L$^1(\mathbb{R}^d, dx)$-UNIQUENESS OF WEAK SOLUTIONS FOR THE FOKKER-PLANCK EQUATION ASSOCIATED WITH A CLASS OF DIRICHLET OPERATORS

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ABSTRACT. The main purpose of this paper is to prove the ($L^1(\mathbb{R}^d, dx), \| \cdot \|_1$)-uniqueness of the Fokker-Planck equation associated with the non-symmetric Dirichlet operator

$$A^* f := \frac{1}{2} \Delta f + b \cdot \nabla f - V f , \quad \forall f \in C^\infty_0(\mathbb{R}^d)$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable locally bounded vector field and $V : \mathbb{R}^d \to \mathbb{R}$ is a locally bounded non-negative potential.

1. Preliminaries

In general, for a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on $L^1(\mathbb{R}^d, dx)$, its adjoint semigroup $\{T^*(t)\}_{t \geq 0}$ is no longer strongly continuous on the dual topological space $L^\infty(\mathbb{R}^d, dx)$ of $L^1(\mathbb{R}^d, dx)$ with respect to the strong dual topology of $L^\infty(\mathbb{R}^d, dx)$.

Recently Wu and Zhang [21] introduced on $L^\infty(\mathbb{R}^d, dx)$ the topology of uniform convergence on compact subsets of $(L^1(\mathbb{R}^d, dx), \| \cdot \|_1)$, denoted by $C(L^\infty, L^1)$. If $\{T(t)\}_{t \geq 0}$ is a $C_0$-semigroup on $L^1(\mathbb{R}^d, dx)$ with generator $L$, then $\{T^*(t)\}_{t \geq 0}$ is a $C_0$-semigroup on $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$ with generator $L^*$. Moreover, one can prove that $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$ is a complete locally convex space and that the topological dual of $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$ is $(L^1(\mathbb{R}^d, dx), \| \cdot \|_1)$.

Let $\mathcal{A} : D \to L^\infty(\mathbb{R}^d, dx)$ be a linear operator with its domain $D$ dense in $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$. $\mathcal{A}$ is called a pre-generator in $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$, if there exists some $C_0$-semigroup on $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$ such that its generator $\mathcal{L}$ extends $\mathcal{A}$.

In accord with the uniqueness notion used by Arendt [3], Eberle [8], Röckner [15], Wu [19] and [20], Arendt, Metafune and Pallara [4], Wu and Zhang [21] and others, we can introduce $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$-uniqueness of pre-generators. We say that a pre-generator $\mathcal{A}$ is $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$-unique if $\mathcal{A}$ is closable and its closure $\overline{\mathcal{A}}$ with respect to the topology $C(L^\infty, L^1)$ is the generator of some $C_0$-semigroup on $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$.

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The main result concerning \((L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))\)-uniqueness of pre-generators is (see [21] or [10] for much more general results).

**Theorem 1.1.** Let \(A\) be a linear operator on \((L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))\) whose domain \(D\) (the test-function space) is assumed to be dense in \((L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))\). Assume that there is a \(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) on \((L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))\) such that its generator \(L\) is an extension of \(A\) (i.e., \(A\) is a pre-generator in the space \((L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))\)). The following assertions are equivalent:

(i) \(A\) is \((L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))\)-unique;

(ii) \(D\) is a core of \(L\);

(iii) (Liouville property) for some \(\lambda > \omega\) (\(\omega\) is the constant in the definition of the \(C_0\)-semigroups \(\{T(t)\}_{t \geq 0}\)), if \(h \in D(A^*)\) satisfies \((\lambda I - A^*)h = 0\), then \(h = 0\);

(iv) (uniqueness of weak solutions for the dual Cauchy problem) for every function \(f \in (L^1(\mathbb{R}^d, dx), \| \cdot \|_1)\), the dual Cauchy problem

\[
\begin{align*}
\partial_t u(t, x) &= A^*u(t, x) \\
\quad u(0, x) &= f(x)
\end{align*}
\]

has a \((L^1(\mathbb{R}^d, dx), \| \cdot \|_1)\)-unique weak solution \(u(t, x) = T^*(t)f(x)\).

We remark that if \(A\) is a second order elliptic differential operator with domain \(D = C_0^\infty(\mathbb{R}^d)\), the space of infinitely differentiable functions with compact support in \(\mathbb{R}^d\), then the weak solutions for the dual Cauchy problem in part (iv) of Theorem 1.1 correspond exactly to those in the distribution sense in the theory of partial differential equations and the dual Cauchy problem becomes the Fokker-Planck equation for heat diffusion in the sense of distributions.

The \((L^1(\mathbb{R}^d, dx), \| \cdot \|_1)\)-uniqueness of weak solutions of the Fokker-Planck equation is very important from the point of view of heat diffusion. Usually the energy in a system at time \(t\) is considered to be \(\int u^2(t, x) \, dx\) or \(\int |\nabla u(t, x)|^2 \, dx\). But those are not the physical energy in the Fokker-Planck equation. Indeed, if the initial distribution of heat is \(f(x)\), then the solution \(u(t, x)\) of the Fokker-Planck equation is the distribution of heat at time \(t\) at the position \(x\). Then the total heat (energy) in the system at time \(t\) is \(\int |u(t, x)| \, dx\), i.e. the \(L^1(\mathbb{R}^d, dx)\)-norm of \(u(t, x)\). So the \((L^1(\mathbb{R}^d, dx), \| \cdot \|_1)\)-uniqueness of weak solution of the Fokker-Planck equation has an important physical interpretation.

In this paper we consider the diffusion operator

\[
A^V f := \frac{1}{2} \Delta f + b \cdot \nabla f - V f \quad \forall f \in C_0^\infty(\mathbb{R}^d)
\]

where \(b : \mathbb{R}^d \to \mathbb{R}^d\) is a measurable locally bounded vector field and \(V : \mathbb{R}^d \to \mathbb{R}\) is a locally bounded potential (here \(\cdot \) denotes the inner product in \(\mathbb{R}^d\)).

The study of this operator has attracted much attention both from the people working on Nelson’s stochastic mechanics and from those working on the theory of Dirichlet forms. We are content here only to cite the papers of Meyer and Zheng [14], Carmona [7], Albeverio, Brasche and Röckner [1], Albeverio, Kondratiev and...
Röckner [2], Bogacev, Krylov and Röckner [5], Manca [13], Bogacev, Da Prato, Röckner and Stannat [6], where the reader can find a large number of references.

In the case where $V = 0$, the essential self-adjointness of $A := \frac{1}{2}\Delta + b \cdot \nabla$ in $L^2$ has been completely characterized in the works of Wielens [18] and Liskevitch [12]. $L^1$-uniqueness of $A$ has been studied by Wu [20] and Stannat [16], [17], its $L^p$-uniqueness has been studied by Eberle [8] for $p \in (1, \infty)$ and by Wu and Zhang [21] in the case $p = \infty$.

Our main purpose in this paper is to prove the $(L^1(\mathbb{R}^d, dx), || \cdot ||_1)$-uniqueness of the Fokker-Planck equation associated with the non-symmetric Dirichlet operator $A^V$ in the case where $V \geq 0$. For this purpose we use the equivalence between the $(L^1(\mathbb{R}^d, dx), || \cdot ||_1)$-uniqueness of the Fokker-Planck equation associated with $A^V$ and the $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$-uniqueness of $A^V$ in Theorem 1.1 (see [11] for detailed proofs).

2. THE MAIN RESULTS

At first, we must remark that the Dirichlet operator $(A^V, C^0_0(\mathbb{R}^d))$ is a pre-generator on $L^\infty(\mathbb{R}^d, dx)$. Indeed, if we consider the Feynman-Kac semigroup $\{P_t^V\}_{t \geq 0}$ given by

$$P_t^V f(x) := \mathbb{E}^x 1_{[t, \tau_v]} f(X_t) e^{-\int_0^t V(X_s) \, ds}$$

where $(X_t)_{0 \leq t < \tau_v}$ is the diffusion generated by $A$ and $\tau_v$ is the explosion time, then by [21, Theorem 1.4] $\{P_t^V\}_{t \geq 0}$ is a $C_0$-semigroup on $L^\infty(\mathbb{R}^d, dx)$ with respect to the topology $C(L^\infty, L^1)$. Let $\partial$ be the point at infinity of $\mathbb{R}^d$. If we put $X_t = \partial$ after the explosion time $t \geq \tau_v$, then by Ito’s formula it follows for any $f \in C^0_0(\mathbb{R}^d)$ that

$$f(X_t) - f(x) - \int_0^t A^V f(X_s) \, ds$$

is a local martingale. As it is bounded over bounded times intervals, it is a true martingale. Thus by taking the expectation under $\mathbb{P}_x$, we get

$$P_t^V f(x) - f(x) = \int_0^t P_s^V A^V f(x) \, ds \quad , \forall t \geq 0.$$ 

Therefore $f$ belongs to the domain of the generator $L^V(\infty)$ of the $C_0$-semigroup $\{P_t^V\}_{t \geq 0}$ on $(L^\infty(\mathbb{R}^d, dx), C(L^\infty, L^1))$. As a result, $(A^V, C^0_0(\mathbb{R}^d))$ is a pre-generator on $L^\infty(\mathbb{R}^d, dx)$ and we can apply Theorem 1.1 to study the $(L^1(\mathbb{R}^d, dx), || \cdot ||_1)$-uniqueness of weak solutions of the Fokker-Planck equation associated with this operator.

Denote by $|x| = \sqrt{x \cdot x}$ the Euclidean norm in $\mathbb{R}^d$. If there is some measurable locally bounded function

$$\beta : \mathbb{R}^+ \rightarrow \mathbb{R}$$

such that

$$b(x) \cdot \frac{x}{|x|} \geq \beta(|x|) \quad , \forall x \in \mathbb{R}^d, x \neq 0,$$
then for any initial point \( x \neq 0 \) we have
\[
|X_t| - |x| \geq \int_0^t \left[ \beta(|X_t|) + \frac{d-1}{2|X_t|} \right] dt + \text{a real Brownian motion}, \quad \forall t < \tau_0.
\]
In other words, \(|X_t|\) goes to infinity more rapidly than the one-dimensional diffusion generated by
\[
A_1 f = \frac{1}{2} f'' + \left[ \beta(r) + \frac{d-1}{2r} \right] f'.
\]
This is standard in probability (see Ikeda and Watanabe [9]). We remark that for \( f \in C^\infty_0(0, \infty) \), the one-dimensional operator
\[
A_1^\lambda f = \frac{1}{2} f'' + \left[ \beta(r) + \frac{d-1}{2r} \right] f' - V(r)f
\]
can be written in the Feller form
\[
A_1^\lambda f = \frac{1}{\rho(r)} \left[ \alpha(r)f' \right]' - V(r)f
\]
where
\[
\rho(r) = 2e^{\int_1^{2\beta(t)+\frac{d-1}{2t}} dt} = 2e^{\int_1^{2\beta(t)} dt} e^{\int_{\lambda}^{\lambda+V(t)n} \frac{dr}{\alpha(r)n}} dt = 2e^{d-1} e^{\int_1^{2\beta(t)} dt}
\]
is the speed measure of Feller and
\[
\alpha(r) = r^{d-1} e^{\int_1^{2\beta(t)} dt}
\]
is the scale function of Feller. So we can consider the operator \((A_1^\lambda, C^\infty_0(0, \infty))\) as an operator on the space \( L^\infty(0, \infty; \rho dx) \) which is endowed with the topology \( C(L^\infty(0, \infty; \rho dx), L^1(0, \infty; \rho dx)) \). One of the main results of this paper is

**Theorem 2.1.** The one-dimensional operator \((A_1^\lambda, C^\infty_0(0, \infty))\) is \( L^\infty(0, \infty; \rho dx) \)-unique with respect to the topology \( C(L^\infty(0, \infty; \rho dx), L^1(0, \infty; \rho dx)) \) if and only if both
\[
(*) \quad \int_c^\infty \rho(y) \sum_{n=0}^\infty \phi_n(y) dy = +\infty
\]
and
\[
(**) \quad \int_0^c \rho(x) \sum_{n=0}^\infty \psi_n(x) dx = +\infty
\]
hold, where \( c \in (0, \infty), \lambda > 0 \) and
\[
\phi_n(y) = \int_c^y 1 \frac{1}{\alpha(r_n)} dr_n \int_r^\infty \rho(t_n)[\lambda + V(t_n)] \phi_{n-1}(t_n) dt_n, \quad n \geq 1, \quad \phi_0(y) = 1
\]
and
\[
\psi_n(x) = \int_x^c 1 \frac{1}{\alpha(r_n)} dr_n \int_r^\infty \rho(t_n)[\lambda + V(t_n)] \psi_{n-1}(t_n) dt_n, \quad n \geq 1, \quad \psi_0(x) = 1.
\]
In particular, for \( V = 0 \), the one-dimensional operator
\[
\mathcal{A}_1 f = a(x) f'' + b(x) f'
\]
is \( L^\infty(0, \infty; \rho dx) \)-unique in the topology \( \mathcal{C}(L^\infty(0, \infty; \rho dx), L^1(0, \infty; \rho dx)) \) if and only if both
\[
\int_0^c \rho(x) \, dx \int_c^r \frac{1}{\alpha(r)} \, dr \int_r^c \rho(t) \, dt = +\infty
\]
and
\[
\int_c^0 \rho(x) \, dx \int_x^c \frac{1}{\alpha(r)} \, dr \int_c^r \rho(t) \, dt = +\infty
\]
hold. In the terminology of Feller this means that \( \infty \) and respectively \( 0 \) are no entrance boundaries (see [21, Theorem 4.1, p.590]).

Now we can formulate the most important result of this paper

**Theorem 2.2.** Suppose that there is some mesurable locally bounded function
\[
\beta : \mathbb{R}^+ \to \mathbb{R}
\]
such that
\[
b(x) \frac{x}{|x|} \geq \beta(|x|) \quad , \quad \forall x \in \mathbb{R}^d, x \neq 0.
\]
If the one-dimensional diffusion operator
\[
\mathcal{A}_V^1 = \frac{1}{2} \frac{d^2}{dr^2} + \left[ \beta(r) + \frac{d-1}{2r} \right] \frac{d}{dr} - V(r)
\]
is \( L^\infty(0, \infty; \rho dx) \)-unique with respect to the topology \( \mathcal{C}(L^\infty(0, \infty; \rho dx), L^1(0, \infty; \rho dx)) \), then for any \( f \in L^1(\mathbb{R}^d, dx) \) the Fokker-Planck equation
\[
\begin{cases}
\partial_t u(t,x) = \frac{1}{2} \Delta u(t,x) - \text{div} \left( b(x) u(t,x) \right) - V(x) u(t,x) \\
u(0,x) = f(x)
\end{cases}
\]
has a \( L^1(\mathbb{R}^d, dx) \)-unique weak solution which is given by \( u(t,x) = P_t^V f(x) \).

**References**

[1] S. Albeverio, J. Brasche and M. Röckner, “Dirichlet Forms and Generalized Schrödinger Operators,” Schrödinger Operators (Sønderborg, 1988), 1–42, Lecture Notes in Phys., 345, Springer, Berlin, 1989. MR 1037315 (91c:47103)

[2] S. Albeverio, Yu. G. Kondratiev and M. Röckner, Dirichlet operators via stochastic analysis, J. Funct. Anal., 128 (1995), 102–138. MR 1317712 (96c:31007)

[3] W. Arendt, The abstract Cauchy problem, special semigroups and perturbation, “One Parameter Semigroups of Positive Operators” (R. Nagel, Eds.), Lect. Notes in Math., 1184, Springer, Berlin, 1986. MR 0839450 (88i:47022)

[4] W. Arendt, G. Metafune and D. Pallara, Schrödinger operators with unbounded drift, J. Operators Theory, 55 (2006), 185–211. MR 2212028 (2007g:35127)

[5] V. I. Bogachev, N. Krylov and M. Röckner, Elliptic regularity and essential self-adjointness of Dirichlet operators on \( \mathbb{R}^n \), Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24 (1997), 451–461. MR 1612385 (99c:35037)

[6] V. I. Bogachev, G. Da Prato, M. Röckner and W. Stannat, Uniqueness of solutions to weak parabolic equations for measures, Bull. London Math. Soc., 39 (2007), 631–640. MR 2346944 (2008e:35162)

[7] R. Carmona, Probabilistic construction of Nelson processes, Probabilistic Methods in Math. Phys. (Katata/Kyoto, 1985), 55–81, Academic Press, Boston, MA, 1987. MR 0933818 (90a:60154)
[8] A. Eberle, $L^p$-uniqueness of non-symmetric diffusion operators with singular drift coefficients I. The finite-dimensional case, J. Funct. Anal., 173 (2000), 328–342. MR 1760618 (2001e:47075)

[9] N. Ikeda and S. Watanabe, “Stochastic Differential Equations and Diffusion Processes,” North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981. MR 0637061 (84b:60080)

[10] L. D. Lemle, “Integrated Semigroups of Operators, Uniqueness of Pre-Generators and Applications,” Doctor degree thesis, Blaise Pascal University of Clermont-Ferrand, 2007.

[11] L. D. Lemle, Domains of uniqueness for $C_0$-semigroups on the dual of a Banach space, Preprint, 2008. arXiv:0806.1428v1

[12] V. Liskevitch, On the uniqueness problem for Dirichlet operators, J. Funct. Anal., 162 (1999), 1–13. MR 1674542 (2000b:35049)

[13] L. Manca, On a class of stochastic semilinear PDEs, Stoch. Anal. Appl., 24 (2006), 399–426. MR 2204720 (2007c:60059)

[14] P. A. Meyer and W. A. Zheng, Construction du processus de Nelson reversible, (French) [Construction of reversible Nelson processes] Séminaire de probabilités, XIX, 1983/84, 12–26, Lect. Notes in Math., Springer, Berlin, 1123 (1984), 12–26. MR 0889465 (89b:60187)

[15] M. Röckner, $L^p$-analysis of finite and infinite dimensional diffusion operators, Stochastic PDE’s and Kolmogorov equations in infinite dimensions (Cetraro, 1998), 65–116, Lecture Notes in Math., 1715, Springer, Berlin, 1999. MR 1731795 (2000k:60129)

[16] W. Stannat, (Nonsymmetric) Dirichlet operators on $L^1$: Existence, uniqueness and associated Markov processes, Ann. Scuole Norm. Sup. Pisa Cl. Sci. (4), 28 (1999), 99–140. MR 1679079 (2000c:60059)

[17] W. Stannat, Time-dependent diffusion operators on $L^1$, J. Evol. Equ., 4 (2004), 463–495. MR 2105273 (2005b:35201)

[18] N. Wielens, The essential selfadjointness of generalized Schrödinger operators, J. Funct. Anal., 61 (1985), 98–115. MR 0779740 (86f:35108)

[19] L. Wu, Uniqueness of Schrödinger operators restricted in a domain, J. Funct. Anal., 153 (1998), 276–319. MR 1614574 (2000c:60117)

[20] L. Wu, Uniqueness of Nelson’s diffusions, Probab. Theory Relat. Fields, 114 (1999), 549–585. MR 1709280 (2000j:60096)

[21] L. Wu and Y. Zhang, A new topological approach to the $L^\infty$-uniqueness of operators and the $L^1$-uniqueness of Fokker-Planck equations, J. Funct. Anal., 241 (2006), 557–610. MR 2271930 (2007j:47144)

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