Orbital angular momentum light frequency conversion and interference with quasi-phase matching crystals

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Abstract: Light with helical phase structures, carrying quantized orbital angular momentum (OAM), has many applications in both classical and quantum optics, such as high-capacity optical communications and quantum information processing. Frequency conversion is a basic technique to expand the frequency range of the fundamental light. The frequency conversion of OAM-carrying light gives rise to new physics and applications such as up-conversion detection of images and generation of high dimensional OAM entanglements. Quasi-phase matching (QPM) nonlinear crystals are good candidates for frequency conversion, particularly due to their high-valued effective nonlinear coefficients and no walk-off effect. Here we report the first experimental second-harmonic generation (SHG) of an OAM-carryed light with a QPM crystal, where a UV light with OAM of 100 $\eta$ is generated. OAM conservation is verified using a specially designed interferometer. With a pump beam carrying an OAM superposition of opposite sign, we observe interesting interference phenomena in the SHG light; specifically, a photonics gear-like structure is obtained that gives direct evidence of OAM conservation, which will be very useful for ultra-sensitive angular measurements. Besides, we also develop a theory to reveal the underlying physics of the phenomena. The methods and theoretical analysis shown here are also applicable to other frequency conversion processes, such as sum frequency generation and difference-frequency generation, and may also be generalized to the quantum regime for single photons.

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Orbital angular momentum (OAM) in light is a very useful degree of freedom that has no dimensional limitation, and has been widely studied in both classical and quantum optics fields since first introduced by Allen in 1992 [1]. It has been shown [1] that a light beam with a helical phase $e^{i\alpha}$ in the azimuthal direction carries $l\hbar$ units of OAM per photon. An OAM-carried light has been widely used in many fields, such as optical manipulation [2–5],...
optical trapping [6], optical tweezers [7], optical vortex knots [8], imaging [9], astronomy [10], free-space information transfer and communications [11], and also quantum information processing [12–19].

The interaction of an OAM-carried light with a matter, such as a nonlinear crystal [12, 20–23] and atomic vapor [24–26], produces many interesting phenomena in contrast to those obtained using Gaussian beams. Allen and his associates [20, 21] have demonstrated the OAM transformation and conservation in frequency conversion in a LBO crystal. In Ref. [23], the authors have presented both theoretical and experimental study of generating a kind of quasi-nondiffracting focused beam with a flower-like OAM light. Zeilinger’s group [12] has generated a high-dimensional OAM entanglement in the spontaneous parametric down-conversion processes in a nonlinear crystal. In all these nonlinear interaction processes, the total OAM conservation of all lights plays a very important role. The frequency conversion of the OAM-carried lights will be very useful in up-conversing detection of images [27] and generating of an OAM-carried light from a fundamental OAM-carried light at special wavelength (in the UV or mid-infrared frequency domains), which are hard to produce with traditional method. For nonlinear processes with crystals, the advantages of high-valued effective nonlinear coefficients and no walk-off effect with quasi-phase matching (QPM) compared with birefringence phase matching make QPM crystals good candidates for frequency conversion of an OAM-carried light. Then some important questions are coming naturally: can we use QPM crystals for nonlinear frequency conversion of an OAM-carried light? If the total OAM of all lights is conserved in such nonlinear processes? Is frequency conversion of an OAM superposition state possible? So far, no experimental work has been reported on such frequency conversion processes, although one theoretical study about the process of sum frequency generation (SFG) and second-harmonic generation (SHG) [28] has appeared.

In this work, we report the first experimental generation of an OAM-carrying UV light through SHG with a QPM type-I PPKTP crystal. We demonstrate the conservation of OAM in the SHG process, which concurs with the theory of Ref. [28]. Moreover, we observe a very interesting interference phenomenon by transforming the pump light into a hyper-superposition of polarization and OAM states. We directly see a photonic gear-like structure in the SHG light beam, which is coincides with the study presented in Ref. [23]. This phenomenon can be regarded as direct evidence of OAM conservation. The photonic gear can be rotated by rotating the pump-beam polarization, an effect that can be used for ultra-sensitive angular measurements. These observations can be well explained by the theory we have developed. The method we demonstrate here provides a new way to generate an OAM-carried light via frequency conversion by using QPM crystals. Moreover, because of its low diffraction, UV light can enhance the resolution of OAM light-based imaging. Using the SHG process in OAM light-based ultra-sensitive angular measurements [18], resolutions can be further enhanced by a factor of 2. Our approach may also be used in sum frequency generation (SFG) or difference-frequency generation (DFG) [29] at the single-photon level. This will be very useful for quantum information processing using the OAM degrees of freedom of photons.
2. Demonstration of OAM conservation

We first demonstrate OAM conservation in the SHG process. Figure 1 shows the different blocks used in our experiments; blocks (a), (b), and (c) are used in the OAM conservation demonstration. Block-(a) is used to generate an OAM-carried light using vortex phase plates (VPPs, from RPC photonics). Block-(b) performs frequency conversion and comprises two lenses (both have the same focus length of 125mm), a type-I PPKTP crystal, and a UV transparent filter used to remove the pump light. The 1 mm × 2 mm × 10 mm PPKTP crystal, supplied by Raicol Crystals, was designed for SHG from 795 nm to 397.5 nm. Both end faces are anti-reflection coated for these two wavelengths. The measured nonlinear conversion efficiency for the PPKTP crystal is 1%/W for a Gaussian pump mode. In our experiments, we produce 1 μW SHG light with 10 mW pump light. The high efficiency frequency conversion of the OAM-carried light could be realized by putting the crystal into an external cavity [23]. The laser we used is from a continuous wave Ti: sapphire laser (Coherent, MBR 110, less than 100 KHz line width when locked). The measured phase matching temperature of the crystal is 64.3°C. The temperature of the crystal is controlled with a semiconductor Peltier cooler with stability of ± 2 mK. Block-(c) is a specially designed balanced interferometer for generating a light in a superposition of OAM states of the form $|l\rangle + e^{i\beta}|-l\rangle$ given an input state $|l\rangle$, where $\beta$ is the relative phase of the two arms of the interferometer. The input light is
set to be 45° linear polarized by setting the half wave plate fast axes at 22.5° relative to the vertical axes, the 45° polarized light is separated by the polarization beam splitter, the light reflected from PBS passes through the quarter wave plate with its fast axes 45° relative to the vertical axes, then the back reflected light from mirror 4 has horizontal polarization, and transmits through the PBS. From import port and the output port of the interferometer, the reflection times of the beams in the two arms have a difference of one, therefore if one beam is in state $|l\rangle$, the other will be in state $|-l\rangle$. The output of this interferometer, taken by a commercial charge coupled device (CCD) camera, has an intensity distribution showing a radially symmetric pattern with $2l$ maxima arranged in a ring. Counting the numbers of maxima in the pattern yields the value of OAM for the pump and SHG beam.

![Fig. 2. Experimental demonstration of OAM conservation in the SHG process. Column (a) and (b) give the intensity profile and interference pattern for the pump beam; Column (c) and (d) give the corresponding image for the SHG light. The top and bottom sets of panels are for pump beams carrying OAM of 2 and 20, respectively. Corresponding theoretical images are paired below the experimental images.](image)

As shown in Ref. [28], the OAM of the SHG light is doubled compared with that of the pump beam if the Gouy phases are the same for both. The experimental results are shown in Fig. 2 paired with corresponding results from theoretical simulations. In the top set of panels in Fig. 2, image-(a) gives the intensity profile of the pump beam with $l = 2$, image-(b) gives the interference pattern obtained by directing the beam into block-(c), image-(c) gives the intensity profile for the SHG beam, and image-(d) gives the corresponding interference...
pattern. We found that the number of maxima of the SHG light is exactly twice the number of the pump beam, which verifies the conservation of OAM in the SHG process. The bottom set of panels is similar to the top set, but with a pump beam with \( l = 20 \). The pump beam has 40 maxima in the interference pattern; the corresponding SHG light has 80 maxima. The value of \( l \) can be further increased in principle; it is limited by the dimensions of the crystal. For crystals with larger cross-sections, \( l \) can be much larger.

### 3. Theoretical analysis of OAM superposition state conversion

When we use a pump beam with a hyper-superposition of polarization and OAM states for SHG, a photonic gear-like structure is obtained. Before showing the experimental results, we first give a detailed theoretical description. We use quantum mechanics to describe the transformation of a light in block-(d) or (e), and a configuration similar to that presented in Refs. [17, 30, 31] is used in our experiment. We assume that the input beam of the interferometer is in a Gaussian mode and is polarized in the horizontal direction, the input state can be expressed as

\[
|\Psi_{in}\rangle = |H\rangle \otimes |0\rangle,
\]

where \( |H\rangle \) denotes the polarization degree of freedom and \( |0\rangle \) represents the OAM degrees of freedom. After passing through block-(d) (or block-(e)), the light is transformed into the state

\[
|\Psi_{out}(\theta, \delta)\rangle = \frac{1}{\sqrt{2}}[(\cos(2\delta)|H\rangle - \sin(2\delta)|V\rangle) \otimes |l\rangle - e^{-i(4\theta + \pi l)}(\sin(2\delta)|H\rangle + \cos(2\delta)|V\rangle) \otimes |l\rangle],
\]

where \( \theta \) and \( \delta \) are the angles of the fast axis of the half-wave plate (HWP) with respect to the vertical axis at the respective input and output ports of the block, and \( l \) is the OAM quantum number imprinted on the two counter-propagated beams in the interferometer. The output SHG light is in the form (See Appendix A for detail)

\[
E_{SHG} \propto \sin^2(2\delta)LG_{0}^{2l} - e^{-i8\theta}\cos^2(2\delta)LG_{0}^{-2l} + \Gamma \sin(4\delta)e^{-i(4\theta + \pi l)}LG_{l}^{0},
\]

where \( \Gamma \) is a constant of renormalization. This expression shows that the output of the SHG light is in a superposition of OAM states of \( 2l \), \( 0 \), and \( -2l \) depending on the angle of \( \delta \). We now focus on the case \( \delta = \pi / 8 \); apart from a relative phase of \( 8\theta \), the first two terms have the same amplitudes. As mentioned before, an interference pattern with \( 4l \) maxima is generated in the intensity distribution of the outer ring, giving direct evidence of OAM conservation in the SHG process. A more interesting thing is that the interference pattern can be rotated if the phase \( \theta \) is changed, indicating that the total phase of the pump beam is preserved in the SHG process. This behaviour is similar to a mechanical gear: the pattern rotates through angle \( \pi / 2l \) if \( \theta \) changes by \( \pi / 4 \), which can be exploited for ultra-sensitive measurements of angles. Furthermore, by changing \( \delta \), we can switch easily between states \( |2l\rangle \), \( |-2l\rangle \), and \( |2l\rangle + e^{i8\theta}|-2l\rangle \), thus presenting a means to perform optical switching between different OAM states (The third term in Eq. (3) can be removed by spatial filtering).

### 4. Experimental results for conversion of an OAM superposition state

Based on the above theoretical analysis, we performed our experiments using blocks (b), (d) and blocks (b), (e). The experimental and theoretical results using the former setup for lights with \( l = 2 \) and \( l = 20 \) are shown in Fig. 3 rows (a) and (b), respectively. The pump power is
about 10 mW, and the temperature of the PPKTP is kept at the QPM point. The first and third images are the respective interference patterns of the pump light and corresponding SHG light observed by a CCD camera directly after block-(b); the second and fourth images are the corresponding theoretical patterns and show good agreement with the experimental results. As the Appendix A shows, the generated modes of the SHG beam are not standard Laguerre-Gaussian (LG) modes at the position where they are emitted. After propagating to the far-field, they evolve into the standard LG modes. There are nine maxima in the SHG intensity profile for \( l = 2 \), created via the interference among modes \( LG_0^4, LG_0^4 \), and \( LG_2^0 \). For small OAM, the diffraction of the \( LG_0^0 \) mode is close to the \( LG_2^2 \) mode (the first two modes have the same diffraction property), hence both will overlap partially in the far-field. For large OAM, however, these two modes have different diffraction properties, hence almost separate from each other. In addition, the intensity redistribution makes the mode \( LG_0^0 \) hard to be observed. For \( l = 20 \), the interference from state \( |40\rangle + e^{i\theta} | -40 \rangle \) can be seen very clearly, but the \( LG_0^0 \) mode is blurred and hard to be seen; only a dim point can be distinguished at the centre, and hence we cannot observe the multi-ring structure. By rotating the angle of the HWP in the input port of the interferometer in block-(d), a rotation in the output image is observed. We also find that the image of the SHG light is clearer than the input; this is because waves of shorter wavelength are diffracted less.

![Image](image1.png)

**Fig. 3.** SHG results with an input light in state \( |l\rangle + e^{i\theta} | -l \rangle \) with (a) \( l = 2 \) and (b) \( l = 20 \) generated via the VPP using block-(d). The first and third images of each row are the respective interference patterns for the pump light and the corresponding SHG light obtained directly from the block-(b) using CCD camera. The second and fourth images are the corresponding theoretical patterns.

To generate a light with high OAM, or a general superposition of OAM, we also perform the experiment using a spatial light modulator (SLM, PLUTO, which has an active area of 15.36 \( \times \) 8.64 mm\(^2\), a pixel pitch size of 8 \( \mu m \), and a total number pixels of 1920 \( \times \) 1080).). The setup for generating the pump state is shown in block-(e). By applying this configuration, we can generate a pump light in the state \( |l\rangle + e^{i\theta} | -l \rangle \) or \( |l\rangle + e^{i\theta} | -l \rangle \); the first state is the same as that prepared using VPPs, the second is an asymmetrical state. Using this configuration, the two counter-propagated beams have the same optical length with an intrinsically stable phase between them. The results are shown in Fig. 4. The first image in each row is the phase
Fig. 4. SHG results with pump states of the form $|l\rangle + e^{i\theta}|-l\rangle$ and $|l\rangle + e^{i\theta}|-l\rangle$ generated using block-(e) via an SLM. Rows (a)–(e) present images corresponding to states $|l\rangle + e^{i\theta}|-l\rangle$ with $l = 2, 3, 9, 20, 50$ for the pump beam; row (f) corresponds to state $|l\rangle + e^{i\theta}|-l\rangle$ with $l_1 = 7, l_2 = 8$. The first image in each row is the phase diagram of SLM for generating a specific OAM-carrying light. The second and fourth images are the respective interference patterns for the pump light, projected onto the diagonal polarization direction, and the SHG light, directly observed after block-(b) using CCD camera. The third and fifth images are the corresponding theoretical patterns.
light in the central region (rows (e), (f)) arising from limitations in creating the mode at the SLM (which are arising from high-order LG mode with the same OAM and unmodulated light, respectively). There would be no such artefact if high-quality VPPs were used (see row (b) in Fig. 3 for comparison). In row (e), the OAM of UV is 100, corresponding to 200 maxima in its intensity profile. We cannot increase the OAM further because the SLM cannot operate at high powers; also, our CCD camera has a limited resolution. For the asymmetrical state $|7\rangle + e^{i\theta}|-8\rangle$, as imaged in row (f), the generated modes in the far-field are $LG_{0}^{14}$, $LG_{-10}^{0}$, and $LG_{-1}^{1}$; the interference pattern of the pump has 15 maxima, whereas the SHG light has 30 maxima. The pattern is not sufficiently clear as the LG modes with different absolute values of $l$ have different diffraction properties. Hence the two modes do not completely overlap in the far-field. The central structures in rows (c) and (f) can’t be distinguished because of the image position of the CCD and the very weak intensity of the central structure compared with the outer structures. The central structures appear at sufficient propagation distances as showing in Ref. [23].

5. Conclusion

In summary, two experiments using a type-I QPM PPKTP crystal have been conducted to investigate OAM transformation and conservation in the SHG process. In the first of the two, we verified that OAM is conserved in the SHG by directing the pump and SHG OAM light into a specially designed balanced interferometer. The conservation law is confirmed by counting the maxima in the interference intensity profile. As the QPM crystal has a high-valued effective nonlinear coefficient and no walk-off effect, it provides a new method to generate OAM light by frequency conversion in QPM crystals. The image resolution depends on the wavelength of light used; shorter wavelengths yield better image resolutions. UV OAM light would be suitable for OAM light-based phase imaging. In the second of the experiments, we observed a very interesting interference phenomenon when pumping the PPKTP crystal with a superposition of two OAM states of opposite sign. The output SHG light intensity profile depended on the polarization of the pump light. A photonics gear-like structure is observed that can be rotated when the pump polarization is rotated. This effect can be used for remote sensing, OAM light-based ultra-sensitive angular measurements, and detection of spinning objects [32]. This interference effect can also be used for optical switching between different SHG patterns generated by controlling the polarization of the pump beam. We also gave analytical expressions for propagation of the SHG light for the tight focus approximation. All experimental phenomena can be well explained within the theory we have developed. For SFG and DFG conversions, the method is not limited to just the classical regime, and can be extended into the quantum regime for single photons.

Appendix A

This is the place to show the detail theoretical calculation of Eq. (3) in the main text. Laguerre-Gaussian (LG) modes are characterized by two indices, the azimuthal index $l$ and the radial index $p$, where $l$ is the number of $2\pi$ cycles in phase around the circumference and $p$ is related to the number of radial nodes [1]. The normalized wave function of an LG mode in cylindrical coordinates is given by

$$LG_{l}^{p}(r, \varphi, z) = \frac{2^{|l|}p!}{\pi^{1/4}(|l|+p)!} \frac{1}{w(x)^{1/2}} \frac{2^{2l}r^{l}}{w(x)^{l}|l|!} \exp \left( -\frac{r^2}{w(x)^2} \right) \exp \left( i k \frac{r^2}{2R(x)} \right) \left( \exp \left( -i \left[ 2|l|+1 \right] \sigma(x) \right) \exp \left( -il\alpha \right) \right)$$

(4)
where $w(x)$ is the beam radius at position $x$, which is the axial distance from the beam waist; $L^p_j$ is a generalized Laguerre polynomials; $k = 2\pi / \lambda$ the wave number; $R(x)$ the radius of curvature of the wavefronts for the beam; $\alpha$ denotes the azimuthal angle; and $\zeta(x)$ the Gouy phase, which adds an extra contribution to the phase.

To investigate the frequency conversion processes in QPM crystals, coupled wave functions are used to describe the interaction of these waves [28]

$$
\begin{align*}
\frac{dE_{1z}}{dx} &= iK_1 E^*_2 E_{3z} e^{-i\Delta k x} \\
\frac{dE_{2z}}{dx} &= iK_2 E^*_1 E_{3z} e^{-i\Delta k x} \\
\frac{dE_{3z}}{dx} &= i\gamma K_1 E^*_1 E_{2z} e^{i\Delta k x}
\end{align*}
$$

(5)

where $E_j = LG_j^2(r, \varphi, x) (j=1,2,3)$ represent the amplitudes of the three waves involved and $z$ denotes the polarization; the $K$-coefficients are $K_j = \frac{2\omega d_{33}}{\pi n c} (j=1,2,3)$ with $\omega$ the angular frequency, $n$ the refractive index, $c$ the speed of light, $d_{33}$ the nonlinear coefficient of KTP; the values of the degeneracy factor are $g = 0.5$ for SHG, as frequency is two-fold degenerate and $g = 1$ for SFG and DFG, and $\Delta k = k_{3z} - k_{1z} - k_{2z} - G_n$, where $k$ represents the wavenumber vector. For phase matching, $\Delta k = 0$, which means the momentum mismatch is fully compensated by the reciprocal vector $G_n$ of the QPM crystals.

We use the language of quantum mechanics to describe the transformation of light via blocks (c) or (f). We use a configuration similar to that described in Refs. [17, 30, 31]. We assume a Gaussian spatial mode polarized in the horizontal direction for the input beam of the interferometer. The input state can be expressed in the tensor product form

$$
|\Psi_{in}\rangle = |H\rangle \otimes |0\rangle,
$$

(6)

where $|H\rangle$ represents the polarization degrees of freedom and $|0\rangle$ denotes the OAM degrees of freedom. The function of the half- and quarter-wave plates is to apply a unitary rotation to the polarization degrees of freedom. We use the Jones calculus notation, with convention

$$
|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

(7)

The functions of the quarter- and half-wave plates, whose fast axes are at angles $\varphi$ and $\theta$ with respect to the vertical axis, are given by the respective $2 \times 2$ matrices

$$
\begin{align*}
\hat{U}_{QWP}(\varphi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - \cos(2\varphi) & \sin(2\varphi) \\ \sin(2\varphi) & 1 + \cos(2\varphi) \end{pmatrix}, \\
\hat{U}_{HWP}(\theta) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & -\cos(2\theta) \end{pmatrix},
\end{align*}
$$

(8)

After passing through the plates, the polarization of the beam becomes
\[
\hat{U}_{\text{QPM}}(\varphi) \bullet \hat{U}_{\text{QPM}}(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a(\theta, \varphi) |H\rangle + b(\theta, \varphi) |V\rangle,
\] (9)

where

\[
a(\theta, \varphi) = \frac{1}{\sqrt{2}} [i \cos(2\theta) - \cos(2\varphi - 2\theta)],
\] (10)

\[
b(\theta, \varphi) = \frac{1}{\sqrt{2}} [-i \sin(2\theta) + \sin(2\varphi - 2\theta)]
\]

After propagating through the interferometer in the opposite direction, the output state of the light will be

\[
|\Psi_{\text{out}}(\theta, \varphi)\rangle = a(\theta, \varphi) |H\rangle \otimes |l\rangle + b(\theta, \varphi) |V\rangle \otimes |l\rangle.
\] (11)

This is a hyper-superposition of the polarization and the OAM degrees of freedom, where \(l\) is the quantum number for OAM imprinted on the beam in the different propagation directions. We fix \(\varphi = \pi/4\) to distribute equally the intensity of the input beam. If we neglect the common phase \(e^{i(2\theta + \pi/2)}\), Eq. (11) becomes

\[
|\Psi_{\text{out}}(\theta)\rangle = \frac{1}{\sqrt{2}} \left( |H\rangle \otimes |l\rangle + e^{-i(4\theta + \pi/2)} |V\rangle \otimes |l\rangle \right).
\] (12)

For light passing through a half-wave plate whose fast axis is at an angle \(\delta\) with respect to the vertical axis before pumping the QPM crystal, the states of (12) are transformed to

\[
|\Psi_{\text{out}}(\theta, \delta)\rangle = \frac{1}{\sqrt{2}} \left( (\cos(2\delta) |H\rangle - \sin(2\delta) |V\rangle) \otimes |l\rangle - e^{-i(4\theta + \pi/2)} (\sin(2\delta) |H\rangle + \cos(2\delta) |V\rangle) \otimes |l\rangle \right).
\] (13)

In type-I QPM crystals, only the light polarized in the vertical direction is involved in the SHG process, so the projection of state (13) in the vertical direction is

\[
|\Psi_{\text{out}}(\theta, \delta)\rangle_v = -\frac{1}{\sqrt{2}} \left( (\sin(2\delta) |l\rangle + e^{-i(4\theta + \pi/2)} \cos(2\delta) |l\rangle) \right).
\] (14)

Considering the coupled wave functions in Eq. (5), there will be three pumping conditions arising from the form of Eq. (14), namely the interactions: (a) \(|l\rangle\) with \(|l\rangle\), (b) \(|l\rangle\) with \(|-l\rangle\), and (c) \(|l\rangle\) with \(|-l\rangle\). Therefore, the generated SHG light will be in the form

\[
E_{\text{SHG}} \propto \sin^2(2\delta) LG_0^{2l} - e^{-i4\theta} \cos^2(2\delta) LG_0^{2l} + \Gamma \sin(4\delta) e^{i4\theta} LG_0^{l}.
\] (15)

where \(\Gamma\) is a normalization constant.

For type-II QPM crystals, the generated SHG light has the form

\[
E_{\text{SHG}} \propto \sin(4\delta) LG_0^{2l} - e^{-i4\theta} \sin(4\delta) LG_0^{2l} + \Gamma \cos(4\delta) e^{i4\theta} LG_0^{l}.
\] (16)

Equations (15) and (16) are phenomenological results valid in the far-field approximation. To be more precise, we derive an analytical expression in an approximation valid for tight focusing or short crystal lengths. This means that the crystal length is much smaller than the Rayleigh range of the pump beam or the interaction length is very small because of the tight focusing. At these approximations, the SHG light generated for the three cases for the given spot size can be expressed as [23]
$$E_{\text{SHG}}(r, \alpha, 0) \propto \begin{cases} 
abla G_0(r, \alpha_0, 0) \ast \nabla G_0(r, \alpha_0, 0) & \text{case a} \\ \nabla G_0^{-1}(r, \alpha_0, 0) \ast \nabla G_0^{-1}(r, \alpha_0, 0) & \text{case b} \\ \nabla G_0^{-1}(r, \alpha_0, 0) \ast \nabla G_0^{-1}(r, \alpha_0, 0) & \text{case c} \end{cases}$$ (17)

Cases a and b are the same, including the sign of \( l \). Within the paraxial approximation, the propagation of the generated SHG light through a stigmatic ABCD optical system can be analysed aided by the Collins integral [33]

$$E(r, \alpha, x) = \frac{i}{\lambda B} \exp(-ikx) \int_0^2 \int_0 \left| E_{\text{SHG}}(r, \alpha, 0) \exp \left\{ -\frac{ik}{2B} \left[ A r_0^2 - 2r_0 r \cos(\alpha - \alpha_0) + D r^2 \right] \right\} r_0 dr_0 d\alpha_0 ight]$$ (18)

Combined with Eq. (4) and substituting Eq. (17) into Eq. (18), the field at distant \( x \) from the source point is of the form

$$E_{\text{SHG}}(r, \alpha, x) = \frac{2}{w_0^2} \left( \frac{\sqrt{2}}{w_0} \right)^{2l} \frac{i}{\lambda B} \exp(-ikD r^2 - ikx) \exp(-\frac{k r^2}{4\xi B}) \left| l! \xi^{-l-1} L^0_l \left( \frac{k^2 r^2}{4\xi B} \right) \right|$$ (19)

where \( w_0 \) is the beam waist of the pump, \( k \) and \( \lambda \) are the respective wavenumber and wavelength of the SHG beam, \( L^0_l \) generalized Laguerre polynomials, and

$$\xi = \frac{2}{w_0^2} + \frac{ikA}{B}. \quad (20)$$

In the above derivations, we have used the following integral formulas

$$\int_0^{2\pi} \exp[-i n \theta + ikbr \cos(\theta_i - \theta)] d\theta_i = 2\pi \exp(in(\frac{\pi}{2} - \theta_i)) \cdot J_n(kbr)$$ (21)

$$\int_0^{\infty} \exp(-ax^2) J_n(2bx) x^{\alpha+n+1} dx = \frac{n!}{2} b^{\alpha+n+1} \exp(-\frac{b^2}{a}) L_n^0 \left( \frac{b^2}{a} \right) \quad (22)$$

The ABCD transfer matrix for free space over distance \( x \) reads as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \quad (23)$$

Inserting Eq. (23) into (19), and using the far-field approximation \( (x >> k w_0^2 / 4) \), Eq. (19) reduces to
\[ E_{\text{SHG}}(r, \alpha, z) = \frac{i}{4\lambda} \exp(-ikr) \exp\left(-\frac{r^2}{w^2}\right) \times \left\{ \frac{i^{2l}}{l!} \left(\frac{\sqrt{2}r}{w}\right)^{2l} \exp(-i2l\alpha) \right\} \]

\[
\begin{cases} 
LG_{2l}^0(r, \alpha, x) & \text{case a} \\
LG_{2l}^0(r, \alpha, x) & \text{case c} 
\end{cases} \quad (24)
\]

where \( w = \frac{xw_0}{\sqrt{2}x_s} \) and \( x_s = \frac{kww_0}{4} \) are the spot radius of the SHG beam at \( x \) and the Rayleigh range, respectively. The beam waist of the SHG beam is \( \frac{w_0}{\sqrt{2}} \). Equation (24) shows that the generated SHG beams will evolve into modes \( LG_{2l}^0 \) (case a), \( LG_{-2l}^0 \) (case b), and \( LG_{l}^0 \) (case c) in the far-field.

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