THE APPROXIMATE SOLUTION FOR BENJAMIN-BONA-MAHONY EQUATION UNDER SLOWLY VARYING MEDIUM

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Abstract. In this paper, we investigate the soliton dynamics under slowly varying medium for the BBM equation, that is how the solution of this equation evolve when the time goes. We construct the approximate solution of this equation and prove that the error term due to the approximate solution can be controlled. By using the method of Lyapunov and Weinstein functions, we prove that the approximate solution is stable.

1. Introduction. In this work, we consider the following BBM equation under slowly varying medium

\[(1 - \lambda \partial_x^2)u_t + (u_{xx} - u + a_x u^2)_x = 0, \quad (t, x) \in \mathbb{R}_t \times \mathbb{R}_x, \]  \tag{1.1}

here \(u = u(t, x)\) is a real-valued function and \(\lambda \in (0, 1)\) is a constant. Concerning slowly varying medium \(a_x = a(\varepsilon x)\), we always assume that there exist positive constants \(K\) and \(\gamma\) such that

\[
\begin{cases}
1 < a(r) < 2, & a'(r) > 0, \forall r \in \mathbb{R} \\
0 < a(r) - 1 < Ke^{\gamma r}, & \forall r \leq 0 \\
0 < 2 - a(r) < Ke^{-\gamma r}, & \forall r > 0
\end{cases} \tag{1.2}
\]

Obviously, it is inferred from (1.2) that \(\lim_{r \to -\infty} a(r) = 1\) and \(\lim_{r \to +\infty} a(r) = 2\).

Many relevant works have been done. Zabusky and Kruskal in 1965 [29], Fermi, Pasta and Ulam in 1974 [5] had research on numerical work of the KdV equation. After this work in 1987, Le Veque [12] concluded interaction of nearly equal solitons in the KdV equation. Martel, Merle [16, 17] discussed inelastic interaction of nearly equal solitons and description of two soliton collision for the quartic gKdV equation. At the same time, there exist another papers of the BBM equation. Benjamin, Bona and Mahony [2] made the study of the regularized long wave range equation. Bona in

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1980 [3] got the solitary-wave interaction for this equation. In 1987, Bona, Strauss [4] studied stability and instability of solitary waves of Korteweg-de Vries type. Martel, Merle also made contributions to the BBM equation. Inelastic interaction of nearly equal solitons for the BBM equation was discovered by Martel, Merle [4, 14, 15].

Weinstein [26, 27, 18] considered the Lyapunov stability of ground states of nonlinear dispersive evolution equations and existence and dynamic stability of solitary wave solutions of equations arising in long wave propagations. Of course, Mizumachi [19] got the conclusion of asymptotic stability of solitary wave solutions to the regularized long wave equation.

With the development of nonlinear science, the study of perturbation of integrable equations is occurred. Asano [1] studied wave propagation in non-uniform media in 1974. Karpman and Maslov [9], Kaup and Newell [10] considered the study of perturbed (in time) integrable equations, in particular, they considered the perturbed (in time \( \tau \)) gKdV equation

\[ u_x + (\beta(\varepsilon \tau) u_{xx} + \alpha(\varepsilon \tau) u^m)_x = 0, \quad m = 2, 3, \quad \alpha, \beta > 0. \quad (1.3) \]

The author found that this equation is not conserved anymore, after the transformation

\[ t = \int_0^T \beta(\varepsilon s) ds, \quad \tilde{u}(t, x) = (\alpha/\beta)^{1/2} (\varepsilon \tau) u(\tau, x), \]

\[ \tilde{u}_t + (\tilde{u}_{xx} + \tilde{u}^m)_x = \varepsilon \gamma(ct) \tilde{u} \text{ where } \varepsilon \gamma(ct) = \frac{1}{m-1} \partial_t \log\left(\frac{\alpha}{\beta}\right)(\varepsilon \tau(t)). \quad (1.4) \]

Ko and Kuehl [11] researched the different integrable model with the problem of the lack of energy conservation. Grimshaw [6, 7, 8] took the study of slowly varying solitary waves for the KdV and Schrödinger equations. Muñoz [20, 21, 22, 23, 24] studied the soliton dynamics under slowly varying medium for nonlinear Schrödinger equations and gKdV equation.

We can easily see the similar character between nonlinear Schrödinger equation and gKdV equation: complete integrability. So we have the idea of applying this method into BBM equation. We can do the researches of the dynamics, stabilities, the long behaviors and the existence of pure soliton solution.

Note that, we take the limit of the (1.1),

\[ (1 - \lambda \partial_x^2) u_t + (u_{xx} - u + 1u^2)_x = 0, \quad \text{where } x \to -\infty. \]

\[ (1 - \lambda \partial_x^2) u_t + (u_{xx} - u + 2u^2)_x = 0, \quad \text{where } x \to +\infty. \]

We take the \( T_\varepsilon = \varepsilon^{-\frac{1}{10}} \) and discuss the different behaviors of the region of the time.

**Theorem 1.1** (Dynamics). For all \( 0 < \varepsilon < \varepsilon_0, T_\varepsilon = \varepsilon^{-\frac{1}{10}}, \) \( \rho(t) = \int_{-T_\varepsilon}^t c(\varepsilon s) ds \) \( - T_\varepsilon, \) there exists a small constant \( \varepsilon_0 > 0, \) we have

1. **Existence of soliton-like solution.** There exists a solution \( u \in C(\mathbb{R}, H^1(\mathbb{R})) \) of (1.1) global in time, such that

\[ \lim_{t \to -\infty} \|u(t) - Q_c(x - ct)\|_{H^1(\mathbb{R})} = 0. \quad (1.5) \]

Specially, \( t = -T_\varepsilon \) be satisfying \( \|u(-T_\varepsilon) - Q_c(T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10}. \)

2. **Soliton potential under slowly varying medium.** There exist \( K > 0, \) \( \bar{T}_\varepsilon \in \mathbb{R} \) such that

\[ \|u(\bar{T}_\varepsilon) - \frac{1}{2} Q_c(x - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1}, \quad (1.6) \]
where
\[ |T_c - \tilde{T}_c| < \frac{T_e}{100}, \quad c_\infty = c_{|t=\infty}, \quad \text{with } \rho(t) = \int_{-T_c}^{t} c(\varepsilon s) ds - T_c, \]
and \( Q_c \) is the solution of \((1 + \lambda c)Q''_c + Q^2_c = (1 + c)Q_c.\)

**Theorem 1.2** (Stability). There exist some constants \( K, c_2 > 0 \) and the \( C^1 \)-function \( \rho_2(t) \) defined on \([T_c, +\infty)\) satisfying \( \rho_2(t) = c_2, \) we have
\[ w^+ = u(t) - \frac{1}{2}Q_{c_2}(x - \rho_2(t)), \]
for all \( t \in [T_c, +\infty), \) then
\[ \|w^+\|_{H^1(\mathbb{R})} + |c_2 - c_\infty| \leq K\varepsilon^{\frac{1}{2}}. \] (1.7)

2. Preliminaries. We write the transformation from the classic BBM equation
\[ (1 - \partial_x^2)u_t + (u + u^2)_x = 0, \quad t, x \in \mathbb{R}, \]
and get the so-called BBM equation as follows
\[ (1 - \lambda_0 t^2)u_t + (u_{xx} - u + u^2)_x = 0, \quad (t, x) \in \mathbb{R}_t \times \mathbb{R}_x, \quad \lambda \in (0, 1). \]
Note that solutions for the equation (2.1) are the form
\[ u(t, x) = Q_c(x - ct), \quad Q_c = (1 + c)Q(\sqrt{\frac{1 + c}{1 + \lambda c}}), \]
where the parameter \( c > 0 \) describes the wave speed of the soliton, and
\[ Q(x) = \frac{3}{2} \cosh^{-2} \left( \frac{x}{2} \right) \text{ solves } Q'' + Q^2 = Q. \]

Let \( f : \mathbb{R}^n \times \mathbb{R}^m, U \times V \subset \mathbb{R}^n \times \mathbb{R}^m \) is a neighbourhood of \((x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m.\)
Suppose that \( f \) is continuous in the partial of \( \partial f / \partial y \) of \( U \times V, \) let \( f(x_0, y_0) = 0; \]
\([\det(\partial f / \partial y)](x_0, y_0) \neq 0.\) There exist a neighbourhood \( U_0 \times V_0 \subset U \times V \) and a unique continuous function \( \phi : U_0 \to V_0 \) satisfy:
\( f(x, \phi(x)) = 0, \quad x_0 \in U_0, \quad \text{and } \phi(x_0) = y_0. \)

Next, we recall the numerical character of soliton \( Q \) and operator \( L.\)
For \( Q_c = (1 + c)Q(\sqrt{\frac{1 + c}{1 + \lambda c}}x), \) we have
(i)
\[ (1 + \lambda c)Q''_c + Q^2_c = (1 + c)Q_c. \] (2.5)
\[ AQ_c = (Q_c(x))'' = \frac{1}{1 + c} \left[ Q_c(x) + \frac{1 - \lambda}{2(1 + \lambda c)} x(Q_c(x))' \right]. \] (2.6)
\[ \int_{\mathbb{R}} Q''_c = (1 + c)^{\frac{1}{2}} (1 + \lambda c)^{\frac{1}{2}} \int_{\mathbb{R}} Q^2, \quad \int_{\mathbb{R}} Q^2 = 6. \] (2.7)
\[ \int_{\mathbb{R}} (Q'_c)^2 = (1 + c)^{\frac{1}{2}} (1 + \lambda c)^{-\frac{1}{2}} \int_{\mathbb{R}} (Q')^2 . \] (2.8)
\[ r \int_{\mathbb{R}} Q'' = \frac{2r + 1}{3} \int_{\mathbb{R}} Q'^{r+1}, \quad r \int_{\mathbb{R}} Q'^r = \frac{2r + 1}{3(1 + c)} \int_{\mathbb{R}} Q''^{r+1}. \] (2.9)
\[ \int_{\mathbb{R}} AQ_c = \left[ \frac{1}{1 + c} - \frac{1 - \lambda}{2 (1 + \lambda c)(1 + c)} \right] \int_{\mathbb{R}} Q_c. \] (2.10)
(ii) Set
\[ \phi(x) = - \frac{Q'(x)}{Q(x)}, \quad \phi_c(x) = - \frac{Q'_c(x)}{Q_c(x)} = \sqrt{\frac{1 + c}{1 + \lambda c}} \phi(\sqrt{\frac{1 + c}{1 + \lambda c}}x). \] (2.11)
then
\[
\lim_{x \to -\infty} \phi_c = -\sqrt{\frac{1+c}{1+\lambda c}}, \quad \lim_{x \to +\infty} \phi_c = \sqrt{\frac{1+c}{1+\lambda c}}. \tag{2.12}
\]

Let
\[
Lf = -(1+\lambda c)f'' + (1+c)f - 2Q_c f, \tag{2.13}
\]
then the operator \(L\) satisfies the following two results:
(i) the kernel of \(L\) is spawned by \(Q_c\);
(ii) for all \(h = h(y) \in L^2(\mathbb{R})\) such that \(\int_{\mathbb{R}} hQ_c' = 0\), there exists a unique \(\hat{h} \in H^2(\mathbb{R})\) such that \(\int_{\mathbb{R}} \hat{h}Q_c' = 0\) and \(L\hat{h} = h\). Moreover, if \(h\) is even (resp.odd), then \(\hat{h}\) is even(resp.odd).

Finally, we introduce the definition of (IP) property \(Y\). We say that \(A_c(\varepsilon t, x)\) satisfies the (\(Y\)) property if and only if:
(i) any spatial derivative of \(A_c(\varepsilon t, x)\) is a localized \(Y\)-function;
(ii) there exist \(K, \gamma > 0\) such that \(\|A_c(\varepsilon t, x)\|_{L^\infty(\mathbb{R})} \leq Ke^{-\gamma |t|}\) for all \(t \in \mathbb{R}\).

\(Y\)-function means the set of functions \(f \in C^\infty(\mathbb{R}, \mathbb{R})\) such that
\[
\forall j \in \mathbb{N}, \exists C_j, c_j > 0, \forall x \in \mathbb{R}, |f^{(j)}(x)| \leq C_j (1 + |x|)^{c_j} e^{-|x|}. \tag{2.14}
\]

Due to the definition of (IP), we get \(Q(x) \in Y\). And if \(f_1, f_2 \in Y\),
\[
\int_{\mathbb{R}} Lf_1 f_2 = \int_{\mathbb{R}} Lf_2 f_1, \quad \int_{\mathbb{R}} (1 - \lambda \partial_x^2)f_1 f_2 = \int_{\mathbb{R}} (1 - \lambda \partial_x^2)f_2 f_1. \tag{2.15}
\]

3. Construction of a soliton-like solution. In this section, we construct the approximate solution on the interval \([-T_\varepsilon, T_\varepsilon]\) for the generalized BBM equation under slowly varying medium and discuss the estimate due to the error term of the constructed approximate solution. Via binomial theorem, integrable estimate and cut-off function, we can prove that the error term can be controlled by \(O(\varepsilon^{\frac{3}{2}}e^{-\gamma |t|})\) in \(H^2(\mathbb{R})\). Moreover, the integration of the error term is also controlled.

3.1. Decomposition of the approximate solution. Let
\[
T_\varepsilon = \varepsilon^{-\frac{1}{\min}}, \tag{3.1}
\]
we look for the \(\tilde{u}(t, x)\), the approximate solution for (1.1) on the interval of time \([-T_\varepsilon, T_\varepsilon]\),
\[
y = x - \rho(t), \quad R = \frac{Q_c(x)}{a(\varepsilon \rho(t))}, \tag{3.2}
\]
where
\[
Q_c(x, t) = (1+c)Q\left(\sqrt{\frac{1+c}{1+\lambda c}}(x - \rho(t))\right), \quad \rho(t) = \int_{-T_\varepsilon}^t c(\varepsilon t)ds - T_\varepsilon. \tag{3.3}
\]

The form of \(\tilde{u}(t, x)\) will be the sum of the soliton plus error term:
\[
\tilde{u} = R + w = R + \varepsilon A_c(\varepsilon t, y), \tag{3.4}
\]
where \(A_c(\varepsilon t, x)\) satisfies (IP) property, we want to measure the size of error term produced by defining \(\tilde{u}(t, x)\) in (3.4) about the equation (1.1). Let
\[
S[\tilde{u}] = (1 - \lambda \partial_x^2)\tilde{u}_t + (\tilde{u}_{xx} - \tilde{u} + a_\varepsilon \tilde{u}_x^2)_x, \tag{3.5}
\]
we can get the following results.
**Proposition 3.1** For $\forall t \in [-T_c, T_c]$, the nonlinear decomposition of the error term $S[\hat{u}]$ holds:

$$
S[\hat{u}] = \varepsilon^2 \left[ \frac{a''}{2a^2} (y^2 Q_c^2)_y + \frac{2a'}{a} (yA_c Q_c)_x + (1 - \lambda \partial_x^2)(\Lambda A_c c' + (A_c)_t) \right]
+ \varepsilon [F - (LA_c)_y] + O(\varepsilon^2 e^{-\gamma|t|}),
$$

(3.6)

where

$$
F = (1 - \lambda \partial_x^2) \frac{c'}{a} \Lambda Q_c - (1 - \lambda \partial_x^2) \frac{a'}{a^2} \varepsilon Q_c + \frac{a'}{a^2} (yQ_c^2)_y.
$$

(3.7)

**Proof.** This proposition is proved explicitly in the following four Lemmas. \(\square\)

**Lemma 3.1.** Set

$$
S[\hat{u}] = I + II + III.
$$

where

$$
I = S[R] = (1 - \lambda \partial_x^2)R_t + (R_{xx} - R + a_x R^2)_x,
$$

$$
II = (1 - \lambda \partial_x^2)w_t + (w_{xx} - w + 2a_x w R)_x,
$$

$$
III = \{a_x w^2\}_x.
$$

**Proof.** Recall $\hat{u} = R + w$ and this lemma is just proved by the binomial theorem. \(\square\)

**Lemma 3.2.** Decomposition of $I$

$$
I = \varepsilon^2 \left[ \frac{c'}{a} (1 - \lambda \partial_x^2) \Lambda Q_c - \frac{a'c}{a^2} (1 - \lambda \partial_x^2) Q_c + \frac{a'}{a^2} (yQ_c^2)_x \right]
+ \varepsilon^2 \frac{a''}{2a^2} (y^2 Q_c^2)_x + O_{H^2(\mathbb{R})}(\varepsilon^3).
$$

**Proof.** By $y = x - \rho(t)$, $R = \frac{Q_c(x)}{a(\varepsilon \rho(t))}$ and $\partial_t \rho(t) = c(\varepsilon t)$, we have

$$
I = (1 - \lambda \partial_x^2)R_t + (R_{xx} - R + a_x R^2)_x
= (1 - \lambda \partial_x^2) \frac{(\Lambda Q_c c' \varepsilon - Q'_c a - Q_c a' \varepsilon c)}{a^2} + \frac{1}{a} Q_{e''} - \frac{Q'_c}{a} + \frac{1}{a^2} (a(\varepsilon x)Q_c^2)_x,
$$

via a Taylor expansion

$$
(a(\varepsilon x)Q_c^2)_x = a(\varepsilon \rho(t))(Q_c^2)_x + \varepsilon a'(\varepsilon \rho(t))(yQ_c^2)_x + \frac{1}{2} \varepsilon^2 a''(\varepsilon \rho(t))(y^2 Q_c^2)_x
+ \frac{1}{6} \varepsilon^3 a'''(\varepsilon \rho(t) + \theta y)(y^3 Q_c^2)_x.
$$

In the term of $a'''(\varepsilon \rho(t) + \theta y)(y^3 Q_c^2)_x$, there exists a constant $k$ such that $|a'''| \leq k$, $(y^3 Q_c^2)_x \in Y$, so

$$
(a(\varepsilon x)Q_c^2)_x = a(\varepsilon \rho(t))(Q_c^2)_x + \varepsilon a'(\varepsilon \rho(t))(yQ_c^2)_x
+ \frac{1}{2} \varepsilon^2 a''(\varepsilon \rho(t))(y^2 Q_c^2)_x + O_{H^2(\mathbb{R})}(\varepsilon^3).
$$
Use the same method just like Lemma 3.2, by Taylor expansion.

Lemma 3.4.

\[ I = \frac{(\Lambda Q_x c')^2}{a^2} - Q_x c' a'' c - \lambda \frac{(\Lambda Q_x c')^2}{a} - \frac{Q'' c}{a} a - \frac{Q'' c'}{a} c + \frac{1}{a} Q'' \\
- \frac{Q'}{a} + \frac{1}{a^2} a(Q'_x c_x) + \varepsilon a' (y Q'_c c_x) + \frac{1}{2} \varepsilon^2 a'' (y^2 Q'_c c_x) + O_{H^2(\mathbb{R})}(\varepsilon^3) \]

\[ = \frac{\varepsilon(1 - \lambda \partial_x^2)(\Lambda A_c c' + (A_c)_c) + 2a'(y A_c Q_c)_c}{\varepsilon^2 a^2} (A_c)_c x + O_{H^2(\mathbb{R})}(\varepsilon^3 e^{-\gamma c |t|}). \]

Proof. We compute

\[ II = (1 - \lambda \partial_x^2)w_t + (w_{x} - w + 2a_w R)_x \]

\[ \varepsilon[(1 - \lambda \partial_x^2)(A_c(\varepsilon t, y)_t) + \varepsilon[(A_c)_y - A_c + 2a(\varepsilon x) A_c Q_c)_x. \]

Use the same method just like Lemma 3.2, by Taylor expansion.

\[ II = \varepsilon[(1 - \lambda \partial_x^2)(\Lambda A_c c' + (A_c)_c) + 2a'(y A_c Q_c)_c] \]

\[ + \varepsilon[(A_c)_y - A_c + 2a(y A_c Q_c)_c + O_{H^2(\mathbb{R})}(\varepsilon^3 e^{-\gamma c |t|})] \]

\[ = \varepsilon(1 - \lambda \partial_x^2)(\Lambda A_c c' + (A_c)_c) + 2a'(y A_c Q_c)_c \]

\[ + \varepsilon[-(1 + c)A_c + (1 + \lambda c)(A_c)_y + 2Q_c A_c)_y + O_{H^2(\mathbb{R})}(\varepsilon^3 e^{-\gamma c |t|})] \]

\[ = \varepsilon^2 [(1 - \lambda \partial_x^2)(\Lambda A_c c' + (A_c)_c) + 2a'(y A_c Q_c)_c] - \varepsilon(L A_c)_y + O_{H^2(\mathbb{R})}(\varepsilon^3 e^{-\gamma c |t|}). \]

Lemma 3.3.

\[ III = \{a_c w^2\}_x = \varepsilon^2 a(\varepsilon c) A^2 c_x = \varepsilon^3 a'(\varepsilon c) A^2 c + \varepsilon^2 a_x (A^2 c')^2 = O(\varepsilon^2 e^{-\gamma c |t|}). \]

Proof. Note that \((A^2 c')^2 \in Y\), and \(A_c\) satisfies (IP) property, so

\[ III = \{a_c w^2\}_x = \varepsilon^3 a'(\varepsilon c) A^2 c + \varepsilon^2 a_x (A^2 c')^2 = O(\varepsilon^2 e^{-\gamma c |t|}). \]

Now we collect the estimate from Lemmas 3.2, 3.3 and 3.4. We finally get

\[ S[\bar{u}] = \varepsilon^2 \frac{a''}{2a^2} (y^2 Q^2 c)_y + \frac{a'}{a} (y A_c Q_c)_x + (1 - \lambda \partial_x^2)(\Lambda A_c c' + (A_c)_c) + O(\varepsilon^2 e^{-\gamma c |t|}). \]

Due to Lemma 3.2, 3.3, 3.4, the Proposition 3.1 is proved.
Note that if we want to improve the approximation \( \bar{u} \), the unknown function \( A_c \) must be chosen such that
\[
F - (LA_c)_y = 0, \quad \forall y \in \mathbb{R}.
\] (Ω)

Then the error term will be reduced to the second order quantity
\[
S[\bar{u}] = \varepsilon^2 \left[ \frac{a''}{2a^2} (y^2 Q_c)_y + 2 \frac{a''}{a} (yA_c Q_c)_x + (1 - \lambda \partial^2_x) (\Lambda A_c c' + (A_c)_t) \right] + O(\varepsilon^2 e^{-\gamma|t|}).
\]

We prove such a solvable result in the next part.

3.2. Resolution of \( \Omega \).

Lemma 3.5 (Existence theory for \( \Omega \)). Suppose \( F \in Y \) even and satisfy the orthogonal condition
\[
\int_{\mathbb{R}} FQ_c = 0.
\]

Let \( \beta = \frac{1}{2} \sqrt{\frac{1 + \lambda c}{1 + c}} \int_{\mathbb{R}} F \), the problem of \( \Omega \) has a bounded solution \( A_c \) of the form
\[
A_c = \beta \phi_c + \delta + A_1(y),
\]
where \( \beta, \delta \in \mathbb{R} \) and \( A_1(y) \in Y \), we have
\[
LA_1(y) = H(y) - \beta L \phi_c - \gamma,
\]
where
\[
H(y) = \int_{-\infty}^{y} F(s) ds, \quad \gamma = \int_{-\infty}^{0} F(s) ds.
\]

Without loss of generality, we can suppose the constant term \( \gamma = \sqrt{\frac{1 + \lambda c}{1 + \lambda c}} \beta \).

The problem of \( \Omega \) is solvable if and only if
\[
\int_{\mathbb{R}} [H(y) - \beta(L \phi_c + 1)] Q_c' = \int_{\mathbb{R}} HQ_c' = - \int_{\mathbb{R}} FQ_c = 0.
\]

Namely recall that \( LQ_c' = 0 \), thus there exists a solution \( A_1(y) \) satisfying
\[
\int_{\mathbb{R}} A_1 Q_c' = 0.
\]

Since
\[
\lim_{y \to -\infty} (H(y) - \beta(L \phi_c + \sqrt{\frac{1 + c}{1 + \lambda c}})) = 0,
\]
\[
\lim_{y \to +\infty} (H(y) - \beta(L \phi_c + \sqrt{\frac{1 + c}{1 + \lambda c}})) = \int_{\mathbb{R}} F - 2 \sqrt{\frac{1 + c}{1 + \lambda c}} \beta.
\]

So we get \( A_1(y) \in Y \) provided \( \beta = \frac{1}{2} \sqrt{\frac{1 + \lambda c}{1 + c}} \int_{\mathbb{R}} F \). This finishes the proof. According to the Lemma 3.5, it suffices to verify the orthogonal condition.

Proof. We prove this Lemma in next three Lemmas.

Lemma 3.6. There exists a solution \( A_c \) of the problem (Ω) satisfying (IP) and such that
\[
A_c = \beta(\phi_c - \sqrt{\frac{1 + c}{1 + \lambda c}}) + A_1(y), \quad \lim_{y \to +\infty} A_c = 0.
\]

\[
\beta = \frac{1}{2} \sqrt{\frac{1 + \lambda c}{1 + c}} \int_{\mathbb{R}} F = \frac{1}{2} \sqrt{\frac{1 + \lambda c}{1 + c}} \left[ \frac{c'}{a(1 + c)} (1 - \frac{1 - \lambda}{2(1 + \lambda c)} - \frac{a'}{a^2} c) \right] \int_{\mathbb{R}} Q_c.
\]

Proof. We prove this Lemma in next three Lemmas.
Lemma 3.7 (The imposed condition). To get orthogonal condition $\int_{\mathbb{R}} F Q_c = 0$, the parameters of $c, a$ satisfy the following condition

$$c'[1 + \frac{1}{5} \lambda + \frac{6}{5} \lambda c - \frac{1 - \lambda}{20} (1 + c)^{\frac{1}{2}} (5 - \lambda + 4 \lambda c)]$$

$$- a' \alpha (1 + c)(1 + \frac{1}{5} \lambda + \frac{6}{5} \lambda c) - \frac{2}{5} (1 + c)^{2}(1 + \lambda c) = 0.$$ 

Proof. Note that

$$F = (1 - \lambda \partial_x^2) c \Lambda Q_c - (1 - \lambda \partial_x^2) \frac{a'}{a^2} c Q_c + \frac{a'}{a^2} (y Q_c)^y.$$ 

We just compute these three terms

$$\int_{\mathbb{R}} (1 - \lambda \partial_x^2)(\Lambda Q_c) Q_c, \quad \int_{\mathbb{R}} (1 - \lambda \partial_x^2) Q_c \Lambda Q_c$$

$$= \frac{1}{1 + c} \int_{\mathbb{R}} \left( \frac{1 - \lambda}{1 + \lambda c} Q_c + \frac{\lambda}{1 + \lambda c} Q_c^2 \right) \left( Q_c + \frac{1 - \lambda}{2(1 + \lambda c)} y Q_c' \right).$$

Note that

$$\int_{\mathbb{R}} Q_c y Q_c' = \frac{1}{2} \int_{\mathbb{R}} y d(Q_c^2) = - \frac{1}{2} \int_{\mathbb{R}} Q_c^3,$$

and

$$\int_{\mathbb{R}} Q_c^2 y Q_c' = \frac{1}{3} \int_{\mathbb{R}} y d(Q_c^3) = - \frac{1}{3} \int_{\mathbb{R}} Q_c^3.$$ 

So

$$\int_{\mathbb{R}} (1 - \lambda \partial_x^2)(\Lambda Q_c) Q_c = \frac{1}{1 + c} \left[ \int_{\mathbb{R}} \left( \frac{1 - \lambda}{1 + \lambda c} - \frac{(1 - \lambda)^2}{4(1 + \lambda c)^2} \right) Q_c^2 + \frac{\lambda}{1 + \lambda c} Q_c^3 \right].$$

$$\int_{\mathbb{R}} (1 - \lambda \partial_x^2) Q_c Q_c = \int_{\mathbb{R}} [Q_c(Q_c - \lambda Q_c^2)] = \int_{\mathbb{R}} \left( \frac{1 - \lambda}{1 + \lambda c} Q_c^2 + \frac{\lambda}{1 + \lambda c} Q_c^3 \right).$$

$$\int_{\mathbb{R}} (y Q_c^2) Q_c = \int_{\mathbb{R}} Q_c d(y Q_c^2) = - \int_{\mathbb{R}} y Q_c^2 d(Q_c) = \frac{1}{3} \int_{\mathbb{R}} Q_c^3.$$ 

So put these three parts together and from (2.5)-(2.10) to get orthogonal condition, we impose

$$c'[1 + \frac{1}{5} \lambda + \frac{6}{5} \lambda c - \frac{1 - \lambda}{20} (1 + c)^{\frac{1}{2}} (5 - \lambda + 4 \lambda c)]$$

$$- a' \alpha (1 + c)(1 + \frac{1}{5} \lambda + \frac{6}{5} \lambda c) - \frac{2}{5} (1 + c)^{2}(1 + \lambda c) = 0.$$ 

□

Lemma 3.8.

$$\beta = \frac{1}{2} \sqrt{\frac{1 + \lambda c}{1 + c}} \int_{\mathbb{R}} F = \frac{1}{2} \sqrt{\frac{1 + \lambda c}{1 + c}} \left[ \frac{c'}{a(1 + c)} (1 - \frac{1 - \lambda}{2(1 + \lambda c)} - \frac{a'}{a^2} \right] \int_{\mathbb{R}} Q_c.$$
Proof.

\[ F = (1 - \lambda \partial_x^2) \frac{c'}{a} \Lambda Q_c - (1 - \lambda \partial_x^2) \frac{a'}{a^2} c Q_c + \frac{a'}{a^2} (y Q_c^2)_y. \]

We just compute the

\[ \int_{\mathbb{R}} (1 - \lambda \partial_x^2) \Lambda Q_c, \quad \int_{\mathbb{R}} (1 - \lambda \partial_x^2) Q_c \text{ and } \int_{\mathbb{R}} (y Q_c^2)_y. \]

Because \( a, c \) is independent of \( x \),

\[ \int_{\mathbb{R}} (1 - \lambda \partial_x^2) \Lambda Q_c = \frac{1}{1 + c} \int_{\mathbb{R}} (1 - \lambda \partial_x^2)(Q_c + \frac{1 - \lambda}{2(1 + \lambda c)} y Q'_c) \]

\[ = \frac{1}{1 + c} \left[ \int_{\mathbb{R}} Q_c - \frac{1 - \lambda}{2(1 + \lambda c)} \int_{\mathbb{R}} Q_c \right]. \]

Due to \( \int_{\mathbb{R}} (1 - \lambda \partial_x^2) Q_c = \int_{\mathbb{R}} Q_c - \lambda \int_{\mathbb{R}} Q'_c = \int_{\mathbb{R}} Q_c \text{ and } \int_{\mathbb{R}} (y Q_c^2)_y = 0, \)

\[ \beta = \frac{1}{2} \sqrt{\frac{1 + \lambda c}{1 + c}} \left( \frac{c'}{a(1 + c)} (1 - \frac{1 - \lambda}{2(1 + \lambda c)}) - \frac{a'}{a^2 c} \right) \int_{\mathbb{R}} Q_c. \]

\[ \square \]

**Lemma 3.9.**

\[ \delta = -\beta \sqrt{\frac{1 + c}{1 + \lambda c}}. \]

**Proof.** Finally, to get \( \lim_{y \to +\infty} A_c = 0 \) by (2.5)-(2.10), we choose \( \delta = -\beta \sqrt{\frac{1 + \epsilon}{1 + \lambda \epsilon}}. \)

According to Lemma 3.7, 3.8, 3.9, we have \( A_c = \beta (\phi_c - \sqrt{\frac{1 + \epsilon}{1 + \lambda \epsilon}}) + A_1(y), A_1 \in Y \), this finishes the proof of Lemma 3.6. This proves the problem of \( \Omega \). \( \square \)

### 3.3. Correction to the solution of problem of \( \Omega \).

Consider the cut-off function \( \eta \in C^\infty(\mathbb{R}) \) satisfying the following properties,

\[ \begin{align*}
0 &\leq \eta(s) \leq 1, \quad 0 \leq \eta'(s) \leq 1, \quad \forall s \in \mathbb{R}, \\
\eta(s) &\equiv 0, \quad \forall s \leq -1, \\
\eta(s) &\equiv 1, \quad \forall s \geq 1.
\end{align*} \quad (3.8) \]

Define

\[ \eta_c(y) = \eta(\epsilon y + 2). \quad (3.9) \]

And for \( A_c = A_c(\epsilon t, y) \) is the solution of \( \Omega \), denote

\[ A_\# = \eta_c A_c(\epsilon t, y). \quad (3.10) \]

Now redefine

\[ \tilde{u} = R + w = R + \epsilon A_\#. \quad (3.11) \]

The following Proposition, which deals with the error associated to the cut-off function and the new approximate solution \( \tilde{u} \), is the main result.

**Proposition 3.2.** There exist constants \( \epsilon_0, K > 0 \) such that for all \( 0 < \epsilon < \epsilon_0 \), the following holds.

(i) (a) New behavior. For all \( t \in [-T_\epsilon, T_\epsilon] \),

\[ \begin{align*}
A_\#(\epsilon t, y) &\equiv 0, \quad \forall y \leq -\frac{3}{\epsilon}, \\
A_\#(\epsilon t, y) &\equiv A_c(\epsilon t, y), \quad \forall y \geq \frac{1}{\epsilon}.
\end{align*} \quad (3.12) \]
(b) Integrable solution. For all \( t \in [-T_\varepsilon, T_\varepsilon] \), \( A_\# \in H^1(\mathbb{R}) \) with
\[
\|\varepsilon A_\#\|_{H^1(\mathbb{R})} \leq K\varepsilon^{\frac{1}{2}} e^{-\gamma_\varepsilon |t|}.
\] (3.13)

(ii) The error associated to the new function \( \tilde{u} \) satisfies
\[
\|S[\tilde{u}]\|_{H^2(\mathbb{R})} \leq K\varepsilon^{\frac{3}{2}} e^{-\gamma_\varepsilon |t|},
\] (3.14)
and the integral estimate holds
\[
\int_{\mathbb{R}} \|S[\tilde{u}]\|_{H^2(\mathbb{R})} \, dt \leq K\varepsilon^{\frac{7}{4}}.
\] (3.15)

Proof. The proof of first part about this proposition is similar to the proof of Proposition 4.6 in [17], so we omit this part. We will prove the second part of this proposition in the next Lemma. \( \Box \)

**Lemma 3.10.**
\[
S[\tilde{u}] = I + II' + III'.
\]
where
\[
II' = -\varepsilon \eta_c(LA_c)y + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma_\varepsilon |t|}), \quad \|II'\|_{H^2(\mathbb{R})} \leq K\varepsilon^{2} e^{-\gamma_\varepsilon |t|}.
\]

Proof.
\[
III' = \{a(\varepsilon x)w^2\}_x = \varepsilon^3 a(\varepsilon x)\eta_c^2 A_c^2_x + 2\varepsilon^3 a(\varepsilon x)\eta_c \eta_c' A_c^2 + 2\varepsilon^2 a(\varepsilon x)\eta_c^2 A_c A_c'.
\]

With \( A_c, A_c' \in Y, \|\eta_c'\|_{L^2(\mathbb{R})} \leq K\varepsilon^{-1/2}, \) uniformly \( t \in [-T_\varepsilon, T_\varepsilon] \). Moreover, we have the estimate
\[
\|III'\|_{H^2(\mathbb{R})} \leq K\varepsilon^{2} e^{-\gamma_\varepsilon |t|}.
\]

Note that
\[
(1 - \lambda \partial_x^2)(\varepsilon A_\#)_t = \varepsilon(1 - \lambda \partial_x^2)(-\varepsilon \eta_c' A_c - c(A_c)_y \eta_c + \varepsilon(A_c)_t \eta_c - \varepsilon \eta_c \Lambda A_c c')
\]
\[
= \varepsilon^2(1 - \lambda \partial_x^2)(-\varepsilon \eta_c' A_c + (A_c)_t \eta_c - \eta_c \Lambda A_c c') - \varepsilon(1 - \lambda \partial_x^2)(c(A_c)_y \eta_c).
\]
Thus
\[(\varepsilon A_\#)_{xx} - \varepsilon A_\# + 2\varepsilon a_\varepsilon A_\# R)_{x}\]
\[= ((\varepsilon \eta_c A_c)_{xx} - \varepsilon \eta_c A_c + 2\varepsilon a_\varepsilon R \eta_c A_c)_{x}\]
\[= \varepsilon [\eta_c (A_c)_{yy} + 2\varepsilon \eta_c' (A_c)_y + \varepsilon^2 \eta_c'' (A_c)_y] - \varepsilon \eta_c A_c + 2\varepsilon a_\varepsilon R \eta_c A_c]_{x}\]
\[= \varepsilon [\eta_c (A_c)_{yy} - A_c + 2 \frac{a(\varepsilon x)}{a(\varepsilon \rho)} Q_c A_c]_{x} + \varepsilon^2 (2\eta_c' (A_c)_y + \varepsilon \eta_c'' (A_c)_y)\]
\[= \varepsilon \eta_c [(A_c)_{yy} - A_c + 2 \frac{a(\varepsilon x)}{a(\varepsilon \rho)} Q_c A_c]_{x} + \varepsilon^2 (3\eta_c'' (A_c)_y)\]
\[= \varepsilon \eta_c [(A_c)_{yy} - A_c + 2Q_c A_c]_{x} + 2\varepsilon a_\varepsilon \frac{a'}{a} (y Q_c A_c)_y\]
\[= \varepsilon^2 (3\eta_c'' (A_c)_y + 3\eta_c' (A_c)_y + 2\eta_c Q_c A_c) + O(\varepsilon^3 \eta_c (y^2 Q_c A_c)_y).\]

From the (IP) property to estimate as follows
\[\left\| 2\varepsilon^2 \frac{a'}{a} (y Q_c A_c)_y \right\|_{H^2(\mathbb{R})} \leq K \varepsilon^2 e^{-\gamma |t|}, \quad \left\| O(\varepsilon^3 \eta_c (y^2 Q_c A_c)_y) \right\|_{H^2(\mathbb{R})} \leq K \varepsilon^3.\]
\[\left\| \varepsilon^2 (3\eta_c'' (A_c)_y + 3\eta_c' (A_c)_y + 2\eta_c Q_c A_c) \right\|_{H^2(\mathbb{R})} \leq K \varepsilon^2 e^{-\gamma |t|}.\]

Therefore
\[((\varepsilon A_\#)_{xx} - \varepsilon A_\# + 2\varepsilon a_\varepsilon A_\# R)_{x}\]
\[= \varepsilon \eta_c [(A_c)_{yy} - A_c + 2Q_c A_c]_{x} + O_{H^2(\mathbb{R})}(\varepsilon^2 e^{-\gamma |t|} + \varepsilon^3).\]

So, we get
\[II' = -\varepsilon \eta_c (L A_c)_y + O_{H^2(\mathbb{R})}(\varepsilon^2 e^{-\gamma |t|}).\]

Note that
\[S[\tilde{u}] = \varepsilon [F - \eta_c (L A_c)_y] + O_{H^2(\mathbb{R})}(\varepsilon^2 e^{-\gamma |t|})\]
\[= \varepsilon (1 - \eta_c) F + O_{H^2(\mathbb{R})}(\varepsilon^2 e^{-\gamma |t|}).\]

For any \( t \in [-T_\varepsilon, T_\varepsilon], \) \( 1 - \eta_c \subseteq (-\infty, -\frac{1}{2}], \) \( \| F \|_{H^2(\mathbb{R})} \leq Ke^{-|y| - \varepsilon |t|}. \) So we gain
\[\| \varepsilon (1 - \eta_c) F \|_{H^2(\mathbb{R})} \leq Ke^{-\frac{1}{2} - \gamma |t|} \ll Ke^{10},\]
\[\| S[\tilde{u}] \|_{H^2(\mathbb{R})} \leq K \varepsilon^2 e^{-\gamma |t|}.\]

(3.15) is just from integration of the formula (3.14).
\[
\int_{\mathbb{R}} \| S[\tilde{u}] \|_{H^2(\mathbb{R})} dt \leq K \varepsilon^2.
\]

This finishes the second proof of Proposition 3.2.
4. First stability results. In this section, our aim is to prove that the approximate solution \( \tilde{u} \) describes the actual dynamics of interaction on the interval \([-T_\varepsilon, T_\varepsilon]\). We analyze the energy conservation with the method of Lyapunov energy function.

**Proposition 4.1.** There exist constants \( K, k, \varepsilon > 0 \), assume \( u(t) \) is a \( H^1(\mathbb{R}) \) solution of (1.1) in a vicinity of \( t = T_\varepsilon \) satisfying

\[
\|u(-T_\varepsilon) - \tilde{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq Ke^k
\]

Then, there exists a \( C^1 \)-function \( \rho_1(t) \), for all \( t \in [-T_\varepsilon, T_\varepsilon] \), we have

\[
\|u(t + \rho_1(t)) - \tilde{u}\|_{H^1(\mathbb{R})} \leq Ke^k, \quad |\rho_1'| \leq Ke^k
\]

Recall that \( u(t) \) is globally well-defined in \( H^1(\mathbb{R}) \). Since

\[
\|u(-T_\varepsilon) - \tilde{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq Ke^k
\]

by continuity of time in \( H^1(\mathbb{R}) \), there exists \(-T_\varepsilon < T^* < T_\varepsilon \) with

\[T^* = \sup\{T \in [-T_\varepsilon, T_\varepsilon], \forall t \in [-T_\varepsilon, T], \exists r(t) \text{ with } \|u(t + r(t)) - \tilde{u}\|_{H^1(\mathbb{R})} \leq Ke^k\}\]

The aim is to prove \( T^* = T_\varepsilon \) that for \( K \) large enough. To achieve this, we argue by assuming \( T^* < T_\varepsilon \) and reaching on a contradiction with the definition of \( T^* \) by proving some independent estimates for

\[
\|u(t + r(t)) - \tilde{u}\|_{H^1(\mathbb{R})} \leq Ke^k \text{ on } [-T_\varepsilon, T^*].
\]

**Lemma 4.1.** There exists a unique \( C^1 \)-function \( \rho_1(t) \), for all \( t \in [-T_\varepsilon, T^*] \), \( \exists K > 0 \), we have

\[
z = u(t + \rho_1(t)) - \tilde{u} \quad \text{satisfies} \quad \int_\mathbb{R} (1 - \lambda \partial_x^2) z Q_c' = 0.
\]

Moreover

\[
(1 - \lambda \partial_x^2) z_t + (1 + \rho_1') \{ z_{xx} - z + a_x[(\tilde{u} + z)^2 - \tilde{u}^2]\}_x
\]

\[
- \rho_1' (1 - \lambda \partial_x^2) \tilde{u}_t + (1 + \rho_1') S[\tilde{u}] = 0.
\]

(4.2)

\[
|\rho_1'| < \frac{K}{c} \left[\|z\|_{L^2(\mathbb{R})} + \varepsilon\|z\|_{L^2(\mathbb{R})} + \|S[\tilde{u}]\|_{L^2(\mathbb{R})}\right].
\]

(4.3)

**Proof.** (4.1) is a consequence of the Implicit Function Theorem. Note that where \( z_t = (1 + \rho_1') u_t - \tilde{u}_t \) is the solution of the equation (1.1) and \( \tilde{u} \) is the approximate solution. Substituting \( z_t \) into the former formula, we can get (4.2).

We take time derivative of orthogonal formula in (4.1)

\[
\int_\mathbb{R} (1 - \lambda \partial_x^2) z_t Q_c' + \int_\mathbb{R} (1 - \lambda \partial_x^2) z \Lambda Q_c' c \varepsilon - \int_\mathbb{R} (1 - \lambda \partial_x^2) z Q_{cc'} c = 0
\]

Replace \( z_t \) with (4.2)

\[
- \int_\mathbb{R} (1 + \rho_1') \{ z_{xx} - z + a_x[(\tilde{u} + z)^2 - \tilde{u}^2]\}_x Q_c' + \int_\mathbb{R} \rho_1' (1 - \lambda \partial_x^2) \tilde{u}_t Q_c'
\]

\[- (1 + \rho_1') \int_\mathbb{R} S[\tilde{u}] Q_c' + \int_\mathbb{R} (1 - \lambda \partial_x^2) z \Lambda Q_c' c \varepsilon - \int_\mathbb{R} (1 - \lambda \partial_x^2) z Q_{cc'} c = 0
\]

Integrating by parts

\[
(1 + \rho_1') \int_\mathbb{R} (z_{xx} - z + a_x[(\tilde{u} + z)^2 - \tilde{u}^2]) Q_c' + \int_\mathbb{R} (1 - \lambda \partial_x^2) (\rho_1' \tilde{u}_t + cz_x) Q_{cc'}
\]

\[- (1 + \rho_1') \int_\mathbb{R} S[\tilde{u}] Q_c' + \int_\mathbb{R} (1 - \lambda \partial_x^2) z \Lambda Q_c' c \varepsilon = 0
\]
Moreover
\[ \| \int_{\mathbb{R}} (z_{xx} - z + a_{\varepsilon}((\tilde{u} + z)^2 - \tilde{u}^2))Q_{e}(z_{xx} - z + a_{\varepsilon}((\tilde{u} + z)^2 - \tilde{u}^2))\|_{L^{2}(\mathbb{R})} \leq K\|z\|_{L^{2}(\mathbb{R})}, \]
\[ \int_{\mathbb{R}} (1 - \lambda \partial_{x}^{2})(\lambda \partial_{x}^{2})Q_{e} = \frac{\alpha_1}{\alpha} \int_{\mathbb{R}} (1 - \lambda \partial_{x}^{2})(\lambda \partial_{x}^{2})^2 + O(\|z\|_{L^{2}(\mathbb{R})}). \]
So
\[ |\rho_1'| < \frac{K}{c} \|z\|_{L^{2}(\mathbb{R})} + \varepsilon \|z\|_{L^{2}(\mathbb{R})} + \|S[\tilde{u}]\|_{L^{2}(\mathbb{R})}, \]
this finishes Lemma 4.1. \hfill \square

**Lemma 4.2** (Almost conservation of energy). Consider the constructed approximate solution,
\[ \partial_{t}E_{a}[\tilde{u}](t) = -\int_{\mathbb{R}} (\tilde{u}_{xx} - \tilde{u} + a_{\varepsilon}\tilde{u}^2)(1 - \lambda \partial_{x}^{2})^{-1}S[\tilde{u}]. \quad (4.4) \]
In particular, there exists $K > 0$ such that
\[ |E_{a}[\tilde{u}](t) - E_{a}[\tilde{u}](T_{\varepsilon})| \leq K\varepsilon \frac{1}{c}. \quad (4.5) \]

**Proof.**
\[ \int_{\mathbb{R}} (\tilde{u}_{xx} - \tilde{u} + a_{\varepsilon}\tilde{u}^2)(1 - \lambda \partial_{x}^{2})^{-1}S[\tilde{u}] = \int_{\mathbb{R}} (\tilde{u}_{xx} - \tilde{u} + a_{\varepsilon}\tilde{u}^2)u_{t} = -\partial_{t} \frac{1}{2} \int_{\mathbb{R}} \tilde{u}_{x}^2 - \partial_{t} \frac{1}{2} \int_{\mathbb{R}} \tilde{u}_{x}^2 + \frac{1}{3} \partial_{t} \int_{\mathbb{R}} a_{\varepsilon}\tilde{u}^3 = -\partial_{t}E_{a}[\tilde{u}](t). \]
From Cauchy-Schwarz inequality, we have
\[ |\partial_{t}E_{a}[\tilde{u}](t)| \leq K\|S[\tilde{u}]\|_{L^{2}(\mathbb{R})}, \]
after integration
\[ |E_{a}[\tilde{u}](t) - E_{a}[\tilde{u}](T_{\varepsilon})| \leq K \int_{\mathbb{R}} \|S[\tilde{u}]\|_{L^{2}(\mathbb{R})}. \]
We can get (4.5) from (3.15). \hfill \square

**Lemma 4.3.** There exists $K > 0$, we have
\[ \int_{\mathbb{R}} (1 - \lambda \partial_{x}^{2})Q_{e}z \leq \frac{K}{c} |z|_{H^{1}(\mathbb{R})}^{2} + c e^{-r_{1}\varepsilon} |z|_{L^{2}(\mathbb{R})} + \|z\|_{H^{1}(\mathbb{R})}^{2}. \quad (4.6) \]

**Proof.** For the equation of (1.1), the conserved energy
\[ E = \frac{1}{2} \int_{\mathbb{R}} u_{x}^2 + \frac{\lambda}{2} \int_{\mathbb{R}} u^2 - \frac{1}{3} \int_{\mathbb{R}} a_{\varepsilon}u^3 \]
\[ E_{a}[\tilde{u} + z](t) = E_{a}[\tilde{u}](t) - \int_{\mathbb{R}} z(\tilde{u}_{xx} - \tilde{u} + a_{\varepsilon}\tilde{u}^2) \]
\[ + \frac{1}{2} \int_{\mathbb{R}} (z_{x}^2 + z^2) - \frac{1}{3} \int_{\mathbb{R}} a_{\varepsilon}(u^3 - \tilde{u}^3 - 3\tilde{u}^2 z). \]
Note that
\[ \int_{\mathbb{R}} z(\tilde{u}_{xx} - \tilde{u} + a_{\varepsilon}\tilde{u}^2)(t) = \int_{\mathbb{R}} z(\tilde{u}_{xx} - \tilde{u} + a_{\varepsilon}\tilde{u}^2)(-T_{\varepsilon}) + E_{a}[\tilde{u}](t) \]
\[ - E_{a}[\tilde{u}](T_{\varepsilon}) + O(\|z\|_{H^{1}(\mathbb{R})}^{2}), \]
where
\[
\int_{\mathbb{R}} z(\tilde{u}_{xx} - \tilde{u} + a_x \tilde{u}^2)(t) = c \int_{\mathbb{R}} (1 - \lambda \partial_x^2) \tilde{u} z + O(\varepsilon e^{-r_1|t|}) z^2_{L^2(\mathbb{R})}.
\]
So, we have
\[
\int_{\mathbb{R}} (1 - \lambda \partial_x^2) \tilde{u} z \leq \frac{K}{c} \left[ \varepsilon \frac{1}{2} + \varepsilon e^{-r_1|t|} z^2_{L^2(\mathbb{R})} + \varepsilon^2 \right],
\]
proof is finished.

We consider the Lyapunov energy function, let
\[
F(t) = \frac{1}{2} \int_{\mathbb{R}} [z_x^2 + z^2 + c(1 - \lambda \partial_x^2) z z] - \frac{1}{3} \int_{\mathbb{R}} a_x (u^3 - \tilde{u}^3 - 3 \tilde{u}^2 z).
\]
Due to
\[
\frac{1}{2} \int_{\mathbb{R}} c(1 - \lambda \partial_x^2) z z = \frac{1}{2} c \int_{\mathbb{R}} [z^2 - \lambda z d z_z] = \frac{1}{2} c \int_{\mathbb{R}} (z^2 + \lambda z_x^2)
\]
So
\[
F(t) = \frac{1}{2} \int_{\mathbb{R}} [z_x^2 + z^2 + c(z^2 + \lambda z_x^2)] - \frac{1}{3} \int_{\mathbb{R}} a_x (u^3 - \tilde{u}^3 - 3 \tilde{u}^2 z).
\]
This completes the proof of Lemma 4.3.

**Lemma 4.4.** There exist constants $K, v_0 > 0$, for all $t \in [-T_x, T_x]$, we have
\[
F(t) \geq v_0 \|z\|^2_{H^1(\mathbb{R})} - K \int_{\mathbb{R}} (1 - \lambda \partial_x^2) Q_c z^2 - K (\varepsilon e^{-r_1|t|} + \varepsilon^2) \|z\|^2_{L^2(\mathbb{R})} - K \|z\|^3_{L^2(\mathbb{R})}
\]

**Proof.**
\[
F(t) = \frac{1}{2} \int_{\mathbb{R}} [z_x^2 + z^2 + c(z^2 + \lambda z_x^2) - 2a_x \tilde{u} z^2] - \frac{1}{3} \int_{\mathbb{R}} a_x (u^3 - \tilde{u}^3 - 3 \tilde{u}^2 z - 3 \tilde{u} z^2),
\]
where
\[
\left| \frac{1}{3} \int_{\mathbb{R}} a_x (u^3 - \tilde{u}^3 - 3 \tilde{u}^2 z - 3 \tilde{u} z^2) \right| \leq K \|z\|^3_{L^2(\mathbb{R})}.
\]
We use the Taylor expansion of $a_x$ for the term of
\[
\frac{1}{2} \int_{\mathbb{R}} [z_x^2 + z^2 + c(z^2 + \lambda z_x^2) - 2a_x \tilde{u} z^2],
\]
thus
\[
\frac{1}{2} \int_{\mathbb{R}} [z_x^2 + z^2 + c(z^2 + \lambda z_x^2) - 2a_x \tilde{u} z^2] = \frac{1}{2} \int_{\mathbb{R}} [z_x^2 + z^2 + c(z^2 + \lambda z_x^2) - 2Q_c z^2] - \varepsilon \frac{a'}{a} \int_{\mathbb{R}} y Q_c z^2 + O(\varepsilon^2 \|z\|^2_{L^2(\mathbb{R})}).
\]
Note that
\[
\left| \varepsilon \frac{a'}{a} \int_{\mathbb{R}} y Q_c z^2 \right| \leq K \varepsilon e^{-r_1|t|} \|z\|^2_{L^2(\mathbb{R})},
\]
\[
\frac{1}{2} \int_{\mathbb{R}} [z_x^2 + z^2 + c(z^2 + \lambda z_x^2) - 2Q_c z^2] \geq v_0 \|z\|^2_{H^1(\mathbb{R})} - K \int_{\mathbb{R}} (1 - \lambda \partial_x^2) Q_c z^2.
\]
So we can get (4.7).
Lemma 4.5. There exist constants $K, K^*, k > 0$, for all $t \in [-T_\varepsilon, T_\varepsilon]$, we have
\[ |F(t) - F(-T_\varepsilon)| \leq K(K^*)^3 \varepsilon^{3k-\frac{3k}{m}} + K(K^*)\varepsilon^{2k} \]
\[ + K \int_{-T_\varepsilon}^t \varepsilon e^{-\gamma|s|} \|z(s)\|_{L^2(\mathbb{R})}^2 ds. \] (4.8)

Proof.
\[ F'(t) = \int_{\mathbb{R}} (z_x z_{xt} + z z_t) + \frac{1}{2} c' \varepsilon \int_{\mathbb{R}} (\lambda z_x^2 + z^2) + c \int_{\mathbb{R}} (\lambda z_x z_{xt} + z z_t) \]
\[ - \int_{\mathbb{R}} a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2 - 2\tilde{u}z)\tilde{u}_t - \int_{\mathbb{R}} a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2)z_t. \]
Due to
\[ \int_{\mathbb{R}} z_x z_{xt} = \int_{\mathbb{R}} z_x dz_t = - \int_{\mathbb{R}} z_{xx} z_t, \]
we have
\[ F'(t) = - \int_{\mathbb{R}} z_t[(1 + \lambda c)z_{xx} - (1 + c)z + a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2)] \]
\[ - \int_{\mathbb{R}} a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2 - 2\tilde{u}z)\tilde{u}_t + \frac{1}{2} c' \varepsilon \int_{\mathbb{R}} (\lambda z_x^2 + z^2). \] (4.9)
Replace $z_t$ with (4.2)
\[ F'(t) = (1 + \rho_1')c \int_{\mathbb{R}} z_x[(1 + \lambda c)z_{xx} - (1 + c)z + a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2)] \]
\[ - \rho_1' \int_{\mathbb{R}} \tilde{u}_t[(1 + \lambda c)z_{xx} - (1 + c)z + a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2)] \] (4.10)
\[ + (1 + \rho_1') \int_{\mathbb{R}} (1 - \lambda \partial_x^2)^{-1} S(\tilde{u})[(1 + \lambda c)z_{xx} - (1 + c)z + a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2)] \]
\[ - \int_{\mathbb{R}} a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2 - 2\tilde{u}z)\tilde{u}_t + \frac{1}{2} c' \varepsilon \int_{\mathbb{R}} (\lambda z_x^2 + z^2). \] (4.11)
Note the term of (4.9), we find that
\[ \int_{\mathbb{R}} z_x z_{xx} = \frac{1}{2} \int_{\mathbb{R}} dz_x^2 = 0, \int_{\mathbb{R}} z_xx = \frac{1}{2} \int_{\mathbb{R}} dz_x^2 = 0. \]
So, we can simplify (4.9) into
\[ F'(t) = (1 + \rho_1')c \int_{\mathbb{R}} z_x[a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2)] \]
and integrate by parts,
\[ (1 + \rho_1')c \int_{\mathbb{R}} z_x[a_\varepsilon((\tilde{u} + z)^2 - \tilde{u}^2)] = (1 + \rho_1')c \int_{\mathbb{R}} [a_\varepsilon(z^2 + 2\tilde{u}z)] z_x \]
\[ = (1 + \rho_1')c \int_{\mathbb{R}} [a_\varepsilon(z^3 + z \tilde{u} dz_x^2)] \]
\[ = - (1 + \rho_1')c \int_{\mathbb{R}} [\varepsilon a_\varepsilon (\frac{z^3}{3} + \varepsilon a_\varepsilon \tilde{u}^2 + a_\varepsilon \tilde{u}_x z^2)], \]
so
\[ |(4.9) + (1 + \rho_1')c \int_{\mathbb{R}} a_\varepsilon \tilde{u}_x z^2| \leq K \varepsilon e^{-\gamma|t|} \|z\|_{L^2(\mathbb{R})} + K \varepsilon \|z\|_{H^1(\mathbb{R})}. \]
Moreover

\[-\rho_1' \int_R \tilde{u}_t[(1+\lambda c)z_{xx} - (1+c)z + a_x(\tilde{u} + z)^2 - \tilde{u}^2)]\]

\[= -\rho_1' \int_R (\tilde{u}_t + c\tilde{u}_x)[(1+\lambda c)z_{xx} - (1+c)z + a_x(z^2 + 2\tilde{u}z)]\]

\[+ \rho_1 c \int_R a_x \tilde{u}_x z^2 + \rho_1' c \int_R \tilde{u}_x [(1+\lambda c)z_{xx} - (1+c)z + 2a_x \tilde{u}z],\]

where

\[\int_R \tilde{u}_x z_{xx} = -\int_R \tilde{u}_x z_{x} = \int_R \tilde{u}_{xxx} z,\]

\[\int_R 2a_x \tilde{u}_x z = \int_R a_x z \tilde{u}_t = \int_R z(a_x \tilde{u}_t^2)_x - \varepsilon \int_R a_x' z \tilde{u}_t^2.

Finally, we simplify the formula of (4.12),

\[\int_R a_x \tilde{u}_x z^2 + \rho_1' c \int_R [(1+\lambda c) \tilde{u}_{xx} - (1+c) \tilde{u} + a_x \tilde{u}^2]_x - \rho_1' \varepsilon \int_R a_x' z \tilde{u}_t^2.\]

Note that

\[\|\tilde{u}_t + c\tilde{u}_x\|_{L^2(\mathbb{R})} \leq Ke^{-\gamma \epsilon |t|},\]

\[\|(1+\lambda c)\tilde{u}_{xx} - (1+c) \tilde{u} + a_x \tilde{u}^2\|_{L^2(\mathbb{R})} \leq Ke^{-\gamma \epsilon |t|}.\]

So

\[|(4.10) - \rho_1' c \int_R a_x \tilde{u}_x z^2| \leq K\varepsilon |\rho_1'| e^{-\gamma \epsilon |t|} \|z\|_{H^1(\mathbb{R})}.\]

Integrating by parts for the term of (4.11),

\[\int_R (1 + \rho_1') \int_R (1-\lambda \partial_x^2)^{-1} S[\tilde{u}][(1+\lambda c)z_{xx} - (1+c)z + a_x((\tilde{u} + z)^2 - \tilde{u}^2)]\]

\[= \int_R (1 + \rho_1') \int_R (1-\lambda \partial_x^2)^{-1} z([(1+\lambda c)S[\tilde{u}]_{xx} - (1+c)S[\tilde{u}] + 2a_x \tilde{u}S[\tilde{u}] + a_x z S[\tilde{u}]]).\]

We can get the conclusion form (3.15) and (4.2),

\[|(4.11)| \leq K\|z\|_{L^2(\mathbb{R})} \cdot \|S[\tilde{u}]\|_{H^2(\mathbb{R})}.\]

Finally, we simplify the formula of (4.12),

\[\frac{1}{2} c' \varepsilon \int_R (\lambda z_x^2 + z^2) - \int_R a_x z^2 \tilde{u}_t\]

\[= \frac{1}{2} c' \varepsilon \int_R (\lambda z_x^2 + z^2) - \int_R a_x (\tilde{u}_t + c\tilde{u}_x)^2 + c \int_R a_x \tilde{u}_x z^2.\]

Because of

\[\|\tilde{u}_t + c\tilde{u}_x\|_{L^2(\mathbb{R})} \leq Ke^{-\gamma \epsilon |t|},\]

so

\[|(4.12) - c \int_R a_x \tilde{u}_x z^2| \leq K\varepsilon e^{-\gamma \epsilon |t|} \|z\|_{L^2(\mathbb{R})}^2.\]

From the estimates of (4.9)-(4.12) and the integration of $F(t)$, we can gain

\[|F(t) - F(-T_{\varepsilon})| \leq K(K^*)^3 e^{3K^* \frac{1}{2\varepsilon}} + K(K^*) e^{2K} + K \int_{-T_{\varepsilon}}^t \varepsilon e^{-\gamma |s|} \|z(s)\|_{L^2(\mathbb{R})}^2 ds.\]
This finishes the proof. □

From the lemmas 4.3, 4.4 and due to \( F(-T_\varepsilon) < K\varepsilon^{2k} \),
\[
\|z\|^2_{L^2(\mathbb{R})} \leq K \int_{\mathbb{R}} (1 - \lambda \partial^2_x)Q_\varepsilon z|^2 + K\varepsilon^{2k} + K(K^*)^3 \varepsilon^{3k-\frac{3}{10}} \nonumber
+ K(K^*)\varepsilon^{2k} + K \int_{-T_\varepsilon}^t \varepsilon e^{-\gamma \varepsilon |s|} \|z(s)\|^2_{L^2(\mathbb{R})} ds.
\]
According to
\[
\int_{\mathbb{R}} (1 - \lambda \partial^2_x)Q_\varepsilon z \leq \frac{K}{c} \left[ \varepsilon^2 + \varepsilon e^{-\gamma \varepsilon |t|} \|z\|_{L^2(\mathbb{R})} + \|z\|_{H^1(\mathbb{R})} \right],
\]
\[
\|z\|^2_{L^2(\mathbb{R})} \leq K\varepsilon^{2k} + K(K^*)^3 \varepsilon^{3k-\frac{3}{10}} + K(K^*)\varepsilon^{2k} + K \int_{-T_\varepsilon}^t \varepsilon e^{-\gamma \varepsilon |s|} \|z(s)\|^2_{L^2(\mathbb{R})} ds.
\]
Using the Gronwall inequality, we conclude that for some large constant \( K > 0 \),
\[
\|z\|^2_{H^1(\mathbb{R})} \leq K\varepsilon^{2k} + K(K^*)^3 \varepsilon^{3k-\frac{3}{10}} + K(K^*)\varepsilon^{2k}.
\]
Obviously, when \( K^* \) large enough, we obtain that for all \( t \in [-T_\varepsilon, T_\varepsilon] \),
\[
\|z\|^2_{H^1(\mathbb{R})} \leq K\varepsilon^{2k}.
\]
This estimate contradicts the definition of \( T^* \) and concludes the proof of Proposition 4.1.

In this section, our aim is to prove that the approximate solution \( \tilde{u} \) describe the actual dynamics. We analyze the energy conservation and the first stability results with the method of Lyapunov energy function.

5. Stability. In this section, we give the stability results of the error term between the exact solution and the constructed approximate solution with the methods of Virial estimates and the new mass and energy conservation when \( t \) goes to \( +\infty \).

**Proposition 5.1** (Stability). Assume \( \exists t_1 > \frac{1}{2} T_\varepsilon \) satisfying \( \|u(t_1) - \frac{1}{2} Q_\varepsilon(x)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{\frac{1}{2}} \). Here \( u(t) \) is a \( H^1(\mathbb{R}) \) solution of the equation (1.1). Then, for all \( t > t_1 \), there exist a constant \( K > 0 \) and a function \( \rho_2(t) \) defined on \( [t_1, +\infty) \), \( \forall t > t_1 \), we have
\[
\sup\|u(t) - \frac{1}{2} Q_\varepsilon(x - \rho_2(t))\|_{H^1(\mathbb{R})} \leq K\varepsilon^{\frac{1}{2}}, where \rho_2(t) = c_2. \tag{5.1}
\]

The proof of this proposition is similar to the proof of Proposition 4.1. Let us recall that for large time \( t > T_\varepsilon \), the soliton-like solution is far away from the region where \( a_\varepsilon \) varies. In particular, the stability and asymptotic stability properties will follow from the fact that in this region (1.1) behaves like the gBBM equation,
\[
(1 - \lambda \partial^2_x)u + (u_{xx} - u + 2u^2)_x = 0, where (t, x) \in [+T_\varepsilon, +\infty) \times \mathbb{R}. \tag{5.2}
\]
Obviously, the argument is so sketchy, now we give the rigorous statement. Let us assume that for some fixed \( K > 0, \)
\[
\|u(t_1) - \frac{1}{2} Q_\varepsilon(x)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{\frac{1}{2}}. \tag{5.3}
\]
In order to simplify the calculations, note that the function \( v = 2u \) solves,
\[
(1 - \lambda \partial^2_x)v + (v_{xx} - v + \frac{a_\varepsilon}{2}v^2)_x = 0, (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{5.4}
\]
At the same time, (5.2) is equal to the inequality
\[ |v(t_1) - Q_c(x)|_{H^1(\mathbb{R})} \leq K^* \varepsilon^{1/2}. \]

Now we make the research of the stability of (5.3) and rename \( v = u \).

Let \( T^* = \sup \{ t > t_1 | \forall t' \in [t_1, t), \exists \rho_2(t') \text{ s.t. } |u(t_1) - Q_c(x - \rho_2(t'))|_{H^1(\mathbb{R})} \leq K^* \varepsilon^{1/2} \} \).

The aim is to prove \( T^* = +\infty \). Therefore, for the sake of contradiction, in what follows we shall suppose \( T^* < +\infty \). The first step to reach a contradiction is now to decompose the solution on \([t_1, T^*)\) using modulation theory around the soliton. In particular, we find a special \( \rho_2(t) \) satisfying its definition but with
\[ \forall t \in [t_1, T^*), \sup \|u(t_1) - Q_c(x - \rho_2(t'))\|_{H^1(\mathbb{R})} \leq \frac{1}{2} K^* \varepsilon^{1/2}, \]
a contradiction with the definition of \( T^* \).

**Lemma 5.1.** There exist \( C^1 \)-functions \( \rho_2(t), c_2(t) \) defined on \([t_1, T^*)\), with \( c_2(t) > 0 \), the function \( z(t) \) given by,
\[ z(t) = u(t, x) - R(t, x), \quad \text{(5.4)} \]
where
\[ R(t, x) = Q_{c_2(t)}(x - \rho_2(t)), \quad y = x - \rho_2(t). \]

For all \( t \in [t_1, T^*] \), we have
\[ \int_{\mathbb{R}} (1 - \lambda \partial_x^2) R(t, x) z(t, x) dx = \int_{\mathbb{R}} (1 - \lambda \partial_x^2) R(t, x) (x - \rho_2(t)) z(t, x) dx = 0. \quad \text{(5.5)} \]

\[ (1 - \lambda \partial_x^2) z_1 + \{z_{xx} - z + \frac{\alpha_x}{2}[(z + R)^2 - R^2] + \frac{\alpha_x}{2}(1 - 1) R^2\} z + (1 - \lambda \partial_x^2) \Lambda Q_{c_2} c'_2 = 0. \quad \text{(5.6)} \]

\[ \frac{|c'_2|}{c_2} \leq K \int_{\mathbb{R}} e^{-\gamma |x - \rho_2(t)|} z^2 + Ke^{-\gamma |t|} + Ke^{-\gamma |t|} \|z\|_{H^1(\mathbb{R})}. \quad \text{(5.7)} \]

**Proof.** The proof of (5.5) is just like the formula (4.1) in the Lemma 4.1 with implicit function theorem.

According to
\[ (1 - \lambda \partial_x^2)(R + z)_t + (R_{xx} + z_{xx} - R - z + \frac{\alpha_x}{2} (R + z)^2)_x = 0, \]
\[ (1 - \lambda \partial_x^2) z_1 + (z_{xx} - z + \frac{\alpha_x}{2}[(z + R)^2 - R^2] + \frac{\alpha_x}{2}(1 - 1) R^2)_x + (1 - \lambda \partial_x^2) R_0 + (R_{xx} - R + R^2)_x = 0. \]

Where \( R_0 = \Lambda Q_{c_2} c'_2 - c_2 Q_{c_2} \) and \( R \) is the solution of \( (1 - \lambda \partial_x^2) R_t + (R_{xx} - R + R^2)_x = 0 \).

So, we take time derivative of (5.5)
\[ \int_{\mathbb{R}} (1 - \lambda \partial_x^2)(\Lambda Q_{c_2} c'_2 - c_2 Q_{c_2}) z dx + \int_{\mathbb{R}} (1 - \lambda \partial_x^2) R z_1 dx = 0. \]

Replace \( z_1 \) with (5.6)
\[ \int_{\mathbb{R}} (1 - \lambda \partial_x^2) \Lambda Q_{c_2} c'_2 (z - R) dx - \int_{\mathbb{R}} (1 - \lambda \partial_x^2) c_2 Q_{c_2} z dx \]
\[ - \int_{\mathbb{R}} \{z_{xx} - z + \frac{\alpha_x}{2}[(z + R)^2 - R^2] + \frac{\alpha_x}{2}(1 - 1) R^2\}_x R = 0 \]

Note that
\[ \int_{\mathbb{R}} z_{xxx} R = - \int_{\mathbb{R}} z_{xx} R_x = \int_{\mathbb{R}} z R_{xx} = - \int_{\mathbb{R}} z R_{xxx}, \quad \int_{\mathbb{R}} z_x R = - \int_{\mathbb{R}} z R_x, \]
So
\[
\int_{\mathbb{R}} (1 - \lambda \partial_x^2) \Lambda Q_{c_2} c_2'(z - R) dx + \int_{\mathbb{R}} z_x Q_{c_2}^2 - \int_{\mathbb{R}} \left( \frac{a_z}{2} [(z + R)^2 - R^2] + \frac{a_z}{2} - 1 \right) R x \, dx = 0.
\]
Moreover
\[
\int_{\mathbb{R}} \{ \frac{a_z}{2} [(z + R)^2 - R^2] + \frac{a_z}{2} - 1 \} R x \, dx = - \frac{1}{2} \int_{\mathbb{R}} a_z R_x z^2 + \frac{1}{2} \int_{\mathbb{R}} R^2 (a_z z)_x + \frac{\varepsilon}{6} \int_{\mathbb{R}} a_z' R^3.
\]
(5.6) can be simplified into the following form
\[
\int_{\mathbb{R}} (1 - \lambda \partial_x^2) \Lambda Q_{c_2} c_2'(z - R) dx - \frac{1}{2} \int_{\mathbb{R}} (a_z z - z)_x Q_{c_2}^2 + \frac{1}{2} \int_{\mathbb{R}} a_z R_x z^2 - \frac{\varepsilon}{6} \int_{\mathbb{R}} a_z' R^3 = 0,
\]
where
\[
|\varepsilon \int_{\mathbb{R}} a_z' R^3| \leq K \varepsilon e^{-\gamma \epsilon |t|},
\]
\[
|\int_{\mathbb{R}} (a_z z - z)_x R^2| \leq K \|z\|_{H^1(\mathbb{R})} e^{-\gamma \epsilon |t|},
\]
\[
|\int_{\mathbb{R}} a_z R_x z^2| \leq K \int_{\mathbb{R}} e^{-\gamma |y|} z^2.
\]
According to the Lemma 3.7, scaling
\[
\int_{\mathbb{R}} (1 - \lambda \partial_x^2) \Lambda Q_{c_2} (R - z) dx = \frac{1}{c_2} \int_{\mathbb{R}} Q_{c_2}^2,
\]
so
\[
\frac{|c_2'|}{c_2} \leq K \int_{\mathbb{R}} e^{-\gamma |x - \rho_2(t)|} z^2 + K \varepsilon e^{-\gamma \epsilon |t|} + K e^{-\gamma \epsilon |t|} \|z\|_{H^1(\mathbb{R})}.
\]
So, we take time derivative of
\[
\int_{\mathbb{R}} (1 - \lambda \partial_x^2) R(t, x)(x - \rho_2(t)) z(t, x) dx = 0,
\]
We can get
\[
\int_{\mathbb{R}} (1 - \lambda \partial_x^2) z_y R + \int_{\mathbb{R}} (1 - \lambda \partial_x^2) z y R_t - \int_{\mathbb{R}} (1 - \lambda \partial_x^2) z c_2 R = 0.
\]
Then
\[
\int_{\mathbb{R}} (1 - \lambda \partial_x^2) z R = 0,
\]
we replace \( z_t \), \( R_t \) and integrate by parts,
\[
c_2' \int_{\mathbb{R}} (1 - \lambda \partial_x^2) y \Lambda Q_{c_2} (z - R) - \int_{\mathbb{R}} (1 - \lambda \partial_x^2) c_2 y Q_{c_2} z
\]
\[+ \int_{\mathbb{R}} [z_{xx} - z + \frac{a_z}{2} ((z + R)^2 - R^2) + \frac{a_z}{2} - 1] R^2 (yR)_x = 0.
\]
Satisfying
\[- \int_{\mathbb{R}} (1 - \lambda \partial_x^2) c_2 y Q_{c_2} z + \int_{\mathbb{R}} (z_{xx} - z)(yR)_x = - \int_{\mathbb{R}} (R^2) x y z + 3 \int_{\mathbb{R}} R x z - \int_{\mathbb{R}} R z = 0.
\]
Therefore
\[
| - \int_{\mathbb{R}} (1 - \lambda \partial_x^2) c_2 y Q_{c_2} z + \int_{\mathbb{R}} (z_{xx} - z)(yR)_x | \leq K e^{-\gamma |y|} \|z\|_{L^2(\mathbb{R})}.
\]
Combining with the estimate of 

This finishes the proof of Lemma 5.1.

Proof of (5.8) is over.

Note that

We can obtain

Due to

So

Combining with the estimate of \( \frac{|c_1^2|}{c_2} \), finally,

This finishes the proof of Lemma 5.1.

\( \square \)

\textbf{Lemma 5.2}. For all \( t \in [t_1, T^*] \), we gain the boundary estimate,

\[
E_a[R](t) - E_a[R](t_1) + c_2(M[R](t) - M[R](t_1)) \leq \frac{1}{5} [(1 + c_2(t_1)) \frac{3}{2}(1 + \lambda c_2(t_1)) \frac{3}{2} - (1 + c_2(1)) \frac{3}{2}(1 + \lambda c_2(1)) \frac{3}{2}] \int R^2 + O(e^{-\gamma e|t|}). \tag{5.8}
\]

\textbf{Proof}. Due to \( \int_R (Q'_c)^2 = (1 + c) \frac{3}{2}(1 + \lambda c)^{-\frac{1}{2}} \int_R (Q')^2 \),

\[
\int_R (Q')^2 = \int_R Q^2 - \frac{2}{3} \int_R Q^3 = \frac{1}{5} \int_R Q^2.
\]

\[
M[R] = \frac{1}{2} \int_R \lambda R_x^2 + \frac{1}{2} \int_R R^2 = \frac{1}{2} (1 + c_2) \frac{3}{2}(1 + \lambda c_2)^{-\frac{1}{2}} \left[ \frac{(1 + c_2)}{5} + (1 + \lambda c_2) \right] \int R^2,
\]

\[
E_a[R] = \frac{1}{2} \int_R R_x^2 + \frac{\lambda}{2} \int_R R^2 - \frac{1}{3} \int_R \frac{a_x}{2} R^3 = \frac{1}{2} (1 + c_2) \frac{3}{2}(1 + \lambda c_2)^{-\frac{1}{2}} \left[ \frac{(1 + c_2)}{5} + (1 + \lambda c_2) \right] \int R^2 + \frac{1}{3} \int_R (1 - \frac{a_x}{2}) R^3
\]

Obviously, we get

\[
\left| \frac{1}{3} \int_R (1 - \frac{a_x}{2}) R^3 \right| \leq Ke^{-\gamma e|t|}.
\]

Note that

\[
E_a[R] + c_2 M[R] = \frac{1}{5} (1 + c_2) \frac{3}{2}(1 + \lambda c_2) \frac{3}{2} \int R^2 + O(e^{-\gamma e|t|})
\]

Proof of (5.8) is over. \( \square \)
Note that $M[u] = \frac{1}{2} \int_{\mathbb{R}} \lambda u_x^2 + \frac{1}{2} \int_{\mathbb{R}} u^2$, for all $t \in [t_1, T^*]$, we have
\[
\partial_t M[u] = \int_{\mathbb{R}} \lambda u_x u_{xt} + \int_{\mathbb{R}} u_t = \int_{\mathbb{R}} (1 - \lambda \partial_x^2) u_t u = - \int_{\mathbb{R}} (a_x u_x^2) u = - \frac{1}{3} \int_{\mathbb{R}} a_x' u^3 \leq 0.
\]
Namely $M[u](t) - M[u](t_1) \leq 0$, thus the mass is not conserved. In order to avoid this problem, we shall introduce a virial-type identity.

**Lemma 5.3. Virial estimate**
Let $\phi(x) \in C(\mathbb{R})$ be an even function satisfying the following properties:
(1) $\phi'(X) \leq 0, \forall x \in [0, +\infty)$.
(2) $\phi(X) \equiv 1, \forall x \in [0, 1]$.
(3) $\phi(X) = e^{-x}, \forall x \in [2, +\infty)$.
(4) $e^{-x} \leq \phi(X) \leq 3e^{-x}, \forall x \in [0, +\infty)$.

Note that, set $\psi = \int_0^x \phi$, it is an odd function. Moreover, for all $|x| \geq 2$,
\[
\psi(\pm \infty) - \psi(|x|) = e^{-|x|}.
\]

Finally, for all $A > 0$, denote
\[
\psi_A(x) = A(\psi(\infty) + \psi(\frac{x}{A})) > 0, \quad e^{-|x|/A} \leq \psi_A(x) \leq 3e^{-|x|/A}.
\]

We omit the proof of Lemma 5.3, because it is the same as the lemma in the paper [20].

**Lemma 5.4.** There exist constants $K, A_0, \delta, \gamma = \gamma(c_\infty, A_0)$, for all $t \in [t_1, T^*]$, we have
\[
\partial_t \int_{\mathbb{R}} z^2 \psi_A(x - \rho_2(t)) \leq K A_0 e^{-\gamma t} \|z\|_{H^1(\mathbb{R})} - \delta \int_{\mathbb{R}} (z^2 + z_x^2) e^{-\frac{|x - \rho_2(t)|}{A_0}}.
\]

**Proof.** According to the (5.6)
\[
\partial_t \int_{\mathbb{R}} z^2 \psi_A(y) = 2 \int_{\mathbb{R}} z_{yt} \psi_A(y) - \rho_2' \int_{\mathbb{R}} z^2 \psi_A'
\]
\[
\partial_t \int_{\mathbb{R}} z^2 \psi_A(y) = \int_{\mathbb{R}} [z_{xx} - z + \frac{\alpha_x}{2} (z^2 + 2zR)] [(1 - \lambda \partial_x^2)^{-1} z \psi_A(y)] x
\]
\[
+ \int_{\mathbb{R}} (\frac{\alpha_x}{2} - 1)R^2 [(1 - \lambda \partial_x^2)^{-1} z \psi_A(y)] x
\]
\[
- 2 \int_{\mathbb{R}} z \psi_A(y) \Lambda Q_{c_2} c_2' - \rho_2' \int_{\mathbb{R}} z^2 \psi_A',
\]
where
\[
\int_{\mathbb{R}} [z_{xx} - z + \frac{\alpha_x}{2} (z^2 + 2zR)] [(1 - \lambda \partial_x^2)^{-1} z \psi_A(y)] x \leq \delta \int_{\mathbb{R}} (z^2 + z_x^2) e^{-\frac{|x|}{A_0}}.
\]

Therefore
\[
-2 \int_{\mathbb{R}} z \psi_A(y) \Lambda Q_{c_2} c_2' - \rho_2' \int_{\mathbb{R}} z^2 \psi_A' \leq - \frac{\delta}{10} \int_{\mathbb{R}} (z^2 + z_x^2) e^{-\frac{|x|}{A_0}}.
\]
Simplifying
\[
\int_{\mathbb{R}} \left( \frac{\alpha}{2} - 1 \right) R^2 \left[ (1 - \lambda \partial_x^2)^{-1} z \psi_A(t) \right] x = \int_{\mathbb{R}} \left( \frac{\alpha}{2} - 1 \right) R^2 \left[ (1 - \lambda \partial_x^2)^{-1} z \psi_A(t) + (1 - \lambda \partial_x^2)^{-1} z \psi_A(t) \right] x.
\]
Obviously
\[
\left| \int_{\mathbb{R}} \left( \frac{\alpha}{2} - 1 \right) R^2 \left[ (1 - \lambda \partial_x^2)^{-1} z \psi_A(t) \right] x \right| \leq KA_0 e^{-\gamma \varepsilon |t|} \|z\|_{H^1(\mathbb{R})}.
\]
(5.9) can be proved by the plus of these estimates.

\[\square\]

**Corollary 1.** For the formula (5.8) in the Lemma 5.2, we have another conclusion,
\[
|E_a[R](t) - E_a[R](t_1) + c_2(M[R](t) - M[R](t_1))| < K \|z(t)\|_{H^1(\mathbb{R})} + KA_0 \|z(t_1)\|_{H^1(\mathbb{R})} + KA_0 \varepsilon e^{-\gamma \varepsilon t_1}.
\]
(5.10)

**Proof.** Taking the \(A_0\) large enough in formula (5.7) and Lemma 5.4, we take the integration of (5.9),
\[
|c_2(t) - c_2(t_1)| \leq KA_0 \|z(t)\|_{L^2(\mathbb{R})} + KA_0 \|z(t_1)\|_{L^2(\mathbb{R})} + KA_0 \varepsilon e^{-\gamma \varepsilon t_1}.
\]
Let
\[
f(c_2(t)) = (1 + c_2(t))^{\frac{3}{2}} (1 + \lambda c_2(t))^{\frac{1}{2}} - (1 + c_2(t))^{\frac{3}{2}} (1 + \lambda c_2(t))^{\frac{1}{2}}.
\]
Via Taylor expansion of the former formula at \(c_2(t_1),\)
\[
f(c_2(t)) = H(c_2(t) - c_2(t_1)) + O((c_2(t) - c_2(t_1))^2).
\]
where
\[
H = \frac{1}{2} (1 + c_2(t_1))^{\frac{3}{2}} (1 + \lambda c_2(t_1))^{-\frac{1}{2}} (5 + 6 \lambda c_2(t_1) + \lambda)
\]
is a limited quality. So, (5.10) can be obtained from the formula (5.8).

\[\square\]

**Lemma 5.5.** For all \(t \in [t_1, T^*],\) we have
\[
E_a[u] + c_2 M[u] = E_a[R] + c_2 M[R] + F_2(t) + O(e^{-\gamma \varepsilon |t|} \|z\|_{H^1(\mathbb{R})}),
\]
(5.11)

where
\[
F_2(t) = \frac{1}{2} \int_{\mathbb{R}} \left[ z_x^2 + z^2 + c_2(\lambda z_x^2 + z^2) \right] - \frac{1}{3} \int_{\mathbb{R}} \alpha z \left( (R + z)^3 - R^3 - 3R^2 z \right).
\]

**Proof.** Due to \(J_{\mathbb{R}} (1 - \lambda \partial_x^2) R z = 0,\)
\[
E_a[u] = \frac{1}{2} \int_{\mathbb{R}} u_x^2 + \frac{1}{2} \int_{\mathbb{R}} u_z^2 - \frac{1}{3} \int_{\mathbb{R}} \alpha z \left( (R + z)^3 - R^3 - 3R^2 z \right).
\]
The term \(\int_{\mathbb{R}} \alpha z R^2 z\) satisfies \(\int_{\mathbb{R}} \alpha z R^2 z \leq Ke^{-\gamma \varepsilon |t|} \|z\|_{H^1(\mathbb{R})},\)
\[
M[u] = \frac{1}{2} \int_{\mathbb{R}} \lambda u_x^2 + \frac{1}{2} \int_{\mathbb{R}} u_x^2 = \frac{1}{2} \int_{\mathbb{R}} \lambda (R_x + z_x)^2 + \frac{1}{2} \int_{\mathbb{R}} (R + z)^2
\]
\[
= M[R] + M[z] + \int_{\mathbb{R}} (\lambda R_z z_x + R z) = M[R] + M[z].
\]
So, this finishes the proof.

\[\square\]
Lemma 5.6. There exist constants $K$, $\gamma$, $\lambda_0 > 0$, for all $t \in [t_1, T^*]$, 
\[ F_2(t) \geq \lambda_0 \int_\mathbb{R} (x^2 + z^2) + O(e^{-\gamma t|t|}z^2_{H^1(\mathbb{R})}) + O(\|z\|_{H^2(\mathbb{R})}^3). \] (5.12)

Proof.

\[ F_2(t) = \frac{1}{2} \int_\mathbb{R} ((1 + \lambda c_2)x^2 + (1 + c_2)z^2) - \frac{1}{3} \int_\mathbb{R} \frac{ae}{2}((R + z)^3 - R^3 - 3R^2z). \]

Due to

\[ |\int_\mathbb{R} (\frac{ae}{2} - 1)Rz^2| = O(e^{-\gamma t|t|}z^2_{H^1(\mathbb{R})}), \]

we consider the function

\[ \phi_{R_0}(t, x) = \phi((x - \rho_2(t))/R_0), \]

obviously, when $|x - \rho_2(t)| < R_0$, we have

\[ 2 - ae \leq K e^{-\gamma t|x|} \leq K e^{\gamma R e^{-\gamma t} \rho_2(t)}. \]

We can also get $\phi_{R_0}(t, x) > e^{-R_0}$, on the region of $|x - \rho_2(t)| > R_0$. Moreover, for some constants $K, \gamma > 0$ satisfying $|(1 - \phi_{R_0})Qc_2| < K e^{-\gamma R_0}$,

\[ \frac{1}{2} \int_\mathbb{R} (1 + c_2)z^2 - \int_\mathbb{R} Rz^2 = \frac{1}{2} \int_\mathbb{R} \phi_{R_0}(1 + c_2)z^2 - \int_\mathbb{R} \phi_{R_0} Rz^2 + \frac{1}{2} \int_\mathbb{R} (1 - \phi_{R_0})z^2. \]

Indeed

\[ \frac{1}{2} \int_\mathbb{R} (1 - \phi_{R_0})(1 + c_2)z^2 - \frac{1}{2} \int_\mathbb{R} (1 - \phi_{R_0})Rz^2 \geq \frac{1}{2} (1 + c_2 - K e^{-\gamma R_0}) \int_\mathbb{R} (1 - \phi_{R_0})z^2, \]

so

\[ F_2(t) \geq \frac{1}{2} \int_\mathbb{R} \phi_{R_0}((1 + \lambda c_2)x^2 + (1 + c_2)z^2) + \frac{1}{2} \int_\mathbb{R} (1 - \phi_{R_0})((1 + \lambda c_2)x^2 + (1 + c_2)z^2) \]

\[ \geq \frac{1}{2} (1 + c_2) - K e^{-\gamma R_0}z^2 - \frac{1}{2} \int_\mathbb{R} (1 - \phi_{R_0})z^2 \]

\[ \geq \frac{1}{2} (1 + c_2) - K e^{-\gamma R_0}z^2 + O(\|z\|^3_{H^2(\mathbb{R})}) + O(\|z\|^3_{H^2(\mathbb{R})}). \]

Taking $R_0$ large enough, there exists $\lambda_0 > 0$ from the Anderson localization argument, we have

\[ F_2(t) \geq \lambda_0 \int_\mathbb{R} (z^2 + z^2) - K e^{\gamma R_0 e^{-\gamma t} \rho_2(t)} \int_\mathbb{R} (1 - \phi_{R_0})z^2 \]

\[ + O(e^{-\gamma t|t|}z^2_{H^1(\mathbb{R})}) + O(\|z\|^3_{H^2(\mathbb{R})}). \]

Finally, taking $\varepsilon$ smaller if necessary, we have (5.12). \qed
Now we prove that our assumption $T^* < +\infty$ leads to a contradiction. Indeed, from Lemma 5.5, 5.6, for all $t \in [t_1, T^*]$ and for $K > 0$

\[
\|z\|_{H^1(\mathbb{R})}^2 \leq K F_2(t) + E_a[u](t) - E_a[u](t_1) + c_2(M[u](t) - M[u](t_1)) \\
+ E_a[R](t_1) - E_a[R](t) + c_2(M[R](t_1) - M[R](t)) \\
+ K \varepsilon \sup_{t \in [t_1, T^*]} e^{-\gamma|t|}\|z\|_{L^2(\mathbb{R})} + K \sup_{t \in [t_1, T^*]} e^{-\gamma|t|}\|z\|_{L^2(\mathbb{R})}^3.
\]

From Lemma 5.1, Lemma 5.2, Corollary 5.1 and the energy conservation, we have

\[
\|z\|_{H^1(\mathbb{R})}^2 \leq K \varepsilon + c_2[M[u](t) - M[u](t_1)] \\
+ K \varepsilon \sup_{t \in [t_1, T^*]} e^{-\gamma|t|}\|z\|_{L^2(\mathbb{R})} + K \sup_{t \in [t_1, T^*]} e^{-\gamma|t|}\|z\|_{L^2(\mathbb{R})}^3.
\]

We know that the mass function is non-increasing

\[
M[u](t) - M[u](t_1) \leq 0.
\]

Taking $\varepsilon$ small and $D_0 = D_0(K)$ large enough, we have $\|z\|_{H^1(\mathbb{R})} \leq \frac{1}{4} D_0 \varepsilon$, which contradicts the definition of $T^*$. So, we obtain $T^* = +\infty$ and the conclusion is that,

\[
\sup_{t > t_1} \|u(t) - \frac{1}{2} Q c_2(t) (x - \rho_2(t))\|_{H^1(\mathbb{R})} \leq K \varepsilon^\frac{1}{4}.
\]

Proposition 5.1 is proved.

6. Main theorems. In this section, we summarize the dynamics, stability results and give the estimate of the soliton solution on the boundary of time.

Proof of theorem 1.1. The proof of (1.5) is similar to Theorem 3.2 in the paper [13], which gives identical estimate of the time. It is omitted here.

From the definition, we know

\[
\tilde{u}(-T_\varepsilon) - Q_c(x + T_\varepsilon) = R(-T_\varepsilon) - Q_c(x + T_\varepsilon) + w(T_\varepsilon).
\]

From the Proposition 3.2, we can get

\[
\|w(-T_\varepsilon)\|_{H^1(\mathbb{R})} = \|z A_h\|_{H^1(\mathbb{R})} \leq K \varepsilon^\frac{1}{4} e^{-\gamma|t|} \leq K \varepsilon^{10}.
\]

So

\[
\|R(-T_\varepsilon) - Q_c(x + T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K \varepsilon^{10}.
\]

In a similar way, we get

\[
\|\hat{u}(T_\varepsilon) - \frac{1}{2} Q c_\infty (x - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \leq K \varepsilon^{10}.
\]

According to the Proposition 4.1, let $k = \frac{1}{2}$, then

\[
\|u(T_\varepsilon + \rho_1(T_\varepsilon)) - \hat{u}(T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K \varepsilon^\frac{1}{4}, \ |\rho_1| \leq K \varepsilon^{-\frac{1}{2}} < \frac{T_\varepsilon}{100}.
\]

Via the triangle inequality

\[
\|u(T_\varepsilon + \rho_1(T_\varepsilon)) - \frac{1}{2} Q c_\infty (x - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \leq K \varepsilon^\frac{1}{4}.
\]

Let

\[
s(t) = t + \rho_1(t),
\]

then

\[
s'(t) = 1 + \rho_1'(t) > \frac{99}{100}.
\]
so
\[ s[-T_\varepsilon, T_\varepsilon] \subseteq \frac{99}{100} [-T_\varepsilon, T_\varepsilon]. \]

Let
\[ \tilde{T}_\varepsilon = T_\varepsilon + \rho_1(T_\varepsilon), \]

obviously, we have
\[ |T_\varepsilon - \tilde{T}_\varepsilon| < \frac{T_\varepsilon}{100}, \]

(1.6) is proved.

Proof of Theorem 1.2. Due to Corollary 5.1 and Lemma 5.6, we know
\[ |c_2(t) - c_2(t_1)| \leq KA_0 \|z\|_{L^2(R)}^2 + KA_0 \|z(t_1)\|_{L^2(R)}^2 + KA_0 \varepsilon e^{-\gamma \varepsilon t_1}, \]

\[ \|z\|_{H^1(R)}^2 \leq \frac{1}{4} D_0^2 \varepsilon. \]

Obviously
\[ |c_2 - c_\infty| \leq K \varepsilon^2. \]

Note that
\[ \sup_{t \geq t_1} \|u(t) - \frac{1}{2} Q_{c_2(t)}(x - \rho_2(t))\|_{H^1(R)} \leq K \varepsilon^2 \]

in Lemma 5.6, it completes the proof of Theorem 1.2.

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REFERENCES

[1] N. Asano, Wave propagation in non-uniform media, Progr. Theoret. Phys. Suppl., 55 (1974), 52–79.
[2] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersion systems, Philo. Trans. Roy. Soc. London Ser. A, 272 (1972), 47–78.
[3] J. L. Bona, W. G. Pritchard and L. R. Scott, Solitary-wave interaction, Phys. Fluids, 23 (1980) 438–441.
[4] J. L. Bona, P. E. Souganidis and W. A. Strauss, Stability and instability of solitary waves of Korteweg-de Vries type, Proc. Roy. Soc. London Ser. A, 411 (1987), 395–412.
[5] E. Fermi, J. Pasta and S. Ulam, Studies of nonlinear problems I, AMS, Editors: A. C. Newell, (1974), 143–156.
[6] R. Grimshaw, Slowly varying solitary waves. I. Korteweg - de Vries equation, Proc. Roy. Soc. London Ser. A, 368 (1979), 359–375.
[7] R. Grimshaw, Slowly varying solitary waves. II. Nonlinear Schrödinger equation, Proc. Roy. Soc. London Ser. A, 368 (1979), 377–388.
[8] R. H. J. Grimshaw and S. R. Pudjaprasetya, Generation of secondary solitary waves in the variable-coefficient Korteweg-de Vries equation, Stud. Appl. Math., 112 (2004), 271–279.
[9] V. I. Karpman and E. M. Maslov, Perturbation theory for solitons, Zh. Eksper. Teoret. Fiz., 73 (1977), 281–291.
[10] D. J. Kaup and A. C. Newell, Soliton as particles, oscillators, and in slowly changing media a singular perturbation theory, Proc. R. Soc. Lond., 361 (1978), 413–446.
[11] K. Ko and H. H. Kuehli, Korteweg-de Vries soliton in a slowly varying medium, Phys. Rev. Lett., 40 (1978), 233–236.
[12] R. LeVeque, On the interaction of nearly equal solitons in the KdV equation, SIAM. J. Appl. Math., 47 (1987), 254–262.
[13] Y. Martel, Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations, Amer. J. Math., 127 (2005), 1103–1140.
[14] Y. Martel and F. Merle, Inelastic interaction of nearly equal solitons for the bbm equation, *Discrete. Cont. Dyn.*, 27 (2010), 487–532.

[15] Y. Martel and F. Merle, Description of the Inelastic Collision of Two Solitary Waves for the BBM Equation, *Arch. Ration. Mech. An.*, 196 (2010), 517–574.

[16] Y. Martel and F. Merle, Inelastic interaction of nearly equal solitons for the quartic gKdV equation, *Invent. Math.*, 183 (2011), 563–648.

[17] Y. Martel and F. Merle, Description of two soliton collision for the quartic gKdV equation, *Ann. Math.*, 174 (2011), 757–857.

[18] J. Miller and M. Weinstein, Asymptotic stability of solitary waves for the regularized long wave equation, *Comm. Pure Appl. Math.*, 49 (1996), 399–441.

[19] T. Mizumachi, Asymptotic stability of solitary wave solutions to the regularized long wave equation, *J. Differ. Equations*, 200 (2004), 312–341.

[20] C. Muñoz, On the soliton dynamics under a slowly varying medium for generalized KdV equations, *Anal. PDE.*, 4 (2011), 573–638.

[21] C. Muñoz, Dynamics of soliton-like solutions for slowly varying, generalized gKdV equations: refraction vs. Reflection, *SIAM J. Math. Anal.*, 44 (2012), 1–60.

[22] C. Muñoz, Inelastic character of solitons of slowly varying gKdV equations, *Commun. Math. Phys.*, 314 (2012), 817–852.

[23] C. Muñoz, On the soliton dynamics under slowly varying medium for Nonlinear Schrodinger equations, *Math. Ann.*, 353 (2012), 867–943.

[24] C. Muñoz, Sharp inelastic character of slowly varying NLS solitons, preprint, arXiv:1202.5807v2.

[25] A. C. Newell, Solitons in mathematics and physics, *Society for Industrial and Applied Mathematics*, (1985), DOI: 10.1137/1.9781611970227.

[26] M. I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, *Comm. Pure Appl. Math.*, 39 (1986), 51–68.

[27] M. I. Weinstein, Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagations, *Commun. Part. Diff. Eq.*, 12 (1987), 1133–1173.

[28] J. Wright, Soliton production and solutions to perturbed Korteweg-de Vries equations, *Phys. Rev. A*, 21 (1980), 335–339.

[29] N. J. Zabusky and M. D. Kruskal, Interaction of “solitons” in a collisionless plasma and recurrence of initial states, *Phys. Rev. Lett.*, 15 (1965), 240–243.