Frobenius restricted varieties in numerical semigroups

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Abstract

The common behaviour of many families of numerical semigroups led up to defining, firstly, the Frobenius varieties and, secondly, the (Frobenius) pseudo-varieties. However, some interesting families are still out of these definitions. To overcome this situation, here we introduce the concept of Frobenius restricted variety (or $R$-variety). We will generalize most of the results for varieties and pseudo-varieties to $R$-varieties. In particular, we will study the tree structure that arise within them.

Keywords: $R$-varieties; Frobenius restricted number; varieties; pseudo-varieties; monoids; numerical semigroups; tree (associated to an $R$-variety).

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1 Introduction

In [11], the concept of (Frobenius) variety was introduced in order to unify several results which have appeared in [1], [3], [16], and [17]. Moreover, the work made in [11] has allowed to study other notables families of numerical semigroups, such as those that appear in [7], [9], [12], and [13].

There exist families of numerical semigroups which are not varieties but have a similar structure. For example, the family of numerical semigroups with maximal embedding dimension and fixed multiplicity (see [15]). The study of this family, in [2], led to the concept of $m$-variety.

In order to generalize the concepts of variety and $m$-variety, in [8] were introduced the (Frobenius) pseudo-varieties. Moreover, recently, the results obtained in [8] allowed us to study several interesting families of numerical semigroups (for instance, see [10]).

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In this work, our aim will be to introduce and study the concept of \(R\)-variety (that is, Frobenius restricted variety). We will see how it generalizes the concept of pseudo-variety and we will show that there exist significant families of numerical semigroups which are \(R\)-varieties but not pseudo-varieties.

Let \(N\) be the set of nonnegative integers. A numerical semigroup is a subset \(S\) of \(N\) such that it is closed under addition, contains the zero element, and \(N \setminus S\) is finite.

It is well known (see [14, Lemma 4.5]) that, if \(S\) and \(T\) are numerical semigroups such that \(S \subseteq T\), then \(S \cup \{\max(T \setminus S)\}\) is another numerical semigroup. We will denote by \(F_T(S) = \max(T \setminus S)\) and we will call it as the Frobenius number of \(S\) restricted to \(T\).

An \(R\)-variety is a non-empty family \(\mathcal{R}\) of numerical semigroups that fulfills the following conditions.

1. \(\mathcal{R}\) has a maximum element with respect to the inclusion order (that we will denote by \(\Delta(\mathcal{R})\)).
2. If \(S, T \in \mathcal{R}\), then \(S \cap T \in \mathcal{R}\).
3. If \(S \in \mathcal{R}\) and \(S \neq \Delta(\mathcal{R})\), then \(S \cup \{F_{\Delta(\mathcal{R})}(S)\} \in \mathcal{R}\).

In Section 2 we will see that every pseudo-variety is an \(R\)-variety. Moreover, we will show that, if \(V\) is a variety and \(T\) is a numerical semigroup, then \(V_T = \{S \cap T \mid S \in V\}\) is an \(R\)-variety. In fact, we will prove that every \(R\)-variety is of this form.

Let \(\mathcal{R}\) be an \(R\)-variety and let \(M\) be a submonoid of \((\mathbb{N}, +)\). We will say that \(M\) is an \(R\)-monoid if it can be expressed as intersection of elements of \(\mathcal{R}\). It is clear that the intersection of \(\mathcal{R}\)-monoids is another \(\mathcal{R}\)-monoid and, therefore, we can define the \(\mathcal{R}\)-monoid generated by a subset of \(\Delta(\mathcal{R})\). In Section 3 we will show that every \(\mathcal{R}\)-monoid admits a unique minimal \(\mathcal{R}\)-system of generators. In addition, we will see that, if \(M\) is an \(\mathcal{R}\)-monoid and \(x \in M\), then \(M \setminus \{x\}\) is another \(\mathcal{R}\)-monoid if and only if \(x\) belongs to the minimal \(\mathcal{R}\)-system of generators of \(M\).

In Section 4 we will show that the elements of an \(R\)-variety, \(\mathcal{R}\), can be arranged in a tree with root \(\Delta(\mathcal{R})\). Moreover, we will prove that the set of children of a vertex \(S\), of such a tree, is equal to \(\{S \setminus \{x\} \mid x\text{ is an element of the minimal }\mathcal{R}\text{-system of generators of }S\text{ and }x > F_{\Delta(\mathcal{R})}(S)\}\). This fact will allow us to show an algorithmic process in order to recurrently build the elements of an \(R\)-variety.

Finally, in Section 5 we will see that, in general and contrary to what happens with varieties and pseudo-varieties, we cannot define the smallest \(R\)-variety that contains a given family \(\mathcal{F}\) of numerical semigroups. Nevertheless, we will show that, if \(\Delta\) is a numerical semigroup such that \(S \subseteq \Delta\) for all \(S \in \mathcal{F}\), then there exists the smallest \(R\)-variety (denoted by \(\mathcal{R}(\mathcal{F}, \Delta)\)) containing \(\mathcal{F}\) and having \(\Delta\) as maximum (with respect the inclusion order). Moreover, we will prove that \(\mathcal{R}(\mathcal{F}, \Delta)\) is finite if and only if \(\mathcal{F}\) is finite. In such a case, that fact will allow us to compute, for a given \(\mathcal{R}(\mathcal{F}, \Delta)\)-monoid, its minimal \(\mathcal{R}(\mathcal{F}, \Delta)\)-system of
generators. In this way, we will obtain an algorithmic process to determine all the elements of \( R(\mathcal{F}, \Delta) \) by starting from \( \mathcal{F} \) and \( \Delta \).

Let us observe that the proofs, of some results of this work, are similar to the proofs of the analogous results for varieties and pseudo-varieties. However, in order to get a self-contained paper, we have not omitted several of such proofs.

2 Varieties, pseudo-varieties, and \( R \)-varieties

It is said that \( M \) is a submonoid of \( (\mathbb{N}, +) \) if \( M \) is a subset of \( \mathbb{N} \) which is closed for the addition and such that \( 0 \in M \). It particular, if \( S \) is a submonoid of \( (\mathbb{N}, +) \) such that \( \mathbb{N} \setminus S \) is finite, then \( S \) is a numerical semigroup.

Let \( A \) be a non-empty subset of \( \mathbb{N} \). Then it is denoted by \( \langle A \rangle \) the submonoid of \( (\mathbb{N}, +) \) generated by \( A \), that is,

\[
\langle A \rangle = \{ \lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, \ a_1, \ldots, a_n \in A, \ \lambda_1, \ldots, \lambda_n \in \mathbb{N} \}.
\]

It is well known (see for instance [14, Lemma 2.1]) that \( \langle A \rangle \) is a numerical semigroup if and only if \( \gcd(A) = 1 \).

Let \( M \) be a submonoid of \( (\mathbb{N}, +) \) and let \( A \subseteq \mathbb{N} \). If \( M = \langle A \rangle \), then it is said that \( A \) is a system of generators of \( M \). Moreover, it is said that \( A \) is a minimal system of generators of \( M \) if \( M \neq \langle B \rangle \) for all \( B \subset A \). It is a classical result that every submonoid \( M \) of \( (\mathbb{N}, +) \) has a unique minimal system of generators (denoted by \( \text{msg}(M) \)) which, in addition, is finite (see for instance [14, Corollary 2.8]).

Let \( S \) be a numerical semigroup. Being that \( \mathbb{N} \setminus S \) is finite, it is possible to define several notable invariants of \( S \). One of them is the Frobenius number of \( S \) (denoted by \( F(S) \)) which is the greatest integer that does not belong to \( S \) (see [6]). Another one is the genus of \( S \) (denoted by \( g(S) \)) which is the cardinality of \( \mathbb{N} \setminus S \).

Let \( S \) be a numerical semigroup different from \( \mathbb{N} \). Then it is obvious that \( S \cup \{F(S)\} \) is also a numerical semigroup. Moreover, from [14, Proposition 7.1], we have that \( T \) is a numerical semigroup with \( g(T) = g + 1 \) if and only if there exist a numerical semigroup \( S \) and \( x \in \text{msg}(S) \) such that \( g(S) = g, \ x > F(S) \), and \( T = S \setminus \{x\} \). This result is the key to build the set of all numerical semigroups with genus \( g + 1 \) when we have the set of all numerical semigroups with genus \( g \) (see [14, Proposition 7.4]).

In [11] it was introduced the concept of (Frobenius) variety in order to generalize the previous situation to some relevant families of numerical semigroups.

It is said that a non-empty family of numerical semigroups \( V \) is a (Frobenius) variety if the following conditions are verified.

1. If \( S, T \in V \), then \( S \cap T \in V \).
2. If \( S \in V \) and \( S \neq \mathbb{N} \), then \( S \cup \{F(S)\} \in V \).

However, there exist families of numerical semigroups that are not varieties, but have a very similar behavior. By studying these families of numerical semigroups, we introduced in [8] the concept of (Frobenius) pseudo-variety.
It is said that a non-empty family of numerical semigroups \( \mathcal{P} \) is a \emph{(Frobenius) pseudo-variet y} if the following conditions are verified.

1. \( \mathcal{P} \) has a maximum element with respect to the inclusion order (that we will denote by \( \Delta(\mathcal{P}) \)).

2. If \( S, T \in \mathcal{P} \), then \( S \cap T \in \mathcal{P} \).

3. If \( S \in \mathcal{P} \) and \( S \neq \Delta(\mathcal{P}) \), then \( S \cup \{F(S)\} \in \mathcal{P} \).

From the definitions, it is clear that every variety is a pseudo-variet y. Moreover, as a consequence of [8, Proposition 1], we have the next result.

**Proposition 2.1.** Let \( \mathcal{P} \) be a pseudo-variet y. Then \( \mathcal{P} \) is a variety if and only if \( \mathbb{N} \in \mathcal{P} \).

The following result asserts that the concept of \( R \)-variet y generalizes the concept of pseudo-variet y.

**Proposition 2.2.** Every pseudo-variet y is an \( R \)-variet y.

**Proof.** Let \( \mathcal{P} \) be a pseudo-variet y. In order to prove that \( \mathcal{P} \) is an \( R \)-variet y, we have to show that, if \( S \in \mathcal{P} \) and \( S \neq \Delta(\mathcal{P}) \), then \( S \cup \{F(S)\} \in \mathcal{P} \). Since \( \mathcal{P} \) is a pseudo-variet y, we know that \( S \cup \{F(S)\} \in \mathcal{P} \). Thus, to finish the proof, it is enough to see that \( F(S) = F_{\Delta(\mathcal{P})}(S) \). On the one hand, it is clear that \( F_{\Delta(\mathcal{P})}(S) \leq F(S) \). On the other hand, since \( S \cup \{F(S)\} \in \mathcal{P} \), then we have that \( F(S) \in \Delta(\mathcal{P}) \). Therefore, \( F(S) \in \Delta(\mathcal{P}) \setminus S \) and, consequently, \( F(S) \leq F_{\Delta(\mathcal{P})}(S) \).

In the next example we see that there exist \( R \)-varieties that are not pseudo-varieties.

**Example 2.3.** Let \( \mathcal{R} \) be the set formed by all numerical semigroups which are contained in the numerical semigroup \( \langle 5, 7, 9 \rangle \). It is clear that \( \mathcal{R} \) is an \( R \)-variet y. However, since \( S = \langle 5, 7, 9 \rangle \setminus \{5\} \in \mathcal{R} \), \( S \neq \Delta(\mathcal{R}) = \langle 5, 7, 9 \rangle \), \( F(S) = 13 \), and \( S \cup \{13\} \notin \mathcal{R} \), we have that \( \mathcal{R} \) is not a pseudo-variet y.

Generalizing the above example, we can obtain several \( R \)-varieties, most of which are not pseudo-varieties.

1. Let \( T \) be a numerical semigroup. Then \( \mathcal{L}_T = \{S \mid S \text{ is a numerical semigroup and } S \subseteq T\} \) is an \( R \)-variet y. Observe that \( \mathcal{L}_T \) is the set formed by all numerical subsemigroups of \( T \).

2. Let \( S_1 \) and \( S_2 \) be two numerical semigroups such that \( S_1 \subseteq S_2 \). Then \( [S_1, S_2] = \{S \mid S \text{ is a numerical semigroup and } S_1 \subseteq S \subseteq S_2\} \) is an \( R \)-variet y.

3. Let \( T \) be a numerical semigroup and let \( A \subseteq T \). Then \( \mathcal{R}(A, T) = \{S \mid S \text{ is a numerical semigroup and } A \subseteq S \subseteq T\} \) is an \( R \)-variet y. Observe that both of the previous examples are particular cases of this one.
Remark 2.4. Let $p, q$ be relatively prime integers such that $1 < p < q$. Let us take the numerical semigroups $S_1 = \langle p, q \rangle$ and $S_2 = \frac{S_1}{2} = \{ s \in \mathbb{N} | 2s \in S_1 \}$. In [4, 5], Kunz and Waldi study the family of numerical semigroups $\{S_1, S_2\}$, which is an $R$-variety but not a pseudo-variety.

The next result establishes when an $R$-variety is a pseudo-variety.

**Proposition 2.5.** Let $R$ be an $R$-variety. Then $R$ is a pseudo-variety if and only if $F(S) \in \Delta(R)$ for all $S \in R$ such that $S \neq \Delta(R)$.

**Proof. (Necessity.)** If $R$ is a pseudo-variety and $S \in R$ with $S \neq \Delta(R)$, then $S \cup \{F(S)\} \in R$. Therefore, $F(S) \in \Delta(R)$.

**(Sufficiency.)** In order to show that $R$ is a pseudo-variety, it will be enough to see that $S \cup \{F(S)\} \in R$ for all $S \in R$ such that $S \neq \Delta(R)$. For that, since $F(S) \in \Delta(R)$, then it is clear that $F_{\Delta(R)}(S) = F(S)$ and, therefore, $S \cup \{F(S)\} = S \cup \{F_{\Delta(R)}(S)\} \in R$.

An immediate consequence of Propositions 2.1 and 2.5 is the following result.

**Corollary 2.6.** Let $R$ be an $R$-variety. Then $R$ is a variety if and only if $\mathbb{N} \in R$.

Our next purpose, in this section, will be to show that to give an $R$-variety is equivalent to give a pair $(V, T)$ where $V$ is a variety and $T$ is a numerical semigroup. Before that we need to introduce some concepts and results.

Let $S$ be a numerical semigroup. Then we define recurrently the following sequence of numerical semigroups.

- $S_0 = S$,
- if $S_\ell \neq \mathbb{N}$, then $S_{\ell+1} = S_\ell \cup \{F(S_\ell)\}$.

Since $\mathbb{N} \setminus S$ is a finite set with cardinality equal to $g(S)$, then we get a finite chain of numerical semigroups $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_{g(S)} = \mathbb{N}$. We will denote by $C(S)$ the set $\{S_0, S_1, \ldots, S_{g(S)}\}$ and will say that it is the chain of numerical semigroups associated to $S$. If $\mathcal{F}$ is a non-empty family of numerical semigroups, then we will denote by $C(\mathcal{F})$ the set $\bigcup_{S \in \mathcal{F}} C(S)$.

Let $\mathcal{F}$ be a non-empty family of numerical semigroups. We know that there exists the smallest variety containing $\mathcal{F}$ (see [11]). Moreover, by [11, Theorem 4], we have the next result.

**Proposition 2.7.** Let $\mathcal{F}$ be a non-empty family of numerical semigroups. Then the smallest variety containing $\mathcal{F}$ is the set formed by all finite intersections of elements of $C(\mathcal{F})$.

Now, let $\mathcal{R}$ be an $R$-variety. By applying repeatedly that, if $S \in \mathcal{R}$ and $S \neq \Delta(\mathcal{R})$, then $S \cup \{F_{\Delta(\mathcal{R})}(S)\} \in \mathcal{R}$, we get the following result.

**Lemma 2.8.** Let $\mathcal{R}$ be an $R$-variety. If $S \in \mathcal{R}$ and $n \in \mathbb{N}$, then $S \cup \{x \in \Delta(\mathcal{R}) | x \geq n\} \in \mathcal{R}$.
We are ready to show the announced result.

**Theorem 2.9.** Let $\mathcal{V}$ be a variety and let $T$ be a numerical semigroup. Then \( \mathcal{V}_T = \{ S \cap T \mid S \in \mathcal{V} \} \) is an $R$-variety. Moreover, every $R$-variety is of this form.

**Proof.** By Proposition 2.4 we know that, if $\mathcal{V}$ is a variety, then $\mathbb{N} \in \mathcal{V}$ and, therefore, $T$ is the maximum of $\mathcal{V}_T$ (that is, $T = \Delta(\mathcal{V}_T)$). On the other hand, it is clear that, if $S_1, S_2 \in \mathcal{V}_T$, then $S_1 \cap S_2 \in \mathcal{V}_T$.

Now, let $S \in \mathcal{V}$ such that $S \cap T \neq T$ and let us have $t = F_T(S \cap T)$. In order to conclude that $\mathcal{V}_T$ is an $R$-variety, we will see that $(S \cap T) \cup \{ t \} \in \mathcal{V}_T$.

First, let us observe that $t = \max(T \setminus (S \cap T)) = \max(T \setminus S)$. Then, because $S \in \mathcal{V}$ and $\mathcal{V}$ is a variety, we can easily deduce that $\bar{S} = S \cup \{ t, \rightarrow \} \in \mathcal{V}$. Moreover, $(S \cap T) \cup \{ t \} \subseteq (S \cap T) \cup (\{ t, \rightarrow \} \cap T) = \bar{S} \cap T$. Let us see now that $\bar{S} \cap T \subseteq (S \cap T) \cup \{ t \}$. In other case, there exists $t' > t$ such that $t' \in T$ and $t' \notin S$, in contradiction with the maximality of $t$. Therefore, $(S \cap T) \cup \{ t \} = \bar{S} \cap T$ and $\bar{S} \in \mathcal{V}$. Consequently, $(S \cap T) \cup \{ t \} \in \mathcal{V}_T$.

Let $\mathcal{R}$ be an $R$-variety and let $\mathcal{V}$ be the smallest variety containing $\mathcal{R}$. To conclude the proof of the theorem, we will see that $\mathcal{R} = \mathcal{V}_{\Delta(R)}$. It is clear that $\mathcal{R} \subseteq \mathcal{V}_{\Delta(R)}$. Thus, let us see the reverse one. For that, we will prove that, if $S \in \mathcal{V}$, then $S \cap \Delta(\mathcal{R}) \in \mathcal{R}$. In effect, by Proposition 2.7 we have that, if $S \in \mathcal{V}$, then there exist $S_1, \ldots, S_k \in C(\mathcal{R})$ such that $S = S_1 \cap \cdots \cap S_k$. Therefore, $S \cap \Delta(\mathcal{R}) = (S_1 \cap \Delta(\mathcal{R})) \cap \cdots \cap (S_k \cap \Delta(\mathcal{R}))$. Since $\mathcal{R}$ is an $R$-variety, then $\mathcal{R}$ is closed under finite intersections. Therefore, to see that $S \cap \Delta(\mathcal{R}) \in \mathcal{R}$, it is enough to show that $S_i \cap \Delta(\mathcal{R}) \in \mathcal{R}$ for all $i \in \{ 1, \ldots, k \}$. Since $S_i \in C(\mathcal{R})$, then it is clear that there exist $S'_i \in \mathcal{R}$ and $n_i \in \mathbb{N}$ such that $S_i = S'_i \cup \{ n_i, \rightarrow \}$. Therefore, $S_i \cap \Delta(\mathcal{R}) = S'_i \cup \{ x \in \Delta(\mathcal{R}) \mid x \geq n_i \} \in \mathcal{R}$, by applying Lemma 2.8.

The above theorem allows us to give many examples of $R$-varieties starting from already known varieties.

1. Let us observe that, if $\mathcal{V}$ is a variety and $T \in \mathcal{V}$, then $\mathcal{V}_T = \{ S \cap T \mid S \in \mathcal{V} \} = \{ S \in \mathcal{V} \mid S \subseteq T \}$ is an $R$-variety contained in $\mathcal{V}$. Thus, for instance, we have that the set formed by all Arf numerical semigroups, which are contained in a certain Arf numerical semigroup, is an $R$-variety.

2. Observe also that, if $\mathcal{V}$ is a variety and $T$ is a numerical semigroup such that $T \notin \mathcal{V}$, then $\mathcal{V}_T = \{ S \cap T \mid S \in \mathcal{V} \}$ is an $R$-variety not contained in $\mathcal{V}$ (because $T \notin \mathcal{V}_T$ and $T \notin \mathcal{V}$). Let us take, for example, the variety $\mathcal{V}$ of all Arf numerical semigroups and $T = \langle 5, 8 \rangle \notin \mathcal{V}$. In such a case, $\mathcal{V}_T$ is the $R$-variety formed by the numerical semigroups which are the intersection of an Arf numerical semigroup and $T$.

**Corollary 2.10.** Let $\mathcal{R}$ be an $R$-variety and let $U$ be a numerical semigroup. Then $\mathcal{R}_U = \{ S \cap U \mid S \in \mathcal{R} \}$ is an $R$-variety.

**Proof.** By applying Theorem 2.9 we have that there exist a variety $\mathcal{V}$ and a numerical semigroup $T$ such that $\mathcal{R} = \mathcal{V}_T = \{ S \cap T \mid S \in \mathcal{V} \}$. Therefore,
\[ \mathcal{R}_U = \{ S \cap T \cap U \mid S \in \mathcal{V} \} = \mathcal{V}_{T \cap U}, \] which is clearly an \( R \)-variety (by Theorem 2.9 again).

The next result says us that Theorem 2.9 remains true when variety is replaced with pseudo-variety.

**Corollary 2.11.** Let \( \mathcal{P} \) be a pseudo-variety and let \( T \) be a numerical semigroup. Then \( \mathcal{P}_T = \{ S \cap T \mid S \in \mathcal{P} \} \) is an \( R \)-variety. Moreover, every \( R \)-variety is of this form.

**Proof.** By Proposition 2.2, we know that, if \( \mathcal{P} \) is a pseudo-variety, then \( \mathcal{P} \) is an \( R \)-variety. Thereby, by applying Corollary 2.10 we conclude that \( \mathcal{P}_T \) is an \( R \)-variety.

Now, by Theorem 2.9 we know that, if \( \mathcal{R} \) is an \( R \)-variety, then there exist a variety \( \mathcal{V} \) and a numerical semigroup \( T \) such that \( \mathcal{R} = \mathcal{V}_T \). To finish the proof, it is enough to observe that all varieties are pseudo-varieties.

Let us see an illustrative example of the above corollary.

**Example 2.12.** From [8, Example 7], we have the pseudo-variety

\[ \mathcal{P} = \{ \langle 5, 6, 8, 9 \rangle, \langle 5, 6, 9, 13 \rangle, \langle 5, 6, 8 \rangle, \langle 5, 6, 13, 14 \rangle, \langle 5, 6, 9 \rangle, \langle 5, 6, 14 \rangle, \langle 5, 6, 13 \rangle, \langle 5, 6, 19 \rangle, \langle 5, 6 \rangle \} \].

Thereby, we have that \( \mathcal{P}_T \) is an \( R \)-variety for each numerical semigroup \( T \).

### 3 Monoids associated to an \( R \)-variety

In this section, \( \mathcal{R} \) we will be an \( R \)-variety. Now, let \( M \) be a submonoid of \((\mathbb{N}, +)\). We will say that \( M \) is an \( R \)-monoid if it is the intersection of elements of \( \mathcal{R} \).

The next result is easy to proof.

**Lemma 3.1.** The intersection of \( \mathcal{R} \)-monoids is an \( \mathcal{R} \)-monoid.

From the above lemma we have the following definition: let \( A \subseteq \Delta(\mathcal{R}) \). We will say that \( \mathcal{R}(A) \) is the \( \mathcal{R} \)-monoid generated by \( A \) if \( \mathcal{R}(A) \) is equal to the intersection of all the \( \mathcal{R} \)-monoids which contain the set \( A \). Observe that \( \mathcal{R}(A) \) is the smallest \( \mathcal{R} \)-monoid which contains the set \( A \) (with respect to the inclusion order). The next result has an easy proof too.

**Lemma 3.2.** If \( A \subseteq \Delta(\mathcal{R}) \), then \( \mathcal{R}(A) \) is equal to the intersection of all the elements of \( \mathcal{R} \) which contain the set \( A \).

Let us take \( A \subseteq \Delta(\mathcal{R}) \). If \( M = \mathcal{R}(A) \), then we will say that \( A \) is an \( \mathcal{R} \)-system of generators of \( M \). Moreover, we will say that \( A \) is a minimal \( \mathcal{R} \)-system of generators of \( M \) if \( M \neq \mathcal{R}(B) \) for all \( B \subseteq A \). The next purpose in this section will be to show that every \( \mathcal{R} \)-monoid has a unique minimal \( \mathcal{R} \)-system of generators. For that, we will give some previous lemmas. We can easily deduced the first one from Lemma 3.2.
Lemma 3.3. Let $A, B$ be two subsets of $\Delta(R)$ and let $M$ be an $R$-monoid. We have that

1. if $A \subseteq B$, then $R(A) \subseteq R(B)$;
2. $R(A) = R(\langle A \rangle)$;
3. $R(M) = M$.

If $M$ is an $R$-monoid, then $M$ is a submonoid of $(\mathbb{N}, +)$. Moreover, as we commented in Section 2, we know that there exists a finite subset $A$ of $M$ such that $M = \langle A \rangle$. Thereby, by applying Lemma 3.3 we have that $M = R(M) = R(\langle A \rangle) = R(A)$. Consequently, $A$ is a finite $R$-system of generators of $M$. Thus, we can establish the next result.

Lemma 3.4. Every $R$-monoid has a finite $R$-system of generators.

In the following result, we characterize the minimal $R$-systems of generators.

Lemma 3.5. Let $A \subseteq \Delta(R)$ and $M = R(A)$. Then $A$ is a minimal $R$-system of generators of $M$ if and only if $a \notin R(A \setminus \{a\})$ for all $a \in A$.

Proof. (Necessity.) If $a \in R(A \setminus \{a\})$, then $A \subseteq R(A \setminus \{a\})$. Thus, by Lemma 3.3 we get that $M = R(A) \subseteq R(R(A \setminus \{a\})) = R(A \setminus \{a\}) \subseteq R(A) = M$. Therefore, $M = R(A \setminus \{a\})$, in contradiction with the minimality of $A$.

(Sufficiency.) If $A$ is not a minimal $R$-system of generators of $M$, then there exists $B \subsetneq A$ such that $R(B) = M$. Then, by Lemma 3.3 if $a \in A \setminus B$, then $a \in M = R(B) \subseteq R(A \setminus \{a\})$, in contradiction with the hypothesis. 

The next result generalizes an evident property of submonoids of $(\mathbb{N}, +)$. More concretely, every element $x$ of a submonoid $M$ of $(\mathbb{N}, +)$ is expressible as a non-negative integer linear combination of the generators of $M$ that are smaller than or equal to $x$.

Lemma 3.6. Let $A \subseteq \Delta(R)$ and $x \in R(A)$. Then $x \in R(\{a \in A \mid a \leq x\})$.

Proof. Let us suppose that $x \notin R(\{a \in A \mid a \leq x\})$. Then, from Lemma 3.2 we know that there exists $S \in R$ such that $\{a \in A \mid a \leq x\} \subseteq S$ and $x \notin S$. By applying now Lemma 2.8 we have that $\bar{S} = S \cup \{m \in \Delta(R) \mid m \geq x + 1\} \in R$. Observe that, obviously, $A \subseteq \bar{S}$ and $x \notin \bar{S}$. Therefore, by applying once again Lemma 2.8 we get that $x \notin R(A)$, in contradiction with the hypothesis.

We are now ready to show the above announced result.

Theorem 3.7. Every $R$-monoid admits a unique minimal $R$-system of generators. In addition, such a $R$-system is finite.

Proof. Let $M$ be an $R$-monoid and let $A, B$ be two minimal $R$-systems of generators of $M$. We are going to see that $A = B$. For that, let us suppose that $A = \{a_1 < a_2 < \cdots\}$ and $B = \{b_1 < b_2 < \cdots\}$. If $A \neq B$, then
there exists $i = \min\{k \mid a_k \neq b_k\}$. Let us assume, without loss of generality, that $a_i < b_i$. Since $a_i \in M = \mathcal{R}(A) = \mathcal{R}(B)$, by Lemma 3.6 we have that $a_i \in \mathcal{R}\{b_1, \ldots, b_{i-1}\}$. Because $\{b_1, \ldots, b_{i-1}\} = \{a_1, \ldots, a_{i-1}\}$, then $a_i \in \mathcal{R}\{a_1, \ldots, a_{i-1}\}$, in contradiction with Lemma 3.6. Finally, by Lemma 3.4 we have that the minimal $\mathcal{R}$-system of generators is finite. □

If $M$ is a $\mathcal{R}$-monoid, then the cardinality of the minimal $\mathcal{R}$-system of generators of $M$ will be called the $\mathcal{R}$-range of $M$.

**Example 3.8.** Let $S, T$ be two numerical semigroups such that $S \subseteq T$. We define recurrently the following sequence of numerical semigroups.

- $S_0 = S$, 
- if $S_i \neq T$, then $S_{i+1} = S_i \cup \{F_T(S_i)\}$.

Since $T \setminus S$ is a finite set, then we get a finite chain of numerical semigroups $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = T$. We will denote by $C(S, T)$ the set $\{S_0, S_1, \ldots, S_n\}$ and will say that it is the chain of $S$ restricted to $T$. It is clear that $C(S, T)$ is an $R$-variety. Moreover, it is also clear that, for each $i \in \{1, \ldots, n\}$, $S_i$ is the smallest element of $C(S, T)$ containing $F_T(S_{i-1})$. Therefore, $\{F_T(S_{i-1})\}$ is the minimal $C(S, T)$-system of generators of $S_i$ for all $i \in \{1, \ldots, n\}$. Let us also observe that the empty set, $\emptyset$, is the minimal $C(S, T)$-system of generators of $S_0$. Therefore, the $C(S, T)$-range of $S_i$ is equal to 1, if $i \in \{1, \ldots, n\}$, and 0, if $i = 0$.

It is well known that, if $M$ is a submonoid of $(\mathbb{N}, +)$ and $x \in M$, then $M \setminus \{x\}$ is another submonoid of $(\mathbb{N}, +)$ if and only if $x \in \text{msg}(M)$. In the next result we generalize this property to $\mathcal{R}$-monoids.

**Proposition 3.9.** Let $M$ be an $\mathcal{R}$-monoid and let $x \in M$. Then $M \setminus \{x\}$ is an $\mathcal{R}$-monoid if and only if $x$ belongs to the minimal $\mathcal{R}$-system of generators of $M$.

**Proof.** Let $A$ be the minimal $\mathcal{R}$-system of generators of $M$. If $x \not\in A$, then $A \subseteq M \setminus \{x\}$. Therefore, $M \setminus \{x\}$ is a $\mathcal{R}$-monoid containing $A$ and, consequently, $M = \mathcal{R}(A) \subseteq M \setminus \{x\}$, which is a contradiction.

Conversely, by Theorem 3.7 we have that, if $x \in A$, then $\mathcal{R}(M \setminus \{x\}) \neq \mathcal{R}(A) = M$. Thereby, $\mathcal{R}(M \setminus \{x\}) = M \setminus \{x\}$. Consequently, $M \setminus \{x\}$ is a $\mathcal{R}$-monoid. □

Let us illustrate the above proposition with an example.

**Example 3.10.** Let $T$ be a numerical semigroup and let $A \subseteq T$. Then we know that $\mathcal{R}(A, T) = \{S \mid S$ is a numerical semigroup and $A \subseteq S \subseteq T\}$ is an $R$-variety. By applying Proposition 3.9, we easily deduce that, if $S \in \mathcal{R}(A, T)$, then the minimal $\mathcal{R}(A, T)$-system of generators of $S$ is $\{x \in \text{msg} \mid x \notin A\}$.

From Theorem 2.9 we know that every $R$-variety is of the form $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$, where $\mathcal{V}$ is a variety and $T$ is a numerical semigroup. Now, our purpose is to study the relation between $\mathcal{V}$-monoids and $\mathcal{V}_T$-monoids.
Proposition 3.11. Let $M$ be a submonoid of $(\mathbb{N}, +)$ and let $T$ be a numerical semigroup. Then $M$ is a $\mathcal{V}_T$-monoid if and only if there exists a $\mathcal{V}$-monoid $M'$ such that $M = M' \cap T$.

Proof. (Necessity.) If $M$ is a $\mathcal{V}_T$-monoid, then there exists $F \subseteq \mathcal{V}_T$ such that $M = \bigcap_{S \in F} S$. But, if $S \in F$, then $S \in \mathcal{V}_T$ and, consequently, there exists $S' \in \mathcal{V}$ such that $S = S' \cap T$. Now, let $F' = \{S' \in \mathcal{V} \mid S' \cap T \in F\}$ and let $M' = \bigcap_{S' \in F'} S'$. Then it is clear that $M'$ is a $\mathcal{V}$-monoid and that $M = M' \cap T$.

(Sufficiency.) If $M'$ is a $\mathcal{V}$-monoid, then there exists $F' \in \mathcal{V}$ such that $M' = \bigcap_{S' \in F'} S'$. Let $F = \{S' \cap T \mid S' \in F'\}$. Then it is clear that $F \subseteq \mathcal{V}_T$ and that $\bigcap_{S \in F} S = M' \cap T$. Therefore, $M' \cap T$ is a $\mathcal{V}_T$-monoid.

Observe that, as a consequence of the above proposition, we have that the set of $\mathcal{V}_T$-monoids is precisely given by $\{M \cap T \mid M$ is a $\mathcal{V}$-monoid$\}$.

Corollary 3.12. Let $T$ be a numerical semigroup. If $A \subseteq T$, then $\mathcal{V}_T(A) = \mathcal{V}(A) \cap T$.

Proof. By Proposition 3.11 we know that $\mathcal{V}(A) \cap T$ is a $\mathcal{V}_T$-monoid containing $A$. Therefore, $\mathcal{V}_T(A) \subseteq \mathcal{V}(A) \cap T$.

Let us see now the opposite inclusion. By applying once more Proposition 3.11 we deduce that there exists a $\mathcal{V}$-monoid $M$ such that $\mathcal{V}_T(A) = M \cap T$. Thus, it is clear that $A \subseteq M$ and, thereby, $\mathcal{V}(A) \subseteq M$. Consequently, $\mathcal{V}(A) \cap T \subseteq M \cap T = \mathcal{V}_T(A)$.

From Corollary 3.12, we have that the set formed by the $\mathcal{V}_T$-monoids is $\{\mathcal{V}(A) \cap T \mid A \subseteq T\} = \{M \cap T \mid M$ is a $\mathcal{V}$-monoid and its minimal $\mathcal{V}$-system of generators is including in $T\}$. Moreover, observe that, if $T \in \mathcal{V}$, then $\mathcal{V}_T(A) = \mathcal{V}(A)$ and, therefore, in such a case the set formed by all the $\mathcal{V}_T$-monoids coincides with the set formed by all the $\mathcal{V}$-monoids that are contained in $T$.

For some varieties there exist algorithms that allow us to compute $\mathcal{V}(A)$ by starting from $A$. Thereby, we can use such results in order to compute $\mathcal{V}_T(A)$. Let us see two examples of this fact.

Example 3.13. An LD-semigroup (see [12]) is a numerical semigroup $S$ fulfilling that $a + b - 1 \in S$ for all $a, b \in S \setminus \{0\}$. Let $\mathcal{V}$ the set formed by all LD-semigroups. In [12] it is shown that $\mathcal{V}$ is a variety. Let $T = \{5, 7, 9\}$ (observe that $T \notin \mathcal{V}$). By Theorem 2.4 we know that $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is an $R$-variety. Let us suppose that we can compute $\mathcal{V}_T$($\{5\}$).

In [12] we have an algorithm to compute $\mathcal{V}(A)$ by starting from $A$. By using such algorithm, in [12] Example 33 it is shown that $\mathcal{V}$$\{5\} = \langle 5, 9, 13, 17, 21 \rangle$. Therefore, by applying Corollary 3.12 we have that $\mathcal{V}_T$$\{5\} = \langle 5, 9, 13, 17, 21 \rangle \cap \langle 5, 7, 9 \rangle = \langle 5, 9, 17, 21 \rangle$.

Example 3.14. An PL-semigroup (see [17]) is a numerical semigroup $S$ fulfilling that $a + b + 1 \in S$ for all $a, b \in S \setminus \{0\}$. Let $\mathcal{V}$ the set formed by all PL-semigroups. In [17] it is shown that $\mathcal{V}$ is a variety and it is given an algorithm to compute $\mathcal{V}(A)$.
by starting from $A$. Let $T = \langle 4, 7, 13 \rangle$ (observe that $T \in \mathcal{V}$). By Theorem 2.9 we know that $\mathcal{V}_T = \{ T \cap S \mid S \in \mathcal{V} \}$ is an $R$-variety. Let us suppose that we can compute $\mathcal{V}_T(\langle 4, 7 \rangle)$.

From \cite{7} Example 48, we know that $\mathcal{V}(\langle 4, 7 \rangle) = \langle 4, 7, 9 \rangle$. Thus, by applying Corollary 3.12 we have that $\mathcal{V}_T(\langle 4, 7 \rangle) = \langle 4, 7, 9 \rangle \cap \langle 4, 7, 13 \rangle = \langle 4, 7, 13 \rangle$.

Let $T$ be a numerical semigroup. We know that, if $M$ is a $\mathcal{V}_T$-monoid, then there exists a $\mathcal{V}$-monoid, $M'$, with minimal $\mathcal{V}$-system of generators contained in $T$, such that $M = M' \cap T$. The next result says us that, in this situation, the minimal $\mathcal{V}$-system of generators of $M'$ is just the minimal $\mathcal{V}_T$-system of generators of $M$.

**Proposition 3.15.** Let $A \subseteq T$. Then $A$ is the minimal $\mathcal{V}_T$-system of generators of $\mathcal{V}_T(A)$ if and only if $A$ is the minimal $\mathcal{V}$-system of generators of $\mathcal{V}(A)$.

**Proof.** (Necessity.) Let us suppose that $A$ is not the minimal $\mathcal{V}$-system of generators of $\mathcal{V}(A)$. That is, there exists $B \subsetneq A$ such that $\mathcal{V}(B) = \mathcal{V}(A)$. Then, from Corollary 3.12 we have that $\mathcal{V}_T(A) = \mathcal{V}(A) \cap T = \mathcal{V}(B) \cap T = \mathcal{V}_T(B)$. Therefore, $A$ is not the minimal $\mathcal{V}_T$-system of generators of $\mathcal{V}_T(A)$.

(Sufficiency.) Let us suppose that $A$ is not the minimal $\mathcal{V}_T$-system of generators of $\mathcal{V}_T(A)$. Then $\mathcal{V}_T(B) = \mathcal{V}_T(A)$ for some subset $B \subsetneq A$. On the other hand, due to $A$ is the minimal $\mathcal{V}$-system of generators of $\mathcal{V}(A)$, from Lemma 3.5 we have an element $a \in A$ such that $a \notin \mathcal{V}(B)$. Consequently, $a \in \mathcal{V}(A) \cap T$ and $a \notin \mathcal{V}(B) \cap T$. Finally, from Corollary 3.12 $\mathcal{V}_T(A) = \mathcal{V}(A) \cap T \neq \mathcal{V}(B) \cap T = \mathcal{V}_T(B)$, which is a contradiction. \qed

We finish this section with two examples that illustrate the above proposition.

**Example 3.16.** Let $\mathcal{V}$ be such as in Example 3.14 and let $T = \langle 4, 6, 7 \rangle$. From \cite{12} Example 26, we know that $\mathcal{V}(\langle 4, 7, 10 \rangle) = \langle 4, 7, 10, 13 \rangle$ and, moreover, that $\{4\}$ is its minimal $\mathcal{V}$-system of generators. Then, from Proposition 3.15 $\{4\}$ is the minimal $\mathcal{V}$-system of generators of $\mathcal{V}_T(\langle 4, 7, 10 \rangle) = \langle 4, 7, 10, 13 \rangle \cap \langle 4, 6, 7 \rangle$.

**Example 3.17.** Let $\mathcal{V}$ be such as in Example 3.14 and let $T = \langle 3, 4 \rangle$. From \cite{7} Example 44, we know that $\{3\}$ is the minimal $\mathcal{V}$-system of generators of $S = \langle 3, 7, 11 \rangle$. Therefore, by Proposition 3.15 $\{3\}$ is the minimal $\mathcal{V}_T$-system of generators of $S \cap T$.

4 The tree associated to an $R$-variety

Let $V$ be a non-empty set and let $E \subseteq \{ (v, w) \in V \times V \mid v \neq w \}$. It is said that the pair $G = (V, E)$ is a graph. In addition, the vertices and edges of $G$ are the elements of $V$ and $E$, respectively.

Let $x, y \in V$ and let us suppose that $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ is a sequence of different edges such that $v_0 = x$ and $v_n = y$. Then, it is said that such a sequence is a path (of length $n$) connecting $x$ and $y$. 

11
Let $G$ be a graph. Let us suppose that there exists $r$, vertex of $G$, such that it is connected with any other vertex $x$ by a unique path. Then it is said that $G$ is a tree and that $r$ is its root.

Let $x, y$ be vertices of a tree $G$ and let us suppose that there exists a path that connects $x$ and $y$. Then it is said that $x$ is a descendant of $y$. Specifically, it is said that $x$ is a child of $y$ when $(x, y)$ is an edge of $G$.

From now on in this section, let $\mathcal{R}$ denote an $R$-variety. We define the graph $G(\mathcal{R})$ in the following way,

- $\mathcal{R}$ is the set of vertices of $G(\mathcal{R})$;
- $(S, S') \in \mathcal{R} \times \mathcal{R}$ is an edge of $G(\mathcal{R})$ if and only if $S' = S \cup \{F_{\Delta(\mathcal{R})}(S)\}$.

If $S \in \mathcal{R}$, then we can define recurrently (such as we did in Example 3.8) the sequence of elements in $\mathcal{R}$,

- $S_0 = S$,
- if $S_i \neq \Delta(\mathcal{R})$, then $S_{i+1} = S_i \cup \{F_{\Delta(\mathcal{R})}(S_i)\}$.

Thus, we obtain a chain (of elements in $\mathcal{R}$) $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = \Delta(\mathcal{R})$ such that $(S_i, S_{i+1})$ is an edge of $G(\mathcal{R})$ for all $i \in \{0, \ldots, n-1\}$. We will denote by $C_\mathcal{R}(S)$ the set $\{S_0, S_1, \ldots, S_n\}$ and will say that it is the chain of $S$ in $\mathcal{R}$. The next result is easy to prove.

**Proposition 4.1.** $G(\mathcal{R})$ is a tree with root $\Delta(\mathcal{R})$.

Observe that, in order to recurrently construct $G(\mathcal{R})$ starting from $\Delta(\mathcal{R})$, it is sufficient to compute the children of each vertex of $G(\mathcal{R})$. Let us also observe that, if $T$ is a child of $S$, then $S = T \cup \{F_{\Delta(\mathcal{R})}(T)\}$. Therefore, $T = S \setminus \{F_{\Delta(\mathcal{R})}(T)\}$. Thus, if $T$ is a child of $S$, then there exists an integer $x > F_{\Delta(\mathcal{R})}(S)$ such that $T = S \setminus \{x\}$. As a consequence of Propositions 3.9 and 4.1 and defining $F_{\Delta(\mathcal{R})}(\Delta(\mathcal{R})) = -1$, we have the following result.

**Theorem 4.2.** The graph $G(\mathcal{R})$ is a tree with root equal to $\Delta(\mathcal{R})$. Moreover, the set formed by the children of a vertex $S \in \mathcal{R}$ is $\{S \setminus \{x\} \mid x \in \text{msg}(S), x > F_{\Delta(\mathcal{R})}(S)\}$.

We can reformulate the above theorem in the following way.

**Corollary 4.3.** The graph $G(\mathcal{R})$ is a tree with root equal to $\Delta(\mathcal{R})$. Moreover, the set formed by the children of a vertex $S \in \mathcal{R}$ is $\{S \setminus \{x\} \mid x \in \text{msg}(S), x > F_{\Delta(\mathcal{R})}(S) \text{ and } S \setminus \{x\} \in \mathcal{R}\}$.

We illustrate the previous results with an example.

**Example 4.4.** We are going to build the $R$-variety $\mathcal{R} = \langle 5, 6, 7 \rangle = \{S \mid S \text{ is a numerical semigroup and } \langle 5, 6 \rangle \subseteq S \subseteq \langle 5, 6, 7 \rangle\}$. Observe that, if $S \in \mathcal{R}$ and $x \in \text{msg}(S)$, then $S \setminus \{x\} \in \mathcal{R}$ if and only if $x \notin \{5, 6\}$. Moreover, the maximum of $\mathcal{R}$ is $\Delta = \langle 5, 6, 7 \rangle$. By applying Corollary 4.3 we can recurrently build $G(\mathcal{R})$ in the following way.
\[ \langle 5, 6, 7 \rangle \text{ has got a unique child, which is } \langle 5, 6, 7 \rangle \setminus \{7\} = \langle 5, 6, 13, 14 \rangle. \] Moreover, \( F_\Delta(\langle 5, 6, 13, 14 \rangle) = 7. \]

\[ \langle 5, 6, 13, 14 \rangle \text{ has got two children, which are } \langle 5, 6, 13, 14 \rangle \setminus \{13\} = \langle 5, 6, 14 \rangle \text{ and } \langle 5, 6, 13, 14 \rangle \setminus \{14\} = \langle 5, 6, 13 \rangle. \] Moreover, \( F_\Delta(\langle 5, 6, 14 \rangle) = 13 \) and \( F_\Delta(\langle 5, 6, 13 \rangle) = 14. \)

\[ \langle 5, 6, 13 \rangle \text{ has not got children.} \]

\[ \langle 5, 6, 14 \rangle \text{ has got a unique child, which is } \langle 5, 6, 14 \rangle \setminus \{14\} = \langle 5, 6, 19 \rangle. \] Moreover, \( F_\Delta(\langle 5, 6, 14 \rangle) = 14. \)

\[ \langle 5, 6, 19 \rangle \text{ has got a unique child, which is } \langle 5, 6, 19 \rangle \setminus \{19\} = \langle 5, 6 \rangle. \] Moreover, \( F_\Delta(\langle 5, 6 \rangle) = 19. \)

\[ \langle 5, 6 \rangle \text{ has not got children.} \]

Therefore, in this situation, \( G(\mathcal{R}) \) is given by the next diagram.

\[ \begin{array}{c}
\langle 5, 6, 7 \rangle \\
\mathcal{R} = \left[ \langle 5, 6, 7 \rangle, \langle 5, 6, 13, 14 \rangle \right] \\
\end{array} \]

Observe that, if we represent the vertices of \( G(\mathcal{R}) \) using their minimal \( \mathcal{R} \)-systems of generators, then we have that \( G(\mathcal{R}) \) is given by the following diagram.

\[ \begin{array}{c}
\mathcal{R}(\{7\}) \\
\mathcal{R}(\{13, 14\}) \\
\mathcal{R}(\{14\}) \quad \mathcal{R}(\{13\}) \\
\mathcal{R}(\{19\}) \\
\mathcal{R}(\emptyset) \\
\end{array} \]

Let us observe that the \( R \)-variety \( \mathcal{R} = \left[ \langle 5, 6 \rangle, \langle 5, 6, 7 \rangle \right] \) depict in the above example is finite and, therefore, we have been able to build all its elements in a finite number of steps. If the \( R \)-variety is infinite, then it is not possible such situation. However, as a consequence of Theorem 4.2, we can show an algorithm in order to compute all the elements of the \( R \)-variety when the genus is fixed.
**Algorithm 4.5.** INPUT: A positive integer \( g \).
OUTPUT: \( \{ S \in \mathcal{R} \mid g(S) = g \} \).
1. If \( g < g(\Delta(\mathcal{R})) \), then return \( \emptyset \).
2. Set \( A = \{ \Delta(\mathcal{R}) \} \) and \( i = g(\Delta(\mathcal{R})) \).
3. If \( i = g \), then return \( A \).
4. For each \( S \in A \), compute the set \( B_S \) formed by all elements of the minimal \( \mathcal{R} \)-system of generators of \( S \) that are greater than \( F_{\Delta(\mathcal{R})}(S) \).
5. If \( \bigcup_{S \in A} B_S = \emptyset \), then return \( \emptyset \).
6. Set \( A = \bigcup_{S \in A} \{ S \setminus \{ x \} \mid x \in B_S \} \), \( i = i + 1 \), and go to (3).

We illustrate the operation of this algorithm with an example.

**Example 4.6.** Let \( \Delta = \langle 4, 6, 7 \rangle = \{ 0, 4, 6, 7, 8, 10, \rightarrow \} \). It is clear that \( g(\Delta) = 5 \). Let \( \mathcal{R} = \{ S \mid S \) is a numerical semigroup and \( \{ 4, 6 \} \subseteq S \subseteq \Delta \} \). We have that \( \mathcal{R} \) is an infinite \( \mathcal{R} \)-variety because \( \langle 4, 6, 2k + 1 \rangle \in \mathcal{R} \) for all \( k \in \{ 5, \rightarrow \} \). By using Algorithm 4.5, we are going to compute the set \( \{ S \in \mathcal{R} \mid g(S) = 8 \} \).

- \( A = \{ \Delta \} \), \( i = 5 \).
- \( B_{\Delta} = \{ 7 \} \).
- \( A = \{ \langle 4, 6, 11, 13 \rangle \} \), \( i = 6 \).
- \( B_{\langle 4, 6, 11, 13 \rangle} = \{ 11, 13 \} \).
- \( A = \{ \langle 4, 6, 13, 15 \rangle, \langle 4, 6, 11 \rangle \} \), \( i = 7 \).
- \( B_{\langle 4, 6, 13, 15 \rangle} = \{ 13, 15 \} \) and \( B_{\langle 4, 6, 11 \rangle} = \emptyset \).
- \( A = \{ \langle 4, 6, 15, 17 \rangle, \langle 4, 6, 13 \rangle \} \), \( i = 8 \).
- The algorithm returns \( \{ \langle 4, 6, 15, 17 \rangle, \langle 4, 6, 13 \rangle \} \).

Our next purpose in this section will be to show that, if \( \mathcal{R} \) is an \( \mathcal{R} \)-variety and \( T \in \mathcal{R} \), then the set formed by all the descendants of \( T \) in the tree \( G(\mathcal{R}) \) is also an \( \mathcal{R} \)-variety. It is clear that, if \( S, T \in \mathcal{R} \), then \( S \) is a descendant of \( T \) if and only if \( T \in C_{\mathcal{R}}(S) \). Therefore, we can establish the following result.

**Lemma 4.7.** Let \( \mathcal{R} \) be an \( \mathcal{R} \)-variety and \( S, T \in \mathcal{R} \). Then \( S \) is a descendant of \( T \) if and only if there exists \( n \in \mathbb{N} \) such that \( T = S \cup \{ x \in \Delta(\mathcal{R}) \mid x \geq n \} \).

As an immediate consequence of the above lemma, we have the next one.

**Lemma 4.8.** Let \( \mathcal{R} \) be an \( \mathcal{R} \)-variety and \( S, T \in \mathcal{R} \) such that \( S \neq T \). If \( S \) is a descendant of \( T \), then \( F_{\Delta(\mathcal{R})}(S) = F_T(S) \).

Now we are ready to show the announced result.
Theorem 4.9. Let $\mathcal{R}$ be an $R$-variety and $T \in \mathcal{R}$. Then

$$\mathcal{D}(T) = \{ S \in \mathcal{R} \mid \text{S is a descendant of T in the tree } G(\mathcal{R}) \}$$

is an $R$-variety.

Proof. Clearly, $T$ is the maximum of $\mathcal{D}(T)$. Let us see that, if $S_1, S_2 \in \mathcal{D}(T)$, then $S_1 \cap S_2 \in \mathcal{D}(T)$. Since, from Lemma 4.7, we know that there exist $n_1, n_2 \in \mathbb{N}$ such that $T = S_i \cup \{ x \in \Delta(\mathcal{R}) \mid x \geq n_i \}$, $i = 1, 2$, it is sufficient to show that $T = (S_1 \cap S_2) \cup \{ x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\} \}$. It is obvious that $(S_1 \cap S_2) \cup \{ x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\} \} \subseteq T$. Let us see now the opposite inclusion. For that, let $t \in T$ such that $t \notin S_1 \cap S_2$. Then $t \notin S_1$ or $t \notin S_2$ and, therefore, $t \in \{ x \in \Delta(\mathcal{R}) \mid x \geq n_1 \}$ or $t \in \{ x \in \Delta(\mathcal{R}) \mid x \geq n_2 \}$. Thereby, $t \in \{ x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\} \}$. Consequently, $T \subseteq (S_1 \cap S_2) \cup \{ x \in \Delta(\mathcal{R}) \mid x \geq \min\{n_1, n_2\} \}$. By applying again Lemma 4.7, we can assert that $S_1 \cap S_2 \in \mathcal{D}(T)$.

Finally, let $S \in \mathcal{D}(T)$ such that $S \neq T$. From Lemma 4.8, $S \cup \{ F_T(S) \} = S \cup \{ F_{\Delta(\mathcal{R})}(S) \} \in \mathcal{R}$ and, in consequence, $S \cup \{ F_T(S) \} \in \mathcal{D}(T)$. \[\Box\]

From the previous comment to [8, Example 7], we know that, if $\mathcal{V}$ is a variety and $T \in \mathcal{V}$, then $\mathcal{D}(T)$ is a pseudo-variety and, moreover, every pseudo-variety can be obtained in this way. Therefore, there exist $R$-varieties which are not the set formed by all the descendants of an element belonging to a variety.

The following result shows that an $R$-variety can be obtained as the set formed by intersecting all the descendants, of an element belonging to a variety, with a numerical semigroup.

Corollary 4.10. Let $\mathcal{V}$ be a variety, let $\Delta \in \mathcal{V}$, and let $T$ be a numerical semigroup. Let $\mathcal{D}(\Delta) = \{ S \mid S \text{ is a descendant of } \Delta \text{ in } G(\mathcal{V}) \}$ and let $\mathcal{D}(\Delta, T) = \{ S \cap T \mid S \in \mathcal{D}(\Delta) \}$. Then $\mathcal{D}(\Delta, T)$ is an $R$-variety. Moreover, every $R$-variety can be obtained in this way.

Proof. If $\mathcal{V}$ is a variety, then $\mathcal{V}$ is an $R$-variety and, by applying Theorem 4.9, we have that $\mathcal{D}(\Delta)$ is an $R$-variety. From Corollary 2.10 we conclude that $\mathcal{D}(\Delta, T)$ is an $R$-variety.

If $\mathcal{R}$ is an $R$-variety, by Theorem 2.0 we know that there exist a variety $\mathcal{V}$ and a numerical semigroup $T$ such that $\mathcal{R} = \{ S \cap T \mid S \in \mathcal{V} \}$. Now, it is clear that $\mathcal{V} = \mathcal{D}(\mathbb{N}) = \{ S \mid S \text{ is a descendant of } \mathbb{N} \text{ in } G(\mathcal{V}) \}$. Therefore, we have that $\mathcal{R} = \{ S \cap T \mid S \in \mathcal{D}(\mathbb{N}) \} = \mathcal{D}(\mathbb{N}, T)$. \[\Box\]

In the next result we see that the above corollary is also true when we write pseudo-variety instead of variety.

Corollary 4.11. Let $\mathcal{P}$ be a pseudo-variety, let $\Delta \in \mathcal{P}$, and let $T$ be a numerical semigroup. Let $\mathcal{D}(\Delta) = \{ S \mid S \text{ is a descendant of } \Delta \text{ in } G(\mathcal{P}) \}$ and let $\mathcal{D}(\Delta, T) = \{ S \cap T \mid S \in \mathcal{D}(\Delta) \}$. Then $\mathcal{D}(\Delta, T)$ is an $R$-variety. Moreover, every $R$-variety can be obtained in this way.
Proof. If \( \mathcal{P} \) is a pseudo-variety, then \( \mathcal{P} \) is an \( R \)-variety and, by applying Theorem 4.9 we have that \( \mathcal{D}(\Delta) \) is an \( R \)-variety as well. Now, from Corollary 2.10 we have that \( \mathcal{D}(\Delta, T) \) is an \( R \)-variety.

That every \( R \)-variety can be obtained in this way is an immediate consequence of Corollary 4.10 and having in mind that each variety is a pseudo-variety.

We conclude this section by illustrating the above corollary with an example.

Example 4.12. Let \( \mathcal{P} \) the pseudo-variety which appear in Example 2.12 In [8, Example 7] it is shown that \( G(\mathcal{P}) \) is given by the next subtree.

\[
\begin{align*}
\langle 5, 6, 8, 9 \rangle & & \langle 5, 6, 9, 13 \rangle & & \langle 5, 6, 8 \rangle \\
\langle 5, 6, 13, 14 \rangle & & \langle 5, 6, 9 \rangle & & \\
\langle 5, 6, 14 \rangle & & \langle 5, 6, 13 \rangle & & \\
\langle 5, 6, 19 \rangle & & \langle 5, 6 \rangle & & \\
\langle 5, 6 \rangle & & \langle 5, 6 \rangle & & \\
\end{align*}
\]

By applying Corollary 4.11 we have that, if \( T \) is a numerical semigroup, then \( \mathcal{R} = \{ S \cap T \mid S \in \mathcal{D}(\langle 5, 6, 13, 14 \rangle) \} \) is an \( R \)-variety. Let us observe that \( \mathcal{D}(\langle 5, 6, 13, 14 \rangle) = \{ \langle 5, 6, 13, 14 \rangle, \langle 5, 6, 14 \rangle, \langle 5, 6, 13 \rangle, \langle 5, 6, 19 \rangle, \langle 5, 6 \rangle \}. \)

5 The smallest \( R \)-variety containing a family of numerical semigroups

In [11, Proposition 2] it is proved that the intersection of varieties is a variety. As a consequence of this, we have that there exists the smallest variety which contains a given family of numerical semigroups.

On the other hand, in [8] was shown that, in general, the intersection of pseudo-varieties is not a pseudo-variety. Nevertheless, in [8, Theorem 4] it is proved that there exists the smallest pseudo-variety which contains a given family of numerical semigroups.

Our first objective in this section will be to show that, in general, we cannot talk about the smallest \( R \)-variety which contains a given family of numerical semigroups.

Lemma 5.1. Let \( \mathcal{F} \) be a family of numerical semigroups and let \( \Delta \) be a numerical semigroup such that \( S \subseteq \Delta \) for all \( S \in \mathcal{F} \). Then there exists an \( R \)-variety \( \mathcal{R} \) such that \( \mathcal{F} \subseteq \mathcal{R} \) and \( \max(\mathcal{R}) = \Delta \).
Proof. Let $\mathcal{R} = \{ S \mid S $ is a numerical semigroup and $ S \subseteq \Delta \}$. From Item 1 in Example 2.3 we have that $\mathcal{R}$ is an $R$-variety. Now, it is trivial that $\mathcal{F} \subseteq \mathcal{R}$ and $\max(\mathcal{R}) = \Delta$.

The proof of the next lemma is straightforward and we can omit it.

**Lemma 5.2.** Let $\{ \mathcal{R}_i \}_{i \in I}$ be a family of $R$-varieties such that $\max(\mathcal{R}_i) = \Delta$ for all $i \in I$. Then $\bigcap_{i \in I} \mathcal{R}_i$ is an $R$-variety and $\max(\bigcap_{i \in I} \mathcal{R}_i) = \Delta$.

The following result says us that there exists the smallest $R$-variety which contains a given family of numerical semigroups and has a certain maximum.

**Proposition 5.3.** Let $\mathcal{F}$ be a family of numerical semigroups and let $\Delta$ be a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$. Then there exists the smallest $R$-variety which contains $\mathcal{F}$ and with maximum equal to $\Delta$.

**Proof.** Let $\mathcal{R}$ be the intersection of all the $R$-varieties containing $\mathcal{F}$ and with maximum equal to $\Delta$. From Lemmas 5.1 and 5.2 we have the conclusion.

We will denote by $\mathcal{R}(\mathcal{F}, \Delta)$ the $R$-variety given by Proposition 5.3. Now we are interested in describe the elements of such an $R$-variety.

**Lemma 5.4.** Let $S_1, S_2, \ldots, S_n, \Delta$ be numerical semigroups such that $S_i \subseteq \Delta$ for all $i \in \{1, \ldots, n\}$. Then $F(\Delta)(S_1 \cap \cdots \cap S_n) = \max \{ F(\Delta)(S_1), \ldots, F(\Delta)(S_n) \}$.

**Proof.** We have that $F(\Delta)(S_1 \cap \cdots \cap S_n) = \max (\Delta \setminus (S_1 \cap \cdots \cap S_n)) = \max ((\Delta \setminus S_1) \cup \cdots \cup (\Delta \setminus S_n)) = \max \{ \max(\Delta \setminus S_1), \ldots, \max(\Delta \setminus S_n) \} = \max \{ F(\Delta)(S_1), \ldots, F(\Delta)(S_n) \}$.

Let us recall that, if $S$ and $\Delta$ are numerical semigroups such that $S \subseteq \Delta$, then we defined $C(S, \Delta)$ in Example 2.3 (that is, the chain of $S$ restricted to $\Delta$). If $\mathcal{F}$ is a family of numerical semigroups such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$, then we will denote by $C(\mathcal{F}, \Delta)$ the set $\bigcup_{S \in \mathcal{F}} C(S, \Delta)$.

**Theorem 5.5.** Let $\mathcal{F}$ be a family of numerical semigroups and let $\Delta$ be a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$. Then $\mathcal{R}(\mathcal{F}, \Delta)$ is the set formed by all the finite intersections of elements in $C(\mathcal{F}, \Delta)$.

**Proof.** Let $\mathcal{R} = \{ S_1 \cap \cdots \cap S_n \mid n \in \mathbb{N} \setminus \{0\} \text{ and } S_1, \ldots, S_n \in C(\mathcal{F}, \Delta) \}$. Having in mind that $\mathcal{R}(\mathcal{F}, \Delta)$ is an $R$-variety which contains $\mathcal{F}$ and with maximum equal to $\Delta$, we easily deduce that $\mathcal{R} \subseteq \mathcal{R}(\mathcal{F}, \Delta)$.

Let us see now that $\mathcal{R}$ is an $R$-variety. On the one hand, it is clear that $\Delta = \max(\mathcal{R})$ and that, if $S, T \in \mathcal{R}$, then $S \cap T \in \mathcal{R}$. On the other hand, let $S \in \mathcal{R}$ such that $S \neq \Delta$. Then $S = S_1 \cap \cdots \cap S_n$ for some $S_1, \ldots, S_n \in C(\mathcal{F}, \Delta)$. Now, from Lemma 5.4 we have that $F(S) = \max \{ F(S_1), \ldots, F(S_n) \}$ and, thus, $F\Delta(S_i) \leq F(S)$ for all $i \in \{1, \ldots, n\}$. Let us observe that, if $F(S) > F(S_i)$, then $S \cup \{ F(S) \} = S_i$. Moreover, if $F(S) = F(S_i)$, then we get $S \cup \{ F(S) \} = S_i \cup \{ F(S) \} \in C(\mathcal{F}, \Delta)$. Therefore, $S \cup \{ F(S) \} \in C(\mathcal{F}, \Delta)$ for all $i \in \{1, \ldots, n\}$. Since $S \cup \{ F(S) \} \cap \cdots \cap S_n \cup \{ F(S) \}$, then $S \cup \{ F(S) \} \in \mathcal{R}$. Consequently, $\mathcal{R}$ is an $R$-variety.

Finally, since $\mathcal{R}$ is an $R$-variety which contains $\mathcal{F}$ and with maximum equal to $\Delta$, then $\mathcal{R}(\mathcal{F}, \Delta) \subseteq \mathcal{R}$ and, thereby, we conclude that $\mathcal{R} = \mathcal{R}(\mathcal{F}, \Delta)$.
Let us observe that, if $\mathcal{F}$ is a finite family, then $C(\mathcal{F}, \Delta)$ is a finite set and, therefore, $\mathcal{R}(\mathcal{F}, \Delta)$ is a finite $R$-variety.

**Lemma 5.6.** Let $\mathcal{R}$ and $\mathcal{R}'$ be two $R$-varieties. If $\mathcal{R} \subseteq \mathcal{R}'$, then $\Delta(\mathcal{R}) \subseteq \Delta(\mathcal{R}')$.

*Proof.* If $\mathcal{R} \subseteq \mathcal{R}'$, then $\Delta(\mathcal{R}) \subseteq \mathcal{R}'$. Therefore, $\Delta(\mathcal{R}) \subseteq \Delta(\mathcal{R}')$. \hfill $\square$

The next example shows us that, in general, we cannot talk about the smallest $R$-variety which contains a given family of numerical semigroups.

**Example 5.7.** Let $\mathcal{F} = \{(5,6), (5,7)\}$. As a consequence of Lemma 5.6, the candidate to be the smallest $R$-variety which contains $\mathcal{F}$ must have as maximum the numerical semigroup $(5,6,7)$ (that is, the smallest numerical semigroup containing $(5,6)$ and $(5,7)$). Thus, the candidate to be the smallest $R$-variety which contains $\mathcal{F}$ is $\mathcal{R}(\mathcal{F}, (5,6,7))$.

Let us see now that $\mathcal{R}(\mathcal{F}, (5,6,7)) \not\subseteq \mathcal{R}(\mathcal{F}, (5,6,7,8))$ and, in this way, that there does not exist the smallest $R$-variety which contains $\mathcal{F}$. In order to do it, we will show that $(5,6,7,8) \not\in \mathcal{R}(\mathcal{F}, (5,6,7,8))$. In fact, by applying Theorem 5.5, if $(5,6,7) \not\in \mathcal{R}(\mathcal{F}, (5,6,7,8))$, then we deduce that there exist $S_1 \in C((5,6),(5,6,7,8))$ and $S_2 \in C((5,7),(5,6,7,8))$ such that $S_1 \cap S_2 = (5,6,7)$. Since $S_1 \in C((5,6),(5,6,7,8))$, then there exists $n_1 \in \mathbb{N}$ such that $S_1 = (5,6) \cup \{x \in (5,6,7,8) \mid x \geq n_1\}$. Moreover, $(5,6,7) \subseteq S_1$ and, thereby, $n_1 \leq 7$. Consequently, $8 \in S_1$. By an analogous reasoning, we have that $8 \in S_2$ too. Consequently, $8 \in S_1 \cap S_2 = (5,6,7)$, which is false.

Let $\mathcal{R}$ be an $R$-variety. We will say that $\mathcal{F}$ (subset of $\mathcal{R}$) is a system of generators of $\mathcal{R}$ if $\mathcal{R} = \mathcal{R}(\mathcal{F}, \Delta(\mathcal{R}))$. Let us observe that, in such a case, $\mathcal{R}$ is the smallest $R$-variety which contains $\mathcal{F}$ and with maximum equal to $\Delta(\mathcal{R})$.

We will say that an $R$-variety, $\mathcal{R}$, is finitely generated if there exists a finite set $\mathcal{F} \subseteq \mathcal{R}$ such that $\mathcal{R} = \mathcal{R}(\mathcal{F}, \Delta(\mathcal{R}))$ (that is, if $\mathcal{R}$ has a finite system of generators). As a consequence of Theorem 5.5 we have the following result.

**Corollary 5.8.** An $R$-variety is finitely generated if and only if it is finite.

From now on, $\mathcal{F}$ will denote a family of numerical semigroups and $\Delta$ will denote a numerical semigroup such that $S \subseteq \Delta$ for all $S \in \mathcal{F}$. Our purpose is to give a method in order to compute the minimal $\mathcal{R}(\mathcal{F}, \Delta)$-system of generators of a $\mathcal{R}(\mathcal{F}, \Delta)$-monoid by starting from $\mathcal{F}$ and $\Delta$.

If $A \subseteq \Delta$, then for each $S \in \mathcal{F}$ we define

$$
\alpha(S) = \begin{cases} 
S, & \text{if } A \subseteq S, \\
S \cup \{x \in \Delta \mid x \geq x_S\}, & \text{if } A \nsubseteq S,
\end{cases}
$$

where $x_S = \min\{a \in A \mid a \notin S\}$.

As a consequence of Lemma 3.2 and Theorem 5.5 we have the next result.

**Lemma 5.9.** The $\mathcal{R}(\mathcal{F}, \Delta)$-monoid generated by $A$ is $\bigcap_{S \in \mathcal{F}} \alpha(S)$.

Recalling that $\mathcal{R}(\mathcal{F}, \Delta)(A)$ denotes the $\mathcal{R}(\mathcal{F}, \Delta)$-monoid generated by $A$, we have the following result.
Proposition 5.10. If $A \subseteq \Delta$, then $B = \{x_S \mid S \in \mathcal{F} \text{ and } A \nsubseteq S\}$ is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$-system of generators of $\mathcal{R}(\mathcal{F}, \Delta)(A)$.

Proof. Let us observe that, if $S \in \mathcal{F}$, then $A \subseteq S$ if and only if $B \subseteq S$. Moreover, if $A \nsubseteq S$, then $\min\{a \in A \mid a \notin S\} = \min\{b \in B \mid b \notin S\}$. Therefore, by applying Lemma 5.9 we have that $\mathcal{R}(\mathcal{F}, \Delta)(A) = \mathcal{R}(\mathcal{F}, \Delta)(B)$. Consequently, in order to prove that $B$ is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$-system of generators of $\mathcal{R}(\mathcal{F}, \Delta)(A)$, will be enough to see that, if $C \subset B$, then $\mathcal{R}(\mathcal{F}, \Delta)(C) \neq \mathcal{R}(\mathcal{F}, \Delta)(A)$.

In effect, if $C \subset B$, then there exists $S \in \mathcal{F}$ such that $x_S \notin C$ and, thereby, we have that $C \subseteq S$ or that $\min\{c \in C \mid c \notin S\} > x_S$. Now, by applying once more time Lemma 5.9, we easily deduce that $x_S \notin \mathcal{R}(\mathcal{F}, \Delta)(C)$. Since $x_S \in B \subseteq A$, then we get that $A \nsubseteq \mathcal{R}(\mathcal{F}, \Delta)(C)$ and, therefore, $\mathcal{R}(\mathcal{F}, \Delta)(C) \neq \mathcal{R}(\mathcal{F}, \Delta)(A)$. 

As an immediate consequence of the above proposition we have the next result.

Corollary 5.11. Every $\mathcal{R}(\mathcal{F}, \Delta)$-monoid has $\mathcal{R}(\mathcal{F}, \Delta)$-range less than or equal to the cardinality of $\mathcal{F}$.

We will finish this section by illustrating its content with an example.

Example 5.12. Let $\mathcal{F} = \{\langle 5, 7, 9, 11, 13 \rangle, \langle 4, 10, 11, 13 \rangle\}$ and $\Delta = \{4, 5, 7\}$. We are going to compute the tree $G(\mathcal{R}(\mathcal{F}, \Delta))$.

First of all, to compute the minimal $\mathcal{R}(\mathcal{F}, \Delta)$-system of generators of $\langle 4, 5, 7 \rangle$, we apply Proposition 5.10 with $A = \{4, 5, 7\}$. Since $x_{\langle 5, 7, 9, 11, 13 \rangle} = 4$ and $x_{\langle 4, 10, 11, 13 \rangle} = 5$, then $\{4, 5\}$ is such a minimal $\mathcal{R}(\mathcal{F}, \Delta)$-system. Now, because $\mathcal{F}_\Delta(\langle 4, 5, 7 \rangle) = -1$, and by applying Theorem 4.2 we get that $\langle 4, 5, 7 \rangle$ has two children, $\langle 4, 5, 7 \rangle \setminus \{4\} = \langle 5, 7, 8, 9, 11 \rangle$ (with $\mathcal{F}_\Delta(\langle 5, 7, 8, 9, 11 \rangle) = 4$) and $\langle 4, 5, 7 \rangle \setminus \{5\} = \langle 4, 7, 9, 10 \rangle$ (with $\mathcal{F}_\Delta(\langle 4, 7, 9, 10 \rangle) = 5$).

Now, if we take $A = \{5, 7, 8, 9, 11\}$ in Proposition 5.10, then we have that $x_{\langle 5, 7, 9, 11, 13 \rangle} = 8$ and $x_{\langle 4, 10, 11, 13 \rangle} = 5$. Thus, we conclude that $\{5, 8\}$ is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$-system of $\langle 5, 7, 8, 9, 11 \rangle$. Moreover, since $\mathcal{F}_\Delta(\langle 5, 7, 8, 9, 11 \rangle) = 4$, then Theorem 4.2 asserts that $\langle 5, 7, 8, 9, 11 \rangle \setminus \{5\} = \langle 7, 8, 9, 10, 11, 12, 13 \rangle$ (with $\mathcal{F}_\Delta(\langle 7, 8, 9, 10, 11, 12, 13 \rangle) = 5$) and $\langle 5, 7, 8, 9, 11 \rangle \setminus \{8\} = \langle 5, 7, 9, 11, 13 \rangle$ (with $\mathcal{F}_\Delta(\langle 5, 7, 9, 11, 13 \rangle) = 8$) are the two children of $\langle 5, 7, 8, 9, 11 \rangle$.

With $A = \{4, 7, 9, 10\}$, we get that $\{4, 7\}$ is the minimal $\mathcal{R}(\mathcal{F}, \Delta)$-system of $\langle 4, 7, 9, 10 \rangle$. By recalling that $\mathcal{F}_\Delta(\langle 4, 7, 9, 10 \rangle) = 5$, we conclude that $\langle 4, 7, 9, 10 \rangle$ has only one child, that is $\langle 4, 7, 9, 10 \rangle \setminus \{7\} = \langle 4, 9, 10, 11 \rangle$ (with $\mathcal{F}_\Delta(\langle 4, 9, 10, 11 \rangle) = 7$).

By repeating the above process, we get the whole tree $G(\mathcal{R}(\mathcal{F}, \Delta))$. 

19
Now, we are going to represent the vertices of $G(\mathcal{R}(\mathcal{F}, \Delta))$ using their minimal $\mathcal{R}(\mathcal{F}, \Delta)$-systems of generators. Moreover, we add to each vertex the corresponding Frobenius number restricted to $\Delta$. Thus, we clarify all the steps to build the tree $G(\mathcal{R}(\mathcal{F}, \Delta))$.

6 Conclusion

We have been able to give a structure to certain families of numerical semigroups which are not (Frobenius) varieties or (Frobenius) pseudo-varieties. For that we have generalized the concept of Frobenius number to the concept of restricted Frobenius number and, then, we have defined the $\mathcal{R}$-varieties (or (Frobenius) restricted variety).

After studying relations among varieties, pseudo-varieties, and $\mathcal{R}$-varieties, we have introduced the concepts of $\mathcal{R}$-monoid and minimal $\mathcal{R}$-system of generators of a $\mathcal{R}$-monoid, which lead to associate a tree with each $\mathcal{R}$-variety and, in consequence, obtain recurrently all the elements of an $\mathcal{R}$-variety.

Finally, although in general it is not possible to define the smallest $\mathcal{R}$-variety that contains a given family $\mathcal{F}$ of numerical semigroups, we have been able to
give an alternative when we fix in advance the maximum of the smallest $R$-variety.

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