Lower bounds for uniform read-once threshold formulae in the randomized decision tree model

Nikos Leonardos∗†

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Abstract

We investigate the randomized decision tree complexity of a specific class of read-once threshold functions. A read-once threshold formula can be defined by a rooted tree, every internal node of which is labeled by a threshold function $T_k^n$ (with output 1 only when at least $k$ out of $n$ input bits are 1) and each leaf by a distinct variable. Such a tree defines a Boolean function in a natural way. We focus on the randomized decision tree complexity of such functions, when the underlying tree is a uniform tree with all its internal nodes labeled by the same threshold function. We prove lower bounds of the form $c(k, n)^d$, where $d$ is the depth of the tree. We also treat trees with alternating levels of AND and OR gates separately and show asymptotically optimal bounds, extending the known bounds for the binary case.

1 Introduction

Boolean decision trees constitute one of the simplest computational models. It is therefore intriguing when the complexity of a function is still unknown. A notable example is the recursive majority-of-three function. This function can be represented by a uniform ternary tree of depth $d$, such that every internal node has three children and all leaves are on the same level. The function computed by interpreting the tree as a circuit with internal nodes labeled by majority gates (with output 1 only when at least two of the three inputs are 1) is $\text{maj}_d$, the recursive majority-of-three of depth $d$.

This function seems to have been given by Ravi Boppana (see Example 1.2 in [15]) as an example of a function that has deterministic complexity $3^d$, while its randomized complexity is asymptotically smaller. Other functions with this property are known. Another notable example is the function $\text{nand}_d$, first analyzed by Snir [17]. This is the function represented by a uniform binary tree of depth $d$, with the internal nodes labeled by nand gates. Equivalently, the internal nodes can be labeled by AND and OR gates, alternately at each level.

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†nikos.leonardos@gmail.com
A simple randomized framework that can be used to compute both \text{maj}_d and \text{nand}_d is the following. Start at the root and, as long as the output is not known, choose a child at random and evaluate it recursively. Algorithms of this type are called \textit{directional}. For \text{maj}_d, the directional algorithm computes the output in \((8/3)^d\) queries. It was noted by Boppana and also in [15] that better algorithms exist for \text{maj}_d. (See [1] for a more recent study of directional algorithms.) Interestingly, Saks and Wigderson show that the directional algorithm is optimal for the \text{nand}_d function and show that its zero-error randomized decision tree complexity is \(\Theta((\frac{1+\sqrt{33}}{4})^d)\). Their proof uses a bottom up induction and generalized costs. Their method of generalized costs allows them to charge for a query according to the value of the variable. In this work we use this method to show that the directional algorithm is optimal for uniform AND-OR trees where each gate receives \(n\) bits.

More generally, there have been studied functions that can be represented by formulae involving threshold functions as connectives. A threshold \(k\)-out-of-\(n\) function, denoted \(T_n^k\), is a Boolean function of \(n\) arguments that has value 1 if at least \(k\) of the \(n\) Boolean input values are 1. A threshold formula can be defined as a rooted tree with labeled nodes; each internal node is labeled by a threshold function and each leaf by a variable. If each variable appears exactly once the formula is called \textit{read-once}; if only AND and OR connectives appear, it is called a \textit{Boolean} read-once formula. A formula represents a Boolean function in a natural way: the tree is evaluated as a circuit where each internal node is the corresponding threshold gate. If the formula is read-once, the function it represents is also called read-once. If no OR gate is an input to another OR gate, and the same for AND gates, then the formula is non-degenerate (see Theorem 2.2 in [6]) and uniquely represents the corresponding function. Thus, we may define the depth \(d\) of \(f\) as the maximum depth of a leaf in the unique tree-representation. Note also that, for read-once formulae, the set of variables coincides with the set of leaves in their representation as a tree. We prove lower bounds for the subclass that is represented by uniform trees (full and complete trees with all their leaves at the same level and with each internal node having the same number of children).

\textbf{Related work and our results.} Heiman and Wigderson [7] managed to show that for every Boolean read-once function \(f\) we have \(R(f) \in \Omega(D(f)^{0.51})\), where \(R(f)\) and \(D(f)\) are the randomized and deterministic complexity of \(f\) respectively. For the subclass of such functions that their representing tree is binary, the question was investigated more recently in [2]. Heiman, Newman, and Wigderson [6] showed that read-once formulae with threshold gates have zero-error randomized complexity \(\Omega(N/2^d)\) (here \(N\) is the number of variables and \(d\) the depth of a canonical tree-representation of the read-once function).

Santha showed in [16] that when considering a notion of balanced read-once formulae with AND and OR gates, the zero-error and bounded-error complexities are intimately related. This holds, in particular, for the class of uniform formulae that we consider. Furthermore, it is not hard to verify that our lower bounds for uniform threshold read-once formulae also hold for bounded-error algorithms if multiplied by \((1 - 2\delta)^d\), where \(\delta\) is the error allowed. Consequently, we focus on zero-error complexity in this work.

With respect to specific read-once functions, in contrast to the exact asymp-
totic bounds we have for \( \text{nand}_d \) (also in the quantum model \([3]\)) there had been no progress on the randomized decision tree complexity of \( \text{maj}_d \) for several years. However, recent papers have narrowed the gap between the upper and lower bounds for recursive majority. An \( \Omega((7/3)^d) \) lower bound was showed in \([9]\). Jayram, Kumar, and Sivakumar, proved their bound using tools from information theory and a top down induction. Furthermore, they presented a non-directional algorithm that improves the \( O((8/3)^d) \) upper bound. Landau, Nachmias, Peres, and Vanniasegaram \([10]\), showed how to remove the information theoretic notions from the proof in \([9]\), keeping its underlying structure the same. Magniez, Nayak, Santha, and Xiao \([12]\), significantly improved the lower bound to \( \Omega((5/2)^d) \) and the upper bound to \( O(2.64946^d) \). Subsequently, in \([11]\), the lower bound was further improved to \( \Omega(2.55^d) \), building upon the techniques of \([15]\). The bound was further improved with a computer-assisted proof in \([13]\). The currently known best lower bound is \( \Omega(2.59^d) \) from \([5]\).

The above methods work by reducing the complexity of \( \text{maj}_d \) to a problem of constant size. It is interesting to note that when this constant-size problem relates to \( \text{maj}_3 \) (i.e., the majority of three bits), all these methods obtain the \( 2.5^d \) lower bound. Furthermore, in order to make further progress, analysis of analogous problems of greater depth is required. In particular, in \([11]\), an intermediate problem between \( \text{maj}_1 \) and \( \text{maj}_2 \) is constructed and analyzed with a reasonable case analysis (without the need of computer assistance) to obtain a \( 2.55^d \) lower bound; in \([13]\), the \( 2.55^d \) bound is surpassed by solving depth-3 and depth-4 problems using a computer; finally, \([5]\) obtains the currently best bound via the solution of a linear program (obtained using a computer) which relates to \( \text{maj}_3 \). Pushing any of the above further seems computationally intractable and will yield possibly modest improvements. We believe a human-readable solution can shed more light into the problem.

In this work we focus our attention to threshold read-once functions which are represented by uniform trees and each internal node is labeled by the same threshold gate. Let us denote by \( F^d_{\wedge,n} \) (resp. \( F^d_{\lor,n} \)) the function represented by a uniform tree of depth \( d \) with AND and OR gates of fan-in \( n \) alternating with each level and the root labeled by an AND gate (resp. an OR gate). Denote by \( F^d_{k,n} \) the function represented by a uniform tree of depth \( d \) with each gate being \( T^k_n \). With respect to these classes of read-once functions we prove the following theorems in the model of randomized decision trees. (See Section 2 for relevant definitions.)

**Theorem 1.** The directional algorithm is optimal for \( F^d_{\wedge,n} \) and \( F^d_{\lor,n} \). In particular, for even \( d \),

\[
R(F^d_{\wedge,n}), R(F^d_{\lor,n}) = \Theta\left(\left[\frac{n-1}{4}\left(1 + \sqrt{1 + \frac{16n}{(n-1)^2}}\right)^d\right]\right).
\]

**Theorem 2.** For \( 1 < k < n \),

\[
R(F^d_{k,n}) = O\left(\left[\frac{n+1}{2}\left(1 + \sqrt{1 - \frac{8k(n-k+1)}{(n-k+2)(k+1)(n+1)}}\right)^d\right]\right).
\]

**Theorem 3.** For \( 1 < k < n \),

\[
R(F^d_{k,n}) = \Omega\left(\left[\frac{n+1}{2}\left(1 + \sqrt{1 - \frac{3k(n-k+1)+(n+1)^2}{(n+1)^2}}\right)^d\right]\right).
\]
Note that the last two expressions are symmetric in \( k \) around \((n + 1)/2\), reflecting the fact that \( R(F_{k,n}^d) = R(F_{n+1-k,n}^d) \) (as can be seen by negating and pushing the negations down to the leaves). Setting \( n = 2 \) in Theorem 1 we retrieve the bound of Saks and Wigderson for nand. Similarly, for \( k = 2 \) and \( n = 3 \), Theorems 2 and 3 give known bounds for \( \text{maj}_d \): the \((8/3)^d\) upper bound of the directional algorithm and the \( 2.5^d \) lower bound respectively. We expect neither the upper bound nor the lower bound to be optimal. Our interest is mostly in Theorem 3 and also Theorem 1. Theorem 2 is obtained by a straightforward analysis of the directional algorithm and is stated mostly for reference and comparison with the bound of Theorem 3.

We note that one could obtain similar bounds using the methodology of [9, 13] or [5]. In particular, formal descriptions of feasible solutions to the corresponding linear programs defined in [5], can be used along with their Composition Theorem to obtain lower bounds for the class of functions we consider.

2 Definitions and notation

In this section we introduce basic concepts related to decision tree complexity and the tree-functions that we consider. The reader can find a more complete exposition of decision tree complexity in the survey of Buhrman and de Wolf [4].

2.1 Definitions pertaining to trees

For a rooted tree \( T \), the \textit{depth} of a node is the number of edges on the path to the root; the \textit{depth} of the tree is the maximum depth of a leaf; the \textit{level} \( i \) of a tree consists of all nodes of depth \( i \); the \textit{height} of a node is the number of edges on the longest path between the node and any descendant leaf. We call a tree \textit{uniform} if all the leaves are on the same level and all internal nodes have the same number of children.

Consider the uniform tree of depth \( d \) in which every internal node has \( n \) children and is labeled by \( T_{k,n}^d \) (the \( k \)-out-of-\( n \) threshold gate) with \( 1 < k < n \). We denote both the tree and the corresponding read-once threshold function by the same symbol \( F_{k,n}^d \) as it will be clear from the context what it refers to.

For the case of AND or OR gates, we consider uniform trees with the root labeled by an AND (resp. OR) gate and subsequent levels of the tree are labeled by OR (resp. AND) and AND (resp. OR) gates alternately. When each gate receives \( n \) inputs, we denote these trees by \( F_{\wedge,n}^d \) and \( F_{\vee,n}^d \), when the root is labeled by an AND and OR respectively.

The inputs considered hard for the functions we consider are the \textit{reluctant} inputs ([15]). Call an input to a \( T_{k,n}^d \) gate \textit{reluctant}, if it has exactly \( k \) or \( k-1 \) ones. Call an input to a threshold read-once formula \textit{reluctant}, if it is such that the input to every gate is reluctant. A \textit{reluctant distribution} has only reluctant inputs in its support.

We will consider trees that have all leaves in the two greatest levels. We now define the reluctant distribution over such trees that we will work with. We focus on \textit{positive inputs} only; i.e., inputs that evaluate to true. The definition is recursive. If the tree consists of a single variable, then all the mass in on 1. Otherwise, consider a tree \( T \) of depth \( d \) with all leaves in levels \( d - 1 \) and \( d \) (level \( d - 1 \) might not have any leaves if \( T \) is uniform). Obtain a tree \( T' \) by
removing all the children of an internal node \( u \) with label \( T_k^d \) in level \( d - 1 \). Let \( \mu' \) denote the reluctant distribution for \( T' \). Consider any reluctant input \( x \) to \( T \) and let \( b \) be the value that \( u \) (as a gate) outputs. Let \( x' \) be the corresponding (relunctant) input to \( T' \); i.e., \( x' \) assigns \( b \) to \( u \) and agrees with \( x \) everywhere else. Define \( \mu(x) = \mu'(x')/\binom{n}{b} \) if \( b = 1 \) and \( \mu(x) = \mu'(x')/\binom{n}{k-1} \) if \( b = 0 \). The reluctant distribution for \( F_{k,n}^d \), \( F_{\land,n}^d \), and \( F_{\lor,n}^d \), is denoted \( \mu_{k,n}^d \), \( \mu_{\land,n}^d \), and \( \mu_{\lor,n}^d \), respectively. Note that although we will restrict the analysis over positive inputs, the algorithms are still required to be zero-error over any input.

### 2.2 Definitions pertaining to decision trees

A deterministic Boolean decision tree \( Q \) over \( \{0,1\}^n \) is a rooted and ordered binary tree. Each internal node is labeled by \( i \in [n] = \{1,2,\ldots,n\} \) and each leaf by a value from \( \{0,1\} \). An assignment or an input is a member of \( \{0,1\}^n \). The output \( Q(x) \) of \( Q \) on an input \( x \) is defined recursively as follows. Start at the root and let its label be \( i \). If \( x_i = 0 \), we continue with the left child of the root; if \( x_i = 1 \), we continue with the right child of the root. We continue recursively until we reach a leaf. We define \( Q(x) \) to be the label of that leaf. When we reach an internal node, we say that \( Q \) queries the corresponding variable and reads its value. We say that \( Q \) computes a Boolean function \( f : \{0,1\}^n \rightarrow \{0,1\} \), if for all \( x \in \{0,1\}^n \), \( Q(x) = f(x) \). The cost of \( Q \) on input \( x \), \( \text{cost}(Q; x) \), is the number of variables queried when the input is \( x \). The cost of \( Q \), \( \text{cost}(Q) \), is its depth, the maximum distance of a leaf from the root. The deterministic complexity, \( D(f) \), of a Boolean function \( f \) is the minimum cost over all Boolean decision trees that compute \( f \).

A randomized Boolean decision tree \( Q_R \) is a distribution \( p \) over deterministic decision trees. On input \( x \), a deterministic decision tree is chosen according to \( p \) and evaluated. The cost of \( Q_R \) on input \( x \) is \( \text{cost}(Q_R; x) = \sum_Q p(Q) \text{cost}(Q; x) \). The cost of \( Q_R \), \( \text{cost}(Q_R) \), is \( \max_x \text{cost}(Q_R; x) \). A randomized decision tree \( Q_R \) computes a Boolean function \( f \) (with zero error), if \( p(Q) > 0 \) only when \( Q \) computes \( f \). The randomized complexity, \( R(f) \), of a Boolean function \( f \) is the minimum cost over all randomized Boolean decision trees that compute \( f \).

We are going to take a distributional view on randomized algorithms. Let \( \mu \) be a distribution over \( \{0,1\}^n \) and \( Q_R \) a randomized decision tree. The expected cost of \( Q_R \) under \( \mu \) is \( \text{cost}_\mu(Q_R) = \sum_Q \mu(x) \text{cost}(Q_R; x) \). The expected complexity under \( \mu \), \( R_\mu(f) \), of a Boolean function \( f \) is the minimum expected cost under \( \mu \) of any randomized Boolean decision tree that computes \( f \). Clearly, \( R(f) \geq R_\mu(f) \), for any \( \mu \), and thus we can prove lower bounds on randomized complexity by providing lower bounds for the expected cost under any chosen distribution.

### 3 The method of generalized costs for read-once threshold formulae

Our goal is to prove a lower bound on the expected cost (under the reluctant distribution we defined in the previous section) of any randomized decision tree \( Q_R \) that computes a uniform threshold read-once function. The high-level outline of our proof is as follows. Given any decision tree \( Q_R \) computing the function \( F_{k,n}^d \), we define a randomized decision tree \( Q_R \) that computes \( F_{k,n}^{d-1} \).
In each step we shrink to a leaf the parent of \( n \). For this, we start with a function \( F \) on all leaves are on levels \( d \). We extend the previous cost-related definitions as follows. The cost of a decision in which \( \phi \) and distribution \( \mu \) is known,

\[
\text{cost}(Q; \phi; x) = \sum_{i \in S} \phi(x; i), \quad \text{where } S = \{ i | i \text{ is queried by } Q \text{ on input } x \}.
\]

The cost of a randomized decision tree \( Q_R \) on input \( x \) under cost-function \( \phi \) is

\[
\text{cost}(Q_R; \phi; x) = \sum_{Q} p(Q) \text{cost}(Q; \phi; x),
\]

where \( p \) is the corresponding distribution over deterministic decision trees. Finally, the expected cost of a randomized decision tree \( Q_R \) under cost-function \( \phi \) and distribution \( \mu \) is

\[
\text{cost}_\mu(Q_R; \phi) = \sum_x \mu(x) \text{cost}(Q_R; \phi; x).
\]

We will mostly work with simple cost functions (as was also the case in [15]), in which \( \phi(x, i) \) depends only on \( x_i \) and \( i \). In particular, we may express such a function with a pair \( c = (c_0, c_1) \), where \( c_0, c_1 \) are functions mapping each variable to a real number and \( c_x(i) = \phi(x, i) \). We call such cost functions local. Defining \( \text{cost}(Q; c; x) = \sum_{i \in S} c_x(i) \), for a local cost-function \( c \), we use both expressions interchangeably in the above quantities.

### 3.2 The method and some preliminary definitions

Consider a tree \( T \) of depth \( d + 1 \) such that all internal nodes have degree \( n \) and all leaves are on levels \( d \) and \( d + 1 \). If we treat every internal node as a \( T_k^n \)-gate, this tree represents a function \( F \) and has an associated reluctant distribution \( \mu \). Suppose \( Q \) is a randomized decision tree that computes \( F \) and \( c = (c_0, c_1) \) a local cost-function. We define a process that shrinks \( T \) to a smaller tree \( T' \) (of depth \( d \) or \( d + 1 \)) and also a corresponding randomized decision tree \( Q' \) that computes the function \( F' \) represented by \( T' \).
for a “more expensive” local cost-function $c'$ and the reluctant distribution $\mu'$ for $F'$,

$$\text{cost}_\mu(Q; c) \geq \text{cost}_{\mu'}(Q'; c').$$

(1)

The main ingredient in this framework is the shrinking process. It entails removing $n$ leaves, which we identify with $[n]$, so that their parent $u$ would become a leaf in $T'$. Given an algorithm $Q$ for $F$ we obtain an algorithm $Q'$ for $F'$ as follows. On input $(x_{u}, y)$, $Q'$ first chooses uniformly at random $z \in \binom{[n-1]}{k-1}$ and $i \in [n]$ and then simulates $Q$ on $x = (z_1, \ldots, z_i-1, x_u, z_i, \ldots, z_{n-1}, y)$. Clearly, if $Q$ computes $F$, then $Q'$ is a randomized decision tree that computes $F'$.

Our goal is to determine the “most expensive” cost function $c'$ for $T'$ for which we can argue (1). To that end, it will be useful to express $\text{cost}_{\mu'}(Q'; c')$ in terms of $Q$ and $\mu$. For an input $x$ for $F$ and a leaf $i$ of $T$, let $b = T^n_k(x_1, \ldots, x_n)$ and define a cost function $\psi$ for $F$ as follows.

$$\psi(x; i) = \begin{cases} 
\frac{c_1'(i)}{k}, & \text{if } i \not\in [n]; \\
\frac{c_1'(u)}{k}, & \text{if } i \in [n] \text{ and } x_i = b = 1; \\
\frac{c_2'(u)}{(n-k+1)}, & \text{if } i \in [n] \text{ and } x_i = b = 0; \\
0, & \text{if } i \in [n] \text{ and } x_i \neq b.
\end{cases}$$

(2)

Informally, $\psi$ agrees with $c'$ outside $[n]$, is zero when $Q'$ will never query $i$, and in the rest of the cases has value so that the following proposition holds.

**Proposition 4.** $\text{cost}_{\mu'}(Q'; c') = \text{cost}_\mu(Q; \psi)$.

**Proof.** It suffices to prove the equality for a deterministic $Q$. The expected cost of any variable outside $[n]$ is clearly the same for both algorithms. Thus, it suffices to show that the contribution of $u$ to the cost of $Q'$ equals that of the variables in $[n]$ to the cost of $Q$. Intuitively, a query of $Q$ to a variable $i \in [n]$ such that $x_i = b$, leads to a query of $Q'$ to $u$ with probability $1/k$ or $1/(n-k+1)$, according to $b = 1$ or $b = 0$ respectively. More formally, let $\chi_i(x)$ be the indicator function of the event that $Q$ queries $i$ when the input is $x$. We consider the inputs with $x_u = 1$.

Recalling the recursive distribution of the reluctant distribution $\mu$, the contribution of the leaves in $[n]$ to the cost of $Q$ can be written

$$\sum_y \sum_{z,i} \frac{\mu'(x_{u}, y)}{\binom{n-1}{k-1}} \cdot \chi_i(x) \cdot \frac{c_1'(u)}{k}.$$ 

In the expression above, $x = (z_1, \ldots, z_{i-1}, x_u, z_i, \ldots, z_{n-1}, y)$, as in the description of $Q'$. Note that although each $x$ is encountered $k$ times, each $i \in [n]$ such that $z_i = 1$ contributes exactly once (those with $z_i = 0$ contribute 0 anyway). The contribution of $u$ to the cost of $Q'$ is

$$\sum_y \mu(x_{u}, y) \sum_{z,i} \frac{1}{\binom{n-1}{k-1}} \cdot \chi_i(x) \cdot c_1'(u),$$

since there are $n$ choices for $i$ and $\binom{n-1}{k-1}$ choices for $z$. Recalling $\binom{n}{k} = \binom{n-1}{k-1}$, the equality of the two expressions follows.

The inputs with $x_u = 0$ can be analysed in the same way, substituting $c_0'$ for $c_1'$ in both expressions and $n-k+1$ for $k$ in the first one. \qed
Let us go back to our plan: given a local cost-function \( c \) for \( F \), determine the most expensive local cost-function \( c' \) for \( F' \) for which we can argue (1). We will work with local cost-functions that charge the same value for discovering a 0 and the same for discovering a 1 at a given level. That is, if \( i \) and \( j \) are leaves at the same level, then \( c_0(i) = c_0(j) \) and \( c_1(i) = c_1(j) \).

We shall define \( c' \) so that it agrees with \( c \) for all \( i \notin [n] \). Defining a cost-function \( \phi(x; i) = c_{x_i}(i) - \psi(x; i) \) for \( i \in [n] \) and \( \phi(x; i) = 0 \) otherwise (where \( \psi \) is the one obtained from Proposition 4), we have

\[
\text{cost}_\mu(Q; c) \geq \text{cost}_\mu(Q'; c') \iff \text{cost}_\mu(Q; \phi) \geq 0.
\]

Clearly, we only need to verify \( \text{cost}_\mu(Q; \phi) \geq 0 \) with respect to the leaves in \([n]\). Thus, we are led to study decision trees over \( \binom{[n]}{k} \) that are non-empty (i.e., query at least one variable). Note also that we don’t care about what these algorithms output. Their only task is to achieve cost \( \mu(Q; \phi) < 0 \) for our choice of \( \phi \). Given a cost function \( c \) for \( F \), what is the most expensive \( c' \) for \( F' \) so that \( \text{cost}_\mu(Q; \phi) \geq 0 \) for any such decision tree?

The quantity \( P(k, n) \). Consider the \( T_k^n \) function. Looking into the definition of cost function \( \phi \) above, note that it charges an algorithm a positive value for reading a 0 and a negative value for a 1. Furthermore, the greater the negative value is, the more expensive \( c' \) is. We are interested in the values defined below.

**Definition 1.** Let \( \nu_{k,n} \) be the uniform distribution over \( \binom{[n]}{k} \). For \( \eta \in \mathbb{R}_+ \) and \( 0 < k \leq n \), define a local cost-function \( c_\eta = (1, -\eta) \),

\[
P_\eta(k, n) = \min_Q \left\{ \text{cost}_{\nu_{k,n}}(Q; c_\eta) \right\} \quad \text{and} \quad P(k, n) = \max \{ \eta : P_\eta(k, n) \geq 0 \},
\]

where \( Q \) ranges over all non-empty decision trees.

The relevance of this definition is demonstrated by Proposition 5 below.

**Remark.** We note that, for any given algorithm \( Q \), its cost is a linear function of \( \eta \): \( \text{cost}_{\nu_{k,n}}(Q; c_\eta) = A - \eta B \), where \( A \) is the expected number of 0s and \( B \) the expected number of 1s that \( Q \) reads under distribution \( \nu_{k,n} \).

It follows that \( P_\eta(k, n) \) is continuous as a function of \( \eta \) and so for \( \eta = P(k, n) \) we must have \( P_\eta(k, n) = 0 \). Furthermore, since we consider non-empty algorithms only, \( P_\eta(k, n) \) is decreasing as a function of \( \eta \).

We call an algorithm \( Q \) with \( \text{cost}_{\nu_{k,n}}(Q; c_\eta) = P_\eta(k, n) \), an optimal algorithm for \( P_\eta(k, n) \); if \( \eta = P(k, n) \), an optimal algorithm for \( P(k, n) \).

The normalized local cost-function \( c_\eta \) charges an algorithm 1 for reading a 0 and pays the algorithm \( \eta \) when it discovers a 1. What is the greatest value of \( \eta \) for which \( \text{cost}_{\nu_{k,n}}(Q; c_\eta) \geq 0 \) for any \( Q \)?

We now discuss how lower bounds for \( P(k, n) \) and \( P(n - k + 1, n) \) lead to lower bounds for threshold read-once functions. Suppose \( c_0(i) = c_0 \) and \( c_1(i) = c_1 \), for all \( i \in [n] \), and write \( c_1' = c_1'(u) \) and \( c_0' = c_0'(u) \). If \( \alpha \leq P(k, n) \) and \( \beta \leq P(n - k + 1, n) \), then we may set

\[
\begin{pmatrix}
  c_1' \\
  c_0'
\end{pmatrix} = \begin{pmatrix}
  k & \alpha k \\
  \beta(n - k + 1) & n - k + 1
\end{pmatrix} \begin{pmatrix}
  c_1 \\
  c_0
\end{pmatrix}.
\]

(4)
Proposition 5. \( \text{cost}_\mu(Q; c) \geq \text{cost}_{\mu'}(Q'; c') \). 

Proof. Recall the definition of cost function \( \phi \) and the related expression (3). Note that \( \phi \) charges 0 for \( i \notin [n] \). For \( i \in [n] \), \( \phi \) charges \( c_0 \) for reading a 0 and \(-\alpha c_0\) for reading a 1 over the support of \( \nu_{k,n} \) and, respectively, \(-\beta c_1\) and \( c_1 \) over the support of \( \nu_{k-1,n} \). An assignment on the variables outside \([n]\), determines an algorithm over \( \{0,1\}^n \). The conditional distribution of the variables in \([n]\), given such an assignment and a value for \( u \), is either \( \nu_{k,n} \) or \( \nu_{k-1,n} \). It follows that \( \text{cost}_\mu(Q; \phi) \) is lower-bounded by a convex combination of \( c_0 P(\alpha, k) \geq 0 \) and \( c_1 P(n-k+1, n) \geq 0 \).

Applying this process repeatedly, shrinking all sibling leaves to their parent, we reduce a tree of depth \( d \) to one of depth \( d-1 \). After \( d \) such steps and starting with a local cost-function that charges \( c_1 \) and \( c_0 \) for reading 1 and 0 respectively, we are left with a single node and the cost function defined by

\[
\begin{pmatrix}
    c' \\
    c'_0
\end{pmatrix} = \Gamma_{k,n}^d \begin{pmatrix}
    c_1 \\
    c_0
\end{pmatrix}, \quad \text{where} \quad \Gamma_{k,n} = \begin{pmatrix}
    k & \alpha k \\
    \beta(n-k+1) & n-k+1
\end{pmatrix}.
\]

(5)

We will obtain a lower bound in the order of \( \lambda^d \), where \( \lambda \) is the largest eigenvalue. Denoting the trace of \( \Gamma_{k,n} \) by \( T \) and its determinant by \( D \),

\[
\lambda = \frac{T}{2} + \sqrt{\frac{T^2}{4} - D}, \quad \text{where} \quad T = n + 1 \quad \text{and} \quad D = (1 - \alpha \beta)k(n-k+1).
\]

(6)

To accomplish this, we will set \( c_1 = 1 \) and \( c_0 \) so that \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is an eigenvector for \( \lambda \). The matrices considered are positive and it can be shown using a theorem of Perron (see Theorem 8.2.2 in [8]) that \( c_0 \) is positive. As long as \( c_0 \) is also \( \Theta(1) \), our asymptotic bounds are not affected. We note for reference that for a matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( b \neq 0 \), \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is an eigenvector for its greatest eigenvalue \( \lambda \).

3.3 Properties and estimations of \( P(k, n) \)

Given (6), we seek a lower bound of \( P(k, n) \). We start with some monotonicity properties of \( P(k, n) \). Note that, by symmetry, we may consider algorithms that read the variables in the order \( x_1, x_2, \ldots, x_n \).

Intuitively, if we increase \( n \), we should be able to increase \( \eta \) as it becomes harder for an algorithm to discover a 1. This is captured in the next statement.

Theorem 6. For any \( \eta > 0 \) and \( 0 < k \leq n \), \( P_\eta(k, n) < P_\eta(k, n+1) \).

Proof. Consider any algorithm \( Q \) over \( \{0,1\}^{n+1} \) and define a randomized algorithm \( Q' \) over \( \{0,1\}^n \) as follows. Algorithm \( Q' \) on input \( (x_1, \ldots, x_n) \), chooses \( i \in [n+1] \) and simulates \( Q \) on input \( (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \). We claim that the cost of \( Q' \) is less than the cost of \( Q \). This is not hard to see as \( Q' \) will gain \(-\eta \) whenever \( Q \) queries a 1 and also gains \(-\eta \), but with positive probability (when \( i = 1 \) for example) will pay 0 when \( Q \) queries \( i \) and pays 1.

Similarly, increasing both \( k \) and \( n \) by 1, and since \( k/n < (k+1)/(n+1) \), one might expect that we would need to decrease \( \eta \).

Theorem 7. For any \( \eta > 0 \) and \( 0 < k \leq n \), \( P_\eta(k, n) > P_\eta(k+1, n+1) \).
Proof. Consider any algorithm $Q$ over $\{0,1\}^n$ and define a randomized algorithm $Q'$ over $\{0,1\}^{n+1}$ as follows. Algorithm $Q'$ chooses $i \in [k+1]$ and simulates $Q$, but when it encounters the $i$-th 1 while querying a variable $j$, $Q'$ continues with the rest of the variables as if $x_j$ was skipped or missing.

More formally, for $i \in [k+1]$, let $Q_i$ be the algorithm which results from $Q$ as follows. Consider a path $(u_1,x_1 = b_1,u_2,x_2 = b_2,\ldots,u_j,x_j = 1,u_{j+1})$ in $Q$, such that $x_j = 1$ is the $i$-th query that returned 1 (i.e., exactly $i$ of the bits $x_1,\ldots,x_j$ are 1 and $x_j = 1$), and let $T$ denote the sub-tree rooted at node $u_j$ of $Q$ that queries $x_j$. Let $T'$ be identical to $T$, but with every query shifted by 1; that is, every node of $T$ labeled by $\ell \in [n]$ is replaced with $\ell + 1 \in [n+1]$. Let $Q_i$ be the tree which results from $Q$ by replacing the (possibly empty) sub-tree rooted at node $u_{j+1}$ with $T'$ and applying this transformation for every such path in $Q$. Call the edge $x_j = 1$ a skipped edge. Finally, $Q'$ is the uniform distribution over the $Q_i$’s.

It suffices to show that $Q'$ has strictly smaller cost than $Q$. Consider a path $P$ in $Q$ of length $m$ with $\ell 1$’s. The probability $P$ is executed in $Q$ is $\binom{n-m}{k-\ell}/\binom{n}{k}$.

There is a natural map of a path in $Q_i$ to a path in $Q$: ignore a skipped edge (if it exists) and shift back by one any shifted label. We compute the probability of the paths in $Q'$ that map to $P$ by counting the number of pairs $(i,x)$, such that the execution path of $Q_i$ on input $x$ is mapped on $P$.

\[
\left[ m \binom{n-m}{k-\ell} + (k+1-\ell) \binom{n+1-m}{k+1-\ell} \right]/\left[ (k+1) \binom{n+1}{k+1} \right].
\]

The second term of the sum is the number of such pairs without skipped edges; equivalently, with $i > \ell$ and the first $m$ bits of $x$ consistent with $P$. The first term counts the rest of the pairs $(i,x)$, since there are $m$ choices for the skipped 1 and this choice determines $i$. This probability can be verified to equal $\binom{n-m}{k-\ell}/\binom{n}{k}$. Furthermore, since $Q$ is non-empty, there is at least one skipped edge in $Q_i$ taken with positive probability, implying $Q'$ has strictly smaller cost.

Corollary 8. An optimal algorithm for $P(k,n)$ stops if the first query reads 1.

Proof. Consider an optimal tree for $P(k,n)$. By Theorem 7, for $\eta = P(k,n)$, we have $P_\eta(k-1,n-1) > P_\eta(k,n) = 0$. Thus, descending further after reading a 1 will add a positive expected cost.

Using the above property of optimal algorithms, we can determine $P(1,n)$ and $P(n,n+1)$.

Lemma 9. For all $n > 1$, $P(1,n) = (n-1)/2$.

Proof. Theorem 6 determines the optimal decision tree $Q$ for $P(1,n)$; it queries the variables one by one until a 1 is read. This is because, for $\eta = P(1,n)$, by the theorem $P_\eta(1,n-1) < 0$; thus, stopping after a 0 is read is not optimal. For $\eta > 0$,

\[
\text{cost}_{V_1,n}(Q; c_\eta) = \sum_{i=1}^{n} \frac{1}{n} \cdot (i-1-\eta) = \frac{n-1}{2} - \eta.
\]

It follows that $\text{cost}_{V_1,n}(Q; c_\eta) = 0$ if and only if $\eta = (n-1)/2$.

Lemma 10. For all $n > 1$, $P(n,n+1) = 1/(2n)$.
Proof. By Corollary 8, the optimal (non-empty) tree stops if \( x_1 = 1 \) and if \( x_1 = 0 \) it proceeds to discover all 1s. Its cost, for \( \eta > 0 \), is \(-\eta n / (n+1) + (1-\eta n) / (n+1)\). This equals zero if and only if \( \eta = 1/(2n) \).

We can now prove the following lower bound on \( P(k, n) \).

**Theorem 11.** For \( 0 < k < n \), \( \frac{n-k}{2k} \leq P(k, n) \).

**Proof.** The intuition of the inductive proof below is that if \( P(k, n) \) is smaller than \( (n-k)/(2k) \), then \( P_{n-k}(k, n-1) \) should be absurdly small.

Note first that for any \( k > 0 \) and \( n = k + 1 \) the statement follows (with equality) from Lemma 10. Suppose—towards a contradiction—that for a lexicographically least pair \((k, n)\) with \( n > k + 1 \), \( P(k, n) < (n-k)/(2k) \). Thus,

\[
P_{n-k}(k, n-1) < P_{n-k}(k, n) < 0,
\]

where the first inequality follows from Theorem 6 and the second from our assumption that \( P(k, n) < (n-k)/(2k) \). Furthermore, by the minimality of \((k, n)\),

\[
P(k-1, n-1) \geq \frac{n-k}{2(k-1)} > \frac{n-k}{2k},
\]

which in turn implies

\[
0 < P_{n-k}(k-1, n-1).
\]

This last inequality implies that an optimal decision tree for \( P_{n-k}(k, n) \) will not descent further after reading \( x_1 = 1 \). We conclude that

\[
P_{n-k}(k, n) = \frac{n-k}{n} \cdot \left[ 1 + P_{n-k}(k, n-1) \right] - \frac{n-k}{2k} \cdot \frac{k}{n},
\]

Suppose \( Q \) is an optimal algorithm for \( P_{n-k}(k, n-1) \). There are constants \( u \) and \( v \) such that, for any \( \eta > 0 \),

\[
\text{cost}_{\nu_k,n-1}(Q; c_\eta) = u - \eta \cdot v.
\]

Combining (7) and (8) and substituting the above expression with \( \eta = (n-k)/(2k) \), we obtain after simplification

\[
u - \frac{n-k}{2k} \cdot v < -\frac{1}{2}
\]

On the other hand, by the minimality of \((k, n)\), \( P(k, n-1) \geq (n-1-k)/(2k) \). This implies

\[
u - \frac{n-1-k}{2k} \cdot v \geq 0.
\]

The above two inequalities are satisfiable only if \( v > k \). However, \( v \) is the expected number of 1s read by \( Q \), which is at most \( k \); a contradiction.

4 **Bounds for uniform read-once functions**

We begin with an analysis of the (straightforward) directional algorithm. We then show that this algorithm is optimal for uniform AND-OR trees. Finally, we obtain bounds for uniform threshold read-once functions.
4.1 Analysis of the directional algorithm

A simple directional algorithm evaluates a node by evaluating its children in a randomly chosen order. To analyze this algorithm on a reluctant input, let $\Phi_d$ denote the expected cost to evaluate a node of height $d$ when the output is 1 and $\Psi_d$ when it is 0. Consider a node with exactly $k$ of its children evaluating to 1. The number of its children that will be evaluated until its value is determined by the directional algorithm follows a negative hypergeometric distribution. The expected number of children that will be evaluated to 0 is $\frac{n-k}{n} \cdot k \cdot \frac{1}{k+1}$. (An elementary exposition of this distribution and calculation of its mean can be found in [14, Chapter 7].) Similarly, when dealing with a node with exactly $n-k+1$ of its children evaluating to 0, the expected number of children that will be evaluated to 1 is $\frac{k-1}{n} \cdot n \cdot \frac{n-k+1}{n-k+2}$. Thus, along with $\Phi_0 = c_0$ and $\Psi_0 = c_1$, we obtain the recurrence

$$
\begin{align*}
\left( \Phi_d \right) & = \Delta_{k,n} \left( \Phi_{d-1} \right), \quad \text{where} \quad \Delta_{k,n} = \left( \frac{k}{n-k+1} \cdot \frac{n-k+1}{n-k+2} \right).
\end{align*}
$$

(9)

4.2 Uniform alternating AND-OR trees

In this subsection we prove Theorem 1, by showing that the directional algorithm for the alternating AND-OR tree satisfies the same recurrence as the one we obtain with the method of generalized costs. In particular, we show that the recurrences (4) and (9) coincide for uniform alternating AND-OR trees.

Let us define the analogous quantities $\Phi_d^\land$ and $\Psi_d^\land$ (resp. $\Phi_d^\lor$, $\Psi_d^\lor$) for the expected cost when evaluating a node of height $d$ and labeled by an AND (resp. OR) gate to 1 and 0 respectively. Recall Equation (5) and note that $\Delta_{n,n} = \Gamma_{n,n}$ and $\Delta_{1,n} = \Gamma_{1,n}$, since $P(n,n) = 0$ and $P(1,n) = (n-1)/2$ (Lemma 9). Thus, Equation (9) instantiates to

$$
\begin{align*}
\left( \Phi_d^\land \right) & = \Gamma_{n,n} \left( \Phi_{d-1}^\land \right) \quad \text{and} \quad \left( \Phi_d^\lor \right) = \Gamma_{1,n} \left( \Phi_{d-1}^\lor \right). \quad \text{(10)}
\end{align*}
$$

Defining $A = \Gamma_{n,n}$ and $B = \Gamma_{1,n}$, we have

$$
\begin{align*}
\left( \Phi_d^{2d} \right) & = (AB)^d \binom{c_1}{c_0} \quad \text{and} \quad \left( \Psi_d^{2d} \right) = (BA)^d \binom{c_1}{c_0}. \quad \text{(11)}
\end{align*}
$$

With respect to the lower bound, in view of Equations (4), (5), and (6), we obtain that if there is a decision tree $Q$ for $F_{2d}^{2d}$ of cost $C$, then there is a decision tree for a single variable of cost $C$ under the cost-function

$$
\binom{c_1}{c_0} = (AB)^d \binom{c_1}{c_0}, \quad \text{where} \quad AB = \left( \frac{n}{2} \cdot \frac{n(n-1)}{2(n+1)^2} \right).
$$

Comparing the above with Equation (11), the first part of Theorem 1 follows.

To obtain the asymptotic bound, observe that the trace $T$ and the determinant $D$ of $AB$ are $T = n + (n+1)^2/4$ and $D = n^2$. Let $c_1 = 1$ and
\[ c_0 = (\lambda - n)/(n(n-1)/2), \] so that \( \binom{c_1}{c_0} \) is an eigenvector for the greatest eigenvalue \( \lambda \) of \( AB \), where

\[ \lambda = n + \frac{(n-1)^2}{8} + \frac{(n-1)^2}{8} \sqrt{1 + \frac{16n}{(n-1)^2}} = \left[ \frac{n-1}{4} \left(1 + \sqrt{1 + \frac{16n}{(n-1)^2}}\right)\right]^2. \]

Since \( c_0 \) is \( \Theta(1) \) and positive, we obtain \( R(\mathcal{F}_{d}^{k,n}) = \Theta(\lambda^d) \). Bounds for trees of odd depth, if desired, can be obtained via the recurrence relations (10).

### 4.3 Uniform threshold read-once functions

With respect to the upper bound for \( F_{k,n}^d \), we obtain directly from (9)

\[ \begin{pmatrix} \Phi_d \\ \Psi_d \end{pmatrix} = \Delta_{k,n} \begin{pmatrix} c_1 \\ c_0 \end{pmatrix}. \]

We set \( c_1 = 1 \) and \( c_0 = (\lambda - k)/(n-k+1) \), so that \( \binom{c_1}{c_0} \) is an eigenvector for the greatest eigenvalue \( \lambda \) of \( \Delta_{k,n} \), where

\[ \lambda = n + 1 + \frac{2k(n-k+1)}{(n-k+2)(k+1)(n+1)}. \]

Since \( c_0 = \Omega(1) \), Theorem 2 follows.

We now turn to the lower bound. From the bounds on \( P(k,n) \) we obtained in the preceding section it follows we can apply the method of generalized costs with

\[ \Gamma_{k,n} = \begin{pmatrix} k + \frac{n-k}{2} \\ n - k + 1 \end{pmatrix}. \]

Its trace and determinant are \( T = n + 1 \) and \( D = k(n-k+1) - (k-1)(n-k)/4 \) respectively. Again, with \( \lambda \) the largest eigenvalue,

\[ \lambda = \frac{n+1}{2} + \frac{n+1}{2} \sqrt{1 - \frac{3k(n-k+1)+n}{(n+1)^2}}, \]

we set \( c_1 = 1 \) and \( c_0 = (\lambda - k)/(n-k+1) \). Since \( \binom{c_1}{c_0} \) is an eigenvector for \( \lambda \) and \( c_0 = O(1) \), \( R(F_{k,n}^d) = \Omega(\lambda^d) \) and we have obtained Theorem 3.

### 5 Future directions

The main motivation for this work has been the recursive majority-of-three function. Although the value \( P(2,3) \) leads to the known \( 2.5^d \) lower bound, one can define an analogous problem on trees of some height \( t \). Informally, \( P^t(2,3) \) could ask for the greatest value \( \eta \) so that any algorithm for \( \text{maj}_t \) would have a non-negative cost under \( c_\eta \). Such a problem can be studied either rigorously or computationally (say using an LP-based approach as in [5]).

Another direction is to generalize the current work to unbalanced functions, in the spirit of [7] or [6]. A first step in this direction would be to obtain the results of [2] using the generalized-costs method.
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