A CHARACTERIZATION OF PL 4-MANIFOLDS ADMITTING SIMPLE CRYSTALLIZATIONS *

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Abstract

Simple crystallizations have been recently introduced by Basak and Spreer. In the present paper, we prove that simply-connected PL 4-manifolds admitting simple crystallizations are characterized by $k(M) = 3\beta_2(M)$ (where $\beta_2(M)$ is the second Betti number of $M$ and $k(M)$ is its gem-complexity, i.e. the non-negative number $p - 1$, $2p$ being the minimum order of a crystallization of $M$). Moreover, we also prove that their regular genus is twice their second Betti number.

As a consequence, both the PL invariants gem-complexity and regular genus turn out to be additive within the class of all PL 4-manifolds admitting simple crystallizations (in particular: within the class of all “standard” simply-connected PL 4-manifolds).

Key words: PL 4-manifold, coloured graph, coloured triangulation, handle-decomposition, simple crystallization, regular genus, gem-complexity.

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1. Introduction and main results

For any PL $n$-manifold $M^n$, it is known the existence of a contracted triangulation, i.e. a pseudocomplex triangulating $M^n$, whose 0-skeleton consists of exactly $n + 1$ vertices.

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1Roughly speaking, pseudocomplexes extend the notion of simplicial complexes, since a set of vertices may determine more than one face. In the literature, a similar notion is also given by the term simplicial poset.
Note that contracted triangulations of a PL n-manifold $M$ may be seen as an intermediate notion, between simplicial complexes (where a pair of distinct simplices can intersect in at most one face) and singular triangulations (where it is required that only the interior of the cells are open simplices, and usually a single 0-simplex is present).

What makes contracted triangulations particularly user-friendly is the possibility of representing them by means of their dual graphs, which turn out to be a special kind of edge-coloured graphs, called crystallizations.

In fact, the purely combinatorial nature of the representing objects has enabled to obtain interesting results in PL-topology by means of crystallization theory\footnote{See \cite{18} or \cite{1} for a survey on crystallizations by the Italian school that started the theory, and \cite{3}, \cite{4}, \cite{23}, \cite{16} for some recent results by different authors which are contributing to its development.} Among them, we recall: the definition of a suitable graph-defined PL-invariant, called regular genus (\cite{19}), and the related classification theorems, especially in dimension 4 and 5 (see, for example, \cite{13}, \cite{7} and \cite{14}); the automatic generation and classification of catalogues of PL 3- and 4-manifolds, for increasing order of the associated crystallizations (see \cite{11} and \cite{12}, together with their references). In particular, the classifying algorithm for PL manifolds in dimension 4 appears to be a promising approach to the problem of detecting different PL structures on the same topological 4-manifold (\cite{12} Section 5).

In this setting, Basak and Spreer (\cite{4}) recently introduced the notion of simple crystallization, i.e. a crystallization of a PL 4-manifold, so that the 1-skeleton of the associated contracted triangulation equals the 1-skeleton of a 4-simplex: since simple crystallizations are isolated global minima in the process of automatic reduction of crystallizations, they may be useful in order to algorithmically prove the PL-equivalence of different triangulations of the same (simply connected) topological 4-manifold (\cite{4} Section 1).

Aim of the present paper is to investigate the properties of PL 4-manifolds admitting simple crystallizations.

The key step of the analysis is the proof that any contracted triangulation associated to such a crystallization induces a particular type of handle-decomposition lacking in 1-handles and 3-handles (i.e. a special handlebody decomposition, according to \cite{22} p. 59)): see Proposition 2\footnote{Note that the existence of a special handlebody decomposition is related to Kirby problem n. 50.}.

As a consequence, various combinatorial properties of simple crystallizations are obtained (Proposition 3 and Proposition 4), which enable to get the main result of the paper, stating that PL 4-manifolds admitting simple crystallizations are characterized by a strong and easy relation between the gem-complexity $k(M)$ (where $k(M) = k$ means that $2(k + 1)$ is the minimum order of a crystallization of $M$: see Definition 1) and the second Betti number $\beta_2(M)$; moreover, they satisfy another nice relation between the regular genus $G(M)$ and the second Betti number, too:

**Theorem 1** Let $M$ be a closed simply-connected PL 4-manifold. Then:

$M$ admits a simple crystallization iff $k(M) = 3\beta_2(M)$.

Moreover, if $M$ admits a simple crystallization, then $G(M) = 2\beta_2(M)$.

In virtue of Theorem 1, both the invariants gem-complexity and regular genus turn out to be additive with respect to connected sum within the class of all PL 4-manifolds.
admitting simple crystallizations (in particular: within the class of all “standard” simply-connected PL 4-manifolds): see Proposition 7.

Note that subadditivity of both gem-complexity and regular genus is known to hold in any dimension, while the additivity has been conjectured, but the problem is still open (with the exception, for regular genus, of the low-dimensional cases). In particular, in dimension four, additivity of regular genus (resp. of gem-complexity) - at least in the simply-connected case - would imply the 4-dimensional Smooth Poincaré Conjecture: see [17, Remark 1] (resp. [12, Remark 2]). From this viewpoint, our results about additivity for 4-manifolds admitting simple crystallizations appears to be significant, in connection with the problem of the existence of simple crystallizations for any given simply-connected PL 4-manifold (see Proposition 9 for some particular families getting a negative answer and for relationships with 4-dimensional crystallization catalogues).

2. Basic notions on coloured triangulations of PL manifolds

Edge-coloured graphs are a representation tool for the whole class of piecewise linear (PL) manifolds, without restrictions about dimension, connectedness, orientability or boundary properties. In the present work, however, we will deal only with closed, connected and orientable PL-manifolds of dimension $n = 4$; hence, we will briefly review basic notions and results of the theory with respect to this particular case.

A 5-coloured graph (without boundary) is a pair $(\Gamma, \gamma)$, where $\Gamma = (V(\Gamma), E(\Gamma))$ is a regular multigraph (i.e. it may include multiple edges, but no loop) of degree five and $\gamma : E(\Gamma) \to \Delta_4 = \{0, 1, 2, 3, 4\}$ is a proper edge-coloration (i.e. it is injective when restricted to the set of edges incident to any vertex of $\Gamma$).

The elements of the set $\Delta_4$ are called the colours of $\Gamma$; thus, for every $i \in \Delta_4$, an $i$-coloured edge is an element $e \in E(\Gamma)$ such that $\gamma(e) = i$. For every $i, j, k \in \Delta_4$ let $\Gamma_i$ (resp. $\Gamma_{ijk}$) (resp. $\Gamma_{ij}$) be the subgraph obtained from $(\Gamma, \gamma)$ by deleting all the edges of colour $i$ (resp. $c \in \Delta_4 - \{i, j, k\}$) (resp. $c \in \Delta_4 - \{i, j\}$). The connected components of $\Gamma_i$ (resp. $\Gamma_{ijk}$) (resp. $\Gamma_{ij}$) are called $i$-residues (resp. $\{i, j, k\}$-coloured residues) (resp. $\{i, j\}$-coloured cycles) of $\Gamma$, and their number is denoted by $g_i$ (resp. $g_{ijk}$) (resp. $g_{ij}$). A 5-coloured graph $(\Gamma, \gamma)$ is called contracted iff, for each $i \in \Delta_4$, the subgraph $\Gamma_i$ is connected (i.e. iff $g_i = 1 \forall i \in \Delta_4$).

Every 5-coloured graph $(\Gamma, \gamma)$ may be thought of as the combinatorial visualization of a 4-dimensional labelled pseudocomplex $K(\Gamma)$, which is constructed according to the following instructions:

- for each vertex $v \in V(\Gamma)$, take a 4-simplex $\sigma(v)$, with vertices labelled $0, 1, 2, 3, 4$;
- for each $j$-coloured edge between $v$ and $w$ ($v, w \in V(\Gamma)$), identify the 3-dimensional faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertex labelled $j$, so that equally labelled vertices coincide.

In case $K(\Gamma)$ triangulates a (closed) PL 4-manifold $M$, then $(\Gamma, \gamma)$ is called a gem (gem = graph encoded manifold) representing $M$.

The construction of $K(\Gamma)$ directly ensures that, if $(\Gamma, \gamma)$ is an order $2p$ gem of $M$, then:
(a) $M$ is orientable (resp. non-orientable) iff $\Gamma$ is bipartite (resp. non-bipartite);

(b) there is a bijection between $i$-labelled vertices (resp. 1-simplices whose vertices are labelled $\Delta_4 - \{i, j, k\}$) (resp. 2-simplices whose vertices are labelled $\Delta_4 - \{i, j\}$) of $K(\Gamma)$ and $i$-residues (resp. $\{i, j, k\}$-coloured residues) (resp. $\{i, j\}$-coloured cycles) of $\Gamma$;

(c) $\chi(\vert K(\Gamma) \vert) = -3p + \sum_{i,j} c_{ij} - \sum_{i,j,k} c_{ijk} + \sum_i g_i$;

(d) $2c_{ijk} = c_{ij} + c_{ik} + c_{jk} - p$ for each triple $(i, j, k) \in \Delta_4$.

Finally, a gem representing a (closed) PL 4-manifold $M$ is a crystallization of $M$ when it is also a contracted graph; by the above property (b), this is equivalent to requiring that the associated pseudocomplex $K(\Gamma)$ contains exactly five vertices (one for each label $i \in \Delta_4$). Pezzana Theorem and its subsequent improvements ([18]) prove that every PL-manifold admits a crystallization.

As already recalled, catalogues of PL manifolds have been obtained both in dimension three (see [21], [10] and [11] for the 3-dimensional orientable case and [8], [9] and [2] for the non-orientable one) and four ([12]). They are constructed with respect to a suitable graph-defined PL invariant, which measures how “complicated” is the representing combinatorial object:

**Definition 1.** Given a PL $n$-manifold $M^n$, its **gem-complexity** is the non-negative integer $k(M^n) = p - 1$, where $2p$ is the minimum order of a crystallization of $M^n$.

Note that, as proved in [12] Proposition 7, if $M$ is assumed to be a handle-free PL 4-manifold (i.e.: if it admits neither the orientable nor the non-orientable $S^{n-1}$-bundle over $S^1$ as a connected summand), then $k(M) = p - 1$, where $2p$ is the order of a crystallization of $M$ lacking in 2-dipoles (i.e. pairs of parallel edges coloured by $\Delta_4 - \{i, j, k\}$, whose endpoints belong to different $\{i, j, k\}$-residues) and $\rho$-pairs (i.e.: pairs of distinct $i$-coloured edges both belonging to at least three common bicoloured cycles).

Crystallizations with these properties are called rigid dipole-free crystallizations; they are exactly the elements considered in the existing crystallization catalogues in dimension four.

As mentioned in Section 1, some of the most interesting results of crystallization theory are related to a graph-based invariant for PL $n$-manifolds, called regular genus and introduced in [19]. It extends to arbitrary dimension the classical notion of Heegaard genus of a 3-manifold and relies on the existence of a particular type of embedding into a surface for graphs representing manifolds of arbitrary dimension.

As far as the 4-dimensional case is concerned, it is well-known that, if $(\Gamma, \gamma)$ is an order 2p crystallization of an orientable PL 4-manifold $M$, then for every cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 4)$ of $\Delta_4$ there exists a so-called regular embedding $\iota_\varepsilon : \Gamma \to F_{\varepsilon}$.
where $F_\varepsilon$ is a closed orientable surface whose genus - denoted by $\rho_\varepsilon(\Gamma)$ - may be directly computed by the following formula (see [19] for details):

$$\sum_{i \in \mathbb{Z}_5} g_{\varepsilon_i,\varepsilon_{i+1}} - 3p = 2 - 2\rho_\varepsilon(\Gamma).$$

(1)

**Definition 2.** The *regular genus* of a bipartite 5-coloured graph $\Gamma$ is defined as the minimum genus of a surface into which $\Gamma$ regularly embeds:

$$\rho(\Gamma) = \min_{\varepsilon} \{\rho_\varepsilon(\Gamma)\};$$

the *regular genus* of a PL 4-manifold $M$ is defined as the minimum regular genus of a crystallization of $M$:

$$\mathcal{G}(M) = \min_{\{\rho(\Gamma) / (\Gamma, \gamma) \text{ crystallization of } M\}}.$$

For the purpose of the present paper, it is worthwhile to note that, if the PL 4-manifold $M$ is assumed to be simply-connected, the following relation involving the regular genus and the second Betti number $\beta_2(M)$ of $M$ always holds (see equality (5) in [12], or [15, Proposition 2]):

$$\beta_2(M) \leq \left\lfloor \frac{\mathcal{G}(M)}{2} \right\rfloor,$$

(2)

where $[x]$ denotes the integer part of $x$.

### 3. 4-manifolds admitting simple crystallizations

In [3], the notion of *simple crystallization* of a (simply-connected) PL 4-manifold is introduced:

**Definition 3.** A 4-dimensional pseudocomplex $K$ triangulating a PL 4-manifold $M$ is said to be *simple* if any pair of vertices belongs to at most one 1-simplex. A *simple crystallization* of a PL 4-manifold $M$ is a crystallization $(\Gamma, \gamma)$ of $M$ whose associated pseudocomplex $K(\Gamma)$ is simple.

As pointed out in [3, Section 1], a crystallization $(\Gamma, \gamma)$ of a PL 4-manifold is simple if and only if *the 1-skeleton of $K(\Gamma)$ equals the 1-skeleton of a single 4-simplex*. With the notations introduced in the previous section, this is equivalent to require $g_{ijk} = 1$ for any distinct $i, j, k \in \Delta_4$.

As a direct consequence of Definition 3, any PL 4-manifold $M$ admitting a simple crystallization turns out to be simply-connected.

Moreover, in [3] the authors prove the existence of simple crystallizations for any “standard” simply-connected PL 4-manifold (i.e. $S^4$, $\mathbb{C}P^2$, $S^2 \times S^2$ and the K3-surface, together with their connected sums, possibly by taking copies with reversed orientation, too).
Interesting information about simple crystallizations arise by taking into account the handle decomposition induced by the associated coloured triangulations. In fact, for any crystallization $(\Gamma, \gamma)$ of a PL 4-manifold $M$ and for any partition $\{\{i, j, k\}, \{r, s\}\}$ of $\Delta_4$, then $M$ admits a decomposition of type $M = N(i, j, k) \cup \partial N(r, s)$, where $N(i, j, k)$ (resp. $N(r, s)$) denotes a regular neighbourhood of the subcomplex $K(i, j, k)$ (resp. $K(r, s)$) of $K(\Gamma)$ generated by the vertices labelled $\{i, j, k\}$ (resp. $\{r, s\}$) and $\phi$ is a boundary identification. In the particular case of a simple crystallization, it is easy to prove that this decomposition yields a so-called special handle decomposition of $M$, i.e. a handle-decomposition lacking in 1-handles and 3-handles (see [22, Section 3.3]):

**Proposition 2** Let $(\Gamma, \gamma)$ be a simple crystallization of a (simply-connected) PL 4-manifold $M$. Then, for any partition $\{\{i, j, k\}, \{r, s\}\}$ of $\Delta_4$, the coloured triangulation $K(\Gamma)$ of $M$ induces a handle decomposition of $M$ consisting of one 0-handle, $g_{rs} - 1$ 2-handles and one 4-handle.

**Proof.** Let us fix an arbitrary partition $\{\{i, j, k\}, \{r, s\}\}$ of $\Delta_4$. Since $(\Gamma, \gamma)$ is assumed to be a simple crystallization, then $K(r, s)$ consists of exactly one 1-simplex; hence, $N(r, s) \cong \mathbb{D}^4$ trivially follows. On the other hand, the assumption implies that also $K(j, k)$ and $K(i, k)$ and $K(i, j)$ each consists of exactly one 1-simplex; hence, $K(i, j, k)$ consists of $g_{rs}$ 2-simplices, all having the same boundary. It is not difficult to check that, if a ‘small’ regular neighbourhood of one (arbitrarily fixed) 2-simplex of $K(i, j, k)$ is considered as a 0-handle $H^{(0)} = \mathbb{D}^4$, then the regular neighbourhoods of the remaining $g_{rs} - 1$ 2-simplices of $K(i, j, k)$ may be considered as $g_{rs} - 1$ 2-handles attached on its boundary. Hence, $N(i, j, k) = H^{(0)} \cup (H^{(2)}_1 \cup \cdots \cup H^{(2)}_{g_{rs}-1})$. Moreover, $\partial N(i, j, k) = \partial N(r, s) = S^3$. The proof is completed by noting that the boundary identification $\phi$ between $N(i, j, k)$ and $N(r, s)$ is $\mathbb{D}^4$ is nothing but the attachment of a 4-handle:

$$M = N(i, j, k) \cup \partial N(r, s) = [H^{(0)} \cup (H^{(2)}_1 \cup \cdots \cup H^{(2)}_{g_{rs}-1})] \cup \partial \mathbb{D}^4 = H^{(0)} \cup (H^{(2)}_1 \cup \cdots \cup H^{(2)}_{g_{rs}-1}) \cup H^{(4)}.$$  

$\square$

The following proposition collects some combinatorial properties of simple crystallizations, involving the second Betti number of the represented PL 4-manifold.

**Proposition 3** Let $(\Gamma, \gamma)$ be an order $2p$ simple crystallization of a (simply-connected) PL 4-manifold $M$. Then:

$^3$Recall that every closed PL 4-manifold $M$ admits a handle-decomposition

$$M = H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{r_1}) \cup (H^{(2)}_1 \cup \cdots \cup H^{(2)}_{g_{rs}-1}) \cup (H^{(3)}_1 \cup \cdots \cup H^{(3)}_{g_3}) \cup H^{(4)}$$

where $H^{(0)} = \mathbb{D}^4$ and each $p$-handle $H^{(p)}_i = \mathbb{D}^p \times \mathbb{D}^{4-p}$ $(1 \leq p \leq 4, 1 \leq i \leq r_p)$ is endowed with an an embedding (called attaching map) $j^{(p)}_i : \partial \mathbb{D}^p \times \mathbb{D}^{4-p} \to \partial (H^{(0)} \cup \cdots (H^{(p-1)}_i \cup \cdots \cup H^{(p-1)}_{r_{p-1}}))$.

$^3$Note that $(\Gamma, \gamma)$ needs not to be simple, since the described decomposition is based only on the contractedness of $\Gamma$: see for example [5], [6] and [14] for applications in the 4-dimensional boundary case, or [13] and [7] for its 5-dimensional extension.
(a) \( p = 1 + 3\beta_2(M) \);
(b) \( g_{ij} = 1 + \beta_2(M), \quad \forall i, j \in \Delta_4 \);
(c) \( \rho_\varepsilon(\Gamma) = 2\beta_2(M), \quad \text{for any cyclic permutation } \varepsilon \text{ of } \Delta_4 \).

Proof. First of all we point out that, if a PL 4-manifold \( M \) admits a handle decomposition of the type described in Proposition 2 then the second Betti number of \( M \) must coincide with the number of 2-handles, i.e. \( g_{rs} = 1 \). Since such a decomposition exists for any simple crystallization and for any partition \( \{\{i, j, k\}, \{r, s\}\} \) of \( \Delta_4 \), statement (b) easily follows.

Let now apply property (c) of Section 2 to an order 2p simple crystallization \( (\Gamma, \gamma) \) of a PL 4-manifold \( M \) (which - as it is well-known - is simply-connected, and therefore orientable):

\[
\chi(M) = 2 + \beta_2(M) = -3p + 10(1 + \beta_2(M)) - 10 + 5,
\]
from which \( 3\beta_2(M) = p - 1 \) (i.e. statement (a)) directly follows.

Finally, let us apply equality (1) to an order 2p simple crystallization \( (\Gamma, \gamma) \), by making use of the above statements (a) and (b), too:

\[
5(1 + \beta_2(M)) - 3(1 + 3\beta_2(M)) = 2 - 2\rho_\varepsilon(\Gamma),
\]
from which \( \rho_\varepsilon(\Gamma) = 2\beta_2(M) \) directly follows, for any cyclic permutation \( \varepsilon \) of \( \Delta_4 \).

Propositions 2 and 3 imply that simple crystallizations realize both gem-complexity and regular genus of the represented 4-manifolds and satisfy further combinatorial conditions:

**Proposition 4** Let \( (\Gamma, \gamma) \) be an order 2p simple crystallization of a (simply-connected) PL 4-manifold \( M \). Then:

(a) \( k(M) = p - 1 \);
(b) \( G(M) = \rho_\varepsilon(\Gamma), \quad \text{for any cyclic permutation } \varepsilon \text{ of } \Delta_4 \);
(c) \( \Gamma \) is rigid and dipole-free.

Proof. For any order 2p crystallization of a PL 4-manifold \( M \), property (d) of Section 2 yields \( 2\sum_{i<j<k} g_{ijk} = 3\sum_{i<j} g_{ij} - 10p \); if, further, \( M \) is assumed to be simply connected, by making use of property (c) of Section 2, together with \( g_i = 1 \ \forall i \in \Delta_4 \) and \( \beta_1(M) = \beta_3(M) = 0 \), the Euler characteristic computation gives

\[
2 + \beta_2(M) = 5 - \frac{1}{3} \sum_{i<j<k} g_{ijk} + \frac{1}{3}p.
\]

Since \( g_{ijk} \geq 1 \) trivially holds, we have \( 3\beta_2(M) \leq p - 1 \), which proves \( k(M) \geq 3\beta_2(M) \) (already stated as equality (3) of [12]). Let now suppose \( (\Gamma, \gamma) \) to be a simple crystallization. In virtue of Proposition 3(a), \( k(M) \leq p - 1 = 3\beta_2(M) \) holds, too. Hence, the equality \( k(M) = 3\beta_2(M) = p - 1 \) is established.
Statement (b) is a direct consequence of Proposition 3(c), by making use of equation (2) of Section 2.

Statement (c) is a direct consequence of statement (a), since dipoles and ρ-pairs may always be eliminated, yielding another crystallization of M, with strictly less order: see [12, Proposition 7(a)].

We are now able to prove our Main Theorem (Theorem 1, stated in Section 1).

Proof of Theorem 1.

As far as the first statement is concerned, note that - in virtue of Proposition 3(a) and Proposition 4(a) - all PL 4-manifolds admitting a simple crystallization do satisfy condition \( k(M) = 3\beta_2(M) \); hence, only the reversed implication has to be proved. Then, let \( M \) be a simply-connected PL 4-manifolds satisfying \( k(M) = 3\beta_2(M) \) and let \( (\Gamma, \gamma) \) be a crystallization of \( M \) realizing its gem-complexity (i.e.: \( \#V(\Gamma) = 2(k(M) + 1) = 6\beta_2(M) + 2 \)). The above equation (3) yields

\[
2 + \beta_2(M) = 5 - \frac{1}{3} \sum_{i<j<k} g_{ijk} + \frac{1}{3}(3\beta_2(M) + 1),
\]

and therefore

\[
\sum_{i<j<k} g_{ijk} = 10
\]

directly follows. Since each summand is at least equal to one, \( g_{ijk} = 1 \) is proved to hold for any triple \( i, j, k \in \Delta_4 \). Hence, \( (\Gamma, \gamma) \) turns out to be a simple crystallization, as required.

The second statement is a direct consequence of Proposition 3(c) and Proposition 4(b).

Remark 1 Actually, the above proof makes use of the weaker assumption \( \beta_1(M) = 0 \) (instead of the simply-connectedness of \( M \)), in order to check the existence of a simple crystallization of \( M \) when \( k(M) = 3\beta_2(M) \) holds. Hence, for each orientable PL 4-manifold, the following implication may be stated:

\[
\beta_1(M) = 0 \quad \text{and} \quad k(M) = 3\beta_2(M) \implies M \text{ admits a simple crystallization.}
\]

Remark 2 By making use of relations (b) and (c) included in the proof of [12, Proposition 12], it is not difficult to prove that, if \( \pi_1(M) \) is assumed to be trivial, then equality \( G(M) = 2\beta_2(M) \) implies the existence of a crystallization \( (\Gamma, \gamma) \) of \( M \) and a permutation \( \varepsilon \) of \( \Delta_4 \) so that \( \rho_\varepsilon(\Gamma) = 2\beta_2(M) \) and \( g_{i\varepsilon_i \varepsilon_{i+2} \varepsilon_{i+3}} = 1 \forall i \in \Delta_4 \). However, in general, this does not imply that \( (\Gamma, \gamma) \) is simple, since at least one \( g_{i\varepsilon_i \varepsilon_{i+1} \varepsilon_{i+2}} > 1 \) may occur. Note that, for example, all rigid dipole-free order 16 crystallizations satisfy relation \( G(M) = 2\beta_2(M) \), while \( g_{ijk} = 2 \) for exactly a triple \( \{\hat{i}, \hat{j}, \hat{k}\} \subset \Delta_4 \) and \( g_{ijk} = 1 \forall \{i, j, k\} \neq \{\hat{i}, \hat{j}, \hat{k}\} \).
The characterization of PL 4-manifolds admitting a simple crystallization, proved in the first statement of our Main Theorem, has the following consequence about possible different PL structures on the same TOP 4-manifold:

**Proposition 5** Let \( M \) and \( M' \) be two PL 4-manifolds, with \( M \cong_{\text{TOP}} M' \) and \( M \not\cong_{\text{PL}} M' \). If both \( M \) and \( M' \) admit a simple crystallization, then \( k(M) = k(M') \).

**Proof.** It is sufficient to note that \( M \cong_{\text{TOP}} M' \) obviously implies \( \beta_2(M) = \beta_2(M') \), and to make use of Proposition 3(a), together with Proposition 4(a).

\( \square \)

**Remark 3** In [12, Section 3] an algorithm is described, for the generation of all rigid dipole-free crystallizations of PL 4-manifolds up to a fixed gem-complexity \( k \). Proposition 4(c) ensures that such a catalogue must contain all simple crystallizations of PL 4-manifolds whose second Betti number does not exceed \( \frac{k}{3} \). Actually these catalogues have been generated up to gem-complexity 9 ([12]), hence they present all simple crystallizations of any PL 4-manifold \( M \) with \( \beta_2(M) \leq 3 \). Moreover, Proposition 5 guarantees that simple crystallizations representing two distinct PL structures on the same topological 4-manifold must appear at the same level in the above crystallization catalogues.

### 4. Further results on simple crystallizations

As already pointed out in Section 3, Basak and Datta produced a simple crystallization of the \( K_3 \)-surface ([8, Section 7]), and hence simple crystallizations for any “standard” simply-connected PL 4-manifold are proved to exist.

Our Main Theorem has the following consequences about the computation of both PL-invariants gem-complexity and regular genus for such 4-manifolds:

**Proposition 6** Let \( M \cong_{\text{PL}} \#(r \mathbb{C}P^2)\#(r'(-\mathbb{C}P^2))\#(s(S^2 \times S^2))\#(tK_3) \) with \( r, r', s, t \geq 0 \). Then,

\[
k(M) = 3(r + r' + 2s + 22t) \quad \text{and} \quad G(M) = 2(r + r' + 2s + 22t).\]

In particular: \( k(K_3) = 66 \) and \( G(K_3) = 44 \).

**Proof.** It is sufficient to apply Theorem 1, by taking into account the values of the second Betti number of each connected summand.

\( \square \)

Moreover, we are able to prove the additivity of both the above invariants under connected sum, within the class of PL 4-manifolds admitting a simple crystallization (and, in particular, for “standard” simply-connected PL 4-manifolds).

**Proposition 7** Let \( M \) and \( M' \) be two (simply-connected) PL 4-manifolds admitting a simple crystallization. Then:

\[
k(M \# M') = k(M) + k(M') \quad \text{and} \quad G(M \# M') = G(M) + G(M').\]
Proof. It is well-known a general construction - called *graph-connected sum* - yielding, from any gem $\Gamma$ (resp. $\Gamma'$) of the PL $n$-manifold $M$ (resp. $M'$), a gem $\Gamma \# \Gamma'$ of $M \# M'$: see [15] for details.

On the other hand, it not difficult to check that, if both $\Gamma$ and $\Gamma'$ are simple crystallizations, then $\Gamma \# \Gamma'$ is, too. Hence, since simple crystallizations do always realize both gem-complexity and regular genus of the represented PL 4-manifolds (Proposition $\textbf{4}$(a) and (b)), the thesis easily follows.

\[\square\]

**Remark 4** Note that the relation $k(M \# M') \leq k(M) + k(M')$ (resp. $\mathcal{G}(M \# M') \leq \mathcal{G}(M) + \mathcal{G}(M')$) can be stated for all PL $n$-manifolds by direct estimation of $k(M \# M')$ (resp. of $\mathcal{G}(M \# M')$) on the gem $\Gamma \# \Gamma'$, when $\Gamma, \Gamma'$ are assumed to be gems of $M, M'$ realizing gem-complexity (resp. regular genus) of the represented $n$-manifolds. Moreover - as pointed out in section 1 - the additivity of both gem-complexity and regular genus under connected sum has been conjectured, and the associated (open) problem is significant especially in dimension four.

In [20, Corollary 4], two classes of closed (not necessarily orientable) 4-manifolds have been detected, for which additivity of regular genus holds. It is not difficult to check that the first one (characterized by relation $\mathcal{G}(M) = 2\chi(M) - 4$) comprehends - in virtue of Theorem 1 - all 4-manifolds admitting a simple crystallization, while the second one (characterized by relation $\mathcal{G}(M) = 1 - \frac{\chi(M)}{2}$) consists - in virtue of [14, Proposition 2] - of connected sums of 3-sphere bundles over $S^1$.

We conclude the paper by reporting two results already proved in [12]. The first one concerns the existence of simple crystallizations of standard simply-connected PL 4-manifolds with $\beta_2 \leq 2$, and has been obtained as a direct consequence of 4-dimensional crystallization catalogues:

**Proposition 8** ([12, Proposition 17])

- $S^4$ and $\mathbb{C}P^2$ admit a unique simple crystallization;
- $S^2 \times S^2$ admits exactly 267 simple crystallizations;
- $\mathbb{C}F^2 \# \mathbb{C}F^2$ admits exactly 583 simple crystallizations;
- $\mathbb{C}F^2 \# (-\mathbb{C}F^2)$ admits exactly 258 simple crystallizations.

\[\square\]

Finally, Proposition $\textbf{5}$ allows to relate the existence of simple crystallizations with known results and open problems about exotic structures on “standard” simply-connected PL 4-manifolds:

**Proposition 9** ([12, Proposition 18])

- $S^4$ and $\mathbb{C}P^2$ admit a unique simple crystallization;
- $S^2 \times S^2$ admits exactly 267 simple crystallizations;
- $\mathbb{C}F^2 \# \mathbb{C}F^2$ admits exactly 583 simple crystallizations;
- $\mathbb{C}F^2 \# (-\mathbb{C}F^2)$ admits exactly 258 simple crystallizations.
(a) Let $M$ be $S^4$ or $\mathbb{CP}^2$ or $S^2 \times S^2$ or $\mathbb{CP}^2 \# \mathbb{CP}^2$ or $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$; if an exotic PL-structure on $M$ exists, then the corresponding PL-manifold does not admit a simple crystallization.

(b) Let $\bar{M}$ be a PL 4-manifold TOP-homeomorphic but not PL-homeomorphic to $\mathbb{CP}^2 \#_2 (-\mathbb{CP}^2)$; then, either $\bar{M}$ does not admit a simple crystallization, or $M$ admits an order 20 simple crystallization (i.e.: $k(M) = 9 = k(\mathbb{CP}^2 \#_2 (-\mathbb{CP}^2))$).

(c) Let $r \in \{3, 5, 7, 9, 11, 13\} \cup \{r = 4n - 1 \mid n \geq 4\} \cup \{r = 4n - 2 \mid n \geq 23\}$; then, infinitely many simply-connected PL 4-manifolds with $\beta_2 = r$ do not admit a simple crystallization.

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