1. Introduction.

1.1. Concentration inequalities. Let \((S_i)_{i=1,2,...}\) be a random process of one-dimensional random walk on the real line. \(S_n\) can be formulated as \(S_n = X_1 + X_2 + ... + X_n\); \(X_i = S_i - S_{i-1}\). If some conditions are satisfied, among which independence is crucial, \(S_n\) converges to a Gaussian distribution in a \(\sqrt{n}\) neighborhood of its mean — central limit theorem (CLT). For \(S_n\) beyond the neighborhood, i.e., \(|S_n - E(S_n)| \gg \sqrt{n}\), Chernoff inequality (\(1.1\)), also referred as Chernoff bound,

\[
Pr(|S_n - E(S_n)| \geq \lambda) \leq e^{-c\frac{\lambda^2}{n}}
\]

tells us how unlikely this occurrence is. In this note, by "Chernoff bound" we refer to the tail probability inequalities of the form (1.1)—the square-exponential decay probability bound. Chernoff inequality is extended to the more general context of bounded martingale difference (say, \(E(X_n | S_{n-1}) = 0\)), referred as Azuma-Hoeffding inequality \([4]\) (see \([15], [28]\) for the survey and references therein). Refinements and extensions followed (e.g. \([33], [28], [6], [7], [18], [25]\)). Known as concentration inequalities, these prove very useful and have a wide variety of applications in computer science, combinatorics, information theory (see e.g. \([17], [3], [31], [30]\)).

1.2. Branching random walk (BRW). In the literature, a branching random walk on the real line is described as follows (see, for example, \([11], [8], [9]\)). In generation zero, an initial particle at the origin on the real line \(\mathbb{R}\). It splits into a random number of child particles who form generation one. The children's displacements, relative to their parent, correspond to a point process on \(\mathbb{R}\). The children in turn split too to form the second generation, and so on. If the average split number (branching factor) is greater than one, with positive probability the number of the descendants grows exponentially through generations. Current BRW studies typically address models where the offspring’s behavior is independent of that of their previous generation (e.g. \([8], [9], [10], [22], [19]\)). The law of large numbers and central limit theorem type results about the distribution of
position are established, under some conditions of independence (for instance, i.i.d of branching and walking). Yet unlike random walk, Chernoff bound is not known so far even in the case of i.i.d aforementioned, while the minimal (and maximal) is studied by many.

Considering i.i.d. offspring (and hence independent of the parent’s position), and i.i.d. displacement, Harris [20] conjectured that the distribution of the descendants’ position of the nth generation approaches Gaussian distribution. This conjecture was proved by [32], [23]. Its extended generation-dependent versions, where offspring and displacement distribution are dependent on generation n, were proved by [12], [24], to mention a few. Problems where offspring and displacement are dependent of parents’ positions are studied in adhoc approaches though (e.g. [34]). Concerning the deviation from the expectation, [9] proved that for any δ > 0 the number of particles locating beyond μ − δ is zero almost surely, μ denoting the scaled expectation. In another direction of estimating the rareness of the large deviation, extremum is well studied (e.g. [16] [29], [5], [13] [22], [1] [2]). [16] showed tightness for $M_n - EM_n$, where $M_n$ is minimum of nth generation. [29] gave probability bound for the deviation of $M_n$, i.e. $Pr(M_n - Med_n > x) < e^{-\delta x}$, where $Med_n$ is the median of $M_n$. So far probability bound for deviation $\epsilon$ of appropriate scale ($\epsilon > 0$) is established. Little is known as to how sparse the population is at a distance of $O(\sqrt{n})$ far from the mean in terms of concentration inequalities.

2. Chernoff bound for BRW. Throughout we consider BRW on N for notational simplicity, though the argument applies to $\mathbb{R}$. We start by a new setting for BRW, which is more general in three respects. First, in our framework both subcritical and supercritical BRW are treated equally. While traditionally subcritical BRW is considered trivial because a branching random walk process almost surely ends with zero population when $n$ is large, we see that the probability space for the survival BRW processes is well defined with infinitely large number of ancestors. Second, the underlying random walks between siblings are not assumed to be independent. Third, branching factor (birth rate) is not assumed identical across generations and siblings. The only major requirement is the independence between birth-rate and birth-place. Specifically, we define a BRW as a sequence of pairs

$$(m_i(u), p_i(x|u))_{i=1,2,\ldots,n}$$

where $m_i(u)$ is the expectation of offsprings (branching factor) of a parent at position $u$, and $p_i(x|u)$ is offsprings’ displacement pdf, the probability (or proportion) density function, or mass probability in discrete cases.

Starting from one initial ancestor (1st parent), a realization of branching random walk process is a random rooted tree. Each node in the tree is associated with a position which equals to the sum of the displacements of its previous parents and the displacement of itself, $u$. As a parent located at $u$, this node in turn produces some number of children with $u$ as the birth-place. Infinite BRW processes with such initial ancestors make a forest, a probability space of the BRW. In other words, the probability space of the BRW can be interpreted as a forest grows from infinite roots —initial ancestors. Let $S_n = X_1 + X_2 + \ldots + X_n$ be the position of a leaf in the nth generation, where $X_i$ is displacement (step size relative from the birth place) of its ith parent. $(X_1, X_2, \ldots, X_n)$ is called a spine (a path from the origin to $S_n$). The central limit theorem for BRW states that almost surely a randomly chosen tree has a Gaussian “canopy”; namely, $S_n$ on the tree has Gaussian distribution. Let $u_i$ be an individual’s position, i.e. $X_1 + \ldots + X_i$, at generation $i$. At position $u_i$, the mean of birth rate is $m_i(u)$ (generally the mean of birth rate may be dependent on the birth place).

Let $M_0$ be the number of initial ancestors (can be infinity), and $M_i(u)$ the total population of ith generation at position $u$, in the BRW forest. The population, at $i + 1$ generation, produced by
those $M_i(u)$ particles is calculated by

$$M_i(u) \sum_x m_i(u)p_{i+1}(x)$$

where

$$M_i(u) \cdot m_i(u)p_{i+1}(x)$$

is population at $u + x$, $m_i(u)$ is the mean of birth-rate of a (parent) particle at $u$ in $i$th generation, and $\sum_x p_{i+1}(x) = 1$. Note, $p_{i+1}(x)$ is dependent of $u$ which we drop off for notational simplicity; given $u$, $p_{i+1}(x)$ is viewed as a function of $x$ only, though. With

$$M_0 \cdot m_1(u_1)p_2(x) \quad (u_1 = X_1)$$

being the population (of 2nd generation) at $u_1 + x$ produced by 1st generation, we have by induction the total population at generation $n$ (note, $p_1(u_1) = 1$)

$$\sum_{x_1} \sum_{x_2} ... \sum_{x_n} M_0 p_1(x_1)m_1(u_1)p_2(x_2)...m_{n-1}(u_{n-1})p_n(x_n)$$

and the law of spine $(x_1, x_2, ..., x_n)$

(2.1)

$$\frac{p_1(x_1)m_1(u_1)p_2(x_2)...m_{n-1}(u_{n-1})p_n(x_n)}{\sum_{x_1} \sum_{x_2} ... \sum_{x_n} p_1(x_1)m_1(u_1)p_2(x_2)...m_{n-1}(u_{n-1})p_n(x_n)}$$

which can be interpreted as the proportion of $(x_1, x_2, ..., x_n)$ in the whole forest. As aforementioned earlier in this section, we are concerned with BRW where birth-rate is independent of birth position; i.e. $m(u)$ is not dependent on $x$. In this case, the law of spine (2.1) is reduced to

(2.2) $$p_1(x_1)p_2(x_2)...p_n(x_n)$$

which is the probability distribution of the random walk $X_1 + X_2 + ... + X_n$ where $Pr(X_i = x | u_{i-1}) = p_i(x)$. This observation turns questions about BRW into questions about random walk (without branching).

Remark.

- As far as the stochastic behavior at generation $n$ is concerned, random walk can be viewed as a special BRW (with branching factor = 1), or random walk is a special BRW where birth-rate is independent of birth-place.

- A BRW of spatial homogeneity in branching (i.e. $m_n(u) = m_n$) can be studied as a random walk process (without branching).

In the following, we refer the random walk by $(p_i)_{i=1,2,...n}$ (note this random walk is not necessarily the underlying random walk of the original BRW, though under the condition of independence between siblings’ motion, the underlying random walk is $(p_i)$). The following theorem presents a Chernoff bound for branching random walk. An immediate corollary of this theorem establishes that, under the condition of i.i.d. which Harris (1960) considered, Chernoff bound holds.

Theorem 2.1. For BRW $(m_i, p_i)_{i=1,2,...,n}$, if (a) $m_i$ is position-independent, and (b) Chernoff bound holds for the random walk $(p_i)$, then

$$Pr(|Q_n(α) - na| \geq λ) \leq 2e^{-\frac{λ^2}{2}}$$
where na is the expected position\(^1\) for an individual in nth generation, \(Q_n(\alpha)\) is \(\alpha\) quantile and \(\alpha \in [e^{-\frac{1}{2}c\lambda^2/n}, 1 - e^{-\frac{1}{2}c\lambda^2/n}]\). Throughout, generation index \(n\) will be omitted when no confusion can arise.

The theorem conveys the same idea as Chernoff inequalities for random walk, stating that “almost surly” (with as small as about \(e^{-O(1)\lambda^2/n}\) chance of exception) a BRW tree’s almost all leaves (about \(1 - e^{-O(1)\lambda^2/n}\) proportion) are concentrating around the expected position within area of \(\sqrt{n}\) order (when \(\lambda\) is set \(O(\sqrt{n})\)).

**Proof.** Let \(z_1^{(n)}, z_2^{(n)}, \ldots\) be an enumeration of the positions of the particles (leaves) in the nth generation and \(Z^{(n)}\) be its population; i.e. \(Z^{(n)} = \{z_1^{(n)}, z_2^{(n)}, \ldots\}\). There should be an index variable \(\tau\) for trees in the above notations which we omit. Define

\[
F(t) = \frac{\sum_i 1_{\{z_i \leq t\}}}{Z^{(n)}}
\]

namely, the cumulative distribution function of the data set \(\{z_1^{(n)}, z_2^{(n)}, \ldots\}\). The \(1 - \alpha\) quantile is

\[
Q_n(1 - \alpha) = \inf\{t : F(t) \geq 1 - \alpha\}
\]

Let \(\tau\) be a tree of \(Z\) leaves and \(p_{(\tau)}(\lambda)\) be the proportion of positions which \(\geq na + \lambda\) (in the nth generation), i.e.

\[
p_{(\tau)}(\lambda) = \frac{\sum_{i=1}^{Z} 1_{\{z_{i(\tau)} \geq na + \lambda\}}}{Z}
\]

By the definitions,

\[
p_{(\tau)}(\lambda) \geq \alpha \iff Q_n(1 - \alpha) \geq na + \lambda
\]

We have \(E(p_{(\tau)}(\lambda)) = Pr(S_n - na \geq \lambda)\), because

\[
E(p_{(\tau)}(\lambda)) = \sum_{i=1}^{Z} \frac{E(1_{\{z_i - na \geq \lambda\}})}{Z} = \sum_{i=1}^{Z} \frac{Pr(S_n - na \geq \lambda)}{Z} = Pr(S_n - na \geq \lambda)
\]

where \(\tau\) is a tree of population \(Z\) (of generation \(n\)). In other words, the expectation of \(p_{(\tau)}(\lambda)\) over all the trees of the same population is \(Pr(S_n - na \geq \lambda)\). The reason \(Z\) does not appear in \(E(p_{(\tau)}(\lambda))\) is that for a randomly selected leaf \(z_i\) from a tree \(E(1_{\{z_i - na \geq \lambda\}})\) is independent of the size of the tree because, by the hypothesis (a), branching is independent of walking. Branching is also independent from leaf indexing so that \(E(1_{\{z_i - na \geq \lambda\}})\) is the same for any \(i\). Below we will use \(k\) instead of capital \(Z\) for readability in summation.

On the other hand, denoting the number of trees of size \(k\) by \(n_k\), we have

\[
\sum_{i=1}^{k} E(1_{\{z_i - na \geq \lambda\}}) = \frac{1}{k} \sum_{i=1}^{n_k} \sum_{\tau_k} 1_{\{z_{i(\tau_k)} - na \geq \lambda\}}
\]
Note $n_k$ is very large for we have infinity initial ancestors. Suppose there are $N_t$ trees in the forest, $n_1$ trees of population 1, $n_2$ trees of population 2, ... and so on. In view of above equations, we have

$$Pr(S_n - na \geq \lambda) = \sum_{k=1,2,...} \frac{n_k}{N_t} \sum_{i=1}^k \sum_{\tau_k} \frac{1\{z_i(\tau_k) - na \geq \lambda\}}{n_k}$$

(2.3) and then (2.4)

$$= \frac{1}{N_t} \sum_{k=1,2,...} \sum_{\tau_k} \frac{1\{z_i(\tau) - na \geq \lambda\}}{k}$$

$$= \frac{1}{N_t} \sum_{\tau} p(\tau)(\lambda)$$

$$\geq \frac{1}{N_t} \sum_{\tau} p(\tau)(\lambda) 1\{p(\tau)(\lambda) \geq \alpha\}$$

$$\geq \frac{1}{N_t} \sum_{\tau} \alpha$$

$$= \alpha \cdot Pr(p(\tau)(\lambda) \geq \alpha)$$

$$= \alpha \cdot Pr(Q_n(1 - \alpha) \geq na + \lambda)$$

By Chernoff bound on the left-hand side of the above, for a certain constant $c > 0$

$$\alpha \cdot Pr(Q_n(1 - \alpha) \geq na + \lambda) \leq e^{-c \lambda^2 / n}$$

Choosing $\alpha = e^{-\frac{1}{2}c \lambda^2 / n}$, we have

$$Pr(Q_n(1 - e^{-\frac{1}{2}c \lambda^2 / n}) - na \geq \lambda) \leq e^{-\frac{1}{2}c \lambda^2 / n}$$

(2.5) and therefore

$$Pr(Q_n(\alpha) - na \geq \lambda) \leq e^{-\frac{1}{2}c \lambda^2 / n}$$

(2.6)

for $\alpha \leq 1 - e^{-\frac{1}{2}c \lambda^2 / n}$.

Similarly, for $\alpha \geq e^{-\frac{1}{2}c \lambda^2 / n}$

$$Pr(Q_n(\alpha) - na \leq -\lambda) \leq e^{-\frac{1}{2}c \lambda^2 / n}$$

(2.7)

The claim follows.

\[\Box\]

Remark.

- The Chernoff bound holds for both supercritical and subcritical BRW.
- If displacements of siblings are independent of each other, with step size $X_i$ then, $p_i$ and the law of $X_i$ are identical (note, generally they are not equal). In other words, if $S_n = X_1 + ... + X_n$ is a random walk with $(p_i)$ as the probability density of the increment, then the distribution of $S_n$ has the same "shape" as the forest of BRW $(m_i, p_i)$. Bear in mind, probability space of random walk is interpreted as the forest of BRW of $(1, p_i)$. 

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In CSP (constraints satisfaction problem) (e.g. K-SAT [26]), enumerating the whole problem instances can be formulated as a branching random walk. In this BRW the forest has only one tree because every tree is the same; \( m_i \) is not random given birth place, and in addition branching factor is large (say \( (2n)^k \)). Because of dependence of branching and position, the BRW can not be reduced to random walk; (2.1) can not be reduced to (2.2). The concentration inequalities for this BRW are developed in a separate paper ([27]).

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