Blow-up profile of the focusing Gross-Pitaevskii minimizer under self-gravitating effect

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Abstract

We consider a Bose-Einstein condensate in a 2D dilute Bose gas, with an external potential and an interaction potential containing both of the short-range attractive self-interaction and the long-range self-gravitating effect. We prove the existence of minimizers and analyze their behavior when the strength of the attractive interaction converges to a critical value. The universal blow-up profile is the unique optimizer of a Gagliardo-Nirenberg interpolation inequality.

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1 Introduction

Since the first observation in 1995 in the Nobel Prize winning works of Cornell, Wieman, and Ketterle [18], the Bose-Einstein condensation has been studied intensively in the last decades. When the interaction is attractive, it is a remarkable that the condensate may collapse, as noted in experiments [5, 20, 13]. In the present paper, we will study this collapse phenomenon in a rigorous model.

We consider a Bose-Einstein condensate in a 2D dilute Bose gas, with an external potential $V : \mathbb{R}^2 \to \mathbb{R}$ and an interaction potential $\omega(x - y)$ containing both of the short-range attractive self-interaction and the long-range self-gravitating effect. For simplicity, we take

$$\omega(x) = -a\delta_0(x) - \frac{g}{|x|}.$$  \hspace{1cm} (1)

Here $\delta_0$ is the Dirac-delta function at 0, $a > 0$ is the strength of the attractive interaction and $g > 0$ is the gravitational constant, which will be set = 1 for simplicity.
(our results are valid for all \( g > 0 \)). The energy of the condensate is described by the Gross-Pitaevskii functional

\[
E_a(u) = \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx + \int_{\mathbb{R}^2} V(x)|u(x)|^2 \, dx - \frac{a}{2} \int_{\mathbb{R}^2} |u(x)|^4 \, dx - \int \int \frac{|u(x)|^2 |u(y)|^2}{|x-y|} \, dx \, dy.
\]

We are interested in the existence and properties of minimizers of the minimization problem

\[
E(a) = \inf \left\{ E_a(u) \mid u \in H^1(\mathbb{R}^2), \|u\|_{L^2} = 1 \right\}. \tag{2}
\]

Note that by the diamagnetic inequality \(|\nabla u| \geq |\nabla| u||
we can always restrict the consideration to the case \( u \geq 0 \).

For the systems of small scales, gravity is often omitted as it is normally much weaker than other forces. However, in the context of ultra-cold gas, the self-gravitating effect has gained increasing interest in physics. In particular, it is crucially relevant to the study of an analog of a black hole in a Bose-Einstein condensate, see e.g. [4, 14, 10, 6]. In the present paper, we will explain some interesting effects of gravity in the instability of the condensate.

By a simple scaling argument, we can see that \( E(a) = -\infty \), if and only if \( a \geq a^* \), where \( a^* > 0 \) the optimal constant in the Gagliardo-Nirenberg inequality:

\[
\int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \geq \frac{a^*}{2} \int_{\mathbb{R}^2} |u(x)|^4 \, dx, \quad \forall u \in H^1(\mathbb{R}^2), \|u\|_{L^2} = 1. \tag{3}
\]

Indeed, it is well-known that (see e.g. [9, 21, 18])

\[
a^* = \int_{\mathbb{R}^2} |Q|^2 \tag{4}
\]

where \( Q \) is the unique positive, radial solution to the nonlinear Schrödinger equation

\[
-\Delta Q + Q - Q^3 = 0, \quad Q \in H^1(\mathbb{R}^2). \tag{5}
\]

Moreover, the normalized function \( Q_0 = Q/\|Q\|_{L^2} \) is the unique (up to translations and dilations) optimizer for the interpolation inequality (3). Indeed,

\[
1 = \int_{\mathbb{R}^2} |Q_0|^2 = \int_{\mathbb{R}^2} |\nabla Q_0|^2 = \frac{a^*}{2} \int_{\mathbb{R}^2} |Q_0|^4. \tag{6}
\]

In [11], Guo and Seiringer studied the collapse phenomenon of the Bose-Einstein without the self-gravitating effect (i.e. \( g = 0 \) in (1)). They proved that with trapping potentials like \( V(x) = |x|^p \), \( p > 0 \), the Gross-Pitaevskii minimizer always exist when \( a < a^* \) and they blow-up (possibly up to translations and dilations) to \( Q_0 \) as \( a \uparrow a^* \). More precisely, if \( u_a \) is a minimizer for \( E(a) \), then

\[
(a^* - a)^{-\frac{1}{2}} u_a \left( x(a^* - a)^{-\frac{1}{2}} \right) \to \beta Q_0(\beta x) \tag{7}
\]
strongly in $L^2(\mathbb{R}^2)$, where

$$\beta = \left( \frac{p}{2} \int_{\mathbb{R}^2} |x|^p |Q(x)|^2 dx \right)^{\frac{1}{p+2}}.$$ 

The result in [11] has been extended to other kinds of external potentials, e.g. ring-shaped potentials [12], periodic potentials [22], and Newton-like potentials [19].

In the present paper, we will consider the existence and blow-up property of the Gross-Pitaevskii minimizers in the case of having long-range self-gravitating interaction. It turns out that the self-gravitating interaction leads to interesting effects. For example, if the external potential $V$ is not singular enough, then the self-gravitating interaction is the main cause of the instability and the details of the blow-up phenomenon are more or less irrelevant to $V$. This situation is very different from the case without gravity studied in [11, 12, 22, 19]. The precise form of our results will be provided in the next section.

2 Main results

Our first result is

**Theorem 1** (Existence). Let $V : \mathbb{R}^2 \to \mathbb{R}$ satisfy one of the following three conditions:

(V1) (Trapping potentials) $V \geq 0$, $V \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $V(x) \to \infty$ as $|x| \to \infty$;

(V2) (Periodic potentials) $V \in C(\mathbb{R}^2)$ and $V(x+z) = V(x)$ for all $x \in \mathbb{R}^2$, $z \in \mathbb{Z}^2$;

(V3) (Attractive potentials) $V \leq 0$ and $V \in L^p(\mathbb{R}^2) + L^q(\mathbb{R}^2)$ with $p, q \in (1, \infty)$.

Then there exists a constant $a_* < a^*$ such that $E(a)$ in (2) has a minimizer for all $a \in (a_*, a^*)$. We can choose $a_*=0$ in cases (V1) and (V3). Moreover, $E(a) = -\infty$ for all $a \geq a^*$.

Except the case of trapping potentials, the proof of the existence is non-trivial. Even in the case $V \equiv 0$, the Gross-Pitaevskii functional is translation-invariant and some mass may escape to infinity, leading to the lack of compactness. The existence result will be proved by the concentration-compactness method of Lions [16, 17].

Now we turn to the blow-up behavior of minimizers when $a \uparrow a^*$. First, we consider the case when the negative part of the external potential $V$ has no singular point, or it has some singular points but the singularity is weak. More precisely, we will assume

$$V \in L^1_{\text{loc}}(\mathbb{R}^2), \quad V(x) \geq -C \sum_{j \in J} \frac{1}{|x - z_j|^p}, \quad 0 < p < 1$$

for a finite set $\{z_j\}_{j \in J} \subset \mathbb{R}^2$. We have
Theorem 2 (Blow-up for weakly singular potentials). Assume (8). Then

$$\limsup_{a \uparrow a^*} E(a)(a^* - a) = \frac{a^*}{4} \left( \int \int \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} \, dx \, dy \right)^2. \quad (9)$$

Moreover, let \( a_n \uparrow a^* \) and let \( u_n \geq 0 \) be an approximate minimizer for \( E(a_n) \), i.e. \( E(a_n)/E(a_n) \rightarrow 1 \). Then there exist a subsequence \( u_{n_k} \) and a sequence \( \{x_k\} \subset \mathbb{R}^2 \) such that

$$\lim_{k \rightarrow \infty} (a^* - a_{n_k})u_{n_k}(x_k + (a^* - a_{n_k})x) = \beta Q_0(\beta x) \quad (10)$$

strongly in \( H^1(\mathbb{R}^2) \) where

$$\beta = \frac{a^*}{2} \int \int \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} \, dx \, dy. \quad (11)$$

Finally, if \( V(x) = |x|^q \) for \( q > 0 \), or \( V(x) = -|x|^{-q} \) for \( 0 < q < 1 \), and if \( u_n \) is a minimizer for \( E(a_n) \) (which exists by Theorem 1), then the convergence (10) holds true for the whole sequence and for \( x_k = 0 \), i.e.

$$\lim_{n \rightarrow \infty} (a^* - a_n)u_n((a^* - a_n)x) = \beta Q_0(\beta x). \quad (12)$$

We observe that in Theorem 2 the details of the blow-up phenomenon is essentially irrelevant to \( V \). This is an interesting effect of the self-gravitating interaction. In the case without gravity studied in [11, 12, 22, 19], the blow-up behavior depends crucially on the local behavior of \( V \) around its minimizers/singular points, which can be seen from (7). Heuristically, if the condensate shrinks with a length scale \( \varepsilon \rightarrow 0 \), then the self-gravitating interaction is of order \( \varepsilon^{-1} \), while the external potential is at most \( \varepsilon^{-p} \) (since \( V \) is not singular than \( |x|^{-p} \)). Therefore, the contribution of the external potential can be ignored to the leading order.

Now we come to the case when the external potential is more singular. We will assume

$$V(x) = -h(x) \sum_{j=1}^{J} |x - z_j|^{-p_j} \quad (12)$$

with a finite set \( \{z_j\}_{j \in J} \subset \mathbb{R}^2 \), with

$$0 < p_j < 2, \quad p = \max_{j \in J} p_j \geq 1$$

and with

$$h \in C(\mathbb{R}^2), \quad C \geq h \geq 0, \quad h_0 = \max\{h(z_j) : p_j = p\} > 0.$$ 

Let us denote the set

$$Z = \{z_j : p_j = p, h_j = h_0\}$$

which contains the most singular points of \( V \). We have
Theorem 3 (Blow-up for strongly singular potentials). Assume (12). Then

\[
\lim_{a \uparrow a^*} E(a)(a^* - a)^{\frac{2}{p-2}} = \frac{\beta^2}{a^*} - \beta^p A
\]

where

\[
A = \begin{cases} 
    h_0 \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} dx + \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} dx dy, & \text{if } p = 1 \\
    h_0 \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} dx, & \text{if } p > 1
\end{cases}
\]

(14)

and

\[
\beta = \left( \frac{pa^* A}{2} \right)^{\frac{1}{p-2}}.
\]

Moreover, let \(a_n \uparrow a^*\) and let \(u_n \geq 0\) be an approximate minimizer for \(E(a_n)\), i.e. \(E(a_n)/E(a_n) \to 1\). Then there exist a subsequence \(u_{n_k}\) and a point \(x_0 \in \mathbb{Z}\) such that

\[
\lim_{n \to \infty} (a^* - a_{n_k})^{\frac{2}{p-2}} u_{n_k} \left( x_0 + x(a^* - a_n)^{\frac{1}{p-2}} \right) = \beta Q_0(\beta x)
\]

strongly in \(H^1(\mathbb{R}^2)\). Finally, if \(\mathcal{Z}\) has a unique element, then the convergence (15) holds for the whole sequence \(\{u_n\}\).

In contrast of Theorem 2, Theorem 3 says that if the external potential \(V\) is singular enough, then its local behavior close its singular points determines the details of the blow-up profile. In particular, when \(p > 1\), the convergences (13) and (15) are similar to the results in [19] when there is no gravity term included in the energy functional. This means that in this case, the effect of the self-gravitating interaction is negligible to the leading order.

Note that the proof of the blow-up result (7) in [11] is based on the analysis of the Euler-Lagrange equation associated to the minimizers. This approach has been used also in follow-up papers [12, 22]. In the present paper, we will use another approach, which has the advantage that we can treat approximate minimizers as well (in principle there is no Euler-Lagrange equation for an approximate minimizer).

More precisely, as in [19], we will prove the blow-up results by the energy method. First, we prove that approximate minimizers must be an optimizing sequence for the Gagliardo-Nirenberg inequality (3). Then by the concentration-compactness argument, up to subsequences, translations and dilations, this sequence converges to an optimizer for (3), which is of the form \(bQ_0(bx + x_0)\) for some \(b > 0\) and \(x_0 \in \mathbb{R}^2\). Then we determine \(b\) and \(x_0\) by matching the asymptotic formula for \(E(a)\).

**Heuristic argument.** Now let us explain the heuristic ideas of our analysis of the blow-up phenomenon. For simplicity, consider the case \(V(x) = -|x|^{-p}\) with \(0 < p < 2\) and the energy functional becomes

\[
\mathcal{E}_a(u) = \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^p} dx - \frac{a}{2} \int_{\mathbb{R}^2} |u(x)|^4 dx - \iint \frac{|u(x)|^2|u(y)|^2}{|x-y|} dx dy.
\]
If the minimizer $u_a$ of $\mathcal{E}_a(u)$ converges to $Q_0$ under the length-scaling $\ell$, i.e. $u_a(x) \approx \ell Q_0(\ell x)$, then

\[
\mathcal{E}_a(u_a) \approx \mathcal{E}_a(\ell Q_0(\ell \cdot)) = \ell^2 \int_{\mathbb{R}^2} |\nabla Q_0(x)|^2 dx - \ell^p \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} dx - \frac{a \ell^2}{2} \int_{\mathbb{R}^2} |Q_0(x)|^4 dx - \ell \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} dxdy
\]

\[
= \ell^2 \left(1 - \frac{a}{a^*}\right) - \ell^p \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x|^p} dx - \ell \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} dxdy. \quad (16)
\]

We want to choose $\ell$ to minimize the right side of (16) (since $u_a$ minimizes $\mathcal{E}_a(u)$). It is not hard to guess that when $a \uparrow a^*$, then $\ell \to \infty$ and the exact behavior of $\ell$ depends on $p$ as follows.

- If $p < 1$ then $\ell^p \ll \ell$, and the term of order $\ell^p$ does not contribute to the leading order. The value of $\ell$ is essentially determined by minimizing

\[
\ell^2 \left(1 - \frac{a}{a^*}\right) - \ell \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} dxdy,
\]

i.e.

\[
\ell \approx \frac{a^*}{2(a^* - a)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} dxdy.
\]

This is the situation covered in Theorem 2.

- If $p > 1$, then $\ell^p \gg \ell$, and the value of $\ell$ is essentially determined by minimizing

\[
\ell^2 \left(1 - \frac{a}{a^*}\right) - \ell^p \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} dx
\]

i.e.

\[
\ell \approx \left[\frac{a^* p}{2(a^* - a)} \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} dx\right]^{\frac{1}{p-1}}.
\]

On the other hand, if $p = 1$, then $\ell^p = \ell$ and the value of $\ell$ is essentially determined by minimizing

\[
\ell^2 \left(1 - \frac{a}{a^*}\right) - \ell \left[\int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|} dx + \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} dxdy\right]
\]

i.e.

\[
\ell \approx \frac{a^*}{2(a^* - a)} \left[\int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|} dx + \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2 |Q_0(y)|^2}{|x - y|} dxdy\right].
\]

These situations are covered in Theorem 3.

**Varying the gravitation constant.** As mentioned, here we consider the interaction of the form (11), with the gravitation constant $g = 1$ for simplicity. Clearly,
our results hold for any constant $g > 0$ (independent of $a$), up to easy modifications. It might be interesting to ask what happens when $g \to 0$ or $g \to \infty$, at the same time as $a \to a^\ast$. By following the above heuristic discussion, if $V(x) = -|x|^{-p}$ with $0 < p < 2$ and $u_a(x) \approx \ell Q_0(\ell x)$, then

$$E_a(u_a) \approx \ell^2 \left(1 - \frac{a}{a^\ast}\right) - \ell^p \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} \, dx - \ell g \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} \, dxdy. \quad (17)$$

Next we minimize the right side of (17). If $\ell g \ll \ell^p$, namely $g \ll \ell^p - 1$, then we can ignore the gravitation term and the optimal value of $\ell$ is $\sim (a^\ast - a)^{-\frac{1}{2-p}}$. This suggests that the threshold for the gravitation effect to be visible in the blow-up profile is $g \sim (a^\ast - a)^{-\frac{1}{2-p}}$. More precisely, we can expect the following:

- If $g \gg (a^\ast - a)^{-\frac{1}{2-p}}$, then the contribution of the $\ell^p$-term can be ignored to the leading order, and
  $$\ell \approx \frac{a^\ast g}{2(a^\ast - a)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} \, dxdy.$$  
  Thus the blow-up profile is determined completely by the gravitation term if either $p < 1$ and $g \to 0$ slowly enough, or $p \geq 1$ and $g \to +\infty$ fast enough.

- If $g \ll (a^\ast - a)^{-\frac{1}{2-p}}$, then the contribution of the $\ell g$-term can be ignored to the leading order, and
  $$\ell \approx \left[\frac{a^\ast \ell}{2(a^\ast - a)} \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} \, dx\right]^{\frac{1}{2-p}}.$$  
  Thus the blow-up profile is determined only by the attractive potential term (i.e. no gravitational effect) if either $p \leq 1$ and $g \to 0$ fast enough, or $p > 1$ and $g \to +\infty$ slow enough.

- If $g \sim (a^\ast - a)^{-\frac{1}{2-p}}$, then both potential term and gravitation term enter the determination of the blow-up profile.

Although our representation will focus on the case when $g$ is independent of $a$, as stated in Theorem 2 and Theorem 3, the interested reader can prove the above assertions when $g$ is dependent on $a$ by following our analysis below.

**Organization of the paper.** In the rest, we will prove Theorems 1, 2, 3 in Sections 3, 4, 5, respectively. Also, for the reader’s convenience, we recall in Appendix A the Concentration-Compactness Lemma and a standard result on the compactness of the optimizing sequences for the Gagliardo-Nirenberg inequality [3], which are useful in our proof.

**Acknowledgement.** I thank the referee for the interesting suggestion of considering the case when the gravitational constant $g$ depends on $a$, leading to an improvement on the representation of the paper.
3 Existence

In this section we prove Theorem 1. We will always denote by $C$ a universal, large constant.

We start with a preliminary result, which is the upper bound in (9).

**Lemma 4.** For all $V \in L^1_{\text{loc}}(\mathbb{R}^2)$, we have the upper bound

$$
\limsup_{a \to a^*} E(a) (a^* - a) \leq -\frac{a^*}{4} \left( \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} \, dx \, dy \right)^2.
$$

**(Proof.** As in [11] we use the trial function

$$
u_\ell(x) = A_\ell \varphi(x - x_0) \ell Q_0(\ell(x - x_0))$$

where $x_0 \in \mathbb{R}^2$, $0 \leq \varphi \in C^\infty_c(\mathbb{R}^2)$ with $\varphi(x) = 1$ for $|x| \leq 1$, and $A_\ell > 0$ is a normalizing factor to ensure $\|u\|_{L^2} = 1$.

Using (6) and the fact that both $Q_0$ and $|\nabla Q_0|$ are exponentially decay (see [9], Proposition 4.1]), we have

$$A_\ell = \left( \int_{\mathbb{R}^2} |\varphi(x)|^2 \ell^2 |Q_0(\ell x)|^2 \, dx \right)^{-1/2} = O(\ell^{-\infty}),$$

$$\int |\nabla u_\ell|^2 = \ell^2 \int |\nabla Q_0|^2 + O(\ell^{-\infty}),$$

$$\int |u_\ell|^4 = \ell^2 \int |Q_0|^4 + O(\ell^{-\infty}) = \ell^2 \frac{2}{a^*} \int |\nabla Q_0|^2 + O(\ell^{-\infty})$$

and

$$\iint \frac{|u_\ell(x)|^2|u_\ell(y)|^2}{|x-y|} \, dx \, dy = \ell \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} \, dx \, dy + O(\ell^{-\infty}),$$

where $O(\ell^{-\infty})$ means that this quantity converges to 0 faster than $\ell^{-k}$ when $\ell \to \infty$ for all $k = 1, 2, \ldots$. Moreover, since $x \mapsto V(x)|\varphi(x - x_0)|^2$ is integrable and $\ell^2 |Q_0(\ell(x - x_0))|^2$ converges weakly to Dirac-delta function at $x_0$ when $\ell \to \infty$, we have

$$\int_{\mathbb{R}^2} V|u|^2 = |A_\ell|^2 \int_{\mathbb{R}^2} V(x)|\varphi(x - x_0)|^2 |Q_0(\ell(x - x_0))|^2 \ell^2 \, dx \to V(x_0)$$

for a.e. $x_0 \in \mathbb{R}^2$. Using (6) we thus obtain

$$E(a) \leq \mathcal{E}_a(u) = \ell^2 \left( 1 - \frac{a}{a^*} \right) + V(x_0) - \ell \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} \, dx \, dy + o(1)_{\ell \to \infty}.$$ 

Choosing $\ell = \lambda(a^* - a)^{-1}$ with a constant $\lambda > 0$ and take $a \uparrow a^*$, we obtain

$$\limsup_{a \to a^*} E(a) (a^* - a) \leq \frac{\lambda^2}{a^*} - \lambda \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} \, dx \, dy.$$  \hspace{1cm} (19)

Choosing the optimal value

$$\lambda = \frac{a^*}{2} \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x-y|} \, dx \, dy$$

leads to the desired result. \hfill \square
Next, we have the a simple lower bound for \( E_a(u) \).

**Lemma 5.** For all \( u \in H^1(\mathbb{R}^2) \) with \( \|u\|_{L^2} = 1 \), we have
\[
E_a(u) \geq \left( 1 - \frac{a}{a^*} - \varepsilon \right) \int_{\mathbb{R}^2} |\nabla u|^2 + \int_{\mathbb{R}^2} V|u|^2 - \frac{C}{\varepsilon}, \quad \forall \varepsilon > 0.
\]

**Proof.** Take arbitrarily \( \varepsilon > 0 \). By the Gagliardo–Nirenberg inequality (3), we have
\[
\frac{a}{2} \int |u|^4 \leq \frac{a}{a^*} \int |\nabla u|^2.
\]
Moreover, by the Hardy-Littlewood-Sobolev inequality [15, Theorem 4.3], Hölder’s inequality and the Gagliardo–Nirenberg inequality (3) again we have
\[
\int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u(x)|^2|u(y)|^2}{|x-y|} \, dx \, dy \leq C \|u\|_{L^{8/3}}^4 \leq C \|u\|_{L^4}^2 \|u\|_{L^2}^2 \leq \varepsilon \int |\nabla u|^2 + \frac{C}{\varepsilon}
\]
for all \( \varepsilon > 0 \). Combining these estimates, we obtain the desired lower bound. \( \square \)

Now we go to the proof of Theorem 1. From Lemma 4 and the fact that \( a \mapsto E(a) \) is non-increasing, we deduce that \( E(a) = -\infty \) for all \( a \geq a^* \).

Next, consider \( a < a^* \). We distinguish three cases when the external potential \( V \) satisfies (V1), (V2) or (V3), respectively.

**Lemma 6** (Trapping potentials). Assume that (V1) holds. Then \( E(a) \) has a minimizer for all \( a \in (0, a^*) \).

**Proof.** From (21) and the assumption \( V \geq 0 \), we have \( E(a) > -\infty \). Moreover, if \( \{u_n\} \) is a minimizing sequence for \( E(a) \), then \( \|u_n\|_{H^1} \) and \( \int V|u_n|^2 \) are bounded. By Sobolev’s embedding, after passing to a subsequence if necessary, we can assume that \( u_n \) converges to a function \( u_0 \) weakly in \( H^1(\mathbb{R}^2) \) and pointwise.

For every \( R > 0 \), \( u_n \to u_0 \) strongly in \( L^2(B(0, R)) \) by Sobolev’s embedding. Therefore,
\[
\int_{|x| \leq R} |u_0|^2 = \lim_{n \to \infty} \int_{|x| \leq R} |u_n|^2 = 1 - \lim_{n \to \infty} \int_{|x| > R} |u_n|^2 \\
\geq 1 - \left( \inf_{|z| > R} V(z) \right)^{-1} \lim_{n \to \infty} \int_{|x| > R} V|u_n|^2 \geq 1 - C \left( \inf_{|z| > R} V(z) \right)^{-1}.
\]

Taking \( R \to \infty \) and \( V(x) \to \infty \) as \( |x| \to \infty \), we obtain
\[
\int_{\mathbb{R}^2} |u_0|^2 \geq \lim_{R \to \infty} \int_{|x| \leq R} |u_0|^2 \geq 1.
\]
Since we have known \( u_n \to u_0 \) weakly in \( H^1(\mathbb{R}^2) \), we can conclude that \( u_n \to u_0 \) strongly in \( L^2(\mathbb{R}^2) \). By Sobolev’s embedding again, \( u_n \to u_0 \) strongly in \( L^p(\mathbb{R}^2) \) for all \( p \in [2, \infty) \). Consequently, \( \|u_0\|_{L^2} = 1 \),
\[
\int |u_n|^4 \to \int |u_0|^4
\]
and, by the Hardy-Littewood-Sobolev inequality,
\[ \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} \, dx \, dy \to \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u_0(x)|^2|u_0(y)|^2}{|x-y|} \, dx \, dy. \]

Moreover, by Fatou’s lemma we have
\[ \int |\nabla u_n|^2 \geq \int |\nabla u_0|^2 + o(1) \]
as \( u_n \to u \) weakly in \( H^1(\mathbb{R}^2) \) and
\[ \int V|u_n|^2 \geq \int V|u_0|^2 + o(1) \]
as \( u_n \to u_0 \) pointwise. In summary,
\[ \mathcal{E}_a(u_n) \geq \mathcal{E}_a(u_0) + o(1). \]

Since \( u_n \) is a minimizing sequence, we conclude that \( E(a) \geq \mathcal{E}_a(u_0) \), i.e. \( u_0 \) is a minimizer.

**Lemma 7** (Periodic potentials). Assume that (V2) holds. Then there exists \( a_* < a^* \) such that \( E(a) \) has a minimizer for all \( a \in (a_*, a^*) \).

**Proof.** Since \( V \) is continuous and periodic, it is uniformly bounded. From (21), we have \( E(a) > -\infty \) for all \( a \in (0, a^*) \). Moreover, by Lemma 4 we have \( E(a) \to -\infty \) as \( a \uparrow a^* \). Therefore, we can find \( a_* < a^* \) such that for all \( a \in (a_*, a^*) \) we have
\[ E(a) < \inf V. \]

Now we prove \( E(a) \) has minimizers for all \( a \in (a_*, a^*) \). We will use the concentration-compactness method of Lions [16, 17]. For the reader’s convenience, we summary all we need in Lemma 13 in Appendix.

Let \( \{u_n\} \) be a minimizing sequence for \( E(a) \). From (21), \( u_n \) is bounded in \( H^1 \). Hence, we can apply Concentration-Compactness Lemma 13 to the sequence \( \{u_n\} \). Up to subsequences, one of the three cases in Lemma 13 must occur.

**No-vanishing.** Assume that the vanishing case (ii) in Lemma 13 occurs. Then we have
\[ \int |u_n|^4 \to 0, \quad \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} \, dx \, dy \to 0 \]
(in the latter we used the Hardy-Littewood-Sobolev inequality). Combining with the obvious lower bound
\[ \int V|u_n|^2 \geq \inf V \]
we find that
\[ E(a) = \lim_{n \to \infty} \mathcal{E}_a(u_n) \geq \inf V. \]
However, this contradicts to (23). Thus the vanishing case cannot occur.

**No-dichotomy.** Assume that the dichotomy case (ii) in Lemma [13] occurs, i.e. we can find two sequences \(u_n^{(1)}, u_n^{(2)}\) such that

\[
\begin{align*}
&\int_{\mathbb{R}^2} |u_n^{(1)}|^2 \to \lambda, \quad \int_{\mathbb{R}^2} |u_n^{(2)}|^2 \to 1 - \lambda, \\
&\text{dist}(\text{supp}(u_n^{(1)}), \text{supp}(u_n^{(2)})) \to +\infty; \\
&\|u_n - u_n^{(1)} - u_n^{(2)}\|_{L^p} \to 0, \quad \forall p \in [2, \infty); \\
&\int_{\mathbb{R}^2} (|\nabla u_n|^2 - |\nabla u_n^{(1)}|^2 - |\nabla u_n^{(2)}|^2) \geq o(1).
\end{align*}
\]

Since \(\|u_n - u_n^{(1)} - u_n^{(2)}\|_{L^p} \to 0\) and \(\text{dist}(\text{supp}(u_n^{(1)}), \text{supp}(u_n^{(2)})) \to +\infty\) we have

\[
\int_{\mathbb{R}^2} |u_n|^4 = \int_{\mathbb{R}^2} |u_n^{(1)} + u_n^{(2)}|^4 + o(1) = \int_{\mathbb{R}^2} (|u_n^{(1)}|^4 + |u_n^{(2)}|^4) + o(1)
\]

and

\[
\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} \, dx \, dy
\]

\[
= \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u_n^{(1)}(x) + u_n^{(2)}(x)|^2|u_n^{(1)}(y) + u_n^{(2)}(y)|^2}{|x-y|} \, dx \, dy + o(1)
\]

\[
= \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(|u_n^{(1)}(x)|^2 + |u_n^{(2)}(x)|^2)(|u_n^{(1)}(y)|^2 + |u_n^{(2)}(y)|^2)}{|x-y|} \, dx \, dy + o(1)
\]

\[
= \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u_n^{(1)}(x)|^2|u_n^{(1)}(y)|^2 + |u_n^{(2)}(x)|^2|u_n^{(2)}(y)|^2}{|x-y|} \, dx \, dy + o(1)
\]

Here we have used the Hardy-Littlewood-Sobolev inequality in (26) and used the decay of Newton potential, i.e. \(|x-y|^{-1} \to 0\) as \(|x-y| \to \infty\), to remove the cross-term

\[
\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u_n^{(1)}(x)|^2|u_n^{(2)}(y)|^2 + |u_n^{(2)}(x)|^2|u_n^{(1)}(y)|^2}{|x-y|} \, dx \, dy
\]

in (27).

Moreover, since \(V\) is bounded we get

\[
\int_{\mathbb{R}^2} V \left( |u_n|^2 - |u_n^{(1)}|^2 - |u_n^{(2)}|^2 \right) = \int_{\mathbb{R}^2} V \left( |u_n|^2 - |u_n^{(1)} + u_n^{(2)}|^2 \right) \to 0
\]

In summary, we have the energy decomposition

\[
E_n(u_n) \geq E_{\alpha}(u_n^{(1)}) + E_{\alpha}(u_n^{(2)}) + o(1).
\]

Next, we use \(u_n^{(1)}/\|u_n^{(1)}\|_{L^2}\) as a trial state for \(E(a)\). By the variational principle, we have

\[
E(a) \leq \mathcal{E} \left( \frac{u_n^{(1)}}{\|u_n^{(1)}\|_{L^2}} \right) = \lambda^{-1} \left( \int |\nabla u_n^{(1)}|^2 + \int V |u_n^{(1)}|^2 \right) - \lambda^{-2} \left( \frac{a}{2} \int |u_n^{(1)}|^4 + \iint \frac{|u_n^{(1)}(x)|^2|u_n^{(1)}(y)|^2}{|x-y|} \, dx \, dy \right) + o(1).
\]
In the latter equality, we have used \( \|u_n^{(1)}\|_{L^2}^2 \to \lambda \) (and \( u_n^{(1)} \) is bounded in \( H^1 \)). The above inequality can be rewritten as
\[
\lambda^2 E(a) + (1 - \lambda) \left( \int |\nabla u_n^{(1)}|^2 + \int |V| |u_n^{(1)}|^2 \right) \leq \mathcal{E}_a(u_n^{(1)}) + o(1).
\]  
(31)

By ignoring the kinetic energy on the left side and the obvious bound
\[
\int V|u_n^{(1)}|^2 \geq (\inf V) \int |u_n^{(1)}|^2 = \lambda(\inf V) + o(1)
\]
we find that
\[
\lambda^2 E(a) + \lambda(1 - \lambda) \inf V \leq \mathcal{E}_a(u_n^{(1)}) + o(1).
\]
Similarly, since \( \|u_n^{(1)}\|_{L^2}^2 \to 1 - \lambda \) we get
\[
(1 - \lambda)^2 E(a) + \lambda(1 - \lambda) \inf V \leq \mathcal{E}_a(u_n^{(2)}) + o(1).
\]
Summing the latter inequalities gives
\[
\mathcal{E}_a(u_n^{(1)}) + \mathcal{E}_a(u_n^{(2)}) \geq (\lambda^2 + (1 - \lambda)^2) E(a) + 2\lambda(1 - \lambda) \inf V + o(1).
\]
Inserting this into (29) and using \( \mathcal{E}_a(u_n) \to E(a) \) we arrive at
\[
E(a) \geq (\lambda^2 + (1 - \lambda)^2) E(a) + 2\lambda(1 - \lambda) \inf V
\]
which is equivalent to
\[
E(a) \geq \inf V
\]
because \( \lambda \in (0, 1) \). But again, it is a contradiction to (23). Thus the dichotomy case cannot occur.

**Compactness.** Now we can conclude that the compactness case (i) in Lemma 13 must occur, i.e. there exists a sequence \( \{z_n\} \subset \mathbb{R}^2 \) such that \( \tilde{u}_n = u_n(. + z_n) \) converges to some \( u_0 \) weakly in \( H^1(\mathbb{R}^2) \) and strongly in \( L^p(\mathbb{R}^2) \) for all \( p \in [2, \infty) \). Since the Lebesgue measure is translation-invariant, we have \( \|u_0\|_{L^2} = 1 \),
\[
\int_{\mathbb{R}^2} |u_n|^4 = \int_{\mathbb{R}^2} |u_n(x + z_n)|^4 \, dx = \int_{\mathbb{R}^2} |u_0|^4 + o(1)_{n \to \infty},
\]
\[
\int_{\mathbb{R}^2} V|u_n|^2 = \int_{\mathbb{R}^2} V(x + z_n)|u_n(x + z_n)|^2 \, dx = \int_{\mathbb{R}^2} V(x + z_n)|u_0(x)|^2 \, dx + o(1)_{n \to \infty}.
\]
by the boundedness of \( V \), and
\[
\int_{\mathbb{R}^2} \frac{|u_n(x)|^2|u_n(y)|^2}{|x - y|} \, dx \, dy = \int_{\mathbb{R}^2} \frac{|u_n(x + z_n)|^2|u_n(y + z_n)|^2}{|x - y|} \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} \frac{|u_0(x)|^2|u_0(y)|^2}{|x - y|} \, dx \, dy + o(1)_{n \to \infty}
\]
by the Hardy-Littlewood-Sobolev inequality.
Moreover, since \( \nabla u_n \rightarrow \nabla u \) weakly in \( L^2(\mathbb{R}^2) \) we have
\[
\int_{\mathbb{R}^2} |\nabla u_n|^2 = \int_{\mathbb{R}^2} |\nabla u_n(x + z_n)|^2 dx \geq \int_{\mathbb{R}^2} |\nabla u_0|^2 + o(1)_{n \rightarrow \infty}.
\]
In summary, since \( u_n \) is a minimizing sequence, we conclude that
\[
E_a(u_n) \geq \int_{\mathbb{R}^2} |\nabla u_0|^2 + \int V(\cdot + z_n)|u_0|^2 - \frac{a}{2} \int |u_0(x)|^4 - \int \int_{\mathbb{R}^2} \frac{|u_0(x)|^2|u_0(y)|^2}{|x - y|} dxdy + o(1)_{n \rightarrow \infty}. \tag{32}
\]
To finish, we use the periodicity of \( V \). We can write
\[
z_n = y_n + z \quad \text{with} \quad y_n \in [0,1]^2, \quad z \in \mathbb{Z}^2.
\]
Since \( y_n \) is bounded, up to a subsequence, we can assume that \( y_n \rightarrow y_0 \) in \( \mathbb{R}^2 \). Thus by the periodicity of \( V \) and the Lebesgue Dominated Convergence, we have
\[
\int V(x + z_n)|u_0(x)|^2 dx = \int V(x + y_n)|u_0(x)|^2 dx \rightarrow \int V(x + y_0)|u_0(x)|^2 dx.
\]
Thus (32) reduces to
\[
E_a(u_n) \geq \int_{\mathbb{R}^2} |\nabla u_0|^2 + \int V(\cdot + y_0)|u_0|^2 - \frac{a}{2} \int |u_0(x)|^4 - \int \int_{\mathbb{R}^2} \frac{|u_0(x)|^2|u_0(y)|^2}{|x - y|} dxdy + o(1)_{n \rightarrow \infty} - E_a(u_0(\cdot - y_0)) + o(1)_{n \rightarrow \infty}.
\]
Since \( u_n \) is a minimizing sequence, we conclude that \( u_0(\cdot - y_0) \) is a minimizer for \( E(a) \).

**Lemma 8 (Attractive potentials).** Assume that (V3) holds. Then \( E(a) \) has a minimizer for all \( a \in (0, a^*) \).

**Proof.** First, if \( V \equiv 0 \), then we can follow the proof of the periodic case and use the strict inequality \( E(a) < 0 \) (i.e. (23) holds true with \( a_* = 0 \)). To prove \( E(a) < 0 \), we can simply take the trial function
\[
u_\ell(\ell x) = \ell Q_0(\ell x)
\]
and use the variational principle
\[
E(a) \leq E(u_\ell) = \ell^2 \left(1 - \frac{a}{a^*}\right) - \ell \int \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} dxdy
\]
with \( \ell > 0 \) sufficiently small.
Thus \( (21) \) reduces to
\[
\varepsilon \int |\nabla u|^2 + \int V|u|^2 \geq -C_{\varepsilon}, \quad \forall \varepsilon > 0, \quad \forall u \in H^1(\mathbb{R}^2), \|u\|_{L^2} = 1.
\]
(33)

Thus \( (21) \) reduces to
\[
\mathcal{E}_a(u) \geq \left( 1 - \frac{a}{a^*} - \varepsilon \right) \int \mathbb{R}^2 |\nabla u|^2 - C_{\varepsilon}, \quad \forall \varepsilon > 0, \quad \forall u \in H^1(\mathbb{R}^2), \|u\|_{L^2} = 1.
\]

This ensures that \( E(a) > -\infty \). Moreover, if \( \{u_n\} \) is a minimizing sequence for \( E(a) \), then \( u_n \) is bounded in \( H^1(\mathbb{R}^2) \). By Sobolev’s embedding, after passing to a subsequence if necessary, we can assume that \( u_n \) converges to a function \( u_0 \) weakly in \( H^1(\mathbb{R}^2) \) and pointwise.

**Energy decomposition.** Now following the concentration-compactness argument, we will show that
\[
\mathcal{E}_a(u_n) \geq \mathcal{E}_a(u_0) + \mathcal{E}_a^\infty(u_n - u_0) + o(1)
\]
(34)

with \( \mathcal{E}_a^\infty(\varphi) \) the energy functional without the external potential, i.e.
\[
\mathcal{E}_a^\infty(\varphi) = \int |\nabla \varphi|^2 - \frac{a}{2} \int |\varphi|^4 - \iint \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x - y|} dx dy.
\]

Indeed, since \( u_n \to u_0 \) weakly in \( H^1(\mathbb{R}^2) \), we have \( \nabla u_n \to \nabla u \) weakly in \( L^2 \) and hence
\[
\|\nabla u_n\|_{L^2}^2 = \|\nabla u_0\|_{L^2}^2 + \|\nabla (u_n - u_0)\|_{L^2}^2 + 2\langle \nabla u_0, \nabla (u_n - u_0) \rangle = \|\nabla u_0\|_{L^2}^2 + \|\nabla (u_n - u_0)\|_{L^2}^2 + o(1).
\]

Moreover, since \( u_n \to u_0 \) weakly in \( H^1(\mathbb{R}^2) \) and \( V \in L^p(\mathbb{R}^2) + L^q(\mathbb{R}^2) \) with \( 1 < p < q < \infty \) we have (see \[15\] Theorem 11.4)
\[
\int V|u_n|^2 \to \int V|u_0|^2.
\]

Moreover, since \( u_n \to u_0 \) pointwise, we have
\[
\int |u_n|^4 = \int |u_0|^4 + \int |u_n - u_0|^4 + o(1)
\]

by Brezis-Lieb’s refinement of Fatou’s lemma \[3\]
\[
\iint \frac{|u_n(x)|^2|u_n(y)|^2}{|x - y|} dx dy = \iint \frac{|u_0(x)|^2|u_0(y)|^2}{|x - y|} dx dy + \iint \frac{|u_n(x) - u_0(x)|^2|u_n(y) - u_0(y)|^2}{|x - y|} dx dy + o(1)
\]

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by a nonlocal version of the Brezis-Lieb lemma in [2, Lemma 2.2]. Thus (34) holds true.

**No-vanishing.** Next, we show that \( u_0 \not\equiv 0 \). Assume by contradiction that \( u_0 \equiv 0 \). Then (34) implies that

\[
E(a) = \lim_{n \to \infty} E_a(u_n) \geq \lim_{n \to \infty} E_a^\infty(u_n) \geq E^\infty(a) \tag{35}
\]

where

\[
E^\infty(a) = \inf \{ E^\infty_a(\varphi) \mid \varphi \in H^1(\mathbb{R}^2), \| \varphi \|_{L^2} = 1 \}.
\]

On the other hand, we have proved that \( E^\infty(a) \) has a minimizer \( \varphi_0 \) (this is the case when the external potential vanishes). We can assume \( \varphi_0 \geq 0 \) by the diamagnetic inequality. By a standard variational argument, \( \varphi_0 \) solves the Euler-Lagrange equation

\[
-\Delta \varphi_0 - a\varphi_0^3 - 2(|\varphi_0|^2 * |x|^{-1})\varphi_0 = \mu \varphi_0 \tag{36}
\]

for a constant \( \mu \in \mathbb{R} \) (the Lagrange multiplier). Consequently, we have \( (-\Delta + |\mu|)\varphi_0 \geq 0 \), and hence \( \varphi_0 > 0 \) by [15, Theorem 9.9].

From the facts that \( V \leq 0 \), \( V \not\equiv 0 \) and \( \varphi_0 > 0 \), we have

\[
E_a(\varphi_0) - E^\infty_a(\varphi_0) = \int V |\varphi_0|^2 < 0.
\]

Therefore, by the variational principle,

\[
E(a) < E^\infty(a). \tag{37}
\]

Thus (35) cannot occur, i.e. we must have that \( u_0 \not\equiv 0 \).

**Compactness.** It remains to show that \( \| u_0 \|_{L^2} = 1 \). We assume by contradiction that \( \| u_0 \|_{L^2} = \lambda \in (0, 1) \). Then similarly to (30) we have

\[
E(a) \leq E \left( \frac{u_0}{\| u_0 \|_{L^2}} \right) = \lambda^{-1} \left( \int |\nabla u_0|^2 + \int V |u_0|^2 \right) - \lambda^{-2} \left( \frac{a}{2} \int |u_0|^4 + \frac{1}{2} \iint \frac{|u_0(x)|^2|u_0(y)|^2}{|x - y|} \, dx \, dy \right) \leq \lambda^{-1} E_a(u_0).
\]

Thus

\[
E_a(u_0) \geq \lambda E(a). \tag{38}
\]

Similarly, using \( \| u_n - u_0 \|^2 \to 1 - \lambda \) we get

\[
E^\infty_a(u_n - u_0) \geq (1 - \lambda) E^\infty(a) + o(1). \tag{39}
\]

Inserting (38) and (39) into (34) and using \( E_a(u_n) \to E(a) \), we get

\[
E(a) \geq \lambda E(a) + (1 - \lambda) E^\infty(a).
\]
which is equivalent to $E(a) = E^\infty(a)$ as $\lambda \in (0, 1)$. However, it contradicts to (37).

Thus we conclude that $\|u_0\|_{L^2} = 1$. Now (39) becomes $E^\infty_a(u_n - u_0) \geq o(1)$, which, together with (34), implies that

$$E(a) = \lim_{n \to \infty} E_a(u_n) \geq E_a(u_0).$$

Thus $u_0$ is a minimizer for $E(a)$. This ends the proof of Lemma 8. \square

The proof of Theorem 1 is complete.

4 Blow-up: weakly singular potentials

In this section we prove Theorem 2. We will always assume that $V$ satisfies (8), i.e. $V \in L^1_\text{loc}(\mathbb{R}^2)$, $V(x) \geq -C\sum_{j \in J} |x - z_j|^p$, $0 < p < 1$.

To simplify the notation, let us denote by $u_a$ the approximate minimizer for $E_a$, i.e. $E_a(u_a)/E_a(a) \to 1$ and write $a \uparrow a^*$ instead of $a_n \uparrow a^*$. Also, we denote

$$\ell_a = (a^* - a)^{-1}, \quad \beta = \frac{a^*}{2} \int \int \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} dxdy.$$

We will always consider the case when $a$ is sufficiently close to $a^*$. We start with

**Lemma 9 (A-priori estimate).** We have

$$C\ell_a^2 \geq \int |\nabla u_a|^2 \geq \frac{\ell_a^2}{C}, \quad \int |\nabla u_a|^2 - \frac{a}{2} \int |u_a|^4 \leq C\ell_a.$$

**Proof.** From Lemma 4 and the assumption $E_a(u_a)/E(a) \to 1$ we have the sharp upper bound

$$E_a(u_a) \leq E(a)(1 + o(1)) \leq -\frac{\beta^2}{a^*}\ell_a(1 + o(1)). \quad (40)$$

Now we go to the lower bound. We recall an elementary result, whose proof follows by a simple scaling argument (see e.g. [19, Proof of Lemma 6] for details).

**Lemma 10.** For every $0 < q < 2$, $y \in \mathbb{R}^2$ and $\varepsilon > 0$, we have

$$\int |u(x)|^2 |x - y|^q dx \leq \varepsilon \int |\nabla u|^2 + C_q\varepsilon^{-q/(2-q)} \int |u|^2, \quad \forall u \in H^1(\mathbb{R}^2).$$

Since $V$ satisfies (5), Lemma 11 implies that

$$\int V|u_a|^2 \geq -\varepsilon \int |\nabla u_a|^2 - C_p\varepsilon^{-p/(2-p)}, \quad \forall \varepsilon > 0. \quad (41)$$
Combining with Lemma 5, we find that
\[ E_a(u_a) \geq \left( 1 - \frac{a}{a^*} - \varepsilon \right) \int |\nabla u_a|^2 - C \varepsilon^{-1} - C_p \varepsilon^{-p/(2-p)}, \quad \forall \varepsilon > 0. \] (42)

From (42), choosing 
\[ \varepsilon = \frac{1}{2} \left( 1 - \frac{a}{a^*} \right) \]
and using \( p < 1 \) (i.e. \( \varepsilon^{-p/(2-p)} \ll \varepsilon^{-1} \) for \( \varepsilon > 0 \) small) we obtain the lower bound
\[ E(u_a) \geq \frac{1}{2} \left( 1 - \frac{a}{a^*} \right) \int |\nabla u_a|^2 - \frac{C}{a^* - a} \geq -C \ell_a. \]

Comparing the latter estimate with the upper bound (40) we also obtain
\[ \int_{\mathbb{R}^2} |\nabla u_a|^2 \leq C \ell_a^2. \]

On the other hand, in (42) we can choose
\[ \varepsilon = \gamma \left( 1 - \frac{a}{a^*} \right) \]
for a constant \( \gamma > 1 \) to get
\[ E_a(u_a) \geq -(\gamma - 1) \left( 1 - \frac{a}{a^*} \right) \int |\nabla u|^2 - \frac{C}{\gamma (a^* - a)}. \]

If \( \gamma \) is sufficiently large, we can use latter estimate and the upper bound (40) to deduce that
\[ \int_{\mathbb{R}^2} |\nabla u_a|^2 \geq \frac{\ell_a^2}{C}. \]

Finally, inserting the upper bound \( \int |\nabla u|^2 \leq C \ell_a^2 \) into (41) and (22), then optimizing over \( \varepsilon > 0 \) we have
\[ - \int V |u_a|^2 \leq o(\ell_a), \] (43)
(we used \( p < 1 \)) and
\[ \iint \frac{|u_a(x)|^2 |u_a(y)|^2}{|x-y|} \, dx \, dy \leq C \ell_a. \] (44)

Combining with the upper bound \( E_a(u_a) \leq C \ell_a \) we conclude that
\[ \int |\nabla u_a|^2 - \frac{a}{2} \int |u_a|^4 = E_a(u_a) - \int V |u_a|^2 + \iint \frac{|u_a(x)|^2 |u(y)|^2}{|x-y|} \, dx \, dy \leq C \ell_a. \]
Now we are ready to give

Proof of Theorem 2. Denote

\[ u_a(x) = \ell_a \varphi_a(\ell_a x). \]  

(45)

Then \( \| \varphi_a \|_{L^2} = 1 \) and by Lemma 9

\[ C \geq \int |\nabla \varphi_a|^2 \geq \frac{1}{C}, \quad \int |\nabla \varphi_a|^2 - \frac{a}{2} \int |\varphi_a|^4 \leq C \ell_a^{-1} \to 0. \]

Thus \( \varphi_a \) is an optimizing sequence for the Gagliardo-Nirenberg inequality (3). By Lemma 14 in Appendix A.2, there exist a subsequence of \( \varphi_a \) (still denoted by \( \varphi_a \) for simplicity), a sequence \( \{x_a\} \subset \mathbb{R}^2 \) and a constant \( b > 0 \) such that

\[ \varphi_a(x + x_a) \to bQ_0(bx) \]

strongly in \( H^1(\mathbb{R}^2) \).

Now we determine \( b \). Since \( \varphi_a(x + x_a) \to bQ_0(bx) \) strongly in \( H^1(\mathbb{R}^2) \), we have

\[ \int |\nabla \varphi_a|^2 \to b^2 \int |\nabla Q_0|^2 = b^2 \]

and

\[ \iint \frac{|\varphi_a(x)|^2|\varphi_a(y)|^2}{|x - y|} \, dx \, dy \to b \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} \, dx \, dy = \frac{2\beta b}{a^*}. \]

Combining with (43) and the Gagliardo-Nirenberg inequality (3) we have

\[ \mathcal{E}_a(u_a) \geq \left( 1 - \frac{a}{a^*} \right) \int |\nabla u_a|^2 - \int \iint \frac{|u_a(x)|^2|u_a(y)|^2}{|x - y|} \, dx \, dy + o(\ell_a) \]

\[ = \ell_a^2 \left( 1 - \frac{a}{a^*} \right) \int |\nabla \varphi_a|^2 - \ell_a \iint \frac{|\varphi_a(x)|^2|\varphi_a(y)|^2}{|x - y|} \, dx \, dy + o(\ell_a) \]

\[ = \frac{\ell_a}{a^*} \left( b^2 - 2\beta b + o(1) \right) = -\frac{\ell_a}{a^*} \left( \beta^2 - (b - \beta)^2 + o(1) \right). \]

Comparing with the upper bound

\[ \mathcal{E}_a(u_a) \leq -\frac{\ell_a}{a^*}(\beta^2 + o(1)) \]

in Lemma 1, we conclude that \( b = \beta \) and

\[ \mathcal{E}_a(u_a) = -\frac{\ell_a}{a^*}(\beta^2 + o(1)). \]

Thus we have proved that

\[ \varphi_a(x + x_a) \to \beta Q_0(\beta x) \]

(46)
strongly in $H^1(\mathbb{R}^2)$ which is equivalent to (10) and

$$E(a) = \mathcal{E}_a(u_a)(1 + o(1)) = -\frac{\ell_a}{a^*}(\beta^2 + o(1)).$$

which is equivalent to (11).

Finally, consider the cases when $V(x) = |x|^q$ for $q > 0$ or $V(x) = -|x|^{-q}$ for $0 < q < 1$. Then by Theorem 1, $E(a)$ has a minimizer $u_a \geq 0$. Moreover, by the rearrangement inequalities [15, Chapter 3], we deduce that

$$\mathcal{E}_a(u_a) \geq \mathcal{E}_a(u_a^*)$$

where $u_a^*$ is the symmetric decreasing rearrangement of $u_a$, and the equality occurs if and only if $u_a = u_a^*$ (since $|x|^q$ is strictly symmetric increasing and $|x|^{-q}$ is strictly symmetric decreasing). Since $u_a$ is a minimizer for $E(a)$, we conclude that $u_a$ must be radially symmetric decreasing. Consequently, $\varphi_a$ is also radially symmetric decreasing. From the convergence (11) and the fact that $Q_0$ is radially symmetric decreasing, it is easy to deduce that $\varphi_a \rightarrow Q_0$ strongly in $H^1(\mathbb{R}^2)$. This is equivalent to (11).

This ends the proof of Theorem 2.

5 Blow-up: strongly singular potentials

In this section we prove Theorem 3. Recall that $V$ satisfies assumption (12), i.e.

$$V(x) = -h(x)\sum_{j=1}^J |x - z_j|^{-p_j}, \quad 0 < p_j < 2, \quad h \in C(\mathbb{R}^2), \quad C \geq h \geq 0$$

and

$$p = \max_{j \in J} p_j \geq 1, \quad h_0 = \max\{h(z_j) : p_j = p\} > 0.$$ 

Also we denote

$$Z = \{x_j : p_j = p, h(x_j) = h_0\}.$$ 

Again, we denote by $u_a$ the approximate minimizer for $E(a)$, i.e. $\mathcal{E}_a(u_a)/E(a) \rightarrow 1$ and write $a \uparrow a^*$ instead of $a_n \uparrow a^*$. Also, we denote

$$\ell_a = (a^* - a)^{-\frac{1}{p-2}}, \quad \beta = \left(\frac{pa^* A}{2}\right)^{\frac{1}{p-2}}$$

with $A$ defined in (14).

Since $V$ is sufficiently singular, the upper bound in Lemma 4 is not optimal when $a \uparrow a^*$. Instead, we have

Lemma 11. We have

$$\limsup_{a \uparrow a^*} E(a)\ell_a^{-p} \leq \inf_{\lambda > 0} \left[\frac{\lambda^2}{a^*} - \lambda^p A\right] = \frac{\beta^2}{a^*} - \beta^p A.$$
Proof. Let $x_j \in \mathbb{Z}$, i.e. $p_j = p$ and $h(x_j) = h_0$. For every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that

$$V(x) \leq (\varepsilon - h_0)|x - x_j|^{-p}, \quad \forall |x - x_j| \leq 2\eta_\varepsilon.$$ 

We choose

$$u_\ell(x) = A_\ell \varphi(x - x_j)\ell Q_0(\ell(x - x_j))$$

where $0 \leq \varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi(x) = 1$ for $|x| \leq \eta_\varepsilon$, $\varphi(x) = 0$ for $|x| \geq 2\eta_\varepsilon$, and $A_\ell > 0$ is a suitable factor to make $\|u_\ell\|_{L^2} = 1$.

Similarly to the proof of Lemma 4, since $Q_0$ and $|\nabla Q_0|$ decay exponentially we have

$$A_\ell = 1 + O(\ell^{-\infty}),$$

$$\int |\nabla u_\ell|^2 = \ell^2 \int |\nabla Q_0|^2 + O(\ell^{-\infty}),$$

$$\int |u_\ell|^4 = \ell^2 \frac{2}{a^s} \int |\nabla Q_0|^2 + O(\ell^{-\infty})$$

and

$$\iint \frac{|u_\ell(x)|^2|u_\ell(y)|^2}{|x - y|} \, dx \, dy = \ell \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} \, dx \, dy + O(\ell^{-\infty}).$$

Moreover, the choice of $\varphi$ ensures that

$$V(x)\varphi(x - x_j) \leq (\varepsilon - h_0)|x - x_j|^{-p}\chi_{\{|x - x_j| \leq \eta_\varepsilon\}}.$$

Therefore,

$$\int_{\mathbb{R}^2} V|u_\ell|^2 \leq (\varepsilon - h_0)A_\ell^2 \int_{|x - x_j| \leq \eta_\varepsilon} \frac{|Q_0(\ell(x - x_j))|^2}{|x - x_j|^p} \ell^2 \, dx$$

$$= (\varepsilon - h_0)\ell^p \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} \, dx + O(\ell^{-\infty}).$$

Putting all together, then using $E(a) \leq E_\alpha(u_\ell)$ and (4) we obtain

$$E(a) \leq \ell^2 \left(1 - \frac{a}{a^s}\right) - \ell \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} \, dx \, dy + (\varepsilon - h_0)\ell^p \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} \, dx + O(\ell^{-\infty}).$$

Choosing $\ell = \lambda \ell_a = \lambda(a^s - a) = \lambda \ell_a^{-p}$ for a constant $\lambda > 0$ we find that

$$E(a)\ell_a^{-p} \leq \frac{\lambda^2}{a^s} - \lambda \ell_a^{-p} \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} \, dx \, dy + \lambda^p(\varepsilon - h_0) \int_{\mathbb{R}^2} \frac{|Q_0(x)|^2}{|x|^p} \, dx + O(\ell_a^{-\infty}).$$

Taking the limit $\ell_a \to \infty$, using $p \geq 1$ and then taking $\varepsilon \to 0$, we conclude that

$$\limsup_{a \to a^s} E(a)\ell_a^{-p} \leq \frac{\lambda^2}{a^s} - \lambda^p A, \quad \forall \lambda > 0. \quad (47)$$

It is straightforward to see that the right side of (47) is smallest when

$$\lambda = \beta = \left(\frac{pa^s A}{2}\right)^{-\frac{1}{p-1}}$$

This ends the proof.
Next, we have

**Lemma 12** (A-priori estimates).

\[
C\ell_a^2 \geq \int_{\mathbb{R}^2} |\nabla u_a|^2 \geq \frac{\ell_a^2}{C}, \quad \int |\nabla u_a|^2 - \frac{a}{2} \int |u_a|^4 \leq C\ell_a^p, \quad \int V|u_a|^2 \leq -\frac{\ell_a^p}{C}.
\]

**Proof.** The proof strategy is similar to Lemma 9. By the choice of \( V \), we have

\[
V(x) \geq -C \sum_{j \in J} \frac{1}{|x - z_j|^p} - C.
\]

Therefore,

\[
\int V|u_a|^2 \geq -\varepsilon \int |\nabla u_a|^2 - C_p\varepsilon^{-p/(2-p)}, \quad \forall \varepsilon > 0 \quad (48)
\]

by Lemma 10 and

\[
E_a(u_a) \geq \left(1 - \frac{a}{a^*} - \varepsilon\right) \int |\nabla u_a|^2 - C\varepsilon^{-1} - C_p\varepsilon^{-p/(2-p)}, \quad \forall \varepsilon > 0. \quad (49)
\]

by Lemma 5. Choosing

\[
\varepsilon = \frac{1}{2} \left(1 - \frac{a}{a^*}\right)
\]

and using the upper bound in Lemma 11 \( p \in [1, 2) \) we get (with \( p \in [1, 2) \))

\[
-\frac{\ell_a^p}{C} \geq E(a)(1 + o(1)) \geq E_a(u_a) \geq C\ell_a^{p-2} \int |\nabla u_a|^2 - C_p\ell_a^p
\]

Thus

\[
\int |\nabla u_a| \leq C\ell_a^2.
\]

Moreover, in (49) choosing

\[
\varepsilon = \gamma \left(1 - \frac{a}{a^*}\right)
\]

with \( \gamma \) sufficiently large, we get

\[
\int |\nabla u_a| \geq \frac{\ell_a^2}{C}.
\]

Next, by (21),

\[
\int_{\mathbb{R}^2} V|u_a|^2 \leq E_a(u_a) - \left(1 - \frac{a}{a^*} - \varepsilon\right) \int_{\mathbb{R}^2} |\nabla u_a|^2 + C\varepsilon^{-1}.
\]

Choosing again

\[
\varepsilon = \gamma \left(1 - \frac{a}{a^*}\right)
\]

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with $\gamma$ sufficiently large, then using $E_a(u_a) \leq -\ell_a^p / C$ and $\int |\nabla u_a|^2 \leq C \ell_a^2$, we obtain
\[
\int_{\mathbb{R}^2} V |u_a|^2 \leq -\frac{\ell_a^p}{C}.
\]

Finally, inserting the bound $\int |\nabla u_a| \leq C \ell_a^2$ into (18) and (22), then optimizing over $\varepsilon > 0$ we have
\[
\int V |u_a|^2 \geq -\frac{\ell_a^p}{C}, \quad \iint \frac{|u_a(x)|^2 |u_a(y)|^2}{|x-y|} \, dx \, dy \leq C \ell_a^p.
\]
Thus
\[
\int |\nabla u_a|^2 - \frac{a}{2} \int |u_a|^4 = E_a(u_a) - \int V |u_a|^2 + \iint \frac{|u_a(x)|^2 |u_a(y)|^2}{|x-y|} \, dx \, dy \leq C \ell_a^p.
\]

Now we go to

**Proof of Theorem**. From the estimate $\int V |u_a|^2 \leq -\ell_a^p / C$ in Lemma 12 and the simple bound
\[
V(x) \geq -C \sum_{j \in J} \frac{1}{|x-z_j|^{p_j}},
\]
we obtain that, up to a subsequence of $u_a$, there exists $i \in \{1, 2, ..., J\}$ such that
\[
\int_{\mathbb{R}^2} \frac{|u_a(x)|^2}{|x-z_i|^{p_i}} \, dx \geq \frac{\ell_a^p}{C}.
\]
Define
\[
u_a(x) = \ell_a \varphi_a(\ell_a(x-z_i)).
\]
By Lemma 12, $\varphi_a$ is bounded in $H^1(\mathbb{R}^2)$, $\|\varphi_a\|_{L^2} = 1$ and
\[
\int |\nabla \varphi_a|^2 \geq \frac{1}{C}, \quad \int |\nabla \varphi_a|^2 - \frac{a}{2} \int |\varphi_a|^4 \leq C \ell_a^{p-2} \to 0.
\]
Thus $\varphi_a$ is an optimizing sequence for the Gagliardo-Nirenberg inequality (3). By Lemma 14 in Appendix A.2, up to a subsequence of $\varphi_a$, there exist a sequence $\{x_a\} \subset \mathbb{R}^2$ and a constant $b > 0$ such that
\[
\varphi_a(x+x_a) \to b Q_0(bx) \quad \text{strongly in } H^1(\mathbb{R}^2).
\]
Next, we prove $p_i = p$. From (50), we have
\[
\ell_a^p \int_{\mathbb{R}^2} \frac{|\varphi_a(x)|^2}{|x|^{p_i}} \, dx \geq \frac{\ell_a^p}{C}.
\]
This implies that \( p_i = p \) because \( \ell_a \to \infty \) and \( \int |\varphi_a|^2/|x|^p \) is bounded (as \( \varphi_a \) is bounded in \( H^1(\mathbb{R}^2) \)).

Moreover, since \( \varphi_a(x + x_a) \to bQ_0(bx) \) in \( H^1(\mathbb{R}^2) \) and \( 1 \leq p < 2 \), we find that

\[
\frac{1}{C} \leq \int \frac{|\varphi_a(x)|^2}{|x|^p} \, dx \leq \int \frac{|\varphi_a(x + x_a)|^2}{|x + x_a|^p} \, dx = \int \frac{|bQ_0(bx)|^2}{|x + x_a|^p} \, dx + o(1).
\]

Since \( Q_0 \) decays exponentially, we conclude that \( x_a \) is bounded. Up to a subsequence, we can assume that \( x_a \to x_\infty \). Thus we have

\[
\varphi_a(x + x_\infty) \to bQ_0(bx) \text{ strongly in } H^1(\mathbb{R}^2).
\]

Finally, we determine \( x_\infty \) and \( b \). This will be done by considering the sharp lower bound for \( E(a) \). By the assumptions on \( V \) and \( p = p_i \geq p_j \), we have

\[
V(x) \geq -\varepsilon + h(x_i) - C \sum_{j \neq i} \frac{1}{|x - z_j|^p} - C_\varepsilon, \quad \forall \varepsilon > 0.
\]

Therefore, using \( u_a(x) = \ell_a \varphi_a(\ell_a(x - z_i)) \) we get

\[
\int V|u_a|^2 = \int V(\ell_a^{-1}x + z_i)|\varphi_a(x)|^2 \, dx \\
\geq -\varepsilon \ell_a^p(\varepsilon + h(x_i)) \int \frac{|\varphi_a(x)|^2}{|x|^p} \, dx + \sum_{j \neq i} \int \frac{|\varphi_a(x)|^2}{|\ell_a^{-1}x + z_i - z_j|^p} \, dx - C_\varepsilon.
\]

Using \( \varphi_a(x) \to bQ_0(b(x - x_\infty)) \) in \( H^1(\mathbb{R}^2) \) and \( Q_0 \) decays exponentially, we have

\[
\int \frac{|\varphi_a(x)|^2}{|x|^p} \, dx = b_p \int \frac{|Q_0(x - bx_\infty)|^2}{|x|^p} \, dx + o(1)
\]

and

\[
\int \frac{|\varphi_a(x)|^2}{|\ell_a^{-1}x + z|^p} \, dx = o(\ell_a^p), \quad \forall z \neq 0.
\]

Thus

\[
\int V|u_a|^2 \geq -\varepsilon \ell_a^p(\varepsilon + h(x_i)) \int \frac{|Q_0(x - bx_\infty)|^2}{|x|^p} \, dx + o(\ell_a^p) - C_\varepsilon, \quad \forall \varepsilon > 0. \tag{52}
\]

Moreover, using again \( u_a(x) = \ell_a \varphi_a(\ell_a(x - z_i)) \) and \( \varphi_a(x) \to bQ_0(b(x - x_\infty)) \) in \( H^1(\mathbb{R}^2) \), we have

\[
\int |
abla u_a|^2 = \ell_a^2 b^2 \int |\nabla Q_0|^2 + o(\ell_a^2) = \ell_a^2 b^2 + o(\ell_a^2) + o(1) \tag{53}
\]

and

\[
\int \int \frac{|u_a(x)|^2|u_a(y)|^2}{|x - y|} \, dx \, dy = \ell_a b \int \int \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} \, dx \, dy + o(\ell_a). \tag{54}
\]

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In summary, from the Gagliardo-Nirenberg inequality (3) and (52), (53), (54) we have
\[ E(a(u_a)) \geq (1 - \frac{a}{a^*}) \int |\nabla u_a|^2 + \int V|u_a|^2 - \int \frac{|u_a(x)|^2|u_a(y)|^2}{|x - y|} \, dx \, dy \]
= \frac{\ell_p^a b^2}{a^*} - \ell_a b (\varepsilon + h(x_i)) \int \frac{|Q_0(x - bx)\,^p|}{|x|^p} \, dx
- \ell_a b \int \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} \, dx \, dy + o(\ell_p) - C_\varepsilon.

Here we have used \( p \geq 1 \), so that \( \ell_p \geq \ell_a \). Taking the limit \( \ell_a \to \infty \), then taking \( \varepsilon \to 0 \) and using the assumption \( E(u_a)/E(a) \to 1 \) we obtain
\[ \liminf_{a \to a^*} E(a)\ell_p^{-p} \geq \frac{b^2}{a^*} - b^p B \] (55)
where
\[ B = \begin{cases} h(x_i) \int_{\mathbb{R}^2} \frac{|Q_0(x - bx)\,^p|}{|x|^p} \, dx + \iint \frac{|Q_0(x)|^2|Q_0(y)|^2}{|x - y|} \, dx \, dy, & \text{if } p = 1 \\ h(x_i) \int_{\mathbb{R}^2} \frac{|Q_0(x - bx)\,^p|}{|x|^p} \, dx, & \text{if } p > 1 \end{cases} \] (56)

Finally, using \( h(x_i) \leq h_0 \) and the rearrangement inequality, we have
\[ h(x_i) \int_{\mathbb{R}^2} \frac{|Q_0(x - bx)\,^p|}{|x|^p} \, dx \leq h_0 \int_{\mathbb{R}^2} \frac{|Q_0(x)\,^p|}{|x|^p} \, dx \]
where the equality occurs if and only if \( h(x_i) = h_0 \) and \( x_\infty = 0 \) (here \( Q_0 \) is symmetric decreasing and \( |x|^{-p} \) is strictly symmetric decreasing). Thus \( B \leq A \) and hence (55) implies that
\[ \liminf_{a \to a^*} E(a)\ell_p^{-p} \geq \frac{b^2}{a^*} - b^p A. \]
Comparing the latter estimate with the upper bound in Lemma 11, we conclude that \( b = \beta \) and \( A = B \), i.e. \( h(x_i) = h_0 \) and \( x_\infty = 0 \). Thus \( x_i \in Z \) and \( u(x) \to \beta Q_0(\beta x) \) strongly in \( H^1(\mathbb{R}^2) \)
which is equivalent to (13). If \( Z \) has a unique element, then we obtain the convergence for the whole sequence \( u_a \) by a standard argument.

The proof is complete. \( \Box \)

A Appendix

A.1 Concentration-compactness lemma

For the reader’s convenience, we recall the following fundamental result of Lions [16, 17].
Lemma 13 (Concentration-compactness). Let $N \geq 1$. Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ with $\|u_n\|_{L^2} = 1$. Then there exists a subsequence (still denoted by $\{u_n\}$ for simplicity) such that one of the following cases occurs:

(i) (Compactness) There exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + z_n)$ converges to a function $u_0$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*)$.

(ii) (Vanishing) $u_n \to 0$ strongly in $L^p$ for all $p \in (2, 2^*)$.

(iii) (Dichotomy) There exist $\lambda \in (0, 1)$ and two sequences $\{u_n^{(1)}\}, \{u_n^{(2)}\}$ in $H^1(\mathbb{R}^N)$ such that

\[
\begin{cases}
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n^{(1)}|^2 = \lambda, & \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n^{(2)}|^2 = 1 - \lambda, \\
\lim_{n \to \infty} \text{dist}(\text{supp}(u_n^{(1)}), \text{supp}(u_n^{(2)})) = +\infty; \\
\lim_{n \to \infty} \|u_n - u_n^{(1)} - u_n^{(2)}\|_{L^p} = 0, & \forall p \in [2, 2^*); \\
\liminf_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u_n^{(1)}|^2 - |\nabla u_n^{(2)}|^2) \geq 0.
\end{cases}
\]

Here $2^*$ is the critical power in Sobolev’s embedding, i.e. $2^* = 2N/(N - 2)$ if $N \geq 3$ and $2^* = +\infty$ if $N \leq 2$.

Proof. The result is essentially taken from [16, Lemma III.1], with some minor modifications that we will explain.

First, the original notion of the compactness case in [16, Lemma I.1] reads

\[
\lim_{R \to \infty} \int_{|x| \leq R} |u_n(x + x_n)|^2 dx = 1. \tag{57}
\]

However, since $u_n$ is bounded in $H^1(\mathbb{R}^N)$, the statement (i) follows from (57) easily.

Next, the original notion of the vanishing case in [16, Lemma I.1] reads

\[
\lim_{R \to \infty} \sup_{y \in \mathbb{R}^N} \int_{|x| \leq R} |u_n(x + y)|^2 dx = 0.
\]

This and the boundedness in $H^1$ implies that $u_n \to 0$ strongly in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*)$, as explained in [17, Lemma I.1].

Finally, in the dichotomy case, (iii) follows from [16, Lemma III.1] (the original statement has a parameter $\varepsilon \to 0$) and a standard Cantor’s diagonal argument. 

\[\square\]

A.2 Optimizing sequences of Gagliardo-Nirenberg inequality

In this section we recall

Lemma 14 (Compactness of optimizing sequences for Gagliardo-Nirenberg inequality). If $f_n \geq 0$ is a bounded sequence in $H^1(\mathbb{R}^2)$, $\|f_n\|_{L^2} = 1$ and

\[
\int |\nabla f_n|^2 - \frac{a^*}{2} \int |f_n|^4 \to 0, \quad \liminf \|\nabla f_n\|_{L^2} > 0,
\]

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then up to subsequences and translations, \( f_n \) converges strongly in \( H^1(\mathbb{R}^2) \) to \( bQ_0(bx) \) for some constant \( b > 0 \).

This is a standard result, see e.g. [2] and the references therein. Since the result can be proved easily by the Concentration-Compactness Lemma 13, let us provide it below for the reader’s convenience.

**Proof.** We apply to the sequence \( f_n \).

**No-vanishing.** If the vanishing case occurs, then \( \|f_n\|_{L^4} \to 0 \), but this contradicts to the assumption in Lemma 14.

**No-dichotomy.** Assume that the dichotomy case occurs. Then we can find two sequences \( f_{n}^{(1)}, f_{n}^{(2)} \) such that

\[
\int |f_{n}^{(1)}|^2 \to \lambda \in (0, 1), \quad \int |f_{n}^{(2)}|^2 \to 1 - \lambda,
\]

\[
\int |\nabla f_n|^2 \geq \int |\nabla f_{n}^{(1)}|^2 + \int |\nabla f_{n}^{(2)}|^2 + o(1)
\]

and

\[
\int |f_n|^4 = \int |f_{n}^{(1)}|^4 + \int |\nabla f_{n}^{(2)}|^4 + o(1).
\]

On the other hand, using the Gagliardo-Nirenberg inequality (3) for \( f_{n}^{(j)}/\|f_{n}^{(j)}\|_{L^2} \) we obtain

\[
\int |\nabla f_{n}^{(1)}|^2 \geq \frac{a^*}{2\lambda} \int |f_{n}^{(1)}|^4, \quad \int |\nabla f_{n}^{(2)}|^2 \geq \frac{a^*}{2(1 - \lambda)} \int |f_{n}^{(2)}|^4.
\]

Thus

\[
\int |\nabla f_n|^2 \geq \int |\nabla f_{n}^{(1)}|^2 + \int |\nabla f_{n}^{(2)}|^2 + o(1)
\]

\[
\geq \frac{a^*}{2\lambda} \int |f_{n}^{(1)}|^4 + \frac{a^*}{2(1 - \lambda)} \int |f_{n}^{(2)}|^4 + o(1)
\]

\[
\geq \frac{a^*}{2} \max\left(\frac{1}{\lambda}, \frac{1}{1 - \lambda}\right) \left( \int |f_{n}^{(1)}|^4 + \int |\nabla f_{n}^{(2)}|^4 \right) + o(1)
\]

\[
= \frac{a^*}{2} \max\left(\frac{1}{\lambda}, \frac{1}{1 - \lambda}\right) \int |f_n|^4 + o(1).
\]

However, this again contradicts to the assumption in Lemma 14 since \( \lambda \in (0, 1) \). Thus the dichotomy case does not occur.

**Compactness.** Now up to subsequences and translations, \( f_n \) converges to a function \( f \) weakly in \( H^1(\mathbb{R}^2) \) and strongly in \( L^s(\mathbb{R}^2) \) for all \( s \in [2, \infty) \). Therefore, \( \|f\|_{L^2} = 1 \) and

\[
\int |\nabla f_n|^2 \geq \int |\nabla f|^2 + o(1), \quad \int |f_n|^4 \to \int |f|^4.
\]
On the other hand, by the assumption in Lemma 14, we have
\[ \int |\nabla f|^2 - \frac{a^*}{2} \int |f|^4 \leq \int |\nabla f_n|^2 - \frac{a^*}{2} \int |f_n|^4 \to 0. \]

Thus \( f \) is an optimizer for the Gagliardo-Nirenberg inequality (3). Moreover, we must have \( \int |\nabla f_n|^2 \to \int |\nabla f|^2 \), i.e. \( f_n \) converges to \( f \) strongly in \( H^1(\mathbb{R}^2) \).

Finally since \( Q_0 \) is the unique optimizer for (3) up to translations and dilations, we have \( f(x + z_0) = bQ_0(bx) \) for constants \( b > 0 \) and \( z_0 \in \mathbb{R}^2 \). Thus \( f_n(x + z_0) \to f(x + z_0) = bQ_0(bx) \) in \( H^1(\mathbb{R}^2) \). This ends the proof.

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