EXAMPLES OF TOPOLOGICAL SPACES WITH ARBITRARY COHOMOLOGY JUMP LOCI

BOTONG WANG

Abstract. Given any subvariety of a complex torus defined over $\mathbb{Z}$ and any positive integer $k$, we construct a finite CW complex $X$ such that the $k$-th cohomology jump locus of $X$ is equal to the chosen subvariety, and the $i$-th cohomology jump loci of $X$ are trivial for $i < k$.

1. Introduction

The motivation of this note is to explore the consequence of the results of [3] and [1], that the cohomology jump loci of a smooth complex quasi-projective variety are unions of torsion translates of subtori.

Given a topological space $M$, we denote by $L(M)$ the space of rank one local systems on $M$, which is equal to $\text{Hom}(\pi_1(M), \mathbb{C}^*)$ and has a group structure. When $\pi_1(M)$ is finitely generated, $L(M)$ has a complex variety structure which is isomorphic to the cartesian product of $(\mathbb{C}^*)^{b_1(M)}$ and a discrete finite abelian group. In $L(M)$, there are some canonically defined subvarieties called the cohomology jump loci (or characteristic varieties). They are defined to be $\Sigma^r_i(M) = \{ \rho \in L(M) \mid \dim H^i(M, L_\rho) \geq r \}$, where $L_\rho$ is the rank one local system on $M$ associated to the representation $\rho$. They are always varieties defined over $\mathbb{Z}$. When $r = 1$, we omit $r$ and just write $\Sigma^i(M)$.

The main result of this note is the following.

**Theorem 1.1.** Fix a positive integer $n$. Let $Z$ be any (not necessarily irreducible) subvariety of $(\mathbb{C}^*)^n$, which is defined over $\mathbb{Z}$, and let $k$ be any positive integer. There exists a finite CW complex $M$, such that $L(M) = (\mathbb{C}^*)^n$, $\Sigma^k(M) \cup \{1\} = Z \cup \{1\}$, and $\Sigma^i(M)$ is either empty or equal to $\{1\}$ for $i < k$.

We prove the theorem in the rest of this note. The proof is divided into two parts, $k = 1$ and $k \geq 2$. When $k = 1$, this is a group theoretic problem, and is essentially known (for example, [3] Lemma 10.3). For completeness, we include the proof still. In fact, the examples for the case $k \geq 2$ are inspired by the case $k = 1$.

When $k \geq 2$ and $n$ is even, and when $Z$ is not a union of torsion translates of subtori, our examples are homotopy $(k - 1)$-equivalent to an abelian variety, but not homotopy $k$-equivalent to any quasi-projective variety. This shows that the result of [1] puts genuine higher homotopy obstruction to the possible homotopy types of quasi-projective varieties.

Carlos Simpson has kindly pointed out to us that the construction for $k \geq 2$ has appeared in [4]. Thus, this note is a self contained proof of some essentially known result.
2. Constructing the examples for $k \geq 2$

We assume $k \geq 2$ though out this section.

Under the notation of Theorem 1.1, we start with a real torus $M_0 = (S^1)^n$. Let $N_0$ be the universal cover of $M_0$, and let the covering map be $p_0 : N_0 \to M_0$. Fixing an origin $O \in M_0$, we attach a $k$-sphere $S^k$ to $M_0$ at $O$, obtaining a new space $M_1$. In other words, $M_1$ is the wedge sum of $M_0$ and $S^k$. We call such a $k$-sphere. Let $p_1 : N_1 \to M_1$ be the universal covering map. Then $N_1$ is obtained from $N_0$ by attaching infinitely many $k$-spheres parametrized by $\mathbb{Z}^n$. Denote the $k$-sphere in $N_1$ corresponding to $J \in \mathbb{Z}^n$ by $B_J$. Suppose the subvariety $Z$ of $(\mathbb{C}^*)^n$ is defined by Laurent polynomials $f_i(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$, for $1 \leq i \leq r$, where $x_j$’s are the coordinates of $(\mathbb{C}^*)^n$. Suppose $f_i(x_1, \ldots, x_n) = \sum_{j \in \Lambda_i} a_j^i x^j$, where $\Lambda_i$ is a finite subset of $\mathbb{Z}^n$, $a_j^i$ are integers and $x^j$ is the product of $x_j$’s under the usual multi-index notation. For example, if $n = 2$ and $J = (−2, 3)$, then $x^J = x_1^{-2} x_2^3$.

The CW complex $M$ satisfying the property of Theorem 1.1 will be obtained by gluing $(k+1)$-cells to $M_1$ corresponding to the Laurent polynomials $f_1, \ldots, f_r$. To explain how to attach the $(k+1)$-cell corresponding to one Laurent polynomial $f_i$, it is easier to work on $N_1$. Fix a $J_0 \in \mathbb{Z}^n$. We can attach a $(k+1)$-cell $e^{k+1}_{J_0}$ to $N_1$, such that $\partial e^{k+1}_{J_0}$ represents the cycle $\sum_{j \in \Lambda_i} a_j^i B_{J_0+j}$ in $H_k(N_1, \mathbb{Z})$. Denote the new space by $N_{2,J_0}$. Notice that $N_1$ is $(k-1)$-connected. Hence $N_{2,J_0}$ is uniquely determined up to homotopy. Define $N_2$ to be the coproduct of $N_{2,J_0}$ over $N_1$ for all $J_0 \in \mathbb{Z}^n$. Then $N_2$ is obtained from $N_1$ by attaching infinitely many $(k+1)$-cells parametrized by $\mathbb{Z}^n$. Suppose we attach these $(k+1)$-cells in a compatible way. Then the Galois action on $N_1$ by $\mathbb{Z}^n$ extends to $N_2$. Now, let $M_2$ be the quotient of $N_2$ by the Galois action $\mathbb{Z}^n$. By our construction, $M_2$ is obtained by attaching one $(k+1)$-cell to $M_1$. In the same way, we can attach more $(k+1)$-cells corresponding to other Laurent polynomials $f_2, \ldots, f_r$. Let $M$ be the space we obtained this way, and let $N$ be the cover of $M$ with Galois group $\mathbb{Z}^n$. Denote the covering map by $p : M \to N$.

By our construction, there is a natural isomorphism $\pi_1(M) \cong \pi_1(M_0)$. Since $M_0 = (S^1)^n$, there is a natural isomorphism $L(M) \cong (\mathbb{C}^*)^n$, and the isomorphism is induced by an isomorphism of the underlying scheme over $\mathbb{Z}$.

**Proposition 2.1.** Under the above isomorphism, we have the following.

1. $\Sigma^i(M) = \{1\}$ for $0 \leq i \leq \min\{n, k-1\}$, $\Sigma^i(M) = \emptyset$ for $n < i \leq k-1$.
2. $\Sigma^k(M) = \mathbb{Z} \cup \{1\}$ when $k \leq n$, and $\Sigma^k(M) = \mathbb{Z}$ when $k > n$.
3. $M$ is homotopy $(k-1)$-equivalent to the real torus $(S^1)^n$.
4. $M$ is not homotopy $k$-equivalent to any quasi-projective variety.

The statements (1) and (3) are obvious from the construction. Moreover, (4) follows from (2) and [1]. To prove this, one has to argue that the cohomology jump locus $\Sigma^i$ is a homotopy $i$-equivalence invariant. We leave this to the reader. We will prove statement (2) in the rest of this section.

We will compute the cohomology jump loci of $M$ via Alexander modules. Recall that $p : N \to M$ is the covering map with Galois action by $\mathbb{Z}^n$. Then
in a natural way, the homology groups $H_i(N, \mathbb{Z})$ become $\mathbb{Z}^n$-modules. Notice that $\mathbb{Z}^n$ is naturally identified with $H_1(M, \mathbb{Z})$, and in a natural way $L(M) = \text{Hom}_k(H_1(M, \mathbb{Z}), \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. Thus, the group ring of $\mathbb{Z}^n$ is naturally identified with $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, which is the coordinate ring of the underlying $\mathbb{Z}$-scheme of $(\mathbb{C}^*)^n$. As $\mathbb{Z}^n$-module, $H_1(N, \mathbb{Z})$ has now a natural $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module structure. These $H_1(N, \mathbb{Z})$, together with the $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module structures are called Alexander modules.

Denote the support of the Alexander modules $H_i(N, \mathbb{Z})$ in $(\mathbb{C}^*)^n$ by $\mathcal{V}^i(M)$. Then the cohomology jump loci and the support of the Alexander modules are closely related by the following theorem of Papadima and Suciu.

**Theorem 2.2** ([2] Theorem 3.6).

$$\bigcup_{i=0}^{l} \Sigma^i(M) = \bigcup_{i=0}^{l} \mathcal{V}^i(M)$$

for any integer $l \geq 0$.

Since $M$ is homotopy $(k-1)$-equivalent to the real torus $(S^1)^n$, $N$ is homotopy $(k-1)$-equivalent to a point. Therefore, $\mathcal{V}^{0}(M) = \{1\}$ and $\mathcal{V}^{i}(M) = \emptyset$ for $1 \leq i \leq k - 1$. According to Theorem 2.2 $\Sigma^i(M) \subset \{1\}$.

By construction, $H_k(N_1, \mathbb{Z})$ is a free $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module of rank one. Recall that $N$ is obtained from $N_1$ by attaching many $(k+1)$-cells. Each $(k+1)$ gives a relation in $H_k(N_1, \mathbb{Z})$ as $\mathbb{Z}$-module. After attaching infinitely many cells, $H_k(N_2, \mathbb{Z})$ is isomorphic to $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]/(f_1)$ as $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-modules. This can be proved by a standard Mayer-Vietoris sequence argument. We leave this to the reader. Similarly, $H_k(N, \mathbb{Z}) \cong \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]/(f_1, \ldots, f_r)$. Thus, $\mathcal{V}^k(M) = \mathbb{Z}$. Now, according to Theorem 2.2 $\Sigma^k(M) \cup \{1\} = \mathbb{Z} \cup \{1\}$.

To check whether $1 \in \Sigma^k(M)$ is to compute whether $H^k(M, \mathbb{C}) = 0$. This can be done directly from the construction. In fact, $H^k(M, \mathbb{C}) \neq 0$ when $k \leq n$ and $H^k(M, \mathbb{C}) = 0$ when $k > n$. Thus we have proved the proposition.

**Remark 2.3.** One can try to use the same construction for $k = 1$. It works except that the circle (or 1-shpere) we attach may create extra elements in $H_1(M, \mathbb{Z})$. In fact, it works when $Z$ does not contain any torsion point.

### 3. Group theoretic first cohomology jump loci

It is an easy and well known fact that for a topological space $M$ and $\rho \in L(M)$, $H^1(M, L_\rho) \cong H^1(\pi_1(M), \mathbb{C}_\rho)$. Here the representation $\rho : \pi_1(M) \to \mathbb{C}^*$ gives $\mathbb{C}$ a $\pi_1(M)$-module structure, and we emphasize the $\pi_1(M)$-module structure on $\mathbb{C}$ by writing $\mathbb{C}_\rho$. Therefore, the first cohomology jump loci of $M$ only depend on $\pi_1(M)$.

**Definition 3.1.** Let $G$ be a finitely presented group. We denote the character variety $\text{Hom}(G, \mathbb{C}^*)$ by $L(G)$, and define the group theoretic cohomology jump loci $\Sigma^i(G) = \{\rho \in L(G) \mid H^i(G, \mathbb{C}_\rho) \neq 0\}$. 
The isomorphism between the cohomology of local system and the group cohomology shows that the natural isomorphism \( L(M) \cong L(\pi_1(M)) \) induces an isomorphism between subvarieties \( \Sigma^1(M) \cong \Sigma^1(\pi_1(M)) \). Given a finitely presented group \( G \), there is a standard process to construct a finite CW complex with fundamental group \( G \). Therefore, the case \( k = 1 \) of Theorem 2.2 is equivalent to the following.

**Proposition 3.2.** Let \( Z \) be any (not necessarily irreducible) subvariety of \( (\mathbb{C}^\ast)^n \) defined over \( \mathbb{Z} \). There is a finitely presented group \( G \) such that \( L(G) = (\mathbb{C}^\ast)^n \) and \( \Sigma^1(G) = Z \cup \{1\} \).

Before proving the Proposition, we give an algorithm to compute the first group theoretic cohomology jump loci, which is slightly different from [5].

\[ H^1(G, \mathbb{C}_\rho) \] can be computed by the quotient of 1-cycles by 1-boundaries, i.e., \( H^1(G, \mathbb{C}_\rho) = Z^1(G, \mathbb{C}_\rho)/B^1(G, \mathbb{C}_\rho) \), where

\[ Z^1(G, \mathbb{C}_\rho) = \{ \tau \in \text{Hom}_{\text{set}}(G, \mathbb{C}) \mid \tau(ab) = \rho(a)\tau(b) + \tau(a) \text{ for any } a, b \in G \} \]
\[ B^1(G, \mathbb{C}_\rho) = \{ \tau \in \text{Hom}_{\text{set}}(G, \mathbb{C}) \mid \tau(a) = \rho(a)\tau(1) - \tau(1) \text{ for any } a \in G \}. \]

Denote the commutator subgroup of \( G \) by \( G' \) and denote the abelianization of \( G \) by \( \text{Ab}(G) \), then a straightforward computation shows the following.

**Lemma 3.3.** When \( \rho \neq 1 \),

\[ H^1(G, \mathbb{C}_\rho) \cong \{ \tau \in \text{Hom}(G', \mathbb{C}) \mid \tau(sa) = \rho(s)\tau(a) \text{ for any } s, a \in G' \}. \]

Given a finitely presented group \( G \), we define an element \( g \in G \) to be in the “torsion free metabelian kernel”, if \( g \in G' \), and the image of \( g \) in \( \text{Ab}(G') \) is torsion. It is very easy to check that the torsion free metabelian kernel forms a normal subgroup. We define the quotient of \( G \) by its torsion free metabelian kernel to be the torsion free metabelianization of \( G \), denoted by \( \text{TFM}(G) \). The following is a direct consequence of the previous lemma.

**Corollary 3.4.** The first cohomology jump locus of \( G \) only depends on \( \text{TFM}(G) \). More precisely, the natural isomorphism \( \pi_1(G) \cong \text{TFM}(G) \) induces an isomorphism \( \Sigma^1(G) \cong \Sigma^1(\text{TFM}(G)) \).

Given a finitely presented group \( G \), we define the first Alexander module of \( G \) to be the kernel of the natural map \( \text{TFM}(G) \to \text{Ab}(G) \), or equivalently the commutator subgroup of \( \text{TFM}(G) \). Denote the first Alexander module of \( G \) by \( \text{Alex}(G) \). Then we have a short exact sequence of groups,

\[ 0 \to \text{Alex}(G) \to \text{TFM}(G) \to \text{Ab}(G) \to 0. \]

This short exact sequence induces an action of \( \text{Ab}(G) \) on \( \text{Alex}(G) \), which gives \( \text{Alex}(G) \) a \( \mathbb{Z}[\text{Ab}(G)] \)-module structure. As an analog of Theorem 2.2 the following follows from Lemma 3.3.

**Corollary 3.5.** Suppose \( b_1(G) > 0 \). The coordinate ring of the underlying \( \mathbb{Z} \)-scheme of \( L(G) \) is naturally isomorphic to \( \mathbb{Z}[\text{Ab}(G)] \). Under this isomorphism,

\[ \Sigma^1(G) = \text{Supp}(\text{Alex}(G)) \cup \{1\}. \]
Proof of Proposition 3.2. Now, we are ready to construct the example satisfying the condition in the proposition. As before, we assume the defining equations of $Z$ to be $f_1, \ldots, f_r$, where $f_i \in \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$.

We start with $G_0$, which we define to be the free group of $n$ generators $g_1, \ldots, g_n$. Let $G_1$ be the torsion free metabelianization of $G_0$. The choice of the generators of $G$ gives a natural isomorphism $\Ab(G_1) \cong (\mathbb{Z})^n$. Let $x_1, \ldots, x_n$ be the natural coordinates on $(\mathbb{Z})^n$. Then $\text{Alex}(G_1)$ has a natural $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module structure. Since $G_0$ is the fundamental group of $n$-loops, we can realize $\text{Alex}(G_1) \cong \text{Alex}(G_0)$ as the first homology group of the integral “net” $N \subset \mathbb{R}^n$. Here $N = \bigcup_{1 \leq i \leq n} (\mathbb{Z}^n + l_i)$, where $l_i$ is the $i$-th coordinate axis. $\mathbb{Z}^n$ acts on $N$ by translation. Hence $\mathbb{Z}^n$ also acts on $\text{Alex}(G_1) \cong H_1(N, \mathbb{Z})$. In fact, the $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module structure on $\text{Alex}(G_1)$ can be interpreted this way.

As a $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module, $H_1(N, \mathbb{Z})$ is generated by the unit squares in each coordinate planes, with a proper choice of orientation. Denote these squares by $\gamma_{ij}$, $1 \leq a < b \leq n$. More canonically, we will allow $a \leq b$ and put the condition $\gamma_{ab} + \gamma_{ba} = 0$. Each unit cubic induces a relation between these generators. In fact, $H_1(N, \mathbb{Z}) \cong \bigoplus_{1 \leq a < b \leq n} / I$, where $I$ is the ideal generated by elements $\gamma_{ab} + \gamma_{ba}$ and $(x_a - 1)\gamma_{bc} + (x_b - 1)\gamma_{ca} + (x_c - 1)\gamma_{ab}$ for all $1 \leq a, b, c \leq n$. It is not hard to see that as a $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module, $\text{Alex}(G_1)$ has rank $n - 1$.

Since $H_1(N, \mathbb{Z})$ is a $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module, for each $1 \leq i \leq r$ and $1 \leq a, b \leq n$, $h_{ab}^i \overset{\text{def}}{=} f_1(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})\gamma_{ab}$ is a well-defined element in $H_1(N, \mathbb{Z})$. By our construction, the isomorphism $\text{Alex}(G_1) \cong H_1(N, \mathbb{Z})$ sends $g_0 g_0 g_0^{-1} g_0^{-1}$ to $\gamma_{ab}$. Denote the element in $\text{Alex}(G_1)$ corresponding to $h_{ab}^i$ by $\bar{h}_{ab}^i$. Let $H_2$ be the normal subgroup of $G_2$ generated by $\bar{h}_{ab}^i$ for all $1 \leq i \leq r$ and $1 \leq a, b \leq n$. Now, define $G_3$ to be the quotient $G_2 / H_2$. Then $\Ab(G_3) = \Ab(G_2)$, and by construction, we have an isomorphism of $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-modules,

$$\text{Alex}(G_3) \cong \text{Alex}(G_2) \otimes_{\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]} \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] / (f_1, \ldots, f_r).$$

Since $\text{Alex}(G_2)$ is of rank $n - 1$ as a $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$-module, $\text{Supp}(\text{Alex}(G_2)) = L(X)$. Therefore, $\text{Supp}(\text{Alex}(G_3))$ is the subvariety defined by $(f_1, \ldots, f_r)$, that is $Z$.

Finally, by Corollary 3.5 $\Sigma^1(G_3) = Z \cup \{1\}$. Thus we have finished the proof of the proposition.\qed

Remark 3.6. For a general connected finite CW-complex $X$, $\text{Hom}(\pi_1(X), \mathbb{C}^*)$ is a direct product of a complex torus with a finite abelian group. So one can ask a more general question, “for a finite abelian group $A$, and any (not necessarily irreducible) variety $Z \subset (\mathbb{C}^*)^n \times A$, does Theorem 1.1 hold?” The answer is yes, and we briefly describe how to adjust our previous constructions to this situation.

First, we can write the coordinate ring of $\text{Hom}(\pi_1(X), \mathbb{C}^*)$ as

$$R \overset{\text{def}}{=} \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, y_1, \ldots, y_m] / (y_1^i, \ldots, y_m^i).$$

Suppose $Z$ is defined by ideal $I = (f_1, \ldots, f_r)$ of $R$.

In the case of $k \geq 2$, we start with $M_0 = (S^1)^n \times Y$, where $Y$ is an Eilenberg-MacLane space of type $K(A, 1)$. Attaching a $k$-sphere to $M_0$, we obtain $M_1$. Then,
similar to the previous construction in section 2, we can attach \( r (k+1) \)-cells to \( M_1 \) corresponding to the functions \( f_1, \ldots, f_r \). Denote the resulting space by \( M_2 \). Now, it follows from Theorem \ref{thm:cohomology_jump_loci} and the same argument as before that \( M_2 \) satisfies the cohomology jump loci property in Theorem \ref{thm:cohomology_jump_loci}. Notice that \( Y \) can be chosen to be a finite-type CW-complex, but it may not be finite. Therefore, even though \( M_2 \) may not be a finite CW-complex, it is of finite type. However, by replacing \( M_2 \) by its \( k' \)-skeleton, where \( k' \) is sufficiently large, it serves as the example of Theorem \ref{thm:cohomology_jump_loci}.

In the case of \( k = 1 \), again it suffices to talk about group theoretic cohomology jump loci. We start with the group

\[
G_0 = \langle g_1, \ldots, g_n, h_1, \ldots, h_m \mid h_1^{l_1} = 1, \ldots, h_m^{l_m} = 1 \rangle.
\]

Let \( G_1 = \text{TFM}(G_0) \). Then \( \text{Alex}(G_1) = \text{Alex}(G_0) \) is a finitely generated \( R \)-module, locally with positive rank. Similar to what we did earlier in this section, by taking a quotient of \( G_1 \) by the subgroup corresponding to \( I \cdot \text{Alex}(G_1) \), we obtain a group \( G_2 \). Then \( \text{Alex}(G_2) \cong \text{Alex}(G_1) \otimes_R R/I \). According to Corollary \ref{cor:Alex_infty}, we have

\[
\Sigma^1(G_2) = Z \cup \{1\}.
\]

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