System reduction using modified inverse distance measure and modified Cauer continued fraction method

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Abstract. A useful technique for obtaining the stable approximated model is proposed by combining modified inverse distance measure and modified Cauer continued fraction method. Modified inverse distance measure is based on pole clustering technique used to get the suitable dominant poles of the approximated model and modified Cauer continued fraction method has the advantages of approximation at the steady state as well as initial transient situations. Effectiveness of the proposed method is validated through considering numerical cases. The suggested method is computer oriented and simple in calculation.

Keywords - Order reduction, pole clustering, inverse distance measure, Cauer continued fraction, stability, integral of square of error.

1. Introduction

A mathematical analysis of linear time invariant systems may results to higher order differential equations. To reduce the computational complexity and easy controller design it is suitable to reduce such type of Higher Order System (HOS) in to Lower Order System (LOS). A number of methods are available in the literature for lower order modeling of Single-Input Single-Output (SISO) and Multi-Input Multi-Output (MIMO) Linear Time Invariant (LTI) systems in frequency and time domains [1-10 & 11-18]. Some of the techniques related to Perturbation [11] and aggregation [12] are based on the time response matching of HOS and LOS by considering the dominant Eigen values of the HOS and some other suitable parameters. Also, Modal analysis [13], Balanced realization [14], Optimal order reduction [15], Projection based method [16], Orthogonal decomposition [17], Laguerre polynomials [18] etc. are other methods available for order reduction of HOS in time domain.

All the above reduction methods proposed are computationally difficult, mathematically involved and not applicable for reduction of all HOSs. In this article a computationally simple, efficient, and computer oriented technique is suggested for lower order modeling of SISO and MIMO systems. The suggested technique is a combination of Modified Inverse Distance Measure (MIDM) and modified Cauer continued fraction method [19]. The MIDM method is utilizing the concept of pole clustering and is based on the
Inverse Distance Measure (IDM) proposed by Sinha [9] and Vishwakarma [8]. The method proposed by [8] is computationally lengthy as it uses several iterations depend upon the order of HOS and so suffers from a problem of difficult calculation and data feeding.

2. Statement of the suggested technique

The denominator of the Reduced Order Model (ROM) is synthesized by condensed cluster centers separately obtained for real and imaginary poles of HOS. The condensed cluster centers also called cluster centers are determined by proposed MIDM method. The numerator of the ROM is obtained by the modified Cauer continued fraction method [19] has the advantages of approximation at the steady state as well as initial transient situations. The method is explained as follows:

2.1. Synthesisation of denominator polynomial using method (MIDM) [21]

Let an nth order system is mathematically expressed as

\[ H_n(s) = \frac{P_n(s)}{Q_n(s)} = \frac{P_0 + P_1s + P_2s^2 + \ldots + P_{n-1}s^{n-1}}{Q_0 + Q_1s + Q_2s^2 + \ldots + Q_n s^n} \]  \hspace{1cm} (1)

The poles of this system are: \( \sigma_1, \sigma_2, \ldots, \sigma_n \) such that \( |\sigma_1| < |\sigma_2| < \ldots < |\sigma_n| \).

Suppose the required \( r \)-th order system is

\[ H_r(s) = \frac{P_r(s)}{Q_r(s)} = \frac{P_0 + P_1s + P_2s^2 + \ldots + P_{r-1}s^{r-1}}{Q_0 + Q_1s + Q_2s^2 + \ldots + Q_r s^r} \]  \hspace{1cm} (2)

The denominator of the required reduced order system can be obtained as follows:

**Step-I:** Find out the poles of the HOS and select the suitable pole clusters according to [21].

**Step-II:** To obtain the \( r \)-th order reduced system; 'r' pole cluster centers are required. The pole cluster center is the most dominant pole in that cluster and obtained as:

1. Let there are \( r \) poles \( \sigma_1', \sigma_2', \ldots, \sigma_r' \) in the \( j \)-th cluster such that \( |\sigma_1'| < |\sigma_2'| < \ldots < |\sigma_r'| \).
2. Set \( v = 1 \)
3. Compute the pole cluster center using [9]

\[ c_v = \left[ \frac{1}{\sum_{k=1}^{r} \left( \frac{1}{|\sigma_k'|} \right)} \right]^{-1} \]  \hspace{1cm} (3)

4. Determine the most dominant pole cluster center using the (4) given by [21].

\[ \sigma_v = -\lambda - \left[ \log(1 + c_v) \right] \times (r \times n) \]  \hspace{1cm} (4)

Where, \( \lambda \) = dominant pole in each cluster

**Step-III:** Now the denominator polynomial \( Q_r(s) \) of the ROM can be synthesized by considering different cases of pole clusters -

**Case (i)** - Pole clusters of the HOS having real poles only

\[ Q_r(s) = (s - \sigma_1)(s - \sigma_2) \ldots (s - \sigma_r) \]  \hspace{1cm} (5)

In equation (5), \( \sigma_1, \sigma_2, \ldots, \sigma_r \) are dominant poles centers obtained from (4).
Case (ii) – Pole clusters of HOS having real and complex both.
\[ Q_r(s) = (s - \sigma_1)(s - \sigma_2) \ldots (s - \sigma_{r-2}) \left( \frac{s^*}{s - \phi_1} \right) \left( \frac{s*}{s - \phi_1} \right) \]  

(6)

Where, \( \phi_i \) and \( \phi_1 \) are real and imaginary parts of complex conjugate cluster centers respectively.

Case (iii) – Pole clusters of HOS having complex pole clusters only
\[ Q_r(s) = \left( \frac{s^*}{s - \phi_1} \right) \ldots \left( \frac{s^*}{s - \phi_{r/2}} \right) \left( \frac{s^*}{s - \phi_{r/2}} \right) \]  

(7)

Now, the denominator polynomial of the ROM i.e. \( Q_r(s) \) is written as
\[ Q_r(s) = q_0 + q_1s + q_2s^2 + \ldots + q_rs^r \]  

(8)

2.2. Modified Cauer Continued Fraction

Step-1: Obtain the denominator polynomial \( Q_r(s) \) using the method MIDM described in section (2.1).

Step-2: Modified Cauer Continued Fraction for obtaining the numerator polynomial \( P_r(s) \) can be described as-

(i) Calculate the first ‘r’ quotients \( d_1, D_1, d_2, D_2 \) using the algorithm [19].

Fig. 1: Modified Routh array [19]
(ii) A new modified Routh array for \( r = 6 \) is built as shown in Fig. 1. First two rows are directly created from the coefficients of \( H_r(s) \). Remaining entries in the array are determined by [19].

So, ROM numerator polynomial is obtained as

\[
P_r(s) = p_0 + p_1s + p_2s^2 + \ldots + p_r s^r
\]

(9)

3. Numerical Examples

**Numerical Example 1:** Let a 4\(^{th}\) order system taken from Mittal [25].

\[
H_4(s) = \frac{s^3 + 7s^2 + 24s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 24}
\]

The poles of the HOS taken from [25] are: \(-1, -2, -3, -4\)

For obtaining 2\(^{nd}\) order ROM, two poles obtained using MIDM given in Section (2.1) as \( \sigma_1' = -1.0459 \), and \( \sigma_2' = -3.0808 \).

Hence, Denominator polynomial \( Q_2(s) \) is obtained as

\[
Q_2(s) = s^2 + 4.1267s + 3.2222
\]

Numerator polynomial can be obtained from step-2 of Section (2.2) as:

\[
d_1 = 1 \begin{bmatrix} 24 & 50 & 35 & 10 & 1 \\ 24 & 24 & 7 & 1 \\ 26 & 28 & 9 & 1 \end{bmatrix}
\]

And so changed Routh array is

\[
d_1 = 1 \begin{bmatrix} 3.2222 & 4.1267 & 1 \\ 3.2222 & 1 & \end{bmatrix}
\]

Fig. 2: Modified Routh array for Example 1

Therefore, 2\(^{nd}\) order model obtained is as

\[
H_2(s) = \frac{s + 3.2222}{s^2 + 4.1267s + 3.2222}
\]
Table 1: The ISE and IAE Comparison for Example 1

| Reduction Methods     | Reduced Models                                                                 | ISE    | IAE    |
|-----------------------|-------------------------------------------------------------------------------|--------|--------|
| Proposed Method       | $H_2(s) = \frac{s + 3.2222}{s^2 + 4.1267s + 3.2222}$                        | 0.00508| 0.1129 |
| Pal [23]              | $R_2(s) = \frac{16.008s + 24}{30s^2 + 42s + 24}$                          | 0.2689 | 0.8054 |
| Prasad and Pal [6]    | $R_2(s) = \frac{s + 34.2645}{s^2 + 239.08082s + 34.2645}$                 | 1.4584 | 1.000  |
| Krishnamurthy [22]    | $R_2(s) = \frac{155658.6152s + 40320}{65520s^2 + 75600s + 40320}$        | 1.6533 | 2.4090 |
| Shieh and Wei [24]    | $R_2(s) = \frac{s + 0.43184}{s^2 + 41.17368s + 0.43184}$                  | 1.9171 | 10.0702|

The performance indices i.e. Integral Square Error (ISE) and Integral of Absolute magnitude of Error (IAE) are calculated and compared between original and reduced order system are given in Table 1. The unit step and frequency response comparisons of ROM i.e. $H_2(s)$ and higher order model $H_4(s)$ are also shown in Fig. 3 respectively.

4. Extension to MIMO System

**Numerical Example 2:** Let 6th order HOS transfer function matrix taken from [20] described as

$$[H(s)] = \frac{1}{Q(s)} \begin{bmatrix} \beta_{11}(s) & \beta_{12}(s) \\ \beta_{21}(s) & \beta_{22}(s) \end{bmatrix}$$
Where, \( Q(s) = s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000 \)

\[
\begin{align*}
\beta_{11} &= 2s^5 + 70s^4 + 762s^3 + 3610s^2 + 7700s + 6000 \\
\beta_{12} &= s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400 \\
\beta_{21} &= s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000 \\
\beta_{22} &= s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000
\end{align*}
\]

The poles of this system are: \(-1, -2, -3, -5, -10, -20\)

The 2\(^{nd}\) order reduced model using proposed method is obtained as

\[
[H_2(S)] = \frac{1}{(s^2 + 6.1167s + 5.2595)} \begin{bmatrix}
2s + 5.2595 & s + 2.6297 \\
2s + 2.6297 & s + 5.2595
\end{bmatrix}
\]

Fig. 4: Frequency and step response comparisons Example 2

Table 2: Error index comparison for Example 2

| \(h_{ij} = \frac{a_{ij}(s)}{Q_2(s)}\); \((i = 1, 2; j = 1, 2)\) | ISE \textbf{Suggested Technique} | ISE \[2\] | ISE \[6\] |
|---|---|---|---|
| \(h_{11}(s)\) | 0.04519 | 0.038713 | 0.135505 |
| \(h_{12}(s)\) | 0.02957 | 0.028153 | 0.002446 |
| \(h_{21}(s)\) | 0.00896 | 0.007419 | 0.040013 |
The error index ISE is calculated between the original i.e. $H_6(s)$ and reduced model i.e. $H_2(s)$ and given in Table II. Also the frequency response of 2nd order approximated model and original model is shown in Fig. 4.

5. Conclusions
A new method for reducing a HOS is suggested by combining MIDM technique and modified Cauer continued fraction method. The suggested method has been elaborated using two different types of models having SISO and MIMO. The algorithm used for approximation is very simple in calculation and takes very little computation time. The proposed method of approximation is equally applicable in SISO as well as MIMO systems. Step and frequency response comparison of original HOS and ROM are given in Fig. 3 and Fig. 4 respectively, and follow the pattern of original system. For both numerical examples the error indices comparison via MATLAB are given in Tables 1 and 2 respectively.

6. References

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