Enumerating Isolated Cliques in Temporal Networks

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Isolation is a concept from the world of clique enumeration that is mostly used to model communities that do not have much contact to the outside world. Herein, a clique is considered isolated if it has few edges connecting it to the rest of the graph. Motivated by recent work on enumerating cliques in temporal networks, we lift the isolation concept to this setting. We discover that the addition of the time dimension leads to six distinct natural isolation concepts. Our main contribution is the development of fixed-parameter enumeration algorithms for five of these six clique types employing the parameter “degree of isolation”. On the empirical side, we implement and test these algorithms on (temporal) social network data, obtaining encouraging preliminary results.

Keywords: Community detection; Dense Subgraphs; Social network analysis; Time-evolving data; Enumeration algorithms; Fixed-parameter tractability.

1 Introduction

“Isolation is the one sure way to human happiness.” – Glenn Gould

Clique detection and enumeration is a fundamental primitive of complex network analysis. In particular, there are numerous approaches (both from a more theoretical and from a more heuristic side) for listing all maximal cliques (that is, fully-connected subgraphs) in a graph. It is well-known that finding a maximum-cardinality clique is computationally hard (NP-hard, hard in the approximation sense and hard when parameterized by the clique cardinality). Hence, heuristic approaches usually govern computational approaches to clique finding and enumeration. From now on, we focus on the case of enumerating maximal cliques. There have been numerous efforts to provide both theoretical guarantees and practically useful algorithms. In particular, to simplify (in a computational sense) the task on the one hand and to enumerate more meaningful maximal cliques (for specific application contexts) on the other hand, Ito and Iwama introduced and investigated the enumeration of maximal cliques that are “isolated”.

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1 Network and Graph are used interchangeably.
Roughly speaking, isolation means that the connection of the maximal clique to the rest of the graph is limited, that is, there are few edges with one endpoint in the clique and one endpoint outside the clique; indeed, the degree of isolation can be controlled by choosing specific values of a corresponding isolation parameter. For instance, think of social networks where one wants to spot more or less segregated sub-communities with little interaction to the world outside but intensive interaction inside the community. We mention in passing that recently there have been (only) theoretical studies on the concept of “secludedness” [2, 3, 5] which is somewhat similar to the older isolation concept: whereas for isolation one requests “few outgoing edges”, for secludedness one asks for “few outneighbors”; while finding isolated cliques becomes tractable [12], finding secluded ones remains computationally hard [2].

Ito and Iwama [12] showed that in static networks isolated cliques can be enumerated efficiently; the only exponential factor in the running time depends on the “isolation parameter”, and so fairly isolated cliques can be enumerated quite quickly. In follow-up work, the isolation concept then was significantly extended and more thorough experimental studies (also with financial networks) have been performed [11, 13]. However, analyzing complex networks more and means studying time-evolving networks. Hence, computational problems known from static networks also need to be solved on temporal networks (mathematically, these are graphs with fixed vertex set but a time-dependent edge set) [10, 14, 15]. Thus, not surprisingly, the enumeration of maximal cliques has recently been lifted to the temporal setting [1, 6, 13, 19]. While getting algorithmically more challenging than in the static network case, nevertheless the empirical results that have been achieved are encouraging. In this work, we now fill a gap by proposing to lift also the isolation concept to the temporal clique enumeration context, otherwise using the same modeling of temporal cliques as in previous work.

Since we believe that enumerating isolated cliques has its most important applications in community detection scenarios, we focus on only two of three basic isolation concepts described by Komusiewicz et al. [13] for the static setting. More specifically, we only consider “maximal isolation” (every vertex has small outdegree) and “average isolation” (vertices have small outdegree on average), but do not study “minimal isolation” (at least one vertex has small outdegree). Nevertheless we still face a richer modeling than in the static case since isolation can happen in two “dimensions”: vertices and time; for both we can consider maximum and average isolation. With this distinction, we end up with eight natural ways to model isolation, where two “pairs” of isolation models turn out to be equivalent, finally leaving six different temporal isolation concepts for further study.

Our main contributions are as follows: First, as indicated above, we already do some conceptual work with identifying six, mathematically formalized concepts of isolation for temporal networks. Second, building on and extending the algorithmic framework of Komusiewicz et al. [13] for static networks, for small isolation values we provide efficient algorithms for five of our six isolated clique enumeration models and prove worst-case performance bounds for them. In this context, a main algorithmic contribution is the development of tailored subroutines (that are only partially shared between different isolation concepts). Finally, on the empirical side we contribute an encouraging first experimental analysis of our algorithms based on social network data. Our preliminary experiments indicate significant differences (mostly in terms of practical running time) but also (sometimes surprising) accordances between the concepts.

2In terms of the language of parameterized algorithmics, we show that these cases are fixed-parameter tractable when parameterized by isolation value.
2 Preliminaries

In this section we first give some basic notation and terminology. We then recall the isolation concept for static graphs and transfer it to temporal graphs. Lastly, we give some motivating examples that are tailored to the different temporal isolation concepts arising and try to give an intuitive understanding of the differences between the various temporal isolation models.

Static Graphs. Graphs in this paper are assumed to be undirected and simple. To clearly distinguish them from temporal graphs, they are sometimes referred to as static graphs. Let \( G = (V, E) \) be a static graph. We denote the vertex set of \( G \) with \( V(G) \) and the edge set of \( G \) with \( E(G) \). For \( v \in V(G) \) we use \( \text{deg}_G(v) \) for the number of edges ending at \( v \). For \( v \in A \subseteq V \), \( \text{outdeg}_G(v, A) \) denotes the number of edges with one endpoint \( v \) and the other one outside of \( A \). Further \( \text{outdeg}_G(A) := \sum_{v \in A} \text{outdeg}_G(v, A) \). We use \( \delta_G(A) := \min_{v \in A} \text{deg}_G(v) \) for the minimum degree. In all these notations, we omit the index \( G \) if there is no ambiguity.

Temporal Graphs and Temporal Cliques. A temporal graph is a tuple \( \mathcal{G} = (V, E_1, E_2, \ldots, E_\tau) \) of a vertex set \( V \) and \( \tau \) edge sets \( E_i \subseteq \binom{V}{2} \). The graphs \( G_i := (V, E_i) \) are called the layers of \( \mathcal{G} \). The time edge set \( E(\mathcal{G}) \) (or \( E \) if \( \mathcal{G} \) is clear from the context) is the disjoint union \( \bigsqcup_{i=1}^{\tau} E_i \) of the edge sets of the layers of \( \mathcal{G} \). For any \( 1 \leq a \leq b \leq \tau \) we define the (static) graphs \( \bigcup_{i=a}^{b} G_i := (V, \bigcup_{i=a}^{b} E_i) \) and \( \bigcap_{i=a}^{b} G_i := (V, \bigcap_{i=a}^{b} E_i) \).

Following the definition of Viard et al. [18], a \( \Delta \)-clique (for some \( \Delta \in \mathbb{N} \)) of \( \mathcal{G} \) is a tuple \((C, [a, b])\) with \( C \subseteq V \) and \( 1 \leq a \leq b \leq \tau \) such that \( C \) is a clique in \( \bigcup_{i=a}^{\tau} G_i \) for all \( t \in [a, b - \Delta] \).

It is easily observed that \((C, [a, b])\) is a \( \Delta \)-clique in \( \mathcal{G} \) if and only if \((C, [a, b - \Delta])\) is a 0-clique in \( \mathcal{G}' = (V, E'_1, \ldots, E'_\tau) \) where \( E'_i := \bigcup_{t=i}^{i+\Delta} E_t \). Due to this, we will only concern ourselves with \( \Delta = 0 \) and simply refer to 0-cliques as temporal cliques.

Temporal Isolation. We first introduce the isolation concepts for static graphs and then describe how we transfer them to the temporal setting. In a (static) graph \( G \), a clique \( C \subseteq V(G) \) is called \( \text{avg-c-isolated} \) if \( \text{outdeg}_G(C) < c \cdot |C| \) where \( c \in \mathbb{Q} \) is some positive number [12]. Further it is called \( \text{max-c-isolated} \) if \( \max_{v \in C} \text{outdeg}_G(v) < c \). Clearly max-c-isolation implies avg-c-isolation.

Moving to temporal graphs, we want to define an isolation concept for temporal cliques. Recall that a temporal clique consist of a vertex set and a time interval. We apply the isolation requirement both on a vertex and on a time level, meaning that for each dimension we can either require the average outdegree (as in the static avg-c-isolation) or the maximum outdegree (as in the static max-c-isolation) to be small. To make this more clear, we give some examples. For instance, we can require that, on average over all layers, the maximum outdegree in a layer is small. Or we can require that the average outdegree must be small in every single layer. Note that the ordering of the requirements for the time dimension and the vertex dimension also matters. Requiring the average outdegree to be small in every layer is different from requiring that, on average over all vertices, the maximum degree over all time steps must be small. Having two isolation requirements (avg and max) for two dimensions with two possible orderings, we arrive at eight canonical temporal isolation types. However it turns out that if we use the same requirement for both dimensions, they behave commutatively, so it boils down to
We now prove some important facts that we will make use of in the correctness proofs of our isolation types. We define the set of usually-max-isolation. The next isolation concept, would be possible to have some distinguished "bridge" vertices inside the clique with relatively much outside activity, in which some or even all vertices have many outgoing edges, as long as the entire graph, only allowing a limited number of communications to occur at any given moment. The next isolation concept, usually-max-isolation, can be seen as allowing short bursts of activity, in which some or even all vertices have many outgoing edges, as long as the entire clique is isolated most of the time. Again, if we reorder the terms, we get a less restrictive concept (max-usually-isolation). Here, the bursts of activity may happen at different times for different vertices. Finally, usually-avg-isolation is the least restrictive of these notions, only limiting the total number of outside contacts over all vertices and layers that are part of the temporal clique.

3 Basic Facts

We now prove some important facts that we will make use of in the correctness proofs of our algorithms. Our first observation concerns the relation between different isolation types. It
is easily checked using the definitions above.

**Observation 1.** Let $G = (V, E_1, \ldots, E_t)$ be a temporal graph. The following implication diagram holds for any $a \leq b$, any clique $C$ in $\bigcap_{i=a}^{b} G_i$, and any $c > 0$:

\[
\begin{array}{c}
(C, [a, b]) \text{ alltime-max-c-isolated} \quad \rightarrow \quad (C, [a, b]) \text{ avg-alltime-c-isolated} \\
(C, [a, b]) \text{ usually-max-c-isolated} \quad \rightarrow \quad (C, [a, b]) \text{ alltime-avg-c-isolated} \\
(C, [a, b]) \text{ max-usually-c-isolated} \quad \rightarrow \quad (C, [a, b]) \text{ usually-avg-c-isolated}
\end{array}
\]

Note that Observation 1 does not hold for maximal isolated temporal cliques: A maximal alltime-max-c-isolated clique is not necessarily a maximal usually-avg-c-isolated clique.

Next, we state two lemmata limiting the minimal size of isolated cliques, helping us to confine the search space for our algorithms. They are inspired by the ideas employed by Komusiewicz et al. [13] in the static setting.

**Lemma 2.** Let $G$ be a static graph and let $C$ be a clique in $G$. Then, any avg-c-isolated subset $C' \subseteq C$ has size $|C'| > \delta(C) - c + 1$.

**Proof.** Suppose $|C'| \leq \delta(C) - c + 1$. Then any vertex $w \in C'$ has outdeg$(w, C') = \text{deg}(w) - (|C'| - 1) \geq \delta(C) - (|C'| - 1) \geq c$. Thus, $C'$ is not avg-c-isolated. \hfill $\Box$

**Lemma 3.** Let $C$ be a clique in $G \cap := \bigcap_{i=1}^{t} G_i$ for some $G_i = (V, E_i)$. Then any subset $C' \subseteq C$ for which $(C', [1, t])$ is usually-avg-c-isolated has size $|C'| > \delta_{G_i}(C) - c + 1$.

**Proof.** Suppose not. Then $|C'| \leq \delta_{G_i}(C) - c + 1 \leq \delta_{G_i}(C) - c + 1$ for all $i$. By Lemma 2, $C'$ is not avg-c-isolated in any $G_i$ and thus certainly not usually-avg-c-isolated. \hfill $\Box$

Next, we show that some vertices must always be contained in vertex-maximal (and thus also in maximal) isolated cliques. This will allow us to refrain from searching for maximal isolated cliques that do not contain these vertices.

**Lemma 4.** Let $C$ be a clique in $G \cap := \bigcap_{i=1}^{t} G_i$ for some $G_i = (V, E_i)$ and let $C' \subseteq C$ be such that $(C', [1, t])$ is a vertex-maximal usually-avg-c-isolated temporal clique. Then $C'$ contains all vertices $v \in C$ that fulfill

\[ \sum_{i=1}^{t} \text{deg}_{G_i}(v) \leq t(\delta_{G_i}(C) + |C'| + 1). \]
Proof. We prove this statement by contraposition. Suppose \((*)\) holds for some \(v \in C \setminus C'\). Let \(k := |C|\) and \(k' := |C'|\). Then we have that
\[
\sum_{i=1}^{t} \text{outdeg}_{G_i}(v, C) = \sum_{i=1}^{t} \left( \text{deg}_{G_i}(v) - (k - 1) \right) \leq t(\delta_{G_{\cap}}(C) - k + k' + 2)
\]
and thus
\[
\sum_{i=1}^{t} \text{outdeg}_{G_i}(C' \cup \{v\}) = \sum_{i=1}^{t} \left( \text{outdeg}_{G_i}(C') + (k - k' - 1) + \text{outdeg}_{G_i}(v, C) - k' \right)
\]
\[
< t \left( c(k' + \delta_{G_{\cap}}(C) - k' + 1) \right)
\]
\[
= t \left( c(k' + 1) + \delta_{G_{\cap}}(C) - c - k' + 1 \right)
\]
\[
< ct(k' + 1),
\]
where the last inequality is due to Lemma 3. Thus, \(C' \cup \{v\}\) is usually-avg-c-isolated.

Lemma 5. Let \(C\) be a clique in \(G_{\cap} := \bigcap_{i=1}^{t} G_i\) for some \(G_i = (V, E_i)\) and let \(C' \subseteq C\) such that \((C', [1, t])\) is a vertex-maximal usually-avg-c-isolated temporal clique. Let \(\tilde{C} \subseteq C\) consist of the \(\delta_{G_{\cap}}(C) - c + 2\) vertices \(v\) with the lowest values of \(\sum_{i=1}^{t} \text{deg}_{G_i}(v)\). Then \(\tilde{C} \subseteq C'\).

Proof. Let \(k := |C|\) and \(k' := |C'|\). For any \(v \in C'\) and \(i\), we have that
\[
\text{outdeg}_{G_i}(v, C') \geq \text{outdeg}_{G_{\cap}}(v, C')
\]
\[
= \text{outdeg}_{G_{\cap}}(v, C) + k - k'
\]
\[
\geq \delta_{G_{\cap}}(C) - k' + 1 =: d.
\]
Suppose for contradiction that there exists \(u \in \tilde{C} \setminus C'\). Then, for each \(v \in C' \setminus \tilde{C}\), we have that
\[
\sum_{i=1}^{t} \text{outdeg}_{G_i}(v, C') = \sum_{i=1}^{t} \left( \text{deg}_{G_i}(v) - k' + 1 \right)
\]
\[
\geq \sum_{i=1}^{t} \left( \text{deg}_{G_i}(u) - k' + 1 \right)
\]
\[
> t \left( \delta_{G_{\cap}}(C) + 2 \right) =: t\tilde{d},
\]
where the last inequality is due to Lemma 3. Furthermore, we have that
\[
h := |C' \setminus \tilde{C}| \geq |C'\| - (|\tilde{C}| - 1) = k' - \delta_{G_{\cap}}(C) + c - 1.
\]
Thus, we get that
\[
\sum_{i=1}^{t} \text{outdeg}_{G_i}(C') = \sum_{v \in C'} \sum_{i=1}^{t} \text{outdeg}_{G_i}(v, C')
\]
\[
> k'td + h(t\tilde{d} - td)
\]
\[
= t \left( k'd + (k' + 1) \right)
\]
\[
\geq tk'(d + h)
\]
\[
= tk'c,
\]
contradicting the isolation of \(C'\).
Table 1: Running time of our maximal isolated temporal clique enumeration algorithms for the different temporal isolation types.

| Isolation Type       | Running Time                                      |
|----------------------|---------------------------------------------------|
| alltime-avg          | $O(c^2\tau^2 \cdot |V| \cdot |E|)$          |
| alltime-max          | $O(2.89^c c\tau \cdot |E|)$                      |
| avg-alltime          | $O((5.78^c c\tau^3 \cdot |E|)$                      |
| max-usually          | $O(2.89^c c\tau^3 \cdot |E|)$                      |
| usually-avg          | $O((5.78^c c\tau^3 \cdot |E|)$                      |

Lemma 6. Let $C$ be a clique and $C' \subseteq C$ a maximal avg-$c$-isolated subset. Set $\hat{C} \subseteq C$ to be the $\delta(C) - c + 2$ vertices of minimal degrees. Then $\hat{C} \subseteq C'$.

Proof. This is a special case of Lemma 5 for $t = 1$.

4 Enumerating Maximal Isolated Temporal Cliques

In this section, we present efficient algorithms to enumerate maximal isolated temporal cliques for five out of the six introduced temporal isolation concepts (all except usually-max)\(^3\). These algorithms have fixed-parameter tractable (FPT) running times for the isolation parameter $c$, that is, for fixed $c$, the running time is a polynomial whose degree does not depend on $c$.\(^4\)

Formally, we show the following result, whose proof will be given in Section 4.3.

Theorem 7. Let a temporal graph $G$, an isolation type $I \in \mathcal{I} \setminus \{\text{usually-max}\}$, and an isolation parameter $c \in \mathbb{Q}$ be given, then all maximal $I$-$c$-isolated temporal cliques in $G$ can be enumerated in FPT-time for the isolation parameter $c$. The specific running times depend on $I$ and are given in Table 1.

Our algorithms are inspired by the algorithms for static isolated clique enumeration [12, 13] and build upon the fact that every maximal $I$-$c$-isolated temporal clique $(C, [a, b])$ is contained in some vertex-maximal $c$-isolated clique $C'$ of $G_\cap := \bigcap_{t=a}^b G_t$ (by Observation 1). Algorithm 1 constitutes the top level algorithm. Here, we iterate over all possible time windows $[a, b]$ and apply the so-called trimming procedure developed by Ito and Iwama [12] to $G_\cap$ to obtain, for each so-called pivot vertex $v$, a set $C_v \subseteq N[v]$ containing all avg-$c$-isolated cliques of $G_\cap$ that contain $v$. Subsequently, we enumerate all maximal cliques within $C_v$ and test each of them for maximal $I$-$c$-isolated subsets. For this step, we employ Lemmas 2, 5, and 6 to quickly skip over irrelevant subsets. The details depend on the choice of $I$, as does the strategy for the last step, that is, removing non-maximal elements from the result set. Remember that we have to pay attention to both, time- and vertex-maximality. For the latter we can, in most cases, adapt an idea by Komusiewicz et al. [13].

We proceed by describing the subroutines isolatedSubsets() (Line 11 of Algorithm 1) and isMaximal() (Line 18 of Algorithm 1). Then we prove correctness of our algorithms and, finally, analyze their running times.

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\(^3\)The reader may wonder why usually-max-isolation was dropped here. The answer is that, even though the same approach also works for usually-max-isolation, we found no way to limit the work that would be required in the isolatedSubsets() subroutine significantly below $O(2^n)$.

\(^4\)The isolation parameter $c$ only influences the leading constant of the polynomial running time but not the degree of the polynomial, that is, the running time is $f(c) \cdot \text{poly}(|G|)$ for some function $f$. 

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Algorithm 1: Enumerating maximal $I$-$c$-isolated cliques for $I \in \mathcal{I} \setminus \{\text{usually-max}\}$

**Input:** A temporal graph $\mathcal{G} = (V, E_1, \ldots, E_\tau)$, a $c \in \mathbb{Q}$, and an isolation type $I \in \mathcal{I} \setminus \{\text{usually-max}\}$.

**Output:** All maximal $I$-$c$-isolated cliques in $\mathcal{G}$.

```plaintext
result ← {}  
foreach $a = 1, \ldots, \tau$ do 
  foreach $b = a, \ldots, \tau$ do 
    /* Here we are looking for cliques with lifetime $[a, b]$. */ 
    $G_\cap \leftarrow \bigcap_{i=a}^{b} G_i$  
    Sort vertices by ascending degree in $G_\cap$  
    foreach vertex $v$ do 
      /* Vertex $v$ is the pivot vertex. */ 
      $C_v \leftarrow$ candidate set for pivot $v$ after trimming stage (in $G_\cap$)  
      $k \leftarrow \lfloor \deg_{G_\cap}(v) - c + 2 \rfloor$ /* By Lemma 2, all isolated cliques are at least this large. */  
      $C \leftarrow$ set of all maximal cliques of size at least $k$ in $C_v \subseteq G_\cap$  
      foreach $C \in C$ do 
        subsets ← $I$-isolatedSubsets($C$, $[a, b]$, $\deg_{G_\cap}(v)$)  
        result ← result $\cup \{(C, [a, b]) | C \in \text{subsets}\}$  
      end 
    end 
  end 
end 
foreach $(C, [a, b]) \in \text{result}$ do 
  if $I$-isMaximal($C$, $[a, b]$) then 
    output $(C, [a, b])$ 
  end 
end
```

4.1 Enumerating Isolated Subsets

We now discuss the isolatedSubsets() subroutine of Algorithm 1 (Line 11). While the details depend on the isolation type, there are two main flavors. For alltime-max-isolation (Function 2) and max-usually-isolation (Function 4), it is possible to determine a single vertex that must be removed in order to obtain an isolated subset. By repeatedly doing so, one either reaches an isolated subset or the size threshold set by Lemma 2. In particular, each maximal clique contains at most one maximal isolated subset.

For usually-avg-isolation (Function 5), avg-alltime-isolation (Function 3), and alltime-avg-isolation (Function 1), multiple vertices are removal candidates. However, their number is upper-bounded by Lemma 5 and Lemma 6, respectively. We therefore build a search tree, iteratively exploring removal sets of growing size. The case of alltime-avg-isolation (Function 3) is somewhat special, as here the set of removal candidates is different for each layer.

4.2 Checking for Maximality

We now discuss the isMaximal() subroutine of Algorithm 1 (Line 18), which in turn uses an isVertexMaximal() subroutine. Note that, while each temporal clique $(C, [a, b])$ returned by isolatedSubsets() is vertex-maximal within its respective set $C_v$, it may be not vertex-maximal with regard to the entire graph. Moreover, we need to check for maximality with regard to cliques with a larger time window. The naive approach of pairwise comparing all elements of the result set is feasible but inefficient. Instead, for alltime-max-isolation, alltime-avg-isolation,
Function 1: alltime-avg-isolatedSubsets($C, [a, b], \delta$)

1. \[ d := \lfloor |C| - \delta + c - 2 \rfloor \] /* By Lemma 5, we can only remove the top $d$ vertices. */
2. $D' \leftarrow \{\emptyset\}$
3. result $\leftarrow \emptyset$
4. while $D' \neq \emptyset$ do
5. \[ D \leftarrow D' \]
6. $D' \leftarrow \emptyset$
7. foreach $D \in D$ do
8. \[ C' \leftarrow C \setminus D \]
9. if \( \exists i \in [a, b] : \sum_{v \in C'} \deg_{G_i}(v) \geq |C'| \cdot (|C'| - 1 + c) \) then
10. take $i$ to be smallest possible
11. if $|C'| > \delta - c + 2$ then
12. \[ E \leftarrow \text{the $d$ vertices of $C'$ that have the highest degrees in $G_i$} \]
13. \[ D' \leftarrow D' \cup \{D \cup \{e\} | e \in E \setminus D\} \]
14. end
15. end
16. else
17. result $\leftarrow$ result $\cup \{C'\}$
18. end
19. end
20. return result

Function 2: alltime-max-isolatedSubsets($C, [a, b], \delta$)

1. $\forall v \in C : s_v := \max_{i \in [a, b]} \deg_{G_i}(v)$
2. \[ k := \lfloor \delta - c + 2 \rfloor \] /* By Lemma 2, all isolated cliques are at least this large. */
3. while $|C| \geq k$ do
4. /* Remove offending vertices until we either succeed or fail. */
5. if \( \exists v \in C : s_v \geq |C| - 1 + c \) then
6. \[ C \leftarrow C \setminus \{v\} \]
7. end
8. else
9. return $\{C\}$
10. end
11. return $\emptyset$

and avg-alltime-isolation it is sufficient to only check whether the time window can be extended in either direction (see Function 6), and whether a larger clique exists within the same time window (i.e., checking vertex-maximality). Except for the case of alltime-avg-isolation (Function 8), the latter can again be implemented more efficiently than by using pairwise comparisons (Function 9). We modify the maximality test developed by Komusiewicz et al. [13] which searches for cliques within the common neighborhood of $C$ and then checks whether these can be used to build a larger isolated clique.

This modified vertex-maximality test also works for the cases of max-usually-isolation and usually-avg-isolation, but here we cannot avoid checking all time windows because isolation of $(C, [a, b])$ does not imply isolation of, say, $(C, [a, b - 1])$ (Function 7).
Function 3: \text{avg-alltime-isolatedSubsets}(C, [a, b], \delta)

1. \forall v \in C : s_v := \max_{i \in [a, b]} \deg_{G_i}(v)
2. \text{d} := |C| - \delta + c - 2 \quad \text{/* By Lemma 8, we can only remove the top d vertices. */}
3. \{v_i | 1 \leq i \leq d\} := \text{the d vertices in } C \text{ with the highest values of } s_v
4. \textbf{D}' \leftarrow \{\emptyset\}
5. \text{result} \leftarrow \emptyset
6. \textbf{while } \textbf{D}' \neq \emptyset \textbf{ do}
7. \textbf{D} \leftarrow \textbf{D}'
8. \textbf{D}' \leftarrow \emptyset
9. \textbf{foreach } D \in \textbf{D} \textbf{ do}
10. \textbf{C}' \leftarrow C \setminus D
11. \text{if } \sum_{v \in \textbf{C}'} s_v \geq |\textbf{C}'| \cdot (|\textbf{C}'| - 1 + c) \text{ then}
12. \quad \text{\textbf{j} := max\{0, i | v_i \in D\}}
13. \quad \textbf{D}' \leftarrow \textbf{D}' \cup \{D \cup \{v_i\} | j < i \leq d\}
14. \text{else}
15. \quad \text{\textbf{result} := \textbf{result} \cup \{C \setminus D\}}
16. \textbf{end}
17. \textbf{end}
18. \textbf{end}
19. \textbf{return result}

Function 4: \text{max-usually-isolatedSubsets}(C, [a, b], \delta)

1. \forall v \in C : s_v := \sum_{i \in [a, b]} \deg_{G_i}(v)
2. \text{\textbf{k} := \lfloor \delta - c + 2 \rfloor \quad \text{/* By Lemma 2, all isolated cliques are at least this large. */}
3. \textbf{while } |C| \geq \textbf{k} \textbf{ do}
4. \quad \text{if } \exists v \in C : s_v \geq (b - a + 1)(|C| - 1 + c) \text{ then}
5. \quad \quad \textbf{C} \leftarrow C \setminus \{v\}
6. \quad \text{\textbf{end}}
7. \textbf{else}
8. \quad \text{\textbf{return } \{C\}}
9. \textbf{end}
10. \textbf{end}
11. \textbf{return } \emptyset

4.3 Correctness

We now show the correctness of our algorithms. We first prove that the \text{isolatedSubsets()} functions (Functions 1 to 5) behave as intended.

Lemma 8. Let \(G = (V, E_1, \ldots, E_T)\) be a temporal graph, \(c \in \mathbb{Q}\), and \(I \in \mathcal{I} \setminus \{\text{usually-max}\}\). Let \(C\) be a clique in \(G\) := \(\bigcap_{i=a}^b G_i\) and \(\delta = \delta_{G_i}(C)\). Then \(I\)-\text{isolatedSubsets}(C, [a, b], \delta) returns all maximal sets \(\tilde{C} \subseteq C\) such that \((\tilde{C}, [a, b])\) is I-\text{c}-isolated.

\textbf{Proof.} For the sake of brevity, we will simply write that some set \(X \subseteq C\) is, say, alltime-avg-isolated to denote that \((X, [a, b])\) is alltime-avg-isolated.

\textbf{Case 1: } I = \text{alltime-avg (Function 1).} Let \(\tilde{C} \subseteq C\) be any maximal subset which is alltime-avg-c-isolated, and suppose the algorithm is currently checking \(C'\) with \(\tilde{C} \subset C' \subseteq C\). Let \(i \in [a, b]\) be the first layer in which \(C'\) is not avg-c-isolated. By Lemma 2 we have that \(|C'| \geq |\tilde{C}| + 1 > \delta(C) - c + 2\), thus the algorithm executes Lines 12 and 13. Note that \(\tilde{C}\) is avg-
Function 5: usually-avg-isolatedSubsets($C$, $[a, b]$, $\delta$)

```plaintext
forall $v \in C : s_v := \sum_{i \in [a, b]} \deg_{C_i}(v)$
d := |C| - $\delta + c - 2$ /* By Lemma 5, we can only remove the top $d$ vertices. */
foreach $v_i \mid 1 \leq i \leq d$ := the $d$ vertices in $C$ with the highest values of $s_v$
$D'$ ← $\{\emptyset\}$
result ← $\emptyset$
while $D' \neq \emptyset$
do
$D ← D'$
$D' ← \emptyset$
foreach $D \in D$
do
$C' ← C \setminus D$
if $\sum_{v \in C} s_v \geq (b - a + 1) \cdot |C'| \cdot (|C'| - 1 + c)$ then
  $j := \max\{0, k \mid v_k \in D\}$
  $D' ← D' \cup \{D \cup \{v_k\} \mid j < k \leq d\}$
else
  result ← result $\cup \{C \setminus D\}$
end
end
end
return result
```

Function 6: $I$-isMaximal($C$, $[a, b]$) - version for $I \in \{\text{alltime-avg, alltime-max, avg-alltime}\}$

```plaintext
for $(a', b') \in \{(a - 1, b), (a, b + 1)\}$ do
  if $(C, [a', b'])$ is isolated clique then
    return false
  end
end
return $I$-isVertexMaximal($C$, $[a, b]$)
```

c-isolated in layer $i$, and let $\tilde{C}' \supseteq \tilde{C}$ be a maximal avg-c-isolated superset. Clearly $\tilde{C} \subseteq \tilde{C}' \subseteq C'$. By Lemma 6, we have that $C' \setminus \tilde{C}'$ is a subset of the set $E$ containing the $d$ highest-degree vertices of $C'$ in layer $i$. Consequently, the algorithm will add some set $C'' \subseteq C'$ with $C'' \supseteq \tilde{C}' \supseteq \tilde{C}$ to $D'$. By recursively applying the same argument to $C''$, we deduce that the algorithm will at some point reach $\tilde{C}$.

Case 2: $I = \text{alltime-max}$ (Function 2). By Lemma 2 and Observation 1 we have that all alltime-max-c-isolated subsets of $C$ have at least size $k$. If $C$ contains an alltime-max-c-isolated subset $\tilde{C}$, then $\tilde{C}$ by definition does not contain any vertex $v$ with $s_v \geq |\tilde{C}| - 1 + c$. Thus, by removing such vertices, we either reach the unique maximal alltime-max-c-isolated subset $\tilde{C}$ of $C$, or, if we reach size $k$, may conclude that no such subset exists.

Case 3: $I = \text{avg-alltime}$ (Function 3). Let $B := \{v_i \mid 1 \leq i \leq d\}$ and let $\tilde{C} \subseteq C$ be any maximal avg-alltime-c-isolated subset. Note that any subset of $C$ is avg-alltime-c-isolated if and only if the same set was avg-c-isolated in a static graph where each vertex’ degree was set to $\max_{i \in [a, b]} \deg_{G_i}(v)$. By applying Lemma 3 to this auxiliary graph, we see that $\tilde{C}$ must contain $C \setminus B$. Thus we observe analogously to Case 1 that the algorithm will at some point reach $\tilde{C}$.

Case 4: $I = \text{max-usually}$ (Function 4). Works analogously to Case 2.

Case 5: $I = \text{usually-avg}$ (Function 5). Let $B := \{v_i \mid 1 \leq i \leq d\}$. By Lemma 5 we
Function 7: $I$-isMaximal($C, [a, b]$) - version for $I \in \{\text{max-usually, usually-avg}\}$

```
1 for $a' = a \ldots 1$ do
2     if $C$ is not a clique in $G_{a'}$ then
3         break
4     end
5 for $b' = b \ldots \tau$ do
6     if $C$ is not a clique in $G_{b'}$ then
7         break the inner loop
8     end
9     if $(C, [a', b'])$ $I$-c-isolated and $(a, b) \neq (a', b')$ then
10       return false
11     end
12     if not $I$-isVertexMaximal($C, [a', b']$) then
13       return false
14     end
15   end
16 end
17 return true
```

Function 8: alltime-avg-isVertexMaximal($C, [a, b]$)

```
1 if the result set contains any $(C', [a, b])$ with $C' \supset C$ then
2     return false
3 end
4 return true
```

have that any maximal usually-avg-$c$-isolated subset of $C$ must contain $C \setminus B$. Note that the loop generates all possible sets $D \subseteq B$ except those, for which $C \setminus B'$ has already found to be usually-avg-$c$-isolated for some $B' \subset B$. Therefore, all maximal usually-avg-$c$-isolated subsets of $C$ are added to the result set.

Next, we prove that the function isVertexMaximal() (Functions 8 and 9) behaves as intended.

**Lemma 9.** Let $G = (V, E_1, \ldots, E_\tau)$ be a temporal graph, let $c \in \mathbb{Q}$, and let $I \in \mathcal{I}\setminus\{\text{usually-max}\}$. Let $(C, [a, b])$ be an $I$-c-isolated clique in $G$. Then $I$-isVertexMaximal($C, [a, b]$) returns true if and only if $(C, [a, b])$ is a vertex-maximal $I$-c-isolated clique.

**Proof.**

**Case 1:** $I = \text{alltime-avg}$ (Function 8). In this case the algorithm simply performs a pairwise comparison of all cliques in this time window and is thus trivially correct.

**Case 2:** $I \in \{\text{alltime-max, avg-alltime, max-usually, usually-avg}\}$ (Function 9). If the algorithm returns false, then it has found a larger $I$-c-isolated clique $(C \cup D, [a, b])$. So suppose conversely that there is $C' \supset C$ for which $(C', [a, b])$ is an $I$-c-isolated clique. Then clearly $C' \subseteq C \cup S$ and thus also $C' \subseteq C \cup D$ for some $D \in \mathcal{D}$. Let $x := |D \setminus C'| < |D|$ and let $X \subseteq D$ be the set of the first $x$ vertices that the algorithm removes from $D$. Then it is not difficult to check for each of the four isolation types in question that $(C \cup D \setminus X, [a, b])$ is at least as $I$-isolated as $(C', [a, b])$. Thus the algorithm will not remove more than $x$ vertices from $D$ and instead return false.

Lastly, we show that the function isMaximal() (Functions 6 and 7) behaves as intended.
Let $c \in \mathbb{Q}$ in the output by Lemma 10.

**Proposition 11**
Now we have all the necessary pieces to prove the correctness of Algorithm 1.

Let $C \in \mathcal{S}$ a maximal clique. So it remains to show that all such cliques are in fact found by the algorithm. To this end, let $(C, [a, b])$ be any maximal $I$-isolated clique in $G_{\cap} = \bigcap_{a \leq t \leq b} G_t$ by Observation 1. Ito and Iwama [12] showed that we then have $C \subseteq C_v$ where $v \in C$ is of minimum degree. Further $|C| \geq |C_v| - k$ by Lemma 2. Thus, $C \subseteq C'$ for some $C' \in C$, so $(C, [a, b])$ is added to the result set by Lemma 8. Finally, $(C, [a, b])$ is also included in the output by Lemma 10.

---

**Lemma 10.** Let $G = (V, E_1, \ldots, E_r)$ be a temporal graph, let $c \in \mathbb{Q}$, and let $I \in \mathcal{I} \setminus \{\text{usually-max}\}$. Let $(C, [a, b])$ be a $I$-c-isolated clique in $G$. Then $I$-isMaximal$(C, [a, b])$ returns true if and only if $(C, [a, b])$ is a maximal $I$-c-isolated clique.

**Proof.** Case 1: $I \in \{\text{alltime-max, alltime-avg, avg-alltime}\}$ (Function 9). By Lemma 9 it only remains to show that the function returns false if there exists an $I$-c-isolated clique $(C', [a', b'])$ with $a' < a$ or $b' > b$. Suppose without loss of generality that $a' < a$. Then $[a - 1, b] \subseteq [a', b']$ and thus $(C', [a - 1, b])$ is $I$-c-isolated.

Case 2: $I \in \{\text{max-usually, usually-avg}\}$ (Function 7). Since the function systematically tries all possible time windows $[a', b'] \subseteq [a, b]$ for which $C$ is a clique, the correctness follows from Lemma 10.

Now we have all the necessary pieces to prove the correctness of Algorithm 1.

**Proposition 11 (Correctness of Algorithm 1).** Let $G = (V, E_1, \ldots, E_r)$ be a temporal graph, let $c \in \mathbb{Q}$, and let $I \in \mathcal{I} \setminus \{\text{usually-max}\}$. Then, Algorithm 7 outputs exactly all maximal $I$-c-isolated temporal cliques.

**Proof.** By Lemma 8 and Lemma 10 every element of the output is in fact a maximal $I$-c-isolated clique. So it remains to show that all such cliques are in fact found by the algorithm. To this end, let $(C, [a, b])$ be any maximal $I$-c-isolated clique. Then, $C$ is an avg-c-isolated clique in $G_{\cap} = \bigcap_{a \leq t \leq b} G_t$ by Observation 1. Ito and Iwama [12] showed that we then have $C \subseteq C_v$ where $v \in C$ is of minimum degree. Further $|C| \geq |C_v| - k$ by Lemma 2. Thus, $C \subseteq C'$ for some $C' \in C$, so $(C, [a, b])$ is added to the result set by Lemma 8. Finally, $(C, [a, b])$ is also included in the output by Lemma 10.
4.4 Running Time Analysis

We will now estimate the time complexity of the different algorithms in terms of $\tau$, $|V|$, $|E|$, and the isolation parameter $c$.

We start estimating the running time of Algorithm 1 in terms of $T_{\text{isolatedSubsets}()}^{(I)}$ and $T_{\text{isMaximal}()}^{(I)}$, which shall denote the running times of the $I$-isolatedSubsets() and $I$-isMaximal() subroutines, respectively, since they are the parts of the running time that depend on the isolation type.

**Lemma 12.** Let $G = (V, E_1, \ldots, E_\tau)$ be a temporal graph, let $c \in \mathbb{Q}$, and let $I \in \mathcal{I} \setminus \{\text{usually-max}\}$. Algorithm 1 runs in $O \left( 2^c c^2 \tau \cdot |E| + T_{\text{isolatedSubsets}()}^{(I)} + T_{\text{isMaximal}()}^{(I)} \right)$ time.

**Proof.** We first investigate the running time of the first part of Algorithm 1 (the part up until Line 10). When iterating over all time windows $[a, b)$, each iteration of the inner loop extends the time window by one layer only. Because of this, we can compute the intersection graphs in $O(\tau \cdot |E|)$ time overall by using incremental updates.

All sorting steps of the algorithm can be done by bucket sort using constant time per time window and pivot vertex. Computing $C_v$ for all vertices of $G_v$ takes $O(c^3 \cdot |E_v|)$ time [12, Lemmata 3.9 and 3.13]. Computing $C$ from $C_v$ takes $O(|E(C_v)| + c \cdot |V| + 2c^2)$ time [13, Proof of Proposition 1]. The overall number of steps (not counting isolatedSubsets() and isMaximal()) is thus $O \left( \tau \cdot |E| + \sum_v \sum_a \left( c^3 \cdot |E_v| + \sum_{\text{pivot} v} (|E(C_v)| + c \cdot |V| + 2c^2) \right) \right)$.

Note that $\sum_v |E_v| \leq \sum_v |E_a| \leq |E|$. Further, we assume that the algorithm is implemented to disregard vertices that have degree zero in $G_a$ and thus also in $\bigcap_{t=a}^b G_t$. Because of this assumption, we can also record the following observation. If we sum $\deg_{G_a}(v) + 1$ over all time windows $[a, b]$ and pivot vertices $v$, then the result is at most $\sum_v \sum_a \sum_{\text{pivot} v} (\deg_{G_a}(v) + 1) \in O(\sum_v \sum_a |E_v|) \subseteq O(\tau \cdot |E|)$ by the handshake lemma. Of course, the same estimation is valid when summing over any of the following: $1 \leq |C| \leq |C_v| \leq \deg_{G_a}(v) + 1 \leq \deg_{G_a}(v) + 1$.

Another key observation is that $\sum_v |E(C_v)| \in O(c^3 \cdot |E_v|)$ [12, Lemma 3.13]. Using this, if we sum $|E(C_v)|$ over all time windows and pivot vertices, we get at most $\sum_v \sum_a \sum_{\text{pivot} v} |E(C_v)| \in O(\tau \sum_v c^3 \cdot |E_v|) \subseteq O(c^3 \tau \cdot |E|)$. Again, this also applies to $|E(C)| \leq |E(C_v)|$.

Employing these observations, the above running time can be bounded by $O(\tau \cdot |E| + c^3 \tau \cdot |E| + c^2 \tau \cdot |E| + 2c^2 \tau \cdot |E|) \subseteq O(2^c c^2 \tau \cdot |E|)$. □

Now we analyze the running time $T_{\text{isolatedSubsets}()}^{(I)}$ of the $I$-isolatedSubsets() subroutine (Functions 1 to 5) depending on the isolation type $I$.

**Lemma 13.** $T_{\text{isolatedSubsets}()}^{(I)} \in \begin{cases} O(2^c c^2 \tau \cdot |E|) & \text{if } I \in \{\text{alltime-max, max-usually}\}, \\
O(2^c c^2 \tau \cdot |E|) & \text{if } I \in \{\text{usually-avg, avg-alltime}\}, \\
O(c^2 \tau^2 \cdot |E|) & \text{if } I \in \{\text{alltime-avg}\}. \end{cases}$

**Proof.** Keep in mind the observations made in the proof of Lemma 12, which are also useful here. Additionally, note that within any iteration of Algorithm 1, $C$ contains at most $2^{c |C_v| - s}$ elements of size $s$ for each $k \leq s \leq |C_v|$ and at most $2^{c |C_v| - k} \leq 2^c$ elements overall.
Case 1: \(I \in \{\text{alltime-max, max-usually}\}\). Computing \(s_v\) takes \(O(\tau \cdot |E|)\) overall (again, using incremental updating between time windows).

For each call, the loop runs at most \(|C| - k \leq |C| - (\delta + 1) + c \leq c\) times, each needing \(O(|C|)\) time. Since there are \(|C|\) calls per time window and pivot, the overall time is in \(O(\tau \cdot |E| + 2^c c \tau \cdot |E|) \subseteq O(2^c c \tau \cdot |E|)\).

\[
\sum_s 2^{|C_v| - s}d^2 \tau \leq O \left( \sum_s 2^{|C_v| - s}c^s + c - C_v - 1 \right) \leq O \left( \sum_s c^{c - 1} \tau \right) \subseteq O(2^c c) \tau
\]
time per time window and pivot, giving at most \(O(c^2 \tau \cdot |E|)\) time overall.

Case 3: \(I \in \{\text{usually-avg, avg-alltime}\}\). Computing \(s_v\) again takes \(O(\tau \cdot |E|)\) time overall.

Here, there are \(2^d\) possible options for \(D\), each tested in constant time. Thus, the loop needs

\[
\sum_s 2^{|C_v| - s}2^d \leq O \left( \sum_s 2^{|C_v| - s}2^s + c - C_v - 1 \right) \leq O \left( \sum_s 2^c \right) \subseteq O(2^c)
\]
time per time window and pivot (again \(s = |C|\)). In total, this gives \(O(2^c c \tau \cdot |E|)\).

Finally, we analyze the running time \(T^{(f)}_{\text{isMaximal}}\) of the \(\text{isMaximal}\) subroutine (Functions 6 and 7) depending on the isolation type.

**Lemma 14.** \(T^{(f)}_{\text{isMaximal}} \in \begin{cases} 
O(2.89^c \tau \cdot |E|) & \text{if } I = \text{alltime-max}, \\
O(2.89^c \tau^3 \cdot |E|) & \text{if } I = \text{max-usually}, \\
O(2^c \cdot |V| \cdot |E|) & \text{if } I = \text{alltime-avg}, \\
O(5.78^c \tau \cdot |E|) & \text{if } I = \text{avg-alltime}, \\
O(5.78^c \tau^3 \cdot |E|) & \text{if } I = \text{usually-avg}.
\end{cases}\)

**Proof.** Case 1: \(I = \text{alltime-max}\). Each call to \(\text{isolatedSubsets}\) returns at most one clique, thus for each time window and pivot \(v\) there are at most \(|C| \leq 2^c\) cliques to be checked. Each call to isMaximal takes \(O(|C|)\) time, in addition to one call to isVertexMaximal.

Regarding isVertexMaximal, the size of \(S \subseteq N(v) \setminus C\) is at most \(\deg(v) + 1 - |C| < c\) and finding it takes \(c \cdot |C|\) time. Computing \(D\) takes \(O(3^c/3)\) time \([17]\) and \(D\) has size at most \(3^c/3\). For each \(D \in D\) we need \(O(|D|) \subseteq O(c)\) time.

Altogether, each call to isMaximal takes \(O(c \cdot |C| + 3^c/3 c)\) time, giving an overall running time of \(O(2^c c \tau \cdot |E| + 2^c 3^c/3 c \cdot |E|) \subseteq O(2.89^c c \tau \cdot |E|)\).

Case 2: \(I = \text{max-usually}\). Again at most \(2^c\) cliques need to be checked for each time window and pivot \(v\).

For each call we need \(O(\tau \cdot |E(C)|)\) to determine the layers where \(C\) is a clique and \(O(\tau^2 \cdot |C|)\) for the isolation check. Further, there are \(\tau^2\) calls to isVertexMaximal. Of these, each takes \(O(3^c/3 c)\) time as for alltime-max-isolation.
Thus the total time per call is $O(\tau|E(C)| + \tau^2|C| + 3^{c/3}c\tau^2)$, giving an overall time bound of $O(2^{c^2\tau^2} \cdot |E| + 2^{\tau^3/2} \cdot |E| + 2^{c^3/3}c^{\tau^3} \cdot |E|) \subseteq O(2.89\cdot |E|)$.  

**Case 3:** $I = \text{alltime-avg}$. Each call to $\text{isolatedSubsets()}$ returns at most $2^d \leq 2^c$ cliques, therefore there are at most $2^{2c}$ cliques to be checked per time window and pivot. Each call to $\text{isMaximal}$ takes $O(|C|)$ time, in addition to one call to $\text{isVertexMaximal}$.  

Within $\text{isVertexMaximal}$, we only need to check against cliques for the same time window, of these there are at most $\sum v |C| \cdot |C_v|$ many, each of size at most $|C_v|$. The time needed to check a clique for maximality is linear in the total size of this set, i.e. $O(\sum v |C| \cdot |C_v|) \subseteq O(|C| \cdot |E_v|)$.  

In total, each call to $\text{isMaximal}$ takes $O(|C| \cdot |E_v|)$ time, and the total time taken is thus $O(2^{2c} \cdot |V| \cdot |E|)$.  

**Case 4:** $I = \text{usually-avg}$. Each call to $\text{isolatedSubsets()}$ returns at most $2^d \leq 2^c$ cliques. Apart from this extra factor, the analysis is identical to the max-usually-isolation case. Thus the total time is $O(2.89\cdot 2^{c^2}c\tau^3 \cdot |E|) \subseteq O(5.78\cdot |E|)$.  

**Case 5:** $I = \text{avg-alltime}$. Each call to $\text{isolatedSubsets()}$ returns at most $2^d \leq 2^c$ cliques. Apart from this extra factor, the analysis is identical to the alltime-max-isolation case. Thus the total time is $O(2.89\cdot 2^{c^2}c\tau^3 \cdot |E|) \subseteq O(5.78\cdot c\tau \cdot |E|)$.  

Now it is straightforward to check that Lemmas 13 and 14 together with Lemma 12 imply the running times given in Table 1. This together with Proposition 11 completes the proof of Theorem 7.

5 Experimental Evaluation

In this section, we empirically evaluate the running times of our enumeration algorithms for maximal isolated temporal cliques (Algorithm 1) on several real-world temporal graphs. In particular, we investigate the effect of different isolation concepts as well as different values for isolation parameter $c$ and $\Delta$ (see the definition of $\Delta$-cliques in Section 2) on the running time and on the number of cliques that are enumerated. We also draw some comparisons concerning running times to a state-of-the-art algorithm to enumerate maximal (non-isolated) temporal cliques by Bentert et al. [1].

5.1 Setup and Statistics

We implemented our algorithms\(^\text{5}\) in Python 3.6.8 and carried out experiments on an Intel Xeon E5-1620 computer clocked at 3.6 GHz and with 64 GB RAM running Debian GNU/Linux 6.0. The given times refer to single-threaded computation. Bentert et al. [1] implemented their algorithm in Python 2.7.12.  

For the sake of comparability we tested our implementation on four freely available data sets, three of which were also used by Bentert et al. [1]:

- Face-to-face contacts between high school students (“highschool-2011”, “highschool-2012”, “highschool-2013”) [6, 7, 16],

\(^5\)The code of our implementation is freely available at https://www.akt.tu-berlin.de/menue/software/
Table 2: Statistics for the data sets used in our experiments. The lifetime $\tau$ of a graph is the difference between the largest and smallest time stamp on an edge in the graph. The resolution $r$ indicates how often edges were measured.

| Data Set          | # Vertices $|V|$ | # Edges $|\mathcal{E}|$ | Resolution $r$ (in s) | Lifetime $\tau$ (in s) |
|-------------------|--------------|------------------------|-----------------------|------------------------|
| highschool-2011   | 126          | 28,560                 | 20                    | 272,330                |
| highschool-2012   | 180          | 45,047                 | 20                    | 729,500                |
| highschool-2013   | 327          | 188,508                | 20                    | 363,560                |
| tij_pres_LH10     | 73           | 150,126                | 20                    | 259,180                |

- Spatial proximity between persons in a hospital (“tij_pres_LH10” [8]).

We list the most important statistics of the data set in Table 2. We chose five roughly exponentially increasing values $\varepsilon$, 1, 5, 25, 125 for the isolation parameter $c$, where $\varepsilon := 0.001$ effectively requires complete isolation and $125 \approx |V|$ imposes little or no restriction. We chose our $\Delta$-values in the same fashion as Bentert et al. [1]. In order to limit the influence of time scales in the data and to make running times comparable between instances, the chosen $\Delta$-values of 0, $5^3$, and $5^5$ were scaled by $L/(5 \cdot |\mathcal{E}|)$, where $L$ is the temporal graph’s lifetime in seconds [8, Section 5.1].

5.2 Experimental Results

In Figs. 1 to 4 the number of maximal isolated temporal cliques and the running time are plotted for each of the five isolation types and a range of isolation values $c$. Missing values indicate that the respective instance exceeded the time limit of 1 hour. In general, the different isolation types produce surprisingly similar output. This suggests that the degrees of the vertices forming an isolated temporal clique are typically rather similar and remain constant over the lifetime of the clique. Unsurprisingly, raising the value of $c$ increases the number of cliques as the isolation restriction is weakened. However, this effect ceases roughly at $c = 5$. Increasing $c$ further does not produce additional cliques, suggesting that the vertices in temporal cliques we found in the data sets mostly have out-degree at most five. Furthermore, we can generally observe that the number of cliques decreases with increasing values of $\Delta$, which might seem unexpected at first glance, but is a consequence from finding many small cliques (with both few vertices and short time intervals) for small $\Delta$-values that “merge together” for larger $\Delta$-values. This behavior is consistent across all data sets we investigated.

Regarding running time, our algorithm is generally slower than the non-isolated clique enumeration algorithm by Bentert et al. [1], even for small values of $c$. For comparison, the algorithm by Bentert et al. [1] solved the instances “highschool-2011”, “highschool-2012”, and “highschool-2013” for the same values for $\Delta$ that we considered in less than 17 Seconds per instance. We believe that the two main reasons for our algorithm to be slower are the following. On the one hand, the maximality check we perform is much more complicated than the one of the algorithm of Bentert et al. [1], which is an issue that also occurs in the static case [11, 13]. On the other hand, we have to explicitly iterate through more or less all possible intervals in which we could find an isolated temporal clique, which seems unavoidable in our setting. A particular consequence of this is that our algorithm is not output sensitive, that is, the running time can be much larger than the number of maximal isolated temporal cliques in the input graph. In the case of (non-isolated) temporal clique enumeration, there are ways to circumvent these issues and in particular, the algorithm of Bentert et al. [1] is output sensitive. Both algorithms have
a similar running time behavior with respect to $\Delta$, that is, the running time increases with $\Delta$, once $\Delta$ reaches moderately large values. Since higher values of $\Delta$ create a more dense graph after the preprocessing step, this behavior is expected. The algorithm of Bentert et al. [1] is slow for very small values of $\Delta$ that are close to zero (compared to itself for larger values of $\Delta$). We do not observe this phenomenon in most of our algorithms. In the variants for max-usually and usually-avg, however, we experience a similar issue for small values of $c$, especially visible in the “tij_pres_LH10” data set (Fig. 4), where the running time is surprisingly high for $\Delta = 0$ and $c = \varepsilon$. A possible explanation is that the usually-variants use a different maximality check than the alltime-variants, which may be the reason for this behavior. Interestingly, no universal trend arises for the running time taken per resulting clique with respect to $c$, which stands in contrast to our theoretical running time analysis.

To get a more fine-grained picture, we tested intermediate values for $\Delta$ and $c$ on the “tij_pres_LH10” data for avg-alltime-isolation. The results are shown in Fig. 5 and Fig. 6. For increasing values of $\Delta$, the number of cliques drops while the running time per clique rises. For fixed $\Delta$ and increasing $c$, the situation is very different. Here, both number of cliques and running time per clique quickly rise and subsequently level off around $c = 5$.

6 Conclusion

We have lifted the concept of isolation from the static to the temporal setting, introducing six different types of temporal isolation. For five out of those we developed algorithms and showed that enumerating maximal temporally isolated cliques is fixed-parameter tractable with respect to the isolation parameter. This leaves one case (usually-max-isolation) open for future research.

From an algorithm engineering perspective there is still room for improvement. So far the practical running times make it hard to analyze larger data sets as done for example by Bentert et al. [1]. Another possibility to approach this issue it to shift focus from the enumeration of all maximal temporally isolated cliques to the “detection” problem, that is, to “only” search for one large temporally isolated cliques (if one exists). Depending on the application, this might still be a task worth investigating. It could allow for better heuristic improvement such as pruning rules that remove parts of the input in which large cliques can be ruled out.

Finally, as in the static case, it would be natural to apply the isolation concepts to further community models such as for example temporal $k$-plexes [1].

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Figure 1: Plot for the data set “highschool-2011” showing the number of cliques (top) and the computing time per clique (bottom) for the different temporal isolation types and different values of $c$ and $\Delta$. The different $\Delta$-values are visualized by the different markers, with circles, triangles and squares denoting values of 0, $5^3$, and $5^5$ respectively.

Figure 2: Plot for the data set “highschool-2012” (see also description of Fig. 1).
Figure 3: Plot for the data set “highschool-2013” (see also description of Fig. 1).

Figure 4: Plot for the data set “tij_pres_LH10” (see also description of Fig. 1).
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