ON THE GEOMETRY OF BORDER RANK ALGORITHMS FOR MATRIX MULTIPLICATION AND OTHER TENSORS WITH SYMMETRY

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Abstract. We establish basic information about border rank algorithms for the matrix multiplication tensor and other tensors with symmetry. We prove that border rank algorithms for tensors with symmetry (such as matrix multiplication and the determinant polynomial) come in families that include representatives with normal forms. These normal forms will be useful both to develop new efficient algorithms and to prove lower complexity bounds. We derive a border rank version of the substitution method used in proving lower bounds for tensor rank. We use this border-substitution method and a normal form to improve the lower bound on the border rank of matrix multiplication by one, to $2n^2 - n + 1$. We also point out difficulties that will be formidable obstacles to future progress on lower complexity bounds for tensors because of the “wild” structure of the Hilbert scheme of points.

1. Introduction

Ever since Strassen discovered in 1969 [30] that the standard algorithm for multiplying matrices is not optimal, it has been a central question to determine upper and lower bounds for the complexity of the matrix multiplication tensor $M_{(n)} \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$. In the language of algebraic geometry, this amounts to determining the smallest value $r$ such that the matrix multiplication tensor lies on the $r$-th secant variety of the Segre variety $\text{Seg}(\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1})$—see below for definitions. This value of $r$ is called the border rank of $M_{(n)}$ and is denoted $R(M_{(n)})$.

The main contribution of this article is the observation that one can simplify this study by restricting one’s search to border rank algorithms of a very special form. This special class of algorithms is of interest in its own right and we develop basic language to study them.

From the perspective of algebraic geometry, our restriction amounts to reducing the study of the Hilbert scheme of points to the punctual Hilbert scheme (those schemes supported at a single point). We expect it to be useful in other situations.

While motivated by the complexity of matrix multiplication, our work fits into both the larger study of the structure of secant varieties of homogeneous varieties (e.g., [31, 9]) and the study of the geometry of tensors (e.g., [19]).

Overview. In §2 we define secant varieties and a variety of border rank algorithms. In §3 we prove our main normal form lemma and show that it applies to the problems of studying the Waring border rank of the determinant and the tensor border rank of the matrix multiplication operator. To better study the normal forms, in §4 we define subvarieties of secant varieties associated to certain constructions that have already appeared in the literature [27, 7]. In §5 we prove a border rank version of the substitution method as used in [11] and apply it to show $R(M_{(n)}) \geq 2n^2 - n + 1$, an improvement by one over the previous lower bound of [26].

Key words and phrases. matrix multiplication complexity, border rank, tensor, commuting matrices, Strassen’s equations, MSC 15.80.

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Notation. Throughout this paper, $\mathbb{V}, A, B, C, U, V, W, M$ denote complex vector spaces and $X \subset \mathbb{P} \mathbb{V}$ denotes a projective variety. If $v \in \mathbb{V}$, we let $[v] \in \mathbb{P} \mathbb{V}$ denote the corresponding point in projective space. For a variety $X$, $X^{(r)} = X^{*r}/\mathcal{G}_r$ denotes the $r$-tuples of points of $X$, where $\mathcal{G}_r$ is the group of permutations on $r$ elements. The vector space of linear maps $U \to V$ is denoted $U^* \otimes V$.

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2. Secant varieties

Let $X \subset \mathbb{P} \mathbb{V}$ be a variety, let

$$\sigma_r^0(X) = \bigcup_{x_1, \ldots, x_r \in X} \langle x_1, \ldots, x_r \rangle \subset \mathbb{P} \mathbb{V}$$

denote the points of $\mathbb{P} \mathbb{V}$ on secant $\mathbb{P}^{r-1}$'s of $X$, and let $\sigma_r(X) := \overline{\sigma_r^0(X)}$ denote its Zariski closure, the $r$-th secant variety of $X$, where $\langle x_1, \ldots, x_r \rangle$ denotes the projective linear space spanned by the points $x_1, \ldots, x_r$ (usually it is a $\mathbb{P}^{r-1}$).

In this paper we are primarily concerned with the case $\mathbb{P} \mathbb{V} = \mathbb{P}(A \otimes B \otimes C)$ and $X = \text{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C)$ is the Segre variety of rank one tensors. The above-mentioned question about the matrix-multiplication tensor is the case $A = U^* \otimes V, B = V^* \otimes W, C = W^* \otimes U$, and $M_{(U, V, W)} \in A \otimes B \otimes C$ is the matrix multiplication tensor. When $U, V, W = \mathbb{C}^n$, we denote $M_{(U, V, W)}$ by $M(n)$. The question is: What is the smallest $r$ such that $[M(U, V, W)] \in \sigma_r(\text{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C))$? Bini [3] showed that this $r$, called the border rank of $M(U, V, W)$ indeed governs its complexity. The border rank of a tensor $T$ is denoted $\mathbf{R}(T)$. The smallest $r$ such that a tensor $[T] \in \mathbb{P}(A \otimes B \otimes C)$ is in $\sigma_r^0(\text{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C))$ is called the rank of $T$ and is denoted $\mathbf{R}(T)$.

Remark 2.1. It is expected that the rank of $M(n)$ is greater than its border rank when $n > 2$, and more generally we expect that for “most” tensors $T$ with a large symmetry group, $\mathbf{R}(T) < \mathbf{R}(T)$. In the case of matrix multiplication we have the following evidence: $\mathbf{R}(M(n)) \geq 2n^2 - O(n)$ and $\mathbf{R}(M(n)) \geq 3n^2 - o(n^2)$ [26, 20]. Moreover, 19 $\leq \mathbf{R}(M(3)) \leq 23$ while 16 $\leq \mathbf{R}(M(3)) \leq 20$ with inequalities proved respectively in [4, 17], this article, and [29].

Definition 2.2. Let $X \subset \mathbb{P} \mathbb{V}$ be a projective variety. By an $X$-border rank $r$ algorithm for $z \in \mathbb{P} \mathbb{V}$, we mean a curve $E_t$ in the Grassmannian $G(r, V)$ such that $z \in \mathbb{P} E_0$ and for $t > 0$, $E_t$ is spanned by $r$ points of $X$. (This includes the possibility of $E_t$ being stationary.) In particular, $z$ admits an $X$-border rank $r$ algorithm if and only if $z \in \sigma_r(X)$. When $X = \text{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C)$, we just refer to border rank algorithms. We will say such an $E_0$ realizes $z$ as point of $\sigma_r(X)$.

Remark 2.3. Instead of taking a curve in the Grassmannian we may and sometimes will just take a convergent sequence $E_{t_n}$.

Define the incidence variety

$$S_r^0(X) := \{(v, ([x_1], \ldots, [x_r])) | v \in \langle x_1, \ldots, x_r \rangle \} \subset \mathbb{P} \mathbb{V} \times X^{(r)}$$

a “Nash”- type blow up of it

$$S_r(X) := \{(v, ([x_1], \ldots, [x_r]), \langle x_1, \ldots, x_r \rangle) | v \in \langle x_1, \ldots, x_r \rangle, \dim\langle x_1, \ldots, x_r \rangle = r \} \subset \mathbb{P} \mathbb{V} \times X^{(r)} \times G(r, V)$$

and the abstract secant variety
\[ S_r(X) := \overline{S^0_r(X)}. \]
We have maps
\[ \rho \xleftarrow{} \pi \]
\[ G(r, V) \quad \sigma_r(X) \]
where the map \( \pi \) is surjective.

When discussing rank algorithms, a point \( p \in \sigma_r^0(X) \) is called identifiable if there is a unique collection of \( r \) points of \( X \) such that \( p \) is in their span. When discussing border rank realizations, the \( r \)-plane \( E_0 \) is the more important object, which motivates the following definition:

**Definition 2.4.** We say \( [v] \in \sigma_r(X) \) is Grassmann-border-identifiable if \( \rho \pi^{-1}([v]) \) is a point.

We will be mostly interested in the case when \( X = G/P \subset \mathbb{P}V \) is a homogeneous variety and \( [v] \) has a nontrivial symmetry group \( G_v \subset G = G_X \). In this case \([v]\) is almost never Grassmann-border-identifiable. Indeed, if \( z \in \pi^{-1}([v]) \), then the orbit closure \( G_v \cdot z \) is also in \( \pi^{-1}([v]) \). Hence, to be Grassmann-border-identifiable, \( G_v \) would have to act trivially on \( \rho(z) \).

### 3. The Normal Form Lemma

By [9] Lemma 2.1] in any border rank algorithm with \( X = G/P \) homogeneous, we may assume there is one stationary point \( x \in X \) with \( x \in \mathbb{P}E_t \) for all \( t \).

The following Lemma is central:

**Lemma 3.1** (Normal form lemma). Let \( X = G/P \subset \mathbb{P}V \) and let \( v \in V \) be such that \( G_v \) has a single closed orbit \( O_{\min} \) in \( X \). Then the \( G_v \)-orbit closure of any border rank \( r \) algorithm \( v \) contains a border rank \( r \) algorithm \( E = \lim_{t \to 0}(x_1(t), \ldots, x_r(t)) \) where there is a stationary point \( x_1(t) \equiv x_1 \) lying in \( O_{\min} \).

If moreover every orbit of \( G_{v, x_1} \) contains \( x_1 \) in its closure, we may further assume that all other \( x_j(t) \) limit to \( x_1 \).

**Proof.** The proof of the first statement follows from the same methods as the proof of the second, hence we focus on the latter. We prove we can have all points limiting to the same point \( x_1(0) \).

By [9] Lemma 2.1 this is enough to conclude.

We work by induction. Say we have shown that \( x_1(t), \ldots, x_q(t) \) all limit to the same point \( x_1 \in O_{\min} \). We will show that our curve can be modified so that the same holds for \( x_1(t), \ldots, x_{q+1}(t) \).

Take a curve \( g_t \in G_{v, x_1} \) such that \( \lim_{t \to 0} g_t x_{q+1}(0) = x_1 \). For each fixed \( \epsilon \), acting on the \( x_j(t) \) by \( g_t \), we obtain a border rank algorithm for which \( g_t x_i(t) \to x_1(0) \) for \( i \leq q \) and \( g_t x_{q+1}(t) \to g_t x_{q+1}(0) \).

Fix a sequence \( \epsilon_n \to 0 \). Claim: we may choose a sequence \( t_n \to 0 \) such that

- \( \lim_{n \to \infty} g_{\epsilon_n} x_{q+1}(t_n) = x_1(0) \), and
- \( \lim_{n \to \infty} g_{\epsilon_n} x_1(t_n), \ldots, g_{\epsilon_n} x_r(t_n) \) contains \( v \).

The first point holds as \( \lim_{t \to 0} g_t x_{q+1}(0) = x_1 \). The second follows as for each fixed \( \epsilon_n \), taking \( t_n \) sufficiently small we may assure that a ball of radius \( 1/n \) centered at \( v \) intersects \( g_{\epsilon_n} x_1(t_n), \ldots, g_{\epsilon_n} x_r(t_n) \). Considering the sequence \( \bar{x}_i(t_n) := g_{\epsilon_n} x_i(t_n) \) we obtain the desired border rank algorithm.

Our main interest consists of the following two examples:
3.1. The determinant polynomial. Let \( v_n : \mathbb{P}W \to \mathbb{P}(S^nW) \) denote the Veronese re-embedding of \( PW \). When \( W = E \otimes F = C^n \otimes C^m \), the space \( V = S^nW \) is the home of the determinant polynomial. Write \( X = v_n(\mathbb{P}W) \subset \mathbb{P}(S^nW) \) and \( v = \det_n \) for the determinant. Here \( G_X = GL_{n^2} \) and \( G_v \simeq (SL(E) \times SL(F)) \times \mathbb{Z}_2 \). The group \( G_v \) has a unique closed orbit \( O_{\min} = v_n(Seg(\mathbb{P}E \times \mathbb{P}F)) \) in \( X \). Moreover, for any \( z \in v_n(Seg(\mathbb{P}E \times \mathbb{P}F)) \), \( G_{\det_n,z} \), the group preserving both \( \det_n \) and \( z \), is isomorphic to \( P_E \times P_F \), where \( P_E, P_F \) are the parabolic subgroups of matrices with zero in the first column except the \((1,1)\)-slot, and \( z \) is in the \( G_{\det_n,z} \)-orbit closure of any \( q \in v_n(\mathbb{P}W) \).

3.2. The matrix multiplication tensor. Set \( A = U^* \otimes V, B = V^* \otimes W, C = W^* \otimes U \). The space \( V = A \otimes B \otimes C \) is the home of the matrix multiplication tensor, \( X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) = Seg(\mathbb{P}(U^* \otimes V) \times \mathbb{P}(V^* \otimes W) \times \mathbb{P}(W^* \otimes U)) \subset \mathbb{P}(A \otimes B \otimes C) \), and \( v = M_{(U,V,W)} = Id_U \otimes Id_V \otimes Id_W \in (U^* \otimes V) \otimes (V^* \otimes W) \otimes (W^* \otimes U) \) is the matrix multiplication tensor. (\( Id_U \in U^* \otimes U \) denotes the identity map and in the expression for \( M_{(U,V,W)} \) we re-order factors.) Here \( G_X = GL(A) \times GL(B) \times GL(C) \) and \( G_{M_{(U,V,W)}} = GL(U) \times GL(V) \times GL(W) \), and both are slightly larger by a finite group if some of the dimensions coincide.

Proposition 3.2. Let

\[
\mathcal{K} := \{(\mu \otimes v \otimes w \otimes \omega \otimes \omega) \in Seg(\mathbb{P}(U^* \otimes V \times PV \times PW \times PW^* \otimes PU) \mid \mu(u) = \omega(v) = \omega(v) = 0)\}
\]

Then \( \mathcal{K} \) is the unique closed \( G_{M_{(U,V,W)}} \)-orbit in \( Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \).

Moreover, if \( k \in \mathcal{K}, \) then \( G_{M_{(U,V,W)},k} \), the group preserving both \( M_{(U,V,W)} \) and \( k \), is such that for every \( p \in Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), k \in G_{M_{(U,V,W)},k} \cdot p \).

Note that \( Seg(\mathbb{P}(U^* \otimes U^*))_0 := \{[u \otimes \alpha] \mid \alpha(u) = 0\} \subset \mathfrak{psl}(U) \) is the closed orbit in the adjoint representation and \( \mathcal{K} \) is isomorphic to \( Seg(Seg(\mathbb{P}(U^* \otimes U^*)_0) \times Seg(\mathbb{P}(V^* \otimes V^*)_0) \times Seg(\mathbb{P}(W^* \otimes W^*)_0) \).

Proof. It is enough to prove the last statement. We will prove that \( k \) is the unique closed orbit under \( G_{M_{(U,V,W)},k} \). This is enough to conclude as the closure of any orbit must contain a closed orbit. Notice that fixing \( k = [(\mu \otimes v) \otimes (v \otimes w) \otimes (w \otimes u)] \) is equivalent to fixing a partial flag in each \( U, V \) and \( W \) consisting of a line and a hyperplane containing it.

Let \([a \otimes b \otimes c] \in Seg(\mathbb{P}(A \times \mathbb{P}B \times \mathbb{P}C)). \) If \([a] \notin Seg(\mathbb{P}U^* \times \mathbb{P}V) \) then the orbit is not closed, even under the torus action on \( V \) that is compatible with the flag. So without loss of generality, we may assume \([a \otimes b \otimes c] \in Seg(\mathbb{P}(U^* \otimes V \times PV \times PW \times PW^* \times PU)). \) Write \( a \otimes b \otimes c = (\mu' \otimes v') \otimes (v' \otimes w') \otimes (w' \otimes u'). \) If, for example \( v' = v \), we may act with an element of \( GL(V) \) that preserves the partial flag and sends \( v' \) to \( v + ev' \). Hence \( v \) is in the closure of the orbit of \( v' \). As \( G_{M_{(U,V,W)},k} \) preserves \( v \) we may continue, reaching \( k \) in the closure.

Remark 3.3. Proposition 3.2 combined with the normal form Lemma allows the argument of [JS] to be simplified tremendously, as it vastly reduces the number of cases. In particular, it eliminates the need for the erratum.

4. Local versions of secant varieties

In this section we introduce several higher order generalizations of the tangent star at a point of a variety and the tangential variety. The generalizations are subvarieties of the \( r \)-th secant variety of a projective variety. We restrict our discussion to projective varieties \( X \subset \mathbb{P}V \), however the discussion can be extended to arbitrary embedded schemes. Our initial motivation was to provide language to discuss the normal form of the main lemma, but the discussion is useful in a wider context; special cases have already been used in [17, 27].

We exhibit local properties of the \( r \)-th secant variety using the language of smoothable schemes of length \( r \) supported at one point (i.e. local). Their moduli space is in the principal component
of the Hilbert scheme of subschemes of length \( r \) of \( X \). This component is an algebraic variety, a compactification of \( r \)-tuples of distinct points of \( X \), i.e. \( (X^{(r)} - D) \), where \( D \) is the big diagonal. It parametrizes smoothable schemes, i.e. schemes that arise as degenerations of \( r \) distinct points with reduced structure. More formally, an ideal \( I \) defines a smoothable scheme if there exists a flat family \( I_t \) with the fiber \( I \) for \( t = 0 \), where for \( t \neq 0 \), \( I_t \) is the ideal of \( r \) distinct points.

4.1. **Areoles and buds.** Recall that a scheme \( S \) supported at \( 0 \in \mathbb{C}^n \) corresponds to an ideal \( I \subset \mathbb{C}[x_1, \ldots, x_n] \) whose only zero is \((0)\). Define the span of \( S \) to be \( \langle S \rangle := \text{Zeros}(I_1) \) where \( I_1 \subset I \) is the homogeneous degree one component. This definition depends on the embedding of \( S \).

We start by recalling the definition of the areole from [7, Section 5.1].

**Definition 4.1 (Areole).** Let \( p \in X \). The \( r \)-th open areole at \( p \) is

\[
\mathfrak{a}^o_r(X,p) := \bigcup \{ (R) \mid R \text{ is smoothable in } X, \text{ supported at } p \text{ and } \text{length}(R) \leq r \},
\]

the \( r \)-th areole at \( p \) is the closure:

\[
\mathfrak{a}_r(X,p) := \overline{\mathfrak{a}^o_r(X,p)},
\]

and the \( k \)-th areole variety of \( X \) is

\[
\mathfrak{a}_r(X) := \bigcup_{p \in X} \mathfrak{a}_r(X,p).
\]

The areole can be regarded as a generalization of a tangent space. Indeed, consider \( r = 2 \) and a smooth point \( p \in X \). Up to isomorphism there is only one local scheme of length two: \( \text{Spec} \mathbb{C}[x]/(x^2) \), and the embedded tangent space at \( p \) may be identified with linear spans of such schemes, supported at \( p \). In particular, if \( X \) is smooth then \( \mathfrak{a}_2(X) = \tau(X) \), the tangential variety of \( X \).

Another, differential geometric, definition of a tangent line is as a limit of secant lines. This motivates the following.

**Definition 4.2 (Greater Areole).** The \( r \)-th open greater areole at \( p \) is

\[
\tilde{\mathfrak{a}}^o_r(X,p) := \bigcup_{x_j(t) \in X} \lim_{t \to 0} \langle x_1(t), \ldots, x_r(t) \rangle,
\]

the \( r \)-th greater areole at \( p \) is the closure:

\[
\tilde{\mathfrak{a}}_r(X,p) := \overline{\tilde{\mathfrak{a}}^o_r(X,p)},
\]

and the \( r \)-th greater areole variety of \( X \) is

\[
\tilde{\mathfrak{a}}_r(X) := \bigcup_{p \in X} \tilde{\mathfrak{a}}_r(X,p).
\]

**Remark 4.3.** The difference between the areole and the greater areole is related to the difference of border rank and smoothable rank - the latter was introduced in [28]. Indeed, points in the \( r \)-th areole belong to a linear span of a scheme hence are of smoothable rank at most \( r \).

The normal form in Lemma 3.1 can be restated in the following way. If a point \( v \) satisfies all assumptions of the Lemma, then it belongs to the \( r \)-th secant variety if and only if it belongs to the \( r \)-th greater areole \( \tilde{\mathfrak{a}}_r(X,p) \) for a point \( p \in \mathcal{O}_{\text{min}} \).

**Lemma 4.4.** \( \mathfrak{a}^o_r(X,p) \subset \tilde{\mathfrak{a}}^o_r(X,p) \) and \( \mathfrak{a}_r(X,p) \subset \tilde{\mathfrak{a}}_r(X,p) \).
Proof. It is enough to show the first inclusion. Let \( v \in \mathfrak{a}_r^n(X, p) \). Then, by definition, there exists a scheme \( S \), smoothable in \( X \), supported at \( p \) such that \( v \in \langle S \rangle \). As \( S \) is smoothable in \( X \) we may find a family of points \( x_i(t) \) for \( i = 1, \ldots, k \), such that \( S \) is their limit as \( t \to 0 \). We may also assume that \( \langle x_1(t), \ldots, x_r(t) \rangle \) is of constant dimension. Let \( E := \lim_{t \to 0} \langle x_1(t), \ldots, x_r(t) \rangle \) that is also of dimension \( r - 1 \). The linear span \( \langle S \rangle \) of the limit is contained in the limit \( E \) of linear spans. By definition \( E \) is contained in \( \mathfrak{a}_r^n(X, p) \). \( \square \)

Remark 4.5. The relation between the linear span of the limit and the limit of linear spans can be viewed as a special case of upper semi-continuity of Betti numbers under deformation \([13, III.12.8]\), i.e. the number of equations of fibers in a flat family of given degree can only jump up in the limit.

The cases when the areole equals the greater areole are of particular interest.

The following proposition is well-known, however usually stated in different language. It is essentially due to Grothendieck \([12]\) and played an important role in the construction of the Hilbert scheme. Recently, it was crucial in \([7]\). The proof of the following proposition follows from \([7, \text{Theorem 5.7}]\).

**Proposition 4.6.** Suppose that \( p \) is a point of a variety \( X \) embedded by at least an \((r - 1)\)-st Veronese embedding. Then:

\[
\mathfrak{a}_r(X, p) = \tilde{\mathfrak{a}}_r(X, p).
\]

Motivated by applications, we restricted ourselves in the definition of areole to smoothable schemes. It may seem that the classification of such local schemes should be easy, that they should all be ‘almost’ like \( \text{Spec} \mathbb{C}[x]/(x^r) \). As we present below, the story is much more interesting.

In the Hilbert scheme, the locus of schemes supported at \( p \) and isomorphic to \( \text{Spec} \mathbb{C}[x]/(x^r) \) is relatively open. Such schemes are called aligned \([15]\) or curvilinear. The schemes in the closure in the Hilbert scheme of the locus of aligned schemes, i.e., schemes that arise as degeneration of aligned schemes, are called alignable. For small values of \( r \) all local smoothable schemes are alignable. An example of a scheme that is alignable, but not aligned is \( \text{Spec} \mathbb{C}[x, y]/(x^2, y^2) \).

A local scheme \( \text{Spec} \mathbb{C}[x_1, \ldots, x_n]/I \) is called Gorenstein if the ideal \( I \) is an apolar ideal of a polynomial \( f \) in the dual variables. That is, the \( x_i \) are differential operators on the dual space of polynomials and \( I \) is the ideal of differential operators annihilating \( f \).

The Hilbert function of a local scheme with a maximal ideal \( \mathfrak{m} \) assigns to \( k \) the dimension of \( \mathfrak{m}^k/\mathfrak{m}^{k+1} \). It is usually presented as a finite sequence, as by convention, we omit the infinite string of zeros after the last nonzero entry.

The last nonzero value of the Hilbert function for a Gorenstein scheme must be equal to 1, because if the form \( f \) is of degree \( d \), the pairing with differential operators of degree \( d \) provides a surjection onto \( \mathbb{C} \).

**Example 4.7.** The scheme \( \text{Spec} \mathbb{C}[x, y]/(x^2, xy, y^2) \) is not Gorenstein, as its Hilbert function equals \((1, 2)\). The scheme \( \text{Spec} \mathbb{C}[x, y]/(x^2 - y^2, xy) \) is Gorenstein as the ideal is apolar to \( X^2 + Y^2 \), where \( x(X) = 1, x(Y) = 0 \) etc..

Schemes that are local and smoothable do not have to be alignable \([6]\). Even more: there exist subvarieties of the Hilbert scheme corresponding to local, smoothable schemes that are of higher dimension than the component of alignable schemes. When \( X \) is \( n \)-dimensional the dimension of locus of alignable schemes equals \((r - 1)(n - 1)\). Already for \( r = 12 \) and \( n = 5 \) there exists another family (also of dimension 44) of smoothable schemes supported at \( p \) that are not alignable. For \( r = 16 \) and \( n = 7 \) there exists a family of dimension 104 of smoothable schemes.
supported at $p$. Summing over different $p$ we obtain a family of dimension $111 = 7 \cdot 16 - 1$, i.e. a divisor in the Hilbert scheme! All this motivates one more definition.

**Definition 4.8** (Bud). The $r$-th open bud at $p$ is

$$b_r^p(X,p) := \bigcup \{ R \mid R = \text{Spec} \mathbb{C}[x]/(x^r) \text{ and supported at } p \},$$

the $r$-th bud at $p$ is the closure:

$$b_r(X,p) := \overline{b_r^p(X,p)},$$

and the $r$-th bud variety of $X$ is

$$b_r(X) := \bigcup_{p \in X} b_r(X,p).$$

Note that the bud $b_r(X,p)$ contains the linear spans of all alignable schemes of given length that are supported on $p$. These schemes do not have to be Gorenstein. However, of course the aligned schemes are Gorenstein.

**Example 4.9.** In [21] we showed that when $\dim A = \dim B = \dim C = m$, then $b_m(\text{Seg}(PA \times PB \times PC)) = GL(A) \times GL(B) \times GL(C) \cdot [T_N] \subset \mathbb{P}(A \otimes B \otimes C)$, where $T_N$ is a tensor such that $T_N(A^*) \subset B \otimes C$ corresponds to the centralizer of a regular nilpotent element.

Both areoles and buds generalize the tangent star [31], which is the case $r = 2$.

**Proposition 4.10.** Let $p$ be a point of a variety $X \subset \mathbb{P}V$. Then

$$b_2(X,p) = a_2(X,p) = \tilde{a}_2(X,p) = T_p^*X,$$

where $T_p^*X$ denotes the tangent star of $X$ at $p$.

**Proof.** The first equality follows by definition as any local scheme of length two is isomorphic to $\mathbb{C}[x]/(x^2)$ i.e., aligned. The second equality is a special case of Proposition 4.6 as any variety is its own first Veronese re-embedding. The third is just the definition of the tangent star. $\square$

Thus when $r = 2$, $b_2(X) = a_2(X) = \tilde{a}_2(X)$. Moreover, $b_2^0(X) = b_2(X)$ because all alignable schemes of length two are aligned.

When $r = 3$, [3 Thm. 1.11] shows that when $X = G/P$ is generalized cominuscule, they still coincide: $b_3(G/P) = a_3(G/P) = \tilde{a}_3(G/P)$. Moreover, $b_3^0(G/P) = b_3(G/P)$ because all the points in the bud give aligned schemes.

We now bound the dimensions of all these varieties. Recall that for an $n$-dimensional variety, $\dim \sigma_r(X) \leq rn + r - 1$.

**Proposition 4.11.** Let $p$ be a point of an $n$ dimensional homogeneous variety $X \subset \mathbb{P}^N$. Then

$$\dim \tilde{a}_r(X,p) \leq rn - n + r - 2,$$

$$\dim a_r(X,p) \leq rn - n + r - 2,$$

$$\dim b_r(X,p) \leq (r - 1)n,$$

and hence,

$$\dim \tilde{a}_r(X) \leq rn + r - 2,$$

$$\dim a_r(X) \leq rn + r - 2,$$

$$\dim b_r(X) \leq rn.$$
Proposition 4.13. Jelisiejew for pointing us towards the following result:

4.2. Important and difficult challenge.

\[ \sigma^{r} \text{ of the secant variety when } \sigma^{r} \text{ is at most a divisor. Since } X \text{ is homogeneous, the fibers of the projection of the intersection to the small diagonal are all isomorphic, hence are of dimension at most } nr - n - 1. \text{ The inequality follows.} \]

Remark 4.12. The only point in the proof where we used that \( X \) was homogeneous was to have equi-dimensional fibers. We expect that the inequalities remain true for any smooth \( X \).

While the areole and greater areole have the same expected dimension, in many cases (for example \( r \leq 9 \)) the areole is of strictly smaller dimension than expected, often coinciding with the bud. Further, the areole and the greater areole are not expected to be irreducible. The problem of distinguishing between the areole and greater areole appears to be important and difficult. On the other hand the bud is irreducible, as the locus of aligned schemes is irreducible in the Hilbert scheme.

By the inequalities in Proposition 4.11, \( \tilde{a}_{r}(X), a_{r}(X), \) and \( b_{r}(X) \) are all proper subvarieties of the secant variety when \( \sigma_{r}(X) \) has the expected dimension. Finding the equations of any of the above varieties when \( r > 2 \), even in the case of Segre or Veronese varieties is another important and difficult challenge.

4.2. The bud and local differential geometry. We thank Jaroslaw Buczyński and Joachim Jelisiejew for pointing us towards the following result:

Proposition 4.13. Let \( X \subset \mathbb{P}V \) be a projective variety and let \( p \in X \) be a smooth point. Then

\[
\mathcal{b}_{r}(X, p) = \bigcup_{x(t) \in X, x(t) \neq p} \langle x(0), x'(0), \ldots, x^{(r-1)}(0) \rangle,
\]

where the union is taken over all curves \( x(\cdot) \) smooth at \( p \).

In the language of differential geometry, \( \mathcal{b}_{r}(X, p) \) is the \( (r - 1) \)-st osculating cone to \( X \) at \( p \). Its span is the \( (r - 1) \)-st osculating space.

Remark 4.14. We obtain the same variety if we take the union over analytic curves.

Proof. Given a smooth curve \( C \), for each \( r \), one has the embedded aligned scheme of length at most \( r \) supported at \( p \), with span \( \langle x(0), x'(0), \ldots, x^{(r-1)}(0) \rangle \), determined by it.

Given an aligned scheme \( S \), we claim there exists a curve that contains it, is smooth at \( p \), and is contained in \( X \). Let \( m \) be the maximal ideal defining \( p \) in \( O(X)_p \), the local ring of \( p \in X \). Let \( J \) be the ideal defining \( S \) in \( O(X)_p \). Since the tangent space of \( S \) is one-dimensional, \( (J + m^2)/m^2 \) is a hyperplane in \( m/m^2 = T^*_pX \). Let \( f_1, \ldots, f_{\dim X - 1} \in J \) span this hyperplane. Locally we may write \( f_i = h_i/s_i \) with \( h_i \in I(S) \) and \( s_i \in \mathbb{C}[V] \) with \( s_i(p) \neq 0 \). Let \( U = X \setminus \cup_i \text{Zeros}(s_i) \). Then the \( h_i \) are lifts of the \( f_i \) to \( \mathbb{C}[U] \). Consider the subscheme (possibly reducible, non-reduced) \( Z \subset U \) they define. The Zariski tangent space \( T_pZ \) is one dimensional, so \( Z \) must also be locally one dimensional at \( p \) (at most one-dimensional because the local dimension is at most the dimension of \( T_pZ \), and at least because we used \( \dim X - 1 \) equations), hence a component through \( p \) must be a curve, smooth at \( p \), and this curve has the desired properties.

For a subvariety \( X \subset \mathbb{P}V \) and a smooth point \( x \in X \), there is a sequence of differential invariants called the fundamental forms \( \mathbb{F}T_1x \to N^j_xX \), where \( N^j_xX \) is the \( j \)-th normal space. After making choices of splittings and ignoring twists by line bundles, write \( V = \hat{x} \oplus \hat{y} \).
Let $X = \text{Seg}(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)$. See [25] §2.2 or [9] for a quick introduction. Adopt the notation $\mathbb{F}_k : T_x X \rightarrow T_x X$ is the identity map.

Let $X = G/P \subset \mathbb{P}V$ be generalized cominuscule. (This is a class of homogeneous varieties that includes Grassmannians, Veroneses and Segre varieties.) Then the only projective differential invariants of $X$ at a point are the fundamental forms, and these are easily (in fact pictorially) determined [25].

Let $X$ be generalized cominuscule, let $p = [v]$ and let $v_1, \ldots, v_r \in \mathbb{T}_p X$. Then calculations in [9] show that a general point of the bud $b_r(X,p)$ is

$$[v + \sum_{k=1}^{r-1} \sum_{j_1+\cdots+j_k=r-1} \mathbb{F}_k(v_{i_1}, \ldots, v_{i_{r-1}})].$$

Example 4.15. When $X = v_d(\mathbb{P}W)$, and $p = [w^d]$, then elements of $\mathbb{T}_p X$ are of the form $w^{d-1}u$ and

$$\mathbb{F}_k(w^{d-1}u_1, \ldots, w^{d-1}u_k) = w^{d-k}u_1 \cdots u_k.$$

Thus a general point of the bud is of the form

$$[\sum_{k=0}^{r-1} \sum_{i_1+\cdots+i_k=r-k} w^{d-k}u_{i_1} \cdots u_{i_k}].$$

Example 4.16. The fundamental forms of Segre varieties are well-known. In particular, for a $k$-factor Segre, the last nonzero fundamental form is $\mathbb{F}_k$. The second fundamental form at $[a_1 \otimes \cdots \otimes a_k]$ is spanned by the quadrics generating the ideal of $\mathbb{P}(A_1/a_1) \sqcup \cdots \sqcup \mathbb{P}(A_k/a_k) \subset \mathbb{P}(A_1/a_1 \otimes \cdots \otimes A_k/a_k) \simeq \mathbb{P}T_{[a_1 \otimes \cdots \otimes a_k]} \text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k)$. A general point of $b_r(\text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k), [a_1 \otimes \cdots \otimes a_k])$ is of the form

$$[\sum_{i_1+\cdots+i_k=r-1} a_{i_1,i_2} \otimes \cdots \otimes a_{k,i_k}]$$

where $a_{j,i,j} \in A_j$ are arbitrary elements with $a_{j,0} = a_j$.

4.3. Examples of points not in the open $r$-bud of generalized cominuscule varieties. Let $X \subset \mathbb{P}V$ be generalized cominuscule.

Our examples are constructed from parametrized curves $x_j(t)$ in $X$. The general procedure to obtain the scheme to which the points degenerate as $t \to 0$ is as follows:

1. A Zariski open subset of $X$ has a rational parametrization, and a priori we are dealing with $r \dim X$ different $C[t]$ coefficients, but in practice the number is much smaller.

2. For example, in the $k$-factor Segre, one is immediately reduced to $rk$ coefficients. Any scheme of length $r$ can be embedded into a space of dimension $r - 1$. So we work in a space of dimension $r - 1$.

3. The desired scheme is given by the ideal $(I,t)$, which may be considered as a subscheme of $C^u$ for some $u \leq r - 1$.

As our curves are given parametrically, the first two steps are instances of the implicitization problem. For the third, one substitutes $t = 0$ into a set of generators.

We already saw that the first possible example of a point not in the open bud is when $r = 4$. Let $r = 4$, and consider $p = \mathbb{F}_2(v_1, v_2) + v_3$, where $v_j \in T_x X$. When $X = \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$,

$$p = a_1 \otimes b_1 \otimes c_4 + a_1 \otimes b_2 \otimes c_1 + a_4 \otimes b_1 \otimes c_1 + \sum_{\sigma \in S_3} a_{\sigma(1)} \otimes b_{\sigma(2)} \otimes c_{\sigma(3)}.$$
When $X = v_3(\mathbb{P}W)$, then $p = xyz + x^2w$.

In the Segre case, $p$ is in the span of the limit 4-plane of the following four curves:

\[ x_0(t) = a_1 \otimes b_1 \otimes c_1, \]
\[ x_1(t) = (a_1 + ta_2 + t^2a_4) \otimes (b_1 + tb_2 + t^2b_4) \otimes (c_1 + tc_2 + t^2c_4), \]
\[ x_2(t) = (a_1 + ta_3) \otimes (b_1 + tb_3) \otimes (c_1 + tc_3), \]
\[ x_3(t) = (a_1 - t(a_2 + a_3)) \otimes (b_1 - t(b_2 + b_3)) \otimes (c_1 - t(c_2 + c_3)). \]

If we set $b_j = c_j = a_j$, we obtain the corresponding curves in the Veronese $v_3(\mathbb{P}A)$.

Consider the affine open subset $\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3$ where the coordinate $a_1 \otimes b_1 \otimes c_1$ is nonzero. All of the curves belong to it. Since the coefficients in $\mathbb{C}[t]$ appearing for each $j$ is the same for $a_j, b_j, c_j$, we may reduce to $\mathbb{C}^3$.

We have reduced to four curves of the form: $(y_1(t), y_2(t), y_3(t))$: $(0, 0, 0), (t, 0, t^2), (0, t, 0), (-t, -t, 0)$. They satisfy the equation $y_3 = y_1(y_2 + t)$, so we may focus on the first two coordinates. Hence we have four points in the projective plane - a complete intersection of two quadrics. The equations in $I$ are now of the form:

\[ y_1(y_1 - t - 2y_2), y_2(2y_1 + t - y_2). \]

Substituting $t = 0$ we obtain the annihilators of the nondegenerate quadratic form $Y_j^2 + Y_j Y_2 + Y_2^2$ (where $Y_j$ is dual to $y_j$), i.e., the limiting scheme is isomorphic to $\text{Spec} \mathbb{C}[y_1, y_2]/(y_1 y_2, y_2^2 - y_1)$, that is Gorenstein alignable, but not aligned. In particular, it is in the bud, but not the open bud. The Hilbert function equals $(1, 2, 1)$.

**Example 4.17** (The Coppersmith-Winograd tensor). The (second) Coppersmith-Winograd tensor generalizes the example above. It is

\[ \tilde{T}_{q,CW} := \sum_{j=1}^{q} (a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0) + a_0 \otimes b_0 \otimes c_{q+1} + a_0 \otimes b_{q+1} \otimes c_0 + a_{q+1} \otimes b_0 \otimes c_0 \in \mathbb{C}^{q+2} \otimes \mathbb{C}^{q+2} \otimes \mathbb{C}^{q+2} \]

It equals

\[
\lim_{t \to 0} \frac{1}{t^2} \left[ \sum_{i=1}^{q} \frac{1}{t^2} (a_0 + ta_i) \otimes (b_0 + tb_i) \otimes (c_0 + tc_i) \right.
\]

\[
- \frac{1}{t^3} (a_0 + t^2 (\sum_{i=1}^{q} a_i)) \otimes (b_0 + t^2 (\sum_{j=1}^{q} b_j)) \otimes (c_0 + t^2 (\sum_{j=1}^{q} c_j))
\]

\[
+ \left[ \frac{1}{t^3} - \frac{q}{t^2} \right] (a_0 + t^3 a_{q+1}) \otimes (b_0 + t^3 b_{q+1}) \otimes (c_0 + t^3 c_{q+1}) \].
\]

The Coppersmith-Winograd tensors are symmetric, the first corresponds to the polynomial $x(y_1^2 + \cdots + y_2^2)$, and the second (which is above), the polynomial $x(xz + y_1^2 \cdots + y_q^2)$. These polynomials have symmetric ranks respectively $2q + 1$ and $2q + 3$ (respectively shown in [24, 10]). In [22] we showed these agree with their tensor ranks - thus the Conen conjecture [11], that the rank and symmetric rank of a symmetric tensor agree, holds for these tensors. Moreover, since our border rank algorithm is symmetric and matches the lower bound, the border rank version of the Conen conjecture [8] holds for these tensors as well.

Since the tensor is symmetric, we may immediately reduce to $\mathbb{C}^{q+2}$ and work in the open set where $a_0 = 1$. Then in the resulting $\mathbb{C}^{q+1}$ the curves are:

\[(t, 0, \cdots, 0), (0, t, 0, \cdots, 0), \cdots, (0, \cdots, 0, t, 0), (t^2, \cdots, t^2, 0), (0, \cdots, 0, t^3)\]
We see that \( y_{q+1} = t^3 - (\sum_{i=1}^{q} y_i) t^2 + y_1 y_2 q - (\sum_{i=1}^{q} y_i) t/q + (\sum_{i=1}^{q} y_i^2)/q - y_1 y_2/q \), so we are reduced to the curves
\[
(0, \ldots, 0), \ (t, 0, \ldots, 0), \ (0, t, 0, \ldots, 0), \ldots, (0, \ldots, 0, t), (t^2, \ldots, t^2)
\]
which satisfy the equations \( y_i(y_i - t) - y_j(y_j - t) \) and \( y_i(ty_i - t^2 - (t - 1)y_j) \) for all \( i \neq j \). Hence in the limit we obtain the Gorenstein scheme given by the annihilators of the nondegenerate quadric, namely \( \text{Spec } \mathbb{C}[y_1, \ldots, y_{q-1}]/(y_iy_j, y_i^2 - y_j^2)_{1 \leq i < j \leq q-1} \).

**Remark 4.18.** The schemes that we obtain are artifacts of the border rank algorithms we choose and are not intrinsic to the tensors. Even if we restrict to schemes/algorithms of minimal degree their uniqueness is related to generalized identifiability questions.

### 4.4. The bud and matrix multiplication.

If one could make the stronger statement that if \( R(M_{(n)}) = r \), then \( M_{(n)} \in \mathfrak{b}_r(\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), k) \) for some \( k \in K \), then one could probably prove \( R(M_{(n)}) \geq 2n^2 - 1 \). We calculated this for \( n = 3 \). The full proof involves many cases. Here we present the two extreme cases to illustrate the idea.

Recall that a point of the open bud is of the form
\[
\sum_{3 \leq i+j+k \leq 2} a_i \otimes b_j \otimes c_k
\]
for some \( a_i \in A, b_j \in B, \) and \( c_k \in C \).

**Proposition 4.19.** Say \( M_{(n)} \in \mathfrak{b}_{2n^2-2}(\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), k) \). Then one cannot have any two of the three sets of vectors \( \{a_1, \ldots, a_{n^2}\}, \{b_1, \ldots, b_{n^2}\}, \{c_1, \ldots, c_{n^2}\} \), linearly independent, nor can one have any of \( \dim(a_1, \ldots, a_{2n^2-n-1}), \dim(b_1, \ldots, b_{2n^2-n-1}), \) or \( \dim(c_1, \ldots, c_{2n^2-n-1}) \) less than \( n^2 \).

**Proof.** For the first case, assume without loss of generality (by cyclic symmetry) that \( a_1, \ldots, a_{n^2} \) and \( c_1, \ldots, c_{n^2} \) are linearly independent. Let \( \alpha \in \langle a_1, \ldots, a_{n^2-1} \rangle^4 \) be non-zero so \( \alpha(a_{n^2}) \neq 0 \) and similarly \( \gamma \in \langle c_1, \ldots, c_{n^2-1} \rangle^4, \gamma(c_{n^2}) \neq 0 \). Consider the proper left ideal \( M_{(n)} \alpha \subset B \). (Here \( A, B, C \) are all the same algebra of \( n \times n \) matrices, but we distinguish them for clarity.) Then, viewing matrix multiplication as \( C \times A \rightarrow B, M_{(n)} \alpha \subset B \) is the span of \( b_1, \ldots, b_{n^2-1} \) because for each choice of \( \gamma \in C \), the vector:
\[
(\gamma(c_{n^2}), \ldots, \gamma(c_1))\begin{pmatrix}
\alpha(a_{n^2}) & 0 & \cdots & 0 \\
\alpha(a_{n^2+1}) & \alpha(a_{n^2}) & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(a_{2n^2-2}) & \cdots & \alpha(a_{n^2}) \\
\end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n^2-1} \\
\end{pmatrix} \in B.
\]

appears in \( M_{(n)} \alpha \). On the other hand, the right ideal \( \gamma M_{(n)} \subset B \), is the span of the vectors, for each choice of \( \tilde{\alpha} \in A^* \),
\[
(b_{n^2-1}, \ldots, b_1)\begin{pmatrix}
\gamma(c_{n^2}) & 0 & \cdots & 0 \\
\gamma(c_{n^2+1}) & \gamma(c_{n^2}) & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\gamma(c_{2n^2-2}) & \cdots & \gamma(c_{n^2}) \\
\end{pmatrix}\begin{pmatrix} \tilde{\alpha}(a_1) \\ \tilde{\alpha}(a_2) \\ \vdots \\ \tilde{\alpha}(a_{n^2-1}) \\
\end{pmatrix},
\]
which is also the span of \( b_1, \ldots, b_{n^2-1} \), a contradiction.

For the second case, assume \( \dim(a_1, \ldots, a_{2n^2-n-1}) < n^2 \). There exists a nonzero \( \alpha \in \langle a_1, \ldots, a_{2n^2-n-1} \rangle^4 \). Then the non-zero left ideal \( \alpha M_{(n)} \) is a subspace of \( \langle c_1, \ldots, c_{n-1} \rangle \), but the smallest dimension of a left ideal is \( n \), a contradiction. \( \square \)
5. A border rank analog of the AFT theorem

We recall the Alexeev-Forbes-Tsimerman variant of the substitution method, as rephrased in [21]:

**Proposition 5.1.** [1] Appendix B, [5] Chapter 6] Fix a basis \( a_1, \ldots, a_n \) of \( A \). Write \( T = \sum_{i=1}^{a} a_i \otimes M_i \), where \( M_i \in B \otimes C \). Let \( R(T) = r \) and \( M_1 \neq 0 \). Then there exist constants \( \lambda_2, \ldots, \lambda_m \), such that the tensor

\[
\tilde{T} := \sum_{j=2}^{m} a_j \otimes (M_j - \lambda_j M_1) \in A^1 \otimes B \otimes C,
\]

has rank at most \( r - 1 \). Moreover, if \( \text{rank}(M_1) = 1 \) then for any choices of \( \lambda_j \) we have \( R(\tilde{T}) \geq r - 1 \).

Here is a border rank version:

**Lemma 5.2.** Let \( T \in A \otimes B \otimes C \). Then there exists a hyperplane \( \mathcal{H}_A \subset A^* \) such that \( R(T|_{\mathcal{H}_A \times B^* \times C^*}) \leq R(T) - 1 \).

In other words, there exists \( a \in A \) such that image \( T' \) of \( T \) under the projection \( A \otimes B \otimes C \to (A/a) \otimes B \otimes C \) has \( R(T') \leq R(T) - 1 \).

**Proof.** First note this is true for rank (AFT). Let \( T_t \) be a curve with \( \lim_{t \to 0} T_t = T \). For each \( t \) the statement is true for some \( \mathcal{H}_t \), and by the proof in [21], one sees that the \( \mathcal{H}_t \) will vary smoothly with \( t \). Since projective space is compact there is a limiting \( \mathcal{H}_0 \) and the statement holds with \( \mathcal{H}_A = \mathcal{H}_0 \).

One could in principle apply this, alternating the roles of \( A, B, C \), to potentially obtain border rank bounds up to \( \text{dim} \ A + \text{dim} \ B + \text{dim} \ C - 3 \), but the need to test all possible hyperplanes makes such a use unlikely in practice.

**Corollary 5.3.** Let \( M^{\text{red}}_{(n)} = M_{(n)} - \sum_{j} x^n_j \otimes y^n_j \otimes z^j_1 \) be a reduced matrix multiplication operator. Then \( R(M_{(n)}) \geq R(M^{\text{red}}_{(n)}) + 1 \).

**Proof.** There are \( n \) non-zero orbits in \( A \) under \( G_{M_{(n)}} \). Say the element \( x \) of \( A \) that works in Lemma 5.2 has rank \( r \). If we act on it by an element of \( G_{M_{(n)}} \), the new element still works, so it will work for points in the orbit closure of \( G_{M_{(n)}} \cdot x \), which contain the rank one elements. Finally all rank one elements are equivalent to \( x^n_1 \).

6. A new lower bound for the border rank of matrix multiplication

**Theorem 6.1.** When \( n \geq 3 \), \( R(M^{\text{red}}_{(n)}) \geq 2n^2 - n \).

When \( n = 2 \), it was shown in [22] that \( R(M^{\text{red}}_{(2)}) = 5 \).

**Theorem 6.2.** Combined with Corollary 5.3 implies:

**Theorem 6.2.** Let \( n \geq 2 \), then \( R(M_{(n)}) \geq 2n^2 - n + 1 \).

**Remark 6.3.** The result was already shown for \( n = 2 \), where it is optimal, in [13, 14].

**Remark 6.4.** The state of the art in other small cases is \( 16 \leq R(M_{(3)}) \leq 20 \) (the upper bound appears in [29]), and \( 29 \leq R(M_{(4)}) \leq 49 \) (with the upper bound due to Strassen).

**Proof of Theorem 6.2** We use the Koszul flattenings defined in [26] which were used to prove \( R(M_{(n)}) \geq 2n^2 - n \). Write \( M^{\text{red}}_{(n)} \in C^{n^2-1} \otimes B \otimes C = \tilde{A} \otimes B \otimes C \). Consider for any \( T \in \tilde{A} \otimes B \otimes C \), the Koszul flattening \( T'_{(n)} : \Lambda^{n-1} \tilde{A} \otimes B^* \to \Lambda^n \tilde{A} \otimes C \) given by:

\[
T = \sum_{j} t_{ijj} a_i \otimes b_j \otimes c_k f_1 \wedge \cdots \wedge f_{n-1} \otimes \beta \mapsto \sum_{\beta(b_j)} t_{ijj} a_i \wedge f_1 \wedge \cdots \wedge f_{n-1} \otimes c_k.
\]
For matrix multiplication and for the reduced matrix multiplication this map factors, $B = V^* \otimes W$ and $C = W^* \otimes U$, where $U, V, W = \mathbb{C}^n$ to $(M_{(1,1,n)})_A^{n-1} \otimes \text{Id}_W$ and $(M_{(1,1,n)}^{\text{red}})_A^{n-1} \otimes \text{Id}_W$ respectively, where $(M_{(1,1,n)}^{\text{red}})_A^{n-1} : \Lambda^{n-1} \tilde{A} \otimes V \to \Lambda^n \tilde{A} \otimes U$. As in [26], we obtain the best result by restricting to an $A' := \mathbb{C}^{2n-1} \subset \tilde{A}$. Define the map

$$\phi : \tilde{A} \to A'$$

$$x_j^i \mapsto e_{i+j-1}.$$

Since $(a \otimes b \otimes c)^{\wedge(n-1)}$ has rank $(\binom{2n-2}{n-1})$ (its image is $a \wedge \Lambda^{n-1}(\mathbb{C}^{2n-1}/a) \otimes c$), it will suffice to prove, for $n \geq 3$, that

$$\frac{\text{rank}(M_{(1,1,n)}^{\text{red}})_A^{n-1}}{\binom{2n-2}{n-1}} \geq 2n - 1$$

We claim that $\text{rank}(M_{(1,1,n)}^{\text{red}})_A^{n-1} = n(2n-1) - 1$, which will prove the result when $n \geq 3$.

To prove the claim we establish some notation. Write $e_S = e_{s_1} \cdot \cdots \cdot e_{s_{n-1}}$, where $S \subset [2n-1]$ has cardinality $n - 1$. Our map is

$$e_S \otimes v_k \mapsto \sum_{\{m(m,k) = (n,1)\}} \phi(u^m \otimes v_k) \wedge e_S \otimes u_m = \sum_{\{m(m,k) = (n,1)\}} e_{m+k-1} \wedge e_S \otimes u_m$$

Index a basis of the source by pairs $(S, k)$, with $k \in [n]$, and the target by $(P, l)$ where $P \subset [2n-1]$ has cardinality $n$ and $l \in [n]$. We define an order relation on the target basis vectors in the following way. For $(P_1, l_1)$ and $(P_2, l_2)$, set $l = \min\{l_1, l_2\}$, and declare $(P_1, l_1) < (P_2, l_2)$ if and only if

1. In lexicographic order, the set of $l$ minimal elements of $P_1$ is strictly after the set of $l$ minimal elements of $P_2$ (i.e. the smallest element of $P_2$ is smaller than the smallest of $P_1$ or they are equal and the second smallest of $P_2$ is smaller or equal etc. up to $l$-th), or
2. the $l$ minimal elements in $P_1$ and $P_2$ are the same, and $l_1 < l_2$.

Note that $([n], 1)$ is the unique minimal element for this relation and $([n], n)$ is the unique maximal element. Note further that

$$e_{n+1} \wedge \cdots \wedge e_{2n-1} \otimes u_n \mapsto e_n \wedge \cdots \wedge e_{2n-1} \otimes v_1$$

i.e., that

$$([n + 1, \ldots, 2n - 1], n) \mapsto ([n, \ldots, 2n - 1], 1)$$

We will prove the claim by showing that the image is the span of all basis elements $(P, l)$ except the maximal element $([n], n)$. We work by induction using the relation, the base case that $([n], 2n-1), 1)$ in the image has been established. Let $(P, l)$ be any basis element other than the maximal, and assume all $(P', l')$ with $(P', l') < (P, l)$ have been shown to be in the image. Write $P = (p_1, \ldots, p_n)$ with $p_i < p_{i+1}$. Consider the image of $(P \setminus \{p_l\}, 1 + p_l - l)$ which is

$$\sum_{\{m(m,1+p_l-l) = (n,1)\}} \phi(u^m \otimes v_{1+p_l-l}) \wedge e_{P \setminus \{p_l\}} \otimes u_m = \sum_{\{m(m,1+p_l-l) = (n,1)\}} e_{p_l-l+m} \wedge e_{P \setminus \{p_l\}} \otimes u_m.$$

In particular, taking $m = l$ we see $(P, l)$ is among the summands, as long as $(P, l)$ is not the maximal element. If $m < l$, the contribution to the summand is a $(P', m)$ where the first $m$ terms of $P'$ equal the first of $P$, so by condition (2), $(P', m) < (P, l)$. If $m > l$, the summand is a $(P'', m)$ where the first $l - 1$ terms of $P$ and $P''$ agree, and the $l$-th terms are respectively $p_l$ and $p_l - l + m$ so by condition (1) $(P'', m) < (P, l)$. $\square$
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