An Algebraic Weak Factorisation System on 01-Substitution Sets: A Constructive Proof

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September 8, 2014

Abstract
We will construct an algebraic weak factorisation system on 01-substitution sets such that the \( R \)-algebras are precisely the Kan fibrations together with a choice of Kan filling operation. The proof is based on Garner’s small object argument for algebraic weak factorization systems. In order to ensure the proof is valid constructively, rather than applying the general small object argument, we give a direct proof based on the same ideas.

1 Introduction

1.1 Aims
We will construct an algebraic weak factorisation system on \( 01\text{Sub} \) such that the \( R \)-algebras are precisely the Kan fibrations together with a choice of Kan filling operation. It will be algebraically free in the sense of [2]. However rather than applying the result in [2], we will give a direct construction based on the same ideas. The construction is also similar to Kan completion, as referred to in [1]. This approach has three main advantages.

Firstly, this allows us to ensure that the proof holds in a constructive setting (such as the set theory CZF), whereas it is not clear whether the proof in [2] is valid constructively.

Secondly, it allows us to see explicitly the objects involved in the construction, which may be useful if one wanted to use the ideas here in computer implementations.

Thirdly, we will see that the proof uses only countable colimits, not requiring the full cocompleteness of \( 01\text{Sub} \). Although \( 01\text{Sub} \) is cocomplete, this property should make it easier to apply the proof in other contexts. For example, if one were to define “01-substitution assemblies” by analogy with Stekelenburg’s recent work on simplicial assemblies in [7] this may be useful.

We will also suggest how to construct stable path objects. This should yield a constructive model of homotopy type theory including the computational rule for identity, which is absent in [1].

1.2 Cubical Sets and 01-Substitution Sets
Cubical sets were developed by Bezem, Coquand and Huber in [1] as a constructive model of homotopy type theory, based on the simplicial set model due to Voevodsky.
Pitts showed in [5], following earlier work by Staton, that the category of cubical sets is equivalent to a category based on nominal sets, called $01$-substitution sets. In this paper we will work over this category, $01\text{Sub}$ of $01$-substitution sets, using the notation and definitions of open box and Kan fibration that appear in [5]. For a good introduction to nominal sets, on which $01$-substitution sets are based, see [4].

## 1.2.1 A Note on Nominal Sets in a Constructive Setting

As in [1] we require the assumption that the set $A$ of names has decidable equality.

In [4] heavy use is made of the existence of least finite support. Constructively this can’t be guaranteed to exist; the proof of theorem 6.4 provides an example where the freshness relation is not decidable and hence there cannot be a least support that is also a finite set. In fact one can show that in general one cannot even prove constructively that any least support exists, finite or otherwise. In practice, however, most of the uses of least finite support in [4] can be viewed as a “notational device” to make definitions and statements of theorems more concise, and these can be easily rephrased to work in a constructive setting.

## 1.3 Algebraic Weak Factorisation Systems

Weak factorisation systems are widely used in homotopical algebra. Algebraic weak factorisation systems (originally called natural weak factorisation systems) are a variation developed by Grandis and Tholen in [3]. Garner showed in [2] that a form of the small object argument can be used to construct awfs’s from a diagram of left maps. However, the proof uses transfinite arguments that may be problematic constructively. Riehl used awfs’s in [6] as the main ingredient in the theory of algebraic model structures; the same paper contains a comprehensive introduction to awfs’s.

## 2 Construction of Functorial Factorisation

### 2.1 Construction of Factorisation

We first define the functor $K : 01\text{Sub}^2 \to 01\text{Sub}$ that will provide the objects of the functorial factorisation $01\text{Sub}^2 \to 01\text{Sub}$.

Fix $f : X \to Y$. We will define $Kf$ in $\omega$ stages. For each $n \in \omega$, we will define inductively $K_n f \in 01\text{Sub}$. We will simultaneously define $\rho_n : K_n f \to Y$ and $\lambda_n : K_n f \to K_{n+1} f$. We will also ensure that $\rho_{n+1} \circ \lambda_n = \rho_n$.

If we have already defined $K_m f$ for $m < n$ then we define $\operatorname{colim}_{m < n} K_m f$ to be the colimit over $K_m f$ for $m < n$ together with the maps $\lambda_m$. We now define $K_n f$, assuming that $K_m f$ has already been defined for $m < n$. We first define nominal sets $K^+_{n} f$, $K^0_{n} f$, $K^A_{n} f$ and $K^−_{n} f$.

1. Define $K^+_{n} f$ to be pairs $(u, y)$ where $u$ is a $1$-open box in $\operatorname{colim}_{m < n} K_m f$ over $y$. The action of permutations on $K^+_{n} f$ is defined componentwise.

2. Define $K^0_{n} f$ to be the subset of $[A]K^+_{n} f$ consisting of $(a)(u, y)$ where $u$ is a $1$-open $A, a$-box in $\operatorname{colim}_{m < n} K_m f$ over $y$. 

2
3. Define $K^1_n f$ analogously to $K^1_n f$ but for 0-open boxes.

4. Define $K^2_n f$ analogously to $K^1_n f$ but for 0-open boxes.

We now define $K_n f$ to be the disjoint union $X \amalg K^1_n f \amalg K^2_n f \amalg K^3_n f \amalg K^4_n f$. Note that there is a natural injection of $K^1_{n-1} f$ into $K^1_n f$, and similarly for the other components of the disjoint union. Hence we can define $\lambda_{n-1}$ componentwise. $\rho_n$ is also defined componentwise in the obvious way.

The action of permutations on $K_n f$ is defined componentwise, but we still need to define the action of substitutions. If $x$ is in the “copy” of $X$ in $K_n f$, then define $x(a := i)$ to be the same as in $X$ itself. If $(u, y)$ is an element of $K^1_n f$, with $u$ an $A, a$ box then define $(u, y)(a' := i)$ by cases. In the below, we write $\lambda_{m<n}$ to mean the natural injection of $\text{colim}_{m<n} K_m f$ into $K_n f$.

$$
(u, y)(a' := i) := \begin{cases} 
\lambda_{m<n}(u(a', i)) & \text{if } (a', i) \in A \times 2 \setminus (a, 1) \\
(a)(u, y) \in K^1_n f & \text{if } a' = a \text{ and } i = 1 \\
(u(a' := i), y(a' := i)) \in K^1_n f & \text{otherwise}
\end{cases}
$$

(1)

Substitution for $K^1_n f$ is defined similarly.

If $(a)(u, y)$ is an element of $K^1_n f$, with $u$ a 1-open $A, a$-box, then we define $(a)(u, y)(a' := i)$ as follows. Note that we may assume without loss of generality that $a' \neq a$.

$$
(a)(u, y)(a' := i) := \begin{cases} 
\lambda_{m<n}(u(a', i)(a := 1)) & \text{if } a' \in A \\
(a)(u(a' := i), y(a' := i)) & \text{otherwise}
\end{cases}
$$

(2)

To check that this does give a well defined function, we note that in both cases $a$ is fresh for $(a)(u, y)(a' := i)$. Substitution for $K^2_n f$ is defined similarly.

To show that $K_n f$ is a 01-substitution set, we need to check the following axioms (see [5]).

\[ \pi(x(a := i)) = \pi x(\pi a := i) \]  

(3)  

\[ a # x(a := i) \]  

(4)  

\[ a # x \Rightarrow x(a := i) = x \]  

(5)  

\[ a # a' \Rightarrow x(a := i)(a' := i') = x(a' := i')(a := i) \]  

(6)

These can be checked by induction on $n$, splitting into cases depending on which disjoint component of $K_n f$ $x$ lies in. For (3) and (4) it is necessary in places to use the freshness and adjacency conditions respectively in the definition of open box.

We now define $K f$ to be the colimit over all $K_n f$ (together with the injections $\lambda_n$). We also have an obvious morphism $\lambda_f : X \to K f$ given by inclusion $X$ into $K_0 f$, and a morphism $\rho_f : K f \to Y$ using the $\rho_n$.

### 2.2 Rank is WellDefined

Note that for every $n$ there is a canonical map $\lambda_{n<\omega} : \text{colim}_{m<n} K_m f \to K f$. In the following lemma we show that for every $x \in K f$ there is a least $n$ such that $x$ appears in the image of such a map. Note that constructively, we are forced to find this $n$ explicitly, rather than appealing to the well ordering of $\omega$. 


Lemma 2.1. For every $x \in Kf$, there is a least $n$ such that there is $x' \in Knf$ such that $\lambda_n(x') = x$. Furthermore, $x'$ is uniquely determined.

Proof. We will show by induction that for every element $x$ of $\coprod_{m<\omega} Knf$, the statement holds for the equivalence class $[x]$.

Suppose that we have proved the statement for $[y]$ for all $y \in \coprod_{m'<m} Knf$. Now given $x \in Knf$, note that $x$ must belong to one of the five components in the disjoint union forming $Knf$. If $x \in X$, then note that we can take $n = 0$ and $x'$ to be $x$.

If $x$ is of the form $(u, y)$ where $u$ is a 1-open $A$, $a$-box, then note that for every $(a', i) \in A \times 2 \setminus (a, 1)$, we have that $u(a', i) \in \colim_{m'<m} K_{m'}f$. Given $(a', i)$ in $A \times 2 \setminus (a, 1)$, let $w$ be a representative of the equivalence class $u(a', i)$. Note that $w$ is an element of $K_{m'}f$ for some $m' < m$. Hence by induction there is $n$ least such that there is $z$ in $K_nf$ with $z$ equivalent to $w$. Note that $z$ and $n$ are independent of the choice of equivalence class representative $w$ and so they depend only on $(a', i)$. We write them now as $za',i$ and $na',i$. Since $\{ na', i \mid (a', i) \in A \times 2 \setminus (a, 1) \}$ is a finite subset of $\omega$ it must have a greatest element, $N$. Then we can take $n$ in the statement of the lemma to be $N + 1$. We take $x'$ to be given by $(z, y)$ where $z$ is the open box with $z(a', i) := za',i$.

If $x$ belongs to one of the remaining 3 components, then the proof is similar to that of $Knf$.

Definition 2.2. Given $x \in Kf$, we refer to the $n$ in lemma 2.1 as the rank of $x$.

2.3 Functoriality of $K$

In section 2.1 we defined $K$ on the objects of $01\text{Sub}^2$. We now complete the construction of the functor by defining the action of $K$ on the morphisms of $01\text{Sub}^2$.

Given $f : X \to Y$ and $g : U \to V$ objects of $01\text{Sub}^2$, recall that a morphism from $f$ to $g$ is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{h} & U \\
\downarrow f & & \downarrow g \\
Y & \xrightarrow{k} & V
\end{array}
\]  

(7)

We also need to check that the following diagram commutes (in order to ensure that the overall factorisation $01\text{Sub}^2 \to 01\text{Sub}^3$ is a functor).

\[
\begin{array}{ccc}
X & \xrightarrow{h} & U \\
\downarrow \lambda_f & & \downarrow \lambda_g \\
Kf & \xrightarrow{K(h, k)} & Kg \\
\downarrow \rho_f & & \downarrow \rho_g \\
Y & \xrightarrow{k} & V
\end{array}
\]  

(8)
We will simultaneously check this while defining $K(h, k)$. One can also check at the same time that $K(h, k)$ is invariant and preserves substitution, i.e., that it actually is a morphism in $01\text{Sub}$.

We define $K(h, k) : Kf \to Kg$ by induction on rank. If $x \in Kf$ has rank 0, then it is an element of $X$. We define $K(h, k)(x)$ to be $h(x)$. Note that this precisely ensures that the upper square of (8) commutes.

If $x$ is of the form $(u, y)$ where $u$ is a 1-open $A, a$-box, then note that for each $(a', i) \in A \times 2 \setminus (a, 1)$ we have that $u(a', i)$ is of strictly lower rank than $x$, and so we may assume that $K(h, k)(u(a', i))$ has already been defined. Note that we have a 1-open $A, a$-box given by $K(h, k) \circ u$. Furthermore, note that by applying the lower square of (8) to each $u(a', i)$ we have that $K(h, k) \circ u$ is an open box over $k(y)$. Hence $(K(h, k) \circ u, k(y))$ is an element of $Kg$, and so we can take $K(h, k)(x)$ to be $(K(h, k) \circ u, k(y))$. Note that we can now easily see that the lower square of (8) holds “locally” at $x$.

Given $\langle a \rangle(u, y) \in K^+f$, we define $K(h, k)$ to be $\langle a \rangle(K(h, k) \circ u, k(y))$.

We can similarly define $K(h, k)$ on elements of $K^0f$ and $K^-f$.

Finally note that one can easily show by induction that $K$ preserves composition and identities.

3 Monad and Comonad Structure

3.1 $L$ is a Comonad

We define the comonad $L$ as follows. We need to define a comultiplication morphism $\Sigma : L \to L^2$. Given a morphism $f : X \to Y$, recall that $Lf = \lambda_f : X \to Kf$. This means that $L^2f$ is the morphism $\lambda_{\lambda_f} : X \to K\lambda_f$. We will define $\sigma_f$ such that $\Sigma f$ will be the commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{\lambda_f} & X \\
\vline & \downarrow \lambda_{\lambda_f} & \\
Kf & \xrightarrow{\sigma_f} & K\lambda_f
\end{array}
$$

(9)

We inductively define $\sigma_f : Kf \to K\lambda_f$, by taking it to be the identity on $X$ (note that this implies that (9) is commutative). Then given an element, $(u, y)$ of $K^0f$, send it to $(\sigma_f \circ u, (u, y))$ in $K\lambda$. Given an element $\langle a \rangle(u, y)$ of $K^+f$, send it to $\langle a \rangle(\sigma_f \circ u, (u, y))$. (And similarly for the remaining components of $K^nf$). Checking that $L$ is indeed a comonad amounts to checking the following
three diagrams commute:

\[
\begin{array}{c}
Kf \\ \downarrow \sigma_f \\
\rho_f \\
K\lambda_f
\end{array}
\]

(10)

\[
\begin{array}{c}
Kf \\ \downarrow \sigma_f \\
K\lambda_f
\end{array}
\]

(11)

\[
\begin{array}{c}
K\lambda_{\lambda_f} \\ \downarrow \sigma_{\lambda_f} \\
K\lambda_f
\end{array}
\]

(12)

All three can be verified by induction on rank.

3.2 R is a Monad

To define the monad R, we need to define a multiplication map \( \Pi : R^2 \rightarrow R \). Given \( f : X \rightarrow Y \), \( Rf \) is of the form \( \rho_f : Kf \rightarrow Y \), and so \( R^2f \) is of the form \( \rho_{\rho_f} : K\rho_f \rightarrow Y \). We will define \( \pi_f \) so that \( \Pi_f \) is the commutative diagram

\[
\begin{array}{c}
K\rho_f \\ \downarrow \rho_{\rho_f} \\
Y
\end{array}
\]

\[
\begin{array}{c}
Kf \\ \downarrow \rho_f \\
Y
\end{array}
\]

(13)

We define \( \pi_f \) inductively. If \( x \in K\rho \) is already an element of \( Kf \), then we can take \( \pi_fx \) to be \( x \). If \( x \in K\rho_f \) is of the form \( (u, y) \), then note that \( \pi_f \circ u \) is an open box over \( y \), and so we can take \( \pi_fx \) to be \( (\pi_f \circ u, y) \). To check that this is a monad, we need to verify the commutativity of the following diagrams.
As before, each diagram can be checked by induction on rank.

### 4 Kan Fibrations

Every algebraic weak factorisation system gives rise to a weak factorisation system \((\bar{\mathcal{L}}, \bar{\mathcal{R}})\) where \(\bar{\mathcal{L}}\) and \(\bar{\mathcal{R}}\) are the retract closures of \(L\)-maps and \(R\)-maps respectively (the fact that this is indeed a weak factorisation system follows from standard results in homotopical algebra). We will check that \(\bar{\mathcal{R}} = \mathcal{R}\) and show that this class is precisely the class of Kan fibrations in the sense of \([5]\), and in fact \(\mathcal{R}\) algebra structures correspond exactly to Kan filling operations.

**Lemma 4.1.** Suppose that \(f: X \to Y\) is a map in \(\mathbf{01Sub}\) and that we are given an algebra structure for \(f\) over \(\mathcal{R}\) regarded as a pointed endofunctor (that is, we only require the first of the identities for algebras). Then we can produce Kan filling operators \(f \uparrow\) and \(f \downarrow\) for \(f\) in a canonical way.

**Proof.** A pointed endofunctor algebra structure for \(f\) is precisely a map \(g\) satisfying the following commutative diagrams.

\[
\begin{array}{c}
Kf \\ \rho_f \downarrow \\
X \\
\end{array} \quad \begin{array}{c}
\lambda_f \\ \downarrow \\
Kf \\
\end{array} \quad g
\]

\[
\begin{array}{c}
Kf \\ \rho_f \downarrow \\
X \\
\end{array} \quad \begin{array}{c}
\lambda_f \\ \downarrow \\
Kf \\
\end{array} \quad g
\]

Let \((u, y)\) be an open box in \(X\) over \(y\) in \(Y\). Note that we can regard \((u, y)\) as an element of \(Kf\). Define \(f \uparrow (u, y)\) to be \(g((u, y))\). One can easily check that this is a filler for \((u, y)\) and that \(f \uparrow\) respects the uniformity conditions by...
applying the above two diagrams and the fact that \( g \) preserves permutations and substitutions (since it is a morphism in 01Sub). Similarly for \( f \downarrow \).

**Lemma 4.2.** Suppose that we are given \( f: X \to Y \) and uniform Kan filling operations \( f \uparrow \) and \( f \downarrow \) for \( f \). We can produce an \( R \)-algebra structure on \( f \) in a canonical way.

**Proof.** Define \( g: Kf \to X \) by induction. For \( k \) a rank 0 element of \( Kf \), take \( g(x) \) to be \( x \). For \((u, y)\) in \( K^\uparrow f \), note that we may assume by induction that \( g \circ u \) is defined and that it is an open box over \( y \) in \( X \). Define \( g((u, y)) \) to be \( f \uparrow (g \circ u, y) \). Note that this is a morphism in 01Sub (ie preserves permutations and substitutions) by applying the uniformity conditions. We need to check that the following diagrams commute.

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda_f} & Kf \\
\downarrow{g} & & \downarrow{g} \\
X & & X
\end{array}
\quad (19)
\]

\[
\begin{array}{ccc}
Kf & \xrightarrow{g} & X \\
\rho_f & & \downarrow{f} \\
Y & = & Y
\end{array}
\quad (20)
\]

\[
\begin{array}{ccc}
K\rho_f & \xrightarrow{\pi_f} & Kf \\
\downarrow{K(g, 1)} & & \downarrow{g} \\
Kf & \xrightarrow{g} & X
\end{array}
\quad (21)
\]

The first follows easily from the definition of \( g \), and the remaining two can be shown by induction on rank.

**Theorem 4.3.** For \( f: X \to Y \), \( R \)-algebra structures on \( f \) are in precise correspondence to pairs \( f \uparrow \) and \( f \downarrow \) of Kan filling operators.

**Proof.** Observe that the construction in the proof of lemma 4.2 is a left inverse to the construction in the proof of lemma 4.1, and is also a right inverse when restricted to (monad) \( R \)-algebra structures.

**Theorem 4.4.** Let \( f: X \to Y \) be a morphism in 01Sub. Then the following are equivalent.

1. \( f \) is an element of \( \bar{R} \)
2. \( f \) admits a pointed endofunctor \( R \)-algebra structure
3. \( f \) is a Kan fibration
4. \( f \) admits a (monad) \( R \)-algebra structure

**Proof.** For (1) \( \Rightarrow \) (2), this follows easily from the fact that \((\bar{L}, \bar{R})\) is a wfs and in particular that \( L \triangleleft R \). (See lemma 2.8 of [6])

For (2) \( \Rightarrow \) (3), apply lemma 4.1

For (3) \( \Rightarrow \) (4), apply lemma 4.2

Finally note that (4) \( \Rightarrow \) (1) is trivial.
Remark 4.5. The implication (2) \(\Rightarrow\) (4) in theorem 4.4 is to be expected, since it holds in general for awfs’s generated with Garner’s version of the small object argument (see lemma 2.30 of [6]).

5 The Generating Left Maps

In [2], Garner shows how to define an awfs from a generating diagram of L-maps. The resulting awfs is algebraically free over the diagram. Since the construction here is constructive variant of Garner’s argument the same should be true here. In this section we define a diagram based on box inclusions and check that indeed the awfs is algebraically free over the diagram.

5.1 Definition of the Diagram

We now define the diagram of generating L-maps. This is based on open box inclusions for cubical sets translated into 01-substitution sets.

Let \(J\) be the category defined as follows. The objects of \(J\) are 4-tuples \((i, A, a, B)\), where \(i \in 2\), \(A\) and \(B\) are finite subsets of \(A\) and \(a \in A\). A morphism \((i, A, a, B) \rightarrow (i', A', a', B')\) is a pair \((f, g)\) such that \(i = i'\), \(f : A' \rightarrow A\) is a bijection with \(f(a') = a\) and \(g : B' \rightarrow B\) is a morphism in the category of names and substitutions, i.e a function \(B' \rightarrow B \amalg 2\) which is “injective where defined.” (Note that \(f\) and \(g\) are in the opposite direction to \((f, g)\)).

Definition 5.1. Given a function \(f : A \rightarrow A \amalg 2\), say \(f\) is injective where defined if the function \(f|_{f^{-1}(A)}\) is an injection.

Definition 5.2. Given \(A\) a finite subset of \(A\), define the standard \(A\)-cube, \(\square_A\), to be the set of functions \(A \rightarrow A \amalg 2\) that are injective where defined. We make \(\square_A\) a 01-substitution set by defining

\[
(\pi.f)(a) := \begin{cases} 
\pi(f(a)) & f(a) \in A \\
 f(a) & f(a) \in 2
\end{cases}
\]

and

\[
(f(a' := i))(a) := \begin{cases} 
f(a) & f(a) \in 2 \\
 f(a) & f(a) \in A \text{ and } f(a) \neq a' \\
i & f(a) = a'
\end{cases}
\]

Note that this gives a functor from the opposite of the category of names and substitutions to 01Sub. Given \(f : B \rightarrow A\), define \(\square_f : \square_A \rightarrow \square_B\) by sending \(g \in \square_A\) to \(g \circ f\). (This corresponds to the functor to cubical sets given by the Yoneda lemma)

Definition 5.3. Given \(A\) a finite subset of \(A\) and \(a \in A\), define the standard 1-open \(A, a\)-box, \(\sqcup_{A,a}\), in 01Sub to be the subset of \(\square_A\) consisting of \(f\) such that for some \((a', i) \in A \times 2 \setminus \{(a, 1)\}\), \(f(a') = i\). Note that this is a subobject of \(\square_A\) by inclusion, in 01Sub.

The standard 0-open \(A, a\)-box, \(\sqcap_{A,a}\), is defined analogously.

Note that the morphism \(\square_f : \square_A \rightarrow \square_B\) given above restricts to a morphism \(\sqcup_{A,a} \rightarrow \sqcup_{B,b}\) if \(f(b) = a\).
We define a functor \( J : \mathcal{J} \to 01\text{Sub}^2 \) as follows. On an object \((1, A, a, B)\), define \( J(1, A, a, B) \) to be the map from \( \sqcup_{A,a} \ast \Box_B \) to \( \Box_A \ast \Box_B \) defined by taking the product of the inclusion \( \sqcup_{A,a} \hookrightarrow \Box_A \) with the identity on \( \Box_B \).

Given a morphism \((f, g) : (1, A, a, B) \to (1', A', a', B')\), define \( J(f, g) \) “componentwise.”

### 5.2 Algebraic Freeness

We will show that the awfs is algebraically free on \( J \). This means we need to find \( \eta : J \to L^-\text{Map} \) over \( 01\text{Sub}^2 \) such that the following is an isomorphism of categories (see [2]).

\[
\text{R-Map} \xrightarrow{\text{lift}} \text{L-Map} \cong \mathcal{J}^h
\]  

(24)

We require that \( \eta(1, A, a, B) \) is an \( L^-\)-coalgebra structure on \( J(1, A, a, B) \). We now fix \((1, A, a, B)\) and refer to \( J(1, A, a, B) \) as \( \iota \). To find \( \eta(1, A, a, B) \) we need to find \( h : \Box_A \ast \Box_B \to K\iota \) satisfying the following commutative diagrams.

\[
\begin{align*}
\sqcup_{A,a} \ast \Box_B & \xrightarrow{h} \Box_A \ast \Box_B \\
\Box_A \ast \Box_B & \xrightarrow{\lambda_\iota} K\iota \\
\Box_A \ast \Box_B & \xrightarrow{\rho_\iota} \Box_A \ast \Box_B \\
\iota & \xrightarrow{\sigma_\iota} K\lambda_\iota
\end{align*}
\]  

(25)

(26)

(27)

If \((f, g) \in \Box_A \ast \Box_B \) is already an element of \( \sqcup_{A,a} \ast \Box_B \), then define \( h(f, g) \) to be \((f, g)\). Note that this precisely ensures that (25) commutes.

If \( f \) is defined everywhere on \( A \), then let \( u \) be the 1-open \( f(A) \ast f(a) \)-box defined by

\[
u(f(a'), i) := f(f(a') := i)
\]  

(28)

We then define

\[
h(f, g) := (u, (f, g))
\]  

(29)

If \( f \) is defined everywhere on \( A \setminus a \) and \( f(a) = 1 \), then let \( b \) be a fresh variable, define \( f' \) as follows

\[
f'(a') :=
\begin{cases}
  f(a') & a' \in A \setminus a \\
  b & a' = a
\end{cases}
\]  

(30)

Then define \( u' \) as for \( u \) in (28), but with \( f' \) in place of \( f \), and we can now define \( h(f, g) \in K^{+\iota} \) to be

\[
h(f, g) := \langle b \rangle(u', (f', g))
\]  

(31)
This completes the definition of $\eta(1, A, a, B) = h$. We now show that (24) is an isomorphism by exhibiting an inverse.

The elements of $\mathcal{F}^\circ$ are of the form $(g, \phi)$ where $g : X \to Y$ is a morphism in $01\text{Sub}$ and $\phi$ is lifting data for $g$ against $J$. We will use $\phi$ to define Kan filling operators on $g$ and apply lemma 4.1 to get an $R$-algebra structure on $g$.

Let $(u, y)$ be an open box in $X$ over $y$ in $Y$. Let $u$ be a 1-open $A, a$-box, and let $C$ be a finite support for $u$. Let $B := C \setminus A$ and note that $B$ is a finite set. For any $(a', i) \in A \times 2 \setminus (1, a)$, we can define $(a', i) \in \sqcup_{a, A}$ by

$$
(a', i)(a'') := \begin{cases} 
i & a'' = a' \\
ii & a'' \neq a'
\end{cases}
$$

Now note that there is a unique morphism $\tilde{u} : \sqcup_{a, A} \to X$ such that for each $(a', i) \in A \times 2 \setminus (1, a)$, $\tilde{u}((a', i)) = u(a', i)$. Also note that there is a unique morphism $\tilde{g} : \Box_A \ast \Box_B \to Y$ such that $\tilde{g}(1_A, 1_B) = y$. (Both of these results can be viewed as translating the Yoneda lemma from cubical sets to 01-substitution sets). These maps together make a commutative square:

\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Y) at (0,-2) {$Y$};
  \node (BoxABoxB) at (-1,0) {$\Box_A \ast \Box_B$};
  \node (BoxABoxBBoxABoxB) at (1,0) {$\sqcup_{a, A}$};
  \draw[->] (X) -- (BoxABoxB) node[pos=0.5,above] {$\tilde{u}$};
  \draw[->] (Y) -- (BoxABoxB) node[pos=0.5,above] {$g$};
  \draw[->] (BoxABoxB) -- (BoxABoxBBoxABoxB) node[pos=0.5,above] {$\tilde{u}$};
  \draw[->] (BoxABoxBBoxABoxB) -- (X) node[pos=0.5,above] {$g$};
  \end{tikzpicture}

Then, applying the lifting data $\phi$ yields a diagonal filler $j : \Box_A \ast \Box_B \to X$. One can check that $j(1_A, 1_B)$ is a filler for $(u, y)$.

To show that this gives a Kan filling operation, we first need to check that it is well defined. So let $C'$ be another finite support for $u$. Since the intersection of two finite supports is also a finite support, we may assume without loss of generality that $C' \subseteq C$. Hence also $B' \subseteq B$ and the inclusion $i : B' \to B$ induces a morphism $(1, A, a, B) \to (1, A, a, B')$ in $\mathcal{F}$. Applying the coherence condition for $\phi$ to $i$, then shows that the fillers we get using $B$ and using $B'$ are equal.

Similar arguments show that the filling operation we get satisfies the uniformity conditions.

Exactly the same argument allows us to construct a filling operator for 0-open boxes. Applying lemma 4.1 then gives us an $R$-algebra structure on $g$.

Finally, one can check that this is in fact an inverse to the map (24).

6 Path Objects

6.1 Name Abstraction

The cubical sets used in [1] to implement identity types correspond in 01-substitution sets to name abstraction.

Definition 6.1. Given $f : X \to Y$, define the name abstraction over $f$, $[A]_f : X$ as follows:

$$[A]_f X := \{a) x \in [A]_X \mid a \# f(x)\}$$

(34)
Theorem 6.2. Let \( f : X \to Y \) be a fibration. Let \( \Delta : X \to X \times_Y X \) be the diagonal map. Then there is a morphism \( h : K\Delta \to [A]fX \) over \( X \times_Y X \).

Proof. We define \( h \) by induction on rank.

Elements of rank 0 in \( K\Delta \) are of the form \( x \) for \( x \in X \). Let \( a \) be fresh for \( x \) and define \( h(x) \) to be \( \langle a \rangle x \).

Suppose that \( (u, (x_0, x_1)) \) is an element \( K^\uparrow \Delta \). We may assume that \( h \) is already defined on the image of \( u \). Let \( A, a \) be such that \( u \) is an \( A, a \)-box. Let \( b \) be fresh for \( u, x_0 \) and \( x_1 \). Define an \( (A \cup \{b\}, a) \)-box, \( u' \) in \( X \) as follows. Note that we may assume without loss of generality that for each \( a', i \in A \times 2 \setminus \{a, 1\} \), \( h(u(a', i)) \) is of the form \( \langle b \rangle z_{a', i} \), then define

\[
 u'(a', i) := \begin{cases} 
 z_{a', i} & \text{if } a' \in A \\
 x_0 & \text{if } a' = b \text{ and } i = 0 \\
x_1 & \text{if } a' = b \text{ and } i = 1 
\end{cases} \quad (35)
\]

Let \( y := f(x_0) \) (which is also the value of \( f(x_1) \)) and note that \( u' \) is an open box over \( y \) and that \( b \# y \). Define \( h((u, (x_0, x_1))) \) to be \( \langle b \rangle (f \uparrow u') \).

\( h \) is defined similarly for \( K^\downarrow f, K^\uparrow f \) and \( K^\downarrow f \).

Theorem 6.3. Let \( f : X \to Y \) be a morphism and let \( \Delta : X \to X \times_Y X \) be the diagonal map. Then there is a morphism \( k : [A]fX \to K\Delta \) over \( X \times_Y X \).

Proof. Given \( (a)x \in [A]X \), let \( u \) be the 0-open \( \{a\}, a \)-box given by \( u(a,0) := x(a := 0) \). Note that

\[
 f(x(a := 0)) = f(x)(a := 0) = f(x) \quad \text{since } a \# f(x) \quad (36)
\]

So \( (x(a := 0), x) \) is an element of \( X \times_Y X \), with \( (u, (x(a := 0), x)) \), an open box over it. Hence \( (a)(u, (x(a := 0), x)) \) is an element of \( K^\uparrow \Delta \). Define \( k((a)x) \) to be \( \langle a \rangle (u, (x(a := 0), x)) \).

One can adapt the proof in \([1]\) that identity types are Kan fibrations to show that the projection map \([A]fX \to X \times_Y X \) is an R-map.

In order to show that \([A]fX \) is a path object on \( X \) over \( f \), it remains only to show that the reflexivity map \( r : X \to [A]fX \) has the left lifting property with respect to every R-map. \( (r(x)) \) is defined to be Fresh \( a \) in \( (a)x \).

We will see later (corollary \([6,12]\)) that assuming classical logic this is a path object. However, we show now that this cannot be done constructively. In particular, we will show that one cannot show constructively that \( r : X \to [A]fX \) is always a left map.

Theorem 6.4. It cannot be proved constructively that for every fibrant 01-substitution set, \( X \), the reflexivity morphism \( r : X \to [A]X \) has the left lifting property with respect to every R-map.

Proof. We will show this by assuming \( r \) does have this property and deriving the weak limited principle of omniscience (WLPO), that is, for any \( f : \omega \to 2 \) the statement “\( f(n) = 0 \) for all \( n \)” is either true or false. (In particular this implies the existence of noncomputable functions)
Let \( f : \mathbb{N} \rightarrow 2 \). Let \( X \) be the fibrant 01-substitution set of continuous functions to \([0,1]\).

We construct a continuous function \( \bar{f} : [0,1] \rightarrow [0,1] \). Given \( x \in [0,1] \) we will construct a Cauchy sequence \((y_n)_{n<\omega}\). Let \( n < \omega \). If \( f(m) = 0 \) for all \( m < n \), then let \( y_n := 0 \). If there is \( m < n \) such that \( f(m) = 1 \), then there is a least \( m_0 \) such that \( f(m_0) = 1 \). Let \( y_n := 2^{-m_0}q \), where \( q \) is chosen such that \( |q - x| < 2^{-n} \). Note that in classical logic \( \bar{f} \) could have been defined \( \bar{f}(x) = \begin{cases} 0 & f(n) = 0 \text{ for all } n \\ 2^{-n}x & \text{there is } n \text{ least such that } f(n) = 1 \end{cases} \) (37)

Now note that we can view \( \bar{f} \) as an element of \( X \) dependent on (at most) a single name, say, \( a \). So we have that \( \langle a \rangle \bar{f} \in [A]X \).

Now by assumption, \( r : X \rightarrow [A]X \) has the left lifting property with respect to the \( R \)-map \( \rho_r \), giving us a diagonal filler, \( j \), in the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda_r} & K\rho_r \\
\downarrow r & & \downarrow [A]X \\
[A]X & \xrightarrow{\rho_r} & [A]X
\end{array}
\] (38)

Now note that \( j(\langle a \rangle \bar{f}) \) has rank 0 if and only if \( \langle a \rangle \bar{f} \) is in the image of \( r \). However, this is true if and only if \( a \) is fresh for \( \bar{f} \), which is the case precisely when \( \bar{f} \) is constant with respect to \( a \), which in turn happens precisely when \( f \) is constantly 0. Since the rank of \( j(\langle a \rangle \bar{f}) \) is either zero or nonzero, we deduce that either \( f \) is constantly 0 or \( f \) is not constantly 0, as required.

\[\Box\]

6.2 Using the Awfs Structure

Note that we can trivially generate path objects using the awfs structure itself. For any fibration \( f : X \rightarrow Y \), if \( \Delta : X \rightarrow X \times Y \) is the diagonal map, then \( K\Delta \) is a path object. However, this construction does not appear to be preserved by substitution, so it is unsuitable for implementing identity types.

6.3 Labelled Name Abstraction

In this section we give a new construction of path object that is valid constructively and preserved by substitution. The basic idea is to add “labels” to the “name abstraction” identity types. Any side of a cube that has been “labelled” in guaranteed to be degenerate (although there may be additional degenerate sides that are not labelled). In order to make the notation and proofs slightly easier we use some notions from the construction of the awfs. In particular, we have already checked that \( K\Delta \) is a 01-substitution set, so we know, for instance that “commutativity of substitution” holds, ie \( z(a := i)(a' := i') = z(a' := i')(a := i) \).

**Definition 6.5.** Let \( f : X \rightarrow Y \) be a fibration and let \( a \in A \). Define the subset \( P_x^a X \) of \( K\Delta \) of pre-normal forms in direction \( a \) inductively as follows.

1. If \( x \in X \) and \( a \# x \) then \( a \) is a pre-normal form in direction \( a \).
2. If \( u \) is a 1-open, \( a \)-box over \( (x_1, x_2) \), \( a \# x_1, u(a, 0) = x_1 \) and for every \( a', i \in (A \setminus a) \times 2, u(a', i) \) is a pre-normal form in direction \( a \), then \( (u, (x_1, x_2)) \) is a pre-normal form in direction \( a \).

Note that \( P^0_X \) is closed under permutations, and every substitution of the form \( (a' := i) \) where \( a' \neq a \).

An element of \( K_\Delta \) is a normal form if it is equal to \( z(a := 1) \) where \( z \) is some pre-normal form in direction \( a \).

We refer to the set of normal forms as \( P_Y X \). Note that it is closed under permutations and substitutions, and so we can view it as a 01-substitution set in the natural way.

**Lemma 6.6.** Let \( z, z' \) be pre-normal forms in direction \( a \). If \( z(a := 1) = z'(a := 1) \) then \( z = z' \).

**Proof.** Note that if \( z \) is an element of \( X \) with \( a \# z \), then \( z(a := 1) = z \). If \( z \) is an open box in direction \( a \), then \( z(a := 1) \) is an element of \( K^+ \Delta \).

Therefore, if \( z(a := 1) = z'(a := 1) \), then either \( z \) and \( z' \) are both elements of \( X \), or they are both elements of \( K^+ \Delta \).

In the former case, we have \( z(a := 1) = z \) and \( z'(a := 1) = z' \) (since \( a \# z \) and \( a \# z' \)) and so \( z = z' \) as required.

In the latter case, we have that \( z \) and \( z' \) are of the form \( (u, (x_1, x_2)) \) and \( (u', (x_1', x_2')) \) respectively. Then \( z(a := 1) \) and \( z'(a := 1) \) are the elements of \( K^+ \Delta, (a)(u, (x_1, x_2)) \) and \( (a)(u', (x_1', x_2')) \). Since \( (a)(u, (x_1, x_2)) = (a)(u', (x_1', x_2')) \), we have \( (u, (x_1, x_2)) = (u', (x_1', x_2')) \) as required. \( \square \)

**Corollary 6.7.** For every normal form \( w \) and every \( a \in A \) fresh, there is a unique pre-normal form \( z \) in direction \( a \) such that \( z(a := 1) = w \).

**Theorem 6.8.** The restriction of \( \rho_\Delta : K_\Delta \to X \times_Y X \) to \( P_Y X \) is a fibration.

**Proof.** Let \( v \) be a 1-open, \( a \)-box in \( P_Y X \) over \( (x_1, x_2) \). Let \( b \) be a fresh variable. By applying corollary 6.7 we get a 1-open, \( a \)-box \( v' \) over \( P^0_X \) satisfying the adjacency conditions by lemma 6.6 and such that for each \( (a', i) \in A \times 2 \setminus (a, 1) \), \( v'(a', i) \) is a pre-normal form in direction \( b \). Note that \( \pi_2 \circ \rho_\Delta \circ v' \) is an open box in \( X \) (where \( \pi_2 \) is the second projection). Let \( A' \) be \( A \cup \{ b \} \). We extend \( \pi_2 \circ \rho_\Delta \circ v' \) to a 1-open \( A', b \)-box \( v'' \) by setting \( v''(b, 0) = x_1 \) and \( v''(b, 1) = x_2 \).

Let \( A'' \) be \( A' \setminus a \). We define a 1-open \( A'', b \)-box, \( w \) as follows. Define \( w(b, 0) \) to be \( x_1(a := 1) \). For \( (a', i) \in A'' \times 2 \), define \( w(a', i) \) to be \( v'(a', i)(a := 1) \). Note that \( (w, (x_1(a := 1), f^+ v'')) \) is a pre-normal form.

We now form a 1-open \( A', b \)-box, \( w' \) as follows. Set \( w'(b, 0) \) to be \( x_1 \). Set \( w'(a, 1) \) to be \( (w, (x_1(a := 1), f^+ v'')) \). For \( (a', i) \in A \times 2 \setminus (a, 1) \), define \( w'(a', i) \) to be \( v'(a', i) \).

Finally this allows to define the Kan filler of \( v \) to be

\[
(w', (x_1, f^+ v''))(b := 1)
\]

**Lemma 6.9.** There is a nominal set morphism \( h : P^0_Y X \ast A \to X \) such that for every \( z \) in direction \( a \) and \( b \) fresh, for \( a' \neq a, b \), \( h(z(a' := i), b) = h(z, b)(a' := i) \), \( \rho_\Delta z = (h(z, b)(b := 0), h(z, b)(b := 1)) \), for \( x \in X \), \( h(x, b) = x \).
Proof. We define $h$ by induction on the construction of $P^0 Y X$. We define $h(x, b)$ to be $x$. Suppose we are given a pre-normal form of the form $(u, (x_1, x_2))$, where $u$ is a 1-open $A, a$-box and $b$ a fresh name. Let $A' := A \cup \{ b \}$. We form a 1-open $A', a$-box, $v$ in $X$ as follows. If $(a', i)$ is an element of $A \times 2 \setminus (a, 1)$, then we may assume by induction that $h(u(a', i), b)$ has already been defined. Let $v(a', i)$ be $h(u(a', i), b)$. Let $v(0, 0)$ be $x_1$ and let $v(1, 1)$ be $x_2$. Now define $h((u, (x_1, x_2)), b)$ to be $f \uparrow v$.

To illustrate $h$, consider when $x_2$ is a path with $x_1$ as one of the endpoints. In this case, $h$ is a homotopy from the constant path at $x_1$ to the path $x_2$.

Lemma 6.10. There is a nominal set morphism $k : P^0 Y X \ast \mathbb{A} \to P^0 Y X$ such that for $a' \neq a, b, k(z(a' := i), b) = k(z, b)(a' := i)$ for $x \in X$ we have $k(x, b) = x$ and such that if $z \in P^0 Y X$ and $\rho_\Delta(z) = (x_1, x_2)$, then $k(z, b)(b := 0) = x_1$ and $k(z, b)(b := 1) = z$. $k$ will also be “direction preserving.”

Proof. We define $k(x, b)$ to be $x$ for $x \in X$. Now suppose we are given a pre-normal form of the form $(u, (x_1, x_2))$, where $u$ is a 1-open $A, a$-box and $b$ a fresh name. Let $A' := A \cup \{ b \}$ as before. We form a 1-open $A', a$-box $v$. Given $(a', i) \in A \times 2 \setminus (a, 1)$, set $v(a', i)$ to be $h(u(a', i), b)$. Set $v(0, 0)$ to be $x_1$, and set $v(1, 1)$ to be $(u, (x_1, x_2))$. Finally, define $k((u, (x_1, x_2)), b)$ to be $(v, (x_1, h((u, (x_1, x_2)), b)))$, where $h$ is as in lemma 6.9.

Theorem 6.11. The inclusion $r : X \to P r Y X$ is an element of $\mathcal{L}$.

Proof. We need to define a copointed endofunctor coalgebra on $r$.

We first define a morphism $l : P^0 Y X \ast \mathbb{A} \to Kr$, ensuring that if $z \in P^0 Y X$ is in direction $a$, then $a$ is fresh for $l(z, b)$. As before, we ensure that $l$ is a nominal set morphism that preserves substitutions $(a' := i)$ for $a' \neq a, b$. We will also ensure that $\rho_r(l(z, b)) = k(z, b)(a := 1)$.

Define $l(x, b)$ to be $x$. Given $(u, (x_1, x_2))$ and $b$, where $u$ is an $A, a$-box, let $A' := (A \setminus a) \cup \{ b \}$. We define a 1-open $A', b$-box, $v$ as follows. Given $(a', i) \in (A \setminus a) \times 2$, define $v(a', i)$ to be $h(u(a', i), b)$. Define $v(0, 0)$ to be $x_1$. Then define $l((u, (x_1, x_2)), b)$ to be $(v, k((u, (x_1, x_2)), b)(a := 1))$, where $k$ is as in lemma 6.10.

This now allows us to define the coalgebra map $c : P r Y X \to Kr$ by defining $c(z(a := 1))$ (for $z$ a prenormal form in direction $a$) to be $l(z, b)(b := 1)$, where $b$ is any fresh variable (so $c((u, (x_1, x_2))(a := 1))$ will be an element of $K^+ r$).

Note that this does give a coalgebra structure, since we have

$$\rho_r(c(z(a := 1))) = \rho_r(l(z, b)(b := 1))$$
$$= \rho_r(l(z, b)(b := 1))$$
$$= k(z, b)(a := 1)(b := 1)$$
$$= k(z, b)(b := 1)(a := 1)$$
$$= z(a := 1) \quad (40)$$

Corollary 6.12. Assume that excluded middle holds. Then $[\mathbb{A}] r X$ is a path object.
Proof. We write $r'$ for the reflexivity map $X \to [A]fX$.

We will first define a map $g : [A]fX \to PYX$ such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{r} & X \\
\downarrow & & \downarrow \\
[A]fX & \xrightarrow{g} & PYX \\
\downarrow & & \downarrow \\
X \times X & \xrightarrow{r'} & X \times X
\end{array}
\]  

(41)

Define

\[X_0 := \{(x, a) \mid a \# f(x)\}\]  

(42)

We first define a morphism $g_0 : X_0 \to P_0YX$. Applying excluded middle, we have that every $x \in X$ has a least finite support, Supp$(x)$. We define $g_0(a, x)$ by induction on $|\text{Supp}(x)|$. If $a \# x$, then define $g_0(a, x) := x$. Otherwise, let $A := \text{Supp}(x)$, and note that $a \in A$. We define a 1-open $A$, a-box $v$ as follows.

Let $v(a, 0) := x(a := 0)$. For $(a', i) \in (A \setminus a) \times 2$, we may assume by induction that $g_0((a) x(a' := i))$ has already been defined. Let $v(a', i) := g(a, x(a' := i))$. We then define $g_0(a, x)$ to be $(v, (x(a := 0), x))$.

We now define $g((a)x)$ to be $g_0(a, x)(a := 1)$.

We also have a morphism $PYX \to [A]fX$ given by projection. This induces a morphism $Kr \to Kr'$, which combined with the result before and the coalgebra structure on $r$ gives a commutative diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{r} & X \\
\downarrow & & \downarrow \\
[A]fX & \xrightarrow{g} & PYX & \xrightarrow{Kr} & Kr' \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
X \times X & \xrightarrow{r'} & X \times X & \xrightarrow{Kr'} & X \times X
\end{array}
\]  

(43)

However, this easily gives us a coalgebra structure on $r'$.

\[\square\]

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