\]

In order to write the equations obtained by higher-order formal derivatives, we observe that, for any function \( \varepsilon \mapsto g_\varepsilon \in C^\infty([0,1]; D(G)) \), one has,

\[
\left( (A + \varepsilon G)g_\varepsilon \right)^{(n)} = \sum_{j=0}^{\infty} \binom{n}{j} (A + \varepsilon G)^{(j)}g_\varepsilon^{(n-j)} = (A + \varepsilon G)g_\varepsilon^{(n)} + nGg_\varepsilon^{(n-1)},
\]

as well as

\[
\left( f(g_\varepsilon) \right)^{(n)} = f'(g_\varepsilon)g_\varepsilon^{(n)} + r(f,g;n-1,\varepsilon),
\]

\[
\left( B(g_\varepsilon) \right)^{(n)} = B'(g_\varepsilon)g_\varepsilon^{(n)} + r(B,g;n-1,\varepsilon),
\]

where \( r(B,g;n-1,\varepsilon) \) is a finite linear combination of terms of the type

\[
B^{(j)}(g_\varepsilon)(g_\varepsilon^{(n_1)}, \ldots, g_\varepsilon^{(n_j)}),
\]

with \( j \in \{2, \ldots, n\}, n_1 + \cdots + n_j = n, n_1, \ldots, n_j \geq 1 \). In particular, note that \( r(B,g;n-1,\varepsilon) \) depends only on terms of order at most \( n-1 \) and does not involve \( G \). Completely similar considerations clearly hold also for \( r(f,g;n-1,\varepsilon) \).

Therefore one has, for every integer \( n \geq 1 \),

\[
\begin{align*}
  du_\varepsilon^n + (A + \varepsilon G)u_\varepsilon^n \, dt + nGu_\varepsilon^{n-1} \, dt \\
  = f'(u_\varepsilon)u_\varepsilon^n \, dt + \varphi_n(u_\varepsilon) \, dt + B'(u_\varepsilon)u_\varepsilon^n \, dW + \Phi_n(u_\varepsilon) \, dW,
\end{align*}
\]

\[u_\varepsilon^n(0) = 0,\]

\[1\text{These terms can be written in a more explicit way by the formula for the derivatives of higher order of a composite function (see, e.g., \cite{2} p. 272), but it is not important for our purposes.}\]
where $\varphi_n(u\varepsilon)$ and $\Phi_n(u\varepsilon)$ are obtained by $r(\cdot,\cdot,n-1,e)$ in the obvious way.

Assuming that $H(m,p)$ holds for a fixed $m \geq 1$, we are going to show that, given any $n \in \{1,\ldots,m\}$, (3.2) admits a unique mild solution $u^n \in C^0(D(G^{m-n}))$ under suitable differentiability assumptions on $f$ and $B$. Let us begin with the equation for first-order formal derivatives. From now on we shall make the

**Assumption 3.1.** The maps

$$f(\omega,t,\cdot): H \rightarrow D(G^{m-1}),$$

$$B(\omega,t,\cdot): H \rightarrow \mathcal{L}_j(H;\mathcal{L}^2(U;D(G^{m-1})))$$

are of class $C^1$ for every $(\omega,t) \in \Omega \times [0,T]$.

The derivatives of $f$ and $B$ in the sense of this assumption will be denoted by $f'$ and $B'$, respectively. Note that the Lipschitz continuity of $f$ and $B$ in (2.3) implies that $f$ and $B$ in this assumption are in fact of class $C^1$, not just $C^1$. More explicitly, the uniform Lipschitz continuity of $B(\omega,t)$ from $H$ to $\mathcal{L}^2(U;D(G^m))$ implies its uniform Lipschitz continuity, with the same Lipschitz constant, also with codomain $\mathcal{L}^2(U;D(G^{m-1}))$, as $D(G^m)$ is clearly contractively embedded in $D(G^{m-1})$. Therefore the norm in $\mathcal{L}_j(H;\mathcal{L}^2(U;D(G^{m-1})))$ of $B'$ is bounded by its Lipschitz constant, uniformly with respect to $(\omega,t) \in \Omega \times [0,T]$. Completely similar considerations hold for $f$.

**Proposition 3.2.** Let $p > 0$, $u_0 \in L^p(\Omega, \mathcal{F}_0; H)$ and assumption $H(m,p)$ be satisfied. Then equation (3.1) admits a unique mild solution $u^\varepsilon_\omega \in C^0(\mathcal{D}(G^{m-1}))$ for all $\varepsilon \in [0,1]$.

**Proof.** By Proposition 2.4 $S_{A+\varepsilon C}$ is a strongly continuous semigroup of quasi-contractions on $D(G^{m-1})$ for all $\varepsilon \in [0,1]$, and by Proposition 2.7 there exists a unique mild solution $u_\varepsilon \in C^0(\mathcal{D}(G^m))$ to (1.1). The latter implies that $Gu_\varepsilon \in L^p(\Omega;L^1(0,T;D(G^{m-1})))$. Moreover, since $f'(u\varepsilon)$ and $B'(u\varepsilon)$ satisfy assumption $H(m-1,p)$, the claim follows by Proposition 2.7. 

Now are now going to establish well-posedness for (3.2). We need further differentiability assumptions on $f$ and $B$, that will be in force throughout the rest of this section.

**Assumption 3.3.** For each $j \in \{1,m-1\}$, the functions

$$f^{(j)}(\omega,t,\cdot): H \rightarrow D(G^{m-j-1}),$$

$$B^{(j)}(\omega,t,\cdot): H \rightarrow \mathcal{L}_j(H;\mathcal{L}^2(U;D(G^{m-j-1})))$$

are of class $C^1$ for every $(\omega,t) \in \Omega \times [0,T]$, with derivative bounded uniformly with respect to $(\omega,t)$.

Note that, as a consequence of this assumption,

$$B^{(m)}(\omega,t,\cdot): H \rightarrow \mathcal{L}_m(H;\mathcal{L}^2(U;H))$$

is continuous and bounded uniformly with respect to $(\omega,t)$.

**Theorem 3.4.** Let $n \in \{1,\ldots,m\}$ and $p > 0$. If $u_0 \in L^p(\Omega, \mathcal{F}_0; D(G^m))$, then (3.2) admits a unique mild solution $u^\varepsilon_n \in C^{n/m}(D(G^{m-n}))$.

**Proof.** Recall that $S_{A+\varepsilon C}$ is a strongly continuous semigroup of quasi-contractions on $D(G^k)$ for every $k \in \{0,\ldots,m\}$ for all $\varepsilon \in [0,1]$, and $u_\varepsilon \in C^0(D(G^m))$. Moreover, Proposition 3.2 yields that the claim is true for $n = 1$. Therefore it suffices to show that if the claim is true for all $n \in \{1,\ldots,n'-1\}$, then it is true also for $n = n'$. To this purpose, note that $f'$ and $B'$ satisfy assumption $H(m-n,p)$ for every $n \in \{1,\ldots,m\}$. Moreover, each term of $\Phi_n(u\varepsilon)$ is of the type

$$B^{(j)}(u\varepsilon)(u^n_{n_1},\ldots,u^n_{n_j}), \quad n_1 + \cdots + n_j = n, \quad j \leq n,$$
hence the differentiability assumptions on $B$ imply
\[
\|B^{(j)}(u_{\varepsilon}^n, \ldots, u_{\varepsilon}^p)\|_{L^p(J; L^2(U; D(G^{m - j})) \cap C^1)} \\
\leq \|B^{(j)}(u_{\varepsilon})\|_{L^p(J; L^2(U; D(G^{m - j})))} \|u_{\varepsilon}^n\| \cdots \|u_{\varepsilon}^p\|.
\]
Since $D(G^{m - n})$ is contractively embedded in $D(G^{m - j})$ for every $j \in \{2, \ldots, n\}$, we have, by Hölder’s inequality,
\[
\|B^{(j)}(u_{\varepsilon}^n, \ldots, u_{\varepsilon}^p)\|_{L^p(J; L^2(U; D(G^{m - n})))} \\
\leq \sqrt[p]{\|u_{\varepsilon}^n\|_{L^p(J; H)} \cdots \|u_{\varepsilon}^p\|_{L^p(J; H)}} \\
\leq h \|u_{\varepsilon}^n\|_{C^{p/n}_1} \cdots \|u_{\varepsilon}^p\|_{C^{p/n}_1},
\]
where
\[
n = \frac{n_1}{p} + \cdots + \frac{n_j}{p}.
\]
By the inductive reasoning mentioned above, since obviously $n_1, \ldots, n_j < n$ for every $j \leq n$, we conclude that indeed $\Phi_n(u_{\varepsilon}) \in L^{p/n}(\Omega; L^2(0, T; L^2(U; D(G^{m - n}))))$. Applying an entirely similar reasoning to $\varphi_n(u_{\varepsilon})$ we infer that the maps $x \mapsto f'(u_{\varepsilon}) x + \varphi_n(u_{\varepsilon})$ and $x \mapsto B'(u_{\varepsilon}) x + \Phi_n(u_{\varepsilon})$ satisfy assumption $H(m - n, p/n)$, hence the proof is completed invoking Proposition 2.7.

### 4 First-order differentiability

Let Assumption $3.1$ as well as the hypotheses of Proposition $3.2$ be in force throughout this section.

**Theorem 4.1.** The map $\varepsilon \mapsto u_{\varepsilon}$ is of class $C^1$ from $[0, 1]$ to $C^p(D(G^{m - 1}))$ and $u_{\varepsilon}' = u_1$ for all $\varepsilon \in [0, 1]$.

**Proof.** Let us set, for any $\varepsilon \in [0, 1]$ and $h > 0$,
\[
w_{\varepsilon, h} := \frac{u_{\varepsilon} + h - u_1 - u_{\varepsilon}}{h}.
\]
We are going to show that $w_{\varepsilon, h}$ converges to zero in $C^p(D(G^{m - 1}))$ as $h \to 0$. For the sake of simplicity we shall assume that $f = 0$, as the calculations regarding this term are entirely similar to those for $B$, and in fact quite simpler. Elementary manipulations involving the Duhamel formula show that $w_{\varepsilon, h}$ is the unique mild solution to the equation
\[
dw_{\varepsilon, h} + (A + \varepsilon G)w_{\varepsilon, h} \, dt + G(u_{\varepsilon} + h - u_{\varepsilon}) = \left(\frac{B(u_{\varepsilon} + h) - B(u_1)}{h}ight) - B'(u_{\varepsilon}) u_{\varepsilon}' \, dW,
\]
\[w_{\varepsilon, h}(0) = 0,
\]
where, by the mean value theorem,
\[
B(u_{\varepsilon} + h) - B(u_{\varepsilon}) = \int_0^1 B'(u_{\varepsilon} + \theta(u_{\varepsilon} + h - u_{\varepsilon}))(u_{\varepsilon} + h - u_{\varepsilon}) \, d\theta,
\]
hence also
\[
\frac{B(u_{\varepsilon} + h) - B(u_1)}{h} - B'(u_{\varepsilon}) u_{\varepsilon}' = \int_0^1 \left(\frac{B'(u_{\varepsilon} + \theta(u_{\varepsilon} + h - u_{\varepsilon}))(u_{\varepsilon} + h - u_{\varepsilon})}{h} - B'(u_{\varepsilon}) u_{\varepsilon}'\right) \, d\theta
\]
\[= \int_0^1 B'(u_{\varepsilon} + \theta(u_{\varepsilon} + h - u_{\varepsilon})) w_{\varepsilon, h} \, d\theta
\]
\[+ \int_0^1 (B'(u_{\varepsilon} + \theta(u_{\varepsilon} + h - u_{\varepsilon})) - B'(u_{\varepsilon}))(u_{\varepsilon}') \, d\theta.
\]
By Proposition 2.7 we have that $G(u_{\varepsilon+h} - u_{\varepsilon}) \to 0$ in $L^p(\Omega; L^1(0,T; D(G^{m-1})))$ as $h \to 0$. Moreover, since $u_{\varepsilon+h} \to u_{\varepsilon}$ in $C^p$, there exists a sequence $h'$ such that $\sup_{[0,T]} |u_{\varepsilon+h'} - u_{\varepsilon}| \to 0$ a.s., hence, by continuity of $B'$,

$$
\lim_{h' \to 0} B'(u_\varepsilon + \theta(u_{\varepsilon+h'} - u_{\varepsilon}))x = B'(u_\varepsilon)x
$$

for all $\theta \in [0,1]$ and $x \in H$ a.e. in $\Omega \times [0,T]$. The dominated convergence theorem then implies, by the boundedness of $B'$, that

$$
\lim_{h' \to 0} \int_0^1 B'(u_\varepsilon + \theta(u_{\varepsilon+h'} - u_{\varepsilon}))x d\theta = B'(u_\varepsilon)x
$$

for all $x \in H$ a.e. in $\Omega \times [0,T]$. Similarly,

$$
\lim_{h' \to 0} \int_0^1 (B'(u_\varepsilon + \theta(u_{\varepsilon+h'} - u_{\varepsilon})) - B'(u_\varepsilon))u_\varepsilon^j x d\theta = 0
$$

a.e. in $\Omega \times [0,T]$, as well as in $L^p(\Omega; L^2(0,T; L^2(U; D(G^{m-1}))))$ again by dominated convergence. Therefore $u_{\varepsilon,h'}$ converges in $C^p(D(G^{m-1}))$ to the unique mild solution $v$ to the equation

$$
dv + (A + \varepsilon G)v dt = B'(u_\varepsilon)v dW, \quad v(0) = 0.
$$

By linearity it immediately follows that $v = 0$. The proof is completed observing that from every sequence $h$ converging to zero we can extract a subsequence $h'$ such that $u_{\varepsilon,h'} \to 0$, hence $u_{\varepsilon,h} \to 0$ as $h \to 0$.

\section{Higher-order differentiability}

Let us assume that Assumptions 3.1 and 3.3 and the hypotheses of Proposition 3.2 are fulfilled.

\begin{theorem}
The map $\varepsilon \mapsto u_\varepsilon$ is of class $C^k$ from $[0,1]$ to $C^{p/k}(D(G^{m-k}))$ and $u_\varepsilon^{(k)} = u_\varepsilon^k$ for all $k \in \{1, \ldots, m\}$ and $\varepsilon \in [0,1]$.
\end{theorem}

\begin{proof}
Let $\varepsilon \in [0,1]$. We have already proved that $u_\varepsilon' = u_\varepsilon^1$ and that $u_\varepsilon^k \in C^{p/k}(D(G^{m-k}))$ for every $k \in \{1, \ldots, m\}$, hence we can achieve the proof by induction as follows: assuming that $u_\varepsilon^j = u_\varepsilon^{(j)}$ for all $j \in \{1, \ldots, k\}$, we are going to show that $u_\varepsilon^{k+1} = u_\varepsilon^{(k+1)}$. As before, we shall assume for simplicity that $f = 0$. One has

$$
du_{\varepsilon+h}^k + (A + \varepsilon G)u_{\varepsilon+h}^k dt + hGu_{\varepsilon+h}^k dt + kGu_{\varepsilon+h}^{k-1} dt = (B'(u_{\varepsilon+h})u_{\varepsilon+h}^k + \Phi_k(u_{\varepsilon+h})) dW,
$$

$$
du_{\varepsilon+h}^k + (A + \varepsilon G)u_{\varepsilon+h}^k dt + kGu_{\varepsilon+h}^{k-1} dt = (B'(u_{\varepsilon})u_{\varepsilon}^k + \Phi_k(u_{\varepsilon})) dW,
$$

$$
du_{\varepsilon+h}^{k+1} + (A + \varepsilon G)u_{\varepsilon+h}^{k+1} dt + (k+1)Gu_{\varepsilon+h}^k dt = (B'(u_{\varepsilon})u_{\varepsilon}^{k+1} + \Phi_{k+1}(u_{\varepsilon})) dW,
$$

hence

$$
w_{\varepsilon,h}^k := \frac{u_{\varepsilon+h}^k - u_\varepsilon^k}{h} - u_{\varepsilon,h}^{k+1}
$$

is the unique mild solution to

$$
du_{\varepsilon,h}^k + (A + \varepsilon G)u_{\varepsilon,h}^k dt + G(u_{\varepsilon+h}^k - u_{\varepsilon}^k) dt + kGu_{\varepsilon,h}^{k-1} dt
$$

$$
= \left(B'(u_{\varepsilon+h})u_{\varepsilon+h}^k - B'(u_{\varepsilon})u_{\varepsilon}^k\right) \frac{u_{\varepsilon+h}^k - u_\varepsilon^k}{h} - B'(u_{\varepsilon})u_{\varepsilon}^{k+1}\right) dW
$$

$$
+ \left(\Phi_k(u_{\varepsilon+h}) - \Phi_k(u_{\varepsilon})\right) \frac{u_{\varepsilon+h}^k - u_\varepsilon^k}{h} - \Phi_{k+1}(u_{\varepsilon}) dW,
$$

$$
w_{\varepsilon,h}^k(0) = 0.
$$
Writing
\[ B'(u_{\varepsilon+h})u_{\varepsilon+h}^k - B'(u_{\varepsilon})u_{\varepsilon}^k = B'(u_{\varepsilon})(u_{\varepsilon+h}^k - u_{\varepsilon}^k) + (B'(u_{\varepsilon+h}) - B'(u_{\varepsilon}))u_{\varepsilon+h}^k \]
yields
\[
\frac{B'(u_{\varepsilon+h})u_{\varepsilon+h}^k - B'(u_{\varepsilon})u_{\varepsilon}^k}{h} = B'(u_{\varepsilon})u_{\varepsilon}^{k+1}
\]
thus also
\[
dw_{\varepsilon,h}^k + (A + \varepsilon G)w_{\varepsilon,h}^k\,dt + G(u_{\varepsilon+h}^k - u_{\varepsilon}^k)\,dt + kGw_{\varepsilon,h}^{k-1}\,dt
\]
\[
= B'(u_{\varepsilon})w_{\varepsilon,h}^k\,dW + \left( \frac{\Phi_k(u_{\varepsilon+h}) - \Phi_k(u_{\varepsilon})}{h} - \Phi_{k+1}(u_{\varepsilon}) \right)
\]
\[
+ \left( \frac{B'(u_{\varepsilon+h}) - B'(u_{\varepsilon}))u_{\varepsilon+h}^k}{h} \right)\,dW.
\]

The identities
\[
B(u_{\varepsilon})^{(k)} = B'(u_{\varepsilon})u_{\varepsilon}^{(k)} + \Phi_k(u_{\varepsilon}),
\]
\[
B(u_{\varepsilon})^{(k+1)} = B'(u_{\varepsilon})u_{\varepsilon}^{(k+1)} + \Phi_{k+1}(u_{\varepsilon})
\]
\[
= (B'(u_{\varepsilon})u_{\varepsilon}^{(k)})' + (\Phi_k(u_{\varepsilon}))'
\]
\[
= B'(u_{\varepsilon})u_{\varepsilon}^{(k+1)} + B''(u_{\varepsilon})(u_{\varepsilon}^k, u_{\varepsilon}^1) + (\Phi_k(u_{\varepsilon}))'
\]
imply
\[
\Phi_{k+1}(u_{\varepsilon}) = B''(u_{\varepsilon})(u_{\varepsilon}^k, u_{\varepsilon}^1) + (\Phi_k(u_{\varepsilon}))',
\]
from which it follows that
\[
\frac{\Phi_k(u_{\varepsilon+h}) - \Phi_k(u_{\varepsilon})}{h} - \Phi_{k+1}(u_{\varepsilon}) + \left( \frac{B'(u_{\varepsilon+h}) - B'(u_{\varepsilon}))u_{\varepsilon+h}^k}{h} \right) \rightarrow 0
\]
in \( L^2(U; H) \) as \( h \rightarrow 0 \) for all \( (\omega, t) \in \Omega \times [0, T] \). Moreover, it follows by the inductive assumption that, as \( h \rightarrow 0 \), \( u_{\varepsilon+h}^k \rightarrow u_k^k \) and \( u_{\varepsilon,h}^{k-1} \rightarrow 0 \) in \( C^{(k)}(D(G^{m-k})) \), which in turn implies that
\[
G(u_{\varepsilon+h}^k - u_k^k) \rightarrow 0 \quad \text{and} \quad Gw_{\varepsilon,h}^{k-1} \rightarrow 0
\]
in \( C^{(k)}(D(G^{m-k-1})) \). Therefore \( u_{\varepsilon,h}^k \) converges in \( C^{(k+1)}(D(G^{m-k-1})) \) to the unique mild solution to the equation
\[
dv + (A + \varepsilon G)v\,dt = B'(u_{\varepsilon})v\,dW, \quad v(0) = 0,
\]
which is zero by linearity.

\[\square\]

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