Robust and Reliable Transfer of a Quantum State Through a Spin Chain

Zhao-Ming Wang, Mark S. Byrd, Bin Shao, and Jian Zou

1 Department of Physics, Southern Illinois University, Carbondale, Illinois 62901-4401
2 Department of Physics, Beijing Institute of Technology, Beijing, 100081, China
3 Department of Computer Science, Southern Illinois University, Carbondale, Illinois 62901-4401

We present several protocols for reliable quantum state transfer through a spin chain. We use a simple two-spin encoding to achieve a remarkably high fidelity transfer for an arbitrary quantum state. The fidelity of the transfer also decreases very slowly with increasing chain length. We find that we can also increase the reliability by taking advantage of a local memory and/or confirm transfer using a second spin-chain.

PACS numbers: 03.67.Hk, 03.67.Pp, 75.10.Jm

I. INTRODUCTION

One of (Di Vincenzo’s) seven requirements of a quantum computer is the transmission of flying qubits, meaning, we must be able to transmit information between components of a quantum computing device. In the interior of a such a device, where short-distance communication is required, a spin chain is a promising candidate for the transfer of information.

Reliable transfer through a spin chain has been studied extensively since the original proposal by Bose [1]. In that proposal, a spin state at one end of the chain is allowed to evolve freely under constant couplings until it arrives after some time at the other end of the chain. Such a system is simple and does not require couplings to be precisely tuned or to be switched on and off. This is desirable for experimental considerations where controllability can be a definite problem.

It is indeed somewhat surprising that a spin state can be reliably transferred through a chain. However, it certainly can be done when particular conditions are met. For example, with finely-tuned, yet fixed couplings, a variety of networks will allow for perfect transfer [2, 3]. There are also methods using two chains which allow for perfect transfer [4, 5]. In this case the perfect transfer is conditioned on the outcome of two measurements, one from each chain. Other methods require a wave packet to be constructed at the beginning of the chain so that the state can be transferred reliably [6, 7].

In each of these scenarios, the state is transferred using an always-on Heisenberg exchange interaction with nearest-neighbor interactions between the spins. The Heisenberg exchange interaction is readily available in many different experimental systems. Thus its use is well-motivated. However, its experimental viability is important for another reason—it enables universal quantum computation on decoherence-free subspaces and noiseless subsystems without the need for individual control over physical qubits. (See [8] and references therein.) There are several documented instances of this. One promising proposal uses two spins, or qubits, to encode one logical qubit. In our case, the universality condition, prompted the following question, “Can a decoherence-free, or noiseless, encoding be used to enable the reliable propagation of a spin state through a spin chain?” In this article, we answer this question and show that a particular state can be used to reliably transfer quantum information over long distances through a spin chain. Our proposal uses encoded states which provide reliable state transfer over relatively long distances through an unmodulated spin chain. This makes our proposal a prime candidate for use in experimental systems where this sort of state transfer is required. Namely within a solid-state quantum computing device.

II. THE HAMILTONIAN AND THE CALCULATION OF THE FIDELITY

The Hamiltonian of a one dimension anisotropic Heisenberg XY model can be described by

$$H = -\frac{J}{2} \sum_{i=1}^{N-1} (\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + J_z \sum_{i=1}^{N-1} \sigma_i^z\sigma_{i+1}^z - h \sum_{i=1}^{N} \sigma_i^z.$$ 

where $J$ is the exchange constant for the $xy$ components, $J_z$ is the exchange constant for the $z$ component, and $h$ is external magnetic field along $z$ direction. The $\sigma_i^{x,y,z}$ are the Pauli operators acting on the $i^{th}$ spin. We will use a ferromagnetic coupling and take $J = 1.0$ throughout this paper. Furthermore, we consider only nearest-neighbor interactions and an open ended chain which is the most natural and practical geometry for this system.

Note that for this Hamiltonian, the $z$-component of the total spin, $\sigma^z = \sum \sigma^z_i$, is a conserved quantity, which indicates that the system contains a fixed number of magnon excitations. When there is only one magnon excitation, the time evolution of the initial state is not affected by the $\sigma^x - \sigma^y$ interaction, whereas for the two-magnon excitations it is. This can be quite complicated [10] due to the magnon interactions arising from a nonzero $J_z$. For these reasons, we let $J \gg J_z$.

The Hamiltonian can be diagonalized by means of the Jordan-Wigner transformation that maps spins to one-dimensional spinless fermions with creation operator de-
defined by \( c_i^\dagger = \frac{j-1}{N} \sigma_i^x \), where \( \sigma_i^x = \frac{1}{2} (\sigma_i^x + i\sigma_i^y) \) denotes the spin raising operator at site \( i \). The action of \( c_i^\dagger \) is to flip the spin at site \( i \) from down to up and \( c_i, c_i^\dagger \) satisfy the anticommutation relations \( \{c_i, c_j^\dagger \} = \delta_{im} \).

The creation operator evolves as \( i^{11} \)

\[
c_j^\dagger(t) = \sum_{l=1}^{N} f_{j,i}(t) c_l^\dagger,
\]

where \( N \) is the number of spins,

\[
f_{j,i}(t) = \frac{2}{N + 1} \sum_{m=1}^{N} \sin(q_m j) \sin(q_m l) e^{-iE_m t},
\]

\( E_m = 2h - 2J \cos(q_m) \), and \( q_m = \pi m / (N + 1) \). Eq. \( 11 \) indicates that the excitation which, initially created in site \( j \), is generally distributed over all the sites. At time \( t_1 \) the probability of the excitation being at site \( l \) is \( |f_{j,i}(t_1)|^2 \) with the normalization condition \( \sum_{l=1}^{N} |f_{j,i}(t)|^2 = 1 \).

When the number of magnon excitations is more than one, the time evolution of the creation operators is given by \( 12 \)

\[
\prod_{m=1}^{M} c_{j_m}^\dagger(t) = \sum_{l_1 \ldots l_M} \det \begin{bmatrix} f_{j_1, l_1} & f_{j_2, l_2} & \cdots & f_{j_M, l_M} \\ f_{j_1, I_{M+1}} & f_{j_2, I_{M+1}} & \cdots & f_{j_M, I_{M+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{j_1, I_{M+M}} & f_{j_2, I_{M+M}} & \cdots & f_{j_M, I_{M+M}} \end{bmatrix} \prod_{m=1}^{M} c_{l_m}^\dagger,
\]

where \( M \) is the number of the excitations. The row \( \{j_1, j_2, \ldots, j_M\} \) denotes the sites where the excitations are created and \( \{l_1, l_2, \ldots, l_M\} \) \( (l_1 < l_2 < \cdots < l_M) \) denotes an ordered set of \( M \) different indices from \( \{1, 2, \ldots, N\} \). For the states presented here, \( M \leq 2 \).

Our procedure is as follows. First we cool the system to the ferromagnetic ground state \( |0\rangle \), where all spins are down. Then we encode the state \( |\varphi(0)\rangle = \alpha |0_L\rangle + \beta |1_L\rangle \) at one end of the chain. The initial state of the whole system is then \( |\Phi(0)\rangle = (\alpha |0_L\rangle + \beta |1_L\rangle) \otimes |0\rangle \). Note that we are using \( |L\rangle \) to denote a logical basis state, emphasizing that our physical spins are encoded into a logical qubit.

**III. DFS ENCODINGS**

We will consider several different encodings with the potential for reliable information transfer and will provide the best overall solution at the end of our analysis. In each case we are motivated to consider a state encoded in a DFS given the universality properties of the states with respect to the Heisenberg exchange interaction. We will consider two, three, and four-qubit DFSs which encode one logical qubit using a subsystem of three or four physical qubits, respectively.

The two-qubit encoding uses \( |0_L\rangle = |01\rangle \) and \( |1_L\rangle = |10\rangle \). For three-qubit encoding \( 13 \), \( |0_L\rangle = \alpha_0(|010\rangle - |100\rangle) + \beta_0(|011\rangle - |101\rangle) / \sqrt{2} \), \( |1_L\rangle = \alpha_1(2|001\rangle - |010\rangle - |100\rangle) / \sqrt{6} + \beta_1(-2|110\rangle + |011\rangle + |101\rangle) / \sqrt{6} \). The notation \( \alpha_0, \beta_0 \) means that the arbitrary superposition of the state \( (|010\rangle - |100\rangle) / \sqrt{2} \) and state \( (|011\rangle - |101\rangle) / \sqrt{2} \) does not change the state \( |0_L\rangle \) and \( \alpha_1, \beta_1 \) has the same meaning. For the four-qubit encoding \( 14 \), \( |0_L\rangle = (|0101\rangle + |1010\rangle - |0110\rangle - |1001\rangle) / 2, |1_L\rangle = (2|0011\rangle + 2|1100\rangle - |0110\rangle - |1001\rangle - |0101\rangle - |1010\rangle) / \sqrt{2} \). The three-qubit and four-qubit DFSs protect a single logical qubit from collective errors of any type (bit-flip, phase-flip or both) with the three-qubit encoding being the most efficient \( 13, 14 \).

The fidelity between the received state and ideal state \( |\varphi(0)\rangle \) is defined by \( F = \langle \varphi(0) | \rho(t) | \varphi(0) \rangle \), where \( \rho(t) \) is the reduced density matrix at the receiving end. (We let our ideal final state be represented by the same vector as our initial state although it is actually at the end, rather than the beginning, of the chain.)

For example, for the two-qubit encoding, the fidelity at sites \( N - 1, N \) is \( F = |\alpha^* A_N + \beta^* A_{N-1} | \) where \( A_l = \beta f_1 + \alpha f_2 \). Similarly we can calculate the fidelity for the three- and four-qubit encodings, although the expressions are understandably much more complicated.

From the initial state, \( |\Phi(0)\rangle \), the system undergoes the time evolution given by Eqs. \( 11-13 \). The transmission amplitude \( f_{j,i}(t) \) describes the propagation and the dispersion characterizes the state transfer. Note also that if the initial state involves a fixed excitation number, the magnetic field will only produce a global phase which will not affect the fidelity. (See, for example, the fidelity of the two-spin encoding above.) So for a fixed excitation number, we will neglect the effect of magnetic field be fixed \( h = 1.0 \).

**FIG. 1: Length dependence of the maximum fidelity that can be obtained when transferring a quantum state \( |0_L\rangle + |1_L\rangle / \sqrt{2} \) from one end of the chain to the other end.**
IV. DFS-ENCODING RESULTS

In Fig. 1 we compare the maximum fidelity for logical state transfer when \( |\Phi(0)\rangle = (|0_L\rangle + |1_L\rangle)/\sqrt{2} \) for different logical/encoded states. Throughout this article the maximum fidelity is found in the time interval \([0,100]\) since the first peak, which is the maximum, is obtained in this time interval for \( N \leq 80 \). For the 3-qubit encoding, we use 3-qubit(1) and 3-qubit(2) to signify one excitation or two excitations in the chain, for example a state in 3-qubit(1) has \( \alpha_0 = \alpha_1 = 1 \).

For the two-spin encoding with \( N = 4,5 \), \( F_{\text{max}} \approx 1 \) thus near-perfect state transfer can be obtained for these values of \( N \). However, for these and almost all other logical states using two, three, or four spins, the maximum decreases quickly with increasing \( N \). There is one exception in the space of the 3-qubit encoding. For a particular state, the fidelity decreases very slowly with increasing \( N \). We will explore this particular case, which shows impressive variance in the fidelity in Fig. 1 and show how it can be used to reliably transfer an arbitrary qubit state through the chain.

V. THREE QUBITS

It turns out that two excitations in the chain do not allow for reliable transfer. So we will consider the 3-qubit(1) DFS. In Fig. 2 we plot the maximal fidelity, \( F_{\text{max}} \), as a function of \( N \) and \( \theta \) when \( \phi = 0 \) and \( t \in [0,100] \) for an arbitrary initial state \( \cos(\phi)/2|0_L\rangle + \sin(\phi)e^{i\phi}|1_L\rangle \). For \( N \) from 6 to 50, a maximum is achieved at \( \theta = 2\pi/3 \). And surprisingly, \( F_{\text{max}} \) decreases very slowly with increasing \( N \) for a wide range of \( \theta \). For example, in the range \( \theta \in [0.5\pi,0.8\pi] \), for any site \( N \leq 50 \), \( F_{\text{max}} > 0.8 \). So if the state is encoded in this range, the fidelity is exceptionally large for a quite long chain. In fact, we have found that \( F \approx 0.7 \) after traversing a spin chain of two hundred spins! Therefore, we can achieve a very reliable state transfer since the fidelity is large and provides a significant robustness to errors in the initial encoding, or during transport, since a variation in the encoded state does not significantly affect the long distance encoding, or during transport, since a variation in the encoded state does not significantly affect the long distance trend in the fidelity. It still decreases very slowly over long distances. We next show how to take advantage of this remarkable state.

VI. EFFICIENT ENCODING FOR ENCODED QUBITS

When \( \phi = 0, \theta = 2\pi/3 \) the 3-qubit(1) encoding can be written as \( |\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |100\rangle) = \frac{1}{\sqrt{2}}(|01\rangle_{13} - |10\rangle_{13} \otimes |0\rangle_2 \), i.e. the first and third spins are in a singlet state. In order to show the high-fidelity transfer of this state, in Fig. 3 we plot the time evolution of the fidelity at every site of a chain of length \( N = 48 \). At \( t = 0 \) the initial state is encoded at the sites of the first and third spins, then it evolves freely under the Heisenberg Hamiltonian. The time dependence is given by \( |\Phi(t)\rangle = 1/\sqrt{2} |\sum_{i=1}^{N}(f_{3,i} - f_{1,i})|e^{i\theta}|0\rangle \). At time \( t \), the fidelity at the \( i-2,i-1,i \) sites in the interior of the chain is \( F \approx 0.5 \) implying only partial information is located at these sites. However, at the end of the chain (\( i = 48 \)), the fidelity shows a peak (\( F_i \approx 0.86, i \approx 25 \)). After this, the wave is reflected by the boundary, and starts to propagate back. This behavior can be interpreted as a wave which broadens inside the chain, but when arriving at the boundary it becomes narrower which enhances the fidelity. Thus we have an end-effect of the chain. From Fig. 3 the oscillation of the state between boundaries is \( T \approx 50 \) and at time \( t_k \approx 25 + (k - 1)T \), a maximum is achieved at the end of the chain \( N = 48 \), where \( k \) denotes the kth peak. For example, at time \( t = 75 \), the state will travel to the other end once more (the second peak \( F_2 \approx 0.76 \)), but \( F_2 < F_1 \) which shows some reduc-
tion of fidelity with each pass, but still relatively high.

We have shown that the state \( 1/\sqrt{2} |001\rangle - |100\rangle \) can be transferred through the chain with high fidelity even when the chain is quite long. For quantum communication, we need to transfer an arbitrary state with high fidelity. In this case it is important to realize that an encoding which uses: \( |0_L\rangle = |00\rangle \) and \( |1_L\rangle = 1/\sqrt{2} |001\rangle - |100\rangle \) can reliably transfer the state since the vacuum state is fixed throughout. Using this encoding we can fully utilize the extremely reliable state \( |\Psi\rangle_L = \alpha |000\rangle + \beta/\sqrt{2} |001\rangle - |100\rangle \).

\[ F_{av} = \frac{1}{3} \sum \frac{1}{3} \text{Re} \left[ \frac{G_N - G_{N-2}}{\sqrt{2}} \right] + \frac{1}{3} \left( \frac{G_N - G_{N-2}}{\sqrt{2}} \right)^2 + \frac{1}{3} \left( 1 - |G_N|^2 - |G_{N-1}|^2 - |G_{N-2}|^2 \right). \]  

where \( G_i = \frac{1}{\sqrt{2}} \left( f_{3,i}(t) - f_{1,i}(t) \right) \). The results are maximized over the time interval \( [0,100] \) and magnetic field \( h \in [0,2] \). (Unlike transferring the \( |1_L\rangle \) state, here the magnetic field can be adjusted to enhance the fidelity since the phase is now not negligible.) For single-spin encoding, \( F_{av} \) decreases relatively quickly with increasing \( N \). However, using our scheme, \( F_{av} \) decreases very slowly with increasing \( N \) and \( F_{av} \) is always relatively high even in a long chain \( (N \geq 80) \). For example \( N \approx 70 \), \( F_{av} \) is still greater than 0.9.

\[ X_L^{(1,3)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]  

(From this it is straight-forward to obtain the logical CNOT which is also to be performed on spins one and three.) The state of the chain is then \( |\Psi\rangle = \alpha |0_L\rangle \otimes |1_L\rangle + \beta |1_L\rangle \otimes |0_L\rangle \) where the first factor for the first chain and the second for the second chain. We now let both chains evolve freely. After the same amount of time as above, we perform a logical CNOT operation to decode the operation. Performing a measurement on the last three spins of the second chain and finding \( |000\rangle \) will confirm that the state has been reliably sent through the chain. This confirmation, along with the very high fidelity of our protocol, provides a high probability of reliable transfer, along with confirmation.

\section{VII. Protocols for improving reliability}

Here we present two protocols for increasing the reliability of state transfer using our encoded state. One is due to Giovannneti et al. \[13,16\] and the other is based on, but a generalization of, a protocol by Burgarth and Bose \[4\].

In the first protocol we consider, Giovannetti et al. \[13,16\] showed that the reliability of the the one-spin encoding can be enhanced using a memory. In this protocol, the receiver swaps the state at the end of the chain to a quantum memory for decoding at a later time. This process is repeated for later times, with each swap and storage increasing our overall chance of success. Fig. 3 shows the variation of the fidelity from which we may infer the chance of success for our protocol. If we perform the swap operation as in Ref. \[13\] at \( t = 25 \), the probability that the \( |1_L\rangle \) state has been swapped to the first memory is \( \eta = \frac{1}{46} | \langle 1_L | \langle 0 | e^{-iHt} | 1_L \rangle_L | 12,3 \rangle 0 \rangle^2 \approx 0.86^2 \), which corresponds to the square of the first peak value at \( N = 48 \). Performing additional swap operations at some later optimal time will increase our already large probability for success, just as it does in the original protocol for the single-spin encoding.

We next provide a protocol which can confirm if a state was indeed transferred appropriately, which is a generalization of that presented in Ref. \[4\]. We begin with two spin chains which are initially decoupled and proceed as follows. First the logical state is encoded into the first and third site of the first chain. Then a logical X gate is performed on spins one and three of the second chain conditioned on the logical state of the first chain being zero \[17\] (using the standard ordered basis). The form of the logical X is

\[ X_L^{(1,3)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]

\section{VIII. Conclusions}

We have presented several results for a decoherence-free/noiseless subspace encoded qubit which is transferred through an unmodulated spin chain. Although many of the encoded states were not more reliable than
the single-spin encoding, we have found remarkable results. A most striking result is that when the initial state is a 3-qubit state, \( \cos(\frac{\theta}{2})|0_L\rangle + \sin(\frac{\theta}{2})e^{i\phi}|1_L\rangle \), with \( \theta \) near \( \frac{2\pi}{3} \) and \( \phi \) near zero, is sent through the spin chain, the fidelity is incredibly high. Surpassing any known result so far. Even out to two hundred spins, the fidelity is quite high (\( \approx 0.7 \)). For transferring an arbitrary state, we have found a very high fidelity based on these results. For example, when \( N \leq 70 \) the fidelity \( F \geq 0.9 \). This is a remarkable result for a simple two-spin encoded state and experimentally viable due to its Heisenberg-mediated transfer with control assumed only on two of the three at the ends of spins of the chain.

Furthermore, we have shown that our protocol can be combined with a protocol using a local memory to enhance the fidelity beyond an already impressive value. We have also presented a protocol for confirmation of the receipt of the state at the other end. We therefore believe this is by far the best protocol to date for the transfer of a quantum state through an unmodulated spin chain.

**Acknowledgments**

This material is based upon work supported by NSF-Grant No. 0545798 to MSB. ZMW thanks the China Scholarship Council. We acknowledge C. A. Bishop and Y.-C. Ou for helpful discussions.

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