CONDITIONS OF DISCRETENESS OF THE SPECTRUM FOR SCHRÖDINGER OPERATOR AND SOME OPTIMIZATION PROBLEMS FOR CAPACITY AND MEASURES

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Abstract. For the Schrödinger operator $H = -\Delta + V(x)\cdot$, acting in the space $L_2(\mathbb{R}^d) (d \geq 3)$, with $V(x) \geq 0$ and $V(\cdot) \in L_{1,loc}(\mathbb{R}^d)$, we obtain some constructive conditions for discreteness of its spectrum. Basing on the Mazya-Shubin criterion for discreteness of the spectrum of $H$ and using the isocapacity inequality and the concept of base polyhedron for the harmonic capacity, we have estimated from below the cost functional of an optimization problem, involved in this criterion, replacing a submodular constrain (in terms of the harmonic capacity) by a weaker but additive constrain (in terms of a measure). By this way we obtain an optimization problem, which can be considered as an infinite-dimensional analogue of the optimal covering problem. We have solved this problem for the case of a non-atomic measure. This approach enables us to obtain for the operator $H$ some sufficient conditions for discreteness of its spectrum in terms of non-increasing rearrangements, with respect to measures from the base polyhedron, for some functions connected with the potential $V(x)$. We construct some counterexamples, which permit to compare our results between themselves and with results of other authors.

1. Introduction

In the present paper we consider the Schrödinger operator $H = -\Delta + V(x)\cdot$, acting in the space $L_2(\mathbb{R}^d) (d \geq 2)$. In what follows we assume that $d \geq 3$, $V(x) \geq 0$ and $V(\cdot) \in L_{1,loc}(\mathbb{R}^d)$. Physically $V(x)$ is the potential of an external electric field.

A wide literature is dedicated to investigation of the spectrum for this operator. In particular, the case of discrete spectrum is interesting where it consists only of isolated eigenvalues of $H$ with finite multiplicities. With the point of view of Quantum Mechanics in this case an electron can move only in a compact neighborhood of an atom kernel along an orbit from a discrete collection of orbits. The simplest condition for discreteness of the
spectrum is: \( \lim_{|x| \to \infty} V(x) = \infty \) \([6]\). Physically this means that there is a potential barrier at infinity. The first sufficient and necessary condition for discreteness of the spectrum to \( H \) in the case of a semi-bounded below potential was obtained by A.M. Molchanov \([14]\). For the one-dimensional case \( (d = 1) \) this criterion has the form: for any \( r > 0 \) \( \lim_{|x| \to \infty} \int_x^{x+r} V(t) \, dt = \infty \).

But in the multi-dimensional case \( (d \geq 2) \) the Molchanov’s criterion is more complicated, because it involves so called “negligible” subsets of \( \mathbb{R}^d \), i.e., ones having a small harmonic capacity. In the paper \([13]\) V. Mazya and M. Shubin have improved significantly the Molchanov’s result. In order to formulate the result from \([13]\), let us introduce some notations. Consider in \( \mathbb{R}^d \) an open domain \( G \) satisfying the conditions:

(a) \( G \) is bounded and star-shaped with respect to any point of an open ball \( B_{\rho}(0) \) \( (\rho > 0) \) contained in \( G \);

(b) \( \text{diam}(G) = 2 \).

As it was noticed in \([13]\), condition (a) implies that \( G \) can be represented in the form

\[
G = \{ x \in \mathbb{R}^d : x = r\omega, \, |\omega| = 1, \, 0 \leq r < r(\omega) \},
\]

where \( r(\omega) \) is a positive Lipschitz function on the standard unit sphere \( S^{d-1} \subset \mathbb{R}^d \). For \( r > 0 \) and \( y \in \mathbb{R}^d \) denote

\[
G_r(y) := \{ x \in \mathbb{R}^d : r^{-1}x \in G \} + \{ y \}.
\]

Denote by \( N_\gamma(y, r) \) \( (\gamma \in (0, 1)) \) the set of all compact sets \( F \subseteq G_r(y) \) satisfying the condition

\[
\text{cap}(F) \leq \gamma \text{cap}(\bar{G}_r(y)),
\]

where \( \text{cap}(F) \) is the harmonic capacity.

**Theorem 1.1.** \([13]\) The spectrum of the operator \( H \) is discrete, if for some \( r_0 > 0 \) and for any \( r \in (0, r_0) \) the condition

\[
\lim_{y \to \infty} \inf_{F \in N_\gamma(y, r)} \int_{G_r(y) \setminus F} V(x) \, dx = \infty,
\]

is satisfied, where

\[
\forall r \in (0, r_0) : \gamma(r) \in (0, 1) \quad \text{and} \quad \lim_{r \to 0} r^{-2} \gamma(r) = \infty,
\]

In \([13]\) also a necessary condition for discreteness of the spectrum was obtained, which is close to sufficient one.

Notice that condition \( (1.4) \) of Theorem \([1.1]\) is hardly verifiable, because in order to test it, one needs to solve a difficult optimization problem, whose cost functional is the set function \( \mathcal{I}(F) = \int_{G_r(y) \setminus F} V(x) \, dx \) and the constrain \( F \in N_\gamma(y, r) \) is submodular (because “cap” is a submodular set function \( \text{(definition (2.2))} \)). In the present paper we estimate the cost functional from below using the isocapacity inequality \( \text{(2.3)} \) and replacing \( F \in N_\gamma(y, r) \) by a weaker but additive constrain. To this end we also use the concept of base polyhedron for the harmonic capacity \( \text{(definition (2.4))} \). By this way
on the base of Theorem 1.1 we obtain some sufficient conditions for discreteness of the spectrum in terms of measures, which permit a reformulation in terms of non-increasing rearrangements of some functions connected with the potential $V(x)$.

Let us notice that in the papers [1], [20], [9] and [4] some constructive sufficient conditions for discreteness of the spectrum for $H$ have been found. In [9] also the case of a unbounded below potential was studied and in [4] also the case of a matrix-valued potential was considered. For a scalar non-negative potential $V(x)$ the result of [4] is most general among all these conditions, but our results are more general (see Remarks 3.9, 3.15 and Example 5.4).

Let us make an overview of our results. Theorem 3.1 is obtained by a direct use of the isocapacity inequality and yields a sufficient condition of discreteness of the spectrum for $H$ in terms of an optimization problem involving Lebesgue measure instead of the capacity. This problem can be considered as an infinite-dimensional analogue of the optimal covering problem [15]. We formulate it in a more general form - for any non-negative measure (Problem 3.2), with the purpose to use its solution for derivation of results involving the base polyhedron of harmonic capacity. Theorem 3.3 solves Problem 3.2 for a non-atomic measure ([8]). In Proposition 3.4 we obtain a two-sided estimate for the solution of Problem 3.2 obtained in Theorem 3.3 via a non-increasing rearrangement of the function $W(x)$ taking part in the cost functional. This estimate enables us to formulate most of our conditions of discreteness of the spectrum for $H$ in terms of non-increasing rearrangements. In such formulations these conditions are easier verifiable than ones obtained by a direct use of Theorem 3.3. Theorem 3.6 yields a condition for discreteness of the spectrum based immediately on Theorems 3.1, 3.3 and Theorem 3.7 is reformulation of it in terms of the rearrangement of the potential $V(x)$ with respect to Lebesgue measure.

All the rest our results are based on the concept of base polyhedron for harmonic capacity. Condition for discreteness of the spectrum given in Theorem 3.10 involves the optimization problem, formulated in Problem 3.2, with measures from the base polyhedron. Condition for discreteness of the spectrum given in Theorem 3.11 involves only measures from the base polyhedron, which are equivalent to Lebesgue measure, and it is formulated in terms of non-increasing rearrangements, with respect to these measures, for products of $V(x)$ and the densities of Lebesgue measure with respect to them. Corollary 3.12 is based on Theorem 3.11 and Proposition A.2 where a constructive description of a part $M_f(y, r)$ of the base polyhedron is obtained (see also Definition 2.2). In turn, Proposition A.2 rests on isocapacity inequality (2.3) and the description, obtained in [2] for the core of convex distortions of probability measures. Theorem 3.14, based on Corollary 3.12, yields a easier verifiable condition for discreteness of the spectrum for $H$ in terms of rearrangements of the potential $V(x)$ with respect to Lebesgue measure on cubes from $m$-adic partitions of unit cubes. In the formulation
of Theorem 3.14 we use our concept of \((\log m, \theta)\)-dense system of subsets of a unit cube (Definition 2.5), which permits to place cubes from \(m\)-adic partitions by their sizes. We need this placement because, roughly speaking, on these cubes the lower bound of relative Lebesgue measure of the sets of points, where \(V(x)\) tends to infinity as the cubes go to infinity, depends on their sizes. The property of \((\log m, \theta)\)-density ensures that on balls with any centers and arbitrary small radius a similar dependence on the radius is conserved for the lower bound of relative measure of the sets, mentioned above. This circumstance enables us to use Corollary 3.12 in the proof of Theorem 3.14.

The paper is organized as follows. After this Introduction, in Section 2 (Preliminaries) we introduce some concepts and notations used in the paper. In Section 3 we formulate the main results, in Section 4 we prove them. In Section 5 we construct examples of \((\log m, \theta)\)-dense systems of sets and some counterexamples, which permit to compare our results between themselves and with results of other authors. Section A is Appendix, where we formulate and prove Proposition A.1 on existence of the non-empty base polyhedron for harmonic capacity and Proposition A.2 mentioned above.

2. Preliminaries

Let us come to agreement on some notations and terminology. Let \(\Omega\) be an open and bounded domain in \(\mathbb{R}^d\). We denote by \(\Sigma_B(\Omega)\) the \(\sigma\)-algebra of all Borel subsets of \(\Omega\). By \(\Sigma_L(\Omega)\) we denote the \(\sigma\)-algebra of all Lebesgue measurable subsets of \(\Omega\), i.e., it is the Lebesgue completion of \(\Sigma_B(\Omega)\) by the Lebesgue measure \(\text{mes}_d\). If \((X, \Sigma, \mu)\) is a measure space, we call all sets from \(\Sigma\) \(\mu\)-measurable and if \(X = \bar{\Omega} \subseteq \mathbb{R}^d\), \(\mu = \text{mes}_d\) and \(\Sigma = \Sigma_L(\Omega)\), we simply call them measurable. If a measure is absolutely continuous with respect to Lebesgue measure, we simply call it absolutely continuous. By \(B_r(y)\) we denote the open ball in \(\mathbb{R}^d\) whose radius and center are \(r > 0\) and \(y\).

Let us recall the definition of the harmonic (or Newtonian) capacity\(^1\) of a compact set \(E \subset \mathbb{R}^d\) ([13]):

\[
(2.1) \quad \capa(E) := \inf \left\{ \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx : u \in C^\infty(\mathbb{R}^d), \, u \geq 1 \text{ on } E, \right.
\]

\[
\left. \quad u(x) \to 0 \text{ as } |x| \to \infty \right\}.
\]

It is known [3] that the set function “\(\capa\)” can be extended in a suitable manner from the set of all compact subsets of the space \(\mathbb{R}^d\) to the set of all Borel subsets of it. It is known ([12], [11]) that the set function “\(\capa\)” is submodular (concave) in the sense that for any pair of sets \(A, B \in \Sigma_B(\Omega)\)

\[
(2.2) \quad \capa(A \cup B) + \capa(A \cap B) \leq \capa(A) + \capa(B).
\]

\(^1\)In the Russian literature it is often called Wiener capacity.
the \textit{isocapacity inequality} is valid:

\begin{equation}
∀ F ∈ \Sigma_B(\overline{Ω}) : \text{mes}_d(F) \leq c_d (\text{cap}(F))^{d/(d-2)}
\end{equation}

with \(c_d = (d(d-2)(\text{mes}_d(B_1(0))))^{2/d} - d/(d-2)\), which comes as identity if \(F\) is a closed ball.

By \(M(\overline{Ω})\) denote the set of all additive set functions on \(\Sigma_B(\overline{Ω})\) (we shall call them briefly “measures”) and by \(M^+(\overline{Ω})\) denote the set of all non-negative measures from \(M(\overline{Ω})\). In the theory of coalition games ([18], [16], [10]) the concept of the \textit{core} of a game is used. Following to [5], we define for the harmonic capacity on \(\overline{Ω}\) a dual concept of the \textit{base polyhedron} \(BP(\overline{Ω})\):

\begin{equation}
BP(\overline{Ω}) := \{\mu \in M^+(\overline{Ω}) : \mu(A) \leq \text{cap}(A) \text{ for all } A ∈ \Sigma_B(\overline{Ω}) \text{ and } \mu(\overline{Ω}) = \text{cap}(\overline{Ω})\}.
\end{equation}

This set is nonempty, convex, and compact in the weak*-topology (Proposition A.1). If \(Ω = \overline{B}_r(y)\) (see (1.2), (1.1)), we shall write briefly \(M^+(y, r)\) and \(BP(y, r)\). Denote by \(BP_{eq}(\overline{Ω})\) the subset of \(BP(\overline{Ω})\) consisting of Radon measures which are equivalent to Lebesgue measure \(\text{mes}_d\) ([7]).

Let us recall the concept of \textit{measure preserving mapping} ([2], Definition 2).

**Definition 2.1.** Let \((Ω, A, Q)\) be a probability space and \(λ\) be Lebesgue measure on \([0, 1]\). A measurable function \(s : Ω → [0, 1]\) is called \textit{measure preserving}, if \(λ(B) = Q(s^{-1}(B))\) for any Borel subset of \([0, 1]\). Denote by \(S(Ω, Q)\) the collection of all such functions.

Suppose that \(Ω = \overline{B}_r(y)\), \(A = Σ_L(\overline{B}_r(y))\) and \(Q\) is the normalized Lebesgue measure \(m_{d,r}\) on \(\overline{B}_r(y)\), defined by:

\begin{equation}
m_{d,r}(A) := \frac{\text{mes}_d(A)}{\text{mes}_d(\overline{B}_r(y))} \quad (A ∈ Σ_L(\overline{B}_r(y))).
\end{equation}

We shall need the following set of measures, which is a part of \(BP_{eq}(\overline{B}_r(y))\) (Proposition A.2):

**Definition 2.2.** Consider the function \(f(t) = t^t(d-2)/d \quad (t ∈ [0, 1])\) and denote by \(M_f(y, r)\) the set of absolute continuous measures on \(\overline{B}_r(y)\), whose densities run over the following convex set:

\begin{equation} \text{Co}(y, r) := \text{cap}(\overline{B}_r(0)) \cdot \text{co}(\{f' ∘ s : s ∈ S(\overline{B}_r(y), m_{d,r})\}), \end{equation}

where “\(\text{co}''\)” denotes the convex hull and the closure is taken for the \(L_1(\overline{B}_r(y), m_{d,r})\) topology.

**Remark 2.3.** We shall consider the unit cube \(Q = [-1, 1]^d \subset \mathbb{R}^d\) and the dilation of it \(Q_r := r ∙ Q\ (r > 0)\) and the translation of the latter \(Q_r(y) := Q_r + \{y\}\).

Denote by \([x]\) the integer part of a real number \(x\). In the sequel we need the following concepts:
**Definition 2.4.** We call a subset of $\mathbb{R}^d$ a *regular parallelepiped*, if it has the form $\times_{k=1}^{d}[a_k, b_k].$

**Definition 2.5.** Suppose that $m > 1$ and $\theta \in (0, 1).$ A sequence $\{D_n\}_{n=1}^{\infty}$ of subsets of a cube $Q_1(y)$ is said to be a $(\log_m, \theta)$-dense system in $Q_1(y),$ if

(a) each $D_n$ is a finite union of regular parallelepipeds;
(b) for any cube $Q_r(z) \subseteq Q_1(y)$ with $r \in \left(0, \min\{1, \frac{1}{m^2}\}\right)$ there is $j \in \{1, 2, \ldots, \left[\log_m \left(\frac{1}{m^2}\right)\right]\}$ such that for some regular parallelepiped $\Pi \subseteq D_j$ there is a cube $Q_{\theta r}(s),$ contained in $\Pi \cap Q_r(z).$

### 3. Main results

Denote by $M_{\gamma}(y, r)$ $(\gamma \in (0, 1))$ the collection of all Borel sets $F \subseteq G_r(y)$ satisfying the condition $\text{mes}_d(F) \leq \gamma \text{mes}_d(G_r(y)),$ where the domain $G_r(y)$ is defined by (1.2), (1.1). A direct use of isocapacity inequality (2.3) leads to the following claim:

**Theorem 3.1.** Suppose that for some $r_0 > 0$ and any $r \in (0, r_0)$ the condition

\begin{equation}
\lim_{|y| \to \infty} \inf_{F \in M_{\gamma}(y, r)} \int_{G_r(y) \setminus F} V(x) \, dx = \infty
\end{equation}

is satisfied with a function $\tilde{\gamma}(r)$ satisfying the condition

\begin{equation}
\forall r \in (0, r_0) : \tilde{\gamma}(r) \in (0, 1) \text{ and } \limsup_{r \to 0} r^{-2(2-d)/\gamma} \tilde{\gamma}(r) = \infty.
\end{equation}

Then the spectrum of the operator $H = -\Delta + V(x)$ is discrete.

In order to represent the expression in left hand side of (3.1) in a more constructive form, we need to solve the following optimization problem:

**Problem 3.2.** Let $(X, \Sigma, \mu)$ be a measure space with a non-negative measure $\mu$ and $W(x)$ be a non-negative function defined on $X$ and belonging to $L_1(X, \mu).$ For $t \in (0, \mu(X))$ consider the collection $\mathcal{E}(t, X, \mu)$ of all $\mu$-measurable sets $E \subseteq X$ such that $\mu(E) \geq t.$ The goal is to find the quantity

\begin{equation}
I_W(t, X, \mu) = \inf_{E \in \mathcal{E}(t, X, \mu)} \int_E W(x) \mu(dx).
\end{equation}

In the formulation of next claim we shall use the following notations. For the measure space and the function $W(x),$ introduced in Problem 3.2 consider the quantity:

\begin{equation}
J_W(t, X, \mu) := \int_{K_W(t, X, \mu)} W(x) \mu(dx) + (t - \kappa_W(t, X, \mu)) W_*(t, X, \mu),
\end{equation}

where $W_*(t, X, \mu)$ is the non-decreasing rearrangement of the function $W(x),$ i.e.,

\begin{equation}
W_*(t, X, \mu) := \sup\{s > 0 : \lambda_*(s, W, X, \mu) < t\} \quad (t > 0)
\end{equation}
with
\[ \lambda_*(s, W, X, \mu) = \mu(L_*(s, W, X)), \]
\[ L_*(s, W, X) = \{ x \in X : W(x) \leq s \}. \]
Furthermore,
\[ \mathcal{K}_W^-(t, X, \mu) = L_*(s^-, W, X)|_{s=W^*(t, X, \mu)}, \]
and
\[ \kappa_W^-(t, X, \mu) = \mu(\mathcal{K}_W^-(t, X, \mu)), \]
where
\[ L_*(s^-, W, X) = \bigcup_{u<s} L_*(u, W, X) = \{ x \in X : W(x) < s \}. \]

The following claim solves Problem 3.2 for a non-atomic measure:

**Theorem 3.3.** Suppose that, in addition to conditions of Problem 3.2, the measure \( \mu \) is non-atomic. Then

(i) for any \( t \in (0, \mu(X)) \) there exists a \( \mu \)-measurable set \( \tilde{K} \subseteq X \) such that
\[ \mu(\tilde{K}) = t, \]
for the quantity \( J_W(t, X, \mu) \), defined by (3.4)-(3.9), the representation
\[ J_W(t, X, \mu) = \int_{\tilde{K}} W(x) \mu(dx) \]
is valid and
\[ \forall x \in \tilde{K} : W(x) \leq W^*(t, X, \mu), \]
\[ \forall x \in X \setminus \tilde{K} : W(x) \geq W^*(t, X, \mu); \]

(ii) the equality
\[ I_W(t, X, \mu) = J_W(t, X, \mu) \]
is valid.

In the next claim we shall obtain a two-sided estimate for the solution \( J_W(t, X, \mu) \) of Problem 3.2 via a non-increasing rearrangement of the function \( W(x) \) on \( X \). This rearrangement is following:
\[ \bar{W}^*(t, X, \mu) := \sup\{ s > 0 : \lambda^*(s, W, X, \mu) \geq t \} \quad (t > 0) \]
where
\[ \lambda^*(s, W, X, \mu) = \mu(L^*(s, W, X, \mu)), \]
\[ L^*(s, W, X) = \{ x \in X : W(x) \geq s \}. \]

The promised claim is following:

\[ \text{In this notation we use the accent “bar”, because in the literature ones denote by } W^* \text{ the non-increasing rearrangement with the strong inequalities in its definition.} \]
Proposition 3.4. Suppose that, in addition to conditions of Problem 3.2 and Theorem 3.3, the measure $\mu$ is finite. Then for $\theta > 1$ and $t \in (0, \mu(X))$ the estimates

\begin{equation}
J_W(\mu(X) - t/\theta, X, \mu) \geq \frac{(\theta - 1)t}{\theta} \bar{W}^*(t, X, \mu),
\end{equation}

\begin{equation}
J_W(\mu(X) - t, X, \mu) \leq (\mu(X) - t) \bar{W}^*(t, X, \mu)
\end{equation}

are valid.

Remark 3.5. If $X = \bar{\Omega}$, where $\Omega$ is an open domain in $\mathbb{R}^d$ and $\mu = \text{mes}_d$, we shall omit $\mu$ in the notations introduced above, i.e., to write $J_W(t, \bar{\Omega})$, $W_*(t, \bar{\Omega})$, $\bar{W}^*(t, \bar{\Omega})$, $\lambda_*(s, W, \bar{\Omega})$ and $\lambda^*(s, W, \bar{\Omega})$. In the case where $\Omega = G_r(y)$ we shall write $J_W(t, y, r, \mu)$, $W_*(t, y, r, \mu)$, $\bar{W}^*(t, y, r, \mu)$, $\lambda_*(s, W, y, r, \mu)$ and $\lambda^*(s, W, y, r, \mu)$. If $G_r(y)$ is a ball $B_r(y)$, we shall use the same notations, if they could not lead to a confusion. If $\mu = \text{mes}_d$, we shall omit $\mu$ in the last notations too.

On the base of Theorems 3.1 and 3.3 we obtain the following claim:

Theorem 3.6. Suppose that for some $r_0 > 0$ and any $r \in (0, r_0)$ the condition

\begin{equation}
\lim_{|y| \to \infty} J_V(\sigma(r), y, r) = \infty
\end{equation}

is satisfied, where $\sigma(r) = (1 - \tilde{\gamma}(r)) \text{mes}_d(G_r(0))$ and $\tilde{\gamma}(r)$ satisfies conditions (3.2). Then the spectrum of the operator $H = -\Delta + V(x)$ is discrete.

Proposition 3.4 enables us to reformulate Theorem 3.6 in terms of the non-increasing rearrangement of the potential $V(x)$, defined above.

Theorem 3.7. Suppose that for some $r_0 > 0$ and any $r \in (0, r_0)$ the condition

\begin{equation}
\lim_{|y| \to \infty} \bar{V}^*(\hat{\delta}(r), y, r) = \infty
\end{equation}

is satisfied with $\hat{\delta}(r) = \hat{\gamma}(r) \text{mes}_d(G_r(0))$ and $\hat{\gamma}(r)$ satisfies conditions (3.2). Then the spectrum of the operator $H = -\Delta + V(x)$ is discrete.

Remark 3.8. Notice that condition (3.20) is easier verifiable than condition (3.19). On the other hand, estimates (3.17) and (3.18) imply that these conditions are equivalent in the following sense: for some function $\hat{\gamma}(r)$ satisfying conditions (3.2) the condition (3.19) is satisfied with $\sigma(r) = (1 - \tilde{\gamma}(r)) \text{mes}_d(G_r(0))$ if and only if for some function $\hat{\gamma}(r)$ satisfying conditions (3.2) the condition (3.20) is satisfied with $\hat{\delta}(r) = \hat{\gamma}(r) \text{mes}_d(G_r(0))$.

Remark 3.9. From Theorem 6 of [4] the following criterion of discreteness of the spectrum of the operator $H = -\Delta + V(x)$ follows (in our notations):
if for some numbers \( \delta > 0, c \in (0,1) \) and \( r_0 > 0 \) and for any \( y \in \mathbb{R}^d, r \in (0, r_0) \) the condition
\[
(3.21) \quad \lambda^* \left( \frac{\delta}{\text{mes}_d(Q_r(0))} \right) \int_{Q_r(y)} V(x) \, dx, V, r, y \geq c \text{mes}_d(B_r(0))
\]
is fulfilled, then the discreteness of the spectrum of the operator \( H \) is equivalent to the condition: \( \lim_{|y| \to \infty} \int_{Q_r(y)} V(x) \, dx = \infty \) for some (hence for every) \( r > 0 \). Recall that the cube \( Q_r(y) \) is defined in Remark 2.3. It is easy to see that as a sufficient condition this criterion follows from Theorem 3.7, with \( \mathcal{G}_r(y) = Q_{r_1}(y - r\vec{a}) \) \( (r_1 = 2r/\sqrt{d}, \vec{a} = (d^{-1/2}, d^{-1/2}, \ldots, d^{-1/2})) \), if one takes \( \tilde{\gamma}(r) \equiv c \).

The concept of base polyhedron for harmonic capacity enables us to obtain some conditions of discreteness of the spectrum for Schrödinger operator, covering potentials which do not satisfy conditions of Theorem 3.7. For \( \mu \in M^+(y, r) \) denote by \( M^+_{\gamma}(y, r) \) \( (\gamma \in (0,1)) \) the collection of all Borel sets \( F \subseteq \mathcal{G}_r(y) \) satisfying the condition
\[
(3.22) \quad \mu(F) \leq \gamma \mu(\mathcal{G}_r(y)).
\]
The following claim is valid:

**Theorem 3.10.** Suppose that for some \( r_0 > 0 \) and any \( r \in (0, r_0) \)
\[
(3.23) \quad \lim_{|y| \to \infty} \sup_{\mu \in \text{BP}(y, r)} \inf_{F \in M^+_{\gamma}(y, r)} \int_{\mathcal{G}_r(y) \setminus F} V(x) \, dx = \infty,
\]
where \( \gamma(r) \) satisfies the condition (1.5). Then the spectrum of the operator \( H = -\Delta + V(x) \) is discrete.

Since each measure \( \mu \in \text{BP}_{eq}(y, r) \) is equivalent to the Lebesgue measure \( \text{mes}_d \), the Lebesgue completion of \( \Sigma_B(\mathcal{G}_r(y)) \) by \( \mu \) coincides with \( \Sigma_L(\mathcal{G}_r(y)) \). Hence we can consider the complete measure space \( (\mathcal{G}_r(y), \Sigma_L(\mathcal{G}_r(y)), \mu) \). For \( \mu \in \text{BP}_{eq}(y, r) \) denote by \( \alpha_\mu(x) \) density of the measure \( \text{mes}_d \) with respect to \( \mu \), i.e.,
\[
(3.24) \quad \alpha_\mu := \frac{d \text{mes}_d}{d \mu}.
\]

The following claim is based on Theorems 3.10, 3.3 and Proposition 3.4:

**Theorem 3.11.** Suppose that the condition
\[
(3.25) \quad \lim_{|y| \to \infty} \sup_{\mu \in \text{BP}_{eq}(y, r)} \tilde{Z}_{\mu}^*(\psi_\mu(r), y, r, \mu) = \infty
\]
is satisfied for some \( r_0 > 0 \) and any \( r \in (0, r_0) \), where
\[
(3.26) \quad Z_\mu(x) = \alpha_\mu(x) V(x), \quad \psi_\mu(r) = \gamma(r) \mu(\mathcal{G}_r(0)) \quad \text{and} \quad \gamma(r) \text{ satisfies conditions (1.5).}
\]
Then the spectrum of the operator \( H = -\Delta + V(x) \) is discrete.
The following consequence of the previous theorem we obtain replacing the set $BP_{eq}$ by its part $M_f(y, r)$ (see Definition 2.2 and Proposition A.2):

**Corollary 3.12.** Suppose that the condition

$$
(3.27) \quad \lim_{|y| \to \infty} \sup_{\mu \in M_f(y, r)} Z^*_\mu(\psi_\mu(r), B_r(y), \mu) = \infty
$$

is satisfied for some $r_0 > 0$ and any $r \in (0, r_0)$, where $Z_\mu$ is defined by (3.26), $\psi_\mu(y, r) = \gamma(r)\mu(B_r(y))$ and $\gamma(r)$ satisfies conditions (1.5). Then the spectrum of the operator $H = -\Delta + V(x)$ is discrete.

On the base of Corollary 3.12 we shall formulate a easier verifiable sufficient condition for discreteness of the spectrum of $H$ in terms of rearrangements of the potential $V(x)$ on cubes of $m$-adic partitions of unit cubes with respect to Lebesgue measure. Recall that we have defined the cube $Q_r(y)$ in Remark 2.3. Consider a covering of the space $\mathbb{R}^d$ by the cubes $Q_1(\vec{l})$ ($\vec{l} \in \mathbb{Z}^d$) and for any $\vec{l} \in \mathbb{Z}^d$ consider a sequence $\{D_j(\vec{l})\}_{j=1}^\infty$ of subsets of $Q_1(\vec{l})$. Furthermore, for some integers $n > 0$ and $m > 1$ consider the $m$-adic partition of each cube $Q_1(\vec{l})$: $\{Q(\vec{\xi}, n)\}_{\vec{\xi} \in \Xi_n(\vec{l})}$, where $Q(\vec{\xi}, n) = Q_{m-n}(\vec{\xi})$ and $\Xi_n(\vec{l}) = \{\vec{\xi} \in m^{-n} \cdot \mathbb{Z}^d : Q(\vec{\xi}, n) \subseteq Q_1(\vec{l})\}$. Denote

$$
(3.28) \quad \Xi_n(\vec{l}, j) = \{\vec{\xi} \in m^{-n} \cdot \mathbb{Z}^d : Q(\vec{\xi}, n) \subseteq D_j(\vec{l})\}.
$$

**Remark 3.13.** For brevity we shall write in the next theorem and in its proof $Z^*_\mu(u, y, r), V^*(u, y, r)$ and $\overline{V}^*(u, \vec{\xi}, n)$ instead of

$$
Z^*_\mu(u \cdot \mu(B_r(y)), B_r(y), \mu), \quad V^*(u \cdot \text{mes}_d(Q_r(y)), Q_r(y))
$$

and $\overline{V}^*(u \cdot \text{mes}_d(Q(\vec{\xi}, n)), Q(\vec{\xi}, n))$.

The promised theorem is following:

**Theorem 3.14.** Suppose that, $\theta \in (0, 1)$ and for each $\vec{l} \in \mathbb{Z}^d$ the sequence $\{D_j(\vec{l})\}_{j=1}^\infty$ forms a $(\log_m, \theta)$- dense system in $Q_1(\vec{l})$. Furthermore, suppose that

$$
(3.29) \quad \forall \vec{l} \in \mathbb{Z}^d, \ n \in \mathbb{N}, \ j \in \{1, 2, \ldots, n\} : \ \Xi_n(\vec{l}, j) \neq \emptyset.
$$

Let $\gamma(r)$ be a nondecreasing monotone function satisfying condition (1.5). If for any natural $n$ the condition

$$
(3.30) \quad \lim_{|\vec{l}| \to \infty} \min_{\vec{\xi} \in \bigcup_{j=1}^n \Xi_n(\vec{l}, j)} \overline{V}^*(\gamma(m^{-n}), \vec{\xi}, n) = \infty
$$

is satisfied, then the spectrum of the operator $H = -\Delta + V(x)$ is discrete.

**Remark 3.15.** We shall construct a family of potentials $V(x)$ (see Section 5 Examples 5.4, 5.5) such that conditions of Theorem 3.14 are satisfied for them (hence the spectrum of $H$ is discrete), but they do not satisfy condition (3.21) of [3], and this family contains potentials which do not satisfy condition (3.20) of Theorem 3.7.
Recall that we use brief notations indicated in Remark 3.5.

4.1. Proof of Theorem 3.1

Proof. Consider the following quantities connected with the domain $G$ having the form (1.1):

\begin{equation}
\bar{r} := \max_{\omega \in \mathcal{S}^{d-1}} r(\omega), \quad r_m := \min_{\omega \in \mathcal{S}^{d-1}} r(\omega),
\end{equation}

\begin{equation}
G := \left( \frac{\bar{r}}{r_m} \right)^{d}.
\end{equation}

we see from (1.1), (1.2), (4.1) and (4.2) that $G r(y) \subseteq B_{\bar{r}} \cdot r(y)$ and $\text{mes}_d(B_{\bar{r}} \cdot r(y)) \leq G \cdot \text{mes}_d(G r(y))$.

Let us define

\begin{equation}
\gamma(r) = \left( \frac{\bar{r}(r)}{G} \right)^{(d-2)/d}.
\end{equation}

In view of (3.2), the function $\gamma(r)$ satisfies condition (1.5). Suppose that $F \in \mathcal{N}_{\gamma(r)}(y, r)$, where the collection $\mathcal{N}_{\gamma}(y, r)$ is defined by (1.3). In view of the isocapacity inequality (2.3) and the fact that it comes as identity for $F = B_{\bar{r}}(y)$, we have, taking into account (4.3):

\begin{align*}
\frac{\text{mes}_d(F)}{\text{mes}_d(G r(y))} & \leq \frac{\text{mes}_d(F)}{\text{mes}_d(B_{\bar{r}}(y))} \leq G \left( \frac{\text{cap}(F)}{\text{cap}(B_{\bar{r}}(y))} \right)^{d/(d-2)} \\
& \leq G \left( \frac{\text{cap}(F)}{\text{cap}(G r(y))} \right)^{d/(d-2)} \leq G(\gamma(r))^{d/(d-2)} = \bar{r}(r),
\end{align*}

i.e., $F \in \mathcal{M}_{\bar{r}(r)}(y, r)$. Thus, $\mathcal{N}_{\gamma(r)}(y, r) \subseteq \mathcal{M}_{\bar{r}(r)}(y, r)$. Hence

\begin{align*}
\inf_{F \in \mathcal{N}_{\gamma(r)}(y, r)} \int_{B_r(y) \setminus F} V(x) \, dx \geq \inf_{F \in \mathcal{M}_{\bar{r}(r)}(y, r)} \int_{B_r(y) \setminus F} V(x) \, dx.
\end{align*}

Therefore in view of condition (3.1), condition (1.4) is satisfied. Hence by Theorem 1.1 the spectrum of the operator $H$ is discrete.

Theorem 3.1 is proven. \hfill $\Box$

4.2. Proof of Theorem 3.3

Proof. In addition to the notations (3.3)-(3.10) let us denote

\begin{equation}
\mathcal{K}_W(t, X, \mu) = \mathcal{L}_s(s, X, \mu)|_{s=W(t, X, \mu)},
\end{equation}

\begin{equation}
\kappa_W(t, X, \mu) = \mu(\mathcal{K}_W(t, X, \mu)).
\end{equation}

For the brevity we shall omit $W$, $X$ and $\mu$ in the brackets and in the subscripts of all the notations mentioned above.

(i) It is easy to show that the function $\lambda(s)$ is non-decreasing, right continuous, for any $s > 0 \mathcal{L}_s(s) \setminus \mathcal{L}_s(s^-) = \{ x \in X : W(x) = s \}$ and
\[ \mu(L_*(s) \setminus L_*(s^-)) = \lambda_*(s) - \lambda_*(s^-). \] Furthermore, in view of definition (3.5),
\[ \kappa^-(t) = \lim_{s \uparrow W_*(t)} \lambda_*(s) \leq t \leq \lim_{s \downarrow W_*(t)} \lambda_*(s) \leq \kappa(t). \]

By Proposition 4.1, there exists a \( \mu \)-measurable set \( D \) contained in \( K(t) \setminus K^-(t) \) such that \( \mu(D) = t - \kappa^-(t) \), hence (5.11) is valid with \( \tilde{K} = K^-(t) \cup D \). Furthermore, it is clear that \( W(x) < W_*(t) \) for \( x \in K^-(t) \), \( W(x) > W_*(t) \) for \( x \in X \setminus K(t) \) and \( W(x) = W_*(t) \) for \( x \in K(t) \setminus K^-(t) \). These circumstances and (3.4) imply that the representation (3.12) is valid and the set \( \tilde{K} \) has the properties (3.13).

(ii) In view of (3.12), (3.11) and definition (3.3) the inequality \( I(t) \leq J(t) \) is valid. Let us prove the inverse inequality. Using again definition (3.3), let us take an arbitrary \( \epsilon > 0 \) and a set \( E \in \mathcal{E}(t, X, \mu) \), i.e.,
\[ \mu(E) \geq t, \]
such that
\[ \int_E W(x) \mu(dx) \leq I(t) + \epsilon. \]

We have:
\[ \int_E W(x) \mu(dx) = \int_{E \cap \tilde{K}} W(x) \mu(dx) + \int_{E \setminus \tilde{K}} W(x) \mu(dx). \]

In view of (3.13)-b,
\[ \int_{E \setminus \tilde{K}} W(x) \mu(dx) \geq W_*(t) \mu(E \setminus \tilde{K}). \]

On the other hand, in view of (3.11) and (4.1),
\[ \mu(E) = \mu(E \setminus \tilde{K}) + \mu(E \cap \tilde{K}) \geq \mu(\tilde{K}) = \mu(\tilde{K} \setminus E) + \mu(\tilde{K} \cap E). \]

Hence \( \mu(E \setminus \tilde{K}) \geq \mu(\tilde{K} \setminus E) \) and we have in view of (4.5), (4.6), (3.13)-a, (4.7) and (3.12):
\[ I(t) + \epsilon \geq \int_E W(x) \mu(dx) \geq \int_{E \cap \tilde{K}} W(x) \mu(dx) + W_*(t) \mu(\tilde{K} \setminus E) \geq \int_{E \cap \tilde{K}} W(x) \mu(dx) + \int_{\tilde{K} \setminus E} W(x) \mu(dx) = \int_\tilde{K} W(x) \mu(dx) = J(t). \]

Since \( \epsilon > 0 \) is arbitrary, from the last estimate follows inequality \( I(t) \geq J(t) \). Thus, equality (3.14) is valid. Theorem 3.3 is proven. \( \square \)

In the proof of Theorem 3.3 we have used the following claim, which is Sierpinski’s theorem on non-atomic measures:
Proposition 4.1. Suppose that conditions of Problem 3.2 and Theorem 3.3 are satisfied. Let $F \subseteq X$ be a $\mu$-measurable set such that $\mu(F) > 0$. Then for any $t \in (0, \mu(F))$ there exists a $\mu$-measurable subset $F_t$ of $F$ such that $\mu(F_t) = t$.

4.3. Proof of Proposition 3.4.

Proof. Like above, we shall omit $W$, $X$ and $\mu$ in the brackets and in the subscripts of the notations (3.4)-(3.10) and (3.15)-(3.16). By (3.15)-(3.16),

(4.8) \[ \forall \epsilon > 0 \exists s_0 > \bar{W}^*(t) - \epsilon : \mu(L^*(s_0)) \geq t. \]

Denote

(4.9) \[ \sigma(t) = \mu(X) - t \]

and $\tilde{t} = \frac{1}{\theta}t$. By claim (i) of Theorem 3.3 there exists a $\mu$-measurable set $\tilde{K} \subseteq X$ such that

(4.10) \[ J(\sigma(\tilde{t})) = \int_{\tilde{K}} W(x) \mu(dx), \quad \mu(\tilde{K}) = \sigma(\tilde{t}). \]

Let us estimate:

\[ \mu(X) + \mu(\tilde{K} \cap L^*(s_0)) \geq \mu(\tilde{K} \cup L^*(s_0)) + \mu(\tilde{K} \cap L^*(s_0)) = \mu(L^*(s_0)) + \mu(\tilde{K}) \geq t + \sigma(\tilde{t}) = \mu(X) + (\theta - 1)\tilde{t}, \]

therefore

\[ \mu(\tilde{K} \cap L^*(s_0)) \geq (\theta - 1)\tilde{t}. \]

Then taking into account (4.8), (3.16)-b and the condition $W(x) \geq 0$, we have:

\[ J(\sigma(\tilde{t})) = \int_{\tilde{K}} W(x) \mu(dx) \geq \int_{\tilde{K} \cap L^*(s_0)} W(x) \mu(dx) \geq (\theta - 1)\tilde{t} \cdot s_0 \geq (\theta - 1)\tilde{t}(\bar{W}^*(t) - \epsilon). \]

Since $\epsilon > 0$ is arbitrary, we get the desired estimate (3.17).

Let us prove estimate (3.18). Using again claim (i) of Theorem 3.3 we can choose a $\mu$-measurable set $\tilde{K} \subseteq X$ such that

(4.11) \[ \mu(\tilde{K}) = \sigma(t), \]

(4.12) \[ \forall x \in \tilde{K} : W(x) \leq W_*(t), \quad \forall x \in X \setminus \tilde{K} : W(x) \geq W_*(t) \]

and

(4.13) \[ J(\sigma(t)) = \int_{\tilde{K}} W(x) \mu(dx) \leq W_*(t) \mu(\tilde{K}) = W_*(t)\sigma(t), \]

On the other hand, by (4.12)-b and (3.16)-b, $X \setminus \tilde{K} \subseteq L^*(W_*(t))$. Furthermore, in view of (4.11) and (4.9), $\mu(X \setminus \tilde{K}) = t$. Hence by definition (3.15), $W_*(t) \leq \bar{W}^*(t)$. This circumstance together with (4.13) imply the desired estimate (3.18).
4.4. **Proof of Theorem 3.6**

**Proof.** It is known that the Lebesgue measure $\text{mes}_d$ is non-atomic. Then by Theorem 3.3 with $\mu = \text{mes}_d$, $X = \mathcal{G}_r(y)$ and $W(x) \equiv V(x)$, we obtain, taking into account the inclusion

$$
\{ E = \mathcal{G}_r(y) \setminus F : F \in \mathcal{M}_{\tilde{\gamma}(r)}(y, r) \} \subset \mathcal{E}(\sigma(r), \mathcal{G}_r(y), \text{mes}_d),
$$

that

$$
\inf_{F \in \mathcal{M}_{\tilde{\gamma}(r)}(y, r)} \int_{\mathcal{G}_r(y) \setminus F} V(x) \, dx \geq \inf_{E \in \mathcal{E}(\sigma(r), \mathcal{G}_r(y), \text{mes}_d)} \int_E V(x) \, dx = J_V(\sigma(r), y, r),
$$

where the collection $\mathcal{E}(t, X, \mu)$ is defined in the formulation of Problem 3.2 (in our case $\Sigma = \Sigma_L(\mathcal{G}_r(y))$). This estimate, condition (3.19) and Theorem 3.1 imply the desired claim. Theorem 3.6 is proven. $\Box$

4.5. **Proof of Theorem 3.7**

**Proof.** Let us take $\theta > 1$, $\tilde{\gamma}(r) = \hat{\gamma}(r)/\theta$ and $\sigma(r) = (1 - \hat{\gamma}(r))\text{mes}_d(\mathcal{G}_r(0))$. Since $\hat{\gamma}(r)$ satisfies conditions (3.2), then $\tilde{\gamma}(r)$ satisfies these conditions. In view of estimate (3.17) (Proposition 3.4) with $W(x) = V(x)$, $X = \mathcal{G}_r(y)$ and $t = \hat{\delta}(r)$, we have:

$$
(4.14) \quad J_V(\sigma(r), y, r) \geq \frac{\theta - 1}{\theta} \hat{\delta}(r) \hat{V}^*(\hat{\delta}(r), y, r).
$$

This estimate, condition (3.20) and Theorem 3.6 imply the desired claim. Theorem 3.7 is proven. $\Box$

4.6. **Proof of Theorem 3.10**

**Proof.** By definition (2.4) of the base polyhedron for the harmonic capacity on $\tilde{\Omega} = \tilde{\mathcal{G}}_r(y)$, for any $\mu \in \text{BP}(y, r)$ and any Borel set $F \subseteq \mathcal{G}_r(y)$

$$
\frac{\mu(F)}{\mu(\mathcal{G}_r(y))} \leq \frac{\text{cap}(F)}{\text{cap}(\mathcal{G}_r(y))}.
$$

Hence, in view of definitions (1.3) and (3.22), the inclusion

$$
\mathcal{N}_{\gamma(r)}(y, r) \subseteq \mathcal{M}^\mu_{\gamma(r)}(y, r)
$$

is valid. This circumstance implies that

$$
\inf_{F \in \mathcal{N}_{\gamma(r)}(y, r)} \int_{\mathcal{G}_r(y) \setminus F} V(x) \, dx \geq \sup_{\mu \in \text{BP}(y, r)} \inf_{F \in \mathcal{M}^\mu_{\gamma(r)}(y, r)} \int_{\mathcal{G}_r(y) \setminus F} V(x) \, dx.
$$

The last estimate and condition (3.23) imply that condition (1.4) of Theorem 1.1 is fulfilled. Hence the spectrum of the operator $H$ is discrete. Theorem 3.10 is proven. $\Box$
4.7. Proof of Theorem 3.11

Proof. Let us take $\theta > 1$ and denote

$$\tilde{\gamma}(r) = \gamma(r)/\theta, \quad \sigma(r) = (1 - \tilde{\gamma}(r))\mu(G_r(0)).$$

We have, taking into account the inclusions $BP_{eq}(y, r) \subseteq BP(y, r)$,

$$\{E = G_r(y) \setminus F : F \in M^\mu_{\tilde{\gamma}(r)}(y, r)\} \subseteq \mathcal{E}(\sigma(r), G_r(y), \mu),$$

that

$$\sup_{\mu \in BP(y, r)} \inf_{E \in \mathcal{M}^\mu_{\tilde{\gamma}(r)}(y, r)} \int_{G_r(y) \setminus F} V(x) \, dx \geq$$

$$\sup_{\mu \in BP_{eq}(y, r)} \inf_{E \in \mathcal{M}^\mu_{\tilde{\gamma}(r)}(y, r)} \int_{G_r(y) \setminus F} Z_\mu(x) \, \mu(dx) \geq$$

$$\sup_{\mu \in BP_{eq}(y, r)} \inf_{E \in \mathcal{E}(\sigma(r), G_r(y), \mu)} \int_{E} Z_\mu(x) \, \mu(dx).$$

Recall that the collection $\mathcal{E}(t, X, \mu)$ is defined in the formulation of Problem 3.2 (in our case $\Sigma = \Sigma_{L}(G_r(y)))$ and the function $Z_\mu(x)$ is defined by (3.26), (3.24). Since $V \in L^1_{1,loc}(\mathbb{R}^d)$, $Z_\mu(x)$ belongs to $L^1_{G_r(y), \mu}$ for any $\mu \in BP_{eq}(y, r)$. Hence since the Lebesgue measure $\text{mes}_d$ is is non-atomic and $\sigma$-finite, and each measure from $BP_{eq}(y, r)$ is absolute continuous with respect to $\text{mes}_d$, then $BP_{eq}(y, r)$ consists of non-atomic measures ([8], Theorem 2,4). Furthermore, in view of definition (2.4) and boundedness of the domain $G_r(y)$, each measure from $BP_{eq}(y, r)$ is finite. Hence using Theorem 3.3 and estimate (3.17) (Proposition 3.4), we obtain:

$$\sup_{\mu \in BP_{eq}(y, r)} \inf_{E \in \mathcal{E}(\sigma(r), G_r(y), \mu)} \int_{E} Z_\mu(x) \, \mu(dx) =$$

$$\sup_{\mu \in BP_{eq}(y, r)} \int_{Z^*_\mu(\sigma(r), y, r, \mu)} \frac{\theta - 1}{\theta} \psi_\mu(r) \sup_{\mu \in BP_{eq}(y, r)} Z^*_\mu(\psi_\mu(r), y, r, \mu).$$

The last estimate, (4.15), condition (3.25) and Theorem 3.10 imply the desired claim. Theorem 3.11 is proven. □

4.8. Proof of Corollary 3.12

Proof. The inclusion $M_f(y, r) \subseteq BP_{eq}(y, r)$ (Proposition A.2) implies:

$$\sup_{\mu \in BP_{eq}(y, r)} \int_{Z^*_\mu(\psi_\mu(r), y, r, \mu)} \geq \sup_{\mu \in M_f(y, r)} Z^*_\mu(\psi_\mu(r), y, r, \mu).$$

This estimate, condition (3.27) and Theorem 3.11 imply the desired claim. □
4.9. **Proof of Theorem [3.14]**

**Proof.** In this proof and in the lemmas, applied in it, we shall use the brief notations indicated in Remark [3.13]. Let us take a ball $B_r(y)$ and consider on it the probability measure $m_{d,r}$, defined by (2.5). Consider the function $s_{r,y} : B_r(y) \to [0, 1]$, defined in the following manner:

$$s_{r,y}(x) := m_{d,r}\{s \in B_r(y) : P_1s \leq P_1x}\},$$

where $P_1$ is the following operator $P_1 : \mathbb{R}^d \to \mathbb{R}$:

$$P_1x := x_1.$$  

It is easy to check, that $s_{r,y}$ is a measure preserving mapping (Definition 2.1). Denote by $y^{-}$ the left point of the two-point set $\partial(B_r(y)) \cap ((I - \tilde{P}_1)^{-1}(I - \tilde{P}_1)y)$, where $\tilde{P}_1$ is the orthogonal projection in $\mathbb{R}^d$ on the first coordinate axis, i.e., for $x = (x_1, x_2, \ldots, x_d)$ $\tilde{P}_1x := (x_1, 0, \ldots, 0)$. Consider the function $z = F_r(x) := \frac{x - \bar{x}}{r}$, which maps bijectively the ball $B_r(y)$ onto the ball $B_1(e)$ with $e = (1, 0, \ldots, 0)$. It is easy to see that

$$s_{r,y}(x) = s_{1,e}(F_r(x)).$$

Consider the function $f(t) = t^{(d-2)/d}$ and the absolutely continuous measure $\mu_s$ on $B_r(y)$, whose density is $f' \circ s_{r,y}$, i.e., this measure belongs to $M_f(r, y)$ (Definition 2.2). This means that for any set $A \in \Sigma_L(B_r(y))$

$$\mu_s(A) = \text{cap}(B_r(0)) \int_A f'(s_{r,y}(x))m_{d,r}(dx),$$

After change of the variable $z = F_r(x)$ we get, taking into account (4.18):

$$\mu_s(A) = \text{cap}(B_r(0)) \int_{F_r(A)} f'(s_{1,e}(z))m_{d,1}(dz).$$

Consider the function $Z_{\mu_s}(x)$, defined by (3.20) and (3.23) with $\mu = \mu_s$, i.e.,

$$Z_{\mu_s}(x) = V(x)\left(f'(s_{r,y}(x))\right)^{-1} = d/(d-2)V(x)\left(s_{r,y}(x)\right)^2/d.$$  

Using Lemma 4.2 with $t = \gamma(\bar{r}, K) (\bar{r} = r/\sqrt{d})$, where

$$\gamma(\rho, K) = K\gamma(\rho/m^2)\rho^d (K > 0).$$

and Lemma 4.3 taking in these lemmas $W(x) = V(x)$, we get that for some $\kappa, \delta \in (0, 1)$, $K > 0$ and any $y \in \mathbb{R}^d$, $r \in (0, 1)$ there are $\bar{y}(y, r) \in \mathbb{Z}^d$, a cube $Q_{\gamma}(\bar{y}) \subseteq B_r(y) \cap Q_j(\bar{y}(y, r))$ and $j \in \{1, 2, \ldots, n\}$ with $n = \lfloor \log_m \left(\frac{1}{\mu r^2} \right) \rfloor + 2$ such that

$$Z_{\mu_s}^*(\gamma(r), y, r) \geq \delta \cdot \tilde{V}^*(\gamma(\bar{r}, K), \bar{y}(y, r), \bar{r}) \geq \delta \min_{\xi, Q(\xi, n) \subseteq F_j} \tilde{V}^*(\gamma(m^{-n}), \xi, n),$$

where $\gamma(r) = K\gamma(r/\sqrt{d}, K)$ and $F_j$ is a non-empty union of cubes $Q(\xi, n)$ such that $F_j \subseteq Q_{\gamma}(\bar{y}) \cap D_j(\bar{y}(y, r))$. Notice that since the function $\gamma(r)$ satisfies conditions (1.5), the function $\gamma(\bar{r})$ satisfies this condition too for
some \( r_0 > 0 \). Then in view of condition (3.29), estimate (4.23) implies:
\[
\tilde{Z}_{\mu_s}^* (\tilde{\gamma}(r), y, r) \geq \delta \min_{\xi \in \cup_{j=1}^n \Xi_j} \tilde{V}^* (\gamma(m^n), \tilde{\xi}, n).
\]
Since
\[
B_r(y) \cap Q_1(\tilde{l}(y, r)) \neq \emptyset
\]
for any \( y \in \mathbb{R}^d \), this estimate, inclusion \( \mu_s \in M_f(y, r) \) and condition (3.30) imply that condition (3.27) of Corollary 3.12 is satisfied with \( \gamma(r) = \gamma(r) \). Then the spectrum of the operator \( H = -\Delta + V(x) \) is discrete. Theorem 3.14 is proven. \( \square \)

In the proof of Theorem 3.14 we have used the following claims:

**Lemma 4.2.** For some \( \kappa, \delta \in (0, 1) \) and any ball \( B_r(y) \) with \( r \in (0, 1) \) there are \( \tilde{l} = \tilde{l}(y, r) \in \mathbb{Z}^d \) and a cube
\[
Q_\tilde{l}(\tilde{y}) \subseteq B_r(y) \cap Q_1(\tilde{l}(y, r))
\]
with \( \tilde{r} = r/\sqrt{d} \) such that for any nonnegative function \( W \in L_1(B_r(y)) \) and \( t \in (0, 1) \) the inequality
\[
\tilde{Z}_{\mu_s}^* (\kappa t, y, r) \geq \delta \cdot W^*(t, \tilde{y}, \tilde{r})
\]
is valid, where \( \mu_s \) is the measure on \( B_r(y) \), defined by (4.19) with \( f(t) = t(d-2)/d \) and \( Z_{\mu_s}(x) = W(x)(f'(s_{r,y}(x)))^{-1} \) with the function \( s_{r,y}(x) \), defined by (4.16).

**Proof.** Let us take \( \delta > 0 \) and consider the set
\[
\Pi_\delta(r, y) := \{ x \in B_r(y) : (f'(s_{r,y}(x)))^{-1} = d/(d-2)(s_{r,y}(x))^{2/d} \geq \delta \},
\]
or in view of (4.18),
\[
\Pi_\delta(r, y) = \{ x \in B_r(y) : P_1(x - y) \geq \sigma(\delta) r \},
\]
where \( \sigma(\delta) = P_1(s_{1,e}^{-1}((d-2)(\delta/d)^{d/2})) \).

By (4.21) and (4.26), we have for \( N > 0 \):
\[
\{ x \in B_r(y) : V(x) \geq N/\delta \} \cap \Pi_\delta(r, y) \subseteq \{ x \in B_r(y) : Z_{\mu_s}(x) \geq N \}.
\]
The numbers \( \delta \) and \( N \) will be chosen below. Consider the cube \( Q_{r_1}(y_1) \) inscribed in the ball \( B_r(y) \), i.e., \( r_1 = 2r/\sqrt{d} \) and \( y_1 = y - r\vec{a} \) with \( \vec{a} = (d^{-1/2}, d^{-1/2}, \ldots, d^{-1/2}) \). In what follows we shall choose \( \delta > 0 \) such that
\[
Q_{r_1}(y_1) \subseteq \Pi_\delta(r, y).
\]
This condition is equivalent to \( \sigma(\delta) \leq 1 - d^{-1/2} \). Since \( r \in (0, 1) \), the cube \( Q_{r_1}(y_1) \) intersects not more than \( 2d \) adjacent cubes \( Q_1(\tilde{l}_k) (\tilde{l}_k \in \mathbb{Z}^d) \) and among them there is a cube \( Q_1(\tilde{l}_k) \) such that for some \( \tilde{y} \in Q_1(\tilde{l}_k) \) \( Q_{\tilde{l}}(\tilde{y}) \subseteq Q_1(\tilde{l}_k) \cap Q_{r_1}(y_1) \) with \( \tilde{r} = r_1/2 = r/\sqrt{d} \). Thus, since \( Q_{r_1}(y_1) \subseteq B_r(y) \) for any \( y \in \mathbb{R}^d \) and \( r \in (0, 1) \) we can choose \( \tilde{l}(y, r) \in \mathbb{Z}^d \) such that inclusion (4.24) is valid. Let us take \( M = W^* (t, \tilde{y}, r) \) for \( t \in (0, 1) \). Then
by definitions (3.15)-(3.16), for any $\epsilon > 0$ there is $s \in (M - \epsilon, M]$ such that 
$$\text{mes}_d\{x \in Q_\varepsilon(y) : W(x) \geq s\} \geq t \cdot \text{mes}_d(Q_\varepsilon(0)).$$
Hence 
$$\text{mes}_d\{x \in Q_\varepsilon(y) : W(x) \geq M - \epsilon\} \geq t \cdot \text{mes}_d(Q_\varepsilon(0)).$$

Notice that in view of (4.28) and the inclusion $Q_\varepsilon(y) \subseteq Q_{r_1}(y_1)$, the inclusion $Q_\varepsilon(y) \subseteq \Pi_\delta(r, y)$ is valid. Then inclusion (4.27) with $N = \delta(M - \epsilon)$, definitions (4.19), (4.26) and equality $\min_{t \in [0, 1]} f'(t) = (d - 2)/d$ imply that for $t \in (0, 1)$ 
$$\mu_*\{x \in B_r(y) : Z_{\mu_*}, W(x) \geq \delta(M - \epsilon)\} \geq$$
$$\frac{d - 2}{d} \frac{\text{cap}(B_r(0))}{\text{mes}_d(B_r(0))} \text{mes}_d\{x \in Q_\varepsilon(y) : W(x) \geq M - \epsilon\} \geq$$
$$\frac{d - 2}{d} \text{cap}(B_r(0)) t \cdot m_{d,r}(Q_\varepsilon(y)) \geq \frac{\delta d - 2}{d} t \cdot \mu_* (Q_\varepsilon(y)) =$$
$$\delta q \frac{d - 2}{d} t \cdot \mu_*(B_r(y)),$$
where, in view of (4.20), the quantity $q = \frac{\mu_* (Q_\varepsilon(y))}{\mu_*(B_r(y))}$ does not depend on $r$ and $y$. Denote $\kappa = \delta q \frac{d - 2}{d}$. Then the last estimate imply that $Z_*^*(\kappa t, y, r) \geq \delta(M - \epsilon)$. Since $\epsilon > 0$ is arbitrary, we obtain the desired inequality (4.25). \qed

**Lemma 4.3.** Suppose that in a cube $Q_1(\bar{l})$ ($\bar{l} \in \mathbb{Z}^d$) there is is a sequence of subsets $\{D_n(\bar{l})\}_{n=1}^{\infty}$ forming in it a $(\log_m, \theta)$-dense system. Let $\gamma : (0, r_0) \to \mathbb{R}$ be a monotone nondecreasing function with $r_0 = \min\{1, 1/(m^2 \theta)\}$. Then for some $K > 0$ and for any cube $Q_r(y) \subset Q_1(\bar{l})$ there are $j \in \{1, 2, \ldots, n\}$ with 
$$n = \left\lceil \log_m \left(\frac{1}{\theta r} \right) \right\rceil + 2$$
and a non-empty set $F_j \subset Q_r(y) \cap D_j(\bar{l})$, which is a union of cubes 
$$Q_\xi(n) (\xi \in m^{-n} \cdot \mathbb{Z}^d),$$
such that for any nonnegative function $W \in L_1(Q_r(y))$ the inequality 
$$\widetilde{W}^* (\gamma(r, K), y, r) \geq \min_{\xi : Q_\xi(n) \subseteq F_j} \widetilde{W}^* (\gamma(m^{-n}), \xi, n)$$
is valid, where the function $\tilde{\gamma}(r, K)$ is defined by (4.22).

**Proof.** Let us take $r \in (0, r_0)$. Then by Definition 2.5 there is 
$$j \in \{1, 2, \ldots, n\}$$
such that for some regular parallelepiped $\Pi \subset D_j(\bar{l})$ there is a cube 
$$Q_{\theta r}(s) \subset \Pi \cap Q_r(y).$$
Definition (4.29) implies that 
$$m^{-(n-1)} < \theta r \leq m^{-(n-2)}.$$
Using Lemma 4.3 and inclusion (4.31) we get:

\( \bar{W}^* (\gamma (r, K), y, r) \geq \bar{W}^* (K \gamma (\theta r/m^2), s, \theta r) \) (4.33)

On the other hand, in view of (4.31) and the left inequality of (4.32), the set \( \{ \xi : Q(\xi, n) \subseteq Q_{\theta r}(s) \} \) is not empty. Denote

\( F_j := \bigcup_{\xi : Q(\xi, n) \subseteq Q_{\theta r}(s)} Q(\xi, n) \). (4.34)

Then the set \( Q_{\theta r}(s) \setminus F_j \) is a union of 2d regular parallelepipeds, each of them is the cartesian product of a face of the cube \( Q_{\theta r}(s) \) and an interval, whose length is less than \( m^{-n} \). Hence, in view of the right inequality (4.32),

\[ \text{mes}_d(Q_{\theta r}(s)) - \text{mes}_d(F_j) < 2d \left( \frac{m^{-n+2}}{m^{-n}} \right)^{d-1} \]

Since the set \( F_j \) is not empty, then in view of (4.34), \( \text{mes}_d(F_j) \geq m^{-nd} \). Hence

\[ \frac{\text{mes}_d(Q_{\theta r}(s))}{\text{mes}_d(F_j)} < \frac{2d \left( \frac{m^{-n+2}}{m^{-n}} \right)^{d-1} m^{-n}}{m^{-nd}} + 1 = 2d \cdot m^{2(d-1)} + 1. \]

Notice that the function \( \bar{W}^*(t, \Omega) \) is non-increasing by \( t \). Continuing estimate (4.33) and using Lemmas 4.4, 4.5, definition (4.34) and the right inequality (4.32), we obtain: taking \( K = (2d \cdot m^{2(d-1)} + 1)^{-1} \):

\[ \bar{W}^*(\gamma (r, K), y, r) \geq \min_{\xi : Q(\xi, n) \subseteq F_j} \bar{W}^*(\gamma (m^{-n}), \xi, n), \]

i.e., the desired inequality (4.30) is proven. \( \square \)

In the proof of Lemma 4.3 we have used the following lemmas:

**Lemma 4.4.** Let \( \Omega \subseteq \mathbb{R}^d \) be a measurable set having the form \( \Omega = \bigcup_{n=1}^{N} \Omega_n \), where \( \Omega_n \) are measurable sets such that \( \text{mes}_d(\Omega_k \cap \Omega_l) = 0 \) for \( k \neq l \). If \( W(x) \) is a non-negative measurable function defined in \( \Omega \), then the inequality is valid for \( t \in (0, 1) \):

\[ \bar{W}^*(t \cdot \text{mes}_d(\Omega), \Omega) \geq \min_{1 \leq n \leq N} \bar{W}^*(t \cdot \text{mes}_d(\Omega_n), \Omega_n). \] (4.35)

**Proof.** By definition (3.15) of the non-increasing rearrangement \( \bar{W}^* \), for any \( n \in \{1, 2, \ldots, N\} \) and \( \epsilon > 0 \) there is \( s_n > 0 \) such that

\[ s_n > \bar{W}^*(t \cdot \text{mes}_d(\Omega_n), \Omega_n) - \epsilon \]

and \( \lambda^*(s_n, W, \Omega_n) \geq t \cdot \text{mes}_d(\Omega_n) \). Let us take \( s_0 = \min_{1 \leq n \leq N} s_n \). Then taking into account that the functions \( \lambda^*(s, W, \Omega_n) \) are non-increasing, we
get:
\[ \lambda^*(s_0, W, \Omega) = \sum_{n=1}^{N} \lambda^*(s_n, W, \Omega_n) \geq N \sum_{n=1}^{N} \lambda^*(s_n, W, \Omega_n) \geq t \sum_{n=1}^{N} \text{mes}_d(\Omega_n) = t \cdot \text{mes}_d(\Omega). \]

Hence
\[ \bar{W}^*(t \cdot \text{mes}_d(\Omega), \Omega) \geq s_0 > \min_{1 \leq n \leq N} \bar{W}^*(t \cdot \text{mes}_d(\Omega_n), \Omega_n) - \epsilon. \]

Since \( \epsilon > 0 \) is arbitrary, we get the desired inequality (4.35). □

**Lemma 4.5.** Let \( \Omega_1 \) and \( \Omega_2 \) be measurable subsets of \( \mathbb{R}^d \) such that \( \Omega_1 \subseteq \Omega_2 \) and \( W(x) \) be a non-negative measurable function defined on \( \Omega_2 \). Then for any \( t > 0 \) the inequality \( \bar{W}^*(t, \Omega_1) \leq \bar{W}^*(t, \Omega_2) \) is valid.

**Proof.** In view of the inclusion \( \Omega_1 \subseteq \Omega_2 \) and definition (3.16), \( \lambda^*(s, W, \Omega_1) \leq \lambda^*(s, W, \Omega_2) \). Hence
\[ \{ s > 0 : \lambda^*(s, W, \Omega_1) \geq t \} \subseteq \{ s > 0 : \lambda^*(s, W, \Omega_2) \geq t \}. \]

This inclusion and definition (3.15) imply the desired claim. □

### 5. Some examples

First of all, consider some examples of the \((\log m, \theta)-\)dense system (Definition 2.5).

**Example 5.1.** Consider the classical middle third Cantor set \( C \subset [0, 1] \), Let \( I_{n,k} (n = 1, 2, \ldots), k = 1, 2, \ldots, 2^n-1 \) be the closures of intervals adjacent to \( C \). It is known that they are disjoint and for any fixed \( n \) and each \( k \in \{1, 2, \ldots, 2^n-1\} \) \( \text{mes}_1(I_{n,k}) = 3^{-n} \). For fixed \( n \) we shall number the intervals \( I_{n,k} \) from the left to the right. Denote \( D_n = \bigcup_{k=1}^{2^n-1} I_{n,k} \). Let us show that the sequence \( \{D_n\}_{n=1}^{\infty} \) forms in \([0, 1]\) a \((\log_3, 1/9)-\)dense system. Let us take \( \theta = 1/9, \ r \in (0, \min\{1, 1/(3^2\theta)\}) = (0, 1) \), an interval \( Q_r(y) = [y, y + r] \) and the natural number \( n = \left\lceil \log_3 \left( \frac{1}{r\theta} \right) \right\rceil \). Then \( n \geq 2 \) and
\[ 3^{-(n+1)} < r\theta \leq 3^{-n}. \]

Consider two cases:

- a) there are \( j \in \{1, 2, \ldots, n-1\}, \ k \in \{1, 2, \ldots, 2^{j-1}\} \) such that \( \text{mes}_1(Q_r(y) \cap I_{j,k}) > 3^{-n} \);

- b) for all \( j \in \{1, 2, \ldots, n-1\}, \ k \in \{1, 2, \ldots, 2^{j-1}\} \) \( \text{mes}_1(Q_r(y) \cap I_{j,k}) \leq 3^{-n} \).

In the case a) in view of the right inequality (5.1), \( \text{mes}_1(Q_r(y) \cap I_{j,k}) > r\theta \), hence there is a real \( z \) such that \( Q_{r\theta}(z) \subset Q_r(y) \cap I_{j,k} \).
Consider the case b). Let \((c_{n,k}, d_{n,k}) (k = 1, 2, \ldots, 2^n - 1)\) be the intervals forming the set \((0, 1) \setminus \bigcup_{j=1}^{n-1} D_j\). By the construction of Cantor set \(C\), \(d_{n,k} - c_{n,k} = 3^{-(n-1)}\) and each interval \((c_{n,k}, d_{n,k})\) contains an unique interval \(I_{n,k} = [c_{n,k} + 3^{-n}, d_{n,k} - 3^{-n}] \subset D_n\). Assumption b) implies that for some \(k \in \{1, 2, \ldots, 2^n - 1\}\) or the left edge \(y\) of the interval \(Q_r(y = [y, y + r])\) belongs to \([c_{n,k} - 3^{-n}, c_{n,k}]\), or the right its edge \(y + r\) belongs to \([d_{n,k}, d_{n,k} + 3^{-n}]\).

On the other hand, the left inequality of (5.1) (with (5.3)) where \(\theta = \{ 1, 2, \ldots, \theta \}\) of the right inequality (5.1), mes \(Q_r(y) \cap I_{n,k}\). Therefore in the case b) there is a real \(z\) and \(k \in \{1, 2, \ldots, 2^n - 1\}\) such that \(Q_{r\theta}(z) \subset Q_r(y) \cap I_{n,k}\). Thus, the sequence \(\{D_n\}_{n=1}^{\infty}\) satisfies the conditions of Definition 2.5 with \(d = 1, m = 3, \theta = 1/9\) and \(\Pi = I_{j,k}\).

**Example 5.2.** Consider a cube \(Q_1(y) \subset \mathbb{R}^d\), represented in the form \(Q_1(y) = Q_1(y_1) \times Q_1(y_2)\), where \(Q_1(y_1) \subset \mathbb{R}^{d_1}\) and \(Q_1(y_2) \subset \mathbb{R}^{d_2}\). Let \(\{D_n\}_{n=1}^{\infty}\) be a sequence of subsets of the cube \(Q_1(y_1)\), forming in it a \((\log m, \theta)\)-dense system. It is easy to see that the sequence \(\{D_{n} \times Q_1(y_2)\}_{n=1}^{\infty}\) forms in \(Q_1(y)\) a \((\log m, \theta)\)-dense system too.

**Example 5.3.** Let \(\{D_n^{(i)}\}_{i=1}^{\infty}\) be a finite collection of sequences of sets such that each of them forms a \((\log m, \theta)\)-dense system \((m > 1, \theta \in (0, 1))\) in a cube \(Q_1(y_1) \subset \mathbb{R}^d\). Consider the following sequence of subsets of the cube \(Q_1(y) = \times_{i=1}^{I} Q_1(y_i) \subset \mathbb{R}^d\) \((y = (y_1, y_2, \ldots, y_I), d = \sum_{i=1}^{I} d_i)\):

\[
\mathcal{D}_N = \bigcup_{\vec{n} \in \mathcal{E}_N} \times_{i=1}^{I} D_{n_i}^{(i)},
\]

where

\[
\mathcal{E}_N = \{\vec{n} = (n_1, n_2, \ldots, n_I) \in \mathbb{N}^I : \max_{1 \leq i \leq I} n_i = N\}
\]

Let us show that the sequence \(\{\mathcal{D}_N\}_{N=1}^{\infty}\) forms a \((\log m, \theta)\)-dense system in the cube \(Q_1(y)\). It is clear that for this sequence the condition (a) of Definition 2.5 is satisfied. Let us show that also condition (b) of this definition is satisfied for it. Let us take a cube \(Q_r(z) = \times_{i=1}^{I} Q_r(z_i) \subset Q_1(y)\) with \(r \in (0, \min\{1, \frac{1}{m\theta}\})\). By the condition (b) of the above mentioned definition applied to \(i\)-th sequence of sets, there is \(j_i \in \{1, 2, \ldots, n\}\) with \(n = \lceil \log_m \left( \frac{1}{\theta} \right) \rceil\) such that for some regular parallelepiped \(\Pi_i \subseteq D_{n_i}^{(i)}\) there is a cube \(Q_{\theta r}(s_i)\), contained in \(\Pi_i \cap Q_r(z_i)\). Denote \(Q_{\theta r}(s) = \times_{i=1}^{I} Q_{\theta r}(s_i)\) \((s = (s_1, s_2, \ldots, s_I), \Pi = \times_{i=1}^{I} \Pi_i\). Let us notice that \(\Pi \subseteq \mathcal{D}_J\) with

\[
J = \max_{1 \leq i \leq I} j_i \leq n.
\]

It is easy to see that \(Q_{\theta r}(s) \subseteq Q_r(z) \cap \Pi\). This means that the sequence \(\{\mathcal{D}_N\}_{N=1}^{\infty}\) satisfies condition (b) of Definition 2.5.
Now we shall construct some counterexamples connected with conditions of discreteness of the spectrum of the operator \( H = -\Delta + V(x) \), obtained above.

**Example 5.4.** Here we shall construct an example of the potential \( V(x) \geq 0 \) which satisfies conditions of Theorem 3.14 (hence the spectrum of the operator \( H = -\Delta + V(x) \cdot x \) is discrete), but the condition (3.21) of the criterion from [4], formulated in Remark 3.9, is not satisfied for it. Let us return to the sequence \( \{D_n\}_{n=1}^{\infty} \) of subsets of the interval \([0, 1]\) and considered in Example 5.1 and the following sequence of subsets of the cube \( Q_1(0) \):

\[
D_n = D_n \times [0, 1]^{d-1}.
\]

Consider also the translations of the cube \( Q_1(0) \) and the sets \( D_n \) by the vectors \( \vec{l} = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d \):

\[
D_n(\vec{l}) = D_n + \{\vec{l}\}.
\]

The arguments of Examples 5.1 and 5.2 imply that for any fixed \( \vec{l} \in \mathbb{Z}^d \) the sequence \( \{D_n(\vec{l})\}_{n=1}^{\infty} \) forms in \( Q_1(\vec{l}) \) a \((\log_3 3, 1/9)\)-dense system. For \( \beta \in (0, 1) \) consider on \( \mathbb{R} \) the 1-periodic function \( \theta_\beta(x) \), defined on the interval \((0, 1]\) in the following manner:

\[
\theta_\beta(x) = \begin{cases} 
1 & \text{for } x \in (0, \beta], \\
0 & \text{for } x \in (\beta, 1].
\end{cases}
\]

Let us take

\[
\alpha \in \left(0, 2\right).
\]

Consider the following function, defined on \((0, 1]:

\[
\Sigma_{N, p, \alpha}(x) := \begin{cases} 
0 & \text{for } x \in (0, 1] \setminus \bigcup_{n=1}^{\infty} D_n, \\
N \theta_\beta(3^p x) |_{\beta=3^{-\alpha n}} & \text{for } x \in D_n (n = 1, 2, \ldots)
\end{cases}
\]

\((N > 0, p \in \mathbb{N})\). Recall that we denote by \( P_1 \) the operator, defined by (4.17). Consider a function \( \mathcal{N} : \mathbb{Z}^d \to \mathbb{R}_+ \), satisfying the condition

\[
\mathcal{N}(\vec{l}) \geq 1, \quad \mathcal{N}(\vec{l}) \to \infty \quad \text{for} \quad \|\vec{l}\| \to \infty,
\]

where \( \|\vec{l}\| = \max_{1 \leq i \leq d} |l_i| \). Let us construct the desired potential in the following manner:

\[
V_\alpha(x) := \Sigma_{N, p, \alpha}(P_1(x - \vec{l})) |_{N=\mathcal{N}(\vec{l}), \beta=\|\vec{l}\| \to \infty+1} \quad \text{for} \quad \vec{l} \in \mathbb{Z}^d
\]

and \( x \in Q_1(\vec{l}) \).

It is clear that \( V_\alpha \in L_{\infty, \text{loc}}(\mathbb{R}^d) \). Let us prove that the potential \( V_\alpha(x) \) satisfies all the conditions of Theorem 3.14. Let us take a natural \( n \),

\[
j \in \{1, 2, \ldots, n\}
\]

and a cube

\[
Q(\vec{\xi}, n) \subseteq D_j(\vec{l})
\]
of the 3-adic partition of \( Q(\vec{l}) \). Let us notice that
\[
P_1(Q(\vec{l}, n)) = [k 3^{-n}, (k + 1) 3^{-n}]
\]
for some \( k \in \mathbb{Z} \). Then taking into account definitions (5.4)-(5.8), (5.10) and the 1-periodicity of \( \theta_\beta(t) \), we get for \( |\vec{l}|_\infty > n \):
\[
(5.13) \quad \text{mes}_d\left( \{ x \in Q(\vec{l}, n) : V_\alpha(x) > 0 \} \right) = 3^{-(d-1)n}\text{mes}_1(\{ x \in [k 3^{-n}, (k + 1) 3^{-n}] : \theta_\beta(3^p x)|_{3^{-\alpha_j}, p=|\vec{l}|_\infty+1} > 0 \})
\]
\[
= 3^{-(d-1)n}\text{mes}_1(\{ x \in [0, 3^{-n}] : \theta_\beta(3^p x)|_{3^{-\alpha_j}, p=|\vec{l}|_\infty+1} > 0 \})
\]
\[
= 3^{-(d-1)n}\text{mes}_1(\{ t \in [0, 3|\vec{l}|_\infty+1-n] : \theta_\beta(t)|_{3^{-\alpha_j}} > 0 \})
\]
\[
= 3^{-dn}\text{mes}_1(\{ t \in [0, 1] : \theta_\beta(t)|_{3^{-\alpha_j}} > 0 \})
\]
\[
= 3^{-\alpha_j}\text{mes}_d(Q(\vec{l}, n)) \geq 3^{-\alpha n}\text{mes}_d(Q(\vec{l}, n)).
\]

Therefore in view of definitions (3.15)-(3.16), (5.8) and (5.10),
\[
V_\alpha^*(3^{-\alpha n}, \vec{l}, n) \geq N'(\vec{l}),
\]
if conditions (5.11) and (5.12) are satisfied. This estimate and conditions (5.7), (5.9) imply that condition (3.30) of Theorem 3.14 is satisfied for the potential \( V_\alpha(x) \) with \( \gamma(r) = r^\alpha \) satisfying condition (1.5). Hence the spectrum of the operator \( H = -\Delta + V_\alpha(x) \) is discrete.

Let us show that the condition (3.21) of the criterion from [4] is not satisfied for the potential \( V_\alpha(x) \). To this end for any natural \( n \) consider \( \vec{l}_n \in \mathbb{Z}^d \) such that \( |\vec{l}_n|_\infty = n \) and take a cube \( Q(\vec{l}_n, n) \subset D_n(\vec{l}_n) \). In view of (5.6), (5.8), (5.10) and (5.9)-a, \( \int_{Q(\vec{l}_n, n)} V_\alpha(x) \, dx > 0 \). Then taking \( \delta > 0 \), we have:
\[
\{ x \in Q(\vec{l}_n, n) : V_\alpha(x) \geq \frac{\delta}{\text{mes}_d(Q(\vec{l}_n, n))} \int_{Q(\vec{l}_n, n)} V_\alpha(x) \, dx \} \subseteq \{ x \in Q(\vec{l}_n, n) : V_\alpha(x) > 0 \}.
\]

On the other hand, if we take in (5.13) \( j = n \) and \( \vec{l} = \vec{l}_n \), we get
\[
\text{mes}_d(\{ x \in Q(\vec{l}_n, n) : V_\alpha(x) > 0 \}) = 3^{-\alpha n}\text{mes}_d(Q(\vec{l}_n, n)).
\]

These circumstances mean that the sequence
\[
\text{mes}_d\left( \{ x \in Q(\vec{l}_n, n) : V_\alpha(x) \geq \frac{\delta}{\text{mes}_d(Q(\vec{l}_n, n))} \int_{Q(\vec{l}_n, n)} V_\alpha(x) \, dx \} \right)
\]
tends to zero as \( n \to \infty \). This means that condition (3.21) is not satisfied for the potential \( V_\alpha(x) \).
Example 5.5. Consider the potential $V_α(x)$ constructed in the previous Example 3.4, but now

\[ (5.14) \quad 2(d - 2)/d < α < 2. \]

As we have shown there, the conditions of Theorem 3.11 are satisfied for this potential (hence the spectrum of the operator $H = -Δ + V_α(x)$ is discrete). My goal is to show that condition (3.20) of Theorem 3.7 is not satisfied for $V_α(x)$. Denote $ψ(r) = r^α$. Then in view of the left inequality of (5.14), $lim_{r \to 0} r^{-2(d - 2)/d} ψ(r) < ∞$. Let us take a function $\tilde{ψ}(r)$ satisfying condition (3.2) for some $r_0 > 0$. Then $lim_{r \to r_0} \tilde{ψ}(r) = ∞$. Hence for some positive decreasing sequence $\{r_j\}_{j=1}^∞$ tending to zero

\[ (5.15) \quad lim_{j \to ∞} \frac{\tilde{ψ}(r_j)}{ψ(r_j)} = ∞. \]

In order to prove that condition (3.20) is not satisfied for the potential $V_α(x)$, it is sufficient to find a sequence of points $y_j ∈ ℝ^d$ such that

\[ (5.16) \quad lim_{j \to ∞} |y_j| = ∞ \quad \text{and} \quad limsup_{j \to ∞} \tilde{V}_α^*(\tilde{ψ}(r_j) mes_d(Q_{r_j}(y_j)), Q_{r_j}(y_j)) < ∞. \]

Let us choose an increasing sequence of natural numbers $\{n_j\}_{j=1}^∞$ such that

\[ (5.17) \quad 3^{-(n_j + 1)} \leq r_j < 3^{-n_j}. \]

Consider the vectors $\bar{t} ∈ ℤ^d$ of the form $\bar{t} = (l, 0, \ldots, 0)$, where $l ∈ ℤ$ will be chosen below. Consider the intervals $I_{n_j,l} = [a_{n_j,l}, b_{n_j,l}] ⊂ D_{n_j}$ and the cubes $Q_{j,l} = Q_{3^{-n_j}}(\tilde{y}_{j,l})$, where $\tilde{y}_{j,l} = (a_{n_j,l} + l, 0, \ldots, 0)$. Then $P_l(Q_{j,l}) = I_{n_j,l} + l$. In view of the right inequality of (5.17), $Q_{r_j}(y_{j,l}) ⊂ Q_{j,l}$.

Then using definitions (5.10) and (5.6), (5.8), Lemma 5.6, the left inequality of (5.17) and taking $l = n_j$, we have:

\[ (5.18) \quad mes_d(\{x ∈ Q_{r_j}(y_{j,l}) : V_α(x) > 0\}) ≤ \]

\[ \quad mes_d(\{x ∈ Q_{j,l} : V_α(x) > 0\}) = \]

\[ \quad 3^{-n_j(d-1)}mes_1(\{x ∈ I_{n_j,l} : \theta_β(3^{l+1}x)|_{β=ψ(3^{-n_j+1})}\}) ≤ \]

\[ \quad ψ(3^{-n_j+1})(3^{-n_jd} + 3 \cdot 3^{-n_j(d-1)3^{l-1}}) = \]

\[ \quad 2ψ(3^{-n_j+1})3^{-n_j+1}3^{-n_j+1}3^{d} ≤ 2 \cdot 3^dψ(r_j)mes_d(Q_{r_j}(y_{j,l})). \]

On the other hand, in view of (5.15),

\[ \exists J > 0 \quad ∀ j ≥ J : \quad 2 \cdot 3^dψ(r_j) < \tilde{ψ}(r_j). \]

This circumstance, estimate (5.18) and definitions (5.10) and (5.6), (5.8) imply that

\[ ∀ j ≥ J : \quad \tilde{V}_α^*(\tilde{ψ}(r_j) mes_d(Q_{r_j}(y_j)), Q_{r_j}(y_j)) = 0, \]

where $y_j = y_{j,n_j}$. This means that relation (5.16) is valid, i.e., the potential $V_α(x)$ does not satisfy condition (3.20).
In the previous consideration we have used the following claim:

**Lemma 5.6.** For any interval \([a,b]\) and \(S > 0\) consider the quantity
\[
M(S, \beta, a, b) = \text{mes}_1 \{ x \in [a, b] : \theta_{\beta}(Sx) > 0 \},
\]
where \(\theta_{\beta}(x)\) is the 1-periodic function, defined by (5.6). It satisfies the inequalities
\[
M(S, \beta, a, b) \leq \beta(b - a) + 2\beta/S,
\]
\[
M(S, \beta, a, b) \geq \beta(b - a) - 2\beta/S;
\]

**Proof.** Taking into account the 1-periodicity of \(\theta_{\beta}(x)\), we obtain:
\[
M(S, \beta, a, b) \leq M(S, \beta, [Sa]/S, ([Sb] + 1)/S) = \frac{S}{S - 1} M(1, \beta, [Sa], [Sb] + 1) = \frac{\beta}{S} ([S] + 1 - [Sa]) \leq \beta(b - a) + 2\beta/S,
\]
\[
M(S, \beta, a, b) \geq M(S, \beta, ([Sa] + 1)/S, [Sb]/S) = \frac{S}{S - 1} M(1, \beta, [Sa] + 1, [Sb]) = \frac{\beta}{S} ([Sb] - 1 - [Sa]) \geq \beta(b - a) - 2\beta/S.
\]

\[\Box\]

**Appendix A. Base polyhedron of harmonic capacity**

The following claim on the base polyhedron of harmonic capacity, defined by (2.4), is valid:

**Proposition A.1.** Suppose that a domain
\[
\Omega \subset \mathbb{R}^d
\]
is open and bounded. Then the base polyhedron \(\text{BP}^*(\bar{\Omega})\) of the harmonic capacity on \(\bar{\Omega}\) is nonempty, convex and weak*-compact.

**Proof.** As it is known ([11]) (p. 537), the harmonic capacity on \(\Omega\) is a non-negative, monotone and bounded set functions and \(\text{cap}(\emptyset) = 0\). Recall that it is submodular, i.e., property (2.2) is valid. Let us consider the set function which is called dual to “cap”: \(\text{cap}^*(A) = \text{cap}(\bar{\Omega}) - \text{cap}(A^c) (A \in \Sigma_B(\bar{\Omega}))\), where \(A^c = \bar{\Omega}\setminus A\). It is easy to show that “\(\text{cap}^*\)” is a non-negative, monotone and bounded set function and \(\text{cap}^*(\emptyset) = 0\), but it is supermodular in the sense that for any pair of sets \(A, B \in \Sigma_B(\bar{\Omega})\) \(\text{cap}^*(A \cup B) + \text{cap}^*(A \cap B) \geq \text{cap}^*(A) + \text{cap}^*(B)\). As it is known ([10], Proposition 1), for the set functions of this kind the collection of measures
\[
\text{Core}^*(\bar{\Omega})) = \{ \mu \in M(\bar{\Omega}) : \mu(A) \geq \text{cap}^*(A) \text{ for all } A \in \Sigma_B(\bar{\Omega}) \text{ and } \mu(\bar{\Omega}) = \text{cap}^*(\bar{\Omega}) \}
\]
is nonempty, convex and weak*-compact. It is called the core of the set function “\(\text{cap}^*\)”. Since “\(\text{cap}^*\)” is non-negative, it is clear that \(\text{Core}^*(\bar{\Omega}) \subseteq \)
Then the set function \( f \) comes as identity for \( \Phi \) to it measure \( \mu \). Hence in view of (5.4) and definition (2.4) with \( \Omega = \bar{B}(y) \), it is supermodular. Consider the core of core \( \mathcal{C}(B_r(y)) \) with \( \Omega = B_r(y) \), hence it is submodular (2, 10). Therefore the conjugate \( f^*(m_{d,r}) \), i.e., the following collection of additive probability measures: \( \mathcal{C}(B_r(y)) \mathcal{C}(B_y(y)) \). It is easy to show that \( \mathcal{C}(B_r(y), m_{d,r}) \) is contained in \( \mathcal{C}(B_r(y), m_{d,r}) \). This circumstance and inclusion (5.3) imply that the set \( \mathcal{C}(y, r) \), defined by (5.4), is contained in \( \mathcal{C}(y, r) \). It is easy to show that \( \mathcal{C}(y, r) \subset L_1(B_r(y), m_{d,r}) \). Furthermore, since \( f'(t) = (d-2)/2d+2/2d \) then \( \inf_{t \in [0,1]} f'(t) > 0 \). Hence all the measures having the densities in \( \mathcal{C}(y, r) \) are equivalent to the Lebesgue measure. The proposition is proven. \( \square \)
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