ENTIRE DOWNWARD TRANSLATING SOLITONS TO THE MEAN CURVATURE FLOW IN MINKOWSKI SPACE

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ABSTRACT. In this paper, we study entire translating solutions $u(x)$ to a mean curvature flow equation in Minkowski space. We show that if $\Sigma = \{(x, u(x))| x \in \mathbb{R}^n\}$ is a strictly spacelike hypersurface, then $\Sigma$ reduces to a strictly convex rank $k$ soliton in $\mathbb{R}^{k,1}$ (after splitting off trivial factors) whose “blowdown” converges to a multiple $\lambda \in (0,1)$ of a positively homogeneous degree one convex function in $\mathbb{R}^k$. We also show that there is nonuniqueness as the rotationally symmetric solution may be perturbed to a solution by an arbitrary smooth order one perturbation.

1. INTRODUCTION

Let $\mathbb{R}^{n,1}$ be the Minkowski space with Lorentz metric
\[
\bar{g} = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.
\]
We will say that a hypersurface $\Sigma = \{(x, u(x))| x \in \Omega\} \subset \mathbb{R}^{n,1}$ is strictly spacelike if $u \in C^1(\Omega)$ and $|Du| \leq c_0 < 1$ in $\Omega$.

Ecker and Huisken [5] studied the mean curvature flow with forcing term in Minkowski space and proved longtime existence. More specifically, they studied the equation
\[
\frac{\partial u}{\partial t} = \sqrt{1 - |Du|^2} \left[ \text{div} \left( \frac{Du}{\sqrt{1 - |Du|^2}} \right) - H \right],
\]
where $u = u(x,t)$ is the height function and $H = H(x)$ is the forcing term. Later, M. Aarons [1] proved the following convergence result.

**Theorem 1.1.** ([1]) Let $M_0$ be a smooth spacelike hypersurface with bounded curvature. Suppose $M_0$ never intersects future null infinity $I^+$ or past null infinity $I^-$. Then $M_t$ converges under the flow (1.1) to a convex downward translating soliton, that is, an entire solution of $H = c + a \frac{1}{\sqrt{1 + |\nabla u|^2}}$, $a < 0$.

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After a rescaling, we may assume $a = -1$ and consider $H = C - \frac{1}{\sqrt{1-|Du|^2}}$, where $C > 1$ is a constant. We obtain

$$\text{div} \left( \frac{\nabla u}{\sqrt{1-|Du|^2}} \right) = C - \frac{1}{\sqrt{1-|Du|^2}}$$

or in nondivergence form

$$\left( \delta_{ij} + \frac{u_iu_j}{1-|Du|^2} \right) u_{ij} = C \sqrt{1-|Du|^2} - 1.$$  

Notice that any $u = \vec{v} \cdot x + b, |v| = \sqrt{1 - \frac{1}{C^2}}$ (a maximal hypersurface) is a solution of (1.3). The existence of a unique (up to translation) radial symmetric solution of (1.3) was shown by Ju, Lu and Jian [9].

Aarons [1] in fact conjectured that any solution $u$ of (1.2) is either rotationally symmetric about some point $x_0$ or is a hyperplane. However, this conjecture is not correct. Let $x' = (x_1, \ldots, x_k)$ and set $u(x) = \sum_{i=k+1}^{n} a_i x_i + h(x')$ where $h$ is strictly convex in $x'$. Then $u$ satisfies (1.3) if and only if $h$ satisfies

$$\sum_{i,j=1}^{k} \left( \delta_{ij} + \frac{h_ih_j}{1-|a|^2 - |Dh|^2} \right) h_{ij} = C \sqrt{1-|a|^2} \sqrt{1 - \frac{|Dh|^2}{1-|a|^2}} - 1.$$  

Now let $\tilde{h} = \frac{1}{\sqrt{\lambda}} h(\lambda x), \tilde{C} = \lambda C > 1$ with $\lambda = \sqrt{1-|a|^2}$. Then $\tilde{h}$ satisfies

$$\sum_{i,j=1}^{k} \left( \delta_{ij} + \frac{\tilde{h}_i\tilde{h}_j}{1-|D\tilde{h}|^2} \right) \tilde{h}_{ij} = \tilde{C} \sqrt{1-|D\tilde{h}|^2} - 1.$$  

Thus $\tilde{h}$ is a rank $k$ solution of (1.3) in $\mathbb{R}^k$ with $C$ replaced by $\tilde{C} = \lambda C > 1$.

In fact, we will show a splitting theorem analogous to what Choi-Treibergs [4] proved for spacelike constant mean curvature hypersurfaces.

**Theorem 1.2.** Let $u$ be a strictly spacelike solution of (1.2) and let $\Sigma = \{(x, u(x))|x \in \mathbb{R}^n\}$ be the graph of $u$. Then $\Sigma$ is convex with uniformly bounded second fundamental form. Moreover after an $R^{n,1}$ rigid motion, $R^{n,1}$ splits as a product $R^{k,1} \times R^{n-k}$ such that $\Sigma$ also splits as a product $\Sigma^k \times R^{n-k}$ where $\Sigma^k = R^{n,1} \cup R^{k,1}$ is a strictly convex graphical solution in $\mathbb{R}^{k,1}$.

Thus, it is natural to ask if Aarons conjecture correct for $u$, a strictly convex solution of (1.2) in $\mathbb{R}^n$? In other words, is $\Sigma = \{(x, u(x))|x \in \mathbb{R}^n\}$ rotationally symmetric? The answer is no.
Theorem 1.3. Let $f \in C^2(S^{n-1}_C)$, $\tilde{C} = \sqrt{1 - (\frac{1}{C})^2}$. Then there exists an entire strictly spacelike hypersurface $u$ satisfying equation (1.2) such that

$$u(x) \to \tilde{C}|x| - \frac{n-1}{C^2} \log |x| + f(\tilde{C}x) \text{ as } |x| \to \infty.$$ 

As in the work of Treibergs [12] and Choi-Treibergs [4], the blow-down of a convex strictly spacelike solution $V_u = \lim_{r \to \infty} u(rx)/r$ converges uniformly on compact subsets to the space $\tilde{C}Q$ of convex homogeneous degree one convex functions whose gradient has magnitude $\tilde{C}$ wherever defined. It was shown in [4] that the space $Q$ is in one to one correspondence with the set of lightlike directions

$$L_u := \{x \in S^{n-1} : V_u(x) = 1\}.$$ 

It may be possible that any cone in $\tilde{C}Q$ arises as the blow-down of a solution to (1.2) but we have not shown this.

An outline of the paper is as follows. In section 2 we show the strictly spacelike assumption implies that the graph $\Sigma$ is mean convex. Then in section 3 we show $\Sigma$ is in fact convex and then prove the splitting Theorem 1.2. In section 4 we study the blow-down $V_u$ and finally in section 5 following [12], we construct counterexamples for the radial cone in $\tilde{C}Q$ and prove Theorem 1.3.

2. Strictly spacelike implies mean convex

Let $a^{ij} = \delta_{ij} + \frac{u_i u_j}{w^2}$, where $w = (1 - |Du|^2)^{1/2}$; then equation (1.3) becomes

(2.1) \[ a^{ij} u_{ij} = Cw - 1 \]

Then $w_i = -\frac{u_k u_{ki}}{w}$ and

(2.2) \[
\begin{align*}
w_{ij} &= -\frac{u_k u_{kij}}{w} - \frac{u_k u_{kij}}{w} + \frac{u_k u_{kij}}{w^2} \\
&= -\frac{u_k u_{kij}}{w} - \frac{1}{w} \left( u_k u_{kij} + \frac{u_k u_{kij} u_{ij}}{w^2} \right) \\
&= -\frac{u_k u_{kij}}{w} - \frac{1}{w} a_{kl} u_k u_{ij}.
\end{align*}
\]

Lemma 2.1. Suppose $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$ is a strictly spacelike hypersurface, and $u(x)$ satisfies equation (1.3). Then $\Sigma$ is mean convex, that is $H \geq 0$.

Proof. We differentiate equation (2.1) with respect to $x_k$ to obtain

(2.3) \[ (a^{ij})_k u_{ij} + a^{ij} u_{ij,k} = Cw_k. \]
Since
\[(a^{ij})_k u_{ij} = \left( \frac{u_{ik}u_j}{w^2} + \frac{u_{ij}u_{jk}}{w^2} - 2 \frac{u_{ij}u_{ik}w_k}{w^3} \right) u_{ij} \]
\[= \frac{2}{w} \left( \frac{u_{ik}u_j}{w} + \frac{u_{ij}u_{il}u_{lk}}{w_3} \right) u_{ij} \]
\[= \frac{2}{w} \left( -w_i u_{ik} - \frac{u_{ij}u_{lk}}{w^2} w_j \right) \]
\[= -\frac{2}{w} \left( \delta_{ij} + \frac{u_{ij}u_{lk}}{w^2} \right) w_i u_{kj} \]
\[= -\frac{2}{w} a^{ij} w_i u_{kj}, \]
and \(u_{ijk} = u_{kij}\), this gives
\[(2.5) \quad a^{ij} u_{kij} - \frac{2}{w} a^{ij} w_i u_{kj} = C w_k.\]
Multiplying (2.5) by \(\frac{u_k}{w}\) and using \(\frac{u_k u_{kij}}{w} = -w_{ij} - \frac{1}{w} a^{kl} u_{ki} u_{lj}\), we obtain
\[(2.6) \quad a^{ij} w_{ij} - 2 \frac{a^{ij} w_i w_j}{w} + C \frac{u_k}{w} \frac{w_k}{w} = -\frac{1}{w} a^{ij} a^{kl} u_{ki} u_{lj}.\]

We now observe that since \(|A|^2 = \frac{1}{w} a^{ij} a^{kl} u_{ki} u_{lj}\) we can rewrite (2.6) as
\[(2.7) \quad a^{ij} \left( \frac{1}{w} \right)_{ij} + C \frac{u_k}{w} \left( \frac{1}{w} \right)_k = |A|^2 \frac{1}{w}.\]

The Omori-Yau maximum principle (see for example [13], [11]) implies that \(\frac{1}{w}\) achieves its maximum at infinity and moreover, there exists a sequence \(\{P_N\}\) such that \(\frac{1}{w}(P_N) \to \sup \frac{1}{w}, |\nabla (\frac{1}{w})| (P_N) < 1/N\), and \(\left( \frac{1}{w} \right)_{ij} (P_N) \geq -1/N \delta_{ij}\). Therefore,
\[(2.8) \quad \frac{1}{n} H^2 \frac{1}{w} \leq |A|^2 \frac{1}{w} \leq C_1 \frac{1}{N} \frac{1}{w} \quad \text{at } P_N.\]
Thus \(H(P_N) \to 0\) at infinity. Since \(H = C - \frac{1}{w}\) we obtain \(\inf H = 0\). \(\square\)

3. Mean convexity implies convexity and constant rank

In this section, we will use ideas due to Hamilton [7] to prove that under the strictly spacelike assumption, \(\Sigma\) is in fact convex. We use the following approximation of Heidusch[8].

**Definition 3.1.** The \(\delta\)-approximation to the function \(\min(x_1, x_2)\) is given by
\[\mu_2(x_1, x_2) = \frac{x_1 + x_2}{2} - \sqrt{\left( \frac{x_1 - x_2}{2} \right)^2 + \delta^2}\]
for any $\delta > 0$. The $\delta$-approximation to the function $\min(x_1, x_2, \cdots, x_n)$ is defined recursively by

$$
\mu_n = \frac{1}{n} \sum_{i=1}^{n} \mu_2(x_i, \mu_{n-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n))
$$

The following lemma is elementary (see [1]).

**Lemma 3.2.** For every $\delta > 0$ and $n \geq 2$ we have:

1. $\mu_n$ is smooth, symmetric, monotonically increasing and concave.
2. $\frac{\partial \mu_n}{\partial x_i} \leq 1$.
3. $\min(x_1, \cdots, x_n) - n\delta \leq \mu_n \leq \min(x_1, \cdots, x_n)$.
4. For $x \in \mathbb{R}^n$ we have

$$
\mu_n \leq \sum_{i=1}^{n} \frac{\partial \mu_n}{\partial x_i} x_i \leq \mu_n + n\delta,
$$

and $\sum_{i=1}^{n} \frac{\partial \mu_n}{\partial x_i} x_i^2 \geq \mu_n^2 - n\delta^2 - \frac{n\delta}{4} \sum_{1 \leq i < j \leq n} |x_i + x_j|$.

**Lemma 3.3.** Assume $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$ is mean convex, and $u$ satisfies equation (1.3). Then the principal curvatures of $A$ are nonnegative, i.e. $\Sigma$ is convex.

**Proof.** Let $p$ be a fixed point in $\Sigma$ (we may assume $p = (0, 0)$) and let $r$ be the distance function from $p$ restricted to the geodesic ball $B^\Sigma(p, a)$ of radius $a$ centered at $p$ (in the induced metric on $\Sigma$). Let $f(x) = |A|^2 = \sum_{i,j} h_{ij}^2$. By a well-known calculation (see equation (2.24) of [3])

\[(3.1) \quad \frac{1}{2} \Delta \left( \sum_{i,j} h_{ij}^2 \right) = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij} h_{ij} + \left( \sum_{i,j} h_{ij}^2 \right)^2 - \left( \sum_{i,j,k} h_{ij} h_{jm} h_{mi} \right) H
\]

\[(3.2) \quad H_{ij} = \nabla_i \nabla_j H = \nabla_i (\nabla_j (\nu, e_{n+1})) = \nabla_i \langle h_{jk} \tau_k, e_{n+1} \rangle = -h_{ijk} u_k - h_{ik} h_{jk} \nu^{n+1}.
\]

In the following, we will denote $\nu^{n+1}$ by $V$.

\[(3.3) \quad \frac{1}{2} \Delta f = \sum_{i,j,k} h_{ijk}^2 - h_{ij} h_{ijk} u_k - h_{ij} h_{ik} h_{jk} V + f^2 - \left( \sum_{i,j,m} h_{ij} h_{jm} h_{mi} \right) (C - V)
\]

\[\geq -\frac{f(V^2 - 1)}{4} + f^2 - Cf^{3/2}
\]

\[\geq -\frac{C^2}{4} f + f^2 - Cf^{3/2} \geq \frac{1}{2} f^2 - C_1 f
\]
Let $\eta(x) = a^2 - r^2$ and set $g = \eta^2 f$. Then in $B^n(p, a)$,
\begin{equation}
\frac{1}{2}(\eta^{-2} g)^2 \leq C_1 \eta^{-2} g + \Delta (\eta^{-2} g) = C_1 \eta^{-2} g + \eta^{-2} \Delta g - 2\eta^{-3} < \nabla \eta, \nabla g > + g \Delta (\eta^{-2}) .
\end{equation}
At the point $x$ where $g$ assumes its maximum, $\nabla g = 0$ and $\Delta g \leq 0$. Since $R_{ii} \geq -\frac{\mu^2}{4} \geq -\frac{C_2}{4}$, we have by Lemma 1 of [13] that $\Delta \eta^2 \leq C_3(1 + r^2)$. Hence at $\nabla g$,
\begin{equation}
\frac{1}{2} g^2 \leq C_1 \eta^2 g + g \eta^4 \Delta (\eta^{-2}) = C_1 \eta^2 g - 2g \eta \Delta \eta + 6g|\nabla \eta|^2
\end{equation}
It follows that $g(x) \leq C_5 a^4$. Therefore, let $a \to \infty$ we get,
\begin{equation}
|A|^2 \leq C_6 .
\end{equation}

Next we will show that the smallest principal curvature $\lambda_{\min}$ of $\Sigma$ is nonnegative. Let $\mu_n(\lambda_1, \cdots, \lambda_n) = F(\gamma^k h_{kl}, \gamma^l)$, assume $\mu_n$ achieves its minimum at an interior point $x_0$. Then at this point we have
\begin{equation}
\begin{align*}
\Delta \mu_n &= F^{ij} h_{ij} + F^{rl, st} h_{rlk} h_{stk} \\
&= F^{ij}(H_{ij} - H h_{ij}^2 + h_{ij} h_{ik}) + F^{rl, st} h_{rlk} h_{stk} \\
&\leq F^{ij} \nabla_k h_{ij} \langle \tau_k, e_{n+1} \rangle - F^{ij} h_{ij}^{n+1} - H F^{ij} h_{ij}^2 + (\mu_n + n\delta)|A|^2 \\
&\leq \langle \nabla k \mu_n, e_{n+1} \rangle - \mu_n^2 + n\delta^2 + \frac{n\delta}{4} \sum_{1 \leq i \neq j \leq n} |\lambda_i + \lambda_j| \\
&\quad + H \left( -\mu_n^2 + n\delta^2 + \frac{n\delta}{4} \sum_{1 \leq i \neq j \leq n} |\lambda_i + \lambda_j| \right) + (\mu_n + n\delta)|A|^2.
\end{align*}
\end{equation}
Thus,
\begin{equation}
0 \leq -\mu_n^2 + n\delta^2 + \frac{n\delta}{4} \sum_{1 \leq i \neq j \leq n} |\lambda_i + \lambda_j| + (\mu_n + n\delta)|A|^2 .
\end{equation}
Letting $\delta \to 0$ we find,
\begin{equation}
|A|^2 \leq C_6 ,
\end{equation}
which implies that $\lambda_{\min} \geq 0$.

Since we have already proven that $|A|^2$ is bounded, we can again apply the Omori-Yau maximum principle (this time on $\Sigma$) and show that, if $\mu_n$ achieves its minimum at infinity then $\mu_n \geq 0$. This completes the proof that mean convexity implies convexity.
\[ \square \]

Now that we have proved convexity, we prove the splitting Theorem 1.2 of the introduction.
Proof of Theorem 1.2. Suppose that for some unit vector $\vec{v}$ and some $x_0 \in \mathbb{R}^n$, $D^2 u(x_0) = 0$. Applying an isometry (boost transformation) of $\mathbb{R}^{n,1}$, may assume $x_0 = 0$, $\vec{v} = e_n$, $Du(0) = 0$, $u_{nn}(0) = 0$ and $u_{ij}$ is nonnegative. Rewrite (1.3) as
\[(3.9) \quad \Delta u = -\frac{u_iu_j}{1-|Du|^2} u_{ij} + C \sqrt{1-|Du|^2} - 1 \]
Differentiating (3.9) twice in the $x_n$ direction, we can apply the argument of Korevaar (a special case of [10]) exactly as in Theorem 3.1 of Choi-Treibergs [4] to conclude $u_{nn} \equiv 0$ and $\Sigma$ is ruled by lines parallel to the $x_n$ axis. Therefore $\Sigma = \Sigma^{n-1} \times \mathbb{R}^1$ and also $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$. Therefore $u = ax_n + h(x')$, $x' = (x_1, \ldots, x_{n-1})$ where
\[(3.10) \quad \sum_{i,j=1}^{n-1} (\delta_{ij} + \frac{h_i h_j}{1-|a|^2-|Dh|^2}) h_{ij} = C \sqrt{1-|a|^2} \frac{1}{1-|a|^2} - 1 . \]
Now let $\tilde{h} = \frac{1}{\lambda^2} h(\lambda x)$, $\tilde{C} = \lambda C > 1$ with $\lambda = \sqrt{1-|a|^2}$. Then $\tilde{h}$ satisfies
\[(3.11) \quad \sum_{i,j=1}^{n-1} (\delta_{ij} + \frac{\tilde{h}_i \tilde{h}_j}{1-|D\tilde{h}|^2}) \tilde{h}_{ij} = \tilde{C} \sqrt{1-|D\tilde{h}|^2} - 1 . \]
Proceeding inductively completes the proof of Theorem 1.2.

4. The asymptotic cone at infinity.

In this section, we will study the asymptotic behavior of $u$ at infinity.

Proposition 4.1. Let $u$ be a convex space like solution of (1.3). Assume $u(0) = 0$ and denote $u^h(x) = \frac{u(hx)}{h}$. Define $V_u(x) = \lim_{h \to \infty} u^h(x)$ then $V_u(x)$ exists for all $x$ and is a positively homogeneous degree one convex function. Moreover for all $x \in \mathbb{R}^n$ and $\delta > 0$ there exists $y \in \mathbb{R}^n$ so that $|y - x| = \delta$ and $|V_u(x) - V_u(y)| = \sqrt{1 - \frac{1}{C^2}} \delta$. In particular $|D V_u(x)| = \sqrt{1 - \frac{1}{C^2}}$ at every point of differentiability of $V_u(x)$.

Proof. Note that since $u$ is convex, $0 = u(0) \geq u(hx) - \sum_{i=1}^{n} h x_i u_{x_i}(hx)$ we get $\frac{d}{dh} u^h(x) \geq 0$. Then $V_u(x)$, the projective boundary values (blow-down) of $u$ at infinity in the terminology of Treibergs [12] and Choi-Treibergs [4], is well-defined, strictly spacelike, convex on $\mathbb{R}^n$ and satisfies
\[V_u(\lambda x) = \lambda V_u(x), \quad \lambda > 0, \]
\[|V_u(x) - V_u(y)| \leq \sqrt{1 - \frac{1}{C^2}} |x - y| . \]
Claim: for all \( x \in \mathbb{R}^n \) and \( \delta > 0 \) there exists \( y \in \mathbb{R}^n \) so that \( |y - x| = \delta \) and \( |V_u(x) - V_u(y)| = \sqrt{1 - \frac{1}{C^2}} \delta \). Suppose the claim is false. Then there exists \( x \in \mathbb{R}^n \) and \( \varepsilon > 0 \) such that

\[
V_u(y) \leq V_u(x) + (1 - 2\varepsilon)\sqrt{1 - \frac{1}{C^2}} \delta \quad \forall y \in \partial B(x, \delta).
\]

Since \( u^h(x) \to V_u(x) \) uniformly on compact subsets, we may choose \( h_0 \) large so that for all \( h > h_0 \),

\[
u^h(y) \leq V_u(x) + (1 - \varepsilon)\sqrt{1 - \frac{1}{C^2}} \delta \quad \forall y \in \partial B(x, \delta).
\]

Now \( u^h(y) \) satisfies

\[
(4.1) \quad H^h := \text{div} \left( \frac{Du^h}{\sqrt{1 - |Du^h|^2}} \right) = h(C - \frac{1}{\sqrt{1 - |Du^h|^2}}) \geq 0 \quad \text{in} \ B(x, \delta)
\]

We now make use of the radial solutions of the maximal surface equation \( H = 0 \) introduced by Bartnik and Simon [2]. Consider the barrier

\[
w(y) = V_u(x) + (1 - \varepsilon)\sqrt{1 - \frac{1}{C^2}} \delta + \int_0^{[y-x]} h \sqrt{t^{2n-2} + h^2} \, dt - \sqrt{1 - \frac{1}{C^2}} \int_0^\delta h \sqrt{t^{2n-2} + h^2} \, dt.
\]

Note that on \( \partial B(x, \delta) \),

\[
w(y) - u^h(y) \geq (1 - \sqrt{1 - \frac{1}{C^2}}) \int_0^\delta h \sqrt{t^{2n-2} + h^2} \, dt > 0.
\]

Hence by the maximum principle, \( u^h(y) < w(y) \) in \( B(x, \delta) \). In particular at \( y = x \),

\[
u^h(x) < V_u(x) + (1 - \varepsilon)\sqrt{1 - \frac{1}{C^2}} \delta - \sqrt{1 - \frac{1}{C^2}} \int_0^\delta h \sqrt{t^{2n-2} + h^2} \, dt.
\]

Now let \( h \to \infty \) to conclude \( V_u(x) \leq V_u(x) - \varepsilon \sqrt{1 - \frac{1}{C^2}} \delta \), a contradiction, so the claim is proven and the proposition is complete. \( \square \)

5. CONSTRUCTION OF COUNTEREXAMPLES.

We will follow Treiberg’s idea (see [12]) to construct counterexamples, more precisely we will construct solution to equation (1.2) such that \( u(x) \to \tilde{C} |x| - \frac{1}{C^2} \log |x| + f \left( \frac{\varepsilon x}{|x|} \right) \), as \( |x| \to \infty \), where \( \tilde{C} = \sqrt{1 - \frac{1}{C^2}} \) and \( f \in C^2(S^{n-1}) \).
We extend the function $f$ to $\mathbb{R}^n \setminus \{0\}$ by defining $f(\bar{C}x) = f\left(\frac{\bar{C}x}{|x|}\right)$. Since $f \in C^2$, we have for all $x, y \in S^{n-1}$:

(5.1) $|f(\bar{C}x) - f(\bar{C}y) - Df(\bar{C}y)(\bar{C}x - \bar{C}y)| \leq M|\bar{C}x - \bar{C}y|^2 = -2\bar{C}M x \cdot (\bar{C}x - \bar{C}y)$.

Let $p_1(\bar{C}y) = Df(\bar{C}y) + 2M\bar{C}y$ and $p_2(\bar{C}y) = Df(\bar{C}y) - 2M\bar{C}y$, so that

(5.2) $p_1(\bar{C}y) \cdot (\bar{C}x - \bar{C}y) \leq f(\bar{C}x) - f(\bar{C}y) \leq p_2(\bar{C}y) \cdot (\bar{C}x - \bar{C}y)$.

Now let $\psi(x)$ denote the rotationally symmetric solution to (1.2) (see [9]). We know that $\psi(x) \to \bar{C}|x| - \frac{n-1}{C^2} \log |x| + o(1)$ as $|x| \to \infty$. Let $z_1(x; \bar{C}y) = f(\bar{C}y) - p_1(\bar{C}y) \cdot \bar{C}y + \psi(x + p_1(\bar{C}y))$ and $z_2(x; \bar{C}y) = f(\bar{C}y) - p_2(\bar{C}y) \cdot \bar{C}y + \psi(x + p_2(\bar{C}y))$. Then by equation (5.2) we have

(5.3) $f(\bar{C}x) \geq z_1(rx; \bar{C}y) - \bar{C}r + \frac{n-1}{C^2} \log r$ as $r \to \infty$, $x, y \in S^{n-1}$, and

(5.4) $f(\bar{C}x) \leq z_2(rx; \bar{C}y) - \bar{C}r + \frac{n-1}{C^2} \log r$ as $r \to \infty$, $x, y \in S^{n-1}$.

Therefore,

(5.5) $\lim_{r \to \infty} z_1(rx; \bar{C}y) - \bar{C}r + \frac{n-1}{C^2} \log r \leq f(\bar{C}x) - \bar{C}r + \frac{n-1}{C^2} \log r$

for $x \in S^{n-1}$.

Let $q_1(x) = \sup_{y \in S^{n-1}} z_1(x; \bar{C}y)$ and $q_2(x) = \inf_{y \in S^{n-1}} z_2(x; \bar{C}y)$. Then, $q_1(x) \leq q_2(x)$ and $q_i(x)$ $(i=1,2)$ tends to $f(\bar{C}x) + \bar{C}r - \frac{n-1}{C^2} \log r$ as $r \to \infty$.

**Lemma 5.1.** There exists a smooth solution $u$ to the Dirichlet problem

(5.6) $\begin{cases} a^{ij}u_{ij} - Cw + 1 = 0 \text{ in } G \\ u = 0 \text{ on } \partial G \end{cases}$

where $G$ is a convex $C^{2,\alpha}$ domain in $\mathbb{R}^n$.

**Proof.** Let $d = \text{diam}(G)$ be the diameter of $G$. For any $y \in \partial G$, we can choose coordinates such that $y = (y_1, 0, \cdots, 0)$ and $G \subset \{x \mid |x_1| \leq y_1\}$, where $0 < y_1 \leq d/2$. Let $\bar{u}(x) = \bar{C}x_1 - \bar{C}y_1$, $\bar{u} \equiv 0$. Then $u \leq \bar{u}$ in $G$ and $u$ satisfies

(5.7) $a^{ij}u_{ij} - Cw_u + 1 = 0$
By the maximum principle, any solution $u$ to the Dirichlet problem (5.6) satisfies
\begin{equation}
(5.8) \quad u - \bar{u} \leq u \leq \bar{u} \quad \text{on } G.
\end{equation}
so $|Du(x)| \leq \tilde{C}$ on $\partial \bar{G}$. Combined with equation (2.7) we conclude that
\begin{equation}
(5.9) \quad |Du(x)| \leq \tilde{C} \quad \text{on } \bar{G}.
\end{equation}
Now it is standard (see [6]) to prove that a smooth solution $u \in C^{2,\alpha}(\bar{G})$ exists. \qed

Finally, we will find a sandwiched solution $u$ such that $q_1 \leq u < q_2$.

Let $\phi$ be a strictly spacelike hypersurface $q_1 \leq \phi < q_2$ so that $\phi(0) = q_1(0)$ and $G_m = \phi^{-1}((\infty, m))$ is a convex domain with $C^{2,\alpha}$ boundary. By lemma 5.1 we know there is an analytic solution $u_m$ to the Dirichlet problem
\begin{equation}
(5.10) \quad a^{ij}u_{ij} - Cw + 1 = 0 \quad \text{on } G_m
\end{equation}
\begin{equation}
\quad u = m \quad \text{on } \partial G_m.
\end{equation}
Therefore, we find a sequence of finite solutions $u_m$ with $q_1 \leq u_m < q_2$ defined on convex domains $G_m$ which exhaust $\mathbb{R}^n$.

Next, let $K$ be a compact subset of $\mathbb{R}^n$. Then, by equation (5.9) there are constants $r_1 < r_2$ so that for sufficiently large $m$ we have
\begin{equation}
\text{dist}_m(0, x) < r_1, \text{ for all } x \in K,
\end{equation}
\begin{equation}
\text{dist}_m(0, x) < r_2, \text{ for all } x \in \partial G_m,
\end{equation}
where $\text{dist}_m(0, x)$ is the intrinsic distance between the points $(0, u_m(0))$ and $(x, u_m(x))$ on $\Sigma_m = \{(x, u_m(x)|x \in G_m)\}$.

At last, following the proof of Lemma 3.3, we find $u_m$ has uniform $C^3$ bounds on compact subsets. Hence, a subsequence can be extracted that converges to a global solution of equation (1.2). Moreover, $\lim_{m \to \infty} u_{m_j} = u$ satisfies $u(x) \to \tilde{C}|x| - \frac{n-1}{C^2} \log |x| + f(\tilde{C}x)$ as $|x| \to \infty$. Thus we have proved

**Theorem 5.2.** Let $f \in C^2(S^{n-1}_{\tilde{C}})$. Then there exists an entire strictly spacelike hypersurface $u$ satisfying equation (1.2) such that
\begin{equation}
\quad u(x) \to \tilde{C}|x| - \frac{n-1}{C^2} \log |x| + f(\tilde{C}x) \text{ as } |x| \to \infty.
\end{equation}
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