1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{R}^+$ denote the set of all positive real numbers and let $\chi$ be a real valued function defined on $\mathbb{R}^+$. For $\nu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the algebraic moments of order $\nu$ is defined by

$$m_\nu(\chi, u) := \sum_{k=-\infty}^{+\infty} \chi(e^{-k}u)(k - \log u)^\nu, \quad \forall \ u \in \mathbb{R}^+.$$

In a similar way, we can define the absolute moment of order $\nu$ as

$$M_\nu(\chi, u) := \sum_{k=-\infty}^{+\infty} |\chi(e^{-k}u)||k - \log u|^{\nu}, \quad \forall \ u \in \mathbb{R}^+.$$

We define $M_\nu(\chi) := \sup_{u \in \mathbb{R}^+} M_\nu(\chi, u)$. We say that $\chi$ is a kernel if it satisfies the following conditions:

(i) for every $u \in \mathbb{R}^+$, $\sum_{k=-\infty}^{+\infty} \chi(e^{-k}u) = 1$. 

Mathematics Subject Classification(2010): 41A25, 26A15, 41A35.
(ii) for some \( \nu > 0 \),
\[
M_\nu(\chi, u) = \sup_{u \in \mathbb{R}^+} \sum_{k=-\infty}^{+\infty} |\chi(e^{-k}u)||k - \log u|^\nu < +\infty.
\]

Let \( \Phi \) denote the set of all functions satisfying conditions (i) and (ii). For \( t \in \mathbb{R}^+ \), \( \chi \in \Phi \) and \( w > 0 \), the exponential sampling series for a function \( f : \mathbb{R}^+ \to \mathbb{R} \) is defined by (6)
\[
(S_\chi^w f)(t) = \sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^k/w).
\]

(1.1)

It is easy to see that the series \( S_\chi^w f \) is well defined for \( f \in L^\infty(\mathbb{R}^+) \). Using the above sampling series \( S_\chi^w f \) one can reconstruct the functions which are not Mellin-band limited. Recently, Bardaro et.al. [4] pointed out that the study of Mellin-band limited functions are different from that of Fourier-band limited functions. Mamedov was the first person who studied the Mellin theory in [17] and then Butzer et.al. further developed the Mellin’s theory and studied its approximation properties in [10, 11, 9, 13]. The reconstruction using exponential sampling formula was first studied by Butzer and Jansche in [6]. The pointwise and uniform convergence of the series \( S_\chi^w f \) for continuous functions was analysed in [6] and the convergence of \( S_\chi^w f \) was studied in Mellin-Lebesgue spaces. Recently Bardaro et.al. studied various approximation results using Mellin transform which can be seen in [5, 2, 3, 4, 6]. To improve the rate of convergence, a linear combination of \( S_\chi^w f \) was taken in [1]. The exponential sampling series with the sample points which are exponentially spaced on \( \mathbb{R}^+ \) has been obtained as solution of some mathematical model related to light scattering, Fraunhofer diffraction and radio astronomy (see [8, 15, 16, 18]).

The approximation of discontinuous functions by classical sampling operators was first initiated by Butzer et.al.[12]. Further, the Kantorovich sampling series for discontinuous signals was analysed in [14]. Inspired by these works we analyse the behaviour of exponential sampling series (1.1) as \( w \to \infty \) for discontinuous functions at the jump discontinuities, i.e., at a point \( t \) where the limits
\[
f(t+0) := \lim_{p \to 0^+} f(t+p),
\]
and
\[
f(t-0) := \lim_{p \to 0^+} f(t-p)
\]
exists and are different. For a kernel \( \chi \), we define the functions
\[
\psi_\chi^+(u) := \sum_{k<\log u} \chi(ue^{-k}),
\]
and
\[
\psi_\chi^-(u) := \sum_{k>\log u} \chi(ue^{-k}).
\]

We observe that \( \psi_\chi^+(u) \) and \( \psi_\chi^-(u) \) are recurrent functions with fundamental interval \([1,e]\). Now we shall recall the definition of Mellin transform. Let \( L^p(\mathbb{R}^+) \), \( p \in [1,\infty) \) be the set of all Lebesgue measurable and \( p \)-integrable functions defined on \( \mathbb{R}^+ \). For \( c \in \mathbb{R} \), we define the space
\[
X_c = \{ f : \mathbb{R}^+ \to \mathbb{C} : f(\cdot)(\cdot)^{c-1} \in L^1(\mathbb{R}^+) \}.
\]
equipped with the norm
\[ \| f \|_{\mathcal{X}_c} = \| f(\cdot)(\cdot)^{c-1} \|_1 = \int_{0}^{+\infty} |f(y)|y^{c-1}dy. \]

For \( f \in \mathcal{X}_c \), its Mellin transform is defined by
\[ \mathcal{F}_M(s) := \int_{0}^{+\infty} y^{s-1}f(y) \, dy, \quad (s = c + it, t \in \mathbb{R}). \]

We say that a function \( f \in \mathcal{X}_c \cap C(\mathbb{R}^+) \), \( c \in \mathbb{R} \) is Mellin band-limited in the interval \([-\kappa, \kappa]\), if \( \mathcal{F}_M(c + it) = 0 \) for all \( |t| > \kappa \), \( \kappa \in \mathbb{R}^+ \).

The paper is organized as follows. In section 2, we prove the representation lemma for the exponential sampling series (1.1) and using this lemma we analyse the approximation of discontinuous functions by \( S^\chi_w f \) in Theorem 2, Theorem 3 and Theorem 5. Further we analyse the degree of approximation for the sampling series (1.1) in-terms of logarithmic modulus of smoothness in section 3. In section 4, we study the round-off and time jitter errors for these sampling series. In example section, we have given a construction of a family of Mellin-band limited kernels such that \( \chi(1) = 0 \) for which \( S^\chi_w f \) converge at any jump discontinuities. Further, the convergence at discontinuity points of the sampling series \( S^\chi_w f \) has been tested numerically and numerical results are provided in Tables 1, 2 and 3.

2. APPROXIMATION OF DISCONTINUOUS SIGNALS

For any given bounded function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \), we first prove the following representation lemma for the sampling series \( S^\chi_w f \). Throughout this section we assume that the right and left limits of \( f \) at \( t \in \mathbb{R}^+ \) exist and are finite.

**Lemma 1.** For a given bounded function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) and a fixed \( t \in \mathbb{R}^+ \), let \( h_t : \mathbb{R}^+ \rightarrow \mathbb{R} \) be defined by
\[ h_t(x) = \begin{cases} f(x) - f(t-0), & \text{if } x < t \\ f(x) - f(t+0), & \text{if } x > t \\ 0, & \text{if } x = t. \end{cases} \]

Then the following holds:
\[ (S^\chi_w f)(t) = (S^\chi_w h_t)(t) + f(t-0) + \psi^\chi(t^w)[f(t+0) - f(t-0)] + \chi(1)[f(t) - f(t-0)], \]
if \( w \log(t) \in \mathbb{Z} \) and
\[ (S^\chi_w f)(t) = (S^\chi_w h_t)(t) + f(t-0) + \psi^\chi(t^w)[f(t+0) - f(t-0)], \]
if \( w \log(t) \notin \mathbb{Z} \).
Proof. Let \( w \log(t) \in \mathbb{Z} \), and \( w > 0 \). Then, we can write

\[
(S^X_{w}h_t)(t) = \sum_{k<w \log(t)} \chi(e^{-ktw})(f(e^{\frac{t}{w}}) - f(t-0)) + \sum_{k>w \log(t)} \chi(e^{-ktw})(f(e^{\frac{t}{w}}) - f(t+0))
\]

\[
= \sum_{k<w \log(t)} \chi(e^{-ktw})f(e^{\frac{t}{w}}) + \sum_{k\geq w \log(t)} \chi(e^{-ktw})f(e^{\frac{t}{w}}) - f(t-0) \sum_{k<w \log(t)} \chi(e^{-ktw})
\]

\[
- f(t+0) \sum_{k> w \log(t)} \chi(e^{-ktw}) - \chi(1)f(t)
\]

\[
= (S^X_{w}f)(t) - f(t-0) \sum_{k< w \log(t)} \chi(e^{-ktw}) = (S^X_{w}f)(t) - f(t-0) \sum_{k\geq w \log(t)} \chi(e^{-ktw}) - \chi(1)f(t).
\]

Adding and subtracting \( f(t-0) \) \( \sum \frac{\chi(e^{-ktw})}{k \geq w \log(t)} \) in the above equation and rearranging all terms, we obtain

\[
(S^X_{w}f)(t) = (S^X_{w}h_t)(t) + f(t-0) \left( \sum_{k<w \log(t)} \chi(e^{-ktw}) + \sum_{k\geq w \log(t)} \chi(e^{-ktw}) \right)
\]

\[
+ [f(t+0) - f(t-0)] \sum_{k\geq w \log(t)} \chi(e^{-ktw}) + \chi(1)f(t) - f(t-0)\chi(1)
\]

\[
= (S^X_{w}h_t)(t) + f(t-0) \sum_{k=-\infty}^{\infty} \chi(e^{-ktw}) + [f(t+0) - f(t-0)] \sum_{k\geq w \log(t)} \chi(e^{-ktw})
\]

\[
\chi(1)[f(t) - f(t-0)].
\]

Hence using the condition that \( \sum_{k=-\infty}^{\infty} \chi(e^{-kt}) = 1 \), we can easily obtain

\[
(S^X_{w}f)(t) = (S^X_{w}h_t)(t) + f(t-0) + \psi^{\chi}(t)w[f(t+0) - f(t-0)] + \chi(1)[f(t) - f(t-0)].
\]

Now let \( w \log(t) \notin \mathbb{Z} \) and \( w > 0 \). Then repeating the same computations, we easily obtain

\[
(S^X_{w}f)(t) = (S^X_{w}h_t)(t) + f(t-0) + [f(t+0) - f(t-0)] \sum_{k\geq w \log(t)} \chi(e^{-ktw})
\]

\[
= (S^X_{w}h_t)(t) + f(t-0) + [f(t+0) - f(t-0)]\psi^{\chi}_{-w}(t).
\]

□

Before proving the approximation of discontinuous functions by \( S^X_{w}f \), we recall the following theorem proved in \([6]\) for continuous functions on \( \mathbb{R}^+ \).

Theorem 1. Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) be a bounded function and \( \chi \in \Phi \). Then \( (S^X_{w}f)(t) \) converges to \( f(t) \) at any point \( t \) of continuity. Moreover, if \( f \in C(\mathbb{R}^+) \), then we have

\[
\lim_{w \rightarrow \infty} \| f - S^X_{w}f \|_{\infty} = 0.
\]

Now we analyse the behaviour of the exponential sampling series at jump discontinuity at \( t \in \mathbb{R}^+ \) when \( w \log(t) \in \mathbb{Z} \).
Theorem 2. Let \( f : \mathbb{R}^+ \to \mathbb{R} \) be a bounded signal and let \( t \in \mathbb{R}^+ \) be a point of non-removable jump discontinuity of \( f \). For a given \( \alpha \in \mathbb{R} \), the following statements are equivalent:

\[
(i) \quad \lim_{w \to \infty} (S_w^X f)(t) = \alpha f(t + 0) + [1 - \alpha - \chi(1)] f(t - 0) + \chi(1) f(t),
\]

\[
(ii) \quad \psi^-_\chi(1) = \alpha,
\]

\[
(iii) \quad \psi^+_\chi(1) = 1 - \alpha - \chi(1).
\]

Proof. First, we prove that \((i) \iff (ii)\). In view of the representation Lemma [1], we have

\[
(S_w^X f)(t) = (S_w^X h_t)(t) + f(t - 0) + \psi^-_\chi(t^w) [f(t + 0) - f(t - 0)] + \chi(1) [f(t) - f(t - 0)],
\]

for any \( w > 0 \) such that \( w \log(t) \in \mathbb{Z} \). Since \( h_t \) is bounded and continuous at zero and using Theorem [1], we obtain

\[
\lim_{w \to \infty} (S_w^X h_t)(t) = 0.
\]

Thus, we have

\[
\lim_{w \to \infty} (S_w^X f)(t) = f(t - 0) + \left( \lim_{w \to \infty} \psi^-_\chi(t^w) \right) [f(t + 0) - f(t - 0)] + \chi(1) [f(t) - f(t - 0)].
\]

Now, we have

\[
\lim_{w \to \infty} \psi^-_\chi(t^w) = \lim_{w \to \infty} \left( \sum_{k > w \log(t)} \chi(e^{-k t^w}) \right).
\]

As \( \psi^-_\chi \) is recurrent with fundamental domain \([1, e]\), we get

\[
\psi^-_\chi(t^w) = \psi^-_\chi(1), \quad \forall w, t \text{ such that } w \log(t) \in \mathbb{Z}.
\]

Therefore, we have

\[
\lim_{w \to \infty} (S_w^X f)(t) = \psi^-_\chi(1) f(t + 0) + [1 - \psi^-_\chi(1) - \chi(1)] f(t - 0) + \chi(1) f(t).
\]

Now \((i) \iff \alpha f(t + 0) + [1 - \alpha - \chi(1)] f(t - 0) + \chi(1) f(t)\)

\[
= \psi^-_\chi(1) f(t + 0) + [1 - \psi^-_\chi(1) - \chi(1)] f(t - 0) + \chi(1) f(t)
\]

\[
\iff \psi^-_\chi(1) (f(t + 0) - f(t - 0)) = \alpha (f(t + 0) - f(t - 0))
\]

\[
\iff \psi^-_\chi(1) = \alpha
\]

\[
\iff (ii) \text{ holds.}
\]

Since \( \sum_{k = -\infty}^{+\infty} \chi(e^{-k t^w}) = 1 \), we have

\[
\psi^+_\chi(1) = 1 - \chi(1) - \psi^-_\chi(1).
\]

This implies that \((ii) \iff (iii)\). Hence, the proof is completed. \( \square \)

Next we analyse the behaviour of the exponential sampling series at jump discontinuity at \( t \in \mathbb{R}^+ \) when \( w \log(t) \notin \mathbb{Z} \).
Theorem 3. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a bounded signal and let $t \in \mathbb{R}^+$ be a point of non-removable jump discontinuity of $f$. Let $\alpha \in \mathbb{R}$. Then the following statements are equivalent:

(i) $\lim_{w \to \infty, w \log(t) \notin \mathbb{Z}} (S^w f)(t) = \alpha f(t + 0) + (1 - \alpha) f(t - 0)$,

(ii) $\psi^-_\chi(u) = \alpha, \quad u \in (1, e)$

(iii) $\psi^+_\chi(u) = 1 - \alpha, \quad u \in (1, e)$.

Proof. Using representation Lemma 1, we obtain

$$\lim_{w \to \infty, w \log(t) \notin \mathbb{Z}} (S^w f)(t) = f(t - 0) + \left( \lim_{w \to \infty, w \log(t) \notin \mathbb{Z}} \psi^-_\chi(t^w) \right) [f(t + 0) - f(t - 0)].$$

Thus, we obtain

$$(i) \iff \alpha f(t + 0) + (1 - \alpha) f(t - 0) = f(t - 0) + \left( \lim_{w \to \infty, w \log(t) \notin \mathbb{Z}} \psi^-_\chi(t^w) \right) [f(t + 0) - f(t - 0)]$$

$$\iff \alpha [f(t + 0) - f(t - 0)] = \left( \lim_{w \to \infty, w \log(t) \notin \mathbb{Z}} \psi^-_\chi(t^w) \right) [f(t + 0) - f(t - 0)]$$

$$\iff \alpha = \lim_{w \to \infty, w \log(t) \notin \mathbb{Z}} \psi^-_\chi(t^w)$$

$$\iff \alpha = \psi^-_\chi(u), \quad \forall u \in (1, e)$$

$$\iff (ii) \text{ holds.}$$

Let $w \log(t) \notin \mathbb{Z}$. Then, we have

$$\psi^+_\chi(t^w) + \psi^-_\chi(t^w) = 1.$$

Thus, we obtain

$$(ii) \iff \psi^+_\chi(u) = 1 - \alpha, \quad u \in (1, e).$$

□

The results in the above Theorem 3 was proved by assuming that $\psi^-_\chi(u)$ is constant on $(1, e)$. In what follows, we show that if $\psi^-_\chi(u)$ is not a constant on $(1, e)$, then the exponential sampling series can not converge at jump discontinuities.

Theorem 4. Let $\chi$ be a kernel such that $\psi^-_\chi(u)$ is not constant on $(1, e)$. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a bounded signal with a non-removable jump discontinuity at $t \in \mathbb{R}^+$. Then $(S^w f)(t)$ does not converge point-wise at $t$.

Proof. Suppose not. Then $\lim_{w \to \infty, w \log(t) \notin \mathbb{Z}} (S^w f)(t) = \ell$, for some $\ell \in \mathbb{R}^+$. By the uniqueness of the limit and Lemma 1, we obtain

$$\ell = f(t - 0) + \lim_{w \to \infty} \psi^-_\chi(t^w) [f(t + 0) - f(t - 0)].$$
Since \( f(t + 0) - f(t - 0) \neq 0 \), we obtain
\[
\frac{\ell - f(t - 0)}{f(t + 0) - f(t - 0)} = \lim_{w \to \infty} \psi_{\chi}^{-}(t^{w}).
\]
The above expression gives a contradiction. Indeed, if
\[
\lim_{w \to \infty} \psi_{\chi}^{-}(t^{w}) = C,
\]
where \( C \) is a constant, then it fails to satisfy that \( \psi_{\chi}^{-} \) is recurrent and not a constant, hence the theorem proved. \( \square \)

Finally, the more general theorem of the exponential sampling series at jump discontinuity at \( t \in \mathbb{R}^{+} \) for any bounded signal can be proved.

**Theorem 5.** Let \( f : \mathbb{R}^{+} \to \mathbb{R} \) be a bounded signal and let \( t \in \mathbb{R}^{+} \) be a point of non-removable jump discontinuity of \( f \). Let \( \alpha \in \mathbb{R} \). Suppose that the kernel \( \chi \) satisfies the additional condition that \( \chi(1) = 0 \). Then, the following statements are equivalent:

(i) \( \lim_{u \to \infty} (S_{w}^{\chi} f)(t) = \alpha f(t + 0) + (1 - \alpha) f(t - 0) \),

(ii) \( \psi_{\chi}^{-}(u) = \alpha, \quad u \in [1, e) \)

(iii) \( \psi_{\chi}^{+}(u) = 1 - \alpha, \quad u \in [1, e) \).

Moreover, if in addition we assume that \( \chi \) is continuous on \( \mathbb{R}^{+} \), then the above statements are equivalent to the following statements:

(iv) \[
\int_{0}^{1} \chi(u)u^{2k\pi i} du = \begin{cases} 
0, & \text{if } k \neq 0 \\
\alpha, & \text{if } k = 0 
\end{cases}
\]

(v) \[
\int_{1}^{\infty} \chi(u)u^{2k\pi i} du = \begin{cases} 
0, & \text{if } k \neq 0 \\
1 - \alpha, & \text{if } k = 0 
\end{cases}
\]

**Proof.** Proceeding along the lines proof of Theorem 2 and Theorem 3 we see that (i), (ii) and (iii) are equivalent. Let \( \chi \) be continuous on \( \mathbb{R}^{+} \) and let
\[
\chi_{0}(u) = \begin{cases} 
\chi(u), & \text{for } u < 1 \\
0, & \text{for } u \geq 1 
\end{cases}
\]

Then, we have
\[
\psi_{\chi}^{-}(u) = \sum_{k > \log u} \chi(ue^{-k}) = \sum_{k \in \mathbb{Z}} \chi_{0}(ue^{-k}).
\]

Therefore, \( \psi_{\chi}^{-} \) is recurrent continuous function with the fundamental interval \([1, e]\). Using Mellin-Poisson summation formula, we obtain
\[
\psi_{\chi}^{-}(u) = \sum_{k = -\infty}^{+\infty} \overline{\chi_{0}}(2k\pi i) u^{-2k\pi i} = \sum_{k = -\infty}^{+\infty} \left( \int_{0}^{1} \chi(u)u^{2k\pi i} du \right) u^{-2k\pi i}.
\]
Therefore, we obtain
\[ \psi^{-}_\chi(u) = \alpha, \forall u \in [1, e) \]
\[ \iff \widehat{\chi}_{M}(2k\pi i) = \begin{cases} 0, & \text{if } k \neq 0 \\ \alpha, & \text{if } k = 0 \end{cases} \]
\[ \iff \int_{0}^{1} \chi(u)u^{2k\pi i}du = \begin{cases} 0, & \text{if } k \neq 0 \\ \alpha, & \text{if } k = 0 \end{cases} \]
This implies that \((ii) \iff (iv)\). Finally using the condition
\[ \sum_{k=-\infty}^{+\infty} \chi(e^{-k}u) = 1 \iff \widehat{\chi}_{M}(2k\pi i) = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases} \]
the equivalence between \((iv)\) and \((v)\) can be established easily. Thus the proof is completed. \(\square\)

**Remark 1.** Let \(f : \mathbb{R}^+ \rightarrow \mathbb{R}\) be a bounded signal with a removable discontinuity \(t \in \mathbb{R}^+\), i.e. \(f(t+0) = f(t-0) = \ell\). Then we have
\[ (i) \lim_{w \rightarrow \infty} (S^{\chi}_{w,f})(t) = \ell + \chi(1)[f(t) - \ell], \]
\[ (ii) \lim_{w \rightarrow \infty} (S^{\chi}_{w,f})(t) = \ell, \]
\[ (iii) \text{If } \chi(1) = 0, \text{ then } \lim_{w \rightarrow \infty} (S^{\chi}_{w,f})(t) = \ell. \]

### 3. Degree of Approximation

In this section, we estimate the order of convergence of the exponential sampling series by using the logarithmic modulus of continuity. Let \(C(\mathbb{R}^+)\) denote the space of all real valued bounded continuous functions on \(\mathbb{R}^+\) equipped with the supremum norm \(\|f\|_{\infty} := \sup_{x \in \mathbb{R}^+} |f(x)|\). We say that a function \(f : \mathbb{R}^+ \rightarrow \mathbb{R}\) is log-uniformly continuous if the following hold: for a given \(\epsilon > 0\), there exists \(\delta > 0\) such that \(|f(p) - f(q)| < \epsilon\) whenever \(|\log(p) - \log(q)| < \delta\), for any \(p, q \in \mathbb{R}^+\). The subspace consisting of all bounded log-uniformly continuous functions on \(\mathbb{R}^+\) is denoted by \(C(\mathbb{R}^+)\). Let \(f \in C(\mathbb{R}^+)\). Then the logarithmic modulus of continuity is defined by
\[ \omega(f, \delta) := \sup\{|f(p) - f(q)| : \text{ whenever } |\log(p) - \log(q)| \leq \delta, \delta \in \mathbb{R}^+\}. \]
The logarithmic modulus of continuity satisfies the following properties:
\[ (a) \omega(f, \delta) \rightarrow 0, \text{ as } \delta \rightarrow 0. \]
\[ (b) \omega(f, c\delta) \leq (c + 1)\omega(f, \delta), \text{ for every } \delta, c > 0. \]
\[ (c) |f(p) - f(q)| \leq \omega(f, \delta) \left(1 + \frac{|\log(p) - \log(q)|}{\delta}\right). \]
Further properties of logarithmic modulus of continuity can be seen in [2]. In the following theorem, we obtain the order of convergence for the exponential sampling series when \(M_{\nu}(\chi) < \infty\) for \(0 < \nu < 1\).
Theorem 6. Let \( \chi \in \Phi \) be a kernel such that \( M_{\nu}(\chi) < \infty \) for \( 0 < \nu < 1 \) and \( f \in C(\mathbb{R}^+) \). Then for sufficiently large \( w > 0 \), the following hold:

\[
|(S^w_{\nu}f)(t) - f(t)| \leq \omega(f, w^{-\nu})[M_{\nu}(\chi) + 2M_0(\chi)] + 2^{\nu+1}\|f\|_{\infty}M_{\nu}(\chi)w^{-\nu},
\]

for every \( t \in \mathbb{R}^+ \).

Proof. Let \( t \in \mathbb{R}^+ \) be fixed. Then using the condition \( \sum_{k=-\infty}^{+\infty} \chi(e^{-ktw}) = 1 \), we obtain

\[
|(S^w_{\nu}f)(t) - f(t)| = \left| \sum_{k=-\infty}^{+\infty} \chi(e^{-ktw})(f(e^{\frac{k}{w}}) - f(t)) \right|
\]

\[
\leq \left( \sum_{k - w \log t < \frac{\nu}{2}} + \sum_{k - w \log t \geq \frac{\nu}{2}} \right) |\chi(e^{-ktw})||f(e^{\frac{k}{w}}) - f(t)| =: I_1 + I_2.
\]

Let \( 0 < \nu < 1 \). Then we have

\[
\omega \left( f, \left| \frac{k}{w} - \log t \right| \right) \leq \omega \left( f, \left| \frac{k}{w} - \log t \right|^\nu \right).
\]

Therefore, using the above inequality and the property (c), we obtain

\[
I_1 \leq \sum_{k - w \log t < \frac{\nu}{2}} \left| \chi(e^{-ktw}) \right| \omega \left( f, \left| \frac{k}{w} - \log t \right|^\nu \right)
\]

\[
\leq \sum_{k - w \log t < \frac{\nu}{2}} \left| \chi(e^{-ktw}) \right| \left( 1 + w^\nu \left| \frac{k}{w} - \log t \right|^\nu \right) \omega(f, w^{-\nu})
\]

\[
\leq \sum_{k - w \log t < \frac{\nu}{2}} \left| \chi(e^{-ktw}) \right| \omega(f, w^{-\nu}) + \sum_{k - w \log t < \frac{\nu}{2}} \left| \chi(e^{-ktw}) \right| w^\nu \left| \frac{k}{w} - \log t \right|^\nu \omega(f, w^{-\nu})
\]

\[
\leq \omega(f, w^{-\nu}) \sum_{k - w \log t < \frac{\nu}{2}} \left| \chi(e^{-ktw}) \right| + \omega(f, w^{-\nu}) \sum_{k - w \log t < \frac{\nu}{2}} \left| \chi(e^{-ktw}) \right| \left| k - w \log t \right|^\nu.
\]

In view of the conditions \( M_0(\chi) \) and \( M_{\nu}(\chi) \), we easily obtain

\[
I_1 \leq \omega(f, w^{-\nu})[M_0(\chi) + M_{\nu}(\chi)].
\]

Now we estimate \( I_2 \). Since \( \left| k - w \log t \right| \geq \frac{\nu}{2} \), we have

\[
\frac{1}{\left| k - w \log t \right|^\nu} \leq 2^\nu w^{-\nu}, \quad 0 < \nu < 1.
\]
The total round-off or quantization error is defined by

\[ \xi = \sum_{k=-\infty}^{+\infty} |\chi(e^{-k/w})| \]

Hence, we obtain

\[
I_2 \leq 2\|f\|_{\infty} \sum_{|k-w \log t| \geq \frac{w}{2}} |\chi(e^{-k/w})| \leq 2\|f\|_{\infty} \sum_{|k-w \log t| \geq \frac{w}{2}} \left| \frac{k-w \log t}{k-w \log t} \right|^\nu |\chi(e^{-k/w})|
\]

\[
\leq 2^{\nu+1}\|f\|_{\infty} w^{-\nu} \sum_{|k-w \log t| \geq \frac{w}{2}} |\chi(e^{-k/w})| |k-w \log t|^{\nu} \leq 2^{\nu+1}\|f\|_{\infty} w^{-\nu} M_\nu(\chi) < \infty.
\]

On combining the estimates \( I_1 \) and \( I_2 \), we get the desired estimate. \( \square \)

4. Round-Off and Time Jitter Errors

This section is devoted to analyse round off and time jitter errors connected with exponential sampling series \((1.1)\). The round-off error arises when the exact sample values \( f(e^{\frac{k}{w}}) \) are replaced by approximate close ones \( \bar{f}(e^{\frac{k}{w}}) \) in the sampling series \((1.1)\). Let \( \xi_k = f(e^{\frac{k}{w}}) - \bar{f}(e^{\frac{k}{w}}) \) be uniformly bounded by \( \xi \), i.e., \( |\xi_k| \leq \xi \), for some \( \xi > 0 \). We are interested in analysing the error when \( f(t) \) is approximated by the following exponential sampling series:

\[
(S_w^\chi f)(t) = \sum_{k=-\infty}^{+\infty} \chi(e^{-k/w}) \bar{f}(e^{\frac{k}{w}}).
\]

The total round-off or quantization error is defined by

\[ (Q\xi f)(t) := |(S_w^\chi f)(t) - (S_w^\chi f)(t)|. \]

**Theorem 7.** For \( f \in C(\mathbb{R}^+) \), the following hold:

(i) \( \|Q\xi f\|_{C(\mathbb{R}^+)} \leq \xi M_0(\chi) \)

(ii) \( \|f - S_w^\chi \bar{f}\|_{C(\mathbb{R}^+)} \leq C \omega \left( f, \frac{1}{w} \right) + \xi M_0(\chi) \), where \( C = M_0(\chi) + M_1(\chi) \).

**Proof.** The error in the approximation can be split as

\[ |f(t) - (S_w^\chi \bar{f})(t)| \leq |f(t) - (S_w^\chi f)(t)| + (Q\xi f)(t) := I_1 + (Q\xi f)(t). \]

The term \( I_1 \) is the error arising if the actual sample value is used and the total round-off or quantization error can be evaluated by

\[
\|Q\xi f\|_{C(\mathbb{R}^+)} = \sup_{t \in \mathbb{R}^+} \left| \sum_{k=-\infty}^{+\infty} \chi(e^{-k/w}) f(e^{\frac{k}{w}}) - \sum_{k=-\infty}^{+\infty} \chi(e^{-k/w}) \bar{f}(e^{\frac{k}{w}}) \right|
\]

\[
= \sup_{t \in \mathbb{R}^+} \sum_{k=-\infty}^{+\infty} \left| \xi_k \chi(e^{-k/w}) \right| \leq \xi M_0(\chi).
\]

In view of Theorem 4 ([6], page no.7), we have

\[
I_1 \leq M_0(\chi) \omega(f, \delta) + \frac{\omega(f, \delta)}{w \delta} M_1(\chi).
\]
On combining the estimates $I_1$ and $I_2$, we get
\[
\| f - S_w^{\chi} \|_{C(\mathbb{R}^+)} \leq \left( M_0(\chi) + \frac{M_1(\chi)}{w \delta} \right) \omega(f, \delta) + \xi M_0(\chi).
\]
Choosing $\delta = \frac{1}{w}$, we obtain
\[
\| f - S_w^{\chi} \|_{C(\mathbb{R}^+)} \leq C \omega(f, \frac{1}{w}) + \xi M_0(\chi),
\]
where $C = M_0(\chi) + M_1(\chi)$. Hence, the proof is completed. \hfill \Box

The time-jitter error occurs when the function $f(t)$ being approximated from samples which are taken at perturbed nodes, i.e., the exact sample values $f(e^{\frac{k}{w}} + \varrho_k)$ in the sampling series (1.1). So we are interested in analysing time jitter error and the approximation behaviour when $f(t)$ is approximated by the sampling series
\[
\sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^{\frac{k}{w}} + \varrho_k).
\]
We assume that the values $\varrho_k$ are bounded by a small number $\varrho$, i.e., $|\varrho_k| \leq \varrho$, for all $k \in \mathbb{Z}$ and for some $\varrho > 0$. The total time jitter error is defined by
\[
J_{\varrho}f(t) := \left| \sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^{\frac{k}{w}}) - \sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^{\frac{k}{w}} + \varrho_k) \right|.
\]

**Theorem 8.** For $f \in C^{(1)}(\mathbb{R}^+)$, the following hold:

(i) $\| J_{\varrho}f \|_{C(\mathbb{R}^+)} \leq \varrho \| f' \|_{C(\mathbb{R}^+)} M_0(\chi)$

(ii) $\left\| f(.) - \sum_{k=-\infty}^{+\infty} \chi(e^{-k}w)f(e^{\frac{k}{w}} + \varrho_k) \right\|_{C(\mathbb{R}^+)} \leq C \omega\left( f, \frac{1}{w} \right) + \varrho \| f' \|_{C(\mathbb{R}^+)} M_0(\chi),$

where $C = M_0(\chi) + M_1(\chi)$.

**Proof.** Applying the mean value theorem, error can be estimated by
\[
\| J_{\varrho}f \|_{C(\mathbb{R}^+)} \leq \sup_{k \in \mathbb{Z}} \{ \sup_{t \in \mathbb{R}^+} |f(e^{\frac{k}{w}}) - f(e^{\frac{k}{w}} + \varrho_k)| \} \sup_{t \in \mathbb{R}^+} \sum_{k=-\infty}^{+\infty} |\chi(e^{-k}tw)|
\]
\[
\leq |\varrho_k| \| f' \|_{C(\mathbb{R}^+)} \sup_{t \in \mathbb{R}^+} \sum_{k=-\infty}^{+\infty} |\chi(e^{-k}tw)| \leq \varrho \| f' \|_{C(\mathbb{R}^+)} M_0(\chi).
\]

From the above estimates it is clear that the jitter error essentially depends on the smoothness of function $f$. For $f \in C^{(1)}(\mathbb{R}^+)$, the associated approximation error is estimated by
\[
\left| f(t) - \sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^{\frac{k}{w}} + \varrho_k) \right| \leq \left| f(t) - \sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^{\frac{k}{w}}) \right|
\]
\[
+ \left| \sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^{\frac{k}{w}}) - \sum_{k=-\infty}^{+\infty} \chi(e^{-k}tw)f(e^{\frac{k}{w}} + \varrho_k) \right|
\]
\[
\leq |f(t) - S_w^{\chi}f(t)| + J_{\varrho}f(t).
\]
Again using Theorem 4 ([6], page no.7), we have
\[ |f(t) - S^*_w f(t)| \leq M_0(\chi) \omega(f, \delta) + \frac{\omega(f, \delta)}{w \delta} M_1(\chi). \]

Using the above estimate and $J_{\varrho} f$, we obtain
\[ \| f(. - \sum_{k=-\infty}^{+\infty} \chi(e^{-k}(.\omega)) f(e^{\frac{k}{n}} + \varrho_k) \|_{C(\mathbb{R}^+)} \leq M_0(\chi) \omega(f, \delta) + \frac{\omega(f, \delta)}{w \delta} M_1(\chi) + \varrho \| f' \|_{C(\mathbb{R}^+)} M_0(\chi). \]

Hence, the proof is completed.

5. EXAMPLES OF THE KERNELS

In this section, we provide certain examples of the kernel functions which will satisfy our assumptions. First we give the family of Mellin-B spline kernels. The Mellin B-spline of order $n$ is given by
\[ \bar{B}_n(x) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \frac{n}{2} + \log x - j \right)^{n+1}, \quad x \in \mathbb{R}^+ \]

It can be easily seen that $\bar{B}_n(x)$ is compactly supported for every $n \in \mathbb{N}$. The Mellin transform of $\bar{B}_n$ (see [6]) is
\[ \left[ \bar{B}_n \right]_M(c + it) = \left( \frac{\sin\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} \right)^n, \quad t \neq 0. \]

The Mellin’s-Poisson summation formula ([3]) is defined by
\[ (i)^j \sum_{k=-\infty}^{+\infty} \chi(e^k x)(k - \log u)^j = \sum_{k=-\infty}^{+\infty} \frac{d^j}{dt^j}[\chi]_M(2k\pi i) x^{-2k\pi i}, \quad \text{for } k \in \mathbb{Z}. \]

We need the following lemma (see [6]).

**Lemma 2.** The condition $\sum_{k=-\infty}^{+\infty} \chi(e^{-k} x u) = 1$ is equivalent to
\[ [\chi]_M(2k\pi i) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{otherwise}. \end{cases} \]

Moreover $m_j(\chi, u) = 0$ for $j = 1, 2, \cdots, n$ is equivalent to $\frac{d^j}{dt^j}[\chi]_M(2k\pi i) = 0$ for $j = 1, 2, \cdots, n$ and $\forall \ k \in \mathbb{Z}$.

Using the above Lemma, we obtain
\[ \left[ B_n \right]_M(2k\pi i) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{otherwise}. \end{cases} \]

Using Mellin’s-Poisson summation formula, it is easy to see that $\bar{B}_n(x)$ satisfies the condition (i). As $\bar{B}_n(x)$ is compactly supported, the condition (ii) is also satisfied. Next we
consider the Mellin Jackson kernels. For \( x \in \mathbb{R}^+, \beta \in \mathbb{N}, \gamma \geq 1 \), the Mellin Jackson kernels are defined by

\[
J^{-1}_{\gamma, \beta}(x) := d^{-1}_{\gamma, \beta} x^{-c} \text{sinc}^{2 \beta} \left( \frac{\log x}{2\gamma \beta \pi} \right),
\]

where

\[
d^{-1}_{\gamma, \beta} := \int_0^\infty \text{sinc}^{2 \beta} \left( \frac{\log x}{2\gamma \beta \pi} \right) \frac{du}{u}.
\]

One can easily verify that the Mellin Jackson kernels also satisfies conditions (i) and (ii) (see [6]). We can analyse the convergence of the exponential sampling series with jump discontinuity associated with these kernels only for the case given in Theorem 2 and we observe that \( \chi(1) \neq 0 \). So Theorem 5 can not be applied for these kernels. In order to obtain the convergence of the exponential sampling series at jump discontinuity \( t \in \mathbb{R}^+ \) of the given bounded signal \( f : \mathbb{R}^+ \to \mathbb{R} \), we need to construct suitable kernels. One such construction is given in the following theorem.

**Theorem 9.** Let \( \chi_a, \chi_b \) be two continuous kernels supported respectively in the intervals \([e^{-a}, e^a]\) and \([e^{-b}, e^b]\). Let \( \alpha \in \mathbb{R} \) be fixed. We define \( \chi : \mathbb{R}^+ \to \mathbb{R}^+ \) by

\[
\chi(u) := (1 - \alpha)\chi_a(2ue^{-a-1}) + \alpha \chi_b(2ue^b), \quad u \in \mathbb{R}^+.
\]

Then \( \chi \) is a kernel satisfying conditions (i), (ii) and \( \chi(1) = 0 \). Moreover, the corresponding exponential sampling series \( S_{\chi}^w f, w > 0 \) based upon \( \chi \) satisfy (i) of Theorem 5 with parameter \( \alpha \) for a given bounded signal \( f : \mathbb{R}^+ \to \mathbb{R} \) at any non-removable discontinuity \( t \in \mathbb{R}^+ \) of \( f \).

**Proof.** The Mellin transform of \( \chi(u) \) is

\[
\hat{\chi}_M(s) = \int_0^\infty (1 - \alpha)\chi_a(2te^{-a-1})t^{s-1}dt + \int_0^\infty \alpha \chi_b(2te^b)t^{s-1}dt
\]

\[
= (1 - \alpha)[\hat{\chi}_a M(s)] \left( \frac{e^{(1+a)}}{2} \right)^s + \alpha[\hat{\chi}_b M(s)] \left( \frac{e^{-b}}{2} \right)^s.
\]

It is simple to check that \( \chi \) satisfies condition (ii). Now we show that kernel satisfies the condition (i). We obtain

\[
[\hat{\chi}_M(2k\pi i) = (1 - \alpha)[\hat{\chi}_a M(2k\pi i)] \left( \frac{e^{(1+a)}}{2} \right)^{2k\pi i} + \alpha[\hat{\chi}_b M(2k\pi i)] \left( \frac{e^{-b}}{2} \right)^{2k\pi i}.
\]

As \( \chi_a \) and \( \chi_b \) satisfies condition (i), we have

\[
[\hat{\chi}_a M(2k\pi i) = [\hat{\chi}_b M(2k\pi i) = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases}
\]

For suitable choices of \( a \) and \( b \), we obtain

\[
[\hat{\chi}_M(2k\pi i) = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases}
\]
Therefore, $\chi$ satisfies condition (i) and we can easily see that $\chi(1) = 0$. Now, we obtain

\[
\int_0^1 \chi(u)u^{2k\pi i - 1}du = \alpha \int_0^1 \chi_b(2e^bu)u^{2k\pi i - 1}du
\]

\[
= \alpha [\chi_b]_{M}(2k\pi i) \left(\frac{e^{-i2k\pi i}}{2^{2k\pi i}}\right)
\]

\[
= \begin{cases} 
0, & \text{if } k \neq 0 \\
\alpha, & \text{if } k = 0.
\end{cases}
\]

Therefore, the condition (iv) of Theorem 5 is satisfied, hence the proof is completed. \(\square\)

Now we test numerically the approximation of discontinuous function

\[
f(t) = \begin{cases} 
\frac{11}{2t^2 + 1}, & t < \frac{3}{2} \\
3, & \frac{3}{2} \leq t < \frac{7}{2} \\
2, & \frac{7}{2} \leq t < \frac{11}{2} \\
\frac{12}{1 + 2t}, & t \geq \frac{11}{2}
\end{cases}
\]

by exponential sampling series at its jump discontinuities $t = \frac{3}{2}$, $t = \frac{7}{2}$ and $t = \frac{11}{2}$. We consider a linear combination of Mellin B-spline kernels defined by (see Fig. 1)

\[
\chi(t) = \frac{1}{4} \bar{B}_2(2te^{-2}) + \frac{3}{4} \bar{B}_2(2te),
\]

where $\bar{B}_2$ is given by

\[
\bar{B}_2(t) = \begin{cases} 
1 - \log t, & 1 < t < e \\
1 + \log t, & \frac{1}{e} < t < 1 \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly the exponential sampling series $S_{w^*}\chi f$ based on $\chi(t)$ satisfies the conditions (i), (ii) and $\chi(1) = 0$. We also observe that the condition (i) of Theorem 5 is satisfied with $\alpha = \frac{3}{4}$. From Theorem 5 and Theorem 9, we have that at the discontinuity points of $f$, the sampling series $S_{w^*}\chi f$ converges to $\frac{3}{4}f(t + 0) + \frac{1}{4}f(t - 0)$. The convergence of the sampling series $S_{w^*}\chi f$ at discontinuity points $t = \frac{3}{2}$, $t = \frac{7}{2}$ and $t = \frac{11}{2}$ of the function $f$ has been tested and numerical results are presented in Tables 1, 2 and 3.
Figure 1. Plot of the kernel \( \chi(t) = \frac{1}{4} B_2(2te^{-2}) + \frac{3}{4} B_2(2te) \).

Table 1. Approximation of \( f \) at the jump discontinuity point \( t = \frac{3}{2} \) by the exponential sampling series \( S^X_\chi f \) based on \( \chi(t) \) for different values of \( w > 0 \). The theoretical limit of \( (S^X_\chi f) \left( \frac{3}{2} \right) \) as \( w \to \infty \) is \( \frac{3}{4} f \left( \frac{3}{2} + 0 \right) + \frac{1}{4} f \left( \frac{3}{2} - 0 \right) = 2.75 \).

| \( w \) | 5    | 10   | 20   | 50   | 100  | 200  |
|-------|------|------|------|------|------|------|
| \( S^X_\chi f \) | 3.0036 | 2.8669 | 2.8059 | 2.7717 | 2.7608 | 2.7554 |

Table 2. Approximation of \( f \) at the jump discontinuity point \( t = \frac{7}{2} \) by the exponential sampling series \( S^X_\chi f \) based on \( \chi(t) \) for different values of \( w > 0 \). The theoretical limit of \( (S^X_\chi f) \left( \frac{7}{2} \right) \) as \( w \to \infty \) is \( \frac{3}{4} f \left( \frac{7}{2} + 0 \right) + \frac{1}{4} f \left( \frac{7}{2} - 0 \right) = 2.25 \).

| \( w \) | 5    | 10   | 20   | 50   | 100  | 200  |
|-------|------|------|------|------|------|------|
| \( S^X_\chi f \) | 2.25  | 2.25  | 2.25  | 2.25  | 2.25  | 2.25  |

Table 3. Approximation of \( f \) at the jump discontinuity point \( t = \frac{11}{2} \) by the exponential sampling series \( S^X_\chi f \) based on \( \chi(t) \) for different values of \( w > 0 \). The theoretical limit of \( (S^X_\chi f) \left( \frac{11}{2} \right) \) as \( w \to \infty \) is \( \frac{3}{4} f \left( \frac{11}{2} + 0 \right) + \frac{1}{4} f \left( \frac{11}{2} - 0 \right) = 1.25 \).

| \( w \) | 5    | 10   | 20   | 50   | 100  | 200  |
|-------|------|------|------|------|------|------|
| \( S^X_\chi f \) | 1.25  | 1.25  | 1.25  | 1.25  | 1.25  | 1.25  |
Figure 2. Approximation of $f(t)$ by $S^\chi_w f$ based on $\chi(t) = \frac{1}{4} B_2(2te^{-2}) + \frac{3}{4} B_2(2te)$ for $w = 5$.

| $w$ | 5   | 10  | 20  | 50  | 100 | 200 |
|-----|-----|-----|-----|-----|-----|-----|
| $S^\chi_w f$ | 1.0492 | 1.1420 | 1.1939 | 1.2271 | 1.2384 | 1.2442 |
Figure 3. Approximation of $f(t)$ by $S^\chi_w f$ based on $\chi(t) = \frac{1}{4} \bar{B}_2(2te^{-2}) + \frac{3}{4} \bar{B}_2(2te)$ for $w = 10$. 
Acknowledgments. The first two authors are supported by DST-SERB, India Research Grant EEQ/2017/000201. The third author P. Devaraj has been supported by DST-SERB Research Grant MTR/2018/000559.

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