Self-interacting Elko dark matter with an axis of locality

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This communication is a natural and nontrivial continuation of the 2005 work of Ahluwalia and Grumiller on Elko. Here we report that Elko breaks Lorentz symmetry in a rather subtle and unexpected way by containing a ‘hidden’ preferred direction. Along this preferred direction, a quantum field based on Elko enjoys locality. In the form reported here, Elko offers a mass dimension one fermionic dark matter with a quartic self-interaction and a preferred axis of locality. The locality result crucially depends on a judicious choice of phases.

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I. INTRODUCTION

The particle nature of dark matter is still unsettled. What we do know is that it is expected to be endowed with a self interaction [1, 2, 3, 4], and that it defines, or couples to, an axis that has come to be known as the axis of evil [5, 6, 7]. The latter aspect is most likely to be settled by Planck. The indicated self-interaction would ordinarily suggest that dark matter is some sort of scalar field. However, as shown in [8, 9], the Elko quantum field is endowed with mass-dimension one, a property that allows for un-suppressed Elko self-interaction. Further consequences of the mass dimensionality of Elko are that its possible interactions with the mass-dimension-three-half Dirac and Majorana fields are suppressed by one order of unification scale and that it cannot enter the SM doublets. This, along with the fact that Elko does not carry the standard U(1) gauge invariance, renders Elko a natural dark matter candidate [8, 9].

Here we report that Elko breaks Lorentz symmetry in a rather subtle and unexpected way by containing a ‘hidden’ preferred direction. All inertial frames that move with a constant velocity along this direction are physically equivalent. Along this direction, a quantum field based on Elko enjoys locality.

Our discourse begins with a review of the SM matter fields in Sec. I A. In section I B we recapitulate the known problems with the interpretation of Majorana spinors as c-numbers, and argue that these problems evaporate under a more careful examination [8, 9]. The pace is deliberately slow. The discussion is designed to provide the right setting for the taken departure. Sections II and III form the core of this communication. The discussion on the Elko dual presented in Sec. II B is a significant addition to the previous work on Elko [8, 9]. The dramatically changed locality structure arises from certain phases and identifications introduced in the Elko spinors at rest (see Eqns. 16a-16d). Section II C reminds that Elko satisfies the Klein-Gordon, but not the Dirac, equation. The Elko spin sums are given in Sec. II D. These spin sums are needed for examining the locality structure of the Elko quantum fields and had to be re-evaluated due to the mentioned changes in the Elko rest spinors. These carry the seeds of the mentioned preferred direction. Section III formally introduces the Elko quantum fields. Section III A makes an argument to identify Elko with self-interacting dark matter that is endowed with an axis of locality. In the form reported here, Elko offers a dimension-one fermionic dark matter with self-interaction and a preferred axis of locality. The locality result crucially depends on a judicious choice of phases. The paper ends with summarising remarks and questions in Sec. IV An appendix provides supplementary information.
A. The matter field underlying the SM

The matter field underlying the SM is a four-component spinor field \[10\] with historical origin in Dirac’s celebrated 1928 paper \[11\]

\[
\Psi(x) = \sum_{\sigma} \int \frac{d^4p}{(2\pi)^3} \frac{1}{\sqrt{2E(p)}} \left[ \begin{array}{c} u(x; p, \sigma) \ a(p, \sigma) \\ v(x; p, \sigma) \ b^\dagger(p, \sigma) \end{array} \right] 
\]

(1)

where \(\sigma\) takes the values \pm 1/2. The zero-momentum coefficient functions may be symbolically written as

\[
u(0, 1/2) = \begin{pmatrix} \uparrow \\ \uparrow \end{pmatrix}, \quad \nu(0, -1/2) = \begin{pmatrix} \downarrow \\ \downarrow \end{pmatrix} \]

(2)

\[
v(0, 1/2) = \begin{pmatrix} \downarrow \\ \downarrow \end{pmatrix}, \quad v(0, -1/2) = \begin{pmatrix} -\uparrow \\ -\uparrow \end{pmatrix} \]

(3)

where

\[
\uparrow \overset{\text{def}}{=} \sqrt{m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \downarrow \overset{\text{def}}{=} \sqrt{m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

(4)

in the ‘polarisation basis’. In the helicity basis, these are eigenspinors of the helicity operator with a specific choice of phases. These phases are determined, e.g., by the locality condition \[37\].

Without any reference to the Dirac equation (see Ref. \[10\] for a detailed argument), the coefficient functions are determined from the condition that under the homogeneous Lorentz transformations the field components superimpose with other field components via spacetime-independent elements (of \(4 \times 4\) matrices). These matrices must furnish a finite dimensional representation of the homogeneous Lorentz group.

The coefficient functions for arbitrary momentum are obtained by the action of the boost

\[
u(p, \sigma) = \kappa \nu(0, \sigma) \]

(5)

where \(\kappa := \kappa_r \oplus \kappa_\ell\). The explicit expressions for \(\kappa_r\) and \(\kappa_\ell\) are given below.

The only non-trivial freedom that \(\Psi(x)\) still carries is the specialisation to the case where \(b^\dagger(p, \sigma)\) is identified with \(a^\dagger(p, \sigma)\). Otherwise, the Poincaré spacetime symmetries along with the symmetries of charge-conjugation, parity and time-reversal and the demand of locality uniquely determine the field \(\Psi(x)\). Seen in this light the field coefficients \(u(p, \sigma)\) and \(v(p, \sigma)\) are eigenspinors of the \(\gamma^\mu p_\mu\) operator with eigenvalues \(+m\) and \(-m\) respectively.

The annihilation of the field \(\Psi(x)\) by the Dirac operator \((i\gamma^\mu \partial_\mu - mI)\) follows as a result of this structure. The Dirac equation is not assumed. Rather, it emerges as a direct consequence of the merger of quantum mechanics and Poincaré spacetime symmetries for spin one half. The apparent simplicity of the Dirac field can be somewhat misleading to the uninitiated. For instance, a change in sign in the right-hand side of the expression for \(v(0, -1/2)\) in Eqn. \[3\] yields a quantum field that is nonlocal when \(b^\dagger(p, \sigma)\) is identified with \(a^\dagger(p, \sigma)\). Even though the mentioned change in phase does not destroy the locality in the original field, it does violate spacetime symmetries in a hidden way. A systematic study of such subtle loss of symmetries and locality remains largely unexplored.

For historical reasons the field \(\Psi(x)\) is known as the Dirac field, while the identification of \(b^\dagger(p, \sigma)\) with \(a^\dagger(p, \sigma)\) yields what has come to be known as the Majorana field \[11, 12\]. The coefficient functions \(u(p, \sigma)\) and \(v(p, \sigma)\) are the usual Dirac spinors. They can be interpreted as being a direct sum of the \(r\)-type and \(\ell\)-type Weyl spinors with specific helicities and phases.

B. Majorana spinors: A critique

History clearly demarcates the introduction of the Majorana field. It was introduced in 1937 under the pen of Ettore Majorana \[12\]. As regards Majorana spinors, we (i.e, the authors) do not know of their historical birth.

While in the operator formalism of quantum field theory Dirac spinors can be treated as c-numbers, it is curious that Majorana spinors must be treated as G-numbers. This is deemed necessary, due to what are considered otherwise unavoidable problems (consider for instance Aitchison and Hay’s attempt to construct a Hamiltonian density \[13\]). What further adds to the problem is that taken by itself a Majorana spinor is nothing but a Weyl spinor in the four-component form. As shown by Ahluwalia and Grumiller \[8, 9\] both of these problems can be circumvented. The
solution to the first problem is found by acting the Dirac operator on a Majorana spinor and finding that, apart from a numerical factor, this operation yields another Majorana spinor (but not the same as the original one). The action of the Dirac operator again returns back the original Majorana spinor. This suggests that the problem is not with the Majorana spinors but with the Lagrangian density Aitchison and Hey assumed [13]. The latter of the two mentioned problems also has a similar solution. The usual set of Majorana spinors are a set of two spinors, and both of these have eigenvalue of unity under the operation of charge conjugation operator. This is the self-conjugate set. However, as pointed out in Ref. [8, 9] there also exists the anti self-conjugate set. Once that is done the complete set of four spinors — the Elko (for Eigenspinoren des Ladungs-Konjugationsoperators) — span the four-dimensional representation space of spin one half and come to par with the Dirac spinors.

Thus we note again that while in the operator formalism of quantum field theory Dirac spinors can be treated as c-numbers, it is curious that Majorana spinors must be treated as G-numbers but now ask if this distinction hides something fundamental about Majorana spinors, and what the Grassmann aspect has to do with their fermionicness. After all the operator formalism lends itself much more directly to the quantum mechanical description and fermionic aspect of Dirac particles, using Dirac spinors, is implemented through the anticommutation relations associated with the creation and destruction operators. Why is that Majorana spinors, without giving them the Grassmann character, have not found their place in a quantum field operator. In an attempt to answer this question we are led to a better understanding of the canonical wisdom, and this allows us the necessary departure in which Majorana spinors as c-numbers play an important role.

We now move towards greater preciseness by rephrasing some of the canonical wisdom and by examining it in greater detail. As we proceed through the critique we shall find that the mentioned obstacles can be surmounted. We do not refrain from making explicit the cost at which this happens. Whether or not one ought to accept the cost, in part, is a matter of experiments to decide. At the very least we shall know as to what it is that we reject if we choose to confine to the canonical wisdom alone.

In the received wisdom, the Majorana spinors arise as follows. If $\phi_\ell$ is a massive Weyl spinor of $\ell$-type, then $\sigma_2 \phi_\ell^*$ transforms as a $r$-type Weyl spinor. For this reason [14, p.20], we can construct a special type of four-component spinor called a Majorana spinor

$$\psi_M = \begin{pmatrix} -\sigma_2 \phi_\ell^* \\ \phi_\ell \end{pmatrix}$$

(6)

It is self-conjugate under charge conjugation. For $\phi_\ell$ there are two choices, a positive helicity and a negative helicity. As such, we have two rather than four four-component spinors. Thus the folklore: the Majorana spinor is a Weyl spinor in four component form [14]. It is self evident, and remains unquestioned in our discourse.

An immediate sign of trouble appears if one naively introduces a Lagrangian density $\mathcal{L}_M = \bar{\psi}_M (i \gamma^\mu \partial_\mu - m) \psi_M$. The usual route at this stage is to treat the Weyl spinorial components as Grassmann numbers, otherwise one encounters the often quoted problems [13, App. P]. The Ahluwalia-Grumiller work in references [8, 9] strongly indicates that this approach may be hiding certain fundamental properties of Majorana spinors. Or, to put it more precisely, having gone the Grassmannian route we may have escaped a rich and fertile ground where Majorana spinors are treated as c-number spinors. To unearth these aspects here we will treat the massive Weyl spinors as 2-component eigenspinors of the helicity operator [15, p. 111] and the fermionic statistics shall be implemented through the canonical field operator formalism [10, 10], and not by treating them as Grassmann fields [38]. The Elko formalism was born in this spirit and it attended to a widespread, but rarely spoken, discontent with abandoning Majorana spinors as c-numbers play an important role.

The assertion about reduction in the degrees of freedom for Majorana spinors also faces trouble if one notes that the relevant charge conjugation operator has not one, but two, real eigenvalues, $+1$ (giving the usual self-conjugate Majorana spinors), and $-1$. There is no physical or mathematical reason to abandon, or project out, the latter. The sense in which the folklore still survives is that by an appropriate similarity transformation half of these (i.e., those
corresponding to the positive eigenvalue) can be morphed into real 4-component spinors, while those corresponding to the negative eigenvalue can be transformed into pure imaginary 4-component spinors.

II. ELKO: THE DEPARTURE FROM GRASSMANN INTERPRETATION OF MAJORANA SPINORS

The Grassmann interpretation of the Majorana spinors is elegant. It is mathematically sound. It has found wide spread applications in modern quantum field theory. Yet, it breaks with the tradition of field operatic formalism which would have required these spinors to be c-number coefficient functions in a field. To implement this programme Ahluwalia and Grumiller undertook a new effort in references [8, 9]. They introduced a complete set of dual helicity eigenspinors of the charge conjugation operator for spin one half. In their formalism there are four, rather than two, 4-components spinors. This is the first point of departure. For the mentioned set of spinors they introduced the name Elko. The new name, as already mentioned, was taken from German and stood for Eigenspinoren des Ladungs konjugation operators. It was necessitated to mark the distinct physical and mathematical content of the introduced departure; and to avoid confusion with the literature on Majorana spinors and fields where Grassmannian interpretation reigns.

Grassmannisation of Majorana spinors is a deep and conceptually nontrivial element of theoretical landscape. The quantum-mechanical field it introduces is not a quantum field in the sense of Weinberg (specifically, in the sense of Weinberg’s monograph [10]). At the same time the uniqueness of Dirac field, modulo its specilisation to the Majorana field, also implies that the programme we embark upon shall necessarily contain an element that breaks Lorentz invariance in some way. This feature had remained hidden in our previous discourse. We now explicitly unearth it.

In this communication we confine our primary attention to spin one half, but we construct Elko in such a way that the procedure immediately generalises to all spins. This is facilitated by the use of Wigner’s time reversal operator Θ, rather than the Pauli matrix σ₂ that appears in Ramond’s primer in the context of Majorana spinors. We shall use the phrase Elko for spinors as well as the quantum fields constructed from them. The context shall be assumed to remove any ambiguity.

A. Construction of Elko

To construct Elko it is first necessary to introduce the charge conjugation operator. This we do as follows. Under parity, P, x → x' = -x, φ → -φ and σ → σ. Consequently, an examination of Eqs. (9) and (11), yields κ_ℓ P ↔ κ_r. This observation suffices to give the action of parity on the r ⊕ ℓ representation space up to a phase

\[ S(P) = \exp[i\vartheta] \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} :_\gamma^0, \vartheta \in \mathbb{R} \]  

(7)

With \( p := p(\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)) \), the \( \mathcal{R} = \{ \theta \to \pi - \theta, \phi \to \pi + \phi, p \to p \} \). If care is taken that the eigenvalues of the helicity operator change sign under \( P \), the arguments given in Ref. [9] fix the phase \( \exp[i\vartheta] \) to be i. The \( S(P) \) now has four doubly-degenerate eigenspinors, carrying opposite eigenvalues of \( S(P) \) — call these \( u \) and \( v \) sectors. The operator

\[ C = \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix} K \]  

(8)

where \( K \) is the complex conjugation operator, formally interchanges the opposite parity sectors: \( u \leftrightarrow v \). It is apparent that \( C \) is the standard charge conjugation operator of Dirac. In the context of Eqn. [5] Wigner’s time reversal operator \( \Theta \) is defined as \( \Theta J \Theta^{-1} = -J^* \) where \( J \) are a set of rotation generators for the representation space under consideration. For spin one half, \( \Theta [\sigma/2] \Theta^{-1} = -[\sigma/2]^* \). We use the realisation

\[ \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

To construct Elko, let \( \phi_\ell(p) \) be a \( \ell \)-type Weyl spinor of spin one half. Under a Lorentz boost, it transforms as
In that case the right- and left-transforming components are necessarily endowed with the same helicity. For Elko, the

\[ \chi(p) \]

with

\[ \kappa_\ell = \exp \left( -\frac{\sigma}{2} \cdot \varphi \right) = \sqrt{\frac{E + m}{2m}} \left( I - \frac{\sigma \cdot p}{E + m} \right) \]  \hspace{1cm} (9)

The \( \epsilon \) is defined as \( p|_{\gamma = 0} \), and not as \( p|_{\gamma = 0} \). This restriction can be removed, if necessary (for example, by working in ‘polarisation basis’ which then comes with its own subtleties). The basic results remain unaltered. In the usual notation, the boost parameter \( \varphi \) is defined as

\[ \cosh \varphi = \frac{E}{m} = \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \sinh \varphi = \frac{p}{m} = \gamma \beta, \quad \hat{\varphi} = \hat{p} \]  \hspace{1cm} (10)

By \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) we denote the Pauli matrices. The symbol \( I \) represents an identity matrix, while in what follows \( 0 \) shall be used for a null matrix (their dimensionality shall be apparent from the context). For \( \epsilon(p) \) we have two possibilities: \( \sigma \cdot \hat{p} \phi_\ell^+(p) = \pm \phi_\ell^+(p) \).

Following Ref. [9] we now note that under a Lorentz boost, \( \vartheta \Theta \phi_\ell^+(p) \) transforms as a \( r \)-type Weyl spinor, \( \left[ \vartheta \Theta \phi_\ell^+(p) \right] = \kappa_r [\vartheta \Theta \phi_\ell^+(\epsilon)] \), with

\[ \kappa_r = \exp \left( +\frac{\sigma}{2} \cdot \varphi \right) = \sqrt{\frac{E + m}{2m}} \left( I + \frac{\sigma \cdot p}{E + m} \right) \]  \hspace{1cm} (11)

where \( \vartheta \) is an unspecified phase to be determined below. The helicity of \( \vartheta \Theta \phi_\ell^+(p) \) is opposite to that of \( \phi_\ell(p) \),

\[ \sigma \cdot \hat{p} \left[ \vartheta \Theta (\phi_\ell^+(p))^* \right] = \mp \left[ \vartheta \Theta (\phi_\ell^+(p))^* \right] \]  \hspace{1cm} (12)

The argument that led to two Majorana spinors, now instead takes us to their cousins, the four 4-component spinors with the general form

\[ \chi(p) = \left( \begin{array}{c} \vartheta \Theta \phi_\ell^+(p) \\ \phi_\ell(p) \end{array} \right) \]  \hspace{1cm} (13)

The \( \chi(p) \) become eigenspinors of the charge conjugation operator, Elko, with real eigenvalues if the phase \( \vartheta \) is restricted to \( \pm i \)

\[ C \chi(p)|_{\vartheta = \pm i} = \pm \chi(p)|_{\vartheta = \pm i} \]  \hspace{1cm} (14)

One can motivate the well-known Dirac spinors in a parallel manner; as eigenspinors of the parity operator \( S(P) \). In that case the right- and left-transforming components are necessarily endowed with the same helicity. For Elko, the right- and left-transforming components carry opposite helicity. So, whereas Dirac spinors may exist as eigenspinors of the helicity operator, the Elko cannot. This eventually reflects in many results that we arrive at.

To give Elko a concrete form, we adopt the global phases so that ‘at rest’ the \( \ell \)-type Weyl spinors take the form \[ \phi_\ell^+(\epsilon) = \sqrt{m} \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \]  \hspace{1cm} (15a)

\[ \phi_\ell^-(\epsilon) = \sqrt{m} \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix} \]  \hspace{1cm} (15b)

Equations (15a, 15b), when coupled with Eqn. (13), allow us to explicitly introduce the self-conjugate spinors \( (\vartheta = +i) \) and anti self-conjugate spinors \( (\vartheta = -i) \) at rest \[ \xi(-,+)(\epsilon) := + \chi(\epsilon)|_{\phi_\ell^+(\epsilon) \rightarrow \phi_\ell^+(\epsilon), \vartheta = +i} \]  \hspace{1cm} (16a)

\[ \xi(+,-)(\epsilon) := + \chi(\epsilon)|_{\phi_\ell^+(\epsilon) \rightarrow \phi_\ell^-(\epsilon), \vartheta = +i} \]  \hspace{1cm} (16b)

\[ \zeta(-,+)(\epsilon) := + \chi(\epsilon)|_{\phi_\ell^-(\epsilon) \rightarrow \phi_\ell^+(\epsilon), \vartheta = -i} \]  \hspace{1cm} (16c)

\[ \zeta(+,-)(\epsilon) := - \chi(\epsilon)|_{\phi_\ell^+(\epsilon) \rightarrow \phi_\ell^-(\epsilon), \vartheta = -i} \]  \hspace{1cm} (16d)
For comparison with equations (21), the above in ‘polarisation basis’ may be written as

\[ \xi_{(-,+)}(\epsilon) = \begin{bmatrix} \iota \\ -\iota \end{bmatrix}, \quad \xi_{(+,-)}(\epsilon) = \begin{bmatrix} -\iota \\ \iota \end{bmatrix} \]

\[ \zeta_{(-,+)}(\epsilon) = \begin{bmatrix} \iota \\ -\iota \end{bmatrix}, \quad \zeta_{(+,-)}(\epsilon) = -\begin{bmatrix} -\iota \\ \iota \end{bmatrix} \]  

(17) (18)

The \( \uparrow \) and \( \downarrow \) differ from \( \uparrow \) and \( \downarrow \) of Eqn. 4 by certain phases (otherwise, as is appropriate for the ‘polarisation basis,’ they are identical when \( \theta \) and \( \phi \) are set to zero). In the context of Weinberg’s work on the uniqueness of Dirac field (modulo its ‘Majoranisation’ in the sense of the 1937 original paper of Majorana [12]), a comparison with Eqns. (21, 22) already tells us that a quantum field that fully respects Lorentz symmetries cannot be built in terms of \( \xi \) and \( \zeta \) Elko spinors. The task then is to unearth this violation, and see how strong, or how weak, the said violation is.

The \( \xi(p) \) and \( \zeta(p) \) for an arbitrary momentum are now readily obtained

\[ \xi(p) = \kappa \xi(\epsilon), \quad \zeta(p) = \kappa \zeta(\epsilon), \quad \kappa := \kappa_r \oplus \kappa_\ell \]

(19)

B. A systematic construction of Elko dual, orthonormality, and completeness

The norm of Elko under the Dirac dual \( \chi(p) := [\chi(p)]^\dagger \gamma^0 \) identically vanishes. However, it is more appropriate to seek a ‘metric’ \( \eta \) such that the product \( [\chi(p)]^\dagger \eta \chi(p) \) — with \( \chi(p) \) as any one of the four Elko spinors — remains invariant under an arbitrary Lorentz transformation. This requirement can be readily shown to translate into the following constraints on \( \eta \)

\[ [J_i, \eta] = 0, \quad \{K_i, \eta\} = 0 \]  

(20)

Since the only property of the generators of rotations and boosts that enters the derivation of the above constraints is that \( J^j = J \) and \( K^j = -K \), the result applies to all finite dimensional representations of the Lorentz group. It need not be restricted to Elko alone. Seen in this light, there is no non-trivial solution for \( \eta \) either for the \( r \)-type or the \( \ell \)-type Weyl spinors. For \( r \oplus \ell \) representation space, the most general solution is found to carry the form

\[ \eta = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & a & -im(a+b) \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix} \]  

(21)

It is now convenient to introduce the notation \( \chi_1(p) := \xi_{(-,+)}(p) \), \( \chi_2(p) := \xi_{(+,-)}(p) \), \( \chi_3(p) := \zeta_{(-,+)}(p) \), and \( \chi_4(p) := \zeta_{(+,-)}(p) \). Then sixteen values of \([\chi_i(p)]^\dagger \eta \chi_j(p)\) as \( i \) and \( j \) vary from 1 to 4 are given in Table 1.

TABLE I: The values of \([\chi_i(p)]^\dagger \eta \chi_j(p)\) evaluated using \( \eta \). The \( i \) runs from 1 to 4 along the rows and \( j \) does the same across the columns.

| \( i \) \( j \) | \( 0 \) | \( -im(a+b) \) | \( -im(a-b) \) | \( 0 \) |
|---|---|---|---|---|
| \( i \) \( 0 \) | \( 0 \) | \( \eta \gamma^0 \) | \( -im(a-b) \) | \( 0 \) |
| \( im(a+b) \) | \( 0 \) | \( 0 \) | \( -im(a-b) \) | \( 0 \) |
| \( -im(a-b) \) | \( 0 \) | \( 0 \) | \( im(a+b) \) | \( 0 \) |
| \( 0 \) | \( -im(a-b) \) | \( -im(a+b) \) | \( 0 \) | \( 0 \) |

(22)

To allow for the possibility of parity covariance we set \( b = a \) (this treats \( r \) and \( \ell \) Weyl spaces on the same footing). To make the invariant norms real, we give \( a \) and \( b \) the common value of \( \pm i \); resulting in \( \eta = \pm i \gamma^0 \). In what follows the choice of the signs shall be dictated by the convenience of book keeping.

Guided by these results we now introduce the Elko dual

\[ \tilde{\chi}_{(r,\pm)}(p) := \mp i \left[ \chi_{(\pm,\mp)}(p) \right]^\dagger \gamma^0 \]  

(23a)

Under the new dual the orthonormality relations read

\[ \tilde{\xi}_\alpha(p) \tilde{\xi}_{\alpha'}(p) = +2m \delta_{\alpha\alpha'} \]  

(23a)

\[ \tilde{\zeta}_\alpha(p) \tilde{\zeta}_{\alpha'}(p) = -2m \delta_{\alpha\alpha'} \]  

(23b)
along with $\bar{\xi}_\alpha(p) \zeta_{\alpha'}(p) = 0$, and $\bar{\zeta}_\alpha(p) \xi_{\alpha'}(p) = 0$. The dual helicity index $\alpha$ ranges over the two possibilities: $\{+, -\}$ and $\{-, +\}$, and $-\{\pm, \mp\} := \{\mp, \pm\}$. The completeness relation

$$\frac{1}{2m} \sum_\alpha \left[ \xi_\alpha(p) \bar{\xi}_{\alpha}(p) - \zeta_\alpha(p) \bar{\zeta}_{\alpha}(p) \right] = \mathbb{I}$$

establishes that we need to use both the self-conjugate as well as the anti self-conjugate spinors to fully capture the relevant degrees of freedom.

C. Elko satisfy the Klein-Gordon, not Dirac, equation

Because we are going to encounter several unexpected results, we pause to examine the behaviour of $\xi(p)$ and $\zeta(p)$ spinors under the action of the operator $\gamma^\mu p_\mu$. This brute force exercise serves the pedagogic purpose of countering some of prejudices that some readers may inevitably carry from their prior studies. Additionally, in the context of Aitchison and Hey’s concern that one encounters a problem in constructing a Lagrangian density for Majorana spinors if they are not treated as Grassmann variables [13, App. P], we provide the origin of that concern and offer a solution.

We already have explicit expressions for $\xi(p)$ and $\zeta(p)$ spinors. On these we act $\gamma^\mu p_\mu$ and find the following identities

$$\gamma^\mu p_\mu \xi_{\{+, -\}}(p) = +im \xi_{\{+, -\}}(p) \quad \Leftrightarrow \quad \gamma^\mu p_\mu \chi_1(p) = +im \chi_2(p) \quad (25a)$$

$$\gamma^\mu p_\mu \xi_{\{+,-\}}(p) = -im \xi_{\{+,-\}}(p) \quad \Leftrightarrow \quad \gamma^\mu p_\mu \chi_2(p) = -im \chi_1(p) \quad (25b)$$

$$\gamma^\mu p_\mu \zeta_{\{+, -\}}(p) = -im \zeta_{\{+, -\}}(p) \quad \Leftrightarrow \quad \gamma^\mu p_\mu \chi_3(p) = -im \chi_4(p) \quad (25c)$$

$$\gamma^\mu p_\mu \zeta_{\{+,-\}}(p) = +im \zeta_{\{+,-\}}(p) \quad \Leftrightarrow \quad \gamma^\mu p_\mu \chi_4(p) = +im \chi_3(p) \quad (25d)$$

Applying $\gamma^\nu p_\nu$ to Eqn. (25) from the left and then using (25) on the resulting right hand side, and repeating the same procedure for the remaining equations we get

$$\left( \gamma^\nu \gamma^\mu p_\nu p_\mu - m^2 \right) \xi_{\{\mp, \pm\}}(p) = 0, \quad \left( \gamma^\nu \gamma^\mu p_\nu p_\mu - m^2 \right) \zeta_{\{\pm, \pm\}}(p) = 0. \quad (26)$$

Now using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^\mu\nu$, yields the Klein-Gordon equation (in momentum space) for the $\xi(p)$ and $\zeta(p)$ spinors. Aitchison and Hey’s concern is thus overcome. The problem resides in the approach of constructing the “simplest candidates for a kinematic spinor term.”

D. Elko spin sums: a preferred axis

We now look at the spin sums in Eqn. (24) separately. These evaluate to

$$\sum_\alpha \xi_\alpha(p) \bar{\xi}_{\alpha}(p) = m \{G(p) + \mathbb{I}\} \quad (27a)$$

$$\sum_\alpha \zeta_\alpha(p) \bar{\zeta}_{\alpha}(p) = m \{G(p) - \mathbb{I}\} \quad (27b)$$

which together define $G(p)$. A direct evaluation of the left hand side of the above equations gives

$$G(p) = i \begin{pmatrix} 0 & 0 & 0 & -e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \end{pmatrix} \quad (28)$$

For later reference, we note that $G(p)$ is an odd function of $p$

$$G(p) = -G(-p) \quad (29)$$

But since $G(p)$ is independent of $p$ and $\theta$, it is more instructive to translate the above expression into

$$G(\phi) = -G(\pi + \phi) \quad (30)$$
This serves to define a preferred axis, \( z \). Another hint for a preferred axis arises when one notes that the Elko spinorial structure does not enjoy covariance under usual local \( U(1) \) transformation with phase \( \exp(i\alpha(x)) \). However, \( U_E(1) = \exp(i\gamma_0\alpha(x)) \) — and not \( U_M(1) = \exp(i\gamma_5\alpha(x)) \) as one would have thought \( \text{[17, p. 72]} \) — preserves various aspects of the Elko structure. Similar comments apply to the non-Abelian gauge transformations of the SM.

For a comparison with the Dirac counterpart (App. A 1), we define \( g^\mu := (0, g) \) with \( g = -[1/\sin(\theta)]\partial_p/\partial\phi = (\sin \phi, -\cos \phi, 0) \). Note may be taken that \( g^\mu \) is a unit spacelike four-vector, \( g_\mu g^\mu = -1 \). Furthermore, \( g_\mu p^\mu = 0 \). In terms of \( g^\mu \), \( \mathcal{G}(p) \) may be written as

\[
\mathcal{G}(p) = \gamma^5(\gamma_1 \sin \phi - \gamma_2 \cos \phi) = \gamma^5 \gamma_\mu g^\mu
\]

This gives Eqns. (27b) and (27a), the form

\[
\sum_\alpha \xi_\alpha(p) \tilde{\xi}_\alpha(p) = m \left[ \gamma^5 \gamma_\mu g^\mu + 1 \right]
\]

\[
\sum_\alpha \zeta_\alpha(p) \tilde{\zeta}_\alpha(p) = m \left[ \gamma^5 \gamma_\mu g^\mu - 1 \right]
\]

The \( \gamma^\mu \), in the Weyl realisation, are taken to be

\[
\gamma^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^i := \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^5 := -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

III. ELKO FERMIONIC FIELDS OF MASS DIMENSION ONE: LAGRANGIAN DENSITIES

Confining to the Elko frame, we now examine the physical and mathematical content of two quantum fields \[45\]

\[
\Lambda(x) \overset{\text{def}}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(p)}} \sum_\alpha \left[ a_\alpha(p) \tilde{\xi}_\alpha(p)e^{-ip_\mu x^\mu} + b_\alpha^\dagger(p) \zeta_\alpha(p)e^{+ip_\mu x^\mu} \right]
\]

and,

\[
\lambda(x) \overset{\text{def}}{=} \Lambda(x)|_{b_\alpha(p) \rightarrow a_\alpha(p)}
\]

We assume that the annihilation and creation operators satisfy the fermionic anticommutation relations

\[
\{a_\alpha(p), a_{\alpha'}^\dagger(p')\} = (2\pi)^3 \delta^3(p - p') \delta_{\alpha\alpha'},
\]

\[
\{a_\alpha(p), a_{\alpha'}(p')\} = 0, \quad \{b_\alpha^\dagger(p), b_{\alpha'}^\dagger(p')\} = 0.
\]

Similar anticommutators are assumed for the \( b_\alpha(p) \) and \( b_\alpha^\dagger(p) \). The adjoint field \( \tilde{\Lambda}(x) \) is defined as

\[
\tilde{\Lambda}(x) \overset{\text{def}}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(p)}} \sum_\alpha \left[ a_\alpha^\dagger(p) \tilde{\xi}_\alpha(p)e^{+ip_\mu x^\mu} + b_\alpha(p) \tilde{\zeta}_\alpha(p)e^{-ip_\mu x^\mu} \right]
\]

The results contained in Eqns. (28a), (28d), assure us that it is the Klein-Gordon, and not the Dirac, operator that annihilates the fields \( \Lambda(x) \) and \( \lambda(x) \). The associated Lagrangian densities are

\[
\mathcal{L}^\Lambda(x) = \partial^\mu \tilde{\Lambda}(x) \partial_\mu \Lambda(x) - m^2 \tilde{\Lambda}(x)\Lambda(x), \quad \mathcal{L}^\lambda(x) = \mathcal{L}^\Lambda(x)|_{\lambda \rightarrow \lambda}
\]

The mass dimensionality of these Elko fields is thus one, and not three half.
A. Identification of Elko with dark matter

These results open up an entirely new and unexpected possibility for the dark matter sector. The primary observations that suggest this are five fold:

1. Due to mismatch in mass dimensionality of $D_\lambda = 1$ and $D_\psi = 3/2$, the new fermionic fields cannot enter the SM doublets.

2. The Lagrangian densities associated with Elko fields, do not carry the gauge symmetries of the SM (see remarks above Eqn. (31)).

3. The dimension four interactions of the $\Lambda(x)$ and $\lambda(x)$ with the standard model fields are restricted to those with the SM Higgs doublet $\phi(x)$. These are

$$L_{\text{int}}(x) = \phi(x) \left[ a_1 \tilde{\Lambda}(x)\Lambda(x) + a_2 \tilde{\lambda}(x)\lambda(x) + a_3 \left( \tilde{\Lambda}(x)\lambda(x) + \tilde{\lambda}(x)\Lambda(x) \right) \right]$$

where the $a'$s are unknown coupling constants and.

4. By virtue of their mass dimensionality the new dark matter fields are endowed with dimension four self interactions

$$L_{\text{self}}(x) = b_1 \left( \tilde{\Lambda}(x)\Lambda(x) \right)^2 + b_2 \left( \tilde{\lambda}(x)\lambda(x) \right)^2 + b_3 \left[ \left( \tilde{\Lambda}(x)\lambda(x) \right)^2 + \left( \tilde{\lambda}(x)\Lambda(x) \right)^2 \right], \quad (39)$$

where the $b'$s are unknown coupling constants. Observational evidence suggests that dark matter needs to be self interacting [1, 2, 3, 4].

5. The Elko fields are endowed with an intrinsic preferred axis. Tentative evidence already exists for such an axis [5, 6, 7].

Combined, the enumerated Elko properties not only render Elko dark with respect to the SM matter fields but they also endow it various observationally-attractive properties. It is worth emphasising that all of these properties are intrinsic to Elko, and arise in a natural way.

B. The locality structure of Elko

The canonically conjugate momenta to the Elko fields are

$$\Pi(x) = \frac{\partial L^\Lambda}{\partial \dot{\Lambda}} = \frac{\partial}{\partial t} \tilde{\Lambda}(x), \quad (40)$$

and similarly $\pi(x) = \frac{\partial}{\partial t} \tilde{\lambda}(x)$. The calculational details for the two fields now differ significantly. We begin with the evaluation of the equal time anticommutator for the $\Lambda(x)$ and its conjugate momentum, and find

$$\{\Lambda(x, t), \Pi(x', t)\} = i \int \frac{d^3 p}{(2\pi)^3 2m} e^{i p \cdot (x-x')} \sum_{\alpha} \left[ \xi_\alpha(p) \bar{\xi}_\alpha(-p) - \bar{\zeta}_\alpha(-p) \zeta_\alpha(p) \right].$$

or, equivalently

$$\{\Lambda(x, t), \Pi(x', t)\} = i \delta^3(x - x') \mathbb{I} + i \int \frac{d^3 p}{(2\pi)^3} e^{i p \cdot (x-x')} \mathcal{G}(p).$$

The anticommutators for the particle/antiparticle annihilation and creation operators suffice to yield the remaining locality conditions,

$$\{\Lambda(x, t), \Lambda(x', t)\} = 0, \quad \{\Pi(x, t), \Pi(x', t)\} = 0.$$
Since the integral on the right hand side of Eqn. (42) vanishes only along the $\pm \hat{z}_e$ axis, the preferred axis also becomes the axis of locality.

For the equal time anticommutator of the $\lambda(x)$ field with its conjugate momentum, we find
\[
\{\lambda(x, t), \pi(x', t)\} = i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2m} \sum_\alpha \left[ e^{ip(x-x')} (\xi_\alpha(p) \tilde{\xi}_\alpha(-p) - \xi_\alpha(-p) \tilde{\xi}_\alpha(p)) \right].
\] (44)

Which, using the same argument as before, yields
\[
\{\lambda(x, t), \pi(x', t)\} = i \delta^3(x - x') \mathbb{I} + i \int \frac{d^3p}{(2\pi)^3} e^{ip(x-x')} \mathcal{G}(p).
\] (45)

The difference arises in the evaluation of the remaining anticommutators. The equal time $\lambda-\lambda$ anticommutator reduces to
\[
\{\lambda(x, t), \lambda(x', t)\} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2mE(p)} e^{ip(x-x')} \sum_\alpha \left[ \xi_\alpha(p)\tilde{\xi}_\alpha^T(-p) + \xi_\alpha(-p)\tilde{\xi}_\alpha^T(p) \right].
\] (46)

Now using explicit expressions for $\xi_\alpha(p)$ and $\xi_\alpha(p)$ we find that $\Omega(p)$ identically vanishes. Eqn. (46) then implies
\[
\{\lambda(x, t), \lambda(x', t)\} = 0.
\] (47)

Finally, the equal time $\pi-\pi$ anticommutator simplifies to
\[
\{\pi(x, t), \pi(x', t)\} = \int \frac{d^3p}{(2\pi)^3} \frac{E(p)}{2m} e^{-ip(x-x')} \sum_\alpha \left[ \left( \tilde{\xi}_\alpha(p) \right)^T \tilde{\xi}_\alpha(-p) + \left( \tilde{\xi}_\alpha(-p) \right)^T \tilde{\xi}_\alpha(p) \right],
\]

yielding
\[
\{\pi(x, t), \pi(x', t)\} = 0.
\] (48)

Equations (42,43) and (45,48) establish that $\Lambda(x)$ and $\lambda(x)$ are local quantum fields in the direction perpendicular to the Elko plane; i.e. along the preferred axis $\hat{z}_e$. We propose to call $\hat{z}_e$ as the axis of locality in the dark sector.

IV. CONCLUDING REMARKS

Modulo its specilisation to the Majorana field, Weinberg’s monographic work [10] establishes the uniqueness of the Dirac quantum field for spin one half particles. Seen from that perspective the Ahluwalia-Grumiller work on Elko in 2005 was unexpected. At this stage two things happened. On the one hand Elko found significant interest among mathematical physicists and cosmologists [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. In these papers one dealt with Elko as spinors and not as a quantum field. So no contradiction with Weinberg’s theorem-like work occurred. Not unexpectedly, Gillard and Martin showed that if Elko were to be taken as ‘good’ quantum fields, Poincaré symmetries had to be violated in some form or other [32]. The results presented in this communication explicitly confirm this and show that the violation occurs in a rather subtle way. This done, Elko now stands as a natural dark matter candidate. By virtue of its mass dimensionality it allows an unsuppressed quartic self coupling. Additionally, it points towards the existence of a preferred axis, along which the Elko quantum fields become local. Both of these aspects can be used to distinguish it from other candidates for the dark-matter sector. Its darkness with respect to the SM matter and gauge fields is built into its intrinsic mass dimension.

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APPENDIX A: APPENDIX

1. Dirac spin sums and a ‘misleading’ derivation of Dirac equation

With a minor departure from the historical path, the Dirac counterpart of (32a) and (32b) may be constructed as follows. Instead of (6), we start with

\[ \psi_D \overset{\text{def}}{=} \left( \begin{array}{c} \phi_r \\ \phi_\ell \end{array} \right) \]  \hspace{1cm} (A1)

The helicities of \( \phi_r \) and \( \phi_\ell \) are identical and are determined by requiring that \( \psi_D \) be eigenspinors of the parity operator \( S(P) \). Again, there are four independent rest spinors (these differ from those mentioned in Sec. (IA) only in that we now work in the ‘helicity basis’)

\[ u_{+1/2}(\epsilon) = \left( \begin{array}{c} \phi_r^+ (\epsilon) \\ \phi_\ell^+ (\epsilon) \end{array} \right), \hspace{1cm} u_{-1/2}(\epsilon) = \left( \begin{array}{c} \phi_r^- (\epsilon) \\ \phi_\ell^- (\epsilon) \end{array} \right) \]  \hspace{1cm} (A2)

\[ v_{+1/2}(\epsilon) = \left( \begin{array}{c} \phi_r^- (\epsilon) \\ -\phi_\ell^- (\epsilon) \end{array} \right), \hspace{1cm} v_{-1/2}(\epsilon) = \left( \begin{array}{c} -\phi_r^+ (\epsilon) \\ \phi_\ell^+ (\epsilon) \end{array} \right) \]  \hspace{1cm} (A3)

The \( u(p) \) and \( v(p) \) for an arbitrary momentum are obtained via the action of the boost \( \kappa \)

\[ u(p) = \kappa u(\epsilon), \hspace{1cm} v(p) = \kappa v(\epsilon) \]  \hspace{1cm} (A4)

These lead to the spin sums

\[ \sum_\beta u_\beta(p) \bar{\tau}_\beta(p) = m \left[ \frac{\gamma_\mu p^\mu}{m} + I \right] \]  \hspace{1cm} (A5a)

\[ \sum_\beta v_\beta(p) \bar{\tau}_\beta(p) = m \left[ \frac{\gamma_\mu p^\mu}{m} - I \right] \]  \hspace{1cm} (A5b)

where \( \beta \) takes two values: +1/2 and –1/2. As before, the right hand sides in the above expression simply express the result of a direct evaluation of the left hand sides. These are covariant.

We thus see that in the Dirac construct, whether it be at the level of spinors (or at the quantum field theoretic level), no preferred frame is introduced. For Majorana spinors, and Elko, the conclusion is both unexpected and inevitable. This difference — as pertaining to the existence of a preferred frame — between the Dirac and Majorana spinors, along with their cousins Elko, to our knowledge is completely unknown. This conclusion carries distinct echoes of the unpublished notes [33] which eventually, in collaboration with Grumiller, led to the discovery reported in references [8, 9].

If we multiply Eqn. (A5a) by \( u_\beta'(p) \) from the right, and use \( u_\beta(p) u_\beta'(p) = 2 m \delta_{\beta\beta'} \); and carry out a similar exercise with Eqn. (A5a), then after a minor rearranging we obtain

\[ (\gamma_\mu p^\mu - m) u(p) = 0 \]  \hspace{1cm} (A6)

\[ (\gamma_\mu p^\mu + m) v(p) = 0 \]  \hspace{1cm} (A7)

These are indeed Dirac equations in momentum space. With \( p^\mu \rightarrow i \partial^\mu \) and

\[ \psi(x) \overset{\text{def}}{=} \begin{cases} u(p) \exp(-ip_\mu x^\mu) \\ v(p) \exp(+ip_\mu x^\mu) \end{cases} \]  \hspace{1cm} (A8)

these yield the well-known Dirac equation in the configuration space

\[ (i\gamma_\mu \partial^\mu - m) \psi(x) = 0 \]  \hspace{1cm} (A9)

To associate these with the dynamics of spin one half spinors, particularly in a quantum field theoretic context (where \( \psi(x) \) is now elevated to a spinor field \( \Psi(x) \)), requires that, in addition, the vacuum expectation value, \( \langle |T[\Psi(x)\bar{\Psi}(x)]| \rangle \), be proportional to the relevant Green’s function. That is, it is not sufficient to find an operator, such as \( (i\gamma_\mu \partial^\mu - m) \), or the Klein Gordon operator, that annihilates \( \Psi(x) \) for it to serve in the Lagrangian.
density of the field $\Psi(x)$. It must also satisfy the said requirement. This will become abundantly clear from what follows in the context of Elko.

While we do consider the above ‘derivation’ of Dirac equation misleading, it does serve to tell us that the Dirac spinors are eigenspinors of $\gamma_\mu p^\mu$ with eigenvalues $\pm m$.

\[
\gamma_\mu p^\mu u(p) = +mu(p), \quad \gamma_\mu p^\mu v(p) = -mv(p)
\]

The Elko counterpart is

\[
\mathcal{G}(p)\xi(p) = +\xi(p), \quad \mathcal{G}(p)\zeta(p) = -\zeta(p)
\]

It again emphasises that identities such as these should not be mistaken for dynamical equations. In particular, $\mathcal{G}(p)$, unlike its Dirac counterpart $\gamma_\mu p^\mu$, contains no time derivative.

## 2. Elko time ordering and propagators

The mass dimensionality of a field can also be deciphered from constructing the Feynman-Dyson propagator. This involves defining a time ordering operator. However, the existence of a preferred direction mentioned above makes it unclear as to how, and if, Elko construct modifies this definition. In what follows we first adopt the standard definition of the fermionic time ordering operator, and then invoke a consistency argument to re-define the Elko time ordering.

Let $T$ be the standard fermionic time ordering operator. Then, a straightforward calculation yields

\[
\langle |T[\Lambda(x') \xrightarrow{\Lambda} (x)] | \rangle = \int \frac{d^4p}{(2\pi)^3} \frac{1}{2mE(p)} \times \sum_\alpha \left[ \theta(t' - t)\xi_\alpha(p) \tilde{\xi}_\alpha(p)e^{-ip_\mu(x'' - x''')} - \theta(t - t')\zeta_\alpha(p) \tilde{\zeta}_\alpha(p)e^{+ip_\mu(x'' - x''')} \right]
\]

(A12)

where the step function $\theta(t)$ equals unity for $t > 0$ and vanishes for $t < 0$.

Using the spin sums (27a) and (27b), setting $q^\mu = (q^0, q = p)$, and using the standard integral representation for the $\theta(t)$, Eqn. (A12) simplifies to

\[
\langle |T[\Lambda(x') \xrightarrow{\Lambda} (x)] | \rangle = i \int \frac{d^4q}{(2\pi)^4} e^{-iq_\mu(x'' - x''')} \left[ \frac{\mathbb{I} + \mathcal{G}(q)}{q_\mu q^\mu - m^2 + i\epsilon} \right]
\]

(A13)

To decipher the mass dimensionality, let $\mathcal{D}_\Lambda$ be the mass dimensionality of $\Lambda(x)$. Then the left-hand side of the above equation has mass dimension $2\mathcal{D}_\Lambda$. As for the right hand side, the mass dimensionality is 2. This gives $\mathcal{D}_\Lambda = 1$.

Similarly, a simple computation shows that \[ \langle |T_{\#}[\Lambda(x') \xrightarrow{\Lambda} (x)] | \rangle = \langle |T_{\#}[\Lambda(x') \xrightarrow{\Lambda} (x)] | \rangle \]. As such, $\mathcal{D}_\Lambda = 1$.

Applying the operator $[\partial^\mu \partial'_\mu + m^2]$ from the left on both sides of Eqn. (A14) gives

\[
\langle |T_{\#}[\Lambda(x') \xrightarrow{\Lambda} (x)] | \rangle = -i\delta^4(x'' - x''')
\]

(A15)

In comparison, for the Dirac field

\[
\langle |T[\Psi(x') \Psi(x) ] | \rangle = i \int \frac{d^4q}{(2\pi)^4} e^{-iq_\mu(x'' - x''')} \left[ \frac{\mathbb{I} + \gamma_\mu q_\mu}{q_\mu q^\mu - m^2 + i\epsilon} \right]
\]

(A16)

This well-known result gives, $\mathcal{D}_\Psi = \frac{3}{2}$. The reader is reminded that the $\gamma^\mu q_\mu$ structure appears here through the spins sums which, in the logical framework of this communication, do not invoke any wave equation or a Lagrangian density. Applying the operator $[i\gamma^\mu \partial'_\mu - m]$ from the left on both sides of Eqn. (A16) yields

\[
[i\gamma^\mu \partial'_\mu - m] \langle |T[\Psi(x') \Psi(x) ] | \rangle = i\delta^4(x'' - x''')
\]

(A17)
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36 This part of the result remains the same as that reported in Refs. 8-8.
37 In fact $\Psi(x)$ ceases to transform correctly under Poincaré transformations if these phases are chosen at random.
38 The Grassmann field is not to be confused with a quantum field of the operator formalism. The latter carries a precise meaning as explained in detail by Weinberg [10, Ch. 5]. This distinction, however (to a reader’s confusion), seems to disappear in the closing volume of the Weinberg’s trilogy [34, pp. 58-59].
39 The question that this exercise cannot be implemented with self conjugate spinors alone shall be addressed below.
40 An exception being Weinberg’s trilogy on the theory of quantum fields [10, 34, 35].
41 The quote here is from Ramond’s primer.
42 In a separate calculation we have taken the rest spinors $\phi^+_e(\epsilon)$ to be eigenspinors of the operator $(\sigma/2) \cdot \vec{z}$ and confirmed the results reported here. In that event the $\phi^e_k$ must carry certain global phases.
43 If one wishes, one could replace the $\pm$ signs (after the symbol $\Rightarrow$) that appear in Eqs. 16a and 16d, by four different phases of the form exp$(i\theta_{\pm})$ to be eigenspinors of the operator $(\sigma/2) \cdot \vec{z}$ and confirmed the results reported here. In that event the $\phi^e_k$ must carry certain global phases.
44 The accompanying $x_e$ and $y_e$ axis help to define a preferred frame.
45 If one prefers, one may wish to consider $\lambda(x)$ and $\lambda(x)$ as two mathematical objects; and not identify them as ‘quantum fields’ in the sense of Weinberg’s discourse in the opening volume of his trilogy on quantum fields. Similar remarks apply to the use of terminology that we borrow from the well-known quantum fields.
46 We parenthetically remark that, the interactions with the standard model gauge fields – with $F^{SM}_{\mu\nu}(x)$ symbolically repre-
senting the associated field strength tensors – through Pauli terms

\[ \mathcal{L}^{\text{Pauli}}(x) = \bar{x}(x) [\gamma^\mu, \gamma^\nu] \lambda(x) F^{\text{SM}}_{\mu\nu}(x), \text{ etc.} \]

may in principle exist. However, we consider them to have vanishing coupling strength as \( \mathcal{L}^\Lambda(x) \) and \( \mathcal{L}^\lambda(x) \) do not carry invariance under SM gauge transformations.

[47] The substitution through \( q^\mu \) requires some discussion; see Sec. 6.2 of Ref. [10] for details.