Canonical forms for matrices of Saletan contractions

Dmytro R Popovych
Faculty of Mechanics and Mathematics, National Taras Shevchenko University of Kyiv, building 7, 2, Academician Glushkov Av., 03127 Kyiv, Ukraine
E-mail: devius@gmail.com

Abstract. We show that each Saletan (linear) contraction can be realized, up to change of bases of the initial and the target Lie algebras, by a matrix-function that is completely defined by a partition of the dimension of Fitting component of its value at the limit value of the contraction parameter. The codimension of the Fitting component and this partition constitute the signature of the Saletan contraction. We study Saletan contractions with Fitting component of maximal dimension and trivial one-part partition. All contractions of such kind in dimension three are completely classified.

1. Introduction
Historically, the first extensively studied kind of contractions of Lie algebras, after Segal introduced the general notion of contractions [1], was the class of Saletan (linear) contractions. Contractions of Lie algebras became known as a tool of theoretical physics after the famous papers by Inönü and Wigner [2,3] on an important specific subclass of linear contractions. Note that Inönü and Wigner planned to consider the whole class of linear contractions but they erroneously claimed in [2] that any linear contraction is diagonalizable. Even though Inönü and Wigner corrected their considerations in the next paper [3], they proceeded to exclusively study diagonalizable linear contractions, which due to their contribution are now called Inönü–Wigner contractions. The effectiveness of such contractions in applications is ensured by their close connection to subalgebras of initial algebras. More precisely, in modern terms the main result of [2], which is Theorem 1 at [2, p. 513], can be reformulated in the following way: Any Inönü–Wigner contraction of a Lie algebra \( g \) to a Lie algebra \( g_0 \) is associated with a subalgebra of \( g \), say \( s \), and starting with an arbitrary subalgebra of the algebra \( g \) one can construct an Inönü–Wigner contraction of this algebra. In the contracted algebra \( g_0 \) there exists an Abelian ideal \( i \) such that the factor-algebra \( g_0/i \) is isomorphic to \( s \).

A thorough study of linear contractions was conducted by Saletan in the course of preparation of his doctoral thesis and was published in [4]. In particular, Saletan obtained a simplified form for matrices of linear contractions up to reparametrization and basis change, derived a criterion for a linear matrix-function to be a contraction matrix, and gave the expression for the Lie bracket of the contracted Lie algebra. He also studied iterated linear contractions, related characteristics of the contraction matrix with the subalgebraic structure of the initial algebra, and discussed linear contractions of representations of Lie algebras.

Further studies by other authors extended rather than deepened Saletan’s results. Thus, Inönü–Wigner contractions of three- and four-dimensional Lie algebras were classified [5,6] due to the subalgebraic structure of these algebras being known [7]. Following Saletan, contractions...
realized by matrix-functions of the generalized form \( A\varepsilon + B\varepsilon^p \), where \( A \) and \( B \) are constant matrices and \( \varepsilon \) is the contraction parameter, were considered in a similar fashion [8–10]. Linear contractions of general algebraic structures were studied in [11].

In contrast to the above studies, this paper is aimed to enhance the original results by Saletan. We find the canonical form of Saletan contraction matrices, which creates the basis for introducing the notion of Saletan contraction’s signature, for developing an algorithm for computation of Saletan contractions, and for posing new problems in this field.

The structure of the paper is the following: Basic notions and results on contractions and, specifically, on Saletan contractions are presented in Section 2. The main result of the paper, Theorem 1, which deals with the canonical form of Saletan contraction matrices, is proved in Section 3. After defining the notion of Saletan signature, we relate the signature of a Saletan contraction with the nested chain of subalgebras of the initial algebra that corresponds to this contraction. In Section 4 we carry out a preliminary study of Saletan contractions associated with chains of nested subalgebras of the maximal possible length, which coincides with the algebra dimension. Then we exhaustively describe such contractions between three-dimensional Lie algebras over the complex (resp. real) field. In the final section, we discuss obtained results and propose new problems for the further investigation.

2. Basic notions and auxiliary results

Given a finite-dimensional vector space \( V \) over the field \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \), by \( \mathcal{L}_n = \mathcal{L}_n(\mathbb{F}) \) we denote the set of all possible Lie brackets on \( V \), where \( n = \dim V < \infty \). Each element \( \mu \in \mathcal{L}_n \) corresponds to a Lie algebra with the underlying space \( V, \mathfrak{g} = (V, \mu) \). Fixing a basis \( \{e_1, \ldots, e_n\} \) of \( V \) leads to a bijection between \( \mathcal{L}_n \) and the set of structure constant tensors

\[
\mathcal{C}_n = \left\{ (c^k_{ij}) \in \mathbb{F}^{n^3} \mid c^k_{ij} + c^k_{ji} = 0, c^k_{ij}c^l_{ik} + c^l_{jk}c^k_{ij} + c^l_{jk}c^k_{ij} = 0 \right\}.
\]

The structure constant tensor \((c^k_{ij}) \in \mathcal{C}_n\) associated with a Lie bracket \( \mu \in \mathcal{L}_n \) is given by the formula \( \mu(e_i, e_j) = c^k_{ij}e_k \). Here and in what follows, the indices \( i, j, k, i', j', k', p \) and \( q \) run from 1 to \( n \) and the summation convention over repeated indices is assumed. The right action of the group \( \text{GL}(V) \) on \( \mathcal{L}_n \), which is conventional for the physical literature, is defined as

\[
(U \cdot \mu)(x, y) = U^{-1}(\mu(Ux, Uy)) \quad \forall U \in \text{GL}(V), \forall \mu \in \mathcal{L}_n, \forall x, y \in V.
\]

Definition 1. Given a Lie bracket \( \mu \in \mathcal{L}_n \) and a continuous matrix function \( U : (0, 1] \to \text{GL}(V) \), we construct the parameterized family of Lie brackets \( \mu_\varepsilon = \mu \circ U_\varepsilon, \varepsilon \in (0, 1) \). Each Lie algebra \( \mathfrak{g}_\varepsilon = (V, \mu_\varepsilon) \) is isomorphic to \( \mathfrak{g} = (V, \mu) \). If the limit

\[
\lim_{\varepsilon \to +0} \mu_\varepsilon(x, y) = \lim_{\varepsilon \to +0} U_\varepsilon^{-1} \mu(U_\varepsilon x, U_\varepsilon y) =: \mu_0(x, y)
\]

exists for any \( x, y \in V \), then \( \mu_0 \) is a well-defined Lie bracket. The Lie algebra \( \mathfrak{g}_0 = (V, \mu_0) \) is called a one-parametric continuous contraction (or simply a contraction) of the Lie algebra \( \mathfrak{g} \). We call a limiting process that provides \( \mathfrak{g}_0 \) from \( \mathfrak{g} \) with a matrix function a realization of the contraction \( \mathfrak{g} \to \mathfrak{g}_0 \).

The notion of contraction is extended to the case of an arbitrary algebraically closed field in terms of orbit closures in the variety of Lie brackets, see, e.g., [12–16].

If a basis \( \{e_1, \ldots, e_n\} \) of \( V \) is fixed, then the operator \( U_\varepsilon \) can be identified with its matrix \( U_\varepsilon \in \text{GL}_n(\mathbb{F}) \), which is denoted by the same symbol, and Definition 1 can be reformulated in terms of structure constants. Let \( C = (c^k_{ij}) \) be the tensor of structure constants of the algebra \( \mathfrak{g} \) in the basis chosen. Then the tensor \( C_\varepsilon = (c^k_{ij,\varepsilon}) \) of structure constants of the algebra \( \mathfrak{g}_\varepsilon \) in this
basis is the result of the action by the matrix $U_\varepsilon$ on the tensor $C$, $C_\varepsilon = C \circ U_\varepsilon$. In term of components this means that
\[
c_{\varepsilon,i,j}^k = (U_\varepsilon)^i_l (U_\varepsilon)^j_l (U_\varepsilon^{-1})_l^k c_{l,j}^{k'}.
\]
Then Definition 1 is equivalent to that the limit
\[
\lim_{\varepsilon \to +0} c_{\varepsilon,i,j}^k =: c_{0,i,j}^k
\]
eexists for all values of $i$, $j$ and $k$ and, therefore, $c_{0,i,j}^k$ are components of the well-defined structure constant tensor $C_0$ of the Lie algebra $\mathfrak{g}_0$. The parameter $\varepsilon$ and the matrix-function $U_\varepsilon$ are called a contraction parameter and a contraction matrix, respectively.

The following useful assertion is obvious.

**Lemma 1.** If the matrix $U_\varepsilon$ of a contraction $\mathfrak{g} \to \mathfrak{g}_0$ is represented in the form $U_\varepsilon = \hat{U}_\varepsilon \tilde{U}_\varepsilon$, where $\hat{U}$ and $\tilde{U}$ are continuous functions from $[0,1]$ to $\text{GL}_n(\mathbb{F})$ and the function $\hat{U}$ has a limit $\hat{U}_0 \in \text{GL}_n(\mathbb{F})$ at $\varepsilon \to +0$, then $\hat{U}_\varepsilon \tilde{U}_0$ also is a matrix of the contraction $\mathfrak{g} \to \mathfrak{g}_0$.

**Remark 1.** Lemma 1 implies that $\hat{U}_\varepsilon$ is a matrix of the equivalent contraction $\mathfrak{g} \to \tilde{\mathfrak{g}}_0$, where $\tilde{\mathfrak{g}}_0 = (V, \mu_0 \circ \hat{U}_0^{-1})$ is the algebra isomorphic to $\mathfrak{g}_0$ with respect to the matrix $U_0^{-1}$.

Historically, the first contractions studied were the ones realized by linear matrix-functions.

**Definition 2.** A realization of a contraction with a matrix-function that is linear in the contraction parameter is called a Saletan (linear) contraction [4].

This class of contractions includes the Inönü–Wigner contractions [2–4].

The matrix of any linear contraction has a well-defined limit at $\varepsilon = 0$. This is why in contrast to the general definition of contractions, in the case of a linear contraction its matrix-function $U_\varepsilon$ can be assumed to be defined on the closed interval $[0,1]$. Then it is convenient to represent the matrix $U_\varepsilon$ in the form $U_\varepsilon = (1-\varepsilon)U_0 + \varepsilon U_1$, where $U_0$ and $U_1$ are the values of $U_\varepsilon$ at $\varepsilon = 0$ and $\varepsilon = 1$, respectively [4]. By the definition of contraction matrix, the matrix $U_1$ is nonsingular, and, for proper contractions, the matrix $U_0$ is necessarily singular.

There exist specific reparametrizations that preserve the class of Saletan contractions [4]. Let $U_\varepsilon = B + \varepsilon A$ be the matrix of a Saletan contraction. We fix $\lambda > -1$ and consider the matrix-function $U_\varepsilon$ on the interval $[0, (1+\lambda)^{-1}]$ instead of $[0,1]$. Then
\[
B + \varepsilon A = (1-\varepsilon)B + \varepsilon(A + \lambda B) = (1-\varepsilon) \left( B + \frac{\varepsilon}{1-\lambda \varepsilon} (A + \lambda B) \right).
\]
The multiplier $(1-\varepsilon)$ is not essential since its limit at $\varepsilon = 0$ equals 1. Removing this multiplier and denoting $\varepsilon/(1-\lambda \varepsilon)$ by $\hat{\varepsilon}$, we obtain the well-defined linear matrix-function
\[
\hat{U}_\varepsilon = B + \hat{\varepsilon}(A + \lambda B), \quad \hat{\varepsilon} \in [0,1],
\]
which realizes the same Saletan contraction as $U_\varepsilon$.

### 3. Canonical forms of Saletan contraction matrices

We denote the $m \times m$ unit matrix by $E^m$, and $m \times m$ Jordan block with an eigenvalue $\lambda$ by $J_\lambda^n$.

**Theorem 1.** Up replacing the algebras $\mathfrak{g}$ and $\mathfrak{g}_0$ with isomorphic ones, every Saletan contraction $\mathfrak{g} \to \mathfrak{g}_0$ is realized by a matrix of the canonical form
\[
E^{n_0} \oplus J_\lambda^{n_1} \oplus \cdots \oplus J_\lambda^{n_s}, \quad \text{or, equivalently,} \quad E^{n_0} \oplus J_0^{n_1} \oplus \cdots \oplus J_0^{n_s} + \varepsilon E^n,
\]
where $n_0 + \cdots + n_s = n$. 


Definition 3. Theorem 1 means that any Saletan contraction can be realized by a matrix of the form $A S_\varepsilon B$, where $A$ and $B$ are constant nonsingular matrices and the linear function $S_\varepsilon$ is in the canonical form (1). Then the tuple $(n_0; n_1, \ldots, n_s)$, where $n_1, \ldots, n_s$ constitute a partition of the dimension $n - n_0$ of the Fitting null component relative to $U_0$ and $n_0 \in \{0, \ldots, n\}$, is called the signature of this Saletan contraction.

Due to containing a partition of $n - n_0$, the signature of a Saletan contraction is defined up to permutation of its parts excluding the first one. Saletan contraction with signature $(n)$ is improper, i.e., the contracted algebra is isomorphic to the initial one. So, for a proper Saletan contraction we necessarily have $n_0 < n$. Inönü–Wigner contractions are associated with Saletan signatures of the form $(n_0; 1, \ldots, 1)$. The Saletan signature $(0; 1, \ldots, 1)$ corresponds to the trivial contraction to the Abelian algebra.

The necessary and sufficient condition for the algebra $\mathfrak{g}$ to be contracted by the linear matrix-function $U_\varepsilon$ [4] is

$$U_0^2[x,y]^0 - U_0[U_0x,y]^0 - U_0[x,U_0y]^0 + [U_0x,U_0y]^0 = 0 \quad \forall x, y \in V. \quad (2)$$

Here and in what follows $[\cdot, \cdot]^0$ and $[\cdot, \cdot]^1$ denote the projections of the Lie brackets $[\cdot, \cdot]$ on the subspaces $V_0$ and $V_1$, respectively, which are not, in general, Lie brackets. Then the contracted Lie bracket is defined by

$$[x, y]_0 = W_1^{-1}[U_0x, U_0y]^1 - W_0[x, y]^0 + [U_0x, y]^0 + [x, U_0y]^0 \quad \forall x, y \in V.$$
The use of the canonical form of $U_z$ simplifies analysis of both the necessary and sufficient conditions and properties of the contracted Lie bracket. In particular, then $W_1^{-1} = E^{n-n_0}$. We would like to emphasize that changing the basis of the underlying space without applying Lemma 1 can simplify the matrix $W_1$ only to its Jordan form.

**Remark 2.** If $U_0$ is the value of the matrix of a well-defined Saletan contraction of the Lie algebra $g$ at $\varepsilon = 0$, then each power of $U_0$ is the value of the matrix of another well-defined Saletan contraction $g$ at $\varepsilon = 0$. The image $\text{im} U_0$ of $U_0$ is a subalgebra of $g$. Combining the above two claims, we have that for each $m = 0, 1, 2, \ldots$ the image $s_m := \text{im} U_0^m$ of $U_0^m$ is also a subalgebra of $g$, and $s_m = V_1$ if $m \geq m_0 := \max(n_1, \ldots, n_s)$ [4]. In other words, the matrix of any Saletan contraction is associated with the chain of nested subalgebras

$$s_0 := g \supset s_1 \supset s_2 \supset \cdots \supset s_{m_0} = V_1.$$ 

The dimensions of these subalgebras are completely defined by the contraction signature,

$$\dim s_m = n - l_1 - \cdots - l_m, \quad m = 0, \ldots, m_0, \quad \text{where} \quad l_m := |\{n_i \mid n_i \geq m, i = 1, \ldots, s\}|.$$ 

In particular, $\dim s_{m_0} = n_0$. The above relation establishes necessary conditions of consistency between the structure of a Lie algebra and signatures of its Saletan contractions.

**Example 1.** Consider the real three-dimensional orthogonal algebra $so(3)$ with the canonical commutation relations $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$. The algebra $so(3)$ has no two-dimensional subalgebras. Therefore, the only possible signature of a proper Saletan contraction of $so(3)$ is $(1, 1, 1)$. The first canonical form of the contraction matrix with this signature is $E^1 \oplus J^1_1 \oplus J^1_2$, which realizes the single In"on"u–Wigner contraction of $so(3)$, which is to the Euclidean algebra $e(2)$ defined by the nonzero commutation relations $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$. This implies that the other proper contraction of $so(3)$, which is to the Heisenberg algebra $h(1)$, cannot be realized as a Saletan contraction, cf. [4].

### 4. Saletan contractions with signature $(0; n)$

There are two different ways of studying Saletan contractions. Given a fixed pair of Lie algebras, one can check whether there exists a Saletan contraction between these algebras and then try to describe all possible Saletan contractions for this pair. The other way is to describe all possible contractions which are realized by contraction matrices with certain signature. A disadvantage of this way is the necessity of classifying Lie algebras that satisfy specific constraints.

In this section we consider contractions with the signature $(0; n)$. Choosing this signature poses the most restrictive constraints on the structure of the initial Lie algebra $g$ compared to other Saletan signatures, cf. equation (2). In particular, the algebra $g$ should contain a nested chain of $n$ nonzero Lie subalgebras and, in general, this condition on $g$ is not sufficient.\(^1\)

For the signature $(0; n)$ we have $V_0 = V$ and we can set

$$U_0 = J^n_0.$$ 

Then $[\cdot, \cdot]^0 = [\cdot, \cdot]$ and the Saletan condition (2) takes the form

$$[U_0 x, U_0 y] - U_0 [U_0 x, y] = U_0 ([x, U_0 y] - U_0 [x, y]) \quad \forall x, y \in V.$$ 

or, equivalently, $[\text{ad}_{U_0 x}, U_0] = U_0 [\text{ad}_x, U_0]$ for any $x \in V$. Here and in what follows, $[A, B]$ denotes the commutator of operators $A$ and $B$, $[A, B] := AB - BA$. Specifying this condition for

\(^1\) This strongly differs from the case of In"on"u–Wigner contractions, for which there exists a one-to-one correspondence with proper subalgebras of $g$. 

5
basis elements, for which \( U_0 = J_0^n \), we derive \([\text{ad}_{e_i}, U_0] = U_0 [\text{ad}_{e_{i+1}}, U_0] = \cdots = U_0^{n-i} [\text{ad}_{e_n}, U_0] \), i.e.,

\[
[\text{ad}_{e_i}, U_0] = U_0^{n-i}[A, U_0],
\]

(3)

where we use the notation \( A = (a_{ij}) := \text{ad}_{e_i} \). For each fixed \( i \), the equation (3) can be considered as inhomogeneous linear system of algebraic equations with respect to entries of the matrix \( \text{ad}_{e_i} \).

A particular solution of this system is given by \( U_0^{n-i}A \), since \([U_0^k A, U_0] = U_0^k AU_0 - U_0^k = A = U_0^k[A, U_0] \). The solution space of the corresponding homogeneous system \([\text{ad}_{e_i}, U_0] = 0 \) coincides with the space of matrices commuting with \( U_0 \), which is spanned by the powers of \( U_0 \) due to \( U_0 \) being a single Jordan block. Therefore, the general solution of the system (3) is

\[
\text{ad}_{e_i} = t_0^{n-i}A + b_{ij}t_0^{n-j}
\]

with parameters \( b_{ij} \), where \( b_{nj} = 0 \) as \( \text{ad}_{e_n} = A \) by definition and \( a_{in} = 0 \) as \( Ae_n = [e_n, e_n] = 0 \). Recall that we assume the summation convention over repeated indices. The Lie bracket is skew-symmetric, which implies

\[
[e_i, e_n] + [e_n, e_i] = \text{ad}_{e_i} e_n + \text{ad}_{e_n} e_i = b_{ij} e_j + a_{ji} e_j = 0,
\]

i.e., \( b_{ij} + a_{ji} = 0 \). In other words, the commutation relations of the algebra \( \mathfrak{g} \) are

\[
[e_i, e_j] = \text{ad}_{e_i} e_j = U_0^{n-i}Ae_j - a_{ki} U_0^{n-k} e_j = a_{kj} e_{k+i-n} - a_{ki} e_{k+j-n} = (a_{p+n-i,j} - a_{p+n-j,i})e_p.
\]

(4)

Here and in what follows, if an index goes beyond the index interval \( \{1, \ldots, n\} \), then the corresponding object is assumed zero. Thus, in view of (4) the skew-symmetric property of the Lie bracket obviously holds for any pair of elements of \( \mathfrak{g} \). Note that the number of essential parameters in the above commutation relations does not exceed \( n(n-1) \). The Jacobi identity imposes more constraints in the form of a system of quadratic equations with respect to entries of the matrix \( A \),

\[
(a_{p+n-i,j} - a_{p+n-j,i})(q_{q+n-k,p} - a_{q+n-p,k}) +
(a_{p+n-j,k} - a_{p+n-k,j})(q_{q+n-i,p} - a_{q+n-p,i}) +
(a_{p+n-k,i} - a_{p+n-i,k})(q_{q+n-j,p} - a_{q+n-p,j}) = 0.
\]

Unfortunately, we were not able to solve this system for an arbitrary dimension of the underlying space.

The contracted Lie bracket is defined by \([x, y]_0 = [U_0 x, y] + [x, U_0 y] - U_0 [x, y] \) for all \( x, y \in V \). Hence, the commutation relations of the contracted algebra \( \mathfrak{g}_0 \) are

\[
[e_i, e_j]_0 = [e_i, e_j] + [e_i, e_{j-1}] - U_0 [e_i, e_j] = (a_{p+n-i+1,j} - a_{p+n-j,i-1})e_p + (a_{p+n-i,j-1} - a_{p+n-j+1,i})e_p - (a_{p+n-i,j} - a_{p+n-j,i})e_p = (a_{p+n-i,j-1} - a_{p+n-j,i})e_p.
\]

In particular, \([e_n, e_j]_0 = (a_{p,j-1} - a_{p+n-j-1,j})e_p \). Consider the matrix \( A_0 = (a_{0,ij}) \), where \( a_{0,ij} = a_{i,j-1} - a_{i+n-j,n-1} \). In terms of \( A \) and \( J_0^n \) we have the representation

\[
A_0 = AJ_0^n - \sum_{i=0}^{n-1} a_{n-i,n-1}(J_0^n)^i.
\]
Roughly speaking, the matrix $A_0$ is obtained from the matrix $A$ by shifting the columns of $A$ to the right, filling of the first column by zeros and subtracting a specific linear combination of powers of $J^n_0$ that gives zeros in the last column of $A_0$. The structure of the algebra $\mathfrak{g}_0$ is defined in terms of the matrix $A_0$ in the same way as the structure of the algebra $\mathfrak{g}$ is defined in terms of the matrix $A$ since $a_{p+n-j,j-1} - a_{p+n-j,j-1} = a_{0,p+n-j,j} - a_{0,p+n-j,j}$. This is consistent with Lemma 3 of [4]. Indeed, as the algebra $\mathfrak{g}_0$ can be contracted by the same matrix $U = J^n_0$, its structure constants satisfy the same constraints imposed by the Saletan conditions (2). Lemma 3 of [4] also implies that $n$ iterations of this contraction leads to the Abelian algebra.

We exhaustively study the case $n = 3$. There are three essential relations among the commutation relations (4) with $n = 3$,

\[
[e_3, e_1] = a_{p1} e_p, \\
[e_3, e_2] = a_{p2} e_p, \\
[e_1, e_2] = (a_{32} - a_{21}) e_1 - a_{31} e_2,
\]

and the single Jacobi identity $[e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] = 0$. Collecting coefficients of basis elements in the Jacobi identity and making additional arrangements, we obtain the following system of equations on entries of the matrix $A$:

\[
a_{31}a_{21} = 0, \quad a_{31}a_{12} = 0, \quad a_{31}(a_{11} - a_{22}) = 0, \quad a_{21}(2a_{32} - a_{21}) = 0, \\
a_{32}(a_{11} - a_{22}) + a_{12}a_{22} = 0.
\]  

A consequence of the system is $a_{21}(a_{11} + a_{22}) = 0$.

In order to simplify the form of the matrix $A$, we can use the transition to a Lie algebra isomorphic to $\mathfrak{g}$ or, equivalently, changing the basis of the underlying space. In view of problem’s statement, admitted basis changes are those whose matrices commute with the matrix $U_0 = J^n_0$. Therefore, each of such matrices is a linear combination of powers of $U_0$,

\[
S = \gamma(E + \alpha U_0 + \beta U_0^2),
\]

where $\alpha$, $\beta$ and $\gamma$ are arbitrary constants with $\gamma \neq 0$ and $E$ is the $3 \times 3$ identity matrix. The inverse of $S$ is

\[
S^{-1} = \gamma^{-1}(E - \alpha U_0 + (\alpha^2 - \beta) U_0^2).
\]

The expressions for entries of the transformed matrix $\tilde{A}$ follow from those for the transformed Lie brackets $[e_3, e_1] \sim$ and $[e_3, e_2] \sim$. We have

\[
[e_3, e_1] \sim = S^{-1} [Se_3, Se_1] = \gamma(a_{11} - \alpha a_{32} - \beta a_{31}) e_1 + \gamma a_{21} e_2 + \gamma a_{31} e_3, \\
[e_3, e_2] \sim = S^{-1} [Se_3, Se_2] = \gamma(a_{12} + \alpha(a_{11} - a_{22}) - \beta a_{21}) e_1 + \gamma(a_{22} + \alpha a_{12} - \beta a_{31}) e_2 + \gamma(a_{32} + \alpha a_{31}) e_3,
\]

i.e.,

\[
\tilde{a}_{11} = \gamma(a_{11} - \alpha a_{32} - \beta a_{31}), \quad \tilde{a}_{12} = \gamma(a_{12} + \alpha(a_{11} - a_{22}) - \beta a_{21}), \\
\tilde{a}_{21} = \gamma a_{21}, \quad \tilde{a}_{22} = \gamma(a_{22} + \alpha(a_{21} - a_{32}) - \beta a_{31}), \\
\tilde{a}_{31} = \gamma a_{31}, \quad \tilde{a}_{32} = \gamma(a_{32} + \alpha a_{31}).
\]

The contracted algebra $\mathfrak{g}_0$ is defined by the commutation relations

\[
[e_3, e_1]_0 = -a_{32} e_1, \\
[e_3, e_2]_0 = (a_{11} - a_{22}) e_1 + (a_{21} - a_{32}) e_2 + a_{31} e_3, \\
[e_1, e_2]_0 = a_{31} e_1.
\]
We study possible cases of the solutions of the system (5) up to allowed basis changes.

If \( a_{31} \neq 0 \), then the system (5) implies that \( a_{21} = a_{12} = 0 \) and \( a_{11} = a_{22} = 0 \). Selecting certain values of the parameters \( \alpha, \beta \), and \( \gamma \) of the basis transformation, we can set \( a_{32} = 0, a_{11} = a_{22} = 0 \) and \( a_{31} = -1 \). In other words, the commutation relations of the algebra \( g \) take the form \([e_1, e_2] = e_2, [e_1, e_3] = e_3 \) and \([e_2, e_3] = 0\). Hence the basis elements \( e_2 \) and \( e_3 \) span the maximal Abelian ideal of the algebra \( g \), and the element \( e_1 \) acts on this ideal as the identity operator, i.e. the algebra \( g \) is the almost Abelian algebra associated with the identity operator, which is denoted by \( g_{3,3} \) in Mubarakzyanov’s classification of three-dimensional Lie algebras [17].

In contrast to Example 1, in what follows we mostly use Mubarakzyanov’s notations. For the contracted algebra \( g_0 \), the commutation relations are: \([e_3, e_1]_0 = 0, [e_2, e_3]_0 = e_3, [e_2, e_1]_0 = e_1\). Therefore, this algebra is isomorphic to the initial algebra \( g \). An isomorphism is established by a permutation of the basis elements. This means that the contraction is improper.

Suppose that \( a_{31} = 0 \) and \( a_{21} \neq 0 \). The solution of the system (5) gives \( a_{32} = \frac{1}{2} a_{21} \), \( a_{21} = -a_{11}, a_{22} = -a_{11} \) and \((a_{21} - a_{12})a_{11} = 0\). The constants \( a_{11}, a_{22}, a_{12} \) and \( a_{21} \) can be set to \( 0, 0, 0 \), and \(-2\), respectively, by changing the basis with an appropriate matrix \( S \). As a result, we obtain the canonical commutation relations of the algebra \( sl(2, \mathbb{R}) \), \([e_1, e_2] = e_1, [e_2, e_3] = e_3, [e_1, e_3] = 2e_2\). The contracted algebra \( g_0 \) is isomorphic to the algebra \( g_{3,3} \), which can be seen from its commutation relations, \([e_1, e_2]_0 = 0, [e_2, e_3]_0 = e_2, [e_1, e_3]_0 = e_1\).

In the case \( a_{31} = a_{21} = 0 \) and \( a_{32} \neq 0 \) the system (5) is reduced to the single equation \( a_{32}(a_{11} - a_{22}) + a_{12}a_{22} = 0 \). Carrying out an admitted basis transformation, we select certain values of the parameters \( \alpha \) and \( \gamma \) of the transformation matrix \( S \) in order to set \( a_{22} = 0 \) and \( a_{32} = -1 \). Then the above equation implies that \( a_{11} = 0 \). Finally, the commutation relations of \( g \) take the form \([e_1, e_3] = 0, [e_2, e_3] = e_3 - a_{12} e_1, [e_2, e_1] = e_1 \), i.e., \( g \) is an almost Abelian algebra associated with the matrix

\[
\begin{pmatrix}
1 & -a_{12} \\
0 & 1
\end{pmatrix}.
\]

The contracted algebra has the same commutation relations as in the previous case, \( g_0 \sim g_{3,3} \). If \( a_{12} = 0 \), then the contraction \( g \to g_0 \) is improper since \( g \sim g_{3,3} \). For \( a_{12} \neq 0 \), the contraction is equivalent to the unit fall\(^2\) the matrix associated with the algebra \( g \sim g_{3,2} \), and the resulting matrix defines the algebra \( g_0 \sim g_{3,3} \).

The last case is given by \( a_{31} = a_{21} = a_{32} = 0 \). The single equation remaining in the system (5) is \( a_{12}a_{22} = 0 \). The commutation relations of the initial and the contracted algebras are respectively

\[
\begin{align*}
[e_3, e_1] &= a_{11} e_1, & [e_3, e_1]_0 &= 0, \\
[e_3, e_2] &= a_{12} e_1 + a_{22} e_2, & [e_3, e_2]_0 &= (a_{11} - a_{22}) e_1, \\
[e_1, e_2] &= 0, & [e_1, e_2]_0 &= 0.
\end{align*}
\]

Consider subcases depending on values of the remaining parameters. If \( a_{11} \neq a_{22} \), then by selecting a proper value of \( \alpha \) in the transformation matrix \( S \) we can set \( a_{12} = 0 \). The parameter \( \beta \) is not essential here, and we can choose the zero value for it. The parameter \( \gamma \) can be used for scaling a nonzero linear combination of \( a_{11} \) and \( a_{22} \) (e.g., \( a_{11} - a_{22} \)) to the unity. As a result, we have the contraction of the almost Abelian algebra \( g = g_{3,4} \) associated with the diagonal (but not proportional to the identity matrix) matrix to the three-dimensional Heisenberg algebra \( h(1) = g_{3,1} \). If \( a_{11} = a_{22} \), the contracted algebra is Abelian, i.e., we have the

\(^2\) The classical Lie algebras \( h(1), e(2), sl(2, \mathbb{R}) \) and \( so(3) \) are denoted by Mubarakzyanov as \( g_{3,1}, g_{3,5}, g_{3,6} \) and \( g_{3,7} \) respectively.

\(^3\) In the case of \( 2 \times 2 \) Jordan blocks, the only possible unit fall is the replacement of the value 1 in the \((1,2)\)th entry by 0.
trivial contraction of an almost Abelian Lie algebra (one of $\mathfrak{g}_{3.1}$, $\mathfrak{g}_{3.2}$, $\mathfrak{g}_{3.3}$ and $3\mathfrak{g}_1$, depending on values of $a_{11} = a_{22}$ and $a_{12}$).

**Proposition 1.** Saletan contractions with the signature $(0; 3)$ realize only the following contractions between three-dimensional Lie algebras: the proper contractions $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}_{3.3}$, $\mathfrak{g}_{3.2} \to \mathfrak{g}_{3.3}$, $\mathfrak{g}_{3.4} \to \mathfrak{g}_{3.1}$, and $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 \to \mathfrak{g}_{3.3}$, the trivial contractions of $\mathfrak{g}_{3.1}$, $\mathfrak{g}_{3.2}$ and $\mathfrak{g}_{3.3}$ to $3\mathfrak{g}_1$, as well as the improper contractions $\mathfrak{g}_{3.3} \to \mathfrak{g}_{3.3}$ and $3\mathfrak{g}_1 \to 3\mathfrak{g}_1$.

5. Conclusion

The main result of the paper is Theorem 1, which describes the canonical form of Saletan contractions. The proved existence of the canonical form for each Saletan contraction gives a specific finite tuple of non-negative integers which corresponds to this contraction and is called its signature. The signature of a Saletan contraction completely defines its canonical form. Introducing the notion of signature leads to posing several interesting problems, which are related to Saletan contractions.

Thus, for each Lie algebra the set of its possible Saletan contractions is partitioned into the subsets corresponding to different Saletan signatures. This allows us to pose the problem on describing Saletan contractions with a fixed signature. The well-known Inönü–Wigner contractions constitute a subclass of Saletan contractions, which is singled out by Saletan signatures of the form $(n_0; 1, \ldots, 1)$. Therefore, the study of Saletan contractions includes, as its simplest part, the study of Inönü–Wigner contractions. Inönü–Wigner contractions of three- and four-dimensional Lie algebras were exhaustively classified in [5] and [6], respectively. The Saletan contractions with other signatures do not have a connection with algebraic structure of initial and contracted algebras as direct as Inönü–Wigner contractions do. This is why the description of general Saletan contractions is a much more difficult problem.

Given a Lie algebra, another problem is finding the tuples of non-negative integers that can be signatures of Saletan contractions of this algebra. As shown in Remark 2, powers of the value of Saletan matrix at limit value of the contraction parameter form a nested chain of subalgebras of the initial algebra and signature components are expressed in the terms of the dimensions of these subalgebras. This claim relates the signatures to the subalgebraic structure of the initial algebra. At the same time, the presence of a nested chain of subalgebras does not imply the existence of the Saletan contraction associated with this chain. Additional constraints that admit no clear algebraic interpretation should be taken into account. Furthermore, even provided that a corresponding contraction exists, there is no known procedure to construct this contraction from the chain of subalgebras. This significantly differs from Inönü–Wigner contractions since there exists an algorithm to construct a well-defined Inönü–Wigner contraction starting from any subalgebra of the initial algebra. The study of Saletan signatures resembles the study of signatures of generalized Inönü–Wigner contractions [18–22]. Recall that the signature components of a generalized Inönü–Wigner contraction are diagonal entries of a diagonal differentiation of the algebra to be contracted, but the converse is not true.

The notion of Saletan signature may serve as a basis for an algorithm of exhaustive classification of Saletan contractions, at least in the case of lowest dimensions. It is known [22–25] that all contractions between three-dimensional complex (resp. real) Lie algebras (except the only contraction $\mathfrak{so}(3) \to \mathfrak{h}(1)$ in the real case) are realized by usual Inönü–Wigner contractions. The contraction $\mathfrak{so}(3) \to \mathfrak{h}(1)$ is realized as a generalized Inönü–Wigner contraction, but not as a Saletan one. In dimension four, the number of contractions that cannot be realized as usual Inönü–Wigner contractions increases crucially. Moreover, there is one (resp. two) contraction between four-dimensional complex (resp. real) Lie algebras that cannot be realized as generalized Inönü–Wigner contractions. Thus, the question whether these contractions can be realized as Saletan contractions is the most interesting problem on Saletan contractions of four-dimensional Lie algebras.
Acknowledgments
The author is grateful for the hospitality and financial support provided by the University of Cyprus. The author thanks Roman Popovych for productive and helpful discussions. The research was supported by the Austrian Science Fund (FWF), project P25064.

References
[1] Segal I E 1951 A class of operator algebras which are determined by groups Duke Math. J. 18 221
[2] İnönü E and Wigner E P 1953 On the contraction of groups and their representations Proc. Nat. Acad. Sci. U. S. A. 39 510
[3] İnönü E and Wigner E P 1954 On a particular type of convergence to a singular matrix Proc. Nat. Acad. Sci. U. S. A. 40 119
[4] Saletan E J 1961 Contraction of Lie groups J. Math. Phys. 2 1
[5] Conatser C W 1972 Contractions of the low-dimensional Lie algebras J. Math. Phys. 13 196
[6] Huddleston P L 1978 İnönü–Wigner contractions of the real four-dimensional Lie algebras J. Math. Phys. 19 1645
[7] Paterna J and Winternitz P 1977 Subalgebras of real three and four-dimensional Lie algebras J. Math. Phys. 18 1449
[8] Kupeczyński M 1969 On the generalized Saletan contractions Comm. Math. Phys. 13 154
[9] Lévy-Nahas M 1967 Deformation and contraction of Lie algebras J. Math. Phys. 8 1211
[10] Minuša F and Ikushima A 1978 Structure of contracted Lie algebras Bull. Kyusha Inst. Tech. Math. Natur. Sci. no 25, 1
[11] Caruñena J F, Grabowski J and Marmo G 2001 Constructions: Nijenhuis and Saletan tensors for general algebraic structures J. Phys. A 34 3769
[12] Burde D 1999 Degenerations of nilpotent Lie algebras J. Lie Theory 9 193
[13] Burde D 2005 Degenerations of 7-dimensional nilpotent Lie algebras Comm. Algebra 33 1259 (Preprint arXiv:math.RA/0409275)
[14] Burde D and Steinhoff C 1999 Classification of orbit closures of 4-dimensional complex Lie algebras J. Algebra 214 729
[15] Grunewald F and O’Halloran J 1988 Varieties of nilpotent Lie algebras of dimension less than six J. Algebra 112 315
[16] Lauret J 2003 Degenerations of Lie algebras and geometry of Lie groups Differential Geom. Appl. 18 177
[17] Mubarakzjanov G M 1963 On solvable Lie algebras Izv. Vysš. Učehn. Zaved. Matematika no 1(32) 114
[18] Doebner H D and Melsheimer O 1967 On a class of generalized group contractions Nuovo Cimento A (10) 49 306
[19] Hegerfeldt G C 1967 Some properties of a class of generalized İnönü–Wigner contractions Nuovo Cimento A (10) 51 439
[20] Lyhmus J H 1969 Limit Lie groups (contracted Lie groups) Second Summer School on the Problems of the Theory of Elementary Particles (Otepää, 1967), Part IV (Inst. Fiz. i Astronom. Akad. Nauk Eston SSR, Tartu) p 3 [in Russian]
[21] Popovych D R and Popovych R O 2009 Equivalence of diagonal contractions to generalized IW-contractions with integer exponents Linear Algebra Appl. 431 1096 (Preprint arXiv:0812.4667)
[22] Popovych D R and Popovych R O 2010 Lowest dimensionai example on non-universality of generalized İnönü-Wigner contractions J. Algebra 324 2742 (Preprint arXiv:0812.1705)
[23] Campoamor-Stursberg R 2008 Some comments on contractions of Lie algebras Adv. Stud. Theor. Phys. 2 865
[24] Nesterenko M and Popovych R 2006 Contractions of low-dimensional Lie algebras J. Math. Phys. 47 123515 (Preprint arXiv:math-ph/0608018)
[25] Weimar-Woods E 1991 The three-dimensional real Lie algebras and their contractions J. Math. Phys. 32 2028