Rotating Dilaton Black Holes

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Abstract

We consider the axially symmetric coupled system of gravitation, electromagnetism and a dilaton field. Reducing from four to three dimensions, the system is described by gravity coupled to a non-linear σ-model. We find the target space isometries and use them to generate new solutions. It seems that it is only possible to generate rotating solutions from non-rotating ones for the special cases when the dilaton coupling parameter $a = 0, \pm \sqrt{3}$. For those particular values, the target space symmetry is enlarged.

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Introduction

Dilaton Black holes, i.e. black hole solutions with a scalar field (dilaton) and electromagnetic fields present, show qualitatively different behavior as compared to solutions in pure Einstein-Maxwell gravity \[1,2\]. In this letter we report on new classes of rotating dilaton black hole solutions. We derive these solutions using a generating technique that goes back to Neugebauer, Kramer and Geroch \[3,4\]. This technique was recently used in \[5\] to generate axion/dilaton solutions. One result of our investigation is that the space of solutions becomes drastically reduced when the axion is not present. In fact, using our approach it seems possible to generate rotating solutions from non-rotating ones only for certain values of the dilaton coupling parameter.

The $\sigma$ model action

We consider the following action:

\[
S = \frac{1}{16\pi} \int \sqrt{-g} d^4x \left( -^{(4)}R + 2\partial_\mu \phi \partial^\mu \phi - e^{-2a\phi} F_{\mu\nu} F^{\mu\nu} \right) 
\]

where \(^{(4)}R\) is the curvature scalar formed from the space-time metric \(g_{\mu\nu}\), \(\phi\) is the dilaton and \(F_{\mu\nu}\) is the electromagnetic field strength. The coupling of the dilaton is governed by the real parameter \(a\) \[1\]. The most general four-dimensional axi-symmetric metric may be written:

\[
^{(4)}ds^2 = f(dt - \omega_i dx^i)^2 - f^{-1} h_{ij} dx^i dx^j,
\]

where \(h_{ij}\), \(f\) and \(\omega_i\) are independent of \(t\). As suggested by (2), we focus on a three dimensional hypersurface embedded in four space. We rewrite the action \[\[\]\] in terms of the intrinsic and extrinsic curvatures, \(^{(3)}R_{ij}\) and \(K_{ij}\), as \[\[\]\]:
where $N$ is the laps function and $\tilde{h}_{ij} = -f^{-1}h_{ij}$ is the metric on the hypersurface perpendicular to the normal vector field $\vec{n}$ defined by $n_\mu dx^\mu = f^{1/2}(dt - \omega_i dx^i)$. We further define the twist potential $\chi$ through

$$\partial_{[ij} \omega_{k]} = - (\sqrt{h} f^2)^{-1} \epsilon^{ijk} \partial_i \chi$$

and the electric and magnetic potentials $v$ and $u$ through

$$F_{i0} \equiv \frac{1}{\sqrt{2}} \partial_i v, \quad F^{ij} \equiv \frac{1}{\sqrt{2h}} \epsilon^{2\alpha\phi} f^3 \epsilon^{ijk} \partial_k u,$$

where indices now are raised and lowered by the metric $h_{ij}$. After a rescaling of the 3-metric $\tilde{h}_{ij}$ in (3), we may write the total action (1) as the action for a 3D nonlinear $\sigma$-model coupled to gravity:

$$S = \int (\langle R(h) - G_{AB} \partial_i \Phi^A \partial^i \Phi^B \rangle \sqrt{h} d^3 x).$$

The $\sigma$-model target space is coordinatized by the fields $\{\Phi^A \} \equiv \{\phi, f, \chi, u, v, \}$ and the target space metric can be read off from the line-element

$$dl^2 = 2d\phi^2 + \frac{1}{2f^2} (df^2 + (d\chi + vdu - udv)^2) - \frac{1}{f} (e^{-2a\phi} dv^2 + e^{2a\phi} du^2).$$

**Isometries**

The action (6) and the corresponding field equations are invariant under target space isometries, i.e. transformations of $\Phi^A$ that leave (7) invariant. This fact may be utilized to generate new solutions from known ones. To do so we must first find the isometries, i.e. find the Killing vectors that generate the infinitesimal isometries and then exponentiate to find the finite transformations.

Letting $\xi^\alpha$ denote the Killing vectors (denumbered by $\alpha$), Killings equation reads

$$\xi^\alpha_{(B:C)} = 0,$$
where : denotes (target space) covariant derivative. The solutions to (8) differ depending on the value of the dilaton coupling parameter $a$. For arbitrary values of $a$, we find four Killing vectors:

\[
\begin{align*}
\vec{\xi}^1 &= 2f\partial_f + 2\chi\partial_\chi + u\partial_u + v\partial_v \\
\vec{\xi}^2 &= v\partial_\chi + \partial_u \\
\vec{\xi}^3 &= u\partial_\chi - \partial_v \\
\vec{\xi}^4 &= \partial_\chi .
\end{align*}
\]  

(9)

A fifth Killing vector,

\[
\vec{\xi}^5 = \frac{1}{a}\partial_\phi - u\partial_u + v\partial_v .
\]

(10)

is only defined for $a \neq 0$. When $a = 0$ we find instead

\[
\hat{\vec{\xi}}^5 = v\partial_u - u\partial_v .
\]

(11)

The vectors $\vec{\xi}^1, ..., \vec{\xi}^5$ form a closed algebra, and for generic $a$’s, these exhaust the solutions. For the special values $a = 0$ and $a^2 = 3$ there are more solutions, however. For $a = 0$ we find the additional solutions

\[
\begin{align*}
\hat{\vec{\xi}}^6 &= \partial_\phi \\
\hat{\vec{\xi}}^7 &= 2fu\partial_f + (u\chi + \frac{1}{2}v(v^2 + u^2) - vf)\partial_\chi + \\
&\quad + \frac{1}{2}(u^2 - 3v^2 + 2f)\partial_u + (2uv - \chi)\partial_v \\
\hat{\vec{\xi}}^8 &= 2fv\partial_f + (v\chi - \frac{1}{2}u(u^2 + u^2) + uf)\partial_\chi + \\
&\quad + \frac{1}{2}(v^2 - 3u^2 + 2f)\partial_v + (2uw + \chi)\partial_u \\
\hat{\vec{\xi}}^9 &= -4f\chi\partial_f + (-2f(u^2 + v^2) + \frac{1}{2}(u^2 + v^2)^2 + 2(f^2 - \chi^2))\partial_\chi - \\
&\quad - (2u\chi + u^2v + v^3 - 2fv)\partial_u + (-2v\chi + uw^2 + u^3 - 2fu)\partial_v .
\end{align*}
\]

(12)

These vectors also form a closed algebra, where $\hat{\vec{\xi}}^6$ commutes with all generators and $\hat{\vec{\xi}}^7, ..., \hat{\vec{\xi}}^9$ form a sub-algebra. When $\phi = 0$, $\hat{\vec{\xi}}^7$ and $\hat{\vec{\xi}}^8$ generate Harrison type transformations and commute to $\hat{\vec{\xi}}^9$ which generates Ehlers type transformations \[.\]
Finally and most interestingly, for \( a^2 = 3 \) we have the following additional set of Killing vectors (for definiteness we choose \( a = \sqrt{3} \)):

\[
\tilde{\xi}_6 = -\sqrt{3}u\partial_\phi + 2fu\partial_f + (u(\chi - uv) - e^{-2\sqrt{3}\phi}fv)\partial_\chi + \\
(2u^2 + e^{-2\sqrt{3}\phi}f)\partial_u - (\chi + uv)\partial_v
\]

\[
\tilde{\xi}_7 = \sqrt{3}v\partial_\phi + 2fv\partial_f + (v(\chi + uv) + e^{2\sqrt{3}\phi}fu)\partial_\chi + \\
(2v^2 + e^{2\sqrt{3}\phi}f)\partial_v + (\chi - uv)\partial_u
\]

\[
\tilde{\xi}_8 = \sqrt{3}uv\partial_\phi + 2\chi f\partial_f + (e^{2\sqrt{3}\phi}fu^2 + e^{-2\sqrt{3}\phi}fv^2 + u^2v^2 + \chi^2 - f^2)\partial_\chi - \\
(ffe^{-2\sqrt{3}\phi} + u^2v - u\chi)\partial_u + (fue^{2\sqrt{3}\phi} + uv^2 + v\chi)\partial_v.
\]  

(13)

For this case the full multiplication table is given in fig.1. In this table, we again find a subalgebra generated by the last three vectors \( \tilde{\xi}_6, \ldots, \tilde{\xi}_8 \). Here \( \tilde{\xi}_6 \) and \( \tilde{\xi}_7 \) generate (dual) Harrison type transformations and \( \tilde{\xi}_8 \) generates an Ehlers type transformation.

**The finite transformations**

To find the finite transformations, we exponentiate the generators (9), (10), (11), (12) and (13) multiplied by the corresponding group parameters \( \lambda_\alpha \). An isometry then acts on the fields as follows:

\[
\Phi'^A = exp\{\lambda_\alpha \tilde{\xi}_\alpha\} \Phi^A. 
\]  

(14)

For a general \( a \neq 0 \) we find from (9), (10):

\[
\begin{align*}
n = 1 : \phi' &= \phi, \ f' = e^{2\lambda_2}f, \ \chi' = e^{2\lambda_1}\chi, \ u' = e^{\lambda_1}u, \ v' = e^{\lambda_1}v \\
n = 2 : \phi' &= \phi, \ f' = f, \ v' = v, \ \chi' = \chi + v\lambda_2, \ u' = u + \lambda_2 \\
n = 3 : \phi' &= \phi, \ f' = f, \ u' = u, \ \chi' = \chi + u\lambda_3, \ v' = v - \lambda_3 \\
n = 4 : \phi' &= \phi, \ f = f', \ u' = u, \ v' = v, \ \chi' = \chi + \lambda_4 \\
n = 5 : f' &= f, \ \chi' = \chi, \ \phi' = \phi + a^{-1}\lambda_5, \ u' = e^{-\lambda_5}u, \ v' = e^{\lambda_5}v.
\end{align*}
\]  

(15)

For \( a = 0 \) two additional transformations are:
\[ n = 5 : \phi' = \phi, f' = f, \chi' = \chi, u' = \cos(\hat{\lambda}_5)u + \sin(\hat{\lambda}_5)v, v' = \cos(\hat{\lambda}_5)v - \sin(\hat{\lambda}_5)u, \]
\[ n = 6 : \phi' = \phi + \hat{\lambda}_6, f = f', u' = u, v' = v, \chi' = \chi. \quad (16) \]

The finite transformations that correspond to \( \tilde{\xi}_7, \ldots, \tilde{\xi}_9 \) in (12) are more complicated. We do not give them here, since the minimal coupling of a dilaton to gravity and electromagnetism reduces to Einstein-Maxwell theory which has been studied in our type of approach in [3].

For \( a^2 = 3 \) the finite transformations are (16) above with \( a = \pm \sqrt{3} \) along with those that correspond to \( \tilde{\xi}_6, \ldots, \tilde{\xi}_8 \) in (13). From the multiplication table in Fig.1 we see that \( \tilde{\xi}_8 \) commutes with all other generators. This means that the corresponding group element is known once the finite transformations generated by \( \tilde{\xi}_6 \) and \( \tilde{\xi}_7 \) have been found:

\[ \exp\{(\tilde{\lambda}_8)^2\tilde{\xi}_8\} = \exp\{\tilde{\lambda}_8\tilde{\xi}_6\}\exp\{\tilde{\lambda}_8\tilde{\xi}_7\}\exp\{-\tilde{\lambda}_8\tilde{\xi}_6\}\exp\{-\tilde{\lambda}_8\tilde{\xi}_7\}. \quad (17) \]

The finite transformations that follow from \( \tilde{\xi}_6 \) are

\[ U_{\pm}^{-1} = U_{\pm}^{-1} - \lambda_6, \quad v' = v - \Xi\lambda_6, \quad F' = F, \quad \Xi' = \Xi, \quad (18) \]

where (for \( a = +\sqrt{3} \))

\[ F \equiv \exp\{2\phi/\sqrt{3}\}f, \quad \Xi \equiv \chi + uv \quad U_{\pm} \equiv 2u \pm \sqrt{2} F \exp\{-4\phi/\sqrt{3}\}. \quad (19) \]

The finite transformations generated by \( \tilde{\xi}_7 \) are very similar, due to the "duality" symmetry \( u \to v, v \to -u, \phi \to -\phi \). They are

\[ V_{\pm}^{-1} = V_{\pm}^{-1} - \lambda_7, \quad u' = u + \Xi\lambda_7, \quad F' = F, \quad \Xi' = \Xi, \quad (20) \]

where

\[ F \equiv \exp\{-2\phi/\sqrt{3}\}f, \quad \Xi \equiv \chi - uv \quad V_{\pm} \equiv 2v \pm \sqrt{2} F \exp\{4\phi/\sqrt{3}\}. \quad (21) \]
New Solutions

We are now in position to generate new solutions from old ones using the isometry transformations just derived. The technique is to start from a known seed solution and then apply one or several transformations to it. It turns out that the transformations (15) and (16) do not lead to interesting new solutions in general. We will thus focus on the Harrison (and Ehlers)-type transformations (17), (18), (20). Below we briefly present several examples. We will give a fuller description elsewhere [8].

Example 1.

We first use a general vacuum solution as seed solution: \( u_0 = v_0 = \phi_0 = 0 \). The original field variables are thus \( f_0 \), and \( \chi_0 \). Acting on these with (18) and (20) we find

\[
\begin{align*}
\phi &= \frac{\sqrt{3}}{4} \ln(1 - 2f_0\lambda^2), \\
f &= f_0/\sqrt{1 - 2f_0\lambda^2} \\
\chi &= \left(\frac{1 - f_0\lambda^2}{1 - 2f_0\lambda^2}\right) \chi_0, \\
u &= \lambda \chi_0, \\
v &= \frac{f_0\lambda}{1 - 2f_0\lambda^2}.
\end{align*}
\]

(22)

The transformation generated by (20) results if we make the substitution \( u \rightarrow v, v \rightarrow -u, \phi \rightarrow -\phi \). If the original solution has fields that go to zero at infinity, so will the new one. It is also clear that whereas we generate matter fields using these transformations, we cannot generate a rotating solution from a non-rotating one.

Example 2.

As a second seed solution, we use the static dilaton black-hole solution [1], for \( a = \sqrt{3} \):

\[
\begin{align*}
f_0 &= \frac{\Delta}{\sigma^2}, \\
\exp\{2\sqrt{3}\phi_0\} &= \sigma^2, \\
v_0 &= \frac{Q}{r}, \\
u_0 &= 0, \\
\chi_0 &= 0,
\end{align*}
\]

(23)

where

\[
\begin{align*}
\Delta &\equiv \left(1 - \frac{r_+}{r}\right) \\
\sigma^2 &\equiv \left(1 - \frac{r_-}{r}\right)^{\frac{3}{2}} \\
r_+r_- &= 2Q^2, \\
-2(2M - r_+) &= r_-.
\end{align*}
\]

(24)
with $M$ and $Q$ the mass and charge of the black hole, respectively. Applying the transformations (20) to the fields in (23) we obtain

$$\exp\{2\sqrt{3}\phi\} = \sigma^2 \left[ \left(1 - \frac{2\lambda Q}{r}\right)^2 - 2\lambda \Delta \right]^{-\frac{3}{2}}, \quad f = \Delta (\sigma^2)^{-1} \left[ \left(1 - \frac{2\lambda Q}{r}\right)^2 - 2\lambda \Delta \right]^{-\frac{3}{2}},$$

$$\chi = 0, \quad u = 0, \quad v = \left[ \left(1 - \frac{2\lambda Q}{r}\right)^2 - 2\lambda \Delta \right]^{-1} \left(\frac{Q}{r} \left(1 - \frac{2\lambda Q}{r}\right) + \lambda \Delta \right).$$  \hspace{1cm} (25)

Again, no rotation. However, the seed solution is asymmetric in $u$ and $v$, and will therefore be treated differently by the two transformations (18) and (20). In fact, the result of (18) is:

$$\exp\{2\sqrt{3}\phi_1\} = (\sigma^2)^{-2} \left((\sigma)^2 - 2\lambda^2 \Delta\right)^{-\frac{3}{2}}, \quad f_1 = \Delta \left((\sigma)^2 - 2\lambda^2 \Delta\right)^{-\frac{3}{2}}$$

$$\chi_1 = -\frac{Q\lambda \Delta}{r} \left((\sigma)^2 - 2\lambda^2 \Delta\right)^{-1}, \quad u_1 = \Delta \lambda \left((\sigma)^2 - 2\lambda^2 \Delta\right)^{-1}, \quad v_1 = \frac{Q}{r}. \hspace{1cm} (26)$$

We see that this produces a rotating solution. When $r \to \infty$ the rotation goes to zero while $f$ and $u$ go to constant values. After a rescaling generated by $\tilde{\xi}_1$ with parameter $\frac{1}{2} \ln(\sqrt{1 - 2\lambda^2})$ and a $\xi_2$-transformation with parameter $-\lambda/(1 - 2\lambda^2)^{3/4}$ we obtain the following asymptotically flat rotating dilaton black-hole solution:

$$\phi = \phi_1, \quad f = Af_1, \quad u = A^{\frac{3}{2}} u_1 - A^{-\frac{3}{2}} \lambda, \quad v = A^{\frac{3}{2}} v_1$$

$$\chi = A \chi_1 - A^{-1} \lambda v_1 \Rightarrow \omega_\theta = \omega_r = 0, \quad \omega_\phi = \left(2\lambda QA^{-1}\right) \cos \theta.$$  \hspace{1cm} (27)

where $A \equiv \sqrt{1 - 2\lambda^2}$. This is a Taub-NUT type solution. That type of a solution has also been found in dilaton-axion gravity for $a = 1$. \[9\]

**Example 3.**

As our final example, we apply the formulae in (22) using the Kerr solution as a seed solution. The starting point is thus

$$\phi_0 = 0, \quad f_0 = \frac{\Delta - \rho^2 \sin^2 \theta}{\Sigma} \quad \chi_0 = \frac{2M \rho \cos \theta}{\Sigma}, \quad u_0 = v_0 = 0,$$

where the coordinates are $t, r, \varphi, \theta,$ and
\[ \Delta \equiv r(r - 2M) + \rho^2, \quad \Sigma \equiv r^2 + \rho^2 \cos^2 \theta. \]  

(29)

Here \( M \) is the mass and \( \rho \) is the angular momentum parameter. The (diagonal) three-metric is

\[ h_{rr} = \frac{\Delta - \rho^2 \sin^2 \theta}{\Delta}, \quad h_{\theta\theta} = \Delta - \rho^2 \sin^2 \theta, \quad h_{\varphi\varphi} = \Delta \sin^2 \theta. \]  

(30)

From (22) we find

\[ \exp\{2\sqrt{3}\phi_1\} = \left[ 1 - \frac{\Delta - \rho^2 \sin^2 \theta}{\Sigma} \right]^{-\frac{3}{2}}, \quad f_1 = \frac{\Delta - \rho^2 \sin^2 \theta}{\sqrt{\Sigma^2 - 2\lambda^2 \Sigma \Delta - \rho^2 \sin^2 \theta}}, \]

\[ \chi_1 = \frac{2M\rho \cos \theta (\Sigma^2 - \lambda^2 \Sigma \Delta - \rho^2 \sin^2 \theta)}{\Sigma (\Sigma^2 - 2\lambda^2 \Sigma \Delta - \rho^2 \sin^2 \theta)}, \quad u_1 = \frac{2M \rho \lambda \cos \theta}{\Sigma}, \quad v_1 = \frac{\lambda (\Delta - \rho^2 \sin^2 \theta)}{\Sigma - 2\lambda^2 (\Delta - \rho^2 \sin^2 \theta)}. \]

A \( \xi_1 \) rescaling with parameter \( \frac{1}{2} \ln(A) \) and a \( \tilde{\xi}_3 \) transformation with parameter \( \lambda/A^{\frac{3}{2}} \), \( A \) as in example 2, lead to the final asymptotically flat result

\[ \phi = \phi_1, \quad f = Af_1, \quad \chi = A\chi_1 + \lambda A^{-1} u_1, \quad u = A^{\frac{1}{2}} u_1, \quad v = A^{\frac{1}{2}} v_1 - \lambda A^{-\frac{3}{2}}. \]  

(31)

This is the rotating dilaton black hole solution found in [10].

Discussion

Since the transformations (15) give trivial new solutions from vacuum solutions and they exhaust the symmetries for generic \( a \)'s, rotating dilaton black hole solutions seem harder to come by than the axion-dilaton ones [3]. It is clear from our discussion that there are families of rotating solutions for \( a = 0, \pm \sqrt{3} \), however. We have given some examples, and more examples are easily available. One may use e.g. the solutions (26) and (31) as seed solutions. The special values of \( a \) correspond to minimal couplings of the dilaton field. For \( a = 0 \) the coupling to the \( F^2 \) term in the lagrangian is 1 in \( D = 4 \). The value \( a^2 = 3 \) corresponds to minimal coupling in \( D = 5 \), and it has previously been encountered in many cases as resulting from dimensional reduction [10,11].
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FIG. 1. The commutators $[\tilde{\xi}_\alpha, \tilde{\xi}_\beta]$ of the $a^2 = 3$ Killing vectors.

| $\tilde{\xi}_\alpha$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------------|----|----|----|----|----|----|----|
| $\tilde{\xi}_2$      | $-2\tilde{\xi}_4$ | 0  | $-\tilde{\xi}_3$ | 0  | $-\tilde{\xi}_1$ | $-\tilde{\xi}_5$ | 0  | $-\tilde{\xi}_7$ |
| $\tilde{\xi}_3$      | 0  | $-2\tilde{\xi}_4$ | $\tilde{\xi}_2$ | $\tilde{\xi}_1$ | $-\tilde{\xi}_5$ | 0  | $-\tilde{\xi}_7$ |
| $\tilde{\xi}_4$      | 0  | 0  | $-\tilde{\xi}_3$ | 0  | $-\tilde{\xi}_1$ | $-\tilde{\xi}_5$ | $\tilde{\xi}_6$ | 0  |
| $\tilde{\xi}_5$      | 0  | 0  | $\tilde{\xi}_3$ | $\tilde{\xi}_2$ | $\tilde{\xi}_1$ | 0  | 0  | 0  |
| $\tilde{\xi}_6$      | $-\tilde{\xi}_7$ | 0  | 0  | $-\tilde{\xi}_8$ | 0  | 0  | 0  | 0  | 0  |
| $\tilde{\xi}_7$      | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |