CW SIMPLICIAL RESOLUTIONS OF SPACES
WITH AN APPLICATION TO LOOP SPACES

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Abstract. We show how a certain type of CW simplicial resolutions of spaces by
wedges of spheres may be constructed, and how such resolutions yield an obstruction
theory for a given space to be a loop space.

1. Introduction

A simplicial resolution of a space $X$ by wedges of spheres is a simplicial space $W$ such that
(a) each space $W_n$ is homotopy equivalent to a wedge of spheres, and (b) for each $k \geq 1$, the augmented simplicial group $\pi_k(W) \to \pi_k(X)$ is acyclic (see §3.3 below). Such resolutions, first constructed by Chris Stover in [St, §2], are dual to the
“unstable Adams resolutions” of [BK, I, §2], and have a number of applications: see
§3.10 below and [St, DKSS, DKS1, B1, B3, B5, B7].

However, the Stover construction yields very large resolutions, which do not lend
themselves readily to computation, and no other construction was hitherto available.
In particular, it was not clear whether one could find minimal resolutions of this type.
The purpose of this note is to show that any space $X$ has simplicial resolutions by
wedges of spheres, which may be constructed from purely algebraic data, consisting
of an (arbitrary) simplicial resolution of $\pi_\ast X$ as a $\Pi$-algebra — that is, as a graded

Theorem A. Every free simplicial $\Pi$-algebra resolution of a realizable $\Pi$-algebra $\pi_\ast X$
is realizable topologically as a simplicial resolution by wedges of spheres.

and in fact such resolutions can be given a convenient “CW structure” (§3.15). There
is an analogous result for maps (Theorem 3.24).

Since no such resolution of a non-realizable $\Pi$-algebra can be realized (see §3.16
below), this completely determines which free simplicial $\Pi$-algebra resolutions are re-
alisable.

The Theorem implies that in the spectral sequences of [St, B1, DKSS] we can work
with minimal resolutions, and allows us to identify the higher homotopy operations of
[B1, B3, B7] as lying in appropriate cohomology groups (compare [B6, 4.17] and [B8,
§6]). A generalization of Theorem A to other model categories appears in [B9].

As an application of such CW resolutions, we describe an obstruction theory for de-
ciding whether a given space $X$ is a loop space, in terms of higher homotopy operations.
One such theory was given in [B7], but the present approach does not require a given
$H$-space structure on $X$, and may be adapted also to the existence of $A_n$-structures
(and thus subsumes [B6]). It is summarized in

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Theorem B. A space $X$ with trivial Whitehead products is homotopy equivalent to a loop space if and only if the higher homotopy operations of §5.10 below vanish coherently.

1.1. Notation and conventions. $\mathcal{G}$ will denote the category of groups, $\mathcal{T}$ that of topological spaces, and $\mathcal{T}_*$ that of pointed topological spaces with base-point preserving maps. The full subcategory of 0-connected spaces will be denoted by $\mathcal{T}_c \subset \mathcal{T}_*$. The category of simplical sets will be denoted by $\mathcal{S}$ and that of pointed simplicial sets by $\mathcal{S}_*$. We shall use boldface letters: $X$, $S^n$, . . . to denote objects in any of these four categories. If $f : X \to Y$ is a map in one of these categories, we denote by $f^# : \pi_*X \to \pi_*Y$ the induced map in the homotopy groups.

1.2. Organization. In section 2 we review some background on simplicial objects and bisimplicial groups, and in section 3 we recall some facts on $\Pi$-algebras, and prove our main results on CW resolutions of spaces by wedges of spheres: Theorem A (=Theorem 3.20) and Theorem 3.24. In section 4 we define a certain cosimplicial simplicial space up-to-homotopy, which can be rectified if and only if $X$ is a loop space. In section 5 we construct a certain collection of face-codegeneracy polyhedra, which are used to define the higher homotopy operations referred to in Theorem B (=Theorem 5.12). We also show how the theorem may be used in reverse to calculate a certain tertiary operation in $\pi_*S^7$.

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2. SIMPLICIAL OBJECTS

We first provide some definitions and facts on simplicial objects:

2.1. Definition. Let $\Delta$ denote the category of ordered sequences $n = \langle 0, 1, \ldots, n \rangle$ $(n \in \mathbb{N})$, with order-preserving maps. A simplicial object over a category $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$, usually written $X_\bullet$, which may be described explicitly as a sequence of objects $\{X_k\}_{k=0}^\infty$ in $\mathcal{C}$, equipped with face maps $d_i^k : X_k \to X_{k-i}$ and degeneracies $s_j^k : X_k \to X_{k+1}$ (usually written simply $d_i$, $s_j$, for $0 \leq i, j \leq k$), satisfying the usual simplicial identities ([Ma, §1.1]). If $I = (i_1, i_2, \ldots, i_r)$ is some multi-index, we write $d_I$ for $d_{i_1} \circ d_{i_2} \circ \cdots \circ d_{i_r}$, with $d_{\emptyset} := id$; and similarly for $s_I$. An augmented simplicial object is one equipped with an augmentation $\varepsilon : X_0 \to Y$ (for $Y \in \mathcal{C}$), with $\varepsilon d_0 = \varepsilon d_1$.

The category of simplicial objects over $\mathcal{C}$ is denoted by $s\mathcal{C}$. We write $s_{(n)} \mathcal{C}$ for the category $n$-simplicial objects over $\mathcal{C}$ (that is, objects of the form $\{X_k\}_{k=0}^n$, with the relevant face maps and degeneracies), and denote the truncation functor $s\mathcal{C} \to s_{(n)} \mathcal{C}$ by $\tau_n$.

For technical convenience in the next two sections we shall be working mainly in the category of simplicial groups, denoted by $\mathcal{G}$ (rather than $s\mathcal{G}$); objects in $\mathcal{G}$ will be denoted by capital letters $X$, $Y$, and so on. A simplicial object $X_\bullet = (X_0, X_1, \ldots)$ in $s\mathcal{G}$ is thus a bisimplicial group, which has an external simplicial dimension (the $n$ in $X_n \in \mathcal{G}$), as well as the internal simplicial dimension $k$ (inside $\mathcal{G}$), which we shall denote by $(X_n)_k^{int}$, if necessary.
2.2. Simplicial sets and groups. The standard $n$-simplex in $S$ is denoted by $\Delta[n]$, generated by $\sigma_n \in \Delta[n]$. $\Delta[n]$ denotes the sub-object of $\Delta[n]$ generated by $d_i\sigma_n$ ($0 \leq i \leq n$). The simplicial $n$-sphere is $S^n := \Delta[n]/\Delta[n]$, and the $n$-disk is $D^n := CS^{n-1}$.

Let $F : S \to G$ denote the (dimensionwise) free group functor of $[M2, \S 2]$, and $G : S \to G$ be Kan's simplicial loop functor (cf. $[Ma, \text{Def. 26.3}]$), with $W : G \to S$ the Eilenberg-Mac Lane classifying space functor (cf. $[Ma, \S 21]$). Recall that if $S : T \to S$ is the singular set functor and $\| - \| : S \to T$ the geometric realization functor (see $[Ma, \S 1,14]$), then the adjoint pairs of functors

$$T \overset{S}{\leftarrow} S \overset{G}{\rightarrow} G$$

induce isomorphisms of the corresponding homotopy categories (see $[Q1, I, \S 5]$), so that for the purposes of homotopy theory we can work in $G$ rather than $T$.

2.4. Definition. In particular, $S^n := F S^{n-1} \in G$ for $n \geq 1$ (and $S^0 := G S^0$ for $n = 0$) will be called the $n$-dimensional $G$-sphere, in as much as $[S^n, GX]_G \cong \pi_n X = [S^n, X]$ for any Kan complex $X \in S$. Similarly, $D^n := F D^{n-1}$ will be called the $n$-dimensional $G$-disk.

2.5. Definition. In any complete category $C$, the matching object functor $M : S^{op} \times sC \to C$, written $M_{\Delta[n]} X_\bullet$ for a (finite) simplicial set $A \in S$ and $X_\bullet \in sC$, is defined by requiring: (a) $M_{\Delta[n]} X_\bullet := X_n$, and (b) if $A = \text{colim}_i A_i$, then

$$M_{A_k} X_\bullet = \text{lim}_i M_{A_i} X_\bullet \text{ (see }[DKS2, \S 2.1]).}$$

In particular, if $A_k$ is the subcomplex of $\Delta[n]$ generated by the last $(n-k+1)$ faces $(d_k\sigma_n, \ldots, d_n\sigma_n)$, we write $M_{A_k} X_\bullet$ for $M_{\Delta[n]} X_\bullet$ explicitly,

$$(2.6) \quad M_{A_k} X_\bullet = \{(x_k, \ldots, x_n) \in (X_{n-1})^{n+1} \mid d_i x_j = d_{j-1} x_i \text{ for all } k \leq i < j \leq n\}.$$  

and the map $\delta_k : X_n \to M_{A_k} X_\bullet$ is defined by the inclusion $A_k \hookrightarrow \Delta[n]$.

2.7. Remark. Note that for $X \in G$ and $A \in S$ we have $M_A X \cong \text{Hom}_G(F A, X) \in sG$ (cf. $[Q2, \S 2]$), so for $X_\bullet \in sG$ also $(M_A X)_k \cong \text{Hom}_G(F A_\bullet(X_\bullet^\text{int})$, in each simplicial dimension $k$.

2.8. Definition. $X_\bullet \in sG$ is called fibrant if each of the maps $\delta_n : X_n \to M_n X_\bullet$ ($n \geq 0$) is a fibration in $G$ (that is, a surjection onto the identity component — see $[Q1, II, 3.8]$). This is just the condition for fibrancy in the Reedy model category, (see $[R]$), as well as in that of $[DKST]$, but we shall not make explicit use of either.

By analogy with Moore's normalized chains (cf. $[Ma, 17.3]$) we have:

2.9. Definition. Given $X_\bullet \in sG$, we define the $n$-cycles object of $X_\bullet$, written $Z_n X_\bullet$, to be the fiber of $\delta_n : X_n \to M_n X_\bullet$, so $Z_n X_\bullet = \{x \in X_n \mid d_i x = 0 \text{ for } i = 0, \ldots, n\}$ (cf. $[Q1, I, \S 2]$). Of course, this definition really makes sense only when $X_\bullet$ is fibrant.
(§2.8). Similarly, the \( n \)-chains object of \( X_\bullet \), written \( C_nX_\bullet \), is defined to be the fiber of \( \delta^1_n : X_n \to M^1_nX_\bullet \).

If \( X_\bullet \in sG \) is fibrant, the map \( d^n_0 = d_0|_{C_nX_\bullet} : C_nX_\bullet \to Z_{n-1}X_\bullet \) is the pullback of \( \delta_n : X_n \to M_nX_\bullet \) along the inclusion \( \iota : Z_{n-1}X_\bullet \to M_nX_\bullet \) (where \( \iota(z) = (z, 0, \ldots, 0) \)), so \( d^n_0 \) is a fibration (in \( G \)), fitting into a fibration sequence
\[
Z_nX_\bullet \xrightarrow{j_n} C_nX_\bullet \xrightarrow{d^n_0} Z_{n-1}X_\bullet.
\]

2.11. Proposition. For any fibrant \( X_\bullet \in sG \), the inclusion \( \iota : C_nX_\bullet \hookrightarrow X_n \) induces an isomorphism \( \iota_* : \pi_*C_nX_\bullet \cong \pi_n(\pi_*X_\bullet) \) for each \( n \geq 0 \).

Proof. (a) First note that if \( j : A \hookrightarrow B \) is a trivial cofibration in \( S \), then \( j^* : M_BX_\bullet \to M_AX_\bullet \) has a natural section \( r : M_AX_\bullet \to M_BX_\bullet \) (with \( j^* \circ r = id \)) for any \( X_\bullet \in sG \). This is because by remark 2.7, \( (M_AX_\bullet)_k \cong \text{Hom}_G(FA, (X_\bullet)_k^{\text{int}}) \) for \( A \in S \); since \( FA \) is fibrant in \( G \), we can choose a left inverse \( \rho : FB \to FA \) for \( Fj : FA \hookrightarrow FB \), so \( j^* : (M_BX_\bullet)_k^{\text{int}} \to (M_AX_\bullet)_k^{\text{int}} \) has a right inverse \( \rho^* \), which is natural in \( (X_\bullet)_k^{\text{int}} \); so these maps \( \rho^* \) fit together to yield the required map \( r \).

This need not be true in general if \( j \) is not a weak equivalence, as the example of \( M_1^1X_\bullet \to M_1^0X_\bullet \) shows.

(b) Given \( \eta \in C_n\pi_mX_\bullet \) represented by \( h : S^m \to X_n \) with \( d_kh \sim 0 \) (1 \( \leq k \leq n \)), consider the diagram:

\[
\begin{array}{ccc}
S^m & \xrightarrow{h} & X_n \\
\downarrow & & \downarrow \\
\sim 0 & \xrightarrow{j^*} & \sim 0 \\
\end{array}
\]

in which \( j^* \) is a fibration by (a) if \( k \geq 1 \), so the lower left-hand square is in fact a homotopy pullback square (see §1). By descending induction on \( 1 \leq k \leq n-1 \), (starting with \( \delta^1_n = d_n \)), we may assume \( \delta^{k+1}_n \circ h : S^m \to M^1_kX_\bullet \) is nullhomotopic in \( C \), as is \( d_k \circ h \), so the induced pullback map \( \delta^k_n \circ h : S^m \to M^k_nX_\bullet \) is also nullhomotopic by the universal property. We conclude that \( \delta^1_n \circ h \sim 0 \), and since \( \delta^1_n : X_n \to M^1_nX_\bullet \) is a fibration by (a), we can choose \( h : S^m \to X_n \) so that \( \delta^1_nh = 0 \). Thus \( h \) lifts to \( C_nX_\bullet = \text{Fib}(\delta^1_n) \), and \( \iota_* \) is surjective.

(c) Finally, the long exact sequence in homotopy for the fibration sequence
\[
C_nX_\bullet \xrightarrow{j_n} X_n \xrightarrow{\delta^n_0} M^1_nX_\bullet
\]
implies that \( \iota_\# : \pi_*C_nX_\bullet \to \pi_*X_n \) is monic, so \( \iota_* : \pi_*C_nX_\bullet \to C_n(\pi_*X_\bullet) \) is, too. \( \square \)
2.12. Definition. The dual construction to that of \([3, 7]\) yields the colimit

\[ L_n X \bullet := \coprod_{0 \leq i \leq n-1} X_{n-1}/\sim, \]

where for any \( x \in X_{n-2} \) and \( 0 \leq i \leq j \leq n-1 \) we set \( s_j x \) in the \( i \)-th copy of \( X_{n-1} \) equivalent under \( \sim \) to \( s_i x \) in the \( (j+1) \)-st copy of \( X_{n-1} \). \( L_n X \bullet \) has sometimes been called the “\( n \)-th latching object” of \( X \bullet \). The map \( \sigma_n : L_n X \bullet \to X_n \) is defined \( \sigma_n x_{(i)} = s_i x \), where \( x_{(i)} \) is in the \( i \)-th copy of \( X_{n-1} \).

3. \( \Pi \)-algebras and resolutions

In this section we recall some definitions and prove our main results on \( \Pi \)-algebras and resolutions:

3.1. Definition. A \( \Pi \)-algebra \( G \) is a graded group \( G_* = \{G_k\}_{k=1}^\infty \) (abelian in degrees \( > 1 \)), together with an action on \( G_* \) of the primary homotopy operations (i.e., compositions and Whitehead products, including the “\( \pi_1 \)-action” of \( G_1 \) on the higher \( G_n \)'s, as in \([W, X, \S 7]\)), satisfying the usual universal identities. See \([B3, \S 2.1]\) for a more explicit description. These are algebraic models of the homotopy groups \( \pi_* X \) of a space (or Kan complex) \( X \), in the same way that an algebra over the Steenrod algebra models its cohomology ring. The category of \( \Pi \)-algebras is denoted by \( \Pi\text{-Alg} \).

We say that a space (or Kan complex, or simplicial group) \( X \) realizes an (abstract) \( \Pi \)-algebra \( G_* \) if there is an isomorphism of \( \Pi \)-algebras \( G_* \cong \pi_* X \). (There may be non-homotopy equivalent spaces realizing the same \( \Pi \)-algebra – cf. \([B3, \S 7.18]\]). Similarly, an abstract morphism of \( \Pi \)-algebras \( \phi : \pi_* X \to \pi_* Y \) (between realizable \( \Pi \)-algebras) is realizable if there is a map \( f : X \to Y \) such that \( \pi_* f = \phi \).

3.2. Definition. The free \( \Pi \)-algebra generated by a graded set \( T_* = \{T_k\}_{k=1}^\infty \) is \( \pi_* W \), where \( W = \bigvee_{k=1}^\infty \bigvee_{\tau \in T_k} S^k(\tau) \) (and we identify \( \tau \in T_k \) with the generator of \( \pi_k W \) representing the inclusion \( S^k(\tau) \hookrightarrow W \)).

If we let \( \mathcal{F} \subset \Pi\text{-Alg} \) denote the full subcategory of free \( \Pi \)-algebras, and \( \Pi \) the homotopy category of wedges of spheres (inside \( ho T_* \) or \( ho S_* \) – or equivalently, the homotopy category of coproducts of \( G \)-spheres in \( ho G \)), then the functor \( \pi_* : \Pi \to \mathcal{F} \) is an equivalence of categories. Thus any \( \Pi \)-algebra morphism \( \varphi : G_* \to H_* \) is realizable (uniquely, up to homotopy), if \( G_* \) and \( H_* \) are free \( \Pi \)-algebras (actually, only \( G_* \) need be free).

3.3. Definition. Let \( T : \Pi\text{-Alg} \to \Pi\text{-Alg} \) be the “free \( \Pi \)-algebra” comonad (cf. \([M3, \S I]\)), defined \( TG_* = \coprod_{k=1}^\infty \coprod_{g \in G_*} \pi_* S^k(\theta_g) \). The counit \( \varepsilon = \varepsilon_{G_*} : TG_* \to G_* \) is defined by \( \iota_{(g)} \mapsto g \) (where \( \iota_{(g)} \) is the canonical generator of \( \pi_* S^k(\theta_g) \)), and the comultiplication \( \vartheta = \vartheta_{G_*} : TG_* \to T^2 G_* \) is induced by the natural transformation \( \vartheta : id_{\mathcal{F}} \to T|_{\mathcal{F}} \) defined by \( x_k \mapsto \iota_{(x_k)} \).

3.4. Definition. An abelian \( \Pi \)-algebra is one for which all Whitehead products vanish.

These are indeed the abelian objects of \( \Pi\text{-Alg} \) – see \([B3, \S 2]\). In particular, if \( X \) is an \( H \)-space, then \( \pi_* X \) is an abelian \( \Pi \)-algebra (cf. \([W, X, (7.8)]\)).
3.5. Definition. A simplicial $\Pi$-algebra $A_\bullet$ is called free if for each $n \geq 0$ there is a graded set $T^*_n \subseteq A_n$ such that $A_n$ is the free $\Pi$-algebra generated by $T^*_n$ (§3.2), and each degeneracy map $s_j : A_n \to A_{n+1}$ takes $T^*_n$ to $T^*_{n+1}$.

A free simplicial resolution of a $\Pi$-algebra $G_\ast$ is defined to be an augmented simplicial $\Pi$-algebra $A_\bullet \to G_\ast$, such that

(i) $A_\bullet$ is a free simplicial $\Pi$-algebra,

(ii) in each degree $k \geq 1$, the homotopy groups of the simplicial group $(A_\bullet)_k$ vanish in dimensions $n \geq 1$, and the augmentation induces an isomorphism $\pi_0(A_{\bullet})_k \cong G_k$.

Such resolutions always exist, for any $\Pi$-algebra $G_\ast$—see [Ql] II, §4], or the explicit construction in [B1], §4.3.

3.6. Definition. For any $X \in G$, a simplicial object $W_\bullet \in sG$ equipped with an augmentation $\varepsilon : W_0 \to X$ is called a resolution of $X$ by spheres if each $W_n$ is homotopy equivalent to a wedge of $G$-spheres, and $\pi_\ast W_\bullet \to \pi_\ast X$ is a free simplicial resolution of $\Pi$-algebras.

3.7. Example. One example of such a resolution by spheres is provided by Stover’s construction; we shall need a variant in $G$ (as in [B7, §5]), rather than the original version of [B1, §2], in $T_\ast$. (The argument from this point on would actually work equally well in $T_\ast$; but we have already chosen to work in $G$, in order to facilitate the proof of Proposition 2.11).

Define a comonad $V : G \to G$ for $G \in G$ by

\[(3.8) \quad VG = \coprod_{k=0}^{\infty} \coprod_{\phi \in \text{Hom}_G(S^k,G)} S^k_{\phi} \cup \coprod_{k=0}^{\infty} \coprod_{\phi \in \text{Hom}_G(D^{k+1},G)} D^{k+1}_{\phi},\]

where $D^{k+1}_\phi$, the $G$-disc indexed by $\Phi : D^{k+1} \to G$, is attached to $S^k_{\phi}$, the $G$-sphere indexed by $\phi = \Phi|_{\partial D^{k+1}}$, by identifying $\partial D^{k+1} := F\partial D^k$ with $S^k$ (see §3.3 above). The coproduct here is just the (dimensionwise) free product of groups; the counit $\varepsilon : VG \to G$ of the comonad $V$ is “evaluation of indices”; and the comultiplication $\partial : VG \hookrightarrow V^2 G$ is as in §3.3.

Now given $X \in G$, define $Q_\bullet \in sG$ by setting $Q_n = V^{n+1}X$, with face and degeneracy maps induced by the counit and comultiplication respectively (cf. [G, App., §3]). The counit also induces an augmentation $\varepsilon : Q_\bullet \to X$; and this is in fact a resolution of $X$ by spheres (see [Ql, Prop. 2.6]).

3.9. Remark. Note that we need not use the $G$-sphere and disk $S^k$ and $D^k$ of §2.3 in this construction; we can replace it by any other homotopy equivalent cofibrant pair of simplicial groups, so in particular by $(FD^k, FS^{k-1})$ for any pair of simplicial sets $(D^k, S^{k-1}) \sim (D^k, S^{k-1})$.

3.10. The Quillen spectral sequence. A resolution by spheres $W_\bullet \to X$ is in fact a resolution (i.e., cofibrant replacement) for the constant simplicial object $cX_\bullet \in sG$ (i.e., $c(X)_n = X$, $d_i = s_j = id_X$) in an appropriate model category structure on $sG$—see [DKSI] and [B3]. However, we shall not need this fact; for our purposes it
suffices to recall that for any bisimplicial group $\mathbf{W} \in s\mathcal{G}$, there is a first quadrant spectral sequence with

$$E^{2}_{s,t} = \pi_{s}(\pi_{t} \mathbf{W}) \Rightarrow \pi_{s+t} \text{diag} \mathbf{W}.$$  

(3.11)

converging to the diagonal $\text{diag} \mathbf{W} \in \mathcal{G}$, defined $(\text{diag} \mathbf{W})_{k} = (\mathbf{W}_{k})_{k}^{\text{int}}$ (see [Q2]). Thus if $\mathbf{W} \rightarrow X$ is a resolution by spheres, the spectral sequence collapses, and the natural map $\mathbf{W}_{0} \rightarrow \text{diag} \mathbf{W}$ induces an isomorphism $\pi_{*}X \cong \pi_{*}(\text{diag} \mathbf{W})$. Combined with the fact that $\pi_{*} \mathbf{W}$ is a resolution (in $s\Pi\text{-Alg}$) of $\pi_{*}X$, this simple result has many applications – see for example [B1], [DKSS], and [St].

3.12. Definition. A $CW$ complex over a pointed category $\mathcal{C}$ is a simplicial object $R_{\bullet} \in s\mathcal{C}$, together with a sequence of objects $R_{n} \ (n = 0, 1, \ldots)$ such that $R_{n} \cong R_{n} \Pi L_{n}R_{\bullet}$ (Remark 3.2.5), and $d_{i}^{n}|_{R_{n}} = 0$ for $1 \leq i \leq n$. The objects $(R_{n})_{n=0}^{\infty}$ are called a $CW$ basis for $R_{\bullet}$, and $d_{n}^{n} := d_{0}|_{R_{n}}$ is called the $n$th attaching map for $R_{\bullet}$.

One may then describe $R_{\bullet}$ explicitly in terms of its CW basis by

$$R_{n} \cong \coprod_{0 \leq \lambda \leq n} \coprod_{I \in \mathcal{I}_{\lambda,n}} \tilde{R}_{n-\lambda}$$

(3.13)

where $\mathcal{I}_{\lambda,n}$ is the set of sequences $I$ of $\lambda$ non-negative integers $i_{1} < i_{2} < \ldots < i_{\lambda}$ ($< n$), with $s_{I} = s_{i_{1}} \circ \ldots \circ s_{i_{\lambda}}$ the corresponding $\lambda$-fold degeneracy (if $\lambda = 0$, $s_{I} = \text{id}$). See [B2, 5.2.1] and [Ma, p. 95(i)].

Such CW bases are convenient to work with in many situations; but they are most useful when each basis object is free, in an appropriate sense. In particular, if $\mathcal{C} = \Pi\text{-Alg}$, we have the following

3.14. Definition. A $CW$ resolution of a $\Pi$-algebra $G_{\bullet}$ is a CW complex $A_{\bullet} \in s\Pi\text{-Alg}$, with CW basis $(\tilde{A}_{n})_{n=0}^{\infty}$ and attaching maps $d_{0}^{n} : \tilde{A}_{n} \rightarrow Z_{n-1}A_{\bullet}$, such that each $\tilde{A}_{n}$ is a free $\Pi$-algebra, and each attaching map $d_{0}^{n}|_{\tilde{A}_{n}}$ is onto $Z_{n-1}A_{\bullet}$ (for $n \geq 0$, where we let $d_{0}^{n}$ denote the augmentation $\varepsilon : A_{\bullet} \rightarrow G_{\bullet}$ and $Z_{-1}A_{\bullet} := G_{\bullet}$). Compare [B2, §5].

Every $\Pi$-algebra has a CW resolution (§3.14), as was shown in [B1, 4.4]: for example, one could take the graded set of generators $T_{n}^{*}$ for $\tilde{A}_{n}$ to be equal to the graded set $\pi_{*}Z_{n-1}A_{\bullet}$.

3.15. Definition. $Q_{\bullet} \in s\mathcal{G}$ is called a $CW$ resolution by spheres of $X \in \mathcal{G}$ if $Q_{\bullet} \rightarrow X$ is a resolution by spheres (Def. 3.9), and $Q_{\bullet}$ is a CW complex with CW basis $(\tilde{Q}_{n})_{n=0}^{\infty}$, such that each $\tilde{Q}_{n} \in \mathcal{F}$ (i.e., $\tilde{Q}_{n}$ is homotopy equivalent to a wedge of spheres). The concept is defined analogously for $X \in \mathcal{S}$ or $X \in \mathcal{T}_{c}$.

3.16. Remark. Closely related to the problem of realizing abstract $\Pi$-algebras (§3.1) is that of realizing a free simplicial $\Pi$-algebra $A_{\bullet} \in s\Pi\text{-Alg}$: this is because, as noted in §3.5, every $G_{\bullet} \in \Pi\text{-Alg}$ has a free simplicial resolution $A_{\bullet} \rightarrow G_{\bullet}$; if it can be realized by a simplicial space $\mathbf{W}_{\bullet} \in s\mathcal{T}_{c}$ – or equivalently, via (2.3), by a bisimplicial space or group – then the spectral sequence (3.11) implies that $\pi_{*}\text{diag} \mathbf{W} \cong G_{\bullet}$. However, not every $\Pi$-algebra is realizable (see [B5, §8] or [B4, Prop. 4.3.6]).

It would nevertheless be very useful to know the converse: namely, that any free resolution of a realizable $\Pi$-algebra is itself realizable. This was mistakenly quoted as a
theorem in [B3, §6], where it was needed to make the obstruction theory for realizing
Π-algebras described there of any practical use – and appeared as a conjecture in [B3
§4], in the context of an obstruction theory for a space to be an H-space.

In order to show that this conjecture is in fact true, we need several preliminary
results:

3.17. Proposition. Every CW resolution $A_\bullet \to \pi_*X$ of a realizable Π-algebra
embeds in $\pi_*Q_\bullet$ for some resolution by spheres $Q_\bullet \to X$.

Proof. To simplify the notation, we work here with topological spaces, rather than
simplicial groups, changing back to $G$ if necessary via the adjoint pairs of $\mathcal{B}_2.2$

Given a free simplicial Π-algebra resolution $A_\bullet \to J_*$ with CW basis $(\tilde{A}_n)_{n=0}^\\infty$, where
$J_* = \pi_*X$ for some $X \in T_\bullet$, and $A_n$ is the free Π-algebra generated by the graded set $T_n^\vee$, let $\mu$ denote the cardinality of $\prod_{n=0}^\infty T_n^\vee$, and set
$X' := X \vee \bigvee_{\lambda<\mu} D_n$.

Define new “spheres” and “disks” of the form $\hat{S}^n := S^n \vee \bigvee_{\lambda<\mu} D_n$ and $\hat{D}^n := \hat{S}^n \vee D^n$. (This is to ensure that there will be at least $\mu$ different representatives
for each homotopy class in $\pi_*X'$ or $\pi_*\hat{S}^n$.)

By remark §3.3 above, if we use the construction of §3.7 in $T_\bullet$ (or in $G$, mutatis
mutandis) with these “spheres” and “disks”, and apply it to the space $X'$, rather than
to $X$, we obtain a resolution by spheres $Q_\bullet \to X'$.

We define $\phi : A_\bullet \hookrightarrow \pi_*Q_\bullet$ by induction on the simplicial dimension; it suffices to
produce for each $n \geq 0$ an embedding $\hat{\phi}_n : \hat{A}_n \to C_n \pi_*Q_\bullet$ commuting with $d_0$. If we
denote $\varepsilon^\mathcal{A} : A_0 \to \pi_*X \cong \pi_*X'$ by $d_0^\mathcal{A} : \mathcal{C}_0A_\bullet \to Z_{-1}A_\bullet := A_{-1}$ and set
$\phi_{-1} = \text{id}_{\pi_*X}$, then we may assume by induction we have a monomorphism $\phi_{n-1} : \hat{A}_{n-1} \hookrightarrow \pi_*Q_{n-1}$
taking generators to generators, and commuting with face and degeneracy maps).

For each Π-algebra generator $\iota_\alpha$ in $(\hat{A}_n)_{k,1}$, if $d_0(\iota_\alpha) \neq 0$ then $\hat{\phi}_{n-1}(d_0(\iota_\alpha)) \in Z_{n-1}\pi_kQ_\bullet$ is represented by some $g : \hat{S}^k \to Q_{n-1}$, and we can choose distinct (though
perhaps homotopic) maps $g$ for different generators $\iota_\alpha$ by our choice of $\hat{S}^k$. Then by $\mathcal{B}_3.8$ there is a wedge summand $\hat{S}_g^k$ in $Q_n = \bigvee Q_{n-1}$ (with no disks attached), and the corresponding free Π-algebra coproduct summand $\pi_*\hat{S}_g^k$ in $\pi_*Q_n$, generated
by $\iota_g$; has $d_0(\iota_g) = [g] \in \pi_kQ_{n-1}$ and $d_i(\iota_g) = \iota_{d_i-1g} = 0 \in \pi_{i+1}Q_{n-1}$ for $1 \leq i \leq n$ by $\mathcal{B}_3.7$, since $[g] = \phi_{n-1}(d_0(\iota_\alpha)) \in Z_{n-1}\pi_kQ_\bullet$ and thus $d_i[g] = [d_i g] = 0$, and spheres indexed by nullhomotopic maps have disks attached to them. We see that $\iota_g \in C_n \pi_kQ_\bullet$, so we may define $\hat{\phi}_n(\iota_\alpha) = \iota_g$.

If $d_0(\iota_\alpha) = 0$, then all we need are enough distinct Π-algebra generators in $Z_n\pi_*Q_\bullet$: we cannot simply take $\iota_g$ for nullhomotopic $g : \hat{S}^k \to Q_{n-1}$, because of the attached
disks; but we can proceed as follows:

Since $\tilde{D}^k = CS^k \vee D^k$ and $X' = X \vee \bigvee_{i=0}^\infty \bigvee_{\lambda<\mu} D_i$, we have $\mu$ distinct nonzero maps $F_\lambda : \tilde{D}^k \to X'$ with $F_\lambda|_{CS^k} = *$. Define $H_+ = F_{\lambda}$, $H_- = *$; then $S_{H_+}^k := \tilde{D}^k \cup_{S_{g_{\lambda}}^k} \tilde{D}^k_-$ is, up to homotopy, a sphere wedge summand in $Q_0$, and thus $\iota_{H_+} \in \pi_kQ_0$ is a Π-algebra generator mapping to 0 under the augmentation. Similarly,
define $S_{H_-}^k := \tilde{D}^k_+ \cup_{S_{g_{\lambda}}^k} \tilde{D}^k_-$ in $Q_1$ by $G^+ = *$, $G_- = * \perp 1^k$ where $\iota^k$ is a
homeomorphism onto the summand $D^k$ in $\tilde{D}^k_+$. Then $G_{\lambda} \sim *$ and $G_{\lambda} \neq *$ but $H \circ G = *$; thus $\iota_{H_\lambda}$ is a Π-algebra generator in $Z_1\pi_kQ_\bullet$. By thus alternating the $+$ and $-$ we produce $\mu$ distinct Π-algebra generators in $Z_n\pi_*Q_\bullet$ for each $n$. \[\blacksquare]
3.18. Remark. The referee has suggested an alternative proof of this Proposition, which may be easier to follow: rather than “fattening” the spheres and disks, we can modify the Stover construction of (3.8) by using μ copies of each sphere or disk for each φ ∈ Hom_G(S^k, G) or Φ ∈ Hom_G(D^k+1, G), respectively. The proof of [31, Prop. 2.6] still goes through, and so does the argument for embedding A_• in π_+Q_• above.

3.19. Proposition. Any free simplicial Π-algebra A_• has a (free) CW basis (A_n)_{n=0}^∞.

Proof. Start with A_0 = A_0. For n ≥ 1, assume A_n = \bigsqcup_{k=0}^{∞} \prod_{τ \in T^k_n} π_*S^k. By Definition 3.3, T^n_π \cong T^n_π \cup \bigcup_{0 ≤ λ ≤ n} \bigcup_{i \in I_{λ,n}} T^n_{π_+λ} (as in §3.13), so we can set A_n = \bigsqcup_{k=0}^{∞} \prod_{τ \in T^k_n} π_*S^k; but d_i|A_n need not vanish for i ≥ 1.

However, given τ ∈ T^k_π, we may define τ_i ∈ (A_n)_k ind inductively, starting with τ_0 = τ, by τ_{i+1} = τ_iσ_{n-i-1}d_{n-i}^-{i-1} (face and degeneracy maps taken in the external direction); we find that τ := τ_n is in C_nA_•. If we define φ : T^n_π \to A_n by φ(τ) = τ, by the universal property of free Π-algebras this extends to a map φ : A_n \to A_n, which together with the inclusion σ_n : L_nA_• \to A_n yields a map ψ : A_n \to A_n which is an isomorphism by the Hurewicz Theorem (cf. [37, Lemma 2.5]). Thus we may set A_• := φ(A_n), that is, the free Π-algebra generated by {τ}_τ∈T^n_π. Compare [31, §3].

3.20. Theorem. Every free simplicial Π-algebra resolution A_• → π_+X of a realizable Π-algebra π_+X is itself realizable by a CW resolution R_• → X in sG.

Proof. By Propositions 3.17 and 3.19 we may assume A_• has a (free) CW basis (A_n)_n=0^∞, and that there is a resolution by spheres Q_• → X (in sG) and an embedding of simplicial Π-algebras φ : A_• → Q_•. We may also assume that Q_• is fibrant (§2.8), with εQ : Ω Q_0 \to X a fibration. We shall actually realize φ by a map of bisimplicial groups f : R_• → Q_•.

Note that once R_• has been defined through simplicial dimension n, for any k ≥ 0 we have a commutative diagram

\[
\begin{array}{ccccccccc}
\pi_kC_nR_• & (d_0)^# & π_kZ_{n-1}R_• & (j_{n-1})^# & π_kC_{n-1}R_• & (d_0)^# & π_kZ_{n-2}R_• & (j_{n-2})^# & π_kC_{n-2}R_• \\
\downarrow t_• & \cong & \downarrow \rho_{n-1} & \downarrow \text{inc.} & \downarrow t_• & \cong & \downarrow \rho_{n-2} & \downarrow \text{inc.} & \downarrow t_• & \cong \\
C_nπ_kR_• & d_0^n & Z_{n-1}π_kR_• & \text{inc.} & C_nπ_kR_• & d_0^{n-1} & Z_{n-2}π_kR_• & \text{inc.} & C_nπ_kR_•
\end{array}
\]

(obtained by fitting together three of the long exact sequences of the fibrations (2.10)). The vertical maps are induced by the inclusions C_nR_• \hookrightarrow R_n, and so on – see Proposition 2.11.

The only difficulty in constructing R_• is that Proposition 2.11 does not hold for Z_n – i.e., the maps ρ_n in the above diagram in general need not be isomorphisms – so we may have an element in Z_nA_• represented by α ∈ C_nπ_+R_• = π_+C_nR_• with (d_0^n)^#(α) ≠ 0 (but of course (j_{n-1})^#(d_0^n)^#(α) = 0). In this case we could not have β ∈ π_+C_{n+1}R_• = C_{n+1}A_• with (j_{n+1})^#(d_0^{n+1})^#(β) = α, so π_+R_• would not be acyclic.

It is in order to avoid this difficulty that we need the embedding φ, since by definition this cannot happen for Q_•: we know that d_0^n : C_nπ_+Q_• \to Z_{n-1}π_+Q_• is surjective for each n > 0, so ρ_{n-1} : π_+Z_{n-1}Q_• \to Z_{n-1}π_+Q_• is, too, which implies that for each
for $0 < m < n$.

We now construct $R_\ast$ by induction on the simplicial dimension:

(i) First, choose a fibration $\varepsilon^R: R_0 \to X$ realizing $\varepsilon^A: A_0 \to \pi_* X$. By \((3.2.1)\), there is a map $f_0': R_0 \to Q_0$ realizing $\phi_0$, so $\varepsilon^Q \circ f_0' \sim \varepsilon^R$; since $\varepsilon^Q$ is a fibration, we can change $f_0'$ to $f_0: R_0 \to Q_0$ with $\varepsilon^Q \circ f_0 = \varepsilon^R$.

(ii) Let $Z_0R_\ast$ denote the fiber of $\varepsilon^R$. Since $\varepsilon^R = \varepsilon^A$ is a surjection, we have $\pi_* Z_0 R_\ast = \ker(\varepsilon^R_\#) = Z_0 A_\ast$, and $d_0^Q$ maps $C_1 A_\ast$ onto $Z_0 A_\ast$, so $d_0^A: \tilde{A}_1 \to A_0$ factors through $\pi_* Z_0 R_\ast$, and we can thus realize it by a map $\tilde{d}_0^R: \tilde{R}_1 \to Z_0 R_\ast$. Set $R'_1 := \tilde{R}_1 \amalg L_1 R_\ast$ (so $\pi_* R'_1 \cong A_1$), with $\delta_1': R'_1 \to M_1 R_\ast = R_0 \times R_0$ equal to $(d_0^R, 0) \amalg \Delta$, and change $\delta_1'$ to a fibration $\delta_1: R_1 \to M_1 R_\ast$. Again we can realize $\phi_1: A_1 \to \pi_* Q_1$ by $f_1: R_1 \to Q_1$ with $\delta_1 \circ f_1 = f_0 \circ \delta_1'$, since $\delta_1$ is a fibration; so we have defined $\tau_1 Z_0 R_\ast \to \pi_* Q_1$, realizing $\tau_1 \phi$.

(iii) Now assume we have $\tau_n f: \tau_n R_\ast \to \tau_n Q_\ast$ realizing $\tau_n \phi$, with Property \((3.2.1)\) holding for $Z_m R_\ast$ for $0 < m < n$.

For each $\Pi$-algebra generator $\alpha \in \tilde{A}_n + 1$ (in degree $k$, say), \((3.2.1)\) implies that $d_0^{n+1}(\alpha) \in \ker(\tilde{d}_0^R) = \ker((\tilde{d}_0^R)_\#) \subset (C_n A_\ast)_k = \pi_k C_n R_\ast$, so by the exactness of \((3.2.2)\) we can choose $\beta \in \pi_k Z_n R_\ast$ such that $(j_n)_\# \beta = d_0^{n+1}(\alpha)$. This allows us to define $\tilde{d}_0^R: \tilde{R}_{n+1} \to Z_n R_\ast$, so that $(j_n)_\# (\tilde{d}_0^R)_\#$ realizes $(\inc \circ \tilde{d}_0^A): A_{n+1} \to C_n A_\ast$, as well as $\tilde{f}_{n+1}: \tilde{R}_{n+1} \to C_{n+1} Q_\ast$ realizing $\phi_{n+1} | A_{n+1}$. Because $\tilde{A}_{n+1} = \pi_* \tilde{R}_{n+1}$ is a free $\Pi$-algebra, this implies the homotopy-commutativity of the outer rectangle in

\[
\begin{array}{ccc}
\tilde{R}_{n+1} & \xrightarrow{\tilde{f}_{n+1}} & C_{n+1} Q_\ast \\
\downarrow & & \downarrow \\
\tilde{d}_0^R & & \tilde{d}_0^Q \\
\downarrow & \downarrow & \downarrow \\
Z_n R_\ast & \xrightarrow{Z_n f} & Z_n Q_\ast \\
\downarrow & \downarrow & \downarrow \\
C_n R_\ast & \xrightarrow{C_n f} & C_n Q_\ast \\
\end{array}
\]
This completes the inductive construction of $f$. By (3.21) this implies $(Z_n f) \# \circ (d^R_0)\# = (d^Q_0)\# \circ (f_{n+1})\#$.

3.24. Theorem. If $K \xrightarrow{\varphi} \pi_* X \text{ and } L \xrightarrow{\varphi} \pi_* Y$ are two free simplicial $\Pi$-algebra resolutions, $g : X \to Y$ is a map in $\mathcal{G}$, and $\varphi : K \to L$ is a morphism of simplicial $\Pi$-algebras such that $\varepsilon^K \circ \varphi_0 = \pi_* g \circ \varepsilon^K$, then $\varphi$ is realizable by a map $f : A \to B$ in $s\mathcal{G}$.

Proof. Choose free CW bases for $K$ and $L$, and realize the resulting CW resolutions by $A$ and $B$, respectively, where (as in the proof of Theorem 3.20) we may assume $d_0 : C_n B \to Z_n B$ is a fibration for each $n \geq 0$. $f : A \to B$ will be defined by induction on $n$: $\varphi_0 : K_0 \to L_0$ may be realized by a map $f_0 : A_0 \to B_0$ (3.22), and since $\varepsilon^B$ is a fibration and $\varepsilon^B \circ f_0^\prime \sim g \circ \varepsilon^A$, we can choose a realization $f_0$ for $\varphi_0$ such that $\varepsilon^B \circ f_0 = g \circ \varepsilon^A$.

In general, $\varphi_n : \varphi_n|_{K_n} : K_n \to C_n L_n$ may be realized by a map $\bar{f}_n : \bar{A}_n \to C_n B_n$ (Proposition 2.11), and since $d_0 : C_n B_n \to Z_{n-1} B_n$ is a fibration, we may choose $\bar{f}_n$ so $d_0 \circ \bar{f}_n = Z_{n-1} f \circ d_0 : \bar{A}_n \to Z_{n-1} B_n$. By induction this yields a map $f_n = L_n f \circ \bar{f}_n : A_n \to L_n B_n$ such that $\delta^B_n \circ f_n = M_n f \circ \delta^A_n : A_n \to M_n B_n$, so $f$ is indeed a simplicial morphism (realizing $\varphi$).
4. The simplicial bar construction

As an application of Theorem 3.2, we describe an obstruction theory for determining whether a given space $X$ is, up to homotopy, a loop space (and thus a topological group -- see [YH, §3]). In the next two sections we no longer need to work with simplicial groups, so we revert to the more familiar category of topological spaces; we can still utilize the results of the previous section via the adjoint pairs of $(2,3)$.

4.1. Definition. A $\Delta$-cosimplicial object $E^\Delta$ over a category $C$ is a sequence of objects $E^0, E^1, \ldots$, together with coface maps $d^i : E^n \to E^{n+1}$ for $1 \leq i \leq n$ satisfying $d^id^j = d^jd^i$ for $i < j$ (cf. [RS]). Given an ordinary cosimplicial object $E^\bullet$ (cf. [BK, X, 2.1]), we let $E^\Delta$ denote the underlying $\Delta$-cosimplicial object (obtained by forgetting the codegeneracies).

4.2. The cosimplicial James construction. Given a space $X \in \mathcal{T}$, we define a $\Delta$-cosimplicial space $U^\Delta = U(X)^\Delta$ by setting $U^n = X^{n+1}$ (the Cartesian product), and $d^i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, *, x_i, \ldots, x_n)$. Note that $\operatorname{colim} U(X)^\Delta \cong JX$ (the James reduced product construction), and

4.3. Fact. If $(X,m)$ is a (strictly) associative $H$-space, we can extend $U^\Delta$ to a full cosimplicial space $U^\bullet$ by setting $s^i(x_0, \ldots, x_n) = (x_0, \ldots, m(x_j, x_{j-1}), \ldots, x_n)$.

4.4. Definition. Let $A^\bullet$ be a CW-resolution of the $\Pi$-algebra $\pi_*X = \pi_*U^0$, as in §3.14. We construct a $\Delta$-cosimplicial augmented simplicial $\Pi$-algebra $(E^\bullet)^\Delta \to \pi_*U^\Delta$, such that each $E^n$ is a CW-resolution of $\pi_*U^n = \pi_*(X^{n+1})$, with CW-basis $\{E^n\}_{r=0}^\infty$. We start by setting $E^0_r = \bar{C}^0_r = A_r$ for all $r \geq 0$, and then define $E^n_r$ by a double induction (on $r \geq 0$ and then on $n \geq 0$) as

$$E^n_r = \coprod_{0 \leq \Delta \leq n} \coprod_{\ell \in \mathcal{J}_{\lambda,n}} [\tilde{C}^{n-\lambda}]_{\ell},$$

where $\mathcal{J}_{\lambda,n}$ is as in (3.13) and $\bar{C}^m_0 = 0 = \bar{C}^0_r$ for all $m,r \geq 0$.

The coface maps $d^i : E^{n-1}_r \to E^n_r$ are determined by the cosimplicial identities and the requirement that $d^i[\tilde{C}^{n-\lambda}]_{(i_1,\ldots,i_n,i)}$ be an isomorphism onto $[\bar{C}^{n-\lambda}]_{(i_1,\ldots,i_n,i)}$ if $i > i_n$.

The only summand in (1.5) which is not defined is thus $[\bar{C}^n_r]_0$, which we denote simply by $\bar{C}^n_r$. We require that it be an $n$-th cross-term in the sense that $d^i|\bar{C}^n_r$ does not factor through the image of any coface map $d^i : E^{n-1}_r \to E^{n-1}_r$. Other than that, $\bar{C}^n_r$ may be any free $\Pi$-algebra which ensures that (1.3) defines a CW-resolution for a CW-resolution $E^n_r \to \pi_*U^n$. We shall call the double sequence $((\bar{C}^n_r)_{r=1}^\infty)_{n=1}^\infty$ a cross-term basis for $(E^\bullet)^\Delta$.

Note that $A^\bullet$ is a retract of $E^2_\bullet$ in two different ways (under the two coface maps $d^0$, $d^1$), corresponding to the fact that $X$ is a retract of $X \times X$ in two different ways; the presence of the cross-terms $\bar{C}^2_r$ indicates that $A^\bullet \times A^\bullet$ is a resolution of $\pi_*X^2$, but not a free one, while $A^\bullet \Pi A^\bullet$ is a free simplicial $\Pi$-algebra, but not a resolution.

Similarly, $X \times X$ embeds in $X^3$ in three different ways, and so on.
4.6. Example. For any $A_\bullet \to \pi_*X$ we may set $\tilde{C}^2 = \bigsqcup_{i,j} S_p^{i,j} \to A_0^{(i)} \bigsqcup S_q^{j} \to A_0^{(j)}$, with $d_0[S^{p+q-1}] = [t_x, t_y]$ (in the notation of §3.3). The higher cross-terms $\tilde{C}^n = 0$ for $n \geq 3$, since any $k$-th order cross-term element $z$ in $\bigsqcup_{j=0}^n A_0^{(j)}$ is a sum of elements of the form $z = \zeta^# \cdot [(r^1_{(x)}, r^2_{(x)}, \ldots, r_k_{(x)}), \ldots]$, and then

$$z = d_0(\zeta^# \cdot [(r^1_1, r^2_{(x_2)}, \ldots, s_0 r_{(x_3)}, \ldots, s_0 r_{(x_3)}]).$$

4.7. Definition. Let $h(W_\bullet)_\Delta \to U^*_\Delta$ be the $\Delta$-cosimplicial augmented simplicial space up-to-homotopy which corresponds to $(E_\bullet)_\Delta \to \pi_*U^*_\Delta$ via §3.2. Thus the various (co)simplicial morphisms exist, and satisfy the (co)simplicial identities, only in the homotopy category (we may choose representatives in $\mathcal{T}_\bullet$, but then the identities are satisfied only up to homotopy). Each $\tilde{W}^n_r$ is homotopy equivalent to a wedge of spheres, and has a wedge summand $\tilde{W}^n_r \hookrightarrow \tilde{W}^n_r$ corresponding to the $CW$-basis free $\Pi$-algebra summand $\tilde{E}^n_r \hookrightarrow \tilde{E}^n_r$. We let $\tilde{C}^n_r$ denote the wedge summand of $\tilde{W}^n_r$ corresponding to $\tilde{C}^n_r \hookrightarrow \tilde{E}^n_r$.

4.8. Definition. An simplicial space $V_\bullet \in s\mathcal{T}_\bullet$ is called a rectification of a simplicial space up-to-homotopy $hW_\bullet$ if $V_n \cong W_n$ for each $n \geq 0$, and the face and degeneracy maps of $V_\bullet$ are homotopic to the corresponding maps of $hW_\bullet$. See [DKS, §2.2], e.g., for a more precise definition; for our purposes all we require is that $\pi_*V_\bullet$ be isomorphic (as a simplicial $\Pi$-algebra) to $\pi_*(hW_\bullet)$. Similarly for rectification of $(\Delta)$-cosimplicial objects, and so on.

By considering the proof of Theorem 3.20, we see that we can make the following

4.9. Assumption. $(E_\bullet)_\Delta$ maps monomorphically into $\pi_*V_\bullet(U^*_\Delta)$, and $h(W_\bullet)_\Delta \to U^*_\Delta$ can be rectified so as to yield a strict $\Delta$-cosimplicial augmented simplicial space $(W_\bullet)_\Delta \to U^*_\Delta$ realizing $(E_\bullet)_\Delta \to \pi_*U^*_\Delta$.

4.10. Definition. Now assume that $\pi_*X$ is an abelian $\Pi$-algebra (Def. 3.4) – this is the necessary $\Pi$-algebra condition in order for $X$ to be an $H$-space – and let $\mu : \pi_*X \times \pi_*X \to \pi_*X$ be the morphism of $\Pi$-algebras defined levelwise by the group operation (see [B3, §2]). This $\mu$ is of course associative, in the sense that $\mu \circ (\mu, id) = \mu \circ (id, \mu) : \pi_*(X^3) \to \pi_*X$, so it allows one to extend the $\Delta$-cosimplicial $\Pi$-algebra $F^*_{\Delta} := \pi_*(U^*_\Delta)$ to a full cosimplicial $\Pi$-algebra $F^*$, defined as in §3.3.

Since $E^n_r \hookrightarrow F^n = \pi_*U^n$ is a free resolution of $\Pi$-algebras, the codegeneracy maps $s^j : F^n \to F^{n-1}$ induce maps of simplicial $\Pi$-algebras $s^j : E^n_r \to E^{n-1}_r$, unique up to simplicial homotopy, by the universal property of resolutions (cf. [D1], p. 1.14 & II, §2, Prop. 5). Note, however, that the individual maps $s^j_r : E^n_r \to E^{n-1}_r$ are not unique, in general; in fact, different choices may correspond to different $H$-multiplications on $X$.

These maps $s^j$ make $(E_\bullet)_\Delta \to F^*_{\Delta}$ into a full cosimplicial augmented simplicial $\Pi$-algebra $E^*_\bullet \to F^*$, and thus $hW^*_\bullet \to U^*_\Delta$ into a cosimplicial augmented simplicial space up-to-homotopy (for which we may assume by [4.3] that all simplicial identities, and all the cosimplicial identities involving only the coface maps, hold precisely).

4.11. Proposition. The cosimplicial simplicial space up-to-homotopy $hW^*_\bullet$ of §4.10 may be rectified if and only if $X$ is homotopy equivalent to a loop space.
5. Polyhedra and higher homotopy operations

As in [36, §4], there is a sequence of higher homotopy operations which serve as obstructions to such a rectification, and these may be described combinatorially in terms of certain polyhedra, as follows:

5.1. Definition. The \( N \)-permutohedron \( P^N \) is defined to be the convex hull in \( \mathbb{R}^N \) of the points \( p_\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(N)) \), where \( \sigma \) ranges over all permutations \( \sigma \in \Sigma_N \) (cf. [2, §9]). It is \( (N - 1) \)-dimensional.

For any two integers \( 0 \leq n < N \), the corresponding \((N, n)\)-face-codegeneracy polyhedron \( P^N_n \) is a quotient of the \( N \)-permutohedron \( P^N \) obtained by identifying two vertices \( p_\sigma \) and \( p_{\sigma'} \) to a single vertex \( \bar{p}_\sigma = p_{\sigma'} \) of \( P^N_n \) whenever \( \sigma = (i, i+1)\sigma' \) and \( (i, i + 1) \) is an adjacent transposition.

Since each facet \( A \) of \( P^N \) is uniquely determined by its vertices (see below), the facets in the quotient \( P^N_n \) are obtained by collapsing those of \( P^N \) accordingly.

Note that \( P^N_{N-1} \) is the \( N \)-permutohedron \( P^N \), and in fact the quotient map \( q: P^N \to P^N_n \) is homotopic to a homeomorphism (though not a combinatorial isomorphism, of course) for \( n \geq 1 \). On the other hand, \( P^N_0 \) is a single point.

5.2. Fact. From the description of the facets of the permutohedron given in [36], we see that any edge connecting a vertex \( p_\sigma \) to any vertex of the form \( p_{(i,i+1)\sigma} \) (unless \( \sigma(i), \sigma(i+1) > n \), in which case the edge is degenerate).

More generally, let \( \bar{p}_\sigma \) be any vertex of \( P^N_n \). The facets of \( P^N_n \) containing \( \bar{p}_\sigma \) are determined as follows:
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Let \( P = \langle 1, 2, \ldots, \ell_1 \mid \ell_1 + 1, \ldots, \ell_2 \mid \ldots \mid \ell_{i-1} + 1, \ldots, \ell_i \mid \ldots \mid \ell_{r-1} + 1, \ldots, N \rangle \) be a partition of \( 1, \ldots, N \) into \( r \) consecutive blocks, subject to the condition that for each \( 1 \leq j < r \) at least one of \( \sigma(\ell_j), \sigma(\ell_{j+1}) \) is \( \leq n \). Denote by \( n_i \) the number of \( j \)'s in the \( i \)-th bloc (i.e., \( \ell_{i-1} + 1 \leq j \leq \ell_i \)) such that \( \sigma(j) \leq n \). Then \( P^N \) will have a subpolyhedron \( Q(P) \) (containing \( p_n \)) which is isomorphic to the product

\[
P_{n_1}^\ell \times P_{n_2}^{\ell_1-\ell_1} \times \cdots \times P_{n_i}^{\ell_i-\ell_{i-1}} \times \cdots \times P_{n_r}^{N-\ell_{r-1}}.
\]

This follows from the description of the facets of the \( N \)-permutahedron in \([B3, \S 4.3]\).

We denote by \((P^N_n(k))\) the union of all facets of \( P^N_n \) of dimension \( \leq k \). In particular, for \( n \geq 1 \) we have \( \partial P^N_n := (P^N_n)^{(N-2)} = S^{N-2} \), since the homeomorphism \( \tilde{q}: P^N \to P^N_n \) preserves \( \partial P^N_n \).

5.3. Factorizations. Given a cosimplicial simplicial object \( E^* \) as in \([\S 10]\), any composite \( E \)-codegeneracy map \( \psi: E_{m+\ell}^{n+k} \to E_\ell^k \) has a (unique) canonical factorization of the form \( \psi = \phi \circ \theta \), where \( \theta: E_{m+\ell}^{n+k} \to E_{m+\ell}^k \) may be written \( \theta = s^{j_1} \circ s^{j_2} \circ \ldots \circ s^{j_n} \) for \( 0 \leq j_1 < j_2 < \ldots < j_n < n+k \) and \( \phi: E_{m+\ell}^k \to E_\ell^k \) may be written \( \phi = d_{i_1} \circ d_{i_2} \circ \ldots \circ d_{i_n} \) for \( 0 \leq i_1 < i_2 < \ldots < i_n \leq m+\ell \).

Let \( D(\psi) \) denote the set of all possible (not necessarily canonical) factorizations of \( \psi \) as a composite of face and codegeneracy maps: \( \psi = \lambda_{n+m} \circ \cdots \circ \lambda_1 \). We define recursively a bijective correspondence between \( D(\psi) \) and the vertices of an \((n+m)\)-permutahedron \( P^{n+m} \) as follows (compare \([B3, \text{Lemma 4.7}]\)):

The canonical factorization \( \psi = d_{i_1} \circ d_{i_2} \circ \ldots \circ d_{i_n} \circ s^{j_1} \circ s^{j_2} \circ \ldots \circ s^{j_n} \) corresponds to the vertex \( p_{\sigma} \). Next, assume that the factorization \( \psi = \lambda_{n+m} \circ \cdots \circ \lambda_1 \) corresponds to \( p_{\sigma} \). Then the factorization corresponding to \( p_{\sigma} \), for \( \sigma = (i, i+1)\sigma' \), is obtained from \( \psi = \lambda_1 \circ \cdots \circ \lambda_{n+m} \) by switching \( \lambda_i \) and \( \lambda_{i+1} \), using the identity \( s^j \circ s^l = s^{j-l} \circ s^j \) for \( i > j \) if \( \lambda_i \) and \( \lambda_{i+1} \) are both codegeneracies, and the identity \( d_i \circ d_j = d_{j-1} \circ d_i \) for \( i < j \) if they are both face maps.

Passing to the quotient face-codegeneracy polyhedron, we see that the vertices of \( P^{n+m} \) are now identified with factorizations of \( \psi \) of the form

\[
E_{m+\ell}^{n+k} \xrightarrow{s^{j_1}} E_{m+\ell}^{n+k-1} \cdots E_{m+\ell}^{n+1} \xrightarrow{s^{j_1}} E_{m+\ell}^{n+1} \xrightarrow{\theta_1} E_{m_1}^{n_1} \cdots E_{m_1}^{n_1} \xrightarrow{s^{0_1}} \cdots E_{m_1}^{n_1} \xrightarrow{s^{0_1}} E_{m_1}^{n_1} \xrightarrow{\theta_0} E_n^m
\]

where \( \theta_i \) is a composite of face maps (i.e., we do not distinguish the different ways of decomposing \( \theta_i \) as \( d_{k_1} \circ \cdots \circ d_{k_r} \)). The collection of such factorizations of \( \psi \) will be denoted by \( D(\psi) / \sim \), where \( \sim \) is the obvious equivalence relation on \( D(\psi) \). We shall denote the face-codegeneracy polyhedron \( P^{n+m} \) with its vertices so labelled by \( P^{n+m}(\psi) \). An example for \( \psi = d_0 d_1 s^0 s^1 \) appears in Figure 4.

5.5. Notation. For \( \psi: E_{m+\ell}^{n+k} \to E_\ell^k \) as above, we denote by \( \mathcal{C}(\psi) \) the collection of all composite face-codegeneracy maps \( \rho: E_{m+\ell}^{(n+k)(\rho)} \to E_{\ell(\rho)}^k \) such that \( \rho \) is of the form \( \rho = \xi_{t} \circ \cdots \circ \xi_{s} \) (\( 1 \leq s \leq t \leq \nu \)) for some decomposition \( \psi = \xi_{\nu} \circ \cdots \circ \xi_{1} = \theta_0 \circ s^{(0)} \circ \cdots \circ s^{(0)} \circ \theta_1 \circ \cdots \circ \theta_i \circ s^{(i)} \circ \cdots \circ s^{(i)} \) of \( (5.4) \). That is, we allow only those subsequences \( \lambda_{h}, \ldots, \lambda_{a} \) of a factorization \( \psi = \lambda_{n+m} \circ \cdots \circ \lambda_1 \) in \( D(\psi) \) which are compatible with the equivalence relation \( \sim \) in the sense that \( \lambda_{b+1} \) and \( \lambda_{b} \) are not both face maps, and similarly for \( \lambda_{a-1} \) and \( \lambda_{a} \). Such a \( \rho \) will be called allowable.
5.6. Higher homotopy operations. Given a cosimplicial simplicial space up-to-homotopy \( hW^* \) as in \( \S 4.2 \), we now define a certain sequence of higher homotopy operations. First recall that the half-smash of two spaces \( X, Y \in T_r \) is \( X \times Y := (X \times Y)/(X \times \{\ast\}) \); if \( X \) is a suspension, there is a (non-canonical) homotopy equivalence \( X \times Y \simeq X \wedge Y \vee X \).

5.7. Definition. Given a composite face-codegeneracy map \( \psi : W^{n+k}_{m+\ell} \to W^k_{\ell} \) as above, a compatible collection for \( C(\psi) \) and \( hW^* \) is a set \( \{g^\rho\}_{\rho \in C(\psi)} \) of maps \( g^\rho : P^{n(\rho)+m(\rho)}_{m(\rho)}(\rho) \times W^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)} \to W^{k(\rho)}_{\ell(\rho)} \) for each \( \rho \in C(\psi) \), satisfying the following condition:

Assume that for such a \( \rho \in C(\psi) \) we have some decomposition

\[ \rho = \xi_\nu \circ \cdots \circ \xi_1 = \theta_0 \circ s^{n_0} \circ \cdots \circ \theta_{\ell_1} \circ \cdots \circ \theta_1 \circ s^{n_1} \circ \cdots \circ s^{n_{\ell_1}} \circ \cdots \circ s^{n_{\ell_r}} \]

in \( D(\rho)/\sim \), as in \( \S 5.4 \), and let

\[ \mathbb{P} = \langle 1, \ldots, \ell_1 \mid \ldots \mid \ell_{i-1} + 1, \ldots, \ell_i \mid \ldots \mid \ell_{r-1} + 1, \ldots, \nu \rangle \]

be a partition of \( \langle 1, \ldots, \nu \rangle \) as in \( \S 5.2 \), yielding a sequence of composite face-codegeneracy maps \( \rho_i \in C(\rho) \subseteq C(\psi) \) for \( i = 1, \ldots, r \).

Let \( Q(\mathbb{P}) \cong P^{\ell_1}_{n_1}(\rho_1) \times \cdots \times P^{\ell_{r-1}}_{n_{r-1}}(\rho_{r-1}) \times P^{\ell_r}_{n_r}(\rho_r) \) be the corresponding sub-polyhedron of \( P^{n(\rho)+m(\rho)}_{m(\rho)}(\rho) \). Then we require that \( g^\rho|_{Q(\mathbb{P}) \times W^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)}} \) be the composite of the corresponding maps \( g^{\rho_i} \) in the sense that

\[ (5.8) \quad g^\rho(x_1, \ldots, x_r, w) = g^{\rho_1}(x_1, g^{\rho_2}(x_2, \ldots, g^{\rho_r}(x_r, w) \ldots)) \]

for \( x_i \in P^{\ell_i}_{n_i}(\rho_i) \) and \( w \in W^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)} \).

We further require that if \( \rho = \lambda_1 \) is of length 1, then \( g^\rho \) must be in the prescribed homotopy class of the face or codegeneracy map \( \lambda_1 \). Thus in particular, for each
vertex $\bar{p}_\sigma$ of $P_{n+m}^n(\psi)$, indexed by a factorization $\psi = \xi_\nu \circ \cdots \circ \xi_1$ in $D(\psi)/\sim$, the map $g^\rho|_{\bar{p}_\sigma} \times W_{m+k}^{n+\ell}$ represents the class $[\xi_\nu \circ \cdots \circ \xi_1]$.  

5.9. **Fact.** Any compatible collection of maps $\{g^\rho\}_{\rho \in C(\psi)}$ for $C(\psi)$ induces a map $f = f^\psi : \partial P_{n+m}^{n+m} \times W_{m+k}^{n+\ell} \to W_{\ell}^k$ (since all the facets of $\partial P_{n+m}^{n+m}$ are products of face-codegeneracy polyhedra of the form $P_{n+\rho}^{\rho+m}(\rho)$ for $\rho \in C(\psi)$, and condition (5.3) guarantees that the maps $g^\rho$ agree on intersections).

5.10. **Definition.** Given $hW_\ast^\psi$ as in §4.10, for each $k \geq 2$ and each composite face-codegeneracy map $\psi : W_{m+k}^{n+\ell} \to W_\ell^k$, the $k$-th order homotopy operation associated to $hW_\ast^\psi$ and $\psi$ is a subset $\langle \psi \rangle$ of the track group $[\Sigma^{n+k}W_{m+k}, W_\ell^k]$, defined as follows:

Let $S \subseteq [\partial P_{n+m}^{n+m} \times W_{m+k}^{n+\ell}, W_\ell^k]$ be the set of homotopy classes of maps $f = f^\psi : \partial P_{n+m}^{n+m} \times W_{m+k}^{n+\ell} \to W_\ell^k$ which are induced as above by some compatible collection $\{g^\rho\}_{\rho \in C(\psi)}$ for $C(\psi)$.

Now choose a splitting

$$\partial P_{n+m}^{n+m}(\psi) \times W_{m+k}^{n+\ell} \cong S^{n+m-2} \times W_{m+k}^{n+\ell} \cong (S^{n+m-2} \wedge W_\ell^k) \vee W_\ell^k$$

and let $\langle \psi \rangle \subseteq [\Sigma^{n+m-2}W_{m+k}, W_\ell^k]$ be the image of the subset $S$ under the resulting projection.

It is clearly a necessary condition in order for the subset $\langle \psi \rangle$ to be non-empty that all the lower order operations $\langle \rho \rangle$ vanish (i.e., contain the null class) for all $\rho \in C(\psi) \setminus \{\psi\}$ – because otherwise the various maps $g^\rho : P_{m(\rho)}^{n(\rho)+m(\rho)}(\rho) \times W_{m+k(\rho)}^{n+k(\rho)}(\rho) \to W_\ell^{k(\rho)}(\rho)$ cannot even extend over the interior of $P_{m(\rho)}^{n(\rho)+m(\rho)}(\rho)$. A sufficient condition is that the operations $\langle \rho \rangle$ vanish coherently, in the sense that the choices of compatible collections for the various $\rho$ be consistent on common subpolyhedra (see [35, §5.7] for the precise definition, and [33, §5.9] for the obstructions to coherence).

On the other hand, if $hW_\ast^\psi$ is the cosimplicial simplicial space up-to-homotopy of $\Sigma_{\Delta^n} \{E_n\}$ (corresponding to the cosimplicial simplicial II-algebra $(E_n)_\Delta$ with the CW-basis $(E_r)_{r,n=0}$), then the vanishing of the homotopy operation $\langle \psi \rangle_{\bar{c}_n(\psi)}$ – with $\psi$ restricted to the $(n,r)$-cross-term – implies the vanishing of $\langle \psi \rangle$, for any $\psi : W_{m+k}^{n+\ell} \to W_\ell^k$ (assuming lower order vanishing). This is because outside of the wedge summand $C_r$, the map $\psi$ is determined by the maps $\rho \in C(\psi)$ and the coface and degeneracy maps of $hW_\ast^\psi$, which we may assume to ∞-homotopy commute by induction and [13] respectively.

We may thus sum up the results of this section, combined with Proposition 1.1 in:

5.12. **Theorem.** A space $X \in T_\ast$, for which $\pi_\ast X$ is an abelian II-algebra, is homotopy equivalent to a loop space if and only if all the higher homotopy operations $\langle \psi \rangle_{\bar{c}_n(\psi)}$ defined above vanish coherently.

5.13. **Remark.** As observed in §1.2, for any $X \in T_\ast$ the space $JX$ is the colimit of the $\Delta$-cosimplicial space $U(X)_\Delta$, and in fact the $n$-th stage of the James construction, $J_n X$, is the (homotopy) colimit of the $(n-1)$-coskeleton of $U_\Delta$. Thus if we think of the sequence of higher homotopy operations “in the simplicial direction” as obstructions to the validity of the identity [35, Thm. 5.7(*)] (up to ∞-homotopy
choose a \( X \) then the \( n \)-th cosimplicial dimension corresponds to verifying this identity for \( f \circ i_A : A \to FB \) of James filtration \( n + 1 \) (cf. [3], §2).

In particular, if we fix \( k = \ell = 0, \ n = 1 \) and proceed by induction on \( m \), we are computing the obstructions for the existence of an \( H \)-multiplication on \( X \), as in [3].

(Thus if \( X \) is endowed with an \( H \)-space structure to begin with, they must all vanish.) Observe that the face-codegeneracy polyhedron \( P^1 \) is an \( (n - 1) \)-cube, as in Figure 2, rather than the \( (n - 1) \)-simplex we had in [3, §1] so the homotopy operations we obtain here are more complicated. This is because they take value in the homotopy groups of spheres, rather than those of the space \( X \).

As a corollary to Theorem 5.12 we may deduce the following result of Hilton (cf. [4], Theorem C):

5.14. Corollary. If \( \langle X, m \rangle \) is a \( (p - 1) \)-connected \( H \)-space with \( \pi_i X = 0 \) for \( i \geq 3p \), then \( X \) is a loop space, up to homotopy.

Proof. Choose a CW-resolution of \( \pi_* X \) which is \( (p - 1) \)-connected in each simplicial dimension, and let \( E^* \) be as in §4.4. By definition of the cross-term \( \Pi \)-algebras \( C_r^n \) in §4.4 they must involve Whitehead products of elements from all lower order cross-terms; but since \( X \) is an \( H \)-space by assumption, all obstructions of the form \( \langle \psi|_{C_2} \rangle \) vanish (see §5.13). Thus, the lowest dimensional obstruction possible is a third-order operation \( \langle \psi|_{C_2} \rangle \) \((r \geq 2)\), which involves a triple Whitehead product and thus takes value in \( \pi_i W^k \) for \( i \geq 3p \). If we apply the \( (3p - 1) \)-Postnikov approximation functor to \( hW^* \) in each dimension, to obtain \( hZ^* \), all obstructions to rectification vanish, and from the spectral sequence of §3.10 we see that the obvious map \( X = \|W^k\| \to \|Z^k\| \) induces an isomorphism in \( \pi_i \) for \( i < 3p \). Since \( \|Z^k\| \) is a loop space by Theorem 5.12 so is its \( (3p - 1) \)-Postnikov approximation, namely \( X \).

5.15. Example. The 7-sphere is an \( H \)-space (under the Cayley multiplication, for example), but none of the 120 possible \( H \)-multiplications on \( S^7 \) are homotopy-associative; the first obstruction to homotopy-associativity is a certain “separation element” in \( \pi_21 S^7 \) (cf. [4], Theorem 1.4 and Corollary 2.5)].
Since $\pi_*S^7$ is a free $\Pi$-algebra, it has a very simple CW-resolution $A_* \to \pi_*S^7$, with $A_0 \cong \pi_*S^7$ (generated by $i^7$), and $A_r = 0$ for $r \geq 1$. A cross-term basis (§4.4) for the cosimplicial simplicial $\Pi$-algebra $E_*^\bullet$ of §4.11 is then given in dimensions $< 24$ by:

- $C_1^1 \cong \pi_*S^{13}$, with $d_0i^3 = [d^0i^1, d^1i^7]$;
- $C_2^2 \cong \pi_*S^{19}$, with $d_0i^9 = [d^0i^{13}, s_0d^2d^1i^7] - [d^1i^{13}, s_0d^2d^1i^7] + [d^2i^{13}, s_0d^1d^0i^7]$;
- $C_3^n$ is at least 24-connected for all other $n, r$.

We set $s_r^i|_{C_2^n} = 0$ for all $n \leq 2$; this determines $E_*^\bullet$ in degrees $\leq 21$ and cosimplicial dimensions $\leq 2$.

By Remark 5.13, the two secondary operations $\langle d_0s^0|_C^1 \rangle$ and $\langle d_1s^0|_C^1 \rangle$ must vanish; on the other hand, by Corollary 5.14 all obstructions to $S^7$ being a loop space are in degrees $\geq 21$, so the only relevant cross-term is $C_2^2$, with three possible third-order operations $\langle \psi|_{C_2^2} \rangle$, for $\psi = d_0d_1s^0s^1$, $d_0d_2s^0s^1$, or $d_1d_2s^0s^1$. The corresponding face-codegeneracy polyhedra $P^4_2(\psi)$ is as in Figure 4.

It is straightforward to verify that the operations $\langle \psi|_{C_2^2} \rangle$ are trivial for $\psi = d_0d_2s^0s^1$ or $d_1d_2s^0s^1$ (in fact, many of the maps $g^\rho$, for $\rho \in C(\psi)$, may be chosen to be null). On may also show that there is a compatible collection $\{g^\rho\}_{\rho \in C(\psi)}$ for $\varphi = d_0d_1s^0s^1$, in the sense of §4.7, so that the corresponding subset $\langle \varphi|_{C_2^2} \rangle \subseteq \pi_21S^7$ is non-empty; in fact, it contains the only possible obstruction to the 21-Postnikov approximation for $S^7$ to be a loop space.

The existence of the tertiary operation $\langle \varphi|_{C_2^2} \rangle$ corresponds to the fact that the element $[[i^7, i^7], i^7] - [[i^7, i^7], i^7] + [[i^7, i^7], i^7] \in \pi_21S^7$ is trivial “for three different reasons”: because of the Jacobi identity, because all Whitehead products vanish in $\pi_*S^7$, and because of the linearity of the Whitehead product — i.e., $[0, \alpha] = 0$.

On the other hand, we know that there is a 3-primary obstruction to the homotopy-associativity of any $H$-multiplication on $S^7$, namely the element $\sigma_1^\#r \tau_1 \in \pi_21S^7$ (see [1], Theorem 2.6). We deduce that $0 \not\in \langle \varphi|_{C_2^2} \rangle$, and in fact (modulo 3) this tertiary operation consists exactly of the elements $\pm \bar{\sigma}_1^\#r \tau_1$.

For a detailed calculation of such higher order operations using simplicial resolutions of $\Pi$-algebras, see [3], §4.13.

5.16. Remark. Our approach to the question of whether $X$ is a loop space is clearly based on, and closely related to, the classical approaches of Sugawara and Stasheff (cf. §32, §32, §31). One might wonder why Stasheff’s associahedra $K_i$ (cf. §31, §2.6) do not show up among the face-codegeneracy polyhedra we describe above. Apparently this is because we do not work directly with the space $X$, but rather with its simplicial resolution, which may be thought of as a “decomposition” of $X$ into wedges of spheres.

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