A REMARK ON THE SCHRÖDINGER SMOOTHING EFFECT

by

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Abstract. — We prove the equivalence between the smoothing effect for a Schrödinger operator and the decay of the associate spectral projectors. We give two applications to the Schrödinger operator in dimension one.

1. Introduction

Let $d \geq 1$, and consider the linear Schrödinger equation

\begin{align}
\begin{cases}
i \partial_t u = H u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0, x) = f(x) \in L^2(\mathbb{R}^d),
\end{cases}
\end{align}

where $H$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$.

By the Hille-Yoshida theorem, the equation (1.1) admits a unique solution $u(t) = e^{-itH}f \in C(\mathbb{R}; L^2(\mathbb{R}^d))$. Under suitable conditions on $H$, this solution enjoys a local gain of regularity (in the space variable): For all $T > 0$ there exists $C > 0$ so that

\[
\left( \int_0^T \|\Psi(x)(H)^\frac{\gamma}{2} \|_{L^2(\mathbb{R}^d)}^2 e^{-itH}f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R}^d)},
\]

for some weight $\Psi$ and exponent $\gamma > 0$.

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This phenomenon has been discovered by T. Kato [7] in the context of KdV equations. For the Schrödinger equation in the case $H = -\Delta$, it has been proved by P. Constantin- J.-C. Saut [2], P. Sjölin [11], L. Vega [12] and K. Yajima [13]. The variable coefficients case has been obtained by S. Doi [3, 4, 5, 6]. The more general results are due to L. Robbiano-C. Zuily [9, 10] for equations with obstacles and potentials.

Let $H$ be a self-adjoint operator on $L^2(\mathbb{R}^d)$. It can be represented thanks to the spectral measure by

$$H = \int \lambda dE_\lambda.$$

In the sequel we moreover assume that $H \geq 0$. For $N \geq 0$, we can then define the spectral projector $P_N$ associated to $H$ by

$$(1.2) \quad P_N = 1_{[N,N+1]}(H) = \int 1_{[N,N+1]}(\lambda) dE_\lambda.$$

Our main result is a characterisation of the smoothing effect by the decay of the spectral projectors. Denote by $\langle H \rangle = (1 + |H|^2)^{1/4}$.

**Theorem 1.1 (Smoothing effect vs. decay). —**

Let $\gamma > 0$ and $\Psi \in C(\mathbb{R}^d, \mathbb{R})$. Then the following conditions are equivalent

(i) There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$

$$(1.3) \quad \left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^{27} e^{-itH}f\|^2_{L^2(\mathbb{R}^d)} dt \right)^{1/2} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}.$$

(ii) There exists $C_2 > 0$ so that for all $N \geq 1$ and $f \in L^2(\mathbb{R}^d)$

$$(1.4) \quad \|\Psi P_N f\|_{L^2(\mathbb{R}^d)} \leq C_2 N^{-7/2} \|P_N f\|_{L^2(\mathbb{R}^d)}.$$

The interesting point is that we can take the same function $\Psi$ and exponent $\gamma > 0$ in both statements (1.3) and (1.4).

By the works cited in the introduction, in the case $H = -\Delta$ on $\mathbb{R}^d$, (1.3) is known to hold with $\gamma = \frac{1}{2}$ and $\Psi(x) = \langle x \rangle^{-\frac{1}{2}}$, for any $\nu > 0$.

There is also a class of operators $H$ on $L^2(\mathbb{R}^d)$ for which (1.3) is well understood. Let $V \in C^\infty(\mathbb{R}, \mathbb{R}_+)$, and assume that for $|x|$ large enough $V(x) \geq C(x)^k$ and that for any $j \in \mathbb{N}^d$, there exists $C_j > 0$ so that $|\partial_j^l V(x)| \leq C_j (x)^{k-j}$. Then L. Robbiano and C. Zuily [9] show that the smoothing effect (1.3) holds for the operator $H = -\Delta + V(x)$, with $\gamma = \frac{1}{k}$ and $\Psi(x) = \langle x \rangle^{-\frac{1}{k}}$, for any $\nu > 0$. 
We now turn to the case of dimension $d = 1$, and consider the operator $H = -\Delta + V(x)$. We make the following assumption on $V$

**Assumption 1.** — We suppose that $V \in C^\infty(\mathbb{R}, \mathbb{R}_+)$, and that there exist $2 < m \leq k$ so that for $|x|$ large enough

(i) There exists $C > 1$ so that $\frac{1}{C} \langle x \rangle^k \leq V(x) \leq C \langle x \rangle^k$.

(ii) $V''(x) > 0$ and $x V'(x) \geq mV(x) > 0$

(iii) For any $j \in \mathbb{N}$, there exists $C_j > 0$ so that $|\partial_x^j V(x)| \leq C_j \langle x \rangle^{k-|j|}$.

For instance $V(x) = \langle x \rangle^k$ with $k > 2$ satisfies Assumption 1.

It is well known that under Assumption 1, the operator $H$ has a self-adjoint extension on $L^2(\mathbb{R})$ (still denoted by $H$) and has eigenfunctions $(e_n)_{n \geq 1}$ which form an Hilbertian basis of $L^2(\mathbb{R})$ and satisfy

$$He_n = \lambda_n^2 e_n, \quad n \geq 1,$$

with $\lambda_n \to +\infty$, when $n \to +\infty$.

For $N \in \mathbb{N}$ the spectral projector $P_N$ defined in (1.2) can be written in the following way. Let $f = \sum_{n \geq 1} \alpha_n e_n \in L^2(\mathbb{R})$, then

$$P_N f = \sum_{N \leq \lambda_n^2 < N+1} \alpha_n e_n.$$

Observe that we then have $f = \sum_{N \geq 0} P_N f$.

For such a potential, we can remove the spectral projectors in (1.4) and deduce from Theorem 1.1

**Corollary 1.2.** —

Let $\gamma > 0$ and $\Psi \in C(\mathbb{R}, \mathbb{R})$. Let $H = \Delta + V(x)$ so that $V(x) = x^2$ or $V(x)$ satisfies Assumption 1. Then the following conditions are equivalent

(i) There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R})$

$$\left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^\frac{1}{2} e^{-iH} f \|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R})}.$$

(ii) There exists $C_2 > 0$ so that for all $n \geq 1$

$$\|\Psi e_n\|_{L^2(\mathbb{R})} \leq C_2 \lambda_n^{-\gamma}, \quad \forall n \geq 1.$$

The statements (1.5) and (1.6) were obtained by K. Yajima & G. Zhang in [16] when $\Psi$ is the indicator of a compact $K \subset \mathbb{R}$ and with $\gamma = \frac{1}{k}$. 


The statement (1.5) holds for $\Psi(x) = \langle x \rangle^{1/2 - \nu}$, by [9], but as far as we know, the bound (1.6) with this $\Psi$ was unknown.

With Theorem 1.1 we are also able to prove the following smoothing effect for the usual Laplacian $\Delta$ on $\mathbb{R}$.

**Proposition 1.3.** — Let $\Psi \in L^2(\mathbb{R})$. Then there exists $C > 0$ so that for all $f \in L^2(\mathbb{R})$

$$\left( \int_0^{2\pi} \| \Psi(x) \langle \Delta \rangle^{1/2} e^{-it\Delta} f \|_{L^2(\mathbb{R})}^2 dt \right)^{1/2} \leq C \| \Psi \|_{L^2(\mathbb{R})} \| f \|_{L^2(\mathbb{R})}.$$ 

From the works cited in the introduction, we have

$$\left( \int_{\mathbb{R}} \| \Psi(x) \langle \Delta \rangle^{1/2} e^{-it\Delta} f \|_{L^2(\mathbb{R})}^2 dt \right)^{1/2} \leq C \| f \|_{L^2(\mathbb{R})},$$

for $\Psi(x) = \langle x \rangle^{1/2 - \nu}$, for any $\nu > 0$. Hence Proposition 1.3 shows that we can extend the class of the weights, but we are only able to prove local integrability in time.

**Notation.** — We use the notation $a \lesssim b$ if there exists a universal constant $C > 0$ so that $a \leq Cb$.

2. Proof of the results

We define the self adjoint operator $A = [H]$ (entire part of $H$) by

$$A = \int |\lambda| dE_\lambda.$$ 

Notice that we immediately have that $A - H$ is bounded on $L^2(\mathbb{R}^d)$.

The first step in the proof of Theorem 1.1 is to show that we can replace $e^{-itH}$ by $e^{-itA}$ in (1.3)

**Lemma 2.1.** — Let $\gamma > 0$ and $\Psi \in C(\mathbb{R}^d, \mathbb{R})$. Then the following conditions are equivalent

(i) There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$

$$\left( \int_0^{2\pi} \| \Psi(x) \langle H \rangle^{1/2} e^{-itH} f \|_{L^2(\mathbb{R}^d)}^2 dt \right)^{1/2} \leq C_1 \| f \|_{L^2(\mathbb{R}^d)}.$$ 

(ii) There exists $C_2 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$

$$\left( \int_0^{2\pi} \| \Psi(x) \langle H \rangle^{1/2} e^{-itH} f \|_{L^2(\mathbb{R}^d)}^2 dt \right)^{1/2} \leq C_2 \| f \|_{L^2(\mathbb{R}^d)}.$$
Proof. — We assume (2.1) and we prove (2.2). Let \( f \in L^2(\mathbb{R}^d) \) and define \( v = e^{-itH}f \). This function solves the problem

\[
(i\partial_t - A)v = (H - A)v, \quad v(0, x) = f(x).
\]

Then by the Duhamel formula

\[
e^{-itH}f = v = e^{-itA}f - i\int_0^t e^{-i(s-t)A}(H - A)v ds
\]

Therefore by (2.1) and Minkowski

\[
\|\Psi\langle H\rangle_\gamma e^{-itH}v\|_{L^2_2} \lesssim \|\Psi\langle H\rangle_\gamma e^{-itA}v\|_{L^2_2} + \int_0^{2\pi} \|\Psi\langle H\rangle_\gamma e^{-i(s-t)A}(H - A)v\|_{L^2_2} ds
\]

(2.3)

Now use that the operator \( (H - A) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is bounded, and by (2.3) we obtain

\[
\|\Psi\langle H\rangle_\gamma e^{-itH}v\|_{L^2_2} \lesssim \|f\|_{L^2},
\]

which is (2.2).

The proof of the converse implication is similar.

Proof of Theorem 1.1 — The proof is based on Fourier analysis in time. This idea comes from [8] and has also been used in [16], but this proof was inspired by [1].

\((i) \implies (ii)\) : To prove this implication, we use the characterisation (2.1). From (1.2) and the definition of \( A \), \( e^{-itA}P_N f = e^{-itN}P_N f \). Hence it suffices to replace \( f \) with \( P_N f \) in (1.3) and (1.4) follows.

\((ii) \implies (i)\) : Again we will use Lemma 2.1. We assume (2.2) and we first prove that

\[
\|\Psi\langle A\rangle_\gamma e^{-itA}f\|_{L^2_2(0,2\pi;L^2(\mathbb{R}^d))} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.
\]

Write \( f = \sum_{N \geq 0} P_N f \), then

\[
\Psi\langle A\rangle_\gamma e^{-itA}f = \sum_{N \geq 0} e^{-itN}(N) \Psi\langle A\rangle_\gamma P_N f.
\]
Now by Parseval in time
\[ \| \Psi \langle A \rangle \frac{\xi}{2} e^{-itA} f \|^2_{L^2(0,2\pi)} \lesssim \sum_{N \geq 0} |N|^\gamma |\Psi \rangle_{P_N f}|^2, \]
and by integration in the space variable and (1.1)
\[ \| \Psi \langle A \rangle \frac{\xi}{2} e^{-itA} f \|^2_{L^2(0,2\pi;L^2(\mathbb{R}^d))} \lesssim \sum_{N \geq 0} \langle N \rangle^\gamma \| \Psi P_N f \|^2_{L^2(\mathbb{R}^d)} \]
\[ \lesssim \sum_{N \geq 0} \| P_N f \|^2_{L^2(\mathbb{R}^d)} = \| f \|^2_{L^2(\mathbb{R}^d)}, \]
which yields (2.4).

Now since the operator \( \langle A \rangle - \frac{\gamma}{2} \langle H \rangle^{\gamma/2} \) is bounded on \( L^2 \) and commutes with \( e^{-itA} \), we have by (2.4)
\[ \| \Psi \langle H \rangle \frac{\xi}{2} e^{-itA} f \|^2_{L^2(0,2\pi;L^2(\mathbb{R}^d))} = \]
\[ = \| \Psi \langle A \rangle \frac{\xi}{2} e^{-itA} \langle A \rangle - \frac{\gamma}{2} \langle H \rangle^{\gamma/2} f \|^2_{L^2(0,2\pi;L^2(\mathbb{R}^d))} \]
\[ \lesssim \| \langle A \rangle - \frac{\gamma}{2} \langle H \rangle^{\gamma/2} f \|^2_{L^2(\mathbb{R}^d)} \]
\[ \lesssim \| f \|^2_{L^2(\mathbb{R}^d)}, \]
which is (2.1).

**Proof of Corollary 1.2.** — Let \( V \) satisfy Assumption 1. Then by [14, Lemma 3.3] there exists \( C > 0 \) such that
\[ |\lambda_{n+1}^2 - \lambda_n^2| \geq C\lambda_n^{1-\frac{m}{2}}, \]
for \( n \) large enough. This implies that \( [\lambda_n^2] < [\lambda_{n+1}^2] \) for \( n \) large enough, because \( m > 2 \) and \( \lambda_n \to +\infty \). As a consequence
\[ P_N f = \alpha_n e_n, \quad \text{with } n \text{ so that } N \leq \lambda_n^2 < N + 1, \]
and this yields the result.

We now consider \( V(x) = x^2 \). In this case, the eigenvalues are the integers \( \lambda_n^2 = 2n + 1 \), and the claim follows.

**Remark 2.2.** — With this time Fourier analysis, we can prove the following smoothing estimate for \( H \) which satisfies Assumption 11
\[ \| \langle H \rangle^{\theta(q,k)} e^{-itH} f \|_{L^p(\mathbb{R};L^2(0,T))} \lesssim \| f \|_{L^2(\mathbb{R})}, \]
where $\theta$ is defined by

$$
\theta(q, k) = \begin{cases} 
\frac{2}{k} \left( \frac{1}{2} - \frac{1}{q} \right) & \text{if } 2 \leq q < 4, \\
\frac{1}{2k} - \eta & \text{for any } \eta > 0 \text{ if } q = 4, \\
\frac{1}{2} - \frac{2}{3} (1 - \frac{1}{q}) (1 - \frac{1}{k}) & \text{if } 4 < q < \infty, \\
\frac{4 - k}{6k} & \text{if } q = \infty.
\end{cases}
$$

This was done in [16] with a slightly different formulation.

**Proof of Proposition 1.3** — By Theorem 1.1, we have to prove that the operator $T$ defined by

$$
T f(x) = N^{\frac{1}{2}} \Psi(x) 1_{[N,N+1]}(-\Delta) f(x),
$$

is continuous from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ with norm independent of $N \geq 1$. By the usual $TT^*$ argument, it is enough to show the result for $TT^*$.

The kernel of $T$ is $K(x, y) = N^{\frac{1}{2}} \Psi(x) F_N(x - y)$ where

$$
F_N(u) = \frac{1}{2\pi} \int e^{iu\xi} 1_{[\sqrt{N}, \sqrt{N+1}]}(|\xi|) d\xi = 4 \cos(D_N u) \frac{\sin(C_N u)}{u},
$$

with $C_N = (\sqrt{N+1} - \sqrt{N})/2$ and $D_N = (\sqrt{N+1} + \sqrt{N})/2$.

The kernel of $TT^*$ is given by

$$
\Lambda(x, z) = \int K(x, y) \overline{K}(z, y) dy,
$$

and by Parseval and (2.5)

$$
\Lambda(x, z) = N^{\frac{1}{2}} \Psi(x) \Psi(z) \int F_N(x - y) \overline{F_N(z - y)} dy \\
= \frac{1}{4} N^{\frac{1}{2}} \Psi(x) \Psi(z) \int e^{i(x-z)\xi} 1_{[\sqrt{N}, \sqrt{N+1}]}(|\xi|) d\xi \\
= \pi N^{\frac{1}{2}} \Psi(x) \Psi(z) \cos(D_N(x - z)) \frac{\sin(C_N(x - z))}{x - z}.
$$

Now, since $C_N \lesssim 1/\sqrt{N}$ and $|\sin(x)| \leq |x|$, we deduce that $|\Lambda(x, z)| \leq C|\Psi(x)||\Psi(z)|$ (independent of $N \geq 1$), and $TT^*$ is continuous for $\Psi \in L^2(\mathbb{R})$.

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