ON THE GRÜSS INEQUALITY FOR UNITAL 2-POSITIVE LINEAR MAPS.

SRIRAM BALASUBRAMANIAN

Abstract. In a recent work, Moslehian and Rajić have shown that the Grüss inequality holds for unital $n$-positive linear maps $\phi : A \to B(H)$, where $A$ is a unital $C^*$-algebra and $H$ is a Hilbert space, if $n \geq 3$. They also demonstrate that the inequality fails to hold, in general, if $n = 1$ and question whether the inequality holds if $n = 2$. In this article, we provide an affirmative answer to this question.

1. Introduction

A classical theorem of Grüss (see [G]) states that if $f$ and $g$ are bounded real valued integrable functions on $[a, b]$ and $m_1 \leq f(x) \leq M_1$ and $m_2 \leq g(x) \leq M_2$ for all $x \in [a, b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \left( \int_a^b f(x)dx \right) \left( \int_a^b g(x)dx \right) \right| \leq \frac{1}{4} \alpha \beta,$$

where $\alpha = (M_1 - m_1)$ and $\beta = (M_2 - m_2)$.

A generalized operator version of the Grüss inequality was given by Renaud in [R], where he proved the following result.

Theorem 1. Let $A, B \in B(H)$ and suppose that their numerical ranges are contained in disks of radii $R$ and $S$ respectively. If $T \in B(H)$ is a positive operator with $\text{Tr}(T) = 1$, where Tr stands for the trace, then

$$|\text{Tr}(TAB) - \text{Tr}(TA)\text{Tr}(TB)| \leq 4RS.$$

If $A, B$ are normal, then the constant 4 on the right hand side can be replaced by 1.

Among other operator versions of the Grüss inequality, of particular interest to us are those of Perić and Rajić (see [PR]), where they prove the Grüss inequality for completely bounded maps, and Moslehian and Rajić (see [MR]), where they prove the Grüss inequality for $n$-positive unital linear maps, for $n \geq 3$. In [MR], the authors show that the inequality fails to hold in general, if $n = 1$ and question whether it holds for the case $n = 2$. The main result of this article gives an affirmative answer to this question.

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Before we state the main result, we shall introduce some notation and definitions. Throughout this article, $\mathcal{A}$ will denote a unital $C^*$-algebra, $M_n(\mathcal{A})$ the $C^*$-algebra of $n \times n$ matrices over $\mathcal{A}$, $H$ and $K$ complex Hilbert spaces and $B(H)$ the $C^*$-algebra of bounded operators on $H$. The notations $e$, 1 will denote the unit elements in $\mathcal{A}$ and $B(H)$ respectively and $\phi : \mathcal{A} \to B(H)$, a unital linear map, i.e. a linear map such that $\phi(e) = 1$. The map $\phi$ is said to be positive if $\phi(a)$ is positive in $B(H)$ for all positive $a \in \mathcal{A}$. For more details, see [P]. It is easy to see that the map $\phi_n : M_n(\mathcal{A}) \to M_n(B(H))$ defined by $\phi_n((a_{ij})) = (\phi(a_{ij}))$ is unital and linear for each $n \in \mathbb{N}$. The map $\phi$ is said to be $n$-positive if $\phi_n$ is a positive map, completely positive if $\phi_n$ is $n$-positive for all $n \in \mathbb{N}$ and completely bounded if $\sup_{n \in \mathbb{N}} \|\phi_n\| < \infty$.

The main result of this article is the following.

**Theorem 2.** Let $\mathcal{A}$ be a $C^*$-algebra with unit $e$. If $\phi : \mathcal{A} \to B(H)$ is a unital $2$-positive linear map, then

$$\|\phi(ab) - \phi(a)\phi(b)\| \leq \left( \inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right) \left( \inf_{\mu \in \mathbb{C}} \|b - \mu e\| \right).$$

for all $a, b \in \mathcal{A}$.

To prove Theorem 2 we use the well-known theorems of Stinespring, Russo-Dye, Fuglede-Putnam, and the result due to Choi (see Lemma 3).

2. **Preliminaries**

In this section we include some lemmas which will be used in the sequel. Observe that if $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-algebras and $\gamma : \mathcal{A} \to \mathcal{B}$ is a unital $n$-positive linear map, then it is $m$-positive for all $m = 1, 2, \ldots, n$. In particular $\gamma$ is positive. It is well known that positive maps are $*$-preserving. i.e. $\gamma(a^*) = \gamma(a)^*$ for all $a \in \mathcal{A}$. Moreover $\|\gamma\| = 1$.

**Lemma 1.** If $P, Q, R \in B(H)$, then $A = \begin{pmatrix} P & R \\ R^* & Q \end{pmatrix} \succeq 0$ in $M_2(B(H))$ if and only if $P, Q \succeq 0$ and $\|(Rx, y)\| \leq \langle Px, x \rangle \langle Qy, y \rangle$, for all $x, y \in H$. Moreover, if $A \succeq 0$, then $\|R\|^2 \leq \|P\|\|Q\|$.

**Lemma 2.** Let $A = \begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \in B(H \oplus K)$. If $R \in B(K)$ be invertible, then the following statements are equivalent.

(i) $A \succeq 0$

(ii) $T, R \succeq 0$ and $T \succeq SR^{-1}S^*$.

The above two lemmas are well known. Their proofs can be found in [A].

**Lemma 3** (Choi). Let $\mathcal{U}$ and $\mathcal{V}$ be $C^*$-algebras and $\phi : \mathcal{U} \to \mathcal{V}$ be a positive linear map. If $x, y \in \mathcal{U}$ and $\begin{pmatrix} x & y \\ y^* & x \end{pmatrix} \succeq 0$, then $\begin{pmatrix} \phi(x) & \phi(y) \\ \phi(y^*) & \phi(x) \end{pmatrix} \succeq 0$.

For a proof of Lemma 3 please see Corollary 4.4 of [C].
Proposition 1. If $B$ is a unital $C^*$-algebra and $\phi : B \to B(H)$ is a unital 2-positive linear map, then
\[
\|\phi(ab) - \phi(a)\phi(b)\|^2 \leq \|\phi(aa^*) - \phi(a)\phi(a)^*\|\|\phi(b^*b) - \phi(b)^*\phi(b)\|,
\]
for all unitaries $a, b \in B$.

Proof. Since $\phi$ is positive, recall that $\phi(x^*) = \phi(x)^*$ for all $x \in B$. Let $a, b \in B$ be unitary. Consider the matrix
\[
A = \begin{pmatrix}
  a^*a & a^*b & a^* & a^*(a^*b) \\
  b^*a & b^*b & b^* & b^*(a^*b) \\
  a & b & a^*a & a^*b \\
  (b^*a)a & (b^*a)b & b^*a & b^*b
\end{pmatrix}.
\]

Since $a, b$ are unitaries, it follows that $R = b^*b = e$ and
\[
T = \begin{pmatrix}
  a^*a & a^*b & a^* \\
  b^*a & b^*b & b^* \\
  a & b & a^*a
\end{pmatrix} = \begin{pmatrix}
  a^*(a^*b) \\
  b^*(a^*b) \\
  a^*b
\end{pmatrix} \begin{pmatrix}
  (b^*a)a & (b^*a)b & b^*a \\
  b^*a & b^*b
\end{pmatrix} = SS^* = SR^{-1}S^*.
\]

Thus Lemma 2 implies that $A \succeq 0$. This is equivalent to
\[
\begin{pmatrix}
  a^*a & a^*b & a^* & a^*(a^*b) \\
  b^*a & b^*b & b^* & b^*(a^*b) \\
  a & b & a^*a & a^*b \\
  (b^*a)a & (b^*a)b & b^*a & b^*b
\end{pmatrix} \succeq 0.
\]

By Lemma 3 applied to the unital positive map $\phi_2$ and the $2 \times 2$ block matrix in equation (3), it follows that
\[
\begin{pmatrix}
  \phi(a^*a) & \phi(a^*b) & \phi(a^*) & \phi(a^*(a^*b)) \\
  \phi(b^*a) & \phi(b^*b) & \phi(b^*) & \phi(b^*(a^*b)) \\
  \phi(a) & \phi(b) & \phi(a^*a) & \phi(a^*b) \\
  \phi((b^*a)a) & \phi((b^*a)b) & \phi(b^*a) & \phi(b^*b)
\end{pmatrix} \succeq 0.
\]

This in turn implies that
\[
\begin{pmatrix}
  \phi(a^*a) & \phi(a^*b) & \phi(a^*) \\
  \phi(b^*a) & \phi(b^*b) & \phi(b^*) \\
  \phi(a) & \phi(b) & \phi(a^*a)
\end{pmatrix} \succeq 0.
\]

By Lemma 2 and the fact that $\phi(a^*a) = \phi(e) = 1$, equation (5) is equivalent to
\[
\begin{pmatrix}
  \phi(a^*a) & \phi(a^*b) \\
  \phi(b^*a) & \phi(b^*b)
\end{pmatrix} - \begin{pmatrix}
  \phi(a)^* \\
  \phi(b)^*
\end{pmatrix} \begin{pmatrix}
  \phi(a) \\
  \phi(b)
\end{pmatrix} \succeq 0,
\]

i.e.
\[
\begin{pmatrix}
  \phi(a^*a) - \phi(a)^*\phi(a) & \phi(a^*b) - \phi(a)^*\phi(b) \\
  \phi(b^*a) - \phi(b)^*\phi(a) & \phi(b^*b) - \phi(b)^*\phi(b)
\end{pmatrix} \succeq 0.
\]

An application of Lemma 1 to the operator matrix in equation (4) yields
\[
\|\phi(a^*b) - \phi(a)^*\phi(b)\|^2 \leq \|\phi(a^*a) - \phi(a)^*\phi(a)\|\|\phi(b^*b) - \phi(b)^*\phi(b)\|
\]
for all unitaries $a, b \in B$. Replacing $a$ by $a^*$ in (8) completes the proof. \(\square\)
The following three theorems are well known.

**Theorem 3** (Fuglede-Putnam). Let \( A \) be a \( C^* \)-algebra. If \( x, y \in A \) are such that \( x \) is normal and \( xy = yx \), then \( x^* y = yx^* \).

For more on the Fuglede-Putnam theorem, please see [B].

**Theorem 4** (Stinespring’s Dilation Theorem). If \( B \) is a unital \( C^* \)-algebra and \( \phi : B \to B(H) \) is a unital completely positive map, then there exist a Hilbert space \( K \), an isometry \( v : H \to K \) and a unital \( * \)-homomorphism \( \pi : B \to B(K) \) such that \( \phi(x) = v^* \pi(x)v \) for all \( x \in B \).

For a proof of Stinespring’s dilation theorem, please see [P].

**Theorem 5** (Russo-Dye). Let \( A \) be a unital \( C^* \)-algebra. If \( a \in A \) is such that \( \|a\| < 1 \), then \( a \) is a convex combination of unitary elements in \( A \).

For a proof and more on the Russo-Dye theorem, please see [B].

### 3. The Proof

This section contains the proof of our main result, i.e. Theorem 2. The following theorem and corollary lead us to it.

**Theorem 6.** If \( a, b \) are commuting normal elements in the unital \( C^* \)-algebra \( A \) and \( \phi : A \to B(H) \) is a unital positive linear map, then

\[
\|\phi(ab) - \phi(a)\phi(b)\| \leq \left( \inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right) \left( \inf_{\mu \in \mathbb{C}} \|b - \mu e\| \right),
\]

i.e. the Grüss inequality holds for such \( a, b \in A \).

**Proof.** The proof is adapted from [PR]. Let \( \lambda, \mu \in \mathbb{C} \). Since \( a, b \) are commuting normal elements in the \( C^* \)-algebra \( A \), it follows from Theorem 3 that the \( C^* \)-subalgebra of \( A \), say \( B \), generated by \( a, b \) and \( e \) is commutative. Since the restricted map \( \phi : B \to B(H) \) is positive and \( B \) is commutative, it follows that \( \phi : B \to B(H) \) is in fact completely positive (see e.g. [P]). By Theorem 4 it follows that there exist a Hilbert space \( K \), an isometry \( v : H \to K \) and a unital \( * \)-homomorphism \( \pi : B \to B(K) \) such that \( \phi(x) = v^* \pi(x)v \) for all \( x \in B \). Since \( \pi \) is a unital \( * \)-homomorphism, it is completely positive and hence is a complete contraction. In particular \( \|\pi\| \leq 1 \). It follows that

\[
\|\phi(ab) - \phi(a)\phi(b)\| = \|\phi((a - \lambda e)(b - \mu e)) - \phi(a - \lambda e)\phi(b - \mu e)\|
\]

\[
= \|v^* \pi((a - \lambda e)(b - \mu e))v - v^* \pi(a - \lambda e)vv^* \pi(b - \mu e)v\|
\]

\[
= \|v^* \pi(a - \lambda e)\pi(b - \mu e)v - v^* \pi(a - \lambda e)vv^* \pi(b - \mu e)v\|
\]

\[
= \|v^* \pi(a - \lambda e)(1 - vv^*)\pi(b - \mu e)v\|
\]

\[
\leq \|a - \lambda e\|\|1 - vv^*\|\|b - \mu e\|
\]

\[
\leq \|a - \lambda e\|\|b - \mu e\|.
\]

The proof is complete by taking infimums on the above inequality first with respect to \( \lambda \) and then with respect to \( \mu \). \( \square \)
Remark 1. It is easy to see that if $A$ is commutative or $\phi$ is completely positive, in the statement of Theorem 6, then the entire proof of Theorem 6 goes through with $B$ replaced by $A$, for arbitrary $a$ and $b$, i.e. the Grüss inequality (9) holds if $A$ is commutative or $\phi$ is completely positive.

Corollary 1. If $\phi$ and $a$ are as in Theorem 6, then
\[
\|\phi(aa^*) - \phi(a)\phi(a)^*\| \leq \left( \inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right)^2.
\]

Proof. The proof follows by taking $b = a^*$ in Theorem 6. \hfill \square

Proof of Theorem 2. Recall $a$, $b$, $A$, $H$ and $\phi$ from the statement of Theorem 2. Let $\epsilon > 0$. By Theorem 5, there exist unitary elements $u_1, \ldots, u_k$ and $v_1, \ldots, v_\ell$ in $A$ such that
\[
a = \frac{a}{\|a\| + \epsilon} = \sum_{i=1}^k \alpha_i u_i \quad \text{and} \quad b = \frac{b}{\|b\| + \epsilon} = \sum_{j=1}^\ell \beta_j v_j,
\]
where $\alpha_i, \beta_j \geq 0$ and $\sum_{i=1}^k \alpha_i = \sum_{j=1}^\ell \beta_j = 1$. It follows from Proposition 1 and Corollary 1 that
\[
\frac{1}{\|a\| + \epsilon} \left[ \phi(ab) - \phi(a)\phi(b) \right] \leq \left[ \sum_{i=1}^k \alpha_i \beta_j \left\| \phi(u_i v_j) - \phi(u_i)\phi(v_j) \right\| \right] \leq \left[ \sum_{i=1}^k \alpha_i \beta_j \right] \left( \inf_{\lambda \in \mathbb{C}} \|u_i - \lambda e\| \right) \left( \inf_{\mu \in \mathbb{C}} \|v_j - \mu e\| \right)
\]
\[
\leq \left[ \sum_{i=1}^k \alpha_i \beta_j \right] \|u_i\| \|v_j\| \leq \left[ \sum_{i=1}^k \alpha_i \beta_j \right] \left( \sum_{j=1}^\ell \beta_j \right) = 1.
\]

Letting $\epsilon \to 0$ above yields,
\[
\|\phi(ab) - \phi(a)\phi(b)\| \leq \|a\| \|b\|.
\]
Let $\lambda, \mu \in \mathbb{C}$ be arbitrary. It follows from equation (12) that
\[
\|\phi(ab) - \phi(a)\phi(b)\| = \|\phi((a - \lambda e)(b - \mu e)) - \phi(a - \lambda e)\phi(b - \mu e)\|
\leq \|(a - \lambda e)\|(b - \mu e)\|.
\]
Taking infimums in the above inequality, first with respect to $\lambda$ and then with respect to $\mu$ completes the proof. \(\square\)

The Grüss inequality fails, in general, when $\phi$ in Theorem 2 is assumed only to be positive, i.e. when $n = 1$, as the following example shows. We point out that [MR] contains an example of such a map $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$.

**Example:** Let $k \geq 2$, $\beta = \{e_1, e_2, \ldots, e_k\}$ be an orthonormal set in $H$, $E = \text{span}(\beta)$, and $\theta : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ denote the transpose map. It is well known that $\theta$ is a unital positive linear map, which is not 2-positive (see [TT]). Define $\phi : M_k(\mathbb{C}) \to \mathcal{B}(H)$ by $\phi(a) = \left( \begin{array}{cc} \theta(a) & 0 \\ 0 & 1 \end{array} \right)$. The block structure is with respect to the orthogonal decomposition $E \oplus E^\perp$ of $H$. Here $1$ denotes the identity operator and $0$ denotes the zero operator. It is easy to see that $\phi$ is a unital positive linear map which is not 2-positive.

Let $a = \left( \begin{array}{cc} 1 & 3 \\ 3 & 3 \end{array} \right) \oplus 0_{k-2} \in M_k(\mathbb{C})$ and $b = \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right) \oplus 0_{k-2} \in M_k(\mathbb{C})$. A simple computation shows that the eigenvalues of $a$ belong to $\{0, 2 \pm \sqrt{10}\}$ and those of $b$ belong to $\{0, 1, 3\}$. Since $a$ and $b$ are normal, it follows from [S] that,
\[
\inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| = \sqrt{10} \quad \text{and} \quad \inf_{\mu \in \mathbb{C}} \|b - \mu e\| = \frac{3}{2}.
\]
Moreover
\[
\phi(ab) - \phi(a)\phi(b) = \left( \left( \begin{array}{cc} 1 & 3 \\ 9 & 9 \end{array} \right) \oplus 0_{k-2} \right) \oplus 1 - \left( \left( \begin{array}{cc} 1 & 9 \\ 3 & 9 \end{array} \right) \oplus 0_{k-2} \right) \oplus 1
\]
\[
= \left( \left( \begin{array}{cc} 0 & -6 \\ 6 & 0 \end{array} \right) \oplus 0_{k-2} \right) \oplus 0.
\]
Thus,
\[
\|\phi(ab) - \phi(a)\phi(b)\| = 6 > \sqrt{10} \cdot \frac{3}{2} = \left( \inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right) \left( \inf_{\mu \in \mathbb{C}} \|b - \mu e\| \right).
\]

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Department of Mathematics, IIT Madras, Chennai - 600036, India.
E-mail address: bsriram@iitm.ac.in, bsriram30@yahoo.co.in