FORMAL KILLING FIELDS FOR MINIMAL LAGRANGIAN SURFACES IN COMPLEX SPACE FORMS

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Abstract. The differential system for minimal Lagrangian surfaces in a $2c$-dimensional, non-flat, complex space form is an elliptic system defined on the bundle of oriented Lagrangian planes. This is a 6-symmetric space associated with the Lie group $\text{SL}(3, \mathbb{C})$, and the minimal Lagrangian surfaces arise as the primitive maps. Utilizing this property, we derive the differential algebraic inductive formulas for a pair of loop algebra $\mathfrak{sl}(3, \mathbb{C})[[\lambda]]$-valued canonical formal Killing fields. As a result, we give a complete classification of the (infinite sequence of) Jacobi fields for the minimal Lagrangian system. We also obtain an infinite sequence of higher-order conservation laws from the components of the formal Killing fields.

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1. Introduction

1.1. Minimal Lagrangian surface. For an immersed Lagrangian submanifold in a Kähler manifold, the vanishing of the mean curvature is equivalent to that the associated section of $(n,0)$-form is parallel along the submanifold. From the well known identity in Kähler geometry, that the Ricci 2-form is up to constant scale the curvature form of the canonical line bundle, the minimality condition implies that the restriction of Ricci 2-form to the Lagrangian submanifold must also vanish. When coupled with the Lagrangian condition, they form an over-determined system of differential equations which is generally not compatible.

In case the ambient manifold is Kähler-Einstein and the Ricci 2-form is a constant multiple of the symplectic form, the minimal Lagrangian equation is involutive and, at least locally, it admits many solutions, [1].

1.1.1. Special Lagrangian submanifold. In relation to the developments in string theory, special Lagrangian submanifolds in Calabi-Yau (or Ricci-flat Kähler) manifolds have received much attention recently. As a particular case of calibrated geometry, the various aspects of the geometry of special Lagrangian varieties have been studied from the analytic perspectives, including deformation problem [23], gluing constructions [14] [15] [24], and singularity analysis [17] [8], etc. We refer to [19] [21] for the further references.

From a different perspective, the special Lagrangian submanifolds in $\mathbb{C}^m$ with non-trivial second order symmetries have been studied by in-depth analyses of the structure equations in [3] [16].
1.1.2. **Minimal Lagrangian surface.** In the 2-dimensional case, Schoen and Wolfson gave a variational analysis of area minimizing (Hamiltonian stationary) Lagrangian surfaces, [26]. They proved the existence of area minimizer with the particular forms of admissible conical singularities. Haskins and Kapouleas gave a gluing construction of the compact high-genus special Legendrian surfaces in the 5-sphere, [14]. Under the Hopf map $S^3 \rightarrow \mathbb{CP}^2$, these surfaces are mapped to minimal Lagrangian surfaces. In [22], the minimal Lagrangian surfaces in the hyperbolic complex space form $CH^2$ were studied in relation to the surface group representations in $SU(1, 2)$. For the integrable system aspects of the theory on minimal Lagrangian tori, we refer to [9] and the references therein.

1.2. **Formal Killing fields.**

1.2.1. **Polynomial Killing field.** One of the characteristic structural properties of a harmonic torus in a symmetric space is the existence of an associated polynomial Killing field, [7]. For the case of a minimal Lagrangian torus in $\mathbb{CP}^2$, a polynomial Killing field can be considered as a higher-order Gauß map which takes values in the polynomial loop algebra $sl(3, \mathbb{C})[[\lambda]]$ (here $\lambda$ denotes the spectral parameter). The corresponding spectral curve is a branched triple covering of $\mathbb{CP}^1$, and a minimal Lagrangian torus linearizes on its Jacobian. From this construction, the relevant spectral curve theory of finite type integration can be applied to the study of a minimal Lagrangian torus.

1.2.2. **Formal Killing fields.** For a general minimal Lagrangian surface in a 2-dimensional, non-flat, complex space form, it turns out that the local analytic data can be packaged into a pair of canonical formal Killing fields, which take values in the formal loop algebra $sl(3, \mathbb{C})[[\lambda]]$. For a minimal Lagrangian torus for example, each of these formal Killing fields would factor and reduce to a polynomial Killing field up to scaling by an element in $\mathbb{C}[[\lambda^6]]$.

The original idea of canonical formal Killing fields (for CMC surfaces) is due to Pinkall and Sterling, [25].

1.3. **Results.**

1.3.1. **A pair of canonical formal Killing fields.** We give a systematic derivation of the differential algebraic inductive formulas for a pair of canonical formal Killing fields for the differential system for minimal Lagrangian surfaces in a 2-dimensional, non-flat, complex space form, Thm 7.2, Thm 7.4. This clarifies the somewhat ad-hoc recursion formulas appeared in [25], [12].

1.3.2. **Jacobi fields and pseudo-Jacobi fields.** In the course of analysis, we find two different kinds of Jacobi fields for the minimal Lagrangian system.

Jacobi fields (ordinary), which correspond to the generalized symmetries of the minimal Lagrangian system, are defined by the operator, [47],

$$\partial_{\xi} \partial_{\bar{\xi}} + \frac{3}{2} \gamma^2.$$

Here the notations $\partial_{\xi}, \partial_{\bar{\xi}}$ denote the covariant derivatives with respect to the unitary (1, 0)-form $\xi$, and its complex conjugate $\bar{\xi}$ respectively, [19], and $4\gamma^2$ is the holomorphic
sectional curvature of the ambient complex space form. This shows that, when restricted to a minimal Lagrangian surface, Jacobi fields are the eigenfunctions of Laplacian of the induced Riemannian metric with eigenvalue $6\gamma^2$. A relevant observation is that the Jacobi operator depends only on the induced metric of the surface, similarly as in some of the calibrated geometries, [23].

Pseudo-Jacobi fields, which correspond to the generalized symmetries of the elliptic Tzitzeica equation underlying the minimal Lagrangian system, are defined by the operator, (48),

$$\partial_{\xi} \partial_{\bar{\xi}} + \frac{1}{2}(\gamma^2 + 4\|\cdot\|^2).$$

Here $\|\cdot\|^2$ is the squared norm of the associated Hopf differential, §2.3.1.

Applying the previous results from [11][10] on the elliptic Tzitzeica equation, we give a complete classification of the infinite sequence of (pseudo) Jacobi fields, Thm 6.6 Cor 7.3 Cor 7.5. As a corollary, this implies that a minimal Lagrangian torus in $\mathbb{CP}^2$ admits a pair of spectral curves.

1.3.3. Conservation laws. We also obtain an infinite sequence of higher-order conservation laws from the components of the canonical formal Killing fields, Thm 8.1.

1.4. Recursion. We shall give a description of the two 3-step recursion relations embedded in the structure equation for the formal Killing fields. This is the main technical ingredient for our construction of the canonical formal Killing fields.

1.4.1. Previous works. The partial differential equation which locally describes the minimal Lagrangian surfaces in a $2\gamma$-dimensional, non-flat, complex space form is the elliptic Tzitzeica equation, (53). An infinite sequence of higher-order symmetries and conservation laws were determined in the original works [11][10] via a recursion modeled on the associated formal Killing field equation. On the other hand, this recursion process left the problem of integrating a sequence of exact differential 1-forms. We show that they can be solved differential algebraically without involving integration.

1.4.2. Recursive structure equation. For the case at hand, a formal Killing field is a (twisted) loop algebra $\mathfrak{sl}(3,\mathbb{C})[[\lambda]]$-valued function $X_\lambda$ on the infinite prolongation space of the minimal Lagrangian system, which satisfies the Killing field equation (62). In terms of an adapted basis of $\mathfrak{sl}(3,\mathbb{C})[[\lambda]]$, the recursive component-wise structure equation (64) can be summarized in the following infinite schematic diagram of period 6, Fig.1.

Here the nodes \{p, b, c, f, a, g, s, r\} are the coefficients of a formal Killing field. Two nodes are connected by an arrow only if there exists a first order differential relation from the structure equation between them. From a node, one moves to the right by applying $\partial_{\xi}$, and to the left by applying $\partial_{\bar{\xi}}$. The upper indices are designated to match (roughly) the jet orders of the coefficients.

The structure equation shows that the right-arrows are differential, which means that the coefficients \{f, a, g, p\} are obtained from the left-adjacent term(s) by $\partial_{\xi}$ operation.
But, the left-arrows decrease the jet order. In addition, note that
\[ \partial_\xi p^{6n+4} = iyb^{6n+5} + 2i\lambda h c^{6n+5}, \]
\[ \partial_\xi s^{6n+8} = -iyt^{6n+9} - i\lambda s^{6n+9}. \]

In order to continue the recursion process, one needs to solve for the coefficients \( \{b^*, c^*, s^*, t^*\} \).

1.4.3. Differential algebraic inductive formulas. The main idea of construction is to impose the constraint in terms of the characteristic polynomial of \( X_{\lambda}; \)
\[ \det(\mu I_3 + X_{\lambda}) = \mu^3 + c\lambda^3, \]
for a constant \( c \in \mathbb{C}^*. \) This allows one to solve for \( \{b^*, c^*, s^*, t^*\} \) differential algebraically not just by using the left-adjacent terms, but by using all of the lower-order terms (the left hand side terms in the diagram above). The relevant explicit formulas using the truncated formal Killing fields are given in §7.3

1.4.4. Jacobi fields and conservation laws. The structure equation (64) implies that the sequence of coefficients \( \{a^{6n+7}\} \) are Jacobi fields, and the sequence of coefficients \( \{p^{6n+4}\} \) are pseudo-Jacobi fields. From this, it follows that the 6-step recursion in the schematic diagram in Fig.1 can be understood as the union of two 3-step recursions between Jacobi fields and pseudo-Jacobi fields.

The structure equation (64) also implies that an infinite sequence of conservation laws can be assembled from the components of the formal Killing fields, Eq.(76), Eq.(77).

In this way, the canonical formal Killing fields provide an efficient method to organize the (infinitely prolonged) local analytic invariants of minimal Lagrangian surfaces.

1.5. Contents. In §2 the exterior differential system for minimal Lagrangian surfaces is defined as a homogeneous differential system on the bundle of Lagrangian 2-planes, and we record the basic structure equations. In §3 we compute the space of classical Jacobi fields and conservation laws which arise from the infinitesimal action of the group of Kähler isometries of the ambient space form. In hindsight, the recursion relations for the formal Killing fields are already implicit in the structure equation for the classical Killing fields. In §4 we determine the structure equation for the infinite prolongation of the minimal Lagrangian system. A branched triple cover is introduced for the field extension to accommodate the higher-order Jacobi fields. In §5 we record two useful lemmas on the
rigidity property of the associated $\partial \bar{\xi}$-equation. They are applied in §6 to prove a complete classification of the (pseudo) Jacobi fields. §7 contains the main results of the paper. We use the determinantal identities from the characteristic polynomial to derive the explicit differential algebraic recursion for the canonical formal Killing fields associated with a pair of natural initial data. In §8, we also read off the formal Killing fields an infinite sequence of higher-order conservation laws.

1.6. Remarks.

1.6.1. This is a continuation of the joint work [12]. We suspect that the analysis carried out in [12] can be extended to the primitive harmonic maps in general.

1.6.2. In the analytic approach, the transition from the minimal surfaces in the 3-sphere to the minimal Lagrangian surfaces in $\mathbb{CP}^2$ is nontrivial, [26]. From our point of view, this amounts to replacing the underlying Lie algebra from $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{sl}(3, \mathbb{C})$. The present work may provide a basis to introduce the further results from the integrable system theory to the study of minimal Lagrangian surfaces in this uniform perspective.

2. Minimal Lagrangian surfaces in a complex space form

After a brief summary of the structure equation for a $2_\mathbb{C}$-dimensional complex space form, we give an analytic description of the differential equation for minimal Lagrangian surfaces as a homogeneous exterior differential system defined on the $\text{U}(2)/\text{SO}(2)$-bundle of oriented Lagrangian planes. The basic structure equations established in §2.2 together with their infinite prolongation in §4 will be the basis of our analysis for the minimal Lagrangian system.

2.1. Complex space form. We will summarize the basic formulas of Kählerian geometry for a $2_\mathbb{C}$-dimensional complex space form. We refer to [2] for the further related details.

In order to avoid repetitions, we agree on the following range for the indices:

$$1 \leq A, B, C \leq 2.$$ 

We use the Einstein summation convention for repeated indices.

2.1.1. $2_\mathbb{C}$-dimensional complex space form. Let $M$ be the $2_\mathbb{C}$-dimensional simply connected complex space form of constant holomorphic sectional curvature $4\gamma^2$. In the case $4\gamma^2 = 0$ and $M = \mathbb{C}^2$, it turns out that the minimal Lagrangian surfaces in $M$ are equivalent to the holomorphic curves in $\mathbb{C}^2$ under a different covariant constant complex structure (this is explained by the fact that $\mathbb{C}^2$ is hyperKähler, [18, p.148]). We shall restrict ourselves to the case

$$\gamma^2 \neq 0,$$

where the differential equation for minimal Lagrangian surfaces is genuinely nonlinear.

The squared expression $\gamma^2$ is introduced for the sake of convenience, for the quantity $\gamma$ (which is $\frac{1}{2}$-times the square root of the holomorphic sectional curvature) appears
frequently in the analysis. We adopt the following convention for $\gamma$:

$$
\gamma = \begin{cases} 
+\sqrt{\gamma^2} & \text{if } \gamma^2 > 0 \\
i \sqrt{-\gamma^2} & \gamma^2 < 0.
\end{cases}
$$

Here $i = \sqrt{-1}$ denotes the unit imaginary number.

2.1.2. Unitary coframe bundle. Let $U(2)$ be the group of 2-by-2 unitary matrices. Let

$$
U(2) \longrightarrow \mathcal{F} \\
\pi \downarrow \\
M
$$

be the principal $U(2)$-bundle of unitary coframes. An element $u \in \mathcal{F}$ is by definition a Hermitian isometry

$$
u : T_{\pi(u)}M \to \mathbb{C}^2,$$

where $\mathbb{C}^2$ is the standard $2\mathbb{c}$-dimensional Hermitian vector space. The structure group $U(2)$ acts on $\mathcal{F}$ on the right by

$$
u \to g^{-1} \circ \nu, \quad \text{for } g \in U(2).$$

Let $(\zeta^1, \zeta^2)^{t}$ be the $\mathbb{C}^2$-valued tautological 1-form on $\mathcal{F}$. The structure group $U(2)$ acts on $(\zeta^1, \zeta^2)^{t}$ on the right by

$$
\begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} \to g^{-1} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}, \quad \text{for } g \in U(2).
$$

By definition, the Kähler structure on $M$ is given by the pair

$$
\begin{align*}
\mathbb{g} & := \zeta^A \circ \overline{\zeta}^A \quad \text{(Riemannian metric)}, \\
\omega & := \frac{i}{2} \zeta^A \wedge \overline{\zeta}^A \quad \text{(symplectic form)}.
\end{align*}
$$

Let $\mathfrak{u}(2)$ be the space of 2-by-2 skew-Hermitian matrices, which is the Lie algebra of $U(2)$. There exists a unique $\mathfrak{u}(2)$-valued connection 1-form $(\zeta^A_B)$ on $\mathcal{F}$ such that the following structure equations hold:

$$
\begin{align*}
d\zeta^A & = -\zeta^A_B \wedge \zeta^B, \\
\zeta^A_B & = -\overline{\zeta}_B^A.
\end{align*}
$$

The curvature 2-forms $\Omega^A_B$ are then defined by

$$
\Omega^A_B = d\zeta^A_B + \zeta^A_C \wedge \zeta^C_B.
$$

For the case at hand, the curvature forms of the complex space form $M$ are given by

$$
\Omega^A_B = \gamma^2 \left( \zeta^A \wedge \overline{\zeta}^B + \delta^A_B \sum_{C=1}^{2} \zeta^C \wedge \overline{\zeta}^C \right).
$$

Here $\delta^A_B$ is the Kronecker delta.
2.1.3. *Real structure equation.* The above treatment considers $M$ as a 2-dimensional complex Hermitian manifold. On the other hand, note the isomorphism

$$U(2) = SO(4) \cap Sp(2, \mathbb{R}).$$

Here the Lie groups $U(2), SO(4)$ (special orthogonal group), $Sp(2, \mathbb{R})$ (symplectic group) are considered as the real subgroups of $SL(4, \mathbb{R})$. Accordingly, it will be convenient for the analysis of Lagrangian surfaces to consider $M$ as a 4-dimensional real manifold equipped with the pair $(g, \omega)$ given by (1).

To this end, we decompose the structure equations for $\{\zeta^A, \overline{\zeta}_b^A\}$ into the real, and imaginary parts as follows.

Set

$$\zeta^A = \omega^A + i\mu^A,$$

$$\left(\begin{array}{c}
\overline{\zeta}_1^A \\
\overline{\zeta}_2^A \\
\end{array}\right) = \left(\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}\right) + i \left(\begin{array}{c}
\beta^1 - 3\gamma^2 \theta_0 \\
\beta^2 \\
-\beta^1 - 3\gamma^2 \theta_0 \\
\end{array}\right),$$

for the set of real 1-forms $\{\omega^A, \mu^A, \rho, \beta^A, \theta_0\}$. In terms of these 1-forms, the structure equations (2), (3), (4) are written as follows:

$$d\omega^1 = -\rho \wedge \omega^2 + (\beta^1 \wedge \mu^1 + \beta^2 \wedge \mu^2) - 3\gamma^2 \theta_0 \wedge \mu^1,$$

$$d\omega^2 = +\rho \wedge \omega^1 + (\beta^2 \wedge \mu^1 - \beta^1 \wedge \mu^2) - 3\gamma^2 \theta_0 \wedge \mu^2,$$

$$d\mu^1 = -\rho \wedge \mu^2 - (\beta^1 \wedge \omega^1 + \beta^2 \wedge \omega^2) + 3\gamma^2 \theta_0 \wedge \omega^1,$$

$$d\mu^2 = +\rho \wedge \mu^1 - (\beta^2 \wedge \omega^1 - \beta^1 \wedge \omega^2) + 3\gamma^2 \theta_0 \wedge \omega^2,$$

$$d\rho = \gamma^2(\omega^1 \wedge \omega^2 + \mu^1 \wedge \mu^2) + 2\beta^1 \wedge \beta^2,$$

$$d\theta_0 = -(\mu^1 \wedge \omega^1 + \mu^2 \wedge \omega^2),$$

$$d\beta^1 = -2\rho \wedge \beta^2 + \gamma^2(\mu^1 \wedge \omega^1 - \mu^2 \wedge \omega^2),$$

$$d\beta^2 = +2\rho \wedge \beta^1 + \gamma^2(\mu^2 \wedge \omega^1 + \mu^1 \wedge \omega^2).$$

2.2. *Exterior differential system.* With this preparation, we proceed to describe the differential system for minimal Lagrangian surfaces.

2.2.1. *Bundle of Lagrangian 2-planes.* An immersed surface $x : \Sigma \hookrightarrow M$ is Lagrangian if $x^*\omega = 0$.

This is by definition a first order constraint on $\Sigma$ that its tangent space at each point is a Lagrangian subspace of the tangent bundle $TM$.

Under the standard representation, the unitary group $U(2)$ acts transitively on the set of oriented Lagrangian subspaces (2-planes) in $\mathbb{C}^2$, with $SO(2) = U(2) \cap SL(2, \mathbb{R})$ as the stabilizer subgroup. The Grassmannian of Lagrangian subspaces in dimension 2 is the homogeneous space,

$$\text{Lag}(\mathbb{C}^2) = U(2)/SO(2).$$
From the general theory of principal bundles, it follows that the $U(2)/SO(2)$-bundle of oriented Lagrangian 2-planes in $TM$ is given by

$$X := \mathcal{F}/SO(2) \to M.$$  

This is summarized by the following diagram:

\begin{center}
\begin{tikzcd}
& \mathcal{F} \\
X \arrow[ru, SO(2)] \arrow[rd, U(2)] & & M \\
\mathcal{F}/SO(2) \arrow[ru] \arrow[rd] & & \end{tikzcd}
\end{center}

2.2.2. Differential system for Lagrangian surfaces. In view of the diagram, an immersed oriented Lagrangian surface in $M$ admits a unique tangential lift to $X$. Such tangential lifts are characterized as the integral surfaces of the canonical contact differential system on $X$.

This is expressed analytically as follows. Consider the real structure equation (5). By a standard moving frame analysis, set the contact ideal $I_0$ on $X$ generated by

$$I_0 := \langle \mu^1, \mu^2, d\mu^1, d\mu^2 \rangle.$$

Note that the differential ideal $I_0$ is originally defined on $\mathcal{F}$. But, it is invariant under the induced action by the structure group $SO(2)$ and $I_0$ descends on $X$ as a well defined differential ideal. By construction, an immersed oriented integral surface of $I_0$ in $X$ corresponds to a possibly singular oriented Lagrangian surface in $M$.

An immersed integral surface of $I_0$ in $X$ may become singular under the projection $X \to M$ to the original complex space form. The formulation of Lagrangian surfaces in $M$ in terms of the canonical contact differential system $I_0$ on $X$ naturally extends the set of admissible Lagrangian surfaces, see §4.2.

2.2.3. Differential system for minimal Lagrangian surfaces. The condition that an immersed Lagrangian surface is minimal (i.e., its mean curvature vector vanishes) is a second order constraint. We therefore need to augment the contact ideal $I_0$ to express this additional minimality condition.

From Eqs.(5), we claim that this is equivalent to the vanishing of the connection 1-form $\theta_0$,

$$\theta_0 = 0.$$  

To see this, consider the structure equation (5) adapted to a Lagrangian surface $\Sigma \hookrightarrow M$. With the 1-forms $\{\mu^1, \mu^2\}$ being set to 0, the induced Riemannian metric on $\Sigma$ is given by

$$I := (\omega^1)^2 + (\omega^2)^2.$$
For the condition of minimality, note that the second fundamental form of $\Sigma$ can be identified with the symmetric cubic differential

$$\mathbb{II} := \omega^A \circ \text{Im}(\zeta^A_B) \circ \omega^B.$$  

The mean curvature vector vanishes when the corresponding trace vanishes,

$$\text{tr}_I(\mathbb{II}) = 0.$$  

This is equivalent to the vanishing of the 1-form $\theta_0$, which is up to constant scale the trace of $\text{Im}(\zeta^A_B)$.

**Proposition 2.1.** Let $X \to M$ be the $\text{U}(2)/\text{SO}(2)$-bundle of oriented Lagrangian 2-planes. Let $I_0$ be the canonical contact differential system on $X$, (6). The differential system for minimal Lagrangian surfaces is given by

$$I := \langle \mu^1, \mu^2, \theta_0, \phi^+, \phi^- \rangle,$$

where

$$\phi^+ = \beta^1 \wedge \omega^1 + \beta^2 \wedge \omega^2,$$

$$\phi^- = \beta^2 \wedge \omega^1 - \beta^1 \wedge \omega^2.$$  

The structure equation (5) shows that the differential ideal $I$, originally defined on $F$, is invariant under the induced action by the structure group $\text{SO}(2)$. The differential ideal $I$ is well defined on $X$.

An immersed minimal Lagrangian surface in $M$ admits a unique tangential lift to $X$ as an integral surface of $I$. Conversely, an immersed integral surface of $I$ in $X$ projects to a possibly singular (branched) minimal Lagrangian surface in $M$.

**Proof.** For the last sentence, see §4.2.

Note from the structure equation that

$$d\theta_0, d\mu^A \equiv 0 \mod I,$$

and $I$ is differentially closed.

In terms of Cartan’s theory of exterior differential systems, the differential system $(X, I)$ is involutive and the local moduli space of solutions depends on two arbitrary real functions of 1 variable, [4].

2.2.4. Complexified structure equation. The differential system for minimal Lagrangian surfaces under consideration is an integrable extension over the elliptic Tzitzeica equation, [11]. The characteristic directions are complex and they induce a complex structure on a minimal Lagrangian surface. In order to utilize this property, we introduce another set of complex differential forms on $F$ adapted to $I$.

Set

$$\xi := \omega^1 + i \omega^2,$$

$$\theta_1 := \mu^1 - i \mu^2,$$

$$\eta_2 := \beta^1 - i \beta^2.$$  

Note
\[ \phi^+ - i\phi^- = \eta_2 \wedge \xi, \]
and the ideal (7) can be written as
(9) \[ I = (\theta_0, \theta_1, \eta_2 \wedge \xi). \]

In terms of these complex 1-forms, the structure equation (5) simplifies to the following set of equations:
(10) \[
\begin{align*}
    d\xi &= i\rho \wedge \xi - 3\gamma^2 \theta_0 \wedge \overline{\theta}_1 - \theta_1 \wedge \overline{\eta}_2, \\
    d\theta_0 &= -\frac{1}{2} (\theta_1 \wedge \xi + \overline{\theta}_1 \wedge \overline{\xi}), \\
    d\theta_1 &= -i\rho \wedge \theta_1 - \eta_2 \wedge \xi + 3\gamma^2 \theta_0 \wedge \overline{\xi}, \\
    d\eta_2 &= -2i\rho \wedge \eta_2 + \gamma^2 \theta_1 \wedge \overline{\xi}, \\
    d\rho &= \frac{i}{2} (\gamma^2 \xi \wedge \overline{\xi} - 2\eta_2 \wedge \overline{\eta}_2 - \gamma^2 \theta_1 \wedge \overline{\theta}_1).
\end{align*}
\]

From now on, the differential analysis for minimal Lagrangian surfaces will be carried out based on this structure equation.

Let us mention here a relevant notation which will be frequently used. For a scalar function \( f : \mathcal{F} \to \mathbb{C} \), the covariant derivatives are written in the upper-index notations,
(11) \[
    df \equiv f^\xi \xi + f^\overline{\xi} \overline{\xi} + f^0 \theta_0 + f^1 \theta_1 + f^\theta_1 + f^2 \eta_2 + f^\eta_2 \mod \rho.
\]

2.3. **Hopf differential.** The induced local geometric structures on a minimal Lagrangian surface consists of a triple of data called admissible triple, Defn2.2 in the below. In particular, it contains a holomorphic cubic differential called **Hopf differential** which arises as a complexified version of the second fundamental form. Hopf differential will play an important role in our study of the minimal Lagrangian system.

2.3.1. **Hopf differential.** Let \( x : \Sigma \hookrightarrow X \) be an immersed integral surface of the differential system for minimal Lagrangian surfaces. This is expressed analytically by
\[
    \theta_0, \theta_1 = 0, \quad \eta_2 \wedge \xi = 0,
\]
on the induced \( \text{SO}(2) \)-bundle \( x^* \mathcal{F} \to \Sigma \). It follows that the structure equation (10) restricted to \( x^* \mathcal{F} \) becomes
(12) \[
\begin{align*}
    d\xi &= i\rho \wedge \xi, \\
    0 &= \eta_2 \wedge \xi, \\
    d\eta_2 &= -2i\rho \wedge \eta_2, \\
    d\rho &= \frac{i}{2} (\gamma^2 \xi \wedge \overline{\xi} - 2\eta_2 \wedge \overline{\eta}_2).
\end{align*}
\]

In particular, the cubic differential
\[ \eta_2 \circ \xi^2 \]
is invariant under the action by the structure group \( \text{SO}(2) \) and becomes a well defined holomorphic cubic differential on \( \Sigma \).
Let $K \to \Sigma$ denote the canonical line bundle of $(1,0)$-forms.

**Definition 2.1.** Let $x : \Sigma \hookrightarrow X$ be an immersed integral surface of the differential system for minimal Lagrangian surfaces. Consider the induced structure equation (12). The **Hopf differential** of $x$ is the holomorphic cubic differential

$$\mathbb{I} = \eta_2 \circ \xi^2 \in H^0(\Sigma, K^3).$$

The **umbilic divisor** $\mathcal{U} = (\mathbb{I})_0$ is defined as the zero divisor of $\mathbb{I}$.

When $\Sigma$ is compact, by Riemann-Roch theorem we have

$$\deg(\mathcal{U}) = 6 \text{ genus}(\Sigma) - 6.$$

**2.3.2. Admissible triple.** Suppose that the integral surface $\Sigma \hookrightarrow X$ is the tangential lift of an immersed oriented minimal Lagrangian surface in $M$. By definition of the tautological forms on $\mathcal{F}$, this implies the independence condition

$$\xi \wedge \bar{\xi} \neq 0,$$

and the induced Riemannian metric on $\Sigma$ is given by

$$I = \xi \circ \bar{\xi}.$$

It follows that $x^*\mathcal{F} \to \Sigma$ can be identified with the principal SO(2)-bundle of the induced metric, and that $\xi$ is the tautological unitary $(1,0)$-form, and $\rho$ is the Levi-Civita connection form.

From the second equation of (12), there exists a coefficient $h_3$ defined on $x^*\mathcal{F}$ such that

$$\eta_2 = h_3 \xi.$$

The Hopf differential is now written as

$$\mathbb{I} = h_3 \xi^3.$$

From the fourth equation of (12), the Gauss equation, the curvature $R$ of the induced metric $I$ satisfies the compatibility equation

$$R = \gamma^2 - 2h_3 \bar{h}_3.$$

Summarizing the analysis so far, we give a definition of the compatible Bonnet data for minimal Lagrangian surfaces.

**Definition 2.2.** Let $M$ be a $2c$-dimensional complex space form of constant holomorphic sectional curvature $4\gamma^2$. An admissible triple for a minimal Lagrangian surface in $M$ consists of a Riemann surface, a conformal metric, and a holomorphic cubic differential which satisfy the compatibility equation (15).

An analogue of the classical Bonnet theorem can be stated as follows. The proof is by a standard ODE argument and is omitted.
Theorem 2.2. Let \( \Sigma \) be a Riemann surface. Let \((\mathcal{I}, \mathcal{II})\) be a pair of a conformal metric and a holomorphic cubic differential on \( \Sigma \) such that \((\Sigma, \mathcal{I}, \mathcal{II})\) form an admissible triple for a minimal Lagrangian surface in a complex space form \( M \). Let \( \pi : \tilde{\Sigma} \to \Sigma \) be the simply connected universal cover. Then there exists a minimal Lagrangian immersion \( \tilde{x} : \tilde{\Sigma} \hookrightarrow M \) which realizes \((\pi^* \mathcal{I}, \pi^* \mathcal{II})\) as the induced Riemannian metric and the Hopf differential. Such an immersion \( \tilde{x} \) is unique up to motion by the Kähler isometries of the ambient complex space form \( M \).

2.3.3. Cube root of Hopf differential. The analysis of the canonical formal Killing fields to be addressed in \( \S 7 \) will inevitably involve the use of the object 

\[
\sqrt[3]{h_3},
\]

or, equivalently, the cube root of Hopf differential

\[
\sqrt[3]{\mathcal{I}}.
\]

It is generally a multi-valued holomorphic 1-form on a minimal Lagrangian surface. In order to better accommodate this, we introduce the triple cover of a minimal Lagrangian surface defined by Hopf differential.

Definition 2.3. Let \( x : \Sigma \hookrightarrow X \) be an immersed integral surface of the differential system for minimal Lagrangian surfaces. Let \( \mathcal{II} \in H^0(\Sigma, \mathcal{K}^3) \) be the Hopf differential \( (\text{13}) \). The triple cover

\[
\nu : \hat{\Sigma} \to \Sigma
\]

associated with \( \mathcal{II} \) is the Riemann surface of the complex curve

\[
\Sigma' = \left\{ \kappa \in \mathcal{K} \mid \kappa^3 = \mathcal{II} \right\} \subset \mathcal{K}.
\]

The cube root \( \omega = \sqrt[3]{\mathcal{II}} \) of the Hopf differential is the holomorphic 1-form on \( \hat{\Sigma} \) obtained by the pull-back of restriction of the tautological 1-form on \( K \) to \( \Sigma' \). By construction, the projection \( \nu \) is a triple covering branched over the umbilics of degree 1 or 2 \( \text{mod } 3 \).

3. Classical Killing fields

Let

\[ G = \text{SU}(3), \quad \text{or } \text{SU}(1,2) \]

be the group of Kähler isometries of the complex space form \( M \), depending on the sign of the holomorphic sectional curvature \( 4\gamma^2 \). The induced action of \( G \) on \( X \) is transitive, and by construction the differential system \((X, \mathcal{I})\) is \( G \)-invariant, i.e., the group \( G \) acts as a symmetry of the differential system for minimal Lagrangian surfaces. In this section, we examine the classical Killing fields generated by the corresponding infinitesimal action of the Lie algebra of \( G \),

\[ \mathfrak{g} = \text{su}(3), \quad \text{or } \text{su}(1,2). \]

The structure equation for classical Killing fields reveals its recursive, symmetrical property when it is written in terms of an adapted basis of \( \mathfrak{g} \), Eq.\( (18) \). From this, we extract two kinds of Jacobi fields called classical Jacobi fields, and their variants classical pseudo-Jacobi fields, as well as the recursion relations between them, \( \S 3.2.3 \).
The structure equation also shows that there exists the eight dimensional space of
classical conservation laws. Combining these results, we deduce that Noether’s theorem
holds and there exists a canonical isomorphism between the classical (pseudo) Jacobi
fields and the classical (local) conservation laws, Cor 3.6.

3.1. Structure equation. The Killing field equation, (17) in the below, characterizes the
classical Killing fields generated by the infinitesimal action of the Lie algebra \( \mathfrak{g} \) on the
homogeneous space \( X \). When restricted to a minimal Lagrangian surface, it becomes the
recursive structure equation (18).

We will find that Eq.(18) encodes many of the important local structural properties of
the minimal Lagrangian system. In particular, a pair of the embedded recursion relations
will serve as the model for the construction of the canonical formal Killing fields.

3.1.1. Complexified Maurer-Cartan form. Recall the complexified di
fferential forms (8) and
their structure equation (10). The left invariant, \( \mathfrak{g} \)-valued Maurer-Cartan form of \( G \) can be
written in terms of these 1-forms by

\[
\psi = \psi_+ + \psi_0 + \psi_-, \tag{16}
\]

where

\[
\psi_- = \frac{1}{2} \begin{bmatrix}
\gamma(\xi + i\theta_1) & -\gamma(\xi - i\theta_1) \\
-i\eta_2 & -\eta_2 & -i\eta_2
\end{bmatrix},
\]

\[
\psi_0 = \begin{bmatrix}
2i\gamma^2\theta_0 & \cdot & \cdot \\
\cdot & -i\gamma^2\theta_0 & \rho \\
\cdot & -\rho & -i\gamma^2\theta_0
\end{bmatrix},
\]

\[
\psi_+ = \frac{1}{2} \begin{bmatrix}
\gamma(\xi + i\theta_1) & -\gamma(\xi - i\theta_1) \\
i\eta_2 & i\eta_2 & i\eta_2
\end{bmatrix}.
\]

It satisfies the structure equation

\[
d\psi + \psi \wedge \psi = 0.
\]

3.1.2. Classical Killing fields.

Definition 3.1. Let \( \mathcal{F}/\SO(2) = X \to M \) be the bundle of oriented Lagrangian planes. A
classical Killing field is an \( \SO(2) \)-equivariant, \( \mathfrak{g} \)-valued function \( X \) on \( \mathcal{F} \) which satisfies
the Killing field equation

\[
dX + [\psi, X] = 0. \tag{17}
\]

By definition, the space of classical Killing fields is isomorphic to \( \mathfrak{g} \).

Definition 3.2. Let \( (X, I) \) be the differential system for minimal Lagrangian surfaces. A
vector field \( V \in H^0(TX) \) is a classical symmetry if it preserves the differential ideal \( I \)
under the Lie derivative,

\[
L_V I \subset I.
\]
The algebra of classical symmetries is denoted by \( \Xi^{(0)} \).
It is clear that a classical Killing field generates a classical symmetry. In fact, a classical symmetry is necessarily generated by a classical Killing field. We thus have;

**Proposition 3.1.**

\[ \mathcal{Z}^{(0)} \simeq g. \]

The proof follows by a direct computation, and is omitted.

3.1.3. **Recursive structure equation.** For the ensuing analysis, it will be useful to have Eq. (17) written component-wise. Consider the explicit decomposition of the Lie algebra \( g \) in Fig. 2.

\[
\begin{bmatrix}
-2ia & b + f + g - s & ib - if + ig + is \\
-b + f + g + s & ic + ia - it & -p + c + t \\
-ib - if + ig - is & p + c + t & -ic + ia + it \\
\end{bmatrix}
\]

**Figure 2.** Decomposition of the Lie algebra \( g = \text{su}(3), \text{ or } \text{su}(1, 2) \)

Here \( \{ p, b, c, f, a, g, s, t \} \) are the scalar coefficients which satisfy the following reality conditions:

\[
g = \begin{cases} 
\text{su}(3) & \text{then } a = \bar{a}, p = \bar{p}, t = -\bar{c}, s = -\bar{b}, g = -\bar{f}, \\
\text{su}(1, 2) & \text{then } a = \bar{a}, p = \bar{p}, t = -\bar{c}, s = +\bar{b}, g = +\bar{f}.
\end{cases}
\]

The Killing field equation (17) then implies

(18)

\[
\begin{align*}
& dp \equiv (iyb + 2ih_3c)\xi + (iy s + 2ih_3t)\bar{\xi}, \\
& dB + ib\rho \equiv ih_3 f \xi + \frac{i}{2} \gamma p \bar{\xi}, \\
& dc - 2ic\rho \equiv iy f \xi + ih_3 p \bar{\xi}, \\
& df - if\rho \equiv \frac{3i}{2} \gamma a \xi + (iy c + ih_3 b) \bar{\xi}, \\
& da \equiv iy g \xi + iy f \bar{\xi}, \\
& dg + ig\rho \equiv (-iy t - ih_3 s) \xi + \frac{3i}{2} \gamma a \bar{\xi}, \\
& ds - is\rho \equiv \frac{i}{2} \gamma p \xi - ih_3 g \bar{\xi}, \\
& dt + 2it\rho \equiv ih_3 p \xi - iy g \bar{\xi}, \quad \text{mod } I.
\end{align*}
\]

Here we applied the substitution, (14),

\[ \eta_2 \equiv h_3 \xi, \quad \bar{\eta}_2 \equiv \bar{h}_3 \bar{\xi} \quad ("\text{mod } I"). \]

The resulting structure equation should be understood as restricted to a minimal Lagrangian surface.
Let us introduce the relevant notations for partial derivatives. For a scalar function \( u \) on \( F \), the notations \( \partial_\xi u, \partial_\tau u \) would mean
\[
\partial_\xi u = u_\xi := \xi\text{-coefficient of } du, \\
\partial_\tau u = u_\tau := \tau\text{-coefficient of } du, \quad "\text{mod } \varGamma".
\]
For instance, the above structure equation shows that
\[
a_\xi = iy_g, \quad a_\tau = iy_f.
\]

3.2. Classical Jacobi fields and pseudo-Jacobi fields. From the apparent recursive symmetry of Eq.(18), we extract two different kinds of Jacobi fields for the minimal Lagrangian system; classical Jacobi fields and classical pseudo-Jacobi fields.

3.2.1. Classical Jacobi field. Consider the coefficient \( a \) in Eq.(18). One finds
\[
a_\xi = iy_g, \\
a_{\xi\xi} = iy_g = -\frac{3}{2} \gamma^2 a.
\]
Motivated by this, we make the following definition.

**Definition 3.3.** A scalar function \( A : X \to \mathbb{C} \) is a **classical Jacobi field** if it satisfies
\[
A_{\xi\xi} + \frac{3}{2} \gamma^2 A = 0.
\]

The \( \mathbb{C} \)-vector space of classical Jacobi fields is denoted by \( \mathfrak{J}^{(0)} = \mathfrak{J}^0 \).

An examination of Eq.(17) shows that the coefficient \( a \) generates the classical Killing field \( X \) in the sense that if the \( a \)-component vanishes then the corresponding classical Killing field \( X \) vanishes. It follows that the induced map
\[
a : g^\mathbb{C} \longrightarrow \mathfrak{J}^{(0)}
\]
is injective. Here \( g^\mathbb{C} = g \otimes \mathbb{C} = \mathfrak{sl}(3, \mathbb{C}) \) is the complexification of \( g \).

We claim that this is in fact an isomorphism, and the \( a \)-components of the (complexified) classical Killing fields generate the classical Jacobi fields.

**Proposition 3.2.**
\[
\mathfrak{J}^{(0)} \cong g^\mathbb{C}.
\]
The claim follows by a direct computation. Rather than a complete proof, we give a brief sketch of ideas.

Let \( A \) be a classical Jacobi field, which is defined on \( X \). Denote the covariant derivatives of \( A \) by
\[
dA = A^\xi_\xi + A^\tau_\tau + A^0 \theta _0 + A^1 \theta _1 + A^2 \eta _2 + A^\varpi _\varpi.
\]
We adopt the similar notation for the successive derivatives of \( A^\xi, A^{\xi\xi}, \ldots, \) etc.
Applying the covariant derivative operators (19), one finds

\[ \partial_\xi A = A_\xi = A^{\xi} + A^2 h_3, \]
\[ \partial_\xi (A_\xi) = A_\xi = A^{\xi} + A^2 \bar{h}_3 + (A^2 \bar{\xi} + A^2 \bar{\xi} h_3) h_3. \]

The Jacobi equation (20) becomes

\[ (A^{\xi \bar{\xi}} + \frac{3}{2} \gamma^2 A) + A^{\xi \bar{\xi}} + A^{\xi \bar{\xi}} h_3 + A^{\xi \bar{\xi}} \bar{h}_3 = 0. \]

Since \( A \) is defined on \( X \) while \( h_3, \bar{h}_3 \) are the prolongation variables, each coefficient of \( \{h_3, \bar{h}_3, h_3 \bar{h}_3\} \) in (22) must vanish separately. As a result, a classical Jacobi field must satisfy;

\[ A^{\xi \bar{\xi}} + \frac{3}{2} \gamma^2 A = 0, \]
\[ A^{\xi \bar{\xi}} = A^{\xi \bar{\xi}} = 0. \]

From this, a standard over-determined PDE analysis shows that \( A \) is the a-component of a (complexified) classical Killing field.

3.2.2. Classical pseudo-Jacobi fields. In an analogy with the classical Jacobi field case, consider next the coefficient \( p \) in Eq.(18). One finds

\[ p_\xi = i \gamma b + 2 i h_3 c, \]
\[ p_{\xi \bar{\xi}} = i \gamma b_{\bar{\xi}} + 2 i h_3 c_{\bar{\xi}} = - \frac{1}{2} (\gamma^2 + 4 h_3 \bar{h}_3) p. \]

**Definition 3.4.** A scalar function \( P : X \rightarrow \mathbb{C} \) is a classical pseudo-Jacobi field if it satisfies

\[ P_{\xi \bar{\xi}} + \frac{1}{2} (\gamma^2 + 4 h_3 \bar{h}_3) P = 0. \]

The \( \mathbb{C} \)-vector space of classical pseudo-Jacobi fields is denoted by \( \mathfrak{z}^{(0)} = \mathfrak{z}^{(0)} \).

By the similar argument as above, the \( \mathfrak{p} \)-components of the (complexified) classical Killing fields generate the classical pseudo-Jacobi fields.

**Proposition 3.3.**

\[ \mathfrak{z}^{(0)} \simeq g^\mathbb{C}. \]

As a consequence of the isomorphisms (21), (24), we have

**Corollary 3.4.**

\[ \mathfrak{z}^{(0)} \simeq \mathfrak{z}^{(0)} \simeq g^\mathbb{C}. \]
3.2.3. **Recursion relations.** The structure equation (18) contains two 3-step recursion relations between the classical pseudo-Jacobi field $p$ and the classical Jacobi field $a$. We record and emphasize these structures here, for they are the main technical ingredients in the construction of the formal Killing fields later on.

**[From $p$ to $a$]**

Let the classical pseudo-Jacobi field $p$ be given. From (18), suppose (either of) the equations

$$
\partial_\xi b = \frac{i}{2} \gamma p, \quad \partial_\xi c = i\bar{h}_3 p
$$

were solved. Differentiating $b, c$,

$$
\partial_\xi b = i\bar{h}_3 f, \quad \partial_\xi c = i\gamma f,
$$

and one gets $f$. Differentiating $f$,

$$
\partial_\xi f = \frac{3i}{2} \gamma a,
$$

and one gets the classical Jacobi field $a$.

The process can be summarized by the following diagram.

$$
p \xrightarrow{\partial_\xi^{-1}} b, c \xrightarrow{\partial_\xi} f \xrightarrow{\partial_\xi} a.
$$

**[From $a$ to $p$]**

In a similar way, starting from the classical Jacobi field $a$, one may reach $p$ by the following process.

$$
a \xrightarrow{\partial_\xi} g \xrightarrow{\partial_\xi^{-1}} s, t \xrightarrow{\partial_\xi} p.
$$

Note that by combining the two processes, one gets a 6-step recursion relation for classical (pseudo) Jacobi fields.

3.3. **Classical conservation laws.** Another important invariants of the minimal Lagrangian system are conservation laws. They also can be read off Eq. (18).

3.3.1. **Definition.** Let $(\Omega^*(X), d)$ be the de-Rham complex of $C$-valued differential forms on $X$. Since $I$ is a differential ideal closed under the exterior derivative, the quotient complex

$$(\Omega^*, d)$$

is well defined, where $\Omega^* = \Omega^*(X)/I$, and $d = d \mod I$. Let $H^q(\Omega^*, d)$ be the cohomology at $\Omega^q$. The set

$$
\{ H^q(\Omega^*, d) \}_{q=0}^2
$$

is called the characteristic cohomology of the differential system $(X, I)$.

**Definition 3.5.** Let $(X, I)$ be the differential system for minimal Lagrangian surfaces. A **classical conservation law** is an element of the 1-st characteristic cohomology $H^1(\Omega^*, d)$ of $(X, I)$. The $C$-vector space of classical conservation laws is denoted by

$$
C^{(0)} = C^0 := H^1(\Omega^*, d).
$$
Let \( C^{(0)}_{\text{loc}} \) denote the space of local classical conservation laws of \( I \) restricted to a small contractible open subset of \( X \).

3.3.2. Classical conservation laws from classical Killing fields. From the structure equation (18), consider the 1-form

\[
\varphi_a = b\xi + s\bar{\xi}.
\]

One finds that

\[
d\varphi_a \equiv 0 \mod I,
\]

and the 1-form \( \varphi_a \) represents a classical conservation law.

Let \([\varphi_a] \in C^{(0)}_{\text{loc}}\) denote the class represented by \( \varphi_a \) (which is globally defined on \( X \)). We claim that the associated map

\[
\mathfrak{g}^C \cong \mathfrak{g}^{(0)} \rightarrow C^{(0)}_{\text{loc}}
\]

given by

\[
a \rightarrow [\varphi_a]
\]

is an isomorphism.

In order to verify this claim, and for other computational purposes, we introduce a differentiated version of conservation laws.

3.4. Classical differentiated conservation laws. From the exact sequence

\[
0 \rightarrow I \rightarrow \Omega'(X) \rightarrow \Omega'(X)/I = \Omega' \rightarrow 0,
\]

at least locally we have

\[
C^{(0)}_{\text{loc}} \cong H^2(I, d).
\]

Here \( H^*(I, d) \) denotes the cohomology of the complex \((\Omega^*(I), d)\).

**Definition 3.6.** Let \((X, I)\) be the differential system for minimal Lagrangian surfaces. A classical differentiated conservation law is an element in the 2-nd cohomology \( H^2(I, d) \). The \( \mathbb{C} \)-vector space of classical differentiated conservation laws is denoted by

\[
\mathcal{H}^{(0)} = \mathcal{H}^0 := H^2(I, d).
\]

Note from the definition of classical conservation laws that a representative 1-form \( \varphi \) of a class \([\varphi] \in C^{(0)}_{\text{loc}}\) is defined up to exact 1-forms. By taking the exterior derivative “d” and considering the differentiated conservation laws, we eliminate this ambiguity, which is practically important in the actual computation of conservation laws.

Moreover, in a geometric situation as in the present case, the ideal \( I \) is often equipped with the relevant structures which enable one to determine a subspace of \( \Omega^2(I) \) in which a differentiated conservation law has the unique representative. In other words, one of the advantages of the differentiated version of conservation laws is that it may lead to an analogue of Hodge theorem; the appropriately normalized differentiated conservation laws can be considered as the harmonic representatives.
Remark 3.5. Consider the symplectic form of $M$, (1):

$$\omega = \frac{1}{2} \zeta^A \wedge \overline{\zeta}^A = \omega^1 \wedge \mu^1 + \omega^2 \wedge \mu^2 = d\theta_0.$$

Thus the class $[\omega]$ is trivial in $H^2(I, d)$.

We shall first establish the isomorphism

$$H^{(0)} \simeq \mathfrak{g}^C$$

by a direct differential analysis. Then, by finding a section (up to constant scale) of the natural map

(26) \[ [\varphi_a] \in C^{(0)}_{loc} \longrightarrow [d\varphi_a] \in H^{(0)}, \]

we show that there exists an isomorphism

$$C^{(0)}_{loc} \simeq H^{(0)}.$$

3.4.1. Initial reduction. Let $\Phi \in \Omega^2(I)$ be a 2-form in the ideal. Up to addition by $d(\Omega^1(I))$, an exact 2-form in the ideal of the form $d(f_0\theta_0 + f_1\theta_1 + f_1\overline{\theta_1})$ for scalar coefficients $f_0, f_1, f_1$, a computation shows that one may write

(27) \[ \Phi = A\Psi + \theta_0 \wedge \sigma, \]

where $A$ is a scalar function, $\sigma \in \Omega^1(X)$, and

$$\Psi = \text{Im}(\theta_1 \wedge \xi) = -\frac{i}{2}(\theta_1 \wedge \xi - \overline{\theta_1} \wedge \overline{\xi}).$$

Note

(28) \[ d\Psi = 3iy^2 \theta_0 \wedge (\xi \wedge \overline{\xi} + \theta_1 \wedge \overline{\theta_1}) \equiv 0 \mod \theta_0. \]

A 2-form $\Phi \in \Omega^2(I)$ normalized as in (27) is called a reduced 2-form. Let

$$H^{(0)} \subset \Omega^2(I)$$

denote the subspace of such reduced 2-forms. By construction, $H^{(0)}$ is transversal to the subspace of exact 2-forms $d(\Omega^1(I)) \subset \Omega^2(I)$. It follows that we have an isomorphism

$$H^{(0)} = \{ \text{closed 2-forms in } H^{(0)} \}.$$
3.4.2. Structure equation. The equation for the classical differentiated conservation laws is now reduced to
\[ d\Phi = 0, \]
for \( \Phi \in H^{(0)}. \) From this, a direct differential analysis yields the following closed structure equation\(^2\)
\begin{equation}
(29)
\end{equation}
\[
\begin{pmatrix}
A \\
A^\xi \\
A^\zeta \\
A^1 \\
A^{1,1} \\
A^{1,1} \\
P
\end{pmatrix}
= i \rho +
\begin{pmatrix}
A_\xi & A^\zeta & A^1 & A^1 & \cdots \\
A^{1,1} & -\frac{3}{2} \gamma^2 A & -3 \gamma^2 A^1 & P & A^1 & \cdots \\
-\frac{3}{2} \gamma^2 A & A^{1,1} & -3 \gamma^2 A^1 & -P & A^1 & \cdots \\
P & 3 \gamma^2 A^\zeta & A^{1,1} & -\frac{3}{2} \gamma^2 A & -A^\zeta & \cdots \\
-\frac{3}{2} \gamma^2 A^\zeta & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\gamma^2 A^1 & -\gamma^2 A^1 & -\gamma^2 A^1 & 2P & \cdots & \cdots \\
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta \\
\theta_0 \\
\theta_1 \\
\eta_2 \\
\eta_1
\end{pmatrix}
\]
Here the upper-script \( A^{\ast \ast} \) denotes the covariant derivative as before.

One finds that the closed linear differential system for \( \{ A, A^\xi, A^\zeta, A^1, A^{1,1}, A^{1,1}, P \} \) is compatible, i.e., \( d^2 = 0 \) is a formal identity. By the existence and uniqueness theorem of ODE, the space of (local) classical differentiated conservation laws is eight dimensional.

Note that, at this stage, the reduced 2-form \( \Phi \) is normalized to
\begin{equation}
(30)
\Phi = A \Psi + i \theta_0 \wedge \left( A^\xi \xi - A^\zeta \zeta + A^1 \theta_1 - A^{1,1} \theta_1 \right).
\end{equation}

3.5. Noether’s theorem. Observe that the coefficients of \( \Phi \) are determined by the coefficient \( A \) (or \( P \)) and its successive derivatives. Moreover, Eq.(29) shows that
\[
\begin{cases}
A \text{ is a classical Jacobi field}, \\
P \text{ is a classical pseudo-Jacobi field}.
\end{cases}
\]
It follows from this that we have the isomorphisms
\[ \mathcal{H}^{(0)} \simeq \mathcal{Z}^{(0)} \simeq \mathcal{C}^{(0)} \simeq \mathfrak{g}. \]
In particular, this implies that the classical differentiated conservation laws are globally defined on \( X \).

Corollary 3.6 (Noether’s theorem for classical conservation laws). Let \( M \) be the simply connected \( 2\mathcal{C} \)-dimensional complex space form of constant holomorphic sectional curvature \( 4 \gamma^2 \).
Let \( G = \text{SU}(3) \), or \( \text{SU}(1, 2) \) be the group of Kähler isometries of \( M \). Let \( \mathfrak{g} \) be its Lie algebra. Then there exist the isomorphisms
\[ (\mathfrak{g}^{(0)})^C \simeq \mathcal{Z}^{(0)} \simeq \mathcal{C}^{(0)} \simeq \mathcal{H}^{(0)} \simeq \mathfrak{g} \simeq C_{\text{loc}}^{(0)}. \]
Here \( C_{\text{loc}}^{(0)} \) denotes the space of local conservation laws of \( I \) restricted to a small contractible open subset of \( X \).

\(^2\)This part of the analysis is more or less mechanical, and shall be omitted.
Proof. For a classical Jacobi field $A$, set
\[ \varphi_A = \frac{i}{\delta y^2} \left( A^r \theta_{-1} - A^s \theta_{1} + A^t \theta_1 - A^u \theta_1 \right). \]
Then $d\varphi_A = \Phi$, where $\Phi$ is given by (30). This establishes the isomorphism $\mathcal{H}^{(0)} \simeq C^{(0)}_{\text{loc}}$. □

For the global consideration of conservation laws including the contribution from the cohomology of the ambient space $X$, we refer to [5, p.584, Theorem 1].

Remark 3.7 (Moment conditions). Consider a closed integral curve $\Gamma \subset X$ of $I$. Such $\Gamma$ corresponds to a pair $(\gamma, \Pi)$ of a closed curve $\gamma \subset M$ equipped with a field of oriented Lagrangian 2-planes $\Pi$ tangent to $\gamma$ such that the associated $(2,0)$-vector field is covariant constant. The integrals of the classical conservation laws on $\Gamma$ yield the moment conditions for $(\gamma, \Pi)$ to bound a minimal Lagrangian surface, with the given boundary $\gamma$ and the prescribed tangent planes $\Pi$ along $\gamma$. Compare [13].

3.6. Higher-order extension. In the remainder of the paper, the analysis of the classical objects in this section will be extended to their higher-order analogues in the setting of the infinite prolongation of the differential system $(\tilde{X}, \tilde{I})$.

The extension process relies on the familiar extension of the Maurer-Cartan form
\[ \psi \longrightarrow \psi_\lambda = \lambda \psi_+ + \psi_0 + \lambda^{-1} \psi_-, \]
obtained by inserting the spectral parameter $\lambda$. This leads to the generalization of the $g$-valued classical Killing field $X$ to the loop algebra $g^C[[\lambda]]$-valued formal Killing field $X_\lambda$.

In practice, this amounts to expanding the scalar coefficients $\{p, b, c, f, a, g, s, t\}$ of $X$ to the appropriate formal series in $\lambda$.

We will find that this extension is also adapted so that the structure equation for the formal Killing field contains the higher-order version of the recursion relations discussed in §3.2.3. We will be able to read off the higher-order (pseudo) Jacobi fields and conservation laws from the structure equation for $X_\lambda$.

As a first step to the higher-order analysis, we begin by introducing the infinite prolongation of the differential system $(\tilde{X}, \tilde{I})$, and record the basic structure equations.

4. Prolongation

The minimal Lagrangian surfaces under consideration are locally described by the elliptic Tzitzeica equation, which is a well known example of an integrable elliptic equation. It possesses an infinite sequence of higher-order symmetries and conservation laws, [10]. In order to access the corresponding structures for the minimal Lagrangian system, it becomes necessary to introduce the infinite prolongation.

In §4.1 we determine the basic structure equations for the infinite prolongation. The commutation relations for the dual frame of vector fields will be used in the next sections for the classification of Jacobi fields. In §4.2 we define the triple cover of the infinite prolongation to support the triple cover of a minimal Lagrangian surface defined by
Hopf differential. In §4.3, we introduce a sequence of adapted functions called balanced coordinates and record their basic structure equations.

In hindsight, the hidden symmetries of the minimal Lagrangian system begin to emerge in the form of a weighted homogeneity with respect to the balanced coordinates. To be precise, this is a non-local symmetry associated with an integrable extension. We do not pursue to formulate this properly in this paper, and the notion of weighted homogeneity would suffice for our purposes. We mention that this is a part of an infinite hierarchy of non-local symmetries called spectral symmetry.

4.1. Infinite prolongation.

4.1.1. Infinite sequence of $\mathbb{P}^1$-bundles. Set $(X^{(0)}, I^{(0)}) = (X, I)$ be the original differential system. Inductively define $(X^{(k+1)}, I^{(k+1)})$ as the prolongation of $(X^{(k)}, I^{(k)})$ such that

$$\pi_{k+1,k} : X^{(k+1)} \to X^{(k)}$$

is the bundle of oriented integral 2-planes of $(X^{(k)}, I^{(k)})$, and that the differential ideal $I^{(k+1)}$ is generated by $\pi_{k+1,k}^* I^{(k)}$ and the restriction of the canonical contact ideal to $X^{(k+1)} \subset \text{Gr}^+ (2, TX^{(k)})$.

For each $k \geq 0$, the prolongation space $X^{(k+1)}$ is a smooth manifold. The projection $\pi_{k+1,k}$ is a smooth submersion with two dimensional fibers isomorphic to $\mathbb{CP}^1$, see §4.2 for further details.

**Definition 4.1.** The infinite prolongation $(X^{(\infty)}, I^{(\infty)})$ of the differential system $(X, I)$ for minimal Lagrangian surfaces is defined as the projective limit

$$\begin{align*}
X^{(\infty)} &= \lim_{\leftarrow} X^{(k)}, \\
I^{(\infty)} &= \bigcup_{k \geq 0} I^{(k)}.
\end{align*}$$

Let $\pi_{\infty,k} : X^{(\infty)} \to X^{(k)}$ be the associated projection. Here we identify $I^{(k)}$ with its image $\pi_{\infty,k}^* I^{(k)} \subset \Omega^*(X^{(\infty)})$. By construction, the sequence of Pfaffian systems $I^{(k)}$ satisfy the inductive closure conditions

$$dI^{(k)} \equiv 0 \mod I^{(k+1)}, \ k \geq 1.$$

4.1.2. Associated principal $\mathbb{SO}(2)$-bundles. For the purpose of computation, we also introduce the following associated bundles.

Consider the principal $\mathbb{SO}(2)$-bundle $\Pi : \mathcal{F} \to X = \mathcal{F}/\mathbb{SO}(2)$. Set $\mathcal{F}^{(k)} = \Pi^* X^{(k)}$ for $k \geq 1$, and define

$$\mathcal{F}^{(\infty)} = \lim_{\leftarrow} \mathcal{F}^{(k)}.$$ 

The higher-order differential analysis will be practically carried out on the principal $\mathbb{SO}(2)$-bundle

$$\mathcal{F}^{(\infty)} \to X^{(\infty)}$$

in an $\mathbb{SO}(2)$-equivariant manner so that it has a well defined meaning on $X^{(\infty)}$. For simplicity, we continue to use $I^{(k)}, I^{(\infty)}$ to denote the corresponding differential ideals on $\mathcal{F}^{(k)}, \mathcal{F}^{(\infty)}$. 

4.1.3. Zariski open sets. Let $X^{(1)}_0 \subset X^{(1)}$ be the open subset defined by the independence condition

$$X^{(1)}_0 = \{ E \in X^{(1)} \mid (\xi \wedge \bar{\xi})_{\mid E} \neq 0 \}. \tag{31}$$

For $k \geq 1$, inductively define the corresponding sequence of open subsets

$$X^{(k+1)}_0 = \pi_{k+1}^{-1}(X^{(k)}_0). \tag{32}$$

Let

$$\lim \leftarrow X^{(k)}_0 = X^{(\infty)}_0 \subset X^{(\infty)}$$

denote their projective limit. Let $F^{(k)}_0 \subset F^{(k)}, F^{(\infty)}_0 \subset F^{(\infty)}$ denote the associated open subsets.

Dually, consider the complementary open subset

$$X^{(1)}_\infty = \{ E \in X^{(1)} \mid (\eta_2 \wedge \bar{\eta}_2)_{\mid E} \neq 0 \}. \tag{33}$$

Its associated open subsets $X^{(k)}_\infty, X^{(\infty)}_\infty, F^{(k)}_\infty, F^{(\infty)}_\infty$ are defined similarly as above. Note that the pair $[X^{(\infty)}_0, X^{(\infty)}_\infty]$ form an open covering of $X^{(\infty)}$ (and similarly for $[F^{(\infty)}_0, F^{(\infty)}_\infty]$ and $F^{(\infty)}_\infty$).

The analysis of the infinitely prolonged differential system for minimal Lagrangian surfaces will be carried out on $F^{(\infty)}_0$, which is one half of the open covering of $F^{(\infty)}$. But we remark that it can be equally well carried out on the other half of the open covering $F^{(\infty)}_\infty$. Since the formulation of the structure equation on $F^{(\infty)}_\infty$ is almost identical to that on $F^{(\infty)}_0$, we shall present only the $F^{(\infty)}_0$-part of the analysis and omit the $F^{(\infty)}_\infty$-part.

Let us give instead an indication on how the dual prolongation process on $F^{(1)}_\infty$ would proceed. Given the 2-form $\eta_2 \wedge \xi$ in the ideal $I$, recall the prolongation variable $h_3$ defined by (14). Introduce the new prolongation variable $p_3$ such that

$$\xi = p_3 \eta_2$$

on the integral 2-planes in $F^{(1)}_\infty$ (and hence $p_3 = h_3^{-1}$ on $F^{(1)}_0 \cap F^{(1)}_\infty$). By switching from $h_3$ to $p_3$, it is clear that one may proceed analogously to the infinite prolongation on $F^{(\infty)}_\infty$, such that the associated formulas agree on the intersection $F^{(\infty)}_0 \cap F^{(\infty)}_\infty$.

4.1.4. Infinitely prolonged structure equation. Recall the induced structure equation (12) on an immersed minimal Lagrangian surface. By the standard prolongation process of the exterior differential system theory, [4, Chapter VII], set

$$\eta_2 = h_3 \xi$$

for a prolongation variable $h_3$. From the third equation of (12), we get

$$dh_3 + 3i h_3 \rho \equiv 0 \mod \xi.$$  

Inductively define the higher-order derivatives of $h_3$ by the equations

$$dh_j + ij h_j \rho = h_{j+1} \xi + T_j \bar{\xi}, \quad j = 3, 4, \ldots., \tag{33}$$
where

\[(34)\]

\[T_3 = 0,\]

\[T_{j+1} = \sum_{s=0}^{j-3} a_{j,s} h_{j-s} \partial_\xi R, \quad \text{for} \quad j \geq 3,\]

\[a_{j,s} = \frac{(j + 2s + 3)(j - 1)}{2(j - 1)} \binom{s + 2}{j},\]

\[\partial_\xi R = \delta_0 \gamma^2 - 2h_{3+s}^3 + h_3.\]

The formula for the sequence \(\{T_{j+1}\}\) is uniquely determined by requiring that \(d(dh_{j}) = 0\) for \(j = 3, 4, \ldots\). This implies the recursive relation

\[(35)\]

\[T_{j+1} = \partial_\xi T_j + \frac{j}{2} \partial_\xi h_j.\]

For example,

\[T_3 = 0, \quad T_4 = \frac{3}{2} h_3(\gamma^2 - 2h_3^3), \quad T_5 = \frac{7}{2} \gamma^2 h_4 - 10h_3^3 h_4.\]

**Remark 4.1.** Similarly as in (19), we use the subscript notation \(f_\xi\) or \(\partial_\xi f\) to denote the \(\xi\)-coefficient of \(df\) (and similarly for \(f_\overline{\xi}\) or \(\partial_\overline{\xi} f\)).

Based on Eqs. (33), (34), we define the following set of differential forms and vector fields on \(\mathcal{F}_{0}^{(\infty)}\):

a) **Differential forms:**

\[(36)\]

\[\eta_j = dh_j + \text{i}jh_j \rho \quad \text{for} \quad j \geq 3,\]

\[\theta_j = \eta_j - h_{j+1} \xi, \quad \text{for} \quad j \geq 2.\]

On the open subset \(\mathcal{F}_{0}^{(k)}\), the set of 1-forms

\[\{\rho; \xi, \overline{\xi}, \theta_0, \theta_1, \overline{\theta}_1, \ldots, \theta_{k+1}, \overline{\theta}_{k+1}; \eta_{k+2}, \overline{\eta}_{k+2}\}\]

form a coframe. On the open subset \(\mathcal{F}_{0}^{(\infty)}\), the set of 1-forms

\[(37)\]

\[\{\rho; \xi, \overline{\xi}, \theta_0, \theta_1, \overline{\theta}_1, \theta_2, \overline{\theta}_2, \ldots\}\]

form a coframe. By construction,

\[I^{(\infty)} = \langle \theta_0, \theta_1, \overline{\theta}_1, \theta_2, \overline{\theta}_2, \ldots \rangle.\]

b) **Vector fields:**

\[\partial_\xi := \text{total derivative with respect to } \xi \mod I^{(\infty)},\]

\[\partial_\overline{\xi} := \text{total derivative with respect to } \overline{\xi} \mod I^{(\infty)}.\]

On the open subset \(\mathcal{F}_{0}^{(\infty)}\), let

\[(38)\]

\[\{E_\rho; \partial_\xi = E_\xi, \partial_\overline{\xi} = E_{\overline{\xi}}, E_0, E_1, E_2, E_2, \ldots\}\]

denote the dual frame of (37).
In terms of the 1-forms introduced above, we denote the covariant derivatives of $T_j$ by
\[
d T_j + i(j-1)T_j \rho = (\partial_\xi T_j) \xi + (\partial_\bar{\xi} T_j) \bar{\xi} + [T_j^3 \bar{\theta}_3 + \sum_{s=3}^{j-1} T_j^s \theta_s], \quad j \geq 3,
\]
where $T_j^s = E_s(T_j)$.
Recall from (10),
\[
d \xi = i \rho \wedge \xi - 3 \gamma^2 \theta_0 \wedge \bar{\theta}_1 - \theta_1 \wedge \theta_2 - \bar{h}_3 \theta_1 \wedge \bar{\xi},
\]
\[
d \theta_0 = -\frac{1}{2} \left( \theta_1 \wedge \bar{\xi} + \bar{\theta}_1 \wedge \xi \right),
\]
\[
d \theta_1 + i \rho \wedge \theta_1 = -\theta_2 \wedge \xi + 3 \gamma^2 \theta_0 \wedge \bar{\xi},
\]
\[
d \theta_2 + 2i \rho \wedge \theta_2 = -\theta_3 \wedge \xi + 3h_3 \gamma^2 \theta_0 \wedge \bar{\theta}_1 + h_3 \bar{\theta}_1 \wedge \bar{\theta}_2 + (\gamma^2 + h_3 \bar{h}_3) \theta_1 \wedge \bar{\xi},
\]
\[
d \rho = \frac{i}{2} \left( R \xi \wedge \bar{\xi} - 2 \theta_2 \wedge \bar{\theta}_2 - \gamma^2 \theta_1 \wedge \bar{\theta}_1 - 2 \bar{h}_3 \theta_2 \wedge \bar{\xi} + 2h_3 \bar{\theta}_2 \wedge \xi \right).
\]
Extending this, we record the structure equations satisfied by the 1-forms $\theta_j$ on $F_0^{(\infty)}$.

**Lemma 4.2.** For $j \geq 3$,
\[
d \theta_j + ij \rho \wedge \theta_j = -\theta_{j+1} \wedge \xi + 3 \gamma^2 \theta_0 \wedge (T_j \theta_1 + h_{j+1} \bar{\theta}_1) + \frac{j h_j}{2} \left( \gamma^2 \theta_1 \wedge \bar{\theta}_1 + 2 \theta_2 \wedge \bar{\theta}_2 \right)
+ T_j \bar{\theta}_1 \wedge \theta_2 + h_{j+1} \theta_1 \wedge \bar{\theta}_2 + \tau_j' \wedge \xi + \tau_j'' \wedge \bar{\xi}, \quad \text{where}
\]
\[
\tau_j' = h_3 T_j \bar{\theta}_1 - j h_j h_j \bar{\theta}_2,
\]
\[
\tau_j'' = \bar{h}_3 h_{j+1} \theta_1 + j h_j h_j \bar{\theta}_2 - (T_j \bar{\theta}_3 + \sum_{s=3}^{j-1} T_j^s \theta_s).
\]

**Proof.** Direct calculation. We omit the details. \qed

Eqs. (39), (40) imply the following commutation relation for the dual frame.

**Corollary 4.3.** The dual frame (38) satisfies the following commutation relations.
\[
[E_\ell, E_{\bar{\ell}}] = \sum_{j \geq \ell + 1} T_j^\ell E_j, \quad \ell \geq 3.
\]

**Proof.** Let $\theta$ be a differential 1-form, and let $E_a$, $E_b$ be vector fields. The corollary follows from Cartan’s formula
\[
d \theta(E_a, E_b) = E_a(\theta(E_b)) - E_b(\theta(E_a)) - \theta([E_a, E_b]).
\]
\qed

We introduce the following notations in order to keep track of orders ($k \geq 2$):
- $\cal{O}(k)$: functions on $X^{(\infty)}$ that do not depend on $h_j$ for $j > k$.
- $\cal{O}(-k)$: functions on $X^{(\infty)}$ that do not depend on $\bar{h}_j$ for $j > k$. 
Note for example that 
\[ T_j \in O(j - 1). \]

4.2. **Triple cover.**

4.2.1. **Set up.** The triple cover \( \hat{\Sigma} \to \Sigma \) of an immersed integral surface of \((X, I)\) defined in Defn. 2.3 prompts the definition of a global triple cover \( \hat{X}^{(\infty)} \to X^{(\infty)} \) such that the following commutative diagram holds:

\[
\begin{array}{ccc}
\hat{\Sigma} & \longrightarrow & \hat{X}^{(\infty)} \\
\downarrow & & \downarrow \\
\Sigma & \longrightarrow & X^{(\infty)}
\end{array}
\]

It turns out that the triple cover \( \hat{X}^{(\infty)} \) to be constructed also supports the splitting field for the (pseudo) Jacobi equation for the minimal Lagrangian system, see §6.

4.2.2. **Definition.** Let

\[ K^F \to \mathcal{F}, \quad K_{\eta_2}^F \to \mathcal{F}, \]

be the (trivial) complex line bundles generated by the 1-forms \( \xi, \eta_2 \) respectively. Recall the principle \( \text{SO}(2) \)-bundle \( \Pi : \mathcal{F} \to X \), and the structure equation (10). Let

\[ K \to X, \quad K_{\eta_2} \to X, \]

be the induced line bundles respectively. Note from Eq. (10) that \( K_{\eta_2} = K^{-2} \).

Consider \( \mathcal{F}^{(1)} = \Pi^*(X^{(1)}) \). Since an element in \( \mathcal{F}^{(1)} \) is defined by the equation

\[ \eta_2 - h_3 \xi = 0 \]

for a coefficient \( h_3 \), we have

\[ \mathcal{F}^{(1)} \cong \mathbb{P}(K^F \oplus K_{\eta_2}^F) \cong \mathcal{F} \times \mathbb{C}P^1. \]

It follows that

\[ X^{(1)} \cong \mathbb{P}(K^{p+1} \oplus K^{p-2}) \]

for any integer \( p \). In order to define the triple cover, we choose \( X^{(1)} \cong \mathbb{P}(K^3 \oplus \mathbb{C}) \).

**Definition 4.2.** The **triple cover** \( \hat{X}^{(1)} \) of the first prolongation \( X^{(1)} \) is defined by

\[ \hat{X}^{(1)} = \mathbb{P}(K \oplus \mathbb{C}). \]

Define similarly \( \hat{\mathcal{F}}^{(1)} = \mathbb{P}(K^F \oplus \mathbb{C}) \).

Denote the triple covering map

\[ c : \hat{\mathcal{F}}^{(1)} \to \mathcal{F}^{(1)}, \]

which is the standard branched triple cover when restricted to the fibers;

\[ \mathbb{C}P^1 \to \mathbb{C}P^1, \]

\[ [x, y] \mapsto [x^3, y^3]. \]
It is branched at the two points $0 = [0, 1]$ and $\infty = [1, 0]$. By construction, $c$ descends to the triple covering map

$$c : \hat{X}^{(1)} \to X^{(1)}.$$  

Let $\hat{I}^{(1)} = c^*I^{(1)}$ be the pulled back ideal, and define the new differential system

$$(\hat{X}^{(1)}, \hat{I}^{(1)}).$$

It is clear that the present construction has the naturality so that the triple cover of a minimal Lagrangian surface $\hat{\Sigma} \to \Sigma$ admits a unique lift to $\hat{X}^{(1)}$ as an integral surface of $\hat{I}^{(1)}$.

**Definition 4.3.** For each $2 \leq k \leq \infty$, define the **triple cover** of the prolongations

$$\hat{X}^{(k)} := c^*(X^{(k)}),$$

$$\hat{F}^{(k)} := c^*(F^{(k)}),$$

such that we have the commutative diagram:

$$\begin{array}{ccc}
\hat{X}^{(k)} & \xrightarrow{c^*} & X^{(k)} \\
\downarrow{\pi_{k,0}} & & \downarrow{\pi_{k,0}} \\
\hat{X}^{(1)} & \xrightarrow{c} & X^{(1)}
\end{array}$$

(and similarly for $\hat{F}^{(k)}, \hat{F}^{(1)}$, etc). Set the ideals $\hat{I}^{(k)} := c^*_x(I^{(k)})$. Denote the pulled-back canonical bundles by

$$\pi_{k,0}^*K := K \to X^{(k)},$$

$$c^*_xK := \hat{K} \to \hat{X}^{(k)}.$$

The following proposition summarizes the construction so far.

**Proposition 4.4.** Let $x : \Sigma \hookrightarrow X$ be an immersed integral surface of the differential system for minimal Lagrangian surfaces. Let

$$x^{(k)} : \Sigma \hookrightarrow X^{(k)}, \ 1 \leq k \leq \infty,$$

be the prolongation of $x$. Let $\nu : \hat{\Sigma} \to \Sigma$ be the triple cover defined by the Hopf differential of $x$, **Definition 2.3**. There exists a lift $\hat{x}^{(1)} : \hat{\Sigma} \hookrightarrow \hat{X}^{(1)}$ and the associated sequence of prolongations

$$\hat{x}^{(k)} : \hat{\Sigma} \hookrightarrow \hat{X}^{(k)}, \ 2 \leq k \leq \infty,$$

such that

a) each $\hat{x}^{(k)}$ is integral to $\hat{I}^{(k)},$

b) $x^{(k)} \circ \nu = c_k \circ \hat{x}^{(k)}$.

The lift $\hat{x}^{(1)}$ and its prolongation sequence $\{\hat{x}^{(k)}\}$ are uniquely determined by these properties.
4.3. Balanced coordinates. As is often the case with integrable differential equations, the infinite prolongation of the minimal Lagrangian system supports a preferred ring of functions called balanced coordinates. We will find that many of the higher-order objects we are interested in, such as higher-order Jacobi fields and conservation laws, admit the weighted homogeneous expressions in terms of these functions.

4.3.1. Set up. Recall the open subsets $X_0^{(k)}, X_\infty^{(k)}$, $1 \leq k \leq \infty$, from (31) and below. Let $X_1^{(1)} \subset X^{(1)}$ be their intersection,

$$X_1^{(1)} := X_0^{(1)} \cap X_\infty^{(1)} = \{ E \in X^{(1)} \mid h_3|E| \neq 0, \infty \}.$$ 

Inductively define the sequence of open subsets $X_1^{(k+1)} := \pi_{k+1}^{-1}(X_1^{(k)}) \subset X^{(k+1)}$. Let

$$X_1^{(k)} := X_1^{(k)} \cap \{ h_j \neq \infty, 4 \leq j \leq k + 2 \} \subset X_1^{(k)}, \quad k \geq 2.$$ 

Denote the corresponding open subsets on the triple cover by

$$\hat{X}_1^{(k)}, \hat{X}_\infty^{(k)}, \hat{X}_1^{(k)}, \hat{X}_\infty^{(k)}, \text{etc.}$$

The balanced coordinates to be constructed will be defined on $\hat{X}_\infty^{(\infty)}$.

Let $\{F_0^{(k)}, F_\infty^{(k)}, F_s^{(k)}, F_*^{(k)}, F^0_0, F^0_\infty, F^0_*\}$ denote the associated open sets.

4.3.2. Definition. With this preparation, we introduce a preferred set of functions on $\hat{X}_\infty^{(\infty)}$, which are adapted for the prolongation structure of the minimal Lagrangian system.

**Definition 4.4.** A sequence of balanced coordinates $z_j : \hat{X}_\infty^{(\infty)} \to \mathbb{C}$, $j \geq 4$, are defined by

$$z_j := h_3^{-\frac{j}{2}} h_j.$$ 

It is clear that these functions, originally defined on $\hat{X}_\infty^{(\infty)}$, are invariant under the action of the structure group $SO(2)$. They are well defined on $\hat{X}_\infty^{(\infty)}$.

4.3.3. Structure equation. We record the structure equation for the balanced coordinates. Set

$$r := \frac{h_3}{n_3} \xi,$$

$$\omega := h_3^\frac{1}{2} \xi,$$

$$\zeta_j := h_3^{-\frac{j}{2}} \theta_j, \quad j \geq 0,$$

$$\hat{\theta}_j := h_3^{-\frac{j}{2}} T_j, \quad j \geq 3.$$ 

They are all invariant under the action of the structure group $SO(2)$ and hence well defined on $\hat{X}_\infty^{(\infty)}$. This enables to express the ideal $\hat{I}^{(\infty)}$ on $\hat{X}_\infty^{(\infty)}$ by

$$\hat{I}^{(\infty)} = \langle \theta_0, \zeta_j, \tilde{\zeta}_j \rangle,$$

and the generators are defined on $\hat{X}_\infty^{(\infty)}$ itself.

---

3They form a coordinate system when restricted to a fiber of the projection $\hat{X}_\infty^{(\infty)} \to \hat{X}^{(1)}$. 

Note the following structure equations:

\[ dr \equiv \frac{r}{2}(z_4 \omega + \overline{z_4 \omega}), \]

\[ dz_j \equiv \left( z_{j+1} - \frac{j}{3} z_4 z_j \right) \omega + \hat{T}_j r^{-\frac{j}{2}} \omega \mod \hat{I}^{(\infty)}, \quad \text{for } j \geq 4, \]

\[ \hat{T}_{j+1} = \partial_\omega \hat{T}_j + \frac{j-1}{3} z_4 \hat{T}_j + \frac{j}{2} (\gamma^2 - 2r^2) z_j, \quad \text{for } j \geq 3. \]

Here we set \( z_3 = 1 \) for convenience, and denote by \( \partial_\omega \) the operator

\[ \partial_\omega := h_3^{-\frac{1}{2}} \partial_\xi. \]

From these formulas, we note an important property of \( \hat{T}_j \).

**Definition 4.5.** The spectral weights of the balanced coordinates are

\[ \text{weight}(z_j) = j - 3, \quad \text{weight}(\overline{z}_j) = -(j - 3). \]

Set weight(\( r \)) = 0. Assign the weights for the 1-forms

\[ \text{weight}(\omega) = -1, \quad \text{weight}(\overline{\omega}) = +1. \]

**Lemma 4.5.** By definition,

\[ \hat{T}_j \in \mathbb{C}[r^2, z_4, z_5, z_6, \ldots]. \]

It is weighted homogeneous of weight \( j - 4 \) under the spectral weight (43).

**Proof.** From the identities

\[ \partial_\omega z_j = z_{j+1} - \frac{j}{3} z_4 z_j, \]

\[ \partial_\omega (r^2) = r^2 z_4, \]

the operator \( \partial_\omega \) increases the spectral weight by +1 when acting on \( \mathbb{C}[r^2, z_4, z_5, z_6, \ldots] \). Note the initial term \( \hat{T}_4 = \frac{3}{2} (\gamma^2 - 2r^2) \) is of weight 0. The rest follows from the inductive formula for \( \hat{T}_j \) in (42). \( \square \)

**4.4. Order vs. spectral weight.** The prolongation variable \( h_3 \) represents the second fundamental form of the minimal Lagrangian surface, and hence it is a second order object. Consequently, the jet order of the balanced coordinate \( z_j \) is \( j - 1 \).

However, for the sake of convenience, we re-define the order of the functions \( z_j, \overline{z}_j \) as follows:

\[ \begin{array}{c|c|c}
\hline
z_j & \text{order} & \text{spectral weight} \\
\hline
j & j - 3 & -(j - 3) \\
\overline{z}_j & j & -(j - 3) \\
\hline
\end{array} \]
5. Two lemmas

We record two useful lemmas regarding the $\partial_{\xi}$-equation on $\hat{X}^{(\infty)}$. Among other things, they will be applied to give a classification of the higher-order Jacobi fields in $\hat{X}^{(\infty)}$.

Lemma 5.1 is a variant of the fact that the minimal Lagrangian system is not Darboux integrable at any order. Lemma 5.3 is a rigid property of the polynomial ring $\mathbb{C}[z_4, z_5, z_6, ...]$ under the $\partial_{\xi}$-operator.

5.1. Lemma 5.1

Lemma 5.1. Let $f : U \subset \hat{X}^{(\infty)} \to \mathbb{C}$ be a scalar function on an open subset $U \subset \hat{X}^{(\infty)}$ such that

$$\partial_{\xi}f = c h_{\xi}^{-3}$$

for a constant $c$. From the structure equation, this implies that for some $k > 0$,

$$df \equiv c h_{\xi}^{-\frac{1}{3}} \xi \mod \xi, \theta_0, \theta_1, \theta_1, \theta_2, \theta_3, \theta_4, \ldots \theta_k,$$

i.e., such $f$ does not depend on any of the conjugate variables $\overline{h}_j, j \geq 3$. Then $c = 0$ necessarily and $f$ is a constant function.

Corollary 5.2. Let $f : U \subset \hat{X}^{(\infty)} \to \mathbb{C}$ be a scalar function on an open subset $U \subset \hat{X}^{(\infty)}$ such that

$$\partial_{\xi}f = 0.$$

Then $f$ is a constant function.

The corollary states that the infinitely prolonged minimal Lagrangian differential system is not Darboux integrable, see [6] for a discussion of Darboux integrability (in the hyperbolic case). Roughly, this means that no matter how many times one differentiates, the minimal Lagrangian system under consideration, $\gamma^2 \neq 0$, does not admit a Weierstraß type of holomorphic representation formula. This is in contrast with the case $\gamma^2 = 0$ where the minimal Lagrangian system is equivalent to the holomorphic differential system for complex curves in $\mathbb{C}^2$.

A proof of Lem. 5.1 can be obtained by a direct adaptation of the proof of the corresponding lemma for the differential system for constant mean curvature surfaces given in [12]. Although the analysis goes by straightforward computations, it is a little involved and, in order to avoid repetition, let us content ourselves with a brief description of the relevant ideas.

We shall apply induction on $k$. Assume $k \geq 5$. The case $k \leq 4$ can be checked by a direct computation.

Let $I$ be the Pfaffian system generated by

$$I = \langle \overline{\partial}_2, \overline{\partial}_1, \theta_0, \xi, \theta_1, \theta_2, ... \theta_k \rangle.$$

Our claim is that there is no nonzero closed 1-form $\alpha$ of the form

$$df = \alpha = c h_{\xi}^{-\frac{1}{3}} \xi + \theta, \quad \theta \in I.$$
Denote a 1-form in \( I \) by
\[
\theta = a_2 \theta_2 + a_1 \theta_1 + a_0 \theta_0 + a_2 \xi + a_1 \theta_1 + \sum_{j=2}^{k} a_j \theta_j.
\]

From the equation (which follows from the identity \( d^2 = 0 \))
\[
d\alpha \equiv 0 \mod I,
\]
collect \( \theta_3 \wedge \xi \)-terms and one gets
\[
a_2 = -\sum_{j=4}^{k} a_j T_j^3.
\]

Let \( I^{(1)} \) be the Pfaffian system generated by
\[
I^{(1)} = \langle \theta_1, \theta_0, \xi, \theta_1, \theta_2, \ldots \theta_k \rangle,
\]
where we now set
\[
\bar{\theta}_1 = \theta_1, \theta_0 = \theta_0, \theta_1 = \theta_1, \theta_2 = \theta_2, \theta_3 = \theta_3, \theta_j = \theta_j - T_j^2 \bar{\theta}_2, \quad \text{for } j \geq 4.
\]

Denote a (new) 1-form in \( I^{(1)} \) by
\[
\theta = a_1 \bar{\theta}_1 + a_0 \theta_0 + a_2 \xi + a_1 \theta_1 + \sum_{j=2}^{k} a_j \theta_j.
\]

Now our claim is that there is no nonzero closed 1-form \( \alpha \) of the form
\[
\alpha = ch_3^{-1} \xi + \theta, \quad \theta \in I^{(1)}.
\]

Repeat the similar computations and, by induction argument, solve for the sequence of coefficients \( \{a_1, a_0, a_2, a_1, a_2, \ldots \} \). Continuing this process, one arrives at the following normal form for the closed 1-form \( \alpha \):
\[
df = \alpha = c \left( h_3^{-1} \xi + f^2 \xi + \sum_{j=-2}^{k} f^j \theta_j \right), \quad (\theta_{-2} = \bar{\theta}_2, \text{ etc}),
\]
for the given constant \( c \), where
\[
f^j \in O(k - 1) \quad \text{for } j \geq 4.
\]

- Suppose \( c = 0 \). Then \( df = 0 \) and \( f \) is a constant.
- Suppose \( c \neq 0 \). We show that this leads to a contradiction. Applying the commutation relation (41),
\[
[E_k, \partial \xi] = \sum_{j \geq k+1} T_j^k E_j,
\]
to the given function \( f \), one gets
\[
\partial \xi (f^k) = E_k (ch_3^{-1} f) = 0.
\]
By the induction hypothesis, this implies that \( f^k \) is a constant multiple of \( h_3^{-\frac{k}{3}} \).

– Suppose \( f^k = 0 \). Then \( f \in O(k - 1) \) and, again by the induction hypothesis, \( f \equiv \text{constant} \). Hence \( c = 0 \), a contradiction.

– Suppose \( f^k \neq 0 \). Applying the commutation relation

\[
[E_{k-1}, \partial_\xi] = \sum_{j \geq k} T_{k-1,j}^1 E_j
\]

to the given function \( f \), one gets (since \( k \geq 5 \))

\[
-\partial_\xi (f^{k-1}) = T_{k}^{k-1} f^k.
\]

It is easily checked that, using the induction hypothesis, this forces \( f^k = 0 \), a contradiction.

5.2. Lemma 5.3. Consider the polynomial rings in the balanced coordinates,

\[
R := \mathbb{C}[z_4, z_5, z_6, ...] \quad \text{and} \quad \overline{R} := \mathbb{C}[\overline{z}_4, \overline{z}_5, \overline{z}_6, ...].
\]

Recall the spectral weights in Defn 4.5. Define the associated sequences of polynomial vector spaces filtered by the spectral weight as follows:

\[
\begin{align*}
\mathcal{P}_d & = \{ \text{weighted homogeneous polynomials of degree } d \geq 0 \text{ in } R \}, \\
\mathcal{P}_d & = \bigoplus_{i=0}^d P_i \subset \mathcal{R}, \\
\mathcal{P}_d(k) & = \mathcal{P}_d \cap O(k) \subset R \cap O(k), \\
\mathcal{Q}_d & = \mathcal{P}_d \oplus (h_3^3)P_{d+1}, \\
\mathcal{Q}_d & = \bigoplus_{i=0}^d Q_i, \\
\mathcal{Q}_d(k) & = \mathcal{Q}_d \cap O(k).
\end{align*}
\]

The following lemma, which is an application of Lem 5.1, records a characteristic rigidity property of the subspace \( \mathcal{P}_d(k) \) under the differential operator \( h_3^3 \partial_\xi \).

**Lemma 5.3.** Let \( v \in O(k), k \geq 4 \). Suppose

\[ h_3^\frac{1}{3} \partial_\xi v \in \mathcal{Q}_d(k). \]

Then

\[ v \in \mathcal{P}_{d+1}(k). \]

**Corollary 5.4.** Let \( u \in O(k), k \geq 4 \). Let \( u^k = E_k(u) = \frac{\partial u}{\partial h_3}. \) Suppose

\[ h_3^\frac{1}{3} \partial_\xi (h_3^\frac{1}{3} u^k) \in \mathcal{Q}_d(k). \]

Then \( h_3^\frac{1}{3} u^k \in \mathcal{P}_{d+1}(k) \), and hence

\[ u \in \mathcal{P}_{d+(k-2)}(k) \mod O(k - 1). \]

**Proof.** Substitute \( v = h_3^\frac{1}{3} u^k \) from Lem 5.3. \( \Box \)
Proof of Lem. 5.3 Consider the case $k = 4$. For functions in $O(4)$, we have the commutation relation $[E_4, \partial_\xi] = 0$. Hence by applying $E_4$ repeatedly to $\partial_\xi v$, we get

$$h_3^4 \partial_\xi E_4^m(v) = 0$$

for some $m \leq d + 1$. By Cor 5.2, $v$ is a polynomial in $z_4$ and one may write

$$v = v_m z_4^m + v_{m-1} z_4^{m-1} + \ldots + v_1 z_4 + v_0,$$

where $v_j \in O(3)$ with $v_m$ being a constant. Substitute this to the given equation for $h_3^4 \partial_\xi v$, and one gets the recursive equations

$$h_3^4 \partial_\xi v_j + (j + 1)v_{j+1} \frac{3}{2} R = c'_j \gamma^2 + c''_j h_3 \bar{h}_3, \quad j = m, m - 1, \ldots,$$

for constants $c'_j, c''_j$ (set $v_{m+1} = 0$). Since $v_m$ is a constant and

$$h_3^4 \partial_\xi z_4 = \frac{3}{2} R = \frac{3}{2} (\gamma^2 - 2 h_3 \bar{h}_3),$$

whereas $v_j \in O(3)$, an inductive argument using Lem. 5.1 for $j$ from $m - 1$ to 0 shows that all the coefficients $v_j$ must be constant. This shows

$$v \in P_{d+1}(4).$$

Applying the induction argument, suppose the claim is true up to $O(k - 1)$. Let $v \in O(k)$. For functions in $O(k)$, we have the commutation relation $[E_k, \partial_\xi] = 0$. Hence, similarly as above, $v$ is a polynomial in $z_k$ and one may write

$$v = v_m z_k^m + v_{m-1} z_k^{m-1} + \ldots + v_1 z_k + v_0,$$

where $v_j \in O(k - 1)$, $m(k - 3) \leq d + 1$, and $v_m$ being a constant. Substitute this to the given equation for $h_3^4 \partial_\xi v$, and one gets the recursive equations

$$h_3^4 \partial_\xi v_j + (j + 1)v_{j+1} \hat{T}_k \in Q_{d-j(k-3)}(k-1), \quad j = m, m - 1, \ldots.$$

By Lem. 4.5, $\hat{T}_k \in Q_{k-4}(k - 1)$. It follows by the similar inductive argument as above, with decreasing $j$ from $m - 1$ to 0, that

$$v_j \in P_{d-j(k-3)+1}(k-1), \quad m - 1 \geq j \geq 0.$$  

This shows $v \in P_{d+1}(k)$. \hfill \Box

6. Jacobi fields

From the general theory of differential equations, [20], a generating function of symmetry of a differential equation is characterized as an element in the kernel of its linearization. For the minimal Lagrangian system, it turns out that the generating functions of symmetries are Jacobi fields.
Since the normal bundle of a Lagrangian surface is canonically isomorphic to the cotangent bundle, the corresponding Jacobi operator (i.e., the defining equation for Jacobi fields) reduces to the second order operator
\[ \partial_\xi \partial_\xi + \frac{3}{2} \gamma^2. \]

In particular, Jacobi fields correspond to the eigenfunctions of Laplacian of the induced metric.

In this section, we give a complete classification of the (pseudo) Jacobi fields for the minimal Lagrangian system based on the results of §5. An infinite sequence of higher-order (pseudo) Jacobi fields will be constructed in the next section by the similar recursion procedure as discussed in §3.2.3.

In §6.1, we show that a Jacobi field is a generating function of symmetry for the minimal Lagrangian system. In §6.2, we prove a splitting theorem that a Jacobi field decomposes as the sum of a classical Jacobi field and a higher-order Jacobi field. Combining this with the results from [11, 10], we obtain a complete classification of Jacobi fields in §6.3. We find that a higher-order (≥ 5) Jacobi field exists at each odd order 5, or 1 mod 6.

The similar analysis applies to the classification of pseudo-Jacobi fields. A pseudo-Jacobi field corresponds to a symmetry of the elliptic Tzitzeica equation underlying the minimal Lagrangian system. We find that a higher-order (≥ 4) pseudo-Jacobi field exists at each even order 4, or 2 mod 6.

6.1. Jacobi fields and pseudo-Jacobi fields.

6.1.1. Definition. Motivated by Defns 3.3, 3.4, we give a general definition of (pseudo) Jacobi fields on \( \hat{X}^{(\infty)} \).

**Definition 6.1.** A scalar function \( A : \hat{X}^{(\infty)} \to \mathbb{C} \) is a **Jacobi field** if it satisfies the linear Jacobi equation
\[ \mathcal{E}(A) := \partial_\xi \partial_\xi A + \frac{3}{2} \gamma^2 A = 0. \]

The \( \mathbb{C} \)-vector space of Jacobi fields is denoted by \( \mathcal{J}^{(\infty)} \).

A scalar function \( P : \hat{X}^{(\infty)} \to \mathbb{C} \) is a **pseudo-Jacobi field** if it satisfies the linear pseudo-Jacobi equation
\[ \mathcal{E}'(P) := \partial_\xi \partial_\xi P + \frac{1}{2} (\gamma^2 + 4 h^T h) P = 0. \]

The \( \mathbb{C} \)-vector space of pseudo-Jacobi fields is denoted by \( \mathcal{J}'^{(\infty)} \).

A higher-order **Jacobi field** is a Jacobi field in the ring of the balanced coordinates \( R \oplus \overline{R} \). The subspace of higher-order Jacobi fields is denoted by \( \mathcal{J}^{(\infty)}_h \).

\[ \text{It turns out that the infinite sequence of higher-order (pseudo) Jacobi fields belong to } R \oplus \overline{R} \subset C^{\infty}(\hat{X}^{(\infty)}). \]
Let $\mathcal{J}^{(k)} \subset \mathcal{J}^{(\infty)}$ be the subspace of Jacobi fields of order $\leq k + 2$ defined on $\hat{X}^{(k)}$. Let $\mathcal{J}_h^{(k)} = \mathcal{J}^{(k)} \cap \mathcal{J}_h^{(\infty)}$. The space of Jacobi fields of order $k + 2$ is defined as the quotient space
\[
\mathcal{J}^k = \mathcal{J}^{(k)}/\mathcal{J}^{(k-1)}, \quad k \geq 1,
\]
\[
\mathcal{J}^0 = \mathcal{J}^{(0)} = \{ \text{classical Jacobi fields} \}.
\]

The corresponding subspaces of pseudo-Jacobi fields are denoted by $\mathcal{J}_h^{(\infty)}$, $\mathcal{J}^{(k)}$, $\mathcal{J}_h^{(k)}$, $\mathcal{J}^k$, $\mathcal{J}^{(0)}$.

6.1.2. Stability lemma. Before we proceed to the analysis of Jacobi fields and symmetries, let us record the useful formulas related to the stability of the ring of balanced coordinates under the Jacobi operators $E, E'$.

**Lemma 6.1.** Consider the polynomial ring $\mathcal{R} = \mathbb{C}[z_4, z_5, ...]$ of balanced coordinates. We have,
\[
E(z_j) = \hat{T}_{j+1} - \frac{j}{3} (z_j \hat{T}_4 + z_4 \hat{T}_j) + \frac{3}{2} \gamma^2 z_j,
\]
\[
E'(z_j) = \hat{T}_{j+1} - \frac{j}{3} (z_j \hat{T}_4 + z_4 \hat{T}_j) + \frac{1}{2} (\gamma^2 + 4r^2) z_j.
\]
It follows that
\[
E(\mathcal{R}), E'(\mathcal{R}) \subset \mathcal{R} \oplus (h_3 \mathcal{R}).
\]
In particular, the operators $E, E'$ preserve the spectral weight when acting on $\mathcal{R}$.

Note $\hat{T}_4 = \frac{3}{2}(\gamma^2 - 2r^2)$.

6.1.3. Jacobi field and symmetry. It was observed in §3 that there exists an isomorphism between the classical Jacobi fields and the classical symmetries. The higher-order analogue of this isomorphism is true, and a Jacobi field uniquely determines a (vertical) symmetry vector field of $(\hat{X}^{(\infty)}, \hat{I}^{(\infty)})$.

Consider the coframe of $\hat{F}^{(\infty)}$,
\[
\{ \rho, \xi, \bar{\xi}, \theta_0, \theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2, ... \},
\]
and its dual frame
\[
\{ E_\rho, E_\xi, E_{\bar{\xi}}, E_{\theta_0}, E_{\theta_1}, E_{\bar{\theta}_1}, E_{\theta_2}, E_{\bar{\theta}_2}, ... \}.
\]
By a vector field on $\hat{X}^{(\infty)}$, we mean a vector field (derivation) on $\hat{F}^{(\infty)}$ of the form
\[
V = V_\xi E_\xi + V_\bar{\xi} E_{\bar{\xi}} + V_0 E_0 + \sum_{j=1}^{\infty} (V_j E_j + V_{\bar{j}} \bar{E}_j)
\]
which is invariant under the action of the structure group $SO(2)$ of the principal bundle $\hat{F}^{(\infty)} \to \hat{X}^{(\infty)}$. We denote the set of vector fields on $\hat{X}^{(\infty)}$ by $H^0(T\hat{X}^{(\infty)})$.

**Definition 6.2.** A vector field $V \in H^0(T\hat{X}^{(\infty)})$ is a symmetry of the differential system $(\hat{X}^{(\infty)}, \hat{I}^{(\infty)})$ if the formal Lie derivative $L_V$ preserves the ideal $\hat{I}^{(\infty)}$,
\[
L_V \hat{I}^{(\infty)} \subset \hat{I}^{(\infty)}.
\]
The \( \mathbb{C} \)-algebra of symmetry vector fields is denoted by \( \mathfrak{S} \).

A symmetry \( V \in \mathfrak{S} \) is **vertical** when \( V_\xi = V_\overline{\xi} = 0 \) and it has no (horizontal) \( E_\xi, E_\overline{\xi} \) components. The subspace of vertical symmetries is denoted by \( \mathfrak{S}_v \).

We wish to give an analytic characterization of symmetry. Consider a vertical symmetry

\[
V = V_0 E_0 + \sum_{j=1}^{\infty} (V_j E_j + V_j \overline{E}_j).
\]

We claim that \( V_0 \) is necessarily a Jacobi field, and that it is the generating function of symmetry in the sense that \( V_j, V_j' \)’s are determined by \( V_0 \) and its successive derivatives.

**Step 0.** The condition that the Lie derivative \( \mathcal{L}_V \theta_0 \equiv 0 \mod \hat{I}^{(\infty)} \) shows that

\[
\text{A. As a consequence, the condition that the Lie derivatives } \mathcal{L}_V \theta_0, \mathcal{L}_V \overline{\theta}_0 \equiv 0 \mod \hat{I}^{(\infty)} \text{ shows that }
\]

\[
dV_0 - V_\mathcal{\mathcal{L}} \left( \frac{1}{2} (\theta_1 \wedge \xi + \overline{\theta}_1 \wedge \overline{\xi}) \right) \equiv dV_0 - \frac{1}{2} (V_1 \xi + V_1 \overline{\xi}) \equiv 0 \mod \hat{I}^{(\infty)}.
\]

One gets

\[
dV_0 \equiv \frac{1}{2} (V_1 \xi + V_1 \overline{\xi}) \mod \hat{I}^{(\infty)}.
\]

**Step 1.** By a similar computation, the condition that the Lie derivatives \( \mathcal{L}_V \theta_1, \mathcal{L}_V \overline{\theta}_1 \equiv 0 \mod \hat{I}^{(\infty)} \) shows that

\[
dV_1 + iV_1 \rho \equiv V_2 \xi - 3\gamma^2 V_0 \overline{\xi},
\]

\[
dV_1 - iV_1 \rho \equiv V_2 \overline{\xi} - 3\gamma^2 V_0 \xi, \quad \text{mod } \hat{I}^{(\infty)}.
\]

Note that Eq. (50) and Eq. (51) imply that \( V_0 \) is a Jacobi field.

**Step j.** The rest of the coefficients \( V_j, V_j', j \geq 2 \), are similarly determined by \( V_0 \) and its successive derivatives by the condition that the Lie derivatives \( \mathcal{L}_V \theta_j, \mathcal{L}_V \overline{\theta}_j \equiv 0 \mod \hat{I}^{(\infty)} \). These equations imply that

\[
dV_j + j iV_j \rho \equiv \left( V_{j+1} + [V_j, V_j, \ldots V_0] \right) \xi + [V_{j-1}, V_{j-1}, \ldots V_0] \overline{\xi},
\]

\[
dV_j - j V_j \rho \equiv \left( V_{j+1} + [V_j, V_j, \ldots V_0] \right) \overline{\xi} + [V_{j-1}, V_{j-1}, \ldots V_0] \xi, \quad \text{mod } \hat{I}^{(\infty)}.
\]

Here \( [V_j, V_j, \ldots V_0] \), etc, is a generic notation for an expression which is linear in \( \{ V_k, V_k \}_k \) with coefficients in the ring \( \mathbb{C}[h_3, \overline{h}_3, h_4, \overline{h}_4, \ldots] \).

In fact, the following converse of this analysis is true.

**Proposition 6.2.** The generating function of a vertical symmetry of the differential system for minimal Lagrangian surfaces is a Jacobi field. Conversely, a Jacobi field \( A \) uniquely determines a vertical symmetry \( V \) of the form (49) with the generating function \( V_0 = A \). As a consequence, there exists a canonical isomorphism

\[
\mathfrak{S}^{(\infty)} \simeq \mathfrak{S}_v.
\]

**Proof.** The compatibility of the recursive defining equations (50), (51), and (52) can be checked from the formula for \( T_j \) and its differential consequences. We omit the details. \( \square \)
6.1.4. \textit{Interpretation of pseudo-Jacobi field}. The pseudo-Jacobi fields correspond to the symmetries of the elliptic Tzitzeica equation underlying the minimal Lagrangian system. The symmetries of the elliptic Tzitzeica equation are classified in \cite{11,10}.

Away from the umbilic divisor on a minimal Lagrangian surface, take a local holomorphic coordinate $z$ such that

$$dz = h_3^4 \xi,$$

and the Hopf differential is normalized to $\mathbb{I} = (dz)^3$. Set accordingly,

$$\xi = e^u dz, \quad h_3 = e^{-\frac{u}{2}}$$

for a real scalar function $u$. The connection 1-form $\rho$ is given by $\rho = \frac{1}{2}(u_z dz - u_z d\bar{z})$, and it follows that the curvature $R$ is,

$$R = -2e^{-u} u_{\bar{z}z}.$$

The compatibility equation (15) then translates to the elliptic Tzitzeica equation

(53)

$$u_{\bar{z}z} + \frac{1}{2} \left( \gamma^2 e^u - 2e^{-2u} \right) = 0.$$ 

On the other hand, consider the pseudo-Jacobi operator (48). From $dz = h_3^4 \xi$, one gets $\partial_z = h_3^{-4} \partial_\xi$. Hence

$$\partial_\xi \partial_{\bar{z}} = e^{-u} \partial_z \partial_{\bar{z}},$$

and the pseudo-Jacobi operator translates to

$$\mathcal{E}' = e^{-u} \left( \partial_z \partial_{\bar{z}} + \frac{1}{2} \left( \gamma^2 e^u + 4e^{-2u} \right) \right).$$

Up to scaling by $e^{-u}$, this is the linearization of (53) and the claim follows.

6.1.5. \textit{Examples}.

\textbf{Example 6.3.} A direct computation show that

$$z_4$$

is a pseudo-Jacobi field (of order 4 and degree 1), and

$$z_5 - \frac{5}{3} z_4^2$$

is a Jacobi field (of order 5 and degree 2).

This example provides a hint for the existence of higher-order (pseudo) Jacobi fields.
6.2. **Splitting theorem.** For a nonlinear differential equation, it is a stringent condition to admit a higher-order symmetry, not to mention an infinite number of them. The corresponding Jacobi equation would imply a variety of compatibility conditions for the given differential equation, and generically one expects that the space of Jacobi fields is trivial.

In this section, we explore the constraints given by the Jacobi equation of the minimal Lagrangian system by a repeated application of Lem.5.1, Lem.5.3, and Cor.5.4. As a result, we obtain a rough normal form for Jacobi fields and this implies that the space of Jacobi fields splits into the direct sum of the classical Jacobi fields and the higher-order Jacobi fields.

A similar splitting theorem holds for the pseudo-Jacobi fields.

6.2.1. **Symbol lemma.** We start with a lemma on the normal form for the highest order term of a Jacobi field.

**Lemma 6.4.** Let \( A \in \mathcal{J}(k) \subset O(k + 2) \) be a (pseudo) Jacobi field. Suppose \( A^{k+2} = E_{k+2}(A) \neq 0 \). Then \( A \) is at most linear in the highest order variable \( z_{k+2} = h_3^{k+2} \), and, up to constant scale, it admits the normal form
\[
A = z_{k+2} + O(k + 1).
\]

**Proof.** Since \( A \in O(k + 2) \), we have
\[
A_{\xi} = h_{k+3}A^{k+2} \mod O(k + 2),
A_{\xi,\xi} = h_{k+3}\partial_{\xi}(A^{k+2}) \mod O(k + 2),
\]
\[
\equiv 0 \mod O(k + 2), \quad \text{for } \mathcal{E}(A) = 0 \quad \text{or} \quad \mathcal{E}'(A) = 0.
\]

This forces \( \partial_{\xi}(A^{k+2}) = 0 \). By Cor.5.2, \( A^{k+2} \) is a constant multiple of \( h_3^{k+2} \). \( \square \)

6.2.2. **Splitting theorem.** We refine Lem.6.4 to the splitting theorem for (pseudo) Jacobi fields with the help of Cor.5.4.

Recall \( \mathcal{R} = C[z_4, z_5, z_6, ...] \), \( \overline{\mathcal{R}} = C[z_4, \overline{z}_5, \overline{z}_6, ...] \).

**Proposition 6.5.** The space of Jacobi fields splits into the direct sum
\[
\mathcal{J}^{(\infty)} = \mathcal{J}_h^{(\infty)} \oplus \mathcal{J}^{(0)}
\]

of the higher-order Jacobi fields and the classical Jacobi fields. By definition,
\[
\mathcal{J}_h^{(\infty)} \subset \mathcal{R} \oplus \overline{\mathcal{R}}
\]
and the space of higher-order Jacobi fields is generated by the un-mixed weighted homogeneous polynomial Jacobi fields.

A similar splitting theorem holds for the space of pseudo-Jacobi fields.

**Proof.** Consider first the Jacobi field case. Let \( A \in O(k) \) be a Jacobi field for \( k \geq 5 \). By Lem.6.4 above, we may set
\[
A = z_k + u(1), \quad u(1) \in O(k - 1).
\]
Applying the Jacobi operator, one finds
\[ -E(z_k) \equiv h_3^2 \partial_z (h_3^{(k-1)} u_{(1)}^{k-1}) z_k \mod O(k - 1). \]
By Lem [6.1], \( E(z_k) \in Q_{k-3}(k) \). By Cor [5.4], one gets
\[ u(1) = p(1) + O(k - 2) \]
for some \( p(1) \in P_{k-3}(k - 1) \).
Suppose by induction we arrive at the formula
\[ A = z_k + p(1) + p(2) + \ldots + p(j) + u(j+1), \quad u(j+1) \in O(k - (j + 1)), \]
where each \( p(i) \in P_{k-3}(k - i) \) such that
\[ q(i) := E(z_k + p(1) + p(2) + \ldots + p(i)) \in O(k - i). \]
By Lem [6.1], one finds \( q(j) \in Q_{k-3}(k - j) \). Applying the Jacobi operator to the refined normal form (54), one gets
\[ -q(j) \equiv h_3^2 \partial_z (h_3^{(k-j)} u_{(j+1)}^{k-j}) z_{k-j} \mod O(k - (j + 1)). \]
By Cor [5.4], one may write
\[ u(j+1) = p(j+1) + u(j+2), \quad p(j+1) \in P_{k-3}(k - (j + 1)), \]
\[ u(j+2) \in O(k - (j + 2)). \]
Continuing this process, we arrive at the normal form
\[ A = p + u_{(k-3)}, \quad u_{(k-3)} \in O(3), \]
\[ p = z_k + p(1) + p(2) + \ldots + p(k-4), \]
where \( p(k-4) \in P_{k-3}(4) \) such that
\[ q(k-4) := E(p) \in Q_{k-3}(4). \]
Since \( E \) is a real operator, the complex conjugate of this argument implies that the Jacobi field \( A \) decomposes into
\[ A = f + g, \]
where \( f \) is an un-mixed pure polynomial in \( z_j, \overline{z}_j \), and \( g \) is a function on \( \hat{X}^{(1)} \).
Now, the Jacobi operator \( E \) preserves the spectral weight. Since \( E(g) \in O(4) \cap O(-4) \) is at most linear in \( z_4, \overline{z}_4 \), this implies (note \( E(z_4) = Rz_4 \),
\[ E(c'z_4 + c''\overline{z}_4) = R(c'z_4 + c''\overline{z}_4) = -E(g), \]
for some constants \( c', c'' \), where \( c'z_4 + c''\overline{z}_4 \) denotes the terms of spectral weight \( \pm 1 \) in the polynomial \( f \).
A short computation shows that this forces \( c', c'' = 0 \), and \( f \in R \oplus \overline{R} \) is a higher-order Jacobi field. Hence, Eq. (55) also implies that \( E(g) = 0 \) and consequently,
\[ h_3 E_3(g), \overline{h}_3 E_3(g) \equiv \text{const}. \]
It is easily checked from this that $E_3(g), E_3(g) = 0$ and $g$ is necessarily a classical Jacobi field defined on $X$.

Consider next the pseudo-Jacobi field case. By the similar argument as above, a pseudo-Jacobi field $P$ decomposes into

$$P = f + g,$$

where $f$ is an un-mixed pure polynomial in $z_{j}, \bar{z}_{j}$, and $g$ is a function on $\tilde{X}^{(1)}$. Since $E'(g) \in O(4) \cap O(-4)$ is at most linear in $z_4, \bar{z}_4$ and $z_4, \bar{z}_4$ are pseudo-Jacobi fields, it follows that

$$E'(f) = 0,$$

and hence $E'(g) = 0$.

By the similar argument as above, this implies that $g$ is necessarily a classical pseudo-Jacobi field. □

6.3. Classification. Combining the results of §§6.1, 6.2 we state a complete classification of (pseudo) Jacobi fields.

The classification consists of the following steps. We first show, by a direct computation, that there exist no even order Jacobi fields in $R \oplus \overline{R}$. This enables to apply the classification results for elliptic Tzitzeica equation in [11, 10], and we find that a nontrivial higher-order Jacobi field exists at order 5, or 1 mod 6 only. The infinite sequence of higher-order Jacobi fields of the admissible orders will be constructed in §7.

The classification of pseudo-Jacobi fields follows from §§6.1.4, Prop.6.5, and [11, 10].

**Theorem 6.6.** The infinitely prolonged differential system $(\tilde{X}^{(\infty)}, \tilde{I}^{(\infty)})$ for minimal Lagrangian surfaces in a $2\mathbb{C}$-dimensional, non-flat, complex space form admits an infinite sequence of higher-order (pseudo) Jacobi fields as follows.

a) There exists a unique (up to constant scale) nontrivial weighted homogeneous polynomial Jacobi field in $R$ of degree $d \geq 2$ for each

$$d \equiv 2, 4 \mod 6,$$

(hence of order $\equiv 5, 1 \mod 6$). The classical Jacobi fields, and these higher-order Jacobi fields and their complex conjugates generate the space of Jacobi fields $\mathcal{J}^{(\infty)}$.

b) There exists a unique (up to constant scale) nontrivial weighted homogeneous polynomial pseudo-Jacobi field in $R$ of degree $d \geq 2$ for each

$$d \equiv 1, 5 \mod 6,$$

(hence of order $\equiv 4, 2 \mod 6$). The classical pseudo-Jacobi fields, and these higher-order pseudo-Jacobi fields and their complex conjugates generate the space of pseudo-Jacobi fields $\mathcal{J}'^{(\infty)}$.

We present the proof of the theorem in the following two subsections.
6.3.1. No even order Jacobi fields. Recall $E(R) \subset R \oplus (h_3\bar{h}_3)R$. Let $F = \frac{3}{2}x_0z_{2k} + \ldots \in R$ be an even order weighted homogeneous polynomial Jacobi field of weight $2k - 3$ (for a constant $\frac{3}{2}x_0$). Consider the expansion,

\[ F = \frac{3}{2}x_0z_{2k} + z_{2k-1}(\begin{array}{c} x_1z_4 \\ x_2z_5 + y_2z_4^2 \\ x_3z_6 + y_3z_5z_4 + \ldots \\ x_4z_7 + y_4z_6z_4 + \ldots \end{array}) + \ldots + z_{k+3}(\begin{array}{c} x_{k-3}z_k \\ x_{k-2}z_{k+1} + y_{k-2}z_kz_4 + \ldots \\ x_{k-1}z_{k+1} + y_{k-1}z_{k+1}z_4 + \ldots \end{array}) \]

(56)

Here \( \{x_i, y_j\} \) are constant coefficients.

First, by considering the $z_{2k}$ term in $E(F)$, we get

\[ x_1 + (a_{2k,0} + \frac{3}{2} - \frac{2k}{3}a_{3,0})x_0 = 0. \]

(57)

In order to extract the compatibility equations imposed on the $x_i$-coefficients only, from now on we compute modulo the curvature $R = \gamma^2 - 2h_3\bar{h}_3$. It means that we identify

\[ h_3\bar{h}_3 \equiv \frac{\gamma^2}{2}. \]

Consider the Jacobi equation

\[ E(F) \mod R, \]

and collect the equations from the coefficients of the monomials $z_{2k}, z_{2k-1}z_4, z_{2k-2}z_5, \ldots$ up to $z_{k+2}z_{k+1}$, i.e. the terms with the coefficient $x_i$'s (let's call them the principal terms). It is easily checked that when acted on by the Jacobi operator, the terms not appearing in the above expansion do not have any contribution to the principal terms. Also, since $\partial_\omega z_4 \equiv 0 \mod R$, by computing mod $R$ we eliminate the contributions from the $y_j$ coefficients, as claimed above. It follows that one may evaluate

\[ E([\text{principal terms}]) \mod R, \]

and check only the principal terms in the image. This would yield a set of linear equations among the coefficients $x_i$'s.
Recall the following formulas:

\[ T_{j+1} = \sum_{s=0}^{j-3} a_{js} h_{j-s} \partial_{\xi}^s R, \quad \text{for } j \geq 3, \]

\[ a_{js} = \frac{(j + 2s + 3)(j - 1)}{2(j - 1)(s + 2)}, \]

\[ \partial_{\xi}^s R = \delta_{1s} \gamma^2 - 2\beta_{3+3}\beta_3, \]

\[ dz_j \equiv \left( z_{j+1} - \frac{j}{3} z_4 z_j \right) \omega + \hat{T}_j r^{-\frac{3}{2}} \omega \mod \hat{I}^{(\infty)}, \quad \text{for } j \geq 4. \]

Note that, for \( j \geq 4, \)

\[ \partial_{\xi} \partial_{\xi} (z_j) \equiv \hat{T}_{j+1} - \frac{j}{3} (z_4 \hat{T}_j) \mod R, \]

\[ \hat{T}_{j+1} \equiv -\gamma^2 a_{j,j-3} z_j + \) (quadratic terms in \( z_j \)'s) \mod R. \]

Since the principal terms except \( z_{3k} \) are quadratic in the balanced coordinates \( z_j \)'s, the term \( -\gamma^2 a_{j,j-3} z_j \) is the only form of contribution to the principal terms from \( \partial_{\xi} z_{j+1} \) when \( j + 1 < 2k \).

With this preparation, a direct computation yields the following formulas for the principal terms. We record only the relevant terms (here we set the scaling factor \( \gamma^2 = 1 \) temporarily for simplicity).

\[-E(z_{2k}) = (a_{2k,2k-3} - \frac{3}{2}) z_{2k} + (a_{2k,2k-4} + a_{2k,1} - \frac{2k}{3} a_{2k-1,2k-4}) z_4 z_{2k-1} \]

\[ + \sum_{j=2}^{k-2} (a_{2k,2k-j-3} + a_{2k,j}) z_{j+3} z_{2k-j}, \]

\[-E(z_4 z_{2k-1}) \equiv \left\{ \begin{array}{l}
(a_{4,1} + a_{2k-1,2k-4} - \frac{3}{2}) z_4 z_{2k-1}, \\
(a_{2k-2,2k-5}) z_5 z_{2k-2}, \\
\vdots
\end{array} \right. \]

\[-E(z_{j+3} z_{2k-j}) \equiv \left\{ \begin{array}{l}
(a_{j+3,j-1}) z_{j+2} z_{2k-(j-1)} \\
(a_{j+3,j} + a_{2k-j,2k-(j+3)} - \frac{3}{2}) z_{j+3} z_{2k-j}, \\
(a_{2k-(j+1),2k-(j+4)}) z_{j+4} z_{2k-(j+1)}, \\
\vdots
\end{array} \right. \]

\[-E(z_{k+1} z_{k+2}) \equiv \left\{ \begin{array}{l}
(a_{k,k-3}) z_k z_{k+3} \\
(2a_{k+1,k-2} + a_{k+2,k-1} - \frac{3}{2}) z_{k+1} z_{k+2}, \mod R. \\
\end{array} \right. \]
Note $a_{j+3,j} = \frac{3}{2}$ for all $j \geq 0$. Hence, except for the terms from $E(z_{2k})$ and the last term $z_{k+1}z_{k+2}$, all the terms have the equal coefficient $\frac{3}{2}$.

The resulting set of linear equations on the coefficients $x_j$'s are the following system of three term relations, including the initial equation \(57\).

\[
\begin{align*}
\cdot \cdot \cdot & \quad x_1 + (a_{2k,2k-3} + a_{2k,0} - k)x_0 = 0, \\
\cdot \quad x_1 & + x_2 + (a_{2k,2k-4} + a_{2k,1} - k)x_0 = 0, \\
x_1 & + x_2 + x_3 + (a_{2k,2k-5} + a_{2k,2})x_0 = 0, \\
\vdots & \\
x_{j-1} & + x_j + x_{j+1} + (a_{2k,2k-j-3} + a_{2k,j})x_0 = 0, \\
\vdots & \\
x_{k-4} & + x_{k-3} + x_{k-2} + (a_{2k,k} + a_{2k,k-3})x_0 = 0, \\
x_{k-3} & + x_{k-2} + x_{k-2} + (a_{2k,k-1} + a_{2k,k-2})x_0 = 0.
\end{align*}
\]

(58)

It is left to show that this system of $k-1$ linear equations on the set of $k-1$ coefficient \((x_0, x_1, \ldots, x_{k-2})\) has full rank. Set $t_j$ be the coefficient of $x_0$ of the $j$-th equation above, i.e.,

\[
t_0 = a_{2k,2k-3} + a_{2k,0} - k, \\
t_1 = a_{2k,2k-4} + a_{2k,1} - k, \\
t_j = a_{2k,2k-j-3} + a_{2k,j}, \quad \text{for } j \geq 2.
\]

A direct computation shows that the determinant $\chi_k$ of the $(k-1)$-by-$(k-1)$ matrix for the set of linear equations is given by

\[
\pm \chi_k = \sum_{j=0}^{k-2} \epsilon'_k t_j,
\]

where

\[
\epsilon'_k = \begin{cases} 
-2 & \text{when } j \equiv k \mod 3 \\
+1 & \text{otherwise}.
\end{cases}
\]

A computation mod 3 shows that

\[
\pm \chi_k = \begin{cases} 
\frac{1}{2} & \text{when } k \equiv 0 \mod 3 \\
1 & \text{otherwise}.
\end{cases}
\]

6.3.2. Proof of Thm. 6.6

b) By Prop. 6.5 and the analysis in §6.1.4 the higher-order pseudo-Jacobi fields correspond to the ‘Jacobi fields’ for the elliptic Tzitzeica equation studied in [11, 10]. It follows from the classification result in [10, Theorem 8.1].

a) From the analysis above, there exist no even order Jacobi fields.

We remark, without giving the proofs, that the higher-order analogues of the results in §§3.3, 3.4 are true: the space of (differentiated) conservation laws injects into the space of Jacobi fields under the natural symbol map given by the differential of the associated spectral sequence; see §8 Thus, Prop. 6.5 also implies the corresponding splitting theorem for the higher-order conservation laws.
In particular, a higher-order conservation law of the minimal Lagrangian system corresponds to a higher-order conservation law of the elliptic Tzitzeica equation. From this, the induction argument using the recursion operators $P, N$ in [10] shows that there are no Jacobi fields (for the minimal Lagrangian system) of degree $0 \mod 6$ (or of order $3 \mod 6$).

The sequence of Jacobi fields of the given degree $d \equiv 2, 4 \mod 6$ will be constructed in §7.

7. Formal Killing fields

Recall that the original differential system for minimal Lagrangian surfaces is defined on the bundle of Lagrangian planes $X \to M$. It is a 6-symmetric space associated with the Lie group $SL(3, \mathbb{C})$, and the minimal Lagrangian surfaces arise as the primitive harmonic maps. From the theory of integrable systems, this implies that the $g$-valued Maurer-Cartan form $\psi$, (16), admits an extension to the $g^C[\lambda^{-1}, \lambda]$-valued ($g^C = sl(3, \mathbb{C})$) extended Maurer-Cartan form $\psi_\lambda$, (60) in the below, by inserting the spectral parameter $\lambda$. The structure equation for the minimal Lagrangian system shows that $\psi_\lambda$ is compatible and satisfies the Maurer-Cartan equation.

In this section, we give a construction of the corresponding $g^C[[\lambda]]$-valued canonical formal Killing fields to utilize this aspect of symmetry of the minimal Lagrangian system. The construction relies on the pair of 3-step recursions between Jacobi fields and pseudo-Jacobi fields which are embedded in the structure equation for the formal Killing fields. We give the differential algebraic inductive formulas for the pair of formal Killing fields that correspond to two particular sets of initial data. As a consequence, we will be able to read off the infinite sequence of higher-order (pseudo) Jacobi fields and conservation laws from the components of the formal Killing fields.

In hindsight, the recursion relations were anticipated from the structure equation for the classical Killing field (18). Note that the 6-step recursion introduced in [10] is the union of these two 3-step recursions when translated to our setting.

In §7.1 we record the structure equation for the formal Killing fields with respect to the extended Maurer-Cartan form. In §7.2 we determine the first few terms of the formal Killing fields for the two initial ansätze given by Exam.6.3. The relevant observation is that the coefficient $h_3$ of Hopf differential provides the lower-end terms for the formal Killing fields, which allow one to truncate the terms of negative $\lambda$-degrees. With this preparation, we give in §7.3 the inductive formula for the formal Killing field for each set of the initial data.

7.1. Structure equation. In this section, we introduce the extended Maurer-Cartan form $\psi_\lambda$ and record the recursive structure equation (mod $\hat{I}^{(\infty)}$) for the coefficients of the $g^C[[\lambda]]$-valued formal Killing field with respect to $\psi_\lambda$.

---

5 The obstruction to the application of the recursion operators $P, N$ lies in the space of higher-order conservation laws of even weight. This vanishes by the results from [11, 10].
7.1.1. **Extended Maurer-Cartan form.** Consider the $g$-valued 1-form $\psi$, Eq. (16). Evaluating mod $\hat{I}^{(\infty)}$ (i.e., $\eta_2 = h_3 \xi, \bar{\eta}_2 = \bar{h}_3 \bar{\xi}$), it becomes

$$\psi = \psi_+ + \psi_0 + \psi_-,$$

where

$$\psi_- = \frac{1}{2} \begin{bmatrix} \cdot & -\gamma & i\gamma \\ \gamma & i h_3 & -h_3 \\ -i\gamma & -h_3 & -i h_3 \end{bmatrix} \xi,$$

$$\psi_0 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \rho \\ -\rho & \cdot & \cdot \end{bmatrix},$$

$$\psi_+ = \frac{1}{2} \begin{bmatrix} 0 & -\gamma & -i\gamma \\ \gamma & i \bar{h}_3 & \bar{h}_3 \\ i\gamma & \bar{h}_3 & -i \bar{h}_3 \end{bmatrix} \bar{\xi}.$$

Let $\lambda \in \mathbb{C}^*$ be the auxiliary spectral parameter.

**Definition 7.1.** The **extended Maurer-Cartan form** is the $g^C[\lambda^{-1}, \lambda]$-valued 1-form on $\hat{F}^{(\infty)}$ given by

$$(60) \quad \psi_\lambda := \lambda \psi_+ + \psi_0 + \lambda^{-1} \psi_-.$$

The extended 1-form $\psi_\lambda$ takes values in the Lie algebra $g$ when $\lambda$ is a unit complex number. It satisfies the structure equation

$$(61) \quad d\psi_\lambda + \psi_\lambda \wedge \psi_\lambda \equiv 0 \mod \hat{I}^{(\infty)}.$$

7.1.2. **Formal Killing field.** Recall the decomposition of the Lie algebra $g^C$ given in Fig. 2. By expanding each of the scalar coefficients $\{p, b, c, f, a, g, s, t\}$ as a series in $\mathbb{C}[[\lambda]]$, we give an abridged definition of the formal Killing fields associated with the extended Maurer-Cartan form.

**Definition 7.2.** Let $\psi_\lambda$, (60), be the extended Maurer-Cartan form. A **formal Killing field** is a function $\mathcal{X}_\lambda : \hat{F}^{(\infty)} \to g^C[[\lambda]]$, such that;

a) it satisfies the Killing field equation

$$(62) \quad d\mathcal{X}_\lambda + [\psi_\lambda, \mathcal{X}_\lambda] \equiv 0 \mod \hat{I}^{(\infty)},$$

We will find that the canonical formal Killing fields to be constructed are defined on the open subset $\hat{F}_{\ast}^{(\infty)} = \{h_3 \neq 0, \infty, \bar{h}_j \neq \infty \forall j \geq 4\} \subset \hat{F}^{(\infty)}$. 

b) its components are the formal series in $\lambda$ given explicitly by

$$
\begin{align*}
\mathbf{p} &= \sum_{k} b^{6k+4} \lambda^{6k+2}, \quad \mathbf{a} = \sum_{k} a^{6k+7} \lambda^{6k+5}, \\
\mathbf{b} &= \sum_{k} b^{6k+5} \lambda^{6k+3}, \quad \mathbf{g} = \sum_{k} g^{6k+2} \lambda^{6k}, \\
\mathbf{c} &= \sum_{k} c^{6k+5} \lambda^{6k+3}, \quad \mathbf{s} = \sum_{k} s^{6k+3} \lambda^{6k+1}, \\
\mathbf{f} &= \sum_{k} f^{6k+6} \lambda^{6k+4}, \quad \mathbf{t} = \sum_{k} t^{6k+3} \lambda^{6k+1}.
\end{align*}
$$

(63)

Here the sums are over the integer index $k$ from 0 to $\infty$.

7.1.3. **Recursive structure equation.** When the Killing field equation (62) is expanded as a series in $\lambda$, it implies the following recursive structure equation.

[$n$-th equation]

$$
\begin{align*}
d\mathbf{p}^{6n+4} &= (iy\mathbf{b}^{6n+5} + 2ih_3\mathbf{c}^{6n+5})\xi + (iy\mathbf{s}^{6n+3} + 2ih_3\mathbf{t}^{6n+3})\xi, \\
d\mathbf{b}^{6n+5} + ib^{6n+5} \rho &= ih_3\mathbf{f}^{6n+6}\xi + \frac{i}{2}y\mathbf{f}^{6n+4}\xi, \\
d\mathbf{c}^{6n+5} - 2ic^{6n+5} \rho &= iy\mathbf{f}^{6n+6}\xi + ih_3\mathbf{p}^{6n+4}\xi, \\
d\mathbf{f}^{6n+6} - if^{6n+6} \rho &= \frac{3i}{2}y\mathbf{a}^{6n+7}\xi + (iy\mathbf{c}^{6n+5} + ih_3\mathbf{b}^{6n+5})\xi, \\
d\mathbf{a}^{6n+7} &= iy\mathbf{g}^{6n+8}\xi + iy\mathbf{f}^{6n+6}\xi, \\
d\mathbf{g}^{6n+8} + ig^{6n+8} \rho &= (-iy\mathbf{f}^{6n+9} - ih_3\mathbf{c}^{6n+9})\xi + \frac{3i}{2}y\mathbf{a}^{6n+7}\xi, \\
d\mathbf{s}^{6n+9} - is^{6n+9} \rho &= \frac{i}{2}yp^{6n+10}\xi - ih_3\mathbf{g}^{6n+8}\xi, \\
d\mathbf{t}^{6n+9} + it^{6n+9} \rho &= ih_3\mathbf{p}^{6n+10}\xi - iy\mathbf{g}^{6n+8}\xi, \qquad \text{(mod} \hat{I}^{(\infty)})
\end{align*}
$$

(64)

The relevance of this formal structure equation for the analysis of the minimal Lagrangian system lies in the following observation.

**Lemma 7.1.** Suppose the coefficients $\{p^{6n+4}, b^{6n+5}, c^{6n+5}, f^{6n+6}, a^{6n+7}, g^{6n+8}, s^{6n+9}, t^{6n+9}\}$ satisfy the recursive structure equation (64). Then,

a) $p^{6n+4}$ is a pseudo-Jacobi field.

b) $a^{6n+7}$ is a Jacobi field.

The lemma indicates that one may obtain a canonical sequence of (pseudo) Jacobi fields by solving the structure equation (64).

7.2. **Initial analysis.** Recall from Exam[5.3] that $z_4$ is a pseudo-Jacobi field, and $z_5 = \frac{2}{3}z_4^2$ is a Jacobi field. We start the process of solving for the canonical formal Killing fields by determining the first few terms generated by these initial data. By Lem[5.3] the coefficients of the resulting formal Killing fields are the elements in the polynomial ring $C[z_4, z_5, ...]$, up to scaling by the appropriate powers of $h_4^{\frac{1}{3}}$.

It turns out that the pair of formal Killing fields generated by $z_4$, and $z_5 - \frac{5}{3}z_4^2$ are sufficient to cover all of the infinite sequence of higher-order (pseudo) Jacobi fields.
7.2.1. Case $p^4 = z_4$. Set $g^2 = 0$. By inspection, set

\[ s^3 = -\frac{3i}{2} y h_3^{-\frac{1}{2}}, \quad t^3 = \frac{3i}{2} h_3^\frac{1}{2}. \]

Differentiating this, we get

\[ p^4 = z_4 \]

as expected.

Solving the equation $\partial_z b^5 = \frac{i}{2} y z_4$, we get

\[ b^5 = -\frac{i}{3y} h_3^{-\frac{1}{2}} (z_5 - \frac{5}{3} z_4^2). \]

From the equation $\partial_z p^4 = iy b^5 + 2i h_3 c^5$, this implies

\[ c^5 = -\frac{i}{3} h_3^{-\frac{1}{2}} (z_5 - \frac{7}{6} z_4^3). \]

Successive derivatives of $b^5$ give

\[ f^6 = -\frac{1}{3y} h_3^{-\frac{1}{2}} (z_6 - \frac{14}{3} z_5 z_4 + \frac{35}{9} z_4^2), \]

\[ a^7 = \frac{2i}{y^2} (z_7 - 7z_6 z_4 - \frac{14}{3} z_5^2 + \frac{245}{9} z_5 z_4^2 - \frac{455}{27} z_4^4), \]

\[ g^8 = \frac{2}{9y^3} h_3^{-\frac{1}{2}} (z_8 - \frac{28}{3} z_7 z_4 - \frac{49}{3} z_6 z_5 + \frac{455}{9} z_6 z_4^2 + 70z_5 z_4 - \frac{5005}{27} z_5 z_4^2 + \frac{7280}{81} z_4^3). \]

Proceed with the similar computation as above by solving (by inspection) the associated $\partial_z$ equation, and we get

\[ s^9 = \frac{2i}{27y^4} h_3^{-\frac{1}{2}} (z_9 - 11z_8 z_4 - \frac{79}{3} z_7 z_5 + \frac{689}{9} z_7 z_4^2 - 16z_6^2 + 286z_6 z_5 z_4 - \frac{3380}{9} z_6 z_4^3 + \frac{1976}{27} z_5^3 - \frac{22360}{27} z_5 z_4^2 + \frac{108680}{81} z_5 z_4^3 - \frac{380380}{729} z_4^6), \]

\[ t^9 = \frac{4i}{27y^4} h_3^{-\frac{1}{2}} (z_9 - 12z_8 z_4 - \frac{76}{3} z_7 z_5 + \frac{758}{9} z_7 z_4^2 - \frac{33}{2} z_6^2 + \frac{901}{3} z_6 z_5 z_4 - \frac{3770}{9} z_6 z_4^3 + \frac{1847}{27} z_5^3 - \frac{47255}{54} z_5 z_4^2 + \frac{120380}{81} z_5 z_4^3 - \frac{432250}{729} z_4^6), \]

\[ p^{10} = \frac{4}{27y^4} (z_{10} - \frac{43}{3} z_9 z_4 - \frac{112}{3} z_8 z_5 + \frac{1118}{9} z_8 z_4^2 - \frac{175}{3} z_7 z_6 + \frac{4979}{9} z_7 z_5 z_4 - \frac{21164}{27} z_7 z_4^3 + \frac{1066}{3} z_6^2 z_4 + \frac{4550}{9} z_6 z_5^2 - \frac{116324}{27} z_6 z_5 z_4^2 + \frac{301340}{81} z_6 z_4^4 - \frac{165776}{81} z_5^3 z_4 + \frac{286520}{27} z_5^2 z_4^2 - \frac{3151720}{243} z_5 z_4^3 + \frac{9509500}{2187} z_4^6), \]

By Lem 7.1, $a^7$ is a Jacobi field, and $p^{10}$ is a pseudo-Jacobi field.
7.2.2. Case \(a^5 = z_5 - \frac{5}{3} z_4^2\). For the formal Killing field generated by the Jacobi field \(z_5 - \frac{5}{3} z_4^2\), it is convenient to lower the upper indices of the formal Killing field coefficients by 2 to match the order. The resulting structure equation is recorded as follows.

\[
\begin{align*}
p &= \sum p^{6k+2} \lambda^{6k}, & a &= \sum a^{6k+5} \lambda^{6k+3}, \\
b &= \sum b^{6k+3} \lambda^{6k+1}, & g &= \sum g^{6k+6} \lambda^{6k+4}, \\
c &= \sum c^{6k+3} \lambda^{6k+1}, & s &= \sum s^{6k+7} \lambda^{6k+5}, \\
f &= \sum f^{6k+4} \lambda^{6k+2}, & t &= \sum t^{6k+7} \lambda^{6k+5}.
\end{align*}
\]

([\(n\)-th equation']

\[
\begin{align*}
dp^{6n+2} &= (iyb^{6n+3} + 2ih_3c^{6n+3}) \xi + (iyg^{6n+1} + 2ih_3f^{6n+1}) \overline{\xi}, \\
db^{6n+3} + ib^{6n+3} \rho &= ih_3f^{6n+4} \xi + \frac{1}{2} yp^{6n+2} \overline{\xi}, \\
dc^{6n+3} - 2ic^{6n+3} \rho &= iyf^{6n+4} \xi + i\overline{h_3p^{6n+2} \xi}, \\
df^{6n+4} - if^{6n+4} \rho &= \frac{3i}{2} yu^{6n+5} \xi + (iyc^{6n+3} + i\overline{h_3b^{6n+3}}) \overline{\xi}, \\
da^{6n+5} &= iyg^{6n+6} \xi + iyf^{6n+4} \overline{\xi}, \\
dg^{6n+6} + ig^{6n+6} \rho &= (-iyf^{6n+7} - ih_3g^{6n+7}) \xi + \frac{3i}{2} yu^{6n+5} \overline{\xi}, \\
ds^{6n+7} - is^{6n+7} \rho &= \frac{1}{2} yp^{6n+8} \xi - i\overline{h_3g^{6n+6} \xi}, \\
dt^{6n+7} + 2it^{6n+7} \rho &= ih_3p^{6n+8} \xi - iyg^{6n+6} \overline{\xi},
\end{align*}
\]

(mod \(\hat{\mathfrak{i}}(\infty)\)).

We proceed to solve for the first few terms.

Let \(p^2 = 0\). By inspection, set

\[b^3 = -\frac{9}{2} y_3^{-\frac{1}{2}}, \quad c^3 = \frac{9}{4} y_3^{-\frac{1}{2}} h_3^\frac{1}{2}.
\]

Differentiating these equations successively, one gets

\[f^4 = \frac{3i}{2} y_3^{-\frac{1}{2}} z_4, \quad a^5 = z_5 - \frac{5}{3} z_4^2, \quad g^6 = -\frac{i}{2} y_3^{-\frac{1}{2}} \left(z_6 - 5z_5z_4 + \frac{40}{9} z_4^3\right).
\]

Note that \(a^5\) is as expected.
A similar computation as in the previous case yields,

\[
s^7 = \frac{1}{3\gamma} h_3^{\frac{1}{3}} \left( z_7 - 6z_6z_4 - \frac{16}{3} z_3^2 + \frac{220}{9} z_5z_4^2 - \frac{385}{27} z_4^3 \right),
\]

\[
t^7 = \frac{2}{3\gamma^2} h_3^{\frac{1}{3}} \left( z_7 - 7z_6z_4 - \frac{29}{6} z_3^2 + \frac{250}{9} z_5z_4^2 - \frac{935}{54} z_4^3 \right),
\]

\[
p^8 = \frac{2i}{3\gamma^2} \left( -\frac{26}{3} z_7z_4 - \frac{50}{3} z_6z_5 + \frac{418}{9} z_8z_4 - \frac{616}{9} z_7z_4^2 - \frac{1540}{9} z_5z_4^3 + \frac{6545}{81} z_4^4 \right),
\]

\[
b^9 = \frac{2}{9\gamma^2} h_3^{\frac{1}{4}} \left( z_9 - 12z_8z_4 - \frac{74}{3} z_7z_5 + \frac{748}{9} z_9z_4^2 - \frac{17z_7z_4^2}{3} + \frac{902}{3} z_6z_5z_4 - \frac{3740}{9} z_6z_4^3 \right)
+ \frac{1760}{27} z_3^2 - \frac{32320}{27} z_5z_4^2 + \frac{118745}{81} z_5z_4^3 - \frac{425425}{729} z_4^6,\]

\[
c^9 = \frac{2}{9\gamma^2} h_3^{\frac{1}{4}} \left( z_9 - 11z_8z_4 - \frac{77}{3} z_7z_5 + \frac{682}{9} z_9z_4^2 - \frac{33z_7z_4^2}{2} + \frac{286z_5z_4^3}{3} - \frac{374z_6z_4^3}{9} \right)
+ \frac{1892}{27} z_3^2 - \frac{22066}{27} z_5z_4^2 + \frac{107525}{81} z_5z_4^3 - \frac{752675}{1458} z_4^6.
\]

Successively differentiating \(b^9, c^9\), one finally gets,

\[
f^{10} = \frac{2i}{9\gamma^3} h_3^{\frac{1}{4}} \left( z_{10} - \frac{44}{3} z_9z_4 - \frac{110}{3} z_8z_5 + \frac{1144}{9} z_8z_4^2 - \frac{176}{3} z_7z_6 + \frac{1672}{3} z_7z_5z_4 - \frac{21692}{27} z_7z_4^3 \right)
+ \frac{363z_6z_4^3 + 4466}{9} z_6z_5^2 + \frac{118184}{27} z_6z_5z_4^2 + \frac{309485}{81} z_6z_4^3 - \frac{164560}{81} z_5z_4^3
+ \frac{871420}{81} z_5z_4^3 - \frac{1075250}{81} z_5z_4^3 + \frac{9784775}{50} z_4^7,\]

\[
a^{11} = \frac{4}{27\gamma^4} \left( z_{11} - \frac{55}{3} z_{10}z_4 - \frac{154}{3} z_9z_5 + \frac{176}{9} z_9z_4^2 - \frac{286}{3} z_8z_6 + \frac{2948}{3} z_8z_5z_4 - \frac{41140}{27} z_8z_4^3 \right)
- \frac{176}{3} z_2^2 + \frac{14014}{9} z_7z_6z_4 + \frac{9482}{27} z_7z_6z_4^2 - \frac{268532}{27} z_7z_5z_4^2 + \frac{247775}{27} z_7z_4^4
+ \frac{12199}{9} z_6z_5^2 - \frac{173723}{27} z_6z_5z_4 + \frac{158950}{9} z_6z_5z_4^2 + \frac{5344460}{27} z_6z_5z_4^3 - \frac{10343905}{243} z_6z_4^5
- \frac{164560}{81} z_5^2 + \frac{11133980}{243} z_5z_4^2 + \frac{36171410}{243} z_5z_4^3 + \frac{20101925}{2187} z_5z_4^4 - \frac{283758475}{6561} z_4^6\right).\]

By Lem 7.1 \(a^5, a^{11}\) are Jacobi fields, and \(p^8\) is a pseudo-Jacobi field.

### 7.3. Inductive formulas

Based on the initial analyses given above, we give the differential algebraic inductive formulas for the respective formal Killing fields:

- **7.3.1.** Case \(p^4 = z_4\),
- **7.3.2.** Case \(a^5 = z_5 - \frac{5}{3}z_4^2\).

Note from Eqs. (64), (66) that one needs to solve for the coefficients \(\{s^*, t^*\}\), and \(\{b^*, c^*\}\).

#### 7.3.1. Case \(p^4 = z_4\)

Assume the initial data from §7.2.1.
[Formulas for $s^{6n+3}, t^{6n+3}$]. Suppose all the coefficients up to $g^{6n+2}$ are known, $n \geq 1$. We give a formula for $\{s^{6n+3}, t^{6n+3}\}$.

Set the truncated formal Killing field

\[
X_{6n+2} := \begin{bmatrix}
-2ia & b + f + g - s & ib - if + ig + is \\
-b + f + g + s & ic + ia - it & -p + c + t \\
-ib - if + ig - is & p + c + t & -ic + ia + it \\
\end{bmatrix},
\]

where

\[
\begin{align*}
p &= \sum_{k=0}^{n} f^{6k+4} \lambda^{6k+2}, & a &= \sum_{k=0}^{n-1} f^{6k+7} \lambda^{6k+5}, \\
b &= \sum_{k=0}^{n-1} f^{6k+5} \lambda^{6k+3}, & g &= \sum_{k=0}^{n} f^{6k+2} \lambda^{6k}, \\
c &= \sum_{k=0}^{n-1} f^{6k+5} \lambda^{6k+3}, & s &= \sum_{k=0}^{n} f^{6k+3} \lambda^{6k+1}, \\
f &= \sum_{k=0}^{n-1} f^{6k+6} \lambda^{6k+4}, & t &= \sum_{k=0}^{n} f^{6k+3} \lambda^{6k+1}.
\end{align*}
\]

(67)

Here the unknown coefficients are $s^{6n+3}, t^{6n+3}, p^{6n+4}$. The determinant is given by

\[
\det(X_{6n+2}) = i(4gsp - 4fga - 4b^2c - 4f^2t + 4g^2c + 4s^2t + 2a^3 - 2ap^2 + 8act - 4bsa + 4bfp).
\]

Expanding as a series in $\lambda$, let us denote

\[
\det(X_{6n+2}) := \sum_{j=0}^{3n} x^{6j+3}_{6n+2} \lambda^{6j+3}.
\]

Consider now the derivative

\[
\partial_\xi(\det(X_{6n+2})).
\]

The structure equation shows that this term stems from the absence of $b^{6n+5}, c^{6n+5}$-terms in $X_{6n+2}$. Hence only the terms that contain $p^{6n+4}$ contribute to $\partial_\xi(\det(X_{6n+2}))$.

From the determinant formula above, one finds by checking the $\lambda$-degree that

\[
\partial_\xi x^{6j+3}_{6n+2} = 0, \quad \text{for } j \leq n.
\]

By Corollary 5.2 and weighted homogeneity, this implies that

\[
x^{6j+3}_{6n+2} = 0, \quad \text{for } j \leq n.
\]

Consider the term $x^{6n+3}_{6n+2}$ of the highest $\lambda$-degree among these. We have

\[
x^{6n+3}_{6n+2} = 4i(2s^3t^3 s^{6n+3} + (s^3)^2 t^{6n+3}) + y^{6n+3}_{6n+2},
\]

(68)

where $y^{6n+3}_{6n+2} \in O(6n + 2)$.

On the other hand, we have

\[
\partial_\xi g^{6n+2} = -ih_3 s^{6n+3} - iy^{6n+3}_{6n+2},
\]

(69)

Combining (68), (69), one gets

\[
s^{6n+3} = \frac{i}{27} h_3^{-1} \left(9\gamma \partial_\xi g^{6n+2} + h_3^2 y^{6n+3}_{6n+2} \right),
\]

(70)

\[
t^{6n+3} = \frac{i}{27} y^{6n+3}_{6n+2} \left(18\gamma \partial_\xi g^{6n+2} - h_3^2 y^{6n+3}_{6n+2} \right).
\]
[Formulas for $b_{6n-1}, c_{6n-1}$]. Suppose all the coefficients up to $p^{6n-2}$ are known, $n \geq 1$. We give a formula for $\{b_{6n-1}, c_{6n-1}\}$.

Given the truncated formal Killing field $X_{6n+2}$ as above, let
\[
\det(\mu I_3 + X_{6n+2}) = \mu^3 + \sigma_2(X_{6n+2})\mu + \det(X_{6n+2})
\]
be the characteristic polynomial. For the case at hand, we utilize $\sigma_2(X_{6n+2})$. It is given by the formula
\[
\sigma_2(X_{6n+2}) = 3a^2 + p^2 - 4ct - 4bs - 4fg.
\]
Expanding as a series in $\lambda$, let us denote
\[
\sigma_2(X_{6n+2}) := \sum_{j=0}^{2n} x_{6n+2}^j \lambda^j + 4.
\]
Consider the derivative $\partial_\xi (\sigma_2(X_{6n+2}))$. By the similar argument as above, one finds that
\[
\partial_\xi x_{6n+2}^{6j-2} = 0, \quad \text{for } j \leq n,
\]
and hence
\[
x_{6n+2}^{6j-2} = 0, \quad \text{for } j \leq n.
\]
Consider the term $x_{6n+2}^{6n-2}$. Then
\[
x_{6n+2}^{6n-2} = -4s^3 b_{6n-1} - 4t c_{6n-1} + y_{6n+2}^{6n-2}.
\]
Here $y_{6n+2}^{6n-2} \in \mathcal{O}(6n-2)$.

On the other hand, we have
\[
\partial_\xi p^{6n-2} = i\gamma b_{6n-1} + 2ih c_{6n-1}.
\]
Combining (71), (72), one gets
\[
b_{6n-1} = \frac{i}{9} \left( -3\partial_\xi p^{6n-2} + h_3^i y_{6n+2}^{6n-2} \right),
\]
\[
c_{6n-1} = -\frac{i}{18} h_3^i \left( 6\partial_\xi p^{6n-2} + h_3^i y_{6n+2}^{6n-2} \right).
\]

**Theorem 7.2.** Given the ansatz $p^4 = z_4$ and the initial data described in §7.2.1,

a) there exists a $\mathbb{C}[[\lambda]]$-valued canonical formal Killing field $X(p^4)$ which extends these data. The coefficients of its components are generated by the structure equation (64), and the differential algebraic inductive formulas (70), (73). Equivalently, $X(p^4)$ is determined by the constraint,
\[
\det(\mu I_3 + X(p^4)) = \mu^3 + \left( \frac{27}{2} \gamma^2 \right) \lambda^3.
\]
Here $I_3$ denotes the 3-by-3 identity matrix.

b) each coefficient of $X(p^4)$ is an element in the polynomial ring $\mathbb{C}[z_4, z_5, \ldots]$ up to scaling by appropriate powers of $h_3^i$. 


Corollary 7.3. Given the formal Killing field $X(p^4)$, the sequence of coefficients
$$\{ p^{6n+4}, p^{6n+4} \}_{n=0}^\infty$$
are distinct higher-order pseudo-Jacobi fields, and the sequence of coefficients
$$\{ d^{6n+7}, d^{6n+7} \}_{n=0}^\infty$$
are distinct higher-order Jacobi fields.

7.3.2. Case $a^5 = z_5 - \frac{5}{3}z_4^2$. Recall that we follow (65), and (66).
Assume the initial data from §7.2.2. By the same analysis as in §7.3.1 we obtain the corresponding formal Killing field $X(a^5)$.

Theorem 7.4. Given the ansatz $a^5 = z_5 - \frac{5}{3}z_4^2$, and the initial data described in §7.2.2,

a) there exists a $\mathfrak{g}^C[[\lambda]]$-valued canonical formal Killing field $X(a^5)$ which extends these data. The coefficients of its components are determined by the structure equation (66), and the constraint,
$$\det(\mu I_3 + X(a^5)) = \mu^3 - \left(\frac{729}{4} \gamma^4\right) \lambda^3.$$

b) each coefficient of $X(a^5)$ is an element in the polynomial ring $\mathbb{C}[z_4, z_5, ...]$ up to scaling by appropriate powers of $h^\frac{1}{3}$.

Corollary 7.5. Given the formal Killing field $X(a^5)$, the sequence of coefficients
$$\{ d^{6n+5}, d^{6n+5} \}_{n=0}^\infty$$
are distinct higher-order Jacobi fields, and the sequence of coefficients
$$\{ p^{6n+8}, p^{6n+8} \}_{n=0}^\infty$$
are distinct higher-order pseudo-Jacobi fields.

8. Higher-order conservation laws

Recall that the classical conservation laws are defined as the elements in the 1-st characteristic cohomology of the quotient complex
$$(\Omega'(X)/I, d).$$
Generalizing this, the conservation laws of the minimal Lagrangian system are defined as the elements in the 1-st characteristic cohomology of the quotient complex of the infinitely prolonged differential system $(\hat{X}^{(\infty)}, \hat{I}^{(\infty)})$,
$$(\Omega'(\hat{X}^{(\infty)})/\hat{I}^{(\infty)}, \hat{d}),$$
where $\hat{d} = d \mod \hat{I}^{(\infty)}$.

In this section, we give a description of the infinite sequence of higher-order conservation laws generated by the canonical formal Killing fields $X(p^4), X(a^5)$. 
8.1. **Definition.** Let \((\Omega^e(\hat{X}^{(\infty)}), d)\) be the de-Rham complex of \(\mathbb{C}\)-valued differential forms on \(\hat{X}^{(\infty)}\). Let
\[
(\Omega^e = \Omega^e(\hat{X}^{(\infty)})/\hat{I}^{(\infty)}, d)
\]
be the quotient space equipped with the induced differential \(d = d \mod \hat{I}^{(\infty)}\). The prolongation sequence of Pfaffian systems \(\hat{I}^{(k)}\) satisfy the inductive closure conditions
\[
d\hat{I}^{(k)} ≡ 0 \mod \hat{I}^{(k+1)}, \quad k ≥ 1.
\]
It follows that \(\hat{I}^{(\infty)} = \bigcup_{k=0}^{\infty} \hat{I}^{(k)}\) is formally Frobenius, and \((\Omega^e, d)\) becomes a complex. Let \(H^q(\Omega^e, d)\) be the cohomology at \(\Omega^e\). The set
\[
\{ H^q(\Omega^e, d) \}_{q=0}^\infty
\]
is called the characteristic cohomology of the differential system \((\hat{X}^{(\infty)}, \hat{I}^{(\infty)})\).

**Definition 8.1.** Let \((\hat{X}^{(\infty)}, \hat{I}^{(\infty)})\) be the triple cover of the infinite prolongation of the differential system for minimal Lagrangian surfaces. A **conservation law** is an element of the 1-st characteristic cohomology \(H^1(\Omega^e, d)\) of \((\hat{X}^{(\infty)}, \hat{I}^{(\infty)})\). The \(\mathbb{C}\)-vector space of conservation laws is denoted by
\[
C^{(\infty)} := H^1(\Omega^e, d).
\]
Let \(C^{(\infty)}_{\text{loc}}\) denote the space of local conservation laws of \(\hat{I}^{(\infty)}\) restricted to a small contractible open subset of \(\hat{X}^{(\infty)}\).

For simplicity, we shall suppress the global issues and identify \(C^{(\infty)} ≃ C^{(\infty)}_{\text{loc}}\).

Note by definition that the classical conservation laws \(C^{(0)} \subset C^{(\infty)}\).

8.1.1. **Spectral sequence.** Consider the filtration by the subspaces
\[
F^p\Omega^e = \text{Image}[I^{(\infty)} \wedge \hat{I}^{(\infty)} \wedge \cdots : \Omega^e(\hat{X}^{(\infty)}) \to \Omega^e(\hat{X}^{(\infty)})].
\]
From the associated graded \(F^p\Omega^e/F^{p+1}\Omega^e\), a standard construction yields the spectral sequence
\[
(E_r^{p,q}, d_r), \quad d_r \text{ has bidegree } (r, 1-r), \quad r ≥ 0.
\]
From the fundamental theorem [5] p562, Theorem 2 and Eq.(4)], at least locally the following sub-complex is exact,
\[
0 \to E_0^{0,1} \hookrightarrow E_1^{1,1} \to E_1^{2,1}.
\]
Here, by definition, the first piece is given by
\[
E_0^{0,1} = \{ \varphi \in \Omega^1(\hat{X}^{(\infty)}) | d\varphi ≡ 0 \mod \hat{I}^{(\infty)} \} / d\Omega^0(\hat{X}^{(\infty)}) + \Omega^1(\hat{I}^{(\infty)})
\]
\[
= H^1(\Omega^e(\hat{X}^{(\infty)})/\hat{I}^{(\infty)}, d)
\]
\[
= C^{(\infty)},
\]
and it is the space of conservation laws.
8.1.2. $E^{1,1}_1$. The second piece $E^{1,1}_1$ is called the space of cosymmetries. In the present case, the differential system is formally self-adjoint and this is the space of Jacobi fields, 

$$E^{1,1}_1 = \mathfrak{J}^{(\infty)}.$$ 

The differential $d_1 : E^{0,1}_1 \hookrightarrow E^{1,1}_1$ can be considered as the symbol map for conservation laws.

The space $E^{1,1}_1$ admits the following analytic description. Let $\Phi$ be a 2-form which represents a class in $E^{1,1}_1$. By definition, one may write

$$\Phi \equiv A\Psi - \theta_0 \wedge \sigma \mod F^2\Omega^2,$$

for a scalar coefficient $A$ and a 1-form $\sigma$, where

$$\Psi = \text{Im}(\theta_1 \wedge \xi) = -\frac{i}{2}(\theta_1 \wedge \xi - \bar{\theta}_1 \wedge \bar{\xi}).$$

Recall

$$d\Psi = 3i\gamma^n \partial_\xi \theta_0 \wedge (\xi \wedge \bar{\xi} + \theta_1 \wedge \bar{\theta}_1) \equiv 0 \mod \theta_0.$$

We wish to show that the coefficient $A$ is a Jacobi field. Differentiating $\Phi$, one gets

$$0 \equiv dA \wedge \Psi - d\theta_0 \wedge \sigma \mod \theta_0, F^2\Omega^3.$$

Since $d\theta_0 = -\frac{1}{2}(\theta_1 \wedge \xi + \bar{\theta}_1 \wedge \bar{\xi})$, this implies that

$$\sigma \equiv -i\left((\partial_\xi A)\xi - (\partial_{\bar{\xi}} A)\bar{\xi}\right) \mod \hat{I}^{(\infty)}.$$

With the given $\sigma$, the coefficient of $\theta_0 \wedge \xi \wedge \bar{\xi}$-term in $d\Phi$ then shows that

$$\mathcal{E}(A) = 0$$

and $A$ is a Jacobi field. We thus have the isomorphism $E^{1,1}_1 \simeq \mathfrak{J}^{(\infty)}$.

8.2. Conservation laws from formal Killing fields. A question arises as to if the infinite sequence of Jacobi fields for the minimal Lagrangian system indeed correspond to the sequence of conservation laws, i.e., if the symbol map 

$$d_1 : \mathcal{C}^{(\infty)}_{\text{loc}} \rightarrow E^{0,1}_1 \hookrightarrow E^{1,1}_1$$

is surjective and a higher-order version of Noether’s theorem holds for the minimal Lagrangian system. We show that, from the two formal Killing fields constructed in the previous section, we are able to assemble an infinite sequence of higher-order conservation laws. It is likely that they are nontrivial and, considering their spectral weights, the higher-order Noether’s theorem holds for the minimal Lagrangian system.

[Formal Killing field $X(p^n)$] Recall the structure equation (64). Set

$$\varphi_n := \theta^{6n+5} \xi + s^{6n+3} \bar{\xi}.$$

The structure equation shows that

$$d\varphi_n \equiv 0 \mod \hat{I}^{(\infty)}.$$
and $\phi_n$ represents a conservation law.

**[Formal Killing field $X(a^5)]** Recall the structure equation (66). Set

$$(77) \quad \phi'_n := b^{6n+3} \xi + s^{6n+1} \xi.$$

The structure equation shows that

$$d\phi'_n \equiv 0 \mod \hat{I}^{(\infty)},$$

and $\phi_n'$ represents a conservation law.

**Theorem 8.1.** Let $X(p^4), X(a^5)$ be the formal Killing fields generated from the initial data $p^4 = z_4, a^5 = z_5 - \frac{5}{3} z_4^2$ respectively in §7. Then the associated sequence of 1-forms

$$\phi_n, \phi'_n, \quad n = 0, 1, 2, \ldots,$$

represent the higher-order conservation laws.

It remains to verify that these conservation laws are indeed nontrivial. But, Thm[8.1] points to the relevant questions such as periods, residues, and the related application of the higher-order conservation laws to the global problems for minimal Lagrangian surfaces.

**Remark 8.2.** Suppose the conservation laws $[\phi_n], [\phi'_n]$ are nontrivial. Then, by an analysis of the differential $d_1$ as in §8.1.2 the spectral weight count shows that,

$$d_1([\phi_n]) = a^{6n+7}, \quad d_1([\phi'_n]) = a^{6n+5},$$

up to constant scale, where $d_1$ is the symbol map $d_1: C^\infty_{\text{loc}}(\mathbb{R}) \to E^{1,1} \cong \hat{I}^{(\infty)}$.

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