Enumeration of 2-uniform maps on the torus

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Abstract

A map is said to be 1-uniform if the automorphism group of the map acts transitively on the vertex set. A 2-uniform map is a map on a surface having 2 distinct transitivity classes of vertices under the action of the automorphism group. The classification of 1-uniform maps on the torus is known. In this article, we classify 2-uniform maps on the torus up to isomorphism.

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1 Introduction

A map is a connected 2-dimensional cell complex on a surface. Equivalently, it is a cellular embedding of a connected graph on a surface. For a map \( K \), let \( V(K) \) be the vertex set of \( K \) and \( u \in V(K) \). The faces containing \( u \) form a cycle (called the face-cycle at \( u \)) \( C_u \) in the dual graph of \( K \). That is, \( C_u \) is of the form \((F_{1,1},\ldots,F_{1,n_1})\cdots(F_{k,1},\ldots,F_{k,n_k})-F_{1,1}\), where \( F_{i,\ell} \) is a regular \( p_{i,\ell} \)-gon for \( 1 \leq \ell \leq n_i, 1 \leq i \leq k, p_r \neq p_{r+1} \) for \( 1 \leq r \leq k-1 \) and \( p_k \neq p_1 \). In this case, the vertex \( u \) is said to be of type \([p_1^{m_1},\ldots,p_k^{m_k}]\) (addition in the suffix is modulo \( k \)). A map \( K \) is said to be \( \ell \)-semiequivelar if \( V(K) = V_i \sqcup V_2 \sqcup \cdots \sqcup V_\ell \) such that the type of the vertices of \( V_i \) (\( 1 \leq i \leq \ell \)) is same and the types of the vertices in \( V_i \) and \( V_j \) (\( i \neq j, 1 \leq i, j \leq \ell \)) are different. A \( \ell \)-semiequivelar map is said to be a semiequivelar map if \( k = 1 \). Similarly, a \( \ell \)-semiequivelar map is said to be an equivelar map if it consists of single type of faces with all the vertices are of same type.

Two maps \( K_1 \) and \( K_2 \) are isomorphic if there exists a map \( f : K_1 \to K_2 \) such that \( f \mid_{V(K_1)} : V(K_1) \to V(K_2) \) is a bijection and \( f(\sigma) \) is a cell in \( K_2 \) if and only if \( \sigma \) is a cell in \( K_1 \). In particular, if \( K_1 = K_2 \), then \( f \) is called an automorphism. The automorphism group \( Aut(K) \) of \( K \) is the group consisting of automorphisms of \( K \). A vertex-transitive or 1-uniform map (or tiling) is a map (or tiling) on a closed surface (or on the plane) on which the automorphism group acts transitively on the set of vertices. The 2-uniform tilings (or map) are the generalization of vertex-transitive maps. A 2-uniform tiling (or map) is a tiling (or a map) of the surface having 2 distinct transitivity classes of vertices under the action of the automorphism group.

Throughout the last few decades there have been many results about maps and semi-equivelar maps that are highly symmetric. In particular, there has been recent interest in the study of discrete objects using combinatorial, geometric, and algebraic approaches,
with the topic of symmetries of maps receiving a lot of interest. There is a great history of work surrounding maps on the Euclidean plane $\mathbb{R}^2$ and the 2-dimensional torus. When working with discrete symmetric structures on a torus, many of the ideas follow the concepts introduced by Coxeter and Moser in [5]. A surjective mapping $\eta: X \rightarrow Y$ from a map $X$ to a map $Y$ is called a covering if it preserves adjacency and sends vertices, edges, faces of $X$ to vertices, edges, faces of $Y$ respectively. That is, let $G \leq \text{Aut}(X)$ be a discrete group acting on a map $X$ properly discontinuously ([15, Chapter 2]). This means that each element $g$ of $G$ is associated to an automorphism $h_g$ of $X$ onto itself, in such a way that $h_g h_h$ for any two elements $g$ and $h$ of $G$, and $G$-orbit of any vertex $u \in V(X)$ is locally finite. Then, there exists $\Gamma \leq \text{Aut}(X)$ such that $Y = X/\Gamma$. In such a case, $X$ is called a cover of $Y$. A map $X$ is called regular if the automorphism group of $X$ acts transitively on the set of flags of $X$. Clearly, if a semi-equivelar map is not equivelar then it cannot be regular.

A combinatorial surface is a kind of surface which is a discretization of a surface. It usually means a piecewise linear surface made by simplicial complexes or polygons. Piecewise linear structure sits between the category of smooth manifolds and the category of topological manifolds. The classification of these structures is a classical problem. Therefore, it is interesting to ask what would be the complete list of discrete structures (known as maps) on a surface. As of now, the only significant study on the classification of maps on the sphere, projective plane, torus, Klein bottle, non-orientable surfaces of genus 3 and orientable surfaces of genus 2, 3, 4 has been done by many researchers. More precisely, the well known eleven types of semiregular tiling of the plane are the examples of vertex transitive semiequivelar maps on Euclidean plane. All semi-equivelar and vertex-transitive maps on the 2-sphere are known. These are the boundaries of Platonic and Archimedean solids and two infinite families of types (namely, of types $[4^2, n^1]$ and $[3^3, m^1]$ for $4 \neq n \geq 3$, $m \geq 4$) [9, 11]. Similarly, there are infinitely many types of semi-equivelar and vertex-transitive on the real projective plane [2, 9]. Thus, there are infinitely many types of semi-equivelar and vertex-transitive maps on the 2-sphere and the real projective plane. Torus is a quotient space of Euclidean plane.

One can construct infinitely many semi-equivelar maps on this closed surface as a quotient of one of the eleven semiregular tilings by a group $\Gamma$ generated by two translations [26, 27]. Karabas and Nedela have obtained the census of Archimedean maps on orientable surface of Euler characteristic $-2$. In 2012, they have classified all the Archimedean maps on orientable surface of Euler characteristic $-2$, $-4$, $-6$ [19]. List of these maps can be found in Karabas’s web page [20]. Kurth, Negami, Brehm and Kühnel have studied all equivelar maps of type $(6^6)$, $(4^4)$, $(6^3)$ on the torus and given complete classification independently [1, 3, 23, 24]. In 2015 - 2019, we have worked on the characterizations of semi-equivelar maps along with the groups of symmetries on the torus and the Klein bottle [7, 8, 16, 18]. We know that if the Euler characteristic $\chi(M)$ of a surface $M$ is negative then the number of semi-equivelar maps on $M$ is finite [2, 5]. We know from [11, 12, 21] that there are twenty 2-uniform tiling of types

\[ [3^6, 3^3, 4^2], [3^6, 3^2, 4^1, 3^1, 4^1], [3^4, 6^1, 3^2, 6^2], [3^2, 4^2, 3^1, 4^1, 6^1, 4^1], [3^3, 4^2, 3^2, 4^1, 3^1, 4^1], \]
\[ [3^6, 3^2, 4^1, 12^1], [3^1, 4^1, 6^1, 4^1, 6^1, 12^1], [3^2, 4^1, 3^1, 4^1, 3^1, 6^1, 4^1], [3^2, 6^2, 3^1, 6^1, 3^1, 6^1], \]
\[ [3^1, 4^1, 3^1, 12^1, 3^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 3^1, 6^1], [3^3, 4^2, 4^1], [3^6, 3^4, 6^1] \]

on the plane (see in Section 2). Thus, we have the following question:

**Question 1.1.** Does there exist any map 2-uniform map on the torus? If so, how many?

In this context, we know from [1, 7, 10, 18, 25, 26] the following results.
Proposition 1.2. ([7, 10]) Let $Y$ be a semiequivelar map on the torus. If the type of $Y$ is $[3^6], [6^3], [4^4]$ or $[3^2, 4^2]$ then $Y$ is vertex-transitive.

Proposition 1.3. ([1, 18]) Let $X = [3^6], [4^4], [6^3], [3^3, 4^2], [3^2, 4^1, 3^1, 4^1], [3^4, 6^1], [4^1, 4^1, 12^1], [3^1, 6^1, 3^1, 6^1], [3^1, 4^1, 6^1, 4^1], [3^1, 12^1]$ or $[4^1, 8^2]$. Then we have classification of maps of type $X$ on a given number of vertices up to isomorphism.

Proposition 1.4. ([17]) Let $X = [3^6], [4^4], [6^3], [3^3, 4^2], [3^2, 4^1, 3^1, 4^1], [3^4, 6^1], [4^1, 4^1, 12^1], [3^1, 6^1, 3^1, 6^1], [3^1, 4^1, 6^1, 4^1], [3^1, 12^1]$ or $[4^1, 8^2]$. Then we have classification of vertex-transitive maps (1-uniform maps) of type $X$ on a given number of vertices up to isomorphism.

Thus, we know the classification of 1-uniform maps on the torus. In this article, we classify 2-uniform toroidal maps. Suppose $E$ is a tiling on $\mathbb{R}^2$ and $X$ is a toroidal obtained as an orbit space of $E$ by the action of some discrete fixed point free group of rank 2. Then, the number of vertex orbits of $X$ under the action of automorphism group of $X$ will be at least the number of vertex orbits of $E$ under the automorphism group of $E$ since any automorphism of $X$ can be lifted to an automorphism of $E$. Hence, if $X$ has two vertex orbits then number of vertex orbits of $E$ will be at most two. Thus, $E$ must belong to either 2-uniform tilings of the plane or the 1-uniform tilings of the plane.

Let $X$ be a 2-uniform toroidal map where $X$ is quotient of either some 1-uniform tiling of $\mathbb{R}^2$ or some 2-uniform tiling of $\mathbb{R}^2$. We enumerate the number of 2-uniform maps on a torus for a fixed vertex number for each of these tiling. In particular, we prove the following:

Theorem 1.5. Let $\Phi_{\ell} (v)$ denote the number of 2-uniform toroidal maps of type same as type of $E_{\ell}$ (see in Section 2) with number of vertices $v$. Then,

(i) $\Phi_1 (v) = \begin{cases} 0, & \text{if } 12 \nmid v \\ f_1 (\frac{v}{12}), & \text{otherwise.} \end{cases}$

(ii) $\Phi_2 (v) = \begin{cases} 0, & \text{if } v \nmid v_0 \\ \frac{1}{2} \left[f_1 (\frac{v}{v_0}) + f_5 (\frac{v}{v_0})\right], & \text{otherwise,} \end{cases}$

for $(\ell, v_0) = (2, 8), (5, 7), (7, 7), (11, 12), (14, 12)$.

(iii) $\Phi_3 (v) = \begin{cases} 0, & \text{if } v_0 \nmid v \\ \frac{1}{2} \left[f_1 (\frac{v}{v_0}) + f_3 (\frac{v}{v_0})\right], & \text{otherwise,} \end{cases}$

for $(\ell, v_0) = (3, 4), (4, 3), (8, 4), (12, 3), (13, 4), (15, 3)$.

(iv) $\Phi_4 (v) = \begin{cases} 0, & \text{if } 8 \nmid v \\ \frac{1}{2} \left[f_2 (\frac{v}{8}) + f_6 (\frac{v}{8})\right], & \text{otherwise.} \end{cases}$

(v) $\Phi_5 (v) = \begin{cases} 0, & \text{if } v_0 \nmid v \\ f_5 (\frac{v}{v_0}), & \text{otherwise,} \end{cases}$

for $(\ell, v_0) = (6, 14), (17, 18), (20, 18)$.

(vi) $\Phi_6 (v) = \begin{cases} 0, & \text{if } v_0 \nmid v \\ f_3 (\frac{v}{v_0}), & \text{otherwise,} \end{cases}$

for $(\ell, v_0) = (18, 5), (19, 5)$.

(vii) $\Phi_7 (v) = \begin{cases} 0, & \text{if } v_0 \nmid v \\ g (\frac{v}{v_0}), & \text{otherwise,} \end{cases}$

for $(\ell, v_0) = (9, 12), (10, 8)$. 
(vii) \( \Phi_{25}(v) = 0 \) for all \( v \in \mathbb{N} \).

(ix) \( \Phi_{\ell}(v) = \begin{cases} 
0 & \text{if } v_0 \nmid v \\
(f_3(\frac{v}{v_0}) - f_5(\frac{v}{v_0})) & \text{otherwise,}
\end{cases} \)

for \((\ell, v_0) = (21, 6), (22, 6), (24, 3)\).

(x) \( \Phi_{23}(v) = \begin{cases} 
0 & \text{if } 4 \nmid v \\
\frac{1}{4}[\vartheta(\frac{v}{4}) - g(\frac{v}{4}) - f_2(\frac{v}{4}) + f_3(\frac{v}{4})] & \text{otherwise,}
\end{cases} \)

(xi) \( \Phi_{26}(v) = \begin{cases} 
0 & \text{if } 12 \nmid v \\
\frac{1}{2}[f_1(\frac{v}{12}) - f_5(\frac{v}{12})] & \text{otherwise,}
\end{cases} \)

(xii) \( \Phi_{27}(v) = \begin{cases} 
0 & \text{if } 4 \nmid v \\
\frac{1}{2}[\vartheta(\frac{v}{4}) - f_2(\frac{v}{4})] & \text{otherwise,}
\end{cases} \)

where for \( n \in \mathbb{N} \),

1. \( \vartheta(n) := \sum_{d \mid n} d \),

2. \( f_1(n) := \begin{cases} 
0 & \text{if } m_j \equiv 1 \pmod{2} \text{ for some } j \in \{0, 1, 2, \ldots, n_2\} \\
\prod_{i=1}^{n_1}(k_i + 1) & \text{otherwise,}
\end{cases} \)

where \( n = 2^{m_0} \cdot 3^{k_0} \cdot \prod_{i=1}^{n_1} p_i^{k_i} \cdot \prod_{j=1}^{n_2} q_j^{m_j} \) with \( p_i \) and \( q_j \) are primes such that \( p_i \equiv 1 \pmod{3} \) for \( i \in \{0, 1, \ldots, n_1\} \) and \( q_j \equiv 2 \pmod{3} \) for \( j \in \{0, 1, \ldots, n_2\} \),

3. \( f_2(n) := \begin{cases} 
0 & \text{if } m_j \equiv 1 \pmod{2} \text{ for some } j \in \{0, 1, 2, \ldots, n_2\} \\
\prod_{i=1}^{n_1}(k_i + 1) & \text{otherwise,}
\end{cases} \)

where \( n = 2^{m_0} \cdot 3^{k_0} \cdot \prod_{i=1}^{n_1} p_i^{k_i} \cdot \prod_{j=1}^{n_2} q_j^{m_j} \) such that \( p_i \) and \( q_j \) are primes with \( p_i \equiv 1 \pmod{4} \) for \( i \in \{0, 1, \ldots, n_1\} \) and \( q_j \equiv 3 \pmod{4} \) for \( j \in \{0, 1, \ldots, n_2\} \),

4. \( f_3(n) := \begin{cases} 
\prod_{i=1}^{n_1}(k_i + 1) & \text{if } k_0 = 0 \\
(2k_0 - 1) \prod_{i=1}^{n_1}(k_i + 1) & \text{otherwise.}
\end{cases} \)

where \( n = 2^{k_0} \cdot \prod_{i=1}^{n_1} p_i^{k_i} \) such that \( p_i \) is any odd prime for \( i \in \{0, 1, \ldots, n_1\} \),

5. \( f_5(n) := \begin{cases} 
0 & \text{if } k_i \equiv 1 \pmod{2} \text{ for some } i \in \{0, 1, 2, \ldots, n_1\} \\
1 & \text{otherwise.}
\end{cases} \)

where \( n = 2^{k_0} \cdot 3^m \cdot \prod_{i=1}^{n_1} p_i^{k_i} \) such that \( p_i \) is any prime other than 3 for \( i \in \{0, 1, \ldots, n_1\} \),

6. \( f_6(n) := \begin{cases} 
0 & \text{if } k_i \equiv 1 \pmod{2} \text{ for some } i \in \{0, 1, 2, \ldots, n_1\} \\
1 & \text{otherwise.}
\end{cases} \)

where \( n = 2^{m_0} \cdot 3^{k_0} \cdot \prod_{i=1}^{n_1} p_i^{k_i} \) such that \( p_i \) is any prime other than 2 for \( i \in \{0, 1, \ldots, n_1\} \),

7. \( g(n) := \sum_{d \mid n} 2 + \sum_{2 \nmid d} 1 \).

In Tables 1, 2, 3, we give explicit number of 2-uniform maps for given number of vertex. These values are calculated using SageMath.
2 Examples

We present twenty 2-uniform and seven Archimidean tilings of the plane. We use these in our proof of Theorem 1.5.
Figure 1: 2-uniform tilings of the plane.
3 2-uniform toroidal maps

Let $X_{i0}$ be a toroidal map with minimal number of vertices. Then, $X_{i0} = \overline{E_i}$, where $H_i$ is the group of translations of $E_i$. Now, $\text{Aut}(X_0) = \text{Nor}(H_i)/H_i$. Hence, the number of vertex orbits of $X_0$ under the action of $\text{Aut}(X_{i0})$ is same as the number of vertex orbits of $E_i$ under the action of $\text{Nor}(H_i)$. Clearly, $\text{Nor}(H_i) = \text{Aut}(E_i)$. Since, $V(E_i)$ has 2 $\text{Aut}(E_i)$-orbits for $i = 1, 2, 3, \ldots, 20$, $X_{i0}$ is a 2-uniform for $i = 1, 2, 3, \ldots, 20$.

Let $X$ be an $n$ sheeted cover of $X_0$, where $n \in \mathbb{N}$ obtained as an orbit space of some tiling $E$ on $\mathbb{R}^2$. Then, we can write $X = \overline{E_K}$ where $K \leq \text{Aut}(E)$ which has no fixed points. So, $K$ contains only translations and glide reflections. Since, $X$ is orientable, $K \leq H$. Choose some origin $O$ in $E$. Let $H = \langle \gamma, \delta \rangle$, where $\gamma : z \mapsto z + A$ and $\delta : z \mapsto z + B$ where $A$ and $B$ are two linearly independent translation vectors of $E$ originating from $O$. Then, $K = \langle w_1, w_2 \rangle$ where $w_1 = \gamma^a \circ \delta^b$ and $w_2 = \gamma^c \circ \delta^d$ for some $a, b, c, d \in \mathbb{Z}$. Clearly, $w_1 : z \mapsto z + (aA + bB)$ and $w_2 : z \mapsto z + (cA + dB)$. Let $\tau, \sigma, \psi$ denote the $180^\circ$, $60^\circ$ and $90^\circ$ rotation in $E$ with respect to $O$ respectively. For Figures 1a to 2g except $E_{16}$ and $E_{23}$, let $R_1$, $R_2$, $R_3$ denote...
| \((v, \Phi_{\ell}(v))\) | \((v, \Phi_{\ell}(v))\) | \((v, \Phi_{\ell}(v))\) | \((v, \Phi_{\ell}(v))\) | \((v, \Phi_{\ell}(v))\) |
|----------------|----------------|----------------|----------------|----------------|
| \(\ell = 1\) | \(\ell = 11, 14\) | \(\ell = 2\) | \(\ell = 5, 7\) | \(\ell = 4, 12, 15\) |
| \((v, 0)\) s.t. 12 \(\nmid v\) | \((v, 0)\) s.t. 12 \(\nmid v\) | \((v, 0)\) s.t. 8 \(\nmid v\) | \((v, 0)\) s.t. 7 \(\nmid v\) | \((v, 0)\) s.t. 3 \(\nmid v\) |
| (12, 1) | (24, 0) | (8, 1) | (14, 0) | (6, 2) |
| (36, 1) | (36, 1) | (24, 1) | (21, 1) | (9, 3) |
| (48, 1) | (48, 1) | (32, 1) | (28, 1) | (12, 5) |
| (60, 0) | (60, 0) | (40, 0) | (35, 0) | (15, 4) |
| (72, 0) | (72, 0) | (48, 0) | (42, 0) | (18, 7) |
| (84, 2) | (84, 1) | (50, 1) | (49, 1) | (21, 5) |
| (96, 0) | (96, 0) | (64, 0) | (56, 0) | (24, 10) |
| (108, 1) | (108, 1) | (72, 1) | (63, 1) | (27, 8) |
| (120, 0) | (120, 0) | (80, 0) | (70, 0) | (30, 10) |
| (132, 0) | (132, 0) | (88, 0) | (77, 0) | (33, 7) |
| (144, 1) | (144, 1) | (96, 1) | (84, 1) | (36, 17) |
| (156, 2) | (156, 1) | (104, 1) | (91, 1) | (39, 8) |
| (168, 0) | (168, 0) | (112, 0) | (98, 0) | (42, 13) |
| (180, 0) | (180, 0) | (120, 0) | (105, 0) | (45, 14) |
| (192, 1) | (192, 1) | (128, 1) | (112, 1) | (48, 19) |
| (204, 0) | (204, 0) | (136, 0) | (119, 0) | (51, 10) |
| (216, 0) | (216, 0) | (144, 0) | (126, 0) | (54, 21) |
| (228, 2) | (228, 1) | (152, 1) | (133, 1) | (57, 11) |
| (240, 0) | (240, 0) | (160, 0) | (140, 0) | (60, 24) |
| (252, 2) | (252, 1) | (168, 1) | (147, 1) | (63, 18) |
| (264, 0) | (264, 0) | (176, 0) | (154, 0) | (66, 19) |
| (276, 0) | (276, 0) | (184, 0) | (161, 0) | (69, 13) |
| (288, 0) | (288, 0) | (192, 0) | (168, 0) | (72, 35) |
| (300, 1) | (300, 1) | (200, 1) | (175, 1) | (75, 17) |
| (312, 0) | (312, 0) | (208, 0) | (182, 0) | (78, 22) |
| (324, 1) | (324, 1) | (216, 1) | (189, 1) | (81, 22) |
| (336, 2) | (336, 1) | (224, 1) | (196, 1) | (84, 31) |
| (348, 0) | (348, 0) | (232, 0) | (203, 0) | (87, 16) |
| (360, 0) | (360, 0) | (240, 0) | (210, 0) | (90, 38) |
| (372, 2) | (372, 1) | (248, 1) | (217, 1) | (93, 17) |
| (384, 0) | (384, 0) | (256, 0) | (224, 0) | (96, 36) |
| (396, 0) | (396, 0) | (264, 0) | (231, 0) | (99, 26) |
| (408, 0) | (408, 0) | (272, 0) | (238, 0) | (102, 28) |
| (420, 0) | (420, 0) | (280, 0) | (245, 0) | (105, 26) |
| (432, 1) | (432, 1) | (288, 1) | (252, 1) | (108, 50) |
| (444, 2) | (444, 1) | (296, 1) | (259, 1) | (111, 20) |
| (456, 0) | (456, 0) | (304, 0) | (266, 0) | (114, 31) |
| (468, 2) | (468, 1) | (312, 1) | (273, 1) | (117, 30) |
| (480, 0) | (480, 0) | (320, 0) | (280, 0) | (120, 50) |
| (492, 0) | (492, 0) | (328, 0) | (287, 0) | (123, 22) |
| (504, 0) | (504, 0) | (336, 0) | (294, 0) | (126, 50) |
| (516, 2) | (516, 1) | (344, 1) | (301, 1) | (129, 23) |
| (528, 0) | (528, 0) | (352, 0) | (308, 0) | (132, 45) |
| (540, 0) | (540, 0) | (360, 0) | (315, 0) | (135, 42) |
| (552, 0) | (552, 0) | (368, 0) | (322, 0) | (138, 37) |
| (564, 0) | (564, 0) | (376, 0) | (329, 0) | (141, 25) |
| (576, 1) | (576, 1) | (384, 1) | (336, 1) | (144, 69) |
| (588, 3) | (588, 2) | (392, 2) | (343, 2) | (147, 30) |
| (600, 0) | (600, 0) | (400, 0) | (350, 0) | (150, 48) |

Table 1: Enumeration of 2-uniform toroidal maps
\begin{array}{cccccccc}
\ell = 16 & \ell = 9 & \ell = 10 & \ell = 6 & \ell = 17, 20 & \ell = 18, 19 \\
\hline
(v, 0) \text{ s.t.
. \; \hline
(8, 1) & (12, 1) & (8, 1) & (14, 1) & (18, 1) & (5, 1) \\
(16, 1) & (24, 3) & (16, 3) & (28, 0) & (36, 0) & (10, 1) \\
(24, 0) & (36, 2) & (24, 2) & (42, 1) & (54, 1) & (15, 2) \\
(32, 1) & (48, 5) & (32, 5) & (56, 1) & (72, 1) & (20, 3) \\
(40, 1) & (60, 2) & (40, 2) & (70, 0) & (90, 0) & (25, 2) \\
(48, 0) & (72, 6) & (48, 6) & (84, 0) & (108, 0) & (30, 2) \\
(56, 0) & (84, 2) & (56, 2) & (98, 0) & (126, 0) & (35, 2) \\
(64, 1) & (96, 7) & (64, 7) & (112, 0) & (144, 0) & (40, 5) \\
(72, 1) & (108, 3) & (72, 3) & (126, 1) & (162, 1) & (45, 3) \\
(80, 1) & (120, 6) & (80, 6) & (140, 0) & (180, 0) & (50, 2) \\
(88, 0) & (132, 2) & (88, 2) & (154, 0) & (198, 0) & (55, 2) \\
(96, 0) & (144, 10) & (96, 10) & (168, 1) & (216, 1) & (60, 6) \\
(104, 1) & (156, 2) & (104, 2) & (182, 0) & (234, 0) & (65, 2) \\
(112, 0) & (168, 6) & (112, 6) & (196, 0) & (252, 0) & (70, 2) \\
(120, 0) & (180, 4) & (120, 4) & (210, 0) & (270, 0) & (75, 4) \\
(128, 1) & (192, 9) & (128, 9) & (224, 1) & (288, 1) & (80, 7) \\
(136, 1) & (204, 2) & (136, 2) & (238, 0) & (306, 0) & (85, 2) \\
(144, 1) & (216, 9) & (144, 9) & (252, 0) & (324, 0) & (90, 3) \\
(152, 0) & (228, 2) & (152, 2) & (266, 0) & (342, 0) & (95, 2) \\
(160, 1) & (240, 10) & (160, 10) & (280, 0) & (360, 0) & (100, 6) \\
(168, 0) & (252, 4) & (168, 4) & (294, 0) & (378, 0) & (105, 4) \\
(176, 0) & (264, 6) & (176, 6) & (308, 0) & (396, 0) & (110, 2) \\
(184, 0) & (276, 2) & (184, 2) & (322, 0) & (414, 0) & (115, 2) \\
(192, 0) & (288, 14) & (192, 14) & (336, 0) & (432, 0) & (120, 10) \\
(200, 2) & (300, 3) & (200, 3) & (350, 1) & (450, 1) & (125, 3) \\
(208, 1) & (312, 6) & (208, 6) & (364, 0) & (468, 0) & (130, 2) \\
(216, 0) & (324, 4) & (216, 4) & (378, 1) & (486, 1) & (135, 4) \\
(224, 0) & (336, 10) & (224, 10) & (392, 0) & (504, 0) & (140, 6) \\
(232, 1) & (348, 2) & (232, 2) & (406, 0) & (522, 0) & (145, 2) \\
(240, 0) & (360, 12) & (240, 12) & (420, 0) & (540, 0) & (150, 4) \\
(248, 0) & (372, 2) & (248, 2) & (434, 0) & (558, 0) & (155, 2) \\
(256, 1) & (384, 11) & (256, 11) & (448, 0) & (576, 0) & (160, 9) \\
(264, 0) & (396, 4) & (264, 4) & (462, 0) & (594, 0) & (165, 4) \\
(272, 1) & (408, 6) & (272, 6) & (476, 0) & (612, 0) & (170, 2) \\
(280, 0) & (420, 4) & (280, 4) & (490, 0) & (630, 0) & (175, 4) \\
(288, 1) & (432, 15) & (288, 15) & (504, 1) & (648, 1) & (180, 9) \\
(296, 1) & (444, 2) & (296, 2) & (518, 0) & (666, 0) & (185, 2) \\
(304, 0) & (456, 6) & (304, 6) & (532, 0) & (684, 0) & (190, 2) \\
(312, 0) & (468, 4) & (312, 4) & (546, 0) & (702, 0) & (195, 4) \\
(320, 1) & (480, 14) & (320, 14) & (560, 0) & (720, 0) & (200, 10) \\
(328, 1) & (492, 2) & (328, 2) & (574, 0) & (738, 0) & (205, 2) \\
(336, 0) & (504, 12) & (336, 12) & (588, 0) & (756, 0) & (210, 4) \\
(344, 0) & (516, 2) & (344, 2) & (602, 0) & (774, 0) & (215, 2) \\
(352, 0) & (528, 10) & (352, 10) & (616, 0) & (792, 0) & (220, 6) \\
(360, 1) & (540, 6) & (360, 6) & (630, 0) & (810, 0) & (225, 6) \\
(368, 0) & (552, 6) & (368, 6) & (644, 0) & (828, 0) & (230, 2) \\
(376, 0) & (564, 2) & (376, 2) & (658, 0) & (846, 0) & (235, 2) \\
(384, 0) & (576, 18) & (384, 18) & (672, 1) & (864, 1) & (240, 14) \\
(392, 1) & (588, 3) & (392, 3) & (686, 1) & (882, 1) & (245, 3) \\
(400, 2) & (600, 9) & (400, 9) & (700, 0) & (900, 0) & (250, 3) \\
\end{array}

Table 2: Enumeration of 2-uniform toroidal maps
| $(v, \Phi_\ell(v))$ | $(v, \Phi_\ell(v))$ | $(v, \Phi_\ell(v))$ | $(v, \Phi_\ell(v))$ | $(v, \Phi_\ell(v))$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\ell = 21, 22$ | $\ell = 23$     | $\ell = 24$     | $\ell = 26$     | $\ell = 27$     |
| $(v, 0)$ s.t. $6 \mid v$ | $(v, 0)$ s.t. $4 \mid v$ | $(v, 0)$ s.t. $3 \mid v$ | $(v, 0)$ s.t. $12 \mid v$ | $(v, 0)$ s.t. $4 \mid v$ |
| $(6, 0)$ | $(4, 0)$ | $(3, 0)$ | $(12, 0)$ | $(4, 0)$ |
| $(12, 1)$ | $(8, 0)$ | $(6, 1)$ | $(24, 0)$ | $(8, 1)$ |
| $(18, 1)$ | $(12, 1)$ | $(9, 1)$ | $(36, 0)$ | $(12, 2)$ |
| $(24, 2)$ | $(16, 1)$ | $(12, 2)$ | $(48, 0)$ | $(16, 3)$ |
| $(30, 2)$ | $(20, 1)$ | $(15, 2)$ | $(60, 0)$ | $(20, 2)$ |
| $(36, 2)$ | $(24, 2)$ | $(18, 2)$ | $(72, 0)$ | $(24, 6)$ |
| $(42, 2)$ | $(28, 2)$ | $(21, 2)$ | $(84, 1)$ | $(28, 4)$ |
| $(48, 5)$ | $(32, 3)$ | $(24, 5)$ | $(96, 0)$ | $(32, 7)$ |
| $(54, 2)$ | $(36, 3)$ | $(27, 2)$ | $(108, 0)$ | $(36, 6)$ |
| $(60, 2)$ | $(40, 3)$ | $(30, 2)$ | $(120, 0)$ | $(40, 8)$ |
| $(66, 2)$ | $(44, 3)$ | $(33, 2)$ | $(132, 0)$ | $(44, 6)$ |
| $(72, 5)$ | $(48, 6)$ | $(36, 5)$ | $(144, 0)$ | $(48, 14)$ |
| $(78, 2)$ | $(52, 3)$ | $(39, 2)$ | $(156, 1)$ | $(52, 6)$ |
| $(84, 2)$ | $(56, 5)$ | $(42, 2)$ | $(168, 0)$ | $(56, 12)$ |
| $(90, 4)$ | $(60, 6)$ | $(45, 4)$ | $(180, 0)$ | $(60, 12)$ |
| $(96, 6)$ | $(64, 7)$ | $(48, 6)$ | $(192, 0)$ | $(64, 15)$ |
| $(102, 2)$ | $(68, 4)$ | $(51, 2)$ | $(204, 0)$ | $(68, 8)$ |
| $(108, 3)$ | $(72, 8)$ | $(54, 3)$ | $(216, 0)$ | $(72, 19)$ |
| $(114, 2)$ | $(76, 5)$ | $(57, 2)$ | $(228, 1)$ | $(76, 10)$ |
| $(120, 6)$ | $(80, 9)$ | $(60, 6)$ | $(240, 0)$ | $(80, 20)$ |
| $(126, 4)$ | $(84, 8)$ | $(63, 4)$ | $(252, 1)$ | $(84, 16)$ |
| $(132, 2)$ | $(88, 8)$ | $(66, 2)$ | $(264, 0)$ | $(88, 18)$ |
| $(138, 2)$ | $(92, 6)$ | $(69, 2)$ | $(276, 0)$ | $(92, 12)$ |
| $(144, 10)$ | $(96, 14)$ | $(72, 10)$ | $(288, 0)$ | $(96, 30)$ |
| $(150, 2)$ | $(100, 7)$ | $(75, 2)$ | $(300, 0)$ | $(100, 14)$ |
| $(156, 2)$ | $(104, 9)$ | $(78, 2)$ | $(312, 0)$ | $(104, 20)$ |
| $(162, 3)$ | $(108, 10)$ | $(81, 3)$ | $(324, 0)$ | $(108, 20)$ |
| $(168, 6)$ | $(112, 13)$ | $(84, 6)$ | $(336, 1)$ | $(112, 28)$ |
| $(174, 2)$ | $(116, 7)$ | $(87, 2)$ | $(348, 0)$ | $(116, 14)$ |
| $(180, 4)$ | $(120, 16)$ | $(90, 4)$ | $(360, 0)$ | $(120, 36)$ |
| $(186, 2)$ | $(124, 8)$ | $(93, 2)$ | $(372, 1)$ | $(124, 16)$ |
| $(192, 9)$ | $(128, 15)$ | $(96, 9)$ | $(384, 0)$ | $(128, 31)$ |
| $(198, 4)$ | $(132, 12)$ | $(99, 4)$ | $(396, 0)$ | $(132, 24)$ |
| $(204, 2)$ | $(136, 12)$ | $(102, 2)$ | $(408, 0)$ | $(136, 26)$ |
| $(210, 4)$ | $(140, 12)$ | $(105, 4)$ | $(420, 0)$ | $(140, 24)$ |
| $(216, 8)$ | $(144, 21)$ | $(108, 8)$ | $(432, 0)$ | $(144, 45)$ |
| $(222, 2)$ | $(148, 9)$ | $(111, 2)$ | $(444, 1)$ | $(148, 18)$ |
| $(228, 2)$ | $(152, 14)$ | $(114, 2)$ | $(456, 0)$ | $(152, 30)$ |
| $(234, 4)$ | $(156, 14)$ | $(117, 4)$ | $(468, 1)$ | $(156, 28)$ |
| $(240, 10)$ | $(160, 21)$ | $(120, 10)$ | $(480, 0)$ | $(160, 44)$ |
| $(246, 2)$ | $(164, 10)$ | $(123, 2)$ | $(492, 0)$ | $(164, 20)$ |
| $(252, 4)$ | $(168, 22)$ | $(126, 4)$ | $(504, 0)$ | $(168, 48)$ |
| $(258, 2)$ | $(172, 11)$ | $(129, 2)$ | $(516, 1)$ | $(172, 22)$ |
| $(264, 6)$ | $(176, 20)$ | $(132, 6)$ | $(528, 0)$ | $(176, 42)$ |
| $(270, 6)$ | $(180, 19)$ | $(135, 6)$ | $(540, 0)$ | $(180, 38)$ |
| $(276, 2)$ | $(184, 17)$ | $(138, 2)$ | $(552, 0)$ | $(184, 36)$ |
| $(282, 2)$ | $(188, 12)$ | $(141, 2)$ | $(564, 0)$ | $(188, 24)$ |
| $(288, 13)$ | $(192, 30)$ | $(144, 13)$ | $(576, 0)$ | $(192, 62)$ |
| $(294, 2)$ | $(196, 14)$ | $(147, 2)$ | $(588, 1)$ | $(196, 28)$ |
| $(300, 3)$ | $(200, 21)$ | $(150, 3)$ | $(600, 0)$ | $(200, 45)$ |

Table 3: Enumeration of 2-uniform toroidal maps
the line passing through $O$ in the direction of the vectors $A$, $A - B$ and $B$. Let $R_4$, $R_5$ and $R_6$ denote the perpendicular bisectors of $R_1$, $R_2$, $R_3$ respectively. For Figures 1p and 2c, let $R_1'$, $R_2'$ denote the line passing through $O$ in the direction of the vectors $A$ and $B$. Let $R_3'$ and $R_4'$ denote the perpendicular bisectors of $R_1'$ and $R_2'$ respectively. Let $r_1$, $r_2$, $r_3$, $r_4$, $r_5$, $r_6$, $r_1'$, $r_2'$, $r_3'$ and $r_4'$ denote the reflections of $E$ about the lines $R_1$, $R_2$, $R_3$, $R_4$, $R_5$, $R_6$, $R_1'$, $R_2'$, $R_3'$ and $R_4'$ respectively. Then the corresponding hermite normal form of $X$ is given by the matrix \[
abla \frac{a}{b} \frac{0}{d} \] where $a$, $d \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{0\}$ with $ad = n$ and $0 \leq b < d$.

From [14] we know that

\[
\text{Aut} \left( \frac{E}{K} \right) = \frac{\text{Nor}(K)}{K}. \tag{1}
\]

This means, for any automorphism $h \in \text{Aut}(E)$, $h \in \text{Aut}(X)$ if and only if $h$ normalizes $K$. In the following lemmas we find out the conditions on the matrix of $X$ for which an automorphism $h \in \text{Nor}(K)$. We also count the total number of such maps. Before going into the details of the proofs let’s observe the symmetries of the tilings and note which symmetries are responsible for determining level of symmetry of a map in other words presence of which symmetries in the automorphism group of the map changes the number of vertex orbits of the map.

All the tilings have translations along the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and point reflection with respect to $O$. Additionally,

For $i = 2, 5, 6, 7, 11, 14, 17, 20, 21, 22, 24$ and $26 E_i$ has the following bunch of symmetries and their compositions, $60^\circ$ rotation, $r_i$ for $i = 1, 2, \ldots, 6$. Here all of these functions are taking part in reducing orbits.

For $i = 3, 4, 8, 12, 13, 15, 18, 19 E_i$ has the following bunch of symmetries and their compositions, reflection about horizontal and vertical lines through $O$. These functions are taking part in reducing orbits.

For $i = 1$ and $25 E_i$ has the following bunch of symmetries and their compositions, $60^\circ$ rotation. This is also taking part in reducing orbits.

For $i = 16$ and $27 E_i$ has the following bunch of symmetries and their compositions, $90^\circ$ rotation, $r_i'$ for $i = 1, 2, 3, 4$. These are also taking part in reducing orbits.

For $i = 23 E_i$ has the following bunch of symmetries and their compositions, Glide reflection whose component translation and reflection are not symmetries of the tiling and reflection about two corner lines. Glide reflection is taking part in reducing orbits but the reflection about the corner lines are not.

Now, for $n \in \mathbb{N}$, let us define,

\[
\rho_1(n) := \#\{x \in \mathbb{Z}_n : 1 + x + x^2 = 0\}
\]

\[
\rho_2(n) := \#\{x \in \mathbb{Z}_n : 1 + x^2 = 0\}
\]

\[
\rho_3(n) := \#\{x \in \mathbb{Z}_n : x^2 + 2x = 0\}
\]

\[
\rho_4(n) := \#\{x \in \mathbb{Z}_n : 1 - x^2 = 0\}
\]

\[
\rho_5(n) := \#\{x \in \mathbb{Z}_n : x^2 + 2x = 0 \text{ and } 1 - x^2 = 0\}
\]

\[
\rho_6(n) := \#\{x \in \mathbb{Z}_n : x^2 + 1 = 0 \text{ and } 1 - x^2 = 0\}
\]

\[
\rho_7(n) := \#\{x \in \mathbb{Z}_n : x^2 + 1 = 0 \text{ and } 2x = 0\}
\]
Lemma 3.1. If $X$ be a $n$ sheeted toroidal cover of $X_0$, then the total number of maps having \( \sigma \) in the automorphism group of $X$ is equal to

\[
f_1(n) := \sum_{d|n} \rho_1 \left( \frac{d^2}{n} \right).
\]

Proof. Let $X$ be a toroidal map of type $K$ with matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. Then $X = \frac{E}{\Gamma}$ with $\Gamma = \langle w_1, w_2 \rangle$ where $w_1 : z \mapsto z + (aA + bB)$ and $w_2 : z \mapsto z + dB$. Now, $\sigma$ sends $A$ to $B$ and $B$ to $B - A$. Hence, $\sigma \in Aut(E)$.

Claim 3.1. (i) $a \mid b$, (ii) $a \mid d$ and (iii) $ad \mid (a^2 + ab + b^2)$.

We look for the conditions on $a, b$ and $d$ so that $\sigma \in \text{Nor}(\Gamma)$. For that, it is enough to check if $\sigma^{-1}w_1\sigma, \sigma^{-1}w_2\sigma$ belong to the lattice of $\Gamma$. We know that conjugation of a translation by a rotation $i$ is again a translation by the rotated vector $i(w)$. So, $\sigma^{-1}w_1\sigma = \sigma(w_1) = \sigma(aA + bB) = a\sigma(A) + b\sigma(B) = aB + b(B - A) = -bA + (a + b)B$. $\sigma^{-1}w_2\sigma = \sigma(w_2) = \sigma(cA + dB) = c\sigma(A) + d\sigma(B) = dB + d(B - A) = -dA + (c + d)B$. Now, $\sigma(w_1)$ and $\sigma(w_2)$ belong to the lattice of $\Gamma_2$ provided that there exists integers $m_1, m_2, m_3, m_4$ such that

\[
-bA + (a + b)B = m_1(aA + bB) + m_2dB \quad \text{and} \quad -(aA + bB) = m_3(aA + bB) + m_4dB.
\]

Since, $A$ and $B$ are linearly independent, we have a system of linear equations: $m_1a = -b, m_1b + m_2d = a + b, m_3a = -d$ and $m_3b + m_4d = d$. Solving these equations, we get, $m_1 = -\frac{b}{a}, m_2 = \frac{a^2 + ab + b^2}{ad}, m_3 = -\frac{d}{a}$ and $m_4 = \frac{a + b}{a}$. Since, $m_1, m_2, m_3$ and $m_4$ are integers, we must have $a \mid b, a \mid d$ and $ad \mid (a^2 + ab + b^2)$. This completes the proof.

Now, $a \mid b$ and $a \mid d$ implies $b = ax$ and $d = ay$ where $0 \leq x < y$. Using the last condition, we get $a.ay \mid (a^2 + a.ax + ax.ax)$, or, $y \mid (1 + x + x^2)$ where $y = \frac{d^2}{n}$. This implies $1 + x + x^2$ has a solution in $\mathbb{Z}_{\frac{d^2}{n}}$. So, the total number of maps is given by the number of solutions of the polynomial $1 + x + x^2$ in $\mathbb{Z}_{\frac{d^2}{n}}$ satisfying $n \mid d^2$ for every divisor $d$ of $n$. Hence, we get the formula.

\[
\square
\]

Lemma 3.2. If $X$ be a $n$ sheeted toroidal cover of $X_0$, then the total number of maps having $\psi$ in the automorphism group of $X$ is equal to

\[
f_2(n) := \sum_{d|n} \rho_2 \left( \frac{d^2}{n} \right).
\]

Proof. Let $X$ be a toroidal map of type $K$ with matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. Then $X = \frac{E}{\Gamma}$ with $\Gamma = \langle w_1, w_2 \rangle$ where $w_1 : z \mapsto z + (aA + bB)$ and $w_2 : z \mapsto z + dB$. Now, $\tau$ sends $A$ to $B$ and $B$ to $-A$. Clearly, $\tau \in Aut(E)$.

Claim 3.2. (i) $a \mid b$, (ii) $a \mid d$ and (iii) $ad \mid (a^2 + b^2)$.

Here, $\tau^{-1}w_1\tau = \tau(w_1) = \tau(aA + bB) = a\tau(A) + b\tau(B) = aB + b(-A) = -bA + aB$. $\tau^{-1}w_2\tau = \tau(w_2) = \tau(dB) = d\tau(B) = d(-A) = -dA$. By applying similar arguments used in Lemma 3.1, $\tau(w_1)$ and $\tau(w_2)$ belong to the lattice of $\Gamma$ if and only if $a \mid b, a \mid d$ and $ad \mid (a^2 + b^2)$.

Using similar argument in the Lemma 3.1 the total number of maps is given by the number of solutions of the polynomial $1 + x^2$ in $\mathbb{Z}_{\frac{d^2}{n}}$ satisfying $n \mid d^2$ for every divisor $d$ of $n$. Hence, we get the formula.

\[
\square
\]
Lemma 3.3. If $X$ be a $n$-sheeted toroidal cover of $X_0$, then the total number of maps having $r_1$ in the automorphism group of $X$ is equal to

$$f_3(n) := \sum_{d|n \atop n|d^2} \rho_3 \left( \frac{d^2}{n} \right).$$

Proof. Let $X$ be a toroidal map of type $K$ with matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. Then $X = \mathcal{E} \Gamma$ with $\Gamma = \langle w_1, w_2 \rangle$ where $w_1 : z \mapsto z + (aA + bB)$ and $w_2 : z \mapsto z + dB$. $r_1$ sends $A$ to $A$ and $B$ to $A - B$. Hence, $r_1 \in \text{Aut}(E)$. \hfill $\square$

Claim 3.3. $r_1 \in \text{Nor}(\Gamma)$ if and only if (i) $a \mid b$, (ii) $a \mid d$ and (iii) $ad \mid (b^2 + 2ab)$.

We know that conjugation of a translation $w$ by a reflection $i$ is again a translation by the reflected vector $i(w)$. So, $r_1^{-1} w_1 r_1 = r_1(w_1) = r_1(aA + bB) = ar_1(A) + br_1(B) = aB + b(A-B) = bA + (a-b)B$. By applying similar method used in Lemma 3.1, we get $r_1(w_1), r_1(w_2) \in \mathbb{Z}(aA + bB) + \mathbb{Z}dB$ if and only if $a \mid b$, $a \mid d$ and $ad \mid (b^2 + 2ab)$.

Using (i) and (ii) from Claim 3.3, we get $b = ax$ and $d = ay$ where $0 \leq x \leq y$ and using (iii), we get $y \mid (x^2 + 2x)$ where $y = \frac{d^2}{n}$. Hence, the total number of required maps is given by the number of solutions of the polynomial $x^2 + 2x$ in $\mathbb{Z}_{\frac{d^2}{n}}$ satisfying $n \mid d^2$ for every divisor $d$ of $n$. Hence, we get the formula.

Lemma 3.4. If $X$ be a $n$-sheeted toroidal cover of $X_0$, then the total number of maps having $r_2$ in the automorphism group of $X$ is equal to

$$f_4(n) := \sum_{d|n \atop n|d^2} \rho_4 \left( \frac{d^2}{n} \right).$$

Proof. Observe that $r_2$ sends $A$ to $B - A$ and $B$ to $B$. Hence, $r_2 \in \text{Aut}(E)$.

Claim 3.4. $r_2 \in \text{Nor}(\Gamma)$ if and only if (i) $a \mid b$, (ii) $a \mid d$ and (iii) $ad \mid (b^2 - a^2)$.

Using (i) and (ii) from Claim 3.4, we get $b = ax$ and $d = ay$ where $0 \leq x \leq y$ and using (iii), we get $y \mid (x^2 + 2x)$ where $y = \frac{d^2}{n}$. Hence, the total number of required maps is given by the number of solutions of the polynomial $x^2 - 1$ in $\mathbb{Z}_{\frac{d^2}{n}}$ satisfying $n \mid d^2$ for every divisor $d$ of $n$. Hence, we get the formula. \hfill $\square$

Lemma 3.5. If $X$ be a $n$-sheeted toroidal cover of $X_0$, then the total number of maps having $r_1$ and $r_2$ in the automorphism group of $X$ is equal to

$$f_5(n) := \sum_{d|n \atop n|d^2} \rho_5 \left( \frac{d^2}{n} \right).$$
Proof. If $r_1$ and $r_2$ belong to $\text{Aut}(X)$, then by Claim 3.3, we have the conditions, $a \mid b, a \mid d, ad \mid (b^2 + 2ab)$ and $ad \mid (b^2 - a^2)$. Using these, we get $y \mid (x^2 + 2x)$ and $y \mid (1 - x^2)$ where $y = \frac{d^2}{n}$ and $0 \leq x < y$. This implies that the polynomials $x^2 + x$ and $1 - x^2$ have a common solution in $\mathbb{Z}_{\frac{d^2}{n}}$. So, the total number of maps is given by the number of solutions of $\text{gcd}(x^2 + 2x, 1 - x^2)$ in $\mathbb{Z}_{\frac{d^2}{n}}$ satisfying $n \mid d^2$ for every divisor $d$ of $n$. Hence, we get the formula. \hfill $\square$

Remark 3.6. $r_1$ and $r_2$ generates the isotropy group of the tiling hence the function $f_5$ counts number of $n$-sheeted maps having highest possible order isotropy group in other words $\mathcal{F}(X) = \mathcal{F}(E_i)$.

Lemma 3.7. If $X$ be a $n$ sheeted toroidal cover of $X_0$, then the total number of maps having $\psi$ and $r'_1$ in the automorphism group of $X$ is equal to

$$f_6(n) := \sum_{\substack{d|n \mod d^2}} \rho_6 \left( \frac{d^2}{n} \right).$$

Proof. If $\psi$ and $r'_1$ belong to $\text{Aut}(X)$, then by Claim 3.4 the conditions, $a \mid b, a \mid d, ad \mid (a^2 + b^2)$ and $ad \mid (b^2 - a^2)$. Using these, we get $y \mid (x^2 + 1)$ and $y \mid (1 - x^2)$ where $y = \frac{d^2}{n}$ and $0 \leq x < y$. This implies that the polynomials $x^2 + 1$ and $1 - x^2$ have a common solution in $\mathbb{Z}_{\frac{d^2}{n}}$. So, the total number of maps is given by the number of solutions of $\text{gcd}(x^2 + 1, 1 - x^2)$ in $\mathbb{Z}_{\frac{d^2}{n}}$ satisfying $n \mid d^2$ for every divisor $d$ of $n$. Hence, we get the formula. \hfill $\square$

Lemma 3.8. If $X$ be a $n$ sheeted toroidal cover of $X_0$, then the total number of maps having $r'_1$ in the automorphism group of $X$ is equal to

$$f_7(n) := \sum_{\substack{d|n \mod d^2}} \rho_4 \left( \frac{d^2}{n} \right).$$

Proof. Observe that $r'_1$ sends $A$ to $B$ and $B$ to $A$. Hence, $r'_1 \in \text{Aut}(E)$.

Claim 3.5. $r'_1 \in \text{Nor}(\Gamma)$ if and only if (i) $a \mid b$, (ii) $a \mid d$ and (iii) $ad \mid (b^2 - a^2)$.

$$r_{1}^{-1}w_1r'_1 = r'_1(w_1) = r'_1(aA + bB) = ar'_1(A) + br_2(B) = aB + bA = bA + aB.$$  

Using (i) and (ii) from Claim 3.5, we get $b = ax$ and $d = ay$ where $0 \leq x \leq y$ and using (iii), we get $y \mid (x^2 + 2x)$ where $y = \frac{d^2}{n}$. Hence, the total number of required maps is given by the number of solutions of the polynomial $x^2 - 1$ in $\mathbb{Z}_{\frac{d^2}{n}}$ satisfying $n \mid d^2$ for every divisor $d$ of $n$. Hence, we get the formula. \hfill $\square$

Observe that $r_1 \in \text{Aut}(X)$ if and only if $r_4 \in \text{Aut}(X)$ and $r_2 \in \text{Aut}(X)$ if and only if $r_5 \in \text{Aut}(X)$.

Lemma 3.9. If $X$ be an $n$ sheeted toroidal cover of $X_0$, then the total number of maps having $r_3$ in the automorphism group of $X$ is equal to

$$f_8(n) := \sum_{d|n} \# \{ b \in \mathbb{Z}_d : dk - 2b = a, ad = n, k \in \mathbb{Z}_d \}.$$
Proof. Let $X$ be a toroidal map of type $E_i$ where $i = 1$, with matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. Then $X = E_\Gamma$ with $\Gamma = \langle w_1, w_2 \rangle$ where $w_1 : z \mapsto z + (aA + bB)$ and $w_2 : z \mapsto z + dB$. Now, $R_3$ sends $A$ to $-B$ and $B$ to $-A$. Hence, $R_3 \in \text{Aut}(E)$. Then, $R_3^{-1}w_1R_3 = R_3(w_1) = R_3(aA + bB) = aR_3(A) + bR_3(B) = a(-B) + b(-A) = -bA - aB$. $R_3^{-1}w_2R_3 = R_3(w_2) = R_3(dB) = dR_3(B) = d(-A) = -dA$. By applying similar arguments used in the previous lemma, $R_1(w_1)$ and $R_1(w_2)$ belong to the lattice of $\Gamma$ if and only if $d \mid (a + 2b)$. We need to find the number of solutions of the linear diophantine equation $dk - 2b = \frac{n}{d}$ in $\mathbb{Z}_d$ for every divisor $d$ of $n$. Hence, we get the formula. \[ \]

Observe that $r_3 \in \text{Aut}(X)$ if and only if $r_6 \in \text{Aut}(X)$.

**Lemma 3.10.** If $X$ be a $n$ sheeted toroidal cover of $X_0$, then the total number of maps having $r_i'$ in the automorphism group of $X$ is equal to $f_4(n)$.

**Proof.** Let $X$ be a toroidal map of type $K$ with matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. Then $X = E_\Gamma$ with $\Gamma = \langle w_1, w_2 \rangle$ where $w_1 : z \mapsto z + (aA + bB)$ and $w_2 : z \mapsto z + dB$. Now, $r_i'$ sends $A$ to $B$ and $B$ to $-A$. Hence, $r_i' \in \text{Aut}(E)$. Then, $R_i'^{-1}w_1R_i' = R_i'(w_1) = R_i'(aA + bB) = aR_i'(A) + bR_i'(B) = aA + b(-B) = aA - bB$. $R_i'^{-1}w_2R_i' = R_i'(w_2) = R_i'(dB) = dR_i'(B) = d(-B) = -dB$. By applying similar arguments used in the previous lemma, $R_2(w_1)$ and $R_2(w_2)$ belong to the lattice of $\Gamma$ if and only if $a \mid b$, $a \mid d$ and $ad \mid (b^2 - a^2)$. Rest of the proof is exactly same as Lemma 3.4. \[ \]

Observe that $r_i'^2 \in \text{Aut}(X)$ if and only if $r_i'^4 \in \text{Aut}(X)$.

**Note 3.11.** The number of maps obtained in above Lemmas are not up to isomorphism. They are number of distinct maps with corresponding properties.

To get number of two uniform toroidal maps up to isomorphism we need the following concepts.

**Definition 3.12.** For a toroidal map $X$ and a vertex $v \in X$ the isotropy group of $X$ is the group of all automorphisms of $X$ which fixes the vertex $v$. We denote it by $\mathcal{F}(X)$.

Observe that the isotropy group of the tilings $E_i$ is $D_6$ for $i \in \{2, 5, 6, 7, 11, 14, 17, 20, 21, 22, 24, 26\}$, $D_4$ for $i \in \{16, 27\}$, $Z_4$ for $i = 23$, $Z_4$ for $i = 1, 25$ and $Z_2 \times Z_2$ for $i \in \{3, 4, 8, 12, 13, 15, 18, 19\}$. Now, for those tilings which have isotropy group $D_6$, the isotropy group of a corresponding map $X$ will be subgroup of $D_6$. So possible orders of $\mathcal{F}(X)$ will be 2, 4, 6 and 12. Order 1 is not possible because point reflection is always present in $\text{Aut}(X)$ and it has order 2. There are $\vartheta(n)$ many distinct $n$-sheeted map exists. However, in this counting some cases counted more than once. If $\mathcal{F}(X)$ has order 2 it is counted 6 times (just by 60 degree rotations). If $\mathcal{F}(X)$ has order 4 then it is counted 3 times. If $\mathcal{F}(X)$ has order 6 it is counted 2 times and if $\mathcal{F}(X)$ has order 12 then it is counted only once. In general, if $\mathcal{F}(X)$ has order $d$ then it is counted $12/d$ times. This is caused because the choice of basis of the map is not unique, by applying different symmetries if the tiling on the basis we get other representation of the same map. Similarly, for those tilings which have isotropy group $D_4$, the isotropy group of a corresponding map $X$ will be subgroup of $D_4$. So possible orders of $\mathcal{F}(X)$ will be 2, 4 and 8. If $\mathcal{F}(X)$ has order $d$ then it is counted $8/d$ times. And for those tilings which have isotropy group $Z_4$ and $Z_6$, the isotropy group of a corresponding map $X$ will be subgroup of $Z_4$ and $Z_6$ respectively. So possible orders of $\mathcal{F}(X)$ will be 2, 4 or 2, 6. If $\mathcal{F}(X)$ has order $d$ then it is counted $4/d$ or $6/d$ times respectively. For the latter case order
of $\mathcal{F}(X)$ cannot be 3 since in that case the group will be generated by $120^\circ$ rotation and presence of $120^\circ$ rotation will ensure the presence of $60^\circ$ rotation.

Now, some more merging will happen due to presence of some additional symmetry in the automorphism group of the map. In $E_{23}$ glide reflection will be present and we have the following,

**Lemma 3.13.** Let $X$ be a $n$ sheeted toroidal map of type $[3^2, 4^1, 3^1, 4^1]$ represented by the matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. Then glide reflection will be present in $\text{Aut}(X)$ if and only if $d \mid 2b$ and number of distinct $n$-sheeted maps having glide in its automorphism group is

$$g(n) := \sum_{\frac{d|n}{2|d}} 2 + \sum_{\frac{d|n}{2|d}} 1.$$  

*Proof.* Corresponding matrix for glide reflection is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Now the proof follows along the same lines as in above Lemmas. □

**Lemma 3.14.** Let $X$ be a $n$ sheeted toroidal map of type $[3^2, 4^1, 3^1, 4^1]$ represented by the matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$ having $\psi$ and glide reflection in its isotropy group. Then number of such distinct maps will be

$$h(n) := \sum_{\frac{d|n}{n|d^2}} \rho_{\tau}\left(\frac{d^2}{n}\right).$$

*Proof.* One can proof it in similar fashion like Lemma 3.1. □

**Lemma 3.15.** Let $X$ be a $n$ sheeted toroidal map of type $[3^2, 4^1, 3^1, 4^1]$ represented by the matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$ such that after applying glide reflection and $90^\circ$ rotation on $X$ we get same map. Then number of distinct such maps are

$$\alpha(n) = f_4(n) - h(n).$$

*Proof.* Similarly like other above lemmas. □

*Proof of Theorem 1.5.* First, we will calculate $f_i$, for each $i \in \{1, 2 \ldots, 7\}$.

Calculation of $f_1(n)$, $n \in \mathbb{N}$.

**Claim 3.6.** $\rho_1(1) = 1$ and for $k \in \mathbb{N}$, $f_1(2^k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{2} \\ 1 & \text{if } k \equiv 0 \pmod{2} \end{cases}$

$\rho_1(1) = 1$ since $x = 0$ is the solution for $1 + x + x^2 \equiv 0 \pmod{1}$. Now, observe that $\rho_1(2^k) = 0$ for all $k \in \mathbb{N}$ since $1 + x + x^2$ is odd for every $x \in \mathbb{N} \cup \{0\}$ and $2^k \nmid (1 + x + x^2)$.

Case 1: If $k = 2k_0$, then $2k_0 \leq 2i$ since $n \mid d^2$. Then, $\frac{d^2}{n} = 2^{2i-2k_0}$ where $0 \leq 2i - 2k_0 \leq 2k_0$. Let $k' = i - k_0$. Then $0 \leq k' \leq k_0$. Therefore,

$$f_1(2^k) = \sum_{k'=0}^{k_0} \rho_1(2^{2k'}) = \rho_1(1) + \rho_1(2^1) + \rho_1(2^2) + \cdots + \rho_1(2^{2k_0})$$

$$= 1 + 0 + \cdots + 0 = 1.$$
Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 2^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[
f_1(2^k) = \sum_{k'=1}^{k_0} \rho_1(2^{2k'-1}) = \rho_1(2) + \rho_1(2^3) + \cdots + \rho_1(2^{2k_0-1}) = 0 + 0 + \cdots + 0 = 0.
\]

Hence, our Claim 3.6 is proved.

**Claim 3.7.** \( f_1(3^k) = 1 \) for all \( k \in \mathbb{N} \).

\( \rho_1(3) = 1 \) since 1 is the only solution of the equation \( 1 + x + x^2 \) in \( \mathbb{Z}_3 \).

We want to show that \( \rho_1(3^k) = 0 \) for all \( k \in \mathbb{N} \) with \( k \geq 2 \). When \( x = 3l \) where \( l \in \mathbb{N} \), \( 1 + x + x^2 = 1 + 3l + 9l^2 \). When \( x = 3l + 1 \) where \( l \in \mathbb{N} \), \( 1 + x + x^2 = 3(1 + 3l + 3l^2) \). When \( x = 3l + 2 \) where \( l \in \mathbb{N} \), \( 1 + x + x^2 = 1 + 3(3k^2 + 5k + 2) \). So, \( 3^k \nmid (1 + x + x^2) \) for all \( x \in \mathbb{N} \).

Hence, our Claim 3.7 is proved.

**Claim 3.8.** Let \( p \) be any prime with \( p \not\equiv 2, 3 \). Then, \( \rho_1(p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod{3} \\ 0 & \text{if } p \equiv 2 \pmod{3} \end{cases} \)

If \( p \) is any odd prime except 3, then \( x^2 + x + 1 \equiv 0 \pmod{p} \) has a solution if and only if \( y^2 \equiv -3 \pmod{p} \) has a solution. Using Legendre symbols [4, Chap. 9], \( \left( -\frac{3}{p} \right) = \left( -\frac{1}{p} \right) \left( \frac{3}{p} \right) \).

If \( p \equiv 1 \pmod{4} \), \( \left( -\frac{1}{p} \right) = 1 \) and \( \left( \frac{3}{p} \right) = \left( \frac{p}{3} \right) \) and if \( p \equiv 3 \pmod{4} \), \( \left( -\frac{1}{p} \right) = -1 \) and \( \left( \frac{3}{p} \right) = -\left( \frac{p}{3} \right) \).

So, \( \left( -\frac{3}{p} \right) = \left( \frac{p}{3} \right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3} \\ -1 & \text{if } p \equiv 2 \pmod{3} \end{cases} \).

Hence, the Claim 3.8 is proved.

**Claim 3.9.** \( f_1(p^k) = k + 1 \) if \( p \equiv 1 \pmod{3} \).

Let \( p \) be an odd prime with \( p \equiv 1 \pmod{3} \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = p^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[
f_1(p^k) = \sum_{k'=0}^{k_0} \rho_1(p^{2k'}) = \rho_1(1) + \rho_1(p^2) + \rho_1(p^4) + \cdots + \rho_1(p^{2k_0}) = 1 + 2 + \cdots + 2 = 2k_0 + 1.
\]
Claim 3.10. Hence, our Claim 3.9 is proved.

Claim 3.11. Following, \(n\) be the prime decomposition of \(f\) of separate congruences \(f\) is multiplicative. Using Result 3.1 we can conclude that \(\rho_1\) is multiplicative. Using this we prove the following.

Claim 3.11. \(f_1\) is multiplicative.

Let \(m\) and \(n\) be natural numbers such that \(\gcd(m, n) = 1\). We want to show that \(f_1(mn) = f_1(m)f_1(n)\). We can write any divisor \(d\) of \(mn\) uniquely as \(d = d_1d_2\) such that \(d_1 | m, d_2 | n\) such that \(\gcd(d_1, d_2) = 1\). Then, \(\gcd(d_1^2, d_2^2) = 1\) and \(\gcd\left(\frac{d_1^2}{m}, \frac{d_2^2}{n}\right) = 1\).
\[ f_1(mn) = \sum_{d|mn, \frac{m^2}{d^2} | n} \rho_1 \left( \frac{d^2}{mn} \right) = \sum_{d_1|d^2, d_2|m} \rho_1 \left( \frac{d_1^2 d_2^2}{mn} \right) = \sum_{d_1|m, d_2|n} \rho_1 \left( \frac{d_1^2 d_2^2}{mn} \right) \]

\[ = \sum_{d_1|m, d_2|n} \rho_1 \left( \frac{d_1^2}{m} \right) \rho_1 \left( \frac{d_2^2}{n} \right) \]  

[Since \( \rho_1 \) is multiplicative]

\[ = \sum_{d_1|m} \rho_1 \left( \frac{d_1^2}{m} \right) \sum_{d_2|n} \rho_1 \left( \frac{d_2^2}{n} \right) = f_1(m) f_1(n). \]

Let \( n = 2^{m_0} 3^{k_0} p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i} q_1^{m_1} q_2^{m_2} \cdots q_{n_2}^{m_{n_2}} \) where \( p_i \) and \( q_j \) are primes with \( p_i \equiv 1 \pmod{3} \) where \( i \in \{0, 1, \ldots, n_1\} \) and \( q_j \equiv 2 \pmod{3} \) where \( j \in \{0, 1, \ldots, n_2\} \). Then

\[ f_1(n) = \begin{cases} 0 & \text{if } m_j \equiv 1 \pmod{2} \text{ for some } j \in \{0, 1, 2, \ldots, n_2\} \\ \prod_{i=1}^{n_1} (k_i + 1) & \text{otherwise.} \end{cases} \]

Let us calculate \( f_2(n) \).

**Claim 3.12.** \( \rho_2(1) = 1 \) and \( f(2^k) = 1 \) for all \( k \in \mathbb{N} \).

\( \rho_2(1) = 1 \) since \( x = 0 \) is the solution for \( 1 + x^2 \equiv 0 \pmod{1} \).

\( \rho_2(2) = 1 \) since \( 1 \) is the only solution of \( 1 + x^2 = 0 \) in \( \mathbb{Z}_2 \). Now, \( \rho_2(2^k) = 0 \) for all \( k \in \mathbb{N} \) with \( k \geq 2 \). This is because when \( x = 2l \) for some \( l \in \mathbb{N} \), \( 1 + x^2 = 1 + 4l^2 \). So, \( 2^k \nmid 1 + x^2 \) for all \( k \).

Let \( n = 2^k \). Then, \( d = 2^i \), \( 0 \leq i \leq k \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 2^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[ f_2(2^k) = \sum_{k'=0}^{k_0} \rho_2(2^{2k'}) = \rho_2(1) + \rho_2(2^2) + \rho_2(2^4) + \cdots + \rho_2(2^{2k_0}) \]

\[ = 1 + 0 + \cdots + 0 = 1. \]

Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 2^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[ f_2(2^k) = \sum_{k'=1}^{k_0} \rho_2(2^{2k'-1}) = \rho_2(2) + \rho_2(2^3) + \cdots + \rho_2(2^{2k_0+1}) \]

\[ = 1 + 0 + \cdots + 0 = 1. \]

Hence, our Claim 3.12 is proved.

**Claim 3.13.** For \( k \in \mathbb{N} \), \( f_2(3^k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{2} \\ 1 & \text{if } k \equiv 0 \pmod{2} \end{cases} \)
Claim 3.16. Then \( i \equiv (\mod 4) \) where \( \rho_f(y) \) is any odd prime except 3, then, \( \rho_f(3^k) = \rho_f(1) + \rho_f(3^2) + \rho_f(3^4) + \cdots + \rho_f(3^{2k_0}) \)

\[ = 1 + 0 + \cdots + 0 = 1. \]

Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 3^{2i-2k_0} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[ f_2(3^k) = \sum_{k'=0}^{k_0} \rho_2(3^{2k'}) = \rho_2(3) + \rho_2(3^3) + \cdots + \rho_2(3^{2k_0+1}) \]

\[ = 0 + 0 + \cdots + 0 = 0. \]

Hence, our Claim 3.13 is proved.

Claim 3.14. Let \( p \) be any prime with \( p \neq 2, 3 \). Then, \( \rho_2(p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod{4} \\ 0 & \text{if } p \equiv 3 \pmod{4} \end{cases} \)

If \( p \) is any odd prime except 3, then, \( x^2 + 1 \equiv 0 \pmod{p} \) has a solution if and only if \( y^2 \equiv -4 \pmod{p} \) has a solution. Using Legendre symbols \( \left( \frac{-4}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{4}{p} \right) = \left( \frac{-1}{p} \right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases} \). Hence, the Claim 3.14 is proved.

Claim 3.15. \( f_2(p^k) = k + 1 \) if \( p \equiv 1 \pmod{4} \) and \( f_2(p^k) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{2} \\ 0 & \text{if } k \equiv 1 \pmod{2} \end{cases} \), where \( p \equiv 3 \pmod{4} \).

Following similar calculations used in Claim 3.9, one can prove this Claim 3.15.

Then from Result 3.1, we can say that \( f_2 \) is multiplicative.

Let \( n = 2^{k_0}3^{m_0}p_1^{k_1}p_2^{k_2} \cdots p_{n_1}^{m_1}q_1^{m_2}q_2^{m_2} \cdots q_{n_2}^{m_2} \) where \( p_i \) and \( q_j \) are primes with \( p_i \equiv 1 \pmod{4} \) where \( i \in \{0, 1, \ldots, n_1\} \) and \( q_j \equiv 3 \pmod{4} \) where \( j \in \{0, 1, \ldots, n_2\} \).

Then

\[ f_2(n) = \begin{cases} 0 & \text{if } m_j \equiv 1 \pmod{2} \text{ for some } j \in \{0, 1, 2, \ldots, n_2\} \\ \prod_{i=1}^{n_1}(k_i + 1) & \text{otherwise.} \end{cases} \]

Let us calculate \( f_3(n) \) where \( n \in \mathbb{N} \).

Claim 3.16. \( \rho_3(1) = 1 \) and \( \rho_3(2) = 1 \), \( \rho_3(4) = 2 \) and \( \rho_3(2^k) = 4 \) for all \( k \in \mathbb{N} \) with \( k \geq 3 \).

\( \rho_3(1) = 1 \) since \( x = 0 \) is the solution for \( x^2 + 2x \equiv 0 \pmod{1} \).

\( \rho_3(2) = 1 \) since 0 is the only solution of 0 in \( \mathbb{Z}_2 \). \( \rho_3(4) = 2 \) since 0 and 2 are the solutions of \( x^2 + 2x = 0 \) in \( \mathbb{Z}_4 \).

Now, \( x^2 + 2x \equiv 0 \pmod{2^k} \) \( \implies (x + 1)^2 \equiv 1 \pmod{2^k} \). Putting \( y = x + 1 \), we get the congruence \( y^2 \equiv 1 \pmod{2^k} \). We know that the congruence \( y^2 \equiv 1 \pmod{2^k} \) has exactly four incongruent solutions. Hence, \( \rho_3(2^k) = 4 \).
Claim 3.17. \( f_3(2^k) = 2k - 1 \) when \( k \in \mathbb{N} \)

Let \( n = 2^k \). Then, \( d = 2^i, \ i \leq k \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 2^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[ f_3(2^k) = \sum_{k' = 0}^{k_0} \rho_3(2^{2k'}) = \rho_3(1) + \rho_3(2^2) + \rho_3(2^4) + \cdots + \rho_3(2^{2k_0}) = 1 + 2 + 4 + \cdots + 4 = 4(k_0 - 1) + 3 = 4k_0 - 1. \]

Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 2^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[ f_3(2^k) = \sum_{k' = 1}^{k_0} \rho_3(2^{2k'-1}) = \rho_3(2) + \rho_3(2^3) + \cdots + \rho_3(2^{2k_0+1}) = 1 + 4 + \cdots + 4 = 4k_0 + 1. \]

Hence, our Claim 3.17 is proved.

Claim 3.18. Let \( p \) be any odd prime. Then, \( f_3(p^k) = k + 1 \) where \( k \in \mathbb{N} \).

The congruence \( x^2 + 2x \equiv 0 \pmod{p} \) has solution if and only if \( y^2 \equiv 4 \pmod{p} \) has a solution. Since, \( p \) is odd we have \( \left( \frac{4}{p} \right) = 1 \). Hence, \( x^2 + 2x \equiv 0 \pmod{p} \) has exactly two solutions for all odd prime \( p \). This implies \( x^2 + 2x \equiv 0 \pmod{p^k} \) has 2 solutions for all \( k \geq 1 \).

So, \( \rho(p^k) = 2 \) for all \( k \geq 1 \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = p^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[ f_3(p^k) = \sum_{k' = 0}^{k_0} \rho_3(p^{2k'}) = \rho_3(1) + \rho_3(p^2) + \rho_3(p^4) + \cdots + \rho_3(p^{2k_0}) = 1 + 2 + \cdots + 2 = 2k_0 + 1. \]

Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = p^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 - 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[ f_3(p^k) = \sum_{k' = 1}^{k_0} \rho_3(p^{2k'-1}) = \rho_3(p) + \rho_3(p^3) + \cdots + \rho_3(p^{2k_0+1}) = 2 + 2 + \cdots + 2 = 2k_0 + 2. \]

Hence, our Claim 3.18 is proved.

Let \( n = 2^{k_0}p_1^{k_1}p_2^{k_2} \cdots p_{n_1}^{k_{n_1}} \) where \( p_i \) is any odd prime for \( i \in \{0, 1, \ldots, n_1\} \).

Then,

\[ f_3(n) = \begin{cases} 
\prod_{i=1}^{n_1} (k_i + 1), & \text{if } k_0 = 0 \\
(2k_0 - 1) \prod_{i=1}^{n_1} (k_i + 1), & \text{otherwise}. 
\end{cases} \]

Let us calculate \( f_3(n) \) for \( n \in \mathbb{N} \).

Claim 3.19. \( \rho_4(1) = 1, \ \rho_4(2) = 1, \ \rho_3(4) = 2 \) and \( \rho_3(2^k) = 4 \) for all \( k \in \mathbb{N} \) with \( k \geq 3 \).
\[ \rho_4(1) = 1 \] since \( x = 0 \) is the solution for \( x^2 + 2x \equiv 0 \pmod{1} \). \( \rho_4(2) = 1 \) since 1 is the only solution of 0 in \( \mathbb{Z}_2 \). \( \rho_3(4) = 2 \) since 1 and 3 are the solutions of \( x^2 + 2x = 0 \) in \( \mathbb{Z}_4 \).

Now, \( x^2 - 1 \equiv 0 \pmod{2^k} \implies x^2 \equiv 1 \pmod{2^k} \). We know that the congruence \( x^2 \equiv 1 \pmod{2^k} \) has exactly four incongruent solutions for \( k \geq 3 \). Hence, \( \rho_3(2^k) = 4 \).

**Claim 3.20.** \( f_4(2^k) = 2k - 1 \) when \( k \in \mathbb{N} \). Let \( p \) be any odd prime. Then, \( f_4(p^k) = k + 1 \) where \( k \in \mathbb{N} \).

Proceeding in similar way as in the proof of Claims 3.17 and 3.18 one can proof Claim 3.20 and for \( n = 2^{k_0} p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \) where \( p_i \) is any odd prime for \( i \in \{0, 1, \ldots, n_1\} \) we get

\[
 f_4(n) = \begin{cases} 
 \prod_{i=1}^{n_1}(k_i + 1) & \text{if } k_0 = 0 \\
 (2k_0 - 1) \prod_{i=1}^{n_1}(k_i + 1) & \text{otherwise.} 
\end{cases}
\]

Let us calculate \( f_5(n) \).

**Claim 3.21.** \( \rho_5(1) = 1 \) and \( \rho_5(3) = 1 \).

\( \rho_5(1) = 1 \) since \( x = 0 \) is the solution for the congruences \( x^2 + 2x \equiv 0 \pmod{1} \) and \( x^2 + 2x \equiv 0 \pmod{1} \). \( \gcd\{(x^2 + 2x), (x^2 - 1)\} = x - 1 \) in \( \mathbb{Z}_3 \). Since \( x = 1 \) is the only solution in \( \mathbb{Z}_3 \), \( \rho_5(3) = 1 \).

**Claim 3.22.** \( f_5(3^k) = 1 \) for all \( k \in \mathbb{N} \).

Since \( \gcd\{(x^2 + 2x), (x^2 - 1)\} = 1 \) in \( \mathbb{Z}_3 \) where \( k \in \mathbb{N} \setminus \{1\} \), we have \( \rho_5(3^k) = 0 \) for all \( k \) when \( n = 3^k \). Then, \( d = 3^i \), \( 0 \leq i \leq k \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( d^2 \equiv 3^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[
 f_5(3^k) = \sum_{k'=0}^{k_0} \rho_5(3^{2k'}) = \rho_5(1) + \rho_5(3^2) + \rho_5(3^4) + \cdots + \rho_5(3^{2k_0})
 = 1 + 0 + \cdots + 0 = 1.
\]

Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( d^2 \equiv 3^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[
 f_5(3^k) = \sum_{k'=1}^{k_0} \rho_5(3^{2k'-1}) = \rho_5(3) + \rho_5(3^3) + \cdots + \rho_5(3^{2k_0+1})
 = 1 + 0 + \cdots + 0 = 1.
\]

Hence, our Claim 3.22 is proved.

**Claim 3.23.** If \( p \) in any prime other than 3, then \( f_5(p^k) = \begin{cases} 
 0 & \text{if } k \equiv 1 \pmod{2} \\
 1 & \text{if } k \equiv 0 \pmod{2} 
\end{cases} \)

Since \( \gcd\{(x^2 + 2x), (x^2 - 1)\} = 1 \) in \( \mathbb{Z}_{p^k} \), we have \( \rho_5(p^k) = 0 \) for all \( k \in \mathbb{N} \). Let \( n = p^k \), \( k \in \mathbb{N} \). Then, \( d = p^i \), \( 1 \leq i \leq k \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( d^2 \equiv p^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[
 f_5(p^k) = \sum_{k'=0}^{k_0} \rho_5(p^{2k'}) = \rho_5(1) + \rho_5(p^2) + \rho_5(p^4) + \cdots + \rho_5(p^{2k_0})
 = 1 + 0 + \cdots + 0 = 1.
\]
Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = p^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[
f_5(p^k) = \sum_{k'=1}^{k_0} \rho_5(p^{2k'-1}) = \rho_5(p) + \rho_5(p^3) + \cdots + \rho_5(p^{2k_0-1}) = 0 + 0 + \cdots + 0 = 0.
\]

Hence, our Claim 3.23 is proved.

Let \( n = 2^{k_0}3^m p_1^{k_1} p_2^{k_2} \cdots p_{n_1}^{k_{n_1}} \) where \( p_i \) is any prime other than 3 for \( i \in \{0, 1, \ldots, n_1\} \). Then,

\[
f_5(n) = \begin{cases} 
0 & \text{if } k_i \equiv 1 \pmod{2} \text{ for some } i \in \{0, 1, 2, \ldots, n_1\} \\
1 & \text{otherwise.}
\end{cases}
\]

Let us calculate \( f_6(n) \) for all \( n \in \mathbb{N} \).

**Claim 3.24.** \( \rho_6(1) = 1 \), \( \rho_6(2) = 1 \) and \( f_6(2^k) = 1 \) for all \( k \in \mathbb{N} \).

\[ \rho_6(1) = 1 \] since \( x \equiv 0 \) is the solution for the congruences \( x^2+1 \equiv 0 \pmod{1} \) and \( x^2-1 \equiv 0 \pmod{1} \). Since \( x^2+1 = x^2-1 \) in \( \mathbb{Z}_2 \), \( \gcd\{(x^2+1), (x^2-1)\} = x^2+1 \). Since \( x = 1 \) is the only solution of \( x^2+1 \) in \( \mathbb{Z}_2 \), \( \rho_6(2) = 1 \).

Since \( \gcd\{(x^2+1), (x^2-1)\} = 1 \) in \( \mathbb{Z}_{2^k} \) where \( k \in \mathbb{N} \setminus \{1\} \), we have \( \rho_6(2^k) = 0 \) for all \( k \).

Let \( n = 2^k \). Then, \( d = 2^i \), \( 0 \leq i \leq k \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 2^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[
f_6(2^k) = \sum_{k'=0}^{k_0} \rho_6(2^{2k'}) = \rho_6(1) + \rho_6(2^2) + \rho_6(2^4) + \cdots + \rho_6(2^{2k_0}) = 1 + 0 + \cdots + 0 = 1.
\]

Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = 2^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[
f_6(2^k) = \sum_{k'=1}^{k_0} \rho_6(2^{2k'-1}) = \rho_6(2) + \rho_6(2^3) + \cdots + \rho_6(2^{2k_0+1}) = 1 + 0 + \cdots + 0 = 1.
\]

Hence, our Claim 3.24 is proved.

**Claim 3.25.** If \( p \) in any odd prime and \( k \in \mathbb{N} \), then \( f_6(p^k) = \begin{cases} 
0 & \text{if } k \equiv 1 \pmod{2} \\
1 & \text{if } k \equiv 0 \pmod{2}
\end{cases} \)

Since \( \gcd\{(x^2+1), (x^2-1)\} = 1 \) in \( \mathbb{Z}_{p^k} \), we have \( \rho_3(p^k) = 0 \) for all \( k \in \mathbb{N} \). Let \( n = p^k \). Then, \( d = p^i \), \( i \leq k \).

Case 1: If \( k = 2k_0 \), then \( 2k_0 \leq 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = p^{2i-2k_0} \) where \( 0 \leq 2i - 2k_0 \leq 2k_0 \). Let \( k' = i - k_0 \). Then \( 0 \leq k' \leq k_0 \). Therefore,

\[
f_6(p^k) = \sum_{k'=0}^{k_0} \rho_3(p^{2k'}) = \rho_3(1) + \rho_3(p^2) + \rho_3(p^4) + \cdots + \rho_3(p^{2k_0}) = 1 + 0 + \cdots + 0 = 1.
\]
Case 2: If \( k = 2k_0 + 1 \), then \( 2k_0 < 2i \) since \( n \mid d^2 \). Then, \( \frac{d^2}{n} = p^{2i-2k_0-1} \) where \( 1 \leq 2i - 2k_0 - 1 \leq 2k_0 + 1 \). Let \( k' = i - k_0 \). Then \( 1 \leq k' \leq k_0 \). Therefore,

\[
f_6(p^k) = \sum_{k' = 1}^{k_0} \rho_3(p^{2k' - 1}) = \rho_3(p) + \rho_3(p^3) + \cdots + \rho_3(p^{2k_0+1}) = 0 + 0 + \cdots + 0 = 0.
\]

Hence, our Claim 3.25 is proved.

Let \( n = 2^{a_0}3^{k_0}p_1^{k_1}p_2^{k_2}\cdots p_{n_1}^{k_{n_1}} \) where \( p_i \) is any prime other than 2 for \( i \in \{0, 1, \ldots, n_1\} \). Then,

\[
f_6(n) = \begin{cases} 
0 & \text{if } k_i \equiv 1 \pmod{2} \text{ for some } i \in \{0, 1, 2, \ldots, n_1\} \\
1 & \text{otherwise}.
\end{cases}
\]

Let us calculate \( f_8(n) \) for \( n \in \mathbb{N} \).

**Claim 3.26.** \( f_8(1) = 1 \) and \( f_8(2) = 1 \).

If \( n = 1 \), then \( a = 1 \) and \( d = 1 \). Since \( b = 0 \) satisfies \( d \mid a + 2b \), \( f_8(1) = 1 \).

Case 1: When \( a = 1 \) and \( d = 2 \), \( \gcd(d, -2) = 2 \) and \( 2 \mid a \). So, the equation \( dk - 2b = a \) has no solution.

Case 2: When \( a = 2 \) and \( d = 1 \), \( \gcd(d, -2) = 1 \) and \( 1 \mid a \). So, the equation \( dk - 2b = a \) has 1 solution in \( \mathbb{Z}_d \). So, \( f_8(2) = 1 \).

**Claim 3.27.** \( f_8(2^k) = 2k - 1 \) for \( k \geq 2 \).

If \( n = 2^k \), then \( a = 2^i \), \( 0 \leq i \leq k \) and \( d = 2^{k-i} \). When \( a = 1 \) and \( d = 2^n \), \( \gcd(d, -2) = 2 \) and \( 2 \mid a \). So, the equation \( dk - 2b = a \) has no solution. If \( a = 2^i \), \( 1 \leq i \leq n-1 \), \( \gcd(d, -2) = 2 \) and \( 2 \nmid a \). So, the equation \( dk - 2b = a \) has 2 solutions for each \( i \). When \( a = 2^n \) and \( d = 1 \), \( \gcd(d, -2) = 1 \) and \( 1 \mid a \). So, the equation \( dk - 2b = a \) has 1 solution in \( \mathbb{Z}_d \). So, \( f_8(2^n) = 0 + 2 + 2 + \cdots + 1 = 2(k - 1) + 1 = 2k - 1 \).

**Claim 3.28.** \( f_8(p) = 1 \) and \( f_8(p^k) = k + 1 \) for \( k \geq 2 \).

When \( n = p \), we have two cases. Case 1: When \( a = 1 \) and \( d = p \), \( \gcd(d, -2) = 1 \) and \( 1 \nmid a \). So, the equation \( dk - 2b = a \) has 1 solution.

Case 2: When \( a = p \) and \( d = 1 \), \( \gcd(d, -2) = 1 \) and \( 1 \mid a \). So, the equation \( dk - 2b = a \) has 1 solution in \( \mathbb{Z}_d \). So, \( f_8(p) = 2 \).

If \( n = p^k \), then \( a = p^i \), \( 0 \leq i \leq k \) and \( d = p^{k-i} \). For any \( i \), \( a = p^i \), \( \gcd(d, -2) = 1 \) and \( 1 \nmid a \). So, the equation \( dk - 2b = a \) has 1 solution each. So, \( f_8(p^k) = 1 + 1 + \cdots + 1 = k + 1 \).

Let \( n = 2^{k_0}p_1^{k_1}p_2^{k_2}\cdots p_{n_1}^{k_{n_1}} \) where \( p_i \) is any odd prime for \( i \in \{0, 1, \ldots, n_1\} \). Then,

\[
f_8(n) = \begin{cases} 
\prod_{i=1}^{n_1} (k_i + 1) & \text{if } k_0 = 0 \\
(2k_0 - 1) \prod_{i=1}^{n_1} (k_i + 1) & \text{otherwise}.
\end{cases}
\]

Note that,

\[
f_3(n) = f_4(n) = f_7(n) = f_8(n) \forall n \in \mathbb{N}.
\]

Now we are going to find out exact formulas for number of 2-uniform maps.

Let \( X \) be a 2-orbital \( n \)-sheeted toroidal cover of \( X_0 \) which is obtained as an orbit space of \( E_1 \). Then, \( X = \frac{E_1}{K_1} \) where \( K_1 \leq H_1 \) with \( H_1 = \langle \gamma, \delta \rangle \) and \( K_1 = \langle w_1, w_2 \rangle \) as defined before Lemma 3.1. Since \( X \) is a 2-orbital map, \( V(X) \) forms 2-Aut(\( X \)) orbits. Using Result (1), we can say that \( V(E_1) \) forms 2-\( \mathrm{Nor}(K_1) \) orbits. So, we have to look for possible groups, \( G \leq \mathrm{Aut}(E_1) \) such that \( G \leq \mathrm{Nor}(K_1) \) and \( V(E_1) \) forms 2-\( G \) orbits. Now, the translations
\(\gamma, \delta \in \text{Nor}(K_1)\) since \(\gamma^{-1}w_1\gamma = w_1, \ \gamma^{-1}w_2\gamma = w_2, \ \delta^{-1}w_1\delta = w_1, \ \delta^{-1}w_2\delta = w_2 \in K_1\).

We know that conjugation of a translation by rotation is again a translation by the rotated vector. So, \(\tau^{-1}w_1\tau = \tau(w_1) = \tau(aA + bB) = a\tau(A) + b\tau(B) = a(-A) + b(-B) = -aA - bB = -(aA + bB) \in K_1.\) Similarly, \(\tau^{-1}w_2\tau \in K_1.\) Hence, \(\tau \in \text{Nor}(K_1).\) Let \(G_1 = \langle \sigma, \delta, \tau \rangle.\) Then, \(V(E_1)\) forms \(6G_1\) orbits. Observe that \(\sigma \in \text{Aut}(E_1).\) Then, from Claim 3.1, we can say that \(\sigma \in \text{Nor}(K_1)\) if and only if \(a \mid b, \ a \mid d\) and \(ad \mid (a^2 + ab + b^2).\) Let \(G_2 = \langle \gamma, \delta, \tau, \sigma \rangle.\) Then, \(V(E_1)\) forms \(2G_2\) orbits. Since \(E_1\) has no reflection in its automorphism group, the only possible group is \(G_2.\) So the isotropy group has order 6. Hence, here we are counting every map exactly once. Then, using Lemma 3.1, we can say that the total number of \(2\)-orbital \(n\) sheeted maps up to isomorphism is \(f_1(n).\)

Let \(X\) be an \(n\) sheeted \(2\)-orbital cover of \(X_0\) obtained as an orbit space of \(E_i\) where \(i = 2, 5, 7, 11, 14\) (see Fig. 1). Then, \(X = \frac{E_i}{K_i}\) where \(K_i \leq H_i\) with \(H_i = \langle \gamma, \delta \rangle\) and \(K_i = \langle w_1, w_2 \rangle\) as defined before Lemma 3.1. Since \(X\) is a \(2\)-orbital map, \(V(X)\) forms \(2\)-\text{Aut}(X) orbits. Using Result (1), we can say that \(V(E_i)\) forms \(2\)-\text{Nor}(K_i) orbits. Now \(V(E_i)\) forms 4 and 6 orbits for \(i = 2, 5, 7\) and \(i = 11, 14\) respectively when only \(\tau\) and translations present in \(\text{Nor}(K_i).\) If \(J(X)\) contains \(\tau\) and \(r_j\) for some \(j \in \{1, 2, \ldots, 6\}\) then \(V(E_i)\) has 3 \(\text{Nor}(K_i)\) orbits. If \(J(X)\) contains \(\sigma\) then \(V(E_i)\) has 2 \(\text{Nor}(K_i)\) orbits. And if \(J(X)\) is equals to isotropy group of the tiling then also it has 2 orbits. Hence number of \(n\)-sheeted \(2\)-uniform maps up to isomorphism there are \(\frac{f_1(n) - f_3(n) + f_5(n)}{2}.\)

Let \(X\) be an \(n\) sheeted \(2\)-orbital cover of \(X_0\) obtained as an orbit space of \(E_i\) where \(i = 3, 4, 8, 12, 13, 15\) (see Fig. 1). In these cases every map is \(2\)-orbital. So, it is enough to find total number of \(n\)-sheeted maps up to isomorphism. Isotropy group of these tilings is \(\mathbb{Z}_2 \times \mathbb{Z}_2\). So, possible orders \(J(X)\) will be 2 and 40. If it is 4 then we count it once in the collection of distinct maps and if it is 2 then we count it twice. Let \(C(n)\) and \(D(n)\) denoted number of maps up to isomorphism having order of isotropy group 2 and 4 respectively. Then \(2C(n) + D(n) = \vartheta(n)\). Now, \(D(n)\) is also number of maps having \(r_1\) in its automorphism group. Hence, \(D(n) = f_7(n)\). Thus, number of \(2\)-uniform maps up to isomorphism if given by \(C(n) + D(n) = \frac{\vartheta(n) + f_5(n)}{2} = \frac{\vartheta(n) + f_3(n)}{2}.\)

Let \(X\) be an \(n\) sheeted \(2\)-orbital cover of \(X_0\) obtained as an orbit space of \(E_i\) where \(i = 9, 10\) (see Fig. 1). Then, \(X = \frac{E_i}{K_i}\) where \(K_i \leq H_i\) with \(H_i = \langle \gamma, \delta \rangle\) and \(K_i = \langle w_1, w_2 \rangle\) as defined before Lemma 3.1. Since \(X\) is a \(2\)-orbital map, \(V(X)\) forms \(2\)-\text{Aut}(X) orbits. Using Result (1), we can say that \(V(E_i)\) forms \(2\)-\text{Nor}(K_i) orbits. In these two tilings we have two additional symmetries. We denote them by \(\mathcal{G}_1\) and \(\mathcal{G}_2\) and define them as follows. \(\mathcal{G}_1\) is the function obtained by rotating the vertical stack of two squares by \(90^\circ\) followed by a translation of that stack to its adjacent horizontal array of two squares in \(E_9.\) \(\mathcal{G}_2\) is the function obtained by reflecting \(X_{10}\) about some line and then translating it by distance \(|A|/2 \) or \(|B|/2\) along \(O\bar{A}\) or \(O\bar{B}\) respectively. The maps will be \(2\)-uniform if and only if \(\mathcal{G}_1\) and \(\mathcal{G}_2\) present in the corresponding automorphism group. Similarly as in Lemma 3.13 \(\mathcal{G}_1\) and \(\mathcal{G}_2\) will present in the map represented by \(\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}\) if and only if \(d \mid 2b\) and number of such distinct \(n\)-sheeted maps will be given by \(g(n)\). Since in these tilings there does not exists any other symmetries like rotation or reflection about lines so this \(g(n)\) is giving number of maps up to isomorphism. Thus number of \(n\)-sheeted \(2\)-uniform maps up to isomorphism is given by \(g(n)\).

Let \(X\) be an \(n\) sheeted \(2\)-orbital cover of \(X_0\) obtained as an orbit space of \(E_i, \ i = 18, 19\) (see Fig. 1). Then, \(X = \frac{E_i}{K_i}\) where \(K_i \leq H_i\) with \(H_i = \langle \gamma, \delta \rangle\) and \(K_i = \langle w_1, w_2 \rangle\) as defined before Lemma 3.1. Since \(X\) is a \(2\)-orbital map, \(V(X)\) forms \(2\)-\text{Aut}(X) orbits. Using Result (1), we can say that \(V(E_i)\) forms \(2\)-\text{Nor}(K_i) orbits. It can be observed that these type of
maps will be 2-uniform if and only if \( r_1' \) or \( r_2' \) present in the automorphism group. We are counting them exactly once since their isotropy group is same as isotropy group of the tiling. Thus number of \( n \)-sheeted 2-uniform maps up to isomorphism is given by \( f_2(n) = f_3(n) \).

Let \( X \) be an \( n \) sheeted 2- orbital cover of \( X_0 \) obtained as an orbit space of \( E_i \). \( i = 6, 17, 20 \) (see Fig. 1). Then, \( X = \frac{E_{16}}{K_1} \) where \( K_1 \) is as above. Similarly, we can say that \( V(E_i) \) forms 2-\( \text{Nor}(K_i) \) orbits. For these types of maps one can observe that \( V(E_i) \) has 2-\( \text{Nor}(K_i) \) orbits if \( \mathcal{F}(X) = \mathcal{F}(E_i) \). Hence number of 2-uniform maps up to isomorphism is \( f_2(n) \).

Let \( X \) be an \( n \) sheeted 2- orbital cover of \( X_0 \) obtained as an orbit space of \( E_{16} \) (see Fig. 1). Then, \( X = \frac{E_{16}}{K_1} \) where \( K_1 \leq H_{16} \) with \( H_{16} = \langle \gamma, \delta \rangle \) and \( K_{16} = \langle w_1, w_2 \rangle \) as defined before Lemma 3.1. Since \( X \) is a 2- orbital map, \( V(X) \) forms 2-\( \text{Aut}(X) \) orbits. Using Result (1), we can say that \( V(E_{16}) \) forms 2-\( \text{Nor}(K_{16}) \) orbits. Now \( V(E_i) \) forms 4-\( \text{Nor}(K_i) \) orbits when only \( \tau \) and translations present in \( \text{Nor}(K_i) \). If \( \mathcal{F}(X) \) contains \( \tau \) and \( r_j \) for some \( j \in \{1, 2, \ldots, 6\} \) then \( V(E_i) \) has 2 \( \text{Nor}(K_i) \) orbits. If \( \mathcal{F}(X) \) contains \( \sigma \) then \( V(E_i) \) has 1 \( \text{Nor}(K_i) \) orbit. And if \( \mathcal{F}(X) \) equals to isotropy group of the tiling then also it has 2 orbits. Hence number of \( n \)-sheeted 2-uniform maps up to isomorphism there are \( f_2(n) - f_6(n) + f_6(n) = f_2(n) + f_6(n) \).

Let \( X \) be an \( n \) sheeted 2- orbital cover of \( X_0 \) obtained as an orbit space of \( E_{23} \). Then, \( X = \frac{E_{23}}{K_3} \) where \( K_3 \leq H_{23} \) with \( H_{23} = \langle \gamma, \delta \rangle \) and \( K_{23} = \langle w_1, w_2 \rangle \) as defined before Lemma 3.1. Since \( X \) is a 2- orbital map, \( V(X) \) forms 2-\( \text{Aut}(X) \) orbits. Using Result (1), we can say that \( V(E_{23}) \) forms 2-\( \text{Nor}(K_{23}) \) orbits. Here, a map \( Y \) will be 1 orbital if and only if \( \psi \) or glide reflection present in \( \text{Aut}(Y) \) So whenever neither \( \psi \) nor glide present \( \text{Aut}(Y) \), \( Y \) will be 2- orbital. Now, if both \( \psi \) and glide present in \( \text{Aut}(X) \) then we count it once in the collection of distinct \( n \)-sheeted maps. If any one of \( \psi \) or glide present in \( \text{Aut}(X) \) we count it twice. If none of them present then whenever for a map \( Y \) after applying \( \psi \) and glide we get equal maps we count it twice and otherwise we count it 4 times. Let \( A(n) \) denotes number of \( n \)-sheeted maps up to isomorphism on which after applying \( \psi \) and glide we get equal maps, and \( B(n) \) denotes the number of maps up to isomorphism for which we get different maps. In both of these countings of \( A(n) \) and \( B(n) \) we considering those maps whose automorphism group does not contains \( \psi \) and glide. Then we have,

\[
2A(n) + 4B(n) = \vartheta(n) - f_2(n) - g(n) + h(n) \quad (\ast)
\]

By Lemma 3.15 we have \( 2A(n) = \alpha(n) \implies A(n) = \frac{\alpha(n)}{2} \). Putting the value of \( A(n) \) in \( \ast \) we get, \( B(n) = \frac{1}{4}[\vartheta(n) - f_2(n) - g(n) + h(n) - \alpha(n)] \). Thus number of 2-uniform maps up to isomorphism is given by,

\[
A(n) + B(n) = \frac{1}{4}[\vartheta(n) - g(n) - f_2(n) + h(n) + \alpha(n)] = \frac{1}{4}[\vartheta(n) - g(n) - f_2(n) + f_3(n)].
\]

Let \( X \) be an \( n \) sheeted 2- orbital cover of \( X_0 \) obtained as an orbit space of \( E_{25} \). Then, \( X = \frac{E_{25}}{K_{25}} \) where \( K_{25} \leq H_{25} \) with \( H_{25} = \langle \gamma, \delta \rangle \) and \( K_{25} = \langle w_1, w_2 \rangle \) as defined before Lemma
3.1. Since \(X\) is a 2-orbital map, \(V(X)\) forms \(2-\text{Aut}(X)\) orbits. Using Result (1), we can say that \(V(E_{25})\) forms \(2-\text{Nor}(K_{25})\) orbits. It can be observed that from the figure that \(\gamma, \delta, \tau, \sigma \in \text{Aut}(E_{25})\). Then, following the calculations done for the tiling \(E_1\), we can say that \(G_1 = \langle \gamma, \delta, \tau, \sigma \rangle \leq \text{Nor}(K_{25})\). It is easy to see that \(V(E_{25})\) forms \(3-G_1\) orbits. Let \(G_2 = \langle \gamma, \delta, \tau, \sigma \rangle\). If \(G_{25} \leq \text{Nor}(K_{25})\) and we observe that \(V(E_{25})\) forms \(1-G_2\) orbit. Since there are no other symmetries in \(\text{Aut}(E_{25})\), the total number of 2-orbital \(n\)-sheeted maps is given by 0.

Let \(X\) be an \(n\) sheeted 2-orbital cover of \(X_0\) obtained as an orbit space of \(E_{26}\). Then, \(X = \frac{E_{26}}{K_{26}}\) where \(K_{26} \leq H_{26}\) with \(H_{26} = \langle \gamma, \delta \rangle\) and \(K_{26} = \langle w_1, w_2 \rangle\) as defined before Lemma 3.1. Since \(X\) is a 2-orbital map, \(V(X)\) forms \(2-\text{Aut}(X)\) orbits. Using Result (1), we can say that \(V(E_{26})\) forms \(2-\text{Nor}(K_{26})\) orbits. Now \(V(E_i)\) forms \(6-\text{Nor}(K_i)\) orbits when only \(\tau\) and translations present in \(\text{Nor}(K_i)\). If \(\mathcal{J}(X)\) contains \(\tau\) and \(r_j\) for some \(j \in \{1, 2, \ldots, 6\}\) then \(V(E_i)\) has \(3-\text{Nor}(K_i)\) orbits. If \(\mathcal{J}(X)\) contains \(\sigma\) then \(V(E_i)\) has \(2-\text{Nor}(K_i)\) orbits. And if \(\mathcal{J}(X)\) is equals to isotropy group of the tiling then also it has 1 orbits. Hence number of \(n\)-sheeted 2-uniform maps up to isomorphism there are \(f_1(n) - f_5(n)\).

Let \(X\) be an \(n\) sheeted 2-orbital cover of \(X_0\) obtained as an orbit space of \(E_{27}\). Then, \(X = \frac{E_{27}}{K_{27}}\) where \(K_{27} \leq H_{27}\) with \(H_{27} = \langle \gamma, \delta \rangle\) and \(K_{27} = \langle w_1, w_2 \rangle\) as defined before Lemma 3.1. Since \(X\) is a 2-orbital map, \(V(X)\) forms \(2-\text{Aut}(X)\) orbits. Using Result (1), we can say that \(V(E_{27})\) forms \(2-\text{Nor}(K_{27})\) orbits. Now \(V(E_i)\) forms \(2-\text{Nor}(K_i)\) orbits when only \(\tau\) and translations present in \(\text{Nor}(K_i)\). If \(\mathcal{J}(X)\) contains \(\tau\) and \(r'_1\) then \(V(E_i)\) has \(2-\text{Nor}(K_i)\) orbits but instead of \(r'_1\) if it contains \(r'_2\) then it will become one orbital. If \(\mathcal{J}(X)\) contains \(\psi\) then \(V(E_i)\) has \(1-\text{Nor}(K_i)\) orbits. And if \(\mathcal{J}(X)\) is equals to isotropy group of the tiling then also it has 1 orbit. Hence number of \(n\)-sheeted 2-uniform maps up to isomorphism there are \(f_1(n) - f_5(n)\) + \(\frac{1}{2} [\vartheta(n) - f_7(n) - f_4(n) - f_2(n) + 2f_6(n)] = \frac{1}{2} [\vartheta(n) - f_2(n)]\) since, \(f_7(n) = f_4(n)\).

Now, to give vertex number as inputs in the above functions note that number of vertex of a map is equals to number of sheets of the covering times number of vertex of the initial map. Observe that the initial map has following number of vertices, for \((i, v_0) = (1, 12), (2, 8), (3, 4), (4, 3), (5, 7), (6, 14), (7, 7), (8, 4), (9, 12), (10, 8), (11, 12), (12, 3), (13, 4), (14, 12), (15, 3), (16, 8), (17, 18), (18, 5), (19, 5), (20, 18), (21, 6), (22, 6), (23, 4), (24, 3), (25, 6), (26, 12), (27, 4)\) where \((i, v_0)\) denotes the initial map of type same as \(E_i\) has \(v_0\) many vertices. Using the above observation we get the formulas stated in Theorem 1.5. This completes the proof.

\(\square\)

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