Quillen equivalence of singular model categories

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Abstract

Let $R$ be a left-Gorenstein ring. We construct a Quillen equivalence between singular contraderived model category and singular coderived model category introduced by Becker [Adv. Math., (2014) 187-232]. As an application, we explicitly give an equivalence $K_{\text{ex}}(\mathcal{P}) \simeq K_{\text{ex}}(\mathcal{I})$ for the homotopy categories of exact complexes of projective and injective modules.

Key Words: Model category, Quillen equivalence, left-Gorenstein ring.

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1. Introduction

The notion of recollement of triangulated categories was introduced by Beilinson, Bernstein and Deligne [2] with an idea that one category can be viewed as being “glued together” from two others. In [18], Krause established a recollement $K_{\text{ex}}(\mathcal{I}) \rightleftarrows K(\mathcal{I}) \rightleftarrows D(R),$ where $K(\mathcal{I})$ (resp. $K_{\text{ex}}(\mathcal{I})$) is the homotopy category of complexes (resp. exact complexes) of injective modules, and $D(R)$ is the derived category. Recently, Becker [1] found a Bousfield localization of model categories, and then he recovered Krause’s recollement and got a dual one $K_{\text{ex}}(\mathcal{P}) \rightleftarrows K(\mathcal{P}) \rightleftarrows D(R).$

Iyengar and Krause [17] proved that for a commutative noetherian ring $R$ with a dualizing complex $D$, there is a triangle-equivalence $D \otimes_R - : K(\mathcal{P}) \to K(\mathcal{I}).$ Note that there are equivalences $K^c(\mathcal{P}) \simeq D^f(R)$ and $K^c(\mathcal{I}) \simeq D^f(R)$ between the subcategories of compact objects $K^c(\mathcal{P})$ and $K^c(\mathcal{I})$, and the derived category $D^f(R)$ of complex whose homology is bounded and finitely generated; see Jørgensen [10] and Krause [18]. Iyengar-Krause equivalence can also be considered as a generalization of Grothendieck duality. If $R$ is left-Gorenstein [3] (i.e. a ring such that any left $R$-module has finite projective dimension if and only if it has finite injective dimension), Chen [6] established an equivalence $K(\mathcal{P}) \simeq K(\mathcal{I})$ via relative derived categories with respect to the so-called balanced pairs.

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An interesting question is raised naturally: is there an equivalence $K_{ex}(\mathcal{P}) \simeq K_{ex}(\mathcal{I})$ for the homotopy categories of exact complexes, which appear on the left end of the above recollements? However, the methods of both Iyengar-Krause and Chen seem not effective in this case.

By Beligiannis [3], when $R$ is left-Gorenstein, the homotopy category $K_{ex}(\mathcal{P})$ is triangle-equivalent to the singularity category $D_{sg}(R)$, which is defined as a Verdier quotient $D^b(R)/K^b(\mathcal{P})$ of bounded derived category modulo the bounded homotopy category of complexes of projective modules. It is well known that over a left-Gorenstein ring $R$, the exact complex of projective (injective) modules is precisely the totally acyclic complex of projectives (injectives). We note that Bergh, Jorgensen and Oppermann [5] also studied the equivalence between the homotopy category of totally acyclic complexes $K_{tac}(\mathcal{P})$ and singularity category $D_{sg}(R)$ for an artin ring or a commutative noetherian local ring.

Recall that a model structure $\mathcal{M}$ on an abelian category is three distinguished classes of maps, called weak equivalences, cofibrations and fibrations respectively, satisfying a few axioms. For a model category, the associated homotopy category $Ho(\mathcal{M})$ is constructed by formally inverting the weak equivalences, i.e. localization with respect to weak equivalences. Recently, Becker [1] realized $K_{ex}(\mathcal{P})$ (resp. $K_{ex}(\mathcal{I})$) as the homotopy category of singular contraderived (resp. singular coderived) model category. For details on model categories, we refer to the original source of Quillen [19], as well as [8, 14, 15].

In this paper, we are inspired to construct a Quillen equivalence between singular contraderived model category and singular coderived model category, and then we give an equivalence $K_{ex}(\mathcal{P}) \simeq K_{ex}(\mathcal{I})$. To illustrate the main result of the paper, we need some notations. Let $Ch(R)$ be the category of $R$-complexes. Following [12], let $exP$ (resp. $exI$) be the subcategory of all exact complexes of projective (resp. injective) modules, and $(exP)^\perp$ (resp. $(exI)^\perp$) be the right (resp. left) orthogonal. In the language of Hovey’s correspondence [16], the singular contraderived model structure on $Ch(R)$ is denoted by a triple $\mathcal{M}_{sing}^{ctr} = (exP, (exP)^\perp, Ch(R))$, and the singular coderived model structure is denoted by $\mathcal{M}_{sing}^{co} = (Ch(R),^\perp(exI), exI)$.

Let $X$ be any complex. We denote by $\Omega : Ch(R) \to Mod(R)$ the functor given by $\Omega(X) = X_0/Imd_X^1$, and $\Theta : Ch(R) \to Mod(R)$ the functor given by $\Theta(X) = Ker d_0^X$. We define $\Lambda : Mod(R) \to Ch(R)$ to be a functor which sends every module to a stalk complex concentrated on degree zero.

**Theorem 1.1.** Let $R$ be a left-Gorenstein ring, $F = \Lambda \Omega$ and $G = \Lambda \Theta$ be functors on $Ch(R)$. Then $(F, G) : (Ch(R), \mathcal{M}_{sing}^{ctr}) \to (Ch(R), \mathcal{M}_{sing}^{co})$ is a Quillen equivalence between the singular contraderived model category and singular coderived model category.

By the fundamental results on homotopy categories of model categories (see e.g. [15, Theorem 1.2.10]), one has triangle-equivalences $Ho(\mathcal{M}_{sing}^{ctr}) \simeq K(\mathcal{P})$ and $Ho(\mathcal{M}_{sing}^{co}) \simeq K(\mathcal{I})$; see [1] or Corollary [22] below. Moreover, a Quillen equivalence of model categories yields an adjoint
equivalence of corresponding homotopy categories. Hence we have the following, which gives an affirmative answer to the above question.

**Corollary 1.2.** Let $R$ be a left-Gorenstein ring. Then there is an equivalence $F' : K_{ex}(\mathcal{P}) \to K_{ex}(\mathcal{I})$ which is defined on objects by first taking $F$, and then taking fibrant replacement (= a special $\text{ex}\tilde{\mathcal{I}}$-preenvelope); the inverse $G' : K_{ex}(\mathcal{I}) \to K_{ex}(\mathcal{P})$ is defined on objects by first taking $G$, and then taking cofibrant replacement (= a special $\text{ex}\tilde{\mathcal{P}}$-precover).

**Question.** More recently, Gillespie [13] established “Gorenstein version” of the aforementioned recollements, i.e. $K_{ex}(\mathcal{G}\mathcal{I}) \equiv K(\mathcal{G}\mathcal{I}) \equiv D(R)$ and $K_{ex}(\mathcal{G}\mathcal{P}) \equiv K(\mathcal{G}\mathcal{P}) \equiv D(R)$, where $\mathcal{G}\mathcal{I}$ and $\mathcal{G}\mathcal{P}$ denote the class of Gorenstein injective and Gorenstein projective modules, respectively. If the underlying ring is left-Gorenstein, it follows from [9] that $\mathbf{K}(\mathcal{G}\mathcal{P}) \simeq \mathbf{K}(\mathcal{G}\mathcal{I})$. Recently, we realize this equivalence in the framework of cotorsion triples [21].

However, we do not know if it is true that $K_{ex}(\mathcal{G}\mathcal{P}) \simeq K_{ex}(\mathcal{G}\mathcal{I})$. We remark that one can not get an answer by simply restricting the equivalent functor $K(\mathcal{G}\mathcal{P}) \simeq K(\mathcal{G}\mathcal{I})$ in [9], or by the methods in [21].

2. The proof of the theorem

First, we recall some basic notations and facts, which are needed in the following. Throughout the paper, let $R$ be a left-Gorenstein ring. All modules are left $R$-modules. A complex $X$ means a sequence of modules $\cdots \to X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \to \cdots$ with $d_n^X \cdot d_{n+1}^X = 0$.

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. A pair of classes $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{A}$ is a cotorsion pair provided that $\mathcal{X} = \perp \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^\perp$, where $\perp \mathcal{Y} = \{ X \mid \text{Ext}_A^1(X, Y) = 0, \ \forall \ Y \in \mathcal{Y} \}$ and $\mathcal{X}^\perp = \{ Y \mid \text{Ext}_A^1(X, Y) = 0, \ \forall \ X \in \mathcal{X} \}$.

The cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is complete provided that for any $M \in \mathcal{A}$, there exist short exact sequences $0 \to Y \to X \xrightarrow{f} M \to 0$ and $0 \to M \xrightarrow{g} Y' \to X' \to 0$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$. In this case, for any $N \in \mathcal{X}$, $\text{Hom}_A(N, f) : \text{Hom}_A(N, X) \to \text{Hom}_A(N, M)$ is surjective since $\text{Ext}_A^1(N, Y) = 0$, and then $f : X \to M$ is said to be a special $\mathcal{X}$-precover of $M$. Dually, $g : M \to Y'$ is called a special $\mathcal{Y}$-preenvelope of $M$.

The cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is resolving if $\mathcal{X}$ is closed under taking kernels of epimorphisms between objects of $\mathcal{X}$, i.e. for any short exact sequence $0 \to X' \to X \xrightarrow{\xi} X'' \to 0$ with $X, X'' \in \mathcal{X}$, we have $X' \in \mathcal{X}$. We say $(\mathcal{X}, \mathcal{Y})$ is coresolving if $\mathcal{Y}$ satisfies the dual. We say $(\mathcal{X}, \mathcal{Y})$ is hereditary if it is both resolving and coresolving. By [1, Corollary 1.1.12], a complete cotorsion pair is resolving if and only if it is coresolving.

By the correspondence of Beligiannis-Reiten [4] or Hovey [16, Theorem 2.2], an abelian model structure on $\mathcal{A}$ is equivalent to a triple $(\mathcal{A}_c, \mathcal{A}_{tri}, \mathcal{A}_f)$ of subcategories, for which $\mathcal{A}_{tri}$ is thick and both $(\mathcal{A}_c, \mathcal{A}_f \cap \mathcal{A}_{tri})$ and $(\mathcal{A}_c \cap \mathcal{A}_{tri}, \mathcal{A}_f)$ are complete cotorsion pairs. In this case, $\mathcal{A}_c$ is the class of cofibrant objects, $\mathcal{A}_{tri}$ is the class of trivial objects and $\mathcal{A}_f$ is the class of fibrant objects. The model structure is called “abelian” since it is compatible with the abelian structure
of the category in the following way: (trivial) cofibrations are monomorphisms with (trivially) cofibrant cokernel, (trivial) fibrations are epimorphisms with (trivially) fibrant kernel, and weak equivalences are morphisms which factor as a trivial cofibration followed by a trivial fibration.

For convenience, we will use the triple $(\mathcal{A}_c, \mathcal{A}_{tri}, \mathcal{A}_f)$ to denote the corresponding model structure. The following is immediate from [1] or [12, Theorem 4.7].

**Lemma 2.1.** On the category $\text{Ch}(R)$ of complexes, there is a singular contraderived model structure $\mathcal{M}^{\text{ctr}}_{\text{sing}} = (\text{ex}\mathcal{P}, (\text{ex}\mathcal{P})^\perp, \text{Ch}(R))$, and a singular coderived model structure $\mathcal{M}^{\text{co}}_{\text{sing}} = (\text{Ch}(R), 1^\perp(\text{ex}\mathcal{I}), \text{ex}\mathcal{I})$.

For a bicomplete abelian category $\mathcal{A}$ with the model structure $\mathcal{M} = (\mathcal{A}_c, \mathcal{A}_{tri}, \mathcal{A}_f)$, the associated homotopy category $\text{Ho}(\mathcal{M})$ is constructed by formally inverting the weak equivalences, i.e. the localization with respect to weak equivalences. The homotopy category of an abelian model category is always a triangulated category. There is an equivalence of categories $\text{Ho}(i) : \mathcal{A}_f/\omega = \mathcal{A}_c/\sim \to \text{Ho}(\mathcal{M})$ induced by the inclusion functor $i : \mathcal{A}_f \to \mathcal{A}$, where $\mathcal{A}_{f} = \mathcal{A}_{c} \cap \mathcal{A}_{f}$; see e.g. [15, Section 1.2].

We use $\mathcal{P}$ (resp. $\mathcal{I}$) to denote the subcategory of contractible complexes of projective (resp. injective) modules. It is well known that a complex $P \in \mathcal{P}$ if and only if $P$ is exact and each $\text{Ker}d^n_P$ is a projective module; similarly, complexes in $\mathcal{I}$ are characterized. Note that for any chain maps $f$ and $g$, if $g - f$ factors through an object in $\omega = \mathcal{A}_{c} \cap \mathcal{A}_{tri} \cap \mathcal{A}_{f}$; see e.g. [15, Section 1.2].

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**Corollary 2.2.** There are equivalences $\text{Ho}(\mathcal{M}^{\text{ctr}}_{\text{sing}}) \simeq K_{\text{ex}}(\mathcal{P})$ and $\text{Ho}(\mathcal{M}^{\text{co}}_{\text{sing}}) \simeq K_{\text{ex}}(\mathcal{I})$.

Let $F = \Lambda \Omega$ and $G = \Lambda \Theta$ be functors on $\text{Ch}(R)$, where $\Omega$ and $\Theta$ are functors from $\text{Ch}(R)$ to $\text{Mod}(R)$ such that for any $X \in \text{Ch}(R)$, $\Omega(X) = X_0/\text{Im}d^X_1$ and $\Theta(X) = \text{Ker}d^X_0$. Let $\Lambda : \text{Mod}(R) \to \text{Ch}(R)$ be a functor which sends every module to a stalk complex concentrated on degree zero.

In the rest of the paper, we are devoted to prove the theorem stated in Introduction. The proof is divided into the following.

**Lemma 2.3.** Let $X, Y$ be any $R$-complexes, and $f : X \to Y$ a monomorphism of complexes. If $f$ is a quasi-isomorphism, then $\Omega(f)$ is also a monomorphism of $R$-modules.

**Proof.** We consider the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \text{Ker}d^X_0/\text{Im}d^X_1 & \to & X_0/\text{Im}d^X_1 & \to & X_0/\text{Ker}d^X_0 & \to & 0 \\
\downarrow{\Omega(f)} & & \downarrow{\Omega(f)} & & \downarrow{\Omega(f)} & & \downarrow{\Omega(f)} & & \\
0 & \to & \text{Ker}d^Y_0/\text{Im}d^Y_1 & \to & Y_0/\text{Im}d^Y_1 & \to & Y_0/\text{Ker}d^Y_0 & \to & 0
\end{array}
\]
Since $f$ is a quasi-isomorphism, we have an isomorphism induced by $f$:

$$H_0(f) : H_0(X) = \text{Ker}d_0^X/\text{Im}d_1^X \rightarrow \text{Ker}d_0^Y/\text{Im}d_1^Y = H_0(Y).$$

Since the chain map $f$ is monic, then the induced map of modules $X_0/\text{Ker}d_0^X \cong \text{Im}d_0^X \rightarrow \text{Im}d_0^Y \cong Y_0/\text{Ker}d_0^Y$ is also monic. Hence, by the “Five Lemma” for the above diagram, we get that $\Omega(f) : X_0/\text{Im}d_1^X \rightarrow Y_0/\text{Im}d_1^Y$ is a monomorphism. We mention that it is also direct to check injectivity of $\Omega(f)$ by diagram chasing. □

For model categories $\mathcal{C}$ and $\mathcal{D}$, recall that an adjunction $(F, G) : \mathcal{C} \rightarrow \mathcal{D}$ is a Quillen adjunction if $F$ is a left Quillen functor, or equivalently $G$ is a right Quillen functor. That is, $F$ preserves cofibrations and trivial cofibrations, or $G$ preserves fibrations and trivial fibrations.

Let $X$ be a complex. By [12, Proposition 3.3], $X \in (ex\tilde{P})^\perp$ if each map $f : P \rightarrow X$ is null homotopic whenever $P \in ex\tilde{P}$; dually, $X \in (ex\tilde{L})^\perp$ if each map $f : X \rightarrow I$ is null homotopic whenever $I \in ex\tilde{L}$.

**Proposition 2.4.** $(F, G) : (\text{Ch}(R), \mathcal{M}^{\text{co}}_{\text{sing}}) \rightarrow (\text{Ch}(R), \mathcal{M}^{\text{co}}_{\text{sing}})$ is a Quillen adjunction.

**Proof.** Let $X, Y$ be any $R$-complexes. It follows from [11, Lemma 3.1] that $(\Omega, \Lambda) : \text{Ch}(R) \rightarrow \text{Mod}(R)$ and $(\Lambda, \Theta) : \text{Mod}(R) \rightarrow \text{Ch}(R)$ are adjunctions. Then we have the following natural isomorphisms: $\text{Hom}_{\text{Ch}(R)}(F(X), Y) \cong \text{Hom}_{\text{R}}(\Omega(X), \Theta(Y)) \cong \text{Hom}_{\text{Ch}(R)}(X, G(Y))$. This implies that $(F, G) : \text{Ch}(R) \rightarrow \text{Ch}(R)$ is an adjunction.

Then, it suffices to show that $F$ preserves cofibration and trivial cofibration. Let $f : X \rightarrow Y$ be a cofibration in $\mathcal{M}^{\text{co}}_{\text{sing}}$, i.e. $f$ is a monomorphism with $\text{Coker}f \in ex\tilde{P}$. Then $f$ is a quasi-isomorphism, and by Lemma 2.3, $\Omega(f)$ is monic. Then, we have an exact sequence

$$0 \rightarrow F(X) \xrightarrow{F(f)} F(Y) \rightarrow F(\text{Coker}f) \rightarrow 0.$$

Since every complex is a cofibrant object in $\mathcal{M}^{\text{co}}_{\text{sing}}$, this implies that $F(f)$ is a cofibration.

Now suppose $f : X \rightarrow Y$ is a trivial cofibration in $\mathcal{M}^{\text{co}}_{\text{sing}}$, i.e. $f$ is a monomorphism with $\text{Coker}f \in ex\tilde{P} \cap (ex\tilde{P})^\perp = \tilde{P}$. Then we have an exact sequence

$$0 \rightarrow F(X) \xrightarrow{F(f)} F(Y) \rightarrow F(\text{Coker}f) \rightarrow 0.$$

Note that $\Omega(\text{Coker}f)$ is a projective module. For any complex $I \in ex\tilde{L}$, it is easy to show that any chain map $F(\text{Coker}f) = \Lambda\Omega(\text{Coker}f) \rightarrow I$ is null homotopic, and then $F(\text{Coker}f) \in (ex\tilde{L})^\perp$. Thus $F(f)$ is a trivial cofibration in $\mathcal{M}^{\text{co}}_{\text{sing}}$. This completes the proof. □

Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories, and $(F, G) : \mathcal{C} \rightarrow \mathcal{D}$ is a Quillen adjunction. Then $(F, G) : \mathcal{C} \rightarrow \mathcal{D}$ is called a Quillen equivalence if and only if it satisfies the Quillen condition: for all cofibrant object $X$ in $\mathcal{C}$ and fibrant object $Y$ in $\mathcal{D}$, a map $f : FX \rightarrow Y$ is a weak equivalence in $\mathcal{D}$ if and only if the associated map $\varphi(f) : X \rightarrow GY$ is a weak equivalence in $\mathcal{C}$, see for example [13, Definition 1.3.12]. A Quillen adjunction $(F, G) : \mathcal{C} \rightarrow \mathcal{D}$ is a Quillen equivalence if and only if $(LF, RG) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$ is an adjoint equivalence of homotopy categories (see e.g.
where $LF$ is the left derived functor defined on objects by first taking cofibrant replacement and then applying the functor $F$, and $RG$ is the right derived functor defined on objects by first taking fibrant replacement and then applying the functor $G$; see \cite{model_categories} Definition 1.3.6. We refer to \cite{quillen_homotopy} Section 5] or \cite{model_categories} Section 1.1] for the notions of the cofibrant and fibrant replacement functors.

By \cite{left-derived-adjunctions}, there is a useful criterion for checking the given Quillen adjunction is a Quillen equivalence. Specifically, we need to show that $F$ reflects weak equivalences between cofibrant objects in $\mathcal{M}_{\text{sing}}^\text{ctr}$ (i.e. complexes in $\text{ex}\tilde{\mathcal{P}}$) and, for every fibrant object $Y$ in $\mathcal{M}_{\text{sing}}^\text{co}$ (i.e. $Y \in \text{ex}\tilde{\mathcal{I}}$) the composition $FQG(Y) \xrightarrow{F(q)} FG(Y) \xrightarrow{\varepsilon} Y$ is a weak equivalence, where $\varepsilon$ is the counit of the adjunction $(F,G)$, and $q : QG(Y) \rightarrow G(Y)$ is a cofibrant replacement of $G(Y)$.

\textbf{Lemma 2.5.} Let $X$, $Y$ be complexes in $\text{ex}\tilde{\mathcal{P}}$, and $f : X \rightarrow Y$ a chain map. If $f$ is a weak equivalence in $\mathcal{M}_{\text{sing}}^{\text{ctr}}$, then $F(f)$ is a weak equivalence in $\mathcal{M}_{\text{sing}}^{\text{co}}$.

\textit{Proof.} In the model category $(\text{Ch}(R),\mathcal{M}_{\text{sing}}^{\text{ctr}})$, we can factor $f : X \rightarrow Y$ as a trivial cofibration $i : X \rightarrow Z$ followed by a trivial fibration $p : Z \rightarrow Y$. By Proposition 2.4, $F(i)$ is a trivial cofibration in $\mathcal{M}_{\text{sing}}^{\text{co}}$, and then $F(i)$ is a weak equivalence.

In the exact sequence $0 \rightarrow X \xrightarrow{i} Z \rightarrow \text{Coker} i \rightarrow 0$, $X \in \text{ex}\tilde{\mathcal{P}}$ and $\text{Coker} i \in \text{ex}\tilde{\mathcal{P}} \cap (\text{ex}\tilde{\mathcal{P}}) \perp$. Then $Z \in \text{ex}\tilde{\mathcal{P}}$. Moreover, it follows from the exact sequence $0 \rightarrow \text{Ker} p \rightarrow Z \xrightarrow{p} Y \rightarrow 0$ that $\text{Ker} p \in \text{ex}\tilde{\mathcal{P}}$. Note that $p$ is a trivial fibration, then $\text{Ker} p \in (\text{ex}\tilde{\mathcal{P}}) \perp$. Hence, $\text{Ker} p \in \tilde{\mathcal{P}} = \text{ex}\tilde{\mathcal{P}} \cap (\text{ex}\tilde{\mathcal{P}}) \perp$, and $\Omega(\text{Ker} p)$ is a projective module.

We consider the push-out diagram of $\Omega(\text{Ker} p) \rightarrow \Omega(Z)$ along $\Omega(\text{Ker} p) \rightarrow I$, where $I$ is an injective envelope of $\Omega(\text{Ker} p)$:

\begin{center}
\begin{tikzpicture}
  \node (A1) at (0,0) {$0$};
  \node (A2) at (1,0) {$\Omega(\text{Ker} p)$};
  \node (A3) at (2,0) {$\Omega(Z)$};
  \node (A4) at (3,0) {$\Omega(Y)$};
  \node (A5) at (4,0) {$0$};
  \node (B1) at (0,-1) {$0$};
  \node (B2) at (1,-1) {$I$};
  \node (B3) at (2,-1) {$J$};
  \node (B4) at (3,-1) {$\Omega(Y)$};
  \node (B5) at (4,-1) {$0$};
  \node (C1) at (0,-2) {$0$};
  \node (C2) at (1,-2) {$L$};
  \node (C3) at (2,-2) {$L$};
  \node (C4) at (3,-2) {$0$};
  \draw[->] (A1) -- (A2);
  \draw[->] (A2) -- (A3);
  \draw[->] (A3) -- (A4);
  \draw[->] (A4) -- (A5);
  \draw[->] (B1) -- (B2);
  \draw[->] (B2) -- (B3);
  \draw[->] (B3) -- (B4);
  \draw[->] (B4) -- (B5);
  \draw[->] (C1) -- (C2);
  \draw[->] (C2) -- (C3);
  \draw[->] (C3) -- (C4);
  \draw[->] (C2) -- (C3);
\end{tikzpicture}
\end{center}

Note that $R$ is left-Gorenstein, then the injective module $I$ is of finite projective dimension. It follows from the exact sequence $0 \rightarrow \Omega(\text{Ker} p) \rightarrow I \rightarrow L \rightarrow 0$ that $L$ is of finite projective dimension. Then for any complex $E \in \text{ex}\tilde{\mathcal{I}}$, $\text{Hom}_R(L,E)$ is also exact. Hence, by \cite{dualizing complexes} Lemma 2.4, we get that every map $\Lambda(L) \rightarrow E$ is null homotopic, and then $\Lambda(L) \in \perp(\text{ex}\tilde{\mathcal{I}})$. From the middle column of the above diagram, we have an exact sequence of complexes: $0 \rightarrow F(Z) \xrightarrow{\Lambda(j)}$.
\( \Lambda(J) \rightarrow \Lambda(L) \rightarrow 0 \), which implies that \( \Lambda(j) \) is a trivial cofibration in \( \mathcal{M}^\text{co}_{\text{sing}} \). Moreover, we have \( F(p) = \Lambda(q) \cdot \Lambda(j) \).

Recall that a complex \( I \in \text{ex}\tilde{I} \) is called totally acyclic (of injectives) if for any injective module \( M \), the complex \( \text{Hom}_R(M, I) \) remains exact. Dually, totally acyclic complex of projectives is defined. We note that over a left-Gorenstein ring, every injective module is of finite projective dimension, hence the category of all totally acyclic complexes of injectives and \( \text{ex}\tilde{I} \) coincide; the dual for totally acyclic complexes of projectives also holds.

By the completeness of the cotorsion pair \( \langle \text{ex}\tilde{I}, \text{co}\tilde{I} \rangle \), for \( \Lambda(I) \) there is an exact sequence 0 \( \rightarrow \Lambda(I) \rightarrow E \rightarrow D \rightarrow 0 \) with \( E \in \text{ex}\tilde{I} \) and \( D \in \text{co}\tilde{I} \). Since every complex in \( \text{ex}\tilde{I} \) is totally acyclic, we have \( \Lambda(I) \in \langle \text{ex}\tilde{I} \rangle \), and then \( E \in \text{ex}\tilde{I} \cap \langle \text{co}\tilde{I} \rangle \). Now we consider the following push-out diagram:

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
0 & \downarrow & \Lambda(I) & \downarrow \Lambda(q) & \downarrow F(Y) & \downarrow 0 \\
& \downarrow r & \Lambda(J) & \downarrow F(Y) & \downarrow 0 \\
0 & \downarrow & E & \downarrow & C & \downarrow & F(Y) & \downarrow 0 \\
& & D & & D & & D & & D \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

where \( r \) and \( s \) are respectively trivial cofibration and trivial fibration in \( \mathcal{M}^\text{co}_{\text{sing}} \). Hence \( \Lambda(q) = sr \) is a weak equivalence in \( \mathcal{M}^\text{co}_{\text{sing}} \). Then \( F(f) = F(p) \cdot F(i) = \Lambda(q) \cdot \Lambda(j) \cdot F(i) \) is a weak equivalence, as desired. \( \square \)

**Lemma 2.6.** Let \( Y \) be an exact complex of injective \( R \)-modules. Then \( \varepsilon : FG(Y) \rightarrow Y \) is a weak equivalence in \( \mathcal{M}^\text{co}_{\text{sing}} \), where \( \varepsilon \) is the counit of the adjoint pair \( (F, G) \).

**Proof.** For \( Y, \ G(Y) = \Lambda\Theta(Y) = \cdots \rightarrow 0 \rightarrow \text{Ker}d_Y^0 \rightarrow 0 \rightarrow \cdots \) is a stalk complex with \( \text{Ker}d_Y^0 \) concentrated in degree zero. It is easy to see that \( FG(Y) = G(Y) \). Then the map \( \varepsilon : FG(Y) \rightarrow Y \) is given by \( \varepsilon_0 : \text{Ker}d_Y^0 \rightarrow Y_0 \) being a natural embedding and \( \varepsilon_i = 0 \) for any \( i \neq 0 \). Let \( C = \text{Coker}\varepsilon \).

Then \( C = \cdots \rightarrow Y_2 \overset{d_Y^2}{\rightarrow} Y_1 \overset{d_Y^1}{\rightarrow} \text{Im}d_Y^0 \overset{i}{\rightarrow} Y_0 \overset{d_Y^{-1}}{\rightarrow} Y_{-1} \rightarrow \cdots \), where \( i \) is an embedding. Let \( Y_- = \cdots \rightarrow Y_2 \overset{d_Y^2}{\rightarrow} Y_1 \rightarrow 0 \) be a hard truncation, \( D = 0 \rightarrow \text{Im}d_Y^0 \overset{i}{\rightarrow} Y_0 \overset{d_Y^{-1}}{\rightarrow} Y_{-1} \overset{i}{\rightarrow} Y_{-2} \rightarrow \cdots \). Then there is an exact sequence of complexes 0 \( \rightarrow Y_- \rightarrow C \rightarrow D \rightarrow 0 \).

Let \( E \) be any \( R \)-complex in \( \text{ex}\tilde{I} \). Since \( R \) is left-Gorenstein, \( E \) is totally acyclic, and then for any \( Y_i, \text{Hom}_R(Y_i, E) \) is an exact complex. By [7 Lemma 2.4], the complex \( \text{Hom}_R(Y_-, E) \) is exact. Note that \( D \) is an exact sequence, and then \( \text{Hom}_R(D, E_i) \) is an exact complex for any \( i \in \mathbb{Z} \). By [7 Lemma 2.5], the complex \( \text{Hom}_R(D, E) \) is exact. Then it follows from the short
exact sequence

$$0 \to \text{Hom}_R(D, E) \to \text{Hom}_R(C, E) \to \text{Hom}_R(Y \otimes D, E) \to 0$$

that the complex $\text{Hom}_R(C, E)$ is exact. This implies that every map from $C$ to any complex in $ex\tilde{I}$ is null homotopic. Then $C \in \perp \text{ex}\tilde{I}$. Hence, $\varepsilon : FG(Y) \to Y$ is a trivial cofibration in $\mathcal{M}_{\text{sing}}^c$, and moreover, $\varepsilon$ is a weak equivalence.

Recall that a module $M$ is Gorenstein projective if $M$ is a syzygy of a totally acyclic complex of projective modules; and dually, Gorenstein injective modules are defined; see [2]. We use $\mathcal{GP}$ to denote the class of Gorenstein projective modules. By [3], over a left-Gorenstein ring $(\mathcal{GP}, \mathcal{W})$ is a complete cotorsion pair, where $\mathcal{W}$ is the class of modules with finite projective (injective) dimension. We proved more in [20, Theorem 2.7] by showing that the cotorsion pair $(\mathcal{GP}, \mathcal{W})$ is cogenerated by a set, i.e. there exists a set $S$ such that $\mathcal{W} = \{S\}^\perp$. This also implies the completeness of $(\mathcal{GP}, \mathcal{W})$, and generalizes Hovey’s Gorenstein projective model structure of $\text{Mod}(R)$ (see [16, Theorem 8.3, 8.6]) from Iwanaga-Gorenstein rings to left-Gorenstein rings.

**Lemma 2.7.** Let $Y$ be an exact complex of injective $R$-modules. Then $F(q) : FQG(Y) \to FG(Y)$ is a weak equivalence in $\mathcal{M}_{\text{sing}}^c$, where $q : QG(Y) \to G(Y)$ is a cofibrant replacement in the model category $(\text{Ch}(R), \mathcal{M}_{\text{sing}}^c)$.

**Proof.** For $Y$, $G(Y) = FG(Y) = \cdots \to 0 \to \text{Ker}d_0^Y \to 0 \to \cdots$. By the completeness of the cotorsion pair $(\mathcal{GP}, \mathcal{W})$, there is an exact sequence of $R$-modules $0 \to W \to M \to \text{Ker}d_0^Y \to 0$ with $M \in \mathcal{GP}$ and $W \in \mathcal{W}$. Consider the totally acyclic complex of $M$, we have a short exact sequence $0 \to K \to P \xrightarrow{q} G(Y) \to 0$, see the following diagram

\[
\begin{array}{cccccccc}
K = \cdots & \to & P_1 & \to & K_0 & \xrightarrow{\pi} & P_1 & \to & P_2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow W & & \downarrow & & \downarrow & & \\
P = \cdots & \to & P_1 & \to & P_0 & & P_1 & \to & P_2 & \to & \cdots \\
\downarrow M & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
G(Y) = \cdots & \to & 0 & \to & \text{Ker}d_0^Y & \to & 0 & \to & 0 & \to & \cdots
\end{array}
\]

Let $K_{\otimes} = \cdots \to P_2 \to P_1 \to \text{Ker}\pi \to 0$ and $K_{\otimes 0} = 0 \to W \to P_1 \to P_2 \to \cdots$. Then there is a short exact sequence of complexes $0 \to K_{\otimes 0} \to K \to K_{\otimes 0} \to 0$. Let $T$ be any complex in $ex\tilde{P}$. Note that $T$ is totally acyclic. Then it follows from [7, Lemma 2.5] that the complex $\text{Hom}_R(T, K_{\otimes 0})$ is exact, and this implies that $K_{\otimes 0} \in (ex\tilde{P})^\perp$. Note that $H_i(K) = 0$ for any $i \neq -1$, then $K_{\otimes 0}$ is an exact complex. For any morphism $f : T \to K_{\otimes 0}$, we consider the
Let $s_i = 0$ for any $i < 0$. Since $d^K_1 : P_1 \to \text{Ker} \pi$ is an epic and $T_0$ is a projective module, there is a map $s_0 : T_0 \to P_1$ such that $f_0 = d^K_1 s_0$. Since 
\[ d^K_1(f_1 - s_0 d^T_1) = d^K_1 f_1 - d^K_1 s_0 d^T_1 = d^K_1 f_1 - f_0 d^T_1 = 0, \]
then $f_1 - s_0 d^T_1 : T_1 \to \text{Ker} d^K_1$, and there exists a map $s_1 : T_1 \to P_2$ such that $f_1 - s_0 d^T_1 = d^K_2 s_1$.

Analogous to comparison theorem, we inductively get homotopy maps $\{s_i\}$ such that $f$ is null homotopic. Then $K_{c0} \in (\text{ex} \tilde{\mathcal{P}})^\perp$. Thus, we have $K \in (\text{ex} \tilde{\mathcal{P}})^\perp$. Note that for any object in the model category $(\text{Ch}(R), \mathcal{M}^\text{ctr}_{\text{sing}})$, its cofibrant replacement is precisely a special $\text{ex} \tilde{\mathcal{P}}$-precovers. Then it follows from the short exact sequence $0 \to K \to P \xrightarrow{q} G(Y) \to 0$ that $P$ is a cofibrant replacement of $G(Y)$, and we can set $QG(Y) = P$.

Note that $F(K) = \cdots \to 0 \to W \to 0 \to \cdots$. Since $W$ is a module of finite projective dimension, for any complex $E \in \text{ex} \tilde{\mathcal{L}}$, $\text{Hom}_R(W, E)$ is exact. This implies that $F(K) \in (\text{ex} \tilde{\mathcal{L}})$. For $F(K)$, there is an exact sequence $0 \to F(K) \to I \to L \to 0$ with $I \in \text{ex} \tilde{\mathcal{L}}$ and $L \in (\text{ex} \tilde{\mathcal{L}})$. Similar to the above argument, we consider the following push-out diagram:

\[
\begin{array}{ccccccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F(K) & \longrightarrow & F(P) & \longrightarrow & F(q) \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
0 & \longrightarrow & I & \longrightarrow & J & \longrightarrow & GF(Y) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \longrightarrow & L & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

It follows that $F(q) = pi$ is a weak equivalence in $\mathcal{M}^\text{co}_{\text{sing}}$, where $i$ and $p$ are trivial cofibration and trivial fibration in $\mathcal{M}^\text{co}_{\text{sing}}$, respectively. This completes the proof. \hfill \Box

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References

[1] H. Becker, Models for singularity categories, Adv. Math., 254 (2014) 187-232.
[2] A.A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque, vol. 100, 1982.
[3] A. Beligiannis, The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co)stabilization. Comm. Algebra, 28 (2000) 4547-4596.
[4] A. Beligiannis, I. Reiten, Homological and homotopical aspects of torsion theories. Mem. Amer. Math. Soc. 188(883), 2007.
[5] P.A. Bergh, D.A. Jorgensen, S. Oppermann, The Gorenstein defect category, Quart. J. Math. 66 (2015) 459-471.
[6] X.W. Chen, Homotopy equivalences induced by balanced pairs, J. Algebra 324 (2010) 2718-2731.
[7] L.W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions-A functorial description with applications, J. Algebra 302 (2006), 231-279.
[8] W.G. Dwyer, J. Spalinski, Homotopy Theories and Model Categories, Handbook of algebraic topology (Amsterdam), North-Holland, Amsterdam, 1995, pp. 73-126.
[9] E.E. Enochs, O.M.G. Jenda, Relative Homological Algebra, De Gruyter Expositions in Mathematics no. 30, New York: Walter De Gruyter, 2000.
[10] P. Jørgensen, The homotopy category of complexes of projective modules, Adv. Math. 193 (2005), 223-232.
[11] J. Gillespie, The flat model structure on Ch(R), Trans. Amer. Math. Soc. 356 (2004) 3369-3390.
[12] J. Gillespie, Cotorsion pairs and degreewise homological model structures, Homol. Homotopy Appl. 10 (2008) 283-304.
[13] J. Gillespie, Gorenstein complexes and recollements from cotorsion pairs, Adv. Math. 291 (2016) 859-911.
[14] P.S. Hirschhorn, Model Categories and Their Localizations, Mathematical Surveys and Monographs vol. 99, Amer. Math. Soc., 2003.
[15] M. Hovey, Model Categories, Mathematical Surveys and Monographs vol. 63, Amer. Math. Soc., 1999.
[16] M. Hovey, Cotorsion pairs, model category structures, and representation theory, Math. Z. 241 (2002) 553-592.
[17] S. Iyengar, H. Krause, Acyclicity versus total acyclicity for complexes over Noetherian rings, Doc. Math. 11 (2006) 207-240.
[18] H. Krause, The stable derived category of a Noetherian scheme, Compos. Math. 141 (5) (2005) 1128-1162.
[19] D.G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics no. 43, Springer-Verlag, 1967.
[20] W. Ren, Gorenstein projective modules and Frobenius extensions, Sci. China Math., 61(7) (2018) 1175-1186.
[21] W. Ren, Applications of cotorsion triples, Comm. Algebra, in press. arXiv:1404.7598v5