Structural dynamic analysis using wavelets

C A Torres Oviedo¹ and J E Quiroga Mendez¹
¹Universidad Industrial de Santander, Bucaramanga, Colombia

E-mail: carlos_andres_torres@outlook.com, jabib@uis.edu.co

Abstract. In this paper, wavelet approach is studied to determine the torsional natural frequencies of a simple discretized structure excited by white noise, emulating environmental conditions. In this work, the implementation of this approach is described and the analytical results of one case are compared with the frequencies determined using Fourier analysis. Results have shown a good agreement between the values reported by Fourier analysis and the obtained by using Wavelet functions.

1. Introduction

Structures in operation are exposed to change due to material aging, the effect of environmental conditions in materials [1], and loss of stiffness due to earthquakes [2]. Therefore, in order to preserve structural integrity, it must be continuously monitoring the system to guarantee safety conditions and normal operation [3]. In this paper, the dynamic response, particularly the natural frequencies, of a beam is studied using Wavelets and its results are compared with the information provided by an FFT analysis. In some previous works, it has been reported the use of Wavelets to investigate the condition of the structure, e.g. [4], proposed an algorithm based on the Wavelet Morlet to predict abnormal conditions in a structure using as fault indicators wavelet coefficients, which are related with physical parameters and load conditions. In [5], it is determined the nodal response of a five-floor building by the Morlet Wavelet applying Continuous Wavelet Transform (CWT). The obtained results are compared with experimental data providing a close agreement with non-stationary, transients and nonlinear signals. Finally, [6] demonstrated the flexibility of CWT in the nodal analysis of a system because the transformation to the frequency domain is not limited to one function, as is the case of Fourier, instead of this, many other functions as the different wavelet mothers can be used to define natural frequencies.

2. Theoretical framework

In order to verify the effectivity of the Wavelet approach in the determination of the natural frequencies, a simple structure is proposed. The studied structure represents a simply supported square beam (0.1 x 0.1 m), discretized in five nodes and six elements where the element’s weight is distributed in each of the different nodes as shown in Figure 1. The considered degrees of freedom in the model is presented in the same figure, and they correspond to the vertical deflections $U_1, U_2, \ldots, U_5$, caused by the environmental excitation. A simplified analysis, based on the superposition principle, is tackled by considering only the linear and elastic range of the material, which is homogeneous and isotropic. Because small deformations are expected, it is assumed straight segments of constant area. The relation between excitation and dynamic response in the structure is expressed in Equation (1).
Figure 1. Schematic representation of the studied beam.

\[
[F] = [K] \ast [U] 
\]

Where \([U]\) is the displacement vector, \([K]\) is the stiffness matrix and \([F]\) is the external forces vector. For a prismatic element with displacements in its boundaries produced by unitary forces generating shear forces, bending moments and axial force, so the terms of the square stiffness matrix can be defined by energetic methods, in our this the Castigliano theorem as shown in Equation (2).

\[
[K] = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\]

(2)

Where \(E\) is Young’s modulus, \(A\) is the cross-sectional area of the beam and \(L\) is the element length. In the case of excitation, the dynamic analysis is performed using white noise, which it is also used to contaminate the amplitude in the nodes (1% of the amplitude of the excitation signal) simulating the different disturbances present in an actual measurement.

Because of structural strain energy dissipation, the damping is proportional to stiffness and due to friction with air, the damping is proportional to mass, in [7] is proposed combining the previous phenomena in a general damping matrix “C” as presented in Equation (3), which is employed in this study assuming a nodal damping of 5%.

\[
C = M \left\{ \sum_{j=0}^{N-1} a_j (M^{-1} K)^j \right\}
\]

(3)

In the Equation (3), \(j\) expresses the number of degrees of freedom and \(N\) the number of nodes. Finally, using the D’Alembert principle and Matlab a solution for the acceleration of the system is obtained using the motion equation as follows:

\[
m\ddot{u} + c\dot{u} + ku = p(t)
\]

(4)

2.1. Modal approximation solution

An approximate solution to the Equation (4) can be obtained neglecting the damping, so Equation (4) is reduced to Equation (5).

\[
\ddot{M}u(t) + Ku(t) = 0
\]

(5)

Where one solution can be stated as shown Equation (6),

\[
u(t) = \Phi_1 e^{\lambda_1 t}
\]

(6)
Here, $\Phi_i$ are the real eigenvectors and $\lambda_i$ are the eigenvalues, which correspond to the nodal shapes and natural frequencies respectively. Then, orthogonality modes properties can be defined like shows Equations (7) and (8).

$$\Phi^T M \Phi = [m_i]$$ (7)

$$\Phi^T K \Phi = [k_i]$$ (8)

Where $m_i$ represent the modal masses, $k_i$ is the modal stiffness and the superindex $T$ is the matrix transpose. Finally, the natural frequencies for the non-damping case of each mode can be determined by Equation (9).

$$\omega_i^2 = k_{e,i}/m_i$$ (9)

A modulated pulse is used to generate the power spectral density (PSD) of each signal obtained from nodal excitation.

2.2. Wavelet dynamic analysis

Wavelet analysis transforms the time domain signal to a multiresolution time-scale representation. Wavelet transforms are based on small wavelets (mother wavelets) with limited duration. The scaled-version wavelets allow analyzing the signal in a different scale (multiresolution) [8].

In this work, the mother wavelet biorthogonal named “Morlet” represented by $\psi_\sigma(t)$ in Equation (10), is used based on the results obtained by [4–6]. In these works, Wavelet analysis with Morlet provided structural and mechanical elements natural frequencies under ambient excitations. The wavelet is defined as a constant $K_\sigma$ subtracted from a plane wave and then localised by a Gaussian window.

$$\psi_\sigma(t) = C_\sigma \pi^{-\frac{1}{4}} e^{-\frac{1}{2} \sigma^2} (e^{i\sigma t} - k_\sigma)$$ (10)

where:

$$\psi_{a,b}(t) = a^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right)$$ (13)
Here, α is the scale parameter which is related to the frequency and b is the wavelet translation term applied to the signal. Thus, translation and scale are represented by Equation (13); The CWT is applied to each nodal acceleration vector obtaining vectors named scales, which are converted to frequencies using the Equation (14).

\[ F_\alpha = \frac{f_c}{\alpha \Delta} \]  

(14)

where \( \alpha \) is the scale, \( \Delta \) is the sampling period, \( f_c \) is the wavelet central frequency and \( F_\alpha \) is the pseudo-frequency associated to the scale \( \alpha \) [9]. The relation between scales and \( F_\alpha \) is presented in Figure 2, for the values used in this study.

![Figure 2. Relation between frequencies and scales in the CWT.](image)

3. Analysis of results

Figure 3 illustrates the power spectral density of the average acceleration signal for the five nodes of the studied beam. The signals are obtained by exciting simultaneously all nodes.

![Figure 3. PSD of the acceleration signals of the five nodes at the studied beam.](image)
In Table 1 is presented the frequency peak values of the PSD by FFT (Fast Fourier Transform) and using the modal approximation:

| Frequencies   | By modal approach | By FFT approach | Difference per percentage |
|---------------|------------------|----------------|---------------------------|
| 2.099 Hz      | 2.051 Hz         |                | 2.296%                    |
| 8.388 Hz      | 8.613 Hz         |                | 2.679%                    |
| 18.756 Hz     | 18.87 Hz         |                | 0.604%                    |
| 32.487 Hz     | 32.4 Hz          |                | 0.269%                    |
| 46.482 Hz     | 46.76 Hz         |                | 0.598%                    |

Now, CWT is applied to the nodal response as it was implemented for the Fourier case. As a result, five response matrices (one for each node) are obtained. Each matrix is composed of the variables: coefficients, time and scale, as depicted in Figure 4. In this figure, red areas correspond to the peak values of the CWT coefficients of the fifth node. In this node, three different scales are identified as peak values (11, 17 and 40). Additionally, in the third node, two extra scales (7 and 161) are also considered as the maximum coefficients values. Finally, by using Equation (14) the selected scales are expressed as frequencies.

One important factor to mention is that peak values are not present with the same amplitude in all responses in CWT domain, these peaks depend on the position of the measurement with respect to the structure geometry and the mode in the given instant of time. It is also observed discontinues oscillations in which peaks appear, i.e. the dynamic behavior of the frequencies, which exist during periods of time, decrease in the level of correlation and reappear again in another or several periods.

CWT provides a frequency range for some time interval in which peak energy is located, at this position the dynamic information is available (considering that CWT provides time variable frequency ranges, thus, for different time exist probability of determining specific frequencies but never exact frequencies and the instants at the same time by the uncertainty principle) [10]. In Table 2 is presented the average between range limits in which the windows have separated the components i.e. these features and dynamic response are time-dependent.
Table 2. Comparison of natural frequencies by modal and CWT approach.

| Frequencies   | By modal approach | By CWT approach | Difference per percentage |
|---------------|-------------------|-----------------|---------------------------|
| 2.099 Hz      | 2.120 Hz          | 0.990%          |
| 8.388 Hz      | 8.531 Hz          | 1.702%          |
| 18.756 Hz     | 20.074 Hz         | 7.023%          |
| 32.487 Hz     | 31.023 Hz         | 4.507%          |
| 46.482 Hz     | 48.75 Hz          | 4.879%          |

4. Conclusions
In this paper, a structural dynamic analysis is proposed based on the Wavelet transform via Morlet mother wavelet. Natural frequencies are detected in the time-scale-frequency plane, however, the time information contained in the CWT supply an enormous quantity of data, most of them redundant. Thus, this method results impractical for determining the dynamic characteristics in complex structures. On the other hand, although the frequency resolution it is not as good as Fourier, it is possible to track the frequency content with the CWT.

References
[1] Rodríguez Caro D A, Vera López E and Muñoz Barajas H M 2016 Diseño e implementación de un sistema de adquisición y monitoreo de datos (shm) para un rectificador de protección catódica usado en ductos Respuestas 21 45–55
[2] Ayestarán A, Graciano C and González-Estrada O A 2017 Resistencia de vigas esbeltas de acero inoxidable bajo cargas concentradas mediante análisis por elementos finitos Rev. UIS Ing. 16 61–70
[3] Sánchez Acevedo H G, Uscátegui J and Gómez S 2017 Metodología para la detección de fallas en una estructura entramada metálica empleando las técnicas de análisis modal y PSO Rev. UIS Ing. 16 43–50
[4] Noh Hae Young, Nair Krishnan K, Kiremidjian Anne S and Loh C H 2009 Application of time series based damage detection algorithms to the benchmark experiment at the National Center for Research on Earthquake Engineering (NCREE) in Taipei, Taiwan Smart Structures and Systems 5(1) 95-117
[5] Thai-Hoa Le and Yukio Tamura 2009 Modal Identification of ambient vibration structure using frequency domain decomposition and wavelet transform Proc. The Seventh Asia-Pacific Conference on Wind Engineering (Taipei: APCWE)
[6] Pislaru C, Freeman J M and Ford D G 2003 Modal parameter identification for CNC machine tools using Wavelet Transform Int. J. Mach. Tools Manuf. 43 987–93
[7] Caughey T K 1960 Classical normal modes in damped linear dynamic systems J. Appl. Mech. 27 269
[8] Yunlong Sheng 1996 The transforms and applications handbook ed Alexander D Poularikas (Boca Raton: CRC Press) Chapter 10
[9] Behnam Feizifar, Mahmoud Reza Haghifam, Soodabeh Soleymani 2011 Application of continuous wavelet transform for fault location in combined overhead line and cable distribution networks 7th International Conference on Electrical and Electronics Engineering (ELECO) (Bursa: IEEE) 1 142-146
[10] Michel Misiti, Yves Misiti, Georges Oppenheim, Jean-Michel Poggi 2009 Wavelet Toolbox™ 4. User’s Guide (Massachusetts: The MathWorks Inc.)