ON THE BOUNDEDNESS OF THRESHOLD OPERATORS IN $L_1[0, 1]$ WITH RESPECT TO THE HAAR BASIS

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Abstract. We prove a near-unconditionality property for the normalized Haar basis of $L_1[0, 1]$.

1. Introduction

Let $(e_i)$ be a semi-normalized basis for a Banach space $X$. For a finite subset $A \subset \mathbb{N}$, let $P_A(\sum a_i e_i) := \sum_{i \in A} a_i e_i$ denote the projection from $X$ onto the span of basis vectors indexed by $A$. Recall that $(e_i)$ is an unconditional basis if there exists a constant $C$ such that, for all finite $A \subset \mathbb{N}$, $\|P_A\| \leq C$.

We say that $(e_i)$ is near-unconditional if for all $0 < \delta \leq 1$ there exists a constant $C(\delta)$ such that for all $x = \sum a_i e_i$ satisfying the normalization condition $\sup |a_i| \leq 1$, and for all finite $A \subseteq \{i : |a_i| \geq \delta\}$,

$$\|P_A(x)\| \leq C(\delta)\|x\|.$$ 

Every unconditional basis is near-unconditional, and it is easy to check that a near-unconditional basis is unconditional if and only if $C(\delta)$ can be chosen to be independent of $\delta$.

It was proved in \cite{1} that a basis is near-unconditional if and only if the thresholding operators $G_\delta(x) := \sum_{|a_i| \geq \delta} a_i e_i$ satisfy, for some constant $C_1(\delta)$,

$$\|G_\delta(x)\| \leq C_1(\delta)\|x\|,$$

and that the class of near-unconditional bases strictly contains the important class of quasi-greedy bases, defined by Konyagin and Temlyakov.

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as the class of bases for which $C_1(\delta)$ may be chosen to be independent of $\delta$.

Elton [2] proved that every semi-normalized weakly null sequence contains a subsequence which is a near-unconditional basis for its closed linear span. On the other hand, Maurey and Rosenthal [6] gave an example of a semi-normalized weakly null sequence with no unconditional subsequence.

By a theorem of Paley [7], the Haar system is an unconditional basis of $L_p[0,1]$ for $1 < p < \infty$. For $p = 1$, on the other hand, a well-known example (see e.g. [4]) shows that the normalized Haar basis is not unconditional. The same example, which we now recall, also shows that the Haar basis fails to be near-unconditional.

Define $h_0 = 1_{[0,1]}$, and for $k \in \mathbb{N}$, set

$$h_1^{(k)} = 2^{k-1}(1_{[0,2^{-k})} - 1_{[2^{-k},2^{1-k})}).$$

Observe that for any $n \in \mathbb{N}$

$$\left\| h_0 + \sum_{k=1}^{2n} h_1^{(k)} \right\| = 1,$$

and for some constant $c > 0$

$$\left\| h_0 + \sum_{k=1}^{2n} h_1^{(2k)} \right\| > cn.$$

So, setting $f_n = h_0 + \sum_{k=1}^{2n} h_1^{(k)}$, and $A_n = \{0, 2, 4, \ldots, 2n\}$, we have $\|f_n\| = 1$ and $\|P_{A_n}(f_n)\| \geq cn$, which witnesses the failure of near-unconditionality with $\delta = 1$. In this example the nonzero coefficients of $f_n$ are equal and they lie along the left branch of the Haar system. Our main result shows that in a certain sense every example of the failure of near-unconditionality must be of this type.

We state our main result precisely below but the idea is as follows. Suppose that the Haar coefficients of $f \in L_1[0,1]$, $\delta$, and $A$ are as stated in the definition of near-unconditionality. We show that there is an enlargement $B \supseteq A$ such that $\|P_B(f)\| \leq C(\delta)\|f\|$ and we provide an explicit construction of $B$. Roughly speaking, the ‘added’ coefficients in $B \setminus A$ are those which lie along a segment of a branch of the Haar system such that the coefficient of the maximal element of the segment (with respect to the usual tree ordering) belongs to $A$ and all the coefficients of $f$ along the segment are approximately equal to each other (to within some prescribed multiplicative factor of $1 + \varepsilon$). For $f_n$ and the sets $A_n$, the enlargements are $B_n = \{0, 1, 2, \ldots, 2n\}$, and so $P_{B_n}(f_n) = f_n$, which renders the example harmless. Here the enlargement is as large
as possible. The interest of our result, however, resides in the fact that, for certain \( f \) and \( A \), the enlargement will often be trivial, i.e., \( B = A \), or quite small.

The normalized Haar basis is not a quasi-greedy basis of \( L_1[0,1] \), i.e., the Thresholding Greedy Algorithm fails to converge for certain initial vectors. In a remarkable paper \([3]\) Gogyan exhibited a weak thresholding algorithm which produces uniformly bounded approximants converging to \( f \) for all \( f \in L_1[0,1] \). The proof of our main theorem uses results and techniques from \([3]\). We have chosen to reprove some of these results to achieve what we hope is a self-contained and accessible presentation.

2. Notation and basic facts

We denote the dyadic subintervals of \([0,1] \) by \( \mathcal{D} \), and put \( \overline{\mathcal{D}} = \mathcal{D} \cup \{[0,2]\} \). We think of \( \mathcal{D} \) and \( \overline{\mathcal{D}} \) being partially ordered by “\( \subset \)”. We denote by \( I^+ \) and \( I^- \) the left and the right half subinterval of \( I \in \mathcal{D} \), respectively. \( I^+ \) and \( I^- \) are then the direct successors of \( I \), while the set \( \text{succ}(I) = \{ J \in \mathcal{D} : J \subset I \} \) is called the successors of \( I \in \mathcal{D} \). The predecessors of an \( I \in \mathcal{D} \) is the set \( \text{pred}(I) = \{ J \in \overline{\mathcal{D}} : J \supseteq I \} \).

It follows that the \( \text{pred}(I) \) is a linearly ordered set. If \( I \subseteq J \) are in \( \overline{\mathcal{D}} \) we put \( [I,J] = \{ K \in \overline{\mathcal{D}} : I \subseteq K \subseteq J \} \).

Let \( S \subset \overline{\mathcal{D}} \) be finite and not empty. Then \( S \) contains elements \( I \) which are minimal in \( S \), i.e., there is no \( J \in S \) for which \( J \subset I \). We put in this case \( S' = S \setminus \{ I \in S : I \text{ is minimal in } S \} \).

Inductively we define \( S^{(n)} \) for \( n \in \mathbb{N}_0 \), by \( S^{(0)} = S \), and, assuming \( S^{(n)} \) has been defined, we put \( S^{(n+1)} = (S^{(n)})' \). Since \( S \) was assumed to be finite, there is an \( n \in \mathbb{N} \), for which \( S^{(n)} = \emptyset \), and we define the order of \( S \) by \( \text{ord}(S) = \min \{ n \in \mathbb{N} : S^{(n)} = \emptyset \} - 1 = \max \{ n \in \mathbb{N} : S^{(n)} \neq \emptyset \} \) and for \( I \in S \) we define the order of \( I \) in \( S \) to be the (unique) natural number \( m \in [0, \text{ord}(S)] \), for which \( I \in S^{(m)} \setminus S^{(m+1)} \), and we denote it by \( \text{ord}(I,S) \).

\((h_I : I \in \overline{\mathcal{D}})\) denotes the \( L_1 \)-normalized Haar basis, i.e.,

\[ h_{[0,2]} = 1_{[0,1]} \text{ and } h_I = 2^n(1_{I^+} - 1_{I^-}), \text{ if } I \in \mathcal{D}, \text{ with } m(I) = 2^{-n}, \]
m denoting the Lebesgues measure. If \( f \in L_1[0,1] \) we denote the coefficients of \( f \) with respect to \((h_I)\) by \( c_I(f) \), and thus

\[
(1) \quad f = \sum_{I \in \mathcal{D}} c_I(f)h_I \quad \text{for } f \in L_1[0,1].
\]

From the normalization of \((h_I)\) it follows that

\[
(2) \quad c_I(f) = \int_{I^+} f \, dx - \int_{I^-} f \, dx.
\]

For \( f \in L_1[0,1] \) the support of \( f \) with respect to the Haar basis is the set

\[
\text{supp}_H(f) = \{I \in \mathcal{D} : c_I(f) \neq 0\}.
\]

We will use the following easy inequalities for \( f \in L_1[0,1] \) and \( I, J \in \mathcal{D} \), with \( J \subseteq I \), i.e.,

\[
(3) \quad \|f\|_I := \int_I |f| \, dx \geq \int_J |f| \, dx \geq \int_{J^+} f \, dx - \int_{J^-} f \, dx = |c_J(f)|.
\]

For a finite set \( \mathcal{S} \subset \mathcal{D} \) we denote by \( P_{\mathcal{S}} \) the canonical projection from \( L_1[0,1] \) onto the span of \((h_I : I \in \mathcal{S})\):

\[
P_{\mathcal{S}} : L_1 \rightarrow L_1, \quad f \mapsto \sum_{I \in \mathcal{S}} c_I(f)h_I.
\]

If \( \mathcal{S} \) is cofinite \( P_{\mathcal{S}} \) is defined by \( \text{Id} - P_{\mathcal{D}\setminus\mathcal{S}} \). We will use the fact that the Haar system is a monotone basis with respect to any order, which is consistent with the partial order “\( \subset \)”. It follows therefore that the projections

\[
S_I : L_1[0,1] \rightarrow L_1[0,1], \quad f \mapsto f - \sum_{J \subseteq I} c_J(f)h_J,
\]

are bounded linear projections with \( \|S_I\| \leq 1 \), for all \( I \in \mathcal{D} \). Moreover we observe that

\[
(4) \quad \|S_I(f)\|_I = \left\| \sum_{J \in \text{pred}(I)} c_J(f)h_J \right\|_I \\
\leq \sum_{J \in \text{pred}(I)} |c_J(f)|\|h_J\|_I \\
= \sum_{J \in \text{pred}(I)} |c_J(f)|\frac{m(I)}{m(J)} \leq \sup_{J \in \text{pred}(I)} |c_J(f)|.
\]
For $f \in L_1[0,1]$, $\varepsilon > 0$, and $\mathcal{A} \subset \text{supp}_H(x)$ we define

$$\mathcal{A}_\varepsilon = \left\{ J \in \overline{D} : \exists I \in \mathcal{A}, \ I \subseteq J \text{ and } \left| \frac{c_K(f) - c_I(f)}{c_I(f)} \right| < \varepsilon, \ \text{for all } K \in [I,J] \right\}.$$  

(5)

Since $\mathcal{A}_\varepsilon$ depends on $\varepsilon$ and the family $(c_I(f) : I \in \overline{D})$, we also write $\mathcal{A}_\varepsilon(f)$ instead of only $\mathcal{A}_\varepsilon$ to emphasize the dependence on $f$.

We are now ready to state our main result;

**Theorem 1.** There is a universal constant $C$ so that for $f \in L_1$, $\delta, \varepsilon > 0$ and $A \subset \{ I \in D : |c_I(f)| \geq \delta \}$, there is an $E \subset D$, with $A \subset E \subset A_{\varepsilon}(f)$, so that

$$\|P_E(f)\| \leq C \frac{\log^2(1/\delta)}{\varepsilon^2} \|f\|.$$  

(6)

**Remark 2.** The proof of Theorem 1 yields an explicit, albeit laborious, description of $\mathcal{E}$.

### 3. Proof of the main Result

We will first state and prove several Lemmas.

**Lemma 3.** Let $f \in L_1[0,1]$ and let $K, J$ and $I$ be elements of $\overline{D}$, and assume that $K$ is a direct successor of $J$, which is a direct successor of $I$. Then

$$\|f\|_{I \setminus K} \geq \left| \frac{|c_I(f)| - |c_J(f)|}{2} \right|.$$  

(7)

**Proof.** We first note that from the monotonicity property of the Haar basis we deduce that

$$\|f\|_{I \setminus J} = \left\| \sum_{L \in \text{pred}(I \setminus J)} c_L(f) h_L|_{I \setminus J} + c_{I \setminus J}(f) h_{I \setminus J} + \sum_{L \in \text{succ}(I \setminus J)} c_L(f) h_L \right\|$$

$$\geq \left\| \sum_{L \in \text{pred}(I \setminus J)} c_L(f) h_L|_{I \setminus J} \right\| = \|S_{I \setminus J}(f)\|_{I \setminus J}$$

(8)

and similarly we obtain

$$\|f\|_{J \setminus K} \geq \|S_{J \setminus K}(f)\|_{J \setminus K}.$$  

$S_I(f)$ takes a constant value $H$ on $I$. Denote by $a$ the value of $c_I(f) h_I$ on $I \setminus J$ and denote by $b$ the value of $c_J(f) h_J$ on $J \setminus K$, and let $\delta = m(I)$. Then we compute
\[ \|f\|_{I\setminus K} = \|f\|_{I\setminus J} + \|f\|_{J\setminus K} \]
\[ \geq \|S_{I\setminus J}(f)\|_{I\setminus J} + \|S_{J\setminus K}(f)\|_{J\setminus K} \]
\[ = \|S_{I}(f)+c_I(f)h_I\|_{I\setminus J} + \|S_{J}(f)+c_J(f)h_J\|_{J\setminus K} \]
\[ = \frac{\delta}{2}|H + a| + \frac{\delta}{4}|H - a + b| \]
\[ \geq \frac{\delta}{4}|H + a| + \frac{\delta}{4}|H - a| - b| \]
\[ \geq \frac{\delta}{4}|2a - b| \geq \frac{\delta}{4}|2|a| - |b||. \]

Our claim follows then if we notice that \(|a|\delta = |c_I(f)|\) and \(|b|\delta = 2|c_J(f)|\).

\[ \square \]

We iterate Lemma 3 to obtain the following result.

**Lemma 4.** Let \( f \in L_1[0, 1] \) and let \( K, J \) and \( I \) be elements of \( \overline{D} \), and assume that \( K \) is a successor of \( J \), which is a successor of \( I \). Then

\[ \|f\|_{I\setminus K} \geq \left\{ c_I(f) - |c_J(f)| \right\}. \]

**Proof.** First we can, without loss of generality, assume that \( K \) is a direct successor of \( J \), we write \([K, I]\) as \([K, I] = \{I_{n+1}, I_n, I_{n-1}, \ldots I_0\}\), with \( K = I_{n+1} \subset I_n = J \subset I_{n-1} \subset \ldots I_0 = I \), so that \( I_{m+1} \) is a direct successor of \( I_m \) for \( m = 0, 1, 2, \ldots, n \). From Lemma 3 we obtain

\[ \|f\|_{I\setminus K} = \sum_{j=0}^{n} \|f\|_{I_j\setminus I_{j+1}} \]
\[ \geq \frac{1}{2} \left( \|f\|_{I_0\setminus I_1} + \|f\|_{I_1\setminus I_2} \right) + \frac{1}{2} \left( \|f\|_{I_1\setminus I_2} + \|f\|_{I_2\setminus I_3} \right) \]
\[ \ldots + \frac{1}{2} \left( \|f\|_{I_{n-1}\setminus I_n} + \|f\|_{I_n\setminus I_{n+1}} \right) \]
\[ = \frac{1}{2} \sum_{j=0}^{n-1} \|f\|_{I_j\setminus I_{j+2}} \]
\[ \geq \sum_{j=0}^{n-1} \left\{ \frac{|c_I(f)| - |c_{I_{j+1}}(f)|}{4} \right\} \geq \left\{ \frac{|c_I(f)| - |c_J(f)|}{4} \right\} \]

which finishes the proof of our assertion. \[ \square \]

**Lemma 5.** Assume that \( f, g \in L_1[0, 1] \) and \( F \subset \overline{D} \) and that the following properties hold for some \( \alpha, \varepsilon \in (0, 1) \)

a) \( \text{supp}_H(f) \cap \text{supp}_H(g) = \emptyset \),
b) For every $I \in \mathcal{F}$ there is a $J \in \text{succ}(I)$, so that

$$[J, I] \cap \mathcal{F} = \{I\}$$

$$|c_J(f)| \geq \alpha$$

$$|c_I(g) - c_J(f)| \geq \varepsilon |c_J(f)|.$$

Then

(10) $$\|f + g\| \geq \frac{\alpha \varepsilon}{6} |\mathcal{F}|.$$

In order to prove Lemma 5 we will first show the following observation.

**Proposition 6.** Let $\mathcal{F} \subset \mathcal{D}$, and define the following partition of $\mathcal{F}$ into sets $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$

$$\mathcal{F}_0 = \{I \in \mathcal{F} : I \text{ is minimal in } \mathcal{F}\} = \{I \in \mathcal{F} : \text{succ}(I) \cap \mathcal{F} = \emptyset\};$$

$$\mathcal{F}_1 = \{I \in \mathcal{F} : \text{succ}(I) \cap \mathcal{F} \text{ has exactly one maximal element}\}, \text{ and }$$

$$\mathcal{F}_2 = \{I \in \mathcal{F} : \text{succ}(I) \cap \mathcal{F} \text{ has at least two maximal element}\}.$$

Then

(11) $$|\mathcal{F}_2| < |\mathcal{F}_0|.$$

**Proof.** In order to verify (11) we first show for $I \in \mathcal{F}_2$ that

(12) $$\left|\{J \in \mathcal{F}_2 : J \subseteq I\}\right| < \left|\{J \in \mathcal{F}_0 : J \subset I\}\right|.$$

Assuming that (12) is true for all $I \in \mathcal{F}_2$, we let $I_1, I_2, \ldots I_t$ be the maximal elements of $\mathcal{F}_2$. Since the $I_j$’s are pairwise disjoint, observe that

$$|\mathcal{F}_2| = \sum_{j=1}^{t} \left|\{I \in \mathcal{F}_2 : I \subseteq I_j\}\right| < \sum_{j=1}^{t} \left|\{J \in \mathcal{F}_0 : J \subset I_j\}\right| \leq |\mathcal{F}_0|.$$

We now prove (12) by induction on $n = \left|\{J \in \mathcal{F}_2 : J \subseteq I\}\right|$. If $n = 1$ then $I$ must have at least two successor, say $J_1$ and $J_2$ in $\mathcal{F}$ which are incomparable, and thus there are elements $I_1, I_2 \in \mathcal{F}_0$ so that $I_1 \subset J_1$ and $I_2 \subset J_2$. Assume that our claim is true for $n$, and assume that $\left|\{J \in \mathcal{F}_2 : J \subseteq I\}\right| = n + 1 \geq 2$. We denote the maximal elements of $\{J \in \mathcal{F}_2 : J \subseteq I\}$ by $I_1, I_2, \ldots I_m$. Either $m \geq 2$, then it follows from the induction hypothesis, and the fact that $I_1, I_2, \ldots I_m$ are incomparable, that

$$\left|\{J \in \mathcal{F}_2 : J \subseteq I\}\right| = 1 + \sum_{j=1}^{m} \left|\{J \in \mathcal{F}_2 : J \subseteq I_j\}\right|.$$
\[ \leq 1 + \sum_{j=1}^{m} \left( |\{ J \in \mathcal{F}_0 : J \subseteq I_j \}| - 1 \right) \]
\[ \leq |\{ J \in \mathcal{F}_0 : J \subseteq I \}| - 1 < |\{ J \in \mathcal{F}_0 : J \subseteq I \}|. \]

Or \( m = 1 \), and if \( \tilde{I} \) is the only maximal element of \( \{ J \in \mathcal{F}_2 : J \not\subseteq I \} \), then by the definition of \( \mathcal{F}_2 \) there must be a \( J_0 \in \mathcal{F}_0 \) with \( J_0 \subset I \setminus \tilde{I} \), and we deduce from our induction hypothesis that
\[ |\{ J \in \mathcal{F}_2 : J \subseteq I \}| = 1 + |\{ J \in \mathcal{F}_2 : J \subseteq \tilde{I} \}| \]
\[ < 1 + |\{ J \in \mathcal{F}_0 : J \subseteq \tilde{I} \}| \leq |\{ J \in \mathcal{F}_0 : J \subseteq I \}|, \]
which finishes the proof of the induction step, and the proof of (12).

Proof of Lemma 6. Assume now that \( \alpha, \varepsilon > 0 \) and \( f, g \in L_1[0, 1] \), and \( \mathcal{F} \subset \mathcal{D} \) are given satisfying (a), (b). Let \( \mathcal{F}_0, \mathcal{F}_1, \) and \( \mathcal{F}_2 \) the subsets of \( \mathcal{F} \) introduced in Proposition 6. We distinguish between two cases.

Case 1. \( |\mathcal{F}_0| \geq \frac{1}{6}|\mathcal{F}| \).

Fix \( I \in \mathcal{F}_0 \), and let \( J \in \text{succ}(I) \) be chosen so that condition (b) is satisfied. It follows then from condition (a) and (3)
\[ \| f + g \|_I \geq \| f + g \|_J \geq |c_J(f)| \geq \alpha. \]

Since all the elements in \( \mathcal{F}_0 \) are disjoint it follows that
\[ \| f + g \| \geq \sum_{I \in \mathcal{F}_0} \| f + g \|_I \geq |\mathcal{F}_0| \alpha \geq |\mathcal{F}| \alpha \frac{4}{6}. \]

Case 2. \( |\mathcal{F}_0| < \frac{1}{6}|\mathcal{F}| \).

Applying (11) we obtain that
\[ |\mathcal{F}_1| = |\mathcal{F}| - |\mathcal{F}_0| - |\mathcal{F}_2| > |\mathcal{F}| - 2|\mathcal{F}_0| > \frac{2}{3}|\mathcal{F}|. \]

Fix \( I \in \mathcal{F}_1 \), and let \( J \in \text{succ}(I) \) satisfy the conditions in (c), and let \( \tilde{I} \), be the unique maximal element of \( \text{succ}(I) \cap \mathcal{F} \). It follows that \( J \not\subseteq I \) and, since by condition (b) \( \tilde{I} \not\in [J, I] \), we deduce that \( J \not\subseteq \tilde{I} \) which implies that either \( \tilde{I} \not\subseteq J \) or \( \tilde{I} \cap J = \emptyset \). In the first case we deduce from Lemma 4 and condition (b) that
\[ \| f + g \|_{I \setminus \tilde{I}} \geq \left| \frac{|c_J(f)| - |c_{J}(f)|}{4} \right| \geq \varepsilon \frac{|c_J(f)|}{4} \geq \frac{\varepsilon \alpha}{4}. \]

In the second case we deduce from (3) and condition (c) that
\[ \| f + g \|_{I \setminus \tilde{I}} \geq \| f + g \|_J \geq |c_J(f)| \geq \alpha. \]
We conclude therefore from the fact that the sets \( I \setminus \tilde{I} \), with \( I \in \mathcal{F}_1 \), are pairwise disjoint and therefore
\[
\|f + g\| \geq \sum_{I \in \mathcal{F}_1} \|f + g\|_{I \setminus \tilde{I}} \geq |\mathcal{F}_1| \frac{\varepsilon \alpha}{4} \geq \frac{2}{3} \varepsilon \alpha |\mathcal{F}| = \frac{\alpha \varepsilon}{6} |\mathcal{F}|.
\]

\( \square \)

In order to formulate our next step we introduce the following Symmetry Operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). For that assume that \( f \in L_1[0,1] \) and \( I \in \mathcal{D} \). We define the following two functions \( \mathcal{L}_1(f, I) \) and \( \mathcal{L}_2(f, I) \) in \( L_1[0,1] \). For \( \xi \in [0,1] \) we put
\[
\mathcal{L}_1(f, I)(\xi) = \begin{cases} 
  f(\xi) & \text{if } \xi \notin I^- \\
  f(\xi - \frac{m(I)}{2}) & \text{if } \xi \in I^- 
\end{cases}
\]
\[
\mathcal{L}_2(f, I)(\xi) = \begin{cases} 
  f(\xi) & \text{if } \xi \notin I^+ \\
  f(\xi + \frac{m(I)}{2}) & \text{if } \xi \in I^+
\end{cases}
\]

Note that \( \mathcal{L}_1(f, I) \) restricted to \( I^- \) is a shift of \( f \) restricted to \( I^+ \), and vice versa \( \mathcal{L}_2(f, I) \) restricted to \( I^+ \) is a shift of \( f \) restricted to \( I^- \).

We will use this symmetrization only for \( f \in L_1[0,1] \) and \( I \in \mathcal{D} \), for which \( c_I(f) = 0 \). We observe in that case that letting \( f' = \mathcal{L}_1(f, I) \) or \( f'' = \mathcal{L}_2(f, I) \), and any \( J \in \mathcal{D} \)
\[
(13) \quad c_J(f') = \begin{cases} 
  c_J(f) & \text{if } J \supsetneq I \text{ (here we use that } c_I(f) = 0) \\
  c_J(f) & \text{if } J \cap I = \emptyset \\
  0 & \text{if } J = I \\
  c_J(f) & \text{if } J \subset I^+ \text{ and } f' = L_1(f, I), \text{ or} \\
  c_J(f) & \text{if } J \subset I^- \text{ and } f' = L_2(f, I), \\
  c_{J-m(I)/2}(f) & \text{if } J \subset I^- \text{ and } f' = L_1(f, I) \\
  c_{J+m(I)/2}(f) & \text{if } J \subset I^+ \text{ and } f' = L_2(f, I).
\end{cases}
\]

Moreover it follows that
\[
(14) \quad \|L_1(f, I)\| = \|f\| + \Delta(f, I) \text{ and } \|L_2(f, I)\| = \|f\| - \Delta(f, I),
\]
with \( \Delta(f, I) = \|f\|_{I^+} - \|f\|_{I^-} \).

**Lemma 7.** Assume that \( f, g \in L_1[0,1] \), so that \( \text{supp}_H(f), \text{supp}_H(g) \subset \mathcal{D} \) and \( I \in \mathcal{D} \) are given. Suppose that \( c_I(f) = c_I(g) = 0 \) and that the following properties hold:

a) \( \text{supp}_H(f) \cap \text{supp}_H(g) = \emptyset \),

b) \( \|f\| > 0 \), and thus, by (a), also \( \|f + g\| > 0 \).
Then for either \( f' = L_1(f, I) \) and \( g' = L_1(g, I) \), or \( f' = L_2(f, I) \) and \( g' = L_2(g, I) \) it follows that

\[
\text{supp}_H(f') \cap \text{supp}_H(g') = \emptyset \quad \text{and} \quad I \notin \text{supp}_H(f') \cup \text{supp}_H(g');
\]

(15) for any \( J \in \mathcal{D} \) it follows that

\[
c_J(f') \in \{ c_I(f') : I \in \text{supp}(f) \} \cup \{ 0 \} \quad \text{and} \quad c_J(g') \in \{ c_I(g') : I \in \text{supp}(g) \} \cup \{ 0 \},
\]

(16) \( \| f' + g' \| > 0 \) and \( \frac{\| f \|}{\| f + g \|} \leq \frac{\| f' \|}{\| f' + g' \|} \).

(17)

\[
\text{Proof.} \quad \text{It follows immediately from (13) that (16) and (15) are satisfied for either of the possible choices of } f' \text{ and } g'.
\]

To satisfy (17) we will first consider the case that \( (f + g)[0,1] \setminus I^+ \equiv 0 \). In this case it follows from (a) that \( c_J(f) = c_J(g) = 0 \) for all \( J \in \mathcal{D} \), with \( J \subset [0,1] \setminus I^+ \), and thus by (b) \( f|I^+ \neq 0 \) and \( (g+f)|I^+ \neq 0 \). If we choose \( f' = L_1(f, I) \) and \( g' = L_1(g, I) \) we obtain that \( \| f' + g' \| > 0 \), \( \| f' \| = 2\| f \| \) and \( \| f' + g' \| = 2\| f + g \| \).

A similar argument can be made if \( (f + g)[0,1] \setminus I^- \equiv 0 \).

If neither of the two previously discussed cases occurs we conclude that

\[
\| f + g \| = \| f + g \|_{[0,1] \setminus I} + \| f + g \|_{I^+} + \| f + g \|_{I^-} > \| f + g \|_{I^+} - \| f + g \|_{I^-}
\]

and

\[
\| f + g \| = \| f + g \|_{[0,1] \setminus I} + \| f + g \|_{I^+} + \| f + g \|_{I^-} > \| f + g \|_{I^-} - \| f + g \|_{I^+}
\]

which implies by (14) that in either of the two possible choices for \( f' \) and \( g' \) it follows that \( \| f' + g' \| > 0 \).

Finally, if \( \Delta(f, I)\| f + g \| \geq \Delta(f + g, I)\| f \| \), we choose \( f' = L_1(f, I) \) and \( g' = L_1(f, I) \) and note that since in this case we have

\[
(\| f \| + \Delta(f, I))\| f + g \| \geq (\| f + g \| + \Delta(f + g, I))\| f \|
\]

it follows that

\[
\frac{\| f' \|}{\| f' + g' \|} = \frac{\| f \| + \Delta(f, I)}{\| f + g \| + \Delta(f + g, I)} \geq \frac{\| f \|}{\| f + g \|}.
\]

If \( \Delta(f, I)\| f + g \| < \Delta(f + g)\| f \| \) and thus \( -\Delta(f, I)\| f + g \| > -\Delta(f + g)\| f \| \), we choose \( f' = L_2(f, I) \) and \( g' = L_2(f, I) \) and note that since in this case we have

\[
(\| f \| - \Delta(f, I))\| f + g \| > (\| f + g \| - \Delta(f + g, I))\| f \|,
\]
and it follows that
\[
\frac{\|f'\|}{\|f' + g'\|} = \frac{\|f\| - \Delta(f, I)}{\|f + g\| - \Delta(f + g, I)} > \frac{\|f\|}{\|f + g\|}
\]
which finishes the verification of (17) and the proof of our claim. \(\square\)

Assume now that \(f, g \subset L_1[0, 1], \|f\| > 0\), are such that \(\text{supp}_H(f)\) and \(\text{supp}_H(g)\) are finite and disjoint subsets of \(D\). We also assume that

\[
c_{[0,1]}(f) = \int_0^{1/2} f(x) \, dx - \int_{1/2}^1 f(x) \, dx = 0 \quad \text{and} \quad c_{[0,1]}(g) = \int_0^{1/2} g(x) \, dx - \int_{1/2}^1 g(x) \, dx = 0.
\]

Define:

\[
\mathcal{F}(f) = \{I \in D : c_I(f) = 0 \text{ but } c_{I+}(f) \neq 0 \text{ or } c_{I-}(f) \neq 0\}
\]
and make the following assumption

\[
\text{supp}_H(g) \cap \mathcal{F}(f) = \emptyset.
\]

Let \(\mathcal{F}(f) = (I_i)_{i=1}^n\), where \(m(I_1) \leq m(I_2) \leq \cdots \leq m(I_n)\). First ‘symmetrize’ the pair \((f, g)\) on \(I_1\) to obtain a pair \((f_1, g_1)\) satisfying \(\|f\|/\|f + g\| \leq \|f_1\|/\|f_1 + g_1\|\). Note that \(\mathcal{F}(f_1) = \mathcal{F}(f)\). Now symmetrize \((f_1, g_1)\) on \(I_2\) to obtain \((f_2, g_2)\) satisfying \(\|f_1\|/\|f_1 + g_1\| \leq \|f_2\|/\|f_2 + g_2\|\). Note that if \(I \in \mathcal{F}(f_2)\) satisfies \(m(I) < m(I_2)\), then \((f_2, g_2)\) on \(I\) is a ‘copy’ of \((f_1, g_1)\) on \(I_1\). Hence, \(f_2\) and \(g_2\) are automatically symmetric on \(I\). On the other hand, if \(I \in \mathcal{F}(f_2)\) satisfies \(m(I) \geq m(I_2)\) then \(I = I_j\) for some \(j \geq 2\). Now symmetrize \((f_2, g_2)\) on \(I_2\) to obtain \((f_3, g_3)\) satisfying \(\|f_2\|/\|f_2 + g_2\| \leq \|f_3\|/\|f_3 + g_3\|\). Note that if \(I \in \mathcal{F}(f_3)\) satisfies \(m(I) < m(I_3)\) then \((f_3, g_3)\) on \(I\) is a copy of \((f_2, g_2)\) on \(I_1\) or \(I_2\). Hence \(f_3\) and \(g_3\) are automatically symmetric on \(I\). Continuing in this way, we finally obtain, after symmetrizing on \(I_n\), a pair \((f_n, g_n)\) such that \(f_n\) and \(g_n\) are symmetric on each \(I \in \mathcal{F}(f_n)\) and \(\|f\|/\|f + g\| \leq \|f_n\|/\|f_n + g_n\|\).

Setting \(\tilde{f} = f_n\) and \(\tilde{g} = f_n\), the following conditions hold:

\[
\begin{align*}
c_{[0,2]}(\tilde{f}) &= c_{[0,2]}(\tilde{g}) = c_{[0,1]}(\tilde{f}) = c_{[0,1]}(\tilde{g}) = 0, \\
\text{For all } I \in \mathcal{F}(\tilde{f}) \text{ it follows that} \\
c_I(\tilde{g}) &= 0 \quad \text{and} \quad \\
\tilde{f}(x) &= \tilde{f}(x - m(I^+)) \quad \text{and} \quad \tilde{g}(x) = \tilde{g}(x - m(I^+)), \quad \text{if } x \in I^-,
\end{align*}
\]

and

\[
\text{supp}_H(\tilde{f}) \cap \text{supp}_H(\tilde{g}) = \emptyset.
\]
for any \( J \in D \) it follows that
\[
c_J(\bar{f}) \in \{c_I(f) : I \in \text{supp}(f)\} \cup \{0\} \quad \text{and} \quad c_J(\bar{g}) \in \{c_I(g) : I \in \text{supp}(g)\} \cup \{0\},
\]
(25) \[\|\bar{f} + \bar{g}\| > 0 \text{ and } \frac{\|f\|}{\|f + g\|} \leq \frac{\|\bar{f}\|}{\|\bar{f} + \bar{g}\|}.\]

**Lemma 8.** Assume that \( f, g \in L_1[0, 1] \) are such that \( \text{supp}_H(f) \) and \( \text{supp}_H(g) \) are finite disjoint subsets of \( D \setminus \{0, 1\} \), and that \( \text{supp}_H(g) \cap \mathcal{F}(f) = \emptyset \), where \( \mathcal{F}(f) \subset D \), was defined above. Assume moreover that for some \( \alpha > 0 \), we have

(26) \[|c_J(f)| \geq \alpha, \text{ for all } J \in \text{supp}_H(f), \quad \text{and} \quad |c_J(g)| \leq 1, \text{ for all } J \in \text{supp}_H(g).\]

Then

(27) \[\|f\| \leq \left(\frac{5}{\alpha} + 1\right)\|f + g\|.
\]

**Proof.** Let \( \bar{f} \) and \( \bar{g} \) be the elements in \( L_1[0, 1] \) constructed from \( f \) and \( g \) as before satisfying the conditions (21), (22), (23), (24), and (25). Note also that (24) implies that \( |c_J(\bar{f})| \geq \alpha \), for all \( J \in \text{supp}_H(\bar{f}) \), and \( |c_J(\bar{g})| \leq 1 \), for all \( J \in \text{supp}_H(\bar{g}) \).

By (25) it is enough to show (27) for \( \bar{f} \) and \( \bar{g} \) instead of \( f \) and \( g \).

We will deduce our statement from the following

**Main Claim.** For all \( I \in \mathcal{F}(\bar{f}) \) it follows that

(28) \[\|\bar{f}\|_I \leq \left(\frac{5}{\alpha} + 1\right)\|\bar{f} + \bar{g}\|_I - 2\alpha - 8.\]

Assuming the Main Claim we can argue as follows. Using (21) it follows that out side of \( J = \bigcup_{I \in \mathcal{F}(\bar{f})} I \) \( \bar{f} \) is vanishing. Thus, we can choose disjoint sets \( I_1, I_2, \ldots, I_n \), in \( \mathcal{F}(\bar{f}) \) so that \( \bar{f} \) vanishes outside of \( \bigcup_{j=1}^n I_j \), and (28) yields

\[\|\bar{f}\| = \sum_{j=1}^n \|\bar{f}\|_{I_j} \leq \left(\frac{5}{\alpha} + 1\right)\sum_{j=1}^n \|\bar{f} + \bar{g}\|_{I_j} \leq \left(\frac{5}{\alpha} + 1\right)\|\bar{f} + \bar{g}\|,
\]

which proofs our wanted statement.

In order to show the Main Claim let \( I \in \mathcal{F}(\bar{f}) \) and denote \( k = \text{ord}(I, \mathcal{F}(\bar{f})). \) We will show the inequality (28) by induction for all \( k \).

First assume that \( k = 0. \) From (22) and (3) we obtain

(29) \[\|\bar{f} + \bar{g}\|_I = 2\|\bar{f} + \bar{g}\|_{I^+} \geq 2|c_{I^+}(\bar{f} + \bar{g})| = 2|c_{I^+}(\bar{f})| \geq 2\alpha.
\]
Using (4), (23) and (27) we obtain

\[ \|S_I(\tilde{g})\|_I \leq 1. \]  

From the definition of \( F(\tilde{f}) \), and the assumption that \( \text{ord}(I, F(\tilde{f})) = 0 \), (23) we deduce that if \( J \in \text{supp}_H(\tilde{f}) \) with \( J \subset I^+ \) or \( J \subset I^- \), then \([J, I^+] \subset \text{supp}_H(\tilde{f})\), or \([J, I^-] \subset \text{supp}_H(\tilde{f})\), respectively. But, using (23), this implies that if \( J \in \text{supp}_H(\tilde{g}) \), with \( J \subset I \), then \( \text{succ}(J) \cap \text{supp}_H(\tilde{f}) = \emptyset \). We deduce therefore from the monotonicity properties of the Haar basis that

\[ \|\tilde{f} + \tilde{g}\|_I \geq \|f + \sum_{J \in D, J \notin I} c_J(\tilde{g})h_J\|_I = \|\tilde{f} + S_I(\tilde{g})\|_I. \]  

We therefore conclude

\[ \left(\frac{5}{\alpha} + 2\right)\|\tilde{f} + \tilde{g}\|_I \geq \left(\frac{5}{\alpha} + 1\right)\|\tilde{f} + \tilde{g}\|_I + \|\tilde{f} + S_I(\tilde{g})\|_I \quad \text{(by (31))} \]

\[ \geq 10 + 2\alpha + \|\tilde{f}\|_I - \|S_I(\tilde{g})\|_I \quad \text{(by (29))} \]

\[ \geq \|f\|_I + 9 + 2\alpha \quad \text{(by (30))}, \]

which proves our claim in the case that \( \text{ord}(I, F(\tilde{f})) = 0 \).

Assume that (28) holds for all \( I \in F \) with \( \text{ord}(I, F(\tilde{f})) < k \), for some \( k \in \mathbb{N} \), and assume that \( I \in F(\tilde{f}) \) with \( \text{ord}(I, F(\tilde{f})) = k \). By the symmetry condition in (22) the number of elements \( J \) of \( F(\tilde{f}) \) for which \( J \subset I \) and \( \text{ord}(J, F) = k - 1 \) is even, half of them being subsets of \( I^+ \), the other being subsets of \( I^- \). We order therefore these sets into \( J_1, J_2, \ldots, J_{2s} \), for some \( s \in \mathbb{N} \), with \( J_i \subset I^+ \) and \( J_{i+i} \subset I^- \), for \( i = 1, 2, \ldots, s \). We note that the \( J_i, i = 1, 2, \ldots \), are pairwise disjoint and that all the \( J \in F(\tilde{f}) \), with \( F \subset I \), and \( \text{ord}(J, F(\tilde{f})) \leq k - 2, \) are subset of some of the \( J_i, i = 1, 2, \ldots, 2s \).

From our induction hypothesis we deduce that

\[ \|\tilde{f}\|_{J_i} \leq \left(\frac{5}{\alpha} + 2\right)\|\tilde{f} + \tilde{g}\|_{J_i} - 2\alpha - 8 \text{ for } i = 1, 2, \ldots, 2s. \]  

We define \( D = I^+ \setminus \bigcup_{i=1}^{s} J_i \) and

\[ \phi = S_{J_1}(S_{J_2}(\ldots S_{J_s}(\tilde{f}) \ldots)) = \sum_{J \in \mathcal{J}} c_J(\tilde{f})h_J \]

\[ \gamma = S_{J_1}(S_{J_2}(\ldots S_{J_s}(\tilde{g}) \ldots)) = \sum_{J \in \mathcal{J}} c_J(\tilde{g})h_J \]

with

\[ \mathcal{J} = \{ J \in D : \forall j=1,2,\ldots,s \quad J \not\subset I_j \}. \]
It follows that
\[ \phi|_D = \tilde{f}|_D \text{ and } \gamma|_D = \tilde{g}|_D, \]
(33) implies that
\[ \|\gamma\|_{J_i} \leq 1, \text{ for } i = 1, 2 \ldots s, \]
and since for any \( J \in D \), with \( J \subseteq D \), for which \( c_J(\phi) \neq 0 \), we have \([J, I^+] \subseteq \text{supp}_H(\phi)\) (otherwise there would be an \( K \in \mathcal{F}(\tilde{f}) \) with \( K \subset I^+ \) and \( K \supseteq J_i \), for some \( i \in \{1, 2 \ldots s\} \), or \( K \subset D \)) it follows from the monotonicity property of the Haar system and (4) that
\[ \|\phi + \gamma\|_{I^+} \geq \|\phi + S_{I^+}(\gamma)\|_{I^+} \geq \|\phi\|_{I^+} - 1. \]
(35) It follows that
\[ \|\phi + \gamma\|_{I^+ \setminus D} \leq \|\phi\|_{I^+ \setminus D} + \|\gamma\|_{I^+ \setminus D} = \|\phi\|_{I^+ \setminus D} + \sum_{i=1}^{s} \|\gamma\|_{J_i} \leq \|\phi\|_{I^+ \setminus D} + s \]
and
\[ \|\tilde{f} + \tilde{g}\|_D = \|\phi + \gamma\|_D = \|\phi + \gamma\|_{I^+} - \|\phi + \gamma\|_{I^+ \setminus D} \geq \|\phi\|_{I^+} - 1 - \|\phi\|_{I^+ \setminus D} - s = \|\phi\|_D - s - 1. \]
This implies together with (32) that
\[ \|\tilde{f}\|_{I^+} = \|\tilde{f}\|_D + \sum_{i=1}^{s} \|\tilde{f}\|_{J_i} = \|\phi\|_D + \sum_{i=1}^{s} \|\tilde{f}\|_{J_i} \leq \|\tilde{f} + \tilde{g}\|_D + s + 1 + \left(\frac{5}{\alpha} + 2\right) \sum_{i=1}^{s} \|\tilde{f} + \tilde{g}\|_{J_i} - s(2\alpha + 8) \leq \left(\frac{5}{\alpha} + 2\right) \|\tilde{f} + \tilde{g}\|_{I^+} - 7s - 2\alpha s + 1. \]
By the symmetry condition (22) we also obtain that
\[ \|\tilde{f}\|_{I^-} \leq \left(\frac{5}{\alpha} + 2\right) \|\tilde{f} + \tilde{g}\|_{I^+} - 7s - 2\alpha s + 1. \]
Adding these two inequalities yields our Main Claim since \( s \geq 1 \). \( \Box \)
Theorem 9. Let \( h \in L_1[0,1] \), with \( \supp_H(h) \subset \text{succ}([0,1]) \), and let \( 0 < \varepsilon < 1 \), \( 0 < \alpha \leq 1 \) and \( b \in \mathbb{R}^+ \). Assume that \( S \subset \mathcal{D} \), is such that
\[
|c_I(h)| \geq \alpha b, \text{ if } I \in S, \text{ and } |c_I(h)| \leq b, \text{ if } I \notin S.
\]
Then
\[
\|P_S(\varepsilon)h\| \leq \frac{42}{\alpha^2 \varepsilon} \|h\|.
\]

Proof. After rescaling we can assume that \( b = 1 \). Put \( f = P_S(\varepsilon)h \) and \( g = h - P_S(\varepsilon)h \). We note that \( f \) and \( g \) satisfy the assumptions of Lemma 5 with \( \mathcal{F} = \{ I \in \mathcal{D} : I \notin S_\varepsilon \text{ or } I^+ \in S_\varepsilon \text{ or } I^- \in S_\varepsilon \} \).

Indeed, condition (a) of Lemma 5 is clearly satisfied, and in order to verify (b) let \( I \in \mathcal{F} \). Without loss of generality we can assume that \( I^+ \in S_\varepsilon \). Thus there is a \( J \in S \), with \( J \subset I \), and so that \( J \) is maximal with that property. It follows therefore from the definition of \( S_\varepsilon \) that \( [J, I^+] \subset S_\varepsilon \), and thus \( [J, I] \cap \mathcal{F} = \{ I \} \), \( |c_J(f)| = |c_J(h)| \geq \alpha \), and
\[
|c_I(g) - c_J(f)| = |c_I(h) - c_J(h)| \geq \varepsilon |c_J(f)|.
\]
Lemma 5 yields that
\[
\|f + g\| \geq \frac{\alpha \varepsilon}{6} |\mathcal{F}|.
\]
Setting
\[
\bar{g} = g - \sum_{I \in \mathcal{F}} c_I(g) h_I,
\]
then, by our assumption on \( h \),
\[
\|f + \bar{g}\| \leq \|f + g\| + \|\bar{g} - g\|
\]
\[
\leq \|f + g\| + \left\| \sum_{I \in \mathcal{F}} c_I(h) h_I \right\|
\]
\[
\leq \|f + g\| + |\mathcal{F}|
\]
\[
\leq \left(1 + \frac{6}{\alpha \varepsilon}\right) \|f + g\|.
\]
Note that since \( \mathcal{F} = \mathcal{F}(f) \) (where \( \mathcal{F}(f) \) was defined in (19)) the pair \( f \) and \( \bar{g} \) satisfies the assumption of Lemma 8 and we deduce that
\[
\|h\| = \|f + g\|
\]
\[
\geq \frac{\alpha \varepsilon}{\alpha \varepsilon + 6} \|f + \bar{g}\|
\]
\[
\geq \frac{\alpha \varepsilon}{\alpha \varepsilon + 6} \frac{\alpha}{\alpha \varepsilon + 6} \|f\|
\]
\[
\geq \frac{\alpha^2 \varepsilon}{42} \|f\| = \frac{\alpha^2 \varepsilon}{42} \|P_S(\varepsilon)h\|.
\]
which implies our claim.

**Corollary 10.** Let \( f \in L_1[0, 1] \), with \( \text{supp}_H(f) \subset \text{succ}([0, 1]) \), \( A \subset \mathcal{D} \), \( 0 < \varepsilon < 1 \), \( \rho \in \mathbb{R}^+ \). Put \( B = A \cap \{ I \in \mathcal{D} : \rho < |c_I(f)| \leq 2\rho \} \).

Then there exists \( C \subset \mathcal{D} \), with \( B \subseteq C \subseteq B_\varepsilon(f) \), so that

\[
\|P_C(f)\| \leq \frac{45738}{\varepsilon} \|f\|.
\]

**Proof.** We first apply Theorem 9 to the set \( S = \{ J \in \mathcal{D} : |c_J(f)| > 3\rho \} \), the numbers \( b = 3\rho \), \( \alpha = 1 \) and \( \varepsilon = \frac{1}{3} \). It follows that

\[
(36) \quad \|P_{S_\varepsilon}(f)\| \leq 120\|f\|.
\]

Note that

\[
S_\varepsilon = \left\{ J \in \mathcal{D} : \exists I \in S, I \subset J \forall K \in [I, J] \left\{ |c_I(f) - c_K(f)| \leq \frac{1}{3} |c_I(f)| \right\} \right\} \subset \{ J \in \mathcal{D} : |c_J(f)| > 2\rho \}
\]

Put \( B^{(1)} := \mathcal{D} \setminus S_\varepsilon \), and \( g = P_{B^{(1)}}(f) \) then,

\[
(37) \quad \|g\| \leq 121\|f\|
\]

and

\[
(38) \quad \{ J \in \mathcal{D} : |c_J(f)| \leq 2\rho \} \subseteq B^{(1)} \subseteq \{ J \in \mathcal{D} : |c_J(f)| \leq 3\rho \}.
\]

Then we apply Theorem 9 again, namely to the function \( g \), the set

\[
B^{(2)} = \{ I \in \mathcal{D} : I \in A, \rho < |c_I(g)| \leq 2\rho \},
\]

and the numbers \( b = 3\rho \), \( \alpha = \frac{1}{3} \). We deduce that for each \( \varepsilon \in (0, 1) \)

\[
(39) \quad \|P_{B^{(2)}_\varepsilon}(g)\| \leq \frac{378}{\varepsilon} \|g\|.
\]

Here we mean by \( B^{(2)}_\varepsilon \), to be precise, the set \( B^{(2)}_\varepsilon(g) \). Since for every \( I \in \mathcal{D} \), with \( c_I(g) \neq 0 \), it follows that \( c_I(g) = c_I(f) \), we deduce that

\[
B^{(2)}_\varepsilon(g) = \left\{ J \in \mathcal{D} : \exists I \in A, I \subset J \forall K \in [I, J] |c_I(f) - c_K(f)| \leq \varepsilon |c_I(f)| \right\} \subseteq \left\{ J \in \mathcal{D} : \exists I \in A, I \subset J \forall K \in [I, J] |c_I(f) - c_K(f)| \leq \varepsilon |c_I(f)| \right\} = B^{(2)}_\varepsilon(f).
\]

Letting therefore \( C = B^{(2)}_\varepsilon(y) \), we deduce our claim from (37), (39) and the fact that \( B \subseteq B^{(1)} \).

We are now in the position to prove Theorem 1.

\[\square\]
Proof of Theorem \[1\] Let \( f \in L_1 \), and \( \varepsilon, \delta > 0 \). We can assume that \( \varepsilon < \frac{1}{3} \) and that \( \text{supp}_H(f) \subset \text{succ}([0,1]) \), with \( |c_I(f)| \leq 1 \), for all \( I \in \text{supp}_H(f) \). We choose \( m_0 \in \mathbb{N} \), so that \( 2^{-m_0} < \delta \leq 2^{1-m_0} \), which implies that \( m_0 \leq \log_2(2/\delta) \).

For each \( m = 1, 2, \ldots, m_0 \), we apply Corollary \[10\] to the function \( \rho = 2^{-m} \), the set \( A \cap \{ I \in D : \delta < |c_I(f)| \} \). We put \( C_\varepsilon = 45738/\varepsilon \) and

\[
\mathcal{B}^{(m)} = A \cap \{ I \in D : 2^m \vee \delta < |c_I(f)| \leq 2^{1-m} \} \text{ for } m = 1, 2 \ldots m_0
\]

and deduce that there are sets \( \mathcal{C}_m, \mathcal{B}^{(m)} \subseteq \mathcal{C}^{(m)} \subseteq \mathcal{B}^{(m)}(f) \), so that

\[
(40) \quad \| P_{\mathcal{C}^{(m)}}(f) \| \leq C_\varepsilon \|f\| \text{ for } m = 1, 2 \ldots m_0.
\]

Since \( \varepsilon \leq \frac{1}{3} \), it follows for \( i, j \in \{ 1, 2, 3 \ldots m_0 \} \), with \( |i - j| \geq 2 \), that \( \mathcal{B}^{(i)}_\varepsilon \cap \mathcal{B}^{(j)}_\varepsilon(f) = \emptyset \), and thus, that \( \mathcal{C}^{(i)} \cap \mathcal{C}^{(j)} = \emptyset \).

We let \( \mathcal{F} = \bigcup_{m=1, \text{m odd}}^{m_0} \mathcal{C}^{(m)} \). It follows from \((40)\) that

\[
(41) \quad \| P_\mathcal{F}(f) \| \sum_{m=1, \text{m odd}}^{m_0} \| P_{\mathcal{C}^{(m)}}(f) \| \leq C_\varepsilon \left\lfloor \frac{m_0}{2} \right\rfloor \leq C_\varepsilon \log \left( \frac{1}{\delta} \right).
\]

We are now applying again Corollary \[10\] to the function \( g = f - P_\mathcal{F}(f) \) and the set \( \tilde{A} = (A \cap \{ I \in D : \delta < |c_I(f)| \}) \setminus \mathcal{F} \), and find sets \( \tilde{\mathcal{C}}^{(j)} \), with \( \tilde{\mathcal{B}}^{(j)} \subset \tilde{\mathcal{C}}^{(j)} \subset \tilde{\mathcal{B}}^{(j)}(g) \), where

\[
\tilde{\mathcal{B}}^{(m)} = \tilde{A} \cap \{ I \in D : 2^m \vee \delta < |c_I(f)| \leq 2^{1-m} \} \text{ for } m = 1, 2 \ldots m_0,
\]

so that

\[
(42) \quad \| P_{\tilde{\mathcal{C}}^{(m)}}(g) \| \leq C_\varepsilon \|g\| \text{ for } m = 1, 2 \ldots m_0.
\]

We note that for every odd \( m \) in \( \{ 1, 2 \ldots m_0 \} \) the set \( \tilde{\mathcal{C}}^{(m)} \) is empty and that therefore the \( \tilde{\mathcal{C}}^{(m)} \)'s are pairwise disjoint. We also note that \( \tilde{\mathcal{B}}^{(m)}_\varepsilon(g) \cap \mathcal{F} = \emptyset \) (since \( \tilde{\mathcal{B}}^{(m)}_\varepsilon(g) \subseteq \text{supp}_H(g) \) which is disjoint from \( \mathcal{F} \)) and thus that \( \mathcal{C}^{(m)} \cap \mathcal{F} = \emptyset \), for all \( m = 1, 2 \ldots m_0 \). Putting now \( \mathcal{E} = \mathcal{F} \cup \bigcup_{m=1, \text{m even}}^{m_0} \tilde{\mathcal{C}}^{(m)} \) we obtain

\[
\| P_\mathcal{E}(f) \| \leq \| P_\mathcal{F}(f) \| + \| P_\varepsilon(f - P_\mathcal{F}(f)) \|
\leq \| P_\mathcal{F}(f) \| + \sum_{m=1}^{m_0} \| P_{\tilde{\mathcal{C}}^{(m)}}(g) \|
\leq C_\varepsilon \log \left( \frac{1}{\delta} \right) + \left\lfloor \frac{m_0}{2} \right\rfloor C_\varepsilon \|g\|
\leq C_\varepsilon \log \left( \frac{1}{\delta} \right) + C_\varepsilon \log \left( \frac{1}{\delta} \right) \left( 1 + C_\varepsilon \log \left( \frac{1}{\delta} \right) \right)
\]

which proves our claim. \( \square \)
Our next example provides a lower bound for the constant on the right side of (6).

**Example 11.** For \( n \in \mathbb{N} \) and \( \delta = 2^{-2n} \) we claim that there is a function \( f \in L_1 \) and an \( A \subset \text{supp}_H(f) \), so that for any \( 0 < \varepsilon < 1 \) it follows that \( \mathcal{A}_\varepsilon(f) = A \) and

\[
\|P_\varepsilon(f)\| \geq \log \left( \frac{1}{\delta} \right) \|f\|.
\]

Indeed, we define \( h_0 = 1_{[0,1]} \), and for \( k \in \mathbb{N} \) and \( j = 1, 2 \ldots, 2^{k-1} \) we put

\[
h_j^{(k)} = 2^{k-1} \left( 1_{[(2j-2)^{-2}, (2j-1)^{-2})} - 1_{[(2j-1)^{-2}, (2j)^{-2})} \right).
\]

We observe that for any \( n \in \mathbb{N} \)

\[
\left\| h_0 + \sum_{k=1}^{2n} h_j^{(k)} \right\| = 1
\]

and for some universal constant \( c > 0 \).

\[
\left\| h_0 + \sum_{k=1}^{2n} h_j^{(2k)} \right\| > cn.
\]

We secondly observe that the joint distribution of the sequence

\[
h_0, \frac{1}{2} (h_1^{(2)} + h_2^{(2)}), \frac{1}{4} (h_1^{(4)} + h_2^{(4)} + h_3^{(4)} + h_4^{(4)}), \frac{1}{8} (h_1^{(6)} + h_2^{(6)} + \ldots + h_8^{(6)}), \ldots,
\]

is equal to the joint distribution of \( h_0, h_1^{(1)}, h_1^{(2)}, \ldots \).

It follows therefore that

\[
\left\| h_0 + \sum_{k=1}^{2n} 2^{-k} \sum_{j=1}^{2^k} h_j^{2k} \right\| = \left\| h_0 + \sum_{k=1}^{2n} h_j^{(k)} \right\| = 1, \text{ and}
\]

\[
\left\| h_0 + \sum_{k=1}^{n} 2^{-2k} \sum_{j=1}^{2^k} h_j^{4k} \right\| \geq cn.
\]

Therefore if we choose

\[
f = h_0 + \sum_{k=1}^{2n} 2^{-k} \sum_{j=1}^{2^k} h_j^{2k}
\]

and \( \mathcal{A} = \{ h_0 \} \cup \{ h_j^{4k} : k = 1, 2 \ldots n \text{ and } j = 1, 2 \ldots 2^{2k} \} \), we obtain for \( \delta = 2^{-2n} \), and any \( 0 < \varepsilon < 1 \) that \( \mathcal{A}_\varepsilon(f) = \mathcal{A} \) and \( \|P_\mathcal{A}(f)\| \geq cn \sim \log(1/\delta) \).
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