ON THE MODULI SPACE OF SEMI-STABLE PLANE SHEAVES WITH HILBERT POLYNOMIAL $P(m) = 6m + 2$

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Abstract. We study the Simpson moduli space of semi-stable sheaves on the complex projective plane that have dimension 1, multiplicity 6 and Euler characteristic 2. We describe concretely these sheaves as cokernels of morphisms of locally free sheaves and we stratify the moduli space according to the types of sheaves that occur.

CONTENTS

1. Introduction 1
2. The open stratum 2
3. The codimension 3 stratum 4
4. The codimension 5 stratum 8
5. The codimension 7 stratum 17
6. The codimension 9 stratum 20
7. The moduli space is the union of the strata 21
References 24

Acknowledgements. The author was supported by the Consiliul Național al Cercetării Științifice, an agency of the Romanian Government, grant PN II–RU 169/2010 PD–219.

1. Introduction

Let $M_{P^2}(r, \chi)$ denote the moduli space of Gieseker semi-stable sheaves on $\mathbb{P}^2(\mathbb{C})$ with Hilbert polynomial $P(m) = rm + \chi$, $r$ and $\chi$ being fixed integers, $r \geq 1$. Le Potier [6] found that $M_{P^2}(r, \chi)$ is an irreducible projective variety of dimension $r^2 + 1$, smooth at points given by stable sheaves and rational if $\chi \equiv 1$ or 2 mod $r$. In [2], [9] and [10] a complete description of semi-stable sheaves giving points in $M_{P^2}(4, \chi)$, $M_{P^2}(5, \chi)$ and $M_{P^2}(6,1)$ was found. These moduli spaces were shown to have natural stratifications given by cohomological conditions on the sheaves involved. Here we are concerned with $M_{P^2}(6, 2)$. We describe all semi-stable sheaves giving points in $M_{P^2}(6, 2)$ and we decompose this moduli space into five strata: an open stratum $X_0$; a locally closed stratum that is the disjoint union of two irreducible locally closed subsets $X_1$ and $X_2$, each of codimension 3; a locally closed stratum that is the disjoint union of two irreducible locally closed subsets $X_3$ and $X_4$, each of codimension 5; an irreducible locally closed stratum $X_5$ of codimension 7 and a closed irreducible stratum $X_6$ of codimension 9. For some of these
sets we have concrete geometric descriptions: $X_1$ is a certain open subset inside a fibre bundle with fibre $\mathbb{P}^2$ and base $N(3, 4, 3) \times \mathbb{P}^2$, where $N(3, 4, 3)$ is the moduli space of semi-stable Kronecker modules $f: 4\mathcal{O}(-2) \to 3\mathcal{O}(-1)$; $X_2$ is an open subset of a fibre bundle with fibre $\mathbb{P}^2$ and base $Y \times N(3, 2, 3)$, where $Y$ is the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2 and $N(3, 2, 3)$ is the moduli space of semi-stable Kronecker modules $f: 2\mathcal{O}(-1) \to 3\mathcal{O}$; $X_5$ is an open subset of a fibre bundle with fibre $\mathbb{P}^2$ and base $\mathbb{P}^2 \times Y$; the closed stratum $X_6$ is isomorphic to the universal sextic in $\mathbb{P}^2 \times \mathbb{P}(S^6V^*)$.

The following table contains a description of each $X_i$ by cohomological conditions. The third column of the table lists all sheaves giving points in $X_i$. The sets $W$ of morphisms $\varphi$ are acted upon by the algebraic groups of automorphisms of sheaves and in each case, apart from $X_6$, the geometric quotient is $X_i$. The points given by properly semi-stable sheaves are all in $X_6$, which is why this stratum cannot be a geometric quotient of the set of morphisms. The table below is organised as the table in the introduction to [10], to which we generally refer for notations and conventions.

Let $C \subset \mathbb{P}^2$ denote an arbitrary smooth sextic curve and let $P_i$ denote distinct points on $C$. The generic sheaves in $X_1$ are of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_6 - P_7)$, where $P_1, \ldots, P_6$ are not contained in a conic curve. The generic sheaves in $X_3$ have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5)$, where $P_1, P_2, P_3$ are non-colinear. The generic sheaves in $X_4$ are of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_5)$, where $P_1, \ldots, P_5$ are in general linear position. The generic sheaves in $X_5$ are of the form $\mathcal{O}_C(2)(P_1 - P_2 - P_3)$. The sheaves giving points in $X_6$ are of the form $\mathcal{O}_C(1)$, in this case $C$ need not be smooth.

2. The open stratum

**Proposition 2.1.** Every sheaf $\mathcal{F}$ giving a point in $M_{22}(6, 2)$ and satisfying the condition $h^1(\mathcal{F}) = 0$ also satisfies the condition $h^0(\mathcal{F}(-1)) = 0$. For these sheaves $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$ or 1. The sheaves from the first case are given by resolutions of the form

(i) \[ 0 \to 4\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O} \to \mathcal{F} \to 0, \]

where $\varphi$ is not equivalent, modulo the action of the natural group of automorphisms, to a morphism represented by a matrix of the form

\[
\begin{bmatrix}
\ast & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\ast & \ast & 0 & 0 \\
\ast & \ast & \ast & 0 \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\ast & \ast & \ast & 0 \\
\ast & \ast & \ast & 0 \\
\ast & \ast & \ast & 0 \\
\ast & \ast & \ast & \ast \\
\end{bmatrix}.
\]

The sheaves in the second case are precisely the sheaves with resolution of the form

(ii) \[ 0 \to 4\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus 2\mathcal{O} \to \mathcal{F} \to 0, \]

where $\varphi_{12} = 0$, $\varphi_{11}$ is semi-stable as a Kronecker $V$-module and $\varphi_{22}$ has linearly independent entries.

**Proof.** The first statement follows from 6.4 [7]. The rest of the proposition follows by duality from 4.3 op.cit. \[ \square \]
| $X_0$ | $h^0(\mathcal{F}(-1)) = 0$  
$h^1(\mathcal{F}) = 0$  
$h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$ | $0 \to 4\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 2\mathcal{O} \to \mathcal{F} \to 0$  
$\varphi$ is not equivalent to a morphism of any of the forms  
$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 & 0 \\ * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ |
|---|---|---|
| $X_1$ | $h^0(\mathcal{F}(-1)) = 0$  
$h^1(\mathcal{F}) = 0$  
$h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$ | $0 \to 4\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus 2\mathcal{O} \to \mathcal{F} \to 0$  
$\varphi_{12} = 0$, $\varphi_{11}$ and $\varphi_{22}$ are semi-stable as Kronecker modules |
| $X_2$ | $h^0(\mathcal{F}(-1)) = 0$  
$h^1(\mathcal{F}) = 1$  
$h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$ | $0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \to \mathcal{F} \to 0$  
$\varphi$ is not equivalent to a morphism of any of the forms  
$\begin{bmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & 0 \end{bmatrix}$ |
| $X_3$ | $h^0(\mathcal{F}(-1)) = 0$  
$h^1(\mathcal{F}) = 1$  
$h^0(\mathcal{F} \otimes \Omega^1(1)) = 2$ | $0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \to \mathcal{F} \to 0$  
$\varphi_{13} = 0$, $\varphi_{12} \neq 0$ and does not divide $\varphi_{11}$  
$\varphi_{23}$ has linearly independent maximal minors |
| $X_4$ | $h^0(\mathcal{F}(-1)) = 1$  
$h^1(\mathcal{F}) = 1$  
$h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$ | $0 \to \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \to \mathcal{F} \to 0$  
$\varphi$ is not equivalent to a morphism of any of the forms  
$\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$ |
| $X_5$ | $h^0(\mathcal{F}(-1)) = 1$  
$h^1(\mathcal{F}) = 2$  
$h^0(\mathcal{F} \otimes \Omega^1(1)) = 4$ | $0 \to 2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1) \to \mathcal{F} \to 0$  
$\varphi_{11}$ has linearly independent entries  
$\varphi_{22} \neq 0$ and does not divide $\varphi_{32}$ |
| $X_6$ | $h^0(\mathcal{F}(-1)) = 2$  
$h^1(\mathcal{F}) = 3$  
$h^0(\mathcal{F} \otimes \Omega^1(1)) = 6$ | $0 \to \mathcal{O}(-4) \oplus \mathcal{O} \to 2\mathcal{O}(1) \to \mathcal{F} \to 0$  
$\varphi_{12}$ has linearly independent entries |
Let $\mathbb{W}_0 = \text{Hom}(4\mathcal{O}(-2), 2\mathcal{O}(-1) \oplus 2\mathcal{O})$ and let $W_0 \subset \mathbb{W}_0$ be the set of morphisms $\varphi$ from 2.1(i). Let

$$G_0 = (\text{Aut}(4\mathcal{O}(-2)) \times \text{Aut}(2\mathcal{O}(-1) \oplus 2\mathcal{O}))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_0$. Let $X_0 \subset M_{\mathbb{P}^2}(6, 2)$ be the set of stable-equivalence classes of sheaves $\mathcal{F}$ as in 2.1(i). This set is open and dense.

**Proposition 2.2.** There exists a categorical quotient of $W_0$ by $G_0$ and it is isomorphic to $X_0$.

**Proof.** We have a canonical morphism $\rho: W_0 \rightarrow X_0$ mapping $\varphi$ to the stable-equivalence class of $\text{Coker}(\varphi)$. As at 4.2.1 [2], $\rho(\varphi_1) = \rho(\varphi_2)$ if and only if $G\varphi_1 \cap G\varphi_2 \neq \emptyset$. Thus any $G_0$-invariant morphism of varieties $f: W_0 \rightarrow Y$ factors through a unique map $g: X_0 \rightarrow Y$. To show that $\rho$ is a categorical quotient map we use the method of 3.1.6 [2]. For any sheaf $\mathcal{F}$ giving a point in $X_0$ we need to obtain resolution 2.1(i) in a natural manner from the Beilinson spectral sequence converging to $\mathcal{F}$. We prefer, instead, to work with the Beilinson sequence of the dual sheaf $\mathcal{G} = \mathcal{F}^0(1)$, which gives a point in $M_{\mathbb{P}^2}(6, 4)$.

Diagram (2.2.3) [2] takes the form

$$
\begin{array}{ccc}
2\mathcal{O}(-2) & 0 & 0 \\
0 & 2\mathcal{O}(-1) & \varphi_4 \rightarrow 4\mathcal{O}
\end{array}
$$

The exact sequence (2.2.5) [2] takes the form

$$0 \rightarrow 2\mathcal{O}(-2) \xrightarrow{\varphi_3} \text{Coker}(\varphi_4) \rightarrow \mathcal{G} \rightarrow 0.$$

According to (2.2.4) [2], $\varphi_4$ is injective. We now easily get the exact sequence dual to 2.1(i):

$$0 \rightarrow 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \rightarrow 4\mathcal{O} \rightarrow \mathcal{G} \rightarrow 0. \quad \Box$$

**Proposition 2.3.** If $\mathcal{F}$ is properly semi-stable and $P_\mathcal{F}(t) = 6t + 2$, then $\mathcal{F}$ gives a point in $X_0$.

3. THE CODIMENSION 3 STRATUM

Let $\mathbb{W}_1 = \text{Hom}(4\mathcal{O}(-2) \oplus \mathcal{O}(-1), 3\mathcal{O}(-1) \oplus 3\mathcal{O})$ and let $W_1 \subset \mathbb{W}_1$ be the set of morphisms $\varphi$ from 2.1(ii). Let

$$G_1 = (\text{Aut}(4\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \text{Aut}(3\mathcal{O}(-1) \oplus 2\mathcal{O}))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_1$. Let $X_1 \subset M_{\mathbb{P}^2}(6, 2)$ be the set of stable-equivalence classes of sheaves $\mathcal{F}$ as in 2.1(ii).

**Proposition 3.1.** There exists a geometric quotient $W_1/G_1$ and it is a proper open subset inside a fibre bundle over $N(3, 4, 3) \times \mathbb{P}^2$ with fibre $\mathbb{P}^{20}$. Moreover, $W_1/G_1$ is isomorphic to $X_1$. In particular, $X_1$ is irreducible and has codimension 3.
Proof. The first statement can be proved identically as 2.2.2 [9]. Let \( W'_1 \) be the locally closed subset of \( W_1 \) given by the following conditions: \( \varphi_{12} = 0, \varphi_{11} \) is semi-stable as a Kronecker \( V \)-module, \( \varphi_{22} \) has linearly independent entries. Let \( \Sigma \subset W'_1 \) be the \( G_1 \)-invariant subset given by the condition
\[
\varphi_{21} = \varphi_{22} u + v \varphi_{11}, \quad u \in \text{Hom}(4\mathcal{O}(-2), \mathcal{O}(-1)), \quad v \in \text{Hom}(3\mathcal{O}(-1), 2\mathcal{O}).
\]
As at loc.cit., we can construct a vector bundle \( Q \) over \( N(3, 4, 3) \times \mathbb{P}^2 \) of rank 21 such that \( P(Q) \) is a geometric quotient of \( W'_1 \setminus \Sigma \) modulo \( G_1 \). Then \( W_1/G_1 \) is a proper open subset of \( P(Q) \).

Let \( F \) give a point in \( X_1 \) and let \( G = \mathcal{F}^0(1) \). The Beilinson tableau (2.2.3) [2] for \( G \) takes the form
\[
\begin{array}{c}
2\mathcal{O}(-2) \xrightarrow{\varphi_1} \mathcal{O}(-1) \rightarrow 0
\end{array}
\]
\[
\begin{array}{c}
0 \rightarrow 3\mathcal{O}(-1) \xrightarrow{\varphi_4} 4\mathcal{O}
\end{array}
\]
As in the proof of 2.2.3 [9], we have \( \text{Ker}(\varphi_1) \simeq \mathcal{O}(-3) \) and an exact sequence
\[
0 \rightarrow \mathcal{O}(-3) \oplus 3\mathcal{O}(-1) \rightarrow 4\mathcal{O} \rightarrow G \rightarrow \text{Coker}(\varphi_1) \rightarrow 0,
\]
yielding the resolution
\[
0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus 3\mathcal{O}(-1) \rightarrow \mathcal{O}(-1) \oplus 4\mathcal{O} \rightarrow G \rightarrow 0.
\]
Since \( H^1(G) = 0 \), we see that \( \mathcal{O}(-3) \) can be canceled yielding the dual of resolution 2.1(ii). \( \square \)

**Proposition 3.2.** The sheaves \( G \) from \( X^0_1 \) are precisely the non-split extension sheaves of the form
\[
0 \rightarrow \mathcal{E} \rightarrow G \rightarrow \mathbb{C}_x \rightarrow 0,
\]
where \( \mathbb{C}_x \) is the structure sheaf of a point \( x \in \mathbb{P}^2 \), \( \mathcal{E} \) gives a point in \( \text{M}_{\mathbb{P}^2}(6, 3) \) and satisfies the conditions \( h^0(\mathcal{E}(-1)) = 0, h^1(\mathcal{E}) = 1 \).

The generic sheaves \( G \) from \( X^0_1 \) are of the form \( \mathcal{O}_C(3)(-P_1 - \cdots - P_6 + P_7) \), where \( P_i \) are seven distinct points on a smooth sextic curve \( C \subset \mathbb{P}^2 \) and \( P_1, \ldots, P_6 \) are not contained in a conic curve.

By duality, the generic sheaves in \( X_1 \) are of the form \( \mathcal{O}_C(1)(P_1 + \cdots + P_6 - P_7) \).

**Proof.** Assume that \( G \) gives a point in \( X^0_1 \), i.e. \( G \simeq \text{Coker}(\varphi^\tau) \) for some morphism \( \varphi \) as in 2.1(ii). From the snake lemma we get an extension
\[
0 \rightarrow \mathcal{E} \rightarrow G \rightarrow \mathbb{C}_x \rightarrow 0,
\]
where \( x \) is the common zero of the entries of \( \varphi_{22} \) and \( \mathcal{E} \) has a resolution
\[
0 \rightarrow \mathcal{O}(-3) \oplus 3\mathcal{O}(-1) \xrightarrow{\psi} 4\mathcal{O} \rightarrow \mathcal{E} \rightarrow 0,
\]
\( \psi_{12} = \varphi_{11}^\tau \). From 5.3 [7] we know that \( \mathcal{E} \) gives a point in \( \text{M}_{\mathbb{P}^2}(6, 3) \) and satisfies the cohomological conditions from the proposition. Conversely, any such sheaf \( \mathcal{E} \) is the cokernel of an injective morphism \( \psi \) for which \( \psi_{12} \) is semi-stable as a Kronecker \( V \)-module. Given a non-split extension of \( \mathbb{C}_x \) by \( \mathcal{E} \), we apply the horseshoe lemma to the above resolution.
of $\mathcal{E}$ and to the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(-1)$. The map $\mathcal{O}(-1) \to \mathbb{C}_x$ lifts to a map $\mathcal{O}(-1) \to \mathcal{G}$ because $H^1(\mathcal{E}(1)) = 0$. We obtain a resolution

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus 3\mathcal{O}(-1) \to \mathcal{O}(-1) \oplus 4\mathcal{O} \to \mathcal{G} \to 0.$$  

Since $\text{Ext}^1(\mathcal{C}_x, 4\mathcal{O}) = 0$, we can deduce, as in the proof of 2.3.2 [9], that the morphism $\mathcal{O}(-3) \to \mathcal{O}(-3)$ is non-zero. We cancel $\mathcal{O}(-3)$ to get the dual to resolution 2.1(ii).

Let $X_{10} \subset X_1$ be the open subset of points given by sheaves $\mathcal{F} = \text{Coker}(\varphi)$ for which the maximal minors of $\varphi_{11}$ have no common factor. Let $X_{10}^D \subset M_{\mathbb{P}^2}(6, 4)$ be the dual subset. According to [1], propositions 4.5 and 4.6, the sheaves $\text{Coker}(\psi_{12})$, where the maximal minors of $\psi_{12}$ have no common factor, are precisely the sheaves with resolution of the form $\mathcal{I}_Z(3)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 6 not contained in a conic curve. It follows that the sheaves $\mathcal{G}$ giving points in $X_{10}$ are precisely the non-split extensions of $\mathbb{C}_x$ by $\mathcal{I}_Z(3)$, where $\mathcal{I}_Z \subset \mathcal{O}_C$ is the ideal sheaf of a subscheme $Z$ as above contained in a sextic curve $C$. Take $C$ to be smooth and take $Z$ to be the union of six distinct points different from $x$. Then $\mathcal{G} \simeq \mathcal{O}_C(3)(-P_1 - \cdots - P_6 + x).$  

\textbf{Proposition 3.3.} Let $\mathcal{F}$ be a sheaf giving a point in $M_{\mathbb{P}^2}(6, 2)$ and satisfying the conditions $h^0(\mathcal{F}(-1)) = 0$, $h^1(\mathcal{F}) = 1$. Then $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$ or 2. The sheaves in the first case are precisely the sheaves with resolution of the form

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \to \mathcal{F} \to 0,$$

where $\varphi$ is not equivalent to a morphism of any of the following forms:

$$\varphi_1 = \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & 0 & 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & 0 & 0 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & 0 & 0 \end{bmatrix}.$$

\textit{Proof.} Let $\mathcal{F}$ give a point in $M_{\mathbb{P}^2}(6, 2)$ and satisfy the cohomological conditions from the hypothesis. Write $m = h^0(\mathcal{F} \otimes \Omega^1(1))$. As in the proof of 2.1.4 [9], the Beilinson free monad for $\mathcal{F}$ leads to a resolution

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus m\mathcal{O}(-1) \to (m - 1)\mathcal{O}(-1) \oplus 3\mathcal{O} \to \mathcal{F} \to 0$$

in which $\varphi_{13} = 0$. As $\mathcal{F}$ maps surjectively onto $\text{Coker}(\varphi_{11}, \varphi_{12})$, we have $m \leq 3$. If $m = 3$, then $\text{Coker}(\varphi_{11}, \varphi_{12})$ has slope $-1/3$, so the semi-stability of $\mathcal{F}$ gets contradicted. Thus $m = 1$ or 2. Assume for the rest of this proof that $m = 1$. We have a resolution

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \to \mathcal{F} \to 0.$$  

The conditions imposed on $\varphi$ follow from the semi-stability of $\mathcal{F}$. Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we need to show that there are no destabilising subsheaves. Assume that $\mathcal{E} \subset \mathcal{F}$ is a destabilising subsheaf. We may take $\mathcal{E}$ to be semi-stable. As $\mathcal{F}$ is generated by global sections, we have $h^0(\mathcal{E}) < h^0(\mathcal{F})$. Thus $\mathcal{E}$ gives a point in $M_{\mathbb{P}^2}(r, 1)$ or $M_{\mathbb{P}^2}(r, 2)$ for some $r$, $1 \leq r \leq 5$. According to 2.3 the situation in which $P_\mathcal{E}(t) = 3t + 1$ is unfeasible. Moreover, we have $h^0(\mathcal{E}(-1)) = 0$, $h^0(\mathcal{E} \otimes \Omega^1(1)) \leq 1$. From the results in [2] and [9] we see that $\mathcal{E}$ may have one of the following resolutions:

$$(1) \quad 0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{E} \to 0,$$
As in the proof of 2.2.4 \cite{9} we may assume that $\varphi$ in which $\alpha$ is injective on global sections. For the first four resolutions $\alpha$ must be injective and we get the contradictory conclusions that $\varphi \sim \varphi_1$, $\varphi \sim \varphi_2$ or $\varphi \sim \varphi_3$. If $E$ has resolution (5), then $\beta$ cannot be injective, hence $\alpha$ is not injective, hence $Ker(\alpha) \simeq Ker(\beta) \simeq O(-1)$ and we conclude, as in the case of resolution (4), that $\varphi \sim \varphi_3$. If $E$ has resolution (6), then, again, $Ker(\alpha) \simeq O(-1) \simeq Ker(\beta)$, which is absurd, because $O(-1)$ cannot be isomorphic to a subsheaf of $3O(-2)$. For resolution (7) we arrive at a contradiction in a similar manner. 

Let $W_2 = Hom(O(-3) \oplus O(-2) \oplus O(-1), 3O)$ and let $W_2 \subset W_2$ be the set of morphisms $\varphi$ from proposition \[3.3\]. Let 

\[ G_2 = (Aut(O(-3) \oplus O(-2) \oplus O(-1)) \times Aut(3O))/\mathbb{C}^* \]

be the natural group acting by conjugation on $W_2$. Let $X_2 \subset M_{g2}(6,2)$ be the set of stable-equivalence classes of sheaves of the form $Coker(\varphi)$, $\varphi \in W_2$.

**Proposition 3.4.** There exists a geometric quotient $W_2/G_2$, which is isomorphic to $X_2$. In particular, $X_2$ is irreducible and has codimension 3.

**Proof.** Diagram (2.2.3) \[2\] for a sheaf $F$ giving a point in $X_2$ takes the form

\[
\begin{array}{c}
4O(-2) \xrightarrow{\varphi_1} 3O(-1) \xrightarrow{\varphi_2} O \\
0 \quad O(-1) \xrightarrow{\varphi_4} 3O
\end{array}
\]

As in the proof of 2.2.4 \[9\], we may assume that $\varphi_1$ and $\varphi_2$ are given by

\[
\varphi_1 = \begin{bmatrix}
-Y & -Z & 0 & 0 \\
X & 0 & -Z & 0 \\
0 & X & Y & 0
\end{bmatrix}, 
\quad \varphi_2 = \begin{bmatrix}
X & Y & Z
\end{bmatrix}.
\]

Thus $Ker(\varphi_1) \simeq O(-3) \oplus O(-2)$ and $Im(\varphi_1) = Ker(\varphi_2)$. The exact sequence (2.2.5) \[2\] takes the form

\[
0 \rightarrow O(-3) \oplus O(-2) \xrightarrow{\varphi_5} Coker(\varphi_4) \rightarrow F \rightarrow 0.
\]

By 2.2 \[2\], $\varphi_4$ is injective. Clearly $\varphi_5$ lifts to a morphism $\varphi'_5: O(-3) \oplus O(-2) \rightarrow 3O$. We obtain the resolution

\[
0 \rightarrow O(-3) \oplus O(-2) \oplus O(-1) \xrightarrow{\varphi} 3O \rightarrow F \rightarrow 0,
\]

Each of these resolutions must fit into a commutative diagram like diagram (*) at 3.1 \[10\] in which $\alpha$ is injective on global sections. For the first four resolutions $\alpha$ must be injective and we get the contradictory conclusions that $\varphi \sim \varphi_1$, $\varphi \sim \varphi_2$ or $\varphi \sim \varphi_3$.
\[ \varphi = \begin{bmatrix} \varphi_5 & \varphi_4 \end{bmatrix}. \]

This proves that the map \( W_2 \to X_2 \) is a categorical quotient. According to [11], remark (2), p. 5, \( X_2 \) is normal. Applying [12], theorem 4.2, we conclude that the map \( W_2 \to X_2 \) is a geometric quotient. \( \square \)

4. The codimension 5 stratum

**Proposition 4.1.** The sheaves \( F \) giving points in \( \mathcal{M}_{\mathbb{P}^2}(6, 2) \) and satisfying the cohomological conditions

\[ h^0(F(-1)) = 0, \quad h^1(F) = 1, \quad h^0(F \otimes \Omega^1(1)) = 2 \]

are precisely the sheaves with resolution of the form

\[ 0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \to F \to 0, \]

where \( \varphi_{12} \neq 0, \varphi_{13} = 0, \varphi_{11} \) is not divisible by \( \varphi_{12} \) and \( \varphi_{23} \) has linearly independent maximal minors.

**Proof.** At 3.3 we proved that a sheaf \( F \) giving a point in \( \mathcal{M}_{\mathbb{P}^2}(6, 2) \) and satisfying the above cohomological conditions has a resolution as in the proposition. The conditions imposed on \( \varphi \) follow from the semi-stability of \( F \).

Conversely, assume that \( F \) has a resolution as in the proposition. Assume that there is a destabilising subsheaf \( E \subset F \). We may assume that \( E \) is semi-stable. From the snake lemma we obtain an extension

\[ 0 \to \mathcal{F}' \to F \to \mathcal{O}_Z \to 0, \]

where \( Z \) is the zero-dimensional scheme of length 2 given by the ideal \((\varphi_{11}, \varphi_{12})\) and \( \mathcal{F}' \) has a resolution

\[ 0 \to \mathcal{O}(-4) \oplus 2\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \to \mathcal{F}' \to 0 \]

in which \( \psi_{12} = \varphi_{23} \). According to 5.2 [10], \( \mathcal{F}' \) gives a point in \( \mathcal{M}_{\mathbb{P}^2}(6, 0) \) and the only subsheaf of \( \mathcal{F}' \) of slope zero, if there is one, must be of the form \( \mathcal{O}_L(-1) \) for a certain line \( L \subset \mathbb{P}^2 \). It follows that \( \mathcal{E} \) must have Hilbert polynomial \( P_E(t) = 2t + 1, t + 2 \) or \( t + 1 \). If \( P_E(t) = 2t + 1 \), then \( \mathcal{E} \) is the structure sheaf of some conic curve \( C \subset \mathbb{P}^2 \). We obtain a commutative diagram with exact rows and injective vertical maps

\[ \begin{array}{ccccccc}
0 & & \to & \mathcal{O}(-2) & & \to & \mathcal{O} & & \to & \mathcal{O}_C & & \to & 0 . \\
& & \downarrow{\beta} & & & & \downarrow{\alpha} & & & & & & \\
0 & & \to & \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) & & \to & \mathcal{O}(-1) \oplus 3\mathcal{O} & & \to & \mathcal{F} & & \to & 0
\end{array} \]

Taking into account the possible canonical forms for \( \beta \), we see that \( \varphi \) is represented by a matrix having one of the following forms:

\[
\begin{bmatrix}
\star & 0 & 0 & 0 \\
\star & 0 & * & * \\
\star & 0 & * & * \\
\star & * & * & *
\end{bmatrix},
\begin{bmatrix}
\star & \star & 0 & 0 \\
\star & \star & * & 0 \\
\star & * & * & 0 \\
\star & * & * & * \\
\star & * & * & *
\end{bmatrix},
\begin{bmatrix}
\star & \star & 0 & 0 \\
\star & \star & * & 0 \\
\star & * & * & 0 \\
\star & * & * & *
\end{bmatrix}.
\]
In each of these situations the hypothesis on $\varphi$ gets contradicted. If $P_\mathcal{E}(t) = t + 1$, then $\mathcal{E}$ is the structure sheaf of some line $L \subset \mathbb{P}^2$ and we obtain a contradiction as above. The case in which $P_\mathcal{E}(t) = t + 2$ is not feasible because in this case $\mathcal{E} \simeq \mathcal{O}_L(1)$, yet $H^0(\mathcal{E}(-1))$ must vanish because the corresponding group for $\mathcal{F}$ vanishes.

Let $W_3 = \text{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1), \mathcal{O}(-1) \oplus 3\mathcal{O})$ and let $W_3 \subset W_3$ be the set of morphisms $\varphi$ from proposition $4.1$. Let

$$G_3 = \left( \text{Aut}(\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1)) \times \text{Aut}(\mathcal{O}(-1) \oplus 3\mathcal{O}) \right) / \mathbb{C}^*$$

be the natural group acting by conjugation on $W_3$. Let $X_3 \subset M_{\mathbb{P}^2}(6, 2)$ be the set of stable-equivalence classes of sheaves of the form $\text{Coker}(\varphi), \varphi \in W_3$.

**Proposition 4.2.** The generic sheaves in $X_3$ have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5)$, where $C \subset \mathbb{P}^2$ is a smooth sextic curve, $P_i$ are five distinct points on $C$ and $P_1, P_2, P_3$ are non-collinear. In particular, $X_3$ lies in the closure of $X_1$. Moreover, $X_3$ also lies in the closure of $X_2$.

**Proof.** Let $X_{30} \subset X_3$ be the open subset given by the following conditions: the equation $\det(\varphi) = 0$ determines a smooth sextic curve $C \subset \mathbb{P}^2$, the scheme $Z$ from $4.1$ consists of two distinct points $P_1, P_3$, the maximal minors of $\varphi_{23}$ have no common factor and the subscheme $Y \subset \mathbb{P}^2$ they determine consists of three distinct points $P_1, P_2, P_3$, which are also distinct from $P_4$ and $P_5$. Let $\mathcal{F}$ give a point in $X_{30}$. According to $5.2$ [10], the sheaf $\mathcal{F}'$ from $4.1$ is isomorphic to $\mathcal{O}_C(2)(-P_1 - P_2 - P_3)$, hence $\mathcal{F}$ is isomorphic to $\mathcal{O}_C(-P_1 - P_2 - P_3 + P_4 + P_5)$. Conversely, we must show that any such sheaf $\mathcal{F}$ gives a point in $X_{30}$. We claim that $\mathcal{F}(1)$ has a global section that does not vanish at $P_4$ or $P_5$. The argument can be found at $2.3.2$ [3] and it will be reproduced here for the sake of completeness. Let $\varepsilon_i : H^0(\mathcal{O}_Z) \to \mathbb{C}$ be the linear form of evaluation at $P_i, i = 4, 5$. Let $\delta : H^0(\mathcal{O}_Z) \to H^1(\mathcal{O}_C(3)(-Y))$ be the connecting homomorphism arising from the exact sequence

$$0 \to \mathcal{O}_C(3)(-Y) \to \mathcal{F}(1) \to \mathcal{O}_Z \to 0.$$

We must show that each $\varepsilon_i$ is not orthogonal to $\ker(\delta)$ or, which is the same, that each $\varepsilon_i$ is not in the image of the dual map $\delta^*$. By Serre duality $\delta^*$ is the restriction morphism

$$H^0(\mathcal{O}_C(Y)) = H^0(\mathcal{O}_C(-3)(Y) \otimes \omega_C) \to H^0((\mathcal{O}_C(-3)(Y) \otimes \omega_C)|_Z) = H^0(\mathcal{O}_C(Y)|_Z).$$

We have the identity $H^0(\mathcal{O}_C(Y)) = H^0(\mathcal{O}_C) = \mathbb{C}$. This follows from the fact that the connecting homomorphism associated to the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(Y) \to \mathcal{O}_Y \to 0$$

is injective. By Serre duality, this is equivalent to saying that the restriction morphism

$$H^0(\mathcal{O}_C(3)) = H^0(\mathcal{O}_C \otimes \omega_C) \to H^0((\mathcal{O}_C \otimes \omega_C)|_Y) = H^0(\mathcal{O}_C(3)|_Y)$$

is surjective, and this is obvious. The claim now easily follows. We may now apply the horseshoe lemma to the extension

$$0 \to \mathcal{O}_C(2)(-Y) \to \mathcal{F} \to \mathcal{O}_Z \to 0.$$
and to the resolutions

\[0 \to \mathcal{O}(-4) \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \to \mathcal{O}(-1) \to \mathcal{O}_Z \to 0,\]

\[0 \to \mathcal{O}(-4) \to \mathcal{I}_Z(2) \to \mathcal{O}_C(2)(-Y) \to 0.\]

Here \(\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^2}\) is the ideal sheaf of \(Y\). We obtain the resolution

\[0 \to \mathcal{I}_Y(2) \to \mathcal{O}(2) \to \mathcal{O}_Y \to 0.\]

As in the proof of 2.3.2 \([9]\), we can show that the morphism \(\mathcal{O}(-4) \to \mathcal{O}(-4)\) above is non-zero. The argument uses the fact that Ext\(^1\)(\(\mathcal{O}_Z, \mathcal{I}_Y(2)\)) = 0. The vanishing of this group follows from the vanishing of Hom(\(\mathcal{O}_Z, \mathcal{O}_Y\)) and of Ext\(^1\)(\(\mathcal{O}_Z, \mathcal{O}(2)\)), in view of the long Ext-sequence associated to the exact sequence

\[0 \to \mathcal{I}_Y(2) \to \mathcal{O}(2) \to \mathcal{O}_Y \to 0.\]

Canceling \(\mathcal{O}(-4)\) and taking into account that \(\mathcal{I}_Y(2) \simeq Coker(\psi)\) for some morphism \(\psi: 2\mathcal{O}(-1) \to 3\mathcal{O}\) that is represented by a matrix with linearly independent maximal minors generating the ideal of \(Y\) (cf. the proof of 2.3.4(i) \([9]\)), we obtain a resolution

\[0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \overset{\varphi}{\to} \mathcal{O}(-1) \oplus 3\mathcal{O} \to \mathcal{F} \to 0,\]

in which \(\varphi_{13} = 0, \varphi_{23} = \psi\) and \(\varphi_{11}, \varphi_{12}\) generate the ideal of \(Z\). It is clear now that \(\mathcal{F}\) gives a point in \(X_{30}\).

To show that \(X_3\) is included in \(\overline{X}_1\) we choose a point in \(X_3\) represented by the sheaf

\(\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5)\).

We may assume that the line through \(P_1\) and \(P_2\) intersects \(C\) at six distinct points \(P_1, P_2, Q_1, Q_2, Q_3, Q_4\), which are also distinct from \(P_4\) and \(P_5\). Then

\(\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5) \simeq \mathcal{O}_C(1)(Q_1 + Q_2 + Q_3 + Q_4 - P_3 + P_4 + P_5)\).

Clearly, we can find points \(R_i\) on \(C\) converging to \(Q_i, 1 \leq i \leq 4\), which are distinct from \(P_3\) and such that \(R_1, R_2, R_3, R_4, P_4, P_5\) do not lie on a conic curve. According to 3.2 the sheaves

\(\mathcal{O}_C(1)(R_1 + R_2 + R_3 + R_4 - P_3 + P_4 + P_5)\)

represent points in \(X_1\). These points converge to the chosen point in \(X_3\). Thus \(X_3 \subset \overline{X}_1\).

Taking into account the description, found at 3.3, of sheaves giving points in \(X_2\), it is clear that for generic \(\varphi \in W_3\) and for \(t \in \mathbb{C}^*\) in a neighbourhood of zero the morphism \(\varphi + t\pi\) is injective and its cokernel gives a point in \(X_2\). Here \(\pi\) is projection onto the last component followed by injection into the first component. Clearly \([Coker(\varphi + t\pi)]\) converges to \([Coker(\varphi)]\) as \(t\) tends to 0. Thus \(X_3 \subset \overline{X}_2\). \(\square\)

**Proposition 4.3.** There exists a geometric quotient \(W_3/G_3\) and it is a proper open subset inside a fibre bundle with fibre \(\mathbb{P}^{22}\) and base \(Y \times N(3, 2, 3)\), where \(Y\) is the Hilbert scheme of zero-dimensional subschemes of \(\mathbb{P}^2\) of length 2. Moreover, \(W_3/G_3\) is isomorphic to \(X_3\).

**Proof.** The construction of \(W_3/G_3\) is identical to the construction of the quotient at 3.2.3 \([9]\). Let \(W_3 \subset W_3\) be the locally closed subset given by the conditions of 4.1 except injectivity. Let \(\Sigma \subset W_3\) be the \(G_3\)-invariant subset given by the condition

\[\varphi_{21} = \varphi_{22}u + v\varphi_{11}, \quad u \in \text{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2), 2\mathcal{O}(-1)), \quad v \in \text{Hom}(\mathcal{O}(-1), 3\mathcal{O}).\]
As at loc.cit., we can construct a vector bundle \( F \) over \( Y \times N(3, 2, 3) \) of rank 23 such that \( \mathbb{P}(F) \) is a geometric quotient of \( W' \setminus \Sigma \) modulo \( G_3 \). Then \( W_3/G_3 \) is a proper open subset of \( \mathbb{P}(F) \).

Let \( \mathcal{F} \) give a point in \( X_3 \). The Beilinson tableau (2.2.3) \(^2\) for \( \mathcal{F} \) has the form

\[
\begin{array}{cccc}
4\mathcal{O}(-2) & \rightarrow & 4\mathcal{O}(-1) & \rightarrow & \mathcal{O} \\
0 & \rightarrow & 2\mathcal{O}(-1) & \rightarrow & 3\mathcal{O} \\
\end{array}
\]

As at 6.5 \(^\text{[10]}\), we have \( \ker(\varphi_3) \simeq \mathcal{O}(-4) \) and \( \ker(\varphi_2)/\text{Im}(\varphi_1) \simeq \mathcal{O}_Z \) for a scheme \( Z \subset \mathbb{P}^2 \) of dimension zero and length 2. The exact sequence (2.2.5) \(^2\) takes the form

\[
0 \rightarrow \mathcal{O}(-4) \rightarrow \text{Coker}(\varphi_4) \rightarrow \mathcal{F} \rightarrow \mathcal{O}_Z \rightarrow 0.
\]

We claim that \( \mathcal{F}(1) \) has a global section which maps to a global section of \( \mathcal{O}_Z \) that generates this sheaf as an \( \mathcal{O}_{\mathbb{P}^2} \)-module. We have \( h^0(\text{Coker}(\varphi_5)(1)) = 7 \), \( h^0(\mathcal{F}(1)) = 8 \), hence \( \mathcal{F}(1) \) has a global section mapping to a non-zero section \( s \) of \( \mathcal{O}_Z \). Consider an extension

\[
0 \rightarrow \mathbb{C}_{z_1} \rightarrow \mathcal{O}_Z \rightarrow \mathbb{C}_{z_2} \rightarrow 0,
\]

where \( z_1, z_2 \) are not necessarily distinct points in \( \mathbb{P}^2 \). If \( s \) maps to zero in \( \mathbb{C}_{z_2} \), then \( s \) generates \( \mathbb{C}_{z_1} \). Let \( \mathcal{F}' \) be the preimage of \( \mathbb{C}_{z_1} \) in \( \mathcal{F} \). We apply the horseshoe lemma to the extension

\[
0 \rightarrow \text{Coker}(\varphi_5) \rightarrow \mathcal{F}' \rightarrow \mathbb{C}_{z_1} \rightarrow 0
\]

and to the resolutions

\[
0 \rightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-1) \rightarrow 3\mathcal{O} \rightarrow \text{Coker}(\varphi_5) \rightarrow 0,
\]

\[
0 \rightarrow \mathcal{O}(-3) \rightarrow 2\mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \rightarrow \mathbb{C}_{z_1} \rightarrow 0.
\]

We obtain the resolution

\[
0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \rightarrow \mathcal{O}(-1) \oplus 3\mathcal{O} \rightarrow \mathcal{F}' \rightarrow 0
\]

from which we get the relation \( h^0(\mathcal{F}') = 4 \). This is absurd, \( h^0(\mathcal{F}') \) cannot exceed \( h^0(\mathcal{F}) \). Thus the image of \( s \) in \( \mathbb{C}_{z_2} \) is non-zero. When \( z_1 = z_2 \) this is enough to conclude that \( s \) generates \( \mathcal{O}_Z \). When \( z_1 \neq z_2 \) we revert the roles of \( z_1 \) and \( z_2 \) in the above argument to deduce that \( s \) also does not vanish at \( z_1 \), so \( s \) generates \( \mathcal{O}_Z \). We can now apply the horseshoe lemma to the extension

\[
0 \rightarrow \text{Coker}(\varphi_5) \rightarrow \mathcal{F} \rightarrow \mathcal{O}_Z \rightarrow 0
\]

to the above resolution of \( \text{Coker}(\varphi_5) \) and to the resolution

\[
0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}_Z \rightarrow 0.
\]

We obtain a resolution of the form

\[
0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \rightarrow \mathcal{O}(-1) \oplus 3\mathcal{O} \rightarrow \mathcal{F} \rightarrow 0
\]

Since \( h^1(\mathcal{F}) = 1 \), the morphism \( \mathcal{O}(-4) \rightarrow \mathcal{O}(-4) \) above is non-zero. Canceling \( \mathcal{O}(-4) \) we arrive at resolution \([4]\). In view of the method at 3.1.6 \(^2\), we have proven that the canonical bijective map \( W_3/G_3 \rightarrow X_3 \) is an isomorphism. \( \square \)
Proposition 4.4. The sheaves \( F \) giving points in \( M_{22}(6,2) \) and satisfying the cohomological conditions \( h^0(\mathcal{F}(-1)) = 1, h^1(\mathcal{F}) = 1 \) are precisely the sheaves having resolution of the form
\[
0 \to \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \to \mathcal{F} \to 0,
\]
where \( \varphi \) is not equivalent to a morphism represented by a matrix having one of the following forms:
\[
\begin{bmatrix}
* & 0 & 0 \\
* & * & 0 \\
* & * & *
\end{bmatrix}, \quad \begin{bmatrix}
* & * & 0 \\
* & * & 0 \\
* & * & *
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & * \\
* & * & * \\
* & * & *
\end{bmatrix}, \quad \begin{bmatrix}
0 & * & * \\
* & * & * \\
* & * & *
\end{bmatrix}.
\]

Proof. Let \( F \) give a point in \( M_{22}(6,2) \) and satisfy the cohomological conditions from the proposition. Write \( m = h^0(\mathcal{F} \otimes \Omega^1(1)) \). The Beilinson free monad (2.2.1) \([2]\) for \( F \) reads
\[
0 \to \mathcal{O}(-2) \to 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \to (m + 2)\mathcal{O}(-1) \oplus 3\mathcal{O} \to \mathcal{O} \to 0
\]
and gives the resolution
\[
0 \to \mathcal{O}(-2) \to 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \to \Omega^1 \oplus (m - 1)\mathcal{O}(-1) \oplus 3\mathcal{O} \to \mathcal{F} \to 0.
\]
Using the Euler sequence and arguing as at 2.1.4 \([2]\) we arrive at a resolution
\[
0 \to \mathcal{O}(-2) \xrightarrow{\eta} \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} (m - 1)\mathcal{O}(-1) \oplus 3\mathcal{O} \to \mathcal{F} \to 0,
\]
where
\[
\eta = \begin{bmatrix}
0 \\
0 \\
\eta_{31}
\end{bmatrix}, \quad \varphi = \begin{bmatrix}
\varphi_{11} & \varphi_{12} & 0 \\
\varphi_{21} & \varphi_{22} & \varphi_{23}
\end{bmatrix}.
\]
As at loc.cit., the entries of \( \eta_{31} \) span \( V^* \), hence \( m \geq 3 \). From the fact that \( F \) maps surjectively onto \( \text{Coker}(\varphi_{11}, \varphi_{12}) \) we get the reverse inequality. Thus \( m = 3, \text{Coker}(\eta_{31}) \simeq \Omega^1(1) \) and we have a resolution
\[
0 \to \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus \Omega^1(1) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 3\mathcal{O} \to \mathcal{F} \to 0
\]
in which \( \varphi_{13} = 0 \). Arguing as at loc.cit., we can show that \( \text{Coker}(\varphi_{23}) \simeq \mathcal{O}(1) \), so we arrive at a resolution as in the proposition. The conditions imposed on \( \varphi \) follow from the semi-stability of \( \mathcal{F} \).

Conversely, we assume that \( \mathcal{F} \) has a resolution as in the proposition and we must show that there are no destabilising subsheaves. Write
\[
\psi = \begin{bmatrix}
\varphi_{11} & \varphi_{12}
\end{bmatrix} = \begin{bmatrix}
q_1 & \ell_{11} & \ell_{12} \\
q_2 & \ell_{21} & \ell_{22}
\end{bmatrix}.
\]
As noted at 4.1 \([10]\), the conditions on \( \varphi \) in the proposition are equivalent to saying that
\[
\begin{vmatrix}
\ell_{11} & \ell_{12} \\
\ell_{21} & \ell_{22}
\end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix}
q_1 & \ell_{11} \\
q_2 & \ell_{21}
\end{vmatrix}, \quad \begin{vmatrix}
q_1 & \ell_{12} \\
q_2 & \ell_{22}
\end{vmatrix}
\]
are linearly independent in \( S^3 V^*/(\ell_{11}\ell_{22} - \ell_{12}\ell_{21})V^* \). Thus the maximal minors of \( \psi \) cannot have a quadratic common factor. It follows that \( \text{Ker}(\psi) \simeq \mathcal{O}(-4) \), if the maximal minors of \( \psi \) have a linear common factor, or \( \text{Ker}(\psi) \simeq \mathcal{O}(-5) \), if they have no common factor. From the snake lemma we have an exact sequence
\[
0 \to \text{Ker}(\psi) \to \mathcal{O}(1) \to \mathcal{F} \to \text{Coker}(\psi) \to 0.
\]
Assume that $\text{Ker}(\psi) \simeq \mathcal{O}(-4)$. Because of the conditions on $\psi$ it is easy to check that $\text{Coker}(\psi)$ has zero-dimensional torsion of length at most 1. Assume that $\text{Coker}(\psi)$ has no zero-dimensional torsion. Then $\text{Coker}(\psi) \simeq \mathcal{O}_L(1)$ for a line $L \subset \mathbb{P}^2$ and we have an extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L(1) \longrightarrow 0,$$

where $C \subset \mathbb{P}^2$ is a quintic curve. Let $\mathcal{F}' \subset \mathcal{F}$ be a non-zero subsheaf of multiplicity at most 5. Denote by $\mathcal{C}$ its image in $\mathcal{O}_L(1)$ and put $\mathcal{K} = \mathcal{F}' \cap \mathcal{O}_C(1)$. Let $\mathcal{A}$ be a sheaf as in 3.1.2 [9]. If $\mathcal{C} = 0$, then $p(\mathcal{F}') \leq 0$ because $\mathcal{O}_C(1)$ is stable. We may, therefore, assume that $\mathcal{C} \neq 0$. We can estimate the slope of $\mathcal{F}'$ as at loc.cit.:

$$P_{\mathcal{F}'}(t) = P_{\mathcal{K}}(t) + P_{\mathcal{C}}(t)$$

$$= P_{\mathcal{A}}(t) - h^0(\mathcal{A}/\mathcal{K}) + P_{\mathcal{O}_L(1)}(t) - h^0(\mathcal{O}_L(1)/\mathcal{C})$$

$$= (5 - d)t + \frac{d^2 - 5d}{2} + t + 2 - h^0(\mathcal{A}/\mathcal{K}) - h^0(\mathcal{O}_L(1)/\mathcal{C}),$$

where $d$ is an integer, $1 \leq d \leq 4$. Thus

$$p(\mathcal{F}') = \frac{1}{6 - d} \left( \frac{d^2 - 5d}{2} + 2 - h^0(\mathcal{A}/\mathcal{K}) - h^0(\mathcal{O}_L(1)/\mathcal{C}) \right) \leq \frac{d^2 - 5d + 4}{2(6 - d)} < \frac{1}{3} = p(\mathcal{F}).$$

We see that in this case $\mathcal{F}$ is stable. Assume next that $\text{Coker}(\psi)$ has a zero-dimensional subsheaf $\mathcal{T}$ of length 1. Let $\mathcal{E}$ be the preimage of $\mathcal{T}$ in $\mathcal{F}$. According to 3.1.5 [9], $\mathcal{E}$ gives a point in $M_{\mathbb{P}^2}(5,1)$. Let $\mathcal{F}'$ and $\mathcal{C}$ be as above. If $\mathcal{C} \subset \mathcal{T}$, then $\mathcal{F}' \subset \mathcal{E}$, hence $p(\mathcal{F}') \leq p(\mathcal{E}) < p(\mathcal{F})$. If $\mathcal{C}$ is not a subsheaf of $\mathcal{T}$, then we can estimate the slope of $\mathcal{F}'$ as above concluding again that it is less than the slope of $\mathcal{F}$.

Assume now that $\text{Ker}(\psi) \simeq \mathcal{O}(-5)$. We have an extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0,$$

where $C \subset \mathbb{P}^2$ is a sextic curve and $\mathcal{T}$ is a zero-dimensional sheaf of length 5. Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf of multiplicity at most 5, let $\mathcal{T}'$ be its image in $\mathcal{T}$ and put $\mathcal{K} = \mathcal{F}' \cap \mathcal{O}_C(1)$. As above, we have

$$P_{\mathcal{F}'}(t) = P_{\mathcal{K}}(t) + h^0(\mathcal{T}')$$

$$= P_{\mathcal{A}}(t) - h^0(\mathcal{A}/\mathcal{K}) + h^0(\mathcal{T}')$$

$$= (6 - d)t + \frac{d^2 - 5d - 6}{2} - h^0(\mathcal{A}/\mathcal{K}) + h^0(\mathcal{T}'),$$

$$p(\mathcal{F}') = -\frac{d + 1}{2} + \frac{h^0(\mathcal{T}') - h^0(\mathcal{A}/\mathcal{K})}{6 - d} \leq -\frac{d + 1}{2} + \frac{5}{6 - d},$$

where $d$ is an integer, $1 \leq d \leq 5$. We see from this that $p(\mathcal{F}') < p(\mathcal{F})$ except, possibly, when $d = 5$ and $P_{\mathcal{F}'}(t) = t + 1$ or $t + 2$, i.e. when $\mathcal{F}'$ is isomorphic to $\mathcal{O}_L$ or $\mathcal{O}_L(1)$ for a line $L \subset \mathbb{P}^2$. These situations can easily be ruled out. If, say, $\mathcal{O}_L$ were a subsheaf of $\mathcal{F}$,
then we would get a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}(-1) & \rightarrow & \mathcal{O} & \rightarrow & \mathcal{O}_L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \alpha & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) & \rightarrow & 2\mathcal{O}(-1) \oplus \mathcal{O}(1) & \rightarrow & \mathcal{F} & \rightarrow & 0
\end{array}
\]
\[
\begin{array}{ccccccccc}
\alpha & \downarrow & & \downarrow & & & & \\
\beta & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \mathcal{O}(-1) & \rightarrow & \mathcal{O} & \rightarrow & \mathcal{O}_L & \rightarrow & 0
\end{array}
\]
in which \( \alpha \) is injective, because it is injective on global sections. Thus \( \beta \) is also injective, which is absurd. We conclude that \( \mathcal{F} \) is stable. \( \square \)

Let \( W_4 = \text{Hom}(\mathcal{O}(-3) \oplus 2\mathcal{O}(-2), 2\mathcal{O}(-1) \oplus \mathcal{O}(1)) \) and let \( W_4 \subset W_4 \) be the set of morphisms \( \varphi \) from proposition 4.4. Let
\[
G_4 = (\text{Aut}(\mathcal{O}(-3) \oplus 2\mathcal{O}(-2)) \times \text{Aut}(2\mathcal{O}(-1) \oplus \mathcal{O}(1))) / \mathbb{C}^*
\]
be the natural group acting by conjugation on \( W_4 \). Let \( X_4 \subset \mathbb{M}_{\mathbb{P}^2}(6, 2) \) be the set of stable-equivalence classes of sheaves of the form \( \text{Coker}(\varphi), \varphi \in W_4 \).

**Proposition 4.5.** There exists a geometric quotient \( W_4/G_4 \), which is isomorphic to \( X_4 \). In particular, \( X_4 \) is irreducible and has codimension 5.

**Proof.** The Beilinson diagram (2.2.3) \[2\] for the dual sheaf \( \mathcal{G} = \mathcal{F}^\vee(1) \) giving a point in \( \mathbb{M}_{\mathbb{P}^2}(6, 4) \) has the form
\[
\begin{array}{cccccc}
3\mathcal{O}(-2) & \xrightarrow{\varphi_1} & 3\mathcal{O}(-1) & \xrightarrow{\varphi_2} & \mathcal{O} \\
\end{array}
\]
\[
\begin{array}{cccccc}
\mathcal{O}(-2) & \xrightarrow{\varphi_3} & 5\mathcal{O}(-1) & \xrightarrow{\varphi_4} & 5\mathcal{O} \\
\end{array}
\]
As in the proof of 2.2.4 \[9\], we have \( \text{Ker}(\varphi_2) = \text{Im}(\varphi_1) \) and \( \text{Ker}(\varphi_1) \simeq \mathcal{O}(-3) \). Combining the exact sequences (2.2.4) and (2.2.5) \[2\] we get the resolution
\[
0 \rightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus 5\mathcal{O}(-1) \rightarrow 5\mathcal{O} \rightarrow \mathcal{G} \rightarrow 0.
\]
As in the proof of 2.1.4 \[9\], we have \( \text{Coker}(\psi) \simeq \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \oplus \Omega^1(1) \). We get the resolution
\[
0 \rightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \oplus \Omega^1(1) \xrightarrow{\varphi} 5\mathcal{O} \rightarrow \mathcal{G} \rightarrow 0.
\]
As at loc.cit., we have \( \text{Coker}(\varphi_{13}) \simeq 2\mathcal{O} \oplus \mathcal{O}(1) \). We finally arrive at the resolution dual to resolution 4.4:
\[
0 \rightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \rightarrow 2\mathcal{O} \oplus \mathcal{O}(1) \rightarrow \mathcal{G} \rightarrow 0.
\]
This proves that the map \( W_4 \rightarrow X_4 \) is a categorical quotient. According to \[11\], remark (2), p. 5, \( X_4 \) is normal. Applying \[12\], theorem 4.2, we conclude that the map \( W_4 \rightarrow X_4 \) is a geometric quotient. \( \square \)

**Proposition 4.6.** The generic sheaves in \( X_4 \) are of the form \( \mathcal{O}_C(1)(P_1 + \cdots + P_5) \), where \( C \subset \mathbb{P}^2 \) is a smooth sextic curve and \( P_i \) are five distinct points on \( C \), no three of which are colinear. In particular, \( X_4 \) lies in the closure of \( X_1 \).
Proof. Let \( X_{40} \subset X_4 \) be the subset defined by the following conditions: the sextic curve \( C \) given by the equation \( \det(\varphi) = 0 \) is smooth, the conic curve \( F \) given by the equation \( f = \ell_{11}\ell_{22} - \ell_{12}\ell_{21} = 0 \) is irreducible, there are constants \( c_1, c_2 \in \mathbb{C} \) such that the cubic curve \( G \) with equation
\[
q_1 \ell_{11} + c_2 q_2 \ell_{22} = 0
\]
meets \( F \) at six distinct points \( P_1, \ldots, P_6 \) (notations as at 4.4). Let \( F = \text{Coker}(\varphi) \) give a point in \( X_{40} \). Performing, possibly, column operations on the matrix representing \( \varphi \) we may assume that \( c_1 = 0, c_2 = 1 \) and that \( P_6 \) is given by the equations \( \ell_{12} = 0, \ell_{22} = 0 \). Then \( \text{Coker}(\psi) \simeq \mathcal{O}_Z \), where \( Z \) is the union of \( P_1, \ldots, P_5 \). As at 4.4, \( F \) is an extension of \( \mathcal{O}_Z \) by \( \mathcal{O}_C(1) \), hence \( F \simeq \mathcal{O}_C(1)(P_1 + \cdots + P_5) \). Since \( P_1, \ldots, P_5 \) are on the irreducible conic \( F \), no three of them are colinear.

Conversely, we must show that every sheaf of the form \( \mathcal{O}_C(1)(P_1 + \cdots + P_5) \) gives a point in \( X_{40} \). Let \( F \subset \mathbb{P}^2 \) be a conic curve containing \( P_1, \ldots, P_5 \). Because these points are assumed to be in general linear position, \( F \) is irreducible. Choose a sixth point \( P_6 \in F \) distinct from the others. Let \( G \subset \mathbb{P}^2 \) be a cubic curve meeting \( F \) precisely at \( P_1, \ldots, P_6 \) (for example, the union of the three lines \( P_1P_2, P_3P_4, P_5P_6 \)). Choose equations \( f = 0, g = 0 \) for \( F, G \). Choose equations \( \ell_{12} = 0, \ell_{22} = 0 \) for \( P_6 \). We may write \( f = \ell_{11}\ell_{22} - \ell_{12}\ell_{21}, \ g = q_1\ell_{22} - q_2\ell_{12} \) for some \( \ell_{11}, \ell_{21} \in V^* \) and \( q_1, q_2 \in S^2V^* \). Let \( \psi : \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \to 2\mathcal{O}(-1) \) be the morphism represented by the matrix
\[
\begin{bmatrix}
q_1 & \ell_{11} & \ell_{12} \\
q_2 & \ell_{21} & \ell_{22}
\end{bmatrix}.
\]

We have \( \text{Coker}(\psi) \simeq \mathcal{O}_Z \), where \( Z \) is the union of \( P_1, \ldots, P_5 \). By construction, the maximal minors of \( \psi \) have no common factor, hence \( \text{Ker}(\psi) \simeq \mathcal{O}(-5) \). We apply the horseshoe lemma to the extension
\[
0 \to \mathcal{O}_C(1) \to \mathcal{F} \to \mathcal{O}_Z \to 0,
\]
to the standard resolution of \( \mathcal{O}_C(1) \) and to the resolution
\[
0 \to \mathcal{O}(-5) \to \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\psi} 2\mathcal{O}(-1) \to \mathcal{O}_Z \to 0.
\]
We claim that the morphism \( 2\mathcal{O}(-1) \to \mathcal{O}_Z \) lifts to a morphism \( 2\mathcal{O}(-1) \to \mathcal{F} \). To see this let \( \alpha : H^0(2\mathcal{O}) \to H^0(\mathcal{O}_Z) \) be the induced morphism and let \( \delta : H^0(\mathcal{O}_Z) \to H^1(\mathcal{O}_C(2)) \) be the connecting homomorphism associated to the exact sequence
\[
0 \to \mathcal{O}_C(2) \to \mathcal{O}_C(2)(P_1 + \cdots + P_5) \to \mathcal{O}_Z \to 0.
\]
We must show that \( \delta \circ \alpha = 0 \). We will show that \( \alpha^* \circ \delta^* = 0 \). Taking duals in the above resolution of \( \mathcal{O}_Z \) we obtain the resolution
\[
0 \to 2\mathcal{O}(-3) \to 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \to \mathcal{O}(1) \to \mathcal{E}x^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2}) \to 0.
\]
The induced map on global sections
\[
V^* \simeq H^0(\mathcal{O}(1)) \to H^0(\mathcal{E}x^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2})) \simeq \text{Ext}^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2}) \simeq H^0(\mathcal{O}_Z)^*
\]
can be identified with \( \delta^* \) because, by Serre duality, \( \delta^* \) is the restriction homomorphism
\[
V^* \simeq H^0(\mathcal{O}_C(1)) \simeq H^0(\mathcal{O}_C(2)^* \otimes \omega_C) \simeq H^1(\mathcal{O}_C(2))^* \to H^0(\mathcal{O}_Z)^*.
\]
The induced map
\[ \text{Ext}^2(O_Z, \omega_{P^2}) \simeq H^0(\text{Ext}^2(O_Z, \omega_{P^2})) \to H^2(2O(-3)) \simeq \text{Ext}^2(2O, \omega_{P^2}) \]
can be identified with \( \alpha^* \). It is clear now that we have \( \alpha^* \circ \delta^* = 0 \), proving the claim. We obtain the resolution
\[ 0 \to O(-5) \to O(-5) \oplus O(-3) \oplus 2O(-2) \to 2O(-1) \oplus O(1) \to F \to 0. \]
Since \( \text{Ext}^1(O_Z, O(1)) = 0 \), the argument at 2.3.2 [9] applies to show that the morphism \( O(-5) \to O(-5) \) above is non-zero. Canceling \( O(-5) \) we obtain a resolution that places \( \mathcal{F} \) in \( X_{40} \).

The inclusion \( X_4 \subset X_1 \) follows from the fact that any sheaf of the form \( O_C(1)(P_1 + \cdots + P_5) \) as above is the limit of a sequence of sheaves \( O_C(1)(P_1 + \cdots + P_6 - P_7) \) as at 3.2.

**Proposition 4.7.** \( X_4 \) lies in the closure of \( X_2 \).

**Proof.** The argument can be found at 2.1.6 [9] and can be traced back to 3.2.3 [2]. Let \( Y \subset M_{P^2}(6, 4) \) be the subset of stable-equivalence classes of sheaves \( \mathcal{G} \) satisfying the conditions \( h^0(\mathcal{G}(-1)) = 1, h^0(\mathcal{G}(-2)) = 0 \). We claim that for any such sheaf we have the relation \( h^0(\mathcal{G} \otimes \Omega^1) = 0 \). To see this denote \( m = h^0(\mathcal{G} \otimes \Omega^1) \) and consider the Beilinson diagram (2.2.3) [2] for \( \mathcal{G}(-1) \):

\[
8O(-2) \xrightarrow{\varphi_1} (m+10)O(-1) \xrightarrow{\varphi_2} 3O.
\]

As \( \varphi_4 \) is injective, we have \( m = 0 \) or \( 1 \). If \( m = 1 \), then \( \text{Coker}(\varphi_4) \simeq O_L \) for a line \( L \subset P^2 \).

The exact sequence (2.2.5) [2] reads
\[ 0 \to \ker(\varphi_1) \to O_L \to \mathcal{G}(-1) \to \ker(\varphi_2)/\text{Im}(\varphi_1) \to 0. \]

The map \( O_L \to \mathcal{G}(-1) \) is zero because \( p(O_L) > p(\mathcal{G}(-1)) \) and both sheaves are semistable. Thus \( O_L \simeq \ker(\varphi_1) \), which is absurd.

Using the Beilinson monad for \( \mathcal{G}(-1) \) we see that \( Y \) is parametrised by an open subset \( M \) inside the space of monads
\[ 0 \to 8O(-1) \xrightarrow{A} 10O \oplus O(1) \xrightarrow{B} 3O(1) \to 0 \]
satisfying \( B_{12} = 0 \). Consider the map \( \Phi: M \to \text{Hom}(10O, 3O(1)) \) defined by \( \Phi(A, B) = B_{11} \). Using the vanishing of \( H^1(\mathcal{G}(1)) \) for an arbitrary sheaf \( \mathcal{G} \) giving a point in \( Y \) (cf. 2.1.3 [2]), we can prove that \( M \) is smooth and that \( \Phi \) has surjective differential at every point. This further leads to the conclusion that the set of monads in \( M \) whose cohomology sheaf \( \mathcal{G} \) satisfies the relation \( h^1(\mathcal{G}) = 1 \) is included in the closure of the set of monads for which \( h^1(\mathcal{G}) = 0 \). Thus \( X^D_4 \) lies in the relative closure of \( X^D_2 \cup X^D_3 \) in \( Y \). It follows that \( X_4 \subset X_2 \cup X_3 \). Since \( X_3 \subset X_2 \), the conclusion follows.  \( \square \)
5. The codimension 7 stratum

**Proposition 5.1.** The sheaves $\mathcal{F}$ giving points in $M_{P^2}(6,2)$ and satisfying the conditions $h^0(\mathcal{F}(-1)) = 1$, $h^1(\mathcal{F}) = 2$ are precisely the sheaves having resolution of the form

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{ij}$ has linearly independent entries, $\varphi_{22} \neq 0$ and does not divide $\varphi_{32}$.

**Proof.** Let $\mathcal{F}$ give a point in $M_{P^2}(6,2)$ and satisfy the cohomological conditions from above. Put $m = h^0(\mathcal{F} \otimes \Omega^1(1))$. Let $\mathcal{G} = \mathcal{F}^\oplus(1)$. The Beilinson monad for $\mathcal{G}$ gives the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow 4\mathcal{O}(-2) \oplus (m + 2)\mathcal{O}(-1) \longrightarrow \Omega^1 \oplus (m - 3)\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Using the Euler sequence and arguing as at 2.1.4 [9] we arrive at a resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus (m + 2)\mathcal{O}(-1) \xrightarrow{\varphi} (m - 3)\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

$$\psi = \begin{bmatrix} 0 \\ 0 \\ \psi_{31} \end{bmatrix}, \quad \varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \end{bmatrix}.$$

Arguing as in the proof of 3.2.5 [9], we see that, modulo operations on rows and columns, $\psi_{31}$ is represented by a matrix of the form

$$\begin{bmatrix} X & Y & Z & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & X & Y & Z & \cdots \end{bmatrix}^T.$$

Thus $m \geq 4$. From the fact that $\mathcal{G}$ maps surjectively onto $\text{Coker}(\varphi_{11}, \varphi_{12})$ we get the reverse inequality. Thus $m = 4$, $\text{Coker}(\psi_{31}) \simeq 2\Omega^1(1)$ and we obtain a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\Omega^1(1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

in which $\varphi_{13} = 0$. Dually, we have the resolution

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\Omega^1 \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Combining with the standard resolution of $\Omega^1$ yields the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 6\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From the semi-stability of $\mathcal{F}$ we see that $\text{rank}(\varphi_{12}) = 5$, cf. argument at 2.1.4 [9]. Canceling $5\mathcal{O}(-2)$ we obtain the desired resolution of $\mathcal{F}$. The conditions imposed on $\varphi$ follow from the semi-stability of $\mathcal{F}$.

Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we must show that there are no destabilising subsheaves. From the snake lemma we get an extension

$$0 \longrightarrow J_Z(2) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}_x \longrightarrow 0,$$

where $J_Z \subset \mathcal{O}_C$ is the ideal sheaf of a zero-dimensional subscheme $Z$ of length 2 inside a sextic curve $C$ and $\mathcal{C}_x$ is the structure sheaf of a point. Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf of
multiplicity at most 5, let \( \mathcal{C} \) be its image in \( \mathbb{C}_2 \) and \( \mathcal{K} = \mathcal{F}' \cap \mathcal{J}_Z(2) \). With the notations of [4, 4] we have

\[
P_{\mathcal{F}'}(t) = P_{\mathcal{K}}(t) + h^0(\mathcal{C})
= P_{\mathcal{A}}(t) - h^0(\mathcal{A}/\mathcal{K}) + h^0(\mathcal{C})
= (6 - d)t + \frac{d^2 - 7d + 6}{2} - h^0(\mathcal{A}/\mathcal{K}) + h^0(\mathcal{C})
\]

for some integer \( d, 1 \leq d \leq 5 \), hence

\[
p(\mathcal{F}') = \frac{1 - d}{2} + \frac{h^0(\mathcal{C}) - h^0(\mathcal{A}/\mathcal{K})}{6 - d} \leq \frac{1 - d}{2} + \frac{1}{6 - d} < p(\mathcal{F}).
\]

We conclude that \( \mathcal{F} \) is stable. \( \square \)

Let \( \mathcal{W}_5 = \text{Hom}(2\mathcal{O}(−3) \oplus \mathcal{O}(−1), \mathcal{O}(−2) \oplus \mathcal{O} \oplus \mathcal{O}(1)) \) and let \( \mathcal{W}_5 \subset \mathcal{W}_5 \) be the set of morphisms \( \varphi \) from proposition 5.1. Let \( G_5 = (\text{Aut}(2\mathcal{O}(−3) \oplus \mathcal{O}(−1)) \times \text{Aut}(\mathcal{O}(−2) \oplus \mathcal{O} \oplus \mathcal{O}(1)))/\mathbb{C}^* \) be the natural group acting by conjugation on \( \mathcal{W}_5 \). Let \( X_5 \subset M_{\mathbb{P}^2}(6, 2) \) be the set of stable-equivalence classes of sheaves of the form \( \text{Coker}(\varphi), \varphi \in \mathcal{W}_5 \).

**Proposition 5.2.** There exists a geometric quotient of \( \mathcal{W}_5 \) by \( G_5 \) and it is isomorphic to a proper open subset inside a fibre bundle with fibre \( \mathbb{P}^4 \) and base \( \mathbb{P}^2 \times Y \), where \( Y \) is the Hilbert scheme of zero-dimensional subschemes of \( \mathbb{P}^2 \) of length 2. Moreover, \( \mathcal{W}_5/G_5 \) is isomorphic to \( X_5 \).

**Proof.** The construction of \( \mathcal{W}_5/G_5 \) is entirely analogous to the construction of the quotient at 3.2.3 [9].

Let \( \mathcal{F} \) give a point in \( X_5 \) and let \( \mathcal{G} = \mathcal{F}^0(1) \). The Beilinson tableau (2.2.3) [2] for \( \mathcal{G} \) takes the form

\[
\begin{array}{ccc}
4\mathcal{O}(−2) & \xrightarrow{\varphi_1} & 4\mathcal{O}(−1) \xrightarrow{\varphi_2} \mathcal{O} \\
2\mathcal{O}(−2) & \xrightarrow{\varphi_3} & 6\mathcal{O}(−1) \xrightarrow{\varphi_4} 5\mathcal{O}
\end{array}
\]

As at 6.5 [10], we have \( \text{Ker}(\varphi_1) \simeq \mathcal{O}(−4) \) and \( \text{Ker}(\varphi_2)/\text{Im}(\varphi_1) \simeq \mathcal{O}_Z \) for a scheme \( Z \subset \mathbb{P}^2 \) of dimension zero and length 2. The exact sequence (2.2.5) [2] takes the form

\[
0 \rightarrow \mathcal{O}(−4) \xrightarrow{\varphi_5} \text{Coker}(\varphi_4) \rightarrow \mathcal{G} \rightarrow \mathcal{O}_Z \rightarrow 0.
\]

As at 3.2.5 [9], we have \( \text{Coker}(\varphi_3) \simeq 2\Omega^1(1) \). This, together with (2.2.4) [2], gives the resolution

\[
0 \rightarrow \mathcal{O}(−4) \oplus 2\Omega^1(1) \rightarrow 5\mathcal{O} \rightarrow \text{Coker}(\varphi_5) \rightarrow 0.
\]

We claim that \( \mathcal{G}(1) \) has a global section which maps to a global section of \( \mathcal{O}_Z \) that generates this sheaf as an \( \mathcal{O}_{\mathbb{P}^2} \)-module. To show this we argue as at 4.3. By 2.1.3 [2], the group \( H^1(\mathcal{G}(1)) \) vanishes, hence we have \( h^0(\mathcal{G}(1)) = 10 \). Since \( h^0(\text{Coker}(\varphi_5)(1)) = 9 \), we
see that \( \mathcal{G}(1) \) has a global section mapping to a non-zero section \( s \) of \( \mathcal{O}_Z \). We have an exact sequence

\[
0 \longrightarrow \mathbb{C}_{z_1} \longrightarrow \mathcal{O}_Z \longrightarrow \mathbb{C}_{z_2} \longrightarrow 0,
\]

where \( z_1, z_2 \) are not necessarily distinct points in \( \mathbb{P}^2 \). If \( s \) maps to zero in \( \mathbb{C}_{z_2} \), then \( s \) generates \( \mathbb{C}_{z_1} \). Let \( \mathcal{F}' \) be the preimage of \( \mathbb{C}_{z_1} \) in \( \mathcal{F} \). We can apply the horseshoe lemma to the extension

\[
0 \longrightarrow \text{Coker}(\varphi_5) \longrightarrow \mathcal{F}' \longrightarrow \mathbb{C}_{z_1} \longrightarrow 0,
\]
to the above resolution of \( \text{Coker}(\varphi_5) \) and to the standard resolution of \( \mathbb{C}_{z_1} \) tensored with \( \mathcal{O}(-1) \). We obtain the exact sequence

\[
0 \longrightarrow \mathcal{O}(-3) \overset{\psi}{\longrightarrow} \mathcal{O}(-4) \oplus 2\mathcal{O}(-2) \oplus 2\Omega^1(1) \longrightarrow \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.
\]

We have \( h^1(\mathcal{F}') = h^2(\text{Coker}(\psi)) = 3 \), hence \( h^0(\mathcal{F}') = 4 \). On the other hand, \( H^0(\text{Coker}(\psi)) \) vanishes, hence \( h^0(\mathcal{F}') \geq 5 \). This is absurd, so an exact sequence as above cannot exist. Thus the image of \( s \) in \( \mathbb{C}_{z_2} \) is non-zero and the claim follows as at 3.1.6 [2]. We can now combine the resolutions of \( \mathcal{O}_Z \) and of \( \text{Coker}(\varphi_5) \) from above to get the exact sequence

\[
0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\Omega^1(1) \longrightarrow \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.
\]

The map \( \mathcal{O}(-4) \to \mathcal{O}(-4) \) is non-zero because \( h^1(\mathcal{G}) = 1 \). We may cancel \( \mathcal{O}(-4) \) to get the resolution

\[
0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\Omega^1(1) \longrightarrow \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.
\]

We saw at 5.1 how this leads to a morphism \( \varphi \in W_5 \) such that \( \mathcal{F} \simeq \text{Coker}(\varphi) \). We conclude, as at 3.1.6 [2], that the canonical bijective map \( W_5/G_5 \to X_5 \) is an isomorphism.

**Proposition 5.3.** The generic sheaves from \( X_5 \) are precisely the non-split extension sheaves

\[
0 \longrightarrow \mathcal{J}_Z(2) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,
\]

where \( \mathcal{J}_Z \subset \mathcal{O}_C \) is the ideal sheaf of a zero-dimensional scheme \( Z \) of length 2 inside a sextic curve \( C \subset \mathbb{P}^2 \) and \( \mathbb{C}_x \) is the structure sheaf of a point \( x \in \mathbb{P}^2 \) that is not in the support of \( Z \).

There is a dense open subset of \( X_5 \) consisting of the isomorphism classes of all sheaves of the form \( \mathcal{O}_C(2)(P_1 - P_2 - P_3) \), where \( C \subset \mathbb{P}^2 \) is a smooth sextic curve and \( P_1, P_2, P_3 \) are distinct points on \( C \). In particular, \( X_5 \) lies in the closure of \( X_3 \) and also in the closure of \( X_4 \).

**Proof.** Let \( \mathcal{F} = \text{Coker}(\varphi) \) give a point in \( X_5 \), where \( \varphi \) is a morphism as at 5.1. Let \( x \in \mathbb{P}^2 \) be the point given by the ideal generated by the entries of \( \varphi_{11} \), let \( Z \subset \mathbb{P}^2 \) be the subscheme given by the equations \( \varphi_{22} = 0, \varphi_{32} = 0 \) and let \( C \subset \mathbb{P}^2 \) be the curve given by the equation \( \det(\varphi) = 0 \). We saw at 5.1 that \( \mathcal{F} \) is a non-split extension of \( \mathbb{C}_x \) by \( \mathcal{J}_Z(2) \). Let \( X_{50} \subset X_5 \) be the open subset given by the condition that \( x \) be not a subscheme of \( Z \). To show that every extension as in the proposition gives a point in \( X_{50} \) we combine the resolutions

\[
0 \longrightarrow \mathcal{O}(-4) \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \longrightarrow \mathbb{C}_x \longrightarrow 0
\]

and

\[
0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{J}_Z(2) \longrightarrow \mathcal{J}_Z(2) \longrightarrow 0.
\]
Here \( I_Z \subset \mathcal{O}_{\mathbb{P}^2} \) is the ideal sheaf of \( Z \). We obtain the resolution
\[
0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \rightarrow \mathcal{O}(-2) \oplus I_Z(2) \rightarrow \mathcal{F} \rightarrow 0.
\]
The group \( \text{Ext}^1(\mathcal{C}_x, I_Z(2)) \) vanishes because \( x \) is not in \( Z \), so we can apply the argument at 2.3.2 \cite{9} to deduce that the morphism \( \mathcal{O}(-4) \rightarrow \mathcal{O}(-4) \) in the above complex is non-zero. Canceling \( \mathcal{O}(-4) \) we obtain the resolution
\[
0 \rightarrow 2\mathcal{O}(-3) \rightarrow \mathcal{O}(-2) \oplus I_Z(2) \rightarrow \mathcal{F} \rightarrow 0,
\]
which shows that \( \mathcal{F} \) gives a point in \( X_{50} \).

Clearly every sheaf of the form \( \mathcal{O}_C(2)(P_1 - P_2 - P_3) \) is the limit of a sequence of sheaves of the form \( \mathcal{O}_C(2)(-Q_1 - Q_2 - Q_3 + Q_4 + Q_5) \) as at \cite{4} (make \( Q_1 \) converge to \( Q_4 \)). Thus \( X_5 \subset X_3 \). To prove that \( X_5 \subset X_4 \) fix a sheaf \( \mathcal{F} = \mathcal{O}_C(2)(Q_1 - P_2 - P_3) \) in \( X_5 \). Choosing \( \mathcal{F} \) general enough, we may assume that \( Q_1, P_2, P_3 \) are non-colinear and that the line \( P_2P_3 \) meets \( C \) at six distinct points \( P_2, P_3, Q_2, Q_3, Q_4, Q_5 \). Then
\[
\mathcal{O}_C(2)(Q_1 - P_2 - P_3) \cong \mathcal{O}_C(1)(Q_1 + \cdots + Q_5).
\]
Clearly, we can choose five distinct points \( R_i \) on \( C \) converging to \( Q_i \), \( 1 \leq i \leq 5 \), such that no three among them are colinear. According to proposition \cite{13} \( \mathcal{O}_C(1)(R_1 + \cdots + R_5) \) gives a point in \( X_4 \). Thus \( \mathcal{O}_C(2)(Q_1 - P_2 - P_3) \) lies in the closure of \( X_4 \).

\[ \square \]

6. The codimension 9 stratum

\textbf{Proposition 6.1.} The sheaves \( \mathcal{F} \) in \( M_{\mathbb{P}^2}(6, 2) \) satisfying the condition \( h^1(\mathcal{F}(1)) > 0 \) are precisely the sheaves with resolution of the form
\[
0 \rightarrow \mathcal{O}(-4) \oplus \mathcal{O} \overset{\varphi}{\rightarrow} 2\mathcal{O}(1) \rightarrow \mathcal{F} \rightarrow 0,
\]
where \( \ell_1, \ell_2 \) are linearly independent one-forms. These sheaves are precisely the sheaves \( \mathcal{J}_x(2) \), where \( \mathcal{J}_x \subset \mathcal{O}_C \) is the ideal sheaf of a point \( x \) on a sextic curve \( C \subset \mathbb{P}^2 \).

\textit{Proof.} This statement follows by duality from \cite{10}, proposition 6.1. \( \square \)

Let \( \mathcal{W}_6 = \text{Hom}(\mathcal{O}(-4) \oplus \mathcal{O}, 2\mathcal{O}(1)) \) and let \( W_6 \subset \mathcal{W}_6 \) be the set of morphisms \( \varphi \) from \( \mathcal{O} \) to \( 2\mathcal{O}(1) \).

Let
\[
G_6 = (\text{Aut}(\mathcal{O}(-4) \oplus \mathcal{O}) \times \text{Aut}(2\mathcal{O}(1)))/\mathbb{C}^*
\]
be the natural group acting by conjugation on \( \mathcal{W}_6 \). Let \( \mathcal{X}_6 \subset M_{\mathbb{P}^2}(6, 2) \) denote the set of stable-equivalence classes of sheaves of the form \( \text{Coker}(\varphi) \), \( \varphi \in W_6 \).

\textbf{Proposition 6.2.} There exists a geometric quotient \( W_6/G_6 \), which is isomorphic to the universal sextic \( \Sigma \subset \mathbb{P}^2 \times \mathbb{P}(S^6 V^*) \). Moreover, \( W_6/G_6 \) is isomorphic to \( \mathcal{X}_6 \), so this is a smooth closed subvariety of \( M_{\mathbb{P}^2}(6, 2) \) of codimension 9.

\textit{Proof.} For the first part of the proposition we notice, as at 3.2 \cite{2} or at 3.2.5 \cite{9}, that the map \( W_6 \rightarrow \Sigma \) defined by
\[
\begin{bmatrix} f_1 & \ell_1 \\ f_2 & \ell_2 \end{bmatrix} \mapsto (x, (f_1\ell_2 - f_2\ell_1)),
\]
is an \( \mathbb{C}^* \)-equivariant mapping.

\[ \square \]
x being given by the equations $\ell_1 = 0$, $\ell_2 = 0$, is a geometric quotient map. The canonical morphism $\rho: W_0 \to X_0$, $\rho(\varphi) = [\text{Coker}(\varphi)]$, determines a bijective morphism

$$v: \Sigma \to X_0, \quad v(x, f) = [J_x(2)],$$

where $J_x \subset O_C$ is the ideal sheaf of $x$ on the curve $C$ given by the equation $f = 0$. As at 6.5 [10], in order to show that $v^{-1}$ is a morphism, we need to construct the pair $(x, C)$ starting from $E^1(J_x(2))$. For technical reasons we will work, instead, with $E^1(J_x^0)$. Denote $G = J_x^0$ and notice that $G$ gives a point in $M_{\mathbb{F}^2}(6, 10)$ and is an extension of the form

$$0 \to O_C(3) \to G \to C_x \to 0.$$

The Beilinson tableau (2.2.3) [2] for $G$ has the form

$$2O(-2) \xrightarrow{\varphi_1} O(-1) \xrightarrow{\varphi_4} 0.$$

$$6O(-2) \xrightarrow{\varphi_3} 15O(-1) \xrightarrow{\varphi_4} 10O$$

Since $G$ is semi-stable and maps surjectively onto $\text{Coker}(\varphi_1)$ we see that $\text{Coker}(\varphi_1) \simeq C_y$ for a point $y \in \mathbb{P}^2$ and $\text{Ker}(\varphi_1) \simeq O(-3)$. The exact sequence (2.2.5) [2] reads

$$0 \to O(-4) \xrightarrow{\varphi_3} \text{Coker}(\varphi_4) \to G \to C_y \to 0.$$

Denote $G' = \text{Coker}(\varphi_5)$. The exact sequence (2.2.4) [2] yields the resolution

$$0 \to 6O(-2) \xrightarrow{\psi} O(-4) \oplus 15O(-1) \xrightarrow{\varphi'} 10O \to G' \to 0.$$

We have $h^0(G') = 10$, hence $H^0(G) = H^0(G')$. The global sections of $G$ generate $O_C(3)$ and $G'$ is generated by its global sections. Thus $G' = O_C(3)$. The maximal minors of any matrix representing $\varphi'$ generate the ideal of $C$ because the Fitting support of $G'$ is $C$. It is clear that $x = y$. In conclusion, we have obtained the pair $(x, C) \in \Sigma$ from $E^1(G)$ by performing algebraic operations.

**Proposition 6.3.** $X_6$ lies in the closure of $X_5$.

**Proof.** Any generic sheaf $O_C(2)(-P)$ in $X_6$, with $C \subset \mathbb{P}^2$ a smooth sextic curve and $P \in C$, is the limit of a sequence of sheaves of the form $O_C(2)(P_1 - P_2 - P)$ as at 5.3 (make $P_1$ converge to $P_2$).

## 7. The moduli space is the union of the strata

In this final section we shall prove that $M_{\mathbb{F}^2}(6, 2)$ is the union of the locally closed subsets $X_1, \ldots, X_6$ we found above.

**Proposition 7.1.** There are no sheaves $\mathcal{F}$ giving points in $M_{\mathbb{F}^2}(6, 2)$ and satisfying the conditions $h^0(\mathcal{F}(-1)) = 0$, $h^1(\mathcal{F}) = 2$.

**Proof.** Assume that there is such a sheaf $\mathcal{F}$. Put $m = h^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson monad for the dual sheaf $\mathcal{G} = \mathcal{F}^0(1)$ gives the resolution

$$0 \to 2O(-2) \to 4O(-2) \oplus (m + 2)O(-1) \xrightarrow{\varphi} mO(-1) \oplus 4O \to \mathcal{G} \to 0.$$
Since \( G \) maps surjectively onto \( \text{Coker}(\varphi_{11}) \), we have \( m \leq 3 \). The rest of the proof is exactly as at 3.1.3 \([9]\). Let \( \psi : 2\mathcal{O}(-2) \to (m+2)\mathcal{O}(-1) \) denote the morphism occurring in the above complex. In the case \( m = 3 \), say, there are three possible canonical forms for \( \psi \) given at loc.cit., each leading to a contradiction. \( \square \)

**Proposition 7.2.** There are no sheaves \( F \) giving points in \( M_{62}(6,2) \) and satisfying the cohomological conditions

\[
\begin{align*}
\text{h}^0(F(-1)) & \leq 1, \\
\text{h}^1(F) & \geq 3, \\
\text{h}^1(F(1)) & = 0.
\end{align*}
\]

**Proof.** The argument is the same as at 7.2 \([10]\) with notational differences only. Assume that \( F \) gives a point in \( M_{62}(6,2) \) and satisfies the conditions \( \text{h}^0(F(-1)) = 0, \text{h}^1(F) \geq 3 \). Write \( p = \text{h}^1(F), m = \text{h}^0(F \otimes \Omega^1(1)) \). The Beilinson free monad for \( F \) reads

\[
0 \to 4\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \to (m+2)\mathcal{O}(-1) \oplus (p+2)\mathcal{O} \xrightarrow{\psi} p\mathcal{O} \to 0,
\]

\[
\psi = \begin{bmatrix} \eta & 0 \end{bmatrix},
\]

and yields a resolution

\[
0 \to 4\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} \text{Ker}(\eta) \oplus (p+2)\mathcal{O} \to F \to 0
\]

in which \( \varphi_{12} = 0 \). We have \( m + 2 - p = \text{rank}(\text{Ker}(\eta)) \leq 4 \) because \( \varphi \) is injective. Thus

\[
\text{h}^0(F(1)) = 3(p+2) + \text{h}^0(\text{Ker}(\eta)(1)) - m \geq 2(p+2) \geq 10
\]

forcing \( \text{h}^1(F(1)) \geq 2 \). Assume, instead, that \( \text{h}^0(F(-1)) = 1 \). The Beilinson monad for the dual sheaf \( G = F^0(1) \) reads

\[
0 \to p\mathcal{O}(-2) \to (p+2)\mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \to m\mathcal{O}(-1) \oplus 5\mathcal{O} \to \mathcal{O} \to 0
\]

and leads to a resolution

\[
0 \to p\mathcal{O}(-2) \to \mathcal{O}(-3) \oplus (p-1)\mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \xrightarrow{\varphi} (m-3)\mathcal{O}(-1) \oplus 5\mathcal{O} \to G \to 0
\]

in which \( \varphi_{13} = 0 \). Since \( G \) maps surjectively onto \( \text{Coker}(\varphi_{11}, \varphi_{12}) \), we have \( m - 3 \leq p \). Dualising the above resolution we get a monad for \( F \) of the form

\[
0 \to 5\mathcal{O}(-2) \oplus (m-3)\mathcal{O}(-1) \to (m+2)\mathcal{O}(-1) \oplus (p-1)\mathcal{O} \oplus \mathcal{O}(1) \xrightarrow{\psi} p\mathcal{O} \to 0,
\]

\[
\psi = \begin{bmatrix} \eta & 0 & 0 \end{bmatrix}.
\]

The exact sequence

\[
0 \to 5\mathcal{O}(-2) \oplus (m-3)\mathcal{O}(-1) \to \text{Ker}(\eta) \oplus (p-1)\mathcal{O} \oplus \mathcal{O}(1) \to F \to 0
\]

gives the estimate

\[
\text{h}^0(F(1)) = 3(p-1) + 6 + \text{h}^0(\text{Ker}(\eta)(1)) - (m-3) \geq 3p + 6 - m \geq 2p + 3 \geq 9.
\]

We deduce that \( \text{h}^1(F(1)) \geq 1 \). \( \square \)

**Proposition 7.3.** Let \( F \) be a sheaf giving a point in \( M_{62}(6,2) \) and satisfying the condition \( \text{h}^1(F(1)) = 0 \). Then \( \text{h}^0(F(-1)) \) = 0 or 1.
Proof. Let \( \mathcal{F} \) give a point in \( M_{\mathbb{P}^2}(6,2) \) and satisfy the condition \( h^0(\mathcal{F}(-1)) \geq 2 \). As at 2.1.3 \([2]\), there is an injective morphism \( \mathcal{O}_C \to \mathcal{F}(-1) \) for a curve \( C \subset \mathbb{P}^2 \). This curve has degree 5 or 6, otherwise \( \mathcal{O}_C \) would destabilise \( \mathcal{F}(-1) \). Assume that \( \deg(C) = 5 \). The quotient sheaf \( \mathcal{C} = \mathcal{F}/\mathcal{O}_C(1) \) has Hilbert polynomial \( P(t) = t + 2 \) and zero-dimensional torsion \( \mathcal{T} \) of length at most 1 (the pull-back in \( \mathcal{F} \) of \( \mathcal{T} \) would be a destabilising subsheaf if its length were at least 2). If \( \mathcal{T} = 0 \), then \( \mathcal{C} \cong \mathcal{O}_L(1) \) for a line \( L \subset \mathbb{P}^2 \). We get that \( h^0(\mathcal{F}(-1)) = 2 \) and that the morphism \( \mathcal{O}(1) \to \mathcal{O}_L(1) \) lifts to a morphism \( \mathcal{O}(1) \to \mathcal{F} \). The horseshoe lemma leads to the resolution

\[
0 \to \mathcal{O}(-4) \oplus \mathcal{O} \to 2\mathcal{O}(1) \to \mathcal{F} \to 0.
\]

Thus \( h^1(\mathcal{F}(1)) = 1 \). Assume now that \( \mathcal{T} \) has length 1. Let \( \mathcal{F}' \subset \mathcal{F} \) be the pull-back of \( \mathcal{T} \). According to 3.1.5 \([2]\), we have \( h^0(\mathcal{F}'(-1)) = 1 \). Since \( \mathcal{F}/\mathcal{F}' \cong \mathcal{C}/\mathcal{T} \cong \mathcal{O}_L \) for a line \( L \subset \mathbb{P}^2 \), we get \( h^0(\mathcal{F}(-1)) = 1 \), contradicting our choice of \( \mathcal{F} \).

Assume now that \( \mathcal{C} \) is a sextic curve. The quotient sheaf \( \mathcal{T} = \mathcal{F}/\mathcal{O}_C(1) \) is zero-dimensional of length 5. Let \( \mathcal{T}' \subset \mathcal{T} \) be a subsheaf of length 4 and let \( \mathcal{F}' \) be its preimage in \( \mathcal{F} \). We claim that \( \mathcal{F}' \) gives a point in \( M_{\mathbb{P}^2}(6,1) \). If this were not the case, then \( \mathcal{F}' \) would have a destabilising subsheaf \( \mathcal{F}'' \), which may be assumed to be semi-stable. By proposition 2.3 \( \mathcal{F} \) is stable, so we have the inequalities \( 1/6 < p(\mathcal{F}'') < 1/3 \). This leaves only two possibilities: that \( \mathcal{F}'' \) give a point in \( M_{\mathbb{P}^2}(5,1) \) or in \( M_{\mathbb{P}^2}(4,1) \). In the first case \( \mathcal{F}/\mathcal{F}'' \) is isomorphic to the structure sheaf of a line, hence \( h^0(\mathcal{F}(-1)) = h^0(\mathcal{F}'(-1)) = 0 \) or 1, cf. \([9]\). This contradicts our choice of \( \mathcal{F} \). In the second case \( \mathcal{F}/\mathcal{F}'' \) is easily seen to be semi-stable, hence it is isomorphic to the structure sheaf of a conic curve. We get \( h^0(\mathcal{F}(-1)) = h^0(\mathcal{F}'(-1)) = 0 \), cf. \([2]\), contradicting our choice of \( \mathcal{F} \). This proves the claim, i.e. that \( \mathcal{F}' \) is semi-stable. We have \( h^0(\mathcal{F}'(-1)) \geq 1 \) so, according to \([10]\), there are two possible resolutions for \( \mathcal{F}' \):

\[
0 \to 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \to \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \to \mathcal{F}' \to 0
\]

or

\[
0 \to \mathcal{O}(-4) \oplus \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O}(1) \to \mathcal{F}' \to 0.
\]

Assume that \( \mathcal{F}' \) has the first resolution. We apply the horseshoe lemma to the extension

\[
0 \to \mathcal{F}' \to \mathcal{F} \to \mathbb{C}_x \to 0,
\]

to the given resolution of \( \mathcal{F}' \) and to the resolution

\[
0 \to \mathcal{O}(-1) \to 2\mathcal{O} \to \mathcal{O}(1) \to \mathbb{C}_x \to 0.
\]

As \( h^0(\mathcal{F}'(-1)) = 1 \) and \( h^0(\mathcal{F}(-1)) \geq 2 \), the morphism \( \mathcal{O}(1) \to \mathbb{C}_x \) lifts to a morphism \( \mathcal{O}(1) \to \mathcal{F} \). We obtain the resolution

\[
0 \to \mathcal{O}(-1) \to 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O} \to \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O}(1) \to \mathcal{F} \to 0.
\]

The morphism \( \mathcal{O}(-1) \to 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \) occurring above is zero and \( \text{Ext}^1(\mathbb{C}_x, \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)) = 0 \), so we can argue as at 2.3.2 \([9]\) to conclude that \( \mathcal{F} \) is a trivial extension of \( \mathbb{C}_x \) by \( \mathcal{F}' \). This contradicts the semi-stability of \( \mathcal{F} \). Assume, finally, that \( \mathcal{F}' \) has the second resolution. We can apply the horseshoe lemma as above, leading to the resolution

\[
0 \to \mathcal{O}(-1) \to \mathcal{O}(-4) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O} \to \mathcal{O} \oplus 2\mathcal{O}(1) \to \mathcal{F} \to 0.
\]

We see from this that \( h^1(\mathcal{F}(1)) = 1 \). }
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