We present an alternative method for constructing a consistent perturbative low energy canonical formalism for higher order time-derivative theories, which consists in applying the standard Dirac method to the first order version of the higher order Lagrangian, augmented by additional perturbative Hamiltonian constraints. The method is purely algebraic, provides the dynamical formulation directly in phase space and can be used in singular theories without the need of initially fixing the gauge. We apply it to two paradigmatic examples: the Pais-Uhlenbeck oscillator and the Bernard-Duncan scalar field with self-interaction. We also compare the results, both at the classical and quantum level, with the ones corresponding to a direct perturbative construction applied to the exact higher order theory, after incorporating the projection to the space of physical modes. This comparison highlights the soundness of the present formalism.

Keywords: higher order theories; perturbative canonical formalism

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1. Introduction

Higher order time-derivative (HOTD) theories, including nonlocal theories, have a long history in physics. They continually reappear in new models of physical interest, particularly in the form of effective Lagrangians that describe small corrections to well established theories. For example, nonlocal effective field theories emerge when high energy degrees of freedom are integrated out\cite{1}. HOTD terms also appear in higher derivative gravity\cite{2}, noncommutative field theory\cite{3}, models derived from string theory\cite{4}, effective models for meson nucleon interactions\cite{5}, etc. Our concern in this work is the discussion of HOTD as perturbations of well established standard second order theories (to be called precursors), instead of considering them as fundamental ones. Then it is appropriate to view the resulting HOTD description as an effective field theory, valid within a given energy range, where a perturbative description is perfectly acceptable, besides of constituting a natural setting for ex-
tracting physical predictions.

Classical HOTD theories introduce more degrees of freedom than their precursors, which is better reflected in their Hamiltonian structure. One characteristic of these theories is the Ostrogradsky instability, the existence of unphysical runaway solutions, not expandible in powers of the parameter codifying the HOTD contribution. At the quantum level the changes in the canonical structure produce important differences, associated to several pathologies. Even when the HOTD terms are considered as small corrections to their precursors, as it is done here, their effect is qualitatively significant. We understand here the number of degrees of freedom as the number of initial conditions that must be given to fully determine the behavior of the system at any time. Naturally, this depends on the degree of the time derivative in the equation of motion, irrespective of the fact that we might, for example, have only one coordinate $x(t)$.

The most traditional approach for a canonical formalism for HOTD theories was developed by Ostrogradsky in 1850 in the context of regular theories. This construction is equivalent to finding a second order Lagrangian by introducing the appropriate auxiliary variables in the HOTD theories via Lagrange multipliers, and by applying the Dirac method. The canonical formalism thus obtained highlights the well known problems of these theories, such as instability, Hamiltonians unbounded from below and lack of unitarity at the quantum level. The Ostrogradsky method was extended to HOTD singular theories by Nesterenko.

However, the HOTD terms usually appear in the Lagrangian as small corrections, labeled by a small parameter $\gamma$. For this reason, perturbative schemes have been developed which identify the correct low-energy degrees of freedom contained in the theory and avoid the high-energy ones that produce all the inconvenience. A typical recent example of this is the Myers-Pospelov model, whereby HOTD operators of dimension five are introduced as small corrections to standard electrodynamics in order to describe possible minute signals of Lorentz invariance violation. An analogous situation occurs in Lorentz violating extensions for the Standard Model, which generalizes the previous situation to the standard model of particles plus gravity. HOTD theories are also present in the analysis of precision tests of electroweak interactions. Those research topics are the subject of a considerable number of experimental observations with ever increasing precision. A very useful toy model for exploring the different aspects of such perturbative methods is the Bernard-Duncan field, a generalization to field theory of the Pais-Uhlenbeck quantum mechanical model, a paradigmatic example of HOTD theories.

There are two main approaches for reducing a higher order Lagrangian to one containing only the low energy modes. One of them is based on the use of field transformations containing derivatives. The other approach is based on a Lagrangian which contains HOTD up to a certain order. The dynamics can be reduced to a second order one by introducing perturbative constraints. Within a Lagrangian perspective, these constraints become naturally generated by equations.
of motion. One of the explored possibilities is to eliminate the higher order derivatives in the Lagrangian using the equations of motion. Given that in general the equations of motion can not be introduced in the Lagrangian without distorting the variational principle, this approach can be applied only to certain special cases to obtain an approximate second order Lagrangian. Another possibility, explored in Refs. 12, is to project the Lagrangian constraints into the phase space of momenta and coordinates, and consider these projections as Hamiltonian constraints, restricting the Ostrogradsky Hamiltonian. The key this approach is that the perturbative Lagrangian constraints are truly projectable and form a set of second class constraints. But, as has been shown in Ref. 24, Lagrangian and Hamiltonian constraints are not directly related, and in general not projectable in the case of gauge theories. If there is a gauge symmetry this approach is not directly applicable, because the first class constraints are not projectable. For this reason it is necessary to first fix the gauge at the Lagrangian level to proceed with the construction.

Once the role of the Lagrangian constraints in the reduction of the phase space is appreciated, the emphasis of the problem shifts to finding more efficient methods of calculating the iterative steps which are required to obtain the sought approximation to a given order in $\gamma$. One of these alternatives is proposed in Refs. 4, 5, where the construction is performed basically in the coordinates-velocities (CV) space, avoiding an explicit projection of constraints. The perturbative Lagrangian constraints are directly implemented on the Noether energy of the HOTD theory, which become the time evolution generator in the constrained CV space. To find the canonical structure this constrained energy is considered as the corresponding Hamiltonian, and the dynamics is written in terms of generalized brackets among coordinates and velocities. Imposing that these brackets reproduce the perturbative equations of motion to the order considered, the corresponding algebra is determined. This step requires the solution of a set of second order differential equations, which become very involved beyond the first order in $\gamma$ and whose solution involves a good amount of guess work. The algebra in the CV space is subsequently rewritten in terms of canonical coordinates and momentum. Ref. 26 includes a very clear review of this work, together with that concerning the general problem of HOTD systems. There, the need to add constraints in order to make perturbative sense of HOTD theories is also emphasized. Constrained HOTD theories become free from the diseases that plague unconstrained HOTD ones. This has also been previously remarked in Ref. 27. Finally we mention Ref. 11 where the use of iterative solutions of the equation of motion is focused on the obtention of the final symplectic form in terms of the variables $q, \dot{q}$, or equivalently $\pi_0$ in the notation of Ref. 12. The substitution of equations of motion into the Lagrangian, which is normally forbidden, is justified here in virtue of the detailed construction of Ref. 12 which shows that this substitution really amounts to strongly imposing a set of second class constraints which lead to Dirac brackets. In particular the method proposed in Ref. 12 can be applied only to regular higher order Lagrangians. The above methods are based on the projection of the Lagrangian constraints into the phase
space of momenta and coordinates, and subsequently considering these projections as Hamiltonian constraints. But, as has been shown in Ref. [24], Lagrangian and Hamiltonian constraints are not directly related, and, what is more, in general they are not projectable. In principle, this restricts the applicability of this approach. At least, in the case of higher order gauge theories it is necessary to implement a gauge fixing before applying these approaches.

Previous arguments made clear that HOTD theories, in particular when considered as corrections to standard ones, require the imposition of perturbative constraints at a given level. This implementation is closely related to the basic question of which are the appropriate Feynman rules to calculate a given low-energy process. To this end it is necessary to understand the difference between HOTD theories in the context of effective field theories and their use as HOTD theories per se. In Ref. [18] such a difference is explored in the context of the Bernard-Duncan model[16]. In brief, given the full HOTD Lagrangian, the Mathews' theorem [28] leads to the Feynman rules as read directly from this Lagrangian, which will imply the use of the full propagator as is proved in Ref. [16]. Nevertheless, as we know, this theory has all the problems of the HOTD theories. Hence, to make it consistent as a perturbation of the standard scalar field, we should expand the propagator in powers of the coupling constant associated to the higher order term. With this manipulation we obtain a perturbative expression for the propagator of the usual scalar field, and because of the disappearance of the high-energy poles, the ghost degrees of freedom are no more present. The important point to be stressed here is that a perturbative expansion of the propagator of the higher order theory provides us with a reference to test the soundness of an effective Lagrangian.

In this paper we propose an alternative construction, which amounts to the application of the well established Dirac method [25] to the first order version of the HOTD theory, augmented by additional perturbative Hamiltonian constraints. This construction leads to the perturbative canonical formalism in a very systematic and simple way, working from the beginning in the corresponding phase space. The construction is purely algebraic and does not require the solution of any system of differential equations. Also, the method can be directly applied to gauge theories without the need of initially fixing the gauge, as would be the case in the constructions of Refs. [4], [12].

The organization of the paper is the following. In the next section we introduce the concept of perturbative Hamiltonian constraints, which allows the application of the Dirac method to obtain a consistent canonical formulation, exact to a given order in the perturbative parameter. There we also show that the additional (perturbative) contraints can be consistently implemented in the Hamiltonian formulation. The method is illustrated in the third section with the construction of a canonical formalism of arbitrary order in the perturbative parameter for the Pais-Uhlenbeck oscillator. The fourth section applies the method to the Bernard-Duncan theory[16] with a $\phi^4$ interaction, both at the classical and quantum level. Finally, the fifth section discusses the two particle quantum scattering in the model of the
previous section in order to compare the exact canonical theory, which contains ghosts, with the well defined perturbative one obtained with our method. We show that our canonical formalism recovers the results obtained by using perturbative propagators and ruling out the unphysical states in the exact, pathological, theory. The last section contains a summary of the main results and some general comments. In the Appendix we consider a simple model to illustrate how our method compares with the proposal in Ref. 4.

2. Extended Dirac approach: Hamiltonian perturbative constraints

Any HOTD Lagrangian can be rewritten as a first order one by introducing an adequate number of auxiliary variables, via Lagrange multipliers. Once this is done we can calculate the corresponding momenta using the usual definition. Some of these relations allow us to write a restricted set of velocities as functions of the coordinates and the momenta, while others yield constraints. The usual procedure to consistently define the dynamics is the use of the Dirac approach, and in this way we obtain the canonical formalism, which in the case of a HOTD theory is plagued with several pathologies. In our case, the HOTD terms turn out to be scaled by a parameter $\gamma$, which we assume small, in such a way that the HOTD contributions are considered as perturbations over a standard theory. Here it is possible to go directly to a perturbative Hamiltonian construction, because some of the relations that come from the definition of the momenta and the constraints are inhomogeneous in the perturbative parameter. These relations, multiplied by a power of $\gamma$, provide perturbative constraints valid up to a given power of $\gamma$. These new constraints can be treated as Hamiltonian ones, and added to the set of original primary constraints. Thus, besides those primary constraints generated by the definition of the momenta, this approach requires the introduction of additional primary constraints further imposed according to the order of $\gamma$ to which we decide to incorporate the HOTD corrections. From now on the procedure follows as in the usual Dirac analysis of constrained systems.

In the following we show that this is a consistent way of considering such constraints. To make this point clear, let us assume that we start with a Lagrangian $L = L(q, \dot{q})$, where $p = \partial L/\partial \dot{q}$, and that we want to introduce an external phase-space constraint, i.e. one not generated by the definition of the momenta, $f(q, p) = 0$. The most straightforward way of doing this is by reformulating the theory in an enlarged space $(q, p)$, where the auxiliary variable $p$ corresponds to the momentum. If the second order Lagrangian is regular, the procedure for this enlargement is given in Ref. 29, and leads to a well known first order Lagrangian of the form

$$L = p_i \dot{q}_i - H(q, p),$$

where $H(q, p)$ is the corresponding Hamiltonian. Here we can impose the additional
constraint by using a Lagrange multiplier. Thus we set
\[ L = p_i \dot{q}^i - H(q, p) + \lambda f(q, p). \] (2)
Next we apply the Dirac procedure to this Lagrangian, with \( q^i, p_i \) and \( \lambda \) considered as coordinates in the extended space. The definition of the momenta yields three primary constraints
\[ \pi_{qi} - p_i \simeq 0, \quad \pi^i_p \simeq 0, \quad \pi_\lambda \simeq 0, \] (3)
with the extended Hamiltonian
\[ H_p = H(q, p) - \lambda f(q, p) + u^i (\pi_{qi} - p_i) + v^i \pi^i_p + w\pi_\lambda, \] (4)
where \( u^i, v^i, w \) are arbitrary functions. The consistency under time evolution of the primary constraints fixes two of the arbitrary functions
\[ \{ \pi_{qi} - p_i, H \} \simeq 0 \rightarrow v_i = -\frac{\partial H}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i}, \] (5)
\[ \{ \pi^i_p, H \} \simeq 0 \rightarrow u^i = \frac{\partial H}{\partial p_i} - \lambda \frac{\partial f}{\partial p_i}, \] (6)
where \( \simeq \) indicates a weak equation. These equations generate a secondary constraint
\[ \{ \pi_\lambda, H \} \simeq 0 \rightarrow f \simeq 0. \] (7)
Thus, at this level we have the Hamiltonian
\[ H_D = H(q, p) - \lambda f(q, p) + \left( \frac{\partial H}{\partial p_i} - \lambda \frac{\partial f}{\partial p_i} \right) (\pi_{qi} - p_i) - \left( \frac{\partial H}{\partial q^i} - \lambda \frac{\partial f}{\partial q^i} \right) \pi^i_p + w\pi_\lambda, \] (8)
with the set of constraints
\[ \pi_{qi} - p_i \simeq 0, \quad \pi^i_p \simeq 0, \quad \pi_\lambda \simeq 0, \quad f \simeq 0. \] (9)

The Poisson bracket of an arbitrary function \( M(q, p) \) with \( H_D \) can be simply written as
\[ \{ M, H_D \} = \{ M, H \}' - \lambda \{ M, f \}', \] (10)
where \( \{ , \}' \) is the Poisson bracket in the \((q, p)\) subspace. Thus, the consistency condition for the secondary constraint \( f \) is
\[ \{ f, H_D \} = \{ f, H \}'. \] (11)
If \( \{ f, H \}' \simeq 0 \) there are no more constraints, and hence \( \pi_\lambda \) is a first class constraint. It generates an orbit of equivalent configurations and we can choose any point of this orbit by imposing a gauge fixing. It is convenient to take \( \lambda = 0 \). Besides this, the first two constraints in Eq. (3) are second class. We finally get, in the reduced space defined by the gauge fixing and these two second class constraint, that the dynamics is described by the usual Hamiltonian \( H_D^R = H(q, p) \), with only the constraint we want to impose, \( f(q, p) \simeq 0 \).
If \( \{f,H\}' \) is not weakly zero there is a new constraint

\[
\{\{f,H_D\}' , H_D\} = \{\{f,H\}' , H\}' - \lambda \{\{f,H\}' , f\}', \tag{12}
\]

whose consistency condition fixes the remaining arbitrary function \( w \)

\[
\{\{f,H\}' , H\}' - \lambda \{\{f,H\}' , f\}' \lambda \{\{f,H\}' , f\}' + \lambda^2 \{\{f,H\}' , f\}' - w \{\{f,H\}' , f\}' . \tag{13}
\]

Now we have the following set of second class constraints

\[
\pi_{qi} - p_i \simeq 0, \quad \pi_p \simeq 0, \quad \pi_\lambda \simeq 0, \quad f \simeq 0, \tag{14}
\]

\[
\{\{f,H\}' , H\}' - \lambda \{\{f,H\}' , f\}' \simeq 0. \tag{15}
\]

Using the first three constraints in (14) and the last one in (15) to partially reduce the phase space we get

\[
H^R_D = H(q,p) - \frac{\{\{f,H\}' , H\}'}{\{\{f,H\}' , f\}'} f, \tag{16}
\]

with \( f \simeq 0 \). In both cases, \( \{f,H\}' = 0 \) or \( \{f,H\}' \neq 0 \), the final result corresponds to considering the original Hamiltonian in the \((q,p)\) phase space, plus the external constraint as a Hamiltonian one in the Dirac approach.

\[
H(q,p) \quad \text{plus} \quad f(q,p) = 0 \quad \leftrightarrow \quad H_{Dirac} = H(q,p) + u f(q,p) . \tag{17}
\]

This construction shows that in the regular case the inclusion of external constraints involving momenta is equivalent to considering these constraints as primary ones in the Dirac formalism. The key for this demonstration is the construction of a first order Lagrangian with a \((q,p)\) configuration space. In the case of a singular Lagrangian, this first order Lagrangian with a \((q,p)\) configuration space can also be constructed following an extension of the scheme given in Lanczos\(^{29}\), developed in Ref.\(^{30}\). Using this construction and following the preceding discussion, we can show that in any case external constraints involving coordinates and momenta can be incorporated as primary Hamiltonian constraints in the framework of the Dirac method. In particular, this justifies the insertion of the perturbative constraints in the Dirac formalism, together with the ones naturally generated by the definition of the momenta.

In the following we illustrate the proposal by applying it to two systems: (i) the Pais-Uhlenbeck oscillator and (ii) the higher order scalar theory discussed in Ref.\(^{16}\) plus a self-interacting term.
We implement the proposal in the framework of the Pais-Uhlenbeck oscillator, which is a regular system. For arbitrary theories described by a first-order Lagrangian we have a well-defined consistent canonical approach, given by the Dirac method. For this reason, our first step will be to rewrite any higher order theory in terms of a first order Lagrangian by introducing auxiliary degrees of freedom together with the corresponding Lagrange multipliers. In this way we obtain a first order singular Lagrangian to which we can apply the Dirac method to obtain the canonical formalism. When we eliminate the auxiliary variables in this canonical formalism, by implementing the corresponding second class Hamiltonian constraints, we recover the standard Ostrogradsky approach. An alternative approach in the case of the Pais-Uhlenbeck oscillator based on complex canonical transformations plus subsequent reality conditions is presented in Ref. 31.

The second-order Lagrangian defining the Pais-Uhlenbeck oscillator is

\[ L = \frac{\dot{x}^2}{2} - \frac{\omega^2 x^2}{2} - \frac{\gamma}{2} \ddot{x}^2. \]  
(18)

The introduction of the additional degree of freedom \( z \), via the corresponding constraint, leads to the first-order Lagrangian

\[ L = \frac{z^2}{2} - \frac{\omega^2 x^2}{2} - \frac{1}{2} \gamma \dot{z}^2 + \lambda (z - \dot{x}), \]  
(19)

with coordinates \( x, z, \lambda \). The corresponding equations of motion are

\[ z + \gamma \ddot{z} + \lambda = 0, \]  
(20)

\[ -\omega^2 x + \dot{\lambda} = 0, \]  
(21)

\[ z - \dot{x} = 0. \]  
(22)

They imply only one perturbative primary constraint at order \( \gamma^n \)

\[ \gamma^n (z + \lambda) = 0, \]  
(23)

which leads to the secondary constraint

\[ \gamma^n (\dot{z} + \dot{\lambda}) = \gamma^n (\dot{z} + \omega^2 x) = 0. \]  
(24)

One can easily verify that eliminating \( z \), and \( \lambda \) from the resulting equations of motion produces the equation of motion for \( x \) directly obtained from Eq. (18). Now that we have a first-order theory, we can construct the canonical formalism using the Dirac approach. The corresponding momenta \( p_k : p, \pi, \kappa \) are

\[ p = \frac{\partial L}{\partial \dot{x}} = -\lambda \rightarrow \sigma_2 = \lambda + p \simeq 0, \]  
(25)

\[ \pi = \frac{\partial L}{\partial \dot{z}} = -\gamma \dot{z}, \]  
(26)

\[ \kappa = \frac{\partial L}{\partial \dot{\lambda}} = 0 \rightarrow \sigma_1 = \kappa \simeq 0. \]  
(27)
This set of equations gives the transformation of the momenta into coordinates and velocities. The equation (25) defines a perturbative constraint of order \( n \)

\[
\Phi^n = \gamma^n \pi \simeq 0,
\]

which means that we take \( \gamma^n = 0, \ m > n \). In this way the velocity \( \dot{z} \) is expressed in terms of the associated momentum. There are two primary constraints, and one perturbative primary constraint is imposed. The primary Hamiltonian is

\[
H = \frac{\pi^2}{2\gamma} - \frac{z^2}{2} + \frac{\omega^2 x^2}{2} - \lambda z + u(x + \lambda) + v\gamma^n \pi + w\kappa.
\]

The time evolution of the exact primary constraints and the perturbative one yields

\[
\{p + \lambda, H\} = -\omega^2 x + w \rightarrow w = \omega^2 x, \tag{30}
\]

\[
\{\kappa, H\} = z - u \rightarrow u = z, \tag{31}
\]

\[
\gamma^n \{\pi, H\} = \gamma^n (z + \lambda) \rightarrow \Psi^n = \gamma^n (z + \lambda) \simeq \gamma^n (z - p) \simeq 0. \tag{32}
\]

Two arbitrary functions are fixed, and a secondary constraint is generated. The Hamiltonian becomes

\[
H = \frac{\pi^2}{2\gamma} - \frac{z^2}{2} + \frac{\omega^2 x^2}{2} + z_1 p + v\gamma^n \pi + \omega^2 x\kappa. \tag{33}
\]

The consistency conditions for the secondary constraints generates a pair of towers of constraints of decreasing order in \( \gamma \), starting from the initial \( \Phi^n, \Psi^n \)

\[
\frac{d\Phi^n}{dt} \simeq 0 \rightarrow \Psi^n, \quad \frac{d\Psi^n}{dt} \simeq 0 \rightarrow \Phi^{n-1}, \quad \frac{d\Phi^{n-1}}{dt} \simeq 0 \rightarrow \Psi^{n-1}, \ldots \tag{34}
\]

according to

\[
\{H, \Phi^{n-m}\} = \Psi^{n-m}, \quad \{H, \Psi^{n-m}\} = \Phi^{n-m-1}. \tag{35}
\]

The general expression for the constraints can be written in terms of the Fibonacci polynomials (22)

\[
F_m(y) = \sum_{i=0}^{[m+1]/2} (-1)^i \left( \begin{array}{c} m+1 \cr i \end{array} \right) y^i, \quad F_0(y) = 1, \quad F_{-1}(y) = 1, \tag{36}
\]

where \([a]\) means the integer part of \( a \), and are

\[
\Phi^n = \gamma^n \pi, \tag{37}
\]

\[
\Psi^{n-m} = \gamma^{n-m} (z F_m (\gamma \omega^2) - p F_{m-1} (\gamma \omega^2)) \quad 0 \leq m \leq n, \tag{38}
\]

\[
\Phi^{n-m} = \gamma^{n-m} (\pi F_{m-1} (\gamma \omega^2) - x \gamma \omega^2 F_{m-2} (\gamma \omega^2)) \quad 1 \leq m \leq n. \tag{39}
\]

The Fibonacci polynomials satisfy

\[
F_{m+1}(y) = F_m(y) - y F_{m-1}(y), \quad F_0(y) = 1, \quad F_{-1}(y) = 1, \tag{40}
\]

which imply the recurrence relations

\[
\Psi^{n-m} = \gamma \Psi^{n-m-1} + \omega^2 \Psi^{n-m+1} \quad 0 \leq m \leq n, \quad \Psi^{-1} = \Psi^{n+1} = 0, \tag{41}
\]

\[
\Phi^{n-m} = \gamma \Phi^{n-m-1} + \omega^2 \Phi^{n-m+1} \quad 1 \leq m \leq n, \quad \Phi^{-1} = \Phi^{n+1} = 0. \tag{42}
\]
Using these recurrence relations we can express the complete set of constraints as proportional to the last ones, $\Psi^0$ and $\Phi^0$. Thus we finally get

$$\Psi_m = \omega^{-2m} F_{m-1} (\gamma \omega^2) \Psi^0,$$

$$\Phi_m = \omega^{-2m} F_{m-1} (\gamma \omega^2) \Phi^0,$$

with

$$\Psi^0 = z F_n (\gamma \omega^2) - p F_{n-1} (\gamma \omega^2),$$

$$\Phi^0 = \pi F_{n-1} (\gamma \omega^2) - x \gamma \omega^2 F_{n-2} (\gamma \omega^2).$$

It is important to emphasize that the final perturbative constraints $\Psi^0$, $\Phi^0$ imply all the remaining ones up to $\Psi^n$, $\Phi^n$, and for this they are taken as the independent perturbative constraints to the order considered.

For simplicity, the illustration of the complete Hamiltonian construction is restricted here to the case $n = 1$, defined by the external perturbative constraint $\gamma \pi \approx 0$. The non-perturbative constraints (25,27) remain valid for all orders. In this case the two independent perturbative constraints become

$$\Phi^0 \equiv \chi_1 = \pi - \gamma \omega^2 x,$$

$$\Psi^0 \equiv \chi_2 = z - (1 + \gamma \omega^2) p.$$

The system is now second class, and it is very easy to compute from Eq. (33) the Dirac Hamiltonian in the constrained space

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 (1 - \gamma \omega^2) x^2,$$

(together with the corresponding Dirac bracket

$$\{p, x\}_D = 1 - (1 + \gamma \omega^2) (1 + \gamma \omega^2) \gamma \omega^2 \approx (1 - \gamma \omega^2).$$

It is clear that $x$ and $p$ are not canonical conjugate variables. Instead we can use

$$x \to x, \quad p \to \tilde{p} = (1 - \gamma \omega^2) p,$$

which satisfy

$$\{\tilde{p}, x\}_D = (1 - \gamma \omega^2) \{p, x\}_D \approx 1.$$

Thus we get

$$\tilde{H} = \frac{1}{2} \tilde{p}^2 + \frac{1}{2} \omega^2 (1 + \omega^2 \gamma) x^2,$$

to first order in $\gamma$. Now we can read the frequency of the oscillator in the Hamiltonian, which coincides with that obtained from the perturbative equation of motion to the order considered.
4. Higher order scalar field theory: Bernard-Duncan field with a $\varphi^4$ interaction

We now consider a higher order field theory, given by a Lagrangian density which is a self-interacting generalization of the Pais-Uhlenbeck oscillator

$$\mathcal{L} = -\frac{1}{2} \varphi \Box \varphi - \frac{1}{2} m^2 \varphi + \frac{\gamma}{2} \varphi \Box^2 \varphi - \frac{\alpha}{4} \varphi^4,$$

$$\Box = \partial^\mu \partial_\mu.$$  (54)

The HOTD equation of motion is

$$\Box \varphi + m^2 \varphi - \gamma \Box^2 \varphi + \alpha \varphi^3 = 0.$$  (55)

Using an iterative procedure for the Pais-Uhlenbeck model, the above equation reduces to the following second order equation

$$\Box \varphi + m^2 (1 - \gamma m^2) \varphi + (1 - 4 \gamma m^2) \alpha \varphi^3 - 3 \gamma \alpha^2 \varphi^5 + 6 \gamma \alpha \varphi (\partial_\mu \varphi \partial^\mu \varphi) = 0.$$  (56)

This equation is of first order in $\gamma$.

We deal with this model following the prescription in the previous subsection, so that the Lagrangian (54) is first rewritten as a first-order one with respect to the time derivatives, by introducing an auxiliary variable $\psi = \partial_0 \varphi$ with a Lagrange multiplier $\lambda$

$$\mathcal{L}' = \frac{1}{2} \psi^2 - \frac{1}{2} \varphi \left( m^2 - \Delta \right) \varphi + \varphi \Delta \psi + \frac{\gamma}{2} \varphi \Delta^2 \varphi - \frac{\alpha}{4} \varphi^4 + \lambda (\dot{\varphi} - \psi) + \frac{\gamma}{2} \dot{\psi}^2.$$  (57)

The definition of the canonical momenta establishes

$$\pi_\psi = \gamma \dot{\psi},$$  (58)

and gives two Hamiltonian primary constraints

$$\chi_1 \equiv \pi_\lambda \simeq 0,$$  (59)

$$\chi_2 \equiv \pi_\varphi - \lambda \simeq 0.$$  (60)

In the first place we will consider this theory without any approximation in $\gamma$, and in the next subsection it will be reformulated using the Hamiltonian constraint approach.

4.1. The $\gamma$-non-perturbative formalism

Here we construct the canonical formalism using the Dirac approach. The primary Hamiltonian density becomes

$$\mathcal{H} = -\frac{1}{2\gamma} \pi_\psi^2 - \frac{1}{2} \psi (1 - 2 \gamma \Delta) \psi + \frac{1}{2} \varphi \left( m^2 - \Delta - \gamma \Delta^2 \right) \varphi + \frac{\alpha}{4} \varphi^4 + \lambda \psi + w (\pi_\varphi - \lambda) + u \pi_\lambda,$$

where $w$ and $u$ are arbitrary functions. The consistency conditions for the primary constraints only fix two of the arbitrary functions

$$\{\pi_\varphi - \lambda, H\} = - \left( m^2 - \Delta - \gamma \Delta^2 \right) \varphi - \alpha \varphi^3 - u \simeq 0,$$  (62)

$$\{\pi_\lambda, H\} = - \dot{\psi} + w \simeq 0.$$  (63)
where $H = \int d^3x \mathcal{H}$. No further constraints are generated, and thus the Hamiltonian density becomes

$$\mathcal{H} = \frac{1}{2\gamma}\pi_\psi^2 - \psi \left(\frac{1}{2} + \gamma \Delta\right) \psi + \left(\frac{\varphi}{2} - \pi_\lambda\right) \left(m^2 - \Delta - \gamma \Delta^2\right) \varphi + \frac{\alpha}{4}\varphi^4 + \psi \pi_\varphi - \alpha \pi_\lambda \varphi^3.$$  

(64)

The primary constraints are second class

$$\{\chi_1(x), \chi_2(x')\} = \delta(x - x'),$$  

(65)

and therefore we can directly use the reduced Hamiltonian density

$$\tilde{\mathcal{H}} = \frac{1}{2\gamma}\pi_\psi^2 + \psi \pi_\varphi - \frac{1}{2}\psi \left(1 + 2\gamma \Delta\right) \psi + \frac{1}{2}\varphi \left(m^2 - \Delta - \gamma \Delta^2\right) \varphi + \frac{\alpha}{4}\varphi^4,$$  

(66)

with the Dirac brackets

$$\{\varphi(x,t), \pi_\varphi(y,t)\}_D = \delta(x - y),$$  

(67)

$$\{\psi(x,t), \pi_\psi(y,t)\}_D = \delta(x - y).$$  

(68)

The theory discussed by Bernard and Duncan corresponds to $\alpha = 0$. We will consider now this case, which can be treated in an exact way. To do this, it is useful to decompose the field $\varphi$ in modes with well defined frequency and with covariant normalization

$$\varphi(x,t) = \int d^3p \left( a_p e^{i(p \cdot x - \omega_p t)} + a_p^\dagger e^{-i(p \cdot x - \omega_p t)} \right)$$

$$+ \frac{1}{\sqrt{2m_p}} \left( b_p e^{i(p \cdot x - \Omega_p t)} + b_p^\dagger e^{-i(p \cdot x - \Omega_p t)} \right).$$  

(69)

The characteristic frequencies result

$$\omega_p = \left(p^2 + \sqrt{1 + 4m^2 \gamma} - 1\right)^{1/2} / 2\gamma,$$  

(70)

$$\Omega_p = \left(p^2 - \sqrt{1 + 4m^2 \gamma} + 1\right)^{1/2} / 2\gamma.$$  

(71)

When $-1 < 4m^2 \gamma < 0$, both frequencies are real. When $4m^2 \gamma < -1$ both frequencies have an imaginary part. In the case $\gamma > 0$, both terms $\left(\sqrt{1 + 4m^2 \gamma} \pm 1\right)$, are positive so that $\omega_p$ is always real, while $\Omega_p$ turns out to be imaginary when $p^2 < \left(\sqrt{1 + 4m^2 \gamma} + 1\right) / 2\gamma$. Those sectors of the theory with complex frequencies give rise to runaway solutions.

From the canonical brackets

$$\{\varphi(x,t), \pi_\varphi(x',t)\} = i\delta(x - x'), \quad \{\psi(x,t), \pi_\psi(x',t)\} = i\delta(x - x'),$$  

(72)
together with the following equations of motion arising from (66)
\[ \dot{\phi} = \psi, \quad \dot{\psi} = \frac{1}{\gamma} \pi \phi, \quad \pi = (1 + 2 \gamma \Delta) \psi - \dot{\pi} \phi, \] (73)
which allow us to express the remaining fields \( \psi, \pi \) in terms of \( a_p, a_{p'}, b_p, b_{p'} \), we get
\[ \{ a_p, a_{p'} \} = \frac{1}{\sqrt{1 + 4 \gamma m^2}} \delta^3 (p - p'), \] (74)
\[ \{ b_p, b_{p'} \} = -\frac{1}{\sqrt{1 + 4 \gamma m^2}} \delta^3 (p - p'), \] (75)
\[ \{ a_p, a_{p'} \} = \{ a_{p}^\dagger, a_{p'}^\dagger \} = \{ b_p, b_{p'} \} = \{ b_{p}^\dagger, b_{p'}^\dagger \} = 0, \] (76)
which is the same result obtained in Ref. [16] with \( \gamma \to -\gamma \) and a different definition for the auxiliary field. Let us recall that any Poisson bracket between \( a_p, a_{p'} \) and \( b_p, b_{p'} \) is zero. With the above normalization the number operator is
\[ N = \sqrt{1 + 4 \gamma m^2} \int d^3 p \left[ a_{p}^\dagger a_p + b_{p}^\dagger b_p \right]. \] (77)

In terms of the momentum space fields the Hamiltonian becomes
\[ H = \frac{1}{2} \int d^3 p \left( (a_p a_{p'}^\dagger + a_{p'} a_p) \omega_p + (b_p b_{p'}^\dagger + b_{p'} b_p) \Omega_p \right). \] (78)

From here the canonical quantization is straightforward. The algebra of the creation and annihilation operators is
\[ \left[ a_p, a_{p'}^\dagger \right] = \frac{1}{\sqrt{1 + 4 \gamma m^2}} \delta^3 (p - p') \] (79)
\[ \left[ b_p, b_{p'}^\dagger \right] = -\frac{1}{\sqrt{1 + 4 \gamma m^2}} \delta^3 (p - p') \] (80)
\[ \left[ a_p, a_{p'} \right] = \left[ a_{p}^\dagger, a_{p'}^\dagger \right] = \left[ b_p, b_{p'} \right] = \left[ b_{p}^\dagger, b_{p'}^\dagger \right] = 0 \] (81)
and the normal ordered Hamiltonian operator results
\[ \hat{H}_N = \int d^3 p \left( a_{p}^\dagger a_p \omega_p + b_{p}^\dagger b_p \Omega_p \right), \] (82)
which acts on one-particle states according to
\[ \hat{H}_N a_{k}^\dagger |0,0\rangle = \frac{\omega_k}{\sqrt{1 + 4 \gamma m^2}} a_{k}^\dagger |0,0\rangle \] (83)
\[ \hat{H}_N b_{k}^\dagger |0,0\rangle = -\frac{\Omega_k}{\sqrt{1 + 4 \gamma m^2}} b_{k}^\dagger |0,0\rangle \] (84)
Due to the minus sign in Eqs. (80,84), the requirement of energy positivity implies a negative metric for states containing type \( b \) particles, spoiling physical unitarity and a consistent probabilistic interpretation. A normalized state with \( n \) particles
(s particles of type a, with momenta $p_1, \ldots, p_s$ and $(n - s)$ particles of type b with momenta $p_{s+1}, \ldots, p_n$) in the $\alpha = 0$ case is given by

$$|\Phi_{a,b}(p_1, p_2, \ldots, p_n) = (1 + 4\gamma m^2) \bar{\varphi} \left( \prod_{i=1}^{s} a_{p_i}^\dagger \right) \left( \prod_{i=s+1}^{n} b_{p_i}^\dagger \right) |0, 0\rangle. \quad (85)$$

The expression (84) shows that the minus sign in the commutation relations (80) leads to negative contributions for the energy from the excitations of type b.

### 4.2. The perturbative Hamiltonian constraint approach

Here we go back to the self-interacting case $\alpha \neq 0$. The expression for $\pi_\varphi$, equation (58), is non homogenous in $\gamma$, and hence generates a Hamiltonian perturbative constraint of order $\gamma^n$

$$\gamma^n \pi_\varphi = 0, \quad (86)$$

where $n$ is the order of the perturbative approximation we want to achieve. Thus, the primary Hamiltonian density for a perturbative approach of order $n$ in $\gamma$ is

$$\mathcal{H} = \frac{1}{2} \left[ \varphi^2 - \frac{1}{2} \bar{\varphi} \left( 1 + 2\gamma \Delta \right) \psi + \frac{1}{2} \bar{\varphi} \left( m^2 - \Delta - \gamma \Delta^2 + \frac{\alpha}{2} \varphi^2 \right) \varphi \right. \left. + \psi \pi_\varphi - \pi_\lambda \left( m^2 - \Delta - \gamma \Delta^2 + \alpha \varphi^2 \right) \varphi + \nu \gamma^n \pi_\varphi \right. \quad (87)$$

and the full set of constraints, primary and secondary, are given by (59) and (60) together with the chain

$$\Phi_0 = \gamma^n \pi_\varphi \simeq 0 \quad (88)$$

$$\Psi_0 = \gamma^n (\psi - \pi_\varphi) \simeq 0 \quad (89)$$

$$\Phi_1 = \gamma^{n-1} (\gamma \left( m^2 - \Delta + \alpha \varphi^2 \right) \varphi + \pi_\varphi) \simeq 0 \quad (90)$$

$$\Psi_1 = \gamma^{n-1} \left( \gamma \left( \Delta + m^2 + 3\alpha \varphi^2 \right) + 1 \right) \psi - \pi_\varphi \simeq 0 \quad (91)$$

which are obtained by requiring

$$\{ H, \Phi_{n-m} \} = \Psi^{n-m}, \quad \{ H, \Psi^{n-m} \} = \Phi^{n-m-1} \quad (92)$$

As a simple illustration, let us consider the case $n = 1$. The above chain produces the additional perturbative constraints

$$\Phi_1 = \gamma \left( m^2 - \Delta + \alpha \varphi^2 \right) \varphi + \pi_\varphi = 0 \quad (93)$$

$$\Psi_1 = \gamma \left( \Delta + m^2 + 3\alpha \varphi^2 + 1 \right) \psi - \pi_\varphi = 0 \quad (94)$$

which have to be considered together with the original ones (59) and (60).

The set (59), (60), (93) and (94) corresponds to four second class constraints, leading to the Dirac brackets

$$\{ \varphi, \pi_\varphi \}_D = \{ \varphi, \pi_\varphi \} - \{ \varphi, \Psi_1 \} C_{\varphi_\Phi_1}^{-1} \{ \Phi_1, \pi_\varphi \} - \{ \varphi, \chi_2 \} C_{\chi_\Psi_1}^{-1} \{ \Psi_1, \pi_\varphi \} - \{ \varphi, \chi_2 \} C_{\chi_\Psi_1}^{-1} \{ \Psi_1, \pi_\varphi \}, \quad (95)$$

\[ \]
where \( C_{ab} \) is the standard matrix of the Poisson brackets among the second class constraints. The final result in the \((\varphi, \pi_\varphi)\) reduced phase space is

\[
\{\varphi, \pi_\varphi\}_D = 1 - \gamma (m^2 - \Delta + 3\alpha\varphi^2),
\]

together with the Hamiltonian density

\[
\mathcal{H} = \frac{1}{2} \pi_\varphi (1 - 2\gamma \Delta) \pi_\varphi + \frac{1}{2} m^2 (1 + \gamma m^2) \varphi^2 - \frac{1}{2} (1 + 2\gamma m^2) \varphi \Delta \varphi (1 + 4\gamma m^2) \varphi^4 + \frac{\alpha}{4} + \frac{1}{2} \gamma \alpha^2 \varphi^6 - \gamma \alpha \left(3\varphi^2 (\nabla \varphi)^2 + 2\varphi^3 \Delta \varphi\right).
\]

The final Dirac bracket (96) is not canonical. To express the Hamiltonian density in terms of canonical variables, maintaining the original field \( \varphi \), it is necessary to apply a non canonical transformation

\[
\tilde{\varphi} = \varphi, \\
\tilde{\pi}_\varphi = (1 + \gamma (m^2 - \Delta + 3\alpha\varphi^2)) \pi_\varphi,
\]

such that

\[
\{\tilde{\varphi}, \tilde{\pi}_\varphi\}_D = 1 + \mathcal{O}(\gamma^2).
\]

In terms of this new momentum, which we call again \( \pi_\varphi \) in an abuse of notation, the Hamiltonian density (97) becomes

\[
\mathcal{H} = \frac{1}{2} \pi_\varphi (1 - 2\gamma (m^2 + 3\alpha\varphi^2)) \pi_\varphi + \frac{1}{2} m^2 (1 + \gamma m^2) \varphi^2 - \frac{1}{2} (1 + 2\gamma (m^2 + \alpha\varphi^2)) \varphi \Delta \varphi + \frac{\alpha}{4} (1 + 4\gamma m^2) \varphi^4 + \frac{1}{2} \gamma \alpha^2 \varphi^6.
\]

It is straightforward to verify that this Hamiltonian density yields the same equation of motion (56) as the original Lagrangian density (54), to first order in \( \gamma \). To close the discussion, we obtain from Eq. (101) the effective Lagrangian density to first order in \( \gamma \) in the configuration space

\[
\mathcal{L}_D = \pi_\varphi \dot{\varphi} - \mathcal{H} = -\left(\frac{1}{2} m^2 \varphi \Delta \varphi + m^2 (1 - \gamma m^2) \varphi^2 \varphi \right)^{\alpha (1 + 2\gamma m^2) \varphi^4 + 2\gamma \alpha \varphi^3 \varphi + \alpha^2 \gamma \varphi^6}.
\]

This Lagrangian density has to be compared with the exact one, given by Eq. (54). The effect of the HOTD term in this effective Lagrangian has been to produce modifications in the mass of the field \( \varphi \), in the coupling constant of the self-interaction \( \varphi^4 \), and has also generated two new interaction terms, one of order \( \gamma \varphi^4 \) with a derivative coupling and another of order \( \gamma \varphi^6 \). Normalizing the kinetic term via the substitution

\[
\varphi \rightarrow \tilde{\varphi} = (1 + \gamma m^2) \varphi,
\]

we get

\[
\mathcal{L}_D = -\frac{1}{2} \tilde{\varphi} \Delta \tilde{\varphi} - \frac{1}{2} m^2 (1 - \gamma m^2) \tilde{\varphi}^2 - \frac{\alpha}{4} \tilde{\varphi}^4 - \gamma \alpha \tilde{\varphi}^3 \varphi - \frac{1}{2} \alpha^2 \gamma \varphi^6.
\]
Before closing this subsection it is interesting to compare the final Lagrangian density \( (104) \), obtained via the perturbative Hamiltonian constraint method, with different alternatives previously proposed to reduce \( (54) \) to an effective first order form.

One such alternative is the double zero method \( 23 \). In this case we can use the equation for motion \( (55) \) to generate such a term, which leads to

\[
\tilde{L}_{DZ} = -\frac{1}{2} \phi \Box \phi - \frac{1}{2} m^2 \phi + \frac{\gamma}{2} \phi^2 \phi - \frac{\alpha}{4} \phi^4 - \frac{1}{2} \gamma \left( \Box \phi + m^2 \phi + \alpha \phi^3 \right)^2, \tag{105}
\]

and thus the effective Lagrangian to first order in \( \gamma \) results identical to \( (102) \), so that after the redefinition \( (103) \) reproduces \( (104) \).

Another possibility is to implement an appropriate derivative transformation in \( (54) \), to first order in \( \gamma \), given by

\[
\phi \rightarrow \phi = \left( \tilde{\phi} + \frac{1}{2} \gamma \Box \tilde{\phi} \right), \tag{106}
\]

which yields the following first-order Lagrangian density to order \( \gamma \)

\[
\mathcal{L} = -\frac{1}{2} \left( 1 + \gamma m^2 \right) \tilde{\phi} \Box \tilde{\phi} - \frac{1}{2} m^2 \tilde{\phi}^2 - \frac{\alpha}{4} \tilde{\phi}^4 - \frac{1}{2} \gamma \alpha \phi^3 \Box \tilde{\phi}. \tag{107}
\]

We can rewrite this expression in terms of the original field \( \phi \) perturbatively, to first order in \( \gamma \), by using equation \( (55) \)

\[
\tilde{\phi} \simeq \left( \phi - \frac{1}{2} \gamma \Box \phi \right) = \phi + \frac{1}{2} \gamma \left( m^2 \phi + \alpha \phi^3 \right). \tag{108}
\]

In this way we get the following Lagrangian density, to first order in \( \gamma \)

\[
\tilde{\mathcal{L}}_G = -\left( \frac{1}{2} + \gamma m^2 \right) \left( \phi \Box \phi + m^2 \left( 1 - \gamma m^2 \right) \phi^2 \right.
\]

\[
+ \frac{\alpha}{2} \left( 1 + 2 \gamma m^2 \right) \phi^4 + 2 \gamma \alpha \phi^3 \Box \phi + \alpha^2 \gamma \phi^6 \left. \right). \tag{109}
\]

This Lagrangian density is identical to \( (102) \) which also reproduces the one given by the double zero method. Summarizing, to first order in \( \gamma \) we have

\[
\mathcal{L}_D = \mathcal{L}_{DZ} = \tilde{\mathcal{L}}_G. \tag{110}
\]

5. The two-particle scattering in the Bernard-Duncan scalar field with a \( \phi^4 \) interaction

In this section we discuss the dispersion of two scalar particles, with a dynamics described by the Lagrangian density \( (54) \), which contains the interaction term

\[
\mathcal{L}_{\text{int}} = -\frac{\alpha}{4} \phi^4. \tag{111}
\]

To test the perturbative Hamiltonian constraint approach we analyze this quantum process following two different approaches. In the first place we directly compute the scattering amplitude at first order in \( \gamma \) using the results of the perturbative
Hamiltonian constraint method. After this we obtain the expression for the scattering amplitude exact in $\gamma$ from the Lagrangian (54), from which we derive the corresponding amplitude at first order in $\gamma$. We assume that the quantum Hamiltonians are defined with the normal order product, so that the tadpole diagrams are not considered.

It is interesting to study the dispersion of two scalar particles at first order in $\gamma$, which in the effective theory involves not only corrections to the original vertices, but also the new derivative vertex. To proceed with the first calculation we consider this process using the Feynman rules derived from the effective Lagrangian (102). It corresponds to the effective Hamiltonian (101), obtained using the perturbative Hamiltonian constraints approach, for which we have the following algebra for the creation and annihilation operators

$$\left[a_p, a_{p'}^\dagger\right] = \left(1 - 2\gamma m^2\right) \delta(p - p'),$$

such that the normalized two $a$-particle in and out states are

$$|\Phi_{\text{in}}\rangle = \frac{1}{\left(1 - 2\gamma m^2\right)} a_{p_1}^\dagger a_{p_2}^\dagger |0\rangle,$$

$$\langle \Phi_{\text{out}}| = \langle 0| a_{p_1'} a_{p_2'} \frac{1}{\left(1 - 2\gamma m^2\right)}.$$

According to this, the first order contributions to the scattering amplitude (see

$$\frac{i \left(1 - 2\gamma m^2\right)}{p^2 - m^2 \left(1 - \gamma m^2\right)}$$

$$-6i\alpha$$

$$6i\alpha \gamma \left(p_1'^2 + p_2'^2 + p_1^2 + p_2^2\right)$$

$$-360i\gamma\alpha^2$$

Fig. 1. Feynman rules for the effective scalar theory. Note that now we have two additional vertices, a $\varphi^4$ derivative vertex and a $\varphi^6$ vertex.
Fig. 1, are the ones given by the $\alpha^4$ vertex

$$S_{fi}^{(1)} = -\frac{3i (2\pi)^4 \alpha}{2 \sqrt{\omega_{p_1} \omega_{p_2} \omega_{p'_1} \omega_{p'_2}}} \delta^4 (p'_1 + p'_2 - p_1 - p_2) + O(\gamma^2),$$ (115)

and the derivative vertex

$$S_{fi}^{(1)} = \frac{6i (2\pi)^4 \alpha \gamma m^2}{\sqrt{\omega_{p_1} \omega_{p_2} \omega_{p'_1} \omega_{p'_2}}} \delta^4 (p'_1 + p'_2 - p_1 - p_2),$$ (116)

such that the total first order contribution results

$$S_{fi}^{(1)} = -\frac{3i (2\pi)^4 \alpha (1 - 4\gamma m^2)}{2 \sqrt{\omega_{p_1} \omega_{p_2} \omega_{p'_1} \omega_{p'_2}}} \delta^4 (p'_1 + p'_2 - p_1 - p_2).$$ (117)

At second order we also have two contributions (see Fig. 2). One of them corresponds to one loop with two $\alpha^4$ type vertices

$$S_{fi}^{(2)} = -\frac{9\alpha^2 (1 - 4\gamma m^2)}{8 \sqrt{\omega_{p_1} \omega_{p_2} \omega_{p'_1} \omega_{p'_2}}} \left( F(p_1 + p_2) + F(p'_1 - p_1) + F(p'_2 - p_1) \right) \times \delta^4 (p'_1 + p'_2 - p_1 - p_2),$$ (118)

$$F(p) = \int \frac{d^4 k}{k^2 - m^2 (1 - \gamma m^2)} \frac{1}{(p - k)^2 - m^2 (1 - \gamma m^2)},$$ (119)

Fig. 2. Feynman diagrams contributing to the two-particle scattering in the effective scalar theory, at first order in $\gamma$ and one loop approximation.
and the other to one \( \alpha^4 \) type vertex and one derivative vertex

\[
S_{fi}^{(2)} = -\frac{9\gamma\alpha^2}{2\sqrt{\omega_{p_1}\omega_{p_2}\omega_{p_1}\omega_{p_2}}}(G(p_1 + p_2) + G(p'_1 - p_1) + G(p'_2 - p_1)) \\
\times \delta^4(p'_1 + p'_2 - p_1 - p_2),
\]

\[G(p) \approx \int d^4k \frac{-\left(2m^2 + k^2 + (p - k)^2\right)}{k^2 - m^2 (1 - \gamma m^2)} \frac{1}{(p - k)^2 - m^2 (1 - \gamma m^2)}.\]

The complete expression for the scattering amplitude in this approximation is

\[
S_{fi}^{(2)} = -\frac{9\alpha^2(1 - 4\gamma m^2)}{2\sqrt{\omega_{p_1}\omega_{p_2}\omega_{p_1}\omega_{p_2}}}
\delta^4(p'_1 + p'_2 - p_1 - p_2) \\
\times \left((F + \gamma G)(p_1 + p_2) + (F + \gamma G)(p'_1 - p_1) + (F + \gamma G)(p'_2 - p_1)\right),
\]

\[(F + \gamma G)(p) = \int d^4k \frac{1}{k^2 - m^2 (1 - \gamma m^2)} \frac{1 - \left(2m^2 + k^2 + (p - k)^2\right)}{(p - k)^2 - m^2 (1 - \gamma m^2)}.\]

Note that \( F_b \) contains a quadratic divergence and a logarithmic divergence. One should regularize this quantity in order to renormalize the theory. However, the point we are interested in is comparing the results for the scattering amplitude using two different methods, computing in the effective Hamiltonian theory constructed using perturbative Hamiltonian constraints in the usual fashion and computing in the exact HOTD theory, but approximating the propagator and restricting the initial and final states. For this reason we do not make explicit here the renormalization issues.

We now consider the same process, but in the framework of the exact Lagrangian \(^{54}\). To compare with the results from the perturbative Hamiltonian constraints approach, in terms of the Fourier decomposition \(^{69}\), we consider only the asymptotic states corresponding to physical, \( a^- \)-type, particles, not including negative norm ghost contributions in the initial and final configurations. Thus, according to \(^{55}\), the \( in \) and \( out \) states are respectively

\[
|\Phi_{in}\rangle = \sqrt{1 + 4\gamma m^2}a_{p_1}^\dagger a_{p_2}^\dagger |0\rangle, \quad (124)
\]

\[
\langle\Phi_{out}| = \langle 0|a_{p'_1}^\dagger a_{p'_2}^\dagger \sqrt{1 + 4\gamma m^2}. \quad (125)
\]

Hence, after adding all contributions up to first order in \( \gamma \) see Figs. \(^{33,41}\), the
scattering amplitude at first and second order in $\alpha$ are respectively

\[ S_{fi}^{(1)} = -\frac{\alpha}{4} \int d^4x \langle 0 | a_{p_1'} a_{p_2'} : \varphi^4(x) : a_{p_1}^\dagger a_{p_2}^\dagger | 0 \rangle = -\frac{3i}{2 \sqrt{\omega_{p_1} \omega_{p_2} \omega_{p_1'} \omega_{p_2'}}} \delta^4(p'_1 + p'_2 - p_1 - p_2), \]

\[ S_{fi}^{(2)} = \frac{1}{2!} \left( -\frac{i\alpha}{4} \right)^2 \int d^4x_1 \int d^4x_2 \langle 0 | a_{p_1'} a_{p_2'} T \left( : \varphi^4(x_1) : : \varphi^4(x_2) : \right) a_{p_1}^\dagger a_{p_2}^\dagger | 0 \rangle = -\frac{9\alpha^2}{2 \sqrt{\omega_{p_1} \omega_{p_2} \omega_{p_1'} \omega_{p_2'}}} \delta^4(p'_1 + p'_2 - p_1 - p_2)
\times \left( \tilde{F}(p_1 + p_2) + \tilde{F}(p_1' - p_1) + \tilde{F}(p_2' - p_1) \right), \]

where

\[ \tilde{F}(p) = \int d^4k \frac{1}{\gamma k^4 + k^2 - m^2} \frac{1}{\gamma (k - p)^4 + (k - p)^2 - m^2}. \]

\[ \frac{i}{\gamma p^4 + p^2 - m^2} \]

\[ -6i\alpha \]

Fig. 3. Feynman rules for the HOTD Bernard-Duncan theory.

\[ + \text{Perm.} \]

\[ + \text{Perm.} \]

Fig. 4. Feynman diagrams contributing to the two-particle scattering in the HOTD Bernard-Duncan theory, to first order in $\gamma$ and one loop approximation.
At first order in $\gamma$ they reduce to (117) and (122), with $\tilde{F}(p) = (F + \gamma G)(p) + O(\gamma^2)$. Both computations, the one based on perturbative Hamiltonian constraints and the one obtained from a perturbative expansion of the propagator of the exact Lagrangian, where we have used a procedure similar to the one applied by Weinberg in Ref. 18, yield the same result at the order $\gamma$ considered. This is evidence of the soundness of the effective Hamiltonian construction proposed here.

6. Final remarks

In this article we have presented an alternative method for constructing a consistent effective Hamiltonian formalism for higher order Lagrangians, which gives a correct low energy approximation. It includes higher energy scale effects under the form of perturbative corrections in the framework of a second order theory, so that it is free from the pathological behavior characteristic of HOTD theories. This method results from an application of the standard Dirac procedure to deal with constrained systems which, together with the constraints generated by the definition of the momenta and their consistency conditions, incorporates a new set of perturbative constraints generated by the inhomogeneous relations in the perturbative parameter contained in the set of the original momenta definitions and constraints. The addition of these new constraints projects the dynamics of the original HOTD system into a stable subspace consistent with the chosen order of approximation. Our method is purely algebraic and does not require to solve any system of differential equations. Also, it can be directly applied to gauge systems without the necessity of initially fixing the gauge. Moreover, we conjecture that the brackets obtained from this procedure in phase space, when rewritten in the CV space via the corresponding equations of motion, would provide a solution for the system of differential equations (A.10), which is the starting point of the Eliezer-Woodard formulation 4. The resulting second order effective Lagrangians produce unitary theories after quantization. The study of the possible breaking of Lorentz covariance induced by the perturbative description is beyond the scope of the present work 16.

At the classical level this Hamiltonian formalism yields canonical equations of motion equivalent to the usual perturbative approximation for the Lagrangian equations of motion of the exact HOTD theory. Furthermore, from this canonical construction we can also derive a well behaved effective Lagrangian formalism. At the quantum level we recover the results obtained using the perturbative expression for the propagators, provided that the asymptotic state space is restricted to the physical one, ruling out the ghosts. The difference with this last approach is that now we have a well defined canonical formalism, instead of a non-consistent one where ghosts must be forbidden because asymptotic states and approximate propagators are used. This new formulation provides a consistent effective theory, which allows the implementation of all the usual manipulations for the construction of a quantum field theory.

An important characteristic of our approach is that it makes no use at all of the
Lagrangian equations of motion or any Lagrangian constraints to obtain the corresponding effective theory. This is a significant difference with previous proposals, particularly the methods presented in Refs. 4 and 12. Moreover, our formalism can deal with singular theories in a straightforward way, since we only have to add the corresponding non-perturbative relations to the total set of constraints and work within the Dirac framework in the usual way. Another characteristic is that the procedure is not an iterative one. At the start we choose the order of the approximation, which is defined by the chosen perturbative constraints, and the Dirac algorithm converges directly to the corresponding effective theory.

As working examples, we have applied the method to two paradigmatic models in HOTD theories: the Pais-Uhlenbeck oscillator and the Bernard-Duncan scalar field, in which we include a $\varphi^4$ interaction. The first example clearly shows the main features of the perturbative Hamiltonian constraint approach, and allows us to express the perturbative constraints in a very simple and closed way, in terms of Fibonacci polynomials. In the other example, closed expressions for the constraints are much more difficult to write and for this reason we restricted the construction only to first order in the perturbative parameter. In both cases the canonical equations of motion are equivalent to the Lagrangian perturbative ones.

From the Hamiltonian thus constructed we can obtain a second order effective Lagrangian. The quantum theory can be obtained either from the Hamiltonian formalism or from the effective Lagrangian approach, via the Matthews’ theorem which is well established for Lagrangians of the form here obtained.16 The calculation of the two-particle scattering shows that the effective theory constructed on the basis of perturbative Hamiltonian constraints gives the same results as the exact higher order theory, provided that in this last theory the space of states is restricted to the physical one, ruling out ghost states, and that the propagator is considered in terms of a perturbative expansion. Similar results are obtained in Ref. 18 although there a direct substitution of leading order equations of motion was performed in the Lagrangian to get the effective theory. As it has been clearly shown in Ref. 23 this procedure can be considered to be correct only when it effectively results in the addition of a double zero term to the original Lagrangian, or when a suitable equivalent derivative field transformation is found.

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Appendix A.

In this Appendix we explore the connection between Lagrangian and Hamiltonian perturbative constraints, in the context of a simple example of HOTD theory. Let
Perturbative Hamiltonian constraints for higher order theories

us consider the Lagrangian

\[ L_g = \frac{1}{2} (\dot{q}_i)^2 - \frac{1}{2} \omega^2 (q_i)^2 + \frac{\gamma}{2} \epsilon_{ij} \dot{q}_i \dot{q}_j, \quad i = 1, 2 \]  

(A.1)

For simplicity, we will construct the Hamiltonian formalism only to first order in \( \gamma \).

The exact equations of motion are

\[ \ddot{q}_i + \gamma \epsilon_{ij} q^{(3)}_j + \omega^2 q_i = 0 \]  

(A.2)

which, to first order in \( \gamma \), reduce to

\[ \ddot{q}_i = \gamma \omega^2 \epsilon_{ij} \dot{q}_j - \omega^2 q_i \]  

(A.3)

In this case, the approach of Jaen, Llosa and Molina \(^{12}\) and the one of Eliezer and Woodard \(^4\) are equivalent, as stated in Ref. \(^5\). For this reason we will consider only the second one, which is more adequate to establish the comparison. Following Ref. \(^4\) we calculate the Noether energy

\[ E(q, \dot{q}) = \frac{1}{2} \dot{q}_i^2 + \frac{1}{2} \omega^2 q_i^2 + \gamma \epsilon_{ij} \dot{q}_i \dot{q}_j, \]  

(A.4)

which is taken as the time evolution generator in the coordinate-velocity (CV) space. In principle \( E(q, \dot{q}) \) should be a projection on the CV space of the Hamiltonian which lives in a not yet known phase space. To first order in \( \gamma \) the above equation yields

\[ E_1 = \frac{1}{2} \dot{q}_i^2 + \frac{1}{2} \omega^2 q_i^2 + \gamma \omega^2 \epsilon_{ij} q_i \dot{q}_j. \]  

(A.5)

The method further assumes the existence of a fundamental bracket \( \{q_i, q_j\} \) in terms of which we can describe the temporal evolution, such that

\[ \dot{q}_i = \{q_i, E_1\} = \{q_i, q_j\} \frac{\partial E_1}{\partial q_j} + \{q_i, \dot{q}_j\} \frac{\partial E_1}{\partial \dot{q}_j}, \]  

(A.6)

\[ \ddot{q}_i = \{\dot{q}_i, E_1\} = \{q_i, q_j\} \frac{\partial E_1}{\partial q_j} + \{\dot{q}_i, \dot{q}_j\} \frac{\partial E_1}{\partial \dot{q}_j}. \]  

(A.7)

These brackets must also satisfy the consistency conditions

\[ \{\dot{q}_i, q_j\} + \{q_i, \dot{q}_j\} = \frac{d}{dt} \{q_i, q_j\}, \]  

(A.8)

\[ \{\dot{q}_i, \dot{q}_j\} = \left( \frac{1}{2} \frac{d^2}{dt^2} + 2 \omega^2 \right) \{q_i, q_j\} \]  

(A.9)

and thus we obtain the following system of differential equations defining the basic objects in our example

\[ \dot{q}_i = (\omega^2 q_j + \gamma \omega^2 \epsilon_{jk} \dot{q}_k) \{q_i, q_j\} + (\dot{q}_j - \gamma \omega^2 \epsilon_{jk} q_k) \{q_i, \dot{q}_j\}, \]  

\[ \ddot{q}_i = (\omega^2 q_j + \gamma \omega^2 \epsilon_{jk} \dot{q}_k) \left( \frac{d}{dt} \{q_i, q_j\} - \{q_i, \dot{q}_j\} \right) \]  

\[ + (\dot{q}_j - \gamma \omega^2 \epsilon_{jk} q_k) \left( \frac{1}{2} \frac{d^2}{dt^2} + 2 \omega^2 \right) \{q_i, q_j\}, \]  

(A.10)
where $\dot{q}_i$ is given by Eq. (A.3). Following this approach, it is necessary to find the adequate solution to the above system, which in this case must be analytical in $q_i$ and $\dot{q}_i$. Next, from this solution the relation between canonical momenta, coordinates and velocities must be inferred. Once this is achieved, the canonical formalism to first order in $\gamma$ is constructed by projecting the energy $E_1$ together with the brackets among coordinates and velocities in the phase space just defined. Additional simplification of the system (A.10) can be achieved by making the following ansatz

$$\{q_i, q_j\} = \gamma A_{ij}, \quad \{q_i, \dot{q}_j\} = \delta_{ij} + \gamma B_{ij},$$

(A.11)

to first order in $\gamma$. Even with the above simplification, some guess work has to be done in order to solve the Eqs. (A.10). It is clear that the complexity of the basic equations (A.10) will rapidly increase either when higher order approximations are considered or when more complicated systems are studied. We consider this as a shortcoming of the method proposed in Ref. [4].

For the purpose of comparing the method based on Lagrangian constraints, in the Eliezer and Woodard incarnation, with the one based on the Dirac approach, the expressions already obtained are enough. Now we will deal with the problem using the Dirac method augmented with perturbative Hamiltonian constraints. To apply the procedure we first rewrite the Lagrangian (A.1) in first order form by introducing the coordinates $z_i$ via the auxiliary coordinates $\lambda_i$

$$L_g = \frac{1}{2} (z_i)'^2 - \frac{1}{2} \omega^2 (q_i)^2 + \frac{\gamma}{2} \epsilon_{ij} z_i \dot{z}_j + \lambda_i (z_i - \dot{q}_i).$$

(A.12)

The canonical momenta are

$$p_{q_i} = -\lambda_i, \quad p_{\lambda_i} = 0, \quad \pi_{z_i} = \frac{\gamma}{2} \epsilon_{ij} \dot{z}_j.$$  

(A.13)

So we have six exact primary constraints, plus two perturbative ones

$$\gamma \pi_{z_i} = 0.$$  

(A.14)

Following with the Dirac method we demand the consistency of the constraints under time evolution and we finally arrive to a Dirac Hamiltonian. Once the second class constraints are imposed as strong relations it reduces to

$$H_C = \frac{1}{2} p_i^2 + \frac{1}{2} \omega^2 q_i^2,$$  

(A.15)

with the Dirac brackets

$$\{q_i, p_j\}_D = \delta_{ij}, \quad \{p_i, p_j\}_D = 0, \quad \{q_i, q_j\}_D = \gamma \epsilon_{ij}.$$  

(A.16)

To project this formalism in the CV space and compare with the Eliezer-Woodard approach, we perform the transformation $(q, \dot{q}) \to (q, p)$ which is given by the canonical equations of motion

$$\dot{q}_i = \{q_i, H_C\} = p_i + \gamma \omega^2 \epsilon_{ij} q_j$$  

(A.17)
Projecting $H_C$ we obtain

$$H_C = \frac{1}{2}\dot{q}_i^2 + \frac{1}{2}\omega^2 q_i^2 + \gamma\omega^2\epsilon_{ij}\dot{q}_i\dot{q}_j,$$

(A.18)

which coincides with the Noether energy restricted to the CV subspace (A.5). From the Dirac brackets (A.16) and the transformation (A.17) we can compute

$$\{q_i, \dot{q}_j\} = \{q_i, p_j + \gamma\omega^2\epsilon_{jk}q_k\} = \delta_{ij},$$

(A.19)

$$\{\dot{q}_i, \dot{q}_j\} = \{p_i + \gamma\omega^2\epsilon_{im}q_m, p_j + \gamma\omega^2\epsilon_{jk}q_k\} = 2\gamma\omega^2\epsilon_{ij}. \quad (A.20)$$

It is straightforward to verify that this set of brackets is indeed a solution of the set of differential equations (A.10).

References

1. A. O. Barvinsky, C. A. Vilkovisky, Nucl. Phys. B 282, 163 (1987); B 333, 471 (1990).
2. K. S. Stelle, Phys. Rev. D 16, 953 (1977); J. Julve and M. Tonin, Nuovo Cim. B 46, 137 (1978).
3. A. Connes, M. R. Douglas and A. Schwarz, JHEP 9802, 003 (1998); N. Seiberg and E. Witten, JHEP 9909, 032 (1999); N. Seiberg, L. Susskind and N. Toumbas, JHEP 0006, 044 (2000).
4. D. A. Eliezer and R.P. Woodard, Nucl. Phys. B 325, 389 (1989).
5. R. P. Woodard, Lect. Notes Phys. 720, 403 (2007).
6. H. Hata, Phys. Lett. B 217, 438 (1989); Nucl. Phys. B 329, 698 (1990).
7. P. Kristensen, C. Moller, K. Dan. Vidensk. Selsk. Mat-Fys. Medd. 27, 7 (1952).
8. M. Ostrogradsky Mem. Acad. St. Petersbourg, VI 4, 385 (1850).
9. J. M. Pons, Letters in Math. Phys. 17, 181 (1989).
10. V. V. Nesterenko, J. Phys. A: Math. Gen. 22, 1673 (1989).
11. T. C. Cheng, P.M. Ho and M.C. Yeh, Nucl. Phys. B 625 , 151 (2002).
12. X. Jaen, J. Llosa and A. Molina, Phys. Rev. D 34, 2302 (1986).
13. R C. Myers, M Pospelov, Phys. Rev. Lett. 90, 211601 (2003).
14. D. Colladay and V. A. Kostelecký, Phys. Rev. D 55, 6760 (1997); Phys. Rev. D 58, 116002 (1998); V. A. Kostelecký, Phys. Rev. D 69, 105009 (2004); V. A. Kostelecký and M. Mewes, Phys. Rev. D 80, 015020 (2009).
15. G. Altarelli, R. Barbieri and F. Caravaglios, Int. J. Mod.Phys. A13, 1031 (1998).
16. C. Bernard and A. Duncan, Phys. Rev. D 11, 848 (1975).
17. A. Pais and G. E. Uhlenbeck, Phys. Rev. 79, 145 (1950).
18. S. Weinberg, Phys. Rev. D 77, 123541 (2008).
19. H. Georgi, Nucl. Phys. B 361, 339 (1991).
20. C. Grosse-Knetter, Phys. Rev. 49, 6709 (1994).
21. D. Barua and S. N. Gupta, Phys. Rev. D 16, 413 (1977).
22. G. Schafer, Phys. Lett. 100a, 128 (1984).
23. B. M. Barker and R. F. O’Connell, Phys. Lett. A 78, 231 (1980).
24. C. Batlle, J. Gomis, J. M. Pons, N. Roman-Roy, J. Math. Phys. 27, 2953 (1986).
25. P. A. M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University Press, N. Y., 1964; E. C. G. Sudarshan and N. Mukunda, Classical Dynamics: A Modern Perspective, Wiley, New York, 1974; K. Sundermeyer, Constrained Dynamics, Springer-Verlag, New York, 1982.
26. J. Z. Simon, Phys. Rev. D 41, 3720 (1990).
27. H. J. Bhabha, Phys. Rev. 70, 759 (1946).
28. P. T. Matthews, Phys. Rev. 76, 684 (1949).
29. C. Lanczos. The Variational Principles of Mechanics, University of Toronto Press (1949).
30. H. Montani, R. Montemayor, Phys. Rev. D 58, 125018 (1998).
31. A. Déctor, H. A. Morales-Técotl, L. F. Urrutia and J. D. Vergara, SIGMA 5, 053 (2009).
32. M.E. Hoggat, M. Bicknell, Fibonacci Quart. 11 (1973) 457; K Dilcher, Fibonacci Quart. 25, 300 (1982); Yuan Yi and Wengpeng Zhang, Fibonacci Quart. 40, 314 (2002).