Relativistic Corrections for Polarized $J/\Psi$-Production in b-decay

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**Abstract**

We analyse the structure of relativistic corrections in the inclusive production of polarized $J/\Psi$ from b-quark decay. The analysis is performed not only for the production channel in which the $c\bar{c}$ pair is a color sinlet, but also for the channels in which the $c\bar{c}$ is a color octet. We find that the correction in the color-singlet channel at the tree-level is completely determined by the decay constant of $J/\Psi$, while in the color-octet channels the corrections are characterized by three matrix elements defined in NRQCD, whose numerical values are unknown. We discuss the impact of these corrections on the polarized $J/\Psi$-production, and the impact is so significant that the predictions based on the analysis for the considered process may be unreliable. Finally, we propose an integrated spin observable to measure the polarization of $J/\Psi$.

PACS numbers: 13.25.Hw, 14.40.Gx, 12.38.Bx, 13.85.Ni
Quarkonium systems generally are thought as simpler than light hadrons and it may be easier to handle them in the framework of QCD. However, only recently we have been able to treat their decays and productions rigorously, based on a factorization with non-relativistic QCD (NRQCD) [1]. In the framework of the NRQCD factorization the effect of short distance is handled with perturbative QCD and the effect of long distance is parameterized with NRQCD matrix elements. The factorization is performed by utilizing the fact that a heavy quark inside a quarkonium moves with a small velocity \( v \) in the quarkonium rest frame and an expansion in \( v \) can be employed. In this factorization, inclusive productions of a quarkonium, e.g., like \( J/\Psi \), can be imagined at the leading order of \( v \) as the following: The \( c^- \) and \( \bar{c}^- \) quark are produced at a same space-time point, and then this pair is transformed into the \( J/\Psi \). The production of the \( c\bar{c} \) pair can be calculated with perturbative QCD, while the transformation is a nonperturbative process, which can be described by matrix elements defined in NRQCD. At higher order of \( v \) various effects are taken into account, e.g., relativistic effect, and the effect of the produced \( c\bar{c} \) pair which does not have the same quantum numbers as those of \( J/\Psi \). For \( J/\Psi \) the transition of a \( c\bar{c} \) pair in a color-octet state happens at higher orders in the small velocity expansion, but in some cases, the produced \( c\bar{c} \) pair is more likely in a color-octet state than in a color-singlet state, hence the production of \( J/\Psi \) through a color-octet \( c\bar{c} \) pair is not negligible, and has an important contribution to the total production rate. Including this contribution one is able to fitting experimental results of the total production rate obtained in Tevatron [2–4].

With the NRQCD factorization, not only the total production rate of a quarkonium can be predicted, but also the polarization of the quarkonium, if it has a spin. Recent preliminary measurement [5] at Tevatron by CDF shows that the produced \( J/\Psi \) is polarized in the way which is unexpected from theoretical predictions. Several attempts to explain the discrepancy are made [6,7]. It should be kept in mind that these predictions are only based the theoretical analysis on at the leading order in \( \alpha_s \) and at the leading order in \( v \). For charmonia the velocity is not very small, typically \( v^2 \approx 0.3 \), and one-loop effects are generally substantial. Adding effects at higher orders of \( \alpha_s \) and of \( v \), the predictions may be changed significantly. In this work we study the effects of higher orders in \( v \) on the polarization of \( J/\Psi \) produced in b-quark decay, to see how important the effects are.

In the framework of heavy quark effective theory inclusive decays of a b-flavored hadron
can be thought approximately as inclusive decays of a free b-quark. The inclusive b-quark decay into a $J/\Psi$ were studied at the leading order in $v$ in [8,9], in [10] the one-loop correction from QCD for the unpolarized decay was studied. In these studies it is already shown that the color-octet $c\bar{c}$ pair plays an important role. In this work we will not only considered the color-singlet contribution but also the color-octet contribution. With the NRQCD factorization, the polarized decay width for the process

$$b \rightarrow J/\Psi + X$$

can be written as:

$$\Gamma_\lambda(J/\Psi) = \sum_n C_n(b \rightarrow c\bar{c}[n] + X)\langle 0|O^{J/\Psi}[n]|0 \rangle,$$  

in which the coefficients $C_n(b \rightarrow c\bar{c}[n] + X)$ describe the production of a $c\bar{c}$ pair in a state $n$, the matrix elements $\langle 0|O^{J/\Psi}[n]|0 \rangle$ characterize the transition of a $c\bar{c}$ pair in $n$-state into the $J/\Psi$. The coefficients can be calculated with perturbative QCD because the production of a $c\bar{c}$ pair is a short-distance process, while the matrix elements represent nonperturbative effects and they are defined with operators in NRQCD. These matrix elements are scaled by the power of $v$ with the rule of power counting in $v$ [11]. The index $\lambda$ stands for the helicity of the $J/\Psi$, $\lambda = L$ is for the longitudinal polarization, $\lambda = T$ is for the transversal polarization. We will use the matching procedure proposed in [1] to identify the operators at the next-to-leading order in $v$ and to calculate the corresponding coefficients.

To do the matching we need to consider the process

$$b(p_b) \rightarrow c(p_1) + \bar{c}(p_2) + X,$$

where the momenta are given in the brackets. With a Lorentz transformation we can boost the $c\bar{c}$ pair into its rest-frame and denote in the rest-frame the three-momentum of $c$ and of $\bar{c}$ as $q$ and $-q$ respectively. The total decay width for the process in Eq.(3) can be written

$$\Gamma(b \rightarrow c\bar{c} + X) = \int \frac{d^3q}{(2\pi)^3} \hat{\Gamma}(q)$$

The matching condition reads:

$$\hat{\Gamma}(q) = \sum_n C_n(b \rightarrow c\bar{c}[n] + X)\langle 0|O^{c\bar{c}}[n]|0 \rangle.$$.  

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In the above equation, the $q$-dependence in the right hand side is only contained in the matrix elements, in which the hadronic state is replaced by the partonic $c\bar{c}$ state. With Eq.(5) one can identify the matrix elements appearing in Eq.(2) and can determine the corresponding coefficients.

The effective weak Hamiltonian for $b$-quark decay is:

$$ H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{q=s,d} \{ V_{cb} V_{cq}^* \frac{1}{3} C_1(\mu) \bar{c} \gamma^\mu (1 - \gamma_5) c \bar{q} \gamma_\mu (1 - \gamma_5) b \\
 + C_8(\mu) \bar{c} T^a \gamma^\mu (1 - \gamma_5) c \bar{q} T^a \gamma_\mu (1 - \gamma_5) b \} \}.$$  \hspace{1cm} (6)

We neglected the contributions of QCD penguin operators in $H_{\text{eff}}$. $T^a (a = 1, \cdots 8)$ is SU(3) color-matrix. The coefficients $C_1$ and $C_8$ are related to the usual $C_{\pm}$ by

$$ C_1(\mu) = 2C_+(\mu) - C_-(\mu), \quad C_8(\mu) = C_+(\mu) + C_-(\mu). \hspace{1cm} (7) $$

With the one-loop evolution one can obtain

$$ \frac{C_1(m_b)}{C_8(m_b)} \approx 0.18 \hspace{1cm} (8) $$

This indicates that the $c\bar{c}$ pair is produced more likely in a color-octet state than in a color-singlet state. Hence the color-octet contribution is significant and should be included in the decay width. With the effective Hamiltonian it is straightforward to calculate $\hat{\Gamma}(q)$ and to determine the coefficients in Eq.(5). However, the power counting of $v$ for matrix elements with partonic states is different than that for matrix elements with hadronic states. Before presenting our results, we briefly discuss the identification of operators in Eq.(2). In general the matrix elements in Eq.(2) take the form

$$ \langle 0|O^H[\mathcal{K}_n]|0\rangle = \sum_X \langle 0|\mathcal{K}_n|H + X\rangle \langle H + X|\mathcal{K}_n'|0\rangle, \hspace{1cm} (9) $$

where $H$ denotes a quarkonium. $\mathcal{K}_n$ and $\mathcal{K}_n'$ are operators defined in NRQCD. The general constraints can be obtained by considering symmetries of QCD or NRQCD. Because QCD respects charge conjugation- and parity symmetry, the operators must be transformed in the same way under C- and P transformation. The operators $\mathcal{K}_n$ and $\mathcal{K}_n'$ can be classified with the weight $j$ of $SU(2)$ representations. Since the total angle momentum is conserved, the operators must have the same weight. By considering the approximated symmetry, spin-symmetry of NRQCD, further constraints can be obtained. With these constraints, one can identify the operators appearing in Eq.(2).
Now we give our results for $\Gamma_\lambda(J/\Psi)$. We write

$$\Gamma_\lambda(J/\Psi) = \Gamma_\lambda^{(1)} + \Gamma_\lambda^{(8)},$$

(10)

where the index 1 or 8 stands for color singlet contributions or for color-octet contributions, respectively. The leading order in $v$ for color singlet contributions is $v^0$, while the leading order for color-octet contributions is $v^4$. We expand the quantities:

$$\Gamma_\lambda^{(1)} = \Gamma_\lambda^{(1,v^0)} + \Gamma_\lambda^{(1,v^2)} + \mathcal{O}(v^4),$$

$$\Gamma_\lambda^{(8)} = \Gamma_\lambda^{(8,v^4)} + \Gamma_\lambda^{(8,v^6)} + \mathcal{O}(v^8).$$

(11)

For color-singlet contributions at the leading order, there is only one matrix element representing the transformation of a $c\bar{c}$ pair in $^3S_1$ state into the $J/\Psi$. At this order we have:

$$\Gamma_\lambda^{(1,v^0)} = \frac{C_1^2 G_2^2 |V_{cb}|^2}{432 \pi m_b^2 m_c} (m_b^2 - 4m_c^2)^2 \langle 0 | O_1^{J/\Psi}(^3S_1) | 0 \rangle \cdot 4m_c^2,$$

$$\Gamma_\lambda^{(1,v^2)} = \frac{C_1^2 G_2^2 |V_{cb}|^2}{432 \pi m_b^2 m_c} (m_b^2 - 4m_c^2)^2 \langle 0 | O_1^{J/\Psi}(^3S_1) | 0 \rangle \cdot m_b^2,$$

(12)

where we used $|V_{cd}|^2 + |V_{cs}|^2 \approx 1$. At order of $v^2$, there is also one matrix element which represents the effect of the relative movement of the $c\bar{c}$ pair inside $J/\Psi$. The results reads:

$$\Gamma_\lambda^{(1,v^2)} = \frac{C_1^2 G_2^2 |V_{cb}|^2}{432 \pi m_c} m_b(m_b^2 - 4m_c^2) \langle 0 | P_1^{J/\Psi}(^3S_1) | 0 \rangle \cdot \frac{m_c^2}{m_b^2} G_{\lambda}(\frac{m_c^2}{m_b^2})$$

(13)

with

$$G_{L}(y) = -\frac{3 + 92y + 208y^2 + 576y^3}{6(1 + 4y)^2},$$

$$G_{T}(y) = \frac{2y(3 - 68y - 304y^2 - 960y^3)}{3(1 + 4y)^2}.$$

(14)

The definition of the matrix elements $\langle 0 | O_1^{J/\Psi}(^3S_1) | 0 \rangle$ and $\langle 0 | P_1^{J/\Psi}(^3S_1) | 0 \rangle$ can be found in \cite{1}. These matrix elements can only be calculated nonperturbatively and they must be calculated with the same accuracy in order of $v$ if one uses both to make predictions.

For color octet contributions at the leading order in $v$, the $c\bar{c}$ pair can be in $^1S_0$, $^3S_1$ and $^3P_1$ states, these states can be transformed into a $J/\Psi$ through emission or absorption of soft gluons. Correspondingly, we have in the matching three types of matrix elements, these matrix elements can be reduced to three matrix elements with the symmetries mentioned before. They are:
\sum_X \langle 0 | \chi^T a | J/\Psi + X \rangle \langle J/\Psi + X | \psi T^a \chi | 0 \rangle

= \frac{1}{3} \varepsilon_i (\lambda) \varepsilon_j^* (\lambda) \langle 0 | O_s^{J/\Psi} (1 S_0) | 0 \rangle (15)

\sum_X \langle 0 | \chi^T a \sigma^i | J/\Psi + X \rangle \langle J/\Psi + X | \psi T^a \sigma^j \chi | 0 \rangle

= \frac{1}{3} \varepsilon_i (\lambda) \varepsilon_j^* (\lambda) \langle 0 | O_s^{J/\Psi} (3 S_1) | 0 \rangle (16)

\sum_X \langle 0 | \chi^T a (\sigma \times (-\frac{i}{\sqrt{2}} \mathbf{D}))^i \psi | J/\Psi + X \rangle \langle J/\Psi + X | \psi T^a \sigma \times (-\frac{i}{\sqrt{2}} \mathbf{D})_i \chi | 0 \rangle

= \frac{1}{3} (\delta_{ij} \varepsilon (\lambda) \cdot \varepsilon^*(\lambda) - \varepsilon_i (\lambda) \varepsilon_j^* (\lambda) \langle 0 | O_s^{J/\Psi} (3 P_1) | 0 \rangle (17)

In the above equations \( \varepsilon_i (\lambda) (i = 1, 2, 3) \) denotes the polarization vector of \( J/\Psi \) in its rest-frame. The field \( \psi (\chi) \) is the field in NRQCD for the \( c(\bar{c}) \) quark. \( \mathbf{D} \) is the spacial part of the covariant derivative \( D^\mu \). The definition of the operator \( O_s^{J/\Psi} (1 S_0), O_s^{J/\Psi} (3 S_1) \) and \( O_s^{J/\Psi} (3 P_1) \) can be found in [1]. The rotation invariance leads to Eq.(15), while spin symmetry is used for Eq.(16) and Eq.(17). The leading order of these matrices is at \( v^4 \). It should be noted that Eq.(16) and Eq.(17) also hold at order of \( v^6 \), because to keep the correct spin configuration the spin of the \( c \bar{c} \) pair must be flipped twice through the interaction which violates the spin symmetry, this can only happen at order of \( v^8 \). With the above identification we obtain the color-octet contribution at the leading order:

\[ \Gamma_L^{(8, v^4)} = \frac{G_F^2 |V_{cb}|^2}{288 \pi m_b m_c} (m_b^2 - 4m_c^2)^2 \cdot \left \{ m_b^2 \langle 0 | O_s^{J/\Psi} (1 S_0) | 0 \rangle + m_b^2 \langle 0 | O_s^{J/\Psi} (3 S_1) | 0 \rangle + 8 \langle 0 | O_s^{J/\Psi} (3 P_1) | 0 \rangle \right \} \]

\[ \Gamma_T^{(8, v^4)} = \frac{G_F^2 |V_{cb}|^2}{288 \pi m_b m_c} (m_b^2 - 4m_c^2)^2 \cdot \left \{ m_b^2 \langle 0 | O_s^{J/\Psi} (1 S_0) | 0 \rangle + 4m_c^2 \langle 0 | O_s^{J/\Psi} (3 S_1) | 0 \rangle + \frac{m_b^2 + 4m_c^2}{m_c^2} \langle 0 | O_s^{J/\Psi} (3 P_1) | 0 \rangle \right \}. \] (18)

At order of \( v^6 \) for the color-octet contributions various matrix elements appear at the first look. For example, the matrix element

\[ \sum_X \langle 0 | \chi^T a \sigma^i (-\frac{i}{\sqrt{2}})^2 \mathbf{D} \mathbf{D}^i \psi | J/\Psi + X \rangle \langle J/\Psi + X | \psi T^a \chi | 0 \rangle + h.c.. \] (19)

To take a close look at the matrix element we consider

\[ B^{ijk} = \sum_X \langle 0 | \chi^T a \sigma^i (-\frac{i}{\sqrt{2}})^2 \mathbf{D} \mathbf{D}^i \psi | J/\Psi + X \rangle \langle J/\Psi + X | \psi T^a \sigma^j \chi | 0 \rangle + h.c.. \] (20)
At order of \( v^6 \) the spin symmetry still can be used for the matrix element, it leads:

\[
B_{ijlk}^{\prime} = \varepsilon_i(\lambda)\varepsilon_j^*(\lambda) \cdot A_{lk}.
\] (21)

By rotation invariance, \( A_{lk} \) is proportional to \( \delta_{lk} \). Because the tensor \( \bar{D}^{(l \leftrightarrow k)} \) is symmetric and trace-less, we conclude that \( B_{ijlk}^{\prime} \) is zero at this order, hence also the matrix element in Eq.(19). This can also be understood as the following: Because the spin symmetry holds, the total orbit angle moment is conserved. The \( X \) state in the bra in Eq.(20) is a \( l = 0 \) state because of the conservation, while the \( X \) state in the ket is a \( l = 2 \) state. Therefore, the matrix element is excluded by the definition of the sum over \( X \) state. By considering all constraints, only three types of matrix elements remain, and they can be further reduced to three matrix elements:

\[
\sum_X \langle 0 | \chi^T a (-\frac{i}{2} \bar{D})^2 \psi | J/\Psi + X \rangle \langle J/\Psi + X | \psi^T a^T \chi | 0 \rangle \text{ h.c.}
\]

\[
= \frac{2}{3} \varepsilon_i(\lambda)\varepsilon_j^*(\lambda) \langle 0 | P_{8}^{I/\Psi}(1S_0) | 0 \rangle \tag{22}
\]

\[
\sum_X \langle 0 | \chi^T a \sigma^i (-\frac{i}{2} \bar{D})^2 \psi | J/\Psi + X \rangle \langle J/\Psi + X | \psi^T a^T \sigma^j \chi | 0 \rangle \text{ h.c.}
\]

\[
= \frac{2}{3} \varepsilon_i(\lambda)\varepsilon_j^*(\lambda) \langle 0 | P_{8}^{I/\Psi}(3S_1) | 0 \rangle \tag{23}
\]

\[
\sum_X \langle 0 | \chi^T a (\sigma \times (-\frac{i}{2} \bar{D}))^i (-\frac{i}{2} \bar{D})^j \psi | J/\Psi + X \rangle \langle J/\Psi + X | \psi^T a \sigma \times (-\frac{i}{2} \bar{D})^j \chi | 0 \rangle \text{ h.c.}
\]

\[
= \frac{2}{3} (\delta_{ij} \varepsilon(\lambda) \varepsilon^*(\lambda) - \varepsilon_i(\lambda)\varepsilon_j^*(\lambda)) \langle 0 | P_{8}^{I/\Psi}(3P_1) | 0 \rangle \tag{24}
\]

The matrix elements \( \langle 0 | P_{8}^{I/\Psi}(1S_0) | 0 \rangle \), \( \langle 0 | P_{8}^{I/\Psi}(3S_1) | 0 \rangle \) and \( \langle 0 | P_{8}^{I/\Psi}(3P_1) | 0 \rangle \) can be obtained by summing over the helicity and by the contraction over the indices in the above equations. With these matrix elements the results for the color-octet contribution at order of \( v^6 \) are:

\[
\Gamma^{(8,\sigma^6)}_{\lambda} = \frac{G_F^2 V_{ud}^2}{288\pi m_c^2} m_b (m_b^2 - 4m_c^2)
\cdot \left\{ F(m_b^2 \frac{m_c^2}{m_b^2} ) \cdot \langle 0 | P_{8}^{I/\Psi}(1S_0) | 0 \rangle + G(m_b^2 \frac{m_c^2}{m_b^2} ) \cdot \langle 0 | P_{8}^{I/\Psi}(3S_1) | 0 \rangle 
\right.
\]

\[
+ H(m_b^2 \frac{m_c^2}{m_b^2} ) \cdot \frac{1}{m_c^2} \langle 0 | P_{8}^{I/\Psi}(3P_1) | 0 \rangle \}
\]

(25)

with

\[
F(y) = -\frac{7 + 108y + 144y^2 + 320y^3}{6(1 + 4y)^2},
\]

\[
H_L(y) = -\frac{4y(1 + 84y + 240y^2 + 704y^3)}{6(1 + 4y)^2},
\]

7
\[ H_T(y) = \frac{-7 + 84y + 144y^2 + 704y^3}{6(1 + 4y)^2}. \]  

(26)

The function \( G_\lambda(y) \) is given in Eq.(14). The above results indicate that the correction in the color-octet contributions is to take the effect of the relative movement of the \( c\bar{c} \) pair inside the \( J/\Psi \) into account. This is similar as the case with the color-singlet contribution. Hence, the total corrections are just relativistic corrections characterized by four matrix elements.

With the results given above one may predict the polarization of the produced \( J/\Psi \) and the total decay width \( \Gamma = 2\Gamma_T + \Gamma_L \), if one knows the numerical values of the eight matrix elements. Among the four matrix elements at the leading order in \( v \) the best known matrix element is \( \langle 0|O^{J/\Psi}_{1}(3S_1)|0 \rangle \), which is calculated with potential models and with lattice QCD, while the other three are extracted from experimental data. The value of \( \langle 0|O^{J/\Psi}_{8}(3S_1)|0 \rangle \) is rather well determined by direct \( J/\Psi \) production at large transverse momentum in \( pp \) collisions [2–4]. However the uncertainty for this determination is large and it can be at the level of 100%. The other two are not well determined, only certain combination of them is know with a large uncertainty. In [14] a re-analysis of experimental data from Tevatron and from Hera is performed, in which some higher-orders effects due to multiple-gluon initial-states radiation are considered. It is shown that the values of these matrix elements can be changed substantially by including these effects. Based on the leading order results an analysis for predictions of the polarized \( J/\Psi \) is given [3]. The four matrix elements at the next-to-leading order in \( v \) are completely unknown. With the power-counting in \( v \), we only know that they are suppressed by \( v^2 \) relatively to the corresponding matrix elements at the leading order in \( v \). All of these prevents us from a detailed prediction for the polarization and for the total decay width. Nevertheless, one can still see the impact of these corrections. For the color-singlet contributions, if one neglects the one-loop QCD correction and uses the vacuum saturation, then with \( H_{\text{eff}} \) in Eq.(6) one has only one constant representing the nonperturbative effect in the process in Eq.(1):

\[ \langle J/\Psi|\bar{c}\gamma^\mu c|0 \rangle = -if_{J/\Psi}M_{J/\Psi}(\varepsilon^\mu(\lambda))^*, \]  

(27)

where \( f_{J/\Psi} \) is the leptonic decay constant of \( J/\Psi \) and is related to the leptonic decay width:

\[ \Gamma(J/\Psi \to \ell^+\ell^-) = \frac{16\pi\alpha^2_{\text{em}}}{27M_{J/\Psi}f_{j/\Psi}^2}. \]  

(28)

In the above equation, the only approximation is to neglect effects of higher orders in \( \alpha_{\text{em}} \) and lepton masses, relativistic corrections are automatically included. In NRQCD the vacuum
saturation brings uncertainty at order of \( v^4 \). Therefore the color-singlet contributions can be written as:

\[
\Gamma^{(1)}_T = \frac{C_4^2 G_F^2 |V_{cb}|^2}{144 \pi m_b^3} (m_b^2 - M_{J/\Psi}^2)^2 M_{J/\Psi}^2 f_{J/\Psi}^2 + O(\alpha_s^2) + O(v^4),
\]

\[
\Gamma^{(1)}_L = \frac{C_4^2 G_F^2 |V_{cb}|^2}{144 \pi m_b^3} (m_b^2 - M_{J/\Psi}^2)^2 m_b f_{J/\Psi}^2 + O(\alpha_s^2) + O(v^4),
\]

(29)

where the relativistic correction calculated before is contained in the \( J/\Psi \)-mass \( M_{J/\Psi} \) and in the decay constant. Comparing the results in Eq.(29) with that in Eq.(12) we may see how large the relativistic correction is. For this we define:

\[
R^{(1)}_\lambda = \frac{\Gamma^{(1,v^0)}_\lambda}{\Gamma^{(1)}_\lambda},
\]

(30)

where \( \Gamma^{(1,v^0)}_\lambda \) is given in Eq.(12) and the results in Eq.(29) are used for \( \Gamma^{(1)}_\lambda \). To obtain numerical values of \( R^{(1)}_\lambda \), we take \( m_b = 4.7 \text{GeV} \) and \( m_c = 1.5 \text{GeV} \). \( f_{J/\Psi} \) is determined by the leptonic decay to be 405MeV. For the matrix element \( \langle 0 | O_{J/\Psi} | 0 \rangle \), its value is determined based on a potential model in [12] to be \( 1.16 \text{(GeV)}^3 \), which is in agreement with a calculation of lattice QCD in [13]. With these values we obtain:

\[
R^{(1)}_L \approx R^{(1)}_T \approx 1.6.
\]

(31)

This indicates that the correction is negative and very large. Another way to see the impact of the correction is to compare the coefficients in the front of the matrix element. We take the above numerical values for quark masses and obtain:

\[
\Gamma^{(1)}_L = \frac{C_4^2 G_F^2 |V_{cb}|^2}{432 \pi} \cdot \left\{ 24.3 \langle 0 | O_{J/\Psi}^{(3S_1)} | 0 \rangle - 52.4 \frac{1}{m_c^2} \langle 0 | P_{J/\Psi}^{(3S_1)} | 0 \rangle \right\} + O(v^4),
\]

\[
\Gamma^{(1)}_T = \frac{C_4^2 G_F^2 |V_{cb}|^2}{432 \pi} \cdot \left\{ 9.90 \langle 0 | O_{J/\Psi}^{(3S_1)} | 0 \rangle - 11.4 \frac{1}{m_c^2} \langle 0 | P_{J/\Psi}^{(3S_1)} | 0 \rangle \right\} + O(v^4).
\]

(32)

With these results the relativistic correction can be at the level of 60% for \( \Gamma^{(1)}_L \) and at the level of 35% for \( \Gamma^{(1)}_T \) if one takes

\[
\frac{1}{m_c^2} \langle 0 | P_{J/\Psi}^{(3S_1)} | 0 \rangle \approx v^2 \langle 0 | O_{J/\Psi}^{(3S_1)} | 0 \rangle, \quad v^2 \approx 0.3.
\]

(33)

Similarly we obtain for the color-octet contributions:

\[
\Gamma^{(8)}_L = \frac{C_8^2 G_F^2 |V_{cb}|^2}{288 \pi} \cdot \left\{ 24.4 \langle 0 | O_{J/\Psi}^{(1S_0)} | 0 \rangle - 68.3 \frac{1}{m_c^2} \langle 0 | P_{J/\Psi}^{(1S_0)} | 0 \rangle \right\}
\]
If one assumes that there are similar relations among the color-octet matrix elements like that in Eq. (33), then the correction is very large. For example, the correction for the $^1S_0$ production channel can be at the level of 80%, and the correction for the $^3P_1$ production channel can be at the level of 70%. With the above discussions one may conclude that the predictions based on the leading- and next-to-leading order in the small velocity expansion for the process are unreliable.

The last subject of this work is to propose an integrated observable to measure the spin of the produced $J/\Psi$. Usually, one looks at the leptonic decay of $J/\Psi$ to measure the spin. At a fixed momentum $P$ of $J/\Psi$ one measures the distribution:

$$\frac{d\Gamma}{d\cos \theta}(J/\Psi \rightarrow \ell^+ \ell^-) \propto 1 + \alpha \cos^2 \theta,$$

where $\theta$ is the angle between $P$ and $k$, $k$ is the momentum of the lepton in the $J/\Psi$-rest frame. The parameter $\alpha$ is predicted as:

$$\alpha = \frac{\Gamma_T - \Gamma_L}{\Gamma_T + \Gamma_L}.$$

This may have a disadvantage that it will be hard to determine the distribution if the number of events with a fixed $P$ is small, hence the parameter $\alpha$. We propose to use an integrated spin observable to overcome this disadvantage. For this purpose we define the density matrix $R_{ij}$ of the produce $J/\Psi$. For arbitrary polarization the decay width can be written:

$$\Gamma(b \rightarrow J/\Psi + X) = \frac{1}{4\pi} \int d\Omega \varepsilon_i R_{ij}(\hat{P}) \varepsilon_j,$$

where $\Omega$ is the solid angle of $P$, $\hat{P}$ denotes the direction of $P$. Similarly we can define the density matrix $\rho_{ij}$ for the leptonic decay. Any observable $O$ can be then predicted by

$$\langle O \rangle = \frac{1}{N} \int \frac{d\Omega}{4\pi} \int \frac{d\Omega}{4\pi} O \cdot \rho_{ij}(\hat{k}) \cdot R_{ji}(\hat{P}),$$
where \( N \) is a normalization factor so that \( \langle 1 \rangle = 1 \). A simple calculation leads:
\[
\rho_{ij}(\hat{k}) = \frac{1}{3}\delta_{ij} - \frac{1}{2}(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}),
\]
(39)
where \( \rho_{ij} \) is normalized. \( R_{ij} \) can be written:
\[
R_{ij}(\hat{P}) = a\delta_{ij} + ib\varepsilon_{ijk}\hat{P}_k + c\hat{P}_i\hat{P}_j,
\]
\[
a = \Gamma_T, \quad c = \Gamma_L - \Gamma_T.
\]
(40)

The anti-symmetric part exits due to that parity is violated, and it is irrelevant here because the parity is conserved in the leptonic decay. With the tensor structure one can construct the integrated spin observable \( O_P \), and predict its value with Eq.(38):
\[
O_P = (\hat{P} \cdot \hat{k})^2 - \frac{1}{3}, \quad \langle O_P \rangle = \frac{2(\Gamma_T - \Gamma_L)}{15\Gamma}.
\]
(41)
If \( \langle O_P \rangle = 0 \) the \( J/\Psi \) is unpolarized. If one know the values of the matrix elements discussed before, one can obtain the value for \( \langle O_P \rangle \) to compare with the measured in experiment. One may also obtain the invariance \( \langle O_P^2 \rangle \) to determine the statistical error of \( \langle O_P \rangle \).

We summarize our work: We have analyzed the relativistic corrections for the polarized \( J/\Psi \)-production in \( b \)-quark decay. We have calculated the perturbative coefficients and identified the matrix elements at the next-to-leading order in \( \nu \). We find that the corrections can be very large in the color-singlet production channel and as well as in the color-octet production channels. For the color-singlet production channel the correction is determined by the leptonic decay constant of \( J/\Psi \). These corrections are so large that the predictions for the considered process may be unreliable, if one only keeps several leading terms in the small velocity expansion. An integrated spin observable is proposed to measure the \( J/\Psi \) polarization. If detailed information of the eight matrix elements is known, numerical values of polarized decay widths and the observable may be obtained.

Acknowledgment: The author would like to thank Prof. K.T. Chao and Prof. Y.Q. Chen for discussions. This work is supported by National Science Foundation of P.R. China and by the Hundred Young Scientist Program of Sinica Academia of P.R.China.
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