Groupoid sheaves as Hilbert modules*

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Abstract

We provide a new characterization of the notion of sheaf on an étale groupoid $G$, in terms of a particular kind of Hilbert module on the quantale $O(G)$ of the groupoid. All the theory is developed in the context of the more general class of quantales known as stable quantal frames, of which examples are easy to construct because their category is algebraic. The homomorphisms of our Hilbert modules are necessarily adjointable and thus form a strongly self-dual category. By restriction we obtain, for any stable quantal frame, two isomorphic categories of sheaves whose morphisms are related by the duality.

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1 Introduction

Étale groupoids can be identified with inverse quantal frames [22], which are unital involutive quantales of a kind that is closely related to inverse semigroups. Similarly, actions of étale groupoids can be described in terms of a natural class of quantale modules [23] and, in particular, so can groupoid sheaves: a sheaf on an étale groupoid $G$, whose quantale is $O(G)$, is identified in [23] with an étale $O(G)$-locale, by which is meant a left $O(G)$-module $X$ satisfying the following three conditions:

1. $X$ is a locale;

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2. Writing $B$ for the unital involutive subquantale (and also a locale)

$$B = \downarrow(e_Q) = \{ a \in \mathcal{O}(G) \mid a \leq e_Q \},$$

(this is isomorphic to the locale $G_0$ of objects of $G$) there is a (necessarily unique and join preserving) monotone and $B$-equivariant map $\zeta : X \to B$ such that $\zeta(x)x = x$ for all $x \in X$ — this means that $G$ acts on an open map $p : X \to G_0$ such that $\zeta = p_!$, and, accordingly, $X$ is called an open $\mathcal{O}(G)$-locale;

3. The set of local sections of $X$, which is defined by

$$\Gamma_X = \{ s \in X \mid x \leq s \Rightarrow x = \zeta(x)s \},$$

satisfies $\bigvee \Gamma_X = 1$ (that is, the local sections cover $X$).

The notion of étale $\mathcal{O}(G)$-locale is a generalization of the notion of étale $B$-locale which is given in [24] for a locale $B$. However, there is another characterization of sheaves on locales in [24] that can be considered more elegant and, in particular, does not even require one to assume that the modules $X$ are locales: a module on a locale $B$ corresponds to a sheaf on $B$ if and only if there is a “symmetric bilinear form” $\langle - , - \rangle : X \times X \to B$ and a subset $\Gamma \subset X$ such that for all $x \in X$ we have

$$x = \bigvee_{s \in \Gamma} \langle x , s \rangle s .$$

In other words, the sheaves on $B$ are the Hilbert $B$-modules (in the sense of [18]) that are equipped with a “Hilbert basis”. The relation to the previous description is given by the conditions $\zeta(x) = \langle x , x \rangle$ and $\langle x , y \rangle = \zeta(x \land y)$; the conditions that $X$ is a locale and $\Gamma \subset \Gamma_X$ are automatically satisfied.

In this paper we shall see that sheaves on étale groupoids can be described in a similar way: if $G$ is an étale groupoid then the $G$-sheaves can be identified with the Hilbert $\mathcal{O}(G)$-modules equipped with “Hilbert bases”, the only difference with respect to the locale case being that we also have to impose $\bigvee \Gamma = 1$ as an axiom.

Let us look at some useful properties of this characterization of groupoid sheaves:

**More general quantales.** The theory works well when applied to the class of stable quantal frames (see [22]), which subsumes that of inverse quantal frames and forms an algebraic category, as such providing us with a theory of sheaves that readily applies to stable quantale frames obtained, say, from
presentations by generators and relations (some aspects of the theory apply even to more general classes of quantales). This may include, for instance, adaptations to stable quantal frames of the quantale presentations which can be found in [11, 14, 15, 17, 21].

Of course, reasons for wanting general theories of sheaves on quantales will vary depending on the application. For instance, sheaves on quantales related to those of [14] may provide good grounds on which to study first order modal logic. More ambitiously, a good notion of structure sheaf on the quantale $\text{Max} A$ (see [15]) of a $C^*$-algebra $A$ might provide a way of doing away with the excessive abundance of quantale homomorphisms that seems to be responsible for the bad properties of the functor $\text{Max}$ [11, 12].

Matrix models of sheaves. The characterization of sheaves provided in this paper relates directly to the description originally given in [1] of the sheaves on a locale $B$ in terms of $B$-valued matrices (see also [2, §2] or [9, pp. 502–513]): a projection matrix $M : \Gamma \times \Gamma \to B$ arises as the restriction of $\langle - , - \rangle : X \times X \to B$, and in [24] it is shown that any projection matrix $M : \Gamma \times \Gamma \to B$ yields a $B$-module (a sheaf) directly by $X = B^\Gamma M$. We shall see that this matrix representation of sheaves still applies when $B$ is replaced by a more general quantale, in particular a stable quantal frame. The resulting matrices are “quantal sets” in the sense of [20], which in turn are an adaptation of the quantal sets on right-sided quantales of [16] (related developments can be found in [3] or, for quantaloids, in [5, 6] and, in a setting of ordered sheaves, in [8] and references therein).

Hence, the results in the present paper also provide a good concrete example of such matrix based formulations of sheaves on quantales — another interesting example is the description of sheaves on Grothendieck sites in terms of sheaves on quantaloids (see [7, 26] or [25, Section 3.6]), which however takes place in a different setting where, in particular, the quantaloid structure has to be equipped with a nucleus that plays the role of a Grothendieck topology; in other words, quantaloids alone are not “spaces”, whereas in the present paper quantales suffice.

We also remark that in our relation between sheaves and modules, the module actions are the natural generalizations of the restriction maps of sheaves on locales. In particular, the actions do not in general restrict to actions on the sets of local sections, which contrasts with other attempts at generalizing restriction maps [5, 6, 16, 20].

Sheaf morphisms and duality. In [23] two (isomorphic) descriptions of the category of sheaves on an étale groupoid $G$ have been given: one, whose
morphisms are maps of locales whose inverse images are module homomorphisms, is the category $\mathcal{O}(G)$-$\mathbf{LH}$, which should be thought of as the analogue of the category $\mathbf{LH}/B$ in the case of a locale $B$; the other, whose morphisms are the direct images of the maps of the former, is $\mathcal{O}(G)$-$\mathbf{Sh}$, and it should be thought of as the actual category of sheaves on $\mathcal{O}(G)$ because its morphisms are closely related to natural transformations of sheaves. In particular, this gives us two quantale-based representations of étendues, due to \cite{10, Theorem VIII.3.3} (see also \cite{9}).

We shall see that our categories of Hilbert modules are strongly self-dual (due to the existence of Hilbert adjoints of homomorphisms) and that $\mathcal{O}(G)$-$\mathbf{LH}$ and $\mathcal{O}(G)$-$\mathbf{Sh}$ are related by this duality. Namely, if $f$ is a morphism in $\mathcal{O}(G)$-$\mathbf{LH}$ then the relation to its direct image $f_!$ (which is a morphism of $\mathcal{O}(G)$-$\mathbf{Sh}$) is given by the condition

$$\langle f_!(x), y \rangle = \langle x, f^*(y) \rangle.$$  

Hence, in addition to being categorical adjoints, the direct image $f_!$ and the inverse image $f^*$ are Hermitian adjoints. This generalizes the situation of \cite{24} for locales, and furthermore it still applies when the quantales $\mathcal{O}(G)$ are replaced by arbitrary stable quantal frames.

We remark that this paper will hardly mention groupoids explicitly, beyond what has been done in this introduction, because it builds directly on the results of \cite{23}. As further background we shall assume from the reader knowledge of locales, inverse semigroups, quantales and their modules, mostly as described in \cite{22, Section 2} and references therein.

The rest of the paper is organized into three more sections. In section 2 we begin by reviewing elementary facts about Hilbert modules and introducing some new ones. The rest of the paper is then entirely based on Hilbert modules: section 3 deals with an auxiliary notion of supported module for supported quantales in the sense of \cite{22}, and section 4 addresses a notion of sheaf for stable quantal frames; in particular, the main theorem (4.17) shows that for those stable quantal frames that are quantales of étale groupoids (the inverse quantal frames) we obtain the usual notion of groupoid sheaf.

A word of caution: in order to avoid unnecessary clutter, the lemmas, theorems, and even examples in this paper do not state explicitly all the conditions that are assumed concerning the quantales involved, which are instead stated at the beginning of the sections or subsections and kept uniformly throughout them. Hence, when being referred from another part of the paper to a paragraph that involves a quantale $Q$, it may useful, if in doubt, to search upwards through the section until the first heading where the conditions assumed about $Q$ are stated explicitly.
2 Hilbert modules

The terminology “Hilbert module” was introduced in [18] as an adaptation to the context of quantales of the notion of Hilbert C*-module (see [13]). Hilbert modules have subsequently been used as a means of relating results from the theory of operator algebras to quantales (see, e.g., [19]) and in the applications to sheaves in [24], where the notion of Hilbert “basis” appears (not an actual basis in the module theoretic sense, but a basis in the topological sense of locale theory).

Throughout this section $Q$ is an arbitrary, but otherwise fixed, unital involutive quantale.

Basic definitions and examples. Let us begin by recalling the notion of Hilbert $Q$-module due to [18]:

Definition 2.1 By a pre-Hilbert $Q$-module will be meant a (unitary) left $Q$-module $X$ equipped with a binary operation

$$\langle -, - \rangle : X \times X \to Q,$$

called the inner product, which for all $x, x_i, y \in X$ and $a \in Q$ satisfies the following axioms:

(2.2) $$\langle ax, y \rangle = a \langle x, y \rangle$$

(2.3) $$\langle \bigvee_i x_i, y \rangle = \bigvee_i \langle x_i, y \rangle$$

(2.4) $$\langle x, y \rangle = \langle y, x \rangle^*.$$

By a Hilbert $Q$-module will be meant a pre-Hilbert $Q$-module whose inner product is non-degenerate:

(2.5) $$\langle x, - \rangle = \langle y, - \rangle \Rightarrow x = y.$$

We remark that, in particular, inner products are “sesquilinear forms”:

(2.6) $$\langle x, \bigvee a_i y_i \rangle = \bigvee \langle x, y_i \rangle a_i^*.$$

Example 2.7 $Q$ itself is a pre-Hilbert $Q$-module with the inner product defined by

$$\langle a, b \rangle = ab^*.$$

The inner product is non-degenerate if $Q$ is unital. More generally, if $S$ is a set then the set $Q^S$ of maps $v : S \to Q$ is a left $Q$-module with the usual
function module structure given by pointwise joins and multiplication on the left, and it is a pre-Hilbert module with the inner product \( \langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} \) given by the standard dot product formula

\[
\mathbf{v} \cdot \mathbf{w} = \bigvee_{s \in S} v_s w_s^* .
\]

(We adopt, for functions in \( Q^S \) and their values, the same notation as for vectors and their components in linear algebra.)

**Example 2.8** If \( p : X \to B \) is a local homeomorphism of locales then \( X \) is an étale \( B \)-locale (see [24]) with \( \zeta = p_! \) and it is a Hilbert \( B \)-module with the inner product defined by \( \langle x, y \rangle = \zeta(x \land y) \). The local sections of \( p \), which are in bijection [24] with the elements \( s \in X \) such that

\[
\forall x \in X \quad x \leq s \Rightarrow x = \zeta(x)s ,
\]

(2.9)

can be described equivalently as the elements \( s \in X \) such that

\[
\forall x \in X \quad \langle x, s \rangle s \leq x .
\]

(2.10)

The equivalence is easily proved:

- If \( s \) satisfies (2.9) and \( x \in X \) then \( x \land s \leq s \) and we have \( \langle x, s \rangle s = \zeta(x \land s)s = x \land s \leq x \).
- Conversely, if \( s \) satisfies (2.10) and \( x \leq s \) then \( x = \zeta(x)x = \langle x, x \rangle x \leq \langle x, s \rangle s \leq x \).

**Hilbert sections.** Now we shall extend the theory of Hilbert bases of [24] so as to apply to general Hilbert \( Q \)-modules. We begin with the following definition, where the terminology “sections” is motivated by (2.10).

**Definition 2.11** Let \( X \) be a pre-Hilbert \( Q \)-module. By a Hilbert section of \( X \) is meant an element \( s \in X \) such that \( \langle x, s \rangle s \leq x \) for all \( x \in X \). The set of all the Hilbert sections of \( X \) is denoted by \( \Gamma_X \). We say that the module \( X \) has enough sections if for all \( x \in X \) we have the equality

\[
x = \bigvee_{s \in \Gamma_X} \langle x, s \rangle s .
\]

Any set \( \Gamma \subset X \) such that \( x = \bigvee_{s \in \Gamma} \langle x, s \rangle s \) for all \( x \in X \) is called a Hilbert basis (in particular, we have \( \Gamma \subset \Gamma_X \) and \( \Gamma \) is a set of \( Q \)-module generators for \( X \)).
The name “Hilbert basis” is suggested by the obvious formal resemblance with the properties of a Hilbert basis of a Hilbert space. Of course, a Hilbert basis in our sense is not an actual basis as in linear algebra because there is no freeness, and thus one might be better off calling such a basis a Hilbert system of generators. But for the sake of simplicity and following [24] we shall retain the shorter terminology.

Example 2.12 The Hilbert $B$-module $\mathcal{O}(X)$ determined by a local homeomorphism of locales $p : X \to B$ has enough sections. Furthermore, $\Gamma_X$ is an actual basis of $X$ in the sense of locale theory (the analogue for locales of a basis of a topological space), and, in particular, $\bigvee \Gamma_X = 1$.

Example 2.13 Seeing $Q$ itself as a Hilbert $Q$-module with $\langle a, b \rangle = ab^*$ as in [2.7] the set of Hilbert sections is $\Gamma_Q = \{ s \in Q \mid s^*s \leq e \}$. It is a Hilbert basis, and so is the singleton $\Gamma = \{ e \}$.

Example 2.14 The condition $\bigvee \Gamma_X = 1$ of [2.12] does not necessarily hold over more general quantales. In order to see this let $R$ be the unital involutive quantale whose involution is trivial and whose order and multiplication table are the following (cf. [22] Example 4.21):

If we regard $R$ as a Hilbert $R$-module with $\langle x, y \rangle = xy^*$ then $R$ has enough sections (because it is a unital quantale) but $\Gamma_R = \{ 0, e \}$.

The existence of a Hilbert basis has useful consequences. In particular the inner product is necessarily non-degenerate:

**Lemma 2.15** Let $X$ be a pre-Hilbert $Q$-module and let $\Gamma \subset X$. If $\Gamma$ is a Hilbert basis then the following properties hold, for all $x, y \in X$.

1. If $\langle x, s \rangle = \langle y, s \rangle$ for all $s \in \Gamma$ then $x = y$. (Hence, $X$ is a Hilbert module.)

2. $\langle x, y \rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \langle s, y \rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \langle y, s \rangle^*$. (“Parseval’s identity”.)

Conversely, $\Gamma$ is a Hilbert basis if $\langle -, - \rangle$ is non-degenerate and [2] holds.
Proof. Assume that $\Gamma$ is a Hilbert basis. The two properties are proved as follows.

1. If $\langle x, s \rangle = \langle y, s \rangle$ for all $s \in \Gamma$ then $x = \bigvee_{s \in \Gamma} \langle x, s \rangle s = \bigvee_{s \in \Gamma} \langle y, s \rangle s = y$.
2. $\langle x, y \rangle = \langle \bigvee_{s \in \Gamma} \langle x, s \rangle s, y \rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \langle s, y \rangle$.

For the converse assume that $\langle -, - \rangle$ is non-degenerate and that 2 holds. Then for all $x, y \in X$ we have

\[
\left\langle \bigvee_{s \in \Gamma} \langle x, s \rangle s, y \right\rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \langle s, y \rangle = \langle x, y \rangle,
\]
and by the non-degeneracy we obtain $\bigvee_{s \in \Gamma} \langle x, s \rangle s = x$. 

Matrix representations. Every Hilbert module with a Hilbert basis has a “metric” associated to that basis:

**Lemma 2.16** If $X$ is a Hilbert $Q$-module with a Hilbert basis $\Gamma$ the $Q$-valued matrix $M : \Gamma \times \Gamma \to Q$ defined by $m_{st} = \langle s, t \rangle$ satisfies $M^* = M^2 = M$ (i.e., it is a projection matrix).

Proof. We have $M = M^*$ by definition of the inner product, and $M = M^2$ follows from 2.13.2.

This has a converse, namely every projection matrix has an associated Hilbert module with a Hilbert basis:

**Lemma 2.17** Let $S$ be a set and $M : S \times S \to Q$ a $Q$-valued projection matrix. Then the subset of $Q^S$

\[
Q^S M = \{ vM \mid v \in Q^S \}
\]

(we regard $v$ as a “row vector” — cf. 2.17) is a Hilbert $Q$-module whose inner product is the dot product of $Q^S$,

\[
\langle v, w \rangle = v \cdot w = \bigvee_{s \in S} v_s w_s^*,
\]

it has a Hilbert basis $\Gamma$ consisting of the functions $\tilde{s} : S \to Q$ defined, for each $s \in S$, by $\tilde{s}_t = m_{st}$ ($\tilde{s}$ is the “$s^{th}$-row” of $M$), and for all $s, t \in S$ and $v \in Q^S$ we have

\[
\langle v, \tilde{t} \rangle = (vM)_t, \tag{2.18}
\]
\[
\langle \tilde{s}, \tilde{t} \rangle = m_{st}. \tag{2.19}
\]
Proof. The assignment \( j : v \mapsto vM \) is a \( Q \)-module endomorphism of \( Q^S \), and \( Q^S M \) is its image, hence a submodule of \( Q^S \). Next note that \( \Gamma \) is a subset of \( Q^S M \) because for each \( s \in S \) we have \( \tilde{s} = \tilde{s}M \in Q^S M \):

\[
\tilde{s}_t = m_{st} = (M^2)_{st} = \bigvee_{u \in S} m_{su} m_{ut} = \bigvee_{u \in S} \tilde{s}_u m_{ut} = (\tilde{s}M)_t .
\]

Now we prove (2.18):

\[
\langle v, \tilde{t} \rangle = v \cdot \tilde{t} = \bigvee_u v_u \tilde{t}_u = \bigvee_u v_u m_{t_u}^* = \bigvee_u v_u m_{ut} = (vM)_t .
\]

In particular, if \( v \in Q^S M \) we have \( vM = v \), hence \( \langle v, \tilde{t} \rangle = v_t \), and (2.19) is an immediate consequence:

\[
\langle \tilde{s}, \tilde{t} \rangle = \tilde{s}_t = m_{st} .
\]

Finally, \( \Gamma \) is a Hilbert basis, since for all \( v \in Q^S M \) we have

\[
\left( \bigvee_t \langle v, \tilde{t} \rangle \right)_s = \left( \bigvee_t v_t \tilde{t}_s \right) = \bigvee_t v_t \tilde{t}_s = \bigvee_t v_t m_{ts} = (vM)_s = v_s . \]

Theorem 2.20 (Representation) Any Hilbert \( Q \)-module with a Hilbert basis arises, up to isomorphism, as in (2.17).

Proof. Let \( X \) be a Hilbert \( Q \)-module with a Hilbert basis \( \Gamma \), let \( M \) be the matrix defined by \( m_{st} = \langle s, t \rangle \) for all \( s, t \in \Gamma \), and let \( \varphi : Q^\Gamma \to X \) be the \( Q \)-module quotient defined by \( \varphi(v) = \bigvee_{s \in \Gamma} v_s s \). By (2.15) we have, for all \( v, w \in Q^\Gamma \), the following series of equivalences:

\[
\varphi(v) = \varphi(w) \iff \forall t \in \Gamma \langle \varphi(v), t \rangle = \langle \varphi(w), t \rangle
\]

\[
\iff \forall t \in \Gamma \left( \bigvee_{s \in \Gamma} v_s \langle s, t \rangle \right) = \left( \bigvee_{s \in \Gamma} w_s \langle s, t \rangle \right)
\]

\[
\iff \forall t \in \Gamma \left( \bigvee_{s \in \Gamma} v_s \langle s, t \rangle \right) = \left( \bigvee_{s \in \Gamma} w_s \langle s, t \rangle \right)
\]

\[
\iff \forall t \in \Gamma (vM)_t = (wM)_t
\]

\[
\iff vM = wM .
\]

Hence, \( \varphi \) factors uniquely through the quotient \( v \mapsto vM : Q^\Gamma \to Q^\Gamma M \) and an isomorphism of \( Q \)-modules \( Q^\Gamma M \cong X \).
**Adjointable maps.** Similarly to Hilbert C*-modules, the module homomorphisms which have “operator adjoints” play a special role:

**Definition 2.21 (Paseka [18])** Let $X$ and $Y$ be pre-Hilbert $Q$-modules. A function

$$\varphi : X \to Y$$

is *adjointable* if there is another function $\varphi^\dagger : Y \to X$ such that for all $x \in X$ and $y \in Y$ we have

$$\langle \varphi(x), y \rangle = \langle x, \varphi^\dagger(y) \rangle .$$

(The notation for $\varphi^\dagger$ in [18] is $\varphi^*$, but we want to avoid confusion with the notation for inverse image homomorphisms of locale maps.)

We note that if $\varphi$ is adjointable and $Y$ is a Hilbert $Q$-module (i.e., the bilinear form of $Y$ is non-degenerate) then $\varphi$ is necessarily a homomorphism of $Q$-modules [18]: we have

$$\langle \varphi \left( \bigvee a_i x_i \right), y \rangle = \langle \bigvee a_i x_i, \varphi^\dagger(y) \rangle = \bigvee a_i \langle x_i, \varphi^\dagger(y) \rangle$$

$$= \bigvee a_i \langle \varphi(x_i), y \rangle = \bigvee \langle a_i \varphi(x_i), y \rangle$$

and thus by the non-degeneracy of $\langle -, - \rangle_Y$ we conclude that

$$\varphi \left( \bigvee a_i x_i \right) = \bigvee a_i \varphi(x_i) .$$

Conversely, and similarly to the situation in [24] where $Q$ was a locale, the homomorphisms of Hilbert $Q$-modules with enough sections are necessarily adjointable. In order to prove this only the domain module need have enough sections:

**Theorem 2.22** Let $X$ and $Y$ be pre-Hilbert $Q$-modules such that $X$ has a Hilbert basis $\Gamma$ (hence, $X$ is a Hilbert module), and let $\varphi : X \to Y$ be a homomorphism of $Q$-modules. Then $\varphi$ is adjointable with a unique adjoint $\varphi^\dagger$, which is given by

$$\varphi^\dagger(y) = \bigvee_{t \in \Gamma} \langle y, \varphi(t) \rangle t .$$

10
Proof. Let \( x \in X, y \in Y \), and let us compute \( \langle x, \varphi^\dagger(y) \rangle \) using \((2.23)\):

\[
\langle x, \varphi^\dagger(y) \rangle = \left\langle \bigvee_{s \in \Gamma} (x, s), \bigvee_{t \in \Gamma} (y, \varphi(t)) \right\rangle
= \bigvee_{s, t \in \Gamma} (x, s) (s, t)^\dagger (y, \varphi(t))
= \bigvee_{t \in \Gamma} (x, t) \varphi(t), y
= \left\langle \varphi \left( \bigvee_{t \in \Gamma} (x, t) \right), y \right\rangle
= \langle \varphi(x), y \rangle.
\]

This shows that \( \varphi^\dagger \) is adjoint to \( \varphi \), and the uniqueness is a consequence of the non-degeneracy of the inner product of \( X \).

**Definition 2.24** The category of Hilbert \( Q \)-modules with enough sections, denoted by \( Q\text{-}HMB \), is the category whose objects are those Hilbert \( Q \)-modules for which there exist Hilbert bases and whose morphisms are the homomorphisms of \( Q \)-modules (equivalently, the adjointable maps).

**Corollary 2.25** The assignment from homomorphisms \( \varphi \) to their adjoints \( \varphi^\dagger \) is a strong self-duality \((\cdot)^\dagger : (Q\text{-}HMB)^{op} \to Q\text{-}HMB \).

### 3 Supported modules

Our goal now is to study the categories of modules on quantales of étale groupoids, in particular the categories of Hilbert modules that are equipped with Hilbert bases. We shall for now do this in a general way, namely addressing arbitrary stably supported quantales, of which the quantales of étale groupoids are special examples (see [22]). This has two purposes: one is to rationalize the presentation in order to achieve a clear understanding of how the various axioms interact with each other; and the other is that by doing so one is paving the way for obtaining possible extensions of the theory developed in this paper in a way that may be applicable to theories of sheaves on quantales of more general types.
Supported quantales. As in the previous section, \( Q \) is a fixed but arbitrary unital involutive quantale, and we shall further denote by \( B \) its involutive unital subquantale \( \downarrow(e) = \{ a \in Q \mid a \leq e \} \), where \( e \) is the multiplicative unit.

Definition 3.1 By a support on \( Q \) is meant a monotone map \( \varsigma : Q \to B \) satisfying the following conditions for all \( a \in Q \):

\[
\begin{align*}
\varsigma(a) & \leq aa^* \\
(3.3) \quad a & \leq \varsigma(a)a .
\end{align*}
\]

The support is stable, and the quantale is stably supported, if in addition we have, for all \( a, b \in Q \):

\[
\varsigma(ab) = \varsigma(a\varsigma(b)) .
\]

This definition differs from those of \[22\] only because we are not requiring supports to be sup-lattice homomorphisms. This has no consequences for us, as we shall see. In particular, it is still true, if \( Q \) is supported, that the following equality holds for all \( a \in Q \),

\[
a1 = \varsigma(a)1 ,
\]

and that the subquantale \( B \) is a locale with \( b \land c = bc \) and \( b^* = b \) for all \( b, c \in B \). Equally true is the uniqueness of stable supports, which are necessarily given by the following formulas,

\[
\begin{align*}
(3.4) \quad \varsigma(a) & = a1 \land e \\
(3.5) \quad \varsigma(a) & = aa^* \land e ,
\end{align*}
\]

and the fact that a support is stable if and only if

\[
\varsigma(a1) \leq \varsigma(a)
\]

for all \( a \in Q \). Moreover, every stable support is \( B \)-equivariant, i.e., for all \( a \in Q \) and \( b \in B \) we have \( \varsigma(ba) = b\varsigma(a) \), and in fact this condition is equivalent to stability:

Theorem 3.6 A support is stable if and only if it is a homomorphism of \( B \)-modules.

Proof. A stable support necessarily preserves joins because \( \varsigma : Q \to B \) is left adjoint to \( b \mapsto b1 : B \to Q \). Hence, stable supports are homomorphisms of \( B \)-modules. In the converse direction, we only need to prove that equivariance implies stability, which, as stated above, is equivalent to the condition that \( \varsigma(a1) \leq \varsigma(a) \) for all \( a \in Q \). Since, as also stated above, any support satisfies \( a1 = \varsigma(a)1 \) (this uses both \(3.2\) and \(3.3\)), if \( \varsigma \) is equivariant it follows that we have \( \varsigma(a1) = \varsigma(\varsigma(a)1) = \varsigma(a)\varsigma(1) \leq \varsigma(a)e = \varsigma(a) \).
Modules on supported quantales. From now on \( Q \) is a fixed but arbitrary supported quantale. We begin with a simple but useful property of arbitrary modules on \( Q \):

**Lemma 3.7** Let \( X \) be a left \( Q \)-module. Then for all \( a \in Q \) we have

\[
a1_X = \varsigma(a)1_X = aa^*1_X = aa^*a1_X = aa^*aa^*1_X.
\]

**Proof.** Let \( a \in Q \). The axioms of supported quantales give us

\[
a1_X \leq \varsigma(a)1_X \leq aa^*1_X \leq \varsigma(a)aa^*1_X \leq aa^*aa^*1_X \leq aa^*a1_X \leq a1_X. \]

Let us introduce a notion of support for modules that is formally similar to that of unital involutive quantales if we replace \( ab^* \) by \( \langle a, b \rangle \):

**Definition 3.8** By a **supported** \( Q \)-module is meant a pre-Hilbert \( Q \)-module \( X \) equipped with a monotone map \( \varsigma : X \to B \) such that the following properties hold for all \( x \in X \):

\[
\varsigma(x) \leq \langle x, x \rangle \\
x \leq \varsigma(x)x.
\]

The support is called *stable*, and the module is said to be **stably supported**, if in addition the support is \( B \)-equivariant; that is, the following condition holds for all \( b \in B \) and \( x \in X \):

\[
\varsigma(bx) = b \land \varsigma(x).
\]

**Lemma 3.9** Let \( X \) be a stably supported \( Q \)-module. The map \( b \mapsto b1_X \) from \( B \) to \( X \) is right adjoint to \( \varsigma : X \to B \). (In particular, \( \varsigma \) preserves joins.)

**Proof.** The unit of the adjunction follows from \( x \leq \varsigma(x)x \leq \varsigma(x)1_X \), and the co-unit follows from the \( B \)-equivariance: \( \varsigma(b1_X) = b \land \varsigma(1_X) \leq b \).

Analogously to supports of quantales, there are several alternative definitions of stability:

**Lemma 3.10** Let \( X \) be a supported \( Q \)-module. The following conditions are equivalent:
1. $\varsigma(ax) = \varsigma(a\varsigma(x))$ for all $a \in Q$ and $x \in X$;
2. $\varsigma(ax) \leq \varsigma(a)$ for all $a \in Q$ and $x \in X$;
3. $\varsigma(a1_X) \leq \varsigma(a)$ for all $a \in Q$;
4. The support of $X$ is stable.

Proof. 2 and 3 are of course equivalent. Let us prove the equivalence of 1 and 2. First assume 1. Then for all $a \in Q$ and $x \in X$ we have

$$\varsigma(ax) = \varsigma(a\varsigma(x)) \leq \varsigma(a) = \varsigma(ax).$$

Now assume 2. Then we have

$$\varsigma(ax) \leq (\varsigma(ax)x) \leq \varsigma(a) \leq \varsigma(ax) = \varsigma(ax),$$

and thus 1 holds. Now assume again 1, and let $b \in B$. Then

$$\varsigma(bx) = \varsigma(b\varsigma(x)) = \varsigma(b \wedge \varsigma(x)) = b \wedge \varsigma(x),$$

and thus we see that 1 implies 4. Finally, assume that 4 holds. Then for all $a \in Q$ we have, using 3.7

$$\varsigma(a1_X) = \varsigma(a)1_X = \varsigma(a) \wedge \varsigma(1_X) \leq \varsigma(a),$$

and thus 4 implies 3. 

Modules on stably supported quantales. From now on $Q$ will be assumed to be a stably supported quantale.

Lemma 3.11 Any supported $Q$-module is necessarily stably supported.

Proof. For all $a \in Q$ and $x \in X$ we necessarily have $\varsigma(ax) \leq \varsigma(a)$, as the following sequence of (in)equalities shows,

$$\varsigma(ax) \leq \langle ax, ax \rangle \wedge e = a\langle x, ax \rangle \wedge e \leq a1_Q \wedge e = \varsigma(a),$$

where the latter equality holds because $Q$ is stably supported. 

We obtain similar properties to those of stable supports of quantales regarding uniqueness of supports, and analogous formulas for $\varsigma$ when we formally replace $ab^*$ by $\langle a, b \rangle$: 14
Lemma 3.12 Let $X$ be a (necessarily stably) supported $Q$-module. The following properties hold:

1. For all $x, y \in X$ we have $\varsigma(\langle x, y \rangle) \leq \varsigma(x) = \varsigma(\langle x, x \rangle) = \varsigma(\langle x, 1 \rangle)$.
2. For all $x \in X$ and $a \in Q$ we have $\varsigma(x)a = \langle x, 1 \rangle \wedge a$.
3. For all $x \in X$ we have $\varsigma(x) = \langle x, 1 \rangle \wedge e$.
4. For all $x \in X$ we have $\varsigma(x) = \langle x, x \rangle \wedge e$.
5. $X$ does not admit any other support.

Proof. First we prove 1. Let $x, y \in X$. Using the stability of the support of $Q$ we have

$$\varsigma(\langle x, y \rangle) \leq \varsigma(\varsigma(x)\langle x, y \rangle) = \varsigma(\varsigma(x)\langle x, 1 \rangle) \leq \varsigma(\varsigma(x)) = \varsigma(x).$$

On the other hand, using the inequality just proved we obtain

$$\varsigma(x) = \varsigma(\varsigma(x)) \leq \varsigma(\langle x, x \rangle) \leq \varsigma(\langle x, 1 \rangle) \leq \varsigma(x),$$

thus proving 1. Now let us prove 2. We have $\varsigma(x)a \leq \langle x, 1 \rangle$ because

$$\varsigma(x)a \leq \langle x, x \rangle 1 \leq \langle x, 1 \rangle 1 = \langle x, 1 \rangle = \langle x, 1 \rangle.$$  

Since we also have $\varsigma(x)a \leq a$ we obtain the inequality

$$\varsigma(x)a \leq \langle x, 1 \rangle \wedge a,$$

and the converse inequality is proved as follows, using 1

$$\langle x, 1 \rangle \wedge a \leq \varsigma(\langle x, 1 \rangle \wedge a)(\langle x, 1 \rangle \wedge a) \leq \varsigma(\langle x, 1 \rangle)a = \varsigma(x)a.$$

Making $a = e$ we obtain 3 (which immediately implies that the support of $X$ is unique), and for 4 it suffices to prove the inequality $\langle x, x \rangle \wedge e \leq \varsigma(x)$, again using 1

$$\langle x, x \rangle \wedge e = \varsigma(\langle x, x \rangle \wedge e) \leq \varsigma(\langle x, x \rangle) \wedge \varsigma(e) = \varsigma(x) \wedge e = \varsigma(x).$$

One consequence of this is that the existence of a support (when $Q$ is stably supported) is a property of a pre-Hilbert $Q$-module rather than extra structure. In fact this uniqueness is even “pointwise”, in the following sense:
Lemma 3.13 Let $X$ be a (necessarily stably) supported $Q$-module, and let $x \in X$ and $b \in B$ be such that
\[ b \leq \langle x, x \rangle \]
\[ x \leq bx . \]
Then we necessarily have $b = \varsigma(x)$.

Proof. Using the $B$-equivariance of the support of $X$ we obtain
\[ \varsigma(x) \leq \varsigma(bx) = b \wedge \varsigma(x) \leq b , \]
and, conversely, $b = \varsigma(b) \leq \varsigma(\langle x, x \rangle) = \varsigma(x)$. \hfill \Box

Finally, if there exists a Hilbert basis we obtain:

Theorem 3.14 Any Hilbert $Q$-module with enough sections is a stably supported $Q$-module.

Proof. Define $\varsigma(x) = \langle x, x \rangle \wedge e$ for all $x \in X$, and let $\Gamma$ be a Hilbert basis of $X$. By definition, in order to verify that $\varsigma$ is a support it only remains to be seen that $x \leq \varsigma(x)x$ for all $x \in X$. Let then $s \in \Gamma$. We have, using the properties of the stable support of $Q$,
\[ \langle x, s \rangle = \varsigma(\langle x, s \rangle)\langle x, s \rangle = (\langle x, s \rangle \langle x, s \rangle^* \wedge e)\langle x, s \rangle \leq (\langle x, x \rangle \wedge e)\langle x, s \rangle = \varsigma(x)\langle x, s \rangle = \langle \varsigma(x)x, s \rangle . \]
Hence, we have $x \leq \varsigma(x)x$ due to 2.15 and by 3.11 we conclude that $X$ is stably supported. \hfill \Box

4 Quantale sheaves

Let us now address the specific case where $Q$ is a fixed stable quantal frame. As we shall see, in this case the theory of sheaves on étale groupoids (equivalently, on inverse quantal frames) generalizes nicely to the extent that we obtain two isomorphic categories, $Q$-$LH$ and $Q$-$Sh$, which are related by the self-duality of $Q$-$HMB$. 

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Stable quantal frames. By a stable quantal frame is meant a stably supported quantale which is also a locale. Equivalently, this is a locale $Q$ equipped with the structure of a unital involutive quantale such that, defining the sup-lattice homomorphism $\varsigma : Q \to \downarrow (e_Q)$ by

$$\varsigma(a) = a1 \land e,$$

the following conditions are satisfied:

$$\varsigma(a) \leq aa^*, \quad a \leq \varsigma(a).$$

Local sections of supported modules. From now on $Q$ is a fixed but arbitrary stable quantal frame, and as before we shall write $B$ for the locale $\varsigma Q = \downarrow (e_Q)$.

In imitation of the local sections of local homeomorphisms (cf. 2.8) we define:

**Definition 4.1** Let $X$ be a (necessarily stably) supported $Q$-module. By a local section of $X$ is meant an element $s \in X$ such that $\varsigma(x \land s)s \leq x$ for all $x \leq s$. The set of local sections of $X$ is denoted by $\Gamma^\ell_X$.

**Lemma 4.2** Let $X$ be a supported $Q$-module.

1. $\Gamma^\ell_X = \{s \in X \mid \varsigma(x \land s)s \leq x \text{ for all } x \in X\}$.
2. $\Gamma^\ell_X$ is downwards closed.
3. $\Gamma_X \subset \Gamma^\ell_X$.
4. $\Gamma_X = \Gamma^\ell_X$ if and only if every local section $s$ is a join $s = \bigvee_i t_i$ of Hilbert sections.

**Proof.** 1 is proved in the same way as the equivalence of (2.9) and (2.10) in 2.8 if $s$ is a local section and $x \in X$ then $x \land s \leq s$ and thus we have $\varsigma(x \land s)s = x \land s \leq x$; conversely, if $s$ satisfies $\varsigma(x \land s)s \leq x$ for all $x \in X$ then if $x \leq s$ we have $x = \varsigma(x)x \leq \varsigma(x)s = \varsigma(x \land s)s \leq \varsigma(x \land s)s \leq x$. Now 2 is an immediate consequence because if $s$ is a local section and $t \leq s$ we have $\varsigma(x \land t)t \leq \varsigma(x \land s)s \leq x$. Similarly, 3 follows from the inequality $\varsigma(x \land s) \leq \langle x, s \rangle$. In order to prove the nontrivial implication in 4 let $s$ be a local section and let $I \subset \Gamma_X$ be such that $s = \bigvee I$. Let $t$ and $u$ be arbitrary elements of $I$. For all $x \in X$ we have $\varsigma(x \land s)s \leq x$, and thus also $\varsigma(x \land t)u \leq x$. In particular, making $x = t$ we obtain $\varsigma(t)u \leq t$. The conclusion that $s$ is a Hilbert section follows immediately, since for all $x \in X$ we have $\langle x, s \rangle s = \bigvee_{t, u \in I} \langle x, t \rangle u$ and $\langle x, t \rangle u = \langle x, \varsigma(t)u \rangle = \langle x, t \rangle \varsigma(t)u \leq \langle x, t \rangle t \leq x$. 

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**Modules on stable quantal frames.** If a Hilbert $Q$-module $X$ is a locale then it is an open $B$-locale in the sense of [24], that is, it is also a $B$-module equipped with a homomorphism of $B$-modules $\varsigma : X \to B$ such that $\varsigma(x)x = x$. Its set of local sections is $\Gamma^\ell_X$, and $X$ defines a sheaf on $B$ if and only if $\bigvee \Gamma^\ell_X = 1$. In particular, it defines a sheaf on $B$ if it satisfies $\bigvee \Gamma_X = 1$, in which case we also have $\Gamma_X = \Gamma^\ell_X$ due to [4.2][3]. The condition $\bigvee \Gamma_X = 1$ is not trivial (cf. stable quantal frame $R$ in [2.14]), it has other pleasant technical consequences to be seen below, and the examples of modules obtained from groupoid sheaves satisfy it. These facts suggest the following notion of quantale sheaf, where we do not require $X$ to be a locale because, as we shall see, this is automatic.

**Definition 4.3** By a *sheaf on $Q$*, or a *$Q$-sheaf*, will be meant a Hilbert $Q$-module $X$ with a Hilbert basis $\Gamma$ such that $\bigvee \Gamma = 1$. We define the *local inner product* of $X$ (cf. 4.5) to be the function

$$\langle -, - \rangle^\ell : X \times X \to B$$

defined by $\langle x, y \rangle^\ell = \langle x, y \rangle \land e$.

**Lemma 4.4** Let $a, b \in Q$ with $b \leq e$. Then $ba \land e = b \land a$.

**Proof.** It has been seen in [22, Lemma 3.4-5]) that $bc = b1 \land c$ for all $c$, and thus

$$ba \land e = (b1 \land a) \land e = (b1 \land e) \land a = b \land a .$$

**Theorem 4.5** Let $X$ be a $Q$-sheaf with a Hilbert basis $\Gamma$. Then the local inner product makes $X$ a Hilbert $B$-module and $\Gamma$ is also a Hilbert basis with respect to this Hilbert module structure.

**Proof.** The operation $\langle -, - \rangle^\ell$ is of course symmetric, and it preserves joins in the left variable because $\langle - , - \rangle$ does and $Q$ is a locale:

$$\langle \bigvee_i x_i, y \rangle^\ell = \bigvee_i \langle x_i, y \rangle \land e = \left( \bigvee_i \langle x_i, y \rangle \right) \land e = \bigvee_i \langle x_i, y \rangle \land e = \bigvee_i \langle x_i, y \rangle^\ell .$$

We show that $\langle -, - \rangle^\ell$ is $B$-equivariant in the left variable, using [4.4]

$$\langle bx, y \rangle^\ell = \langle bx, y \rangle \land e = b \langle x, y \rangle \land e = b \land \langle x, y \rangle = b \land e \land \langle x, y \rangle = b \land \langle x, y \rangle^\ell .$$
Hence, \( X \) together with the local inner product is a pre-Hilbert \( B \)-module, and it remains to be seen that \( \Gamma \) is a Hilbert basis with respect to this pre-Hilbert \( B \)-module structure; that is, we must prove that for all \( x \in X \) we have \( x = \bigvee_{s \in \Gamma} \langle x, s \rangle s \). One inequality is trivial:
\[
x = \bigvee_{s \in \Gamma} \langle x, s \rangle s \geq \bigvee_{s \in \Gamma} (\langle x, s \rangle \wedge e) s = \bigvee_{s \in \Gamma} \langle x, s \rangle \wedge s .
\]

In order to prove the other inequality first we use the condition \( ba = b1 \wedge a \) that holds for all \( b \in B \) and \( a \in Q \) [22, Lemma 3.4-5]:
\[
(4.6) \quad \left( \bigvee_{s \in \Gamma} \langle x, s \rangle \wedge s, t \right) = \bigvee_{s \in \Gamma} \langle x, s \rangle \wedge \langle s, t \rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \wedge 1 \wedge \langle s, t \rangle \\
(4.7) \quad = \bigvee_{s \in \Gamma} (\langle x, s \rangle \wedge e) 1 \wedge \langle s, t \rangle .
\]

Now recall the following inequality from [22, Lemma 4.17]:
\[
(a \wedge e) 1 \geq \bigvee_{yz \leq a} y \wedge z .
\]

Applying this to the right hand side of (4.7) we obtain
\[
(4.8) \quad \bigvee_{s \in \Gamma} (\langle x, s \rangle \wedge e) 1 \wedge \langle s, t \rangle \geq \bigvee_{s \in \Gamma} \bigvee_{yz \leq \langle x, s \rangle} y \wedge z \wedge \langle s, t \rangle .
\]

A particular choice of \( y \) and \( z \) for which \( yz^* \leq \langle x, s \rangle \) is to take \( y = \langle x, t \rangle \) and \( z = \langle s, t \rangle \), and thus with these values of \( y \) and \( z \) the right hand side of (4.8) is greater than or equal to
\[
\bigvee_{s \in \Gamma} y \wedge z \wedge \langle s, t \rangle = \bigvee_{s \in \Gamma} \langle x, t \rangle \wedge \langle s, t \rangle \wedge \langle s, t \rangle = \bigvee_{s \in \Gamma} \langle x, t \rangle \wedge \langle s, t \rangle \\
= \langle x, t \rangle \wedge \bigvee_{s \in \Gamma} \langle s, t \rangle = \langle x, t \rangle \wedge \left( \bigvee_{s \in \Gamma} s, t \right) \\
= \langle x, t \rangle \wedge \langle 1, t \rangle = \langle x, t \rangle .
\]

Hence, we have concluded that \( \bigvee_{s \in \Gamma} \langle x, s \rangle \wedge s, t \rangle \geq \langle x, t \rangle \) for all \( t \in \Gamma \), which finally gives us:
\[
\bigvee_{s \in \Gamma} \langle x, s \rangle \wedge s \geq x .
\]

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Corollary 4.9  Any $Q$-sheaf $X$ is a locale and $\zeta : X \to B$ is the direct image $p_!$ of a local homeomorphism $p : X \to B$ (whose inverse image homomorphism is defined by $p^*(b) = b1_X$).

We remark that the condition that $Q$ should be a locale is necessary. Otherwise any inverse quantale would be a locale (contradicting [22, Example 4.23]) because if we regard an inverse quantale $Q$ as a Hilbert $Q$-module with $\langle a, b \rangle = ab^*$ the inverse monoid of partial units $I(Q)$ is a join-dense Hilbert basis.

Categories of sheaves.  Now we shall study the two promised categories of $Q$-sheaves.

Definition 4.10  Let $X$ and $Y$ be $Q$-sheaves. We define the following classes of morphisms.

- A direct image homomorphism $\varphi : X \to Y$ is a homomorphism of left $Q$-modules that preserves local sections (equivalently, Hilbert sections) and supports: $\varphi(\Gamma_X) \subset \Gamma_Y$ and $\zeta(\varphi(x)) = \zeta(x)$ for all $x \in X$.

- An inverse image homomorphism $\psi : Y \to X$ is a homomorphism of left $Q$-modules that is also a homomorphism of locales: $\psi(1_Y) = 1_X$ and $\psi(y \land z) = \psi(y) \land \psi(z)$ for all $y, z \in Y$.

These classes are obtained from each other by taking adjoints, and within these classes the “operator adjoints” coincide with categorical adjoints:

Theorem 4.11  Let $X$ and $Y$ be $Q$-sheaves, and let $\varphi : X \to Y$ be a homomorphism of left $Q$-modules. Then $\varphi$ is a direct image homomorphism if and only if $\varphi^\dagger : Y \to X$ is an inverse image homomorphism. Furthermore, if $\varphi$ is a direct image homomorphism then $\varphi^\dagger$ is its right adjoint $\varphi_*$ (conversely, if $\psi$ is an inverse image homomorphism then $\psi^\dagger$ is left adjoint to $\psi$), and $\varphi$ coincides with the direct image homomorphism $f_! : X \to Y$ of a map of locales $f : X \to Y$.

Proof. The equation $\langle \varphi(x), y \rangle = \langle x, \varphi^\dagger(y) \rangle$ implies, when we take the meet with $e$ on both sides, that $\varphi^\dagger$ is also the adjoint of $\varphi$ with respect to the local inner products: $\langle \varphi(x), y \rangle^\ell = \langle x, \varphi^\dagger(y) \rangle^\ell$. In other words, both inner products define the same adjoints of $Q$-module homomorphisms. Let $\varphi : X \to Y$ be a direct image homomorphism. Noting that $\varphi$ is a direct image homomorphism if and only if it is a $Q$-module homomorphism and a sheaf homomorphism in the sense of [24, Definition 5], we conclude, from [24, Theorem 12], that
\( \varphi^\dagger \) coincides with the right adjoint of \( \varphi \). The converse statement, that the adjoint \( \psi^\dagger \) of an inverse image homomorphism is both left adjoint to \( \psi \) and a direct image homomorphism, follows from \cite{24} Lemma 1 and Theorem 11, and thus we also have \( \psi^\dagger = f^! \) for the map \( f \) defined by \( f^* = \psi \).

Hence, the two classes of homomorphisms just described are related to each other by the strong self-duality of \( Q \text{-HMB} \). The direct image homomorphisms send local sections to local sections and correspond to natural transformations of sheaves on \( B \), whereas the inverse image homomorphisms are inverse image homomorphisms of continuous maps between étale spaces over \( B \). This justifies the following terminology and notation:

**Definition 4.12** The category \( Q \text{-Sh} \) is that whose objects are the \( Q \)-sheaves and whose arrows \( h : X \to Y \) are the direct image homomorphisms.

The category \( Q \text{-LH} \) is that whose objects are the \( Q \)-sheaves and whose arrows \( f : X \to Y \), called *continuous maps*, or simply *maps*, are defined to be the maps of locales such that \( f^* : Y \to X \) is an inverse image homomorphism.

Hence, \( Q \text{-LH} \) should be thought of as the category of “étale spaces over \( Q \)”, whereas \( Q \text{-Sh} \) is the actual category of “sheaves on \( Q \)”. In this setting the usual equivalence between sheaves and local homeomorphisms becomes an isomorphism:

**Corollary 4.13** The categories \( Q \text{-Sh} \) and \( Q \text{-LH} \) are isomorphic.

**Back to groupoid sheaves.** Finally, we close the circle by proving that if \( Q = \mathcal{O}(G) \) for an étale groupoid \( G \) then the notion of \( Q \)-sheaf that we have introduced in this paper indeed coincides with that of étale \( Q \)-locale of \cite{23}, and thus with that of \( G \)-sheaf. In other words, in addition to being a stable quantal frame now we shall require \( Q \) to satisfy the condition \( \bigvee I(Q) = 1 \), where we recall from \cite{22} that \( I(Q) \) is the set of *partial units* of \( Q \), which are defined to be the elements \( s \in Q \) such that \( ss^* \leq e \) and \( s^*s \leq e \). Equivalently, the partial units can be identified with the (images of the) local bisections of \( G \), where a local section \( s : U \to G_1 \) of the domain map \( d : G_1 \to G_0 \) of \( G \) is said to be a *local bisection* if by composing with the range map \( r : G_1 \to G_0 \) we obtain a regular monomorphism \( r \circ s : U \to G_0 \) of locales.

**Lemma 4.14** Let \( X \) be an étale \( Q \)-locale. The action of \( Q \) on \( X \) restricts to a monoid action \( I(Q) \times \Gamma_X \to \Gamma_X \).
Proof. Let $t \in \Gamma_X$ and $s \in \mathcal{I}(Q)$. We want to prove that $st \in \Gamma_X$. So let $x \leq st$ and let us show that $x = \varsigma(x)st$. First we note that $s^*x \leq s^*st \leq t$, and thus

\[(4.15) \quad \varsigma(s^*x)t = s^*x\]

because $t$ is a section. Secondly, we have $\varsigma(x) \leq \varsigma(st) \leq \varsigma(s) = ss^*$, and thus

\[(4.16) \quad ss^*x = x.\]

Hence, since $s = ss^*s$ and both $\varsigma(x)$ and $ss^*$ belong to $B$, applying (4.15)–(4.16) and the equality $\varsigma(s^*x) = s^*\varsigma(x)s$ from [23, Theorem 4.3-3] we obtain

\[\varsigma(x)st = \varsigma(x)ss^*st = ss^*\varsigma(x)st = ss^*x = x.\]

**Theorem 4.17** $Q$-sheaves and étale $Q$-locales amount to the same thing.

Proof. It is immediate that $Q$-sheaves are étale $Q$-locales, so let us prove the converse. Let $X$ be an étale $Q$-locale. We shall use the construction of Hilbert modules from matrices described in [2.17]. In order to simplify the notation we shall denote by $S$ the set $\Gamma_X$ of local sections of $X$. Let $M : S \times S \rightarrow Q$ be the matrix defined by

\[\mathcal{I}(Q)_{st} = \{a \in \mathcal{I}(Q) \mid \varsigma(a^*) \leq \varsigma(t) \text{ and } at \leq s\}\]

\[(4.19) \quad m_{st} = \bigvee \mathcal{I}(Q)_{st}.\]

The two conditions $\varsigma(a^*) \leq \varsigma(t)$ and $at \leq s$ in (4.18) also imply $\varsigma(a) \leq \varsigma(s)$, since

\[(4.20) \quad \varsigma(a) = \varsigma(a\varsigma(a^*)) \leq \varsigma(a\varsigma(t)) = \varsigma(at) \leq \varsigma(s).\]

Hence, the geometric interpretation of the formula (4.19) is that $m_{st}$ is the union of all the local bisections $a$ that satisfy the following three conditions:

- the domain of $a$ is contained in the domain of the local section $s$;
- the “image of $a$” (i.e., the image of $r \circ a$) is contained in the domain of the local section $t$;
- $a$ acts on $t$ yielding a subsection of $s$.

Figure 1 illustrates this situation for a topological groupoid $G$. (This also suggests a generalization of the logical interpretation of $B$-sets of [4]: the truth values are no longer just locale elements but rather quantale elements.
Figure 1: Action of $a$ on $t$ at an arbitrary point $y = r(a(x)) \in \text{dom}(t)$.

that convey a dynamic notion of “equality” of local sections, which now is defined up to local translations rather than just restriction.)

This geometric motivation makes obvious the fact that for all $a \in \mathcal{I}(Q)_{st}$ we should also be able to obtain $a^*s \leq t$. In order to verify this, first note that from (4.20) it follows that $\varsigma(a) \leq \varsigma(at)$. Hence, from $at \leq s$ we obtain $aa^*s = \varsigma(a)s \leq \varsigma(at)s = at$, and thus

$$a^*s = a^*aa^*s \leq a^*at \leq t.$$ 

Hence, $a^* \in \mathcal{I}(Q)_{ts}$, and we conclude that $M^* = M$. Now let $a \in \mathcal{I}(Q)_{st}$ and $b \in \mathcal{I}(Q)_{tu}$. Then $ab \in \mathcal{I}(Q)_{su}$:

$$\varsigma((ab)^*) = \varsigma(b^*a^*) \leq \varsigma(b^*) \leq u$$

$$abu \leq at \leq s.$$ 

Hence, $m_{at}m_{tu} \leq m_{su}$, and thus $M^2 \leq M$. Now we prove the converse. First note that for all $a \in \mathcal{I}(Q)_{st}$ we have $aa^* \in \mathcal{I}(Q)_{ss}$:

$$\varsigma((aa^*)^*) = \varsigma(aa^*) \leq \varsigma(a) \leq \varsigma(s),$$

$$aa^*s \leq at \leq s.$$ 

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As a consequence we have $a = aa^*a \leq m_{ss}a$, which implies $m_{st} \leq m_{ss}m_{st}$ and thus $M \leq M^2$.

Since $M$ is a projection matrix, by 2.17 we have a Hilbert $Q$-module $Q^S M$, whose inner product is the dot product of $Q^S$, with a Hilbert basis $\Gamma$ consisting of the rows of $M$. Using the same notation as in 2.17 for each $s \in S$ we denote the $s$-row of $M$ by $\tilde{s}$. Now we shall prove that this module is a $Q$-sheaf. The greatest element of $Q^S M$ is $1M$, where $1 : S \to Q$ is the constant function with value 1, and thus the condition $1_{Q^S M} \leq \bigvee \Gamma$ is equivalent to

$$1 \bigvee_{s \in S} m_{st} \leq \bigvee_{s \in S} m_{st} \text{ for all } t \in S. \tag{4.21}$$

(A shorter form is $1M \leq eM$, where $e : S \to Q$ is the constant function with value $e$.) Let us compute the right hand side of (4.21): we have

$$\bigvee_{s \in S} m_{st} = \bigvee_{s \in S} \{ a \in \mathcal{I}(Q) \mid \varsigma(a^*) \leq \varsigma(t) \text{ and } at \leq s \}$$

$$= \bigvee_{s \in S} \{ a \in \mathcal{I}(Q) \mid \exists_{s \in S} \varsigma(a^*) \leq \varsigma(t) \text{ and } at \leq s \}$$

$$= \bigvee_{s \in S} \{ a \in \mathcal{I}(Q) \mid \varsigma(a^*) \leq \varsigma(t) \},$$

where the latter equality is a consequence of the fact that we may always take $s = at$ because $S = \Gamma_X$ and thus $at \in S$, by 4.14. Furthermore we can replace $\mathcal{I}(Q)$ by $Q$ in the above expression because $Q$ is an inverse quantal frame, hence obtaining

$$\bigvee_{s \in S} m_{st} = \bigvee \{ a \in Q \mid \varsigma(a^*) \leq \varsigma(t) \}. \tag{4.22}$$

Then the left hand side of (4.21) equals, using stability of the support of $Q$:

$$1 \bigvee_{s \in S} m_{st} = 1 \bigvee \{ a \in \mathcal{I}(Q) \mid \varsigma(a^*) \leq \varsigma(t) \}$$

$$= \bigvee \{ 1a \in 1\mathcal{I}(Q) \mid \varsigma(a^*) \leq \varsigma(t) \}$$

$$= \bigvee \{ 1a \in 1\mathcal{I}(Q) \mid \varsigma((1a)^*) \leq \varsigma(t) \}$$

$$\leq \bigvee \{ a \in Q \mid \varsigma(a^*) \leq \varsigma(t) \} = \bigvee_{s \in S} m_{st}.$$
It follows that $Q^S M$ is a $Q$-sheaf. Its local inner product $\langle -,- \rangle^\ell$ satisfies, for all $s,t \in S$,

$$\langle \tilde{s}, \tilde{t} \rangle^\ell = \tilde{s} \cdot \tilde{t} \wedge e = m_{st} \wedge e$$

$$= \bigvee \{ a \wedge e \mid a \in I(Q) \text{ and } \varsigma(a^*) \leq \varsigma(t) \text{ and } at \leq s \}$$

$$\leq \bigvee \{ a \wedge e \mid a \in I(Q) \text{ and } \varsigma((a \wedge e)^*) \leq \varsigma(t) \text{ and } (a \wedge e)t \leq s \}$$

$$= \bigvee \{ b \in B \mid \varsigma(b^*) \leq \varsigma(t) \text{ and } bt \leq s \}$$

$$= \bigvee \{ b \in B \mid \varsigma(b^*) \leq \varsigma(t) \text{ and } bt \leq s \} \wedge e$$

$$\leq \bigvee \{ a \in I(Q) \mid \varsigma(a^*) \leq \varsigma(t) \text{ and } at \leq s \} \wedge e$$

$$= m_{st} \wedge e = \langle \tilde{s}, \tilde{t} \rangle^\ell .$$

Hence, all the expressions in the above derivation are equal, and thus

$$\langle \tilde{s}, \tilde{t} \rangle^\ell = \bigvee \{ b \in B \mid \varsigma(b^*) \leq \varsigma(t) \text{ and } bt \leq s \}$$

$$= \bigvee \{ b \in B \mid b \leq \varsigma(t) \text{ and } bt \leq s \}$$

$$= \varsigma(s \wedge t) .$$

Since $\varsigma(s \wedge t)$ is the $B$-valued inner product of $X$ we conclude by the representation theorem 2.20 (or [24, Lemma 5]) that $X \cong B^S M^\ell$, where $M^\ell$ is the matrix defined by $(M^\ell)_{st} = \varsigma(s \wedge t)$. Since this matrix is also that of the local inner product of $Q^S M$ it follows that $B^S M^\ell \cong Q^S M$, and thus $X$ is a $Q$-sheaf.

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