ERGODIC ACTIONS OF THE COMPACT QUANTUM GROUP $O_{-1}(2)$

ALEXANDRU CHIRVASITU, SOULEIMAN OMAR HOCHE

ABSTRACT. Among the ergodic actions of a compact quantum group $\mathcal{G}$ on possibly non-commutative spaces, those that are embeddable are the natural analogues of actions of a compact group on its homogeneous spaces. These can be realized as coideal subalgebras of the function algebra $\mathcal{O}(\mathcal{G})$ attached to the compact quantum group.

We classify the embeddable ergodic actions of the compact quantum group $O_{-1}(2)$, basing our analysis on the bijective correspondence between all ergodic actions of the classical group $O(2)$ and those of its quantum twist resulting from the monoidal equivalence between their respective tensor categories of unitary representations.

In the last section we give counterexamples showing that in general we cannot expect a bijective correspondence between embeddable ergodic actions of two monoidally equivalent compact quantum groups.

Key words: compact quantum group, ergodic action, idempotent state

MSC 2010: 20G42; 16T05; 22D10; 22D25; 22D30

Introduction

Ergodic actions of compact groups on possibly noncommutative operator algebras offer a natural bridge between dynamical systems and non-commutative geometry. The topic has been studied extensively and we could not do justice to the literature, but we mention here the papers [15, 23–25], some of whose material will feature below in various ways.

With the advent of compact quantum groups introduced and studied by Woronowicz in [26–28] the scope of topics pertinent to the study of classical compact groups has expanded to include
these. In this context we mention [6,22], where the authors study ergodic coactions
\begin{equation}
N \to N \otimes A
\end{equation}
of non-commutative “function algebras” \(A\) of compact quantum groups on (typically again non-commutative) operator algebras \(N\), be it \(C^*\) or von Neumann.

Purely quantum phenomena arise: in stark contrast to ordinary compact groups, compact quantum groups can act ergodically on type-III factors ([22, Corollary 3.7]). Moreover, closer in spirit to the contents of this paper, it is explained in [22, §6] that the underlying non-commutative spaces on which a compact quantum group acts ergodically need not be a quotient by a quantum subgroup.

The so-called embeddable ergodic actions constitute a class that is intermediate between fully general and quotients by quantum subgroups. In the language of coactions (0.1) embeddability simply means that there is an embedding \(N \to A\) that respects the right \(A\)-coactions on both sides (see Section 1 below for precise definitions).

In the present paper we study the class of embeddable ergodic actions for the compact quantum group \(O_{-1}(2)\) obtained by “cocycle-twisting” the usual orthogonal group \(O(2)\) and fitting into the family of deformed orthogonal groups \(O_q(2)\) for \(q \in [-1,1]\), classifying such actions in Corollary 2.6.

Using the theory of idempotent states (analogous to idempotent measures on classical locally compact groups) and its relation to embeddable ergodic actions ([12]), the authors of [14] show that for the less-problematic values \(-1 < q \leq 1\) the embeddable ergodic actions of the \(q\)-deformations \(U_q(2)\), \(SU_q(2)\), and \(SO_q(3)\) do in fact all arise as quotients by quantum subgroups. Corollary 2.6 shows that this contrasts markedly with the situation for \(O_{-1}(2)\).

Cocycle deformation does not alter the monoidal category of representations of the compact quantum group [2], and implements an equivalence between the categories of ergodic actions [10].

The natural question arises of whether we also have a natural bijective correspondence between embeddable ergodic actions of two monoidally equivalent compact quantum groups. We will prove below in Section 3 that the answer is negative in general, in the strong sense that even for finite monoidally equivalent quantum groups with equidimensional underlying function algebras the numbers of isomorphism classes of embeddable ergodic actions need not be equal.

The paper is structured as follows.

Section 1 contains preparatory material to be used throughout the paper.

In Section 2 we study the ergodic actions of the cocycle twist \(O_{-1}(2)\) and classify those that are embeddable (Corollary 2.6). We then also describe them in terms of quantum subgroups of \(O_{-1}(2)\) and generalizations thereof (see §2.4).

Finally, Section 3 is concerned with studying to what extent embeddable ergodic actions transport over to a cocycle twist. We will see in Corollary 3.5 that even for finite quantum groups, this can fail in a very strong sense. Along the way, we analyze the cocycle twists of the dihedral groups \(D_K\) analogously to \(O_{-1}(2)\) and as discrete versions of the latter.
Acknowledgements. We thank Uwe Franz for his insights and advice on the contents of this paper.

This project was inspired by an interesting talk on actions of compact quantum groups given by Kenny De Commer at Bedlewo (28 June - 11 July, 2015). We are grateful to him both for his talk and for a very useful subsequent discussion during the workshop “Quantum groups from combinatorics to analysis” (Caen, 2016).

A.C. is grateful for funding through NSF grant DMS-1565226.

S.O.H acknowledge support by MAEDI/MENESR and DAAD through the PROCOPE program.

1. Preliminaries

We will need some background on coalgebras and Hopf algebras; for this, we refer the reader to any of the numerous good sources on the subject: e.g. [17, 19, 21].

Our algebras are all unital, and unless specified otherwise the ‘⊗’ symbol denotes minimal tensor products when placed between operator algebras ($C^*$ or, in rare cases, von Neumann algebras) and the plain, algebraic tensor product when placed between non-topological algebras.

Given functionals $\phi_i$, $i = 1, 2$ on a coalgebra $C$ with comultiplication $\Delta$ we denote by $\phi_1 \ast \phi_2$ their convolution defined by

$$
\Delta: C \rightarrow C \otimes C, \quad \phi_1 \ast \phi_2: C \rightarrow C
$$

1.1. Compact quantum groups. We adopt the notion of compact quantum group introduced by Woronowicz. The present recollection will be very brief, as the theory is quite expansive. We refer the reader to the excellent surveys [16, 29] for background on the topic.

Definition 1.1. A compact quantum group is a pair $(A, \Delta)$ where $A$ is a unital $C^*$-algebra and $\Delta: A \rightarrow A \otimes A$ is a unital $*$-homomorphism which is coassociative:

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta$$

and $A$ satisfies the quantum cancellation properties:

$$\text{Lin}((1 \otimes A) \circ \Delta(A)) = \text{Lin}((A \otimes 1) \circ \Delta(A)) = A \otimes A$$

We denote by $A^*$ the set of states of $A$. One of the most important features of compact quantum groups is the existence of a unique Haar state $h$, i.e a unique state on the $C^*$-algebra $A$ such that

$$(h \otimes id_A) \circ \Delta(a) = (id_A \otimes h) \circ \Delta(a) = h(a)1, \quad \forall a \in A.$$  

(1.1)

A compact quantum group is said of Kac type if $h$ is tracial i.e $h(ab) = h(ba), \forall a, b \in A.$ ♦
The $C^*$-algebra $A$ underlying a compact quantum group has a unique dense Hopf $*$-algebra $A$ (see [16, Theorem 3.2.2]), and much of the theory of compact quantum groups can be phrased purely algebraically, in terms of their underlying Hopf algebras $A$. Abstractly, these objects were introduced in [11] and following that source we use the following terminology to refer to them.

**Definition 1.2.** A CQG algebra is a complex Hopf $*$-algebra $A$ with a state $h : A \to \mathbb{C}$ satisfying (1.1) and which is positive in the sense that $h(a^*a) \geq 0$ for all $a \in A$, with equality only at $a = 0$.

We can largely go back and forth between the $C^*$ and purely algebraic context for studying compact quantum groups (see e.g. the discussion in [11], Sections 4 and 5):

- On the one hand, as mentioned above, for any $C^*$-algebraic compact quantum group as in **Definition 1.1** one can find a unique dense Hopf $*$-subalgebra that meets the criteria of **Definition 1.2**.
- Conversely, a CQG algebra has a universal $C^*$-completion that turns out to satisfy the requirements of **Definition 1.1**.

**Remark 1.3.** The $C^*$ envelope $C(G)$ from the above discussion is sometimes denoted by $C^u(G)$ (for universal), to distinguish it from other completions of $A(G)$ which in general exist and are also compact quantum groups in the sense of **Definition 1.1**. We focus mainly on the universal setting, as sketched above.

For the purposes of this paper it will be convenient to phrase things primarily in terms of Hopf algebras, reverting to their $C^*$ envelopes whenever necessary. We denote compact quantum groups by bold face letters such as $G$, by $C(G)$ the underlying $C^*$-algebra of the compact quantum group and by $A(G)$ its dense CQG algebra.

Moreover, we can also define a canonical von Neumann algebraic version of $G$:

**Definition 1.4.** $L^\infty(G)$ is the von Neumann closure of the GNS representation of $A(G)$ with respect to the Haar state $h$.

Note that $L^\infty(G)$ comes equipped with a coassociative comultiplication arising as the closure of

$$\Delta : A(G) \to A(G) \otimes A(G).$$

When referring to generic compact quantum groups we will sometimes be vague on which context we are in, unless it makes a difference.

Examples abound in the sources mentioned thus far; in this paper, the main compact quantum group is the following “twisted” version of the orthogonal group $O(2)$.

**Definition 1.5.** The compact quantum group $O_{-1}(2)$ is defined as the compact quantum group with underlying CQG algebra with self-adjoint generators $y = (y_{jk})_{1 \leq j, k \leq 2}$ and the relations

1. $y$ is orthogonal, i.e. the generators $y_{jk}$ are self-adjoint and satisfy the unitarity relations

$$y_{jk}y_{lk} + y_{kj}y_{lk} = \delta_{jk} = y_{j1}y_{k1} + y_{j2}y_{k2} \text{ for } j, k = 1, 2;$$

2. $y_{jk}y_{j\ell} = -y_{j\ell}y_{jk}$ and $y_{k\ell}y_{\ell j} = -y_{\ell j}y_{k\ell} \text{ for } k \neq \ell;$


\( y_{jky_{\ell m}} = y_{\ell m}y_{jk} \) for \( j \neq \ell \) and \( k \neq m \).

The coproduct, counit and antipode of \( O_{-1}(2) \) are given by

\[
\Delta(y_{jk}) = \sum_i y_{ji} \otimes y_{ik}, \quad \varepsilon(y_{jk}) = \delta_{jk}, \quad S(y_{jk}) = y_{kj}.
\]

The notion of a quantum subgroup was introduced by Podleś [18] for matrix pseudo-groups.

**Definition 1.6.** Let \( (A, \Delta_A) \) and \( (B, \Delta_B) \) be two compact quantum groups. Then \( (B, \Delta_B) \) is called a quantum subgroup of \( (A, \Delta_A) \), if there is exists a surjective \(-\)algebra homomorphism \( \pi : A \to B \) such that \( \Delta_B \circ \pi = (\pi \otimes \pi) \circ \Delta_A \).

**Definition 1.7.** A right coideal subalgebra or coideal algebra \( C \) in a compact quantum group \( (A, \Delta) \) is a unital \(-\)subalgebra \( C \subset A \) (complete when \( A \) is a \( C^* \)-algebra) such that \( \Delta(C) \subset C \otimes A \).

1.2. **Actions.** We aggregate here material on actions of compact quantum groups on compact non-commutative spaces. The reader may consult [7] for a good survey of the field.

In general, we will denote by \( G \) a compact quantum group realized either as a \( C^* \)-algebra \( C(G) \) as in **Definition 1.1** or as a CQG algebra \( \mathcal{A}(G) \) as described in **Definition 1.2**.

Similarly, we denote by \( X \) a compact quantum (or non-commutative) space, i.e. the object dual to a unital \( C^* \)-algebra \( C(X) \).

**Definition 1.8.** Let \( X \) and \( G \) be as above. A right action \( X \overset{\alpha}{\to} G \) is a \(*\)-morphism

\[
\alpha : C(X) \to C(X) \otimes C(G)
\]

(a coaction of \( C(G) \) on \( C(X) \)) which

- is coassociative in the sense that

\[
(\alpha \otimes \text{Id}_G) \circ \alpha = (\text{Id}_X \otimes \Delta) \circ \alpha,
\]

and

- \( \alpha(C(X))(C \otimes C(G)) \) is dense in \( C(X) \otimes C(G) \).

Keeping with the spirit of translating \( C^* \)-algebraic concepts into purely algebraic ones, we note that given an action \( \alpha \) as above there is a dense \(*\)-subalgebra \( A = \mathcal{A}(\alpha) \) (or more improperly \( \mathcal{A}(X) \), since it depends on \( \alpha \) and not just \( X \)) such that \( \alpha \) is a completion of a comodule algebra structure denoted by the same symbol:

\[
\alpha : A \to A \otimes \mathcal{A}(G).
\]

**Definition 1.9.** Let \( X \overset{\alpha}{\to} G \) and the quantum orbit space \( X/G \) i.e the \( C^* \)-algebra

\[
C(X/G) = \{ a \in C(X) \mid \alpha(a) = a \otimes 1 \}.
\]
Passing to the dense subalgebra $\mathcal{A} = \mathcal{A}(\alpha) \subseteq C(X)$ discussed above, it can be shown that the algebraic version of (1.3) defined by
\[ \mathcal{A}(X/G) := \{ a \in \mathcal{A} \mid \alpha(a) = a \otimes 1 \} \]
is dense in $\mathcal{A}$.

**Definition 1.10.** An action $X \overset{\alpha}{\curvearrowright} G$ is called ergodic if $C(X/G) = \mathbb{C}1$ or equivalently, if $\mathcal{A}(X/G) = \mathbb{C}$.

Let (1.2) be an ergodic algebraic action. It then turns out that there is a unique state $h_\alpha$ on $\mathcal{A}$ that is preserved by the coaction. We can then form the von Neumann closure $L^\infty = L^\infty(\mathcal{A}, h_\alpha)$ of the GNS representation of $\mathcal{A}$, and (1.2) lifts to a coassociative von Neumann algebra morphism
\[ L^\infty \to L^\infty \otimes L^\infty(G) \]
(see Definition 1.4). Once again, we transition freely between the von Neumann algebraic and the purely algebraic setting for ergodic actions. The former features mostly in the classical context in §1.4 below, in order for us to connect with the literature on ergodic actions of compact (plain, non-quantum) groups.

**Definition 1.11.** Let $X \overset{\alpha}{\curvearrowright} G$. One calls $\alpha$ of quotient type if there exists a compact quantum subgroup $H \subset G$ with corresponding quotient map $\pi : C(G) \to C(H)$ and a $\ast$-isomorphism
\[ \theta : C(X) \to C(H \backslash G) =\{ g \in C(G) \mid (\pi \otimes id_G) \Delta(a) = 1_H \otimes a \} \]
such that
\[ (\theta \otimes id_G) \circ \alpha = \Delta \circ \theta, \]
i.e. such that $\theta$ respects the $C(G)$ coactions on the domain and codomain.

Note that actions of quotient type are automatically ergodic. The following definition captures a somewhat broader class of ergodic actions.

**Definition 1.12.** An action $X \overset{\alpha}{\curvearrowright} G$ is embeddable if there exists a faithful coaction-preserving $\ast$-morphism
\[ \theta : C(X) \hookrightarrow C(G) \]

**Remark 1.13.** In other words, embeddable ergodic actions can be realized as coidgebras in the Hopf algebra attached to the quantum group.

1.3. Monoidal equivalence. The following notion of monoidal equivalence was introduced in [5] (see also [10]).

**Definition 1.14.** Two compact quantum groups $G_1 = (A_1, \Delta_1)$ and $G_2 = (A_2, \Delta_2)$ are said to be monoidally equivalent if there exists a bijection $\psi : Irred(G_1) \to Irred(G_1)$ satisfying $\psi(\varepsilon) = \varepsilon$, together with linear isomorphisms
\[ \psi : Mor(x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_k) \to Mor(\psi(x_1) \otimes \cdots \otimes \psi(x_r), \psi(y_1) \otimes \cdots \otimes \psi(y_k)) \]
satisfying the following conditions:
\[ \psi(1) = 1, \quad \psi(S^*) = (\psi(S))^*, \quad \psi(ST) = \psi(S)\psi(T), \quad \psi(S \otimes T) = \psi(S) \otimes \psi(T) \]
for all $S \subset A_1$ and $T \subset A_2$, whenever the formulas make sense. Such a collection of maps $\psi$ is called a monoidal equivalence between $G_1$ and $G_2$.

**Remark 1.15.** The categories $\text{Rep}(G_i)$ of unitary representations of $G_i$ are monoidal $*$-categories, in the sense that there are complex conjugate-linear operators $\text{hom}(x, y) \rightarrow \text{hom}(y, x)$ satisfying the obvious analogues of $*$ structures on $*$-algebras. Keeping this in mind, **Definition 1.14** simply says that $\text{Rep}(G_i)$ are equivalent as monoidal $*$-categories.

**Remark 1.16.** Concrete examples of monoidally equivalent compact quantum groups are given in section 4 of [10]. As we will recall momentarily, our compact quantum group of interest $O_{-1}(2)$ is monoidally equivalent to $O(2)$.

One source of monoidal equivalence is cocycle twisting. [4] is an excellent source for the material that we once more only skim here. As announced above, we work mainly with plain, non-topologized Hopf algebras.

A 2-cocycle on a CQG algebra $H$ is map $\lambda : H \otimes H \rightarrow \mathbb{C}$ with convolution inverse $\lambda^{-1}$ and satisfying certain associativity-like conditions that specialize to it being a cocycle in the usual sense when $H = \mathbb{C}\Gamma$ is the group algebra of a discrete group (see [4, Example 1.3]).

A 2-cocycle allows us to deform the multiplication of $H$. In Sweedler notation $H \ni a \mapsto a_1 \otimes a_2 = \Delta(a) \in H \otimes H$ the deformed multiplication is $a \bullet b = \lambda(a_1, b_1)a_2b_2\lambda^{-1}(a_3, b_3)$.

The cocycle conditions ensure that this equips the underlying space of $H$ with an associative algebra structure, and preserving the comultiplication we obtain another CQG algebra $H^\lambda$ (see [4, §3.3]).

As explained in [4, §3.3], there is a monoidal equivalence $\lambda \triangleright$ between the category of $H$-comodules (i.e. $\text{Rep}(G)$ if $H$ is the CQG algebra of the compact quantum group $G$) and that of $H^\lambda$-comodules.

The instance of cocycle twisting that we are most concerned with here is

**Example 1.17.** Let $H$ be the CQG algebra of the orthogonal group $O(2)$, which surjects onto the CQG algebra $\mathbb{C}Z_2^2$ of the diagonal subgroup of $O(2)$.

Now, the 2-cohomology $H^2(Z_2^2, \mathbb{C})$ is isomorphic to $\mathbb{Z}/2$, and hence we can choose a 2-cocycle that represents the unique non-trivial class. Such a cocycle then precomposes with the surjection $H^\otimes 2 \rightarrow (\mathbb{C}Z_2^2)^\otimes 2$ to give a 2-cocycle on $H$ in the sense of the present subsection. The twist $H^\lambda$ will be precisely the CQG algebra of $O_{-1}(2)$, as described in **Definition 1.5**.

Recall the following paraphrase of [10, Theorem 7.3].
Theorem 1.18. A monoidal equivalence between the categories of representations of two compact quantum groups $G_i$, $i = 1, 2$ induces an equivalence between their categories of ergodic actions.

We denote by $\lambda \triangleright$ all such equivalences arising in this context: the monoidal equivalence between categories of representations, the equivalence between the categories of ergodic actions, etc. Context will suffice to determine the correct interpretation of the symbol $\lambda \triangleright$ in each case.

Theorem 1.18 motivates

Question 1.19. Let $G_1$ and $G_2$ be two monoidally equivalent compact quantum group. Is there a natural bijective correspondence between their embeddable ergodic actions?

Even though the question is rather ill-posed and ambiguous, we will see below that the answer is negative in as strong a sense as possible, even for finite quantum groups.

1.4. Ergodic actions of classical compact groups. Here we recall various generalities on ergodic actions of (ordinary, non-quantum) compact groups on possibly non-commutative operator algebras for later use. Our main references for all of this are the seminal papers [23–25].

We work in the context of actions on von Neumann algebras of compact groups $G$ on von Neumann algebras (as in the papers referenced above). In that setting, an action is ergodic if the fixed-point subalgebra consists of scalars only.

The general theory of ergodic actions of compact groups on von Neumann algebras is developed in [25] and deployed later in [23, 24] for classification purposes. First, we recall the following simple procedure for producing ergodic actions.

Definition 1.20. Let $H \leq G$ be an inclusion of compact groups and $H \triangleright N$ an $H$-action on a von Neumann algebra. The induced representation $\text{Ind}_H^G(N)$ is the von Neumann algebra
$$\{L^\infty(G) \otimes N \ni f : G \to N \mid f(gh^{-1}) = \alpha_h(f(g))\}$$
equipped with the $G$-action given by
$$g \triangleright f = f(g^{-1}\bullet)$$

Induction is the right adjoint to the restriction functor from $G$-actions to $H$-actions, from which it follows immediately that it preserves ergodicity: the induction to $G$ of an ergodic $H$-action is again ergodic.

The following familiar concept will allow us to further explicate the ergodic actions of the compact groups we study.

Definition 1.21. Let $G$ be a compact group, $V$ a finite-dimensional Hilbert space, and $U(V)$ its unitary group. A projective unitary representation of $G$ on $V$ is a map $\pi : G \to U(V)$ such that

$$(1.4) \quad \pi(x)\pi(y) = \lambda(x,y)\pi(xy) \quad \forall x, y \in G$$

where $\lambda : G \times G \to U(V)$ is called the associated multiplier.
It is easy to see that if $V$ is an irreducible projective representation of the compact subgroup $\mathbb{H} \subseteq G$ then $B(V)$ is ergodic over $\mathbb{H}$ and hence $\text{Ind}_{\mathbb{H}}^{G} B(V)$ is an ergodic $G$-action.

**Definition 1.22.** A compact group $G$ is ergodically rigid if all of its ergodic representations are of the form $\text{Ind}_{\mathbb{H}}^{G} B(V)$ for some closed subgroup $\mathbb{H} \subseteq G$ and some irreducible projective $\mathbb{H}$-representation $V$ of $\mathbb{H}$.

Recall [25, Theorem 20] (slightly paraphrased):

**Theorem 1.23.** $G$ is ergodically rigid in the sense of Definition 1.22 if and only if its only ergodic actions are on type-I von Neumann algebras.

Note also that according to [24, Theorem, p. 309] $SU(2)$ is ergodically rigid. Here, we first need the following simple remark.

**Lemma 1.24.** Abelian compact groups are ergodically rigid.

**Proof.** Let $\mathbb{G}$ be a compact abelian group acting ergodically on a von Neumann algebra $M$. For a character $\chi : \mathbb{G} \to \mathbb{S}^1$ we denote by $M_{\chi}$ the spectral subspace of $M$ (i.e. those elements of $M$ which $\mathbb{G}$ scales via $\chi$).

According to [25, Theorem 1 (a)] we have $\dim(M_{\chi}) \leq 1$. For a non-zero $x \in M_{\chi}$ we have

$$0 \neq x^* x \in M_1 = \mathbb{C},$$

meaning that $x$ is a scalar multiple of the identity. Those $\chi$ for which $M_{\chi} \neq 0$ then form a subgroup

$$\overline{\mathbb{G}/\mathbb{H}} \subseteq \mathbb{G}$$

of the character group of $\mathbb{G}$ (for some closed subgroup $\mathbb{H} \subseteq \mathbb{G}$), and we have

$$M \cong \text{Ind}_{\mathbb{H}}^{\mathbb{G}} N$$

for a full-multiplicity ergodic action of $\mathbb{H}$ on a von Neumann algebra $N$ in the sense of [23], i.e. such that for each character $\chi \in \overline{\mathbb{H}}$ the spectral space $N_{\chi}$ has maximal dimension 1.

In turn, [23, Theorem 2] then shows that the full-multiplicity ergodic actions of $\mathbb{H}$ are precisely $B(V)$ for irreducible projective representations $V$.

We also need the following result on the persistence of ergodic rigidity under certain extensions.

**Proposition 1.25.** Let

$$1 \to \mathbb{H} \to \mathbb{G} \to \Gamma \to 1$$

be an extension of a finite group $\Gamma$ by an ergodically rigid compact group $\mathbb{H}$. Then, $\mathbb{G}$ is ergodically rigid.

**Proof.** According to the already-cited [25, Theorem 20], it suffices to prove that for every ergodic action of $\mathbb{G}$ on a von Neumann algebra $M$, the latter is of type I. Furthermore, recall from [25, Corollary 8] that every ergodic action is induced from an ergodic action of a closed subgroup on a factor, so we may as well assume that $M$ is a factor.

Now consider the von Neumann subalgebra $M^\mathbb{H}$ fixed by $\mathbb{H}$. It is acted upon ergodically by $\Gamma$, and hence is finite-dimensional by [25, Theorem 1 (a)].
Let $p$ be a minimal projection of $M^H$. The factor $pMp$ then admits an ergodic action by $H$, and hence, by the assumption of ergodic rigidity, must be of type I. Since $M$ is a factor with a corner $pMp$ of type I, it must itself be of type I. As anticipated above, this finishes the proof via [25, Theorem 20].

We end this section with the following simple consequence of the general theory recalled above; it will be of use to us in the classification results to follow.

**Lemma 1.26.** Let $H_i$, $i = 1, 2$ be two closed subgroups of a compact group $G$ with respective irreducible projective representations $V_i$. Then, the induced representations $M_i = \text{Ind}_{H_i}^G B(V_i)$ are isomorphic if and only if there is an element $g \in G$ such that

- $g^{-1}H_1g = H_2$;
- the pullback through the isomorphism 
  \[ \text{ad}_{g^{-1}} = g^{-1} \bullet g : H_1 \to H_2 \]
  of $B(V_2)$ is isomorphic to the $H_1$-module algebra $B(H_1)$.

**Proof.** The sufficiency of the condition is clear: if an element $g$ satisfying the two conditions exists, then the action of $g$ implements an isomorphism

\[ M_1 = \text{Ind}_{H_1}^G B(V_1) \cong \text{Ind}_{H_2}^G B(V_2) = M_2. \]

Conversely, suppose we have an isomorphism (1.5). First, according to [25, Theorem 7], $L^\infty(G/H_i)$ are the centers of the von Neumann algebras $M_i$ respectively, and are hence $G$-equivariantly isomorphic.

The algebras $C(G/H_i)$ can be extracted as the algebras of norm-continuous elements with respect to the $G$-actions on $M_i$, and are hence once more $G$-equivariantly isomorphic. This translates to a $G$-space homeomorphism $G/H_1 \to G/H_2$. If such a homeomorphism sends the class of $1$ in $G/H_1$ to the class of $g \in G$ in $G/H_2$ then the isotropy group $H_1$ of the former must coincide with the isotropy group $gH_2g^{-1}$ of the latter.

Upon applying $g$, we may now assume that $H_i$ coincide (and hence drop the subscripts $i$ from $H$). The hypothesis is now that

\[ \text{Ind}_{H_1}^G B(V_1) \cong \text{Ind}_{H_2}^G B(V_2) \]

via an isomorphism that identifies the centers $L^\infty(G/H)$ of the two respective sides. The $C^*$-algebras of norm continuity on the two sides of (1.6) are the algebras of continuous sections of the bundles over $G/H$ associated to the actions of $H$ on $B(V_i)$.

The desired conclusion that $B(V_i)$ are isomorphic as $H$-module algebras now follows by evaluating sections of said bundles at the class of $1 \in G$ in $G/H$.

2. Classification results for the compact quantum group $O_{-1}(2)$

In this section we first describe the ergodic actions of $O_{-1}(2)$ and we apply the results of the previous section to obtain the list of embeddable ergodic actions.
2.1. Ergodic actions of $O_{-1}(2)$. In this subsection, we will give the complete list of ergodic coactions of $O_{-1}(2)$. Let's recall [2, Theorem 4.3]:

**Theorem 2.1.** The category of corepresentations of $C(O_{-1}^r)$ is tensor equivalent to the category of representations of $O_n$.

By Theorem 2.1 the compact quantum groups $O_{-1}(2)$ and $O(2)$ are monoidally equivalents and by Theorem 1.18 their respective ergodic actions of $O(2)$ are in bijective correspondence. It thus suffices to classify the ergodic actions of $O(2)$.

In the sequel we will identify $O(2) \cong T \rtimes C_2$, with $T = S^1$ being the circle group, and with the cyclic group $C_2 = \{1, \sigma\}$ acting on $T$ by $\sigma(z) = \bar{z}$. As a first observation, we have

**Theorem 2.2.** The compact group $O(2)$ is ergodically rigid in the sense of Definition 1.22.

*Proof.* Immediate from the expression of $O(2)$ as an extension $T \rtimes C_2$ together with Lemma 1.24 and Proposition 1.25.

We now describe the ergodic actions more explicitly, via Theorem 2.2 and the representation theory of the closed subgroups of $O(2)$. These fall into two classes:

- the closed subgroups $C_k \leq T$, either cyclic of order $k$ or equal to $T$ for $k = \infty$;
- the dihedral groups $D_k = C_k \rtimes C_2$, where again we set $D_k = O(2)$ for $k = \infty$.

All irreducible representations of $C_k$ give rise through the procedure described above, by induction, to the same ergodic action $\alpha^{(k)}$ of $O(2)$ on $L^\infty(O(2)/C_k) = L^\infty(T/C_k) \oplus L^\infty(T/C_k)$, namely

$$\alpha_z(f, g) = (f_z, g_z), \quad \alpha_\sigma(f, g) = (g, f),$$

where $f_z$ denotes the $z$-translate of $f$.

As for the $D_k$, we have the action $\alpha = \beta_0^{(k)}$ on $L^\infty(O(2)/D_k)$ coming from the characters of $D_k$, as well as those induced from $D_k$ from the actions of the latter on $M_2(\mathbb{C})$ given by

$$\alpha_z\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l} c & d \end{pmatrix}, \quad \alpha_\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

for positive integers $0 < l < k$. We denote these $O(2)$-actions by $\beta_{l/2}^{(k)}$ respectively (with $k = \infty$ corresponding to the finite-dimensional ergodic actions of $D_\infty = O(2)$ itself).

All in all, we obtain

**Proposition 2.3.** The full list of mutually non-equivalent ergodic actions of $O(2)$ is

$$\left\{ \beta_{l/2}^{(k)}, \alpha^{(k')}, \text{ where } k, k' \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}, 0 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$  

*Proof.* The fact that this list contains all (isomorphism classes of) ergodic actions follows from Theorem 2.2, while the claim about their being mutually non-isomorphic is a consequence of Lemma 1.26.
2.2. Embeddable ergodic actions of $O_{-1}(2)$. In this subsection we determine the embeddable ergodic actions on $O_{-1}(2)$, based on those of $O(2)$ classified above in Proposition 2.3. The plan for achieving this is as follows.

First, note that by definition an embeddable ergodic action is by definition a comodule $^\ast$-algebra of the CQG algebra $\mathcal{A}_{-1}$ associated to $O_{-1}(2)$ which embeds into $\mathcal{A}_{-1}$ as such (i.e. by an embedding that preserves all of the structure: comodule, algebra, etc.).

Since the twisting equivalence $\lambda \triangleright$ that implements Theorem 1.18 also implements an equivalence between the categories of coideal $^\ast$-algebras over $\mathcal{A}_{-1}$ and the untwisted version $\mathcal{A}$ (algebra of representative functions on the classical group $O(2)$), it will be sufficient to identify the ergodic $O(2)$-action $B$ in the list of Proposition 2.3 for which $\lambda \triangleright B \cong \mathcal{A}$ as $\mathcal{A}_{-1}$ comodule $^\ast$-algebras, and to then also identify the members of that list that embed into $B$.

We will see that there is only one candidate for $B$ (namely $\beta_{1/2}$) using the Peter-Weyl theorem to determine the representation type of the ergodic actions identified in Proposition 2.3 (where by representation type we mean the multiplicities of the various irreducible $O(2)$-representations). Indeed, this is the substance of the following result.

**Proposition 2.4.** The only comodule algebras among those in Proposition 2.3 that are isomorphic to $\mathcal{A}$ as $O(2)$-representations are $\alpha^{(1)} \cong \mathcal{A}$ itself and $\beta_{1/2}$.

**Proof.** The ($\infty$)-superscript $O(2)$-representations are finite-dimensional, so we can discount them for the purposes of this proposition.

For the other members of the list, we will use the Frobenius reciprocity formula

$$\hom_{O(2)}(V, \text{Ind}_{H}^{O(2)} W) \cong \hom_{H}(V, W)$$

for $V \in \text{Rep}_{O(2)}$ and $W \in \text{Rep}_{H}$ in order to compute the multiplicities of various irreducible $O(2)$-representations.

For each $k \geq 1$ we have a 2-dimensional $O(2)$-representation $V_k$ whose restriction to $T$, upon identifying the Pontryagin dual

$$\widehat{T} \cong \mathbb{Z},$$

splits as $k \oplus (-k)$.

Now, for $k \geq 2$, $\alpha^{(k)}$ is induced from the non-trivial cyclic group $C_k \subset T$. Taking $H = C_k$, $W$ to be trivial, and $V = V_1$ in (2.1), the right hand side vanishes and hence so must the left hand side. This means that $V_1$ is not a summand of $\alpha^{(k)}$, $k \geq 2$, and hence these list members can also be dropped as candidates for an isomorphism to $\mathcal{A}$ as $\mathcal{A}$-comodules.

Next we look at the representations $\beta_{0}^{(k)}$ for all $k \geq 1$ induced from the trivial representation of the order-2k dihedral groups $D_k \subset O(2)$. In these cases, (2.1) with $H = D_k$, $W$ trivial and $V$ being the non-trivial character of $O(2)$ annihilates the right hand side, and hence the left hand side too. In conclusion, the non-trivial character of $O(2)$ does not appear in $\beta_{0}^{(k)}$; this disqualifies these representations.
Finally, we consider $\beta^{(k)}_{\ell/2}$ for $\ell > 0$ and $k \geq 2$. Here, we apply (2.1) with $H = D_k$, $W$ the representation of $D_k$ on $M_2$ described in the discussion preceding Proposition 2.3, and $V = V_1$. There are now a few possibilities:

(a) If $\ell > 1$ then the right hand side of (2.1) is zero, so these cases can be discarded;

(b) If $\ell = 1$ and $k \geq 3$ then the right hand side of (2.1) is one-dimensional, because the restriction of $V_1$ to $D_k$ is irreducible. In conclusion $V_1$ appears in $\beta^{(k)}_{\ell/2}$ with multiplicity one, but it appears in $A$ with multiplicity two (by Peter-Weyl, since it is a two-dimensional irreducible representation). Once more, these cases do not qualify for the purposes of the proposition;

(c) Finally, $\ell = 1$ and $k = 2$ is left, in which case one easily checks that the multiplicities match as expected. Indeed, $D_k$ is then the Klein group $\mathbb{Z}_2^2$, and its 4-dimensional representation $W$ that is induced up to $O(2)$ to produce $\beta^{(2)}_{1/2}$ breaks up as a sum of all of its characters.

It follows from the previous paragraph that if the irreducible $O(2)$-representation $V$ is one-dimensional then the right hand side of (2.1) is also one-dimensional, whereas if $V$ is two-dimensional then its restriction to $D_2$ breaks up as a sum of two distinct characters, and hence the right hand side of (2.1) is two-dimensional.

This finishes the proof of the proposition.

Remark 2.5. In the sequel, we will make repeated and implicit use of the fact that in the Frobenius reciprocity formula (2.1), when $V$ and $W$ are algebras in the respective categories of representations, $\text{Ind}^{O(2)}_H W$ is again an algebra in $\text{Rep}_{O(2)}$.

Moreover, (2.1) identifies the subspaces of algebra morphisms (i.e. those morphisms that are multiplicative in addition to being $O(2)$ and $H$-module maps).

We can now record the consequence alluded to above.

Corollary 2.6. The twisting equivalence $\lambda \triangleright$ induces a bijection between

$$\{ \alpha^{(k)}_l, \beta^{(k)}_{l/2} \mid k = \infty \text{ or even }, l = 0 \text{ or odd} \}$$

from Proposition 2.3 and the isomorphism classes of embeddable ergodic actions of $O_{-1}(2)$.

Proof. The function algebra of $O_{-1}(2)$ can be obtained from that of $O(2)$ by twisting the multiplication both on the right and the left, by the cocycle $\lambda$ and its convolution inverse $\lambda^{-1}$. Since $\lambda \triangleright$ by definition twists by $\lambda$ on the right, the $A$-comodule algebra $B$ from the introductory remarks to §2.2 is a twist of $A$ on the left and hence cannot be abelian, and yet must have the same representation type as $A$ as a right $A$-comodule. It must thus be $\beta^{(2)}_{1/2}$ by Proposition 2.4.

In summary, the desired conclusion will follow once we show that the ergodic $O(2)$-actions listed in the statement are precisely those that embed into $\beta^{(2)}_{1/2}$.

Throughout the proof, we denote by $W$ the $D_2$-representation on $M_2$ that gives rise to $\beta^{(2)}_{1/2}$ by induction to $O(2)$. We examine the representations listed in Proposition 2.3 systematically.
Type-\(\alpha\) actions.

\(\alpha^{(\infty)}\) is two-dimensional. Its restriction to \(H = D_2\) embeds into \(W\) as the diagonal subalgebra of the realization of \(W\) as \(2 \times 2\) matrices, and hence \(\alpha^{(\infty)}\) embeds into \(\beta^{(2)}_{1/2}\) by Frobenius reciprocity (2.1).

As for \(\alpha^{(k)}\) for positive integers \(k\), consider first the case when \(k\) is odd. If we had an embedding

\[ \alpha^{(k)} \subseteq \beta^{(2)}_{1/2}, \]

then the Frobenius adjunction (2.1) would turn it into a map

\[ \text{Res}_H^{O(2)} \alpha^{(k)} \to W \]

of algebras in \(\text{Rep}_{D_2}\). The condition that \(k\) be odd then ensures that this map is surjective, since in that case all four characters of \(D_2\) admit unitary eigenvectors in the restriction of \(\alpha^{(k)}\). Since however the left hand side of (2.2) is commutative while the right hand side is not, we obtain a contradiction.

For even \(k\) on the other hand, we can embed \(\alpha^{(k)}\) into \(\beta^{(2)}_{1/2}\) by inducing in stages. First, embed

\[ \text{Res}_{D_2}^{D_k} \text{Ind}_{C_k}^{D_k} \mathbb{C} \subseteq W \]

as the diagonal subalgebra of the \(2 \times 2\) matrix realization of \(W\). Frobenius reciprocity then translates this into an embedding

\[ \text{Ind}_{C_k}^{D_k} \mathbb{C} \subseteq \text{Ind}_{D_2}^{D_k} W. \]

Finally, induce this map further to \(O(2)\).

Type-\(\beta\) actions, \(l = 0\).

\(\beta^{(\infty)}_0\) is simply the trivial representation and hence is embeddable into \(\beta^{(2)}_{1/2}\). We note also that \(\beta^{(k)}_0\) for odd \(k\) can be eliminated in exactly the same way we did \(\alpha^{(k)}\) above.

For even \(k\) \(\beta^{(k)}_0\) is again embeddable into \(\beta^{(2)}_{1/2}\) by the case of even \(\alpha^{(k)}\), since we have

\[ \beta^{(k)}_0 \subseteq \alpha^{(k)}. \]

Type-\(\beta\) actions, \(l > 0\).

Consider the case of \(\beta^{(k)}_{1/2}\) (including \(k = \infty\)) for even positive \(l\). Here we have an embedding

\[ \beta^{(\infty)}_{1/2} \subseteq \beta^{(k)}_{1/2} \]

of algebras in \(\text{Rep}_{O(2)}\), and hence an embedding of the right hand side into \(\beta^{(2)}_{1/2}\) would imply the existence of a morphism of the left hand side into \(W\) in the category \(\text{Rep}_{D_2}\). This is impossible: both the left hand side of (2.3) and \(W\) are \(2 \times 2\) matrix algebras and hence the morphism would have to be one-to-one, but the evenness of \(l\) ensures that when restricted to \(D_2\) the left hand side of (2.3) has a two-dimensional space of invariants.
When $k$ is positive and odd, then for every $l$ we have an even $l'$ such that
\[ l' \equiv l \pmod{k}. \]

We have an embedding
\[ \beta^{(\infty)}_{l/2} \subseteq \beta^{(k)}_{l/2} \]

of algebras in $\text{Rep}_O(2)$ and we can repeat the argument above to conclude that $\beta^{(k)}_{l/2}$ is not embeddable into $\beta^{(2)}_{1/2}$.

For even $k$ (including by abuse the case $k = \infty$ with $D_k = O(2)$) and positive odd $l$ the restriction of $\beta^{(\infty)}_{l/2}$ to $D_k$ embeds into $\text{Ind}_{D_k}^{D_2} W$, and hence $\beta^{(k)}_{l/2}$ is embeddable into $\beta^{(2)}_{1/2}$, as desired.

This concludes the last case and the proof of the result.

\[ \blacksquare \]

2.3. Quotients by quantum subgroups. In this section we identify those embeddable ergodic actions that arise as function algebras of quotients by quantum subgroups of $O_{-1}(2)$.

We denote by $H = A_{-1}$ the Hopf algebra underlying $O_{-1}(2)$. The Hopf $*$-algebra quotients of $H$ (i.e. the function algebras of the quantum subgroups of $O_{-1}(2)$) are classified in [3, Theorem 7.1]. We briefly recall that classification here. The non-trivial quotients are as follows.

- For each $n \in \mathbb{Z}_>0 \cup \{\infty\}$ a quotient isomorphic to the group group algebra $\mathbb{C}D_n$ of the dihedral group of order $2n$ (including $n = \infty$);
- Two families of Hopf algebras $A(n,e)$, $e = \pm 1$, $n \in \mathbb{Z}_0$ of respective orders $4n$.

For each quotient Hopf $*$-algebra $\pi : H \to \mathcal{L}$ we have an associated right coideal $*$-subalgebra
\begin{equation}
\mathcal{A} = \mathcal{A}_\pi = \{ x \in H \mid (\pi \otimes \text{id})\Delta(x) = 1 \otimes x \in \mathcal{L} \otimes H \}. \tag{2.4} \end{equation}

Our first remark identifies those embeddable actions that can be realized as such coideal subalgebras for the quotients $\mathcal{L} = A(n,e)$ from the above classification.

**Proposition 2.7.** Let $n \in \mathbb{Z}_0$. The coideal subalgebras corresponding to $H \to A(n,e)$, $e = \pm 1$ are isomorphic to the ergodic action $\beta^{(2n)}_0$ from Corollary 2.6.

**Proof.** It is easy to see from the proofs of Proposition 2.4 and Corollary 2.6 that $\beta^{(2n)}_0$ are the only embeddable coactions among those in Corollary 2.6 that do not contain the non-trivial one-dimensional comodule of $H$.

On the other hand, it follows from [3, Lemma 7.3 and Theorem 7.1] that the quotients $\pi_{n,e} : H \to A(n,e)$ are those for which the non-trivial grouplike $d \in H$ satisfies $\pi(d) \neq 1$; by the previous paragraph, it follows that the comodule algebras corresponding to the actions $\beta^{(2n)}_0$ are indeed among $A_{\pi_{n,e}}$ defined as in (2.4).

Now consider the simple two-dimensional $H$-comodules $V_k$, $k \in \mathbb{Z}_0$ corresponding to the simple $O(2)$-representations denoted by the same symbols in the proof of Proposition 2.4. It follows from [3, Lemma 7.4] (and its proof) that when regarded as a comodule over $A(n,e)$,
$V_k$ contains the trivial representation precisely when $2n$ divides $k$. This, then, is the sufficient and necessary condition that ensures that $V_k$ appears as a submodule of $\mathcal{A}_{\pi_n,e}$.

The conclusion now follows from the observation that, by Frobenius reciprocity, $V_k$ is similarly embeddable into $\beta^{(2n)}$ as a comodule if and only if $2n|k$.

It now remains to identify those embeddable ergodic actions that correspond to the quantum subgroups $\mathcal{H} \to \mathbb{C}D_n$ for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. According to Corollary 2.6 and Proposition 2.7, these will be among the $\alpha^{(k)}$ and $\beta^{(k)}$ for even $k$ (including $k = \infty$) and odd $l$.

**Proposition 2.8.** Let $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. The $\mathcal{H}$-comodule algebra $\alpha^{(2n)}$ is isomorphic to the right coideal subalgebra $A_{\pi_n}$ of $\mathcal{H}$ associated to the Hopf quotient $\pi_n : \mathcal{H} \to \mathbb{C}D_n$.

**Proof.** As in the proof of Proposition 2.7 above, denote by $V_k$ the simple two-dimensional $\mathcal{H}$-comodules for $k \in \mathbb{Z}_{>0}$. Similarly, let $C_k \subset \mathcal{H}$ be the corresponding $2 \times 2$ matrix coalgebra.

The explicit description of the matrix coalgebras $C_k$ from [3, discussion preceding Proposition 7.1] shows that $C_k$ is contained in $A_{\pi_n}$ when $2n|k$, and intersects $A_{\pi_n}$ trivially otherwise.

The statement is now a consequence of the fact that similarly, the multiplicity of $V_k$ in $\alpha^{(2n)}$ is two when $2n|k$ and zero otherwise. ■

2.4. **Generalized quantum subgroups.** As seen in §2.3 above, the quantum subgroups of $O_{-1}(2)$ do not account for all embeddable ergodic actions of the latter quantum group. We will see here that nevertheless, these ergodic actions can be recovered through what might be deemed “subquotient” quantum groups of $O_{-1}(2)$. To make sense of this, we need to recall some material from [13].

First, consider an arbitrary CQG algebra $\mathcal{H}$. [20, Theorem 1] establishes a one-to-one correspondence between certain coideal subalgebras of $\mathcal{H}$ (which are morally the embeddable actions of the underlying quantum group of $\mathcal{H}$) and the idempotent states on the latter, i.e. those states $\phi$ satisfying $\phi \ast \phi = \phi$ for the convolution product.

An idempotent state is a generalization of a quantum subgroup, since given such a quantum subgroup $\pi : \mathcal{H} \to \mathcal{B}$ the composition $h_B \circ \pi$ is idempotent. For this reason, the coideal subalgebra of $\mathcal{H}$ defined by

$$\text{Im}(\phi \otimes \text{id}) \circ \Delta$$

for an idempotent state $\phi$ can be regarded as a natural generalization of a quotient by a quantum subgroup.

Now suppose $\mathcal{H} = \mathbb{C}\Gamma$ is the group algebra of a discrete group (i.e. the Hopf algebra underlying an abelian compact quantum group). As seen in [13, Theorem 6.2] (for finite groups but the discussion generalizes), the idempotent states on $\mathcal{H}$ are simply the characteristic functions of subgroups of $\Gamma$.

In general, for an arbitrary CQG algebra $\mathcal{H}$ with a quotient $\mathcal{H} \to \mathbb{C}\Gamma$, the characteristic function on a subgroup of $\Gamma$ is an idempotent state on $\mathcal{H}$ and hence corresponds to some coidealgebra of $\mathcal{H}$. With this in mind, we introduce the following term to aid the streamlining of the presentation.
Definition 2.9. Let $\mathcal{H}$ be a CQG algebra, $\pi : \mathcal{H} \rightarrow \mathbb{C}\Gamma$ a quotient group algebra, and $\Omega \subset \Gamma$ a discrete subgroup. We denote
$$A_{\pi,\Omega} = \text{Im}(\phi \otimes \text{id}) \circ \Delta,$$
where $\phi : \mathcal{H} \rightarrow \mathbb{C}$ is the characteristic function on $\Omega \subset \Gamma$ composed with $\pi$.

A tame embeddable ergodic action of the quantum group attached to $\mathcal{H}$ is one that is isomorphic to the coidealgebra $A_{\pi,\Omega}$ for some $\pi : \mathcal{A} \rightarrow \mathbb{C}\Gamma$ and some subgroup $\Omega \subseteq \Gamma$.

This notion allows us to draw the conclusion announced above.

Proposition 2.10. The ergodic actions $\beta^{(k)}_{l/2}$, $l \neq 0$ of $O_{-1}(2)$ listed in Corollary 2.6 are tame in the sense of Definition 2.9.

Proof. Specifically, we will show that all of these are isomorphic to coideal algebras $A_{\pi,\Omega}$ where $\pi : \mathcal{H} \rightarrow \mathbb{C}D_\infty$ is the surjection onto the group algebra of the infinite dihedral group from [3, Theorem 7.1] and $\Omega \leq D_\infty$ are various subgroups.

The discussion in [3, Section 7] introduces the matrix counits $v_{ij}$ for the comodule $V_1$ of $\mathcal{H}$, and the quotient $\mathcal{H} \rightarrow \mathbb{C}D_\infty$ sends $v_{ii}$, $i = 1, 2$ to the two involutions $\sigma_i$ generating $D_\infty$ and annihilates $v_{ij}$, $i \neq j$.

Furthermore, the matrix subcoalgebra $C_k \subset \mathcal{H}$ associated to the simple two-dimensional $\mathcal{H}$-comodule $V_k$, $k \in \mathbb{Z}_{>0}$ is
$$\begin{pmatrix}
(v_{11}v_{22})^m v_1^\varepsilon & (v_{12}v_{21})^m v_2^\varepsilon \\
(v_{21}v_{12})^m v_1^\varepsilon & (v_{22}v_{11})^m v_2^\varepsilon
\end{pmatrix},$$
where $\varepsilon \in \{0, 1\}$ and $k = 2m + \varepsilon$.

Now let $k$ be even or $\infty$ and $\ell$ odd, parametrizing the actions $\beta^{(k)}_{l/2}$ from Corollary 2.6. By simply counting multiplicities of the various $V_i$, the explicit description of the matrix coalgebras $C_k$ now makes it an easy check that $\beta^{(k)}_{l/2}$ is isomorphic as an $\mathcal{H}$-comodule to $A_{\pi,\Omega}$, where the subgroup $\Omega$ of $D_\infty$ is the semidirect product of the subgroup of index $k$ in $\mathbb{Z} \subset D_\infty$ by the order-two group generated by $(g_1g_2)^{\frac{k}{2k}}g_1$.

The conclusion follows from this, since the comodule algebras in Corollary 2.6 are mutually non-isomorphic as comodules. □

In conjunction with Propositions 2.7 and 2.8, this result accounts for all of the ergodic actions of $O_{-1}(2)$ as classified in Corollary 2.6.

3. Counterexamples: dihedral groups

Recall Question 1.19, on whether or not cocycle-twisting in some sense preserves isomorphism classes of embeddable ergodic actions. One possible precise interpretation would be as follows (in the context of compact quantum groups $G_\lambda$ obtained via cocycle deformation for a cocycle $\lambda$).
Question 3.1. Does

\[ \lambda \triangleright : Erg(G_1) \rightarrow Erg(G_2) \]

restrict to an equivalence between subcategories of embeddable ergodic coactions?

We already know that the answer to this version of the question is negative, by examining
the mutual twists \( O(2) \) and \( O_{-1}(2) \) we have been studying:

Corollary 3.2. Let \( \lambda \) be a cocycle which twists \( O(2) \) into \( O_{-1}(2) \). Then, the answer to
Question 3.1 is negative for \( G_1 = O(2) \) and \( G_2 = O_{-1}(2) \).

Proof. This is an immediate consequence of Corollary 2.6.

The question remains however of whether one can implement a more sophisticated equivalence
between the embeddable ergodic actions of two mutual twists. To rule this out, we will observe
below that there are examples of mutually cocycle-twisted finite quantum groups with different
numbers of isomorphism classes of embeddable ergodic actions.

The groups in question will be discrete versions of \( O(2) \), i.e. the dihedral groups \( D_K \) (for
even \( K \)). The contents of this section can thus be regarded as a “discretization” of those of
Section 2. We will mostly omit proofs, as they are almost verbatim recapitulations of those
in the preceding section.

Fix an even positive integer \( K \) (though evenness will only be of relevance to parts of the
discussion below).

Then, for the order-\( 2K \) dihedral group \( D_K \), we preserve the notation \( \alpha^{(k)} \) and \( \beta^{(k)}_{\frac{l}{2}} \) for rep-
resentations induced from subgroups \( C_k \) and \( D_k \) of \( K \). Note that whenever we employ this
notation, the condition \( k|K \) is implicit.

The classification of ergodic actions is perfectly analogous to that in Proposition 2.3 with
a parallel proof, via ergodic rigidity and an appeal to Proposition 1.25 and Lemmas 1.24
and 1.26).

Proposition 3.3. The full list of mutually non-equivalent ergodic actions of \( D_K \) is

\[ \{ \beta^{(k)}_{\frac{l}{2}}, \alpha^{(k')} \} \]

for \( k, k'|K \) and \( 0 \leq l \leq \lfloor \frac{K}{2} \rfloor \).

The usual cocycle used to twist \( O(2) \) into \( O_{-1}(2) \) descends to a cocycle on the function algebra
of \( D_K \subset O(2) \), so it can be used to twist the latter into \( (D_K)_{-1} \). We preserve the notation \( \lambda \)
for the cocycle.

Pursuing the same strategy as for \( O(2) \), we can now classify the embeddable ergodic actions
of \( (D_K)_{-1} \) as an analogue of Corollary 2.6.

Proposition 3.4. The twisting equivalence \( \lambda \triangleright \) induces a bijection between

\[ \{ \alpha^{(k)}, \beta^{(k)}_{\frac{l}{2}} \} \]

from Proposition 3.3 for even \( k|K \) and \( l = 0 \) or odd and the isomorphism classes of embeddable
ergodic actions of \( (D_K)_{-1} \).
Finally, an immediate consequence of this of relevance to Question 1.19 is

**Corollary 3.5.** For infinitely many $K$ the sets of isomorphism classes of embeddable ergodic actions for $D_K$ and $(D_K)^{-1}$ have different cardinalities.

**Proof.** This is a simple numerical estimate based on the classification of embeddable ergodic actions of $(D_K)^{-1}$ from Proposition 3.4. That result shows that the number of isomorphism classes for $(D_K)^{-1}$ grows quadratically with the number of divisors of $K$. On the other hand, for $D_K$, the ergodic actions that are embeddable are the $\alpha$s and those $\beta$s in Proposition 3.3 with $l = 0$. In conclusion, the number of such isomorphism classes grows linearly with the number of divisors of $K$. \(\blacksquare\)

**References**

[1] Saad Baaj and Jonathan Crespo, *Equivalence monoidale de groupes quantiques et k-thorie bivariante* (July 24, 2015), available at http://arxiv.org/abs/1507.06808v1.

[2] Teo Banica, Julien Bichon, and Benoît Collins, *The hyperoctahedral quantum group*, Journal Ramanujan Math Society 22 (2007), 345–384.

[3] Teodor Banica and Julien Bichon, *Quantum groups acting on 4 points*, J. Reine Angew. Math. 626 (2009), 75–114. MR2492990

[4] Julien Bichon, *Hopf-Galois objects and cogroupoids*, Rev. Un. Mat. Argentina 55 (2014), no. 2, 11–69. MR3285340

[5] Julien Bichon, An De Rijdt, and Stefaan Vaes, *Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups*, Comm. Math. Phys. 262 (2006), no. 3, 703–728. MR2202309

[6] Florin P. Boca, *Ergodic actions of compact matrix pseudogroups on C∗-algebras*, Astérisque 232 (1995), 93–109. Recent advances in operator algebras (Orléans, 1992). MR1372527

[7] Kenny De Commer, *Actions of compact quantum groups* (April 1, 2016), available at http://arxiv.org/abs/1604.00159v1.

[8] Jonathan Crespo, *Monoidal equivalence of locally compact quantum groups and application to bivariant K-theory*, Theses, 2015.

[9] Kenny De Commer, *Monoidal equivalence for locally compact quantum groups*, arXiv preprint arXiv:0804.2405 (2008).

[10] An De Rijdt and Nikolas Vander Vennet, *Actions of monoidally equivalent compact quantum groups and applications to probabilistic boundaries*, Annales de l’institut fourier, 2010, pp. 169–216.

[11] Mathijs S. Dijkhuizen and Tom H. Koornwinder, *CQG algebras: a direct algebraic approach to compact quantum groups*, Lett. Math. Phys. 32 (1994), no. 4, 315–330. MR1310296

[12] Uwe Franz and Adam Skalski, *A new characterisation of idempotent states on finite and compact quantum groups*, Comptes Rendus Mathematicque 347 (2009), no. 17, 991–996.

[13] Uwe Franz, Adam Skalski, and Reiji Tomatsu, *Idempotent states on compact quantum groups and their classification on Uq(2), SUq(2), and SOq(3)*, Journal of Noncommutative Geometry 7 (2013), no. 1, 221–254.

[14] R. Høegh Krohn, M. B. Landstad, and E. Størmer, *Compact ergodic groups of automorphisms*, Ann. of Math. (2) 114 (1981), no. 1, 75–86. MR625345

[15] Johan Kustermans and Lars Tuset, *A survey of C∗-algebraic quantum groups. I*, Irish Math. Soc. Bull. 43 (1999), 8–63. MR1741102

[16] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993. MR1243637

[17] Piotr Podleś, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups*, Comm. Math. Phys. 170 (1995), no. 1, 1–20. MR1331688

[18] David E. Radford, *Hopf algebras*, Series on Knots and Everything, vol. 49, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. MR2894855
[20] Pekka Salmi and Adam Skalski, *Idempotent States on Locally Compact Quantum Groups II*, Q. J. Math. 68 (2017), no. 2, 421–431. MR3667207

[21] Moss E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969. MR0252485

[22] Shuzhou Wang, *Ergodic actions of universal quantum groups on operator algebras*, Comm. Math. Phys. 203 (1999), no. 2, 481–498. MR1697607

[23] Antony Wassermann, *Ergodic actions of compact groups on operator algebras. II. Classification of full multiplicity ergodic actions*, Canad. J. Math. 40 (1988), no. 6, 1482–1527. MR990110

[24] , *Ergodic actions of compact groups on operator algebras. III. Classification for SU(2)*, Invent. Math. 93 (1988), no. 2, 309–354. MR948104

[25] , *Ergodic actions of compact groups on operator algebras. I. General theory*, Ann. of Math. (2) 130 (1989), no. 2, 273–319. MR1014926

[26] S. L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), no. 4, 613–665. MR901157

[27] , *Twisted SU(2) group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. 23 (1987), no. 1, 117–181. MR890482

[28] , *Tannaka-Kreın duality for compact matrix pseudogroups. Twisted SU(N) groups*, Invent. Math. 93 (1988), no. 1, 35–76. MR943923

[29] Stanislaw L Woronowicz, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995) 845 (1998), 884.

Department of Mathematics, University at Buffalo, Buffalo, NY 14260-2900, USA

E-mail address: achirvas@buffalo.edu

Laboratoire de Mathématiques de Besançon, Université de Bourgogne Franche-Comté, 16, Route de Gray, 25030 Besançon Cedex, France

E-mail address: hoche.souleiman_omar@univ-fcomte.fr