Plethysms and operads

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Abstract

We introduce the $\mathcal{T}$-construction, an endofunctor on the category of generalized operads as a general mechanism by which various notions of plethystic substitution arise from more ordinary notions of substitution. The construction itself is a generalization of the Giraudo $T$-construction from monoids to operads. We recover several kinds of plethysm as convolution products arising from the homotopy cardinality of the incidence bialgebra of the bar construction of various operads obtained from the $\mathcal{T}$-construction. The bar constructions are simplicial groupoids, and in the special case of the terminal reduced operad $\text{Sym}$, we recover the simplicial groupoid of [9], a combinatorial model for ordinary plethysm in the sense of Pólya, given in the spirit of Waldhausen $S$ and Quillen $Q$ constructions. In some of the cases of the $\mathcal{T}$-construction, an analogous interpretation is possible.

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Introduction

Plethysm is a substitution law in the ring of power series in infinitely many variables. It was introduced by Pólya [38] in unlabelled enumeration theory in combinatorics, motivated as a series analogue of the wreath product of permutation groups. Another notion of plethysm was defined by Littlewood [27] in the context of symmetric functions and representation theory of the general linear groups [30]. It appears also in algebraic topology, in connection with λ-rings [4] and power operations in cohomology [11]. The two notions of plethysm are closely related, as described in [41] and [3]. The present work starts with Pólya’s notion of plethysm, which we proceed to recall, and fits it into a more general framework covering also many variations.

Let $\mathbb{Q}[x]$ be the ring of power series in the infinite set of variables $x = (x_1, x_2, \ldots)$
without constant term. Given $F, G \in \mathbb{Q}[x]$, their \textit{plethystic substitution} is defined as
\[(G \otimes F)(x_1, x_2, \ldots) = G(F_1, F_2, \ldots), \text{ where } F_k(x_1, x_2, \ldots) = F(x_k, x_{2k}, \ldots).\]

The formal power series can be expressed as
\[F(x) = \sum_{\lambda} F_{\lambda} \frac{x^\lambda}{\text{aut}(\lambda)},\]
where \text{aut}(\lambda) are certain symmetry factors (see Section 3 below).

It is well appreciated in combinatorics that working with the combinatorial structures themselves gives a deeper understanding than working with their numbers. This is the so-called objective method, pioneered by Lawvere \cite{lawvere1969}, Joyal \cite{joyal1981}, and Baez–Dolan \cite{baez2004}. In the development of the theory of species, Joyal \cite{joyal1981} presented a combinatorial model for the plethystic substitution of the cycle index series generating function. Specifically, he proved that composition of species corresponds to plethystic substitution of their cycle index series. However, a fully combinatorial construction was only given a few years later by Nava and Rota \cite{nava1986}. They developed the notion of partitional, a functor from the groupoid of partitions to the category of finite sets, and showed that a suitable notion of composition of partitionals yields plethystic substitution of their generating functions, in analogy with composition of species and composition of their exponential generating functions. A variation of this combinatorial interpretation was given shortly after by Bergeron \cite{bergeron1990}, who instead of partitionals considered permutationals, functors from the groupoid of permutations to the category of finite sets. This approach is nicely related to the theory of species and their cycle index series through an adjunction. Later on, Nava \cite{nava1990} studied both partitionals and permutationals from the point of view of incidence coalgebras, and added a third class of functors called linear partitionals.

The bialgebras arising from the various plethystic substitutions are called plethystic bialgebras in the present article. Here is a general definition, as given in \cite{bergeron1990}: the \textit{plethystic bialgebra} is the free polynomial algebra on the linear functionals $A_{\lambda}(F) = F_{\lambda}$ with comultiplication dual to plethystic substitution,
\[\Delta(A_{\lambda})(F, G) = A_{\lambda}(G \otimes F).\]

In this definition, the difference between the three bialgebras of Nava depends on the definition of the linear functionals $A_{\lambda}$, which in turn depend on the definition of the symmetry factors \text{aut}(\lambda).

The present work introduces a construction on operads, called the $\mathcal{T}$-construction, which formalizes the relationship between ordinary substitutions and plethystic substitutions. In particular, this construction produces combinatorial models for the partitional and the linear partitional (also called exponential) cases, but also for other kinds of
plethysm: plethysm of power series with variables indexed by a (locally finite) monoid, introduced by Méndez and Nava [34] in the course of generalizing Joyal’s theory of colored species to an arbitrary set of colors; plethysm in two variables $x, y$; plethysm of series with coefficients in a non-commutative ring, in the style of [6], and plethysm of series with non-commuting variables. All these plethysms and their bialgebras will be explained in Section 5. The $\mathcal{T}$-construction relies on operads and the theory of decomposition spaces and their incidence bialgebras.

The theory of operad has long been a standard tool in topology and algebra [28, 31], and in category theory [26], and it is getting increasingly important also in combinatorics [21, 33]. In the present work, for maximal flexibility, we work with operads in the form of generalized multicategories [26]. This allows us to cover simultaneously notions such as monoids, categories, non-symmetric ops and symmetric operads.

On the other hand, decomposition spaces (certain simplicial spaces) provide a general machinery to objectify the notion of incidence algebra in algebraic combinatorics. They were introduced by Gálvez, Kock and Tonks [16–18] in this framework and they are the same as 2-Segal spaces, introduced by Dyckerhoff and Kapranov [12] in the context of homological algebra and representation theory. To recover the algebraic incidence coalgebra from the categorified incidence coalgebra one takes homotopy cardinality, a cardinality functor defined from groupoids to the rationals.

It was shown in [23] that the two-sided bar construction [32, 46] of an operad is a Segal groupoid, a particular type of decomposition space, and classical constructions of bialgebras arising from operads factor through this construction (see [11, 42, 43] for related constructions). Next we give two relevant examples of bialgebras that arise as incidence bialgebras of operads.

**Example.** Let $Q[[x]]$ be the ring of formal power series in $x$ without constant term, and let $F, G \in Q[[x]]$. The Faà di Bruno bialgebra $\mathcal{F}$ is the free algebra $Q[A_1, A_2, \ldots ]$, where $A_n \in Q[[x]]^*$ is the linear map defined by

$$A_n(F) = \frac{d^nF}{dx^n}.$$  

Its comultiplication is defined to be dual to substitution of power series. That is

$$\Delta(A_n)(F, G) = A_n(G \circ F).$$

It is a result of Joyal [24 §7.4] that this bialgebra can be objectified by using the category of finite sets and surjections $\mathbf{S}$. In the context of Segal spaces and incidence bialgebras the result reads as follows: *the Faà di Bruno bialgebra $\mathcal{F}$ is equivalent to the homotopy cardinality of the incidence bialgebra of the fat nerve $\mathcal{N}\mathbf{S}$ of the category $\mathbf{S}$.* The comultiplication here is given by summing over factorizations of surjections.
Example. In \[9\] a simplicial groupoid \((TS, \text{see Section 6 below})\) which also arises from
the category of surjections was found to play the same role for plethystic substitution: \textit{the
homotopy cardinality of the incidence bialgebra of }\(TS\) \textit{is isomorphic to the (partitional)
plethystic bialgebra.} The comultiplication extracted from this simplicial groupoid can be
interpreted as summing over certain transversals of partitions, as in the work of Nava and Rota \[37\].

Now, it is well-known that \(NS\) is equivalent to the two-sided bar construction of \(\text{Sym}\),
the terminal reduced symmetric operad. This equivalence takes the surjection \(n \rightarrow 1\)
to the unique operation of arity \(n\), and the comultiplication of an operation runs through
all possible 2-step factorizations. For example

\[
\Delta(\Psi) = \Psi \otimes \Psi + 3 \Psi \otimes \Psi.
\]

The starting point of the present work is the observation that also \(TS\) is equivalent
to the two-sided bar construction of an operad. As we shall see, this operad can be
obtained from \(\text{Sym}\) by the aforementioned \(\mathcal{T}\)-construction, which makes sense for any
(nice enough) operad. As stated above, this construction leads to many other flavors of
plethysm, some of which had already been studied in various contexts. For instance, from
\(\text{Ass}\), the reduced associative operad, we obtain the exponential plethystic bialgebra, and
from \(n\)-colored \(\text{Sym}\) or \(\text{Ass}\), we obtain the \(n\)-variables plethystic bialgebra. The results
relating the bialgebras to these operads are explained in Section 5.

Let us give now a brief introduction to the \(\mathcal{T}\)-construction. The word \(\mathcal{T}\)-construction
comes from the simplicial \(T\)-construction \[9\], where \(T\) stands for transversal (in the sense
of Nava–Rota \[37\]), and which is analogous to Waldhausen S and Quillen Q constructions.
By coincidence Giraudo \[20\] had used the same letter \(T\) for a functor from monoids to
nonsymmetric ops. The \(\mathcal{T}\)-construction of the present work encompasses both these
constructions, and the letter \(T\) has been maintained, but now in a fancier font.

Let us first describe Giraudo’s \(T\)-construction \[20\]. Let \((Y, \cdot, 1)\) be a monoid. Then

\[
TY := \bigsqcup_{n \geq 0} MY(n),
\]

where for all \(n \geq 1\),

\[
TY(n) := \{(x_1, \ldots, x_n) \mid x_i \in Y \text{ for all } i = 1, \ldots, n\},
\]

so that the \(n\)-ary operations are \(n\)-tuples of elements of \(Y\). The substitution law in \(TY\),

\[
\circ_i: TY(n) \times TY(m) \rightarrow TY(n + m - 1),
\]

is defined as follows: for all \(x \in TY(n), y \in TY(m)\), and \(i = 1, \ldots, n\),

\[
x \circ_i y := (x_1, \ldots, x_{i-1}, x_i \cdot y_1, \ldots, x_i \cdot y_m, x_{i+1}, \ldots, x_n).
\]
The main technical contribution of the present paper is to upgrade this construction from monoids to the context of $P$-operads (generalized multicategories in Leinster [26] terminology), for $P$ a cartesian monad on a cartesian category $\mathcal{E}$. This level of generality allows to work with symmetric and non-symmetric colored and non-colored operads. A $P$-operad is represented by a span and two arrows

$$
\begin{array}{c}
PQ_0 \\
\downarrow s \quad \downarrow t \\
Q_0 \quad Q_0
\end{array}
\xrightarrow{Q_1 \times_{PQ_0} Q_1} Q_1
$$

where $Q_0$ is thought of as the object of colors, $Q_1$ is thought of as the object of operations, $s$ returns the $P$-configuration of input colors, $t$ returns the output color, $e$ is the unit and $m$ is composition. All these arrows have to satisfy associativity and unit axioms.

For instance, if $\text{Id}$ is the identity monad, then an $\text{Id}$-operad is a category internal to $\mathcal{E}$. The $\mathcal{T}$-construction is in fact a composition of two constructions, one from $P$-operads to (internal) categories and one from categories to $P$-operads. The latter contains the Giraudo $T$-construction for the case $\mathcal{E} = \text{Set}$ if we consider monoids as categories with one object.

However, we will be mainly interested in $\mathcal{E} = \text{Grpd}$. In particular, non-symmetric operads will be considered as $M'$-operads, where $M'$ is the free semimonoidal category monad in $\text{Grpd}$, and symmetric operads as $S'$-operads, where $S'$ is the free symmetric semimonoidal category monad in $\text{Grpd}$. There are two main reasons for working over $\text{Grpd}$: on the one hand, note that unlike non-symmetric operads, symmetric operads cannot be portrayed as $P$-operads in $\text{Set}$, because the free commutative monoid monad is not cartesian; on the other hand, working in $\text{Grpd}$ adapts better with the theory of decomposition spaces and incidence coalgebras. This theory uses weak notions of simplicial groupoids, slice categories, and pullbacks, but by keeping track of fibrancy we can stay within strict notions and strict monads in the style of [45].

In order for the $\mathcal{T}$-construction to work, it is necessary to assume that the monads come equipped with a strength. This notion goes back to work of A. Kock [22] in enriched category theory, but it has turned out to be fundamental for the role monads play in functional programming [35, 44]. In Section 4 we will recall the theory of generalized operads, the notion of strong monad, and the two-sided bar construction in this context.

In Section 2 we will briefly explain Segal groupoids, incidence coalgebras and homotopy cardinality. Section 3 will be devoted to the $\mathcal{T}$-construction, and Section 4 to some examples. Next, in Section 5 we will introduce the bialgebras and state and prove the main results: the equivalence between the homotopy cardinality of the incidence bialgebras of the two-sided bar constructions of operads obtained from the $\mathcal{T}$-construction and the plethystic bialgebras, as well as the Faà di Bruno bialgebra and some of its
variations. Finally, in Section 6 we will prove the equivalence between \( TS \) and \( B\widetilde{Sym} \), and we will characterize some of the two-sided bar constructions as simplicial groupoids similar to \( TS \).

**List of notations**

\((P, \mu, \eta)\) generic strong cartesian monad (1.1.1 and 1.3.1)

\(\text{Id}\) identity monad (1.3.3)

\(M\) free monoid monad (page 8)

\(M'\) free semigroup monad (1.3.5)

\(S\) free symmetric monoidal category monad (1.3.7)

\(S'\) free symmetric semimonoidal category monad (1.3.8)

\(L\) monad \(A \mapsto P1 \times A\) (page 22)

\(Y\) generic (locally finite) monoid (1.3.6)

\(Y\) monad given by \(A \mapsto Y \times A\), for \(Y\) a monoid (1.3.6)

\(TY\) Giraudo \(T\)-construction of \(Y\) (4.1.3)

\(\mathcal{D}_{A,B}\) strength natural transformation (1.3.1)

\(\mathcal{D}_B\) strength for \(A = 1\) (page 22)

\(R_A\) projection \(P1 \times A \mapsto A\) (page 22)

\(\mathcal{B}\) two-sided bar construction (page 16)

\(\mathcal{B}P\) two-sided bar construction relative to a monad \(P\) (page 19)

\(\text{Sym}\) the reduced symmetric operad (1.3.8)

\(\text{Ass}\) the reduced associative operad (1.3.5)

\(\mathcal{E}\) generic ambient cartesian category, mainly \(\text{Set}\) or \(\text{Grpd}\) (1.1)

\(C\) category internal to \(\mathcal{E}\) (page 8)

\(C_0, C_1\) objects and arrows of \(C\), respectively (page 8)

\(Q\) \(P\)-operad internal to \(\mathcal{E}\) (page 8)

\(Q_0, Q_1\) objects and operations of \(Q\) (page 8)

\((Q, \mu^Q, \eta^Q)\) monad on \(\mathcal{E}/Q_0\) defined by the \(P\)-operad \(Q\) (page 15)

\(\mathcal{T}_F C\) \(\mathcal{T}\)-construction from \(C\) to a \(P\)-operad (page 22)

\(\mathcal{T}^P Q\) \(\mathcal{T}\)-construction from a \(P\)-operad to a category \(C\) (page 27)

\(\mathcal{T}_F Q\) \(\mathcal{T}\)-construction from a \(P'\)-operad to a \(P\)-operad (page 32)

\(\text{Grpd}\) category of groupoids and groupoid morphisms

\(\text{Set}\) category of sets and set maps

\(\Delta\) simplex category (page 16)

\(TS\) simplicial groupoid of \([9]\) (page 54)
1 Monads and operads

As mentioned in the introduction, the $T$-construction fits neatly within the context of generalized operads and strong monads. The following discussion of generalized operads is taken from [26]. Let us start by expressing the notions of category and of plain operad in this setting.

A small category $C$ can be described by sets and functions

$$
\begin{array}{ccc}
C_1 & \xrightarrow{m} & C_1 \\
\downarrow{s} & & \downarrow{t} \\
C_0 & \leftarrow & C_0
\end{array}
\quad
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
C_0 & \rightarrow & C_1
\end{array}
$$

where the pullback is taken along $C_1 \xrightarrow{s} C_0 \xleftarrow{t} C_1$, satisfying associativity and identity axioms, which can be expressed with commutative diagrams in $\text{Set}$ (see Appendix A.1). The set $C_0$ is the set of objects and $C_1$ is the set of arrows of $C$. The map $s$ returns the source of an arrow and $t$ returns its target. The maps $m$ and $e$ represent composition and identities.

A non-symmetric operad can be defined in a similar way. Let $M: \text{Set} \rightarrow \text{Set}$ be the free monoid monad: it sends a set $A$ to $\bigcup_{n \in \mathbb{N}} A^n$. Then an operad can be described as consisting of sets and functions

$$
\begin{array}{ccc}
Q_1 & \xrightarrow{m} & Q_1 \\
\downarrow{s} & & \downarrow{t} \\
MQ_0 & \leftarrow & Q_0
\end{array}
\quad
\begin{array}{ccc}
MQ_1 \times_{MQ_0} Q_1 & \rightarrow & Q_1 \\
\downarrow & & \downarrow \\
Q_0 & \rightarrow & Q_1
\end{array}
$$

(1.0.1)

satisfying associativity and identity axioms, which can be expressed with commutative diagrams in $\text{Set}$ (see Appendix A.2) and involve the monad structure on $M$. The set $Q_0$ is the set of objects and $Q_1$ is the set of operations of $Q$. The map $s$ assigns to an operation the sequence of objects constituting its source, and $t$ returns its target. The maps $m$ and $e$ represent composition and identities.

1.1 $P$-operads

The above characterization of non-symmetric $\text{Set}$ operads can be generalized to any ambient category and any monad $P$ as long as they are cartesian. The classical case is $\text{Set}$; we shall be concerned also with $\text{Grpd}$.

Definition 1.1.1. A category is cartesian if it has all pullbacks. A functor is cartesian if it preserves pullbacks. A natural transformation is cartesian if all its naturality squares are pullbacks. A monad $(P, \mu, \eta)$ is cartesian if $P$ is cartesian as a functor and $\mu$ and $\eta$ are cartesian natural transformations.
Given a cartesian category $\mathcal{E}$ and a cartesian monad $(P, \mu, \eta)$, we define $\mathcal{E}(P)$ as the bicategory whose 0-cells are the objects $E$ of $\mathcal{E}$, whose 1-cells $E \to E'$ are spans $PE \leftarrow M \to E'$, and 2-cells are the usual morphisms $M \to N$ between spans:

Given two 1-cells

the composite is given by taking a pullback and using the multiplication $\mu$ of $P$, and the 1-cell identity is given by $\eta$ and id. They are shown in the following diagram:

Composition and identity of 2-cells are obvious. Since composition assumes a global choice of pullbacks, and since the pasting of two chosen pullbacks is not generally a chosen pullback, composition is associative up to coherent isomorphism. The coherence 2-cells are defined using the universal property of the pullback.

**Definition 1.1.2** (Burroni [7]). Let $P$ be a cartesian monad in a cartesian category $\mathcal{E}$. A $P$-operad is a monad in the bicategory $\mathcal{E}(P)$.

This means that a $P$-operad $Q$ consists precisely of objects $Q_0$ and $Q_1$ of $\mathcal{E}$ together with maps $s$, $t$, composition $m$ and identities $e$ as in diagram (1.0.1) satisfying associativity and identity axioms (Appendix A.2). A morphism $Q \to Q'$ of $P$-operads is defined
as a pair of arrows \( Q_0 \xrightarrow{f_0} Q'_0, Q_1 \xrightarrow{f_1} Q'_1 \), satisfying the following diagrams,

\[
\begin{align*}
&\begin{array}{c}
PQ_0 \\
\downarrow_{f_0} \hspace{0.5cm} \downarrow_{f_0} \hspace{0.5cm} \downarrow_{f_0} \hspace{0.5cm} \downarrow_{f_0}
\end{array} & \begin{array}{c}
PQ'_0 \\
\downarrow_{f'_0} \hspace{0.5cm} \downarrow_{f'_0} \hspace{0.5cm} \downarrow_{f'_0} \hspace{0.5cm} \downarrow_{f'_0}
\end{array} \\
&\begin{array}{c}
PQ_1 \\
\downarrow_{f_1} \hspace{0.5cm} \downarrow_{f_1} \hspace{0.5cm} \downarrow_{f_1} \hspace{0.5cm} \downarrow_{f_1}
\end{array} & \begin{array}{c}
PQ'_1 \\
\downarrow_{f'_1} \hspace{0.5cm} \downarrow_{f'_1} \hspace{0.5cm} \downarrow_{f'_1} \hspace{0.5cm} \downarrow_{f'_1}
\end{array}
\end{align*}
\]

\[
Q_0 \xrightarrow{e} Q_1 \hspace{0.5cm} P_{Q_1 \times_{PQ_0} Q_1} \xrightarrow{m} Q_1
\]

regarding compatibility with the spans, identities and composition maps. Notice that this is not an arrow in \( E(P) \). The category of \( P \)-operads is denoted \( P\text{-Operad} \).

### 1.2 Morphisms of spans

In Section 3 we will deal with morphisms between long horizontal composites of spans. It is thus worth to set up a framework for such morphisms: consider the following diagrams, named blocks, made of maps in \( E \),

\[
\begin{align*}
&\begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array} & \begin{array}{c}
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\end{array}
\end{align*}
\]

Notice that (1.2.2) induce isomorphisms of spans if the vertical maps are isomorphisms, since in this case they represent horizontal composition of spans. Diagram (1.2.1) is an isomorphism when all the vertical arrows are isomorphisms, and (1.2.3) are isomorphisms when all the vertical arrows and the span projected away are isomorphisms. Besides, the blocks can be horizontally and vertically attached in the obvious way to get morphisms of longer spans, with the only restriction that the diagrams (1.2.3) can be attached to the right and to the left respectively.

**Lemma 1.2.1.** Any pasting of blocks defines a morphism between the limit of the top row and the limit of the bottom row. Moreover, such a morphism is an isomorphism if it can be constructed from blocks that are isomorphisms.

The morphisms between long spans will be pictured with diagrams
where the left bold part is the limit of the diagram: the upper dot is the limit of the upper row, and same for the bottom row. Observe that the decomposition of a morphism into blocks is not unique, and there may be decompositions of isomorphisms whose blocks are not necessarily isomorphisms. Here is an example that will be used later on.

**Example 1.2.2.** The following diagram represents an isomorphism of composites of spans:

![Diagram](image)

Indeed, it can be expressed by pasting isomorphism blocks:

![Diagram](image)

1.3 Strong monads

We recall now the notion of strong monad [22], which is central in the $\mathcal{T}$-construction. From now on the ambient category $\mathcal{E}$ is required to have a terminal object, hence all finite limits.

**Definition 1.3.1.** Let $(P, \mu, \eta)$ be a monad on $\mathcal{E}$. A **strength** for $P$ is a natural transformation with components $D_{A,B}: A \times PB \to P(A \times B)$, satisfying the following two axioms concerning tensoring with 1 and consecutive applications of $D$,

\[
\begin{align*}
(1 \times A) \times (B \times C) & \xrightarrow{D_{A,B,C}} (A \times B) \times P(C) \xrightarrow{A \times D_{B,C}} A \times (B \times C), \\
1 \times A & \xrightarrow{D_{1,A}} P(1 \times A) \xrightarrow{P_2} PA
\end{align*}
\]
and two axioms concerning compatibility with monad unit and multiplication,

\[
\begin{array}{ccc}
A \times B & \xrightarrow{A \times \eta_B} & A \times PB \\
\downarrow{\eta_{A \times B}} & & \downarrow{D_{A,B}} \\
P(A \times B) & & \\
\end{array}
\quad (1.3.2a)
\]

\[
\begin{array}{ccc}
A \times P^2B & \xrightarrow{D_{A,PB}} & P(A \times PB) \\
\downarrow{A \times \mu_B} & & \downarrow{\mu_{A \times B}} \\
A \times PB & \xrightarrow{D_{A,B}} & P(A \times B) \\
\end{array}
\quad (1.3.2b)
\]

Before seeing some examples of \(P\)-operads and strong monads, we prove the following lemma, which will be useful in Section 3.

**Lemma 1.3.2.** Let \(u\) be the unique morphism \(u: P1 \rightarrow 1\). Then the square

\[
\begin{array}{ccc}
A \times P^21 & \xrightarrow{A \times Pu} & A \times P1 \\
\downarrow{D_{A,1}} & & \downarrow{D_{A,1}} \\
P(A \times P1) & \xrightarrow{P(A \times u)} & P(A \times 1) \\
\end{array}
\quad (1.3.3)
\]

is a pullback.

**Proof.** Observe that if we project the bottom rows of this square to the first component,

\[
\begin{array}{ccc}
A \times P^21 & \xrightarrow{A \times Pu} & A \times P1 \\
\downarrow{D_{A,1}} & & \downarrow{D_{A,1}} \\
P(A \times P1) & \xrightarrow{P(A \times u)} & P(A \times 1) \\
\downarrow{P_{P2}} & & \downarrow{P_{P2}} \\
P^21 & \xrightarrow{P_{u}} & P1 \\
\end{array}
\]

then the lower square is a pullback because \(P\) is cartesian, and the outer square is a pullback because it is a projection, by (1.3.1a). Therefore the upper square is a pullback too. 

Let us see some examples of strong monads.

**Example 1.3.3.** Obviously the identity monad is strong. If we take the identity monad \(Id\) on any cartesian cartesian category \(E\) then a \(Id\)-operad is the same as a category internal to \(E\), and a non colored \(Id\)-operad is a monoid in \(E\). In particular if \(E = \text{Set}\) they are small categories and monoids, respectively.
Example 1.3.4. Let \((M, \mu, \eta)\) be the free monoid monad on the category \(\mathcal{E} = \text{Set}\). As mentioned above, a \(M\)-operad is the same thing as a nonsymmetric operad. Here is the full explicit description of \(M\). Let \(A\) be a set and \(a_0, \ldots, a_n \in A\), then

\[
MA = \bigsqcup_{n \in \mathbb{N}} A^n,
\]

\[
\eta_A(a_0) = (a_0),
\]

\[
\mu_A((a_1, \ldots, a_i), \ldots, (a_j, \ldots, a_n)) = (a_1, \ldots, a_n).
\] (1.3.4)

The operations of a non-symmetric operad \(Q\) will be pictured as

```
\[
\begin{array}{ccc}
  c_1 & c_2 & c_3 \\
  x & & \\
  d & & \\
\end{array}
\]
```

or as

```
\[
\begin{array}{ccc}
  x & & \\
  & & \\
  & & \\
\end{array}
\]
```

where \(c_1, c_2, c_3, d \in Q_0\) and \(x \in Q_1\). The free monoid monad is strong with the following strength:

\[
D_{A,B}: A \times MB \longrightarrow M(A \times B)
\]

\[
(a, (b_1, \ldots, b_n)) \longrightarrow ((a, b_1), \ldots, (a, b_n)).
\]

It is straightforward to check that the diagrams (1.3.2b) and (1.3.2a) are satisfied and clear that \(D_{A,B}\) is injective. This last feature is relevant because to define the \(T\)-construction, in Section 3, it will be necessary that \(D_{1,C}\) is a monomorphism.

Example 1.3.5. The free semigroup monad \(M'\) on \(\text{Set}\) is defined in the same way as the free monoid monad, except that in this case \(M' A = \bigsqcup_{n \geq 1} A^n\). This means that a \(M'\)-operad is a non-symmetric operad without nullary operations. The terminal \(M'\)-operad will be denoted \(\text{Ass}\), which is of course the reduced associative operad. Notice that \(M'\) is also a strong cartesian monad on \(\text{Grpd}\). In this sense the operad \(\text{Ass}\) can also be considered as an \(M'\)-operad in \(\text{Grpd}\), with discrete groupoid of objects and discrete groupoid of operations. The context will suffice to distinguish between \(\text{Set}\) and \(\text{Grpd}\), but in the main applications (Section 5) we will work over \(\text{Grpd}\).

Example 1.3.6. Let \(Y\) be a monoid. Denote by \(Y\) the monad on \(\text{Set}\) given by \(YA = Y \times A\) with unit and multiplication given by those of \(Y\). Then \(Y\) is strong with strength given by the associator of the cartesian product. Therefore in this case the strength is an isomorphism. The same holds if \(Y\) is a monoid in \(\text{Grpd}\) and \(Y\) is then a monad on \(\text{Grpd}\).
Example 1.3.7. Let \((S, \mu, \eta)\) be the free symmetric monoidal category monad on \(\text{Grpd}\). An \(S\)-operad is an operad internal to groupoids, so that it has a groupoid of colors and a groupoid of operations. Let \(A\) be a groupoid and \(\mathcal{G}_n\) the symmetric group on \(n\) elements. The monad \(S\) acts on \(A\) by

\[
SA = \bigsqcup_{n \in \mathbb{N}} A^n // \mathcal{G}_n,
\]

where \(//\) means homotopy quotient \([5, 15]\). Hence it is analogous to \(M\), but we add an arrow

\[
(a_1, \ldots, a_n) \xrightarrow{\sigma} (a_{\sigma 1}, \ldots, a_{\sigma n})
\]

for every element \(\sigma \in \mathcal{G}_n\). The multiplication and unit natural transformations are defined as in (1.3.4) for both objects and operations. Notice that any symmetric operad \(Q\) is in particular an \(S\)-operad, where the groupoid of objects \(Q_0\) is discrete and the groupoid \(Q_1\) has only the arrows coming from the permutations of its source sequence. In other words, a symmetric operad is an \(S\)-operad

\[
SQ_0 \xleftarrow{s} Q_1 \xrightarrow{t} Q_0
\]

such that \(Q_0\) is discrete and \(s\) is a discrete fibration. The operations of symmetric operads will be pictured as

\[
\begin{array}{ccc}
  c_1 & c_2 & c_3 \\
  \downarrow & & \downarrow \\
  x & & \text{or as} & \text{or as} \\
  d & & x & \text{as} \\
\end{array}
\]

where \(c_1, c_2, c_3, d \in Q_0\) and \(x \in Q_1\). The strength for \(S\) is defined the same way as for \(M\),

\[
D_{A,B} : A \times SB \longrightarrow S(A \times B)
\]

\[
(a, (b_1, \ldots, b_n)) \longrightarrow ((a, b_1), \ldots, (a, b_n)),
\]

and it is again a monomorphism, since it is injective both on objects and morphisms.

Observe that symmetric operads cannot be expressed as \(P\)-operads in \(\text{Set}\), since the actions of the symmetric groups have to be encoded necessarily as morphisms in \(Q_1\). Also, the only monad \(P\) one could attempt to use to define them is the free commutative monoid monad, but it is not cartesian.

Example 1.3.8. As for \(M\) and \(M'\), we can remove the empty sequence from \(S\) to get a monad \(S'\) on \(\text{Grpd}\) whose operads do not have nullary operations. We will denote by \(\text{Sym}\) the terminal \(S'\)-operad, which is the reduced commutative operad.
1.4 The two-sided bar construction for \( P \)-operads

The two-sided bar construction for operads is standard \[32\]. In this section we will introduce the construction in the more general setting of \( P \)-operads by using induced monads. Any \( P \)-operad \( Q \) defines a monad \((Q, \mu^Q, \eta^Q)\) on the slice category of \( E \) over \( Q_0 \)

\[
Q : E/Q_0 \longrightarrow E/Q_0,
\]
given by pullback and composition, as shown in the following diagram for an element \( X \overset{f}{\to} Q_0 \) of \( E/Q_0 \)

\[
\begin{array}{ccc}
QX & \xleftarrow{\sim} & X \\
\downarrow & & \downarrow \\
PQ_0 & \xleftarrow{s} & Q_0,
\end{array}
\]

The image of \( f \) is thus the red composite. The multiplication \( \mu^Q \) and the unit \( \eta^Q \) are defined by the following morphisms

\[
\begin{array}{ccccccc}
Q^2X & \xrightarrow{p^2f} & P^2Q_0 & \xleftarrow{PS} & PQ_1 & \xrightarrow{Pt} & PQ_0 & \xleftarrow{s} & Q_1 & \xrightarrow{t} & Q_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mu^QX & \xrightarrow{\mu_X} & P^2X & \xleftarrow{P^2f} & P^2Q_0 & \xleftarrow{\mu Q_0} & PQ_1 \times Q_0 & \xrightarrow{t} & Q_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
QX & \xrightarrow{Pf} & PQ_0 & \xleftarrow{s} & Q_1 & \xrightarrow{t} & Q_0,
\end{array}
\]

\[
\begin{array}{ccccccc}
X & \xrightarrow{f} & Q_0 & \xrightarrow{\eta Q_0} & Q_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
P_QX & \xrightarrow{Pf} & PQ_0 \xleftarrow{s} & Q_1 \xrightarrow{t} & Q_0.
\end{array}
\]

**Definition 1.4.1.** An algebra over the \( P \)-operad \( Q \) is an algebra over the monad \( Q \).

Notice that the category \( E/Q_0 \) has a terminal object, \( Q_0 \xrightarrow{1} Q_0 \), so that there is an algebra over \( Q \) given by the unique arrow \( q : Q_1 \to 1 \). Moreover, since \( E \) has a terminal object, the \( P \)-operad \( P : E/1 \to E/1 \) itself can be represented by the span

\[
P1 \xleftarrow{P1} 1,
\]

and is the terminal \( P \)-operad. Now, the terminal arrow \( u : Q_0 \to 1 \) induces, by postcomposition, a functor \( u_1 : E/Q_0 \to E/1 \). The diagram

\[
\begin{array}{ccccccc}
PQ_0 & \xleftarrow{Q_1} & Q_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
P1 & \xleftarrow{P1} & 1.
\end{array}
\]
represents a natural transformation \( u!Q \xrightarrow{\phi} Pu \) which satisfies that

\[
\begin{array}{cccccccc}
PQ_0^{p^2Q_0} & \xrightarrow{p^2} & PQ_1 & \xrightarrow{Q_1} & Q_0 \\
P_1 & \xrightarrow{p^2} & P_2 & \xrightarrow{p_1} & P_1 \\
\downarrow & & \downarrow & & \downarrow \\
P_1 & \xrightarrow{p_1} & P_1 & \xrightarrow{P_1} & 1 \\
\end{array}
\quad =
\begin{array}{cccccccc}
PQ_0^{p^2Q_0} & \xrightarrow{p^2} & PQ_1 & \xrightarrow{Q_1} & Q_0 \\
P_1 & \xrightarrow{p^2} & P_2 & \xrightarrow{p_1} & P_1 \\
\downarrow & & \downarrow & & \downarrow \\
P_1 & \xrightarrow{p_1} & P_1 & \xrightarrow{P_1} & 1 \\
\end{array}
\]  

\tag{1.4.5}

\[
\begin{array}{cccccccc}
Q_0 & \xrightarrow{Q_0} & Q_0 & \xrightarrow{Q_0} & Q_0 \\
\eta_{Q_0} & \downarrow & \eta & \downarrow & \eta \\
PQ_0 & \xrightarrow{Q_1} & Q_0 & \xrightarrow{1} & 1 \\
P_1 & \xrightarrow{Q_1} & P_1 & \xrightarrow{1} & 1 \\
\end{array}
\]  

\tag{1.4.6}

**Lemma 1.4.2.** The natural transformation \( \phi \) is cartesian.

**Proof.** We will describe the naturality squares of \( \phi \). Let \( H \) be a map in \( \mathcal{E}/Q_0 \), that is, a commutative triangle

\[
X \xrightarrow{h} Y \xleftarrow{g} Q_0.
\]

Consider the diagram

\[
PQ_0 \xrightarrow{Q_1} Q_0 \\
PQ_0 \xrightarrow{PQ_0} \xrightarrow{Q_1} Q_0 \\
P_1 \xrightarrow{PQ_0} \xrightarrow{Q_1} 1 \\
\]

From (1.4.1) it is clear that the pullback square on the right is precisely the definition of \( u!Qg \). From (1.4.1) and (1.4.4) we have that the square on the left is the naturality square for \( \phi \) at \( H \), and moreover that \( \phi_X \) and \( \phi_Y \) are projections. But \( PuH = Ph \) and \( Pg \circ Ph = Pf \), so that the composite square is precisely the definition of \( u!Qf \), which is a pullback. As a consequence, the naturality square is a pullback too.

Given a \( P \)-operad \( Q \), we define its **two-sided bar construction** \([23,32,46]\)

\[
\mathcal{B}Q : \Delta^{op} \rightarrow \mathcal{E}
\]

as the two-sided bar construction of \( Q \), \( \phi \) and the terminal algebra 1. This means that the space of \( n \)-simplicies \( \mathcal{B}_n Q \) is given by

\[
P_n Q, 1,
\]
the inner face maps are given by the monad multiplication $\mu^Q$, the bottom face map is given by $c : Q1 \to 1$ and the top face maps are given by $\phi$ and $\mu$. Similarly, the degeneracy maps are given by $\eta^Q$. Diagrams (1.4.5) and (1.4.6) and the monad axioms for $P$ and $Q$ guarantee that the simplicial identities are satisfied.

In practice, the bar construction of $Q$ will be simply

$$\xymatrix{PQ_0 & PQ_1 & PQ_2 & PQ_3 & \cdots,} \tag{1.4.7}$$

where

(i) $Q_2 := PQ_1 \times Q_1$ and $Q_3 := P^2Q_1 \times PQ_1 \times Q_1$, etc.;

(ii) the bottom face maps $d_0$ are induced by $t$;

(iii) the top face maps $d_n$ are induced by $s$ and $\mu$;

(iv) the inner face maps are induced by $m$ and $\mu$, and

(v) the degeneracy maps are induced by $e$ and $\eta$.

Henceforth we will indiscriminately use this simplicial notation. Let us see some examples.

**Example 1.4.3.** Let $C$ be a small category. Hence $C$ is a ld-operad in $\text{Set}$. Then $BC$ is the nerve of $C$. Moreover, we can consider $C$ as a category internal to $\text{Grpd}$ whose groupoid of objects has as morphisms the isomorphisms of $C$, and whose groupoid of arrows has as morphisms the isomorphisms of the arrow category of $C$. In this case $BC$ is the fat nerve of $C$, whose groupoid of $n$-simplices is the groupoid $\text{Map}(\Delta[n], C)$. In the theory of incidence coalgebras, this is often more interesting than the ordinary nerve, cf. [9, 17, 18].

**Example 1.4.4.** If $Q$ is a symmetric operad, as in Example 1.3.7, then $BQ$ is the usual operadic two-sided bar construction. Its $n$-simplices have as objects forests of $n$-level $Q$-trees, and as morphisms permutations at each level. For example, the following picture...
is an object of $PQ_2$ with $(2! \cdot 2!^2 \cdot 3!^4) \cdot (2!) \cdot (2! \cdot 2!^2)$ automorphisms.

The following result is a reformulation of [46, Proposition 4.4.1] and [23, Proposition 3.3] in the context of $P$-operads.

**Proposition 1.4.5.** The simplicial object $BQ$ is a strict category object.

**Proof.** We have to check that the squares

$$
\begin{array}{ccc}
B_{n+2}Q & \xrightarrow{d_0} & B_{n+1}Q \\
n_{n+2} & \downarrow & \downarrow d_{n+1} \\
B_{n+1}Q & \xrightarrow{d_0} & B_nQ.
\end{array}
$$

(1.4.8)

are pullbacks for $n \geq 0$. We will show the case $n = 0$, the rest are similar. The square is given by

$$
\begin{array}{ccc}
P_{u_1}Q_1 & \xrightarrow{\mu^P_{u_1}} & P_{u_1}Q_1 \\
\sigma & \downarrow & \downarrow \sigma \\
P_{u_1}Q_1 & \xrightarrow{\mu^P_{u_1}} & P_{u_1}Q_1
\end{array}
$$

(1.4.9)

The bottom square is cartesian because it is a naturality square for $\mu^P$, and $P$ is a cartesian monad. The top square is $P$ applied to a naturality square of $\phi$, which is cartesian, by Lemma 1.4.2. Since $P$ preserves pullbacks, the square is cartesian. \qed

This allows to obtain the following result, in the special case where $E = \text{Grpd}$.

**Proposition 1.4.6.** Let $P : \text{Grpd} \to \text{Grpd}$ be a cartesian monad that preserves fibrations. Let $Q$ be a $P$-operad such that $Q_0$ is a discrete groupoid. Then the simplicial groupoid $BQ$ is a Segal groupoid.

**Proof.** It is enough to see that the strict pullbacks 1.4.8 are also homotopy pullbacks. For $n = 0$, notice that $P_{u_1}Q_1 \xrightarrow{\mu^P_{u_1}} P_{u_1}Q_1$ is precisely the map $PQ_1 \xrightarrow{\mu Q_1} PQ_0$. But since $Q_0$ is discrete $m$ is a fibration, which means that $Pm$ is a fibration, because $P$ preserves fibrations. This implies that the square is also a homotopy pullback. Moreover, since pullbacks preserve fibrations the map $P_{u_1}QQ_1 \xrightarrow{\mu^P_{u_1}} P_{u_1}Q_1$ is again a fibration. The same argument then implies that the square for $n = 1$ is also a homotopy pullback, and so on. \qed

Suppose now that $R : E \to E$ is another cartesian monad and that there is a cartesian monad map $P \xrightarrow{\psi} R$. Then we can take the bar construction over $R$

$$
B^RQ : \Delta^{\text{op}} \longrightarrow E
$$
whose \( n \)-simplices are given by

\[
R_u Q^n 1 \quad \text{(or \( RQ^n \))}. \]

In this case all the face maps coincide with the previous ones except the top face map, which is given by

\[
R_u Q^{n+1} \xrightarrow{R(\phi_Q^n)} RP_u Q^n 1 \xrightarrow{R(\psi_u Q^n)} RRU_u Q^n 1 \xrightarrow{\mu_{u Q^n}} R_u Q^n 1. \]

Since \( \psi \) is cartesian the simplicial object \( B^R \) is also a strict category object. Moreover, if \( R \) preserves fibrations, it is a Segal groupoid, for the same reason as \( BQ \) in Proposition 1.4.6. The main example of this bar construction that we will use comes from the natural transformation \( M' \Rightarrow S' \), as in [23].

2 Segal groupoids and incidence coalgebras

Throughout this section, pullbacks and fibres of groupoids refer to homotopy pullbacks and homotopy fibres. A brief introduction to the homotopy approach to groupoids in combinatorics can be found in [15, §3].

2.1 Segal groupoids

A simplicial groupoid \( X: \Delta^{op} \rightarrow \text{Grpd} \) is a Segal space [17, §2.9, Lemma 2.10] if the following square is a pullback for all \( n > 0 \):

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_0} & X_n \\
\downarrow d_{n+1} & & \downarrow d_n \\
X_n & \xrightarrow{d_0} & X_{n-1}.
\end{array}
\]  

(2.1.1)

Segal spaces arise prominently through the fat nerve construction: the fat nerve of a category \( C \) is the simplicial groupoid \( X = NC \) with \( X_n = \text{Fun}([n], C)^\simeq \), the groupoid of functors \( [n] \rightarrow C \). In this case the pullbacks above are strict, so that all the simplices are strictly determined by \( X_0 \) and \( X_1 \), respectively the objects and arrows of \( C \), and the inner face maps are given by composition of arrows in \( C \). In the general case, \( X_n \) is determined from \( X_0 \) and \( X_1 \) only up to equivalence, but one may still think of it as a “category” object whose composition is defined only up to equivalence.

**Remark 2.1.1.** Despite the Segal conditions (2.1.1) require the squares to be homotopy pullbacks, if the top or bottom face maps are fibrations, the ordinary pullbacks are also homotopy pullbacks. In the present work, homotopy pullbacks mostly arise in this way.
2.2 Incidence coalgebras

Let $X$ be a simplicial groupoid. The spans

$$
X_1 \leftarrow d_1 \ X_2 \xrightarrow{\begin{pmatrix} d_2 & d_0 \end{pmatrix}} X_1 \times X_1, \quad X_1 \xleftarrow{s_0} X_0 \rightarrow 1,
$$

define two functors

$$
\Delta: \text{Grpd}_{/X_1} \rightarrow \text{Grpd}_{/X_1 \times X_1}, \quad \epsilon: \text{Grpd}_{/X_1} \rightarrow \text{Grpd}
$$

Recall that upperstar is homotopy pullback and lowershriek is postcomposition. This is the general way in which spans interpret homotopy linear algebra [16].

Segal spaces are a particular case of decomposition spaces [17, Proposition 3.7], simplicial groupoids with the property that the functor $\Delta$ is coassociative with the functor $\epsilon$ as counit (up to homotopy). In this case $\Delta$ and $\epsilon$ endow $\text{Grpd}_{/X_1}$ with a coalgebra structure [17, §5] called the incidence coalgebra of $X$. Note that in the special case where $X$ is the nerve of a poset, this construction becomes the classical incidence coalgebra construction after taking cardinality, as we shall do shortly.

The morphisms of decomposition spaces that induce coalgebra homomorphisms are the so-called CULF functors [17 §4], standing for conservative and unique-lifting-of-factorisations. A Segal space $X$ is CULF monoidal if it is a monoid object in the monoidal category $(\text{Dcmp}_{\text{CULF}}, \times, 1)$ of decomposition spaces and CULF functors [17, §9]. More concretely, it is CULF monoidal if there is a product $X_n \times X_n \rightarrow X_n$ for each $n$, compatible with the degeneracy and face maps, and such that for all $n$ the squares

$$
\begin{array}{ccc}
X_n \times X_n & \xrightarrow{g \times g} & X_1 \times X_1 \\
\downarrow & & \downarrow \\
X_n & \xrightarrow{g} & X_1
\end{array}
$$

are pullbacks [17 §4]. Here $g$ is induced by the unique endpoint-preserving map $[1] \rightarrow [n]$. For example the fat nerve of a monoidal extensive category is a CULF monoidal Segal space. Recall that a category $\mathcal{C}$ is monoidal extensive if it is monoidal $(\mathcal{C}, +, 0)$ and the natural functors $\mathcal{C}/A \times \mathcal{C}/B \rightarrow \mathcal{C}/A + B$ and $\mathcal{C}/0 \rightarrow 1$ are equivalences.

If $X$ is CULF monoidal then the resulting coalgebra is in fact a bialgebra [17 §9], with product given by

$$
\circ: \text{Grpd}_{/X_1} \otimes \text{Grpd}_{/X_1} \xrightarrow{\sim} \text{Grpd}_{/X_1 \times X_1} \xrightarrow{+1} \text{Grpd}_{/X_1}
$$

Briefly, a product in $X_n$ compatible with the simplicial structure endows $X$ with a product, but in order to be compatible with the coproduct it has to satisfy the diagram (2.2.1) (i.e. it has to be a CULF functor).

20
2.3 Homotopy cardinality

A groupoid \( X \) is \textit{finite} if \( \pi_0(X) \) is a finite set and \( \pi_1(x) = \text{Aut}(x) \) is a finite group for every point \( x \). If only the latter is satisfied then it is called \textit{locally finite}. A morphism of groupoids is called finite when all its fibres are finite. The \textit{homotopy cardinality} [5], [16, §3] of a finite groupoid \( X \) is defined as

\[
|X| := \sum_{x\in \pi_0X} \frac{1}{|\text{Aut}(x)|} \in \mathbb{Q},
\]

and the homotopy cardinality of a finite map of groupoids \( A \xrightarrow{p} B \) is

\[
|p| := \sum_{b\in \pi_0B} \frac{|A_b|}{|\text{Aut}(b)|} \delta_b,
\]

in \( \mathbb{Q}_{\pi_0B} \), the vector space spanned by \( \pi_0B \). In this sum, \( A_b \) is the homotopy fibre and \( \delta_b \) is a formal symbol representing the isomorphism class of \( b \). A simple computation shows that \( |1 \xrightarrow{\gamma} B| = \delta_b \).

A Segal space \( X \) is \textit{locally finite} [18, §7] if \( X_1 \) is a locally finite groupoid and both \( s_0: X_0 \to X_1 \) and \( d_1: X_2 \to X_1 \) are finite maps. In this case one can take homotopy cardinality to get a comultiplication

\[
\Delta: \mathbb{Q}_{\pi_0X_1} \to \mathbb{Q}_{\pi_0X_1} \otimes \mathbb{Q}_{\pi_0X_1},
\]

\[
|S \xrightarrow{s} X_1| \mapsto |(d_2, d_0) \circ d_1^*(s)|
\]

and similarly for \( \epsilon \) (cf. [18, §7]). Moreover, if \( X \) is CULF monoidal then \( \mathbb{Q}_{\pi_0X_1} \) acquires a bialgebra structure with the product \( \cdot = |\circ| \). In particular, if we denote by + the monoidal product in \( X \), then \( \delta_a \cdot \delta_b = \delta_{a+b} \) for any \( |1 \xrightarrow{\gamma} X_1| \) and \( |1 \xrightarrow{\gamma} X_1| \). The following result gives a closed formula for the computation of the comultiplication when \( X \) is a Segal space.

**Lemma 2.3.1** ([9, 19]). Let \( X \) be a Segal space. Then for \( f \) in \( X_1 \) we have

\[
\Delta(\delta_f) = \sum_{b\in \pi_0X_1} \sum_{a\in \pi_0X_1} \frac{|\text{Iso}(d_0a, d_1b)\rangle_f|}{|\text{Aut}(b)||\text{Aut}(a)|} \delta_a \otimes \delta_b,
\]

where \( \text{Iso}(d_0a, d_1b) \) is the set of morphisms from \( d_0a \) to \( d_1b \) and \( \text{Iso}(d_0a, d_1b)\rangle_f \) is its homotopy fibre along \( d_1 \).

3 The \( \mathcal{T} \)-construction

Throughout this section \((\mathcal{P}, \mu, \eta)\) is a cartesian strong monad on a cartesian category \( \mathcal{E} \), and category means a category internal to \( \mathcal{E} \). As mentioned in the introduction, the
\( T \)-construction consists of two construction, one from internal categories to \( P \)-operads and another from \( P \)-operads to categories. With the purpose of reducing the diagrams and fibre products, we will use the following notation for the endofunctors and natural transformations featuring in this section,

\[
\begin{align*}
L &: \mathcal{E} \longrightarrow \mathcal{E} \\
F &: \text{Id} \quad > L \\
D &: L \longrightarrow P \\
D_A &: A \times P1 \\nD \quad &: \quad L \quad P \quad R \\
R &: L \longrightarrow \text{Id} \\
R_A &: A \times P1 \\n\end{align*}
\]

\( F_A : A \times 1 \xrightarrow{id \times \eta} LA, \)

Observe that \( L \) is cartesian as a functor. Also, notice that \( R \) and \( F \) are cartesian natural transformations. Finally, by monomorphism we will refer to the 1-categorical notion. In the case of most interest where \( \mathcal{E} \) is \textbf{Set} or \textbf{Grpd}, this means injective on objects and injective on arrows.

### 3.1 From categories to \( P \)-operads

Let \( C \) be a category such that \( D_{C_0} : P1 \times C_0 \rightarrow PC_0 \) is a monomorphism. It is convenient in this section to adopt a simplicial nomenclature. Hence \( C \) will be represented by the span

\[
\begin{array}{ccc}
C_1 & \xleftarrow{\partial_0} & C_0 \\
\downarrow{d_0} & & \downarrow{\partial_1} \\
C_0 & & C_0 \\
\end{array}
\]

C_1 \times_{C_0} C_1 =: C_2 \xrightarrow{d_1} C_1

with the only inconvenience that some of the face maps share their names. Notice that we still call \( e \) the degeneracy map \( s_0 \). We will now construct a \( P \)-operad \( T_PC \) from the category \( C \). To keep notation short, the simplicial nomenclature for \( T_PC \) will be \( \tilde{C}_i \) for the simplices and \( \tilde{d}_i \) for the face maps. The span defining the objects and operations of \( T_PC \) is given by the pullback

\[
\begin{array}{ccc}
\tilde{C}_1 & \xleftarrow{\tilde{d}_1} & \tilde{C}_0 \\
\downarrow{i_1} & & \downarrow{\tilde{d}_0} \\
PC_1 & & LC_0 \\
\downarrow{p_{d_1}} & & \downarrow{D_{C_0}} \\
PC_0 & & PC_0 \\
\downarrow{D_{C_0}} & & \downarrow{R_{C_0}} \\
C_0 & & C_0.
\end{array}
\]

Observe that \( \tilde{C}_0 = C_0 \), so that \( T_PC \) has the same objects as \( C \). Besides, the morphism \( i_1 \) is a monomorphism, since monomorphisms are preserved by pullbacks and \( D_{C_0} \) is a monomorphism.
To define composition we need to specify a map $\tilde{C}_2 \xrightarrow{\tilde{d}_1} \tilde{C}_1$, where $\tilde{C}_2 := \mathcal{P}C_1 \times \mathcal{C}_1$, satisfying the axioms of Appendix [A.1]. However, to describe it we have to express $\tilde{C}_2$ in a way we can naturally use composition in the original category $C_2 \xrightarrow{d_1} C_1$. The following diagram represents an isomorphism

$$\tilde{C}_2 \cong \mathcal{P}^2C_1 \times \mathcal{P} \mathcal{C}_1 \times \mathcal{C}_0 \times \mathcal{P}^21 =: \tilde{C}_2'$$

It is clear that all the squares in (3.1.2) commute. Moreover, the square (A) is cartesian because $R$ and $P$ are cartesian, and the square (B) is the same as (1.3.3) of Lemma 1.3.2.

**Definition 3.1.1.** The composition of $\mathcal{T}_P C$ is given by the following arrow $\tilde{C}_2' \xrightarrow{\tilde{d}_1} \tilde{C}_1$,

$$\tilde{C}_2' \xrightarrow{\tilde{d}_1} \tilde{C}_1$$

It is clear that the diagram commutes: (A) is $P$ applied to a naturality square of $D$; (B) is the definition of $D^2$; (C) and (D) are $P^2$ applied to axioms [A.1.1a] and [A.1.1b] for composition in $C$; (E) and (F) are naturality squares of $\mu$, and (G) is again axiom [1.3.2b] for strong monads. The remaining squares are trivial.

Notice that from this definition it is clear that $\tilde{d}_1$ satisfies axioms [A.2.1a] and [A.2.1b]. Furthermore, there is a map

$$\tilde{C}_2' \xrightarrow{\tilde{d}_1} \mathcal{P}^2C_2,$$

given by the diagram

$$\mathcal{P}^2C_0 \xleftarrow{\mathcal{P}^2d_1} \mathcal{P}^2C_1 \xrightarrow{\mathcal{P}d_0} \mathcal{P}^2C_0 \xleftarrow{\mathcal{P} \mathcal{C}_1} \mathcal{P}^2C_1 \xrightarrow{\mathcal{P} \mathcal{C}_0} \mathcal{C}_0 \times \mathcal{P}^21 \xrightarrow{\mathcal{P} \mathcal{C}_0} \mathcal{C}_0 \xleftarrow{\mathcal{P} \mathcal{C}_0} \mathcal{P}^2C_0 \xrightarrow{\mathcal{P} \mathcal{C}_1} \mathcal{P}^2C_1 \xrightarrow{\mathcal{P} \mathcal{C}_0} \mathcal{C}_0 \times \mathcal{P}^21 \xrightarrow{\mathcal{P} \mathcal{C}_0} \mathcal{C}_0.$$
which clearly makes the square

\[
\begin{array}{ccc}
\tilde{C}_2 & \xrightarrow{i_2} & P^2C_2 \\
\downarrow{\tilde{d}_1} & & \downarrow{p^2d_1} \\
\tilde{C}_1 & \xleftarrow{i_1} & PC_1,
\end{array}
\]

and therefore also the square

\[
\begin{array}{ccc}
\tilde{C}_2 & \xrightarrow{i_2} & P^2C_2 \\
\downarrow{\tilde{d}_1} & & \downarrow{\mu C_1} \\
\tilde{C}_1 & \xleftarrow{i_1} & PC_1,
\end{array}
\]

commute, for the corresponding arrow \(i_2\). This says, roughly speaking, that composition in \(T_P C\) is “the same” as composition in \(P^2C\), as it is clear in most of the examples.

We have to check now that composition is associative (A.2.3). We state first the following lemma. We omit its proof since it is long and unenlightening; it can be found in [10].

**Lemma 3.1.2.** There is a map \(\tilde{C}_3 \xrightarrow{i_3} P^3C_3\) such that the following diagrams commute

\[
\begin{array}{ccc}
\tilde{C}_3 & \xrightarrow{i_3} & P^3C_3 \\
\downarrow{\tilde{d}_1} & & \downarrow{p^3d_1} \\
\tilde{C}_2 & \xrightarrow{i_2} & P^2C_2 \\
\downarrow{\tilde{d}_2} & & \downarrow{p^2d_1} \\
\tilde{C}_1 & \xrightarrow{i_1} & PC_1,
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{C}_3 & \xrightarrow{i_3} & P^3C_3 \\
\downarrow{\tilde{d}_2} & & \downarrow{\mu C_2} \\
\tilde{C}_2 & \xrightarrow{i_2} & P^2C_2, \\
\downarrow{\tilde{d}_1} & & \downarrow{\mu C_2} \\
\tilde{C}_1 & \xrightarrow{i_1} & PC_1,
\end{array}
\]

(3.1.5)

**Proposition 3.1.3.** Composition is associative.

**Proof.** In view of Lemma 3.1.2 there is a diagram

\[
\begin{array}{ccc}
\tilde{C}_3 & \xrightarrow{i_3} & P^3C_3 \\
\downarrow{\tilde{d}_2} & & \downarrow{p^3d_2} \\
\tilde{C}_2 & \xrightarrow{i_2} & P^2C_2, \\
\downarrow{\tilde{d}_1} & & \downarrow{\mu C_2} \\
\tilde{C}_1 & \xrightarrow{i_1} & PC_1,
\end{array}
\]

where

\[
\begin{array}{ccc}
P^3C_3 & \xrightarrow{p^3d_1} & P^3C_2 \\
\downarrow{(A)} & & \downarrow{(B)} \\
P^3C_2 & \xrightarrow{p^3d_1} & P^3C_1 \\
\downarrow{(C)} & & \downarrow{(D)} \\
P^2C_2 & \xrightarrow{p^2d_1} & PC_1 \\
\downarrow{i_2} & & \downarrow{\mu C_1} \\
\tilde{C}_2 & \xrightarrow{i_1} & \tilde{C}_1,
\end{array}
\]
where the four trapeziums are diagrams (3.1.4) and (3.1.5) of Lemma 3.1.2. The inner squares are the following: (A) is \( P^3 \) applied to associativity of \( C \); (B) is \( P \) applied to naturality of \( \mu \) at \( d_1 \); (C) is naturality of \( \mu \) at \( Pd_1 \) and (D) is the associativity law of \( \mu \). Since \( i_1 \) is a monomorphism (3.1.1) and all the inner diagrams commute, so does the outer square, as we wanted to see.

The unit morphism of \( T_P C \) is easier to obtain than composition. Recall that the unit is a morphism \( \tilde{\varepsilon} : C_0 \to \tilde{C}_1 \) such that this diagram (A.2.2) commutes,

\[
\begin{array}{ccc}
C_0 & \xrightarrow{\eta_{C_0}} & C_1 \\
\downarrow{\eta_{C_0}} & \searrow{\tilde{\varepsilon}} & \downarrow{\text{id}} \\
PC_0 & \xrightarrow{P_{\tilde{\varepsilon}}} & PC_0
\end{array}
\]

\textbf{Definition 3.1.4.} The unit of \( T_P C \) is given by the following arrow:

\[
\begin{array}{ccccccccc}
C_0 & \xleftarrow{\eta_{C_0}} & PC_0 & \xrightarrow{\eta_{C_0}} & C_0 & \xrightarrow{\eta_{C_0}} & C_0 & \xrightarrow{\eta_{C_0}} & C_0 \\
\downarrow{\tilde{\varepsilon}} & \downarrow{P_{\varepsilon}} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} \\
\tilde{C}_1 & \xleftarrow{P_{d_1}} & PC_1 & \xleftarrow{PC_0} & PC_1 & \xleftarrow{PC_0} & PC_1 & \xleftarrow{PC_0} & PC_1 \\
\downarrow{PC_0} & \downarrow{P_{\eta_{C_0}}} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} & \downarrow{PC_0} \\
\tilde{C}_1 & \xleftarrow{F_{C_0}} & C_1 & \xleftarrow{C_1} & C_1 & \xleftarrow{C_1} & C_1 & \xleftarrow{C_1} & C_1
\end{array}
\] (3.1.6)

It is clear that all the diagrams commute: (A) and (B) come from \( P \) applied to (A.1.2a) and (A.1.2b), this is \( d_1 \circ \varepsilon = \text{id} = d_0 \circ \varepsilon \); (C) is the compatibility between \( D \) and \( \eta \) (1.3.2a), and (D) is obvious from the definitions of \( R \) and \( F \).

We have to verify now that composition with the unit morphism is the identity (A.2.4). To prove it we will follow the same strategy as for associativity. That is, we will project the diagrams into diagrams in the original category \( C \) containing the corresponding unit axioms. Again, the proof of the following lemma can be found in [10]. Recall first that

\[
C_2 := C_1 \times_{C_0} C_1 \quad \text{and} \quad \tilde{C}_2 := P\tilde{C}_1 \times_{\tilde{C}_0} \tilde{C}_1.
\]

\textbf{Lemma 3.1.5.} We have commutative squares

\[
\begin{array}{ccc}
PC_0 \times \tilde{C}_1 & \xrightarrow{i^*} & PC_0 \times PC_1 \\
\downarrow{P_{\varepsilon} \times \text{id}} & \downarrow{P_{\varepsilon} \times \text{id}} & \downarrow{PC_2} \\
\tilde{C}_2 & \xrightarrow{i_2} & \text{P}^2 C_2
\end{array}
\] (3.1.7a)

\[
\begin{array}{ccc}
\tilde{C}_1 \times C_0 & \xrightarrow{i^*} & PC_1 \times PC_0 \\
\downarrow{\eta_{\tilde{C}_1} \times \eta_{C_0} \tilde{\varepsilon}} & \downarrow{\text{id} \times \text{id} \varepsilon} & \downarrow{PC_2} \\
\tilde{C}_2 & \xrightarrow{i_2} & \text{P}^2 C_2
\end{array}
\] (3.1.7b)
where \( i_1^l \) and \( i_1^r \) are the morphisms corresponding to \( i_1 \).

**Proposition 3.1.6.** The unit morphism \( \tilde{e} \) of \( \mathcal{T}_P C \) satisfies the left and right composition axioms (A.2.4).

**Proof.** For the left composition (A.2.4a), the required commutative triangle is the outline of the diagram

\[
\begin{array}{ccc}
\mathcal{P}C_0 \times \mathcal{C}_1 & \xrightarrow{P \times id} & \mathcal{C}_2 \\
\downarrow \mathcal{P}C_1 & & \downarrow \mathcal{C}_2 \\
\mathcal{P}C_0 & \xrightarrow{id} & \mathcal{C}_0 \\
\end{array}
\]

We have that diagram (A) commutes by definition of \( i_1^l \); (B) is precisely (3.1.7a) of Lemma 3.1.5; (C) is \( P \) applied to the left composition with the unit axiom in the category \( C \); (D) is naturality of \( \mu \) at \( d_1 \); (E) is \( P \) of the unit axiom of \( P \) applied to \( C_2 \), and (F) is the same as (3.1.4). Since \( i_1 \) is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see.

For the right composition (A.2.4b), the required commutative triangle is the outline of the diagram

\[
\begin{array}{ccc}
\mathcal{C}_1 \times \mathcal{C}_0 & \xrightarrow{\eta \times \eta C_0 \tilde{e}} & \mathcal{C}_2 \\
\downarrow \mathcal{P}C_1 \times \mathcal{P}C_0 & & \downarrow \mathcal{P}C_0 \times \mathcal{P}C_1 \\
\mathcal{P}C_0 & \xrightarrow{id \times \mathcal{P}e} & \mathcal{P}C_2 \\
\end{array}
\]

(3.1.9)
We have that diagram (A) commutes by definition of \( i_1 \), (B) is precisely (3.1.7a) of Lemma 3.1.5, (C) is \( P \) applied to the right composition with the unit axiom in the category \( C \); (D) is again naturality of \( \mu \) at \( d_1 \), (E) is the unit axiom of \( P \) applied to \( PC_2 \) and (F) is the same as (3.1.4), as before. Since \( i_1 \) is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see.

The last thing to check is that the construction is functorial. First of all we have to specify how the construction acts on morphisms. Let \( C \) and \( C' \) be two categories and \( C \xrightarrow{f} B \) a functor, that is a diagram

\[
\begin{array}{ccc}
C_0 & \xrightarrow{d_1} & C_1 & \xrightarrow{d_0} & C_0 \\
\downarrow{f_0} & & \downarrow{f_1} & & \downarrow{f_0} \\
B_0 & \xleftarrow{d_1} & B_1 & \xrightarrow{d_0} & B_0
\end{array}
\]

satisfying the commutative squares of 1.1.1. Then \( \mathcal{T}_P f \) is the morphism given by

\[
\begin{array}{cccc}
\mathcal{T}_P C & \xleftarrow{Pd_1} & PC_1 & \xrightarrow{Pd_0} & PC_0 \\
\downarrow{Pf_0} & & \downarrow{Pf_1} & & \downarrow{Pf_0} \\
\mathcal{T}_P B & \xleftarrow{Pd_1} & PB_1 & \xrightarrow{Pd_0} & PB_0
\end{array}
\]

\[
\begin{array}{cccc}
PC_0 & \xleftarrow{DC_0} & LC_0 & \xrightarrow{RC_0} & C_0 \\
\downarrow{Lf_0} & & \downarrow{Lf_0} & & \downarrow{f_0} \\
PB_0 & \xleftarrow{DB_0} & LB_0 & \xrightarrow{RB_0} & B_0
\end{array}
\]

It is a bit tedious but not difficult to see that \( \tilde{f} \) satisfies again the commutative squares of 1.1.1 [10]. Moreover, given another morphism \( B \xrightarrow{g} A \) it is clear that \( \mathcal{T}_P (g \circ f) = \mathcal{T}_P g \circ \mathcal{T}_P f \), just because of the functoriality of \( P \) and \( L \).

Since the construction is functorial, if the strength \( D_A \) is a monomorphism for every object \( A \in \mathcal{E} \) then \( \mathcal{T}_P \) is in fact a functor from categories internal to \( \mathcal{E} \) to \( P \)-operads.

### 3.2 From \( P \)-operads to categories

This construction has a similar structure as the construction above, so we will follow the same steps. Let \( Q \) be a \( P \)-operad,

\[
\begin{array}{ccc}
PQ_0 & \xleftarrow{d_0} & Q_0 \\
\downarrow{d_1} & & \downarrow{d_1} \\
Q_1 & \xrightarrow{d_1} & Q_1
\end{array}
\]

\[
PQ_1 \times_{PQ_0} := Q_2 \xrightarrow{d_1} Q_1
\]

and assume that \( D_{Q_0} : P1 \times Q \rightarrow PQ \) is a monomorphism. We will construct a category \( \mathcal{T}^P Q \) from the \( P \)-operad \( Q \). In this case, the simplicial nomenclature for \( \mathcal{T}^P Q \) will be \( \overline{Q}_i \) for the simplices and \( \overline{d}_i \) for the face maps. The following pullback defines the objects
and arrows of $\mathcal{T}^p Q$:

\[
\begin{array}{c}
\begin{tikzcd}
\bar{C}_1 \\
\downarrow_{\mathcal{L} C_0} \\
\downarrow_{D C_0} \\
C_0 \\
\end{tikzcd}
\end{array}
\]  \quad (3.2.1)

Observe that $\bar{Q}_0 = Q_0$, so that again $\mathcal{T}^p Q$ has the same objects as $Q$. Besides, the morphism $j_1$ is a monomorphism, since monomorphisms are preserved by pullbacks and $D_{Q_0}$ is a monomorphism.

To define composition we need to define a map $\bar{Q}_2 \xrightarrow{\bar{d}_1} \bar{Q}_1$, where $\bar{Q}_2 := \bar{Q}_1 \times_{Q_0} \bar{Q}_1$, satisfying the axioms of Appendix A.2. However, to specify this map we need to express it in a way we can naturally use composition in the original $\mathcal{P}$-operad $Q_2 \xrightarrow{d_1} Q_1$. The following diagram represents an isomorphism

\[
\begin{array}{c}
\begin{tikzcd}
\bar{Q}_2 \\
\downarrow_{\mathcal{L} Q_2} \\
\downarrow_{\mathcal{L} Q_1} \\
Q_1 \\
\end{tikzcd}
\end{array}
\]

It is clear that all the squares in (3.2.2) commute. Moreover, the squares (A) and (B) are cartesian because so is $R$.

**Definition 3.2.1.** The composition of $\bar{Q}$ is given by the following arrow $\bar{Q}_2 \xrightarrow{\bar{d}_1} \bar{Q}_1$:

\[
\begin{array}{c}
\begin{tikzcd}
\bar{Q}_2 \\
\downarrow_{\mathcal{L} Q_2} \\
\downarrow_{\mathcal{L} Q_1} \\
Q_1 \\
\end{tikzcd}
\end{array}
\]

Let us see that all the diagrams commute: (A) is a combination of naturality of $D$ applied to $D_{Q_0}$ and axiom (1.3.1b) concerning consecutive applications of the strength, (B) is naturality of $D$ at $d_1$, (C) is axiom (1.3.2b) for strong monads and (D) and (E) are respectively axioms (A.2.1a) and (A.2.1b) for composition in $Q$. The remaining diagrams are clear.
Notice that from this definition it is clear that $\overline{d}_1$ satisfies axioms (A.1.1a) and (A.1.1b). Furthermore, there is a morphism

$$\overline{Q}_2 \xrightarrow{j'_2} Q_2,$$

given by the diagram

$$Q_0 \leftarrow \text{L}^2Q_0 \xrightarrow{\text{L}DQ_0} \text{LP}Q_0 \xleftarrow{\text{L}d_1} \text{L}Q_1 \rightarrow \text{P}Q_0 \xleftarrow{d_1} \text{Q}_1 \rightarrow \text{d}_0 \rightarrow Q_0$$

$$\text{P}^2Q_0 \xleftarrow{\text{P}d_1} \text{P}Q_1 \rightarrow \text{P}Q_0 \xleftarrow{d_1} \text{Q}_1 \rightarrow \text{d}_0 \rightarrow Q_0,$$

which clearly makes the square

$$\overline{Q}_2 \xrightarrow{j'_2} Q_2$$

$$\overline{Q}_1 \xrightarrow{j_1} Q_1,$$

and therefore also the square

$$\overline{Q}_2 \xrightarrow{j_2} Q_2$$

$$\overline{Q}_1 \xrightarrow{j_1} Q_1,$$

commute, for the corresponding $j_2$. This says, roughly speaking, that composition in $\overline{Q}$ is “the same” as composition in $Q$, as is clear in most of the examples.

We have to check now that composition is associative (A.1.3). As in the previous subsection, the proof of the following lemma is omitted; it can be read in [10].

**Lemma 3.2.2.** There is a morphism $\overline{Q}_3 \xrightarrow{j_3} Q_3$ such that the following diagrams commute

$$\overline{Q}_2 \xrightarrow{j_2} Q_2$$

$$\overline{Q}_1 \xrightarrow{j_1} Q_1,$$

(3.2.4)

$$\overline{Q}_3 \xrightarrow{j_3} Q_3$$

(3.2.5)

**Proposition 3.2.3.** Composition is associative.

**Proof.** In view of Lemma 3.2.2 there is a diagram
The four trapeziums are the commutative diagrams (3.2.4) and (3.2.5) of Lemma 3.2.2 respectively, and the inner square is associativity of composition in \( C \) (A.2.3). Since \( j_1 \) is a monomorphism and all the inner diagrams commute, so does the outer square, as we wanted to see. \( \square \)

The unit morphism of the new category is easier to obtain than composition. Recall that the unit is a morphism \( e : Q_0 \to Q_1 \) such that this diagram (A.1.2) commutes

\[
\begin{array}{ccc}
Q_0 & \xrightarrow{id} & Q_0 \\
\downarrow{id} & & \downarrow{id} \\
Q_0 & \xleftarrow{\eta_1} & Q_1 & \xrightarrow{\epsilon} & Q_0
\end{array}
\]

\textbf{Definition 3.2.4.} The unit of \( \overline{Q} \) is given by the following arrow:

\[
\begin{array}{c}
Q_0 \\
\downarrow{id} \\
Q_0
\end{array}
\xleftarrow{\eta_1}
\begin{array}{c}
Q_0 \\
\downarrow{id} \\
Q_0
\end{array}
\xrightarrow{\epsilon}
\begin{array}{c}
Q_0 \\
\downarrow{id} \\
Q_0
\end{array}
\]

\[ (3.2.6) \]

It is clear that all the diagrams commute: \( (A) \) is obvious from the definitions of \( R \) and \( F \), \( (B) \) is axiom (1.3.2a) for strong monads, and \( (C) \) and \( (D) \) are respectively the unit axioms (A.2.2b) and (A.2.2a) of \( Q \).

We have to check now that composition with the unit morphism is the identity (A.1.4). To prove it we will follow the same strategy as for associativity. That is, we will project the diagrams into diagrams in the original \( P \)-operad \( Q \) containing the corresponding unit axioms. The proof of the following lemma can be found in [10]. Recall first that

\[
Q_2 := PQ_{Q_0} \times Q_1 \quad \text{and} \quad \overline{Q}_2 := Q_1 \times Q_0 Q_1.
\]

\textbf{Lemma 3.2.5.} We have commutative squares

\[
\begin{align*}
Q_0 \times Q_0 & \xrightarrow{j_1^1} PQ_1 \times Q_1 \\
\overline{Q}_2 & \xrightarrow{j_2^1} Q_2,
\end{align*}
\]

\[ (3.2.7a) \]

\[
\begin{align*}
\overline{Q}_1 \times Q_0 & \xrightarrow{j_1^2} Q_1 \times Q_0 \\
\overline{Q}_2 & \xrightarrow{j_2^2} Q_2,
\end{align*}
\]

\[ (3.2.7b) \]

where \( j_1^1 \) and \( j_1^2 \) are the morphisms corresponding to \( j_1 \).

\textbf{Proposition 3.2.6.} The unit morphism \( \overline{e} \) of \( \overline{Q} \) satisfies the left and right composition axioms (A.1.4).
Proof. For the left composition (A.1.4a), the required commutative triangle is the outline of the diagram

\[
\begin{array}{c}
\text{\(Q_0 \times Q_1\)} \\
\downarrow j_1 \ \\
\text{\(PQ_0 \times Q_1\)} \\
\downarrow p \times id \ \\
\text{\(PQ_1 \times Q_1\)} \\
\downarrow d_1 \ \\
\text{\(Q_1\)} \\
\downarrow j_1 \ \\
\text{\(Q_1,\)}
\end{array}
\]

We have that diagram (A) commutes by definition of \(j_1\), (B) is precisely (3.2.7a) of Lemma 3.2.5, (C) is the left composition with unit axiom in the \(P\)-operad \(C\) (A.2.4a), and (D) is the same as (3.2.4). Since \(j_1\) is a monomorphism and all the inner diagrams commute, so does the outer triangle, as we wanted to see.

For the right composition (A.1.4b), the required commutative triangle is the outline of the diagram

\[
\begin{array}{c}
\text{\(Q_1 \times Q_0\)} \\
\downarrow j'_1 \ \\
\text{\(Q_1 \times Q_0\)} \\
\downarrow id \times id \ \\
\text{\(PQ_1 \times Q_1\)} \\
\downarrow d_1 \ \\
\text{\(Q_1\)} \\
\downarrow j_1 \ \\
\text{\(Q_1,\)}
\end{array}
\]

We have that diagram (A) commutes by definition of \(j'_1\), (B) is precisely (3.2.7b) of Lemma 3.2.5, (C) is the right composition with unit axiom in the \(P\)-operad \(Q\) (A.2.4b), and (D) is the same as (3.2.4), as before. Since \(j_1\) is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see.

The last thing to check is that the construction is functorial. First of all we have to specify how the construction acts on morphisms. Let \(Q\) and \(Q'\) be two \(P\)-operads and
$Q \xrightarrow{f} B$ a morphism, that is a diagram

\[
\begin{array}{c}
PQ_0 \xleftarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \\
Pf_0 \downarrow \quad f_1 \quad \downarrow f_0 \\
PB_0 \xleftarrow{d_1} B_1 \xrightarrow{d_0} B_0
\end{array}
\]

satisfying the commutative squares of 1.1.1. Then $\mathcal{T}^P f$ is the functor given by

\[
\begin{array}{c}
\mathcal{T}^P Q \\
\mathcal{T}^P f \\
\mathcal{T}^P B
\end{array}
\]

\[
\begin{array}{ccccccc}
Q_0 & \xleftarrow{f_0} & \xrightarrow{f_0} & \xrightarrow{f_0} & \xrightarrow{f_0} & \xrightarrow{f_0} & \xrightarrow{f_0} \\
LQ_0 & \xrightarrow{d_0} & \xrightarrow{d_0} & \xrightarrow{d_0} & \xrightarrow{d_0} & \xrightarrow{d_0} & \xrightarrow{d_0} \\
PB_0 & \xleftarrow{d_1} & \xrightarrow{d_1} & \xrightarrow{d_1} & \xrightarrow{d_1} & \xrightarrow{d_1} & \xrightarrow{d_1}
\end{array}
\]

It is a bit tedious but not difficult to see that $\mathcal{T}^P f$ satisfies again the commutative squares of 1.1.1. Moreover, given another morphism $B \xrightarrow{g} A$ it is clear that $\mathcal{T}^P(g \circ f) = \mathcal{T}^P g \circ \mathcal{T}^P f$, just because of the functoriality of $P$ and $L$.

Since the construction is functorial, if the strength $D_A$ is a monomorphism for every object $A \in \mathcal{E}$ then $\mathcal{T}^P$ is in fact a functor from $P$-operads to categories internal to $\mathcal{E}$.

### 3.3 The composite construction

Since we have defined a construction from $P$-operads to categories and a construction from categories to $P$-operads, we obtain a composite construction from $P'$-operads to $P$-operads, for $P'$ and $P$ not necessarily the same monad. In particular, since a category is the same as an $\text{Id}$-operad, the composite construction for $P' = \text{Id}$ is the same as the functor from categories to $P$-operads. From now on we will call $\mathcal{T}$-construction any of the three constructions, the context will suffice to distinguish, but we will be mainly interested in landing on a $P$-operad, rather than a category. To keep notation short, we denote by

\[\mathcal{T}_P Q := \mathcal{T}_P \mathcal{T}^{P'} Q\]

the composite construction that produces a $P$-operad from the $P'$-operad $Q$. The monad $P'$ will be always clear from the context.

### 3.4 Finiteness conditions

In Section 5 we will be interested in computing the incidence bialgebra of the bar construction of several $P$-operads in $\mathcal{E} = \text{Grpd}$. Recall that to be able to take the homotopy cardinality, the bar construction has to be locally finite as a simplicial groupoids (in the sense of 18). We will now define the notion of locally finite operad (in the sense of (23)) in the setting of $P$-operads, which is the sufficient condition for its bar construction to

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be locally finite, and we will give sufficient conditions on the $\mathcal{T}$-construction to preserve locally finiteness.

**Definition 3.4.1.** A natural transformation is **finite** if all its components are finite. A monad $(P, \mu, \eta)$ on $\text{Grpd}$ is **locally finite** if $\mu$ and $\eta$ are finite natural transformations. A $P$-operad $Q$ is locally finite if $Q_1$ is locally finite, and the maps $d_1$ and $e$ are finite.

In the special case of $P = \text{Id}$, $P$-operads are just categories, and the notion of locally finite agrees with the standard notion. Notice that $Q$ can be locally finite even if $P$ is not. The condition of $P$ being locally finite will appear in the $\mathcal{T}$-construction.

**Example 3.4.2.** For a classical symmetric or non-symmetric operad, the locally finiteness condition amounts to saying that every operation can be expressed as a composition of operations in a finite number of ways. For instance, the operads $\text{Ass}$ and $\text{Sym}$ are locally finite. For this it is important that nullary operations are excluded. The non-reduced versions, where there is a nullary operation, are not locally finite.

The bar construction of $Q$ is locally finite if $Q$ is locally finite and $P$ preserves locally finite groupoids and finite maps (see Section 1). Also, given another locally finite monad $R$ on $E$ that preserves locally finite groupoids and finite maps, if there is a cartesian monad map $P \xrightarrow{\psi} R$ with $\psi$ finite then the bar construction $B^R$ is also locally finite. Let us see that the $\mathcal{T}$-construction interacts well with finiteness, as long as some simple conditions are satisfied.

**Lemma 3.4.3.** Let $C$ be a locally finite category in $\text{Grpd}$. Let $P$ be a locally finite strong monad with finite strength. Assume moreover that $P$ preserves locally finite groupoids and finite maps. Then the $P$-operad $T_\mathcal{T}P C$ is locally finite.

**Proof.** Recall from diagram 3.1.1 that $\tilde{C}_1$ is defined as the pullback

\[
\begin{array}{ccc}
\tilde{C}_1 & \to & LC_0 \\
\downarrow & & \downarrow \text{D} \text{C}_0 \\
PC_1 & \overset{p\text{d}_0}{\to} & PC_0.
\end{array}
\]

Notice that the pullback and the monomorphism refer to the 1-categorical notions, while the finite map condition is a homotopy notion.

Let us see first that $\tilde{C}_1$ is locally finite. Since $C_1$ is locally finite and $P$ preserves locally finite groupoids, $PC_1$ is locally finite. Now, an automorphism in $\tilde{C}_1$ is a pair of automorphisms $(f, g) \in PC_1 \times LC_0$ coinciding at $PC_0$, but there is only a finite number of $f$’s, since $PC_1$ is locally finite, and for each $f$ at most one $g$, since $D_{C_0}$ is a monomorphism.

We have to prove also that $\tilde{d}_1$ and $\tilde{e}$ are finite maps. This follows directly from their definitions, 3.1.1 and 3.1.4 since all the vertical maps involved in diagrams 3.1.3 and 3.1.6 are finite.
Lemma 3.4.4. Let $P$ be a locally finite strong monad with finite strength, and let $Q$ be a locally finite $P$-operad in $\text{Grpd}$. Then the category $T^P Q$ is locally finite.

Proof. The proof is analogous to the proof of Lemma 3.4.3. \hfill \square

In particular these results imply of course that if $Q$ is a locally finite $P'$-operad and $P$ is a strong monad as in lemmas 3.4.3 and 3.4.4 then $T^P Q$ is locally finite.

4 $\mathcal{T}$-construction for $M^r$ and $S^r$-operads

In this section we unravel the $\mathcal{T}$-construction with some of the main examples. We begin discussing the construction from categories to $M^r$-operads and $S^r$-operads. When the category is just a monoid we get the Giraudo $T$-construction, which we will recall next. Lastly we will treat symmetric and non-symmetric operads.

The choice of working with the reduced version of the operads (excluding nullary operations), is irrelevant for the sake of the $\mathcal{T}$-construction itself, which is abstract enough to work with any operad. The reason for preferring the reduced version is to stay within the realm of locally finite operads. It is straightforward to see that both $M^r$ and $S^r$ satisfy the conditions required in Lemma 3.4.3, that is, they preserve locally finite groupoids and finite maps and their strength is finite. Moreover, it is also easy to see that the cartesian monad map $M^r \Rightarrow S^r$ is finite. Recall also that the operads $\text{Ass}$ and $\text{Sym}$, as well as their colored versions, are locally finite too.

4.1 The $\mathcal{T}$-construction for categories

Let now $C$ be a category internal to $\text{Set}$, represented by the span $C_0 \leftarrow C_1 \rightarrow C_0$, and take the free semigroup monad $M^r$. The set of objects of $\mathcal{T}_{M^r} C$ is again $C_0$, while $\tilde{C}_1$ is given by

\[
\begin{tikzcd}
\tilde{C}_1 & \sim \\
M^r C_1 & \sim \\
M^r C_0 & \sim \sim \\
M^r C_0 & \sim \sim \\
C_0 & \sim \sim
\end{tikzcd}
\]

(4.1.1)

Recall from Example 1.3.7 that the strength is given by

\[
\begin{align*}
D_{C_0} & : LC_0 \xrightarrow{\sim} M^r C_0 \\
(c, (1, \ldots, 1)) & \mapsto ((c, 1), \ldots, (c, 1)).
\end{align*}
\]

(4.1.2)
Therefore, the pullback condition means that the elements in $\tilde{C}_1$ that have input $c_1, \ldots, c_n$ and output $c$ are the sequences of $n$ arrows in $C$ whose sources are $c_1, \ldots, c_n$ and whose targets are all $c$. Hence

$$\tilde{C}_1 = \sum_{(c_1, \ldots, c_n; c)} \prod_{i=1}^{n} \text{Hom}(c_i, c).$$

Substitution in $\mathcal{T}_{M'C}$,

$$\circ_i : \prod_{i=1}^{n} \text{Hom}(c_i, c) \times \prod_{j=1}^{m} \text{Hom}(d_j, c_k) \rightarrow \prod_{i=1}^{k-1} \text{Hom}(c_i, c) \times \prod_{j=1}^{m} \text{Hom}(d_j, c_k) \times \prod_{i=k+1}^{n} \text{Hom}(c_i, c),$$

go
tes as follows: for all $x \in \prod_{i=1}^{n} \text{Hom}(c_i, c)$, $y \in \prod_{j=1}^{m} \text{Hom}(d_j, c_k)$,

$$x \circ_i y := (x_1, \ldots, x_{k-1}, x_k \circ y_1, \ldots, x_k \circ y_m, x_{k+1}, \ldots, x_n).$$

Note that now the composition $\circ$ inside the parenthesis is composition of morphisms of $C$, while $x \circ_i y$ is composition in $M'C$. It is not difficult to see that the composition we get from $3.1.1$ agrees with the one defined above: both use the fact that $\tilde{C}_2$ is a subset of $(M')^2C$ together with $(M')^2\circ$ and the monad multiplication. The identity elements of this operad are given by the identity morphisms of $C$. If the category $C$ has coproducts (+) then

$$\prod_{i=1}^{n} \text{Hom}(c_i, c) = \text{Hom}(c_1 + \cdots + c_n, c),$$

so that the operations of $\mathcal{T}_{M'C}$ are in fact arrows of $C$.

Since $C$ can be considered as a category internal to $\text{Grpd}$, we can also compute $\mathcal{T}_{S'C}$ to get a symmetric operad. It is clear that $\mathcal{T}_{M'C} = \mathcal{T}_{M'C}/\mathcal{G}_n$, where the action of the symmetric group $\mathcal{G}_n$ is given by permutation of tuples, that is

$$\mathcal{G}_n \times \prod_{i=1}^{n} \text{Hom}(c_i, c) \rightarrow \prod_{i=1}^{n} \text{Hom}(c_{\sigma(i)}; c)$$

$$(\sigma, (x_1, \ldots, x_n)) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

It is useful to picture elements $(c_1, \ldots, c_n; c)$ as (picturing $n = 3$)

$$\begin{array}{c}
c_3 \\
\downarrow \\
c_2 \\
\downarrow \\
c_1 \\
\downarrow \\
c \\
\end{array}$$

Under this representation, composition in $\mathcal{T}_{S'C}$ (or $\mathcal{T}_{M'C}$) looks like

$$\begin{array}{c}
c_3^2 \\
\downarrow \\
c_3 \\
\downarrow \\
c_2 \\
\downarrow \\
c_1 \\
\downarrow \\
c \\
\end{array} = \begin{array}{c}
c_3^2 \\
\downarrow \\
c_3 \\
\downarrow \\
c_2 \\
\downarrow \\
c_1 \\
\downarrow \\
c \\
\end{array}$$

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**Example 4.1.1.** Take $C = \{0 \leftrightarrow 1\}$. For any pair of objects of $C$ there is exactly one morphism between them. Hence $\mathcal{T}_{M'}C$ has one operation for each given sequence of inputs and output, so that it is the 2-colored associative operad $\text{Ass}_2$. In the same way $\mathcal{T}_S' C$ is the 2-colored symmetric operad $\text{Sym}_2$. In fact it is straightforward to see that the $\mathcal{T}$-constructions of the discrete connected groupoid of $n$ elements are $\text{Ass}_n$ and $\text{Sym}_n$.

**Example 4.1.2.** Consider now the category $C = \{0 \rightarrow 1\}$. Note that in this case there is either one or no morphism between two objects of $C$. Thus clearly

$$\mathcal{T}_{S'} C(c_1, \ldots, c_n; c) = \begin{cases} (c \rightarrow c_1, \ldots, c \rightarrow c_n) & \text{if } c = 0 \text{ or } c = c_1 = \cdots = c_n = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Of course this operad is a suboperad of the previous one, since this category is a subcategory of the previous one. In particular composition is obvious.

**Example 4.1.3.** We now specialize to the case of categories with only one object, that is monoids, recovering the $\mathcal{T}$-construction of Giraudo. This construction was introduced by Giraudo [20] as a generic method to build combinatorial operads from monoids.

Since a monoid is just a category with one object, it is represented by the span $1 \leftarrow Y \rightarrow 1$, and because the morphism $L1 \xrightarrow{D_1} M'1$ is an isomorphism, we have that $\mathcal{T}_{M'} Y$ is given by

\[
\begin{array}{ccc}
M'Y & \xleftarrow{M's} & M'1 \\
\searrow & & \nearrow \\
M'Y & \xrightarrow{M't} & L1 \\
\downarrow & & \downarrow \xrightarrow{D_1} \\
M'1 & & 1.
\end{array}
\]

It is easy to see that this gives the same operad $TY$ defined in the introduction, since $TY$ is precisely $M'Y$, and both compositions are defined by using composition in $(M')^2Y$ and the monad multiplication.

**Example 4.1.4.** If $Y_1$ is the singleton monoid, then $\mathcal{T}_{M'} Y_1 = \text{Ass}$, the associative operad, and $\mathcal{T}_S'Y_1 = \text{Sym}$, the commutative operad.

### 4.2 The $\mathcal{T}$-construction for operads

In this subsection we will unravel the full $\mathcal{T}$-construction from nonsymmetric operads to $S'$-operads. As we already know, the first ones are the same as $M'$-operads in $\text{Set}$, but we will view them as $M'$-operads in $\text{Grpd}$ with discrete groupoids of objects and
arrows. At the end we will comment on other variations similar to this case, such as from symmetric operads to $\mathcal{S}'$-operads.

Let $Q$ be an $\mathcal{M}'$-operad represented by the span $\mathcal{M}'Q_0 \leftarrow Q_1 \rightarrow Q_0$. As in Example (1.3.4) elements of $Q_1$ will be depicted

\[
\begin{array}{c}
\text{or as}
\end{array}
\]

We apply first the $\mathcal{T}$-construction to get a category $\mathcal{T}^{\mathcal{M}'} Q$:

\[
\begin{array}{ccc}
\mathcal{Q}_1 & \xrightarrow{\sim} & Q_1 \\
\mathcal{L}Q_0 & \xleftarrow{R_{\mathcal{Q}_0}} & Q_0 \\
Q_0 & \xrightarrow{S\mathcal{Q}_0} & SQ_0 & \xleftarrow{t} & Q_0.
\end{array}
\]

The strength morphism is the same as in (4.1.2). Therefore the elements of $\mathcal{Q}_1$ are the elements of $Q_1$ such that all the input objects coincide,

\[
\begin{array}{c}
\text{or}
\end{array}
\]

so that $x$ is an arrow $c \xrightarrow{x} d$ in $\mathcal{T}^{\mathcal{M}'} Q$. Notice that $\mathcal{Q}_2$ is a subset of $Q_2$. Therefore composition in $\mathcal{T}^{\mathcal{M}'} Q$ is the same as composition in $Q$. For example

\[
\begin{array}{c}
\text{where } y \circ (x, x) \text{ is composition in } Q. \text{ Hence the recipe is to repeat } x \text{ for each input of } y \text{ and use composition in } Q. \text{ Now we have to apply again the } \mathcal{T}\text{-construction to get a } \mathcal{S}'\text{-operad from the category } \mathcal{T}^{\mathcal{M}'} Q. \text{ This step was made above for any category: the objects of } \mathcal{Q}_1 \text{ are sequences } (x_1, \ldots, x_n) \text{ of elements } x_i \in \mathcal{Q}_1. \text{ For instance the pair}
\end{array}
\]
is an operation \((x, y)\)

in \(\mathcal{T}_{\mathcal{S}'} Q\). Clearly \(\mathcal{T}_{\mathcal{S}'} Q\) is a symmetric operad, since the groupoid of objects is discrete and the morphisms in the groupoid \(\tilde{Q}_1\) are given by permutation of tuples.

**Example 4.2.1.** If the starting \(\mathcal{M}'\)-operad is \(\text{Ass}\), which is a non-colored operad, then it is easy to see that the monoid \(\mathcal{T}^{\mathcal{M}'} \text{Ass}\) is isomorphic to \((\mathbb{N}^+, \times)\). Therefore the operations of \(\mathcal{T}_{\mathcal{S}'} \text{Ass}\) are sequences of natural numbers and composition is given by multiplication. For example

\[
((2, 3), (4, 7)) \circ (5, 9) = (5 \cdot 2, 5 \cdot 3, 9 \cdot 4, 9 \cdot 7) = (10, 15, 36, 63).
\]

If the starting \(\mathcal{M}'\)-operad is \(\text{Ass}_2\) the 2-colored associative operad, then the category \(\mathcal{T}^{\mathcal{M}'} \text{Ass}_2\) has two objects and a morphism \(\xrightarrow{n}\) for every pair of objects and positive natural number \(n\). Composition is given by multiplication. The operations of \(\mathcal{T}_{\mathcal{S}'} \text{Ass}_2\) are thus sequences of such arrows with the same output.

Suppose we start now from a symmetric operad \(Q\). Recall from Example 4.3.7 that a symmetric operad is an \(\mathcal{S}\)-operad in \(\text{Grpd}\) such that \(Q_0\) is discrete and \(\mathcal{S}' Q_0 \xrightarrow{\sim} Q_1\) is discrete fibration. The \(\mathcal{T}\)-construction to get another \(\mathcal{S}'\)-operad is completely analogous to the previous case, but in this case the groupoid \(\tilde{Q}_1\) inherits morphisms from \(Q\), so that for instance the element

\[
\begin{array}{ccc}
x & \xrightarrow{2!} & \mathcal{S}' Q_1 \\
x & \xrightarrow{3!^2} & y
\end{array}
\]

has \(2! \cdot 3!^2 \cdot 2!\) automorphisms, corresponding to \(2!\) invariant permutations on \((x, x, y)\) and permutations of the inputs. The latter contribution did not appear in the previous case, since \(Q\) was a planar operad. Notice that this means that \(\mathcal{T}_{\mathcal{S}'} Q\) is not a symmetric operad, but just an \(\mathcal{S}'\)-operad in \(\text{Grpd}\).

**Example 4.2.2.** If the starting \(\mathcal{S}'\)-operad is \(\text{Sym}\), which is a non-colored symmetric operad, then it is easy to see that the monoid \(\mathcal{T}^{\mathcal{S}'} \text{Sym}\) is isomorphic to the monoid \((\mathbb{N}^+, \times)\) internal to groupoids where \(\text{Aut}(n) \cong \mathfrak{S}_n\). The objects of \(\mathcal{T}_{\mathcal{S}'} \text{Sym}\) are the same as the objects in \(\mathcal{T}_{\mathcal{S}'} \text{Ass}\), and the morphisms are given by permutation of tuples (as in \(\mathcal{T}_{\mathcal{S}'} \text{Ass}\)) plus the ones given by \(\text{Aut}(n)\) for each \(n\). The colored case is analogous.
4.3 The opposite convention

When dealing with plethysm it will be more natural, from a combinatorial point of view, to apply the $\mathcal{T}$-construction to the opposite category $C^{\text{op}}$. We will now develop this point of view. From a formal perspective there is not much to say, since in the context of internal categories if $C$ is represented by $C_1 C_0 C_0$

then $C^{\text{op}}$ is represented by

and thus the $\mathcal{T}$-construction can be applied the same way. Let us see how $\mathcal{T}_S C^{\text{op}}$ looks like. We have that

$$\tilde{C}_1^{\text{op}} = \sum_{(c_1, \ldots, c_n; c)} \prod_{i=1}^n \text{Hom}_{C^{\text{op}}}(c_i, c) = \sum_{(c_1, \ldots, c_n; c)} \prod_{i=1}^n \text{Hom}_{C}(c, c_i),$$

for each tuple $(c_1, \ldots, c_n; c)$ of elements of $C_0$. In this case elements $(c_1, \ldots, c_n; c)$ can be pictured as (picturing $n = 3$)

and under this representation, composition in $\mathcal{T}_S C^{\text{op}}$ looks like

Furthermore, if $C$ has products then

$$\prod_{i=1}^n \text{Hom}_{C}(c, c_i) = \text{Hom}_{C}(c, c_1 \times \cdots \times c_n).$$

Suppose now that we start from an $M'$-operad $Q$. The first step is the same as before: we obtain a category $\mathcal{T}^{M'} Q$ whose arrows are operations of $Q$. 

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all of whose inputs coincide. Now we take the opposite category $\mathcal{T}^M Q^{\text{op}}$, and depict its arrows as

The $\mathcal{T}$-construction $\mathcal{T}_S Q^{\text{op}}$ has as operations sequences $(x_1, \ldots, x_n)$ of arrows $x_i \in Q_1^{\text{op}}$ with the same output. For instance the pair

in $\mathcal{T}_S Q^{\text{op}}$.

**Remark 4.3.1.** If the starting operad $Q$ is the colored commutative operad or the colored associative operad then the opposite convention does not affect the result, because the category $\mathcal{T}^S_r Q$ (or $\mathcal{T}^S_s Q$) is self dual. In the case of $\text{Ass}$ this self duality means that the monoid $(\mathbb{N}^+, \times)$ is commutative.

## 5 Plethysms and operads

In this section we will present the relation between the several plethystic bialgebras, operads and the $\mathcal{T}$-construction. Some proofs will be omitted, since most of them are similar. The operads involved will be the reduced symmetric operad $\text{Sym}$, the reduced associative operad $\text{Ass}$ and their 2-colored versions. Also, playing the same role as these operads, we will have a locally finite monoid $Y$. On the other hand, the $\mathcal{T}$-constructions will be taken with respect to the monads $S$ and $M$, as in Section 4 and everything will be internal to $\mathcal{E} = \text{Grpd}$.

Notice that both $\text{Ass}$ and $\text{Sym}$ and their colored versions have a discrete groupoid of colors. Since the $\mathcal{T}$-construction does not alter this groupoid and the monads $M$ and $S$ preserve fibrations, Proposition 1.4.6 tells us that all the bar constructions of this section are Segal groupoids. Also, by Lemmas 3.4.3 and 3.4.4 and the discussion of Section 4.
they are locally finite Segal groupoids, so that we can take cardinality to arrive at their incidence bialgebra in the classical sense of vector spaces.

We will start with the classical Faà di Bruno and the classical plethystic bialgebras, then give a brief summary of all the variations we study, and finally introduce these variations.

The following standard notation is used:

- \( x = (x_1, x_2, \ldots) \),
- \( \Lambda \): set of infinite vectors of natural numbers with \( \lambda_i = 0 \) for all \( i \) large enough,
- \( \Lambda \ni \lambda = (\lambda_1, \lambda_2, \ldots) \),
- \( x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots \),
- \( \operatorname{aut}(\lambda) = 1!^{\lambda_1} \lambda_1! \cdot 2!^{\lambda_2} \lambda_2! \cdots \),
- \( \lambda! = \lambda_1! \cdots \lambda_2! \cdots \),
- \( W \): set of finite words of positive natural numbers,
- \( W \ni \omega = \omega_1 \cdots \omega_n \),
- \( x_\omega = x_{\omega_1} \cdots x_{\omega_n} \),
- \( \omega! = \omega_1! \cdots \omega_n! \).

### 5.1 The classical case

The classical Faà di Bruno bialgebra \( F \) [13][24] is obtained from the substitution of power series in one variable. Let \( \mathbb{Q}[x] \) be the ring of formal power series with coefficients in \( \mathbb{Q} \) without constant term. Elements of \( \mathbb{Q}[x] \) are written

\[
F(x) = \sum_{n \geq 1} \frac{F_n}{n!} x^n.
\]

The set \( \mathbb{Q}[x] \) forms a (non-commutative) monoid with substitution of power series. The Faà di Bruno bialgebra \( F \) is the free polynomial algebra \( \mathbb{Q}[A_1, A_2, \ldots] \) generated by the linear maps

\[
A_i : \mathbb{Q}[x] \longrightarrow \mathbb{Q} \\
F \longrightarrow F_i
\]

together with the comultiplication induced by substitution, meaning that

\[
\Delta(A_n)(F, G) = A_n(G \circ F),
\]

and counit given by \( \epsilon(A_n) = A_n(x) \). The comultiplication of the generators can be explicitly described through the (exponential) Bell polynomials \( B_{n,k} \), which count the number of partitions of an \( n \)-element set into \( k \) blocks:

\[
\Delta(A_n) = \sum_{k=1}^{n} A_k \otimes B_{n,k}(A_1, A_2, \cdots).
\]
Theorem 5.1.1 (Joyal, cf. modern reformulation in [15]). The Faà di Bruno bialgebra $\mathcal{F}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $B\text{Sym}$.

Note that $\text{Sym}$ is of course the same as $\mathcal{T}_{\text{id}}\text{Sym}$ and, as explained in Section 4, it is also $\mathcal{T}_{S}$ of the trivial monoid. This connects the Faà di Bruno bialgebra to the $\mathcal{T}$-construction in an analogous way as the plethystic bialgebras.

Let us recall how the classical plethystic substitution works [24, 37, 38]. Let $\mathbb{Q}[x_1, x_2, \ldots]$ be the ring of power series in infinitely many variables without constant term and coefficients in $\mathbb{Q}$. Elements of $\mathbb{Q}[x]$ are written

$$F(x) = \sum_{\lambda \in \Lambda} \frac{F_{\lambda}}{\text{aut}(\lambda)} x^\lambda.$$ 

Given two power series $F, G \in \mathbb{Q}[x]$, their plethystic substitution is defined as

$$(G \otimes F)(x_1, x_2, \ldots) := G(F_1, F_2, \ldots), \quad \text{where}$$

$$F_k(x_1, x_2, \ldots) := F(x_k, x_{2k}, \ldots). \quad (5.1.1)$$

The set $\mathbb{Q}[x]$ forms a (non-commutative) monoid with plethystic substitution. The plethystic bialgebra $\mathcal{P}$ [9, 36] is the free polynomial algebra $\mathcal{P} = \mathbb{Q}[\{A_\lambda\}_\lambda]$ generated by the set maps

$$A_\lambda : \mathbb{Q}[x] \longrightarrow \mathbb{Q}$$

$$F \longmapsto F_\lambda$$

together with the comultiplication induced by substitution, meaning that

$$\Delta(A_\lambda)(F, G) = A_\lambda(G \otimes F),$$

and counit given by $\epsilon(A_\lambda) = A_\lambda(x_1)$. The comultiplication of the generators can be explicitly described through the polynomials $P_{\sigma, \lambda}$, a plethystic version of the Bell polynomials which, in the terminology of Nava–Rota [37], count transversals of partitions.

$$\Delta(A_\sigma) = \sum_\lambda A_\lambda \otimes P_{\sigma, \lambda}(\{A_\mu\}_\mu).$$

Theorem 5.1.2. The plethystic bialgebra $\mathcal{P}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $B\mathcal{T}_{S}\text{Sym}$.

Proof. The comparison between these two incidence bialgebras was made in [9], where the simplicial interpretation of plethysm was established. In Section 6 we will see that indeed $B\mathcal{T}_{S}\text{Sym}$ is equivalent to $\mathcal{T}S$, the simplicial groupoid of [9].
5.2 Overview of variations

We proceed to introduce the variations of the plethystic bialgebra we will explore. For the set of variables \((x_1, x_2, \ldots)\), there are three sources of variations. At the level of power series they are the following:

(i) Commuting or noncommuting variables: of course in the classical case the variables commute. When the variables do not commute we will index them by \(\omega \in W\), rather than \(\lambda \in \Lambda\).

(ii) Commuting or non-commuting coefficients.

(iii) Two types of automorphisms: \(\text{aut}(\lambda)\) or \(\lambda!\) for commuting variables, and \(\omega!\) or 1 for noncommuting variables.

These variations are not independent: if the variables commute then the coefficients commute. Analogous variations can be obtained of the Faà di Bruno bialgebra, except in this case there is only one variable.

At the objective level, these three variations correspond (respectively) to the following choices:

(i) \(\mathcal{T}\)-construction over \(S'\) or over \(M'\).

(ii) Bar construction over \(S'\) or over \(M'\).

(iii) Taking \(\text{Sym}\) or \(\text{Ass}\) as input operads.

The reason why they are not independent is clear here: there is a cartesian natural transformation \(M' \Rightarrow S'\) that allows taking \(B_{S'}\) of a \(M'\)-operad (see Section 1), but no natural transformation in the opposite direction.

Let us give now a brief justification of these correspondences. Consider the following sequence of operations:

\[
a =
\]

We have used the opposite convention (Subsection 4.3), which in this case does not affect the result (Remark 4.3.1). This could be either an operation in one of the following operads:

(i) \(\mathcal{T}_S\text{Sym}\): in this case each operation has automorphisms, coming from the action of the symmetric group on \(\text{Sym}\), and since the \(\mathcal{T}\)-construction is over \(S'\) we can
permute the operations. This means that the isomorphism class of \( a \) is given by \( \lambda = (0, 3, 1, 2) \), since the order of the operations does not matter, and it has \( \text{aut}(\lambda) = 2!^3 3!^3 1!^1 2!^2 2! \) automorphisms. The corresponding bialgebra will be thus \( \mathcal{P} \) and this particular operation corresponds to \( A_{(0,3,1,2)} \), the linear map returning the coefficient of \( x_2^3 x_3 x_2^4 / \text{aut}(\lambda) \).

(ii) \( \mathcal{T}_M \text{Sym} \): in this case the operations have automorphisms again, but since the \( \mathcal{T} \)-construction is over \( M \) we cannot permute them. This means that the isomorphism class of \( a \) is given by \( \omega = (3, 2, 4, 2, 2, 4) \), so that it corresponds to non-commuting variables. Clearly it has \( 3!^2 4!^2 2!^4 \) automorphisms. Now, depending on the bar construction it will correspond to commuting or non-commuting coefficients. This particular operation corresponds to \( A_{(3,2,4,2,2,4)} \), the linear map returning the coefficient of \( x_3 x_2 x_4 x_2 x_2 x_4 / \omega! \).

(iii) \( \mathcal{T}_M \text{Ass} \): in this case the operations do not have automorphisms, and since the \( \mathcal{T} \)-construction is over \( M \) we cannot permute them. This means that the isomorphism class of \( a \) is given by \( \omega = (3, 2, 4, 2, 2, 4) \), so that it corresponds to non-commuting variables, and it has no automorphisms. Now, depending on the bar construction it will correspond to commuting or non-commuting coefficients, as in the previous case. This particular operation corresponds to \( a_{(3,2,4,2,2,4)} \), the linear map returning the coefficient of \( x_3 x_2 x_4 x_2 x_2 x_4 \).

(iv) \( \mathcal{T}_S \text{Ass} \): in this case the operations do not have automorphisms, and since the \( \mathcal{T} \)-construction is over \( S \) we can permute them. This means that the isomorphism class of \( a \) is given by \( \lambda = (0, 3, 1, 2) \), and it has \( \lambda! = 3! \cdot 1! \cdot 2! \) automorphisms. Therefore it corresponds to commuting variables and coefficients. This particular operation corresponds to \( a_{(0,3,1,2)} \), the linear map returning the coefficient of \( x_2^3 x_3 x_2^4 / \lambda! \).

The cases of \( \text{Sym} \) and \( \text{Ass} \) are developed in Subsections 5.3 and 5.4 respectively. In Subsection 5.5 we generalize \( \mathcal{EP} \) to power series in the set of variables \( \{ x_m \mid m \in Y \} \) indexed over a locally finite monoid.

In Subsections 5.3 and 5.4 we also study the Faà di Bruno bialgebra in two variables and the plethystic bialgebra in the two sets of variables \( \{ x_1, x_2, \ldots \}, \{ y_1, y_2, \ldots \} \). For the plethystic case we will only consider commuting variables and coefficients. Let us give a similar digression as above for the plethystic cases. Consider the following 2-colored operation:

\[
a = \begin{array}{c}
\begin{array}{c}
\text{operation}
\end{array}
\end{array}
\]
The isomorphism class of this operation is given by \((\lambda^1, \lambda^2) = ((0, 2, 0, 2), (0, 1, 1, 1))\) (since everything commutes now), and it can either be an operation in \(T^*_S \text{Ass}_2\) or \(T^*_S \text{Ass}_2\).

It thus corresponds to \(A_{(0,2,0,2),(0,1,1,1)} \in \mathcal{P}^2\) or to \(a_{(0,2,0,2),(0,1,1,1)} \in \mathcal{EP}^2\), the linear maps returning the coefficients of \(x_2^2 x_2^4 y_2^3 y_4 / \text{aut}(\lambda^x) \text{aut}(\lambda^y)\).

## 5.3 Bialgebras from \(\text{Sym}\) and \(\text{Sym}_2\)

We have already seen two bialgebras arising from \(\text{Sym}\) in Subsection 5.1, the Faà di Bruno bialgebra \(\mathcal{F}\) and the plethystic bialgebra \(\mathcal{P}\). Let us see now the aforementioned variations.

Replace \(\mathbb{Q}[x]\) by \(\mathbb{Q}\langle\langle x \rangle\rangle\), that is, non-commuting variables. Elements of \(\mathbb{Q}\langle\langle x \rangle\rangle\) are written

\[
F(x) = \sum_{\omega \in \mathcal{W}} \frac{F_{\omega}}{\omega!} x^\omega,
\]

Substitution of power series in \(\mathbb{Q}\langle\langle x \rangle\rangle\) is defined in the same way as before (5.1.1). The plethystic bialgebra with non-commuting variables \(\mathcal{P}^X\) is defined as the free polynomial algebra \(\mathbb{Q}[\{A_\omega\}]\) on the set maps \(A_\omega\) and comultiplication and counit as usual.

**Theorem 5.3.1.** The plethystic bialgebra with non-commuting variables \(\mathcal{P}^X\) is isomorphic to the homotopy cardinality of the incidence bialgebra of \(B^T \mathcal{M}^r \text{Sym}\).

If we take now \(R\langle\langle x \rangle\rangle\) with \(R\) a non-commutative unital ring, then we get the non-commutative plethystic bialgebra with non-commuting variables \(\mathcal{X}\mathcal{P}^X\), which is the free associative algebra \(\mathbb{Q}\langle\{A_\omega\}\rangle\) together with the usual comultiplication and counit. In this case, substitution of power series is defined in the same way but it is not associative. However the comultiplication is still associative. A proof of this can be found in [6] for the one variable case, which will be obtained below.

**Theorem 5.3.2.** The non-commutative plethystic bialgebra with non-commuting variables \(\mathcal{X}\mathcal{P}^X\) is isomorphic to the homotopy cardinality of the incidence bialgebra of \(B^T \mathcal{M}^r \text{Sym}\).

Let us now move forward to power series in two variables. All the results are also valid for any number of variables, but for simplicity and notation we have chosen to show the two variables case. Also, for the bivariate plethystic bialgebras, we will not enter into non-commutativity of the variables or of the coefficients.

Let \(\mathbb{Q}[x, y]\) be the ring of formal power series in the variables \(x\) and \(y\) with coefficients in \(\mathbb{Q}\) without constant term. Elements of \(\mathbb{Q}[x, y]\) are written

\[
F(x, y) = \sum_{n+m \geq 1} \frac{F_{n,m}}{n!m!} x^n y^m.
\]
The set \( \mathbb{Q}[x, y] \times \mathbb{Q}[x, y] \) forms a (non-commutative) monoid with substitution of power series:

\[
\begin{align*}
\left( \mathbb{Q}[x, y] \times \mathbb{Q}[x, y] \right) \times \left( \mathbb{Q}[x, y] \times \mathbb{Q}[x, y] \right) &\twoheadrightarrow \mathbb{Q}[x, y] \times \mathbb{Q}[x, y] \\
\left( (F^1, F^2), (G^1, G^2) \right) &\twoheadrightarrow (G^1(F^1, F^2), G^2(F^1, F^2)).
\end{align*}
\]

We define the Faà di Bruno bialgebra in two variables \( \mathcal{F}^2 \) as the free polynomial algebra \( \mathbb{Q}\{A_{i,n,m}^{i=1,2}\}_{n+m \geq 1} \) generated by the set maps \( A_{i,n,m}^{i} : \mathbb{Q}[x,y] \times \mathbb{Q}[x,y] \to \mathbb{Q} \)

\[
(\mathcal{F}^1, \mathcal{F}^2) \twoheadrightarrow F_{i}^{i}_{n,m}
\]

together with the comultiplication induced by substitution, meaning that

\[
\Delta(A_{i,n,m}^{i})((\mathcal{F}^1, \mathcal{F}^2), (\mathcal{G}^1, \mathcal{G}^2)) = A_{i,n,m}^{i}((\mathcal{G}^1, \mathcal{G}^2) \circ (\mathcal{F}^1, \mathcal{F}^2)),
\]

and counit given by \( \epsilon(A_{i,n,m}^{i}) = A_{i,n,m}^{i}(x,y) \).

**Theorem 5.3.3.** The Faà di Bruno bialgebra in two variables \( \mathcal{F}^2 \) is isomorphic to the homotopy cardinality of the incidence bialgebra of \( \mathcal{B}\text{Sym}_2 \). The same holds for \( n \) variables and \( \text{Sym}_n \).

Notice that \( \text{Sym}_2 \) is the same as \( \mathcal{T}_\mathcal{U}\text{Sym} \) and, as explained in Example [4.1.1] it is also \( \mathcal{T}_\mathcal{S}_\mathcal{C} \), where \( \mathcal{C} = \{0, 1\} \). This connects the Faà di Bruno bialgebra in two variables to the \( \mathcal{T} \)-construction in an analogous way as the plethystic bialgebras.

We can do the same with the power series ring in two sets of infinitely many variables \( \mathbb{Q}\lfloor x, y \rfloor \) with coefficients in \( \mathbb{Q} \). We shall write

\[
X = (x, y), \lambda = (\lambda^1, \lambda^2) \in \Lambda^2, \text{ aut}(\lambda) = \text{ aut}(\lambda^1) \text{ aut}(\lambda^2) \text{ and } X^\lambda = x^{\lambda^1}y^{\lambda^2},
\]

so that elements of \( \mathbb{Q}\lfloor X \rfloor \) are written

\[
F(X) = \sum_{\lambda} \frac{F_\lambda}{\text{ aut}(\lambda)} X^\lambda.
\]

The set \( \mathbb{Q}\lfloor X \rfloor \times \mathbb{Q}\lfloor X \rfloor \) forms a (non-commutative) monoid with plethystic substitution of power series:

\[
\begin{align*}
\left( \mathbb{Q}\lfloor X \rfloor \times \mathbb{Q}\lfloor X \rfloor \right) \times \left( \mathbb{Q}\lfloor X \rfloor \times \mathbb{Q}\lfloor X \rfloor \right) &\twoheadrightarrow \mathbb{Q}\lfloor X \rfloor \times \mathbb{Q}\lfloor X \rfloor \\
\left( (\mathcal{F}^1, \mathcal{F}^2), (\mathcal{G}^1, \mathcal{G}^2) \right) &\twoheadrightarrow (G^1(F^1, F^2), G^2(F^1, F^2))
\end{align*}
\]

The plethystic bialgebra in two variables \( \mathcal{P}^2 \) is defined as the free polynomial algebra \( \mathbb{Q}\{A\lambda_{i=1,2}\}_{i=1,2} \) generated by the set maps

\[
A_{\lambda}^{\lambda} : \mathbb{Q}\lfloor X \rfloor \times \mathbb{Q}\lfloor X \rfloor \twoheadrightarrow \mathbb{Q} \]

\[
(F^1, F^2) \twoheadrightarrow F_{\lambda}^{i}
\]

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together with the comultiplication induced by substitution, meaning that
\[ \Delta(A_\lambda^i)((F^1, F^2), (G^1, G^2)) = A_\lambda^i((G^1, G^2) \circ (F^1, F^2)), \]
and counit given by \( \epsilon(A_\lambda^i) = A_\lambda^i(x, y) \).

**Theorem 5.3.4.** The plethystic bialgebra in two variables \( P^2 \) is isomorphic to the homotopy cardinality of the incidence bialgebra of \( \mathcal{B}T_S \cdot \text{Sym}_2 \).

### 5.4 Bialgebras from \( \text{Ass} \) and \( \text{Ass}_2 \)

Take again \( \mathbb{Q}[x] \), but write now elements of \( \mathbb{Q}[x] \) as
\[ F(x) = \sum_{n \geq 1} f_n x^n. \]

The ordinary Faà di Bruno bialgebra \( OF \) is the free polynomial algebra \( \mathbb{Q}[a_1, a_2, \ldots] \) generated by the linear maps \( a_i(F) = f_i \) together with the comultiplication induced by substitution and counit given by \( \epsilon(a_n) = a_n(x) \), as before.

**Theorem 5.4.1.** The ordinary Faà di Bruno bialgebra \( OF \) is isomorphic to the homotopy cardinality of the incidence bialgebra of \( \mathcal{B}^S \cdot \text{Ass} \).

It is clear that \( F \) and \( OF \) are isomorphic bialgebras, since we have only changed the basis. However their combinatorial meaning is slightly different, and indeed \( \mathcal{B}^S \cdot \text{Sym} \) and \( \mathcal{B}^S \cdot \text{Ass} \) are not equivalent. Note that \( \text{Ass} \) is of course the same as \( T_{\text{id}} \cdot \text{Ass} \) and, as explained in Section 4, it is also \( T_{M^r} \) of the trivial monoid. This connects the ordinary Faà di Bruno bialgebra to the \( T \)-construction.

If we replace above \( \mathbb{Q} \) by \( R \) (a non-commutative unital ring), we obtain the non-commutative Faà di Bruno bialgebra \( \mathcal{X}F \) \((6, 14, 29)\), the free associative unital algebra \( \mathbb{Q}(a_1, a_2, \ldots) \) generated by the set maps \( a_i(F) = f_i \), together with the comultiplication induced by substitution and counit \( \epsilon(a_n) = a_n(x) \), as before. In this case, substitution of power series is not associative, but the comultiplication is still coassociative \( [6] \). It is clear that \( F \) and \( OF \) are the abelianization of \( \mathcal{X}F \) \([6] \).

**Theorem 5.4.2.** The non-commutative Faà di Bruno bialgebra \( \mathcal{X}F \) is isomorphic to the homotopy cardinality of the incidence bialgebra of \( \mathcal{B}^M \cdot \text{Ass} \).

We move now to the plethystic bialgebras. The exponential plethystic bialgebra \( EP \) is the same bialgebra as \( P \), but in this case \( \text{aut}(\lambda) = \lambda! = \lambda_1! \cdot \lambda_2! \cdots \) \([36] \). The generators of this bialgebra will be denoted by \( a_\lambda \).

**Theorem 5.4.3.** The exponential plethystic bialgebra \( EP \) is isomorphic to the homotopy cardinality of the incidence bialgebra of \( \mathcal{B}T_S \cdot \text{Ass} \).
The linear plethystic bialgebra with non-commuting variables $\mathcal{L}P^X$ is the same bialgebra as $P^X$ but without automorphisms of $\omega$. The generators for this bialgebra are denoted $a_\omega$.

**Theorem 5.4.4.** The linear plethystic bialgebra with non-commuting variables $\mathcal{L}P^X$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $B^S T_{M'} \text{Ass}$.

The non-commutative linear plethystic bialgebra with non-commuting variables $\mathcal{L}XP^X$ is the same as $XP^X$ but without automorphisms on $\omega$. We will thus write $a_\omega$ for its generators. Contrary to what it may seem, the non-commutativity simplifies the explicit formula for the comultiplication of the generators. Denote by $|w|$ the length of a word. Let also $W^W_n$ be the set of length $n$ words of words of $W$. Finally, for $k \in \mathbb{N}$ and $\omega = \omega_1 \ldots \omega_n \in W$, define the $k$th Verschiebung operator as

$$k\omega = (k\omega_1) \ldots (k\omega_n).$$

**Proposition 5.4.5.** The comultiplication of $XP^X$ is given by

$$\Delta(a_\nu) = \sum_{\omega \in W} \sum_{\kappa \in W^W} T^k_{\nu, \omega} \left( \prod_{i=1}^{|\omega|} a_{\kappa_i} \right) \otimes a_\omega,$$

where

$$T^k_{\nu, \omega} = \begin{cases} 1 & \text{if } \nu = \sum_{i=1}^n \omega_i \kappa_i \\ 0 & \text{otherwise.} \end{cases}$$

This proposition is analogous to [9, Proposition 3.3].

**Theorem 5.4.6.** The non-commutative linear plethystic bialgebra with non-commuting variables $\mathcal{L}XP^X$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $B^1 T_{M'} \text{Ass}$.

**Proof.** Notice that $B^1 T_{M'} \text{Ass}$ is discrete. Its elements are given by sequences of tuples

$$(m^1_1, \ldots, m^1_{n_1}), \ldots, (m^k_1, \ldots, m^k_{n_k})$$

of elements of positive natural numbers (see Example 4.2.1), but there is only the identity morphisms between them. Thus juxtaposition of sequences gives $B^1 T_{M'} \text{Ass}$ a (non-symmetric) monoidal structure. Sequences containing one tuple are called connected, and form an algebra basis of the incidence bialgebra. The subgroupoid of connected sequences is denoted $B^1 T_{M'} \text{Ass}$. It is clear that $\pi_0 B^1 T_{M'} \text{Ass} = B^1 T_{M'} \text{Ass}$ is isomorphic to $W$, and that $\pi_0 B^1 T_{M'} \text{Ass} = B^1 T_{M'} \text{Ass}$ is isomorphic to $W^W$. Although $\pi_0 B^1 T_{M'} \text{Ass} = B^1 T_{M'} \text{Ass}$
we will keep using the notation $\delta_\omega$ for the isomorphism class of $\omega \in B_1 T_M^r \text{Ass}$. It only remains to compute the comultiplication:

$$\Delta(\delta_\nu) = \sum_{\omega \in \pi_0 B_1^* T_M^r \text{Ass}} \sum_{\kappa \in \pi_0 B_1 T_M^r \text{Ass}} |\text{Iso}(d_0 \kappa, d_1 \omega)_\nu| \delta_\kappa \otimes \delta_\omega.$$  

By the discussion above we only have to check that

$$|\text{Iso}(d_0 \kappa, d_1 \omega)_\nu| = T_{\nu, \omega}^\kappa,$$

but this is clear because there is only one morphism between $d_0 \kappa$ and $d_1 \omega$ and fibering over $\nu$ means taking the subset of those morphisms that give $\nu$ after composing, hence $|\text{Iso}(d_0 \kappa, d_1 \omega)_\nu| = 1$ if $d_1(\kappa, \omega) = \nu$ and 0 otherwise, exactly as $T_{\nu, \omega}^\kappa$. \hfill \Box

Let us now move forward to power series in two variables. Again, all the results are also valid for any number of variables, but for simplicity and notation we have chosen to show the two variables case.

Let $\mathbb{Q}\langle\langle x, y \rangle\rangle$ be the ring of formal power series in the non-commutative variables $x$ and $y$ with coefficients in $\mathbb{Q}$ without constant term. Elements of $\mathbb{Q}\langle\langle x, y \rangle\rangle$ are written

$$F(x, y) = \sum_\omega f_\omega \omega,$$

where $\omega$ is a non-empty word in $x$ and $y$. The set $\mathbb{Q}\langle\langle x, y \rangle\rangle$ forms a non-commutative monoid with substitution of power series.

We define the Faà di Bruno bialgebra in two non-commuting variables $\mathcal{F}^{(2)}$ as the free polynomial algebra $\mathbb{Q}[\{a^i_\omega\}]$ generated by the set maps

$$a^i_\omega : \mathbb{Q}\langle\langle x, y \rangle\rangle \times \mathbb{Q}\langle\langle x, y \rangle\rangle \rightarrow \mathbb{Q}$$

$$(F^1, F^2) \rightarrow f^i_\omega$$

together with the counit given by $\epsilon(a^i_\omega) = a^i_\omega(x, y)$ and the comultiplication induced by substitution.

**Theorem 5.4.7.** The Faà di Bruno bialgebra in two non-commuting variables $\mathcal{F}^{(2)}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $B^S \text{Ass}_2$. 

We obtain the non-commutative Faà di Bruno bialgebra in two non-commuting variables $\mathcal{X}\mathcal{F}^{(2)}$ by taking above power series with coefficients in $R$.

**Theorem 5.4.8.** The non-commutative Faà di Bruno bialgebra in two non-commuting variables $\mathcal{X}\mathcal{F}^{(2)}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $B\text{Ass}_2$. 

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Finally, the exponential plethystic bialgebra in two variables $EP^2$ is the same as $P^2$ but with exponential automorphisms $\text{aut}(\lambda) = \lambda_1!\lambda_2! \cdots$. The generators of this bialgebra are denoted $a_\lambda$.

**Theorem 5.4.9.** The exponential plethystic bialgebra in two variables $EP^2$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $BT_S \cdot \text{Ass}_2$.

### 5.5 $Y$-plethysm and bialgebras from $Y$

In Subsection 5.4 we could have taken the locally finite monoid $(\mathbb{N}, \times)$ instead of $\text{Ass}$, since $T^M \text{Ass} = (\mathbb{N}, \times)$ (Example 4.2.1). In fact, we have indirectly done so in the proof of Theorem 5.4.6. It is the case that the three plethystic bialgebras of Subsection 5.4 can be generalized to any locally finite monoid. In this section we will explain the generalization of $EP$, which arises from $Y$-plethysm, introduced by Méndez and Nava [34] in the context of colored species.

Let $Y$ be a locally finite monoid; this means that any $m \in Y$ there has a finite number of two-step factorizations $m = nk$. This is the same as the finite decomposition property of Cartier–Foata [8]. Consider the ring of formal power series $\mathbb{Q}[x_m|m \in Y]$ without constant term. Following the same conventions as above, the set of variables $\{x_m\}_{m \in Y}$ will be denoted $x$. Elements of $\mathbb{Q}[x]$ are written

$$F(x) = \sum_{\lambda \in \Lambda} \frac{f_\lambda}{\lambda!} x^\lambda,$$

where now the sum is indexed by the subset $\Lambda \subseteq \text{Hom}_{\text{Set}}(Y, \mathbb{N})$ of maps with finite support, and $x^\lambda$ is the obvious monomial, for $\lambda \in \Lambda$. In this case $\lambda! = \prod \lambda_m!$.

The monoid structure of $Y$ defines an operation $x_n \otimes x_m = x_{mn}$, which extends to a binary operation on $\mathbb{Q}[x]$ as

$$(G \otimes F)(x_{m|m \in Y}) := G(F_m|m \in Y), \quad \text{where} \quad F_m(x_{n|n \in Y}) := F(x_{mn}|n \in Y).$$

This substitution operation was introduced in [34] in the context of species colored over a monoid, although their conditions on the monoid are more restrictive. The main example comes from the monoid $(\mathbb{N}^+, \times)$, which gives ordinary plethysm. Another relevant example is $(\mathbb{N}, +)$, which gives $F_k(x) = F(x_k, x_{k+1}, \ldots)$, which appears in [33]. The power series $F_m$ can be described by using the Verschiebung operators: for each $m \in Y$ we define the $m$th Verschiebung operator $V^m$ on $\text{Hom}_{\text{Set}}(Y, \mathbb{N})$ as

$$V^m(Y^\lambda \mapsto \mathbb{N}) = Y \xrightarrow{\lambda} Y \xrightarrow{m} Y \xrightarrow{\lambda} \mathbb{N}.$$
Clearly if $Y = (\mathbb{N}^+, \times)$ this gives the usual Verschiebung operators $[9,36,37]$. The power series $F_m$ can be expressed as

$$F_m(x) = \sum_{\lambda} \frac{f_{\lambda}}{\text{aut}(\lambda)} x^{V_{\lambda m}}.$$ 

As usual, we define the $Y$-plethystic bialgebra $\mathcal{MP}$ as the polynomial algebra $\mathbb{Q}[\{a_\lambda\}_{\lambda}]$ on the set maps $a_\lambda : \mathbb{Q}[x] \to \mathbb{Q}$ defined by $a_\lambda(F) = f_\lambda$, with comultiplication dual to plethystic substitution, that is

$$\Delta(a_\lambda)(F,G) = a_\lambda(G \odot F),$$

and counit given by $\epsilon(a_\lambda) = a_\lambda(x_1)$.

What follows is devoted to express the comultiplication of $\mathcal{MP}$. Consider a list $\mu \in \Lambda^n$ of $n$ infinite vectors, regarded as a representative element of a multiset $\overline{\mu} \in \Lambda^n/\mathfrak{S}_n$. We denote by $R(\mu) \subseteq \mathfrak{S}_n$ the set of automorphisms that maps the list $\mu$ to itself. For example if $\mu = \{\alpha, \alpha, \beta, \gamma, \gamma, \gamma\}$ then $R(\mu)$ has $2! \cdot 1! \cdot 3!$ elements. Notice that if $\mu, \mu' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $R(\mu) \cong R(\mu')$. We may thus refer to $R(\mu)$ for a multiset $\overline{\mu} \in \Lambda^n/\mathfrak{S}_n$ by taking a representative, since we are only interested in its cardinality.

Fix two infinite vectors, $\sigma, \lambda \in \Lambda$, and a list of infinite vectors $\mu \in \Lambda^n$, with $n = |\lambda|$. We define the set of $(\lambda, \mu)$-decompositions of $\sigma$ as

$$T^\mu_{\sigma,\lambda} := \left\{ p : \mu \xrightarrow{\sim} \sum_{m \in Y} \{1, \ldots, \lambda_m\} \mid \sigma = \sum_{\mu \in \mu} V_{q(\mu)}^m \mu \right\},$$

where $p$ is a bijection of $n$-element sets and $q$ returns the index of $p(\mu)$ in the sum. A useful way to visualize an element of this set is as a placement of the elements of $\mu$ over a grid with $\lambda_m$ cells in the $m$th column such that if we apply $V_m^\mu$ to the $m$th column and sum the cells the result is $\sigma$. For example, if $\lambda = (\lambda_{m_1}, \lambda_{m_2}, \lambda_{m_3}) = (2,1,3)$ and $\mu = \{\alpha, \alpha, \beta, \gamma, \gamma, \gamma\}$ the placement

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\gamma & \gamma & \alpha \\
V_{m_1} & V_{m_2} & V_{m_3}
\end{array}
\]

belongs to $T^\mu_{\sigma,\lambda}$ if $\sigma = V_{m_1}^\mu (\gamma + \alpha) + V_{m_2}^\mu (\gamma) + V_{m_3}^\mu (\alpha + \beta + \gamma)$, where the sum is a pointwise vector sum in $\Lambda$. Note that each such placement appears $|R(\mu)|$ times in $T^\mu_{\sigma,\lambda}$. Observe also that if $\mu, \mu' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $T^\mu_{\sigma,\lambda} \cong T^\mu_{\sigma,\lambda}$. We may thus refer to $T^\mu_{\sigma,\lambda}$ for a class $\overline{\mu} \in \Lambda^{|\lambda|}/\mathfrak{S}_{|\lambda|}$ by taking a representative, since we are only interested in its cardinality.
Proposition 5.5.1. The comultiplication of $\mathcal{MP}$ is given by

$$\Delta(\sigma) = \sum_{\lambda} \sum_{\overrightarrow{\mu}} \frac{\text{aut}(\sigma) \cdot |T_{\sigma,\lambda}^{\mu}|}{\text{aut}(\lambda) \cdot \text{aut}(\mu)} \prod_{\mu \in \mu} a_{\mu}. \quad (5.5.1)$$

This proposition is analogous to [9, Proposition 3.3].

Theorem 5.5.2. The $Y$-plethystic bialgebra $\mathcal{MP}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $BT_s^r Y$.

Proof of 5.5.2. Let us compute the homotopy cardinality of the incidence bialgebra of $BT_s^r Y$. First of all, notice that the elements of $B_1 T_s^r Y = S^r T_s^r Y$ are sequences of tuples

$$(m_1^1, \ldots, m_{n_1}^1), \ldots, (m_1^k, \ldots, m_{n_k}^k)$$

of elements of $Y$. Juxtaposition of sequences gives $BT_s^r Y$ a symmetric monoidal structure. Sequences containing only one tuple are called connected, and form an algebra basis of the incidence bialgebra. Since the morphisms between tuples are given by permutations, it is clear that the set of isomorphism classes of connected elements $\pi_0 B_1 T_s^r Y$ is isomorphic to $\Lambda$, the subset of $\text{Hom}_{\text{Set}}(Y, \mathbb{N})$ consisting of maps with finite support. The isomorphism class $\delta_\lambda$ of a connected element $\lambda$ is given by the map $Y^\lambda \to \mathbb{N}$ such that $\lambda_m$ is the number of times $m$ appears in $\lambda$. Be aware that the same notation is used for either the connected elements of $B_1 T_s^r Y$ and the maps representing their isomorphism class. Moreover,

$$\pi_0 B_1 T_s^r Y \cong \sum_n \Lambda^n / \mathcal{S}_n,$$

so that an element $\tau \in \pi_0 B_1 T_s^r Y$ may be identified with a multiset $\overrightarrow{\mu}$ of maps. With these identifications we clearly have

$$|\text{Aut}(\lambda)| = \lambda! \quad \text{and} \quad |\text{Aut}(\tau)| = \text{aut}(\mu),$$

for $\lambda$ connected and $\tau$ not necessarily connected. The left hand sides refer to the automorphisms groups in $B_1 T_s^r Y$, while the right hand sides were introduced above.

The assignment

$$Q_{\pi_0 B_1 T_s^r Y} \to \mathcal{E}\mathcal{P} \quad \delta_\lambda \mapsto a_\lambda \quad \delta_\lambda + \delta_\mu = \delta_{\lambda + \mu} \mapsto a_{\lambda + \mu},$$

for $\lambda$ and $\mu$ connected, defines an isomorphism of algebras. Notice that $\lambda + \mu$ is the monoidal sum in $B_1 T_s^r Y$, which does not correspond to the pointwise sum of their corresponding infinite vectors, since it has two connected components.
We have to compute now the coproduct in $\mathbb{Q}_{\pi_0 B_1 T_s Y}$. It is enough to compute it for connected elements. From Lemma 2.3.1 we have, for $\sigma$ connected,

$$
\Delta(\delta_\sigma) = \sum_{\lambda \in \pi_0 B_1 T_s Y} \sum_{\tau \in \pi_0 B_1 T_s Y} \frac{|\text{Iso}(d_0 \tau, d_1 \lambda)_{\sigma}|}{|\text{Aut}(\lambda)| |\text{Aut}(\tau)|} \delta_\tau \otimes \delta_\lambda.
$$

(5.5.2)

In view of the discussion above, it only remains to show that

$$
|\text{Iso}(d_0 \tau, d_1 \lambda)_{\sigma}| = \text{aut}(\sigma) \cdot |T_{\sigma,\lambda}^\mu|.
$$

Consider representatives for $\tau$ and $\lambda$,

$$
\tau = ((m_1^1, \ldots, m_{n_1}^1), \ldots, (m_k^1, \ldots, m_{n_k}^1))
$$

$$
\lambda = (m_1, \ldots, m_k),
$$

then $d_0 \tau = d_1 \lambda = (1, \ldots, 1)$, $k$ times. This means that

$$
\text{Iso}(d_0 \tau, d_1 \lambda) = \text{Aut}(1, \ldots, 1) \cong S_k.
$$

Any element $\phi \in \text{Iso}(d_0 \tau, d_1 \lambda)$ induces a map between sequences

$$
((m_1^1, \ldots, m_{n_1}^1), \ldots, (m_k^1, \ldots, m_{n_k}^1)) \xrightarrow{\phi} (m_1, \ldots, m_k).
$$

We will express it as a permutation on $\tau$ and write

$$
\phi(\tau) = ((m_1^{\phi(1)^1}, \ldots, m_{n_{\phi(1)}}^{\phi(1)}), \ldots, (m_1^{\phi(k)^1}, \ldots, m_{n_{\phi(k)}}^{\phi(k)})).
$$

Now, consider the subset

$$
\{\phi \in \text{Iso}(d_0 \tau, d_1 \lambda) \mid d_1((\phi(\tau), \lambda)) \simeq \sigma\}.
$$

It is straightforward to see that this subset is isomorphic to

$$
T_{\sigma,\lambda}^\mu := \left\{ p : \mu \overset{\sim}{\longrightarrow} \sum_{m \in M} \{1, \ldots, \lambda_m\} \mid \sigma = \sum_{\mu \in \mu} V^{q(\mu)} \mu \right\},
$$

under the identifications $\tau \rightarrow \mu$ and $\phi \rightarrow p$. The summation of the Verschiebung operators is precisely composition of $\phi(\tau)$ and $\lambda$. Finally, since $\text{Iso}(d_0 \tau, d_1 \lambda)_{\sigma}$ is a homotopy fibre we have that

$$
\text{Iso}(d_0 \tau, d_1 \lambda)_{\sigma} \cong \text{Aut}(\sigma) \times \{\phi \in \text{Iso}(d_0 \tau, d_1 \lambda) \mid d_1((\phi(\tau), \lambda)) \simeq \sigma\} \cong \text{Aut}(\sigma) \times T_{\sigma,\lambda}^\mu
$$

and therefore

$$
|\text{Iso}(d_0 \tau, d_1 \lambda)_{\sigma}| = \text{aut}(\sigma) \cdot |T_{\sigma,\lambda}^\mu|,
$$

as we wanted to see.

This proves also Theorem 5.4.3 by taking the monoid $(\mathbb{N}^+, \times)$. 

\qed
6 Relation with \(TS\)

In this section we will explore the relations between the \(T\)-construction and the simplicial groupoid \(TS\) of [9]. We will first recall how this simplicial groupoid looks like. Then we will prove that \(TS\) and \(BTS_{\mathcal{S}}\) are equivalent simplicial groupoids. This will prove in particular Theorem 5.1.2. Finally we will show that the operads of Section 5 arising from \(Ass\) or \(Sym\) are also equivalent to similar simplicial groupoids.

6.1 The simplicial groupoid \(TS\)

It can be defined through a general construction [9], but we will content ourselves with a brief description: objects in \(T_1S\) and \(T_2S\) (1 and 2-simplices of \(TS\)) are, respectively, diagrams of finite sets and surjections

\[
\begin{align*}
\begin{array}{ccc}
t_{01} & \sim & t'_{01} \\
\downarrow t_{00} & & \downarrow t_{11} \\
t_{11} & \sim & t'_{11}
\end{array}
\end{align*}
\]

Morphisms of such shapes are levelwise bijections \(t_{ij} \sim t'_{ij}\) compatible with the diagram. In general \(T_nS\) is an analogous pyramid, with \(t_{0n}\) in the peak, all of whose squares are pullbacks of sets. The face maps \(d_i\) remove all the sets containing an \(i\) index, and the degeneracy maps \(s_i\) repeat the \(i\)th diagonals. Diagrams whose last set is singleton are called connected. It is not difficult to see that \(TS\) is a Segal groupoid [9].

We will prove now that \(TS \simeq BTS_{\mathcal{S}}\). We will prove the equivalence by constructing an intermediate simplicial groupoid. More precisely, we will find a subsimplicial groupoid of \(TS\) which is equivalent to \(TS\) and isomorphic to \(BTS_{\mathcal{S}}\). First of all we need some notation and elementary results.

**Definition 6.1.1.** Consider the category of finite ordinals \([n] = \{1, \ldots, n\}\) and set maps. We say that a square

\[
\begin{align*}
\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & q & \quad \\
p & \sim & \quad \\
\quad & f & \quad \\
\quad & \quad & \quad \\
\quad & g & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad
\end{array}
\end{align*}
\]

is monotone if it is a pullback of sets, \(p\) is monotone and \(q\) is monotone at each fiber over \(p\), that is, \(q|_{p^{-1}(i)}\) is monotone for all \(i \in [l]\).

**Lemma 6.1.2.** Consider the category of finite ordinals and set maps.

(i) The class of monotone pullback squares is closed under composition of squares.
Given a diagram $[l] \xrightarrow{f} [k] \xleftarrow{g} [n]$, there is a unique monotone square as 6.1.1.

Proof. (i) is clear, and (ii) follows from the fact that we can totally order the pullback,

$$P = \sum_{i \in [k]} [l]_i \times [n]_i,$$

by using the orders of $[l]$ and $[n]$. That is, given $a, b \in P$, then $a < b$ if $p(a) < p(b)$ or $p(a) = p(b)$ and $q(a) < q(b)$.

Consider the full subsimplicial groupoid $\mathcal{V} \subseteq TS$ containing only the simplices whose entries are the finite ordinals $[k]$, whose left-down-arrows and right-arrows are monotone surjections and whose left-down arrows are fiber-monotone in the sense of Definition 6.1.1, and whose pullback squares are monotone. Note that Lemma 6.1.2 ensures that $\mathcal{V}$ is well defined, meaning that the inclusion $\mathcal{V} \hookrightarrow TS$ is a morphism of simplicial groupoids.

**Lemma 6.1.3.** $\mathcal{V} \hookrightarrow TS$ is an equivalence of simplicial groupoids.

Proof. Given an element of $T_nS$,

\[
\begin{array}{cccc}
    & t_{01} & \\
\hline
    t_{00} & \rightarrow & t_{11} & \longrightarrow & t_{n-1,n-1} & \rightarrow & t_{nn}, \\
\end{array}
\]

it is clear we can choose an ordering of the $t_{ii}$ and the $t_{i,i+1}$ such that all the arrows between them are monotone. Then by Lemma 6.1.2 there exists a unique ordering on the rest of the $t_{ij}$'s making the pullback squares monotone. Hence the inclusion is essentially surjective. Since we have taken the full inclusion, the automorphism group of any element of $\mathcal{V}_n$ is equal to to its automorphism group as an element of $T_nS$. Hence the inclusion is an equivalence.

Note that in $\mathcal{V}$ the uniqueness of the monotone squares implies that the Segal maps are in fact isomorphisms,

$$\mathcal{V}_n \cong \mathcal{V}_1 \times \mathcal{V}_0 \cdots \times \mathcal{V}_0 \mathcal{V}_1.$$

In other words, there is a well-defined composition $d_1 : \mathcal{V}_1 \times \mathcal{V}_0 \mathcal{V}_1 \to \mathcal{V}_1$. In view of this we may drop the elements $t_{ij}$ with $j \geq i + 2$ from the diagrams.

**Lemma 6.1.4.** Let $\mathcal{V}$ be the operad whose $n$-ary operations are diagrams

\[
\begin{array}{cccc}
    & [m] & \\
\hline
    [n] & \rightarrow & 1 \\
\end{array}
\]

where $[m] \rightarrow [n]$ is monotone, whose morphisms are entrywise bijections, and whose composition is given by monotone pullback squares. Then $\mathcal{V} \cong BV$.
Proof. The isomorphism is given by
\[
\begin{array}{c}
[m_1] & \leftrightarrow & [m_k] \\
\downarrow & & \downarrow \\
[n_1] & \Rightarrow & [n_1 + \cdots + n_k] \\
\downarrow & & \downarrow \\
1, & \cdots, & [n_2] & \Rightarrow & [k] \\
\end{array}
\]
at the level of 1-simplices and similarly in general. □

Lemma 6.1.5. \( V \) is isomorphic to \( \mathcal{T}_S \text{Sym} \).

Proof. An operation of \( \mathcal{T}_S \text{Sym} \) is a family of operations of \( \text{Sym} \), which is equivalent to a monotone surjection \( [m] \to [n] \). It is also clear that morphisms between operations of \( \mathcal{T}_S \text{Sym} \) are the same as morphisms in \( V \). Thus we only need to see that composition coincides. Let us denote by \( x \) the unique \( x \)-ary operation of \( \text{Sym} \). Thus a general element of \( \mathcal{T}_S \text{Sym} \) is a tuple \( (x_1, \ldots, x_n) \). By definition of the \( \mathcal{T} \)-construction
\[
(x_1, \ldots, x_n) \circ ((y_1^1, \ldots, y_{k_1}^1), \ldots, (y_1^n, \ldots, y_{k_n}^n)) = (y_1^1 \cdot x_1, \ldots, y_{k_1}^1 \cdot x_1, \ldots, y_1^n \cdot x_n, \ldots, y_{k_n}^n \cdot x_n),
\]
which is nothing but the pullback
\[
\begin{array}{c}
\sum_{i,j} y_j^i x_i \\
\downarrow & \downarrow & \downarrow \\
\sum_{i,j} y_j^i & \sum_i x_i \\
\downarrow & \downarrow & \downarrow \\
\sum_i k_i & [n] & \Rightarrow & 1,
\end{array}
\]
the composition of their corresponding operations in \( V \). □

Proposition 6.1.6. The simplicial groupoids \( T \mathcal{S} \) and \( B\mathcal{T}_S \text{Sym} \) are equivalent.

Proof. It is direct from Lemmas 6.1.2, 6.1.3 and 6.1.4. □

6.2 Other \( T \mathcal{S} \)-like simplicial groupoids

We will now see other equivalences between variations of \( T \mathcal{S} \) and some of the bar constructions treated before. First of all we introduce some notation: monotone surjections between ordered sets will be denoted \( a \twoheadrightarrow b \). We will call fiber-ordered surjection \( a \twoheadrightarrow b \) a surjection between finite sets \( f : a \to b \) with an order on \( f^{-1}(r) \) for each \( r \in b \). Test
Example 6.2.1. The simplicial groupoid $B\mathcal{T}_S^r \text{Ass}$ is equivalent to the simplicial groupoid constructed as $TS$ but with the additional structure that all the left-down surjections are fiber-ordered. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

```
    t_{01}
   / \
  t_{00} ----> t_{11}.
```

Clearly isomorphism classes of connected diagrams are again infinite vectors $\lambda = (\lambda_1, \lambda_2, \ldots)$ as in $TS$ and the number of automorphisms of a connected element of class $\lambda$ is precisely $\lambda_1! \cdot \lambda_2! \cdots$, since $t_{01}$ is fixed.

Example 6.2.2. The simplicial groupoid $B\mathcal{T}_M^{s'}^r \text{Ass}$ is equivalent to the simplicial groupoid constructed as $TS$ but with the additional structure that the left-down surjections and the right surjections are fiber-ordered. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

```
    t_{01}
   / \
  t_{00} ----> t_{11}.
```

Observe that for a connected element, $t_{00}$ is totally ordered. Thus the isomorphism classes of connected elements are given by words $\omega = \omega_1 \omega_2 \ldots \omega_n$ where $\omega_i$ is the size of the $i$th fibre. It does not have any automorphisms, since $t_{01}$ and $t_{00}$ are fixed.

Example 6.2.3. The simplicial groupoid $B\mathcal{T}_M^{s'}^r \text{Sym}$ is equivalent to the simplicial groupoid constructed as $TS$ but with the additional structure that the right surjections are fiber-ordered. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

```
    t_{01}
   / \
  t_{00} ----> t_{11}.
```

Observe that for a connected element, $t_{00}$ is totally ordered. Thus the isomorphism classes of connected elements are given by finite words $\omega = \omega_1 \omega_2 \ldots \omega_n$ where $\omega_i > 0$ is the size of the $i$th fibre. It has $\omega! := \omega_1! \omega_2! \cdots \omega_n!$ automorphisms, since $t_{00}$ is fixed.

Example 6.2.4. The simplicial groupoid $B\mathcal{T}_M^{s'}^r \text{Sym}$ is equivalent to the simplicial groupoid constructed as $TS$ but with the additional structure that the right surjections are fiber ordered. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

```
    t_{01}
   / \
  t_{00} ----> t_{11}.
```

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Observe that for a connected element, \( t_{00} \) is totally ordered. Thus the isomorphism classes of connected elements are given by finite words \( \omega = \omega_1\omega_2\ldots\omega_n \) where \( \omega_i > 0 \) is the size of the \( i \)th fibre. It has \( \omega! := \omega_1!\omega_2!\ldots\omega_n! \) automorphisms, since \( t_{00} \) is fixed.

**Example 6.2.5.** The simplicial groupoid \( BT_M'\text{Ass} \) is equivalent to the simplicial groupoid constructed as \( TS \) but with the additional structure that the left-down surjections and the right surjections are fiber-ordered. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

\[
\begin{array}{c}
t_01 \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
A.2 Axioms for \( P \)-operad

Let \( \mathcal{E} \) be a cartesian category and \((P, \mu, \eta)\) a cartesian monad. A \( P \)-multicategory \( Q \) can be described by objects and arrows of \( \mathcal{E} \)

\[
\begin{array}{c}
\text{\( C_0 \times_{C_0} C_1 \rightarrow C_1 \times_{C_0} C_1 \downarrow m \downarrow \)} \\
\text{\( C_0 \times_{C_0} C_1 \downarrow s \rightarrow C_0 \rightarrow \)} \\
\text{\( C_1 \times_{C_0} C_1 \rightarrow C_1 \downarrow m \rightarrow \)} \\
\end{array}
\]

(A.1.4a)

\[
\begin{array}{c}
\text{\( C_1 \times_{C_0} C_0 \rightarrow C_1 \downarrow \)} \\
\text{\( C_1 \times_{C_0} C_0 \downarrow t \rightarrow C_0 \rightarrow \)} \\
\text{\( C_1 \downarrow m \rightarrow \)} \\
\end{array}
\]

(A.1.4b)

where the pullback is taken along \( P Q_1 \rightarrow P Q_0 \leftarrow Q_1 \), satisfying the following commutative diagrams:
\[ PQ_1 \times PQ_0, Q_1 \xrightarrow{p_1} PQ_1 \xrightarrow{p_0} P^2 Q_0 \xrightarrow{m} PQ_0, Q_1 \xrightarrow{p_1} PQ_1 \xrightarrow{m} Q_1 \] (A.2.1a)

\[ Q_0 \xrightarrow{e} Q_1 \xrightarrow{id} Q_0 \] (A.2.2a)

\[ Q_0 \xrightarrow{e} Q_1 \xrightarrow{t} Q_0 \] (A.2.2b)

\[ (P^2 Q_1 \times P^2 Q_0, PQ_1) \times PQ_0, Q_1 \xrightarrow{P Q_1 \times PQ_0, Q_1} PQ_1 \times PQ_0, Q_1 \] (A.2.3)

\[ P^2 Q_1 \times P^2 Q_0, (PQ_1 \times PQ_0, Q_1) \xrightarrow{\mu Q_1 \times \mu Q_0, m} PQ_1 \times PQ_0, Q_1 \xrightarrow{m} Q_1 \]

\[ PQ_0, P Q_0, Q_1 \xrightarrow{p_2} PQ_1 \xrightarrow{m} Q_1 \] (A.2.4a)

\[ Q_1 \times Q_0 Q_0 \xrightarrow{Q_1 \times Q_0 e} PQ_1 \times PQ_0, Q_1 \] (A.2.4b)

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