ON EXPLICIT CONSTRUCTIONS OF EXOTIC SPHERES

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Abstract. We generalize the construction of the Milnor sphere $M^7_{1,-1}$ in [10] through a pull-back procedure and apply it to exhibit explicit nontrivial elements on some equivariant homotopy groups and differentiable structure of total spaces of sphere bundles over the exotic 8-sphere. In particular we prove that there is no new example of exotic 15-sphere if one considers the exotic 8-sphere instead of the standard 8-sphere in [17].

1. Introduction

This paper is concerned with geometrical presentations of manifolds homeomorphic, but not diffeomorphic, to spheres, the, so called, exotic spheres. Using the symmetries of certain maps, we realize these manifolds as isometric quotients of principal bundles with connection metric over standard spheres and derive some topological results. The geometry of such bundles are discussed in [19].

After introducing the construction in [10] through a point of view similar to [8], we present, in section 3, how it can be pull-backed in a way that produces new examples of spheres. We recall from [4] that $Sp(2) = \{(x, y) \in S^7 \times S^7 | \langle x, y \rangle_H = 0\}$ and prove that

Theorem 1.1. Let $\eta : S^8 \rightarrow S^7$ and $b_{10} : S^{10} \rightarrow S^7$ be the maps defined by

$$\eta(\lambda, x, y) = \frac{\lambda + x\bar{x}}{\sqrt{\lambda^2 + |x|^2 + |y|^2}}$$

$$b_{10}(\xi, x, y) = \exp \left( \frac{x}{|x|} \pi \bar{x} y \right),$$

where $(\lambda, x, y)$ and $(\xi, x, y)$ are in the unitary spheres of $\mathbb{R} \times \mathbb{H}^2$ and $\text{Im}\mathbb{H} \times \mathbb{H}^2$, respectively. Then the quotient of

$$E^{11} = \{(x, y) \in S^8 \times S^7 | \langle \eta(x), y \rangle_H = 0\}$$

$$E^{13} = \{(x, y) \in S^{10} \times S^7 | \langle b_{10}(x), y \rangle_H = 0\}$$

where $\langle , \rangle_H$ is the standard hermitian quaternionic product the $S^3$-actions

$$q \ast \begin{pmatrix} \lambda \\ x \\ y \\ d \end{pmatrix} = \begin{pmatrix} \lambda \\ qx \\ qy \bar{q} \\ qd \end{pmatrix}, \quad q \ast \begin{pmatrix} \xi \\ x \\ y \\ d \end{pmatrix} = \begin{pmatrix} \xi \\ qx \\ qc \\ qy \bar{q} \\ qd \end{pmatrix}$$

are the only exotic sphere in dimension 8 and a generator of the subgroup of spheres that bound spin manifolds in dimension 10.

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A further application of the construction results in

**Theorem 1.2.** A homotopy 15-sphere $\Sigma_{15}$ can be realized as the total space of a linear 7-sphere bundle over the exotic 8-sphere with characteristic class $[\alpha] \in \pi_7 SO(8)$ if and only if it can also be realized as the total space of a linear 7-sphere bundle over the standard 8-sphere with the same characteristic class.

In section 4 we will present non-trivial elements in some equivariant homotopy groups of spheres and constraints on actions over some spheres.

In this paper we consider principal $G$-bundle endowed with an additional $G$-action, which we call $\ast$-action and which commute with the principal one. For $p$, an element in the total space, and $q \in G$ we write $pq^{-1}$ and $qp$ for the principal and $\ast$ action, respectively, of $q$ on $p$. Our convention for the principal action in a trivialization chart is $(x, g)q^{-1} = (x, qg)$.

### 2. The Gromoll-Meyer sphere through another point of view

Let $Sp(2)$ be the set of $2 \times 2$ quaternionic matrices respecting the identity $\bar{Q}^TQ = \text{id}$, where $\bar{Q}^T$ is the transpose conjugate of $Q$. For $S^4$ and $S^7$, the unit spheres in $\mathbb{R} \times \mathbb{H}$ and $\mathbb{H} \times \mathbb{H}$, we define $h : S^7 \to S^4$ as

$$h(x, y) = (|x|^2 - |y|^2, 2xy).$$

Following [4], we have a commutative diagram

$$\begin{align*}
Sp(2) & \xrightarrow{pr_2} S^7 \\
\downarrow pr_2 & \downarrow \downarrow \\
S^7 & \xrightarrow{h} S^4
\end{align*}$$

where $pr_i$ is the projection to the $i$-th column. Furthermore, $pr_1, pr_2 : S^7 \to S^7$ are $S^4$-principal bundles with principal actions given by right multiplication of the matrices $\text{diag}(1, \bar{q})$ and $\text{diag}(\bar{q}, 1)$, respectively.

Let $S^4_+ = S^4 - \{(-1, 0)\}$ and $S^4_- = S^4 - \{(1, 0)\}$. Then, $S^7$, as a principal bundle, is identified with

$$S^4_+ \times S^3 \leftrightarrow (S^4_+ \cap S^4_-) \times S^3 \xrightarrow{f_a} S^4_- \times S^3$$

where $f_a(\lambda, x, g) = (\lambda, x, gx/|x|)$. By the other hand, we have the $S^3$-action in $Sp(2)$ defined in [10] as

$$q \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} qa\bar{q} & qc \\ gb\bar{q} & gd \end{pmatrix}.$$ 

It is proved there that the quotient of $Sp(2)$ by this action is diffeomorphic to the exotic Milnor sphere $M^7_{2-1}$. We now present a different proof:

**Theorem 2.1** ([14][15]). The quotient of $Sp(2)$ by the action (2.3) is diffeomorphic to

$$D^4 \times S^3 \leftrightarrow S^3 \times S^3 \xrightarrow{g_a^{-1}f_a} S^3 \times D^4$$

where $f_a, g_a : S^3 \times S^3$ are defined by $f_a(x, y) = (x, xy\bar{v})$ and $g_a(x, y) = (yx\bar{v}, y)$. In particular, it is a generator of $\theta^7 \cong \mathbb{Z}_{28}$.
Proof. From the diagram (2.2) we conclude that
\[\text{Sp}(2) = h^{-1}(S^4_+ \times S^3) \hookrightarrow h^{-1}(S^4_+ \cap S^4) \times S^3 \xrightarrow{f_{ah}} h^{-1}(S^4_+) \times S^3\]
as a principal bundle, where \(f_{ah}(x, y, g) = (x, y, g x y / |x y|)\). Note that, if we define \(a : S^4_+ \cap S^4 \to S^3\) as \(a(\lambda, x) = x/|x|\) when referring that the subindex \(a\) stands for the composition \(a \circ h : h^{-1}(S^4_+ \cap S^4) \to S^3\), as we will admit as a definition. These are the transition maps of their respective bundles. Note also that \(pr_1(qQ) = qpr_1(Q)\), if we define in \(S^7\)
\[q(x, y) = (qx, qy, qg^{-1})\]
Furthermore, \(ah(g(x, y)) = gah(x, y)g^{-1}\), in particular, the action (2.3), in the identification (2.4), is written as
\[\Phi : h^{-1}(S^4_+ \cap S^4) \times S^3 \to S^4_+ \times S^3\]
\[F_\epsilon | F_\epsilon^{-1} \downarrow \downarrow \quad \text{where } F_\epsilon : S^4_+ \cap S^4 \to S^4_+ \cap S^4\]
is defined by \(F_\epsilon(x, y, g) = ha(x, y)(x, y)\). Furthermore, \(F_\epsilon(q(x, y, g)q^{-1}) = r(F_\epsilon(x, y, g))q^{-1}\).

Proof. Define \(F_\epsilon\) by \((x, y, g) \mapsto (g(x, y), g^{-1})\). It is clearly smooth and an involution. One has
\[F_\epsilon^{-1}f_{ah}F_\epsilon(x, y, g) = F_\epsilon(g(x, y), g^{-1}ah(g(x, y))) = (ah(x, y)(x, y), gah(x, y)^{-1})\]
and
\[F_\epsilon(q(x, y, g)q^{-1}) = r(F_\epsilon(x, y, g))q^{-1}\]
\[\square\]
In particular, the action (2.3) defines a \(S^3\)-principal bundle \(pr_1' : \text{Sp}(2) \to \text{Sp}^{-1}(S^4_+ \cup h^{-1}(S^4_+))\). Now, for the diffeomorphisms \(\Psi : D^4 \times S^3 \to h^{-1}(S^4_+)\) and \(\Phi : S^3 \times D^4 \to h^{-1}(S^4_+)\), given by \((x, y) \mapsto (x, (1 - |x|^2)/2y)\) and \((x, y) \mapsto ((1 - |y|^2)/2x, y)\), it is straightforward to check that \(\Phi^{-1}v \nu h \Psi|_{(D^4 \setminus \{0\}) \times S^3} = g_a^{-1}f_a\). \(\square\)

3. A generalization of the Gromoll-Meyer construction

We begin this section by constructing bundles that admit actions like (2.3) and use the same proof to identify its quotient. For this, instead of starting with the total space of a bundle, we start from below.

Let \(G\) be a Lie group and \(M\) a smooth \(G\)-manifold, i.e., a smooth manifold together with a smooth \(G\) action, which we will denote by \((g, x) \mapsto gx\). Let \(\{U_i\}\) be a \(G\)-invariant open cover of \(M\). We call \(\{\phi_{ij} : U_i \cap U_j \to G\}\) a smooth \(*\)-family if \(\phi_{ij}\) is smooth and satisfies the conditions
\[\phi_{hk}(x) = \phi_{ij}(x)\phi_{jk}(x)\]
\[\phi_{ij}(gx) = g\phi_{ij}(x)g^{-1}\]
Consider \( \pi : P \to M \), the \( G \)-principal bundle defined by
\[
P = \bigcup_{f_{ij}} U_i \times G \xrightarrow{\phi_{ij}} U_i = M,
\]
where \( f_{ij}(x, g) = (x, q\phi_{ij}(x)) \) and \( \pi \) is the projection to the first coordinate. The cocycle condition \[ \text{(3.1)} \]
guarantees that it is a well-defined principal bundle. We write, for \( (x, g) \in U_i \times G \) and \( (q, r) \in G \times G \)
\[
(q, r) \cdot (x, g) = q(x, g)r^{-1} = (qx, rgy^{-1})
\]
and note the equivariance \( \pi((q, r) \cdot (x, g)) = q \cdot \pi(x, g) = qx \). We also observe that the principal action of \( \pi \) is given by the \( r \) factor and that condition \[ \text{(3.2)} \]
guarantees that the \( q \) factor defines an action in the whole \( P \). We call the first as the \( \bullet \)-action and the second as the \( \ast \)-action. The bundle \( \pi : P \to M \) equipped with both actions will be called the \( \ast \)-bundle associated with \( \{\phi_{ij}\} \).

We have

**Theorem 3.1.** Let \( \widehat{\phi}_{ij} : U_i \cap U_j \to U_i \cap U_j \) be the map defined by \( \widehat{\phi}_{ij}(x) = \phi_{ij}(x)x \).

Then, \( \{\widehat{\phi}_{ij}\} \) is a family of \( G \)-equivariant diffeomorphism such that
\[
\widehat{\phi}_{jk}\widehat{\phi}_{ij}(x) = \widehat{\phi}_{ik}(x)
\]
whenever \( x \in U_i \cap U_j \cap U_k \). In particular, \( M' = \bigcup_{U_i} U_i \) is a smooth \( G \)-manifold with the actions defined by the restrictions of the \( G \) action on \( M \) to \( U_i \). Furthermore, the quotient of the \( \ast \)-action in \( P \) is \( G \)-equivariantly diffeomorphic to \( M' \) with the action induced by the principal one.

**Proof.** The first part follows from the following lemma, whose proof is straightforward.

**Lemma 3.2.** Let \( M \) be a \( G \)-manifold and \( \alpha, \beta : M \to G \) maps satisfying \[ \text{(3.2)} \].

Then, for \( \alpha\beta : M \to G \) the map defined by the pointwise product of \( \alpha \) and \( \beta \), \( \alpha\beta = \beta\alpha \).

In particular \[ \text{(3.3)} \] follows from \[ \text{(3.1)} \] and \( \widehat{\phi}_{ij}^{-1} = \widehat{\phi}_{ij}^{-1} \) implies that these maps are \( G \)-equivariant diffeomorphisms. The well-definition of the action in \( M' \) follows from the equivariance of \( \widehat{\phi}_{ij} \). The proof of the second statement follows by verbatin the proof of Lemma \[ \text{(2.2)} \] by replacing \( f_c : h^{-1}(S^2) \times S^3 \to h^{-1}(S^2) \times S^3 \) by \( F_c : U_i \times G \to U_i \times G \).

Note that \( Sp(2) \to S^7 \) can be viewed as \( \ast \)-bundles. We remark that we also proved that

**Proposition 3.3.** The \( \ast \)-action induces a \( G \)-principal bundle \( \pi' : P \to M' \) with transition maps \( \{\phi_{ij}^{-1}\} \) such that \( \pi'(pr^{-1}) = r\pi'(p) \) for all \( (p, r) \in P \times G \). Furthermore, the inclusions \( U_i \times \{1\} \subset P \) are sections for both bundles \( \pi \) and \( \pi' \).

Now, exploiting once more the example of Gromoll-Meyer, we notice that \( -h : S^7 \to S^4 \) is also a \( \ast \)-bundle with \( \ast \)-action \( (q, 1) \cdot (x, y) = (qx, qy) \). We want to show how the \( \ast \)-action in \( Sp(2) \) is induced by this. In general, we consider \( \pi : P \to M \), a \( \ast \)-bundle associated with the \( \ast \)-family \( \{\phi_{ij} : U_i \cap U_j \to G\} \), another \( G \)-manifold \( N \) and a \( G \)-equivariant map \( f : N \to M \). We can always define the induced bundle \( \pi_f : f^*P \to \hat{N} \) as
\[
f^*P = \{(x, y) \in N \times P \mid f(x) = \pi(y)\}\

with the projections in the first coordinate, $\pi_f$, and in the second coordinate $f^*: f^*P \to P$. The diagram of an induced bundle is the analogous of (2.2) (compare [4]). We have

**Theorem 3.4.** The family $\{\phi_{ij} \circ f : f^{-1}(U_i) \cap f^{-1}(U_j) \to G\}$ is a $\ast$-family in $N$. Furthermore,

(i) $f^*P$ is $G \times G$ equivariantly diffeomorphic to the $\ast$-bundle associated to this family with $\ast$-action given by $q(x,y) = (qx,qy);

(ii) there exists a $G$-equivariant map $f' : N' \to M'$ such that

\[
\begin{array}{ccc}
N & \xrightarrow{f} & N' \\
\downarrow f & & \downarrow f' \\
M & \xleftarrow{f^{-1}(U_i)} & M'
\end{array}
\] (3.6)

Proof. Writing $P_f = \cup_{f_{ij},j} f^{-1}(U_i)$ and $P$ as (3.3), we define an equivariant diffeomorphism $P_f \to f^*P$ by $l_i : f^{-1}(U_i) \times G \to N \times P$ defined by $(x,g) \mapsto (x,(f(x),g))$ since $(id \times f_{ij})(l_i) = l_i(id \times f_{ij})$. Item (ii) is proved by noticing that $f^* : f^*P \to P$ is $G \times G$-equivariant, in particular, from Proposition 3.3 it descends to a $G$-equivariant map $f' : N' \to M'$ with (3.6) given by the sections in the same proposition, which are common for both bundles.

We note that the same proof of Proposition 1.7 in [11] works in the equivariant case. Given two $\ast$-families $\{\phi_{ij} : U_i \cap U_j \to G\}$ and $\{\phi'_{ij} : U_i \cap U_j \to G\}$, we call them *equivariantly homotopic* if, for each $(i,j)$, $\phi_{ij}$ is homotopic to $\phi'_{ij}$ through a homotopy of equivariant maps.

**Proposition 3.5.** If $\{\phi_{ij}\}$ and $\{\phi'_{ij}\}$ are equivariantly homotopic $\ast$-families then their associated $\ast$-bundles are $G \times G$-equivariantly diffeomorphic.

This proposition allows us to remove the smoothness condition on $\phi_{ij}$ through equivariant approximation theorems (from [1], for example). Also it gives sense to a $G$-equivariant diffeomorphism class of $M'$ even if we consider a cover $\{U_0, U_1\}$ of closed subsets, instead of open ones, as far as $U_0 \cap U_1$ is a smooth subvariety of $M$, since the equivariant diffeomorphism class of $M'$ will not be affected by the choice of an equivariant extension of $\phi_{01}$ to a small neighborhood around $U_0 \cap U_1$. We assume these facts it in the rest of the paper.

4. **Explicit realization of spheres**

Here, we want to use results about Milnor’s plumbing pairing (see [12] for definition) and the construction that we just introduced to prove Theorem 1.1. For this, consider $S^6 \subset \text{Im} \mathbb{H} \times \mathbb{H}$, $S^8 \subset \mathbb{R} \mathbb{H}^2$ and $S^{10} \subset \text{Im} \mathbb{H} \times \mathbb{H}^2$, unitary spheres, and $S^4$ and $S^7$ as above. Define the maps $\tau : S^3 \to SO(3)$, $\eta_4 : S^4 \to S^3$, $\eta_8 : S^8 \to S^7$, and $\eta_{10} : S^{10} \to S^9$.
b : S^6 → S^3 and \( \tilde{b}_{10} : S^{10} → S^7 \) as

\[
(4.1) \quad \tau(q) = (x → qx\bar{q});
\]

\[
(4.2) \quad \eta_4(\lambda y) = (\lambda + x\bar{x}|x|^{-1});
\]

\[
(4.3) \quad \eta_8(\lambda, x, y) = (\lambda + x\bar{x}|x|^{-1}, y);
\]

\[
(4.4) \quad b(\xi, y) = \exp(\pi x\bar{x}|x|^{-2});
\]

\[
(4.5) \quad \tilde{b}_{10}(\xi, x, y) = \left( b \left( \frac{\xi, x}{\sqrt{\xi^2 + |x|^2}} \right) \sqrt{|\xi|^2 + |x|^2}, y \right).
\]

We also write \( u : S^3 → SO(4) \) for the map defined by \( u(q)v = v\bar{q} \) and \( s_k : SO(m) → SO(m + k) \) for the inclusion given by adding a \( k \times k \) identity on the upper left corner. From [18] or [9], we have

**Theorem 4.1.** \( \sigma(\tau\eta_4, s_1u) \) is the non-trivial element in \( \theta^8 \cong \mathbb{Z}_2 \) and \( \sigma(\tau b, s_4u) \) is the generator of the index two subgroup of \( \theta^{10} \cong \mathbb{Z}_4 \) consisting of spheres which bound spin manifolds.

**Proof.** We recall from [2] that \( b : S^6 → S^3 \) is homotopic to the Samelson product \( \langle u, u \rangle : S^6 → S^3 \) and to \( Jr \) so, according to Theorems 3.1 and 5.1 in [18], \( \sigma(\tau b, s_4u) = 2\sigma(s_jub, s_4u) \) is an exotic sphere which bounds a spin manifold. We also note that \( \eta_4 \) is the image of the \( J \)-homomorphism of the non-trivial element of \( \pi_1SO(3) \), in particular, from Theorem 5.2 of [18], the Kervaire Milnor map applied to \( \sigma(\tau\eta_4, s_2u) \) lies in the image of \( \langle \nu', \eta, \nu \rangle \) in \( \pi^8_3/\text{Im}J \), which, according to [20], is non-zero.

By the other hand,

**Proposition 4.2.** The maps \( \eta_8 \) and \( \tilde{b}_{10} \) are equivariant by the \( S^3 \)-actions defined, respectively, on their domains by

\[
q \cdot (\lambda, x, y) = (\lambda, qx, qy\bar{q})
\]

\[
q \cdot (\xi, x, y) = (\xi, qx, qy\bar{q})
\]

and \( (2.5) \).

**Proof of the Theorem [1,7].** We first remark that \( \eta \) and \( b_{10} \) are equivariantly homotopic to \( \eta_8 \) and \( \tilde{b}_{10} \). We do only the case of the 8-sphere, since the other is analogous. According to section 1 and Proposition 5.3, \( pr_1 : Sp(2) → S^7 \) is the \( * \)-bundle defined by \( U_0 = D^4 × S^3_e \) and \( U_1 = S^3_e × D^4_e \) with actions defined by \( (2.3) \) and transition map defined by \( \phi_{01} = \nu h|_{S^2_e × S^2_e} \), where \( S^2_e \) and \( D^4_e \) are to be understood as spheres and discs with radius \( \epsilon \) and \( \epsilon' = (1 − \epsilon^2)^{1/2} \). Following this and analogous identifications, \( \eta_8^{-1}(D^4_e × S^3) = D^5 × S^3_e \) and \( \eta_8^{-1}(S^3_e × D^4_e) = S^4_e × D^4_e \), where \( D^5 × S^3_e \) is identified with \( \{(\lambda, x, y) \in S^8 \mid \lambda^2 + |y|^2 \leq \epsilon^2 \} \) and so on. We have, for \( (\lambda, x, y) \in S^4_e × S^3_e \)

\[
\phi_{01}\eta_8(\lambda, x, y) = \nu h(\eta_4(\epsilon^{-1}\lambda, \epsilon^{-1}x)e, y) = \eta_4(\epsilon^{-1}\lambda, \epsilon^{-1}x)\bar{y},
\]

where the last equality holds since \( \nu h(x, y) = \epsilon^{-2}x\bar{y} \). Now, we are able to consider a more general situation. Consider \( S^k \) and \( S^l \), unitary spheres, equipped with isometric \( G \)-actions and maps \( \alpha : S^k → G, \beta : S^l → G \) satisfying \( (3.2) \). Write \( \Delta_1 : S^k → SO(k + 1) \) and \( \Delta_2 : S^l → SO(l + 1) \) for the homomorphisms defined by the actions. We have,
Lemma 4.3. For \( r : S^k \times S^l \to G \), defined by \( r(x, y) = \alpha(x)\beta(y)^{-1} \), we have
\[
D^{k+1} \times S^l \cup_r S^k \times D^{l+1} = \sigma(\Delta_2\alpha, \Delta_1\beta).
\]

Proof. Write \( \check{r}(x, y) = \alpha(x)\beta(y)^{-1}(x, y) \). Then, for \( g_\beta(x, y) = (\beta(y)x, y) \) and \( f_\alpha(x, y) = (x, \alpha(y)) \), \( f_\alpha(\hat{\alpha} \times id) = (\hat{\alpha} \times id) f_\alpha = \hat{\alpha} \), where \( \hat{\alpha}(x, y) = \alpha(x) \). Being the analogous for \( \beta \) also true. In particular,
\[
\check{r} = \hat{\beta}^{-1}\hat{\alpha} = g_\beta^{-1}(id \times \hat{\beta}^{-1}) f_\alpha(\hat{\alpha} \times id) = g_\beta^{-1}(\hat{\alpha} \times \hat{\beta}^{-1}) f_\alpha
\]
However,
\[
g_\beta^{-1}(\hat{\alpha} \times \hat{\beta}^{-1}) f_\alpha = g_\beta^{-1}(\hat{\alpha} \times id) g_\beta \circ g_\beta^{-1} f_\alpha \circ f_\alpha^{-1}(id \times \hat{\beta}^{-1}) f_\alpha
\]
and the proof would be complete applying again the techniques in \([13]\), once demonstrated that \( g_\beta^{-1}(\hat{\alpha} \times id) g_\beta = (\hat{\alpha} \times id) \) and \( f_\alpha^{-1}(id \times \hat{\beta}^{-1}) f_\alpha = (id \times \hat{\beta}^{-1}) \). Nevertheless,
\[
f_\alpha^{-1}(id \times \hat{\beta}^{-1}) f_\alpha(x, y) = (x, \alpha(x)^{-1}\hat{\beta}^{-1}(\alpha(x)y))
\]
\[
= (x, \alpha(x)^{-1}\alpha(x)\hat{\beta}^{-1}(y)) = (x, \hat{\beta}^{-1}(y)),
\]
and the analogous for the other term. \( \square \)

The proof for the 10-sphere follows by verbatim. \( \square \)

We write \( \Sigma^8 \) and \( \Sigma^{10} \) for these spheres. Before starting the proof of Theorem 1.2 we remark that \([7]\) proves

**Theorem 4.4 ([7]).** The bundle \( pr_1 : Sp(2) \to S^7 \) is the \( * \)-bundle defined by \( b^{-1} : S^6 \to S^3 \) and the action \([2.4]\).

Proof of Theorem 1.2 : Consider \( \Omega \), the Cayley algebra defined as in \([4]\), and recall that \( \pi_7 SO(8) \approx \mathbb{Z} \oplus \mathbb{Z} \) with representants \( f_{ij}(X)Y = X^i Y^j \), where \( X \in S^7 \subset \Omega \) and \( Y \in \Omega \). We write \( \pi_{ij} : S_{ij} \to S^8 \) for the sphere bundle with characteristic map \( f_{ij} \) and observe that \( S_{ij} = D^8 \times S^7 \cup D^8 \times S^7 \) admits an action by the subgroup of isomorphisms of \( \Omega \)
\[
G_2 = \{ g \in SO(8) \mid g(XY) = g(X)g(Y) \}
\]
defined by \( g \cdot (X, Y) = (gX, gY) \). Now, the linear isomorphism \( \Xi : \mathbb{H}^2 \to \Omega \) defined by \( \Xi(x, y) = (y, \bar{x}) \) induces a new multiplicative structure on \( \mathbb{H}^2 \), isomorphic to the Cayley numbers, with the algebra-isomorphism subgroup of \( SO(8) \) isomorphic to \( G_2 \) by the isomorphism \( g \mapsto \Xi^{-1}g\Xi \). And we may consider, for simplicity, the Cayley numbers given by this product. The advantage of it is that \([1.6]\) defines the action of a subgroup \( S^3 \subset G_2 \), so \( \pi_{ij} : S_{ij} \to S^8 \) is \( S^3 \) equivariant. Consider the \( * \)-bundle \( E^{11} = \eta^* Sp(2) \to S^8 \) as in Theorem 1.1 and \( E^{18} = \pi_{ij} E^{11} \to S_{ij} \) and observe that Proposition 4.3 allows us to restrict the 'operation' defined by Theorem 3.1 to a small neighborhood of one fiber of \( \pi_{ij} \), in such way that
\[
(4.9) \quad (S_{ij})' = D_2^8 \times S^7 \hookrightarrow S_2^7 \times S^7 \underbrace{\sigma_{ij} \triangleright (D^8 - D_2^8)}_{\theta} \times S^7 \hookrightarrow S^7 \times S^7 \underbrace{\hat{f}_i}_{\theta} \times D^8 \times S^7,
\]
where \( \theta = \nu \eta_{S^3} : S^7 \to S^3 \) with \( S^7 = \{(0, x, y) \in S^8 \} \). So, if \( S_{ij} \) is a homotopy sphere, \( D_2^8 \times S^7 \) lives in a disc so \( (S_{ij})' = S_{ij}\#(S^{15})' \) with \( (S^{15})' = (S_{1,0})' \). For
$F \colon (X, Y) = (XY, X); f_\beta(X, Y) = (X, XY)$ and $g_\beta(X, Y) = (XY, Y)$, we have a diffeomorphism $F$ defined by

$$
\begin{array}{ccc}
D^8 \times S^7 & \leftarrow & S^7 \times S^7 \\
\downarrow & & \downarrow \\
D^8 \times S^7 & \leftarrow & S^7 \times S^7 \\
\end{array}
\xrightarrow{g_\beta} \xrightarrow{f_\beta} S^7 \times D^8 \\
\xrightarrow{f_\beta} S^7 \times D^8
$$

Now $\pi_{1,0}F|_{D^8 \times S^7}(X, Y) = \pi_{1,0}(X, Y)$, so, it is easy to note that $(S^8)^\prime = D^8 \times S^7 \cup_{g_\beta f_\beta(\delta \times \text{id})} S^7 \times D^8$, where $f_\beta(X, Y) = (X, \theta(X)Y)$ (now thinking of $X$ in the equator of $S^8$ and taking $\theta : S^7 \to S^3$ as above). Note that $f_\beta$ is isotopic to the identity. In fact, for $\Delta : S^3 \to SO(8)$, the homomorphism defined by $(1.6)$, $f_\beta(X, Y) = (X, \Delta(X)Y)$. Since $\pi_8 S^3$ is torsion (20) and $\pi_7 SO(8)$ is torsion free, $\Delta \circ \theta$ is null-homotopic, such homotopy defines an isotopy from $f_\beta$ to the identity. So, $D^8 \times S^7 \cup_{g_\beta f_\beta(\delta \times \text{id})} S^7 \times D^8$ is diffeomorphic to $D^8 \times S^7 \cup_{g_\beta(\delta \times \text{id})} S^7 \times D^8$ which is an element in the image of the Milnor-Munkres-Novikov $\tau$-pairing $\tau_{7,7}$, which is zero, according to [12].

It remains to show that the projection $\pi_i' : (S_\delta)' \to \Sigma^8$ is a bundle projection with characteristic map $f_{ij}$. But, one can see from Theorem 3.4 that $(S_\delta)' = D^8 \times S^7 \cup_{f_{ij}, f_\beta(\delta \times \text{id})} D^8 \times S^7$, for $f_{ij}(X, Y) = (X, f_{ij}(XY))$. Since, $f_\alpha$ is isotopic to the identity, $(S_\delta)' \to \Sigma^8$ is isomorphic to $D^8 \times S^7 \cup_{f_{ij}, f_\beta(\delta \times \text{id})} D^8 \times S^7$ with projection in the first factor, as desired.

5. Actions and equivariant homotopy groups

In this section we write $[N, M]^G$ for the set of equivariant homotopy classes of equivariant maps between $G$-manifolds $N$ and $M$. We also write $\text{Diff}^G(M)$ as the set of equivariant diffeomorphisms of $G$ and $M^G$ as the set of equivariant classes of smooth $G$-manifolds. Consider $G$ as a $G$-manifold with the action given by conjugation and $[M, G]^G$ as a group with multiplication done pointwise. We have

**Proposition 5.1.** There is a homomorphism $\text{DR} : [M, G]^G \to \pi_G \text{Diff}^G(M)$ defined by $[\alpha] \mapsto [\alpha^{-1} \circ \alpha \circ \alpha^{-1}]$, where $\alpha'$ is an equivariant smooth approximation of $\alpha$. Furthermore, if $\pi : P \to M$ is a $\ast$-bundle and $N$ is another $G$-manifold. There is a map $\text{DR}_\pi : [N, M]^G \to M$ which sends $f \in [N, M]^G$ to the quotient of $f^P$ by the induced $\ast$-action.

**Proof.** We remark, from [1], that we can always consider smooth the maps and homotopies. Suppose that $H : M \times I \to G$ defines a smooth homotopy such that $x \mapsto H(x, t)$ satisfies (3.2) for all $t$. Then, $H : M \times I \to M \times I$ defines an equivariant isotopy. We also observe that $\alpha_1 \alpha_2 = \alpha_2 \alpha_1$. The map $\text{DR}_\pi$ is well-defined by Proposition 3.5.

We also observe that

**Lemma 5.2.** If $\text{DR}_\pi(f) \neq N$ then $f$ is not equivariantly homotopic to a constant map.

**Proof.** If $f$ is equivariantly homotopic to a constant map, then, by Proposition 3.5 and Theorem 3.4 $\text{DR}_P(f)$ is the quotient of $N \times G$ by the action defined by
\[ r(x, g) = (rx, rg). \] Taking \( U_0 = N \) in Proposition \( \ref{prop:1} \), we see that \( N \times \{1\} \) is a section for this action. \( \square \)

We write \( S^7 \) for the \( S^3 \)-manifold defined by \( \ref{eq:2.3} \), \( S^7_1 \) for the \( S^1 \times S^3 \)-manifold defined by \( z, q \cdot (x, y) = (qx, qy) \) or any restriction of this action, \( S^{10} \) for the \( S^3 \)-manifold defined by \( \ref{eq:4.7} \), \( S_{10}^1 \) by the \( S^3 \times S^3 \)-manifold defined by \( \langle p, q \rangle \cdot (x, y, z) = (px, qx, qy) \) and \( S^6, S^9 \) and \( S^3_1 \) for any equivariant equator of \( S^7, S^{10} \) and \( S_{10}^1 \), respectively. We have

\begin{customthm}{5.3}
\begin{itemize}
  \item The sets \([S^7, S^4]S^3, [S^8, S^3 \times S^1]S^3 \times S^3\) and \([S^4, S^3]S^3 \times S^3\) contain copies of \( \mathbb{Z} \times \mathbb{Z} \);
  \item \([S^6, S^3]S^3, [S^9, S^3 \times S^3]S^3 \times S^3\) and \([S^7, S^3 \times S^1]S^3 \times S^3\) contains a copies of \( \mathbb{Z} \);
  \item the sets \([S^1, S^3]S^3\) and \([S^8, S^3]S^3\) contain copies of \( \mathbb{Z}_2 \).
\end{itemize}
\end{customthm}

\textbf{Proof.} We recall from \( \ref{prop:6} \) that it is possible to do a connected sum of \( \Sigma^7 \), the sphere in Theorem \( \ref{thm:2.1} \) and \( \#_k \Sigma^7 \) is equivariantly diffeomorphic to \( \#_l \Sigma^7 \) if and only if \( k = \pm l \). Since the \( G \)-manifold \( \#_k \Sigma^7 \) can be realized by the clutching function \( b^{-k} \), we conclude that \( DR([S^6, S^3]S^n) \supset \mathbb{Z} \). We also note that \( DR([S^8, S^3]S^n) \) and \( DR([S^6, S^3]S^n) \) has at least 2 and 3 elements, respectively. Furthermore, for the first one, we can observe that \( \eta_k : S^4 \to S^3 \) is equivariantly homotopic to its group inverse (by the homotopy given in \( \ref{prop:5} \), for example), so, we conclude that the first has exactly two elements. For \([S^7, S^4]S^3\) and \([S^8, S^3]S^3\) we use the same arguments for the maps \( DR_{-h} \) and \( DR_{-h_{\alpha \beta}} \) by observing that \( \#_k \Sigma^7 \) can be realized as the pull-back of \( h : S^7 \to S^3 \) by \( E(b^{-k}) \), where \( E(b^{-1}) : S^7 \to S^4 \) is the equivariant suspension of \( b^{-k} \) analogous to the suspension of \( \eta_4 \) given by \( \eta_k \). The other copy of \( Z \) in \([S^7, S^4]S^3\) is given by the composition of the Hopf map and the maps \( \rho_k \) of \( \ref{thm:8} \). They are equivariant maps and \( \{h \rho_k\} \) is a subgroup in \( \pi_7 S^4 \) that doesn’t contain the homotopy classes of \( E(b^{-k}) \) (the map \( b \) in \( \ref{thm:20} \) is denoted as \( \nu \), we refer to it for details). For \([S^8, S^3 \times S^3]S^3 \times S^3\), \([S^7, S^3 \times S^3]S^3 \times S^3\) and \([S^9, S^3]S^3 \times S^3\), we note that \([N, M_1 \times M_2]G = [N, M_1]G \times [N, M_2]G \). For example, here we have \( S^8 \) as an \( S^3 \times S^1 \) manifold being the first action the conjugation and the second action trivial, for example, since \( \eta_k \) is invariant under the \( S^3 \) action, we have that the elements we found can be actually lifted to these groups. \( \square \)

For the last application of these constructions, we note that \( \beta^{-1} \) in \( \ref{thm:8} \) can be replaced by any \( G \)-equivariant diffeomorphism \( \phi : S^4 \to S^3 \). In particular, recalling (from \( \ref{thm:13} \)) that the sphere glued by \( id \times \phi \) is diffeomorphic to the standard one, we have

\begin{customprop}{5.4}
Let \( S^l \), the unity sphere of \( \mathbb{R}^{l+1} \), be equipped with an isometric \( G \)-action. Then, if \( \phi : S^l \to S^3 \) is a \( G \)-equivariant diffeomorphism and \( \alpha : S^k \to G \), the sphere defined by \( f^{-1}_\alpha(id \times \phi)f_\alpha \) is diffeomorphic to the standard one.
\end{customprop}

\begin{customrem}{5.5 (\ref{thm:3})}
For \( \Sigma^8 \) and \( \Sigma^{10} \) as in Theorem \( \ref{thm:14} \), \( \tau((\Sigma^8), \pi_3 SO(10)) = \theta^{13} \cong \mathbb{Z}_2 \) and \( \tau((\Sigma^8), \pi_1 SO(8)) \) is a subgroup of order two in \( \theta^{9} \cong \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \).
\end{customrem}
We conclude that

**Theorem 5.6.** Let $S^l$ be as in Proposition 5.4. Then, if $\phi : S^l \to S^l$ is $G$-equivariant and $\Delta : G \to SO(l + 1)$ is the homomorphism defined by the action, $\tau([\phi], \Delta, \pi_k G) = 0$ for every $k$.

So, suppose that $S^9$ is equipped with an isometric $S^3$ action and $S^1$ is equipped with an isometric $S^1$ action. Write $\Delta^9$ and $\Delta^7$ the homomorphisms defined by these actions. We have

**Corollary 5.7.**
- If $\phi : S^9 \to S^9$ is an $S^3$-equivariant diffeomorphism, then, the elements of the subgroup $\Delta_9^9 \pi_3 S^3$ must be divisible by 3;
- if $\phi : S^7 \to S^7$ is $S^1$ equivariant, then $\Delta_7^7 \pi_1 S^1$ is null-homotopic.

**Proof.** Let $\iota \in \pi_3 S^3$ and $\nu \in \pi_3 SO(9)$ be generators. Then, by one hand $\tau([\phi], \Delta_9^9(\iota)) = r\tau([\phi], \nu) = r[\Sigma^{13}]$, where $\Sigma^{13}$ is an element of order 3 in $\theta^{13}$. By other hand, from Proposition 5.4, we have $\tau([\phi], \Delta_9^9(\iota)) = 0$, so $r$ must be a multiple of 3. The analogous statement follows for $S^7$. □

We recall that the number $r \in \mathbb{Z}$ in the proof is called the index of the homomorphism. Observe that the index of both $S^3$ actions defined on $S^9$ are 3, and that the $S^1$ action in $S^7$ is the subaction of a semi-free $S^3$ action, so, the inclusion $S^1 \subset SO(8)$ is null-homotopic.

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