IRREDUCIBLE REPRESENTATIONS OF BRAID GROUPS OF CORANK TWO

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Abstract. This paper is the first part of a series of papers aimed at improving the classification by Formanek of the irreducible representations of Artin braid groups of small dimension. In this paper we classify all the irreducible complex representations \( \rho \) of Artin braid group \( B_n \) with the condition \( \text{rank}(\rho(\sigma_i) - 1) = 2 \) where \( \sigma_i \) are the standard generators. For \( n \geq 7 \) they all belong to some one-parameter family of \( n \)-dimensional representations.

1. Introduction.

In his paper [3] Edward Formanek classified all irreducible complex representations of Artin braid groups \( B_n \) of dimension at most \( n - 1 \). This paper is the first in a series of papers aimed at extending this classification to irreducible representations of higher dimensions.

To describe our results, we need the following definition.

Definition 1.1. The corank of the representation \( \rho : B_n \to GL_r(\mathbb{C}) \) is \( \text{rank}(\rho(\sigma_i) - 1) \) where the \( \sigma_i \) are the standard generators of the group \( B_n \).

Remark 1.1. Because the \( \sigma_i \) are conjugate to each other ([2], p.655), the number \( \text{rank}(\rho(\sigma_i) - 1) \) does not depend on \( i \), which justifies the above definition.

The corank of specializations of the reduced Burau representation ([1], p.121; [2], p.338) and of the standard one-dimensional representation is 1.

By the results of Formanek ([3], Theorem 23) almost all of the irreducible complex representations \( B_n \) of degree at most \( n - 1 \) are the tensor product of a one-dimensional representation and a representation of corank 1. He also classified all the irreducible representations of corank 1 (see [3], Theorem 10). For \( n \) large enough they are one of the following.

1. A one-dimensional representation \( \chi(y) : B_n \to \mathbb{C}^* \), \( \chi(y)(\sigma_i) = y \)
2. An irreducible \((n - 1)\)-dimensional specialization of the reduced Burau representation
3. An irreducible \((n - 2)\)-dimensional specialization of the composition factor of the reduced Burau representation

The main goal of this paper is to classify all the irreducible complex representations of corank 2. Apart from a number of exceptions for \(n \leq 6\), they all are equivalent to specializations for \(u \neq 1\), \(u \in \mathbb{C}\) of the following representation \(\rho : B_n \to GL_n(\mathbb{C}[u^\pm1])\), first discovered by Dian-Ming Tong, Shan-De Yang and Zhong-Qi Ma in [6]:

\[
\rho(\sigma_i) = \begin{pmatrix}
I_{i-1} & 0 & u \\
0 & 1 & 0 \\
1 & 0 & I_{n-1-i}
\end{pmatrix},
\]

for \(i = 1, 2, \ldots, n - 1\), where \(I_k\) is the \(k \times k\) identity matrix.

The main tool we use is the friendship graph of a representation. Namely the (full) friendship graph of a representation \(\rho\) of a braid group \(B_n\) is a graph whose vertices are the set of generators \((\sigma_0, \sigma_1, \ldots, \sigma_{n-1})\) of \(B_n\). Two vertices \(\sigma_i\) and \(\sigma_j\) are joined by an edge if and only if \(\text{Im}(\rho(\sigma_i) - 1) \cap \text{Im}(\rho(\sigma_j) - 1) \neq \{0\}\).

Using the braid relations, we investigate the structure of the friendship graph. It turns out that every irreducible representation of \(B_n\) of dimension at least \(n\) and corank 2 the friendship graph is a chain, provided that \(n \geq 6\). This means that \(\sigma_i\) and \(\sigma_j\) are joined by an edge if and only if \(|i - j| = 1\).

For a given friendship graph it is relatively easy to classify all irreducible complex representations of \(B_n\) for which it is the associated friendship graph." When the graph is a chain, we get specializations of the representation discovered by Tong, Yang and Ma.

Now we are going to explain the place of this paper in the coming series. According to [3], Theorem 23, for \(n\) large enough every irreducible complex representation of \(B_n\) of dimension at most \(n - 1\) is a tensor product of a one-dimensional representation and a representation of corank 1. Using similar ideas one can show that for \(n\) large enough every irreducible complex representation of \(B_n\) of dimension at most \(n\) is a tensor product of a one-dimensional representation and a representation of corank 2. Therefore one can use the results of this paper to extend the classification theorem of Formanek to the representations of \(B_n\) of dimension \(n\). The proof of this result will appear elsewhere.
Another result, which will appear elsewhere is that for $n$ large enough there are no irreducible complex representations of $B_n$ of corank 3 and no irreducible complex representations of $B_n$ of dimension $n + 1$.

Based on the above result we would like to make the following two conjectures.

**Conjecture 1.** For every $k \geq 3$ for $n$ large enough there are no irreducible complex representations of $B_n$ of corank $k$.

**Conjecture 2.** For every $k \geq 1$ for $n$ large enough there are no irreducible complex representations of $B_n$ of dimension $n + k$.

We should also note that for the purpose of brevity we did not include in this paper some of the details of the classification of representations of $B_n$ for small $n$. The full proof can be found in our thesis [5], Chapters 6 and 7.

The paper is organized as follows. In section 2 we introduce some convenient notation that will be used throughout the rest of the paper. In section 3 we define the friendship graph of the representation and study its structure. We also study the case when the friendship graph is totally disconnected. In section 4 we prove that for $n \geq 6$ for any irreducible complex representation of $B_n$ of corank 2 and dimension at least $n$ the associated friendship graph is a chain. In section 5 we determine all irreducible representations of corank 2 whose friendship graph is a chain.

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## 2. Notation and preliminary results

Let $B_n$ be the braid group on $n$ strings. It has a presentation

\[ B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, 1 \leq i \leq n-2; \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2 \rangle. \]

**Lemma 2.1.** For the braid group $B_n$ set

\[ \tau = \sigma_1 \sigma_2 \ldots \sigma_{n-1} \text{ and } \sigma_0 = \tau \sigma_{n-1} \tau^{-1}. \]

Then:

1) (\[3\], p.655)

\[ \sigma_{i+1} = \tau \sigma_i \tau^{-1}, \]

for $1 \leq i \leq n-2$;

2) \[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \]

\[ \tau = \sigma_1 \sigma_2 \ldots \sigma_{n-1} \text{ and } \sigma_0 = \tau \sigma_{n-1} \tau^{-1}. \]
\[ \sigma_{i+1} = \tau \sigma_i \tau^{-1}, \]

and

\[ \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2 \]

for all \( i, j \) where indices are taken modulo \( n \).

**Remark 2.2.** Taking into account the above lemma, we also have the following presentation of \( B_n \):

\[ B_n = \langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}; \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2; \sigma_0 = \tau \sigma_{n-1} \tau^{-1} \rangle \]

for all \( i, j \) where indices are taken modulo \( n \) and \( \tau \) is defined as above.

Let \( \rho : B_n \to GL_r(\mathbb{C}) \) be a matrix representation of \( B_n \) with

\[ \rho(\sigma_i) = 1 + A_i, \]

and

\[ \rho(\tau) = T \in GL_r(\mathbb{C}). \]

Then for any \( i \) (indices are modulo \( n \)), the relation

\[ \tau \sigma_i \tau^{-1} = \sigma_{i+1} \]

implies that

\[ TA_i T^{-1} = A_{i+1}. \]

Hence all the \( A_i \) are conjugate to each other, so they have the same rank, spectrum and Jordan normal form.

**Lemma 2.3.** For a representation \( \rho \) of \( B_n \) with

\[ \rho(\sigma_i) = 1 + A_i, \]

we have:

1) \( A_i A_j = A_j A_i \), for \( |i - j| \geq 2 \);

2) \( A_i + A_i^2 + A_i A_{i+1} A_i = A_{i+1} + A_{i+1}^2 + A_{i+1} A_i A_{i+1} \)

for all \( i = 0, 1, \ldots, n - 1 \), where indices are taken modulo \( n \).

**Proof.** This follows easily from the relations on the generators of \( B_n \).

3. **The friendship graph.**

In this section we define and prove some properties of the *friendship graph* which is a finite graph associated with a representation of \( B_n \). Our graphs are *simple-edged*, which means that there is at most one unoriented edge joining two vertices, and no edges joining a vertex to itself.
We assume throughout this section that we have a representation
\[ \rho : B_n \to GL_r(\mathbb{C}) , \]
with
\[ \rho(\sigma_i) = 1 + A_i, \quad (i = 0, 1, \ldots, n - 1) . \]

**Definition 3.1.**

1) \( A_i, A_{i+1} \) are neighbors (indices modulo \( n \)).

2) \( A_i, A_j \) are friends if
\[ \text{Im}(A_i) \cap \text{Im}(A_j) \neq \{0\} . \]

3) \( A_i, A_j \) are true friends if either
   a) \( A_i \) and \( A_j \) are not neighbors, and
   \[ A_iA_j = A_jA_i \neq 0 ; \]
   or
   b) \( A_i \) and \( A_j \) are neighbors, and
   \[ A_i + A_i^2 + A_iA_jA_i = A_j + A_j^2 + A_jA_iA_j \neq 0 . \]

**Lemma 3.1.** If \( A, B \) are true friends, then they are friends.

**Proof.**
1) If \( A \) and \( B \) are not neighbors, then \( AB = BA \neq 0 \), so,
\[ \text{Im}(A) \cap \text{Im}(B) \supseteq \text{Im}(AB) \cap \text{Im}(BA) = \text{Im}(AB) \neq \{0\} . \]

2) If \( A \) and \( B \) are neighbors, then
\[ A(1 + A + BA) = A + A^2 + ABA = B + B^2 + BAB = B(1 + B + AB) \neq 0 , \]
and again
\[ \text{Im}(A) \cap \text{Im}(B) \supseteq \text{Im}(A + A^2 + ABA) \neq \{0\} . \]

**Definition 3.2.** The full friendship graph (associated with the representation \( \rho : B_n \to GL_n(\mathbb{C}) \)) is the simple-edged graph with \( n \) vertices \( A_0, A_1, \ldots, A_{n-1} \) and an edge joining \( A_i \) and \( A_j \) (\( i \neq j \)) if and only if \( A_i \) and \( A_j \) are friends.

The friendship graph is the subgraph with vertices \( A_1, \ldots, A_{n-1} \) obtained from the full friendship graph by deleting \( A_0 \) and all edges incident to it.

Our main interest is the friendship graph, but it is convenient to introduce the full friendship graph as a tool, because of the following lemma.
Lemma 3.2. There is an edge between $A_i$ and $A_j$ in the full friendship graph if and only if there is an edge between $A_{i+k}$ and $A_{j+k}$ where indices are taken modulo $n$. In other words, $\mathbb{Z}_n$ acts on the full friendship graph by permuting the vertices cyclically.

Proof. This follows immediately from the fact that conjugation by $T = \rho(\tau) = \rho(\sigma_1 \ldots \sigma_{n-1})$ permutes $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ cyclically (Lemma 2.1).

Lemma 3.3 (Lemma about friends). Let $A$ and $B$ be neighbors which are not friends. If $C$ is not a neighbor of $A$ and $C$ is a friend of $B$ then $C$ is a true friend of $A$.

Proof. By lemma 3.1, $A$ and $B$ are true not friends, because they are not friends, that is
\[ A + A^2 + ABA = B + B^2 + BAB = 0. \]
Consider $y \in V$ such that $Cy \in \text{Im}(B), Cy = Bz \neq 0$ ($y$ exists because $C$ and $B$ are friends). Then
\[ BACy = BABz = -(B + B^2)z = -(1 + B)Bz \neq 0 \]
because $Bz \neq 0$ and $(1 + B)$ is invertible.
So, $AC = CA \neq 0$; that is, $A$ and $C$ are true friends.

Theorem 3.4. Let $\rho : B_n \to GL_r(\mathbb{C})$ be a representation. Then one of the following holds.
(a) The full friendship graph is totally disconnected (no friends at all).
(b) The full friendship graph has an edge between $A_i$ and $A_{i+1}$ for all $i$.
(c) The full friendship graph has an edge between $A_i$ and $A_j$ whenever $A_i$ and $A_j$ are not neighbors.
Proof. Suppose neither (a) nor (b) holds. Since the graph is not totally disconnected, there is an edge joining some vertices $B$ and $C$. Since (b) does not hold, no neighbors are joined by an edge. Lemma 3.3 implies that there is an edge between $C$ and any neighbor of $B$ which is not a neighbor of $C$. It follows inductively that there is an edge joining $C$ to every vertex which is not a neighbor of $C$. Then (c) holds, because the full friendship graph is a $\mathbb{Z}_n$-graph.

Definition 3.3. The friendship graph (the full friendship graph) is a chain, if the only edges are between neighbors.

Case (b) of the above theorem can be restated as
(b) The full friendship graph contains the chain graph.

Corollary 3.5. For $n \neq 4$, the friendship graph and the full friendship graph are either totally disconnected (no edges) or connected.

Remark 3.6. For $n = 4$ there is a friendship graph which is neither totally disconnected nor connected:

By [5], Lemmas 6.2 and 6.3, every representation of $B_4$ of corank 2 and dimension at least 4, which has this friendship graph, is reducible.

Now consider the case when the friendship graph is totally disconnected (that is, statement (a) of theorem 3.4 holds).

Lemma 3.7. If $A$ and $B$ are neighbors and not friends then:
(a) $A^2B = AB^2$; $BA^2 = B^2A$.
(b) If $x \in \text{Im}(A) \cap \text{Ker}(A - \lambda I)$, then $B(Bx) = \lambda(Bx)$ and $ABx = -(1 + \lambda)x$.

Proof. (a). By lemma 3.1, $A$ and $B$ are not true friends, so

$$A + A^2 + AB = B + B^2 + BAB = 0.$$
Multiplying the left hand side on the right by $B$ and the right hand side on the left by $A$ gives
\[ AB + A^2B + ABAB = 0 = AB + AB^2 + ABAB. \]
Thus, $A^2B = AB^2$; by a symmetric argument $BA^2 = B^2A$.

(b) Let $x = Ay \in \text{Im}(A) \cap \text{Ker}(A - \lambda I)$. Then
\[ B(Bx) = B^2Ay = BA^2y = BAx = \lambda Bx, \]
and
\[ 0 = (A + A^2 + ABA)y = (1 + A + AB)x = (1 + \lambda)x + ABx. \]
Thus, $ABx = -(1 + \lambda)x$.

**Theorem 3.8.** Let $\rho : B_n \to GL_r(\mathbb{C})$, $(n \geq 2)$ be an irreducible representation, whose associated friendship graph is totally disconnected. Then $r = \text{dim} V \leq n - 1$.

**Proof.** If $A_i = 0$, $\rho$ is a trivial representation and $r = 1$.
If $A_i \neq 0$, choose an eigenvalue $\lambda$ for $A_1$ and a non-zero vector
\[ x_1 \in \text{Im}(A_1) \cap \text{Ker}(A_1 - \lambda I). \]
Set $x_2 = A_2x_1, x_3 = A_3x_2, \ldots, x_{n-1} = A_{n-1}x_{n-2}, U = \text{span}\{x_1, x_2, \ldots, x_{n-1}\}$.
By induction and lemma 3.7 (b) $x_i \in \text{Im}(A_i) \cap \text{Ker}(A_i - \lambda I)$.

Let $x_i = A_iy_i$. Then by lemma 3.7 (b) and the fact that $A_iA_j = A_jA_i = 0$, if $i$ and $j$ are not neighbors,
\[ A_{i-1}x_i = A_{i-1}A_ix_{i-1} = -(1 + \lambda)x_{i-1}, \quad i = 2, \ldots, n - 1, \]
\[ A_i x_i = \lambda x_i, \quad i = 1, \ldots, n - 1, \]
\[ A_{i+1} x_i = x_{i+1}, \quad i = 1, \ldots, n - 2, \]
and
\[ A_j x_i = A_jA_iy_i = 0 \quad j \neq i - 1, i, i + 1. \]

Thus $U$ is invariant under $B_n$. Hence $r = \text{dim} U \leq n - 1$, since $\rho$ is irreducible.

**Corollary 3.9.** Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where $r = \text{dim} V \geq n, n \neq 4$.
Then the associated friendship graph is connected.

**Proof.** By corollary 3.5 the friendship graph of $\rho$ is either totally disconnected or connected. By theorem 3.8 it is not disconnected.

**Corollary 3.10.** Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where $r = \text{dim} V \geq n, n \neq 4$. Suppose $\rho(\sigma_i) = 1 + A_i$, where $\text{rank}(A_i) = k$.
Then $r = \text{dim} V \leq (n - 1)(k - 1) + 1$.
In particular, for $k = 2$, $r = \text{dim} V = n$, where $V = \mathbb{C}^n$. 

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Proof. By corollary 3.9, the friendship graph of the representation is connected. Arrange the vertices of the graph in a sequence $A_{i_1}, A_{i_2}, \ldots, A_{i_{n-1}}$ such that each term $A_{i_j}$, $2 \leq j \leq n - 1$, is a friend of one of the terms $A_{i_1}, A_{i_2}, \ldots, A_{i_{j-1}}$. Then

$$\dim(\text{Im}(A_{i_1})) = k$$

$$\dim(\text{Im}(A_{i_1}) + \text{Im}(A_{i_2})) \leq k + k - 1 = 2k - 1$$

$$\ldots$$

$$\dim(\text{Im}(A_{i_1}) + \cdots + \text{Im}(A_{i_{n-1}})) \leq k + (n-2)(k-1) = (n-1)(k-1)+1.$$

Combining Theorem 3.4 and Corollaries 3.9 and 3.10, we get the following

**Theorem 3.11.** Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where $r = \dim V \geq n$, $n \neq 4$. Suppose $\rho(\sigma_i) = 1 + A_i$, where $\text{rank}(A_i) = 2$.

Then $r = n$ and one of the following holds.

(a) The full friendship graph has an edge between $A_i$ and $A_{i+1}$ for all $i$.

(b) The full friendship graph has an edge between $A_i$ and $A_j$ whenever $A_i$ and $A_j$ are not neighbors.

4. For corank 2 the friendship graph is a chain.

In this section, we assume throughout that we have an irreducible representation

$$\rho : B_n \to GL_r(\mathbb{C}),$$

where $r \geq n$, and

$$\rho(\sigma_i) = 1 + A_i, \text{ rank}(A_i) = 2, \; 1 \leq i \leq n - 1.$$

**Theorem 4.1.** Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $r \geq n$ and $n \geq 6$. Let $\text{rank}(A_1) = 2$.

Then $\text{Im}(A_i) \cap \text{Im}(A_{i+1}) \neq \{0\}$ for $1 \leq i \leq n - 2$; that is the friendship graph of $\rho$ contains the chain graph.

**Proof.** Suppose not. Then by Theorem 3.11(b), $\text{Im}(A_i) \cap \text{Im}(A_j) \neq 0$ whenever $A_i$ and $A_j$ are not neighbors. Consider

$$U = \text{Im}(A_1) + \text{Im}(A_2) + \text{Im}(A_3).$$

Since $\text{Im}(A_1) \cap \text{Im}(A_3) \neq 0$, $\dim U \leq 5$.

For $i = 4, \ldots, n - 1$, let $a_i$, $b_i$ be, respectively, nonzero elements of $\text{Im}(A_1) \cap \text{Im}(A_i)$ and $\text{Im}(A_2) \cap \text{Im}(A_i)$. Since $\text{Im}(A_1) \cap \text{Im}(A_2) = 0$,
$a_i$ and $b_i$ are linearly independent, so they are a basis for $\text{Im}(A_i)$, and $\text{Im}(A_i) \subseteq \text{Im}(A_1) + \text{Im}(A_2)$. Thus

$$U = \text{Im}(A_1) + \text{Im}(A_2) + \cdots + \text{Im}(A_{n-1}),$$

which is invariant under $\rho(B_n)$. Thus $r \leq 5$, by the irreducibility of $\rho$, a contradiction with $r \geq n \geq 6$.

**Remark 4.2.** For $n = 5$ and $\rho$ satisfying the hypothesis of theorem 4.1 there are two possible friendship graphs: 1) all neighbors are friends and 2) an exceptional case:

![Friendship Graph](image)

By [5], Theorem 7.1, part 2, every irreducible representation with the above friendship graph is equivalent to the restriction to $B_5$ of the Jones’ representation (see [3], p. 296).

**Lemma 4.3.** Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $r \geq n$, $n \geq 5$, and $\text{rank}(A_1) = 2$. Suppose that the associated friendship graph contains the chain.

Then $r = n$ and the associated friendship graph is the chain (that is, the only edges are between neighbors).

**Proof.** By corollary 3.10, $r = n$. Consider the full friendship graph of $\rho$. Then

$$\text{Im}(A_i) \cap \text{Im}(A_{i+1}) \neq \{0\}$$

for any $i$ where indices are taken modulo $n$. If $\text{Im}(A_i) \cap \text{Im}(A_{i+1})$ is two-dimensional, then $\text{Im}(A_1) = \text{Im}(A_2) = \cdots$, and $\text{Im}(A_1)$ is a two-dimensional invariant subspace, contradicting the irreducibility of $\rho$. Hence $\text{Im}(A_i) \cap \text{Im}(A_{i+1})$ are one-dimensional.

For any $x \in \text{Im}(A_i)$, $x = A_i y$, $x \neq 0$, we have that

$$Tx = TA_i y = TA_i T^{-1}(Ty) = A_{i+1}(Ty) \in \text{Im}(A_{i+1})$$

for $T = \rho(\tau)$. Moreover, $Tx \neq 0$ because $T$ is invertible.

Choose $x_1 \neq 0$ to be a basis vector for $\text{Im}(A_1) \cap \text{Im}(A_2)$. Define $x_{i+1} = T^i x_1$ for $1 \leq i \leq n - 1$. Then $x_i$ is a basis vector for $\text{Im}(A_i) \cap \text{Im}(A_{i+1})$. 

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If for some $i$, $x_i$ is proportional to $x_{i+1}$ then, because a full friendship graph is a $\mathbb{Z}_n$-graph, all the $x_j$ are proportional to $x_1$. Then, because we have 5 or more vertices in the full friendship graph, for any $A_i$ there exists $j$ such that both $A_j$ and $A_{j+1}$ are not neighbors of $A_i$. Then

$$A_iA_j = A_jA_i$$

and

$$A_iA_{j+1} = A_{j+1}A_i.$$ 

So, if $x \in \text{Im}(A_j) \cap \text{Im}(A_{j+1})$ then $A_ix \in \text{Im}(A_j) \cap \text{Im}(A_{j+1})$. But this means that $\text{span}\{x_1\}$ is an invariant subspace and the representation is not irreducible.

So, if the representation is irreducible, then for any $i$, $x_i /\notin \text{span}\{x_{i+1}\}$. From this follows that for any $i$

$$\text{Im}(A_i) = \text{span}\{x_{i-1}, x_i\}$$

and the $n$ vectors $x_0, x_1, \ldots, x_{n-1}$ form a basis of $V$. Then for any two non-neighbors $A_i$ and $A_j$

$$\text{Im}(A_i) \cap \text{Im}(A_j) = \{0\}.$$

Now, we have the following

**Theorem 4.4.** Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where $r \geq n$. Suppose that for any generator $\sigma_i$, $\rho(\sigma_i) = 1 + A_i$, where $\text{rank}(A_i) = 2$.

1) If $n \geq 6$, then $r = n$ and $\rho$ has a friendship graph which is a chain.

2) If $n = 5$, then $r = 5$ and either $\rho$ has a friendship graph which is a chain or $\rho$ has the exceptional friendship graph (see Remark 4.2).

3) If $n = 4$, then either $r = 4$ and $\rho$ has a friendship graph which is a chain; or $\rho$ has one of the following exceptional friendship graphs:

![Exceptional Friendship Graphs](image)

**Proof.** 1) If $n \geq 6$, then by theorem 4.1 the associated friendship graph contains a chain, and, by lemma 4.3 has no other edges and $r = n$.

2) If $n = 5$, then by corollaries 3.9 and 3.10 the friendship graph of $\rho$ is connected and $r = n$. If it contains a chain graph, then, by lemma
3. It has no other edges. If it does not contain a chain graph, we obtain the exceptional case.

3) If \( n = 4 \), then by theorem 3.8 the friendship graph is not totally disconnected. Hence, we have only three possible \( \mathbb{Z}_4 \)-graphs on 4 vertices.

**Remark 4.5.** It is proven in [5], Chapter 6, that any representation of \( B_4 \) with either of the exceptional friendship graphs in 3) of the above theorem is reducible.

## 5. Representations whose friendship graph is a chain

**Definition 5.1.** The **standard representation** is the representation

\[ \tau_n : B_n \rightarrow GL_n(\mathbb{Z}[t^\pm 1]) \]

defined by

\[
\tau_n(\sigma_i) = \begin{pmatrix}
I_{i-1} & 0 & t \\
0 & t & 0 \\
1 & 0 & I_{n-1-i}
\end{pmatrix},
\]

for \( i = 1, 2, \ldots, n - 1 \), where \( I_k \) is the \( k \times k \) identity matrix.

**Theorem 5.1.** Let \( \rho : B_n \rightarrow GL_n(\mathbb{C}) \) be an irreducible representation, where \( n \geq 4 \). Suppose that \( \rho(\sigma_1) = 1 + A_1 \), where \( \text{rank}(A_1) = 2 \), and the associated friendship graph of \( \rho \) is a chain.

Then \( \rho \) is equivalent to a specialization \( \tau_n(u) \) of the standard representation for some \( u \in \mathbb{C}^\ast \).

Before proving the theorem, we will need the following technical lemma:

**Lemma 5.2.** Let \( A \) be a friend and a neighbor of \( B \), \( B \) be a friend and a neighbor of \( C \) and suppose that \( A \) is not a friend of \( C \):

\[ A A A A A
\]

\[
1 \quad 2 \quad 3 \quad \ldots \quad n-2 \quad n-1
\]
Let \( a \neq 0 \) be such that \( \text{span}\{a\} = \text{Im}(A) \cap \text{Im}(B) \), and let \( b = (1 + B)a \). Then:

1) \( \text{span}\{b\} = \text{Im}(C) \cap \text{Im}(B) \).

2) \( (1 + B)b \in \text{span}\{a\} \) and \( (1 + B)b \neq 0 \).

3) The vectors \( a \) and \( b \) are linearly independent.

**Proof.** First of all, notice that the vector \( b \) is non-zero, because \( 1 + B \) is invertible and \( a \neq 0 \).

1) \( b = (1 + B)a \in \text{Im}(B) \), because \( a \in \text{Im}(B) \).

\( A \) and \( C \) are not friends, that is \( CA = 0 \), so \( Ca = 0 \).

Let \( a = Ba_1 \). Then

\[
(1+B)a = (1+B+BC)a = (1+B+BC)Ba_1 = (B+B^2+BCB)a_1 = (C+C^2+CBC)a_1 \in \text{Im}(C);
\]

that is, \( b \in \text{Im}(C) \cap \text{Im}(B) \), and because \( \text{Im}(C) \cap \text{Im}(B) \) is one-dimensional and \( b \neq 0 \),

\[
\text{span}\{b\} = \text{Im}(C) \cap \text{Im}(B).
\]

2) Clearly, \( (1 + B)b \in \text{Im}(B) \).

Note, that \( Ab = 0 \), as \( b \in \text{Im}(C) \) by the above, and \( AC = 0 \). Let \( b = Ba' \). Then

\[
(1+B)b = (1+B+BA)b = (1+B+BA)Ba' = (A+A^2+ABA)a' \in \text{Im}(A).
\]

3) \( a \in \text{Im}(A) \), \( b \in \text{Im}(C) \) by part 1), and \( \text{Im}(A) \cap \text{Im}(C) = \{0\} \) by the hypothesis of the lemma.

**Proof of Theorem 5.1** We include the redundant generator \( \sigma_0 \), and indices are modulo \( n \). Consider \( \text{Im}(A_i) \cap \text{Im}(A_{i+1}) \), which is 0, 1, or 2-dimensional. It is nonzero, because of the hypothesis that the friendship graph is a chain. It is not 2-dimensional, for then

\[
\text{Im}(A_0) = \text{Im}(A_1) = \cdots = \text{Im}(A_{n-1})
\]

would be a 2-dimensional invariant subspace, contradicting the irreducibility of \( \rho \). Hence, \( \text{Im}(A_i) \cap \text{Im}(A_{i+1}) \) is one-dimensional.

Let \( a_0 \) be a basis vector for \( \text{Im}(A_0) \cap \text{Im}(A_1) \). Let

\[
a_1 = (1+A_1)a_0, \ a_2 = (1+A_2)a_1, \ \ldots, \ a_{n-1} = (1+A_{n-1})a_{n-2}.
\]
By induction and lemma 5.2, part 1), $a_i$ is a basis vector for $\text{Im}(A_i) \cap \text{Im}(A_{i+1})$, for $0 \leq i \leq n - 1$. By lemma 5.2, part 3), $a_i$ and $a_{i+1}$ are linearly independent. Thus $\{a_i, a_{i+1}\}$ is a basis for $\text{Im}(A_i)$.

Since
\[
\text{span}\{a_0, \ldots, a_{n-1}\} = \text{Im}(A_1) + \cdots + \text{Im}(A_{n-1})
\]
is invariant under $B_n$ and $\rho$ is an $n$–dimensional irreducible representation, $\{a_0, \ldots, a_{n-1}\}$ is a basis for $\mathbb{C}^n$.

We now wish to determine the action of $\rho(\sigma_1), \rho(\sigma_2), \ldots, \rho(\sigma_{n-1})$ on this basis.

Consider $a_i \in \text{Im}(A_i) \cap \text{Im}(A_{i+1})$. If $j \neq i$, $i + 1$, then $A_j$ is not a neighbor of one of $A_i$, $A_{i+1}$ (since $n \geq 4$), say $A_k$, and then $A_k A_j = A_j A_k = 0$, so $A_j a_i = 0$, and

\[
\rho(\sigma_j) a_i = (1 + A_j) a_i = a_i.
\]

By our construction

\[
\rho(\sigma_{i+1}) a_i = (1 + A_{i+1}) a_i = a_{i+1}
\]

for $0 \leq i \leq n - 2$.

By lemma 5.2, part 2),

\[
\rho(\sigma_i) a_i = (1 + A_i) a_i = u_i a_{i-1},
\]

for $1 \leq i \leq n - 1$, where $u_i \in \mathbb{C}^*$. By the above calculations the matrices of $\rho(\sigma_1), \ldots, \rho(\sigma_{n-1})$ with respect to the basis $a_0, a_1, \ldots, a_{n-1}$ are

\[
\rho(\sigma_i) = \begin{pmatrix}
I_{i-1} & 0 & u_i \\
0 & 1 & 0 \\
1 & 0 & I_{n-1-i}
\end{pmatrix},
\]

for $i = 1, 2, \ldots, n - 1$, where $I_k$ is the $k \times k$ identity matrix, and $u_1, \ldots, u_{n-1} \in \mathbb{C}^*$. Since $\sigma_1, \ldots, \sigma_{n-1}$ are conjugate in $B_n$, the $u_i$ are all equal, and we have the standard representation.

Now let us consider when the standard representation is irreducible.

**Lemma 5.3.** If $u = 1$ then $\tau_n(u)$ is reducible.

**Proof.** If $u = 1$ then the vector $v = (1, 1, 1, \ldots, 1)^T$ is a fixed vector.

**Lemma 5.4.** If $u \neq 1$ then $\tau_n(u)$ is irreducible.
Proof. We need to prove that starting from any non-zero vector 
\[ x = \sum a_i e_i, \]
we can generate the whole space. Obviously, it is enough to show that we can generate one of the standard basis vectors \( e_i \). To do this, take \( i \) such that \( a_i \neq 0 \). Consider the operator
\[ H = A + A^2 + ABA = B + B^2 + BAB, \]
where \( A = \rho(\sigma_{i-1}) \) and \( B = \rho(\sigma_i) \). By a direct calculation \( Hx = (u - 1)a_i e_i \). Because \( u \neq 1 \) the vector \( Hx \) is a non-zero multiple of \( e_i \).

Now, we have the main result of this paper:

**Theorem 5.5** (The Main Theorem). Let \( \rho : B_n \to GL_r(\mathbb{C}) \) be an irreducible representation of \( B_n \) for \( n \geq 6 \). Let \( r \geq n \), and let \( \rho(\sigma_1) = 1 + A_1 \) with \( \text{rank}(A_1) = 2 \).

Then \( r = n \) and \( \rho \) is equivalent to the following representation :
\[ \tau : B_n \to GL_n(\mathbb{C}), \]

\[ \rho(\sigma_i) = \begin{pmatrix} I_{i-1} & 0 & u \\ 0 & 1 & 0 \\ 1 & 0 & I_{n-1-i} \end{pmatrix}, \]

for \( i = 1, 2, \ldots, n-1 \), where \( I_k \) is the \( k \times k \) identity matrix, and \( u \in \mathbb{C}^* \), \( u \neq 1 \). These representations are non-equivalent for different values of \( u \).

**Proof.** By Theorem \[ \[ \text{4.4} \] the friendship graph of \( \rho \) is a chain. Then, by theorem \[ \[ \text{5.1} \] \( \rho \) is equivalent to a standard representation \( \tau(u) \) for some \( u \in \mathbb{C}^* \). By Lemmas \[ \[ \text{5.3} \] and \[ \[ \text{5.4} \] \( u \neq 1 \).

Combining Theorem \[ \[ \text{5.3} \] and the classification theorem of Formanek (see \[ \[ \text{3} \], Theorem 23), we get the following

**Corollary 5.6.** Let \( \rho : B_n \to GL_r(\mathbb{C}) \) be an irreducible representation of \( B_n \) for \( n \geq 7 \). Let \( \text{corank}(\rho) = 2 \).

Then \( \rho \) is equivalent to a specialization of the standard representation \( \tau_n(u) \), for some \( u \neq 1, u \in \mathbb{C}^* \).

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