Morse–Smale complexes on convex polyhedra

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Abstract

Motivated by applications in geomorphology, the aim of this paper is to extend Morse–Smale theory from smooth functions to the radial distance function (measured from an internal point), defining a convex polyhedron in 3-dimensional Euclidean space. The resulting polyhedral Morse–Smale complex may be regarded, on one hand, as a generalization of the Morse–Smale complex of the smooth radial distance function defining a smooth, convex body, on the other hand, it could be also regarded as a generalization of the Morse–Smale complex of the piecewise linear parallel distance function (measured from a plane), defining a polyhedral surface. Beyond similarities, our paper also highlights the marked differences between these three problems and it also relates our theory to other methods. Our work includes the design, implementation and testing of an explicit algorithm computing the Morse–Smale complex on a convex polyhedron.

keywords Morse–Smale complexes, polyhedral surfaces, static equilibrium points, radial function

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1 Introduction

Morse theory, the investigation of the critical points of a smooth function defined on a smooth manifold, has been in the focus of research since the middle of the 20th century [21]. This area has received especially much attention since the seminal paper of Witten [27] directly relating Morse theory to quantum field theory, and has gained applications in many disciplines both within and outside mathematics (cf. e.g. [25]). Following the rapid development of computer graphics, the description of smooth functions on a smooth manifold was extended in several directions:

- The discrete problem [16, 18], which was used to analyze the shape of objects of many different sizes from molecules [3].

- The piecewise linear problem represented, for example, by polyhedral surfaces over the \([x, y]\) plane. In [2, 14] the polyhedral surface has been equipped with the so-called height function, which, in many applications, e.g. to terrains [14], everyday objects [15, 12] is the distance measured from a fixed Euclidean plane \(\mathbb{R}^2\). The restriction of the height function to a polyhedral surface is piecewise linear, so all critical points of the function are located at vertices and the algorithm presented in [14] relies on this fact. We will briefly refer to this case as the piecewise linear problem.

- The point set problem, discussed in [13, 4, 17], where the space \(\mathbb{R}^3\) has been equipped with the function whose value is the distance to the nearest of a finite set of points. The main applications of this approach are surface reconstruction and segmentation. Here, the resulting function is neither smooth nor piecewise linear, but the authors present a way to define its gradient vector field and flow curves. The finite point set also defines a Delaunay diagram and its dual Voronoi diagram, critical points of the distance function are the intersection points of these two diagrams. The flow complex encapsulates the subsets of \(\mathbb{R}^3\) covered by the flow lines ending at the same critical points. We are going to call this case the point set problem. While the flow complex proved to be useful in several applications, its potential is restricted: as noted in [13], the concept of the flow complex can not be used to analyze discrete systems (such as [16, 18]).

As we will point out (see below and also Remark [11]), neither of the above theories is applicable to the construction of Morse theory on a compact, polyhedral surface which we will call the polyhedral problem. In this paper our goal is to analyze the latter, by equipping it with the so-called radial distance function, i.e. the distance measured from a fixed, internal reference point. The radial distance function is, like the function in the point set problem, neither smooth nor piecewise linear, leading to significant differences in comparison with both the smooth and piecewise linear case. In particular, critical points may appear not only at vertices (as in the piecewise linear case) or at smooth points (as in...
the smooth case), but both at vertices and in the interiors of edges and faces of the simplicial complex.

Despite these differences, the piecewise linear and the polyhedral problem are also closely related: in the polyhedral (radial) case, if we let the reference point approach infinity, the radial distance function will approach the parallel distance function and all critical points will approach vertices. (We remark that the transition between the polyhedral and piecewise linear case is far from trivial: if the reference point is not in the interior but at finite distance from the polyhedron, then neither the polyhedral nor the piecewise linear theories will apply.)

While the piecewise linear and the polyhedral problems are mathematically closely related, the applications driving these two models are somewhat different. While parallel distance functions can be efficient models in image processing and serve as good approximations in local cartography [14], radial distance functions are the only option in case of particle shape models in geomorphology and planetology [10, 5, 11, 20]. Also, for global cartography, spherical geometry has to be taken into account.

The relationship between the polyhedral problem and the point set problem is nontrivial at first sight: one might be tempted to regard the former as a special case of the latter. Indeed, by reflecting the reference point to the plane of every face, the resulting Voronoi diagram has the polyhedron as one of its cells. Nevertheless, a significant difference between the two problems is that, unlike in the point set problem, in the polyhedral problem the curves of the flow are restricted to remain on boundary of the polyhedron. In Remark [11] we present an example exploiting this difference and showing that the resulting flow complex might be different from the Morse–Smale complex we are after.

Our current paper is primarily motivated by particle shape modelling in geomorphology. Traditional geological shape descriptors (e.g. axis ratios, roundness) have been recently complemented [5, 11] by a new class, called mechanical shape descriptors which proved to be rather efficient not only in determining the provenance of sedimentary particles [22] but also served surprisingly well in the identification of the shapes of asteroids [5, 8]. Mechanical descriptors rely on the radial distance function measured from the center of mass to which we briefly refer as the mechanical distance function. The number of stable, unstable and saddle-type critical points are called first order mechanical descriptors [11, 20] whereas Morse–Smale complexes belong to second order mechanical descriptors [5, 9]. While the mathematical properties of Morse–Smale complexes associated with a smooth mechanical distance function have been investigated in detail [5, 9], so far there has been little experimental data to test this theory. The main obstacle in obtaining field data is the identification of Morse–Smale complexes on scanned images of particles [20]. These scans can be regarded as compact, polyhedral surfaces which we associate with the radial (mechanical) distance function.

In this paper we present the mathematical background to describe these complexes leading, in a natural way, to an algorithm to perform the above task, i.e. to identify Morse–Smale complexes on the surfaces of polyhedra equipped
with the radial distance function. We remark that a Morse–Smale complex on a surface is naturally associated to the topological graph whose vertices and edges are those of the complex. This graph, called Morse–Smale graph, often appears in applications \[6\], and by the above correspondence, information on one provides equivalent information on the other one.

The structure of the paper is as follows. In Section 2 we give a brief summary of the elementary properties of Morse–Smale complexes defined by a smooth convex body in \(\mathbb{R}^3\). The next two sections deal with the main goal of the paper: by investigating the properties of the radial distance function defined on a polyhedral surface, we build up a theory leading to the definition of a Morse–Smale complex and the related Morse–Smale graph on the surface (cf. Definitions 13 and 16). These definitions may appear, at least at first glance, rather similar to their smooth counterparts (Definitions 5 and 6). Also, as we will see, Morse–Smale graphs on polyhedra and Morse–Smale graphs on smooth convex bodies share most, but not all of their combinatorial properties. Still, there also exist fundamental and significant differences between the smooth, the piecewise linear and the polyhedral problems, so each of these cases require different tools leading to these definitions. We present our mathematical results in two sections: in particular, in Section 3 we find an analogue of the gradient vector field on the polyhedron, and in Section 4 we define ascending integral curves generated by this vector field and explore their properties, yielding the notion of Morse–Smale complex on a polyhedral surface. Then in Section 5 we present an algorithm that computes the Morse–Smale complex generated by a generic convex polyhedron. Finally, in Section 6 we collect our additional remarks and open questions.

In the paper we denote by \(\mathbb{R}^3\) the 3-dimensional Euclidean space. We regard this space as a 3-dimensional real vector space equipped with the usual Euclidean norm, denoted by \(|\cdot|\) or \(\|\cdot\|\). We denote the origin of this space by \(o\), and the unit sphere centered at \(o\) by \(S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}\). We use the notation \(\text{int}(\cdot)\) and \(\partial\) to the interior and the boundary of a set, respectively, and for points \(x, y \in \mathbb{R}^3\), we denote the closed segment with endpoints \(x, y\) by \([x, y]\). Let \(K\) be a convex body (i.e. a compact, convex set with nonempty interior) in \(\mathbb{R}^3\), containing \(o\) in its interior. The radial function \(\rho_K : S^2 \to (0, \infty)\) of \(K\) (cf. 23) is defined by

\[
\rho_K(x) = \max\{\lambda : \lambda x \in K\}.
\]

This function, often appearing in the literature, is the reciprocal of the gauge function of \(K\) 23, and it clearly determines the convex body \(K\). We note that while our considerations are valid if \(o\) is an arbitrary interior point of \(K\) chosen as reference point, in applications in mechanics and geology, it usually coincides with the center of gravity of \(K\).
In this section we consider only convex bodies with \( C^\infty \)-class boundary. In this case the mapping \( u \mapsto \rho_K(u) \) is a \( C^\infty \)-diffeomorphism between \( S^2 \) and \( \partial K \), and hence, we assume that \( \rho_K \) is infinitely many times differentiable on \( S^2 \). This diffeomorphism establishes a natural correspondence between topological properties of the Morse–Smale complex on \( S^2 \) defined by \( \rho_K \) and the Morse–Smale complex on \( \partial K \) defined by Euclidean norm in the sense that central projection from \( S^2 \) to \( \partial K \) maps the first complex to the second one, and vice versa, making possible to introduce these complexes in two equivalent ways. In this section we follow the first approach as it is more convenient for those familiar with Morse theory. Nevertheless, in Sections 3 and 4, since our considerations are usually based on the geometric properties of the convex polyhedron under investigation, we use the second approach.

**Definition 1.** A point \( x \in S^2 \) is called a critical point of \( \rho_K \), if the derivative of \( \rho_K \) at \( x \) is zero, that is if \( (\nabla \rho_K)(x) = 0 \). A noncritical point of \( \rho_K \) is called a regular point of \( \rho_K \). A value \( \alpha \in \mathbb{R} \) is called a critical value of \( \rho_K \) if there is a critical point of \( \rho_K \) satisfying \( \rho_K(x) = \alpha \). A critical point \( x \) is nondegenerate, if the Hessian of \( \rho_K \) at \( x \) is not singular.

Recall the well-known fact that if the Hessian of \( \rho_K \) at a nondegenerate critical point \( x \) has 0, 1 or 2 negative eigenvalue, then \( x \) is a local maximum, a saddle point or a local minimum of \( \rho_K \), respectively.

**Remark 1.** If \( x \) is a critical point of \( \rho_K \), then the point \( \rho_K(x) \in \partial K \) is usually called a (static) equilibrium point of \( K \) (with respect to \( o \)). Furthermore, for any nondegenerate critical point \( x \) of \( \rho_K \), the point \( \rho_K(x) \) is called a stable, or a saddle-type or an unstable equilibrium point of \( K \) if \( x \) is a local minimum, a saddle point or a local maximum of \( \rho_K \), respectively [9].

**Definition 2.** The function \( \rho_K \) is called a Morse function if it is \( C^\infty \)-class, and all its critical points are nondegenerate.

In this paper we assume that \( \rho_K \) is a Morse function. This implies that any critical point of \( \rho_K \) has a neighborhood in \( S^2 \) that does not contain any other critical point, and thus, by the compactness of \( S^2 \), \( \rho_K \) has only finitely many critical points.

**Definition 3.** An integral curve \( c : \mathbb{R} \to S^2 \) is a curve, maximal with respect to inclusion, whose derivative at every point coincides with the value of the gradient of \( \rho_K \) at that point; that is, \( c'(t) = (\nabla \rho_K)(c(t)) \) for all \( t \in \mathbb{R} \). We call \( \text{org}(c) = \lim_{t \to -\infty} c(t) \) the origin, and \( \text{dest}(c) = \lim_{t \to \infty} c(t) \) the destination of \( c \) [14].

We remark that our smoothness conditions and the Picard-Lindelöf Theorem yield that any regular point \( x \) belongs to exactly one integral curve, all integral
curves consists of only regular points, and two integral curves are either disjoint or coincide. Furthermore, the origin and the destination of every integral curve are critical points of $\rho_K$ (cf. e.g. [14]).

**Definition 4.** The descending (resp. ascending) manifold of a critical point $x$, denoted by $D(x)$ (resp. $A(x)$) is the union of $x$ and all integral curves with $x$ as their destination (resp. origin).

The function $\rho_K$ is called Morse–Smale if all ascending and descending manifolds of $\rho_K$ intersect only transversally; or equivalently, if any pair of intersecting ascending and descending 1-manifolds cross.

**Definition 5.** The cells of the Morse–Smale complex generated by $\rho_K$ are the sets obtained by intersecting a descending and an ascending manifold. The 2-, 1-, and 0-dimensional cells of this complex are called faces, edges, and vertices, respectively.

Observe that if $x \neq y$ are critical points of $\rho_K$, then $A(x) \cap D(x) = \{x\}$, and $A(x) \cap D(y)$ is the union of all integral curves with origin $x$ and destination $y$. Clearly, the vertices of the Morse–Smale complex of $\rho_K$ are the critical points of $\rho_K$, and if $x$ is a stable and $y$ is an unstable point, then $A(x) \cap D(y)$ is an open set in $S^2$, implying that it is a face of the complex. On the other hand, the same does not hold if $x$ or $y$ is a saddle point since every saddle point is the origin, and also the destination, of exactly two integral curves, corresponding to edges of the complex. It is well known (cf. e.g. Lemma 1 in [14]) that every face of the Morse–Smale complex is bounded by four edges, and the four critical points in the boundary are a stable, a saddle, an unstable and a saddle point, in this cyclic order, where the two saddle points may coincide.

**Definition 6.** The topological graph $G$ on $S^2$, whose vertices are the critical points of $\rho_K$, and whose edges are the edges of the Morse–Smale complex, is called the Morse–Smale graph generated by $\rho_K$ [6]. This graph is usually regarded as a 3-colored quadrangulation of $S^2$, where the ‘colors’ of the vertices are the three types of a critical point.

### 3 Generating the gradient vector field on a polyhedral surface: local properties of the function $N_{\partial P}$

In the remaining part of the paper we deal with a convex polyhedron $P \subset \mathbb{R}^3$ containing $o$ in its interior. We note that the concepts of equilibrium points and nondegeneracy of convex polyhedra are already established in the literature [7] [19]. The following definition is from [7].

**Definition 7.** Let $P \subset \mathbb{R}^3$ be a convex polyhedron and let $o \in \text{int}(P)$. We say that $q \in \partial P$ is an equilibrium point of $P$ (with respect to $o$) if the plane
$H$ through $q$ and perpendicular to $[o,q]$ supports $P$ at $q$. In this case $q$ is nondegenerate, if $H \cap P$ is the (unique) face, edge or vertex of $P$ that contains $q$ in its relative interior; here, by the relative interior of a vertex we mean the vertex itself. A nondegenerate equilibrium point $q$ is called stable, saddle-type or unstable, if $\dim(H \cap P) = 2, 1$ or $0$, respectively. We call $P$ nondegenerate if all its equilibrium points are nondegenerate. The points of $\partial P$ which are not equilibrium points are called regular points of $\partial P$.

In our investigation, we deal only with nondegenerate convex polyhedra. For any set $S \subset \mathbb{R}^3$, we denote the restriction of the Euclidean norm function onto $S$ by $N_S$, and denote the affine hull of $S$ by $< S >$. Note that since $\partial P$ is piecewise smooth, the gradient of the function $N_{\partial P}$ exists only at interior points of the faces of $P$. To extend this notion to nonsmooth points of $\partial P$, we intend to find the ‘direction and rate of steepest ascent’ for all nonsmooth points $q$ among all unit tangent vectors at $q$. The aim of Section 3 is to generalize the notion of gradient vector field by following this approach, and examine the properties of our generalization. In Section 4 we generalize the notion of integral curves and Morse–Smale complex for our setting.

**Definition 8.** Let $q \in \partial P$. If, for some vector $v \neq o$, the half line $\{q+tv : t \geq 0\}$ intersects $\partial P$ in a nondegenerate segment starting at $q$, we say that $v$ is a tangent vector of $P$ at $q$. If $X$ is a face or an edge of $P$ containing $q$, and the gradient $(\nabla N_{<X>})(q)$ is a tangent vector of $P$ at $q$, we say that this vector is a candidate gradient at $q$.

Note that by definition, if $v = (\nabla N_{<X>})(q)$ is a candidate gradient at $q$ and $v_0 = \frac{v}{|v|}$, then the one-sided directional derivative $\lim_{t \to 0^+} \frac{|q+tv_0|}{t} = |v|$ is strictly positive. Furthermore, the maximum of the one-sided directional derivatives of $N_{\partial P}$ at $q$ is the maximum of the lengths of the candidate gradients at $q$. The main result of this section is the following.

**Theorem 1.** Let $q \in \partial P$. If $q$ is an equilibrium point of $P$, then there is no candidate gradient at $q$. If $q$ is a regular point of $P$, then there is at least one candidate gradient at $q$, and there is a unique candidate gradient at $q$ with maximal length.

**Proof.** We distinguish three cases.

**Case 1:** $q$ is an interior point of a face $F$ of $P$. If $q$ is the orthogonal projection of $o$ onto the plane $< F >$, then $q$ is clearly a stable equilibrium point of $P$, and there is no candidate gradient at $q$. In the opposite case there is a unique candidate gradient at $q$, namely $(\nabla N_{<F>})(q)$, which clearly has maximal length among all candidate gradients.

**Case 2:** $q$ is a relative interior point of an edge $E$ of $P$. Let $F_1$ and $F_2$ denote the two faces of $P$ that contain $E$, and for $i = 1, 2$, let $H_i$ denote the open half plane in $< F_i >$ that is bounded by the line $< E >$, and whose closure contains $F_i$. Furthermore, let $o_i$ denote the orthogonal projection of $o$ onto the plane $< H_i >$, and let $o_E$ be the orthogonal projection of $o$ onto $< E >$. 


Thus, in this case gradients at \( - \) value of \( \pi \) implies that the dihedral angle of \( P \) at \( P \) is \( \pi - (o_1 o_2 o_3) \), which contradicts the convexity of \( P \). Thus, we have \( o_t \neq H_i \) for at least one value of \( i \). Since the gradient of \( N_{< H_i >} \) at any \( x \neq o_t \) points in the direction of \( x - o_t \), this implies that at most one of \( (\nabla N_{< H_i >})(q) \) is a candidate gradient.

First, assume that \( o_1 \in H_1 \) and \( o_2 \in H_2 \). Then \( (\nabla N_{< H_i >})(q) \) is not a candidate gradient for \( i = 1, 2 \). On the other hand, \( (\nabla N_{< E >})(q) \) is a tangent vector of \( P \) at \( q \) if and only if \( q \neq o_E \). Thus, this vector is the unique candidate gradient for \( q \neq o_E \), and there is no candidate gradient at \( q \) if \( q = o_E \).

Now we consider the case that exactly one of the \( o_i \) lies in \( H_i \); say, we have \( o_1 \in H_1 \) and \( o_2 \notin H_2 \). Then, since \( o_2 \notin H > \) yields \( o_2 \notin H > \), \( (\nabla N_{< F_i >})(q) \) is a tangent vector of \( P \) at \( q \). On the other hand, in this case \( |(\nabla N_{< E >})(q)| \), if it exists, is a directional derivative of the function \( N_{< F_i >} \) at \( q \), implying that \( |(\nabla N_{< E >})(q)| \leq |(\nabla N_{< F_i >})(q)| \), with equality if and only if \( o_2 \notin H > \) and \( (\nabla N_{< E >})(q) = (\nabla N_{< F_i >})(q) \).

**Case 3:** \( q \) is a vertex of \( P \). Let \( F_1, F_2, \ldots, F_k \) denote the faces of \( P \) containing \( q \) such that \( k \geq 3 \), and for \( i = 1, 2, \ldots, k \), the set \( E_i = F_i \cap F_{i+1} \) is an edge of \( P \). Then the edges of \( P \) containing \( q \) are \( E_1, E_2, \ldots, E_k \). First, observe that if \( q \) is an equilibrium point of \( F \), then for every tangent vector \( v \) of \( F \), the one-sided directional derivative of \( N_{o_P} \) in the direction of \( v \) is negative, which yields that there is no candidate gradient at \( q \). Thus, we may assume that \( q \) is a regular point of \( F \), which yields that \( (\nabla N_{< E >})(q) \) is a candidate gradient for at least one value of \( i \).

We show that there is at most one value of \( i \) such that \( (\nabla N_{< F_i >})(q) \) is a candidate gradient. Indeed, suppose for contradiction that \( (\nabla N_{< F_i >})(q) \) and \( (\nabla N_{< F_j >})(q) \) are candidate gradients at \( q \), where \( i \neq j \). Let \( E = F_i < F_j \), and let \( H_i \) (resp. \( H_j \)) denote the open half plane in \( < F_i \) (resp. \( < F_j \)) that is bounded by the line \( E \), and whose closure contains \( F_i \) (resp. \( F_j \)). Similarly like in Case 2, we may observe that the fact that \( (\nabla N_{< F_i >})(q) \) and \( (\nabla N_{< F_j >})(q) \) are tangent vectors of \( P \) at \( q \) yields that the orthogonal projection of \( o \) onto \( < F_i \) (resp. \( < F_j \)) is not contained in \( H_i \) (resp. \( H_j \)), and thus, the dihedral angle between \( < o > \cup E \) and \( H_i \) (resp. \( H_j \)) is at least \( \frac{\pi}{2} \). But this contradicts the fact that the intersection of the two supporting half spaces of \( P \), containing \( F_i \) and \( F_j \), respectively, is a convex polyhedral region.

Consider the case that for some value of \( i \), \( v = (\nabla N_{< F_i >})(q) \) is a candidate gradient, i.e. it is a tangent vector of \( P \) at \( q \). Then, clearly, the convexity of \( P \) implies that for any unit tangent vector \( v' \neq \frac{v}{|v|} \) of \( P \) at \( q \), the one-sides directional derivative of \( N_{o_P} \) at \( q \) in the direction of \( v' \) is strictly less than \( |v| \). Thus, in this case \( v = (\nabla N_{< F_i >})(q) \) has maximal length among all candidate gradients at \( q \).

Finally, we deal with the case that \( (\nabla N_{< F_i >})(q) \) is not a tangent vector of \( P \) at \( q \) for any value of \( i \), but there are more than one values of \( i \) such that \( (\nabla N_{< E >})(q) \) is a candidate gradient of maximal length. Let \( i_1 \neq i_2 \) be such
values. For \( j = 1, 2 \), consider the closed half line \( R_{\ell_j} = \{ q + tv_{\ell_j} : t \geq 0 \} \) where \( v_{\ell_j} \) is the unit vector pointing from \( q \) towards the other endpoint of \( E_{\ell_j} \), and let \( H = \langle E_{\ell_1} \cup E_{\ell_2} \rangle \). Note that \( |(\nabla N_{<E_{\ell_1}>})(q)| \) is the directional derivative of \( N_H \) in the direction of \( v_{\ell_j} \), and hence, the equality \( |(\nabla N_{<E_{\ell_1}>})(q)| = |(\nabla N_{<E_{\ell_2}>})(q)| \) implies that \( (\nabla N_H)(q) \) points in the direction of \( v_{\ell_1} + v_{\ell_2} \). On the other hand, the half line \( \{ q + tv_{\ell_1} + v_{\ell_2} : t \geq 0 \} \) intersects \( P \) in a non-degenerate segment, and thus, there is a unit tangent vector \( v \) of \( P \) at \( q \) such that the one-sided directional derivative of \( N_{\partial P}(q) \) in the direction of \( v \) is strictly greater than \( |(\nabla N_{<E_{\ell_j}>})(q)| \) \( (j = 1, 2) \), contradicting the assumption that the latter two vectors have maximal length.

\[ \square \]

**Remark 2.** By the argument in the last paragraph, for any vertex \( q \) of \( P \), the set of tangent vectors \( v \) of \( P \) at \( q \) with the property that in the direction of \( v \) the one-sided derivative of the distance function increases is connected, implying the same statement for the set of tangent vectors in which the distance function decreases.

**Definition 9.** If \( q \) is a regular point of \( P \), then we call the unique candidate gradient at \( q \) with maximal length the extended gradient of \( N_{\partial P} \) at \( q \). If \( q \) is an equilibrium point of \( P \), we say that the extended gradient of \( N_{\partial P} \) at \( q \) is 0. We denote the extended gradient of \( N_{\partial P} \) at \( q \) by \( (\nabla^{\text{ext}} N_{\partial P})(q) \).

Our analysis in Case 2 of the proof of Theorem 1 leads to Definition 10.

**Definition 10.** Let \( E \) be an edge of \( P \), and let \( F_1 \) and \( F_2 \) be the two faces of \( P \) containing \( E \). For \( i = 1, 2 \), let \( H_i \) denote the open half plane in \( \langle F_i \rangle \) that is bounded by the line \( \langle E \rangle \), and whose closure contains \( F_i \), and let \( o_i \) be the orthogonal projection of \( o \) onto \( \langle F_i \rangle \). If \( o_1 \in H_1 \) and \( o_2 \in H_2 \), then for any regular point \( q \) in the relative interior of \( E \), the extended gradient at \( q \) is parallel to \( E \). In this case we say that \( E \) is a followed edge of \( P \). If either \( o_1 \notin H_1 \) or \( o_2 \notin H_2 \), then we say that \( E \) is a crossed edge. In particular, if \( E \) is a crossed edge with \( o_1 \notin H_1 \), then for every point \( q \) in the relative interior of \( E \), the extended gradient at \( q \) points towards the interior of \( F_1 \). In this case we say that \( E \) is crossed from \( F_2 \) to \( F_1 \).

We need the following remark in Section 4.

**Remark 3.** Let \( q \) be a vertex of \( P \) and let \( F_1, F_2, \ldots, F_k \) be the faces of \( P \) containing \( q \) such that for \( i = 1, 2, \ldots, k \), \( E_i = F_{i-1} \cap F_i \) is an edge of \( P \). Assume that all edges \( E_i \) are crossed edges. Then there is some value of \( i \) such that \( E_i \) is crossed from \( F_i \) to \( F_{i+1} \), and a value of \( i \) such that \( E_i \) is crossed from \( F_{i+1} \) to \( F_i \). Indeed, if \( d_i \) denotes the distance of \( o \) and \( |(\nabla N_{<F_i>})(q)| \) decreases. Thus, if \( t \) is a value of \( i \) such that \( d_i \) is minimal, then \( E_{i-1} \) is crossed from \( F_i \) to \( F_{i-1} \), and \( E_i \) is crossed from \( F_i \) to \( F_{i+1} \).

**Remark 4.** As a summary, we collect the following rules to find the extended gradient at any given boundary point \( q \) of \( P \).

- If \( q \) is an interior point of a face \( F \), then \( (\nabla^{\text{ext}} N_{\partial(P)})(q) = (\nabla N_{<F>})(q) \).
• If $q$ is a relative interior point of an edge $E$ of $P$, where $F_1$ and $F_2$ denotes the two faces of $P$ containing $E$, then:
  - if $E$ is a followed edge, then $(\nabla^\text{ext} N_{\partial(P)})(q) = (\nabla N_{<E>})(q)$;
  - if $E$ is a crossed edge from $F_1$ to $F_2$, then $(\nabla^\text{ext} N_{\partial(P)})(q) = (\nabla N_{<F_2>})(q)$.

• If $q$ is a vertex of $P$, then:
  - if $q$ is an unstable point of $P$, then $(\nabla^\text{ext} N_{\partial(P)})(q) = 0$;
  - if there is a face $F$ containing $q$ such that $(\nabla N_{<F>})(q)$ is a tangent vector of $P$ at $q$, then $(\nabla^\text{ext} N_{\partial(P)})(q) = (\nabla N_{<F>})(q)$;
  - in the remaining case there is a followed edge $E$ with endpoint $q$ such that $(\nabla^\text{ext} N_{\partial(P)})(q) = (\nabla N_{<E>})(q)$.

It is also worth noting that for any face $F$, the gradient vector $(\nabla N_{<F>})(q)$ is a positive scalar multiple of the vector pointing from the orthogonal projection of $o$ onto $< F >$ to $q$.

The consequence of Remark 4 is that after the classification of edges as crossed or followed, we can always determine which candidate gradient is the extended gradient. There is no need to compute more than one candidate, which makes our algorithm quicker and also less prone to numerical errors.

4 Morse–Smale complex on a polyhedral surface: global properties of the function $N_{\partial P}$

So far, we have defined a variant of the gradient vector field on the boundary of $P$. This vector field, as already $\partial P$, is not smooth, implying that the definition of integral curve cannot be applied to our case without any change. Our definition is as follows, where we denote the interval $(-\infty, 0)$ by $\mathbb{R}^-$. 
Definition 11. An ascending curve \( c : \mathbb{R}^- \to \partial P \) is a curve maximal with respect to inclusion, such that for any point \( q = c(\tau) \) of \( c \) the right-hand side derivative of \( c \) is equal to the extended gradient at \( q \), namely \( c'(\tau) = (\nabla^\text{ext} N_{\partial(P)})(q) \). We call \( \lim_{\tau \to -\infty} c(\tau) \) the origin and \( \lim_{\tau \to 0^-} c(\tau) \) the destination of the ascending curve \( c \).

As we will see, the origin (resp. destination) of any ascending curve is a stable or a saddle point (resp. a saddle or an unstable point) of \( P \). Thus, for any ascending curve \( c \) with origin \( s \), the first segment of \( c \) is a segment on a line perpendicular to \([o, s]\), and, in particular, the derivative \( c'(\tau) \) tends to zero as the point tends to \( s \). From this, an elementary computation shows that the curve \( c \) is parameterized on an interval unbounded from below. On the other hand, if \( s \) is the destination of \( c \), then the last segment in \( c \) is one not perpendicular to \([o, s]\). Thus, \( c \) is parameterized on an interval bounded from above. This shows that every ascending curve can be parameterized on the interval \( \mathbb{R}^- \) in a unique way.

Figure 2: One of the infinitely many ascending curves between the stable point \( s \) and the unstable point \( u \) on a cube

To study the properties of the ascending curves, we start with some preliminary observations.

Remark 5. Clearly, any point of an ascending curve is regular. On the other hand, to any regular point \( q \in \partial P \), one can assign a ‘direction of greatest descent’ as a direction in which the one-sided directional derivative \( \lim_{t \to 0^+} \frac{|q + tv_0|}{t} = |v| \) is minimal among all unit tangent vectors \( v_0 \) of \( P \) at \( q \). Similarly like in case of extended gradient, it can be shown that this direction exists, with the corresponding derivative being strictly negative, and also that it is the opposite of a candidate gradient \( (\nabla N_{X < >})(q) \) for some face or edge \( X \) of \( P \) containing \( q \). This implies that every regular point of \( P \) belongs to the relative interior of at least one ascending curve of \( P \). Note that unlike in the smooth case, a regular point may belong to more than one ascending curve. This is true, for example, for any regular point in the relative interior of a followed edge, where, in general, three ascending curves meet.

Remark 6. By their definition, no two ascending curves can cross or split. On the other hand, as we have seen in Remark 5, they can merge (cf. Figure 2).
Figure 3: Two ascending curves $c_1$ and $c_2$ merging along a followed edge $E$ at $p_2$

Note that every ascending curve is the union of some segments in the interiors of faces or on followed edges of $P$. Our next theorem is an important tool to describe the geometric properties of such a curve.

**Theorem 2.** Let $S_1, S_2, \ldots, S_k$ be a sequence of consecutive segments in an ascending curve such that $S_i$ lies in the face or followed edge $X_i$ of $P$, where $X_i \neq X_{i+1}$. Let $d_i$ denote the distance of $<X_i>$ from $o$. Then $d_1, d_2, \ldots, d_k$ is a strictly increasing sequence.

**Proof.** For $i = 1, 2, \ldots, k-1$, let $H_i$ denote the plane spanned by $S_i \cup S_{i+1}$, where the existence of $H_i$ follows from the condition that $S_i$ and $S_{i+1}$ are not contained in the same face or edge. For $i = 1, 2, \ldots, k$, let $o_i$ denote the orthogonal projection of $o$ onto $H_i$, and set $S_i = [q_i, q_i']$. Let $C_i$ be the convex angular region bounded by the two half lines starting at $q_i$ and containing $S_i$ and $S_{i+1}$, respectively. By the definition of extended gradient, the plane through $<X_i>$ and perpendicular to the plane $<\{o\} \cup S_i>$ supports $P$. This implies that $o_i$ and $q_{i+1}$ lie in the same open half plane of $H_i$ bounded by $<S_i>$. We obtain similarly that $o_i$ and $q_{i-1}$ lie in the same open half plane of $H_i$ bounded by $<S_{i+1}>$. Thus, we have that $o_i$ is an interior point of $C_i$.

We show that for any value of $i$, the distance of $<S_i>$ from $o$ is equal to $d_i$. Indeed, this is trivial if $X_i$ is a followed edge. On the other hand, if $X_i$ is a face, then our observation follows from the fact that by the properties of extended gradient, the orthogonal projection $o_i$ of $o$ onto $<X_i>$ lies on $<S_i>$. Now we prove the assertion. Let $1 \leq i \leq k-1$. Since $S_i$ and $S_{i+1}$ are consecutive segments on an ascending curve, moving a point $q$ at a constant velocity from $q_i$ to $q_{i+1}$ along $S_i \cup S_{i+1}$, the distance of $q$ from $o$ strictly increases. By the Pythagorean Theorem, the same property holds for the distance of $q$ and $o_i$. Thus, $q_i$ does not separate the orthogonal projection of $o_i$ onto $<S_i>$ from $q_{i-1}$, while it strictly separates the orthogonal projection of $o_i$ onto $<S_{i+1}>$ from $q_{i+1}$. Since $o_i \in C_i$, this yields that $o_i$ is strictly closer to $<S_i>$ than to $<S_{i+1}>$ (Figure illustrates this arrangement). Thus, the inequality $d_i < d_{i+1}$ follows from the observation in the previous paragraph.

\[\square\]

In the following we give an upper bound for the number of segments in any ascending curve.
**Theorem 3.** Every ascending curve is a polygonal curve whose origin and destination are equilibrium points of $P$, and it intersects every face and edge of $P$ in at most one segment.

**Proof.** First, we show that every ascending curve is rectifiable. To do it, it is sufficient to observe that the intersection of every face $F$ of $P$ with an ascending curve is a union of segments contained on lines passing through the orthogonal projection $o_F$ of $o$ onto $< F >$. Since the Euclidean norm $N$ strictly increases along an ascending curve, this implies that the total length of these segments is not more than the distance of the farthest point of $F$ from $o_F$.

Next, we show that every ascending curve is the union of finitely many segments. To do it, it is sufficient to show that it intersects the interior of every face of $P$ in finitely many segments. Suppose for contradiction that there is an ascending curve $c$ and a face $F$ such that $\text{int}(F) \cap c(\mathbb{R}^-)$ has infinitely many components. Since $c(\mathbb{R}^-)$ is rectifiable, this implies that $F$ has a vertex $q$ such that every neighborhood of $q$ contains infinitely many components of $\text{int}(F) \cap c(\mathbb{R}^-)$. Also by rectifiability, this implies that for some closed interval $I$, the arc $c(I)$ starts at an interior point of $F$, crosses each edge of $P$ starting at $q$, and ends up at an interior point of $F$. Note that this implies that each edge of $P$ starting at $q$ is a crossed edge, and the direction of the crossing corresponds to the cyclic order of the faces around $q$. But this contradicts Remark 3.

Finally, the fact that an ascending curve intersects every face and edge in at most one segment follows from Theorem 2 and the fact that it is a polygonal curve. □

Now we are ready to define the ascending and the descending manifolds.

**Definition 12.** The ascending (resp. descending) polyhedral manifold of an equilibrium point $x$ is the union of $x$ and all ascending curves with $x$ as their origin (resp. destination). We denote the ascending (resp. descending) manifold of $x$ by $A(x)$ (resp. $D(x)$).
Our first result is Theorem 4.

**Theorem 4.** If \( P \subset \mathbb{R}^3 \) is a nondegenerate convex polyhedron, then for any unstable point \( y \in \partial P \), the descending manifold \( D(s) \) is open in \( \partial P \).

The proof of Theorem 4 is based on Lemma 1.

**Lemma 1.** Let \( P \) be nondegenerate, and let \( q \) be a vertex of \( P \) which is not an equilibrium point. Let \( q' \) be a point on the ascending curve through \( q \) such that \( q' \) is not a vertex of \( P \), and \( [q, q'] \) belongs to \( \partial P \). Then, for every neighborhood \( U \) of \( q' \) in \( \partial P \) there is a neighborhood \( V \subset \partial P \) of \( q \) such that the ascending curve through a point of \( V \) intersects \( U \).

**Proof.** First, we show that there is a unique candidate gradient at \( q \). Let \( F_1, \ldots, F_k \) denote the faces of \( P \) containing \( q \) in cyclic order. Let \( E_i = F_{i-1} \cap F_i \) for \( i = 1, 2, \ldots, k \). Let us call an edge \( E_t \) descending if the one-sided directional derivative at \( q \) in the direction of \( E_t \) is negative, and let us call it nondescending otherwise. We have seen in the proof of Theorem 4 that if some \( (\nabla N_{<F_i>})(q) \) is a candidate gradient, then \( (\nabla N_{<F_i>})(q) \) is the extended gradient at \( q \), and there is no candidate gradient generated by any other face \( F_j \). Now we show more, namely that apart from the face/edge generating the extended gradient, all nondescending edges are crossed edges. Before doing it, recall that by Remark 2 we may assume that the nondescending edges of \( P \) at \( q \) are \( E_1, E_2, \ldots, E_j \) for some \( 1 \leq j < k \).

Case 1: \( (\nabla^{ext} N_{\partial(P)})(q) = (\nabla N_{<F_i>})(q) \) for some value of \( i \). Then at least one of \( E_i \) and \( E_{i+1} \) is nondescending, and hence, we may assume that \( 1 \leq i \leq j \). First, observe that since \( (\nabla N_{<F_i>})(q) \) points towards the interior of \( F_i \), the lines through \( E_i \) and \( E_{i+1} \) separate this vector from the orthogonal projection of \( o \) onto \( < F_i > \). Thus, \( E_i \) is a crossed edge from \( F_{i-1} \) to \( F_i \), and \( E_{i+1} \) is a crossed edge from \( F_{i+1} \) to \( F_i \), respectively.

We show by induction on \( t \) that \( E_t \) is a crossed edge from \( F_{t-1} \) to \( F_t \), for \( t = 1, 2, \ldots, i \). Note that we have already shown it for \( t = i \). Assume that \( E_{t+1} \) is a crossed edge from \( F_t \) to \( F_{t+1} \) for some \( t \geq 1 \). Let \( o_t \) denote the orthogonal projection of \( o \) onto \( < F_t > \). The fact that \( E_{t+1} \) is crossed from \( F_t \) to \( F_{t+1} \) implies that \( < E_{t+1} > \) does not separate \( o_t \) and \( F_t \). Furthermore, the fact that both \( E_{t+1} \) and \( E_t \) are nondescending yields that the angle between \( E_{t+1} \) and \( [o_t, q] \), as well as the angle between \( E_t \) and \( [o_t, q] \), are both non-acute. But from this we have that \( < E_t > \) separates \( o_t \) and \( F_t \), which yields that \( E_t \) is crossed from \( F_{t-1} \) to \( F_t \) (see Figure 5 for reference.). Repeating this argument for \( i+1 \leq t \leq j \), we obtain that all nondescending edges at \( q \) are crossed.

Case 2: \( (\nabla^{ext} N_{\partial(P)})(q) = (\nabla N_{<F_i>})(q) \) for some value of \( i \). In this case a similar argument yields the desired statement.

We remark that the argument in Case 1 also shows that there are no two consecutive edges \( E_t, E_{t+1} \) which are both descending, and \( E_t \) is crossed from \( F_t \) to \( F_{t-1} \), and \( E_{t+1} \) is crossed from \( F_t \) to \( F_{t+1} \). This observation, combined with the properties of nondescending edges proved above, yields the assertion.
Figure 5: A single step of the induction

Proof of Theorem 4. Let \( c : \mathbb{R}^{-} \rightarrow \partial P \) be an ascending curve with the unstable point \( y \) as its destination, and let \( q = c(\tau) \) be a point of \( c \). We need to show that \( q \) has a neighborhood \( V \) such that for any point \( q' \in V \), the destination of the ascending curve through \( q' \) is \( y \). By Theorem 3, \( c([\tau, 0]) \cup \{y\} \) is the union of finitely many closed segments such that

(i) each segment is a subset of a face or a followed edge of \( P \)

(ii) the endpoint of each segment, apart from \( q \), is either a relative interior point of an edge, or a vertex of \( P \).

Thus, it is sufficient to show that if \([x_j, x_{j+1}]\) is one of these segments with \( x_j \) preceding \( x_{j+1} \) on \( c \), then for every neighborhood \( U \) of \( x_{j+1} \) there is a neighborhood \( V \) of \( x_j \) such that for any \( q' \in V \), the ascending curve through \( q' \) intersects \( U \). This statement is trivial if \( x_j \) is a relative interior point of an edge. Assume that \( x_j \) is a vertex, and let \( x' \) be a relative interior point of \([x_j, x_{j+1}]\). Then, by the above observation, for any neighborhood \( U \) of \( x_j \) there is a neighborhood \( V' \) of \( x' \) such that any ascending curve intersecting \( V' \) intersects \( U \), and by Lemma 1, the same statement holds for \( x_j \) and \( x' \) in place of \( x' \) and \( x_{j+1} \), respectively.

\[ \square \]

Definition 13. The intersection of an ascending and a descending polyhedral manifold, i.e. a set of the form \( A(x) \cap D(y) \) for some (not necessarily distinct) equilibrium points \( x, y \) of \( P \), is called a cell of the Morse–Smale complex of \( P \). The Morse–Smale complex of \( P \) is defined as the family consisting of its cells.

Definition 14. An ascending curve is isolated if it does not belong to both a descending and an ascending polyhedral manifold.

Recall that for generic smooth convex bodies, isolated integral curves correspond to integral curves starting or ending at saddle points of the body. The following example shows that the same statement does not hold for every non-degenerate convex polyhedron \( P \).

Example 1. Let \( H^+ \) and \( H^- \) be two closed half planes in \( \mathbb{R}^3 \) such that their intersection is the line \( L = \{(1, t, 0) : t \in \mathbb{R}\} \), they are symmetric to the \((x, y)\)-plane, and their convex hull \( P_0 \) contains \( o \). Let \( s_1 = (1, 0, 0) \), and let \( H \) be a
plane parallel to the \( z \)-axis such that it intersects the \( x \)-axis at a point \((1+\varepsilon, 0, 0)\) with some sufficiently small \( \varepsilon > 0 \), and passes through the point \( q = (1, 1, 0) \). Let \( s_2 = (x, y, 0) \) be a point of \( H \) with \( y > 1 \), and let \( H' \) be a plane containing the line through \( s_2 \), parallel to the \( z \)-axis, such that \( s_2 \) is a saddle point of the truncation of \( P_0 \) with \( H \) and \( H' \). Then the polygonal curve \([s_1, q] \cup [q, s_2]\) is an ascending curve from \( s_1 \) to \( s_2 \), corresponding to a saddle-saddle connection. Furthermore, if \( o_+ \) is the orthogonal projection of \( o \) onto \( H^+ \), then \( \text{conv}\{o_+, s_1, q\} \) is contained in \( D(s_2) \).

**Figure 6:** The polyhedron bounded by one possible choice of \( H^+, H^-, H \) and \( H' \) in Example 1.

**Remark 7.** Clearly, for any nondegenerate convex polyhedron \( P \) every saddle point \( s \) is a relative interior point of a followed edge \( E \), implying that all ascending curves ending at \( s \) reach it in one of the two directions perpendicular to \( E \), and all ascending curves starting at \( s \) start along the edge \( E \). Thus, for any nondegenerate convex polyhedron, the ascending and the descending manifolds at any saddle point cross.

Based on Example 1, and Remark 7, we introduce the following notion. Before that, we recall that a *simplicial* convex polyhedron \( P \subset \mathbb{R}^3 \) is defined as a convex polyhedron having only triangular faces, and observe that if \( P = \text{conv}\{q_1, q_2, \ldots, q_k\} \) is a nondegenerate convex polyhedron with equilibrium points \( s_1, s_2, \ldots, s_m \), and for any sufficiently small perturbation \( P' = \text{conv}\{q'_1, q'_2, \ldots, q'_k\} \), with the point \( q'_i \) being the perturbation of \( q_i \), \( P' \) is a nondegenerate convex polyhedron with \( m \) equilibrium points \( s'_1, s'_2, \ldots, s'_m \), and for a suitable choice of indices, \( s'_j \) has the same type as that of \( s_j \), and their distance is small.

**Definition 15.** Let \( P \subset \mathbb{R}^3 \) be a simplicial, nondegenerate convex polyhedron, and let the set of equilibrium points of \( P \) be \( \text{Eq}(P) \). If, for any vertex \( q \) of \( P \) with
\( q \in A(x) \cap D(y) \) and \( x, y \in \text{Eq}(P) \), and any sufficiently small perturbation \( P' \) of \( P \), we have \( q' \in A(x') \cap D(y') \), where \( q' \) is the vertex of \( P \) corresponding to \( q \), and \( x', y' \) are the equilibrium points of \( P' \) corresponding to \( x \) and \( y \), respectively, then we say that \( P \) is \textit{generic}, or \textit{Morse–Smale}.

**Theorem 5.** Let \( P \subset \mathbb{R}^3 \) be a generic polyhedron. Then the following holds:

(i) the descending manifold of any saddle point \( s \) of \( P \) contains no vertex of \( P \);

(ii) the descending manifold of any saddle point consists of exactly two ascending curves;

(iii) the 2-dimensional cells of the Morse–Smale complex of \( P \) are bounded by four ascending curves, and the four endpoints of these curves are a stable, a saddle, an unstable and a saddle point in this cyclic order.

(iv) the isolated ascending curves are exactly those starting or ending at a saddle point;

**Proof.** Let \( P \) be generic. Suppose for contradiction that there is an ascending curve \( c \) of \( P \) ending at a saddle point \( s \) that contains a vertex. Let us denote the open segments of \( c(\mathbb{R}^-) \) by \( S_1, S_2, \ldots, S_k \) in consecutive order. Let the endpoints of \( S_i \) be \( q_i \) and \( q_{i+1} \). By the properties of saddle points, \( S_k \) lies in the interior of a face of \( P \). Thus, \( q_k \) is either a vertex or a relative interior point of a crossed edge. In the latter case \( S_{k-1} \) lies in the interior of a face of \( P \). Again, \( q_{k-1} \) is either a vertex, or a relative interior point of a crossed edge of \( P \). Repeating this argument we obtain that the last vertex \( q_i \) of \( P \) on \( c \) has the property that for all \( i < j \leq k \), the point \( q_j \) is a relative interior point of a crossed edge of \( P \). Now, let \( F' \) be the face containing the segment \( [q_i, q_{i+1}] \), and let \( o' \) denote the orthogonal projection of \( o \) onto \( < F' > \). Let us perturb \( q_i \) by moving it in the plane \( < F' > \) to a point \( q'_i \) such that \( q_i, q'_i \) and \( o' \) are not collinear. Then it is easy to see that the ascending curve through \( q'_i \) on the perturbed polyhedron meets the edge of \( s \) at a point different from \( s \), and thus, it cannot end at \( s \). But this contradicts our assumption that \( P \) is generic, implying (i).

Now, assume that \( P \) is a simplicial convex polyhedron with the property that the descending manifold of any saddle point \( s \) contains no vertex other than \( s \). Note that any saddle point \( s \) is a relative interior point of a followed edge. Thus, there are exactly two ascending curves starting at \( s \), and they run on the edge of \( s \) until they reach the endpoints of this edge, implying that any ascending curve starting at a saddle point contains a vertex of \( P \). On the other hand, the ascending curves ending at \( s \) reach it along the two segments in \( \partial P \) perpendicular to the edge of \( P \). By our assumption and the consideration in the previous paragraph, these segments belong to unique ascending curves whose all vertices lie on crossed edges of \( P \). Thus, for any saddle point \( s \) there are exactly two ascending curves starting at \( s \), and these contain vertices of \( P \), and exactly two ascending curves ending at \( s \), and these do not contain vertices of
This implies, in particular, that the other endpoints of these curves are not saddle points. Thus, the ascending curves starting at $s$ end at unstable, and the ones ending at $s$ start at stable points of $P$. From this, (ii) readily follows.

Note that since ascending curves cannot split, the two ascending curves ending at $s$ are necessarily disjoint, but the two curves starting at $s$ may merge later. Nevertheless, the above facts are sufficient to show that, after removing all these curves from $\partial P$, any component is bounded by four arcs of such curves, whose endpoints are a stable, a saddle, an unstable and a saddle point in this cyclic order. Clearly, any such region $R$ is bounded by $4k$ arcs for some $k \in \mathbb{Z}^+$, and the sequence of the types of the endpoints of the corresponding curves are stable, saddle, unstable and saddle repeated $k$ times. Observe that by its definition, $R$ is an open subset of $\partial P$. On the other hand, by Theorem 4 and since $R$ is open, $R \cap D(y)$ is an open subset of $R$ for any unstable point $y$ being the destination of an ascending curve in $\partial P$. Thus, the connectedness of $R$ yields that $k = 1$ and (iii) holds, implying also (iv).

Corollary 1. By Theorem 5, the Morse–Smale complex of a generic convex polyhedron $P$ consists of the following types of cells.

1. The equilibrium points of $P$, called the vertices of the Morse–Smale complex.
2. The closures of the isolated ascending curves, namely the curves starting or ending at a saddle point. These cells are called edges of the complex.
3. for a stable point $s$ and an unstable point $u$, the union of all ascending curves starting at $s$ and ending at $u$, and also the four isolated ascending curves intersecting the boundary of this union in an arc. These cells are called faces of the complex.

We note that a face of a Morse–Smale complex is not necessarily homeomorphic to a disc. This happens if two ascending curves in the boundary of the face merge. The existence of such a face in a Morse–Smale complex is equivalent to the property that the complex is not a CW-decomposition of $\text{bd} P$.

Definition 16. Let $P \subset \mathbb{R}^3$ be a generic convex polyhedron. The topological graph $G$ on $\partial P$ whose vertices are the equilibrium points of $P$, and whose edges are the isolated ascending curves of $P$, is called the Morse–Smale graph generated by $P$. This graph can be regarded as a 3-colored quadrangulation of $S^2$, where the ‘colors’ of the vertices are the three types of a critical point.

5 An algorithm to compute Morse–Smale complex

It is a natural problem to determine the Morse–Smale graph $G$ of a generic convex polyhedron $P$. Our results in Sections 3 and 4 provide an algorithmic answer for this problem:
Step 1: Find the orthogonal projections of $o$ onto the line of each edge of $P$.

Step 2: Find which edge is followed, which edge is crossed.

Step 3: Find the saddle points of $P$.

Step 4: For each saddle point, find the ascending curves ending there. Note that by genericity, there are exactly two such curves for each saddle point, and they intersect only crossed edges.

Step 5: For each saddle point, find the ascending curves starting there.

We implemented the above algorithm and we illustrate it on the following example. Let $P_{ex}$ be a polyhedron constructed as follows:

1. Consider a regular octagon $O_1 \subset \mathbb{R}^3$ with diameter $d = 2$, lying in a plane parallel to the $(x, y)$ plane and with center $(0, 0, 1)$. Denote the vertices of $O_1$ by $v_i$, ($i = 1, 2, \ldots, 8$).

2. Project $O_1$ orthogonally onto the plane $z = -1$ to obtain $O_2$.

3. Rotate $O_2$ by $\varphi = \frac{\pi}{20}$ clockwise and denote the vertices of the rotated octagon by $v_i$, ($i = 9, 10, \ldots, 16$).

4. Add the points $v_0 = (0, 0, 1.2), v_{17} = (0, 0, -1.2)$.

5. Define $P_{ex}$ as the convex hull of the points $\{v_i\}, (i = 0, 1, \ldots, 17)$.

To equip $P_{ex}$ with a radial distance function, we select the point $o_{ex} = (0.5, 0.5, 0.5)$ as origin (in this case the origin, as a center of mass, corresponds to inhomogeneous material distribution). Figure 7 shows the ascending curves of $P_{ex}$ and we can see that $P_{ex}$ is generic.

6 Concluding remarks and summary

Before summarizing our results we make some general remarks and state an open question.

6.1 Some remarks and a question

Remark 8. We have seen in Theorem 4 that for any nondegenerate convex polyhedron, the descending manifold of any unstable point is open. Nevertheless, there are even generic convex polyhedra with stable points whose ascending manifolds are not open. This happens, for example, if two ascending curves in the boundary of a 2-cell of the Morse–Smale complex merge. Figure 8 shows a face of $P$ where this situation can arise.
Figure 7: Isolated ascending curves on $P_{\text{ex}}$ with respect to $o_{\text{ex}}$. Red, green and blue points represent unstable, stable and saddle-type equilibria, respectively. Stable-saddle isolated ascending curves are shown in green, saddle-unstable isolated ascending curves are shown in red.

Before our next remark, let us recall a few concepts from Baire category theory. A topological space is called a Baire space, if the union of every countably many closed sets with empty interior has empty interior. In particular, the family of convex polyhedra with at most $n$ vertices is a complete metric space, and thus, it is a Baire space \cite{23}. A set in a complete metric space is called typical or residual if its complement is a countable union of nowhere dense sets. In particular, if a set in a complete metric space is open and everywhere dense, it is typical.

Remark 9. Let $\mathcal{P}_m$ denote the family of convex polyhedra in $\mathbb{R}^3$ with at most $m$ vertices. Observe that $\mathcal{P}_m$ is a complete metric space with respect to Hausdorff distance, and thus, it is a Baire space. Furthermore, a sufficiently small perturbation of the vertices of a generic convex polyhedron with $m$ vertices yields a generic polyhedron with $m$ vertices, and any convex polyhedron with at most $m$ vertices can be approximated arbitrarily well by such a polyhedron. Thus, in Baire category sense, a typical convex polyhedron in $\mathcal{P}_m$ with at most $m$ vertices is a generic convex polyhedron with $m$ vertices.

Remark 10. The computation of Morse–Smale complexes is a fast growing area in data visualization and some algorithms have even benefited from GPU acceleration \cite{24}. Apparently, the speed of such algorithms is essential, we make some simple observations about our algorithm. Note that if $P$ has $n$ vertices, it has at most $2n - 4$ faces and $3n - 6$ edges, and each vertex, edge, face contains at most one equilibrium point. Thus, Steps 1-3 in our algorithm can be carried
Figure 8: The saddle points $h_1$ and $h_2$ are destinations of two isolated ascending curves from the stable point $s$. They are also origins of isolated ascending curves passing through the vertex $q$. If $q$ is unstable then the 2-cell bounded by the previously mentioned isolated ascending curves is open, otherwise it is not.

out in $O(n)$ time, and by Theorem 3 Steps 4-5 can be carried out in $O(n^2)$ time. Clearly, at least $\Omega(n)$ steps are necessary to find the Morse–Smale graph of $P$. While the exact magnitude of a fastest algorithm that solves this problem is an open question (see below), we note that on all test examples (with a couple hundred vertices per polyhedron on average) our algorithm, running on a single core of a standard laptop CPU, computed the Morse–Smale complex well under a second.

Problem 1. Find the exact magnitude of a fastest algorithm that finds the Morse–Smale graph of any generic convex polyhedron.

Remark 11. Let $X \in \mathbb{R}^3$ be a finite point set, and for any point $q \in \mathbb{R}^3$, set $h : \mathbb{R}^3 \to \mathbb{R}, h(q) = \min\{|q-x|^2 : x \in X\}$. This function is a piecewise smooth function which is not smooth exactly at the boundary points of the Voronoi cells of the points of $X$. In [17], the authors considered the gradient vector field in $\mathbb{R}^3$ defined by $h$, and examined the properties of the 3-dimensional Morse–Smale complex associated to it. An important tool of their description was a geometric interpretation of the direction of the steepest ascent, which can be stated as follows: Consider a point $q \in \mathbb{R}^3$. Then there is a unique smallest dimensional Voronoi object $V$ (vertex, edge, face or cell) in the Voronoi decomposition of $\mathbb{R}^3$ by $X$ which contains $q$. Let $D$ be Delaunay object in the Delaunay decomposition of conv($X$) by $X$ dual to $V$, and let $x$ be the point of $D$ closest to $q$, which is called the driver of $q$. Then the direction of the steepest ascent of $h$ at $q$ is the direction of the vector $q-x$. In the point set problem mentioned in the introduction, these directions, viewed as a (not everywhere smooth) vector field in $\mathbb{R}^3$, are used to define ascending curves through noncritical points, which is then applied to define the flow complex associated to the point set.

The results in [17] are especially interesting if we recall that if $P$ is a convex polyhedron in $\mathbb{R}^3$ with $x \in \text{int}(P)$, then $P$ is the Voronoi cell of the point set $X$ consisting of $x$ and its reflected copies to all face planes of $P$. Furthermore, for any $x \in \mathbb{R}^3$ the gradient vector field defined by $q \mapsto |q-x|$ coincides with the
normalized gradient vector field defined by \( q \mapsto ||q - x||^2 \). On the other hand, an important difference between this model and our setting is that in the point set problem, the direction of steepest ascent is considered among all directions in \( \mathbb{R}^3 \), whereas in our paper it is restricted to \( \partial P \).

We illustrate this difference by the following example (see Figure 9) which shows that in the flow complex defined by \( P \) in the previous paragraph, an ascending curve may leave \( \partial P \). First, let the reference point in \( P \) be the origin \( o \). Let \( W \) be a (convex) wedge bounded by the half planes \( H_1 \) and \( H_2 \) perpendicular to the \((x, y)\)-plane such that \( W \) is symmetric to the \((y, z)\)-plane and it contains \( o \) in its interior. For \( i = 1, 2 \), let \( o_i \) denote the reflection of \( o \) to the plane containing \( H_i \). These points are in the \((x, y)\)-plane. Let \( L = H_1 \cap H_2 \). Then, with respect to the point set \( \{o, o_1, o_2\} \), \( L \) is a Voronoi edge whose associated Delaunay face is the triangle with vertices \( o, o_1, o_2 \). Thus, if \( d \) denotes the midpoint of the segment \([o_1, o_2]\), then for any \( q \in L \) the direction of the steepest ascent points in the direction of \( q - d \) from \( q \), and thus, it leaves the wedge \( W \). The direction of steepest ascent would coincide with \( L \) iff \( L \) intersects the triangle with vertices \( o, o_1, o_2 \). Now, we construct \( P \) by truncating \( W \) with planes in such a way that the obtained convex polyhedron \( P \) has an edge lying in \( L \), providing the desired example.

![Figure 9: The wedge \( W \) of Remark 11 truncated by two planes parallel to the \((x, y)\)-plane and one parallel to the \((x, z)\)-plane, forming the polyhedron \( P \). \( P \)'s intersection with the \((x, y)\)-plane is shown with dashed lines.](image)

6.2 Summary

Motivated by geophysical applications, in this paper we extended Morse–Smale theory to a convex polyhedron \( P \), interpreted as a scalar (radial) distance measured from a point \( o \) in its interior. We defined the polyhedral Morse–Smale
complex and the Morse–Smale graph $G$ associated with $P$ and $o$ and we built an algorithm to compute $G$. In geophysical applications, $o$ is the center of mass of the particle and the scanned image of the particle is a polyhedron, so this algorithm associates one topological graph to a scanned particle.

In classical Morse–Smale theory the Morse–Smale complex is defined by a function $f : \mathcal{M} \rightarrow \mathbb{R}$, acting on the compact manifold $\mathcal{M}$ and in the classical case $f$ is considered to be smooth. From the abstract point of view, the natural extension of this theory is to extend the class of functions $f$ to be considered and this path was followed in [2] and [14] where the authors considered piecewise linear functions. Here we took a slightly different approach: instead of defining a new class of functions we would discuss, we considered the geometric image of the function as a given object. In our case, this image was a convex polyhedron and the function $f$ was defined on the sphere as a radial distance, measured from a point. We hope that this geometric approach may lead to other potential generalizations of Morse–Smale theory.

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References

[1] Vladimir I. Arnold. Topological classification of Morse functions and generalisations of Hilbert’s 16-th problem. *Math. Phys. Anal. Geom.*, 10(3):227–236, 2007.

[2] T. F. Banchoff. Critical points and curvature for embedded polyhedral surfaces. *Amer. Math. Monthly*, 77:475–485, 1970.

[3] Frédéric Cazals, Frédéric Chazal, and Thomas Lewiner. Molecular shape analysis based upon the Morse-Smale complex and the Connolly function.
In *Proceedings of the nineteenth annual symposium on Computational geometry*, pages 351–360, 2003.

[4] Tamal K. Dey, Joachim Giesen, and Samrat Goswami. Shape segmentation and matching with flow discretization. In *Algorithms and data structures*, volume 2748 of *Lecture Notes in Comput. Sci.*, pages 25–36. Springer, Berlin, 2003.

[5] Gábor Domokos. Natural numbers, natural shapes. *Axiomathes*, 32:743–763, 2022.

[6] Gábor Domokos, Philip Holmes, and Zsolt Lángi. A genealogy of convex solids via local and global bifurcations of gradient vector fields. *J. Nonlinear Sci.*, 26(6):1789–1815, 2016.

[7] Gábor Domokos, Flórián Kovács, Zsolt Lángi, Kriszta Regős, and Péter T. Varga. Balancing polyhedra. *Ars Math. Contemp.*, 19(1):95–124, 2020.

[8] Gábor Domokos and Zsolt Lángi. The isoperimetric quotient of a convex body decreases monotonically under the Eikonal abrasion model. *Mathematika*, 65(1):119–129, 2019.

[9] Gábor Domokos, Zsolt Lángi, and Tímea Szabó. A topological classification of convex bodies. *Geom. Dedicata*, 182:95–116, 2016.

[10] Gábor Domokos, András Árpád Sipos, Gyula Szabó, and Péter Várkonyi. Formation of sharp edges and planar areas of asteroids by polyhedral abrasion. *The Astrophysical Journal*, 699(1):L13–L16, jun 2009.

[11] Gábor Domokos, András Árpád Sipos, Tímea Szabó, and Péter Várkonyi. Pebbles, shapes and equilibria. *Math. Geosci.*, 42:Article No. 29, 2010.

[12] Shen Dong, Peer-Timo Bremer, Michael Garland, Valerio Pascucci, and John C Hart. Spectral surface quadrangulation. In *ACM SIGGRAPH 2006 Papers*, pages 1057–1066. 2006.

[13] Herbert Edelsbrunner. Surface reconstruction by wrapping finite sets in space. In *Discrete and computational geometry*, volume 25 of *Algorithms Combin.*, pages 379–404. Springer, Berlin, 2003.

[14] Herbert Edelsbrunner, John Harer, and Afra Zomorodian. Hierarchical Morse-Smale complexes for piecewise linear 2-manifolds. volume 30, pages 87–107. 2003. ACM Symposium on Computational Geometry (Medford, MA, 2001).

[15] Wei Feng, Jin Huang, Tao Ju, and Hujun Bao. Feature correspondences using Morse Smale complex. *Vis. Comput.*, 29(1):53–67, 2013.

[16] Robin Forman. A user’s guide to discrete Morse theory. *Séminaire Lotharingien de Combinatoire*, 48:B48c, 2002.
[17] Joachim Giesen and Matthias John. The flow complex: a data structure for geometric modeling. *Comput. Geom.*, 39(3):178–190, 2008.

[18] Attila Gyulassy, Peer-Timo Bremer, Bernd Hamann, and Valerio Pascucci. A practical approach to Morse-Smale complex computation: Scalability and generality. *IEEE Trans. Vis. Comput. Graph.*, 14(6):1619–1626, 2008.

[19] Zsolt Lángi. A solution to some problems of Conway and Guy on monostable polyhedra. *Bull. Lond. Math. Soc.*, 54(2):501–516, 2022.

[20] Balázs Ludmány and Gábor Domokos. Identification of primary shape descriptors on 3d scanned particles. *Period. Polytech. Electr. Eng. Comput. Sci.*, 62(2):59–64, 2018.

[21] J. Milnor. *Morse theory*. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[22] Tímea Novák-Szabó, András Árpád Sipos, Sam Shaw, Duccio Bertoni, Alessandro Pozzebon, Edoardo Grottoli, Giovanni Sarti, Paolo Ciavola, Gábor Domokos, and Douglas J. Jerolmack. Universal characteristics of particle shape evolution by bed-load chipping. *Science Advances*, 4(3), 2018.

[23] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[24] Varshini Subhash, Karran Pandey, and Vijay Natarajan. GPU parallel computation of Morse-Smale complexes. In *2020 IEEE Visualization Conference (VIS)*, pages 36–40, 2020.

[25] K. Uhlenbeck. Morse theory by perturbation methods with applications to harmonic maps. *Trans. Amer. Math. Soc.*, 267(2):569–583, 1981.

[26] Péter Várkonyi and Gábor Domokos. Static equilibria of rigid bodies: Dice, pebbles, and the Poincare-Hopf theorem. *J. Nonlinear Sci.*, 16:255–281, 2006.

[27] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.