A Solution to Qi’s Conjecture on a Double Inequality for a Function Involving the Tri- and Tetra-Gamma Functions

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Abstract: In the paper, the author gives a solution to a conjecture on a double inequality for a function involving the tri- and tetra-gamma functions, which was first posed in Remark 6 of the paper “Complete monotonicity of a function involving the tri- and tetragamma functions” (2015) and repeated in the seventh open problem of the paper “On complete monotonicity for several classes of functions related to ratios of gamma functions” (2019).

Keywords: digamma function; trigamma function; double optimal inequality

1. Introduction

It is common knowledge that the classical Euler’s gamma function [1,2] is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

for $x > 0$ and the digamma function [3] is defined as the logarithmic derivative of the gamma function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$ 

The functions $\psi, \psi', \psi'', \psi''', \ldots$ are known as polygamma functions [4].

Very recently, in the paper [5], F. Qi and R. P. Agarwal surveyed some results related to the function $\psi'^2 + \psi''$. They also posed eight open problems. The goal of the paper is to find a solution of the seventh open problem which was first posed as a conjecture in Remark 6 of the paper [6].

The seventh open problem states that the double inequality

$$\frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^\alpha < [\psi'(x)]^2 + \psi''(x) < \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^\beta$$

holds on $(0, \infty)$ if and only if $\alpha \geq 6/5$ and $\beta \leq 1$.

2. The Key Lemmas

In this section, we prove two important lemmas.
Lemma 1. Let $\Delta(x) = [\psi'(x)]^2 + \psi''(x)$ and

$$
\phi(a, x) = \Delta(x) - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^a
$$

for $x > 0$ and $a \in \mathbb{R}$. Then

$$
\lim_{x \to +\infty} \phi(a, x) = 0, \quad (1)
$$

$$
\lim_{x \to 0^+} x^4 \phi(a, x) = 0, \quad (2)
$$

$$
\lim_{x \to 0^+} \frac{d[x^4 \phi(a, x)]}{dx} = -2 + \frac{5a}{3}. \quad (3)
$$

Proof of Lemma 1. To prove (1) it suffices to show $\lim_{x \to +\infty} \Delta(x) = 0$ which follows from the double inequality

$$
\frac{x^2 + 12}{12x^4(1 + x)^2} < \Delta(x) < \frac{x^2 + 4x + 12}{12x^4(1 + x)^2}
$$

(see [5], p. 9). Making use of the previous double inequality yields

$$
\lim_{x \to 0^+} \left\{ x^4 \Delta(x) - \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^a \right\} = 0 = \lim_{x \to 0^+} \left[ x^4 \phi(a, x) \right]
$$

which is (2).

Simple calculation brings

$$
\frac{d}{dx} \left[ x^4 \phi(a, x) \right] = 4x^3 \Delta(x) + x^4 \left[ 2\psi'(x)\psi''(x) + \psi'''(x) \right] + a \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{a-1} \frac{x + 10}{6(1 + x)^3}.
$$

Denote

$$
\delta(x) = 4x^3 \Delta(x) + x^4 \left[ 2\psi'(x)\psi''(x) + \psi'''(x) \right].
$$

To prove (3) it suffices to show

$$
\lim_{x \to 0^+} \delta(x) = \lim_{x \to 0^+} 4x^3 \left( [\psi'(x)]^2 + \psi''(x) \right) + x^4 \left( 2\psi'(x)\psi''(x) + \psi'''(x) \right) = -2.
$$

The function $\delta(x)$ can be rewritten as $\delta(x) = \delta_1(x) + \delta_2(x)$ where

$$
\delta_1(x) = 2x^3 \psi'(x) \left( 2\psi'(x) + x\psi''(x) \right),
$$

$$
\delta_2(x) = x^3 \left( 4\psi'(x) + x\psi''(x) \right). \quad (4)
$$

Making use of well known formulas:

for polygamma functions

$$
\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^2}, \quad \psi''(x) = -2 \sum_{n=0}^{\infty} \frac{1}{(n + x)^3}, \quad \psi'''(x) = 6 \sum_{n=0}^{\infty} \frac{1}{(n + x)^4},
$$

and for values of the Riemann zeta function [7]

$$
\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.20205..., \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}
$$
\[ 2x^3 \psi'(x) < 2x + 2x^3 \sum_{n=1}^{\infty} \frac{1}{n^2} = 2x + \frac{x^3 \pi^2}{3} \]

and

\[ |2\psi'(x) + x\psi''(x)| = \left| 2 \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} - 2x \sum_{n=1}^{\infty} \frac{1}{(n+x)^3} \right| \leq \frac{\pi^2}{3} + 2x \zeta(3). \]

\[ \lim_{x \to 0^+} \delta_1(x) = \lim_{x \to 0^+} 2x^3 (\psi'(x) + x\psi''(x)) = 0. \]

The equality (4) can be rewritten as

\[ \delta_2(x) = 4x^3 \left( \frac{-2}{x^3} - 2 \sum_{n=1}^{\infty} \frac{1}{(n+x)^3} \right) + x^4 \left( 6 \frac{1}{x^4} + 6 \sum_{n=1}^{\infty} \frac{1}{(n+x)^4} \right) \]

\[ = -2 - 8x^3 \sum_{n=1}^{\infty} \frac{1}{(n+x)^3} + 6x^4 \sum_{n=1}^{\infty} \frac{1}{(n+x)^4}. \]

Because of

\[ \sum_{n=1}^{\infty} \frac{1}{(n+x)^3} < \zeta(3) \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{(n+x)^4} < \zeta(4) = \frac{\pi^4}{90}, \]

we obtain \( \lim_{x \to 0^+} \delta_2(x) = -2. \) The proof of Lemma 1 is complete.

**Lemma 2.** Let

\[ s(x) = \phi \left( 6 \frac{6}{x^2}, x \right) = \Delta(x) - \frac{1}{x^4} \left[ \frac{x^2 + 12x + 12}{12(1+x)^2} \right]^{6/5} \]

for \( x > 0. \) Then \( s(x) > 0 \) for \( x > 0. \)

**Proof of Lemma 2.** Consider three cases

(a) \( 5.7 \leq x < \infty, \)

(b) \( 1.27 \leq x < 5.7, \)

(c) \( 0 < x \leq 1.27. \)

The case (a).

In the paper (see [5], p. 9), it was presented that

\[ \Delta(x) > \frac{1}{x^4} \left[ \frac{x^2 + 12}{12(1+x)^2} \right] \] (5)

for \( x > 0. \) So the case (a) will be done if we show

\[ \frac{1}{x^4} \left[ \frac{x^2 + 12}{12(1+x)^2} \right] \geq \left[ \frac{x^2 + 4x + 12}{12(1+x)^2} \right]^{6/5}, \]

which is equivalent to

\[ s_1(x) = \log(x^2 + 12) - \frac{6}{5} \log(x^2 + 4x + 12) + \frac{1}{5} \log(12) + \frac{2}{5} \log(1 + x) > 0 \]
for $5.7 \leq x < \infty$. In order to prove $s_1(x) > 0$ it is sufficient to show $s_1(5.7) > 0$ and $s_1'(x) > 0$ for $5.7 < x < \infty$. The inequality $s_1(x) > 0$ is equivalent to

$$g(x) = 12(x^2 + 12)^5(1 + x)^2 - (x^2 + 4x + 12)^6 > 0.$$  

Easy computation gives $g(5.7) \approx 1.06 \times 10^9 > 0$, so $s_1(5.7) > 0$.

Differentiation yields

$$s_1'(x) = \frac{x(22x^2 + 40x - 216)}{5(x^2 + 12)(x + 1)(x^2 + 4x + 12)}.$$  

Because of

$$22x^2 + 40x - 216 = 2(11x^2 + 20x - 108) = 2 \left[ x - \frac{2}{11} \left( \sqrt{322} - 5 \right) \right] \left[ x + \frac{2}{11} \left( \sqrt{322} + 5 \right) \right]$$

$$= 2(x + 4.1717\ldots)(x - 2.3535\ldots)$$

the function $s_1'(x)$ is positive for $x > \frac{2}{11} \left( \sqrt{322} - 5 \right)$. The proof of the case (a) is complete.

The case (b). Let $1.27 \leq x < 5.7$. Using the following formulas

$$\psi'(x) = \frac{1}{x^2} + \psi'(1 + x), \quad \psi''(x) = -\frac{2}{x^3} + \psi''(1 + x)$$

for $x > 0$ yields

$$\Delta(x) = [\psi'(x)]^2 + \psi''(x) = \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + \Delta(1 + x).$$

Making use of the inequality (see [5], p. 9)

$$\Delta(x) > \frac{1}{x^4} \left[ \frac{x^2 + 12}{12(x + 1)^2} \right].$$

yields

$$\Delta(x) > z(x) = \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + \frac{1}{(x + 1)^4} \left[ \frac{(x + 1)^2 + 12}{12(x + 2)^2} \right].$$

To prove the case (b) it suffices to show

$$\alpha(x) = z(x) - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(x + 1)^2} \right]^{6/5} > 0.$$  

It will be done if we prove

1. $z(x) > 0 \quad \text{for} \quad 1.27 \leq x < 5.7,$
2. $F(1.27) \geq 0,$
3. $F'(x) > 0 \quad \text{for} \quad 1.27 \leq x < 5.7$

where

$$F(x) = \log(z(x)) + 4 \log(x) - \frac{6}{5} \log \left[ \frac{x^2 + 4x + 12}{12(x + 1)^2} \right].$$
Differentiating $F$ yields

$$F'(x) = \frac{-\frac{4}{x} + \frac{6}{x^2} - \frac{4}{x^2} \psi'(1 + x) + \frac{2}{x} \psi''(1 + x) + \left[-\frac{2x^3 - 7x^2 - 44x - 63}{6(x + 1)^3(x + 2)^3}\right]}{\frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + \frac{1}{(x + 1)^3} \left[\frac{(x + 1)^2 + 12}{12(x + 2)^2}\right]} + \frac{4}{x} + \frac{6}{5} \left[\frac{2x + 20}{(x + 1)(x^2 + 4x + 12)}\right].$$

It is clear that, $F(1.27) \geq 0$ is equivalent to $\alpha(1.27) \geq 0$.

In the paper [8], inequality (2.3) it was established that

$$\psi'(1 + x) > \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^4} + \frac{1}{42x^5} - \frac{1}{30x^6} = \frac{210x^8 - 105x^7 + 35x^6 - 7x^4 + 5x^2 - 7}{210x^9} \tag{6}$$

for $x > 0$.

Using (6) gives

$$\psi'(2.27) = \frac{1}{2.27^2} + \psi'(3.27) > \frac{1}{2.27^2} + \frac{210 \times 2.27^6 - 105 \times 2.27^7 + 35 \times 2.27^6 - 7 \times 2.27^4 + 5 \times 2.27^2 - 7}{210 \times 2.27^9} > 0.551 > 0.55. \tag{7}$$

By (7), it follows that

$$z(x) > \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} 0.55 + \frac{1}{(x + 1)^4} \left[(x + 1)^2 + 12\right] = \frac{60x^8 + 408x^7 + 821x^6 + 274x^5 - 949x^4 - 960x^3 + 324x^2 + 720x + 240}{60x^4(x + 1)^4(x + 2)^2}.$$

Putting $x = 1.27$ in $q(x)$ yields $z(1.27) > q(1.27) = 0.0950575080326270 > 0.095$.

Because of

$$\left[1.27^4 \times 0.095\right]^5 - \left[\frac{1.27^2 + 4 \times 1.27 + 12}{12 \times 2.27^2}\right]^6 \approx 1.58 \times 10^{-04} > 0$$

we have $\alpha(1.27) > 0$. So $F(1.27) > 0$.

Next, we show $z(x) > 0$ for $1.27 < x < 5.7$. It is well known [9] that

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2} \text{ for } x > 0.$$

This implies that

$$z(x) > z_1(x) = \frac{1}{x^3} - \frac{2}{x^4} + \frac{2}{x^3} \left(\frac{1}{x + 1} + \frac{1}{2(x + 1)^2}\right) + \frac{1}{(x + 1)^4} \left[(x + 1)^2 + 12\right] \frac{12x^4(x + 1)^4(x + 2)^2}{x^6 + 2x^5 + 25x^4 + 72x^3 + 156x^2 + 144x + 48} > 0.$$
Finally, we show $F'(x) > 0$ for $1.27 \leq x < 5.7$. The inequality $F'(x) > 0$ is equivalent to

$$G(x) = -\frac{4}{x^5} + \frac{6}{x^4} - \frac{4}{x^3} \psi(1 + x) + \frac{2}{x^2} \psi'(1 + x) - \frac{2x^3 + 7x^2 + 44x + 63}{6(x + 1)^3(x + 2)^3} +$$

$$\left[ \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + \frac{(x + 1)^2 + 12}{12(x + 1)^4(x + 2)^2} \right] \times$$

$$\left[ \frac{4}{x^4} + \frac{6}{5} \left( \frac{2x + 20}{(x + 1)(x^2 + 4x + 12)} \right) \right] > 0.$$ 

The inequality $G(x) > 0$ may be rearranged as

$$G(x) = \psi'(1 + x) \left[ \frac{20x^3 + 124x^2 + 560x + 240}{5x^3(x + 1)(x^2 + 4x + 12)} \right] + \frac{2 \psi''(1 + x)}{x^2} -$$

$$\frac{1}{30x^3(x + 1)^3(x + 2)^3(x^2 + 4x + 12)} \times \left( 60x^9 + 1044x^8 + 9267x^7 + 47554x^6 + 147543x^5 + 285108x^4 + 344260x^3 + 254352x^2 + 105984x + 18624 \right) > 0.$$ 

In the paper [10], by using asymptotic expansion, it was deduced that

$$\psi''(x) = -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + \frac{1}{6x^6} - \frac{\theta}{6x^8} \text{ for } x > 0, \quad 0 \leq \theta \leq 1.$$ 

This implies

$$\psi''(x) \geq -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + \frac{1}{6x^6} - \frac{1}{6x^8} \text{ for } x > 0. \quad (8)$$ 

Utilizing (6), (8) we obtain that $G(x) \geq G_1(x)$, where

$$G_1(x) = \left( \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^4} + \frac{1}{42x^5} - \frac{1}{30x^6} \right) \left[ \frac{20x^3 + 124x^2 + 560x + 240}{5x^3(x + 1)(x^2 + 4x + 12)} \right] -$$

$$\frac{2}{x^2} \left[ \frac{1}{(1 + x)^2} + \frac{1}{(1 + x)^3} + \frac{1}{2(1 + x)^4} - \frac{1}{6(1 + x)^6} + \frac{1}{6(1 + x)^8} \right] -$$

$$\frac{1}{30x^3(x + 1)^3(x + 2)^3(x^2 + 4x + 12)} \times \left( 60x^9 + 1044x^8 + 9267x^7 + 47554x^6 + 147543x^5 + 285108x^4 + 344260x^3 + 254352x^2 + 105984x + 18624 \right).$$ 

Rewriting $G_1(x)$ yields $G_1(x) = \varphi(x)/\phi(x)$, where

$$\varphi(x) = 385x^{19} + 1435x^{18} + 2100x^{17} + 18662x^{16} + 153421x^{15} + 477255x^{14} + 718276x^{13} + 568208x^{12} + 232044x^{11} + 38256x^{10} - 46764x^9 - 314648x^8 - 1233916x^7 - 2806448x^6 - 3866740x^5 - 3395384x^4 - 1933504x^3 - 697344x^2 - 145600x - 13440,$$

$$\phi(x) = 1050x^{12}(x + 1)^3(x + 2)^3(x^2 + 4x + 12).$$ 

To show the positivity of the function $G_1(x)$ it suffices to prove $\varphi(x) > 0$. By virtue of $x \geq 1.27$ we see that $\varphi(x) \geq n_1(x) \geq n_2(x) \geq n_3(x)$ where the functions $n_1(x), n_2(x), n_3(x)$ are derived in Appendix A. It is obvious that the proof of $\varphi(x) > 0$ will be done if we show $n_3(1.27) > 0, n_3'(1.27) > 0, n_3''(1.27) > 0$ on $(1.27, 5.7)$.
A direct differentiation yields

\[ n_3'(x) = \frac{14197117854302775}{274877906944} x^5 + \frac{44097108486849535}{161330847079861} x^4 + \frac{1424065194179289}{2147483648} x^3 - \frac{2147483648}{328178783611353} x^2 + \frac{7951559879725953}{536870912} x + \frac{8589934592}{536870912} . \]

and

\[ n_3''(x) = \frac{70985589271513875}{274877906944} x^4 + \frac{44097108486849535}{4839926541239583} x^3 + \frac{1424065194179289}{1073741824} x^2 - \frac{1073741824}{328178783611353} x - \frac{4839926541239583}{536870912} . \]

From \( n_3'''(x) > 0, n_3'(1.27) \approx 3.96 \times 10^{66} \) we deduce \( n_3(x) \) is a convex function on \((1.27, 5.7)\).

The proof of the case (b) follows from \( n_3(1.27) \approx 2.12 \times 10^{66} > 0, n_3'(1.27) \approx 2.19 \times 10^{66} > 0. \)

The case (c). We have \( 0 < x \leq 1.27. \) Using

\[ [\psi'(x)]^2 = \frac{1}{x^4} + \frac{2}{x^2} \psi'(1 + x) + [\psi'(1 + x)]^2, \]

\[ \psi''(x) = -\frac{2}{x^3} + \psi''(1 + x) \]

gives

\[ s(x) = \phi \left( \frac{6}{5}, x \right) = \Delta(x) - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{6/5} \]

\[ = [\psi'(x)]^2 + \psi''(x) - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{6/5} \]

\[ = \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + [\psi'(1 + x)]^2 + \psi''(1 + x) - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{6/5} \]

\[ = \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + \Delta(1 + x) - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{6/5}. \]

Replacing \( x \) by \( 1 + x \) in (5) yields

\[ \Delta(1 + x) \geq \frac{1}{(1 + x)^4} \left[ \frac{(1 + x)^2 + 12}{12(2 + x)^2} \right]. \]

So

\[ s(x) = \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + \Delta(1 + x) - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{6/5} \geq t(x), \]

where

\[ t(x) = \frac{1}{x^4} - \frac{2}{x^3} + \frac{2}{x^2} \psi'(1 + x) + \frac{(1 + x)^2 + 12}{12(1 + x)^4(2 + x)^2} - \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{6/5}. \]
The inequality \( t(x) \geq 0 \) is equivalent to
\[
    r(x) = 12(1 + x)^4(2 + x)^2(1 - 2x) + x^4((1 + x)^2 + 12) + \\
    24x^2(1 + x)^4(2 + x)^2\psi'(1 + x) - 12(1 + x)^4(2 + x)^2 \times \\
    (x^2 + 4x + 12)/(12(1 + x)^2) \right)^{6/5} \geq 0.
\]

Making use of the following inequality (see [11], p. 6)
\[
    \psi'(x) > \frac{1}{x^2 + \left(1 + \frac{1}{|\psi(1)|}\right)}
\]
we come to the conclusion that \( r(x) \geq h(x) \), where
\[
    h(x) = 12(1 + x)^4(2 + x)^2(1 - 2x) + 24x^2(1 + x)^4(2 + x)^2(1/(1 + x)^2 + \\
    1/(2 + x)^2 + 1/(3 + x)^2 + 1/(3 + x + 1/1.64)) + x^4((1 + x)^2 + 12) - \\
    12(1 + x)^4(2 + x)^2 \left(\frac{x^2 + 4x + 12}{12(1 + x)^2}\right)^{6/5}.
\]

(We used \( \psi'(1) = \pi^2/6 > 1.64. \))

The inequality \( h(x) \geq 0 \) is equivalent to
\[
    T(x) = \log(12(1 + x)^4(2 + x)^2(1 - 2x) + 24x^2(1 + x)^4(2 + x)^2(1/ \\
    (1 + x)^2 + 1/(2 + x)^2 + 1/(3 + x)^2 + 1/(3 + x + 1/1.64)) + x^4((1 + x)^2 + \\
    12)) - \log(12(1 + x)^4(2 + x)^2) - \log\left(\frac{x^2 + 4x + 12}{12(1 + x)^2}\right)^{6/5} \geq 0.
\]

Differentiating \( T(x) \) yields \( T'(x) = \varrho(x)/\tau(x) \), where
\[
    \varrho(x) = -190374x^{13} - 4286906x^{12} - 45015548x^{11} - 282667076x^{10} - \\
    1146875462x^9 - 3046852634x^8 - 5055591704x^7 - 4276245096x^6 + \\
    386216448x^5 + 4183878144x^4 + 3096950400x^3 + 875102976x^2 + \\
    292571136x
\]

and
\[
    \tau(x) = 5(41x + 148)(x + 1)(x + 2)(x + 3)(x^2 + 4x + 12) \times \\
    (-67x^9 - 412x^8 + 1054x^7 + 20728x^6 + 98457x^5 + \\
    251532x^4 + 404436x^3 + 425376x^2 + 252144x + 63936).
\]

To prove \( T(x) \geq 0 \) it suffices to show \( T(0) \geq 0 \), \( T(1.27) \geq 0 \), \( \tau(x) > 0 \), \( \varrho(0) = 0 \), \( \varrho'(0) > 0 \), \( \varrho(1.27) < 0 \), \( \varrho(x) = 0 \) has only one real root in \((0, 1.27) \). It is evident that if \( h(1.27) > 0 \) then \( T(1.27) > 0 \). Some calculations give \( T(0) = 0 \), \( h(1.27) \approx 2.70 \). So \( T(1.27) > 0 \). The inequality \( \tau(x) > 0 \) is equivalent to
\[
    d_1(x) = -67x^9 - 412x^8 + 1054x^7 + 20728x^6 + \\
    98457x^5 + 251532x^4 + 404436x^3 + 425376x^2 + 252144x + 63936 > 0.
\]
Because of
\[ d_1(x) > d_2(x) = (-67 \times 1.27^2 - 412 \times 1.27 + 1054)x^2 + 20728x^6 + 98457x^5 + 251532x^4 + 404436x^3 + 425376x^2 + 252144x + 63936, \]
and \(-67 \times 1.27^2 - 412 \times 1.27 + 1054 = 422.6957\) we obtain \(\tau(x) > 0\). Direct computation yields \(q(0) = 0, q(1.27) \approx -5.88 \times 10^{10}, q'(0) = 292,571,136\). Further, we prove that \(q(x)\) has only one root in \((0, 1.27)\). Denote \(n_1(x) = q(x)/x\). Because of \(q(x) = 0\) in \((0, 1.27)\) it suffices to show that

1. \(q(x) > 0\) for \(0 < x \leq 0.8\),
2. \(n_1' (x) < 0\) for \(0.8 < x < 1.27\),
3. \(q(1.27) < 0\).

Let \(0 < x \leq 0.6\). First we show \(q(x) > 0\) for \(0 < x \leq 0.8\). It is obvious that
\[ n_1(x) \geq n_4(x) = 0.6^6 (-190374x^6 - 4286906x^5 - 45015548x^4 - 282667076x^3 - 1146875462x^2 - 3046852634x - 5055591704) - 4276245096x^3 + 386216448x^4 + 4183878144x^5 + 3096950400x^2 + 875102976x + 292571136. \]

Differentiation yields
\[ n_1'(x) = -832695876x^5/15625 - 66819454779474x^4/3125 + 24007262662032x^3/15625 + 195501095104788x^2/15625 + 9510755557604x/15625 + 11452328429814/15625, \]
\[ n_1''(x) = -832695876x^4/3125 - 267277819117896x^3/3125 + 72021787986096x^2/15625 + 391002190209576x/15625 + 9510755557604/15625. \]

Now recall from [12,13] that if
\[ f(x) = ax^4 + bx^3 + cx^2 + dx + e \]
and
\[ \delta = 256a^3e^3 - 192a^2bd^2 - 128a^2c^2d^2 + 144a^2cd^2 - 27a^2d^4 + 144ab^2ce^2 - 6b^2ade - 80abc^2de + 18abcd^3 + 16c^4ae - 4c^3d^2 - 27b^6e^2 + 18cde^3 - 4a^3b^3 - 4b^2c^3e + b^2c^2d^2 < 0 \]
then the polynomial \(f(x)\) has two real distinct roots and two complex but not real roots.

Recall [14] the Bolzano Theorem which states: Let \(a < b\) be two real numbers, let \(f(x)\) be continuous function on a closed interval \([a, b]\) such that \(f(a)f(b) < 0\). Then there is a number \(x_0 \in (a, b)\) such that \(f(x_0) = 0\).

Consider the equation \(n_4''(x) = 0\). Direct computation gives
\[ \delta \approx -2.21 \times 10^{66} < 0 \]
and
This implies that there are only two real roots of $n''_s(x) = 0$. Table 1 implies the first root of $n''_s(x) = 0$ is in $(-330,000, -320,000)$ and the second root of $n''_s(x) = 0$ is in $(0.6, 0.7)$.

Because of $n''_s(0) > 0$, $n'_s(0) > 0$, $n_s(0) > 0$ we obtain $\rho(x) > 0$ for $0 < x \leq 0.6$.

Table 1. Values of $n''_s(x)$.

| Points    | Values of $n''_s(x)$ |
|-----------|----------------------|
| $-330,000$| $\approx -8.63 \times 10^{25}$ |
| $-320,000$| $\approx 8.54 \times 10^{24}$ |
| $0.6$     | $\approx 4.28 \times 10^{09}$ |
| $0.7$     | $\approx -3.47 \times 10^{09}$ |

Similarly, for $0.6 < x \leq 0.8$ we deduce

\[
\begin{align*}
n_1(x) & \geq n_s(x) = 8.6^6 \left( -190374 \times 0.8^6 - 4286906 \times 0.8^5 - 45015548 \times 0.8^4 - 282667076 \times 0.8^3 - 1146875462 \times 0.8^2 - 3046852634 \times 0.8 - 5055591704 \right) - 4276245096x^3 + 386216448x^4 + 418387144x^3 + 3096950400x^2 + 875102976x + 292571136.
\end{align*}
\]

Differentiation yields

\[
\begin{align*}
n'_r(x) &= -21381225480x^4 + 1544865792x^3 + 12551634432x^2 + 6193900800x + 875102976.
\end{align*}
\]

Consider the equation $n'_r(x) = 0$. Straightforwardly computing acquires

\[
\delta \approx -3.27 \times 10^5 < 0
\]

and

This implies that there are only two real roots of $n'_r(x) = 0$. Table 2 implies that the first root of $n'_r(x) = 0$ is in $(-0.2, -0.3)$ and the second root of $n'_r(x) = 0$ is in $(0.9, 1)$.

Table 2. Values of $n'_r(x)$.

| Points    | Values of $n'_r(x)$ |
|-----------|----------------------|
| $-0.2$    | $\approx 9.18 \times 10^{07}$ |
| $-0.3$    | $\approx -6.83 \times 10^{07}$ |
| $1$       | $-215,721,480$ |
| $0.9$     | $\approx 3.71 \times 10^{09}$ |

It brings $n'_r(x) > 0$ on $0.6 < x \leq 0.8$ (evidently $n'_r(0) > 0$). Because of $n_r(0.6) \approx 3.53 \times 10^{08} > 0$ we obtain $n_r(x) > 0$ on $0.6 < x \leq 0.8$. So $\rho(x) > 0$ for $0.6 < x \leq 0.8$. Now we prove $n'_1(x) < 0$ for $0.8 < x \leq 1.27$.

\[
\begin{align*}
n'_1(x) &= -2284488x^{11} - 47155966x^{10} - 450155480x^9 - 2544003684x^8 - 9175003696x^7 - 21327968438x^6 - 30333550224x^5 - 21381225480x^4 + 1544865792x^5 + 12551634432x^4 + 6193900800x + 875102976.
\end{align*}
\]
We first show $n'_1(x) < 0$ for $0.85 < x < 1.27$. It is obvious that
\[
n'_1(x) < v(x) = -2284488x^{11} - 47155966x^{10} - 450155480x^9 - 2544003684x^8 - 9175003696x^7 - 21327968438x^6 - 30333550224x^5 - 21381225480x^4 + 1544865792 \times 1.27^3 + 12551634432 \times 1.27^2 + 6193900800 \times 1.27 + 875102976.
\]
Because of $v'(x) < 0$, $v(0.85) \approx -4.26 \times 10^{10}$ we obtain $n'_1(x) < 0$ for $0.85 < x < 1.27$. Next, we show $n'_1(x) < 0$ for $0.8 \leq x \leq 0.85$. Easy to see that
\[
n'_1(x) < v(x) = -2284488x^{11} - 47155966x^{10} - 450155480x^9 - 2544003684x^8 - 9175003696x^7 - 21327968438x^6 - 30333550224x^5 - 21381225480x^4 + 1544865792 \times 0.85^3 + 12551634432 \times 0.85^2 + 6193900800 \times 0.85 + 875102976.
\]
Because of $v'(x) < 0$, $v(0.8) \approx -0.105 \times 10^{20}$ we get $n'_1(x) < 0$ for $0.8 \leq x \leq 0.85$. From $n_1(0.8) \approx 1.67 \times 10^{19}$ and $n_1(1.27) \approx 4.63 \times 10^{20}$ we can conclude that $\varphi(x)$ has only one root in $(0, 1.27)$ which completes the proof of Lemma 2. \[\square\]

3. Proof of the Main Result

In this section, we prove Qi’s Conjecture.

**Theorem 1.** Let $\Delta(x) = [\varphi'(x)]^2 + \varphi''(x)$ for $x > 0$. Then
\[
\frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{\alpha} < \Delta(x) < \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^{\beta},
\]
holds on $(0, \infty)$ if and only if $\alpha \geq 6/5$ and $\beta \leq 1$.

**Proof of Theorem 1.** The upper bound of (9) follows from the following conclusion. Let $x$ be a fixed positive real number. Denote
\[
F(\beta, x) = \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^\beta.
\]
Then $F'_\beta(\beta, x) < 0$. So, $F(\beta, x)$ is a decreasing function in $\beta$ for each fixed $x > 0$. In the paper [5], it was obtained that
\[
\frac{1}{x^4} \frac{x^2 + 12}{12(1 + x)^2} < \Delta(x) < \frac{1}{x^4} \frac{x^2 + 4x + 12}{12(1 + x)^2}
\]
for $x > 0$. From $F(1) \geq \Delta(x)$ and
\[
\lim_{x \to +\infty} x^4 \Delta(x) = -\left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^\beta = \frac{1}{12} - \frac{1}{12\beta}
\]
we can derive that $\beta = 1$ is an optimal constant.

Now we show the lower bound of (9). Lemma 2 implies
\[
\Delta(x) = \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^\alpha > 0
\]
on $(0, \infty)$ for $\alpha \geq 6/5$. The cases (2), (3) of Lemma 2 imply that $\alpha = 6/5$ is the best constant. This completes the proof. \[\square\]
4. Materials and Methods

In this paper, MATLAB software and methods of mathematical analysis were used.

5. Conclusions

The main result of this paper is the Theorem 1. The Theorem 1 says that the double inequality

\[
\frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^\alpha < [\psi'(x)]^2 + \psi''(x) < \frac{1}{x^4} \left[ \frac{x^2 + 4x + 12}{12(1 + x)^2} \right]^\beta
\]

holds on \((0, \infty)\) if and only if \(\alpha \geq 6/5\) and \(\beta \leq 1\) where \(\psi'(x)\) and \(\psi''(x)\) are the tri- and tetra-gamma functions respectively. The double inequality was posed by F. Qi and R. P. Agarwal as the seventh open problem in Remark 6 of the paper [5].

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**Appendix A**

Deriving the polynomials \(n_1(x), n_2(x), n_3(x)\). It is obvious that

\[
n_1(x) = 1.277(385x^{12} + 1435x^{11} + 2100x^{10} + 18662x^9 + 153421x^8 + 47725x^7 + 718276x^6 + 568208x^5 + 32044x^4 + 38256x^3 - 46764x^2 - 314648x - 1233916x^7 - 2806448x^6 - 3866740x^5 - 395384x^4 - 1933504x^3 - 697344x^2 - 145600x - 13440
\]

\[
= \frac{1154932396539119275}{562949953421312} x^{12} + \frac{4304748023463990025}{562949953421312} x^{11} + \frac{562949953421312}{1574907813462435375} x^{10} + \frac{28147976710656}{140737488355328} x^9 + \frac{283104983340120160239}{1574907813462435375} x^8 + \frac{73704824244373999153}{140737488355328} x^7 + \frac{562949953421312}{140737488355328} x^6 - \frac{29516054779357928685}{35184372088832} x^5 - \frac{30383500251087779687}{35184372088832} x^4 + \frac{60856544015076107363}{35184372088832} x^3 - 697344x^2 - 145600x - 13440.
\]
By virtue of $x \geq 1.27$ one may deduce that $n_1(x) \geq n_2(x)$ where

$$n_2(x) = 1.27^3 \left( \frac{1154932396539119275}{562949953421312} x^9 + \frac{4304748023463990025}{562949953421312} x^8 + \frac{1574907813462435757}{283104983340120160239} x^7 + \frac{182548976710656}{737048242443739991533} x^6 + \frac{14730326148044650491}{303835002510877789687} x^5 + \frac{10404651649353497429418550644025}{140737488355328} x^4 + \frac{1191456000}{13440} x^3 \right) - 10404651649353497429418550644025 x^9 + 247588007857060549798248448 x^8 + 6189700196426901374497494795 x^7 + 70940867001374715316302641285 x^6 + 2951605477935728685 x^5 - 351843720888321 x^4 - 6086544010576107363 x^3 - 351843720888321 x^2 - 145600 x - 13440.$$ 

Similarly, we have $n_2(x) \geq n_3(x)$ where

$$n_3(x) = 1.27^3 \left( \frac{110404651649353497429418550644025}{247588007857060549798248448} x^9 + \frac{9695243582352122447439497494795}{247588007857060549798248448} x^8 + \frac{6189700196426901374497494795}{247588007857060549798248448} x^7 + \frac{70940867001374715316302641285}{247588007857060549798248448} x^6 + \frac{309480098921345068724781056}{247588007857060549798248448} x^5 + \frac{834924520509078426781024008335}{247588007857060549798248448} x^4 + \frac{77371252455336267181195264}{247588007857060549798248448} x^3 + \frac{924711257333410630048926774241}{247588007857060549798248448} x^2 + \frac{483503278458516698824704}{247588007857060549798248448} x + \frac{2187851639229258349412352}{247588007857060549798248448} \right) - 10404651649353497429418550644025 x^9 + 247588007857060549798248448 x^8 + 6189700196426901374497494795 x^7 + 70940867001374715316302641285 x^6 + 2951605477935728685 x^5 - 351843720888321 x^4 - 6086544010576107363 x^3 - 351843720888321 x^2 - 145600 x - 13440.$$ 

Similarly, we have $n_2(x) \geq n_3(x)$ where
\[
\begin{align*}
&= \frac{10656355779728833048985112408175}{1237940392853802748999124224} x^6 + \\
&+ \frac{19859572134949100226816117461335}{618970019642690137449562112} x^5 + \\
&+ \frac{2270530507949571081038231535}{483570327848516698824704} x^4 + \\
&+ \frac{17102451940952370969088397565}{77371252455336267181195264} x^3 - \\
&- \frac{2955971695165983842306493551217}{967140656917033397649408} x^2 + \\
&+ \frac{17905321055673993813657464422955}{19342813113836066795298816} x + \\
&- \frac{1407898285901328284374752202495}{19342813113836066795298816}.
\end{align*}
\]

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