Summary. On an empirical level, the most successful alternative to dark matter in bound gravitational systems is the modified Newtonian dynamics, or MOND, proposed by Milgrom. Here I discuss the attempts to formulate MOND as a modification of General Relativity. I begin with a summary of the phenomenological successes of MOND and then discuss the various covariant theories that have been proposed as a basis for the idea. I show why these proposals have led inevitably to a multi-field theory. I describe in some detail TeVeS, the tensor-vector-scalar theory proposed by Bekenstein, and discuss its successes and shortcomings. This lecture is primarily pedagogical and directed to those with some, but not a deep, background in General Relativity.

1 Introduction

There is now compelling observational support for a standard cosmological model. It is most impressive that this evidence is derived from very different observational techniques applied to very different phenomena: from precise measurements of anisotropies in the Cosmic Microwave Background (CMB) [1]: from systematic photometric observations of the light curves of distant supernovae [2, 3, 4]: from redshift surveys mapping the distribution of observable matter on large scale and interpreting that distribution in the context of structure formation by gravitational collapse [5, 6]. Using the standard parameterised Friedmann-Robertson-Walker models (FRW), all of these observations imply a convergence to a narrow range of parameters that characterise the Universe; this convergence is rightly heralded as a remarkable achievement of the past decade.

However, the Universe that we are presented with is strange in its composition: only five percent is the ordinary baryonic matter that we are familiar with; twenty-five percent consists of pressureless dark matter presumed to be fundamental particles that are as yet undetected by other means; and about seventy percent is the even stranger negative pressure dark energy, possibly identified with a cosmological term in Einstein’s field equation, and emerging...
relatively recently in cosmic history as the dominate contributor to the energy density budget of the Universe.

A general sense of unease, primarily with this dark energy, has led a number of people to consider the possibility that gravity may not be described by standard four-dimensional General Relativity (GR) on large scale (see Sami, this volume)—that is to say, perhaps the left-hand-side rather than the right-hand-side of the Einstein equation should be reconsidered. Various possibilities have been proposed—possibilities ranging from the addition of a scalar field with a non-standard kinetic term, K-essence [7]; to gravitational actions consisting of general functions of the usual gravitational invariant, $F(R)$ theories (Sotiriou, this volume and [8, 9]); to braneworld scenarios with leakage of gravitons into a higher dimensional bulk ([10] and Maartens, this volume). But, in fact, there is a longer history of modifying gravity in connection with the dark matter problem—primarily that aspect of the problem broadly described as “missing mass” in bound gravitational systems such as galaxies or clusters of galaxies. The observations of this phenomenology have an even longer history, going back to the discovery of a substantial discrepancy between the dynamical mass and the luminous mass in clusters of galaxies [11].

The precise measurement of rotation curves of spiral galaxies in the 1970’s and 1980’s, primarily by 21 cm line observations which extend well beyond the visible disk of the galaxy [12, 13], demonstrated dramatically that this discrepancy is also present in galaxy systems.

A fundamental, often implicit, aspect of the cosmological paradigm is that this observed discrepancy in bound systems is due to the cosmological dark matter—primarily that aspect of the problem broadly described as “missing mass” in bound gravitational systems such as galaxies or clusters of galaxies. The observations of this phenomenology have an even longer history, going back to the discovery of a substantial discrepancy between the dynamical mass and the luminous mass in clusters of galaxies [11]. The precise measurement of rotation curves of spiral galaxies in the 1970’s and 1980’s, primarily by 21 cm line observations which extend well beyond the visible disk of the galaxy [12, 13], demonstrated dramatically that this discrepancy is also present in galaxy systems.

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The motivation behind considering modifications of gravity as an alternative to CDM is basically the same as that underlying modified gravity as an alternative to dark energy: when a theory, in this case GR, requires the existence of a medium which has not been, or cannot be, detected by means other than its global gravitational influence, i.e., an ether, then it is not unreasonable to question that theory. The primary driver for such proposals has been the direct observation of discrepancies in bound systems—galaxies
and clusters of galaxies—rather than cosmological considerations, such as that of structure formation in an expanding Universe. The most successful of the several suggestions, modified Newtonian dynamics or MOND, has an entirely phenomenological rather than theoretical basis [17, 18, 19]. In accounting for the detailed kinematics of galaxies and galaxy groups, while encompassing global scaling relations and empirical photometric rules, MOND has, with one simple formula and one new fixed parameter, subsumed a wide range of apparently disconnected phenomena.

In this respect it is similar to the early proposal of continental drift by Alfred Wegener in 1912. This suggestion explained a number of apparently disconnected geological and palaeontological facts but had no basis in deeper theory; no one, including Wegener, could conceive of a mechanism by which giant land masses could drift through the oceans of the earth. Hence the idea was met with considerable ridicule by the then contemporary community of geologists and relegated to derisive asides in introductory textbooks. It was decades later, after the development of the modern theory of plate tectonics and direct experimental support provided by the frozen-in magnetic field reversals near mid-oceanic rifts, that the theory underlying continental drift became the central paradigm of geology and recognised as the principal process that structures the surface of the earth [20]. I do not wish to draw a close analogy between MOND and the historical theory of continental drift, but only to emphasise the precedent: an idea can be basically correct but not generally accepted until there is an understandable underlying physical mechanism—until the idea makes contact with more familiar physical concepts.

The search for a physical mechanism underlying modified Newtonian dynamics is the subject here. I begin with a summary of the phenomenological successes of the idea, but, because this has been reviewed extensively before [21], I will be brief. I consider the proposals that have been made for modifications of GR as a basis of MOND. These proposals have led to the current best candidate—the tensor-vector-scalar (TeVeS) theory of Bekenstein [22], a theory that is complicated but free of obvious pathologies. I summarise the successes and shortcomings of the theory, and I present an alternative form of TeVeS which may provide a more natural basis to the theory. I end by a discussion of more speculative possibilities.

2 The phenomenology of MOND

2.1 The basics of MOND

If one wishes to modify Newtonian gravity in an ad hoc manner in order to reproduce an observed property of galaxies, such as asymptotically flat rotation curves, then it would seem most obvious to consider a $1/r$ attraction beyond a fixed length scale $r_0$. Milgrom [17] realized early on that this would
not work so that any modification explaining the systematics of the discrepancy in galaxies cannot be attached to a length scale but to a fixed acceleration scale, $a_0$. His suggestion, viewed as a modification of gravity, was that the true gravitational acceleration $g$ is related to the Newtonian gravitational acceleration $g_n$ as

$$g \mu(|g|/a_0) = g_n$$

where $a_0$ is a new physical parameter with units of acceleration and $\mu(x)$ is a function that is unspecified but must have the asymptotic form $\mu(x) = x$ when $x \ll 1$ and $\mu(x) = 1$ when $x \gg 1$.

The immediate consequence of this is that, in the limit of low accelerations, $g = \sqrt{g_n a_0}$. For a point mass $M$, if we set $g$ equal to the centripetal acceleration $v^2/r$, then the circular velocity is

$$v^4 = GM a_o$$

Fig. 1. The near-infrared Tully-Fisher relation of Ursa Major spirals [25]. The rotation velocity is the asymptotically constant value. The line is a least-square fit to the data and has a slope of 3.9 ± 0.2
in the low acceleration regime. So all rotation curves are asymptotically flat and there is a mass-velocity relation of the form $M \propto v^4$. These are aspects that are built into MOND so they cannot rightly be called predictions. However, in the context of MOND, the aspect of an asymptotically flat rotation curve is absolute. Unambiguous examples of rotation curves (of isolated galaxies) that decline in a Keplerian fashion at a large distance from the visible object would falsify the idea.

The implied mass-rotation velocity relation explains a well-known global scaling relation for spiral galaxies, the Tully-Fisher relation. This is a correlation between the observed luminosity of spiral galaxies and the characteristic rotation velocity, a relation of the form $L \propto v^\alpha$ where $\alpha \approx 4$ if luminosity is measured in the near-infrared. If the mass-to-light ratio of galaxies does not vary systematically with luminosity, then MOND explains this scaling relation. In addition, because it reflects underlying physical law, the relation is absolute. The TF relation should be the same for different classes of galaxies and the logarithmic slope (at least of the MASS-velocity relation) must be 4. Moreover, the relation is essentially one between the total baryonic mass of a galaxy and the asymptotic flat rotational velocity– not the peak rotation velocity but the velocity at large distance. This is the most immediate prediction [23, 24].

The near-infrared TF relation for a sample of galaxies in the Ursa Major cluster (and hence all at nearly the same distance) is shown as a log-log plot in Fig. 1 where the velocity is that of the flat part of the rotation curve [25]. The scatter about the least-square fit line of slope $3.9 \pm 0.2$ is consistent with observational uncertainties (i.e., no intrinsic scatter).

Given the mean $M/L$ in a particular band ($\approx 1$ in the K’ band), this observed TF relation (and eq. 2) tells us that $a_o$ must be on the order of $10^{-8}$ cm/s$^2$. It was immediately noticed by Milgrom that $a_o \approx cH_o$ to within a factor of 5 or 6. This cosmic coincidence suggests that MOND, if it is right, may reflect the effect of cosmology on local particle dynamics.

### 2.2 A critical surface density

It is evident that the surface density of a system $M/R^2$ is proportional to the internal gravitational acceleration. This means that the critical acceleration may be rewritten as a critical surface density:

$$\Sigma_m \approx a_o/G.$$  \hspace{1cm} (3)

If a system, such as a spiral galaxy has a surface density of matter greater than $\Sigma_m$, then the internal accelerations are greater than $a_o$, so the system is in the Newtonian regime. In systems with $\Sigma \geq \Sigma_m$ (high surface brightness or HSB galaxies) there should be a small discrepancy between the visible and classical Newtonian dynamical mass within the optical disk. But in low surface brightness (LSB) galaxies ($\Sigma << \Sigma_m$) there is a low internal acceleration,
so the discrepancy between the visible and dynamical mass would be large. By this argument Milgrom predicted, before the actual discovery of a large population of LSB galaxies, that there should be a serious discrepancy between the observable and dynamical mass within the luminous disk of such systems—should they exist. They do exist, and this prediction has been verified [23].

Moreover, spiral galaxies with a mean surface density near this limit—HSB galaxies—would be, within the optical disk, in the Newtonian regime. So one would expect that the rotation curve would decline in a near Keplerian fashion to the asymptotic constant value. In LSB galaxies, with mean surface density below $\Sigma_m$, the prediction is that rotation curves would rise to the final asymptotic flat value. So there should be a general difference in rotation
curve shapes between LSB and HSB galaxies. In Fig. 2 I show the observed rotation curves (points) of two galaxies, a LSB and HSB, where we see exactly this trend. This general effect in observed rotation curves was pointed out in ref. [27].

It is well-known that rotationally supported Newtonian systems tend to be unstable to global non-axisymmetric modes which lead to bar formation and rapid heating of the system [28]. In the context of MOND, these systems would be those with \( \Sigma > \Sigma_m \), so this would suggest that \( \Sigma_m \) should appear as an upper limit on the surface density of rotationally supported systems. This critical surface density is 0.2 g/cm\(^2\) or 860 M\(_\odot\)/pc\(^2\). A more appropriate value of the mean surface density within an effective radius would be \( \Sigma_m/2\pi \) or 140 M\(_\odot\)/pc\(^2\), and, taking \( M/L_b \approx 2 \), this would correspond to a surface brightness of about 22 mag/arc sec\(^2\). There is such an observed upper limit on the mean surface brightness of spiral galaxies and this is known as Freeman’s law [29]. The existence of such a limit becomes understandable in the context of MOND.

### 2.3 Pressure-supported systems

Of course, spiral galaxies are rotationally supported. But there other galaxies, elliptical galaxies, which are pressure supported—i.e., they are held up against gravity by the random motion of the stars. There are numerous other examples of pressure-supported systems such as globular clusters and clusters of galaxies, and often the observable components of these systems have a velocity dispersion (or temperature) that does not vary much with position; i.e., they are near “isothermal”. With Newtonian dynamics, pressure-supported systems that are nearly isothermal have infinite extent. But in the context of MOND it is straightforward to demonstrate that such isothermal systems are finite with the density at large radii falling roughly like \( 1/r^4 \) [30].

The equation of hydrostatic equilibrium for an isotropic, isothermal system reads
\[
\sigma_r^2 \frac{d\rho}{dr} = -\rho g \tag{4}
\]
where, in the limit of low accelerations \( g = \sqrt{GMa_o/r} \). Here \( \sigma_r \) is the radial velocity dispersion and \( \rho \) is the mass density. It then follows immediately that, in this MOND limit,
\[
\sigma_r^4 = GMa_o \left( \frac{d \ln(\rho)}{d \ln(r)} \right)^{-2}. \tag{5}
\]
Thus, there exists a mass-velocity dispersion relation of the form
\[
(M/10^{11} M_\odot) \approx (\sigma_r/100 \text{ km/s}^{-1})^4
\]
which is similar to the observed Faber-Jackson relation (luminosity-velocity dispersion relation) for elliptical galaxies [31]. This means that a MOND near-isothermal sphere with a velocity dispersion on the order of 100 km/s will
always have a galactic mass. This is not true of Newtonian pressure-supported objects. Because of the appearance of an additional dimensional constant, \( a_o \), in the structure equation (eq. 4), MOND systems are much more constrained than their Newtonian counterparts.

Any isolated system which is nearly isothermal will be a MOND object. That is because a Newtonian isothermal system (with large internal accelerations) is an object of infinite size and will always extend to the region of low accelerations \(< a_o\). At that point \((r_e^2 \approx GM/a_o)\), MOND intervenes and the system will be truncated. This means that the internal acceleration of any isolated isothermal system \((\sigma r^2/r_e)\) is expected to be on the order of or less than \(a_o\) and that the mean surface density within \(r_e\) will typically be \(\Sigma_m\) or less (there are low-density solutions for MOND isothermal spheres, \(\rho < a_o^2/G \sigma^2\), with internal accelerations less than \(a_o\)). It was pointed out long ago that elliptical galaxies do appear to have a characteristic surface brightness [32]. But the above arguments imply that the same should be true of any pressure supported, near-isothermal system, from globular clusters to clusters of galaxies. Moreover, the same \(M - \sigma\) relation (eq. 5) should apply to all such systems, albeit with considerable scatter due to deviations from a strictly isotropic, isothermal velocity field [33].

Most luminous elliptical galaxies are high surface brightness objects which would imply a surface density greater than the MOND limit. This suggests that luminous elliptical galaxies should be essentially Newtonian objects, and, viewed in the traditional way, should evidence little need for dark matter within the effective (or half-light) radius. This does seem to be the case as demonstrated by dynamical studies using planetary nebulae as kinematic tracers [34, 35].

### 2.4 Rotation curves of spiral galaxies

Perhaps the most impressive observational success of MOND is the prediction of the form of galaxy rotation curves from the observed distribution of baryonic matter, stars and gas. Basically, one takes the mean radial distribution of light in a spiral galaxy as a precise tracer of the luminous mass, includes the observed radial dependence of neutral hydrogen (increased by 30% to account for the primordial helium) and assumes all of this is in a thin disk (with the occasional exception of a central bulge component). One then solves the standard Poisson equation to determine the Newtonian force, applies the MOND formula (eq. 1 with a fixed value of \(a_0\)) to determine the true gravitational force and calculates the predicted rotation curve. The mass-to-light ratio of the visible component is adjusted to achieve the best fit to the observed rotation curve.

The results are spectacular considering that this is a one-parameter fit. The solid curves in Fig. 2 are the results of such a procedure applied to a LSB and HSB galaxy; this has been done for about 100 galaxies. The fitted M/L
values are not only reasonable, but demonstrate the same trend with colour that is implied by population synthesis models as we see in Fig. 3 [25, 36].

Fig. 3. MOND fitted mass-to-light ratios for the UMa spirals [25] in the B-band (top) and the K’-band (bottom) plotted against B-V (blue minus visual) colour index. The solid lines show predictions from populations synthesis models [36].

Here I wish to emphasise another observed aspect of galaxy rotation curves– a point that has been made, in particular, by Sancisi [37]. For many objects, the detailed rotation curve appears to be extremely sensitive to the distribution observable matter, even in LSB galaxies where, in the standard interpretation, dark matter overwhelmingly dominates within the optical image. There are numerous examples of this– for example, the LSB galaxy shown in Fig. 2 where we see that the total rotation curve reflects the Newtonian rotation curve of the gaseous component in detail. Another example [37, 38] is the dwarf galaxy, NGC 3657. Fig. 4 shows the surface densities of the baryonic components, stars and gas, compared to the observed rotation curve. Again the dotted and dashed curves are the Newtonian rotation curves of the stellar
and gaseous components and the solid curve is the resulting MOND rotation curve. The agreement with observations is obvious.

**Fig. 4.** The upper panel is the logarithm of the surface density of the gaseous and stellar components of NGC 3657. The lower panel shows the observed rotation curve (points), the Newtonian rotation curves for the stellar (dashed) and gaseous (dotted) components as well as the MOND rotation curve (solid) [37, 38].

For this galaxy, there is evidence from the rotation curve of a central cusp in the density distribution— and, indeed, the cusp is seen in the light distribution. In cases where there is no conspicuous cusp in the light distribution, there is no kinematic evidence for a cusp in the rotation curve. This would appear to make the entire discussion about cusps in halos somewhat irrelevant. But equally striking in this case is the gradual rise in the rotation curve at large radii. This rise is clearly related to the increasing dominance of the gaseous component in the outer regions. The point is clear: the rotation curve reflects the global distribution of baryonic matter, even in the presence of a large discrepancy between the visible and Newtonian dynamical mass. This is entirely understandable (and predicted) in the context of modified gravity in
the form of MOND (what you see is all there is), but remains mysterious in
the context of dark matter

2.5 Clusters of galaxies: a phenomenological problem for MOND?

It has been known for 70 years [11] that clusters of galaxies exhibit a significant
discrepancy between the Newtonian dynamical mass and the observable mass,
although the subsequent discovery of hot X-ray emitting gas goes some way
in alleviating the original discrepancy. For an isothermal sphere of hot gas at
temperature T, the Newtonian dynamical mass within radius \( r_o \), calculated
from the equation of hydrostatic equilibrium, is

\[
M_n = \frac{r_o}{G} \frac{kT}{m} \left( \frac{d \ln(\rho)}{d \ln(r)} \right),
\]

where \( m \) is the mean atomic mass and the logarithmic density gradient is
evaluated at \( r_o \). This dynamical mass turns out to be typically about a factor
of 4 or 5 larger than the observed mass in hot gas and in the stellar content
of the galaxies (see Fig. 5, left [39]).

With MOND, the dynamical mass (eq. 5) is given by

\[
M_m = (Ga_o)^{-1} \left( \frac{kT}{m} \right)^2 \left( \frac{d \ln(\rho)}{d \ln(r)} \right)^2,
\]

(7)
and, using the same value of $a_0$ determined from nearby galaxy rotation curves, turns out to be, on average, a factor of two larger than the observed mass (Fig. 5, right). The discrepancy is reduced but still present. This could be interpreted as a failure [40], or one could say that MOND predicts that the mass budget of clusters is not yet complete and that there is more mass to be detected [39]. The cluster missing mass could, for example, be in neutrinos of mass 1.5 to 2 eV [41], or in “soft bosons” with a large de Broglie wavelength [12], or simply in heretofore undetected baryonic matter. It would have certainly been a falsification of MOND had the predicted mass turned out to be typically less than the observed mass in hot gas and stars.

3 Relativistic MOND

MOND not only allows the form of rotation curves to be precisely predicted from the distribution of observable matter, but it also explains certain systematic aspects of the photometry and kinematics of galaxies and clusters: the presence of a preferred surface density in spiral galaxies and ellipticals—the so-called Freeman and Fish laws; the fact that pressure-supported nearly isothermal systems ranging from molecular clouds to clusters of galaxies are characterised by a specific internal acceleration, $a_o$, [21]; the existence of a TF relation with small scatter—specifically a correlation between the baryonic mass and the asymptotically flat rotation velocity of the form $v^4 \propto M$; the Faber-Jackson relation for ellipticals, and with more detailed modelling, the Fundamental Plane [33]; not only the magnitude of the discrepancy in clusters of galaxies but also the fact that mass-velocity dispersion relation which applies to elliptical galaxies (eq. 5) extends to clusters (the mass-temperature relation). And it accomplishes all of this with a single new parameter with units of acceleration—a parameter determined from galaxy rotation curves which is within an order of magnitude of the cosmologically significant value of $cH_0$. This is why several of us believe that, on an epistemological level, MOND is more successful than dark matter. Further, many of these systematic aspects of bound systems do not have any obvious connection to what has been traditionally called the “dark matter problem”. This capacity to connect seemingly unrelated points is the hallmark of a good theory. However, as I argued in the Introduction, MOND will never be entirely credible to most astronomers and physicists until it makes some contact with more familiar physics—until there is an underlying and understandable physical mechanism for MOND phenomenology. Below I consider that mechanism in terms of possible modifications of the theory of gravity.

3.1 Steps to TeVeS

TeVeS (tensor-vector-scalar) theory [22] is a relativistic theory yielding MOND phenomenology in the appropriate limit. Of course, I do not need to belabour
the advantages of a relativistic theory. It allows one to address a number of issues on which MOND is silent: gravitational lensing, cosmology, structure formation, anisotropies in the CMB. The theory is complicated—considerably more complicated than GR—in that involves additional dynamical elements and is characterised by three additional free parameters and a free function—i.e., a function that is not specified by any a priori considerations but may be adjusted to achieve the desired result. In this sense, TeVeS, like MOND itself, is a phenomenologically driven theory. It is entirely “bottom-up” and thereby differs from what is normally done in gravity theory or cosmology.

As the name implies it is a multi-field theory; i.e., there are fields present other than the usual tensor field $g_{\mu\nu}$ of GR. It appears that any viable theory of MOND as a modification of gravity must be a multi-field theory; no theory based upon a single metric field can work [43]. In TeVeS, the MOND phenomenology appears as a “fifth force” mediated by a scalar field. This fifth force must be designed to fall as $1/r$ and dominate over the usual Newtonian force when the total force is below $a_0$ as shown in Fig. 6.

![Fig. 6. MOND phenomenology as a result of multi-field modifications of gravity. The dashed curve shows the log force resulting from a scalar field with a non-standard Lagrangian as a function of log radius in units of the MOND radius $r_M = \sqrt{GM/a_0}$. The solid line is the usual Einstein-Newton force.](image)

Now if we are proposing a fifth force, then that implies non-geodesic motion and one may naturally ask about the validity of the equivalence principle,
even in its weak form expressing the universality of free fall (there are strong experimental constraints on the composition independence of acceleration in a gravitational field). The weak version of the equivalence principle can be preserved if there is a specific form of coupling between the scalar field a matter— one in which the scalar couples to matter jointly with the gravitational or Einstein metric. This allows for the definition of a physical metric, \( \tilde{g}_{\mu\nu} \) that is distinct from the Einstein metric. In the simplest sort of joint coupling the physical metric is *conformally* related to the Einstein metric, i.e.,

\[
\tilde{g}_{\mu\nu} = f(\phi)g_{\mu\nu}.
\]  

This is the case in traditional scalar-tensor theories such as the Brans-Dicke theory [44]. So the theory remains a metric theory, but particle and photons follow geodesics of the physical metric and not the Einstein metric. Of course, a great part of the beauty of GR is that the gravitational metric is the metric of a 4-D space time with Lorentzian signature—gravitational geometry is physical geometry. It is beautiful, but the world doesn’t have to be that way.

Another ingredient is necessary if the scalar field is to produce MOND phenomenology. In standard scalar-tensor theory, the scalar field Lagrangian is

\[
L_s = \frac{1}{2}\phi,\alpha\phi^{\alpha}.
\]  

Forming the action from this Lagrangian (and the joint coupling with \( g_{\mu\nu} \) to matter) and taking the condition of stationary action leads, in the weak field limit, to the usual Poisson equation for \( \phi \). In other words, the scalar force about a point mass falls as \( 1/r^2 \) as in Brans-Dicke theory. Therefore, MOND requires a non-standard scalar field Lagrangian; for example, something like

\[
L_s = \frac{1}{2l^2}F(l^2\phi,\alpha\phi^{\alpha})
\]  

where \( F(X) \) is an, as yet, unspecified function of the usual scalar invariant and \( l \) is a length scale on the order of the present Hubble scale (\( \approx c/H_0 \)). Bekenstein refers to this as a quadratic Lagrangian theory or AQUAL. The condition of stationary action then leads to a scalar field equation that, in the weak field limit, is

\[
\nabla \cdot [\mu(\phi)|\nabla \phi|/a_0] \nabla \phi = 4\pi G \rho
\]  

where \( a_0 = c^2/l \) and \( \mu = dF(X)/dX \). This we recognise as the Bekenstein-Milgrom field equation [45] which produces MOND like phenomenology if \( \mu(y) = y \) when \( y < 1 \) or \( F(X) = \frac{3}{4}X^3 \). Here, however, we should recall that \( \phi \) is not the total gravitational field but on the scalar component of a two-field theory. Another phenomenological requirement on the free function is that \( F(X) \rightarrow \omega X \) in the limit where \( X \gg 1 \) (or \( \nabla \phi > a_0 \)). That is to say, the scalar field Lagrangian becomes standard in the limit of large field gradients; the theory becomes equivalent to Brans-Dicke theory in this limit.
This guarantees precise $1/r^2$ attraction in the inner solar system, but, to be consistent with post-Newtonian constraints, it is necessary that $\omega > 10^4$.

Looking at the form of $F$ required for MOND phenomenology, we see an immediate problem with respect to cosmology. In the limit of a homogeneous Universe, where $\nabla \phi \to 0$ and the cosmic time derivative, $\dot{\phi}$, dominates the invariant, i.e., $X < 0$. This means that the form of the free function must change in this limit (this is a problem which persists in TeVeS). But there is another more pressing problem which was immediately noticed by Bekenstein and Milgrom. In the MOND limit, small disturbances in the scalar field, scalar waves, propagate acausally; i.e., $V_s = \sqrt{2}c$ in directions parallel to $\nabla \phi$. This is unacceptable; a physically viable theory should avoid the paradoxes resulting from acausal propagation.

The superluminal propagation (or tachyon) problem led Bekenstein to propose a second non-standard scalar-tensor theory for MOND–phase-coupling gravitation or PCG [46]. Here, the scalar field is taken to be complex, $\chi = q e^{i\phi}$ with the standard Lagrangian,

$$L_S = \frac{1}{2} [q_{,\alpha} q^{,\alpha} + q^2 \phi_{,\alpha} \phi^{,\alpha} + 2V(q)]$$

where $V(q)$ is the potential function of the scalar field. The non-standard aspect is that only the phase couples to matter in the usual conformal way,

$$g_{\mu\nu} = e^{-\eta \phi} g_{\mu\nu}. \quad (13)$$

This leads (weak field limit) to the field equation,

$$\nabla \cdot [q^2 \nabla \phi] = \frac{8\pi G \rho}{c^2}. \quad (14)$$

So now we see that $q^2$ replaces the usual MOND interpolating function $\mu$, but now $q$ is given by a second scalar field equation,

$$q^{,\alpha,\alpha} = q_{,\alpha} \phi^{,\alpha} + V'(q). \quad (15)$$

That is to say, the relation between $q^2$ and $\nabla \phi$ is now differential and not algebraic as in AQUAL theory. Bekenstein demonstrated that if $V(q) = -A q^6$ (a negative sextic potential) then the predicted phenomenology is basically that of MOND on a galactic scale.

Obviously the property $dV/dq < 0$ cannot apply for all $q$ because this would lead to instability of the vacuum, but there is a more serious problem: By a suitable redefinition of the fields, it may be shown that, in the limit of very weak coupling ($\eta << 1$) the term on the left-hand side of eq. 15 may be neglected– that is to say, we are left with only the right-hand side and the relation between $q^2$ and $\nabla \phi$ once again becomes algebraic as in AQUAL. In other words, PCG approaches AQUAL in the limit of very weak coupling. This suggests that PCG may suffer from a similar ailment as AQUAL; indeed, there
is a problem, but it appears as the absence of a stable background solution rather than superluminal propagation. But I only mention this because I want to emphasise that the weak coupling limit of PCG is equivalent to the quadratic theory; this turns out to be a significant aspect of TeVeS.

At about the same time it was realized that there is a serious phenomenological problem with AQUAL or PCG or any scalar-tensor theory in which the the relation between the physical and gravitational metrics is conformal as in eqs. 8 or 13. That is, such a theory would predict no enhanced deflection of photons due to the presence of the scalar field. Recall that photons and other relativistic particles follow null geodesics of the physical metric. These are given by the condition that

$$d\tilde{\tau}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = 0.$$  \hspace{1cm} (16)

Now given the conformal relation between the two metrics (eq. 8) you don’t have to be a mathematical genius to see that $d\tilde{\tau} = 0$ corresponds to $d\tau = 0$; i.e., null geodesics of the two metrics coincide which means that photons also follow geodesics of the gravitational metric where the scalar field doesn’t enter (except very weakly as an additional source). Hence the scalar field does not influence the motion of photons!

This has a major observational consequence: It would imply that, for a massive cluster of galaxies, the Newtonian mass one would determine from the kinematics of galaxies (non-relativistic particles) via the virial theorem should be much greater than the mass one would determine from gravitational deflection of photons (relativistic particles). This is, emphatically, not the case. The lensing contradiction is a severe blow to scalar-tensor theories of MOND, at least for those with a conformal coupling.

An obvious solution to this problem is to consider a non-conformal relationship between the Einstein and physical metrics, for example

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} e^{-\eta\phi} - (e^{\eta\phi} - e^{-\eta\phi}) A_\mu A_\nu$$ \hspace{1cm} (17)

where now $A^\mu$ is a normalized vector field, i.e., $A_\mu A^\mu = -1$. Basically, the conformal relation transforms the gravitational geometry by stretching or contracting the 4-D space isotropically but in a space-time dependent way. This disformal transformation, eq. 17, picks out certain directions for additional stretching or contracting. Because we would like space in the cosmological frame to be isotropic (the Cosmological Principle) we should somehow arrange for the vector to point in the time direction in the cosmological frame, which then becomes a preferred frame. In the spirit of the ancient stratified theories, one may propose an a priori non-dynamical vector field postulated to have this property. This may be combined with an AQUAL theory to provide MOND phenomenology with enhanced gravitational lensing; in fact, with the particular transformation given by eq. 17 one can show that the relation between the total weak field force and the deflection of photons is the same as it is in GR. Hence relativistic and non-relativistic particles would both feel the same weak-field force.
The problem with this initial theory is that the non-dynamical vector field quite explicitly violates the principle of General Covariance making it impossible to define a conserved energy-momentum tensor (this has been known for some time [52]). This problem led Bekenstein to endow the vector field with its own dynamics, and, hence, to TeVeS.

3.2 The structure of TeVeS

As the name implies, the theory is built from three fields.

a) The tensor: This is the usual Einstein metric that we are all familiar with. Its dynamics are given by the standard Einstein-Hilbert action of GR:

\[ S_T = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x. \] (18)

It is necessary that the tensor should be the Einstein metric because we want the theory to approach GR quite precisely in the appropriate strong field limits.

b) The scalar: We want the scalar, \( \phi \), to provide a long-range fifth force in the limit of low field gradients. Bekenstein takes the scalar field action to be

\[ S_S = -\frac{1}{16\pi G} \int \left[ \frac{1}{2} q^2 h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + l^{-2} V(q) \right] \sqrt{-g} d^4x. \] (19)

Here I have kept the notation of PCG because the action is, in fact, the weak coupling, or AQUAL limit, of PCG where there is no explicit kinetic term for the field \( q \). In other words, \( q \) behaves as a non-dynamical auxiliary field where \( q^2 \) will play the role of \( \mu \) in the Bekenstein-Milgrom field equation (the fact that this field is non-dynamical does not violate General Covariance because it does not act directly upon particles). I use this bi-scalar notation because I think it is important to realise that the auxiliary field could, in fact, be dynamical. This, in some respects, provides a plausible interpretation of the free function, \( V(q) \), as a potential (let’s call it a pseudo-potential for now). As we see below, this can provide a basis for cosmological dark matter.

Another difference with standard scalar-tensor theory is that the invariant \( h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \) has replaced the usual scalar field invariant \( g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \) where

\[ h^{\alpha\beta} = g^{\alpha\beta} - A^\alpha A^\beta \] (20)

and \( A \) is the normalized vector field described below. Bekenstein has shown that this simple replacement solves the superluminal propagation problem of AQUAL theories of MOND. The speed of scalar waves turns out to be precisely \( c \).

c) The vector: The dynamical normalized vector field is necessary to provide the disformal transformation and the enhanced gravitational lensing. Bekenstein chose to describe its dynamics through the action
\[ S_V = \frac{K}{32\pi G} \int |F^\mu\nu F_{\mu\nu} - 2(\lambda \frac{\mu}{K})(A^\mu A_\mu + 1)\sqrt{-g}d^4x \]  

(21)

where \( F_{\mu\nu} \) is the electromagnetic-like anti-symmetric tensor constructed from \( A \)

\[ F_{\mu\nu} = A_{\nu\mu} - A_{\mu\nu}, \]  

(22)

and \( \lambda \) is a Lagrangian multiplier function which enforces the normalisation condition \( A_\mu A^\mu = -1 \). \( K \) is a new parameter which determines the strength of the vector field coupling.

All of this is combined with the particle action

\[ S_P = -mc \int (-\tilde{g}_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp})^{\frac{3}{2}} dp \]  

(23)

where \( \tilde{g}_{\mu\nu} \) is the physical metric disformally related to the Einstein metric as in eq. 17. This guarantees that the deflection of photons is given by

\[ \delta \theta = \frac{2}{c^2} \int f_\perp dl \]  

(24)

where the integral is over the line-of-sight and \( f_\perp \) is the perpendicular component of the total weak-field force, Newtonian and scalar.

The free parameters of the theory are \( \eta \), the scalar field coupling, \( K \) the vector field coupling, and \( l \) the characteristic length scale determining the MOND acceleration scale \( (a_0 = c^2/l) \). It can be shown that, as the parameters \( \eta \) and \( K \) approach zero, the theory reduces to GR, as it should do. The free function is \( V(q) \) or the pseudo-potential of the auxiliary \( q \) field. I could have absorbed the length scale \( l \) into \( V(q) \) but, following Bekenstein, I choose to express it explicitly in order to render \( V(q) \) unitless.

In the weak-field static limit, the scalar field equation is of the Bekenstein-Milgrom form:

\[ \nabla \cdot (\mu f_s) = 4\pi G \rho \]  

(25)

where, in my notation, the scalar force is given by \( f_s = \eta c^2 \nabla \phi \) and \( \mu = q^2/2\eta^2 \). Making use of these expressions, we may show that the MOND interpolating function is then given by the algebraic relation,

\[ \frac{dV(\mu)}{d\mu} = -\frac{f_s^2}{a_0^2} = -X \]  

(26)

where \( a_0 = c^2/l \). This, of course, necessitates \( V'(\mu) < 0 \) in the static domain.

Now, to obtain MOND phenomenology, it must be the case that \( \mu(X) = \sqrt{X} \) in the low acceleration limit. For example, \( V(\mu) = -\frac{1}{4} \mu^3 \) would work (recalling the relation between \( \mu \) and \( q \) above, we see that this gives rise to the negative sextic potential in PCG). But this leaves us with the old problem of extending AQUAL into the cosmological regime where \( X < 1 \).

Bekenstein chose to solve this problem by taking a free function that provides two separate branches for \( \mu(X) \)- one for static mass concentrations,
Fig. 7. Bekenstein’s trial free function shown, $\mu(X)$ (solid curve) where $X$ is defined as $\eta^2l^2\phi_{,\alpha}\phi^{,\alpha}$. There are two discontinuous branches for cosmology ($X < 0$) and for quasi-static mass concentrations ($X > 0$). The dashed curve shows one possibility for avoiding the discontinuity (eq. 28).

where the spatial gradients of $\phi$ dominate, and one for the homogeneous evolving Universe where the temporal derivative dominates. Specifically,

$$X = \frac{1}{4}\mu^2(\eta^2\mu - 2)^2(1 - \eta^2\mu)^{-1}$$  \hspace{1cm} (27)

($\eta^2$ appears because my definition of $\mu$ differs from Bekenstein’s). This two branch, $\mu(X)$, is shown in Fig. 7, where now we are defining $X$ more generally as $X = \eta^2l^2\phi_{,\alpha}\phi^{,\alpha}$. The corresponding pseudo-potential, $V(\mu)$, is shown in Fig. 8.

If we interpret $V(\mu)$ as the potential of an implicitly dynamical field, it is certainly a rather curious-looking one– with the infinite pit at $\eta^2\mu = 1$. It also illustrates the peculiar aspect of the two-branch form of $\mu$. For cosmological solutions, $\eta^2\mu = 2$ is an attractor; i.e., the $\mu$ field seeks the point where $dV/d\mu = 0$ \[53\]. However, on the outskirts of galaxies $\eta^2\mu \to 0$ as it must to provide the $1/r$ scalar force. So somehow, in progressing from the galaxies to the cosmological background $\eta^2\mu$ must jump from 0 to 2 apparently discontinuously (photons propagating in a cosmological background also have to make this leap). This problem indicates that such a two-branch $\mu(X)$ may not be appropriate, but more on this below.
Fig. 8. The pseudo-potential ($V(\mu)$) corresponding to the $\mu(X)$ shown in Fig. 7

4 TeVeS: Successes, issues and modifications

4.1 Successes of TeVeS

The theory is an important development because it solves several of the outstanding problems of earlier attempts:

1.) While providing for MOND phenomenology in the form of the old non-relativistic Bekenstein-Milgrom theory, it also allows for enhanced gravitational lensing. It does this in the context of a proper covariant theory, albeit by construction—by taking the particular disformal relation between the physical and gravitational metrics given by eq. 17. This aspect of the theory has favourably tested on a sample of observed strong lenses [54], although there are several case with unreasonable implied mass-to-light ratios.

2.) It has been shown [22, 58] that, for TeVeS, the static post-Newtonian effects are identical to those of GR; that is to say, the Eddington-Robertson post-Newtonian parameters are $\gamma = \beta = 1$ as in GR. This provides consistency with a range of Solar System gravity tests such as light deflection and radar echo delay.

3.) Scalar waves propagate causally ($v_s \leq c$). This is true because the new scalar field invariant $h^{\alpha\beta} \phi_\alpha \phi_\beta$ ($h^{\alpha\beta}$ is a new tensor built from the Einstein metric and the vector field (eq. 20)) replaces the standard invariant in the
scalar field Lagrangian (eq. 9). This is a major improvement over the old AQUAL theory, but also one which relies upon the presence of the vector field.

4.) Gravitational waves propagate causally if $\phi > 0$. One can show that the speed of the standard tensor waves is given by $V_g = c e^{-\eta \phi}$. This means that the cosmology must provide $\phi > 0$ in a natural way. Moreover, there is a prediction here which is possibly testable, and that is $V_g < c$. If an event, such as a gamma-ray burst, also produces gravitational radiation (as is likely), the gravitational waves should arrive somewhat later than the gamma rays.

5.) The theory allows for standard FRW cosmology and, at least in the linear regime, for a MONDian calculation of structure formation. Moreover, there is an evolving dark energy (quintessence) which is coupled to the background baryon density, offering a possible solution to the near coincidence of these components at the present epoch. This comes about through the presence of $V(\mu)$ as a negative pressure fluid in the Friedmann equations. The cosmological value of the dark energy density, $V(\mu)$, corresponds to the minimum of an effective potential $V_{eff} = V(\mu) + B(\rho \tau)/\mu$ where $B$ is a function of the product of cosmic time $\tau$ and the baryonic mass density $\rho$ (it is identical in this sense to PCG in a cosmological context).

4.2 Remaining issues

In spite of these important successes there are a number of problems that the theory is yet to confront:

1.) The discontinuous $\mu(X)$. The two discontinuous branches (Fig. 7)—one for cosmology and one for quasi-static mass concentrations—appear awkward, particularly if the free-function is interpreted as a potential of the $\mu$ field. Moreover, this presents very practical problems for gravitational lensing and calculation of structure formation into the non-linear regime. But more seriously, it appears that such two branch $\mu$ may be an intrinsic aspect of a theory with the structure of TeVeS. One could propose that the space-like branch of $\mu$ is simply reversed at the $\mu = 0$ axis (see dotted line in Fig. 7), so, instead of eq. 27, Bekenstein’s free function could be expressed as

$$X = \pm \frac{3\mu^2}{1 - \eta^2 \mu}. \quad (28)$$

However, the pseudo-potential, $V(\mu)$, would then also be double valued which would appear distinctly unphysical if this is really to be identified with the potential of an implicitly dynamical scalar $\mu$ (or $q$). In my opinion, the only solution to this problem is to alter the structure of the theory (see below).

2.) Even given a $\mu(X)$ with two branches, the separation between quasi-static and cosmological phenomena is artificial. Eq. 26, which provides the relation
between the scalar field gradient and $\mu$, should also contain the cosmic time derivative of the scalar field because this is likely to be of the same order as $dV/d\mu$; i.e., eq. 26 should read
\[
\frac{dV}{d\mu} = -\frac{f_s^2}{a_0^2} + \frac{\eta^2 t^2 \dot{\phi}^2}{c^2}.
\] (29)
Therefore the free function, relevant to mass concentrations, may also be thought of as an evolving effective potential (this can actually be an advantage which I make use of below).

3.) This is a preferred frame theory that violates the Lorentz invariance of gravitational phenomena. This is because of the cosmic vector field $\mathbf{A}$. In the cosmic frame, only the time component of $\mathbf{A}$ is non-zero but for frames in relative motion with respect to the CMB spatial components also develop non-zero values, and this has a real effect on particle dynamics. In the Solar System for example, there should be gravitational ether drift effects, such as a polarisation of the earth-moon orbit along the direction of $\mathbf{w}$, the velocity vector with respect to the CMB. Such effects, in conservative theories, are quantified by two post-Newtonian parameters [57], $\alpha_1$ and $\alpha_2$, which enter the effective Lagrangian of an N-body system as the coefficients of terms containing $v \cdot w/c^2$ where $v$ is the velocity with respect to the centre-of-mass of the N-body system. These parameters are experimentally constrained; for example, $\alpha_1 < 10^{-4}$ on the basis of Lunar Laser Ranging [59].
It is important to determine predicted values of $\alpha_1$ and $\alpha_2$ for TeVeS. A reasonable guess is that these post-Newtonian parameters will approach zero as the free parameters of the theory, $\eta$ and $K$ approach zero [61]. That is because in this limit the theory approaches GR, and in GR there are no preferred frame effects. Whether or not the resulting constraints on $\eta$ and $K$ are consistent with other aspects of Solar System and galaxy phenomenology remains to be seen.

4.) In the outer solar system the force is not precisely inverse square. For example, in the context of Bekenstein’s free function, the non-inverse square component of the force is shown, as a function of radius, in Fig. 9 for two different values of the scalar coupling strength, $\eta$. Constraints from planetary motion are shown by the upper limits [61]. Such a deviation, at some level, is an aspect of any multi-field theory of MOND [51], and it may be a problem or it may be a blessing. A non-inverse square component of the force, in the form of a constant acceleration, is indicated by Doppler ranging to both the Pioneer spacecrafts (indicated by the horizontal bar in Fig. 8) [61]. If this effect is confirmed, it would be a major discovery, indicating that gravity is not what we think it is beyond the inner solar system.

5.) As I mentioned in the Introduction, there is compelling evidence for cosmological dark matter—a pressureless fluid which appears to affect early large scale structure formation (evident in the CMB anisotropies) and the more
Fig. 9. The dashed curve is the log of the total force \( f_t = f_s + f_N \), in units of \( 10^{-8} \) cm/s\(^2\) plotted against the log of the radial distance from the sun in astronomical units for TeVeS. The dotted curve is the anomalous force (the non-inverse square force) for Bekenstein’s initial choice of free function with \( \eta = 0.01 \). The long dashed curve is the same but with \( \eta = 0.1 \). Observed constraints on the non-inverse square part of the acceleration are (left to right): from the precession of perihelion of Mercury, and of Icarus, from variation of Kepler’s constant between Earth and Mars, between inner planets and Jupiter, Uranus or Neptune, respectively. The horizontal bar is the Pioneer anomaly range. From reference [62].

recent expansion history of the Universe (evident in the SNIa results). The weight of this evidence implies that a proper theory of MOND should at least simulate the cosmological effects of the apparent dark matter, again not an evident aspect of TeVeS.

In a general sense, the theory, at present, is intricate and misses a certain conceptual simplicity. There are several loose threads which one might hope a theory of MOND to tie up. For example, the MOND acceleration parameter, \( a_0 \), is put in by hand, as an effective length scale \( l \); the observational fact that \( a_0 \approx cH_0 \) remains coincidental. This seems unfortunate because this coincidence suggests that MOND results from the effect of cosmology on local particle dynamics, and, in the theory as it now stands, no such connection is evident.

Finally, by mentioning these problems, I do not wish to imply that TeVeS is fundamentally flawed, but that it is not yet the theory in final form. In this
procedure, building up from the bottom, the approach to the final theory is incremental.

4.3 Variations on a theme: biscalar-tensor-vector theory

The motivation behind this variation is to use the basic elements of TeVeS in order to construct a cosmologically effective theory of MOND. The goals are to reconcile the galaxy scale success of MOND with the cosmological evidence for CDM and to provide a cosmological basis for $a_0$.

There are two essential differences with TeVeS in original form: First, the auxiliary field $q$ is made explicitly dynamical as in PCG. This is done by introducing a kinetic term for $q$ in the scalar action (eq. 19), i.e., $q_{\alpha}q^{\alpha}$. Secondly, one makes use of the preferred frame to separate the spatial and time derivatives of the matter coupling scalar field $\phi$ at the level of the Lagrangian. Basically, this is done by defining new scalar field invariants. If we take the usual invariant to be $I = g^{\alpha\beta}\phi_{\alpha}\phi_{\beta}$ and define $J = A^{\alpha}A^{\beta}\phi_{\alpha}\phi_{\beta}$, $K = J + I$, then we can readily see that $J$ is just the square of the time derivative in the preferred cosmological frame ($\dot{\phi}^2$) and $K$ is the spatial derivative squared in that frame ($\nabla\phi \cdot \nabla\phi$). The scalar field Lagrangian is then taken to be

$$L_s = \frac{1}{2}[q_{\alpha}q^{\alpha} + h(q)K - f(q)J + 2V(q)].$$

(30)

So, separate functions of $q$ multiply the spatial and temporal gradients of $\phi$ in the cosmological frame. This means that the potential for $q$ becomes an effective potential involving the cosmic time derivative, $\dot{\phi}$, for both the homogeneous cosmology and for quasi-static mass concentrations. Indeed, one can show, given certain very general conditions on the free functions, $q$ at a large distance from a mass concentration approaches its cosmological value. There is smooth transition between mass concentrations and cosmology. Moreover, if I take $h(q) \approx q^2$, $f(q) \approx q^6$ and a simple quadratic bare potential $V(q) \approx Bq^2$, I obtain a cosmological realisation of Bekenstein’s PCG with a negative sextic potential but where the coefficient in the potential, and hence $a_0$, is identified with the cosmic $d\phi/dt$.

There are two additional advantages of making $q$ dynamical. First of all, as the $q$ field settles to the evolving potential minimum, oscillations of this field about that minimum inevitably develop. If the bare potential has a quadratic form, then these oscillations constitute CDM in the form of “soft bosons” [63]. Depending upon the parameters of the theory, the de Broglie wavelength of these bosons may be so large that this dark matter does not cluster on the scale of galaxies (but possibly on the scale of clusters). A cosmological effective theory of MOND produces cosmological CDM for free.

A second advantage is that appropriately chosen free functions can reproduce the Pioneer anomaly in the outer Solar System—both the magnitude ($\approx 8 \times 10^{-9}$ cm/s$^2$) and the form—constant beyond 20 AU (see Fig. 10). It does this while being consistent with the form of galaxy rotation curves.
Fig. 10. The Newtonian (dashed curve) and scalar (solid curves) force in the Solar System in the context of the biscalar theory. The different curves correspond to different values of scalar coupling constant $\eta$. This should be compared with Fig. 6 which shows the Newtonian and scalar forces for TeVeS with the initial free function.

Of course the presence of three free functions appear to give the theory considerable arbitrariness, but, in fact, the form of these functions is strongly constrained by Solar System, galaxy and cosmological phenomenology.

Many other modifications of TeVeS are possible. For example, it may only be necessary to make the auxiliary field $q$ explicitly dynamical and choose a more appropriate form of the free function. The number of alternative theories is likely to be severely restricted by the demands imposed by observations—ranging from the solar system, to galaxies, to clusters, to gravitational lensing, to cosmology. The hope is that the number of survivors is not less than one.

5 Conclusions

Here I have outlined the attempts that have been made to define modifications of gravity that may underly the highly successful empirically-based MOND, proposed by Milgrom as an alternative to dark matter in bound self-gravitating systems. These attempts lead inevitably to a multi-field theory of gravity—the Einstein metric to provide the phenomenology of GR in the strong field limit, the scalar field to provide the MOND phenomenology most apparent in the outskirts of galaxies and in low surface brightness systems,
and the vector field to provide a disformal relation between the Einstein and physical metrics– necessary for the observed degree of gravitational lensing. I re-emphasise that this process has been entirely driven by phenomenology and the need to cure perceived pathologies; there remains no connection to more a priori theoretical considerations or grand unifying principles such as General Covariance or Gauge Invariance. It would, of course, be a dramatic development if something like MOND were to emerge as a incidental consequence of string theory or a higher dimensional description of the Universe, but, in my opinion, this is unlikely. It is more probable that an empirically based prescription, such as MOND, will point the way to the correct theory.

The coincidence between the critical acceleration and $cH_0$ (or possibly the cosmological constant) must be an essential clue. MOND must be described by an effective theory; that is, the theory predicts this phenomenology only in a cosmological context. The aspect, and apparent necessity, of a preferred frame invites further speculation: Perhaps cosmology is described by a preferred frame theory (there certainly is an observed preferred frame) with a long range force mediated by a scalar field coupled to a dynamical vector field as well as the gravitational metric. With the sort of bi-scalar Lagrangian implied by TeVeS, the scalar coupling to matter becomes very weak in regions of high field gradients (near mass concentrations). This protects the Solar System from detectable preferred frame effects where the theory essentially reduces to General Relativity. Because we live a region of high field gradients, we are fooled into thinking that General Relativity is all there is. Only the relatively recent observations of the outskirts of galaxies or objects of low surface brightness (or perhaps the Pioneer anomaly) reveal that there may be something more to gravity.

On the other hand, it may well be that we have been pursuing a mirage with tensor-vector-scalar theories. Perhaps the basis of MOND lies, as Milgrom has argued, with modified particle action– modified inertia– rather than modified gravity [64, 65]. For a classical relativist this distinction between modified gravity and modified inertia is meaningless– in relativity, inertia and gravity are two sides of the same coin; one may be transformed into another by a change of frame. But perhaps in the limit of low accelerations, lower than the fundamental cosmological acceleration $cH_0$, that distinction is restored [66].

It is provocative that the Unruh radiation experienced by a uniformly accelerating observer, changes its character at accelerations below $c\sqrt{\Lambda}$ in a de Sitter universe [65]. If the temperature difference between the accelerating observer and the static observer in the de Sitter Universe is proportional to inertia, then we derive an inertia-acceleration relation very similar to that required by MOND [65]. At present this is all very speculative, but it presents the possibility that we may be going down a false path with attempted modifications of GR through the addition of extra fields.

In any case, the essential significance of TeVeS is not that it, at present, constitutes the final theory of MOND. Rather, the theory provides a counter-
example to the often heard claim that MOND is not viable because it has no covariant basis.

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