Some remarks on finite-gap solutions of the Ernst equation

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It is explicitly shown that the class of algebro-geometric (finite-gap) solutions of the Ernst equation constructed several years ago in [1] contains the solutions recently constructed by R. Meinel and G. Neugebauer [2] as a subset.

1. Algebro-geometrical solutions of Ernst equation

The Ernst equations which arises from certain dimensional reduction of 4D Einstein’s equations has the following form:

$$(\mathcal{E} + \bar{\mathcal{E}}) \Delta \mathcal{E} = 2(\mathcal{E}^2 + \bar{\mathcal{E}}^2)$$

where $\mathcal{E}(x, \rho)$ is a complex-valued Ernst potential and

$$\Delta = \partial_x^2 + \frac{1}{\rho} \partial_{\rho} + \partial_{\rho}^2$$

is a cylindrical Laplacian operator. For $\mathcal{E} \in \mathbb{R}$ Ernst equation reduces to the classical Eulers-Darboux equation

$$\Delta \log \mathcal{E} = 0$$

corresponding to static space-times. Denote $x + i\rho$ by $\xi$ and consider the hyperelliptic algebraic curve $L$ of genus $g$ defined by

$$w^2 = (\lambda - \xi)(\lambda - \bar{\xi}) \prod_{j=1}^{g} (\lambda - E_j)(\lambda - F_j)$$

with $\xi = x + i\rho$ symmetric with respect to antiholomorphic involution $\lambda \rightarrow \bar{\lambda}$ that entails for some $m \leq g$

$$E_j = \bar{F}_j, \quad j = 1, \ldots, m$$

$$E_j, F_j \in \mathbb{R} \quad j = m + 1, \ldots, g$$

Introduce on $\mathcal{L}$ the canonical basis of cycles $(a_j, b_j) \quad j = 1, \ldots, g$. Each cycle $a_j$ is chosen to surround the branch cut $[E_j, F_j]$; cycle $b_j$ starts on one bank of the branch cut $[\xi, \bar{\xi}]$, goes on the other sheet through branch cut $[E_j, F_j]$ and comes back. The dual basis of holomorphic differentials $dU_j \quad j = 1, \ldots, g$ is normalized by

$$\oint_{a_j} dU_k = \delta_{jk}$$

Define $g \times g$ matrix of b-periods $B_{jk} = \oint_{b_j} dU_k$ and related $g$-dimensional theta-function $\Theta(z|B)$. Differentials $dU_j$ are linear combinations of non-normalized holomorphic differentials

$$dU_j^0 = \frac{\lambda^{j-1}d\lambda}{w} \quad j = 1, \ldots, g$$

General algebro-geometrical solution of the Ernst equation may be written in many different forms (see [1]). Here it is convenient to use the original form of [1]:

$$\mathcal{E} = \frac{\Theta(U_1^{\infty_1}, K)\Theta(U_1^{\infty_1}, -K)}{\Theta(U_1^{\infty_1}, K)\Theta(U_1^{\infty_1}, -K)} \times \exp\{\Omega_{\infty_1}\}$$

where the new objects are defined as follows: $K$ is a vector of the Riemann constants of $\mathcal{L}$; $D$ is a set (divisor) of $g (\xi, \bar{\xi})$-independent points $D_1, \ldots, D_g$ on $\mathcal{L}$;

$$\left(U_1^{\infty_1}\right)_k \equiv \int_{P_0}^{\infty_1} dU_k - \sum_{j=1}^{g} \int_{P_0}^{D_j} dU_k$$

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with an arbitrary base point \( P_0 \) (entering also vector \( K \)).

It remains to define \( \Omega(\infty^1) - \Omega(\infty^2) \) and vector \( B_\Omega \). Let \( d\Omega(P) \) be an arbitrary locally holomorphic 1-form on \( \mathcal{L} \) with \((\xi, \bar{\xi})\)-independent singularities and related singular parts satisfying the normalization conditions

\[
\int_{a_j} d\Omega = 0 \quad j = 1, \ldots, g
\]

Define its vector of \( b \)-periods

\[
(B_\Omega)_j = \int_{b_j} d\Omega
\]

and require the reality conditions

\[
\Omega(\infty^2) - \Omega(\infty^1) \in \mathbb{R} \quad \text{Re}(B_\Omega)_k = \pm \frac{1}{4}
\]

Now solution (\( 7 \)) is completely defined. If one take \( g = 0 \) then combination of theta-functions in (\( 6 \)) disappears and we get

\[
\mathcal{E} = \exp \left\{ \Omega_0(\infty^2) - \Omega_0(\infty^1) \right\} \in \mathbb{R}
\]

i.e. static solution, which serves as a static background of solution (\( 3 \)). It is easy to show that by an appropriate choice of differential \( d\Omega_0 \) on the Riemann surface \( \mathcal{L}_0 \) given by

\[
w^2 = (\lambda - \xi)(\lambda - \bar{\xi})
\]

one can get arbitrary static solution. Namely, take an arbitrary solution \( \mathcal{E}_0 \in \mathbb{R} \) (for definiteness, asymptotically flat i.e. \( \mathcal{E}_0(\xi = \infty) = 1 \)) satisfying (\( 2 \)) and define 1-form \( d\Omega_0 \) by

\[
d\Omega_0(\lambda, \xi, \bar{\xi}) = \frac{d\lambda}{4} \int_{\infty}^{\xi} d\xi' \frac{\partial}{\partial \lambda} \left( \sqrt{\frac{\lambda - \bar{\xi'}}{\lambda - \xi'}} \right) \frac{\partial \log \mathcal{E}_0(\xi', \bar{\xi'})}{\partial \xi'}
\]

and analogous equation with respect to \( \bar{\xi} \) (which are compatible as a corollary of (\( 2 \)));

\[
d\Omega_0(\lambda) = \frac{d\Omega_0(\lambda)}{d\lambda}
\]

This is a simple example of ”direct scattering procedure” (and analog of Fourier transform): the positions and structure of singularities of \( d\Omega_0 \) carry the whole information about solution \( \mathcal{E}_0 \).

The 1-form \( d\Omega \) on \( \mathcal{L} \) which enters (\( 6 \)) inherits all singularities of \( d\Omega_0 \) on \( \mathcal{L}_0 \) and is assumed to have additional simple poles at the branch points \( E_j \) with the residues \( 1/2 \) and at the branch points \( F_j \) with the residues \(-1/2 \), \( j = 1, \ldots, g \).

Therefore, for fixed genus \( g \) the solution (\( 6 \)) is defined by the following set of data: an arbitrary background solution \( \mathcal{E}_0 \) of the Ehlers-Darboux equation (\( 2 \)) and \((\xi, \bar{\xi})\)-independent points \( \{E_j, F_j, D_j \mid j = 1, \ldots, g \} \).

2. Reduction to Meinel-Neugebauer construction

To obtain the solutions constructed in (\( 2 \)) as a special case of (\( 6 \)) one have to take \( m = g \) i.e. for all \( j = 1, \ldots, g \) one assume

\[
F_j = \bar{E}_j
\]

Then to rewrite solutions (\( 6 \)) in the form of (\( 2 \)) introduce on \( \mathcal{L} \) meromorphic 1-form \( dW \) having the 1st order poles at \( \lambda = \infty^1 \) and \( \lambda = \infty^2 \) with the residues \(-1 \) and \(+1 \) respectively normalized by

\[
\int_{a_j} dW = 0 \quad j = 1, \ldots, g
\]

The following simple identity:

\[
\exp\{W(\tilde{D}) - W(D)\}
\]

\[
\equiv \Theta(U(\infty^2) - U(\tilde{D}) - K) / \Theta(U(\infty^1) - U(D) - K)
\]

\[
\times \Theta(U(\infty^1) - U(D) - K) / \Theta(U(\infty^2) - U(D) - K)
\]

is valid for arbitrary two sets of \( g \) points \( D \) and \( \tilde{D} \) and may be verified by simple comparison of the pole structure of both sides with respect to every \( D_j \) and every \( \tilde{D}_j \).

Thus solution (\( 6 \)) may be rewritten as follows:

\[
\mathcal{E} = \exp \left\{ W(\tilde{D}) + \Omega|_{\infty^2} \right\}
\]

where divisor \( \tilde{D} \) consists of the points \( \tilde{D}_1, \ldots, \tilde{D}_g \) defined by the following system of equations:

\[
U(\tilde{D}) - U(D) = -B_\Omega
\]
The vector in the l.h.s. is understood as

\[(U(\tilde{D}) - U(D))_k = \sum_{j=1}^{g} \int_{D_j} \tilde{D}_j \ dU_k\]

The problem of determining points of \(\tilde{D}\) from \([13]\) is called the Jacobi inversion problem.

Equations \([14]\) may be rewritten in terms of non-normalized basis of holomorphic differentials given by \([1]\) as follows:

\[\sum_{k=1}^{g} \int_{D_k} \frac{\lambda_k^{-1} d\lambda}{w} = \int_{\partial L} \frac{\Omega^{-1} d\lambda}{w} \]

for \(j = 1, \ldots, g\) where \(\partial L\) is the boundary of 4g-sided fundamental polygon \(L\) of surface \(\mathcal{L}\) which is obtained if we cut \(L\) along all basic cycles;

\[\Omega(P) = \int_{P_0}^{P} d\Omega \quad P \in L\]

with arbitrary base point \(P_0 \in L\); choice of \(P_0\) does not influence the r.h.s. of \([16]\). Expression \([13]\) may be easily derived from \([15]\) using the general formula valid for any two 1-forms \(W_{1,2}\) on \(L\) \([1]\):

\[\oint_{\partial L} W_1 dW_2 = \sum_{j=1}^{g} \left\{ A_{W_1} A_{W_2} - B_{W_1} B_{W_2} \right\} \]

(17)

where \(A_{W_{1,2}}\) and \(B_{W_{1,2}}\) are \(a\) and \(b\) periods of the forms \(dW_{1,2}\) (to derive \([16]\) one should take \(dW_1 = d\Omega, \ dW_2 = d\Omega^0\)).

Introducing differential

\[dW^0 \equiv \frac{\lambda^0 d\lambda}{w} \]

(which coincides with \(dW\) up to some combination of holomorphic differentials \([1]\) which provide vanishing of all \(a\)-periods of \(dW\)), and applying \([17]\) to \(d\Omega\) and \(dW_0\), we rewrite \([14]\) as follows:

\[\mathcal{E} = \exp \left\{ \oint_{\partial L} W_0 d\Omega + \int_{\partial L} \Omega dW_0 + \int_{\infty}^{\infty} d\Omega \right\} \]

(18)

Formulas \([16]\) and \([18]\) after identification

\[u_j = \int_{\partial L} \Omega \frac{\lambda_j^{-1} d\lambda}{w} \quad j = 0, \ldots, j - 1 \]

(19)

\[u_g = \oint_{\partial L} \Omega \frac{\lambda_g^{-1} d\lambda}{w} + \int_{\infty}^{\infty} d\Omega \]

(20)

may be rewritten in the following way:

\[\sum_{j=1}^{g} \int_{D_j} \frac{\lambda_j^{-1} d\lambda}{w} = u_j \quad j = 0, \ldots, j - 1 \]

(21)

and

\[\mathcal{E} = \exp \left\{ \sum_{j=1}^{g} \int_{D_j} \frac{\lambda_j^{-1} d\lambda}{w} + u_g \right\} \]

(22)

which precisely coincide with expressions of \([3]\).

Functions \(u_j, \ j = 1, \ldots, g\) satisfy the Laplace equation

\[\Delta u_j = \sigma \]

and the recurrent equations

\[u_j \xi = \frac{1}{2} u_{j-1} + \xi u_{j-1} \xi \]

(23)

as a corollary of the relations

\[\Delta \left( \frac{\lambda^0}{w} \right) \epsilon = \frac{1}{2} \left( \frac{\lambda^{-1}}{w} \right) \epsilon \]

and the residue theorem applied to the contour integral over \(\partial L\).

The static background of solution \([22]\) is given by an arbitrary solution of the Laplace equation \(u_g\) (one could take any other function \(u_{g_j}\), since it would almost uniquely determine the others according to \([23]\)), which may, therefore, be alternatively expressed as

\[u_g = \int_{\infty}^{\infty} d\Omega_0 \]

in terms of the differential \(d\Omega_0\) \([11]\).

Let us show how to choose the parameters of the present construction to get the “dust disc” solution of \([3]\) posed at \((x = 0, \ \rho \leq \rho_0)\).

One should take \(g = 2\), choose some complex \(E_1\) (related to parameter \(\mu\) of \([3]\) and put \(F_1 = E_1; \ E_2 = -F_1; \ F_2 = E_2\). The 1-form \(d\Omega\) should be taken in the form

\[d\Omega(\lambda) = d\tilde{\Omega} + d\tilde{\Omega}\]
where
\[
d\hat{\Omega} \equiv \int_{-i\rho_0}^{i\rho_0} f(\gamma) d\Omega^{(\gamma)}(\lambda) d\gamma
\]  
(24)

with the integral taken along the imaginary axis; 
\(d\hat{\Omega}(\gamma)\) is meromorphic 1-form on \(L\) with vanishing \(\alpha\)-periods and unique pole of the second order at \(\lambda = \gamma\) with leading coefficient equal to 1; \(f(\gamma)\) may be an arbitrary measure satisfying \(\overline{f}(\gamma) = f(\overline{\gamma})\) (for example, \(f \in \mathbb{R}\)). 

\(\log \mathcal{E}_0 = \int_{-i\rho_0}^{i\rho_0} \frac{f(\gamma)d\gamma}{\{(\gamma - \xi)(\gamma - \overline{\xi})\}^{1/2}}\)

Specifying \(f(\gamma)\) in some special way (see \[3\]) one arrives to the “dust disc” solution of \[3\].

Formula \[3\] for the Ernst potential may now be rewritten as follows (see \[3\]):
\[
\mathcal{E} = \frac{\Theta(U|_{\xi}^{\infty} + B_{\hat{\Omega}})}{\Theta(U|_{\xi}^{\infty} + B_{\hat{\Omega}})} \exp \hat{\Omega}|_{\xi}^{\infty},
\]
(25)

where \(2\pi i B_{\hat{\Omega}}\) is the vector of b-periods of \(d\hat{\Omega}\).

3. Summary

We have shown that the solutions of the Ernst equation obtained recently in \[2\] (and, in particular, some partial solution of this class exploited in \[3\] to describe rigidly rotating dust disc) constitute a subclass of the algebro-geometric (finite-gap) solutions found before in \[1\].

In spite of the solutions derived in \[2,3\] are not new, the physical interpretation of special solution of this class proposed in \[3\] would be very interesting if it would really describe the dust disc. However, the rings \(\xi = E_j\) are most probable the singular points of the Weyl scalars, even though the metric coefficients may be finite and differentiable once at these rings. One of the main reasons to expect the singularity of these rings is that the generic asymptotical expansion of \(\mathcal{E}\) at \(\xi = E_j\) contains logarithmic singular terms (this is a standard hypergeometric-like asymptotical expansion near the singularity). By the same reason it is not appropriate for numerical simulation at these points. The only way to prove non-singularity or singularity of solution of \[3\] on the rings would be to find explicit asymptotical expansion of related Weyl scalars at \(\xi = E_j\). Before this is done the physical interpretation of solution \([23,24]\) as the field of rotating dust disc will remain questionable.

Concluding, we express the hope that some of the finite-gap solutions will find reasonable physical application (see also \[3\] for discussion, where, in particular, we describe a solution with toroidal ergosphere).

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