COMPACT RANGE PROPERTY AND OPERATORS ON $C^*$-ALGEBRAS

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Abstract. We prove that a Banach space $E$ has the compact range property (CRP) if and only if for any given $C^*$-algebra $A$, every absolutely summing operator from $A$ into $E$ is compact. Related results for $p$-summing operators ($0 < p < 1$) are also discussed as well as operators on non-commutative $L^1$-spaces and $C^*$-summing operators.

1. INTRODUCTION

A Banach space $E$ is said to have the compact range property (CRP) if every $E$-valued countably additive measure of bounded variation has compact range. It is well known that every Banach space with the Radon-Nikodym property (RNP) has the (CRP) and for dual Banach spaces, the (CRP) were completely characterized as those whose predual do not contain any copies $\ell^1$. For more in depth discussions on Banach spaces with the (CRP), we refer to [9].

The following characterization can be found in [9]: A Banach space $E$ has the (CRP) if and only if every 1-summing operator from $C[0,1]$ into $E$ is compact. Since $C[0,1]$ is a (commutative) $C^*$-algebra, it is a natural question whether $C[0,1]$ can be replaced by any $C^*$-algebras. Let us recall that in [7], it was shown that if $X$ is a Banach space that does not contain any copies of $\ell^1$ then any 1-summing operators from any given $C^*$-algebra into $X^*$ is compact; hinting that, as in commutative case, the (CRP) is the right condition to provide compactness. The present note is an improvement of [7]. Our main result confirms that, if $A$ is a $C^*$-algebra and $E$ is a Banach space that has the (CRP) then every 1-summing from $A$ into $E$ is compact. Our proof relies on factorizations of summing operators used in [7] and properties of integral operators.

There is another well known characterization of spaces with the (CRP) in terms of operators defined on $L^1[0,1]$: a Banach space $E$ has the (CRP) if and only every operator $T$ from $L^1[0,1]$ into $E$ is Dunford Pettis (completely continuous) thus the (CRP) is also referred to as the complete continuity property (CCP). Unlike the 1-summing operators on $C^*$-algebras,
operators defined on non-commutative $L^1$-spaces do not behave the same way as those defined
on $L^1[0,1]$ do. In the last section of this note, we will discuss these operators along with
$C^*$-summing operators studied by Pisier in [3].

Our terminology and notation are standard as may be found in [2] and [4] for Banach
spaces, [5] and [8] for $C^*$-algebras and operator algebras.

2. PRELIMINARIES

In this section, we recall some definitions.

Definition 1. Let $X$ and $Y$ be Banach spaces and $0 < p < \infty$. An operator $T : X \rightarrow Y$ is said to be $p$-summing if there is a constant $C$ such that for any finite sequence $(x_1, x_2, \ldots, x_n)$ of $X$, one has
\[
\left( \sum_{i=1}^{n} \|Tx_i\|^p \right)^{\frac{1}{p}} \leq C \sup \left\{ \left( \sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} ; \ x^* \in X^*, \|x^*\| \leq 1 \right\}.
\]

The smallest constant $C$ for which the above inequality holds is denoted by $\pi_p(T)$ and is
called the $p$-summing norm of $T$.

Definition 2. We say that an operator $T : X \rightarrow Y$ is an integral operator if it admits a factorization:
\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y^* \\
\downarrow{i} & & \downarrow{\beta}
\end{array}
\]
\[
\begin{array}{ccc}
L^\infty(\mu) & \xrightarrow{J} & L^1(\mu)
\end{array}
\]
where $i$ is the natural inclusion from $Y$ into $Y^{**}$, $\mu$ is a probability measure on a compact
space $K$, $J$ is the natural inclusion and $\alpha$ and $\beta$ are bounded linear operators.

We define the integral norm $i(T) : = \inf \left\{ \|\alpha\| \cdot \|\beta\| \right\}$ where the infimum is taken over all such factorizations.

Similarly, we shall say that $T$ is strictly integral if $T$ is integral and on the factorization
above $\beta$ takes its values in $Y$.

It is well known that integral operators are 1-summing but the converse is not true.

If $X = C(K)$ where $K$ is a compact Hausdorff space then it is well known that every
1-summing operator from $X$ into $Y$ is integral.

For more details on the different properties of the classes of operators involved, we refer
to [3].

The following simple fact will be needed in the sequel.
Proposition 3. Let $T : X \to Y$ be a strictly integral operator. If $Y$ has the (CRP) then $T$ is compact.

Proof. The operator $T$ has a factorization $T = \beta \circ J \circ \alpha$ where $\alpha : X \to L^\infty(\mu)$, $J : L^\infty(\mu) \to L^1(\mu)$ and $\beta : L^1(\mu) \to Y$ are as in the above definition. Note that $J$ is 1-summing so $\beta \circ J : L^\infty(\mu) \to Y$ is 1-summing and since $L^\infty(\mu)$ is a $C(K)$-space and $Y$ has the (CRP), $\beta \circ J$ (and hence $T$) is compact. \qed

We recall that a von Neumann algebra $\mathcal{M}$ is said to be $\sigma$-finite if the identity is countably decomposable equivalently if there exist a faithful state $\varphi \in \mathcal{M}_*$. As is customary, for every functional $\varphi \in \mathcal{M}_*$ and $x \in \mathcal{M}$, $x\varphi$ (resp. $\varphi x$) denotes the normal functional $y \to \varphi(yx)$ (resp. $y \to \varphi(xy)$).

3. MAIN RESULT

Theorem 4. For a Banach space $E$, the following are equivalent:

(1) $E$ has the CRP;
(2) Every 1-summing operator $T : C[0,1] \to E$ is compact;
(3) For any given $C^*$-algebra $\mathcal{A}$, every 1-summing operator $T : \mathcal{A} \to E$ is compact.

The equivalence (1) $\iff$ (2) is well known, we refer to \cite{r}, \cite{r} for more details. Clearly (3) $\Rightarrow$ (2) so what we need to show is (1) $\Rightarrow$ (3). For this, it is enough to consider the following particular case (see \cite{r} for this reduction).

Proposition 5. Let $E$ be a Banach space with the (CRP) and $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra. If $T : \mathcal{M} \to E$ is 1-summing and is weak$^*$ to weakly continuous then $T$ is compact.

Proof. The proof is a refinement of the argument used in Proposition 3.2 of \cite{r}. We will include most of the details for completeness. Without loss of generality, we can assume that $E$ is separable.

Let $\delta > 0$. From Lemma 2.3 of \cite{r},

$$\|Tx\| \leq 2(1 + \delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}_*}$$

for every $x \in \mathcal{M}$, where $f$ is a faithful normal state in $\mathcal{M}_*$. If $L^2(f)$ is completion of the prehilbertian space $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ where $\langle x, y \rangle = f\frac{xy^* + y^*x}{2}$ then we have the following factorization:
where \( J \) is the inclusion map, \( \theta(Jx) = \langle \cdot, J(x^*) \rangle \) for every \( x \in M \) and \( L(f_x + x) = Tx \). We recall that \( L \) is a well defined bounded linear map since \( \{ xf + fx; x \in M \} \) is dense in \( M_* \) and \( \| L(xf + fx) \| \leq 4(1 + \delta)\pi_1(T)\|xf + fx\|_{M_*} \). Let \( S := J^* \circ \theta \circ J \).

Claim: \( J \circ L^*: E^* \to L^2(f) \) is compact.

For this, let us consider \( L^*: E^* \to M_\ast \). Since \( E \) is separable, it is isometric to a subspace of \( C[0, 1] \). Let \( I_E \) be the isometric embedding of \( E \) in \( C[0, 1] \) and \( i \) be the natural inclusion of \( C[0, 1] \) into \( C[0, 1]^{**} \).

Define the following map \( \tilde{T} \) from \( M \) into \( C[0, 1] \) by setting \( \tilde{T} = I_E \circ T(x^*) \) for every \( x \in M \). (Here, \( \bar{f} \) is the map \( t \to \bar{f}(t) \) for \( f \in C[0, 1] \) with \( \bar{f}(t) \) being the conjugate of the complex number \( f(t) \)).

Clearly, \( \tilde{T} \) is linear and bounded and it can be shown that \( \tilde{T} \) is 1-summing and is weak\( \ast \) to weakly continuous. In fact, if \( (x_1, x_2, \ldots, x_n) \) is a finite sequence in \( M \) then

\[
\sum_{i=1}^{n} \| \tilde{T}x_i \| = \sum_{i=1}^{n} \| I_E \circ T(x_i^*) \| \\
= \sum_{i=1}^{n} \| I_E \circ T(x_i^*) \| \\
= \sum_{i=1}^{n} \| T(x_i^*) \| \\
\leq \pi_1(T) \sup \left\{ \sum_{i=1}^{n} |\langle x_i^*, \varphi \rangle|, \varphi \in M^*, \| \varphi \| \leq 1 \right\} \\
\leq \pi_1(T) \sup \left\{ \sum_{i=1}^{n} |\langle x_i, \varphi^* \rangle|, \varphi \in M^*, \| \varphi \| \leq 1 \right\}
\]

so \( \tilde{T} \) is 1-summing with \( \pi_1(\tilde{T}) \leq \pi_1(T) \). Moreover if \( (x_\alpha)_{\alpha} \) is a net that converges to zero weak\( \ast \) in \( M \) so does the net \( (x_\alpha^*)_{\alpha} \) and since \( T \) is weak\( \ast \) to weakly continuous, \( (T(x_\alpha^*))_{\alpha} \) converges to zero weakly in \( E \) and hence \( (\tilde{T}(x_\alpha))_{\alpha} \) is weakly null which shows that \( \tilde{T} \) is weak\( \ast \) to weakly continuous.
To complete the proof, consider $E^* \xrightarrow{L^*} M \xrightarrow{i \circ \tilde{T}} C[0,1]^{**}$.

Since $C[0,1]^{**}$ has the Hahn-Banach extension property and $i \circ \tilde{T}$ is 1-summing, $i \circ \tilde{T}$ is an integral operator. Let $K : C[0,1]^* \to M_*$ such that $K^* = i \circ \tilde{T}$ (such operator exists since $i \circ \tilde{T}$ is weak$^*$ to weakly continuous); $K$ is integral ([3]) and since $M_*$ is a complemented subspace of its bidual $M^*$ (see for instance [8]), $K$ is strictly integral and therefore $L \circ K : C[0,1]^* \to E$ is strictly integral and by Proposition 3, $L \circ K$ (and hence $(L \circ K)^* = i \circ \tilde{T} \circ L^*$) is compact.

Let $(U_n)$ be a bounded sequence in $E^*$. There exists a subsequence $(U_{n_k})$ so that $(i \circ \tilde{T} \circ L^*(U_{n_k}))_k$ is norm convergent in $C[0,1]^{**}$. Since $i$ and $I_E$ are isometries, we get that $(T \circ L^*(U_{n_k}))_k$ is norm convergent so

$$\lim_{k,m} \| T(L^*(U_{n_k}))^* - T(L^*(U_{n_m}))^* \| = 0.$$  

As in [4], we get

$$\lim_{k,m} \langle T(L^*(U_{n_k}))^* - T(L^*(U_{n_m}))^*, U_{n_k} - U_{n_m} \rangle = \lim_{k,m} \| J \circ L^*(U_{n_k} - U_{n_m}) \|^2_{L^2(f)} = 0$$

which proves that $(J \circ L^*(U_{n_k}))_k$ is norm-convergent in $L^2(f)$. The proof is complete. \qed

**Theorem 6.** Let $\mathcal{A}$ be a $C^*$-algebra, $E$ be a Banach space and $0 < p < 1$. Every $p$-summing operator from $\mathcal{A}$ into $E$ is compact.

**Proof.** Let $T : \mathcal{A} \to E$ be an operator with $\pi_p(T) < \infty$. One can choose, by the Pietsch Factorization Theorem, a probability space $(\Omega, \Sigma, \mu)$ such that

$$\xymatrix{ \mathcal{A} & E \\
 S & \tilde{\mathcal{T}} \ar^{i_p}[u] \ar^{\tilde{T}}[r] \\
 L^\infty(\mu) & L^p(\mu) \ar^j[u] \ar^{i_p}[u] \ar^{j_p}[u] }$$

where $S$ is a subspace of $L^\infty(\mu)$, $S_p$ is the closure of $S$ in $L^p(\mu)$ and $i_p$ is the restriction of the natural inclusion $j_p$.

Denote by $S_1$ the closure of $S$ in $L^1(\mu)$, by $i_1$ the restriction of the natural inclusion and $i_{1,p}$ the natural inclusion of $S_1$ into $S_p$.

**Claim:** $\tilde{T} \circ i_{1,p} : S_1 \to E$ is weakly compact.

To see this, let $(f_n)_n$ be a bounded sequence in $S_1 \subset L^1(\mu)$. By Komlós’s Theorem, there exists a subsequence $(f_{n_k})_k$ and a function $f \in L^1(\mu)$ such that $\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^m f_{n_k}(\omega) = ...$
f(ω) for a.e. ω ∈ Ω. Since 0 < p < 1,
\[
\lim_{m \to \infty} \left\| \frac{1}{m} \sum_{k=1}^{m} f_{n_k} - f \right\|_p = 0.
\]
This shows that \( f \in S_p \) and \( (\tilde{T} \circ i_{1,p} \left( \frac{1}{m} \sum_{k=1}^{m} f_{n_k} \right) )_m \) converges to \( \tilde{T}(f) \) in \( E \) and the claim follows.

Using the factorization of weakly compact operator [1], \( i_{1,p} \circ \tilde{T} \) factors through a reflexive space and since \( i_1 \circ J \) is 1-summing, the theorem follows from Theorem 4.

**4. Concluding remarks**

Let us recall some definitions

**Definition 7.** Let \( X \) and \( Y \) be Banach spaces. An operator \( T : X \to Y \) is called Dunford-Pettis if \( T \) sends weakly compact sets into norm compact sets.

The following class of operators was introduced by Pisier in [3] as an extension of the \( p \)-summing operators in the setting of \( C^* \)-algebras.

**Definition 8.** Let \( A \) be a \( C^* \)-algebra and \( E \) be a Banach space, \( 0 < p < \infty \). An operator \( T : A \to E \) is said to be \( p \)-\( C^* \)-summing if there exists a constant \( C \) such that for any finite sequence \( (A_1, \ldots, A_n) \) of Hermitian elements of \( A \), one has
\[
\left( \sum_{i=1}^{n} \left\| T(A_i) \right\|_p \right)^{\frac{1}{p}} \leq C \left( \sum_{i=1}^{n} |A_i|^p \right)^{1/p} \left\| A \right\|_A.
\]

Let \( \mathcal{M} \) be a finite von Neumann algebra with a faithful tracial state \( \tau \) and let \( J \) be the canonical inclusion map from \( \mathcal{M} \) into \( L^1(\mathcal{M}, \tau) \). As in the commutative case, we have the following :

**Proposition 9.** Let \( E \) be a Banach space and \( T : L^1(\mathcal{M}, \tau) \to E \) a bounded linear map. Then the following are equivalent:

(i) \( T \) is Dunford-Pettis;

(ii) \( T \circ J \) is compact.

**Proof.** (i) \( \implies \) (ii) is trivial. For the converse, let \( (a_n)_n \) be a weakly null sequence in the unit ball of \( L^1(\mathcal{M}, \tau) \). It is clear that \( (a^*_n)_n \) is also weakly null so without loss of generality, we can
assume that \((a_n)_n\) is a sequence of self-adjoint operators. For each \(n \geq 1\), set \(a_n = \int_{-\infty}^{\infty} t \, d e_t^{(n)}\) the spectral decomposition of \(a_n\) and for every \(N \geq 1\), let
\[
p_{n,N} = \int_{-N}^{N} 1 \, d e_t^{(n)}.
\]
It is clear that for every \(n \geq 1\) and \(N \geq 1\),
\[
\tau(1 - p_{n,N}) = \tau\left(\int_{\{|t|>N\}} 1 \, d e_t^{(n)}\right) \leq \frac{1}{N} \tau(|a_n|).
\]
By the Akeman’s characterization of relatively weakly compact subset in \(L^1(\mathcal{M}, \tau)\) (see for instance [8] Theorem 5.4 p.149), we conclude that for any given \(\epsilon > 0\), there is \(N_0 \geq 1\) such that for every \(n \geq 1\), \(\|a_n(1 - p_{n,N_0})\| \leq \epsilon\). Moreover \((a_n p_{n,N_0})_n\) is a bounded sequence in \(\mathcal{M}\) and since \(T \circ J\) is compact, there is a compact subset \(K_\epsilon\) of \(E\) such that \(\{T(a_n); \, n \in \mathbb{N}\} \subset K_\epsilon + \epsilon B_E\). The proof is complete.

Fix a type \(II_1\) von Neumann algebra \(\mathcal{M}\) such that \(\mathcal{M}\) contains a complemented copy of a Hilbert space \(H\). The space \(H\) is reflexive (and therefore has (CRP) ) but the projection map \(P\) from \(L^1(\mathcal{M}, \tau)\) onto \(H\) can not be Dunford-Pettis.

A very well known property of \(p\)-summing operators is that they are Dunford-Pettis. This is not the case for \(C^*\)-summing operators in general. By Proposition 9, \(P \circ J\) is not compact. We remark that the argument used in [7] requires only that the operator is \(C^*\)-summing and Dunford-Pettis hence since \(J\) is clearly \(C^*\)-summing and \(P \circ J\) is not compact, \(P \circ J\) should not be Dunford-Pettis.

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