The multi-moment map of the nearly Kähler
$S^3 \times S^3$

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Abstract

We describe the multi-moment map associated to an almost Hermitian manifold which admits an action of a torus by holomorphic isometries. We investigate in particular the case of a $T^3$ action on the homogeneous nearly Kähler $S^3 \times S^3$. We find that the multi-moment map in this case acts more-or-less similarly to the moment map of a toric manifold, while the more general case does not.

1 Introduction

An almost Hermitian manifold $(M, g, J)$ is nearly Kähler if $\nabla^g J$ is skew-symmetric. We say a nearly Kähler manifold is strict if it is not Kähler. The minimum dimension admitting strict nearly Kähler manifolds is 6, and there are only a handful of known examples of compact strictly Kähler 6-manifolds. The homogeneous spaces $S^6, S^3 \times S^3, \mathbb{CP}^3$ and $SU_3 / T^2$ admit strict nearly Kähler structures, and there are no other homogeneous strict nearly Kähler 6-manifolds [1]. The only non-homogeneous examples that are known are cohomogeneity one structures on $S^6$ and $S^3 \times S^3$ [2], and these are conjectured to be the only cohomogeneity one examples.

If one wants to look for higher cohomogeneity examples, one could look for strict nearly Kähler 6-manifolds admitting the action of a 3-torus $T^3$ by holomorphic isometries. Of the list of known examples in the previous paragraph, only the homogeneous $S^3 \times S^3$ admits such a symmetry group. The purpose of this paper is to explore this example.

A compact Kähler 6-manifold admitting a $T^3$ action of holomorphic isometries would be toric. Such a manifold could be studied with use of the moment map $\mu$, which is a $T^3$-equivariant map from the manifold to the dual Lie algebra of the torus, $t^*$. Each fiber of $\mu$ is a $T^3$ orbit, and the image of $\mu$ is the polyhedron which is the convex hull of the $\mu$-image of the fixed points of the $T^3$ action.

In general, an almost Hermitian 6-manifold admitting a $T^3$ action of holomorphic isometries would not be toric. However, we can study the multi-moment
map $\nu$ associated to the 3-form $d\omega$. This is a $T^3$-equivariant map from the manifold to the three dimensional vector space $\Lambda^2t^*$, so one can hope that it will have similar properties to the momentum map of a toric 6-manifold. We find that multi-moment map of $S^3 \times S^3$ does have some similar properties and some differences with the momentum map of a toric 6-manifold, while a more generic almost Hermitian manifold can have a rather poorly behaved multi-moment map.

We find that the multi-moment map image $\Delta := \nu(S^3 \times S^3)$ of $S^3 \times S^3$ is convex and that its boundary $\partial \Delta$ contains the 1-skeleton of a regular tetrahedron. However, $\Delta$ bulges beyond the faces of the tetrahedron, and $\partial \Delta$ is smooth away from the vertices. Along $\partial \Delta$, each $\nu$-fiber is a $T^3$ orbit, but in the interior, each fiber contains two orbits. The following table compares the fiber types for the multi-moment map of $S^3 \times S^3$ to the moment map of a toric 3-manifold:

| Fiber of a point in ... | $\mu$ toric 6-manifold | $\nu$ for nearly Kähler $S^3 \times S^3$ |
|------------------------|------------------------|--------------------------|
| a vertex               | $\{\text{point}\}$     | $T^2$                    |
| an edge                | $S^1$                  | $T^3$                    |
| a face                 | $T^2$                  | $T^3$                    |
| the interior           | $T^3$                  | $T^3 \coprod T^3$        |

2 Torus actions on almost Hermitian structures

Let $(M, g, J, \omega)$ be an almost Hermitian manifold. Let $T$ be a torus acting on $M$ by holomorphic isometries. Any vector $X \in t$ induces a vector field $K_X$ on $M$, which is a holomorphic Killing vector field. This means that $\mathcal{L}_{K_X} g = 0 = \mathcal{L}_{K_X} J$.

By the Leibniz rule, this implies that $\mathcal{L}_{K_X} \omega = 0$.

If $(M, g, J, \omega)$ is Kähler, so that $\omega$ is closed, then there exists a moment map $\mu : M \to t^*$ defined by

$$\langle d\mu, X \rangle = -K_X \omega,$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing of $t$ and $t^*$.

If we do not require $(M, g, J, \omega)$ to be Kähler, there is a multi-moment map associated to the closed 3-form $d\omega$ [4]. This is the map $\nu : M \to \Lambda^2t^*$ defined by

$$\langle d\nu, \sum_i X_i \wedge Y_i \rangle = -\sum_i K_{X_i} (K_{Y_i} d\omega), \quad \forall \sum_i X_i \wedge Y_i \in \Lambda^2t,$$

where here $\langle \cdot, \cdot \rangle$ is the natural pairing of $\Lambda^2t$ and $\Lambda^2t^*$. Recall that the Lie derivative acts on differential forms by

$$\mathcal{L}_{X} \tau = d(X \cdot \tau) + X \cdot d\tau.$$
We can use this to simplify our expression for the multimoment map $\nu$:

$$
\left\langle d\nu, \sum_i X_i \wedge Y_i \right\rangle = -\sum_i K_{X_i} (K Y_i \omega - d(K Y_i \omega))
$$

$$
= \sum_i K_{X_i} d(K Y_i \omega) = \sum_i L_{K_{X_i}} (K Y_i \omega) - d(K_{X_i} K Y_i \omega)
$$

$$
= \sum_i d\omega(K_{X_i}, K Y_i).
$$

Here we've used the fact that $L_{K_{X_i}} \omega = 0 = [K_{X_i}, K Y_i]$ and the Leibniz rule to get $L_{K_{X_i}} (K Y_i \omega) = 0$. This equation can be integrated to solve for $\nu$:

$$
\nu \left( \sum_i X_i \wedge Y_i \right) = \sum_i \omega(K_{X_i}, K Y_i) + C
$$

for some constant $C$. Note that we can always choose $C$ to be 0, so we will.

Note that one cannot expect $\nu$ to behave well for an arbitrary Hermitian structure. Motivated by the behaviour of the moment map of toric manifolds, one could expect that $\nu$ is almost everywhere a submersion, which means that the (multi-)moment map locally separates orbits. The following proposition shows that this condition does not always hold:

**Proposition 2.1.** Let $(M, g, J)$ be an almost Hermitian manifold equipped with a torus $\mathbb{T}$ acting by holomorphic isometries. Then there exists a metric $\hat{g}$ related to $g$ by a $\mathbb{T}$-invariant conformal factor such that the multimoment map $\hat{\nu}$ of $(M, \hat{g}, J, \mathbb{T})$ is not a submersion on some open set in $M$.

**Proof.** If $\nu(M) = \{0\}$, then $\hat{g} = g$ satisfies the claimed property. Otherwise, there exists some $p_0 \in M$ with $\nu(p_0) \neq 0$. We can choose a smooth $\mathbb{T}$-invariant function $\phi$ so that $\phi(p) = -\log \|\nu(p)\|$ for all $p$ in some neighbourhood $U$ of $p_0$. Consider the conformally related metric $\hat{g} = e^\phi g$. The multi-moment map with respect to the conformally related Kähler form $\hat{\omega} = e^\phi \omega$ is $\hat{\nu} = e^\phi \nu$. We chose $\phi$ so that $\hat{\nu}$ maps $U$ into the unit sphere in $\Lambda^2 T^*$, so that $\hat{\nu}$ is not a submersion on $U$.

In the rest of the paper, we will describe the multi-moment map for a torus action on the homogeneous nearly Kähler $S^3 \times S^3$. We find that $\nu$ is a submersion near generic orbits, and show other similarities and differences to the moment map of a toric manifold.

## 3 Homogenous nearly Kähler $S^3 \times S^3$

We begin by reviewing the definition of the homogenous nearly Kähler structure on $S^3 \times S^3$, following the work in [2].

We identify $S^3$ with the unit sphere in the quaternions $\mathbb{H}$. For any $p \in S^3$, $T_p S^3 \subset T_p \mathbb{H}$ is the image of $T_i S^3$ by the pushforward of left-multiplication by $p$. 

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Identifying $T_pS^3 \subset T_p\mathbb{H}$ with $p^\bot \subset \mathbb{H}$, this pushforward is simply quaternionic multiplication by $p$. Thus the basis $\{i, j, -k\}$ of $\text{Im} \mathbb{H}$ which is identified with $T_1S^3$ gives a frame for $T_{(p,q)}S^3 \times S^3 = T_pS^3 \oplus T_qS^3$:

$$E_1(p, q) = (pi, 0), \quad F_1(p, q) = (0, qi),$$

$$E_2(p, q) = (pj, 0), \quad F_2(p, q) = (0, qj),$$

$$E_3(p, q) = (-pk, 0), \quad F_3(p, q) = (0, -qk),$$

where $i, j, k$ are imaginary quaternions satisfying $ij = k$.

The almost complex structure for the homogenous nearly Kähler $S^3 \times S^3$ is given in this frame by

$$J = \frac{1}{\sqrt{3}} \sum_{n=1}^3 (-E_n \otimes E^n + F_n \otimes F^n + 2F_n \otimes E^n - 2E_n \otimes F^n).$$

The metric $g$ is given by the average of $g_{\mathbb{H}^2}$ and $g_{\mathbb{H}^2}(J \cdot, J \cdot)$, where

$$g_{\mathbb{H}^2} = \sum_{n=1}^3 \left( (E^n)^2 + (F^n)^2 \right)$$

is the flat metric from $\mathbb{H}^2$ restricted to $S^3 \times S^3$. This gives

$$g = \frac{4}{3} \sum_{n=1}^3 ((E^n)^2 - E^n F^n + (F^n)^2),$$

$$\omega = \frac{4}{\sqrt{3}} \sum_{n=1}^3 E^n \wedge F^n.$$

3.1 Torus actions

For unit quaternions $a, b, c \in S^3$, the map

$$F_{a,b,c} : S^3 \times S^3 \to S^3 \times S^3 : (p, q) \mapsto (ap^{-1}c, bqc^{-1})$$

is a holomorphic isometry [5].

**Lemma 3.1.** The map

$$F : S^3 \times S^3 \times S^3 \to \text{Aut}(S^3 \times S^3, J) \cap \text{Isom}(S^3 \times S^3, g) : (a, b, c) \mapsto F_{a,b,c}$$

is an injective homomorphism.

**Proof.** It is clear from the definition of $F$ that $F_{a,b,c} \circ F_{a',b',c'} = F_{aa',bb',cc'}$, so that $F$ is a homomorphism. To see that $F$ is injective, let $F_{a,b,c} = Id$. Then

$$(1, 1) = F_{a,b,c}(1, 1) = (ac^{-1}, ab^{-1}),$$

so that $a = b = c$. For any $(p, q) \in S^3 \times S^3$,

$$(p, q) = F_{a,a,a}(p, q) = (apa^{-1}, aqa^{-1}).$$

Since this is true for all $p, q \in S^3$, we find that $a$ lies in the center of $S^3$. Since $S^3$ has a trivial center, $a = 1$ as required. 

\[\square\]
Since the projection of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ onto any of its factors is a homomorphism, any abelian subgroup of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ must be a product of abelian subgroups of $\mathbb{S}^3$. But every non-trivial abelian subgroup of $\mathbb{S}^3$ is of the form $\{e^{At}\}_{t \in \mathbb{R}}$ for some unit imaginary quaternion $A$. Thus, a maximal torus in $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ is of the form 

$$\{ (e^{At_1}, e^{Bt_2}, e^{Ct_3}) \}_{(t_1, t_2, t_3) \in \mathbb{R}^3},$$

for some $A, B, C \in \mathbb{S}^2$, identifying $\mathbb{S}^2$ with the unit imaginary quaternions. A routine computation shows that the image of such a torus under $F$ is generated by the Killing vector fields

$$K_1 = (Ap, 0),$$
$$K_2 = (0, Bq),$$
$$K_3 = (-pC, -qC).$$

Since $p \in \mathbb{S}^3$, $Ap = \bar{p}Ap$, so that these Killing vector fields can be written in terms of the frame $(E_1, E_2, E_3, F_1, F_2, F_3)$ as

$$K_1 = ((\bar{p}Ap) \cdot i)E_1 + ((\bar{p}Ap) \cdot j)E_2 - ((\bar{p}Ap) \cdot k)E_3,$$
$$K_2 = ((\bar{q}Bq) \cdot i)F_1 + ((\bar{q}Bq) \cdot j)F_2 - ((\bar{q}Bq) \cdot k)F_3,$$
$$K_3 = (C \cdot i)(E_1 + F_1) + (C \cdot j)(E_2 + F_2) - (C \cdot k)(E_3 + F_3),$$

where $\cdot$ is the dot product on $\mathbb{H}$. This allows us to compute

$$\frac{\sqrt{3}}{4} \omega(K_1, K_2) = (\bar{p}Ap) \cdot (\bar{q}Bq),$$
$$\frac{\sqrt{3}}{4} \omega(K_1, K_3) = (\bar{p}Ap) \cdot C,$$
$$\frac{\sqrt{3}}{4} \omega(K_2, K_3) = (\bar{q}Bq) \cdot C.$$

Choosing the basis $\left\{ \frac{\sqrt{3}}{4} (K_n \wedge K_m)^* \right\}_{1 \leq n < m \leq 3}$ for $\Lambda^2 t^*$, this allows us to write the multi-moment map as

$$\nu(p, q) = ((\bar{p}Ap) \cdot (\bar{q}Bq), (\bar{p}Ap) \cdot C, (\bar{q}Bq) \cdot C).$$

Note that $\nu^{-1}(0)$ is the union of the Lagrangian torus orbits. The example of a Lagrangian torus in [2] can be found with the values $A = B = i$ and $C = j$.

3.2 Behaviour of the multi-moment map

We will first describe the image of the multi-moment map $\nu$. Then we will describe the structure of its fibers.

For $X \in \mathbb{S}^2$, let us define a map

$$\pi_X : \mathbb{S}^3 \to \mathbb{S}^2 : p \mapsto \bar{p}Xp.$$
When $X = i$, this is the usual Hopf fibration. For general $X$, $\pi_X$ also identifies $S^3$ as a $S^1$ bundle over $S^2$.

Define a function
\[ \bar{\nu} : S^2 \times S^2 \to \Lambda^2 t^* : (x, y) \mapsto (x \cdot y, x \cdot C, y \cdot C), \]
so that $\nu = \bar{\nu} \circ (\pi_A \times \pi_B)$. Let $\Delta = \nu(S^3 \times S^3) = \bar{\nu}(S^2 \times S^2)$ with interior $\Delta$.

**Lemma 3.2.** $\Delta = \left\{ (X, Y, Z) : X \in \left[ f_-(Y, Z), f_+(Y, Z) \right] \right\}$, where
\[ f_\pm(Y, Z) = YZ \pm \sqrt{1 - Y^2} \sqrt{1 - Z^2}. \]

**Proof.** Let $C^\perp \leq \text{Im} \ H$ be the plane orthogonal to $C$. Use the orthogonal decomposition $\text{Im} \ H = \mathbb{R}C \oplus C^\perp$ to write any $(x, y) \in S^2 \times S^2$ as
\[ x = (x \cdot C)C + x^\perp, \quad y = (y \cdot C)C + y^\perp, \quad x^\perp, y^\perp \in C^\perp. \]

Then we have the following relations:
\begin{align*}
1 &= x \cdot x = (x \cdot C)^2 + \|x^\perp\|^2, \\
1 &= y \cdot y = (y \cdot C)^2 + \|y^\perp\|^2, \\
x \cdot y &= (x \cdot C)(y \cdot C) + x^\perp \cdot y^\perp.
\end{align*}

If $\bar{\nu}(x, y) = (X, Y, Z)$, then
\[ X = x \cdot y = YZ + x^\perp \cdot y^\perp. \]

By the Cauchy-Schwarz inequality, \[ |x^\perp \cdot y^\perp| \leq \|x^\perp\| \|y^\perp\| = \sqrt{1 - X^2} \sqrt{1 - Y^2}, \]
so that $f_-(Y, Z) \leq X \leq f_+(Y, Z)$. It is clear that by varying $x$ and $y$, any value of $X$ in this range can be attained, proving the claimed result. \qed

**Lemma 3.3.** $\Delta$ is convex.

**Proof.** By the previous lemma, it suffices to prove that $\bar{\nu} f_\pm$ is a convex function. This follows from the computation
\[ \det \circ \text{Hess}(f_\pm) = \left( \frac{X \sqrt{1 - Y^2} + Y \sqrt{1 - X^2}}{\sqrt{1 - X^2} \sqrt{1 - Y^2}} \right)^2 \geq 0. \]

\qed

**Proposition 3.4.** $\partial \Delta$ is contained in the affine variety $0 = F(X, Y, Z) = 2XYZ - X^2 - Y^2 - Z^2 + 1$. The set of singular points of $\partial \Delta$ is
\[ V := \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}. \]
\textbf{Proposition 3.5.} The line segment between any two points in $V$ lies in $\partial \Delta$.

\textit{Proof.} We will show that the line segment between $(1, 1, 1)$ and $(1, -1, -1)$ lies in $\partial \Delta$, with the other line segments following similarly. This line segment is parametrized by

$$\gamma : [-1, 1] \to \mathbb{R} : t \mapsto (1, t, t).$$

Consider the functions $\bar{f}_\pm(X, Y, Z) := f_\pm(Y, Z) - X$. By lemma 3.2,

$$\partial \Delta = \{ \bar{X} \in \mathbb{R} : \bar{f}_+(\bar{X}) = 0 \geq \bar{f}_-(\bar{X}) \text{ or } \bar{f}_+(\bar{X}) \geq 0 = \bar{f}_-(\bar{X}) \}.$$

We compute

$$\bar{f}_\pm \circ \gamma(t) = t^2 - 1 \pm (1 - t^2).$$

Thus for $t \in [-1, 1]$, $\bar{f}_+ \circ \gamma(t) = 0$ and $\bar{f}_- \circ \gamma(t) = 2(t^2 - 1) \leq 0$. This shows that $\gamma(t) \in \partial \Delta$ as required. \hfill \square

By the previous proposition, we find that $\partial \Delta$ contains the 1-skeleton of the regular tetrahedron with vertices $V$. However, the full tetrahedron is properly contained in $\Delta$. In Figure 1, we see that $\Delta$ is a regular tetrahedron with convexly bulging sides:

\textbf{Proposition 3.6.} $\tilde{\nu}$ has three different orbit types according to the following table:

| dim $\text{Span}\{x, y, C\}$ | location on $\Delta$ | $\text{dim}\text{Span}\{x, y\}$ | $\text{dim}\text{Span}\{x, y, C\}$ |
|-----------------|---------------------|-----------------|-----------------|
| 1               | $V$                 | $\{x, y\}$     | $\mathbb{T}^2$ |
| 2               | $\partial \Delta \setminus V$ | $\mathbb{S}^1 \mathbb{S}^1$ | $\mathbb{T}^3 \mathbb{T}^3$ |
| 3               | $\Delta$            | $\mathbb{S}^1 \mathbb{S}^1$ | $\mathbb{T}^3 \mathbb{T}^3$ |

\textit{Proof.} Let $(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2$ such that $\dim \text{Span}\{x, y, C\} \neq 1$. Then one of $x$ or $y$ is not $\pm C$. We will treat the case when $x \notin \{\pm C\}$, with the other case following similarly.

Write $\tilde{\nu}(x, y) = \tau = (\tau_1, \tau_2, \tau_3)$. Let $(x_0, y_0) \in \tilde{\nu}^{-1}(\tilde{\nu}(x, y))$. Thus $\tau_2 = x_0 \cdot C \notin \{\pm 1\}$. This relation defines a circle $S_0$ on $\mathbb{S}^2$ of possible $x$ values. For a fixed $x_0 \in S_0$, the relations $\tau_1 = x_0 \cdot y_0$ and $\tau_3 = y_0 \cdot C$ define two circles $S_1$ and $S_2$ on $\mathbb{S}^2$ centered at $x_0$ and $C$ respectively, which intersect at possible solutions for $y_0$. Two circles can intersect in at most 2 points. If $S_1$ and $S_2$ do not intersect, then $\tilde{\nu}^{-1}(\tau) = \emptyset$, contradicting $\tau \in \Delta$. If $S_1$ and $S_2$ intersect at exactly one point $y_0$, then $y_0$ is a linear combination of the centers $x_0$ and $C$ of $S_1$ and $S_2$. If they intersect at two points, then each intersection point is not a
Figure 1: The multi-moment map image of $S^3 \times S^3$

linear combination of $x_0$ and $C$. Since there is a circle worth of choices for $x_0$, this gives the last two rows of the table.

The remaining points in $S^2 \times S^2$ satisfy $\dim \text{Span}\{x, y, C\} = 1$. This is equivalent to $x, y \in \{\pm C\}$, which define 4 points. To see that these points live in different $\tilde{\nu}$ fibres, the following table evaluates $\tilde{\nu}$ at each of these points:

| $x$ | $y$ | $\tilde{\nu}(x, y)$ |
|-----|-----|---------------------|
| $C$ | $C$ | (1, 1, 1)           |
| $C$ | $-C$| (-1, 1, -1)        |
| $-C$| $C$ | (-1, -1, 1)        |
| $-C$| $-C$| (1, -1, -1)        |

Thus the singleton fibres get mapped to $V$. We've established the correspondence between the first and third rows in the claimed table. The last column follows since $\nu = \tilde{\nu} \circ (\pi_A \times \pi_B)$, where $\pi_A \times \pi_B$ determined a $T^2$ bundle. The second column follows from the description in lemma 3.2 noting that $\partial \Delta$ consists of the points where the Cauchy-Schwarz inequality is an equality, which are the points where $\{x, y, C\}$ are linearly dependent vectors in $\text{Im} \mathbb{H}$.

Note that $\tilde{\nu}^{-1}(\Delta)$ has two connected components determined by the sign of $\det\{x, y, C\}$, while $\tilde{\nu}^{-1}(\partial \Delta)$ is the vanishing locus of $\det\{x, y, C\}$. It follows that $\nu$ is a submersion along $\nu^{-1}(\Delta)$.
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