Aspects of Jamming in Two-Dimensional Frictionless Systems

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In this work we provide an overview of jamming transitions in two dimensional systems focusing on the limit of frictionless particle interactions in the absence of thermal fluctuations. We first discuss jamming in systems with short range repulsive interactions, where the onset of jamming occurs at a critical packing density and where certain quantities show a divergence indicative of critical behavior. We describe how aspects of the dynamics change as the jamming density is approached and how these dynamics can be explored using externally driven probes. Different particle shapes can produce jamming densities much lower than those observed for disk-shaped particles, and we show how jamming exhibits fragility for some shapes while for other shapes this is absent. Next we describe the effects of long range interactions and jamming behavior in systems such as charged colloids, vortices in type-II superconductors, and dislocations. We consider the effect of adding obstacles to frictionless jamming systems and discuss connections between this type of jamming and systems that exhibit depinning transitions. Finally, we discuss open questions such as whether the jamming transition in all these different systems can be described by the same or a small subset of universal behaviors, as well as future directions for studies of jamming transitions in two dimensional systems, such as jamming in self-driven or active matter systems.

1 Introduction

Jamming is defined to occur when a system consisting of a collection of interacting particles passes from some type of flowing state with liquid-like properties to a stuck or rigid state with solid-like properties. An everyday example of jamming occurs in the flow of salt or coffee beans down a funnel. Even when the aperture of the funnel is significantly larger than the diameter of any individual flowing particle, the system can undergo a transition from flowing to clogged. At low particle density, the flow in such systems normally continues unperturbed; however, when the density of particles is high enough, a jamming event can occur when enough particles come into contact with each other to form a rigid structure that halts further flow. The jamming in this case is caused by a collective configuration of particles spanning the gap between the funnel walls. Other types of jammed states can also occur in confined and unconfined geometries for increasing particle densities. The formation of these jammed states differs from the freezing of a liquid into a crystalline solid since the jammed states are generally spatially disordered and can have properties that are very different from those of a typical crystalline solid. For example, they may only be jammed against certain types of perturbations but not others, or they may act like a solid in some directions and a liquid in others, so that they often have aspects of fragility and may only be marginally rigid states. An example of such fragility appears when a small tap or shake applied to a clogged salt shaker or to jammed particles in a hopper allows the system to start flowing again after a jamming event. Another aspect that makes jammed systems differ from typical crystallized solids is that jammed systems generally nonthermal. This means that although a system may enter a jammed state at a given density, there is no guarantee that this configuration represents the lowest energy state of the system. Jammed systems may also appear at first sight to be similar to systems that form a glass state; however, in many of the commonly studied jamming systems such as granular media, there is a complete absence of any type of thermal fluctuations, and the question of whether aspects of jamming can be connected directly to a glass remains open.

The basic idea that increasing the density of a collection of interacting particles can cause a change from some type of flowing state to a jammed state applies to a remarkably wide variety of systems. Examples include highway traffic, where when the density of cars is high enough the system can become congested or jammed\textsuperscript{6}, dense flow of people through a doorway in emergency situations\textsuperscript{3,4}, and the flow of motor proteins through a cell, where at high enough density the proteins get in each others'\textsuperscript{5} way and impede the flow. In materials science it is known that the higher the dislocation density in a material becomes, the harder it is to cause the dislocations to move at yielding\textsuperscript{7}. There are also systems that exhibit depinning transitions where a collection of particles interact with each other as well as with pinning sites in a substrate. In this case some of the particles can be trapped directly by the
pinning sites, and these pinned particles can then impede the motion of other particles not directly trapped at pinning sites, causing a flowing system to become jammed\textsuperscript{2}.

Although all these systems can have vastly different length scales and different types of interactions, there has been growing interest in determining whether they can be described by the general concept of jamming, and whether jamming has universal characteristics that are independent of the different microscopic aspects of the systems. Even though jamming and clogging phenomena have been familiar ever since humans began to handle particulate matter, the idea that jamming could be a distinct transition that can be understood using methods of statistical mechanics was only suggested recently. In 1998, Liu and Nagel proposed a scenario in which a loose assembly of particles such as granular media, bubbles, or emulsions could have a unique point, termed point J, as a function of increasing density that marks the transition to a jammed state\textsuperscript{1}. They also suggested that the physics and dynamics near point J could remain robust to perturbations such as the application of a load, where below a critical load the material can be said to be jammed, while above yield the material is unjammed\textsuperscript{1,5,6}. As a function of temperature, the high temperature state of system will form a liquid that flows or is unjammed, while at lower temperatures the system freezes into a jammed solid-like phase\textsuperscript{7}. Around the same time, Cates \textit{et al.} also proposed that certain systems that are solid or in a jammed state represent a new class of materials called fragile materials that only behave like a solid under certain perturbations and that have a strong memory dependence\textsuperscript{8}. Since these simple but elegant ideas were proposed, there has been a tremendous growth in jamming research, and the idea of jamming as a distinct type of transition is now being applied to an ever wider class of materials, including quantum systems\textsuperscript{9}.

Here we focus on aspects of jamming in two dimensions (2D), primarily in systems where there is an absence of friction and where thermal effects are negligible. Although this is only a small subset of the systems exhibiting jamming that are currently being studied, we show that even these simple 2D systems exhibit a remarkable variety of behaviors, and that there are many open problems and new directions to explore in this area.

\section{Nature of the Transitions and Correlation Lengths}

Ever since a scenario for a unified picture of jamming transitions was first proposed in Ref.\textsuperscript{1}, one of the most popular theoretical and numerical models for investigating jamming in 2D systems has been frictionless disks interacting with harmonic or Hertzian short range repulsion with a well defined range, where the disks have two different radii $r_A$ and $r_B$ with $r_A/r_B = 1.48$. A 2D assembly of monodisperse disks forms a triangular lattice at a density close to 0.9, and for this reason the bidisperse disk systems are used since they avoid crystallization.

A key question is whether point J, the transition into a jammed state, has properties similar to those found for phase transitions observed in equilibrium and certain nonequilibrium systems. The first complication is that phase transitions are typically identified though the breaking of some form of spatial symmetry; however, jamming systems are typically disordered or amorphous whether jammed or not. Since there is little or no change in the spatial structure of the system across jamming, identifying the proper order parameter can be challenging. For second order or continuous phase transitions, there is some length scale that shows a divergence as the critical point between the two phases is approached. In the Ising model, the two phases are a ferromagnetic ordered state and a paramagnetic or disordered state, and at the transition between these phases, various quantities become scale free\textsuperscript{10}. If the system exhibits a continuous phase transition, the correlation length $\xi$ grows as a power law as the density $\phi$ approaches the jamming density $\phi_j$,

$$\xi \propto (\phi_j - \phi)^\nu. \quad (1)$$

In principle, if the scaling behavior can be found and the critical exponents such as $\nu$ can be identified accurately, it would be possible to determine the exact nature of the transition and see if it falls into one of the already known universality classes such as Ising model, directed percolation, or Potts\textsuperscript{11}. Once the nature of the transition has been identified in one system that exhibits jamming, other systems with different microscopic details that also exhibit jamming can be examined to see if they fall into the same universality class or to understand why they do not.

Accurately measuring critical exponents can be difficult even in equilibrium systems, where a variety of complications and subtleties can arise. Such complications include understanding how close it is necessary to be to the critical point to obtain accurate scaling and how large the system must be to avoid finite size effects\textsuperscript{12}. The jamming systems have the extra complication of generally being out of equilibrium; however, even nonequilibrium systems can exhibit continuous phase transitions\textsuperscript{13}. Despite the significant number of theoretical and numerical studies on nonequilibrium systems that exhibit phase transitions\textsuperscript{14,15}, only in recent years have experimental systems been realized that exhibit clear nonequilibrium phase transitions\textsuperscript{16,17}.

An issue that often arises in systems that exhibit jamming is that the jamming density obtained for one type of preparation method can differ from that obtained by preparing the system in a different manner\textsuperscript{18,19}. Furthermore, the measurable quantities in the system can differ depending upon whether the
jamming transition is approached from below by increasing the density \( \phi \) or from above by starting in an already jammed high density state and decreasing \( \phi \) to the jamming density \( \phi_j \). For example, below \( \phi_j \) it is possible to arrange the disks such that no two disks are in contact, making it impossible to measure vibrational spectra, whereas in the jammed state above \( \phi_j \) the system acts as a solid, vibrational spectra can be analyzed. Multiple length scales may come into play, but these scales might be distinguishable only in very large systems\(^{18}\).

It is also possible that jamming exhibits a mixture of first and second order features or that it is weakly first order\(^{19}\).

The initial studies of jamming focused on finding evidence that jamming exhibits some critical properties associated with continuous phase transitions, and concentrated on identifying quantities that exhibit scaling near a critical density or a critical load. O’Hern et al. showed that in a frictionless bidisperse disk system several quantities scale upon approaching a critical jamming density of \( \phi_j \approx 0.8444 \), and suggested that one possible way to directly examine a diverging length scale as jamming is approached would be to push a single probe particle through a random assembly of disks and measure the perturbation distance in the surrounding disks. Well below the jamming density, this perturbation length scale would be small, but the perturbation length would diverge as the system approaches the critical jamming density at which it becomes marginally rigid. Drocco et al.\(^{20}\) subsequently simulated dragging a probe particle through a 2D bidisperse frictionless disk assembly, where overdamped motion of the particles was assumed. Figure 1, adapted from Ref.\(^{20}\), shows the location of the probe particle, the unperturbed particles, and the particles that are experiencing a force due to the motion of the probe particle. In Fig. 1(a), at \( \phi = 0.656 \), well below \( \phi_j \), an empty trail forms behind the probe particle. The trail does not refill with particles because there is no pressure in the unjammed phase. As the probe particle moves through the system, it pushes other particles out of the way, and these particles push other particles they encounter, and so forth, producing a perturbed area that varies in size depending on the density of the packing. In Fig. 1(b) at \( \phi = 0.8 \), there is still a void behind the probe particle since the sample is unjammed; however, the extent of the temporarily perturbed region has increased. In Fig. 1(c) at \( \phi = 0.83 \), the system is just at the jamming density for this size of simulation area, and the motion of the probe particle immediately propagates throughout the sample, indicating that the probe particle is dragging all of the other particles along with it. Another signature that the system has jammed is the disappearance of the trailing void region behind the probe particle. Experiments in 2D disk assemblies with a driven probe particle also found that at low density a void or cavity opens behind the probe particle, while near the jamming density the void disappears\(^{21}\).

Let the velocity of a probe particle in an overdamped

![Figure 1](image1.png)

**Fig. 1** Snapshots of a jamming length scale probe showing the perturbation caused by a single driven particle. Blue: Probe particle driven with a constant force in the positive \( x \)-direction; brown: bidisperse background particles; red: the cluster of background particles in force contact with the driven particle at this instant in time. (a) Unjammed state at \( \phi = 0.656 \). (b) Unjammed state at \( \phi = 0.8 \), where the perturbation has crossed the periodic boundary conditions in the \( x \) direction. In (a) and (b) the dragged particle leaves behind an empty trail. (c) Jammed state at \( \phi = 0.83 \), where the dragged particle no longer leaves a trail and the perturbation wraps around the periodic boundaries. The figure is adapted from Ref.\(^{20}\).

![Figure 2](image2.png)

**Fig. 2** \( N_{dr} \), the average number of grains dragged by the probe particle, vs \( \phi \), showing a divergence near \( \phi_j \approx 0.84 \). The inset shows that this divergence has a power law form suggestive of critical behavior. The figure is adapted from Ref.\(^{20}\).
medium under a constant force $F_D$ in the absence of any other particles be $V_0$. When placed inside a packing above the jamming density, the average probe particle velocity $\langle V \rangle$ drops to $V_0/N$, where $N$ is the total number of grains in the system. In the thermodynamic limit, $\langle V \rangle \to 0$. At a particle density just below jamming, $\langle V \rangle = V_0/N_{dr}$, where $N_{dr} < N$ is the number of grains dragged with the probe particle. In Fig. 2, adapted from Ref.20, we plot $N_{dr}$ as a function of $\phi$, showing a divergence at $\phi_{j} \approx 0.84$. The inset of Fig. 2 shows that the divergence has a power law form $N_{dr} \propto (\phi_{j} - \phi)^{\delta}$. Assuming that the cluster of dragged particles is circular in shape, we have $\delta = v(d+2)$ with $d = 2$, and using the value of $\delta$ extracted from the figure, we obtain the critical exponent $\nu = 0.71 \pm 0.02$.

By examining different quantities, O’Hern et al. found the value $\nu = 0.6 - 0.75$. A number of other numerical studies have been performed on the same bidisperse disk system in order to extract critical exponents. Olsson and Teitel conducted shear simulations and found $\nu = 0.6 \pm 0.05$, Heussinger and Barrat obtained $\nu = 0.8 - 1.0$ in quasistatic shear simulations24, and relaxation studies performed by Head gave $\nu \approx 0.6 \pm 0.05$. More recently, Vagberg et al. carried out extensive finite size scaling on very large 2D systems, and argue that corrections to scaling are very important, producing more accurate calculations of $\nu \approx 1.0 \pm 1$. There are also a number of theoretical models that give diverging correlation lengths as point $J$ is approached from the high density side. Wyart et al. obtain $\nu = 1/2$ using a cutting argument, while Silbert et al. argue for two different exponents, $\nu_L = 0.48$ for the longitudinal length and $\nu_T = 0.24$ for the transverse length. Using k-core percolation, Schwarz et al. found two exponents of 1/2 and 1/4, while field theoretical studies of Henkes and Chakraborty give $\nu = 1/4$. Recently Goodrich et al. presented results suggesting that the cutting length found by Wyart et al. can be understood as a rigidity length and that it is correlated with the longitudinal length found by Silbert et al. Waitukaitis et al. performed an experimental study of a dynamical jamming front induced by uniaxial compression of a packing with a rake. Ahead of the wake is a densification front that has a width which diverges as the jamming density is approached. Analysis of the front width gives an exponent $\nu = 0.65$.

Other studies also find scaling in other quantities, such as the contact number $Z$, where $Z - Z_j = Z_0(\phi - \phi_j)^{\eta}$, with $\eta = 0.5$ in 2D8,32,33. This value has also been observed in experiment. One interesting feature observed by O’Hern et al. is that many of the scaling exponents for the different quantities change depending upon the detailed nature of the interparticle interactions. This is different from the behavior typically observed at critical points, where systems that are in the same universality class have the same exponents regardless of the microscopic interactions between the particles. Diverging quantities near jamming were also observed in 2D lattice model.35

Recently, a new study described jamming in sheared 2D systems in terms of a transition from chaotic behavior below jamming to non-chaotic behavior above jamming. Here, the system exhibits chaotic rearrangements below jamming, but above jamming the system becomes rigid and can no longer move in a chaotic fashion. The authors measure a dynamical length scale $\xi_d$, and showed that as the jamming density of $\phi = 0.841$ is approached, $\xi_d$ decreases as a power law, $\xi_d = (\phi - \phi)^{-\alpha}$, with $1.7 < \alpha < 2.1$. This exponent is considerably different than the other diverging length scale exponents discussed previously; however, the length $\xi_d$ may be relevant to the dynamical rather than the static properties of the jamming transition, and may not be closely related to the diverging length scales considered in the other studies. The association of jamming with a transition from chaotic to non-chaotic motion provides another definition for the onset of jamming. It would be interesting to apply a similar approach to jamming systems in which friction between the particles is relevant and where the jamming density is considerably lower than $\phi = 0.841$. Other studies in 2D systems also find growing dynamical length scales as jamming is approached.

The idea of understanding jamming in terms of dynamical processes can also be studied by applying a periodic drive to the sample and observing whether the system reaches a reversible state. This approach was used by Corte et al. for colloids well below jamming, where a transition from reversible to irreversible behavior was linked to a nonequilibrium phase transition. Schreck et al. recently studied a frictionless disk system using the periodic shear protocol of Ref. and found several different dynamical regimes below $\phi_j$, including a point reversible regime, transient reversible regime, and an irreversible regime. This work suggests that there may be additional length scales or transitions at densities below $\phi_j$ that come into play under driving. For example, Shen et al. found a percolation transition at a density $\phi_p < \phi_j$. The percolation occurs through the formation of a non-rigid system-spanning cluster that causes the onset of non-trivial mechanical responses related to the emergence of correlated particle motions. If there are additional dynamical transitions below $\phi_j$, this could be of importance in connecting jamming with glass transitions. Olsson and Teitel found in 2D systems that the onset of glassy behavior occurs at a density $\phi_g < \phi_j$. It would be interesting to understand whether $\phi_g$ is related to the percolation density $\phi_p$ found by Shen et al. For densities greater than $\phi_j$, studies have suggested that there could be new types of phenomena with different mechanical properties in what is called the deep jamming regime, where certain quantities scale differently than they do close to point $J$ in the marginally jammed regime. If very dense jammed systems exhibit distinct phenomena, this could be relevant for understanding many types of jammed soft matter systems such as...
emulsions\textsuperscript{43} or foams\textsuperscript{44}, where densities well above $\phi_j$ can be accessed.

Although there has been mounting evidence that point J exhibits critical properties, the strong variations in the values of the extracted critical exponents may mean that more than one length scale diverges at point J, or that point J has properties that are different from those normally associated with critical points. Even for the heavily studied bidisperse disk system, the exact nature of the jamming transition remains an open issue. Once this has been settled, the next question will be to determine whether the same type of transition exists in other jamming systems and frictional systems.

Comparisons between theoretical predictions and jamming experiments can be complicated by the existence of frictional effects in the experimental system, both in the interactions between grains and in the motion of the grains against the containing vessel. Typically, 2D jamming densities found in experiments\textsuperscript{21,34,35} are lower than the value $\phi_j = 0.844$ obtained from simulations, and this difference is likely due to friction. There are several possible jamming scenarios for the frictional systems. For example, there could be two separate jamming transitions, a frictional jamming at lower density as well as a point J frictionless jamming at higher density that could be accessed by further compressing the system. There are some studies indicating that frictional jamming can have the same nature as frictionless jamming\textsuperscript{45}.

3 Fluctuations Near Jamming

Another approach to understanding jamming is by analyzing changes in dynamically fluctuating quantities. If the fluctuations increase near jamming, this would be consistent with the idea that jamming has the properties of a second order transition, with scale free fluctuations of all sizes at the critical point. Geng and Behringer studied the fluctuating drag force on a probe particle dragged at constant velocity through packings of different densities\textsuperscript{46}. They found a characteristic packing density $\phi_j \approx 0.653$ below which the force fluctuations become very small. This characteristic packing density is considerably below the ideal point J density $\phi = 0.844$, indicating that friction plays an important role in their system. Nonetheless, as the packing density increases toward $\phi_j$ in this study, the average force $\langle F \rangle$ on the probe increases according to a power law, $\langle F \rangle \propto (\phi_j - \phi)^{1.53}$, while the distribution of force fluctuations develops increasingly long tails that can be fit to an exponential distribution. The power spectrum of the force fluctuation time series has a $1/f^2$ form and the overall noise power increases with increasing density. The distribution of the size of avalanche-like jumps in force is exponential\textsuperscript{47}.

In simulations of a probe particle driven with constant applied force through a frictionless packing, the velocity fluctuations also become increasingly intermittent upon approaching $\phi_j = 0.844$\textsuperscript{20,48}, as shown in Fig. 3(a,b) at $\phi = 0.71$ and $\phi = 0.8414$. The average velocity $\langle V \rangle$ of the probe particle decreases linearly with increasing $\phi$ until $\phi \approx 0.83$, above which $\langle V \rangle$ drops off more rapidly before reaching zero for $\phi > 0.844$\textsuperscript{48}. The distribution of the velocity fluctuations $P(V)$ for $\phi < \phi_j$ has an exponential tail, illustrated in Fig. 3(c) for $\phi = 0.807$\textsuperscript{48}, similar to that found experimentally in Ref.\textsuperscript{47}. As the density increases, there is a crossover in the distribution to a power law form, as illustrated in Fig. 3(d), where the $\phi = 0.8414$ curve is fit by a power law with exponent $\alpha = -2.75$\textsuperscript{48}. It is possible that power law distributions were not observed experimentally since they appear only in the critical region very close to the frictionless point J.

Measurements of the power spectra of the probe particle velocity fluctuations in experiments and simulations show a characteristic knee shape for $\phi < \phi_j$ and $1/f^2$ behavior at high frequency\textsuperscript{47,48}. Very close to $\phi_j$, the form of the spectrum changes to a low frequency $1/f^{1.1}$ characteristic with a persistent $1/f^2$ shape at high frequency. The appearance of $1/f$ noise is consistent with the occurrence of large scale rearrangements on length scales up to the entire system size, which produce excess low frequency noise. For packings with $\phi > \phi_j$, which can be simulated using harmonic particle-particle interactions, a finite threshold force $F_c$ must be applied in order to induce local plastic rearrangements of the jammed grains and move the
probe particle through the packing, and the value of $F_C$ increases with increasing $\phi_j^{84}$. At $\phi = 0.844$, the velocity versus force curves scale as $(V) \propto (F - F_C)^{\beta}$ with $\beta \approx 1.5$, while for $\phi > \phi_j$ the exponent changes to $\beta = 0.5 - 1.0$, indicating a modification in the dynamics of the probe particle. Studies of the plastic depinning of interacting particles driven over random substrates produce velocity-force exponents of $\beta = 1.5 - 2.0$ when the flow is strongly intermittent$^{49}$, while a single particle driven over a random or periodic substrate has $\beta = 0.5^{50}$. This suggests that in the critical region near but below point J, global plastic rearrangements occur as large portions of the particles move in response to the probe particle, while above point J, the system remains jammed and only a smaller localized region of particles near the probe particle undergoes plastic rearrangements when the probe particle depins.

The dynamic force fluctuations near jamming have also been investigated experimentally using an intruder particle driven at constant velocity$^{51}$. A small vibration was used to reduce the effect of friction in these experiments, allowing access to densities close to $\phi = 0.841$. Large scale rotational motions can occur around the intruder particle. As the jamming density is approached in this work, the force needed to keep the intruder particle moving at a constant average velocity diverges, and the intruder particle moves in bursts of activity that become more intermittent as the density increases. By analyzing the distribution of the bursts and using scaling, the authors provide strong evidence for the divergence of several quantities and obtain a critical exponent of $\nu = 1.0$. Event-driven simulations of frictional 2D systems with a driven intruder particle show that as the density increases, the intruder mobility decreases, dropping close to zero for $\phi \approx 0.842$. Kolb et al.$^{21}$ conducted experiments of driven intruder particles and find a divergence in the mean force fluctuations $\Delta F$ as the jamming density is approached, with the form $\Delta F \propto f/(\phi - \phi_j)^{1}$. They also find large scale circular or rotational motions of the surrounding media near the probe particle.

4 Different Types of Particle Geometries

Beyond circular disks, jamming in 2D can also be explored for more complex particle shapes$^{52}$, permitting the realization of different jamming densities and different average particle contact numbers. One example of such a system is “granular polymers” composed of grains strung into a finitely flexible chain with a fixed minimum bending angle between any two grains along the chain. Such chains have been experimentally studied with vibration$^{53,54}$ and shear$^{55}$ apparatus. In experiments on 3D granular polymer arrangements, examination of the density of the system revealed that the final packing density decreases with increasing chain length$^{56}$.

Granular polymers represent an ideal system in which the jamming density $\phi_j$ can be tuned by changing the length of the chains, and they can be used to study whether the same types of jamming or critical behaviors found for the bidisperse disk system still occur. In 2D simulations of non-thermal frictionless granular polymers with a finite bending angle$^{58,59}$, jamming was detected by compressing two walls and identifying the onset of a finite pressure as well as by conducting shear measurements. In Fig. 4(a) we show an image from Ref.$^{58}$ of the unjammed state of a 2D granular polymer chain simulation with 16 monomers per chain, and in Fig. 4(b) we show the same system in a jammed configuration. For comparison, in Fig. 4(c) we illustrate the jammed configuration obtained from compression of a bidisperse disk system. The jammed state of the granular polymers contains a large number of voids that are absent in the jammed bidisperse disk system, indicating that the jamming density is significantly reduced in the granular polymer system. In Fig. 5(a), adapted from Ref.$^{20}$, we show that the pressure $P$ in a bidisperse disk system increases linearly with $\phi$ for $\phi > \phi_j$, as previously observed$^{34}$. For granular polymers of length 6, 8, 10, 16, and 24 grains, Fig. 5(b) shows that the jamming density drops with increasing chain length, and that above jamming the linear scaling of $P$ with $\phi$ is replaced by an exponential scaling$^{59}$. In bidisperse disk systems, the contact number $Z$ scales as jamming is approached from above with an exponent $\beta = 0.5^{34}$, while in the granular polymer system, $\beta = 0.6 - 0.8$, and $\beta$ increases with increas-
cycle, producing a hysteretic response. The plastic events occurring in the bulk of the sample during the first or coordination number. On the other hand, in compressive oc-rangements occur in the bulk, so there is no hysteresis in the pressure several times, above jamming no plastic rearrangement events occur due to plastic rearrangements in the system. Figure adapted from Ref. 29.

Fig. 5 (a) Pressure $P$ versus $\phi$ for the bidisperse disk system. Jamming occurs close to $\phi = 0.84$, and above jamming, $P$ increases linearly with increasing $\phi$. (b) $P$ vs $\phi$ for granular polymer systems with chain lengths of 6 (filled blue triangles), 8 (open green triangles), 10 (filled orange diamonds), 16 (open red squares), and 24 (filled black circles). Above jamming, $P$ does not increase linearly but is better fit by an exponential. (c) $P$ vs $\phi$ for a granular polymer system with chains of length 38 for the initial compression (red) and a subsequent wall compression cycle (black), showing strong hysteresis. Inset: a zoom of the main panel showing that during the first compression cycle, sudden drops in the pressure occur due to plastic rearrangements in the system. Figure adapted from Ref. 29.

There have not yet been any direct measures of a diverging length scale in granular polymer systems, so the question of whether jamming for the granular polymers has a different nature from jamming in bidisperse disk packings remains open. Although the pressure and coordination number scale differently in the granular polymer and bidisperse disk systems, this fact alone is not enough to establish that jamming in the two systems is different and that there is thus more than one type of jamming transition. It is possible that extraction of a true critical length scale in the granular polymer system would require performing measurements on samples of extremely large size.

If bidisperse frictionless disk systems are subjected to cycles of compression and pass through the jamming density several times, above jamming no plastic rearrangement events occur in in the bulk, so there is no hysteresis in the pressure or coordination number. On the other hand, in compressive cycling of the granular polymer systems, strong plastic rearrangements occur in the bulk of the sample during the first cycle, producing a hysteretic response. The plastic events occur in abrupt avalanches, as illustrated in the inset of Fig. 5(c) where a sudden pressure drop event occurs. Such drops are associated with the collapse of the void structures shown in Fig. 4(b). Under repeated compression, the density of the granular polymers gradually increases until eventually the system saturates to a final jamming density $\phi_f$, that is significantly higher than the $\phi_j$ measured during the first compression cycle. At $\phi_f$, the granular polymer system exhibits properties that are more similar to those found at point J for the bidisperse disk system; however, some voids always remain present in the granular polymer packing. The behavior of the granular polymers may be relevant to the case of frictional bidisperse disks, which also jam at a much lower density than frictionless disks. If the frictional packing is subjected to further compression, the frictional contacts can be broken, producing plastic rearrangements accompanied by hysteresis. The hysteresis vanishes when the system undergoes only elastic displacements, which could correspond to reaching the frictional point J. This scenario would imply that there are two jamming transitions, one at lower density associated with more fragile structures such as loops or frictional contacts, and the other at higher density corresponding to the frictionless or fully elastic jamming at point J.

Beyond granular polymers, other types of 2D particle geometries can be considered, such as dumbbell shapes. At first glance, it might seem as if a dumbbell packing would have a lower jamming density than the bidisperse disk system, but this is not necessarily the case; frictionless dumbbells with only a pointlike contact between the two dumbbell halves actually jam at a higher density of $\phi = 0.9$, and the jammed state is an ordered triangular lattice. If the elongation of the dumbbells is varied by increasing the amount of overlap between the two halves of each dumbbell, the jamming density can be modified and the jammed state becomes disordered. Shreck et al. report evidence that different types of scaling occur for a system of distorted dumbbells than for bidisperse disk systems. Mailman considered frictionless elliptical-shaped particles and also found scaling properties different from those of bidisperse disk packings, supporting the idea that such systems exhibit a different type of jamming transition. It would be interesting to extract the critical exponent for a diverging length scale in these systems to see if it is the same as in the bidisperse disk system or if the elongated grain shapes can exhibit multiple jamming behaviors. In addition, an analysis of the force fluctuations in these systems in the jamming regions could show whether the fluctuations are exponentially distributed, like the avalanche and velocity fluctuations of the frictional bidisperse disk systems, or whether they are more consistent with critical phenomena.
5 2D Jamming With Quenched Disorder

There are many examples of systems where a collection of particles interact not only with each other but also with obstacles or pinning sites. Such systems include vortices in type-II superconductors, classical Wigner crystals, colloids in disordered media, and topological defects such as skyrmions. Another example is granular or colloidal media flowing through an array of posts, where it could be expected that as the number of obstacles $N_p$ increases, the density at which jamming occurs should decrease. By adding obstacles and a drift force to a 2D bidisperse disk system which has a clean jamming density of $\phi = 0.84$, it is possible to measure any change in the jamming density and determine whether the properties of the jamming are altered by the obstacles. When $\phi > \phi_j$, a single obstacle ($N_p = 1$) suffices to pin the entire system, while for $\phi < \phi_j$ the particles should be able to flow past a single obstacle. Now suppose $\phi$ is held fixed at a value $\phi < \phi_j$ while $N_p$, the number of obstacles, increases. Once $N_p$ is high enough, the particles cease to flow and the system enters a jammed state. In Fig. 6(a), adapted from Ref. 55, we mark where the jammed and unjammed phases appear on a plot of $N_p/N_j$ vs $\phi$, where $N_j$ is the number of disks at point J in the absence of obstacles. The obstacles are modeled as pinning sites that can each capture at most one disk. A disk trapped at a pinning site can escape from the pinning site if the net force on the disk from other disks and the drift force is high enough, the particles cease to flow and the system becomes a rigid elastic solid.

Between the black and red lines, the system flows plastically when a drift force larger than $F_c$ is applied, with a coexistence of moving and jammed regions in the sample. As the number of obstacles increases, the value of $\phi$ at which jamming occurs initially decreases linearly with $N_p$, until for $N_p/N_j \geq 0.1$, it begins to flatten off as the system becomes much easier to pin. Near point J, the correlation length grows as $\xi \propto (\phi_j - \phi)^{-\nu}$, and the system should jam when $\xi$ becomes larger than the average distance between pinning sites, $l_p \propto \rho_p^{-1/2}$, where $\rho_p$ is the pinning density. Thus, the line demarking the jammed state should follow $\rho_p \propto (\phi_j - \phi)^{2\nu}$. The linear behavior of this line shown in Fig. 6(a) then gives $2\nu = 1$ or $\nu = 1/2$. This exponent is in agreement with certain predictions, but differs from others.

The jammed or pinned states that form for $N_p/N_j > 0.1$ and $\phi < 0.8$ are strongly spatially heterogeneous, as shown in Fig. 7(a) for a sample with $\phi/\phi_j = 0.761$ and $N_p/N_j = 0.415$. In contrast, at the same pinning density but for $\phi/\phi_j = 1.04$, Fig. 7(b) shows that the jammed state is homogeneous. The heterogeneous states that form at lower densities...
Fig. 7 (a) The positions of bidisperse disks driven with a drift force through an obstacle array. (a) The jammed state at $\phi/\phi_j = 0.761$ and $N_p/N_j = 0.415$, where $\phi_j$ is the pin free jamming density. The disk density is highly spatially heterogeneous, with some empty areas and other local clusters that have a density close to 0.844. (b) The jammed state for $\phi/\phi_j = 1.04$ and $N_p/N_j = 0.415$ is homogeneous. Figure adapted from Ref. 55.

differ from the states near point J since they are very fragile, meaning that they are sensitive to the direction of the drift force under which they were prepared. If a system in this fragile regime reaches a jammed state due to a drift force applied along the $x$ direction, but is then exposed to a different drift force applied along a new direction (such as the $y$ direction), the disks may begin to flow again and may or may not organize into a new jammed or pinned state. For densities above the red line in Fig. 6(a), the jammed state is robust under all directions of drift force and local plastic deformations do not occur even when the drift force exceeds $F_c$, and all of the particles begin to move. The lower density heterogeneous states are better described as clogged states due to their directional fragility. These clogged states retain a connection to point J since the local density of the clusters of disks is close to the obstacle-free jamming density of $\phi_j = 0.844$. Thus, the lower density clogged states effectively phase separate into locally jammed regions of the point J density and regions of zero density. The effect of pinning on jamming can be more clearly seen in Fig. 6(b,c). In Fig. 6(b) we plot the critical force $F_c$ necessary for the disks to be depinned versus $\phi/\phi_j$ in the low pinning density regime for $N_p/N_j = 0.00138$, 0.00692, 0.0346, and 0.09267. At $N_p/N_j = 0.00138$ there is only a single pinning site in the system, so $F_c = 0$ for $\phi/\phi_j < 1.0$ and $F_c$ is finite only for $\phi/\phi_j > 1.0$. In this regime, once the system depins it moves as a rigid solid. For increasing $N_p/N_j$, the density $\phi/\phi_j$ at which $F_c$ becomes finite decreases, indicating that the system is effectively jammed at lower densities. The depinning behavior is different in character for $\phi/\phi_j > 1.0$ and $\phi/\phi_j < 1.0$. At the lower disk densities, depinning is associated with large plastic deformations, and the $F_c$ vs $\phi/\phi_j$ curves have a peak value at the transition from plastic to elastic depinning. At low pinning densities, the depinning threshold increases upon crossing the jamming density from below; however, for higher pinning densities the behavior is reversed. The onset of jamming at higher pinning densities can be defined to occur when the system depins rigidly or acts like an elastic solid. Here, the unjammed or liquidlike state at $\phi/\phi_j < 1.0$ is more strongly pinned than the jammed or rigid state at $\phi/\phi_j > 1.0$. This is illustrated in Fig. 6(c) where we plot $F_c$ vs $\phi/\phi_j$ for $N_p/N_j = 0.828$, 0.415, and 0.277. There is a sharp drop in $F_c$ near $\phi/\phi_j$, and the system is much more strongly pinned for $\phi/\phi_j < 1.0$. A very similar phenomenon is believed to be behind what is called the peak effect in type-II superconducting vortex systems. In samples where the peak effect is observed there are a large number of weak pinning sites. When the vortices form a solid structure they are weakly pinned, while when the vortex structure is liquidlike, the vortices can easily adjust their positions to take advantage of the low energy pinning site positions, and the system is strongly pinned. This result shows that the onset of jamming can either increase or decrease the effectiveness of pinning of friction depending on the density or strength of the pinning disorder. Brito et al. investigated 2D jamming in systems where a fraction of the particles are pinned down, and found that as long as the packings are isostatic, the presence of pinning does not modify the jamming transition.

Future directions could include examining periodic arrangements of defects, where the pinning length scale would be much better defined. Additionally, for periodic substrates the jamming density may be a function of the direction of the drift force with respect to the symmetry directions of the substrate. Another approach would be to consider the effects of obstacles or pinning when the jammed state is reached not with a drift force, but by allowing the disk radii to expand until the energy of the relaxed system remains finite; properties such as the shear modulus, vibrational spectrum, and force chains could then be examined.

6 Systems with Longer Range Interactions

Several works have shown that ideas from the jamming of 2D granular packings can be applied successfully to systems in which the particle-particle interactions are intermediate or long ranged, rather than short-ranged as in granular media. For example, in charged colloidal suspensions the interactions can have two length scales, an intermediate range repulsive Yukawa interaction and a shorter range steric repulsive interaction. Dislocations and vortices in type-II superconductors are examples of systems with long range repulsive interactions. A jamming scenario for 2D gliding dislocations was recently explored in Ref. 68, where the dislocations are modeled as point particles with both attractive and repulsive long range strain interactions. Depending on the sign of
of Ref. 1 of Ref. 66; this is redrawn in Fig. 1 of Ref. 66 to show that the inverse density axis in the dislocation system does not end at a point J but extents out to infinity. Finally, the yield stress $\tau_\infty$ grows with the dislocation density as $\tau_\infty \propto \rho^{1/2}$, in agreement with many experiments. An open question is whether the yielding transition for dislocations is similar to the yielding transition for granular disk systems, or whether the long range interactions in the dislocation system fundamentally change the nature of the yielding.

The fact that dislocations jam at any density is attributed to the long range or power law nature of the interactions between them. A similar argument should apply to other systems with long range interactions, such as unscreened charged systems or logarithmically interacting superconducting vortices. The situation can change if some type of screening cuts off the long range interactions. For instance, in polydisperse charged colloidal systems, studies of a probe particle driven through a 3D packing found that below the packing density at which the steric or short-range interactions between the particles become important, a finite threshold force must be applied to move the particle, indicating that the system is forming a weak amorphous solid. In 2D simulations of Yukawa interacting particles, where a single probe particle is driven through the system, there is a finite force threshold for motion even at low densities. In Fig. 8(a) we plot the trajectories of bidisperse Yukawa interacting particles when a probe particle is driven through the bulk. The density is well below the jamming density of particles with a small but finite steric interaction radius; nevertheless, a finite force must be applied in order to move the probe particle, which induces strong localized motion in the surrounding particles. Unlike the granular jamming system just at point J, where rearrangements caused by a local probe particle are spread throughout the entire system, in Fig. 8(a) large rearrangements close to the driven particle serve to localize the disturbance, indicating that further away in the bulk the system is acting like a solid. At low enough densities, the Yukawa particles are so far apart that they no longer interact with each other, and the system is no longer jammed. This suggests that there could be two jamming transitions in Yukawa systems: one at low densities when the longer range interactions become effective, and a second transition at higher densities that is similar to point J for granular media, when the short range steric repulsion interactions become important.

The question of whether systems with intermediate or long range interactions jam can also be approached by analyzing the velocity versus force (V-F) curves obtained with a driven probe particle. If the system is jammed, there should be a finite critical depinning force $F_c$ required to move the probe particle, and the V-F curves may have the form

$$V \propto (F - F_c)^\beta.$$  

In a viscous unjammed fluid, $F_c = 0$ and generally $\beta = 1.0$, although complex or glassy fluids may have additional nonlinear features in the V-F curves that we do not consider here. In Fig. 8(b) we plot the average velocity $V$ of two different sizes of probe particles $V$ vs $F - F_c$ for the Yukawa system from Fig. 8(a). For the large probe particle, which induces strong plastic distortions in the background particles when it moves, we find $\beta = 1.47$ as indicated by the red dashed line. This exponent is close to that observed for plastic depinning of vortices and colloids moving over random disorder. In a system with a lower density of particles, $F_c$ decreases but the V-F scaling persists. For soft particles with a finite range interaction range, once the density is low enough $F_c = 0$ and the system is unjammed. This density corresponds to $\phi_j$ in the bidisperse disk system. If the particle-particle interactions are isotropic, it is possible to define an effective finite particle radius $R$, and the system could be considered jammed when the effective area density of the particles reaches 0.84. We note that even when the density is lower than the value at...
which $F_c = 0$, the V-F curves could still scale as $V \propto F^\beta$; however, the scaling range would diminish in width until eventually $\beta = 0$. For systems with infinite range interactions, such as dislocation assemblies, $F_c$ would decrease but still be finite for arbitrarily low dislocation densities. Another limit occurs when the charge of the probe particle becomes so small compared to the charge of the bulk particles that the probe particle induces virtually no plastic rearrangements in the background particles, which act as fixed obstacles to the probe particle motion. In this case, $F_c$ can still be finite but the scaling exponent becomes $\beta = 1.0$, as shown by the blue curve in Fig. 8(b). It is likely that in this limit, the system has the same behavior as a single particle being driven over a random substrate where $\beta = 0.5$ very close to $F_c$\textsuperscript{39}, with a crossover to $\beta = 1.0$ at higher drives. The $\beta = 0.5$ behavior appears only extremely close to $F_c$, and the measurement in Fig. 8(b) was taken outside this range.

For superconducting vortex matter, there have been numerical\textsuperscript{20} and experimental\textsuperscript{11} studies of 2D systems in which the vortices pass through a constricting geometry. Due to the repulsive vortex-vortex interactions, the vortex motion may cease or be strongly reduced due to jamming behavior. This system resembles hopper flow of granular media\textsuperscript{25}, and demonstrates that the idea of jamming can be applied to other systems. Similar studies could be performed for the flow of charged colloids through funnel geometries.

### 7 Active Matter and Quantum Matter

Another class of systems that has been attracting much recent interest is self-propelled or active matter. Such systems could be comprised of swimming bacteria, Janus colloidal particles, pedestrians, or traffic. There has already been considerable work on understanding clogging or jamming behavior in pedestrian and traffic models.\textsuperscript{34} Here we focus on active matter made of self-driven colloidal particles, where it has been shown that even for systems in which the interactions between the particles are purely repulsive, the addition of self-propulsion can produce a transition to a clustered or self-jammed state.\textsuperscript{76} Henkes et al. considered a 2D system of harmonically repulsive self-propelled disks moving according to overdamped dynamics, which develop directed motion when confined by a circular container. In Fig. 9 we show images from Ref.\textsuperscript{77} highlighting the instantaneous particle velocities in the low density ($\phi = 0.6$) unjammed phase and the higher density ($\phi = 0.95$) jammed phase. Even in the jammed phase, where the net motion of the particles is minimal, there are still small correlated particle motions. This work suggests that active matter jammed phases might exhibit collective motions reminiscent of those found in crawling cells. In a subsequent study of self-driven polar disks with isotropic repulsive forces and no alignment, the system formed a cluster state or phase separated state as the density or activity of the disks increased.\textsuperscript{77} In experimental studies of self clustering of monodisperse active particles, the clusters were termed “living crystals” since the self-jammed clusters can exhibit crystalline order.\textsuperscript{24} Although the living crystals are not strictly jammed, if such a system were placed in a confining environment, the self-clustering effect could cause the system to become effectively jammed or unable to move. This suggests that perhaps another axis for the jamming phase diagram of Ref.\textsuperscript{11} would include a self motility feature such as run length, where at very long run lengths the system forms a cluster.

There have recently been proposals for applying jamming ideas to 2D or 3D quantum systems. Nussinov et al. con-

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**Fig. 9** Images of an active matter system in the liquid phase at $\phi = 0.6$ (left) and in the jammed phase at $\phi = 0.95$ (right), with arrows indicating the instantaneous velocity field. Reprinted with permission from S. Henkes, Y. Fily, and M.C. Marchetti, Phys. Rev. E 84, 040301(R) (2011). Copyright 2011 by the American Physical Society.

**Fig. 10** Jamming phase diagram for quantum systems with axes of inverse density, stress, and inverse mass. Reprinted with permission from Z. Nussinov, P. Johnson, M.J. Graf, and A.V. Balatsky, Phys. Rev. B 87, 184202 (2013). Copyright 2013 by the American Physical Society.
sidered a mapping between classical and quantum systems to examine quantum critical jamming of hard-core bosons undergoing a metal (flowing state) to insulator (jammed state) transition. In Fig. 11 we show a schematic jamming phase diagram for quantum systems from Ref., where at zero temperature a point J is expected to exist. Boninsegni et al. also argued that jamming could occur in a quantum system in a disordered assembly of $^4$He atoms, and that this could enhance the metastability of superfluid glass states. The extension of jamming concepts to quantum and semi-classical systems is a new area and we expect that this will be a growing field.

8 Summary

We have provided an overview of jamming in 2D systems focusing mostly on frictionless particle assemblies that form amorphous packings. There has been considerable evidence mounting that at least in frictionless bidisperse disk systems, jamming has the properties of a continuous phase transition, including critical fluctuations. Despite our extensive work on this system, the exact nature of the transition remains elusive, which could be due to a number of subtle issues. There is also evidence that other types of transitions, detectable via dynamic response, could be occurring below the jamming point J, while for high densities above point J a new type of deep jamming behavior can occur. We discuss jamming in 2D systems with varied particle shapes, such as an assembly of chains which jams at significantly lower densities than the disk systems. New types of jamming or clogging behaviors can occur for disk assemblies in the presence of obstacles or pinning sites, suggesting that there may be a new axis, quenched disorder, on the jamming phase diagram. The ideas of jamming in 2D can also be applied to systems with longer range interactions such as dislocations, vortices in superconductors, and charged colloids, and it remains an open question whether jamming in these systems has the same universal properties as jamming in disk systems. Active matter or self-driven particle systems is another area in which ideas of jamming can be applied, and recent work suggests that yet another axis, activity, can be described and can produce dynamic jamming in an active matter jamming phase diagram. Ideas of jamming are also beginning to be applied to quantum systems. Jamming has become a field in its own right as the ideas of jamming can be applied to a wide variety of situations, and jamming should continue to be an active and growing field for the foreseeable future.

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