Brane cosmology with a van der Waals equation of state

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Abstract

The evolution of a Universe confined onto a 3-brane embedded in a five-dimensional space-time is investigated where the cosmological fluid on the brane is modeled by the van der Waals equation of state. It is shown that the Universe on the brane evolves in such a manner that three distinct periods concerning its acceleration field are attained: (a) an initial accelerated epoch where the van der Waals fluid behaves like a scalar field with a negative pressure; (b) a past decelerated period which has two contributions, one of them is related to the van der Waals fluid which behaves like a matter field with a positive pressure, whereas the other contribution comes from a term of the Friedmann equation on the brane which is inversely proportional to the scale factor to the fourth power and can be interpreted as a radiation field, and (c) a present accelerated phase due to a cosmological constant on the brane.

Key words: brane cosmology, van der Waals fluid, acceleration field

According to the cosmological observations one can distinguish three distinct periods for the Universe that are related to its acceleration field. The first period refers to an accelerated epoch dominated by a scalar field where a rapid expansion of the Universe characterizes its inflationary phase. The next one is related to a decelerated phase dominated by matter fields. This period is followed by a return to an accelerated epoch dominated by a cosmological constant or dark energy.

Recently several authors have investigated the evolution of the Universe within the framework of the one-brane model of Randall and Sundrum [1] where the Universe is confined onto a 3-brane, which is a hyper-surface embedded in a five-dimensional space-time called bulk. For recent reviews on this subject one is referred to the works of Brax and van de Bruck [2] and Langlois [3] and the references therein.

In the present work we investigate the evolution of a Universe confined onto the 3-brane embedded in a five-dimensional space-time in order to describe the three distinct periods of the Universe beginning with an accelerated phase.

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passing through a decelerated epoch and returning to an accelerated period. For that end we model the cosmological fluid on the brane by the van der Waals equation of state. The use of the van der Waals equation of state in cosmological problems was first proposed by Capozziello and co-workers [4, 5], who recognize that this equation of state could describe the transition from a scalar field dominated period to a matter dominated epoch without the need of introducing scalar fields. Recently a model for the Universe as a mixture of a van der Waals fluid with dark energy (modeled as quintessence or Chaplygin gas) was proposed in the work [6] – within the framework of a four-dimensional space-time theory – in order to describe the transition from the accelerated-decelerated-accelerated periods of the Universe.

In this work we show that the Universe on the brane modeled by the van der Waals equation of state evolves in such a manner that the three distinct periods concerning its acceleration field are attained. The initial accelerated epoch is due to the van der Waals fluid which behaves like a scalar field with a negative pressure. The past decelerated period has two contributions: one of them is related to the van der Waals fluid which behaves like a matter field with a positive pressure, whereas the other comes from a term of the Friedmann equation on the brane which is inversely proportional to the scale factor to the fourth power and can be interpreted as a radiation field. The present accelerated period is due to a cosmological constant on the brane.

In the one-brane model of Randall and Sundrum [1] the Universe is confined onto a hyper-surface – called brane – which is embedded in a five-dimensional space-time with coordinates \((t, x^1, x^2, x^3, y)\), called bulk. The brane is located at \(y = 0\) and the line element which describes a spatially flat, homogeneous and isotropic Universe on the brane is given by (see, for example, the reviews [2, 3])

\[
\begin{align*}
    ds^2 &= g_{MN} dx^M dx^N = n(t, y)^2 dt^2 - a(t, y)^2 \delta_{ij} dx^i dx^j - dy^2, \\
    \end{align*}
\]

where \(g_{MN}\) is the metric tensor with signature \((+, -, -, -, -)\). The two functions \(n(t, y)\) and \(a(t, y)\) are determined from the five-dimensional Einstein field equations that read

\[
R_{MN} - \frac{1}{2} R g_{MN} + \Lambda g_{MN} = -\kappa^2 T_{MN}. \tag{2}
\]

Above, \(R_{MN}\) is the five-dimensional Ricci tensor, \(R = R^M_M\) its trace, \(\Lambda\) denotes a bulk cosmological constant, \(\kappa^2 = 8\pi G_5\) is related to the five-dimensional gravitational constant \(G_5\), and \(T_{MN}\) is the five-dimensional energy-momentum tensor.

The five-dimensional energy-momentum tensor \(T_{MN}\) is decomposed into a sum of two terms: one refers to the bulk whereas the other is related to the brane, i.e.,

\[
T^M_N = T^M_N|_{\text{bulk}} + T^M_N|_{\text{brane}}. \tag{3}
\]

The energy-momentum tensor in the bulk is given by

\[
T^M_N|_{\text{bulk}} = \text{diag} (\rho_B, -p_B, -p_B, -p_B, \rho_B), \tag{4}
\]
where the energy density $\rho_B$ and the pressure $p_B$ in the bulk do not depend on the $y$ coordinate. Furthermore, the energy-momentum tensor on the brane reads

$$T_{M N}|_{\text{brane}} = \delta(y) (\sigma g_{\mu \nu} + T_{\mu \nu}) \delta^M \delta^N. \quad (5)$$

On the brane there exist two contributions: one is related to the constant tension on the brane $\sigma$, whereas the other refers to the energy-momentum tensor of the cosmological fluid $T_{\mu \nu}$ which is written as

$$T_{\mu \nu} = \text{diag} (\rho, -p, -p, -p). \quad (6)$$

Above, $\rho$ and $p$ denote the energy density and the pressure of the cosmological fluid on the brane, respectively.

For the line element given by (1), the components of the Einstein field equations (2) become

$$3 \left( \left( \frac{\dot{a}}{a} \right)^2 - n^2 \left( \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right) \right) - 3 \Lambda n^2 = \kappa^2 T_{00}, \quad (7)$$

$$a^2 \left( \left( \frac{a'}{a} \right)^2 + 2 \frac{a' n'}{a n} + \frac{n''}{n} \right) - \frac{a^2}{n^2} \left( \left( \frac{\dot{a}}{a} \right)^2 - \frac{\dot{a} n'}{a n} + \frac{\ddot{a}}{a} \right) + \Lambda \delta_{ij} = \kappa^2 T_{ij}, \quad (8)$$

$$3 \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{a' n'}{a n} - \frac{\dot{a}'}{a} - \frac{\ddot{a}}{a} \right) = \kappa^2 T_{0y}, \quad (9)$$

$$3 \left( \left( \frac{a'}{a} \right)^2 + 3 \frac{a' n'}{a n} - \frac{3}{n^2} \left( \left( \frac{\dot{a}}{a} \right)^2 - \frac{\dot{a} n'}{a n} + \frac{\ddot{a}}{a} \right) \right) + \Lambda = \kappa^2 T_{yy}. \quad (10)$$

In the above equations the dot and the prime refer to a differentiation with respect to the coordinates $t$ and $y$, respectively.

We follow the work [7] and introduce the abbreviations at $y = 0$: $a_0(t) \equiv a(t, y = 0)$ and $n_0(t) \equiv n(t, y = 0)$. Hence, on the brane the line element is written as

$$ds^2 = n_0(t)^2 dt^2 - a_0(t)^2 \delta_{ij} dx^i dx^j. \quad (11)$$

By imposing the gauge condition $n_0(t) = 1$, the time coordinate becomes the proper time on the brane and $a_0(t)$ is identified with the scale factor.

If one assumes that there exists no matter flow along the fifth dimension (see, e.g. [8])—so that $T_{0y} = 0$—the integration of equation (9) leads to

$$n(y, t) = \frac{\dot{a}(t, y)}{a_0(t)}. \quad (12)$$

We integrate equations (7) and (8) over $-\epsilon \leq y \leq +\epsilon$, and by taking the limit $\epsilon \to 0$, it follows the so-called junction conditions:

$$\frac{[a']}{a_0} = -\frac{\kappa^2}{3} (\rho + \sigma), \quad [n'] = -\frac{\kappa^2}{3} \sigma + \frac{2 \kappa^2}{3} \rho + \kappa^2 p. \quad (13)$$
where $[f] = f(0+) - f(0-)\) denotes the jump of the function $f$ across the brane.

If we take into account in equation (9) the junction conditions (13) together with the condition of no matter flow along the fifth dimension – i.e., $T_{05} = 0$ – we get the conservation equation for the energy density on the brane

$$\dot{\rho} + 3 \frac{\dot{a}}{a_0} (\rho + p) = 0.$$  

(14)

In order to analyze the system of Einstein field equations (7) – (10) in the bulk, we follow the work [8] and introduce the function

$$F(t, y) = (a')^2 - \frac{(\dot{a}a)^2}{n^2}.$$  

(15)

In terms of the function $F(t, y)$ the 00, yy and ij - components in the bulk read:

$$F' = -\frac{2a'^3}{3}(\Lambda + \kappa^2 \rho_B), \quad \dot{F} = -\frac{2\dot{a}a^3}{3}(\Lambda + \kappa^2 \rho_B),$$  

(16)

$$\left(\frac{F'}{a'}\right)^* = -2\dot{a}a^2(\Lambda - \kappa^2 p_B),$$  

(17)

respectively. Now we differentiate (16)1 with respect to time $t$ and (16)2 with respect to the coordinate $y$, and get that $\rho_B$ does not depend on time $t$. Furthermore, from the differentiation with respect to time of the expression $F'/a'$ obtained from (16)1 it follows an equation when compared with (17) implies that $p_B = -\rho_B$. Finally, the integration of (16)1 with respect to the coordinate $y$, leads to

$$F = (a'a)^2 - \frac{(\dot{a}a)^2}{n^2} = -\frac{a^4}{6}(\Lambda + \kappa^2 \rho_B) + C.$$  

(18)

Above, $C$ is a constant since $\rho_B$ is time independent.

The Friedmann equation on the brane is obtained from (18) by considering the limit $y \to 0$ and the junction conditions (13), yielding

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{\kappa^4}{18} \sigma \rho + \frac{\kappa^4}{36} \rho^2 + \frac{1}{36}(\kappa^4 \sigma^2 + 6\Lambda + 6\kappa^2 \rho_B) + \frac{C}{a_0^2}.$$  

(19)

From the Friedmann equation one can determine the acceleration equation on the brane. Indeed, the differentiation of (19) with respect to time, yields

$$\frac{\ddot{a}_0}{a_0} = -\frac{\kappa^4}{36} \sigma (\rho + 3p) - \frac{\kappa^4}{36} \rho (2\rho + 3p) + \frac{1}{36}(\kappa^4 \sigma^2 + 6\Lambda + 6\kappa^2 \rho_B) - \frac{C}{a_0^2},$$  

(20)

thanks to the equation (14) for the energy density on the brane. This equation can be obtained also from equation (10) by using the junction conditions (13).

Now we have a system of differential equations for the determination of the energy density $\rho(t)$ and of the scale factor $a_0(t)$ on the brane which is composed by the conservation equation for the energy density (14) and by the Friedmann equation (19) – or by the acceleration equation (20). In order to
find the time evolution of the energy density and of the scale factor from the system of differential equations one has to prescribe initial conditions for these fields and to close the system by choosing an equation of state which relates the pressure of the cosmological fluid to the energy density on the brane, i.e, $p = p(\rho)$.

For the determination of the solution of the system of differential equations described above we introduce the dimensionless quantities

\[ t \equiv \frac{\kappa^2 \sigma}{\sqrt{18}} t, \quad \rho \equiv \frac{\rho}{\sigma}, \quad p \equiv \frac{p}{\sigma}, \quad a_0 \equiv \frac{a_0(t)}{a_0(0)}, \]

and write the conservation equation for the energy density (14), the Friedmann (19) and the acceleration (20) equations in terms of the dimensionless quantities as

\[ \dot{\rho} + 3 \frac{\dot{a}_0}{a_0} (\rho + p) = 0, \]

\[ \left( \frac{\dot{a}_0}{a_0} \right)^2 = \rho + \frac{p^2}{2} + \frac{\lambda}{a_0^3}, \]

\[ \frac{\ddot{a}_0}{a_0} = -\frac{1}{2} (\rho + 3p) - \frac{1}{2} (2\rho + 3p) + \frac{\lambda}{a_0^3}. \]

In equations (23) and (24) the following abbreviations were introduced

\[ \lambda = 9 \left[ \frac{\Lambda}{\kappa^4 \sigma^2} + \frac{\rho_B}{\kappa^2 \sigma^2} + \frac{1}{6} \right], \quad \chi = \frac{18C}{|a_0(0)\kappa^4 \sigma^2|}. \]

The constant $\lambda$ in the Friedmann (23) and acceleration (24) equations can be interpreted as a cosmological constant on the brane. Moreover, it is noteworthy to call attention to the fact that equations (23) and (24) can be written as

\[ \left( \frac{\dot{a}_0}{a_0} \right)^2 = \rho + \rho_{\text{rad}} + \frac{\rho^2}{2} + \frac{\lambda}{3}, \]

\[ \frac{\ddot{a}_0}{a_0} = -\frac{1}{2} (\rho + \rho_{\text{rad}} + 3p + 3p_{\text{rad}}) - \frac{1}{2} (2\rho + 3p) + \frac{\lambda}{3}, \]

thanks to the well-known relationships valid in the four-dimensional case where the energy density of the radiation field $\rho_{\text{rad}}$ scales as $1/a_0^4$ and the radiation pressure is related to the energy density by $p_{\text{rad}} = \rho_{\text{rad}}/3$.

From now on we shall analyze the system of differential equations consisting of the conservation equation for the energy density (23) and of the acceleration equation (24). In order to close this system of differential equations we have to choose an equation of state that relates the pressure to the energy density of the cosmological fluid on the brane. In the four-dimensional case one normally uses a barotropic equation of state $p = w\rho$ with $0 \leq w \leq 1$ to represent a radiation or matter dominated Universe. The barotropic equation of state with $-1 \leq w \leq 0$ is also used to represent the scalar fields: the inflaton in the inflationary period
of the Universe, and the quintessence in the dark energy dominated epoch of the Universe. Other equations of state are also used to model the cosmological fluid, namely the Chaplygin equation of state (see e.g. the work [9] and the references therein) which can describe a transition from a matter dominated period to a cosmological constant dominated epoch and the van der Waals equation of state which can simulate the transition from an inflationary period to a matter field dominated epoch [6]. Here we interested in the description of an accelerated period followed by a decelerated epoch, hence we shall use the van der Waals equation of state

\[ p = \frac{8w}{3} \rho^3 - 3\rho^2, \]  

(28)

where the parameter \( w \) could be identified with the coefficient of proportionality in the barotropic formula, since for small values of the energy density \( p \propto w\rho \).

In classical thermodynamics (see e.g. [10]) the equation (28) is a reduced van der Waals equation of state where the free parameter \( w \) is connected with a dimensionless temperature. Capozziello and co-workers [4, 5] have used a non-reduced form of the van der Waals equation which is characterized by three free parameters instead of only one. In their works [4, 5] they also write a reduced form of the van der Waals equation with only one free parameter which is not similar to the reduced van der Waals equation of state normally found in the literature, since the free parameter multiplies also the term proportional to \( \rho^2 \).

Apart from the initial conditions for the fields of energy density \( \rho(0) \), scale factor \( a_0(0) \), and velocity \( \dot{a}_0(0) \), one has to specify values for the parameters \( \lambda \), \( \chi \) and \( w \) in order to find a solution of the system of differential equations (22) and (24) which is closed by the van der Waals equation of state (28). Here we have specified the initial conditions (by adjusting clocks): \( \rho(0) = 1 \) for the energy density, \( a_0(0) = 1 \) for the scale factor and \( \dot{a}(0) = \sqrt{3/2 + \lambda/3 + \chi} \) for the velocity field, which is a consequence of the Friedmann equation (22).

Furthermore, in order to plot the time evolution of the fields in figures 1 and 2 we have chosen the following values for the parameters \( \lambda = 0.03 \), \( \chi = 0.5 \), and three different values for \( w \), namely, \( w = 0.51 \), \( w = 0.501 \), and \( w = 0.5001 \). Later on we shall discuss how the changes in the parameters \( w \), \( \lambda \) and \( \chi \) affect the solutions of the system of differential equations.

In figure 1 it is plotted the acceleration field \( \ddot{a}_0(t) \) as function of time \( t \) for \( w = 0.51 \) (straight line), \( w = 0.501 \) (dotted line), and \( w = 0.5001 \) (dashed line). We infer from this figure that the acceleration field for these three values of \( w \) describes the distinct accelerated-decelerated-accelerated phases of the Universe. In the first period, which refers to a scalar field dominated epoch, the positive acceleration grows up to a maximum value followed by a decay towards zero. The next phase is related to a matter dominated period where the acceleration is always negative and decays to a maximum negative value followed by a growth towards zero. The third period is a cosmological constant dominated epoch where the acceleration field assumes a positive value. The energy density \( \rho \) and pressure \( p \) fields are plotted in figure 2 as function of time \( t \) for \( w = 0.51 \) (straight line), \( w = 0.501 \) (dotted line), and \( w = 0.5001 \) (dashed line). We conclude from this figure that for the three values of \( w \) the energy density field
Figure 1: Acceleration $\ddot{a}_0$ vs time $t$ for $w = 0.51$ (straight line), for $w = 0.501$ (dotted line), and for $w = 0.5001$ (dashed line).

Figure 2: Energy density $\rho$ and pressure $p$ vs time $t$ for $w = 0.51$ (straight lines), for $w = 0.501$ (dotted lines), and for $w = 0.5001$ (dashed lines).
decays with time whereas the pressure field has two distinct behaviors. At the beginning the pressure is negative so that the van der Waals fluid behaves like a scalar field and it is the responsible for the initial accelerated period. At later times the pressure becomes positive so that the van der Waals fluid behaves like a matter field and it is partially responsible for the decelerated period, since the radiation field which scales as $1/a^{4}$ also contributes to the decelerated phase of the Universe. When $w$ approaches the value of $w = 0.5$, we can also infer from the figures that the pressure field of the van der Waals fluid at the beginning behaves like an inflaton with an equation of state given by $p = -\rho$ and it is the responsible for an increase of the early acceleration.

We have chosen three nearby values for $w$ in order to show how the acceleration, energy density and pressure fields behave for small changes of $w$. We proceed now to discuss how significant changes in the values of $w$ have influence on the behavior of these fields for fixed values of $\lambda$ and $\chi$, here $\lambda = 0.03$ and $\chi = 0.5$. For $w = 0.5$ the energy density remains constant so that the van der Waals fluid behaves like an inflaton with an equation of state given by $p = -\rho$, the acceleration field grows exponentially and in this case there exists only an accelerated phase for the Universe. For values of $w < 0.5$ the energy density grows with time, and we infer that this behavior does not represent a physical solution for the evolution of the Universe. For values of $0.5 < w < 0.58$ the initial accelerated period decreases, since the negative part of pressure of the van der Waals fluid decreases, whereas its positive part increases. Hence in the interval $0.5 < w < 0.58$ the three periods accelerated-decelerated-accelerated are present in the evolution of the Universe. These three periods for the Universe can be found for a larger interval for $w$ by decreasing the amount of radiation, i.e., by decreasing the value of the constant $\chi$. For values of $w > 0.58$ (and $\lambda = 0.03$, $\chi = 0.5$) the acceleration field evolves from a matter dominated Universe where $\ddot{a} < 0$ to a cosmological constant dominated Universe where $\ddot{a} > 0$. In this last case there exists no inflationary period.

Let us comment on the behavior of the fields when we change the values of the parameters $\chi$ and $\lambda$. By increasing the value of $\chi$ there exists a more pronounced predominance of the radiation field which scales as $1/a^{4}$ and the deceleration period begins at earlier times. By decreasing the value of $\lambda$, which is connected to the cosmological constant on the brane, the present accelerated period begins at later times. Moreover, the limit $\lambda \to 0$ leads to an initial accelerated period followed by a decelerated period without a present accelerated period, indicating that $\lambda$ plays the role of the dark energy. We could also obtain the present accelerated period – which is characterized by a dark energy dominated Universe – by following the same methodology of the work [6] and instead of introducing a cosmological constant which represents the dark energy, we could model the dark energy by an equation of state like the Chaplygin or the quintessence equations of state.

As a final comment we have also investigated the solution of the system of differential equations (22) and (24) closed by a barotropic equation of state for the cosmological fluid on the brane, i.e., $p = w\rho$ with $-1 \leq w \leq 1$. The results we have found for the acceleration field by changing the values of $w$ describe
either an accelerated phase, or a decelerated epoch followed by an accelerated period of the Universe, i.e., with the barotropic equation of state it was not possible to describe the three phases accelerated-decelerated-accelerated found here by using the van der Waals equation of state.

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