Homomorphisms Between Algebras of Holomorphic Functions on the Infinite Polydisk

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Abstract
We study the vector-valued spectrum $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$, that is, the set of nonzero algebra homomorphisms from $\mathcal{H}_\infty(B_{c_0})$ to $\mathcal{H}_\infty(B_{c_0})$ which is naturally projected onto the closed unit ball of $\mathcal{H}_\infty(B_{c_0}, \ell_\infty)$, likewise the scalar-valued spectrum $\mathcal{M}_\infty(B_{c_0})$ which is projected onto $B_{\ell_\infty}$. Our itinerary begins in the scalar-valued spectrum $\mathcal{M}_\infty(B_{c_0})$: by expanding a result by Cole et al. (Michigan Math J 39(3):551–569, 1992), we prove that in each fiber, there are $2^c$ disjoint analytic Gleason isometric copies of $B_{\ell_\infty}$. For the vector-valued case, building on the previous result we obtain $2^c$ disjoint analytic Gleason isometric copies of $B_{\mathcal{H}_\infty(B_{c_0}, \ell_\infty)}$ in each fiber. We also take a look at the relationship between fibers and Gleason parts for both vector-valued spectra $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$ and $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$.

Keywords Spectrum · Algebras of holomorphic functions · Homomorphisms of algebras

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1 Introduction

In 1909, David Hilbert wrote an article [23], which is now considered as one of the pioneering works of the theory of analytic functions in infinitely many variables. There he claimed his intention of transferring the main theorems about analytic functions of several variables to the infinite dimensional setting. Following his path, the open unit ball of $c_0$ appears as a natural domain for analytic functions in infinitely many variables since it is the usual infinite dimensional analogue of the polydisk $\mathbb{D}^n$. While Infinite Dimensional Holomorphy has been developed in many directions besides Hilbert’s initial approach, the open unit ball of $c_0$ has continued to be a favorite domain for the study of bounded holomorphic functions.

Our interest here is to describe the set of homomorphisms between algebras of bounded holomorphic functions on the open unit ball of $c_0$. While pursuing this objective, we give a description of the fibers of the spectrum of the algebra of bounded holomorphic functions on the open unit ball of $c_0$ which improves and completes results of [3,4,10]. In order to explain our goal more precisely, we need to recall some definitions and introduce some notation.

For a complex Banach space $X$, let $\mathcal{H}^\infty(B_X)$ be the space of bounded holomorphic functions on $B_X$ (the open unit ball of $X$). The (scalar-valued) spectrum of this uniform algebra is the set $\mathcal{M}_\infty(B_X) = \{ \varphi : \mathcal{H}^\infty(B_X) \to \mathbb{C} \text{ nonzero algebra homomorphisms} \}$. Since $X^*$ is contained in $\mathcal{H}^\infty(B_X)$, there is a natural projection $\pi : \mathcal{M}_\infty(B_X) \to B_{X^{**}}$ given by $\pi(\varphi) = \varphi|_{X^*}$. Through this projection, $\mathcal{M}_\infty(B_X)$ has a fibered structure: for each $z \in B_{X^{**}}$ the set $\pi^{-1}(z) = \{ \varphi \in \mathcal{M}_\infty(B_X) : \pi(\varphi) = z \}$ is called the fiber over $z$. For an infinite dimensional space $X$, the fibers of its spectrum have been studied in several articles, e.g. [2,4,10,17]. Frequently, a useful sub-algebra of $\mathcal{H}^\infty(B_X)$ is considered: $\mathcal{A}_u(B_X)$ is the space of holomorphic functions on $B_X$ which are uniformly continuous. The spectrum $\mathcal{M}_u(B_X) = \{ \varphi : \mathcal{A}_u(B_X) \to \mathbb{C} \text{ nonzero algebra homomorphisms} \}$ is similarly projected onto $B_{X^{**}}$ and the fibering over this set is consequently defined. By [1,11], there is a canonical extension $[f \mapsto \widetilde{f}]$ from $\mathcal{H}^\infty(B_X)$ to $\mathcal{H}^\infty(B_{X^{**}})$ which is an isometric homomorphism of Banach algebras. In this way, for the spectrum $\mathcal{M}_\infty(B_X)$, in the fiber over each $z \in B_{X^{**}}$ there is a distinguished element $\delta_z$ given by $\delta_z(f) = \widetilde{f}(z)$. The extension $[f \mapsto \widetilde{f}]$ is also an isometric algebra homomorphism from $\mathcal{A}_u(B_X)$ to $\mathcal{A}_u(B_{X^{**}})$. Hence, for every $f \in \mathcal{A}_u(B_X)$ the function $\widetilde{f}$ is uniformly continuous in $B_{X^{**}}$ so it can be extended to $B_{X^{**}}$ (the closed unit ball of $X^{**}$). Thus, for the spectrum $\mathcal{M}_u(B_X)$, there is a distinguished element $\delta_z$ in the fiber over $z$ for every $z \in B_{X^{**}}$.

For $\mathcal{A} = \mathcal{H}^\infty(B_X)$ or $\mathcal{A}_u(B_X)$, the spectrum $\mathcal{M}(\mathcal{A})$ is partitioned into equivalence classes called Gleason parts. Since $\mathcal{M}(\mathcal{A})$ is contained in the unit sphere of $\mathcal{A}^*$ it is clear that $|\varphi - \psi| \leq 2$, for every $\varphi, \psi \in \mathcal{M}(\mathcal{A})$. For $\varphi \in \mathcal{M}(\mathcal{A})$, the Gleason part of $\varphi$ is the set

$$G\mathcal{P}(\varphi) = \{ \psi : \rho(\varphi, \psi) < 1 \} = \{ \psi : \| \varphi - \psi \| < 2 \},$$

where $\rho(\varphi, \psi) = \sup\{ |\varphi(f)| : f \in \mathcal{A}, \| f \| \leq 1, \psi(f) = 0 \}$. In $\mathcal{M}(\mathcal{A})$, the usual distance between the elements $|\varphi - \psi|$ is referred as the Gleason metric and $\rho(\varphi, \psi)$ is called the pseudo-hyperbolic distance. General information concerning Gleason
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parts can be seen in [6,20] while the description in the case of the spectrum of $\mathcal{H}^\infty(\mathbb{D})$ can be found in [21,24]. The depiction of Gleason parts for $\mathcal{M}_\infty(B_X)$ and $\mathcal{M}_u(B_X)$ for infinite dimensional $X$ (with a special focus in the case $X = c_0$) was initiated in [3]. Within Infinite Dimensional Holomorphy, Gleason parts have also been used in studying homomorphisms between uniform algebras (for instance in [18,19]).

In [15] we consider, for complex Banach spaces $X$ and $Y$, the set $\mathcal{M}_\infty(B_X, B_Y)$ which we called vector-valued spectrum, defined by

$$\mathcal{M}_\infty(B_X, B_Y) = \{ \Phi : \mathcal{H}^\infty(B_X) \to \mathcal{H}^\infty(B_Y) \text{ nonzero algebra homomorphism} \}.$$  

Every $\Phi \in \mathcal{M}_\infty(B_X, B_Y)$ is continuous with $\| \Phi \| = 1$ since $B_Y$ is connected and $\Phi(1) = 1$. Additionally, $\mathcal{M}_\infty(B_X, B_Y)$ is a weak-star compact subset of the unit sphere of $\mathcal{L}(\mathcal{H}^\infty(B_X), \mathcal{H}^\infty(B_Y))$ (see for instance [15, p. 3]). The vector-valued spectrum is fibered over the closed unit ball of $\mathcal{H}^\infty(B_Y, X^{**}) = \{ f : B_Y \to X^{**} \text{ bounded holomorphic functions} \}$ through the projection (with image $\overline{B}_{\mathcal{H}^\infty(B_Y, X^{**})}$)

$$\xi : \mathcal{M}_\infty(B_X, B_Y) \to \mathcal{H}^\infty(B_Y, X^{**}), \quad \Phi \mapsto \left[ y \mapsto [x^* \mapsto \Phi(x^*)(y)] \right].$$

As in the scalar-valued case, this projection gives rise to a fibered structure of the spectrum. For each $g \in \overline{B}_{\mathcal{H}^\infty(B_Y, X^{**})}$, the fiber over $g$ is the set

$$\mathcal{F}(g) = \{ \Phi \in \mathcal{M}_\infty(B_X, B_Y) : \xi(\Phi) = g \}.$$  

We can associate to each function $g \in \mathcal{H}^\infty(B_Y, X^{**})$ satisfying $g(B_Y) \subset B_{X^{**}}$ a composition homomorphism $C_g \in \mathcal{M}_\infty(B_X, B_Y)$ given by

$$C_g(f) = \tilde{f} \circ g, \quad \text{for all } f \in \mathcal{H}^\infty(B_X),$$

(where $\tilde{f} \in \mathcal{H}^\infty(B_{X^{**}})$ is the canonical extension of $f$ referred to above). It is easy to see that $C_g$ belongs to $\mathcal{F}(g)$ and so each fiber over a function $g \in \mathcal{H}^\infty(B_Y, X^{**})$ with $g(B_Y) \subset B_{X^{**}}$ has a distinguished element.

In a previous article [15], we studied the fibers for this spectrum and exhibited conditions assuring that analytic copies of balls can be injected into some particular fibers. Now, focusing on the case $X = Y = c_0$ we are able to prove that in every fiber we can inject, isometrically for the Gleason metric, infinitely many disjoint analytic copies of the unit ball of $\mathcal{H}^\infty(B_{c_0}, \ell_\infty)$. To achieve this result we need first to produce a similar statement for the scalar-valued spectrum: there are infinitely many disjoint analytic Gleason isometric copies of $B_{\ell_\infty}$ in any fiber of $\mathcal{M}_\infty(B_{c_0})$. This is our main result in Sect. 2. It is done by expanding a construction due to Cole et al. [10, Th. 6.7] giving a complete answer to a question posed in [4] and an improvement of [3, Cor. 3.12]. In Sect. 3 the vector-valued version of this result is presented.
The notion of Gleason part has a version for the vector-valued spectrum. Indeed, for \( \Phi \in \mathcal{M}_\infty(B_X, B_Y) \) we can define

\[
\mathcal{GP}(\Phi) = \{ \Psi : \| \Phi - \Psi \| < 2 \} = \{ \Psi : \sigma(\Phi, \Psi) = \sup_{y \in B_Y} \rho(\delta_y \circ \Phi, \delta_y \circ \Psi) < 1 \}.
\]

As in the scalar-valued case, this leads to a partition of \( \mathcal{M}_\infty(B_X, B_Y) \) into equivalence classes. This concept, with the name of norm vicinity or path component, was previously studied in several articles (see, for instance [5,9,18,22,26,27]). In [15] we have some relationships between fibers and Gleason parts for the spectrum \( \mathcal{M}_\infty(B_X, B_Y) \). Now we continue in this line devoting ourselves to the case \( X = Y = c_0 \). Inspired by what is done in [3] for the scalar-valued spectrum, we begin by studying Gleason parts for the simpler spectrum

\[
\mathcal{M}_{u,\infty}(B_{c_0}, B_{c_0}) = \{ \Phi : A_u(B_{c_0}) \to \mathcal{H}^\infty(B_{c_0}) \text{ nonzero algebra homomorphisms} \}.
\]

Then, we look into Gleason parts for the more complex spectrum \( \mathcal{M}_\infty(B_{c_0}, B_{c_0}) \). These are the topics of Sect. 4.

For general theory about Infinite Dimensional Holomorphy, we refer the reader to the classical books [7,16,28] or to the recently published beautiful monograph [12]. Note that in this last book the space \( \mathcal{H}^\infty(B_{c_0}) \) has a leading role: it is introduced in (and studied from) Chapter 2 while the theory of holomorphic functions on arbitrary Banach spaces is put off until Chapter 15. The exact quotation from Hilbert’s article that we have referred to at the beginning of this Introduction can be seen either in [12, p. 10] or in [29, p. 188].

### 2 Fibers for the Scalar-Valued Spectrum \( \mathcal{M}_\infty(B_{c_0}) \)

We begin by describing the fibers of the scalar-valued spectrum \( \mathcal{M}_\infty(B_{c_0}) \) because, as in [15], we can profit from this knowledge in order to work in the vector-valued framework.

Recall that in the case of the spectrum \( \mathcal{M}_\infty(\mathbb{D}) \), each fiber over an interior point is a singleton (that is, \( \pi^{-1}(z) = \{ \delta_z \} \) for every \( z \in \mathbb{D} \)) and each fiber over a point of the torus contains an analytic copy of the disk \( \mathbb{D} \subset \pi^{-1}(z) \) for every \( |z| = 1 \). The situation in the infinite dimensional setting is truly different. In [10, Th. 6.7], Cole, Gamelin, and Johnson defined an analytic injection

\[
\Phi : B_{\ell_\infty} \times \bar{B}_{\ell_\infty} \to \mathcal{M}_\infty(B_{c_0}),
\]

such that for all \( z \in B_{\ell_\infty} \) the image of \( \Phi(z, -) \) is contained in the fiber \( \pi^{-1}(z) \). As a result, for each \( z \in B_{\ell_\infty} \), the fiber over \( z \) contains an analytic copy of \( B_{\ell_\infty} \). Since the spectrum \( \mathcal{M}_\infty(B_{c_0}) \) is projected onto \( B_{\ell_\infty} \), the question about whether the same result holds for the fibers over those \( z \in S_{\ell_\infty} \) naturally arises. Aron, Falcó, García, and Maestre [4, Th. 2.2], through a different construction managed to produce an
analytic copy of $B_{\ell_\infty}$ in the fiber over $z$, for points $z$ in the infinite torus $\mathbb{T}^\infty$ (that is, those $z$ satisfying $|z_n| = 1$ for all $n \in \mathbb{N}$). They could also extend this construction to boundary points $z$ having infinitely many coordinates with modulus 1. However, the question for boundary points with no (or finitely many) modulus 1 coordinates has remained open. Specifically, we can read in [4, Rmk. 2.10] the following: “... we do not know if $B_{\ell_\infty}$ can be embedded in the fiber over $z$, for $z$ in the unit sphere of $\ell_\infty$ but $|z_n| < 1$ for all $n$, as for example $(\frac{n-1}{n})$”.

Our main result in this section answers this question in the affirmative and completes the picture of the fibers. We obtain it by modifying the construction of Cole et al. to reach points $z$ with a subsequence of coordinates at a positive distance from 1 and then, by means of Gleason isometries between fibers, we manage to get to all $z$’s. In this way we prove that there is an analytic injection from $B_{\ell_\infty}$ into the fiber over $z$, for any $z \in \overline{B}_{\ell_\infty}$. Moreover, as Cole et al. had proved for the fiber over 0, we obtain that the injection is in fact an isometry for the Gleason metric.

It is important to comment that the above result has simultaneously been proved by Choi et al. [8]. However, their argument is not the same; specifically the construction of the injection is different. For us, in order to apply this procedure to the vector-valued spectrum (see Theorem 3.4) both the statement of the theorem and the construction performed in the proof of the theorem are relevant. In fact, we make use of our construction to prove the vector-valued result.

In addition, we can go a step further to obtain not just one analytic copy of the ball $B_{\ell_\infty}$ in each fiber but $2^c$ disjoint copies. Indeed, we prove that for any $z \in \overline{B}_{\ell_\infty}$ and for each $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$ there is an analytic Gleason isometry $\Phi^\eta_z$ from $B_{\ell_\infty}$ into the fiber over $z$ and that any two of these copies are in different Gleason parts.

We want to point out that a previous version of the next theorem is referred to in [3, Rmk. 3.5]. Precisely, at the moment that article was written, the statement of our Theorem 2.1 was that there is one analytic Gleason isometric copy of the ball $B_{\ell_\infty}$ in each fiber. We then obtained this improvement with infinitely many disjoint balls in each fiber. Note also that this current statement turns out to be an improvement of [3, Cor. 3.12] because there it was proved the containment of $2^c$ disks (instead of balls) lying in different Gleason parts in the fiber over each $z$ in the open (instead of closed) unit ball of $\ell_\infty$.

**Theorem 2.1** For every $z \in \overline{B}_{\ell_\infty}$, there is a map

$$\Phi_z: \beta(\mathbb{N}) \setminus \mathbb{N} \times B_{\ell_\infty} \to \pi^{-1}(z) \subset \mathcal{M}_\infty(B_{c_0})$$

satisfying

1. For all $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$, the mapping $\Phi^\eta_z: B_{\ell_\infty} \to \pi^{-1}(z)$ given by $\Phi^\eta_z(w) = \Phi(z(\eta, w))$ is an analytic Gleason isometry.
2. If $\eta_1 \neq \eta_2$ are in $\beta(\mathbb{N}) \setminus \mathbb{N}$, then the images $\Phi^\eta_1_z(B_{\ell_\infty})$ and $\Phi^\eta_2_z(B_{\ell_\infty})$ lie in different Gleason parts.

The proof of this theorem is performed in several steps. To prove that our analytic mappings are isometric (from the Gleason metric of $B_{\ell_\infty}$ to the Gleason metric of the spectrum $\mathcal{M}_\infty(B_{c_0})$) and to deduce that images associated to different elements in
\( \beta(\mathbb{N}) \setminus \mathbb{N} \) lie in different Gleason parts we make use of the following technical lemma regarding infinite products.

**Lemma 2.2** Let \((\alpha_k)\) be an increasing sequence of positive real numbers converging to 1, such that \(\sum_j (1 - \alpha_j) < \infty\). Consider a point \(z = (z_j) \in \overline{B}_{\ell_\infty} \) for which there exists \(\delta > 0\) satisfying \(|z_j - 1| > \delta\) for all \(j\). Then, for each \(N \in \mathbb{N}\) such that \(1 - \alpha_j < \delta/4\) for all \(j \geq N\), the following infinite product converges (to a nonzero number):

\[
\prod_{j=N}^{\infty} \frac{\alpha_j - z_j}{1 - \alpha_j z_j}.
\]

(2)

Additionally, if \((\ell(k))_k\) is an increasing sequence of positive integers converging to \(\infty\), then

\[
\lim_{k \to \infty} \prod_{j=N}^{\ell(k)} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} = \prod_{j=N}^{\infty} \frac{\alpha_j - z_j}{1 - \alpha_j z_j}.
\]

(3)

**Proof** We first show that the product in (2) is convergent. Indeed, this is derived from the inequality:

\[
\left| 1 - \frac{\alpha_j - z_j}{1 - \alpha_j z_j} \right| = \left| \frac{1 + z_j}{1 - \alpha_j z_j} \right| (1 - \alpha_j) \leq \frac{2}{\delta^3/4} (1 - \alpha_j),
\]

where the last term is summable by our hypothesis.

Secondly we study, for each \(k \in \mathbb{N}\), the rate of convergence for the infinite product \(\prod_{j=N}^{\infty} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j}\). We have the following bound independent of \(k\):

\[
\left| 1 - \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} \right| = \left| \frac{1 + \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} \right| (1 - \alpha_j) \leq \frac{2}{\delta/2} (1 - \alpha_j).
\]

Then, for any given \(\varepsilon > 0\), there is a number \(N_1 \in \mathbb{N}\) such that for every \(\tilde{N} \geq N_1\),

\[
\left| \prod_{j=N}^{\infty} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} - \prod_{j=N}^{\tilde{N}} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} \right| < \varepsilon/4,
\]

for all \(k \in \mathbb{N}\), and

\[
\left| \prod_{j=N}^{\infty} \frac{\alpha_j - z_j}{1 - \alpha_j z_j} - \prod_{j=N}^{\tilde{N}} \frac{\alpha_j - z_j}{1 - \alpha_j z_j} \right| < \varepsilon/4.
\]
Now, we can find \( k_0 \in \mathbb{N} \) such that for every \( k \geq k_0 \) we get
\[
\left| \prod_{j=N}^{N_1} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} - \prod_{j=N}^{N_1} \frac{\alpha_j - z_j}{1 - \alpha_j z_j} \right| < \varepsilon / 4.
\]

Finally, take \( k'_0 = \min\{k : \ell(k) \geq N_1\} \) and \( k_1 = \max\{k_0, k'_0\} \). Hence, for \( k \geq k_1 \), the expression
\[
\left| \ell(k) \prod_{j=N}^{\infty} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} - \prod_{j=N}^{\infty} \frac{\alpha_j - z_j}{1 - \alpha_j z_j} \right|
\]
is bounded by
\[
\left| \ell(k) \prod_{j=N}^{\infty} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} - \prod_{j=N}^{\infty} \frac{\alpha_j - z_j}{1 - \alpha_j z_j} \right| + \left| \prod_{j=N}^{N_1} \frac{\alpha_j - \alpha_k z_j}{1 - \alpha_j \alpha_k z_j} - \prod_{j=N}^{N_1} \frac{\alpha_j - z_j}{1 - \alpha_j z_j} \right| < \varepsilon.
\]
Thus, the desired limit is proved.

As we have already mentioned, Lemma 2.2 will be used to prove several Gleason isometries inside the fiber over \( z \). Even if this lemma only works for points \( z \) with coordinates at a positive distance from 1, we can overcome the restriction through the following remark.

**Remark 2.3** Let \( z, w \in \overline{B}_\ell\infty \) such that \( |z_n| = |w_n|, \forall n \in \mathbb{N} \). Then, the fibers \( \pi^{-1}(z) \) and \( \pi^{-1}(w) \) are Gleason isometric.

Indeed, [3, Prop. 1.6] gives sufficient conditions for an automorphism \( \Phi : B_X \rightarrow B_X \) to induce a Gleason isometry in the spectrum. In our particular setting, we can write \( w_n = \lambda_n z_n \) for \( \lambda_n \in \mathbb{T} \) and define the mapping
\[
\theta : \ell_\infty \rightarrow \ell_\infty \quad \theta(x) = (\lambda_n x_n)_n.
\]
Note that \( \theta \) is an isometric linear mapping satisfying \( \theta(B_{c_0}) = B_{c_0} \). Additionally, \( \theta \) coincides with the bitranspose of its restriction to \( c_0 \). It follows that the induced mapping \( \Lambda \theta : \mathcal{M}_\infty(B_{c_0}) \rightarrow \mathcal{M}_\infty(B_{c_0}) \) is an onto Gleason isometry taking the fiber over \( z \) onto the fiber over \( w \).

**Proof of Theorem 2.1** Fix \( (\alpha_k) \) an increasing sequence of positive real numbers converging to 1 such that \( \sum_k (1 - \alpha_k) < \infty \) and take a sequence of non-negative integers
(m_k)_k satisfying m_k + k < m_{k+1}. We first prove the result under additional hypotheses over z. Then, we extend it to any z ∈ \overline{B}_{\ell_\infty}.

Step 1: Let z ∈ \overline{B}_{\ell_\infty} for which there exists \delta > 0 satisfying |z_k - 1| > \delta for all k ∈ \mathbb{N}. Define, for each k ∈ \mathbb{N}, the mapping \Phi_k^z : B_{\ell_\infty} \to B_{\ell_\infty} as follows

\[
(\Phi_k^z(w))_j = \begin{cases} 
\alpha_k z_j, & \text{for } 1 \leq j \leq k \\
0, & \text{for } k + 1 \leq j \leq m_k \\
\alpha_k - w_i, & \text{for } j = m_k + i, 1 \leq i \leq k \\
0, & \text{for } j > m_k + k.
\end{cases}
\]

By identifying \Phi_k^z(w) with \delta \Phi_k^z(w) \in M_\infty(B_{c_0}) we can now define \Phi^z : N \times B_{\ell_\infty} \to M_\infty(B_{c_0})

\[
\Phi^z(k, w) = \Phi_k^z(w).
\]

Since M_\infty(B_{c_0}) is a weak-star compact subset of \(\mathcal{H}^\infty(B_{c_0})^*\), for each w ∈ B_{\ell_\infty} there is a unique continuous extension

\[
\Phi^z(-, w) : \beta(\mathbb{N}) \to M_\infty(B_{c_0}).
\]

We now write \Phi^\eta_z(w) = \Phi_z(\eta, w) for \eta ∈ \beta(\mathbb{N}). Note that, if n ≤ k,

\[
\langle \Phi_k^z(w), e_n \rangle = \alpha_k z_n \xrightarrow{k \to \infty} z_n.
\]

So π(\Phi^\eta_z(w)) = z for every w ∈ B_{\ell_\infty} and \eta ∈ \beta(\mathbb{N}) \setminus \mathbb{N}, meaning that the image of \Phi^\eta_z is contained in \pi^{-1}(z). To see that \Phi^\eta_z is an analytic mapping, note that, for a given f ∈ \mathcal{H}^\infty(B_{c_0}), the sequence (f \circ \Phi_k^z)_k is contained in the weak-star compact set \(\|f\|_{\mathcal{H}^\infty(B_{\ell_\infty})}\). A standard argument then yields that f \circ \Phi^\eta_z is indeed in \(\|f\|_{\mathcal{H}^\infty(B_{\ell_\infty})}\) and, in particular, holomorphic.

Now we need to check that \Phi^\eta_z is an isometry. By [3, Th. 2.4] (see also [10, (6.1)]) and the fact that the mapping [λ ↦ \frac{\mu - \lambda}{\mu - \overline{\lambda}}] (\mu ∈ \mathbb{D}) is an automorphism of the unit disk \mathbb{D}, we know that for all w, v ∈ B_{\ell_\infty} and k ∈ \mathbb{N},

\[
\rho(\delta \Phi_k^z(w), \delta \Phi_k^z(v)) = \sup_{j \in \mathbb{N}} \left| \frac{\Phi_k^z(w)_j - \Phi_k^z(v)_j}{1 - \Phi_k^z(w)_j \Phi_k^z(v)_j} \right| \leq \rho(\delta w, \delta v).
\]

From this, it is easily derived that

\[
\|\Phi^\eta_z(w) - \Phi^\eta_z(v)\| \leq \|\delta w - \delta v\| \text{ for all } w, v ∈ B_{\ell_\infty}, \eta ∈ \beta(\mathbb{N}) \setminus \mathbb{N}.
\]
To prove the reverse inequality we consider, for each $i \leq N$, the mapping

$$G_{i,N}(\omega) = \prod_{j=N}^{\infty} \frac{\alpha_j - \omega_i m_j}{1 - \alpha_j \omega_i m_j}.$$ 

A straightforward computation shows that the modulus of each factor is bounded by 1. Additionally, for every $0 < r < 1$ and $\omega \in r B_{c_0}$ we have that

$$\left| 1 - \frac{\alpha_j - \omega_i m_j}{1 - \alpha_j \omega_i m_j} \right| \leq \left| 1 + \frac{\omega_i m_j}{1 - \alpha_j \omega_i m_j} \right| (1 - \alpha_j) \leq \frac{1 + r}{1 - r} (1 - \alpha_j).$$

It follows that the partial products involved in $G_{i,N}$ converge uniformly on $r B_{c_0}$ for each $0 < r < 1$. Hence, applying a Weierstrass type theorem [12, Th. 2.13] to the partial products, we get that $G_{i,N} \in \mathcal{H}^\infty(B_{c_0})$ with $\|G_{i,N}\| \leq 1$, for all $i \leq N$. We now consider for $k > N$ the composition $G_{i,N}(\Phi^k_z(w))$. Observe that whenever $\lim_k G_{i,N}(\Phi^k_z(w))$ exists, it must coincide with $\Phi^\eta_z(w)(G_{i,N})$ for $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$. If we denote by $\ell(k)$ the maximum $j \in \mathbb{N}$ such that $i + m_j \leq k$, we can write

$$G_{i,N}(\Phi^k_z(w)) = \left( \prod_{j=N}^{\ell(k)} \frac{\alpha_j - \alpha_k z_i m_j}{1 - \alpha_j \alpha_k z_i m_j} \right) \left( \prod_{j=\ell(k)+1}^{k-1} \alpha_j \right) w_j \left( \prod_{j=k+1}^{\infty} \alpha_j \right).$$

Since $\alpha_k \not\rightarrow 1$ there exists $k_0 \in \mathbb{N}$ such that $1 - \alpha_k < \delta/4$ for every $k \geq k_0$. Then, if $N \geq k_0$, by Lemma 2.2, the first factor converges to $C_{i,N}(z) = \prod_{j=N}^{\infty} \frac{\alpha_j - z_i m_j}{1 - \alpha_j z_i m_j}$ as $k$ goes to infinity. Since both the second and the last terms converge to 1, we get for $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$ that

$$\Phi^\eta_z(G_{i,N}) = C_{i,N}(z) \cdot w_i.$$ 

Note that $|C_{i,N}(z)| \leq 1$ for all $N$ and $C_{i,N}(z) \rightarrow 1$ when $N \rightarrow \infty$. Thus, for each $h \in \mathcal{H}^\infty(\mathbb{D})$ with $\|h\| \leq 1$ we have that $h \circ G_{i,N} \in \mathcal{H}^\infty(B_{c_0})$ with $\|h \circ G_{i,N}\| \leq 1$ and

$$|\Phi^\eta_z(w)(h \circ G_{i,N}) - \Phi^\eta_z(v)(h \circ G_{i,N})| = |h(C_{i,N}(z) \cdot w_i) - h(C_{i,N}(z) \cdot v_i)| \xrightarrow{N \rightarrow \infty} |h(w_i) - h(v_i)|.$$ 

This implies $\|\Phi^\eta_z(w) - \Phi^\eta_z(v)\| \geq \|\delta w_i - \delta v_i\|$ for all $i$, where the last norm is taken in $\mathcal{M}_\infty(\mathbb{D})$. Appealing once more to [3, Th. 2.4], or [10, (6.1)] we derive

$$\|\Phi^\eta_z(w) - \Phi^\eta_z(v)\| \geq \|\delta w - \delta v\| \quad \text{for all} \; w, v \in B_{c_\infty},$$

and hence the isometry is proved.

Finally, if $\eta_1, \eta_2$ are in $\beta(\mathbb{N}) \setminus \mathbb{N}$, with $\eta_1 \neq \eta_2$, then there exists an infinite set $A \subset \mathbb{N}$ such that $\mathbb{N} \setminus A$ is also infinite and $\eta_1 \in \mathbb{A}$, while $\eta_2 \in \mathbb{N} \setminus \mathbb{A}$. We now consider
the function

\[ G_{1,N}^{A}(\omega) = \prod_{\substack{j \geq N \atop j \in A}} \frac{\alpha_j - \omega^{1+m_j}}{1 - \alpha_j \omega^{1+m_j}}. \]

Reasoning as above we can see for any \( k \in A \), that \( G_{1,N}^{A}(\Phi_{z}^{k}(0)) = 0 \), so that \( \Phi_{z}^{n_1}(0)(G_{1,N}^{A}) = 0 \). On the other hand, for \( k' \geq N \) in \( \mathbb{N} \setminus A \) we can compute

\[ G_{1,N}^{A}(\Phi_{z}^{k'}(0)) = \prod_{\substack{N \leq j \leq \ell(k') \atop j \in A}} \frac{\alpha_j - \alpha_{k'} z^{1+m_j}}{1 - \alpha_j \alpha_{k'} z^{1+m_j}} \prod_{j \geq \ell(k') + 1 \atop j \in A} \alpha_j. \]

And since the second factor converges to one, we obtain

\[ \Phi_{z}^{n_2}(0)(G_{1,N}^{A}) = \prod_{\substack{j \geq N \atop j \in A}} \frac{\alpha_j - z^{1+m_j}}{1 - \alpha_j z^{1+m_j}}. \]

We then know by Lemma 2.2 that \( \Phi_{z}^{n_2}(0)(G_{1,N}^{A}) \to 1 \) as \( N \to \infty \), from which we conclude that \( \Phi_{z}^{n_1}(0) \) and \( \Phi_{z}^{n_2}(0) \) lie in different Gleason parts. The claim is finally derived from the fact that, by being a Gleason isometry, \( \Phi_{z}^{n_i}(B_{\ell_{\infty}}) \) is contained in a single Gleason part for \( i = 1, 2 \).

Step 2: Let us now consider \( z \in \overline{B_{\ell_{\infty}}} \) for which there is \( \delta > 0 \) and a subsequence \( (z_{n_j}) \) satisfying \( |z_{n_j} - 1| > \delta \). We thus take \( \mathbb{J} = \{ n_j : j \in \mathbb{N} \} \) and perform the same construction as in the previous step restricting to those coordinates in \( \mathbb{J} \). That is, for each \( k \in \mathbb{N} \) we define \( \Phi_{z,\mathbb{J}}^{k} \) as

\[
(\Phi_{z,\mathbb{J}}^{k}(w))_n = \begin{cases} 
\alpha_k z_n, & \text{for } n \neq n_j \text{ for all } j, \ n \leq k, \\
\alpha_k z_{n_j}, & \text{for } n = n_j, \ 1 \leq j \leq k, \\
\frac{\alpha_k - w_i}{1 - \alpha_k w_i}, & \text{for } n = n_j, \ \text{with } j = m_k + i, \ 1 \leq i \leq k, \\
0, & \text{otherwise.}
\end{cases} \tag{4}
\]

As before, we obtain \( \Phi_{z,\mathbb{J}} : \beta(\mathbb{N}) \times B_{\ell_{\infty}} \to \mathcal{M}_{\infty}(B_{c_0}) \) by taking for each \( w \in B_{\ell_{\infty}} \) the unique extension to \( \beta(\mathbb{N}) \) of the mapping \( \Phi_{z,\mathbb{J}}(-, w) \). The result is then achieved by considering the functions

\[ G_{i,N}(\omega) = \prod_{j=N}^{\infty} \frac{\alpha_j - \omega_{n_i+m_j}}{1 - \alpha_j \omega_{n_i+m_j}}. \]
Proceeding as in Step 1 we deduce, for each $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$, that $\Phi^\eta_{z,\lambda}$ is an analytic Gleason isometry projecting over $z$. Furthermore, by considering the corresponding $G^A_{1,N}$ for this case it is readily seen that $\Phi^\eta_{z,\lambda}(B_{\ell_1})$ and $\Phi^\eta_{z,\lambda}(B_{\ell_1})$ lie in different Gleason parts whenever $\eta_1 \neq \eta_2$.

Step 3: If $z \in B_{\ell_1}$ does not satisfy the conditions of Step 2, then $z_k \to \infty$. Fix $\lambda \in \mathbb{T}$, $\lambda \neq 1$. Then, we can apply the procedure of Step 1 to $\lambda z$ to get a mapping $\Phi_{z,\lambda} : \beta(\mathbb{N}) \setminus \mathbb{N} \times B_{\ell_1} \to \pi^{-1}(\lambda z)$ with all the desired properties. Now, through Remark 2.3 the result follows.

**Remark 2.4** For $z \in B_{\ell_1}$, it is worth noting that the image $\Phi^\eta_{z}(B_{\ell_1})$ is disjoint from $\mathcal{GP}(\delta)$. Indeed, we know from the proof of Theorem 2.1 that $\Phi^\eta(0)(G_{1,N}) = 0$, while $\delta(G_{i,N}) \to \infty$ as $N \to \infty$, so that $\rho(\Phi^\eta(0), \delta) = 1$. The remark then follows from the fact that $\Psi^\eta_{z}$ is a Gleason isometry.

### 3 Fibers for the Vector-Valued Spectrum $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$

The vector-valued spectrum $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$ is projected onto $\overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell_1)}$ through the mapping $\xi$ given by $\xi(\Phi)(x)(x^*) = \Phi(x^*)(x)$ for all $x \in B_{c_0}, x^* \in \ell_1$. Our aim is to prove that there are big analytic sets in the fibers given by this projection. In [15, Prop. 4.2, Th. 4.3, Th. 4.6] we have shown different situations where $B_{\mathcal{H}^\infty(B_{c_0})}$ is analytically injected into the fiber $\mathcal{F}(g) \subset \mathcal{M}_\infty(B_X, B_Y)$ for constant functions $g$. Also, in [15, Th. 4.5] we have seen that, if there exists a polynomial on $X$ which is not weakly continuous on bounded sets, then the complex disk $\mathbb{D}$ is analytically inserted in $\mathcal{F}(g) \subset \mathcal{M}_\infty(B_X, B_Y)$ for every $g \in B_{\mathcal{H}^\infty(B_{c_0}, \ell_1)}$. But $c_0$ is the typical example of a space where all the polynomials are weakly continuous on bounded sets. Nevertheless, we will see in Theorem 3.4 that the ball $B_{\mathcal{H}^\infty(B_{c_0}, \ell_1)}$ can be analytically injected in the fiber $\mathcal{F}(g) \subset \mathcal{M}_\infty(B_{c_0}, B_{c_0})$ for every $g \in \overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell_1)}$. Moreover, as in the scalar-valued case, we produce $2^\ell$ analytic Gleason isometric copies of $B_{\mathcal{H}^\infty(B_{c_0}, \ell_1)}$ in the fiber $\mathcal{F}(g)$ with each of these copies lying in different Gleason parts.

Before we continue we recall the fibered description of $\mathcal{M}_\infty(\mathbb{D}, \mathbb{D})$ (as described in [15]) which resembles what happens in the scalar-valued case $\mathcal{M}_\infty(\mathbb{D})$. The spectrum $\mathcal{M}_\infty(\mathbb{D}, \mathbb{D})$ is projected onto $\overline{B}_{\mathcal{H}^\infty(\mathbb{D})}$ and it is one-to-one over the set (g $\in \overline{B}_{\mathcal{H}^\infty(\mathbb{D})}$ : g(\mathbb{D}) $\subset \mathbb{D}$). However, each fiber over a constant function $g$ with $\|g\| = 1$ contains an analytic copy of $B_{\mathcal{H}^\infty(\mathbb{D})}$.

We now return to the infinite dimensional setting. As we previously did in [15], the vector-valued result will be performed by building on the mappings obtained in the scalar-valued case. Note that $\Phi \in \mathcal{F}(g)$ if and only if $\delta_x \circ \Phi \in \pi^{-1}(g(x))$ for all $x \in B_{c_0}$. Hence in order to have, for each $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$, an analytic mapping $\Psi^\eta$ from $B_{\mathcal{H}^\infty(B_{c_0}, \ell_1)}$ into the fiber $\mathcal{F}(g)$ it seems natural to propose

$$\Psi^\eta(h)(f)(x) = \Phi^\eta_{g(x)}(h(x))(f).$$

The problem is that the construction of $\Phi^\eta_{z}$ in Theorem 2.1 is dependent on $z$ (specifically, whether $z$ has a subsequence whose coordinates are far away from 1 and which of those coordinates are used). Thus, to make formula (5) work we need all $z = g(x)$.
to be of the same kind, independently of \( x \in B_{c_0} \). In other words, we need (perhaps after a rotation) a subsequence \((g(x))_{n_k}\), independent of \( x \) with all of its elements at a positive distance from 1. Before proving the existence of such a subsequence, let us first recall some information about the functions \( g \) in \( \overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)} \).

**Remark 3.1** Let \( g \in \overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)} \). Since whenever a holomorphic function attains its maximum in \( B_{c_0} \) it must have constant norm, if there exists \( x_0 \in B_{c_0} \) such that \( \| g(x_0) \| = 1 \) then \( \| g(x) \| = 1 \) for all \( x \in B_{c_0} \). Hence, there are two alternatives for the range of \( g \):

- \( g(B_{c_0}) \subseteq B_{\ell_\infty} \).
- \( g(B_{c_0}) \subseteq S_{\ell_\infty} \).

A function \( g \in \overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)} \) can be viewed as a sequence of functions \( g = (g_n) \) with each \( g_n \in \overline{B}_{\mathcal{H}^\infty(B_{c_0})} \). Let us see now that the expected behavior of \( g \) actually holds.

**Lemma 3.2** Let \( g \in \overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)} \), \( g = (g_n) \). Then at least one of the following occur:

(i) There exists a subsequence \((g_{n_k})\) such that \( g' = (g_{n_k})_k \) satisfies \( g'(B_{c_0}) \subseteq B_{\ell_\infty} \).

(ii) There exist \( \lambda \in \mathbb{T} \) and a subsequence \((g_{n_k})\) such that \( g_{n_k}(x) \to \lambda \) for all \( x \in B_{c_0} \).

**Proof** Applying [12, Th. 2.17] to the sequence \((g_n)\) we know that there exist a subsequence \((g_{n_k})_k\) and a function \( h \in \overline{B}_{\mathcal{H}^\infty(B_{c_0})} \) such that \( g_{n_k}(x) \to h(x) \), for all \( x \in B_{c_0} \). If \( g \) does not satisfy (i), taking into account the comment in the Remark 3.1, we realize that

\[
\sup_{k \geq k_0} |g_{n_k}(x)| = 1 \text{ for all } x \in B_{c_0}, \text{ and } k_0 \in \mathbb{N}.
\]

This implies that there is \( \lambda \in \mathbb{T} \) such that \( h(x) = \lambda \), for all \( x \) and thus \((g_{n_k})_k\) satisfies (ii). \( \square \)

We also need to have an isometry between certain fibers of the vector-valued spectrum, in the spirit of what we stated in Remark 2.3 for the scalar-valued case.

**Lemma 3.3** Let \( g, h \in \overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)} \) such that \( |g_n(x)| = |h_n(x)| \), for all \( n \in \mathbb{N} \) and \( x \in B_{c_0} \). Then:

(i) There exists a sequence \((\lambda_n)\) in \( \mathbb{T} \) such that \( h_n(x) = \lambda_n g_n(x) \) for all \( n \in \mathbb{N} \) and \( x \in B_{c_0} \).

(ii) The fibers \( \mathcal{F}(g) \) and \( \mathcal{F}(h) \) are Gleason isometric.

**Proof** (i) For a given \( n \), if \( g_n \equiv 0 \) the result is trivial. If not, we can find an open subset \( V_n \) of \( B_{c_0} \) such that \( g_n \neq 0 \) on \( V_n \). This implies that \( \frac{h_n}{g_n} \) is a holomorphic function in \( V_n \) having constant modulus 1. This in turn implies by the identity principle that this function is constant and so (i) holds.

(ii) Given numbers \((\lambda_n)\) in \( \mathbb{T} \), let \( \theta : B_{c_0} \to B_{c_0} \) be the mapping defined by \( \theta(x_n) = (\lambda_n x_n) \). A similar argument as the one used in Remark 2.3, this time using [15, Prop. 5.3] instead of [3, Prop. 1.6] yields the desired result. \( \square \)
Now, we have all the necessary ingredients to proceed with the promised result.

**Theorem 3.4** For every $g \in \overline{B_{H^\infty(B_{c_0},e_\infty)}}$, there is a map

$$\Psi_g : \beta(\mathbb{N}) \setminus \mathbb{N} \times B_{H^\infty(B_{c_0},e_\infty)} \to \mathcal{F}(g),$$

satisfying

1. For all $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$, the mapping $\Psi^\eta_g : B_{H^\infty(B_{c_0},e_\infty)} \to \mathcal{F}(g)$, given by $\Psi^\eta_g(h) = \Psi_g(\eta, h)$, is an analytic Gleason isometry.
2. If $\eta_1 \neq \eta_2$ are in $\beta(\mathbb{N}) \setminus \mathbb{N}$, then the images $\Psi^\eta_1_g(B_{H^\infty(B_{c_0},e_\infty)})$ and $\Psi^\eta_2_g(B_{H^\infty(B_{c_0},e_\infty)})$ lie in different Gleason parts.

**Proof** Let $g \in \overline{B_{H^\infty(B_{c_0},e_\infty)}}$, $g = (g_n)_n$. We first assume that $(g_n)$ has a subsequence $(g_{n_k})$ such that $g' = (g_{n_k})_k$ verifies $g'(B_{c_0}) \subset B_{e_\infty}$. We write $\mathbb{J} = \{n_k\}_k$, take the functions $\Phi^k_{\varepsilon,\mathbb{J}}(w)$ from (4) and define

$$\Psi_g : \mathbb{N} \times B_{H^\infty(B_{c_0},e_\infty)} \to \mathcal{M}_\infty(B_{c_0}, B_{c_0})$$

$$\Psi_g(k, h)(f)(x) = \Phi^k_{g(x),\mathbb{J}}(h(x))(f).$$

For each $k \in \mathbb{N}$, the mapping $[(x, x') \mapsto \Phi^k_{g(x),\mathbb{J}}(h(x'))]$ is separately holomorphic on $B_{c_0} \times B_{c_0}$, and, by Hartogs’ Theorem [28, Th. 36.8], it is holomorphic on $B_{c_0} \times B_{c_0}$. By restricting the previous mapping to the diagonal we obtain that $\Psi^k_g(h)(f) = \Psi_g(k, h)(f)$ is holomorphic. A straightforward computation shows that it is also bounded and that $\Psi^k_g$ is well defined. By the weak-star compactness of $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$, for each fixed $h \in B_{H^\infty(B_{c_0},e_\infty)}$, the mapping $\Psi_g(\cdot, h) : \mathbb{N} \to \mathcal{M}_\infty(B_{c_0}, B_{c_0})$ has a unique extension to $\beta(\mathbb{N})$; this induces

$$\Psi_g : \beta(\mathbb{N}) \times B_{H^\infty(B_{c_0},e_\infty)} \to \mathcal{M}_\infty(B_{c_0}, B_{c_0}).$$

Note that for every $\eta \in \beta(\mathbb{N}) \setminus \mathbb{N}$, and $x \in B_{c_0}$ we have the following equality

$$\delta_x \circ \Psi^\eta_g(h) = \Phi^\eta_{g(x),\mathbb{J}}(h(x)). \quad (6)$$

Since we know from Theorem 2.1 that $\pi(\Phi^\eta_{g(x),\mathbb{J}}(h(x))) = g(x)$, the previous equality yields that $\xi(\Psi^\eta_g(h)) = g$ and thus the image of $\Psi^\eta_g$ is contained in $\mathcal{F}(g)$. It is also readily seen that $\Psi^\eta_g$ is a Gleason isometry, as we have that

$$\rho(\delta_x \circ \Psi^\eta_g(h), \delta_x \circ \Psi^\eta_g(h')) = \rho(\Phi^\eta_{g(x),\mathbb{J}}(h(x)), \Phi^\eta_{g(x),\mathbb{J}}(h'(x))) = \rho(\delta h(x), \delta h'(x)), \quad (6)$$

where the last equality derives from the fact that, by Theorem 2.1, $\Phi^\eta_{g(x),\mathbb{J}}$ is a Gleason isometry. If we now take $\eta_1 \neq \eta_2 \in \beta(\mathbb{N}) \setminus \mathbb{N}$, it follows from (6) paired with Theorem 2.1 that the images $\Phi^\eta_1_g(B_{H^\infty(B_{c_0},e_\infty)})$ and $\Phi^\eta_2_g(B_{H^\infty(B_{c_0},e_\infty)})$ lie in different Gleason parts.
Additionally, for any fixed \( f \in \mathcal{H}^\infty(B_{c_0}), x \in B_{c_0} \) and \( \eta \in \beta(\mathbb{N}) \setminus \mathbb{N}, \) the equality in (6) shows that the mapping \( \delta \circ f \circ \Psi_g : B_{\mathcal{H}^\infty(B_{c_0}, \ell^\infty)} \to \mathbb{C} \) can be seen as the composition of the following analytic mappings

\[
B_{\mathcal{H}^\infty(B_{c_0}, \ell^\infty)} \to B_{\ell^\infty} \quad h \mapsto h(x) \quad B_{\ell^\infty} \to \mathbb{C} \quad z \mapsto \Phi_\eta^g(x), f(z)(f),
\]

showing that \( \Psi^g_\eta \) is analytic.

If instead there is no subsequence such that the corresponding \( g' \) satisfies \( g'(B_{c_0}) \subseteq B_{\ell^\infty}, \) then by Lemma 3.2, we can find a subsequence \( (g_{n_k}) \) and a number \( \lambda \in \mathbb{T} \) such that \( g_{n_k}(x) \to \lambda \) for every \( x \in B_{c_0}. \) If \( \lambda \neq 1, \) for each \( x \in B_{c_0} \) there exist \( \delta = \delta(x) > 0 \) and \( k_0 = k_0(x) \in \mathbb{N} \) such that \( |g_{n_k}(x) - 1| > \delta \) for all \( k \geq k_0. \) Let \( \mathbb{J} = \{ n_k : k \in \mathbb{N} \}. \) Taking then

\[
\Psi^k_\eta(h)(f)(x) = \Phi^k_\eta(x), f(h(x))(f),
\]

and proceeding as before we obtain the desired result. Finally, the remaining case, that is, \( \lambda = 1, \) follows from the previous argument combined with Lemma 3.3. \( \square \)

**Remark 3.5** In [4, Lem. 2.9] it was shown that, for each \( b \in \mathbb{D} \) and \( w \in \overline{B}_{\ell^\infty}, \) the fibers over \( w \) and \( (b, w) \) are homeomorphic. Then, it is observed in [3] that this homeomorphism is in fact, a Gleason isometry.

In order to present an extension of this result to the vector-valued spectrum, let us recall how this scalar-valued isometry is built. For that, given \( b \in \mathbb{D}, \) let us denote by \( \Lambda_b : B_{c_0} \to B_{c_0} \) the mapping given by \( \Lambda_b(x) = (b, x). \) Also, we denote by \( S \) the left shift operator from \( c_0 \) onto \( c_0 \) (that is, \( Sx = (x_2, x_3, \ldots) \)).

The construction is the following: any \( \varphi \) in the fiber over \( w \) is mapped to \( \psi \) in the fiber over \( (b, w), \) where \( \psi(f) = \varphi(f \circ \Lambda_b) \) for every \( f \in \mathcal{H}^\infty(B_{c_0}). \) The main step is then to show that if \( \psi \) belongs to the fiber over \( (b, w) \) then, for all \( f, \psi(f) = \psi(f \circ \Lambda_b \circ S). \) We make use of this equality to derive the vector-valued version of the statement:

**Lemma 3.6** Let \( g_1 \in \mathcal{H}^\infty(B_{c_0}) \) such that \( g_1(B_{c_0}) \subseteq \mathbb{D}. \) Then for every \( h \in \overline{B}_{\mathcal{H}^\infty(B_{c_0}, \ell^\infty)} \) we have that \( \mathcal{F}(h) \) is Gleason isometric to \( \mathcal{F}(g_1, h). \)

**Proof** Consider the mapping: \( R_{g_1} : \mathcal{F}(h) \to \mathcal{F}(g_1, h) \) given by

\[
R_{g_1}(\Phi)(f)(x) = \Phi(f \circ \Lambda_{g_1(x)})(x).
\]

To check that \( R_{g_1} \) is a well-defined Gleason isometry take \( f \in \mathcal{H}^\infty(B_{c_0}) \) and \( \Phi \in \mathcal{F}(h). \) For \( x \in B_{c_0} \) it is clear that \( f \circ \Lambda_{g_1(x)} \) belongs to \( \mathcal{H}^\infty(B_{c_0}). \) We need to go a step further and check that \( [x \mapsto \Phi(f \circ \Lambda_{g_1(x)})(x)] \) is a bounded holomorphic mapping in \( B_{c_0}. \) Since the mapping \( [(x, y) \mapsto \Phi(f \circ \Lambda_{g_1(x)})(y)] \) is separately holomorphic for \( x \) and \( y \) in \( B_{c_0}, \) by Hartogs’ Theorem, it is holomorphic as a function of two variables. Thus, it remains so when restricted to the diagonal. Additionally, it is clear that this function is bounded by \( \| f \| \) and that \( R_{g_1}(\Phi) \) is a homomorphism; so \( R_{g_1}(\Phi) \in \mathcal{F}(h). \) \( \square \)
Now we want to see that \( \xi(R_{g_1}(\Phi)) = (g_1, h) \). Take \( x^* \in \ell_1 \) and denote by \( S_1 \) the left shift operator from \( \ell_1 \) onto \( \ell_1 \). Then,

\[
R_{g_1}(\Phi)(x^*)(x) = \Phi(x^* \circ \Lambda_{g_1(x)}(x)) = \Phi(x_1^* g_1(x) + S_1 x^*)(x) \\
= x_1^* g_1(x) + \Phi(S_1 x^*)(x) = x_1^* g_1(x) + h(x)(S_1 x^*) \\
= x^*(g_1(x), h(x)).
\]

This says that \( R_{g_1} \) maps the fiber over \( h \) to the fiber over \( (g_1, h) \) and thus \( R_{g_1} \) is well defined. Next we prove that \( R_{g_1} \) is onto. For that, let \( \Psi \in \mathcal{F}(g_1, h) \) and define the mapping \( \Phi \in \mathcal{M}_\infty(B_{c_0}, B_{c_0}) \) by

\[
\Phi(f)(x) = \Psi(f \circ S)(x).
\]

Reasoning as before, it is readily seen that \( \Phi \) is a homomorphism from \( \mathcal{H}_\infty(B_{c_0}) \) to \( \mathcal{H}_\infty(B_{c_0}) \) and \( \xi(\Phi) = h \). Further, for any \( x \in B_{c_0} \) and \( f \in \mathcal{H}_\infty(B_{c_0}) \) we can apply Remark 3.5 to \( \delta_x \circ \Psi \) obtaining \( \delta_x \circ \Psi(f) = \delta_x \circ \Psi(f \circ \Lambda_{g_1(x)} \circ S) \). It follows that

\[
\delta_x \circ R_{g_1}(\Phi)(f) = \Phi(f \circ \Lambda_{g_1(x)}(x)) = \Psi(f \circ \Lambda_{g_1(x)} \circ S)(x) = \delta_x \circ \Psi(f).
\]

So, \( R_{g_1}(\Phi) = \Psi \) meaning that \( R_{g_1} \) is onto. Finally we derive that \( R_{g_1} \) is a Gleason isometry as a consequence of the Gleason isometry mentioned in Remark 3.5 for the scalar-valued case and the fact that \( S \) and \( \Lambda_b \) are linear contractions satisfying \( S \circ \Lambda_b = Id \).

As in the scalar-valued case, it is important to note that the procedure of the previous lemma can be repeated in several coordinates, not necessarily the first ones. Hence if \( h \in \overline{\mathcal{H}_\infty(B_{c_0}, c_0)} \) and we insert into \( (h_n) \) finitely many coordinates \( g_1, \ldots, g_k \) with \( g_i \in \mathcal{H}_\infty(B_{c_0}) \) and \( g_i(B_{c_0}) \subset \mathbb{D} \) then the fiber over the resulting function is Gleason isometric to the fiber over \( h \).

### 4 Gleason Parts

Before getting into the subject let us recall some information about Gleason parts of general vector-valued spectra. If \( \mathcal{A} \) and \( \mathcal{B} \) are uniform algebras with (scalar-valued) spectra \( \mathcal{M}(\mathcal{A}) \) and \( \mathcal{M}(\mathcal{B}) \) respectively, let us denote by \( \mathcal{M}(\mathcal{A}, \mathcal{B}) \) their vector-valued spectrum, that is the set of nonzero continuous algebra homomorphisms from \( \mathcal{A} \) into \( \mathcal{B} \). For any \( \Phi \in \mathcal{M}(\mathcal{A}, \mathcal{B}) \) it is known that its transpose \( \Phi^* \) maps \( \mathcal{M}(\mathcal{B}) \) into \( \mathcal{M}(\mathcal{A}) \).

Recall that

\[
\mathcal{GP}(\Phi) = \{ \Psi : ||\Phi - \Psi|| < 2 \} = \{ \Psi : ||\Phi^* - \Psi^*|| < 2 \} \\
= \{ \Psi : \sup_{\varphi \in \mathcal{M}(\mathcal{B})} \rho(\Phi^*(\varphi), \Psi^*(\varphi)) < 1 \}.
\]
An element $\Phi$ in $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is then said to be isolated in the Gleason metric if $\mathcal{G}\mathcal{P}(\Phi) = \{\Phi\}.$

Recall that $\varphi \in \mathcal{M}(\mathcal{A})$ is a strong boundary point if, given an arbitrary open neighborhood $V$ of $\varphi,$ there exists $f \in \mathcal{A}$ such that $\varphi(f) = \|f\| = 1$ and $|\psi(f)| < 1$ for all $\psi \in \mathcal{M}(\mathcal{A}) \setminus V.$ It is proved in [18, Th. 6.2] that if $\mathcal{M}(\mathcal{B})$ is connected and $\Phi^*$ is a nonconstant mapping taking strong boundary points of $\mathcal{M}(\mathcal{B})$ to strong boundary points of $\mathcal{M}(\mathcal{A})$ then $\Phi$ forms a singleton hyperbolic vicinity in $\mathcal{M}(\mathcal{A}, \mathcal{B}).$ We do not introduce the involved definition of hyperbolic vicinity but we mention that this implies that the Gleason part of $\Phi$ in $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is a singleton. Since we are focusing solely on the Gleason topology, we can slightly extend the result in the following way. We include the proof for the sake of completeness, even if it is naturally adapted from the one in [18, Th. 6.2].

**Proposition 4.1** Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras and $\Phi \in \mathcal{M}(\mathcal{A}, \mathcal{B}).$ If $\Phi^*$ maps each strong boundary point of $\mathcal{M}(\mathcal{B})$ into a singleton Gleason part of $\mathcal{M}(\mathcal{A})$ then $\Phi$ is isolated in the Gleason metric for $\mathcal{M}(\mathcal{A}, \mathcal{B}).$

**Proof** Suppose that $\Psi, \Phi \in \mathcal{M}(\mathcal{A}, \mathcal{B})$ are in the same Gleason part. Then for all $\varphi \in \mathcal{M}(\mathcal{B})$ we have that

$$\rho(\varphi \circ \Phi, \varphi \circ \Psi) = \rho(\Phi^*(\varphi), \Psi^*(\varphi)) \leq r < 1.$$ 

If $\varphi$ is a strong boundary point our hypothesis tells us that $\Phi^*(\varphi) = \Psi^*(\varphi).$ Thus,

$$\varphi(\Phi(f)) = \varphi(\Psi(f)), \forall \varphi \text{ strong boundary point of } \mathcal{M}(\mathcal{B}), \forall f \in \mathcal{A}.$$ 

Since every element of a uniform algebra attains its norm at a strong boundary point of its spectrum [20, Th. 12.10], we derive that $\Phi(f) = \Psi(f),$ for all $f$ and hence $\Phi = \Psi.$ 

We have begun in [15] the study of relationships between fibers and Gleason parts for the spectrum $\mathcal{M}_\infty(B_X, B_Y).$ Now we want to go deeper for the particular case $\mathcal{M}_\infty(B_{c_0}, B_{c_0}).$ For the scalar-valued spectrum $\mathcal{M}_\infty(B_{c_0})$ the description of its Gleason parts (and their interaction with fibers) was addressed in [3]. There, the starting point was describing the Gleason parts for the simpler spectrum $\mathcal{M}_d(B_{c_0}).$ Here, we follow that path devoting us first to the spectrum $\mathcal{M}_{u,\infty}(B_{c_0}, B_{c_0}).$

### 4.1 Gleason Parts in $\mathcal{M}_{u,\infty}(B_{c_0}, B_{c_0})$

The projection $\xi : \mathcal{M}_\infty(B_{c_0}, B_{c_0}) \rightarrow \overline{B}_{\mathcal{H}_\infty(B_{c_0}, \ell_\infty)}$ can be restricted to $\mathcal{M}_{u,\infty}(B_{c_0}, B_{c_0})$ without modifying its range. Also, as the space of finite type polynomials is dense in $\mathcal{A}_d(B_{c_0})$ and the composition homomorphism $C_g$ is defined from $\mathcal{A}_d(B_{c_0})$ to $\mathcal{H}_\infty(B_{c_0})$ for every $g \in \overline{B}_{\mathcal{H}_\infty(B_{c_0}, \ell_\infty)}$, an analogous argument to [14, p. 10] and [15, Prop. 2.2] yields that $\xi$ is one-to-one when restricted to $\mathcal{M}_{u,\infty}(B_{c_0}, B_{c_0}).$ In other words, $\mathcal{M}_{u,\infty}(B_{c_0}, B_{c_0}) = \{C_g : g \in \overline{B}_{\mathcal{H}_\infty(B_{c_0}, \ell_\infty)}\}.$ That is, each fiber for this spectrum is a singleton. Moreover, for any $x \in B_{c_0}$ and $g \in \overline{B}_{\mathcal{H}_\infty(B_{c_0}, \ell_\infty)},$ we know that

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\[ \delta_x \circ C_g = \delta_{g(x)} \in \mathcal{M}_u(B_{c_0}). \] So, if we denote by \( \rho_u \) the pseudo-hyperbolic distance for the spectrum \( \mathcal{M}_u(B_{c_0}) \), we have

\[ \mathcal{GP}(C_g) = \{ C_h : \sup_{x \in B_{c_0}} \rho_u(\delta_x \circ C_g, \delta_x \circ C_h) < 1 \} = \{ C_h : \sup_{x \in B_{c_0}} \rho_u(\delta_{g(x)}), \delta_{h(x)}) < 1 \} \]

Recall that the equality proved in [3, Th. 2.4] says (using \( \rho_u \) also for the pseudo-hyperbolic distance in the spectrum \( \mathcal{M}_u(\mathbb{D}) \))

\[ \rho_u(\delta_{g(x)}, \delta_{h(x)}) = \sup_{n \in \mathbb{N}} \rho_u(\delta_{g_n(x)}, \delta_{h_n(x)}) = \sup_{n \in \mathbb{N}} \frac{|g_n(x) - h_n(x)|}{1 - |g_n(x)h_n(x)|}, \quad (7) \]

where the last fraction should be replaced by 0 whenever \( g_n(x) = h_n(x) \). The properties of the Gleason part \( \mathcal{GP}(C_g) \) are then closely related to the properties of the associated holomorphic function \( g \).

In that regard, the statement of [15, Prop. 5.1] can be easily adapted to our setting so that we obtain, as in the case of \( \mathcal{M}_\infty(B_X, B_Y) \) a splitting of the fibers in three sets: interior fibers, middle fibers and edge fibers. Elements from different kinds of fibers can not share a Gleason part. Precisely, the situation here (with singleton fibers) is the following:

- Interior fibers are those \( C_g \) with \( \|g\| < 1 \). They all share the same Gleason part.
- Middle fibers are those \( C_g \) with \( g(B_{c_0}) \subset B_{\ell_\infty} \) and \( \|g\| = 1 \).
- Edge fibers are those \( C_g \) with \( g(B_{c_0}) \subset S_{\ell_\infty} \).

Nothing more can be said about interior fibers since there is only one Gleason part for all of them. But within middle fibers and edge fibers we will see that there are a lot of Gleason parts, some of them singleton and some of them containing balls of elements. We begin with two known (or easily deduced) examples of singleton Gleason parts.

**Example 4.2** If \( g : B_{c_0} \rightarrow B_{c_0} \) is a biholomorphic function then \( C_g^* : \mathcal{M}_\infty(B_{c_0}) \rightarrow \mathcal{M}_u(B_{c_0}) \) maps strong boundary points into singleton Gleason parts. Indeed, as it was noted in [15, p. 15], \( C_g^* \) maps strong boundary points to strong boundary points when viewed as a self map of \( \mathcal{M}_\infty(B_{c_0}) \). Additionally, by [3, Prop. 3.6], any strong boundary point in \( \mathcal{M}_\infty(B_{c_0}) \) is projected onto \( T^\infty \). It follows that \( C_g^* : \mathcal{M}_\infty(B_{c_0}) \rightarrow \mathcal{M}_u(B_{c_0}) \) maps strong boundary points into singleton Gleason parts. As a result, any such \( C_g \) is a middle fiber isolated in the Gleason metric. Examples of such functions \( g \) are for instance \( g = Id \) or \( g(x) = \left( \frac{a_n - x_n}{1 - a_n x_n} \right)_n \) with \( (a_n)_n \in B_{c_0} \).

**Example 4.3** Let \( g : B_{c_0} \rightarrow S_{\ell_\infty} \) be a constant function \( g(x) = (a_n)_n \) with \( |a_n| = 1 \) for all \( n \). Then, it is clear by (7) that \( C_g \) is an edge fiber isolated in the Gleason metric.

In view of the previous examples we wonder if there are other kind of singleton Gleason parts. That is, we ask if there exist isolated edge fibers \( C_g \) with \( g \) nonconstant or isolated middle fibers \( C_g \) with \( g \) nonbiholomorphic. In order to answer these questions we first present a result that relates singleton Gleason parts of \( \mathcal{M}_{u, \infty}(B_{c_0}, B_{c_0}) \) with singleton Gleason parts of \( \mathcal{M}(A(\mathbb{D})), \mathcal{H}^\infty(B_{c_0})) \) (where \( A(\mathbb{D}) = A_{u}(\mathbb{D}) \) is the algebra of holomorphic functions in \( \mathbb{D} \) which are continuous in \( \overline{\mathbb{D}} \)).
Lemma 4.4 Let \( g \in \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})}, g = (g_n)_n \). Then, \( C_g \) is isolated in \( \mathcal{M}_{u, \infty}(B_{c^0}, B_{c^0}) \) if and only if \( C_{g_n} \) is isolated in \( \mathcal{M}(A(\mathbb{D}), \mathcal{H}^{\infty}(B_{c^0})) \) for all \( n \in \mathbb{N} \).

**Proof** Let \( g \neq h \in \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})} \). If \( C_{g_n} \) is isolated for every \( n \in \mathbb{N} \), we have that

\[
\sup_{x \in B_{c^0}} \rho_u(\delta_x \circ C_g, \delta_x \circ C_h) = \sup_{x \in B_{c^0}} \sup_{n \in \mathbb{N}} \rho_u(\delta_x \circ C_{g_n}, \delta_x \circ C_{h_n}) = 1.
\]

Thus, \( C_g \) is isolated. Conversely, assume that there exists \( n_0 \in \mathbb{N} \) such that \( C_{g_{n_0}} \) is not isolated in \( \mathcal{M}(A(\mathbb{D}), \mathcal{H}^{\infty}(B_{c^0})) \). Taking \( C_{g_{n_0}} \in \mathcal{GP}(C_{g_{n_0}}) \) with \( \hat{g}_{n_0} \neq g_{n_0} \) and defining \( h \in \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})} \) as \( h_n = g_n \) for \( n \neq n_0 \) and \( h_{n_0} = \hat{g}_{n_0} \) gives us an element in \( \mathcal{GP}(C_g) \) different than \( C_g \). \( \square \)

The previous Lemma together with Example 4.2 (which shows that \( C_{Id} \) is isolated) implies that the composition operator corresponding to any coordinate function \( g_n : B_{c^0} \rightarrow \mathbb{D}, g_n(x) = x_n \) is isolated in \( \mathcal{M}(A(\mathbb{D}), \mathcal{H}^{\infty}(B_{c^0})) \). This can also been seen by easily checking that \( C_{g_n} \) sends strong boundary points of \( \mathcal{M}_{\infty}(B_{c^0}) \) into singleton Gleason parts of \( \mathcal{M}(A(\mathbb{D})) \). Consequently, again by the previous Lemma, any combination of coordinate functions produces an isolated middle fiber:

**Example 4.5** Let \( g : B_{c^0} \rightarrow B_{\ell_{\infty}} \) given by \( g(x) = (x_1)_n \). Then, \( C_g \) is a middle fiber isolated in the Gleason metric, even though \( g \) is not biholomorphic. Clearly, any other selection of \( g \) with \( g_n(x) = x_{n_k} \) for all \( n \) (and any \( k_n \)) has an associated composition homomorphism which is isolated in the Gleason metric.

Lemma 4.4 is also useful in providing an example of an isolated edge fiber associated to a nonconstant function. Indeed, it is readily seen that any Möbius function of a coordinate \( g : B_{c^0} \rightarrow \mathbb{D}, g(x) = \frac{a-x_n}{1-\overline{a}x_n} \) (with \( |a| < 1 \)) has an associated composition homomorphism which is isolated in \( \mathcal{M}(A(\mathbb{D}), \mathcal{H}^{\infty}(B_{c^0})) \). Hence, we can combine a sequence of these functions to obtain a singleton edge fiber:

**Example 4.6** Let \( (a_n)_n \) be an increasing sequence of real numbers with \( 0 < a_n < 1 \), for all \( n \) and \( a_n \rightarrow 1 \). Consider \( g : B_{c^0} \rightarrow \ell_{\infty} \) given by \( g(x) = \left( \frac{a_n-x_n}{1-\overline{a}x_n} \right)_n \). Then \( g(B_{c^0}) \subset S_{\ell_{\infty}} \) and thus \( C_g \) is an edge fiber. Moreover, it is clear that \( g \) is nonconstant and that by the previous argument, \( C_g \) is isolated in \( \mathcal{M}_{u, \infty}(B_{c^0}, B_{c^0}) \).

Let us now see that the fact that \( C_g \) is isolated forces \( g \) to be an extreme point of \( \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})} \). This is stated in the following proposition whose proof is modeled after [15, Ex. 5.5] which, in turn, was adapted from [27, Ex. 2].

**Proposition 4.7** Let \( g \in \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})} \) such that \( C_g \) is isolated in \( \mathcal{M}_{u, \infty}(B_{c^0}, B_{c^0}) \). Then \( g \) is an extreme point of \( \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})} \).

**Proof** Suppose that \( g \in \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})} \) is not an extreme point. Then there exist \( f \neq h \in \overline{B}_{\mathcal{H}^{\infty}(B_{c^0}, \ell_{\infty})} \) such that \( g = (f + h)/2 \). Let \( j(x) = g(x) + k(f(x) - h(x))^2 \) with
0 < k < 1/8. Recall that \( \| j \| = \sup_{n \in \mathbb{N}} \| j_n \| \). Now a quick computation shows that

\[
\| j_n \| = \sup_{x \in B_{c_0}} |g_n(x) + k(f_n(x) - h_n(x))^2|
\]

\[
= \sup_{x \in B_{c_0}} \left| \frac{f_n(x) + h_n(x)}{2} + k(f_n(x) - h_n(x))^2 \right|
\]

\[
\leq \sup_{x \in B_{c_0}} \left| \frac{f_n(x) + h_n(x)}{2} + k|f_n(x) - h_n(x)|^2 \right|
\]

\[
\leq \sup_{x \in B_{c_0}} \left| \frac{f_n(x) + h_n(x)}{2} + 4k \left( 1 - \left| \frac{f_n(x) + h_n(x)}{2} \right|^2 \right) \right| \leq 1.
\]

Thus, \( j \in \overline{B_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)}} \). To check that \( C_j \) is in the same Gleason part as \( C_g \) we compute, for each \( n \in \mathbb{N} \),

\[
\left| \frac{g_n(x) - f_n(x)}{1 - g_n(x) j_n(x)} \right| = \left| \frac{k(f_n(x) - h_n(x))^2}{1 - \frac{f_n(x) + h_n(x)}{2} \left( \frac{f_n(x) + h_n(x)}{2} + k(f_n(x) - h_n(x))^2 \right)} \right|
\]

\[
= \left| \frac{k|f_n(x) - h_n(x)|^2}{1 - \left| \frac{f_n(x) + h_n(x)}{2} \right|^2 - k \left( \frac{f_n(x) + h_n(x)}{2} \right)^2 (f_n(x) - h_n(x))^2} \right|
\]

\[
\leq \frac{k}{1/4 - k} < 1.
\]

So we conclude that \( C_j \in \mathcal{GP}(C_g) \) and thus \( C_g \) is not isolated. \( \Box \)

**Open question 1** Does the converse of the previous proposition hold? That is, for \( g \in \overline{B_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)}} \), is \( g \) being an extreme point equivalent to \( C_g \) being isolated in \( \mathcal{M}_{n, \infty}(B_{c_0}, B_{c_0}) \)?

Note that if we replace \( B_{c_0} \) by \( \mathbb{D} \) the answer is yes. Indeed, combining [27, Cor. 9] with [26, Th. 4.1] and a classical characterization of extreme points proved in 1957 by Arens et al. (see [13, Th. 12] or [25, Ch. 9]) it is obtained that \( g \in \overline{B_{\mathcal{H}^\infty(\mathbb{D})}} \) with \( g(\mathbb{D}) \subset \mathbb{D} \) is an extreme point if and only if \( C_g \) is isolated in the set of composition operators from \( \mathcal{H}^\infty(\mathbb{D}) \) to \( \mathcal{H}^\infty(\mathbb{D}) \). This is equivalent to saying that any \( g \in \overline{B_{\mathcal{H}^\infty(\mathbb{D})}} \) is an extreme point if and only if \( C_g \) is isolated in \( \mathcal{M}(A(\mathbb{D}), \mathcal{H}^\infty(\mathbb{D})) \).

Since it is readily seen that \( g = (g_n)_n \) is a extreme point of \( \overline{B_{\mathcal{H}^\infty(B_{c_0}, \ell_\infty)}} \) if and only if \( g_n \) is a extreme point of \( \overline{B_{\mathcal{H}^\infty(B_{c_0})}} \) for every \( n \), in view of Lemma 4.4 the open question can be reformulated as follows: is \( C_g \) isolated in \( \mathcal{M}(A(\mathbb{D}), \mathcal{H}^\infty(B_{c_0})) \) whenever \( g \) is an extreme point of \( \overline{B_{\mathcal{H}^\infty(B_{c_0})}} \)?

To finish our intended description of Gleason parts for middle/edge fibers, we show examples of Gleason parts containing *balls* of elements.

**Example 4.8** Using that all interior fibers share a Gleason part and that the distance between two homomorphisms is computed by a supremum over the coordinates it is easy to produce examples of *thick* Gleason parts (that is, Gleason parts containing
copies of balls) within middle or edge fibers. Indeed, let \( g \in S_{H^\infty(B_{c_0}, \ell_\infty)} \) (i.e. \( C_g \) is a middle or edge fiber) and consider \( h \in S_{H^\infty(B_{c_0}, \ell_\infty)} \), \( h = (h_n) \) given by \( h_{2n} = g_n \) and \( h_{2n+1} = 0 \) for \( n \in \mathbb{N} \). Now it is clear that \( C_h \) is a middle fiber (resp. edge fiber) whether \( C_g \) is a middle (resp. edge) fiber. Changing odd coordinates of \( h \) to those of any function in \( B_{H^\infty(B_{c_0}, \ell_\infty)} \) we obtain a function whose associated homomorphism is in the same Gleason part as \( C_h \). As a result we have that \( \mathcal{GP}(C_h) \) contains a Gleason isometric copy of \( \{ C_j : j \in B_{H^\infty(B_{c_0}, \ell_\infty)} \} \).

### 4.2 Gleason Parts in \( \mathcal{M}_\infty(B_{c_0}, B_{c_0}) \)

For this larger spectrum our goal is to obtain some knowledge about how fibers and Gleason parts relate to each other. First of all, note that we have already produced an interesting outcome about this relationship in Theorem 3.4. Indeed, the statement of the theorem, translated to Gleason parts’ language, is the following:

For every \( g \in \overline{B}_{H^\infty(B_{c_0}, \ell_\infty)} \), the fiber \( \mathcal{F}(g) \) intersects at least \( 2^c \) different Gleason parts and each of these intersections is thick (since it is a Gleason isometric copy of an infinite dimensional ball).

To complete our intended overview we propose the following questions:

- **(Q1)** Which Gleason parts have elements from different fibers?
- **(Q2)** Which fibers contain singleton Gleason parts?

Before getting into the subject we observe some trivial facts regarding the interaction between Gleason parts for the spectra \( \mathcal{M}_{u, \infty}(B_{c_0}, B_{c_0}) \) and \( \mathcal{M}_\infty(B_{c_0}, B_{c_0}) \).

Whenever \( g, h \in H^\infty(B_{c_0}, \ell_\infty) \) map \( B_{c_0} \) into \( B_{\ell_\infty} \) the composition homomorphisms \( C_g \) and \( C_h \) can be defined in \( \mathcal{M}_{u, \infty}(B_{c_0}, B_{c_0}) \) as well as in \( \mathcal{M}_\infty(B_{c_0}, B_{c_0}) \). Since the distance between evaluation homomorphisms coincide whether it is computed in \( \mathcal{M}_{u}(B_{c_0}) \) or in \( \mathcal{M}_\infty(B_{c_0}) \) we deduce the same identity in the vector-valued case:

\[
\sigma(C_g, C_h) = \sigma_u(C_g, C_h) = \sup_{x \in B_{c_0}} \sup_{n \in \mathbb{N}} \left| \frac{g_n(x) - h_n(x)}{1 - g_n(x)h_n(x)} \right|.
\]

Here \( \sigma \) is the metric defined in (1) and \( \sigma_u \) is its analogue for the spectrum \( \mathcal{M}_{u, \infty}(B_{c_0}, B_{c_0}) \). That is, for \( \Phi, \Psi \in \mathcal{M}_{u, \infty}(B_{c_0}, B_{c_0}) \)

\[
\sigma_u(\Phi, \Psi) = \sup_{x \in B_{c_0}} \rho_u(\delta_x \circ \Phi, \delta_x \circ \Psi).
\]

Recall that we split the fibers in three groups: interior, middle and edge fibers. By [15, Prop. 5.1], all the elements of any Gleason part should belong to the same kind of fibers (that is, all interior or all middle or all edge). For any \( g \in \overline{B}_{H^\infty(B_{c_0}, \ell_\infty)} \) with \( g(B_{c_0}) \subset B_{\ell_\infty} \) we have that both the image of the mapping \( \Psi_g \) given by Theorem 3.4 and the corresponding composition homomorphism \( C_g \) lie in the fiber \( \mathcal{F}(g) \). Additionally, we know that for different values of \( \eta \in \beta(\mathbb{N}) \setminus \mathbb{N} \), the images \( \Psi_g^n(B_{\ell_\infty}) \) lie in different Gleason parts. It is also worth remarking that the images of the mappings

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\(\Psi_g\) are also disjoint from \(\mathcal{GP}(C_g)\). Indeed, this holds by combining the equality (6) with Remark 2.4.

Some of the middle and edge fibers are over functions \(g\) which satisfy that their associated homomorphisms \(C_g\) are isolated in the spectrum \(\mathcal{M}_{u,\infty}(B_{C_0}, B_{C_0})\). In the previous subsection we have obtained some information about these functions \(g\). Let us call them isolated functions.

Note that any \(\Phi \in \mathcal{M}_{\infty}(B_{C_0}, B_{C_0})\) can be restricted to \(\mathcal{M}_{u,\infty}(B_{C_0}, B_{C_0})\) obtaining the composition homomorphism \(C_{\xi(\Phi)}\). For every \(\Phi, \Psi \in \mathcal{M}_{\infty}(B_{C_0}, B_{C_0})\), it is clear that

\[
\sigma(\Phi, \Psi) \geq \sigma_u(C_{\xi(\Phi)}, C_{\xi(\Psi)}).
\]

Hence, if \(C_{\xi(\Phi)}\) and \(C_{\xi(\Psi)}\) do not share a Gleason part for the spectrum \(\mathcal{M}_{u,\infty}(B_{C_0}, B_{C_0})\) then the same holds for \(\Phi\) and \(\Psi\) within the spectrum \(\mathcal{M}_{\infty}(B_{C_0}, B_{C_0})\). In particular, if \(g \in \overline{B}_{\mathcal{H}_{\infty}(B_{C_0}, r_{\infty})}\) is an isolated function then, for every \(\Phi \in \mathcal{F}(g) \subset \mathcal{M}_{\infty}(B_{C_0}, B_{C_0})\), the Gleason part of \(\Phi\) is contained in \(\mathcal{F}(g)\). This is part of the answer to our first guiding question (Q1). We give a complete answer in the following proposition, which is inspired by [3, Prop. 3.3].

**Proposition 4.9** Let \(\Phi \in \mathcal{F}(g) \subset \mathcal{M}_{\infty}(B_{C_0}, B_{C_0})\). The following are equivalent:

(i) \(g\) is not an isolated function.

(ii) \(\mathcal{GP}(\Phi)\) contains elements from different fibers.

**Proof** We just need to prove that (i) implies (ii) since the converse was explained in the paragraph above. If \(g\) is not isolated, by Lemma 4.4 there exists \(n\) such that \(g_n\) is not isolated. We can assume (without loss of generality) that \(g_1\) is not isolated, that is, there exist \(h_1 \in \overline{B}_{\mathcal{H}_{\infty}(B_{C_0}, r_{\infty})}\) such that \(h_1 \neq g_1\) and \(C_{g_1}\) and \(C_{h_1}\) are in the same Gleason part in \(\mathcal{M}(A(\mathbb{D}), \mathcal{H}_{\infty}(B_{C_0}))\). Observe that \(g_1(B_{C_0}) \subset \mathbb{D}\) (and the same holds for \(h_1\)) because if this is not the case, \(g_1\) would be a constant function \(g_1(x) = \lambda \in \mathbb{T}\) meaning that \(C_{g_1}\) should be isolated. Denote by \(\widehat{g} = (g_n)_{n \geq 2} \in \overline{B}_{\mathcal{H}_{\infty}(B_{C_0}, r_{\infty})}\) and consider, as in Lemma 3.6, the mappings

\[
\begin{align*}
R_{g_1} : \mathcal{F}((\widehat{g})) &\rightarrow \mathcal{F}(g) \quad \text{and} \quad R_{h_1} : \mathcal{F}((\widehat{g})) &\rightarrow \mathcal{F}(h_1, \widehat{g}), \\
R_{g_1}(\Phi)(f)(x) &\equiv \Phi(f \circ \Lambda_{g_1(x)})(x) \quad \text{and} \quad R_{h_1}(\Phi)(f)(x) &\equiv \Phi(f \circ \Lambda_{h_1(x)})(x),
\end{align*}
\]

where \(\Lambda_{g_1(x)}(y) = (g_1(x), y)\). As \(R_{g_1}\) is surjective there exists \(\Psi \in \mathcal{F}((\widehat{g}))\) such that \(R_{g_1}(\Psi) = \Phi\). Let us see that \(R_{h_1}(\Psi)\) is in the same Gleason part of \(\Phi\) (but in a different fiber):

\[
\|R_{g_1}(\Psi) - R_{h_1}(\Psi)\| = \sup_{f \in B_{\mathcal{H}_{\infty}(B_{C_0})}} \|R_{g_1}(\Psi)(f) - R_{h_1}(\Psi)(f)\| = \sup_{f \in B_{\mathcal{H}_{\infty}(B_{C_0})}} \sup_{x \in B_{C_0}} \|\Psi(f \circ \Lambda_{g_1(x)} - f \circ \Lambda_{h_1(x)}) (x)\| \\
\leq \sup_{f \in B_{\mathcal{H}_{\infty}(B_{C_0})}} \sup_{x \in B_{C_0}} \|f \circ \Lambda_{g_1(x)} - f \circ \Lambda_{h_1(x)}\| = \sup_{x \in B_{C_0}} \|\delta_{g_1(x)} - \delta_{h_1(x)}\| = \|C_{g_1} - C_{h_1}\| < 2.
\]
This completes the proof.  

With respect to our second guiding question (Q2), the above proposition tells us that the only places to look for singleton Gleason parts are the fibers over isolated functions $g$. We would like to know whether all those fibers contain singleton Gleason parts. Unfortunately, we do not have a complete answer for this but we can contribute with some steps in that direction. Let us recall the four examples of isolated functions (within the middle and the edge fibers) presented in the previous subsection.

- Any biholomorphic function $g$ is isolated by Example 4.2. As it was commented there, $C^*_g$ maps strong boundary points into strong boundary points. Hence, by Proposition 4.1, the Gleason part of $C^*_g$ is singleton in $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$.
- Example 4.3 shows that each constant function $g(x) = \lambda$, with $\lambda \in \mathbb{T}^\infty$, is isolated. Since this is an edge function, the composition homomorphism $C_g$ is not defined in $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$. But we know in this case that there is a singleton Gleason part in the fiber over $g$. Indeed, for the scalar-valued homomorphism $C_g$ is not defined.
- Example 4.5 an isolated middle function $g$ not biholomorphic is presented. By Proposition 4.9, the Gleason part of $C^*_g$ (from $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$) is contained in $\mathcal{F}(g)$ but we do not know if it is singleton.
- The isolated function $g$ of Example 4.6 is a nonconstant edge function. We could not answer whether it exists a singleton Gleason part within its fiber.

In the scalar-valued case, the isolated elements of $\mathcal{M}_u(B_{c_0})$ are exactly all the evaluations $\delta_\lambda$ with $\lambda \in \mathbb{T}^\infty$. As we mentioned above, it is proved in [2, Prop. 3.6] that in the fiber over any $\lambda \in \mathbb{T}^\infty$ there is a singleton Gleason part. Note that the scalar-valued spectrum $\mathcal{M}_\infty(B_{c_0})$ is naturally contained in $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$ and each singleton $\varphi \in \mathcal{M}_\infty(B_{c_0})$ in $\mathcal{M}_\infty(B_{c_0}, B_{c_0})$. This implies that for a constant function $g(x) = \lambda$, with $\lambda \in \mathbb{T}^\infty$, the fiber $\mathcal{F}(g) \subset \mathcal{M}_\infty(B_{c_0}, B_{c_0})$ contains a singleton Gleason part.

**Open question 2** Let $g \in S_{\mathcal{H}_\infty(B_{c_0},\ell_\infty)}$ be an isolated function. Is there a homomorphism $\Phi \in \mathcal{F}(g) \subset \mathcal{M}_\infty(B_{c_0}, B_{c_0})$ such that the Gleason part of $\Phi$ is a singleton?

By the previous explanation we can confine this question to isolated nonconstant edge functions or isolated nonbiholomorphic middle functions $g$. In the latter case it would be interesting to know not only if there is a homomorphism in the fiber over $g$ with singleton Gleason part but also if $C^*_g$ itself has singleton Gleason part.

To finish, we want to add that even though we could not prove the existence of singleton Gleason parts in the fibers over the functions $g$ from Examples 4.5 and 4.6 we can construct other isolated functions which are “middle nonbiholomorphic” or “edge nonconstant” whose fibers do contain singleton parts.

**Example 4.10** Let $g_1 \in S_{\mathcal{H}_\infty(B_{c_0})}$ be an isolated function for $\mathcal{M}(\mathcal{A}(\mathbb{D}), \mathcal{H}_\infty(B_{c_0}))$ such that $g_1(B_{c_0}) \subset \mathbb{D}$. For instance, $g_1(x) = x_1$ or $g_1(x) = \frac{a - x_1}{1 - \overline{a} x_1}$, with $|a| < 1$. Take a biholomorphic mapping $h : B_{c_0} \to B_{c_0}$. Now, from Lemma 3.6 we know that there is a surjective Gleason isometry $R_{g_1} : \mathcal{F}(h) \to \mathcal{F}(g_1, h)$. Since we comment above that
\[ \mathcal{G}(C_h) = \{ C_h \} \text{ and we deduce from Lemma 4.4 that } (g_1, h) \text{ is an isolated function, it is clear due to Proposition 4.9 that } R_{g_1}(C_h) = C_{(g_1, h)} \text{ has singleton Gleason part. Note that } (g_1, h) \text{ is a middle nonbiholomorphic function.} \]

If, instead, we consider \( h \in S_\mathcal{H}^\infty (B_{\mathcal{C}^0}, \ell^\infty) \) a constant function, \( h(x) = \lambda \) with \( \lambda \in T^\infty \), we have seen above that there is a (scalar-valued) homomorphism \( \Phi \in \mathcal{F}(h) \) with singleton Gleason part. Repeating the previous argument it is easily seen that \( R_{g_1}(\Phi) \in \mathcal{F}(g_1, h) \) is isolated. Note that in this case \( (g_1, h) \) is an edge nonconstant isolated function.

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