Analog Schwarzschild-like geometry in fluids with external pressure

Neven Bilić

Theoretical Physics Division, Rudjer Bošković Institute, 10002 Zagreb, Croatia

March 29, 2022

Abstract

We study acoustic geometry in fluids with external pressure. In particular, we examine the conditions under which the acoustic metric mimics a Schwarzschild-like metric. We demonstrate that it is possible to mimic a Schwarzschild-like geometry in a consistent way only in the framework of relativistic acoustic geometry.

1 Introduction

Analog gravity is based on the dynamics of acoustic perturbations in a fluid with a nonhomogeneous flow. Since Unruh’s discovery [1] that a supersonic flow may cause analog Hawking radiation a lot of research on analog gravity has been done (see, e.g., Refs. [2, 3] for reviews).

The acoustic perturbations in a fluid propagate in an effective curved spacetime geometry with the analog metric dubbed the acoustic metric. In general, the acoustic metric $G_{\mu\nu}$ in its relativistic form is given by [4, 5, 6]

$$G_{\mu\nu} = \frac{n}{m^2 c_s w} [g_{\mu\nu} - (1 - c_s^2) u_\mu u_\nu],$$

where $u_\mu$ is the 4-velocity of the fluid flow in the background metric $g_{\mu\nu}$, usually taken to be the flat Minkowski metric, $c_s$ is the speed of sound in units of the speed of light, $n$ is the particle number density, $w$ is the relativistic specific enthalpy, and the mass scale $m$ is introduced to make the conformal factor dimensionless. The fluid is assumed to satisfy the Euler equation with no external pressure and isentropy and particle number conservation are usually assumed. In the nonrelativistic (NR) approach the acoustic metric takes the form [1, 2, 7]

$$G_{\mu\nu}^{\text{NR}} = \rho_{\text{NR}} v_s \left( \begin{array}{cc} v_s^2 - v_i^2 & v_j \\ v_i & -\delta_{ij} \end{array} \right),$$

where $v_s^2 = c_s^2 - 1$. 

*bilic@irb.hr
where $v$ is the velocity of the flow, $v_s \equiv c_s c$ is the speed of sound, and $\rho_{\text{NR}}$ is the mass density. The conformal factor in (2) as it stands is not dimensionless. This fact is usually ignored because the analog spacetime line element will have the right dimension if the metric tensor is multiplied by a constant conformal factor of dimension $L^4 M^{-1} T^{-1}$.

In the context of acoustic geometry, a natural question arises whether it is possible to mimic the Schwarzschild spacetime, or even more generally, a static spacetime of the Schwarzschild-like type. The answer to this question is of considerable scientific importance because Schwarzschild-like metrics includes several metrics of theoretical and phenomenological interest, e.g., the Reissner-Nordström spacetime or the de Sitter and anti-de Sitter spacetimes in static coordinates.

A few attempts were made to mimic the exact form of the Schwarzschild black hole. However, it seems that the corresponding acoustic metric of the form (1) or (2), satisfying the assumptions mentioned above, is not to be found. As noted in Refs. [2, 3], assuming a constant speed of sound, it is possible to construct a NR metric of the form (2) differing from the Schwarzschild metric by a nonconstant conformal factor. With nonisentropic fluids recently studied in Ref. [8], one has more flexibility. However, in a recent paper [9] it was shown that the Schwarzschild metric could be reproduced in a nonisentropic fluid only in the stiff-fluid limit when the sound speed approaches unity. de Oliveira et al. [10] have recently obtained a closed-form expression for an analog Schwarzschild metric in NR setup with a nonconstant speed of sound and a coordinate dependent external force. As we will shortly see, the external force can be equivalently described by a nonvanishing gradient of external pressure. Unfortunately, as we will demonstrate here, the suggested scheme is not consistent, even assuming the presence of an external potential with a nonvanishing gradient.

This paper is dealing basically with two issues: first, we derive a correct form of the acoustic metric in a fluid with external pressure, and second, we explore the possibility to mimic the exact Schwarzschild black hole in such a fluid. Effects of external pressure, or equivalently external potential, have usually been ignored in the analog gravity literature as it is claimed that upon linearization, the fluctuations are insensitive to any external force [2]. Here we will show that external pressure with a nonvanishing gradient may exhibit nontrivial effects in the framework of both relativistic and nonrelativistic acoustic geometry. As to the second issue, we will describe a procedure by which the relativistic acoustic metric exactly reproduces a Schwarzschild-like metric. To this end, we will study a spherically symmetric acoustic metric with a nonvanishing external pressure gradient. We will demonstrate that this procedure will be consistent only if the fluid is essentially relativistic.

**Notation:** We use the $+−−$ signature convention and, as a rule, we work in units $c = h = 1$. The exception is the NR regime in sections 2.3 and 3.2 where we reinstate the speed of light $c$.

The remainder of the paper is organized as follows. In section 2 we discuss the relativistic hydrodynamics of the fluid with external pressure and derive both the relativistic and the nonrelativistic acoustic metrics. Section 3 is devoted to the construction of the Schwarzschild-like geometry as an analog gravity model in a fluid with external pressure. In section 4, we summarize our results and give concluding remarks.
2 Isentropic flow with external pressure

The analog acoustic geometry has been derived under the strict requirements of energy-momentum conservation, particle number conservation, and vanishing vorticity. The first two restrictions are sufficient conditions for adiabaticity. If a stronger restriction of isentropy is assumed together with vanishing vorticity, the velocity field may be expressed as a potential flow, i.e., [11]

\[ wu_\mu = \partial_\mu \theta, \]  

(3)

where \( \theta \) is the velocity potential and \( w \) is the relativistic specific enthalpy. The reverse of the above statement is not true: a potential flow alone implies only vanishing vorticity and implies neither particle number conservation nor isentropy.

Next, we consider an isentropic fluid flow under the influence of an external pressure field. We will show that if the external pressure, or equivalently external potential, has a nonvanishing gradient, the potential flow equation (3) needs to be modified.

2.1 Energy-momentum conservation and Euler equation

The most fundamental assumption is the energy-momentum conservation

\[ T^\mu_\nu;_\nu = 0, \]  

(4)

where we assume that \( T^\mu_\nu \) is the energy-momentum tensor of an ideal relativistic fluid modified to account for the presence of a conservative external force. To this end we add a term of the form \( p^\text{ext} g^\mu_\nu \), i.e., we assume

\[ T^\mu_\nu = (p + \rho)u^\mu u_\nu - pg^\mu_\nu - p^\text{ext} g^\mu_\nu \]  

(5)

where \( p \) and \( \rho \) are the fluid pressure and energy density, respectively, \( p^\text{ext} \) is the external pressure, and \( g^\mu_\nu \) is the background metric. The contraction of (4) with \( u_\mu \) gives

\[ u^\mu \rho^\mu + (p + \rho)u^\mu u_\mu - u^\mu p^\text{ext}_\mu = 0. \]  

(6)

Inserting this into (4) with (5) gives a modified relativistic Euler equation

\[ (p + \rho)u^\nu u_\mu;_\nu - (p + p^\text{ext})_\mu u^\nu (p + p^\text{ext})_\nu u_\mu = 0, \]  

(7)

which differs from the standard expression [11] by two additional terms due to external pressure.

The assumption that an external force modifies the energy-momentum tensor in the form (5) may be justified by two arguments. The first one is based on the Lagrangian formulation. Suppose that the perfect fluid stress tensor corresponds to a Lagrangian \( \mathcal{L}(\varphi, X) \) that depends on the field \( \varphi \) and its kinetic term \( X = g^\mu_\nu \varphi^\mu_\nu \varphi^\nu \). Then, \( T^\mu_\nu \) is obtained by taking the functional derivative of the action with respect to \( g^\mu_\nu \), as usual. Now, one can add to the action an external potential \( U \), i.e., \( S = \int d^4x \sqrt{-g}(\mathcal{L} - U) \), where \( U = U(x) \) is just a function of \( x \) (not a dynamical field). This function serves as a potential for an external force. The functional derivative of \( S \) with respect to \( g^\mu_\nu \) will produce \( T^\mu_\nu \) in the form (5) with \( p = \mathcal{L}, \rho = 2X \mathcal{L}_X - \mathcal{L}, u_\mu = \varphi^\mu_\mu / \sqrt{X} \) and \( p^\text{ext} = -U \).
The second argument is based on the nonrelativistic limit of the Euler equation. In that limit, as we will shortly demonstrate, the energy-momentum conservation (4) with (5) and the Euler equation (7) in turn, yields the Euler equation in the standard nonrelativistic form in which \( p^{\text{ext}} \) appears in the combination \( p + p^{\text{ext}} \) and hence, apparently plays the role of external pressure. Then, the external force is proportional to the gradient of \( p^{\text{ext}} \).

Next, we introduce a few useful thermodynamic relations. For a general thermodynamic system at nonzero temperature \( T \) the first law of thermodynamics may be written as

\[
dp = n dw - nT ds,
\]

where \( n \) is the particle number density, \( s \) is the specific entropy, i.e., the entropy per particle, and \( w \) is the relativistic specific enthalpy defined as

\[
w = \frac{p + \rho}{n}.
\]

Next, we assume that the flow is isentropic, i.e., we set \( ds = 0 \). For an isentropic flow, from Eqs. (8) and (9) it follows that

\[
w_{,\mu} = \frac{p_{,\mu}}{n},
\]

and

\[
n_{,\mu} = \frac{\rho_{,\mu}}{w}.
\]

Using (9) and (10) from Eq. (7) it follows that

\[
u^{\nu}(wu_{,\nu})_{,\nu} - w_{,\mu} = \frac{1}{n}(p^{\text{ext}}_{,\mu} - u^{\nu}p^{\text{ext}}_{,\nu}u_{,\mu}).
\]

Similarly, Eq. (6) with (9) and (11) gives

\[
(nu^{\mu})_{,\mu} - \frac{1}{w} w^{\mu} p^{\text{ext}}_{,\mu} = 0.
\]

Clearly, the usual continuity equation

\[
(nu^{\mu})_{,\mu} = 0
\]

will not hold if the external pressure gradient is nonzero and hence, the particle number is generally not conserved. The particles are locally created or destroyed depending on the sign of \( u^{\mu} p^{\text{ext}}_{,\mu} \). Suppose that \( w \) is positive which, given the definition (9), holds for normal matter satisfying the weak energy condition. Then, Eq. (13) states that the particles are locally created/destroyed if the gradient of the external pressure \( w^{\mu} p^{\text{ext}}_{,\mu} \) is positive/negative. For matter that violates weak energy condition—the so called “phantom matter”—the condition is reversed: the particles will be locally created/destroyed if the gradient of the external pressure is negative/positive.

This situation is similar to the case of a nonisentropic fluid where, as demonstrated in Ref. [8], the particle number conservation is violated for a large class of nonisentropic fluids with nonvanishing entropy gradients.
Nota bene: It may easily be demonstrated (see section 2.4) that in the NR limit $c \to \infty$ the second term in (13) is $1/c^2$ suppressed compared with the first term. Hence, the above-mentioned particle production is purely a relativistic effect. It is worth mentioning that in most applications of thermodynamics and fluid dynamics in cosmology the conservation of both particle number and entropy has been assumed (see, e.g., [12] and references therein).

In the absence of external pressure, some of the equations would further simplify if the enthalpy flow $wu_\mu$ were a gradient of a scalar potential, i.e., if the velocity field satisfied Eq. (3). Then, as a consequence of (3), the left-hand side of (12) would vanish identically. The assumption (3) is automatically satisfied in the field-theory formalism as demonstrated in Ref. [8].

Obviously, equation (3) would not solve (12) if there existed external pressure with a nonvanishing gradient. However, in this case, we show that a suitably modified potential-flow equation can solve Eq. (12).

### 2.2 Modified potential flow

For a stationary flow with external pressure it is possible to define an effective enthalpy function $W$,

$$W = w + \tilde{w},$$

where the time independent function $\tilde{w}$ satisfies

$$u^\nu (\tilde{w}u_\mu)_{,\nu} - \tilde{w}_{,\mu} + \frac{1}{n} (p^\text{ext}{}_{\mu} - u^\nu p^\text{ext}{}_{\nu} u_\mu) = 0.$$  

Then, Eq. (12) takes the form

$$u^\nu (Wu_\mu)_{,\nu} - W_{,\mu} = 0.$$  

This equation will be satisfied if the effective enthalpy flow $Wu_\mu$ is a gradient of a potential as in (3), i.e., if there exist a scalar function $\theta$ such that

$$W_{\mu} = \partial_\mu \theta.$$  

To find $\tilde{w}$ we multiply Eq. (16) by the time-translation Killing vector $\xi^\mu$ and use the fact that $\xi^\mu p^\text{ext}{}_{\mu} = \xi^\mu \tilde{w}_{,\mu} = 0$ since $p^\text{ext}$ and $\tilde{w}$ for a stationary flow do not depend on time. By making use of the Killing equation $\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0$ we find

$$u^\nu \left[ (\tilde{w}\xi^\mu u_\mu)_{,\nu} - \frac{1}{n} \xi^\mu u_\mu p^\text{ext}{}_{\nu} \right] = 0$$  

Since $u^\mu$ is basically an arbitrary timelike field we obtain a simple differential equation for $\tilde{w}$

$$\tilde{w} = \frac{1}{\xi^\mu u_\mu} \int \frac{\xi^\mu u_\mu}{n} dp^\text{ext}.$$  

with solution

$$\tilde{w} = \frac{1}{\xi^\mu u_\mu} \int \frac{\xi^\mu u_\mu}{n} dp^\text{ext}.$$
This integral may be thought of as a line integral along an arbitrary curve starting from a fixed spacetime point and ending at \( x \). The integration curve may be conveniently chosen depending on the symmetry of the flow. For example, for a radial flow in a static spherically symmetric spacetime with \( \xi^\mu = (1; \vec{0}) \) we have

\[
\tilde{w}(r) - \tilde{w}(r_0) = \frac{1}{\gamma\sqrt{g_{00}}} \int_{r_0}^{r} \frac{\gamma\sqrt{g_{00}}}{n} \partial_r p^\text{ext} dr,
\]

where \( r_0 \) is arbitrary, \( \gamma = (1 - v^2)^{-1/2} \) is the usual relativistic factor, and \( v \) is the radial velocity of the fluid.

Next, we derive the acoustic metric assuming a potential flow defined by (18).

### 2.3 Acoustic metric

The acoustic metric is the effective metric perceived by acoustic perturbations propagating in a perfect fluid background. Under certain conditions, the perturbations satisfy the Klein-Gordon equation in curved geometry with metric of the form (1).

We first derive a propagation equation for linear perturbations of an isentropic flow assuming a fixed background geometry. Given some average bulk motion represented by \( p, n, \) and \( u^\mu \), and external pressure \( p^\text{ext} \), following the standard procedure \([5]\), we replace

\[
p \to p + \delta p, \quad n \to n + \delta n, \quad W \to W + \delta W, \quad u^\mu \to u^\mu + \delta u^\mu,
\]

where \( \delta p, \delta n, \delta W, \) and \( \delta u^\mu \) are small disturbances. As we do not include the nonadiabatic perturbations, since for an isentropic flow \( \delta s = 0, \) we may assume that there exists an equation of state \( n = n(W) \). Using this and the replacements (23) in equation (13) at linear order we find

\[
\frac{\partial n}{\partial W} \delta W u^\mu + n \delta u^\mu = \frac{1}{w^2} u^\mu p^\text{ext}_\mu (\delta W - \delta \tilde{w}) + \frac{1}{w} p^\text{ext}_\mu \delta u^\mu
\]

Here and from here on, it is understood that the partial derivatives are taken at fixed \( s \). The adiabatic perturbations \( \delta W, \delta u^\mu, \) and \( \delta \tilde{w} \) may be expressed in terms of the velocity-potential perturbation \( \delta \theta \). From (18) it follows that

\[
\delta W = u^\mu \delta \theta_{,\mu},
\]

\[
W \delta u^\mu = (g^{\mu\nu} - u^\mu u^\nu) \delta \theta_{,\nu}
\]

According to (21), the perturbation of the quantity \( \tilde{w} \) is induced by the perturbation of \( n \) and hence

\[
\delta \tilde{w} = \frac{\partial \tilde{w}}{\partial n} \partial n \delta W u^\nu \delta \theta_{,\nu}.
\]

To simplify the notation, in the following we denote by \( \chi \) the perturbation \( \delta \theta \equiv \chi \). Besides, we introduce an effective speed of sound \( \tilde{c}_s \) defined by

\[
\tilde{c}_s^2 = \frac{n \partial W}{W \partial n} = \frac{1}{1 + \tilde{w}/w} \left( c_s^2 + \frac{n \partial \tilde{w}}{w \partial n} \right),
\]
where \( c_s \) is the usual adiabatic speed of sound defined by
\[
c_s^2 = \frac{\partial p}{\partial \rho} = \frac{n \partial w}{w \partial n}.
\]

Then, combined with (25)-(28), equation (24) takes the form
\[
(f^\mu\nu \chi_{,\nu})_{,\mu} = \frac{1}{wW} \left[ g^{\mu\nu} - \left( 1 + \frac{W}{w} - \frac{n \partial \tilde{w}}{w \partial n} \frac{1}{c_s^2} \right) u^\mu u^\nu \right] p_{,\mu}^{\text{ext}} \chi_{,\nu}
\]
where
\[
f^{\mu\nu} = \frac{n}{W} \left[ g^{\mu\nu} - \left( 1 - \frac{1}{c_s^2} \right) u^\mu u^\nu \right].
\]

Applying the standard procedure we can recast (30) into the form
\[
\frac{1}{\sqrt{-G}} \partial_{\mu} \left( \sqrt{-G} G^{\mu\nu} \partial_{\nu}\chi \right) - \frac{m^2 c_s W}{n^2 w} \left[ g^{\mu\nu} - \left( 1 + \frac{c_s^2}{c_s^2} \right) u^\mu u^\nu \right] p_{,\mu}^{\text{ext}} \chi_{,\nu} = 0.
\]

Here, the matrix
\[
G^{\mu\nu} = \frac{1}{\omega} [g^{\mu\nu} - (1 - \frac{1}{c_s^2}) u^\mu u^\nu],
\]
is the inverse of the effective metric tensor
\[
G_{\mu\nu} = \omega [g_{\mu\nu} - (1 - c_s^2) u_{\mu} u_{\nu}],
\]
with
\[
\omega = \frac{n}{m^2 c_s W},
\]
and the determinant
\[
G \equiv \det G_{\mu\nu} = \omega^4 c_s^2 \det g_{\mu\nu}.
\]
The mass parameter \( m \) in (35) is introduced to make \( G_{\mu\nu} \) dimensionless. Hence, the analog metric, in contrast to the standard expression (1), involves the effective specific enthalpy, the effective speed of sound \( c_s \), and a derivative coupling with the external pressure.

Consider next a stationary flow and time-independent external pressure. Because of Eq. (18), a stationary flow restricts the time dependence of \( \theta \) so that
\[
\theta = mt + g(x)
\]
where \( g \) is an arbitrary time independent function and \( m \) is a constant which may be identified with the mass scale parameter in (35). Equation (37) together with (18) fixes the effective specific enthalpy
\[
W = \frac{m}{u_0} = \frac{m}{\gamma}.
\]

Next, we derive the acoustic metric for a NR fluid with the Newtonian gravitational field and an external pressure field acting on the fluid.
2.4 Nonrelativistic limit

The transition to NR regimes is achieved by transforming the thermodynamic functions as follows:

\[ n \rightarrow \rho_{NR} \frac{m}{c}, \quad \rho \rightarrow \rho_{NR} c^2 + \varepsilon, \quad w \rightarrow mc^2 + mh, \quad (39) \]

where \( \rho_{NR}, \varepsilon, \) and \( h \) are the NR mass, energy, and enthalpy densities, respectively. Next, the NR versions of the continuity and Euler equations are obtained by making use of the background metric in the post-Newtonian gauge \[13, 14, 15\]

\[ ds^2 = \left(1 + 2\frac{\Phi}{c^2}\right)(cdt)^2 - \left(1 - 2\frac{\Phi}{c^2}\right)d\bar{x}^2, \quad (40) \]

where \( \Phi = \Phi(\bar{x}) \) is a Newtonian-like potential. Besides, we make use of the prescription \( v_i \rightarrow v_i/c \) and

\[ \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad u^\mu = \gamma(1/\sqrt{g_{00}}, \bar{v}/c), \quad u_\mu = \gamma(\sqrt{g_{00}}, -\bar{v}/c), \quad \partial_\mu = \left(\frac{\partial}{c\partial t}, \nabla\right). \quad (41) \]

Then, applying this to (7) and keeping the leading terms in \( 1/c^2 \) we find the NR Euler equation

\[ \rho_{NR} \left(\frac{\partial \bar{v}}{\partial t} + \frac{1}{2} \nabla v^2 - \bar{v} \times \nabla \times \bar{v}\right) + \rho_{NR} \nabla \Phi + \nabla (p + p_{ext}) = 0. \quad (42) \]

Assuming, as usual, that the fluid is irrotational, the velocity may be written as a gradient of a scalar potential, i.e.,

\[ \bar{v} = -\nabla \theta, \quad (43) \]

which is a NR version of (3). In this case Eq. (42) simplifies to

\[ \nabla \left(-\frac{\partial \theta}{\partial t} + \frac{1}{2}(\nabla \theta)^2\right) + \nabla \Phi + \frac{1}{\rho_{NR}} \nabla (p + p_{ext}) = 0, \quad (44) \]

which may be reduced to the Bernoulli equation

\[ -\frac{\partial \theta}{\partial t} + \frac{1}{2} v^2 + \Phi + h + V = \text{const}, \quad (45) \]

where we have introduced the NR specific enthalpy \( h \) and the external potential \( V \)

\[ \nabla h = \frac{1}{\rho_{NR}} \nabla p, \quad (46) \]

\[ \nabla V = \frac{1}{\rho_{NR}} \nabla p_{ext}. \quad (47) \]

Clearly, the quantity \( V \) is a potential for an external force \( \bar{F} = \nabla V \) acting on the fluid.

Equations (46) and (47) are the NR versions of equations (10) and (20), respectively.

\[ ^1 \text{Note that we are using the Landau-Lifshitz convention for the sign of the Newton potential, i.e., } \Phi(r) = -GM/r. \]
Next, applying (41) to either (6) or (13) and keeping the leading and next to leading terms in \( v/c \) expansion, we find a modified NR continuity equation
\[
\partial_t \rho_{\text{NR}} + \nabla (\rho_{\text{NR}} \vec{v}) = \frac{1}{c^2} \vec{v} \nabla (p + 2p^\text{ext}) + \frac{3}{c^2} \rho_{\text{NR}} \vec{v} \nabla \Phi + \mathcal{O}(v^4/c^4). \tag{48}
\]
Clearly, the right-hand side is a relativistic correction and hence, in the limit \( c \to \infty \), the NR continuity equation
\[
\partial_t \rho_{\text{NR}} + \nabla (\rho_{\text{NR}} \vec{v}) = 0 \tag{49}
\]
holds.

Now we proceed to derive the NR acoustic metric for a fluid subjected to external potential. Following Visser \[7\], we start from the potential flow equation (43), Bernoulli equation (45), and the NR version of the continuity equation (49). We linearize Eqs. (45) and (49) by replacing
\[
\rho_{\text{NR}} \to \rho_{\text{NR}} + \delta \rho_{\text{NR}}, \quad h \to h + \delta h, \quad \theta \to \theta + \delta \theta, \tag{50}
\]
where the quantities \( \delta \rho_{\text{NR}}, \delta h, \) and \( \delta \theta \) are small acoustic perturbations around some average bulk motion represented by \( \rho_{\text{NR}}, h, \) and \( \theta \). For notation simplicity, in the following equations we shall use \( \phi \) instead of \( \delta \theta \), keeping in mind that \( \phi \) is infinitesimally small. Then, Eqs. (45) and (49) at linear order give
\[
\partial_t \phi + \vec{v} \nabla \phi - \delta h - \delta V = 0, \tag{51}
\]
and
\[
\partial_t \delta \rho_{\text{NR}} + \nabla (\vec{v} \delta \rho_{\text{NR}} - \rho_{\text{NR}} \nabla \phi) = 0. \tag{52}
\]
Owing to Eqs. (46) and (47) the specific enthalpies \( h \) and \( V \) may be regarded as implicit functions of \( \rho_{\text{NR}} \). Then, by making use of Eq. (51) and mathematical identity
\[
\delta h + \delta V = \left( \frac{\partial h}{\partial \rho_{\text{NR}}} + \frac{\partial V}{\partial \rho_{\text{NR}}} \right) \delta \rho_{\text{NR}}, \tag{53}
\]
the variation \( \delta \rho_{\text{NR}} \) can expressed as
\[
\delta \rho_{\text{NR}} = \left( \frac{\partial h}{\partial \rho_{\text{NR}}} + \frac{\partial V}{\partial \rho_{\text{NR}}} \right)^{-1} (\partial_t \phi + \vec{v} \nabla \phi). \tag{54}
\]
Substituting this into (52) we find
\[
\partial_t \left[ \frac{\rho_{\text{NR}}}{\tilde{v}_s^2} (\partial_t \phi + \vec{v} \nabla \phi) \right] + \nabla \left[ \frac{\rho_{\text{NR}}}{\tilde{v}_s^2} (\partial_t \phi + \vec{v} \nabla \phi) \vec{v} - \rho_{\text{NR}} \nabla \phi \right] = 0, \tag{55}
\]
where we have introduced an effective speed of sound squared
\[
\tilde{v}_s^2 = \left( \frac{\partial h}{\partial \rho_{\text{NR}}} + \frac{\partial V}{\partial \rho_{\text{NR}}} \right) \rho_{\text{NR}} = v_s^2 + \frac{\partial V}{\partial \rho_{\text{NR}}} \rho_{\text{NR}}. \tag{56}
\]
Here, \( v_s \) is the usual adiabatic speed of sound defined by
\[
v_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_s. \tag{57}
\]
Equation (55) can be used to construct the acoustic metric in its NR form. Applying the usual procedure [7] we find

$$G_{\mu\nu} = \frac{\rho_{NR}}{\tilde{v}_s} \begin{pmatrix} \tilde{v}_s^2 - v^2 & v_j \\ v_i & -\delta_{ij} \end{pmatrix},$$

(58)

precisely in the form of (2) except that the standard speed of sound $v_s^2$ is replaced by $\tilde{v}_s^2$. As mentioned in Introduction, the conformal factor can be made dimensionless by multiplying it by a suitable dimensionful factor. In the following, we will assume that this factor is absorbed in the mass density $\rho_{NR}$ so that the quantity $\rho_{NR}$ in the equations of the NR acoustic geometry will appear to have the dimension $L/T$, i.e., the dimension of velocity.

Obviously, for a fluid without external potential, the metric (58) coincides with the NR acoustic metric (2).

3 Schwarzschild-like analog geometry

As an application of the formalism presented in section 2 in this section, we proceed toward the construction of a spherically symmetric analog black hole using the approach of de Oliveira et al. [10]. In their approach, the conservation of the particle number was imposed and, to maintain the Euler equation and the correct definition of the speed of sound, it was necessary to introduce an external force field, or equivalently, external pressure with a nonvanishing gradient. Here we demonstrate that it is not possible to simultaneously satisfy the continuity equation and the definition of the speed of sound even in the presence of external pressure with a nonvanishing gradient. We will show that a consistent approach requires giving up the continuity equation so that the violation of the particle number conservation is compensated by external pressure with a nonvanishing gradient.

3.1 Relativistic approach

Consider a relativistic fluid subjected to an external pressure field. To mimic a Schwarzschild-like geometry in analog gravity we assume a spherical symmetry with radial three-velocity $v$. We start from the expression (34) applied to a flow in Minkowski spacetime with spherical symmetry and by suitable coordinate transformation $t = t(T, R), r = r(T, R)$ we will try to mimic a spacetime with line element of the form

$$ds^2 = f dT^2 - \frac{1}{f} dR^2 - R^2 d\Omega^2$$

(59)

where $f$ is a function of $R$ and $f = 0$ at the horizon. We first transform the metric (34) with

$$g_{\mu\nu} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \vartheta),$$

(60)

into diagonal form using a coordinate transformation

$$dT = dt + \frac{(1 - c_s^2) \gamma^2 v}{1 - (1 - c_s^2) \gamma^2} dr,$$

(61)
where we use the following notation:

\[ u_\mu = (\gamma, -\gamma v), \quad \gamma = (1 - v^2)^{-1/2}. \]  

(62)

The transformation (61) is similar to the Gullstrand–Painlevé type of coordinate transformation used recently by Hossenfelder [16] to construct an analog AdS planar black hole. By making use of a new radial coordinate \( R \) defined by

\[ R^2 = \omega r^2, \]  

(63)

where \( \omega \) is given by (35), we obtain the analog line element in the form

\[ ds^2 = \frac{R^2}{r^2} \left[ (\tilde{c}_s^2 - v^2)\gamma^2 dT^2 - \frac{\tilde{c}_s^2}{(\tilde{c}_s^2 - v^2)\gamma^2} \frac{dR^2}{(R')^2} \right] - R^2 d\Omega^2. \]  

(64)

where the prime \( ' \) denotes a derivative with respect to \( r \). To reproduce the metric (59), we demand

\[ f = \frac{R^2}{r^2} (\tilde{c}_s^2 - v^2) \gamma^2 = \frac{r^2(R')^2(\tilde{c}_s^2 - v^2)\gamma^2}{\tilde{c}_s^2}, \]  

(65)

as a constraint on \( r \)-dependent quantities \( R, v \), and \( \tilde{c}_s \). Clearly, the acoustic horizon defined by \( v^2 = \tilde{c}_s^2 \) coincides with the horizon \( f = 0 \) of the Schwarzschild-like black hole. From (65) we find \( v^2 \) and \( \tilde{c}_s^2 \) expressed in terms of \( R \) and \( r \).

\[ v^2 = \frac{\tilde{c}_s^2 - fr^2/R^2}{1 - fr^2/R^2}, \quad \gamma^2 = \frac{1 - fr^2/R^2}{1 - \tilde{c}_s^2}, \]  

(66)

\[ \tilde{c}_s^2 = \frac{r^4}{R^4}(R')^2 \]  

(67)

In addition, we have an equation which connects \( n \) and \( \tilde{c}_s^2 \) through the conformal factor \( \omega \). Using the definition (55) with (38), (63), and (67) we find

\[ n^2 = m^6(1 - v^2)(R')^2 \]  

(68)

The system of three equations (66)-(68) is not complete since we have four variables. We still have two conditions that must be met: the continuity equation (13) and the definition of the effective speed of sound (28).

Consider first the case of zero external pressure. In this case the effective speed of sound \( \tilde{c}_s \) becomes equal to the standard adiabatic speed of sound \( c_s \), the second term in Eq. (13) vanishes, and the continuity equation (14) applies. This equation then implies

\[ n = \frac{\alpha m}{r^2} \frac{(1 - \tilde{c}_s^2)^{1/2}}{(\tilde{c}_s^2 - fr^2/R^2)^{1/2}}, \]  

(69)

where \( \alpha \) is an arbitrary positive dimensionless constant. Then, combining Eqs. (66)- (69) we find a differential equation for \( R \)

\[ \frac{\alpha^2 R^4}{m^4} \left( 1 - \frac{r^2 f}{R^2} \right) - r^8 (R')^4 + r^6 f R^2 (R')^2 = 0. \]  

(70)
This equation will coincide with its NR version derived in Ref. [10] if we identify \( \alpha/m^2 \) with their scale \( k \) and neglect the second term in brackets on the left-hand side of (70). This term is a relativistic correction of the order of \( v^2/c^2 \), as can be shown by applying the NR limit to the second equality in (66). It is important to stress that Eq. (70) has been derived under the requirement of particle number conservation, i.e., assuming Eq. (14).

Next, we discuss what would happen if, instead of the continuity equation (14), we imposed Eq. (28). Although this equation reduces to (29) in the case of nonvanishing external pressure, the following analysis applies to both vanishing and nonvanishing external pressure. In either case, we would obtain a differential equation for \( R \) different from (70). Indeed, from Eqs. (28) with (38), (67), and (68) we find

\[
\gamma' \left(1 - \frac{(R')^2 r^4}{R^4}\right) = -\frac{R'R''r^4}{R^4},
\]

(71)

where

\[
\gamma = \frac{(1 - f r^2 / R^2)^{1/2}}{(1 - r^4 (R')^2 / R^4)^{1/2}}.
\]

(72)

Note that Eq. (71) with (72) holds for a fluid with or without external pressure. Clearly, Eqs. (70) and (71) are substantially different but it is not immediately apparent that these equations are incompatible with each other. To make a comparison, we recast (70) into a form similar to (71) by making use of (66) and the logarithmic derivative. In this way we arrive at a second order differential equation

\[
\gamma' \left(1 - \frac{(R')^2 r^4}{R^4}\right) = \frac{R''}{R} + \frac{1}{2} (R')^2 \left(\frac{r^4}{R^2}\right)' + \left(1 - \frac{(R')^2 r^4}{R^4}\right) \left[\frac{3}{2} + \frac{R'}{R} + \frac{1}{2} \frac{r^2 (R'/R)^2 - f'}{r^2 (R'/R)^2 - f}\right].
\]

(73)

In contrast to Eq. (71), the right-hand side of Eq. (73) involves a dependence on \( f \), and hence, equations (70) and (71) cannot be simultaneously satisfied. In other words, the particle number conservation expressed by the continuity equation (14) and the defining equation for the speed of sound (28) cannot be made compatible with each other. Hence, the scheme that involves external force suggested in Ref. [10] seems to be inconsistent in the relativistic approach. This inconsistency can be demonstrated also in the NR approach which we discuss next in Sec. 3.2.

A consistent approach would require giving up the particle number conservation (14) and adhering to the definition of the speed of sound. As shown in section 2.1, the condition of particle number conservation is altered in the presence of external pressure with a nonvanishing gradient. Hence, with a suitable choice of external pressure, a modified continuity equation (13) can be made compatible with Eq. (28).

To find the explicit expression for the required external pressure gradient we start from Eq. (13) which, assuming a stationary radial flow, takes the form

\[
\frac{w}{r^2} (r^2 n \gamma v)' = \gamma v (p^{\text{ext}})'.
\]

(74)
With the help of (15), (22), and (38), equation (74) can be expressed as a differential equation for the external pressure gradient \((p^\text{ext})'\),

\[
-\frac{\gamma}{n} (p^\text{ext})' = \left(\frac{\gamma^2 r^2 v (p^\text{ext})'}{(\gamma r^2 v n)'}\right)',
\]

(75)

with solution

\[
(p^\text{ext})' = \beta m^2 \frac{(\gamma r^2 v n)'}{\gamma^2 r^2 v (\gamma r^2 v n)},
\]

(76)

where \(\beta\) is a dimensionless integration constant. The quantities \(v\), \(\gamma\), and \(n\) can be expressed in terms of \(r\), \(R\), and \(R'\) via Eqs. (66)–(68). Hence, in conjunction with Eq. (71) which can be solved for \(R\), Eq. (76) yields the external pressure gradient as an explicit function of \(r\).

Comparing Eqs. (74) and (76) we also obtain a closed expression for \(w\),

\[
w = \frac{\beta m^2}{\gamma^2 r^2 v n}.
\]

(77)

Once the function \(R = R(r)\) is obtained as a solution to (71), we can determine the equation of state of the fluid \(w = w(n)\) as a parametric function defined by (68) and (77). Hence we have fixed everything: the velocity profile, the equation of state, and the external pressure gradient.

It would be of considerable interest to study the existence and uniqueness of solutions to Eq. (71). This equation is a rather complicated nonlinear differential equation and it is difficult to analytically prove or disprove the existence and uniqueness of its solutions. However, it would be straightforward to look for a numerical solution on an interval of \(r\), similar to what has been done in Ref. [10] for the nonrelativistic version of Eq. (70) but this task goes beyond the scope of this paper. Nevertheless, even if Eq. (71) has more than one solution or a multivalued function as a solution, one can single out a particular branch. For example, if \(R(r)\) is a solution to (71), it is clear that \(-R(r)\) is also a solution and we can single out the positive one.

### 3.2 Nonrelativistic approach

We now repeat the procedure described in the previous section using the NR form of the acoustic metric (58). In our derivation, unlike in Ref. [10], we assume the existence of an external potential, from the very beginning. We will show that it is not possible to mimic a Schwarzschild-like metric in a NR setup.

For dimensional reasons we introduce an arbitrary constant \(v_0\) of the dimension of velocity and write the metric (59) as

\[
ds^2 = f v_0^2 dT^2 - \frac{1}{f} dR^2 - R^2 d\Omega^2.
\]

(78)

As we are dealing with NR fluid, we assume \(v_0 \ll c\). Assuming spherical symmetry, we transform the NR acoustic metric (58) into a diagonal form using a coordinate transformation

\[
T = t + \int \frac{vd\tau}{\bar{v}_s^2 - v^2},
\]

(79)
and
\[ R^2 = \frac{\rho_{NR}}{v_s} r^2. \]

With this, we obtain the analog line element in the form similar to (64)
\[ ds^2 = \frac{R^2}{r^2} \left[ (\tilde{v}_s^2 - v^2) dT^2 - \frac{\tilde{v}_s^2}{(\tilde{v}_s^2 - v^2)} \frac{dR^2}{(R')^2} \right] - R^2 d\Omega^2. \]

As before, to reproduce the metric (78), we impose
\[ f = \frac{R^2(\tilde{v}_s^2 - v^2)}{r^2 v_0^2} = \frac{r^2(R')^2(\tilde{v}_s^2 - v^2)}{R^2 \tilde{v}_s^2}, \]

as a constraint on \( r \)-dependent quantities \( R, v, \) and \( \tilde{v}_s \). Then, \( \tilde{v}_s^2, v^2, \) and \( \rho_{NR}^2 \) can be expressed in terms of \( r, R, \) and \( R' \):
\[ \frac{\tilde{v}_s^2}{v_0^2} = \frac{r^4}{R^4} (R')^2, \]
\[ \frac{v^2}{v_0^2} = \frac{r^4}{R^4} (R')^2 - \frac{f r^2}{R^2}, \]
\[ \rho_{NR}^2 = v_0^2 (R')^2. \]

Now we impose the continuity equation (49). Assuming time independence we have
\[ \partial_r (r^2 \rho_{NR} v) = 0. \]

This, together with (85) gives another expression for \( v^2 \)
\[ \frac{v^2}{v_0^2} = \frac{k^2}{r^4 (R')^2}, \]
where \( k \) is a constant of dimension of length squared. Plugging this into Eq. (84) we find a differential equation
\[ k^2 R^4 - r^8 (R')^4 + r^6 f R^2 (R')^2 = 0. \]

This equation, previously obtained in Ref. [10], is the NR version of (70).

We still must satisfy equation (56) because it was used in the derivation of the acoustic metric. For a spherically symmetric radial flow we have
\[ \tilde{v}_s^2 = \frac{p'}{\rho_{NR}}, \quad \frac{\partial V}{\partial \rho_{NR}} = \frac{1}{\rho_{NR}} \frac{(p_{ext})'}{\rho_{NR}'}, \]

and, hence,
\[ \tilde{v}_s^2 = v_s^2 + \frac{(p_{ext})'}{\rho_{NR}} = \frac{p'}{\rho_{NR}} + \frac{(p_{ext})'}{\rho_{NR}}. \]

Can we use this equation to determine the external force required to maintain the desired flow as it was done in Ref. [10]? To answer this question we make use of the Euler equation for a stationary radial flow
\[ vu' + \frac{p' + (p_{ext})'}{\rho_{NR}} = 0. \]
Combining this with Eq. (90) we find

\[(v^2)' = -2\tilde{v}_s^2 \frac{f'_{\text{NR}}}{\rho_{\text{NR}}}.\]  

(92)

Then, using (83) and (85) we obtain

\[(v^2)' = -2v_0^2 \frac{r^4}{R} R'R''.\]  

(93)

Note that this equation does not involve a dependence on the external pressure gradient. Furthermore, equation (93) together with Eq. (84) yields another differential equation for \(R\),

\[4r^3 (RR'R'' - (R')^3) + 4r^2 R(R')^2 + 2rfR^2 R' - (2f + rf')R^3 = 0,\]

(94)

which holds for a fluid with or without external pressure. This equation can be obtained directly by way of the NR limit of Eq. (71) with (72).

It may easily be verified that Eqs. (93) and (94) are not equivalent. These two equations are not directly comparable since Eq. (93) does not involve the second derivative of \(R\) with respect to \(r\). To make a comparison, one can divide Eq. (88) by \(R^4\) and take a derivative. Then, it may be seen that the equation thus obtained is substantially different from (94). Hence, the continuity equation (86) and the defining equation for \(\tilde{v}_s\), Eq. (56), cannot be simultaneously satisfied and hence, Eq. (90) cannot be used to determine the external force needed to maintain the desired flow.

As in the previous section, we must give up the continuity equation (14). This does not seem to present a problem because, as shown in Sec. 2.1, external pressure modifies the continuity equation anyway. However, this modification is a relativistic correction of the order of \(v^2/c^2\) as shown by (48), so we can expect that the needed external pressure must be of the order of \(c^2/v^2\) to make its contribution significant. To find the required external pressure gradient we start from Eq. (48) for a spherically symmetric flow and assume that the pressure terms on the right-hand side dominate the Newtonian gravity term. Combining (48) with (91) and neglecting the gravity term, we obtain

\[(p_{\text{ext}}')' = \frac{c^2}{r^2 v}(r^2 \rho_{\text{NR}} v)' + \rho_{\text{NR}} vv'.\]

(95)

One can easily verify that Eq. (95) coincides with the NR limit of (74) including the lowest relativistic correction. From (84) and (85) it follows that the first term on the right-hand side of this expression is of the order of \(c^2v'\), whereas the second term is of the order of \(v_0^2v'\). Hence, the first term is dominant unless the velocity scale \(v_0\) is close to the speed of light. Either way, the external pressure gradient is of the order

\[(p_{\text{ext}}')' \sim c^2v'.\]

(96)

On the other hand, according to (91), the gradients of both pressure and external pressure are of the order

\[p' \sim (p_{\text{ext}}')' \sim v_0^2v'.\]

(97)
Comparing (96) and (97) we conclude that the velocity scale $v_0 \sim c$ and, as a consequence, the velocity of the fluid must be of the order of $v \sim c$. Moreover, this implies that the speed of sound $v_s$ will be close to the speed of light as can be deduced from Eqs. (83) and (89).

Hence, to construct a Schwarzschild-like metric one needs a relativistic radial fluid flow in which $v$ and $v_s$ are both close to the speed of light. This result contradicts the initial assumption that the fluid is nonrelativistic. In this way, we have demonstrated that it is not possible to mimic a Schwarzschild-like metric in the framework of NR acoustic geometry.

4 Summary and conclusions

We have presented a formulation of analog gravity in a fluid subjected to external pressure, or equivalently to external potential. We have shown that the external pressure with a nonvanishing gradient may exhibit nontrivial effects in both relativistic and nonrelativistic approaches. Using this we have investigated the possibility to mimic the exact form of the Schwarzschild black hole in such a fluid. Transforming the metric to a diagonal form we have found the conditions under which the metric exactly reproduces a Schwarzschild-like metric. We have found that this procedure can be carried out consistently if the fluid is essentially relativistic.

The attempt to adapt the relativistic procedure of Sec. 3.1 to a NR fluid has failed because it is not possible to satisfy simultaneously the NR continuity equation and the definition of the speed of sound. The reason behind this failure is that NR fluid dynamics neglects the relativistic effect of particle creation or destruction caused by an external pressure gradient. The considerations in Sec. 3.2 lead to the conclusion that to mimic a Schwarzschild-like metric one needs a relativistic fluid with a radial flow in which the fluid velocity $v$ and the speed of sound $v_s$ are both close to the speed of light. In other words, it is impossible to mimic a Schwarzschild-like metric by analog gravity in the framework of NR acoustic geometry.

One may wonder whether a different definition of the external pressure—e.g., one corresponding to another ideal fluid rather than to one of the vacuum energy type as in Eq. (5)—might resolve the mentioned disagreement with Ref. [10]. Unfortunately, this attempt is unlikely to succeed, mainly for the reason that, as demonstrated here, the NR formalism is inherently incompatible with a Schwarzschild-like analog metric. The formalism which employs the nonrelativistic acoustic geometry in the form derived originally for a single ideal fluid while resorting to the use of an external force, is not consistent. An external force basically amounts to adding another ideal fluid which necessarily requires a reformulation of the acoustic geometry. We have implemented such a reformulation here for the external pressure defined by Eq. (5). If we had used another definition of the external pressure, e.g., by replacing $p$ by $p + p^{\text{ext}}$ in the ideal fluid expression for $T_{\mu\nu}$, we would have obtained the nonrelativistic limit of the Euler equation still in the form (42), so Eqs. (43)-(47) would have remained the same. As the definition of the speed of sound (29) involves the pressure $p$ rather than $p + p^{\text{ext}}$, an effective speed of sound as defined in (56) would appear in the derivation of the nonrelativistic acoustic metric. As a result, the acoustic metric would inevitably have the form (58) and the above-mentioned inconsistency problem would remain.
Acknowledgments

The work of N.B. has been partially supported by the ICTP - SEENET-MTP Project No. NT-03 Cosmology - Classical and Quantum Challenges.

References

[1] W. G. Unruh, Phys. Rev. Lett. 46, 1351 (1981).

[2] C. Barcelo, S. Liberati, and M. Visser, Living Rev. Rel. 8, 12 (2005) [Living Rev. Rel. 14, 3 (2011)]; [gr-qc/0505065].

[3] M. Novello, M. Visser, and G. Volovik (eds.), Artificial Black Holes (World Scientific, Singapore, 2002).

[4] V. Moncrief, Astrophys. J. 235, 1038–1046, (1980).

[5] N. Bilić, Classical Quantum Gravity 16, 3953 (1999), [arXiv:gr-qc/9908002].

[6] M. Visser and C. Molina-Paris, New J. Phys. 12, 095014 (2010), [arXiv:1001.1310 [gr-qc]].

[7] M. Visser, Classical Quantum Gravity 15, 1767-1791 (1998), [arXiv:gr-qc/9712010 [gr-qc]].

[8] N. Bilić, H. Nikolić, Classical Quantum Gravity 35, 135008 (2018), [arXiv:1802.03267].

[9] N. Bilić and H. Nikolić, Universe 7, no.11, 413 (2021), [arXiv:2109.02880 [gr-qc]].

[10] C. C. de Oliveira, R. A. Mosna, J. P. M. Pitelli and M. Richartz, Phys. Rev. D 104, no.2, 024036 (2021), [arXiv:2106.03960 [gr-qc]].

[11] L. D. Landau, E. M. Lifshitz, Fluid Mechanics, (Pergamon, Oxford, 1993) p. 507.

[12] E. N. Saridakis, P. F. Gonzalez-Diaz and C. L. Siguenza, Classical Quantum Gravity 26, 165003 (2009), [arXiv:0901.1213 [astro-ph.CO]].

[13] C. M. Will, Theory and Experiment in Gravitational Physics, (Cambridge University Press, Cambridge, 1981).

[14] L. Blanchet, T. Damour and G. Schaefer, Mon. Not. Roy. Astron. Soc. 242, 289-305 (1990).

[15] E. Poisson and C. M. Will, Gravity, (Cambridge University Press, Cambridge, 2014).

[16] S. Hossenfelder, Phys. Lett. B 752, 13 (2016), [arXiv:1508.00732 [gr-qc]].