PROFINITE COMMENSURABILITY
OF S-ARITHMETIC GROUPS

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Abstract. Given an $S$-arithmetic group, we ask how much information on the ambient algebraic group, number field of definition, and set of places $S$ is encoded in the commensurability class of the profinite completion. As a first step, we show that the profinite commensurability class of an $S$-arithmetic group with CSP determines the number field up to arithmetical equivalence and the places in $S$ above unramified primes. We include some applications to profiniteness questions of group invariants.

1. Introduction

Let $k \subset \mathbb{C}$ be a number field, let $G \subseteq \text{SL}_n$ be an algebraic simply-connected absolutely almost simple $k$-subgroup and let $S$ be a finite set of places of $k$ containing all the infinite ones. The three objects $G$, $k$ and $S$ define what we want to call a standard $S$-arithmetic group $\Gamma = G(O_{k,S})$, where $O_{k,S}$ are the $S$-integers of $k$. We say more generally that an abstract group $\Gamma$ is $S$-arithmetic if there are $G$, $k$ and $S$ as above such that $\Gamma$ can be embedded as a subgroup of $G(k)$ which is commensurable to the standard $S$-arithmetic group $G(O_{k,S})$.

We call $(k, G, S)$ an ambient triple for $\Gamma$. Two ambient triples $(k, G, S)$ and $(l, H, T)$ are called equivalent if there is a field isomorphism $\sigma: k \rightarrow l$ such that $\sigma S = T$ and such that $\sigma G$ is $l$-isomorphic to $H$. Here the set $\sigma S$ consists of all places of $l$ of the form $w = v \circ \sigma^{-1}$ for $v \in S$, and the $l$-group $\sigma G \subseteq \text{SL}_n$ is obtained from $G$ by applying $\sigma$ to the coefficients of the defining polynomials.

An equivalence class $[k, G, S]$ will be called an ambient class. Clearly, if one triple in $[k, G, S]$ is ambient for $\Gamma$, then so is any other triple in $[k, G, S]$. Recall that the $S$-rank of $G$ is defined by

$$\text{rank}_S(G) = \sum_{v \in S} \text{rank}_{k_v}(G)$$

where $\text{rank}_{k_v}(G)$ is the dimension of a maximal $k_v$-split torus in $G$. From now on, we assume $\text{rank}_S(G) \geq 2$ (“higher rank”) as well as $\text{rank}_{k_v}(G) \geq 1$ for all finite $v \in S$ (“no compact $p$-adic factor”) and say for short that $\Gamma$ is a higher rank $S$-arithmetic group. Margulis superrigidity combined with strong approximation implies that a higher rank $S$-arithmetic group $\Gamma$ determines an ambient class $[k, G, S]$ uniquely. Details will be given in Proposition 14.
1.1. Profinite commensurability. We call two groups (abstractly) commensurable if they have isomorphic finite index subgroups. The discussion so far says that ambient classes provide the commensurability classification of higher rank $S$-arithmetic groups: a commensurability class $[\Gamma]$ determines an ambient class $[k, G, S]$ and an ambient class $[k, G, S]$ defines the commensurability class $[G(O_{k,S})]$. These two constructions are well-defined and inverses of one another (Corollary 15). The profinite completion $\hat{\Gamma}$ of a group $\Gamma$ is the projective limit over the inverse system of all finite quotients of $\Gamma$.

**Definition 1.** Two groups are called profinitely commensurable if their profinite completions are commensurable.

It is readily verified that commensurable $S$-arithmetic groups are also profinitely commensurable (Proposition 16). Viewing that the commensurability class of a higher rank $S$-arithmetic group $\Gamma$ remembers the entire ambient information, it is a remarkable observation due to M. Aka [1] that the profinite commensurability class of $\Gamma$ determines neither $k$, nor $G$, nor $S$. Here are some examples.

(i) For $k = \mathbb{Q}(\sqrt[8]{3})$ and $l = \mathbb{Q}(\sqrt[8]{48})$ let $S$ and $T$ consist of all infinite places of $k$ and $l$, respectively, and of the finite places lying over 2 and 3. Then $\text{SL}_3(O_{k,S})$ and $\text{SL}_3(O_{l,T})$ are profinitely commensurable.

(ii) For a real quadratic number field $k$, the groups $\text{Spin}(6,1)(O_k)$ and $\text{Spin}(5,2)(O_k)$ are profinitely commensurable.

(iii) Let $k/\mathbb{Q}$ be Galois and let each of $S$ and $T$ contain precisely one pair of finite places, all four lying over a fixed split prime $p$. Then $\text{SL}_3(O_{k,S})$ and $\text{SL}_3(O_{k,T})$ are profinitely commensurable but not isomorphic unless the two pairs are $\text{Gal}(k/\mathbb{Q})$-conjugate.

To appreciate these examples, be aware that $S$-arithmetic groups are finitely generated and linear, whence residually finite: they embed densely into their profinite completion. All these observations call for a systematic study of higher rank $S$-arithmetic groups from the profinite point of view.

**Question 2.** What are the strongest notions of equivalence between the fields $k$ and $l$, the groups $G$ and $H$ and the sets $S$ and $T$ that can be concluded from $G(O_{k,S})$ and $H(O_{l,T})$ being profinitely commensurable?

Ideally, requiring the three notions of equivalence simultaneously would recover the equivalence relation of profinite commensurability. But there is no a priori reason why profinite commensurability would have such a tripartite description. With Theorem 3 and Theorem 4 below, we offer first results for the fields and the places.

**Theorem 3.** Suppose $G(O_{k,S})$ and $H(O_{l,T})$ have CSP and are profinitely commensurable. Then the fields $k$ and $l$ are arithmetically equivalent.

The shortest way to define arithmetical equivalence of two number fields is to say that they have the same Dedekind zeta function. An example is given by the pair $k = \mathbb{Q}(\sqrt[8]{3})$ and $l = \mathbb{Q}(\sqrt[8]{48})$ from (i) above. We discuss the notion of arithmetical equivalence more thoroughly in Section 2.1.

We define and discuss the congruence subgroup property (CSP) in Section 2.3. But let us already stress here that a well-known conjecture of Serre implies that the assumption of CSP in the theorem should be automatic.
Theorem 3 says that the notion of equivalence in Question 2 between k and l must at least be as strong as arithmetical equivalence. But in fact, for any pair of arithmetically equivalent fields k and l, a generalization of example (i) above gives $S$-arithmetic groups defined over k and l which are profinitely commensurable (Proposition 19). Hence the notion of equivalence between k and l cannot be stronger than arithmetical equivalence, either.

We will see in Theorem 11 (iii) below that arithmetically equivalent fields have equal discriminant. Therefore the same rational primes $p$ ramify in k and l. Let $S_p$ and $T_p$ be the set of places in S and T lying over $p$.

**Theorem 4.** Suppose $G(\mathcal{O}_k,S)$ and $H(\mathcal{O}_l,T)$ have CSP and are profinitely commensurable. Then for each unramified prime $p$, there is a residue degree preserving bijection $S_p \to T_p$.

Similarly as above, this notion of equivalence between $S$ and $T$ is close to optimal (Proposition 20). It could only be improved by an additional assertion on the places over ramified primes.

### 1.2. Applications to profiniteness of group invariants.

Once one has digested that residually finite groups with the same profinite completion must not be isomorphic, it becomes a fair question to ask what invariants of a group are already invariants of the profinite completion. Similarly, one can ask what commensurability invariants are already profinite commensurability invariants. These questions have attracted quite some research efforts recently and shall also be the theme of this section.

Aka’s example (ii) shows that the profinite commensurability class does not determine the group $G$ and hence neither does it determine the real Lie group $G = \prod_{\nu \in \mathbb{N}} G(k_\nu)$ in which an arithmetic group is a lattice. But even if one fixes a $\mathbb{Q}$-group $G$, one can still construct profinitely commensurable $S$-arithmetic groups with different surrounding Lie group $G_S = \prod_{\nu \in S} G(k_\nu)$ by considering $G$ over different fields. Here is an example.

**Proposition 5.** Let $k = \mathbb{Q}(\sqrt[8]{97})$ and $l = \mathbb{Q}(\sqrt[8]{1552})$ and let $S$ and $T$ consist of all places lying over 2 and 97. Then $SL_3(\mathcal{O}_k,S)$ and $SL_3(\mathcal{O}_l,T)$ have isomorphic profinite completions but $(SL_3)_S$ and $(SL_3)_T$ are not isomorphic.

The fields $k$ and $l$ are arithmetically equivalent and have common discriminant $-2^{10}97^4$ so that we inverted precisely the places over ramified primes. To exclude this possibility, let us say that the set $S$ is unramified if it contains no places over ramified (finite) primes. For unramified $S$, we finally obtain from Theorem 3 that the profinite commensurability class determines the Lie group $G_S$.

**Proposition 6.** Assume that the $\mathbb{Q}$-group $G$ has CSP. Then the profinite commensurability class of $[k, G, S]$ determines the Lie group $G_S$ up to isomorphism provided $S$ is unramified.

Here and in the following two results, the assumption of CSP for the $\mathbb{Q}$-group $G$ shall mean that $(k, G, S)$ has CSP for any $k$ and $S$ we consider. Again, this is conjecturally always true and we discuss in Section 2.3 that it is for example known to be true if $G$ is $\mathbb{Q}$-isotropic. Invoking recent work of Kyed–Petersen–Vaes [10], Proposition 6 has the following consequence.
Theorem 7. Assume that the $\mathbb{Q}$-group $G$ has CSP, that $\Gamma$ and $\Lambda$ with ambient triples $(k, G, S)$ and $(l, G, T)$ are profinitely commensurable, and that $S$ and $T$ are unramified. Then $b^{(2)}_n(\Gamma) = 0$ if and only if $b^{(2)}_n(\Lambda) = 0$.

For instance, the theorem applies to the groups $\Gamma = \text{Spin}(3, 2)(O_k, S)$ and $\Lambda = \text{Spin}(3, 2)(O_l, T)$ with $k = \mathbb{Q}(\theta_1)$ and $l = \mathbb{Q}(\theta_2)$ where $\theta_1$ is a root of the first and $\theta_2$ is a root of the second polynomial given in (13) below. For the finite sets $S$ and $T$, any choice is fine provided $S_2 \cup S_{13} \cup S_{191}$ is empty and provided there are residue degree preserving bijections $S_p \to T_p$ for all (finite) $p$. We will see on p. 14 that both $\Gamma$ and $\Lambda$ have a positive $n$-th $\ell^2$-Betti number if and only if $n = 7 + 2|S|$.

In contrast, the groups $\Gamma = \text{Spin}(6, 1)(O_k)$ and $\Lambda = \text{Spin}(5, 2)(O_k)$ from example (ii) are also profinitely commensurable but $b^{(2)}_n(\Gamma) > 0$ if and only if $n = 6$ and $b^{(2)}_n(\Lambda) > 0$ if and only if $n = 10$. So some restriction on the group $G$ will always be necessary when asserting profiniteness properties for $\ell^2$-Betti numbers of $S$-arithmetic groups. We remark that Aka came up with the latter groups as examples of profinitely isomorphic groups with and without Kazhdan’s property (T) [2]. A detailed account including the computation for $\ell^2$-Betti numbers can be found in the Master thesis of N. Stucki [24].

By Selberg’s lemma, an $S$-arithmetic group $\Gamma$ has a torsion-free subgroup $\Gamma_0$ of finite index. In [3, Proposition 6.10], Borel and Serre show that $\Gamma_0$ acts freely and cocompactly on a product of a “bordified” symmetric space and certain Bruhat–Tits buildings. The space is contractible and the quotient is triangulable so that it defines a finite classifying space $B\Gamma_0$. Thus the virtual Euler characteristic $\chi(\Gamma) = \chi(B\Gamma_0)/[\Gamma : \Gamma_0]$ is defined (and well-defined). Theorem 7 has the consequence that the sign of the Euler characteristic is constant throughout the profinitely commensurable groups in question.

Corollary 8. Assume that the $\mathbb{Q}$-group $G$ has CSP, that $\Gamma$ and $\Lambda$ with ambient triples $(k, G, S)$ and $(l, G, T)$ are profinitely commensurable, and that $S$ and $T$ are unramified. Then $\text{sgn}\chi(\Gamma) = \text{sgn}\chi(\Lambda)$.

Here, as usual, the function $\text{sgn}(x)$ takes the values $-1$, $0$ or $1$ according to whether $x$ is $< 0$, $= 0$ or $> 0$. We will explain at the end of the paper that it is possible to drop the assumption of unramified $S$ and $T$ both from Theorem 7 and from Corollary 8 if one assumes instead that the group $G$ splits over $k$ and $l$.

1.3. Outline. The article is organized as follows. In Section 2 we provide necessary background material. To wit, Section 2.1 is a quick report on arithmetically equivalent fields, with characterizations, properties and examples. Section 2.2 formulates strong approximation and defines the congruence completion in an adelic formulation. Section 2.3 surveys the main ideas of the congruence subgroup problem. In Section 3 the proofs of the results presented in this introduction are given. The commensurability classification is given in Section 3.1. Section 3.2 proves the main results and Section 3.3 concludes with the applications to profiniteness of group invariants.

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2. Preliminaries

In this section we collect necessary background material to prepare the proofs of the results from the introduction.

2.1. Arithmetically equivalent number fields. Let $k$ be a number field. The decomposition type of a rational prime $p$ in $k$ is the tuple

$$A_k(p) = (f_1, \ldots, f_r)$$

consisting of the residue degrees $f_i = f(p_i|p)$ of all primes $p_i$ lying over $p$ in ascending order: $f_1 \leq \cdots \leq f_r$.

**Definition 9.** Two number fields $k$ and $l$ are called arithmetically equivalent if $A_k(p) = A_l(p)$ for all but finitely many rational primes $p$.

We will next see that this is only one out of several characterizations of this notion. Let $N/\mathbb{Q}$ be any Galois extension containing both $k$ and $l$ and let $U$ and $V$ be the subgroups of $G = \text{Gal}(N/\mathbb{Q})$ corresponding to $k$ and $l$, respectively. We denote by $1_U^G$ and $1_V^G$ the characters of the coset permutation representations $\mathbb{C}[G/U]$ and $\mathbb{C}[G/V]$. Equivalently, $1_U^G$ and $1_V^G$ are the characters induced from the trivial (one-dimensional) characters $1_U$ and $1_V$. Finally, let $\zeta_k$ and $\zeta_l$ be the Dedekind zeta functions of $k$ and $l$.

**Theorem 10.** The following are equivalent.

(i) The fields $k$ and $l$ are arithmetically equivalent.

(ii) We have $A_k(p) = A_l(p)$ for all rational primes $p$.

(iii) We have $1_U^G = 1_V^G$.

(iv) We have $\zeta_k = \zeta_l$.

The result can be found in [8, Theorem III.1.3, p. 77]. Note that statement (iii) gives an entirely group theoretic characterization of arithmetical equivalence which will be useful in a moment. We include the proof of the corresponding equivalence (i) $\iff$ (iii) as it is brief and instructive.

**Proof.** (i) $\Rightarrow$ (iii). Given an unramified prime $\mathfrak{Q}$ of $N$ over a rational prime $p$, reduction mod $\mathfrak{Q}$ defines an isomorphism from the decomposition group $D(\mathfrak{Q}) = \{\sigma \in G : \mathfrak{Q}^\sigma = \mathfrak{Q}\}$ to the Galois group $\text{Gal}(O_N/\mathfrak{Q} / \mathbb{Z}/p)$ of the residue field. The latter group is cyclic, canonically generated by the Frobenius automorphism $x \mapsto x^p$. The unique preimage $F(\mathfrak{Q}) \in D(\mathfrak{Q})$ is called the Frobenius element of $\mathfrak{Q}$. By Chebotarev’s density theorem, each element $g \in G$ is a Frobenius element $g = F(\mathfrak{Q})$ for infinitely many primes $\mathfrak{Q}$ in $N$. Thus we can choose $\mathfrak{Q}$ above some unramified $p$ with $A_k(p) = A_l(p)$. The element $F(\mathfrak{Q})$ permutes the set $G/U$ and the cycle lengths of this permutation, in ascending order, coincide with the decomposition type $A_k(p)$ of $p$ in $k$. [8, Theorem I.1.12, p.11]. Hence $1_U^G(g)$ is the number of primes $p_i$ in $k$ over $p$ with residue degree $f_i = 1$. Since $A_k(p) = A_l(p)$, it follows that $1_U^G(g) = 1_V^G(g)$.

(iii) $\Rightarrow$ (i). The above shows more generally that the residue degrees $f_i$ in $k$ above any unramified prime $p$ satisfy the relation $1_U^G(g^a) = \sum_{f_i|a} f_i$ for
any positive integer $s$. Thus all residue degrees above unramified primes can be computed recursively from the numbers $1_{G}^{U}(g^s)$. □

Many field invariants turn out to be invariant under arithmetical equivalence because they can be calculated from the character $1_{G}^{U}$ alone. Here are some examples [8, Theorem III.1.4].

**Theorem 11.** The following invariants coincide for arithmetically equivalent number fields:

(i) degree over $\mathbb{Q}$,
(ii) Galois closure,
(iii) discriminant,
(iv) signature.

Recall that the signature of a number field is the pair $(r, s)$ where $r$ and $s$ is the number of real and complex places.

**Proof.** Retaining the setting and notation from Theorem 10, we obtain $[k : \mathbb{Q}] = [G : U] = 1_{G}^{U}(1)$ which shows (i). The Galois closure of $k$ corresponds to the normal core of $U$ which is precisely the kernel of the permutation representation $G \rightarrow \text{GL}(\mathbb{C}[G/U])$. Apparently, this kernel is the set of all $\sigma \in G$ satisfying $1_{G}^{U}(\sigma) = 1_{G}^{U}(1)$ which proves (ii). Artin’s conductor-discriminant formula says the discriminant of $k$ is given by the Artin conductor of $1_{G}^{U}$ [15, Korollar VII.11.8, p. 557]. This demonstrates (iii).

Complex conjugation defines an element $c \in G$. The value $1_{G}^{U}(c)$ is the number of cosets $gU$ fixed by $c$. The identity $cgU = gU$ says precisely that $cg$ and $g$ restrict to the same transformations on $k$; equivalently $g(k) \subset \mathbb{R}$. Thus $1_{G}^{U}(c) = r$ is the number of real embeddings, and $s = (1_{G}^{U}(1) - r)/2$ is the number of complex embeddings of $k$ up to conjugation. This shows (iv). □

**Example 12.** We conclude this excursion on arithmetically equivalent fields with a brief report on examples of low degree. According to [17], number fields of degree $\leq 6$ over $\mathbb{Q}$ are arithmetically solitary, meaning any pair of arithmetically equivalent fields is already conjugate over $\mathbb{Q}$. In degree seven, the fields generated by a root of the polynomials $X^7 - 7X^3 + 3$ and $X^7 + 14X^4 - 42X^2 - 21X + 9$, respectively, are examples of arithmetically equivalent fields which are not isomorphic [25]. More generally, one can construct infinitely many pairs of irreducible degree seven polynomials which define arithmetically equivalent but non-isomorphic number fields, including totally real ones, many of them with different class numbers [5]. I am grateful to Guillermo Mantilla for making me aware of the explicit polynomials

$$X^7 - 18X^5 - 28X^4 + 10X^3 + 24X^2 - 2 \quad \text{and} \quad X^7 - 3X^6 - 15X^5 + 51X^4 - 19X^3 - 41X^2 + 13X + 11$$

which, as we will see on p. [13] define a pair of non-isomorphic totally real number fields $k$ and $l$ that are even adelically equivalent, meaning the adele rings $\mathbb{A}_k$ and $\mathbb{A}_l$ are isomorphic (as $\mathbb{A}_\mathbb{Q}$-algebras or, equivalently, as topological rings). Equivalently, there is a bijection $\varphi$ between the finite places of $k$ and $l$ such that $k_v \cong \mathbb{Q}_p$, $l_{\varphi(v)}$ for all $v | p$ and all $p$ [8, Theorem 2.3,(b),
p. 237]. Since \([k_v : \Q_p] = e_v f_v\) is the product of ramification index and residue degree, this shows that adelic equivalence is a stronger equivalence relation than arithmetical equivalence.

In degree eight, examples of arithmetically and adelically equivalent fields occur among simple radical extensions. If \(a\) is any square-free integer with \(|a| \geq 3\), then the fields \(k = \Q(\sqrt{a})\) and \(l = \Q(\sqrt{2\sqrt{a}})\) are arithmetically equivalent and not conjugate \([9]\). If (and only if) in addition \(a = 2, 7, 14, 15 \mod 16\), the fields \(k, l\) are adelically equivalent. The eighth root \(\sqrt[8]{a}\) can be replaced by any \(2^n\)-th root \(\sqrt[n]{a}\) for \(r \geq 3\), creating examples of arithmetically and adelically equivalent fields of arbitrarily large degree.

2.2. Adeles, strong approximation, and the congruence completion. Let \(\mathbb{A}_{k,S}\) be the ring of \(S\)-truncated adeles of \(k\): the restricted product \(\prod_{v \notin S} k_v\) consisting of all elements in \(\prod_{v \notin S} k_v\) with almost all coordinates in the valuation ring \(O_v\) of \(k_v\). The ring \(\mathbb{A}_{k,S}\) becomes a locally compact topological ring if we declare that rectangular sets \(\prod_{v \in E \setminus S} U_v \times \prod_{v \notin E} O_v\) for finite \(E \supset S\) and open \(U_v \subset k_v\) form a basis of the topology. We have a diagonal embedding \(k \to \mathbb{A}_{k,S}\) which is dense according to the strong approximation theorem \([20]\) Theorem 1.5, p. 14]. Here it was important that \(S\) is always nonempty as it contains the infinite places of \(k\). The subspace topology of the open and closed subset \(\prod_{v \notin S} O_v \subset \mathbb{A}_{k,S}\) is generated by the open sets \(\prod_{v \in E \setminus S} U_v \cap O_v \times \prod_{v \notin E} O_v\), so that strong approximation implies that the closure of \(O_{k,S}\) under the embedding \(k \to \mathbb{A}_{k,S}\) is the product \(\prod_{v \notin S} O_v\).

The latter has the equivalent algebraic description \(\prod_{v \notin S} O_v = \widehat{O}_{k,S}\), where \(\widehat{O}_{k,S}\) is the profinite completion of the ring \(O_{k,S}\) defined as

\[
\widehat{O}_{k,S} = \varprojlim_{a \neq 0} O_{k,S}/a,
\]

the projective limit along the inverse system of all factor rings by nonzero ideals \(a \subset O_{k,S}\).

To see this, we observe that the canonical map \(O_{k,S} \to \widehat{O}_{k,S}\) is an embedding. The embedding is moreover dense by strong approximation, which in this setting is just a reformulation of the Chinese remainder theorem. The nonzero ideals \(a \subset O_{k,S}\) form a fundamental system of neighborhoods of zero for the subspace topology. This topology agrees with the topology of \(O_{k,S}\) as a subset of \(\prod_{v \notin S} O_v\) because the sets \(\prod_{v \in E \setminus S} \mathbb{P}^n_v \times \prod_{v \notin E} O_v\) are such a fundamental system in \(\prod_{v \notin S} O_v\), where \(n_v \geq 0\) and \(\mathbb{P}_v \subset O_v\) is the maximal ideal. Thus both \(\widehat{O}_{k,S}\) and \(\prod_{v \notin S} O_v\) are completions of the topological ring \(O_{k,S}\) (with respect to the canonical uniform structure) and therefore the identity map on \(O_{k,S}\) extends uniquely to an isomorphism \(\widehat{O}_{k,S} \cong \prod_{v \notin S} O_v\).

Similarly, for a \(k\)-subgroup \(G \subset \SL_n\), we define the \(\mathbb{A}_{k,S}\)-points \(G(\mathbb{A}_{k,S})\) as the restricted product \(\prod_{v \notin S} G(k_v)\) where almost all coordinates lie in \(G(O_v)\). We thus have a diagonal embedding of the \(k\)-rational points \(G(k) \subset G(\mathbb{A}_{k,S})\) and \(G\) is said to have the strong approximation property (with respect to \(S\)) if this embedding is dense. Arguing as above, we see that strong approximation implies that \(G(O_{k,S})\) has closure \(\prod_{v \notin S} G(O_v)\) under this embedding. The strong approximation theorem due to Kneser–Platonov \([20]\) Theorem 7.12, p. 427] asserts that for a \(k\)-simple, simply-connected
group $G$, strong approximation with respect to $S$ is equivalent to $G_S$ being noncompact. For the groups $G$ under our consideration, this condition is guaranteed as we assume $\text{rank}_S G \geq 2$. The embedding $G(O_{k,S}) \subset \prod_{v \notin S} G(O_v)$ endows $G(O_{k,S})$ with a subspace topology. In this topology, a fundamental system of neighborhoods of the unit element $e \in G(O_{k,S})$ consists of matrices whose entries differ from the entries of the unit matrix by elements in $O_{k,S}$ with certain prescribed minimal $v$-adic valuations for finitely many $v$. Algebraically, these neighborhoods can thus be described as the kernel $\Gamma(a)$ of the homomorphism $G(O_{k,S}) \to G(O_{k,S}/a)$ for the nonzero ideal $a \subset O_{k,S}$ given by the product of the prime ideal powers corresponding to the minimal $v$-adic valuations. The groups $\Gamma(a)$ are also known as the principal congruence subgroups of $G(O_{k,S})$ where a congruence subgroup would be any subgroup that contains some $\Gamma(a)$. Arguing as above, we see that the identity map on $G(O_{k,S})$ extends to a canonical isomorphism of topological groups

$$G(O_{k,S}) \cong \prod_{v \notin S} G(O_v)$$

where $G(O_{k,S})$ is given by the projective limit $\varprojlim_{a \neq 0} G(O_{k,S}/a)$. Consequently, $G(O_{k,S})$ is called the congruence completion of $G(O_{k,S})$. Note that from the functorial viewpoint, $G$ preserves products. One can therefore memorize the above isomorphism more suggestively as saying that $G$ is “continuous at the zero ideal”:

$$\varprojlim_{a \neq 0} G(O_{k,S}/a) = G \left( \lim_{a \neq 0} O_{k,S}/a \right).$$

2.3. The congruence subgroup problem. The congruence subgroups show that an $S$-arithmetic group $\Gamma = G(O_{k,S})$ comes with a wealth of finite index subgroups. This raises the question whether they actually provide for all (or essentially all) finite index subgroups of $\Gamma$. To state the precise, modern version of the problem, recall that $\hat{\Gamma}$ denotes the profinite completion of $\Gamma$, the projective limit along the inverse system of all finite quotient groups of $\Gamma$. Since the congruence completion $\overline{\Gamma}$ is a profinite group as well, the universal property of $\hat{\Gamma}$ yields a canonical homomorphism $\pi : \hat{\Gamma} \to \overline{\Gamma}$ that restricts to the identity on $\Gamma$. It is always an epimorphism because the image is dense, as it contains $\Gamma$, and closed because $\hat{\Gamma}$ is compact and $\overline{\Gamma}$ is Hausdorff. The kernel $C(k, G, S)$ of $\pi$ is known as the congruence kernel of the $k$-group $G$ with respect to $S$. We thus have a short exact sequence

$$1 \to C(k, G, S) \to \hat{\Gamma} \xrightarrow{\pi} \overline{\Gamma} \to 1.$$
all \(k\)-isotropic ones. The most notorious open case occurs when \(G\) is an anisotropic inner form of type \(A_n\).

For \((k,G,S)\) of our consideration, it is expected more precisely that \(C(k,G,S)\) should be isomorphic to the group of roots of unities \(\mu_k \subset k^*\) if \(k\) is totally imaginary and should be trivial if \(k\) is not. Raghunathan informs us in [22, p. 304] that the proof of this more precise picture is meanwhile complete for \(k\)-isotropic \(G\). So at least if \(k\) has a real place and \(\text{rank}_k G \geq 1\), we are in the convenient situation that every finite index subgroup of \(\Gamma = G(O_{k,S})\) is a congruence subgroup. In this case, strong approximation and CSP combine to the elegant canonical isomorphism

\[
G(O_{k,S}) \cong G(O_{k,S}).
\]

For more information on CSP, we refer the reader to the survey article [21].

### 3. Proofs

In this section we prove the results that were outlined in the introduction. We start by recalling how superrigidity and strong approximation imply the commensurability classification of higher rank \(S\)-arithmetic groups in Section 3.1. In Section 3.2, we show how profinite commensurability of two such groups implies similarities of the corresponding \(k\) and \(S\) by proving Theorems 3 and 4 together with the accompanying Propositions 19 and 20 which give a partial converse. Section 3.3 concludes with the proofs of the applications to profiniteness questions of group invariants.

#### 3.1. Commensurability classification

We show that ambient classes classify commensurability classes of higher rank \(S\)-arithmetic groups.

**Proposition 14.** Any two ambient triples \((k,G,S)\) and \((l,H,T)\) of a higher rank \(S\)-arithmetic group \(\Gamma\) are equivalent.

**Proof.** Fix an embedding \(\Gamma \leq G(k)\) commensurable with \(G(O_{k,S})\). Since \((l,H,T)\) is an ambient triple as well, we have another embedding \(\delta: \Gamma \to H(l)\) commensurable with \(H(O_{l,T})\). Thus there is a finite index subgroup \(\Gamma_0 \leq \Gamma \cap G(O_{k,S})\) which is mapped by \(\delta\) to a finite index subgroup in \(H(O_{l,T})\).

The group \(\delta(\Gamma_0)\) is Zariski dense in \(H\) according to [14, Proposition I.3.2.10, p. 64]. By Margulis superrigidity [14, Theorem (C), p. 259], there is a homomorphism \(\sigma: k \to l\), an \(l\)-epimorphism \(\eta: \hat{G} \to H\), and a homomorphism \(\nu: \Gamma_0 \to Z(H)\) to the (finite) center of \(H\) such that \(\delta(\gamma) = \nu(\gamma) \cdot (\sigma^0(\gamma))\) for all \(\gamma \in \Gamma_0\). Here \(\sigma^0: G(k) \to G(l)\) is the group isomorphism given by applying \(\sigma\) to the matrix entries. The subgroup \(\Gamma_1 = \ker \nu\) still has finite index in \(\Gamma\), and we have \(\delta(\gamma) = \eta(\sigma^0(\gamma))\) for all \(\gamma \in \Gamma_1\). As we can swap the roles of \(\Gamma\) and \(\Lambda\), there is also a homomorphism \(l \to k\), showing that \(\sigma\) is a field isomorphism. Moreover, since \(H\) is simply-connected, the central \(l\)-isogeny \(\eta\) is in fact an \(l\)-isomorphism.

It remains to show that \(\sigma S = T\). To this end, note that according to [14, Lemma I.3.1.1.(iv), p. 60], the group \(\eta \circ \sigma^0(G(O_{k,S})) = \eta(G(O_{l,S}))\) is commensurable with \(H(O_{l,S})\). As the homomorphism \(\eta \circ \sigma^0\) restricts to \(\delta\) on \(\Gamma_1\), we have that \(H(O_{l,S})\) is commensurable with \(H(O_{l,T})\). This says that the intersection \(H(O_{l,S}) \cap H(O_{l,T}) = H(O_{l,S \cap T})\) has finite index both in \(H(O_{l,T})\) and in \(H(O_{l,T})\). Therefore the proof is complete once we show...
that for general \( S_1 \subseteq S_2 \) satisfying our assumptions, the group \( H(O_{l,S_2}) \) has infinite index in \( H(O_{l,S_1}) \) unless \( S_1 = S_2 \). So suppose \( S_2 \setminus S_1 \neq \emptyset \). Then \( H_{S_2 \setminus S_1} = \prod_{v \in S_2 \setminus S_1} H(\ell_v) \) is a nontrivial product of noncompact \( p \)-adic \( \text{Lie} \) groups because we assume that \( H \) is \( \ell_v \)-isotropic for all finite places \( v \in S_2 \).

The assumption \( \text{rank}_S H \geq 2 \) says in particular that \( H_{S_1} \) is noncompact which, as discussed in Section 2.2, is equivalent to \( H \) having the strong approximation property with respect to \( S_1 \). The density of \( H(\ell) \) in \( H(A_{l,S_1}) \) clearly implies that the diagonal embedding of \( H(O_{l,S_2}) \) into \( H_{S_2 \setminus S_1} \) is dense. On the other hand, \( H(O_{l,S_1}) \) lies in the subgroup \( H_{S_2 \setminus S_1} = \prod_{v \in S_2 \setminus S_1} H(\ell_v) \) of \( H_{S_2 \setminus S_1} \), which is compact hence closed because \( H_{S_2 \setminus S_1} \) is Hausdorff. Since a compact subgroup of a noncompact group has infinite index, we conclude

\[
[H(O_{l,S_2}) : H(O_{l,S_1})] \geq [H_{S_2 \setminus S_1} : H(O_{l,S_1})] \geq [H_{S_2 \setminus S_1} : K_{S_2 \setminus S_1}] = \infty. \quad \Box
\]

If the embeddings \( \Gamma \to G(k) \) and \( \Gamma \to H(\overline{k}) \) are specified, then Margulis superrigidity says additionally that the isomorphism \( \sigma: k \to l \) and the \( l \)-isomorphism \( \eta: G \to H \) exhibiting the equivalence of ambient triples are unique with the property that the diagram

\[
\begin{array}{ccc}
G(k) & \xrightarrow{\sigma} & G(l) \\
\downarrow \cong & & \downarrow \eta \\
\hat{\Gamma} & \xrightarrow{\delta} & H(l)
\end{array}
\]

commutes for a finite index subgroup \( \Gamma_1 \leq \Gamma \). So it is fair to say that the ambient triple of a higher rank \( S \)-arithmetic group is unique up to unique equivalence. The above proposition shows in fact that ambient classes classify higher rank \( S \)-arithmetic groups up to abstract commensurability.

**Corollary 15.** Let \( \Gamma \) and \( \Lambda \) be higher rank \( S \)-arithmetic groups with ambient triples \( (k, G, S) \) and \( (l, H, T) \). Then \( \Gamma \) and \( \Lambda \) are abstractly commensurable if and only if \([k, G, S] = [l, H, T]\).

**Proof.** If \([k, G, S] = [l, H, T]\), then there are isomorphisms \( \sigma: k \to l \) and \( \eta: G \to H \) such that \( \sigma S = T \). Thus \( \Gamma \) is commensurable with \( G(O_{k,S}) \), which is isomorphic to \( \eta(G(O_{l,T})) \) which is commensurable with \( H(O_{l,T}) \) which is commensurable with \( \Lambda \). Thus \( \Gamma \) is abstractly commensurable with \( \Lambda \).

Conversely, let \( \delta: \Gamma_1 \to \Lambda_1 \) be an isomorphism where \([\Gamma : \Gamma_1] < \infty \) and \([\Lambda : \Lambda_1] < \infty \). Then \( \delta \) yields an embedding of \( \Gamma_1 \) into \( H(\overline{l}) \) commensurable with \( H(O_{l,T}) \). From Proposition 13 we conclude \([k, G, S] = [l, H, T]\). \( \Box \)

Finally, we show that commensurability is a stronger equivalence relation than profinite commensurability for the groups of interest.

**Proposition 16.** Let \( \Gamma \) and \( \Lambda \) be commensurable finitely generated residually finite groups. Then \( \Gamma \) and \( \Lambda \) are profinitely commensurable.

**Proof.** Let \( H \) be a group which has finite index embeddings into \( \Gamma \) and \( \Lambda \). Then the closures \( \overline{\Gamma}^H \) and \( \overline{\Lambda}^H \) are open in \( \Gamma \) and \( \Lambda \) according to [23] Proposition 3.2.2, p. 84]. So the cosets of these closures form disjoint open covers of the compact groups \( \widehat{\Gamma} \) and \( \widehat{\Lambda} \) which implies that \( \overline{\Gamma}^H \) and \( \overline{\Lambda}^H \) have
finite index in $\hat{\Gamma}$ and $\hat{\Lambda}$. Finally, $\hat{\mathbb{H}}_\Gamma \cong \hat{\mathbb{H}} \cong \hat{\mathbb{H}}_\Lambda$ because the profinite topologies on $\Gamma$ and $\Lambda$ induce the full profinite topology on any finite index subgroup. □

### 3.2. Consequences of profinite commensurability

As we just saw, the commensurability class of a higher rank $S$-arithmetic group $\Gamma$ can be identified with an ambient class $[k, G, S]$. Aka’s main result [1, Theorem 3], translated to our setting, says that the set of all higher rank $S$-arithmetic groups $\Lambda$ with $\hat{\Lambda} \cong \hat{\Gamma}$ lies in a finite union of ambient classes. The key proposition from which the theorem is concluded will also be the basis of our further investigations. To state it, let $g_p$ be the Lie algebra of the $p$-adic Lie group

$$G_p = \prod_{v | p, v \notin S} G(k_v).$$

We call $g_p$ the $p$-algebra of $G$ (or of $\Gamma$). Clearly, it is well-defined up to $\mathbb{Q}_p$-isomorphism.

**Proposition 17** ([1, Proposition 14]). The $p$-algebras $g_p$ depend only on $\hat{\Gamma}$.

One sentence on the proof: apply Margulis superrigidity to show that $g_p$ is isomorphic to the Lie algebra of a maximal $p$-adic analytic quotient of $\hat{\Gamma}$. To prepare the proof of Theorem 3, it is helpful to recall from [1] how to extract information on $k, G$ and $S$ from the algebras $g_p$. One first verifies that the simple ideals of $g_p$ are precisely the Lie algebras $\text{Lie}_{\mathbb{Q}_p} (G(k_v))$ of the factors $G(k_v)$ for $v \mid p$ and $v \notin S$. Hence the number of simple ideals of $g_p$ is maximal if and only if $p$ is split and $S$-unrelated, meaning $p$ does not lie below any place in $S$. This notion is well-defined because $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes the places of the Galois closure $N$ of $k$ lying over $p$ so that $S$ and $\sigma S$ have the same unrelated primes. Primes that are split and $S$-unrelated always exist because a prime $p$ splits in $k$ if and only if it splits in $N$ [15, Aufgabe I.9.4, p. 62], hence infinitely many primes $p$ are split in $k$ by Chebotarev’s density theorem. Thus the maximal number of simple ideals occurring among the $g_p$ equals $[k : \mathbb{Q}]$. For every prime $p$, the dimension of $g_p$ is given by

$$\dim_{\mathbb{Q}_p} g_p = \dim G \cdot \sum_{v | p, v \notin S} [k_v : \mathbb{Q}_p].$$

Here $\dim G = \dim_{\mathbb{C}} G(\mathbb{C})$ is the absolute dimension of $G$. Hence $\dim_{\mathbb{Q}_p} g_p$ takes on the maximal value $\dim G \cdot [k : \mathbb{Q}]$ if and only if $p$ is $S$-unrelated. This discussion allows the following conclusion.

**Corollary 18** ([1, Corollary 16]). The profinite completion $\hat{\Gamma}$ determines the numbers $[k : \mathbb{Q}]$ and $\dim G$ as well as the set of all $S$-unrelated primes.

The observation that $p$ is split and $S$-unrelated if and only if $g_p$ has maximal number of ideals also shows that $\hat{\Gamma}$ identifies almost all rational primes that split in $k$. As mentioned above, a rational prime splits in $k$ if and only if it splits in the Galois closure $N$. Since a Galois extension is uniquely determined by any finitely complemented subset of the set of split primes [15, Satz 13.9, p. 572], the profinite completion $\hat{\Gamma}$ also determines the
Galois closure $N$ of $k$ [11, Corollary 16 (1)]. In view of Theorem [11 (iii)], we now prove a sharpening of this result.

Proof of Theorem [3]. We have a 1-1 correspondence from the set $\{h_1, \ldots, h_r\}$ of simple ideals of $\mathfrak{p}_p$ to the set of places above $p$ and outside $S$ which sends an ideal $h_i$ to a place $v_i \mid p$ satisfying $h_i \cong_{\mathbb{Q}_p} \text{Lie}_{\mathbb{Q}_p} G(k_{v_i})$. The place $v_i$ determines the dimension of $h_i$ by

$$\dim_{\mathbb{Q}_p} h_i = \dim_{\mathfrak{p}_p} \text{Lie}_{\mathbb{Q}_p} G(k_{v_i}) = \dim G : [k_{v_i} : \mathbb{Q}_p] = \dim G \cdot e_{v_i} \cdot f_{v_i}$$

where $e_{v_i}$ and $f_{v_i}$ denote ramification index and residue degree of $v_i$. All but finitely many prime numbers $p$ are both unramified in $k$ and $S$-unrelated.

Hence in almost all cases the unordered tuple

$$A_p = \left\{ \frac{\dim_{\mathbb{Q}_p} h_1}{\dim G}, \ldots, \frac{\dim_{\mathbb{Q}_p} h_r}{\dim G} \right\}$$

gives the decomposition type of $p$ in $k$. $\square$

Next we show that any pair of arithmetically equivalent fields leads to examples of profinitely commensurable higher rank $S$-arithmetic groups.

Proposition 19. For arithmetically equivalent fields $k$ and $l$, let $S$ and $T$ contain precisely all infinite places and all places over ramified primes. Then $\text{SL}_3(\mathcal{O}_{k,S})$ and $\text{SL}_3(\mathcal{O}_{l,T})$ are profinitely commensurable.

The proof will additionally reveal that the groups are profinitely isomorphic whenever $k$ and $l$ are not totally imaginary.

Proof. Since $k$ and $l$ are arithmetically equivalent, Theorem [11 (iii)] says that for each rational prime $p$ there exists a residue degree preserving bijection $\varphi_p$ from the places $v \mid p$ of $k$ to the places $w \mid p$ of $l$. Whenever $p$ is unramified in $k$ (equivalently in $l$, Theorem [11 (iii)]), we have $k_v \cong_{\mathbb{Q}_p} l_{\varphi_p(v)}$ for all $v \mid p$. Indeed, an unramified extension of $\mathbb{Q}_p$ is uniquely determined by the residue field [11 Proposition II.§4.9, p. 49] which in both cases is the cyclic extension of $\mathbb{Z}/p$ with degree $f_v$. We thus also have $\mathcal{O}_v \cong \mathcal{O}_{\varphi_p(v)}$ for the valuation rings. Let us now first assume that $k$ (equivalently $l$, Theorem [11 (iv)]) is not totally imaginary. Then according to the discussion in Section 2.3, the congruence kernels $C(k, \text{SL}_3, S)$ and $C(l, \text{SL}_3, T)$ are trivial. Therefore the isomorphisms described in Sections 2.2 and 2.3 together with the bijections $\varphi_p$ assemble to an isomorphism

$$\text{SL}_3(\mathcal{O}_{k,S}) \cong \prod_{v \notin S} \text{SL}_3(\mathcal{O}_v) \cong \prod_{w \notin T} \text{SL}_3(\mathcal{O}_w) \cong \text{SL}_3(\mathcal{O}_{l,T}).$$

If $k$ is totally imaginary, we explained in Section 2.3 that we have a short exact sequence

$$1 \rightarrow \mu_k \rightarrow \text{SL}_3(\mathcal{O}_{k,S}) \xrightarrow{\pi_k} \prod_{v \notin S} \text{SL}_3(\mathcal{O}_v) \rightarrow 1$$

and similarly for $l$ and $T$. In a profinite group, the intersection of all open normal subgroups is trivial. Thus we can find an open, hence finite index subgroup $U \subset \text{SL}_3(\mathcal{O}_{k,S})$ which intersects the finite group $\mu_k$ trivially. This has the effect that $\pi_k$ embeds $U$ into $\prod_{v \notin S} \text{SL}_3(\mathcal{O}_v)$ as a finite index subgroup by surjectivity. The group $\pi_k(U)$ is compact, thus closed because
profinite groups. Therefore \( \pi_k(U) \), being closed and of finite index, is open in \( \prod_{v \in S} SL_3(O_v) \). Repeating the same construction for \( l \) and \( T \), we obtain an open subgroup \( \pi_k(V) \) in \( \prod_{w \in T} SL_3(O_w) \). Identifying the two products by the bijections \( \varphi_p \) as above, we can form the intersection \( \pi_k(U) \cap \pi_\ell(V) \) which is still of finite index. Thus the preimages of this intersection under \( \pi_k \) and \( \pi_\ell \) are isomorphic finite index subgroups. \( \square \)

Theorem 4 implies that the profinite commensurability class determines the set of places \( S \) of an \( S \)-arithmetic group, at least above unramified primes:

**Proof of Theorem 4.** By Theorem 3 and Theorem 10 (ii), all decomposition types of all rational primes \( p \) are equal in \( k \) and \( l \). For an unramified prime \( p \), the dimensions of the simple ideals in \( g_p \) yield the residue degrees of the places lying over \( p \) and outside \( S \) as in the proof of Theorem 3. Thus, also the residue degrees of the places in \( S_p \) are determined by \( g_p \). \( \square \)

The next proposition says that Theorem 4 cannot be improved, except possibly by a statement about places over ramified primes.

**Proposition 20.** Let \( k \) and \( l \) be arithmetically equivalent fields. For finitely many unramified \( p \), let \( S_p \) and \( T_p \) be sets of places in \( k \) and \( l \) above \( p \) such that there are residue degree preserving bijections \( S_p \to T_p \). If \( S \) and \( T \) consist of all the sets \( S_p \) and \( T_p \), all infinite places, and all places over ramified primes, then \( SL_3(O_{k,S}) \) and \( SL_3(O_{l,T}) \) are profinitely commensurable.

**Proof.** By default, let us set \( S_p = T_p = \emptyset \) for the remaining \( p \) and let us denote the discriminant of \( k \) by \( \Delta_k \). Assume first that \( k \) is not totally imaginary. Incorporating Theorem 5, we have similarly as in Proposition 19

\[
SL_3(O_{k,S}) \cong \prod_{p \mid \Delta_k} \prod_{v \not\in S_p} SL_3(O_v) \cong \prod_{p \mid \Delta_k} \prod_{w \not\in T_p} SL_3(O_w) \cong SL_3(O_{l,T}).
\]

If \( k \) is totally imaginary, the same construction as in Proposition 19 yields that the groups are profinitely commensurable. \( \square \)

Both in Proposition 19 and in Proposition 20, we used the group \( G = SL_3 \) for simplicity but any other \( k \)- and \( l \)-isotropic group \( G \) satisfying our assumptions would work equally fine.

### 3.3. Profiniteness of group invariants

We construct an example of two profinitely isomorphic \( S \)-arithmetic groups with the same underlying \( \mathbb{Q} \)-group but different surrounding Lie groups.

**Proof of Proposition 4.** The fields \( k = \mathbb{Q}(\sqrt{97}) \) and \( l = \mathbb{Q}(\sqrt{16 \cdot 97}) \) are arithmetically equivalent by our discussion in Example 12. The common discriminant of \( k \) and \( l \) equals \(-2^{10}97^2 \), thus \( 2 \) and \( 97 \) are the only ramified primes. By Proposition 19, they define a pair of profinitely isomorphic \( S \)-arithmetic groups \( SL_3(O_{k,S}) \) and \( SL_3(O_{l,T}) \). According to Perlis [17, p. 351], the prime \( 2 \) decomposes into four different places both in \( k \) and \( l \) with ramification indices 1, 1, 2, 4 and 2, 2, 2, 2, respectively. The prime \( 97 \) is totally ramified in \( k \) and \( l \). Let \( k_{97} \) and \( l_{97} \) be the completions of \( k \) and
with respect to the unique place lying over 97. Then \( \text{SL}_3(\mathcal{O}_k,S) \) and \( \text{SL}_3(\mathcal{O}_{l,T}) \) are lattices in the Lie groups

\[
(\text{SL}_3)_S = \prod_{v \in S} \text{SL}_3(k_v) = \text{SL}_3(\infty) \times \text{SL}_3(k_{97}) \times \prod_{v \mid 2} \text{SL}_3(k_v) \quad \text{and}
\]

\[
(\text{SL}_3)_T = \prod_{w \in T} \text{SL}_3(l_w) = \text{SL}_3(\infty) \times \text{SL}_3(l_{97}) \times \prod_{w \mid 2} \text{SL}_3(l_w),
\]

respectively, where \( \text{SL}_3(\infty) = \text{SL}_3(\mathbb{R}) \times \text{SL}_3(\mathbb{R}) \times \text{SL}_3(\mathbb{C}) \times \text{SL}_3(\mathbb{C}) \). Both groups are products of a real Lie group, a 97-adic Lie group and a 2-adic Lie group. In \( (\text{SL}_3)_S \) the Lie algebra of the 2-adic factor has simple ideals of dimensions 8, 8, 16 each. Thus \( (\text{SL}_3)_S \) and \( (\text{SL}_3)_T \) cannot be isomorphic. □

We will now see that examples of this type can be ruled out by assuming that the set \( S \) is unramified.

**Proof of Proposition 6.** If \( [k, G, S] \) and \( [l, G, T] \) are profinitely commensurable, then \( k \) and \( l \) are arithmetically equivalent by Theorem 3. By Theorem 4, there are residue degree preserving bijections \( \varphi_p : S_p \to T_p \) for all unramified primes \( p \). As we already discussed in the proof of Proposition 1, we thus obtain field isomorphisms \( k_v \cong l_{\varphi_p(v)} \) for all \( v \in S_p \) with unramified \( p \). Since by assumption \( S_p = T_p = \emptyset \) whenever \( p \) is ramified, we thus obtain

\[
G_S = \prod_{v|\infty} G(k_v) \times \prod_p \prod_{v \in S_p} G(k_v) \cong \prod_{w|\infty} G(l_w) \times \prod_p \prod_{w \in T_p} G(l_w) = G_T. \quad \Box
\]

We now prove Theorem 7 which gives a consequence of these observations for the \( \ell^2 \)-cohomology of \( S \)-arithmetic groups.

**Proof of Theorem 7.** Kye–Petersen–Vaes [16, Theorem B] showed that the \( \ell^2 \)-Betti numbers \( b_n^{(2)}(\Gamma) \) of a lattice \( \Gamma \leq G \) in a locally compact group \( G \) with Haar measure \( \mu \) are given by \( b_n^{(2)}(\Gamma) = b_n^{(2)}(G,\mu) \cdot \mu(\Gamma\setminus G) \). Here \( b_n^{(2)}(G,\mu) \) is the \( \ell^2 \)-Betti number of the locally compact group \( G \) as defined by Petersen [18] and \( \mu(\Gamma\setminus G) > 0 \) is the induced \( G \)-invariant measure on the quotient space. The assertion of the theorem is therefore immediate from Proposition 6. □

Here are some explanations on the example below Theorem 7. To begin with, one can establish that the polynomials in [13] define arithmetically equivalent number fields by means of Perlis’ criterion [16] which states that \( k \) and \( l \) of prime degree \( p \) are arithmetically equivalent if and only if \( [k : l : \mathbb{Q}] < p^2 \). The arithmetic equivalence of \( k \) and \( l \) already implies that \( k \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong_{\mathbb{Q}_p} l \otimes_{\mathbb{Q}} \mathbb{Q}_p \) for all unramified \( p \). To see that \( k \) and \( l \) are adelically equivalent, it thus remains to analyze the completions at ramified primes. Since the common discriminant of \( k \) and \( l \) is \( 2^6 \cdot 13^4 \cdot 191^2 \), we only need to compare \( k \otimes_{\mathbb{Q}} \mathbb{Q}_p \) to \( l \otimes_{\mathbb{Q}} \mathbb{Q}_p \) for \( p = 2, 13, 191 \). With the Database of Local Fields due to Jones–Roberts [6], it is conveniently verified that these \( \mathbb{Q}_p \)-algebras are likewise isomorphic. Interestingly, the two fields \( k \) and \( l \) have
non-isomorphic integral trace forms as can be seen by the methods of [13]. Of course this implies in particular that they are not isomorphic.

Along the lines of the proof of Proposition [19] we see that the groups $\text{Spin}(3, 2)(\mathcal{O}_k,S)$ and $\text{Spin}(3, 2)(\mathcal{O}_{l,T})$ are profinitely commensurable, and in fact profinitely isomorphic. The assumption that $S$ contains no places over ramified primes effects that the full algebra $k \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is contained in $k_{k,S}$ for $p = 2, 13, 191$ and similarly for $l$ and $T$ so that $\prod_{v \mid S} \text{Spin}(3, 2)(\mathcal{O}_v) \cong \prod_{w \notin T} \text{Spin}(3, 2)(\mathcal{O}_w)$ by the adelic equivalence of $k$ and $l$.

Finally, we check for which $n$ the $\ell^2$-Betti number $b^{(2)}_n(Sp(3, 2))$ is nonzero. For this question one can clearly suppress the Haar measures from the notation. The real semisimple Lie group

$$G = \prod_{v \mid \infty} \text{Spin}(3, 2)(\mathbb{R}) = (\text{Spin}(3, 2)(\mathbb{R}))^7$$

has maximal compact subgroup $K = (\text{Spin}(3) \times \text{Spin}(2))^7$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ for which we have rank$_{\mathbb{C}} \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \text{rank}_{\mathbb{C}} \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$. This shows that $G$ has discrete series representations from which it follows that $b^{(2)}_n(G) > 0$ if and only if $n = 21$. In our terms, this can be concluded from [19, Corollary 11] though the calculation is due to Borel [3]. The degree $n = 21$ is precisely the middle dimension of the associated symmetric space. For a finite place $v$ of $k$, the totally disconnected locally compact group $\text{Spin}(3, 2)(k_v)$ can only have a positive $\ell^2$-Betti number in degree two which is the $k_v$-rank of $\text{Spin}(3, 2)$ [19, Corollary 13]. In fact, in this case we have $b^{(2)}_2(\text{Spin}(3, 2)(k_v)) = \nu(\text{St})$ where $\nu$ is the Plancherel measure on the unitary dual $\hat{G}$ and $\text{St}$ is the Steinberg module. The latter is a discrete series representation and those representations form precisely the atoms of the Plancherel measure so that indeed $b^{(2)}_2(\text{Spin}(3, 2)(k_v)) > 0$. Petersen’s Küneth formulas [13, Theorems 6.5 and 6.7] for products of locally compact groups thus show that all $\ell^2$-Betti numbers of $(\text{Spin}(3, 2))_S$ vanish, except in degree $n = 21 + 2(|S| - 7)$ when

$$b^{(2)}_{7+2|S|}((\text{Spin}(3, 2))_S) = b^{(2)}_{21}(G) \prod_{v \in S} b^{(2)}_{2}(\text{Spin}(3, 2)(k_v)) > 0.$$ 

We remark that, strictly speaking, [19, Corollary 13] contains the assumption that the residue field of $k_v$ should be large (of order $> \frac{1233}{24}$). But A. Valette informs us that it should be possible to drop this assumption by transferring methods of Borel–Wallach from finite-dimensional to infinite-dimensional representations although this has not been written up yet.

We conclude easily from Theorem 7 that the Euler characteristic has the same sign for all the profinitely commensurable groups it deals with.

*Proof of Corollary* [8] Just as for ordinary Betti numbers, we have the alternating sum formula $\chi(\Gamma) = \sum_{n \geq 0} (-1)^n b^{(2)}_n(\Gamma)$, see [12, Theorem 1.35.(2), p.37] or [7, Theorem 2.30]. But as we discussed above, $S$-arithmetic subgroups of simple algebraic groups have a nonzero $\ell^2$-Betti number in at most one degree. Theorem 7 says that this degree is the same for the groups under consideration.
Finally, we explain why we need no assumptions on $S$ and $T$ in Theorem 10 and Corollary 11 if $G$ splits over $k$ and $l$. By Theorem 9 and Theorem 11 (iv), the real and complex factors in the Lie group products $G_S$ and $G_T$ are isomorphic. The $p$-adic factors in $G_S$ and $G_T$ may differ, as we saw in Proposition 9, but their number is the same. Indeed, for each rational prime $p$ we know that the number of places over $p$ is equal in $k$ and $l$ by Theorem 8 and Theorem 10 (ii). Also, the number of places in $k$ above $p$ and outside $S$ is equal to the number of places in $l$ above $p$ and outside $T$ because this number equals the number of simple ideals in $\mathfrak{g}_p$. Put together, this implies $|S_p| = |T_p|$ for all $p$, thus $|S| = |T|$. Since we assume that $G$ splits over $k$ and $l$, we have $\text{rank}_k G = \text{rank}_w G = \text{rank}_C G = r$ for all places $v$ in $k$ and $w$ in $l$. Thus by the Künneth formula, the only nonzero $\ell^2$-Betti number of the totally disconnected factor in $G_S$ and $G_T$ sits in the same degree $r([S] - [\{ v | \infty \}])$. Again by the Künneth formula, $b_n^{(2)}(G_S) = 0$ if and only if $b_n^{(2)}(G_T) = 0$ and this implies the result. Note that instead of requiring that $G$ be split over $k$ and $l$, it would actually have sufficed to assume that $G$ splits over all local fields corresponding to the finite places in $S$ and $T$.

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