Simple approximations for the ruin probability in the risk model with stochastic premiums and a constant dividend strategy

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Abstract We deal with a generalization of the risk model with stochastic premiums where dividends are paid according to a constant dividend strategy and consider heuristic approximations for the ruin probability. To be more precise, we construct five- and three-moment analogues to the De Vylder approximation. To this end, we obtain an explicit formula for the ruin probability in the case of exponentially distributed premium and claim sizes. Finally, we analyze the accuracy of the approximations for some typical distributions of premium and claim sizes using statistical estimates obtained by the Monte Carlo methods.

Keywords Risk model with stochastic premiums, constant dividend strategy, ruin probability, net profit condition, De Vylder approximation, Monte Carlo method

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1 Introduction

The ruin probability of an insurance company is one of the main risk measures considered in risk theory, and the problems of its calculation and approximation have attracted a lot of attention recently (see, e.g., [2, 16, 22, 25, 27–29] and references therein). Risk models where shareholders receive dividends from their insurance company have been of great interest to researchers since De Finetti first considered dividend strategies in insurance dealing with a binomial model [12]. The classical risk model and its various modifications with different dividend strategies are investigated

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in a number of papers (see, e.g., [1, 4, 8, 10, 11, 15, 20, 21, 26, 27, 30] and references therein).

It is well known that explicit formulas for the ruin probability can be derived only in a few special cases even for the classical Cramér–Lundberg risk model, so numerous heuristic approximations for this function have been proposed and studied (see [2, 3, 5, 9, 13, 16, 17, 28, 29]). So-called simple approximations, which use only some moments of the distribution of claim sizes and do not take into account its tail behavior, form a special class of approximations for the ruin probabilities.

The De Vylder approximation, which is introduced in [13] for the classical risk model, is supposed to be one of the most successful simple approximations. It is based on the heuristic idea to replace the investigated risk process by a risk process with exponentially distributed claim sizes such that the first three moments coincide (see also [16, 28, 29] for details). Thus, to apply the De Vylder approximation, we need to calculate only the first three moments of the distribution of the claim sizes. Despite its simplicity, the approximation gives surprisingly good results when the initial surplus is not too small, especially when the distribution of claim sizes is light-tailed. This fact was explained later by Grandell [17] after analyzing the simple approximations from a mathematical viewpoint. Analogues to the De Vylder approximation are constructed in the risk model with additional funds [23] and some risk models with reinsurance [7, 19].

The present paper deals with a generalization of the risk model with stochastic premiums where dividends are paid according to a constant dividend strategy. In what follows, we suppose that all stochastic objects we use below are defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the usual conditions. In the risk model with stochastic premiums (see, e.g., [6, 22]), premium sizes form a sequence \((\bar{Y}_i)_{i \geq 1}\) of non-negative independent and identically distributed (i.i.d.) random variables (r.v.’s) with cumulative distribution function (c.d.f.) \(F_{\bar{Y}}(y) = \mathbb{P}[\bar{Y}_i \leq y]\), and the number of premiums on the time interval \([0, t]\) is a Poisson process \((\bar{N}_t)_{t \geq 0}\) with constant intensity \(\bar{\lambda} > 0\). Similarly, claim sizes form a sequence \((Y_i)_{i \geq 1}\) of i.i.d. r.v.’s with c.d.f. \(F_Y(y) = \mathbb{P}[Y_i \leq y]\), and the number of claims on the time interval \([0, t]\) is a Poisson process \((N_t)_{t \geq 0}\) with constant intensity \(\lambda > 0\). Thus, the total premiums and claims on \([0, t]\) equal \(\sum_{i=1}^{\bar{N}_t} \bar{Y}_i\) and \(\sum_{i=1}^{N_t} Y_i\), respectively. Note that here \(\sum_{i=1}^{\bar{N}_t} \bar{Y}_i = 0\) if \(\bar{N}_t = 0\), and \(\sum_{i=1}^{N_t} Y_i = 0\) if \(N_t = 0\). In what follows, we also assume that the r.v.’s \((Y_i)_{i \geq 1}\) and \((\bar{Y}_i)_{i \geq 1}\) have finite expectations \(\mu > 0\) and \(\bar{\mu} > 0\), respectively, and \((Y_i)_{i \geq 1}\), \((\bar{Y}_i)_{i \geq 1}\), \((N_t)_{t \geq 0}\) and \((\bar{N}_t)_{t \geq 0}\) are mutually independent.

Moreover, we make the additional assumption that the insurance company pays dividends to its shareholders according to a constant dividend strategy, which implies that dividends are paid continuously at a rate \(d > 0\). The strategy can be considered as a multi-layer dividend strategy where the number of layers is equal to one (see, e.g., [27])). Next, we denote a non-negative initial surplus of the insurance company by \(x\), and let \(X_t(x)\) be its surplus at time \(t\) provided that the initial surplus is \(x\). Then the surplus process \((X_t(x))_{t \geq 0}\) is defined by the equality

\[
X_t(x) = x + \sum_{i=1}^{\bar{N}_t} \bar{Y}_i - \sum_{i=1}^{N_t} Y_i - dt, \quad t \geq 0.
\]
Next, let \( \tau(x) = \inf\{t \geq 0 : X_t(x) < 0\} \) be the ruin time for the risk process \((X_t(x))_{t \geq 0}\) defined by (1). For \( x \geq 0 \), the infinite-horizon ruin probability is defined by

\[
\psi(x) = \mathbb{E}[\mathbb{1}(\tau(x) < \infty) | X_0(x) = x],
\]

where \( \mathbb{1}(\cdot) \) is the indicator function. Note that the ruin probability is a special case of the expected discounted penalty function, which is introduced in [14] and also called the Gerber–Shiu function.

Thus, it is easily seen that the risk model described above is a special case of the model with stochastic premiums and a multi-layer dividend strategy investigated in [27], although in that paper it is assumed that the number of layers is more than one. In [27], piecewise integro-differential equations for the Gerber–Shiu function and the expected discounted dividend payments until ruin are derived. In addition, the model is studied in detail in the case of exponentially distributed claim and premium sizes. In particular, explicit formulas for the ruin probability as well as for the expected discounted dividend payments are obtained.

The aim of the present paper is to construct analogues to the De Vylder approximation for the ruin probability in the risk model described above and analyze the accuracy of these approximations. The rest of the paper is organized as follows. In Section 2, we obtain an explicit formula for the ruin probability in the case of exponentially distributed premium and claim sizes. We use this formula in Section 3, where we derive five- and three-moment analogues to the De Vylder approximation. Finally, Section 4 is devoted to numerical illustrations. To be more precise, we deal with some typical distributions of premium and claim sizes and apply the results obtained in Section 3. To analyze the accuracy of the approximations, we use statistical estimates obtained by the Monte Carlo methods.

## 2 An explicit formula for the ruin probability in the case of exponentially distributed premium and claim sizes

From now on, we suppose that the net profit condition holds, which in this model means that

\[
\bar{\lambda} \bar{\mu} > \lambda \mu + d.
\]

Theorem 1 below is a special case of Theorem 1 in [27], where it is formulated and proved for the Gerber–Shiu function in the model where the number of layers is more than one. It is easy to check that the assertion of the theorem remains true if the number of layers equals one.

**Theorem 1.** Let the surplus process \((X_t(x))_{t \geq 0}\) be defined by (1) under the above assumptions, and let \( F_Y(y) \) be continuous on \( \mathbb{R}_+ \). Then the function \( \psi(x) \) is differentiable on \( \mathbb{R}_+ \) and satisfies the integro-differential equation

\[
\frac{d}{dx} \psi'(x) + (\bar{\lambda} + \lambda) \psi(x) = \bar{\lambda} \int_0^\infty \psi(x + y) \, dF_Y(y)
\]

\[
+ \lambda \int_0^x \psi(x - y) \, dF_Y(y) + \lambda (1 - F_Y(x)), \quad x \geq 0.
\]
Remark 1. To solve equation (3), we use the following two boundary conditions. Firstly, using standard considerations (see, e.g., [22, 24, 28]) it can be easily shown that \( \lim_{x \to \infty} \psi(x) = 0 \) provided that the net profit condition (2) holds. Secondly, it is obvious that \( \psi(0) = 1 \) for this risk model. Although equation (3) is not solvable analytically in the general case, we can find explicit expressions for the corresponding ruin probability in some special cases. The uniqueness of the required solution to equation (3) should be justified in each case.

Assume now that the premium and claim sizes are exponentially distributed, i.e. their probability density functions (p.d.f.’s) are

\[
f_Y(y) = \frac{1}{\mu} e^{-y/\mu}, \quad y \geq 0,
\]

respectively. In this case, the integro-differential equation (3) can be reduced to a linear differential equation with constant coefficients.

Lemma 1. Let the surplus process \( (X_t(x))_{t \geq 0} \) be defined by (1) under the above assumptions, and let the premium and claim sizes be exponentially distributed with means \( \bar{\mu} \) and \( \mu \), respectively. Then for all \( x \geq 0 \), \( \psi(x) \) is a solution to the differential equation

\[
d\bar{\mu}\mu \psi'''(x) + (d(\bar{\mu} - \mu) + \bar{\mu}\mu(\bar{\lambda} + \lambda))\psi''(x) + (\bar{\lambda}\bar{\mu} - \lambda\mu - d)\psi'(x) = 0.
\]

The proof of Lemma 1 is similar to the proof of Lemma 1 in [27]. An explicit formula for the ruin probability is given in Theorem 2 below.

Theorem 2. Let the surplus process \( (X_t(x))_{t \geq 0} \) follow (1) under the above assumptions, and let premium and claim sizes be exponentially distributed with means \( \bar{\mu} \) and \( \mu \), respectively. If the net profit condition (2) holds, then

\[
\psi(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} \quad \text{for all} \quad x \geq 0,
\]

where

\[
\begin{align*}
z_1 &= -\frac{(d(\bar{\mu} - \mu) + \bar{\mu}\mu(\bar{\lambda} + \lambda)) + \sqrt{D}}{2d\bar{\mu}\mu}, \\
z_2 &= -\frac{(d(\bar{\mu} - \mu) + \bar{\mu}\mu(\bar{\lambda} + \lambda)) - \sqrt{D}}{2d\bar{\mu}\mu}, \\
D &= (d(\bar{\mu} + \mu) + \bar{\mu}\mu(\lambda - \bar{\lambda}))^2 + 4\bar{\lambda}\bar{\mu}^2\mu^2, \\
C_1 &= \frac{\bar{\lambda}\bar{\mu}(\bar{\mu} + \mu)(dz_2 + \bar{\lambda}) + d\bar{\lambda}\mu(\bar{\mu}z_1 - 1)}{d\bar{\lambda}\bar{\mu}^2(z_2 - z_1)}.
\end{align*}
\]

and

\[
C_2 = -\frac{\bar{\lambda}\bar{\mu}(\bar{\mu} + \mu)(dz_1 + \bar{\lambda}) + d\bar{\lambda}\mu(\bar{\mu}z_2 - 1)}{d\bar{\lambda}\bar{\mu}^2(z_2 - z_1)}.
\]

Proof. By Lemma 1, \( \psi(x) \) is a solution to (4) for all \( x \geq 0 \). The characteristic equation corresponding to (4) has the form

\[
d\bar{\mu}\mu z^3 + (d(\bar{\mu} - \mu) + \bar{\mu}\mu(\bar{\lambda} + \lambda))z^2 + (\bar{\lambda}\bar{\mu} - \lambda\mu - d)z = 0.
\]

Proof.
The discriminant of the equation
\[ d\tilde{\mu}\mu z^2 + (d(\tilde{\mu} - \mu) + \tilde{\mu}\mu(\tilde{\lambda} + \lambda))z + (\tilde{\lambda}\tilde{\mu} - \lambda\mu - d) = 0 \] (7)
is equal to
\[
(d(\tilde{\mu} - \mu) + \tilde{\mu}\mu(\tilde{\lambda} + \lambda))^2 - 4d\tilde{\mu}\mu(\tilde{\lambda}\tilde{\mu} - \lambda\mu - d)
= (d(\tilde{\mu} + \mu) + \tilde{\mu}\mu(\tilde{\lambda} + \lambda))^2 + 4\tilde{\lambda}\tilde{\mu}\mu^2 \mu^2,
\]
which is obviously positive and coincides with the constant D introduced above. Therefore, \(z_1\) and \(z_2\) defined in the assertion of the theorem are two real roots of equation (7).

By the net profit condition (2), we conclude that \(\tilde{\lambda}\tilde{\mu} - \lambda\mu - d > 0\) and
\[
d(\tilde{\mu} - \mu) + \tilde{\mu}\mu(\tilde{\lambda} + \lambda) = \mu(\tilde{\lambda}\tilde{\mu} - \lambda\mu - d) + \lambda\mu^2 + \lambda\tilde{\mu}\mu + d\tilde{\mu} > 0,
\]
which implies that \(z_1 < 0\) and \(z_2 < 0\) by Vieta’s theorem. Consequently, \(z_1 < 0\), \(z_2 < 0\) and \(z_3 = 0\) are roots of equation (6), from which we deduce that
\[
\psi(x) = C_1 e^{z_1x} + C_2 e^{z_2x} + C_3 \quad \text{for all} \quad x \geq 0
\]
with some constants \(C_1\), \(C_2\) and \(C_3\). Since the net profit condition (2) holds, by Remark 1, we have \(\lim_{x \to \infty} \psi(x) = 0\), which yields \(C_3 = 0\). So we obtain (5).

To determine the constants \(C_1\) and \(C_2\), we use the following two conditions.
Firstly, substituting (5) into the equality \(\psi(0) = 1\) we get
\[
C_1 + C_2 = 1. \tag{8}
\]
Secondly, letting \(x = 0\) in (3) we obtain
\[
d\psi'(0) + (\tilde{\lambda} + \lambda)\psi(0) = \tilde{\lambda} \int_0^\infty \psi(y) dF_\tilde{X}(y) + \lambda. \tag{9}
\]
Since
\[
\psi'(x) = C_1 z_1 e^{z_1x} + C_2 z_2 e^{z_2x} \quad \text{for all} \quad x \geq 0
\]
and
\[
\frac{1}{\tilde{\mu}} \int_0^\infty \psi(u) e^{-u/\tilde{\mu}} du = -\frac{C_1}{\tilde{\mu} z_1 - 1} - \frac{C_2}{\tilde{\mu} z_2 - 1},
\]
from (9) we get
\[
C_1 \left( dz_1 + \frac{\tilde{\lambda}}{\tilde{\mu} z_1 - 1} \right) + C_2 \left( dz_2 + \frac{\tilde{\lambda}}{\tilde{\mu} z_2 - 1} \right) = -\tilde{\lambda}. \tag{10}
\]
Taking into account that
\[
(\tilde{\mu} z_1 - 1)(\tilde{\mu} z_2 - 1) = \tilde{\mu}^2 z_1 z_2 - \tilde{\mu}(z_1 + z_2) + 1
= \tilde{\mu}^2 \frac{\tilde{\lambda}\tilde{\mu} - \lambda\mu - d}{d\tilde{\mu}\mu} + \tilde{\mu} \frac{d(\tilde{\mu} - \mu) + \tilde{\mu}\mu(\tilde{\lambda} + \lambda)}{d\tilde{\mu}\mu} + 1 = \frac{\tilde{\lambda}\tilde{\mu}(\tilde{\mu} + \mu)}{d\mu},
\]
we find the constants \(C_1\) and \(C_2\) from the system of linear equations (8) and (10), which always has a unique solution. Applying arguments similar to those in the proof of Theorem 3 in [27] we can show that the function \(\psi(x)\) that we found is a unique solution to (3) satisfying the required conditions, which completes the proof. \(\Box\)
3 Analogues to the De Vylder approximation

3.1 An auxiliary result

Let the process \((U_t)_{t \geq 0}\) be defined by

\[
U_t = \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i - \sum_{i=1}^{\bar{N}_t} Y_i - d t, \quad t \geq 0.
\]  

We construct two analogues to the De Vylder approximation replacing the process \((U_t)_{t \geq 0}\) by a process \((\tilde{U}_t)_{t \geq 0}\) with exponentially distributed premium and claim sizes. Since in this risk model the process \((\tilde{U}_t)_{t \geq 0}\) is determined by five parameters, which we denote by \(\tilde{\lambda}_0, \tilde{\mu}_0, \lambda_0, \mu_0\) and \(d_0\), five equalities are required to determine these parameters. Consequently, we need the first five moments of \((U_t)_{t \geq 0}\).

Lemma 2. Let the process \((U_t)_{t \geq 0}\) be defined by (11) under the above assumptions, \(\mathbb{E}[\bar{Y}_i^5] < \infty\) and \(\mathbb{E}[\bar{Y}_i^5] < \infty\). Then for all \(t \geq 0\), we have

\[
\mathbb{E}[U_t] = (\tilde{\lambda} \tilde{\mu} - \lambda \mu - d) t, \quad \text{(12)}
\]

\[
\mathbb{E}[U_t^2] = (\mathbb{E}[U_t])^2 + (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^2] + \lambda \mathbb{E}[Y_i^2]) t, \quad \text{(13)}
\]

\[
\mathbb{E}[U_t^3] = (\mathbb{E}[U_t])^3 + 3 \mathbb{E}[U_t] \cdot (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^2] + \lambda \mathbb{E}[Y_i^2]) t + (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^3] - \lambda \mathbb{E}[Y_i^3]) t, \quad \text{(14)}
\]

\[
\mathbb{E}[U_t^4] = (\mathbb{E}[U_t])^4 + 6 (\mathbb{E}[U_t])^2 \cdot (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^2] + \lambda \mathbb{E}[Y_i^2]) t + 4 \mathbb{E}[U_t] \cdot (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^3] - \lambda \mathbb{E}[Y_i^3]) t + 3 (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^3] - \lambda \mathbb{E}[Y_i^3]) t^2 \quad \text{(15)}
\]

\[
\mathbb{E}[U_t^5] = (\mathbb{E}[U_t])^5 + 10 (\mathbb{E}[U_t])^3 \cdot (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^2] + \lambda \mathbb{E}[Y_i^2]) t + 10 (\mathbb{E}[U_t])^2 \cdot (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^3] - \lambda \mathbb{E}[Y_i^3]) t + 15 \mathbb{E}[U_t] \cdot (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^2] + \lambda \mathbb{E}[Y_i^2]) t^2 + 5 \mathbb{E}[U_t] \cdot (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^2] + \lambda \mathbb{E}[Y_i^2]) t + 10 (\tilde{\lambda} \mathbb{E}[\bar{Y}_i^3] - \lambda \mathbb{E}[Y_i^3]) t^2 \quad \text{(16)}
\]

Proof. Let \(M_{\bar{Y}}(s)\) and \(M_Y(s)\) be the moment generating functions of the r.v.’s \(\bar{Y}_i\) and \(Y_i\), respectively, provided that they exist in some neighborhood of \(s = 0\). Furthermore, we denote the moment generating function of \((U_t)_{t \geq 0}\) by \(M(s)\). An easy computation shows that

\[
M(s) = \mathbb{E}[e^{s U_t}] = \exp\{\tilde{\lambda} t (M_{\bar{Y}}(s) - 1) + \lambda t (M_Y(-s) - 1) - dt s\}.
\]

Taking the first five derivatives of \(M(s)\) yields

\[
M'(s) = (\tilde{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt) M(s),
\]

\[
M''(s) = \left((\tilde{\lambda} t M''_{\bar{Y}}(s) - \lambda t M''_Y(-s) - dt)^2 + \tilde{\lambda} t M''_{\bar{Y}}(s) + \lambda t M''_Y(-s)\right) M(s),
\]
\[
M'''(s) = \left((\tilde{\lambda} t M''_Y(s) - \lambda t M'_Y(-s) - dt)^3
\right.
\]
\[+ 3(\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)(\tilde{\lambda} t M''_Y(s) + \lambda t M''_Y(-s))
\]
\[+ \tilde{\lambda} t M'''_Y(s) - \lambda t M'''_Y(-s)
\}
M(s).
\]
\[
M^{(IV)}(s) = \left((\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)^4
\right.
\]
\[+ 6(\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)^2(\tilde{\lambda} t M''_Y(s) + \lambda t M''_Y(-s))
\]
\[+ 4(\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)(\tilde{\lambda} t M''_Y(s) - \lambda t M''_Y(-s))
\]
\[+ 3(\tilde{\lambda} t M''_Y(s) + \lambda t M''_Y(-s))^2
\]
\[+ \tilde{\lambda} t M^{(IV)}_Y(s) + \lambda t M^{(IV)}_Y(-s)
\}
M(s).
\]

and
\[
M^{(V)}(s) = \left((\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)^5
\right.
\]
\[+ 10(\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)^3(\tilde{\lambda} t M''_Y(s) + \lambda t M''_Y(-s))
\]
\[+ 10(\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)^2(\tilde{\lambda} t M''_Y(s) - \lambda t M''_Y(-s))
\]
\[+ 15(\tilde{\lambda} t M'_Y(s) - \lambda t M'_Y(-s) - dt)(\tilde{\lambda} t M''_Y(s) + \lambda t M''_Y(-s))^2
\]
\[+ 5(\tilde{\lambda} t M''_Y(s) - \lambda t M''_Y(-s) - dt)(\tilde{\lambda} t M^{(IV)}_Y(s) + \lambda t M^{(IV)}_Y(-s))
\]
\[+ 10(\tilde{\lambda} t M''_Y(s) + \lambda t M''_Y(-s))(\tilde{\lambda} t M^{(V)}_Y(s) - \lambda t M^{(V)}_Y(-s))
\]
\[+ \tilde{\lambda} t M^{(V)}_Y(s) - \lambda t M^{(V)}_Y(-s)
\}
M(s).
\]

Since \(\mathbb{E}[U^k_i] = M^{(k)}(0)\) for all integer \(k \geq 1\), substituting \(s = 0\) into the formulas above gives (12)–(16).

If the moment generating functions of \(\tilde{Y}_i\) and \(Y_i\) do not exist, we can obtain (12)–(16) by a direct computation of the required expectations provided that \(\mathbb{E}[\tilde{Y}^5_i] < \infty\) and \(\mathbb{E}[Y^5_i] < \infty\), which completes the proof.

In what follows, we use the following constants:
\[
\gamma_2 = \tilde{\lambda} \mathbb{E}[\tilde{Y}^2] + \lambda \mathbb{E}[Y^2], \quad \gamma_3 = \tilde{\lambda} \mathbb{E}[\tilde{Y}^3] - \lambda \mathbb{E}[Y^3],
\]
\[
\gamma_4 = \tilde{\lambda} \mathbb{E}[\tilde{Y}^4] + \lambda \mathbb{E}[Y^4], \quad \gamma_5 = \tilde{\lambda} \mathbb{E}[\tilde{Y}^5] - \lambda \mathbb{E}[Y^5].
\]

### 3.2 A five-moment approximation

To construct a five-moment analogue to the De Vylder approximation, we replace the process \((U_t)_{t \geq 0}\) by a process \((\tilde{U}_t)_{t \geq 0}\) with exponentially distributed premium and claim sizes such that
\[
\mathbb{E}[U^k_i] = \mathbb{E}[\tilde{U}^k_i], \quad k = 1, 2, 3, 4, 5.
\]
**Theorem 3** (a five-moment analogue to the De Vylder approximation). Let the surplus process \( (X_t(x))_{t \geq 0} \) be defined by (1) under the above assumptions, \( \mathbb{E}[Y_i^5] < \infty, \mathbb{E}[Y_i^5] < \infty, \) and let the net profit condition (2) hold. Then the ruin probability is approximately equal to

\[
\psi_{DV5}(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} \quad \text{for all} \quad x \geq 0,
\]

where

\[
z_1 = \frac{-(d_0(\check{\mu}_0 - \mu_0) + \check{\mu}_0 \mu_0(\check{\lambda}_0 + \lambda_0)) + \sqrt{D}}{2d_0 \check{\mu}_0 \mu_0},
\]

\[
z_2 = \frac{-(d_0(\check{\mu}_0 - \mu_0) + \check{\mu}_0 \mu_0(\check{\lambda}_0 + \lambda_0)) - \sqrt{D}}{2d_0 \check{\mu}_0 \mu_0},
\]

\[
D = (d_0(\check{\mu}_0 + \mu_0) + \check{\mu}_0 \mu_0(\lambda_0 - \check{\lambda}_0))^2 + 4\check{\lambda}_0 \lambda_0 \check{\mu}_0^2 \mu_0^2,
\]

\[
C_1 = \frac{\check{\lambda}_0 \check{\mu}_0(\check{\mu}_0 + \mu_0)(dz_2 + \check{\lambda}_0) + d_0 \check{\lambda}_0 \mu_0(\check{\mu}_0 z_1 - 1)}{d_0 \check{\lambda}_0 \check{\mu}_0^2(z_2 - z_1)},
\]

\[
C_2 = -\frac{\check{\lambda}_0 \check{\mu}_0(\check{\mu}_0 + \mu_0)(dz_1 + \check{\lambda}_0) + d_0 \check{\lambda}_0 \mu_0(\check{\mu}_0 z_2 - 1)}{d_0 \check{\lambda}_0 \check{\mu}_0^2(z_2 - z_1)},
\]

and the constants \( \check{\lambda}_0, \check{\mu}_0, \lambda_0, \mu_0 \) and \( d_0 \) are defined by the following equalities:

\[
\mu_0 = -\frac{5\gamma_3 \gamma_4 - 3\gamma_2 \gamma_5}{40\gamma_3^2 - 30\gamma_2 \gamma_4} + \frac{\sqrt{(5\gamma_3 \gamma_4 - 3\gamma_2 \gamma_5)^2 + (4\gamma_3 \gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2 \gamma_4)}}{|40\gamma_3^2 - 30\gamma_2 \gamma_4|},
\]

\[
\check{\mu}_0 = \frac{5\gamma_3 \gamma_4 - 3\gamma_2 \gamma_5}{40\gamma_3^2 - 30\gamma_2 \gamma_4} + \frac{\sqrt{(5\gamma_3 \gamma_4 - 3\gamma_2 \gamma_5)^2 + (4\gamma_3 \gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2 \gamma_4)}}{|40\gamma_3^2 - 30\gamma_2 \gamma_4|},
\]

\[
\lambda_0 = \frac{3\check{\mu}_0 \gamma_2 - \gamma_3}{6\check{\mu}_0^2(\check{\mu}_0 + \mu_0)}, \quad \check{\lambda}_0 = \frac{\gamma_2 - 2\lambda_0 \check{\mu}_0^2}{2\check{\mu}_0^2},
\]

\[
d_0 = \check{\lambda}_0 \check{\mu}_0 - \lambda_0 \mu_0 - (\check{\lambda} \check{\mu} - \lambda \mu - d),
\]

provided that \( \check{\lambda}_0 > 0, \check{\mu}_0 > 0, \lambda_0 > 0, \mu_0 > 0, d_0 > 0, 4\gamma_3^2 - 3\gamma_2 \gamma_4 \neq 0 \) and

\[
(5\gamma_3 \gamma_4 - 3\gamma_2 \gamma_5)^2 + (4\gamma_3 \gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2 \gamma_4) > 0.
\]

**Proof.** Taking into account that the kth moments of the r.v.’s that are exponentially distributed with means \( \check{\mu}_0 \) and \( \mu_0 \) equal \( k! \check{\mu}_0^k \) and \( k! \mu_0^k \), respectively, from Lemma 2 we conclude that (17) is equivalent to the system of equations (19)–(23):

\[
\check{\lambda}_0 \check{\mu}_0 - \lambda_0 \mu_0 - d_0 = \check{\lambda} \check{\mu} - \lambda \mu - d, \quad (19)
\]

\[
2\check{\lambda}_0 \check{\mu}_0^2 + 2\lambda_0 \mu_0^2 = \gamma_2, \quad (20)
\]

\[
6\check{\lambda}_0 \mu_0^3 - 6\lambda_0 \mu_0^3 = \gamma_3, \quad (21)
\]

\[
24\check{\lambda}_0 \mu_0^4 + 24\lambda_0 \mu_0^4 = \gamma_4, \quad (22)
\]
Now our aim is to find the constants $\bar{\lambda}_0$, $\bar{\mu}_0$, $\lambda_0$, $\mu_0$ and $d_0$ from this system. From (20) we have $2\bar{\lambda}_0\bar{\mu}_0^2 = \gamma_2 - 2\lambda_0\mu_0^2$. Substituting this into equations (21)–(23) we get

\begin{align}
3\gamma_2\bar{\mu}_0 - 6\lambda_0\mu_0^2(\bar{\mu}_0 + \mu_0) &= \gamma_3, \tag{24} \\
12\gamma_2\bar{\mu}_0^2 - 24\lambda_0\mu_0^2(\bar{\mu}_0^2 - \mu_0^2) &= \gamma_4, \tag{25} \\
60\gamma_2\bar{\mu}_0^3 - 120\lambda_0\mu_0^2(\bar{\mu}_0^3 + \mu_0^3) &= \gamma_5. \tag{26}
\end{align}

Next, from (24) we have $6\lambda_0\mu_0^2(\bar{\mu}_0 + \mu_0) = 3\gamma_2\bar{\mu}_0 - \gamma_3$. Substituting this into equations (25)–(26) we obtain

\begin{align}
12\gamma_2\bar{\mu}_0\mu_0 + 4\gamma_3(\bar{\mu}_0 - \mu_0) &= \gamma_4 \tag{27} \\
60\gamma_2\bar{\mu}_0\mu_0(\bar{\mu}_0 - \mu_0) + 20\gamma_3((\bar{\mu}_0 - \mu_0)^2 + \bar{\mu}_0\mu_0) &= \gamma_5. \tag{28}
\end{align}

Multiplying (27) by $(-5(\bar{\mu}_0 - \mu_0))$ and adding (28) we get

\begin{equation}
20\gamma_3\bar{\mu}_0\mu_0 + 5\gamma_4(\bar{\mu}_0 - \mu_0) = \gamma_5. \tag{29}
\end{equation}

Note that (27) and (29) form a system of two equations, which are linear with respect to variables $\bar{\mu}_0\mu_0$ and $\bar{\mu}_0 - \mu_0$. Solving this system we obtain

\begin{equation}
\bar{\mu}_0\mu_0 = \frac{4\gamma_3\gamma_5 - 5\gamma_4^2}{80\gamma_3^2 - 60\gamma_2\gamma_4} \tag{30}
\end{equation}

and

\begin{equation}
\bar{\mu}_0 - \mu_0 = \frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{20\gamma_3^2 - 15\gamma_2\gamma_4} \tag{31}
\end{equation}

provided that $4\gamma_3^2 - 3\gamma_2\gamma_4 \neq 0$.

Substituting the expression for $\bar{\mu}_0$ from (31) into (30) gives

\begin{equation}
\mu_0^2 + \frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{20\gamma_3^2 - 15\gamma_2\gamma_4}\mu_0 - \frac{4\gamma_3\gamma_5 - 5\gamma_4^2}{80\gamma_3^2 - 60\gamma_2\gamma_4} = 0, \tag{32}
\end{equation}

from which we have

\begin{equation}
\mu_0 = -\frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{40\gamma_3^2 - 30\gamma_2\gamma_4} \pm \sqrt{\frac{(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4)}{40\gamma_3^2 - 30\gamma_2\gamma_4}} \tag{33}
\end{equation}

provided that $(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4) > 0$ and $4\gamma_3^2 - 3\gamma_2\gamma_4 \neq 0$. Hence, taking into account (31) we get

\begin{equation}
\bar{\mu}_0 = \frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{40\gamma_3^2 - 30\gamma_2\gamma_4} \pm \sqrt{\frac{(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4)}{40\gamma_3^2 - 30\gamma_2\gamma_4}} \tag{34}
\end{equation}
Since both $\bar{\mu}_0$ and $\mu_0$ must be positive, from the expressions for $\bar{\mu}_0$ and $\mu_0$ we deduce that we can take only the values of the parameters given in the assertion of the theorem (otherwise, if we take the values with “$-$”, at least one of the parameters $\bar{\mu}_0$ or $\mu_0$ will be negative). Finally, we obtain the corresponding values of the parameters $\lambda_0$, $\bar{\lambda}_0$ and $d_0$ from (24), (20) and (19), respectively, provided that all the values are positive.

Thus, we have a new process $\{\bar{U}_t\}_{t \geq 0}$ with exponentially distributed premium and claim sizes, which is completely determined by $\bar{\lambda}_0$, $\bar{\mu}_0$, $\lambda_0$, $\mu_0$ and $d_0$. Now the assertion of the theorem follows immediately from Theorem 2.

3.3 A three-moment approximation

From the assertion of Theorem 3 it is clear that its conditions are quite restrictive. In particular, some of the parameters $\bar{\lambda}_0$, $\bar{\mu}_0$, $\lambda_0$, $\mu_0$ and $d_0$ can be negative, and numerical computations show that such situations happen quite often. Hence, it is impossible to construct the five-moment approximation in those cases. Namely for this reason we also consider a simplified three-moment analogue to the De Vylder approximation. To construct it, we replace the process $\{U_t(x)\}_{t \geq 0}$ by the process $\{\bar{U}_t\}_{t \geq 0}$ with exponentially distributed premium and claim sizes such that

$$\mathbb{E}[U_t^k] = \mathbb{E}[\bar{U}_t^k], \quad k = 1, 2, 3,$$

(33)

and the following proportionality conditions hold:

$$\frac{\bar{\mu}}{\mu} = \frac{\bar{\mu}_0}{\mu_0} \quad \text{and} \quad \frac{\bar{\lambda}}{\lambda} = \frac{\bar{\lambda}_0}{\lambda_0}.$$

(34)

In particular, condition (34) implies that $\bar{\lambda}\bar{\mu}/\lambda\mu = \bar{\lambda}_0\bar{\mu}_0/\lambda_0\mu_0$. This means that the ratio between the expected premiums and the expected claims per unit time remains the same, which seems to be natural.

**Theorem 4** (a three-moment analogue to the De Vylder approximation). *Let the surplus process $\{X_t(x)\}_{t \geq 0}$ is defined by (1) under the above assumptions, $\mathbb{E}[\bar{Y}_t^3] < \infty$, $\mathbb{E}[Y_t^3] < \infty$, and let the net profit condition (2) hold. Then the ruin probability is approximately equal to

$$\psi_{DV3}(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} \quad \text{for all} \quad x \geq 0,$$

(35)

where $z_1$, $z_2$, $C_1$ and $C_2$ are defined as in Theorem 3 and the constants $\bar{\lambda}_0$, $\bar{\mu}_0$, $\lambda_0$, $\mu_0$ and $d_0$ are defined by the following equalities:

$$\mu_0 = \frac{\gamma_3 \mu (\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)}{3\gamma_2 (\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)}, \quad \lambda_0 = \frac{9\gamma_3^3 \bar{\lambda}(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)^2}{2\gamma_3^2 (\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)^3},$$

$$\bar{\mu}_0 = \frac{\gamma_3 \mu (\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)}{3\gamma_2 (\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)}, \quad \bar{\lambda}_0 = \frac{9\gamma_3^3 \bar{\lambda}(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)^2}{2\gamma_3^2 (\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)^3},$$

$$d_0 = \bar{\lambda}_0\bar{\mu}_0 - \lambda_0\mu_0 - (\bar{\lambda}\bar{\mu} - \lambda\mu - d),$$

provided that $\gamma_3(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3) > 0$ and $d_0 > 0$. 
Proof. From Lemma 2 we conclude that (33) is equivalent to the system of equations (19)–(21), and from (34) we get

\[ \tilde{\lambda}_0 \tilde{\mu}_0^2 = \lambda_0 \mu_0^2 \frac{\tilde{\lambda} \tilde{\mu}^2}{\lambda \mu^2} \quad \text{and} \quad \tilde{\lambda}_0 \tilde{\mu}_0^3 = \lambda_0 \mu_0^3 \frac{\tilde{\lambda} \tilde{\mu}^3}{\lambda \mu^3}. \]  

(36)

Substituting (36) into (20) and (21) we obtain

\[ 2\lambda_0 \mu_0^2 \left( \frac{\tilde{\lambda} \tilde{\mu}^2}{\lambda \mu^2} + 1 \right) = \gamma_2 \]  

(37)

and

\[ 6\lambda_0 \mu_0^3 \left( \frac{\tilde{\lambda} \tilde{\mu}^3}{\lambda \mu^3} - 1 \right) = \gamma_3. \]  

(38)

Dividing (38) by (37) we easily get the expression for \( \mu_0 \) provided that \( \gamma_3 (\tilde{\lambda} \tilde{\mu}^3 - \lambda \mu^3) > 0 \). Next, substituting this expression into (37) we obtain the expression for \( \lambda_0 \).

The constants \( \tilde{\mu}_0 \) and \( \tilde{\lambda}_0 \) are determined from (34). Finally, we can find \( d_0 \) from (19).

Applying Theorem 2 for exponentially distributed premium and claim sizes yields the assertion of the theorem.

Comparing the assertions of Theorems 3 and 4 we deduce that the conditions of Theorem 4 are much less restrictive.

Remark 2. Instead of conditions (34), we can consider the following more general conditions:

\[ \frac{\bar{\mu}}{\mu} = v_1 \frac{\mu_0}{\bar{\mu}_0} \quad \text{and} \quad \frac{\bar{\lambda}}{\lambda} = v_2 \frac{\lambda_0}{\bar{\lambda}_0}, \]  

(39)

where \( v_1 > 0 \) and \( v_2 > 0 \). The corresponding approximation for the ruin probability is calculated using the same formula (35), but the constants \( \tilde{\lambda}_0, \tilde{\mu}_0, \lambda_0, \mu_0 \) and \( d_0 \) are defined by the following equalities:

\[ \mu_0 = \frac{\gamma_3 \mu_1 (\tilde{\lambda} \tilde{\mu}^2 + \lambda \mu^2 v_1 v_2)}{3\gamma_2 (\tilde{\lambda} \tilde{\mu}^3 - \lambda \mu^3 v_1 v_2)}, \quad \lambda_0 = \frac{9\gamma_3^2 \lambda_2 (\tilde{\lambda} \tilde{\mu}^3 - \lambda \mu^3 v_1 v_2)^2}{2\gamma_2^2 (\tilde{\lambda} \tilde{\mu}^2 + \lambda \mu^2 v_1 v_2)^3}, \]

\[ \tilde{\mu}_0 = \frac{\gamma_3 \tilde{\lambda} (\tilde{\lambda} \tilde{\mu}^2 + \lambda \mu^2 v_1 v_2)}{3\gamma_2 (\tilde{\lambda} \tilde{\mu}^3 - \lambda \mu^3 v_1 v_2)}, \quad \tilde{\lambda}_0 = \frac{9\gamma_3^2 \lambda (\tilde{\lambda} \tilde{\mu}^3 - \lambda \mu^3 v_1 v_2)^2}{2\gamma_2^2 (\tilde{\lambda} \tilde{\mu}^2 + \lambda \mu^2 v_1 v_2)^3}, \]

\[ d_0 = \tilde{\lambda}_0 \tilde{\mu}_0 - \lambda_0 \mu_0 - (\tilde{\lambda} \tilde{\mu} - \lambda \mu - d), \]

provided that \( \gamma_3 (\tilde{\lambda} \tilde{\mu}^3 - \lambda \mu^3 v_1 v_2) > 0 \) and \( d_0 > 0 \). Conditions (39) enable us to consider different cases by changing the values of the coefficients \( v_1 \) and \( v_2 \) and choose those ones that approximate the ruin probability more accurately. Nevertheless, note that for some values of \( v_1 \) and \( v_2 \), the corresponding approximations give not so good results, and consequently, should not be applied.

4 Numerical illustrations

4.1 A statistical estimate for the ruin probability

To analyze the accuracy of the approximations proposed in Section 3, we will need a statistical estimate for the ruin probability obtained by the direct simulation of the
surplus process \((X_t(x))_{t \geq 0}\) using the Monte Carlo methods. To this end, we use the approach described in [23]. Let \(N\) be the total number of simulations of \((X_t(x))_{t \geq 0}\). To get the corresponding statistical estimate \(\hat{\psi}(x)\) for the ruin probability \(\psi(x)\), we divide the number of simulations leading to ruin by the total number of simulations \(N\). To find the number of simulations \(N\), which is necessary in order to calculate the ruin probability with the required accuracy and reliability, we apply the following proposition, which follows immediately from Hoeffding’s inequality (see [18]).

**Proposition 1.** Let the surplus process \((X_t(x))_{t \geq 0}\) be defined by (1) under the above assumptions. Then for any \(\varepsilon > 0\), we have

\[
\mathbb{P}\left[|\psi(x) - \hat{\psi}(x)| > \varepsilon\right] \leq 2e^{-2\varepsilon^2 N}.
\]

In all examples below, we set \(\varepsilon = 0.005\) and \(2e^{-2\varepsilon^2 N} = 0.005\). Therefore, we get \(N = 119830\). Moreover, let \(\bar{\lambda} = 2.3\), \(\bar{\mu} = 0.2\), \(\lambda = 0.1\), \(\mu = 3\) and \(d = 0.05\).

4.2 Gamma distributions for the premium and claim sizes

Let the p.d.f. of \(\bar{Y}_i\) be

\[
f_{\bar{Y}}(y) = \frac{1}{\Gamma(\bar{\alpha})\bar{\beta}^{\bar{\alpha}}} y^{\bar{\alpha} - 1} e^{-y/\bar{\beta}}, \quad y \geq 0,
\]

where \(\bar{\alpha} > 0\), \(\bar{\beta} > 0\) and \(\bar{\alpha}\bar{\beta} = \bar{\mu}\), and let the p.d.f. of \(Y_i\) be

\[
f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha - 1} e^{-y/\beta}, \quad y \geq 0,
\]

where \(\alpha > 0\), \(\beta > 0\) and \(\alpha\beta = \mu\).

Then

\[
\mathbb{E}[\bar{Y}_i] = \bar{\alpha}\bar{\beta} = \bar{\mu}, \quad \mathbb{E}[\bar{Y}_i^2] = \bar{\alpha}(\bar{\alpha} + 1)\bar{\beta}^2,
\]

\[
\mathbb{E}[\bar{Y}_i^3] = \bar{\alpha}(\bar{\alpha} + 1)(\bar{\alpha} + 2)\bar{\beta}^3,
\]

\[
\mathbb{E}[\bar{Y}_i^4] = \bar{\alpha}(\bar{\alpha} + 1)(\bar{\alpha} + 2)(\bar{\alpha} + 3)\bar{\beta}^4,
\]

\[
\mathbb{E}[\bar{Y}_i^5] = \bar{\alpha}(\bar{\alpha} + 1)(\bar{\alpha} + 2)(\bar{\alpha} + 3)(\bar{\alpha} + 4)\bar{\beta}^5,
\]

and analogous formulas hold for the moments of \(Y_i\).

Therefore, we get

\[
\gamma_2 = \bar{\lambda}\bar{\alpha}(\bar{\alpha} + 1)\bar{\beta}^2 + \lambda\alpha(\alpha + 1)\beta^2,
\]

\[
\gamma_3 = \bar{\lambda}\bar{\alpha}(\bar{\alpha} + 1)(\bar{\alpha} + 2)\bar{\beta}^3 - \lambda\alpha(\alpha + 1)(\alpha + 2)\beta^3,
\]

\[
\gamma_4 = \bar{\lambda}\bar{\alpha}(\bar{\alpha} + 1)(\bar{\alpha} + 2)(\bar{\alpha} + 3)\bar{\beta}^4 + \lambda\alpha(\alpha + 1)(\alpha + 2)(\alpha + 3)\beta^4,
\]

\[
\gamma_5 = \bar{\lambda}\bar{\alpha}(\bar{\alpha} + 1)(\bar{\alpha} + 2)(\bar{\alpha} + 3)(\bar{\alpha} + 4)\bar{\beta}^5 - \lambda\alpha(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)\beta^5.
\]

A number of numerical examples show that the conditions of Theorem 3 hold provided that \(\alpha\) is very close to 1. So the five-moment approximation can be constructed only in those cases. We now consider two examples.
Table 2. Results of computations: the gamma distributions for the premium and claim sizes, $\tilde{\alpha} = 2$, $\tilde{\beta} = 0.1$, $\alpha = 1$ and $\beta = 3$

| $x$ | $\hat{\psi}(x)$ | $\psi_{DV5}(x)$ | $(\frac{\psi_{DV5}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%$ | $\psi_{DV3}(x)$ | $(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%$ |
|-----|----------------|----------------|---------------------------------|----------------|---------------------------------|
| 1   | 0.6912         | 0.6832         | -1.15%                          | 0.6766         | -2.11%                          |
| 2   | 0.6205         | 0.6270         | 1.05%                           | 0.6210         | 0.08%                           |
| 3   | 0.5681         | 0.5754         | 1.29%                           | 0.5700         | 0.34%                           |
| 5   | 0.4870         | 0.4846         | -0.50%                          | 0.4802         | -1.41%                          |
| 7   | 0.4040         | 0.4081         | 1.01%                           | 0.4045         | 0.12%                           |
| 10  | 0.3149         | 0.3154         | 0.17%                           | 0.3128         | -0.66%                          |
| 15  | 0.2024         | 0.2053         | 1.42%                           | 0.2037         | 0.66%                           |
| 20  | 0.1374         | 0.1336         | -2.73%                          | 0.1327         | -3.38%                          |
| 30  | 0.0584         | 0.0566         | -3.09%                          | 0.0563         | -3.58%                          |
| 50  | 0.0098         | 0.0102         | 3.63%                           | 0.0101         | 3.45%                           |

Example 1. Let now $\tilde{\alpha} = 2$, $\tilde{\beta} = 0.1$, $\alpha = 1$ and $\beta = 3$.

If we construct the five-moment analogue to the De Vylder approximation, by Theorem 3, we get $\hat{\lambda}_0 \approx 3.923743, \hat{\mu}_0 \approx 0.132632, \lambda_0 \approx 0.099996, \mu_0 \approx 3.000027, d_0 \approx 0.110423$, and consequently,

$$\psi_{DV5}(x) \approx 0.255492 e^{-29.147189x} + 0.744508 e^{-0.085895x}$$

for all $x \geq 0$.

For the corresponding three-moment approximation using conditions (34), by Theorem 4, we have $\hat{\lambda}_0 \approx 2.129067, \hat{\mu}_0 \approx 0.205450, \lambda_0 \approx 0.092568, \mu_0 \approx 3.081744, d_0 \approx 0.042145$, and hence,

$$\psi_{DV3}(x) \approx 0.262882 e^{-48.085872x} + 0.737118 e^{-0.085730x}$$

for all $x \geq 0$.

Table 1 presents the results of computations for some values of $x$. Next, Table 2 shows the values of the relative approximation errors $(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%$ for the three-moment approximations constructed using conditions (39) for different $\nu_1$ and $\nu_2$. Note that here we chose some values of $\nu_1$ and $\nu_2$ that give more or less good

Table 2. Values of $(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%$ for different $\nu_1$ and $\nu_2$: the gamma distributions for the premium and claim sizes, $\tilde{\alpha} = 2$, $\tilde{\beta} = 0.1$, $\alpha = 1$ and $\beta = 3$

| $x$ | $\nu_1=0.7$ | $\nu_2=1.5$ | $\nu_1=0.9$ | $\nu_2=1.2$ | $\nu_1=1.1$ | $\nu_2=1.1$ | $\nu_1=1.5$ | $\nu_2=0.7$ | $\nu_1=2$ | $\nu_2=0.2$ | $\nu_1=5$ | $\nu_2=0.05$ | $\nu_1=10$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|------------|-------------|-------------|-------------|-------------|
| 1   | -3.48%      | -2.36%      | -2.21%      | -1.00%      | -1.41%      | -0.35%      | 0.52%       |             |            |             |             |             |             |
| 2   | -1.29%      | -0.17%      | 0.00%       | 1.20%       | 0.70%       | 1.84%       | 2.72%       |             |            |             |             |             |             |
| 3   | -1.02%      | 0.09%       | 0.26%       | 1.43%       | 0.97%       | 2.08%       | 2.93%       |             |            |             |             |             |             |
| 5   | -2.69%      | -1.65%      | -1.46%      | -0.38%      | -0.76%      | 0.28%       | 1.05%       |             |            |             |             |             |             |
| 7   | -1.13%      | -0.12%      | 0.09%       | 1.12%       | 0.80%       | 1.80%       | 2.52%       |             |            |             |             |             |             |
| 10  | -1.82%      | -0.89%      | -0.67%      | 0.26%       | 0.04%       | 0.95%       | 1.57%       |             |            |             |             |             |             |
| 15  | -0.39%      | 0.44%       | 0.70%       | 1.48%       | 1.42%       | 2.21%       | 2.67%       |             |            |             |             |             |             |
| 20  | -4.26%      | -3.58%      | -3.29%      | -2.70%      | -2.60%      | -1.97%      | -1.68%      |             |            |             |             |             |             |
| 30  | -4.21%      | -3.75%      | -3.99%      | -3.13%      | -2.70%      | -2.33%      | -2.36%      |             |            |             |             |             |             |
| 50  | 3.31%       | 3.31%       | 3.86%       | 3.45%       | 4.61%       | 4.45%       | 3.76%       |             |            |             |             |             |             |
results. Choosing some other values of the coefficients leads to extremely bad approximations. In addition, analyzing the relative approximation errors in Table 2 we conclude that it is difficult to decide which of the approximations is better: choosing $ν_1$ and $ν_2$ that yield smaller errors for some values of the initial surplus results in larger errors for other values.

**Example 2.** Let now $\tilde{\alpha} = 4$, $\tilde{\beta} = 0.05$, $\alpha = 3$ and $\beta = 1$.

In this case, the conditions of Theorem 3 do not hold, so we can construct only the three-moment analogue to the De Vylder approximation. By Theorem 4, we obtain $\tilde{\lambda}_0 \approx 4.871659$, $\tilde{\mu}_0 \approx 0.111879$, $\lambda_0 \approx 0.211811$, $\mu_0 \approx 1.678181$, $d_0 \approx 0.079577$, and therefore,

$$
\psi_{DV3}(x) \approx 0.221130 e^{-55.405586x} + 0.778870 e^{-0.132881x} \quad \text{for all} \quad x \geq 0.
$$

Table 3 presents the results of computations for some values of $x$, whereas Table 4 shows the values of the relative approximation errors $\left( \frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1 \right) \cdot 100\%$ for the three-moment approximations constructed using conditions (39) for different $ν_1$ and $ν_2$.

**Table 3.** Results of computations: the gamma distributions for the premium and claim sizes, $\tilde{\alpha} = 4$, $\tilde{\beta} = 0.05$, $\alpha = 3$ and $\beta = 1$

| $x$ | $\hat{\psi}(x)$ | $\psi_{DV3}(x)$ | $(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%$ |
|-----|-----------------|-----------------|-----------------------------------------------------|
| 1   | 0.6690          | 0.6820          | 1.94%                                               |
| 2   | 0.6046          | 0.5971          | −1.24%                                              |
| 3   | 0.5301          | 0.5228          | −1.38%                                              |
| 5   | 0.4039          | 0.4008          | −0.77%                                              |
| 7   | 0.3039          | 0.3073          | 1.10%                                               |
| 10  | 0.2065          | 0.2062          | −0.13%                                              |
| 15  | 0.1056          | 0.1061          | 0.50%                                               |
| 20  | 0.0535          | 0.0546          | 2.17%                                               |
| 30  | 0.0150          | 0.0145          | −3.60%                                              |

**Table 4.** Values of $\left( \frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1 \right) \cdot 100\%$ for different $ν_1$ and $ν_2$: the gamma distributions for the premium and claim sizes, $\tilde{\alpha} = 4$, $\tilde{\beta} = 0.05$, $\alpha = 3$ and $\beta = 1$

| $x$ | $ν_1=0.7$ | $ν_1=0.9$ | $ν_1=1.1$ | $ν_1=1.5$ | $ν_1=2$ | $ν_1=5$ | $ν_1=10$ |
|-----|-----------|-----------|-----------|-----------|---------|---------|----------|
|     | $ν_2=1.5$ | $ν_2=1.2$ | $ν_2=0.6$ | $ν_2=0.7$ | $ν_2=0.2$ | $ν_2=0.05$ | $ν_2=0.03$ |
| 1   | 0.82%     | 1.73%     | 1.68%     | 2.85%     | 2.51%   | 3.42%   | 3.97%    |
| 2   | −2.30%    | −1.44%    | −1.47%    | −0.39%    | −0.68%  | 0.17%   | 0.68%    |
| 3   | −2.41%    | −1.58%    | −1.58%    | −0.55%    | −0.81%  | 0.01%   | 0.50%    |
| 5   | −1.75%    | −0.96%    | −0.93%    | 0.01%     | −0.17%  | 0.59%   | 1.02%    |
| 7   | 0.16%     | 0.91%     | 0.99%     | 1.84%     | 1.73%   | 2.45%   | 2.84%    |
| 10  | −0.97%    | −0.31%    | −0.17%    | 0.53%     | 0.53%   | 1.15%   | 1.45%    |
| 15  | −0.21%    | 0.33%     | 0.58%     | 1.03%     | 1.22%   | 1.69%   | 1.85%    |
| 20  | 1.60%     | 2.01%     | 2.37%     | 2.58%     | 2.96%   | 3.30%   | 3.31%    |
| 30  | −3.86%    | −3.72%    | −3.17%    | −3.46%    | −2.74%  | −2.71%  | −2.96%   |
4.3 Hyperexponential distributions for the premium and claim sizes

Let

\[ F_Y(y) = \bar{p}_1 F_{Y,1}(y) + \bar{p}_2 F_{Y,2}(y) + \cdots + \bar{p}_k F_{Y,k}(y), \quad y \geq 0, \]

where \( \bar{k} \geq 1, \bar{p}_j > 0, F_{Y,j} \) is the c.d.f. of the exponential distribution with mean \( \bar{\mu}_j \) for all \( 1 \leq j \leq \bar{k} \), \( \sum_{j=1}^{\bar{k}} \bar{p}_j = 1 \), \( \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j = \bar{\mu} \), and let

\[ F_Y(y) = p_1 F_{Y,1}(y) + p_2 F_{Y,2}(y) + \cdots + p_k F_{Y,k}(y), \quad y \geq 0, \]

where \( k \geq 1, p_j > 0, F_{Y,j} \) is the c.d.f. of the exponential distribution with mean \( \mu_j \) for all \( 1 \leq j \leq k \), \( \sum_{j=1}^{k} p_j = 1 \), \( \sum_{j=1}^{k} p_j \mu_j = \mu \).

Then

\[
\mathbb{E}[\bar{Y}_i] = \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j = \bar{\mu}, \quad \mathbb{E}[\bar{Y}_i^2] = 2 \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^2, \quad \mathbb{E}[\bar{Y}_i^3] = 6 \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^3, \\
\mathbb{E}[\bar{Y}_i^4] = 24 \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^4, \quad \mathbb{E}[\bar{Y}_i^5] = 120 \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^5,
\]

and analogous formulas hold for the moments of \( Y_i \).

Hence, we obtain

\[
\gamma_2 = 2 \left( \bar{\lambda} \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^2 + \lambda \sum_{j=1}^{k} p_j \mu_j^2 \right), \quad \gamma_3 = 6 \left( \bar{\lambda} \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^3 - \lambda \sum_{j=1}^{k} p_j \mu_j^3 \right), \\
\gamma_4 = 24 \left( \bar{\lambda} \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^4 + \lambda \sum_{j=1}^{k} p_j \mu_j^4 \right), \quad \gamma_5 = 120 \left( \bar{\lambda} \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^5 - \lambda \sum_{j=1}^{k} p_j \mu_j^5 \right).
\]

**Example 3.** Let now \( \bar{k} = 2, \bar{p}_1 = 0.75, \bar{p}_2 = 0.25, \bar{\mu}_1 = 0.1, \bar{\mu}_2 = 0.5, k = 2, \\
p_1 = 0.8, p_2 = 0.2, \mu_1 = 2.8 \) and \( \mu_2 = 3.8 \).

If we construct the five-moment analogue to the De Vylder approximation, by Theorem 3, we obtain \( \bar{\lambda}_0 \approx 10.626422, \bar{\mu}_0 \approx 0.141004, \lambda_0 \approx 0.082185, \mu_0 \approx 3.245591, d_0 \approx 1.121624, \) and consequently,

\[
\psi_{DV}(x) \approx 0.236453 e^{-2.683647x} + 0.763547 e^{-0.079854x} \quad \text{for all} \quad x \geq 0.
\]

For the corresponding three-moment approximation using conditions (34), by Theorem 4, we have \( \bar{\lambda}_0 \approx 2.738661, \bar{\mu}_0 \approx 0.190975, \lambda_0 \approx 0.119072, \mu_0 \approx 2.864627, d_0 \approx 0.071919 \) (here we use conditions (34)), and therefore,

\[
\psi_{DV}(x) \approx 0.228569 e^{-3.768023x} + 0.771431 e^{-0.080413x} \quad \text{for all} \quad x \geq 0.
\]

Table 5 presents the results of computations for some values of \( x \), whereas Table 6 shows the values of \( \frac{\psi_{DV,1}(x)}{\psi_{DV,2}(x)} - 1 \) · 100% for the three-moment approximations constructed using conditions (39) for different \( \nu_1 \) and \( \nu_2 \).
Table 5. Results of computations: the hyperexponential distributions for the premium and claim sizes, \(\bar{k} = 2, \bar{p}_1 = 0.75, \bar{p}_2 = 0.25, \bar{\mu}_1 = 0.1, \bar{\mu}_2 = 0.5, k = 2, p_1 = 0.8, p_2 = 0.2, \mu_1 = 2.8\) and \(\mu_2 = 3.8\)

| \(x\) | \(\hat{\psi}(x)\) | \(\psi_{DV5}(x)\) | \((\frac{\psi_{DV5}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%\) | \(\psi_{DV3}(x)\) | \((\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%\) |
|-------|-----------------|-------------------|---------------------|-----------------|---------------------|
| 1     | 0.6988          | 0.7211            | 3.20%               | 0.7118          | 1.87%               |
| 2     | 0.6365          | 0.6519            | 2.43%               | 0.6568          | 3.19%               |
| 3     | 0.5909          | 0.6010            | 1.71%               | 0.6061          | 2.58%               |
| 5     | 0.5023          | 0.5122            | 1.97%               | 0.5160          | 2.74%               |
| 7     | 0.4253          | 0.4366            | 2.67%               | 0.4394          | 3.32%               |
| 10    | 0.3367          | 0.3436            | 2.06%               | 0.3452          | 2.54%               |
| 15    | 0.2210          | 0.2305            | 4.31%               | 0.2309          | 4.51%               |
| 20    | 0.1542          | 0.1546            | 0.26%               | 0.1545          | 0.17%               |
| 30    | 0.0670          | 0.0696            | 3.84%               | 0.0691          | 3.17%               |
| 50    | 0.0141          | 0.0141            | -0.09%              | 0.0138          | -1.84%              |

Table 6. Values of \((\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%\) for different \(v_1\) and \(v_2\): the hyperexponential distributions for the premium and claim sizes, \(\bar{k} = 2, \bar{p}_1 = 0.75, \bar{p}_2 = 0.25, \bar{\mu}_1 = 0.1, \bar{\mu}_2 = 0.5, k = 2, p_1 = 0.8, p_2 = 0.2, \mu_1 = 2.8\) and \(\mu_2 = 3.8\)

| \(x\) | \(v_1=0.4\) | \(v_1=0.5\) | \(v_1=0.7\) | \(v_1=0.9\) | \(v_1=1.1\) | \(v_1=1.5\) | \(v_1=2\) |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|-----------|
| 1     | -2.81%      | -1.14%      | 0.28%       | 0.73%       | 1.37%       | 4.60%       | 4.25%     |
| 2     | -1.47%      | 0.19%       | -0.01%      | 1.72%       | 2.59%       | 2.18%       | 2.77%     |
| 3     | -1.98%      | -0.36%      | -0.60%      | 1.16%       | 2.01%       | 1.12%       | 1.93%     |
| 5     | -1.67%      | -0.10%      | -0.26%      | 1.41%       | 2.22%       | 1.35%       | 2.17%     |
| 7     | -0.95%      | 0.57%       | 0.49%       | 2.08%       | 2.86%       | 2.09%       | 2.87%     |
| 10    | -1.47%      | -0.04%      | 0.01%       | 1.45%       | 2.17%       | 1.57%       | 2.27%     |
| 15    | 0.83%       | 2.14%       | 2.41%       | 3.65%       | 4.27%       | 3.94%       | 4.54%     |
| 20    | -2.97%      | -1.85%      | -1.38%      | -0.42%      | 0.08%       | 0.03%       | 0.50%     |
| 30    | 0.73%       | 1.59%       | 2.51%       | 3.04%       | 3.35%       | 3.86%       | 4.11%     |
| 50    | -2.62%      | -2.36%      | -0.62%      | -1.02%      | -1.12%      | 0.43%       | 0.22%     |

Example 4. Let now \(\bar{k} = 3, \bar{p}_1 = 0.2, \bar{p}_2 = 0.5, \bar{p}_3 = 0.3, \bar{\mu}_1 = 0.1, \bar{\mu}_2 = 0.15, \bar{\mu}_3 = 0.35, k = 3, p_1 = 0.1, p_2 = 0.4, p_3 = 0.5, \mu_1 = 1, \mu_2 = 2.7\) and \(\mu_3 = 3.64\).

In this case, the conditions of Theorem 3 do not hold, so we can construct only the three-moment analogue to the De Vylder approximation. By Theorem 4, we get \(\bar{\lambda}_0 \approx 2.112044, \bar{\mu}_0 \approx 0.217677, \lambda_0 \approx 0.091828, \mu_0 \approx 3.265162, d_0 \approx 0.049911\) (here we use conditions (34)), and therefore,

\[
\psi_{DV3}(x) \approx 0.252988 e^{-39.790359x} + 0.747012 e^{-0.077929x} \quad \text{for all} \quad x \geq 0.
\]

Table 7 presents the results of computations for some values of \(x\), whereas Table 8 shows the values of \((\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%\) for the three-moment approximations constructed using conditions (39) for different \(v_1\) and \(v_2\).

4.4 Lomax distributions for the premium and claim sizes

Let

\[
F_\tilde{y}(y) = 1 - \left(\frac{\bar{\beta}}{y + \bar{\beta}}\right)^{\tilde{\alpha}}, \quad y \geq 0,
\]
Table 7. Results of computations: the hyperexponential distributions for the premium and claim sizes, $\bar{k} = 3$, $\bar{p}_1 = 0.2$, $\bar{p}_2 = 0.5$, $\bar{p}_3 = 0.3$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.15$, $\bar{\mu}_3 = 0.35$, $k = 3$, $p_1 = 0.1$, $p_2 = 0.4$, $p_3 = 0.5$, $\mu_1 = 1$, $\mu_2 = 2.7$ and $\mu_3 = 3.64$

| $x$  | $\hat{\psi}(x)$ | $\psi_{DV3}(x)$ | $(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%$ |
|------|------------------|------------------|-------------------------------------------------|
| 1    | 0.6949           | 0.6910           | −0.56%                                           |
| 2    | 0.6368           | 0.6392           | 0.38%                                            |
| 3    | 0.5861           | 0.5913           | 0.88%                                            |
| 5    | 0.5035           | 0.5059           | 0.49%                                            |
| 7    | 0.4306           | 0.4329           | 0.55%                                            |
| 10   | 0.3446           | 0.3427           | −0.56%                                           |
| 15   | 0.2299           | 0.2321           | 0.95%                                            |
| 20   | 0.1594           | 0.1572           | −1.35%                                           |
| 30   | 0.0694           | 0.0721           | 3.91%                                            |

Table 8. Values of $(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1) \cdot 100\%$ for different $v_1$ and $v_2$: the hyperexponential distributions for the premium and claim sizes, $\bar{k} = 3$, $\bar{p}_1 = 0.2$, $\bar{p}_2 = 0.5$, $\bar{p}_3 = 0.3$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.15$, $\bar{\mu}_3 = 0.35$, $k = 3$, $p_1 = 0.1$, $p_2 = 0.4$, $p_3 = 0.5$, $\mu_1 = 1$, $\mu_2 = 2.7$ and $\mu_3 = 3.64$

| $x$  | $v_1=0.5$ | $v_1=0.7$ | $v_1=0.9$ | $v_1=1.1$ | $v_1=1.5$ | $v_1=2$ | $v_1=3$ | $v_2=2.1$ | $v_2=1.5$ | $v_2=1.1$ | $v_2=1$ | $v_2=0.5$ | $v_2=0.2$ | $v_2=0.1$ |
|------|---------|---------|---------|---------|---------|--------|--------|---------|---------|---------|--------|--------|--------|--------|
| 1    | −3.93%  | −1.89%  | −0.94%  | −0.16%  | 0.19%   | 0.19%  | 0.19%  | 0.69%   | −1.89%  | −0.94%  | −0.16%  | 0.19%   | 0.19%  | 0.19%  | 0.69%   |
| 2    | −2.96%  | −0.94%  | 0.00%   | 0.77%   | 1.12%   | 0.98%  | 1.47%  |         | −0.94%  | 0.00%   | 0.77%   | 1.12%   | 0.98%  | 1.47%  |         |
| 3    | −2.42%  | −0.42%  | 0.52%   | 1.27%   | 1.63%   | 1.50%  | 1.98%  |         | −0.42%  | 0.52%   | 1.27%   | 1.63%   | 1.50%  | 1.98%  |         |
| 5    | −2.69%  | −0.77%  | 0.13%   | 0.86%   | 1.21%   | 1.12%  | 1.58%  |         | −0.77%  | 0.13%   | 0.86%   | 1.21%   | 1.12%  | 1.58%  |         |
| 7    | −2.51%  | −0.66%  | 0.21%   | 0.90%   | 1.26%   | 1.20%  | 1.65%  |         | −0.66%  | 0.21%   | 0.90%   | 1.26%   | 1.20%  | 1.65%  |         |
| 10   | −3.41%  | −1.69%  | −0.87%  | −0.24%  | 0.12%   | 0.11%  | 0.53%  |         | −1.69%  | −0.87%  | −0.24%  | 0.12%   | 0.11%  | 0.53%  |         |
| 15   | −1.65%  | −0.09%  | 0.67%   | 1.23%   | 1.61%   | 1.68%  | 2.06%  |         | −0.09%  | 0.67%   | 1.23%   | 1.61%   | 1.68%  | 2.06%  |         |
| 20   | −3.61%  | −2.26%  | −1.59%  | −1.12%  | −0.75%  | −0.60% | −0.26% |         | −2.26%  | −1.59%  | −1.12%  | −0.75%  | −0.60% | −0.26% |         |
| 30   | 2.14%   | 3.18%   | 3.72%   | 4.05%   | 4.47%   | 4.79%  | 5.07%  |         | 3.18%   | 3.72%   | 4.05%   | 4.47%   | 4.79%  | 5.07%  |         |

where $\bar{\alpha} > 1$, $\bar{\beta} > 0$ and $\bar{\beta}/(\bar{\alpha} - 1) = \bar{\mu}$, and let

$$F_Y(y) = 1 - \left(\frac{\beta}{y + \beta}\right)^\alpha, \quad y \geq 0,$$

where $\alpha > 1$, $\beta > 0$ and $\beta/(\alpha - 1) = \mu$. In what follows, we assume that $\bar{\alpha} > 5$, $\alpha > 5$ and both $\bar{\alpha}$ and $\alpha$ are integer. Then

$$\mathbb{E}[\bar{Y}_i] = \frac{\bar{\beta}}{\bar{\alpha} - 1} = \bar{\mu}, \quad \mathbb{E}[\bar{Y}_i^2] = \frac{2\bar{\beta}^2}{(\bar{\alpha} - 2)(\bar{\alpha} - 1)},$$

$$\mathbb{E}[\bar{Y}_i^3] = \frac{6\bar{\beta}^3}{(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)}, \quad \mathbb{E}[\bar{Y}_i^4] = \frac{24\bar{\beta}^4}{(\bar{\alpha} - 4)(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)},$$

$$\mathbb{E}[\bar{Y}_i^5] = \frac{120\bar{\beta}^5}{(\bar{\alpha} - 5)(\bar{\alpha} - 4)(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)},$$

and analogous formulas hold for the moments of $Y_i$. 
Consequently, we have

\[
\begin{align*}
\gamma_2 &= 2 \left( \frac{\bar{\lambda} \bar{\beta}^2}{(\bar{\alpha} - 2)(\bar{\alpha} - 1)} + \frac{\lambda \beta^2}{(\alpha - 2)(\alpha - 1)} \right), \\
\gamma_3 &= 6 \left( \frac{\bar{\lambda} \bar{\beta}^3}{(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)} - \frac{\lambda \beta^3}{(\alpha - 3)(\alpha - 2)(\alpha - 1)} \right), \\
\gamma_4 &= 24 \left( \frac{\bar{\lambda} \bar{\beta}^4}{(\bar{\alpha} - 4)(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)} + \frac{\lambda \beta^4}{(\alpha - 4)(\alpha - 3)(\alpha - 2)(\alpha - 1)} \right), \\
\gamma_5 &= 120 \left( \frac{\bar{\lambda} \bar{\beta}^5}{(\bar{\alpha} - 5)(\bar{\alpha} - 4)(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)} - \frac{\lambda \beta^5}{(\alpha - 5)(\alpha - 4)(\alpha - 3)(\alpha - 2)(\alpha - 1)} \right).
\end{align*}
\]

Note that the Lomax distribution, which is also called the Pareto type II distribution, is a heavy-tailed distribution in contrast to the gamma and hyperexponential distributions. This can be the reason why the conditions of Theorem 3 do not hold for this distribution, at least as a number of numerical examples indicate. However, the three-moment approximation can be applied provided that the conditions of Theorem 4 hold.

**Example 5.** Let now \(\bar{\alpha} = 6\), \(\bar{\beta} = 1\), \(\alpha = 6\) and \(\beta = 15\).

In this case, we can construct only the three-moment analogue to the De Vylder approximation. By Theorem 4, we get \(\bar{\lambda}_0 = 1.035\), \(\bar{\mu}_0 \approx 0.333333\), \(\lambda_0 = 0.045\), \(\mu_0 = 5\), \(d_0 = 0.01\) (here we use conditions (34)), and therefore,

\[
\psi_{DV3}(x) \approx 0.313466 e^{-105.137225x} + 0.686534 e^{-0.062775x} \quad \text{for all} \quad x \geq 0.
\]

Table 9 presents the results of computations for some values of \(x\), whereas Table 10 shows the values of \(\left( \frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1 \right) \cdot 100\%\) for the three-moment approximations constructed using conditions (39) for different \(\nu_1\) and \(\nu_2\).

### Table 9. Results of computations: the Lomax distributions for premium and claim sizes, \(\bar{\alpha} = 6\), \(\bar{\beta} = 1\), \(\alpha = 6\) and \(\beta = 15\)

| \(x\) | \(\hat{\psi}(x)\) | \(\psi_{DV3}(x)\) | \(\left( \frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1 \right) \cdot 100\\%\) |
|------|-----------------|------------------|----------------------------------|
| 1    | 0.6881          | 0.6448           | −6.30%                           |
| 2    | 0.6391          | 0.6055           | −5.25%                           |
| 3    | 0.5899          | 0.5687           | −3.59%                           |
| 5    | 0.5086          | 0.5016           | −1.37%                           |
| 7    | 0.4429          | 0.4424           | −0.10%                           |
| 10   | 0.3638          | 0.3665           | 0.73%                            |
| 15   | 0.2643          | 0.2677           | 1.32%                            |
| 20   | 0.1887          | 0.1956           | 3.67%                            |
| 30   | 0.1025          | 0.1044           | 1.92%                            |
| 50   | 0.0301          | 0.0298           | −1.16%                           |
Table 10. Values of \( \left( \frac{\psi_D V_3(x)}{\psi(x)} - 1 \right) \cdot 100\% \) for different \( \nu_1 \) and \( \nu_2 \): the Lomax distributions for the premium and claim sizes, \( \bar{\alpha} = 6, \bar{\beta} = 1, \alpha = 6 \) and \( \beta = 15 \)

| \( x \) | \( \nu_1=0.9 \) | \( \nu_1=1.1 \) | \( \nu_1=1.5 \) | \( \nu_1=2 \) | \( \nu_1=5 \) | \( \nu_1=7 \) | \( \nu_1=10 \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
|       | \( \nu_2=1.1 \) | \( \nu_2=0.9 \) | \( \nu_2=0.6 \) | \( \nu_2=0.4 \) | \( \nu_2=0.2 \) | \( \nu_2=0.15 \) | \( \nu_2=0.1 \) |
| 1     | -6.78% | -5.94% | -5.11% | -4.56% | -3.24% | -3.03% | -2.89% |
| 2     | -5.73% | -4.90% | -4.07% | -3.52% | -2.21% | -2.00% | -1.86% |
| 3     | -4.07% | -3.24% | -2.40% | -1.85% | -0.55% | -0.33% | -0.20% |
| 5     | -1.84% | -1.02% | -0.19% | 0.36%  | 1.64%  | 1.85%  | 1.99%  |
| 7     | -0.56% | 0.24%  | 1.06%  | 1.60%  | 2.85%  | 3.05%  | 3.19%  |
| 10    | 0.29%  | 1.06%  | 1.85%  | 2.37%  | 3.55%  | 3.75%  | 3.88%  |
| 15    | 0.92%  | 1.62%  | 2.36%  | 2.84%  | 3.90%  | 4.08%  | 4.20%  |
| 20    | 3.30%  | 3.94%  | 4.63%  | 5.09%  | 6.04%  | 6.20%  | 6.32%  |
| 30    | 1.65%  | 2.14%  | 2.69%  | 3.06%  | 3.74%  | 3.86%  | 3.95%  |
| 50    | -1.26% | -1.06% | -0.76% | -0.55% | -0.38% | -0.34% | -0.29% |

5 Conclusion

The results of computations presented in Tables 1, 3, 5, 7 and 9 indicate that both approximations yield very small relative errors. Although the existence of exponential moments of the distributions of the premium and claim sizes is not required to construct the approximations, it is easily seen that the relative errors are smaller when those distributions do not have heavy tails.

The construction of the five-moment approximation is based on the classical approach, where we only require that the first five moments of the processes coincide without any additional assumptions. The numerical illustrations show that this approach gives very good results, but unfortunately, the conditions that are necessary for its construction are too restrictive. A definite advantage of the three-moment approximation is that the corresponding conditions are much less restrictive, but the construction of this approximation requires two additional conditions, which are also based on some heuristic assumptions. Nevertheless, there is no reason to assert that one of the approximations is more accurate than the other one: the corresponding relative errors vary for different values of the initial surplus.

The relative errors for some three-moment approximations using more general conditions (39) are given in Tables 2, 4, 6, 8 and 10. The analysis of the errors shows that the accuracy of those approximations is more or less the same: choosing \( \nu_1 \) and \( \nu_2 \) that yield smaller errors for some values of the initial surplus leads to larger errors for other values. Nonetheless, for some other values of \( \nu_1 \) and \( \nu_2 \), the corresponding approximations can give not so good results. Therefore, the choice of coefficients \( \nu_1 \) and \( \nu_2 \) should be controlled using other methods that enable us to approximate the ruin probability.

Finally, note that although the numerical examples considered above are not sufficient to make conclusions about the accuracy of the suggested approximations in general and it would be highly desirable to have a tool to control the accuracy in terms of parameter values, those illustrations enable us to outline some general tendencies.
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References

[1] Albrecher, H., Hartinger, J.: A risk model with multilayer dividend strategy. N. Am. Actuar. J. 11, 43–64 (2007) MR2380719. https://doi.org/10.1080/10920277.2007.10597447
[2] Asmussen, S., Albrecher, H.: Ruin Probabilities. World Scientific, Singapore (2010) MR2766220. https://doi.org/10.1142/9789814282536
[3] Avram, F., Banik, A.D., Horvath, A.: Ruin probabilities by Padé’s method: simple moments based mixed exponential approximations (Renyi, De Vylder, Cramér–Lundberg), and high precision approximations with both light and heavy tails. Eur. Actuar. J. 9, 273–299 (2019) MR3982210. https://doi.org/10.1007/s13385-018-0180-8
[4] Badescu, A., Landriault, D.: Recursive calculation of the dividend moments in a multi-threshold risk model. N. Am. Actuar. J. 12, 74–88 (2008) MR2485710. https://doi.org/10.1080/10920277.2008.10597501
[5] Beekman, J.A.: A ruin function approximation. Trans. – Soc. Actuar. 21, 41–48 (1969)
[6] Boikov, A.V.: The Cramér–Lundberg model with stochastic premium process. Theory Probab. Appl. 47, 489–493 (2003) MR1975908. https://doi.org/10.1137/S0040585X9797987
[7] Burnecki, K., Teuerle, M.A., Wilkowska, A.: De Vylder type approximation of the ruin probability for the insurer-reinsurer model. Math. Appl. 47, 5–24 (2019) MR3988929. https://doi.org/10.14708/ma.v47i1.6417
[8] Chi, Y., Lin, X.S.: On the threshold dividend strategy for a generalized jump-diffusion risk model. Insur. Math. Econ. 48, 326–337 (2011) MR2820045. https://doi.org/10.1016/j.insmatheco.2010.11.006
[9] Choi, S.K., Choi, M.H., Lee, H.S., Lee, E.Y.: New approximations of ruin probability in a risk process. Qual. Technol. Quant. Manag. 7, 377–383 (2010)
[10] Cossette, H., Marceau, E., Marri, F.: Constant dividend barrier in a risk model with a generalized Farlie–Gumbel–Morgenstern copula. Methodol. Comput. Appl. Probab. 13, 487–510 (2011) MR2822392. https://doi.org/10.1007/s11009-010-9168-9
[11] Cossette, H., Marceau, E., Marri, F.: On a compound Poisson risk model with dependence and in the presence of a constant dividend barrier. Appl. Stoch. Models Bus. Ind. 30, 82–98 (2014) MR3191344. https://doi.org/10.1002/asmb.1928
[12] De Finetti, B.: Su un’impostazione alternativa dell teoria colletiva del rischio. Trans. XV Int. Congr. Actuar. 2, 433–443 (1957)
[13] De Vylder, F.: A practical solution to the problem of ultimate ruin probability. Scand. Actuar. J. 1978, 114–119 (1978)
[14] Gerber, H.U., Shiu, E.S.W.: On the time value of ruin. N. Am. Actuar. J. 2, 48–72 (1998) MR1988433. https://doi.org/10.1080/10920277.1998.10595671
[15] Gerber, H.U., Shiu, E.S.W., Smith, N.: Methods for estimating the optimal dividend barrier and the probability of ruin. Insur. Math. Econ. 42, 243–254 (2008) MR2392086. https://doi.org/10.1016/j.insmatheco.2007.02.002
[16] Grandell, J.: Aspects of Risk Theory. Springer, New York (1991) MR1084370. https://doi.org/10.1007/978-1-4613-9058-9
[17] Grandell, J.: Simple approximations of ruin probabilities. Insur. Math. Econ. 26, 157–173 (2000) MR1787834. https://doi.org/10.1016/S0167-6687(99)00050-5
[18] Hoeffding, W.: Probability inequalities for sums of bounded random variables. J. Am. Stat. Assoc. 58, 13–30 (1963) MR0144363
[19] Hu, X., Duan, B., Zhang, L.: De Vylder approximation to the optimal retention for a combination of quota-share and excess of loss reinsurance with partial information. Insur. Math. Econ. 76, 48–55 (2017) MR3698186. https://doi.org/10.1016/j.insmatheco.2017.06.007
[20] Landriault, D.: Constant dividend barrier in a risk model with interclaim-dependent claim sizes. Insur. Math. Econ. 42, 31–38 (2008) MR2392066. https://doi.org/10.1016/j.insmatheco.2006.12.002
[21] Lin, X.S., Sendova, K.P.: The compound Poisson risk model with multiple thresholds. Insur. Math. Econ. 42, 617–627 (2008) MR2404318. https://doi.org/10.1016/j.insmatheco.2007.06.008
[22] Mishura, Y., Ragulina, O.: Ruin Probabilities: Smoothness, Bounds, Supermartingale Approach. ISTE Press – Elsevier, London (2016) MR3643478
[23] Mishura, Y., Ragulina, O., Stroev, O.: Practical approaches to the estimation of the ruin probability in a risk model with additional funds. Mod. Stoch. Theory Appl. 1, 167–180 (2014) MR3316485. https://doi.org/10.15559/15-VMSTA18
[24] Mishura, Y.S., Ragulina, O.Y., Stroev, O.M.: Analytic property of infinite-horizon survival probability in a risk model with additional funds. Theory Probab. Math. Stat. 91, 131–143 (2015)
[25] Navickienė, O., Sprindys, J., Šiaulys, J.: Ruin probability for the bi-seasonal discrete time risk model with dependent claims. Mod. Stoch. Theory Appl. 6, 133–144 (2019) MR3935430. https://doi.org/10.15559/18-vmsta118
[26] Ragulina, O.: The risk model with stochastic premiums, dependence and a threshold dividend strategy. Mod. Stoch. Theory Appl. 4, 315–351 (2017) MR3739013. https://doi.org/10.15559/17-vmsta89
[27] Ragulina, O.: The risk model with stochastic premiums and a multi-layer dividend strategy. Mod. Stoch. Theory Appl. 6, 285–309 (2019) MR4028078. https://doi.org/10.15559/19-vmsta136
[28] Rolski, T., Schmidli, H., Schmidt, V., Teugels, J.: Stochastic Processes for Insurance and Finance. John Wiley & Sons, Chichester (1999) MR1680267. https://doi.org/10.1002/9780470317044
[29] Schmidli, H.: Risk Theory. Springer, Cham (2018) MR3753610. https://doi.org/10.1007/978-3-319-72005-0
[30] Shi, Y., Liu, P., Zhang, C.: On the compound Poisson risk model with dependence and a threshold dividend strategy. Stat. Probab. Lett. 83, 1998–2006 (2013) MR3079035. https://doi.org/10.1016/j.spl.2013.05.008