Noncommutative Differential Geometry and Classical Field Theory on Finite Groups *

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Abstract
Plan of this report is given below
1. Motivation from Physical and Mathematical Point of View
2. Differential Calculi on Finite Groups
3. Metrics
4. Lagrangian Field Theory and Symplectic Structure
5. Scalar Field Theory and Spectral of Finite Groups

I Introduction

Finite groups provide a type of simple, nevertheless characterizing enough models for noncommutative geometry (NCG) from both physical and mathematical point of views.

Physically, both the problem which geometry would possess a physical realization, hence being descended to the real world, and the problem which geometry could be abstracted out from the realistic recognition of the physical laws are profound and ever being asked by generations of physicists who pursue the perfect unity between the laws and the languages. Microscopic description of the space-time, whose picture, after the great success of general relativity, was legalized as a differential Lorentzian 4-manifold, was considered bit by bit by some outstanding theorists, which can be categorized into two lines according to Madore

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1. Modification of the structure of space-time, which can be traced back to Dirac [2], Snyder [3], Yang [4];

2. Extension of space-time, to which immediately one could recall Kaluza and Klein [5], Manton [6] and, appearing ten years ago, Connes and Lott [7].

Along both lines, noncommutative geometry will play a crucial role as the linguistic foundation, once if the geometric picture deviates from a manifold, being differential or just being topological. And finite groups are by far among those which appear in both circumstances as the most convenient and manipulable specific models.

Mathematically, the famous *Gelfand-Naimark theorem* bridges (locally) compact Hausdorff spaces on the geometric bank and (unital) commutative $C^*$-algebras on the algebraic bank [8]. The philosophy that people learn from this relation is that geometric information is able to be encoded in algebraic structure; therefore, the generalization of this result to noncommutative regime gives birth to noncommutative geometry [9][1]. To mathematicians, NCG is not only a novel method employing fully the power of algebraic tools, whose origin could be even retrieved to Descartes, but also providing a class of completely new math objects, hence a completely new scientific scope. Within all kinds of approaches towards the implementation of the NCG idea, the noncommutative differential geometry on finite groups is, in our understanding, the most accessible, however by no means being trivial.

This report is organized in the following way. First in Sect. II, mathematical foundation of NCG over finite groups is established. Then Section III is dedicated to consider metrics in NCG over finite groups. As physical applications, lagrangian field theory over finite groups is derived in Section IV, together with an induced (multi-)symplectic structure, while harmonic analysis of finite groups is considered in Sect. V. Some open discussions are put in Sect. VI.

II Differential Calculi on Finite Groups

Mathematical framework of NCG for finite sets, especially for finite groups, is developed in a series of work [10], and here the formalism mainly follows [11].
II.1 Group Function

Let $G$ be a finite set. $\mathcal{A}(G)$ is the algebra of complex functions on $G$, whose multiplication is defined pointwisely. The basis of $\mathcal{A}(G)$ is a collection of delta functions

$$e^g(a) = \delta_a^g, \forall g, a \in G$$

and the algebraic structure of $\mathcal{A}(G)$ is characterized by

$$e^g \cdot e^h = e^{g\delta_h}, \sum_g e^g = 1$$

where 1 is the multiplication unit of $\mathcal{A}(G)$. A group structure can be implemented on $G$ by extending $\mathcal{A}(G)$ to be a Hopf algebra, namely for all $f, g \in \mathcal{A}(G)$ and $a, b \in G$,

$$M : \mathcal{A}(G) \otimes \mathcal{A}(G) \to \mathcal{A}(G), M(f, g)(a) = f(a)g(a), M(f, g) \equiv f \cdot g \equiv fg$$

$$\Delta : \mathcal{A}(G) \to \mathcal{A}(G) \otimes \mathcal{A}(G), \Delta f(a, b) = f(m(a, b)), m : G \otimes G \to G, (a, b) \mapsto m(a, b) =: ab$$

$$1' : \mathcal{A}(G) \to \mathbb{C}, f \mapsto f(e), e \in G$$

$$s : \mathcal{A}(G) \to \mathcal{A}(G), s(f)(a) = f(\iota(a)), \iota \in ISO(G)$$

in which $e$ is a specified element in $G$ and $ISO(G)$ is the isomorphism group of $G$ as a set. It is easy to verify this follow claim.

**Proposition 1** \[11\] $G$ is a group under $(m, e, \iota)$ iff $\mathcal{A}(G)$ is a Hopf algebra under $(M, \Delta, 1', s)$, i.e. the following consistent conditions are satisfied

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

(1)

$$(1' \otimes id)\Delta = id = (id \otimes 1')\Delta$$

(2)

$$M(id \otimes s)\Delta = 1' = M(s \otimes id)\Delta$$

(3)

**Remark 1** Eqs.(1)(2)(3) as properties of Hopf algebra correspond to associativity, existence of unit and existence of inverse map, which are nothing but group axioms. Therefore, the group structure on $G$ can be encoded into the structure of Hopf algebra.

Left and right actions on $G$ are set isomorphisms of $G$ defined as

$$L_g(g') := gg', R_gg' := g'g, \forall g, g' \in G$$

which can be pulled back to be $L_g^*, R_g^*$ acting on $\mathcal{A}(G)$ in the canonical way.

Note here we introduce the notation $\hat{g} := \iota(g)$ and will use below.
II.2 First Order Differential Forms

Definition 1 A first order differential calculus is a pair \((M, d)\), in which \(M\) is a bi-module over \(\mathcal{A}(G)\) and \(d\) is a linear homomorphism called differential \(d : \mathcal{A}(G) \to M\) satisfying Leibnitz rule

\[
d(f f') = d(f) f' + f d(f'), \forall f, f' \in \mathcal{A}(G) \tag{4}
\]

The action of \(1\) on \(M\) is required to be identity map. The elements in \(M\) are called first order differential forms.

Universal first order differential is defined by \(\Omega^1_u(A(G)) := \mathcal{A}(G) \otimes \mathcal{A}(G)/\sim\), and \(d_u f = 1 \otimes f - f \otimes 1\), \(d_u f(a, b) = f(b) - f(a)\) where the equivalent relation is given by \(F(a, a) \sim 0\), \(\forall F \in \mathcal{A}(G) \otimes \mathcal{A}(G)\).

Proposition 2 (Universality of \((\Omega^1_u, d_u)\)) \[13\]

There exists a linear homomorphism \(\mathcal{R}\) from \(\Omega^1_u\) to any \(M\), which is referred as reduction map, such that the follow diagram commutes

\[
\begin{array}{ccc}
\mathcal{A}(G) & \xrightarrow{\cdot} & (\Omega^1_u, d) \\
\downarrow & & \downarrow \mathcal{R} \\
(M, d) & &
\end{array}
\]

Remark 2 The structure of \((M, d)\) can be descended from the universal forms by reduction, namely a collection of annihilation relation \(e^g d(e^h) = 0 = \mathcal{R}(e^g d_u(e^h))\) for some pairs \((g, h)\).

Left actions are pulled back onto \(\Omega^1_u\) by

\((L^*_g \omega)(a, b) = \omega(ga, gb), \forall g, a, b \in G, \omega \in \Omega^1_u\)

To generic \(M\), left actions are defined by the following rules

\(L^*_g(fv) = L^*_g(f)L^*_g(v), L^*_g d = dL^*_g\)

for all \(g \in G, f \in \mathcal{A}(G), v \in M\). Similar for inducing right actions onto \(M\). Left-invariant forms in \(\Omega^1_u\) are defined by

\((L^*_g \omega)(a, b) = \omega(a, b)\)

It is obviously that all left-invariant forms form a linear subspace in \(\Omega^1_u\).

Proposition 3 \[13\] The space of left-invariant forms is \((|G| - 1)\)-dimensional, whose basis is

\[\chi^g = (s \otimes id) \Delta(e^g), g \neq e\]

This basis is also a module basis.

Similarly, right-invariant forms are defined as

\[\eta^g = (id \otimes s) \Delta(e^{id(g)})\]

Below just left-invariant forms will be used.
Proposition 4

\[ d_u f = \sum_{g \in G} \partial_g f \chi^g, \text{ where } \partial_g = R_g^* - \text{id} \text{ and } G' := G \setminus \{e\}; \]

\[ d \omega \text{ in which } \omega \] is that \( \lambda \) for any interpolation \( \lambda \); however, Eqs. (4) are compatible iff \( \lambda = 1 \). Moreover, even if deformed Leibnitz rule Eq. (3) is adopted, \( \{ \partial_g \} \) is not able to be extended to be a left \( \mathcal{A}(G) \)-module, which is the well-known property of \( \text{Der}(C^\infty(V)) \) of a differential manifold \( V \).

A reduction is left-invariant, if there is a subset \( G'' \subset G' \) such that \( \mathcal{R}(\chi^g) = 0 \), for all \( g \in G' \setminus G'' \). We will not distinguish left invariant basis in \( M \) and in universal forms below.

II.3 High Order Forms

The space of universal \( p \)-forms is given by \( \Omega^p_u(\mathcal{A}(G)) = \bigotimes^{(p+1)} \mathcal{A}(G)/\sim \), where the equivalent relation is that \( \omega(a_0, a_1, ..., a_p) \sim 0 \) whenever \( a_i = a_{i+1} \) for one \( i \in \{0, 1, 2, ..., p - 1\} \), and the differential is \( d_u : \Omega^{p-1}_u \rightarrow \Omega^p_u \),

\[ (d_u \omega_{p-1})(a_0, a_1, ..., a_p) = \sum_{k=0}^{p} (-)^k \omega_{p-1}(a_0, a_1, ..., \hat{a}_k, ..., a_p) \]

in which \( \omega_{p-1}(a_0, a_1, ..., \hat{a}_k, ..., a_p) = \omega_{p-1}(a_0, a_1, ..., a_{k-1}, a_{k+1}, ..., a_p) \).

Universal differential algebra is \( \Omega^*_u(\mathcal{A}(G)) = \bigoplus_p \Omega^p_u(\mathcal{A}(G)) \) where \( \Omega^0_u(\mathcal{A}(G)) := \mathcal{A}(G) \), together with the differential \( d \). A direct calculation shows that

Proposition 5

\[ d \circ d = 0 \]

\[ d(\omega^{(p)} \omega') = d(\omega^{(p)}) \omega' + (-)^p \omega^{(p)} d(\omega') \]

in which \( \omega^{(p)} \in \Omega^p_u, \omega' \in \Omega^*_u \).

Proposition 6

The cohomology \( (\Omega^*_u(G), d) \) is trivial.

Proof:

In fact, consider \( \omega_p \in Z^p(\mathcal{A}(G)) \), i.e. \( d\omega_p = 0 \), define \( \eta_{p-1} \in \Omega^{p-1}_u(\mathcal{A}(G)) \) by

\[ \eta_{p-1}(a_1, a_2, ..., a_p) = \omega_p(e, a_1, a_2, ..., a_p) \]
One can check then that $dη_{p-1} = ω_p$. □

**Remark 4** Non-trivial topology will emerge if reduction is extended to $Ω^∗(A(G))$ by requiring $d$ to be nilpotent and graded Leibnitz still. The resulting quotient algebra is a differential calculus on $A(G)$, which is denoted as $Ω^∗(A(G))$. In this paper, only the extensions of left-invariant reductions will be considered.

**III Metrics and Covariant Reductions**

**Definition 2** An involution $†$ on $Ω^∗(A(G))$ is defined as

$$f^†(g) = \overline{f(g)}, (dω(p))^† = (-)^p d(ω(p)^†), (ωω')^† = ω'^†ω^†$$

∀$f \in A(G)$, $ω(p) \in Ω^p$, $ω, ω' \in Ω^∗$.

**Lemma 1** i) For any $g \in G''$, $(χ^g)^† = -χ^g$;

ii) $X^† = X ⇔ R^g(\overline{X^g}) + X^g = 0$;

iii) If $X \in Ω^1(G)$ is of real coefficients, to which we refer as real, and exact, then

$$X^† = X.$$  (7)

Therefore, a reduction is compatible with involution unless $R(χ^g) = 0 ⇔ R(\overline{χ^g}) = 0$.

**Definition 3** A metric on $G$ is given by $ξ : Ω^1 ⊗ A(G) \rightarrow A(G)$, if ∀$X, Y, Z \in Ω^1$, $a \in A(G)$, $ξ$ satisfies

$$ξ(X + Y, Z) = ξ(X, Z) + ξ(Y, Z), \quad ξ(X, Y + Z) = ξ(X, Y) + ξ(X, Z)$$  (8)

$$ξ(X, aY) = aξ(X, Y)$$  (9)

$$ξ(X^†, X) \text{ is real.}$$  (10)

**Remark 5** i) $ξ$ can be characterized by a set of functions $ξ^{gh}$; in fact, for ∀$X, Y \in Ω^1$

$$ξ(X, Y) = ξ(Xgχ^g, Yhχ^h) = ξ(χ^gR_gX_g, Y_hχ^h) = ξ(χ^g, (R_gX_g)Y_hχ^h)$$

$$= (R_gX_g)Y_hξ(χ^g, χ^h) =: (R_gX_g)Y_hξ^{gh}$$

ii) Similarly, $ξ(Y, X) = X_g(R_hY_h)ξ^{hg}$; therefore, $ξ(X, Y) ≠ ξ(Y, X)$ usually.

A Hermitian structure on $G$ can be identified as $ξ(X^†, Y)$.

**Theorem 1** (Symmetries of metric)

i) $ξ^{gh}$ are real functions;

ii) $ξ^{gh} = ξ^{hg}$. 

6
Proof:

\[ \xi(X^\dagger, X) = \xi((-\chi^g), X_\chi^h) = -X_\chi^h \xi^g h \]

Let \(X\) be real, then Eq. (10) implies that \(\xi^g h\) is real. So \(\xi(X, Y), \xi(X^\dagger, Y)\) are real, if \(X, Y\) are real 1-forms. Let \(X\) be exact, and it can be decomposed into two real exact 1-forms as \(X = X^R + iX^I\), with \(X^\dagger = X^R - iX^I\) followed from Eq.(7)

\[ \xi(X^\dagger, X) = \xi(X^R - iX^I, X^R + iX^I) \]

Therefore, \(\xi(X^R, X^I) - \xi(X^I, X^R)\) has to vanish.

Remark 6 (Consistency)

Now three algebraic structures, reduction, involution and metric, are specified to \(G\). For each reduction, there is a constrain on metric which singles out some metrics compatible with this reduction, together with involution, i.e. \(R(\chi^g) = 0 \Rightarrow \xi(\chi^g, \chi^h) = 0, \xi(\chi^h, \chi^g) = 0, \forall h \in G\); following theorem \(\xi^g h\), there we have that \(\xi(\chi^g, \chi^h) = 0, \xi(\chi^h, \chi^g) = 0, \forall h \in G\). On the other hand, if \(\exists g, s.t. \forall h, \xi^g h = 0 = \xi^h g\), we can induce a reduction which set \(\chi^g = 0\); also following the theorem above, \(\xi^g h = \xi^h g = 0\), implying that \(\chi^g = 0\).

Definition 4 A reduction, involution and a metric are compatible, if

\[ \chi^g = 0 \Rightarrow \chi^g = 0, \xi^g h = \xi^h g = \xi^g h = \xi^h g = 0 \]

Let \(\sigma : G \rightarrow G\) be a bijection, the metric/Hermitian structure is said to be \(\sigma\)-covariant if

\[ \sigma^* (\xi(X^\dagger, Y)) = \xi(\sigma^*(X^\dagger), \sigma^*(Y)), \forall X, Y \in \Omega^1, \]

which implies that

\[ \xi^g h (\sigma(g)) = \xi(\sigma^*(\chi^g), \sigma^*(\chi^g))(g), \forall g \in G \]

(11)

Note that such transformations have no infinitesimal forms. Below we consider three type of bijections.

- \(\sigma\) is a left-translation \(L_h\),

\[ (\mathbb{L}) \Rightarrow \xi^g h (hg) = \xi^g h (g) \]

i.e. \(\xi^g h\)'s are constant functions on \(G\), which can be denoted as \(\eta^g h\);
• \( \sigma \) is a right-translation \( R_h \),
\[ (11) \Rightarrow \eta \tilde{g}_1 g_2 = \eta^{Ad_h(\tilde{g}_1) Ad_h(g_2)}; \]

• \( \sigma \in \text{Aut}(G) \),
\[ (11) \Rightarrow \eta \tilde{g}_1 g_2 = \eta^{\sigma^{-1}(\tilde{g}_1) \sigma^{-1}(g_2)} \]

Define \( \gamma^{gh} := \eta^{h} \) which just is a change of symbols, then our analysis can be summarized as
\[ \gamma^{gh} = \gamma^{hg}, \gamma^{gh} = \gamma^{\sigma(g) \sigma(h)}, \]
which implies that if for any \( g \in G, \sigma \in \text{Aut}(G) \), if a reduction scheme sets \( R(\chi^g) = 0 \), then \( \chi^{\sigma(g)} \) has to vanish.

**Definition 5** A reduction is covariant, if \( G'' \) contains \( \text{Aut}(G) \)-orbits only.

A covariant metric should be defined upon a covariant reduction. In Table I, covariant reductions for some finite (generated) groups are listed.

|   |   |   |   |
|---|---|---|---|
|   |   |   |   |

**Table I: Covariant Reductions for Some Finite (Generated) Groups**

|   |   |   |   |
|---|---|---|---|
|   |   |   |   |

**IV Lagrangian Field Theory**

The following two sections are dedicated to physics of classical field theory on finite groups. First integral on \( \mathcal{A}(G) \) is given by
\[ \int_G f = \frac{1}{|G|} \sum_{g \in G} f(g), \forall f \in \mathcal{A}(G) \]
Action functional of a scalar field is $S \in (\mathcal{A}(G) \oplus \Omega^1(\mathcal{A}(G)))^*_\mathbb{R}$. Following continuum case, a locality principle is stated as that there exists a lagrangian density $\mathcal{L}$, such that

$$S(f, \omega) = \int_G \mathcal{L}(f, \omega),$$

and

$$\frac{\partial S}{\partial f(a)} = \frac{\partial \mathcal{L}(a)}{\partial f} = \frac{\partial \mathcal{L}(a)}{\partial \omega_g(a)}, \quad \frac{\partial S}{\partial \omega_g} = \frac{\partial \mathcal{L}}{\partial \omega_g}(a)$$

where $\omega \in \Omega^1$ will be take to be $df$ eventually.

**Variation principle** acting on the space of $\Omega^0 \otimes \Omega^1$ by

$$\begin{align*}
\hat{\delta} \mathcal{L}(a) &= \frac{\partial \mathcal{L}(a)}{\partial f} \hat{\delta}(f(a)) + \frac{\partial \mathcal{L}(a)}{\partial \omega_g} \hat{\delta}(\partial_g f(a)) \\
&= \frac{\partial \mathcal{L}(a)}{\partial f} \hat{\delta}(f(a)) + \partial_g [R^*_g (\frac{\partial \mathcal{L}}{\partial \omega_g}) \hat{\delta} f](a) - (\partial_g R^*_g \frac{\partial \mathcal{L}}{\partial \omega_g})(a) \hat{\delta}(f(a)) \\
&= (\frac{\partial \mathcal{L}(a)}{\partial f}) + (\partial_g \frac{\partial \mathcal{L}}{\partial \omega_g}) \hat{\delta}(f(a)) + \partial_g [R^*_g (\frac{\partial \mathcal{L}}{\partial \omega_g}) \hat{\delta} f](a)
\end{align*}$$

where $\lambda$ in Eq.(3) is chosen to be zero, and an identity $R^*_g \partial_g + \partial_g = 0$ is used.

**Remark 7** $\hat{\delta}(f(a))$ is differential form upon the space of field variables, $\hat{\delta}((\partial_g f)(a)) = \hat{\delta}(f(a)) = \hat{\delta}(f(a)) - \hat{\delta}(f(a))$. Introduce a form-valued function $\hat{\delta} f(a) = \hat{\delta}(f(a))$, then $\hat{\delta}((\partial_g f)(a)) = (\partial_g \hat{\delta} f)(a)$.

Equation of motion can be read out

$$E(a) := \frac{\partial \mathcal{L}(a)}{\partial f} + (\partial_g \frac{\partial \mathcal{L}}{\partial \omega_g})(a) = 0$$

Twice acting $\hat{\delta}$ on lagrangian gives the result

$$\hat{\delta}^2 \mathcal{L}(a) = \hat{\delta}(E(a)) \wedge \hat{\delta}(f(a)) + \partial_g [\hat{\delta}(R^*_g (\frac{\partial \mathcal{L}}{\partial \omega_g})) \wedge \hat{\delta} f](a) = 0$$

in which the second term contains a multi-symplectic structure over finite groups.

**Theorem 2** (Noether theorem on finite groups)

If $\mathcal{L}$ is invariant under a (vertical) continue transformation $\sigma$ so that $\delta_\sigma f$ is an infinitesimal (not a symbolic differential in above discussion), i.e. $\delta_\sigma \mathcal{L} = \partial_g k^g$, then we have

$$\partial_g J^g = 0, J^g := (R^*_g (\frac{\partial \mathcal{L}}{\partial \omega_g})) \delta_\sigma f + k^g$$

**Proof:**

In fact,

$$\delta_\sigma \mathcal{L} = \partial_g k^g = \frac{\partial \mathcal{L}}{\partial f} \delta_\sigma f + \frac{\partial \mathcal{L}}{\partial \omega_g} \delta_\sigma \partial_g f$$

$$= (\partial_g (R^*_g (\frac{\partial \mathcal{L}}{\partial \omega_g}))) \delta_\sigma f + (R^*_g R^*_g (\frac{\partial \mathcal{L}}{\partial \omega_g})) \partial_g \delta_\sigma f$$
\[\partial g((R^g \frac{\partial L}{\partial \omega_g}) \delta_{\sigma_f})\]

Remark 8 The lesson of this theorem is that there exists a subtlety for the relation between conservation laws and symmetries, namely space-time symmetry, since being discrete, e.g. hyper-cubic symmetry \(\mathbb{Z}_2^d \times S_d\) does imply a conservation currents as in continuum.

V Scalar Field Theory and the Spectral of Finite Groups

It is a novel fact that the spectral of Laplacian operator on a Riemannian manifold, encodes much, if not all, geometric information of this manifold [14]. In this section, the spectral of finite groups and their relations to representation theory are explored by defining Laplacian on these groups following the results of Sect. [4, 11, 15] and computing its eigenvalues.

A linear scalar field \(\phi\) is an element in \(\mathcal{A}(G)\). Let \(G\) be a finite group supported with a left-invariant reduction calculus \(G''\) and a covariant metric \(\xi\), then a classical scalar field theory is defined by the following action

\[S(\phi, d\phi) = \int_G (\xi(d\phi^1, d\phi) + V(\phi))\]
\[= -\int_G (\gamma_{g'g} \partial_g \phi \partial_{g'} \phi - V(\phi))\]
\[= -\int_G (\phi(\gamma_{g'g} \partial_g \partial_{g'} \phi) - V(\phi))\]

in which \(V(\phi)\) is a generic local function of \(\phi\). Laplacian \(\Delta_G\) specified to \((G, G'', \xi)\) is defined as

\[\Delta_G = \gamma_{g'g} \partial_g \partial_{g'}\]

and Laplacian equation is

\[\Delta_G \phi = \lambda^2 \phi\]

Eigenvalues \(\lambda^2\) form the spectral of \(G\). Detailed computations are given below for some examples appeared in table [6].

V.1 Spectral of \(\mathbb{Z}_2\)

Laplacian equation is given by

\[\partial \partial \phi = \mu^2 \phi\]
\[\left( R^* + \left( \frac{\lambda^2}{2} - 1 \right) \right) \phi = 0\]
\[\begin{pmatrix} \frac{\lambda^2}{2} - 1 & 1 \\ 1 & \frac{\lambda^2}{2} - 1 \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(1) \end{pmatrix} = 0\]
where two elements in $\mathbb{Z}_2$ are labeled by 0, 1 and 1 as subscription is omitted. Spectral of $\mathbb{Z}_2$ are computed to be

$$\lambda^2 = 0, 4$$

which corresponds to trivial and nontrivial representations of $\mathbb{Z}_2$ respectively, namely

$$\phi_{\lambda^2=0} = (1, 1)/\sqrt{2}$$
$$\phi_{\lambda^2=4} = (1, -1)/\sqrt{2}$$

### V.2 Spectral of $\mathbb{Z}_3$

Define $\mathbb{Z}_3 = \{0, +, -\}$ and the covariant metric on $\mathbb{Z}_3$ is

$$[\gamma_{gg'}] = \begin{pmatrix} \gamma & \gamma' \\ \gamma' & \gamma \end{pmatrix}$$

where $\gamma, \gamma'$ are free real parameters. Laplacian equation reads

$$(2\gamma\partial_+\partial_- + \gamma'(\partial_+\partial_+ + \partial_-\partial_-) - \lambda^2)\phi = 0$$
$$((2\gamma + \gamma')(\partial_+ + \partial_-) + \lambda^2)\phi = 0$$

Introduce symbols $\alpha := 2\gamma + \gamma'$, $\beta := 2\alpha - \lambda^2$, then Eq. (14) takes the form that

$$(\alpha(R^*_+ + R^*_-) - \beta)\phi = 0$$

Adopt matrix form,

$$\Phi := (\phi(0), \phi(+)\phi(-))^T$$

$$\square := \begin{pmatrix} -\beta & \alpha & \alpha \\ \alpha & -\beta & \alpha \\ \alpha & \alpha & -\beta \end{pmatrix}$$

Then Laplacian equation becomes

$$\square\Phi = 0$$

whose $\text{det}(\square) = -\beta^3 + 3\alpha^2\beta + 2\alpha^3$. Equation $\text{det}(\square) = 0$ gives the spectral

$$\beta = 2\alpha, -\alpha, -\alpha$$

$$\Leftrightarrow \lambda^2 = 0, 3\alpha, 3\alpha$$

Eigenvectors are

$$\phi_0 = (1, 1, 1),$$
$$\phi_{3\alpha}^{(1)} = (1, \omega, \omega^2),$$
$$\phi_{3\alpha}^{(2)} = (1, \omega^2, \omega),$$

corresponding to the three irreducible representations of $\mathbb{Z}_3$, where $\omega := -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the cubic root of unity.
V.3 Spectral of $\mathbb{Z}_4$

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$, and reduction is chosen to be $\{\chi^1, \chi^2, \chi^3\}$. Then metric with four free parameters is

$$[\gamma_{gg'}] = \begin{pmatrix} \gamma^1 & \gamma^2 & \gamma^3 \\ \gamma^2 & \gamma^4 & \gamma^2 \\ \gamma^3 & \gamma^2 & \gamma^1 \end{pmatrix}$$

together with Laplacian equation is given by

$$[2\gamma^1 \partial_1 \partial_3 + \gamma^4 \partial_2^2 + 2\gamma^2 (\partial_2 \partial_1 + \partial_2 \partial_3) + \gamma^3 (\partial_3^2 + \partial_1^2)] \phi = \lambda^2 \phi$$

$$(2(\gamma^1 + \gamma^3) (\partial_1 + \partial_3) + 2(\gamma^4 + \gamma^2 - \gamma^3) \partial_2 + \lambda^2) \phi = 0$$

whose spectral are

$$\lambda^2 = 0, 4(\gamma^1 + \gamma^4 + 2\gamma^2), 4(\gamma^1 + \gamma^4 + 2\gamma^2), 8(\gamma^1 + \gamma^3) \quad (15)$$

V.4 Spectral of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$

$\mathbb{Z}_2 \otimes \mathbb{Z}_2 = \{0, a, b, ab\}$, with reduction $\{\chi^a, \chi^b, \chi^{ab}\}$. Then two-parameter metric is

$$[\gamma_{gg'}] = \begin{pmatrix} \gamma' & \gamma & \gamma' \\ \gamma & \gamma & \gamma' \\ \gamma' & \gamma' & \gamma \end{pmatrix}$$

Laplacian equation

$$[\gamma (\partial_a^2 + \partial_b^2 + \partial_{ab}^2) + 2\gamma' (\partial_a \partial_b + \partial_b \partial_{ab} + \partial_{ab} \partial_a)] \phi = \lambda^2 \phi$$

$$(\partial_a + \partial_b + \partial_{ab} + \frac{\lambda^2}{2(\gamma + \gamma')}) \phi = 0$$

$$\frac{\lambda^2}{2(\gamma + \gamma')} = 0, 4, 4, 4$$

in contrasting with the result in Eq.(15).

V.5 Spectral of $\mathbb{Z}_5$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, and reduction is $\{\chi^1, \chi^2, \chi^3, \chi^4\}$.

$$[\gamma_{gg'}] = \begin{pmatrix} a & b & b & d \\ b & a & d & b \\ b & d & a & b \\ d & b & b & a \end{pmatrix}$$

Laplacian equation

$$[2a(\partial_1 \partial_4 + \partial_3 \partial_2) + 2b(\partial_1 \partial_2 + \partial_1 \partial_3 + \partial_2 \partial_4 + \partial_3 \partial_4) + d(\partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2)] \phi = \lambda^2 \phi$$
\[(2a + 2b + d) \sum_{i=1}^{4} \partial_i + \lambda^2)\phi = 0\]

Spectral \(\chi^2 = \frac{\lambda^2}{2a + 2b + d} = 0, 5, 5, 5\)

V.6 Spectral of \(\mathbb{Z}_6\)

There are two different six-element groups, \(Z_6 \text{ vs } S_3\), \(\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}\), and reduction is chose to be \(\{\chi^1, \chi^3, \chi^5\}\)

\[\gamma^{gg'} = \begin{pmatrix} \gamma_1 & \gamma & \sigma \\ \gamma & \gamma^3 & \gamma \\ \sigma & \gamma & \gamma^1 \end{pmatrix}\]

Laplacian equation

\[\begin{align*}
&\{2\gamma^1 \partial_0 \partial_1 + 4\gamma^3 \partial_0^2 + 2\gamma(\partial_0 \partial_1 + \partial_3 \partial_4) + \sigma(\partial_0^2 + \partial_1^2)\phi = \lambda^2 \phi \\
&(2(\gamma^1 + \gamma + \sigma)(\partial_0 + \partial_3) + 2(\gamma^3 + 2\gamma)(\partial_0 + \partial_4) + \lambda^2)\phi = 0
\end{align*}\]

Spectral

\[
\lambda^2 = 0, 3(2\gamma^1 + \sigma), 3(2\gamma^1 + \sigma), 4(2\gamma^1 + 2\sigma + \gamma^3 + 4\gamma), 2\gamma^1 + 4\gamma + 4\gamma^3 - \sigma, 2\gamma^1 + 4\gamma + 4\gamma^3 - \sigma
\] (16)

which contain two bi-degenerates.

V.7 Spectral of \(S_3\)

\(S_3 := \{e, a, a^2, \gamma, \gamma a, \gamma a^2\}\), which is the most simple non-Abelian group. Differential on \(S_3\) is given by

\[df = \partial_i f \chi^i + \partial_{\gamma a} f \chi^{\gamma a} + \partial_{\gamma a^2} f \chi^{\gamma a^2}\]

and metric is given by

\[
\begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{02} \\
\gamma_{10} & \gamma_{11} & \gamma_{12} \\
\gamma_{20} & \gamma_{21} & \gamma_{22}
\end{pmatrix}
= \begin{pmatrix}
\eta & \eta' & \eta' \\
\eta' & \eta & \eta' \\
\eta' & \eta' & \eta
\end{pmatrix}
\]

where \(\partial_0 := \partial_\gamma, \partial_1 := \partial_{\gamma a}, \partial_2 := \partial_{\gamma a^2}\). Laplacian equation over \(S_3\) is

\[(\eta(\partial_0 \partial_0 + \partial_1 \partial_1 + \partial_2 \partial_2) + \eta'(\partial_0 \partial_1 + \partial_0 \partial_2 + \partial_1 \partial_2 + \partial_2 \partial_0) + \partial_2 \partial_0 + \partial_2 \partial_0) - \lambda^2)\phi = 0\]

\[(3\eta'(\partial_0 + \partial_2) - 2(\eta + 2\eta')(\partial_0 + \partial_1 + \partial_2) - \lambda^2)\phi = 0\] (17)

Let \(c := 2(\eta + 2\eta'), d := 6\eta + 6\eta' - \lambda^2\), Eq. (17) is transformed as

\[(3\eta' (R^*_{\gamma a} + R^*_{\gamma a^2}) - c(R^*_{\gamma a} + R^*_{\gamma a} + R^*_{\gamma a^2}) + d)\phi = 0\]
Again define matrix formulated Laplacian equation as

$$
\Box := \begin{pmatrix}
d & 3\eta' & 3\eta' & -c & -c & -c \\
3\eta' & d & 3\eta' & -c & -c & -c \\
3\eta' & 3\eta' & d & -c & -c & -c \\
-c & -c & -c & d & 3\eta' & 3\eta' \\
-c & -c & -c & 3\eta' & d & 3\eta' \\
-c & -c & -c & 3\eta' & 3\eta' & d
\end{pmatrix},
$$

$$\Phi := (\phi(e), \phi(a), \phi(a^2), \phi(\gamma), \phi(\gamma a), \phi(\gamma a^2))^T$$

$$\Box \Phi = 0$$

Then

$$det(\Box) = -9c^2d^4 + d^6 + 108c^2d^3\eta' - 486c^2d^2\eta'^2 - 54d^4\eta'^2 + 972c^2d\eta'^3 + 108d^3\eta'^3 - 729c^2\eta'^4 + 729d^2\eta'^4 - 2916d\eta'^5 + 2916\eta'^6$$

Vanishing of determinant of $\Box$ gives rise to

$$d = 3\eta', 3\eta', 3\eta', 3\eta', -6\eta' - 3c, -6\eta' + 3c$$

$$\Leftrightarrow \lambda^2 = 6\eta + 3\eta', 6\eta + 3\eta', 6\eta + 3\eta', 12\eta + 24\eta', 0$$

which contain a quadruple-degenerate, contrasting with the spectral in Eq. (16). Eigenvectors are organized into three inequivalent irreducible representations of $S_3$

$$\Phi_{trivial} = 1 \oplus 1$$

$$\Phi_{alternative} = 1 \oplus (-1)$$

$$(\Phi_2)_1 = \begin{pmatrix} t \oplus 0 & 0 \oplus t \\ 0 \oplus \bar{t} & \bar{t} \oplus 0 \end{pmatrix}$$

in which any function in $\mathcal{A}(S_3)$ is written as $f = (f(e), f(a), f(a^2), f(\gamma), f(\gamma a), f(\gamma a^2)) =: f_+ \oplus f_-$, with $f_+ := (f(e), f(a), f(a^2))$, $f_- := (f(\gamma), f(\gamma a), f(\gamma a^2))$, and $1 := (1, 1, 1), t := (1, \omega, \omega^2)$.

**VI Discussion**

In conclusion, after geometric structures, i.e. differential calculi and metric, being introduced onto finite groups, field theory is able to be defined, as well as harmonic analysis is generalized on the category of finite groups.

The topics of high-tensor fields and spinor fields are not touched in this report. In fact, Yang-Mills field can be introduced as one-form [10]: however, to define fermions on a generic finite group will meet obstacles caused by the possible non-Abelian nature of groups. For Abelian cases, a formulation of fermions
is given in [15].

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