Non-equilibrium Luttinger liquid: Zero-bias anomaly and dephasing

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The interest in one-dimensional (1D) interacting electron systems is due to their fascinating physical properties and potential applications in nanoelectronics. A variety of experimental realizations of quantum wires includes carbon nanotubes, quantum Hall edges, semiconductor structures, polymer fibers, and metallic nanowires. At equilibrium, the physics of 1D electrons has been thoroughly explored. The main peculiarity of the system is the formation of a strongly correlated state, Luttinger liquid (LL), commonly described in terms of collective bosonic excitations [1]. A hallmark of LL correlations is a strong, power-law suppression of tunneling current at low bias — zero-bias anomaly (ZBA) [2].

On the other hand, little is known about LL away from equilibrium. Theoretical efforts so far focused on nonlinear transport through a single impurity [3, 4]. In this work we consider the problem of tunneling into a LL out of equilibrium. It is important to emphasize a remarkable property of the LL: the absence of relaxation towards equilibrium. Once non-equilibrium energy distributions of left- and right-movers are created, they propagate along the wire arbitrarily long without thermalization (within the clean LL model, i.e. when back-scattering by interaction or impurities, spectral nonlinearity, and momentum dependence of interaction are neglected) [5].

Below we address such a situation and analyze the tunneling density of states (TDOS) of a non-equilibrium LL wire. We show that while the energy relaxation rate is zero, the ZBA in TDOS is smeared out by dephasing originating from electron-electron scattering.

We begin by discussing possible experimental realizations of a non-equilibrium LL. Throughout most of the paper we assume that the interaction strength interpolates adiabatically between its value in the LL (central part of the wire where the measurements are performed) and zero near the electrodes. The assumption of adiabaticity is not entirely innocent; at the end, we briefly discuss the case of sharp switching of interaction and ensuing modifications.

The simplest setup is shown in Fig. 1a. A long clean LL is adiabatically coupled to two electrodes with different potentials, $\mu_L - \mu_R = \epsilon V$ and different temperatures $T_L$ (where $\eta = L, R$ stands for left- and right-movers) [7]. A particularly interesting situation arises when one of temperatures is much larger than the other, e.g., $T_L = 0$ and $T_R$ finite. Then the ZBA at $\mu_L$, where the distribution function has a sharp step, is broadened solely by dephasing originating from electron-electron scattering.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Schematic view of setups for measurement of the TDOS of a LL out of equilibrium (see text for details).}
\end{figure}

A more complex situation arises if electrons coming from reservoirs with different potentials mix by impurity scattering. Two different realizations of such devices are shown in Figs. 1b,c. In the first case, Fig. 1b, the mixture of left and right movers coming from reservoirs with $\mu_L \neq \mu_R$ is caused by impurities which are located in the non-interacting part of the wires. In the second setup, Fig. 1c: the LL wire is attached to two metallic wires which are themselves biased. We assume that these electrodes are diffusive but sufficiently short, so that inter-
The key features of the functional bosonization are: (i) it retains explicitly both fermionic and bosonic degrees of freedom; (ii) for \( V_0 = 0 \) it straightforwardly reduces to the model of non-interacting fermions. This makes the functional bosonization an appropriate tool for the development of the theory of a non-equilibrium LL.

Decoupling the interaction by Hubbard-Stratonovich transformation via a bosonic field \( \phi \), we obtain the action

\[
S[\psi, \phi] = i \sum_{n=R,L} \psi_n^\dagger (\partial_{\eta} - \phi) \psi_n - \frac{1}{2} \delta V_0^{-1} \phi, \tag{2}
\]

where \( \partial_{R,L} = \partial_t \pm \nu \partial_x \). It is convenient to perform a rotation in Keldysh space, thus decomposing fields into classical and quantum components, \( \psi, \tilde{\phi} = (\psi_+ \pm \psi_-)/\sqrt{2} \) (where + and - label the fields on two branches of the contour) and analogously for \( \phi \). We further introduce vector notations by combining \( \phi \) and \( \phi \) in a 2-vector \( \tilde{\phi} \).

The Green function of interacting electrons can be presented in the form \( G_n(t - t', x - x') = \langle \eta \phi(x, t; x', t') \rangle \), where \( G_{\eta\phi} \) is the Green function of non-interacting fermions in an external field \( \phi \), the averaging goes with the weight \( Z_\phi e^{-\frac{1}{2} \phi^2 + i \phi \Theta, 0} \) and \( Z_\phi \) is a sum of vacuum diagrams (fermionic loops) in the field \( \phi \). The special feature of 1D geometry is that the coupling between the fermionic and bosonic fields can be eliminated by a gauge transformation, \( \psi_n(x, t) \rightarrow \psi_n(x, t) e^{i \Theta_n(x, t)} \), with \( \Theta_n = \sigma_0 \theta + \sigma_1 \tilde{\theta} \), if we require

\[
i \partial_t \tilde{\theta} = \phi. \tag{3}
\]

As a result, \( G_{\eta\phi} \) can be cast in the form

\[
G_{\eta\phi}(x, t; x', t') = e^{i \Theta_n(t)} G_{\eta 0}(x, t; x', t') e^{-i \Theta_n(t')},
G_{\eta 0} = \begin{pmatrix} G_{\eta 0}^R & G_{\eta 0}^K \\ 0 & G_{\eta 0}^L \end{pmatrix}. \tag{4}
\]

Here \( G_{\eta 0} \) is a Green function of free fermions, with the Keldysh component \( G_{\eta 0}^K(\epsilon) = [1 - 2n_\eta(\epsilon)] G_{\eta 0}^R(\epsilon) - G_{\eta 0}^L(\epsilon) \), and \( n_\eta(\epsilon) \) is fermionic distribution function.

To proceed further, we use the random-phase approximation (RPA), within which \( Z_\phi \) is Gaussian, \( \log Z_\phi = -\frac{1}{2} \tilde{\phi}^T \Pi \tilde{\phi} \). This is an exact relation at equilibrium \( \Pi \), which is crucial for the LL being an exactly solvable problem. It remains exact for the non-equilibrium setup of Fig. 11, where both distributions \( n_\eta \) are of Fermi-Dirac form. On the other hand, RPA becomes an approximation for more general non-equilibrium situations (cf. Figs. 11,c); we discuss its status and possibility of an exact solution at the end.

The polarization operator of free fermions is given by

\[
\Pi = \Pi_R + \Pi_L, \quad \Pi_{R,L} = \Pi_{R,L}^c \langle B_{\eta}(\omega) \rangle, \quad \Pi_{R,L} = \frac{1}{2\pi} \omega \mp v_F q, \quad \Pi_{R,L}^c = \frac{1}{2\pi} \omega \pm v_F q.
\]

In order to find the TDOS \( \nu(\epsilon) = \nu_R(\epsilon) + \nu_L(\epsilon) \) at a point \( x \) one needs to calculate the retarded ("r") and advanced ("a") single particle Green function, \( \nu_\eta(\epsilon) = \frac{1}{2\pi} [G_\eta^R(\epsilon, x, x) - G_\eta^A(\epsilon, x, x)] \). To do it away from equilibrium we employ the Keldysh formalism. The Keldysh Green function reads \( G_\eta(x, t; x', t') = -i \langle T_K \psi_\eta(x, t) \psi_\eta^\dagger(x', t') \rangle \), where the time ordering is along the Keldysh contour. We proceed by following the lines of functional bosonization approach [11, 12, 13]. While fully equivalent to the conventional bosonization technique for the case of clean equilibrium LL, this method is advantageous in more complicated situations. In particular, the efficiency of the functional bosonization for the analysis of transport and interference phenomena in a disordered LL was recently demonstrated [9].
where $\omega_\pm = \omega \pm i\delta$. The function

$$B_\eta^v(\omega) = \frac{2}{\omega} \int_{-\infty}^{\infty} d\eta \eta (2 - \eta (\epsilon - \omega) - \eta (\epsilon + \omega)), \quad (6)$$

is related to the distribution function $N^\nu_{\eta}(\omega)$ of electron-hole excitations moving with velocity $v$ in direction $\eta$, $B_\eta^v(\omega) = 1 + 2 N^\nu_{\eta}(\omega)$. At equilibrium, $B_\eta^v(\omega) = B_{eq}(\omega) = 1 + 2 N_{eq}(\omega)$, where $N_{eq}(\omega)$ is the Bose distribution. Since fermions are free after the gauge transformation, no relaxation of the distribution functions $n_\eta(\epsilon)$ takes place.

Performing the averaging over $\phi$, we express the TDOS in terms of the correlation function of the gauge fields

$$\frac{2\nu_\eta(\epsilon)}{v_0} = 1 + 2i \int_{-\infty}^{\infty} dt n_\eta(t) \exp (-I^{(\eta)}_{\theta\bar{\theta}}(t)) \sin(I^{(\eta)}_{\theta\bar{\theta}}). \quad (7)$$

Here $v_0$ is the bare (non-interacting) density of states, $n_\eta(t)$ is the Fourier transform of $n_\eta(\epsilon)$,

$$I^{(\eta)}_{\theta\bar{\theta}}(t) = \int (d\omega)(dq) (1 - \cos(\omega t)) \theta^{(\eta)}_{\omega q} e^{-|\omega|/\Lambda},$$

$$I^{(\eta)}_{\theta\bar{\theta}} = 2 \int (d\omega)(dq) \sin(\omega t) \bar{\theta}^{(\eta)}_{\omega q} e^{-|\omega|/\Lambda}, \quad (8)$$

and $\Lambda$ is an ultraviolet cutoff. To calculate $I^{(\eta)}_{\theta\bar{\theta}}$ and $J^{(\eta)}_{\theta\bar{\theta}}$ one needs to resolve Eq. (3) and to express the gauge field $\theta$ in terms of the Hubbard-Stratonovich filed $\phi$,

$$\bar{\theta}_\eta = G_{\eta\phi}\phi, \quad (9)$$

where $G_{\eta\phi}$ is the Green function of free bosons with the Keldysh component $G^K_{\eta\phi} = (G^R_{\eta\phi} - G^L_{\eta\phi})B^v_\eta$. The latter serves to reproduce correctly the distribution function $N^\nu_{\eta}$ in the $\theta\bar{\theta}$ correlation function, ensuring, in particular, the fluctuation-dissipation theorem at equilibrium.

The correlation functions of $\bar{\theta}$ fields can be now readily found. For the $\theta\bar{\theta}$ component (which is independent of the distribution functions) we get

$$\langle \theta^{(\eta)}_{\omega q} \rangle (R,L) = \frac{1}{2(\omega_+ + q)^2} = \frac{iV_0}{2} \frac{\omega \pm v q}{(\omega_+ + v q)(\omega_+ - u^2 q^2)^2},$$

where $u = v(1 + V_0/\pi v)^{1/2}$ is the sound velocity. This yields for the $q$-integrated propagator (which enters Eq. (8))

$$\int (dq) \theta^{(\eta)}_{\omega q} = \pi \gamma / 2 \omega, \quad \text{where} \quad \gamma = (K - 1)^2 / 2K,$$

and $K = v_\eta(\epsilon) = (1 + V_0/\pi v)^{-1/2}$ is the conventional dimensionless parameter characterizing the LL interaction strength. Evaluation of the $\omega$ integral leads to

$$I^{(\eta)}_{\theta\bar{\theta}}(t) = \gamma \arctan (1 + t^2 \Lambda^2). \quad (10)$$

At equilibrium $\gamma \theta^{(\eta)}_{\omega q} = \pi \gamma / 2 \omega B_{eq}(\omega)$. For $T = 0$ this yields

$$I^{(\eta)}_{\theta\bar{\theta}}(t) = \frac{\gamma}{2} \log (1 + t^2 \Lambda^2). \quad (11)$$

Substituting Eqs. (11), (10) into Eq. (7), we reproduce the famous power-law behavior of TDOS,

$$\nu_{eq}(\epsilon) \sim \nu_0 (\epsilon / \Lambda)^\gamma. \quad (12)$$

At finite $T$ the long-time behavior of $I^{(\eta)}_{\theta\bar{\theta}}$ is modified,

$$I^{(\eta)}_{\theta\bar{\theta}}(t) \simeq \pi \gamma T t \equiv t / 2 \tau_\phi. \quad (13)$$

The ZBA dephasing rate is thus $1 / \tau_\phi = 2 \pi \gamma T$ and agrees with the one relevant to Aharonov-Bohm effect [8, 10]. While $1 / \tau_\phi$ contributes to smearing of $\nu_{eq}(\epsilon)$, for $\gamma \lesssim 1$ this is not particularly important, as the singularity is anyway smeared on the scale of $T$ due to the distribution function $n_\eta(t)$ in Eq. (7).

We turn now to the non-equilibrium situation. It is important to emphasize that the functional bosonization procedure can be also applied to a (clean) LL with spatially dependent interaction constant $K(x)$ (such a model was considered in [12]). Since the interaction can still be gauged out, the conclusion about independence of the fermionic distribution $n_\eta(\epsilon)$ on $x$ retains its validity. On the other hand, the boson sector will be, most generally, characterized by different distribution functions $B^u_\eta(\omega)$, $B^v_\eta(\omega)$ corresponding to the modes propagating with velocities $u$ and $v$ (coexistence of such modes is clear from the correlation function (11)). While the $u$-excitations are conventional plasmons, the $v$-mode describes bare electron-hole pairs (thus moving with velocity $v$ of non-interacting fermions). The corresponding formalism for higher-dimensional diffusive system was developed in [16] (where the analog of the $v$-mode was termed “ghosts”); for an analysis of energy relaxation in disordered LL in this framework, see Ref. [17]. We find

$$\int (dq) \theta^{(\eta)}_{\omega q} = \frac{\pi}{2 \omega} [\gamma B^v_{R}(\omega) + (\gamma + 2) B^R_{R}(\omega) - 2 B^v_{R}(\omega)]$$

and analogously for $\theta^{(\eta)}_{\bar{\eta}}$.

While the distribution $B^v_\eta$ of bare electron-hole pairs is determined by the fermion distribution function $n_\eta$, see Eq. (9), the plasmon distribution $B^v_\eta$ may differ, depending on the setup. We consider first the “adiabatic” situation when $K(x)$ changes slowly on the scale $l_\phi$ determined below. Then plasmons with relevant wave vectors are not backscattered by modulation of the interaction $K(x)$, so that $B^u_\eta = B^v_\eta \equiv B_\eta$ is given by Eq. (9). In this situation Eq. (13) reduces to

$$\int (dq) \theta^{(\eta)}_{\omega q} = \pi \gamma [B^v_{R}(\omega) + B^v_{L}(\omega)] / 2 \omega. \quad (14)$$

For the setup of Fig. 1A, Eq. (14) yields $I^{(\eta)}_{\theta\bar{\theta}} = \pi \gamma (T_L + T_R) t / 2$. Therefore, the ZBA dips in TDOS of both chiral sectors (separated by $\mu_L - \mu_R = eV$) get broadened (in addition to the thermal smearing) by the dephasing rate

$$1 / \tau_\phi = \pi \gamma (T_L + T_R). \quad (15)$$
If one of $T_{R,L}$ is much lower than the other, so that e.g. $T_L \ll 1/\tau_\phi$, the broadening of the corresponding ZBA minimum (near $\mu_L$) is determined by the dephasing. For the “fully non-equilibrium” setups of Fig.1b,c the fermionic distributions $n_\eta(\epsilon)$ are of a double-step form,

$$n_\eta(\epsilon) = a_n n_0(\epsilon-) + (1-a_n)n_0(\epsilon+),$$  \hfill (16)

where $n_0(\epsilon)$ is the zero-T Fermi distribution (we assume $T_\eta \ll eV$), $0 < a_n < 1$, and $\epsilon_{\pm} = \epsilon \pm V/2$. For the distribution (16), one finds the bosonic distribution function

$$\omega B^u_\eta(\omega) = [a_n^2 + (1-a_n)^2] \omega B_{eq}(\omega) + a_n(1-a_n) \times \left[ (\omega + eV)B_{eq}(\omega + eV) + (\omega - eV)B_{eq}(\omega - eV) \right].$$

This yields the TDOS (see Fig. 2)

$$\nu_\eta(\epsilon) \simeq a_n \nu_{eq}(\max\{\epsilon-, 1/2\tau_\phi^{(n)}\})$$

$$+ (1-a_n) \nu_{eq}(\max\{\epsilon+, 1/2\tau_\phi^{(n)}\}),$$  \hfill (17)

with the non-equilibrium ZBA dephasing rate

$$1/\tau_\phi^{(n)} = c_\eta \pi \gamma eV$$  \hfill (18)

and the numerical prefactor $c_\eta = a_R(1-a_R) + a_L(1-a_L)$.

If $K(x)$ varies fast, the plasmon distribution $B^u_\eta$ becomes spatially dependent, while $B^u_\eta$ remains unchanged. In the limit when the interaction is turned on as a sharp (on the scale $l_p$) step, the plasmons are scattered accordingly to the Fresnel law [13], with a reflection coefficient $R = (1-K^2)/(1+K)^2$. This yields a boundary condition for the distributions $B^u_\eta$; e.g. for a sharp change on the r.h.s. of the wire $B^u_\eta(\omega) = (1-R)B^u_\eta(\omega) + RB^u_R(\omega)$, and similarly for the left boundary. In this way it is easy to treat the situation with sharp switching of the interaction on one or both sides [20]. The results (17, 18) retain their validity but with modified numerical prefactors $c_\eta$.

To summarize, we have analyzed several setups in which non-equilibrium LL can be observed. Using the functional bosonization formalism, we have developed a theory of tunneling into such systems. While energy relaxation is absent, the dips of split ZBA are broadened by dephasing, Eqs. (15), (16), yielding a finite quasiparticle life time. We reiterate that RPA is exact for the setup of Fig. 1a, but not for the setups of Figs. 1b,c. In particular, the RPA value of the prefactor $c_\eta$ in Eq. (15) for $1/\tau_\phi$ should be considered as an approximation. One can expect that RPA becomes controllable for weak interaction, $\gamma \ll 1$. Also, it remains to be seen whether exact results can be obtained for a generic non-equilibrium setup. The key observation is that the sum of vacuum diagrams (log $Z_\phi$) can be cast in the form analogous to the generating function in the counting statistics problem [18, 19]. Work in these directions is underway.

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