Clopper-Pearson Bounds from HEP Data Cuts∗

Bernd A. Berg
Department of Physics, The Florida State University, Tallahassee, FL 32306, USA.
E-mail: berg@hep.fsu.edu

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Abstract

For the measurement of $N_s$ signals in $N$ events rigorous confidence bounds on the true signal probability $p_{\text{exact}}$ were established in a classical paper by Clopper and Pearson [Biometrica 26, 404 (1934)]. Here, their bounds are generalized to the HEP situation where cuts on the data tag signals with probability $P_s$ and background data with likelihood $P_b < P_s$. The Fortran program which, on input of $P_s$, $P_b$, the number of tagged data $N_Y$ and the total number of data $N$, returns the requested confidence bounds as well as bounds on the entire cumulative signal distribution function, is available on the web. In particular, the method is of interest in connection with the statistical analysis part of the ongoing Higgs search at the LEP experiments.

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1 Introduction

The general theory of confidence bounds (or fiducial intervals) was developed by Fisher [1], Neyman and Pearson [2]. We consider a particular problem which is of interests when cuts are used to analyze high energy physics data. Typically, a neural network or some other method of performing the cuts results in probabilities (efficiencies) to tag signals more likely than background events. For instance by means of Monte Carlo (MC) simulations, these probabilities can normally be calculated. Let $P_s$ be the probability to tag a signal and $P_b$ be the likelihood to tag a background event, $0 < P_b < P_s < 1$. Out of a total number of $N$ data one gets in this way

$$N^Y \text{ tagged data.}$$

(1)

It is easy to find from this the mean expectation for the signal probability. Assume that there are $N_s$ signals and $N_b$ background events in the data. Then we have

$$N = N_s + N_b \quad \text{and} \quad N^Y = P_s N_s + P_b N_b$$

and these two equations solve for

$$p_{\text{mean}} = \frac{N_s}{N} = \frac{N^Y - P_b N}{N (P_s - P_b)} .$$

(2)

The question is, what are the implied confidence limits on the signal probability?

The special case $P_s = 1$ and $P_b = 0$ (sure signal detection) has been treated by Clopper and Pearson [3] in 1934. After briefly reviewing their approach in the next section, we derive and illustrate the general case in section 3. This is, in part, based on Ref.[4]. In particular, the method is valid when the number of tagged data is small and returns the probability $P_0$ for the case that there is no signal, i.e. that the exact signal probability is $p_{\text{exact}} = 0$. This is of interest for the statistical analysis of the ongoing Higgs search at LEP [5]. Discovery of the Higgs particle on the 5$\sigma$ level would mean $P_0 \leq 0.287 \times 10^{-6}$. Subsection 3.3 explains the use of the corresponding Fortran programs and how to download them from the web. Conclusions follow in the final section [4].
The Clopper–Pearson Confidence Limits

Let $p$ be the likelihood that a data point is a signal. For $N$ measurements the probability to observe $k$ signals is given by the binomial coefficient

$$b_N(k, p) = \binom{N}{k} p^k q^{N-k} = \frac{N!}{k!(N-k)!} p^k q^{N-k} \quad \text{with} \quad q = p - 1.$$  \ (3)

The probability to observe $k \geq N_s$ signals is given by

$$P_{k \geq N_s}(p) = \sum_{k=N_s}^{N} b_N(k, p)$$  \ (4)

and the probability to observe $k \leq N_s$ signals by

$$P_{k \leq N_s}(p) = \sum_{k=0}^{N} b_N(k, p).$$  \ (5)

For $N = 26$ and $N_s = 10$ the functional forms of $P_{k \geq N_s}(p)$ and $P_{k \leq N_s}(p)$ are depicted in figure 1.

Assume that $N_s$ signals are found in $N$ measurements and that a probability $Q^c < 0.5$ (typical values are $Q^c = 0.16$ or $Q^c = 0.025$) is given. We can solve equation (4) for

![Figure 1: The probability functions $P_{k \geq N_s}(p)$ to observe $k \geq N_s$ signals in $N$ events (solid line) and $P_{k \leq N_s}(p)$ to observe $k \leq N_s$ signals in $N$ events (dotted line) are depicted for $N = 26$ and $N_s = 10$. Symmetric 68% confidence bounds ala Clopper-Pearson are also indicated.](image_url)
$P_{k \geq N_s}(p_{-}) = Q^c$ and $p_{-}$ is a lower bound on the true signal probability $p_{\text{exact}}$, such that the likelihood to find $k \geq N_s$ signals in $N$ measurement is smaller than $Q^c$ for every $p_{\text{exact}} < p_{-}$.

Correspondingly, we can solve equation (5) for $P_{k \leq N_s}(p_{+}) = Q^c$ and $p_{+}$ is an upper bound on the true signal probability $p_{\text{exact}}$, such that the likelihood to find $k \leq N_s$ signals in $N$ measurement is smaller than $Q^c$ for every $p_{\text{exact}} > p_{+}$. Together, this combines into the Clopper–Pearson bounds: The probability to find the true signal probability in the range

$p_{-} \leq p_{\text{exact}} \leq p_{+}$ is larger or equal to $P^c = 1 - 2Q^c$.  

(6)

In more details the meaning of the inequality is discussed in [4]. For $P^c = 0.68$ ($Q^c = 0.16$) the $p^\pm$ values are indicated in figure [1]. Approximately, this range corresponds to the confidence of a $1\sigma$ error bar. Similarly the confidence range corresponding to a $2\sigma$ error bar, etc., can be found.

3 Confidence Limits from Data Cuts

We are interested in the situation where signal and background data can no longer be distinguished unambiguously. Instead, a neural network or other device yields statistical information by tagging signals with efficiency $P_s$ and background data with probability $P_b$, as discussed in the introduction.

Applying the cuts to all $N$ data results in $N^Y$ tagged data ($0 \leq N^Y \leq N$), composed of $N^Y = N^Y_s + N^Y_b$, where $N^Y_s$ is the number of tagged signals and $N^Y_b$ is the number of tagged background data. Of course, the values for $N^Y_s$ and $N^Y_b$ are not known. Our task is to determine confidence limits for the signal probability $p$ from the sole knowledge of $N^Y$.

We proceed by writing down the probability density of $N^Y$ for given $p$ and, subsequently, generalizing the Clopper-Pearson method.

First, assume fixed $N_s$. The probability densities of $N^Y_s$ and $N^Y_b$ are binomial and thus the probability density for $N^Y$ is given by the convolution

$$P^Y(N^Y|N_s) = \sum_{N^Y_s + N^Y_b = N^Y} b_{N_s}(N^Y_s, P_s) b_{N_b}(N^Y_b, P_b), \ N_b = N - N_s.$$  

(7)

Proof: For a signal event the probability to be tagged is $P_s$, so $b_{N_s}(N^Y_s, P_s)$ is the probability to tag $N^Y_s$ out of the $N_s$ signals. Similarly, the probability for a background event to become tagged is $P_b$ and $b_{N_b}(N^Y_b, P_b)$ is the probability to tag $N^Y_b$ of the $N_b = N - N_s$ background events. As these two probabilities are independent, the likelihood that precisely
The probability to tag \( k \) events becomes
\[
b_N^Y(k, p) = \sum_{N_s=0}^{N} b_N(N_s, p) P^Y(k|N_s) .
\] (8)

Fourier transformation of the convolution (7) allows for an efficient numerical calculation of the \( P(k|N_s) \) coefficients. In analogy with equations (4) and (5) we find the probabilities to tag \( k \geq N^Y \) and \( k \leq N^Y \) events to be
\[
P^Y_{k \geq N^Y}(p) = \sum_{k=N^Y}^{N} b_N^Y(k, p) \quad \text{and} \quad P^Y_{k \leq N^Y}(p) = \sum_{k=0}^{N^Y} b_N^Y(k, p) .
\] (9)

For \( N = 35, N^Y = 12, P_s = 0.8 \) and \( P_b = 0.05 \) the functions \( P^Y_{k \geq N^Y}(p) \) and \( P^Y_{k \leq N^Y}(p) \) are depicted in figure 2. The 68% confidence range (4) is also indicated in figure 2, where the

Figure 2: The probability functions \( P^Y_{k \geq N^Y}(p) \) to find \( k \geq N^Y \) tags in \( N \) events and \( P^Y_{k \leq N^Y}(p) \) to find \( k \leq N^Y \) tags in \( N \) events (9) are depicted for \( N = 35, N^Y = 12, P_s = 0.8 \) and \( P_b = 0.05 \). Symmetric 68% confidence bounds, found for \( p_- = 0.274 \) and \( p_+ = 0.521 \), are also indicated.
bound values $p_{\pm}$ are now defined as solutions of the equations

$$Q^c = P_{Y}^{k \geq N_Y}(p_-) \quad \text{and} \quad Q^c = P_{Y}^{k \leq N_Y}(p_+).$$  \hspace{1cm} (10)

The range $[p_-, p_+]$, obtained with $Q^c = 0.16$, guarantees the standard one error bar confidence probability of 68\% for every true signal probability $p_{\text{exact}}$. For almost all values the actual confidence will be better. However, the bounds cannot be improved without violating the requested confidence probability for the case that $p_{\text{exact}}$ happens to agree with either $p_-$ or $p_+$. In the same way, bounds calculated with $Q^c = 0.023$ ensure the standard two error bar confidence level of 95.4\% or better, and so on.

### 3.1 Data Sets with Few Signals

As outlined, data sets with few signals are of of particular interest in high energy physics. Let us replace $N_Y = 12$ of figure 2 by $N_Y = 3$. The resulting graph is depicted in figure 3. From the $P_{Y}^{k \geq N_Y}(p)$ curve we read off the finite probability $P_0 = 0.254$ for the likelihood that the true signal probability $p_{\text{exact}} = 0$ generates $k \geq N_y$ tags. Due to this probability the

![Graph of functions](image)

Figure 3: The same functions as in figure 2 are depicted, but for $N_Y = 3$ instead of $N_Y = 12$ tags. The lower 68\% confidence bound does not exist anymore, instead the likelihood for $p_{\text{exact}} = 0$ has become $P_0 = 0.254$. The upper 68\% confidence bound is found at $p_+ = 0.149.$

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lower 68% confidence bound $p_-$ disappears, whereas the upper $p_+$ bound does still exist. In passing let us note that for the data of figure [2] we have $P_0 = 0.69 \times 10^{-7}$, i.e. there $p_{\text{exact}} = 0$ is ruled out on the $5\sigma$ level.

### 3.2 Signal Probability Distributions

To avoid frequentist objections, the cumulative signal distribution function $F(p)$ is, in the opinion of the author, best defined as the *expectation of the researcher* to find the true signal probability $p_{\text{exact}}$ in the range $0 \leq p_{\text{exact}} \leq p$. Our approach allows to estimate upper and lower bounds for the cumulative signal distribution function

$$F(p) = \int_{-\infty}^{p} dp' f(p')$$

where $f(p')$ is the signal probability density. \hspace{1cm} (11)

Equation (10) implies

$$F_1(p) = 1 - P_{k \leq N_Y}(p) = \sum_{k=N_Y}^{N} b_N(k,p) \leq F(p) \leq F_2(p) = P_{k \geq N_Y}(p) = \sum_{k=N_Y}^{N} b_N(k,p) \hspace{1cm} (12)$$

![Figure 4: Upper and lower bounds, $F_2(p)$ and $F_1(p)$, for the cumulative signal distribution function $F(p)$. The values used for $N, N_Y, P_s$ and $P_b$ are identical with those of figure [2].](image)

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and note that \( F(0) = F_2(0) \). Figure 4 shows the result for the same data which were used in figure 2. Any reasonable Bayesian estimates (which involves additional \textit{a-priori} assumptions) should give a function \( F(p) \) which is sandwiched between \( F_1(p) \) and \( F_2(p) \).

It is instructive to define peaked distribution functions [7] by

\[
F_{\text{peaked}}^i(p) = \begin{cases} 
F_i(p) & \text{for } F_i(p) \leq 0.5 \\
1 - F_i(p) & \text{for } F_i(p) \geq 0.5,
\end{cases} \quad (i = 1, 2).
\] (13)

Using the same data as in figure 4, \( F_{\text{peaked}}^1(p) \) and \( F_{\text{peaked}}^2(p) \) are depicted in figure 5. The advantages of using peaked distribution functions instead of conventional cumulative distribution functions are:

1. The ordinate becomes enlarged by a factor of two.

2. The estimated medians are located at the peaks and the probability content of the distribution is instructively displayed.

3. Bounds like those of figure 4 are easily read off.

![Peaked distribution functions](image)

Figure 5: Peaked distribution functions \( F_{\text{peaked}}^2(p) \) and \( F_{\text{peaked}}^1(p) \) as defined in equation (13). The values used for \( N, N^Y, P_s \) and \( P_b \) are identical with those of figures 2 and 4.
The probability densities corresponding to the cumulative distribution functions (12) are the derivatives of the $F_i(p)$ with respect to $p$

$$f_i(p) = \frac{dF_i(p)}{dp} \quad (i = 1, 2) .$$

(14)

Their numerical calculation is straightforward when analytical expressions for the derivatives of the binomial coefficients in equation (8) are used. Figure 6 exhibits the results for $f_1(p)$ and $f_2(p)$ corresponding to $F_1(p)$ and $F_2(p)$ of figure 4. At $p = 0$ the probability densities have $\delta$-function contributions

$$f_i(p) = F_i(0) \delta(p) + \ldots , \quad (i = 1, 2) .$$

(15)

In case of figure 6 the $F_i(0)$ coefficients are practically zero. However, for the probability densities corresponding to figure 3 there would be a substantial contribution: $F_1(0) = 0.096$ and $F_2(0) = 0.254$ in that case.

### 3.3 The Fortran Code

The Fortran code which produces the illustrations of this paper is available on the web. Start at the author’s homepage [8] www.hep.fsu.edu/~berg and follow the research and

![Figure 6: Probability densities $f_i(p)$ corresponding (14) to the cumulative distribution function of figure 4. The values used for $N$, $N^Y$, $P_s$ and $P_b$ are identical with those of figure 4.](image)
from there the Clopper-Pearson hyperlink. Load down all files of the Fortran Programs subdirectory into an empty directory. Any standard Fortran 77 compiler should then be able to compile the cp1.f, cp2.f and cp3.f programs. Be aware that the program files include (via include Fortran statements) some of the other files you downloaded. Running one of the programs produces a data files with the name of that program and a .d extension. Subsequently, gnuplot users can produce the graphical presentations of this paper by using the *.plt driver files, as listed in the following.

| program | generates file | use with gnuplot driver |
|---------|----------------|------------------------|
| cp1.f   | cp1.d          | cp1.plt                |
| cp2.f   | cp2.d          | cp2.plt, cp4.plt, cp5.plt, cp6.plt |
| cp3.f   | cp3.d          | cp3.plt                |

Here, the gnuplot file number corresponds to the figure number of this paper. To get encapsulated postscript files, the comment signs in front of the first two rows of each gnuplot file have to be eliminated.

4 Conclusions

We have calculated confidence limits, and corresponding limits of the entire cumulative distribution function, for an unknown true signal likelihood \( p_{\text{exact}} \). The only input used are the efficiencies \( P_s \) for tagging signals, the probabilities \( P_b \) for tagging background events, the number \( N_Y \) of tagged data and the total number of data \( N \). In particular, the method allows to deal with the situation where only few signals occur and yields then a finite probability for the likelihood that \( k \geq N_Y \) tags are observed if the true signal probability is \( p_{\text{exact}} = 0 \). In real life the probabilities \( P_s \) and \( P_b \) are most likely estimators by themselves, i.e. quantities with error bars. This causes no major problem, one just has to apply our confidence calculations to an appropriate sample and to average over the results.

References

[1] R.A. Fisher, Proc. Camb. Phil. Soc. 26, 528 (1930); Proc. Roy. Soc. A139, 343 (1933).

[2] J. Neyman and E.S. Pearson, Phil. Trans. Roy. Soc. A231, 289 (1933).
[3] C.J. Clopper and E.S. Pearson, Biometrika 26, 404 (1934). For a textbook treatment see S. Brandt, *Statistical and Computational Methods in Data Analysis* (North-Holland, 1983).

[4] B.A. Berg and J. Riedler, Comp. Phys. Commun. 107, 39 (1997).

[5] http://alephwww.cern.ch/ALPUB/seminar/wds/

[6] For the numerical evaluation of equations (9) use $\sum_{k=0}^{N} b_y^N(k, p) = 1$ together with the partial sums from either $\sum_{k=N^Y}^{N} b_y^N(k, p)$ or $\sum_{k=0}^{N^Y} b_y^N(k, p)$, but not both. Normally $N^Y < N/2$ and $\sum_{k=0}^{N^Y} b_y^N(k, p)$ will be used.

[7] B.A. Berg, *Introduction to Monte Carlo Simulations and Their Statistical Analysis*, in preparation.

[8] The address of the authors homepage and its tree structure are expected to be stable, whereas the absolute address where the programs are located is likely to change.