Invariant Monotone Coupling Need Not Exist

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Abstract. We show by example that there is a Cayley graph and two invariant random subgraphs $X$ and $Y$ of it such that there exists a monotone coupling between them in the sense that $X \subset Y$ but no such coupling can be invariant. Here, “invariant” means that the distribution is invariant under group multiplications.

§1. Introduction.

There are several models when one is selecting a random subset of vertices or edges of a given graph $G = G(V,E)$ according to some distribution. Formally these are $2^V$-valued random objects where $V$ is the vertex set of $G$ (which can be replaced by $E$, the set of edges). We can look at this as a $0,1$ labeling of the vertices; then it is natural to allow more general label sets $\Lambda$ replacing $\{0,1\} = 2$.

We are interested in particular in Cayley graphs, and in that case most naturally occurring examples have an extra common feature: invariance. This means that their distribution is invariant under the group multiplication of the base graph. More formally, if $G$ is a right Cayley graph of the group $\Gamma$, then the random object $R$ is invariant if for any finite $\{v_1,\ldots,v_n\}$ and $\gamma \in \Gamma$, the distribution of $(R(\gamma v_1),\ldots,R(\gamma v_n))$ does not depend on $\gamma$. This motivates an investigation of invariant random subgraphs in general. This was done, for example, in [2].

In this context, our result is a counterexample. To explain it, we first need to recall the notion of coupling.

Definition 1.1. If $S_1,S_2$ are random objects taking values in $\Delta_1,\Delta_2$ respectively, then a coupling of them is a random pair $(\tilde{S}_1,\tilde{S}_2)$ taking values in $\Delta_1 \times \Delta_2$ such that $\tilde{S}_i$ has the same distribution as $S_i$.

Intuitively this means that we manage to produce the two objects using the same random source so that pointwise comparison makes sense. Proofs using coupling arguments are usually very conceptual and fit well with probabilistic intuition.

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A very simple instance of this is the simultaneous coupling of all Bernoulli$(E,p)$ percolations corresponding to possible parameters $p \in [0,1]$ which we briefly recall. A Bernoulli$(E,p)$-percolation is obtained by putting i.i.d. 0,1 labels on the edges, where for a given edge $e$ its label is 1 with probability $p$ and 0 otherwise. Note that replacing the edge set $E$ with the set of vertices $V$ in the above definition is well defined and we denote that process by Bernoulli$(V,p)$. There is a strong intuition that “the bigger $p$ is, the bigger the subgraph with label 1”. We can make this intuition have a precise formal meaning as follows: Put first i.i.d. uniform (from $[0,1]$) labels on the edges, which we call $U$. Then for each $p$ define a 0,1 label $U_p$ so that if an edge has $U$ label $U(e)$, its $U_p$ label is 1 if $U(e) \leq p$ and 0 otherwise. Clearly as a distribution $U_p$ is nothing but a Bernoulli$(E,p)$-percolation, and for $p \leq p^+$ we have $U_p \subset U_{p^+}$.

This is an example of what is called monotone coupling. For the definition assume that the label set $\Lambda$ is partially ordered by $\preceq$.

**Definition 1.2.** If $X$ and $Y$ are random $\Lambda$-labelings of the same graph, then we say that a coupling $(\tilde{X},\tilde{Y})$ of $X$ and $Y$ is a **monotone coupling** if $\tilde{X} \preceq \tilde{Y}$.

The next two examples we mention are related to open questions which motivates the question we are going to ask.

The first is the case of Wired and Free Uniform Spanning Forest measures (WUSF and FUSF respectively) see [3]. These processes both can be considered as natural generalizations of the Uniform Spanning Tree (easily defined on finite graphs) to infinite graphs. It is known that there is a monotone coupling where the Free one dominates the Wired one. However in general it is still open if there is an invariant monotone coupling.

There are partial results which show that for certain classes of graphs there indeed exists an invariant monotone coupling between the FUSF and WUSF. For example, Lewis Bowen [4] showed it for Cayley graphs of residually amenable groups, while recently Russell Lyons and Andreas Thom (personal communication, [8]) showed it for the Cayley graphs of so-called sofic groups.

The second example is random walk in random environment. In [1], David Aldous and Russell Lyons considered a continuous time nearest-neighbour random walk $RW(t,\mu)$, with jumps governed by Poisson clocks on the edges with rates given by a distribution $\mu$. The walks start at the origin $o$ of the Cayley graph and we are interested in how different environments affect the return probabilities. In [1] they showed that if for two random environments $\mu_1, \mu_2$ (different clock frequencies in this case) there exists a monotone coupling $\mu_1 \leq \mu_2$, which is itself invariant then

$$E_{\mu_1}(P(RW(t,\mu_1) = o)) \geq E_{\mu_2}(P(RW(t,\mu_2) = o)).$$
We may ask what happens if we drop the condition for the coupling being invariant. Is it enough for example that the marginals are invariant? Note that in [1] they actually dealt with so called unimodular processes but this condition always holds for invariant processes on Cayley graph (this fact is the Mass Transport Principle which we prove later).

Oded Schramm and Russell Lyons asked (unpublished, [7]) the following, note that a positive answer would immediately settle the above problems.

**Question 1.3.** Let $X$ and $Y$ be invariant subgraphs of a Cayley graph $\Gamma$, so that there exists a monotone coupling between them. Does it follow that there exists a monotone coupling between them which is also invariant?

It is known that the answer to the above question is “yes” if the Cayley graph is amenable. In this paper we show by an example that in full generality the answer is “no”. In our case, the graph will be $T_3 \Box C_n$ for $n$ large enough (here $T_3$ is the 3-regular tree, and $C_n$ is the cycle of length $n$, we define $T_3 \Box C_n$ in Section 3) which is a Cayley graph for $(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2) \times \mathbb{Z}_n$, where $H \ast K$ is the free product of $H$ and $K$. For simplicity we make an assumption about $n$ which is not optimal. See Remark 1.6 at the end of this section for an explanation.

**Theorem 1.4.** If $n \geq 1050$, then there exist two invariant random $\{0,1\}$-labelings $X$ and $Y$ of $T_3 \Box C_n$ so that there is a coupling $(\tilde{X}, \tilde{Y})$ of them for which $\tilde{X} \leq \tilde{Y}$ holds, but no such coupling can be invariant.

The proof will be more succinct if we first show a similar result with labels different from $\{0,1\}$. In this case the (partially ordered) label set will be the power set $\mathcal{P}(S)$ of some finite set $S$. Note also that in this case we can use a tree as the underlying Cayley graph:

**Lemma 1.5.** If $n \geq 1050$ and $|S| = n$, then there exist $\mathcal{P}(S)$-labelings $\mathcal{X}$ and $\mathcal{Y}$ of $T_3$ so that there is a coupling $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$ of them for which $\tilde{\mathcal{X}} \subset \tilde{\mathcal{Y}}$ holds, but no such coupling can be invariant.

Although the examples themselves might be artificial, they will have some “nice” properties as well. So if we want to add some extra conditions to Question 1.3 to get an affirmative answer, then we know for sure that these nice properties will not work (at least not alone). For a discussion of these, see the end of the last section.

We summarize some conventions we use: When $\mathcal{S}$ is a random object and $\mu$ is its distribution, we often will just express this by saying that $\mathcal{S}$ is a copy of $\mu$ and in a similar way with a further abuse of notation, if $\mathcal{T}$ is a random object with the same distribution as $\mathcal{S}$, we will also say that $\mathcal{S}$ is a copy of $\mathcal{T}$. We also note that one way to specify a
probability measure is to describe a random object with the given measure. We will do it without further comments.

If the graph $G$ is understood, $V(G)$ will be its set of vertices and $E(G)$ its set of edges. We use right Cayley graphs and then left multiplications are graph automorphisms.

**Remark 1.6.** The condition that $n \geq 1050$ we made in Theorem 1.4 was meant to ensure the following: If $S$ is a finite set of cardinality $n$ and $\alpha_1, \alpha_2, \ldots, \alpha_{20}$ are i.i.d. uniform elements of $S$ and $\beta_1, \ldots, \beta_9$ are also i.i.d. uniform elements of $S$ (we emphasize that we make no extra assumption on the joint distribution of the full family $\alpha_1, \ldots, \alpha_{20}, \beta_1, \ldots, \beta_9$) then with probability at least $\frac{2}{3}$ the random elements $\alpha_1, \ldots, \alpha_{20}$ are all distinct and the random elements $\beta_1, \ldots, \beta_9$ are all distinct as well (but it may happen that some $\beta_i = \alpha_j$). It is easy to see that if $\prod_{i=1}^{19} (1 - \frac{i}{n}) \geq \frac{5}{6}$ (which is true for $n \geq 1050$) then this holds.

§2. The Mass-Transport Principle and Ends.

This section owes a lot to the exposition in [6]. An effective tool in showing that there is no invariant random process on a Cayley graph satisfying a certain requirement is the so-called Mass-Transport Principle. Recall that $\Lambda$ is the label set, which will always be finite in our case. The “space of configurations” $\Lambda^V$ or $\Lambda^E$ will be naturally equipped with the product $\sigma$-algebra. Assume that $\mathcal{R}$ is a probability measure on $\Omega$ so that $\Omega$ is either $\Lambda^V$ or $\Lambda^E$. Let $F : V \times V \times \Omega \to [0, \infty]$ be a diagonally invariant measurable function (meaning that $F(x, y, \omega) = F(\gamma x, \gamma y, \gamma \omega)$ for all $\gamma \in \Gamma$). The quantity $F(x, y, \omega)$ is often called the mass sent by $x$ to $y$ or the mass received by $y$ from $x$ and then $F$ thought to describe a “mass transport” among the vertices which may depend on some randomness created by $\mathcal{R}$. The Mass-Transport Principle says that if $\mathcal{R}$ is invariant, then for the identity $o \in V$ the expected overall mass $o$ receives is the same as the expected overall mass it sends out.

Now we formalize and prove this:

**Theorem 2.1.** If $\mathcal{R}$ and $F$ are as above, $\mathcal{R}$ is invariant, $f(x, y) := \mathbf{E}_\mathcal{R} F(x, y, \ast)$, then

$$\sum_{x \in V} f(o, x) = \sum_{x \in V} f(x, o).$$

To prove it, first observe that the invariance of $\mathcal{R}$ implies that $f$ is also diagonally invariant. This implies that $f(o, x) = f(x^{-1} o, x^{-1} x) = f(x^{-1}, o)$ and this finishes the proof since inversion is a bijection.

This means that in order to show that a random process with a given property cannot be invariant, it is enough to show that the property in question allows us to define a mass transport contradicting the above equality.
From now until Section 3, the base graph is always $T_3$, the 3-regular tree. If $v$ is a vertex, then $J(v)$ will denote the set of edges for which $v$ is one of the endpoints. A “ray” is a one-sided infinite path (i.e., a sequence of vertices $v_0, \ldots, v_n, \ldots$ so that there is no repetition and $v_i$ and $v_{i+1}$ are adjacent). We call two rays equivalent if their symmetric difference is finite. An equivalence class is then called an end. If we fix an end $\xi$, then for any vertex $v$ there is a unique ray $v = v_0^\xi, v_1^\xi, \ldots, v_n^\xi, \ldots$ so that the ray starts at $v$ and belongs to the equivalence class $\xi$. Let the unique edge joining $v$ with $v_1^\xi$ be $e_{v \to v_1^\xi}$ and let us denote $J(v) \setminus \{e_{v \to v_1^\xi}\}$ as $J^\xi(v)$. Observe that for distinct vertices $v_1, v_2$, we have

$$J^\xi(v_1) \cap J^\xi(v_2) = \emptyset.$$ 

This will be important in constructing a monotone coupling of our processes and it also implies that an end cannot be determined using invariant processes. The intuition is simple: given an end $\xi(\omega)$ (which is “somehow determined” by a configuration $\omega$) a vertex $v$ could send mass 1 to each of the two vertices that are the other endpoints of the two edges in $J^{\xi(\omega)}(v)$. In this way the overall mass sent out is 2 while the overall mass received is 1. To make this precise in a general setting we have to deal with measurability issues related to how a configuration $\omega$ determines an end $\xi(\omega)$ but this is not needed for our purposes.

§3. The Fixed-End Trick.

As we have indicated, an end cannot be determined using invariant processes in a tree, and Steve Lalley (unpublished [5]) proposed a way to exploit this fact to settle Question 1.3. Here we present a simpler version of the idea, see the last paragraph in this section for the original one. Given an end $\xi$ in $T_3$, we shall define a $\{0, 1\}^2$-labeling $(X^\xi, Y^\xi)$ so that its components, $X^\xi$ and $Y^\xi$, are invariant and $(X^\xi, Y^\xi)$ is a monotone coupling of them, i.e.,

$$X^\xi \leq Y^\xi.$$ 

Let $\{\eta(e)\}_{e \in E}$ be a Bernoulli($E, \frac{1}{2}$) label. For a vertex $v$, let

$$X^\xi(v) := \max\{\eta(e); e \in J^\xi(v)\},$$

while

$$Y^\xi(v) := \max\{\eta(e); e \in J(v)\}.$$ 

It is clear that $Y^\xi$ itself is an invariant labeling.
However $X^\xi$ is also invariant since the family $X^\xi(v)\;v \in V$ is actually i.i.d.! This is because of the observation from the last section that $J^\xi(v_1)$ and $J^\xi(v_2)$ are disjoint for $v_1 \neq v_2$. So $X^\xi$ itself is actually $\text{Bernoulli}(V, \frac{3}{4})$. Since the monotone coupling of these processes was defined using an end (a non-invariant step), it is reasonable that maybe these processes already witness Theorem 1.4.

However, the construction below — which is due to Yuval Peres (unpublished [9]) — shows that there exists an invariant monotone coupling between $X^\xi$ and $Y^\xi$. Let $\{\eta(e)\}_{e \in E}$ be as above.

For each vertex $v$ with $J(v) = \{e_1(v), e_2(v), e_3(v)\}$, define $\hat{X}(v) := 0$ iff $\{\eta(e_1) = \eta(e_2) = \eta(e_3)\}$ and $\hat{X}(v) := 1$ otherwise. Then $\hat{X}(v) \leq Y^\xi(v)$ and the coupling $(\hat{X}, Y^\xi)$ is clearly invariant. Moreover, $\hat{X}$ is a $\text{Bernoulli}(V, \frac{3}{4})$ vertex labeling. To see this, one way is to prove by induction on $k + l$ that for any set of distinct vertices $\{v_1, \ldots, v_k, w_1, \ldots, w_l\}$ we have $P(\hat{X}(v_1) = 1, \ldots, \hat{X}(v_k) = 1, \hat{X}(w_1) = 0, \ldots, \hat{X}(w_l) = 0) = (\frac{3}{4})^k(\frac{1}{4})^l$. In doing this we may assume that these vertices span a connected subgraph and then the induction can proceed by picking a leaf of the spanned subtree; the details are left to the reader.

Although the above processes could be coupled in an invariant way, it is clear that the idea leaves us a lot of freedom to use other partially ordered sets and other monotone operations (instead of taking maxima we could take the sum, for example, which was Lalley’s original suggestion). But it seems that other examples are difficult to analyze from the point of view of Question 1.3. With our next construction, however, it will be very succinct why a monotone coupling cannot be invariant.

§4. Set Valued Labels on $T_3$.

In this section, we describe an example that will prove Lemma 1.5.

Let $S$ be a finite set with $|S| = n \geq 1050$ and let $\mathcal{P}(S)$ denote its power set. We will use $\mathcal{P}(S)$ as a label set with inclusion as a partial order. The two invariant $\mathcal{P}(S)$-labelings $\mathcal{Y}_S$ and $\mathcal{X}_S$ of the vertices of $T_3$ are defined as follows (for the rest of the section we drop the subscript $S$ but in the next section we use it again).

For $\mathcal{Y}$, first let $\lambda$ be a labeling of the edges of $T_3$ with independent uniform labels from $S$, then for a vertex $v$, let $\mathcal{Y}(v) := \bigcup_{e \in J(v)} \{\lambda(e)\}$.

For $\mathcal{X}$, first define a distribution $\nu$ on $\mathcal{P}(S)$; to get a copy of $\nu$ first pick a uniform $(x_1, x_2) \in S \times S$ and take $\{x_1\} \cup \{x_2\}$. Finally let $\{\mathcal{X}(v)\}_{v \in V(T_3)}$ be a labeling of the vertices with i.i.d. copies of $\nu$.

Remark 4.1. Observe that if $\hat{\mathcal{Y}}$ is any copy of $\mathcal{Y}$ then the following is true: if $v_0$ is any vertex with neighbors $v_1, v_2, v_3$, then any $s \in \hat{\mathcal{Y}}(v_0)$ is also contained in at least one of the
\(\hat{Y}(v_i)\)'s for \(i \in \{1, 2, 3\}\).

By fixing an end \(\xi\) it is easy to find a monotone coupling of \(X\) and \(Y\) just as in Lalley's example.

However there cannot be any invariant monotone coupling as we will show now. Let \((X^*, Y^*)\) be any monotone coupling of \(X\) and \(Y\). We will show that using this monotone coupling we can define a mass transport \(F\) which contradicts the Mass Transport Principle, showing that the coupling cannot be invariant. To define the mass transport we have to say for every pair \((v_0, v)\) of vertices and every possible configuration \(\omega\) (defined in terms of \((X^*, Y^*)\)) the value \(F(v_0, v, \omega) \in [0, \infty]\). The dependence on \(\omega\) will be through an event \(E(v_0)\) which we define now.

**Definition 4.2.** First, let \(v_1, v_2, v_3\) be the neighbors of \(v_0\) and \(v_4, \ldots, v_9\) be the vertices at graph distance 2 from \(v_0\) (in any order).

We say that \(E_1(v_0)\) holds if for each \(1 \leq i, j \leq 3, i \neq j\), we have \(|Y^*(v_j)| = 3, Y^*(v_j) \cap Y^*(v_i) = \emptyset\).

We say that \(E_2(v_0)\) holds if for each \(0 \leq i, j \leq 9, i \neq j\), we have \(|X^*(v_i)| = 2\) and \(X^*(v_i) \cap X^*(v_j) = \emptyset\).

Finally, let \(E(v_0) := E_1(v_0) \cap E_2(v_0)\).

Note the connection with the condition in Remark 1.6: the labels \(X(v_0), \ldots X(v_9)\) can be identified with \(\{\alpha_1, \alpha_2\}, \ldots, \{\alpha_{19}, \alpha_{20}\}\), while the edge labels of those 9 edges which are relevant in the \(Y\) labels of \(v_0, v_1, v_2, v_3\) can be identified with \(\beta_1, \ldots, \beta_9\). Then the condition we made on \(n\) ensures that \(P(E(v_0)) \geq \frac{2}{3}\).

Now we are ready to define the mass transport \(F : V \times V \times \Omega \rightarrow [0, \infty]\). If \(E(v_0)\) does not hold, then set \(F(v_0, v, \omega) := 0\) for each vertex \(v\). If \(E(v_0)\) holds, then let \(F(v_0, v, \omega) := 1\) if \(v_0\) is connected to \(v\) and \(X^*(v_0) \cap Y^*(v) \neq \emptyset\), while in every other case, set \(F(v_0, v, \omega) := 0\).

We show that the expected mass the origin sends out is at least \(\frac{4}{3}\), while the mass it receives is not greater than 1 (even point-wise).

To prove this we show first that if \(E(v_0)\) holds, then the mass \(v_0\) sends out is exactly 2. This combined with the fact that \(P(E(v_0)) \geq \frac{2}{3}\) implies the first part of the claim. Let \(X^*(v_0) := \{s_1, s_2\}\); observe that \(E_2(v_0)\) implies \(s_1 \neq s_2\). By monotonicity of the coupling, \(\{s_1, s_2\} \subset Y^*(v_0)\), so by Remark 4.1 there exist neighbors \(v_0(s_1), v_0(s_2)\) of \(v_0\) so that \(Y^*(v_0(s_i))\) contains \(s_i\). Since \(Y^*(v_1), Y^*(v_2), Y^*(v_3)\) are pairwise disjoint sets (by \(E_1(v_0)\)), there can be at most two of them which non-trivially intersect \(\{s_1, s_2\}\), and this implies \(v_0(s_1) \neq v_0(s_2)\). By definition, \(v_0(s_1)\) and \(v_0(s_2)\) are exactly the vertices receiving non-zero mass from \(v_0\).
To prove that the expected mass \( v_0 \) receives is at most 1, assume that \( v_0 \) receives non-zero mass from \( v_1 \) and \( v_2 \). First of all, \( v_1 \) sends out non-zero mass only if \( E(v_1) \) (and in particular \( E_2(v_1) \)) holds. Since \( v_0 \) and \( v_2 \) are both within distance 2 from \( v_1 \), the event \( E_2(v_1) \) implies that \( \{a_1, a_2\} := X^*(v_1), \{b_1, b_2\} := X^*(v_2) \), and \( \{c_1, c_2\} := X^*(v_0) \) are pairwise disjoint and each has size 2. By the condition for the mass transport, \( Y^*(v_0) \) contains one of the \( a_i \)'s, one of the \( b_i \)'s and — by the monotonicity of the coupling — \( \{c_1, c_2\} \) as well. But this would mean that \( Y^*(v_0) \) has at least four distinct elements, which is impossible.

This mass transport violates the Mass Transport Theorem, so no monotone coupling of \( X \) and \( Y \) can be invariant.

Remark 4.3. Observe that an end \( \xi \) can be identified by the orientation on the edges given as follows: orient the edges in \( J^\xi(v) \) away from \( v \). Then the out-degree of a vertex is always 2 while the in-degree is always 1. The mass transport above has some similarity with this end: if for a vertex \( v \) we define \( J(X^*,Y^*)(v) \) to be the set of edges connecting \( v \) with vertices receiving non-zero mass from \( v \) and we orient the edges in \( J(X^*,Y^*)(v) \) away from \( v \), then the out-degree of a vertex is either 0 or 2 and the in-degree is either 0 or 1.

§5. The \( \{0,1\} \)-labels on \( T_3 \square C_n \).

Now we prove Theorem 1.4. The Cayley graph we use is \( T_3 \square C_n \). Here \( C_n \) is the cycle of length \( n \), and in general for graphs \( G \) and \( H \) their Cartesian product \( G \square H \) is the graph with vertex set \( V(G \square H) = V(G) \times V(H) \) and two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are connected in \( G \square H \) iff either \( u_1 = v_1 \) and \( u_2 \) is adjacent with \( v_2 \) in \( H \), or \( u_2 = v_2 \) and \( u_1 \) is adjacent with \( v_1 \) in \( G \). It is easy to see that if \( G_1, G_2 \) are Cayley graphs of \( \Gamma_1, \Gamma_2 \) respectively then \( G_1 \square G_2 \) is a Cayley graph of \( \Gamma_1 \times \Gamma_2 \). Note that \( T_3 \) is a Cayley graph of \( \mathbb{Z}_2^3 \) := \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( C_n \) is a Cayley-graph of \( \mathbb{Z}_n \), so \( T_3 \square C_n \) is a Cayley graph of \( \mathbb{Z}_2^3 \times \mathbb{Z}_n \). If we want to check the invariance of a process defined on \( T_3 \square C_n \) it is enough to check invariance under group multiplication from \( \mathbb{Z}_2^3 \) and \( \mathbb{Z}_n \) since the direct components generate the full group.

The processes we define can be considered as very faithful copying of the previous processes. In the previous section, the set \( S \) whose subsets were used as labels was not important besides its cardinality \(|S|\). Now it will be convenient to choose it to be \( S := V(C_n) \). If \( Z \) is any \( S \)-labeling of the vertices of \( T_3 \), then let \( \text{lift}(Z) \) be the following \( \{0,1\} \)-labeling of \( T_3 \square C_n \): for a vertex \((u,v) \in V(T_3 \square C_n)\), let \( \text{lift}(Z)(u,v) := 1 \) if \( v \in Z(u) \), otherwise let \( \text{lift}(Z)(u,v) := 0 \). Note that this function lift from the \( S \) labelings of \( T_3 \) to the \( \{0,1\} \)-labelings of \( T_3 \square C_n \) is invertible.
Consider the previously defined processes $X_S, Y_S$. Let $X := \text{lift}(X_S)$ and $Y := \text{lift}(Y_S)$. We claim that these witness the truth of Theorem 1.4.

First, the invariance of $X$ and $Y$ under group multiplication from $Z_2^3$ follows from the fact that $X$ and $Y$ were invariant on $T_3$ and the invariance under $Z_n$ follows from the fact that for a fixed vertex $v_0$ the distribution of $X' (Y)$ is invariant under any permutation of $S$.

Second, there exists a monotone coupling of $X, Y$ since if $(X^*, Y^*)$ is any monotone coupling of $X, Y$, then $(\text{lift}(X^*), \text{lift}(Y^*))$ is clearly a monotone coupling.

Third, if $(X^*, Y^*)$ was an invariant monotone coupling then $(\text{lift}^{-1}(X^*), \text{lift}^{-1}(Y^*))$ would have been an invariant coupling of $X, Y$, which is impossible as we have seen.

It would be nice to have some natural condition on random subgraphs under which the answer to Question 1.3 would be affirmative. Note that our example is $k$-dependent, meaning: a random subgraph $Z$ is said to be $k$-dependent if for vertex sets $S_1, S_2, \ldots, S_m$ whose pairwise distances are all at least $k$, the random objects $F_i := Z \upharpoonright S_i, 1 \leq i \leq m$, are independent. So assuming $k$-dependence is certainly not enough.

With slight modifications we can exclude other conditions as well. Observe that the mass transport we used would still work (in the sense that $E(v)$ would have probability greater than $\frac{1}{2}$) if we “perturbed” our processes with a $\text{Bernoulli}(V, \epsilon)$ process for $\epsilon > 0$ small enough (meaning that we change the original random subgraphs on those vertices where $\text{Bernoulli}(V, \epsilon)$ turns out to be 1). In this way we see that assuming the following condition does not work either: a random subgraph $Z$ is said to have uniform finite energy if there exists an $\epsilon \in (0, 1)$ so that for a vertex $v$ we have $\epsilon < P(Z(v) = 1 | Z \upharpoonright V - \{v\}) < 1 - \epsilon$.

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