An upper bound to multiscale roughness-induced adhesion enhancement

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Abstract

Recently Guduru and coworkers have demonstrated with neat theory and experiments that both increase of strength and of toughness are possible in the contact of a rigid sphere with concentric single scale of waviness, against a very soft material. The present note tries to answer the question of a multiscale enhancement of adhesion, considering a Weierstrass series to represent the multiscale roughness, and analytical results only are used. It is concluded that the enhancement is bounded for low fractal dimensions but it can happen, and possibly to very high values, whereas it is even unbounded for high fractal dimensions, but it is also much less likely to occur, because of separated contacts.

Keywords:
Roughness, Adhesion, Guduru’s theory, Fuller and Tabor’s theory

1. Introduction

Guduru and collaborators (Guduru (2007), Guduru & Bull (2007), Waters et al (2009)) have recently considered a model in which a sphere has a superposed waviness, as defined by the axisymmetric form

\[ f(r) = \frac{r^2}{2R} + A \left(1 - \cos \frac{2\pi r}{\lambda}\right) \]  

i.e. with concentric waviness, where \( R \) is the sphere radius, \( \lambda \) is wavelength of roughness (see an example in Fig.1). Guduru also shows that similar results are obtained if a plane roughness is assumed, similar to the function above.
by with $x$–coordinate rather than $r$. Guduru shows that very significant (one order of magnitude) increase of strength as well as toughness can be obtained by adding roughness, i.e. with respect to the smooth case. It should be immediately remarked that Jin et al (2011) have since then shown that some of the enhancement obtained by Guduru is specific to this assumption (either axisymmetric or purely 1D roughness), and therefore we may expect much less enhancement for, say, random roughness.

![Fig.1 The Guduru sphere for $R = \lambda = A = 1$](image)

The concentric waviness permits a quite simple exact axisymmetric analysis, assuming a simply connected contact area develops. Already for a single waviness as in Guduru (2007), there are some limitations for this solution to hold, as clearly for ”sufficiently” large amplitude of roughness a realistic solution will show some separated contacts. Also, Waters et al (2009) have clarified that much of the enhancement comes from the assumption of JKR regime, and therefore one needs to check also the ”Tabor parameter”.

We shall here try to repeat some of the Guduru (2007) aspects of the solution, in the context of a multiscale roughness, as it is more likely to occur in practical cases, using for simplicity a Weierstrass series instead of a single sinusoid, which was used in related contexts in Ciavarella et al (2000) without adhesion for the fully separated regime, and by Afferrante et al (2015) with adhesion, but with limited results concerning loading phase. Specifically, we assume

$$f(r) = f_0(r) + g_0 \sum_{n=0}^{\infty} \gamma^{(D-2)n} \cos \left( 2\pi \gamma^n r / \lambda_0 \right)$$

(2)
where $f_0(r)$ is a “smooth profile” defining function, which is a convex punch for example $f_0(r) = \frac{r^2}{2R}$ - we introduce this to avoid having to deal with a fully periodic surface, for which the ”smooth” behaviour is itself more difficult to define. If $\gamma > 1$ and $D > 1$, eq. (2) defines, in a plane section, a plane fractal surface of fractal dimension $D$ (the real surface dimension will be one unit higher), where we have

$$g_n = g_0\gamma^{(D-2)n}, \quad \lambda_n = \lambda_0\gamma^{-n}$$

(3)

and hence the radius at given scale $n$ is $R_n = \frac{1}{g_n} \left( \frac{\lambda_n}{2\pi} \right)^2 = \frac{1}{g_0} \frac{\lambda_0^2}{4\pi^2} \gamma^{-Dn}$.

Fig. 2 plots some examples of rough spheres so produced. Notice that the roughness may equally be present in the other body, although Guduru for his experiments considered a rough rigid sphere against an elastic nominally flat material.
Fig. 1 The Weierstrass sphere for $R = g_0 = \lambda = 1$, $D = 1.2$. For (a) $\gamma = 2$ and (b) $\gamma = 4$

2. Some results

Waters et al (2009) give a good summary of Guduru’s theory and experiments: it is shown that the load oscillates when it crosses a crest of a wave, and this results in highly “wavy” curves. We will not give detailed account of this theory, as we shall instead concentrate on an asymptotic expansion solution (which permits, by joining all the minima and maxima of the resulting function, also to obtain an “envelope” solution) given by Kesari et al (2011) for small wavelength, in particular $\lambda << a$, where $a$ is the contact area radius.

Kesari et al (2011) suggest that if roughness is described by a function $\lambda_0 \varrho (r/\lambda_0)$, where the dimensionless function $\varrho (r/\lambda_0)$ can be expanded in Fourier series. Here, we shall use directly the Kesari result as a special case for the Weierstrass series in order to get deterministic results for the maxima and minima. Weierstrass is in fact a restricted form of Fourier series as we shall consider $\gamma$ as integer and

$$\varrho (\xi) = \sum_{n=0}^{\infty} a_n \cos (2\pi \gamma^n \xi)$$ (4)

According to the Kesari et al (2011) expansion, the equilibrium curves are described by load $P_K (a)$ and approach $h_K (a)$

$$P_K (a) = P_M (a) - E^* \sqrt{2\pi a^3 \lambda_0 \rho (a/\lambda_0)}$$ (5)

$$h_K (a) = h_M (a) - \sqrt{\frac{\pi a \lambda}{2}} \rho (a/\lambda_0)$$ (6)

where $E^*$ is plane strain elastic modulus, $h_M (a)$, $P_M (a)$ correspond to the smooth profile solution, and for $\xi = a/\lambda_0$, the function $\rho (\xi)$ is given by

$$\rho (\xi) = \sum_{n=0}^{\infty} \sqrt{2\pi \gamma^n} \left[ -a_n \sin \left( 2\pi \gamma^n \xi - \frac{\pi}{4} \right) \right]$$ (7)

Guduru’s case is recovered when $a_0 = A/\lambda = A/\lambda_0$, and the macroscopic shape $f_0 (r)$ is Hertzian parabola. To find the envelope, one simply needs to
take the maxima and minima of the equilibrium curve, which are trivial for a single sinusoid. In fact, in this case

\[ P_K(a) = P_M(a) \pm 2\pi E^* \frac{A}{\lambda_0} \sqrt{a^3 \lambda_0} \]  
\[ h_K(a) = h_M(a) \pm \pi \frac{A}{\lambda_0} \sqrt{a \lambda_0} \]

Before proceeding further, let us notice that an interesting feature emerges in general, and that is that the smooth profile solution \( h_M(a), P_M(a) \) contains a profile-independent contribution (which essentially is the flat punch solution term in the JKR process), and a profile dependent part \( h_{M,\text{profile}}(a), P_{M,\text{profile}}(a) \). With this separation, for example using 2.12a, 2.13a of Kesari et al (2011), one can derive at the quite general expressions for the Weierstrass series roughness

\[ P_K(a) = P_{M,\text{profile}}(a) - a^{3/2} \sqrt{8\pi w E^*} \left( 1 \pm \frac{1}{\alpha_0 \sqrt{\pi}} \sum_{n=0}^{\infty} \sqrt{\gamma^n + 1} \left[ \gamma^{(D-2)n} \sin \left( 2\pi \gamma^n \xi - \frac{\pi}{4} \right) \right] \right) \]
\[ h_K(a) = h_{M,\text{profile}}(a) - a^{1/2} \sqrt{\frac{2\pi w}{E^*}} \left( 1 \pm \frac{1}{\alpha_0 \sqrt{\pi}} \sum_{n=1}^{\infty} \sqrt{\gamma^n + 1} \left[ \gamma^{(D-2)n} \sin \left( 2\pi \gamma^n \xi - \frac{\pi}{4} \right) \right] \right) \]
\[ \alpha_0 = \sqrt{\frac{2w \lambda_0}{\pi^2 E^* g_0^2}} \]

is the parameter Johnson (1995) introduced for the JKR adhesion problem of a nominally flat contact with a single scale sinusoidal waviness of amplitude \( g_0 \) and wavelength \( \lambda_0 \).

In the general case, if we had used a Fourier representation of roughness, we would not have known how the maxima and minima of the various Fourier components could combine. But as here we are considering a Weierstrass series and we can take \( \gamma >> 1 \), then the maxima and minima simply sum
algebraically, leading to the envelope (assuming \( \sqrt{\gamma^n + 1} \simeq \gamma^{n/2} \))

\[
P_{K,env}(a) = P_{M,profile}(a) - a^{3/2} \sqrt{8\pi w E^*} \left( 1 \pm \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \right) \tag{13}
\]

\[
h_{K,env}(a) = h_{M,profile}(a) - a^{1/2} \sqrt{2\pi w \over E^*} \left( 1 \pm \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \right) \tag{14}
\]

where we introduced a scale-dependent Johnson parameter

\[
\alpha_n = \alpha_0 \left( \gamma^{(2D-3)n} \right)^{-1/2} = \frac{\alpha_0}{\gamma^{(D-3/2)n}} \tag{15}
\]

The series defined by the sum of the Johnson parameters converges for all \( D < 1.5 \) as

\[
\lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{\alpha_n} = \frac{1}{\alpha_0} \lim_{N \to \infty} \sum_{n=0}^{N} \gamma^{(D-3/2)n} = \frac{1}{\alpha_0} \lim_{N \to \infty} \left( \frac{\gamma^{(D-3/2)(1+N)} - 1}{\gamma^{(D-3/2)} - 1} \right) = \frac{1}{\alpha_0} \left( \frac{1}{1 - \gamma^{(D-3/2)}} \right) \tag{16}
\]

which is the case of common interest for fractal surfaces, and which is the case where we can expect more easily enhancement anyway since alternative solutions during loading phase only (Afferrante et al, 2015) show that only in this case of \( D < 1.5 \) we expect a limit contact area due to infinite roughness: for higher fractal dimension, the contact resembles increasingly that obtained in the absence of adhesion, as in the classical (Ciavarella et al., 2000) solution for the Weierstrass profile.

With respect to the smooth surface therefore, it is easy to show that we have obtained the amplification factor for pull off as

\[
F(\alpha_0, \gamma, D) = \left( 1 + \frac{1}{\alpha_0 \sqrt{\pi}} \left( \frac{1}{1 - \gamma^{(D-3/2)}} \right) \right)^2 \tag{17}
\]

which is plotted in Fig.3 for representative values.
Fig. 3 Pull-off amplification factor $F(\alpha_0, \gamma, D)$ for $\alpha_0 = 0.56$ (a) and $\alpha_0 = 1$ (b). In both cases, thick solid line is $\gamma = 5$, and dashed line is $\gamma = 10$.

We have taken in Fig. 3 values of $\gamma$ relatively high, as otherwise there is no guarantee that our procedure in estimating the cumulative scale effect was accurate. Therefore, we chose $\gamma = 5$ and $\gamma = 10$ while Fig. 3a and Fig. 3b distinguish for the initial value of the Johnson’s parameter. As it is evident also from Fig. 3, this factor can be extremely high for the fractal dimensions near $D = 1.5$, and this is not so surprising considering that we found the amplification doesn’t converge for higher $D$. Notice also that the for the chosen parameters of $\alpha_0 = 0.56, 1$, the amplification for a single wave of roughness were respectively 4 and $\left(1 + \frac{1}{\sqrt{\pi}}\right)^2 = 2.45$, and therefore the
multiscale additional enhancement is really significant.

Obviously similar remarks can be made regarding the factor \( \alpha_0 \): the lower this factor (and therefore the lower the adhesion effect in the Johnson solution of a single sinusoid), the higher the amplification. So it would seem from this analysis that the highest amplifications would occur for low \( \alpha_0 \) and high fractal dimensions, which is exactly the case where we expect more likely separated contact! This is simply an indication that separation is more and more likely the higher the amplification is expected to be in the simply connected contact area model we are assuming.

3. Discussion

There are various reasons why the amplifications in adhesion predicted by the theory are limited. We shall discuss them separately in order.

3.1. JKR assumption

JKR theory is strictly valid, in the classical JKR case of a sphere, when \( \mu > 5 \), but JKR works well for \( \mu > 0.3 \) in practice: below \( \mu = 0.3 \), the behaviour approaches that of a rigid sphere. In the original case of the sphere, JKR and rigid theory only differ by a small prefactor in the pull-off loads, but Tabor parameter was shown in Waters et al (2009) to limit enhancement of the Guduru problem considerably, namely \( \mu < 1 \) at the scale of the sphere completely destroys the enhancement. Following Waters et al (2009), we can define, for multiscale roughness, a scale-dependent Tabor parameter

\[
\mu_n = \frac{\sigma_{th}}{E^*} \left( \frac{9 R_n}{2\pi l_a} \right)^{1/3} = \mu_0 \gamma^{-D_n/3} \tag{18}
\]

where \( \sigma_{th} \) is theoretical strength of the material, \( E^* \) elastic modulus, \( l_a = w/E^* \) and \( R_n \) which is not a radius of a sphere, nevertheless can serve to estimate the role of elastic deformation at the \( n-th \) scale. It is clear that, even if at scale 0 the Tabor parameter is well in the JKR regime, with finer scales of roughness, the Tabor parameter would tend to reduce quite fast. For example, with \( \gamma = 10 \) and even with low \( D = 1.2 \), this reduces Tabor parameter at the macroscale to 1/10 just with one additional scale of roughness. Therefore, this factor alone will limit very substantially practical evidence of multiscale enhancement.
3.2. Loading dependence

In the original Guduru (2007) problem of a single sinusoid, the condition \( \alpha_0 > 0.56 \) was seen to correspond to self-flattening of waviness, irrespective of the applied load, as in that case it was shown to correspond also to imposing monotonicity of the profile, so that the solution was obtained without need of a proper, sufficiently high loading stage. Hence, the amplifications with this range was likely to occur. Obviously, this is the range where a single waviness, even from the equations above, amplification is lower than 4. But for multiscale roughness, we could increase this. Unfortunately, we do not know if \( \alpha_0 > 0.56 \) guarantees that self-flattening occurs on all scales. A first consideration seems to suggest that, for \( D < 1.5 \), as \( \alpha_n \) increase, a fortiori there should be self-flattening on all scales. However, we should check if there is also monotonicity of the profile — as otherwise, even if \( \alpha_n > 0.56 \), separated points in the profile may simply not be in a condition to jump into contact. Hence, it is useful therefore to extend Guduru’s analysis of monotonicity to Weierstrass, by taking the derivative of the surface function

\[
z'(r) = f'_0(r) + g_0 \frac{2\pi}{\lambda_0} \sum_{n=0}^\infty \gamma^{(D-1)n} \sin \left(2\pi \gamma^n r / \lambda_0\right) 
\]

(19)

The second term of this is obviously is related to the full contact pressure under adhesionless conditions of the Weierstrass profile contact (Ciavarella et al., 2000)

\[
p(x) = \bar{p} + \sum_{n=0}^{\infty} p_n^* \cos \left(2\pi \gamma^n x / \lambda_0\right), \quad \bar{p} - \hat{p} \leq p(x) \leq \bar{p} + \hat{p} 
\]

(20)

being

\[
p_n^* = \pi E^* g_0 \frac{\gamma^{(D-1)n}}{\lambda_0} = p_0^* \gamma^{(D-1)n} 
\]

(21)

\[
\hat{p} = \sum_{n=0}^{\infty} p_n^* = p_0^* \sum_{n=0}^{\infty} \gamma^{(D-1)n} 
\]

(22)

For \( \gamma > 1 \) and \( D > 1 \), the series (22) does not converge. This suggests that the monotonicity condition which Guduru could guarantee for just one

\[\text{Footnote 1: Indicating that there is no finite value of mean pressure } \bar{p} \text{ that is sufficient to ensure complete contact between a fractal rigid surface of the Weierstrass form and an elastic half-plane in the adhesiveless case.}\]
term of waviness, independently on loading, becomes increasingly more difficult to satisfy. Already with 2 scales, we find that one needs to rely on the loading process to find "complete" (simply connected) contact over the contact area. The amplification factor above could however be considered an upper bound, which could be reached upon sufficient pre-loading. Adhesion will certainly show pressure-sensitivity.

3.3. Kesari’s envelope validity

We have obtained the amplification factors under the implicit assumptions that Kesari’s envelope works. In the single scale waviness of Guduru, this was true for low values of $\beta_G = \frac{\lambda^3 E^*}{2\pi w R^2}$. Here, we can define a scale-dependent $\beta_n$ looking at the scale $n$ and the waviness at scale $n+1$,

$$\beta_n = \frac{\lambda_{n+1}^3 E^*}{2\pi w R^2_n} = \frac{16\pi^4 E^* g_0^2}{2\pi w} \frac{\gamma^{3n-3+2D_n}}{\lambda_0}$$

which rapidly goes to zero only for low fractal dimensions, and this suggests the Kesari envelope is increasingly more appropriate for this, most important case. Therefore, at least this assumption is not particularly restrictive.

4. Conclusions

We have attempted to extend the Guduru model to multiscale roughness, using a Weierstrass series. We found some estimates of the potential amplification, which is higher than that of the single scale of waviness. We find in particular that the potential amplification is bounded for $D < 1.5$ and is unbounded otherwise. However, many limitations suggest this amplification is often impractical to reach: the assumption of JKR regime becomes increasingly invalid for finer scales, the monotonicity of the profile, needed to guarantee simply connected contact area, is also very unpractical to reach, and the highest amplifications occur exactly where the assumption of a simply connected area is most difficult to satisfy. Finally, a true 1D or axisymmetric roughness is less common than random roughness, although one can contrive some systems to wrinkle only in one direction and therefore it is not unconceivable. When the roughness is random, this further reduces the expected amplifications.
5. References

Afferrante, L., Ciavarella, M., & Demelio, G. (2015). Adhesive contact of the Weierstrass profile. In Proc. R. Soc. A (Vol. 471, No. 2182, p. 20150248). The Royal Society.

Ciavarella, M., Demelio, G., Barber, J. R., & Jang, Y. H. (2000). Linear elastic contact of the Weierstrass profile. In Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences (Vol. 456, No. 1994, pp. 387-405). The Royal Society.

Guduru, P.R. (2007). Detachment of a rigid solid from an elastic wavy surface: theory J. Mech. Phys. Solids, 55, 473–488
Guduru, P.R., Bull, C. (2007). Detachment of a rigid solid from an elastic wavy surface: experiments J. Mech. Phys. Solids, 55, 473–488

Jin, C., Khare, K., Vajpayee, S., Yang, S., Jagota, A., & Hui, C. Y. (2011). Adhesive contact between a rippled elastic surface and a rigid spherical indenter: from partial to full contact. Soft Matter, 7(22), 10728-10736.

Johnson, K. L., K. Kendall, and A. D. Roberts. (1971). Surface energy and the contact of elastic solids. Proc Royal Soc London A: 324. 1558.

Kesari, H., Doll, J. C., Pruitt, B. L., Cai, W., & Lew, A. J. (2010). Role of surface roughness in hysteresis during adhesive elastic contact. Philosophical Magazine & Philosophical Magazine Letters, 90(12), 891-902.

Kesari, H., & Lew, A. J. (2011). Effective macroscopic adhesive contact behavior induced by small surface roughness. Journal of the Mechanics and Physics of Solids, 59(12), 2488-2510.

Waters, J.F. Leeb, S. Guduru, P.R. (2009). Mechanics of axisymmetric wavy surface adhesion: JKR–DMT transition solution, Int J of Solids and Struct 46 5, 1033–1042