Some Existence Results for a Singular Elliptic Problem via Bifurcation Theory

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1 Introduction

The aim of this paper is to prove existence of solutions pairs $(\lambda, u)$ of the singular problem:

\[
\begin{cases}
-\Delta u = \lambda u - \frac{1}{u} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\nabla u \cdot \nu = 0 & \text{on } \partial \Omega
\end{cases}
\] (P)

where $\Omega$ is a bounded smooth open subset of $\mathbb{R}^N$ and $\nu$ denotes the unit normal defined on $\partial \Omega$ – we are therefore considering a Neumann boundary condition. This research is motivated by the increasing interest on singular problem that has been showing up in the literature in recent years (starting more or less from the year 2000). If we confine ourselves on elliptic problems, studying Problem (P) seems a natural continuation of the paper [5] by Montenegro Silva, where a two solutions result is proved for the Dirichlet problem with the semilinear term $\lambda u^p - \frac{1}{u^\alpha}$, with $0 < p < 1$ and $0 < \alpha < 1$.

In our case the singular term has the exponent $\alpha = 1$. Notice that the minus sign in front of the singular term makes the problem quite more challenging than the plus sign. In the latter problem indeed one can exploit the convexity of the corresponding term in the Euler-Lagrange functional, which allows to treat any positive exponent $\alpha$ (using suitable tricks, see e.g. [1]). If one makes some tests in the radial case it is not difficult to realize that the Dirichlet problem has no reasonable solutions for $\alpha = 1$. Actually solutions starting from zero are forced to stick at zero and not allowed to emerge (in contrast with the $\alpha < 1$ case). On this respect see Remark (3.9). For this reason we are lead to consider the corresponding Neumann problem (P).

In this alternative setting we need to mention the paper [3] by Del Pino, Manásevich, and Montero which deals with the ODE case ($N = 1$) in the periodic case with a more general, non autonomous, singular term $f(u, x)$ (singular in $u$ and $T$-periodic in $x$). Using topological degree arguments the authors prove for instance that the equation:

\[ -\ddot{u} = \lambda u - \frac{1}{u^\alpha}, \quad u(x) > 0, \quad u(x + T) = u(x), \]

where $\alpha \geq 1$ and $h$ is $T$-periodic, has a solution provided $\lambda \not= \frac{\mu_k}{4}$ for all $k$. Here $\mu_k$ denote the eigenvalues of a suitable linearized problem which arises in a natural

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way from the problem. In the variational case the results of [3] can be seen as deriving from the existence of two global bifurcation branches originating from the pairs \((\mu_k/2, 1/\sqrt{\mu_k/2})\) (the second element of the pair is a constant function).

Following this idea we prove a local bifurcation result for (P), (see Theorem (2.1)) showing that there exist two bifurcation branches of solutions, emanating from the pairs \((\mu_k/2, 1/\sqrt{\mu_k/2})\). In Theorem (3.3) of section 3 we consider the radial case and, exploiting a continuation argument for the nodal regions of the solutions, we can prove that one of the two branches is global and bounded in \(\lambda\).

2 A local bifurcation result

Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^N\) with smooth boundary.

2.1 Theorem. Let \(\hat{\mu} > 0\) be an eigenvalue of the following Neumann problem:

\[
\begin{align*}
-\Delta u &= \mu u \quad \text{in } \Omega, \\
\nabla u \cdot \nu &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(\(\nu\) denotes the normal to \(\partial \Omega\)).

Then there exists \(\rho_0 > 0\) such that for all \(\rho \in [0, \rho_0]\) there exist two distinct pairs \((\lambda_{1,\rho}, u_{1,\rho})\) and \((\lambda_{2,\rho}, u_{2,\rho})\) such that, for \(i = 1, 2\):

\[
(\lambda_{i,\rho}, u_{i,\rho}) \text{ are solutions of (P) }, \int_{\Omega} \left( u_{i,\rho} - \frac{1}{\sqrt{\lambda_{i,\rho}}} \right)^2 dx = \rho^2, \lambda_{i,\rho} \rightarrow \frac{\mu_k}{2}.
\]

Proof. We start by introducing some changes of variables. First of all notice that, for all \(\lambda > 0\), Problem (P) has the constant solution \(u(x) = \frac{1}{\sqrt{\lambda}}\). If we seek for solutions of the form \(u = \frac{1}{\sqrt{\lambda}} + z\) we easily end up with the equivalent problem on \(z\):

\[
\begin{align*}
-\Delta z &= 2\lambda z - h_\lambda(z) \quad \text{in } \Omega, \\
\sqrt{\lambda}z &= -1 \quad \text{in } \Omega, \\
\nabla z \cdot \nu &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(P0)

where \(h_\lambda : [-\frac{1}{\sqrt{\lambda}}, +\infty] \rightarrow \mathbb{R}\) is defined by:

\[h_\lambda(s) = \frac{\lambda \sqrt{\lambda} s^2}{1 + \sqrt{\lambda} s}.\]

Now we consider another simple transformation: \(v := \sqrt{\lambda}z\), so that (P0) turns out to be equivalent to:

\[
\begin{align*}
-\Delta v &= \lambda (2v - h_\lambda(v)) \quad \text{in } \Omega, \\
v &= -1 \quad \text{in } \Omega, \\
\nabla v \cdot \nu &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(P1)

Choose \(s_0 \in ]-1, 0]\) (for instance \(s_0 := 1/2\)) and define \(\tilde{h}_1 : \mathbb{R} \rightarrow \mathbb{R}\) by:

\[\tilde{h}_1(s) = \frac{\lambda \sqrt{\lambda} s^2}{1 + \sqrt{\lambda} s}.
\]
\[ h_1(s) := \begin{cases} h_1(s_0) + h'_1(s_0)(s - s_0) \quad & \text{if } s \geq s_0, \\ h_1(s_0) + \frac{h''_1(s_0)}{2}(s - s_0)^2 \quad & \text{if } s \leq s_0. \end{cases} \]

Then \( \tilde{h}_1 \in C^2(\mathbb{R} \times \mathbb{R}) \) and \( \tilde{h}'_1(0) = \tilde{h}''_1(0) = 0 \). Let \( \tilde{H}_1 : \mathbb{R} \to \mathbb{R} \) denote the primitive function for \( h_1 \) (i.e. \( \tilde{H}'_1 = h_1 \)) such that \( \tilde{H}_1(0) = 0 \). Moreover we consider the functional \( \tilde{I} : W^{1,2}(\Omega) \to \mathbb{R} \) defined by
\[ \tilde{I}(v) := Q(v) + \tilde{H}_1(v) \]
where:
\[ Q(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} v^2 \, dx, \quad \tilde{H}_1(v) := \int_{\Omega} \tilde{H}_1(v) \, dx. \]

Using a classical bifurcation result for potential operators (see e.g. [8]) we get that there exists \( \rho_0 > 0 \) such that for all \( \rho \in [0, \rho_0[ \) there are two distinct pairs \((\lambda_1, \rho, v_1, \rho)\) and \((\lambda_2, \rho, v_2, \rho)\) which solve:
\[
\begin{cases}
-\Delta v = \lambda \left( 2v - \tilde{h}_1(v) \right) & \text{in } \Omega, \\
\nabla v \cdot \nu = 0 & \text{on } \partial \Omega
\end{cases}
\]
and such that
\[ \int_{\Omega} v_{i, \rho}^2 \, dx = \rho^2 , \quad \lim_{\rho \to 0} \lambda_{i, \rho} = \frac{\mu_k}{2} \quad i = 1, 2. \]

Now using a standard regularity argument we can find a constant \( K \) such that, for any solution \( w \) of (3):
\[ \|v\|_{\infty} \leq \|2v - \tilde{h}_1(v)\|_2 \leq K\|v\|_2. \]

Then, for \( \rho_0 \) small, \( v_{i, \rho} > s_0, i = 1, 2 \), so \( v_{i, \rho} \) actually solve (P1). Going backwards and setting \( u_{i, \rho} := \frac{1}{\sqrt{\lambda_{i, \rho}}} + \sqrt{\lambda_{i, \rho}} v_{i, \rho} \), we find the desired solutions of (P).

3 A global bifurcation result for radial solutions

We consider the case \( N = 2 \) and \( \Omega = B(0, R) = \{ x \in \mathbb{R}^2 : \|x\| < R \} \). We look for radial solutions i.e. \( z(x, y) = w(\|x, y\|) \). Actually with similar arguments we could have considered the general case \( N \geq 2 \). Given \( R > 0 \), it is therefore convenient to introduce the Hilbert space:
\[ E := \left\{ w : [0, R] \to \mathbb{R} : \int_0^R \rho \tilde{u}^2 \, d\rho < +\infty \right\} \]
endowed with \( (v, w)_E := \int_0^R \rho \tilde{v} \tilde{w} \, d\rho + \int_0^R \rho v w \, d\rho \) and for \( \lambda_j > 0 \) the set:
\[ W_\lambda := \left\{ w \in E : 1 + \sqrt{\lambda w(\rho)} > 0 \right\}, \quad \mathcal{W} := \{ (\lambda, w) \in \mathbb{R} \times E : w \in W_\lambda \}. \]
It is clear that $\|w\|_\infty \leq \text{constant} \|w\|_E$ so $W$ is open in $E$ and $W$ is open in $\mathbb{R} \times E$. As well known the search for radial solutions leads the equation:

$$
\begin{aligned}
\ddot{w} + \frac{\dot{w}}{\rho} &= -\lambda w - \frac{\lambda w}{1 + \sqrt{\lambda w}} =: f_\lambda(u), \\
\dot{w}(0) &= \dot{w}(R) = 0.
\end{aligned}
$$

(RP)

By the above we mean that:

$$(\lambda, w) \in W, \quad \int_0^R \rho \dot{w} \delta d\rho = \int_0^R \rho f_\lambda(w) \delta d\rho \quad \forall v \in E. \quad (4)$$

It is standard to check that “weak solutions”, i.e. solutions to (4) actually solve (RP) in a classical sense.

It is clear that $(\lambda, 0)$ is a solution for (RP) for any $\lambda \in \mathbb{R}$. We call “nontrivial” solution a pair $(\lambda, w)$ with $w \neq 0$ such that (RP) holds.

3.1 Remark. If $(\lambda, w)$ is a nontrivial solution then $\lambda > 0$. To see this it suffices to multiply (RP) by $u$ and integrate over $[0, R]$. Actually this property is true in the general case (not just in the radial problem).

We shall use the following simple inequality.

3.2 Remark. Let $0 < a < b < +\infty$. We have:

$$
\frac{b-a}{b} \leq \ln \left( \frac{b}{a} \right) \leq \frac{b-a}{a}. \quad (5)
$$

We have indeed:

$$
\ln \left( \frac{b}{a} \right) = \ln \left( 1 + \frac{b-a}{a} \right) \leq \frac{b-a}{a}
$$

and

$$
\ln \left( \frac{b}{a} \right) = -\ln \left( \frac{a}{b} \right) = -\ln \left( 1 + \frac{a-b}{b} \right) \geq -\frac{a-b}{b} = \frac{b-a}{b}.
$$

Now let us suppose that a solution $(\lambda, w)$ exists so we can find some properties and estimates on $w$. Arguing as in the proof of Lemma 2.2 in [2] we have that either $w = 0$ or $[0, R]$ can be split as the union of a finite number of subintervals $[\rho_1, \rho_2]$ where $w$ has one of the following behaviors:

(A) $w(r_1) > 0$, $\dot{w}(r_1) = 0$, $\dot{w} < 0$ in $[r_1, r_2]$, and $w(r_2) = 0$;

(B) $w(r_1) = 0$, $\dot{w} < 0$ in $[r_1, r_2]$, $\dot{w}(r_2) = 0$, and $w(r_2) < 0$;

(C) $w(r_1) < 0$, $\dot{w}(r_1) = 0$, $\dot{w} > 0$ in $[r_1, r_2]$, and $w(r_2) = 0$;

(D) $w(r_1) = 0$, $\dot{w} > 0$ in $[r_1, r_2]$, $\dot{w}(r_2) = 0$, and $w(r_2) > 0$. 

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (2,0) -- (3,0) -- (3,1) -- (2,1) -- cycle;
\draw (0,0) -- (0,2);
\draw (2,0) -- (2,2);
\end{tikzpicture}
\end{center}
So let $w: [r_1, r_2] \to \mathbb{R}$ be as in one of the above cases. Multiplying (RP) by $\dot{w}$ gives

$$\frac{1}{2} \ddot{w} + \dot{w} \frac{\dot{w}^2}{\rho} = \frac{d}{dp} F_\lambda(w)$$

Let $p := \dot{w}^2$ the previous equation can be written as:

$$\frac{1}{2} \dot{p} + \frac{p}{\rho} = \frac{d}{dp} F_\lambda(w)$$

which is equivalent to

$$\frac{d}{dp} (\rho^2 p) = 2 \rho^2 \frac{d}{dp} F_\lambda(w) \rho^2 = 2 \rho^2 \frac{d}{dp} F_1 \left( \sqrt{\lambda} w \right).$$

We integrate between $\rho_1$ and $\rho_2$, where $r_1 \leq \rho_1 \leq \rho_2 \leq r_2$:

$$\rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) = 2 \rho_2^2 F(w(\rho_2)) - 2 \rho_1^2 F(w(\rho_1)) - \int_{\rho_1}^{\rho_2} 4 \sigma F_\lambda(w(\sigma)) \, d\sigma$$

Notice that $F_\lambda$ is increasing on $\left[-\frac{1}{\sqrt{\lambda}}, 0\right]$ and decreasing on $[0, +\infty]$, so:

$$\sigma \mapsto F_\lambda(w(\sigma))$$

is increasing (decreasing) in cases (A) and (C) (in cases (B) and (D)).

We hence get, in cases (A) and (C):

$$-2(\rho_2^2 - \rho_1^2) F_\lambda(w(\rho_2)) \leq - \int_{\rho_1}^{\rho_2} 4 \sigma F_\lambda(w(\sigma)) \, d\sigma \leq -2(\rho_2^2 - \rho_1^2) F_\lambda(w(\rho_1))$$

while in cases (B) and (D):

$$-2(\rho_2^2 - \rho_1^2) F_\lambda(w(\rho_1)) \leq - \int_{\rho_1}^{\rho_2} 4 \sigma F_\lambda(w(\sigma)) \, d\sigma \leq -2(\rho_2^2 - \rho_1^2) F_\lambda(w(\rho_2))$$

So in cases (A) and (C) we have:

$$2\rho_2^2 (F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1))) \leq \rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) \leq 2\rho_2^2 (F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)))$$

and in cases (B) and (D):

$$2\rho_2^2 (F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1))) \leq \rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) \leq 2\rho_2^2 (F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)))$$

Now we estimate $w(\rho)$ - we need to take into account all the four cases (A),(B),(C),(D).

Case (A). We replace $\dot{\rho} := r_1$, $\rho_0 := r_2$ and let $h := \dot{w}(\dot{\rho}) > 0$. We use (6) with $\rho_1 = \dot{\rho}$ and $\rho_2 = \sigma \in [\dot{\rho}, \rho_0]$:

$$2\dot{\rho}^2 (F_\lambda(w(\sigma)) - F_\lambda(h)) \leq \sigma^2 \dot{w}(\sigma)^2 \leq 2\sigma^2 (F_\lambda(w(\sigma)) - F_\lambda(h)).$$

Then we take the square root and divide:

$$\frac{\dot{\rho}^2}{\sigma} \leq \frac{-\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \leq \sqrt{2}.$$
Integrate on $[\bar{\rho}, \rho]$ getting:

$$\sqrt{2} \bar{\rho} \ln \left( \frac{\rho}{\bar{\rho}} \right) \leq -\Phi_{\lambda,h}(w(\rho)) + \Phi_{\lambda,h}(h) \leq \sqrt{2}(\rho - \bar{\rho})$$

where $\Phi_{\lambda,h} : [0,h] \to \mathbb{R}$ is defined by:

$$\Phi_{\lambda,h}(s) := \int_{0}^{s} \frac{d\xi}{\sqrt{F_{\lambda}(\xi) - F_{\lambda}(h)}}$$

(it is simple to check the the integral converges at $\xi = h$). So we deduce:

$$\Phi_{\lambda,h}^{-1} \left( \Phi_{\lambda,h}(h) - \sqrt{2}(\rho - \bar{\rho}) \right) \leq w(\rho) \leq \Phi_{\lambda,h}^{-1} \left( \Phi_{\lambda,h}(h) - \sqrt{2}\bar{\rho} \ln \left( \frac{\rho}{\bar{\rho}} \right) \right)$$

which we prefer to write as

$$\Phi_{\lambda,h}^{-1} \left( \Phi_{\lambda,h}(h) + \sqrt{2}(\rho - \bar{\rho}) \right) \leq w(\rho) \leq \Phi_{\lambda,h}^{-1} \left( \Phi_{\lambda,h}(h) + \sqrt{2}\bar{\rho} \ln \left( \frac{\rho}{\bar{\rho}} \right) \right)$$

(8)

In particular, taking $\rho = \rho_{0}$, which gives $w(\rho_{0}) = 0$, (and using (5)) we have:

$$\sqrt{2} \frac{\rho_{0} - \bar{\rho}}{\rho_{0}} \leq \sqrt{2}\bar{\rho} \ln \left( \frac{\rho_{0}}{\bar{\rho}} \right) \leq \Phi_{\lambda,h}(h) \leq \sqrt{2}(\rho_{0} - \bar{\rho}) \enspace .$$

(9)

Moreover taking $\rho_{1} = \bar{\rho}$ and $\rho_{2} = \rho_{0}$ in (6) we have:

$$\sqrt{2} \frac{\bar{\rho}}{\rho_{0}} \sqrt{-F_{\lambda}(h))} \leq -\dot{w}(\rho_{0}) \leq \sqrt{2}\sqrt{-F_{\lambda}(h))}$$

(10)

Case (B). We rename $\rho_{0} := r_{1}$, $\bar{\rho} := r_{2}$ and let $h := w(\bar{\rho}) < 0$. We use (7) with $\rho_{1} = \sigma \in [\rho_{0}, \bar{\rho}]$ and $\rho_{2} = \bar{\rho}$:

$$2\rho^{2}(F_{\lambda}(h) - F_{\lambda}(w(\sigma))) \leq -\sigma^{2}\dot{w}(\sigma)^{2} \leq 2\sigma^{2}(F_{\lambda}(h) - F_{\lambda}(w(\sigma))).$$

We change sign and proceed as in case (A):

$$2\sigma^{2}(F_{\lambda}(w(\sigma)) - F_{\lambda}(h)) \leq \sigma^{2}\dot{w}(\sigma)^{2} \leq 2\rho^{2}(F_{\lambda}(w(\sigma)) - F_{\lambda}(h)).$$

Take the square root and divide:

$$\sqrt{2} \leq \frac{-\dot{w}(\sigma)}{\sqrt{F_{\lambda}(w(\sigma)) - F_{\lambda}(h)}} \leq \sqrt{2}\frac{\bar{\rho}}{\sigma}$$

Integrate on $[\rho, \rho_{0}]$:

$$\sqrt{2}(\rho - \rho) \leq -\Phi_{\lambda,h}(h) + \Phi_{\lambda,h}(w(\rho)) \leq \sqrt{2}\bar{\rho} \ln \left( \frac{\rho}{\bar{\rho}} \right)$$

defining $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$ as in case (A). Applying $\Phi_{\lambda,h}^{-1}$ we get that (8) holds in case (B) too. In particular, taking $\rho = \rho_{0}$ (and using (5)):

$$\sqrt{2}(\rho - \rho_{0}) \leq -\Phi_{\lambda,h}(h) \leq \sqrt{2}\bar{\rho}_{0} \ln \left( \frac{\rho_{0}}{\bar{\rho}_{0}} \right) \leq \sqrt{2}\bar{\rho}_{0}(\rho_{0} - \rho_{0}) \enspace .$$

(11)
and taking $\rho_1 = \rho_0$ and $\rho_2 = \bar{\rho}$ in (7) we have:

$$\sqrt{2} \sqrt{-F_{\lambda}(h)} \leq -\dot{\rho}(\rho_0) \leq \sqrt{2} \frac{\bar{\rho}}{\rho_0} \sqrt{-F_{\lambda}(h)}$$

(12)

Case (C). We rename $\bar{\rho} := r_1$, $\rho_0 := r_2$ and let $h := w(\bar{\rho}) < 0$. Using (6) with $\rho_1 = \bar{\rho}$ and $\rho_2 = \sigma \in ]\bar{\rho}, \rho_0]$ we obtain the same inequality of case (A). After taking the square root and dividing:

$$\sqrt{2} \frac{\bar{\rho}}{\sigma} \leq \frac{\dot{\rho}(\sigma)}{\sqrt{F_{\lambda}(w(\sigma)) - F_{\lambda}(h)}} \leq \sqrt{2}.$$

We integrate between $\bar{\rho}$ and $\rho \in [\bar{\rho}, \rho_0]$ getting:

$$\sqrt{2} \rho \ln \left( \frac{\rho}{\bar{\rho}} \right) \leq \Phi_{\lambda,h}(w(\rho)) - \Phi_{\lambda,h}(h) \leq \sqrt{2} (\rho - \bar{\rho})$$

with $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$ defined as above. So we deduce:

$$\Phi_{\lambda,h}^{-1} \left( \Phi_{\lambda,h}(h) + \sqrt{2} \rho \ln \left( \frac{\rho}{\bar{\rho}} \right) \right) \leq w(\rho) \leq \Phi_{\lambda,h}^{-1} \left( \Phi_{\lambda,h}(h) + \sqrt{2} (\rho - \bar{\rho}) \right)$$

(13)

In particular, taking $\rho = \rho_0$ (and using (5)):

$$\sqrt{2} \frac{\bar{\rho}}{\rho_0} \left( \rho_0 - \bar{\rho} \right) \leq \sqrt{2} \rho \ln \left( \frac{\rho_0}{\bar{\rho}} \right) \leq -\Phi_{\lambda,h}(h) \leq \sqrt{2} (\rho_0 - \bar{\rho}).$$

(14)

Moreover taking $\rho_1 = \bar{\rho}$ and $\rho_2 = \rho_0$ in (6) we have:

$$\sqrt{2} \frac{\bar{\rho}}{\rho_0} \sqrt{-F_{\lambda}(h)} \leq \dot{\rho}(\rho_0) \leq \sqrt{2} \sqrt{-F_{\lambda}(h)}$$

(15)

Case (D). We rename $\rho_0 := r_1$, $\bar{\rho} := r_2$ and let $h := w(\bar{\rho}) > 0$. Using (7) with $\rho_1 = \sigma \in ]\rho_0, \bar{\rho}]$ and $\rho_2 = \bar{\rho}$ we obtain the same inequalities of case (B). When we take the square root and divide:

$$\sqrt{2} \leq \frac{\dot{\rho}(\sigma)}{\sqrt{F_{\lambda}(w(\sigma)) - F_{\lambda}(h)}} \leq \sqrt{2} \frac{\bar{\rho}}{\sigma}.$$

Integrate on $[\rho, \rho_0]$:

$$\sqrt{2} (\bar{\rho} - \rho) \leq \Phi_{\lambda,h}(h) - \Phi_{\lambda,h}(w(\rho)) \leq \sqrt{2} \bar{\rho} \ln \left( \frac{\bar{\rho}}{\rho} \right)$$

with the usual definition of $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$. Applying $\Phi_{\lambda,h}^{-1}$ we obtain that (13) holds in case (D) too. In particular, taking $\rho = \rho_0$ (and using (5)):

$$\sqrt{2} \left( \bar{\rho} - \rho_0 \right) \leq \Phi_{\lambda,h}(h) \leq \sqrt{2} \bar{\rho} \ln \left( \frac{\rho}{\bar{\rho}} \right) \leq \sqrt{2} \frac{\bar{\rho}}{\rho_0} (\rho - \rho_0)$$

(16)

and taking $\rho_1 = \rho_0$ and $\rho_2 = \bar{\rho}$ in (7) we have:

$$\sqrt{2} \sqrt{-F_{\lambda}(h)} \leq \dot{\rho}(\rho_0) \leq \sqrt{2} \frac{\bar{\rho}}{\rho_0} \sqrt{-F_{\lambda}(h)}$$

(17)
Now we have:
\[
\sqrt{2}\Phi_{\lambda,h}(h) = \int_0^h \frac{d\xi}{\sqrt{F(\sqrt{\lambda}\xi)} - F(\sqrt{\lambda}h)} = \int_0^1 \frac{h\ d\sigma}{\sqrt{F(\sigma\sqrt{\lambda}h) - F(\sqrt{\lambda}h)}} = \frac{1}{\sqrt{\lambda}} \Phi(\sqrt{\lambda}h)
\]
where:
\[
\Phi(s) := \int_0^1 \frac{s\ d\sigma}{\sqrt{F(\sigma s) - F(s)}} = \text{sgn}(s) \int_0^1 \sqrt{s^2 F(\sigma s) - F(s)}\ d\sigma.
\]
With simple computations:
\[
\lim_{s \to 0} s^2 F(\sigma s) - F(s) = 1 - \sigma^2,
\]
\[
\lim_{s \to +\infty} s^2 F(\sigma s) - F(s) = 2(1 - \sigma^2),
\]
and
\[
\lim_{s \to -1+} s^2 F(\sigma s) - F(s) = 0.
\]
So we deduce that
\[
\lim_{h \to 0^+} \Phi_{\lambda,h}(h) = \frac{\pi}{2\sqrt{2}\lambda}, \quad \lim_{h \to +\infty} \Phi_{\lambda,h}(h) = \frac{\pi}{2\sqrt{\lambda}},
\]
\[
\lim_{h \to 0^-} \Phi_{\lambda,h}(h) = -\frac{\pi}{2\sqrt{2}\lambda}, \quad \lim_{h \to -1^+} \Phi_{\lambda,h}(h) = 0.
\]

To state the main result we need some notation, which we take from [2, 6]. For \(k \in \mathbb{N}, k \geq 1\), we consider
\[
\mathcal{S} := \{ (\lambda, w) \in \mathcal{W} : \lambda, w \} \text{ is a solution to }\{\text{RP}\}
\]
\[
\mathcal{S}^+_k := \{ (\lambda, w) \in \mathcal{S} : w \text{ has } k \text{ nodes in } ]0, R[, \ w(0) > 0 \},
\]
\[
\mathcal{S}^-_k := \{ (\lambda, w) \in \mathcal{S} : w \text{ has } k \text{ nodes in } ]0, R[\}, \ w(0) < 0 \}.
\]
We also consider the two eigenvalue problems:
\[
\ddot{w} + \frac{\dot{w}}{\rho} = -\mu w, \quad \dot{w}(0) = \dot{w}(R) = 0. \quad (\text{RP}^*)
\]
\[ \ddot{v} + \frac{\dot{v}}{\rho} = -\nu v, \quad \dot{v}(0) = 0, v(R) = 0. \]  
(RP0)

It is clear that \( w \neq 0 \) and \( \mu \neq 0 \) solve (RP*) if and only if, for some integer \( k \geq 1 \):

\[ \mu = \mu_k := \left( \frac{y_k}{R} \right)^2 \]  
(20)

where \( y_k \) denotes the \( k \)-th nontrivial zero of \( J'_0 \) and \( J_0 \) is the first Bessel function, and

\[ w = \alpha w_k, \quad \alpha \in \mathbb{R}, \quad w_k(\rho) := J_0 \left( \frac{y_k}{R} \rho \right). \]  
(21)

For the sake of completeness we can agree that \( \mu_0 = 0 \) and \( w_0(\rho) = J_0(0) \). In the same way \( v \neq 0 \) and \( \nu \) solve (RP0) if and only if, for some integer \( k \geq 1 \):

\[ \nu = \nu_k := \left( \frac{z_k}{R} \right)^2 \]  
(22)

where \( z_k \) is the \( k \)-th zero of \( J_0 \) and

\[ v = \alpha v_k, \quad \alpha \in \mathbb{R}, \quad v_k(\rho) := J_0 \left( \frac{z_k}{R} \rho \right). \]  
(23)

Notice that \( \nu_k < \mu_k < \nu_{k+1} \) for all \( k \).

**3.3 Theorem.** Let \( \mu_k > 0 \) be an eigenvalue for (RP*). Then \( S^+_k \) is a connected set and

- \( (\mu_k/2, 0) \in \overline{S_k^+} \);
- \( 0 < \inf \{ \lambda \in \mathbb{R} : \exists w \in E \text{ with } (\lambda, w) \in S_k^+ \} \);
- \( \sup \{ \lambda \in \mathbb{R} : \exists w \in E \text{ with } (\lambda, w) \in S_k^+ \} < +\infty \);
- \( S_k^+ \) is unbounded and contains a sequence \( (\lambda_n, w_n) \) such that \( \|w_n\|_E \to \infty \) and

\[ \lim_{n \to \infty} \lambda_n = \begin{cases} \mu_k/2 & \text{if } k \text{ is even}, \\ \nu_{(k+1)/2} & \text{if } k \text{ is odd}. \end{cases} \]  
(24)

The proof of (3.3) will be obtained from some preliminary statements.

**3.4 Remark.** If \( (\lambda, w) \in S^+ \) (resp. \( (\lambda, w) \in S^+ \)), and \( 0 = \rho_0 < \rho_1, \ldots, \rho_k < \rho_{k+1} = R, \rho_1, \ldots, \rho_k \) being the nodal points of \( w \), then:

\[ \rho_{i+1} - \rho_i \geq \frac{\pi}{4\sqrt{\lambda}} \text{ for } i \text{ even} \quad (\text{resp. for } i \text{ odd}). \]  
(25)

This is easily seen using the right hand sides of the inequalities \( 9 \), \( 14 \), and \( 18 \).

**3.5 Lemma.** For any integer \( k \) there exist two constants \( \underline{\lambda}_k \) and \( \overline{\lambda}_k \) such that

\[ (\lambda, w) \in S^+_k \cup S^-_k \Rightarrow 0 < \underline{\lambda}_k \leq \lambda \leq \overline{\lambda}_k < +\infty \]  
(26)

**Proof.** Take any subinterval \([r_1, r_2]\) as in cases (A)–(D) and consider the first eigenvalue \( \bar{\mu} = \bar{\mu}(r_1, r_2) \) for the mixed type boundary condition:

\[
\begin{cases}
-(\rho \dot{w})' = \mu w & \text{on } ]r_1, r_2[ \\
\dot{w}(r_1) = 0, w(r_2) = 0 & \text{(resp. } w(r_1) = 0, \dot{w}(r_2) = 0) 
\end{cases}
\]

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in cases (A), (C) (resp. cases (C), (D)). We can choose an eigenfunction $\tilde{e}$ corresponding to $\tilde{\mu}$ so that $z \tilde{e} > 0$ in $[r_1, r_2]$. Multiplying (RP) by $\tilde{e}$ and integrating over $[r_1, r_2]$ yields:

$$
\tilde{\mu} \int_{r_1}^{r_2} \rho z \tilde{e} \, d\rho = \lambda \int_{r_1}^{r_2} \rho z \tilde{e} \left(1 + \frac{1}{1 + \sqrt{z}}\right) \, d\rho.
$$

This implies:

$$
\lambda \int_{r_1}^{r_2} \rho z \tilde{e} \, d\rho \leq \tilde{\mu} \int_{r_1}^{r_2} \rho z \tilde{e} \, d\rho \leq 2\lambda \int_{r_1}^{r_2} \rho z \tilde{e} \, d\rho
$$

which gives $\frac{\tilde{\mu}}{2} \leq \lambda \leq \tilde{\mu}$. Now since $]r_1, r_2[ \subset ]0, R[$ we have $\tilde{\mu} \geq \bar{\mu}[0, R]$. On the other side since $w$ has $k$ nodal points we can choose $r_1, r_2$ such that $r_2 - r_1 \geq R/k$, which implies $\frac{\bar{\mu}}{2} \leq \sup_{b-a=R/k} \frac{\bar{\mu}}{2} < +\infty$. This proves (26). 

**3.6 Lemma.** Let $(\lambda_n, w_n)$ be a sequence in $\mathcal{S}_k^+$. Then we can consider $0 < \rho_{1,n} < \cdots < \rho_{k,n} < R$ to be the nodes of $w_n$ and set $\rho_{0,n} := 0$, $\rho_{k+1,n} := R$; in this way $w_n(\rho) > 0$ on $]\rho_{1,n}, \rho_{1,n+1}[ \text{ if } i \text{ is even and } w_n(\rho) < 0 \text{ on } ]\rho_{1,n+1}, \rho_{1,n}[ \text{ if } i \text{ is odd. The following facts are equivalent:}

(a) $\lim_{n \to \infty} \sup_{\rho \in [0, R]} w_n(\rho) = +\infty$;

(b) $\lim_{n \to \infty} \inf_{\rho \in [0, R]} (1 + \lambda_n w_n(\rho)) = 0$;

(c) $\lim_{n \to \infty} \sup_{\rho \in [\rho_{i,n}, \rho_{i+1,n}]} w_n(\rho) = +\infty$ if $i$ is even;

(d) $\lim_{n \to \infty} \inf_{\rho \in [\rho_{i,n}, \rho_{i+1,n}]} (1 + \lambda_n w_n(\rho)) = 0$ if $i$ is odd;

(e) $\lim_{n \to \infty} \rho_{i+1,n} - \rho_{i,n} = 0$ if $i$ is odd;

Moreover, if any of the above holds, then (24) holds.

**Proof.** We can assume, passing to a subsequence that $\lambda_n \to \lambda \in [\lambda_k, \lambda^0]$. First notice that for all $i$ even (corresponding to $w > 0$) we have:

$$
\rho_{i+1,n} - \rho_{i,n} \geq \frac{\pi}{4\sqrt{k}}
$$

as we can infer from (9) or (16) and the behaviour of $\Phi_{\lambda,h}(h)$ in (18). Let

$$
h_{i,n} := \max_{\rho_{i,n} \leq \rho \leq \rho_{i+1,n}} w(\rho) \text{ for } i \text{ even, } h_{i,n} := \min_{\rho_{i,n} \leq \rho \leq \rho_{i+1,n}} w(\rho) \text{ for } i \text{ odd.}
$$

Then for any $i$ even:

$$
h_{i,n} \to +\infty \Leftrightarrow \Phi_{\lambda_n,h_{i,n}}(h_{i,n}) \to \frac{\pi}{2\sqrt{\lambda}} \Leftrightarrow w(\rho_{i,n}) \to +\infty \Leftrightarrow w(\rho_{i+1,n}) \to -\infty.
$$
This can be deduced from (18), (10), and (17). In the same way, using (19), (12), and (15) we get that, for $i$ odd:

$$1 + \sqrt{\lambda_n} h_{i,n} \rightarrow 0 \iff \Phi_{\lambda_n,h_{i,n}}(h_{i,n}) \rightarrow 0 \iff \hat{w}(\rho_{i,n}) \rightarrow -\infty \iff \hat{w}(\rho_{i+1,n}) \rightarrow +\infty.$$ 

Now we prove our claims. Let $i \in \{0, \ldots, k\}$ with $i$ even (resp. odd) and suppose that $h_i \to +\infty$ (resp. $1 + \sqrt{\lambda_n} h_i \to 0$). Then $F_{\lambda_n}(h_i) \to +\infty$ (resp. $F_{\lambda_n}(h_i) \to -\infty$) and by (10), (17), (12), (15) we get that:

$$\hat{w}_n(\rho_{i,n}) \to +\infty, \quad \hat{w}_n(\rho_{i+1,n}) \to -\infty \quad (\hat{w}_n(\rho_{i,n}) \to -\infty, \quad \hat{w}_n(\rho_{i+1,n}) \to +\infty)$$

which in turn implies:

$$F_{\lambda_n}(h_{i-1,n}) \to -\infty \text{ (resp. } +\infty) \quad F_{\lambda_n}(h_{i+1,n}) \to -\infty \text{ (resp. } +\infty)$$

(with the obvious exceptions when $i - 1 < 0$ or $i + 1 > k$). So we get:

$$1 + \sqrt{\lambda_n} h_{i-1,n} \rightarrow 0 \quad (h_{i-1,n} \to +\infty), \quad 1 + \sqrt{\lambda_n} h_{i+1,n} \rightarrow 0 \quad (h_{i+1,n} \to +\infty).$$

This shows that the property $|F_{\lambda_n}(h_i)| \to +\infty$ “propagates” from the $i$-th interval to the previous and to the next one. From this it is easy to deduce that (a)–(d) are all equivalent. To prove that they are equivalent to (e) just use (9),(11),(14),(16), depending on the case, noticing that $\rho_{1,n} \geq \frac{\pi}{4\pi \lambda_k}$, as from (25) (this would not be possible if we were considering $S_k^-$).

Finally suppose that $(\lambda_n, w_n)$ verifies any of (a)–(e). Then $\|w_n\|_\infty \to +\infty$. Let $\tilde{w}_n := \frac{w_n}{\|w_n\|_\infty}$. We can suppose that $\tilde{w}_n \to \tilde{w}$ in $E$ and that:

$$\rho_{1,n} \to \rho_1, \quad \rho_{2j-1,n} \to \rho_j, \quad \rho_{2j,n} \to \rho_j \quad 1 \leq j \leq k/2, \quad \rho_{k,n} \to R \text{ if } k \text{ is odd,}$$

where $0 = \rho_0 < \rho_1 < \cdots < \rho_h < \rho_h + 1 = R$ and $h = \lfloor k/2 \rfloor$ (so $\rho_1 = R$ when $k = 1$). It is not difficult to prove that $\hat{w}(\rho) > 0$ in $[\rho_i, \rho_{i+1}]$ if $i = 0, \ldots, h,$ $\hat{w}(\rho_i) = \cdots = \hat{w}(\rho_h) = 0$, $\hat{w}'(0) = 0$ and $\hat{w}'(R)$ is even if $k$ is even while $\hat{w}'(R) = 0$ is $k$ odd. Moreover for any $i = 0, \ldots, h$:

$$-(\rho u')' = \hat{\lambda} \hat{w} \quad \text{on } [\rho_i, \rho_{i+1}]$$

Now we can rearrange $\hat{w}$ defining $\hat{w} := \sum_{j=0}^h (-1)^j \alpha_j \hat{w}_{\rho_j}$, where $\alpha_1 = 1$ and $\alpha_j \hat{w}'(\rho_j) = \alpha_{j+1} \hat{w}'(\rho_j)$, $j = 1, \ldots, h$. In this way $(\hat{\lambda}, \hat{w})$ is an eigenvalue – eigenfunction pair relative for problem (21) if $k$ is even and of (23) if $k$ is odd. Since $\hat{w}$ has $h = k/2$ nodal points for $k$ even and $h + 1 = (k + 1)/2$ if $k$ is odd, then (24) holds.

\textbf{Proof of (3.3).} If $\varepsilon \in [0, 1]$ we set:

$$O_\varepsilon := \left\{ (\lambda, w) \in E : \varepsilon < \lambda < \varepsilon^{-1}, 1 + \sqrt{\lambda} w(\rho) > \varepsilon, w(\rho) < \varepsilon^{-1} \forall \rho \in [0, R] \right\}$$

Clearly $O_\varepsilon$ is an open set with $O_\varepsilon \subset W$. Moreover $(\mu_{k/2}, 0) \in O_\varepsilon$ if $\varepsilon$ is sufficiently small. Define $\tilde{h}_{\lambda,\varepsilon}$ as in (2) with $\varepsilon = \varepsilon$ and let $\hat{h}_{\lambda}(s) := \tilde{h}_{\lambda}(\sqrt{\lambda}s)$. Using (6) we get there that there exists a pair $(\lambda_\varepsilon, w_\varepsilon)$ in $\partial O_\varepsilon$, with $w_\varepsilon$ having $k$ nodal points, which solves Problem (RP) with $h_{\varepsilon,\lambda} := \hat{h}_{\varepsilon}(\lambda, \cdot)$ instead of $h_{\lambda}$. Since $(\lambda, w) \in \partial O_\varepsilon \Rightarrow$
\[ h_{\epsilon}(\lambda, w) = h_{\lambda}(w), \] we get that \((\lambda_{\epsilon}, w_{\epsilon}) \in S^{+}_k\). For \(\epsilon\) small we have \(\epsilon < \lambda_k \leq \bar{\lambda}_k < \epsilon^{-1}\) so we get \(w_{\epsilon} \in \partial \{ 1 + \sqrt{\lambda_{\epsilon}} w > \epsilon, w < \epsilon^{-1} \}\) i.e. there exists a point \(\rho_{\epsilon} \in [0, R]\) such that

\[
\text{either } 1 + \sqrt{\lambda_{\epsilon}} w_{\epsilon}(\rho_{\epsilon}) = \epsilon \quad \text{or} \quad w_{\epsilon}(\rho_{\epsilon}) = \epsilon^{-1}.
\]

We can find a sequence \(\epsilon_n \to 0\) such that the corresponding \((\lambda_n, w_n) := (\lambda_{\epsilon_n}, w_{\epsilon_n})\) verify one of the above properties for all \(n \in \mathbb{N}\). If the first one holds for all \(n\), then \((\lambda_n, w_n)\) verifies (b) of Lemma (3.6); in the second case \((\lambda_n, w_n)\) verifies (a) of Lemma (3.6). Then by Lemma (3.6) \(\|w_n\|_{\infty} \to \infty\) and (24) holds. This proves the Theorem.

### 3.7 Remark.
As a consequence of Theorem (3.3) we get that for any \(h \geq 1\) integer and any \(\lambda\) strictly between \(\lambda_h\) and \(\lambda_{2h}/2\) there exists \(u\) such that \((\lambda, u)\) solves (P). The same is true for all \(\lambda\) strictly between \(\nu_h\) and \(\lambda_{2h-1}/2\).

### 3.8 Remark.
The proof of (P) fails if we follow the bifurcation branch \((\lambda_{\rho}, w_{\rho})\) with \(w_{\rho}(0) < 0\). In this case it seems possible that the branch tends to a point \((\lambda, \tilde{w})\) where \(\sqrt{\lambda} \tilde{w}(0) = -1\) (but \(\sqrt{\lambda} \tilde{w}(0) > -1\) for \(\rho > 0\)). This phenomenon, if true, would be worth studying.

### 3.9 Remark.
The computations of this section show that, if \(\Omega\) is the ball, then there are no solutions for the Dirichlet problem. It is indeed impossible to construct a (nontrivial) solution \((\lambda, w)\) for (RP) with \(w(R) = 0\).

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