Analytical Differential Calculus with Integration

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Abstract

Differential lambda-calculus was first introduced by Thomas Ehrhard and Laurent Regnier in 2003. Despite more than 15 years of history, little work has been done on a differential calculus with integration. In this paper, we shall propose a differential calculus with integration from a programming point of view. We show its good correspondence with mathematics, which is manifested by how we construct these reduction rules and how we preserve important mathematical theorems in our calculus. Moreover, we highlight applications of the calculus in incremental computation, automatic differentiation, and computation approximation.

2012 ACM Subject Classification Software and its engineering → General programming languages; Software and its engineering → Functional languages

Keywords and phrases Differential Calculus, Integration, Lambda Calculus, Incremental Computation, Adaptive Computing

1 Introduction

Differential calculus has more than 15 years of history in computer science since the pioneer work by Thomas Ehrhard and Laurent Regnier [9]. It is, however, not well-studied from the perspective of programming languages; we would expect the profound connection of differential calculus with important fields such as incremental computation, automatic differentiation and self-adjusting computation just like how mathematical analysis connects with mathematics. We want to understand what is the semantics of the derivative of a program and how we can use these derivatives to write a program. That is, we wish to have a clear description of derivatives and introduce integration to compute from operational derivatives to the program.

The two main lines of the related work are the differential lambda-calculus [9, 8] and the change theory [7, 4, 5]. On one hand, the differential lambda-calculus uses linear substitution to represent the derivative of a term. For example, given a term $x \cdot x$ (i.e., $x^2$), with the differential lambda-calculus, we may use the term $\partial x \cdot x \cdot 1$ to denote its derivative at 1. As there are two alternatives to substitute 1 for $x$ in the term $x \cdot x$, it gives $(1 \cdot x) + (x \cdot 1)$ (i.e., $2x$) as the derivative (where $+$ denotes 'choice').

Despite that the differential lambda-calculus provides a concise way to analyze the alternatives of linear substitution on a lambda term, there is a gap between analysis on terms and computation on terms. For instance, let $+$ denote our usual addition operator, and $+$ denote the choice of linear substitution. Then we have that $\frac{\partial x \cdot x}{\partial x} \cdot 1 = (1 + x) + (x + 1)$, which is far away from the expected $1 + 1$. Moreover, it offers no method to integrate over a derivative, say $\frac{\partial y}{\partial x} \cdot y$.

On the other hand, the change theory gives a systematic way to define and propagate...
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The main idea is to define the change of function $f$ as $\text{Derive } f$, satisfying

$$ f(x + \Delta x) = f(x) \oplus (\text{Derive } f) x \Delta x, $$

where $\oplus$ denotes an updating operation. It reads that the change over the input $x$ by $\Delta x$ results in the change over the result of $f(x)$ by $\Delta x$. While change theory provides a general way to describe changes, the changes it described are differences (deltas) instead of derivatives. It is worth noting that derivative is not the same as delta. For example, by change theory, we can deduce that $f(x)$ will be of the form of $x \ast x + C$ if we know $(\text{Derive } f) x \Delta x = 0$ or $\Delta x$, but we cannot deduce this form if we just know that its derivative is $2 \ast x$, because change theory has no concept of integration or limits.

Although a bunch of work has been done on derivatives [9, 8, 7, 4, 20, 17, 22, 10, 1], there is unfortunately, as far as we are aware, little work on integration. It may be natural to ask what a derivative really means if we cannot integrate it. If there is only a mapping from a term to its derivative without its corresponding integration, how can we operate on derivatives with a clear understanding of what we actually have done?

In this paper, we aim at a new differential framework, having dual mapping between derivatives and integrations. With this framework, we can manifest the power of this dual mapping by proving, among others, three important theorems, namely the Newton-Leibniz formula, the Chain Rule and the Taylor’s theorem.

Our key idea can be illustrated by a simple example. Suppose we have a function $f$ mapping from a term to its derivative without its corresponding integration, how can we operate on derivatives with a clear understanding of what we actually have done?

We prove three important theorems and highlight their practical application for incremental computation, automatic differentiation, and computation approximation. We prove three important theorems and highlight their practical application for incremental computation, automatic differentiation, and computation approximation. We prove the Taylor’s formula:

$$ f(x + \Delta x) - f(x) = A\Delta x + o(\Delta x), $$

where $A$ is a Jacobian matrix to represent its derivative, which satisfies the equation

$$ f(x + \Delta x) = f(x) + \sum_{k=0}^{\infty} \frac{1}{k!} (f^{(k)})(x_0) (t \oplus t_0)^k. $$

Different from that one of the differential lambda-calculus [9], this Taylor’s theorem manifests res-
In this section, we shall give a clear definition of our calculus with both derivatives and integration. We explain important insights in our design, and prove some useful properties and theorems that will be used later.

2.1 Syntax

Our calculus, as defined in Figure 1, is an extension of the simply-typed lambda calculus [21]. Besides the usual constant, variable, lambda abstraction, function application, and tuple, it introduces five new operations: addition $\oplus$, subtraction $\ominus$, multiplication $\ast$, derivative $\frac{\partial t}{\partial x}$, and integration $\int t \, dx$. The three binary operations, namely $\oplus$, $\ominus$, and $\ast$, are generalizations of those from our mathematics. Intuitively, $x \oplus \Delta$ is for updating $x$ with change $\Delta$, $\ominus$ for canceling updates, and $\ast$ for distributing updates. We build up terms from terms of base types (such as $\mathbb{R}$, $\mathbb{C}$), and on each base type we require these operations satisfy the following properties:

- The addition and multiplication are associative and commutative, i.e., $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, $a \oplus b = b \oplus a$, $(a \ast b) \ast c = a \ast (b \ast c)$, $a \ast b = b \ast a$.
- The addition and the subtraction are cancellable, i.e., $(a \oplus b) \ominus b = a$ and $(a \ominus b) \oplus b = a$.

Figure 1 Calculus Syntax

ults of computation instead of analysis on occurrence of terms. It can be used for approximation of a function computation. (Section 5)
The multiplication is distributive over addition, i.e., \( a * (b \oplus c) = a * b \oplus a * c \).

**Example 1 (Basic Operations on Real Numbers).** For real numbers \( r_1, r_2 \in \mathbb{R} \), we have the following definitions.

\[
\begin{align*}
    r_1 \oplus r_2 & = r_1 + r_2 \\
    r_1 \ominus r_2 & = r_1 - r_2 \\
    r_1 \odot r_2 & = r_1 r_2
\end{align*}
\]

We use \( \frac{\partial t_1}{\partial x} \bigg|_{t_2} \) to denote derivative of \( t_1 \) over \( x \) at point \( t_2 \), and \( \int_{t_1}^{t_2} t \, dx \) to denote integration of \( t \) over \( x \) from \( t_1 \) to \( t_2 \).

### 2.2 Typing

As defined in Figure 1, we have base types (denoted by \( B \)), tuple types, function types, and sum type. To make our later typing rules easy to understand, we introduce the following type notations.

\[
\text{Type } T^* ::= B \quad \text{base type} \\
| (T^*, T^*, \ldots, T^*) \quad \text{product type} \\
| T \to T^* \quad \text{arrow type}
\]

\( T^* \) means the types that are addable (i.e., updatable through \( \oplus \)). We view the addition between functions, tuples and base type terms as valid, which will be showed by our reduction rules later. But here, we forbid the addition and subtraction between sum types because we
view updates such as inl \(0 \oplus\) inr \(1\) as invalid. If we want to update the change to a term of sum types anyway, we may do case analysis such as case \(t\) of inl \(x_1 \Rightarrow\) inl \((x_1 \oplus ...)\) | inr \(x_2 \Rightarrow\) \((x_2 \oplus ...).\)

Next, we introduce two notations for derivatives on types:

\[
\frac{\partial T}{\partial B} = T, \\
\frac{\partial T}{\partial (T_1, T_2, ..., T_n)} = (\frac{\partial T}{\partial T_1}, \frac{\partial T}{\partial T_2}, ..., \frac{\partial T}{\partial T_n}).
\]

The first notation says that with the assumption that differences (subtraction) of values of base types are of base types, the derivative over base types has no effect on the result type. And, the second notation resembles partial differentiation. Note that we do not consider derivatives on functions because even for functions on real numbers, there is no good mathematical definition for them yet. Therefore, we do not have a type notation for \(\frac{\partial T}{\partial f:T_1 \rightarrow T_2}\). Besides, because we forbid the addition and subtraction between the sum types, we will view the differentiation of the sum types as invalid, so we do not have notations for \(\frac{\partial T}{\partial (T_1 + T_2)}\) either.

Figure 2 shows the typing rules for the calculus. The typing rules for constant, variable, lambda abstraction, function application, tuple, and projection are nothing special. The typing rules for addition and subtraction are natural, but the rest three kinds of rules are more interesting. Rule TMUL the typing rule for \(t_1 * t_2\). If \(t_1\) is a derivative of \(T_1\) over \(T_2\), and \(t_2\) is of type \(T_2\), then multiplication will produce a term of type \(T_1\). This may be informally understood from our familiar equation \(\Delta Y \Delta X = \Delta Y\). Rule TDER shows introduction of the derivative type through a derivative operation, while Rule TINT cancellation of the derivative type through an integration operation.

### 2.3 Semantics

We will give a two-stage semantics for the calculus. At the first stage, we assume that all the constants (values and functions) over the base types are interpretable in the sense there is a default well-defined interpreter to evaluate them. At the second stage, the important part of this paper, we define a set of reduction rules and use the full reduction strategy to compute their normal form, which enjoys good properties of soundness, confluence, and strong normalization.

More specifically, after the full reduction of a term in our calculus, every subterm (now in a normal form of interpretable types) outside the lambda function body will be interpretable on base types, which will be proved in the appendix. In other words, our calculus helps to reduce a term to a normal form which is interpretable on base types, and leave the remaining evaluations to interpretation on base types. We will not give reduction rules to the operations on base types because we do not want to touch on implementations of primitive functions on base types.

For simplicity, in this paper we will assume that the important properties such as the Newton-Leibniz formula, the Chain Rule, and the Taylor’s theorem, are satisfied by all the primitive functions and their closures through addition, subtraction, multiplication, derivative and integration. This assumption may seem too strong, since not all primitive functions on base types meet this assumption. However, it would make sense to start with the primitive functions meeting these requirements to build our system, and extend it later with other primitive functions.
2.4 Interpretable Types and Terms

Here, a term is interpretable means it can be directly interpreted by a base type interpreter. We use $B$ to denote the base type, over which its constants are interpretable. To make this clear, we define interpretable types as follows.

▶ **Definition 2 (Interpretable Type).** Let $B$ be base types. A type $iB$ is interpretable if it is generated by the following grammar:

\[
iB ::= \begin{align*}
  & B & \text{base type} \\
  & iB \to iB & \text{function type}
\end{align*}
\]

Constants of interpretable types can be both values or primitive functions of base types. For example, we can use $\sin(x)$, $\cos(x)$, $\text{square}(x)$ as primitive functions in our calculus.

Next, we consider terms that are constructed from constants and variables of interpretable types. These terms are interpretable by a default evaluator under an environment mapping variables to constants. Formally, we define the following interpretable terms.

▶ **Definition 3 (Interpretable Terms).** A term is an interpretable if it belongs to $it$.

\[
it ::= \begin{align*}
  & c & \text{constants of } iB \\
  & x & \text{variable of } iB \\
  & \lambda x : iB. it & \text{lambda abstraction} \\
  & it \ it & \text{function application} \\
  & it \oplus it & \text{addition} \\
  & it \odot it & \text{subtraction} \\
  & it \ast it & \text{multiplication} \\
  & \frac{\partial it}{\partial x} & \text{derivative} \\
  & \int it \ it \ dx & \text{integration}
\end{align*}
\]

2.5 Reduction Rules

Our calculus is an extension of simply-typed lambda-calculus. Our lambda abstraction and application are nothing different from the simply-typed lambda calculus, and we have the reduction rule:

\[(\lambda x : T. t)t_1 \to t[t_1/x].\]

We use an $n$-tuple to model structured data and projection $\pi_j$ to extract $j$-th component from a tuple, and we have the following reduction rule:

\[\pi_j(t_1, t_2, \ldots, t_n) \to t_j.\]

Similarly, we have reduction rules for the case analysis:

\[\text{case } (\text{inl } t) \text{ of } \text{inl } x_1 \Rightarrow t_1 | \text{inr } x_2 \Rightarrow t_2 \to t_1[t/x_1]\]

\[\text{case } (\text{inr } t) \text{ of } \text{inl } x_1 \Rightarrow t_1 | \text{inr } x_2 \Rightarrow t_2 \to t_2[t/x_2]\]

Besides, we introduce fix-point operator to deal with recursion:

\[\text{fix } f \to f (\text{fix } f)\]
It is worth noting that tuples, having a good correspondence in mathematics, should be understood as structured data instead of high-dimensional vectors because there are some operations that are different from those in mathematics. As will be seen later, there is difference between our multiplication and matrix multiplication, and derivative and integration operations that are different from those in mathematics. As will be seen later, there is difference between our multiplication and matrix multiplication, and derivative and integration operations that are different from those in mathematics. As will be seen later, there is difference between our multiplication and matrix multiplication, and derivative and integration operations that are different from those in mathematics. As will be seen later, there is difference between our multiplication and matrix multiplication, and derivative and integration operations that are different from those in mathematics. As will be seen later, there is difference between our multiplication and matrix multiplication, and derivative and integration operations that are different from those in mathematics.

The core reduction rules in our calculus are summarized in Figure 3, which define three basic cases for both reducing derivative terms and integration terms. For derivative, we use \( \frac{\partial}{\partial x} \mid_{t_0} \) to denote the derivative of \( t \) over \( x \) at point \( t_0 \), and we have four reduction rules:

- **Rule EAppDer1** is to distribute point \( t_0 : B \) into a tuple. This resembles the case in mathematics; if we have a function \( f \) defined by \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T \), its derivative will be \( \left( \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}, \ldots, \frac{\partial f_n}{\partial x} \right)^T \). For example, if we have a function \( f : \mathbb{R} \to (\mathbb{R}, \mathbb{R}) \) defined by \( f(x) = (x, x \times x) \), then its derivative will be \( (1, 2 \times x) \).

- **Rule EAppDer2** is similar to Rule EAppDer1.

- **Rule EAppDer3** is to distribute point \( t_0 : B \) into a lambda abstraction. Again this is very natural in mathematics. For example, for function \( f(x) = \lambda y : B . x \times y \), then we would have its derivative on \( x \) as \( \lambda y : B . y \).

- **Rule EAppDer4** is to deal with partial differentiation, similar to the Jacobian matrix in mathematics (as shown in the introduction). For example, if we have a function that maps a pair \( (x, y) \) to \( (x \times x, x \times y \oplus y) \), which may be written as \( \lambda z : (\mathbb{B}, \mathbb{B}) . (\pi_1 z \times \pi_1 z, (\pi_1 z \times \pi_2 z \oplus \pi_2 z)) \) then we would have its derivative \( \frac{\partial (f \mid_{x, y})}{\partial x} \mid_{(x, y)} \) as \( ((2 \times x, y), (0, x \oplus 1)) \).

Similarly, we can define four reduction rules for integration. Rules EAppInt1,EAppInt2 and EAppInt3 are simple. Rule EAppInt4 is worth more explanation. It is designed to...
\[ (t_{11}, ..., t_{1n}) \oplus (t_{21}, ..., t_{2n}) \rightarrow (t_{11} \oplus t_{21}, ..., t_{1n} \oplus t_{2n}) \] (EAPPADD1)

\[ (\lambda x : T. t_1) \oplus (\lambda y : T. t_2) \rightarrow \lambda x : T. (t_1 \oplus t_2[y/x]) \] (EAPPADD2)

\[ (t_{11}, ..., t_{1n}) \odot (t_{21}, ..., t_{2n}) \rightarrow (t_{11} \odot t_{21}, ..., t_{1n} \odot t_{2n}) \] (EAPPSUB1)

\[ (\lambda x : T. t_1) \odot (\lambda y : T. t_2) \rightarrow \lambda x : T. (t_1 \odot t_2[y/x]) \] (EAPPSUB2)

\[ t_0 : B \]
\[ \frac{\partial t}{\partial y} \bigg|_{x} dx = t[t_2/y] \odot t[t_1/y] \]

when \( t_1 \) and \( t_2 \) are tuples:

\[ \int_{(t_{11}, t_{12}, ..., t_{1n})}^{(t_{21}, t_{22}, ..., t_{2n})} \frac{\partial t}{\partial y} \bigg|_{x} dx = t[(t_{21}, t_{22}, ..., t_{2n})/y] \odot t[(t_{11}, t_{12}, ..., t_{1n})/y]. \]

So we design the rule to have

\[ \int_{t_{1j}}^{t_{2j}} \frac{\partial t[(t_{21}, ... t_{2(j-1)}, x_j, t_{(j+1)}, ..., t_{1n})/y]}{\partial x_j} dx_j = \int_{(t_{12}, ..., t_{1(j-1)}, t_{(j+1)}, ..., t_{1n})}^{(t_{22}, ..., t_{2(j-1)}, t_{(j+1)}, ..., t_{2n})} \frac{\partial t}{\partial y} \bigg|_{x_j} dx_j. \]

Notice that under our evaluation rules on derivative, \( \pi_j(\frac{\partial t}{\partial y} \bigg|_{x=x_j}) \) will be equal to the derivative of \( t \) to its \( j \)-th parameter \( x_j \), so the integration will lead us to the original \( t \).

Finally, we discuss the reduction rules for the three new binary operations, as summarized in Figure 4. The addition \( \oplus \) is introduced to support the reduction rule of integration. It is also useful in proving the theorem and constructing the formula. We can understand the two reduction rules for addition as the addition of high-dimension vectors and functions respectively. Similarly, we can have two reduction rules for subtraction \( \odot \). The operator \( \ast \) was introduced as a powerful tool for constructing the Chain Rule and the Taylor’s theorem. The first two reduction rules can be understood as multiplications of a scalar with a function and a high-dimension vector respectively, while the last one can be understood as the multiplication on matrix. For example, we have

\[ ((1, 4), (2, 5), (3, 6)) \ast (7, 8, 9) = (50, 122) \]

which corresponds to the following matrix multiplication.

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
7 & 8 & 9
\end{pmatrix}
= 
\begin{pmatrix}
50 & 122
\end{pmatrix}
\]
It is worth noting that while they are similar, * is different from the matrix multiplication operation. For example, we cannot write \( x \) as an \( m \)-dimensional vector (or \( m \times 1 \) matrix) in Taylor’s theorem because no matrix \( A \) is well-performed under \( A \times x \times x \), but we can write Taylor’s Theorem easily under our framework. In the matrix representation, the number of rows of the first matrix and the number of columns of the second matrix must be equal so that we can perform multiplication on them. This means, we can only write case \( m = 1 \)'s Taylor’s theorem in matrices, while our version can perform for any tuples.

### 2.6 Normal Forms

| Normal Form   | nf ::= | \( nb \) | normal form on \( iB \) |
|---------------|--------|----------|-------------------|
|               |        | (nf, nf, ..., nf) | tuple |
|               |        | \( \lambda x : T.t \) | function, \( t \) cannot be further reduced |
|               |        | \( \text{inl/inr} \ nf \) | injection |

| Normal Forms on \( iB \) | nb ::= | \( c \) | constants on \( iB \) |
|---------------------------|--------|----------|-------------------|
|                           |        | \( x \) | variables on \( iB \) |
|                           |        | \( nb \ nf \) | primitive function application |
|                           |        | \( \text{nf} \oplus \text{nf} \) \( \text{nf} \oplus \text{nf} \) | addition |
|                           |        | \( \text{nf} \oplus \text{nf} \) \( \text{nf} \oplus \text{nf} \) | subtraction |
|                           |        | \( \text{nf} \times \text{nf} \) | multiplication |
|                           |        | \( \frac{\partial}{\partial x} \text{nf} \) | derivative |
|                           |        | \( \int_{\text{nf}} \text{nf} \ dx \) | integration |

**Figure 5** Normal Forms

In our calculus, base type stands in a very special position, and we may involve many evaluations under the context of some free variables of an interpretable type. So for simplicity, we will use full reduction\[1\] but allow free variables of interpretable types (i.e., \( iB \)) in our normal form. Figure 5 defines our normal form. It basically consists of the normal forms on interpretable types, the tuple normal form, and the function normal form.

We have an interesting result about about the normal form of a term of an interpretable terms.

**Lemma 4** (Interpretability). All the normal forms of terms of interpretable types are interpretable terms. That is, given a term \( t : iB \), if \( t \) is in normal form, then \( t \) is an interpretable term.

**Proof.** We prove that a normal form \( t \) is interpretable by induction on the form of \( t \).

- Case \( \lambda x : T.t \). Because \( \lambda x : T.t \) is of type \( iB \), \( T \) must be of type \( iB \). Notice that the function body \( t \) has a free variable \( x \) of type \( iB \). By induction, we know \( t \) is an interpretable term, therefore, \( \lambda x : T.t \) is interpretable.

- Case \( (nf, \text{nf}, ..., \text{nf}) \). This case is impossible, because it is not of type \( iB \). Using the same technique, we can prove the cases for \( \text{inl/inr} \ nf \).

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\[1\] By full reduction, we mean that a term can be reduced wherever any of its subterms can be reduced by a reduction rule.
We need to talk a bit more on equality because we do not consider reduction or calculation where a normal form variations.

Lemma 7. ▶

2.8 Term Equality

Proof. Based on the equality of terms of base types, we can prove it by induction. ▶

2.7 Properties

Next, we prove some properties of our calculus. The proof is rather routine with some small variations.

Lemma 5 (Properties). This calculus has the properties of progress, preservation and confluence. Moreover, if a term \( t \) does not contain subterms \( \text{fix} \ t' \), then \( t \) is strong normalizable.

Proof. Full proof is in the Appendix [23, 25, 29, 45], which is adapted from the standard proof. ▶

2.8 Term Equality

We need to talk a bit more on equality because we do not consider reduction or calculation on primitive functions. This notion of equality has little to do with our evaluation but has a lot to do with the equality of primitive functions. Using this notion of equality, we can compute the result from completely different calculations. This will be used in our later proof of the three theorems.

Since we have proved the confluence property, we know that every term has at most one normal form after reduction. Thus, we can define our equality based on their normal forms; the equality between unnormalizable terms is undefined.

Definition 6 (Term Equality). An open term \( t_1 \) is said to be equal to a term \( t_2 \), if and only if for all free variables \( x_1, x_2, \ldots, x_n \) in \( t_1 \) and \( t_2 \), for all closed and weak-normalizable term \( u_i \) whose type is the same as that of \( x_i \), we have \( t_1[u_1/x_1, \ldots, u_n/x_n] = t_2[u_1/x_1, \ldots, u_n/x_n] \).

A closed-term \( t_1 = t_2 \), if their normal forms \( n_1 \) and \( n_2 \) have the relation that \( n_1 = n_2 \), where a normal form \( n_1 \) is said to be equal to another normal form \( n_2 \), if they satisfy one of the following rules:

- (1) \( n_1 \) is a of type \( iB \), then \( n_2 \) has to be of the same type, and under the base type interpretation, \( n_1 \) is equal to \( n_2 \);
- (2) \( n_1 \) is \( (t_1, t_2, \ldots, t_n) \), then \( n_2 \) has to be \( (t_1', t_2', \ldots, t_n') \), and \( \forall j \in [1, n], t_j \) is equal to \( t_j' \);
- (3) \( n_1 \) is \( \lambda x : T.t \), then \( n_2 \) has to be \( \lambda y : T.t' \) (\( y \) can be \( x \)), and \( n_1 \) \( x \) is equal to \( n_2 \) \( x \).
- (4) \( n_1 \) is \( \text{inl} \ t_1' \), then \( n_2 \) has to be \( \text{inl} \ t_2' \), and \( t_1' \) is equal to \( t_2' \).
- (5) \( n_1 \) is \( \text{inr} \ t_1' \), then \( n_2 \) has to be \( \text{inr} \ t_2' \), and \( t_1' \) is equal to \( t_2' \).

Lemma 7. The equality is reflexive, transitive and symmetric for weak-normalizable terms.

Proof. Based on the equality of terms of base types, we can prove it by induction. ▶
The first important theorem we will give is the Newton-Leibniz’s formula, which ensures we can not prove equality between them.

Example 14

Calculation example related to derivative and integration.

The following shows the calculation of how $f^{(2,3)}_{(0,0)} \frac{\partial (f(y))}{\partial y} |_{x} dx$ comes equal with $f y[(2,3)/y] \oplus f y[(0,0)/y]$

$$f^{(2,3)}_{(0,0)} \frac{\partial (f(y))}{\partial y} |_{x} dx = \left\{ \text{Rule EAppInt3} \right\} f^{0}_{0} f(x,0) dx_1 \oplus f^{3}_{0} f^{0}(x) dx_1$$

$$= \left\{ \text{Rule EAppDeq3} \right\} f^{0}_{0} f(x,0) dx_1 \oplus f^{3}_{0} f^{0}(x) dx_1$$

$$= \left\{ \text{Projection} \right\} f^{0}_{0} f(x,0) dx_1 \oplus f^{3}_{0} f^{0}(x) dx_1$$

$$= \left\{ \text{Function Application} \right\} f^{0}_{0} f(x,0) dx_1 \oplus f^{3}_{0} f^{0}(x) dx_1$$
Theorem 15 (Newton-Leibniz). Let $t$ contain no free occurrence of $x$, and both $\int_{t_1}^{t_2} \frac{\partial t}{\partial y} |_x dx$ and $t[t_2/y] \odot t[t_1/y]$ are well-typed and weak-normalizable. Then we have

$$\int_{t_1}^{t_2} \frac{\partial t}{\partial y} |_x dx = t[t_2/y] \odot t[t_1/y].$$

Proof. If $t_1$, $t_2$ or $t$ is not closed, then we need to prove $\forall u_1, ..., u_n$, we have

$$(\int_{t_1}^{t_2} \frac{\partial t}{\partial y} |_x dx)[u_1/x_1, ..., u_n/x_n] = (t[t_2/y] \odot t[t_1/y])[u_1/x_1, ..., u_n/x_n].$$

By freezing $u_1, ..., u_n$, we can apply the substitution $[u_1/x_1, ..., u_n/x_n]$ to make every term closed. So, for simplicity, we will assume $t$, $t_1$ and $t_2$ to be closed.

We prove this by induction on types.

Case: $t_1, t_2$ and $t$ are of base types. By the confluence lemma, we know there exists the normal form $t', t'_1$ and $t'_2$ of the term $t$, $t_1$ and $t_2$. Also, we know $\int_{t_1}^{t_2} \frac{\partial t}{\partial y} |_x dx = \int_{t'_1}^{t'_2} \frac{\partial t'}{\partial y} |_x dx$ and $t[t_2/y] \odot t[t_1/y] = t'[t'_2/y] \odot t'[t'_1/y]$. Since on base types we have $\int_{t'_1}^{t'_2} \frac{\partial t'}{\partial y} |_x dx = t'[t'_2/y] \odot t'[t'_1/y]$, we have $\int_{t_1}^{t_2} \frac{\partial t}{\partial y} |_x dx = t[t_2/y] \odot t[t_1/y]$.

Case: $t_1, t_2$ are of base types, $t$ is of type $(T_1, T_2, ..., T_n)$. By the confluence lemmas, there exist a normal form $(t'_{11}, t'_{12}, ..., t'_{1n})$ for $t$. Using Rules (EAppInt1) and (EAppDer1), we know

$$\int_{t_1}^{t_2} \frac{\partial t}{\partial y} |_x dx = \int_{t_1}^{t_2} \frac{\partial (t'_{11}, t'_{12}, ..., t'_{1n})}{\partial y} |_x dx$$

$$= \int_{t_1}^{t_2} \frac{\partial t'}{\partial y} |_x dx$$

$$= \int_{t_1}^{t_2} \frac{\partial t'}{\partial y} |_x dx = (t'[t_2/y] \odot t'[t_1/y])$$

On the other hand, we have

$$t[t_2/y] \odot t[t_1/y] = (t'_{11}[t_2/y], t'_{11}[t_1/y]) \odot ...$$
By induction, we have \( \forall j \in [1, n], \int_{t_1}^{t_2} (\frac{\partial t_j}{\partial y})_{x} |x| dx = t'_j[t_2/y] \cup t'_j[t_1/y], \) so we have proven the case.

- Case: \( t_1, t_2 \) are of base types, \( t \) is of type \( A \rightarrow B \). By Lemma \ref{lem:var} we can use \( \lambda z : A.t z \) (for simplicity, we use \( \lambda z : A.t' \) where \( t' = t_2/z \)) to substitute for \( t \), where \( z \) is a fresh variable. Now, we have for any \( u \),

\[
\int_{t_1}^{t_2} \left( \frac{\partial (\lambda z : A.t')}{\partial y} \right)_{x} |x| dx = \lambda z : A. (\int_{t_1}^{t_2} \left( \frac{\partial t'_j}{\partial y} \right)_{x} |x| dx) u
\]

and on the other hand, since \( z \) is free in \( t_1 \) and \( t_2 \), we have

\[
(\int t_2/y) \cup t_1/y) u = (\int (\lambda z : A.t'[t_1/y] \cup \lambda z : A.t'[t_2/y])) u
\]

By induction (on \( B \)), we know \( \int_{t_1}^{t_2} \left( \frac{\partial u[z]}{\partial y} \right)_{x} |x| dx = (t'[u/z])[t_2/y] \cup (t'[u/z])[t_1/y], \) thus we have proven the case.

- Case: \( t_1, t_2 \) are of type base types, \( t \) is of type \( T_1 + T_2 \). This case is impossible because the right-hand term is not well-typed.

- Case: \( t_1, t_2 \) are of type \( (T_1, T_2, \ldots, T_n) \), \( t \) is of any type \( T \). By using the confluence lemma, we know there exist the normal forms \((t'_1, t''_2, \ldots, t'_n)\) and \((t''_1, t''_2, \ldots, t''_n)\) for \( t_1 \) and \( t_2 \) respectively.

Applying Rules (EAppDer3) and (EAppInt3), we have

\[
\int_{t_1}^{t_2} \left( \frac{\partial \pi_j}{\partial y} \right)_{x} |x| dx = \int_{t_1}^{t_2} \pi_j (t'_1, t'_2, \ldots, t'_n, x_1, t'_1(x_1), \ldots, t'_n(x_1))/x) |x| dx_1 \oplus \cdots \oplus \int_{t_1}^{t_2} \pi_n (t''_1, t''_2, \ldots, x_n)/x) |x| dx_n
\]

Notice that there is no occurrence of \( x \) in \( t \), so we have

\[
\int_{t_1}^{t_2} \left( \frac{\partial \pi_j}{\partial y} \right)_{x} |x| dx_1 (t'_1, t'_2, \ldots, t'_2/j-1, x, t'_1(j+1), \ldots, t'_n)/x) |x| dx_j
\]

By induction (on the case where \( t_1, t_2 \) are of type \( T_j \), \( t \) is of type \( T \)), we have

\[
\int_{t_1}^{t_2} \left( \frac{\partial \pi_j(y)}{\partial x_j} \right)_{x_j} |x_j| dx_j
\]

Note that the last equation holds because \( x_j \) is a fresh variable and \( t \) has no occurrence of \( x_j \).
Now we have the following calculation.

\[
\begin{align*}
  f_{t_1}^{t_2} \frac{\partial f}{\partial t} \big|_x dx &= \{ \text{all the above} \} \\
  &\quad \left( (t([t'_1], t'_2, \ldots, t'_n)/y)) \oplus (t([t'_1], t'_2, \ldots, t'_n)/y)) \oplus \cdots \oplus \right. \\
  &\quad \left. (t([t'_1], t'_2, \ldots, t'_2)/y)) \oplus (t([t'_1], t'_2, \ldots, t'_1)/y)) \right) \\
  &= \{ \text{Lemma } 13 \} \\
  t[t_2/y] &\oplus t[t_1/y]
\end{align*}
\]

Thus we have proven the theorem.

\textbf{Application: Incremental Computation}

A direct application is incrementalization [15, 7, 11]. Given a function \( f(x) \), if the input \( x \) is changed by \( \Delta \), then we can obtain its incremental version of \( f(x) \) by \( f'(x, \Delta) \).

\[ f(x \oplus \Delta) = f(x) \oplus f'(x, \Delta) \]

where \( f' \) satisfies that

\[ f'(x, \Delta) = \int_x^{x \oplus \Delta} \frac{\partial f(x)}{\partial x} \big|_x dx. \]

\textbf{Example 16} (Averaging a Pair of Real numbers). As a simple example, consider the average of a pair of real numbers

\[ \text{average} :: (\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R} \]

\[ \text{average} = x. (\pi_1(x) + \pi_2(x))/2 \]

Suppose that we want to get an incremental computation of \( \text{average} \) at \( x = (x_1, x_2) \) when the first element \( x_1 \) is changed to \( x_1 + d \) while the second component \( x_2 \) is kept the same. The incremental computation is defined by

\[ \text{inc}(x, d) = \text{average}(x, (d, 0)) = \int_x^{x \oplus (d, 0)} \frac{\partial \text{average}(x)}{\partial x} \big|_x dx = \frac{d}{2} \]

which is efficient.

\section{Chain Rule}

The Chain Rule is another important theorem of the relation between function composition and derivatives. This Chain Rule in our calculus has many important applications in automatic differentiation and incremental computation. We first give an example to get some taste, before we give and prove the theorem.

\textbf{Example 17} (chain rule). Consider two functions \( f \) and \( g \) on real numbers, usually defined in mathematics as \( f(x, y) = (x + y, x \ast y, y) \) and \( g(x, y) = (x + y, y) \). In our calculus, they are defined as follows.

\[ f :: (\mathbb{R}, \mathbb{R}) \rightarrow (\mathbb{R}, \mathbb{R}, \mathbb{R}) \]

\[ f = \lambda x : (\mathbb{R}, \mathbb{R}). (\pi_1(x) \oplus \pi_2(x), \pi_1(x) \ast \pi_2(x), \pi_2(x)) \]

\[ g :: (\mathbb{R}, \mathbb{R}) \rightarrow (\mathbb{R}, \mathbb{R}) \]

\[ g = \lambda x : (\mathbb{R}, \mathbb{R}). (\pi_1(x) \oplus \pi_2(x), \pi_2(x)) \]
We demonstrate that for any \( r_1, r_2, r_3, r_4 \in \mathbb{R} \), we have
\[
\frac{\partial f(g(x))}{\partial x}\big|_{(r_3, r_4)} \ast (r_1, r_2) = \frac{\partial f(x)}{\partial x}\big|_{g^{-1}(r_3, r_4)} \ast (\frac{\partial g(x)}{\partial x}\big|_{(r_3, r_4)} \ast (r_1, r_2))
\]
by the following calculation. First, for the LHS, we have:
\[
\frac{\partial f(g(x))}{\partial x}\big|_{(r_3, r_4)} \ast (r_1, r_2) = \frac{\partial f(g(x))}{\partial x}\big|_{(r_3, r_4)} \ast (r_1, r_2)
\]
\[
= \{ \text{Rule EAPPDER3} \}
\frac{\partial f(g(x_1, r_3))}{\partial x_1}\big|_{r_3 \ast r_4} \ast (r_1 \oplus \frac{\partial f(g(x_1, r_3))}{\partial x_1}\big|_{r_3 \ast r_4} \ast (r_1 \ast r_2)
\]
\[
= \{ \text{Application} \}
\frac{\partial (x_1 \oplus (r_4 \cdot (r_3 \oplus r_4^1)) \cdot r_4)}{\partial x_1}\big|_{r_3 \ast r_1} \oplus \frac{\partial (r_3 \cdot (r_4 \cdot (r_3 \oplus r_4^1)) \cdot r_4)}{\partial x_1}\big|_{r_3 \ast r_2}
\]
\[
= \{ \text{Lemma 11} \}
(1, r_4, 0) \ast r_1 \oplus (2, r_3 \oplus 2 \ast r_4, 1) \ast r_2
\]
\[
= \{ \text{Rule EAPPMULT1 and Rule APPADD1} \}
(r_1 \oplus (2 \ast r_2), r_4 \ast r_1 \oplus ((r_3 \oplus (2 \ast r_4)) \ast r_2)
\]
Now, for the RHS, we calculate with the following two steps.
\[
\frac{\partial g(x)}{\partial x}\big|_{(r_3, r_4)} \ast (r_1, r_2) = \frac{\partial g(x)}{\partial x}\big|_{(r_3, r_4)} \ast (r_1, r_2)
\]
\[
= \{ \text{Rule EAPPDER3} \}
\frac{\partial g(x_1, r_3)}{\partial x_1}\big|_{r_3 \ast r_1} \oplus \frac{\partial g(x_1, r_3)}{\partial x_1}\big|_{r_3 \ast r_2}
\]
\[
= \{ \text{Application} \}
\frac{\partial (x_1 \oplus (r_4 \cdot (r_3 \oplus r_4^1)) \cdot r_4)}{\partial x_1}\big|_{r_3 \ast r_1} \oplus \frac{\partial (r_3 \cdot (r_4 \cdot (r_3 \oplus r_4^1)) \cdot r_4)}{\partial x_1}\big|_{r_3 \ast r_2}
\]
\[
= \{ \text{Lemma 11} \}
(1, 0) \ast r_1 \oplus (1, 1) \ast r_2
\]
\[
= \{ \text{Rule EAPPMULT1 and Rule APPADD1} \}
(r_1 \oplus r_2, r_2)
\]
\[
\frac{\partial f(g(x))}{\partial x}\big|_{(r_3, r_4)} \ast (r_1 \oplus r_2, r_2)
\]
\[
= \{ \text{Application and Rule EAPPDER3} \}
\frac{\partial f(g(x_1, r_3 \oplus r_4))}{\partial x_1}\big|_{r_3 \oplus r_4} \oplus \frac{\partial f(g(x_1, r_3 \oplus r_4))}{\partial x_1}\big|_{r_3 \oplus r_4} \ast (r_1 \oplus r_2, r_2)
\]
\[
= \{ \text{Application} \}
\frac{\partial (x_1 \oplus (r_4 \cdot (r_3 \oplus r_4^1)) \cdot r_4)}{\partial x_1}\big|_{r_3 \oplus r_4} \oplus \frac{\partial (r_3 \cdot (r_4 \cdot (r_3 \oplus r_4^1)) \cdot r_4)}{\partial x_1}\big|_{r_3 \oplus r_4} \ast (r_1 \oplus r_2, r_2)
\]
\[
= \{ \text{Lemma 11} \}
((1, r_4, 0), (1, (r_3 \oplus r_4, 1))) \ast (r_1 \oplus r_2, r_2)
\]
\[
= \{ \text{Rule EAPPMULT4, Rule APPADD1 and Lemma 11} \}
(r_1 \oplus (2 \ast r_2), r_4 \ast r_1 \oplus ((r_3 \oplus (2 \ast r_4)) \ast r_2)
\]
\[
\square
\]

---

**Theorem 18** (Chain Rule). Let \( f : T_1 \rightarrow T, g : T_2 \rightarrow T_1 \). If both \( \frac{\partial f(g(x))}{\partial x}\big|_{(g(t), t)} \ast (\frac{\partial g(x)}{\partial x}\big|_{(g(t), t)} \ast t) \) and \( \frac{\partial f(g(x))}{\partial x}\big|_{(g(t), t)} \ast (\frac{\partial g(x)}{\partial x}\big|_{(g(t), t)} \ast t) \) are well-typed and weak-normalizable. Then for any \( t, t_1 : T_2 \), we have
\[
\frac{\partial f(g(x))}{\partial x}\big|_{(g(t), t)} \ast t = \frac{\partial f(g(x))}{\partial x}\big|_{(g(t), t)} \ast (\frac{\partial g(x)}{\partial x}\big|_{(g(t), t)} \ast t).
\]

**Proof.** Like in the proof of Theorem 15 for simplicity, we assume that \( f, g, t \) and \( t_1 \) are closed. Furthermore, we assume that \( t \) and \( t_1 \) are in normal form. We prove this by induction on types.
Now we can calculate as follows:

\[
g z = (\pi_1(g z), \pi_2(g z), \ldots, \pi_n(g z))
\]

\[
= ((\lambda b' : B. \pi_1(g b')) z, (\lambda b' : B. \pi_2(g b')) z, \ldots, (\lambda b' : B. \pi_n(g b')) z)
\]

and for any \( j \), we notice that \( \pi_j(g b') \) has only one free variable of base type, so it can be reduced to a normal form, say \( E_j \), of base type. Let \( g_j \) be \( \lambda b' : B. E_j \), then we have

\[
g z = (g_1 z, g_2 z, \ldots, g_n z).
\]

Next, we deal with the term \( f \):

\[
f = \lambda a : T_1. (f a)
\]

\[
f = \lambda a : T_1. ((\lambda y_1 : B. \lambda y_2 : B, \ldots, \lambda y_n : B. (f (y_1, y_2, \ldots, y_n))) \pi_1(a) \pi_2(a) \ldots \pi_n(a))
\]

and we know that \( f (y_1, y_2, \ldots, y_n) \) only contains base type free variables, so it can be reduced to a base type normal form, say \( N \), so we have

\[
f = \lambda a : T. ((\lambda y_1 : B. \lambda y_2 : B, \ldots, \lambda y_n : B. N) \pi_1(a) \pi_2(a) \ldots \pi_n(a)).
\]

Now, we can calculate as follows:

\[
\frac{\partial f(x)}{\partial x} \bigg|_{t_1 \ast t} = \frac{\partial (\lambda a : T. (\lambda y_1 : B. \lambda y_2 : B, \ldots, \lambda y_n : B. N) \pi_1(a) \pi_2(a) \ldots \pi_n(a)) (g_1, g_2, \ldots, g_n)}{\partial x} \bigg|_{t_1 \ast t}
\]

\[
= \frac{\partial (\lambda y_1 : B. \lambda y_2 : B, \ldots, \lambda y_n : B. N) (g_1, g_2, \ldots, g_n)}{\partial x} \bigg|_{t_1 \ast t}
\]

\[
= \frac{\partial N(g_1, g_2, \ldots, g_n, t_1)}{\partial y} \bigg|_{g_1, t_1, \ldots, g_n, t_1} (g_1, t_1, \ldots, g_n, t_1)
\]

\[
= \left( \frac{\partial N}{\partial y_1} \right)_{g_1, t_1, y_1, g_2, \ldots, g_n, t_1} \bigg|_{g_1, t_1, \ldots, g_n, t_1} \left( \frac{\partial N}{\partial y_2} \right)_{g_1, t_1, y_2, g_2, \ldots, g_n, t_1} \bigg|_{g_1, t_1, \ldots, g_n, t_1} \left( \frac{\partial N}{\partial y_n} \right)_{g_1, t_1, y_n, g_2, \ldots, g_n, t_1} \bigg|_{g_1, t_1, \ldots, g_n, t_1}
\]

Notice that by the base type interpretation, \( f(g_1(x), g_2(x), \ldots, g_n(x)) = f'_1(g_1(x), g_2(x), \ldots, g_n(x)) + f'_2(g_1(x), g_2(x), \ldots, g_n(x)) + \ldots + f'_n(g_1(x), g_2(x), \ldots, g_n(x)) \) is \( g'_1(x) + f'_2(g_1(x), g_2(x), \ldots, g_n(x)) + g'_2(x) + \ldots + f'_n(g_1(x), g_2(x), \ldots, g_n(x)) + g'_n(x) \) where \( f'_i \) means the derivative of \( f \) to its \( j \)-th parameter, so we get the following and prove the case:

\[
\frac{\partial N(g_1, g_2, \ldots, g_n, t_1)}{\partial y} \bigg|_{g_1, t_1, \ldots, g_n, t_1} = \left( \frac{\partial N}{\partial y_1} \right)_{g_1, t_1, y_1, g_2, \ldots, g_n, t_1} \bigg|_{g_1, t_1, \ldots, g_n, t_1} \left( \frac{\partial N}{\partial y_2} \right)_{g_1, t_1, y_2, g_2, \ldots, g_n, t_1} \bigg|_{g_1, t_1, \ldots, g_n, t_1} \left( \frac{\partial N}{\partial y_n} \right)_{g_1, t_1, y_n, g_2, \ldots, g_n, t_1} \bigg|_{g_1, t_1, \ldots, g_n, t_1}
\]
Case $T_2$ is base type, $T_1$ is any type, $T$ is $A \rightarrow B$. We prove that for any $u$ of type $A$, we have $$(\partial (f \circ x) u |_{t_1} * t) = (\partial f u |_{y} |_{g(t_1)} * (\partial g z |_{z} |_{t_1} * t)) u.$$ First, let $f' = \lambda x : T_1. (f \circ x) u$, $g' = g$, then by induction we have

$$\frac{\partial f'(g' x)}{\partial x}|_{t_1} * t = \frac{\partial f'}{\partial y}|_{(g' t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)$$

that is, we have

$$\frac{\partial f (f x) u}{\partial x}|_{t_1} * t = \frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)$$

Then, we prove $$(\frac{\partial f (f x) u}{\partial x} |_{t_1} * t) u = (\frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)) u$$

Next, we prove $$(\frac{\partial f (f x) u}{\partial x} |_{t_1} * t) u = \frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)$$

For simplicity, we assume $T_1$ to be $(B, B, B, ..., B)$ of $n$-tuples (the technique below can be applied to any $T_1$ type which makes the term well-typed).

On one hand, by substituting $(g_1 z, g_2 z, ..., g_n z)$ for $g z$, we have

$$\frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)) u = \frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)$$

Since

$$f(g_1 t_1, g_2 t_1, ..., g_n t_1) = \lambda a : A. f(g_1 t_1, g_2 t_1, ..., g_n t_1) a$$

which will be denoted as $\lambda A. A^*_t$, we continue the calculation as follows.

$$\frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)) u = \frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)$$

On the other hand, we have

$$\frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)) u = \frac{\partial f u}{\partial y}|_{(g t_1)} * (\frac{\partial g z}{\partial z} |_{t_1} * t)$$

Therefore, we have proven the case.
The Chain Rule provides another way to compute the derivatives. There are many applications. Thus we have proven the theorem.

Application: Automatic Differentiation

Example 19

differentiation \[10\].

The rest of the proof is similar to that for the case when \( T = A \to B \). We have proved the case when \( T \) is base type, and we assume that \( T \) has type \((T_1, T_2, \ldots, T_n)\). Suppose the normal form of \( t_1 \) is \((t'_1, t'_2, \ldots, t'_{n_1})\) and the normal form of \( t \) is \((t'_1, t'_2, \ldots, t'_{2n})\), Then

\[
\frac{\partial (g(x))}{\partial y} \bigg|_{t_1} \ast t = \frac{\partial \sigma_j (f(w))}{\partial y} \bigg|_{(g \ast t_1) \ast (t_1 \ast t)}
\]

On the other hand, we can use Lemma 12 (i.e., \( t_1 \ast (t_2 \oplus t_3) = (t_1 \ast t_2) \oplus (t_1 \ast t_3) \)) to do the following calculation.

Now by induction using \( f' = f, g' = \lambda x : T_3: g((t'_{11}, t'_{12}, \ldots, t'_{1(j-1)}, x, t'_{1(j+1)}, \ldots, t'_{1n_1}), we have\)

\[
\frac{\partial (g(x'))}{\partial y} \bigg|_{t_1} \ast (g' \ast t'_j) = \frac{\partial f}{\partial x} \bigg|_{t_1} \ast (g' \ast t'_j)
\]

Therefore by Lemma 10 we have proven the case. Thus we have proven the theorem.
First of all, let \( t_1 \) and \( t_2 \) two pairs, then it is easy to prove that \( \frac{\partial (t_1, t_2)}{\partial x} |_{t_2} = \frac{\partial t_1}{\partial x} |_{t_2} \). Next, we can perform automatic differentiation on \( \text{magSqr} \) by the following calculation.

\[
\frac{\partial (\text{magSqr} x)}{\partial x} |_{(a, b) * t} = \frac{\partial \text{sqrt}(\text{magSqr}(\pi_x x))}{\partial t} |_{(a, b) * t} = \frac{\partial \text{sqrt} u}{\partial y} |_{\pi_1(a, b) * (\frac{\partial \pi_x x}{\partial n})} |_{(a, b) * t} + \frac{\partial (\text{sqrt} u)}{\partial y} |_{\pi_2(a, b) * (\frac{\partial \pi_x x}{\partial n})} |_{(a, b) * t} = 2 * a * ((1, 0) * t) + 2 * b * ((0, 1) * t)
\]

Now, because the theorem applies for any \( t \) of pair type, we use \((1, 0)\) and \((0, 1)\) to substitute for \( t \) respectively, and we will get \( \frac{\partial (\text{magSqr} x)}{\partial x} |_{(a, b)} = (2 * a, 2 * b) \), which means its derivative to \( a \) is \( 2 * a \) and its derivative to \( b \) is \( 2 * b \).

## 5 Taylor’s Theorem

In this section, we discuss Taylor’s Theorem, which is useful to give an approximation of a \( k \)-order differentiable function around a given point by a polynomial of degree \( k \). In programming, it is important and has many applications in approximation and incremental computation. We first give an example and then we prove the theorem.

First of all, we introduce some high-order notations.

\[
\begin{align*}
\frac{\partial^nt}{\partial x^n} |_{t_2} &= t_1^n & \frac{\partial^n t_1}{\partial x^n} |_{t_2} &= \frac{\partial^2 a_{n-1} t_1}{\partial x^2} |_{t_2} \\
t * t_1^n &= t & t * t_1^n &= (t * t_1) * t_1^{n-1} \\
f^0 &= f & f^n &= \langle f^n \rangle^{n-1} \\
(\lambda x : T. t)' &= \lambda x : T. \frac{\partial t}{\partial x}
\end{align*}
\]

▶ Example 20 (Taylor). Consider a function \( f \) on real numbers, usually defined in mathematics as \( f(x, y) = (2 * x + y, 3 * x + x + y) \). In our calculus, it is defined as follows.

\[
f :: (\mathbb{R}, \mathbb{R}) \rightarrow (\mathbb{R}, \mathbb{R}) \\
f = \lambda x : (\mathbb{R}, \mathbb{R}).(2 * \pi_1(x) * \pi_2(x), 3 * \pi_1(x) * \pi_1(x) + \pi_2(x))
\]

The following expand the Taylor’s theorem up to 2-order derivative.

\[
\begin{align*}
f(C_1, C_2) &= (2 * C_1 * C_2, 3 * C_1 * C_1 + C_1) \\
f(0, 0) &= (0, 0) \\
f' (0, 0) * (C_1, C_2) &= \left\{ \text{Application} \right\} \frac{\partial (2 * \pi_1(x) * \pi_2(x), 3 * \pi_1(x) * \pi_1(x) + \pi_2(x))}{\partial x} |_{(0, 0)} * (C_1, C_2) \\
&= \left\{ \text{Rule EAPPDEK3} \right\} \left( \frac{\partial (2 * \pi_1(x) * \pi_2(x), 3 * \pi_1(x) * \pi_1(x) + \pi_2(x))}{\partial x} \right) |_{(0, 0)} * (C_1, C_2) \\
&= \left\{ \text{Lemma 11} \right\} \left( (0, 0), (0, 1) \right) * (C_1, C_2) \\
&= \left\{ \text{Rule EAPP_MUL, Rule EAPP_ADD1} \right\} \left(0, C_2\right)
\end{align*}
\]
\[
\frac{f''(0,0)}{(0,0)} = \left\{ \text{Application} \right\} \frac{\partial}{\partial x_2} \left( \frac{\partial^{(2+\pi_1(x)+\pi_2(x))}}{\partial x_2} \right) |_{(0,0)}
\]

\[
= \left\{ \text{Rule EAppDER3} \right\} \frac{\partial^{(2+\pi_1(x)+\pi_2(x))}}{\partial x_2} |_{(0,0)} \frac{\partial^{(2+\pi_1(x)+\pi_2(x))}}{\partial x_2} |_{(0,0)}
\]

\[
= \left\{ \text{Rule EAppDER3} \right\} \frac{\partial^{(2+\pi_1(x)+\pi_2(x))}}{\partial x_2} |_{(0,0)} \frac{\partial^{(2+\pi_1(x)+\pi_2(x))}}{\partial x_2} |_{(0,0)}
\]

\[
= \left\{ \text{Lemma 11} \right\} \frac{\partial^{(2+\pi_1(x)+\pi_2(x))}}{\partial x_2} |_{(0,0)} \frac{\partial^{(2+\pi_1(x)+\pi_2(x))}}{\partial x_2} |_{(0,0)}
\]

\[
(f''(0,0)) \cdot (C_1, C_2)^2 = \left\{ \text{Rule EAppMUL4, Rule EAppADD1} \right\} \frac{(2 \ast C_2, 6 \ast C_1), (2 \ast C_1, 0)}{(2 \ast C_1, 0)} \ast (C_1, C_2)
\]

\[
= \left\{ \text{Rule EAppMUL4} \right\} \frac{(2 \ast C_2 \ast C_1, 6 \ast C_1 \ast C_1) \oplus (2 \ast C_1 \ast C_2, 0)}{(2 \ast C_1 \ast C_2, 0)} \ast (C_1, C_2)
\]

Thus we have \( f(C_1, C_2) = (2 \ast C_1 \ast C_2, 3 \ast C_1 \ast C_1 \oplus C_2, 3 \ast C_1 \ast C_1) = f(0,0) \oplus \frac{1}{2}(f'(0,0)) \ast (C_1, C_2) \oplus \frac{1}{2}((f''(0,0)) \ast (C_1, C_2)) \ast (C_1, C_2)
\)

**Theorem 21** (Taylor’s Theorem). If both \( f \) and \( \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!} \ast (t \ominus t_0)^k \) are weak-normalizable, then

\[
f = \sum_{k=0}^{\infty} \frac{1}{k!}(f^{(k)}(t_0)) \ast (t \ominus t_0)^k.
\]

**Proof.** Like in the proof of Theorem 15 for simplicity, we assume that \( f, g, t \) and \( t_1 \) are closed. Furthermore, we assume that \( t \) and \( t_1 \) are in normal form. We prove it by induction on the type of \( f : T \to T' \).

**Case** \( T' \) is a base type. \( T \) must contain no \( \rightarrow \) by our typing, so for simplicity, we suppose \( T \) to be \((B, B, ..., B)\). Using the same technique in Theorem 18 we assume \( f \) to be

\[
f = \lambda x : T. (\lambda x_1 : B. \lambda x_2 : B. ... \lambda x_n : B. N) \pi_1(x) \pi_2(x) ... \pi_n(x)
\]

(denoted by \( f = \lambda \alpha : T. t_2 \) later), \( t \) to be \((t_{11}, t_{12}, ..., t_{1n})\), and \( t_0 \) to be \((t_{21}, t_{22}, ..., t_{2n})\), where each \( t_{ij} \) is a normal form of base type. Then we have

\[
(f^{(n)}(t_0)) \ast (t \ominus t_0)^n
\]

\[
= \frac{\partial f^{(n)}}{\partial x_2} |_{(0,0)} \ast (t \ominus t_0)^n
\]

\[
= \left\{ \text{Rule EAppADD1} \right\} \frac{(t_{21} \ominus t_{11}) \ast (t_{21} \ominus t_{22}) ... \ominus t_{2n}}{(t_{21} \ominus t_{11}) \ominus (t_{21} \ominus t_{22}) ... \ominus t_{2n}} \ast (t \ominus t_0)^n
\]

\[
= \left\{ \text{Rule EAppADD1} \right\} \frac{(t_{21} \ominus t_{11}) \ominus (t_{21} \ominus t_{22}) ... \ominus t_{2n}}{(t_{21} \ominus t_{11}) \ominus (t_{21} \ominus t_{22}) ... \ominus t_{2n}} \ast (t \ominus t_0)^n \ominus (t \ominus t_0)^{n-1}
\]

\[
= \left\{ \text{Rule EAppADD1} \right\} \frac{(t_{21} \ominus t_{11}) \ominus (t_{21} \ominus t_{22}) ... \ominus t_{2n}}{(t_{21} \ominus t_{11}) \ominus (t_{21} \ominus t_{22}) ... \ominus t_{2n}} \ast (t \ominus t_0) \ominus (t \ominus t_0)^{n-1}
\]

\[
= \left\{ \text{Rule EAppADD1} \right\} \frac{(t_{21} \ominus t_{11}) \ominus (t_{21} \ominus t_{22}) ... \ominus t_{2n}}{(t_{21} \ominus t_{11}) \ominus (t_{21} \ominus t_{22}) ... \ominus t_{2n}} \ast (t \ominus t_0) \ominus (t \ominus t_0)^{n-1}
\]
want to know where the point will be if we slightly change the radius

Taylor’s Theorem has many applications. Here we give an example of using Taylor’s Theorem

Thus we have proven the theorem.

Appication: Polynomial Approximation

enables us to make a fast polynomial approximation.

As seen in the above, every time we decompose a $\frac{\partial}{\partial x_j}|_{(t_1, t_2, \ldots, t_n)}$, apply Rule EAppDer1, and then make reduction with Rule EAppMul3 to lower down the exponent of $(t \odot t_0)^n$. Finally, we will decompose the last derivative and get the term $t_2$ in the form of $t_2 |_{(t_1' / x_1, t_2' / x_2, \ldots, t_n' / x_n)}$ where $\forall j \in [1, n], t_j'$ is either $t_j$ or $x_j$.

Note that on base type we assume that we have Taylor’s Theorem:

$$f(x_0 + h) = f(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} \right)^k f(x_0)$$

where $x_0$ and $h$ is an $n$-dimensional vector, and $x_j, h_j$ is its projection to its $j$-th dimension.

So we have $(f^{(k)} t_0) \odot (t \odot t_0)^k$ corresponds to the $k$-th addend $\frac{1}{k!} \left( \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} \right)^k f(x_0)$.

- Case: $T'$ is function type $A \rightarrow B$. Similar to the proof in Theorem [18] for all $u$ of type $A$, we define $f^* = \lambda x : T. f x u$, and by using the inductive result on type $B$, we can prove the case similarly as that in Theorem [18]

- Case: $T'$ is a tuple type $(T_1, T_2, T_3, \ldots)$. Just define $f^* = \lambda x : T. \pi_j(f x)$ to use inductive result. The rest is simple.

- Case: $T'$ is a tuple type $T_1 + T_2$. This case is impossible because the right hand is not well-typed.

Thus we have proven the theorem. ▼

**Application: Polynomial Approximation**

Taylor’s Theorem has many applications. Here we give an example of using Taylor’s Theorem for approximation. Suppose there is a point $(1, 0)$ in the polar coordinate system, and we want to know where the point will be if we slightly change the radius $r$ and the angle $\theta$. Since it is extremely costive to compute functions such as $\sin()$ and $\cos()$, Taylor’s Theorem enables us to make a fast polynomial approximation.
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Example 22. Let function polar2catesian be defined by

\[
polar2catesian :: (\mathbb{R}, \mathbb{R}) \rightarrow (\mathbb{R}, \mathbb{R})
polar2catesian(r, \theta) = (r \ast \cos(\theta), r \ast \sin(\theta))
\]

We show how to expand \( polar2catesian(r, \theta) \) at \((1, 0)\) up to 2nd-order derivative. Since

\[
\frac{\partial (polar2catesian(x))}{\partial x} \bigg|_{(1,0)} = \frac{\partial (r \ast \cos(\theta) \ast x \ast \sin(\pi x))}{\partial x} \bigg|_{(1,0)} = \left( \frac{\partial (r \ast \cos(0) \ast x \ast \sin(0))}{\partial x} \bigg|_{1}, \frac{\partial (1 \ast \cos(x) \ast 1 \ast \sin(\pi))}{\partial x} \bigg|_{0} \right) = ((1, 0), (0, 1))
\]

we have

\[
\frac{\partial (polar2catesian(x))}{\partial x} \bigg|_{(1,0)} \ast (\Delta r, \Delta \theta) = (\Delta r, \Delta \theta).
\]

Again, we have

\[
\frac{1}{2} \frac{\partial^2 (polar2catesian(x))}{\partial x^2} \bigg|_{(1,0)} \ast (\Delta r, \Delta \theta)^2 = ((0, 0), (0, 1)), ((0, 1), (-1, 0))) \ast (\Delta r, \Delta \theta)^2 = (-\frac{1}{2} \Delta \theta^2, \Delta r \ast \Delta \theta)
\]

Combining the above, we can use \((\oplus \Delta r \oplus \frac{1}{2} \Delta \theta^2, \Delta \theta \oplus \Delta r \ast \Delta \theta)\) to make an approximation to \( polar2catesian(1 \ast \Delta r, \Delta \theta) \).

6  Discussion

In this section, we make remarks on generality of our approach, and on how to deal with discrete derivatives in our context.

6.1  More Theorems and Applications

We keep many mathematical structures in our calculus. As a result, we can prove more theorems under this framework. We select the most important three, but there are many other theorems that hold in our system:

\[
\begin{align*}
&= (t_1 \oplus t_2) \oplus (t_2 \ominus t_3) \oplus \ldots \oplus (t_{n-1} \ominus t_n) = t_1 \ominus t_n, \\
&= \frac{\partial t_1 \oplus t_2}{\partial x} \bigg|_{t_3} = \frac{\partial t_1}{\partial x} \bigg|_{t_3} \oplus \frac{\partial t_2}{\partial x} \bigg|_{t_3}, \\
x \in B, &\frac{\partial t_1 \ast x}{\partial x} \bigg|_{t_3} = \frac{\partial t_1}{\partial x} \bigg|_{t_3} \ast t_3 \ast t_2 \ast [t_3/x] \ominus t_1 \ominus [t_3/x] \ast \frac{\partial t_3}{\partial x} \bigg|_{t_3}, \\
&\forall t_1, \text{if } t_1 \text{ contains no free } x, \frac{\partial t_1 \ast x}{\partial x} \bigg|_{t_2} = t_1.
\end{align*}
\]

Associated with each of these theorems is a bunch of applications. For lack of space, we only discuss three theorems in detail.

Now it is natural to ask whether all the theorems on base types have correspondence in our system. The answer is that it depends on the mathematical structure of the base types. In our proof, we assume the commutative law and associative law of addition and multiplication, and the distributive law of multiplication. We can construct a counterexample under this case. Suppose there is a strange law on a base type that \( \forall x, y, x \ast y = y \), which is interpreted by our system as \( t_1 \ast t_2 = t_2 \). Now let \( t_1 \) be \( (r_1, r_2), (r_3, r_4) \), and \( t_2 \) be \( (r_5, r_6) \). Then

\[
\begin{align*}
t_1 \ast t_2 &= \ (r_1 \ast r_5 \ominus r_3 \ast r_6, r_2 \ast r_5 \ominus r_4 \ast r_6) \\
&= \ (r_5 \ominus r_6, r_5 \ominus r_6)
\end{align*}
\]

which does not equal to \( t_2 \). This means that our system does not preserve this strange law.
Normal Form\[\text{nf} ::= \text{nb} \mid \lambda x : T.t\]

| \text{Interpretable Normal Forms} \text{nb} ::= \text{c} | \text{constants of interpretable type} |
| \text{nb \ nf} | \text{primitive functions application} |
| \text{nb \oplus \ nf} | \text{nf \oplus \ nb} | \text{addition on interpretable type} |
| \text{nb \ominus \ nf} | \text{nf \ominus \ nb} | \text{subtraction on interpretable type} |
| \text{x} | \text{interpretable types variables} |
| \frac{\Delta \text{nb}}{\Delta x} | \Delta x, \text{nf}\,\text{nf} | \text{derivative on base type} |

**Figure 6** Discrete normal form

In our design of the calculus, we touch little on details of base types. So for some strange base types, we may not be able to preserve its mathematical structure. But as for the widely used \( \mathbb{R} \) and \( \mathbb{C} \), our system preserves most of their important theorems.

It is interesting to note that it is impossible to prove these theorems using the theory of change [7], because the theory of change does not tell difference between smooth functions and non-continuous functions and use the same calculation for them. In our calculus, we distribute these calculation to base types step by step, and use these calculation (such as on base types, we have \( f_{a_2} \frac{\partial f}{\partial y} |_{x} = f(a_2 \oplus f(a_1)) \)) to prove our theorems.

### 6.2 Discrete Derivatives

We can define discrete version of our calculus, where we represent changes as discrete deltas instead of through derivatives and integrations. We will show the equivalence between our discrete version and change theory [7] by implementing function \( \text{Derive} \) in our calculus.

The normal form this time is defined in Figure 6. We use the term \( \frac{\Delta \text{t}}{\Delta x} |_{t, t} \) to represent discrete derivative. This time we can easily manipulate values of base types because we only require the operator \( \oplus \) and \( \ominus \) to be well-defined. Also notice that this time we can implement derivatives on function type.

To show that our discrete version can be used to implement the change theory [7] it is sufficient to consider terms of base types or function types, without need to consider tuples and the operator * and \( \int \). We want to use our calculus to implement function \( \text{Derive} \) which satisfies the equation \((\text{Derive } f)\ x \Delta x = f(x \oplus \Delta x) \ominus f(x)\).

For interpretation of derivatives on base types, we just require they satisfy \( \frac{\Delta f}{\Delta y} |_{t_1, t_2} = \Delta f y |_{t_1, t_2} = f(t_1 \ominus t_2/y) \ominus t_1/y \). Then similarly to Newton-Leibniz Theorem we can prove \( \frac{\Delta f(y)}{\Delta y} |_{x, \Delta x} = f(x \oplus \Delta x) \ominus f(x) \) (where \( f \) does not contain free \( y \)), which is our version of function \( \text{Derive} \).

To see this clear, in change theory, we write function \( \text{Derive} \) and the system will automatically calculate it by using the rules:

\[
\begin{align*}
\text{Derive } c & = 0 \\
\text{Derive } x & = \Delta x \\
\text{Derive}(\lambda x : T.t) & = \lambda x : T. \lambda dx : \Delta T. \text{Derive}(t) \\
\text{Derive}(s\ t) & = \text{Derive}(s) \ t \text{ Derive}(t)
\end{align*}
\]

In our calculus, one writes \( \Delta f y |_{x, \Delta x} \), and the system will automatically calculate the
Analytical Differential Calculus with Integration

following rules:

\[
\begin{align*}
\frac{\Delta y}{\Delta x} |_{x, \Delta x} &= 0 \\
\frac{\Delta^2 y}{\Delta x^2} |_{x, \Delta x} &= \Delta x \\
\frac{\Delta y}{\Delta x} |_{t_0, t_1} &= \lambda y : T.(\frac{\Delta}{\Delta x}|_{t_0, t_1}) \\
(\lambda x. \Delta x, \frac{\Delta y}{\Delta x} |_{x, \Delta x}) t_1 t_2 &= \lambda y : T. t (t_1 \oplus t_2) \ominus \lambda y : T. t t_1
\end{align*}
\]

Notice that the first three rules have good correspondence, while the last one is a bit different. This is because in change theory’s definition, we have \(\Delta(A \rightarrow B) = A \rightarrow \Delta A \rightarrow \Delta B\), while in our calculus, we have \(\Delta(A \rightarrow B) = A \rightarrow \Delta B\). We, fortunately, can achieve the same effect through Newton-Leibniz Formula.

7 Related Work

Differential Calculus and The Change Theory

The differential lambda-calculus [9, 8] has been studied for computing derivatives of arbitrary higher-order programs. In the differential lambda-calculus, derivatives are guaranteed to be linear in its argument, where the incremental lambda-calculus does not have this restriction. Instead, it requires that the function should be differentiable. The big difference between our calculus and differential lambda calculus is that we perform computation on terms instead of analysis on terms.

The idea of performing incremental computation using derivatives has been studied by Cai et al. [7], who give an account using change structures. They use this to provide a framework for incrementally evaluating lambda calculus programs. It is shown that the work can be enriched with recursion and fix-point computation [4]. The main difference between our work and change theory is that we describe changes as mathematical derivatives while the change theory describe changes as (discrete) deltas.

Incremental/Self-Adaptive Computation

Paige and Koenig [20] present derivatives for a first-order language with a fixed set of primitives for incremental computation. Blakeley et al. [17] apply these ideas to a class of relational queries. Koch [15] guarantees asymptotic speedups with a compositional query transformation and delivers huge speedups in realistic benchmarks, though still for a first-order database language. We have proved Taylor’s theorem in our framework, which provides us with another way to perform finite difference on the computation.

Self-adjusting computation [2] or adaptive function programming [3] provides a dynamic approach to incrementalization. In this approach, programs execute on the original input in an enhanced runtime environment that tracks the dependencies between values in a dynamic dependence graph; intermediate results are memoized. Later, changes to the input propagate through dependency graphs from changed inputs to results, updating both intermediate and final results; this processing is often more efficient than recomputation. Mathematically, self-adjusting computations corresponds to differential equations (The derivative of a function can be represented by the computational result of function), which may be a future work of our calculus.

Automatic Differentiation

Automatic differentiation [13] is a technique that allows for efficiently computing the derivative of arbitrary programs, and can be applied to probabilistic modeling [16] and machine learning [6]. This technique has been successfully applied to some higher-order languages [22, 10]. As pointed out in [4], while some approaches have been suggested [19, 14], a general theoretical framework for this technique is still a matter of open research. We prove the chain rule inside our framework, which lays a foundation for our
calculus to perform automatic differentiation. And with more theorems in our calculus, we expect more profound applications in differential calculus.

## Conclusion

In this paper, we propose an analytical differential calculus which is equipped with integration. This calculus, as far as we are aware, is the first one that has well-defined integration, which has not appeared in both differential lambda calculus and the change theory. Our calculus enjoys many nice properties such as soundness and strong normalizing (when $\text{fix}$ is excluded), and has three important theorems, which have profound applications in computer science. We believe the following directions will be important in our future work.

- **Adding more theorems.** We may wish to write programs on many specialized base types besides $\mathbb{R}$ and $\mathbb{C}$. As we have demonstrated in this paper, our calculus preserves many important computational structures on base types. Therefore, it is possible to extend our system with theorems having one unique mathematical structures and use these theorems to optimize computation.

- **Working on Derivatives on functions.** We did not talk about derivatives on continuous functions because we have not had a good mathematical definition for them from perspective of computation. But derivatives on functions would be useful; it would be nice if we could use $\int_0^1 \frac{d(a_1,a_2)}{dx} |_x dx$ to compute $a_1 \ast a_2 \oplus (a_1 \oplus a_2)$.

- **Manipulating differential equations.** Differential equations would be very useful for users to program dynamic systems directly; one may write differential equations on data structures without writing the primitive forms of functions. It could be applied in many fields such as self-adjusting computation or self-adaptive system construction.

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A Appendix-Calculus Property

A.1 Progress

Lemma 23 (Progress). Suppose t is a well-typed term (Allow free variables of interpretable type iB), then t is either a normal form or there is some t’ such that t→t’.

Proof. We prove this by induction on form of t.

Case c.
It is a normal form.

Case t_1 ⊕ t_2.

It is well-typed if and only if t_1 has the same type T with t_2. If t_1 or t_2 is not a normal form, we make reductions on t_1 or t_2. If either t_1 or t_2 is nb, then t_1 ⊕ t_2 is a nb. For other cases of normal forms, we have

\[(t_{11}, t_{12}, ..., t_{1n}) ⊕ (t_{21}, t_{22}, ..., t_{2n}) \rightarrow (t_{11} ⊕ t_{21}, t_{12} ⊕ t_{22}, ..., t_{1n} ⊕ t_{2n})\]
\[(\lambda x : T.t_1) ⊕ (\lambda y : T.t_2) \rightarrow \lambda x : T.t_1 ⊕ (t_2[x/y])\]

Case t_1 ⊖ t_2.

It is the same case with the t_1 ⊕ t_2.

Case x.
Then x is an interpretable type free variable, otherwise it is not well-typed. An interpretable type free variable is a normal form.

Case inl/inr t.
If t is not a normal form, then we can make reduction in t, else this term itself is a normal form.

Case case t of inl x_1 ⇒ t_1 | inr x_2 ⇒ t_2.
To be well-typed t has to be the type of T_1 + T_2. If t is not a normal form, then we can make reduction in t, else t has to be inl/inr t’. So we can make reduction to t_1[t'/x_1] or t_2[t'/x_2]

Case λx : T.t.
It is a normal form if t can’t be further reduced.

Case t_1 t_2.
If t_1 is not a normal form then we make reductions on t_1.
If t_1 is a normal form, then t_1 has to be λx : T.t, or nb. For the former case we have

\[(\lambda x : T.t)t_1 \rightarrow t[t_1/x]\]

For the latter case, t_2 must be a nf, or it can make further reductions. So t_1 t_2 is a nb.

Case \int_{t_1}^{t_2} t dx.
If t_1 or t_2 is not a normal form then we can make reductions on t_1 or t_2.
If both t_1 and t_2 are normal forms, then t_1 and t_2 have to be (nf,nf,...,nf) or base type to be well-typed. If it is the former case.

\[\int_{(t_{11}, t_{12}, ..., t_{1n})}^{(t_{21}, t_{22}, ..., t_{2n})} t dx \rightarrow \int_{t_{11}}^{t_{21}} \pi_1(t[(x_1, t_{12}, ..., t_{1n})/x])dx_1 \oplus \int_{t_{12}}^{t_{22}} \pi_2(t[(x_1, t_{22}, ..., t_{1n})/x])dx_2 \oplus \dots \int_{t_{1n}}^{t_{2n}} \pi_n(t[(x_1, t_{2n}, ..., t_{1n})/x])dx_n\]
If it is the latter case, let us inspect $t_3$. If $t_3$ is not a normal form, then we can make reductions on $t_3$ (notice that we only introduce a base type free variable into $t_3$).

If $t_3$ is a normal form, and if $t_1, t_2$ and $t_3$ are $\text{nb}$, then $\int_{t_1}^{t_2} t_3 dx$ is a normal form. For other cases of normal forms:

$$ t_1, t_2 : B \\
\int_{t_1}^{t_2} (t_{11}, t_{12}, ..., t_{1n}) dx \rightarrow (\int_{t_1}^{t_2} t_{11} dx, \int_{t_1}^{t_2} t_{12} dx, ..., \int_{t_1}^{t_2} t_{1n} dx) $$

$$ t_1, t_2 : B \\
\int_{t_1}^{t_2} \lambda y : T_2 t dx \rightarrow \lambda y : T_2 \int_{t_1}^{t_2} t dx $$

$$ t_1, t_2 : B \\
\int_{t_1}^{t_2} \text{inl/inr } t dx \rightarrow \text{inl/inr } \int_{t_1}^{t_2} t dx $$

- Case $(t_1, t_2, ..., t_n)$.
  If $t_1$ is not a normal form, then we make reductions on $t_1$. If all the $t_i$ are normal forms, then $t$ is a normal form.

- Case $\pi_j(t_1)$.
  If $t_1$ is not a normal form, then we make reductions on $t_1$.
  If $t_1$ is a normal form, then it has to be $(\text{nf}, \text{nf}, ..., \text{nf})$ to be well-typed, then we have

$$ \pi_j(t_1', t_2', ..., t_n') \rightarrow t_j' $$

- Case $\frac{\partial t}{\partial x}|_{t_2}$.
  If $t_2$ is not a normal form, then we make reductions on $t_2$.
  If $t_2$ is a normal form, then it has to be $(t_1, ..., t_n)$ or an $\text{nb}$. If it is the form case:

$$ \forall i, (t_1, t_2, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n) \text{is written as } t_i $$

$$ \frac{\partial t}{\partial x}|_{t_1, t_2, ..., t_n} \rightarrow (\frac{\partial t_{i+1}}{\partial x}|_{t_1}, \frac{\partial t_{i+2}}{\partial x}|_{t_1}, ..., \frac{\partial t_n}{\partial x}|_{t_1}) $$

If it is the latter case, if $t_1$ is not a normal form, then we can make reductions on $t_1$ (notice that we only introduce a base type free variable into $t_1$). If $t_1$ is a $\text{nb}$, then $t$ is a $\text{nb}$, else we have

$$ t_0 : B \\
\frac{\partial t_{i+1}}{\partial x}|_{t_0} \rightarrow (\frac{\partial t_{i+1}}{\partial x}|_{t_0}, \frac{\partial t_{i+2}}{\partial x}|_{t_0}, ..., \frac{\partial t_n}{\partial x}|_{t_0}) $$

$$ t_0 : B \\
\frac{\partial (\lambda y : T) x}{\partial x}|_{t_0} \rightarrow \lambda y : T \frac{\partial x}{\partial x}|_{t_0} $$

$$ t_0 : B \\
\frac{\partial \text{inl/inr } x}{\partial x}|_{t_0} \rightarrow \text{inl/inr } \frac{\partial x}{\partial x}|_{t_0} $$

- Case $t_1 * t_2$.
  If $t_1$ or $t_2$ is not a normal form, then we can make reductions on $t_1$ or $t_2$.
  If both $t_1$ and $t_2$ are normal forms, $t_2$ has to be $(t_1, ..., t_n)$ or a $\text{nb}$. If it is the former case, $t_1$ has also to be $(t_1, ..., t_n)$, then we have

$$ t_1 : (t_{11}, t_{12}, ..., t_{1n}), t_2 : (t_{21}, t_{22}, ..., t_{2n}) \\
t_1 * t_2 \rightarrow (t_{11} * t_{21}) \oplus (t_{12} * t_{22}) \oplus ... \oplus (t_{1n} * t_{2n}) $$
If $t_2$ is a $\mathbf{nb}$, if $t_1$ is a $\mathbf{nb}$, then $t_1 \ast t_2$ is a $\mathbf{nb}$, else we have

\[
\frac{t_2 : \mathcal{B}}{(\lambda x : T.t) \ast t_2 \rightarrow \lambda x : T.(t \ast t_2)}
\]

\[
\frac{t_0 : \mathcal{B}}{(t_1, t_2, \ldots, t_n) \ast t_0 \rightarrow (t_1 \ast t_0, t_2 \ast t_0, \ldots, t_n \ast t_0)}
\]

Let

- Case $\text{fix } f$.

Then we have $\text{fix } f \rightarrow f \,(\text{fix } f)$

\section{A.2 Preservation}

\begin{lemma}[Preservation under substitution]\label{lem:presubstitution}
If $\Gamma, x : S \vdash t : T$ and $\Gamma \vdash s : S$, then we have $\Gamma \vdash t[s/x] : T$.
\end{lemma}

\begin{proof}
First we prove preservation under substitution.

\begin{itemize}
  \item Case $c$.
    If $c[s/x] = c$, therefore $\Gamma \vdash t[s/x] : \mathcal{B}$
  \item Case $t_1 \oplus t_2$.
    Suppose $\Gamma, x : S \vdash t_1 \oplus t_2 : T$, then we have $\Gamma, x : S \vdash t_1 : T, t_2 : T$, based on induction we have $\Gamma \vdash t_1[s/x] \oplus t_2[s/x] : T$, therefore $\Gamma \vdash (t_1 \oplus t_2)[s/x] : T$.
    Using the same techniques we can prove the case of $t_1 \ominus t_2, t_1 \ast t_2, t_1, t_2, \lambda x : T.t, \frac{\partial t}{\partial x}, \lambda x : T.t, \frac{\partial t}{\partial x}$, $t_0 : \mathcal{B}$, $\pi_j(t)$ and $\text{fix } f, \text{inl/inr } t$, case $t$ of $\text{inl } x_1 \Rightarrow t_1 \text{ inr } x_2 \Rightarrow t_2$.
  \item Case $y$.
    If $y = x$ then $y[s/x] = s$, so $\Gamma \vdash y[s/x] : T$.
    If $y$ is other than $x$, then $y[s/x] = y$, so $\Gamma \vdash y[s/x] : T$.
\end{itemize}

Then we prove the preservation

\begin{lemma}[Preservation]\label{lem:preservation}
If $t : T$ and $t \rightarrow t'$, then $t' : T$. (Allowing free variable of $i\mathcal{B}$)
\end{lemma}

\begin{proof}
\begin{itemize}
  \item Case $(\lambda x : T.t)t_1 \rightarrow t[t_1/x]$:
    It is straightforward by using the Lemma\ref{lem:presubstitution}
  \item Case $\text{fix } f \rightarrow f \,(\text{fix } f)$
    Suppose $\Gamma \vdash f : A \rightarrow A$, then $\Gamma \vdash \text{fix } f : A$ and $\Gamma \vdash f \,(\text{fix } f) : A$, so they have the same type.
  \item Case $\pi_j(t_1, t_2, \ldots, t_n) \rightarrow t_j$
    Suppose $\Gamma \vdash (t_1, t_2, \ldots, t_n) : (T_1, T_2, \ldots, T_n)$, then $\Gamma \vdash \pi_j(t_1, t_2, \ldots, t_n) : T_j$ and $\Gamma \vdash t_j : T_j$, so they have the same type.
  \item Case

\[
\frac{t_0 : \mathcal{B}}{\frac{\partial (t_1, t_2, \ldots, t_n)}{\partial x}|_{t_0} \rightarrow \left( \frac{\partial t_1}{\partial x}|_{t_0}, \frac{\partial t_2}{\partial x}|_{t_0}, \ldots, \frac{\partial t_n}{\partial x}|_{t_0} \right)}
\]

Suppose $\Gamma \vdash \frac{\partial (t_1, t_2, \ldots, t_n)}{\partial x}|_{t_0} : (T_1, T_2, \ldots, T_n)$, then $\Gamma, x : \mathcal{B} \vdash t_j : T_j$, then $\Gamma \vdash \frac{\partial t_1}{\partial x}|_{t_0}, \frac{\partial t_2}{\partial x}|_{t_0}, \ldots, \frac{\partial t_n}{\partial x}|_{t_0} : (T_1, T_2, \ldots, T_n)$, so they have the same type.

Using the same technique, we can prove the case.
\end{proof}
\[ t_0 : B \]
\[ (t_1, t_2, \ldots, t_n) * t_0 \rightarrow (t_1 * t_0, t_2 * t_0, \ldots, t_n * t_0) \]

\[ \int_{t_1}^{t_2} \{ t_{11}, t_{12}, \ldots, t_{1n} \} dx \rightarrow (\int_{t_1}^{t_2} t_{11} dx, \int_{t_1}^{t_2} t_{12} dx, \ldots, \int_{t_1}^{t_2} t_{1n} dx) \]

= Case

\[ \frac{\partial (\lambda y. T.)}{\partial x}|_{t_0} \rightarrow \lambda y : T, \frac{\partial t}{\partial x}|_{t_0} \rightarrow A \rightarrow B \]

Suppose \( \Gamma \vdash \frac{\partial (\lambda y. T.)}{\partial x}|_{t_0} : A \rightarrow B \), then \( \Gamma, y : A \vdash \frac{\partial t}{\partial x}|_{t_0} : B \), therefore \( \Gamma \vdash \lambda y : T, \frac{\partial t}{\partial x}|_{t_0} : A \rightarrow B \), so they have the same type.

Using the same techniques, we can prove the case

\[ \int_{t_1}^{t_2} \lambda y : T_2. t dx \rightarrow \lambda y : T_2, \int_{t_1}^{t_2} t dx \]

\[ t_2 : B \]

= Case \((t_{11}, t_{12}, \ldots, t_{1n}) \oplus (t_{21}, t_{22}, \ldots, t_{2n}) \rightarrow (t_{11} \oplus t_{21}, t_{12} \oplus t_{22}, \ldots, t_{1n} \oplus t_{2n}) \)

From \( \Gamma \vdash (t_{11}, t_{12}, \ldots, t_{1n}) : T_1, \) we have \( \Gamma \vdash (t_{11}, t_{12}, \ldots, t_{1n}) : T_1, (t_{21}, t_{22}, \ldots, t_{2n}) : T_2. \) Suppose \( T \) is \((T_1, T_2, \ldots, T_n) \), then we have \( \Gamma \vdash t_{11} : T_1, t_{21} : T_1. \) Therefore \( \Gamma \vdash t_{11} \oplus t_{21} : T_1. \) So we have \( \Gamma \vdash (t_{11} \oplus t_{21}, t_{12} \oplus t_{22}, \ldots, t_{1n} \oplus t_{2n}) : (T_1, T_2, \ldots, T_n) = T. \)

Using the same techniques, we can prove the following rules

\( \lambda x : T.t_1 \oplus \lambda y : T.t_2 \rightarrow \lambda x : T.\{t_1 \oplus t_2[x/y]\} \)

\( (t_{11}, t_{12}, \ldots, t_{1n}) \oplus (t_{21}, t_{22}, \ldots, t_{2n}) \rightarrow (t_{11} \oplus t_{21}, t_{12} \oplus t_{22}, \ldots, t_{1n} \oplus t_{2n}) \)

\( \lambda x : T.t_1 \oplus \lambda y : T.t_2 \rightarrow \lambda x : T.\{t_1 \oplus t_2[x/y]\} \)

and reduction for case \( t \) of \( \text{inl} \) \( x_1 \Rightarrow t_1 \) \( \text{inr} \) \( x_2 \Rightarrow t_2. \)

= Case

\[ \forall i, (t_1, t_2, \ldots, t_{i-1}, x_i, t_{i+1}, \ldots, t_n) \text{is written as } t_i \]

\[ \frac{\partial t_i}{\partial x}|_{t_0} \rightarrow (\frac{\partial (t_{i1}/x)|_{t_1}}{\partial x}, \frac{\partial (t_{i2}/x)|_{t_2}}{\partial x}, \ldots, \frac{\partial (t_{in}/x)|_{t_n}}{\partial x}) \]

Let’s suppose \( \frac{\partial t_i}{\partial x}|_{t_0} \) has the type \( \frac{\partial t_i}{\partial x}|_{t_0} \), then we have \( \Gamma, x : T_0 \vdash t : T. \) Suppose that \( T_0 = (T_1, T_2, \ldots, T_n) \), then we have \( \Gamma, x : T_1 \vdash t[t_1/x] : T. \) Thus \( \Gamma \vdash \frac{\partial (t[t_1/x]/x)|_{t_1}}{\partial x}, \frac{\partial (t[t_2/x]|_{t_2}}{\partial x}, \ldots, \frac{\partial (t[t_n/x]|_{t_n}}{\partial x} = \frac{\partial t}{\partial x}. \)

= Case \( \int_{t_{11}}^{t_{12}} \pi_1(t([x_1, t_{12}, \ldots, t_{1n}] / x)) dx_1 \oplus \cdots \oplus \int_{t_{1n}}^{t_{2n}} \pi_n(t([t_{21}, t_{22}, \ldots, x_n] / x)) dx_n \)

Let’s suppose \( \Gamma, x : T_0 \vdash t : \frac{\partial t}{\partial x}|_{t_0} \) and \( \Gamma \vdash \int_{t_{11}}^{t_{12}} \pi_1(t([x_1, t_{12}, \ldots, t_{1n}] / x)) dx_1 \oplus \cdots \oplus \int_{t_{1n}}^{t_{2n}} \pi_n(t([t_{21}, t_{22}, \ldots, x_n] / x)) dx_1 \oplus \cdots \oplus \int_{t_{1n}}^{t_{2n}} \pi_n(t([t_{21}, t_{22}, \ldots, x_n] / x)) dx_n \rightarrow T: T. \)

Assume that \( T_0 = (T_1, T_2, \ldots, T_n) \). Then for all \( j \), we have \( \Gamma, x : T_j \vdash t[t_{j1}, \ldots, t_{jn}/x] : \frac{\partial t}{\partial x}|_{t_0}. \) Thus \( \Gamma, x : T_j \vdash \int_{t_{j1}}^{t_{jn}} \pi_j(t([t_{j2}, \ldots, t_{jn}/x] / x)) dx_1 \oplus \cdots \oplus \int_{t_{1n}}^{t_{2n}} \pi_n(t([t_{21}, t_{22}, \ldots, x_n] / x)) dx_1 \oplus \cdots \oplus \int_{t_{1n}}^{t_{2n}} \pi_n(t([t_{21}, t_{22}, \ldots, x_n] / x)) dx_n \rightarrow T: T. \)

Using the same technique, we can prove the case.

\[ t_1 : (t_{11}, t_{12}, \ldots, t_{1n}), t_2 : (t_{21}, t_{22}, \ldots, t_{2n}) \]

\[ t_1 * t_2 \rightarrow (t_{11} * t_{21}) \oplus (t_{12} * t_{22}) \oplus \cdots \oplus (t_{1n} * t_{2n}) \]

Therefore, we prove the preservation of the system.
A.3 Confluence

Define a binary relation \( \rightarrow \) by induction on relation on terms.

\[ M \rightarrow M' \]

\[ M \rightarrow M', N \rightarrow N' \]

\[ M \uplus N \rightarrow M' \uplus N', M \ominus N \rightarrow M' \ominus N', M * N \rightarrow M' * N', \frac{\partial M}{\partial x} |_{N} \rightarrow \frac{\partial M'}{\partial x} |_{N'} \]

\[ \forall j \in [1, n], M_{j} \rightarrow M'_{j} \]

\[ (M_{1}, M_{2}, ..., M_{n}) \rightarrow (M'_{1}, M'_{2}, ..., M'_{n}) \]

\[ M_{1} \rightarrow M'_{1}, M_{2} \rightarrow M'_{2}, M_{3} \rightarrow M'_{3} \]

\[ \int_{M_{3}}^{M_{2}} \int_{M_{1}}^{M_{0}} dx, \text{ case } M_{1} \text{ of } \text{inl } x_{1} \Rightarrow M_{2} \text{ } \text{inr } x_{2} \Rightarrow M_{3} \Rightarrow \text{ case } M'_{1} \text{ of } \text{inl } x_{1} \Rightarrow M'_{2} \text{ } \text{inr } x_{2} \Rightarrow M'_{3} \]

\[ M_{1} \rightarrow M'_{1}, M_{2} \rightarrow M'_{2}, M_{3} \rightarrow M'_{3} \]

\[ \text{ case } \text{inr } M_{1} \text{ of } \text{inl } x_{1} \Rightarrow M_{2} \text{ } \text{inr } x_{2} \Rightarrow M_{3} \Rightarrow M'_{3}[M'_{1}/x_{2}] \]

\[ M_{1} \rightarrow M'_{1}, M_{2} \rightarrow M'_{2}, M_{3} \rightarrow M'_{3} \]

\[ \text{ case } \text{inl } M_{1} \text{ of } \text{inl } x_{1} \Rightarrow M_{2} \text{ } \text{inr } x_{2} \Rightarrow M_{3} \Rightarrow M'_{3}[M'_{1}/x_{1}] \]

\[ \lambda x : T.M \Rightarrow \lambda x : T'.M', \pi_{j}(M) \Rightarrow \pi_{j}(M'), \text{ inl/inr } M \Rightarrow \text{ inl/inr } M \]

\[ \forall j \in [1, n], M_{j} \rightarrow M'_{j} \]

\[ \pi_{j}(M_{1}, M_{2}, ..., M_{n}) \Rightarrow M'_{j} \]

\[ M \Rightarrow M', N \Rightarrow N' \]

\[ (\lambda x : T.M)N \Rightarrow M'[N'/x] \]

\[ \forall j \in [1, n], M_{1}, M_{2}, ..., M_{n} \rightarrow M'_{1}, M_{2}, ..., M_{n} \rightarrow M'_{2}, M_{1}, M_{2}, ..., M_{n} \rightarrow M'_{3}, M_{1}, M_{2}, ..., M_{n} \rightarrow M'_{4}, M_{1}, M_{2}, ..., M_{n} \rightarrow M'_{5}, M_{1}, M_{2}, ..., M_{n} \rightarrow M'_{6} \]

\[ \lambda x : T.M \oplus (\lambda y : T.N) \Rightarrow \lambda x : T'.M' \oplus N'[y/x] \]

\[ M \Rightarrow M', N \Rightarrow N' \]

\[ (\lambda x : T.M) \oplus (\lambda y : T.N) \Rightarrow \lambda x : T'.M' \oplus N'[y/x] \]

\[ \forall j \in [1, n], M_{j} \rightarrow M'_{j}, N \Rightarrow N', N : B \]

\[ \frac{\partial(M_{1}, M_{2}, ..., M_{n})}{\partial x} |_{N} \rightarrow \left( \frac{\partial M'_{1}}{\partial x} |_{N'}, \frac{\partial M'_{2}}{\partial x} |_{N'}, ..., \frac{\partial M'_{n}}{\partial x} |_{N'} \right) \]

\[ \frac{\partial \lambda y : T.M}{\partial x} |_{N} \rightarrow \lambda y : T.\frac{\partial M'}{\partial x} |_{N'} \]
Lemma 26 (Preservation of →). If \( N : B, N \rightarrow N' \), then \( N' : B \).

**Proof.** If we name the one-step relation of our calculus as \( \rho \), and its transitive closure as \( \rho^* \), then we have \( \rightarrow \subseteq \rho^* \). So we have \( N \rho^* N' \). Notice that we have the preservation property of our calculus, thus we have \( N' : B \).

Lemma 27 (→ under substitution). \( M \rightarrow M', N \rightarrow N' \), then we have \( M[N/x] \rightarrow M'[N'/x] \)

**Proof.** Induction on \( M \rightarrow M' \)

- Case \( M \rightarrow M' \), make induction on the form of \( M \).
  - Subcase c, then \( c[N/x] = c = c[N'/x] \), using \( M \rightarrow M \) we have \( M[N/x] \rightarrow M[N'/x] \).
  - Subcase \((t_1, t_2, ..., t_n)\), using induction we have \( t_i[N/x] \rightarrow t_i[N'/x] \). Then using \( \forall i, M_i \rightarrow M'_i \Rightarrow (M_1, M_2, ..., M_n) \rightarrow (M'_1, M'_2, ..., M'_n) \) we have \( M[N/x] \rightarrow M[N'/x] \).

Using the same technique, we can prove the subcase of \( t \oplus t, t \star t, \lambda x : T.t, t \cdot \frac{\partial}{\partial x_i} t, \int t dx, \pi_j(t), \int M_1 \cdots M_3 dx \rightarrow \int M'_1 \cdots M'_3 dx, \text{inl/inr } M, \text{case } \text{inr } M_1 \text{ of } \text{inl } x_1 \Rightarrow M_2 \text{ inr } x_2 \Rightarrow M_3 \).

- Subcase variable \( y \), if \( y = x \) then \( y[N/x] = N, y[N'/x] = N' \), then \( y[N/x] \rightarrow y[N'/x] \), if \( y \) is not \( x \) then same as the subcase c.

The rest cases can be divided into three categories.
Case relation based on the relation of subterms.

Subcase $M N \rightarrow M' N'$, using induction we have $M[K/x] \rightarrow M'[K'/x], N[K/x] \rightarrow N'[K'/x]$, using $M \rightarrow M', N \rightarrow N' \Rightarrow M N \rightarrow M' N'$ we have $(M N)[K/x] \rightarrow (M' N')[K'/x]$.

Subcases $M \oplus N \rightarrow M' \oplus N', M \otimes N \rightarrow M' \otimes N', M \ast N \rightarrow M' \ast N'$, $\frac{\partial M}{\partial x}[N] \rightarrow \frac{\partial M'}{\partial x}[N']$, $(M_1, M_2, ..., M_n) \rightarrow (M_1', M_2', ..., M_n')$, $\lambda x : T.M \rightarrow \lambda x' : T.M', \pi_j(M) \rightarrow \pi_j(M')$, $\text{fix } M \rightarrow \text{fix } M'$, $\text{inl/inr } M \rightarrow \text{inl/inr } M'$: same as $M N \rightarrow M' N'$.

Case reduction changes the structure

Subcase $(M_{11}, M_{12}, ..., M_{1n}) \oplus (M_{21}, M_{22}, ..., M_{2n}) \rightarrow (M_{11}' \oplus M_{12}' \oplus M_{22}' \oplus ..., M_{1n}' \oplus M_{2n}')$, using induction we have $\forall i \in [1, 2], \forall j \in [1, n], M_{ij}[K/x] \rightarrow M_{ij}'[K'/x]$, so we have $((M_{11}, M_{12}, ..., M_{1n}) \oplus (M_{21}, M_{22}, ..., M_{2n}))[K/x] \rightarrow (M_{11}' \oplus M_{12}' \oplus M_{22}' \oplus ..., M_{1n}' \oplus M_{2n}')[K'/x]$.

Subcases $(M_{11}, M_{12}, ..., M_{1n}) \oplus (M_{21}, M_{22}, ..., M_{2n}) \rightarrow (M_{11}' \oplus M_{12}' \oplus M_{22}' \oplus ..., M_{1n}' \oplus M_{2n}')$, $\frac{\partial (M_{11}, M_{12}, ..., M_{1n})}{\partial x}[N] \rightarrow \frac{\partial (M_{11}', M_{12}', ..., M_{1n}')}{\partial x}[N'], \frac{\partial M}{\partial y}[T.M][x] \rightarrow \frac{\partial M'}{\partial y}[T.M'][x], \int M_{ij} \lambda y : T_2.M_{ij}dx \rightarrow \int M_{ij}' \lambda y : T_2,M_{ij}'dx, \int M_{ij}N_{ij}dx \rightarrow \int M_{ij}'N_{ij}dx, (\lambda x : T.M)[N] \rightarrow \lambda x : T'(M'\ast N'), (M_{11}, M_{12}, ..., M_{1n}) \ast N \rightarrow (M_{11}' \ast M_{12}', ..., M_{1n}' \ast N'), (M_{11}, M_{12}, ..., M_{1n}) \ast (M_{21}, M_{22}, ..., M_{2n}) \rightarrow M_{11}' \ast M_{12}' \oplus ... \oplus M_{1n}' \ast M_{2n}'. \text{fix } M \rightarrow M' \text{ (fix } M')$: same as $(M_{11}, M_{12}, ..., M_{1n}) \oplus (M_{21}, M_{22}, ..., M_{2n}) \rightarrow (M_{11}' \oplus M_{12}' \oplus M_{22}' \oplus ..., M_{1n}' \oplus M_{2n}')$.

Case reduction involves substitution

Subcase $(\lambda x : T.M)N \rightarrow M'[(N'/x)]$, by induction hypothesis, we have $M[K/y] \rightarrow M'[K'/y], N[K/y] \rightarrow N'[K'/y]$, thus $((\lambda x : T.M)N)[K/y] = ((\lambda x : T.M)[K/y])N[K/y] \rightarrow M'[K'/y][(N'[K'/y])/x]$, and we have $M'[N'/x][K'/y] = M'[K'/y][(N'[K'/y])/x]$. Therefore we prove the case.

Subcase\text{$i \in \{1, 2, ..., n\}$, $M_{1i}'$, $M_{2i}'$, $M_{1i}'[x_i]$, $M_{2i}'[x_i]$ is written as $M_i \ast \frac{\partial M_i}{\partial x_i}[x_i] \rightarrow \frac{\partial (M_i[x_i])}{\partial x_i}[x_i]$, $\frac{\partial (M_i[x_i])}{\partial x_i}[x_i]$}注意到

$$M_i[M_i'[x_i]/[K'/y]] = M_i[K'/y][(M_i'[K'/y])/x]$$

$$= (M_i[K'/y])[M_i'[K'/y]/[x_i]], ..., M_{i+1}[(K'/y)\cdot x_i][K'/y], M_{i+1}[K'/y], ..., M_n[K'/y])/x]$$

$$= (M_i[K'/y])[(M_i'[K'/y]/[x_i])$$

Using induction, we know that $M_i[K/y] \rightarrow M_i'[K'/y]$, so we prove the case.

Subcase $(\lambda x : T.M) \oplus (\lambda y : T.N) \rightarrow \lambda x : T.M' \oplus (\lambda y : T.N)$, $\lambda x : T.M' \oplus (\lambda y : T.N)$: same as $(\lambda x : T.M)N \rightarrow M'[N'/x]$.

Subcases $\int M_{11} \ast M_{12} \ast ... \ast M_{1n} M_{0} dx \rightarrow \frac{\partial M_{11}}{\partial x_1} \pi_1(M_0[M_{11}'/x])dx_1 \oplus ... \oplus \int M_{1n} \ast M_{2n} \ast M_{11}' \ast M_{12}' \ast ... \ast M_{1n}'[x]/x_{11}$, case $\text{inl } M_1 \ast inl \ x_1 \rightarrow \text{inl } x_2 \rightarrow M_3 \rightarrow M_3[M_1'[x_2/x]]$, case $\text{inr } M_1 \ast inr \ x_1 \rightarrow \text{inr } x_2 \rightarrow M_3 \rightarrow M_3[M_1'[x_2/x]]$: same as $\forall i \in \{1, 2, ..., n\}, M_{1i}', M_{2i}', M_{1i}'[x_i], M_{1i+1} \ast ... \ast M_{1n}'$ is written as $M_i' \ast \frac{\partial M_i}{\partial x_i}[x_i] \rightarrow \frac{\partial (M_i[x_i])}{\partial x_i}[x_i]$, $\frac{\partial (M_i[x_i])}{\partial x_i}[x_i]$.
Thus we complete the proof.

\begin{lemma}[diamond property] For \( M \to M_1, M \to M_2 \), there exists a \( M_3 \), such that \( M_1 \to M_3, M_2 \to M_3 \).
\end{lemma}

\begin{proof}
We do induction on the case of \( M \to M_1 \).

- Case \( M \to M \)
  Then we choose \( M_3 \) as \( M_2 \).

- Case \( M N \to M' N' \)
  \begin{itemize}
    \item Subcase \( M \to M_2 \) as \( M \to M \)
      Then we choose \( M_3 \) as \( M_1 \).
    \item Subcase \( M \to M_2 \) as \( M N \to M'' N'' \)
      Then we use the induction hypothesis, we have \( M^* \) that \( M' \to M^*, M'' \to M^* \), and we have \( N^* \) that \( N' \to N^*, N'' \to N^* \), so we choose \( M_3 \) as \( M^* N^* \).
    \item Subcase \( M \to M_2 \) as \( M = \lambda x : T.P, (\lambda x : T.P)N \to P''[N''/x] \).
      Then we first have that \( M = \lambda x : T.P \) Then \( M' = \lambda x : T.P' \), so we choose \( M_3 = P''[N''/x] \).
  \end{itemize}

- Case \( M \oplus N \to M' \oplus N' \)
  \begin{itemize}
    \item Subcase \( M \to M_2 \) as \( M \to M \)
      Then we choose \( M_3 \) as \( M_1 \).
    \item Subcase \( M \to M_2 \) as \( M \to M'', N \to N'' \Rightarrow M \oplus N \to M'' \oplus N'' \); same as 1.
    \item Subcase \( M \to M_2 \) as \( (\lambda x : T.M) \oplus (\lambda y : T.N) \to \lambda x : T.M' \oplus N'[y/x] \)
      Then \( M \to M_1 \) must be \( (\lambda x : T.M) \oplus (\lambda y : T.N) \to (\lambda x : T.M'') \oplus (\lambda y : T.N'') \), Then we choose \( M_3 \) to be \( \lambda x : T.M' \oplus N'[y/x] \).
    \item Subcase \( M \to M_2 \) as \( (M_{11}, M_{12}, \ldots M_{1n}) \oplus (M_{21}, M_{22}, \ldots M_{2n}) \to (M'_{11} \oplus M'_{21}, M'_{12} \oplus M'_{22}, \ldots M'_{1n} \oplus M'_{2n}) \); same as \( (\lambda x : T.M) \oplus (\lambda y : T.N) \to \lambda x : T.M' \oplus N'[y/x] \).
  \end{itemize}

All the other cases are similar to the case of application and \( \oplus \), except that we may have more subcases on these cases, but the extra subcases are all similar to 2.

\end{proof}

\begin{lemma}[Confluence] One term has at most one normal form.
\end{lemma}

\begin{proof}
The relation \( \to \) has the diamond property, and reduction relation \( \rho \) satisfy that \( \rho \subseteq \to \subseteq \rho^* \). Also notice that \( \to^* \) has the diamond property, and \( \to^* = \rho^* \), so the relation \( \rho^* \) has the diamond property. This is what results in the confluence.
\end{proof}
A.4 Strong normalization

Here we write $\rightarrow$ as $\rho^*$. $t$ and $t'$

► Lemma 30 (existence of $\nu$). $t$ is strongly normalisable if and there is a number $\nu(t)$ which bounds the length of every normalisation sequence beginning with $t$.

Proof. See P27 in Proofs and Types [12].

► Definition 31. We define a set $\text{RED}_T$ by induction on the type $T$.

1. For $t$ of base type, $t$ is reducible if it is strongly normalisable.
2. For $t$ of type $(T_1, T_2, ..., T_n)$, $t$ is reducible iff $\forall j, \pi_j(t)$ is reducible.
3. For $t$ of type $U \rightarrow V$, $t$ is reducible if, for all reducible $u$ of type $U$, $t u$ is reducible of type $V$.
4. For $t$ of type $T_1 + T_2$, $t$ is reducible iff, $\text{case } t \text{ of } \text{inl } x \Rightarrow 0 | \text{inr } x_2 \Rightarrow 0$ is reducible term of base type.

► Definition 32. $t$ is neutral if $t$ is not of the form $(t_1, t_2, ..., t_n)$ or $\lambda x : T.t$ or $\text{inl/inr } t$

We will verify the following 3 properties by induction on types.

(CR 1) If $t \in \text{RED}_T$, then $t$ is strongly normalisable.
(CR 2) If $t \in \text{RED}_T$ and $t \rightarrow t'$, then $t' \in \text{RED}_T$.
(CR 3) If $t$ is neutral, and whenever we convert a redex of $t$ we obtain a term $t' \in \text{RED}_T$, then $t \in \text{RED}_T$.

Case base type
(CR 1) is a tautology.
(CR 2) If $t$ is strongly normalisable then every term $t'$ to which $t$ reduces is also.
(CR 3) A reduction path leaving $t$ must pass through one of the terms $t'$, which are strongly normalisable, and so is finite. In fact, it is immediate that $\nu(t)$ is equal to the greatest of the numbers $\nu(t') + 1$, as $t'$ varies over the (one-step) conversions of $t$.

Case tuple type
(CR 1) Suppose that $t$, of type $(T_1, T_2, ..., T_n)$, is reducible; then $\pi_i(t)$ is reducible and by induction hypothesis (CR 1) for $T_i$, $\pi_i(t)$ is strongly normalisable. Moreover, $\nu(t) \leq \nu(\pi_i(t))$. Since to any reduction sequence $t, t_1, t_2, ..., t_n$, one can apply $\pi_i(t)$ to construct a reduction sequence $\pi_1(t), \pi_1(t_1), \pi_1(t_2), ...$ (in which the $\pi_1(t)$ is not reduced). So $\nu(t)$ is finite, and $t$ is strongly normalisable.
(CR 2) If $t \rightarrow t'$, then $\forall j, \pi_j(t) \rightarrow \pi_j(t')$. By the induction hypothesis for type $T_j$ on (CR 2), we have $\forall j, \pi_j(t')$ is reducible, so $t'$ is reducible.
(CR 3) Let $t$ be neutral and suppose all the $t'$ one step from $t$ are reducible. Applying a conversion inside $\pi_j(t)$, the result is a $\pi_j(t')$, since $\pi_j(t)$ cannot itself be a redex (it is not a tuple), and $\pi_j(t')$ is reducible, since $t'$ is. But as $\pi_j(t)$ is neutral, and all the terms one step from $\pi_j(t)$ are reducible, the induction hypothesis (CR 3) for $T_j$ ensures that $\pi_j(t)$ is reducible, so $t$ is reducible.

Case arrow type
(CR 1) If $t$ is reducible of type $U \rightarrow V$, let $x$ be a variable of type $U$. $\Rightarrow$ And we have $\nu(t) \leq \nu(t x)$
(CR 2) If $t \rightarrow t'$, and $t$ is reducible, take $u$ reducible of type $U$; then $t u$ is reducible and $t u \rightarrow t' u$. The induction hypothesis (CR 2) for $V$ gives that $t' u$ is reducible. So $t'$ is reducible.
(CR 3) Let \( t \) be neutral and suppose all the \( t' \) one step from \( t \) are reducible. Let \( u \) be a reducible term of type \( U \); we want to show that \( t u \) is reducible. By induction hypothesis (CR 1) for \( U \), we know that \( u \) is strongly normalisable; so we can reason by induction on \( \nu(u) \).

In one step, \( t u \) converts to

1. \( t u \) with \( t' \) one step from \( t \); but \( t' \) is reducible, so \( t u \) is.
2. \( t u' \), with \( u' \) one step from \( u \). \( u' \) is reducible by induction hypothesis (CR 2) for \( U \), and \( \nu(u') < \nu(u) \); so the induction hypothesis for \( u' \) tells us that \( t u' \) is reducible.
3. There is no other possibility, for \( t u \) cannot itself be a redex (\( t \) is not of the form \( \lambda x : T.t \)).

Case sum type

(CR 1) If \( t \) is reducible of type \( T_1 + T_2 \), Then we have \( \nu(t) \leq \nu(\text{case } t \text{ of } \text{inl } x_1 \Rightarrow 0 | \text{inr } x_2 \Rightarrow 0) \)
(CR 2) same as tuple type.
(CR 3) same as arrow type.

▶ **Lemma 33.** If \( t_1, t_2, ..., t_n \) are reducible terms, then so is \((t_1, t_2, ..., t_n)\)

**Proof.** Because of (CR 1), we can reason by induction on \( \nu(t_1) + \nu(t_2) + ... + \nu(t_n) \) to show that \( \pi_j(t_1, t_2, ..., t_n) \), is reducible. This term converts to

1. \( t_j \), then it is reducible.
2. \((t_1, ..., t_{k-1}, t'_k, t_{k+1}, ..., t_n)\), based on induction, it is reducible.

▶ **Lemma 34.** If for all reducible \( u \) of type \( U \), \( t[u/x] \) is reducible, then so is \( \lambda x : T.t \).

**Proof.** To show \( \lambda x : T.t u \) is reducible, we make reductions on \( \nu(u) + \nu(t) \), \( \lambda x : T.t u \), which can be reduced to

1. \( t[u/x] \), then it is reducible.
2. \((\lambda x : T.t') u \) or \((\lambda x : T.t) u' \), based on induction we know it is reducible.

▶ **Lemma 35.** If \( t \) is reducible, then so is \( \text{inl}/\text{inr } t \).

**Proof.** Same as the case \( \lambda x : T.t \).

▶ **Lemma 36.** If for all reducible \( t_1 \) and \( t_2 \) of type \( T_1 \) and \( T_2 \), we have \( t_3[t_1/x_1] \) and \( t_4[t_2/x_2] \) are reducible, and \( t \) is reducible term of type \( T_1 + T_2 \), then so is \( \text{case } t \text{ of } \text{inl } x_1 \Rightarrow t_3 | \text{inr } x_2 \Rightarrow t_4 \).

**Proof.** Same as the case \( \lambda x : T.t \).

▶ **Lemma 37.** If \( t_1 \) and \( t_2 \) are reducible terms of \( T \), then so is \( t_1 \oplus t_2 \).

**Proof.** We prove this by induction on type.

- Case base type, then it can only be reduced to \( t'_1 \oplus t'_2 \), so \( \nu(t_1 \oplus t_2) = \nu(t_1) + \nu(t_2) \), Therefore it is strongly normalisable, and thus reducible.
- Case \((T_1, T_2, ..., T_n)\), we make induction on \( \nu(t_1) + \nu(t_2) \), \( t_1 \oplus t_2 \) can be reduced to.
1. Subcase \( (t_1 \oplus t_2, \ldots, t_{2n}) \). Because \( t_1 \) and \( t_2 \) are reducible, then \( \forall j, t_{ij} \) is reducible. Case \( t_1 \oplus t_2, \ldots, t_{2n} \) is reducible.

2. Subcase \( (t_1' \oplus t_2) \) or \( (t_1 \oplus t_2') \). Based on induction, we know it is reducible.

Case \( A \to B \) for all reducible term \( u \) of type \( A \), we make induction on \( \nu(t_1) + \nu(t_2) + \nu(u) \).

1. Subcase \( (\lambda x : T.t_1) \oplus (\lambda y : T.t_2) \). Based on induction on types we have \( (\lambda x : T.t_1) \oplus (\lambda y : T.t_2) \) reducible terms based on induction. So we have \( \lambda x : T.t_1 \oplus (\lambda y : T.t_2)' \) is reducible.

2. Subcase \( (t_1' \oplus t_2) \) or \( (t_1 \oplus t_2') \). Based on induction we can prove the case.

Lemma 38. If \( t_1 \) and \( t_2 \) are reducible terms of \( T \), then so is \( t_1 \oplus t_2 \).

Proof. Same as \( \oplus \).

Lemma 39. If \( t_1 \) and \( t_2 \) are reducible terms of \( \frac{\partial T}{\partial T} \) and \( T_2 \), then so is \( t_1 \ast t_2 \).

Proof. We prove this by induction on types.

Case \( T_1 \): base type, \( T_2 \): base type: same as the case of \( \oplus \).

Case \( T_1 \): \( (T_1, T_2, \ldots, T_n) \), \( T_2 \): base type: same as the case of \( \oplus \).

Case \( T_1 \): \( A \to B \), \( T_2 \): base type: same as the case of \( \oplus \).

Case \( T_1 \): \( (T_1, T_2, \ldots, T_n), T_2 \): \( (T_1', T_2', \ldots, T'_n) \)

Suppose \( t_1 : (t_1, t_{21}, \ldots, t_{2n}), t_2 : (t_2, t_{22}, \ldots, t_{2n}) \), we do induction on \( \nu(t_1) + \nu(t_2) \).

Subcase \( t_1 \ast t_2 \). Because \( t_1 \) and \( t_2 \) is reducible, then so is \( \forall j, t_{ij} \). Based on induction on types we have \( \forall j, t_{ij} \ast t_{2j} \) is reducible, then so is \( (t_1 \ast t_2), t_{22} \ast t_{2n} \).

Subcase \( t_1' \ast t_2 \) or \( t_1 \ast t_2' \), based on induction we know it is reducible.

Lemma 40. If \( t_1 \) and \( t_2 \) are reducible terms of \( T_1 \) and \( T_2 \), and for all reducible \( u \) of type \( T_2 \), we have that \( t_1[u/x] \) is reducible then so is \( \frac{\partial T_1}{\partial x} \frac{\partial T_2}{\partial x} \).

Proof. We prove this by induction on types.

Case \( T_1 \): \( (T_1, T_2, \ldots, T_n) \) or \( A \to B \) or base type, \( T_2: \) base type. Same as the case \( t_1 \ast t_2 \).

Case \( T_1 \): \( (T_1, T_2, \ldots, T_n), T_2 \): \( (T_1', T_2', \ldots, T'_n) \), we make induction on \( \nu(t_1) + \nu(t_2) \).
Subcase \( \forall t_i, (t_1, t_2, ..., t_{i-1}, x_i, t_{i+1}, ..., t_n) \) is written as \( t_{i*} \),
\[
\frac{\partial}{\partial x_i} [t_1, t_2, ..., t_{i-1}, x_i, t_{i+1}, ..., t_n] \rightarrow (\frac{\partial t_1}{\partial x_i} |_{t_1}, \frac{\partial t_2}{\partial x_i} |_{t_1}, ..., \frac{\partial t_n}{\partial x_i} |_{t_n}),
\]
ote that \( t_{i*} \) is reducible so based on induction. We have that \( \frac{\partial t_j}{\partial x_j} |_{t_j} \) is reducible. Note that this induction is based on the hypothesis \( (t_1, t_2, ..., t_{i-1})[u/x_i] \) is reducible for all the reducible \( u \) of type \( T'_j \), and \( (t_1, t_2, ..., t_{i-1})[u/x_j] = (t_1, t_2, ..., t_{i-1}) [u/x_j] \) because \( t \) has no occurrence of \( x_j \), and it is easy to show that \( (t_1, t_2, ..., t_{i-1})[u/x_j] \) is a reducible term of type \( T_2 \), so we finish the induction, then we have
\[
(\frac{\partial t_1}{\partial x_1} |_{t_1}, \frac{\partial t_2}{\partial x_2} |_{t_2}, ..., \frac{\partial t_n}{\partial x_n} |_{t_n}) \text{ is reducible.}
\]
Subcase \( \frac{\partial t_i}{\partial x} |_{t_2} \) or \( \frac{\partial t_j}{\partial x} |_{t'_2} \), based on induction we have the proof.

\[\blacksquare\]

\textbf{Lemma 41.} If \( t_1, t_2 \) and \( t_3 \) are reducible terms of \( T_1, T_2, T_3 \), and for all reducible \( u \) of type \( T_1 \), we have \( t_3[u/x] \) is reducible then so is \( \int_{t_1}^{t_2} t_3 dx \).

\textbf{Proof.} Same as the case of \( \frac{\partial}{\partial x} \ldots \) \[\blacksquare\]

\textbf{Lemma 42.} If \( t_1, t_2 \) and \( t_3 \) are reducible terms of \( T_1, T_2, T_3 \), and for all reducible \( u_1 \) of type \( T_1 \), \( u_2 \) of type \( T_2 \), we have that \( t_2[u_1/x_1] \) and \( t_3[u_2/x_2] \) are reducible then so is \( \text{case } t_1 \text{ of } \text{inl } x_1 \Rightarrow t_2 | \text{inr } x_2 \Rightarrow t_3 \).

\textbf{Proof.} Same as the case of \( \frac{\partial}{\partial x} \ldots \) \[\blacksquare\]

\textbf{Lemma 43.} Let \( t \) be any term (not assumed to be reducible), and suppose all the free variables of \( t \) are among \( x_1, \ldots, x_n \) of types \( U_1, \ldots, U_n \). If \( u_1, \ldots, u_n \) are reducible terms of types \( U_1, \ldots, U_n \), then \( t[u_1/x_1, \ldots, u_n/x_n] \) is reducible.

\textbf{Proof.} By induction on \( t \). We write \( t[u/x] \) for \( t[u_1/x_1, \ldots, u_n/x_n] \).

1. \( t \) is \( x_i \), then \( t[u/x_i] \) is reducible.
2. \( t \) is \( c \), then \( t \) has no free variable, and \( c \) itself is reducible, so it is reducible.
3. \( t \) is \( (t_1, t_2, \ldots, t_n) \), based on induction we prove \( t_j[u/x] \) is reducible, based on the lemma we know it is reducible.
4. \( t \) is \( t \oplus t, t \odot t, t * t, \text{inl} / \text{inr } t \) or \( \pi_j(t) \): same as the case \( (t_1, t_2, \ldots, t_n) \).
5. \( t \) is \( \lambda y : T.t , t \) by induction we have \( t[u/y, v/y] \) is reducible, then by lemma we have \( \lambda y : T. (t[u/y]) \) is reducible, so \( (\lambda y : T.t)[u/y] \) is reducible.
6. \( t \) is \( \frac{\partial t}{\partial x} |_{t}, \text{case } t \text{ of } \text{inl } x_1 \Rightarrow t_1 | \text{inr } x_2 \Rightarrow t_2 \) or \( \int_{t_1}^{t_2} t dx \): same as the case \( \lambda y : T.t \).

\[\blacksquare\]

\textbf{Theorem 44.} All terms are reducible.

\textbf{Proof.} For arbitrary term \( t \), apply the Lemma 43 to \( t[x_1/x_1, \ldots, x_n/x_n] \) and we get the result.

\[\blacksquare\]

\textbf{Corollary 45.} All terms are strongly normalisable.
Appendix-Lemmas

Lemma 46. If \( t_1 \rho \ast t'_1, t_2 \rho \ast t'_2 \), then \( t_1[t_2/x] \rho \ast t'_1[t'_2/x] \).

Proof. Using the confluence property, it is easy to see.

Lemma 47. If \( t_1 = t'_1, t_2 = t'_2 \), then \( t_1 \odot t_2 = t'_1 \odot t'_2 \).

Proof. if \( t_1 \) or \( t_2 \) is not closed, then we use the substitution \([u_1/x_1, \ldots, u_n/x_n]\) to make it closed. For simplicity of notation, we just use \( \rho \).

Based on the equality definition, we can assume that \( t_1, t_2, t'_1, t'_2 \) are all normal forms and we prove this by induction on types.

- Case \( t_1 \) is of base type. Then \( t_2, t'_1 \) and \( t'_2 \) have to be base type to be well-typed. And for base type normal forms, we have \( t_1 \odot t_2 = t'_1 \odot t'_2 \).

- Case \( t_1 \) is \( A \rightarrow B \) type, let’s suppose \( t_1 : \lambda x : T.t_3, t'_1 : \lambda x' : T.t'_3, t_2 : \lambda y : T.t_4, t'_2 : \lambda y' : T.t'_4 \).

If \( t_1 \) or \( t_2 \) or \( t'_1 \) or \( t'_2 \)’s normal form are not \( \lambda x : T.t \), then we know their normal form are all interpretable in base type, thus we have \( t_1 \odot t_2 = t'_1 \odot t'_2 \).

Else for all \( u \)

\[
(t_1 \odot t_2) u = (\lambda x : T.t_3 \odot \lambda y : T.t_4) u = (\lambda x : T.t_3 \odot t_4[x/y]) u = t_3[u/x] \odot t_4[u/y]
\]

Similarly, we have \((t'_1 \odot t'_2) u' = t'_3[u/x'] \odot t'_4[u/y']\).

And notice that because \( t_1 = t'_1 \), so \( t_1 u = t'_1 u \), so \( t_3[u/x] = t'_3[u/x] \), based on induction of type \( B \), we have \( t_3[u/x] \odot t_4[u/y] = t'_3[u/x] \odot t'_4[u/y] \), so we have proven the case.

- Case \( t_1 \) is of type \( (T_1, T_2, \ldots, T_n) \). Then we suppose \( t_1 : (t_{11}, t_{12}, \ldots, t_{1n}), t'_1 : (t'_{11}, t'_{12}, \ldots, t'_{1n}), t_2 : (t_{21}, t_{22}, \ldots, t_{2n}), t'_2 : (t'_{21}, t'_{22}, \ldots, t'_{2n}) \).

Then

\[
t_1 \odot t_2 = (t_{11} \odot t_{12} \odot \ldots \odot t_{1n}) \odot (t_{21} \odot t_{22} \odot \ldots \odot t_{2n}) = (t_{11} \odot t_{21} \odot t_{12} \odot t_{22} \odot \ldots \odot t_{1n} \odot t_{2n}).
\]

Similarly we have \( t'_1 \odot t'_2 = (t'_{11} \odot t'_{12} \odot \ldots \odot t'_{1n}) \odot (t'_{21} \odot t'_{22} \odot \ldots \odot t'_{2n}) \), and based on induction we have \( \forall j, t_{ij} \odot t_{2j} = t'_{ij} \odot t'_{2j} \), so we have \( t_1 \odot t_2 = t'_1 \odot t'_2 \).

Case \( t_1 \) is of type \( T_1 + T_2 \). This case is impossible because it is not well-typed.

Lemma 48. For a term \( t \), for any subterm \( s \), if the term \( s' = s \), then \( t[s'/s] = t \). (We only substitute the subterm \( s \), but not other subterms same as \( s \))

Proof. We prove this by induction. We first substitute for all the free variables in \( t \). Then

\[
t[s'/s][u_1/x_1, \ldots, u_n/x_n] = t[u_1/x_1, \ldots, u_n/x_n][s'[u_1/x_1, \ldots, u_n/x_n]/s[u_1/x_1, \ldots, u_n/x_n]]
\]

notice that \( s'[u_1/x_1, \ldots, u_n/x_n] = s[u_1/x_1, \ldots, u_n/x_n] \) because \( s' = s \), and we just substitute for some of the free variables in \( s' \) and \( s \). So we only need to prove that for a closed-term \( t \), for any subterm \( s \), if the term \( s' = s \), then \( t[s'/s] = t \).

And notice that if we choose the subterm \( s \) to be the \( t \) itself, then we have \( t[s'/s] = s' = s = t \), And we prove the case. So we next make induction on the form of \( t \).
Case $t$ is $(t_1, t_2, ..., t_n)$
Using induction, we know $t_i[s'/s] = t_i$, and we want to prove

$$(t_1, t_2, ..., t_n) = (t_1[s'/s], t_2[s'/s], ..., t_n[s'/s])$$

And because we have the transitive property of equality, then we just reduce both of them to normal forms, then by definition we know they equal to each other, thus we prove the case.

Using the same technique, we can prove the case $\lambda x : T.t$ and $\text{inl}/\text{int} t$.

Case $t$ is $c$
Then it has no subterm except $c$ itself, if $s' = c$, then $c[s'/c] = s'$, thus we prove the case.
Using the same technique, we can prove the case $x$.

Case $t$ is $t_1 \oplus t_2$
Using induction we have $t_1[s'/s] = t_1, t_2[s'/s] = t_2$, and we have proven Lemma [10] that if $t_1 = t'_1, t_2 = t'_2$, then $t_1 \oplus t_2 = t'_1 \oplus t'_2$, thus we have proven the case.
Using the same technique, we can prove the case $t \otimes t, t * t, \pi_j(t)$.

Case $t$ is $t_1 t_2$
We want to prove if $t_1 = t'_1, t_2 = t'_2$, then $t_1 t_2 = t'_1 t'_2$.
By definition we know if $t_1 = t'_1$, then $t_1 t'_2 = t'_1 t'_2$.

Then we have proven $t_1 t'_2 = t_1 t_2$. Using confluence property, we can reduce the $t_1$ to $\lambda x : A.t$ or a $\text{nb}$. If it is the former case, then we using induction we have $t[t'_{2}/x] = t[t/2/x]$ . Thus we have $t_1 t'_2 = t_1 t_2$. If it is the latter case, then $t_2$ normal form can be is interpretable, and on base type interpretation we have if $x = x'$, then $f(x) = f(x')$. Thus we prove the case.

Case $t$ is $\frac{\partial t}{\partial x_i} |_{t_2}$
If $t_2$ is base type, then we can use the techniques for how we proved the case $t_1 * t_2$.
If $t_2$ is of type $(T_1, T_2, ..., T_n)$, we can reduce the $t_2$ and $t'_2$ to the normal forms $(t_1, t_2, ..., t_n)$, and $(t'_1, t'_2, ..., t'_n)$, and then we have

$$\forall i, (t_1, t_2, ..., t_{i-1}, x_i, t_{i+1, ...}, t_n) \text{is written as } t_n \ni \frac{\partial t}{\partial (t_1, t_2, ..., t_n)} \rightarrow (\frac{\partial t[t_j/x]}{\partial x_j} |_{t_1}, \frac{\partial t[t_j/x]}{\partial x_2} |_{t_2}, ..., \frac{\partial t[t_j/x]}{\partial x_n} |_{t_n})$$

, using induction we have $t[t_j/x] = t[t_j'/x]$, then based on induction we have $\frac{\partial t[t_j/x]}{\partial x_j} |_{t_j} = \frac{\partial t[t_j'/x]}{\partial x_j} |_{t'_j}$. Thus we have proven the case.
Using the same technique, we can prove the case $\int_{t_1}^{t_2} t_3 dx$ and case $t$ of $\text{inl}$ $x_1 \Rightarrow t_1 | \text{inr } x_2 \Rightarrow t_2$.
Thus we have proven the lemma.

**Lemma 49.** If $t_1 * (t_2 \oplus t_3)$ and $(t_1 * t_2) \oplus (t_1 * t_3)$ are weak-normalizable, then $t_1 * (t_2 \oplus t_3) = (t_1 * t_2) \oplus (t_1 * t_3)$

**Proof.** if $t_1, t_2$ and $t_3$ are not closed, then we use the substitution $[u_1/x_1, ..., u_n/x_n]$ to make it closed. For simplicity of notation, we just use $t_1, t_2$ and $t_3$ to be the closed-term of themselves.

Because of the confluence and normalization property of the system, we can assume that $t_1, t_2, t_3$ are all normal formss and we prove this by induction on type.
Case: $t_1$, $t_2$ and $t_3$ are of base type. Then based on base type interpretation, we have $t_1 \ast (t_2 \oplus t_3) = (t_1 \ast t_2) \oplus (t_1 \ast t_3)$.

Case: $t_1$ is of type $A \rightarrow B$, $t_2$ and $t_3$ are of base type. Suppose $t_1$ is $\lambda x : A.t$.

If $t_1$'s normal form is not $\lambda x : A.1$, then we notice that $t_1 = \lambda x : A.1$, use the Lemma \[\] we know we can use $\lambda x : A.1$ x to substitute for $t_1$, Thus we can suppose $\lambda x : A$. Then we have for all $u$ of type $A$,

\[
\begin{align*}
\lambda x & : A.t \ast (t_2 \oplus t_3) u \\
& = (\lambda x : A.t) \ast (t_2 \oplus t_3) u \\
& = \lambda x : A.(t \ast (t_2 \oplus t_3)) u \\
& = t[u/x] \ast (t_2 \oplus t_3)
\end{align*}
\]

And

\[
\begin{align*}
(t_1 \ast t_2) \oplus (t_1 \ast t_3) u \\
& = ((\lambda x : A.t) \ast t_2 \oplus (\lambda x : A.t) \ast t_3) u \\
& = (\lambda x : A.(t \ast t_2) \oplus \lambda x : A.(t \ast t_3)) u \\
& = (\lambda x : A.(t \ast t_2) \oplus (t \ast t_3)) u \\
& = (t[u/x] \ast t_2) \oplus (t[u/x] \ast t_3)
\end{align*}
\]

Based on induction on type B, we have $t[u/x] \ast (t_2 \oplus t_3) = (t[u/x] \ast t_2) \oplus (t[u/x] \ast t_3)$, Therefore we prove the case.

Case: $t_1$ is of type $(T_1, T_2, \ldots, T_n)$, $t_2$ and $t_3$ are of base type. Suppose $t_1$ is $(t_1', t_2', \ldots, t_n')$.

Then

\[
\begin{align*}
t_1 \ast (t_2 \oplus t_3) \\
& = (t_1', t_2', \ldots, t_n') \ast (t_2 \oplus t_3) \\
& = (t_1' \ast (t_2 \oplus t_3), t_2' \ast (t_2 \oplus t_3), \ldots, t_n' \ast (t_2 \oplus t_3))
\end{align*}
\]

\[
\begin{align*}
(t_1 \ast t_2) \oplus (t_1 \ast t_3) \\
& = (t_1', t_2', \ldots, t_n') \ast t_2 \oplus (t_1', t_2', \ldots, t_n') \ast t_3 \\
& = (t_1' \ast t_2 + t_2' + t_3) \oplus (t_1' \ast t_3 + t_2' + t_3) \\
& = (t_1' \ast t_2 + t_2' + t_3) \oplus (t_1' \ast t_3 + t_2' + t_3)
\end{align*}
\]

And based on induction we have $t_1' \ast (t_2 \oplus t_3) = (t_1' \ast t_2) \oplus (t_1' \ast t_3)$, so we have $t_1 \ast (t_2 \oplus t_3) = (t_1 \ast t_2) \oplus (t_1 \ast t_3)$.

Case: $t_1$ is of type $T_1 + T_2$, $t_2$ and $t_3$ are of base type: this case is not possible because the righthand term is not well-typed.

Case: $t_1$ is of type $(T_1, T_2, \ldots, T_n)$, $t_2$ and $t_3$ are of type $(T_1', T_2', \ldots, T_n')$. Suppose $t_1 : (t_1', t_2', \ldots, t_n'), t_2 : (t_2', t_2', \ldots, t_2')$ and $t_3 = (t_3', t_3', \ldots, t_3')$.

Then

\[
\begin{align*}
t_1 \ast (t_2 \oplus t_3) \\
& = (t_1', t_2', \ldots, t_n') \ast ((t_2', t_2', \ldots, t_2') \oplus (t_3', t_3', \ldots, t_3')) \\
& = t_1' \ast (t_2' \oplus t_3') \oplus t_1' \ast (t_2' \oplus t_3') \oplus \ldots \oplus t_n' \ast (t_2' \oplus t_3')
\end{align*}
\]

And we have

\[
\begin{align*}
(t_1 \ast t_2) \oplus (t_1 \ast t_3) \\
& = (t_1', t_2', \ldots, t_n') \ast (t_2 \ast (t_2 \oplus t_3) \oplus (t_3 \ast (t_2 \oplus t_3))) \\
& = t_1' \ast (t_2' \oplus t_3') \oplus t_1' \ast (t_2' \oplus t_3') \oplus \ldots \oplus t_n' \ast (t_2' \oplus t_3')
\end{align*}
\]
(t_1 \odot t_2) \oplus (t_1 \odot t_3)
= (t'_{11}, t'_{12}, \ldots, t'_{1n}) \oplus (t'_{21}, t'_{22}, \ldots, t'_{2n}) \oplus (t'_{31}, t'_{32}, \ldots, t'_{3n})
= ((t'_{11} \odot t'_{21}) \odot (t'_{31} \odot t'_{32})) \oplus (t'_{12} \odot t'_{32}) \oplus \ldots \oplus (t'_{1n} \odot t'_{2n}) \oplus (t'_{1n} \odot t'_{3n})

Based on induction we have \forall j, (t'_{2j} \oplus t'_{3j}) = ((t'_{1j} \odot t'_{2j}) \oplus (t'_{1j} \odot t'_{3j})), and using Lemma 10. Thus we have proven the theorem.

\textbf{Lemma 50.} If (t_1 \odot t_2) \oplus (t_2 \odot t_3) and t_1 \odot t_3 are weak-normalizable, then (t_1 \odot t_2) \oplus (t_2 \odot t_3) = t_1 \odot t_3

\textbf{Proof.} If t_1, t_2 or t_3 is not closed, then we just substitute them to be closed. Because of the confluence and normalization property, we can assume that t_1, t_2 and t_3 are all normal forms.

Then we make induction on type of t_1.

\begin{itemize}
  \item Case base type
    Then because on base type, we require that (t_1 \odot t_2) \oplus (t_2 \odot t_3) = t_1 \odot t_3. Thus we have proven the case.
  \item Case A \rightarrow B
    Then we need to prove that \forall u, ((t_1 \odot t_2) \oplus (t_2 \odot t_3)) u = (t_1 \odot t_3) u.
    Then we can suppose that t_1, t_2 and t_3 are of the form \lambda a : A.t l a to substitute for t_l.
    Then we have
    \[
    ((t_1 \odot t_2) \oplus (t_2 \odot t_3)) u = ((\lambda a : A.t'_{1l} \odot \lambda a : A.t'_{2l}) \oplus (\lambda a : A.t'_{3l})) u
    = \lambda a : A.((t'_{1l} \odot t'_{2l}) \oplus (t'_{2l} \odot t'_{3l})) u
    = ((t'_{1l}[u/a] \odot t'_{2l}[u/a]) \oplus (t'_{2l}[u/a] \odot t'_{3l}[u/a]))
    = ((\lambda a : A.t'_{1u} u \odot \lambda a : A.t'_{2u}) \oplus (\lambda a : A.t'_{3u} u))
    = ((t_1 u \odot t_2 u) \oplus (t_2 u \odot t_3 u))
    \]
    And similarly we have (t_1 \odot t_3) u = (t_1 u \odot t_3 u).
    Based on induction on type B, we have (t_1 u \odot t_3 u) = ((t_1 u \odot t_2 u) \oplus (t_2 u \odot t_3 u)).
    Thus we have proven the case.
  \item Case (T_{11}, T_{22}, \ldots, T_{nn})
    Let's suppose t_1 to be (t_{11}, t_{12}, \ldots, t_{1n}), t_2 to be (t_{21}, t_{22}, \ldots, t_{2n}) and t_3 to be (t_{31}, t_{32}, \ldots, t_{3n}).
    Then we have
    \[
    ((t_1 \odot t_2) \oplus (t_2 \odot t_3))
    = (((t_{11} \odot t_{21}) \oplus (t_{21} \odot t_{31})), \ldots, ((t_{1n} \odot t_{2n}) \oplus (t_{2n} \odot t_{3n})))
    \]
    And
    \[
    (t_1 \odot t_3)
    = ((t_{11} \odot t_{31}), \ldots, (t_{1n} \odot t_{3n}))
    \]
    Base on induction on type T_l, we have (t_{1l} \odot t_{2l}) \oplus (t_{2l} \odot t_{3l}) = (t_{1l} \odot t_{3l}) Thus we prove the case.
  \item Case T_1 + T_2: This case is not possible because it is not well-typed.

Thus we have proven the theorem.