Coherence and entanglement measures based on Rényi relative entropies

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Abstract. We study systematically resource measures of coherence and entanglement based on Rényi relative entropies, which include the logarithmic robustness of coherence, geometric coherence, and conventional relative entropy of coherence together with their entanglement analogues. First, we show that each Rényi relative entropy of coherence is equal to the corresponding Rényi relative entropy of entanglement for any maximally correlated state. By virtue of this observation, we establish a simple operational connection between entanglement measures and coherence measures based on Rényi relative entropies. We then prove that all these coherence measures, including the logarithmic robustness of coherence, are additive. Accordingly, all these entanglement measures are additive for maximally correlated states. In addition, we derive analytical formulas for Rényi relative entropies of entanglement of maximally correlated states and bipartite pure states, which reproduce a number of classic results on the relative entropy of entanglement and logarithmic robustness of entanglement in a unified framework. Several nontrivial bounds for Rényi relative entropies of coherence (entanglement) are further derived, which improve over results known previously. Moreover, we determine all states whose relative entropy of coherence is equal to the logarithmic robustness of coherence. As an application, we provide an upper bound for the exact coherence distillation rate, which is saturated for pure states.

Keywords: quantum coherence, entanglement, Rényi relative entropies, robustness of coherence, exact coherence distillation, resource theory, maximally correlated states

1. Introduction

Quantum coherence is a root of many nonclassical phenomena and a valuable resource for quantum information processing. Recently, the resource theory of coherence was
established in [1–3] and stimulated increasing attention in the quantum information community; see [4,5] for a review. It turns out that this resource theory is closely related to the well-established resource theory of entanglement [6–14], which plays a crucial role in the development of coherence theory. Understanding the connections between the two resource theories is a focus of ongoing research.

Recently, Streltsov et al. showed that coherence with respect to a reference basis can be converted to entanglement by incoherent operations acting on the system and an incoherent ancilla [6]. Moreover, the maximum entanglement generated in this way defines a coherence measure. Surprisingly, this mapping can establish a one-to-one correspondence between many useful entanglement measures and coherence measures, including those based on the relative entropy, fidelity, and convex-roof construction [6–11]. Although not so obvious, the $l_1$-norm of coherence [2,13] turns out to be the analogue of the negativity under this mapping [12].

Despite these progresses, it is still not clear what coherence measures in general can be derived from entanglement measures in a natural way. A case in point is the family of coherence measures based on Rényi relative entropies [15–19], which includes three of the most important coherence measures, namely, relative entropy of coherence [1,2] (equal to the distillable coherence [3,7]), logarithmic robustness of coherence [13,18,20,21], and geometric coherence [6]. Their entanglement analogues are equally important in the resource theory of entanglement [22]. Although these resource measures have been studied extensively, most previous works focus on individual measures separately, without studying the connections between them, which leads to severe limitation on our understanding about this subject.

In this paper we explore the connections between entanglement and coherence by studying systematically resource measures based on Rényi relative entropies. First, we show that Rényi relative entropies of coherence and entanglement are equal to the corresponding Rényi conditional entropies for maximally correlated states. Interestingly, the same conclusion holds for three variants of entanglement measures based on separable states, positive-partial-transpose (PPT) states, and nondistillable states, respectively. By virtue of this observation, we show that each Rényi relative entropy of coherence is equal to the maximum of the corresponding Rényi relative entropy of entanglement generated by incoherent operations acting on the system and an incoherent ancilla. The generalized CNOT gate turns out to be the common optimal incoherent operation. In this way, we set an operational one-to-one mapping between entanglement measures and coherence measures based on Rényi relative entropies, which complements a similar mapping between measures based on the convex roof [11].

We then prove that all Rényi relative entropies of coherence, including the logarithmic robustness of coherence, are additive. As an implication, all Rényi relative entropies of entanglement are additive for maximally correlated states. In addition, we derive several nontrivial bounds on Rényi relative entropies of coherence and the robustness of coherence, which significantly improve over bounds known before. In particular, our study shows that the logarithmic $l_1$-norm of coherence is a universal
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upper bound for all Rényi relative entropies of coherence. Similar results apply to Rényi relative entropies of entanglement of maximally correlated states. Moreover, we derive analytical formulas for Rényi relative entropies of entanglement of maximally correlated states and bipartite pure states, which reproduce a number of classic results on the relative entropy of entanglement and logarithmic robustness of entanglement in a unified framework.

Furthermore, we clarify the relations between different Rényi relative entropies of coherence and determine all states whose relative entropy of coherence (or distillable coherence) is equal to the logarithmic robustness of coherence. It turns out that for these states all Rényi relative entropies of coherence coincide with the relative entropy of coherence. To achieve this goal, we determine the condition under which Rényi relative entropies are independent of the order parameter, note that they are usually monotonically increasing with this parameter.

As an application, we provide an upper bound for the exact coherence distillation rate, which is saturated for pure states. It turns out that for pure states this rate remains the same under three distinct classes of operations, namely, strictly incoherent operations, incoherent operations, and incoherence-preserving operations. This result parallels a similar result on exact entanglement distillation [17, 23, 24], which further strengthens the connection between the resource theory of coherence and that of entanglement. In addition, we derive a necessary condition under which the exact coherence distillation rate is equal to the distillable coherence, thereby clarifying the relation between exact coherence distillation and approximate distillation with vanishing error asymptotically. Besides, the results presented here play a crucial role in studying secure random number generation via incoherent operations [25].

The rest of this paper is organized as follows. In section 2 we review the basic concepts and known results about Rényi relative entropies together with entanglement measures and coherence measures based on them. In section 3 we establish an operational one-to-one mapping between entanglement measures and coherence measures based on Rényi relative entropies and thereby derive Rényi relative entropies of entanglement of maximally correlated states. In section 4 we prove the additivity of Rényi relative entropies of coherence and the logarithmic robustness of coherence. In section 5 we derive several nontrivial upper and lower bounds for Rényi relative entropies of coherence. In section 6 we investigate the relations between different Rényi relative entropies. In section 7 we clarify the relations between different Rényi relative entropies of coherence. In section 8 we provide an upper bound for the exact coherence distillation rate, which is saturated for pure states. Section 9 summarizes this paper.

2. Preliminaries

In this section we review the basic concepts and known results about two types of Rényi relative entropies together with entanglement measures and coherence measures based on them. A few new results are added for completeness.
2.1. Rényi relative entropies and conditional entropies

The relative entropy between two density matrices $\rho$ and $\sigma$ on a given Hilbert space $\mathcal{H}$ reads

$$ S(\rho\|\sigma) := \text{tr}(\rho \ln \rho) - \text{tr}(\rho \ln \sigma) = -S(\rho) - \text{tr}(\rho \ln \sigma), $$

where “$\ln$” denotes the natural logarithm and $S(\rho) := -\text{tr}(\rho \ln \rho)$ denotes the von Neumann entropy of $\rho$. Although we choose the natural logarithm in this paper, except for section 3, however, the choice of the base for the logarithm does not affect our results explicitly as long as “exp” and “log” take on the same base. The relative entropy $S(\rho\|\sigma)$ reduces to the relative entropy between two probability distributions when both $\rho$ and $\sigma$ are diagonal with respect to a reference basis.

As generalization, consider two types of Rényi relative entropies [15,16] [17, Section 3.1]

$$ S_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \ln \text{tr}(\rho^\alpha \sigma^{1-\alpha}), \quad S_\infty(\rho\|\sigma) := \frac{1}{\alpha - 1} \ln \text{tr}(\sigma^{1-\alpha/2} \rho \sigma^{-1/2})^\alpha, $$

where $\alpha \geq 0$ is known as the order parameter. The power of a positive operator is understood as the power on its support. The second argument $\sigma$ in $S_\alpha(\rho\|\sigma)$ and $S_\infty(\rho\|\sigma)$ can be generalized to positive operators. These Rényi relative entropies have wide applications in quantum information processing [17] and have operational interpretations in connection with quantum hypothesis testing [26].

In the cases $\alpha = 0, 1, \infty$, the definitions of $S_\alpha(\rho\|\sigma)$ and $S_\infty(\rho\|\sigma)$ above are understood as proper limits, all of which are well defined. Hence, the order parameter $\alpha$ for both types of Rényi relative entropies can be regarded to run from 0 to $\infty$. To be concrete,

$$ S_0(\rho\|\sigma) = \lim_{\alpha \to 0} S_\alpha(\rho\|\sigma) = -\ln \text{tr}(\Pi_\rho \sigma), $$

where $\Pi_\rho$ is the projector onto the support of $\rho$; the limit $S_0(\rho\|\sigma) = \lim_{\alpha \to 0} S_\alpha(\rho\|\sigma)$ is derived in [27], but is not needed here. Both $S_\alpha(\rho\|\sigma)$ and $S_\infty(\rho\|\sigma)$ in the limit $\alpha \to 1$. The limits $\lim_{\alpha \to \infty} S_\alpha(\rho\|\sigma)$ and $\lim_{\alpha \to \infty} S_\infty(\rho\|\sigma)$ are written as $S_\infty(\rho\|\sigma)$ and $S_\infty(\rho\|\sigma)$, respectively. The latter $S_\infty(\rho\|\sigma)$ is known as the max relative entropy [15,28,30] and can be expressed as

$$ S_\infty(\rho\|\sigma) = \min \{\ln \lambda | \lambda \sigma \geq \rho\}. $$

The following two special cases of $S_\alpha(\rho\|\sigma)$ are also useful to the current study,

$$ S_{1/2}(\rho\|\sigma) = -\ln F(\rho, \sigma), $$
$$ S_2(\rho\|\sigma) = \ln \text{tr} \left[ (\sigma^{-1/4} \rho \sigma^{-1/4})^2 \right] = \ln \text{tr}(\sigma^{-1/2} \rho \sigma^{-1/2}), $$

where $F(\rho, \sigma) := (\text{tr} \sqrt{\sigma^{1/2} \rho \sigma^{-1/2}})^2$ denotes the fidelity between $\rho$ and $\sigma$. The two relative entropies $S_{1/2}(\rho\|\sigma)$ and $S_2(\rho\|\sigma)$ are known as the min relative entropy and collision relative entropy, respectively [15,28,30].
According to the Araki-Lieb-Thirring inequality \cite{31,32} and the result in \cite{33}, the two types of Rényi relative entropies defined in \cite{2} satisfy the following inequality \cite{15,16,17,section 3.1}

\[ S_\alpha(\rho||\sigma) \geq S_\alpha(\rho||\sigma) \quad \forall \alpha \in [0, \infty]. \]  

(7)

When \( \alpha \in (0, \infty) \) with \( \alpha \neq 1 \), the inequality is saturated if and only if (iff) \( \rho \) and \( \sigma \) commute \cite{26}. Both \( S_\alpha(\rho||\sigma) \) and \( S_\alpha(\rho||\sigma) \) are monotonically increasing (means nondecreasing in this paper) with \( \alpha \). Similar to \( S(\rho||\sigma) \), the Rényi relative entropy \( S_\alpha(\rho||\sigma) \) satisfies the data-processing inequality for \( \alpha \in [0, 2] \) \cite{34}, and \( S_\alpha(\rho||\sigma) \) satisfies the data-processing inequality for \( \alpha \in [\frac{1}{2}, \infty] \) \cite{15,16,35,36,17,lemma 3.1}. In other words, these Rényi relative entropies are contractive under any completely positive and trace-preserving (CPTP) map \( \Lambda \). More precisely, we have

\[ S_\alpha(\Lambda(\rho)||\Lambda(\sigma)) \leq S_\alpha(\rho||\sigma) \quad \forall \alpha \in [0, 2], \]  

(8)

\[ S_\alpha(\Lambda(\rho)||\Lambda(\sigma)) \leq S_\alpha(\rho||\sigma) \quad \forall \alpha \in [\frac{1}{2}, \infty]. \]  

(9)

In addition, \( \exp[(\alpha - 1)S_\alpha(\rho||\sigma)] \) is jointly convex for \( \alpha \in (1, 2] \) and jointly concave for \( \alpha \in (0, 1) \); by contrast, \( \exp[(\alpha - 1)S_\alpha(\rho||\sigma)] \) is jointly convex for \( \alpha \in (1, \infty] \) and jointly concave for \( \alpha \in [\frac{1}{2}, 1] \) \cite{15,37}. To see this, let \( \rho_1, \rho_2, \sigma_1, \sigma_2 \) be four arbitrary quantum states on \( \mathcal{H} \) and \( 0 \leq \lambda \leq 1 \). Consider the two states

\[ \rho := \begin{pmatrix} \lambda \rho_1 & 0 \\ 0 & (1 - \lambda) \rho_2 \end{pmatrix}, \quad \sigma := \begin{pmatrix} \lambda \sigma_1 & 0 \\ 0 & (1 - \lambda) \sigma_2 \end{pmatrix} \]  

(10)

on the composite system \( \mathbb{C}^2 \otimes \mathcal{H} \). Taking the partial trace over the first subsystem yields

\[ \lambda \exp[(\alpha - 1)S_\alpha(\rho_1||\sigma_1)] + (1 - \lambda) \exp[(\alpha - 1)S_\alpha(\rho_2||\sigma_2)] = \exp[(\alpha - 1)S_\alpha(\rho||\sigma)] \]

\[ \geq \exp[(\alpha - 1)S_\alpha(\lambda \rho_1 + (1 - \lambda) \rho_2||\lambda \sigma_1 + (1 - \lambda) \sigma_2)] \quad \forall \alpha \in (1, 2], \]  

(11)

where the inequality follows from the data-processing inequality \cite{8} and the fact that the partial trace is a CPTP map. Therefore, \( \exp[(\alpha - 1)S_\alpha(\rho||\sigma)] \) is jointly convex for \( \alpha \in (1, 2] \). The joint convexity of \( \exp[(\alpha - 1)S_\alpha(\rho||\sigma)] \) for \( \alpha \in (1, \infty] \) follows from the same reasoning. The joint concavity of \( \exp[(\alpha - 1)S_\alpha(\rho||\sigma)] \) for \( \alpha \in [0, 1) \) and \( \exp[(\alpha - 1)S_\alpha(\rho||\sigma)] \) for \( \alpha \in [\frac{1}{2}, 1] \) can also be proved in a similar manner.

Next, we turn to conditional entropies constructed from Rényi relative entropies. Given a bipartite state \( \rho \) shared by Alice (A) and Bob (B), the conditional entropy of A given B have three equivalent definitions,

\[ H(A|B)_\rho := S(\rho_{AB}) - S(\rho_B) = -S(\rho_{AB}||I_A \otimes \rho_B) = -\min_{\sigma_B} S(\rho_{AB}||I_A \otimes \sigma_B), \]  

(12)

where \( \rho_{AB} = \rho \) (the subscripts are omitted if there is no confusion), \( \rho_B = \text{tr}_A(\rho) \), \( I_A \) denotes the identity on \( \mathcal{H}_A \), and the minimization is taken over all quantum states.
\[ H_{\alpha}^{\downarrow}(A|B)_{\rho} := -S_{\alpha}(\rho_{AB}\|I_{A} \otimes \rho_{B}), \quad H_{\alpha}^{\uparrow}(A|B)_{\rho} := -\min_{\sigma_{B}} S_{\alpha}(\rho_{AB}\|I_{A} \otimes \sigma_{B}), \quad (13) \]

\[ \overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho} := -\max_{\sigma_{B}} S_{\alpha}(\rho_{AB}\|I_{A} \otimes \rho_{B}), \quad \overline{H}_{\alpha}^{\uparrow}(A|B)_{\rho} := -\max_{\sigma_{B}} S_{\alpha}(\rho_{AB}\|I_{A} \otimes \sigma_{B}). \quad (14) \]

By definitions and the inequality \( S_{\alpha}(\rho\|\sigma) \geq S_{\alpha}(\rho\|\sigma) \) in (7), these conditional entropies satisfy

\[ H_{\alpha}^{\downarrow}(A|B)_{\rho} \leq H_{\alpha}^{\downarrow}(A|B)_{\rho}, \quad \overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho} \leq \overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho}, \quad (15) \]

The conditional entropy \( H_{\alpha}^{\uparrow}(A|B)_{\rho} \) has a closed formula according to [38],

\[ H_{\alpha}^{\uparrow}(A|B)_{\rho} = \frac{\alpha}{1 - \alpha} \ln \text{tr} \left\{ \left[ \text{tr}_{A}(\rho_{AB}^{\alpha}) \right]^{1/\alpha} \right\}. \quad (16) \]

When \( \rho \) is a classical-quantum state, i.e., it has the form \( \rho = \sum_{a} p_{A}(a) |a\rangle \langle a| \otimes \rho_{B(a)} \), the quantity \( \exp \left[-\overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho}\right] \) expresses the optimal probability of guessing correctly the classical information concerning \( A \) from the quantum system \( B \) [39, theorem 1].

When \( \rho = \rho_{A} \otimes \rho_{B} \) is a tensor product, straightforward calculation shows that the four types of Rényi conditional entropies coincide with each other,

\[ \overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho} = \overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho} = \overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho} = H_{\alpha}^{\downarrow}(A|B)_{\rho} = -S_{\alpha}(\rho_{A}\|I_{A}) = S_{\alpha}(\rho_{A}), \quad (17) \]

where

\[ S_{\alpha}(\rho_{A}) := \frac{1}{1 - \alpha} \text{tr}(\rho_{A}^{\alpha}) \quad (18) \]

is the Rényi \( \alpha \)-entropy of \( \rho_{A} \).

When \( \rho \) is a tripartite pure state shared by \( A, B \) and \( E \) (Eve), Rényi conditional entropies obey the following duality relations.

**Proposition 1** ([15] [35] [38] [17, theorem 5.13]).

\[ H_{\alpha}^{\downarrow}(A|E)_{\rho} + H_{\beta}^{\downarrow}(A|B)_{\rho} = 0, \quad (19) \]

\[ \overline{H}_{\alpha}^{\downarrow}(A|E)_{\rho} + \overline{H}_{\beta}^{\downarrow}(A|B)_{\rho} = 0, \quad (20) \]

\[ \overline{H}_{\alpha}^{\downarrow}(A|E)_{\rho} + \overline{H}_{\beta}^{\downarrow}(A|B)_{\rho} = 0, \quad (21) \]

where (19) holds for \( \alpha, \beta \in [0, 2] \) with \( \alpha + \beta = 2 \), (20) holds for \( \alpha, \beta \in [\frac{1}{2}, \infty] \) with \( \frac{1}{\alpha} + \frac{1}{\beta} = 2 \), and (21) holds for \( \alpha, \beta \in [0, \infty] \) with \( \alpha \beta = 1 \).

The duality relations in proposition 1 can be used to derive inequalities between different Rényi conditional entropies [38, corollary 4] as well as upper and lower bounds for these conditional entropies.
Lemma 1. Suppose $\alpha \in [\frac{1}{2}, \infty]$ and $\rho$ is a bipartite state shared by Alice and Bob. Then

\begin{align}
H^\dagger_\alpha(A|B)_\rho &\leq H^\dagger_1(A|B)_\rho \leq H^\dagger_{\frac{1}{2} - \frac{1}{\alpha}}(A|B)_\rho, \\
\overline{H}^\dagger_\alpha(A|B)_\rho &\leq \overline{H}^\dagger_1(A|B)_\rho \leq \overline{H}^\dagger_{\frac{1}{2} - \frac{1}{\alpha}}(A|B)_\rho, \\
H^\dagger_\alpha(A|B)_\rho &\leq H^\dagger_\alpha(A|B)_\rho \leq H^\dagger_{\frac{1}{2} - \frac{1}{\alpha}}(A|B)_\rho, \\
H^\dagger_\alpha(A|B)_\rho &\leq \overline{H}^\dagger_\alpha(A|B)_\rho \leq H^\dagger_{\frac{1}{2} - \frac{1}{\alpha}}(A|B)_\rho.
\end{align}

The second inequality in each of the four equations is saturated whenever $\rho$ is pure.

Remark 1. The inequalities in lemma 1 were derived in [38, corollary 4]. The first inequalities in the four equations are reproduced from (23). The paper [38] did not discuss the equality conditions. The following proof refines the original proof in [38], so as to show that the second inequalities in the four equations are saturated when $\rho$ is pure.

Proof. Suppose $\alpha \in [\frac{1}{2}, \infty]$. Let $\sigma$ be a purification of $\rho$ that is shared by A, B, and E. Then $H^\dagger_\alpha(A|B)_\rho = H^\dagger_\alpha(A|B)_\sigma$, so that

\begin{equation}
H^\dagger_\alpha(A|B)_\rho = -\overline{H}^\beta_\alpha(A|E)_\sigma \leq -H^\beta_\alpha(A|E)_\sigma = H^\dagger_\gamma(A|B)_\rho
\end{equation}

according to proposition 1 where $\beta = 1/\alpha$ and $\gamma = 2 - \beta = 2 - (1/\alpha)$. This result confirms the first equation in lemma 1 given that the first inequality there is trivial. If $\rho$ is pure, then $\sigma_{AE}$ must be a product state, so that the inequality in (26) is saturated according to (17), which implies that $H^\dagger_\alpha(A|B)_\rho = H^\dagger_{\frac{1}{2} - \frac{1}{\alpha}}(A|B)_\rho$. The other three equations in lemma 1 can be derived in a similar manner.

Lemma 2.

\begin{align}
H^\dagger_\alpha(A|B)_\rho &\leq H^\dagger_\alpha(A|B)_\rho \leq S_\alpha(\rho_A) \quad \forall \alpha \in [0, \infty], \\
\overline{H}^\dagger_\alpha(A|B)_\rho &\leq \overline{H}^\dagger_\alpha(A|B)_\rho \leq S_\alpha(\rho_A) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right].
\end{align}

All the four inequalities are saturated simultaneously for all $\alpha$ iff $\rho$ is a product state.

Remark 2. The inequality $\overline{H}^\dagger_\alpha(A|B)_\rho \leq S_\alpha(\rho_A)$ was derived in [40].

Proof. If $\alpha \in [\frac{1}{2}, \infty]$, then

\begin{equation}
\overline{H}^\dagger_\alpha(A|B)_\rho = -\min_{\sigma_B} S_\alpha(\rho_{AB}\|I_A \otimes \sigma_B) \leq -S_\alpha(\rho_A\|I_A) = -S_\alpha(\rho_A) = S_\alpha(\rho_A),
\end{equation}

where the inequality is due to the monotonicity of $S_\alpha$ under the partial trace. This observation confirms (28) given that the first inequality there is obvious. By the same token, $H^\dagger_\alpha(A|B)_\rho \leq S_\alpha(\rho_A)$ for $\alpha \in [0, 2]$. In addition $H^\dagger_\alpha(A|B)_\rho \leq \overline{H}^\dagger_\alpha(A|B)_\rho \leq S_\alpha(\rho_A)$ for $\alpha \in \left[\frac{1}{2}, \infty\right]$, which confirms (27).
If \( \rho \) is a product state, then the four inequalities in lemma 2 are saturated according to (17). Conversely, if all the four inequalities are saturated for all \( \alpha \), then
\[
H(A|B)_{\rho} = S(\rho_A),
\]
which implies that \( S(\rho_{AB}||\rho_A \otimes \rho_B) = 0 \), so that \( \rho_{AB} = \rho_A \otimes \rho_B \) is a product state.

The following lemma generalizes the Araki-Lieb inequality
\[
H(A|B) \geq -S(\rho_A) \quad [41],
\]
in which (33) was derived in [40].

Lemma 3.
\[
\begin{align*}
H_{\alpha}^{\downarrow}(A|B)_{\rho} & \geq -S_{2-\alpha}(\rho_A) \quad \forall \alpha \in [0, 2], \\
H_{\alpha}^{\uparrow}(A|B)_{\rho} & \geq -S_{\frac{2}{\alpha}}(\rho_A) \quad \forall \alpha \in [0, 2], \\
\overline{H}_{\alpha}^{\downarrow}(A|B)_{\rho} & \geq -S_{\frac{2}{\alpha}}(\rho_A) \quad \forall \alpha \in [0, \infty], \\
\overline{H}_{\alpha}^{\uparrow}(A|B)_{\rho} & \geq -S_{\frac{2}{\alpha-1}}(\rho_A) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right].
\end{align*}
\]

All the four inequalities are saturated simultaneously for all \( \alpha \) iff the system A is independent of the environment of \( \rho \). In particular, all the four inequalities are saturated when \( \rho \) is pure.

Remark 3. When \( \rho \) is pure, the system A is independent of the environment of \( \rho \). However, the converse does not hold in general. For example, when \( \rho = \rho_A \otimes \rho_B \) with \( \rho_A \) a pure state, the system A is independent of the environment of \( \rho \), although \( \rho \) is not necessarily pure.

Proof. Let \( \sigma \) be a purification of \( \rho \) that is shared by A, B, and E. If \( \alpha \in [0, 2] \), then
\[
H_{\alpha}^{\downarrow}(A|B)_{\rho} = -H_{2-\alpha}(A|E)_{\sigma} \geq -S_{2-\alpha}(\rho_A)
\]
according to proposition 1 and lemma 2. If the system A is independent of the environment of \( \rho \), that is, if \( \sigma_{AE} \) is a product state, then the inequality above is saturated according to lemma 2. The other three inequalities in lemma 3 can be derived in a similar manner, and they are saturated when \( \sigma_{AE} \) is a product state by the same token.

Conversely, if all the four inequalities in lemma 3 are saturated for all \( \alpha \), then we have \( H(A|E) = S(\rho_A) = S(\sigma_A) \), which implies that \( S(\sigma_{AE}||\sigma_A \otimes \sigma_E) = 0 \), so that \( \sigma_{AE} = \sigma_A \otimes \sigma_E \). Therefore, the system A is independent of the environment of \( \rho \). 

2.2. Entanglement measures based on Rényi relative entropies

Given a bipartite state \( \rho \) shared by Alice and Bob, we can define two types of Rényi relative entropies of entanglement as
\[
E_{\alpha}^{\downarrow}(\rho) := \min_{\sigma \in \mathcal{A}} S_{\alpha}(\rho||\sigma), \quad E_{\alpha}^{\uparrow}(\rho) := \min_{\sigma \in \mathcal{A}} S_{\alpha}(\rho||\sigma),
\]
where \( \mathcal{A} \) may denote one of the three sets, the set of separable states, that of PPT states, and that of nondistillable states. To simplify the notation, we will drop this superscript if a statement applies to all three choices of \( \mathcal{A} \). Incidentally, Rényi relative entropies are also useful to quantifying quantum correlations [42].
Proposition 2. $E_{r,\alpha}(\rho)$ for $\alpha \in [0, 2]$ and $\mathcal{E}_{r,\alpha}(\rho)$ for $\alpha \in \left[\frac{1}{2}, \infty\right]$ do not increase under local operations and classical communication (LOCC).

This proposition shows that $E_{r,\alpha}(\rho)$ for $\alpha \in [0, 2]$ and $\mathcal{E}_{r,\alpha}(\rho)$ for $\alpha \in \left[\frac{1}{2}, \infty\right]$ are proper entanglement measures. This conclusion follows from the following two facts: First, the Rényi relative entropies $S_{\alpha}(\rho||\sigma)$ for $\alpha \in [0, 2]$ and $S_{\alpha}(\rho||\sigma)$ for $\alpha \in \left[\frac{1}{2}, \infty\right]$ satisfy the data-processing inequality [31, 37, lemma 8.7] [37, lemma 3.4]; see [8] and [9]. Second, the set of separable states is invariant under LOCC, and so are the set of PPT states and that of nondistillable states. Actually, here LOCC can be replaced by CPTP maps that preserve the set $\mathcal{A}$ of concern. Outside these parameter ranges, $E_{r,\alpha}(\rho)$ and $\mathcal{E}_{r,\alpha}(\rho)$ do not satisfy basic requirements for entanglement measures, but they are still useful in our study.

Incidentally, the quantities $\exp[(\alpha - 1) E_{r,\alpha}(\rho)]$ with $\alpha \in (1, 2]$ and $\exp[(\alpha - 1) \mathcal{E}_{r,\alpha}(\rho)]$ with $\alpha \in (1, \infty]$ are convex in $\rho$ due to the joint convexity of the corresponding Rényi relative entropies [31, 37, lemma 3.4]. By contrast, the quantities $\exp[(\alpha - 1) E_{r,\alpha}(\rho)]$ with $\alpha \in [0, 1)$ and $\exp[(\alpha - 1) \mathcal{E}_{r,\alpha}(\rho)]$ with $\alpha \in \left[\frac{1}{2}, 1\right)$ are concave [37, lemma 3.4]. Taking the logarithm, we find that the entanglement measures $E_{r,\alpha}(\rho)$ with $\alpha \in [0, 1)$ and $\mathcal{E}_{r,\alpha}(\rho)$ with $\alpha \in \left[\frac{1}{2}, 1\right)$ are convex in $\rho$.

In the limit $\alpha \to 1$, both Rényi relative entropies of entanglement $E_{r,\alpha}^d(\rho)$ and $\mathcal{E}_{r,\alpha}^d(\rho)$ approach the conventional relative entropy of entanglement [22, 43, 44]

$$E_{r}^d(\rho) := \min_{\sigma \in \mathcal{A}} S(\rho||\sigma).$$

(36)

In another limit $\alpha \to \infty$, the variant $\mathcal{E}_{r,\alpha}^d(\rho)$ approaches the logarithmic robustness of entanglement [28, 29, 45]

$$\mathcal{E}_{r,\infty}^d(\rho) = E_{r,\infty}^d(\rho) := \ln(1 + E_{r}^d(\rho)),$$

(37)

where

$$E_{r}^d(\rho) := \min \left\{ x \left| x \geq 0, \exists \text{ a state } \sigma, \frac{\rho + x\sigma}{1 + x} \in \mathcal{A} \right\}$$

(38)

is the robustness of entanglement (originally called the generalized robustness of entanglement) [22, 46-49]. Here $\sigma$ is an arbitrary quantum state, not necessarily contained in $\mathcal{A}$. In general, $E_{r,\alpha}(\rho)$ and $\mathcal{E}_{r,\alpha}(\rho)$ are monotonically increasing with $\alpha$. Therefore,

$$E_{r,\alpha}(\rho) \leq E_{r,\infty}(\rho) \quad \forall \alpha \in [0, \infty].$$

(39)

The special case $E_{r}(\rho) \leq E_{r,\infty}(\rho)$ is well known [45, 50]. In addition, the min relative entropy of entanglement $E_{r,1/2}^d(\rho)$ is equal to a variant of the geometric (measure of) entanglement [22, 51, 52],

$$E_{r,1/2}^d(\rho) = E_{G}^d(\rho) := -\ln \max_{\sigma \in \mathcal{A}} F(\rho, \sigma),$$

(40)

recall that $S_{1/2}(\rho||\sigma) = -\ln F(\rho, \sigma)$ according to [45]. The measure $E_{G}^d(\rho)$ has a popular variant defined as

$$\mathcal{E}_{G}^d(\rho) := 1 - \max_{\sigma \in \mathcal{A}} F(\rho, \sigma).$$

(41)
In this paper we are more interested in the first variant $E_G(\rho)$ due to its simple connection with Rényi relative entropies of entanglement. It is known that $E_{r,1/2}(\rho)$ and $E_{r,0}(\rho)$ set upper bounds for the asymptotic exact distillation rate of entanglement [17, lemma 8.15], and both bounds are saturated for pure states [23,24] [17, Exercise 8.32].

When $\rho$ is a pure state, $E_r(\rho)$ is equal to the von Neumann entropy of each reduced state [44], while $E_R(\rho)$ is equal to the negativity [53]. Recall that the negativity of a bipartite state $\rho$ is defined as

$$\mathcal{N}(\rho) := \text{tr}(|\rho^T_A|) - 1,$$

where $T_A$ denotes the partial transpose with respect to the subsystem $A$, and $|M| = \sqrt{M^\dagger M}$. For example, let $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_j \sqrt{\lambda_j} |jj\rangle$. Then we have

$$E_r(\rho) = S(\rho_A) = -\sum_j \lambda_j \ln \lambda_j, \quad E_R(\rho) = \mathcal{N}(\rho) = (\text{tr} \sqrt{\rho_A})^2 - 1 = \left(\sum_j \sqrt{\lambda_j}\right)^2 - 1.$$  

The following lemma provides lower bounds for Rényi relative entropies of entanglement in terms of Rényi conditional entropies. The special case (44) is derived in [54].

**Lemma 4.** Any bipartite state $\rho$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ satisfies

$$E_t(\rho) \geq -H(A|B)_\rho, \quad E_{r,\alpha}(\rho) \geq -H^\alpha(A|B)_\rho \quad \forall \alpha \in [0, 2], \quad E_{r,\alpha}(\rho) \geq -\overline{H}^\alpha(A|B)_\rho \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right], \quad E_{R,L}(\rho) \geq -\overline{H}_\infty(A|B)_\rho.$$  

**Proof.** Let $\sigma$ be an arbitrary nondistillable state. Then $\sigma \leq I_A \otimes \sigma_B$ according to proposition 3 below, so that

$$S(\rho\|\sigma) \geq S(\rho\|I_A \otimes \sigma_B),$$

because the relative entropy is monotonically decreasing in the second argument. Therefore,

$$E_{t,d}^{\sigma}(\rho) = \min_{\sigma \in \mathcal{A}} S(\rho\|\sigma) \geq \min_{\sigma \in \mathcal{A}} S(\rho\|I_A \otimes \sigma_B) = -H(A|B)_\rho,$$

where $\mathcal{A}$ could be the set of separable states, that of PPT states, or that of nondistillable states (note that the first two sets are contained in the third one). This observation confirms (44). Equations (45) and (46) follow from the same reasoning, note that Rényi relative entropies $S_\alpha$ with $\alpha \in [0, 2]$ and $\overline{S}_\alpha$ with $\alpha \in \left[\frac{1}{2}, \infty\right]$ are also monotonically decreasing in the second argument [15,16,17, Exercise 5.25]. Equation (47) is the limit $\alpha \rightarrow \infty$ of (46).
Coherence and entanglement measures based on Rényi relative entropies

Figure 1. Illustration of the definition of the Rényi relative entropy of coherence $C_{r,\alpha}(\rho)$. The other variant $C_{r,\alpha}(\rho)$ is defined in a similar way.

The following proposition was proved in [55]. See [56] for a partial converse.

Proposition 3 ([55]). Any nondistillable bipartite state $\sigma$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ satisfies the reduction criterion, that is,

$$\sigma \leq I_A \otimes \sigma_B, \quad \sigma \leq \sigma_A \otimes I_B.$$  \hfill (50)

2.3. Coherence measures based on Rényi relative entropies

Consider a $d$-dimensional Hilbert space $\mathcal{H}$ with a reference basis $\{|j\rangle\}$. A quantum state $\rho$ is incoherent if it is diagonal with respect to the reference basis. The set of incoherent states is denoted by $\mathcal{I}$. A CPTP map $\Lambda$ is incoherence preserving (also called maximally incoherent) if $\Lambda(\rho) \in \mathcal{I}$ whenever $\rho \in \mathcal{I}$. Suppose the CPTP map $\Lambda$ has Kraus representation $\{K_j\}$, that is, $\Lambda(\rho) = \sum_j K_j \rho K_j^\dagger$ for all $\rho$. Then $\{K_j\}$ is incoherent if each Kraus operator $K_j$ maps every incoherent state to an incoherent state, that is $K_j \rho K_j^\dagger / \text{tr}(K_j \rho K_j^\dagger) \in \mathcal{I}$ whenever $\rho \in \mathcal{I}$[1–4]. It is strictly incoherent if in addition $K_j^\dagger \rho K_j / \text{tr}(K_j^\dagger \rho K_j) \in \mathcal{I}$ whenever $\rho \in \mathcal{I}$[3]. A CPTP map is necessarily incoherence preserving if it has an (strictly) incoherent Kraus representation. A pure state of the form $|\psi\rangle = \sum_j c_j |j\rangle$ with $|c_j|^2 = 1/d$ is called maximally coherent because any other state in dimension $d$ can be generated from it under (strictly) incoherent operations [2].

Note that the definition of coherence is basis dependent, and so are many related concepts in the resource theory of coherence, including incoherent states, maximally coherent states, incoherence-preserving operations, and (strictly) incoherent operations. All results about coherence in this paper are stated with respect to a given reference basis.

In analogy to entanglement theory, two families of coherence quantifiers can be defined in terms of Rényi relative entropies [18,19] as illustrated in figure 1.

$$C_{r,\alpha}(\rho) := \min_{\sigma \in \mathcal{I}} S_\alpha(\rho||\sigma), \quad C_{r,\alpha}(\rho) := \min_{\sigma \in \mathcal{I}} S_\alpha(\rho||\sigma),$$  \hfill (51)

where $\mathcal{I}$ denotes the set of incoherent states. Related measures based on Tsallis relative entropies were studied in [57]. Many results presented in this paper still apply if
Rényi relative entropies are replaced by Tsallis relative entropies because the latter are monotonic functions of the former.

Proposition 4. \( C_{r,\alpha}(\rho) \) for \( \alpha \in [0, 2] \) and \( C_{r,\alpha}(\rho) \) for \( \alpha \in [\frac{1}{2}, \infty] \) do not increase under incoherence-preserving operations (including incoherent operations).

This proposition shows that \( C_{r,\alpha}(\rho) \) for \( \alpha \in [0, 2] \) and \( C_{r,\alpha}(\rho) \) for \( \alpha \in [\frac{1}{2}, \infty] \) are proper coherence measures, in analogy to the corresponding entanglement measures. This conclusion follows from two facts: First, the Rényi relative entropies \( S_\alpha(\rho\|\sigma) \) for \( \alpha \in [0, 2] \) and \( S_\alpha(\rho\|\sigma) \) for \( \alpha \in [\frac{1}{2}, \infty] \) satisfy the data-processing inequality [34] [17, lemma 8.7] [37, lemma 3.4]; see (8) and (9). Second, the set of incoherent states is invariant under incoherence-preserving operations. Outside these parameter ranges, \( C_{r,\alpha}(\rho) \) and \( C_{r,\alpha}(\rho) \) do not satisfy basic requirements for coherence measures, but they are still useful in our study.

Incidentally, the quantities \( \exp[(\alpha-1)C_{r,\alpha}(\rho)] \) with \( \alpha \in (1, 2] \) and \( \exp[(\alpha-1)C_{r,\alpha}(\rho)] \) with \( \alpha \in (1, \infty) \) are convex in \( \rho \) due to the joint convexity of the corresponding Rényi relative entropies as shown in [11]. [37, lemma 3.4]. By contrast, the quantities \( \exp[(\alpha-1)C_{r,\alpha}(\rho)] \) with \( \alpha \in [0, 1) \) and \( \exp[(\alpha-1)C_{r,\alpha}(\rho)] \) with \( \alpha \in [\frac{1}{2}, 1) \) are concave [37, lemma 3.4]. Taking the logarithm, we find that the coherence measures \( C_{r,\alpha}(\rho) \) with \( \alpha \in [0, 1) \) and \( C_{r,\alpha}(\rho) \) with \( \alpha \in [\frac{1}{2}, 1) \) are convex in \( \rho \).

In the limit \( \alpha \to 1 \), both measures \( C_{r,\alpha}(\rho) \) and \( C_{r,\alpha}(\rho) \) approach the conventional relative entropy of coherence [12],

\[
C_r(\rho) := \min_{\sigma \in \mathcal{I}} S(\rho\|\sigma) = S(\rho^{\text{diag}}) - S(\rho),
\]

where \( \rho^{\text{diag}} \) is the diagonal part of \( \rho \) with respect to the reference basis. In another limit \( \alpha \to \infty \), the measure \( C_{r,\alpha}(\rho) \) approaches the logarithmic robustness of coherence [18],

\[
C_{r,\infty}(\rho) = C_{R,\infty}(\rho) := \ln(1 + C_R(\rho)),
\]

where

\[
C_R(\rho) := \min \left\{ x \left| x \geq 0, \exists \text{ a state } \sigma, \frac{\rho + x\sigma}{1 + x} \in \mathcal{I} \right. \right\}
\]

is the robustness of coherence, which is an observable coherence measure and has an operational interpretation in connection with the task of phase discrimination [20, 21]. Similar to \( E_{r,\alpha}(\rho) \) and \( E_{r,\alpha}(\rho) \) discussed in section 2.2, \( C_{r,\alpha}(\rho) \) and \( C_{r,\alpha}(\rho) \) are monotonically increasing with \( \alpha \). Therefore,

\[
C_{r,\alpha}(\rho) \leq C_{R,\infty}(\rho) \quad \forall \alpha \in [0, \infty],
\]

which implies the inequality \( C_r(\rho) \leq C_{R,\infty}(\rho) \) derived in [13]. In addition, the min relative entropy of coherence \( C_{r,1/2}(\rho) \) is equal to a variant of the geometric (measure of) coherence,

\[
C_{r,1/2}(\rho) = C_G(\rho) := -\ln \max_{\sigma \in \mathcal{I}} F(\rho, \sigma),
\]
In this paper we are more interested in the first variant $C_\alpha$ connection with Rényi relative entropies of coherence. As shown in section 8, which is closely related to another common variant $[6]$, coherence and entanglement measures based on Rényi relative entropies and $\alpha$ $(\alpha \neq 1)$, where the minimum is attained when $\sigma$ is pure.

An explicit formula for $C_{r,\alpha}(\rho)$ was derived in $[18]$ as reproduced below.

**Proposition 5** ($[18]$).

$$C_{r,\alpha}(\rho) = \frac{1}{\alpha - 1} \ln \| (\rho^\alpha)^{\text{diag}}\|_{1/\alpha} \quad \forall \alpha \in [0, \infty],$$  \hspace{1cm} \quad (58)

where $(\rho^\alpha)^{\text{diag}}$ denotes the diagonal matrix with the same diagonal as $\rho^\alpha$.

**Remark 4.** Note that $C_r(\rho)$ is correctly reproduced in the limit $\alpha \to 1$,

$$\lim_{\alpha \to 1} C_{r,\alpha}(\rho) = \lim_{\alpha \to 1} \frac{1}{\alpha - 1} \ln \| (\rho^\alpha)^{\text{diag}}\|_{1/\alpha} = S(\rho^{\text{diag}}) - S(\rho) = C_r(\rho).$$  \hspace{1cm} \quad (59)

The paper $[18]$ considered $C_{r,\alpha}(\rho)$ only for $\alpha \in [0,2]$, but the formula in (58) is valid for $\alpha \in [0,\infty]$, as demonstrated in the following proof. An alternative proof is presented in the appendix, which is applicable for $\alpha \in [0,2]$.

**Proof.** Suppose $\alpha \geq 0$ and $\alpha \neq 1$. Then

$$C_{r,\alpha}(\rho) = \min_{\sigma \in \mathcal{H}} S_\alpha(\rho||\sigma) = \min_{\sigma \in \mathcal{H}} \frac{1}{\alpha - 1} \ln \text{tr}(\rho^\alpha \sigma^{1-\alpha}) = \min_{\sigma \in \mathcal{H}} \frac{1}{\alpha - 1} \ln \text{tr}[(\rho^\alpha)^{\text{diag}} \sigma^{1-\alpha}],$$  \hspace{1cm} \quad (60)

where the last equality follows from the assumption that $\sigma$ is diagonal in the reference basis. Let $Q = ((\rho^\alpha)^{\text{diag}})^{1/\alpha}$ and $\hat{Q} = Q/\text{tr}(Q)$. Then

$$C_{r,\alpha}(\rho) = \min_{\sigma \in \mathcal{H}} \frac{1}{\alpha - 1} \ln (Q^\alpha \sigma^{1-\alpha}) = \min_{\sigma \in \mathcal{H}} \frac{1}{\alpha - 1} \left[ \ln (\text{tr} Q)^\alpha + \ln \text{tr} (\hat{Q}^\alpha \sigma^{1-\alpha}) \right],$$  \hspace{1cm} \quad (61)

where the minimum is attained when $\sigma = \hat{Q}$.

In the case $\alpha = 2$, (58) reduces to

$$C_{r,2}(\rho) = \ln \left[ \sum_j \left( \sum_k |\rho_{jk}|^2 \right)^{1/2} \right]^2.$$  \hspace{1cm} \quad (62)

In the limit $\alpha \to 0$, (58) yields

$$C_{r,0}(\rho) = - \ln \| (\Pi_\rho)^{\text{diag}} \|,$$  \hspace{1cm} \quad (63)

where $\Pi_\rho$ is the projector onto the support of $\rho$, and $\|M\| = \|M\|_\infty$ denotes the operator norm of $M$.

When $\rho$ is pure, the formulas of $C_{r,\alpha}(\rho)$ and $C_{r,\alpha}(\rho)$ are derived in $[18]$. 

Proposition 6 ([18]). Suppose $\rho = |\phi\rangle\langle\phi|$ is a pure state with $|\phi\rangle = \sum_i a_i |i\rangle$ and $|a_i|^2 = p_i$. Then we have

$$C_{r,\alpha}(\rho) = \begin{cases} \frac{\alpha}{\alpha - 1} \ln\left(\sum_i p_i^\alpha\right) & \text{if } \alpha > 0, \\ -\ln \max_i p_i & \text{if } \alpha = 0, \end{cases} (64)$$

and

$$C_{r,\alpha}(\rho) = \begin{cases} \frac{2\alpha - 1}{\alpha - 1} \ln\left(\sum_i p_i^{2\alpha - 1}\right) & \text{if } \alpha > \frac{1}{2}, \\ -\ln \max_i p_i & \text{if } \alpha = \frac{1}{2}. \end{cases} (65)$$

In the case $\alpha = 1$, the formulas in proposition 6 are understood as proper limits. Alternatively, these formulas can be expressed as follows,

$$C_{r,\alpha}(\rho) = \begin{cases} S_{\frac{1}{\alpha}}(\rho_{\text{diag}}) = S_{\alpha,\alpha}(\rho\|\rho_{\text{diag}}) & \forall \alpha \in [0, \infty], \\ S_{2\frac{1}{\alpha}}(\rho\|\rho_{\text{diag}}) & \forall \alpha \in \left[\frac{1}{2}, \infty\right], \end{cases} (66)$$

and

$$C_{r,\alpha}(\rho) = S_{\frac{2\alpha - 1}{\alpha - 1}}(\rho_{\text{diag}}) = S_{2\frac{1}{\alpha}}(\rho\|\rho_{\text{diag}}) & \forall \alpha \in \left[\frac{1}{2}, \infty\right]. (67)$$

The reasons behind these equalities are explained in theorems 4 and 5 in section 5 and theorem 7 in section 7.

Proposition 6 implies that any pure state $\rho$ satisfies

$$C_{RL}(\rho) = C_{r,\infty}(\rho) = C_{r,2}(\rho) = C_{L}(\rho) = 2 \ln(\text{tr} \sqrt{\rho_{\text{diag}}}). (68)$$

Here $C_{L}(\rho) := \ln(1 + C_{L}(\rho))$ and

$$C_{L}(\rho) := \sum_{j \neq k} |\rho_{jk}| (69)$$

is the $l_1$-norm of coherence [2], which may be seen as the analogue of the negativity in entanglement theory [12,13]. In particular, the $l_1$-norm of coherence can be uniquely characterized by a few simple axioms in a similar way to the negativity. In addition, the $l_1$-norm of coherence is equal to the maximum entanglement, quantified by the negativity, produced by incoherent operations acting on the system and an incoherent ancilla [12].

According to theorem 4 in [21], any state $\rho$ in dimension $d$ satisfies

$$\frac{C_{L}(\rho)}{d - 1} \leq C_{RL}(\rho) \leq C_L(\rho), (70)$$

which implies that

$$\ln\left[\frac{C_{L}(\rho)}{d - 1} + 1\right] \leq C_{RL}(\rho) \leq C_L(\rho). (71)$$

In conjunction with (55), we deduce that

$$C_{r,\alpha}(\rho) \leq C_L(\rho) \quad \forall \alpha \in [0, \infty], (72)$$
which implies that $C_r(\rho) \leq C_L(\rho)$ in particular. On the other hand, the lower bound for $C_{RL}(\rho)$ in (71) in general does not apply to $C_r(\rho)$. For example, when $d = 2$, the upper and lower bounds in (71) coincide, which implies that $C_{RL}(\rho) = C_L(\rho)$. However, the inequality $C_r(\rho) < C_{RL}(\rho) = C_L(\rho)$ is strict except when $\rho$ is either maximally coherent or incoherent (cf. theorem 8 in section 7). To be concrete, consider $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle$. We have

$$C_{RL}(\rho) = C_L(\rho) = \ln[1 + \sin(2\theta)], \quad C_r(\rho) = -\cos^2 \theta \ln \cos^2 \theta - \sin^2 \theta \ln \sin^2 \theta. \quad (73)$$

It is easy to verify that $C_r(\rho) < C_L(\rho)$ when $\theta > 0$ is sufficiently small.

The coherence measures introduced above can be generalized to a bipartite or multipartite system, in which case the reference basis is the tensor product of reference bases for respective subsystems. The following lemma clarifies the relations between entanglement measures and coherence measures based on Rényi relative entropies for a bipartite system. It is an immediate consequence of the definitions and the fact that incoherent states are separable, PPT, and nondistillable. The same conclusion also applies to a multipartite system.

**Lemma 5.**

$$E_\epsilon(\rho) \leq C_\epsilon(\rho), \quad E_{RL}(\rho) \leq C_{RL}(\rho), \quad (74)$$

$$E_{\epsilon,\alpha}(\rho) \leq C_{\epsilon,\alpha}(\rho), \quad E_{\epsilon,\alpha}(\rho) \leq C_{\epsilon,\alpha}(\rho) \quad \forall \alpha \in [0, \infty]. \quad (75)$$

Although coherence measures depend on the choice of local bases (unlike entanglement measures), lemma 5 is applicable to any given choice of local bases. In theorem 4 in the next section, we will show that all the inequalities in lemma 5 are saturated when $\rho$ is a maximally correlated state [58] as long as the corresponding Rényi relative entropies satisfy the data processing inequality. Recall that a maximally correlated state has the form [58]

$$\rho_{MC} := \sum_{jk} \rho_{jk} |jj\rangle\langle kk|. \quad (76)$$

### 3. Connecting entanglement measures and coherence measures

In this section we establish an operational one-to-one mapping between entanglement measures and coherence measures based on Rényi relative entropies. To achieve this goal, we first clarify the relations between these measures and Rényi conditional entropies for maximally correlated states. As applications, we derive several analytical formulas for Rényi relative entropies of entanglement of maximally correlated states, which reproduce a number of classic results on the relative entropy of entanglement and logarithmic robustness of entanglement as special cases. In addition, the results presented here play crucial roles in understanding several topics discussed in the following sections, including the additivity of Rényi relative entropies of coherence (section 4) and the exact coherence distillation rate (section 8).
Our study is inspired by a recent work of Streltsov et al. [6], which provides a general framework for constructing coherence measures from entanglement measures; see also [11,12]. Let \( \rho \) be any density matrix on \( \mathcal{H}_A \) of dimension \( d_A \). If \( \rho \) is coherent, then it can generate entanglement under incoherent operations acting on the system \( \mathcal{H}_A \) and an incoherent density matrix on an ancilla \( \mathcal{H}_B \). Given any entanglement measure \( E \), the maximum entanglement generated in this way defines a coherence measure \( C_E \) according to [6]. More precisely,

\[
C_E(\rho) := \lim_{d_B \to \infty} \left\{ \sup_{\Lambda_i} E\left(\Lambda_i [\rho \otimes |0\rangle \langle 0|]\right) \right\},
\]

where \( d_B \) is the dimension of the ancilla, and the supremum is taken over all incoherent operations \( \Lambda_i \). Interestingly, (77) maps the relative entropy of entanglement, geometric entanglement, and negativity to the relative entropy of coherence, geometric coherence, and \( l_1 \)-norm of coherence, respectively, that is, \( C_E = C_r, C_G, C_l \) when \( E = E_r, E_G, N \) [6,12]. Moreover, it enables establishing a one-to-one mapping between entanglement measures and coherence measures that are based on the convex roof [11]. Surprisingly, the generalized CNOT gate \( U_{\text{CNOT}} \) is the common optimal incoherent operation with respect to all these entanglement measures. Recall that \( U_{\text{CNOT}} \) corresponds to conjugation by the unitary \( U_{\text{CNOT}} \) defined as follows,

\[
U_{\text{CNOT}}|jk\rangle = \begin{cases} 
|j(j + k)\rangle & k < d_A, \\
|jk\rangle & k \geq d_A,
\end{cases}
\]

where the addition is modulo \( d_A \), assuming that \( d_B \geq d_A \). The operation \( U_{\text{CNOT}} \) turns any state \( \rho = \sum_{jk} \rho_{jk} |j\rangle \langle k| \) on \( \mathcal{H}_A \) into a maximally correlated state on \( \mathcal{H}_A \otimes \mathcal{H}_B \) [3,6,58],

\[
\rho_{\text{MC}} = U_{\text{CNOT}} [\rho \otimes |0\rangle \langle 0|] = \sum_{jk} \rho_{jk} |jj\rangle \langle kk|.
\]

It is worth mentioning that any bipartite entangled pure state is equivalent to a maximally correlated state under local unitary transformations.

Here we shall extend the operational connection between entanglement and coherence to measures based on \( \text{Rényi relative entropies} \). By virtue of (77), we can define two families of coherence quantifiers based on the two families of \( \text{Rényi relative entropies} \) of entanglement as illustrated in figure 2

\[
C_{E_{r,\alpha}}(\rho) := \lim_{d_B \to \infty} \left\{ \sup_{\Lambda_i} E_{r,\alpha}\left(\Lambda_i [\rho \otimes |0\rangle \langle 0|]\right) \right\},
\]

\[
C_{E_{G,\alpha}}(\rho) := \lim_{d_B \to \infty} \left\{ \sup_{\Lambda_i} E_{G,\alpha}\left(\Lambda_i [\rho \otimes |0\rangle \langle 0|]\right) \right\}.
\]

According to proposition 2, \( E_{r,\alpha}(\rho) \) for \( \alpha \in [0,2] \) and \( E_{G,\alpha}(\rho) \) for \( \alpha \in [\frac{1}{2}, \infty] \) are proper entanglement measures. Therefore, \( C_{E_{r,\alpha}}(\rho) \) for \( \alpha \in [0,2] \) and \( C_{E_{G,\alpha}}(\rho) \) for \( \alpha \in [\frac{1}{2}, \infty] \) are proper coherence measures. In the limit \( \alpha \to 1 \), both \( E_{r,\alpha} \) and \( E_{G,\alpha} \) approach the
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Coherent & separable state

\[ E_{r,\alpha}(\rho \otimes |0\rangle\langle 0|) \]

incoherent operation

maximize

\[ E_{r,\alpha}(\Lambda_i[\rho \otimes |0\rangle\langle 0|]) \]

coherent & entangled state

\[ A \]

\[ B \]

Figure 2. Illustration of the definition of the coherence measure \( C_{E_{r,\alpha}}(\rho) \) as the maximum Rényi relative entropy of entanglement \( E_{r,\alpha} \) generated by incoherent operations acting on the system and an incoherent ancilla.

relative entropy of entanglement \( E_t \), so \( C_{E_{r,\alpha}} \) and \( E_{r,\alpha} \) reduce to \( C_{E_{r}} \), which is equal to the relative entropy of coherence \( C_r \) according to \( [1] \). In another limit \( \alpha \to \infty \), \( E_{r,\alpha} \) approaches the logarithmic robustness of entanglement \( E_{RL} \), and \( (81) \) takes on the form

\[ C_{E_{RL}}(\rho) := \lim_{d_B \to \infty} \left\{ \sup_{\Lambda_i} E_{RL}(\Lambda_i[\rho \otimes |0\rangle\langle 0|]) \right\}. \tag{82} \]

To achieve our goal, we first show that the inequalities between Rényi relative entropies of entanglement and Rényi conditional entropies as well as Rényi relative entropies of coherence in lemmas 4 and 5 are saturated for maximally correlated states (for the parameter ranges of interest).

**Theorem 1.** Any maximally correlated state \( \rho_{MC} \) satisfies the following relations,

\[ E_t(\rho_{MC}) = C_t(\rho_{MC}) = -H(A|B)_{\rho_{MC}}, \tag{83} \]

\[ E_{r,\alpha}(\rho_{MC}) = C_{r,\alpha}(\rho_{MC}) = -H^\alpha(A|B)_{\rho_{MC}} \quad \forall \alpha \in [0, 2], \tag{84} \]

\[ E_{r,\alpha}(\rho_{MC}) = C_{r,\alpha}(\rho_{MC}) = -\overline{T}_\alpha(A|B)_{\rho_{MC}} \quad \forall \alpha \in \left[ \frac{1}{2}, \infty \right], \tag{85} \]

\[ E_{RL}(\rho_{MC}) = C_{RL}(\rho_{MC}) = -\overline{H}_\infty(A|B)_{\rho_{MC}}. \tag{86} \]

**Remark 5.** Although coherence measures depend on the choice of local bases (unlike entanglement measures), theorem \([1]\) is applicable to any given choice of local bases. In addition, this theorem applies to entanglement measures defined with respect to three type of states, namely, separable states, PPT states, and nondistillable states (see section \([2.2]\)). Similar remarks apply to many other results presented in this paper.

**Proof.** Let \( P \) be the projector onto the space spanned by \(|jj\rangle\) for all \( j \) and define the CPTP map \( \Lambda_P \) by \( \Lambda_P(\rho) := P\rho P + (1 - P)\rho(1 - P) \). Then \( \Lambda_P(\rho_{MC}) = \rho_{MC} \), so that

\[ S(\rho_{MC}||I \otimes \sigma_B) \geq S(\Lambda_P(\rho_{MC})||\Lambda_P(I \otimes \sigma_B)) = S(\rho_{MC}||P(I \otimes \sigma_B)P) \tag{87} \]
Coherence and entanglement measures based on Rényi relative entropies

for any state $\sigma_B$ on $\mathcal{H}_B$. Observing that $P(I \otimes \sigma_B) = \sum_j (\sigma_B)_{jj}|jj\rangle \langle jj|$ is a normalized incoherent state, we conclude that

$$-H(A|B)_{\rho_{MC}} = \min_{\sigma_B} S(\rho_{MC}||I \otimes \sigma_B) \geq \min_{\sigma \in \mathcal{I}} S(\rho_{MC}||\sigma) = C_t(\rho_{MC}). \tag{88}$$

This result confirms (83) given the inequality $-H(A|B)_{\rho_{MC}} \leq E_r(\rho_{MC}) \leq C_t(\rho_{MC})$ according to lemmas 4 and 5.

Equations (84), (85), and (86) can be proved in a similar way. In particular, (87) still applies if $S$ is replaced by $S_\alpha$ with $\alpha \in [0, 2]$ or $S_2$, with $\alpha \in [\frac{1}{2}, \infty]$. Therefore, $-H^+\alpha(A|B)_{\rho_{MC}} \geq C_t,\alpha(\rho_{MC})$ and $-H^{\text{r}}\alpha(A|B)_{\rho_{MC}} \geq C^\text{r},\alpha(\rho_{MC})$ for the given parameter ranges, which imply (84) and (85) in view of lemmas 4 and 5. Finally, (86) is derived from (85) by taking the limit $\alpha \to \infty$. \hfill \square

Remark 6. The equality $E_r(\rho_{MC}) = -H(A|B)_{\rho_{MC}}$ in (83) is known before [17, (8.143)]; also, the equality $E_r(\rho_{MC}) = C_t(\rho_{MC})$ has been derived in [6]. In addition, the relations $E_t,\alpha(\rho_{MC}) = -H_\alpha(A|B)_{\rho_{MC}}$ and $E^\text{r},\alpha(\rho_{MC}) = -H^{\text{r}}\alpha(A|B)_{\rho_{MC}}$ in (84) and (85) were stated in [17, lemma 8.9]. However, the derivation there contains an error, which is fixed here.

Now we can establish an operational connection between entanglement measures and coherence measures based on Rényi relative entropies.

Theorem 2. We have the following relations,

$$C_{E_t}(\rho) = C_t(\rho), \tag{89}$$
$$C_{E_t,\alpha}(\rho) = C_{t,\alpha}(\rho) \quad \forall \alpha \in [0, 2], \tag{90}$$
$$C_{E^\text{r},\alpha}(\rho) = C_{t,\alpha}(\rho) \quad \forall \alpha \in [\frac{1}{2}, \infty], \tag{91}$$
$$C_{E^\text{r},L}(\rho) = C_{R_{\text{L}}}(\rho). \tag{92}$$

Remark 7. According to the following proof, the generalized CNOT gate is the common optimal incoherent operation that achieves the supremums in the definitions of $C_{E_t}(\rho)$, $C_{E_t,\alpha}(\rho)$, $C_{E^\text{r},\alpha}(\rho)$, and $C_{E^\text{r},L}(\rho)$. Here the special case (89) was derived in [6]. In view of the relation $\sum_{1/2}(\rho||\sigma) = -\ln F(\rho, \sigma)$, theorem 2 implies that $C_{E_t}(\rho) = C_G(\rho)$ and $C_{E^\text{r}}(\rho) = \bar{C}_G(\rho)$, which were derived in [6,11] based on different approaches.

Proof. Let $\Lambda_i$ be an arbitrary incoherence-preserving operation acting on the system and the ancilla. Then

$$E_r(\Lambda_i | \rho \otimes |0\rangle \langle 0|) \leq C_t(\Lambda_i | \rho \otimes |0\rangle \langle 0|) \leq C_t(\rho \otimes |0\rangle \langle 0|) = C_t(\rho) \tag{93}$$

according to lemma 5 and proposition 4. By theorem 1, the two inequalities are saturated when $\Lambda_i$ is the generalized CNOT gate, in which case $\Lambda_i | \rho \otimes |0\rangle \langle 0|$ is maximally correlated. This observation confirms (89).

Equations (90), (91), and (92) follow from the same reasoning as above, note that (93) still holds if $E_t$ is replaced by $E_{t,\alpha}$, $E^\text{r},\alpha$, and $E_{R_{\text{L}}}$, while $C_t$ is replaced by $C_{t,\alpha}$, $C_{t,\alpha}$, and $C_{R_{\text{L}}}$ accordingly. \hfill \square
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Theorem 1 is useful not only in connecting entanglement measures and coherence measures based on Rényi relative entropies, but also in studying entanglement measures of maximally correlated states, including bipartite pure states.

**Corollary 1.** Suppose \( \rho \) is a maximally correlated state. Then

\[
E_{r,\alpha}(\rho) = \frac{1}{\alpha - 1} \ln \| (\rho^\alpha)_{\text{diag}} \|_1 \quad \forall \alpha \in [0, 2].
\]

This corollary is a consequence of theorem 1 and proposition 5. In conjunction with (59), we deduce that

\[
\lim_{\alpha \to 1} E_{r,\alpha}(\rho) = S(\rho_{\text{diag}}) - S(\rho) = S(\rho_A) - S(\rho) = E_r(\rho),
\]

which reproduces the relative entropy of entanglement of maximally correlated states \[54][17, (8.143)]\, including bipartite pure states \[44\].

**Corollary 2.** Suppose \( \rho = |\phi\rangle\langle\phi| \) is a bipartite pure state with \( |\phi\rangle = \sum_i a_i |ii\rangle \) and \( |a_i|^2 = p_i \). Then

\[
E_{r,\alpha}(\rho) = \begin{cases} 
\frac{\alpha}{\alpha - 1} \ln \left( \sum_i p_i^{\frac{1}{\alpha}} \right) & \text{if } \alpha > 0, \\
- \ln \max_i p_i & \text{if } \alpha = 0.
\end{cases}
\]

\[
E_{r,\alpha}(\rho) = \begin{cases} 
\frac{2\alpha - 1}{\alpha - 1} \ln \left( \sum_i p_i^{\frac{\alpha}{\alpha - 1}} \right) & \text{if } \alpha > \frac{1}{2}, \\
- \ln \max_i p_i & \text{if } \alpha = \frac{1}{2}.
\end{cases}
\]

This corollary is a consequence of theorem 1 and proposition 6. It reproduces the relative entropy of entanglement of bipartite pure states \[44\] in the limit \( \alpha \to 1 \). In addition, it reproduces the logarithmic robustness of entanglement in another limit \( \alpha \to \infty \) \[16, 18\] and implies that any bipartite pure state \( \rho \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \) satisfies

\[
E_{R_L}(\rho) = E_{r,\alpha}(\rho) = E_{r,2}(\rho) = N_L(\rho) = 2 \ln(\text{tr} \sqrt{\rho_A}),
\]

where \( N_L(\rho) := \ln(1 + N(\rho)) \) is the logarithmic negativity \[22, 53\].

**Corollary 3.** If \( \rho \) is a \( d \times d \) maximally correlated state, then

\[
\frac{N(\rho)}{d - 1} \leq E_R(\rho) \leq N(\rho).
\]

**Proof.** If \( \rho \) is a \( d \times d \) maximally correlated state, then \( \rho \) is supported on a \( d \)-dimensional subspace spanned by \( d \) computational-basis states. Therefore, \( C_{\alpha}(\rho) \leq C_{R}(\rho) \leq C_{l_1}(\rho) \) according to theorem 4 in \[21\]; cf. (70) in section 2.3. Now the corollary follows from the equality \( E_R(\rho) = C_{R}(\rho) \) presented in theorem 1 and the equality \( N(\rho) = C_{l_1}(\rho) \) \[12, 13\], which is straightforward to verify.

**Corollary 3** above implies that \( E_R(\rho) = N(\rho) \) if \( \rho \) is a two-qubit maximally correlated state. This requirement is sufficient but not necessary. Indeed, the equality
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$E_R(\rho) = \mathcal{N}(\rho)$ holds for all Bell-diagonal states, not all of which are maximally correlated. To see this, consider the Bell-diagonal state $\rho_{BD}(p) := \sum_{j=0}^{3} p_j |\Psi_j\rangle\langle\Psi_j|$, where $p = (p_0, p_1, p_2, p_3)$ is a probability distribution with $p_0 \geq 1/2$ and

$$
|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),
$$

$$
|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
$$

(100)

are four Bell states, which form a Bell basis. The Bell-diagonal state $\rho_{BD}(p)$ is maximally correlated iff $p_2 = p_3 = 0$. Calculation shows that

$$
E_R(\rho_{BD}(p)) = \mathcal{N}(\rho_{BD}(p)) = 2p_0 - 1,
$$

(101)

where the equality $E_R(\rho_{BD}(p)) = 2p_0 - 1$ follows from [45, (29)].

4. Additivity of Rényi relative entropies of coherence

In quantum information processing, it is often more efficient to process a family of quantum states collectively. In this context, it is natural to ask whether the resource content of this family is equal to the sum of the resource contents of individual members. Additive resource measures are particularly appealing because they can significantly simplify the task of quantifying resources. By virtue of theorem 1 in this section we prove that all Rényi relative entropies of coherence defined in section 2.3 are additive, as long as they are monotonic under incoherence-preserving operations. Accordingly, Rényi relative entropies of entanglement defined in section 2.2 are additive for maximally correlated states, although they are not additive in general [45].

To achieve our goal, we first recall the additivity properties of Rényi conditional entropies, which can be proved using the duality relations presented in proposition 1.

Proposition 7 ([59, lemma 7]). Any pair of states $\rho_1$ and $\rho_2$ shared by Alice and Bob satisfies the following additivity relations:

$$
H^\uparrow_\alpha(A|B)_{\rho_1 \otimes \rho_2} = H^\uparrow_\alpha(A|B)_{\rho_1} + H^\uparrow_\alpha(A|B)_{\rho_2} \quad \forall \alpha \in [0, \infty],
$$

(102)

$$
\overline{H}^\uparrow_\alpha(A|B)_{\rho_1 \otimes \rho_2} = \overline{H}^\uparrow_\alpha(A|B)_{\rho_1} + \overline{H}^\uparrow_\alpha(A|B)_{\rho_2} \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right].
$$

(103)

Note that the other two types of conditional entropies $H^\downarrow_\alpha(A|B)_{\rho}$ and $\overline{H}^\downarrow_\alpha(A|B)_{\rho}$ are obviously additive. Combining theorem 1 with proposition 7 we can prove the additivity of Rényi relative entropies of coherence, including the logarithmic robustness of coherence.
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Theorem 3.

\[ C_t(\rho_1 \otimes \rho_2) = C_t(\rho_1) + C_t(\rho_2), \]  \tag{104} 
\[ C_{t,\alpha}(\rho_1 \otimes \rho_2) = C_{t,\alpha}(\rho_1) + C_{t,\alpha}(\rho_2) \quad \forall \alpha \in [0, \infty], \]  \tag{105} 
\[ C_{t,\alpha}(\rho_1 \otimes \rho_2) = C_{t,\alpha}(\rho_1) + C_{t,\alpha}(\rho_2) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right], \]  \tag{106} 
\[ C_{RL}(\rho_1 \otimes \rho_2) = C_{RL}(\rho_1) + C_{RL}(\rho_2). \]  \tag{107} 

Theorem 3 is of fundamental interest to understanding the resource theory of coherence and its distinction from the resource theory of entanglement. Recall that most entanglement measures are in general not additive. In addition, theorem 3 can significantly simplify the calculation of Rényi relative entropies of coherence of tensor products of quantum states. Recall that the logarithmic robustness of coherence \( C_{RL}(\rho) \) quantifies the maximum advantage enabled by a quantum state in the task of phase discrimination as measured by the logarithm of the ratio of success probabilities \[20, 21\]. The additivity of the logarithmic robustness of coherence thus has an operational implication: the maximum advantage enabled by a tensor product of quantum states is additive. Theorem 3 also implies the additivity of one variant of the geometric coherence \( C_G(\rho) \), which coincides with \( C_{t,1/2}(\rho) \). Incidentally, the coherence of formation is additive according to \[3\], and the logarithmic \( l_1 \)-norm of coherence is obviously additive. Surprisingly, most useful coherence measures are additive or have additive variants, in sharp contrast with entanglement measures.

Proof. Equation (104) follows from the formula \( C_t(\rho) = S(\rho^{\text{diag}}) - S(\rho) \), which is well known. Similarly, (105) follows from the closed formula of \( C_{t,\alpha}(\rho) \) in proposition 5.

To show (106), let \( \rho_{MC} = \mathcal{U}_{\text{CNOT}}[\rho \otimes |0\rangle \langle 0|] \). Then

\[ C_{t,\alpha}(\rho) = C_{t,\alpha}(\rho \otimes |0\rangle \langle 0|) = C_{t,\alpha}(\rho_{MC}) = -\text{Tr}_\alpha^\dagger(A|B\rangle_{\rho_{MC}}), \]  \tag{108} 

where the last equality follows from theorem 1. Now (106) is an immediate consequence of proposition 7. The same reasoning can also be applied to derive (104) and (107) as well as (105) for \( \alpha \in [0, 2] \). In addition, (107) follows from (106) by taking the limit \( \alpha \to \infty \).

\[ \square \]

The combination of theorems 2 and 3 implies the additivity of the maximum Rényi relative entropies of entanglement generated by incoherent operations acting on the system and an incoherent ancilla.

Corollary 4.

\[ C_{E_t}(\rho_1 \otimes \rho_2) = C_{E_t}(\rho_1) + C_{E_t}(\rho_2), \]  \tag{109} 
\[ C_{E_{t,\alpha}}(\rho_1 \otimes \rho_2) = C_{E_{t,\alpha}}(\rho_1) + C_{E_{t,\alpha}}(\rho_2) \quad \forall \alpha \in [0, 2], \]  \tag{110} 
\[ C_{E_{t,\alpha}}(\rho_1 \otimes \rho_2) = C_{E_{t,\alpha}}(\rho_1) + C_{E_{t,\alpha}}(\rho_2) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right], \]  \tag{111} 
\[ C_{E_{RL}}(\rho_1 \otimes \rho_2) = C_{E_{RL}}(\rho_1) + C_{E_{RL}}(\rho_2). \]  \tag{112} 

Further, the combination of theorem 1 and proposition 7 (or theorem 3) implies the additivity of Rényi relative entropies of entanglement of maximally correlated states. This result is of intrinsic interest to understanding entanglement properties of maximally correlated states.

**Corollary 5.** If \( \rho_1 \) and \( \rho_2 \) are maximally correlated states, then

\[
E_{r}(\rho_1 \otimes \rho_2) = E_{r}(\rho_1) + E_{r}(\rho_2),
\]

\[
E_{r,\alpha}(\rho_1 \otimes \rho_2) = E_{r,\alpha}(\rho_1) + E_{r,\alpha}(\rho_2) \quad \forall \alpha \in [0, 2],
\]

\[
E_{\alpha}(\rho_1 \otimes \rho_2) = E_{\alpha}(\rho_1) + E_{\alpha}(\rho_2) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right],
\]

\[
E_{R_L}(\rho_1 \otimes \rho_2) = E_{R_L}(\rho_1) + E_{R_L}(\rho_2).
\]

This corollary implies the additivity of the geometric entanglement \( E_G(\rho) \), which coincides with \( E_{r,1/2}(\rho) \), for maximally correlated states. The additivity of an alternative geometric measure was considered in [45]. The additivity of the relative entropy of entanglement of maximally correlated states was proven previously in [58]; the special case of maximally correlated generalized Bell-diagonal states was also considered in [45].

5. Upper and lower bounds for Rényi relative entropies of coherence

By virtue of theorem 1, here we derive several nontrivial upper and lower bounds for Rényi relative entropies of coherence, including the logarithmic robustness of coherence. Similar bounds apply to Rényi relative entropies of entanglement of maximally correlated states.

**Theorem 4.** Any state \( \rho \) satisfies

\[
C_{r,\alpha}(\rho) \leq S_{\frac{1}{\alpha}}(\rho_{\text{diag}}) \quad \forall \alpha \in [0, 2],
\]

\[
C_{\alpha}(\rho) \leq S_{\frac{\alpha}{2\alpha-1}}(\rho_{\text{diag}}) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right],
\]

\[
C_{R_L}(\rho) \leq S_{\frac{1}{2}}(\rho_{\text{diag}}), \quad C_G(\rho) \leq -\ln\|\rho_{\text{diag}}\|;
\]

all the upper bounds are saturated if \( \rho \) is pure.

**Proof.** Let \( \rho_{\text{MC}} = U_{\text{CNOT}}[\rho \otimes |0\rangle\langle 0|] \). Then

\[
C_{r,\alpha}(\rho) = C_{r,\alpha}(\rho_{\text{MC}}) = -H_{\alpha}^\dagger(A|B)_{\rho_{\text{MC}}} \leq S_{\frac{1}{\alpha}}((\rho_{\text{MC}})_A) = S_{\frac{1}{\alpha}}(\rho_{\text{diag}}) \quad \forall \alpha \in [0, 2]
\]

according to theorem 1 and lemma 3. The inequality is saturated if \( \rho \) is pure according to lemma 3. By the same token,

\[
C_{r,\alpha}(\rho) = -H_{\alpha}^\dagger(A|B)_{\rho_{\text{MC}}} \leq S_{\frac{\alpha}{2\alpha-1}}((\rho_{\text{MC}})_A) = S_{\frac{\alpha}{2\alpha-1}}(\rho_{\text{diag}}) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right],
\]

and the inequality is saturated if \( \rho \) is pure. Equation (119) follows from (118) by taking the limits \( \alpha \to \infty \) and \( \alpha \to 1/2 \).

\[\square\]
Theorem 5. Any state $\rho$ satisfies

\[
S_{2-\frac{1}{\alpha}}(\rho\|\rho_{\text{diag}}) \leq C_{t,\alpha}(\rho) \leq S_{\alpha}(\rho\|\rho_{\text{diag}}) \quad \forall \alpha \in \left[\frac{1}{2}, 2\right],
\]

\[
S_{2-\frac{1}{\alpha}}(\rho\|\rho_{\text{diag}}) \leq C_{t,\alpha}(\rho) \leq S_{\alpha}(\rho\|\rho_{\text{diag}}) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right],
\]

\[
S_2(\rho\|\rho_{\text{diag}}) \leq C_{R_L}(\rho) \leq S_{\infty}(\rho\|\rho_{\text{diag}});
\]

all the lower bounds in the three equations are saturated if $\rho$ is pure.

Proof. The upper bounds in (122) to (124) are trivial given that $\rho_{\text{diag}}$ is incoherent. To establish the lower bound in (122) for $\alpha \in \left[\frac{1}{2}, 2\right]$, let $\rho_{MC} = U_{\text{CNOT}}[\rho \otimes |0\rangle \langle 0|]$, then

\[
C_{t,\alpha}(\rho) = C_{t,\alpha}(\rho_{MC}) = -H_\alpha^d(A|B)_{\rho_{MC}} \geq -H^d_{2-\frac{1}{\alpha}}(A|B)_{\rho_{MC}} = S_{2-\frac{1}{\alpha}}(\rho\|\rho_{\text{diag}}).
\]

Here the second and third equalities follow from theorem 1 in section 3 and lemma 6 below, respectively; the inequality follows from lemma 1 and is saturated when $\rho$ is pure.

The lower bound for $C_{t,\alpha}(\rho)$ in (123) and the saturation for a pure state can be proved in the same way. Equation (124) follows from (123) by taking the limit $\alpha \to \infty$. \qed

Equation (123) in theorem 5 yields a lower bound for the geometric coherence $C_G(\rho) \geq S_\alpha(\rho\|\rho_{\text{diag}})$. The bounds for $C_{R_L}(\rho)$ in (124) can be expressed more explicitly as

\[
\ln \text{tr}\left\{\left(\rho_{\text{diag}}^{-1/4}\rho\rho_{\text{diag}}^{-1/4}\right)^2\right\} \leq C_{R_L}(\rho) \leq \ln\|\rho_{\text{diag}}^{-1/2}\rho\rho_{\text{diag}}^{-1/2}\|. \tag{126}
\]

Here the lower bound improves over the bound $C_{R_L}(\rho) \geq C_t(\rho) = S(\rho\|\rho_{\text{diag}})$ derived in [13]. Equation (126) implies that

\[
\text{tr}\left\{\left(\rho_{\text{diag}}^{-1/4}\rho\rho_{\text{diag}}^{-1/4}\right)^2\right\} - 1 \leq C_t(\rho) \leq \|\rho_{\text{diag}}^{-1/2}\rho\rho_{\text{diag}}^{-1/2}\| - 1 . \tag{127}
\]

In addition, theorems 1 and 5 enable a simple derivation of Rényi relative entropies of coherence of pure states (for certain parameter ranges); cf. section 2.3. Also, they offer a simple explanation of why the equalities in (66) and (67) hold.

Lemma 6. Let $\rho_{MC} = U_{\text{CNOT}}[\rho \otimes |0\rangle \langle 0|]$. Then

\[
H_\alpha^d(A|B)_{\rho_{MC}} = -S_\alpha(\rho\|\rho_{\text{diag}}), \quad \overline{H}_\alpha^d(A|B)_{\rho_{MC}} = -\overline{S}_\alpha(\rho\|\rho_{\text{diag}}) \quad \forall \alpha \in [0, \infty]. \tag{128}
\]

Proof. According to the definition and lemma 7 below,

\[
-H_\alpha^d(A|B)_{\rho_{MC}} = S_\alpha(\rho_{MC}\|I_A \otimes (\rho_{MC})_B) = S_\alpha(\rho_{MC}\|I_A \otimes \rho_{\text{diag}}) = S_\alpha(\rho\|\rho_{\text{diag}}). \tag{129}
\]

The other equality in lemma 6 follows from a similar reasoning. \qed

The following lemma is proved in the appendix.
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Lemma 7. Let $\rho$ and $\sigma$ be two density matrices on $\mathcal{H}$ with $\sigma$ being diagonal in the reference basis. Let $\rho_{\text{MC}} = U_{\text{CNOT}}[\rho \otimes |0\rangle\langle 0|]$. Then

$$S_{\alpha}(\rho_{\text{MC}}||I_A \otimes \sigma) = S_{\alpha}(\rho||\sigma), \quad S_{\alpha}(\rho_{\text{MC}}||I_A \otimes \sigma) = S_{\alpha}(\rho||\sigma) \quad \forall \alpha \in [0, \infty]. \quad (130)$$

In view of theorem 1, when $\rho$ is a maximally correlated state, theorems 4 and 5 still hold if Rényi relative entropies of coherence are replaced by corresponding Rényi relative entropies of entanglement. For example, the following corollary is a consequence of theorems 1 and 4.

Corollary 6. Any maximally correlated state $\rho$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ satisfies

$$E_{r,\alpha}(\rho) \leq S_{\frac{1}{2}}(\rho^{\text{diag}}) = S_{\frac{1}{2}}(\rho_A) \quad \forall \alpha \in [0, 2]. \quad (131)$$

$$E_{R_{MC},\alpha}(\rho) \leq S_{\frac{1}{2}}(\rho^{\text{diag}}) = S_{\frac{1}{2}}(\rho_A) \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right], \quad (132)$$

$$E_{R_{MC},\alpha}(\rho) \leq S_{\frac{1}{2}}(\rho^{\text{diag}}) = S_{\frac{1}{2}}(\rho_A), \quad E_{G}(\rho) \leq -\ln\|\rho^{\text{diag}}\| = -\ln\|\rho_A\|. \quad (133)$$

All the upper bounds are saturated if $\rho$ is pure.

Note that this corollary yields a simple derivation of the relative entropy of entanglement and robustness of entanglement of bipartite pure states.

6. Relations between Rényi relative entropies

In this section, we determine the condition under which Rényi relative entropies are independent of the order parameter $\alpha$. Remember that usually they are monotonically increasing with the order parameter. The results presented in this section will be used in the next section to study the relations between different Rényi relative entropies of coherence.

For this purpose, we recall the classical case regarding Rényi relative entropies between two probability distributions $p$ and $q$ on $\mathcal{X}$. We assume that the support of $p$ is included in that of $q$ and define the random variables $\ln p(X)$ and $\ln q(X)$ on the support of $p$. Let $\phi(s)$ for $s \geq -1$ be the cumulant generating function of the classical random variable $\ln p(X) - \ln q(X)$, i.e.,

$$\phi(s) := \ln \mathbb{E}_{p,X} \exp\{s[\ln p(X) - \ln q(X)]\}, \quad (134)$$

where $\mathbb{E}_{p,X}$ expresses the expectation with respect to the random variable $X$ under the distribution $p$. Then the Rényi relative entropy $S_{1+s}(p||q)$ can be expressed as $S_{1+s}(p||q) = \phi(s)/s$. Note that $\phi(0) = 0$, we deduce that

$$\phi'(0) = S(p||q), \quad \phi''(0) = 2 \lim_{s \to 0} S'_{1+s}(p||q). \quad (135)$$

The first derivative $\phi'(0)$ expresses the expectation of the variable $\ln p(X) - \ln q(X)$, i.e., the relative entropy $S(p||q)$. The second derivative $\phi''(0)$ expresses the variance of $\ln p(X) - \ln q(X)$, which is called the relative varentropy $V(p||q)$,

$$2 \lim_{s \to 0} S'_{1+s}(p||q) = \phi''(0) = V(p||q) := \mathbb{E}_{p,X}[\ln p(X) - \ln q(X)]^2 - S(p||q)^2. \quad (136)$$
Incidentally, \( V(p\|q) \) plays an important role in the second order analysis and moderate deviation analysis in hypothesis testing \[60, \text{section 9} \] \[61, (34)\]. In conjunction with the monotonicity of \( S_{1+s}(p\|q) \) with \( s \), \((136)\) implies the following proposition.

**Proposition 8.** The following conditions are equivalent.

(A1) \( S_{1+s}(p\|q) = S(p\|q) \), i.e., \( \phi(s) = s\phi'(0) \), for all \( s \geq -1 \).

(A2) \( S_{1+s}(p\|q) = S(p\|q) \), i.e., \( \phi(s) = s\phi'(0) \), for some \( s \geq -1 \) with \( s \neq 0 \).

(A3) \( \lim_{s \to 0} S'_{1+s}(p\|q) = 0 \), i.e., \( \phi''(0) = 0 \).

(A4) \( p \) is a constant times of \( q \) on the support of \( p \).

Now, we consider the quantum scenario in which \( \rho \) and \( \sigma \) are two density matrices with \( \text{supp}(\rho) \leq \text{supp}(\sigma) \). The following analysis also applies to the case in which \( \sigma \) is a positive operator instead of a density matrix. Since \( S_\alpha(\rho\|\sigma) \) and \( \sum_\alpha(\rho\|\sigma) \) are combinations of differentiable functions with respect to \( \alpha \), their derivatives with respect to \( \alpha \) are defined and are denoted by \( S'_\alpha(\rho\|\sigma) \) and \( \sum'_\alpha(\rho\|\sigma) \), respectively. Let \( s = \alpha - 1 \) and define \( \phi(s) := \ln \text{tr}(\rho^{1+s}\sigma^{-s}) \) as the analogue of the classical cumulant generating function. Then \( S_{1+s}(\rho\|\sigma) = \phi(s)/s \) as in the classical case. Calculation shows that [17, Exercise 3.5]

\[
\phi'(s) = \frac{\text{tr}[\rho^{1+s}(\ln \rho - \ln \sigma)\sigma^{-s}]}{\text{tr}(\rho^{1+s}\sigma^{-s})},
\]

\[
\phi''(s) = \frac{\text{tr}[\rho^{1+s}(\ln \rho - \ln \sigma)\sigma^{-s}(\ln \rho - \ln \sigma)] - \left( \frac{\text{tr}[\rho^{1+s}(\ln \rho - \ln \sigma)\sigma^{-s}]}{\text{tr}(\rho^{1+s}\sigma^{-s})} \right)^2}{\text{tr}(\rho^{1+s}\sigma^{-s})},
\]

which implies that

\[
\phi'(0) = \text{tr}[\rho(\ln \rho - \ln \sigma)] = S(\rho\|\sigma)
\]

\[
\phi''(0) = V(\rho\|\sigma) := \text{tr}[\rho(\ln \rho - \ln \sigma)^2] - S(\rho\|\sigma)^2 = \text{tr} \{ \rho(\ln \rho - \ln \sigma - S(\rho\|\sigma))^2 \}. \tag{139}
\]

The relative varentropy \( V(\rho\|\sigma) \) in the quantum setting also plays an important role in the second order analysis and moderate deviation analysis in hypothesis testing \[63, 61, 62, (34)\]. As in the classical case, we still have \( \phi''(0) = 2 \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) \). Suppose \( \rho \) and \( \sigma \) have spectral decompositions \( \rho = \sum_j \lambda_j P_j \) and \( \sigma = \sum_k \mu_k Q_k \), where \( \lambda_j \) and \( \mu_k \) are distinct positive eigenvalues of \( \rho \) and \( \sigma \), respectively. Then

\[
2 \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = \phi''(0) = \left[ \sum_{jk} a_{jk} \lambda_j \left( \frac{\ln \lambda_j}{\mu_k} \right)^2 - \left( \sum_{jk} a_{jk} \lambda_j \ln \frac{\lambda_j}{\mu_k} \right)^2 \right], \tag{141}
\]

where \( a_{jk} = \text{tr}(P_j Q_k) \), which satisfy \( \sum_k a_{jk} = \text{tr}(P_j) \) given that \( \text{supp}(\rho) \leq \text{supp}(\sigma) \).

By virtue of \( (141) \), we can prove the following lemma.

**Lemma 8.** Suppose \( \rho \) is a density matrix and \( \sigma \) is a positive operator with \( \text{supp}(\sigma) \geq \text{supp}(\rho) \). Then

\[
\lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma). \tag{142}
\]
lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = 0 and lim_{\alpha \to 1} S''_\alpha(\rho\|\sigma) = 0 iff \rho commutes with \sigma and is proportional to \Pi_\rho\sigma, where \Pi_\rho is the projector onto the support of \rho.

Proof. Since \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = \lim_{\alpha \to 1} S''_\alpha(\rho\|\sigma) = S(\rho\|\sigma) and \sum_{\alpha} \rho \leq S(\rho\|\sigma) for all \alpha \geq 0, we have
\[
\lim_{\alpha \to 1+0} S'_\alpha(\rho\|\sigma) \leq \lim_{\alpha \to 1+0} S''_\alpha(\rho\|\sigma), \quad \lim_{\alpha \to 1-0} S'_\alpha(\rho\|\sigma) \geq \lim_{\alpha \to 1-0} S''_\alpha(\rho\|\sigma),
\]
which implies (142).

Note that the expression in (141) may be interpreted as the variance of the variable \ln \frac{\lambda_j}{\mu_k} with respect to the probability distribution composed of the components a_{jk}\lambda_j. If \rho commutes with \sigma and is proportional to \Pi_\rho\sigma, then it is straightforward to verify that \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = \lim_{\alpha \to 1} S''_\alpha(\rho\|\sigma) = 0; cf. (145) below.

Conversely, if \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = 0 or \lim_{\alpha \to 1} S''_\alpha(\rho\|\sigma) = 0, then \ln \frac{\lambda_j}{\mu_k} = c for some constant c whenever a_{jk} \neq 0 (as defined after (141)). In that case, the coefficient matrix a_{jk} has at most one nonzero entry in each row and each column. On the other hand, by assumption the support of \rho is contained in the support of \sigma, which implies that \sum_k a_{jk} = \sum_k \text{tr}(P_j Q_k) = \text{tr}(P_j) for each j. Therefore, for each spectral projector \rho of \rho, there exists a spectral projector \rho_{w(j)} of \sigma such that \text{tr}(P_j Q_{w(j)}) = \text{tr}(P_j) and \text{tr}(P_j Q_{m}) = 0 for all m \neq w(j), where w is an injective map from the spectral projectors of \rho to that of \sigma. Consequently, the support of \rho_j is contained in the support of \rho_{w(j)}, so that \rho commutes with \sigma. Furthermore, \lambda_j/\mu_{w(j)} is a constant according to the above discussion. Therefore, \rho is proportional to \Pi_\rho\sigma.

Now, as the quantum analogue of proposition 5, we derive the following theorem, which is very useful to understanding the relations between Rényi relative entropies with different order parameters.

Theorem 6. Suppose \rho is a density matrix and \sigma is a positive operator with \text{supp}(\sigma) \geq \text{supp}(\rho). Then the following conditions are equivalent.

(B1) \quad S_\alpha(\rho\|\sigma) = S(\rho\|\sigma) for all \alpha \geq 0.
(B2) \quad S_\alpha(\rho\|\sigma) = S(\rho\|\sigma) for some \alpha \geq 0 with \alpha \neq 1.
(B3) \quad \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = 0.
(B4) \quad S_\alpha(\rho\|\sigma) = S(\rho\|\sigma) for all \alpha \geq 0.
(B5) \quad S_\alpha(\rho\|\sigma) = S(\rho\|\sigma) for some \alpha \geq 0 with \alpha \neq 1.
(B6) \quad \lim_{\alpha \to 1} S'_\alpha(\rho\|\sigma) = 0.
(B7) \quad \rho commutes with \sigma and is proportional to \Pi_\rho\sigma.

Proof. We shall prove the theorem by establishing the following implications,

(B1) \Rightarrow (B2) \Rightarrow (B3) \Rightarrow (B7) \Rightarrow (B1), \quad (B4) \Rightarrow (B5) \Rightarrow (B6) \Rightarrow (B7) \Rightarrow (B4).

Obviously, (B1) implies (B2). If \rho(\rho\|\sigma) = S(\rho\|\sigma) for some \alpha \geq 0 with \alpha \neq 1, then \rho(\rho\|\sigma) = 0 in the interval [1, \alpha] if \alpha > 1 or [\alpha, 1] if \alpha < 1, given that \rho(\rho\|\sigma)
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is monotonically increasing with \( \alpha \). Therefore, (B2) implies (B3). The implication (B3) \( \Rightarrow \) (B7) is shown in lemma 8. The implications (B4) \( \Rightarrow \) (B5) \( \Rightarrow \) (B6) \( \Rightarrow \) (B7) follow from a similar reasoning.

For the implications (B7) \( \Rightarrow \) (B1) and (B7) \( \Rightarrow \) (B4), note that
\[
S_{\alpha}(\rho\|\sigma) = S_{\alpha}(\rho\|\sigma)
\]
because \( \rho \) commutes with \( \sigma \). Meanwhile, the condition (B7) implies that
\[
\Pi_{\rho} \sigma = c \rho
\]
for some constant \( c > 0 \), so that
\[
\text{tr}(\rho^{1-\alpha}) = \text{tr}(\rho^{1-\alpha}) = c^{1-\alpha} \text{tr}(\rho) = c^{1-\alpha}
\]
for \( \alpha \geq 0 \). Therefore,
\[
S_{\alpha}(\rho\|\sigma) = S_{\alpha}(\rho\|\sigma) = -\ln c \quad \forall \alpha \in [0, \infty],
\]
which implies (B1) and (B4). \( \square \)

As applications of (141) and theorem 6, here we reproduce several well-known folklore results concerning Rényi entropies based on the observation
\[
S_{\alpha}(\rho) = -S_{\alpha}(\rho\|I).
\]
Setting \( \sigma = I \) in (141) yields
\[
\lim_{\alpha \to 1} S'_{\alpha}(\rho) = -\frac{1}{2} \left[ \sum_j m_j \lambda_j (\ln \lambda_j)^2 - \left( \sum_j m_j \lambda_j \ln \lambda_j \right)^2 \right],
\]
where \( \lambda_j \) are the distinct eigenvalues of \( \rho \) and \( m_j \) are the corresponding multiplicities. It follows that \( \lim_{\alpha \to 1} S'_{\alpha}(\rho) = 0 \) iff all nonzero eigenvalues of \( \rho \) are equal, that is, \( \rho \) is proportional to a projector.

Theorem 6 has an analogue for Rényi entropies.

Corollary 7. The following statements concerning a density matrix \( \rho \) are equivalent.
(C1) \( S_{\alpha}(\rho) = S(\rho) \) for all \( \alpha \geq 0 \).
(C2) \( S_{\alpha}(\rho) = S(\rho) \) for some \( \alpha \geq 0 \) with \( \alpha \neq 1 \).
(C3) \( \lim_{\alpha \to 1} S'_{\alpha}(\rho) = 0 \).
(C4) \( \rho \) is proportional to a projector.

7. Relations between Rényi relative entropies of coherence

By virtue of the results presented in previous sections, here we clarify order relations between different Rényi relative entropies of coherence, including the logarithmic robustness of coherence. We then determine all states whose relative entropy of coherence (or distillable coherence) is equal to the logarithmic robustness of coherence or geometric coherence. These results will be useful in understanding the relation between exact coherence distillation and asymptotic coherence distillation as discussed in section 8.

First, the inequality (7) implies that
\[
C_{r,\alpha}(\rho) \geq C_{r,\alpha}(\rho) \quad \forall \alpha \in [0, \infty].
\]
The following theorem establishes inequalities in the opposite direction.
Theorem 7. Any state $\rho$ satisfies
\[ C_{r,\alpha}(\rho) \geq C_{r,2-\frac{1}{\alpha}}(\rho) \quad \forall \alpha \in \left[ \frac{1}{2}, \infty \right], \tag{148} \]
and the two inequalities are saturated when $\rho$ is pure.

Proof. Let $\rho_{\text{MC}} = U_{\text{CNOT}}[\rho \otimes |0\rangle \langle 0|]$. Then
\[ C_{r,\alpha}(\rho) = C_{r,\alpha}(\rho_{\text{MC}}) = -H^\uparrow_{\alpha}(A|B)_{\rho_{\text{MC}}} = C_{r,2-\frac{1}{\alpha}}(\rho_{\text{MC}}) = C_{r,2-\frac{1}{\alpha}}(\rho) \tag{150} \]
according to theorem 1 and lemma 1. In addition, lemma 1 implies that the inequality is saturated when $\rho$ is pure, which can also be verified explicitly by virtue of proposition 6.

Taking the limit $\alpha \to \infty$ in (148) and applying (62), we obtain (149). Again, the inequality $C_{RL}(\rho) \geq C_{r,2}(\rho)$ known previously [13], note that $C_{r,2}(\rho) \geq C_{r}(\rho)$. As a corollary, we get a lower bound for the robustness of coherence,
\[ C_{R}(\rho) \geq \left[ \sum_j \left( \sum_k |\rho_{jk}|^2 \right)^{1/2} \right]^2 - 1. \tag{153} \]

Equation (148) in theorem 7 yields a lower bound for the geometric coherence,
\[ C_{G}(\rho) \geq C_{r,0}(\rho) = -\ln \| (\Pi_{\rho})^{\text{diag}} \|, \tag{151} \]
where the formula for $C_{r,0}(\rho)$ comes from (63) and $\Pi_{\rho}$ is the projector onto the support of $\rho$. This in turn leads to a lower bound for the other variant of the geometric coherence $\tilde{C}_{G}(\rho)$, that is,
\[ \tilde{C}_{G}(\rho) \geq 1 - \| (\Pi_{\rho})^{\text{diag}} \|. \tag{152} \]

Equation (149) improves over the bound $C_{RL}(\rho) \geq C_{r}(\rho)$ known previously [13], note that $C_{r,2}(\rho) \geq C_{r}(\rho)$. As a corollary, we get a lower bound for the robustness of coherence,
\[ C_{R}(\rho) \geq \left[ \sum_j \left( \sum_k |\rho_{jk}|^2 \right)^{1/2} \right]^2 - 1. \tag{153} \]

By virtue of theorem 7 and the inequality $C_{RL}(\rho) \leq C_{L}(\rho)$ [21], we can derive a universal upper bound for all R"enyi relative entropies of coherence.

Corollary 8. Any state $\rho$ satisfies
\[ C_{r,\alpha}(\rho) \leq C_{RL}(\rho) \leq C_{L}(\rho) \quad \forall \alpha \in [0, 2], \tag{154} \]
\[ C_{r,\alpha}(\rho) \leq C_{RL}(\rho) \leq C_{L}(\rho) \quad \forall \alpha \in [0, \infty]. \tag{155} \]

In conjunction with (58), corollary 8 leads to an interesting inequality,
\[ \frac{1}{\alpha - 1} \ln \| (\rho^\alpha)^{\text{diag}} \|_{1/\alpha} \leq C_{L}(\rho) \quad \forall \alpha \in [0, 2], \tag{156} \]
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which is applicable to any density matrix. It is equivalent to the following inequality,

\[
\left\| (\rho^\alpha)^{\text{diag}} \right\|_{1/\alpha}^{\frac{1}{\alpha-1}} \leq C_1(\rho) + 1 = \sum_{j,k} |\rho_{jk}| \quad \forall \alpha \in [0, 2].
\]

(157)

If \( \rho \) is a maximally correlated state, then the logarithmic negativity is equal to the logarithmic l₁-norm of coherence, that is, \( \mathcal{N}_L(\rho) = C_1(\rho) \) \[12,13\]. By virtue of theorem 1 and corollary 8, we can derive a universal upper bound for all Rényi relative entropies of entanglement of maximally correlated states.

**Corollary 9.** Any maximally correlated state \( \rho \) satisfies

\[
E_{r,\alpha}(\rho) \leq E_{R_L}(\rho) \leq \mathcal{N}_L(\rho)
\]

∀ \( \alpha \in [0, 2] \), \( \mathcal{N}_L(\rho) \).

(158)

\[
E_{r,\alpha}(\rho) \leq E_{R_L}(\rho) \leq \mathcal{N}_L(\rho)
\]

∀ \( \alpha \in [0, \infty] \).

(159)

Note that the two equations above still hold if \( \rho \) is subjected to any local unitary transformation.

Now, using theorems 5, 6, and 7, we determine all states whose relative entropy of coherence (or distillable coherence \[3\]) coincides with the logarithmic robustness of coherence or geometric coherence.

**Theorem 8.** The following conditions are equivalent.

(D1) \( C_{R_L}(\rho) = C_{r,\infty}(\rho) = C_r(\rho) \).

(D2) \( C_{r,\alpha}(\rho) = C_r(\rho) \) for some \( \alpha \geq 1/2 \) with \( \alpha \neq 1 \).

(D3) \( C_{r,\alpha}(\rho) = C_r(\rho) \) for all \( \alpha \geq 1/2 \).

(D4) \( C_{r,\alpha}(\rho) = C_r(\rho) \) for all \( \alpha \geq 0 \).

(D5) \( C_{r,\alpha}(\rho) = C_r(\rho) \) for some \( \alpha \geq 0 \) with \( \alpha \neq 1 \).

(D6) \( \rho \) commutes with \( \rho^{\text{diag}} \) and is proportional to \( \Pi_\rho \rho^{\text{diag}} \).

Different Rényi relative entropies of coherence are interesting in different contexts and have different operational interpretations. For example, the relative entropy of coherence is equal to the distillable coherence \[3\], while the geometric coherence upper bounds the exact coherence distillation rate (see section 8). Therefore, theorem 8 is instructive to understanding the connections between different operational tasks in which Rényi relative entropies of coherence play certain roles. For example, theorem 8 is helpful to clarifying the relation between exact coherence distillation and asymptotic coherence distillation.

The combination of theorem 8 and corollary 8 yields the following result.

**Corollary 10.** If \( \rho \) saturates the inequality \( C_r(\rho) \leq C_L(\rho) \), then \( \rho \) commutes with \( \rho^{\text{diag}} \) and is proportional to \( \Pi_\rho \rho^{\text{diag}} \).

As an implication of theorem 8 and corollary 10 when \( \rho \) is pure, \( C_{R_L}(\rho) = C_r(\rho) \) iff all nonzero elements of \( \text{diag}(\rho) \) are equal, in which case \( \rho \) is either incoherent or maximally coherent on the support of \( \rho^{\text{diag}} \) (here \( \text{diag}(\rho) \) is a vector, while \( \rho^{\text{diag}} \) is a
diagonal matrix). Similarly, when $\rho$ is a qubit state, $C_{R_{\alpha}}(\rho) = C_{r}(\rho)$ iff $\rho$ is incoherent or maximally coherent. The same is true if $C_{R_{\alpha}}(\rho)$ is replaced by $C_{L}(\rho)$ given that $C_{L}(\rho) = C_{R_{\alpha}}(\rho)$ in both cases. In general, incoherent states and maximally coherent states can satisfy the conditions in theorem 8, but they are not the only candidates. For example, the conditions can also be satisfied by a weighted direct sum of two maximally coherent states, say

$$\rho = p_{1}(|\psi\rangle\langle\psi|) + p_{2}(|\varphi\rangle\langle\varphi|),$$

where

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |\varphi\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \quad 0 \leq p_{1}, p_{2} \leq 1, \quad p_{1} + p_{2} = 1. \quad (161)$$

When $\rho$ is a maximally correlated state, theorem 8 and corollary 10 still hold if Rényi relative entropies of coherence are replaced by the corresponding Rényi relative entropies of entanglement, while the logarithmic $l_1$-norm of coherence is replaced by the logarithmic negativity. For example, the following corollary is the analogue of corollary 10

**Corollary 11.** If $\rho$ is a maximally correlated state that saturates the inequality $E_{r}(\rho) \leq N_{L}(\rho)$, then $\rho$ commutes with $\rho^{\text{diag}}$ and is proportional to $\Pi\rho^{\text{diag}}$.

**Proof of theorem 8.** We shall prove theorem 8 by establishing the following implications,

$$(D4) \Rightarrow (D3) \Rightarrow (D1) \Rightarrow (D2) \Rightarrow (D5) \Rightarrow (D6) \Rightarrow (D4).$$

The implications $(D4) \Rightarrow (D3)$ and $(D2) \Rightarrow (D5)$ follow from theorem 7 and (147), and the monotonicity of $C_{r,\alpha}, C_{r,\alpha}$ with $\alpha$. The implications $(D3) \Rightarrow (D1)$ and $(D1) \Rightarrow (D2)$ are trivial. The implication $(D6) \Rightarrow (D4)$ follows from lemma 9 below.

It remains to show the implication $(D5) \Rightarrow (D6)$. If $C_{r,\alpha}(\rho) = C_{r}(\rho)$ for some $\alpha < 1$, then $S_{u}(\rho\|\rho^{\text{diag}}) = C_{r}(\rho) = S(\rho\|\rho^{\text{diag}})$. If $C_{r,\alpha}(\rho) = C_{r}(\rho)$ for some $\alpha > 1$, then $S_{2-\frac{1}{\alpha}}(\rho\|\rho^{\text{diag}}) = C_{r}(\rho) = S(\rho\|\rho^{\text{diag}})$ according to theorem 5. Therefore, $\rho$ commutes with $\rho^{\text{diag}}$ and is proportional to $\Pi\rho^{\text{diag}}$ according to theorem 8.$\square$

In the rest of this section, we prove a lemma used in the proof of theorem 8

**Lemma 9.** Suppose $\rho$ is a density matrix that commutes with $\rho^{\text{diag}}$ and satisfies $\Pi\rho^{\text{diag}} = c\rho$ for some positive constant $c$. Then

$$C_{R_{\alpha}}(\rho) = C_{r,\alpha}(\rho) = S_{\alpha}(\rho\|\rho^{\text{diag}}) = S_{\alpha}(\rho\|\rho^{\text{diag}}) = -\ln c \quad \forall \alpha \in [0, \infty], \quad (162)$$

$$C_{r,\alpha}(\rho) = -\ln c \quad \forall \alpha \in \left[\frac{1}{2}, \infty\right]. \quad (163)$$

**Proof.** The equalities $S_{\alpha}(\rho\|\rho^{\text{diag}}) = S_{\alpha}(\rho\|\rho^{\text{diag}}) = -\ln c$ follow from (146). To prove other equalities in the lemma, let $\rho = \sum_{j} \lambda_{j} P_{j}$ be the spectral decomposition of $\rho$ with $\lambda_{j} > 0$. If $\rho$ commutes with $\rho^{\text{diag}}$ and satisfies $\Pi\rho^{\text{diag}} = c\rho$, then $P_{j}^{\text{diag}}$ have mutually orthogonal supports and all nonzero entries of $P_{j}^{\text{diag}}$ are equal to $c$. Suppose $P_{j}^{\text{diag}}$ have
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$n_j$ nonzero entries, then $\sum_j n_j \lambda_j c = \text{tr} \rho = 1$, so that $\sum_j n_j \lambda_j = 1/c$. According to proposition 5

$$C_{\tau,\alpha}(\rho) = \frac{1}{\alpha - 1} \ln \| (\rho^\alpha)^{\text{diag}} \|_{1/\alpha} = \frac{1}{\alpha - 1} \ln \left\| \left( \sum_j \lambda_j^\alpha P_j \right)^{\text{diag}} \right\|_{1/\alpha}$$

$$= \frac{1}{\alpha - 1} \ln \left( \sum_j n_j \lambda_j c^{1/\alpha} \right) = \frac{1}{\alpha - 1} \ln c^{1-\alpha} = -\ln c \quad \forall \alpha \in [0, \infty), \quad (164)$$

which further implies that $C_{\tau,\infty}(\rho) = -\ln c$. According to theorem 5

$$-\ln c = S_{r,2-\frac{1}{\alpha}}(\rho)^{\text{diag}} \leq C_{\tau,\alpha}(\rho) \leq S_{r,\alpha}(\rho)|\rho^{\text{diag}}| = -\ln c \quad \forall \alpha \in \left[ \frac{1}{2}, \infty \right), \quad (165)$$

which implies that $C_{\tau,\alpha}(\rho) = -\ln c$ for $\alpha \geq 1/2$. Alternatively, this result can be derived from theorem 7 and the equality $C_{\tau,\alpha}(\rho) = -\ln c$. Taking the limit $\alpha \to \infty$ yields the equality $C_{R_L}(\rho) = -\ln c$. 

8. Exact coherence distillation

It is known that Rényi relative entropies of entanglement upper bound the exact distillation rate of entanglement [17, lemma 8.15]. Moreover, the bounds based on $E_{r,0}$ and $E_{r,1/2}$ are saturated in the case of pure states [23, 24, 17, Exercise 8.32]. In this section we show that Rényi relative entropies of coherence play the same role in exact coherence distillation as Rényi relative entropies of entanglement play in exact entanglement distillation.

Exact coherent distillation is a procedure for producing perfect maximally coherent states from partially coherent states as illustrated in figure 3. In other words, the goal is to generate maximally coherent states with exactly zero error. By contrast, in conventional asymptotic coherence distillation, the goal is to generate maximally coherent states with a small error that goes to zero asymptotically. For a given state $\rho$, we define the exact coherence distillation length $L_{e,c}(\rho)$ as

$$L_{e,c}(\rho) := \max \{ L | \exists \Lambda_i, \Lambda_i(\rho) = |\Phi_{c,L}\rangle \langle \Phi_{c,L}| \}, \quad (166)$$

where $|\Phi_{c,L}\rangle := \sum_{j=0}^{L-1} \frac{1}{\sqrt{L}} |j\rangle$ is a maximally coherent state in dimension $L$ [24], and $\Lambda_i$ is an incoherent operation. Then, we define the asymptotic exact coherent distillation rate $R_{e,c}(\rho)$ as

$$R_{e,c}(\rho) := \lim_{n \to \infty} \frac{1}{n} \ln L_{e,c}(\rho^\otimes n). \quad (167)$$

Lemma 10.

$$\ln L_{e,c}(\rho) \leq R_{e,c}(\rho) \leq C_{\tau,\alpha}(\rho) \quad \forall \alpha \in [0,2], \quad (168)$$

$$\ln L_{e,c}(\rho) \leq R_{e,c}(\rho) \leq C_{\tau,\alpha}(\rho) \quad \forall \alpha \in \left[ \frac{1}{2}, \infty \right]. \quad (169)$$
Proof. According to the definition of $L_{e,c}(\rho)$, it is straightforward to verify that $L_{e,c}(\rho^\otimes n) \geq L_{e,c}(\rho)^n$. Therefore,

$$R_{e,c}(\rho) = \lim_{n \to \infty} \frac{1}{n} \ln L_{e,c}(\rho^\otimes n) \geq \lim_{n \to \infty} \frac{1}{n} \ln L_{e,c}(\rho)^n = \ln L_{e,c}(\rho).$$

(170)

Let $C$ be any coherent measure that does not increase under incoherent operations. Then $C(\rho_1, \rho_2) \leq C(\rho)$ whenever $|\Phi_{c,L} \rangle \langle \Phi_{c,L}|$ can be generated from $\rho$ by incoherent operations. If, in addition, $C$ satisfies the normalization condition $C(\rho_1) = \ln L$, which is the case for all the coherent measures that appear in lemma 10 then $\ln L_{e,c}(\rho) \leq C(\rho)$. Therefore,

$$R_{e,c}(\rho) \leq \lim_{n \to \infty} \frac{1}{n} C(\rho^\otimes n).$$

(171)

Now lemma 10 follows from the fact that $C_{r,\alpha}$ for $\alpha \in [0, 2]$ and $C_{r,\alpha}(\rho)$ for $\alpha \in [\frac{1}{2}, \infty]$ are additive according to theorem 8.

Recall that both $C_{r,\alpha}$ and $C_{r,\alpha}(\rho)$ are monotonically increasing with $\alpha$ and that $C_{r,0}(\rho) \leq C_{r,1/2}(\rho)$ according to [143]. So the bound $C_{r,0}(\rho)$ on the exact distillation rate is the best among all bounds based on Rényi relative entropies of coherence. Actually, this bound is saturated when $\rho$ is pure, in which case $C_{r,0}(\rho) = C_{r,1/2}(\rho)$.

Theorem 9. Suppose $\rho = |\psi\rangle\langle\psi|$ is a pure state. Then $L_{e,c}(\rho) = [1/p_{\max}]$ and $R_{e,c}(\rho) = -\ln(p_{\max}) = C_{r,0}(\rho)$, where $p_{\max} = p_{\max}(\rho) := \|\text{diag}(\rho)\|_\infty$.

Proof. A pure state $\rho$ can be transformed to another pure state $\sigma$ under incoherent operations if $\text{diag}(\rho)$ is majorized by $\text{diag}(\sigma)$ [4][5][11][15] (the same is true if we consider strictly incoherent operations). In addition, $\text{diag}(\rho)$ is majorized by $\text{diag}(|\Phi_{c,L}\rangle\langle\Phi_{c,L}|)$ if $p_{\max} \leq 1/L$. Therefore, $L_{e,c}(\rho) = [1/p_{\max}]$.

$$R_{e,c}(\rho) = \frac{1}{n} \lim_{n \to \infty} \ln[(p_{\max}(\rho^\otimes n))^{-1}] = \frac{1}{n} \lim_{n \to \infty} \ln[(p_{\max})^{-n}] = -\ln(p_{\max}) = C_{r,0}(\rho).$$

(172)

Note that lemma 10 and theorem 9 still hold if the operation $\Lambda_i$ in the definition of $L_{e,c}(\rho)$ in [166] is only required to be incoherence-preserving instead of being incoherent. In this case, the current proof of lemma 10 still applies after replacing incoherent operations with incoherence-preserving operations. The current proof of
theorem 9 implies that $L_{e,c}(\rho) \geq \lfloor 1/p_{\text{max}} \rfloor$ and $R_{e,c}(\rho) \geq -\ln(p_{\text{max}}) = C_{t,0}(\rho)$, while the opposite inequalities follow from lemma 10. Therefore, for pure states, the exact coherence distillation rate (length) remains the same under three distinct classes of operations, namely, strictly incoherent operations, incoherent operations, and incoherence-preserving operations.

In general, the exact distillation rate $R_{e,c}(\rho)$ is smaller than the distillable coherence, which is equal to the relative entropy of coherence $C_t(\rho)$ [3]. Therefore, exact distillation requires more resources than distillation with negligible small error even asymptotically under incoherence-preserving operations. Consequently, the exact distillation rate of coherence is in general smaller than the coherence cost.

A necessary condition for saturating the inequality $R_{e,c}(\rho) \leq C_t(\rho)$ can be derived from theorem 8 and lemma 10.

**Corollary 12.** If the exact distillation rate of coherence is equal to the distillable coherence, that is, if the bound $R_{e,c}(\rho) \leq C_t(\rho)$ is saturated, then $\rho$ commutes with $\rho^{\text{diag}}$ and is proportional to $\Pi_\rho \rho^{\text{diag}}$.

According to this corollary or theorem 9, when $\rho$ is a pure state, the inequality $R_{e,c}(\rho) \leq C_t(\rho)$ is saturated iff $\rho$ is incoherent or maximally coherent on the support of $\rho^{\text{diag}}$. Similarly, when $\rho$ is a qubit state, the inequality $R_{e,c}(\rho) \leq C_t(\rho)$ is saturated iff $\rho$ is incoherent or maximally coherent.

9. Summary

We proved that Rényi relative entropies of coherence and Rényi relative entropies of entanglement are both equal to the corresponding Rényi conditional entropies for maximally correlated states. By virtue of this observation and the generalized CNOT gate, we established an operational one-to-one mapping between entanglement measures and coherence measures based on Rényi relative entropies. In particular, every Rényi relative entropy of coherence is equal to the maximum Rényi relative entropy of entanglement generated by incoherence-preserving operations (or incoherent operations) acting on the system and an incoherent ancilla. These results significantly strengthen the connection between the resource theory of coherence and that of entanglement. They are also useful to understanding the properties of maximally correlated states themselves. We then proved that all Rényi relative entropies of coherence, including the logarithmic robustness of coherence, are additive. Accordingly, all Rényi relative entropies of entanglement are additive for maximally correlated states. In addition, we derived several nontrivial bounds on Rényi relative entropies of coherence and logarithmic robustness of coherence, which improve over bounds known in the literature, including the inequality $C_t(\rho) \leq C_{R_L}(\rho)$ between the relative entropy of coherence and logarithmic robustness of coherence. Furthermore, we determined all states whose relative entropy of coherence (or distillable coherence) is equal to the logarithmic robustness of coherence or geometric coherence. As an application, we provided an upper bound for the exact
coherence distillation rate based on a special Rényi relative entropy of coherence, which is saturated for pure states.

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Appendix A. Alternative proof of proposition 5

In this appendix we give an alternative proof of proposition 5 for \( \alpha \in [0, 2] \) by virtue of theorem (11) and (16).

\[ C_{r, \alpha}(\rho) = C_{r, \alpha}(\rho_{MC}) = -\frac{\alpha}{\alpha - 1} \ln \text{tr}\left\{ \left[ \text{tr}_A(\rho^\alpha_{MC}) \right]^{1/\alpha} \right\} \]

\[ = \frac{1}{\alpha - 1} \ln \| (\rho^\alpha)^\text{diag} \|_{1/\alpha}. \]  

(A.1)

Here the second inequality follows from theorem (11) the third one from (16), and the last one from the observation that

\[ \text{tr}_A(\rho^\alpha_{MC}) = \sum_j (\rho^\alpha)_{jj} |j\rangle \langle j| = (\rho^\alpha)^\text{diag}. \]  

(A.2)

Appendix B. Proof of lemma 7

Proof. According to the definition of \( S_\alpha \) in (12),

\[ S_\alpha(\rho_{MC}||I_A \otimes \sigma) = \frac{1}{\alpha - 1} \ln \text{tr}\left\{ \rho^\alpha_{MC} \left[ I_A \otimes \sigma^{1-\alpha} \right] \right\} = \frac{1}{\alpha - 1} \ln \text{tr}\left\{ (\rho^\alpha)^\text{diag} \sigma^{1-\alpha} \right\} \]

\[ = \frac{1}{\alpha - 1} \ln \text{tr}(\rho^\alpha \sigma^{1-\alpha}) = S_\alpha(\rho||\sigma), \]  

(B.1)
where the third equality follows from the assumption that $\sigma$ is diagonal in the reference basis. Similarly,

\[
S_\alpha (\rho_{MC} \| I_A \otimes \sigma) = \frac{1}{\alpha - 1} \ln \text{tr} \left\{ \left[ (I_A \otimes \sigma^{\frac{1-\alpha}{2\alpha}}) \rho_{MC} (I_A \otimes \sigma^{\frac{1-\alpha}{2\alpha}}) \right]^\alpha \right\}
\]

\[
= \frac{1}{\alpha - 1} \ln \text{tr} \left\{ \left[ \sum_{jk} \sigma^{\frac{1-\alpha}{2\alpha}}_{jj} \rho_{jk} \sigma^{\frac{1-\alpha}{2\alpha}}_{kk} (|j\rangle \langle k|) \right]^\alpha \right\}
\]

\[
= \frac{1}{\alpha - 1} \ln \text{tr} \left[ (\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha \right] = S_\alpha (\rho \| \sigma).
\] (B.2)

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