Quantum Deformed $su(m|n)$ Algebra and Superconformal Algebra on Quantum Superspace

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Abstract

We study a deformed $su(m|n)$ algebra on a quantum superspace. Some interesting aspects of the deformed algebra are shown. As an application of the deformed algebra we construct a deformed superconformal algebra. From the deformed $su(1|4)$ algebra, we derive deformed Lorentz, translation of Minkowski space, $iso(2,2)$ and its supersymmetric algebras as closed subalgebras with consistent automorphisms.

* Fellow of the Japan Society for the Promotion of Science. Work partially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (# 030083)
1. Introduction

Recently, quantum groups and quantum algebras [1-4] have attracted much attention in theoretical physics and mathematics, such as statistical models, integrable models, conformal field theory and knot theory [5-8]. Also a quantum space has been studied intensively as a non-commutative space representing the quantum group [7, 9]. Differential calculus on the quantum space is very intriguing as an application of the quantum group and useful to show interesting aspects of the quantum groups [10-15].

Further, quantum Lorentz group and q-deformed Lorentz and Poincaré algebras have been constructed on the quantum space [16-19]. The approach was extended to construction of q-deformed conformal and superconformal algebras on the quantum space [20, 21]. On the other hand, through the Drinfeld-Jimbo procedure, another q-deformed Poincaré and conformal algebras were obtained without use of the differential calculus of the quantum space [22, 23, 24]. As well as the deformed algebras, whole quantum deformed analyses including representations of the algebras on the quantum space are very interesting. Therefore, in this paper we consider deformed algebras on the quantum space, generalizing the approach of Ref.[17, 20] to a deformed $su(m|n)$ algebra on the quantum superspace. As an application, we study deformed superconformal algebra and its closed subalgebras with some automorphisms.

This paper is organized as follows. In section two we review the differential calculus on the quantum superspace and construct a deformed $su(m|n)$ algebra consistent with the space. For the algebra, a representation on the fermionic space and a relation to the Drinfeld-Jimbo algebra are shown. In section three, we introduce the conjugate space and derive a simple conjugation of the deformed $su(m|n)$ algebra. In section four we construct a deformed superconformal algebra in four-dimensional space-time and also give a quantum 6-vector. From the deformed $su(1|4)$ algebra, we derive some closed subalgebras with simple consistent automorphisms. Section five is devoted to conclusions and remarks. In Appendix A we study decomposition of the quantum differential calculus and representation of the deformed algebra on a new space and in Appendix B the deformed $su(1|4)$ algebra is shown explicitly.
2. Deformed $su(m|n)$ algebra on quantum superspace

2.1 Quantum superspace

The quantum space is a non-commutative space which represents the quantum groups. Namely, the quantum group transforms covariantly commutation relations of the quantum space. These commutation relations between coordinates $Z^I(x^i, \theta^\alpha)$ ($i = 1 \sim m, \alpha = 1 \sim n$) and derivatives $\partial_I(\partial_i, \partial_\alpha)$ of the quantum superspace are obtained in terms of a $\hat{R}$-matrix for $GL_q(m|n)$ \[11, 15\] as follows,

$$
Z^I Z^J = \hat{R}^{IJ}_{KL} Z^K Z^L, \quad \partial_I \partial_J = \hat{R}^{LK}_{IJ} \partial_K \partial_L,
$$

$$
\partial_J Z^I = \delta^J_L + X \hat{R}^{JK}_{IL} Z^L \partial_K,
$$

where $X$ is a deformation parameter. The $\hat{R}$-matrix should satisfy the Yang-Baxter equation and it involves many deformation parameters in addition to $X$.

In Ref.\[15\], the quantum group matrices transforming the quantum superspace has been studied and also it showed conditions on the parameters for a superdeterminant to be a center. Here we consider the differential calculus with one parameter ($X = 1/q^2$), making use of the following $\hat{R}$-matrix,

$$
\hat{R}^{IJ}_{KL} = \delta^I_L \delta^J_K \{((-q^2)^{\tilde{I}} - q(-1)^{\tilde{I} \tilde{J}})\delta^{IJ} + q(-1)^{\tilde{I} \tilde{J}}\} + (1 - q^2)\delta^I_K \delta^J_L \Theta^{IJ},
$$

where $\Theta^{IJ} = 1$ for $I < J$, otherwise vanishes and $\tilde{I}$ denotes the grassman parity, i.e., $\tilde{I} = 0$ for $I = i$ and $\tilde{I} = 1$ for $I = \alpha$. The above $\hat{R}$-matrix leads to a central superdeterminant. Appearance of $\Theta^{IJ}$ implies that ordering of bosonic and fermionic elements is nontrivial. Although in this paper we study the ordering where any fermionic element follows all of bosonic ones, i.e., $\Theta^{\alpha i} = 1$, we could discuss other ordering.

The $\hat{R}$-matrix leads to the following commutation relations,

$$
Z^I Z^J = -\frac{1}{q} Z^J Z^I, \quad \partial_I \partial_J = -\frac{1}{q} \delta^I_J q \partial_\alpha \partial_i, \quad (I < J),
$$

$$
(\theta^\alpha)^2 = (\partial_\alpha)^2 = 0, \quad \partial_J Z^I = -\frac{1}{q} \delta^I_J Z^J \partial_I, \quad (I \neq J),
$$

$$
\partial_i x^i = 1 + q^{-2} x^i \partial_i + (q^{-2} - 1)(\sum_{i<j} x^j \partial_j + \sum_\alpha \theta^\alpha \partial_\alpha),
$$

See in detail Ref.\[15\]
\[ \partial_{\alpha} \theta^\alpha = 1 - \theta^\alpha \partial_{\alpha} + (q^{-2} - 1) \sum_{\alpha < \beta} \theta^\beta \partial_{\beta}. \]

In eq.(2.3), the commutation relation between \( \partial_I \) and \( Z_I \) depends on other coordinates and derivatives. In Appendix A, we discuss decomposition of the above differential algebra, following Ref.\[25\] and also representation of the deformed algebra on a new space is studied.

The \( \hat{R} \)-matrix is decomposed into two projection operators, a symmetric projector \( S \) and an antisymmetric one \( \mathcal{A} \) as follows,
\[
S = \frac{1}{1 + q^2} (\hat{R} + q^2 1), \quad \mathcal{A} = \frac{-1}{1 + q^2} (\hat{R} - 1).
\]  

(2.4)

### 2.2 Deformed \( su(m|n) \) algebra on quantum superspace

Now we consider deformed \( su(m|n) \) generators on the quantum superspace. First of all, we study generators \( T_{I+1}^I \) and \( T_{I-1}^I \) associated with simple roots, which correspond to \( Z_I \partial_{I+1} \) and \( Z_I \partial_{I-1} \) in the classical limit \( (q \to 1) \). Following Ref.\[17\], we assume that \( T_{I\pm 1}^I \) acts on \( Z_K \) and \( \partial_K \) as follows,
\[
T_{I \pm 1}^I Z^K = a(I, K) Z^K T_{I \pm 1}^I + \delta_{I \pm 1}^K Z^I, \quad T_{I \pm 1}^I \partial_K = b(I, K) \partial_K T_{I \pm 1}^I + C(I) \partial_{I \pm 1}^K. \]  

(2.5)

The actions of \( T_{I \pm 1}^I \) (2.5) should be consistent with the commutation relations of the quantum superspace (2.3). For example, we calculate \( T_{I+1}^I(qZ^J Z^K - (-1)^{\hat{J}\hat{K}} Z^K Z^J) \) \((J < K)\), so that we derive the following consistency condition,
\[
\delta_{I+1}^I(qZ^I Z^K - (-1)^{\hat{J}\hat{K}} a(I, K) Z^K Z^J) + \delta_{I+1}^K Z^I - (-1)^{\hat{J}\hat{K}} Z^I Z^J = 0. \]  

(2.6)

Namely, we obtain \( a(I, I)_{+} = (-1)^{\hat{I}(I+1)}/q \) for \( I = i \) and \( a(I, J)_{+} = 1 \) for \( I > J \) or \( I + 1 < J \). Further, we investigate consistency between (2.5) and the other commutation relations (2.3), so that we obtain actions of \( T_{I+1}^I \) as follows,
\[
T_{i+1}^i x^i = q^{-1} x^i T_{i+1}^i, \quad T_{i+1}^i \partial_i = q \partial_i T_{i+1}^i - q \partial_{i+1}, \quad T_{i+1}^i = q a^{i+1} T_{i+1}^i + x^i, \quad T_{i+1}^i \partial_{i+1} = q^{-1} \partial_{i+1} T_{i+1}^i, \quad T_{\alpha+1}^\alpha \theta^\alpha = q \theta^\alpha T_{\alpha+1}^\alpha, \quad T_{\alpha+1}^\alpha \partial_{\alpha} = q^{-1} \partial_{\alpha} T_{\alpha+1}^\alpha - q^{-1} \partial_{\alpha+1}, \]  

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\[ T_{\alpha+1}^\alpha \theta^{\alpha+1} = q^{-1} \theta^{\alpha+1} T_{\alpha+1}^\alpha + \theta^\alpha, \quad T_{\alpha+1}^\alpha \partial_{\alpha+1} = q \partial_{\alpha+1} T_{\alpha+1}^\alpha, \quad (2.7) \]
\[ T_{i=1}^i x^m = q^{-1} x^m T_{i=1}^i + q^{-1} x^m, \quad T_{i=1}^i \partial_i = q \partial_i T_{i=1}^i, \quad (2.8) \]
\[ T_{i=1}^i \partial_i = q^{-1} \partial_i + q^{-1} \partial_i T_{i=1}^i - q^{-1} \partial_i, \]
\[ T_{\alpha}^\alpha \theta^\alpha = q \theta^\alpha T_{\alpha}^\alpha + q^{-1} \theta^\alpha, \quad T_{\alpha}^\alpha \partial_\alpha = q^{-1} \partial_\alpha T_{\alpha}^\alpha, \]
\[ T_{\alpha}^\alpha \partial_{\alpha+1} = q^{-1} \partial_{\alpha+1} T_{\alpha}^\alpha - q^{-1} \partial_{\alpha+1}, \quad (2.8) \]
\[ T_{i=m}^i x^m = q^{-1} x^m T_{i=m}^i + q^{-1} x^m, \quad T_{i=m}^i \partial_i = q \partial_i T_{i=m}^i, \]
\[ T_{i=m}^i \partial_i = q^{-1} \partial_i T_{i=m}^i - q^{-1} \partial_i + q \partial_i. \]

For the other elements, \( T_{I+1}^I \) satisfy the classical relations, i.e., they commute or anticommute with each other depending on their grassman parities.

Similarly we can derive actions of \( T_{I+1}^i \) as follows,
\[ T_{i}^{i+1} x^i = q^{-1} x^i T_{i}^{i+1} + x^i, \quad T_{i}^{i+1} \partial_i = q \partial_i T_{i}^{i+1}, \]
\[ T_{i}^{i+1} x^i = q x^i T_{i}^{i+1} + x^i, \quad T_{i}^{i+1} \partial_i = q^{-1} \partial_i T_{i}^{i+1} - q^{-1} \partial_i, \]
\[ T_{\alpha}^\alpha \partial_i = q \theta^\alpha T_{\alpha}^\alpha + q^{-1} \theta^\alpha, \quad T_{\alpha}^\alpha \partial_i = q^{-1} \partial_i T_{\alpha}^\alpha, \]
\[ T_{\alpha}^\alpha \partial_{\alpha+1} = q^{-1} \partial_{\alpha+1} T_{\alpha}^\alpha - q^{-1} \partial_{\alpha+1}, \quad (2.8) \]
\[ T_{i=m}^i x^m = q^{-1} x^m T_{i=m}^i + q^{-1} x^m, \quad T_{i=m}^i \partial_i = q \partial_i T_{i=m}^i, \]
\[ T_{i=m}^i \partial_i = q^{-1} \partial_i T_{i=m}^i - q^{-1} \partial_i + q \partial_i. \]

For the other elements, \( T_{I+1}^i \) satisfy the classical relations.

Next, we define Cartan generators \( H_I \) in terms of a commutation relation between \( T_{I+1}^i \) and \( T_{I+1}^i \). In the classical limit, \( H_I \) corresponds to \( Z_{I+1}^i \partial_{I+1} - (-1)^{i(I+1)} Z_i \partial_i \). For example, we can obtain actions of \( T_{i+1}^i T_{i+1}^i \) and \( T_{i+1}^i T_{i+1}^i \) making use of (2.7) and (2.8) as follows,
\[ T_{i+1}^i T_{i+1}^i x^i = q^{-2} x^i T_{i+1}^i T_{i+1}^i + q^{-1} x^i T_{i+1}^i, \]
\[ T_{i+1}^i T_{i+1}^i x^i = q^2 x^i T_{i+1}^i T_{i+1}^i + q^{-1} x^i T_{i+1}^i + x^i, \]
\[ T_{i+1}^i T_{i+1}^i x^i = q^{-2} x^i T_{i+1}^i T_{i+1}^i + q x^i T_{i+1}^i + x^i, \]
\[ T_{i+1}^i T_{i+1}^i x^i = q^2 x^i T_{i+1}^i T_{i+1}^i + q x^i T_{i+1}^i + x^i. \]

Following Ref. [17], we define Cartan generators \( H_i \) \((i = 1 \sim m - 1)\) by linear combination of \( T_{i+1}^i T_{i+1}^i \) and \( T_{i+1}^i T_{i+1}^i \) in order to eliminate the linear terms of \( T \) on the right hand side of (2.9) as follows,
\[ H_i \equiv q T_{i+1}^i T_{i+1}^i - q^{-1} T_{i+1}^i T_{i+1}^i. \]

\[ (2.10) \]
Similarly we define the other Cartan generators $H_0 (= H_{i=m})$ and $H_\alpha$ $(\alpha = 1 \sim n)$ in terms of $T_{\alpha=1}^{i=m}, T_{\alpha+1}^{i=m}, T_{\alpha}^{i+1}$ and $T_{\alpha+1}^{i+1}$ so that linear terms of $T$ disappear in actions of $H_0$ and $H_\alpha$. Thus we have

$$H_\alpha \equiv q^{-1}T_{\alpha}^{i+1}T_{\alpha+1}^{i} - qT_{\alpha+1}^{i}T_{\alpha}^{i+1}, \quad H_0 \equiv \{ T_{\alpha=1}^{i=m}, T_{\alpha+1}^{i=m} \}, \quad (2.11)$$

and actually the Cartan generators act on the quantum superspace as follows,

$$H_i x^i = q^{-2} x^i H_i - q^{-1} x^i, \quad H_i \partial_i = q^2 \partial_i H_i + q \partial_i,$$

$$H_i x^{i+1} = q^2 x^{i+1} H_i + q x^{i+1}, \quad H_i \partial_{i+1} = q^{-2} \partial_{i+1} H_i - q^{-1} \partial_{i+1},$$

$$H_\alpha \partial^\alpha = q^2 \partial^\alpha H_\alpha - q \partial^\alpha, \quad H_\alpha \partial_\alpha = q^{-2} \partial_\alpha H_\alpha + q^{-1} \partial_\alpha,$$

$$H_\alpha \theta^\alpha = q^{-2} \theta^\alpha H_\alpha + q \theta^\alpha, \quad H_\alpha \theta_\alpha = q^2 \theta_\alpha H_\alpha - q \theta_\alpha + 1, \quad (2.12)$$

$$H_0 x^m = q^{-2} x^m H_0 + x^m, \quad H_0 \partial_{i=m} = q^2 \partial_{i=m} H_0 - q \partial_{i=m},$$

$$H_0 \theta^1 = q^{-2} \theta^1 H_0 + \theta^1, \quad H_0 \theta_1 = q^2 \theta_1 H_0 - q \theta_1,$$

and for the other elements the generators satisfy the classical algebra. Note that the definition of $H_0$ (2.11) is never deformed.

Finally, we define the other generators $T_{iJ}^I$ $(J \neq I \pm 1)$ in terms of commutation relations of $T_{i\pm 1}^I$. For example we define $T_{i+2}^I$ as

$$T_{i+2}^I \equiv [T_{i+1}^I, T_{i+2}^I]_h, \quad (2.13)$$

where $[A, B]_h \equiv AB - hBA$. Then we investigate closure of their algebra. Using (2.7) and (2.13), we can easily calculate actions of $T_{i+1}^I T_{i+2}^I$ and $T_{i+2}^I T_{i+1}^I$ on $x^k$ as follows,

$$[T_{i+1}^I T_{i+2}^i + x^j]_q = [T_{i+2}^i T_{i+1}^I, x^j]_q = 0,$$

$$[T_{i+1}^I T_{i+2}^i + x^{i+1}] = (q(q-h)x^{i+1}T_{i+1}^I + (q+q^{-1}-h)x^i)T_{i+1}^I,$$

$$[T_{i+2}^I T_{i+1}^I]_q = (q(q-h)x^{i+1}T_{i+2}^I + (q+q^{-1}-h)x^i)T_{i+1}^I, \quad (2.14)$$

Eq.(2.14) shows that if $h = q$ or $1/q$, $hT_{i+1}^I T_{i+2}^I$ is identified with $T_{i+1}^I T_{i+2}^I$. Here we choose $h = 1/q$, so as to obtain

$$[T_{i+1}^I, T_{i+2}^I]_q = 0. \quad (2.15)$$
Similarly we define the generators $T^K_I$ as follows,

$$T^K_I = T^K_I T^K_I - q_I T^K_I T^K_I, \quad (I < J < K \text{ or } I > J > K), \quad (2.16)$$

where $q_I = q^{J-1}$, i.e., $q_I = 1/q$ for $J = i$ and $q_I = q$ for $J = \alpha$. The definition satisfy the algebraic closure similar to the above. The generators $T^K_I$ ($I < K$) act on the quantum superspace as follows,

$$T^K_I Z^I = q_I Z^I T^K_I, \quad T^K_I Z^K = (-1)^{(i+K)} Z^K T^K_I, \quad (2.16)$$

$$T^K_I Z^K = (-1)^{(i+K)} (q^K_I - 1) Z^K T^K_I + Z^I + (-1)^I \lambda \sum_{I < J < K} Z^J T^K_I, \quad (2.17)$$

$$T^K_I \partial_I = (-1)^{(i+K)} \partial_I T^K_I - (-1)^I \lambda \partial_K T^K_I, \quad (2.17)$$

where $\lambda = q - 1/q$ and $I < J < K$ and for the other elements the generators satisfy the classical relations. Also the generators $T^K_I$ ($I < K$) act on the quantum superspace as follows,

$$T^K_I Z^I = q_I Z^I T^K_I + Z^K, \quad T^K_I Z^K = (-1)^{(i+K)} (q^K_I - 1) Z^K T^K_I, \quad (2.16)$$

$$T^K_I Z^J = (-1)^{(i+K)} Z^J T^K_I + (-1)^I \lambda Z^K T^K_I, \quad (2.17)$$

$$T^K_I \partial_I = (q^K_I - 1) \partial_I T^K_I, \quad T^K_I \partial_J = (-1)^{(i+K)} \partial_J T^K_I, \quad (2.17)$$

$$T^K_I \partial_K = q^{-1} \partial_K T^K_I - q^{2(i-k)+1} \partial_i - \lambda \sum_{i < j < k} q^{2(j-k)} \partial_j T^K_I, \quad (2.18)$$

$$T^K_I \partial_K = q^{-1} \partial_K T^K_I - q^{2(i-k)+1} \partial_i - \lambda \sum_{i < j < k} q^{2(j-k)} \partial_j T^K_I, \quad (2.17)$$

where $I < J < K$, $i < j < k$ and $\alpha < \beta < \gamma$, and for the other elements the generators satisfy the classical relations.

From (2.12), (2.17) and (2.18), we can derive commutation relations of generators through the calculation similar to (2.14) and (2.15). Among the whole algebra, commutation relations of $T^{I}_{I\pm1}$ and $H_I$ are obtained as

$$[T^{I-1}_I, T^{I+1}_I]_{q_I} = [T^{I+1}_I, T^{I-1}_I]_{q_I} = [T^{I}_J, T^{K}_L] = 0, \quad (I, J \neq K, L),$$

$$[H_I, T^{I+1}_I]_{(q_I)^2} = -(q_I)^2 (q + q^{-1}) T^{I+1}_I, \quad [H_I, T^{I+1}_I]_{1/(q_I)^2} = (q_I)^{-2} (q + q^{-1}) T^{I+1}_I,$$

where $I < J < K$. For $I < J < K$, the generators satisfy the classical relations.

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Then they satisfy the following relations,

\[ \{H_I, T_{J+1}\}_1/(q^2) = (q^2)^{-1}T_{J+1}^I, \quad \{H_I, T_{J+1}\}_{q^2} = -qT_{J+1}^I, \quad (I = J \pm 1), \]

\[ [H_0, T_{i=m}^{i=m}] = [H_0, T_{i=m-1}^{i=m}] = 0 \]

\[ [H_0, T_{i=m-1}^{i=m}]_{q^2} = -q^2T_{i=m-1}^{i=m}, \quad [H_0, T_{i=m}^{i=m}]_{1/q^2} = T_{i=m}^{i=m}, \]

\[ [H_0, T_{\alpha=2}^{i=m-1}]_{1/q^2} = T_{\alpha=2}^{i=m-1}, \quad [H_0, T_{\alpha=2}^{i=m}]_{q^2} = -q^2T_{\alpha=2}^{i=m}, \]

where \( H_I \) do not involve \( H_0 \). Further, it is easily shown that the Cartan generators commute with each other and \( T_i^\alpha \) and \( T_i^\alpha \) are nilpotent, i.e., \( (T_i^\alpha)^2 = (T_i^\alpha)^2 = 0 \). The other commutation relations of the whole algebra are obtained by use of (2.19) or the calculation similar to (2.14) and (2.15), through boresomely long calculations. For a concrete example, all commutation relations of the deformed \( su(1|4) \) algebra is found explicitly in Appendix B, while the whole deformed \( su(4) \) algebra has been shown in Ref. [20]. It is remarkable that the deformed \( su(\ell) \) algebra on the quantum bosonic space is represented in the same way as on the quantum fermionic space, except replacing \( q \) by \( 1/q \). This replacement is not essential.

### 2.3 Representation of deformed \( su(2) \) algebra on quantum fermionic space

In this and the following subsections, two interesting topics about the above deformed \( su(\ell) \) algebra are discussed. Here we study the deformed \( su(2) \) algebra on the two-dimensional quantum fermionic space, whose coordinates \((\theta^1, \theta^2)\) and derivatives \((\partial_1, \partial_2)\) satisfy the following commutation relation,

\[ q\theta^1\theta^2 = -\theta^2\theta^1, \quad \partial_1\partial_2 = -q\partial_2\partial_1, \]

\[ q\partial_\alpha\theta^\beta = -\theta^\beta\partial_\alpha, \quad (\alpha \neq \beta), \]

\[ \partial_1\theta^1 = 1 - \theta^1\partial_1 + (q^{-2} - 1)\theta^2\partial_2, \quad \partial_2\theta^2 = 1 - \theta^2\partial_2. \]  

Suppose that we define \( \hat{T}_1^1 \) and \( \hat{T}_1^2 \) like the classical generators using the quantum elements as follows,

\[ \hat{T}_1^1 \equiv \theta^1\partial_2, \quad \hat{T}_1^2 \equiv \theta^2\partial_1. \]  

Then they satisfy the following relations,

\[ \hat{T}_1^1\theta^2 = q^{-1}\theta^2\hat{T}_1^1 + \theta^1, \quad \hat{T}_1^2\theta^1 = q\theta^1\hat{T}_1^2 + \theta^2, \]

\[ \hat{T}_1^1\partial_1 = q^{-1}\partial_1\hat{T}_1^1 - q^{-1}\partial_2, \quad \hat{T}_1^2\partial_2 = q\partial_2\hat{T}_1^2 - q\partial_1. \]
The above relations are nothing but the actions of the generators $T_{2}^{1}$ and $T_{1}^{2}$ of the deformed $su(2)$ algebra on the quantum fermionic space (2.7) and (2.8).

Next, in the similar way to (2.8) we define $\hat{H}$ using $\hat{T}_{2}^{1}$ and $\hat{T}_{1}^{2}$ as follows,

$$\hat{H} \equiv q^{-1}\hat{T}_{1}^{2} \hat{T}_{2}^{1} - q\hat{T}_{1}^{1}\hat{T}_{2}^{2} = q^{-1}\theta^{2} \partial_{2} - q\theta^{1} \partial_{1}. \quad (2.23)$$

The generator $\hat{H}$ satisfies the following relations,

$$\hat{H}\theta^{1} = q^{2}\theta^{1}\hat{H} - q\theta^{1}, \quad \hat{H}\theta^{2} = q^{-2}\theta^{2}\hat{H} + q^{-1}\theta^{1},$$

$$\hat{H}\partial_{1} = q^{-2}\partial_{1}\hat{H} + q^{-1}\partial_{1}, \quad \hat{H}\partial_{2} = q^{2}\partial_{2}\hat{H} - q\partial_{1}. \quad (2.24)$$

The above relations are also exactly same as the actions of $H_{1}$ (2.12) of the deformed $su(2)$ algebra. Therefore the deformed $su(2)$ algebra is completely represented in terms of the coordinates and derivatives of the quantum fermionic space (2.21) and (2.23). Unfortunately, for the bosonic case we can not represent the deformed $su(2)$ algebra by the quantum bosonic coordinates and derivatives.
2.4 Map to Drinfeld-Jimbo basis

In this subsection, we discuss a map between the above deformed algebra and Drinfeld-Jimbo’s algebra. The deformed $su(\ell)$ algebra on the bosonic and the fermionic quantum spaces involves the following deformed $su(2)$ algebra as a closed subalgebra,

\[ [H_\alpha, T^\alpha_{\alpha+1}]q^4 = -q^2(q + q^{-1})T^\alpha_{\alpha+1}, \quad [H_\alpha, T^{\alpha+1}]_{1/q^4} = q^{-2}(q + q^{-1})T^{\alpha+1}_\alpha, \]

\[ [T^{\alpha+1}_\alpha, T^\alpha_{\alpha+1}]q^2 = qH_\alpha, \]  

(2.25)

where if we replace $q$ by $1/q$, we obtain the deformed $su(2)$ algebra on the quantum bosonic space, which consists of $T^i_{i+1}, T^{i+1}_i$ and $H_i$. This algebra (2.25) is identified with a deformed $su(2)$ algebra by Woronowicz, up to a normalization factor [10].

On the other hand, Drinfeld and Jimbo constructed the following deformed $su(2)$ algebra with a deformation parameter $q'$ as follows,

\[ [J_0, J_+] = J_+, \quad [J_-, J_0] = J_- \]

\[ [J_+, J_-] = \frac{1}{2}[2J_0]q', \]  

(2.26)

where $[A]_q = (q^A - q^{-A})/(q - q^{-1})$.

In Ref.[26, 1], a map between the Woronowicz’s and the Drinfeld-Jimbo’s $su(2)$ algebras was discussed. Thus, changing a normalization factor on the map of Ref.[26, 1], we can easily derive a map from the Drinfeld-Jimbo algebra (2.26) with $q' = q$ to (2.25) as follows,

\[ H_\alpha = \lambda^{-1}(1 - q^{-4J_0}), \quad T^\alpha_{\alpha+1} = \sqrt{2}q^{-J_0}J_-, \quad T^{\alpha+1}_\alpha = \sqrt{2}q^{-J_0}J_+. \]  

(2.27)

Under the similar map, the deformed $su(2)$ algebra on the quantum bosonic space is related with the Drinfeld-Jimbo’s algebra with $q' = 1/q$. The above map could be generalized to the deformed $su(\ell)$, because we can relate the Cartan generators and the generators associated with the simple roots of two basis through (2.27).

The map (2.27) leads to actions of $J_0$ and $J_\pm$ on the quantum space. For example, in the case of (2.27) with $\alpha = 1$ we have

\[ q^{-4J_0}\theta_1 = q^2\theta_1 q^{-4J_0}, \quad q^{-4J_0}\theta^2 = q^{-2}\theta^2 q^{-4J_0}, \]

\[ q^{-4J_0}\partial_1 = q^{-2}\partial_1 q^{-4J_0}, \quad q^{-4J_0}\partial_2 = q^2\partial_2 q^{-4J_0}, \]
\[ J_+ \theta^1 = \sqrt{q} \theta^1 J_+ + \sqrt{\frac{q}{2}} \theta^2 q^J_0, \quad J_+ \theta^2 = \frac{1}{\sqrt{q}} \theta^2 J_+, \]
\[ J_+ \partial_1 = \frac{1}{\sqrt{q}} \partial_1 J_+, \quad J_+ \partial_2 = \sqrt{q} \partial_2 J_+ - \sqrt{\frac{q}{2}} \partial_2 J_0, \quad (2.28) \]
\[ J_- \theta^1 = \sqrt{q} \theta^1 J_-, \quad J_- \theta^2 = \frac{1}{\sqrt{q}} \theta^2 J_- + \frac{1}{\sqrt{2q}} \theta^1 J_0, \]
\[ J_- \partial_1 = \frac{1}{\sqrt{q}} \partial_1 J_- - \frac{1}{\sqrt{2q}} \partial_2 J_0, \quad J_- \partial_2 = \sqrt{q} \partial_2 J_. \]

Similarly we can derive actions of the Drinfeld-Jimbo generators on the quantum bosonic space.

In Ref.[26], it was shown maps to other deformed \( su(2) \) algebra, e.g., the two Witten’s algebras [27] and the Fairlie’s algebra [28]. Through the procedure, the deformed algebra obtained here could be related to other algebra and we could derive relations between the quantum space and other algebra.

3. Conjugation

In this section, we introduce another quantum space \( \overline{Z}_I \) conjugate to \( Z^I \). We set up commutation relations of \( \overline{Z}_I \) as follows,
\[ \overline{Z}_L \overline{Z}_K = \hat{R}^{IJ}_{KL} \overline{Z}_J \overline{Z}_I, \quad \overline{Z}_K \overline{Z}_I = \hat{R}^{IJ}_{KL} \overline{Z}_L \overline{Z}_J. \quad (3.1) \]

The latter relation is explicitly written as
\[ \overline{Z}_J \overline{Z}^I = (-1)^{\hat{I}} q \overline{Z}^I \overline{Z}_J, \]
\[ \overline{x}_i x^i = x^i \overline{x}_i + (1 - q^2) \left( \sum_{i<j} x^j \overline{x}_i + \sum_\alpha \theta^\alpha \overline{\theta}_\alpha \right), \quad (3.2) \]
\[ \overline{\theta}_\alpha \theta^\alpha = -q^2 \theta^\alpha \overline{\theta}_\alpha + (1 - q^2) \sum_{\alpha<\beta} \theta^\beta \overline{\theta}_\beta. \]

These commutation relations have a center \( \sum_I Z^I \overline{Z}_I \). Further, we assume that \( \overline{Z}_I \) has the same commutation relations with the generators as \( \partial_I \). For example, we have
\[ T^{i+1}_i \overline{x}_{i+1} = q^{-1} \overline{x}_{i+1} T^{i+1}_k - q^{-1} \overline{x}_i. \quad (3.3) \]

Now we relate \( \overline{Z}_I \) and \( Z^I \) by a conjugation, where \( \overline{Z}_I \) must be proportional to the complex conjugate of \( Z^I \). The conjugation should be consistent with the commutation
relations (2.3) and (3.1). We can find two types of consistent conjugations, depending on the value of $q$. In the case with real $q$, we can consistently relate $Z^I$ and $\overline{Z}_I$ as follows,

$$\overline{Z}_I = g_I(Z^I)^*,$$

(3.4)

where $g_I$’s are diagonal elements of a metric for $su(m_1, m_2\mid n_1, n_2)$ and $*$ implies the complex conjugate. In this case, when taking the conjugation, we have to reverse the order of elements, i.e., $\overline{ab} = \overline{b}\overline{a}$.

On the other hand, when $|q| = 1$, the following conjugation is consistent,

$$\overline{x}_i = ig_iKq^{m-i+1}(x^i)^*, \quad \overline{\theta}_\alpha = g_\alpha q^\alpha K(\theta^\alpha)^*,$$

(3.5)

where $K$ is an arbitrary phase factor. In this case, when taking the conjugation, we do not reverse the order of elements, i.e., $\overline{ab} = \overline{a}\overline{b}$.

Now we consider conjugation of the generators. Here, we restrict ourselves the case with real $q$. We could extend the following approach to the case where $|q| = 1$. Actually, in Ref.[20, 21], the conjugation of the deformed $su(4)$ and $su(1\mid 4)$ algebras was discussed in the case where $|q| = 1$. For example we take conjugation of the relation between $T^i_{i+1}$ and $x^{i+1}$ (2.7), so that we obtain

$$\overline{x}_{i+1}T^i_{i+1} = g_i g_{i+1}\overline{x}_i + g_i g_{i+1}\overline{x}_i.$$

(3.6)

Through comparison with (3.3), eq.(3.6) leads to the following conjugation relation,

$$\overline{T}^i_{i+1} = g_i g_{i+1}\overline{T}^i_{i+1}.$$

(3.7)

Similarly we can derive the conjugation relations of the other generators as follows,

$$\overline{T}^I_J = g_I g_J T^I_J, \quad \overline{H}_I = H_I, \quad \overline{H}_0 = H_0.$$  

(3.8)
4. Deformed superconformal algebra on quantum space

In this section, we study a deformed superconformal algebra as a very interesting application of the deformed $su(m|n)$ algebra on the quantum superspace. The $N=1$ superconformal algebra in the four-dimensional space-time is represented by $su(1|2,2)$, whose deformed algebra is found in Appendix B. Therefore, we need the quantum space which consists of one bosonic and four fermionic elements to represent the algebra. In this section, the bosonic coordinate and derivative are denoted by $x^0$ and $\partial_0$, while $\theta^\alpha$ and $\partial_\alpha$ ($\alpha = 1 \sim 4$) implies the fermionic elements as the previous sections. Further, here we choose the $su(2,2)$ metric as $g = (1, 1, -1, -1)$.

4.1 Quantum 6-vector

In the classical limit, the $su(2,2)$ algebra is isomorphic to $so(4,2)$. In Ref.[29, 20], the isomorphism has been generalized to the quantum case. In the papers, a quantum 6-vector was discussed in terms of bi-spinors. It is an important representation as well as the spinor representation, which corresponds to $\theta^\alpha$. In this subsection, we consider a representation of the deformed superconformal algebra $su_q(1|2,2)$, which includes the quantum 6-vector.

Now we construct the quantum 6-vector in terms of a tensor product of two quantum spinors. For that purpose, we introduce another spinor $\eta^\alpha$ ($\alpha = 1 \sim 4$) and $y^0$. The new quantum superspace $y^0$ and $\eta^\alpha$ represent the deformed $su(1|2,2)$ algebra in the same way as $x^0$ and $\theta^\alpha$. We assume that they satisfy the following commutation relations,

$$Z_1^I Z_2^J = \hat{R}^{IJ}_{KL} Z_2^K Z_1^L, \tag{4.1}$$

where $Z_1 \equiv (x^0, \theta^\alpha)$ and $Z_2 \equiv (y^0, \eta^\alpha)$. Eq.(4.1) is explicitly written as follows,

$$x^0 y^0 = y^0 x^0, \quad x^0 \eta^\alpha = q \eta^\alpha x^0 + (1 - q^2) y^0 \eta^\alpha,$$

$$\theta^\alpha y^0 = q y^0 \theta^\alpha, \quad \theta^\alpha \eta^\alpha = -q^2 \eta^\alpha \theta^\alpha, \tag{4.2}$$

$$\theta^\alpha \eta^\beta = -q \eta^\beta \theta^\alpha + (1 - q^2) \eta^\alpha \theta^\beta, \quad \theta^\beta \eta^\alpha = -q \eta^\alpha \theta^\beta, \quad (\alpha < \beta).$$

We use the projection operators $S$ and $A$ in order to decompose the tensor product $Z_1 Z_2$ into irreducible representations of the deformed $su(1|2,2)$ algebra. Although in
Ref. [20] the antisymmetric projector $A$ was used to decompose the tensor product of two bosonic quantum spinors into the quantum 6-vector, the symmetric projector $S$ is available to derive another quantum 6-vector from a tensor product of two fermionic quantum space. Actually, from $S_{KL}^{I} Z_1^K Z_2^L$ we obtain eleven independent elements as follows,

\[
S^{\alpha\beta} = \theta^\alpha \eta^\beta - q \theta^\beta \eta^\alpha, \quad (\alpha < \beta),
\]

\[
S^{0\alpha} = x^0 \eta^\alpha + q \theta^\alpha y^0, \quad S^{00} = x^0 y^0, \quad (4.3)
\]

where $S^{\alpha\beta}$ is the quantum 6-vector.

From (2.12), (2.17) and (2.18), we can derive actions of the deformed $su(1|2,2)$ generators on $S$. The Cartan generators act on $S$ as follows,

\[
[H_\alpha, S^{\alpha\beta}]_{q^2} = -q S^{\alpha\beta}, \quad [H_\alpha, S^{\alpha+1, \beta}]_{1/q^2} = q^{-1} S^{\alpha+1, \beta},
\]

\[
[H_\alpha, S^I \alpha^{+1}]_{1/q^2} = q^{-1} S^I \alpha^{+1}, \quad [H_\alpha, S^I \alpha]_{q^2} = -q S^I \alpha, \quad (4.4)
\]

\[
[H_0, S^0 J]_{1/q^4} = (1 + q^{-4}) S^0 J, \quad [H_0, S^J \alpha]_{1/q^2} = S^J \alpha,
\]

where $I < \alpha, \alpha + 1 < \beta$ and $J = 0, 1$. Similarly, we obtain actions of $T^L_K (K < L)$ as follows,

\[
[T^\alpha, S^\gamma]_{q^2} = -q \lambda \sum_{\alpha < \beta < \gamma} S^{\alpha\beta} T^\alpha_{\beta},
\]

\[
\{T^0_\beta, S^{0\beta}\}_{1/q^2} = (q + q^{-1}) S^{00} + \lambda \sum_{\alpha < \beta} S^{0\alpha} T^0_{\alpha},
\]

\[
[T^\alpha_\beta, S^I \alpha]_{q} = [T^I_\sigma, S^I \rho]_{(-1)^{\sigma}(I)} = 0, \quad (\sigma \neq \rho),
\]

\[
[T^\alpha_\gamma, S^I \gamma]_{1/q} = S^{I\alpha} - \lambda \sum_{\alpha < \beta < \gamma} S^{I\beta} T^\alpha_{\beta}, \quad (4.5)
\]

\[
[T^I_\beta, S^\beta \gamma]_{1/q} = S^I \gamma - \lambda \sum_{I < \alpha \beta} S^{\alpha\beta} T^I_{\alpha},
\]

\[
[T^I_\gamma, S^{0\gamma}]_{1/q} = -q S^{I\alpha} - \lambda \sum_{< \beta < \gamma} S^{\alpha\beta} T^I_{\beta} + q \lambda \sum_{I < \rho < \alpha} S^{\rho\alpha} T^0_{\rho},
\]

where $I < \alpha < \beta < \gamma$. Further, we have actions of $T^L_K (K < L)$ as,

\[
[T^\beta_\alpha, S^{\alpha\gamma}]_{q} = \{T^\beta_0, S^{0\gamma}\}_{1/q} = S^{\beta\gamma},
\]

\[
[T^\alpha_\tau, S^{\alpha\beta}]_{1/q} = \{T^\alpha_0, S^{0\alpha}\}_{1/q} = [T^\alpha_\gamma, S^{I\alpha}]_{1/q} = 0,
\]

\[
[T^\beta_\alpha, S^{I\alpha}]_{1/q} = S^{I\alpha},
\]

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\[ [T^\gamma_\alpha, S^{\alpha\beta}]_q = -q S^{\beta\gamma} - q \lambda S^{\alpha\gamma} T^\beta_\alpha, \] (4.6)

\[ \{T^\beta_0, S^{0\alpha}\}_1/\lambda = -q S^{\alpha\beta} + \frac{\lambda}{q} S^{0\beta} T^\alpha_0, \]

\[ [T^\gamma_\alpha, S^{I\beta}] = -\lambda S^{I\gamma} T^\beta_\alpha, \quad [T^\beta_I, S^{\alpha\gamma}] = -\lambda S^{\beta\gamma} T^\alpha_I, \]

\[ [T^\gamma_I, S^{\alpha\beta}] = -\lambda S^{\alpha\gamma} T^\beta_I + q \lambda S^{\alpha\beta} T^\gamma_I, \]

where \( I \neq J \) and \( I, J < \alpha < \beta < \gamma \). Note that \( S^{0\alpha} \) has the same relations with the deformed \( su(2,2) \) generators as the quantum 4-spinor \( \theta^\alpha \).

### 4.2 Deformed superconformal algebra

In this section we assign the deformed \( su(1|4) \) generators to the physical superconformal generators (See e.g. Ref. [30]). The assignment should be consistent with the conjugation (3.8). First of all, it is convenient to assign the deformed \( su(2,2) \) generators to a deformed \( so(4,2) \) generators \( M_{\mu\nu} \) (\( \mu, \nu = 0 \sim 5 \)) as follows,

\[
\begin{align*}
M_{12} &= \frac{i}{2} (H_1 + H_3), \quad M_{23} = \frac{i}{2} (T^1_2 + T^2_1 + T^3_4 + T^4_3), \\
M_{31} &= \frac{i}{2} (T^1_2 - T^2_1 + T^3_4 - T^4_3), \quad M_{01} = \frac{i}{2} (T^1_4 - T^4_1 + T^2_3 - T^3_2), \\
M_{02} &= \frac{i}{2} (T^1_4 + T^4_1 - T^2_3 - T^3_2), \quad M_{03} = -\frac{1}{2} (-T^1_3 + T^3_1 + T^2_4 - T^4_2), \\
M_{40} &= \frac{i}{2} (T^1_3 + T^3_1 + T^4_2 + T^2_4), \quad M_{41} = \frac{i}{2} (T^1_2 + T^2_1 - T^3_4 - T^4_3), \\
M_{42} &= \frac{i}{2} (T^1_2 - T^2_1 - T^3_4 + T^4_3), \quad M_{43} = \frac{i}{2} (H_1 - H_3) \\
M_{50} &= \frac{i}{2} (H_1 + 2H_2 + H_3), \quad M_{51} = \frac{i}{2} (T^2_3 + T^3_2 + T^1_4 + T^4_1) \\
M_{52} &= \frac{i}{2} (T^2_3 - T^3_2 - T^1_4 + T^4_1), \quad M_{53} = \frac{i}{2} (-T^1_3 - T^3_1 + T^2_4 + T^4_2) \\
M_{45} &= \frac{i}{2} (T^1_3 - T^3_1 + T^2_4 - T^4_2).
\end{align*}
\] (4.7)

In the classical limit, the generators \( M_{\mu\nu} \) satisfy the following algebra,

\[ [M_{\mu\nu}, M_{\rho\sigma}] = -ig_{\mu\rho} M_{\nu\sigma} + ig_{\nu\rho} M_{\mu\sigma} + ig_{\mu\sigma} M_{\rho\nu} - ig_{\nu\sigma} M_{\rho\mu}, \] (4.8)

where the \( so(4,2) \) metric is choosen \( g_{\mu\nu} = \text{diag} \ (1, -1, -1, -1, -1, 1) \). Under the conjugation (3.8), \( M_{\mu\nu} \) is ‘real’, i.e., they satisfy

\[ \overline{M}_{\mu\nu} = M_{\mu\nu}. \] (4.9)
Next we ‘compactify’ two-dimensional space, e.g., fourth and fifth dimensions. Namely we choose generators for translation $P_\mu \ (\mu = 0 \sim 3)$, conformal boost $K_\mu$ and dilatation $D$ as,

$$\begin{align*}
P_\mu &= M_{4\mu} + M_{5\mu}, \\
K_\mu &= M_{5\mu} - M_{4\mu}, \\
D &= M_{45}.
\end{align*} \ (4.10)$$

Further we assign the supercharges $Q_\alpha, \overline{Q}_{\dot{\alpha}}, S^\alpha, \overline{S}^\dot{\alpha}$ as follows,

$$\begin{align*}
Q_1 &= \sqrt{2}(T^0_1 - iT^0_3), \\
\overline{Q}_1 &= \sqrt{2}(T^1_0 - iT^3_0), \\
Q_2 &= \sqrt{2}(-T^0_2 + iT^0_4), \\
\overline{Q}_2 &= \sqrt{2}(-T^2_0 + iT^4_0), \\
S^1 &= \sqrt{2}(T^0_1 + iT^0_3), \\
S^2 &= \sqrt{2}(T^1_0 + iT^3_0), \\
S^2 &= \sqrt{2}(-T^0_2 - iT^4_0),
\end{align*} \ (4.11)$$

so that in the classical limit they satisfy

$$\begin{align*}
\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} &= 2(\sigma^\mu)_{\alpha\beta}P_\mu, \\
\{\overline{S}^\dot{\alpha}, S^\beta\} &= 2(\sigma^\mu)^{\dot{\alpha}\beta}K_\mu.
\end{align*} \ (4.12)$$

At last we choose the U(1) charge as,

$$A = -\frac{1}{4}(4H_0 + 3H_1 + 2H_2 + H_3). \ (4.13)$$

From the deformed $su(1|4)$ algebra, we can read off the deformed superconformal algebra of the generators defined in the above. However, as the algebra in the basis is very complicated, it is convenient to represent the algebra in the $T-H$ basis. It is interesting that the supercharges are not longer nilpotent. For example, $Q_1$ satisfy the following relations,

$$\begin{align*}
(Q_1)^2 &= 2i(q - 1)T^0_3T^0_1, \\
(Q_1)^3 &= 0.
\end{align*} \ (4.14)$$

The other supercharges satisfy similar relations.

It is very important that in this basis either the deformed Poincaré or the super-Poincaré algebras is not a closed subalgebra of the deformed superconformal algebra. Therefore, we need some contraction procedure \[31\] to obtain a closed Poincaré or super-Poincaré algebra from the above basis \[22, 23, 24, 20, 21\].

We restricted ourselves deformation of the $N=1$ superconformal algebra on four-dimensional space-time in the above. It is easy to extend the above approach to extended superconformal algebra on four-dimensional and two-dimensional space, i.e., $su(N|2, 2)$ and $su(N|1, 1)$. 

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4.3 Closed subalgebra of deformed $su(1|4)$ algebra with automorphism

In this subsection, we investigate closed subalgebras with some simple automorphisms, i.e., other conjugations besides (3.8). In Ref.[20], it was shown that $H_\alpha$, $T^{\alpha+1}_\alpha$ and $T^{\alpha+1}_\alpha$ ($\alpha = 1, 3$) compose a closed subalgebra (2.25) for a deformed Lorentz algebra with the following consistent automorphism,

$$H_1 \leftrightarrow H_3, \quad T^1_2 \leftrightarrow T^4_3, \quad T^2_1 \leftrightarrow T^3_4. \quad (4.15)$$

Next we consider another closed subalgebra consists of $T^1_3, T^1_4, T^2_3$ and $T^2_4$. They satisfy the following relations,

$$[T^1_3, T^2_4] = \lambda [T^1_4, T^2_3], \quad [T^1_4, T^2_3] = 0,$$
$$[T^2_3, T^1_3] = [T^1_4, T^1_3] = [T^2_4, T^2_3] = [T^2_4, T^1_3] = 0. \quad (4.16)$$

In the case where $|q| = 1$, this algebra has the following automorphism,

$$T^1_3 \rightarrow T^1_3, \quad T^2_4 \rightarrow T^2_4, \quad T^1_4 \leftrightarrow T^2_3. \quad (4.17)$$

When we take the above ‘conjugation’ with $|q| = 1$, we have to reverse the order of elements. The fact is opposite to the conjugation discussed in section three. For this closed subalgebra with the conjugation (4.17), we can find translation operators $\tilde{P}_\mu$ ($\mu = 0 \sim 4$) with the Minkowski metric as follows,

$$\begin{pmatrix} \tilde{P}_0 + i\tilde{P}_3 & \tilde{P}_1 + i\tilde{P}_2 \\ \tilde{P}_1 - i\tilde{P}_2 & \tilde{P}_0 - i\tilde{P}_3 \end{pmatrix} \equiv \begin{pmatrix} T^1_3 & T^1_4 \\ T^2_3 & T^2_4 \end{pmatrix}. \quad (4.18)$$

It is easily shown that $\tilde{P}_\mu$ is real under the conjugation (4.17). Further the above algebra has a center $C$ as follows,

$$C = qT^1_3 T^2_4 - T^1_4 T^2_3. \quad (4.19)$$

Although supercharges associated with the translation operators $\tilde{P}_\mu$ correspond to linear combinations of $T^1_0, T^2_0, T^0_3$ and $T^0_4$, unfortunately the supercharges and $\tilde{P}_\mu$ do not compose a closed algebra and their commutation relations involve $H_0$, $H_1$ and $T^2_1$. We are very interested in a closed Poincaré algebra including $\tilde{P}_\mu$. However, we can not find a Lorentz algebra consistent with the conjugation (4.17).
Instead of a deformed Poincaré algebra, we can derive an interesting deformed inhomogeneous $iso(2,2)$ algebra with a metric $\hat{g}_{\mu\nu} = \text{diag}(1,-1,1,-1)$ composed by translation $T^1_3,T^1_4,T^2_3,T^2_4$ and rotation $T^\alpha_{\alpha+1},T^{\alpha+1}_{\alpha},H_\alpha$ ($\alpha = 1,3$). They satisfy a closed subalgebra, which is read off Appendix B. We define a physical basis of translation and rotation as follows,

\[
\left( \hat{P}_0 + \hat{P}_3 \begin{array}{c} \hat{P}_1 + \hat{P}_2 \\ \hat{P}_1 - \hat{P}_2 \end{array} \right) \equiv i \left( \begin{array}{cc} T^1_3 & T^1_4 \\ T^2_3 & T^2_4 \end{array} \right).
\]

\[
\hat{M}_{03} = \frac{i}{2}(H_1 - H_3), \quad \hat{M}_{21} = \frac{i}{2}(H_1 + H_3), \quad \hat{M}_{23} = \frac{i}{2}(T^1_2 + T^2_1 + T^3_4 + T^4_3), \quad \hat{M}_{13} = \frac{i}{2}(T^1_2 - T^2_1 + T^3_4 - T^4_3),
\]

\[
\hat{M}_{02} = \frac{i}{2}(-T^1_2 + T^2_1 + T^3_4 - T^4_3), \quad \hat{M}_{01} = \frac{i}{2}(-T^1_2 - T^2_1 + T^3_4 + T^4_3).
\]

In the case where $|q| = 1$, this deformed algebra of $\hat{P}, \hat{M}$ has the following automorphism,

\[
H_\alpha \to -H_\alpha, \quad T^{\alpha}_{\alpha+1} \to -qT^{\alpha+1}_{\alpha}, \quad T^{\alpha+1}_\alpha \to -\frac{1}{q}T^{\alpha+1}_\alpha, \quad T^\beta_\gamma \to -T^\beta_\gamma,
\]

where $\alpha = 1,3, \beta = 1,2, \gamma = 3,4$. When we take the above conjugation, we also reverse the order of elements. Under the conjugation (4.21) $\hat{P}_\mu$ and $\hat{M}_{\mu\nu}$ are real. We can take away the minus sign of the last equation of (4.21) if we put off $i$ from the definition of $\hat{P}_\mu$ (4.20). Further it is remarkable that $C$ (4.19) is still the center in the deformed $iso(2,2)$ algebra.

At last we consider supersymmetrization of the above deformed $iso(2,2)$ algebra. Supercharges associated with $\hat{P}$ correspond to $T^1_0, T^2_0,T^0_3,T^0_4$. The elements need $H_0$ so that they have closed commutation relations with the generators of the deformed $iso(2,2)$ generators. The superalgebra has the following automorphism consistent with (4.21),

\[
T^1_0 \to fT^1_0, \quad T^2_0 \to \frac{f}{q^2}T^2_0, \quad H_0 \to \frac{1}{q^2}H_0, \quad T^0_\alpha \to \frac{1}{q^3f}T^0_\alpha, \quad T^\alpha_\gamma \to \frac{1}{q^3}T^\alpha_\gamma.
\]

where $\alpha = 3,4$ and $f$ is an arbitrary factor. In the last equation of (4.22), the minus sign is due to the sign of the last equation (4.21). So we can take away both signs simultaneously. Suppose that we define physical supercharges as follows,

\[
\hat{Q}_1 \equiv iT^1_0, \quad \hat{Q}_2 \equiv iq^{-1}T^2_0, \quad \overline{Q}_1 \equiv -q^{-3/2}T^0_3, \quad \overline{Q}_2 \equiv -q^{-3/2}T^0_4.
\]
where we choose $f = 1$. They satisfy classically commutation relations similar to (4.12). Note that $\hat{Q}_\alpha$ and $\overline{Q}_\alpha$ are ‘real’ themselves under the conjugation (4.22), i.e., $\overline{Q}_\alpha$ does not imply the conjugate of $\hat{Q}_\alpha$ and they are independent operators. The approach to the above superalgebra could be extended.

5. Conclusion

We have studied here the deformed $su(m|n)$ algebra on the quantum superspace. Some interesting aspects of the algebra has been shown and we have constructed the deformed superconformal algebra as an application of the deformed $su(m|n)$ algebra. The quantum 6-vector is also obtained from the tensor product of the fermionic quantum 4-spinors. Further we have discussed the closed subalgebras of the deformed $su(1|4)$ algebra with the consistent automorphisms, which include the deformed Lorentz, translation of Minkowski space, $iso(2, 2)$ and its supersymmetric algebras.

It is very important to apply the above approach to deformed $so$ and $sp$ algebras. The deformed $su(m|n)$ algebra obtained here includes lots of interesting algebra, e.g., the extended 2-dim and 4-dim superconformal algebra to be studied and the deformed $iso(2, 2)$ algebra, whose classical space is interesting, e.g., for the O(2) string (See e.g. Ref.[32]). It might be possible to derive a deformed Poincaré algebra with the correct metric and reality, from a large algebra in the similar way to the procedure to find the closed subalgebra with the simple automorphism in section 4.3.

Acknowledgement

The author would like to thank T. Uematsu for numerous valuable discussions and reading the manuscript and P. P. Kulish and R. Sasaki for helpful discussions. He also thanks S. Matsuda and H. Aoyama for encouragements.
Appendix A

Here, we study decomposition of the differential algebra (2.3) following Ref. [25], where decomposition of the bosonic differential algebra was discussed through “renormalization” of the quantum coordinates and derivatives. We define

\[ \mu_i \equiv \partial_i x^i - x^i \partial_i = 1 + (q^{-2} - 1)(\sum_{i \leq j} x^i \partial_j + \sum_\alpha \theta^\alpha \partial_\alpha), \]

\[ \mu_\alpha \equiv \partial_\alpha \theta^\alpha + \theta^\alpha \partial_\alpha = 1 + (q^{-2} - 1)(\sum_{\alpha < \beta} \theta^\beta \partial_\beta). \] (A.1)

They satisfy the following commutation relations,

\[ [\mu_I, Z^J]_{1/q^2} = [\mu_I, \partial_J]_{q^2} = [\mu_i, x^i]_{1/q^2} = [\mu_i, \partial_i]_{q^2} = 0 \]

\[ [\mu_I, Z^J] = [\mu_I, \partial_I] = [\mu_\alpha, \theta^\alpha] = [\mu_\alpha, \partial_\alpha] = 0, \] (A.2)

where \( I < J \), and \( \mu_I \) commute with each other.

Suppose we define new coordinates \( \hat{Z}^I (X^i, \Theta^\alpha) \) and derivatives \( D_\alpha \) as follows,

\[ X^i = \mu_i^{-1/2} x^i, \quad \Theta^\alpha = \mu_\alpha^{-1/2} \theta^\alpha, \quad D_I = \mu_I^{-1/2} \partial_I. \] (A.3)

Then the new coordinates and derivatives satisfy simple relation as

\[ D_i X^i = 1 + q^2 X^i \partial_i, \quad D_\alpha \Theta^\alpha = 1 - \Theta D_\alpha, \]

\[ [\hat{Z}^I, \hat{Z}^J] = [\hat{Z}^I, D_J] = [D_I, D_J] = 0, \] (A.4)

It is remarkable that the fermionic elements \( \Theta^\alpha \) and \( D_\alpha \) satisfy completely the classical algebra. Therefore, we can derive the quantum fermionic coordinates \( \theta^\alpha \) and derivatives \( \partial_\alpha \) from the classical elements \( \Theta^\alpha \) and \( D_\alpha \) as follows,

\[ \theta^\alpha = \sqrt{\mu_\alpha} \Theta^\alpha, \quad \partial_\alpha = \sqrt{\mu_\alpha} D_\alpha, \]

\[ \mu_\alpha = \prod_{\alpha < \beta} (1 + (q^{-2} - 1) \Theta^\beta D_\beta). \] (A.5)

The similar relation has been found for the fermionic q-oscillators in Ref. [33].

Next we study actions of the generators on the new space. For example, we consider the case of the two-dimensional bosonic quantum space with coordinates \( x^i (i = 1, 2) \) and derivatives \( \partial_i \). By definition (A.1), we can easily obtain

\[ [T_2^1, \mu_1] = [T_1^2, \mu_1] = 0, \]
\[ [T^1_2, \mu_2] = (q^{-2} - 1)x^1 \partial_2 = (q^{-1} - 1)(\sqrt{\mu_2} x^1 D_2 + x^1 D_2 \sqrt{\mu_2}), \quad (A.6) \]
\[ [T^2_1, \mu_2] = (1 - q^{-2})x^2 \partial_1 = q^{-1}(q^{-1} - 1)(\sqrt{\mu_2} x^2 \partial_1 + X^2 \partial_1 \sqrt{\mu_2}). \]

Eq. (A.6) leads to the following relations,
\[ [T^1_2, \sqrt{\mu_2}] = (q^{-1} - 1)(\mu_2)^{-1/2} x^1 \partial_2, \quad [T^2_1, \sqrt{\mu_2}] = q^{-1}(q^{-1} - 1)(\mu_2)^{-1/2} x^2 \partial_1. \quad (A.7) \]

Therefore, we obtain actions of \( T^1_2 \) and \( T^2_1 \) on the renormalized coordinates \( X^1 \) and \( X^2 \) as follows,
\[ T^1_2 X^1 = -q^{-1} X^1 T^1_2, \quad T^1_2 X^2 = q X^2 T^1_2 + q \mu X^1 + q(q - 1) \mu X^1 X^2 D_2, \]
\[ T^2_1 X^1 = q^{-1} X^1 T^2_1 + \mu^{-1} X^2, \quad T^2_1 X^2 = q X^2 T^1_2 + (1 - q^{-1}) \mu X^2 X^2 D_1. \quad (A.8) \]

where \( \mu = \sqrt{\mu_1 / \mu_2} \). This result is rather complicated than (2.7) and (2.8). We could obtain actions of \( T^1_2 \) and \( T^2_1 \) on the renormalized fermionic quantum space \( \Theta^\alpha \), which are also complicated.
Appendix B

Here, the whole deformed $su(1|4)$ algebra is explicitly shown.

\[ [T^I_J, T^I_K]_q = T^I_K, \quad (I < J < \text{Kor}I > J > \text{K}). \]

\[ [T^\alpha_{\alpha+1}^\alpha, T^\alpha_{\alpha+1}^\alpha]_q = qH_\alpha, \quad \{T^\alpha_1, T^\alpha_1\} = H_0, \]

\[ [H_\alpha, T^\alpha_{\alpha+1}^\alpha]_q = -q^2(q + q^{-1})T^\alpha_{\alpha+1}, \quad [H_\alpha, T^\alpha_{\alpha+1}^\alpha]_1/q^4 = \frac{q + q^{-1}}{q^2}T^\alpha_{\alpha+1}, \]

\[ [H_\alpha, T^\alpha_J^J]_q = -qT^\alpha_J, \quad [H_\alpha, T^\alpha_{\alpha+1}^\alpha]_1/q^4 = q^{-1}T^\alpha_{\alpha+1}^J, \quad (\alpha + 1 < J < \text{Kor}J < \alpha), \]

\[ [H_\alpha, T^\alpha_J^J]_1/q^2 = q^{-1}T^\alpha_J, \quad [H_\alpha, T^\alpha_{\alpha+1}^\alpha]_1/q^2 = -qT^\alpha_{\alpha+1}, \quad (\alpha + 1 < J < \text{Kor}J < \alpha), \]

\[ [H_0, T^\rho_J^J]_1/q^2 = T^\rho_J, \quad [H_0, T^\rho_J^\rho]_q = -q^2T^\rho_J, \quad (J = 0, 1 \rho > 1), \]

\[ [T^\beta_{\beta+1}^\beta, T^\beta_{\beta+1}^\beta]_q = [T^\beta_{\beta}, T^\beta_{\beta}]_q = [T^\beta_{\alpha}, T^\beta_{\alpha}]_q = 0, \]

\[ [T^I_{\beta}, T^I_{\alpha}]_{-1}^q = [T^\beta_{\beta}, T^\alpha_{\beta}]_{-1}^q = [T^\alpha_{\beta}, T^\beta_{\alpha}]_q = 0, \]

\[ [T^\alpha_{\gamma}, T^I_{\beta}]_q = \lambda T^\alpha_{\gamma}T^I_{\beta}, \quad [T^\beta_{I}, T^\gamma_{\alpha}]_q = \lambda T^\gamma_{\alpha}T^\beta_{I}, \]

\[ [T^\alpha_{\alpha+1}, T^\alpha_{\alpha+1}]_q = T^\alpha_{\alpha+1} - \lambda T^\alpha_{\alpha+1}H_\alpha, \quad (\alpha + 1 < \rho), \]

\[ [T^\alpha_{\alpha}, T^\alpha_{\alpha+1}]_q = q^2T^\rho_{\alpha+1} - \lambda q^2T^\rho_{\alpha+1}H_\alpha, \quad (\alpha + 1 < \rho), \]

\[ [T^\gamma_{\sigma}, T^\gamma_{\rho}]_{1/q} = T^\rho_{\sigma}, \quad (\sigma < \rho, \rho = \gamma - 1) \text{ or } (\rho < \sigma, \sigma = \gamma - 1), \]

\[ [T^\gamma_{\sigma}, T^\gamma_{\rho}]_q = T^\rho_{\sigma} - \lambda T^\rho_{\rho}T^\gamma_{\gamma}, \quad ((\sigma, \rho) = (1,2) \text{ or } (2,1)), \]

\[ [T^\sigma_{\alpha}, T^\sigma_{\alpha}]_{q^{-1}} = -q^{\sigma-\sigma-2}T^\sigma_{\sigma} + \lambda T^\sigma_{\sigma}T^\sigma_{\sigma} + \lambda q^{\rho-\rho+2}T^\rho_{\sigma}(H_1 + H_2) - \lambda^2T^\rho_{\sigma}T^\sigma_{\sigma}H_1 \]

\[ - \lambda^2q^{\rho-\rho+2}T^\rho_{\sigma}H_1H_2, \quad ((\sigma, \rho) = (3,4) \text{ or } (4,3)), \]

\[ [T^\alpha_{\alpha+2}, T^\alpha_{\alpha+2}]_{q^{-2}} = -q^{-1}H_\alpha - qH_{\alpha+1} - \lambda q^{-1}T^\alpha_{\alpha+1}T^\alpha_{\alpha+1} + \lambda q^{-1}T^\alpha_{\alpha+2}T^\alpha_{\alpha+2} \]

\[ + q\lambda H_\alpha H_{\alpha+1} - \lambda^2q^{-1}T^\alpha_{\alpha+1}T^\alpha_{\alpha+1}H_\alpha, \]

\[ [T^\gamma_{\alpha}, T^\gamma_{\alpha}]_{q^{-2}} = -q(2 - q^{-2})H_\gamma - q^{-1}(2 - q^2)H_2 - q(2 - q^2)H_3 - q^{-1}\lambda T^\gamma_{\alpha}T^\gamma_{\alpha} + \lambda q^{-1}T^\gamma_{\alpha}T^\gamma_{\alpha} \]

\[ - q^{-1}\lambda T^\gamma_{\gamma}T^\gamma_{\gamma} + q^{-1}(2 - q^2)\lambda T^\gamma_{\gamma}T^\gamma_{\gamma} + (1 + q^{-2})\lambda T^\gamma_{\gamma}T^\gamma_{\gamma}H_1 - \lambda qH_1H_2 - \lambda q^3H_1H_3 \]

\[ + \lambda q^{-1}(2 - q^{-2})(2 - q^2)H_1H_2 - q + q^{-1})(2 - q^{-1})H_2H_3 - \lambda^2T^\gamma_{\gamma}T^\gamma_{\gamma} \]

\[ - q^{-3}(2 - q^2)\lambda^2T^\gamma_{\gamma}T^\gamma_{\gamma}H_2 - q^2\lambda^2T^\gamma_{\gamma}T^\gamma_{\gamma}H_1 - \lambda q^3T^\gamma_{\gamma}T^\gamma_{\gamma}H_1H_2 + q^2\lambda^3H_1H_2H_3, \]

\[ [T^\gamma_{\gamma}, T^\gamma_{\alpha}]_{1/q} = q^2(\gamma - \alpha - 1)T^\gamma_{\alpha} - \lambda \sum_{\alpha < \beta < \gamma} q^2(\gamma - \beta - 1)T^\beta_{\alpha}, \]

\[ 21 \]
\[
[T_0^\alpha, T_3^\alpha]_q = -q^{2(\gamma-\alpha)-1} T_0^\alpha + \lambda \sum_{\alpha<\beta<\gamma} q^{2(\gamma-\beta)} T_\beta^\alpha T_0^\beta,
\]

\[
\{T_0^\rho, T_1^1\}_q = q T_1^1 - \lambda T_\rho^1 H_0, \quad \{T_1^2, T_0^\rho\}_q = q^3 T_0^\rho - \lambda q^2 T_\rho^1 H_0, \quad (\rho > 1),
\]

\[
\{T_0^\rho, T_0^2\}_q = q T_2^\rho - \lambda T_\rho^2 H_0 - q \lambda T_\rho^1 H_1 - \lambda q^3 T_1^1 T_2^\rho + \lambda^2 T_\rho^2 H_1 H_0 + \lambda^2 q^2 T_\rho^1 T_1^2 H_0,
\]

\[
\{T_2^\rho, T_\rho\}_q = q^3 T_2^\rho - \lambda q^2 T_2^\rho H_0 - \lambda q^3 T_2 H_1 - \lambda q^3 T_1^1 T_2^\rho + \lambda^2 q^2 T_\rho^1 H_1 H_0 + \lambda^2 q^3 T_2^1 T_2^\rho H_0,
\]

\[
\{T_0^2, T_0^0\} = H_0 + q H_1 - \lambda H_1 H_0 + q \lambda T_1^1 T_0^0 - \lambda q^3 T_1^1 T_2^1 + \lambda^2 q^2 T_2^1 T_1^2 H_0,
\]

\[
\{T_3^0, T_3^1\} = q^4 H_0 + q^3 H_1 + q H_2 - \lambda q^2 T_0^1 T_1^0 - q \lambda T_0^0 T_2^0 - q \lambda H_0 H_1 - \lambda H_2 H_0 - \lambda q^3 T_3^1 T_2^2 - \lambda q^3 T_3^1 T_3^1 - q \lambda H_1 H_2 - q^4 \lambda^2 T_2^2 T_1^2 + \lambda^2 q^2 T_3^1 T_2^2 H_1 + \lambda^2 q^2 T_3^2 T_3^1 H_0 + \lambda^2 H_1 H_2 H_0 - \lambda^3 q^2 T_3^2 T_3^2 H_0 + \lambda^3 q^3 T_1^1 T_2^1 H_0,
\]

\[
\{T_4^0, T_4^1\} = -[T_4^1, T_4^1]_{q^2} (q^4 + \lambda H_0) + q^6 H_0 - \lambda q^3 T_1^0 T_0^1 - \lambda q T_2^0 T_2^0 - \lambda q T_0^0 T_3^3 + \lambda q^5 T_1^3 T_3^1 T_1 + \lambda q^7 T_1^2 T_1 + \lambda^2 q^4 T_3^1 T_3^1 H_0 + \lambda^2 q^6 T_2^1 T_2^1 H_0,
\]

where \( I < \alpha < \beta < \gamma \), and for the other commutation relations the generators satisfy the classical algebras, i.e., they commute or anticommute depending on their Grassman parity.
References

[1] V. G. Drinfeld, Sov. Math. Dokl. 32(1985)254.
[2] M. Jimbo, Lett. Math. Phys. 10(1985)63.
[3] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, Algebra and Analysis, 1(1987)178.
[4] P. P. Kulish and N. Yu. Reshetikhin, J. Sov. Math. 23(1983)2435.
[5] H. -D. Doebner and J. -D. Hennig eds., *Quantum Groups*, Lecture Notes in Physics 370(Springer Verlag, 1990).
[6] T. Curtright, D. Fairlie and C. Zachos eds., *Quantum Groups*, Proceedings of the Argonne Workshop (World Scientific, 1990).
[7] L. A. Takhtajan, Adv. Studies Pure Math. 19(1989)435; Lecture Notes in Physics 370(1989)3.
[8] P. P. Kulish ed., *Quantum Groups*, Lecture Notes in Mathematics 1510(Springer Verlag, 1992).
[9] Yu. I. Manin, Commun. Math. Phys. 123(1989)163.
[10] S. L. Woronowicz, Commun. Math. Phys. 111(1987)613; 122(1989)125.
    Publ. RIMS, Kyoto Univ. 23(1987)117.
[11] J. Wess and B. Zumino, Nucl. Phys. B (Proc. Suppl.) 18B(1990)302.
[12] A. Schirrmacher, J. Wess and B. Zumino, Z. Phys. C – Particles and Fields 49(1991)317.
[13] A. Schirrmacher, Z. Phys. C – Particles and Fields 50(1991)321.
[14] U. Carow-Watamura, M. Schlieker and S. Watamura, Z. Phys. C – Particles and Fields 49(1991)439.
[15] T. Kobayashi and T. Uematsu, Z. Phys. C – Particles and Fields 56(1992) 193.
[16] U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, Z. Phys. C – Particles and Fields 48(1990)159; Int. J. Mod. Phys. A6(1991)3081.
[17] W. B. Schmidke, J. Wess and B. Zumino, Z. Phys. C – Particles and Fields \textbf{52}(1991)471.

[18] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, Lett. Math. Phys. \textbf{23}(1991)233.

[19] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, Commun. Math. Phys. \textbf{150}(1992)495.

[20] T. Kobayashi and T. Uematsu, “q-Deformed Conformal and Poincaré Algebra on Quantum 4-spinors” Preprint KUCP-52-revised; to be published in Z. Phys. C – Particles and Fields.

[21] T. Kobayashi and T. Uematsu, “q-Deformed Superconformal Algebra on Quantum Superspace” Preprint KUCP-56.

[22] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoy, Phys. Lett. \textbf{264B} (1991)331.

[23] J. Lukierski and A. Nowicki, Phys. Lett. \textbf{279B}(1992)299.

[24] J. Lukierski, A. Nowicki and H. Ruegg, “Quantum Deformation of Poincaré and Conformal Algebras ”, Preprint BUHEP-91-21(1991).

[25] O. Ogievetsky, Lett. Math. Phys. \textbf{24}(1992)245.

[26] T. Curtright and C. Zachos, Phys. Lett. \textbf{243B}(1990)237.

[27] E. Witten, Nucl. Phys. \textbf{B330}(1990)285.

[28] D. Fairlie, J. Phys. \textbf{A23}(1990)L183.

[29] V. Jain and O. Ogievetsky, Mod. Phys. Lett. \textbf{A7}(1992)2199.

[30] P. van Nieuwenhuizen, Phys. Reports \textbf{68}(1981)189. M. F. Sohnius, Phys. Reports \textbf{128}(1985)39.

[31] E. Celeghini, R. Giacchetti, E. Sorace and M. Tarlini, J. Math. Phys. \textbf{31}(1990) 2548; \textbf{32}(1991) 1155; \textbf{32}(1991) 1159.
[32] M. Ademollo, L. Brink, A. D’Adda, R. D’Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, R. Pettorino and J. H. Schwarz, Nucl. Phys. B111(1976)77.
L. Brink and J. H. Schwarz, Nucl. Phys. B121(1977)285.
H. Ooguri and C. Vafa, Mod. Phys. Lett. A5(1990)1389.

[33] M. Chaichan, P. Kulish and J. Lukierski, Phys. Lett. 262B(1991)43.