On the Preservation of Quasilocality by the Integration-Out Transformation

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Abstract

We demonstrate that the integration-out step of the renormalization group transformation preserves the quasilocality of the effective action. This is shown in the case of a single, real, scalar field on a torus, but the proof holds more generally. The main result can be thought of as showing the flow invariance of the quasilocal subset under the flow generated by the Polchinski equation.

One of the deepest ideas in physics is the renormalization group. This is the concept that one can study a complicated system by analyzing it one scale at a time. Intuitively, it is clear that for this procedure to be practical, one should deal with systems which are at least approximately local. In technical terms, the effective actions must admit some form of derivative expansion. In other words, when one implements the renormalization group, a standard requirement is that the effective actions considered are required to be "quasilocal" using the terminology of [1]. Roughly speaking, (we will give precise definitions later) these are functionals of the form

\[ S[\phi] = \sum_{n=0}^{\infty} \int dp_1 \ldots dp_n \delta(p_1 + \cdots + p_n) G(p_1, \ldots, p_n) \phi(p_1) \ldots \phi(p_n). \]

The goal of this paper is to prove rigorously that the integration-out step of the renormalization group preserves quasilocality. We shall demonstrate this fact in a specific realization, notably for a theory of a single real scalar field. However, it will be clear that the proofs can be easily generalized to other set-ups.

As will be apparent below, one way to view the main result of this work is as a fact in the theory of viability [2] [3]. This is a vast subject in which a great amount of work has been expended. However, the greatest majority of the established results pertain to conditions one puts on the full flow corresponding to the evolution equation (typically, a form of tangency of the flow to the set is considered). There are only a few results [4] [5] which
supply sufficient conditions on the generator, and even then, they require
information on the behaviour of the generator away from the set which is
to be invariant. Below, a viability fact is proven under the assump-
tion that the linear part of the equation preserves the set, while the nonlinear part
of the generator maps the set to itself (see the discussion at the end of the
paper for more details).

As mentioned above, we will deal with the theory of a real scalar field \( \phi \).
To make things rigorous, we need both an infrared and an ultraviolet cut-
off. We shall implement the infrared cutoff by putting periodic boundary
conditions, i.e. we will consider our space of fields to be \( L^2 \) functions on the
d-dimensional torus \( T^d = S^1 \times S^1 \times \cdots \times S^1 \) \(^1\) Of course, we have the usual

Fourier correspondence

\[
\phi(x) \iff a_n = \int_{T^d} \phi(x) e^{-2\pi i nx} dx
\]

\[
\{a_n\}_{n \in \mathbb{Z}^d} \iff \phi(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{2\pi i nx},
\]

where \( nx = n_1 x_1 + \cdots + n_d x_d \), and the measure on \( T^d \) is normalized so that
it has unit volume. Note that the fact that \( \phi \) is real implies that
\( a_{-n} = \overline{a_n} \) for every \( n \in \mathbb{Z}^d \). Moreover, \( \phi \in L^2(T^d) \iff \sum_{n \in \mathbb{Z}^d} |a_n|^2 < \infty \). We shall
denote the space of the Fourier coefficients by \( l^2(\mathbb{Z}^d) \).

Let \( \Lambda, \Lambda' \) be positive numbers with \( \Lambda > \Lambda' \). Consider the following families
of bilinear forms on the space of fields

\[
K_\Lambda(\{a_n\}, \{b_n\}) = \frac{1}{2} \sum_{n \in \mathbb{Z}^d} e^{-\frac{n^2}{\Lambda}} \overline{a_n} b_n
\]

\[
K_{\Lambda,\Lambda'}(\{a_n\}, \{b_n\}) = \frac{1}{2} \sum_{n \in \mathbb{Z}^d} e^{-\frac{n^2}{\Lambda}} - e^{-\frac{n^2}{\Lambda'}} \overline{a_n} b_n.
\]

The exponentials in the formulas above implement the ultraviolet cutoffs.
The exact expressions chosen here are not important. Instead of \( e^{-x^2} \), we
could have chosen any other even, nonnegative function \( \alpha(x) \) on \( \mathbb{R} \), such that
\( \alpha(0) = 1 \), \( \alpha \) is monotone decreasing for \( x \geq 0 \) and decays sufficiently fast
at \( \infty \), as well as \( \alpha \) is differentiable at 0 (this last condition will be needed
later). Moreover, we’ve selected the propagator of the un-cutoff theory to
be that of the massive field \( \frac{1}{n^2 + 1} \). We could have easily replaced it with any

\(^1\)They will be contained in a much smaller space after we impose the ultraviolet cutoffs.
nonnegative sequence which doesn’t grow too fast.

The two bilinear forms above are trivially seen to be positive definite. Moreover, these two forms correspond to two families of diagonal operators on \( l^2(\mathbb{Z}^d) \) which are trace-class. Therefore, by standard results, the bilinear forms above define two families of Gaussian measures on \( l^2(\mathbb{Z}^d) \) whose covariances are the given expressions. We shall denote these measures by \( \mu_\Lambda \) and \( \mu_{\Lambda,\Lambda'} \). Now, while we have defined these to be measures on the space of \( L^2(\mathbb{T}^d) \) functions, they are in fact supported on a much smaller space. Again, by standard results, we can easily see that all these measures are supported on the subspace of elements \( \{a_n\}_{n\in\mathbb{Z}^d} \) for which

\[
|||a_n||| \equiv \sum_{n\in\mathbb{Z}^d} e^{\sqrt{n_1^2 + \cdots + n_d^2}} |a_n|^2 < \infty.
\]

In particular, note that this implies that the measures are supported on a certain subspace of smooth functions on the torus. This subspace of \( l^2(\mathbb{Z}^d) \) topologized by the norm \( ||| \cdot ||| \) shall be denoted by \( V \). It follows that all our measures can be thought to have a common domain which is \( V \).

We can now describe our space of effective actions. Consider first the space \( L^2(\mu_\Lambda) \). As is well-known, any element \( S \) of this space can be written as

\[
S = \sum_{n_1,\ldots,n_k \in \mathbb{Z}^d, k=0,1,\ldots} G(n_1,\ldots,n_k) : a_{n_1} \cdots a_{n_k} :_\Lambda
\]

(1)

where \( : a_{n_1} \cdots a_{n_k} :_\Lambda \) is the Wick product of \( a_{n_1},\ldots,a_{n_k} \) (with respect to the measure \( \mu_\Lambda \)) and the sum converges in \( L^2(\mu_\Lambda) \). Incidentally, note that while the Wick products are an orthogonal basis of \( L^2(\mu_\Lambda) \) they are not normalized and thus the \( L^2 \) norm of the element above is given by

\[
\sum_{n_1,\ldots,n_k,k=0,1,\ldots} k! e^{-n_1^2/n_1^2 + 1} \cdots e^{-n_k^2/n_k^2 + 1} |G(n_1,\ldots,n_k)|^2.
\]

(2)

Note that any functional of the form

\[
\int_{\mathbb{T}^d} \prod_{m=0}^M \left( \prod_{i_1,\ldots,i_m=1,\ldots,d} (\partial_{x_{i_1},\ldots,x_{i_m}}^m \phi)^{\alpha_{i_1,\ldots,i_m}} \right)(x) dx,
\]

(3)

\footnote{In fact, the supports are significantly smaller still, see e.g. example 2.3.6 in \cite{6}. Moreover, the supports actually depend on the parameters \( \Lambda \). However, the space that we chose is sufficient for our purposes.}

\footnote{For the reader’s convenience, the definition of the Wick product, as well as a few useful formulae, are given in Appendix A.}

\footnote{One needs to be careful here, as the \( a_n \)’s are not all independent, since \( \overline{a_n} = a_{-n} \). However, the formula for the \( L^2 \) norm remains valid.}

\[
|||a_n||| \equiv \sum_{n\in\mathbb{Z}^d} e^{\sqrt{n_1^2 + \cdots + n_d^2}} |a_n|^2 < \infty.
\]
where all $\alpha_{i_1,\ldots,i_m} \in \mathbb{N}$ is in fact a well-defined element of $L^2(\mu_\Lambda)$. This is a consequence of the inequality
\[
|\partial_{x_1,\ldots,x_m}^m \phi(x)| \lesssim \sum_{n \in \mathbb{Z}^d} (n_1^2 + \ldots + n_d^2)^m |a_n| \lesssim |||a_n|||^A,
\] (4)
where $\lesssim$ means less than a constant multiple of. It thus follows that the square of the expression above is bounded from above by a constant multiple of $|||a_n|||^A$ for some integer $A$\footnote{Twice the sum of all the $\alpha_{i_1,\ldots,i_m}$ works.}. As follows by Fernique’s theorem, each such expression is integrable, and we have our claim.

We can now make an important

**Definition.** A finite linear combination of functionals of the form (3) is called a local functional. An element in the closure in $L^2(\mu_\Lambda)$ of the set of local functionals is said to be quasilocal. The set of all quasilocal functionals will be denoted by $Q(\Lambda)$.

The next proposition shows that we could have defined the notion of quasilocality differently.

**Proposition.** An element of $L^2(\mu_\Lambda)$ is quasilocal if and only if in (1) we have that
\[
n_1 + \cdots + n_k \neq 0 \implies G(n_1,\ldots,n_k) = 0, \quad \forall n_1,\ldots,n_k \in \mathbb{Z}^d, k \in \mathbb{N}
\] (5)

**Proof.** The only if direction is a consequence of the fact that every local functional satisfies (5). This is done by a direct computation using the facts that all the fields and their derivatives are multiplied at the same point in a local functional, and that the propagator between $a_n$ and $a_m$ (which enters in the Wick product) vanishes unless $n = -m$. The fact that a limit of functionals satisfying (5), itself satisfies (5), follows at once from the expression for the $L^2$ norm given in (2).

To see the other direction, first note that for a fixed $k$, if $G(n_1,\ldots,n_k)$ is a polynomial (in $n_1,\ldots,n_k$) then, a direct computation shows that
\[
\sum_{n_1,\ldots,n_k \in \mathbb{Z}^d} G(n_1,\ldots,n_k) : a_{n_1} \cdots a_{n_k} :_\Lambda
\]
is a local functional. Thus, we will be done if we show that polynomials in $n_1,\ldots,n_k$ are dense in the weighted $l^2(\mathbb{Z}^d \times \cdots \times \mathbb{Z}^d)$, with the weight given by the expression before $|G|^2$ in (2). However, this is a well-known fact about the density in $L^2$ of polynomials for measures with sub-exponential tails (see e.g. [7]). Hence, the proof is complete. \qed
Now, we want to introduce the integration-out operator of the renormalization group. This is meant to act on an element of $L^2(\mu_\Lambda)$, produce an element of $L^2(\mu_{\Lambda'})$, and be given by the following formula

$$I_{\Lambda,\Lambda'}(S)[\psi] = -\ln \left( \int e^{-S[\phi+\psi]} d\mu_{\Lambda}[\phi] \right).$$

(6)

Clearly the expression above cannot be defined on all of $Q(\Lambda)$ and one needs to restrict the space somewhat. Let $b \in \mathbb{R}$ and denote by $L^2_b(\mu_\Lambda)$ the subset of all elements in $L^2(\mu_\Lambda)$ which are bounded from below by $b$. A natural restriction to make (6) well-defined would be to consider only elements in $Q \cap L^2_b(\mu_\Lambda)$. However, for technical reasons, it turns out to be convenient to restrict the space a little further as is given in the following

**Definition.** Let $Q_b(\Lambda)$ be the closure in $L^2(\mu_\Lambda)$ of the bounded elements in $Q \cap L^2_b(\mu_\Lambda)$.

We show in Appendix B that $Q_b$ is rich enough to include every local functional whose integrand (defined there) is bounded from below by $b$.

We can now state our main result which is the following

**Theorem.** $I_{\Lambda,\Lambda'}$ is a continuous map from $Q_b(\Lambda)$ to $Q_b(\Lambda')$.

**Proof.** The fact that (6) is well defined on $Q_b(\Lambda)$ (in fact on $L^2_b(\mu_\Lambda)$) and that the lower bound is preserved by $I_{\Lambda,\Lambda'}$ is immediate. Let $I'_{\Lambda,\Lambda'}(S)[\psi]$ be given by

$$\int S[\phi+\psi] d\mu_{\Lambda}[\phi].$$

By Jensen’s inequality, we have that

$$I_{\Lambda,\Lambda'}(S)[\psi] \leq I'_{\Lambda,\Lambda'}(S)[\psi].$$

Let $f^+$ stand for the positive part of a function. We then have that

$$\sqrt{\int \left( I_{\Lambda,\Lambda'}(S)[\psi] \right)^2 d\mu_{\Lambda}[\psi]} \leq \sqrt{\int \left( I'_{\Lambda,\Lambda'}(S)[\psi] \right)^2 d\mu_{\Lambda}[\psi]} \leq \sqrt{\int \left( I'_{\Lambda,\Lambda'}(S)[\psi] \right)^2 d\mu_{\Lambda}[\psi]}.$$

Since $K_{\Lambda'}(\{a_n\}, \{b_n\}) + K_{\Lambda,\Lambda'}(\{a_n\}, \{b_n\}) = K_\Lambda(\{a_n\}, \{b_n\})$, we have at once that for any $L^1(\mu_\Lambda)$ function $F$ the following equality

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The notation is due to the fact that $I'_{\Lambda,\Lambda'}$ is a formal linearization of the $I_{\Lambda,\Lambda'}$. 

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\[
\int F[\chi]d\mu_\Lambda[\chi] = \int F[\phi + \psi]d\mu_{\Lambda,\Lambda'}[\phi]d\mu_{\Lambda'}[\psi]
\]

holds.\(^7\)

Therefore, again by Jensen, we have the well-known inequality

\[
\int \left( I_{\Lambda,\Lambda'}(S)[\psi] \right)^2 d\mu_{\Lambda'}[\psi] \leq \int \int \left( S[\phi + \psi] \right)^2 d\mu_{\Lambda,\Lambda'}[\phi]d\mu_{\Lambda'}[\psi] = \int S^2[\chi]d\mu_\Lambda[\chi].
\]

(7)

Putting everything together, we get that

\[
\sqrt{\int \left( I_{\Lambda,\Lambda'}(S)[\psi] \right)^2 d\mu_{\Lambda'}[\psi]} \leq |b| + \sqrt{\int S^2[\chi]d\mu_\Lambda[\chi]}.
\]

It follows that \( I_{\Lambda,\Lambda'}(S + c)[\psi] = I_{\Lambda,\Lambda'}(S)[\psi] + c. \)

Using this, and the equivalences

\[
I_{\Lambda,\Lambda'}(S_n) \to I_{\Lambda,\Lambda'}(S_0) \iff I_{\Lambda,\Lambda'}(S_n) - b \to I_{\Lambda,\Lambda'}(S_0) - b
\]

\[
\iff I_{\Lambda,\Lambda'}(S_n - b) \to I_{\Lambda,\Lambda'}(S_0 - b),
\]

we can assume that \( b = 0 \) in what follows, i.e. that all our functions are nonnegative. Now, using (7) we have immediately that \( I'_{\Lambda,\Lambda'}(S_n) \to I'_{\Lambda,\Lambda'}(S_0) \) in \( L^2_{\mu_{\Lambda'}} \). Also, using (7), the fact that \( |e^{-x} - e^{-y}| \leq |x - y| \) for nonnegative \( x \) and \( y \), and Jensen yet again we have that

\[
\int \left( e^{-I_{\Lambda,\Lambda'}(S_n)[\psi]} - e^{-I_{\Lambda,\Lambda'}(S_0)[\psi]} \right)^2 d\mu_{\Lambda'}[\psi] \leq \]

\[
\int \int \left( e^{-S_n[\phi + \psi]} - e^{-S_0[\phi + \psi]} \right)^2 d\mu_{\Lambda,\Lambda'}[\phi]d\mu_{\Lambda'}[\psi] \leq \]

\[
\int \left( S_n[\chi] - S_0[\chi] \right)^2 d\mu_{\Lambda}[\chi],
\]

from which we conclude that \( e^{-I_{\Lambda,\Lambda'}(S_n)} \to e^{-I_{\Lambda,\Lambda'}(S_0)} \) in \( L^2(\mu_{\Lambda'}) \). Passing to subsequences, we get a subsequence \( \{S_{n_i}\}_{i=1}^\infty \) such that \( I_{\Lambda,\Lambda'}(S_{n_i}) \) and

\(^7\)This is of course the fundamental equation behind the renormalization group.
$I_{\Lambda,\Lambda'}(S_{n})$ converge almost everywhere to $I_{\Lambda,\Lambda'}(S_{0})$ and $I_{\Lambda,\Lambda'}(S_{0})$ respectively. Combining this with the fact that $I_{\Lambda,\Lambda'}(S_{n}) \leq I_{\Lambda,\Lambda'}(S_{n})$ and that $I_{\Lambda,\Lambda'}(S_{n}) \rightarrow I_{\Lambda,\Lambda'}(S_{0})$ in $L_{2}^{2}(\mu_{\Lambda'})$, we get that $I_{\Lambda,\Lambda'}(S_{n}) \rightarrow I_{\Lambda,\Lambda'}(S_{0})$ in $L_{2}^{2}(\mu_{\Lambda'})$. Thus, we get a subsequence of $\{I_{\Lambda,\Lambda'}(S_{n})\}_{n=1}^{\infty}$ which converges to $I_{\Lambda,\Lambda'}(S_{0})$. Since we can repeat this argument for any subsequence of the original sequence, it follows that $I_{\Lambda,\Lambda'}(S_{n}) \rightarrow I_{\Lambda,\Lambda'}(S_{0})$, and we have continuity.

It remains to show that a quasilocal $S$ is mapped to a quasilocal one. In view of the definition of $Q_{b}(\Lambda)$ and the above continuity result, we can assume that $S$ is bounded. Now let $F$ stand for a finite subset of $\mathbb{Z}^{d}$ such that $(n_{1}, \ldots, n_{d}) \in F \iff -(n_{1}, \ldots, n_{d}) \in F$. Denote by $P_{F}$ the projection of $V$ onto those components corresponding to $F$ (i.e. the map which sets all the $a_{n}’s$ to zero unless $n \in F$). For an $L^{1}(\mu_{\Lambda})$ function $g$, denote by $g_{F}$ the cylindrical approximation obtained from $g$ by integrating out the components “not in $F$”. More precisely,

$$g_{F}[\phi] = \int g[P_{F}\phi + (I - P_{F})\psi]d\mu_{\Lambda}[\psi],$$

where $I$ stands for the identity map. As is well-known, $g_{F} \rightarrow g$ in $L^{2}(\mu_{\Lambda})$ as $F \uparrow \mathbb{Z}^{d}$. Denote by $\mu_{\Lambda,F}$ the pushforward of $\mu_{\Lambda}$ by $P_{F}$. It follows that if $g^{(n)} \rightarrow g$ in $L^{2}(\mu_{\Lambda})$, then $g_{F}^{(n)} \rightarrow g_{F}$ in $L^{2}(\mu_{\Lambda,F})$. Needless to say, $g_{F}$ can be considered to be a function of only finitely many variables (those in the image of $P_{F}$).

Applying this to $S$, we have that $S_{F}$ is bounded (with the lower bound being $b$), that $S_{F} \rightarrow S$ in $L^{2}(\mu_{\Lambda}$), and, recalling that $S$ is quasilocal, that if

$$S = \sum_{n_{1}+\cdots+n_{k}=0,k=0,1,\ldots} G(n_{1}, \ldots, n_{k}) : a_{n_{1}} \cdots a_{n_{k}} \Lambda,$$

then

$$S_{F} = \sum_{n_{1}+\cdots+n_{k}=0,k=0,1,\ldots} G(n_{1}, \ldots, n_{k}) \left( : a_{n_{1}} \cdots a_{n_{k}} \Lambda \right),$$

with the latter sum converging in $L^{2}(\mu_{\Lambda,F})$. Note that if we consider $S_{F}$ to be a function of finitely many variables then, by virtue of it being in $L^{2}(\mu_{\Lambda,F})$, it should also have an expansion of the form

$$S_{F} = \sum_{n_{1},\ldots,n_{k} \in F,k=0,1,\ldots} \tilde{G}(n_{1}, \ldots, n_{k}) : a_{n_{1}} \cdots a_{n_{k}} \Lambda,F,$$

where $\cdot \cdot \cdot \Lambda,F$ stands for the Wick ordering with respect to the measure $\mu_{\Lambda,F}$. We claim that in this latter expansion, $\tilde{G}(n_{1}, \ldots, n_{k}) = 0$ if $n_{1}+\cdots+$
nk \neq 0$. For obvious reasons, we shall call $L^2(\mu_{\Lambda,F})$ functions which satisfy this property quasilocal as well. Since the space of quasilocal functions is clearly closed, we will be done if we show that $\left( : a_{n_1} \ldots a_{n_k} :_{\Lambda} \right)_F$ is quasilocal. Consider one term in the definition of $\left( : a_{n_1} \ldots a_{n_k} :_{\Lambda} \right)_F$. It is a product of a subcollection of $a_{n_1}, \ldots, a_{n_k}$’s multiplied by a collection of $(K_\Lambda)_{n,m}$’s pairing the remaining indices (see formula (9) in Appendix A). Since $(K_\Lambda)_{n,m}$ is proportional to $\delta_{n,-m}$, it follows that indices which appear in the subcollection of $a_n$’s are obtained from $n_1, \ldots, n_k$ by omitting pairs of opposite ones. Since $n_1 + \cdots + n_k = 0$, this equation is still true for the subcollection. In view of the above, we will be done if we show $(\phi : a_{m_1} \ldots a_{m_l} :_{\Lambda})_F$ is quasilocal if $m_1 + \cdots + m_l = 0$. We now use the fact that

$$\int a_{a_{a_b}}^A a_{a_b}^B d\mu_\Lambda = 0 \quad \text{unless} \quad A = B \quad \text{and} \quad a = -b.$$  

It thus follows that $(a_{m_1} \ldots a_{m_l})_F = 0$ unless the $a_n$’s that get integrated are present in pairs with equal and opposite indices. It follows that the sum of the indices of the $a_n$’s that are left over after integration is done, still is equal to zero. Putting it all together, we have that $\left( : a_{n_1} \ldots a_{n_k} :_{\Lambda} \right)_F$ is equal to a sum of terms proportional to $a_{m_1} \ldots a_{m_l}$ with $m_1 + \cdots + m_l = 0$. It is trivial to see that each such term is quasilocal and we have what we want. Note that this implies that $S_F$ is also quasilocal in the original sense, i.e. considered as a function on the full space. In view of the discussion above, we see that it is enough to show that $I_{\Lambda,\Lambda'}(S_F)$ is quasilocal with respect to $\mu_{\Lambda,F}$.

We have thus reduced the original problem to one defined on functions of finitely many variables. Let us show now that we can further assume that $S_F$ is smooth. To this end, let

$$O(\tau)S_F[\phi] = \int S_F[e^{-\tau} \phi + \sqrt{1 - e^{-2\tau}} \psi]d\mu_{\Lambda,F}[\psi].$$  

It is well-known that $O(\tau)S_F$ is smooth and that $O(\tau)S_F \to S_F$ in $L^2(\mu_{\Lambda,F})$ as $\tau \to 0^+$. Moreover, since $O(\tau)$ is diagonal in the basis of Wick products, we have that $O(\tau)S_F$ preserves quasilocality. We have thus reduced the problem to showing that a $C^\infty$, cylindrical quasilocal function maps to a quasilocal one under $I_{\Lambda,\Lambda'}$.

We now need to streamline our notation. First, since $F$ will be held fixed in the rest of this paper, it shall be omitted (e.g. we’ll just write $S$ instead of $S_F$, $d\mu_\Lambda$ instead of $d\mu_{\Lambda,F}$ and so on). Second, let $\Lambda'(t) = e^{-t}\Lambda$. We shall

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8This is of course the Ornstein-Uhlenbeck semigroup action, hence the notation.

9The exact form here is unimportant. We can take $\Lambda'$ to be any smooth function of $t$ which is equal to $\Lambda$ for $t = 0$ and which has a strictly negative derivative everywhere.
denote by $K_{n,m}(t)$ the propagator of the measure $\mu_{\Lambda,\Lambda'}(t)$, i.e.

$$K_{n,m}(t) = \delta_{n,-m} e^{-\frac{n^2}{\Lambda - \Lambda'}} e^{-\frac{n^2}{\Lambda(0)}}. \tag{10}$$

Also, $S(t)$ will stand for $I_{\Lambda,\Lambda'}(S)$. Additionally, we will denote the operator $I_{\Lambda'(t_1),\Lambda'(t_2)}$ (note that this is the linearized map) by $U(t_2, t_1)$. Finally, the closed subspace of $C^1$ functions (topologized with the usual $||\cdot||_{C^1}$ norm) whose derivative is uniformly continuous will be denoted by $BUC^1$.

We now have the following important

**Lemma.** $S(t)$ satisfies the following equation

$$S(t) = U(t, 0)S + \int_0^t U(t, \tau) \left( 2 \sum_{n,m} \dot{K}_{n,m}(\tau) \frac{\partial S(\tau)}{\partial a_m} \frac{\partial S(\tau)}{\partial a_n} \right) d\tau,$$

where this equation holds as an equation in the Banach space of continuous functions $t \rightarrow f(t)$ where $t \in [0, \infty)$ and $f(t) \in BUC^1$.

The formula above is of course the celebrated Polchinski equation [8] written in the “variation of constants” form.

**Proof of Lemma.** First, note that since $S$ is $C^\infty$, then $S(t)$ is $C^\infty$ for every $t$ and thus certainly $S(t) \in BUC^1$. Now, suppose $f$ is a smooth function. We claim that the map $t \rightarrow U(t, 0)f$ is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$ where $U(t, 0)f = f_t$ is considered as an element of $BUC^1$.

To see this, note first that if $\Delta t > 0$, we have that

$$||f_{t+\Delta t} - f_t||_{C^1} = ||U(t + \Delta t, t)f_t - f_t||_{C^1} \leq \int ||f_t(x + y) - f_t(x)||_{C^1} d\mu_{\Lambda'(t_1),\Lambda'(t_2)}(y).$$

Now, since the upper bound and the modulus of continuity of $f_t$ are non-increasing as functions of $t$, the latter expression goes to zero uniformly in $t$ as $\Delta t \to 0^+$, as can be shown by the usual elementary argument of breaking the integral into two parts, one with small $y$ where uniform continuity of $f_t$ and its derivative is used, and the rest, which goes to zero since $d\mu_{\Lambda'(t_1+\Delta t),\Lambda'(t)}$ converges to a delta function. Continuity of $t \rightarrow f_t$ follows at once.

To see that $t \rightarrow f_t$ is continuously differentiable on $(0, \infty)$, first note that as long as $t_0 > 0$, one can interchange the integral and the derivative with

\footnote{In fact, it is $C^1$ on $[0, \infty)$ but we are not going to need that.}
respect to $t$ in $\frac{df}{dt}\big|_{t=t_0}$. This is a basic fact which follows from dominated convergence. Now, note that

\[
\left| f_{t_0+\Delta t} - f_{t_0} - \frac{df}{dt}\big|_{t=t_0} \right|_{C^1} \leq \left| \frac{d\mu_{\Lambda,\Lambda'(t_0+\Delta t)} - d\mu_{\Lambda,\Lambda'(t_0)}}{\Delta t} \bigg|_{t=t_0} \right|
\]

where $c$ in the last equation is between $t_0$ and $t_0 + \Delta t$. It is trivial to see now that the last integral goes to zero as $\Delta t \to 0$. We thus have differentiability of the map $t \to f_t$. The fact that $t \to \frac{df}{dt}$ is continuous follows essentially in the same way as the continuity of $t \to f_t$.

Now, observe that $g \to -\ln(g)$ and $h \to e^{-h}$ are local diffeomorphisms which are inverses of each other from $BUC^1$ into itself, as long as $g$ is bounded away from 0. Putting everything together, we have that $t \to S(t)$ maps into $BUC^1$, is continuous on $[0, \infty)$, and is continuously differentiable on $(0, \infty)$.

The rest of the proof is a standard argument in the theory of evolution equations in Banach spaces \[9, 10\]. Consider the function $\tilde{S}(\tau) = U(t, \tau)S(\tau)$. By similar arguments to the one above, (when we showed that $t \to S(t)$ is continuous and continuously differentiable) we have that $\tilde{S}(\tau)$ is a continuous function into $BUC^1$ on $[0, \infty)$, and is continuously differentiable on $(0, \infty)$. Moreover, by a direct calculation, we have that

\[
\frac{d\tilde{S}}{d\tau} = U(t, \tau) \left( 2 \sum_{n,m} \hat{K}_{n,m}(\tau) \frac{\partial S(\tau)}{\partial a_m} \frac{\partial S(\tau)}{\partial a_n} \right).
\]

Integrating the above equation between $t_1$ and $t_2$, and then taking the limits $t_1 \to 0^+, t_2 \to t^-$, we get what we want.

The lemma above shows in effect that the function $t \to S(t)$ is a fixed point of the following variation of constants map

\[
\Phi(f)(t) = U(t, 0)S + \int_0^t U(t, \tau) \left( 2 \sum_{n,m} \hat{K}_{n,m}(\tau) \frac{\partial f(\tau)}{\partial a_m} \frac{\partial f(\tau)}{\partial a_n} \right).
\]

Let us make this more precise. Again, we follow the general ideas of the theory of the evolution equations in Banach spaces\[11\]. Consider the metric space $X$ of all continuous functions on $[0, \delta]$ valued in the closed ball of

\[11\] The reader is encouraged here to go over the introductory discussion in \[10\] for the abstract setting behind the argument below.
center 0 and radius $R$ in $\text{BUC}^1$. We shall take $R = 2 ||S||_{C^1}$ and will specify $\delta$ below. The metric on $X$ is given by

$$\rho(f_1, f_2) = \sup_{t \in [0, \delta]} ||f_1(t) - f_2(t)||_{C^1}.$$  

Note that $\Phi(v)(t)$ is indeed in $\text{BUC}^1$. This is due to the regularizing effect of $U(t, \tau)$ inside the integral. In fact, it is straightforward to show that the following inequality $||U(t, \tau) g||_{C^1} \lesssim \frac{||g||_{C^0}}{\sqrt{t-\tau}}$, holds.

Now, note that for any two $\text{BUC}^1$ functions $f_1$ and $f_2$ we trivially have the inequality

$$\left| \sum_{n,m} K_{n,m}(\tau) \left( \frac{\partial f_1(\tau)}{\partial a_m} \frac{\partial f_1(\tau)}{\partial a_n} - \frac{\partial f_2(\tau)}{\partial a_m} \frac{\partial f_2(\tau)}{\partial a_n} \right) \right|_{C^0} \lesssim ||f_1 - f_2||_{C^1},$$

where the implicit constant depends on $R$ in general. Using the inequalities above, it is easy to show that

$$\rho(\Phi(f_1), \Phi(f_2)) \lesssim \left( \int_0^\delta \frac{1}{\sqrt{t-\tau}} d\tau \right) \rho(f_1, f_2).$$

Since the square root is an integrable function, we can choose $\delta$ small enough such that the overall constant in front of $\rho(g_1, g_2)$ is less than $\frac{1}{2}$. Using the fact that $||U(t, 0) S||_{C^1} \leq ||S||_{C^1}$, we have that

$$\sup_{t \in [0, \delta]} ||\Phi(g)(t)||_{C^1} \leq \sup_{t \in [0, \delta]} ||\Phi(g)(t) - \Phi(0)||_{C^1} + \sup_{t \in [0, \delta]} ||\Phi(0)||_{C^1} \leq \frac{1}{2} \rho(g, 0) + \sup_{t \in [0, \delta]} ||U(t, 0) S||_{C^1} < R.$$  

Putting everything together, we have that $\Phi$ is a contraction which takes the space $X$ to itself. We thus have that $t \rightarrow S(t)$ is indeed the unique fixed point of $\Phi$. Moreover, by starting with any point in the space and repeatedly applying $\Phi$, we converge to this fixed point. We are going to take the function $t \rightarrow U(t, 0) S$ as our starting point.

Now, let us say that $t \rightarrow f(t)$ in $X$ is quasilocal if for every $t$, $f(t)$ is quasilocal with respect to the appropriate measure, i.e. with respect to the measure $\mu_{X(t)}$. Note that according to this definition, our starting point of the iteration $t \rightarrow U(t, 0) S$ is quasilocal. We now claim that if $t \rightarrow f(t)$ is quasilocal

\footnote{Recall that $U$ is basically a convolution with a Gaussian. Incidentally, this is where we would need the differentiability of $\alpha$ at 0 in case we decide to replace $e^{-x^2}$ with $\alpha(x)$ in the ultraviolet cutoff.}
then its image by $\Phi$ is quasilocal as well. In view of the above, this will immediately imply that $S(t)$ is quasilocal, for $t \in [0, \delta]$, and thus, we will have that $S(\delta)$ is quasilocal. Repeating the argument\textsuperscript{13}, we can extend this fact to all of $[0, \infty)$ and obtain that $S(t)$ is quasilocal for every $t$. This would conclude the proof of the theorem.

It thus remains to demonstrate the claim. Since $U(t_2, t_1)$ acts diagonally in the Wick expansion, and thus preserves quasilocality, we will be done if we show that

$$2 \sum_{n,m} \hat{K}_{n,m}(\tau) \frac{\partial f(\tau)}{\partial a_m} \frac{\partial f(\tau)}{\partial a_n},$$

is quasilocal if $f(\tau)$ is quasilocal. Therefore, consider the expression

$$\sum_{n,m} \hat{K}_{n,m}(\tau) \int \stackrel{\cdots}{a_{n_1} \ldots a_{n_k}} \Lambda'(\tau) \left( \frac{\partial f(\tau)}{\partial a_m} \frac{\partial f(\tau)}{\partial a_n} \right) d\mu_{\Lambda'(\tau)}, \quad (8)$$

for some indices $n_1, \ldots, n_k$ such that $n_1 + \cdots + n_k \neq 0$. We need to show that the expression above vanishes. Since $\tau$ will be fixed in what remains of the proof, we shall drop it, together with the index $\Lambda'(\tau)$, and will simply write $f$ for $f(\tau)$, $d\mu$ instead of $d\mu_{\Lambda'(\tau)}$, and so on.

Let us assume now that $g$ is smooth (and not just in $BUC^1$). It is easy to show then that its Wick expansion can be differentiated term by term with the result converging to the appropriate derivative.

We now use the fact \textsuperscript{6} that for a smooth function $g$, we have

$$\int \stackrel{\cdots}{a_{n_1} \ldots a_{n_k}} g d\mu \simeq \int \frac{\partial^k g}{\partial a_{n_1} \cdots \partial a_{n_k}} d\mu,$$

where $\simeq$ means equality up to an irrelevant constant. Therefore, we have that \textsuperscript{8} is equal to a finite sum of terms which are schematically of the form

$$\int (f)^{(A)} (f)^{(B)} d\mu,$$

where the bracketed exponents stand for derivatives. It thus follows that if $f_k$ denotes the the $k$-th partial sum of the Wick expansion of $f$ then, since $f_k \to f$ (and thus $(f_k)^{(A)} \to (f)^{(A)}$, $(f_k)^{(B)} \to (f)^{(B)}$ in $L^2(\mu)$), we have that

\textsuperscript{13}Note that $||e^{-S(t)}||_{C^1}$ doesn’t increase as a function of $t$, and thus $||S(t)||_{C^1}$ is bounded from above by some multiple (which is a function of $t$) of $||S||_{C^1}$. This implies that the estimates that went into showing that $\Phi$ is a contraction which preserves the space $X$, can be carried through on any compact subinterval of $[0, \infty)$. 

Putting it all together, we have that

\[ \sum_{n,m} \hat{K}_{n,m} \int \frac{a_{n_1} \cdots a_{n_k}}{a_{m} a_{n}} \left( \frac{\partial f_k}{\partial a_m} \frac{\partial f_k}{\partial a_n} \right) d\mu \rightarrow \sum_{n,m} \hat{K}_{n,m} \int \frac{a_{n_1} \cdots a_{n_k}}{a_{m} a_{n}} \left( \frac{\partial f}{\partial a_m} \frac{\partial f}{\partial a_n} \right) d\mu. \]

Rearranging now in the standard way [11, 12], and using the fact that \( K_{n,m} \) is proportional to \( \delta_{n,-m} \), we have that the expression above is proportional to \( \delta_{n_1+\cdots+n_k,0} = 0 \), since we've assumed that \( n_1 + \cdots + n_k \neq 0 \). The proof is thus complete for the case of a smooth \( f \).

Now, using the fact that \( O(\sigma)f \rightarrow f \) in \( L^4(\mu) \) as \( \sigma \rightarrow 0^+ \), we have that

\[ \sum_{n,m} \hat{K}_{n,m} \int \frac{a_{n_1} \cdots a_{n_k}}{a_{m} a_{n}} \left( \frac{\partial (O(\sigma)f)}{\partial a_m} \frac{\partial (O(\sigma)f)}{\partial a_n} \right) d\mu \rightarrow \sum_{n,m} \hat{K}_{n,m} \int \frac{a_{n_1} \cdots a_{n_k}}{a_{m} a_{n}} \left( \frac{\partial f}{\partial a_m} \frac{\partial f}{\partial a_n} \right) d\mu. \]

Since the Ornstein-Uhlenbeck semigroup preserves quasilocality, and since the limit of a zero sequence is zero, the proof is complete.

We finish by noting that the latter part of the proof above can be considered to be a theorem in viability theory. In effect, we show that a mild solution [9, 10] of the Polchinski’s equation

\[ \frac{\partial S}{\partial t} = 2 \sum_{n,m} \hat{K}_{n,m}(\tau) \frac{\partial^2 S}{\partial a_n \partial a_m} - \frac{\partial S(\tau)}{\partial a_m} \frac{\partial S(\tau)}{\partial a_n}, \]

which starts in \( Q_b \), remains there. It is easy to see that the evolution generated by the linear term in the PDE above preserves quasilocality (this is simply the map \( U(t_1, t_2) \) above). We do have the knowledge that the nonlinear term when acting on a quasilocal \( S \) would give a quasilocal expression. However, it doesn’t seem easy to control this term away from the quasilocal subset which is what is usually required in the viability literature [4, 5]. The proof above circumvents this by using a variation of constants formula and the smoothing effect of the \( U \). It is clear that the argument above does not depend on the precise form of the Polchinski’s equation, but rather on the smoothing effect of the \( U \), and thus generalizes to any setting where the same fact holds.
Appendix A

We gather here a few elementary facts about Wick products. This is standard material \[12, 13\]. The reason we’re giving it here (apart from the reader’s convenience) is that one needs to be a little careful in using the usual formulas in our setting. This is because, due to the way we have defined our measures, the $a_n$’s are not independent random variables (recall that $\overline{a_n} = a_{-n}$).

So, let $\mu$ be a Gaussian measure. Define the Wick-ordered exponential via

$$\exp(i(t_1 a_{n_1} + \cdots + t_k a_{n_k})) = \exp(i(t_1 a_{n_1} + \cdots + t_k a_{n_k}))^{-1},$$

where $K_{n_{\alpha},n_{\beta}} = \int a_{n_{\alpha}} a_{n_{\beta}} d\mu$. Now, the Wick monomial is defined by

$$a_{n_1} \cdots a_{n_k} := \frac{1}{i^k} \frac{\partial^k}{\partial t_1 \cdots \partial t_k} \left( : \exp(i(t_1 a_{n_1} + \cdots + t_k a_{n_k})) : \right)_{t_1,\ldots,t_k=0}.$$

Using these definitions one immediately gets that

$$\int : a_{n_1} \cdots a_{n_k} :: a_{m_1} \cdots a_{m_l} : d\mu_{\Lambda,L} = \begin{cases} 0 & \text{if } k \neq l \\ \sum_{\sigma} K_{n_{\sigma(1)} n_{\alpha(1)}} K_{n_{\sigma(2)} n_{\alpha(2)}} \cdots K_{n_{\sigma(k)} n_{\alpha(k)}} & \text{if } k = l \end{cases},$$

and that

$$a_{n_1} \cdots a_{n_k} := \sum_P \prod_{\{i,j\} \in P} \left( -K_{n_i,n_j} \right) \prod_{l \notin P} a_{n_l}, \quad (9)$$

where $P$ is a collection of pairs of indices from $\{1, \ldots, k\}$.

Appendix B

Recall that a local functional is a linear combination of expressions of the form \[3\]. Clearly, any such expression is of the form

$$\int_{\mathbb{T}^d} \mathcal{L} \left( \phi(x), \partial_{x_1} \phi(x), \ldots, \partial_{x_d}^M \phi(x) \right) dx,$$

where $\mathcal{L}(z_1, \ldots, z_N)$ is some polynomial in $N$ variables.\[14\] Let us call this polynomial the *integrand* of the local functional. In this appendix, we prove the following

\[14\] $N = \sum_{m=0}^M (m+d-1)$. 

Proposition. If $S$ is a local functional whose integrand is bounded from below by $b$, then $S \in Q_b(\Lambda)$.

Note that this would imply that e.g. $\int_T (\partial \phi)^2 + P(\phi)$ is in $Q_b(\Lambda)$ as long as $P(z)$ is a polynomial in $z$ bounded from below by $b$.

Proof. Clearly, it is enough to prove this for the case $b = 0$, which is what we are going to assume below. Let $\varepsilon > 0$. Let $\beta$ be a constant (which we'll assume is greater than 1) such that $|\partial^{m}_{x_{i_1},...,x_{i_m}} \phi(x)| \leq \beta ||| a_n |||$ for all $i_1, \ldots, i_m \in \{1, \ldots, d\}$ and $m \leq M$. Using this estimate, we have that $| \mathcal{L}(\phi(x), \partial_{x_1} \phi(x), \ldots, \partial^{M}_{x_{d},x_{d},...,x_{d}} \phi(x)) | \leq C_1 + C_2 ||| a_n |||^A$ for some constants $A, C_1$ and $C_2$. Choose $R$ such that

$$\sqrt{\int_{B^c(0,R)} \left( C_1 + C_2 ||| a_n |||^A \right)^2 d\mu_{\Lambda}} < \frac{\varepsilon}{4},$$

and

$$(C_1 + C_2(\beta(R + 1))^A) \mu_{\Lambda}(B^c(0,R)) < \frac{\varepsilon}{4},$$

where $B^c(0,R)$ is the complement of the ball of radius $R$ (with respect to the norm $||| \cdot |||$). The existence of such an $R$ follows immediately from Fernique’s theorem. Now, let $T(z_1, \ldots, z_N)$ be a trigonometric polynomial such that

- $T$ is nonnegative.
- $|T| \leq C_1 + C_2(\beta(R + 1))^A$.
- $\sup_{(z_1, \ldots, z_N) \in [-\beta R, \beta R]^N} |T(z_1, \ldots, z_N) - \mathcal{L}(z_1, \ldots, z_n)| < \frac{\varepsilon}{2}.$

To see that such a $T$ exists, first restrict $\mathcal{L}(\phi(x), \partial_{x_1} \phi(x), \ldots, \partial^{M}_{x_{d},x_{d},...,x_{d}} \phi(x))$ to $[-\beta(R + 1), \beta(R + 1)]^N$, and then extend it periodically. If we now take the Cesàro sum of a sufficiently far truncation of the Fourier series of this periodization, we get what we want directly from the properties of the Fejér kernel. Putting everything together, we have that
We thus have that $S$ can be approximated in $L^2(\mu_\Lambda)$ by elements of the form 
\[ \int \L(\phi(x), \ldots, \partial^M_{x_d, x_d, \ldots, x_d} \phi(x)) - T(\phi(x), \ldots, \partial^M_{x_d, x_d, \ldots, x_d} \phi(x)) \] 
\[ \frac{d\mu_\Lambda}{\sqrt{\mu_\Lambda(B(0, R))}} \cdot \ldots \cdot \sqrt{\mu_\Lambda(B(0, R))} \] 
\[ \sqrt{\int_{B^c(0, R)} \left( C_1 + C_2 ||a_n|| |A| \right)^2 d\mu_\Lambda + (C_1 + C_2(\beta(R + 1))A)\mu_\Lambda(B^c(0, R)) < \frac{\epsilon}{2} \sqrt{\mu_\Lambda(B(0, r))} + \frac{\epsilon}{2} < \epsilon. \]

\[ \text{We thus have that } S \text{ can be approximated in } L^2(\mu_\Lambda) \text{ by elements of the form } \]
\[ \int T(\phi(x), \partial_{x_1} \phi(x), \ldots, \partial^M_{x_d, x_d, \ldots, x_d} \phi(x)) dx. \]
Since clearly each such element is bounded and nonnegative, we will have what we want provided we show that each such element is in $Q$. This, in turn, would follow if we show that each element of the form 
\[ \int_{\mathbb{T}^d} e^{i\alpha_0 \phi(x)} e^{i\alpha_1 \partial_{x_1} \phi(x)} \ldots e^{i\alpha_d \partial_{x_d} \phi(x)} dx \] 
is in $Q$, where $\alpha_0, \alpha_1, \ldots, \alpha_d$ are constants. To this end, note that 
\[ \int_{\mathbb{T}^d} \prod_{n=0}^N \left( \prod_{i_1, \ldots, i_n=1, \ldots, d} \left( \sum_{l=0}^L \frac{i^{\alpha_{i_1, \ldots, i_n} \partial^l_{x_{i_1}, \ldots, x_{i_n}} \phi(x)}{l!} \right) \right) dx \]
is in $Q$ for every $L$, converges pointwise to (10), and is bounded (uniformly in $L$) from above by 
\[ e \left( |\alpha_0| + |\alpha_1| + \ldots + |\alpha_d| \right) ||a_n||. \]
By another application of Fernique’s theorem, we have that they converge to (10) in $L^2(\mu_\Lambda)$, and the proof is complete.

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\footnote{We are slightly abusing the terminology here, as quasilocality was defined only for real-valued functions. However, either we trivially extend the definition to complex-valued ones, or we replace the product of exponentials by a product of sines and cosines. The estimates below would still go through.}
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