The Efficiency Gap Does Not Satisfy the Efficiency Principle

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Abstract
We prove that the efficiency gap does not satisfy the efficiency principle. We assume no mathematical background, with the intent that a law scholar can read this short note.

1 Introduction
Partisan gerrymandering is seeing a lot of attention these days in many venues including the mathematical community, the news, and the legal community. The Metric Geometry Gerrymandering Group (MGGG) is a group of mathematicians that recently formed to study the issue of redistricting as well as train mathematicians to be expert witnesses in cases involving gerrymandering. For the first time, a federal court struck down the congressional map in North Carolina due to partisan gerrymandering. And last fall, the Supreme Court heard Gil v. Whitford, a case arguing that partisan gerrymandering occurred in the Wisconsin redistricting that followed the 2010 census.

An important component of the Gil v. Whitford case was that of the efficiency gap. The efficiency gap is a number which can be calculated after an election, and it gives an idea of how fairly the districts were drawn for that election. This number was introduced by Stephanopoulos and McGhee in [6] and has been discussed in legal and mathematical circles since. (See, for example, [1 5 4 2]). It is one of many tools that have been introduced to detect partisan gerrymandering.

In order to discriminate between these various tools, mathematicians and law scholars have proposed various properties that we’d like these tools to have. In [3] McGhee introduced the concept of the efficiency principle and in [5] Stephanopoulos and McGhee argued (incorrectly) that the efficiency gap satisfies the efficiency principle. In this note,
we give simple examples that the non-mathematician can approach to understand why
the efficiency gap does not satisfy the efficiency principle. A scholar already familiar
with how to calculate the efficiency gap can skip Section 2 and jump straight to Section
3 where we show that the efficiency gap is does not satisfy the efficiency principle. In
Section 4 we emphasize the importance of collaboration between the mathematical and
legal communities.

2 Mathematical Definitions

The efficiency gap is based on the concept of a wasted vote. There are two kinds of
wasted votes: the losing vote and the surplus vote. I’ve made a losing vote for candidate
A if I voted for candidate A but candidate B won my district. And I’ve made a surplus
vote if already a majority of the population in my district voted for candidate A, and
I made yet another vote for candidate A on top of that. Both the losing vote and the
surplus vote don’t help my candidate get elected, so in either case my vote is wasted.
The efficiency gap calculates the difference between the votes wasted for candidate A and
the votes wasted for candidate B, and then divides by the total number of votes.

We now make the definition in the above paragraph mathematically precise. In a
general election, suppose that there are n districts and let \( i \in \{1, 2, \ldots, n\} \). Let \( V_i^A \) be
the number of voters who voted for candidate A in district \( i \) and similarly let \( V_i^B \) be the
number of voters who voted for candidate B in district \( i \). We define \( L_i^A \) to be the number
of losing votes for candidate A in district \( i \):

\[
L_i^A = \begin{cases} 
V_i^A & \text{if candidate A lost in district } i \\
0 & \text{otherwise}
\end{cases}
\]

Similarly, we let \( L_i^B \) be the number of losing votes for candidate B:

\[
L_i^B = \begin{cases} 
V_i^B & \text{if candidate A lost in district } i \\
0 & \text{otherwise}
\end{cases}
\]

In order to help the reader unfamiliar with mathematically precise definitions, we
will use the following running example. We suppose there are two candidates: A and B,
and three districts which we number 1, 2, and 3. Our running example is the following
election:

Example 1.

|          | Votes for Candidate A | Votes for Candidate B |
|----------|-----------------------|-----------------------|
| District 1 | 2                     | 5                     |
| District 2 | 2                     | 5                     |
| District 3 | 8                     | 6                     |
For Example 1,

\[
\begin{align*}
V_A^1 &= 2 & V_A^2 &= 2 & V_A^3 &= 8 \\
V_B^1 &= 5 & V_B^2 &= 5 & V_B^3 &= 6
\end{align*}
\]

and thus,

\[
\begin{align*}
L_A^1 &= 2 & L_A^2 &= 2 & L_A^3 &= 0 \\
L_B^1 &= 0 & L_B^2 &= 0 & L_B^3 &= 6
\end{align*}
\]

The surplus votes for candidates A and B in district \( i \) are denoted \( S_A^i \) and \( S_B^i \) respectively. There is some slight nuance in whether there are an even number of voters in a district or an odd number of voters, and to address this nuance we must define the floor function.

**Definition 1.** The floor function \( \lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z} \), upon an input of a real number, outputs the largest integer which is less than or equal to that number:

\[
\lfloor x \rfloor = \max \{ n : n \in \mathbb{Z}, n \leq r \}
\]

**Example 2.** To orient the unfamiliar reader, we calculate:

\[
\begin{align*}
\lfloor 3.5 \rfloor &= 3 \\
\lfloor 4 \rfloor &= 4 \\
\lfloor 0.5 \rfloor &= 0
\end{align*}
\]

If the total number of voters in district \( i \) is \( T_i = 2m \) for an integer \( m \) (that is, \( T_i \) is even), a candidate must get \( m + 1 \) votes to win. Thus, any vote beyond \( m + 1 = \lfloor \frac{T_i}{2} \rfloor + 1 \) is a surplus vote. If the total number of voters in district \( i \) is \( T_i = 2m + 1 \) for an integer \( m \) (that is, \( T_i \) is odd), a candidate must get \( m + 1 \) votes to win. Thus, any vote beyond \( m + 1 = \lfloor \frac{T_i}{2} \rfloor + 1 \) is a surplus vote. Hence, in any case, we have

\[
\begin{align*}
S_A^i &= \begin{cases} 
V_A^i - (\lfloor \frac{T_i}{2} \rfloor + 1) & \text{if candidate A won in district } i \\
0 & \text{otherwise}
\end{cases} \\
S_B^i &= \begin{cases} 
V_B^i - (\lfloor \frac{T_i}{2} \rfloor + 1) & \text{if candidate B won in district } i \\
0 & \text{otherwise}
\end{cases}
\]

Thus, going back to Example 1, we can calculate:

\[
\begin{align*}
T_1 &= 7 \\
T_2 &= 7 \\
T_3 &= 14
\end{align*}
\]
so that

\begin{align*}
S^A_1 &= 0 & S^A_2 &= 0 & S^A_3 &= 0 \\
S^B_1 &= 1 & S^B_2 &= 1 & S^B_3 &= 0
\end{align*}

**Remark 1.** The definition of surplus vote is slightly different in [1], where any vote beyond 50% is considered a surplus vote. This makes for cleaner mathematics, but we believe that the definition given here is the one that law scholars use. We note that our result (that the efficiency gap does not satisfy the efficiency principle) is also true using the definition of surplus vote in [1], with just a slight tweaking of the examples in this note.

Now we can define the wasted votes and the efficiency gap:

**Definition 2.** The wasted votes for candidate $A$ is the sum of the losing votes and the surplus votes, across all districts $1, 2, \ldots, n$:

\[
W^A = \sum_{i=1}^n L^A_i + \sum_{i=1}^n S^A_i
\]

The efficiency gap is the difference between the wasted votes for candidate $A$ and the wasted votes for candidate $B$, divided by the total number of votes:

\[
EG = \frac{W_A - W_B}{T}
\]

where

\[
T = \sum_{i=1}^n T_i
\]

Using the Example [1] from above, we can see that

\begin{align*}
W^A &= L^A_1 + L^A_2 + L^A_3 + S^A_1 + S^A_2 + S^A_3 = 2 + 2 + 0 + 0 + 0 + 0 = 4 \\
W^B &= L^B_1 + L^B_2 + L^B_3 + S^B_1 + S^B_2 + S^B_3 = 0 + 0 + 6 + 1 + 1 + 0 = 8
\end{align*}

and

\[
T = T_1 + T_2 + T_3 = 7 + 7 + 14 = 28
\]

so that

\[
EG = \frac{W_A - W_B}{T} = \frac{4 - 8}{28} = \frac{-4}{28} = -\frac{1}{7}
\]
3 Results

We now show that the efficiency gap does not satisfy the efficiency principle. The following is directly from [5]:

[The efficiency principle] states that a measure of partisan gerrymandering “must indicate a greater advantage for (against) a party when the seat share for that party increases (decreases) without any corresponding increase (decrease) in its vote share” [3]. The principle would be violated, for example, if a party received 55% of the vote and 55% of the seats in one election, and 55% of the vote and 60% of the seats in another election, but a metric did not shift in the party’s favor. The principle would also be violated if a party’s vote share increased from 55% to 60%, its seat share stayed constant at 55%, and a metric did not register a worsening in the party’s position.

First we note that, given the examples the authors stated, their intended definition of efficiency principle should instead read:

The efficiency principle states that a measure of partisan gerrymandering must indicate both

1. a greater advantage for (against) a party when the seat share for that party increases (decreases) without any corresponding increase (decrease) in its vote share

2. and a greater advantage for (against) a party when the vote share decreases (increases) without any corresponding decrease (increase) in the seat share.

We will see that the efficiency gap violates both parts 1 and 2 of the efficiency principle. To show that it violates part 1, consider the following two elections:

| District | Votes for Candidate A | Votes for Candidate B |
|----------|-----------------------|-----------------------|
| District 1 | 2                     | 5                     |
| District 2 | 2                     | 5                     |
| District 3 | 8                     | 6                     |

Table 1: Election 1

First note that in both of these elections, candidate A has 12 votes while candidate B has 16 votes. Also note that in Election 1, candidate A wins one seat while in Election 2, candidate A wins two seats. Thus, if the efficiency gap satisfied part 1 of the efficiency principle, the efficiency gaps of these two elections would be different.
Now note that Election 1 is our first example, and we have already calculated the efficiency gap: $-\frac{1}{7}$. We now calculate the efficiency gap of the second election:

\[
\begin{align*}
W^A &= L_1^A + L_2^A + L_3^A + S_1^A + S_2^A + S_3^A = 0 + 0 + 4 + 0 + 0 + 0 = 4 \\
W^B &= L_1^B + L_2^B + L_3^B + S_1^B + S_2^B + S_3^B = 2 + 3 + 0 + 0 + 0 + 3 = 8 \\
\text{EG} &= \frac{W^A - W^B}{T} = \frac{4 - 8}{28} = -\frac{1}{7} \quad \text{(Election 2)}
\end{align*}
\]

Thus, since the efficiency gaps of Election 1 and Election 2 are both $-\frac{1}{7}$, the efficiency gap does not satisfy part 1 of the efficiency principle.

We now address part 2 of the efficiency principle. Consider the two election schemas below, Elections 3 and 4.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
        & Votes for Candidate A & Votes for Candidate B \\
\hline
District 1 & 2 & 6 \\
District 2 & 8 & 4 \\
\hline
\end{tabular}
\caption{Table 3: Election 3}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
        & Votes for Candidate A & Votes for Candidate B \\
\hline
District 1 & 3 & 7 \\
District 2 & 6 & 4 \\
\hline
\end{tabular}
\caption{Table 4: Election 4}
\end{table}

First note that in Election 3, Candidate A has 10 votes out of 20, while in Election 4 Candidate A has 9 votes out of 20. Thus, Candidate A has a higher percentage of the total votes in Election 3. Also note that in both elections, candidate A wins one seat. Thus, if the efficiency gap satisfied part 2 of the efficiency principle, the efficiency gaps of these two elections would be different.

In calculating the efficiency gap of Election 3, we have:

\[
\begin{align*}
W^A &= L_1^A + L_2^A + S_1^A + S_2^A = 2 + 0 + 0 + 1 = 3 \\
W^B &= L_1^B + L_2^B + S_1^B + S_2^B = 0 + 4 + 1 + 0 = 5 \\
\text{EG} &= \frac{W^A - W^B}{T} = \frac{3 - 5}{20} = -\frac{1}{10}
\end{align*}
\]
And finally, to calculate the efficiency gap of Election 4, we have:

\[ W^A = L_1^A + L_2^A + S_1^A + S_2^A = 3 + 0 + 0 + 0 \]
\[ W^B = L_1^B + L_2^B + S_1^B + S_2^B = 0 + 4 + 1 + 0 \]
\[ EG = \frac{W^A - W^B}{T} = \frac{3 - 5}{20} = -\frac{1}{10} \]

Thus, since the efficiency gaps of Election 1 and Election 2 are both \(-\frac{1}{10}\), the efficiency gap does not satisfy part 2 of the efficiency principle.

4 Final Remarks

We note that the heuristic and data-driven arguments that Stephanopoulos and McGhee make in [5], while incorrect, are not unreasonable. In their training, mathematicians repeatedly encounter reasonable-sounding arguments that can be shown to be mathematically incorrect. We also frequently run into the difficulty of producing a truly random sample. Thus we are accustomed to the need to prove any mathematical claim, as well as the need to prove that a sample is truly random. My sincere hope is that the legal and mathematical communities collaborate more closely on the challenging problem of partisan gerrymandering.

Acknowledgments

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