A First-Landau-Level Laughlin/Jain Wave Function for the Fractional Quantum Hall Effect

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(March 23, 2022)

Abstract

We show that the introduction of a more general closed-shell operator allows one to extend Laughlin’s wave function to account for the richer hierarchies (1/3, 2/5, 3/7 . . .; 1/5, 2/9, 3/13, . . ., etc.) found experimentally. The construction identifies the special hierarchy states with condensates of correlated electron clusters. This clustering implies a single-particle algebra within the first Landau level (LL) identical to that of multiply filled LLs in the integer quantum Hall effect. The end result is a simple generalized wave function that reproduces the results of both Laughlin and Jain, without reference to higher LLs or projection.
One of the more intriguing problems in condensed matter physics has been the explanation of the fractional quantum Hall effect (FQHE) [1,2] and the connection of this phenomenon with the integer case (IQHE) [1,3]. While the IQHE can be understood [4] qualitatively as a noninteracting electron problem, the FQHE must be connected with breaking of the degeneracy of the noninteracting ground state by the electron-electron Coulomb interaction. Laughlin provided a simple and physically appealing ground-state wave function for fractional fillings of the form $1/m$, $m$ an odd integer [5]. Others have attempted to extend his ansatz in order to account for the more complex pattern of minima in the resistivity seen experimentally, particularly the strong features at fractional fillings $1/3$, $2/5$, $3/7$, ...[6-9]. In particular, one effort that is remarkable for its numerical success is that of Jain and his collaborators [8], who introduced excitations into successively higher LLs that were later eliminated by numerical projection. Yet the need for such excitations is somewhat puzzling: they play no role in the final answer, nor is there a natural argument associating the large magnetic gaps of the IQHE with the small Coulomb splittings within the first LL.

In this letter we construct a physically appealing generalization of Laughlin’s ansatz that we believe greatly helps in reconciling the work of Laughlin, Jain, and others. The construction is entirely in the first LL and extends Laughlin’s wave function in a remarkably simple way: a closed $\ell$-shell in his treatment is replaced by a closed $\ell s$-shell, where $2s+1$ counts the number of electrons involved in the underlying clusters (or composites). When such wave functions are recast in their corresponding $(\ell s)j$ form, a shell structure analogous to filled LLs in the IQHE emerges, but with shell gaps determined by a pairing energy associated with the Coulomb force, and not by the magnetic energy. The net result is a simple generalization of Laughlin’s wave function that also reproduces the results of Jain without reference to higher LLs or projection.

We follow Haldane [6] in confining the electrons to the surface of a sphere, where they move under the influence of a perpendicular magnetic field generated by a monopole at the sphere’s center. The Dirac monopole quantization condition requires $\phi = 2S\phi_0$ where $\phi_0 = hc/e$ is the elementary unit of magnetic flux, with $2S$ an integer. The single-particle
wave functions are Wigner D-functions
\[
D_{S,q}^{(L)}(\phi, \theta, 0)
\]
where \( L \) is the Landau level index, \( L = S, S + 1, \ldots \). Thus there are \( 2S + 1 \) degenerate single-particle wave functions in the first LL, \( 2S + 3 \) in the second, etc. The wave functions for the first LL can be written as a monomial of power \( 2S \) in the elementary spinors \( u_q \)
\[
D^{(S)}_{S,q} = \left[ \frac{(2S)!}{(S + q)!(S - q)!} \right]^{1/2} u_{1/2}^{S+q} u_{-1/2}^{S-q} \equiv u_q^{2S}
\]
where
\[
u_q(\phi, \theta) = D_{1/2,q}^{(1/2)}(\phi, \theta) = \begin{cases} 
\cos(\theta/2)e^{i\phi/2}, & q = 1/2 \\
\sin(\theta/2)e^{-i\phi/2}, & q = -1/2 
\end{cases}
\]
(identical to the \( u \) and \( v \) of Haldane [6]). The Laughlin wave function [5,6] for \( N \) electrons corresponds to \( m \) powers of the closed shell, \( m \) odd,
\[
L_m(N) = \left[ \prod_{i<j} u(i) \cdot u(j) \right]^m \equiv \left[ \prod_{i<j} u(i) \cdot u(j) \right] L_{m-1}^{sym}(N)
\]
where the factoring of this expression into a closed-shell operator acting on a symmetric wave function will prove useful later. Here \( u(i) \cdot u(j) = u_{+1/2}(i)u_{-1/2}(j) - u_{-1/2}(i)u_{+1/2}(j) \) is the usual scalar product of two spin-1/2 tensors. Hence Eq. (4a) has total angular momentum zero and is the analog of a translationally invariant state in the plane. As there are \( m(N-1) \) appearance of each spinor \( u(i) \)
\[
2S = m(N-1).
\]
This wave function is identified with fractional filling \( 1/m: N/(2S+1) \rightarrow 1/m \) for large \( N \). Because it contains \( m \) powers of \( u(i) \cdot u(j) \), two-particle angular momenta are restricted to \( J_{ij} \leq 2S - m \). This excludes contributions from the largest Coulomb matrix elements \( \langle (SS)J_{ij}|V_c|(SS)J_{ij} \rangle \) corresponding to short-range electron-electron interactions, thus helping to minimize the energy. The wave function is “incompressible” because any reduction in \( S \), for fixed \( N \), forces electrons into configurations with \( J_{ij} > 2S - m \).
Below we will consider the process of compressing a Laughlin state, labeled by \( N \) and \( m \), by successively reducing the magnetic flux (and thus \( S \)), ultimately reaching the next Laughlin state \(( N, m - 2 \)\). The task before us is to identify other incompressible states encountered in this process - nondegenerate ground states distinguished energetically by their special symmetry. We begin by describing a classical caricature of this problem that may help the reader conceptually. Envision charges living on a one-dimensional regular lattice. If we start with 6 charges and 11 lattice sites, the obvious minimum energy configuration has the charges spaced uniformly, on sites 1,3, ...,11. In this configuration no two charges appear on neighboring sites, a condition analogous to the two-body Laughlin restriction on \( J_{ij} > 2S - m \). Now remove one lattice site at a time and look for similar unique configurations of lowest energy. The first such case arises for 8 sites, with the charges clustered in pairs, occupying sites 1,2,4,5,7,8. This configuration is completely specified by a condition on three-body correlations, that no three charges are allowed to occupy contiguous sites. For 7 sites another such state is found, comprised of clusters of three: the occupied sites are 1,2,3,5,6,7, and no four particles occupy contiguous sites. The pattern of denser states, clustering, and more complicated many-particle correlations is clear.

Now consider the analogous progression through states of increasing density in the FQHE, e.g., the 1/3, 2/5, 3/7, ... hierarchy. Eq. (4a) states that the first term in this series is produced by the action of an antisymmetric closed-shell operator acting on \( L_2^{sym} \), the (symmetric) half-filled shell to which the series converges. This operator produces \( N - 1 \) units of magnetic flux. This suggests constructing analogous antisymmetric operators for the "compressed" states 2/5, 3/7,... corresponding to reduced magnetic flux. A scalar operator that destroys magnetic flux is readily constructed: defining \( d_q = (-1)^{1/2+q} \frac{d}{du_{-q}} \), one finds \( d(i) \cdot d(j) u(i) \cdot u(j) = 2 \). This operator can be applied in such a way that it reduces the magnetic flux by one unit, provided \( N \) is even

\[
[d(1) \cdot d(2) \ldots d(N - 1) \cdot d(N)] L_2^{sym}
\]

(6)

The derivatives produce a condensate of particle pairs that - by necessity due to the
reduced magnetic flux - are unfavorably correlated spatially: \( d(i) \cdot d(j) \) acting on \((u(i)^{N-1} \otimes u(j)^{N-1})_J \) destroys one power of each spinor, reducing \( S \), but does not change \( J_{ij} \). This is reminiscent of our classical configuration of 6 clustered charges on 8 lattice sites, with the important difference that we have not yet imposed any condition that will keep the clusters separated. Such separation is important in minimizing the energy. Now Laughlin’s closed-shell operator separates single electrons

\[
\prod_{i<j=1}^N u(i) \cdot u(j) = A \left[ \prod_{i<j=1}^N u_{-1/2}(i)u_{+1/2}(j) \right] \quad (7a)
\]

where the antisymmetrization operator \( A \) has been introduced to make the corresponding generation for pairs obvious. An operator that produces and separates two-particle clusters is thus

\[
A \left[ \prod_{I<J}^{N/N_c} U_-(I)U_+(J) (d(1) \cdot d(2) \ldots d(N-1) \cdot d(N)) \right] \quad (7b)
\]

where \( N_c=2 \) and \( U_-(I = 1) = u_{-1/2}(1)u_{-1/2}(2), U_-(I = 2) = u_{-1/2}(3)u_{-1/2}(4), \) etc. Note that \( U_-(I)U_+(J) \), acting on a four-particle m-scheme configuration, increases \( 2S \) by one unit, lowers the magnetic quantum numbers of the particles in cluster \( I \), and raises those in cluster \( J \). Thus it spreads the clusters in azimuthal space. The generalization for all pairs in Eq. (7b) displaces each of the clusters relative to one another and increases \( 2S \) by \( N/N_c-1 \). Thus this operator is responsible for a fractional change in \( 2S + 1 \) of \( 1/N_c \) relative to that produced by Laughlin’s closed-shell operator of Eq. (7a).

We introduced an additional quantum number in Eq. (7b), \( N_c \), the clustering size. Clearly, for all \( N_c \) for which \( N/N_c \) is an integer, we can repeat the arguments given above. The generalization of the derivative pairs \( d(1) \cdot d(2) \) to any clustering size is

\[
L_d^I(I) \equiv \prod_{i<j=1}^{N_c} d(i) \cdot d(j), \quad I \text{ represents the set of particles } \{1, \ldots, N_c\}. \]

This provides one factor of \( d(i) \cdot d(j) \) for each possible pairing of particles in the cluster. The corresponding generalization of \( U_-(I) \) is clearly \( u_{-1/2}(1) \ldots u_{-1/2}(N_c) \). Thus for arbitrary \( N_c \) we obtain

\[
A \left[ \prod_{I<J}^{N/N_c} U_-(I)U_+(J) \prod_{I=1}^{N/N_c} L_d^I(I) \right] \quad (7c)
\]
which reduces to Eqs. (7a) and (7b) for \( N_c = 1 \) and 2, respectively. The substitution of Eq. (7c) for (7a) in the right-hand side of Eq. (4a) gives our generalized Laughlin wave function.

Eq. (7c), deduced from rather straightforward physical arguments, can be rewritten in a form that better illustrates its simple connections to Laughlin’s wave function. We introduce an angular momentum \((\ell, m_i)\) for the \( i \)th electron, where \( 2\ell + 1 = \frac{N}{N_c} \) is the number of clusters, and a spin \((s, q_i)\) with \( 2s + 1 = N_c \), the number of electrons in each cluster. Thus the total number of distinct pairs of magnetic indices \((m, q)\) is \((2\ell + 1)(2s + 1) = N\). Then Eq. (7c) can be rewritten and substituted into Eq. (4a) to give

\[
L_{m,N_c}(N) = \left[ \sum_{m',s,q'} \epsilon_{M_1 \ldots M_N} u_{m_1}^{2\ell}(1) \ldots u_{m_N}^{2\ell}(N) d_{q_1}^{2s}(1) \ldots d_{q_N}^{2s}(N) \right] \begin{pmatrix} L^{sym}_{m-1} \end{pmatrix}
\]  

(4b)

where \( \epsilon \) is the antisymmetric tensor with \( N \) indices and \( M_i = (m_i, q_i) \). Thus Eq. (4b) is obtained from Laughlin’s wave function (Eq. (4a)) by replacing a closed \( \ell \) shell \((2\ell + 1 = N)\) by a closed \( \ell s \) shell \(((2\ell + 1)(2s + 1) = N)\), where the spin is associated with the destruction of \( 2s \) units of magnetic flux accompanied by the clustering of sets of \( 2s + 1 \) electrons. Like Eq. (4a), it has total angular momentum zero. This \( \ell s \) operator, written as a Slater determinant, was first introduced in Ref. [10] from arguments based on symmetry. In deriving Eq. (7c) we have shown it is the mathematical manifestation of the underlying clustering.

As the derivatives decrease the number of flux quanta, the corresponding generalization of Eq. (5a) is

\[
2S = (m - 1)(N - 1) + 2\ell - 2s = (m - 1)(N - 1) + \frac{N}{N_c} - N_c
\]

(5b)

where \( m \) is an odd integer and \( N_c \) can take on any integral value that divides evenly into \( N \). For fixed cluster size \( N_c \) and large \( N \), the fractional filling becomes \( N/(2S + 1) \rightarrow N_c/((m - 1)N_c + 1) \). Thus the \( m=3 \) hierarchy corresponds to fractional fillings 1/3, 2/5, 3/7, \ldots; \( m=5 \) gives 1/5, 2/9, 3/13, \ldots; etc. These series converge to 1/\((m - 1) = 1/2, 1/4, \) etc., from the low-density side. One can also consider large \( N \) for a fixed number of clusters \( \bar{N}_c = N/N_c, N/(2S + 1) \rightarrow \bar{N}_c/((m - 1)\bar{N}_c - 1) \). This yields the \( m=3 \) series converging to
the half-filled shell from the high-density side, 1, 2/3, 3/5, \ldots; the $m=5$ series 1/3, 2/7, 3/11, \ldots; etc. For finite $N$, the natural division between the low- and high-density series is $N = N_c^2$ or, equivalently, $\ell = s$.

Thus we see the $m$ series, generated by antisymmetric closed-shell operators of Eq. (7c) acting on the $1/(m-1)$-filled shell, span the fractional fillings between $1/m$ and $1/(m-2)$. If one steps through this evolution for fixed $N$ by gradually reducing the magnetic flux, Eq. (7c) states that the cost of the resulting compression is the condensation of clusters of electrons that are unfavorably correlated spacially: within each cluster the allowed two-electron angular momenta $J_{ij}$ are those of the $m-2$ Laughlin state. The hierarchy states $(m, N_c)$ correspond to those special geometries where the ground state is filled by a condensate of clusters, each containing $N_c$ electrons. Any further compression of the system forces clusters of $N_c + 1$ electrons into unfavorable correlations characteristic of the denser $m-2$ Laughlin state, with a corresponding increase in the energy/particle. In this sense the states are incompressible. Continued compression produces larger clusters, with the final step being the single-cluster $1/(m-2)$ Laughlin state.

The Laughlin states are eigenstates of a two-body interaction that is infinitely repulsive for $J_{12} > 2S - m$. They also exclude three-body correlations for which $J_{123} > 3S - 3m$ and, in general, $J_{1...n} > nS - mn(n-1)/2$. The generalized Laughlin state for a given $(S, N)$ is an eigenfunction of an $N_c + 1$-body interaction that is infinitely repulsive for $J_{1...N_c+1} > (N_c + 1)(S - (m - 2)N_c/2 - 1)$ [11]. This constraint is more severe than the corresponding constraint for the $(m-2)$ Laughlin state. Thus, while the $N_c$ electrons within each cluster are forced into correlations characteristic of the denser $m-2$ Laughlin state, the system maintains correlations among $N_c+1$ or more particles more favorable than those of the $m-2$ Laughlin state. All of this is strikingly similar to our classical lattice caricature.

Eq. (4b) can be easily evaluated using

$$d_q^n \propto (-1)^{\frac{n}{2}+q} \left[ \frac{\kappa!}{(\frac{\kappa}{2}+q)(\kappa/2-q)!} \right]^{1/2} \left[ \frac{(\frac{\kappa}{2}+q_1)!(\frac{\kappa}{2}-q_1)!}{(\frac{n-\kappa}{2}+q+q_1)(\frac{n-\kappa}{2}-q-q_1)!} \right]^{1/2} u_{q+q_1}^{n-\kappa}. \quad (10)$$

Table 1 gives the overlaps with exact wave functions obtained by numerical diagonalization.
We include results through $N = 10$ and $2S = 24$, excluding trivial cases ($N \leq 3$ or $\bar{N} = 2S + 1 - N \leq 3$). The overlaps for $N_c \geq 2$ are $\geq 0.993$ in all cases. The corresponding results for Jain’s projected wave functions on the sphere are given in [8] for $(2S, N) = (11, 6), (19, 6)$, and $(16, 8)$, corresponding to states of filling $2/5, 2/7$, and $2/5$, respectively. Our results are identical to four significant digits in the $2/5$ cases, and differ by $0.0008$ for the $2/7$ state.

The reason for this agreement is that the algebra of filled LLs of the IQHE, which Jain employs, is identical to an algebra that exists entirely within the first LL, imposed by the clustering: the closed $\ell s$ shell of Eq. (4b) can be immediately recoupled in the form $(\ell s)j$, forming a set of $N_c = 2s + 1$ closed $j$ shells with $|\ell - s| \leq j \leq \ell + s$, that is, a shell structure identical to $N_c$ filled LLs of the IQHE, where the first LL contains $2(\ell - s) + 1$ electrons. We have shown that the operations employed by Jain - construction of higher LL wave functions followed by projection - are mimicking the effects of electron clustering within the first LL, a phenomenon we have argued is a natural consequence of the compression of a Laughlin state.

The origin of shell gaps within the first LL is easy to see. As the general case is similar, we illustrate the physics for $N_c = 2$. The operator of Eq. (7c) is then a product of two closed $(\ell s = 1/2)j$ shells, the lower one consisting of single-particle operators $[u^{2\ell}(i) \otimes d(i)]_{j = \ell - 1/2} = u^{2\ell - 1}(i)u(i) \cdot d(i) = u^{2\ell - 1}(i)$. Here we have noted that the derivatives act on $L^{sym}_{m-1}(N)$, for which $u(i) \cdot d(i)$ is the identity operator. The upper shell operators are $[u^{2\ell}(i) \otimes d(i)]_{j = \ell + 1/2} = [u^{2\ell - 1} \otimes [u(i) \otimes d(i)]_{1}]_{j = \ell + 1/2}$. But the operator $[u(i) \otimes d(i)]_{1}$, acting on $L^{sym}_{m-1}$, destroys an antisymmetric $u(i) \cdot u(j)$ pair, replacing it with the symmetric product $[u(i) \otimes u(j)]_{1}$, which then allows the maximum value of $J_{ij}$ to increase by one. Thus electrons $i$ and $j$ can approach more closely, at a corresponding cost in the energy. The general $N_c$ case is similar: each successive filled $(\ell s)j$ shell involves one additional pair-breaking operator: it is the energy of the broken pair that generates the shell gap [11].

In summary, we have shown than a particularly simple generalization of Laughlin’s wave function - replacement of a filled $\ell$ shell by a filled $\ell s$ shell - is a natural consequence of the
clustering of electrons when a Laughlin state is compressed. This clustering implies an \((\ell s)j\) algebra within the first LL level, analogous to multiply filled LLs in the IQHE, but with shell energies indexed by the number of broken \(u(i) \cdot u(j)\) pairs. We have thus reproduced the results of both Laughlin and Jain in a simple first-LL wave function, while providing an appealing physical argument in support of Jain’s construction.

The clustering also offers an interesting perspective on the spectroscopy of excited states in the FQHE. For example, Rezayi and Read [13] accounted for the first excited band of a series of “half-filled” systems as quasiparticle-hole valence excitations of a postulated shell Hamiltonian \(H = \vec{L}^2\). We recognize this as the special case of the \((\ell s)j\) algebra where \(\ell = s\), so that \(j = L = 0, 1, 2, \ldots \) and \(N = 1, 9, 16, \ldots \) All other \(m = 3\) hierarchy states can be arranged in similar series converging to the half-filled shell, corresponding to increasing \(\ell\) with \(\ell - s\) held fixed, and the arguments of Ref. [13] can be applied to each [11]. Thus the shell structure and shell gaps associated with electron clustering provide the starting point for exploring FQHE spectroscopy quite generally.

We thank David Thouless for many helpful discussions, and J. K. Jain for several comments. This work was supported in part by the U.S. Department of Energy.

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TABLE I. Overlaps $|\langle \Psi_{ex} | \Psi_{m,N_c} \rangle|$ for the generalized Laughlin wave functions of Eq. (4b). The third column lists the lesser of $N_c$ and $\tilde{N}_c$, with the latter appearing in (). Both corresponding fractional fillings are provided when $N_c = \tilde{N}_c$. The original Laughlin wave functions are those with $N_c = 1$.

| N  | 2S | $N_c$($\tilde{N}_c$) | m  | filling | $|\langle \Psi_{ex} | \Psi \rangle|$ |
|-----|----|----------------------|----|---------|------------------|
| 4   | 9  | 1                    | 3  | 1/3     | .9980            |
| 4   | 12 | 2                    | 5  | 2/9, 2/7| .9999            |
| 4   | 15 | 1                    | 5  | 1/5     | .9841            |
| 4   | 18 | 2                    | 7  | 2/13, 2/11 | .9992 |
| 4   | 21 | 1                    | 7  | 1/7     | .9741            |
| 5   | 12 | 1                    | 3  | 1/3     | .9991            |
| 5   | 20 | 1                    | 5  | 1/5     | .9974            |
| 6   | 9  | (2)                  | 3  | 2/3     | .9965            |
| 6   | 11 | 2                    | 3  | 2/5     | .9998            |
| 6   | 15 | 1                    | 3  | 1/3     | .9965            |
| 6   | 19 | (2)                  | 5  | 2/7     | .9964            |
| 6   | 21 | 2                    | 5  | 2/9     | .9928            |
| 7   | 18 | 1                    | 3  | 1/3     | .9964            |
| 8   | 12 | (2)                  | 3  | 2/3     | .9982            |
| 8   | 16 | 2                    | 3  | 2/5     | .9996            |
| 8   | 21 | 1                    | 3  | 1/3     | .9954            |
| 9   | 16 | 3                    | 3  | 3/7, 3/5| .9994            |
| 9   | 24 | 1                    | 3  | 1/3     | .9941            |
| 10  | 15 | (2)                  | 3  | 2/3     | .9940            |
| 10  | 21 | 2                    | 3  | 2/5     | .9980            |
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