First- and Second-Order Models
of Recursive Arithmetics

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Abstract. We study a quadruple of interrelated subexponential subsystems of arithmetic $\text{WKL}_0^*$, $\text{RCA}_0^*$, $I\Delta_0$, and $\Delta RA_1$, which complement the similarly related quadruple $\text{WKL}_0$, $\text{RCA}_0$, $I\Sigma_1$, and $\text{PRA}$ studied by Simpson, and the quadruple $\text{WKL}_0^*$, $\text{RCA}_0^*$, $I\Delta_0(\exp)$, and $\text{EFA}$ studied by Simpson and Smith.

We then explore the space of subexponential arithmetic theories between $I\Delta_0$ and $I\Delta_0(\exp)$. We introduce and study first- and second-order theories of recursive arithmetic $\mathcal{AR}_A$ and $\mathcal{ARA}_2$ capable of characterizing various computational complexity classes and based on function algebras $A$, studied by Clote and others.

1 Introduction

Simpson in [20] studied the relations among four relatively weak subsystems of arithmetic: $\text{WKL}_0$, $\text{RCA}_0$, $I\Sigma_1$, and $\text{PRA}$. The first two are second-order, the remaining are first-order. What is remarkable, is that all four theories share the same provably recursive functions, namely the primitive recursive ones.

Later Simpson and Smith [21] found the same kind of relationship among the weaker quadruple of theories: $\text{WKL}_0^*$, $\text{RCA}_0^*$, $I\Delta_0(\exp)$, and $\text{EFA}$ (Elementary Function Arithmetic) where all four share the Kalmar-elementary functions as provably recursive.

The last four theories are discussed in a recent paper by Enayat and Wong [6] as a framework for the unification of model theory of the first and second-order arithmetic.

About the same time (summer of 2016) we have discovered even a weaker framework $\text{WKL}_0^*$, $\text{RCA}_0^*$, $I\Delta_0$, and $\Delta RA_1$ (the theory of provably recursive functions of $I\Delta_0$). The four theories are obtained by dropping the exponentiation from the four theories of Simpson and Smith. We have obtained the same kind of relationships among the theories.

This weakening is presented in sections 2 through 5 of this paper. In order to obtain the result we had to be rather careful with the coding of finite sequences, sets, and trees. We have also decided to add the Cantor’s pairing function to the language of arithmetic and to the basic axioms of PA (Peano Arithmetic).

We do not treat the functions (in extensions of PA) in the usual way as denotations of $n$-ary function symbols, but use the set-theoretical concept as sets of pairs (where the set-theoretical cartesian product is replaced by Cantor’s pairing). This is exactly how the functions are treated in second-order subsystems of arithmetic. We have used this in our reformulation of primitive recursive arithmetic PRA and the two above weaker
systems of recursive arithmetics. Instead of \( n \)-ary function symbols we have set constants (i.e. unary predicates). The advantage is that the respective quadruples of theories share similar language and no rather awkward translation is needed (as in [20]).

We can summarize our contributions as follows. The first one is the generalization obtained by dropping the assumptions of exponentiation and proving similar results as above. The second contribution is the introduction of first- and second-order theories of recursive arithmetics capable of characterizations of various complexity classes. These theories are all subexponential falling between the theories \( I\Delta_0 \) and \( I\Delta_0(\text{exp}) \) and so our study of the interplay of their first- and second-order model theories is an extension of the goal of Enayat and Wong.

Although it is possible to characterize the complexity classes in second-order arithmetic through set-comprehension axioms (see e.g. [25]), we have found it simpler and more convenient to replace the set comprehension by function existence axioms. This calls for the formulation of complexity classes as inductively defined function classes (so called function algebras, see [3]). A typical second-order function existence axiom is, for instance, the composition axiom: \( \forall g, h \in F \exists f \in F \forall x f(x) = g(h(x)) \).

We discuss the function algebras in section 6 where we discuss the ways of defining function algebras \( A(X) \). The basic question we had to solve was how to specify the framework for the definition of the operators of function algebras. We have tried several approaches and have finally settled for the use of the Clausal Language (CL). CL is a subset of PA (extended by definitions with functions) which we have developed in 1997 and use in the teaching of computer programming and verification courses at our university. Computer programming calls for a simple, readable, yet expressive language. CL gives us a uniform treatment of complex recursive schemes needed in operators of function algebras.

In section 7 we assign to each function algebra \( A(X) \) a first-order theory \( ARA_1 \), called the recursive arithmetic of \( A(X) \) where we can talk about the functions of the algebra. The relationship between the two is like the one between the function algebra of primitive recursive functions and the theory PRA.

In section 8 we discuss the provably recursive functions of recursive arithmetics \( ARA_1 \) which we show to be exactly the functions of the algebra \( A(X) \).

In section 9 we present the second order theories \( ARA_2 \) extending the first-order theories \( ARA_1 \) and prove that they share the same provably recursive functions.

In section 10 we characterize some basic complexity classes by means of recursive arithmetics.

2 Preliminaries

When talking about the interplay of first- and second-order arithmetic we have to decide on the often conflicting terminology and notation. We generally prefer Kaye’s [12] over Simpson’s [20]. Since the pairing function is central to the treatment of functions in this paper (all functions are unary, we do not have any introduced functions symbols other than those mentioned in the following paragraph), we have decided on one pairing
function and added it to the basic symbols. This saves the rather annoying constant referral to the theories extending \( L_1 \) with pairing.

### 2.1 Languages and Basic Axioms of our Fragments of First-Order Arithmetic.

The *first-order language of arithmetic* \( L_1 \) consists of the usual symbols 0, \( S \), \( + \), \( \cdot \), \(<\) plus the binary pairing function \((\cdot, \cdot)\) and its associated projection functions \( H \) and \( T \).

We use the modified Cantor’s pairing function defined by:

\[
(x, y) = z \leftrightarrow 2 \cdot z = (x + y) \cdot (x + y + 1) + 2 \cdot x + 2.
\]

This offsets the standard diagonal Cantor’s function by one and makes it a bijection \( \mathbb{N}^2 \mapsto \mathbb{N} \setminus \{0\} \). The further properties of pairing are:

\[
(x, y) = (x', y') \rightarrow x = x' \land y = y' \quad z > 0 \rightarrow \exists x, y < z \quad (x, y) = z.
\]

The symbols \( H \) and \( T \) are for the first (Head) and second (Tail) projection functions:

\[
H((x, y)) = x \quad H(0) = 0 \quad (P3-4)
\]

\[
T((x, y)) = y \quad T(0) = 0. \quad (P5-6)
\]

We let pairing to associate to the right, i.e. \((a, b, c)\) abbreviates \((a, (b, c))\) and drop the unnecessary parentheses in function applications involving pairing: Thus \( g(h(x, y)) \) abbreviates \( g(h((x, y))) \). This does not lead to confusion because apart from the six function symbols in \( L_1 \) we never use other function symbols and throughout the paper whenever we mention the term *function* we mean a special set (see Par. 3.2) and so our functions are effectively unary.

The properties of the usual symbols of arithmetic are:

\[
S(x) \neq 0 \quad S(x) = S(y) \rightarrow x = y \quad (N1-2)
\]

\[
x + 0 = x \quad x + S(y) = S(x + y) \quad (N3-4)
\]

\[
x \cdot 0 = 0 \quad x \cdot S(y) = x \cdot y + x \quad (N5-6)
\]

\[
x < 0 \quad x < S(y) \leftrightarrow x < y \land x = y \quad (N7-8)
\]

\[
x < y \lor x = y \lor y < x \quad 0 < x \rightarrow \exists y < x \quad S(y) = x. \quad (N9-10)
\]

We designate the universal closures of the properties \( N1-10+P0-6 \) by BASIC.

The reader will note that the group \( N1-10 \) deviates from the now standard basic axioms PA\(^-\) (see [12]) and from the basic axioms of Simpson [20]. We have decided on the axioms \( N1-9 \) of Shoenfield [19]. The reasons behind the choice are that the axioms for \( S, +, \cdot \) and \(<\) are actually recurrences (as opposed to the algebraic properties of PA\(^-\)). The recurrences become important in our investigation in section 6 of axiomatization of small fragments of arithmetic with function symbols corresponding to the inductively defined classes of functions (think of primitive recursive functions and their associated theory PRA). With our choice of \( L_1 \) we, for instance, dispense with the annoying translation between the standard language of arithmetic and that of PRA (see for instance [20]).

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2. Preliminaries

The only extensions of the language of arithmetic $L_1$ discussed in this paper are with set constants $X$ (unary predicate symbols). The languages are designated by $L_1(X)$. We will almost always use the set constants in the form $t \in X$ instead of the predicate applications $X(t)$.

2.2 Language of Second-Order Arithmetic. The language for the theories of second-order arithmetic discussed in this paper is the two sorted language $L_2$ which extends $L_1$ with variables ranging over sets and permits set quantification in formulas. For a set variable $X$ and a first-order term $t$ the atomic formula $t \in X$ is in $L_2$. We will use the set identity $X \subseteq Y$ as an abbreviation for $\forall x(x \in X \rightarrow x \in Y)$, $X = Y$ abbreviates $X \subseteq Y \land Y \subseteq X$, and the relation $X < Y$ is an abbreviation for $\forall x(x \in X \rightarrow x < Y)$. We often write the last quantifier as $\forall x \in X x < Y$.

2.3 Structures for $L_1$ and $L_2$. A structure for the language $L_1$ of first-order arithmetic is a tuple $\mathcal{M} = (\mathcal{M}, 0^\mathcal{M}, 1^\mathcal{M}, +^\mathcal{M}, \cdot^\mathcal{M}, -, ^\mathcal{M}, \leq^\mathcal{M}, =^\mathcal{M}, \star^\mathcal{M}, \cdot^\mathcal{M}, H^\mathcal{M}, T^\mathcal{M})$. We designate by $\mathbb{N}$ both the standard structure for $L_1$ satisfying BASIC as well as its domain of natural numbers. A structure for the language $L_2$ of second-order arithmetic $(\mathcal{M}, S)$ with $\mathcal{M}$ a first-order structure for $L_1$ and $S \subseteq P(\mathcal{M})$. The set variables of $L_2$ range over the elements of $S$.

The structures for the first order languages extended with set constants $L_1(X)$ are $(\mathcal{M}, X)$ with subsets of $\mathcal{M}$ assigned as meanings to the constants $X$ (note that we identify the constant symbols with their denotations). Although such structures look similar to the second-order structures our notation is a standard one (see [13] page 3 for this treatment). We have the following obvious theorem:

2.4 Proposition. Let $(\mathcal{M}, X)$ be a first-order structure for $L_1(X)$ and $(\mathcal{M}, S)$ a second-order structure (for $L_2$) such that $\{X\} \subseteq S$. If $\varphi(\vec{v})$ is formula of $L_2$ without second-order quantifiers with all free (first and second-order) variables among $\vec{v}$ then for every $\vec{v} \in \mathcal{M} \cup \{X\}$ we have $(\mathcal{M}, X) \models \varphi(\vec{v})$ iff $(\mathcal{M}, S) \models \varphi(\vec{v})$.

The reader will note the subtlety that the occurrence of the sequence $(\vec{v})$ in the second-order satisfaction relation is an assignment of values to variables of $\varphi(\vec{v})$ whereas in the first-order satisfaction relation the set constants in $\vec{v}$ replace the corresponding set variables in $\varphi$ (which thus becomes a formula in $L_1(X)$) and only the first-order values from $\vec{v}$ are in the assignment.

2.5 Induction and Related Principles. We designate the usual sets of arithmetical formulas $\Sigma_n$, $\Pi_n$, $\Delta_n$ without superscripts when they do not contain set variables or constants, i.e. if they are in the languages $L_1$. The same with superscripts, e.g. $\Sigma^0_n$, include also the formulas with the set variables and constants. We will often call as arithmetical, also the formulas which are only equivalent in some (usually implicitly understood) structure or theory to a formula in the proper syntactic form. In addition to $\Delta_0$ formulas (with superscript or not), which are the usual bounded formulas, we will designate a formula as $\Delta_1$ (possibly with superscript) only relatively to a structure or a theory because
such a formula must satisfy the additional constraint:
\[ \forall x (\varphi(x) \leftrightarrow \psi(x)) \]  
with \( \varphi(x) \in \Sigma_1 \) and \( \psi(x) \in \Pi_1 \) (both possibly with superscripts).

For a formula \( \varphi(x) \) we designate by \( I[\varphi(x)] \) the induction formula:
\[ \varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x). \]

We designate by \( B[\varphi(x, y)] \) the collection formula:
\[ \forall x < a \exists y \varphi(x, y) \rightarrow \exists b \forall x < a \exists y < b \varphi(x, y) \]
and by \( C[\varphi(x)] \) the comprehension formula:
\[ \exists X \forall x (x \in X \leftrightarrow \varphi(x)). \]

In the comprehension formula \( \varphi(x) \) may not contain \( X \) as a parameter. In all three kinds of formulas \( \varphi \) may contain additional parameters.

When we call the three kinds of formulas principles (for example: the induction principles \( I[\varphi(x)] \)) then we understand the formulas to be universally closed.

For \( \Gamma \) one of \( \Sigma_1, \Delta_0 \) (possibly with superscripts) we will designate by \( I[\Gamma], B[\Gamma], \) or \( C[\Gamma] \) the sets of corresponding induction, collection, or comprehension principles for \( \varphi \in \Gamma \).

For \( \Gamma \) one of \( \Delta_0, \Delta_1 \) (possibly with superscripts) we will designate by \( I[\Gamma] \) (\( C[\Gamma] \)) the sets of universal closures of \( \Delta \rightarrow I[\varphi] \) (\( \Delta \rightarrow C[\varphi] \)) for \( \varphi \in \Gamma \).

### 2.6 Fragments of Arithmetic with Limited Induction.

For \( \Gamma \) one of \( \Delta_0, \Delta_1, \Sigma_1 \) (possibly with superscripts) we define the theory \( I[\Gamma] := \text{BASIC} + I[\Gamma] \).

We define the theory \( B\Sigma_1 := I\Delta_0 + B[\Sigma_1] \) (possibly with superscript).

For \( \Gamma \) one of \( \Delta_0, \Delta_1 \) we define the second-order theory \( C\Gamma := \text{BASIC} + \text{IND} + C[\Gamma] \) where the induction axiom IND is the universal closure of
\[ 0 \in X \land \forall x (x \in X \rightarrow S(x) \in X) \rightarrow x \in X. \]  

We say that a first-order theory \( T \) in \( L_1(\overline{X}) \) is inductive if the induction principles of \( T \) hold also for the formulas containing the set constants \( \overline{X} \).

### 3 The Second-Order Theory \( \text{RCA}_0^- \)

For the theory \( C\Delta_0^0 \) we have:

#### 3.1 Proposition.

1. \( C\Delta_0^0 \vdash I[\Delta_0^0] \).
2. \( C\Delta_0^0 + C[\Delta_1^0] \vdash I[\Delta_1^0] \).
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Proof

1) Prove induction for \( \varphi(x) \in \Delta^0_1 \) by comprehension and then use IND.

2) The same as 1) but with \( \Delta^0_1 \)-comprehension.

In the following we will strengthen the weak base theory \( \mathrm{C} \Delta^0_0 \) by axioms asserting existence of functions.

3.2 Functions.

As mentioned above throughout this paper a “function” means a set acting like a function in the set-theoretical sense where instead of set pairs \((x, y)\) we use pairing \((x, y)\). We use the symbols \(f, g, h\), as set variables in second-order contexts or as set (unary predicate) constants in first-order contexts.

Within theories extending \( \mathrm{C} \Delta^0_0 \) we define the property \( f \text{ is a function} \), in writing \( f \in F \) as follows:

\[
f \in F \leftrightarrow \forall x \exists y(x, y) \in f \land \forall x, y, y'(x, y) \in f \land (x, y') \in f \rightarrow y = y' \land 0 \notin f .
\]

We will often abbreviate \((x, y) \in f \) to \( f(x) = y \). Function term \( f(t) \) used in an atomic formula \( \varphi(f(t)) \) should be understood as abbreviation for the unnested form: \( \exists y(f(t) = y \land \varphi(y)) \) or \( \forall y(y = f(t) \rightarrow \varphi(y)) \).

We use functions also in first-order theories in the languages \( L(X) \) where \( f \in F \) should be viewed as a schema of abbreviations for the RHS of the above definition with the metavariable \( f \) ranging over set constants.

The expression \( f \text{ is a function in } \mathcal{N} \) where \( \mathcal{M} \) is a first- or second-order structure means \( f \in F^\mathcal{N} \).

With \( T \) being a first- or second-order theory we say that the function \( f \) is polynomially bounded if there is a term \( t(x) \in L_1 \) such that \( T \) proves \( \forall x, y(f(x) = y \rightarrow y \leq t(x)) \). The function \( f \) is non-growing if \( T \vdash \forall x, y(f(x, p) = y \rightarrow y \leq p) \).

3.3 Some Operators on Functions.

For the set variables \( f, g, \) and \( h \), we introduce the following abbreviations (named on the right) as the universal closures of the following formulas:

\[
\begin{align*}
    h(x) &= z \land g(z) = y \rightarrow f(x) = y & (f := g \circ h) \\
    (g(y) = z \rightarrow f(0, y) = z) \land \\
    (f(x, y) = v \land \ h((x, v), y) = w \rightarrow f(S(x), y) = w) \land \ \\
    f(0) &= 0 \quad (f := \mathrm{PR}(g, h)) \\
    f(0) &= S(0) \land (f(x) = y \rightarrow f(S(x)) = y + y) \. \quad (f := E)
\end{align*}
\]

In the second-order context we will use a name of a function operator as a name of the axioms asserting the closure under the operator:

\[
\begin{align*}
    \forall g, h \in F \exists f \in F_f := g \circ h & \quad (\circ) \\
    \forall g, h \in F \exists f \in F_f := \mathrm{PR}(g, h) & \quad (\mathrm{PR}) \\
    \exists f \in F_f := E . & \quad (E)
\end{align*}
\]
We will see below that the certain function operators are equivalent to set comprehension. The first such equivalence is given in the following theorem:

**3.4 Theorem.** Over $\mathsf{C[\Delta^0_1]}$ the theory $\mathsf{C[\Delta^0_1]}$ is equivalent to the axiom $\circ$.

**Proof** $\mathsf{C[\Delta^0_0]+C[\Delta^0_1]} \vdash \circ$: Take any $(\mathcal{M}, S) \vDash \mathsf{C[\Delta^0_0]+C[\Delta^0_1]}$, any functions $h, g \in S$. For all $w \in \mathcal{M}$ we have

\[
(\mathcal{M}, S) \vDash \exists x, y < w(w = (x, y) \land h(x) = z \land g(z) = y) \iff \\
\forall z, \forall x, y < w(w = (x, y) \land h(x) = z \rightarrow g(z) = y)
\]

and so by $\Delta^0_1$-comprehension with the LHS formula we obtain the desired set $f \in S$ for which

\[
(\mathcal{M}, S) \vDash \forall x, y \in f \iff \exists z(h(x) = z \land g(z) = y) .
\]

$\mathsf{C[\Delta^0_0]+o} \vdash \mathsf{C[\Delta^0_1]}$: Take any $(\mathcal{M}, S) \vDash \mathsf{C[\Delta^0_0]+o}$, any $\Delta^0_0$-formulas $\varphi(\vec{y}, x), \psi(\vec{y}, x)$, any values of possible parameters $\vec{y} \in \mathcal{M} \cup S$ occurring in them, and assume $(\mathcal{M}, S) \vDash \forall x(\exists \vec{y} \varphi(\vec{y}, x) \rightarrow \forall \vec{y} \psi(\vec{y}, x))$. We may assume w.l.o.g. that the variables $\vec{y}$ are paired to a single one $y$. We wish to find an $X \in S$ such that $(\mathcal{M}, S)$ satisfies $X = \{x \mid \exists y \varphi(x, y)\}$.

By $\Delta^0_0$-comprehension we obtain a set $h \in S$ such that

\[
(\mathcal{M}, S) \vDash \forall w \left( w \in h \iff \exists x, y < w(w = (x, y, x) \land \varphi(y, x) \land \forall z < y(\varphi(z, x) \land \neg \varphi(z, x))) \right) .
\]

We have $(\mathcal{M}, S) \vDash \forall x \exists y(\varphi(y, x) \rightarrow \varphi(y, x))$ because for any $x \in \mathcal{M}$ there is either a witness to $\exists y \varphi(y, x)$ or a counterexample to $\forall y \varphi(y, x)$. Thus $h$ is a function yielding pairs $(y, x)$. By $\Delta^0_0$-comprehension we obtain a set $g \in S$ such that

\[
(\mathcal{M}, S) \vDash \forall w \left( w \in g \iff \exists x, y < w(w = ((y, x), z) \land \\
(\varphi(y, x) \rightarrow z = 1) \land (\neg \varphi(y, x) \rightarrow z = 0)) \right) .
\]

Clearly, $g$ is the characteristic function of the formula $\varphi$. From $\circ$ we obtain $f = g \circ h \in S$ and for all $x \in \mathcal{M}$ we have $(\mathcal{M}, S) \vDash f(x) = 1 \iff \exists y \varphi(x, y)$. Thus the desired $X$ is obtained by $\Delta^0_0$-comprehension to satisfy $X = \{x \mid f(x) = 1\}$.

\[\square\]

**3.5 The theory $\mathsf{RCA}^*_0$.** We define $\mathsf{RCA}^*_0 := \mathsf{C[\Delta^0_0]+o}$. In the view of Theorem 3.4 the theory $\mathsf{RCA}^*_0$ is equivalent to $\mathsf{C[\Delta^0_1]}$ and so by Proposition 3.1 $\mathsf{RCA}^*_0 \models \mathsf{I[\Delta^0_1]}$.

How does $\mathsf{RCA}^*_0$ compare to the theory $\mathsf{RCA}^*_0$ (see [21] or [6]) which is equivalent to $\mathsf{I[\Delta^0_1]}(\text{exp}) + \mathsf{C[\Delta^0_1]}$? By the absence of the axiom

\[
\forall x \exists y 2^x = y
\]

where $2^x \not\in y$ stands for a $\Delta^0_0$ formula defining the graph of exponential function. We, namely, have:
3.6 Theorem. Theories $\text{RCA}_0^+$ and $\text{RCA}_0^+ + \text{E}$ are equivalent.

Proof $\text{RCA}_0^+ \vdash \text{RCA}_0^+ + \text{E}$: We work in $\text{RCA}_0^+$ which proves IND and trivially $C[\Delta_0^0]$. By Thm. 3.4 it proves $\sigma$. We define $f := \{ u \mid \exists x, y < u (v = (x, y) \land 2^x \equiv y) \}$ by $\Delta_0^0$ comprehension. From exp we get $f \in \mathcal{P}$, the recurrences in $f := E$, and thus $E$.

$\text{RCA}_0^+ + \text{E} \vdash \text{RCA}_0^+$: We work in $\text{RCA}_0^+ + \text{E}$. From Prop. 3.1 we obtain $I[\Delta_0^0]$. From Thm. 3.4 we get $C[\Delta_0^0]$. From $E$ we get a function $f$. Since $f$ is a function, we have

$$\forall x (\exists y (f(x) = y \land 2^x \equiv y) \leftrightarrow \forall y (f(x) = y \rightarrow 2^x \equiv y))$$

and can use a $\Delta_0^0$ induction (which we obtain from Prop. 3.1) to prove $\forall x \exists y (f(x) = y \land 2^x \equiv y)$ from which exp directly follows. □

The theory $\text{RCA}_0$ (see [20]) is defined as $I\Sigma_1 + C[\Delta_0^0]$. Lemma 2.5 of [21] asserts that $\text{RCA}_0^+ + \text{PR}$ is equivalent to $\text{RCA}_0$. The following is a sharpening by dropping the exponentiation:

3.7 Theorem. Theories $\text{RCA}_0^+$ and $\text{RCA}_0^+ + \text{PR}$ are equivalent.

Proof $\text{RCA}_0^+ \vdash \text{RCA}_0^+ + \text{PR}$: $\text{RCA}_0^+$ trivially proves $C[\Delta_0^0]$. It also proves the closure under composition and primitive recursion (see [20]).

$\text{RCA}_0^+ + \text{PR} \vdash \text{RCA}_0$: We work in $\text{RCA}_0^+ + \text{PR}$ and define

$$h := \{ u \mid \exists x, y, v < u (u = (((x, y), v), v + v) \lor u = ((0, y), 0) \lor u = (0, 0)) \}$$

by $C[\Delta_0^0]$. We obviously have $h \in \mathcal{P}$. We define $g(y) = 1$ by $C[\Delta_0^0]$ and then $f'(0, y) = g(y), f'(S(x), y) = h((x, f'(x, y)), y)$ by primitive recursion. Finally, we define a function $f(x) = f'(x, 0)$ by $C[\Delta_0^0]$. $\Delta_0^0$ induction establishes $f := E$ and hence $E$. By Thm. 3.6 we get $\text{RCA}_0^+$ and then use the above mentioned consequence of Lemma 2.5 of [21]. □

4 The First-Order Part of $\text{RCA}_0^-$

Simpson and Smith’s [21] proved that the first-order part of $\text{RCA}_0^+$ is $I\Delta_0^0(\exp) + B[\Sigma_1]$. In this section we will drop the exponentiation from their proof and show that the first-order part of $\text{RCA}_0^-$ is $B[\Sigma_1]$. This directly determines by Cor. 4.5 the first-order parts not only of $\text{RCA}_0^+$ but also of $\text{RCA}_0$ (see [20]). Recall that the first-order part of a second-order theory $T_2$ is a first-order theory $T_1$ whose theorems are identical to the theorems of $T_2$ expressed in the language of $T_1$.

We have been inspired in Thm. 4.1 by the unpublished proof of Gandy that over $I\Delta_0$ the least number principle for $\Delta_1$ formulas implies $B[\Sigma_1]$ (see [22][10]). This obviates the use of bounded recursion (needing exponentiation) in the proof of Simpson and Smith. The structure of this section is otherwise similar to the corresponding ones in [20][21].

4.1 Theorem. $\text{RCA}_0^- \vdash B[\Sigma_1^0]$. 

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If the first model is countable, so is the expanded model.

4.1 Lemma. We wish to prove \(\forall u\) \((\forall v\exists\bar{u}\exists\bar{v} \varphi(x, y, \bar{z})\) possibly with number and set parameters \(\bar{w}\)\). We wish to prove the principle \(B\exists \exists \varphi(x, y, \bar{z})\). So we take any \(a, \bar{w}\), and assume \(\forall x < a \exists y \exists \bar{z} \varphi(x, y, \bar{z})\). By taking \(v := \max(y, \bar{z})\) we get

\[
\forall x < a \exists y \exists \bar{z} \leq v \varphi(x, y, \bar{z}).
\]

(†)

Suppose we manage to obtain i) a function \(f(x)\) yielding the least bound \(v\) and ii) we find its maximum \(c := \max_{z \in [0, \infty)} f(z)\). Setting \(b := c + 1\) we would then have \(\forall x < a \exists y \exists \bar{z} < b \varphi(x, y, \bar{z})\), and we would get the desired conclusion of the collection by dropping the bound on \(\bar{z}\).

Toward the goal i) we use \(\Delta^0_0\) comprehension to define the set

\[
f = \{ (x, v) \mid \{ x < a \land \exists y, \bar{z} \leq v \varphi(x, y, \bar{z}) \land \forall y, \bar{z} < v \varphi(x, y, \bar{z}) \} \land (x \geq a \land v = 0) \}.
\]

We wish to prove \(f \in T\). That \(v\) is unique is obvious. For the proof of its existence we take any \(x\). If \(x \geq a\), we have \(v := 0\). If \(x < a\) we have \(\exists y, \bar{z} \leq v' \varphi(x, y, \bar{z})\) for some \(v'\) from (†) and by \(\Delta^0_0\) least number principle we get the smallest such \(v\) for which also \((x, v) \in f\).

Toward the goal ii) we suppose that we contrive to define the set \(X = \{ x \mid f(x) = \max_{z \in [x, \infty)} f(z) \}\). Since \(a \in X\), a \(\Delta^0_0\) least number principle gives the least element \(m\) of \(X\). Furthermore, if we succeed in defining \(Y = \{ y \leq m \mid f(m) = \max_{z \in [y, m]} f(z) \}\) we will have \(m \in Y\) and so \(Y\) will have the least element \(n\) for which we have \(f(m) = \max_{z \in [n, \infty)} f(z)\).

Now, if \(n > 0\), then \(f(n^+) > f(m)\) and we would get a contradiction \((n - 1) \in X\). Thus \(n = 0\) and \(c := f(m)\) is the desired maximum of all \(f(z)\).

It remains to define the sets \(X\) and \(Y\). The defining formula for \(X\) is \(\Sigma^0_1\) because it can be written as

\[
\exists v \{ f(x) = v \land (x < a \rightarrow \forall z \in [x, a] \exists w \leq v f(z) = w) \}.
\]

This is equivalent to a \(\Pi^0_1\) formula: \(\forall v \{ f(x) = v \land x < a \rightarrow \forall z \in [x, a] \exists w \leq v f(z) = w \}\) and the set \(X\) is obtained by \(\Delta^0_1\) comprehension. The definition of \(Y\) is similar. \(\square\)

4.2 \(\Delta^0_1\) Definability of Sets. For a model \((\mathcal{M}, S)\) for \(L_2\) we designate by \(\Delta^0_1\)-def \((\mathcal{M}, S)\) the subset of \(P(\mathcal{M})\) consisting of \(\Delta^0_1\)-definable sets \(X\), i.e. such that there is a \(\Sigma^0_1\) formula \(\varphi(x, \bar{v})\) and a \(\Pi^0_1\) formula \(\psi(x, \bar{v})\), possibly with parameters \(\bar{v} \in \mathcal{M} \cup S\), such that \((\mathcal{M}, S) \models \forall x (\varphi(x, \bar{v}) \leftrightarrow \psi(x, \bar{v}))\), and

\[
\text{for all } x \in \mathcal{M} \text{ we have } x \in X \text{ iff } (\mathcal{M}, S) \models \varphi(x, \bar{v}).
\]

(1)

4.3 Lemma. For every model \((\mathcal{M}, S)\) for \(L_2\) we have

\[
\text{if } (\mathcal{M}, S) \models B\Sigma^0_1 \text{ then } (\mathcal{M}, \Delta^0_1\text{-def}(\mathcal{M}, S)) \models RCA^0_0.
\]

If the first model is countable, so is the expanded model.
4.5 Corollary. The first-order part of $\text{RCA}_0^-$ is $\Sigma_1$.

Proof Take a $(\mathcal{M}, S) \models B\Sigma_0^1$ and set $S' := \Delta_0^1$-def$(\mathcal{M}, S)$. We need to verify that $(\mathcal{M}, S')$ satisfies $C[\Delta_0^1]$ and IND. For that we prove the auxiliary claim:

For every $\Delta_0^1$ formula $\theta$ possibly with parameters from $\mathcal{M} \cup S'$ there is a $\Sigma_1^1$-formula $\theta^*$ with parameters from $\mathcal{M} \cup S$ such that $(\mathcal{M}, S') \models \theta \iff \theta^*$.

The proof is by induction on the form of $\theta$ in the negation normal form. If $\theta$ is $t \in X$ or $t \notin X$ with $X$ a set from $S'$ $\Delta_0^1$ defined as in Par. 4.4, then define $\theta^* := \varphi(t)$ or $\theta^* := \neg \varphi(t)$ respectively. In both cases $\theta^*$ is $\Sigma_1^1$ with parameters from $\mathcal{M} \cup S$. The remaining literals $\theta$ cannot have parameters from $S'$ and we set $\theta^* := \theta$. When $\theta$ is $\exists x < t \theta_1(x)$ then we set $\theta^* := \exists x < t \theta^*_1(x)$ which is $\Sigma_1^1$ in $(\mathcal{M}, S')$. When $\theta$ is a disjunction or conjunction we similarly obtain $\theta^*$ directly from IH. The most interesting case is when $\theta$ is $\forall x < t \theta_1(x)$. We can put $\theta^*_1(x)$ (which is without parameters in $S'$) into the form $\exists \bar{z} \theta_2(x, \bar{z})$ with $\theta_2 \in \Delta_0^1$. We then have

$$(\mathcal{M}, S') \models \theta \iff \forall x < t \theta^*_1(x) \iff \forall x < t \exists \bar{z} \theta_2(x, \bar{z}) \iff$$

$$\forall x < t \exists \bar{y} \exists \bar{z} \leq y \theta_2(x, \bar{z}) \iff \exists \bar{y} \forall x < t \exists \bar{y} \exists \bar{z} \leq y \theta_2(x, \bar{z})$$

where the last step is obtained in the direction $\rightarrow$ from $B[\Sigma_1^1]$ and in the direction $\leftarrow$ by predicate calculus. Thus we set $\theta^*$ to the last formula. This ends the proof of the claim.

In order to prove $C[\Delta_1^0]$ assume $(\mathcal{M}, S') \models \forall x (\exists y \varphi(x, y) \leftrightarrow \forall y \psi(x, y))$ for $\Delta_0^1$ formulas $\varphi$ and $\psi$ possibly with parameters from $\mathcal{M} \cup S'$ which we do not show. We assume that the possibly multiple quantifiers on $\bar{y}$ have been contracted to $y$. We wish to show that $X := \{ x \in \mathcal{M} \mid (\mathcal{M}, S') \models \exists y \varphi(x, y) \}$ is in $S'$. From the claim we get

$$(\mathcal{M}, S') \models \exists y \varphi^*(x, y) \leftrightarrow \exists y \varphi(x, y) \leftrightarrow \forall y \psi(x, y) \leftrightarrow$$

$$\neg \exists y \neg \psi(x, y) \leftrightarrow \neg \exists y \neg \psi^*(x, y) \leftrightarrow \forall y \neg \psi^*(x, y)$$

with the leftmost $\Sigma_1^1$ and the rightmost one $\Pi_1^0$ both with parameters at most from $\mathcal{M} \cup S$. Thus $(\mathcal{M}, S') \models \exists y \varphi^*(x, y) \iff \forall y \neg \psi^*(x, y)$ and $X \in \Delta_0^1$-def$(\mathcal{M}, S)$.

In order to prove $(\mathcal{M}, S') \models \text{IND}$ we observe that $(\mathcal{M}, S) \models I[\Delta_0^1]$ as the induction follows from $B[\Sigma_1^1]$ (see [2][10]). We take any $X \in S'$. $X$ is $\Delta_0^1$ definable in $(\mathcal{M}, S)$ and so there is a $\varphi(x)$ s.t. $\Box[\varphi(x)]$ within which we get that $(\mathcal{M}, S')$ satisfies the principle $I[\varphi]$ from which we can derive $(\mathcal{M}, S')$ satisfies the principle $I[x \in X]$, i.e. $\text{IND}$.

In order to finish the proof we observe that there are only countably many sets definable from countably many parameters.

4.4 Theorem. Let $\mathcal{M}$ be a structure for $L_1$. We have $\mathcal{M} \models B\Sigma_1$ iff $(\mathcal{M}, S) \models \text{RCA}_0^-$ for some $S \subseteq \mathcal{P}(\mathcal{M})$.

Proof The direction $\Rightarrow$ follows from Lemma 4.3. In the direction $\Leftarrow$ if $(\mathcal{M}, S)$ satisfies $\text{RCA}_0^-$ then it also satisfies $I[\Delta_0^1]$ by Prop. 3.1 and $B[\Sigma_1^1]$ by Lemma 4.3. For the formulas $L_1$ we thus get that $\mathcal{M}$ satisfies $\text{BASIC} + I[\Delta_0^1] + B[\Sigma_1^1] =: B\Sigma_1$. 

4.5 Corollary. The first-order part of $\text{RCA}_0^-$ is $\Sigma_1$. 

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Proof By Lemmas 3.1 and 4.1 we have $\text{RCA}_0 \vdash \Delta_0, B[\Sigma_1]$. Since $B\Sigma_1 = \Delta_0 + B[\Sigma_1]$, $\text{RCA}_0$ proves $B\Sigma_1$. Vice versa, we need to show that $\text{RCA}_0$ is conservative over $B\Sigma_1$ for sentences of $L_1$. Thus we take any $\phi \in L_1$ not proved by $B\Sigma_1$. This makes $B\Sigma_1 + \neg \phi$ consistent and with a first-order model $M$. By Theorem 4.4 there is a model $(M, S)$ of $\text{RCA}_0$. Hence $\text{RCA}_0 \not\vdash \phi$. □

5 Extension of $\text{RCA}_0$ to $\text{WKL}_0$

In this section we follow the general development of [20,21] and expand models of $\text{RCA}_0$ to those of $\text{WKL}_0$. Once again we have to be careful with the coding because of the absence of exponentiation. To mathematicians the details of coding are mostly immaterial. This is because they work in frameworks with primitive recursion (or at least with the exponentiation) available. The various encodings of finite sets and sequences, trees, terms, and formulas are then invariant. On the other hand, all feasible complexity classes are subexponential. Hence the details of coding are usually relevant.

5.1 Bounded Sets, 0—1-Sequences, and Binary Trees. Within $\text{RCA}_0$ we define a set $X$ bounded iff $\exists b X < b$ (see Par. 2.2). The size $|X|$ of this set is the least such $b$. The graph of the size function $|X| \doteq b$ is $\Delta_0$ defined as a partial function. We use the Ackermann’s encoding of bounded sets with the relation $x \in_{\text{Ack}} y$ holding iff the $x$-th least significant digit of the binary number representation of $y$ is 1. The relation has a $\Delta_0$ definition:

$$x \in_{\text{Ack}} y \iff \exists z, y_1 \leq y \exists y_2 < z (2^x \doteq z \land y = (2 \cdot y_1 + 1) \cdot z + y_2).$$

If $X$ is bounded and $2^{|X|}$ exists, i.e. if $\exists b < |X| \doteq b \land 2^b \doteq y$ for some $y$, then the set $X$ is coded by the number $\sum y \in X 2^y < y$.

A finite sequence $\langle x_0, \ldots, x_{n-1} \rangle$ of length $n$, where $\forall i < n x_i < 2$ is encoded by the number $(1x_0, \ldots, x_{n-1})_2$ which is the code of the set $\{ i \mid i = n \lor (i < n \land x_{n-(i+1)} = 1) \}$ with the size $n + 1$. Thus every non-zero number codes a finite sequence where the empty sequence ($n = 0$), is encoded by the number 1, the sequence $\langle 0100 \rangle$ by the number $20 = (10100)_2$, the sequence $0^i$ of $i$ zeroes is encoded by $2^i$ if it exists, and the sequence $1^i$ by $2^{i+1} - 1$. We will henceforth identify the finite sequences $\tau$ with their codes.

The (finite) sequence length function $|\tau|$ is defined as

$$|\tau| = y \iff \exists p \leq \tau \exists x < p (2^x \doteq p \land \tau = p + x) \lor \tau = 0 \land y = 0.$$

The sequence concatenation function $\sigma \ast \tau$ is defined as

$$\sigma \ast \tau = \rho \iff \exists p \leq \rho \exists x < p (2^{|\tau|} \doteq p \land \tau = p + x \land \sigma > 0 \land \rho = \sigma \cdot p + x) \lor \sigma \cdot \tau \cdot \rho = 0 \land \rho = 0.$$

The relation $\sigma$ is a subsequence of $\tau$, in writing $\sigma \leq \tau$, is $\Delta_0$ defined as

$$\sigma < \tau \iff \exists p \leq \tau (\sigma \ast p = \tau \land \sigma > 0).$$
The sequence \( \sigma \) is a proper subsequence of \( \tau \), in writing \( \sigma < \tau \) if, in addition to \( \sigma < \tau \), we have \( \sigma < \tau \).

For a set \( T \) we define the property of being \( T \) is a (binary) tree, in writing \( T \in \mathcal{T} \), as follows:

\[
T \text{ is a tree} \iff 0 \notin T \land \forall \tau \in T [\forall \sigma < \tau \sigma \in T].
\]

Note that \( 1 \) is the root of a tree \( T \neq \emptyset \) and if \( 1 < \tau \in T \) then the parent of \( \tau \) is \( (\tau - 2) \in T \) and \( 2 \cdot \tau (2 \cdot \tau + 1) \) is the left (right) child of \( \tau \) neither necessarily in \( T \) in which case \( \tau \) is a leaf.

A tree \( S \) is a subtree of the tree \( T \) if \( S \subseteq T \). A tree \( P \) is a branch if it is linearly ordered in \( < \). A tree \( T \) is finite if it is bounded and infinite otherwise. The property \( T \) is an infinite tree will be written as \( T \in \mathcal{T} \).

### 5.2 Monotone formulas
Let \((\mathcal{M}, S) \models \text{RCA}_0^\neg\). A formula \( \varphi(\tau, \overline{v}) \), possibly with parameters \( \overline{v} \), is monotone in \( \tau \) when for all \( \overline{v} \in \mathcal{M} \cup S \) we have

\[
(\mathcal{M}, S) \models \forall \tau, \tau', \overline{v} (\varphi(\tau, \overline{v}) \land \tau < \tau' \implies \varphi(\tau', \overline{v})).
\]

### 5.3 The Theory WKL\(^{-}\)
Denote by WKL\(^{-}\) the theory \( \text{RCA}_0^\neg + \text{WKL} \) where the sentence

\[
\forall T \in \mathcal{T} \exists P \in T (P \text{ is a branch} \land P \subseteq T)
\]

is called the Weak König lemma.

In the following we will show that every countable \((\mathcal{M}, S) \models \text{RCA}_0^\neg\) can be expanded to a model \((\mathcal{M}, S') \models \text{WKL}_0^\neg\). This is done by refining the forcing-like argument from [21] where we add to \( S \) an infinite branch contained in every infinite tree in \( S \).

### 5.4 Generic branches
Let \( \mathcal{N} := \langle \mathcal{M}, S \rangle \) be a model of \( \text{RCA}_0^\neg\). A property \( D \subseteq T^{\mathcal{N}} \) of infinite trees is \( \mathcal{N} \)-definable if there is a formula \( \varphi(T, \overline{v}) \in L_2 \) and parameters \( \overline{v} \in \mathcal{N} \) such that for every \( T \in S \) we have

\[
T \in D \iff \mathcal{N} \models \forall T \in T \land \varphi(T, \overline{v}).
\]

Such a \( D \) is dense if \( \mathcal{N} \models \forall T \in T \exists T' \in D \exists T' \subseteq T \).

A set \( G \subseteq \mathcal{M} \) is a generic branch over infinite trees in \( S \) if for every dense definable property \( D \) we have

\[
(\mathcal{M}, S \cup G) \models G \text{ is an infinite branch} \land \exists T \in D G \subseteq T.
\]

[Lemma 5.5] and [Thm. 5.6] are proved under the assumption that generic branches exist and the [Lemma 5.7] asserts that for countable structures they do.

### 5.5 Lemma
If \((\mathcal{M}, S) \models \text{RCA}_0^\neg\), \( G \) is a generic branch over infinite trees in \( S \), and if \( \varphi(x, \tau) \) with parameters \( \overline{v} \) is monotone in \( \tau \) then the expanded model \((\mathcal{M}, S \cup \{G\})\) satisfies the generic collection:

\[
\forall a (\forall x < a \exists \tau \in G \varphi(x, \tau) \implies \exists \tau \in G \forall x < a \varphi(x, \tau))
\]

for all \( \overline{v} \in \mathcal{M} \cup S \).
5. Extension of $\text{RCA}_0^-$ to $\text{WKL}_0^-$

Proof. Abbreviate $\mathcal{N} := (\mathcal{M}, S)$, $\mathcal{N}' := (\mathcal{M}, \mathcal{U}(G))$, and take a $\varphi(x, r)$ as in the theorem. Take any $a, \bar{v} \in \mathcal{N}$, assume the hypothesis of the special collection, and define the properties $\mathcal{E}$ and $D$ of $T \in \mathcal{S}$:

\[
T \in \mathcal{E} \text{ iff } \mathcal{N} \vDash T(T) \land \exists x < a \forall r \in T \neg \varphi(x, r)
\]

\[
T \in D \text{ iff } \mathcal{N} \vDash T \in \mathcal{E} \lor (T \in T \land \forall T' \in T (T' \subseteq T \rightarrow T' \notin \mathcal{E})).
\]

The property $D$ is dense because for any $T \in T^\mathcal{N}$ when $\mathcal{N} \vDash \exists T' \in T (T' \subseteq T \rightarrow T' \notin \mathcal{E})$ then we choose such a $T'$ to have the property $D$. Otherwise there is no need to do anything because we already have $T \in D$.

Since $G$ is generic, there is a $T \in T^\mathcal{N}$ such that $T \in D$ and $G \subseteq T$. From the assumption we have $T \notin \mathcal{E}$. We take any $x <^\mathcal{M} a$ and consider the set

\[
T' := \{ r \in \mathcal{M} | \mathcal{N} \vDash r \in T \land \neg \varphi(x, r) \}.
\]

$T' \subseteq T$ is a tree by monotonicity of $\varphi$ and it is $\mathcal{N}$-finite because otherwise we would have $T \in \mathcal{E}$. Thus there is a $c_\varepsilon \in \mathcal{M}$ such that $\mathcal{N} \vDash \forall r \in T (r > c_\varepsilon \rightarrow \varphi(x, r))$. We have thus established:

\[
\mathcal{N} \vDash \forall x < a \exists c \forall r \in T (r > c \rightarrow \varphi(x, r)). \tag{†}
\]

Our goal is to find an upper bound of all $c_\varepsilon$ for $x <^\mathcal{M} a$. We could use $\Sigma^0_1$-collection in $\mathcal{N}$ but for that we would need a suitable upper bound on $r$. For reasons we will see below we take $4 \cdot c + 4$ as the bound and specialize (†) to

\[
\mathcal{N} \vDash \forall x < a \exists c \forall r \in [c + 1, 4 \cdot c + 4] (r \in T \rightarrow \varphi(x, r)) \tag{‡}
\]

By applying collection we get a $b \in \mathcal{M}$ such that

\[
\mathcal{N} \vDash \forall x < a \exists c < b \forall r \in [c + 1, 4 \cdot c + 4] (r \in T \rightarrow \varphi(x, r)). \tag{§}
\]

In order to prove the conclusion of the theorem we choose from the infinite $G$ a $r \in T$ such that $\mathcal{N}' \vDash r > 2 \cdot b \land r \in G$. We now take any $x <^\mathcal{M} a$ and use it in (§) to obtain a $c \in \mathcal{M}$ s.t.

\[
\mathcal{N} \vDash c < b \land \forall \sigma \in [c + 1, 4 \cdot c + 4] (r \in T \rightarrow \varphi(x, \sigma)).
\]

We have $\mathcal{N} \vDash |r| \geq |2b| = |b| + 1 \geq |c + 1| + 1$. All sequences $\sigma \in \mathcal{M}$ s.t. $\mathcal{N}$ satisfies $|\sigma| = |c + 1| + 1$ are such that $\mathcal{N} \vDash \sigma \in [2^{c+1}+1, 2^{c+1}+2] \subseteq \{c + 1, 4 \cdot c + 4\}$ and so we can choose one such that $\mathcal{N} \vDash \exists \rho (\tau = \sigma \star \rho \land |\sigma| = |c + 1| + 1)$. But then

\[
\mathcal{N} \vDash \sigma \leq r \land \sigma \in T \land \sigma \in [c + 1, 4 \cdot c + 4] \land \varphi(x, \sigma).
\]

We have $\mathcal{N} \vDash \varphi(x, r)$ by monotonicity and hence $\mathcal{N}' \vDash \varphi(x, r)$. □

5.6 Theorem. If $(\mathcal{M}, S) \vDash \text{RCA}_0^-$ and $G$ is a generic branch over infinite trees in $S$ then $(\mathcal{M}, \mathcal{U}(G)) \vDash \text{BS}^0_1$.
Proof  Abbreviate \( \mathcal{N} := (\mathcal{M}, S) \) and \( \mathcal{N}^\prime := (\mathcal{M}, S \cup \{G\}) \). We prove first a normal form property:

For every \( \Delta^0_0 \)-formula \( \varphi(X) \) possibly with parameters \( \bar{v} \) there is a \( \Delta^0_0 \)-formula \( \bar{\varphi}(\bar{\tau}) \) with the same parameters such that \( \mathcal{N}^\prime \) establishes its monotonicity in \( \tau \) and we have for all \( \bar{v} \in \mathcal{N}^\prime \):

\[
\mathcal{N}^\prime \models \varphi(G) \iff \exists \bar{\tau} \in G \bar{\varphi}(\bar{\tau}) .
\]

The proof is by induction on the form of \( \varphi(X) \) in negation normal form where we omit the straightforward proofs of monotonicity. If \( \varphi(X) \) is \( t \in X \) then set \( \bar{\varphi}(\bar{\tau}) : = \exists \bar{t} \in \bar{G} : t \leq \bar{t} \land t \neq \bar{t} \). For the case when \( \varphi \) is a disjunction \( \varphi \land \psi \) we set \( \bar{\varphi}(\bar{\tau}) : = \bar{\varphi}(\bar{\tau}) \land \bar{\psi}(\bar{\tau}) \). The implication \( \mathcal{N}^\prime \models \varphi(G) \vdash \exists \bar{\tau} \in G \bar{\varphi}(\bar{\tau}) \) follows from Lemmas 5.5 and the converse from predicate calculus. This ends the proof of the normal form property.

In order the prove the conclusion of the theorem it suffices to establish that \( \mathcal{N}^\prime \) satisfies B[\( \Sigma^0_1 \)] and I[\( \Delta^0_0 \)].

Take any \( \Sigma^0_1 \) formula \( \varphi(x, y, \bar{z}, X) \) possibly with number and set parameters \( \bar{w} \). We wish \( \mathcal{N}^\prime \) to satisfy B[\( \exists \bar{z} \varphi(x, y, \bar{z}, G) \)]. So take any \( a, \bar{w} \in \mathcal{N}^\prime \), and assume \( \mathcal{N}^\prime \vdash \forall x < a \exists \bar{y} \exists \bar{z} \varphi(x, y, \bar{z}, G) \). From the normal form property we get

\[
\mathcal{N}^\prime \vdash \forall x < a \exists \bar{y}, \exists \bar{\tau} \in G \bar{\varphi}(x, y, \bar{z}, \bar{\tau}) .
\]

Since \( G \) is infinite, there is for any \( x < M \) a sequence \( \bar{\tau} \in G \) s.t. \( \mathcal{N}^\prime \vdash y, \bar{z} \leq \bar{\tau} \land \bar{\tau}' = \bar{\tau} \). Using monotonicity we get

\[
\mathcal{N}^\prime \vdash \forall x < a \exists \bar{\tau} \in G \exists y, \bar{z} \leq \bar{\tau} \bar{\varphi}(x, y, \bar{z}, \bar{\tau}) .
\]

We now apply Lemma 5.5 to obtain:

\[
\mathcal{N}^\prime \vdash \exists \bar{\tau} \in G \forall x < a \exists y, \bar{z} \leq \bar{\tau} \bar{\varphi}(x, y, \bar{z}, \bar{\tau}) .
\]

From this we get after some easy manipulation with \( b := \tau + 1 \):

\[
\mathcal{N}^\prime \vdash \exists b \forall x < a \exists y < b \exists \bar{\tau} \in G \bar{\varphi}(x, y, \bar{z}, \bar{\tau})
\]
and it remains to apply the normal form property backwards to get
\[ \mathcal{N}' \models \exists b \forall x < a \exists y < b \exists z \varphi(x, y, z, G) \]
as desired.

For the proof of \( \mathbb{I}[\Delta^0_0] \) we take a \( \Delta^0_0 \) formula \( \varphi(x, X) \) possibly with parameters \( \vec{v} \). We take any \( \vec{v} \in \mathcal{N} \) and assume by way of contradiction
\[ \mathcal{N}' \models \varphi(0, G) \land \forall x (\varphi(x, G) \rightarrow \varphi(x + 1, G)) \land \neg \varphi(a, G) \quad (\dagger) \]
for some \( a \in \mathcal{M} \).

For every \( x \in \mathcal{M} \) we have from the normal form property:
\[ \mathcal{N}' \models \exists \tau \in G \, \exists \vec{v}(x, \tau) \text{ iff } \mathcal{N}' \models \forall \tau \in G - \bar{\varphi}(x, \tau). \]
Thus \( \mathcal{N}' \models \forall x \leq a \exists \tau \in G(\bar{\varphi}(x, \tau) \lor \bar{\varphi}(x, \tau)) \) and by \( \Sigma^0_1 \) collection (with \( a := a + 1 \)) we get for some \( b \in \mathcal{M} \):
\[ \mathcal{N}' \models \forall x \leq a \exists \tau < b \exists (\tau \in G \land (\bar{\varphi}(x, \tau) \lor \bar{\varphi}(x, \tau))). \]

\( G \) is an infinite tree, and so there is a \( \sigma \in G \) such that \( \sigma >^M b \) and for any \( x \leq^M a \) we get a \( \tau <^M b, \tau \in G \) such that \( \mathcal{N}' \models \bar{\varphi}(x, \tau) \lor \bar{\varphi}(x, \tau) \). As \( G \) is a branch we have \( \tau <^M \sigma \) and from the monotonicity we obtain \( \mathcal{N}' \models \bar{\varphi}(x, \sigma) \lor \bar{\varphi}(x, \sigma) \). Thus
\[ \mathcal{N}' \models \varphi(x, G) \Rightarrow \mathcal{N}' \models \neg \bar{\varphi}(x, \sigma) \Rightarrow \mathcal{N}' \models \bar{\varphi}(x, \sigma) \Rightarrow \mathcal{N}' \models \varphi(x, G). \]

From (\dagger) we have \( \mathcal{N}' \models \neg \varphi(a, \sigma) \) and by the least number principle in \( \mathcal{N} \) there is a least such \( m \leq^M a \). It cannot be the case that \( m = 0^M \) and so \( \mathcal{N}' \models \bar{\varphi}(m - 1, \sigma) \land \neg \bar{\varphi}(m, \sigma) \) contradicting (\dagger).

**5.7 Lemma.** Let \((\mathcal{M}, S)\) be a countable model of \( \text{RCA}^-_0 \). For every infinite tree \( T \in S \) there is a generic branch \( G \) over infinite trees in \( S \) such that \( G \subseteq T \).

**Proof** Abbreviate \( \mathcal{N} := (\mathcal{M}, S) \) and enumerate all \( \mathcal{N} \)-definable (with parameters) dense sets into a countable sequence \( \{D_i\}_{i \in \mathbb{N}} \).

For every \( b \in \mathcal{M} \) define
\[ T \in \mathcal{E}_b \text{ iff } \mathcal{N} \models T \in T \land \exists \tau \in T(|\tau| = b \land \forall \sigma \in T(|\sigma| = b \rightarrow \sigma = \tau)). \]
The sets \( \mathcal{E}_b \) are dense because given an infinite tree \( T \) there must be a sequence \( \tau \in \mathcal{N} \) s.t.
\[ \mathcal{N} \models \tau \in T \land |\tau| = b \land \exists \tau' \in T(\tau' \subseteq T \land \tau \in T'). \]
We form by \( \Delta^0_0 \)-comprehension an infinite tree \( T' \subseteq T \) such that
\[ \mathcal{N} \models \sigma \in T' \iff \sigma \in T \land (|\sigma| < |\tau| \lor \exists \rho \leq \sigma \sigma = \tau \star \rho). \]
We clearly have \( T' \in \mathcal{E}_b \).
Given an infinite tree $T \in S$, we set $T_0 := T$ and for $i \in \mathbb{N}$ we set $T_{i+1}$ to a $D_i$ dense infinite tree obtained for $T_i$. Clearly, for all $i \in \mathbb{N}$ we have $T_i \in \mathcal{T}^N$. $T_{i+1} \subseteq T_i$, and the set $G := \bigcap_{i \in \mathbb{N}} T_i \subseteq T \subseteq \mathcal{M}$ is an infinite branch because at every level $b \in \mathcal{M}$ it has exactly one sequence and all of them are $\prec$ comparable. For this the order of $\varepsilon_b$ in the enumeration is irrelevant, although for different orders the infinite branches $G$ may differ. Moreover, $G$ is generic because every dense definable set must be $D_i$ for some $i \in \mathbb{N}$ and we have $G \subseteq T_{i+1} \in D_i$. }

5.8 Theorem. Every countable structure $(\mathcal{M}, S) \not\models \text{RCA}_0^-$ can be expanded to a countable structure $(\mathcal{M}, S') \models \text{WKL}_0^-$ with $S \subseteq S'$.

Proof. We will define a sequence of sets $\{S_i\}_{i \in \mathbb{N}}$ such that for all $i, j \in \mathbb{N}$, $i < j$ we will have $S_i \subseteq S_j \subseteq \mathcal{P}(\mathcal{M})$.

For that we need a function $\text{Branch}_{a,b} := G$ for $G$ obtained by Lemma 5.7 with $(\mathcal{M}, S_{(a,b),-1})$ and an $T_b \in S_a$. Here $T_b$ is an infinite tree at the $b$-th position in some fixed enumeration of infinite trees in $\mathcal{T}^{(\mathcal{M},S)}$.

The sets $S_i$ are defined by $S_0 := S$, and $S_{i+1} := \Delta^0_1(\mathcal{M}, S_i \cup \{\text{Branch}_{a,b}\})$ where $a, b$ are such that $i + 1 = (a, b)$.

Complete induction on $i$ establishes

$(\mathcal{M}, S_i) \not\models \text{RCA}_0^-$ and if $i = (a, b)$ then there is an infinite branch $\text{Branch}_{a,b}$ with $\text{Branch}_{a,b} \subseteq T_b$ where $T_b$ is the $b$-th tree $T_b$ in $\mathcal{T}^{(\mathcal{M},S_i)}$. We have $\text{Branch}_{a,b}, T_b \in S_i$.

Indeed, there is nothing to prove when $i = 0$. Otherwise we have $i = (a, b)$ for some $a, b < i$ and the structures $(\mathcal{M}, S_a)$ and $(\mathcal{M}, S_{i-1})$ both satisfy $\text{RCA}_0^-$ by IH. We use the last structure and the infinite tree $T_b \in S_a \subseteq S_{i-1}$ in Lemma 5.7 to obtain a generic branch $\text{Branch}_{a,b} \subseteq T_b$. The structure $(\mathcal{M}, S_{i-1} \cup \{\text{Branch}_{a,b}\})$ satisfies $\text{BS}_1^b$ by Thm. 5.6 and so closing it by $\Sigma^0_1$-definitions by Lemma 4.3 yields $(\mathcal{M}, S_i) \not\models \text{RCA}_0^-$ with $\text{Branch}_{a,b}, T_b \in S_i$.

We now set $S^* := \bigcup_{i \in \mathbb{N}} S_i$ and claim that the structure $\mathcal{N} := (\mathcal{M}, S^*)$ is the desired countable structure extending $S$ and satisfying $\text{WKL}_0$. The extension is trivial: $S = S_0 \subseteq S^*$. That the structure is countable, follows from the fact that it is the result of countably many operations which change a countable structure to another countable one. In order to establish that $\mathcal{N} \not\models \text{RCA}_0^-$, it suffices to show that $\mathcal{N} \not\models \text{IND}, C \Delta^0_0 \circ$. For IND we take a set $X \in S^*$. It appears first in some $S_i$ and $(\mathcal{M}, S_i)$ satisfies IND. Similarly two functions $g, h \in S^*$ appear both in some $S_i$ which is closed under composition. The principle $\text{C}[\varphi]$ for a $\Delta^0_1$ formula $\varphi$ is also similar because all its set parameters must appear in some $S_i$ because there is only finitely many of them. Thus already $(\mathcal{M}, S_i)$ contains the comprehended set.

For $\mathcal{N} \not\models \text{WKL}$ we take a $T \in \mathcal{T}^N$. It appears for the first time at the $b$-th position in some $\mathcal{T}^{(\mathcal{M},S_i)}$. Thus $\text{Branch}_{a,b}$ is an infinite branch in $T$ which is in $S_{(a,b)} \subseteq S^*$. }

5.9 Theorem. $\text{WKL}_0^-$ is $\Pi^1_1$ conservative over $\text{RCA}_0^-$. 
Proof Suppose that \( \forall x \varphi(x) \in \Pi^1_1 \) is not provable in \( \text{RCA}_0^\omega \). Thus there is a countable model \( (M, S) \models \text{RCA}_0^\omega + \exists x \neg \varphi(x) \). Take \( x \in S \) s.t. \( (M, S) \models \neg \varphi(x) \) Expand the model to \( (M, S') \models \text{WKL}_0^\omega \). Since \( S \subseteq S' \), we have \( (M, S') \models \neg \varphi(X) \). Thus \( \text{WKL}_0^\omega \not\models \forall x \varphi(x) \). \( \square \)

5.10 Corollary. The first-order part of \( \text{WKL}_0 \) is the same as that of \( \text{RCA}_0^\omega \), namely \( \text{BSigma}_1 \).

Proof \( \text{WKL}_0 \) is an extension of \( \text{RCA}_0^\omega \) so it proves all of the latter’s theorems in \( L_1 \). Vice versa, take any sentence \( \varphi \in L_1 \) such that \( \text{WKL}_0 \not\vdash \varphi \). Thus \( \text{RCA}_0^\omega \not\vdash \varphi \) by [Thm. 5.9] because \( \varphi \) is trivially \( \Pi^1_1 \). Thus \( \text{WKL}_0 \) and \( \text{RCA}_0^\omega \) have the same theorems in \( L_1 \) which by [Cor. 4.5] are exactly the theorems of \( \text{BSigma}_1 \). \( \square \)

6 Function Algebras

In this section we introduce operators for defining classes of functions over natural numbers by inductive definitions. The classes are called function algebras by Clote in [3] where the reader will find a comprehensive overview of defining classes of functions of computational complexity. We assign to every function algebra \( \mathcal{A}(X) \) a first-order theory \( \mathcal{A}_X \) called the recursive arithmetic of \( \mathcal{A}(X) \). This is similar to the going from the class of primitive recursive functions to the theory PRA (Primitive Recursive Arithmetic) in the form presented in Simpson [20].

However, rather than treating the functions as \( n \)-ary, we work with their pair contractions into unary functions. We have opted for this approach because of its direct connection to the second-order theories of recursive arithmetics which will be discussed in the next section.

6.1 Function Algebras. A function operator \( f := \text{op}(g_1, \ldots, g_n) \) is a mapping that takes \( n \geq 0 \) functions \( g_1, \ldots, g_n \in \mathbb{N} \) and yields a unique function \( f \) in \( \mathbb{N} \). The oracle operator \( f := X_s \) is a mapping that given any set \( X \subseteq \mathbb{N} \) yields the unique function \( f \) such that \((\mathbb{N}, X, f)\) satisfies

\[
\forall x (x \in X \rightarrow f(x) = 1) \land (x \notin X \rightarrow f(x) = 0). \quad (f := X_s)
\]

Clearly, \((\mathbb{N}, X, f) \models f \in \mathcal{F} \).

For every oracle \( X \subseteq \mathbb{N} \) and a \( k \)-tuple of function operators \( \mathcal{A} \) a function algebra \( \mathcal{A}(X) \) is the least set that contains the function \( f := X_s \) and is closed under the operators of \( \mathcal{A} \). We view \( \mathcal{A}(X) \) without \( X \) specified as the class of algebras \( \{ \mathcal{A}(X) \mid X \subseteq \mathbb{N} \} \).

Henceforth, every function operator \( f := \text{op}(g_1, \ldots, g_n) \) will be specified by a formula in \( L_2 \) with no free first-order variables, no-second-order quantifiers, and which contains exactly the set variables \( f, g_1, \ldots, g_n \). The second-order set variables are to be viewed in first-order contexts as meta-variables ranging over the set constants. This effectively turns operators into schemas. We require that any structure \((\mathbb{N}, X, g_1, \ldots, g_n) \models g_1, \ldots, g_n \in \mathcal{F} \) can be uniquely expanded to the structure \((\mathbb{N}, X, g_1, \ldots, g_n, f) \) satisfying \( f := \text{op}(g_1, \ldots, g_n) \) and \( f \in \mathcal{F} \).
6. Function Algebras

Although our algebras are formulated in a general way we are mostly interested in subelementary classes of algebras characterizing some of the main computational complexity classes (see [Par. 10.3]). It turns out that the oracles \( X \) play important role in this and we use them as arguments (input) to the predicates of the complexity classes. This is similar to the approach to computational complexity by finite models (see e.g. [5]) where the arguments are finite models. Finite models contain interpretations of (finite) predicates which are comparable to our oracles. We will thus restrict in our characterizations the oracles to finite subsets of \( \mathbb{N} \).

6.2 Derivations. We fix one class of algebras \( \mathcal{A}(X) \) until the end of the paragraph. Derivation terms (or just a derivations) are the least set of symbols containing the symbol \( X_s \) and the symbol \( \text{op}(d_1, \ldots, d_n) \) for each \( n \)-ary operator \( \text{op} \) and derivations \( d_1, \ldots, d_n \). We fix the derivation terms into the standard enumeration of derivations: \( d_{i_0}, d_1, d_2, \ldots \) where the derivation \( \text{op}(d_1, \ldots, d_n) \) is preceded by the symbols \( d_1, \ldots, d_n \). We identify the derivation terms with their indices in the standard enumeration. Thus for each algebra in the class we have a sequence \( \vec{\mathcal{F}} \) enumerating its functions such that for each \( d \in \mathbb{N} \) the function \( f_d \) has the derivation \( d \). Note that the the enumeration sequence is independent of the value of the oracle.

A typical use of enumerations \( \vec{\mathcal{F}} \) will be in the construction of the first-order structures \( (\mathbb{N}, X, \vec{f}) \) constituting the standard models of the first-order theory called the recursive arithmetic of \( \mathcal{A}(X) \) and designated by \( \mathcal{A} \mathcal{R} \mathcal{A}_1 \). This will be discussed in the following section.

6.3 Clausal Definitions of Functions. With the exception of the operator of bounded minimization (see [Par. 7.3]), all function operators \( f := \text{op}(g_1, \ldots, g_n) \) discussed in this paper are specified by clausal definitions. A clausal definition is obtained by a finite sequence \( C_0, \ldots, C_k \) of finite sets of formulas in \( \mathcal{L}_2 \). The formulas in \( C_i \) are called clauses. The set \( C_k \) consists of the single clause \( \top \to f(x) = y \). The set of clauses \( C_{i+1} \) is obtained from the set \( C_i \) by replacing one incomplete clause \( \varphi(x, \vec{z}) \to f(x) = y \) in \( C_i \) which is such that the formula \( \varphi \) does not contain the variable \( y \). If all clauses in \( C_i \) are complete then \( k = i \) and the universal closure with the first-order quantifiers of the formula \( \bigwedge C_k \), abbreviated by \( f := \text{op}(g_1, \ldots, g_n) \), is the clausal definition of \( f \) (or the schema-of clausal definitions in the first order case).

The selected incomplete clause \( \varphi(x, \vec{z}) \to f(x) = y \) in \( C_i \) is refined by choosing one of the numbered items in the following list. The set \( C_{i+1} \) is then formed to be like \( C_i \) except that the selected clause is replaced by one or two clauses given in the chosen item:

1. \( \varphi(x, \vec{z}) \land g(t(x, \vec{z})) = v \to f(x) = y \) where \( g \) is one of \( g_1, \ldots, g_n \), or \( f, t \) is a term of \( \mathcal{L}_1 \) in at most the indicated variables, and \( v \) is a new variable, i.e. not occurring in \( \varphi \) and different from \( y \),
2. \( \varphi(x, \vec{z}) \land v = 0 \to f(x) = y \) and \( \varphi(x, \vec{z}) \land v = S(w) \to f(x) = y \) where the variable \( v \) is one of \( x, \vec{z} \) and \( w \) is new,
3. \( \varphi(x, \bar{z}) \land v = 0 \rightarrow f(x) = y \) and \( \varphi(x, \bar{z}) \land v = (w_1, w_2) \rightarrow f(x) = y \) where the variable \( v \) is one of \( x, \bar{z} \) and \( w_1, w_2 \) are new.

4. \( \varphi(x, \bar{z}) \land t_1(x, \bar{z}) \) rel \( t_2(x, \bar{z}) \rightarrow f(x) = y \) and \( \varphi(x, \bar{z}) \land t_1(x, \bar{z}) \) rel \( t_2(x, \bar{z}) \rightarrow f(x) = y \) where \( t_1 \) and \( t_2 \) are terms as above and rel is either \( = \) or \( < \).

5. \( \varphi(x, \bar{z}) \land t(x, \bar{z}) = y \rightarrow f(x) = y \) where \( t \) is as above. Note that this clause is complete and cannot be further refined.

The clausal definition of \( f \) is recursive if the variable \( f \) occurs in the antecedent of at least one clause, and explicit otherwise. A first-order variable other than \( x \) and \( y \) occurring in the antecedent of a clause is called a local variable.

The three function operators given in Par. 3.3 are not in the form of clausal definitions, but it is straightforward to bring them in into an equivalent clausal form in any theory extending \( \text{I}\Delta_0^m \). For instance, the operator of primitive recursion has an equivalent strict clausal form:

\[
\left( \top \land x = 0 \land 0 = y \rightarrow f(x) = y \right) \land \\
\left( \top \land x = (u, p) \land v = 0 \land g(p) = z \land z = y \rightarrow f(x) = y \right) \land \\
\left( \top \land x = (u, p) \land v = S(u) \land f(u, p) = z \land h((u, z), p) = u \land u = y \rightarrow f(x) = y \right).
\]

In the following we will not adhere to the strict form of clausal definitions if they can be equivalently rewritten in an obvious way.

6.4 Explicit Clausal Definitions. Provided that we already have the functions \( g_1, \ldots, g_n \) defined, we wish to introduce the function \( f \) specified by an explicit clausal definition \( f := \text{op}(g_1, \ldots, g_n) \) by a definitional extension of a theory \( T \vdash \text{I}\Delta_0^m + g_1, \ldots, g_n \in F \) in the language including \( L_1(g_1, \ldots, g_n) \) so that the extended theory \( T_1 \) proves \( f := \text{op}(g_1, \ldots, g_n) \) and \( f \in F \).

This is easy to achieve for explicit clausal definitions which have the form

\[
\forall x, y, \bar{w}\left( \bigwedge_i \left( \bigwedge_j \varphi_{i,j} \rightarrow f(x) = y \right) \right)
\]

with the formulas \( \varphi_{i,j} \) literals (atomic or their negations), \( \bar{w} \) the local variables, and the variable \( f \) not occurring in the antecedents of clauses.

The reader will note that the refinements of clauses are such that \( T \) proves that for all \( i \neq i' \) we have

\[
\forall x, y, \bar{w}\left( \bigwedge_j \varphi_{i,j} \rightarrow \neg \bigwedge_j \varphi_{i',j} \right)
\]

and

\[
\forall x \exists! y, \bar{w}\left( \bigvee_i \varphi_{i,j} \right).
\]

In other words, for each argument \( x \) there is exactly one clause whose antecedent holds and its local variables plus \( y \) are uniquely determined.
The definitional extension of $T$ to $T_1$ is with the defining axiom:
\[ \forall v \left( v \in f \iff \exists x, y \exists \tilde{w}(v = (x, y) \land \bigvee_i \bigwedge_j \varphi_{i,j}) \right) . \]

$T_1$ then proves $f \in F$ and $f := \text{op}(g_1, \ldots, g_n)$.

### 6.5 Restrictions on Recursive Clausal Definitions.

For a class of algebras $A(X)$ with a recursive function operator $f := \text{op}(g_1, \ldots, g_n)$ we impose two additional constraints on the form of its recursive clauses. They will allow to show that $f$ is primitive recursive in $\tilde{g}$.

Generally, some measure function $m$ must go down in recursive applications. This means that the above operator is actually of the form $f := \text{op}(g_1, \ldots, g_n, m)$ and in every model $(\mathbb{N}, X, g_1, \ldots, g_n, m)$ and for every recursive clause of the form $\varphi_1 \land f(t) = w \land \varphi_2 \to f(x) = y$, the expanded model must satisfy (the first-order universal closure of) $\varphi \to m(t) < m(x)$. The measure function for the operator of primitive recursion $f := \text{PR}(g, h)$ is the identity function $I(x) = x$ and if this is the case we do not explicitly include the measure as an argument to the operator because the above condition is simply $\varphi \to t < x$. As the first restriction on recursive definitions in our function algebras we require that the measure function is the identity $I$.

If $n > 0$ then there is a second restriction on recursive definitions that they must be in a parameterized form where their argument $x$ must be of the form $x = (v, p)$ with $p$ a parameter shipped unchanged to all applications of $h \in \{ f, \tilde{g} \}$ in antecedents of clauses in the form $h(t(v, p, \tilde{z}), p) = w$ or $g_i(p) = w$. The requirement that $x = (v, p)$ is obviously required only in clauses with applications of functions in $\{ f, \tilde{g} \}$ in antecedents. The requirement on the parameterization is an inessential restriction (the functions $g_i$ can always ignore the parameter $p$), it permits the smooth transformation of first-order models of theories for function algebras to second-order models (see Lemma 9.8). The recursive operators used in this paper (primitive recursion, bounded primitive recursion, doubly nested recursion) are all parameterized in this way.

### 6.6 Reduction of Recursive Clausal Definitions to Primitive Recursion.

It should be clear that explicit clausal definitions plus $f := \text{PR}(g, h)$ define all primitive recursive functions. We will now show the converse. So we take an arbitrary recursive clausal definition of $f$ from the functions $g_1, \ldots, g_n$ with the measure $m$ whose set of clauses we designate by $C_0$. We will define $f$ by the operator of primitive recursion and by explicit definitions.

We could do it in way typically employed by logicians, namely by encoding the definition. See, for instance, the treatment of nested ordinal recursion in Rose [18]. Computer scientists usually prefer program transformations over encoding where one definition is effectively translated into a simpler one. In doing this, we will illustrate the construction of computer programs directly in Peano Arithmetic. The main trick which makes this feasible is the offsetting of the Cantor’s pairing function by one as reflected in its property $0 \neq (x, y)$. This gives us a tool for the smooth development of programs directly in PA. This is because we obtain very simple codes of finite sequences of natural numbers.
numbers, called *lists* in computer science. There is, namely, a one-to-one correspondence between lists and natural numbers because for every natural number \( n \) there are unique numbers \( n, x_1, \ldots, x_n \) such that \( x = (x_1, \ldots, x_n, 0) \). Thus \( x \) can be taken as the code of the sequence \( x_1, \ldots, x_n \). The length \( L(x) \) of the list \( x \) is \( n \) and it satisfies the recursive clausal definition \( L(0) = 0 \land L(v, w) = L(w) + 1 \).

The clausal definitions are employed in a slightly more refined form in our programming language CL (Clausal Language). We have been using the language (which comes with an integrated theorem prover for PA) in courses teaching computer programming and program verification for the last twenty years [23].

Returning to the clauses of the above recursive definition in the set \( C_0 \) we let \( j_p \) to designate the number of recursive applications in the \( p \)-th clause of \( C_0 \) (in some fixed ordering of \( C_0 \)) and let \( J := \max_p(j_p) \). For all \( p \) we assume w.l.o.g. that the successive recursive applications in the \( p \)-th clause of \( C_0 \) are numbered as as

\[
f(t_1) = z_1, f(t_2) = z_2, \ldots, f(t_{j_p}) = z_{j_p}.
\]

Technically, we should have designated the terms by \( t_i^{(p)} \) because they depend on \( p \) but we will refrain from doing so in order not to clutter the presentation. Because the results of preceding recursions can be used in succeeding ones, such recursion is called nested recursion.

Just as it was demonstrated for the explicit clausal definitions in [Par. 6.4] for each argument \( x \) the antecedent of exactly one clause in \( C_0 \) holds. For the demonstration just remove all recursive invocations of \( f \) from the antecedents of its clauses. Denote the number of the clause applying to \( x \) by \( C(x) \).

We will translate the clauses in \( C_0 \) into an explicit clausal definition of an auxiliary function \( h \). The function will be invoked in the form \( h(x, c) \) where \( c \) is a list such that if \( p = C(x) \) then \( i := L(c) \leq j_p \) and we have \( c = (z_1, \ldots, z_i, 0) \) for some \( z_1, \ldots, z_i \) which are in that order the values of the first \( i \) recursive calls to \( f \) in the \( p \)-th clause. If \( i < j_p \) then the call \( f_{i+1}(t_{i+1}) = z_{i+1} \) needs to be computed and this will be indicated by the function \( h \) yielding \( (0, t_{i+1}) \) with 0 a tag indicating this. If \( i = j_p \) then all recursive calls in the clause \( p \) have been computed and the value \( y \) of \( f(x) \) can be determined and \( h(x, c) \) will yield \((1, y)\) with the tag 1 indicating that the value of \( f(x) \) has been found.

We are now ready to describe the construction of the clauses for the function \( h \). This is done by successively forming the sets of clauses \( C_1, C_2, \ldots C_k, \ldots \). In forming the set \( C_{k+1} \) we select a clause in the set \( C_k \) to which one of the following numbered transformations applies. The set \( C_{k+1} \) is obtained from \( C_k \) by the replacement of the selected clause by one or more clause specified in the applicable transformation step. If the selected clause is of the form:

1. \( T \land \varphi \rightarrow f(x) = y \) then the replacement clauses are

\[
\begin{align*}
T \land v = 0 & \land v = w \rightarrow h(v) = w \\
T \land v = (x, c_0) & \land \varphi \rightarrow f(x) = y.
\end{align*}
\]
2. \( \varphi_1 \land f(t_{i+1}) = z_{i+1} \land \varphi_2 \rightarrow f(x) = y \) without \( f \) occurring in \( \varphi_1 \) then the replacement clauses are

\[
\varphi_1 \land c_i = 0 \land (0, t_{i+1}) = w \rightarrow h(v) = w \\
\varphi_1 \land c_i = (z_{i+1}, c_{i+1}) \land \varphi_2 \rightarrow f(x) = y ,
\]

3. \( \varphi \land t = y \rightarrow f(x) = y \) without \( f \) occurring in \( \varphi \) then the replacement clause is

\[
\varphi \land (1, t) = w \rightarrow h(v) = w .
\]

The selection and replacement process will eventually terminate with the set \( C_k \) where no clauses are selectable. The clauses in \( C_k \) then explicitly define \( h \). The reader will note that the clauses in \( C_k \) have consequences of the form \( h(v) = w \) instead of \( h(x) = y \) but this is inessential as the variables can be systematically renamed. We also assume w.l.o.g. that the auxiliary variables \( v, w, c_0, \ldots, c_f \) introduced by the transformation are new.

The function \( h \) is used in the following explicitly defined function \( f_1 \) which are easily transformable into the strict clausal form:

\[
f_1(0) = 0 \\
f_1(0, s_1) = (0, s_1) \\
h(x, c) = 0 \rightarrow f_1((x, c), s_1) = ((x, c), s_1) \\
h(x, c) = (0, z) \rightarrow f_1((x, c), s_1) = ((z, 0), (x, c), s_1) \text{ (}\dagger\text{)} \\
h(x, c) = ((t_1, t_2), z) \land s_1 = 0 \rightarrow f_1((x, c), s_1) = ((x, c), s_1) \text{ (}\dagger\text{)} \\
h(x, c) = ((t_1, t_2), z) \land s_1 = 0 \rightarrow f_1((x, c), s_1) = ((x, c), s_1) \text{ (}\dagger\text{)} \\
h(x, c) = ((t_1, t_2), z) \land s_1 = 0 \rightarrow f_1((x, c), s_1) = ((x, c), s_1) \text{ (}\dagger\text{)}
\]

The real work is done in the marked clauses. The remaining ones are the default clauses which make the function \( f \) total although they cannot apply when \( f_1 \) is correctly initialized and used as \( f_1^{\mu(x)}((x, 0), 0) \). This is a notation for the iteration of \( f_1 \) \( \mu(x) \) times. The length of the iteration is given by the function \( \mu(x) \) which will be determined below. The iteration function has the primitive recursive definition \( f_1^{\mu}(s) = s \) and \( f_1^{\mu+1}(s) = f_1 f_1^s(s) \). The function \( \oplus \) used in the clause \( \dagger \) is the list concatenation function with the clausal definition:

\[
(0 \oplus y = y) \land ((v, x) \oplus y = (v, x \oplus y)) .
\]

The argument to the function \( f_1(s) \) is a stack \( s \) which is a nonempty list of non-empty elements of the form \((x, c)\). When \( s = (x, c) \), \( s_1 \) then the top of the stack \((x, c)\) specifies that the \((L(c)+1)\)-th recursive application of \( f \) in its \( C(x) \)-th clause should be computed by \( h(x, c) \). The clause \( \dagger \) applies when there is such an application (because the tag yielded by \( h \) is 0) and its argument is \( z \). The stack is extended by pushing \( z \) on top of it together with the empty list 0 signifying that the first recursive application in the \( C(z) \)-th clause for \( f \) should be computed (if there is such).
The clauses (†₂) and (†₃) apply if \( h(x, c) \) yields \((1, z)\) signifying that the value of \( f(x) \) has been computed to \( z \). In the clause (†₂) the tail \( s_1 \) of the stack \( s \) is empty and we are essentially done. However, due to the fact that the length of iteration function \( \mu(x) \) will give only an upper bound, we yield the same stack by entering an idling loop. When the iteration of \( f_1 \) eventually terminates we will be able to read off the desired value of \( f(x) \) from the stack as follows \( f(x) := T \ h \ H \ f_1^{m(x)}((x, 0), 0) \). The last identity is the desired definition of \( f \) as primitive recursive in the functions \( \bar{g}, m \).

The clause (†₃) applies when the stack \( s_1 \) is not empty and then \( z \) is the value of the \( (L(d) + 1) \)-th recursive application of \( f \) in the \( C(w) \)-th clause for \( f \). The stack \( s \) is popped by removing \((x, c)\), and the value \( z \) extends the list \( d \) before resuming the computation of the \( C(w) \)-th clause.

It remains to find the function \( \mu(x) \) giving the upper bound to the iterations of \( f_1 \). The maximal length of the stack \( s \) computing \( f(x) \) is given by the measure \( m := m(x) \) but during the computation of at most \( J \) recursive applications in the antecedents of clauses for \( f \) the stack will be repeatedly pushed and popped. View the stack \( s \) as coding the tail of the sequence:

\[
(a_0, c_0, (a_1, c_1), \ldots, (a_{m+1-L(s)}, c_{m+1-L(s)}), \ldots, (a_m, c_m)
\]

which starts from the index \( m + 1 - L(s) \) and with \( a_0 = c_0 = a_1 = c_1 = \ldots = a_{m-L(s)} = c_{m-L(s)} = 0 \). With the stack initialized to \((x, 0, 0)\) we have \( a_m = x \). The function \( m'(s) = \sum_{i \leq m} L(c_i) \cdot J_i \) gives the weight of the stack \( s \) during the computation of \( f(x) \). We have \( m'(s) < J^{m+1} \) and the reader can convince themselves that we have \( m'(s) < m'f_1(s) \) until the iteration of \( f_1 \) starts idling by yielding the same stack. To bring the computation to idling it thus suffices to define \( \mu(x) = J^{m'(s)+1} \).

## 7 Recursive Arithmetics

### 7.1 Recursive Arithmetics

Fix a class of function algebras \( \mathcal{A}(X) \). The class specifies a first-order theory in the language \( L_{1}(X, \vec{f}) \), designated by \( \mathcal{ARA}_1 \), and called the recursive arithmetic of \( \mathcal{A}(X) \). The theory consists of the basic axioms BASIC, the oracle axioms \( f_{\chi} := X \) and \( f_{\chi} \not\in F \), and of the operator axioms \( f_{op(d_1, \ldots, d_n)} := \text{op}(f_{d_1}, \ldots, f_{d_n}) \) and \( f_{op(d_1, \ldots, d_n)} \in F \) for each \( n \)-ary operator \( \text{op} \) of \( \mathcal{A}(X) \) and each derivation \( d_1, \ldots, d_n \).

Note that any structure \((\mathbb{N}, X)\) can be uniquely expanded to the structure \((\mathbb{N}, X, \vec{f})\) satisfying \( \mathcal{ARA}_1 \). The structures are called standard models of \( \mathcal{ARA}_1 \). In addition to the standard models, we also admit non-standard models \((\mathcal{M}, X, \vec{f})\) with \( \mathcal{M} \) a model for \( L_1 \) and \( X, \vec{f} \subseteq \mathcal{M} \).

### 7.2 Quasi-Terms and Quasi-Bounded Formulas

We extend the notion of terms to quasi-terms by allowing the expressions \( f(t) \) in positions where a first-order term is permitted. They can be always unnested to the form \( \exists y (f(t) \land q(y)) \) or \( \forall y (y = f(t) \rightarrow q(y)) \). We abbreviate this to \( \exists y = f(t) \ q(y) \) and \( \forall y = f(t) \ q(y) \) respectively, and
call the quantifiers quasi-bounded. Bounded formulas extended with quasi-terms and quasi-bounded quantifiers are called quasi-bounded formulas.

We note that by unnesting the quasi-bounded terms in a quasi-bounded formula we obtain a quasi-bounded formula and the ability to choose the kind of quasi-bounded quantifiers makes the last formula equivalent to a $\Delta^0_1$ formula over any theory proving $\text{BS}^0_1$.

7.3 Operators of $\Delta_0$ Functions. We wish to connect the theory $\text{I} \Delta_0$ with a recursive arithmetic. To that end we will need the following function operators which are the first-order universal closures of the formulas named by the operators:

\[
\begin{align*}
    f(x) &= S(x) & (f := S) \\
    f(x, y) &= x + y \land f(0) = 0 & (f := \oplus) \\
    f(x, y) &= x \cdot y \land f(0) = 0 & (f := \otimes) \\
    (x < y \rightarrow f(x, y) = 1) \land (x \geq y \rightarrow f(x, y) = 0) \land f(0) = 0 & (f := <) \\
    f(x) &= x & (f := I) \\
    f(0, y, z) &= y \land f(S(x), y, z) = z \land f(0) = 0 \land f(x, 0) = 0 & (f := D) \\
    g(x) &= v \land h(x) = w \rightarrow f(x) = (v, w). & (f := \text{P}(g, h))
\end{align*}
\]

The nullary operators of successor, addition, multiplication, identity, (the characteristic function of) order, case-analysis, and the binary pairing operator are not in a strict clausal form. However, any weak theory proving $\text{BASIC}$ permits to bring them into the strict form of explicit clausal definitions.

In order to capture the properties of bounded quantifiers we introduce the unary operator of bounded minimization $f := \mu(g)$ which is the first-order universal closure of:

\[
\begin{align*}
    (f(b, x) &= z \rightarrow z \leq b) \land \\
    (f(b, x) &= z \land y < b \land g(y, x) = 1 \rightarrow z \leq y) \land \\
    (f(b, x) &= z \land z < b \rightarrow g(z, x) = 1) \land \\
    f(0) &= 0.
\end{align*}
\]

Informally, the function $f(b, x) := \mu_{z \leq b}[g(z, x) = 1]$ yields the least $z < b$ satisfying $g(z, x) = 1$ if there is such and $b$ otherwise. Bounded minimization can be brought (by a recursive search for $z$) into an equivalent recursive clausal form, but this apparently requires a theory stronger than $\text{I} \Delta_0^0$ (it is an open problem). We can, however, extend by definition any theory $T \vdash \text{I} \Delta_0^0 + g \in P$. The defining axiom for $f$ is the closure by $\forall v$ of:

\[
\begin{align*}
    v \in f &\leftrightarrow v = (0, 0) \lor \exists h, x, z < v \forall (v = ((h, x), z) \land (z < b \land g(z, x) = 1 \lor z = b) \land \\
    &\forall y < z ((y, x), 1) \not\in g)
\end{align*}
\]
whose RHS is $\Delta^0_0$. By working in $\mathcal{T}$ we prove $f \in \mathcal{F}$. The proof of $0 \not\in f$ and of the uniqueness property is straightforward. The existence condition $\forall x' \exists y(x', y) \in f$ is trivial for $x' = 0$. Otherwise, we have $x' = (b', x)$ for some $b', x$ and we prove by $\Delta^0_0$ induction on $b$:

$$b \leq b' \rightarrow \exists z \leq b((z < b \land g(z, x) = 1 \lor z = b) \land \forall y < z ((y, x), 1) \not\in g).$$

Now $f := \mu(g)$ easily follows.

We define the class $\Delta(X)$ of $\Delta^0_0$-algebras to consists of the operators:

$$S, \oplus, \otimes, \leq, I, D, P, e, \mu.$$

For any $X \subseteq \mathbb{N}$ the algebra $\Delta(X)$ has the standard model $(\mathbb{N}, X, \bar{f})$ with its functions polynomially bounded.

We henceforth require that the operators of every function algebra $\mathcal{A}(X)$ contains the operators of $\Delta(X)$.

The following Lemma 7.5 asserts that the characteristic functions of quasi-bounded formulas of a recursive arithmetic are denotations of its function constants. The lemma needs an auxiliary lemma about terms of such arithmetics. The reader will note that the proofs do not rely on any form of induction holding in the arithmetics. That the induction holds for all quasi-bounded formulas needs the two lemmas and it is asserted by [Thm. 7.7].

7.4 Lemma. For any recursive arithmetic $\mathcal{ARA}_1$, any sequence of variables $\bar{x} := x_0, \ldots, x_n$ and any quasi-term $t(\bar{x})$ (whose variables are all indicated) there is a derivation in $\mathcal{A}(X)$, designated (without the risk of confusion) by $t(\bar{x})$, such that

$$\mathcal{ARA}_1 \vdash \forall \bar{x} f_{t(\bar{x})}(\bar{x}) = t(\bar{x}).$$

Proof. By induction on the structure of the term $t$ while working in the recursive arithmetic $\mathcal{ARA}_1$.

If $t$ is 0 we consider the following informal identity:

$$f_{\mu(S\circ\circ)}(b, x) = \mu_{y<\mu}(f_{S\circ\circ}(y, x) = 1).$$

Thus we can prove $\forall x f_{\mu(S\circ\circ)}(x, x) = 0$ and take the desired derivation $0(\bar{x}) := Z := \mu(S\circ\circ)\circ P(1, I)$.

If $t$ is the variable $x_i$ ($i \leq n$) then we are looking for a derivation $d := x_i(\bar{x})$ s.t. $\forall \bar{x} f_d(\bar{x}) = x_i$. Toward that end we abbreviate $H := D\circ P(Z, I)$. It is easy to see that we have $\forall x f_{H}(x) = H(x)$. Likewise, $T := D\circ P(S\circ\circ, I)$ is the derivation s.t. $\forall x f_T(x) = T(x)$. We introduce the following abbreviations on derivations $T^0 := I$ and $T^{i+1} := T \circ T^i$. Now, if $i = n$ we set $d := T^n$ and set $d := H\circ T^i$ otherwise.

If $t$ is one of $S t_1(\bar{x})$, $f_d t_1(\bar{x})$, $t_1(\bar{x}) + t_2(\bar{x})$, $t_1(\bar{x}) \cdot t_2(\bar{x})$, or $(t_1(\bar{x}), t_2(\bar{x}))$ then we set the corresponding derivations $t(\bar{x})$ to $S t_1(\bar{x})$, $D t_1(\bar{x})$, $\oplus P(t_1(\bar{x}), t_2(\bar{x}))$, $\otimes P(t_1(\bar{x}), t_2(\bar{x}))$, or $P(t_1(\bar{x}), t_2(\bar{x}))$ where the derivations $t_i(\bar{x})$ are obtained from the induction hypothesis.

$\Box$
7.5 Lemma. For any recursive arithmetic \( \mathcal{ARA}_1 \), any quasi-bounded formula \( \varphi(\vec{x}) \) in \( L_1(X, \vec{f}) \) there is a derivation designated (without the risk of confusion) by \( \varphi(\vec{x}) \) such that

\[
\mathcal{ARA}_1 \vdash \forall x f_{\varphi(\vec{x})}(x) \leq 1
\]

\[
\mathcal{ARA}_1 \vdash \forall \vec{x} \varphi(\vec{x}) \leftrightarrow f_{\varphi(\vec{x})}(\vec{x}) = 1.
\]

Proof We do not give the proof that \( f_{\varphi(\vec{x})} \) is a \((0-1)\)-valued function because this will be obvious from the way the derivations are constructed. We work in \( \mathcal{ARA}_1 \) and proceed by induction on the quasi-formulas \( \varphi(\vec{x}) \) whose free variables are among the indicated ones and they are constructed from atomic formulas by negation, disjunction, bounded existential, and quasi-bounded existential quantification.

If \( \varphi \) is \( t_1(\vec{x}) < t_2(\vec{x}) \) we prove:

\[
\forall \vec{x} (t_1(\vec{x}) < t_2(\vec{x}) \leftrightarrow f_{<}(f_{t_1(\vec{x})}(\vec{x}), f_{t_2(\vec{x})}(\vec{x})) = 1)
\]

and it suffices to set the desired derivation \( \varphi(\vec{x}) := \varphi_{\vec{p}}(t_1(\vec{x}), t_2(\vec{x})) \).

In the remaining cases we obtain the desired derivations from IH and in a straightforward way from the properties proved below.

If \( \varphi \) is \( t_1(\vec{x}) = t_2(\vec{x}) \) we note that \( t_1 = t_2 \leftrightarrow t_1 \not< t_2 \land t_2 \not< t_1 \) and prove

\[
\forall \vec{x} (t_1(\vec{x}) = t_2(\vec{x}) \leftrightarrow f_I(f_{<}(f_{t_1(\vec{x})}(\vec{x}), f_{t_2(\vec{x})}(\vec{x})), f_D(f_{<}(f_{t_2(\vec{x})}(\vec{x}), f_{t_1(\vec{x})}(\vec{x})), 1, 0, 0) = 1).
\]

The final case is when \( \varphi(\vec{x}) \) is \( \exists y < t(\vec{x}) \psi(y, \vec{x}) \). We need an auxiliary function obtained by bounded minimization: \( f_{\mu(\psi(\vec{x}))}(b, x) = \mu_{y \leq b} \psi(y, x) = 1 \). We then prove

\[
\forall \vec{x} (\exists y = f_d(t(\vec{x}) \psi(y, \vec{x}) \leftrightarrow f_{\psi}(f_{\mu(\psi(\vec{x}))}(f_{t(\vec{x})}(\vec{x}), f_1(\vec{x})), f_{t(\vec{x})}(\vec{x})) = 1).
\]

\( \square \)

7.6 Lemma. Every recursive arithmetic \( \mathcal{ARA}_1 \) is closed under explicit clausal definitions, i.e. for any explicit clausal definition of \( f \) from the function constants \( f_{d_1}, \ldots, f_{d_k} \), there is a function \( f_d \) satisfying the clauses of the definition (after the replacement \( f := f_d \)).

Proof We take an explicit clausal definition as in the theorem. The clauses of the definition are constructed as in Par. 6.3 into the set \( C_f \) by refinements from the single clause in \( C_0 \). We first reason informally and for each set \( C_i \) \((i \leq k)\) we construct a set \( D_i \) of
clauses with consequents of the form \( f(x) = s \) for some quasi-terms \( s \) composed from \( x, y \), local variables, and from the function constants of \( \text{ARA}_1 \) where we abbreviate \( g_1 \coloneqq f_{d_1}, \ldots, g_n \coloneqq f_{d_n} \). During the process we fold the clauses into a single formula \( f(x) = s \).

We work backwards from \( k \) to 0 and construct \( D_k \coloneqq C_k \). Assuming for \( i < k \) that the set \( D_{i+1} \) has been constructed, we construct the set \( D_i \) according to the form of the clause in \( C_i \) refined into the set \( C_{i+1} \). If the clause was refined by the point 1) and the clause corresponding to the refined clause in \( D_{i+1} \) is \( \varphi(x, \overline{z}) \land g(t(x, \overline{z})) = v \rightarrow f(x) = s(x, \overline{z}, v) \) with \( g \) one of \( g_1, \ldots, g_n \), say \( g_j \), we place the clause \( \varphi(x, \overline{z}) \rightarrow f(x) = s(x, \overline{z}, g_{d_j}(t(x, \overline{z}))) \) into \( D_i \). If the clause was refined by 2) and the clauses corresponding to the refined clauses in \( D_{i+1} \) are \( \varphi(x, \overline{z}) \land v = 0 \rightarrow f(x) = s_1(x, \overline{z}) \) and \( \varphi(x, \overline{z}) \land v = S(w) \rightarrow f(x) = s_2(x, \overline{z}, w) \) we place \( \varphi(x, \overline{z}) \rightarrow f(x) = f_{D}(v, s_1(x, \overline{z}), s_2(x, \overline{z}, f_{D}(v))) \) into \( D_i \), where the function \( f_{D} \) is the predecessor function easily defined in the algebra by bounded minimization. If the clause was refined by 3) and the clauses corresponding to the refined clauses in \( D_{i+1} \) are \( \varphi(x, \overline{z}) \land v = 0 \rightarrow f(x) = s_1(x, \overline{z}) \) and \( \varphi(x, \overline{z}) \land v = (w_1, w_2) \rightarrow f(x) = s_2(x, \overline{z}, w_1, w_2) \) we place \( \varphi(x, \overline{z}) \rightarrow f(x) = f_{D}(v, s_1(x, \overline{z}), s_2(x, \overline{z}, f_{D}(v))) \) into \( D_i \). If the clause was refined by 4) and the clauses corresponding to the refined clauses in \( D_{i+1} \) are \( \varphi(x, \overline{z}) \land t_1(x, \overline{z}) \land t_2(x, \overline{z}) \rightarrow f(x) = s_1(x, \overline{z}) \) and \( \varphi(x, \overline{z}) \land t_1(x, \overline{z}) \land t_2(x, \overline{z}) \rightarrow f(x) = s_2(x, \overline{z}) \) we place

\[
\varphi(x, \overline{z}) \rightarrow f(x) = f_{D}(f_{D}(t_1(x, \overline{z}), t_2(x, \overline{z})), s_2(x, \overline{z}), s_1(x, \overline{z}))
\]

into \( D_i \). Finally, if the clause was refined by 5) then we place \( \varphi(x, \overline{z}) \rightarrow f(x) = t(x, \overline{z}) \) into \( D_i \). The remaining unaffected clauses of \( D_{i+1} \) are just copied to \( D_i \).

At the end we have a single clause \( \top \rightarrow f(x) = s(x) \) in \( D_0 \) and we use Lemma 7.4 to obtain the desired derivation \( d \coloneqq s(x) \) for the function constant \( f_{d} \).

To prove that the clauses for \( f \) with \( f \) replaced by \( f_{d} \) in \( D_k(f) \) are provable we work in \( \text{ARA}_1 \) and show by external induction that successively the clauses in \( D_0(f_{d}), \ldots, D_k(f_{d}) \) are all provable. The clauses \( D_k(f_{d}) \) are the clauses of the explicit clausal definition.

The next theorem asserts that the arithmetic \( \Delta^0_1(X) \) is essentially the inductive theory \( \Gamma^0_1(X) \):

**Theorem.** \( \Delta^0_1 \) proves the induction principles \( \Gamma^0_1(X, \overline{f}) \) and that the functions \( \overline{f} \) are polynomially bounded. Vice versa, any inductive theory \( \Gamma^0_1(X) \) can be extended by definitions to the theory \( \Delta^0_1 \).

**Proof.** Take any formula \( \varphi(b, \overline{y}) \) which is \( \Delta^0_1 \) in the language \( L(X, \overline{f}) \). It suffices to prove the least number principle for \( \varphi \) by working in \( \text{ARA}_1 \). So take any \( \overline{y} \) and assume \( \varphi(b, \overline{y}) \) for some \( b \). By Lemma 7.5, there is a derivation \( d \coloneqq \varphi(b, \overline{y}) \) such that the theory proves that \( f_{d} \) is the characteristic function of \( \varphi \). For \( f_{\mu(d)}(b, x) = \mu_{y<b}(f_{d}(y, x) = 1) \) the theory proves that \( f_{\mu(d)}(S(b), \overline{y}) \) is the least witness for \( \varphi(b, \overline{y}) \).

By external induction on derivations we prove that the functions \( f_{d} \) are polynomially bounded.
The second part of the theorem directly follows from the way the operators of the algebra \( \Delta(X) \) are defined. \( \square \)

8 Provably Recursive Functions of \( \mathcal{A}_{RA_1} \)

8.1 Provably Total Functions of \( \mathcal{A}_{RA_1} \). A \((\Sigma^0_1)\) provably total function of an arithmetic \( \mathcal{A}_{RA_1} \) is any function \( f : \mathbb{N} \times 2^\mathbb{N} \rightarrow \mathbb{N} \) such that there is a \( \Sigma^0_1 \) formula without parameters \( \varphi(x, y) \in L_1(X, \bar{f}) \) such that:

(a) \( \mathcal{A}_{RA_1} \vdash \forall x \exists! y \varphi(x, y) \), and

(b) for all \( X \subseteq \mathbb{N} \) and all \( x \in \mathbb{N} \) the standard model \((\mathbb{N}, X, \bar{f})\) of \( \mathcal{A}_{RA_1} \) satisfies

\[ \varphi(x, f(x, X)) \]

We denote by \( 2^{\mathbb{N}} \) the class of finite subsets of \( \mathbb{N} \). The restriction of a provably total function \( f \) to the domain \( \mathbb{N} \times 2^{\mathbb{N}} \) is called a provably recursive function of \( \mathcal{A}_{RA_1} \). Clearly, \( f \) is in general only recursive in \( X \) but its restriction is recursive (with Ackermann’s coding of finite sets).

Provably total functions of \( \mathcal{A}_{RA_1} \) can be characterized using the ideas of Ferreira [9], which are based on the following special form of Herbrand’s theorem. Its special case was proved by Krajiček, Pudlák, and Takeuti [14].

8.2 Lemma. (Ferreira [9]) Let \( T \) be a universal theory in a first-order language \( L \). Suppose that \( \exists \vec{u} \forall \vec{v} \varphi \) is a consequence of \( T \), with \( \varphi(\vec{u}, \vec{v}, x) \) an existential formula with only the indicated variables free. Then there are terms \( \bar{t}_1(x), \bar{t}_2(x, \vec{v}_1), \ldots, \bar{t}_k(x, \vec{v}_1, \ldots, \vec{v}_{k-1}) \) of \( L \) such that

\[ T \vdash \varphi(\bar{t}_1(x), \vec{v}_1, x) \lor \varphi(\bar{t}_2(x, \vec{v}_1), \vec{v}_2, x) \lor \cdots \lor \varphi(\bar{t}_k(x, \vec{v}_1, \ldots, \vec{v}_{k-1}), \vec{v}_k, x). \]

8.3 Theorem. The class of provably total functions of a recursive arithmetic \( \mathcal{A}_{RA_1} \) is exactly the class of functions \( f \) for which there is a derivation \( d \) in \( \mathcal{A}(X) \) such that all standard models \((\mathbb{N}, X, \bar{f})\) of \( \mathcal{A}_{RA_1} \) satisfy \( f_d(x) = f(x, X) \) for all \( x \in \mathbb{N} \).

Proof Note that the function operators of \( \mathcal{A}(X) \) are universal sentences. For the \( \subseteq \) inclusion of function classes from the claim of the theorem, take any function \( f : \mathbb{N} \times 2^{\mathbb{N}} \rightarrow \mathbb{N} \) such that for some derivation \( d \) for all \( X \subseteq \mathbb{N} \) and all \( x \in \mathbb{N} \) we have \((\mathbb{N}, X, \bar{f}) \models f_d(x) = f(x, X) \) in the standard model of \( \mathcal{A}_{RA_1} \). The \( \Sigma^0_1 \) formula \( \varphi(x, y) \) such that that \( f \) is a provably total function of \( \mathcal{A}_{RA_1} \) is simply \( f_d(x) = y \). We have \( \forall x \exists! y \varphi(x, y) \) in \( \mathcal{A}_{RA_1} \), as \( f_d \in \mathcal{F} \), and the condition (b) is immediate.

For the \( \supseteq \) inclusion, take any function \( f : \mathbb{N} \times 2^{\mathbb{N}} \rightarrow \mathbb{N} \) such that \( f \) is a provably total function of \( \mathcal{A}_{RA_1} \). So there is a \( \Sigma^0_1 \) formula \( \varphi(x, y) \) of \( L_1(X, \bar{f}) \) satisfying conditions a) and b) of [Par. 8.1]. We will prove the theorem if we find a derivation \( d \) of \( f \) s.t. \( \mathcal{A}_{RA_1} \vdash \forall x \varphi(x, f_d(x)) \).

Note that \( \varphi(x, y) \) is \( \exists z \varphi_0 \) for some \( \Delta_0 \) formula \( \varphi_0(x, y, z) \). Hence, \( \exists! y \varphi \) is equivalent in \( \mathcal{A}_{RA_1} \) to \( \exists! \psi \psi \) where \( \psi(x, w) \) is the \( \Delta_0 \) formula \( \exists y, z \leqslant \psi(w) = (y, z) \land \varphi_0(x, y, z) \land \psi(x, w) \)
\(\forall u \succ 0, z \prec q_0(x, y, v)\). By [Lemma 7.3] there is a constant \(h := f_{\psi(x, w)}\) which is the characteristic function of \(\psi(x, w)\). \(\text{ARA}_1\) thus proves \(\exists! w \ h(x, w) = 1\).

Although the \text{BASIC} part of \(\text{ARA}_1\) contains some existential sentences, they can be eliminated at the expense of rewriting a few axioms using a newly introduced (proper) unary function symbol \(\text{Pr}\):

\[
0 < z \rightarrow (H(z), T(z)) = z \land H(z) < z \land T(z) < z \quad (\text{P2}')
\]

\[
\text{Pr}(0) = 0 \land (0 < x \rightarrow S(\text{Pr}(x)) = x \land \text{Pr}(x) < x). \quad (\text{N10}')
\]

The resulting theory \(\text{ARA}'_1\) is a conservative extension of \(\text{ARA}_1\), equivalent to the extension of \(\text{ARA}_1\) by definition \(\text{Pr}(x) = y \leftrightarrow (x = 0 \rightarrow y = 0) \land (x > 0 \rightarrow x = S(y))\) of \(\text{Pr}\). In particular, \(\text{ARA}'_1\) proves \(\exists! w \ h(x, w) = 1\).

By compactness, a finite subset \(T_1 \not\subseteq T_2\) of \(\text{ARA}'_1\) proves \(\exists w \ h(x, w) = 1\). Here, sentences \(T_1\) are universal, i.e., some of \text{BASIC}' axioms, instances of operators of \(\Lambda(X, T)\), and the uniqueness parts \(\forall x \forall y \forall z \forall f(d)(x) = y \land f(d)(x) = z \land f(d) = f(d)\) of axioms \(f_d \in \mathcal{F}\) for some derivations \(d_1, \ldots, d_n\). Sentences \(T_2\) are \(\forall x \exists w \forall f_d(x) = w\) of \(f_d \in \mathcal{F}\). Thus, \(T_1 \models \exists w \exists \vec{u} \forall \vec{v} (\bigwedge_{i=1}^{n} f_d(u_i) = v_i \rightarrow h(x, w) = 1)\).

Let us abbreviate the antecedent to \(\vec{g}(\vec{u}) \triangleq \vec{v}\).

By Lemma 8.2 we have \(n\)-tuples of terms \(\vec{s}_1(x), \ldots, \vec{s}_k(x, \vec{u}_1, \ldots, \vec{u}_{k-1})\) and terms \(t_1(x), \ldots, t_k(x, \vec{u}_1, \ldots, \vec{u}_{k-1})\) such that \(T_1\) (and hence also \(\text{ARA}'_1\)) has as its consequence the disjunction of

\[
\vec{g}(\vec{s}_1(x)) = \vec{v}_1 \rightarrow h(x, t_1(x)) = 1
\]

\[
\vec{g}(\vec{s}_2(x, \vec{u}_1)) = \vec{v}_2 \rightarrow h(x, t_2(x, \vec{u}_1)) = 1
\]

\[
\cdots
\]

\[
\vec{g}(\vec{s}_k(x, \vec{u}_1, \ldots, \vec{u}_{k-1})) = \vec{v}_k \rightarrow h(x, t_k(x, \vec{u}_1, \ldots, \vec{u}_{k-1})) = 1
\]

which can be rewritten as the formula

\[
\vec{g}(\vec{s}_1(x)) = \vec{v}_1 \land \vec{g}(\vec{s}_2(x, \vec{u}_1)) = \vec{v}_2 \land \cdots \land \vec{g}(\vec{s}_k(x, \vec{u}_1, \ldots, \vec{u}_{k-1})) = \vec{v}_k \rightarrow
\]

\[
h(x, t_1(x)) = 1 \lor h(x, t_2(x, \vec{u}_1)) = 1 \lor \cdots \lor h(x, t_k(x, \vec{u}_1, \ldots, \vec{u}_{k-1})) = 1. \quad (\theta_1)
\]

Thus \(\text{ARA}'_1 \vdash \forall \vec{v}_1 \ldots \forall \vec{v}_k \ [\theta_1]\). The last formula can be abbreviated as the quasi-bounded formula

\[
h(x, q_1(x)) = 1 \lor h(x, q_2(x)) = 1 \lor \cdots \lor h(x, q_k(x)) = 1 \quad (\theta_2)
\]

where the quasi-terms \(q_i(x)\) are obtained by substituting the respective left-hand sides from the antecedent of \(\theta_1\) for the variables \(\vec{v}_1, \ldots, \vec{v}_k\), i.e., \(q_1(x)\) is \(t_1(x)\), \(q_2(x)\) is \(t_2(x, g_1(s_{1,2}(x))), \ldots, g_n(s_{1,n}(x)))\), etc.

Note that [Lemma 7.3] can be proved also for \(\text{ARA}'_1\) in \(L_4(\text{Pr}, X, \vec{f})\). In particular, any quasi-term of the form \(\text{Pr}(t(x))\) is computed by the bounded minimization \(\mu_{\prec \text{Pr}}(t(x) < S(S(T)))\) with the derivation \(\text{Pr} := \mu_{\prec \text{Pr}}(t(x) < S(S(T)))\) of \(\text{Pr}(I, I) \circ t(x)\). We can thus replace quasi-terms \(q_i(x)\) in \(\theta_2\) with applications of the respective functions \(f_{q_i(x)}\), thus obtaining an equivalent quasi-bounded formula

\[
h(x, f_{q_1(x)}(x)) = 1 \lor \cdots \lor h(x, f_{q_k(x)}(x)) = 1. \quad (\theta_3)
\]
As this is in the original language $L_1(\mathcal{X}, \bar{f})$, we have $\mathcal{AR}_1 \vdash \{\bar{g}\}$ by conservativity.

Since for each $x$ a unique $w$ satisfies $\psi(x, w)$ and this is one of $f_{q_{1}(x)}(x)$, we can obtain it by simply testing the values $f_{q_{1}(x)}(x)$ one after another, informally:

$$f_D(h(x, f_{q_{1}(x)}(x))), f_D(h(x, f_{q_{2}(x)}(x))), \ldots, f_D(h(x, f_{q_{k}(x)}(x))), f_{q_{1}(x)}(x), f_{q_{2}(x)}(x), \ldots,$$

$$f_{q_{1}(x)}(x)).$$

Since a derivation $e$ of the above function exists in $\mathcal{A}(\mathcal{X})$, so does $d' := \text{Hoe}$ which is such that $\mathcal{AR}_1 \vdash \forall x \phi(x, f_d(x))$, thus completing the proof. \(\square\)

9 Second-Order Recursive Arithmetics

In the second draft we will modify the presentation in this section in order to be able to accommodate the theories $\mathcal{AR}_2 + \text{WKL}$. For the time being we have a problem with the formulation of correct assumptions for the Lemma 9.8. The proofs of the remaining theorems will be simplified once we fix this problem. Also Thm. 9.10 should be generalized to all function subalgebras of PRA.

9.1 Second-Order Theories for Function Algebras. Fix a class of function algebras $\mathcal{A}(\mathcal{X})$. The algebras determine a second-order theory in the language $L_2$ designated by $\mathcal{AR}_2$ and called the second-order recursive arithmetic of $\mathcal{A}(\mathcal{X})$. The theory is axiomatized by BASIC plus the following set existence axioms:

$$\forall g_1, \ldots, g_n \in F \exists f \in F \ f := \text{op}(g_1, \ldots, g_n),$$

one for each operator $f := \text{op}(g_1, \ldots, g_n)$ of $\mathcal{A}(\mathcal{X})$. There is the oracle axiom:

$$\forall X \exists f \in F \ f := X_*$$

and the function comprehension axiom FC:

$$\forall p \forall f \in F \exists X \ X := \text{FC}(f, p)$$

where $X := \text{FC}(f, p)$ abbreviates $\forall x(x \in X \iff f(x, p) = 1)$.

9.2 Lifting of First-Order Sentences to Second-Order. The language of second-order arithmetic $L_2$ is in general not an extension of the first-order language $L_1(\mathcal{X}, \bar{f})$. In order to characterize the relationship of the second-order arithmetic $\mathcal{AR}_2$ with its first-order counterpart $\mathcal{AR}_1$ we will employ a syntactic transformation called lifting.

Fix a language $L_1(\mathcal{X}, \bar{f})$. For $k \in \mathbb{N}$, let $\bar{f}_k$ be the initial part $f_0, \ldots, f_{k-1}$ of $\bar{f}$, and let $\Phi_k$ be the set of definitions of symbols $f \in \bar{f}_k$, i.e., either the oracle axiom $f := X$ or the operator axiom $f := \text{op}(\bar{g})$ of $\mathcal{AR}_1$ respective to the constant $\bar{f}$. Note that $\Phi_0$ is
empty, and if there is \( f_d := \text{op}(f_d_1, \ldots , f_d_n) \) in \( \Phi_k \), then there are also the definitions of \( f_d_1, \ldots , f_d_n \) in \( \Phi_k \).

Let \( \varphi \) be any \( L_1(X, \vec{f}) \) sentence. We define \( \Phi_\varphi := \Phi_k \) and \( \vec{f}_\varphi := \vec{f}_k \) where \( k = d + 1 \) for the highest \( d \) such that \( f_d \) occurs in \( \varphi \), and \( k = 0 \) if no such \( d \) exists. If we treat set constants \( X \) and \( \vec{f}_\varphi \) as set variables of \( L_2 \), then \( \forall X \forall \vec{f}_\varphi ([\bigwedge \Phi_\varphi] \rightarrow \varphi) \) is a formula of \( L_2 \). We write it as \( \varphi \uparrow \), and call it the \textit{lifted form} of \( \varphi \).

\[ 9.3 \text{ Lemma.} \ ARA_2 \models \forall X \exists \vec{f}_k \bigwedge \Phi_k \]

\[ \text{Proof} \quad \text{The claim is easily proved by external induction on } k. \text{ The inductively constructed proofs in } ARA_2 \text{ use its axioms (X),} \]

\[ 9.4 \text{ Relationships of First-Order and Second-Order Arithmetics.} \text{ Lifting enables us to express the connection of a first-order and the respective second-order recursive arithmetics through the following analogues of standard notions of extensions and conservativity.} \]

We say that a second-order theory \( T \) is a \textit{lifted extension} of \( ARA_1 \) if it proves the lifted forms of all its theorems, i.e., for any sentence \( \varphi \in L_1(X, \vec{f}) \) such that \( ARA_1 \models \varphi \) we have \( T \models \varphi \uparrow \).

For a class of sentences \( \Gamma \subseteq L_1(X, \vec{f}) \), a second-order theory \( T \) is \textit{lifted } \( \Gamma \) \textit{conservative over } \( ARA_1 \) if all \( \Gamma \) sentences whose lifted forms are theorems of \( T \) are theorems of \( ARA_1 \), i.e., for any sentence \( \varphi \in \Gamma \) such that \( T \models \varphi \uparrow \) we have \( ARA_1 \models \varphi \).

The following \[ \text{Thm. 9.5} \text{ straightforwardly shows that } ARA_2 \text{ is a lifted extension of } ARA_1. \text{ Conservativity is more involved: We first show how certain models of first-order arithmetics can be extended to second-order models in } \text{Lemma 9.8} \text{ We then show } \Pi_2^0 \text{ conservativity for the special case of polynomially bounded arithmetics in } \text{Thm. 9.10} \]

\[ 9.5 \text{ Theorem.} \ ARA_2 \text{ is a lifted extension of } ARA_1. \]

\[ \text{Proof} \quad \text{Fix a class of function algebras } A(X). \text{ Take any sentence } \varphi \text{ such that } ARA_1 \models \varphi, \text{ and any second-order structure } (M, S) \not\models ARA_2. \text{ Consider } \varphi \uparrow \text{ and the respective } \Phi_\varphi = \Phi_k \text{ for some } k, \text{ which has } X \text{ and } \vec{f}_k \text{ as its free set variables. Choose any sets } Y, \vec{g}_k \in S \text{ so that } (M, S) \models \Phi_\varphi(Y, \vec{g}_k). \]

Since \( (M, S) \) satisfies \( \text{(X)} \) and \( \text{op} \) axioms for operators of \( A(X) \), \( \vec{g}_k \) can be extended by external induction to \( \vec{g}_d \) such that \( (M, S) \models \Phi_d(Y, \vec{g}_d) \) for all \( d \in \mathbb{N} \). Thus \( (M, Y, \vec{g}) \) for \( \vec{g} = \bigcup_{d \in \mathbb{N}} \vec{g}_d \) is a model of \( ARA_1 \), and hence a model of \( \varphi \) as well. Then, however also \( (M, S) \not\models \varphi(Y, \vec{g}_k) \). \]

\[ 9.6 \text{ Theorem.} \ ARA_2 \not\models \text{RCA}_0^- \]

\[ \text{Proof} \quad \text{The two theories share the BASIC axioms. Since } A(X) \text{ includes } \varnothing \text{ along with other operators of } \Delta(X), \text{ we have } ARA_2 \not\models \varnothing. \text{ It thus remains to prove } C[\Delta_0^0] \text{ and IND in } ARA_2. \]

For \( C[\Delta_0^0] \), take a \( \Delta_0^0 \) formula \( \varphi(x, \vec{y}, \vec{X}) \) with all parameters among those in non-empty sequences \( \vec{y} \) or \( \vec{X} \). Let \( \varphi'(x, \vec{y}, \vec{X}) \) be the result of replacing every occurrence
of \( \tau \in X_i \) in \( \varphi(x, \bar{y}, \bar{X}) \) with \((S'(0), \tau) \in X\). Considering \( X \) as the oracle set constant, obtain the characteristic function \( f' \in \vec{f} \) of \( \varphi'(x, \bar{y}) \) from \[ \text{Lemma 9.5} \]

Note that \( \mathcal{ARA}_1 \vdash \forall x \forall y \, f'(x, \bar{y}) \leq 1 \land \forall x \forall y (f'(x, \bar{y}) = 1 \iff \varphi'(x, \bar{y})) \) and denote this formula by \( \psi \). \( \mathcal{ARA}_2 \) proves \( \forall X \, \forall f'_w (\Phi_\psi \Rightarrow \psi) \) by the extension theorem \[ \text{9.5} \] and \( \forall X \, \exists f'_w \Phi'_w \) by \[ \text{Lemma 9.3} \]

Let us now reason in \( \mathcal{ARA}_2 \): Take any numbers \( \bar{y} \) and any sets \( \bar{X}, \bar{X} \), and obtain for every \( X_i \in \bar{X} \) its characteristic function \( g_i \), by \( \{ \text{X} \} \) axiom. By applying the respective operators for \( \Delta(X) \) operators, obtain the function \( g \) such that \( \forall x \forall p \left( (g(x, p) = 1 \iff \forall_i (H(x) = S'(0) \land g T(x) = 1)) \right) \) (cf. Lemmas \[ \text{7.5} \] and \[ \text{7.4} \]. Obtain \( Y \) from \( g \) using the axiom \( \{ \text{FC} \} \) with \( p = 0 \). Notice that \( \forall x ((S'(0), x) \in Y \iff x \in X_i) \). Since \( \forall X \, \exists f'_w \Phi'_w \), take some \( f'_w \) for which \( \Phi'_w (Y, f'_w) \) holds. We then have \( \psi(Y, f'_w) \), and \( f' \) among \( f'_w \) is such that \( f'(x, \bar{y}) = 1 \iff \varphi'(x, \bar{y}, Y) \iff \varphi(x, \bar{y}, X) \) for all \( x \) and \( \bar{y} \). Hence the \( \{ \text{FC} \} \) axiom gives us for \( f' \) and \( p = (y_1, \ldots, y_n) \) a set \( X \) such that \( \forall x (x \in X \iff \varphi(x, \bar{y}, X)) \). This concludes the proof of \( C[\Delta_0^0] \).

Let us prove \( \text{IND} \) in \( \mathcal{ARA}_2 \): Take any \( X \) and assume \( 0 \in X \), and \( \forall x (x \in X \rightarrow S(x) \in X) \). Suppose there is some \( z \notin X \). Obtain the set \( X' \) such that \( \forall x (x \in X' \iff H(x) \notin X) \) from \( C[\Delta_0] \), its characteristic function \( g'(x) = 1 \iff x \in X' \) from \( \{ \text{X} \} \), and its bounded minimization \( f'(b, p) = \mu_{y<_b} g'(y, p) = 1 = \mu_{y<_b} [y \notin X] \) from the respective \( \{ \text{OP} \} \) axiom. Now \( y = f'(z, 0) \) is the least number such that \( y \notin X \). Since \( 0 \in X \), there must be \( u \) such that \( y = S(u) \). By minimality of \( y \), we have \( u \in X \), but then \( S(u) = y \notin X \) contradicts the second assumption of the induction axiom. Thus \( \forall x \in X \).

\[ \square \]

\[ \text{9.7 Corollary.} \] The theories \( \mathcal{RCA}_0^- \) and \( \Delta \mathcal{RA}_2 \) are equivalent.

\[ \text{Proof} \] \( \Delta \mathcal{RA}_2 \vdash \mathcal{RCA}_0^- \) is a special case of \[ \text{Thm. 9.6} \]. For \( \mathcal{RCA}_0^- \vdash \Delta \mathcal{RA}_2 \), \( \mathcal{RCA}_0^- \) includes \( \circ \), and we can easily prove \( \{ \text{X} \} \), \{ \text{FC} \}, and \( \{ \text{OP} \} \) for operators \( \circ, \otimes, \preceq, \ast, I, D, P \) and \( \mu \) within \( \mathcal{RCA}_0^- \) using \( C[\Delta_0] \). For instance, in the latter case, we obtain \( f := \mu(g) \) as

\[
\begin{align*}
w \in f & \iff w = (0, 0) \lor \exists b \leq w \exists x \leq w \exists z \leq b (w = ((b, x), z) \\
& \land ((z, x), 1) \in g \land \forall y < z ((y, x), 1) \notin g \land z = b \land \forall y \leq b ((y, x), 1) \notin g).
\end{align*}
\]

The function value uniqueness part of \( f \in F \) is immediate. The function value existence part follows from \( \text{IND} \) for the set \( X \) such that \( u \in X \iff \forall w \leq u \exists x, b \leq w \exists z \leq b (w = (b, x) \land (w, z) \in f) \) obtained by \( C[\Delta_0] \).

\[ \square \]

\[ \text{9.8 Lemma.} \] If \( (\mathcal{M}, X, \vec{f}) \models \mathcal{ARA}_1 \) and \( I \) is a proper initial segment of \( \mathcal{M} \) such that \( (I, X \cap I, \{ f_d \cap I \}_{d \in \mathbb{N}}) \models \mathcal{ARA}_1 \) then there is a class \( S := FC_I (\mathcal{M}, X, \vec{f}) \) of subsets of \( \mathcal{M} \) such that \( (I, S) \models \mathcal{ARA}_2 \) and \( X \cap I \in S \) as well as \( f \cap I \in F^{(I, S)} \) for all \( f \in \vec{f} \).

\[ \text{Proof} \] Assumptions of this lemma are too weak and do not force the closure of \( S \) under recursive ops of the algebra \( \mathcal{ARA}_1 \), although they are sufficient for the closure under \( \mu \) and explicit ops. A strengthening of the assumptions to semiregular cuts is too strong, it forces \( S \) to be closed under primitive recursion. We are currently looking into some intermediate assumptions and think that we know how to formulate them. The basic problem is that we have to know more about the recursive operators of the algebra.
9.9 Function Comprehension and Standard Systems. The class denoted in the preceding lemma as $\text{FC}_I(M, X, \bar{f})$ is closely related to the standard system

$$\text{SSy}_I(M) = \{ \{ x \mid x \in I, M \vdash x \in_{\text{Ack}} y \} \mid y \in M \}.$$ 

There is the characteristic function of the $\Delta_0$ formula $x \in_{\text{Ack}} y$ among $\bar{f}$. It produces every set from $\text{SSy}_I(M)$ via function comprehension, hence $\text{SSy}_I(M) \subseteq \text{FC}_I(M, X, \bar{f})$.

If additionally $I <^M a \in M$ and $(M, X, \bar{f}) \not\vdash \text{exp}$, then every $\Delta_0$ set of elements less than $a$ is coded in $M$, i.e., for every $\Delta_0^0$ formula $\varphi(x, \bar{z})$ we have $(M, X, \bar{f}) \not\vdash \forall \exists \exists y \forall x < a (\varphi(x, \bar{z}) \leftrightarrow x \in_{\text{Ack}} y)$ by a folklore lemma [10] Lemma IV.2.12 [6] Prop. 2.1. For inductive $X \subseteq M$, the lemma applies to $\Delta_0^0(X, \bar{f})$ formulas as well, hence, in particular, to $f_d(X, \bar{p}) = 1$. Therefore, $\text{FC}_I(M, X, \bar{f}) \subseteq \text{SSy}_I(M)$.

Thus, if $\text{ARA}_1 \not\vdash \text{exp}$, we have $\text{FC}_I(M, X, \bar{f}) = \text{SSy}_I(M)$ for every proper initial submodel $(I, X \cap I, \bar{f} \cap I)$ of any $(M, X, \bar{f}) \models \text{ARA}_1$.

9.10 Theorem. If $\mathcal{A}(X)$ is a class of function algebras with polynomially bounded functions, then the second-order arithmetic $\text{ARA}_2$ is a lifted $\Pi^0_2$ conservative extension of the first-order arithmetic $\text{ARA}_1$.

Proof Take any $\mathcal{A}(X)$ with polynomially bounded functions. $\text{ARA}_2$ is a lifted extension of $\text{ARA}_1$ by the previous theorem.

For lifted $\Pi^0_2$ conservativity, suppose a $\Pi^0_2$ sentence $\forall x \exists y \varphi(x, y)$ is not provable in $\text{ARA}_1$. We need to show that its lifted form is not provable in $\text{ARA}_2$, i.e., to find a model of $\text{ARA}_2$ with sets $X$ and $\bar{f}_X$ satisfying $\Phi_\varphi = \Phi_k$ while not satisfying $\forall x \exists y \varphi(x, y)$.

Since $\text{ARA}_1 \models \forall x \exists y \varphi(x, y)$, the theory $\text{ARA}_1 + \forall y \varphi(e, y)$ with $e$ a new constant is consistent. Moreover, for another new constant $c$ every finite subset of the theory

$$T := \text{ARA}_1 \cup \{ \forall y \varphi(e, y) \} \cup \{ e^k < c \mid k \in \mathbb{N} \}$$

is consistent as well. By compactness, $T$ is consistent, and by completeness, it has a model $\mathcal{N} := (M, X, \bar{f}, e, c)$.

Let $I$ be the $L_1$ structure with the domain $I = \{ x \mid x \in M, M \models x < e^k, k \in \mathbb{N} \}$. $I$ is a proper initial segment of $M$ due to the definition of $I$ and since $I <^M c$. Since the functions of $\mathcal{A}(X)$ are polynomially bounded, $I$ is closed under the functions $\bar{f}$. Moreover, $e \in I$ and $\forall y \varphi(e, y)$ is $\Pi^0_1$- and thus absolute. Hence $(I, X \cap I, \{ f_d \cap I \}_{d \in \mathbb{N}}) \models \text{ARA}_1 + \forall y \varphi(e, y)$.

[Lemma 9.8] now gives us a system of sets $S$ such that $(I, S) \models \text{ARA}_2$ with $\{ X \cap I \} \cup \{ f_d \cap I \}_{d \in \mathbb{N}} \subseteq S$. We thus have $(I, S) \models \Phi_k(X \cap I, \{ f_d \cap I \}_{d \in \mathbb{N}})$, and $(I, S) \models \forall y \varphi(e, y)(X \cap I, \{ f_d \cap I \}_{d \in \mathbb{N}})$. Hence $(I, S) \not\models \varphi^e$, as desired. □

10 Some Function Algebras for Complexity Classes

In this paper we have introduced a general framework for connecting the provable functions of first and second-order recursive arithmetics. This section serves as an illustrative
application where we formulate several subelementary recursive arithmetics capturing some of the main complexity classes. For this reason the section does not contain any theorems and its assertions are mostly only sketched out.

10.1 Space Algebras $Sp(X)$. The function operator $f \eqdef PR(g, h)$ (see Par. 3.3) is bounded when for all $x \in \mathbb{N}$ we have $f(x) \leq b(x)$. This is a semantic condition and so we cannot use it as an operator (it does not always yield a function). We define instead the operator $f \eqdef BPR(g, h)$ of bounded primitive recursion by the following clausal definition with the identity function as measure:

$$g(p) = z \land z \leq p \rightarrow f(0, p) = z$$
$$h((v, f(v, p)), p) = z \land z \leq p \rightarrow f(v + 1, p) = z.$$ 

When we present a non-strict clausal definition like this we trust the reader that they can transform it into a strict one. In this case this means the applications of functions in quasi-terms must be unnested (i.e. $f(v, p)$). The consequents have to be brought to the strict form $f(x) = y$ which involves possibly renaming variables and moving the terms in the arguments of $f$ in the consequents (such terms are called in computer programming patterns) by moving them into antecedents. For instance, in the first clause we put $x = (u, p) \land u = 0$ into the antecedent and add the missing clauses, so called default clauses when $f$ will yield $0$, i.e. $x = (u, p) \land u \neq 0 \land \ldots$, or the clause $f(0) = 0$. After adding the default clauses, the clauses should be conjuncted into one formula and its first-order variables universally closed.

We designate by $Sp(X)$ the class of space algebras obtained by adjoining the operator $BPR$ to the operators of the class $\Delta(X)$. The function $f$ yielded by the operator $BPR$ is non-growing and therefore the functions of the algebra are polynomially bounded (note that the parameter $p$ of bounded primitive recursion can be set to at most a polynomial in $x$). We will see below that the algebra $Sp(X)$ is suitable for the characterization of space complexity classes.

It should be clear that $Sp(X)$ is closed under primitive recursion $f \eqdef PR(g, h)$ which is bounded by $b$ because we can define $f_1 \eqdef BPR(g, h)$ and then $f(x) = f_1(x, \max(b(x), 1))$.

The computation of $f(x, p)$ defined by $BPR$ when done by iteration requires the space sufficient to hold two numbers no larger than $p$ provided $x \leq p$. This is the way computations in space complexity classes are done.

10.2 Time Algebras $Tm(X)$. We define the operator of special nested recursion $f \eqdef SNR(g, h)$ by the following schema of clausal definitions:

$$g(x, p) = (0, z) \land z < x \land f(z, p) = v \land h((x, v), p) = w \land w < x \rightarrow f(x, p) = f(w, p)$$
$$g(x, p) = (1, z) \land z \leq p \rightarrow f(x, p) = z.$$ 

The function $f$ is obviously non-growing and the identity function is its measure.

We designate by $Tm(X)$ the class of time algebras obtained by adjoining the operator $SNR$ to the operators of $\Delta(X)$. The functions of $Tm(X)$ are polynomially bounded.
because the parameter $p$ of special nested recursion can be set to at most a polynomial in $x$.

We call a recursive clausal definition of $f$ bounded nested if identity is its measure function and for some bounding function $p$ we have $f(v) \leq p(v)$ for all $v \in \mathbb{N}$. We will now show the algebras $Tm(X)$ closed under such definitions by defining $f$ by special nested recursion. We first transform the definition of $f$ to the explicit function $h$ as in Par. 6.6 from where we also obtain the constant $J$ giving the maximal nesting of recursions in the clauses of $f$. It basically remains to reduce $J$ to 2. To that end we define $f_1 := \text{SNR}(g_1, h_1)$ where the auxiliary functions have the following explicit clausal definitions:

$$p = (m, b, p') \land v = [x, c']_b \land m \vdash c' = c \land h(x, c) = (0, z) \rightarrow g_1(v, p) = (0, [z, m]_b)$$

$$p = (m, b, p') \land v = [x, c']_b \land m \vdash c' = c \land h(x, c) = (1, z) \rightarrow g_1(v, p) = (1, z)$$

$$p = (m, b, p') \land v = [x, c']_b \land m \vdash c' = c \rightarrow h_1((v, w), p) = [x, m \vdash (c \oplus_m (w, 0))]_b .$$

The argument $v$ in both functions is a pair of numbers $x$ and $c'$ which is not coded by the Cantor’s function but rather as two digits of a number in the base $b$: $[x, y]_b := x \cdot b + y$ which has the pairing property when $x, y < b$. This function, both of its projections, as well as the modified subtraction $\setminus$ (yielding 0 if the result should be negative) are easily derivable in $\Delta(X)$.

The lists $c$ passed to the function $h$ contain at most the values $p := p(x)$ and they grow during the computation of $f_1$ from the length of 0 to the maximal length $J$. They are thus at most $m := (p, \ldots, p, 0)$ large. We now have an explicit definition of $f$ as $f(x) = f_1([x, m]_b, m, b^2)$ where $b := \max(x, m) + 1$. Note that the lower $b$-digit $c'$ of $v$ codes $c$ “backwards” where the list is $c := m \vdash c'$. This makes the measure of $f_1$ the identity function.

It remains to derive the bounded list concatenation $x \oplus_m y$ (abbreviating $\oplus((x, y), m)$) as a non-growing function. Note that we do not have the general concatenation function in $Tm(X)$ because it is bounded by an exponential with the exponent depending on $L(x)$. Since $L(c) \leq J$ it suffices to use the following approximation explicitly defined in $\Delta(X)$:

$$0 \oplus_m y = y$$

$$(z_1, y) = z \land z \leq m \rightarrow (z_1, 0) \oplus_m y = z$$

$$\vdots$$

$$(z_1, \ldots, z_{J-1}, y) = z \land z \leq m \rightarrow (z_1, \ldots, z_{J-1}, 0) \oplus_m y = z .$$

Note that the operator $f := \text{BPR}(g, h)$ of bounded primitive recursion yields a non-growing function and so its definition is a bounded nested one and we have $Sp(X) \subseteq Tm(X)$.

As we have seen in Par. 6.6 a straightforward evaluation of nested recursive definition of $f(x)$ with the identity as measure needs a stack whose length is $x$ and time (length
by looking up the recursive applications in the definition of \( \propto \) the time \( \propto \) can be reduced to \( \propto \) Jones in [11] who has noticed that the exponentially many steps of the stack computation of iteration) exponential in \( \propto \) is bounded by \( \propto \) memoization. Such techniques are called in computer programming memoization. However, the combination of a stack with a look-up table is not necessary, because the course-of-values recursion does the trick directly.

10.3 Classes of Computational Complexity and Function Algebras. We wish to connect the classes of computational complexity with the classes of function algebras \( A(X) \). Toward that goal we present the complexity classes \( C \) in two forms. A type-0 class is a set of subsets of \( \mathbb{N} \). For a set \( P \) in such a class we decide whether or not \( x \in P \) by presenting \( x \) to a computing device (usually a Turing machine) in the binary representation as finite sequences of 0 and 1. A type-1 class is a set of subsets of \( 2^{<\mathbb{N}} \). For a set \( P \) in such a class we decide whether or not \( X \in P \) by presenting to a computing device the binary representation of Ackermann’s encoding of \( X \), i.e., a finite sequence \( x_{|X|-1}, \ldots, x_0 \) such that for all \( i < |X| \) we have \( x_i = 1 \) if \( i \in X \) and 0 otherwise.

A type-0 class \( C \) is 0-characterized by the class \( A(X) \) if \( C \) is the set of \( P \subseteq \mathbb{N} \) such that there is a provably recursive \((0−1)\) valued function \( f \) of \( A\text{RA}_1 \) and \( P = \{ x \in \mathbb{N} \mid f(x, \emptyset) = 1 \} \).

A type-1 class \( C \) is 1-characterized by the class \( A(X) \) if \( C \) is the set of \( P \subseteq 2^{<\mathbb{N}} \) such that there is a provably recursive \((0−1)\) valued function \( f \) of \( A\text{RA}_2 \) and \( P = \{ X \in 2^{<\mathbb{N}} \mid f(|X|, X) = 1 \} \).

The type 0 and 1 characterizations of complexity classes by means of different inputs to function algebras stem from the second author’s cooperation with L. Kristiansen (see e.g. [13]).

10.4 Some Function Algebras Characterizing Complexity Classes. The space class of algebras \( Sp(X) \) 0-characterizes the complexity class \( \text{LINSPACE} \) (i.e. \( \text{Space}(O(n)) \)). The same class 1-characterizes the class \( \text{LOGSPACE} \) (i.e. \( \text{Space}(O(\log(n))) \)). The 0-characterization comes from the early result of Ritchie [17,13] that \( \text{LINSPACE} \) is identical to the predicates of the Grzegorczyk’s class \( \mathcal{E}_2 \) [7,18]. The class \( \mathcal{E}_2 \) is defined by bounded primitive recursion and so are the algebras \( Sp(X) \). For a more detailed discussion see [13].

The time class of algebras \( Tm(X) \) 0-characterizes the complexity class \( \text{ETIME} \) (i.e. \( \text{Time}(2^{O(n)}) \)). The same class 1-characterizes the class \( \text{PTIME} \) (i.e. \( \text{Time}(n^{O(1)}) \)). The characterization of \( \text{PTIME} \) by bounded nested recursion is from [11], the modification to the 0-characterization is obvious because of exponentially more time available (as a
function of input): $x$ vs. $|X|$. There is an old characterization of $ETIME$ by bounded twofold recursion by Monien [163] which is, however, not nested.

The class of algebras $\Delta(X)$ $0$-characterizes the complexity class $LINTH$ (linear time hierarchy) which is the class of $\Delta_0$-definable predicates (see [3]). The same class $1$-characterizes the class $LOGTH$ (log time hierarchy), also known as FOL (first-order logic) in finite model theory.

If we add the nullary operator $f := E$ (see Par. 3.3) to the operators of $\Delta(X)$ we obtain the class of algebras $\Delta e(X)$ which are obviously the algebras of elementary functions with oracles. We can define a subexponential operator $f := \#$ yielding the function $f(x) = 2^{|x|}$ which has the same growth rate as the smash function $x \# y = 2^{|x||y|}$ or the function $\omega_1(x, y) = x^{|y|}$ of Wilkie and Paris [24] where $|x|$ is the size of $x$ in binary representation. Adding the operator to $\Delta(X)$ gets the class $\Delta\#(X)$ which $0$-characterizes the complexity class $PH$ of polynomial time hierarchy (see c.f. [10]). Adding the $\#$ operator to the algebra $Sp(X)$ gets the class $Sp\#(X)$ which $0$-characterizes the complexity class PSPACE (i.e. $Space(n^{\#(1)})$) (see [3]).

Weak König lemma ($WKL^-$) does not seem to be directly usable with the characterization of non-deterministic classes like NP (non-deterministic polynomial time), or NL (nondeterministic log space) because the lemma deals with infinite trees.

There are the following well-known inclusions of the complexity classes

$$LOGSPACE \subseteq LINTH \subseteq LINSPACE \subseteq ETIME$$

and

$$LOGTH \subseteq LOGSPACE \subseteq PTIME \subseteq PH \subseteq PSPACE.$$  

Frustratingly, the questions whether any of the inclusions are strict are the major open problems of computational complexity, although we have $LOGSPACE \subsetneq LINSPACE \subsetneq PSPACE$.

The arithmetic $\Delta eRA_1$ is obviously equivalent to the Elementary function arithmetic $EFA$ and $\Delta\#RA_1$ is a conservative extension of the theory $I \Delta_0(\Omega_1)$ of [24] where $\Omega_1$ states that the function $\omega_1$ is total. Although it is known that the hierarchy $I \Delta_0(\Omega_k)$ is strict and spans the theory $\Delta_0(\exp)$ (see, e.g., [10]), its levels $k > 1$ are not directly connected to any major complexity classes.

## 11 Conclusions and Future Work

For the final version of this paper we plan to tidy up the axioms in BASIC. We think that the languages of arithmetic integrating the four forms of its presentation (by induction on first-order formulas, by recursive arithmetics (both first- and second order), and by second order arithmetics in the style of Friedman and Simpson) should be based on the pairing function as the basic binary function. The language of arithmetics should contain the constant $0$, the symbol $(\cdot, \cdot)$ of pairing, and possibly the relation symbol $<$ as basic. All remaining symbols should be set constants (in the first-order theories) and set variables (in the second-order theories). The axiomatization could be by the pairing
axioms P1−22 as well the quasi-formulas characterizing the successor function with the help of the modified Cantor’s pairing:

\[ S(0) = (0, 0) \quad S(x, S(y)) = (S(x), y) \quad S(x, 0) = (0, S(x)) . \]

From this we get the usual properties of the successor function.

The properties of the set constants + and ⋅, which are in this draft designated by ⊕ and ⊗ respectively, can be axiomatized by recursive quasi-formulas. We can possibly replace the relation symbol < by the set constant <∗ denoting its characteristic function. For syntactic comfort we should use quasi-terms and quasi-formulas as abbreviations for their unnested forms. We did not systematically use them in this draft because we have in our arithmetic languages the symbols +, ⋅, and < available as the standard ones.

The recursive arithmetics introduced in this draft are formulated in such a way that we can add to them as initial functions the hierarchy functions of Grzegorczyk’s hierarchy to characterize the theory Σ₁ and the functions of the Weiner-Schwichtenberg hierarchy for the characterization of fragments Σ₄+₂ of PA (cf. [1]).

For the second-draft of this paper we should present the recursive arithmetics as triples \( \mathcal{A}_{\mathcal{R}_1}, \mathcal{A}_{\mathcal{R}_2}, \) and \( \mathcal{A}_{\mathcal{R}_2} + \mathcal{WKL} \). We think that the last theory can be characterized similarly as the theory \( \mathcal{WKL}_0 \) vs. \( \mathcal{RCA}_0 \). The missing element, completing this to a quadruple of the kind discussed in the introduction, is a theory with induction, say \( \mathcal{I}_{\Delta_0}(e) \) with \( e \) an axiom asserting the totality of some subexponential function. It seems that the space and time arithmetics \( SpRA_1, TrmRA_2 \) cannot be fully characterized in this way. For instance, the smash function \( 2^{[n]} \) does not seem to work except in the cases mentioned in [Par. 10.4].

The obvious area for research is the characterization of major non-deterministic classes (NPTIME, NLOGSPACE) by means of recursive arithmetics. Although non-determinism can be viewed as a search for a path in a tree expressing a particular property, the approach through \( \mathcal{WKL} \) does not seem to work because of the lack of exponentiation (not too many definable trees) and it probably will not be possible to downscale the infinite trees to the finite ones of complexity theory.

Another area for research is the characterization of subexponential second-order models in the form \( (\mathcal{M}, SSy(\mathcal{M})) \). We were not able to do this because there do not seem to be sufficiently many coded sets.

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11. CONCLUSIONS AND FUTURE WORK

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