GIVENTAL’S NON-LINEAR MASLOV INDEX ON LENS SPACES
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1. Introduction

A discriminant point of a contactomorphism \( \phi \) of a co-oriented contact manifold \((V, \xi)\) is a point \( p \) of \( V \) such that \( \phi(p) = p \) and \((\phi^*\alpha)_p = \alpha_p \) for some (hence any) contact form \( \alpha \) for \( \xi \); the discriminant of \((V, \xi)\) is the space of those contactomorphisms that have at least one discriminant point. Givental’s non-linear Maslov index [Gi90] assigns to any contact isotopy \( \{\phi_t\} \) of real projective space \( \mathbb{R}P^{2n-1} \) with its standard contact structure an integer \( \mu(\{\phi_t\}) \) that is defined using generating functions and can be interpreted as an intersection index of the path \( \{\phi_t\} \) in the contactomorphism group with (a certain subspace of) the discriminant. This number only depends on the homotopy class of \( \{\phi_t\} \) with fixed endpoints, and thus defines a map

\[
\mu : \widetilde{\text{Cont}}_0(\mathbb{R}P^{2n-1}) \to \mathbb{Z}
\]
on the universal cover of the identity component of the contactomorphism group. It follows from [Gi90, Theorem 9.1] that \( \mu \) is a quasimorphism, i.e. a homomorphism up to a bounded error (cf. Ben Simon [BeS07]). While quasimorphisms on Hamiltonian groups were studied by several authors, starting with Biran, Entov and Polterovich [BEP04] and Entov and Polterovich [EnP03], Givental’s non-linear Maslov index and its reductions studied by Borman and Zapolsky [BZ15] are the only known non-trivial quasimorphisms on contactomorphism groups.

In [Gi90] Givental also studied intersections with the discriminant in two other related settings. One is a space of Legendrian submanifolds of \( \mathbb{R}P^{2n-1} \), with discriminant given by those Legendrians
that intersect a fixed one. The second setting (that was also studied by Théret [Th98]) is the Hamiltonian group of complex projective space \( \mathbb{CP}^{n-1} \) with the Fubini–Study symplectic form; in this case the discriminant is formed by Hamiltonian diffeomorphisms of \( \mathbb{CP}^{n-1} \) that lift to contactomorphisms of \( S^{2n-1} \) having discriminant points. The applications of the non-linear Maslov index that were discussed already in [Gi90] include a proof of the Arnold conjecture for fixed points of Hamiltonian diffeomorphisms and for Lagrangian intersections in \( \mathbb{CP}^{n-1} \), results about existence of Reeb chords between Legendrians in \( \mathbb{RP}^{2n-1} \) that are Legendrian isotopic to each other, and a proof of the chord and Weinstein conjectures for \( \mathbb{RP}^{2n-1} \).

After the work of Givental, discriminant points appeared again more recently in proofs (based on generating functions) of other contact rigidity results. In [Bh01] Bhupal used the rigidity of discriminant points to define a partial order on the identity component of the group of compactly supported contactomorphisms of the standard contact Euclidean space \( \mathbb{R}^{2n+1} \). Elaborating on the work of Bhupal, the fourth author obtained a new proof of the contact non-squeezing theorem of Eliashberg, Kim and Polterovich [EKP06], and a construction of an integer-valued bi-invariant metric on the identity component of the group of compactly supported contactomorphisms of \( \mathbb{R}^{2n} \times S^1 \) ([Sa11a] and [Sa10] respectively). In these works the role played by discriminant and translated points is made more explicit, and appears to be similar to the one described in [Gi90]. Recall from [Sa11a, Sa12] that a point \( p \) of a contact manifold \( (V,\xi) \) is said to be a translated point of a contactomorphism \( \phi \) with respect to a contact form \( \alpha \) for \( \xi \) if \( p \) is a discriminant point of \( \varphi^\alpha_\eta \circ \phi \) for some real number \( \eta \) (called the time-shift), where \( \varphi^\alpha_\eta \) denotes the Reeb flow. In terms of the Legendrian graph \( \text{gr}(\phi) \) in the contact product \( V \times V \times \mathbb{R} \), discriminant and translated points correspond, respectively, to intersections and Reeb chords between \( \text{gr}(\phi) \) and the diagonal \( \Delta \times \{0\} = \text{gr}(\text{id}) \). By Weinstein’s theorem, if \( \phi \) is \( C^1 \)-close to the identity then \( \text{gr}(\phi) \) can be identified with the 1-jet of a function \( f \) on \( V \); then translated points of \( \phi \) correspond to critical points of \( f \), and discriminant points to critical points of critical value zero. It follows from this local description that for such a \( \phi \) translated points always exist, while discriminant points can be removed by a small perturbation. On the other hand, Givental’s non-linear Maslov index for \( \mathbb{RP}^{2n-1} \) and the bi-invariant metric for \( \mathbb{R}^{2n} \times S^1 \) defined in [Sa10] show that there exist contact isotopies that must intersect the discriminant at least a certain number of times, and these intersections cannot be removed by perturbing the contact isotopy in the same homotopy class with fixed endpoints. This rigidity of discriminant points is also the main ingredient in [Sa11a] for the construction of a contact capacity for domains of \( \mathbb{R}^{2n} \times S^1 \), and for the proof of the contact non-squeezing theorem. Elaborating on this idea, the fourth author, in collaboration with Colin, defined [CS12] bi-invariant (pseudo)metrics (the discriminant metric and the oscillation pseudometric) on the universal cover of the identity component of the contactomorphism group of any compact contact manifold, and, using results from [Gi90], proved that both are unbounded in the case of \( \mathbb{RP}^{2n-1} \). Previously, Givental’s non-linear Maslov index was also used by Eliashberg and Polterovich [EP00] to show that \( \mathbb{RP}^{2n-1} \) is orderable, i.e. it does not admit any positive contractible loop of contactomorphisms\(^1\), and by the fourth author [Sa11c] to prove that any contactomorphism of \( \mathbb{RP}^{2n-1} \) isotopic to the identity has at least \( 2n \) translated points.

In the present article we give an analogue for lens spaces of the construction of Givental’s non-linear Maslov index and its applications. This is the first step of a more general program: study the contact rigidity phenomena mentioned above in the case of prequantizations of symplectic toric manifolds, by extending to the contact case techniques from [Gi95]. Note that although \( \mathbb{RP}^{2n-1} \) and \( S^{2n-1} \) are both prequantizations of \( \mathbb{CP}^{n-1} \), and the former is the quotient of the latter by the

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\(^1\)Recall from [EP00] that a contact isotopy of a co-oriented contact manifold is said to be positive (non-negative) if it moves every point in a direction positively transverse (or tangent) to the contact distribution, and that a compact co-oriented contact manifold \( (V,\xi) \) is said to be orderable if the relation \( \leq \) on \( \text{Cont}_0(V,\xi) \) defined by posing

\[
[\{\phi\}] \leq [\{\psi\}] \text{ if } [\{\psi_t\}] : [\{\phi_t\}]^{-1} \text{ can be represented by a non-negative contact isotopy}
\]

is a partial order. By [EP00, Criterion 1.2.C], this is equivalent to asking that \( (V,\xi) \) does not admit any positive contractible loop of contactomorphisms. Recall also that the oscillation pseudometric on a compact co-oriented contact manifold \( (V,\xi) \) is a metric if and only if \( (V,\xi) \) is orderable [CS12].
antipodal \(\mathbb{Z}_2\)-action, \(S^{2n-1}\) (for \(n > 1\)) is not orderable \([EKP06]\) and does not admit non-trivial quasimorphisms and unbounded bi-invariant metrics \([FPR12]\). In this work we prove that lens spaces (in particular those that are prequantizations of \(\mathbb{C}P^{n-1}\)) behave rather as \(\mathbb{R}P^{2n-1}\); in spite of the fact that the ring structure of the cohomology of general lens spaces is different from that of projective space, which affects the proofs of some of the key properties of the topological invariant that is used in the construction, we show that it is still possible to define the non-linear Maslov index and use it to extend to lens spaces the applications described above.

Let \(k \geq 2\) be an integer and \(w = (w_1, \ldots, w_n)\) an \(n\)-tuple of positive integers that are relatively prime to \(k\). The lens space \(L_k^{2n-1}(w)\) is the quotient of the unit sphere \(S^{2n-1}\) in \(\mathbb{C}^n\) by the free \(\mathbb{Z}_k\)-action generated by the map

\[
(z_1, \ldots, z_n) \mapsto (e^{\frac{2\pi i w_1}{k}z_1}, \ldots, e^{\frac{2\pi i w_n}{k}z_n}).
\]

We equip the lens space \(L_k^{2n-1}(w)\) with its standard contact structure, i.e. the kernel of the contact form whose pullback to \(S^{2n-1}\) is equal to the pullback from \(\mathbb{R}^{2n}\) of the 1-form \(\sum_{j=1}^n (x_j dy_j - y_j dx_j)\). We denote by \(\{r_i\}\) the Reeb flow on \(L_k^{2n-1}\) with respect to this contact form.

Throughout the article, when the weights are not relevant in the discussion we denote the lens space \(L_k^{2n-1}(w)\) simply by \(L_k^{2n-1}\). As usual, we see the universal cover \(\widehat{\text{Cont}}_0(L_k^{2n-1})\) of the identity component of the contactomorphism group as the space of contact isotopies starting at the identity modulo smooth 1-parameter families with fixed endpoints; the group operation is given by \([\{\phi_1\}] \cdot [\{\psi_1\}] = [\{\phi_1 \circ \psi_1\}]\).

Our main result is the following theorem.

1.2. Theorem. For any lens space \(L_k^{2n-1}\) with its standard contact structure there is a map

\[\mu: \widehat{\text{Cont}}_0(L_k^{2n-1}) \to \mathbb{Z}\]

such that \(\mu([\{r_{2\pi it}\}_{t \in [0,1]}]) = 2nl\) for every integer \(l\), and with the following properties:

(i) (Quasimorphism.) For any two elements \([\{\phi_1\}]\) and \([\{\psi_1\}]\) of \(\widehat{\text{Cont}}_0(L_k^{2n-1})\) we have

\[|\mu([\{\phi_1\}] \cdot [\{\psi_1\}] - \mu([\{\phi_1\}]) - \mu([\{\psi_1\}])| \leq 2n + 1\,.
\]

(ii) (Positivity.) If \(\{\phi_1\}\) is a non-negative contact isotopy then \(\mu([\{\phi_1\}]) \geq 0\). If \(\{\phi_1\}\) is a positive contact isotopy then \(\mu([\{\phi_1\}]) > 0\).

(iii) (Relation with discriminant points.) Let \(\{\phi_1\}_{t \in [0,1]}\) be a contact isotopy of \(L_k^{2n-1}\), and \([t_0, t_1]\) a subinterval of \([0,1]\). If \(\mu([\{\phi_1\}_{t \in [t_0, t_1]}]) \neq \mu([\{\phi_1\}_{t \in [t_0, t_1]}])\) then there is \(\ell \in [t_0, t_1]\) such that \(\phi_1\) belongs to the discriminant. If there is only one such \(\ell\) then the following holds: if \(\phi_2\) has only finitely many discriminant points then

\[|\mu([\{\phi_1\}_{t \in [0, t_0]}]) - \mu([\{\phi_1\}_{t \in [0, t_1]}])| \leq 2;\]

if all discriminant points of \(\phi_2\) are non-degenerate \(\mu([\{\phi_1\}_{t \in [0, t_0]}]) - \mu([\{\phi_1\}_{t \in [0, t_1]}])| \leq 1.\]

The map \(\mu: \widehat{\text{Cont}}_0(L_k^{2n-1}) \to \mathbb{Z}\) (the \textit{non-linear Maslov index}) is defined at the beginning of Section 4, using the material that is developed in Sections 2 and 3. The quasimorphism property is proved in Proposition 4.2, positivity in Proposition 4.14 and the relation with discriminant points in Proposition 4.8. The calculation for the Reeb flow is presented in the discussion that leads to Example 4.6.

Theorem 1.2 allows to extend the applications of the non-linear Maslov index to the case of lens spaces, giving the following results (see Section 5).

\(^2\text{Recall from [Sa11c] that a discriminant point } p \text{ of a contactomorphism } \phi \text{ of a contact manifold } (V, \xi = \ker(\alpha)) \text{ is said to be non-degenerate if there are no vectors } X \in T_pV \text{ such that } \phi^*_X X = X \text{ and } dg(X) = 0, \text{ where } g \text{ is the function defined by } \phi^*\alpha = \phi^*\alpha. \text{ A translated point of } \phi \text{ of time-shift } \eta \text{ is said to be non-degenerate if it is a non-degenerate discriminant point of } \phi_{\eta}^* \phi \text{ (where } \phi_{\eta}^* \text{ denotes the Reeb flow).}
1.3. Corollary. Consider a lens space $L^{2n-1}_k$ with its standard contact structure. Then:

(i) $L^{2n-1}_k$ is orderable.
(ii) The discriminant and oscillation metrics on the universal cover of the identity component of the contactomorphism group of $L^{2n-1}_k$ are unbounded.
(iii) Any contactomorphism of $L^{2n-1}_k$ contact isotopic to the identity has at least $n$ translated points with respect to the standard contact form. Moreover, if all translated points are non-degenerate then their number is at least $2n$.
(iv) Any contact form on $L^{2n-1}_k$ defining the standard contact structure has at least one closed Reeb orbit.

Orderability of lens spaces was also proved with different methods by Milin [Mi08] and by the fourth author [Sa11b]. Regarding part (iii), this proves for the standard contact form of lens spaces the non-degenerate and cup-length variants of the following conjecture: if $(V, \xi)$ is a compact contact manifold then any contactomorphism $\phi$ contact isotopic to the identity should have at least as many translated points (with respect to any contact form for $\xi$) as the minimal number of critical points of a smooth function on $V$. This conjecture, formulated in [Sa11c], can be thought of as a contact analogue of the Arnold conjecture on fixed points of Hamiltonian diffeomorphisms. As in the Hamiltonian case, one can consider weaker versions obtained by replacing the lower bound on translated points by the Lusternik–Schnirelmann category or (even weaker) the cup length, or the version where the lower bound is the sum of the Betti numbers if all translated points are assumed to be non-degenerate. Working with $\mathbb{Z}_k$-coefficients (with $k$ prime), for any lens space $L^{2n-1}_k$ the sum of the Betti numbers is $2n$, while the cup-length is $2n$ if $k = 2$ (i.e. for $\mathbb{RP}^{2n-1}$) and $n$ if $k > 2$; the fact that our bound in the general case is just $n$, while the one obtained in [Sa11c] in the case of $\mathbb{RP}^{2n-1}$ is $2n$, is consistent with this difference in cuplengths. On the other hand, since the Lusternik–Schnirelmann category of $L^{2n-1}_k$ is $2n$ for all $k$, we should still have at least $2n$ translated points for all $L^{2n-1}_k$ also in the degenerate case. It might be possible to prove this using Massey products (similarly to [Vi97]), but this goes beyond the scope of the present article (see also Remark 4.13).

1.4. Remark. In the case $k = 2$ our arguments prove, as in [Gi90], the following stronger form of Theorem 1.2: in (i) the bound is $2n$, rather than $2n + 1$, and in (iii) the last bound holds also in the degenerate case. Using this one recovers (see Section 5) the stronger bound in Corollary 1.3(iii) that holds in the case of $\mathbb{RP}^{2n-1}$: any contactomorphism contact isotopic to the identity has at least $2n$ translated points with respect to the standard contact form. Notice also that it is enough to prove Theorem 1.2 in the case when $k$ is prime. Indeed, for any multiple $k'$ of $k$ one then obtains a quasimorphism on $\widehat{\text{Cont}}_0(L^{n-1}_k)$ with the required properties by pulling back $\mu$ by the natural map $\widehat{\text{Cont}}_0(L^{2n-1}_k) \to \widehat{\text{Cont}}_0(L^{2n-1}_{k'})$. Because of this, if $k$ is even then Theorem 1.2 and Corollary 1.3 hold in the same stronger form as in the case $k = 2$. ⋄

As in the case of projective space, it follows from Theorem 1.2 that the asymptotic non-linear Maslov index

$$\mathbb{P}: \widetilde{\text{Cont}}_0(L^{2n-1}_k) \to \mathbb{R}, \quad \mathbb{P}([\phi_t]) = \lim_{m \to \infty} \frac{\mu([\phi_t])^m}{m}$$

is monotone, i.e. $\mu([\phi_t]) \leq \mu([\psi_t])$ whenever $|[\phi_t]| \leq |[\psi_t]|$, and has the vanishing property, i.e. it vanishes on any element that can be represented by a contact isotopy supported in a displaceable set (see Propositions 4.15 and 4.16). In [BZ15] Borman and Zapolsky showed that, in analogy with the symplectic case [Bo13], in certain situations monotone quasimorphisms descend under contact reduction; starting from Givental’s non-linear Maslov index on $\mathbb{RP}^{2n-1}$ they thus obtained induced quasimorphisms on those contact toric manifolds that can be written in a certain way as contact reductions of $\mathbb{RP}^{2n-1}$. Moreover, it is proved in [BZ15] that if a contact manifold admits a non-trivial monotone quasimorphism then it is orderable, and if the prequantization of a symplectic toric manifold admits a non-trivial monotone quasimorphism with the vanishing property (a property that is preserved under contact reduction) then it has a non-displaceable
Finally, the construction of generating functions that we use in this article is not the one from [Gi90], our generalization to lens spaces of Givental’s non-linear Maslov index allows us to extend the class of contact toric manifolds that inherit, by contact reduction, a quasimorphism. We thus obtain the following result (see Section 5).

1.5. Corollary. Let $(W^{2n}, \omega)$ be a compact monotone symplectic toric manifold. Write the moment polytope as $\Delta = \{ x \in \mathbb{R}^d \mid \langle \nu_j, x \rangle + \lambda \geq 0 \text{ for } j = 1, \cdots, d \}$, where $d$ is the number of facets and $\nu_j \in \mathbb{R}^d$ are vectors normal to the facets and primitive in the integer lattice $\mathbb{Z}^d = \ker(\exp: \mathbb{R} \to \mathbb{T}^d)$. Suppose that, for some $k \in \mathbb{N}$, $\sum_{j=1}^{d} \nu_j \in k \mathbb{Z}^d$. Then there is a rescaling $\omega$ of the symplectic form such that the prequantization of $(W, \omega)$ admits a non-trivial monotone quasimorphism with the vanishing property, and so is orderable and contains a non-displaceable pre-Lagrangian toric fibre.

Note however that the link with discriminant points is lost in the reduction process. Therefore, it is not clear if it is possible to use these induced quasimorphisms to obtain the applications (other than orderability) listed in Corollary 1.3.

The original idea of the construction of Givental’s non-linear Maslov index in $\mathbb{R}P^{2n-1}$ is as follows. Given a contact isotopy $\{ \phi_t \}_{t \in [0,1]}$ of $\mathbb{R}P^{2n-1}$, one associates to it a 1-parameter family of functions $f_t: \mathbb{R}P^{2M-1} \rightarrow \mathbb{R}$, for some large $M$, so that critical points of $f_t$ of critical value zero correspond to discriminant points of $\phi_t$. In order to detect intersections of $\{ \phi_t \}$ with the discriminant one then analyzes the changes in topology of the sublevel sets $A_t = \{ f_t \leq 0 \}$. This is done by studying the cohomological index of these sets, i.e. the dimension of the image of the homomorphism $H^\ast(\mathbb{R}P^{2M-1}; \mathbb{Z}_2) \rightarrow H^\ast(A_t; \mathbb{Z}_2)$ induced by the inclusion $A_t \hookrightarrow \mathbb{R}P^{2M-1}$. The non-linear Maslov index of $\{ \phi_t \}_{t \in [0,1]}$ is then defined to be the difference between the cohomological indices of $A_0$ and $A_1$. The key difference in the construction for lens spaces is in the properties of the cohomological index. In the case of projective space the cohomological index satisfies subadditivity (the cohomological index of a union $A \cup B$ is not more than the sum of the cohomological indices of $A$ and $B$) and join additivity (the cohomological index of an equivariant join is equal to the sum of the cohomological indices of the factors). The proofs of both properties use the fact that the cohomological ring of projective space is generated by the class in degree one, and thus they do not go through in the case of lens spaces. However we show that weaker versions of the subadditivity and join additivity properties also hold in the case of general lens spaces, and are enough to define the non-linear Maslov index and prove the properties listed in Theorem 1.2.

In the case $k = 2$, Givental’s proof of the join additivity property uses an equivariant Künneth formula [Gi90, Proposition A.1]. A crucial ingredient in the proof is the fact that the Künneth short exact sequence (of modules over the equivariant cohomology of a point) splits, but it is not clear to us why this fact should be true. In any case, for $k > 2$ the Künneth (or Eilenberg–Moore) spectral sequence does not collapse in general (due to the fact that for $k > 2$ the $\mathbb{Z}_k$-equivariant cohomology of a point has zero divisors) and so we do not even have a Künneth short exact sequence for the equivariant cohomology of a product. We thus present a different proof (even for the case $k = 2$). Our proof is based on the study of a join operation in equivariant homology, which is defined in the same way for all $k$. When $k = 2$ the properties of this operation imply Givental’s join additivity, while for $k > 2$, as the join of two even dimensional generators of the equivariant homology of a point is zero, we only obtain a weaker join quasi-additivity property (Proposition 3.9 (v)). As mentioned above, this property is still strong enough to imply the applications we are interested in.

Finally, the construction of generating functions that we use in this article is not the one from [Gi90], but is an adaptation to our setting of the work of Théret [Th98] (cf. Remark A.6).

\footnote{Since our construction of generating functions is different from Givental’s [Gi90], and we do not have a general uniqueness theorem for generating functions of contact isotopies of projective space, strictly speaking we do not know whether in the special case of projective space our quasimorphism actually coincides with Givental’s. However, all the features and properties are the same.}
The article is organized as follows. In Section 2 we explain how to construct 1-parameter families of generating functions for contact isotopies of lens spaces. In Section 3 we describe the properties of the topological invariant (the cohomological index) that is used to analyze the changes in topology of the sublevel sets of generating functions, deferring the most technical part of the proof of the join quasi-additivity property to Appendix B. In Section 4 we put these ingredients together to define the non-linear Maslov index of a contact isotopy, and prove the properties listed in Theorem 1.2. In Section 5 we review how to use the properties of the non-linear Maslov index to prove Corollaries 1.3 and 1.5, mostly following the corresponding arguments in the case of projective space. In Appendix A we discuss several interpretations of the composition formula that is used in the construction of generating functions.

Throughout the article, when we say that we follow Givental [Gi90] or Théret [Th98] we mean that their arguments, developed for $\mathbb{R}P^{2n−1}$ and $\mathbb{C}P^{n−1}$, can be repeated for lens spaces without any modification other than replacing the action of $\mathbb{Z}_2$ or $S^1$ by the $\mathbb{Z}_k$-action (1.1).

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2. Generating functions

In this section we explain how, given a contact isotopy $\{\phi_t\}_{t \in [0,1]}$ of a lens space $L_k^{2n−1}$, one can define a 1-parameter family of functions $f_t$ on a higher dimensional lens space $L_k^{2n-M}$ so that critical points of $f_t$ of critical value zero correspond to discriminant points of $\phi_t$.

Generating functions for symplectomorphisms of $\mathbb{R}^{2n}$. We start by recalling the definition of generating functions of Lagrangian submanifolds in cotangent bundles. Observe first that any Lagrangian section of the cotangent bundle $T^*B$ of a manifold $B$ is the graph of a closed 1-form on $B$: if this 1-form is exact, given by the differential of a function $F$, then we say that $F$ is a generating function for the Lagrangian. Generalizing this idea, one can associate a generating function to a larger class of Lagrangian submanifolds of $T^*B$ by the following construction, which goes back to Hörmander [Ho71]. Consider a function $F: E \to \mathbb{R}$ defined on the total space of a fibre bundle $p: E \to B$. Let $N^*_E$ be the fibre conormal bundle, i.e. the space of points $(e, \eta)$ of $T^*E$ such that $\eta$ vanishes on the kernel of $dp|_e$. We say that $F$ is a generating function if $dF: E \to T^*E$ is transverse to $N^*_E$. If this condition is satisfied then the set of fibre critical points $\Sigma_F = (dF)^{-1}(N^*_E)$ is a smooth submanifold of $E$, and the map

$$i_F: \Sigma_F \to T^*B, \ e \mapsto (p(e), v^*(e))$$

defined by posing $v^*(e)(X) = dF(\hat{X})$ for $X \in T_{p(e)}B$, where $\hat{X}$ is any vector in $T_eE$ such that $p_*(\hat{X}) = X$, is a Lagrangian immersion. If $i_F$ is an embedding then we say that $F$ is a generating
function for the Lagrangian submanifold $L_F := i_F(\Sigma_F)$ of $T^*B$. In this case, critical points of $F$ are in $1$–$1$ correspondence with intersections of $L_F$ with the zero section.

In our applications, generating functions are always defined on trivial bundles of the form $E = \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}^{2n}$. Denoting the coordinates by $(\zeta, \nu)$ with $\zeta \in \mathbb{R}^{2n}$ and $\nu \in \mathbb{R}^{2nN}$, we then have that $dF$ is transverse to $N_F^\ast$ if and only if 0 is a regular value of the vertical derivative $\frac{\partial F}{\partial \nu}: E \to (\mathbb{R}^{2nN})^\ast$; moreover, $\Sigma_F = (\frac{\partial F}{\partial \nu})^{-1}(0)$ and $i_F(\zeta, \nu) = (\zeta, \frac{\partial F}{\partial \nu}(\zeta, \nu))$.

For a symplectomorphism $\Phi$ of $\mathbb{R}^{2n}$, as in [Vi92] we consider the Lagrangian submanifold $\Gamma_\Phi$ of $T^*\mathbb{R}^{2n}$ which is the image of the graph of $\Phi$ under the symplectomorphism $\tau: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ given by
\[
\tau(x, y, X, Y) = \left(\frac{x + X}{2}, \frac{y + Y}{2}, Y - y, x - X\right)
\]
or, in complex notation,
\[
\tau(z, Z) = \left(\frac{z + Z}{2}, i(z - Z)\right).
\]
We say that a function $F$ is a generating function for $\Phi$ if it is a generating function for $\Gamma_\Phi$. Since $\tau$ sends the diagonal onto the zero section, critical points of a generating function $F$ of $\Phi$ are in $1$–$1$ correspondence with fixed points of $\Phi$.

Any Hamiltonian symplectomorphism $\Phi$ of $\mathbb{R}^{2n}$ such that $\Gamma_\Phi$ is a section of $T^*\mathbb{R}^{2n}$ has a generating function $F: \mathbb{R}^{2n} \to \mathbb{R}$. In order to obtain a generating function for more general Hamiltonian symplectomorphisms we use the following composition formula$^4$.

2.2. Proposition (Composition formula). If $F_1: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \to \mathbb{R}$ and $F_2: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \to \mathbb{R}$ are, respectively, generating functions for symplectomorphisms $\Phi^{(1)}$ and $\Phi^{(2)}$ of $\mathbb{R}^{2n}$, then the function $F_1 \sharp F_2: \mathbb{R}^{2n} \times ((\mathbb{R}^{2n} \times \mathbb{R}^{2nN_1}) \times \mathbb{R}^{2nN_2}) \to \mathbb{R}$ defined by
\[
F_1 \sharp F_2(q; \zeta_1, \zeta_2, \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2\langle \zeta_2 - q, i(\zeta_1 - q)\rangle
\]
(where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^{2n}$) is a generating function for the composition $\Phi = \Phi^{(2)} \circ \Phi^{(1)}$.

In Appendix A we discuss two interpretations of the composition formula in terms of symplectic reduction, a generalization to any even number of factors and the relation to the method of broken trajectories of Chaperon, Laudenbach and Sikorav [Ch84, LS85, Si85, Si87] and to the construction in Givental [Gi90]. Below we present a direct proof.

Proof. Step 1: Criterion for fibre critical points.

The vertical derivative of $F_1 \sharp F_2$ is
\[
(q; \zeta_1, \zeta_2, \nu_1, \nu_2) \mapsto \left(\frac{\partial F_1}{\partial \zeta_1} + 2i(\zeta_2 - q), \frac{\partial F_2}{\partial \zeta_2} - 2i(\zeta_1 - q), \frac{\partial F_1}{\partial \nu_1}, \frac{\partial F_2}{\partial \nu_2}\right)
\]
thus a point $(q; \zeta_1, \zeta_2, \nu_1, \nu_2)$ is a fibre critical point of $F_1 \sharp F_2$ if and only if $(\zeta_j, \nu_j)$ is a fibre critical point of $F_j$ ($j = 1, 2$) and
\[
\begin{cases}
\frac{\partial F_1}{\partial \zeta_1} = -2i(\zeta_2 - q) \\
\frac{\partial F_1}{\partial \zeta_2} = 2i(\zeta_1 - q).
\end{cases}
\]
Since $F_j: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_j} \to \mathbb{R}$ is a generating function for $\Phi^{(j)}$ ($j = 1, 2$), the map
\[
\Sigma_{F_j} \to \mathbb{R}^{2n}, \ (\zeta_j, \nu_j) \mapsto z_j
\]
$^4$Théret’s composition formula in [Th98] is
\[
F_1 \sharp F_2(q; \zeta_1, \zeta_2, \nu_1, \nu_2) = F_1(q + \zeta_2, \nu_1) + F_2(\zeta_1 + \zeta_2, \nu_2) + 2(q - \zeta_1, i\zeta_2).
\]
Although it differs from ours just by a change of variables, in [Th98] it is proved to hold only under the assumption that $F_1$ or $F_2$ has no fibre variables. This is not sufficient for our purposes: in the proof of the quasimorphism property of the non-linear Maslov index (Proposition 4.2) we need to allow fibre variables in both factors.
given by the composition
\[ \Sigma_{F^j} \xrightarrow{i_{F^j}} \Gamma_{\Phi^j} \xrightarrow{\tau^{-1}|_{\phi^j}} \text{gr}(\Phi^j) \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n} \xrightarrow{(z,z') \mapsto z} \mathbb{R}^{2n} \]
is a diffeomorphism. Under the change of variables (2.5) the equations (2.4) become
\[
\begin{align*}
&i(\bar{z}_1 - \Phi(1)(z_1)) = -2i \left( \frac{2 z_2 + \Phi(2)(z_2)}{2} - q \right) \\
&i(\bar{z}_2 - \Phi(2)(z_2)) = 2i \left( \frac{z_1 + \Phi(1)(z_1)}{2} - q \right)
\end{align*}
i.e.
\[
\begin{align*}
\begin{cases} 
\bar{z}_2 = \Phi(1)(z_1) \\
q = \frac{z_1 + \Phi(1)(z_1)}{2}
\end{cases}
\]
Step 2: $F_1 \not\equiv F_2$ is a generating function.

In order to prove that $F_1 \not\equiv F_2$ is a generating function we need to show that 0 is a regular value of the vertical derivative of $F_1 \not\equiv F_2$. This can be seen as follows. The diffeomorphism (2.5) is the restriction to $\Sigma = \Sigma_{F^j} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n N_j}$ of the map
\[
\Sigma \mid_{\Gamma_{\Phi}} \xrightarrow{\tau^{-1}} \text{gr}(\Phi) \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \quad \left(\zeta_j, \nu_j\right) \mapsto \zeta_j + \frac{1}{2i} \frac{\partial F_j}{\partial \zeta_j} \bigg|_{(\zeta_j, \nu_j)}.
\]
Thus, for every $(\zeta_j, \nu_j) \in \Sigma_{F^j} = (\partial F_j/\partial \nu_j)^{-1}(0)$, the restriction to $T_{(\zeta_j, \nu_j)} \Sigma_{F^j} = \ker \left( \left. \frac{\partial F_j}{\partial \nu_j} \right|_{(\zeta_j, \nu_j)} \right)$ of the differential at $(\zeta_j, \nu_j)$ of the map (2.7) is bijective. Since $d\left(\frac{\partial F_j}{\partial \nu_j}\right) \mid_{(\zeta_j, \nu_j)} : \mathbb{R}^{2n} \times \mathbb{R}^{2n N_j} \to \mathbb{R}^{2n N_j}$ is surjective, this implies that the matrix
\[
M_j = \begin{pmatrix}
\frac{\partial^2 F_j}{\partial \zeta^2} + I & \frac{\partial^2 F_j}{\partial \zeta \partial \nu} \\
\frac{\partial^2 F_j}{\partial \zeta \partial \nu} & \frac{\partial^2 F_j}{\partial \nu^2}
\end{pmatrix}
\]
is invertible at every $(\zeta_j, \nu_j)$ that is fibre critical for $F_j$. The differential of (2.3) at $(q; \zeta_1, \zeta_2, \nu_1, \nu_2)$ is given by the matrix
\[
\begin{pmatrix}
-2iI & \frac{\partial^2 F_1}{\partial \zeta^2} & 2i I & \frac{\partial^2 F_1}{\partial \zeta \partial \nu_1} & 0 \\
2i I & -2i I & \frac{\partial^2 F_2}{\partial \zeta^2} & 0 & \frac{\partial^2 F_2}{\partial \zeta \partial \nu_2} \\
0 & \frac{\partial^2 F_1}{\partial \zeta \partial \nu_1} & 0 & \frac{\partial^2 F_1}{\partial \nu^2} & 0 \\
0 & 0 & \frac{\partial^2 F_2}{\partial \zeta \partial \nu_2} & 0 & \frac{\partial^2 F_2}{\partial \nu^2}
\end{pmatrix}
\]
which, by elementary row and column operations, can be brought to the form
\[
\begin{pmatrix}
* & M_1 & 0 \\
* & & M_2
\end{pmatrix}
\]
Since each $M_j$ is invertible, the columns of this matrix span all of $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n N_1} \times \mathbb{R}^{2n N_2}$, proving that 0 is a regular value of the vertical derivative of $F_1 \not\equiv F_2$.

Step 3: $F_1 \equiv F_2$ is a generating function for $\Phi$.

We need to show that the Lagrangian immersion $i_{F_1 \equiv F_2} : \Sigma_{F_1 \equiv F_2} \to T^* \mathbb{R}^{2n}$ induces a diffeomorphism between $\Sigma_{F_1 \equiv F_2}$ and $\Gamma_{\Phi}$. The relation (2.6) for fibre critical points and the fact that the maps (2.5) are diffeomorphisms imply that the map
\[
\Sigma_{F_1 \equiv F_2} \to \mathbb{R}^{2n}, \quad (q; \zeta_1, \zeta_2, \nu_1, \nu_2) \mapsto z_1
\]
is a diffeomorphism. For a fibre critical point $(q; \zeta, \nu)$ we have
\[
\frac{\partial (F_1 \equiv F_2)}{\partial q}(q; \zeta, \nu) = 2i (\zeta_1 - \zeta_2) = i (z_1 + \Phi(1)(z_1) - z_2 - \Phi(2)(z_2)) = i (z_1 - \Phi(z_1))
\]
and so
\[ i_{F_1,F_2}(q;\zeta,\nu) = \left( q, \frac{\partial (F_1 \sharp F_2)}{\partial q}(q,\zeta,\nu) \right) = \left( \frac{z_1 + \Phi(z_1)}{2}, i(z_1 - \Phi(z_1)) \right) = \tau(z_1, \Phi(z_1)). \]

In other words, \( i_{F_1,F_2}: \Sigma_{F_1,F_2} \to T^*\mathbb{R}^{2n} \) is the composition of the diffeomorphism (2.8) with the embedding \( \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}, z_1 \mapsto \tau(z_1, \Phi(z_1)) \), and so it induces a diffeomorphism between \( \Sigma_{F_1,F_2} \) and \( \Gamma_\Phi \). \( \square \)

Proposition 2.2 (as well as the analogous constructions in [Ch84, LS85, Si85, Si87, Th98]) can be used to show that every compactly supported Hamiltonian diffeomorphism of \( \mathbb{R}^{2n} \) has a generating function quadratic at infinity. This class of generating functions is used for instance in the work of Viterbo [Vi92] and Traynor [Tr94]. As we now explain, in our case (similarly to [Gi90] and [Th98]) we use Proposition 2.2 to produce instead conical generating functions for Hamiltonian diffeomorphisms of \( \mathbb{R}^{2n} \) that lift contactomorphisms of lens spaces.

**Generating functions for contact isotopies of lens spaces.** Throughout our discussion, we fix the vector of weights \( \underline{w} \) that defines the action (1.1) of \( \mathbb{Z}_k \) on \( \mathbb{R}^{2n} \), and denote the lens space \( L_k^{2n-1}(\underline{w}) \) by \( L_k^{2n-1} \). On products \( (\mathbb{R}^{2n})^N \) that occur as the domains of definition of various generating functions, we always take the diagonal action of \( \mathbb{Z}_k \) that is given by this same \( \underline{w} \) on each factor, and denote the corresponding lens space by \( L_k^{2nN-1} \). The \( \mathbb{R}^{2n} \)-action on \( \mathbb{R}^{2n} \), or on products of \( \mathbb{R}^{2n} \), is always the radial action. By a contact isotopy we always mean a contact isotopy starting at the identity.

Given a contact isotopy \( \{\phi_t\} \) of \( L_k^{2n-1} \), we obtain a Hamiltonian isotopy \( \{\Phi_t\} \) of \( \mathbb{R}^{2n} \setminus \{0\} \) by first lifting \( \{\phi_t\} \) to a \( \mathbb{Z}_k \)-equivariant contact isotopy of \( S^{2n-1} \), and then lifting this to a Hamiltonian isotopy of the symplectization of the sphere, which we identify with \( \mathbb{R}^{2n} \setminus \{0\} \). We now explain this procedure in more detail. Recall that the symplectization of a co-oriented contact manifold \( (V,\xi) \) is the symplectic submanifold \( SV \) of \( T^*V \) that consists of those covectors that vanish on the contact distribution and are positive with respect to the given co-orientation. Given a contactomorphism of \( V \), its lift to the cotangent bundle restricts to a symplectomorphism of \( SV \); the lift of a contact isotopy of \( V \) is a Hamiltonian isotopy of \( SV \). A choice of a contact form \( \alpha \) for \( \xi \) gives a diffeomorphism \( \mathbb{R} \times V \to SV \), defined by \( (\theta, u) \mapsto e^{\theta}\alpha|_u \). In the special case of \( V = S^{2n-1} \) and \( \alpha = \sum_{j=1}^{n-1} \frac{1}{j} (x_j dy_j - y_j dx_j) \) we further identify \( \mathbb{R} \times V \) with \( \mathbb{R}^{2n} \setminus \{0\} \) by the map \( (\theta, u) \mapsto \frac{1}{\theta} e^{\theta} u \).

The lift of a contactomorphism \( \phi: S^{2n-1} \to S^{2n-1} \) is then the map \( \Phi: \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}^{2n} \setminus \{0\} \) given by the formula
\[ \Phi(z) = \frac{z}{e^{\theta(\frac{z^-}{|z^-|})}} \phi\left( \frac{z^-}{|z^-|} \right), \]
and we extend \( \Phi \) continuously to \( \mathbb{R}^{2n} \) by setting \( \Phi(0) = 0 \). For any contact isotopy \( \{\phi_t\} \) of \( L_k^{2n-1} \) the resulting homeomorphisms \( \Phi_t \) of \( \mathbb{R}^{2n} \) are \( (\mathbb{Z}_k \times \mathbb{R}_{>0}) \)-equivariant, Lipschitz with Lipschitz inverse, and smooth symplectomorphisms on \( \mathbb{R}^{2n} \setminus \{0\} \). We call such maps conical symplectomorphisms of \( \mathbb{R}^{2n} \).

As in [Th98], in order to work with conical symplectomorphisms we must relax the smoothness assumption on our generating functions. Notice first that if \( \Phi \) is a conical symplectomorphism of \( \mathbb{R}^{2n} \) that lifts a \( C^1 \)-small contactomorphism of \( L_k^{2n-1} \) then its generating function \( F: \mathbb{R}^{2n} \to \mathbb{R} \) is \( \mathbb{Z}_k \)-invariant, homogeneous of degree 2, \( C^1 \) with Lipschitz differential and smooth on \( \mathbb{R}^{2n} \setminus \{0\} \). More generally, we say that \( F: E \to \mathbb{R} \), where \( E = \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \), is a conical function if it is \( \mathbb{Z}_k \)-invariant, homogeneous of degree 2 and \( C^1 \) with Lipschitz differential. Such an \( F: E \to \mathbb{R} \) is a conical generating function if it is smooth at all its fibre critical points other than the origin \( (0,0) \in \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \), and \( dF: E \to T^*E \) is transverse to the fibre conormal bundle \( N_E^* \) except possibly at the origin. If this condition is satisfied then the set \( \Sigma_F \) of fibre critical points is a smooth \( (\mathbb{Z}_k \times \mathbb{R}_{>0}) \)-invariant submanifold except possibly at the origin, and the corresponding map \( i_F: \Sigma_F \to T^*\mathbb{R}^{2n} \) is continuous and is a smooth Lagrangian immersion outside the origin. If \( i_F \) is a homeomorphism between \( \Sigma_F \) and \( \Gamma_F \) for a conical symplectomorphism \( \Phi \) then we say that \( F \) is a conical generating function for \( \Phi \) and for the induced contactomorphism \( \phi \) of \( L_k^{2n-1} \).
By a family of conical generating functions we mean a family of conical generating functions \( F_t : E \to \mathbb{R} \), parametrized by \( s \in S \) for some manifold with corners \( S \) (for example \([0, 1]\) or \([0, 1] \times [0, 1]\)), such that the map \((s, x) \mapsto F_s(x)\) is \( C^1 \) with locally Lipschitz differential and is smooth at \((s, x)\) whenever \( x \) is a fibre critical point of \( F_s \) other than the origin.

2.9. **Proposition.** If \( F^{(1)}_t \) and \( F^{(2)}_t \) are families of conical generating functions for contact isotopies \( \phi^{(1)}_t \) and \( \phi^{(2)}_t \) of \( L^{2n-1}_k \), then the functions \( F^{(1)}_t \sharp F^{(2)}_t \) defined as in Proposition 2.2 form a family of conical generating functions for the composition \( \phi^{(2)}_t \circ \phi^{(1)}_t \).

**Proof.** We first show that \( F^{(1)}_t \sharp F^{(2)}_t \) is a family of conical generating functions. It is immediate to see that each \( F^{(1)}_t \sharp F^{(2)}_t \) is \( C^1 \) with Lipschitz differential, \( \mathbb{Z}_k \)-invariant, and homogeneous of degree 2. The property of being smooth at fibre critical points other than the origin is also preserved by the composition formula, as we now explain. Let \((q; \zeta_1, \zeta_2, \nu_1, \nu_2)\) be a fibre critical point of \( F^{(1)}_t \sharp F^{(2)}_t \) other than the origin. From the formula (2.3) we see that, for \( j = 1, 2 \), \( (\zeta_j, \nu_j) \) is a fibre critical point of \( F^{(j)}_t \). Moreover, \((\zeta_j, \nu_j)\) is the origin if and only if the point \( z_j \in \mathbb{R}^{2n} \) that corresponds to it by the bijection (2.5) is also the origin. If this happens for one of \( j = 1 \) or \( j = 2 \) then equation (2.6) implies that \( q \) and both \( z_1 \) and \( z_2 \) are the origin and thus, using the bijection (2.5) once more, that \((q; \zeta_1, \zeta_2, \nu_1, \nu_2)\) is the origin, contrary to our assumptions. Therefore both \((\zeta_1, \nu_1)\) and \((\zeta_2, \nu_2)\) must be different from the origin and so \( F^{(1)}_t \sharp F^{(2)}_t \) is smooth at \((q; \zeta_1, \zeta_2, \nu_1, \nu_2)\). The proof now continues by repeating the proof of Proposition 2.2; notice that derivatives of order higher than one are taken only at fibre critical points. \( \square \)

2.10. **Proposition** (Existence of generating functions). Any contact isotopy \( \{ \phi_t \}_{t \in [0, 1]} \) of \( L^{2n-1}_k \) (starting at the identity) has a family of conical generating functions \( F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R} \).

**Proof.** For \( N \) big enough we can write \( \phi_t = \phi^{(N)}_t \circ \cdots \circ \phi^{(1)}_t \) for \( C^1 \)-small contact isotopies \( \phi^{(j)}_t \). For each \( j \) let \( F^{(j)}_t : \mathbb{R}^{2n} \to \mathbb{R} \) be the corresponding 1-parameter families of conical generating functions. By Proposition 2.9, a family of the form \( F_t := F^{(1)}_t \sharp \cdots \sharp F^{(N)}_t \) (for any choice of parenthesization of the factors) is a family of conical generating functions for \( \{ \phi_t \} \). \( \square \)

2.11. **Remark.** In several places later on we use 1-parameter families of generating functions obtained by a decomposition of the following form. For a contact isotopy \( \{ \phi_t \}_{t \in [0, 1]} \) (starting at the identity), let \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1 \) be a decomposition of the time interval such that all contact isotopies \( \{ \phi_t \circ \phi_{t_{j-1}}^{-1} \}_{t \in [t_{j-1}, t_j]} \) are sufficiently \( C^1 \)-small. Define contact isotopies \( \{ \phi^{(j)}_t \}_{t \in [0, 1]} \) by \( \phi^{(j)}_t = \phi_t \circ \phi_{t_{j-1}}^{-1} \) for \( t \in [t_{j-1}, t_j] \), \( \phi^{(j)}_t = \text{id} \) for \( t \in [0, t_{j-1}] \) and \( \phi^{(j)}_t = \phi_{t_j} \circ \phi_{t_{j-1}}^{-1} \) for \( t \in [t_j, 1] \); then \( \phi_t = \phi^{(N)}_t \circ \cdots \circ \phi^{(1)}_t \).

In Section 4 we use generating functions to define the non-linear Maslov index of a contact isotopy of \( L^{2n-1}_k \). In order to show that the index is well defined we use the fact that generating functions are in some sense unique. The following discussion leads to this result, which we state and prove in Proposition 2.17 below.

We say that a homeomorphism of \( \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \) is a fibre preserving conical homeomorphism if it takes each fibre \( \{ z \} \times \mathbb{R}^{2nN} \) to itself and is \((\mathbb{Z}_k \times \mathbb{R}_{>0})\)-equivariant. By a family of fibre preserving conical homeomorphisms we mean a collection of fibre preserving conical homeomorphism \( \theta_s \), parametrized by \( s \in S \) for some manifold with corners \( S \) (for example \([0, 1]\) or \([0, 1] \times [0, 1]\)), such that \( (s, x) \mapsto \theta_s(x) \) is continuous.

The stabilization of a 1-parameter family \( F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R} \) of conical generating functions by a non-degenerate \( \mathbb{Z}_k \)-invariant quadratic form \( Q : \mathbb{R}^{2nN'} \to \mathbb{R} \) is the 1-parameter family of conical generating functions

\[
F_t \oplus Q : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \times \mathbb{R}^{2nN'} \to \mathbb{R}.
\]
On the set of 1-parameter families of conical generating functions we consider the smallest equivalence relation under which two such families are equivalent if they differ by stabilization or by a 1-parameter family of fibre preserving conical homeomorphisms that restrict to diffeomorphisms on neighborhoods of the fibre critical points.

2.12. Remark. If two 1-parameter families $F_{t}^{(1)}$ and $F_{t}^{(2)}$ are equivalent respectively to $G_{t}^{(1)}$ and $G_{t}^{(2)}$ then $F_{t}^{(1)} \circ F_{t}^{(2)}$ is equivalent to $G_{t}^{(1)} \circ G_{t}^{(2)}$.

Equivalent 1-parameter families of conical generating functions generate the same contact isotopy. In Proposition 2.17 we prove (mostly following Théret’s proof of [Th98, Proposition 4.7]) a partial converse: any two 1-parameter families of conical generating functions for a given contact isotopy of $L_{k}^{-2n-1}$ that are obtained by the construction of Proposition 2.10 are equivalent. The main ingredient is [Th98, Lemma 4.8], whose proof holds also in our situation and gives the following result.

2.13. Lemma. Let $\{ \phi_{t} \}_{t \in [0,1]}$ be a contact isotopy of $L_{k}^{-2n-1}$ (starting at the identity). Suppose that $F_{s,t} : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$, for $(s, t) \in [0,1] \times [0,1]$, is a family of conical generating functions such that for every $(s, t)$ the function $F_{s,t}$ is a conical generating function for $\phi_{t}$. Then there is a family $\theta_{s,t}$ of fibre preserving conical homeomorphisms of $\mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ that restrict to diffeomorphisms on neighborhoods of the fibre critical points and such that $F_{s,t} \circ \theta_{s,t} = F_{0,t}$. In particular, the 1-parameter families $F_{0,t}$ and $F_{1,t}$ are equivalent.

Proof. We search for a vector field $X : [0,1] \times [0,1] \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}$ such that $X_{s,t} : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}$ integrates to $\theta_{s,t}$. Differentiating $F_{s,t} \circ \theta_{s,t} = F_{0,t}$ with respect to $s$ we see that it is enough to require

$$\partial F_{s,t} / \partial s(q, \zeta) + \partial F_{s,t} / \partial \zeta(q, \zeta) X_{s,t}(q, \zeta) = 0.$$  

If $\partial F_{s,t} / \partial \zeta \neq 0$ we can solve this equation by setting $X_{s,t}$ to be an appropriate multiple of $\partial F_{s,t} / \partial \zeta$; the resulting vector field $X_{s,t}$ is automatically Lipschitz and ($\mathbb{Z}_{k} \times \mathbb{R}_{>0}$)-equivariant on its domain of definition.

Let now $\Sigma_{s,t} = \partial F_{s,t} / \partial \zeta(1)$ and $\Sigma = \cup_{s,t} \Sigma_{s,t}$. In order to complete the proof, it suffices to define a homogeneous and $\mathbb{Z}_{k}$-invariant vector field $Y_{s,t}$ satisfying (2.14) in a neighborhood of $\Sigma$, as we can then use a homogeneous and $\mathbb{Z}_{k}$-invariant bump function to interpolate between $Y$ and the solution on the complement of $\Sigma$ described above.

The derivative $\partial F_{s,t} / \partial \zeta$ vanishes along $\Sigma$. After composing with a fibre preserving conical homeomorphism that restricts to a diffeomorphism on neighborhoods of critical points, we may assume that $\partial F_{s,t}$ also vanishes along $\Sigma$. Recall that Hadamard’s lemma gives the following “normal form” for a smooth function vanishing along a submanifold. Let $P$ be a submanifold of a manifold $Q$, $T^{\nu}P \to Q$ be a tubular neighborhood and $\pi : T^{\nu}P \to P$ denote the projection. If $g : Q \to \mathbb{R}$ is a smooth function which vanishes along $P$ then there exists a smooth section $G : T^{\nu}P \to \pi^{\ast}(T^{\nu}P)^{*}$ (necessarily satisfying $G|_{P} = dq|_{T^{\nu}P}$) such that

$$g(\phi(y)) = \langle G(y), y \rangle$$

for $y$ near the zero section. A similar statement holds if $g$ is a section of a vector bundle over $Q$ which vanishes along $P$. Now let $Q = [0,1] \times [0,1] \times L^{2n(N+1)-1}$, $P$ be the image of $\Sigma \setminus \{0,1\} \times \{0,1\} \times \{0\}$ in $Q$, and $E \to Q$ be the bundle obtained by restricting $T^{\nu}T^{\nu} \mathbb{R}^{2nN}$ to $[0,1] \times [0,1] \times S^{2n(N+1)-1}$ and then taking the quotient by the cyclic group action. Let $\phi_{1} : Q \to \mathbb{R}$ and $\phi_{2} : Q \to E$ be obtained by restricting $\partial F_{s,t} / \partial s$ and $\partial F_{s,t} / \partial \zeta$ to $[0,1] \times [0,1] \times S^{2n(N+1)-1}$ and then passing to the quotient. Hadamard’s lemma provides sections

$$G_{1} : T^{\nu}P \to \pi^{\ast}(T^{\nu}P)^{*} \quad \text{and} \quad G_{2} : T^{\nu}P \to \text{Hom}(\pi^{\ast}(T^{\nu}P), \phi^{\ast}E)$$

for $y$ near the zero section.
such that \( g_1(\phi(y)) = \langle G_1(y), y \rangle \) and \( g_2(\phi(y)) = \langle G_2(y), y \rangle \) near the zero section. In order to obtain a \( ((\mathbb{Z}_k \times \mathbb{R}_{>0})\text{-equivariant}) \) solution \( Y \) of \((2.14)\) near \( \Sigma \), it suffices to solve the equation
\[
G_1(y) + (G_2(y))(Y) = 0.
\]
This equation can be solved near the zero section since the transversality assumption on \( \frac{\partial F_{s,t}}{\partial \zeta} \) along \( \Sigma \) ensures that \( G_2(y) \) is an isomorphism if \( y \) is in the zero section, and hence also if \( y \) is near the zero section. \(\square\)

Using Lemma 2.13 we now prove the following statement.

2.15. **Lemma.** Let \( F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \) be a family of conical generating functions for a contact isotopy \( \{ \phi_t \} \). Then \( F_t \not\equiv 0 \) and \( 0 \not\equiv F_t \) are equivalent to \( F_t \).

**Proof.** We prove that \( F_t \not\equiv 0 \) is equivalent to \( F_t \) (the case of \( 0 \not\equiv F_t \) is similar). For each \( s \in [0, 1] \) we define a 1-parameter family \( F_{s,t} : \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \to \mathbb{R} \) by
\[
F_{s,t}(q; \zeta_1, \zeta_2, \nu) = F_t(s \zeta_1 + (1 - s)q, \nu) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle.
\]
Then \( F_{1,t} = F_t \not\equiv 0 \), and \( F_{0,t}(q; \zeta_1, \zeta_2, \nu) = F_t(q, \nu) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle \) is equivalent to \( F_t \) since it differs from it by a stabilization followed by the fibre preserving homeomorphism \( (q; \zeta_1, \zeta_2, \nu) \to (q; \zeta_1 - q, \zeta_2 - q, \nu) \). By arguments as in the proof of Proposition 2.2 one sees that for each \( s \in [0, 1] \) the 1-parameter family \( F_{s,t} \) generates \( \{ \phi_t \} \). By Lemma 2.13 we conclude that \( F_{0,t} \) and \( F_{1,t} \) are equivalent, and thus so are \( F_t \) and \( F_{1,t} \not\equiv 0 \). \(\square\)

Another ingredient of the proof of Proposition 2.17 is the fact that the composition formula defines an associative operation, and thus different choices of parenthetization in the proof of Proposition 2.10 produce equivalent 1-parameter families of generating functions\(^5\).

2.16. **Lemma (Associativity).** Let \( F_t^{(1)} \), \( F_t^{(2)} \) and \( F_t^{(3)} \) be families of conical generating functions for contact isotopies \( \{ \phi_t^{(1)} \} \), \( \{ \phi_t^{(2)} \} \) and \( \{ \phi_t^{(3)} \} \) respectively, constructed as in Proposition 2.10. Then the families \( (F_t^{(1)} \not\equiv F_t^{(2)}) \not\equiv F_t^{(3)} \) and \( (F_t^{(1)} \not\equiv F_t^{(2)}) \not\equiv (F_t^{(2)} \not\equiv F_t^{(3)}) \) are equivalent.

**Proof.** Suppose first that \( \{ \phi_t^{(1)} \} \) is \( C^1 \)-small. Let \( \{ \phi_t^{(2)} \}_{s \in [0, 1]} \) be a 1-parameter family of \( C^1 \)-small contact isotopies from \( \{ \phi_t^{(1)} \} = \{ \phi_t^{(2)} \} \to \{ \phi_t^{(1)} \} = \{ \phi_t^{(2)} \} \), for instance the one given by \( \phi_t^{(2)} = \phi_t^{(1)} \). Denote by \( F_{s,t}^{(2)} : \mathbb{R}^{2n} \to \mathbb{R} \) and by \( F_{s,t}^{(1)} : \mathbb{R}^{2n} \to \mathbb{R} \) the generating functions of \( \phi_t^{(s)} \) and \( \phi_t^{(1)} \) respectively. Then \( F_{1,t}^{(2)} = F_{0,t}^{(2)} = F_t^{(2)} \) and \( F_{0,t}^{(1)} = F_t^{(1)} \). Consider the family of generating functions \( G_{s,t} = (F_t^{(1)} \not\equiv F_t^{(2)}) \not\equiv (F_t^{(2)} \not\equiv F_t^{(3)}) \) for \( \phi_t^{(3)} \circ \phi_t^{(2)} \circ \phi_t^{(1)} \). The 1-parameter families \( \{ G_{s,t} \}_{t \in [0, 1]} \) generate the same Hamiltonian isotopy for all \( s \in [0, 1] \). Thus, by Lemma 2.13, the 1-parameter families \( G_{0,t} \) and \( G_{1,t} \) are equivalent. But \( G_{0,t} = (F_t^{(1)} \not\equiv 0) \not\equiv (F_t^{(2)} \not\equiv F_t^{(3)}) \) and \( G_{1,t} = (F_t^{(1)} \not\equiv F_t^{(2)}) \not\equiv (0 \not\equiv F_t^{(3)}) \), thus by Lemma 2.15 and Remark 2.12 we conclude that the 1-parameter families \( G_{0,t} \) and \( G_{1,t} \) are equivalent respectively to \( F_t^{(1)} \not\equiv (F_t^{(2)} \not\equiv F_t^{(3)}) \) and \( (F_t^{(1)} \not\equiv F_t^{(2)}) \not\equiv F_t^{(3)} \), and so \( F_t^{(1)} \not\equiv (F_t^{(2)} \not\equiv F_t^{(3)}) \) and \( (F_t^{(1)} \not\equiv F_t^{(2)}) \not\equiv F_t^{(3)} \) are equivalent. Finally, if \( \{ \phi_t^{(1)} \} \) is not \( C^1 \)-small then we divide it into small pieces and apply the above step several times. \(\square\)

We now put these ingredients together in order to prove that, up to equivalence, the choices involved in the construction of Proposition 2.10 do not affect the resulting family of conical generating functions.

2.17. **Proposition.** Given a contact isotopy \( \{ \phi_t \} \) of \( L_4^{2n-1} \), any two families of conical generating functions constructed as in Proposition 2.10 are equivalent.

---

\(^5\)This fact is not explained (nor needed) in [Th98], but for us is important because it will be used in the proof of the quasimorphism property (Proposition 4.2).
Proof. Recall that in Proposition 2.10 we constructed a family of conical generating functions for \( \{ \phi_t \} \) by writing \( \phi_t = \phi^{(N)}_t \circ \cdots \circ \phi^{(1)}_t \) for \( C^1 \)-small contact isotopies \( \{ \phi^{(j)}_t \} \) and then applying the composition formula \( N - 1 \) times to the corresponding generating functions \( F^{(j)}_t \). By Lemma 2.16 the resulting family \( F_t \) does not depend, up to equivalence, on the order in which we apply the composition formula to the pieces. We now prove that if \( \phi_t = \psi^{(K)}_t \circ \cdots \circ \psi^{(1)}_t \) is another decomposition into \( C^1 \)-small contact isotopies and \( G^{(j)}_t : \mathbb{R}^{2n} \to \mathbb{R} \) are the corresponding generating functions then \( F^{(1)}_t \circ \cdots \circ F^{(N)}_t \) is equivalent to \( G^{(1)}_t \circ \cdots \circ G^{(K)}_t \). For every \( j = 1, \ldots, N \) and \( l = 1, \ldots, K \) consider the homotopy \( \{ \psi^{(j)}_{s,t} \}_{s \in [0,1]} \) from \( \{ \phi^{(j)}_{0,t} \} = \{ \text{id} \} \) to \( \{ \phi^{(j)}_{1,t} \} = \{ \psi^{(j)}_{1,t} \} \) defined by \( \phi^{(j)}_{s,t} = \phi^{(j)}_{st} \), and the homotopy \( \{ \psi^{(j)}_{s,t} \}_{s \in [0,1]} \) from \( \{ \psi^{(j)}_{0,t} \} = \{ \text{id} \} \) to \( \{ \psi^{(j)}_{1,t} \} = \{ \psi^{(j)}_{1,t} \} \) defined by \( \psi^{(j)}_{s,t} = \psi^{(j)}_{st} \). Let \( F^{(j)}_{s,t} \), \( F^{(j)}_{s,t} \) and \( G^{(j)}_{s,t} \) be generating functions (without fibre variable) for \( \phi^{(j)}_{s,t} \), \( (\phi^{(j)}_{s,t})^{-1} \) and \( \psi^{(j)}_{s,t} \) respectively. For every \( s \), the two 1-parameter families

\[
G^{(1)}_{s,t} \circ \cdots \circ G^{(N)}_{s,t} \circ F^{(N)}_{s,t} \circ \cdots \circ F^{(1)}_{s,t}
\]

and

\[
\overline{F}^{(N)}_{s,t} \circ \cdots \circ \overline{F}^{(1)}_{s,t} \circ \overline{F}^{(N)}_{s,t} \circ \cdots \circ \overline{F}^{(1)}_{s,t}
\]

generate \( \{ \text{id} \} \). For \( s = 0 \) both are of the form \( 0 \circ \cdots \circ 0 \) and thus, by Lemma 2.15, they are equivalent to 0. By Lemma 2.13 we thus have that the two families \( G^{(1)}_{s,t} \circ \cdots \circ G^{(N)}_{s,t} \circ F^{(N)}_{s,t} \circ \cdots \circ F^{(1)}_{s,t} \) and \( \overline{F}^{(N)}_{s,t} \circ \cdots \circ \overline{F}^{(1)}_{s,t} \circ \overline{F}^{(N)}_{s,t} \circ \cdots \circ \overline{F}^{(1)}_{s,t} \) are equivalent to 0. It follows that \( F^{(1)}_{t} \circ \cdots \circ F^{(N)}_{t} \) and \( G^{(1)}_{t} \circ \cdots \circ G^{(N)}_{t} \) are both equivalent to

\[
G^{(1)}_{t} \circ \cdots \circ G^{(K)}_{t} \circ F^{(N)}_{t} \circ \cdots \circ F^{(1)}_{t} \circ F^{(N)}_{t} \circ \cdots \circ F^{(1)}_{t},
\]

and so they are equivalent to each other. \( \square \)

The next result is used in Section 4 to obtain that the non-linear Maslov index descends to a map on the universal cover of the identity component of the contactomorphism group.

2.18. Proposition. Suppose that \( \{ \phi_{0,t} \} \) and \( \{ \phi_{1,t} \} \) are two contact isotopies from the identity to the same contactomorphism \( \phi \) which are homotopic with fixed endpoints. Then there are families \( F_{0,t} \) and \( F_{1,t} \) of conical generating functions for \( \phi_{0,t} \) and \( \phi_{1,t} \) respectively, constructed as in the proof of Proposition 2.10, such that \( F_{0,0} = F_{1,0} \) and \( F_{0,1} \) and \( F_{1,1} \) differ by a fibre preserving conical homeomorphism.

Proof. Let \( \phi_{s,t} \) be a smooth homotopy with fixed endpoints from \( \{ \phi_{0,t} \} \) to \( \{ \phi_{1,t} \} \), and for \( N \) big enough write \( \phi_{s,t} = \phi^{(N)}_{s,t} \circ \cdots \circ \phi^{(1)}_{s,t} \) for \( C^1 \)-small contact isotopies \( \{ \phi^{(j)}_{s,t} \} \), with each family \( \{ \phi^{(j)}_{s,t} \}_{s \in [0,1]} \) depending smoothly on \( s \). For each \( j = 1, \ldots, N \) let \( F^{(j)}_{s,t} : \mathbb{R}^{2n} \to \mathbb{R} \) be the corresponding generating functions, and define \( F_{s,t} = F^{(1)}_{s,t} \circ \cdots \circ F^{(N)}_{s,t} \). For every \( s \) we have that \( F_{s,0} = 0 \circ \cdots \circ 0 \) and that \( F_{s,1} \) generates \( \phi \). In particular \( F_{0,0} = F_{1,0} \) and, by Lemma 2.13, \( F_{0,1} \) and \( F_{1,1} \) differ by a fibre preserving conical homeomorphism. \( \square \)

Let \( \{ \phi_t \}_{t \in [0,1]} \) be a contact isotopy of \( L^{2n-1}_2 \), and consider a family \( F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R} \) of conical generating functions, as constructed in Proposition 2.10. Since all \( F_t \) are homogeneous of degree 2, they are determined by their restrictions \( \overline{F}_t : S^{2n(N+1)-1} \to \mathbb{R} \). Moreover, as the \( F_t \) are invariant by the \( Z_k \)-action, the \( \overline{F}_t \) are also \( Z_k \)-invariant and so they descend to a family of functions

\[
f_t : L^{2n(N+1)-1}_k \to \mathbb{R}
\]

that are \( C^1 \) with Lipschitz differential. We also say that \( f_t \) (as well as the corresponding \( F_t \)) is a family of generating functions for the contact isotopy \( \{ \phi_t \} \).

2.19. Proposition. Let \( f_t : L^{2n(N+1)-1}_k \to \mathbb{R} \) be a family of generating functions for a contact isotopy \( \{ \phi_t \} \) of \( L^{2n-1}_2 \). For every \( t \) there is a 1–1 correspondence between (non-degenerate) discriminant points of \( \phi_t \) that are discriminant points also for the lift of \( \phi_t \) to \( S^{2n-1} \) and (non-degenerate)
critical points of \( f_t \) of critical value zero. This correspondence is given by the restriction of a map 
\( L_k^{2n-1} \to L_k^{2n(N+1)-1} \) that is isotopic to the standard inclusion.

**Proof.** For every \( t \), denote by \( F_t \) the conical function on \( \mathbb{R}^{2n(N+1)} \) inducing \( f_t \). Since \( F_t \) is homogeneous of degree 2 and \( \mathbb{Z}_k \)-invariant, its critical points come in \((\mathbb{Z}_k \times \mathbb{R}_{>0})\)-orbits and all have critical value zero. Such \((\mathbb{Z}_k \times \mathbb{R}_{>0})\)-orbits are in \(1 \to 1\) correspondence with critical points of \( f_t \) of critical value zero. On the other hand, the discriminant points of \( \phi_t \) which are also discriminant points of the lift to \( S^{2n-1} \) correspond to \((\mathbb{Z}_k \times \mathbb{R}_{>0})\)-orbits of fixed points of \( \Phi_t \). We then use the fact that (non-degenerate) discriminant points of \( \phi_t \) correspond to (non-degenerate) critical points of \( F_t \) (see for example [Sa11c, Lemma 3.5]). We now show that the \(1 \to 1\) correspondence is given by the restriction of a map 
\( L_k^{2n-1} \to L_k^{2n(N+1)-1} \) that is isotopic to the standard inclusion.

Recall that, for every \( t \in [0,1] \), the conical map
\[
\mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to T^* \mathbb{R}^{2n} , \ (\xi, \nu) \mapsto (\xi, \frac{\partial F_t}{\partial \xi}(\xi, \nu))
\]
restricts to a homeomorphism \( i_{F_t} \) between the set \( \Sigma_{F_t} \) of fibre critical points and \( \Gamma_{\Phi_t} \). Fix now \( t \in [0,1] \). The required isotopy is the map induced on the quotient by the \((\mathbb{Z}_k \times \mathbb{R}_{>0})\)-action by the composition
\[
[0,t] \times \mathbb{R}^{2n} \to \bigcup_{s \in [0,t]} \{s\} \times \Sigma_{F_s} \to \bigcup_{s \in [0,t]} \{s\} \times \Sigma_{F_s} \subset [0,t] \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN}
\]
\[
(s, \xi) \mapsto (s, \tau (\xi, \Phi_s(\xi))) \mapsto (s, (i_{F_s})^{-1} (\tau (\xi, \Phi_s(\xi)))) .
\]

\( \square \)

**Monotonicity and quasi-additivity of generating functions.** We start with the following monotonicity property, that is used to show that lens spaces are orderable.

**2.20. Proposition** (Monotonicity of generating functions). Let \( \{\phi_t\}_{t \in [0,1]} \) be a contact isotopy of \( L_k^{2n-1} \) (starting at the identity). Assume that, for a subinterval \( I \) of \( [0,1] \), \( \{\phi_t\}_{t \in I} \) is non-negative (respectively positive, non-positive, negative). Then there is a 1-parameter family \( f_t : L_k^{2M-1} \to \mathbb{R} \) of generating functions for \( \{\phi_t\}_{t \in [0,1]} \), obtained by the construction of Proposition 2.10, such that for \( t \in I \) we have \( \frac{\partial f_t}{\partial \tau} \geq 0 \) (respectively \( \frac{\partial f_t}{\partial \tau} > 0 \), \( \frac{\partial f_t}{\partial \tau} \leq 0 \), \( \frac{\partial f_t}{\partial \tau} < 0 \)).

**Proof.** Let \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1 \) be a decomposition of the time interval such that \( I \) is a union of intervals \( [t_{j-1},t_j] \). Let \( f_t : L_k^{2M-1} \to \mathbb{R} \) be a 1-parameter family of generating functions for \( \{\phi_t\} \) that is obtained, using such a decomposition, by the construction of Proposition 2.10 as described in Remark 2.11. If \( \{\phi_t\}_{t \in I} \) is non-negative (respectively positive, non-positive, negative) then the same is true for all \( \{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1},t_j]} \) such that \( [t_{j-1},t_j] \subset I \). We then conclude using the Hamilton–Jacobi equation and the composition formula as in [Sa11c, Lemma 3.6].

\( \square \)

As we have seen in Proposition 2.2, the generating function for a composition is not given by the direct sum of the generating functions of the factors. On the other hand it agrees with it in codimension \( 2n \), in the following sense.

**2.21. Proposition** (Quasiadditivity of generating functions). Suppose that \( F_1 : \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \to \mathbb{R} \) and \( F_2 : \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \to \mathbb{R} \) are (conical) generating functions for the (conical) symplectomorphisms \( \Phi^{(1)} \) and \( \Phi^{(2)} \) respectively. Then there is a (conical) injection
\[
\iota : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \to \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2})
\]
such that \( (F_1 \circ F_2) \circ \iota = F_1 \oplus F_2 \).
Proof. We have
\[F_1 \oplus F_2(\zeta_1, \zeta_2; \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2)\]
and
\[F_1 \neq F_2(q; \zeta_1, \zeta_2; \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle .\]
Thus for the injection \(i(\zeta_1, \zeta_2; \nu_1, \nu_2) = (\zeta_1, \zeta_2, \nu_1, \nu_2)\) we have \((F_1 \neq F_2) \circ i = F_1 \oplus F_2.\)

The above quasiadditivity property is crucial in proving that the non-linear Maslov index is a quasi-morphism (Proposition 4.2). If generating functions had been additive, not just quasiadditive, then at least for real projective space the non-linear Maslov index would have been a homomorphism. Since no non-trivial homomorphisms exists on contactomorphism groups [Ts08, Ry10], this shows that the lack of additivity of the composition formula is not a technical failure but something essential.

To conclude this section we record one more property of the composition formula that is needed later. For a quadratic form \(Q\) we denote by \(i(Q)\) the maximal dimension of a subspace on which \(Q\) is negative semi-definite. We then have the following result.

\[\text{Lemma.} \text{ Let } Q_1: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \rightarrow \mathbb{R} \text{ and } Q_2: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \rightarrow \mathbb{R} \text{ be quadratic forms that are obtained by applying the composition formula to the zero function several times. Then } i(Q_1 \neq Q_2) = i(Q_1) + i(Q_2).\]

Proof. For any quadratic form \(Q: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}\) that is obtained by applying the composition formula to the 0 function several times, the kernel of the associated bilinear symmetric form is equal to the 2-dimensional subspace \(V = \{ (\zeta; \nu) \mid \nu = (\zeta, \cdots, \zeta) \}\) (this can be seen using induction on \(N\)). Denote by \(V'\) the quotient of the domain of \(Q\) by \(V\), and by \(Q': V' \cong \mathbb{R}^{2nN} \rightarrow \mathbb{R}\) the induced non-degenerate quadratic form. Consider now two quadratic forms \(Q_1: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \rightarrow \mathbb{R}\) and \(Q_2: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \rightarrow \mathbb{R}\) obtained by applying the composition formula to the 0 function several times. For the induced non-degenerate quadratic form \((Q_1 \neq Q_2)' : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2} \rightarrow \mathbb{R}\) we have
\[(Q_1 \neq Q_2)'(\zeta_1, \zeta_2, \nu_1, \nu_2) = Q_1'(\nu_1 - (\zeta_1, \cdots, \zeta_1)) + Q_2'(\nu_2 - (\zeta_2, \cdots, \zeta_2)) - 2 \langle \zeta_2, i\zeta_1 \rangle\]
and so the index of \((Q_1 \neq Q_2)'\) is equal to \(i(Q_1') + i(Q_2') + 2n\). We conclude that
\[i(Q_1 \neq Q_2) = 2n + i((Q_1 \neq Q_2)') = i(Q_1) + i(Q_2).\]

\[\square\]

3. The cohomological index

As already outlined in the introduction, the value of the non-linear Maslov index of a contact isotopy \(\{\phi_t\}\) of a lens space \(L^{2n-1}_k\) depends on the changes in topology of the sublevel sets of a 1-parameter family \(f_t: L^{2M-1}_k \rightarrow \mathbb{R}\) of generating functions. As in [Gi90] and [Th98], the topological invariant that we use to analyze these changes is the cohomological index. In this section we review the definition of this invariant and describe some of its properties in the case of lens spaces. Cohomological indices have also been studied in a more general context by Fadell and Rabinowitz [FR78] (see also Remark 3.7).

Continuity of the cohomological index (Proposition 3.9 (ii) below) is important for our applications. Since the sets we need to consider (sublevel sets of generating functions) might not be locally contractible, in order to guarantee continuity we work with Čech cohomology (as for instance in [FR78]). Note that Čech cohomology agrees with singular cohomology on spaces that are paracompact and locally contractible (see [Sp66, Corollary 6.8.8 and Theorem 6.9.1]), in particular on manifolds or, more generally, on CW-complexes. Recall also that the Čech cohomology of a compact subset \(A\) of a manifold can be computed in terms of singular cohomology as
\[(3.1) \quad \tilde{H}^*(A) = \lim_{\cdots \to} H^*(U_j)\]
where $U_j$ is any decreasing sequence of open sets having $A$ as their intersection. Indeed, $\check{H}^*(A) = \lim H^*(U_j)$ (see [Sp66, Theorem 6.6.2]) and since the $U_j$ are open, their Čech cohomology agrees with their singular cohomology.

Recall that we assume that $k$ is prime (cf. Remark 1.4).

### 3.2. Definition

Let $A$ be a paracompact Hausdorff topological space and $\pi: \tilde{A} \to A$ a principal $\mathbb{Z}_k$-bundle with classifying map $g: A \to B\mathbb{Z}_k = L^\infty_k$. The **cohomological index** of $\pi: \tilde{A} \to A$ is the dimension over $\mathbb{Z}_k$ of the image of the induced map $g^*: \check{H}^*(L^\infty_k; \mathbb{Z}_k) \to \check{H}^*(A; \mathbb{Z}_k)$. If $A$ is a subset of a lens space $L^M_k(w)$ then its cohomological index, denoted by $\text{ind}(A)$, is defined to be the cohomological index of the restriction $\pi: \tilde{A} \to A$ of the principal $\mathbb{Z}_k$-bundle $\pi: S^{2M-1}(w) \to L^M_k(w)$.

It follows from the definition that $0 \leq \text{ind}(A) \leq 2M$ if $A \subset L^M_k(w)$, that $\text{ind}(A) = 1$ if $A$ is finite and non-empty, and that $\text{ind}(\emptyset) = 0$.

We now specialize to the case when the prime $k$ is different from 2, leaving to the reader the task of adapting the discussion to the (easier) case of $k = 2$ (cf. Remarks 3.5 and 3.7).

A principal $\mathbb{Z}_k$-bundle $\tilde{A} \to A$ is determined by the Čech cohomology class $\alpha \in \check{H}^1(A; \mathbb{Z}_k)$ that is represented by the transition functions for a choice of local trivializations. The Bockstein homomorphism $B: \check{H}^*(A) \to \check{H}^{*+1}(A)$ (see [Ha02, Section 3.E]) is a derivation whose square is zero, so, setting $\beta = B(\alpha) \in \check{H}^2(A; \mathbb{Z}_k)$, we have

$$B(\alpha^j) = \beta^{j+1} \quad \text{and} \quad B(\beta^j) = 0 \quad \text{for all} \ j \geq 0.$$

A map of principal $\mathbb{Z}_k$-bundles $\tilde{A} \to \tilde{B}$ pulls back the classes $\alpha, \beta$ on the base $B$ to the classes $\alpha, \beta$ on the base $A$.

### 3.3. Lemma

For any $M$-tuple of weights $\underline{w}$ and $0 \leq j \leq 2M - 1$,

$$\check{H}^j(L^M_k(w); \mathbb{Z}_k) \text{ is generated by } \begin{cases} \beta^i & \text{for } j = 2i \\ \alpha^i & \text{for } j = 2i + 1 \end{cases}. \tag{3.4}$$

For a subset $A$ of $L^M_k(w)$,

$$\text{ind}(A) = \dim_{\mathbb{Z}_k} (\text{im } \iota^*)$$

where $\iota^*: \check{H}^*(L^M_k(w); \mathbb{Z}_k) \to \check{H}^*(A; \mathbb{Z}_k)$ is the map on Čech cohomology that is induced by the inclusion $\iota: A \hookrightarrow L^M_k(w)$. Moreover,

$$\text{im } \iota^* \cap \check{H}^j(A; \mathbb{Z}_k) \cong \begin{cases} \mathbb{Z}_k & \text{if } 0 \leq j < \text{ind}(A) \\ 0 & \text{if } j \geq \text{ind}(A) \end{cases}. \tag{3.5}$$

**Proof.** For (3.4), see [Ha02, Example 3E.2]. The equality $\text{ind}(A) = \dim_{\mathbb{Z}_k} (\text{im } \iota^*)$ follows from the facts that the classifying map $g(\underline{w}): L^M_k(w) \to L^\infty_k$ induces a surjection in cohomology (by (3.4)) and that $g(\underline{w}) \circ \iota$ is a classifying map for $A$. The ring structure and the action of the Bockstein homomorphism imply that if $x \in \check{H}^j(L^M_k(w))$ is non-zero and $\iota^*(x) = 0$ then $\iota^*(y) = 0$ for all $y$ with $\deg(y) \geq \deg(x)$; this implies the last statement.

A **lens subspace** of $L^M_k(w)$ is the $\mathbb{Z}_k$-quotient of the intersection of $S^{2M-1}(w)$ with a $\mathbb{Z}_k$-invariant real (hence complex) linear subspace of $\mathbb{C}^{2M}(w)$. Lemma 3.3 implies that the cohomological index of a $2r - 1$ dimensional lens subspace is $2r$.

---

6In our applications $M$ is a multiple of $n$ and the $M$-tuple of weights on $L^M_k(w)$ has the form $\underline{w} = (w', \ldots, w')$ for an $n$-tuple of weights $w'$ on $L^M_k(w)$. However in this section $\underline{w}$ can be any tuple of weights.

7We write $S^{2M-1}(w)$, and similarly $\mathbb{C}^{M}(w)$ or $\mathbb{R}^{2M}(w)$, when we wish to specify the $\mathbb{Z}_k$-action.

8If $\{\psi_{ij}: U_i \cap U_j \to \mathbb{Z}\}$ are lifts of the transition functions $\{\alpha_{ij}: U_i \cap U_j \to \mathbb{Z}\}$, then $\beta$ is represented by the Čech 2-cocycle $\left\{ \frac{1}{k} (\psi_{ij} + \psi_{ji} + \psi_{ji}) \mod k \right\}$. 

3.5. **Remark.** For a subset \( A \) of a real projective space \( \mathbb{RP}^M \),
\[
\text{ind}(A) = \min \{ j \in \mathbb{N} | i^*(x^j) = 0 \}
\]
where \( x \) is the generator of \( H^1(\mathbb{RP}^M; \mathbb{Z}_2) \). Similarly one defines the cohomological index for subsets of complex projective spaces and for principal \( S^1 \)-bundles; in this case the analogue of (3.6) holds for \( x \) a generator of degree two (cf. [Th98]).

3.7. **Remark.** Let \( \pi: \tilde{A} \to A \) be a principal \( G \)-bundle over a paracompact Hausdorff topological space with classifying map \( g: A \to BG \). For any non-zero class \( \eta \in H^*(BG) \) Fadell and Rabinowitz [FR78] define the \( \eta \)-index as the maximal \( j \in \mathbb{N} \) such that \( g^*(\eta^j) \neq 0 \). If \( G = \mathbb{Z}_2 \) and \( \eta \) is the generator of \( H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \) then the \( \eta \)-index just differs by 1 from the index of Definition 3.2. If however \( G = \mathbb{Z}_2 \) with \( k \neq 2 \) and \( \beta \) is a generator of \( H^2(L_k^\infty; \mathbb{Z}_2) \) then the \( \beta \)-index is equal to \( \lfloor \frac{\text{ind}(A)-1}{2} \rfloor \). We use the index from Definition 3.2 rather than the \( \beta \)-index in order to obtain a better bound on the number of translated points (see Section 5): using the \( \beta \)-index we would only prove existence of \( n \) translated points on \( L_k^{2n-1} \), even in the non-degenerate case.

Given subsets \( A \) of \( L_k^{2M-1}(w) \) and \( B \) of \( L_k^{2M'-1}(w') \) with preimages \( \tilde{A} \subset S^{2M-1}(w) \) and \( \tilde{B} \subset S^{2M'-1}(w') \), their \( Z_k \)-join is the subset
\[
A \ast Z_k B \subset L_k^{2(M+M')-1}(w, w')
\]
defined by
\[
A \ast Z_k B = \left\{ \sqrt{t} a, \sqrt{1-t} b \mid a \in \tilde{A}, b \in \tilde{B}, 0 \leq t \leq 1 \right\}
\]
if \( A \) and \( B \) are non-empty. If \( B \) is empty, we define \( A \ast Z_k \emptyset \) to be the image of \( A \) under the natural embedding \( a \mapsto [a, 0] \) of \( L_k^{2M-1}(w) \) into \( L_k^{2(M+M')-1}(w, w') \). We define \( \emptyset \ast Z_k B \) similarly. Finally, the \( Z_k \)-join of the empty sets is empty.

We now describe the properties of the cohomological index that we need for our applications. The proofs of properties (i)-(iv) are easy adaptations of the corresponding proofs in [Gi90, Th98, FR78] and are included for the convenience of the reader. In (v), the lower bound on \( \text{ind}(A \ast Z_k B) \) requires a more involved proof, which we postpone to Appendix B.

3.9. **Proposition.** The cohomological index of subsets of lens spaces has the following properties:

(i) (Monotonicity) If \( A \subset B \subset L_k^{2M-1}(w) \) then \( \text{ind}(A) \leq \text{ind}(B) \).

(ii) (Continuity) Every closed subset \( A \) of \( L_k^{2M-1}(w) \) has a neighborhood \( U \) such that if \( A \subset V \subset U \) then \( \text{ind}(V) = \text{ind}(A) \).

(iii) (Lefschetz property) Let \( A \) be a closed subset of \( L_k^{2M-1}(w) \), and let \( A' = A \cap H \) where \( H \subset L_k^{2M-1}(w) \) is a lens subspace of codimension 2. Then \( \text{ind}(A') \geq \text{ind}(A) - 2 \).

(iv) (Subadditivity) For closed subsets \( A \) and \( B \) of \( L_k^{2M-1}(w) \) we have
\[
\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B) + 1
\]
and
\[
\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B) \quad \text{if \( \text{ind}(A) \) is even or \( \text{ind}(B) \) is even.}
\]

(v) (Join quasi-additivity) For closed subsets \( A \) of \( L_k^{2M-1}(w) \) and \( B \) of \( L_k^{2M'-1}(w') \) we have
\[
| \text{ind}(A \ast Z_k B) - \text{ind}(A) - \text{ind}(B) | \leq 1
\]
and
\[
\text{ind}(A \ast Z_k B) = \text{ind}(A) + \text{ind}(B) \quad \text{if \( \text{ind}(A) \) is even or \( \text{ind}(B) \) is even.}
\]

In particular (Join stability),
\[
\text{ind} \left( A \ast Z_k L_k^{2K-1}(w') \right) = \text{ind}(A) + 2K
\]

3.10. **Remark.** The above properties (ii), (iii), (iv), and (v) are stated for closed subspaces of lens spaces but they hold (and are proved) also for open subsets of lens spaces.
Proof Proposition 3.9. (i) Let $\tilde{A}$ and $\tilde{B}$ be the preimages in $S_k^{2n-1}(w)$ of $A$ and $B$. The result follows from the fact that the restriction to $A$ of the classifying map of $B \to B$ is a classifying map for $\tilde{A} \to A$.

(ii) Let $x$ be a generator of $H^{\text{ind}(A)}(L_k^{2M-1}(w); \mathbb{Z}_k)$. Then $i_{U^*}(x) = 0$. By (3.1), there exists an open neighborhood $U$ of $A$ such that $i_U(x) = 0$, where $i_U : U \to M$ is the inclusion map. By Lemma 3.3, $\text{ind}(U) \leq \text{ind}(A)$. By monotonicity, $\text{ind}(U) \geq \text{ind}(A)$.

(iii) Assume that $\text{ind}(A) \geq 3$; otherwise the inequality is trivial. By continuity of the cohomological index, there exist open neighborhoods $U$ of $A$ and $V$ of $H$ such that $\text{ind}(U) = \text{ind}(A)$, $\text{ind}(V) = \text{ind}(H)$, and $\text{ind}(U \cap V) = \text{ind}(A \cap H)$. We have the following commuting diagram, where $D$ denotes the Poincaré duality isomorphism and $\bullet$ the homology intersection product (see [Do95, VIII.13.5]):

$$
\begin{array}{ccc}
H_*(U) \otimes H_*(V) & \xrightarrow{\bullet} & H_*(U \cap V) \\
\downarrow i_{U*} \otimes i_{V*} & & \downarrow i_{U \cap V*} \\
H_*(L_k^{2M-1}(w)) \otimes H_*(L_k^{2M-1}(w)) & \xrightarrow{D \otimes D} & H_*(L_k^{2M-1}(w)) \\
\downarrow & & \downarrow D \\
H^*(L_k^{2M-1}(w)) \otimes H^*(L_k^{2M-1}(w)) & \cup & H^*(L_k^{2M-1}(w)).
\end{array}
$$

Let $x$ be a class in $H_{\text{ind}(A)-1}(U)$ such that $i_{U*}(x) \neq 0$ (this exists, as homology and cohomology with field coefficients are dually paired) and similarly let $y$ be a class in $H_{2M-3}(V)$ with $i_{V*}(y) \neq 0$. Since $D(i_{U*}(x))$ is a non-zero class in $H^{\leq 2M-3}(L_k^{2M-1}(w))$ and $D(i_{V*}(y))$ is a non-zero multiple of $\beta$, we have $D(i_{U*}(x)) \cup D(i_{V*}(y)) \neq 0$. It follows that $i_{U \cap V*}(x \bullet y) \neq 0$, which shows that $\text{ind}(A \cap H) = \text{ind}(U \cap V) \geq \text{ind}(A) - 2$.

(iv) Assume that $\text{ind}(A) + \text{ind}(B) < 2M$; otherwise the inequality is trivial. By continuity, there exist open neighborhoods $U$ of $A$ and $V$ of $B$ such that $\text{ind}(U) = \text{ind}(A)$, $\text{ind}(V) = \text{ind}(B)$, and $\text{ind}(U \cup V) = \text{ind}(A \cup B)$. By the exact cohomology sequence of the pair

$$
\cdots \to H^*(L_k^{2M-1}(w), U) \xrightarrow{j_{U*}} H^*(L_k^{2M-1}(w), U \cap V) \xrightarrow{j_{U \cap V*}} H^*(U) \to \cdots.
$$

and by Lemma 3.3, the index of $U$ is the lowest degree of a non-zero class in the image of $j_U^*$. A similar statement holds for $V$. Consider the commutative diagram

$$
\begin{array}{ccc}
H^*(L_k^{2M-1}(w), U) \otimes H^*(L_k^{2M-1}(w), V) & \cup & H^*(L_k^{2M-1}(w), U \cup V) \\
\downarrow j_{U*} \otimes j_{V*} & & \downarrow j_{U \cup V*} \\
H^*(L_k^{2M-1}(w)) \otimes H^*(L_k^{2M-1}(w)) & \cup & H^*(L_k^{2M-1}(w)).
\end{array}
$$

Assume first that one of the indices is even, for instance that of $A$. Let $x$ be a class in $H^{\text{ind}(A)}(L_k^{2M-1}(w), U)$ such that $j_{U*}^*(x) \neq 0$, and $y$ a class in $H^{\text{ind}(B)}(L_k^{2M-1}(w), V)$ such that $j_{V*}^*(y) \neq 0$. Since $j_U^*(x)$ is a non-zero multiple of $\beta^{\text{ind}(A)/2}$ and $\text{ind}(A) + \text{ind}(B) < 2M$, it follows that $j_{U \cup V*}^*(x \cup y) = j_U^*(x) \cup j_V^*(y)$ is non-zero and so $\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B)$.

If both $\text{ind}(A)$ and $\text{ind}(B)$ are odd, replace $x$ in the above argument with a class $x' \in H^{\text{ind}(A)+1}(L_k^{2M-1}(w), U)$ such that $j_U^*(x') \neq 0$, to obtain $\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B) + 1$.

(v) The subset $A' = (A \ast_{Z_k} B) \setminus B$ deformation retracts to $A$, and the subset $B' = (A \ast_{Z_k} B) \setminus A$ deformation retracts to $B$. Since $A \ast_{Z_k} B = A' \cup B'$, the subadditivity property (iv) implies that $\text{ind}(A \ast_{Z_k} B) \leq \text{ind}(A) + \text{ind}(B) + 1$ and $\text{ind}(A \ast_{Z_k} B) \leq \text{ind}(A) + \text{ind}(B)$ if at least one of the indices is even. The reverse inequalities in the join quasi-additivity property are proved in Appendix B.

$\square$
3.11. Remark. In the case of real projective spaces the cohomology ring is generated by the generator in degree one, and the above arguments can be adapted to show that properties (iii) and (iv) of Proposition 3.9 hold in the following stronger form:

(iii') If $A$ is a closed subset of $\mathbb{RP}^M$, and $A' = A \cap H$ where $H \subset \mathbb{RP}^M$ is a real projective subspace of codimension one, then $\text{ind}(A') \geq \text{ind}(A) - 1$.

(iv') For closed subsets $A$ and $B$ of $\mathbb{RP}^M$ we have
\[
\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B).
\]

Moreover, we also have the following result (see Remark B.23 or [Gi90]):

(v') For closed subsets $A$ of $\mathbb{RP}^M$ and $B$ of $\mathbb{RP}^{M'}$ we have
\[
\text{ind}(A \ast_{\mathbb{Z}_2} B) = \text{ind}(A) + \text{ind}(B).
\]

Analogous properties hold for the cohomological index of subsets of complex projective spaces (see [Th98]). As we will see in Sections 4 and 5, the weaker properties that we have in the case of lens spaces still suffice to define a non-linear Maslov index and recover the applications we are interested in.

We define the index of a conical function $F: \mathbb{R}^{2M} \to \mathbb{R}$ by
\[
\text{ind}(F) = \text{ind}\left(\{f \leq 0\}\right)
\]
where $f$ is the function on $L^{2M-1}_k(w)$ induced by $F$.

3.12. Remark. If $Q$ is a $\mathbb{Z}_k$-invariant quadratic form on $\mathbb{R}^{2M}$ then $\text{ind}(Q)$ coincides with $i(Q)$, the maximal dimension of a subspace on which $Q$ is negative semi-definite. In particular, $\mathbb{Z}_k$-invariance implies (if $k > 2$) that in this case $\text{ind}(Q)$ is even.

Given functions $f$ and $g$ on $L^{2M-1}_k(w)$ and $L^{2M'-1}_k(w')$ respectively, we write
\[
f \oplus g: L^{2(M+M')-1}_k(w, w') \to \mathbb{R}
\]
for the function induced by the sum $F \oplus G: \mathbb{R}^{2(M+M')} \to \mathbb{R}$, where $F$ and $G$ are the conical functions on $\mathbb{R}^{2M}$ and $\mathbb{R}^{2M'}$ associated to $f$ and $g$.

3.13. Proposition. Let $f: L^{2M-1}_k(w) \to \mathbb{R}$ and $g: L^{2M'-1}_k(w') \to \mathbb{R}$ be continuous functions. Then
\[
\text{ind}\left(\{f \oplus g \leq 0\}\right) = \text{ind}\left(\{f \leq 0\} \ast_{\mathbb{Z}_k} \{g \leq 0\}\right).
\]

Proof. By continuity (Proposition 3.9(ii)) there is a neighborhood $U$ of $\{f \leq 0\} \ast_{\mathbb{Z}_k} \{g \leq 0\}$ with $\text{ind}(U) = \text{ind}(\{f \leq 0\} \ast_{\mathbb{Z}_k} \{g \leq 0\})$. Consider the diagram
\[
\begin{array}{c}
\{f \leq 0\} \ast_{\mathbb{Z}_k} \{g \leq 0\} \\
\downarrow r \\
L^{2M+2M'-1}_k(w, w')
\end{array}
\]$
\begin{array}{c}
U \\
\downarrow j \\
L^{2M-1}_k(w, w')
\end{array}$

By monotonicity, it suffices to show that the inclusion $j$ can be deformed into a map $r$ with image contained in $U$. This can be done as follows. We work $\mathbb{Z}_k$-equivariantly on the preimages in $S^{M+M'-1}(w, w')$. To simplify the formulas, given $x \in S^{2M-1}(w)$ and $y \in S^{2M'-1}(w')$ we write $tx + (1-t)y$ for $(\sqrt{t}x, \sqrt{1-t}y) \in S^{2M+2M'-1}(w, w')$. Moreover we still write $f: S^{2M-1}(w) \to \mathbb{R}$, $g: S^{2M'-1}(w') \to \mathbb{R}$, and $f \oplus g: S^{2(M+M')-1}(w, w') \to \mathbb{R}$ for the composition of the original...
functions $f$, $g$, and $f \oplus g$ with the projections from spheres to lens spaces. With this notation we have
\[(f \oplus g)(tx + (1 - t)y) = tf(x) + (1 - t)g(y).\]
The rough idea for constructing the map $r$ is the following. If a point $tx + (1 - t)y$ is in $\{f \oplus g \leq 0\}$ then at least one of $f(x)$ and $g(y)$ is non-positive. The map $r$ will act as the identity on points $tx + (1 - t)y$ with $f(x)$ and $g(y)$ both non-negative. If $f(x)$ is positive, and thus $g(y)$ is negative, $r$ will move the point $tx + (1 - t)y$ to a point $(1 - s)(tx + (1 - t)y) + sy$, with $s \in [0, 1]$ big enough so that this point is in the chosen neighborhood $U$ of $\{f \leq 0\}$ such that $g \leq 0$. Similarly if $g(y)$ is positive. However, one needs to interpolate between these deformations in order to ensure that the resulting map is continuous. Here are the details. For each $\delta > 0$ consider the map
\[R_\delta : \{f \oplus g \leq 0\} \times [0, 1] \to \{f \oplus g \leq 0\}\]
defined by the expression
\[R_\delta (tx + (1 - t)y, s) = \begin{cases} 
(1 - s)(tx + (1 - t)y) + sy & \text{if } f(x) \geq \delta \\
(1 - s\frac{f(x)}{\delta})(tx + (1 - t)y) + s\frac{f(x)}{\delta}y & \text{if } 0 \leq f(x) \leq \delta \\
(tx + (1 - t)y) & \text{if } f(x) \leq 0 \text{ and } g(y) \leq 0 \\
(1 - s\frac{g(y)}{\delta})(tx + (1 - t)y) + s\frac{g(y)}{\delta}x & \text{if } 0 \leq g(y) \leq \delta \\
(tx + (1 - t)y) + sx & \text{if } g(y) \geq \delta.
\end{cases}\]
Thus $R_\delta$ moves a point $tx + (1 - t)y$ such that $f(x) > 0$ along the segment
\[s \mapsto (1 - s)(tx + (1 - t)y) + sy\]
a portion of the way towards $y$ (note that $g(y)$ must be negative). To ensure continuity, the portion depends on the value of $f$ at $x$. The pasting lemma guarantees the continuity of $R_\delta$. By continuity of $(t, x, y) \mapsto (tx + (1 - t)y)$ and compactness of its domain, for $\delta$ small enough the set $\{f \oplus g \leq 0\} \cap \{tx + (1 - t)y \mid f(x) \leq \delta \text{ or } g(y) \leq \delta\}$ is contained in $U$. For such $\delta$, the image of $R_\delta(\cdot, 1)$ is contained in the preimage of $U$ in $S^{2(M+M')-1}(w, w')$. We take $r$ to be the map induced by $R_\delta(\cdot, 1)$ on the $\mathbb{Z}_k$-orbits.

Proposition 3.13 and Proposition 3.9 (v) imply the following result.

3.14. Corollary. For conical functions $F : \mathbb{R}^{2M} \to \mathbb{R}$ and $G : \mathbb{R}^{2M'} \to \mathbb{R}$ we have
\[|\text{ind}(F \oplus G) - \text{ind}(F) - \text{ind}(G)| \leq 1.\]
Moreover, if either $F$ or $G$ has even index, in particular if either $F$ or $G$ is a $\mathbb{Z}_k$-invariant quadratic form ($k > 2$), then
\[\text{ind}(F \oplus G) = \text{ind}(F) + \text{ind}(G).\]

4. The non-linear Maslov index

Using the construction of generating functions given in Section 2 and the definition and properties of the cohomological index discussed in Section 3, we now define the non-linear Maslov index
\[\mu : \text{Cont}_0(L^{2n-1}_k) \to \mathbb{Z}\]
on the universal cover of the identity component of the contactomorphism group of $L^{2n-1}_k$, and describe the properties that are used in the applications.

Definition and quasimorphism property. As before, $L^{2n-1}_k$ denotes a lens space with any vector of weights. The non-linear Maslov index of a contact isotopy $\{\phi_t\}_{t \in [0, 1]}$ of $L^{2n-1}_k$ starting at the identity is defined by
\[\mu(\{\phi_t\}) = \text{ind}(F_0) - \text{ind}(F_1)\]
where $F_1 : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$ is a 1-parameter family of generating functions for $\{\phi_t\}$ obtained via Proposition 2.10. By Proposition 2.17 and Corollary 3.14, $\mu$ does not depend on the choice of the 1-parameter family of generating functions (as long as it is obtained by the construction of
Proposition 2.10. Moreover, Proposition 2.18 implies that $\mu(\{\phi_t\})$ only depends on the homotopy class of $\{\phi_t\}$ with fixed endpoints, and thus $\mu$ descends to a map $\mu: \text{Cont}_0(L^2_{-1}) \to \mathbb{Z}$.

We now prove that the non-linear Maslov index is a quasimorphism (in the case of projective spaces, see also [Be07] and [Gi05, Theorem 9.1]).

4.1. Example. If a contact isotopy $\{\phi_t\}_{t \in [0,1]}$ has a 1-parameter family of generating functions with no fibre variable then $0 \leq \mu(\{\phi_t\}) \leq 2n$; if moreover $\{\phi_t\}$ is positive then $\mu(\{\phi_t\}) = 2n$, and if it is non-positive then $\mu(\{\phi_t\}) = 0$.

4.2. Proposition (Quasimorphism property). For elements $\{(\phi_t)\}$ and $\{(\psi_t)\}$ of $\text{Cont}_0(L^2_{-1})$ we have

$$|\mu(\{(\phi_t)\}) - \mu(\{(\psi_t)\})| \leq 2n + 1.$$

Proof. If $F_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \to \mathbb{R}$ and $G_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \to \mathbb{R}$ are families of generating functions for $\{\phi_t\}$ and $\{\psi_t\}$ respectively then

$G_t \sharp F_t: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \to \mathbb{R}$

is a family of generating functions for $\{\phi_t \circ \psi_t\}$. Since $\mu(\{(\phi_t)\}) = \mu(\{(\psi_t)\}) = \mu(\{(\phi_t)\} \cdot (\psi_t))$ we have

$$|\mu(\{(\phi_t)\}) - \mu(\{(\psi_t)\})| = |\mu(\{(\phi_t)\}) - \mu(\{(\psi_t)\})| = |\mu(\{(\phi_t)\} \cdot (\psi_t))| = |\mu(\{(\phi_t)\} \cdot (\psi_t)) - \mu(\{(\phi_t)\}) - \mu(\{(\psi_t)\})|$$

where the last equality follows from Lemma 2.22 and Remark 3.12, since $F_0$ and $G_0$ are obtained by applying the composition formula to the 0 function several times. As we have seen in Proposition 2.21, $G_t \sharp F_1$ coincides with $G_1 + F_1$ in codimension 2n. Therefore, using the Lefschetz property from Proposition 3.9 we get

$$|\mu(\{(\phi_t)\}) - \mu(\{(\psi_t)\})| = |\mu(\{(\psi_t)\})| = |\mu(\{(\phi_t)\})| = |\mu(\{(\phi_t)\} \cdot (\psi_t))| = |\mu(\{(\phi_t)\} \cdot (\psi_t)) - \mu(\{(\phi_t)\}) - \mu(\{(\psi_t)\})|.$$

Recall that a quasimorphism $\nu: G \to \mathbb{R}$ is said to be homogeneous if $\nu(x^m) = m \nu(x)$ for all $x \in G$ and $m \in \mathbb{Z}$. Any quasimorphism $\nu: G \to \mathbb{R}$ has an associated homogeneous quasimorphism, defined by

$$\overline{\nu}(x) = \lim_{m \to \infty} \frac{\nu(x^m)}{m}$$

(see for instance [Ca09, Section 2.2.2]). The homogeneous quasimorphism $\overline{\mu}: \text{Cont}_0(L^2_{-1}) \to \mathbb{R}$ associated to the non-linear Maslov index is called the asymptotic non-linear Maslov index$^{10}$.

$^9$Here it is important that our composition formula works also with fibre variables in both factors and is associative (up to equivalence). Notice that just to define the non-linear Maslov index we could have avoided using associativity of the composition formula by fixing a choice of parenthetization in the construction of Proposition 2.10. However, in the proof of the quasimorphism property we need the freedom to change the parenthetization.

$^{10}$In [Gi05, Section 9] the asymptotic non-linear Maslov index of a contact isotopy $\{\phi_t\}_{t \in [0,\infty)}$ starting at the identity is defined as $\overline{\mu}(\{\phi_t\}_{t \in [0,\infty)}) = \lim_{T \to \infty} \frac{\mu(\{\phi_t\}_{t \in [0,T]})}{T}$. Given a contact isotopy $\{\phi_t\}_{t \in [0,1]}$ we can extend
The linear case. We now show that if the lift to $\mathbb{R}^{2n}$ of a contact isotopy $\{\phi_t\}$ of $L_{k}^{2n-1}$ is a loop $\{\Phi_t\}$ in $\text{Sp}(2n;\mathbb{R})$ then $\mu(\{\phi_t\})$ is equal to the linear Maslov index of $\{\Phi_t\}$. Notice first that if $\{\Phi_t\}$ is a path in $\text{Sp}(2n;\mathbb{R})$ starting at the identity then the construction of Section 2 gives a 1-parameter family $Q_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$ of generating functions so that each $Q_t$ is a quadratic form. As in [Th95, Th99], we define

$$\nu(\{\Phi_t\}) = i(Q_0) - i(Q_1)$$

where $i$ denotes the maximal dimension of a subspace on which a quadratic form is negative semi-definite. By the arguments in Section 2, the integer $\nu(\{\Phi_t\})$ is well defined and depends only on the homotopy class (with fixed endpoints) of the path $\{\Phi_t\}$. Moreover, if $\{\Phi_t\}$ is the lift of a contact isotopy $\{\phi_t\}$ of $L_{k}^{2n-1}$ then $\nu(\{\phi_t\}) = \mu(\{\phi_t\})$.

4.3. Proposition. The induced map $\nu: \pi_1(\text{Sp}(2n;\mathbb{R})) \to \mathbb{Z}$ is a group homomorphism, and agrees with the linear Maslov index.

The proof of this proposition is based on the following lemma, which is taken from [Th95, Proposition 35] and whose equivariant version is also used in the proof of the contact Arnold conjecture in Section 5.

4.4. Lemma. If $Q: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$ is a quadratic form generating the identity then there is an isotopy $\{\Psi_s\}_{s \in [0,1]}$ of fibre preserving linear diffeomorphisms of $\mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ such that $\Psi_0$ is the identity and $Q \circ \Psi_1$ is a quadratic form that only depends on the fibre variable. Moreover, if $Q$ is $\mathbb{Z}_k$-invariant then $\{\Psi_s\}_{s \in [0,1]}$ can be chosen to be $\mathbb{Z}_k$-equivariant.

Proof. Write $Q(z) = \frac{1}{2} \langle z, Bz \rangle$ for a symmetric matrix $B = \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}$. Since $Q$ generates the zero section in $T^* \mathbb{R}^{2n}$ we have that $c$ is invertible and $a - bc^{-1}b^T = 0$. Then

$$\Psi_s(\zeta, \nu) = (\zeta', \nu - sc^{-1}b^T \zeta)$$

is an isotopy of fibre preserving linear diffeomorphisms of $\mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ such that $Q \circ \Psi_1$ only depends on the fibre variable, as $Q \circ \Psi_1(\zeta, \nu) = \frac{1}{2} \nu^T c \nu$. If $Q$ is $\mathbb{Z}_k$-invariant then $\{\Psi_s\}_{s \in [0,1]}$ is $\mathbb{Z}_k$-equivariant.

Proof of Proposition 4.3. Let $\{\Phi^{(1)}_t\}$ and $\{\Phi^{(2)}_t\}$ be loops in $\text{Sp}(2n;\mathbb{R})$, based at the identity. If $Q^{(1)}_t$ and $Q^{(2)}_t$ are 1-parameter families of generating quadratic forms for $\{\Phi^{(1)}_t\}$ and $\{\Phi^{(2)}_t\}$ respectively, then $Q^{(1)}_t \neq Q^{(2)}_t$ is a 1-parameter family of generating quadratic forms for $\{\Phi^{(1)}_t \circ \Phi^{(2)}_t\}$, and so

$$\nu(\{\Phi^{(1)}_t\}) \cdot \nu(\{\Phi^{(2)}_t\}) = i(Q^{(1)}_0 \circ Q^{(2)}_0) - i(Q^{(1)}_1 \circ Q^{(2)}_1).$$

By Lemma 4.4, for $l = 1, 2$ and $j = 0, 1$ there is an isotopy $\{\Psi^{(l)}_{s,j}\}_{s \in [0,1]}$ of fibre preserving linear diffeomorphisms such that $\Psi^{(l)}_{0,j}$ is the identity and $Q^{(l)}_j \circ \Psi^{(l)}_{0,j}$ is a quadratic form that does not depend on the base variable, and so is equal to a quadratic form $\overline{Q}^{(l)}_j$ on the fibre. Therefore

$$(Q^{(1)}_j \circ \Psi^{(1)}_{1,j}) \circ (Q^{(2)}_j \circ \Psi^{(2)}_{1,j}) = (0 \circ 0) = (0 \circ 0) \oplus \overline{Q}^{(1)}_j \oplus \overline{Q}^{(2)}_j,$$

and so

$$i(Q^{(1)}_j \circ Q^{(2)}_j) = i(0 \circ 0) + i(\overline{Q}^{(1)}_j) + i(\overline{Q}^{(2)}_j).$$

Together with (4.5) this gives

$$\nu(\{\Phi^{(1)}_t\}) \cdot \nu(\{\Phi^{(2)}_t\}) = i(\overline{Q}^{(1)}_0) + i(\overline{Q}^{(2)}_0) - i(\overline{Q}^{(1)}_1) - i(\overline{Q}^{(2)}_1) = \nu(\{\Phi^{(1)}_t\}) + \nu(\{\Phi^{(2)}_t\}),$$

and thus $\nu: \pi_1(\text{Sp}(2n;\mathbb{R})) \to \mathbb{Z}$ is a homomorphism.

In order to show that $\nu$ agrees with the linear Maslov index, it now suffices to check that it takes the value 2 on the standard loop $t \mapsto \Phi_t = e^{2\pi it} \in \text{Sp}(2n;\mathbb{R})$. For $0 \leq t \leq \frac{1}{4}$ we have the 1-parameter...
family of generating quadratic forms \( Q_t(z) = \frac{\sin(2\pi t)}{1 + \cos(2\pi t)} (z, z) \). Applying the composition formula twice we obtain a 1-parameter family of generating quadratic forms on \( \mathbb{R}^{10} \) determined by the family of \( 10 \times 10 \) symmetric matrices

\[
A_t = \begin{bmatrix}
0 & J & -J & 0 & 0 \\
-J & 0 & J & -J & 0 \\
J & -J & \lambda_t & 0 & 0 \\
0 & -J & 0 & \lambda_t & J \\
0 & J & 0 & -J & \lambda_t \\
\end{bmatrix}, \quad \text{with} \quad \lambda_t = \frac{\sin \left( \frac{2\pi t}{1 + \cos \left( \frac{2\pi t}{3} \right)} \right)}{1 + \cos \left( \frac{2\pi t}{3} \right)}
\]

where we denote by \( J \) the matrix \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). It follows that

\[ \nu(\{\Phi_t\}) = i(A_0) - i(A_1) = 6 - 4 = 2 \]

as required. \( \square \)

4.6. **Example.** Recall that we denote by \( \{r_t\} \) the Reeb flow on \( L^{2n-1}_k \) with respect to the contact form whose pullback to \( S^{2n-1} \) is equal to the pullback from \( \mathbb{R}^{2n} \) of the 1-form \( \sum_{j=1}^{2n} (x_j dy_j - y_j dx_j) \). It follows from the above discussion that for the \( (lk) \)-th iteration of the loop \( \{r_{2\pi t} \}_{t \in [0,1]} \) we have \( \mu(\{r_{2\pi t} \}_{t \in [0,1]}) = 2nl \).

**Relation with discriminant points.** We now show that the way the non-linear Maslov index of a contact isotopy changes for \( t \) varying in a subinterval of \([0,1]\) is related to the changes in the topology of the set of discriminant points. Before stating the results we recall the following fact.

4.7. **Lemma.** Let \( V \) be a compact manifold and \( f_t : V \to \mathbb{R} \), for \( t \in [0,1] \), a 1-parameter family of functions such that the total map \( f : V \times [0,1] \to \mathbb{R} \) is \( C^1 \) with Lipschitz differential. Suppose that \( a \in \mathbb{R} \) is a regular value of \( f_t \) for every \( t \in [0,1] \). Then there is an isotopy \( \theta_t \) of \( V \) such that \( \theta_t(\{f_0 \leq a\}) = \{f_t \leq a\} \).

**Proof.** Consider the open subset

\[ U := \{(x, t) \mid df_t|x| \neq 0 \} \]

of \( V \times [0,1] \). By assumption, the set \( \{(x, t) \mid f_t(x) = a\} \) is contained in \( U \). Fix a Riemannian metric on \( V \). Since the functions \( f_t \) are \( C^1 \) with Lipschitz differential, their gradient flow is well defined and enjoys the usual properties. Note also that the gradient \( \nabla f_t \) is non-zero at a point \( x \) exactly when \( (x, t) \in U \). For every \( t \in [0,1] \) define a vector field \( u_t \) on \( U_t := \{x \in V \mid df_t|x| \neq 0\} \) by \( u_t = \nabla f_t/\|\nabla f_t\|^2 \). Then \( df_t(u_t) \equiv 1 \) on \( U_t \). Take \( \varepsilon > 0 \) small enough so that the closed neighborhood

\[ W := \{(x, t) \in V \times [0,1] \mid |f(x, t) - a| \leq \varepsilon \} \]

of \( \{(x, t) \mid f_t(x) = a\} \) is contained in \( U_t \). Let \( \rho : V \times [0,1] \to \mathbb{R} \) be a smooth function that is supported in \( U_t \) and is equal to 1 on \( W \) and consider the time-dependent vector field \( \{X_t\}_{t \in [0,1]} \) on \( V \) that is given by

\[ X_t|_x = -\rho(x, t) \frac{d}{dt} f_t(x) u_t|_x \]

for \( (x, t) \) in \( U_t \) and that vanishes for \( (x, t) \) outside of \( U_t \). Its flow is an isotopy \( \theta_t \) of \( V \) with the required properties. Indeed

\[ \frac{d}{dt} f_t(\theta_t(x)) = \dot{f}_t(\theta_t(x)) + df_t(X_t)|_{\theta_t(x)} = (1 - \rho(\theta_t(x), t)) \dot{f}_t(\theta_t(x)) \]

thus \( \frac{d}{dt} f_t(\theta_t(x)) = 0 \) if \( (\theta_t(x), t) \in W \), and so each \( \theta_t \) sends \( \{f_0 = a\} \) onto \( \{f_t = a\} \). Since \( \theta_0 \) is the identity, by continuity the isotopy \( \theta_t \) sends \( \{f_0 \leq a\} \) onto \( \{f_t \leq a\} \) for all \( t \). \( \square \)

We can now prove that the non-linear Maslov index detects discriminant points, as described in Theorem 1.2(iii). More precisely we show the following result.
4.8. Proposition. Let \( \{ \phi_t \}_{t \in [0,1]} \) be a contact isotopy of \( L^{2n-1}_k \) (starting at the identity), and \( [t_0, t_1] \) a subinterval of \([0,1]\).

(i) If there are no values of \( t \in [t_0, t_1] \) for which \( \phi_t \) belongs to the discriminant then
\[
\mu([\{ \phi_t \}_{t \in [0,t_0]}]) = \mu([\{ \phi_t \}_{t \in [0,t_1]}]).
\]

(ii) If \( t \) is the only value of \( t \in [t_0, t_1] \) for which \( \phi_t \) belongs to the discriminant then
\[
|\mu([\{ \phi_t \}_{t \in [0,t_1]}]) - \mu([\{ \phi_t \}_{t \in [0,t_0]}])| \leq \text{ind}(\Delta(\phi_t)) + 1
\]
where \( \Delta(\phi_t) \subset L^{2n-1}_k \) is the set of discriminant points of \( \phi_t \). Consequently,
\[
|\mu([\{ \phi_t \}_{t \in [0,t_1]}]) - \mu([\{ \phi_t \}_{t \in [0,t_0]}])| \leq 2n + 1
\]
and moreover if \( \phi_t \) has only finitely many discriminant points then
\[
|\mu([\{ \phi_t \}_{t \in [0,t_1]}]) - \mu([\{ \phi_t \}_{t \in [0,t_0]}])| \leq 2.
\]

(iii) If \( t \) is the only value of \( t \in [t_0, t_1] \) for which \( \phi_t \) belongs to the discriminant, and moreover all discriminant points of \( \phi_t \) are non-degenerate, then
\[
|\mu([\{ \phi_t \}_{t \in [0,t_1]}]) - \mu([\{ \phi_t \}_{t \in [0,t_0]}])| \leq 1.
\]

Proof. Let \( f_t : L^{2M-1}_k \to \mathbb{R} \) be a 1-parameter family of generating functions for \( \phi_t \). If there are no values of \( t \in [t_0, t_1] \) for which \( \phi_t \) belongs to the discriminant then, by Proposition 2.19, zero is a regular value of \( f_t \) for all \( t \in [t_0, t_1] \). Hence, (i) follows from Lemma 4.7.

Suppose now that \( t \) is the unique value of \( t \in [t_0, t_1] \) for which \( \phi_t \) belongs to the discriminant. For any \( \epsilon > 0 \) we have
\[
|\mu([\{ \phi_t \}_{t \in [0,t_1]}]) - \mu([\{ \phi_t \}_{t \in [0,t_0]}])| \leq \text{ind}(\{f_t \leq \epsilon\}) - \text{ind}(\{f_t \leq -\epsilon\}).
\]
This is a consequence of (i), monotonicity of the index, and the fact that, since \( f : L^{2M-1}_k \times [0,1] \to \mathbb{R} \) is continuous, for every \( \epsilon > 0 \) and \( a \in \mathbb{R} \) there exists \( \delta > 0 \) such that for all \( t, t' \) with \( |t - t'| < \delta \) we have \( \{ f_t(x) \leq a \} \subset \{ f_{t'}(x) \leq a + \epsilon \} \).

Let \( C \) be the set of critical points of \( f_t \) with critical value 0. By continuity of the index, there is an open subset \( W \) in \( L^{2M-1}_k \) that contains \( C \) and has the same index. For sufficiently small \( \epsilon > 0 \) we have
\[
\text{ind}(\{f_t \leq \epsilon\}) \leq \text{ind}(\{f_t \leq -\epsilon\} \cup W).
\]
This follows from monotonicity of the index and the fact that, as we now explain, \( \{f_t \leq \epsilon\} \) deformation retracts into \( \{f_t \leq -\epsilon\} \cup W \) (cf. [Vi97, p 548]). Pick \( \delta > 0 \) such that if \( df_t(x) = 0 \) and \( |f_t(x)| \leq \delta \) then \( x \in W \). Consider the disjoint closed sets
\[
V_0 = \{ x \in L^{2M-1}_k : df_t(x) = 0 \text{ or } |f_t(x)| \geq 2\delta \}
\]
and
\[
V_1 = f_t^{-1}([-\delta, \delta]) \setminus W,
\]
and let \( \rho : L^{2M-1}_k \to [0,1] \) be a smooth function that vanishes in a neighborhood of \( V_0 \) and is constant equal to 1 in a neighborhood of \( V_1 \). Fix a metric on \( L^{2M-1}_k \) and consider the vector field \( X = -\rho \nabla f_t / \| \nabla f_t \| \). Writing \( \theta_t \) for the flow of \( X \) we have
\[
\frac{d}{dt} f_t(\theta_t(x)) = -\rho(\theta_t(x)).
\]
Let \( m = \max\{\|X(x)\| : x \in L^{2M-1}_k\} \) and \( d = \text{dist} \left( \{ \rho(x) \leq 1 \} \cap f_t^{-1}([-\delta, \delta]) \right) \setminus W \). Note that \( d > 0 \). For \( \epsilon < \min\{\delta, \frac{d}{2m}\} \) we now prove that
\[
\theta_{2\epsilon}(\{f \leq \epsilon\}) \subset \{f \leq -\epsilon\} \cup W.
\]
Given \( x \in \{f \leq \epsilon\} \) set
\[
s(x) = \inf\{ t \in [0,2\epsilon] : \rho(\theta_t(x)) < 1 \text{ or } t = 2\epsilon \}.
\]
If \( s(x) = 2\epsilon \) then \( \rho(\theta_t(x)) = 1 \) for all \( t \in [0, 2\epsilon] \) and, by (4.11), \( f_2(\theta_{2\epsilon}(x)) = f_2(x) - 2\epsilon \leq -\epsilon \). If \( s(x) < 2\epsilon \) then \( \theta_{\epsilon}(x) \subseteq \{ \rho < 1 \} \cap f_2^{-1}([-\delta, \delta]) \) (as \( \epsilon < \delta \)). Then we must have \( \text{dist}(\theta_{\epsilon}(x), W^\epsilon) \geq d \) and our bound on \( \epsilon \) ensures that the path \( \{ \theta_t(x); t \in [s(x), 2\epsilon] \} \) is entirely contained in \( W \). In particular, \( \theta_{2\epsilon}(x) \in W \). This completes the proof of (4.12) and hence of (4.10).

By (4.9), (4.10), subadditivity of the cohomological index and Proposition 2.19 we have

\[
\text{ind}(\{ f_2 \leq \epsilon \}) \leq \text{ind}(\{ f_2 \leq -\epsilon \} \cup W) \leq \text{ind}(\{ f_2 \leq -\epsilon \}) + \text{ind}(W) + 1
\]

\[
= \text{ind}(\{ f_2 \leq -\epsilon \}) + \text{ind}(C) + 1
\]

\[
= \text{ind}(\{ f_2 \leq -\epsilon \}) + \text{ind}(\Delta(\phi_2)) + 1.
\]

This implies (ii).

As for (iii), if all discriminant points of \( \phi_t \) are non-degenerate then (by Proposition 2.19) all critical points of \( f_2 \) of critical value zero are non-degenerate. Thus \( f_2 \) has only finitely many critical points with critical value zero and zero is an isolated critical value of \( f_2 \). We can choose \( W \) so that \( \{ f_2 \leq -\epsilon \} \cup W \) can be obtained from \( \{ f_2 \leq -\epsilon \} \) by attaching a finite number of disjoint handles.

If \( H \) is a handle and \( A \subseteq L^{2M-1}_k \) then \( \text{ind}(A \cup H) \leq \text{ind}(A) + 1 \) (as the sum of the Betti numbers of \( A \cup H \) is at most 1 more than the sum of Betti numbers of \( A \)) and, unless the index of the handle \( H \) is equal to \( \text{ind}(A) + 1 \), we have \( \text{ind}(A \cup H) = \text{ind}(A) \). Attaching the handles in \( W \) sequentially, starting with those of highest index, we therefore obtain

\[
\text{ind}(\{ f_2 \leq -\epsilon \} \cup W) \leq \text{ind}(\{ f_2 \leq -\epsilon \}) + 1,
\]

which, together with (4.9) and (4.10), concludes the proof of (iii). \( \square \)

4.13. Remark. In order to prove the version of the contact Arnold conjecture with the bound given by the Lusternik–Schnirelmann category (cf. Section 5) we would need to know that the conclusion of Proposition 4.8(iii) holds also in the degenerate case. Using Massey products similarly to [Vi97] it is possible to prove (at least if \( k = 3 \)) that this is the case if, in the notation of the proof above, \( \text{ind}(\{ f_2 \leq \epsilon \}) = 1 \). It is not clear to us whether such arguments can be pushed further to improve this result. \( \diamond \)

Further properties. We now prove the positivity property from Theorem 1.2, and the fact that the asymptotic non-linear Maslov index is monotone and has the vanishing property.

4.14. Proposition (Positivity). If \( \{ \phi_t \} \) is a non-negative (respectively non-positive) contact isotopy then \( \mu(\{ \phi_t \}) \geq 0 \) (respectively \( \mu(\{ \phi_t \}) \leq 0 \)). Moreover, if \( \{ \phi_t \} \) is positive then \( \mu(\{ \phi_t \}) > 0 \).

Proof. It follows from monotonicity of generating functions (Proposition 2.20) and monotonicity of the cohomological index (Proposition 3.9(i)) that if \( \{ \phi_t \} \) is a non-negative (respectively non-positive) contact isotopy then \( \mu(\{ \phi_t \}) \geq 0 \) (respectively \( \mu(\{ \phi_t \}) \leq 0 \)). By Example 4.1, if \( \{ \phi_t \} \) is a small positive contact isotopy then \( \mu(\{ \phi_t \}) = 2n > 0 \). Since (by Proposition 2.20 and Proposition 3.9(i)) the cohomological index does not decrease along a positive contact isotopy, we conclude that \( \mu(\{ \phi_t \}) > 0 \) for any positive contact isotopy \( \{ \phi_t \} \). \( \square \)

We now show that the asymptotic non-linear Maslov index satisfies the following stronger property. Recall from [EP00] that for a contact manifold \( (V, \xi) \) the relation \( \leq \) on \( \text{Cont}_0(V, \xi) \) is defined by posing \( \{ \psi \} \leq \{ \psi \} \) if \( \{ \psi \} \cdot \{ \psi \} \) can be represented by a non-negative contact isotopy. As in [BZ15] we say that a quasimorphism \( \nu \) on \( \text{Cont}_0(V, \xi) \) is monotone if \( \nu(\{ \phi_t \}) \leq \nu(\{ \psi_t \}) \) whenever \( \{ \phi_t \} \leq \{ \psi_t \} \). The proof of the following result is a direct imitation of the proof of the similar statement for real projective spaces that is given in [BZ15].

4.15. Proposition. The asymptotic non-linear Maslov index \( \bar{\mu} \) on \( \text{Cont}_0(L^{2n-1}_k) \) is monotone.
Proof. Suppose that \([\{\phi_t\}] \leq [\{\psi_t\}]\). Since the set of non-negative elements of \(\widetilde{\text{Cont}}_0(L_k^{2n-1})\) is conjugation invariant and closed under multiplication, \([\{\phi_t\}]^m \leq [\{\psi_t\}]^m\) for any \(m \in \mathbb{Z}_{>0}\). By Proposition 4.14, \(\overline{\pi}\) is non-negative on non-negative contact isotopies. Thus \(m \overline{\pi}([\{\psi_t\}]) - m \overline{\pi}([\{\phi_t\}]) = \overline{\pi}([\{\psi_t\}]^m) + \overline{\pi}([\{\phi_t\}]^{-m}) \geq \overline{\pi}([\{\psi_t\}]^m \cdot [\{\phi_t\}]^{-m}) - D \geq -D,

where \(D\) is the error of the quasimorphism. Dividing by \(m\) and taking the limit when \(m \to \infty\) we obtain that \(\overline{\pi}([\{\phi_t\}]) \leq \overline{\pi}([\{\psi_t\}])\). \(\square\)

For the next result we also follow [BZ15]. Recall that a subset \(\mathcal{U}\) of a contact manifold \((V, \xi)\) is said to be displaceable if there exists a contactomorphism \(\psi\) contact isotopic to the identity such that \(\overline{\mathcal{U}} \cap \psi(\mathcal{U}) = \emptyset\). A quasimorphism \(\nu\) on \(\widetilde{\text{Cont}}_0(V, \xi)\) is said to have the vanishing property if \(\nu([\{\phi_t\}]) = 0\) for any contact isotopy \(\{\phi_t\}_{t \in [0,1]}\) that is supported in \([0,1] \times \mathcal{U}\) for a displaceable set \(\mathcal{U}\).

4.16. Proposition. The asymptotic non-linear Maslov index has the vanishing property.

Proof. Suppose that a subset \(\mathcal{U}\) of \(L_k^{2n-1}\) is displaceable by a contactomorphism \(\psi\) contact isotopic to the identity. After taking a \(C^\infty\)-perturbation, we can assume that \(\psi\) has no discriminant points. Let \(\{\psi_t\}_{t \in [0,1]}\) be a contact isotopy from the identity to \(\psi_1 = \psi\), and \(\{\phi_t\}_{t \in [0,1]}\) a contact isotopy supported in \([0,1] \times \mathcal{U}\). We need to show that \(\overline{\pi}([\{\phi_t\}]) = 0\). Observe first that, for every \(m \in \mathbb{N}\) and \(t \in [0,1]\), the contactomorphism \(\psi_{\phi_t}^m\) has no discriminant points. Indeed, assume by contradiction that \(p\) is a discriminant point of \(\psi_{\phi_t}^m\). Since \(\phi_t^m\) is supported in \(\mathcal{U}\) and \(\psi\) has no discriminant points we must have \(p \notin \mathcal{U}\). But then \(\phi_t^m(p) \in \mathcal{U}\) and so \(\psi\phi_t^m(p) \in \mathcal{U} \cap \psi(\mathcal{U})\) contradicting the hypothesis that \(\psi\) displaces \(\mathcal{U}\). Since \(\psi\phi_t^m\) has no discriminant points for all \(t \in [0,1]\), it follows from Proposition 4.8(i) that for the concatenation \(\{\psi_t\} \sqcup \{\psi_{\phi_t}^m\}\) we have

\[
\mu([\{\psi_t\} \sqcup \{\psi_{\phi_t}^m\}]) = \mu([\{\psi_t\}]) .
\]

Since \(\{\psi_t\} \sqcup \{\psi_{\phi_t}^m\}\) and \(\{\psi_t \cdot \phi_t^m\}\) are homotopic, the quasimorphism property (Proposition 4.2) implies that \(\mu([\phi_t^m]) \leq 2n+1\). As this holds for every \(m \in \mathbb{N}\), we conclude that \(\overline{\pi}([\{\phi_t\}]) = 0\). \(\square\)

5. Applications

In this section we use the properties of the non-linear Maslov index to prove the applications listed in Corollaries 1.3 and 1.5. Most of the arguments are taken from [EP00, CS12, Sa11c, BZ15] with only minor changes to adapt them to the case of lens spaces, and are included here for the sake of completeness.

Orderability. As in the case of projective spaces discussed in [EP00], orderability of lens spaces follows from positivity of the non-linear Maslov index and the fact that the non-linear Maslov index is well-defined on the universal cover of the contactomorphism group. Indeed, suppose by contradiction that a lens space \(L_k^{2n-1}\) is not orderable, i.e. it admits a positive contractible loop \(\{\phi_t\}_{t \in [0,1]}\). Since \(\{\phi_t\}_{t \in [0,1]}\) is contractible we have that \(\mu([\{\phi_t\}_{t \in [0,1]}]) = 0\). On the other hand, since \(\phi_t\) is positive, Proposition 4.14 implies that \(\mu([\{\phi_t\}]) > 0\), giving a contradiction.

Unboundedness of the discriminant and oscillation metrics. Recall that any bi-invariant (pseudo-)metric \(d : G \times G \to \mathbb{R}\) on a group \(G\) defines a conjugation-invariant (pseudo-)norm \(\| \cdot \| : G \to \mathbb{R}\) by posing \(\|g\| = d(g, \text{id})\); conversely, any conjugation-invariant (pseudo-)norm \(\| \cdot \| : G \to \mathbb{R}\) defines a bi-invariant (pseudo-)metric by posing \(d(g_1, g_2) = \|g_1g_2^{-1}\|\). The discriminant and oscillation metrics on \(\widetilde{\text{Cont}}_0(V, \xi)\) for a compact co-oriented contact manifold \((V, \xi)\) are defined as follows [CS12]. As proved in [CS12], any element in \(\widetilde{\text{Cont}}_0(V, \xi)\) has a representative \(\{\phi_t\}_{t \in [0,1]}\) that can be written as the concatenation of a finite number of pieces \(\{\phi_t\}_{t \in [t_{j-1}, t_j]}\), \(j = 1, \ldots, L\), such that each piece is embedded, i.e. for every two distinct \(t\) and \(t'\) in \([t_{j-1}, t_j]\) the composition
\( \phi_t \circ \phi_{t-1}^{-1} \) does not have any discriminant point. The discriminant norm of \([\{ \phi_t \}]\) (i.e. its distance to the identity with respect to the discriminant metric) is then defined to be the minimal number of pieces in such a decomposition. Moreover, any element in Cont_0(V, \xi) also has a representative \( \{ \phi_t \}_{t \in [0,1]} \) that can be written as the concatenation of a finite number of embedded pieces such that each piece is either non-negative or non-positive. Let \( L_+ \) and \( L_- \) be respectively the minimal number of non-negative and of non-positive pieces in such a decomposition; the oscillation pseudo-norm of \([\{ \phi_t \}]\) is then defined to be \( L_+ + L_- \). The oscillation pseudo-metric is non-degenerate, hence a metric, if and only if \((V, \xi)\) is orderable.

In [CS12] the non-linear Maslov index has been used to show that the discriminant and oscillation metrics for real projective space are unbounded, hence not equivalent to the trivial metric. The argument, applied to lens spaces, is as follows.

Consider the Reeb flow \( \{ r_t \} \) on \( L^2_n \) with respect to the contact form whose pullback to \( S^{2n-1} \) is equal to the pullback from \( \mathbb{R}^{2n} \) of the 1-form \( \sum_{j=1}^n (x_j dy_j - y_j dx_j) \). We first show that the discriminant norm of the \( 3kl \)-th iteration \( \{ r_{6nl} \}_{t \in [0,1]} \) of the loop \( \{ r_{\pi_l t} \}_{t \in [0,1]} \) is at least \( l + 1 \).

By Example 4.6 we know that \( \mu([\{ r_{6nl} \}_{t \in [0,1]}]) = 6nl \). Let \( \{ \phi_t \}_{t \in [0,1]} \) be a contact isotopy that represents \([\{ r_{6nl} \}_{t \in [0,1]}]\), is a concatenation of embedded pieces and minimizes the discriminant norm. Then \( \mu([\{ \phi_t \}_{t \in [0,1]}]) = 6nl \) by Proposition 4.8(i) and Example 4.1, if we assume \( l > 0 \) then \( \{ \phi_t \}_{t \in [0,1]} \) must intersect the discriminant, and so \( \{ \phi_t \}_{t \in [0,1]} \) has at least two embedded pieces. Suppose now that \( l > 1 \), and write \( \{ \phi_t \}_{t \in [0,1]} \) as a concatenation of \( L \) embedded pieces \( \{ \phi_t \}_{t \in [t_{j-1}, t_j]} \), \( j = 1, \ldots, L \). For each \( j \), since \( \{ \phi_t \}_{t \in [t_{j-1}, t_j]} \) is embedded we have in particular that \( \phi_t \circ \phi_{t_j}^{-1} \) does not have any discriminant point for every \( t \in [t_{j-1}, t_j] \). Consider a value of time \( t \in (t_{j-1}, t_j) \) such that \( \phi_t \circ \phi_{t_j}^{-1} \in [t_{j-1}, t_j] \) is \( C^1 \)-small. By Proposition 4.8(i) and Example 4.1 we have

\[
\mu ([\{ \phi_t \circ \phi_{t_j}^{-1} \}_{t \in [t_{j-1}, t_j]}]) = \mu ([\{ \phi_t \circ \phi_{t_j}^{-1} \}_{t \in [t_{j-1}, t_j]}]) \leq 2n.
\]

Suppose now by contradiction that \( L < l + 1 \). Then

\[
(5.1) \quad \sum_{j=1}^{L} \mu ([\{ \phi_t \circ \phi_{t_j}^{-1} \}_{t \in [t_{j-1}, t_j]}]) \leq 2nL < 2n(l + 1).
\]

On the other hand, by the quasimorphism property (Proposition 4.2) we have

\[
\left| \mu ([\{ \phi_t \}_{t \in [0,1]}]) - \sum_{j=1}^{L} \mu ([\{ \phi_t \circ \phi_{t_j}^{-1} \}_{t \in [t_{j-1}, t_j]}]) \right| \leq (L - 1)(2n + 1) < l(2n + 1),
\]

and thus

\[
\sum_{j=1}^{L} \mu ([\{ \phi_t \circ \phi_{t_j}^{-1} \}_{t \in [t_{j-1}, t_j]}]) > 4nl - l.
\]

This contradicts (5.1), and thus concludes the proof that the discriminant norm of \( \{ r_{6nl} \}_{t \in [0,1]} \) is at least \( l + 1 \).

The fact that the oscillation metric is unbounded can be seen by combining the above argument with positivity of the non-linear Maslov index, as follows. We show that the oscillation norm of the \( 6kl \)-th iteration \( \{ r_{12nl} \}_{t \in [0,1]} \) of the loop \( \{ r_{2\pi_l t} \}_{t \in [0,1]} \) is at least \( l + 1 \). The oscillation norm is the sum of the minimal number of non-negative embedded pieces in a decomposition of a contact isotopy representing the class and the minimal number of non-positive embedded pieces. The minimal number of non-negative embedded pieces is zero, because \( \{ r_{12nl} \}_{t \in [0,1]} \) is a representative of \( \{ r_{12nl} \}_{t \in [0,1]} \) made only of non-negative pieces. So we need to show that the minimal number of non-negative pieces is at least \( l + 1 \). Let \( \{ \phi_t \}_{t \in [0,1]} \) be a contact isotopy that represents \( \{ r_{12nl} \}_{t \in [0,1]} \), is a concatenation of non-negative or non-positive embedded pieces and minimizes the number of non-negative ones. Then \( \mu ([\{ \phi_t \}_{t \in [0,1]}]) = 12nl \). Regarding adjacent embedded pieces of the same sign as a single positive or negative isotopy, let \( L_+ \) and \( L_- \) respectively be the number of non-negative and of non-positive isotopies in the decomposition. If \( L_+ \geq l + 1 \)
then the number of non-negative embedded pieces is also at least \( l + 1 \) and so we are done. Assume thus that \( L_+ < l + 1 \), and so \( L := L_+ + L_- < 2l + 2 \). By the quasimorphism property we then have

\[
\left| \mu\left( \left[ \phi_t \right]_{t \in [0,1]} \right) - \sum_{j=1}^{L} \mu\left( \left[ \phi_t \circ \phi^{-1}_{t_{j-1}} \right]_{t \in [t_{j-1}, t_j]} \right) \right| \leq (L - 1)(2n + 1) < (2l + 1)(2n + 1).
\]

Write \( \sum_{j=1}^{L} \mu\left( \left[ \phi_t \circ \phi^{-1}_{t_{j-1}} \right]_{t \in [t_{j-1}, t_j]} \right) = \mu_+ + \mu_- \), where \( \mu_+ \) and \( \mu_- \) are respectively the sum of the non-linear Maslov indices of the non-negative and of the non-positive pieces. By Proposition 4.14 we have \( \mu_- \leq 0 \), and thus we conclude that

\[
(5.2) \quad \mu_+ > 12nl - (2l + 1)(2n + 1).
\]

Suppose now by contradiction that the number of non-negative embedded pieces is less than \( l + 1 \). Then \( \mu_+ < 2n(l + 1) \). This contradicts (5.2), concluding the proof.

**Contact Arnold conjecture.** Adapting to our case the argument given for \( \mathbb{R}P^{2n-1} \) in [Sa11c] (which in turns is an adaptation of the proof of the Hamiltonian Arnold conjecture for \( \mathbb{C}P^n \) given in [Th98] and [Gi90]) we show that for any contactomorphism of \( L_k^{2n-1} \) which is contact isotopic to the identity the number of translated points (with respect to the standard contact form) is at least \( n \), and at least \( 2n \) if all translated points are assumed to be non-degenerate.

Recall that we denote by \( \{ r_t \} \) the Reeb flow on \( L_k^{2n-1} \). Translated points of a contactomorphism \( \phi \) of \( L_k^{2n-1} \) correspond to discriminant points of the composition \( r_{2\pi t} \circ \phi \), for \( t \) varying in \( [0,1] \), and thus to those discriminant points of \( r_{2\pi t} \circ \phi \) (for \( t \in [0,1] \)) that are also \( (\mathbb{Z}_k \text{-orbits of}) \) discriminant points of the lift to the sphere. Without loss of generality we can assume that \( \phi \) has no discriminant points. Indeed, for some value \( t \) of \( t \in [0,1] \) the composition \( r_{2\pi t} \circ \phi \) has no discriminant points, and its translated points are in 1-1 correspondence with those of \( \phi \). Assume thus that \( \phi \) has no discriminant points, and let \( \{ \phi_t \}_{t \in [0,1]} \) be a contact isotopy from the identity to \( \phi \). We first prove that for the concatenation \( \{ \phi_t \}_{t \in [0,1]} \cup \{ r_{2\pi t} \circ \phi \}_{t \in [0,1]} \) we have

\[
(5.3) \quad \mu\left( \{ \phi_t \}_{t \in [0,1]} \cup \{ r_{2\pi t} \circ \phi \}_{t \in [0,1]} \right) = 2n.
\]

Let \( F_t \) be a 1-parameter family of generating functions for \( \{ \phi_t \}_{t \in [0,1]} \), and \( Q_t \) a 1-parameter family of generating quadratic forms for \( \{ r_{2\pi t} \circ \phi \}_{t \in [0,1]} \). Then \( F_t \triangleright Q_t \) is a 1-parameter family of generating functions for \( \{ r_{2\pi t} \circ \phi \} \), which is homotopic to \( \{ \phi_t \} \cup \{ r_{2\pi t} \circ \phi \} \), and \( F_t \triangleright Q_0 \) is a 1-parameter family of generating functions for \( \{ \phi_t \} \). Thus

\[
\mu\left( \{ \phi_t \}_{t \in [0,1]} \cup \{ r_{2\pi t} \circ \phi \}_{t \in [0,1]} \right) - \mu\left( \{ \phi_t \}_{t \in [0,1]} \right) = \text{ind}(F_t \triangleright Q_0) - \text{ind}(F_t \triangleright Q_1).
\]

By Lemma 4.4 there are isotopies \( \{ \Psi^0_s \}_{s \in [0,1]} \) and \( \{ \Psi^1_s \}_{s \in [0,1]} \) of \( \mathbb{Z}_k \)-equivariant fibre preserving linear homeomorphisms such that \( Q_0 \circ \Psi^0_s = Q_1 \circ \Psi^1_s \) independent of the base variable, and so are equal to quadratic forms \( \overline{Q}_0 \) and \( \overline{Q}_1 \) on the fibre. For \( j = 0, 1 \) we have \( F_t \triangleright (Q_j \circ \Psi^j_s) = (F_t \triangleright 0) \oplus \overline{Q}_j \), and so

\[
\text{ind}(F_t \triangleright Q_j) = \text{ind}(F_t \triangleright(Q_j \circ \Psi^j_s)) = \text{ind}((F_t \triangleright 0) \oplus \overline{Q}_j) = \text{ind}(F_t \triangleright 0) + \text{ind}(\overline{Q}_j)
\]

where the last equality follows from Corollary 3.14. By Example 4.6 we thus have

\[
\text{ind}(F_t \triangleright Q_0) - \text{ind}(F_t \triangleright Q_1) = \text{ind}(\overline{Q}_0) - \text{ind}(\overline{Q}_1) = 2n,
\]

hence (5.3). Knowing this, Proposition 4.8(ii) and the fact that \( \phi \) has no discriminant points imply that either there are at least \( n \) distinct values of \( t \in (0,1) \) at which \( \mu \) jumps, and so \( \phi \) has at least \( n \) translated points (which are necessarily all distinct, since they have different time-shifts), or there is at least one value of \( t \) at which \( \mu \) jumps by more than 2, and thus \( \phi \) has infinitely many translated points. Moreover, if all translated points of \( \phi \) are non-degenerate then Proposition 4.8(iii) implies that there must be at least \( 2n \) of them (all distinct).
**Weinstein conjecture.** Following Givental [Gi90] we show how to use the asymptotic non-linear Maslov index to prove that for any contact form $\alpha$ on $L_k^{2n-1}$ defining the standard contact structure there exist closed Reeb orbits. Denote by $\alpha_0$ the standard contact form, and let $f$ be the function such that $\alpha = e^f \alpha_0$. Then $h = e^{-f}$ is the contact Hamiltonian (with respect to $\alpha_0$) of the Reeb flow $\{\phi_t^\alpha\}$ of $\alpha$ and, for every $m \in \mathbb{N}$, $mh$ is the contact Hamiltonian of $\{\phi_{mt}^\alpha\}$. Discriminant points of $\phi_{mt}^\alpha$ correspond to closed Reeb orbits of $\alpha$ of period $mt$. Since $h > 0$, there is $m \in \mathbb{N}$ such that $mh \geq k$. The constant function $k$ is the contact Hamiltonian of the $k$-th iteration of the Reeb flow associated to the form $\alpha_0$, i.e. of $\{\phi_{2\pi t}^\alpha\}_{t \in [0,1]}$. By Proposition 4.15 and Example 4.6 we then have $\bar{\pi} \left( \{\phi_{mt}^\alpha\}_{t \in [0,1]} \right) \geq \bar{\pi} \left( \{\phi_{2\pi t}^\alpha\}_{t \in [0,1]} \right) = 2n > 0$. It thus follows from Proposition 4.8 that $\alpha$ has at least one closed Reeb orbit.

**Constructing new quasimorphisms via contact reduction, and more applications to orderability and non-displaceability.** In [BZ15] Borman and Zapolsky explain how in certain situations quasimorphisms descend under contact reduction, and use this to show that Givental’s asymptotic non-linear Maslov index on projective spaces induces quasimorphisms on certain pre-quantizations of symplectic toric manifolds. Moreover they obtain applications to orderability and existence of non-displaceable pre-Lagrangian fibres. As already observed in [BZ15, Remark 1.5], our extension of Givental’s non-linear Maslov index to lens spaces allows us to enlarge the class of spaces to which the results of [BZ15] apply.

Consider a contact manifold $(V, \xi)$, and suppose that it is equipped with a non-trivial monotone quasimorphism $\nu : \text{Cont}_0(V, \xi) \to \mathbb{R}$. Following [BZ15] we say that a subset $Y$ of $V$ is subheavy with respect to $\nu$ if $\nu$ vanishes on all elements that can be represented by a contact isotopy generated by an autonomous Hamiltonian that vanishes on $Y$. Suppose now that $(V, \xi)$ is also equipped with a contact $\mathbb{T}^n$-action, and denote by $f_\alpha : V \to \mathbb{R}^n$ the moment map with respect to a $\mathbb{T}^n$-invariant contact form $\alpha$ for $\xi$. Recall that if $\mathbb{T}^n$ acts freely on the level set $f_\alpha^{-1}(0)$ then $\alpha$ induces a contact form $\alpha'$ on the quotient $V' = f_\alpha^{-1}(0)/\mathbb{T}^n$ (see for instance [Ge08, Theorem 7.7.5]). The contact manifold $(V', \xi' = \ker(\alpha'))$ is said to be the contact reduction of $(V, \xi)$ at the level $f_\alpha^{-1}(0)$. By [BZ15, Theorem 1.8], if $f_\alpha^{-1}(0)$ is subheavy with respect to the non-trivial monotone quasimorphism $\nu$ then $\nu$ naturally descend to a non-trivial monotone quasimorphism $\nu' : \text{Cont}_0(V', \xi') \to \mathbb{R}$. Moreover, if $\nu$ has the vanishing property then so does $\nu'$. By [BZ15, Theorem 1.3], if a monotone symplectic toric manifold $(W, \omega)$ is even, i.e. the sum of the normals of the moment polytope $\Delta \subset \mathfrak{t}^*$ is in $2\pi \mathbb{Z}$, then there is a rescaling $\omega'$ of the symplectic form such that the prequantization $(V, \xi)$ of $(W, \omega')$ can be written as contact reduction of a projective space $\mathbb{R}P^{2n-1}$ at a level $f_\alpha^{-1}(0)$ containing the torus $T_{\mathbb{R}P^{2n-1}} = \left\{ [z] \in \mathbb{R}P^{2n-1} \mid |z_1|^2 = \ldots = |z_n|^2 \right\}$.

By [BZ15, Lemma 1.22 and Theorem 1.11 (i)], $T_{\mathbb{R}P^{2n-1}}$ is subheavy with respect to the asymptotic non-linear Maslov index $\bar{\pi}$ and so, by [BZ15, Proposition 1.10(iii)], $f_\alpha^{-1}(0)$ is also subheavy. It follows that $\bar{\pi}$ descends to a non-trivial monotone quasimorphism on $\text{Cont}_0(V, \xi)$ with the vanishing property. By [BZ15, Theorem 1.28], if a contact manifold $(V, \xi)$ admits a non-trivial monotone quasimorphism $\nu : \text{Cont}_0(V, \xi) \to \mathbb{R}$ then it is orderable; by [BZ15, Theorem 1.17], if moreover $(V, \xi)$ is the prequantization of a symplectic toric manifold and the quasimorphism $\nu$ also has the vanishing property then $V$ has a non-displaceable pre-Lagrangian toric fibre. The conclusion is thus that any monotone even symplectic toric manifold has a prequantization that is orderable and has a non-displaceable pre-Lagrangian toric fibre.

In the case of lens spaces, repeating the proof of [BZ15, Lemma 1.22] one sees that $T_{L_k^{2n-1}} = \left\{ [z] \in L_k^{2n-1} \mid |z_1|^2 = \ldots = |z_n|^2 \right\} \subset L_k^{2n-1}(1, \ldots, 1)$ is subheavy with respect to the asymptotic non-linear Maslov index on $L_k^{2n-1}(1, \ldots, 1)$. Consider now a compact monotone symplectic toric manifold $(W^{2n}, \omega)$. Write the moment polytope as $\Delta = \{ x \in \mathfrak{t}^*, \langle \nu_j, x \rangle + \lambda \geq 0 \text{ for } j = 1, \ldots, d \}$, where $d$ is the number of facets and $\nu_j \in \mathfrak{t}$ are
vectors normal to the facets and primitive in the integer lattice $t \in \ker (\exp: t \to T^n)$. Suppose that, for some $k \in \mathbb{N}$,

$$\sum_{j=1}^{d} \nu_j \in k \mathbb{Z}.$$  \hfill (5.4)

Then the same argument as in the proof of [BZ15, Theorem 1.3] shows that there is a rescaling$^{11}$ of the symplectic form such that the prequantization of $(W, \alpha)$ can be written as contact reduction of $L^{2n-1}_k(1, \ldots, 1)$ at a level $f^{(0)}(0)$ containing $T^{L^{2n-1}_k}$. Therefore, such a prequantization admits a non-trivial monotone quasimorphism with the vanishing property, and so it is orderable and it contains a non-displaceable pre-Lagrangian toric fibre.

5.5. Example. A compact monotone symplectic toric manifold $(W, \omega)$ satisfying condition (5.4) can be obtained by the following generalization of [BZ15, Example (i) of page 385]. Consider the $\mathbb{CP}^1$-bundle over $\mathbb{CP}^n$ obtained, for $1 \leq k \leq n$, as the projectivization $\mathbb{P}(1 \oplus \mathcal{O}(k))$ of the direct sum of a trivial line bundle with the bundle $\mathcal{O}(k)$ over $\mathbb{CP}^n$. This manifold can be equipped with a monotone symplectic structure, and the inward normals of the corresponding moment polytope are $\epsilon_1, \ldots, \epsilon_n, \epsilon_{n+1}, -\epsilon_{n+1}, k\epsilon_{n+1} - \epsilon_1 - \ldots - \epsilon_n$.

5.6. Remark. One can show that for any compact monotone symplectic toric manifold $(W, \omega)$ the prequantization of $W$ with appropriately rescaled $\omega$ can be written as a contact reduction of $L^{2n-1}_k(w)$ at a level containing $T^{L^{2n-1}_k(w)}$. However, if $w \neq (1, \ldots, 1)$ we do not know whether $T^{L^{2n-1}_k(w)}$ is subheavy and so we cannot conclude that this prequantization has an induced quasimorphism. In order to prove that $T^{L^{2n-1}_k(1, \ldots, 1)}$ is subheavy one uses the fact that the Clifford torus in $\mathbb{CP}^{n-1}$ is the unique non-displaceable orbit of the standard torus action. A similar statement is not true in general for weighted projective spaces: for example $\mathbb{CP}(1, 3, 5)$ contains a 2-dimensional family of non-displaceable pre-Lagrangian toric fibres [WW13].

APPENDIX A. ON THE CONSTRUCTION OF GENERATING FUNCTIONS

In Proposition 2.2 we proved that if $\Phi^{(1)}$ and $\Phi^{(2)}$ are Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ with generating functions $F_1: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ and $F_2: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ respectively then the function $F_1 \sharp F_2: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}) \to \mathbb{R}$ defined by

$$F_1 \sharp F_2(q; \zeta_1, \zeta_2, \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

is a generating function for the composition $\Phi = \Phi^{(2)} \circ \Phi^{(1)}$. Here we present two alternative proofs of this fact in terms of symplectic reduction, we generalize the composition formula to the case of any even number of factors and discuss its relation with the method of broken trajectories by Chaperon, Laudenbach and Sikorav [Ch84, LS85, Si85, Si87].

Recall that if $V$ is a coisotropic submanifold of a symplectic manifold $(W, \omega)$ then the kernel of the restriction of $\omega$ to $V$ is an integrable distribution. If the space of leaves $V/\sim$ is a manifold then it inherits a symplectic form $\omega$, and is said to be the symplectic reduction of $(W, \omega)$ along $V$. If $L$ is a Lagrangian submanifold of $W$ which is transverse to $V$ then the restriction to $L \cap V$ of the projection $V \to V/\sim$ is a Lagrangian immersion. For instance, consider a fibre bundle $p: E \to B$. The fibre conormal bundle $N^*_E$ is a coisotropic submanifold of $T^*E$, and the symplectic reduction can be identified with $T^*B$. If $F: E \to \mathbb{R}$ is a generating function then the Lagrangian immersion

\footnote{If $\omega$ is rescaled so that $[\omega] = c_1(TW)$ then we can take $a = k$. Indeed, the moment polytope for $(W, k\omega)$ can be written as $\Delta = \{ x \in \mathbb{T}^d \mid (\nu_j, x) + k \geq 0 \text{ for } j = 1, \ldots, d \}$ and the prequantization $(V, \xi)$ of $(W, k\omega)$ corresponds to a cone in $\mathbb{R}^{n+1}$ with primitive inward normals $(\nu_j, x) \in \mathbb{T} \times \mathbb{R}$, $j = 1, \ldots, d$. The contact toric manifold $(V, \xi)$ is a contact reduction of $(S^{2d-1}, \ker \alpha_{std})$ by a subgroup $K$ of $T^d$ containing $\{e_1, \ldots, e_d\} \in \mathbb{R}^d$, $\mathbb{Z} = \mathbb{Z}d$ (because $\sum_{j=1}^{d} (\nu_j, x)$ is the lattice of $\mathbb{T} \times \mathbb{R}$; see [LO2]). In fact it would be enough to take $k \cdot \eta$ where $\eta$ denotes the primitive integral class in the direction of $c_1(TW)$.}
The first interpretation of Proposition 2.2 in terms of symplectic reduction that we present is an adaptation to our composition formula of the discussion in Théret [Th95, Section I.3]. We use three basic properties of generating functions (Lemmas A.1, A.2 and A.3) whose verification is immediate and therefore left to the reader.

**A.1. Lemma.** If $L_1 \subset T^*B_1$ has generating function $F_1: B_1 \times \mathbb{R}^{N_1} \rightarrow \mathbb{R}$ and $L_2 \subset T^*B_2$ has generating function $F_2: B_2 \times \mathbb{R}^{N_2} \rightarrow \mathbb{R}$ then the product $L_1 \times L_2 \subset T^*B_1 \times T^*B_2 \equiv T^*(B_1 \times B_2)$ has generating function $F_1 \circ F_2: (B_1 \times B_2) \times (\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}) \rightarrow \mathbb{R}$.

**A.2. Lemma.** Suppose that $L \subset T^*B$ has generating function $F: B \times \mathbb{R}^N \rightarrow \mathbb{R}$, and consider a symplectomorphism $A_h$ of $T^*B$ of the form $A_h(q,p) = (q,p + dh(q))$ for some function $h: B \rightarrow \mathbb{R}$. Then $A_h(L) \subset T^*B$ has generating function $F + h$.

**A.3. Lemma.** If a Lagrangian submanifold $L$ of $T^*(\mathbb{R}^n \times \mathbb{R}^m)$ has a generating function $F: (\mathbb{R}^n \times \mathbb{R}^m) \times \mathbb{R}^N \rightarrow \mathbb{R}$, then the reduction $\mathcal{L} \subset T^*\mathbb{R}^n$ of $L$ with respect to the isotropic manifold $V = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^N$ has a generating function $\mathcal{F}: \mathbb{R}^n \times (\mathbb{R}^m \times \mathbb{R}^N) \rightarrow \mathbb{R}$, $\mathcal{F}(\zeta_1; \zeta_2; \nu) = F(\zeta_1; \zeta_2; \nu)$.

Since $\Gamma_{id}$ has generating function $\mathbb{R}^{2n} \rightarrow \mathbb{R}$, $q \mapsto 0$, Lemma A.1 implies that the function $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}$ defined by

$$(q, \zeta_1, \zeta_2; \nu_1, \nu_2) \mapsto F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2)$$

is a generating function for $\Gamma_{id} \times \Gamma_{\Phi(1)} \times \Gamma_{\Phi(2)} \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$. By applying Lemma A.2 with

$$h: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad h(q, \zeta_1, \zeta_2) = -2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

we obtain that the function $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}$ defined by

$$(q, \zeta_1, \zeta_2; \nu_1, \nu_2) \mapsto F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

is a generating function for $A_h(\Gamma_{id} \times \Gamma_{\Phi(1)} \times \Gamma_{\Phi(2)}) \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$. The function (A.4) is equal to $F_1 \sharp F_2$, except that in the latter $\zeta_1$ and $\zeta_2$ are fibre variables. Thus, it follows from Lemma A.3 that $F_1 \sharp F_2$ is a generating function for the reduction of

$$L := A_h(\Gamma_{id} \times \Gamma_{\Phi(1)} \times \Gamma_{\Phi(2)}) \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$$

along the coisotropic submanifold

$$V = \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \times (\mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2})$$

of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$. We are left to prove that such reduction is equal to $\Gamma_{\Phi}$. Observe that the reduction $V \rightarrow V/\sim$ sends a point $(q, \zeta_1, \zeta_2, \xi_q, 0, 0)$ to $(q, \xi_q)$. We have

$$L = A_h(\Gamma_{id} \times \Gamma_{\Phi(1)} \times \Gamma_{\Phi(2)})$$

$$= A_h \left( \left\{ \left( q, \frac{z_1 + \Phi(1)(z_1)}{2}, \frac{z_2 + \Phi(2)(z_2)}{2}, 0, i(z_1 - \Phi(1)(z_1)), i(z_2 - \Phi(2)(z_2)) \right) \right\} \right)$$

$$= \left\{ \left( q, \frac{z_1 + \Phi(1)(z_1)}{2}, \frac{z_2 + \Phi(2)(z_2)}{2}, 0, i(z_1 - \Phi(1)(z_1)), i(z_2 - \Phi(2)(z_2)) \right) \right\}$$

$$+ dh(q, \frac{z_1 + \Phi(1)(z_1)}{2}, \frac{z_2 + \Phi(2)(z_2)}{2}, i(z_1 + \Phi(1)(z_1) - z_2 - \Phi(2)(z_2)), i(z_1 - \Phi(1)(z_1) + z_2 + \Phi(2)(z_2) - 2q), i(z_2 - \Phi(2)(z_2) - z_1 - \Phi(1)(z_1) + 2q) \right\}.$$
The intersection $L \cap V$ is given by the points in the above set which satisfy
\[
\begin{align*}
2q &= z_1 - \Phi^{(1)}(z_1) + z_2 + \Phi^{(2)}(z_2) \\
2q &= z_1 + \Phi^{(1)}(z_1) - z_2 + \Phi^{(2)}(z_2)
\end{align*}
\]

hence for which $z_2 = \Phi^{(1)}(z_1)$ and $q = \frac{z_1 + \Phi^{(2)}(z_2)}{2}$. The reduction is thus given by
\[
(L \cap V)/\sim = \left\{ \left( \frac{z_1 + \Phi^{(2)}(z_2)}{2}, i(z_1 - \Phi(z_1)) \right) \right\} = \Gamma \Phi
\]
as we wanted.

**Second interpretation.** The second alternative proof of Proposition 2.2 that we discuss uses symplectic reduction at the level of graphs and is based on the fact, immediate to verify, that the function
\[
h: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}, \quad h(q, \xi_1, \xi_2) = -2 \langle \xi_2 - q, i(\xi_1 - q) \rangle
\]
is a generating function for the symplectomorphism
\[
\sigma: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n}, \quad \sigma(z_0, z_1, z_2) = (z_2, z_0, z_1).
\]
Proposition 2.2 can be deduced from this fact as follows. For simplicity of notation we assume that the generating functions of $\Phi^{(1)}$ and $\Phi^{(2)}$ have no fibre variables (the general case does not present any additional difficulty). Suppose thus that $\Phi^{(1)}$ and $\Phi^{(2)}$ are Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ with generating functions $F_1: \mathbb{R}^{2n} \to \mathbb{R}$ and $F_2: \mathbb{R}^{2n} \to \mathbb{R}$ respectively. Consider the function $F = F_1 + F_2: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \to \mathbb{R}$ defined by
\[
F(q; \xi_1, \xi_2) = F_1(\xi_1) + F_2(\xi_2) + h(q, \xi_1, \xi_2).
\]
Denote the coordinates of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ by $(q, \xi_1, \xi_2; p_0, p_1, p_2)$, and recall that, by the definition of generating function, the Lagrangian submanifold $L_F$ of $T^*\mathbb{R}^{2n}$ generated by $F$ is the symplectic reduction of $dF \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ with respect to the fibre conormal bundle $V = \{ p_1 = p_2 = 0 \}$. The submanifold
\[
V_\Phi := \{(q, \xi_1, \xi_2, p_0, -\partial F_1/\partial \xi_1, -\partial F_2/\partial \xi_2)\}
\]
is also coisotropic, with projection $V_\Phi \to V_\Phi/\sim$ given by $(q, \xi_1, \xi_2, p_0, p_1, p_2) \mapsto (q, p_0)$. Thus, $L_F$ is also equal to the symplectic reduction of $dh \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ with respect to $V_\Phi$. Since
\[
\tau^{-1}(dh) = \text{gr}(\sigma),
\]
our problem is reduced to proving that the reduction of $\text{gr}(\sigma)$ along
\[
\tau^{-1}(V_\Phi) = \{(z_0, z_1, z_2; Z_0, Z_1, Z_2) | Z_1 = (\Phi^{(1)})^{-1}(z_1) \text{ and } Z_2 = (\Phi^{(2)})^{-1}(z_2) \}
\]
is equal to the graph of $\Phi$. But, the projection $\tau^{-1}(V_\Phi) \to \tau^{-1}(V_\Phi)/\sim$ is given by
\[
(z_0, z_1, z_2; Z_0, Z_1, Z_2) \mapsto (z_0, Z_0)
\]
and $\text{gr}(\sigma) \cap \tau^{-1}(V_\Phi)$ is the set of points $(z_0, z_1, z_2, z_2, z_0, z_1)$ such that $z_0 = (\Phi^{(1)})^{-1}(z_1)$ and $z_1 = (\Phi^{(1)})^{-1}(z_2)$. The projection sends such a point to $(z_0, z_2) = (z_0, \Phi(z_0))$.

**Generalization to any even number of factors.** We now show that the second interpretation of Proposition 2.2 in terms of symplectic reduction permits to easily generalize the composition formula to the case of any even number of factors (obtaining an alternative proof of Proposition 2.10). Suppose that $\Phi$ is a Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$ that can be written as a composition
\[
\Phi = \Phi^{(N)} \circ \cdots \circ \Phi^{(1)}
\]
so that $\Gamma_{\Phi^{(i)}} = dF_j$ for functions $F_j: \mathbb{R}^{2n} \to \mathbb{R}$. Assume that $N$ is even, and consider the symplectomorphism $\sigma$ of $\mathbb{R}^{2n} \times (\mathbb{R}^{2n})^N$ defined by
\[
\sigma(z_0, z_1, \ldots, z_N) = (z_N, z_0, z_1, \ldots, z_{N-1}).
\]
A straightforward calculation shows that the function $h: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$ given by
\[
h(q, \xi_1, \ldots, \xi_N) = 2 \sum_{1 \leq j \leq N} (-1)^j \langle \xi_j, iq \rangle + 2 \sum_{1 \leq j < \ell \leq N} (-1)^{j+\ell-1} \langle \xi_j, i\xi_\ell \rangle
\]
is a generating function for $\sigma$.

**A.5. Proposition.** Suppose that, for each $j = 1, \ldots, N$, $\Phi^{(j)}$ is a Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$ with generating function $F_j : \mathbb{R}^{2n} \to \mathbb{R}$. Then the function $F : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$ defined by

$$F(q; \zeta_1, \ldots, \zeta_N) = F_1(\zeta_1) + \ldots + F_N(\zeta_N) + h(q, \zeta_1, \ldots, \zeta_N)$$

is a generating function for the composition $\Phi = \Phi^{(N)} \circ \ldots \circ \Phi^{(1)}$.

**Proof.** Denote the coordinates on $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2nN})$ by $(q, \zeta_1, \ldots, \zeta_N; p_0, p_1, \ldots, p_N)$. The Lagrangian submanifold $L_F$ of $T^*\mathbb{R}^{2n}$ generated by $F$ is the symplectic reduction of $dh \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2nN})$ with respect to the coisotropic submanifold

$$V_\Phi = \{ p_1 = -\frac{\partial F_1}{\partial \zeta_1}(\zeta_1), \ldots, p_N = -\frac{\partial F_N}{\partial \zeta_N}(\zeta_N) \}.$$

Since $\tau^{-1}(dh) = \text{gr}(\sigma)$, our problem is reduced to proving that the reduction of $\text{gr}(\sigma)$ along $\tau^{-1}(V_\Phi) = \{ (z_0, z_1, \ldots, z_N; Z_0, Z_1, \ldots, Z_N) \mid Z_1 = (\Phi^{(1)})^{-1}(z_1), \ldots, Z_N = (\Phi^{(N)})^{-1}(z_N) \}$ is equal to the graph of $\Phi$. But, the projection $\tau^{-1}(V_\Phi) \to \tau^{-1}(V_\Phi)/\sim$ is given by

$$(z_0, z_1, \ldots, z_N; Z_0, Z_1, \ldots, Z_N) \mapsto (z_0, Z_0)$$

and $\text{gr}(\sigma) \cap \tau^{-1}(V_\Phi)$ is the set of points $(z_0, z_1, \ldots, z_N, Z_0, z_1, \ldots, z_{N-1})$ such that $z_{j-1} = (\Phi^{(j)})^{-1}(z_j)$ for $j = 1, \ldots, N$. The projection sends such a point to $(z_0, \Phi(z_0))$. \hfill \Box

**A.6. Remark.** In the case of $\mathbb{R}^{2n-1}$, Givental does not use directly the generating function given by Proposition A.5. Instead he studies a path $-\Phi_t$ starting at $-\text{id}$ by looking at a family of generating functions $F_t$ of the path $\Phi_t$ (starting at the identity). For fibre critical points of $F_t$, we have $q = \frac{z_1 + \Phi_t(z_1)}{2}$. Thus critical points of the restriction of $F_t$ to the fibre over $q = 0$ correspond to fixed points of $-\Phi_t$. So, instead of looking at the whole function $\Phi_t$, Givental only considers the restriction of $F_t$ to the fibre over $q = 0$. \hfill \diamond

**Relation with the method of broken trajectories.** We now discuss the relation between the composition formula of Proposition 2.2 and the construction of generating functions via the method of broken trajectories, due to Chaperon, Laudenbach and Sikorav [Ch84, LS85, Si85, Si87].

The method of broken trajectories is used to construct generating functions for Lagrangian submanifolds of a cotangent bundle $T^*B$ that are Hamiltonian isotopic to the zero section. The idea is to first interpret the symplectic action functional on a space of paths in $T^*B$ as a generating function with infinite dimensional domain, and then to construct a finite dimensional approximation. Recall that the symplectic action functional associated to a time-dependent Hamiltonian $H_t$ on an exact symplectic manifold $(W, \omega = -d\lambda)$ is the functional $A_H$ on the space of paths $\gamma : [0, 1] \to W$ which is defined by

$$A_H(\gamma) = \int_0^1 \lambda(\frac{\partial \gamma}{\partial t}) - H_t(\gamma(t)) \, dt.$$

A path $\gamma$ is a critical point of $A_H$ with respect to variations with fixed endpoints if and only if it is a trajectory of the Hamiltonian flow of $H_t$. Consider now the case where $W$ is a cotangent bundle $T^*B$. Let $E$ be the space of paths $\gamma : [0, 1] \to T^*B$ that begin at the zero section, and see it as the total space of a fibre bundle over $B$ with projection $p : E \to B$ given by $p(\gamma) = \pi(\gamma(1))$, where $\pi$ is the projection of $T^*B$ into $B$. Given a time-dependent Hamiltonian $H_t : T^*B \to \mathbb{R}$, consider the functional $F : E \to \mathbb{R}$ defined by

$$F(\gamma) = A_H(\gamma).$$

The fibre critical points of $F : E \to \mathbb{R}$ are the trajectories of the Hamiltonian flow of $H_t$, and the covector $\nu^*(\gamma)$ associated to a fibre critical point $\gamma$ is the vertical component of $\gamma(1)$. Thus, $F$ generates the image of the zero section by the time-1 map of the Hamiltonian flow of $H_t$. Although $F$ is not a generating function in the usual sense, because its domain is infinite dimensional, a finite
dimensional reduction can be obtained as follows. Let $N$ be an integer. Consider the direct sum $\bigoplus_{i=1}^{N-1} TB \oplus T^* B$, and denote its elements by expressions of the form $e = (q, X, P)$, where $q$ is a point of $B$, $X = (X_1, \ldots, X_{N-1})$ is an $(N-1)$-tuple of vectors $X_j \in T_q B$ and $P = (P_1, \ldots, P_{N-1})$ is an $(N-1)$-tuple of covectors $P_j \in T^*_q B$. Let $U$ be a neighborhood of the zero section of $TB$, and consider the subspace $E_N$ of $\bigoplus_{i=1}^{N-1} TB \oplus T^* B$ that is formed by those elements $e = (q, X, P)$ such that all $X_j$ belong to $U$. If $U$ is sufficiently small then an element $e = (q, X, P)$ of $E_N$ can be interpreted as a broken Hamiltonian trajectory of $H_t$, with $N$ smooth pieces and $N-1$ jumps, as follows. The first smooth piece $\gamma_1$ is obtained by following the Hamiltonian flow of $H_t$ for $t \in [0, \frac{1}{N}]$ from the point $(q, 0)$ to a point of $T^* B$ that we denote by $\eta_1^+$. The second smooth piece $\gamma_2$ starts from a point $\eta_1^-$, which is uniquely determined by $\eta_1^+, X_1$ and $P_1$ in a way that we describe later, and follows the flow of $H_t$ for $t \in \left[\frac{1}{N}, \frac{2}{N}\right]$ to a point $\eta_2^-$. We continue in this way to obtain the whole broken trajectory. In order to describe the jumps we fix a Riemannian metric on $B$, and consider the associated Levi–Civita connections on $TB$ and $T^* B$. The point $\eta_1^+ = (q_1^+, P_1^+)$ is determined by $\eta_1^- = (q_1^-, P_1^-)$, $X_1$ and $P_1$ in the following way. Denote by $X_1 \in T_{\eta_1^-} B$ and $P_1 \in T^*_{\eta_1^-} B$ the vector and the covector obtained by parallel transport of $X_1$ and $P_1$ along the projection to $B$ of the path $\gamma_1$. Since $X_1$ is in $U$, which is assumed to be sufficiently small, we can then define $q_1^+ = \exp_{\eta_1^-}^{q_1^-}(X_1)$ and $P_1^+ = (d\exp_{\eta_1^-}^{q_1^-}\gamma_1^-)^{-1}(P_1)$. The other jumps are defined similarly. Consider the projection $p: E_N \to B$ that sends a point $e = (q, X, P)$ of $E_N$ to the projection $q$ to $B$ of the endpoint $\eta_N$ of the broken Hamiltonian trajectory associated to $e$. Define a function $F: E_N \to \mathbb{R}$ by

$$F(e) = \sum_{j=1}^{N} A_H(\gamma_j) + \sum_{j=1}^{N-1} P_j(X_j). \quad (A.7)$$

Denote the flow of $H_t$ by $\{\varphi_t\}_{t \in [0,1]}$, and assume that, for all $j = 1, \cdots, N$, the symplectomorphism $\varphi_{\frac{1}{N}} \circ (\varphi_{\frac{1}{N}})^{-1}$ is sufficiently $C^1$-small. Then [Si85] the fibre critical points of $F$ are the unbroken trajectories, and the covector $v^*(e)$ associated to a fibre critical point $e$ is given by $v^*(e) = p_N^*$. Thus, $F: E_N \to \mathbb{R}$ is a generating function for the image of the zero section by $\varphi_1$.

If $\varphi_1$ is already sufficiently $C^1$-small then the above construction (for $N = 1$) reduces to the following. The space $E_1$ can be identified with $B$, by associating to a point $q$ of $B$ the Hamiltonian trajectory $\gamma_q$ of $H_t$ starting at $q$. We see $E_1$ as the total space of a fibre bundle over $B$ by the diffeomorphism

$$E_1 \to B, \quad q \mapsto \pi(\gamma_q(1)). \quad (A.8)$$

Then the function $F: E_1 \to \mathbb{R}$, $F(q) = A_H(\gamma_q)$ is a generating function, with respect to the projection $(A.8)$, for the image of the zero section by $\varphi_1$. In other words, the function on $B$ obtained by precomposing $F$ with the inverse of $(A.8)$ is a generating function (with respect to the projection $B \to B$ given by the identity) for the image of the zero section by $\varphi_1$.

Returning to a general $N$, in the case when $B$ is $\mathbb{R}^n$ with the Euclidean metric we can identify $E_N$ with the product $\mathbb{R}^n \times (\mathbb{R}^n)^{N-1} \times (\mathbb{R}^n)^{N-1}$. For an element $e = (q, X_1, \ldots, X_{N-1}, P_1, \ldots, P_{N-1})$ we then have $q_j^+ = q_j^+ + X_j$, $p_j^+ = P_j$ and $P_j(X_j) = \langle P_j, X_j \rangle$.

We now discuss how the composition formula of Proposition 2.2 is related to this construction. We are interested in Lagrangians of $T^* \mathbb{R}^{2n}$ of the form $\Gamma_\Psi = \tau(\text{gr}(\Psi))$, where $\tau: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to T^* \mathbb{R}^{2n}$ is the identification (2.1). Any such Lagrangian is Hamiltonian isotopic to the zero section by the Hamiltonian isotopy of $T^* \mathbb{R}^{2n}$ that corresponds under $\tau$ to a Hamiltonian isotopy of the form $\text{id} \times \Phi_t$. The Hamiltonian function thus satisfies

$$H_t(\tau(z, Z)) = H_t(\tau(z + a, Z)) = H_t(\tau(z, Z) + \langle a, Z \rangle) \quad \text{for all } a \in \mathbb{R}^{2n} \equiv \mathbb{C}^n. \quad (A.9)$$

Suppose now that the flow $\{\Psi_t\}_{t \in [0,1]}$ of $H_t$ is the concatenation of pieces $\{\Psi_t\}_{t \in [0,1]}$ so that each $\Psi(t) := \Psi \circ \Psi_t^{-1}$ has a generating function $F_j: \mathbb{R}^{2n} \to \mathbb{R}$. Using $(A.9)$ we can relate the action terms in $(A.7)$ with the action of Hamiltonian trajectories starting at the zero section, and
thus with the functions $F_j$. Observe first that $A_H(\gamma_1) = F_1(\zeta_1)$ where $\zeta_1 := q_1^+ \equiv \frac{q+\Psi^{(1)}(q)}{2}$. For $j = 2, \cdots, N$ set

$$\tilde{\gamma}_j(t) = \gamma_j(t) + (\frac{ip_j-1}{2}, -P_{j-1}).$$

The $\tilde{\gamma}_j$ are Hamiltonian trajectories starting at $(q_{j-1}^-+X_{j-1}+\frac{ip_j-1}{2}, 0)$, and

$$A_H(\tilde{\gamma}_j) - A_H(\gamma_j) = \int_0^1 \lambda \frac{\partial \gamma_j}{\partial t} - \lambda \frac{\partial \gamma_j}{\partial t} \, dt = - \langle P_{j-1}, q_j^- - (q_{j-1}^-+X_{j-1}) \rangle.$$

Set

$$\zeta_j = \frac{u_j + \Psi^{(j)}(u_j)}{2}$$

where $u_j = q_{j-1}^-+X_{j-1}+\frac{ip_j-1}{2}$. Then $A_H(\tilde{\gamma}_j) = F_j(\zeta_j)$, and so the function $(A.7)$ reduces to

$$F(e) = F_1(\zeta_1) + \cdots + F_N(\zeta_N) + \sum_{j=2}^N \langle P_{j-1}, q_j^- - (q_{j-1}^-+X_{j-1}) \rangle + \sum_{j=1}^{N-1} \langle P_j, q_{j-1}^- - q_j^- \rangle.$$

Set $q_j := q_{j+1}^- - \frac{ip_j}{2}$ for $j = 1, \cdots, N-1$, and consider the change of variables

$$e = (q,X_1, \cdots, X_{N-1}, P_1, \cdots, P_{N-1}) \mapsto (\zeta_1, \cdots, \zeta_N, q_1, \cdots, q_{N-1}).$$

Then

$$\sum_{j=1}^{N-1} \langle P_j, q_{j+1}^- - q_j^- \rangle = -2i(\zeta_2 - q_1), q_1 - \zeta_1) + \sum_{j=2}^{N-1} (-2i(\zeta_{j+1} - q_j), q_j - q_{j-1})$$

$$= -2 \langle \zeta_2 - q_1, i(q_1 - \zeta_1) \rangle - 2 \sum_{j=2}^{N-1} \langle \zeta_{j+1} - q_j, i(q_j - q_{j-1}) \rangle$$

and so the function $(A.7)$ reduces to the function $((F_1 \sharp F_2 \sharp \ldots \sharp F_N)$ obtained by iteratively applying the composition formula of Proposition 2.2.

**APPENDIX B. THE HOMOLOGY JOIN**

Recall that the $\mathbb{Z}_k$-join $A \ast_{\mathbb{Z}_k} B$ of subsets $A$ of $L^2(M-1)(w)$ and $B$ of $L^2(M'-1)(w')$ is the subset of $L^2(M+M'-1)(w,w')$ defined by (3.8). In this section we complete the proof of the join quasi-additivity property of the cohomological index (Part (v) of Proposition 3.9) by proving the following lower bounds on the index of the equivariant join: if $A$ and $B$ are closed, then

$$\text{ind}(A \ast_{\mathbb{Z}_k} B) \geq \begin{cases} \text{ind}(A) + \text{ind}(B) & \text{if at least one of the indices is even,} \\ \text{ind}(A) + \text{ind}(B) - 1 & \text{if both indices are odd.} \end{cases}$$

To prove this lower bound, we develop a join operation on equivariant homology.

The join stability property of Proposition 3.9(v),

$$\text{ind} \left( A \ast_{\mathbb{Z}_k} L^2_k(M-1)(w') \right) = \text{ind}(A) + 2K,$$

is a special case of the join quasi-additivity property, but since several of our applications of the non-linear Maslov index only need this special case, we also give a short direct proof of it.

**B.3. Remark.** By continuity of the index (Proposition 3.9(i)), and since for every neighborhood $O$ of $A \ast_{\mathbb{Z}_k} B$ there exist neighborhoods $U$ of $A$ and $V$ of $B$ such that $O$ contains $U \ast_{\mathbb{Z}_k} V$, it is enough to prove (B.1) and (B.2) when $A$ and $B$ are open subsets of lens spaces.
B.4. Lemma. Let $A$ be an open subset of a lens space $L^{2M-1}_k(w)$. Then the map
\[ \iota_* : H_j(A; \mathbb{Z}_k) \to H_j(L^{2M-1}_k(w); \mathbb{Z}_k) \]
that is induced by the inclusion $\iota : A \hookrightarrow L^{2M-1}_k(w)$ is surjective (with image $\cong \mathbb{Z}_k$) for all $j < \text{ind}(A)$ and is the zero map for all $j \geq \text{ind}(A)$.

**Proof.** Because $A$ is a manifold, its Čech cohomology agrees with its singular cohomology. The result then follows from Lemma 3.3 by the duality between homology and cohomology with field coefficients. \qed

**Proof of the join stability property (B.2).**

Recall that the Thom space of a real vector bundle $\pi : E \to X$ is the space $\text{Th}(\pi) = D(E)/S(E)$, where $S(E)$ and $D(E)$ denote the total spaces of the corresponding unit sphere bundle and closed disk bundle. An orientation of $E$ gives rise to the Thom isomorphism
\[ T : \bar{H}_{j+m} (\text{Th}(\pi)) \to H_j(X) \quad \text{for} \ j \geq 0, \]
where the tilde denotes reduced homology and $m$ is the rank of $\pi : E \to X$ (see for instance [Sp66, Theorem 5.7.10]).

By Remark B.3, we can assume that $A$ is an open subset of $L^{2M-1}_k(w)$. The preimage $\tilde{A}$ of $A$ in $\Sigma^{2M-1}_k(w)$ is a principal $\mathbb{Z}_k$-bundle. Consider the associated vector bundle
\[ (B.5) \quad \pi : \left( \tilde{A} \times \mathbb{R}^{2K}(w') \right)/\mathbb{Z}_k \to A. \]
We claim that there is a cofibre sequence
\[ (B.6) \quad L^{2K-1}_k(w') \hookrightarrow A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w') \xrightarrow{\pi} \text{Th}(\pi), \]
where the first map is the canonical embedding into the join. Indeed, there is a natural $\mathbb{Z}_k$-equivariant homeomorphism
\[ \psi : (\tilde{A} \ast S^{2K-1}(w')) \setminus S^{2K-1}(w') \to \tilde{A} \times (D^{2K}(w')), \]
where $\text{int}(D^{2K}(w')) \subset \mathbb{C}^k(w')$ is the open unit disc and $\mathbb{Z}_k$ acts on $\tilde{A} \times (D^{2K}(w'))$ diagonally. The map that $\psi$ induces on the quotient spaces identifies $(A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w')) \setminus L^{2K-1}_k(w')$ with the total space of the open disk bundle of the vector bundle $(B.5)$ and extends to a homeomorphism
\[ (A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w'))/L^{2K-1}_k(w') \to \text{Th}(\pi), \]
which expresses (B.6) as a cofibre sequence.

The canonical embedding $L^{2K-1}_k(w') \hookrightarrow A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w')$ is injective in homology (for example, because its composition with the classifying map of $A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w')$ is a classifying map for $L^{2K-1}_k(w')$, which is injective in homology). It follows that the long exact sequence associated to (B.6) splits into short exact sequences
\[ 0 \to H_j(L^{2K-1}_k(w')) \to H_j(A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w')) \to H_j(\text{Th}(\pi)) \to 0, \]
and thus the collapse map
\[ q : A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w') \to \text{Th}(\pi) \]
induces an isomorphism in homology in degrees $\geq 2K$. Consider the isomorphism $\ast \ell$ defined by
\[ \ast \ell : H_j(A) \xrightarrow{T^{-1}} H_{j+2K}(\text{Th}(\pi)) \xrightarrow{(q \ast 1)^{-1}} H_{j+2K}(A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w')) \quad \text{for} \ j \geq 0. \]
Since $\ast \ell$ is natural, by applying it to $A$ and to $L^{2M-1}(w)$ we obtain a commuting diagram

\[
\begin{array}{ccc}
H_j(A) & \xrightarrow{\ast \ell} & H_{j+2K}(A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w')) \\
\downarrow \iota_\ast & & \downarrow \iota_{\ast \mathbb{Z}_k \text{id}}_\ast \\
H_j(L^{2M-1}_k(w)) & \xrightarrow{\ast \ell} & H_{j+2K}(L^{2(M+K)-1}_k(w, w'))
\end{array}
\]

Since $\text{ind}(A)$ and $\text{ind}(A \ast_{\mathbb{Z}_k} L^{2K}_k(w'))$ are the lowest degrees of the homology groups on which the corresponding vertical arrows of (B.7) are the zero maps (by Lemma B.4), and since the horizontal arrows of (B.7) are isomorphisms that increase the degree by $2K$, we conclude that $\text{ind}(A \ast_{\mathbb{Z}_k} L^{2K-1}_k(w')) = \text{ind}(A) + 2K$.

The join and the equivariant join. In order to construct a join operation on singular homology we first need to discuss in some detail the join operation on topological spaces. Here, all topological spaces are assumed to be Hausdorff. Let

\[
\Delta^m = \{(t_0, \ldots, t_m) \in \mathbb{R}^{m+1} \mid t_0 + \ldots + t_m = 1\}
\]

be the standard $m$-simplex ($m \geq 0$). The join of two topological spaces $X$ and $Y$ is defined to be the quotient

\[
X \ast Y = (X \times Y \times \Delta^1)/\sim
\]

where the only non-trivial relations are $(x, y, (0, 1)) \sim (x', y, (0, 1))$ and $(x, y, (1, 0)) \sim (x, y', (1, 0))$ for all $x, x' \in X$ and $y, y' \in Y$. We denote the equivalence class of $(x, y, (t_0, t_1))$ in $X \ast Y$ by $t_0x + t_1y$.

Similarly, the join of $(m + 1)$ topological spaces $X_0, \ldots, X_m$ is defined as

\[
X_0 \ast \cdots \ast X_m = (X_0 \times \cdots \times X_m \times \Delta^m)/\sim,
\]

with the class of $(x_0, \ldots, x_m, (t_0, \ldots, t_m))$ denoted by $t_0x_0 + \ldots + t_mx_m$, and with $t_0x_0 + \ldots + t_mx_m = t'_0x'_0 + \ldots + t'_mx'_m$ if and only if $t_j = t'_j$ for all $j$ and $x_j = x'_j$ whenever $t_j \neq 0$.

The join operation is natural in the sense that continuous functions $f_j : X_j \to X'_j$ induce a continuous function $f_0 \ast \cdots \ast f_m : X_0 \ast \cdots \ast X_m \to X'_0 \ast \cdots \ast X'_m$, such that $(g_0 \circ f_0) \ast \cdots \ast (g_m \circ f_m) = (g_0 \ast \cdots \ast g_m) \circ (f_0 \ast \cdots \ast f_m)$. In particular, when the spaces $X_j$ are equipped with an action of a group $G$, their join acquires the $G$-action given by $a \cdot (t_0x_0 + \ldots + t_mx_m) = t_0(a \cdot x_0) + \ldots + t_m(a \cdot x_m)$. The (equivariant) join of principal $G$-bundles is a principal $G$-bundle. If $X$ and $Y$ are spaces equipped with principal $G$-bundles $\tilde{X} \to X$ and $\tilde{Y} \to Y$, we define their $G$-join to be the space $X \ast G Y := (\tilde{X} \ast \tilde{Y})/G$ equipped with the principal bundle $\tilde{X} \ast \tilde{Y} \to X \ast G Y$.

The natural maps

\[
(X \ast Y) \ast Z \xrightarrow{\varphi_1} X \ast Y \ast Z \xrightarrow{\varphi_2} X \ast (Y \ast Z)
\]

given by $\varphi_1 : s_0(t_0x + t_1y) + s_1z \mapsto (s_0t_0)x + (s_0t_1)y + s_1z$ and similarly for $\varphi_2$ are homeomorphisms (because they are continuous proper bijections of Hausdorff spaces). We have similar maps for any parenthetization of any number of factors. We also have the map

\[
\tau : X \ast Y \to Y \ast X, \quad t_0x + t_1y \mapsto t_1y + t_0x.
\]

These maps are natural, in the sense that for any continuous maps $f : X \to X'$, $g : Y \to Y'$ and $h : Z \to Z'$ the following diagrams commute

\[
\begin{array}{ccc}
(X \ast Y) \ast Z & \xrightarrow{(f \ast g) \ast h} & X \ast Y \ast Z \\
\downarrow (f \ast g) \ast h & & \downarrow (f \ast g) \ast h \\
X' \ast (Y' \ast Z') & \xrightarrow{\varphi_2} & X' \ast Y' \ast Z'
\end{array}
\]

and similarly for $\varphi_2$. For principal $G$-bundles over $X, Y, Z$, the associativity and commutativity homeomorphisms (B.8) and (B.9) pass to the quotients and yield homeomorphisms

\[
(X \ast G Y) \ast G Z \to X \ast G Y \ast G Z \leftarrow X \ast G (Y \ast G Z) \quad \text{and} \quad X \ast G Y \to Y \ast G X
\]
that are natural in the sense analogous to (B.10).

**Joins of standard simplices, spheres, and lens spaces.** For the standard unit spheres, we have identifications

\[ \psi_{M, M'} : S^{2M-1} \ast S^{2M'-1} \cong S^{2(M+M')-1} \]

given by

\[
\begin{pmatrix}
  t(z_0, \ldots, z_M) + t'(z'_0, \ldots, z'_{M'}) \\
  t'(z'_0, \ldots, z'_{M'}) \rightarrow (\sqrt{t}z_0, \ldots, \sqrt{t'}z'_0, \ldots, \sqrt{t'}z'_{M'})
\end{pmatrix}
\]

(see maps are continuous proper bijections, hence homeomorphisms). These identifications descend to lens spaces, where we use the same notation

\[
(B.11) \quad \psi_{M, M'} : L_k^{2M-1}(w) \ast_{Z_k} L_k^{2M'-1}(w') \cong L_k^{2(M+M')-1}(w, w').
\]

We have similar identifications for multiple joins.

These identifications are consistent in the sense that the diagrams commute, and similarly for the other parenthetization, where to simplify notation we omitted the weights. By induction we get consistent identifications of iterated multiple joins of lens spaces with higher lens spaces.

Given subsets \( A \subset L_k^{2M-1}(w) \) and \( B \subset L_k^{2M'-1}(w') \), identifying \( A \ast_{Z_k} B \) with a subset of \( L_k^{2M-1}(w) \ast_{Z_k} L_k^{2M'-1}(w') \) by naturality and further with a subset of \( L_k^{2(M+M')-1}(w, w') \) by (B.11), we get the same subset of \( L_k^{2(M+M')-1}(w, w') \) that was described in (3.8).

B.13. Remark. If \( f \) and \( g \) are smooth maps between spheres or lens spaces, then \( f \ast g \) (viewed as a map between higher dimensional spheres or lens spaces) might not be smooth.

For the standard simplices, the identifications

\[
(B.14) \quad \Delta^l \ast \Delta^m \cong \Delta^{l+m+1}
\]

given by

\[
u_0(t_0, \ldots, t_l) + u_1(s_0, \ldots, s_m) \rightarrow (u_0t_0, \ldots, u_0t_l, u_1s_0, \ldots, u_1s_m)
\]

are also consistent, in a sense similar to (B.12). In particular, the composition

\[
(B.15) \quad \Delta^{l+m+n+2} \cong \Delta^l \ast \Delta^m \ast \Delta^n \cong (\Delta^l \ast \Delta^m) \ast \Delta^n \cong \Delta^{l+m+1} \ast \Delta^n \cong \Delta^{l+m+n+2}
\]

is the identity map, and similarly with the other parenthetization.

**The join operation on homology.** Given singular simplices \( \sigma : \Delta^l \rightarrow X \) and \( \mu : \Delta^m \rightarrow Y \) on \( X \) and \( Y \), the identification (B.14) makes their join into a singular \((l+m+1)\)-simplex

\[
\sigma \ast \mu : \Delta^{l+m+1} \rightarrow X \ast Y.
\]

Extending bilinearly, we obtain a map of singular chains

\[
(B.16) \quad \ast : C_l(X) \otimes C_m(Y) \rightarrow C_{l+m+1}(X \ast Y).
\]

Similarly, the triple join gives a map of singular chains

\[
C_j(X) \otimes C_l(Y) \otimes C_m(Z) \rightarrow C_{j+l+m+2}(X \ast Y \ast Z).
\]
The definitions of the join and boundary operations $*$ and $\partial$ directly imply that for any two singular simplices $\sigma: \Delta^l \to X$ and $\mu: \Delta^m \to Y$ we have

$$\partial(\sigma \ast \mu) = \begin{cases} (\partial\sigma) \ast \mu + (-1)^{l+1} \sigma \ast \partial\mu & \text{if } l, m > 0, \\ (\partial\sigma) \ast \mu + (-1)^{l+1} \sigma & \text{if } l > 0, m = 0, \\ \mu - \sigma \ast (\partial\mu) & \text{if } l = 0, m > 0, \\ \mu - \sigma & \text{if } l = m = 0. \end{cases} \quad (B.17)$$

It follows from (B.17) that the chain join (B.16) defines an operation

$$*: H_l(X) \otimes H_m(Y) \to H_{l+m+1}(X \ast Y) \text{ when } l > 0 \text{ and } m > 0.$$ We now show that the same considerations go through also in the case of equivariant homology for principal $G$-bundles, and that moreover if we consider $\mathbb{Z}_k$-coefficients with $k$ dividing the order of $G$ then the induced operation in homology is defined in all degrees.

Let $\tilde{X} \to X$ be a principal $G$-bundle and $C_*(\tilde{X})^G$ the complex of $G$-invariant chains on $\tilde{X}$. There is a canonical isomorphism of complexes

$$\varphi: C_*(X) \xrightarrow{\cong} C_*(\tilde{X})^G$$

which on a singular simplex $\sigma: \Delta^m \to X$ is given by $\varphi(\sigma) = \sum_{g \in G} g \cdot \tilde{\sigma}$, where $\tilde{\sigma}: \Delta^m \to \tilde{X}$ is any lift of $\sigma$. The join operation sends $G$-invariant chains on $\tilde{X}$ and $\tilde{Y}$ to $G$-invariant chains on $\tilde{X} \ast \tilde{Y}$. Defining

$$\sigma \ast_G \mu := \varphi^{-1}(\varphi(\sigma) \ast \varphi(\mu))$$
on simplices, and extending bilinearly, we obtain an equivariant join operation on chains,

$$*_G: C_l(X) \otimes C_m(Y) \to C_{l+m+1}(X \ast}_G Y). \quad (B.18)$$

The definitions of the join operation on chains (B.16) and on equivariant chains (B.18) make sense with arbitrary ring coefficients. We now consider $\mathbb{Z}_k$-coefficients, for $k$ dividing the order of $G$.

**Lemma.** Let $\tilde{X} \to X$ and $\tilde{Y} \to Y$ be principal $G$-bundles. Assume that $k$ divides the order of $G$. Then the equivariant join operation on chains (B.18) satisfies

$$\partial(\sigma \ast_G \mu) = (\partial\sigma) \ast_G \mu + (-1)^{l+1} \sigma \ast_G \partial\mu.$$

**Proof.** By bilinearity, it is enough to consider the case of simplices (rather than chains) $\sigma: \Delta^l \to X$ and $\mu: \Delta^m \to Y$. If $l > 0$ and $m > 0$ the result follows from (B.17). Assume now that $l > 0$ and $m = 0$; the remaining cases are similar. Let $\tilde{\sigma}: \Delta^l \to \tilde{X}$ and $\tilde{\mu}: \Delta^m \to \tilde{Y}$ be any lifts of $\sigma$ and $\mu$. Since we use $\mathbb{Z}_k$-coefficients, we have

\[
\partial(\sigma \ast_G \mu) = \partial(\varphi^{-1}(\varphi(\sigma) \ast \varphi(\mu))) = \varphi^{-1}\left(\partial\left(\sum_{g,h \in G} (g \cdot \tilde{\sigma}) \ast (h \cdot \tilde{\mu})\right)\right) = \varphi^{-1}\left(\sum_{g,h \in G} \left(\partial(g \cdot \tilde{\sigma}) \ast (h \cdot \tilde{\mu}) + (-1)^{l+1} g \cdot \tilde{\sigma}\right)\right) = \varphi^{-1}(\varphi(\partial\sigma) \ast \varphi(\mu)) + \varphi^{-1}(|G|(-1)^{l+1}\varphi(\sigma)) = (\partial\sigma) \ast_G \mu.
\]

Lemma B.19 implies that if $k$ divides the order of $G$ then the equivariant join operation on chains induces an operation on homology:

$$*_G: H_l(X; \mathbb{Z}_k) \otimes H_m(Y; \mathbb{Z}_k) \to H_{l+m+1}(X \ast}_G Y; \mathbb{Z}_k) \quad \text{for all } l \geq 0 \text{ and } m \geq 0.$$
Moreover, the naturality of the chain level formula provides join operations on relative homology
\[ *_{G}: H_{i}(X, A; \mathbb{Z}) \otimes H_{m}(Y, B; \mathbb{Z}) \to H_{i+m+1}(X *_{G} Y, (X *_{G} B) \cup (A *_{G} Y); \mathbb{Z}) \]
such that \[ x_{i} *_{G} y_{m} = x_{i} *_{G} y_{m} \] for any \( x_{i} \in H_{i}(X; \mathbb{Z}) \) and \( y_{m} \in H_{m}(Y; \mathbb{Z}) \), where \( x_{i}, y_{m} \) and \( x_{i} *_{G} y_{m} \) denote the images of \( x_{i}, y_{m} \) and \( x_{i} *_{G} y_{m} \) in the relative homology.

It follows from the consistency of the identifications of the standard simplices (specifically, from (B.15) being the identity map) that the join operation on chains (B.16) is associative in the following sense. For any three singular simplices \( \sigma: \Delta^{j} \to X \), \( \mu: \Delta^{l} \to Y \) and \( \nu: \Delta^{m} \to Z \) we have
\[
(\varphi_{1} \circ (\sigma * \mu) * \nu) = \sigma * \mu * \nu = (\sigma * (\mu * \nu)) ,
\]
where \( \varphi_{1} \) and \( \varphi_{2} \) are the homeomorphisms of (B.8). This further implies that the equivariant join operation on homology is associative in the following sense. For homology classes \( \alpha, \beta, \gamma \) on spaces \( X, Y, Z \) equipped with principal \( G \)-bundles with \( k \) dividing the order of \( G \), we have
\[
(\varphi_{1} \circ ((\alpha * \beta) * \gamma)) = (\alpha * (\beta * \gamma)) = (\alpha * (\beta * \gamma)).
\]
The join operation in homology also satisfies a commutativity property. We postpone this result to at the end of this appendix (Proposition B.24); we do not need it for our applications.

**Computation for lens spaces.** Lemma B.21 and Proposition B.22 contain computations of equivariant joins for lens spaces. For our applications, we only need the “if” direction of Proposition B.22 and we don’t need Lemma B.21.

**B.21. Lemma.** Let \( x_{0} \in H_{0}(L_{k}^{1}; \mathbb{Z}) \) be the homology class of a point. Then \( x_{0} *_{Z_{k}} x_{0} = 0 \).

**Proof.** The class \( x_{0} \in H_{0}(L_{k}^{1}; \mathbb{Z}) \) is represented by the singular simplex sending \( \Delta^{0} \) to \([1] \in L_{k}^{1}\). The class \( x_{0} *_{Z_{k}} x_{0} \in H_{1}(L_{k}^{2}; \mathbb{Z}) \) is represented by \( \sum_{j=0}^{k-1} \sigma_{j} \), where, for \( j = 0, 1, \ldots, k-1 \), \( \sigma_{j}: \Delta^{1} = [0, 1] \to L_{k}^{1} \) be the singular 1-simplex given by
\[
\sigma_{j}(t) = \left[ \sqrt{1 - t}, \sqrt{t e^{j \frac{2 \pi k}{3}}} \right].
\]
The paths \( \sigma_{j} \) all have initial point \([1, 0] \) and end point \([0, 1] \) in \( L_{k}^{1} \). The concatenation \( \sigma_{0} \sigma_{j} \) is a loop in \( L_{k}^{2} \) in the homology class \([\sigma_{j} - \sigma_{0}] \) whose lift to \( S^{3} \) that starts at \([0, 1] \) and ends at \([0, e^{2 \pi j/k}] \). If \( 1 \leq j \leq k-1 \) then the loop \( \sigma_{0} \sigma_{j} \) generates \( \pi_{1}(L_{k}^{2}) \), and \( \sigma_{0} \sigma_{j} = [\sigma_{0} \sigma_{j}]^{l} \) in \( \pi_{1}(L_{k}^{2}) \). In \( H_{1}(L_{k}^{2}) \) we have \([\sigma_{j} - \sigma_{0}] = j[\sigma_{1} - \sigma_{0}] \). So
\[
x_{0} *_{Z_{k}} x_{0} = \sum_{j=0}^{k-1} \sigma_{j} = \sum_{j=0}^{k-1} (\sigma_{j} - \sigma_{0}) = \sum_{j=0}^{k-1} j[\sigma_{1} - \sigma_{0}] = \frac{k(k-1)}{2} [\sigma_{1} - \sigma_{0}] = 0.
\]

**B.22. Proposition.** Suppose that \( x_{m} \) and \( x_{m'} \) are non-zero elements of \( H_{m}(L_{k}^{2M-1}(w); \mathbb{Z}) \) and \( H_{m'}(L_{k}^{2M'-1}(w'); \mathbb{Z}) \). Then the join \( x_{m} *_{Z_{k}} x_{m'} \) is non-zero if and only if \( m \) or \( m' \) is odd.

**Proof.** By functoriality, it suffices to consider the case when the weights \( w \) and \( w' \) are of the form \((1, \ldots, 1)\). Indeed, for any \( w \) (and similarly for \( w' \)), the classifying map of \( L_{k}^{2M-1}(w) \) induces an injection in homology (even over \( \mathbb{Z} \)) and can be obtained as the composition of a map \( L_{k}^{2M-1}(w) \to L_{k}^{2K-1} := L_{k}^{2K-1}(1, \ldots, 1) \) with the classifying map \( L_{k}^{2K-1} \to L_{k}^{\infty} \) for some sufficiently large \( K \).

Let \( \sigma_{0}: \Delta^{0} \to L_{k}^{1} \) be the simplex \( 1 \mapsto [1] \) and let \( \sigma_{1}: \Delta^{1} \to L_{k}^{1} \) be the simplex \((t_{0}, t_{1}) \mapsto [e^{2 \pi i t / k}].\)

As chains, \( \sigma_{0} \) and \( \sigma_{1} \) are closed; denote their homology classes by \( y_{0} = [\sigma_{0}] \) and \( y_{1} = [\sigma_{1}] \).

The standard cell decomposition of \( L_{k}^{2M+1} \) can be described as follows (see for instance [Ha02, Example 2.43]). There is one cell \( e^{j} \) in each dimension \( 0 \leq j \leq 2M + 1 \). The standard inclusion \( L_{k}^{2M-1} \to L_{k}^{2M+1}, [z] \mapsto [z, 0] \), takes the jth cell of \( L_{k}^{2M-1} \) to the jth cell of \( L_{k}^{2M+1} \) for all \( 0 \leq j \leq 2M + 1 \).
2M − 1 and is injective in homology. Moreover, under the identification $L^{2M+1}_k = L^{2M-1}_k *_{Z_k} L^1_k$ of (B.11), we have that $e^{2M} = e^{2M-1} *_{Z_k} e^1$ and $e^{2M+1} = e^{2M-1} *_{Z_k} e^1$.

When $j$ is odd, the $j$th skeleton of the cellular decomposition of $L^{2M+1}_k$ is the lens subspace $L^j_k$, embedded by the standard inclusion $[z] \mapsto [z, 0]$. We denote the $j$th skeleton of the cell complex by $L^j_k$ even when $j$ is even.

Let
\[ \sigma_{2M-1}: (\Delta^{2M-1}, \partial \Delta^{2M-1}) \to (L^{2M-1}_k, L^{2M-2}_k) \]
be a characteristic map of $e^{2M-1}$. Then $[\sigma_{2M-1}]$ is a generator of $H_{2M-1}(L^{2M-1}_k, L^{2M-2}_k; Z_k)$. Let $y_{2M-1}$ be the generator of $H_{2M-1}(L^{2M-1}_k)$ that maps to $[\sigma_{2M-1}]$ under the isomorphism
\[ H_{2M-1}(L^{2M-1}_k, L^{2M-2}_k; Z_k) \to H_{2M-1}(L^{2M-1}_k, L^{2M-2}_k; Z_k). \]

For $j = 0, 1$ the chains $\sigma_{2M-1} *_{Z_k} \sigma_j$ are triangulations of the cells $e^{2M+j}$ relative to their boundary, so they represent generators of $H_{2M+j}(L^{2M+j}_k, L^{2M+j-1}_k; Z_k)$. By the naturality of the equivariant join, $y_{2M-1} *_{Z_k} y_j$ maps to $[\sigma_{2M-1} *_{Z_k} \sigma_j]$ under the isomorphism
\[ H_{2M+j}(L^{2M+j}_k, L^{2M+j-1}_k; Z_k) \to H_{2M+j}(L^{2M+j}_k, L^{2M+j-1}_k; Z_k). \]

It follows that $y_{2M-1} *_{Z_k} y_j$ is a generator of $H_{2M+j}(L^{2M+1}_k; Z_k)$.

Taking iterations, and using associativity to remove the brackets, we conclude that each of the classes $y_1 *_{Z_k} \ldots *_{Z_k} y_1$, $y_1 *_{Z_k} \ldots *_{Z_k} y_1 *_{Z_k} y_0$ and (by a similar argument) $y_0 *_{Z_k} y_1 *_{Z_k} \ldots *_{Z_k} y_1$ is a generator of the homology group in the appropriate dimension.

Let $x_m \in H_m(L^{2M-1}_k; Z_k)$ and $x_m' \in H_{m'}(L^{2M-1}_k; Z_k)$ be non-zero classes. Suppose that $m$ and $m'$ are not both even. Expressing $x_m$ as a non-zero scalar multiple of $y_1 *_{Z_k} \ldots *_{Z_k} y_1$ or $y_0 *_{Z_k} y_1 *_{Z_k} \ldots *_{Z_k} y_1$, and expressing $x_m'$ as a non-zero scalar multiple of $y_1 *_{Z_k} \ldots *_{Z_k} y_1$ or $y_1 *_{Z_k} \ldots *_{Z_k} y_1 *_{Z_k} y_0$, we conclude (by associativity) that $x_m * x_m'$ (in which $y_0$ might occur as a first or last factor but not both) is non-zero.

Since $(y_1 *_{Z_k} \ldots *_{Z_k} y_1 *_{Z_k} y_0) *_{Z_k} (y_0 *_{Z_k} y_1 *_{Z_k} \ldots *_{Z_k} y_1)$ is zero (by associativity and by Lemma B.21), it similarly follows that, if $m$ and $m'$ are both even, then $x_m *_{Z_k} x_m'$ is zero.

**Proof of the lower bounds (B.1) on the index of a join.** By Remark B.3, we can assume that $A$ and $B$ are open subsets of $L^{2M-1}_k(w)$ and of $L^{2M'-1}_k(w')$.

First, suppose that ind$(A)$ or ind$(B)$ is even. By Lemma B.4, there exist classes $\alpha \in H_{\text{ind}(A)-1}(A)$ and $\beta \in H_{\text{ind}(B)-1}(B)$ whose images in $H_{\text{ind}(A)-1}(L^{2M-1}_k(w))$ and in $H_{\text{ind}(B)-1}(L^{2M'-1}_k(w'))$ are non-zero. By Lemma B.22 and the naturality of the equivariant join, $\alpha *_{Z_k} \beta$ is a class in $H_{\text{ind}(A)+\text{ind}(B)-1}(A *_{Z_k} B)$ whose image in $H_{\text{ind}(A)+\text{ind}(B)-1}(L^{2M+M'-1}_k(w, w'))$ is non-zero. By Lemma B.4, this shows that ind$(A *_{Z_k} B) \geq \text{ind}(A) + \text{ind}(B)$.

If ind$(A)$ or ind$(B)$ are both odd, we apply a similar argument to classes $\alpha \in H_{\text{ind}(A)-1}(A)$ and $\beta' \in H_{\text{ind}(B)-2}(B)$ to conclude that ind$(A *_{Z_k} B) \geq \text{ind}(A) + \text{ind}(B) - 1$.

**B.23. Remark.** In the case of projective space, any cell of the standard cellular decomposition is the equivariant join of the cell in the previous degree with a 0-cell. Therefore the proof of Proposition B.22 shows that in this case the join of two generators in any degree is non-zero. It follows from this argument and property (iv') in Remark 3.11 that in the case of projective spaces the cohomological index is join additive: for closed subsets $A$ of $\mathbb{RP}^M$ and $B$ of $\mathbb{RP}^{M'}$ we have ind$(A *_{Z_k} B) = \text{ind}(A) + \text{ind}(B)$.\[\text{ }\]

**Commutativity of the homology join.** We complete our discussion of the join operation on homology with a commutativity property of this operation.
B.24. **Proposition.** Let \( \tilde{X} \to X \) and \( \tilde{Y} \to Y \) be principal \( G \)-bundles and suppose that \( k \) divides the order of \( G \). Let \( \tau : X *_G Y \to Y *_G X \) denote the homeomorphism (B.9). Then for \( \alpha \in H_i(X; \mathbb{Z}_k) \) and \( \beta \in H_m(Y; \mathbb{Z}_k) \) we have

\[
\tau_*(\alpha *_G \beta) = (-1)^{(l+1)(m+1)} \beta *_G \alpha .
\]

**Proof.** We use the geometric interpretation of singular cycles that appears in [Ha02, p. 108–109]. Let \( x \in C_l(X; \mathbb{Z}_k) \) be a cycle, and write \( x = \sum_i x_i \sigma_i \) with \( \sigma_i : \Delta^l \to X \) and \( x_i \neq 0 \). Let

\[
K_x = \left( \prod \Delta^l \right) / \sim
\]

be the disjoint union of one \( l \)-simplex \( \Delta^l \) for each \( \sigma_i \), quotiented by the identification of the facets of the \( \Delta^l \)'s which give rise (via the maps \( \sigma_i \)) to the same singular \((l-1)\)-simplex. Then the singular simplices \( \sigma_i \)'s assemble to give a map \( \sigma : K_x \to X \). We denote by \( \bar{\sigma}_i : \Delta^l_i \to K_x \) the inclusion in the coproduct followed by quotient. Then

\[
\bar{x} := \sum_i x_i \bar{\sigma}_i \in C_l(K_x; \mathbb{Z}_k)
\]

is a cycle and \( \sigma_*(\bar{x}) = x \). Now let \( x = \sum_i x_i \sigma_i \in C_l(X; \mathbb{Z}_k) \) and \( y = \sum_j y_j \mu_j \in C_m(Y; \mathbb{Z}_k) \) be cycles representing the homology classes \( \alpha \) and \( \beta \) respectively. Then

\[
x *_G y = \varphi^{-1} \left( \sum_{g,h \in G} x_i y_j (g \cdot \bar{\sigma}_i) \ast (h \cdot \mu_j) \right)
\]

and

\[
y *_G x = \varphi^{-1} \left( \sum_{g,h \in G} x_i y_j (h \cdot \bar{\mu}_j) \ast (g \cdot \bar{\sigma}_i) \right)
\]

where \( \bar{\sigma}_i \) and \( \bar{\mu}_j \) are some liftings of \( \sigma_i \) and \( \mu_j \). Let \( \nu : K_{x*_G y} \to X *_G Y \) and \( \eta : K_{y*G x} \to Y *_G X \) be the maps geometrically realizing the cycles \( x *_G y \) and \( y *_G x \) via the procedure described above, and \( \bar{x} *_G \bar{y} \in C_{l+m+1}(K_{x*_G y}; \mathbb{Z}_k) \) and \( \bar{y} *_G \bar{x} \in C_{l+m+1}(K_{y*_G x}; \mathbb{Z}_k) \) be the corresponding cycles. Then the following diagram commutes

\[
\begin{array}{ccc}
K_{x*_G y} & \xrightarrow{\nu} & X *_G Y \\
\downarrow T & & \downarrow \tau \\
K_{y*_G x} & \xrightarrow{\eta} & Y *_G X
\end{array}
\]

where \( T \) is the homeomorphism induced by the canonical homeomorphisms

\[
\Delta^{l+m+1} = \Delta^l * \Delta^m \xrightarrow{T} \Delta^m * \Delta^l = \Delta^{l+m+1}
\]

between the top cells of \( K_{x*_G y} \) and \( K_{y*_G x} \). Since for cellular homology with \( \mathbb{Z}_k \)-coefficients we have \( T_* \left( [\bar{x} *_G \bar{y}] \right) = (-1)^{(m+1)(l+1)} [\bar{y} *_G \bar{x}] \), the same holds in singular homology. Hence

\[
\tau_*(\alpha *_G \beta) = \tau_* \nu_* \left( [\bar{x} *_G \bar{y}] \right) = \eta_* T_* \left( [\bar{x} *_G \bar{y}] \right) = \eta_* (-1)^{(l+1)(m+1)} [\bar{y} *_G \bar{x}] = (-1)^{(l+1)(m+1)} \beta *_G \alpha .
\]

\[\square\]

**References**

[BeS07] G. Ben Simon, The nonlinear Maslov index and the Calabi homomorphism. *Commun. Contemp. Math.* 9 (2007), 769–780.

[Bh01] M. Bhupal, A partial order on the group of contactomorphisms of \( \mathbb{R}^{2n+1} \) via generating functions. *Turkish J. Math.* 25 (2001), 125–135.

[BEKP04] P. Biran, M. Entov and L. Polterovich, Calabi quasimorphisms for the symplectic ball. *Commun. Contemp. Math.* 6 (2004), 793–802.

[Bo13] M. S. Borman, Quasi-states, quasi-morphisms, and the moment map. *Int. Math. Res. Not.* 11 (2013), 2497–2533.

[BZ15] M. S. Borman and F. Zapolsky, Quasimorphisms on contactomorphism groups and contact rigidity. *Geom. Topol.* 19 (2015), 365–411.
[Ca09] D. Calegari, *scl*, *MSJ Memoirs*, 20, Mathematical Society of Japan, Tokyo, 2009.

[Ch84] M. Chaperon, Une idée du type “géodésiques brisées”pour les systèmes hamiltoniens. *C. R. Acad. Sci. Paris, Sér. I Math.* **298** (1984), 293–296.

[CS12] V. Colin and S. Sandon, The discriminant and oscillation lengths for contact and Legendrian isotopies. *J. Eur. Math. Soc.* **17** (2015), 1657–1685.

[Do95] A. Dold, *Lectures in Algebraic Topology*. Classics in Mathematics, Springer-Verlag, 1995.

[EKP06] Y. Eliashberg, S.S. Kim and L. Polterovich, Geometry of contact transformations and domains: order-ability vs squeezing. *Geom. and Topol.* **10** (2006), 1635–1747.

[EP00] Y. Eliashberg and L. Polterovich, Partially ordered groups and geometry of contact transformations. *Geom. Funct. Anal.* **10** (2000), 1448–1476.

[EnP03] M. Entov and L. Polterovich, Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.* (2003), 1635–1676.

[EnP06] M. Entov and L. Polterovich, Quasistates and symplectic intersections. *Comment. Math. Helv.* **81** (2006), 75–99.

[FR78] E. Fadell and P. H. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.* **45** (1978), 139–174.

[FPR12] M. Fraer, L. Polterovich and D. Rosen, On Sandon-type metrics for contactomorphism groups. *arXiv:1207.3151*.

[Ge08] H. Geiges, *An introduction to contact topology*. Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.

[Gi89] A. B. Givental, Periodic mappings in symplectic topology. *Funktsional. Anal. i Prilozhen.* **23** (1989), 37–52; translation in *Funct. Anal. Appl.* **23** (1989), 287–300.

[Gi90] A.B. Givental, Nonlinear generalization of the Maslov index. *Theory of singularities and its applications*, 71–103, *Adv. Soviet Math.*, 1, Amer. Math. Soc., Providence, RI, 1990.

[Gi95] A.B. Givental, A symplectic fixed point theorem for toric manifolds. *The Floer memorial volume*, 445–481, *Progr. Math.*, 133, Birkhäuser, Basel, 1995.

[Ha02] A. Hatcher, *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[Ho71] L. Hörmander, Fourier integral operators I. *Acta Math.* **127** (1971), 79–183.

[LS85] F. Laudenbach and J.C. Sikorav, Persistance d’intersection avec la section nulle au cours d’une isotopie géométrique hamiltonienne dans un fibré cotangent. *Invent. Math.* **82** (1985), 349–357.

[Lo2] E. Lerman, Contact toric manifolds. *Journal of Symplectic Geometry* **1** (2002), no. 4, 785–828.

[Mi08] I. Milin, Orderability of contactomorphism groups of lens spaces. *Ph.D. Thesis, Stanford University*.

[Ry10] T. Rybicki, Commutators of contactomorphisms. *Adv. Math.*, **225** (2010), 3291–3326.

[Sa10] S. Sandon, An integer valued bi-invariant metric on the group of contactomorphisms of $\mathbb{R}^{2n} \times S^1$. *J. Topol. Anal.* **2** (2010), 327–339.

[Sa11a] S. Sandon, Contact homology, capacity and non-squeezing in $\mathbb{R}^{2n} \times S^1$ via generating functions. *Ann. Inst. Fourier (Grenoble)* **61** (2011), 145–185.

[Sa11b] S. Sandon, Equivariant homology of generating functions and orderability of lens spaces. *J. Symplectic Geom.* **9** (2011) 123–146.

[Sa11c] S. Sandon, A Morse estimate for translated points of contactomorphisms of spheres and projective spaces. *Geom. Dedicata* **165** (2013), 95–110.

[Sa12] S. Sandon, On iterated translated points for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$. *Internat. J. Math.* **23** (2012), 1250042, 14 pp.

[Si85] J.–C. Sikorav, Sur les immersions lagrangiennes dans un fibré cotangent admettant une phase génératrice globale. *C. R. Acad. Sci. Paris Sér. I Math.* **302** (1986), 119–122.

[Si87] J.–C. Sikorav, Problèmes d’intersections et de points fixes en géométrie hamiltonienne. *Comment. Math. Helv.* **62** (1987), 62–73.

[Sp66] E. Spanier, *Algebraic Topology*. Springer-Verlag, New York, 1966.

[Th95] D. Théret, Utilisation des fonctions génératrices en géométrie symplectique globale. *Ph.D. Thesis, Université Denis Diderot (Paris 7)* (1995).

[Th98] D. Théret, Rotation numbers of Hamiltonian isotopies in complex projective spaces. *Duke Math. J.* **94** (1998), 13–27.

[Th99] D. Théret, A Lagrangian camel. *Comment. Math. Helv.* **74** (1999), 591–614.

[Ty94] L. Traynor, Symplectic Homology via generating functions. *Geom. Funct. Anal.* **4** (1994), 718–748.

[Ts08] T. Tsunboi, On the simplicity of the group of contactomorphisms. *Groups of diffeomorphisms*, 491–504, Adv. Stud. Pure Math., 52, Math. Soc. Japan, Tokyo, 2008.

[V92] C. Viterbo, Symplectic topology as the geometry of generating functions. *Math. Ann.* **292** (1992), 685–710.

[V97] C. Viterbo, Some remarks on Massey products, tied cohomology classes, and the Lusternik-Shnirelman category. *Duke Math. J.* **86** (1997), 547–564.

[WW13] G. Wilson and C.T. Woodward, Quasimap Floer cohomology for varying symplectic quotients. *Canad. J. Math.* **65** (2013), 467–480.
