Some probabilistic results
on width measures of graphs

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Abstract

Fixed parameter tractable (FPT) algorithms run in time $f(p(x)) \text{poly}(|x|)$, where $f$ is an arbitrary function of some parameter $p$ of the input $x$ and poly is some polynomial function. Treewidth, branchwidth, cliquewidth, NLC-width, rankwidth, and booleanwidth are parameters often used in the design and analysis of such algorithms for problems on graphs.

We show asymptotically almost surely (aas), booleanwidth $\beta_w(G)$ is $O(\text{rw}(G) \log \text{rw}(G))$, where $\text{rw}$ is rankwidth. More importantly, we show aas $\Omega(n)$ lower bounds on the treewidth, branchwidth, cliquewidth, NLC-width, and rankwidth of graphs drawn from a simple random model. This raises important questions about the generality of FPT algorithms using the corresponding decompositions.

1 The Introduction

Fixed parameter tractable (FPT) algorithms run in time $f(p(x)) \text{poly}(|x|)$, where $f$ is an arbitrary function of some parameter $p$ of the input $x$ and poly is some polynomial function. Notice that as long as it is safe to assume that $p$ is $O(1)$, the run time is polynomial in the length of the input, even with $f$ exponential or worse. For problems on graphs, parameter $p$ is usually a measure of complexity of some tree decomposition of a graph, which is referred to as the graph's width.

There has been much progress in the development of FPT graph algorithms recently. The attention seems to have shifted from treewidth ($\text{tw}$) \cite{26} \cite{20} to newer width measures \cite{16}: branchwidth ($\text{bw}$) \cite{25}, cliquewidth
It is known [10, 23, 19] graph $G$ has
\[ \text{nlcw}(G) \geq \text{cwd}(G) \geq \text{rw}(G) \] (1)
\[ \text{tw}(G) + 2 \geq \text{bw}(G) + 1 \geq \text{rw}(G) \] (2)

except for some trivial exceptions. There are graphs, whose treewidth is unbounded in cliquewidth [10], as well as graphs, for which cliquewidth is exponential in either rankwidth or booleanwidth [8]. Intriguingly, boolean-width can be both more or exponentially less than rankwidth [8].

There are $O(n)$ FPT algorithms for obtaining treewidth and branchwidth decompositions [6, 5]. For some (presently unknown) $f$, there exists an $f(k)O(n^3)$ algorithm for obtaining rankwidth-$k$ decompositions of a graph on $n$ vertices or certifying their non-existence [17], which also gives the best known approximation of cliquewidth and booleanwidth. Hence, the attraction.

To some extent, however, our understanding of fixed parameter tractability is limited by the very assumption that the parameter is constant. In this paper, we attempt to use probabilistic methods to study the dependence of width measures on the number of vertices of a graph. We show asymptotically almost surely, there are $\Omega(n)$ lower bounds on the treewidth, branchwidth, cliquewidth, NLC-width, and rankwidth of graphs drawn from a simple random model.

## 2 The Definitions

We mention only the definitions of rankwidth and booleanwidth we use in the proofs. For standard definitions of rankwidth and booleanwidth [24, 9, 8], as well as for any other definitions, please follow the references.

We take a more general view of what is a width measure, suggested by Robertson and Seymour [25] and quoted in verbatim from Bui-Xuan et al. [8]: Let $f$ be a cut function of a graph $G$, and $(T, \delta)$ a decomposition tree of $G$. For every edge $uv$ in $T$, $\{X_u, X_v\}$ denotes the 2-partitions of $V$ induced by the leaf sets of the two subtrees we get by removing $uv$ from $T$. The $f$-width of $(T, \delta)$ is the maximum value of $f(X_u)$, taken over every edge $uv$ of $T$. An optimal $f$-decomposition of $G$ is a decomposition tree of $G$ having minimum $f$-width. The $f$-width of $G$ is the $f$-width of an optimal $f$-decomposition of $G$.

In this framework, it is easy to define rankwidth and booleanwidth. In rankwidth, the function $f$ is the cut-rank function:
\[ f_{\text{rw}} = \log_2 |\{Y \subseteq B : \exists X \subseteq A, Y = \triangle_{x \in X} N(x)\}|, \] (3)
where neighborhood $N(x)$ are vertices adjacent to $x$ and $\triangle$ denotes the set difference. This is the base-2 logarithm of the size of the row space over GF(2)-sums, which is the number of pairwise different vectors that are spanned by the rows of the $|A| \times |V \setminus A|$ submatrix of the adjacency matrix of $G$ over GF(2), where $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$. The corresponding discontiguous definition of taking a submatrix will be used throughout the paper. Boolean-width can then be defined similarly with

$$f_{\beta w} = \log_2 |\{Y \subseteq B : \exists X \subseteq A, Y = \cup_{x \in X} N(x)\}|.$$  \hspace{1cm} (4)

Informally, we take the logarithm of the number of distinct unions of the neighbourhoods of vertices. This, without much surprise, is the base-2 logarithm of the size of the row space of a binary matrix with boolean-sums ($1 + 1 = 1$).

Let us now approach rankwidth via the rank of certain submatrices of random matrices over GF(2).

3 A Lemma

Let us initially use a simple model $M(m, n)$ of random $m \times n, m \leq n$ matrices over GF(2), where each element of a matrix is chosen independently to be 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$. More rigorously, this is a family of probability spaces over matrices over GF(2).

First, we state a theorem derived from Blömer, Karp, and Welzl \[4\], which may remind us of Shannon’s switching game \[27\], and use it to derive a simple lemma.

**Theorem 1** (Blömer et al. \[4\]). The probability that an $m \times n, m \leq n$ matrix $M_{m,n}$ drawn randomly from $M(m,n)$ has rank less than $m - d$ is $2^{-\Omega((n-m+d)(m+d))}$.

**Lemma 1.** Asymptotically almost surely, the minimum rank of $\frac{n}{3} \times \frac{2n}{3}$ submatrices of $M_{n,n}$ drawn randomly from $M(n,n)$ is bounded from below by $\Omega(n)$.

**Proof.** Let us denote the minimum rank among $\frac{n}{3} \times \frac{2n}{3}$ submatrices $S_{\frac{n}{3}, \frac{2n}{3}}$ in an $n \times n$ matrix $M_{n,n}$ over GF(2) drawn from $M(n,n)$ by $\mu$ and let us study the probability of $\mu$ being greater than an arbitrary $\frac{n}{6}$. Using Boole’s inequality \[1\], $O(3^{3n})$ bound \[21\ \[25\] on the number of submatrices of interest.
in an $n \times n$ matrix given by binomial coefficient $\binom{n}{k}$ \cite{7}, and Theorem 1 \cite{8}:

$$
\mathbb{P}(\mu \leq \frac{n}{6}) = \mathbb{P}(\text{rank}(N) \leq \frac{n}{6} \ \forall S_{\frac{n}{3}, \frac{2n}{3}} \in M_{n,n}) \leq \sum_{S_{\frac{n}{3}, \frac{2n}{3}} \in M_{n,n}} \mathbb{P}(\text{rank}(S_{\frac{n}{3}, \frac{2n}{3}}) \leq \frac{n}{6}) \leq \left(\frac{n}{\frac{n}{3}}\right) \mathbb{P}(\text{rank}(S_{\frac{n}{3}, \frac{2n}{3}}) \leq \frac{n}{6}) \approx O(3^{3n}2^{-n^2})
$$

Clearly,

$$
\lim_{n \to \infty} 3^{3n}2^{-n^2} = 0.
$$

\hfill \square

4 The Main Result

Now, we can state the main result using a simple model $G(n, \frac{1}{2})$ of random graphs of $n$ vertices, where each edge appears independently with probability $\frac{1}{2}$.

**Theorem 2.** Asymptotically almost surely, the rankwidth of a graph drawn randomly from $G(n, \frac{1}{2})$ is bounded from below by $\Omega(n)$.

**Proof.** The minimax theorem of Robertson and Seymour \cite{25} linking branchwidth and tangles, translated to rankwidth by Oum \cite{23}, implies there exists an edge in any decomposition tree, which corresponds to a partition $(V_1, V_2)$ of $n$ vertices of $G$, such that $\frac{n}{2} \geq |V_1| \geq \frac{n}{3}$ and $\frac{n}{2} \leq |V_2| \leq \frac{2n}{3}$. But then the rank of the $\frac{n}{3} \times \frac{2n}{3}$ submatrix of the minimum rank in the adjacency matrix of graph $G$, given by Lemma 1, is a lower bound on the value of the rankwidth of $G$. Notice we need not consider skew-symmetric matrices in Lemma 1 as the sub-matrix is not (necessarily) symmetric. \hfill \square

Given the trivial upper bound of $n$ on rankwidth of a graph on $n$ vertices, it is easy to see this lower bound is tight:

**Corollary 1.** Asymptotically almost surely, the rankwidth of a graph drawn randomly from $G(n, \frac{1}{2})$ is $\Theta(n)$.

Finally, using the inequalities between the values of the parameters \cite{12}, we can state the following:
Corollary 2. Asymptotically almost surely, the treewidth of a graph drawn randomly from $G(n, \frac{1}{2})$ is $\Theta(n)$.

Corollary 3. Asymptotically almost surely, the branchwidth of a graph drawn randomly from $G(n, \frac{1}{2})$ is $\Theta(n)$.

Corollary 4. Asymptotically almost surely, the cliquewidth of a graph drawn randomly from $G(n, \frac{1}{2})$ is $\Theta(n)$.

Corollary 5. Asymptotically almost surely, the NLC-width of a graph drawn randomly from $G(n, \frac{1}{2})$ is $\Theta(n)$.

It should be noted that Corollary 3 seems to be the first probabilistic result on branchwidth of random graphs.

5 Yet Another Bound

Independently, but still using probabilistic arguments, we can also show:

Theorem 3. Asymptotically almost surely, booleanwidth $\beta_{bw}(G)$ of a graph $G$ is $O(\text{rw}(G) \log \text{rw}(G))$, where $\text{rw}(G)$ is the rankwidth of $G$.

Proof. Bui-Xuan et al. [9, 8] have shown $2^{\beta_{bw}(G(A))}$ is bounded from above by the number of subspaces $GF(2)$-spanned by the rows (resp. columns) of $A$, where $G(A)$ is the graph given by the adjacency matrix $A$. Goldman and Rota [15] have shown the the number of subspaces of a vector space corresponds to the number of partitions of a set. But when we look at the number $c$ of partitions of a set of size $n$, with probability $1 - o(e^{-n})$ [13]:

$$\log c \leq n(\log n - \log \log(n - 1) + O(1)).$$

6 The Conclusions

In this paper, we have used probabilistic methods to study modern width measures of random graphs. We are aware of only a few results in this direction. Prior to the unofficial publication of this draft, Bodlaender and Kloks [20] studied treewidth of random graphs and Johansson [18] studied NLC-width and cliquewidth of random graphs. Independently, Gao [14] studied treewidth of random NK landscapes. Since the unofficial publication of this draft, Lee and Lee [22] have provided very elegant proofs of the our
results and Telle [1] has established the polylogarithmic booleanwidth of random graphs. Our results also complement the theorem of Boliac and Lozin [7], which implies that for each $k > 1$, the number of graphs having $n$ vertices and clique-width at most $k$ is only $2^{\Theta(n \log n)}$.

The results suggest the limits of generality of algorithms designed and analysed using five well-known width measures of graphs, although there clearly are exponentially large classes of graphs, for which they are very appropriate. If, however, the runtime is $f(k) \text{poly}(|x|)$, where $f$ is exponential or worse and there is a $\Omega(n)$ lower bounded to go with $k$, we have not gained much by making the analysis more detailed.

An important goal for further research is the characterisation of graphs with the expected value of some width measures logarithmic in the number of vertices, so as to provide some guidance, where can one apply graph decompositions and fixed parameter tractable algorithms successfully. Could it be that sparse constraint matrices of large classes of integer programs have branchwidth and rankwidth bounded by $O(\log n)$, and hence [12], are solvable in polynomial time, for instance? In random models parametrised with density, it seems interesting to study the behaviour of the expected value of width measures of “hard” instances. Could there be a relationship with high width measures?

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