Very Compact Expressions for Amplitudes

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Abstract

A number theoretic algorithm is given for writing gauge theory amplitudes in a compact manner. It is possible to write down all details of the complete $L$ loop amplitude with two integers, or a complex integer. However, a more symmetric notation requires more integers, five or seven, depending on the type of theory. It is possible that in the symmetric form (or in the non-symmetric form) that a direct (or less direct) recursive algorithm or generating function can be developed to compute these numbers at arbitrary loop order. The existence of this function is implied by the recursive structure of loop amplitudes and their analyticity, i.e. multi-particle poles; a function requiring a finite number of computations such as a polynomial with derivable coefficients is desired.
The computation of gauge theory amplitudes has been preoccupying modern researchers for years, and due to the complexity, requires many tedious calculations by conventional methods. Alternative and more efficient methods for computing these amplitudes are merited. Current well used methods include string-inspired methods, unitarity and factorization conditions, spinor helicity and color decomposition, and the recent Calabi-Yau/gauge theory weak-weak duality correspondence pertaining to deformations of MHV amplitudes. These techniques have been useful in computing tree and one-loop gauge theory amplitudes.

The derivation of the quantum amplitudes in the derivative expansion are presented in [1]-[11].

In this letter, a notation is given so that the typically complicated kinematic expressions for loop amplitudes may be shortened to a few lines. These formulae, in spinor notation, usually require a dozen pages to write down for the known one-loop amplitudes; a few lines is a major improvement in notation. A few lines of integers is convenient for many reasons, including computational ones that are typically performed for relevant cross section calculations.

The analytic pieces of the amplitudes are considered in this work. A modification of the notation can be made to include general products of polylogarithms which arise at multi-loop. Also, these non-analytic terms are redundant in the sense that they are derivable via perturbative unitarity.

The amplitudes are to be constructed from a small set of integers. The integers required to specify the amplitude consist of: (1) two in order to specify the spinor products \( \langle ij \rangle \) in the numerator and denominator via a series of \( n \) equations and \( m \) unknowns, required for each term in the expansion at a given loop order, (2) one to label the internal group quantum numbers, (3) one to label the particle spin numbers including line factors, (4) one to describe the coefficient.

In order to specify a gauge theory amplitude the numbers pertaining the kinematics and quantum numbers pertaining to the \( n \) particle states must be described. The kinematics are associated in spinor helicity notation via,

\[
(\sum [\sigma(i)\bar{\sigma}(j)])^p,
\]

where \( p \) is a number labeling the exponent of the term. For example, an individual term in the amplitude may contain the factors,

\[
[34] \quad ([12] + [48] + [49])^{-2}.
\]
The amplitude contains a series of these factors in one of the additive terms. For an individual term in an $n$-particle amplitude, there are many factors, which may be described in polynomial form via

\[ \sum x_{\sigma(i)} y_{\bar{\sigma}(i)} + |p| = 0 , \]  

with the vectors $\sigma$ and $\bar{\sigma}$ describing the indices in the inner products. For example, the two terms in (3) are described by,

\[ x_1 y_2 + 1 = 0 , \]  

and

\[ x_1 y_2 + x_4 y_8 + x_4 y_9 + 2 = 0 , \]

with another minus sign required for the 2 in (5). A minus sign is required in the last term in the sum; the minus sign is specified by ordering the first term in the series via $i < j$ in $x_i y_j$ for min(i), with min(j) pertaining to the min(i). The information in the series is grouped into a larger number $N$ via another base $q$ expansion. The inner products are described via the expansion,

\[ N = \alpha x^2 + \beta x + \gamma . \]  

The numbers $\alpha$ and $\beta$ in a well-ordered polynomial form are,

\[ \alpha = \sum_{i=1}^{n} \sigma(i) x^{i+1} \quad \beta = \sum_{i=1}^{n} \bar{\sigma}(i) x^{i+1} , \]

and $\gamma$ is an a priori arbitrary integer specifying the power. The numbers $\alpha$ and $\beta$ range in base $n$ up to the maximum value $n^{n+2}$, with the coefficients $\sigma(i)$ and $\bar{\sigma}(i)$ ranging from 1 to $n$. To uniquely specify the kinematic factor, a number $N$ is required which must be written in base $n^{n+2} + 1$ ranging up to

\[ N = 0, \ldots, (n^{n+2} + 1)^3 . \]
The condition in (8) is satisfied for $q$ less than the maximum number. Otherwise, a larger base is required; If for some reason $q$ is larger than $N_{\text{max}}$, which for $n = 10$ is approximately $10^{36}$ (a large pole), then a bound is required for the amplitude, $q_{\text{max}}$, and the base $q_{\text{max}}$ is used. (This bound depends on the number of terms in the additive expansion of the amplitude; it may proved by a partial expansion of the denominators and the multi-particle pole information. The assumption that $q_{\text{max}}$ is less than that in (8) is used.)

This number $N$ is enough to specify one of the kinematic factors in one term of the amplitude's additive expansion. The specification of a complete term requires $m$ factors, which is found via another (superseding) decomposition via a polynomial with numbers labeling the factors,

$$\sum a_i w^i.$$  \hspace{1cm} (9)

Assuming the maximum factor $q_{\text{max}}$, and $m$ terms, the polynomial is specified in base $N_{\text{max}}$ via a number $N_i$, for the $i$th term, bounded by $N_{\text{max}}^{m+1}$.

The full expression describing the amplitude requires a series of these numbers $N_i$, with each number parameterizing a single additive term. Given the three integers, $q_{\text{max}}$, the number of terms $N_{\text{terms}}$, and the maximum number $N_{\text{max}}$ required to specify a term (at a given order), a number $Q$ and $P$ is used via another base reduction to specify the terms, i.e. $Q = \sum a_i x^i$ in base $P$. The kinematics is specified by the pair $(Q, P)$. The bound on the number is $N_{\text{max}}^{N_{\text{terms}}}$. The complex brackets, i.e. $(ij)$, are specified by two more additional numbers $\bar{P}$ and $\bar{Q}$.

The remaining numbers required to specify the amplitude are the prefactors of the individual terms and the particles’ quantum numbers.

The helicity states in pure gauge theory are described by a vector $(\pm 1, \ldots, \pm 1)$, which may be written in base 2 with expansion of $0 \rightarrow 1$ and $1 \rightarrow -1$. The number required is one from 0 to $2^n$, via $\sum_{i=1}^{n} a_i x^i$.

The group theory quantum numbers are described by another number with a similar decomposition. Consider $U(m)$ with $m^2$ generators. The generators of the particle states are parameterized by a mode expansion,

$$\sum_{i=1}^{m^2-1} b_i x^i ,$$  \hspace{1cm} (10)
with $b$, a label ranging from 1 to $m^2$. The number is one from 1 to $m^2$, which requires a number of maximum $N_R = (m^2)^m$. The multiple trace structure, in which there are $L + 1$ traces as in,

$$\Tr \prod_{\sigma(j)} a_1 \cdot \cdots \cdot \Tr \prod_{\sigma(j)} a_{L+1}$$  \hspace{1cm} (11)

ordered via the particles in a permuted set, for example as $\rho = (1, 2, 4, 3, 9, 5, 7, 6, 8)$, is described by the integers from 0 to $L$ (or 1 to $L + 1$) as in,

$$(0, 0, 0, 1, 2, 1, 2, 1, 1),$$  \hspace{1cm} (12)

where the entry is the $j$th particle, and the entry directs the matrix to the appropriate trace. The color decomposition is well-known, and there are $L + 1$ trace structures at loop order $L$. The vector is parameterized by a number in base $n$ with entries 0 to $L$ as in $N_T = \sum c_i x^i$. There are a maximum of $p$ trace structures, and the number $N_T$ ranges from 0 to $L^n$.

The prefactors of the individual terms $g_{l,a}$ require factors of rational numbers, powers of $\pi$ with products of euler-mascheroni constant. There are also divergences, such as $1/\epsilon^n$. These numbers may also be encoded in superseded numbers.

The five numbers $P + iQ$, $P + i\bar{Q}$, $N_S$, $N_G$, and $N_R$ together with the prefactors label the amplitude at a given loop order $L$ and set of quantum numbers. The lowest number of terms at this order is labeled at $N_L(N_S, N_G, N_R)$; this number is a lower bound on the additive complexity. There are $N_L$ prefactors, $G_{L,a}$.

Another form of the terms of the amplitude is simply to express the amplitude as a polynomial directly in terms of the variables $x, y$ and $z$. The terms

$$\left( \sum_{i<j} \alpha_{ij} [ij] \right)^p$$  \hspace{1cm} (13)

in the product are expressed as terms in the polynomial via

$$P_j(z) = p_j z_j \sum_{a < b} \alpha_{ab} x^a y^b,$$  \hspace{1cm} (14)

with the $\alpha_{ij}$ representing the sign of the inner products $[ij]$ with $i < j$, and $p_j$ the exponent of the sum. The polynomial form, as a function of $x, y$ and $z$ may be useful in differential equation applications. The polynomial is
\[ P(x; y; z) = P_j(z) \] (15)

which has an ordering ambiguity in the coefficients \( z_i \) removed by relations between multi-point and multi-genus amplitudes such as factorization. The polynomials have an interpretation in a six-dimensional algebraic variety (and associated polyhedra); this is possibly related to the Calabi-Yau/gauge theory twistor duality at tree level, or to a differential equation at \( n \)-point and loop \( l \) generating the polynomial.

The notational complexity is much simpler than writing the full amplitude. A typical amplitude at one-loop with six external legs may require several pages to write on paper, whereas, perhaps only a few hundred digits are required.

With these numbers, the natural question is how to iterate the numbers from one order in the loop expansion to the next. A number of the order of \( 10^{1000} \) is an approximate guess. The functional bootstrap from a set of numbers at \( L \) loop order for a given set of quantum numbers seems possible via analyticity requirements in the general case. The mathematical derivation of the iteration is quite interesting, if a finite equation or recursive approach can be given; that is, not a set of infinite degree polynomials that label all of the numbers at various loop orders \( L \), and requiring the solutions apriori to determine the coefficients.

An example would be a set of polynomial equations of finite degree in the variables \( P + iQ, \bar{P} + i\bar{Q}, N_S, N_G, N_R \) with \( L \) and \( N_L \), or a generating function with a finite number of initial conditions. Given these numbers, the coefficients \( G_{L,a} \) also can be deduced via multi-loop factorization on intermediate poles.

The generalization to further theories is direct. The spin reps are generalized to the numbers \( (s_1, s_2, \ldots, s_n) \), and are grouped into a single integer. There are no further numbers required in the multiplet description.

Known amplitude examples may be used to investigate the patterns in the numbers. Also, at one-loop the multi-trace subamplitudes are redundant and can be expressed in terms of the leading trace ones; this property should have a number theoretic description.

The full amplitudes in self-dual gauge theory described by \( \mathcal{L} = \text{Tr}G^{\alpha\beta}F_{\alpha\beta} \) with \( F \) the self-dual field strength [13], i.e. the one-loop helicity types \((+,...,+,-,-)\) [14],

\[
A_{n;1} = \frac{g^{n-2}}{192\pi^2} \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \frac{\langle i_1 i_2 \rangle \langle i_2 i_3 \rangle \langle i_3 i_4 \rangle \langle i_4 i_1 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle n1 \rangle},
\] (16)
are a simple gauge theory to investigate in this context (likewise for self-dual gravity). The amplitudes in (16) are maximally compressed, and independent of the non-abelian group. The kinematics is independent of $N_G$ and $N_R$, as well as the helicity configuration $N_S$. The simple integrability could manifest in a more direct fashion on the numbers $(P + iQ, P + iQ)$.

There are further equivalent representations of the amplitudes in terms of integers. There could also be some polynomial (knot) arithmetic associated with the numbers [15] and their relations.
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