SOLUTION OF A LOEWNER CHAIN EQUATION IN SEVERAL COMPLEX VARIABLES

MIRCEA VODA

Abstract. We find a solution to the Loewner chain equation in the case when the infinitesimal generator satisfies $h(0,t) = 0$, $Dh(0,t) = A$ for any $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$. We also study the related classes of spiral-like mappings, mappings with parametric representation and asymptotically spiral-like mappings.

1. Introduction and preliminaries

Subordination chains in several complex variables, the associated differential equations and applications have been studied by various authors (see [13], [8], [12], [9], [10], [3], [2] and the references therein). Initially one assumed that the infinitesimal generators of the subordination chains satisfied the normalization $Dh(0,t) = I$ (and hence the chains satisfied $Df(0,t) = e^tI$). Unlike the one variable situation it is not true that the non-normalized case can be reduced to the $Dh(0,t) = I$ case (see [6, p 413]). Recently there has been interest in working with a more general normalization ([9],[10]) or no normalization at all ([3],[2]). In order to derive the existence of solutions to the Loewner chain equation one needed to make further restrictions on $Dh(0,t)$ ([9],[10]) or to enlarge the “range” of the Loewner chain ([2]).

We will treat the situation when $Dh(0,t) = A$, where $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ is such that $m(A) := \min \{ \Re \langle A(z), z \rangle : \|z\| = 1 \} > 0$. More specifically, we are interested in studying the problems considered by Graham, Hamada, Kohr and Kohr [9] without using the assumption that $k_+(A) < 2m(A)$ (see [9] Theorem 2.3 and [6] Remark 2.8). $k_+(A) := \max \{ \Re \lambda : \lambda \in \sigma(A) \} = \lim_{t \to \infty} \ln \|e^{tA}\|/t$ is the upper exponential (Lyapunov) index of A ($\sigma(A)$ denotes the spectrum of A).

Definition 1.1. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain (Loewner chain) if $f(\cdot,t)$ is holomorphic (univalent) on $B^n$ and $f(z,s) = f(v(z,s,t),t)$, $0 \leq s \leq t$, where $v(\cdot,s,t)$ is a self-map of $B^n$ fixing 0 (in other words, $f(\cdot,s)$ is subordinate to $f(\cdot,t)$) . $v$ is called the transition mapping of the chain.

Let

$N_A = \{ h \in H(B^n) : \Re \langle h(z), z \rangle > 0, Dh(0) = A \}$

where $B^n$ denotes the unit ball in $\mathbb{C}^n$. Let $H_A(B^n)$ be the class of mappings $h : B^n \times [0, \infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in N_A$ for $t \geq 0$ and $h(z,\cdot)$ is measurable on $B^n$.

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\( [0, \infty) \) for \( z \in B^n \). Such mappings will be called infinitesimal generators. We study the existence of solutions for the Loewner chain equation:

\[
(1.1) \quad \frac{\partial f}{\partial t} (z, t) = Df (z, t) h (z, t) \text{ a.e. } t \geq 0, \ z \in B^n
\]

where \( h \in \mathcal{H}_A (B^n) \).

Throughout this paper we let \( n_0 := [k_+ (A) / m (A)] \).

By [9, Theorem 2.1] we know that the Loewner equation for the transition mapping has a solution regardless of the value of \( n_0 \). More precisely, we know that the initial value problem

\[
(1.2) \quad \frac{\partial v}{\partial t} = -h (v, t) \text{ a.e. } t \geq s, \ v (z, s, s) = z, \ s \geq 0
\]

has a unique solution \( v = v(z,s,t) \) such that \( v (., s, t) \) is a univalent Schwarz mapping and \( v(z,s,\cdot) \) is Lipschitz continuous on \([s, \infty)\) locally uniformly with respect to \( z \). Furthermore we know that

\[
(1.3) \quad \frac{\|v(z,s,t)\|}{(1 - \|v(z,s,t)\|)^{\frac{1}{2}}} \leq e^{m(A)(s-t)} \frac{\|z\|}{(1 - \|z\|)^{\frac{1}{2}}}, \ z \in B^n, \ t \geq s \geq 0.
\]

If \( f(z,t) \) is a Loewner chain satisfying (1.1) then \( Df(0,t) = e^{tA} \) and we can write

\[
f(z,t) = e^{tA} \left( z + \sum_{k=2}^{\infty} F_k (z^k, t) \right)
\]

\[
h(z,t) = Az + \sum_{k=2}^{\infty} H_k (z^k, t)
\]

where \( F_k (\cdot, t) \) and \( H_k (\cdot, t) \) are homogeneous polynomial mappings of degree \( k \). We will denote the Banach space of homogeneous polynomial mappings of degree \( k \) from \( \mathbb{C}^n \) to \( \mathbb{C}^n \) by \( \mathcal{P}^k (\mathbb{C}^n) \).

Equating coefficients on both sides of (1.1) we get

\[
(1.4) \quad \frac{dF_k}{dt} (z^k, t) = B_k (F_k (z^k, t)) + N_k (z^k, t), \text{ a.e. } t \in [0, \infty)
\]

where \( B_k \) is a linear operator on \( \mathcal{P}^k (\mathbb{C}^n) \) defined by

\[
B_k (Q_k (z^k)) = kQ_k (Az, z^{k-1}) - AQ_k (z^k)
\]

and \( N_k (\cdot, t) \in \mathcal{P}^k (\mathbb{C}^n) \) is defined by

\[
N_k (z^k, t) = H_k (z^k, t) + \sum_{j=2}^{k-1} jF_j (H_k - j + 1, z^{k-j+1}, z^{j-1}, t).
\]

We will say that a solution \( f (z, t) \) of (1.1) is polynomially bounded (bounded) if \( \{e^{-tA}f (\cdot, t)\}_{t \geq 0} \) is locally polynomially bounded (locally bounded), i.e. for any compact set \( K \subset B^n \) there exists a constant \( C_K \) and a polynomial (constant polynomial) \( P \) such that

\[
\|e^{-tA}f (z, t)\| \leq C_K P (t), \ z \in K, \ t \in [0, \infty).
\]

The solutions of (1.4) will be regarded as functions \( F_k : [0, \infty) \to \mathcal{P}^k (\mathbb{C}^n) \). Consequently we will say that such \( F_k \) are polynomially bounded (bounded) if there exists a polynomial (constant polynomial) \( P \) such that \( \|F_k (t)\| \leq P (t), \ t \geq 0 \).
Proposition 2.1 will show that a polynomially bounded solution of (1.1) can be recovered from its first $n_0$ coefficients and the solution of (1.2). Conversely, Theorem 2.3 will show that by finding polynomially bounded solutions to the first $n_0$ coefficient equations (1.4) we can find a solution of (1.1). These results generalize Poreda [17, Theorem 4.1 and Theorem 4.4]. Finally, after a discussion about the existence of polynomially bounded solutions to the coefficient equations we will obtain the main result, Theorem 2.11 that guarantees the existence of a Loewner chain solution for (1.1).

We also consider what happens to the various classes of univalent mappings that are related to Loewner chains. For convenience we recall the definitions of the classes that we are considering, as given in [9], where the case $n_0 = 1$ is treated.

Let $\Omega \subset \mathbb{C}^n$ be a domain containing the origin.

**Definition 1.2.** We say that $\Omega$ is spirallike with respect to $A$ if $e^{-tA}w \in \Omega$ for any $w \in \Omega$ and $t \geq 0$.

**Definition 1.3.** We say that $\Omega$ is $A$-asymptotically spirallike if there exists a mapping $Q = Q(z, t) : \Omega \times [0, \infty) \to \mathbb{C}^n$ that satisfies the following conditions:

1. $Q(\cdot, t)$ is a holomorphic mapping on $\Omega$, $Q(0, t) = 0$, $DQ(0, t) = A$, $t \geq 0$, and the family $\{Q(\cdot, t)\}_{t \geq 0}$ is locally uniformly bounded on $\Omega$;
2. $Q(z, \cdot)$ is measurable on $[0, \infty)$ for all $z \in \Omega$;
3. the initial value problem
   \begin{equation} \frac{\partial w}{\partial t} = -Q(w, t) \text{ a.e. } t \geq s, \quad w(z, s, s) = z \end{equation}
   has a unique solution $w = w(z, s, t)$ for each $z \in \Omega$ and $s \geq 0$, such that $w(\cdot, s, t)$ is a holomorphic mapping of $\Omega$ into $\Omega$ for $t \geq s$, $w(z, s, \cdot)$ is locally absolutely continuous on $[s, \infty)$ locally uniformly with respect to $z \in \Omega$ for $s \geq 0$, and $\lim_{t \to \infty} e^{tA}w(z, 0, t) = z$ locally uniformly on $\Omega$.

Let $f : B^n \to \mathbb{C}^n$ be a normalized univalent mapping, i.e. such that $f(0) = 0$ and $Df(0) = I$. $S(B^n)$ will denote the class of all such mappings.

**Definition 1.4.** We say that $f$ is spirallike with respect to $A$ if $f(B^n)$ is spirallike. We will use $\hat{S}_A(B^n)$ to denote the class of mappings that are spirallike with respect to $A$.

**Definition 1.5.** We say that $f$ is $A$-asymptotically spirallike if $f(B^n)$ is $A$-asymptotically spirallike. $S_A^0(B^n)$ will denote the class of $A$-asymptotically spirallike mappings.

**Definition 1.6.** We say that $f$ has $A$-parametric representation if there exists a mapping $h \in \mathcal{H}_A(B^n)$ such that $f(z) = \lim_{t \to \infty} e^{tA}v(z, t)$ locally uniformly on $B^n$, where $v$ is the unique locally absolutely continuous solution of the initial value problem
   \begin{equation} \frac{\partial v}{\partial t} = -h(v, t) \text{ a.e. } t \geq 0, \quad v(z, 0) = z, \quad z \in B^n. \end{equation}
   $S_A^0(B^n)$ will denote the class of mappings with $A$-parametric representation.

From [9] we know that when $n_0 = 1$ we have that $S_A(B^n) = S_A^0(B^n)$ and the classes are compact. Also, since $\hat{S}_A(B^n)$ is a closed subset of $S_A(B^n)$ we also have that $\hat{S}_A(B^n)$ is compact when $n_0 = 1$. In Section 3 we will obtain a complete description of the $A$’s for which $\hat{S}_A(B^n)$ is compact (Theorem 3.1) and we will study
how the properties of $S^n_1 (B^n)$ (Example 3.7, Remark 3.9) and $S^n_2 (B^n)$ (Proposition 3.12, Remark 3.14) degenerate when $n_0 > 1$.

Throughout this paper the solutions of (1.1), (1.2), (1.4) and (1.6) are assumed to be locally absolutely continuous in $t$, locally uniformly with respect to $z$.

2. Solution of the Loewner Chain Equation

We will repeatedly use the fact that given $\epsilon > 0$ there exists a constant $C_\epsilon$ such that

\[
\|e^{tA}\| \leq C_\epsilon e^{t(k_+(A)+\epsilon)}, \quad t \geq 0
\]

(this follows immediately from the definition of $k_+(A)$). In fact we can find a polynomial $P_{\lambda}$ such that

\[
\|e^{tA}\| \leq P_{\lambda}(t) e^{t(k_+(A)+\epsilon)}, \quad t \geq 0
\]

(see for example [1] p 61, Exercise 16). Furthermore, if $A$ is normal then

\[
\|e^{tA}\| = e^{t(k_+(A))}, \quad t \geq 0.
\]

Indeed, if we write $A = UDU^*$ where $D$ is a diagonal matrix and $U$ is a unitary matrix then

\[
\|e^{tA}\| = \|U e^{tD} U^*\| = \|e^{tD}\| = e^{t(k_+(D))} = e^{t(k_+(A))}
\]

(for non-Euclidean norms we just get $\|e^{tA}\| \leq C_\epsilon e^{t(k_+(A))}$).

The following is a generalization of [17, Theorem 4.1].

**Proposition 2.1.** If $f(z,t)$ is a polynomially bounded solution of (1.4) such that

\[
f(z,t) = e^{tA} \left( z + \sum_{k=2}^{\infty} F_k \left( z^k, t \right) \right)
\]

then

\[
f(z,s) = \lim_{t \to \infty} e^{tA} \left( v(z,s,t) + \sum_{k=2}^{n_0} F_k \left( v(z,s,t)^k, t \right) \right)
\]

and the limit is locally uniform in $z$.

**Proof.**

\[
f(z,s) = f(v(z,s,t),t) = e^{tA} \left( v(z,s,t) + \sum_{k=2}^{n_0} F_k \left( v(z,s,t)^k, t \right) \right) + e^{tA} R(v(z,s,t),t)
\]

where $R(z,t) = \sum_{k=n_0+1}^{\infty} F_k \left( z^k, t \right)$. From the assumption on $f$, the formula for the remainder of the Taylor series and Cauchy’s formula, we easily get that $\{R(\cdot,t)\}_{t \geq 0}$ is locally polynomially bounded and in fact

\[
\|R(z,t)\| \leq C_\epsilon P(t) \|z\|^{n_0+1}, \quad \|z\| \leq r.
\]

From the above and (1.3) we get

\[
\|e^{tA} R(v(z,s,t),t)\| \leq C_\epsilon e^{t(k_+(A)+\epsilon)} P_\nu(t) \|v(z,s,t)\|^{n_0+1} \leq C_{\epsilon,r,s} e^{t(k_+(A)+\nu-(n_0+1)\nu(A))} P(t), \quad \|z\| \leq r.
\]
Hence, taking $\epsilon$ small enough we can conclude that $e^{tA}R(v(z,s,t), t) \to 0$ locally uniformly.

In order to prove Theorem 2.8 we will need the following lemmas.

**Lemma 2.2.** If $Q_k \in \mathcal{P}^k(\mathbb{C}^n)$ then the following identities hold for $t \in \mathbb{R}$

\begin{align}
\label{2.3}
e^{tA}e^{tB_k}Q_k \left( (e^{-tA}z)^k \right) &= Q_k \left( z^k \right) \\
\label{2.4}e^{tB_k}Q_k \left( (e^{-tA}z)^k \right) &= e^{tB_k}Q_k \left( z^k \right) \\
\label{2.5}e^{tA}Q_k \left( (e^{-tA}z)^k \right) &= Q_k \left( (e^{tA}z)^k \right)
\end{align}

**Proof.** Define $A_k$ on $\mathcal{P}^k(\mathbb{C}^n)$ by $A_k \left( Q_k \left( z^k \right) \right) = AQ_k \left( z^k \right)$. One easily sees that $e^{tA_k} \left( Q_k \left( z^k \right) \right) = e^{tA}Q_k \left( z^k \right)$, $A_kB_k = B_kA_k$ and $(A_k + B_k) \left( Q_k \left( z^k \right) \right) = kQ_k \left( A^z, z^{k-1} \right)$.

For (2.3) it is enough to check that $\phi(t) = e^{t(A_k + B_k)}Q_k \left( (e^{-tA}z)^k \right)$ satisfies $\phi'(t) = 0$. Indeed $\phi'(t) = e^{t(A_k + B_k)} \left[ (A_k + B_k) \left( Q_k \left( (e^{-tA}z)^k \right) \right) - kQ_k \left( e^{-tA}z, (e^{-tA}z)^{k-1} \right) \right] = 0$.

The last two identities follow immediately from the first one. \qed

**Lemma 2.3.** If $F_k, G_k : [0, \infty) \to \mathcal{P}^k(\mathbb{C}^n)$, $k = 2, \ldots, m$ are solutions of (1.4) and the limits

\begin{align}
f(z, s) &:= \lim_{t \to \infty} e^{tA} \left( v(z, s, t) + \sum_{k=2}^m F_k \left( v(z, s, t)^k, t \right) \right) \\
g(z, s) &:= \lim_{t \to \infty} e^{tA} \left( v(z, s, t) + \sum_{k=2}^m G_k \left( v(z, s, t)^k, t \right) \right)
\end{align}

exist locally uniformly in $z \in B^n$ for some $s \geq 0$, then $f(\cdot, s) = g(\cdot, s)$ if and only if $F_k = G_k, k = 2, \ldots, m$.

**Proof.** Assume that $f(\cdot, s) = g(\cdot, s)$ and that there exists $k \in \{2, \ldots, m\}$ such that $F_k \neq G_k$. Let $k_0$ be the minimal such $k$.

Equating the $k_0$-th coefficients of $f(\cdot, s)$ and $g(\cdot, s)$ and taking into account the minimality of $k_0$ we get

\begin{equation}
\label{2.6}
\lim_{t \to \infty} e^{tA} \left( F_{k_0} \left( (e^{(s-t)A}z)^{k_0}, t \right) - G_{k_0} \left( (e^{(s-t)A}z)^{k_0}, t \right) \right) = 0.
\end{equation}

Using (2.4) and the fact that $F_{k_0}$ is a solution of (1.4) we get

\begin{align}
e^{tA}F_{k_0} \left( (e^{(s-t)A}z)^{k_0}, t \right) &= e^{sA}e^{(s-t)B_k}F_{k_0} (z_{k_0}, t) \\
&= e^{sA}e^{sB_k} \left( F_{k_0} (z_{k_0}, 0) + \int_0^t e^{-sB_{k_0}}N_{k_0} (z_{k_0}, s) \, ds \right).
\end{align}

We can get an analogous identity for $G_{k_0}$ and then (2.6) becomes

\begin{equation}
e^{sA}e^{sB_k} \left( F_{k_0} (z_{k_0}, 0) - G_{k_0} (z_{k_0}, 0) \right) = 0.
\end{equation}
Since $F_{k_0}$ and $G_{k_0}$ satisfy the same differential equation with the same initial condition we have $F_{k_0} = G_{k_0}$, thus reaching a contradiction.

Lemma 2.4. If $P$ is a polynomial such that $P(t) \geq 0$ for $t \geq s$ then
\[ \int_s^\infty P(t) \left\| e^{(t-s)A} \right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1 - \|v(z,s,t)\|)^2} dt \leq \frac{Q_{e,A,P}(s)}{\left(1 - \|z\|\right)^{2\alpha m(A) + \epsilon}}, \quad \epsilon > 0 \]
where $Q_{e,A,P}$ is a polynomial of the same degree as $P$.

Proof. Let
\[ \alpha = \frac{k_+(A)}{m(A)} + \epsilon. \]
We can restrict to the case when $\epsilon$ is small enough so that $\alpha < n_0 + 1$. Using (1.3) we see that
\[ \left\| v(z,s,t) \right\|^{n_0+1} \leq \left\| v(z,s,t) \right\|^{\alpha m(A)} \leq \frac{e^{(s-t)\alpha m(A)}}{(1 - \|z\|)^{2\alpha}}. \]
Let $\epsilon'$ be small enough so that
\[ \left\| e^{(t-s)A} \right\| \leq C_{\epsilon'} e^{(t-s)\left(k_+(A) + \epsilon'\right)} \]
and $\delta := \alpha m(A) - k_+(A) - \epsilon' > 0$. Then
\[ \int_s^\infty P(t) \left\| e^{(t-s)A} \right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1 - \|v(z,s,t)\|)^2} dt \leq \frac{C_{\epsilon'}}{(1 - \|z\|)^{2\alpha}} \int_s^\infty P(t) e^{-\delta(t-s)} dt \]
and it is not hard to see that
\[ Q_{e,A,P}(s) := C_{\epsilon'} \int_s^\infty P(t) e^{-\delta(t-s)} dt \]
satisfies our requirements.

Remark 2.5. When $A$ is normal, $P$ is constant and $k_+(A)/m(A) > 1$ we can sharpen the above bound by letting $\epsilon = 0$.

Let $\beta = n_0 + 1 - \alpha$, then using (1.3) we get
\[ \left\| v(z,s,t) \right\|^{n_0+1} \leq \frac{e^{(s-t)\alpha m(A)}}{(1 - \|z\|)^{2\alpha}} \left\| v(z,s,t) \right\|^{\beta} (1 - \|v(z,s,t)\|)^{2(\alpha-1)}. \]
If $A$ is normal we know that
\[ \left\| e^{(t-s)A} \right\| = e^{(t-s)k_+(A)} \]
and hence we get
\[ \int_s^\infty \left\| e^{(t-s)A} \right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1 - \|v(z,s,t)\|)^2} dt \leq \frac{1}{(1 - \|z\|)^{2\alpha}} \int_s^\infty \left\| v(z,s,t) \right\|^{\beta} (1 - \|v(z,s,t)\|)^{2(\alpha-1)} dt. \]
From the proof of [9] Theorem 2.1 we know that
\[ -\frac{1 + \|v(z,s,t)\|}{1 - \|v(z,s,t)\|} \frac{d\|v(z,s,t)\|}{dt} \geq m(A). \]
Using the above inequality it is easy to conclude that
\[
\int_s^\infty \|v(z, s, t)\|^2 \frac{(1 - \|v(z, s, t)\|)^{2(\alpha - 1)}}{d\|v(z, s, t)\|} dt
\]
\[
\leq - \frac{2}{m(A)} \int_s^\infty \|v(z, s, t)\|^{\beta - 1} (1 - \|v(z, s, t)\|)^{2\alpha - 3} \frac{d\|v(z, s, t)\|}{dt} dt
\]
\[
= \frac{2}{m(A)} \int_0^\infty u^{\beta - 1} (1 - u)^{2\alpha - 3} du
\]
\[
\leq \frac{2}{m(A)} \int_0^1 u^{\beta - 1} (1 - u)^{2\alpha - 3} du.
\]
The last integral converges because by our assumptions \(\beta - 1 > -1\) and \(2\alpha - 3 > -1\). This completes the proof of our claim.

**Lemma 2.6.** If \(F_k : [0, \infty) \to \mathcal{P}^k(\mathbb{C}^n)\), \(k = 2, \ldots, m\) are polynomially bounded and the limit
\[
f(z, s) = \lim_{t \to \infty} e^{tA} \left( v(z, s, t) + \sum_{k=2}^m F_k(v(z, s, t)^k, t) \right)
\]
exists locally uniformly in \(z \in B^n\) for some \(s \geq 0\) then \(f(\cdot, s)\) is univalent.

**Proof.** First note that if \(Q \in \mathcal{P}^k(\mathbb{C}^n)\) then
\[
\|Q(z^k) - Q(w^k)\| = \left\| \sum_{j=0}^{k-1} Q(z - w, z^j, w^{k-1-j}) \right\|
\]
\[
\leq \|Q\| \|z - w\| \sum_{j=0}^{k-1} \|z^j\| \|w\|^{k-1-j}.
\]

Using the above and (13) we see that
\[
\left\| \sum_{k=2}^m F_k(v(z_1, s, t)^k, t) - \sum_{k=2}^m F_k(v(z_2, s, t)^k, t) \right\|
\]
\[
\leq C P(t) e^{(s-t)m(A)} \|v(z_1, s, t) - v(z_2, s, t)\|, \|z_1\|, \|z_2\| \leq r
\]
where \(P\) is a polynomial bound on \(F_k\), \(k = 2, \ldots, m\). For sufficiently large \(t\) we get
\[
\left\| \sum_{k=2}^m F_k(v(z_1, s, t)^k, t) - \sum_{k=2}^m F_k(v(z_2, s, t)^k, t) \right\|
\]
\[
< \|v(z_1, s, t) - v(z_2, s, t)\|, \|z_1\|, \|z_2\| \leq r
\]
which implies that for sufficiently large \(t\)
\[
v(z, s, t) + \sum_{k=2}^m F_k(v(z, s, t)^k, t)
\]
is univalent on the ball \(\|z\| \leq r\). Now the conclusion follows easily. \(\square\)

The following consequence together with Theorem 2.8 generalizes [9, Theorem 2.6].

**Corollary 2.7.** All polynomially bounded solutions of (1.1) are Loewner chains.
Proof. This follows from Proposition 2.1 and Lemma 2.6.

\[\square\]

**Theorem 2.8.** If \(F_k, k = 2, \ldots, n_0\) are polynomially bounded solutions of (1.4) then
\[g(z, s) := \lim_{t \to \infty} e^{tA} \left(v(z, s, t) + \sum_{k=2}^{n_0} F_k \left(v(z, s, t)^k, t \right)\right)\]
exists locally uniformly with respect to \(z\) and is a polynomially bounded Loewner chain solution of (1.1). If \(F(t)\) is a polynomial bound for \(F_k, k = 2, \ldots, n_0\) then, given \(\epsilon > 0\), there exists a polynomial \(Q_{\epsilon, A, F}\) of the same degree as \(F\) such that
\[\|e^{-tA}g(z, t)\| \leq \frac{Q_{\epsilon, A, F}(t)}{(1 - \|z\|)^{2m(A) + \epsilon}}, z \in B^n, t \geq 0.\]

Furthermore, if
\[g(z, t) = e^{tA} \left(z + \sum_{k=2}^{\infty} G_k(z^k, t)\right)\]
then \(G_k = F_k, k = 2, \ldots, n_0.\)

**Proof.** Let
\[u(z, s, t) = e^{tA} \left(v(z, s, t) + \sum_{k=2}^{n_0} F_k \left(v(z, s, t)^k, t \right)\right).\]

We begin by showing that \(\lim_{t \to \infty} u(z, s, t)\) exists locally uniformly.

It is easy to see that \(u(z, s, t)\) is locally absolutely continuous in \(t\), so
\[(2.8)\]
\[\|u(z, s, t_1) - u(z, s, t_2)\| = \left\| \int_{t_1}^{t_2} \frac{\partial u}{\partial t} (z, s, t) \, dt \right\| \leq \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} (z, s, t) \right\| \, dt, s \leq t_1 \leq t_2.\]

Now
\[\frac{\partial u}{\partial t} (z, s, t) = e^{tA}A \left(v(z, s, t) + \sum_{k=2}^{n_0} F_k \left(v(z, s, t)^k, t \right)\right) - e^{tA} \left(h(v(z, s, t), t) + \sum_{k=2}^{n_0} kF_k \left(v(z, s, t)^{k-1}, h(v(z, s, t), t), t \right) - \sum_{k=2}^{n_0} \frac{dF_k}{dt} \left(v(z, s, t)^k, t \right)\right).\]

Let
\[R(z, t) = h(z, t) - Az - \sum_{k=2}^{n_0} H_k(z^k, t).\]

Similarly to the proof of [17 Theorem 4.4], a straightforward computation using the assumption that \(F_k, k = 2, \ldots, n_0\) satisfy (1.4), leads to
\[(2.9)\]
\[\frac{\partial u}{\partial t} (z, s, t) = -e^{tA} \left(R(v(z, s, t), t) - \sum_{k=2}^{n_0} kF_k \left(v(z, s, t)^{k-1}, R(v(z, s, t), t)\right)\right)\]
\[-e^{tA} \left(\sum_{k=2}^{n_0} \sum_{l=n_0-k+2}^{n_0} kF_k \left(v(z, s, t)^{k-1}, H_l(v(z, s, t)^l, t), t \right)\right).\]
Using the ideas from the proof of [8] Theorem 1.2 (cf. [9] Lemma 1.2) we get
\begin{equation}
\|R(z,t)\| \leq C_A \frac{\|z\|^{n_0+1}}{(1 - \|z\|)^2}
\end{equation}
From (2.9), (2.10) and (1.3) we get
\begin{equation}
\left| \frac{\partial u}{\partial t} (z, s, t) \right| \leq P(t) \left| e^{tA} \right| \left| v(z, s, t) \right|^{n_0+1} \left(1 - \|v(z, s, t)\|\right)^2
\end{equation}
where \( P \) is a polynomial depending only on \( F \) and \( A \) (because the bounds on \( H_k \) can be chosen to depend only on \( A \)). Substituting this estimate into (2.8) we get
\begin{equation}
\|u(z, s, t_1) - u(z, s, t_2)\| \leq \|e^{sA}\| \int_{t_1}^{t_2} P(t) \left| e^{(t-s)A} \right| \left| v(z, s, t) \right|^{n_0+1} \left(1 - \|v(z, s, t)\|\right)^2 dt.
\end{equation}
Using Lemma 2.4 we can now conclude that \( \lim_{t \to \infty} u(z, s, t) \) exists uniformly on compact subsets.

From the semigroup property for \( v \) we immediately get that
\begin{equation}
g(v(z, s, t), t) = g(z, s), 0 \leq s \leq t
\end{equation}
and by Lemma 2.6 we can conclude that \( g(z, t) \) is a Loewner chain. Differentiating (2.11) with respect to \( t \) and then letting \( s \nearrow t \) we see that \( g \) is a solution of (1.1).

By the same considerations as above we get
\begin{equation}
\|e^{-sA}(u(z, s, t_1) - u(z, s, t_2))\| \leq \int_{t_1}^{t_2} P(t) \left| e^{(t-s)A} \right| \left| v(z, s, t) \right|^{n_0+1} \left(1 - \|v(z, s, t)\|\right)^2 dt.
\end{equation}
Letting \( t_1 = s \) and \( t_2 \to \infty \) and using Lemma 2.4 we get
\begin{equation}
\|e^{-sA}g(z, s)\| \leq \|z\| + \int_s^{\infty} P(t) \left| e^{(t-s)A} \right| \left| v(z, s, t) \right|^{n_0+1} \left(1 - \|v(z, s, t)\|\right)^2 dt
\end{equation}
(P depends on \( A \) and \( F \)). The above guarantees that the solution is polynomially bounded.

The last statement follows from Proposition 2.1 and Lemma 2.3.

For the proof of Theorem 2.11 we will need some basic facts about ordinary differential equations. We will use [5] for this, but we are interested only in the finite dimensional case.

Let \( X \) be a finite dimensional Banach space and \( L \) be a bounded linear operator on \( X \). We consider the equation
\begin{equation}
\frac{dx}{dt} = Lx + f(t), \text{ a.e. } t \geq 0
\end{equation}
where \( f : [0, \infty) \to X \) is a locally Lebesgue integrable function. We know that any (locally absolutely continuous) solution of (2.13) is of the form
\[ x(t) = e^{tL}x(0) + \int_0^t e^{(t-s)L}f(s) \, ds. \]
Note that the local Lebesgue integrability of \( f \) is needed to ensure the differentiability of the solution above, which follows from the Lebesgue differentiation theorem (see [7] Theorem 3.21).
We will use the following notations
\[ \sigma_+ (L) = \{ \lambda \in \sigma (L) : \text{Re} \lambda > 0 \} \]
\[ \sigma_- (L) = \{ \lambda \in \sigma (L) : \text{Re} \lambda \leq 0 \} \]
\[ \sigma_0 (L) = \{ \lambda \in \sigma (L) : \text{Re} \lambda = 0 \} . \]

\( P^+ \), \( P^- \) and \( P^0 \) will denote the spectral projections corresponding to \( \sigma_+ (L) \), \( \sigma_- (L) \) and \( \sigma_0 (L) \) respectively (see e.g. [5] p 19). We say that \( f \) is polynomially bounded if there exists a polynomial \( P \) such that
\[ ||f(t)|| \leq P(t), t \geq 0 \]

Following the proof of [5] Chapter II, Theorem 4.2 it is straightforward to check that if \( f \) is polynomially bounded then to each element \( x^\leq \in P^\leq (X) \) there corresponds a unique polynomially bounded solution of (2.13) that satisfies \( P^\leq x(0) = x^\leq_0 \). This solution is given by the formula
\[ x(t) = e^{(t-t_0)L}x^\leq_0 + \int_0^\infty G_L (t-s) f(s) ds \]

where
\[ G_L (t) = \begin{cases} e^{tL}P^\leq , & t \geq 0 \\ -e^{tL}P^+, & t < 0. \end{cases} \]

Furthermore if \( f \) is bounded and \( \sigma_0 (L) = \emptyset \) then the solution (2.14) is bounded. Note that in order to obtain a polynomial bound on the solution one needs to use (2.9) rather than (2.1).

Remark 2.9. We are only interested in the case when \( X = \mathcal{P}^k (\mathbb{C}^n) \), but the above considerations also apply to the case when \( X \) is not finite dimensional if we replace polynomially bounded by subexponential. We say that \( f \) is subexponential if for any \( \epsilon > 0 \) there exists \( C_\epsilon \) such that \( ||f(t)|| \leq C_\epsilon e^{\epsilon t}, t \geq 0 \). To be able to define the spectral projections we would also need to require that \( \sigma_+ (L), \sigma_- (L) \) lie in different connected components of \( \sigma (L) \).

We will apply the above results to the coefficient equations (1.4) \((X = \mathcal{P}^k (\mathbb{C}^n) \) and we regard the coefficients as functions subject to the maps \( F_k : [0, \infty) \rightarrow \mathcal{P}^k (\mathbb{C}^n) \). We need information about the spectra of the operators \( B_k \). It is known (e.g. [11] pp. 182-183) that if \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a vector whose components are the (not necessarily distinct) eigenvalues of \( A \) then the eigenvalues of \( B_k \) are
\[ \{ (m, \lambda) - \lambda_s : |m| = k, s \in \{1, \ldots, n\} \} \]

where \( m = (m_1, \ldots, m_n) \in \mathbb{N}^n \) and \( |m| = m_1 + \ldots + m_n \). Furthermore, if \( A \) is a diagonal matrix then \( B_k \) is also diagonal and \( z^m e_s \) is an eigenvector corresponding to \( (m, \lambda) - \lambda_s \) (\( e_i, i = 1, \ldots, n \) denote the elements of the standard basis of \( \mathbb{C}^n \)).

Following the terminology from [11] we will say that \( A \) is nonresonant if \( 0 \notin \sigma (B_k) \) for all \( k \) (i.e. if the eigenvalues of \( A \) are nonresonant; see [11] p 180). Otherwise we say that \( A \) is resonant.

Remark 2.10. \( 0 \notin \sigma (B_k) \) for all \( k > n_0 \) (i.e. there are no resonances of order greater than \( n_0 \)). Indeed, if \( m \in \mathbb{N}^n, |m| = k \) and \( \lambda \) is as above then
\[ \text{Re} ((m, \lambda) - \lambda_s) \geq (n_0 + 1) k_+ (A) - k_- (A) > 0 \]
where \( k_- (A) = \min \{ \Re \lambda : \lambda \in \sigma (A) \} \). For the last inequality we used the fact that \( k_- (A) \geq m (A) \) and the definition of \( n_0 \) (which implies that \( k_+ (A) < (n_0 + 1) m (A) \)). In particular note that if \( n_0 = 1 \) then \( A \) is nonresonant.

We will use \( P_k^+ \), \( P_k^\leq \), \( P_k^0 \) to denote the projections associated with \( B_k \). For \( Q \in \mathcal{P}^k (\mathbb{C}^n) \) we will let \( Q^+ := P_k^+ Q \), \( Q^\leq := P_k^\leq Q \) and \( Q^0 := P_k^0 Q \).

**Theorem 2.11.** The equation (1.1) always has a polynomially bounded Loewner chain solution that is uniquely determined by the values of \( F_k^\leq (z^k, 0) \), \( k = 2, \ldots, n_0 \), which can be prescribed arbitrarily. Furthermore, if \( A + \tilde{A} \) is nonresonant then the solution can be chosen to be bounded.

**Proof.** This is an immediate consequence of Theorem 2.8 and of the considerations on solutions of ordinary differential equations from above. Note that \( A + \tilde{A} \) is nonresonant if and only if \( \sigma_0 (B_k) = \emptyset \), \( k \geq 2 \).

**Remark 2.12.** It is not hard to see that if \( A + \tilde{A} \) is resonant, in general, one cannot find a bounded solution (though it will be possible to do this for particular choices of the infinitesimal generator \( h \)). For example, one can choose \( A \) such that \( \sigma_0 (B_2) = \{ 0 \} \) and 0 is a simple eigenvalue for \( B_2 \). In this case we would have \( e^{B_2 \left| P_2^0 (\mathbb{P}^2 (\mathbb{C}^n)) \right|} = I_{P_2^0 (\mathbb{P}^2 (\mathbb{C}^n))} \) and so

\[
F_2^0 (z^2, t) = F_2^0 (z^2, 0) + \int_0^t H_2^0 (z^2, s) \, ds.
\]

In order to get a solution that is not bounded it is enough to choose \( h \) such that \( \phi (t) := \int_0^t H_2^0 (z^2, s) \, ds \) is not bounded on \( [0, \infty) \).

**Definition 2.13.** Let \( \mathcal{F} \subset \prod_{k=2}^{n_0} \mathcal{P}_k^\leq (\mathbb{P}^k (\mathbb{C}^n)) \). We define \( S_A^\mathcal{F} (B^n) \) to be the family of mappings \( f (z) = z + \sum_{k=2}^{n_0} F_k (z^k) \in S (B^n) \) that can be embedded as the first element of a polynomially bounded Loewner chain and such that \( \left( F_k^\leq \right)_{k=2, \ldots, n_0} \in \mathcal{F} \).

We want to study the compactness of the class \( S_A^\mathcal{F} (B^n) \). For this we need the following lemma that can be proved using similar arguments to those in the proof of [13, Lemma 2.8] (cf. [9, Lemma 2.14]).

**Lemma 2.14.** Every sequence of Loewner chains \( \{ f_k (z, t) \} \) such that \( D f_k (0, t) = e^{t A} \) and

\[
\| e^{-t A} f_k (z, t) \| \leq C_r P (t), \quad \| z \| \leq r < 1, \quad t \geq 0,
\]

where \( P (t) \) is a polynomial, has a subsequence that converges locally uniformly on \( B^n \) to a polynomially bounded Loewner chain \( f (z, t) \) for \( t \geq 0 \).

**Theorem 2.15.** If \( \mathcal{F} \subset \prod_{k=2}^{n_0} \mathcal{P}_k^\leq (\mathbb{P}^k (\mathbb{C}^n)) \) is bounded (compact) then \( S_A^\mathcal{F} (B^n) \) is normal (compact).

**Proof.** Let \( f \in S_A^\mathcal{F} (B^n) \) and \( f (z, t) \) be a polynomially bounded Loewner chain such that \( f (z, 0) = f (z) \). Suppose that

\[
f (z, t) = e^{t A} \left( z + \sum_{k=2}^{\infty} F_k (z^k, t) \right).
\]
We know that (see (1.4) and (2.14))
\[ F_k(z^k, t) = e^{tB_k}F^\leq_k(z^k, 0) + \int_0^\infty G_{B_k}(t-s)N_k(s)\,ds. \]

Now it is straightforward to check that if \( F \) is bounded then \( F_{k}, k = 2, \ldots, n_0 \) can be bounded by a polynomial \( F \) that doesn’t depend on \( f \) (it depends only on \( F \) and \( A \)). By Theorem 2.8 we have
\[ \|e^{-tA}f(z, t)\| \leq \frac{Q_{t, A, F}(t)}{(1 - \|z\|^{2m(A)+1})^{\epsilon}}. \]

When \( t = 0 \) the above inequality proves the fact that \( S^F_A(B^n) \) is normal. Furthermore, if \( F \) is also closed we can now argue by contradiction using the previous Lemma to see that \( S^F_A(B^n) \) is also closed. \( \square \)

**Remark 2.16.** It is not hard to see that the results of this section (except for Remark 2.5) remain true for any norm on \( \mathbb{C}^n \). Furthermore, with appropriate modifications (see Remark 2.9 and [15]) the results can be extended to reflexive complex Banach spaces.

3. **Spirallikeness, parametric representation, asymptotical spirallikeness**

We start by answering [12, Open Problem 6.4.13].

**Theorem 3.1.** \( \hat{S}_A(B^n) \) is compact if and only if \( A \) is nonresonant.

**Proof.** If \( f \in \hat{S}_A(B^n) \) we know that
\[ f(z, t) := e^{tA}f(z) = e^{tA} \left( z + \sum_{k=2}^{\infty} F_k(z^k) \right) \]
is a Loewner chain (this follows easily from the definitions). It is clear that \( \hat{S}_A(B^n) \subset S^F_A(B^n) \), where
\[ \mathcal{F} := \left\{ \left( F^\leq_k \right)_{k=2, \ldots, n_0} : f(z) = z + \sum_{k=2}^{\infty} F_k(z^k) \in \hat{S}_A(B^n) \right\}. \]

It is easy to see that \( \hat{S}_A(B^n) \) is closed by using the analytic characterization (3.2) and the fact that \( \mathcal{N}_A \) is compact. Now, by Theorem 2.15 if the coefficients \( F_k \), \( k = 2, \ldots, n_0 \) can be bounded independently of \( f \) then \( \hat{S}_A(B^n) \) is compact. For our particular Loewner chain (3.1) the coefficient equations (1.4) take the simple form 0 = \( B_kF_k + N_k \).

If \( A \) is nonresonant then the operators \( B_k \) are invertible and hence \( F_k = -B_k^{-1}N_k \). Now it is straightforward to see that we can choose bounds for \( F_k, k = 2, \ldots, n_0 \) that don’t depend on \( f \), thus yielding compactness of \( \hat{S}_A(B^n) \).

If \( A \) is resonant then let \( k_0 \leq n_0 \) be the largest \( k \) such that \( B_k \) is singular (by Remark 2.10 \( B_k \) is not singular for \( k > n_0 \)). Let \( h(z) = Az + H_{k_0}(z^{k_0}) \in \mathcal{N}_A \), where \( H_{k_0} \) is chosen such that \( B_{k_0}F_{k_0} + H_{k_0} = 0 \) has a solution. Note that for our particular \( h \) we have \( N_k = 0, k = 2, \ldots, k_0 - 1 \) and \( N_{k_0} = H_{k_0} \). Since \( B_k \),
If \( k > k_0 \) are nonsingular there is no problem in solving for \( F_k, k > k_0 \) and then, using Theorem 2.8, we get that

\[
f(z, s) = \lim_{t \to \infty} e^{tA} \left( v(z, s, t) + \sum_{k=2}^{n_0} F_k \left( v(z, s, t)^k \right) \right)
\]

is a Loewner chain solution of (1.1) with \( h(z, t) = h(z) \). Since \( h \) doesn’t depend on \( t \) we have \( v(z, s, t) = v(z, 0, t - s) \) and this yields that \( f(z, s) = e^{sA} f(z, 0) \). Hence \( f(\cdot, 0) \in \hat{S}_A(B^n) \) and by Theorem 2.8 it’s \( k_0 \)-th coefficient is \( F_{k_0} \).

This construction works with any \( F_{k_0} \) that is a solution of \( B_{k_0} F_{k_0} + H_{k_0} = 0 \). Since \( B_{k_0} \) is singular, the solutions of the equation form a non-trivial affine subspace of \( P^{k_0}(\mathbb{C}^n) \), so in particular there exist solutions of arbitrarily large norm. Now we can conclude that there exist spirallike mappings with arbitrarily large \( k_0 \)-th coefficient. This proves that \( \hat{S}_A(B^n) \) is not compact when \( A \) is resonant.

\[ \square \]

Remark 3.2. Let \( h \in N_A \). By the same ideas as in the proof of the previous theorem we can conclude that if \( A \) is nonresonant then the equation

\[
Df(z) h(z) = Af(z)
\]

has a unique holomorphic solution, which is in fact biholomorphic (because of Corollary 2.7). By Remark 2.10 this generalizes [6, Corollary 4.8]. On the other hand, if \( A \) is resonant, there either is no holomorphic solution (for example if \( H_2 \notin B_2(P^2(\mathbb{C}^n)) \)) or the holomorphic solutions (in fact, biholomorphic) are not unique.

Remark 3.3. As a consequence of the proof of Theorem 3.1 and of Theorem 2.15 we have the following bound for mappings in \( \hat{S}_A(B^n) \):

\[
\|f(z)\| \leq \frac{C_{e,A}}{(1 - \|z\|)^{2/m(A)} + \epsilon}, \quad z \in B^n, \epsilon > 0, f \in \hat{S}_A(B^n)
\]

(cf. [14] Theorem 3.1 and [13] Theorem 12]). Furthermore, if \( A \) is normal the above estimate holds with \( \epsilon = 0 \) (the case \( k_+(A)/m(A) > 1 \) follows using Remark 2.5 while the case \( k_+(A)/m(A) = 1 \) is covered by [14] Corollary 3.1)

Remark 3.4. Let \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) (\( \text{Re} \lambda_i > 0 \)) and \( m \in \mathbb{N}_n \) with \( m_i = 0, i = 1, \ldots, s \), where \( 1 \leq s < n \). Then it is easy to compute that for \( f(z) = z + az^m e_s \) we have

\[
h(z) = [Df(z)]^{-1} Af(z) = Az + a(\lambda_s - \langle m, \lambda \rangle) z^m e_s.
\]

If \( \lambda_s - \langle m, \lambda \rangle = 0 \) we get that \( f \in \hat{S}_A(B^n) \) for any \( a \in \mathbb{C}^n \) generalizing an example from [14] p 57. If \( \lambda_s - \langle m, \lambda \rangle \neq 0 \) then \( f \in \hat{S}_A(B^n) \) for any \( a \) such that

\[
|a| \leq \frac{m(A)}{|\lambda_s - \langle m, \lambda \rangle|}.
\]

This example suggests that in the case when \( A \) is nonresonant a sharp upper growth bound on \( \hat{S}_A(B^n) \) would have to depend on the entire spectrum of \( A \).

Next we extend [14] Corollary 2.2. For simplicity we only treat the 2-dimensional case. \( S^*(B^n) = \hat{S}_f(B^n) \) denotes the class of normalized starlike mappings.
Proposition 3.5. Let $A = \text{diag} (1, \lambda)$, $\text{Re} \lambda \geq 1$. Define $\Phi_{\alpha, \beta} : S^* (B^1) \to \hat{S}_A (B^2)$ by

$$
\Phi_{\alpha, \beta} (f) (z) = \left( f (z_1), \left( \frac{f (z_1)}{z_1} \right)^{\alpha} (f' (z_1))^{\beta} z_2 \right).
$$

If $\alpha \in [0, \text{Re} \lambda]$ and $\beta \in [0, 1/2]$ such that $\alpha + \beta \leq \text{Re} \lambda$ then $\Phi_{\alpha, \beta} (S^* (B^1)) \subset \hat{S}_A (B^2)$.

Proof. We follow the proof of [11, Theorem 2.1]. Let $f \in S^* (B^1)$ and define

$$
F (z, t) = e^{tA} \Phi_{\alpha, \beta} (f) (z) = \left( e^{tA} f (z_1), e^{tA} \left( \frac{f (z_1)}{z_1} \right)^{\alpha} (f' (z_1))^{\beta} z_2 \right).
$$

It is sufficient to check that $F (z, t)$ is a Loewner chain. Because of the particular form of $F$ and by Corollary 2.7 it is enough to check that $F$ satisfies a Loewner chain equation, i.e. that

$$
h (\cdot, t) := [DF (\cdot, t)]^{-1} \frac{\partial F}{\partial t} (\cdot, t) \in \mathcal{H}_A (B^2), \text{ a.e. } t \geq 0.
$$

Let $p (z_1) = f (z_1) / (z_1 f' (z_1))$. Straightforward computations yield that

$$
h (z, t) = \left( z_1 p (z_1), z_2 (\lambda - \alpha - \beta + (\alpha + \beta) p (z_1) + \beta z_1 p' (z_1)) \right).
$$

The same arguments as in the proof of [11, Theorem 2.1] (we are using the fact that $f \in S^* (B^1)$ implies that $\text{Re} p > 0$) show that it is sufficient to check that

$$
q (x) = (\text{Re} \lambda - \alpha - \beta) x^2 - 2 \beta x + \alpha + \beta
$$

is non-negative on $[0, 1]$. This follows by elementary analysis. \hfill $\square$

Remark 3.6. Let $A$ be as in the Proposition above. For $\alpha = \text{Re} \lambda - 1/2$, $\beta = 1/2$ and $f (z) = z / (1 - z)^2$ we can see that $\Phi_{\alpha, \beta} (f) \in \hat{S}_A (B^2)$ attains the asymptotic growth bound from Remark 3.3.

Next we consider the class of mappings with $A$-parametric representation. Unlike the class of spirallike mappings, the class $S^a_\alpha (B^n)$ is not compact when $n_0 > 1$, as we can see from the following example.

Example 3.7. Let $A = \text{diag} (\lambda, 1)$, $\text{Re} \lambda \geq 2$ and define

$$
h (z, t) = \left( \lambda z_1 + a (t) z_2^2, z_2 \right), \quad z = (z_1, z_2) \in B^2.
$$

If for example $|a (t)| \leq 1$, $t \geq 0$ it is easy to check that $h (\cdot, t) \in \mathcal{N}_A$, $t \geq 0$. Then

$$
v (z, t) = \left( e^{\lambda t} \left( z_1 - \left( \int_0^t a (s) e^{(\lambda - 2) s} ds \right) z_2^2 \right), e^{-t} z_2 \right)
$$

is the solution of (1.6). When $\lim_{t \to \infty} e^{tA} v (z, t)$ exists locally uniformly on $B^n$ we get that $f (z) = (z_1 - \left( \int_0^\infty a (s) e^{(\lambda - 2) s} ds \right) z_2^2, z_2) \in S^a_\alpha (B^2)$. Since the second coefficient of the Taylor series expansion can be made arbitrarily large by an appropriate choice of $a (\cdot)$ we conclude that $S^a_\alpha (B^2)$ is not compact.

This example can be generalized for any $A$ by considering $h (z, t) = A z + a (t) H_2 (z^2) \in \mathcal{H}_A (B^n)$ such that $H_2 \subseteq 0$.

Next we consider the class $S^a_\alpha (B^n)$. The following characterization of $A$-asymptotically spirallike mappings is derived from the proofs of [9, Theorem 3.1 and Theorem 3.5].
Proposition 3.8. Let \( f : B^n \to \mathbb{C}^n \) be a holomorphic mapping and

\[
f(z) = z + \sum_{k=2}^{\infty} F_k(z^k).
\]

Then \( f \) is \( A \)-asymptotically spirallike if and only if there exists \( h \in \mathcal{H}_A(B^n) \) such that

\[
f(z) = \lim_{t \to \infty} e^{tA} \left( v(z,t) + \sum_{k=2}^{n_0} F_k\left(v(z,t)^k\right)\right)
\]

locally uniformly on \( B^n \), yielding the desired conclusion (the fact that \( f \) is \( A \)-asymptotically spirallike).

**Proof.** First assume that \( f \) is \( A \)-asymptotically spirallike. Hence there exists a mapping \( Q : f(B^n) \times [0,\infty) \to \mathbb{C}^n \) satisfying the assumptions from Definition 1.3. Let \( \nu \) be the solution of the initial value problem (1.6). By definition it will satisfy

\[
\lim_{t \to \infty} e^{tA} \nu(f(z),0,t) = f(z)
\]

locally uniformly on \( B^n \).

Let \( v \) be defined by \( v(z,s,t) = f^{-1}(\nu(f(z),s,t)), z \in B^n, t \geq s \). Also, let \( h(z,t) = (Df(z))^{-1}Q(f(z),t), z \in B^n, t \geq 0 \). With the same proof as in [9, Theorem 3.5] one sees that \( h \in \mathcal{H}_A(B^n) \) and that \( v \) is the solution of (1.6).

We have

\[
f(z) = \lim_{t \to \infty} e^{tA} \nu(f(z),0,t) = \lim_{t \to \infty} e^{tA} f(v(z,0,t))
\]

locally uniformly on \( B^n \). Like in the proof of Proposition 2.1 we also see that

\[
\lim_{t \to \infty} e^{tA} f(v(z,0,t)) = \lim_{t \to \infty} e^{tA} \left( v(z,0,t) + \sum_{k=2}^{n_0} F_k\left(v(z,0,t)^k\right)\right)
\]

yielding the desired conclusion (the fact that \( f \) is univalent follows from Lemma 2.6).

Now assume that (3.3) holds. The conclusion follows exactly as in the proof of [9, Theorem 3.1].

**Remark 3.9.** From the above characterization of \( \mathcal{S}_A^\alpha(B^n) \) it is easy to see that \( \mathcal{S}_A^\alpha(B^n) \neq \mathcal{S}_A^\beta(B^n) \) when \( n_0 > 1 \).

In Proposition 3.12 we obtain a partial result about the normality of the class \( \mathcal{S}_A^\alpha(B^n) \), but first we need the following lemmas.

**Lemma 3.10.** Let \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and \( h \in \mathcal{H}_A(B^n) \). If \( v = (v_1, \ldots, v_n) \) is the solution of (1.6) then

\[
\|v_t(z,t)\| \leq C \begin{cases} e^{-\Re\lambda_1 t}, & \Re\lambda_1 < 2m(A) \\ (1+t)e^{-\Re\lambda_1 t}, & \Re\lambda_1 = 2m(A) \\ e^{-2m(A)t}, & \Re\lambda_1 > 2m(A) \end{cases}
\]

where \( C \) is a constant that depends on \( A, \lambda_i \) and \( \|z\| \).

**Proof.** Writing \( h = (h_1, \ldots, h_n) \) and \( \tilde{h}_i = h_i - \lambda_i v_i, (1.6) \) yields

\[
\frac{dv_i}{dt} = -\lambda_i v_i - \tilde{h}_i(v,t).
\]
Integrating we get
\[ e^{t \lambda_i} v_i = z_i - \int_0^t e^{s \lambda_i} \tilde{h}_i(v, s) \, ds. \]

Hence
\[
\left\| e^{t \lambda_i} v_i (z, t) \right\| \leq |z_i| + \int_0^t e^{s \Re \lambda_i} \| h(v(z, s), s) - Av(z, s) \| \, ds \\
\leq |z_i| + C_A \| z \| \int_0^t e^{s \Re \lambda_i} \| v(z, s) \|^2 \, ds \\
\leq |z_i| + C_A \| z \| \int_0^t e^{s (\Re \lambda_i - 2m(A))} \, ds \\
\leq \begin{cases} 
C_A \| z \| \lambda_i, & \text{Re} \lambda_i < 2m(A) \\
C_A \| z \| (1 + t), & \text{Re} \lambda_i = 2m(A) \\
C_A \| z \| \lambda_i e^{(\Re \lambda_i - 2m(A))t}, & \text{Re} \lambda_i > 2m(A)
\end{cases}
\]

(0.5)

(3.4) \[ \lim_{t \to \infty} \int_0^t e^{s \lambda} (h(s) + a) \, ds = 0 \]

then \( |a| \leq C \).

**Proof.** We argue by contradiction. Assume that \( |a| > C \). Then
\[ \Re \left( (h(s) + a) \bar{a} \right) \geq |a|^2 - |a| |h(s)| \geq |a| (|a| - C') =: \delta > 0. \]

If \( \Im \lambda = 0 \) we get
\[ \Re \left( \left( \int_0^t e^{s \lambda} (h(s) + a) \, ds \right) \bar{a} \right) \geq t \delta \]

contradicting (3.4).

If \( \Im \lambda \neq 0 \) we can find \( \tau > 0 \) such that
\[ \Re \left( e^{s \lambda} (h(s) + a) \bar{a} \right) \geq \frac{\delta}{2}, \quad s \in \left[ \frac{2k \pi}{\Im \lambda}, \frac{2(k+1) \pi}{\Im \lambda} \right], \quad k \geq 0, \ k \in \mathbb{Z}. \]

From (3.4) we get that
\[ \lim_{t \to \infty} \int_t^{t+\tau} e^{s \lambda} (h(s) + a) \, ds = 0. \]

This is contradicted by
\[ \Re \left( \left( \int_{t_k}^{t_k+\tau} e^{s \lambda} (h(s) + a) \, ds \right) \bar{a} \right) \geq \frac{\delta \tau}{2}, \]
where \( t_k = 2k \pi / \Im \lambda. \) Thus we must have that \( |a| \leq C. \)

**Lemma 3.11.** Let \( \lambda \in \mathbb{C} \) such that \( \Re \lambda \geq 0 \), \( a \in \mathbb{C} \) and \( h : [0, \infty) \to \mathbb{C} \) such that \( |h(t)| \leq C, \ t \geq 0. \) If

(3.4) \[ \lim_{t \to \infty} \int_0^t e^{s \lambda} (h(s) + a) \, ds = 0 \]

then \( |a| \leq C. \)

**Proof.** We argue by contradiction. Assume that \( |a| > C \). Then
\[ \Re \left( (h(s) + a) \bar{a} \right) \geq |a|^2 - |a| |h(s)| \geq |a| (|a| - C') =: \delta > 0. \]

If \( \Im \lambda = 0 \) we get
\[ \Re \left( \left( \int_0^t e^{s \lambda} (h(s) + a) \, ds \right) \bar{a} \right) \geq t \delta \]

contradicting (3.4).

If \( \Im \lambda \neq 0 \) we can find \( \tau > 0 \) such that
\[ \Re \left( e^{s \lambda} (h(s) + a) \bar{a} \right) \geq \frac{\delta}{2}, \quad s \in \left[ \frac{2k \pi}{\Im \lambda}, \frac{2(k+1) \pi}{\Im \lambda} \right], \quad k \geq 0, \ k \in \mathbb{Z}. \]

From (3.4) we get that
\[ \lim_{t \to \infty} \int_t^{t+\tau} e^{s \lambda} (h(s) + a) \, ds = 0. \]

This is contradicted by
\[ \Re \left( \left( \int_{t_k}^{t_k+\tau} e^{s \lambda} (h(s) + a) \, ds \right) \bar{a} \right) \geq \frac{\delta \tau}{2}, \]
where \( t_k = 2k \pi / \Im \lambda. \) Thus we must have that \( |a| \leq C. \)

**Proposition 3.12.** Suppose that \( A \) is normal, nonresonant and \( n_0 = 2 \). Then \( S_A^n (B^n) \) is a normal family. Furthermore, if \( f \in S_A^n (B^n) \) has \( h \in H_A (B^n) \) as an infinitesimal generator (see Proposition 3.8) then \( f \) can be embedded as the first element of a bounded Loewner chain with infinitesimal generator \( h \).
Proof: If $U$ is a unitary matrix, $f \in S_A^n (B^n)$ and $h \in H_A (B^n)$ is an infinitesimal generator for $f$ then it is straightforward to check that $U^* f U \in S_A^n (B^n)$ and that $U^* h U \in H_A^* (B^n)$ is an infinitesimal generator for $U^* f U$. This allows us to assume without loss of generality that $A = \text{diag} (\lambda_1, \ldots, \lambda_n)$ and $\text{Re} \lambda_1 \geq \ldots \geq \text{Re} \lambda_n > 0$ (note that $m (A) = \text{Re} \lambda_n$).

Let $f$ be an $A$-asymptotically spirallike mapping and $h \in H_A (B^n)$ be an infinitesimal generator for $f$. Let $v$ be the solution of (1.6). Also, assume that $f$, $h (\cdot, t)$ and $v (\cdot, t)$ have the following Taylor series expansions:

$$
\begin{align*}
  f (z) &= z + F_2 (z^2) + \ldots \\
  h (z, t) &= Az + H_2 (z^2, t) + \ldots \\
  v (z, t) &= e^{-tA} z + V_2 (z^2, t) + \ldots.
\end{align*}
$$

From (1.6) and then (2.4) one easily gets that

$$
e^{tA} V_2 (z^2, t) = - \int_0^t e^{sA} H_2 \left( (e^{-sA} z)^2, s \right) ds
\begin{equation}
= - \int_0^t e^{-sB^2} H_2 (z^2, s) ds.
\end{equation}
$$

As a consequence of Proposition 3.8, the above equality and (2.4) we have

$$
(3.5) \quad F_2 (z^2) = \lim_{t \to \infty} \left( e^{tA} V_2 (z^2, t) + e^{tA} F_2 \left( (e^{-tA} z)^2 \right) \right)
\begin{equation}
= \lim_{t \to \infty} \left( - \int_0^t e^{-sB^2} H_2 (z^2, s) ds + e^{-tB^2} F_2 (z^2) \right).
\end{equation}
$$

We want to show that $F_2^2$ can be bounded independently of $f \in S_A^n (B^n)$. For this we will show that each of the coefficients $f_{ij}^k$ of the monomials $z_i z_j e_k$ from $F_2^2$ can be bounded independently of $f \in S_A^n (B^n)$.

We know that

$$
B_2 (z_i z_j e_k) = (\lambda_i + \lambda_j - \lambda_k) z_i z_j e_k
$$

and so

$$
e^{tB_2} (z_i z_j e_k) = e^{t(\lambda_i + \lambda_j - \lambda_k)} z_i z_j e_k.
$$

Projecting (3.5) on the subspace generated by $z_i z_j e_k$ we get

$$
f_{ij}^k = \lim_{t \to \infty} \left( - \int_0^t e^{-s(\lambda_i + \lambda_j - \lambda_k)} h_{ij}^k (s) ds + e^{-t(\lambda_i + \lambda_j - \lambda_k)} f_{ij}^k \right)
\begin{equation}
= \lim_{t \to \infty} \left( - \int_0^t e^{-s(\lambda_i + \lambda_j - \lambda_k)} (h_{ij}^k (s) + (\lambda_i + \lambda_j - \lambda_k) f_{ij}^k) ds + f_{ij}^k \right)
\end{equation}
$$

$(h_{ij}^k (s))$ are the coefficients of the monomials $z_i z_j e_k$ from $H_2 (z^2, s)$. Hence

$$
(3.6) \quad \lim_{t \to \infty} \int_0^t e^{-s(\lambda_i + \lambda_j - \lambda_k)} (h_{ij}^k (s) + (\lambda_i + \lambda_j - \lambda_k) f_{ij}^k) ds = 0.
$$

For the coefficients of the monomials of $F_2^2$ we have that $\text{Re} (\lambda_i + \lambda_j - \lambda_k) \leq 0$, hence we can use Lemma 3.11 and the fact that $\lambda_i + \lambda_j - \lambda_k \neq 0$ (since $A$ is nonresonant) to conclude that the coefficients of $F_2^2$ are bounded independently of $f$ ($h_{ij}^k$ are bounded because $N_A$ is compact).
Let \( f(z, t) \) denote the polynomially bounded Loewner chain with infinitesimal generator \( h \) and such that \( F_2^\leq (z^2, 0) = F_2^\leq (z^2) \) (see Theorem 2.11). We will see that \( f = f(\cdot, 0) \). This will show that \( S_A^n (B^n) \subset S_A^n (B^n) \) where

\[
\mathcal{F} = \left\{ F_2^\leq : f(z) = z + F_2 (z^2) + \ldots \in S_A^n (B^n) \right\}.
\]

By Theorem 2.15 this yields the normality of \( S_A^n (B^n) \).

It is enough to check that

\[
0 = f (z) - f (z, 0) = \lim_{t \to \infty} e^{tA} \left( F_2 \left( v(z, t)^2 \right) - F_2 \left( v(z, t)^2, t \right) \right).
\]

We know that (see (1.4), (2.14), (2.15))

\[
F_2 (z^2, t) = e^{tB_2} F_2^\leq (z^2) + \int_0^t G_{B_2} (t - s) N_2 (z^2, s) \, ds
\]

\[
= e^{tB_2} F_2^\leq (z^2) + \int_0^t e^{(t-s)B_2} H_2^\leq (z^2, s) \, ds - \int_t^\infty e^{(t-s)B_2} H_2^+ (z^2, s) \, ds
\]

\[
= F_2^\leq (z^2) + e^{tB_2} \int_0^t e^{-sB_2} \left( H_2^\leq (z^2, s) + B_2 F_2^\leq (z^2) \right) \, ds
\]

\[
- \int_0^\infty e^{-sB_2} H_2^+ (z^2, s + t) \, ds.
\]

Substituting the above in (3.7) we need to verify that

\[
\lim_{t \to \infty} e^{tA} e^{tB_2} \int_0^t e^{-sB_2} \left( H_2^\leq \left( v(z, t)^2, s \right) + B_2 F_2^\leq \left( v(z, t)^2 \right) \right) \, ds = 0
\]

and

\[
\lim_{t \to \infty} e^{tA} \int_0^\infty e^{-sB_2} \left( H_2^+ \left( v(z, t)^2, s \right) - H_2^+ \left( v(z, t)^2, s + t \right) \right) \, ds = 0.
\]

Using (2.5), (3.8) becomes

\[
\lim_{t \to \infty} \int_0^t e^{-sB_2} \left( H_2^\leq \left( \left( e^{tA} v(z, t)^2 \right)^2, s \right) + B_2 F_2^\leq \left( \left( e^{tA} v(z, t)^2 \right)^2 \right) \right) \, ds = 0.
\]

Let \( v = (v_1, \ldots, v_n) \). Separating the monomials in (3.10) it is enough to prove that

\[
\lim_{t \to \infty} e^{t\lambda_i} v_i (z, t) e^{t\lambda_j} v_j (z, t) \int_0^t e^{-s(\lambda_i + \lambda_j - \lambda_k)} c_{ij}^k (s) \, ds = 0
\]

\((c_{ij}^k (s))\) are the coefficients of the monomials in the polynomial \( H_2^\leq (v, s) + B_2 F_2^\leq \) provided that \( \Re (\lambda_i + \lambda_j - \lambda_k) \leq 0 \) and (because of (3.9))

\[
\lim_{t \to \infty} \int_0^t e^{-s(\lambda_i + \lambda_j - \lambda_k)} c_{ij}^k (s) \, ds = 0.
\]

It is now enough to check that \( e^{t\lambda_i} v_i (z, t) \) and \( e^{t\lambda_j} v_j (z, t) \) are bounded on \( \{z\} \times [0, \infty) \), assuming that \( \Re (\lambda_i + \lambda_j - \lambda_k) \leq 0 \). This follows from Lemma 3.10 provided that \( \Re \lambda_i, \Re \lambda_j < 2 \Re \lambda_n \). Assume that this is not the case, so for example \( \Re \lambda_i \geq 2 \Re \lambda_n \). This implies that

\[
\Re (\lambda_i + \lambda_j - \lambda_k) \geq \Re (3 \lambda_n - \lambda_1) > 0
\]

which contradicts \( \Re (\lambda_i + \lambda_j - \lambda_k) \leq 0 \). For the last inequality we used the hypothesis \( n_0 = 2 \) which implies that \( 2 \leq \Re \lambda_1 / \Re \lambda_n < 3 \).
Separating the monomials in (3.9) it is enough to prove that
\[
\lim_{t \to \infty} e^{t \lambda_k} v_i (z, t) v_j (z, t) \int_0^\infty e^{-s (\lambda_i + \lambda_j - \lambda_k)} d_{ij}^k (s, t) \, ds = 0
\]
provided that \( \text{Re} (\lambda_i + \lambda_j - \lambda_k) > 0 \) (\( d_{ij}^k (s, t) \) are the coefficients of the monomials in the polynomial \( H_s^+ ( \cdot, s) - H_s^+ ( \cdot, s + t) \)). Since \( H_s^+ ( \cdot, s) - H_s^+ ( \cdot, s + t) \) can be bounded independently of \( s \) and \( t \) we see that \( \int_0^\infty e^{-s (\lambda_i + \lambda_j - \lambda_k)} d_{ij}^k (s, t) \, ds \) can be bounded independently of \( t \). Hence it is enough to check that
\[
\lim_{t \to \infty} e^{t \lambda_k} v_i (z, t) v_j (z, t) = 0.
\]

If \( \text{Re} \lambda_i, \text{Re} \lambda_j \leq 2 \text{Re} \lambda_n \) then using Lemma 3.10 we have
\[
\| e^{t \lambda_k} v_i (z, t) v_j (z, t) \| \leq C (1 + t)^2 e^{-t \text{Re} (\lambda_i + \lambda_j - \lambda_k)}.
\]

If \( \text{Re} \lambda_i > 2 \text{Re} \lambda_n \) or \( \text{Re} \lambda_j > 2 \text{Re} \lambda_n \) then using Lemma 3.10 again we get
\[
\| e^{t \lambda_k} v_i (z, t) v_j (z, t) \| \leq Ce^{-t \text{Re} (3 \lambda_n - \lambda_k)}.
\]

Since \( \text{Re} (\lambda_i + \lambda_j - \lambda_k) > 0 \) and \( \text{Re} (3 \lambda_n - \lambda_k) \geq \text{Re} (3 \lambda_n - \lambda_1) > 0 \) the above inequalities prove the desired limit. This completes the proof. \( \square \)

**Remark 3.13.** It is not clear whether \( S_\lambda^A (B^n) \) is closed under the assumptions of the above Theorem. Suppose that \( \{ f_k \} \) is a sequence in \( S_\lambda^A (B^n) \) converging to some \( f \in S (B^n) \). Let \( \{ f_k (z, t) \} \) be polynomially bounded Loewner chains such that \( f_k (z, 0) = f_k (z) \). Then by Lemma 2.14 we have that up to a subsequence \( \{ f_k (z, t) \} \) converges to a polynomially bounded Loewner chain \( f (z, t) \) such that \( f (z, 0) = f (z) \). In order to conclude that \( f \in S_\lambda^A (B^n) \) it would be natural to have that \( f \) satisfies (3.3) with \( v \) satisfying \( f (v (z, t), t) = f (z) \). Unfortunately one can find examples when this doesn’t happen.

**Remark 3.14.** (3.6) gives a necessary condition for a mapping \( h \in H_A \) to be the infinitesimal generator associated to some \( f \in S_\lambda^A (B^n) \). It is possible to choose \( h \) such that (3.6) is not satisfied for any \( f \in S_\lambda^A (B^n) \). This means that unlike the \( n_0 = 1 \) case, there exist polynomially bounded Loewner chains for which the first element is not from \( S_\lambda^A (B^n) \).

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