1 Overview

A major theme in the theory of \( p \)-adic deformations of automorphic forms is how \( p \)-adic \( L \)-functions over eigenvarieties relate to the geometry of these eigenvarieties. In this article we prove results in this vein for the ordinary part of the eigencurve (i.e. Hida families). We address how Taylor expansions of one variable \( p \)-adic \( L \)-functions varying over families can detect "bad" geometric phenomena: crossing components of a certain intersection multiplicity and ramification over the weight space. Our methods involve proving a converse to a result of Vatsal relating congruences between eigenforms to their algebraic special \( L \)-values and then \( p \)-adically interpolating congruences using formal models.

1.1 Congruences between Cusp Forms and Special Values

The connection between algebraic parts of special values and congruences between eigenforms was observed by Mazur. The underlying principle is that congruent forms should have congruent special values. General results were proven by Vatsal (23) in a number of different situations in this direction with applications towards nonvanishing theorems in mind. In particular, let \( N > 3 \) and \( k > 1 \). Let \( p > 3 \) be a prime and let \( K \) be an extension of \( \mathbb{Q}_p \) containing the Fourier coefficients of all normalized eigenforms for the congruence subgroup \( \Gamma_1(N) \) and weight \( k \). Let \( T_{N,k} \) be the Hecke algebra of \( S_k(N,\mathcal{O}_K) \). A maximal ideal \( m \) of \( T_{N,k} \) corresponds to a residual Galois representation \( \overline{\rho} \). We will make the assumption that

\[
H_1(\mathbb{H}/\Gamma_1(N), \mathcal{L}_n(\mathcal{O}_K))_{m}^\pm \cong T_{N,k,m}
\]

as \( T_{N,k} \)-modules, where \( \mathcal{L}_n(\mathcal{O}_K) \) is the local system associated to \( L_n(\mathcal{O}_K) \). This isomorphism is unique up to an element in \( \mathcal{O}_K^* \) and a choice of isomorphism corresponds to choosing periods. If \( f \) and \( g \) are two eigenforms with residual representation \( \overline{\rho} \), then we have two \( \mathcal{O}_K \)-algebra homomorphisms, \( \delta_f \) and \( \delta_g \), from \( T_{N,k,m} \) to \( \mathcal{O}_K \). Any congruence satisfied between \( f \) and \( g \) is necessarily satisfied between \( \delta_f \) and \( \delta_g \). Evaluating \( \delta_* \) on the appropriate cycle (maybe extending scalars to include the necessary roots of unity) yields special \( L \)-values. These special values must satisfy any congruence between \( f \) and \( g \).

In this article we prove that the converse is true. We show that if periods \( \Omega_f \) and \( \Omega_g \) can be chosen so that the algebraic special values of both eigenforms are congruent mod \( p^r \), then we have \( f \cong g \mod p^r \). To prove this result, we use the theory of modular symbols introduced by Manin [17] and generalized further by Ash and Stevens [2]. We show that a modular symbol is completely determined by its special values (in fact, finitely many special values are needed) and then use a standard congruence module argument (see [12] or [20]) with a Hecke module made up of modular
symbols. This result can be reinterpreted using the $p$-adic $L$-function constructed in \[18\]. By the uniqueness of the $p$-adic $L$-function, we conclude that we only need to consider the special values $L_{\text{alg}}^\pm(f, \chi, 1)$ as opposed to all critical values between 1 and $k$. Combining the results from the first half of the article with Vatsal's result gives:

**Theorem 1.1.** Let $f$ and $g$ be eigenforms as above and assume that

$$H_1(\mathbb{H}/\Gamma_1(N), L_n(O_K))_m^\pm \cong \mathbb{T}_{N,k,m}.$$ 

Then the following are equivalent:

- The forms $f$ and $g$ are congruent modulo $p^r$.
- There exists periods $\Omega_f^\pm$ and $\Omega_g^\pm$ such that for all Dirichlet characters $\chi$ we have
  $$\tau(\chi) \frac{L(f, \chi, 1)}{2\pi i \Omega_f^\pm} \equiv \tau(\chi) \frac{L(g, \chi, 1)}{2\pi i \Omega_g^\pm} \mod p^r,$$
  where $\tau(\chi)$ denotes the Gauss sum.
- There exists $N > 0$ and periods $\Omega_f^\pm$ and $\Omega_g^\pm$ such that for all Dirichlet characters $\chi$ of character less than $N$ we have
  $$\tau(\chi) \frac{L(f, \chi, 1)}{2\pi i \Omega_f^\pm} \equiv \tau(\chi) \frac{L(g, \chi, 1)}{2\pi i \Omega_g^\pm} \mod p^r,$$
  where $\tau(\chi)$ denotes the Gauss sum.
- There exists $p$-adic $L$-functions defined using the same periods such that $L_p(f, \chi, s) - L_p(g, \chi, s)$ is divisible by $p^r$ (here $L_p(f, \chi, s)$ denotes the cyclotomic $p$-adic $L$-function twisted by the Dirichlet character $\chi$) for all $\chi$.

### 1.2 Crossing components in Hida families

In the second part of this article we prove a geometric analogue to the results of the first part. In the first part we are concerned with congruences between cusp forms of powers of $p$ of level $N$. This corresponds to a geometric picture in $X := \text{Spec} \mathcal{T} \otimes O_K$. The points of co-dimension zero in $X$ correspond to cuspidal eigenforms of level $N$. The points of co-dimension one correspond to residual representation. Let $x_f$ and $x_g$ be co-dimension zero points of $X$ corresponding to eigenforms $f$ and $g$. Then $x_f$ and $x_g$ specialize to the same co-dimension one point $x_\rho$ if and only if $f \equiv g \mod \pi_K$. We define the "intersection multiplicity" of the components $\overline{x_f}$ and $\overline{x_g}$ at $x_\rho$ to be

$$\dim_{O_K/\phi_K} \mathbb{T}_{x_\rho}/(p_f + p_g)$$

where $p_*$ is the prime corresponding to $x_*$. This definition agrees with the algebraic definition provided in \[8\]. The largest power of $\pi_K$ for which $f$ and $g$ are congruent is equal to this intersection multiplicity. The results from the first part may be reformulated to relate congruences between special $L$-values and the intersection multiplicity of $\overline{x_f}$ and $\overline{x_g}$.

This geometric interpretation of congruences between eigenforms suggests analogues for Hida families. Congruences between connected components of Hida families corresponds to crossing components in the generic fiber. Instead of looking at congruences for different powers of $\pi_K$ we will be
interested in the intersection multiplicity of two crossing components. Any two such components correspond to minimal primes of \( \mathbb{T}_m \) where \( m \) is a maximal prime of the big Hecke-Hida algebra.

The \( p \)-adic \( L \)-functions we will be interested can be described as follows. For a fixed Dirichlet character \( \chi \) we will describe \( L_p(\chi, s) \in \mathbb{T}_m \) that interpolates \( L(f, \chi, 1)_{\text{alg}} \) as \( f \) varies over the hida family. This \( L \)-function is obtained by fixing the ”twisted” cyclotomic variable in a slightly modified version of the \( p \)-adic \( L \)-function described in [7]. For a connected component \( C \) of \( \text{Spec}(\mathbb{T}_m) \) we may restrict \( L_p(\chi, s) \) to \( C \), which we call \( L_p(C, \chi, s) \). If the natural projection onto the weight space \( \pi : C \to \text{Spec}\mathcal{O}_K[[T]] \) is etale at a point \( x \), then we may find a small enough affinoid neighborhood \( U \) around \( x \) so that \( L_p(C, \chi, s)|_U \) may be written as a power series \( f_{x, \chi, C}(T) \) in a canonical way. We may now state our main result:

**Theorem 1.2.** Let \( C_1 \) and \( C_2 \) be two components of \( \text{Spec}(\mathbb{T}_m)^{\text{rig}} \) crossing at \( x \). Assume that \( \pi \) is etale at \( x \) and that the weight of \( x \) is a limit of classical weights (e.g. any \( \mathcal{O}_K \)-valued weight.) Let \( I \) be the intersection multiplicity of \( C_1 \) and \( C_2 \) at \( x \). Then \( f_{x, \chi, C_1}(T) \) and \( f_{x, \chi, C_2}(T) \) are congruent modulo \((T - \pi(x))^I\). In other words, the Taylor expansions of \( f_{x, \chi, C_1} \) and \( f_{x, \chi, C_2} \) agree for the first \( I \) terms. For some character \( \chi' \), the two \( L \)-functions differ at the \( I + 1 \)-th term.

To prove this theorem, we reduce the problem to the situation where both components look *almost* like \( \text{Spec}(\mathcal{O}_K[[T]]) \). This involves choosing small affinoid neighborhoods of \( x \) in the rigid fiber and choosing an appropriate formal model in the sense of Raynaud [19]. This formal model is chosen in a way that allows us to remember congruences. When both components look like \( \text{Spec}(\mathbb{Z}_p[[T]]) \), we repeatedly apply the \( p \)-adic Weierstrass preparation theorem to further simplify the situation. Finally, we will apply Theorem 3.5 to a limit of classical weights approaching \( \pi(x) \).

It would be interesting to extend these results to the positive slope part of the eigencurve. There is one technical difficulty that immediately come to mind. The construction of Coleman and Mazur [4] does not come with a formal integral model. The large Hecke-Hida algebra over the integers of a local field allows us to see congruences. Without an integral model that captures all congruences, our methods fail.

### 1.3 Ramification over the weight space

In the final section we describe the behavior of the \( p \)-adic \( L \)-functions at points on Hida families that are ramified above the weight space. Informally our result says that a component is etale over the weight space if and only if no poles are introduced when we differentiate each \( L \)-function along the weight space. More precisely, let \( C \) be a regular connected component of a Hida family and let \( T' \) be any parameter for our weight space \( \text{Spec}(\Lambda) \). Our parameter defines a derivation on the function field of \( \text{Spec}(\Lambda) \) denoted \( \frac{d}{dT'} \). This derivation extends to the function field \( K \) of \( C \). If \( C \) is etale over \( \text{Spec}(\Lambda) \) then \( \frac{d}{dT'} \) will give a derivation on the global functions \( A \) of \( C \). If for some \( x \in C \) there exists \( f \in A \) such that \( \frac{d}{dT'} f \) has a pole at \( x \), then \( x \) must be ramified over the weight space. The largest pole occurring will be one less than the ramification index. Our main result is that it is enough to check if there exists a Dirichlet character such that \( \frac{d}{dT'} L(C, \chi) \) has poles.

**Theorem 1.3.** A regular \( \mathcal{O}_K \)-point \( x \in C^{\text{rig}} \) is ramified over \( \pi(x) \in \text{Spec}(\Lambda_K) \) if and only if there exists a Dirichlet twist such that \( \frac{d}{dT'} L(C, \chi) \) has a pole at \( x \), where \( T \) is a parameter of the
weight space. The ramification index of $x$ over $\pi(x)$ is equal to one more than the largest order pole occurring.

The proof of this theorem is similar to the proof of Theorem 1.2. We first take a small affinoid neighborhood around $x$ which comes naturally equip with a formal model that is isomorphic to $\text{Spec}(\mathcal{O}_K(Y))$. This allows us to apply the $p$-adic Weierstrauss preparation theorem and Theorem 3.5.

1.4 Further Remarks

The results in this article should have generalizations to ordinary families of automorphic forms for larger algebraic groups. A several variable $p$-adic $L$-function was constructed by Dimitrov in [5] that varies over ordinary families of Hilbert modular forms. It seems likely that our geometric methods could be adapted to this context. Even more generally, it seems plausible that one could construct measures using compatible families of automorphic cycles living in Emerton’s completed cohomology (see [3]) that detect ramification over the weight space and crossing.

The author is currently investigating the extension of the results in this paper to points of characteristic $p$. Following the philosophy of [1] we may view these points as the boundary of our Hida families. These points can be regarded as the ordinary part of the spectral halo conjectured by Coleman. In [1] a formal model is constructed for the part of the eigencurve living over the outer Halo of the weight space. It is plausible that $p$-adic $L$-functions can be constructed on the spectral halo and that this formal model could be used to imitate the techniques used in this paper.

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2 Modular Symbols and the Eichler-Shimura Isomorphism

2.1 Modular Symbols and Various Cohomology Theories

Throughout this article we will let $N > 3$ and $\Gamma = \Gamma_1(N)$. In particular, $\Gamma$ is free of torsion. Let $D_0$ be the divisors of $\mathbb{P}(\mathbb{Q})$ of degree 0 with a natural left action of $\Gamma$. For any $\mathbb{Z}[\Gamma]$-module $E$, we let $\Phi(E) = \text{Hom}_\Gamma(D_0, E)$. These are modular symbols with values in $E$. If $E$ also admits a right action of $GL_2(\mathbb{Q})$ (resp. $GL_2(\mathbb{Z})$) we may define a right on $\Phi(E)$ of $GL_2(\mathbb{Q})$ (resp. $GL_2(\mathbb{Z})$). If $\alpha \in \Phi(E)$ and $g \in GL(\mathbb{Q})$ then $\alpha|_g$ sends $(r_1 - r_2)$ to $\alpha(g(r_1) - g(r_2))/g$.

It is known that $\Phi(E) \cong H^1_c(\mathbb{H}/\Gamma, \widetilde{E})$ (see [2 Proposition 4.1]), where $\widetilde{E}$ is the local system associated to $E$. The group $H^1_c(\mathbb{H}/\Gamma, \widetilde{E})$ is defined to be the image of $H^1_c(\mathbb{H}/\Gamma, E)$ in $H^1(\mathbb{H}/\Gamma, E)$. We may compare topological cohomology, which we may interpret as singular or deRham using deRham’s theorem, and group cohomology to get the following commutative diagram whose vertical arrows are isomorphisms.
We call $\phi$ the map that takes a modular symbol to a 1-cocycle. Explicitly this map takes the modular symbol $\alpha$ to the 1-cocycle that sends $g$ to $\alpha(g(x)) - \alpha(x)$ for some $x \in \mathbb{P}(\mathbb{Q})$. The right two vertical maps send a 1-form to $\omega$ to the 1-cocycle $g \mapsto \int g(z_0) \omega$, where $z_0$ can be any number in $\mathbb{H}$. If $\omega$ is compact we may even allow $z_0$ to be in $x \in \mathbb{P}(\mathbb{Q})$. When $E$ is not a $\mathbb{R}$-vector space we must use singular cohomology, but the idea is essentially the same. A more thorough explanation can be found in the appendix of [11].

### 2.2 The Complex Conjugation Involution

Each space in the above diagram has an action induced by the involution $\sigma$ of $\mathbb{H}$ given by $z \mapsto -\bar{z}$. Consider the 1-cocycle $\beta$ defined by a 1-form $\omega_\beta$. Then we get a new 1-cocycle that sends $g$ to

$$
\int_i g(z_0) \omega,
$$

where $z_0$ can be any number in $\mathbb{H}$. If $\omega$ is compact we may even allow $z_0$ to be in $x \in \mathbb{P}(\mathbb{Q})$. When $E$ is not a $\mathbb{R}$-vector space we must use singular cohomology, but the idea is essentially the same. A more thorough explanation can be found in the appendix of [11].

If $V$ is any one of the spaces in the diagram above, we have a decomposition $V = V^+ \oplus V^-$. Here $V^+$ are the elements of $V$ fixed by the action of complex conjugation and $V^-$ are the elements negated by this action. In fact if 2 is invertable in a subring $A$ of $\mathbb{C}$ then we get a diagram whose vertical arrows are isomorphisms.
2.3 The Eichler Shimura Isomorphism

Let \( f \in S_{n+2}(\Gamma) \) and let \( L_n \) be the space of homogeneous polynomials in \( x \) and \( y \) of degree \( n \). The 1-form \( f(z)(x - zy)^n dz \), which takes values in \( L_n \) is invariant under \( \Gamma \) and therefore gives us an element \( \omega_f \) of \( H^1(\mathbb{H}/\Gamma, \widetilde{L}_n(\mathbb{C})) \). The form \( \omega_f \) does not have compact support, but it turns out that \( \omega_f \in H^1(\mathbb{H}/\Gamma, \widetilde{L}_n(\mathbb{C})) \). It turns out that if we take the real part of \( \omega_f \) we get an isomorphism

\[
S_{n+2}(\Gamma) \cong H^1(\mathbb{H}/\Gamma, \widetilde{L}_n(\mathbb{R})) \cong H^1_p.
\]

We easily check that the action of complex conjugation sends \( f(z)(x - zy)^n dz \) to \( -f(-\overline{z})(x + \overline{z}y)^n d\overline{z} \). In particular we see that the projection of \( \omega_f \) onto \( H^1(\mathbb{H}/\Gamma, \widetilde{L}_n(\mathbb{C}))^\pm \) is

\[
\frac{f(z)(x - zy)^n dz \pm f(-\overline{z})(x + \overline{z}y)^n d\overline{z}}{2}.
\]

2.4 Integral Cohomology

Let \( f \) be an eigenform for \( \Gamma \) and let \( \omega_f^\pm \) be the corresponding 1-form in \( H^1(\mathbb{H}/\Gamma, \widetilde{L}_n(\mathbb{C}))^\pm \). We define a modular symbol \( \alpha_f^\pm \) by

\[
\alpha_f^\pm(r_1) - \{r_2\}) = \int_{r_1}^{r_2} \omega_f^\pm.
\]

This gives a Hecke equivariant map \( s : S_k(\Gamma) \to \Phi(L_n(\mathbb{C}))^\pm \). By a theorem of Shimura (see [9, Theorem 4.8]) the subspace of \( \Phi(L_n(\mathbb{C}))^\pm \) that has the same Hecke eigenvalues as \( f \) is one dimensional. If \( A \) is a subring of \( \mathbb{C} \) containing the Hecke eigenvalues of \( f \), there exists periods \( \Omega_f^\pm \) such that

\[
\frac{\alpha_f^\pm}{\Omega_f^\pm} \in \Phi(L_n(A))^\pm.
\]

The map \( s \) is a section of the map \( \phi : \Phi(L_n(\mathbb{C}))^\pm \to H^1(\Gamma, \widetilde{L}_n(\mathbb{C}))^\pm \). We know that \( \phi(L_n(A))^\pm \subset H^1(\Gamma, \widetilde{L}_n(A))^\pm \) by our explicit description of \( \phi \).

3 Congruences Between Cusp Forms and \( L \)-functions

In this section we prove that two cusp forms are congruent if and only if the "algebraic" special values of their \( L \)-functions admit congruences for all twists. It turns out that we only need to consider finitely many twisted \( L \)-functions to determine if there are congruences.

3.1 Special Values of Modular Symbols

Let \( \left[ \frac{a}{q} \right] \) denote the degree zero divisor \( \{ \frac{a}{q} \} - \{\infty\} \). For a primitive Dirichlet character \( \chi \) of conductor \( D \) we define

\[
\Lambda(\chi) = \sum_{m=0}^{D-1} \chi(m) \left[ \frac{m}{q} \right] \in D_0 \otimes \mathbb{Z}[\chi].
\]
Let $E$ be a $\mathbb{Z}_p[\Gamma]$-module and let $\alpha \in E$. Define the special value $L(\alpha, \chi)$ of $\alpha$ to be $\alpha(\Lambda(\chi))$. If $\alpha$ is the modular symbol associated to a cusp form then the special values of $\alpha$ are related to the special values of the forms $L$-functions. The next proposition says that a modular symbol is completely determined by special values and that we have some control over which special values we need to look at. More specifically, let $\epsilon > 0$ and let $r \in \mathbb{Z}$ be prime to $p$. Take $X$ to be the set of primes $q$ larger than $\epsilon$ that satisfy the congruences
\[
q \equiv r \mod p \\
q \equiv 1 \mod N^2.
\]
We will prove that $\alpha$ is determined by its special values for Dirichlet characters with conductor in $X$. This type of result was first observed by Stevens (Lemma 2.1 in [22]) for weight 2 forms.

**Lemma 3.1.** Let $\frac{c}{d}$ be a reduced fraction whose denominator is $1 \mod N$. There exists $\gamma \in \Gamma$ such that the denominators of $\gamma\left(\frac{c}{d}\right)$ and $\gamma(0)$ are in $X$.

**Proof.** First assume that $p \nmid c$. If this is not the case, replace $\frac{c}{d}$ with
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c+d \\ d \end{pmatrix},
\]
whose denominator is prime to $p$. A similar manipulation will guarantee $c$ is coprime to $N$ as well.
Let $l_1$ be a prime number that is congruent to $1$ modulo $N^2$ and congruent to $r$ modulo $p$. We may take $l_1$ large enough to be contained in $X$ and so that $l_1 \nmid c$. Next take $z$ to be a prime satisfying
\[
\begin{align*}
    z &\equiv 1 \mod N^2, \\
    z &\equiv dl_1 \mod Nc, \text{ and} \\
    z &\equiv r \mod p.
\end{align*}
\]
Then $z$ can be written as $yNc + dl_1$. We set $l_2 = Ny$. There is a matrix $\gamma = \begin{pmatrix} t_1 & t_2 \\ l_1 & l_2 \end{pmatrix}$ in $\Gamma$ and we may assume that $t_1$ is divisible by $z$. We see that $\gamma$ satisfies the desired properties. \qed

**Lemma 3.2.** Let $\alpha$ be a module symbol with values in $E$ and assume that $\alpha$ doesn’t map to zero in $H^1(\Gamma, E)$. Let $r$ be prime to $p$ with $r \neq 1 \mod p$ and let $\epsilon > 0$. Then either $L(\alpha, \chi_{\text{triv}}) \neq 0$ or there exists a primitive Dirichlet character $\chi$ whose conductor is in $X$ such that $L(\alpha, \chi) \neq 0$.

**Proof.** First, extend any modular symbol in $\Phi(E)$ to an element of $\text{Hom}_\Gamma(D_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p, E \otimes_{\mathcal{O}} \mathbb{Z}_p)$. For any subset $S$ of $\mathbb{Q}$, we will define
\[
A(S) := \bigoplus_{x \in S} \mathbb{Z}_p[x],
\]
\[
A'(S) := \text{Hom}(A(S), E \otimes_{\mathcal{O}} \mathbb{Z}_p),
\]
\[
\rho_S := \text{Hom}_\Gamma(D_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p, E \otimes_{\mathcal{O}} \mathbb{Z}_p) \rightarrow A'(S),
\]
where $\rho_S$ is the natural map. Note that the map $\rho_Q$ is an injection and that if $S \subset T \subset \mathbb{Q}$ then $\rho_S$ factors through $\rho_T$. Let $S_X$ be the set of rational numbers whose denominator is in $X$. Let
$S'_X = S_X \cap (0,1]$. We will be interested in $\rho_{S_X}$ and $\rho_{S'_X}$.

Let $\alpha$ be a modular symbol with $L(\alpha, \chi) = 0$ for all $\chi$ whose conductor is in $X$. For $m \in X$, we let $S_m$ be the rational numbers $\frac{1}{m}, \ldots, \frac{m-1}{m}$. The $\mathbb{Z}_p$-span of $\Lambda(\chi)$ for all $\chi$ of conductor $m$ is a submodule $M$ of $A(S_m)$. Then $M$ consists of all elements $\Sigma a_i m_i$ where $\Sigma a_i = 0$. That is to say, $M$ is the kernel of the augmentation map on $A(S_m)$. Consider $\left[ \frac{N}{m} \right] \in A(S_m)$. We have

$$
\alpha([\frac{N}{m}]) = \left[ \alpha([\frac{N}{m}]) - \alpha([0]) \right] = \alpha([\frac{N}{m}] - \{0\}),
$$

where $\alpha([0]) = 0$ because $L(\alpha, \chi_{\text{triv}}) = 0$. Let $\gamma_0 = \left( \begin{array}{cc} 1 & 0 \\ -Nk & 1 \end{array} \right)$, where $m = N^2k + 1$. Then

$$
\gamma_0([\frac{N}{m}] - \{0\}) = \{-N\} - \{0\} = [-N] - [0].
$$

There is a parabolic element $\gamma_1 \in \Gamma$ such that $\gamma_1([N]) = [0]$, so $\alpha([-N]) = 0$. It follows that $\alpha(\gamma_0([\frac{N}{m}] - \{0\})) = 0$ and thus $\alpha([\frac{N}{m}] - \{0\}) = \alpha([\frac{N}{m}]) = 0$. Since the span of $M$ and $[\frac{N}{m}]$ is all of $A(S_m)$ we see that $\alpha$ is zero on all of $A(S_m)$. In other words $\rho_{S_m} = 0$.

Let $x \in S_X$. There is a parabolic element $\gamma' \in \Gamma$ such that $\gamma'(x) \in S'_X$. Applying $\gamma'^{-1}$ to the equation $\alpha(\gamma'([x])) = 0$ shows that $\alpha([x]) = 0$. It follows that $\rho_{S_X}(\alpha) = 0$. Now we consider the divisor $\{0\} - \{\frac{c}{d}\}$, where $\frac{c}{d} = \gamma(0)$ for some $\gamma \in \Gamma$. The denominator $c$ is $1 \mod N$, so by \ref{corollary} there exists $\gamma_0 \in \Gamma$ such that $\gamma_0(0)$ and $\gamma_0(\frac{c}{d})$ are in $S_X$. Then we have

$$
\alpha(\{\gamma_0(0)\} - \{\gamma_0(\frac{c}{d})\}) = \alpha([\gamma_0(0)] - [\gamma_0(\frac{c}{d})]) = 0,
$$

since $\alpha$ is in the kernel of $\rho_{S_X}$. This leads to a contradiction, since the cohomology class of $\alpha$ in $H^1(\Gamma, E)$ is represented by the 1-cocycle that sends $\gamma$ to $\alpha(\{0\} - \{\gamma(0)\})$.

$\square$

**Corollary 3.3.** Let $O$ be as in the lemma. Let $\alpha_1$ and $\alpha_2$ be modular symbols that takes values in $L_n(O)$ and let $I$ be an ideal of $O$. The following are equivalent.

- For $r, e$ and $X$ be as in the lemma

$$
L(\alpha_1, \chi) \equiv L(\alpha_2, \chi) \mod I
$$

for all $\chi$ with conductor in $X$.

- $\alpha_1 \equiv \alpha_2 \mod IL_n(O)$.

**Proof.** By (1.16) of \cite{12} we know that $\Phi(L_n(O)) \otimes O/I \cong \Phi(L_n(O/I))$. Consider the image $\beta$ of the modular symbol $\alpha_1 - \alpha_2$ in $\Phi(L_n(O/I))$. If $L(\alpha_1, \chi) \equiv L(\alpha_2, \chi) \mod I$ then $L(\beta, \chi) = 0$. Then from the lemma, we see the first condition implies the second. The other direction is immediate.

$\square$
3.2 Special Values of L-functions

Let \( f(z) \in S_k(\Gamma) \) and write \( f(z) = \Sigma a_n q^n \) where \( q = e^{2i\pi z} \). Then \( L(s, f) \) is defined to be \( \Sigma a_n n^{-s} \). We have

\[
\int_0^\infty f(z)z^m dz = \frac{m!L(m+1, f)}{(-2\pi i)^{m+1}}.
\]

More generally, if \( \chi \) is a Dirichlet character of conductor \( D \) we define \( L(s, f, \chi) \) as \( \Sigma a_n \chi(n)n^{-s} \). This yields

\[
\sum_{a=0}^{D-1} \chi(a) \int_0^\infty f(z)z^m dz = \tau(\chi) \left( \frac{k-2}{m} \right) m! L(m+1, f, \chi) \frac{1}{(-2\pi i)^{m+1}}.
\]

These two integrals tell us the special values of the modular symbol \( s(\omega_f) \). In particular

\[
s(\omega_f)(\Lambda(\chi)) = \left( \ldots, \tau(\chi) \left( \frac{k-2}{m} \right) m! L(m+1, f, \chi) \frac{1}{(-2\pi i)^{m+1}}, \ldots \right).
\]

3.3 Congruences Between Special Values

In this section we use Theorem 3.2 in conjunction with a standard congruence module argument (cf. 20 and 12). Let \( f \) and \( g \) be normalized eigenforms in \( S_k(\Gamma) \) with \( f(z) = \Sigma a_n q^n \) and \( g(z) = \Sigma b_n q^n \). Let \( K \) be a finite extension of \( \mathbb{Q} \) containing the eigenvalues of both eigenforms. Let \( v \) be a prime of \( \mathcal{O}_K \) whose residue characteristic is \( p \). Let \( R = \mathcal{O}_{K(v)} \) be the localization of \( \mathcal{O}_K \) at \( v \) and let \( \pi_v \) be a uniformizer of \( R \). Since 2 is invertible in \( R \) we have

\[
\Phi(L_n(R)) \cong \Phi(L_n(R))^+ \oplus \Phi(L_n(R))^-.
\]

Let \( M^+ = (\mathbb{C} \alpha_f^+ \oplus \mathbb{C} \alpha_g^+) \cap \Phi(L_n(R))^+ \). We remark that \( M^+ \) is invariant under the Hecke operators and has dimension 2. Let \( M_s^+ = M^+ \cap \mathbb{C} \omega^* \) and let \( M_s^+ \) be the projection of \( M^+ \) onto \( \mathbb{C} \omega^* \) (here * can be \( f \) or \( g \)). The spaces \( M_s^+ \) and \( M_s^* \) are free \( R \)-modules of rank one and \( M_s^+ \subset M_s^* \). Picking a basis of \( M^+ \) is equivalent to choosing a period \( \Omega^+ \). The choice of such a period is canonical up to a \( v \)-adic unit as discussed in 23. Let \( \beta^+_s = \frac{\Omega^+}{\Omega^*} \) be the normalized modular symbol. Let us assume that we were able to choose the periods so that

\[
L(\beta^+_s, \chi) \cong L(\beta^+_h, \chi) \mod v^m
\]

for all \( \chi \) with conductor in \( X \). Then by the previous section we know that \( \beta^+_f - \beta^+_g \in v^m \Phi(L_n(R)) \). In particular, we may find \( x \in \Phi(L_n(R)) \) with \( \pi_v^m x = \beta^+_f - \beta^+_g \). As \( x \) is in the space spanned by \( f^+ \) and \( g^+ \) we see that \( M^+ \) contains \( x \). Thus we have an injection

\[
R/\pi_v^m R \rightarrow \frac{M^+}{M_f^+ \oplus M_g^+}, \text{ defined by }
\]

\[
1 \mod \pi_v^m R \rightarrow x \mod M_f^+ \oplus M_g^+.
\]

We get the following equivalence of Hecke modules:
\[
\frac{M^f_+}{M^f_+} \cong \frac{M^f_+ \oplus M^g_+}{M^+} \cong \frac{M^g_+}{M^g_+}, \quad \text{and}
\]
\[
\frac{M^f_+ \oplus M^g_+}{M^+} \cong \frac{M^+}{M^f_+ \oplus M^g_+}.
\]

In particular we find that
\[
\frac{M^f_+}{M^f_+} \otimes R/\pi_v^m R \cong \frac{M^g_+}{M^g_+} \otimes R/\pi_v^m R,
\]
and both of these Hecke modules are equal to \( R/\pi_v^m R \) as an \( R \)-module. The Hecke operator \( T_n \) acts on \( \frac{M^f_+}{M^f_+} \otimes R/\pi_v^m R \) (resp \( \frac{M^g_+}{M^g_+} \otimes R/\pi_v^m R \)) through scalar multiplication by \( a_n \mod \pi_v^m R \) (resp \( b_n \mod \pi_v^m R \)). The isomorphism of Hecke modules then implies \( a_h \cong b_n \mod \pi_v^m R \).

**Theorem 3.4.** Let \( f \) and \( g \) be normalized eigenforms in \( S_k(\Gamma) \). If there exists periods \( \Omega_f^\pm \) and \( \Omega_g^\pm \) (these are defined canonically up to \( p \)-adic unit) that satisfy
\[
(..., \tau(\chi) \binom{k-2}{m} \frac{L(m+1, f, \chi)}{(-2\pi i)^m \Omega_f^\pm}, ...) \cong (... , \tau(\chi) \binom{k-2}{m} \frac{L(m+1, g, \chi)}{(-2\pi i)^m \Omega_g^\pm}, ...) \mod \pi_v^m,
\]
for all characters whose conductor is in \( X \) then \( f \cong g \mod \pi_v^m \).

### 3.4 The one variable cyclotomic \( p \)-adic L-function

We will now describe the construction of cyclotomic \( p \)-adic L-function as described in [7]. The only difference in what we describe is that we allow a fixed tame level of the Dirichlet twists that we interpolate, while [7] only covers the tame level 1 case. This construction is more or less equivalent to the function described in [15]. Let \( f \) be an \( \omega \)-ordinary eigenform of weight \( k \geq 2 \) and level \( N \). We define \( \mathbb{T}_{N,r,k} \) to be the Hecke algebra over \( \mathcal{O}_K \) generated by \( T_l \) for \( l \nmid Np \), \( U_l \) for \( l \mid Np \), and the diamond operators \( \langle a \rangle \) for \( a \mod N \). Let \( \mathfrak{m} \) be the maximal ideal of \( \mathbb{T}_{N,r,k} \) corresponding to the residue of \( f \) modulo \( p \). We will assume that
\[
H_1(\mathbb{H}/\Gamma, \hat{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{m}}^{ord} \cong (\mathbb{T}_{N,r,k})_{\mathfrak{m}}.
\]
This will be true if the residual representation is \( p \)-distinguished.

There is a natural map from \( (\mathbb{T}_{N,r,k})_{\mathfrak{m}} \) to \( \mathbb{C} \) sending each Hecke operator to its eigenvalue on \( f \). Call this map \( \delta_f \) and note that the image of \( \delta_f \) is in \( \mathcal{O}_K[f] \). We also have a natural map from \( H_1(\mathbb{H}/\Gamma, \hat{L}_{k-2}(\mathcal{O}_K))_{\mathfrak{m}}^{ord} \) to \( \mathbb{C} \) that is induced by integrating each cycle along the differential \( \omega_f \). These maps are the same up to a complex period. This is the same period we encountered in the previous section (up to a \( p \)-adic unit.) The choice of period will be determined by the choice of isomorphism from our homology group to our Hecke algebra.

Let \( M > 1 \) be prime to \( p \). We will let \( \Lambda = \mathbb{Z}_p[[T]] \) be the standard Iwasawa algebra. Define \( \Lambda_M \) as \( \Lambda[\mathbb{Z}/p\mathbb{M}^\infty] \). Then \( \Lambda[\mathbb{Z}/p\mathbb{M}^\infty] \cong \lim \mathbb{Z}_p[\mathbb{Z}/Mp^s]^{\mathbb{Z}^\infty} \) where \( 1 + T \) goes to the topological generator \( 1 + p \) of \( 1 + p\mathbb{Z}_p \). Recall that for any \( \mathbb{Z}_p \) module \( A \) we may think of elements of \( \Lambda_M \otimes A \) as measures on
Dirichlet character of tame level \(M\) \(\chi\) character corresponding to \(\tau\) and \(\omega\) sending the open set \((a + ptZ_p, a + MZ)\) in \(Z_p^\times \oplus \mathbb{Z}/MZ^\times\) to \(U_p^{-r}\{\frac{a}{pM}, \infty\} \in H_1(\mathbb{H}/T, \mathbb{L}_{k-2}(\mathcal{O}_K))\). By the definition of \(U_p\) we see that this definition is a well defined measure (in particular it is additive).

When we specialize \(L_{M,m}\) by \(\delta_f\) we obtain an element \(L_{M,m,f}\) of \(\Lambda_M \otimes \mathcal{O}_K[f]\). Specializing at certain \(C_p\) points of \(\Lambda_M \otimes \mathcal{O}_K[f]\) will then give us special values of \(L(f, \chi, s)\), for a primitive Dirichlet character of tame level \(M\). Giving a \(C_p\) point of \(\Lambda_M \otimes \mathcal{O}_K[f]\) is equivalent to giving an element of \(\text{Hom}_{\text{cont}}(Z_p^\times \oplus \mathbb{Z}/MZ^\times, C_p^\times)\). We may interpret any primitive Dirichlet character \(\chi\) of tame level \(M\) as an element in \(\text{Hom}_{\text{cont}}(Z_p^\times \oplus \mathbb{Z}/MZ^\times, C_p^\times)\). Then evaluating \(L_{M,m,f}\) at this character corresponding to \(\chi\) gives \(\tau(\chi)\frac{L(f, \chi, 1)}{\Omega_f}\). If we multiply \(\chi\) by the character \((1 + p) \to (1 + p)^s\), then \(L_{M,m,f}\) evaluates to \(\tau(\chi)s\frac{L(f, \chi^s, 1)}{(2\pi)^s\Omega_f}\) for \(0 \leq s < k-1\) where \(\omega\) is the Tiechmuller character. Note that \(\text{Spec}(\Lambda_M)\) is equal to \(\phi(pM)\) copies of the open unit \(p\)-adic ball. One copy for each character of \(\mathbb{Z}/pM\). For a character \(\psi\) of conductor \(pM\) we let \(L_{\psi,*}\) denote the restriction of \(L_{M,m,*}\) to the unit ball corresponding to \(\psi\).

**Theorem 3.5.** Let \(f\) and \(g\) be two \(p\)-ordinary eigenforms of weight \(k \geq 2\) and level \(Np^r\). Let \(\mathcal{O}_K\) be the ring of integers of an extension of \(\mathbb{Q}_p\) that contains the coefficients of \(f\) and \(g\). Let \(\pi\) be a uniformizer of \(\mathcal{O}_K\). There exists an \(M'\) depending on the weight and level such that the following are equivalent

- The forms \(f\) and \(g\) are congruent modulo \(\pi^t\).
- The \(p\)-adic \(L\)-functions \(L_{M,m,f}\) and \(L_{M,m,g}\) are congruent modulo \(\pi^t\) for all \(M\).
- The \(p\)-adic \(L\)-functions \(L_{\psi,f}\) and \(L_{\psi,g}\) are congruent modulo \(\pi^t\) for all primitive Dirichlet characters \(\psi\).

**Proof.** That congruent forms have congruent \(p\)-adic \(L\)-functions follows from the above discussion. It was originally proven by Vatsal in \([23]\) using the \(p\)-adic Weierstrass preparation theorem. That congruent \(p\)-adic \(L\)-functions implies a congruence of forms is a restatement of the results from the previous section. \(\square\)

**Corollary 3.6.** Using the notation of the Theorem 3.5 the forms \(f\) and \(g\) are congruent modulo \(\pi^t\) if and only if for every Dirichlet character \(\chi\) we have

\[
\tau(\chi)\frac{L(f, \chi, 1)}{\Omega_f} \equiv \tau(\chi)\frac{L(g, \chi, 1)}{\Omega_g} \mod \pi^t.
\]

**Remark.** Theorem 3.4 tells us that the critical values \(1, ... k-1\) of the twisted \(L\)-functions can detect congruences. This corollary says that it is sufficient to look at the critical value at \(s = 1\).

**Proof.** The necessity is already established. Conversely, let us assume that the congruence between the critical values at \(s = 1\) holds for all characters. By Theorem 3.5 is enough to show that \(\pi^tL_{\psi,f} - L_{\psi,g}\), where we view \(L_{\psi,*}\) as an element of \(\mathcal{O}_K[[T]]\). By the Weierstrass preparation theorem we may write

\[
L_{\psi,f} - L_{\psi,g} = \pi^r u(T) P(T)
\]
where \( u(T) \) is a unit and \( P(T) \) is distinguished. Now let \( \chi \) be a primitive character of conductor \( p^s \). The \( \mathbb{C}_{p^s} \)-point of Spec(\( \mathcal{O}_K[[T]] \)) corresponding to \( \chi \) sends \( T \) to \( 1 - \zeta_{p^s} \) for some primitive \( p^s \)-th root of unity. The \( p \)-adic valuation of \( 1 - \zeta_{p^s} \) is \( \frac{1}{\phi(p^s)} \). As \( P(T) \) is distinguished we know that \( v_p(p(1 - \zeta_{p^s})) = \deg(P(T)) \frac{1}{\phi(p^s)} \) if \( s \) is sufficiently large. Putting this together with out hypothesis gives

\[
 tv_p(\pi) \leq v_p(\pi^r u(1 - \zeta_{p^s}) P(1 - \zeta_{p^s}))
\]

\[
 = rv_p(\pi) + \det(P(T)) \frac{1}{\phi(p^s)},
\]

for sufficiently large \( s \). Letting \( s \) tend to \( \infty \) shows that \( t \leq r \).

\[\square\]

4 \( p \)-adic \( L \)-functions on Hida families

In this section we describe a \( p \)-adic \( L \)-function that varies analytically over Hida families whose residual representation are \( p \)-stabilized. We will also interpret the results of the previous section in terms of \( p \)-adic \( L \)-functions. Throughout the remainder of this article we set \( K \) to be a finite extension of \( \mathbb{Q}_p \) and \( \mathcal{O}_K \) to be the ring of integers in \( K \) with uniformizing element \( \pi_K \). We let \( \Lambda_K := \Lambda \otimes \mathcal{O}_K \) be the Iwasawa algebra with coefficients in \( \mathcal{O}_K \).

4.1 Hida Families

In this section we give a summary of Hida theory. For an introduction to the theory with tame level 1 see [11]. For the general situation see [13]. For the most part we adopt the notation of [7]. Let \( M > 0 \) be relatively prime to \( p \). For \( k \geq 2 \), let \( S_k(Np^\infty, \mathcal{O}_K) \) be the union of all weight \( k \) cusp forms for \( \Gamma_1(Np^r) \) that are defined over the \( \mathbb{Z}_p \)-algebra \( \mathcal{O}_K \). There is a natural action of \( \mathbb{Z}/p^r\mathbb{Z} \) on \( S_k(Np^r, \mathcal{O}_K) \) given by the product of the Nebentypus action and the character \( \gamma \to \gamma^k \). These actions are compatible with the inclusion of \( S_k(Np^r, \mathcal{O}_K) \) in \( S_k(Np^{r+s}, \mathcal{O}_K) \) for any \( s > 0 \) so that we have an action of \( \mathbb{Z}_p^r \) on \( S_k(Np^\infty, \mathcal{O}_K) \). In particular \( S_k(Np^\infty, \mathcal{O}_K) \) is a \( \mathcal{O}_K[[\mathbb{Z}_p^r]] \)-module and a \( \Lambda_K \)-module.

For a prime \( p \) of \( \Lambda_K \) of height one we write \( \mathcal{O}(p) := \Lambda_K/p \). We say that \( p \) is classical of weight \( k \) if the residue has characteristic 0 and if the composition \( \kappa_p := 1 + p\mathbb{Z}_p \Lambda_K \to \mathcal{O}(p) \) coincides with the character \( \lambda \to \lambda^k \) on an open subgroup of \( 1 + p\mathbb{Z}_p \).

Let \( \mathbb{T}_N \) be the \( \mathcal{O}_K[[\mathbb{Z}_p^r]] \)-algebra generated by the Hecke operators and diamond operators acting on \( S_k(Np^\infty, \mathcal{O}_K)^{\text{ord}} \). Then \( \mathbb{T}_N \) is a \( \Lambda_K \)-algebra. A height one prime ideal \( p \) of \( \mathbb{T}_N \) is said to be classical of weight \( k \) if it lies above a weight \( k \) prime of \( \Lambda \). Just as before, we set \( \mathcal{O}(p) := \mathbb{T}_N/p \) and define \( \kappa_p \) to be the character from \( 1 + p\mathbb{Z}_p \Lambda_K \to \mathcal{O}(p) \). The fibers of the residues of weight \( k \) primes of \( \Lambda_K \) in \( \mathbb{T}_N \) recover the Hecke algebra acting on weight \( k \) cusp forms of tame level \( N \). More specifically, Hida proved the following theorem in [13].

Theorem 4.1. Using the above notation we have:

- \( \mathbb{T}_N \) is free of finite rank over \( \Lambda_K \).
• Let \( p \in \text{Spec}(\Lambda_K) \) be a classical height one prime of weight \( k \). Then \( \mathbb{T}_N \otimes \mathcal{O}(p) \) is equal to the full Hecke algebra acting on \( S_k(Np^\infty, \mathcal{O}(p))^{\text{ord}}[\kappa_p] \).

The Hecke algebra \( \mathbb{T}_N \) is a semi-local ring. The different maximal ideals correspond to the different representations of \( \text{Gal}(\overline{\mathbb{Q}}, \mathbb{Q}) \) into \( \text{GL}_2(\overline{\mathbb{F}}_p) \) that arise as residues of representations associated to cusp forms of tame level \( N \). The classical height one primes of \( \mathbb{T}_N \) are in one to one correspondence with the Galois conjugacy classes of eigenforms in \( S_k(Np^\infty, \mathcal{O}_K) \). The minimal primes represent maximal families of eigenforms. In particular, Let \( a \) be a minimal prime of \( \mathbb{T}_N \).

Consider the formal power series

\[
f(q) = \sum T_nq^n \text{ with } T(n) \in \mathbb{T}_N/a.
\]

Then the image of \( f(q) \) in \( \mathbb{T}_N/a \otimes \mathcal{O}(p) \), where \( p \) is a classical height one prime, will give the Fourier expansion of the modular form corresponding to \( p \).

### 4.2 The two variable cyclotomic \( p \)-adic \( L \)-function

Let \( m \) be a maximal prime of \( \mathbb{T}_N \). Then the classical height one primes of \( \mathbb{T}_{N,m} \) (the localization of \( \mathbb{T}_N \) at the prime \( m \)) correspond to eigenforms with the same residual representation. Equivalently, the Fourier coefficients of all classical height one primes of \( \mathbb{T}_{N,m} \) are equivalent in the appropriate residue field. From now on we will assume that this residual representation is \( p \)-distinguished and irreducible. This is a necessary condition to use the previous section.

**Theorem 4.2.** Let \( m \) be a maximal prime of \( \mathbb{T}_{N,r,k} \) whose residual representation is irreducible and \( p \)-distinguished. Then \( H_1(\mathbb{H}/\Gamma, \widehat{L}_{k-2}(\mathcal{O}_K))^{\pm \text{ord}}_m = (\mathbb{T}_{N,r,k})_m \) as \( \mathbb{T}_{N,r,k} \)-modules.

**Proof.** [1] Proposition 3.1.1 \( \square \)

We define \( H_1(Np^\infty, \mathcal{O}_K) := \lim H_1(\mathbb{H}/\Gamma_1(Np^r), \mathcal{O}_K)^{\text{ord}}_m \). It is proven in [7] that \( H_1(Np^\infty, \mathcal{O}_K)^{\pm \text{ord}} \) is a free rank 1 \( \mathbb{T}_N \)-module. We also have

**Theorem 4.3.** Let \( p_{N,r,k} \) be the product of all primes of weight \( k \) in \( \mathbb{T}_N \). Then

\[
H_1(Np^\infty, \mathcal{O}_K) \otimes \mathbb{T}_N/p_{N,r,k} \cong H_1(\mathbb{H}/\Gamma_1(Np^r), \widehat{L}_{k-2}(\mathcal{O}_K))^\text{ord}_m.
\]

**Proof.** This is [13] Theorem 1.9 \( \square \)

As in the previous section, we define a measure sending the open set \( (a + p^r\mathbb{Z}_p, a + M\mathbb{Z}) \) in \( \mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z} \) to \( U_p^{-1}\left\{ \frac{a}{p^rM}, \infty \right\} \subset H_1(Np^\infty, \mathcal{O}_K) \). This defines an element

\[
L_p(N, m, M) \in H_1(Np^\infty, \mathcal{O}_K) \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times].
\]

Fix an isomorphism between \( \mathbb{T}_N \) and \( H_1(Np^\infty, \mathcal{O}_K) \). This gives

\[
L_p(N, m, M) \in \mathbb{T}_{m,N} \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times].
\]

By specializing to a classical weight one prime of \( \mathbb{T}_{m,N} \) corresponding to an eigenform \( f \) we recover the \( p \)-adic \( L \) function \( L_{M,m,f} \) (up to a \( p \)-adic unit) described in the previous section. In particular, if \( p \) is the classical height one prime corresponding to \( f \) then the image of \( L_p(N, m, M) \) in \( \mathcal{O}(p) \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times] \) is \( L_{M,m,f}u \) where \( u \) is in \( \mathcal{O}(p)^\times \).
4.3 The one variable $p$-adic $L$-function over the Hida family

The first variable of $L_p(N, m, M)$ is our Hida family and the second variable is the cyclotomic variable, which varies over Dirichlet twists and the different critical values $s = 1, \ldots, k - 1$. In the previous section, we saw that specializing in the first variable recovered the $p$-adic $L$-function of \[18\]. If we specialize in the second variable, we recover a $p$-adic $L$-function that interpolates a fixed special value over our Hida family. This $L$-function won’t be meaningful for small classical weights in $T$, because the classical cyclotomic $p$-adic $L$-function only interpolates $s$-values less than the weight.

Let $\chi$ be a Dirichlet character of level $N = p^r M$ with $N$ prime to $p$. Then $\chi$ gives a homomorphism $\mathbb{Z}/N\mathbb{Z}^\times \to \mathbb{C}_p$ and a homomorphism $\Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times] \to \mathbb{C}_p$. The prime $p_{\chi,s}$ of $\Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times]$ that is the kernel of this map corresponds to twisting the $L$-function by $\chi$ and evaluating that $L$-function at $s = 1$.

The image of $L_p(N, m, M)$ along the map $T_{N,m} \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times] \to T_{N,m} \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times]/p_{\chi,s}$ gives a one variable $p$-adic $L$-function $L_p(N, m, \chi) \in T_{N,m} \otimes \mathcal{O}_K[\chi]$. This $L$-function interpolates the values of the $L$-functions of the cusp forms in our Hida family for a fixed twist.

**Theorem 4.4.** Let $p \in \text{Spec}(T_{N,m})$ be a classical height one prime corresponding to the modular form $f$. The image of $L_p(N, m, \chi)$ in $T_{N,m}/p$ is $L_{\text{alg}}(f, \chi, 1)u$, where $u$ is a $p$-adic unit.

5 Some Geometric Preliminaries

Let $C_1$ and $C_2$ two connected components of $\text{Spec}(T_{N,m})$. It is often the case that structure maps (i.e. the map onto the weight space) $\pi_i$ from $C_i$ to Spec($\Lambda$) are isomorphisms. This is an ideal situation, as functions on Spec($\Lambda_K$) are easily understood through the Weierstrass preparation theorem. However, there are many examples of components whose structure maps are not isomorphisms (e.g. families of CM forms where the class group of the imaginary quadratic field is divisible by some power of $p$). In this section we address this phenomena by finding formal models of small affinoid neighborhoods of $C_i^{\text{rig}}$ whose coordinate ring is isomorphic to $\mathcal{O}_K(T)$. We choose these formal models in a way so that they still carry information about congruences between cusp forms. We address this in the first subsection. In the second subsection we introduce an auxiliary $p$-adic metric on the $\mathcal{O}_K$ points of a schemes over $\mathcal{O}_K$. There is nothing particularly novel here, but it will be convenient for later arguments. Finally, we define intersection multiplicities and explain how we can pass from schemes to rigid varieties. There is no extra difficulty for us to work in a general geometric situation and we do so.

Throughout this section we set $\pi : X \to \text{Spec}(\Lambda_K)$ be a finite morphism. We will assume that $X$ is affine with coordinate ring $R$. By a component of $X$ we refer to a subscheme $C := \text{Spec}(R/\mathfrak{a})$ of $X$ where $\mathfrak{a}$ is a minimal ideal of $X$.

5.1 The inverse function theorem for formal models

Let $C$ be a component of $X$ and let $x \in C$ be a $\mathcal{O}_K$ point. We will assume that $\pi$ is etale at $x$. 
Lemma 5.1. There is an affinoid neighborhood $U$ of $x$ in $C^{rig}$ and $V$ of $\kappa := \pi(x)$ in $\Lambda^{rig}$ such that $U$ and $V$ are isomorphic as rigid varieties. In particular, if $p_x$ is the maximal ideal defining $x$, the affinoid $U$ may be taken to be $\{y \in C | |f(y)| \leq \epsilon, y \in p_x\}$.

Proof. This is a rigid analytic inverse function theorem. It can be deduced from the results of Chapter III section 9 in [21].

Since $C$ is finite over $\text{Spec}(\Lambda)$, we may write $C = \text{Spec}(A)$ with $A = \mathcal{O}_K[[T_0]][T_1, \ldots, T_n]/I$. After a change of variables, we may assume that $x$ corresponds to the point $T = T_i = 0$. By Lemma 5.1 there exists $N > 0$ such that the affinoid variety $X = \text{Sp}((A \otimes \mathcal{K})\langle \frac{1}{p} \rangle^{N} T_i))$ maps isomorphically onto $\text{Sp}(K\langle \frac{1}{p} \rangle^{N} T_0))$. Set $Y_i = \frac{1}{p}^{N}T_i$ so that $\text{Spec}(A(Y_i))$ is a formal model for $X$. Then $Y_i = f_i(Y_0)$ for some $f_i \in K\langle Y_0 \rangle$ (these power series will define the inverse map $\text{Sp}(K\langle Y_0 \rangle) \to X$). The coefficients of $f_i$ tend to zero so we may find $k$ such that $p^k f_i \in \mathcal{O}_K\langle Y_0 \rangle$ for each $k$. In particular we find $f_i \in \mathcal{O}_K\langle \frac{1}{p} \rangle^{k} Y_0)$ and we may assume that $f_i$ has all integral coefficients by replacing $Y_0$ with $\frac{1}{p}^{k} Y_0$.

We now have an explicit description of $X$ as $\text{Sp}(K\langle Y_i \rangle/(Y_i - f_i(Y_0)))$. The ring $A(Y_i)$ injects into $(A \otimes \mathcal{K})\langle \frac{1}{p} \rangle^{N} (Y_i - f_i(Y_0))$. Since the power series $f_i$ have integral coefficients, we have an isomorphism $A(Y_i) \cong \mathcal{O}_K\langle Y_i \rangle/Y_i - f_i(Y_0)$ and the ring $\mathcal{O}_K\langle Y_i \rangle/Y_i - f_i(Y_0)$ is isomorphic to $\mathcal{O}_K\langle Y_0 \rangle$. This is more or less an inverse function theorem for formal models.

Lemma 5.2. Write $C = \text{Spec}(A)$ with $A = \mathcal{O}_K[[T_0]][T_1, \ldots, T_n]/I$ as above. For some $N > 0$ there exists an isomorphism of formal schemes

$$\text{Spec}(A\langle \frac{1}{p} \rangle^{N} T_i)) \to \text{Spec}(\mathcal{O}_K\langle \frac{1}{p} \rangle^{N} T_0))$$

The isomorphism is given by projecting onto the $T_0$ coordinate.

The particular situation we are interested in involves two components $C_1$ and $C_2$ of $X = \text{Spec}(A)$ corresponding to the minimal primes $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Let $x_1$ (resp $x_2$) be a $\mathcal{O}_K$-point of $C_1$ (resp $C_2$)

5.2 $p$-adic Distances and Congruences

We start by discussing $p$-adic distances for affine schemes that are quotients of rings of power series. We work in this generality because it adds no extra difficulty. First let’s consider an open $n$-dimensional ball $B$ centered around the origin of radius $p^{-1}$. The ring of analytic functions converging on this ball is the Tate algebra $\mathcal{O}_K[[p^{-1}T_1, \ldots, p^{-1}T_n]]$. If $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two $\mathcal{O}_K$ points in $B$, a natural choice for the distance between them is $\max |x_i - y_i|/p$. It would be great to translate this definition into something more intrinsic. Let $A$ be some reduced quotient of a Tate algebra over $\mathcal{O}_K$.

Definition 1. Let $R$ be a ring of (topologically) finite type over $\mathcal{O}_K$. Let $I$ be an ideal of $R$. For any prime $p \in \text{Spec}(R)$ let $I(p)$ be the ideal $I + p \mod p$ in $R/p$. If $p$ is a prime that is maximal in $R \otimes \mathbb{Q}_p$, then $R/p$ is a discrete valuation ring (it is a finite extension of $\mathbb{Z}_p$). Let $|I|_p$ be the largest absolute value occurring in $I(p)$, where the absolute value is normalized so $|p|_p = p^{-1}$.

Definition 2. Let $p_1$ and $p_2$ be height one prime ideals with residue characteristic 0. We define $d(p_1, p_2)$ to be $|p_1|_{p_2}$. 

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Lemma 5.3. The following properties of \( d(,\) hold:

1. \( d(p_1, p_2) = d(p_2, p_1) \)
2. \( d(p_1, p_1) = 0 \)
3. Let \((f_1, ..., f_r)\) be any set of generators of \( p_1 \). Then \( d(p_1, p_2) = \max |f_i \mod p_2| \).
4. Suppose \( A = O_K[[T_1, ..., T_n]] \). Assume also that \( p_1 \) corresponds to \((x_1, ..., x_n)\) and \( p_2 \) corresponds to \((y_1, ..., y_n)\). Then \( d(p_1, p_2) = \max |x_i - y_i|_p \).
5. Suppose we have a closed embedding \( f : \text{Spec}(A) \to \text{Spec}(O_K[[T_1, ..., T_n]]) \) (so that \( A \) is a quotient of a Tate algebra.) Then \( d(p_1, p_2) = d(f(p_1), f(p_2)) \).
6. The non-Archemedian triangle inequality holds. That is \( d(p_1, p_3) \leq \max(d(p_1, p_2), d(p_2, p_3)) \).

Since \( O_K[[T_1, ..., T_n]] \) is local with maximal ideal \((\pi_K, T_1, ..., T_n)\), we see that \( A \) is also local with a maximal ideal \( m \). There is an equivalent definition of distance, which easily connects to congruences of cusp forms.

Lemma 5.4. Suppose \( A/ \cong O_K \cong A/p_2 \). Let \( r \) be the largest integer such that \( m^r + p_1 = m^r + p_2 \). Then \( p_1^{\frac{r}{e}} = d(p_1, p_2) \), where \( e \) is the ramification index of \( O_K \) over \( \mathbb{Z}_p \). In particular, the natural map
\[
A \to A(p_1) \cong O_K \to O_K/\pi_K^r
\]
is the same as
\[
A \to A(p_2) \cong O_K \to O_K/\pi_K^r,
\]
and \( r \) is the largest integer for which this is true.

Proof. Not very hard. \( \square \)

Now consider the big Hecke algebra \( T_N \) described in 4.1. Let \( m \) be a maximal ideal of \( T_N \). The ring \( T_{N,m} \) is local, reduced, finite over \( \Lambda \) and \( m \)-adically complete. We will assume that the residue fields of \( T_{N,m} \) and \( \Lambda \) are the same, which is equivalent to saying \( T_{N,m}/m \cong O_K/\pi_K \mathcal{O} \) (if this is not the case we may replace \( K \) with a larger extension). Let \( r_1, ..., r_n \) generate \( T_{N,m} \) as a \( \Lambda \) algebra. If \( r_i \) is a unit, then it is equivalent to an element of \( O_K(T_0) \) modulo \( m \), so we may assume that each \( r_i \) is in \( m \). Thus the \( r_i \) are topologically nilpotent and we have a surjection \( O_K[[T_1, ..., T_n]] \) by sending \( T_i \) to \( r_i \). Geometrically, we can embed the local components of our big Hecke algebra into an \( n \)-dimensional open unit Ball. Summarizing everything gives:

Proposition 1. The local big Hecke algebra \( T_{N,m} \) embeds into an \( n \)-dimensional open unit ball. This mapping is "isometric" with respect to the natural \( p \)-adic metric on the \( p \)-adic ball and the metric \( d(,\) defined above. Let \( p_1, p_2 \in \text{Spec}(T_{N,m}) \) be \( O_K \)-points. If \( d(p_1, p_2) = p^{\frac{r}{e}} \) then the modular forms (not necessarily of classical weight) corresponding to \( p_i \) are congruent modulo \( \pi_K^r \) but not \( \pi_K^{r+1} \).
5.3 Intersection Multiplicities

In this section we will define the intersection multiplicity of two crossing components of a curve over $\mathcal{O}_K$. After proving some basic properties we will reduce our problem to talking about $f(C_i)$ as in the previous section. Let $X$ be a scheme over $\mathcal{O}_K$ such that $X^{\text{rig}}$ has dimension one. Let $C_1$ and $C_2$ be connected components of $X$ and let $x$ be an $\mathcal{O}_K$ point of $X$ whose residue characteristic is 0 (so we may think of $x$ as a point in $X^{\text{rig}}$.) Let $j_i : C_i \to X$ be the natural inclusion. Let $\mathcal{I}_i$ be the $\mathcal{O}_X$-sheaf of ideals that define the component $C_i$.

**Definition 3.** The intersection multiplicity $I(X, C_1, C_2, x)$ of $C_1$ and $C_2$ at $x$ is the $K$ dimension of

$$\left(\mathcal{O}_{C_1}/j_1^*(\mathcal{I}_2)\right)_x = (\mathcal{O}_X/(\mathcal{I}_1 + \mathcal{I}_2))_x = (\mathcal{O}_{C_2}/j_2^*(\mathcal{I}_1))_x.$$ 

**Remark.** The intersection multiplicity is nonzero if and only if both $C_1$ and $C_2$ contain $x$.

Since this definition is Zariski local (and rigid analytic local, we shall see...) we may take an affine neighborhood $U = \text{Spec}(A)$ of $x$. Let $a_1$ and $a_2$ be the minimal prime ideals of $A$ defining the components $C_1$ and $C_2$. Let $p_x$ be the prime corresponding to $x$. Then

$$I(X, C_1, C_2, x) = \dim_{\mathbb{Q}_p}(A/(a_1 + a_2))_{p_x}.$$ 

**Lemma 5.5.** The following properties of intersection multiplicities are true.

1. Let $X'$ be another formal model of $X^{\text{rig}}$. Let $C'_1$ and $C'_2$ be connecting components corresponding to $C_1^{\text{rig}}$ and $C_2^{\text{rig}}$. These components cross at $x'$ whose image in $X^{\text{rig}}$ is $x$. Then $I(X, C_1, C_2, x) = I(X', C'_1, C'_2, x)$. In other words, the intersection multiplicity only depends on the rigid fiber.

2. Recall how stalks are defined for a sheaf on a rigid analytic varieties

$$\mathcal{F}_{X^{\text{rig}}, x} = \lim_{\substack{\text{affinoi} \\ U \in U}} \mathcal{F}(U).$$

Then $I(X, C_1, C_2, x) = \dim_{\mathbb{Q}_p}(\mathcal{O}_{X^{\text{rig}}}/(\mathcal{I}_1^{\text{rig}} + \mathcal{I}_2^{\text{rig}}))$. In other words, we can use the rigid analytic stalks or the Zariski stalks to find intersection numbers.

3. Let $U$ be an affinoi neighborhood of $x$. Let $\mathcal{U}$ be a formal model of $U$ and let $\mathcal{C}_i$ be the formal models of $U \cap C_i^{\text{rig}}$ that are components of $\mathcal{U}$. Then $I(X, C_1, C_2, x) = I(\mathcal{C}_1, \mathcal{C}_2, x)$.

4. Let $i : Y \to X$ be a closed subscheme such that $i(Y)$ contains the generic points of $C_1$ and $C_2$. Then $I(X, C_1, C_2, x)$ is the same as $I(Y, Y \times_X C_1, Y \times_X C_2, i^{-1}(x))$.

**Proof.** The first statement is true because $p$ is invertible in $\mathcal{O}_{X,x}$. The third statement is an immediate consequence of the second statement. To prove the second statement, pick an affinoi neighborhood of $x$ and use the fact that $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X^{\text{rig}}, x}$, where $\hat{A}$ denotes the completion of a local ring $A$ along it’s maximal ideal (see [H].) The last statement is easily checked by picking an affinoi neighborhood of $x$.

\[\square\]
In the simple situation of two rational curves crossing in $O_K(T_0, T_1)$, we can come up with a precise formula for the intersection number. Let $X$ be finite over $\text{Spec}(O_K(T_0))$ and let $X \rightarrow O_K(T_0, T_1)$ be a closed embedding. Let $C_1$ and $C_2$ be two connected components of $X$ that are isomorphic both $O_K(T_0)$. Then we have embeddings $C_1 \rightarrow \text{Spec}(O_K(T_0, T_1))$ that factor through $X$ and we see that $C_i = \text{Spec}(O_K(T_0)T_1/(T_1 - f_i(T_0)))$ with $f_i(T_0) \in O_K(T_0)$. Then we can compute the intersection number:

**Lemma 5.6.** The intersection $I(X, C_1, C_2, x)$ is equal to the largest power of $T_0$ dividing $f_1 - f_2$.

6 Crossing components in Hida families

We are now ready to prove Theorem 1.2. Recall our definition of $\mathbb{T}_{N,m}$ from Section 4.1 as a local component of the Hecke algebra that acts on the space of all cusp forms of tame level $N$. There is a map

$$\pi : \text{Spec}(\mathbb{T}_{N,m}) \rightarrow \text{Spec}(\Lambda_k) \cong \text{Spec}(O_K[[T]])$$

We will assume that $\text{Spec}(\mathbb{T}_{N,m})$ has at least two components $C_1$ and $C_2$. Let $\kappa$ be a $O_K$-point of $\text{Spec}(\Lambda_K)$ that is the $p$-adic limit of classical weights and let $x_1$ (resp $x_2$) be a $O_K$-point of $C_1$ (resp. $C_2$) with $\pi(x_1) = \kappa$. After a change of variables we may take $\kappa$ to be the point $T = 0$. We will assume that the restriction of $\pi$ to $C_1$ (resp $C_2$) is etale at $x_1$ (resp $x_2$). If $C_1$ and $C_2$ cross above $\kappa$ then it is possible to choose $x_1$ and $x_2$ to be the same point when viewed as points of $\text{Spec}(\mathbb{T}_{N,m})$.

6.1 Reducing to the simplest geometric situation

We may write $C_i = \text{Spec}(A_i)$ where $A_i$ is generated as a $\Lambda_K$-algebra by $T_{i,1},...,T_{i,n}$ and we may choose the $T_{i,1}$ so that $x_i$ corresponds to the origin. By applying Lemma 5.2 to $C_1$ and $C_2$ simultaneously we know that there exists $N > 0$ such that

$$B_i := A_i\left(\frac{1}{p} N T_0, \frac{1}{p} N T_{i,1}, ..., \frac{1}{p} N T_{i,n}\right) \cong O_K(\frac{1}{p} N T_0).$$

Let $Y = \frac{1}{p} N T_0$. Then $\text{Spec}(B_i)$ is a connected component of $\text{Spec}(\mathbb{T}_{N,m}(Y))$. To see this, consider the commutative diagram

$$\begin{array}{ccc}
\text{Spec}(B_i) & \longrightarrow & \text{Spec}(\mathbb{T}_{N,m}(Y)) \\
\pi_i^{-1} \uparrow & & \downarrow \\
\text{Spec}(O_K(Y)) & \longrightarrow & \text{Spec}(O_K(Y))
\end{array}$$

From this diagram we see that the generic point of $\text{Spec}(B_i)$ must be sent to a minimal prime in $\text{Spec}(\mathbb{T}_{N,m}(\frac{1}{p} N))$ and that the map $\mathbb{T}_{N,m}(\frac{1}{p} N Y) \rightarrow B_i$ is surjective. Let $Z$ be the scheme theoretic union of $\text{Spec}(B_i)$ inside of $\text{Spec}(\mathbb{T}_{N,m}(\frac{1}{p} N))$. Then $Z$ comes naturally equip with a map to $\text{Spec}(O_K(\frac{1}{p} N Y))$, which we call $\pi$ by abuse of notation. This map is surjective and finite of degree two by our definition of $Z$. Thus

$$Z = \text{Spec}(O_K(\frac{1}{p} N Y)[T]/f(T,Y)),$$
where $T$ is some indeterminate and $f$ is monic of degree two in $T$. As $\pi$ admits two sections (one for each $\text{Spec}(B_i)$) the polynomial $f$ factors into linear terms, i.e.

$$f(T,Y) = (T - g_1(Y))(T - g_2(Y)).$$

Here we have $g_i \in \mathcal{O}_K(\frac{1}{p^N} Y)$ and $g_i$ corresponds to the closed subscheme $\text{Spec}(B_i)$. The points $x_1$ and $x_2$ are the same if and only if $g_1$ and $g_2$ have the same constant term.

There is a natural map $s : Z \to \mathbb{T}_{N,m}$. For any two $\mathcal{O}_K$-points $y_1, y_2$ in $Z$ we have a relation between the distances of these points in the two spaces:

$$d(y_1, y_2) = p^N d(s(y_1), s(y_2)).$$

These points correspond to cusp forms $f_{y_1}$ and $f_{y_2}$. We combine Lemma 1 with this relation to get:

**Lemma 6.1.** If $d(y_1, y_2) = p^e$ then the cusp forms $f_{y_1}$ and $f_{y_2}$ are congruent modulo $\pi_K^e$ but not $\pi_K^{e+1}$. Informally, the distances between points in $Z$ tells us exactly how congruent the corresponding cusp forms are.

### 6.2 An ideal of differences of $L$-values

Recall that for $\chi$ and $k \in \mathbb{N}$ there is a 1-variable $p$-adic $L$-function $L_p(T_{N,m}, \chi) \in \mathbb{T}_{N,m}[\chi]$. Let $L_p(B_i, \chi)$ be the restriction of this function to $\text{Spec}(B_i)$. In particular $L_p(B_i, \chi)$ is in $B_i$. Then for $x \in \text{Spec}(B_i)$ corresponding to a classical modular form $f_x$ we see that $L_p(B_i, \chi)$ evaluated at $x$ is equal to the algebraic part of $L(f_x, \chi, 1)$. The isomorphism $\pi_i : \text{Spec}(B_i[\chi]) \to \text{Spec}(\mathcal{O}_K[\chi](\frac{1}{p^N} T))$ allows us to view $L_p(B_i, \chi)$ as a power series in $T$. In fact, this gives us the Taylor series expansion of $L_p(C_i, \chi)$ expanded around $x_i$.

**Definition 4.** Let $I_L$ be the ideal of $\mathcal{O}_K^{\text{cusp}}(\frac{1}{p^N} T)$ generated by the elements $L_p(B_1, \chi) - L_p(B_2, \chi)$ for all Dirichlet characters $\chi$.

**Lemma 6.2.** Let $\kappa \in \text{Spec}(\mathbb{Z}_p(T))$ corresponds to a classical weight. Then $v_p(I_L(\kappa))$ is equal to the valuation of $g_1(\kappa) - g_2(\kappa) - N$.

**Proof.** Let $y_i$ be the point of $\text{Spec}(B_i)$ lying above $\kappa$. By Proposition 3.5 we know that $v_p(I_L(\kappa))$ tells us exactly how congruent the modular forms $f_{y_i}$ associated to these points are. Then by applying Lemma 6.1 we see that $v_p(I_L(\kappa)) = d(y_1, y_2)p^{-N}$. The coordinates of $y_i$ are $(\kappa, g_i(\kappa))$. Then applying parts four and five of Lemma 5.3 we find that $d(y_1, y_2) = |g_1(\kappa) - g_2(\kappa)|$. The result follows.

### 6.3 Proof of Theorem 1.2

We are now ready to prove Theorem 1.2 by comparing $g_1(Y) - g_2(Y)$ with the ideal $I_L$. The proof involves using Lemma 6.2 to see how congruences behave as we approach the crossing point at classical weights.

**Definition 5.** Define $B_{p^r}$ to be $\text{Spec}(\mathbb{Z}_p(\frac{1}{p^r} Y))$, the neighborhood of radius $\frac{1}{p^r}$ around $\kappa = 0$. 
Lemma 6.3. Let \( f(T) \in \mathbb{Z}_p(Y) \). Let \( x \in \mathbb{Z}_p \) with \( v_p(x) < 1 \). If \( f(x) \neq 0 \) (equivalently \( T - x \) does not divide \( f(T) \)), then there exists a ball \( B_{p^r}(x) \) centred at \( x \) such that \( f(T) \) is equal to a power of \( p \) times a unit when restricted to \( B_{p^r}(x) \). Equivalently, we have \( f(T) = p^s u(T) \) with \( u(T) \) being in \( \mathbb{Z}_p[[p^r(T - x)]] \). Any neighbourhood of \( x \) where \( f(T) \) has no roots will suffice.

Proposition 2. The largest power of \( Y \) dividing the ideal \( I_L \) is \( I(\mathbb{T}_N, C_1, C_2, x_1) \).

Proof. Let \( n = I(\mathbb{T}_N, C_1, C_2, x) \) and let \( m \) be the largest power of \( Y \) contained in \( I_L \). Let \( \chi \) be a Dirichlet character and \( k \in \mathbb{N} \). By Lemma 6.3 and Lemma 5.6 we may replace \( \mathcal{O}_K(Y) \) with a smaller ball \( B_{p^r}(0) \) where

\[
\begin{align*}
g_1(Y) - g_2(Y) &= Y^nu(Y)\pi_K^r \quad (3) \\
L_p(B_1, \chi) - L_p(B_2, \chi) &= Y^{m_\chi,s}v_{\chi,s}(Y)c_{\chi,s}. \quad (4)
\end{align*}
\]

Here \( u(Y), v_{\chi,s}(Y) \) are units and \( c_{\chi,s} \) is a constant in \( \mathcal{O}_K[\chi] \). Note that \( m \min(m_{\chi,s}) \). Pick a sequence \( t_n \) of points in \( B_{p^r}(0) \) that converge to 0 such that each \( t_n \) corresponds to a classical weight. We denote by \( L_p(B_i, \chi)(t_n) \) the function \( L_p(B_i, \chi) \) evaluated at the point \( t_n \). Then by Lemma 6.2 we know that

\[
\begin{align*}
v_p(g_1(t_n) - g_2(t_n)) &\leq v_p(L_p(B_1, \chi)(t_n) - L_p(B_2, \chi)(t_n)) \text{ which gives} \\
mv_p(t_n) + r &\leq mv_p(t_n) + t. \quad (5)
\end{align*}
\]

As \( v_p(t_n) \to \infty \) as \( n \to \infty \) we see that \( n \leq m_{\chi,s} \) and so \( n \leq m \). In particular we find that \( Y^n \) divides \( I_L \). Conversely, we know

\[
mv_p(t_n) \leq m_{\chi,s}v_p(t_n) \leq v_p(L_p(B_1, \chi)(t_n) - L_p(B_2, \chi)(t_n)).
\]

Applying Lemma 6.2 again while \( n \to \infty \) shows \( m \leq n \).

The proof of Theorem 1.2 follows as a corollary.

7 Some examples

In this section we explore two situations where crossings may occur. First, we look at two Hida families of different levels with the same residual representation. Under a suitable hypothesis on the levels, we can determine if the two families will cross in a higher level by looking at the \( L \)-functions on each family modified by appropriate Euler factors. These modified \( L \)-functions were introduced in \([7]\) for trivial tame character using results from \([24]\).

7.1 Components of different level

In this subsection we apply our results on crossing components to the situation described in Section 2.6 of \([7]\). We will briefly summarize the set up. For details and references see \([7]\). Let \( \overline{\mathbb{F}} \) be a modular residual Galois representation and let \( \overline{V} \) be the \( \mathbb{F}_q \) vector space on which \( G_{\overline{Q}} \) acts. We will assume that \( \overline{\mathfrak{p}} \) is odd, irreducible, \( p \)-ordinary, and \( p \)-distinguished. Fix a \( p \)-stabilization of \( \overline{\mathfrak{p}} \). We may assume that \( \mathbb{F}_q \) is the possible extension of \( \mathbb{F}_p \) (i.e. \( \mathbb{F}_q \) is generated by the traces of \( \overline{\mathfrak{p}} \). Let \( N(\overline{\mathfrak{p}}) \) be
the conductor of $\rho$. For a prime $l \neq p$ let $n_l$ be the dimension of the $I_l$ invariant of $V$. Let $\Sigma$ be a finite set of primes not containing $p$. Define

$$N(\Sigma) := N(\rho) \prod_{l \in \Sigma} l^{n_l}.$$ 

For any tame level $N$ we let $T'_{N}$ be the Hecke algebra acting on $S(Np^\infty, \mathcal{O}_K)$ generated by $U_p$ and $T_l$ for all $l \nmid Np$ (explicitly, we are just leaving out the Atkin Lehner operators).

We let $T'_{new}$ denote the Hecke algebra generated by $T_l$ for primes $l \nmid Np$ and $U_l$ for $l | Np$ acting on the subspace of $S(Np^\infty, \mathcal{O}_K)$ consisting of all newforms. Then we have a natural map of $\Lambda$-algebras

$$T'_{N(\Sigma)} \rightarrow \prod_{M | N(\Sigma)} T'_{new}.$$ 

This map becomes an isomorphism after tensoring over $\Lambda$ with its fraction field $\mathcal{L}$. As described by Hida [11] there is a Galois representation $\rho'_M : G_{\mathbb{Q}} \rightarrow T'_{new} \otimes \mathcal{L}$ for any $M$. This gives a Galois representation $\rho' : G_{\mathbb{Q}} \rightarrow T'_{N(\Sigma)} \otimes \mathcal{L}$. We have the following two theorems

**Theorem 7.1.** There exists a unique maximal prime $m$ of $T'_{N(\Sigma)}$ such that the residual representation of the composition

$$G_{\mathbb{Q}} \rightarrow T'_{N(\Sigma)} \rightarrow T_{N(\Sigma), m}$$

is $\rho$. Furthermore, there is a unique maximal prime $n$ of $T_{N(\Sigma)}$ such that the two local Hecke algebras are isomorphism:

$$T_{N(\Sigma), n} \cong T'_{N(\Sigma), m}.$$ 

**Proof.** Cite Wiles and Diamond. \hfill $\square$

**Theorem 7.2.** Let $a$ be a minimal primes ideal of $T_{N(\Sigma), n}$. There exists some $N(a)|N(\Sigma)$ and a minimal prime ideal $a'$ of $T'_{new}$ that makes the following diagram commute.

$$
\begin{array}{ccc}
T_{N(\Sigma), n} & \cong & T'_{N(\Sigma), n} \\
\downarrow & & \downarrow \\
T_{N(\Sigma), n}/a & \rightarrow & T'_{new}/a'
\end{array}
$$

**Proof.** This follows from 7.1 and the isomorphism

$$T'_{N(\Sigma)} \otimes \mathcal{L} \rightarrow \prod_{M | N(\Sigma)} T'_{new} \otimes \mathcal{L}.$$ 

See Proposition 2.5.2 in [7] for more details. \hfill $\square$

**Remark.** We may think of $\text{Spec}(T_{N(\Sigma), n}/a)$ as a family of old forms of level $N(\Sigma)$ and $\text{Spec}(T'_{new}/a')$ as a family of new forms of level $N(a)$. If $x \in \text{Spec}(T_{N(\Sigma), n})$ corresponds to the classical old form $f_x$, then there is a corresponding $x' \in \text{Spec}(T'_{new}/a')$ that is sent to $x$ under the map $\text{Spec}(T'_{new}/a') \rightarrow \text{Spec}(T_{N(\Sigma), n}/a)$. The point $x'$ corresponds to a newform $f_{x'}$ of level $N(a)$. The Fourier coefficients of $f_x$ and $f_{x'}$ agree away from the primes dividing the level $N(\Sigma)$. 21
By Theorem 1.2 we can determine when two components of \( T_{N(\Sigma)} \) by looking at \( p \)-adic \( L \)-functions on each component. It is then natural to ask if we can determine when a family of newforms of level \( M_1 | N(\Sigma) \) will cross a family of newforms of level \( M_2 | N(\Sigma) \) by looking at \( p \)-adic \( L \)-functions. To employ Theorem 1.2 it is necessary to relate our \( p \)-adic \( L \)-functions on \( \text{Spec}(T_{N(\Sigma)}) \) to our \( p \)-adic \( L \)-functions on \( \text{Spec}(\mathcal{T}_{N(a),n}/\mathfrak{a}') \). The former interpolates special values of eigenforms for the Hecke algebra \( T_{N(\Sigma)} \) and the later interpolates special values of eigenforms for the Hecke algebra \( T_{N(a)} \). As these two Hecke algebras only differ at \( l | N(\Sigma) \), it is natural to suspect that the two \( L \)-functions will be the same after introducing some Euler factors for the primes \( l | N(\Sigma) \).

**Definition 6.** Let \( l \neq p \) be a prime and let \( \chi \) be a Dirichlet character of level \( Mp^r \). Define \( E_{N(a)}(\chi,l) \in T_{N(a)} \) as follows:

\[
E_{N(a)}(\chi,l) := \begin{cases} 
1 - \chi(l)T_l^{-1}l^{-3} & \text{if } l \nmid N(a) \\
1 - \chi(l)T_l^{-1} & \text{if } l | N(a)
\end{cases}
\]

We then define

\[
E_\Sigma(a,\chi) := \prod_{l \in \Sigma} E_{N(a)}(\chi,l).
\]

**Remark.** This definition is similar to Definition 2.7.1 and 3.6.1 in [7]. The Euler of [7] varies over a branch of the Hida family and the cyclotomic variable, while our definition only varies over the branch. If the conductor of \( \chi \) is a power of \( p \) then our Euler factor is equal to a specialization of the Euler factor defined in [7].

**Proposition 3.** Let \( \chi \) be a Dirichlet character. There exists a unit \( u \in T_{N(a)}/\mathfrak{a}' \) independent of \( \chi \) Dirichlet character such that

\[
L_p(T_{N(\Sigma)}/\mathfrak{a},\chi) = E_\Sigma(a,\chi)L_p(T_{N(a)}/\mathfrak{a}',\chi).
\]

**Proof.** The proof follows from the computations in beginning of the proof of Theorem 3.6.2 in [7]. The only difference is that we are specializing in the cyclotomic variable and we allow a nontrivial tame conductor.

**Theorem 7.3.** Let \( M_1 \) and \( M_2 \) be two integers dividing \( N(\Sigma) \). Let \( \mathfrak{a}_i \) be a minimal prime ideal of \( T_{new}^{M_i} \). Let \( C_1 \) and \( C_2 \) be the components of \( T_{N(\Sigma)} \) corresponding to \( \mathfrak{a}_1 \) and \( \mathfrak{a}_2 \). The following are equivalent:

- The components \( C_1 \) and \( C_2 \) cross at a point \( x \). We assume that each component is etale at \( x \) over the weight space and the weight \( \kappa \) of \( x \) is the \( p \)-adic limit of classical weights.

- There exists a point \( x_i \) of \( \text{Spec}(T_{new}^{M_i}) \) over \( \kappa \) and a unit \( u \) of \( \Lambda \) such that for all Dirichlet characters \( \chi \) the value of \( uL_p(T_{new}^{M_i}/\mathfrak{a}_1,\chi)E_\Sigma(\mathfrak{a}_1,\chi) \) evaluated at \( x_1 \) is the same as \( L_p(T_{new}^{M_2}/\mathfrak{a}_2,\chi)E_\Sigma(\mathfrak{a}_2,\chi) \) evaluated at \( x_2 \).

**Proof.** This is a consequence of Proposition 3 and Theorem 1.2.
8 Ramification over the weight space

In this section we describe how $p$-adic $L$-functions behave when a Hida family is ramified over the weight space. Recall that $\Lambda_K$ is the ring of power series over $\mathcal{O}_K$. Let $C$ be a component of $\text{Spec}(\mathbb{T}_N)$. Then $C$ is affine and we let $A$ be the coordinate ring. Informally, the main result of this section says that $C$ has ramified points over $\Lambda_K$ if and only if there exists an $L$-function $L_p(C, \chi)$ that acquires singularities after being hit with the differential operator $\frac{d}{dT}$. Here $L_p(C, \chi)$ refers to the $L$-function defined in Section 4.3 restricted to the component $C$. The singularities will be at ramified points.

**Theorem 8.1.** A regular $\mathcal{O}_K$-point $x \in C$ is ramified over $\pi(x) \in \text{Spec}(\Lambda_K)$ if and only if there exists a Dirichlet twist such that $\frac{d}{dT}L(C, \chi)$ has a pole at $x$, where $T$ is a parameter of the weight space. The ramification index of $x$ over $\pi(x)$ is equal to one more than the largest order pole occurring.

**Proof.** First let's assume that $\pi$ is etale at $x$. Informally, this means that a small neighborhood of $x$ looks just like part of $\text{Spec}(\Lambda_K)$. We may assume that $A_x$ is a uniformizing element of $\mathcal{O}_C$ at $x$. Define $\Lambda_{K,\pi(x)}$ to be the completion along the maximal ideal of the stalk of $\mathcal{O}_C$ at $x$. Let $\Lambda_{\Lambda_K}$ be the completion along the maximal ideal of the stalk of $\mathcal{O}_{\Lambda_K}$ at $x$. The natural map from $\Lambda_{K,\pi(x)} \to A_x$ is an etale morphism of complete local rings with isomorphic residue fields. This means the two rings are isomorphic. This isomorphism commutes with the differential operator $\frac{d}{dT}$. In particular there is a map $A \to \Lambda_{K,\pi(x)}$ such that commutes with $\frac{d}{dT}$ and the maximal ideal of $\Lambda_{K,\pi(x)}$ pulls back to $x$. It is then clear that for any $f \in A$ the function $\frac{d}{dT}f$ does not have a pole at $x$.

The converse is more difficult. We begin by making some geometric simplifications similar to those in Section 6.1. Assume that $\pi$ is ramified at $x$. After a change of coordinates we may assume that $\pi(x) = 0$. The ring $A_x$ is a discrete valuation ring since $x$ is a regular point of codimension one. Let $X$ be a uniformizing element of $A_x$. We may assume that $X$ is in $A$ by clearing any denominators. Since $X$ is topologically nilpotent in $A$ we have a map

$$g : C = \text{Spec}(A) \to \text{Spec}(\mathcal{O}_K[[Y]])$$

induced by the ring map sending $Y$ to $X$. This map is etale at $x$ so we may apply Lemma 5.2. In particular, let $Y_1, ..., Y_r$ generate $A$ as a $\mathcal{O}_K[[Y]]$-algebra. There exists $N$ large enough so that

$$g : \text{Spec}(A(\frac{1}{p} Y, \frac{1}{p} Y_i)) \to \text{Spec}(\mathcal{O}_K(\frac{1}{p} Y))$$

is an isomorphism. Setting $T' = \frac{1}{p} Y$ and $Y' = \frac{1}{p} Y$, we have $T' = f(Y')$ where $f(Y') \in \mathcal{O}_K(Y')$. By increasing $N$, we may guarantee that the only zero of $f(Y')$ is at $Y' = 0$. Thus $f(Y') = \pi^e_k u(Y') Y^{e_e}$ where $u(Y')$ is a unit. We may write

$$A(\frac{1}{p} Y, \frac{1}{p} Y_i) \cong \mathcal{O}_K(T', Y')/(\pi^e_k u(Y') Y^{e_e} - T').$$

The ramification of $x$ over $\pi(x)$ is seen to be $e$. We also remark distances of points relate to higher congruences of the corresponding cusp forms. If $x_1$ and $x_2$ are two $\mathcal{O}_K$-points of $A(\frac{1}{p} Y, \frac{1}{p} Y_i)$
corresponding to classical cusp forms $f_{x_1}$ and $f_{x_2}$ that are congruent modulo $\pi^r_K$ then $d(x_1, x_2) = p^{-\frac{N}{\nu_K(p)}}N$. This is more or less the same as Lemma 6.1.

For a Dirichlet character $\chi$ we let $L_p(\chi)$ be the restriction of $L_p(C, \chi)$ to $A(Y', T')[\chi]$. Then we may think of $L_p(\chi)$ as an element of $O_K(Y')[\chi]$ written as

$$\Sigma c_{i,\chi} Y'^i.$$

Let $\alpha$ be an $O_K$-point of our weight space $\text{Spec}(O_K(Y'))$ corresponding to a classical weight. By abuse of notation we will also think of $\alpha$ as an element of $O_K$ for the parameter $T'$. Let $a_1$ and $a_2$ be distinct $O_K$-points in $\pi^{-1}(\alpha) \subset A(Y', T') \cong O_K(Y')$. By abuse of notation we will also think of $a_i$ as an element of $O_K$ for the parameter $Y'$. Both $a_1$ and $a_2$ are roots of $\alpha - \pi^r_K u(Y')Y^e$ and $d(a_1, a_2) = |a_1 - a_2|_p$. Relating $d(a_1, a_2)$ to congruences and then applying Theorem 3.5 gives

$$\log_p d(a_1, a_2) = v_p(a_1 - a_2)$$

$$= \min_{\chi} v_p(L_p(\chi)(a_1) - L_p(\chi)(a_2)) + N$$

$$= \min_{\chi} (-N + v_p(\Sigma c_{i,\chi} (a_1^i - a_2^i))).$$

If we take $\alpha$ to have large enough valuation then both of the $a_i$'s have valuation larger than $N + 1$ (look at the Newton polygon of $\alpha - \pi^r_K u(Y')Y^e$). This means $v_p(a_1^i - a_2^i) > v_p(a_1 - a_2) + N + 1$ whenever $i > 1$.

$$v_p(a_1 - a_2) = \min_{\chi} (-N + v_p(\Sigma c_{i,\chi} (a_1^i - a_2^i)))$$

$$= \min_{\chi} (-N + v_p(c_{1,\chi} (a_1 - a_2)))$$

$$= \min_{\chi} (-N + v_p(c_{1,\chi} + v_p(a_1 - a_2)).$$

Thus there exists a Dirichlet character $\chi_0$ such that $c_{1,\chi_0} \neq 0$. Differentiating the equation

$$T - \pi^r_K u(Y)Y^e = 0$$

with respect to $T$ yields

$$\frac{dY}{dT} = \frac{1}{\pi^r_K Y^{e-1} Y'u' + eu(Y)}.$$

This shows that $\frac{dY}{dT}$ has a pole at $Y = 0$ of order $e - 1$. Since $L_p(\chi)$ has a nonzero linear term we find that $\frac{d}{dT} L_p(\chi)$ has a pole of order $e - 1$.

For the previous result, we choose a parameter for the weight space. The result holds true for any parameter and it would be nice to have a statement that makes no reference to any choice of parameter. This can be achieved using the Gauss-Manin connection, which can be defined without choosing a basis. For an overview of the Gauss-Manin connection see [15] or [16]. More precisely, consider the relative 0-th de Rham cohomology group $H^0_{dR}(C/\text{Spec}(\Lambda_K))$ (see for example [10]). We may identify $H^0_{dR}(C/\text{Spec}(\Lambda_K))$ with $\pi_*(\mathcal{O}_C)$ (here $\mathcal{O}_C$ just denotes the structure sheaf of $C$). Let $U$ be an open subscheme of $C$ such that $\pi|_U$ is etale. Then following [15] there is a Gauss-Manin connection

$$\nabla : H^0_{dR}(C/\text{Spec}(\Lambda_K))|_U \to H^0_{dR}(C/\text{Spec}(\Lambda_K)) \otimes \Omega_{\text{Spec}(\Lambda_K)}|_U.$$
If \( f \in \Gamma(O_C, U) \) and \( T_0 \) is any parameter of the weight space then \( \nabla(f) = \frac{d}{dT_0} f dT_0 \). The map \( \nabla \) makes sense on all of \( \text{Spec}(\Lambda_K) \) when we allow poles in the image.

**Corollary 8.2.** Let \( \kappa \) be a \( O_K \) point of \( \Lambda_K \). The map \( \pi \) is etale at the points above \( \kappa \) if and only if for all Dirichlet characters \( \chi \) we have \( \nabla(L_p(C, \chi)) \in \Gamma((\pi_\ast O_C \otimes \Omega_{\text{Spec}(\Lambda_K)}), V) \) where \( V \) is some Zariski open containing \( \kappa \).

**Proof.** The "if" direction follows from the existence of the Gauss-Manin connection for smooth maps. For the "only if" let \( x \) be a point in \( \pi^{-1}(\kappa) \). Let \( \chi \) be a Dirichlet character. By our hypothesis \( \nabla(L_p(C, \chi)) \in \Gamma((\pi_\ast O_C \otimes \Omega_{\text{Spec}(\Lambda_K)}), \kappa) \). Note that

\[
(\pi_\ast O_C \otimes \Omega_{\text{Spec}(\Lambda_K)})_\kappa \cong A_\kappa \otimes A_{\Lambda_K, \kappa} \otimes \Omega_{\text{Spec}(\Lambda_K), \kappa}.
\]

Choose a parameter \( T_0 \) of the weight space and let \( D \) be the map from \( \Omega_{\text{Spec}(\Lambda_K), \kappa} \) to \( O_{\text{Spec}(\Lambda_K)} \) that sends \( dT_0 \) to 1. Then \( D \circ \nabla \) is the map \( A_\kappa \rightarrow A_\kappa \) given by differentiation with respect to \( T_0 \). There is a natural map \( l : A_\kappa \rightarrow A(x) \), the localization of \( A \) at \( x \). Since \( \nabla(L_p(C, \chi)) \) is contained in \( (\pi_\ast O_C \otimes \Omega_{\text{Spec}(\Lambda_K)})_\kappa \) we see that \( l \circ D \circ \nabla(L_p(C, \chi)) \) is contained in \( A(x) \). This means that \( \frac{d}{dT_0} L_p(C, \chi) \) does not have a pole at \( x \). The corollary then follows from Theorem 8.1.

\[\square\]

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