Embedding truncated skew polynomial rings into matrix rings and embedding of a ring into $2 \times 2$ supermatrices

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Abstract. For an endomorphism $\sigma : R \to R$ with $\sigma^2 = 1$ we prove that the truncated polynomial ring (algebra) $R[w, \sigma]/(w^t)$ embeds into $M_2(R[z]/(z^t))$. For an involution $\sigma$ we exhibit an embedding $R \to M^\sigma_{2, 1}(R)$, where $M^\sigma_{2, 1}(R)$ is the algebra of the so called $(\sigma, 2, 1)$ supermatrices.

1. INTRODUCTION

The main inspiration of the present work are the various embedding results in [SvW] and [MMSvW]. A certain embedding of the two-generated Grassmann algebra $E^{(2)}$ into a $2 \times 2$ matrix algebra over a commutative ring leads to a Cayley-Hamilton identity of degree $2n$ for any $n \times n$ matrix over $E^{(2)}$ (see [SvW]). For a field $K$ (of characteristic zero) let $E^{(m)} = K \langle v_1, \ldots, v_m \mid v_iv_j + v_jv_i = 0 \text{ for all } 1 \leq i \leq j \leq m \rangle$ denote the $m$-generated Grassmann algebra and $M_n(K)$ denote the full $n \times n$ matrix ring ($K$-algebra) over a ring ($K$-algebra) $R$ with identity $I_n \in M_n(R)$. In [MMSvW] a so called constant trace representation ($K$-embedding)

$$\varepsilon^{(m)} : E^{(m)} \to M_{2^m - 1}(K[z_1, \ldots, z_m]/(z_1^2, \ldots, z_m^2)),$$

is presented, where the ideal $(z_1^2, \ldots, z_m^2)$ of the commutative polynomial algebra $K[z_1, \ldots, z_m]$ is generated by the monomials $z_1^2, \ldots, z_m^2$. One of the remarkable consequences of this embedding is a Cayley-Hamilton identity (with coefficients in $K$) of degree $2^m - 1 - n$ for $n \times n$ matrices over $E^{(m)}$.

The induction step in the construction of the above $\varepsilon^{(m)}$ is based on the observation that $E^{(m)}[w, \tau]/(w^2) \cong E^{(m+1)}$ as $K$-algebras, where $\tau : E^{(m)} \to E^{(m)}$ is the natural involution defined by the well known $\mathbb{Z}_2$-grading $E^{(m)} = E^{(m)}_0 \oplus E^{(m)}_1$ and $(w^2)$ is the ideal of the skew polynomial algebra $E^{(m)}[w, \tau]$ generated by $w^2$. The main ingredient of the mentioned induction is a ”Fundamental Embedding” $\mu : R[w, \sigma]/(w^2) \to M_2(R[z]/(z^2))$, which is defined for an arbitrary involution

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\[\sigma : R \rightarrow R\]. In Section 2 we give a far reaching generalization of this "Fundamental Embedding". We note that the idea of considering the truncated polynomial ring \((K\text{-algebra}) R[w, \sigma] / (w^t)\) comes from \([SSz]\).

The use of an endomorphism (involution) \(\sigma : R \rightarrow R\) enables us to give a generalization of the concept of a supermatrix. Supermatrices over the infinitely generated Grassmann algebra play an important role in Kemer's classification of T-prime T-ideals (see \([K]\)). Section 3 contains an embedding of \(R\) into a \(2 \times 2\) supermatrix algebra (over \(R\)) determined by \(\sigma\).

2. TRUNCATED SKEW POLYNOMIAL RINGS AND EMBEDDINGS

For a ring \((K\text{-algebra})\) endomorphism \(\sigma : R \rightarrow R\) let us consider the skew polynomial ring \((K\text{-algebra}) R[w, \sigma]\) in the skew indeterminate \(w\). The elements of \(R[w, \sigma]\) are left polynomials of the form \(f(w) = r_0 + r_1 w + \cdots + r_k w^k\) with \(r_0, r_1, \ldots, r_k \in R\). Besides the obvious addition, we have the following multiplication rule in \(R[w, \sigma]\): \(\sigma(w = \sigma(r)w\) for all \(r \in R\) and

\[(r_0 + r_1 w + \cdots + r_k w^k)(s_0 + s_1 w + \cdots + s_l w^l) = u_0 + u_1 w + \cdots + u_{k+l} w^{k+l},\]

where

\[u_m = \sum_{i+j=m, i \geq 0, j \geq 0} r_i \sigma^j(s_j) \text{ (for } 0 \leq m \leq k + l)\].

If \(\sigma^t = 1\) (such a \(\sigma\) is an automorphism), then \(w^t\) is a central element of \(R[w, \sigma]\): we have \(\sigma^t(r) = r\) and \(w^t r = w^{t-1} \sigma(r) w = \cdots = \sigma^t(r) w^t = r w^t\) for all \(r \in R\), moreover \(w^t\) commutes with the powers of \(w\). Thus the ideal \((w^t) < R[w, \sigma]\) generated by \(w^t\) can be written as \((w^t) = R[w, \sigma]/w^t R[w, \sigma]\). For any element \(f(w) + (w^t)\) of the truncated polynomial ring \((K\text{-algebra}) R[w, \sigma]/(w^t)\) there exists a unique sequence of coefficients \(r_0, r_1, \ldots, r_{t-1} \in R\) such that

\[r_0 + r_1 w + \cdots + r_{t-1} w^{t-1} + (w^t) = f(w) + (w^t)\].

Hence the elements of \(R[w, \sigma]/(w^t)\) can be represented by left polynomials of degree less or equal than \(t - 1\).

The following is called "Fundamental Embedding" in \([MMSzvW]\).

2.1. Theorem \([MMSzvW]\). For an involution \(\sigma : R \rightarrow R\), putting

\[\mu(r_0 + r_1 w + (w^2)) = \begin{bmatrix} r_0 + (z^2) & r_1 z + (z^2) \\ \sigma(r_1) z + (z^2) & \sigma(r_0) + (z^2) \end{bmatrix}\]

(with \(r_0, r_1 \in R\)) gives an embedding \(\mu : R[w, \sigma]/(w^2) \rightarrow M_2(R[z]/(z^2))\).

Now we present the following generalization of Theorem 2.1 (already announced in \([MMSzvW]\)).

2.2. Theorem. For an endomorphism \(\sigma : R \rightarrow R\) with \(\sigma^t = 1\), putting

\[\mu(r_0 + r_1 w + \cdots + r_{t-1} w^{t-1} + (w^t)) = [\sigma^{t-1}(r_{j-i}) z^{j-i} + (z^t)]_{t \times t}\]

gives an embedding \(\mu : R[w, \sigma]/(w^t) \rightarrow M_t(\sigma(R[z]/(z^t)))\), where the difference \(j - i \in \{0, 1, \ldots, t - 1\}\) is taken modulo \(t\), and the element of the factor algebra \(\sigma(R[z]/(z^t))\) in the \((i, j)\) position of the \(t \times t\) matrix \([\sigma^{t-1}(r_{j-i}) z^{j-i} + (z^t)]_{t \times t}\) is \(\sigma^{t-1}(r_{j-i}) z^{j-i} + (z^t)\).

The trace of \([\sigma^{t-1}(r_{j-i}) z^{j-i} + (z^t)]_{t \times t}\) is in the fixed ring \(R^T = \{r \in R | \sigma(r) = r\}\):

\[\text{tr}(\mu(r_0 + r_1 w + \cdots + r_{t-1} w^{t-1} + (w^t))) = \text{tr}([\sigma^{t-1}(r_{j-i}) z^{j-i} + (z^t)]_{t \times t}) = \]
\[ r_0 + \sigma(r_0) + \cdots + \sigma^{t-1}(r_0) + (z^t) \in R^\sigma + (z^t). \]

**Proof.** We only have to prove the multiplicative property of \( \mu \).

In order to avoid confusion, for \( i, j \in \{1, \ldots, t\} \) let \( j \circ i \) denote the modulo \( t \) difference:

\[ j \circ i = \begin{cases} 
    j - i & \text{if } i \leq j \\
    (j - i) + t & \text{if } j \leq i - 1 
\end{cases} . \]

The \((p, q)\) entry in the product of the \( t \times t \) matrices

\[ \mu(r_0 + r_1 w + \cdots + r_{t-1} w^{t-1} + (w^t)) = \left[ \sigma^{i-1}(r_{j \circ i}) z^{j \circ i} + (z^t) \right]_{t \times t} \]

and

\[ \mu(s_0 + s_1 w + \cdots + s_{t-1} w^{t-1} + (w^t)) = \left[ \sigma^{j-1}(s_{q \circ j}) z^{q \circ j} + (z^t) \right]_{t \times t} \]

is

\[ a_{p, q} = \sum_{j=1}^{t} (\sigma^{p-1}(r_{j \circ p}) z^{j \circ p} + (z^t)) \left( \sigma^{j-1}(s_{q \circ j}) z^{q \circ j} + (z^t) \right) = \]

\[ \left( \sum_{j=1}^{t} \sigma^{p-1}(r_{j \circ p}) \sigma^{j-1}(s_{q \circ j}) \right) z^{(j \circ p) + (q \circ j)} + (z^t). \]

Since \( \sigma^t = 1 \), we have \( \sigma^{p-1}(\sigma^{j \circ p}(s_{q \circ j})) = \sigma^{j-1}(s_{q \circ j}) \). It is straightforward to check that if \((j \circ p) + (q \circ j) \leq t - 1 \) holds, then

\[ (j \circ p) + (q \circ j) = q \circ p \text{ and } 0 \leq j \circ p \leq q \circ p. \]

In view of the above observations and using \( i = j \circ p \), we can see that

\[ a_{p, q} = \sigma^{p-1} \left( \sum_{j=1}^{t} r_{j \circ p} \sigma^{j \circ p}(s_{q \circ j}) \right) z^{(j \circ p) + (q \circ j)} + (z^t) = \]

\[ \sigma^{p-1} \left( \sum_{j=1}^{t} r_{j \circ p} \sigma^{j-1}(s_{q \circ (j-1)}) \right) z^{q \circ p} + (z^t) = \sigma^{p-1}(u_{q \circ p}) z^{q \circ p} + (z^t) \]

is the \((p, q)\) entry of \( \mu(u_0 + u_1 w + \cdots + u_{t-1} w^{t-1} + (w^t)) \), where

\[ u_0 + u_1 w + \cdots + u_{t-1} w^{t-1} + (w^t) = (r_0 + r_1 w + \cdots + r_{t-1} w^{t-1} + (w^t))(s_0 + s_1 w + \cdots + s_{t-1} w^{t-1} + (w^t)). \]

holds in \( R[w, \sigma]/(w^t) \).

Since

\[ \sigma(r_0 + \sigma(r_0) + \cdots + \sigma^{t-1}(r_0)) = \sigma(r_0) + \sigma^2(r_0) + \cdots + \sigma^{t-1}(r_0) + \sigma^t(r_0) \]

and \( \sigma^t(r_0) = r_0 \), the trace of \( \left[ \sigma^{i-1}(r_{j \circ i}) z^{j \circ i} + (z^t) \right]_{t \times t} \) is in \( R^\sigma + (z^t) \). \( \square \)

Now consider the free associative \( K \)-algebra \( K \langle x_1, \ldots, x_m, \ldots \rangle \) generated by the (non-commuting) indeterminates \( x_1, \ldots, x_m, \ldots \) and let

\[ S_m(x_1, \ldots, x_m) = \sum_{\pi \in \text{Sym}(m)} \text{sgn}(\pi)x_{\pi(1)} \cdots x_{\pi(m)} \]

be the standard polynomial in \( K \langle x_1, \ldots, x_m, \ldots \rangle \). In the following corollaries we keep the notations and the conditions of Theorem 2.2.

**2.3. Corollary.** If \( R \) is commutative and \( n \geq 1 \) is an integer, then \( S_{2n} = 0 \) is an identity on \( M_n(R[w, \sigma]/(w^t)) \). In particular \( S_{2t} = 0 \) is an identity on \( R[w, \sigma]/(w^t) \).
Proof. The natural extension
\[ \mu_n : M_n(R[w,\sigma]/(w^t)) \to M_n(M_l(R[z]/(z^t))) \cong M_{tn}(R[z]/(z^t)) \]
of \( \mu \) is an embedding. Since \( R[z]/(z^t) \) is also commutative and \( S_{2tn} = 0 \) is an identity on \( M_{tn}(R[z]/(z^t)) \) by the Amitsur-Levitzki theorem (see [Dr, DrF]), the proof is complete. \( \square \)

2.4. Corollary. If \( S_m = 0 \) is an identity on \( R \) and \( n \geq 1 \) is an integer, then \( S_{(m-1)2n^2+1} = 0 \) is an identity on \( M_n(R[w,\sigma]/(w^t)) \). In particular \( S_{(m-1)2n^2+1} = 0 \) is an identity on \( R[w,\sigma]/(w^t) \).

Proof. Since \( S_m = 0 \) is also an identity on \( R[z]/(z^t) \), using \( \mu_n \) and Theorem 5.5 of Domokos [Do] completes the proof. \( \square \)

2.5. Corollary. If \( R \) is a PI-algebra and \( n \geq 1 \) is an integer, then \( M_n(R[w,\sigma]/(w^t)) \) is also a PI-algebra. In particular \( R[w,\sigma]/(w^t) \) is a PI-algebra.

Proof. Since \( R[z]/(z^t) \) is also PI, using \( \mu_n \) and the well known fact that full matrix algebras over a PI-algebra are PI (a special case of Regev’s tensor product theorem), the proof is complete. \( \square \)

2.6. Theorem. If \( R \) is commutative and \( \sigma : R \to R \) is an endomorphism, then the fixed ring (algebra) \( R^\sigma = \{ r \in R \mid \sigma(r) = r \} \) of \( \sigma \) is a central subring (algebra) of \( R[w,\sigma] \) and \( R^\sigma + (z^t) \subseteq Z(R[w,\sigma]/(w^t)) \).

If \( \sigma^t = 1 \) and \( A \in M_n(R[w,\sigma]/(w^t)) \) is an \( n \times n \) matrix, then \( A \) satisfies a “Cayley-Hamilton” identity of the form
\[ A^n + c_1 A^{n-1} + \cdots + c_{tn-1} A + c_{tn} I_n = 0, \]
where \( c_i \in R^\sigma = \{ r \in R \mid \sigma(r) = r \}, 1 \leq i \leq tn \). In particular \( R[w,\sigma]/(w^t) \) is integral over \( R^\sigma \) of degree \( t \).

Proof. The containments \( R^\sigma \subseteq Z(R[w,\sigma]) \) and \( R^\sigma + (z^t) \subseteq Z(R[w,\sigma]/(w^t)) \) are clear. In the rest of the proof we follow the steps of the proof of Theorem 2.1 in [MMSzvW]. Let \( A = [a_{i,j}] \) and take \( \mu_n \) from the proof of Corollary 2.3. The trace of the \( tn \times tn \) matrix \( B = \mu_n(A) \in M_{tn}(R[z]/(z^t)) \) is the sum of the traces of the diagonal \( t \times t \) blocks:
\[ \text{tr}(B) = \sum_{i=1}^n \text{tr}(\mu(a_{i,i})). \]

Theorem 2.2 ensures that \( \text{tr}(\mu(a_{i,i})) \in R^\sigma + (z^t) \) for each \( 1 \leq i \leq n \). For the sake of simplicity we can take \( \text{tr}(\mu(a_{i,i})) \in R^\sigma \). It follows that \( \text{tr}(B) \in R^\sigma \). The coefficients of the characteristic polynomial
\[ \det(xI - B) = c_0 x^n + c_1 x^{n-1} + \cdots + c_{tn-1} x + c_{tn} \in (R[z]/(z^t))[x] \]
of \( B \) determined by the following recursion: \( c_0 = 1 \) and
\[ c_k = -\frac{1}{k} \left( c_{k-1} \text{tr}(B) + c_{k-2} \text{tr}(B^2) + \cdots + c_1 \text{tr}(B^{k-1}) + c_0 \text{tr}(B^k) \right) \]
for \( 1 \leq k \leq tn \) (Newton formulae). In view of
\[ \text{tr}(B^k) = \text{tr}((\mu_n(A))^k) = \text{tr}(\mu_n(A^k)) \in R^\sigma, \]
we deduce that \( c_i \in R^a \) for each \( 0 \leq i \leq tn \). Thus \( \det(xI - B) \in R^a[x] \) and the Cayley-Hamilton identity for \( B \) is of the form
\[
B^{tn} + c_1B^{tn-1} + \cdots + c_{tn-1}B + c_{tn}I_n = 0.
\]
Notice that for \( r \in R \) and \( c \in R^a \) we have \( c\sigma^{i-1}(r) = \sigma^{i-1}(cr) \) and \( c\mu_n(A^k) = \mu_n(cA^k) \) follows from
\[
\mu(cr_0 + cr_1w + \cdots + cr_{t-1}w^{t-1} + (w^t)) = c\mu(r_0 + r_1w + \cdots + r_{t-1}w^{t-1} + (w^t)).
\]
Thus
\[
(\mu_n(A))^{tn} + c_1(\mu_n(A))^{tn-1} + \cdots + c_{tn-1}\mu_n(A) + c_{tn}I_n = 0
\]
holds in \( M_{tn}(R[z]/(z^t)) \) and the injectivity of \( \mu_n \) gives the desired identity. \( \square \)

### 3. SUPERMATRIX ALGEBRAS DETERMINED BY INVOLUTIONS

For an arbitrary endomorphism \( \sigma : R \rightarrow R \) and for the integers \( 1 \leq k \leq n \) a matrix \( A \in M_n(R) \) is called a \( (\sigma, n, k) \)-supermatrix if \( A \) is of the shape
\[
A = \begin{bmatrix}
  A_{1,1} & A_{1,2} \\
  A_{2,1} & A_{2,2}
\end{bmatrix},
\]
where \( A_{1,1} \) is a \( k \times k \) and \( A_{2,2} \) is an \( (n-k) \times (n-k) \) square block, while \( A_{1,2} \) is a \( k \times (n-k) \) and \( A_{2,1} \) is an \( (n-k) \times k \) rectangular block such that \( \sigma(u) = u \) for each entry \( u \) of \( A_{1,1} \) and \( A_{2,2} \) and \( \sigma(u) = -u \) for each entry \( u \) of \( A_{1,2} \) and \( A_{2,1} \). Let \( M_{n,k}^\sigma(R) \) denote the set of \( (\sigma, n, k) \)-supermatrices. If \( R = E \) is the (infinitely generated) Grassmann algebra and \( \tau(g_0 + g_1) = g_0 - g_1 \) is the natural \( E \rightarrow E \) involution, then \( M_{n,k}^\sigma(E) \) is the classical algebra of \( (n,k) \)-supermatrices (see [K]).

#### 3.1. Proposition

The set \( M_{n,k}^\sigma(R) \) is a subring (subalgebra) of \( M_n(R) \).

**Proof.** Straightforward verification. \( \square \)

#### 3.2. Theorem

Let \( \frac{1}{2} \in R \) and \( \sigma : R \rightarrow R \) be an arbitrary endomorphism. For \( r \in R \) the definition
\[
\Theta(r) = \frac{1}{2} \begin{bmatrix}
  r + \sigma(r) & r - \sigma(r) \\
  r - \sigma(r) & r + \sigma(r)
\end{bmatrix}
\]
gives an embedding \( \Theta : R \rightarrow M_2(R) \). If \( \sigma \) is an involution \( (\sigma^2 = 1) \), then \( \Theta \) is an \( R \rightarrow M_{2,1}^\sigma(R) \) embedding.

**Proof.** We give the details of the straightforward proof.
The additive property of \( \Theta \) is clear. In order to prove the multiplicative property of \( \Theta \) take \( r, s \in R \) and compute the product of the \( 2 \times 2 \) matrices \( \Theta(r) \) and \( \Theta(s) \):
\[
\Theta(r) \cdot \Theta(s) = \frac{1}{4} \begin{bmatrix}
  r + \sigma(r) & r - \sigma(r) \\
  r - \sigma(r) & r + \sigma(r)
\end{bmatrix} \begin{bmatrix}
  s + \sigma(s) & s - \sigma(s) \\
  s - \sigma(s) & s + \sigma(s)
\end{bmatrix} =
\frac{1}{4} \begin{bmatrix}
  (r + \sigma(r))(s + \sigma(s)) + (r - \sigma(r))(s - \sigma(s)) & (r + \sigma(r))(s - \sigma(s)) + (r - \sigma(r))(s + \sigma(s)) \\
  (r - \sigma(r))(s + \sigma(s)) + (r + \sigma(r))(s - \sigma(s)) & (r - \sigma(r))(s - \sigma(s)) + (r + \sigma(r))(s + \sigma(s))
\end{bmatrix} =
\frac{1}{4} \begin{bmatrix}
  2rs + 2\sigma(r)s(s) & 2rs - 2\sigma(r)s(s) \\
  2rs - 2\sigma(r)s(s) & 2rs + 2\sigma(r)s(s)
\end{bmatrix} = \Theta(rs).
\]
The injectivity of \( \Theta \) follows from the fact that
\( r + \sigma(r) = s + \sigma(s) \) and \( r - \sigma(r) = s - \sigma(s) \)
imply
\[ 2r = (r + \sigma(r)) + (r - \sigma(r)) = (s + \sigma(s)) + (s - \sigma(s)) = 2s. \]
If \( \sigma \) is an involution, then
\[ \sigma(r + \sigma(r)) = \sigma(r) + \sigma^2(r) = \sigma(r) + r \quad \text{and} \quad \sigma(r - \sigma(r)) = \sigma(r) - \sigma^2(r) = \sigma(r) - r \]
ensure that \( \Theta(r) \in M_{2,1}^\sigma(R). \)

**3.3. Remark.** Taking \( R = M_n(E) \) and the natural extension \( \sigma = \tau_n \), we obtain the well-known embedding
\[ \Theta : M_n(E) \longrightarrow M_{2n}^{\tau_n^2}(M_n(E)) \cong M_{2n,n}^2(E). \]

**REFERENCES**

[Do] M. Domokos, *Eulerian polynomial identities and algebras satisfying a standard identity*, Journal of Algebra 169(3) (1994), 913-928.
[Dr] V. Drensky, *Free Algebras and PI-Algebras*, Springer-Verlag, 2000.
[DrF] V. Drensky and E. Formanek, *Polynomial Identity Rings*, Birkhäuser-Verlag, 2004.
[K] A. R. Kemer, *Ideals of Identities of Associative Algebras*, Translations of Math. Monographs, Vol. 87 (1991), AMS, Providence, Rhode Island.
[MMSzvW] L. Márki, J. Meyer, J. Szigeti and L. van Wyk, *Matrix representations of finitely generated Grassmann algebras and some consequences*, arXiv:1307.0292
[SSz] S. Sehgal and J. Szigeti, *Matrices over centrally \( \mathbb{Z}_2 \)-graded rings*, Beiträge zur Algebra und Geometrie (Berlin) 43(2) (2002), 399-406.

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