TOTAL POSITIVITY IN SPRINGER FIBRES

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CONTENTS

1. The totally positive part of $B_u$.
2. Examples.
3. Partial flag manifolds.
4. A conjecture and its consequences.
5. The map $g \mapsto P_g$ from $G_{\geq 0}$ to $P_{\geq 0}$.

1. The totally positive part of $B_u$

1.1. Let $G$ be a reductive connected algebraic group over $\mathbb{C}$ and let $B$ be the variety of Borel subgroups of $G$. For $g \in G$ the Springer fibre at $g$ is the subvariety $B_g = \{ B \in B; g \in B \}$ of $B$. The Springer fibres can be quite complicated; they play an important role in representation theory (for example in character formulas for finite reductive groups over a finite field). In this paper we assume that a pinning of $G$ is given so that the totally positive part $G_{\geq 0}$ of $G$ and the totally positive part $B_{\geq 0}$ of $B$ are defined (see [L94, 2.2, 8.1]). Recall that $G_{\geq 0}$ is a submonoid and a closed subset of $G$ and $B_{\geq 0}$ is a closed subset of $B$ on which $G_{\geq 0}$ acts. We are interested in the interaction of the theory of total positivity with that of Springer fibres. More precisely, for any $g \in G_{\geq 0}$ we consider the closed subset $B_{g, \geq 0} = B_g \cap B_{\geq 0}$ of $B_g$, which we call the totally positive part of the Springer fibre $B_g$. Let $U$ be the variety of unipotent elements of $G$ and let $U_{\geq 0} = U \cap G_{\geq 0}$. Our main result is that if $g \in U_{\geq 0}$, $B_{g, \geq 0}$ has a surprisingly simple structure, much simpler than that of $B_g$ itself (see Cor. 1.16); namely it has a canonical cell decomposition which is part of the canonical cell decomposition of $B_{\geq 0}$. By a similar argument we show that an analogous result holds for the fixed point set of $g$ on a partial flag manifold intersected with the totally positive part $[L98]$ of that partial flag manifold (see 3.9). In §4 we state a conjecture (see 4.4) on the compatibility of two ways to define a positive structure on the cells of $B_{\geq 0}$ and give some of its consequences; this conjecture extends a result of [L97, §3] which concerns the open dense cell $B_{\geq 0}$ of $B_{\geq 0}$ defined in [L94, 8.8]. Let $G_{\geq 0}$ the open dense sub-semigroup of $G_{\geq 0}$ defined in [L94]. In [L94, 8.9(c)], a map $g \mapsto B_g$ from

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$G_{>0}$ to $B_{>0}$ was defined. In §5 we give a definition of this map which is simpler than that in [L94] and we define an extension of this map to a map from $G_{>0}$ to the set of parabolic subgroups of $G$. As a result we obtain a new definition of $B_{>0}$.

We shall always assume that $G$ is simply laced; the non-simply laced case can be reduced to the simply laced case by descent.

1.2. Recall that we have fixed a pinning of $G$. Thus we are given a maximal torus $T$ of $G$ and a pair $B^+, B^-$ of opposed Borel subgroups of $G$ containing $T$ with unipotent radicals $U^+, U^-$. The pinning also includes root homomorphisms $x_i : C \to U^+$, $y_i : C \to U^-$ indexed by a finite set $I$ (corresponding to simple roots). Let $NT$ be the normalizer of $T$ in $G$ and let $W$ be the Weyl group. For $i \in I$ we set $s_i = y_i(1)x_i(-1)y_i(1) \in NT$; let $s_i$ be the image of $s_i$ in $W$. Now $W$ is a Coxeter group with simple reflections $\{s_i; i \in I\}$; let $w \mapsto |w|$ be the standard length function. Let $w_f$ be the unique element of maximal length of $W$. More generally, for any $H \subset I$ we denote by $w_H$ the longest element in the subgroup $W_H$ of $W$ generated by $\{s_i; i \in H\}$.

Let $\leq$ be the standard partial order on $W$. For $w \in W$ let $I_w$ be the set of sequences $(i_1, i_2, \ldots, i_m)$ in $I$ such that $w = s_{i_1}s_{i_2}\ldots s_{i_m}$, $m = |w|$; let $\text{supp}(w)$ be the set of all $i \in I$ which appear in some (or equivalently any) $(i_1, i_2, \ldots, i_m) \in I_w$. For $w \in W$ we set $\hat{w} = s_{i_1}s_{i_2}\ldots s_{i_m} \in NT$ where $(i_1, i_2, \ldots, i_m) \in I_w$; this is known to be well defined.

For $B, B'$ in $\mathcal{B}$ there is a unique $w \in W$ denoted by $\text{pos}(B, B')$ such that for some $g \in G$ we have $gBg^{-1} = B^+, gB'g^{-1} = \hat{w}B\hat{w}^{-1}$. There is a unique isomorphism $\phi : G \to G$ such that $\phi(x_i(a)) = y_i(a)$, $\phi(y_i(a)) = x_i(a)$ for all $i \in I$, $a \in C$ and $\phi(t) = t^{-1}$ for all $t \in T$. This carries Borel subgroups to Borel subgroups hence induces an isomorphism $\phi : \mathcal{B} \to \mathcal{B}$ such that $\phi(B^+) = B^-$, $\phi(B^-) = B^+$. For $i \in I$ we have $\phi(s_i) = x_i(1)y_i(-1)x_i(1) = s_i^{-1}$. Hence $\phi$ induces the identity map on $W$. We show:

(a) Let $B, B'$ in $\mathcal{B}$ be such that $\text{pos}(B, B') = w$. Then $\text{pos}(\phi(B), \phi(B')) = w_1\hat{w}w_1$.

Assume first that for some $i \in I$ we have $B = B^+, B' = s_iB^+s_i^{-1}$ so that $\text{pos}(B, B') = s_i$. We have

$$\text{pos}(\phi(B), \phi(B')) = \text{pos}(B^-, s_i^{-1}B^-s_i) = \text{pos}(\hat{w}_IB^+\hat{w}_I^{-1}, s_i^{-1}\hat{w}_IB^+\hat{w}_I^{-1}s_i)$$
$$= \text{pos}(B^+, \hat{w}_I^{-1}s_i^{-1}\hat{w}_IB^+\hat{w}_I^{-1}s_i\hat{w}_I) = w_1s_iw_1.$$

It follows that (a) holds when $w = s_i$ for some $i \in I$. The general case can be easily reduced to this special case. This proves (a).

1.3. Let $V$ be an irreducible rational $G$-module over $C$ with a given highest weight vector $\eta$. Let $\beta$ be the basis of $V$ (containing $\eta$) obtained by specializing at $v = 1$ the canonical basis [L90] of the corresponding module over the corresponding quantized enveloping algebra. For any $B \in \mathcal{B}$ let $L_B$ be the unique $B$-stable line in $V$. We can assume that $V$ is such that $B \mapsto L_B$ is a bijection.
From [L94, 8.7] it follows that morphisms $X$ let
\[ \xi \in V, V \]
for any $\xi$ and various algebraic group homomorphisms $\chi : C^* \to T$ and various $a \in R_{\geq 0}$; this is a closed subset of $G$. We have

(a) $\phi(G_{\geq 0}) = G_{\geq 0}$.

By [L94, 3.2] we have in the setup of 1.3:

(b) $G_{\geq 0}V_+ \subset V_+$.

From [L94, 8.7] it follows that

(c) $\phi(B_{\geq 0}) = B_{\geq 0}$.

Let $X_{\geq 0}$ be the set of lines in $X$ which meet $V_+ - \{0\}$. From [L94, 8.17] we have that:

(d) under the bijection 1.3(a), $B_{\geq 0}$ corresponds to $X_{\geq 0}$.

Using (b),(d) we deduce

(e) $g \in G_{\geq 0}, B \in B_{\geq 0} \implies gBg^{-1} \in B_{\geq 0}$.

(This also follows from [L94, 8.12].)

1.6. Recall that $U_{\geq 0} = U \cap G_{\geq 0}$ and that for $u \in U_{\geq 0}$ we have $B_{u, \geq 0} = B_u \cap B_{\geq 0}$. Let $V, V_+$ be as in 1.3. For $u \in U_{\geq 0}$ let $V^u_+ = \{ \xi \in V_+; u^{\xi} = \xi \}$. In particular $V^x_i(a), V^y_i(a)$ are defined for $i \in I, a \in R_{>0}$. We show:

(a) Let $u, u', u''$ in $U_{\geq 0}$ be such that $u = u'u''$. We have $V^u_+ = V^{u'}_+ \cap V^{u''}_+$. Let $\xi \in V^{u'}_+ \cap V^{u''}_+$. Then $u^{\xi} = u'(u''^{\xi}) = u^{\xi'} = \xi$ hence $\xi \in V^u_+$. Conversely, let $\xi \in V^u_+$. By [L94, 6.2, 3.2] for any $b \in \beta$ we have $u^{b'}b \in V_+, u''b - b \in V_+$. Hence for any $\xi', \xi''$ in $V_+$ we have $u^{\xi'} - \xi' \in V_+, u''\xi'' - \xi'' \in V_+$. Thus $u''\xi = \xi + \xi'$ where $\xi' \in V_+ \implies \xi = u\xi = u'(u''\xi) = u'(\xi + \xi')$. We have $u^{\xi} = \xi + \xi'_1, u^{\xi'} = \xi' + \xi'_2$ where $\xi'_1 \in V_+, \xi'_2 \in V_+$. Thus $\xi = \xi + \xi'_1 + \xi' + \xi'_2$ so that $\xi'_1 + \xi' + \xi'_2 = 0$; using 1.3(c) we see that $\xi'_1 = \xi' = \xi'_2 = 0$. Thus we have $u''\xi = \xi, u^{\xi'} = \xi$ and $\xi \in V^{u'}_+ \cap V^{u''}_+$. This proves (a).

We show:
Let $u \in \mathcal{U}_{\geq 0}$. From 1.5(d) we see that under the bijection 1.3(a), $\mathcal{B}_{u, \geq 0}$ corresponds to the set of lines $L$ in $\mathcal{X}_{\geq 0}$ which are $u$-stable. Since $u : V \to V$ is unipotent, for any such $L$, $u$ automatically acts on $L$ as identity. Thus we have:

(c) Under the bijection 1.3(a), $\mathcal{B}_{u, \geq 0}$ corresponds to the set of lines in $\mathcal{X}$ which meet $V^u_+ \setminus \{0\}$.

1.7. Following [L94, 8.15] we define a partition $\mathcal{B}_{\geq 0} = \sqcup_{(v, w) \in W \times W; v \leq w} \mathcal{B}_{0, v, w}$. In [L94, 8.15] it was conjectured that for $v \leq w$,

(a) $\mathcal{B}_{0, v, w}$ is homeomorphic to $\mathbf{R}_{\geq 0}^{[w] - [v]}$.

This conjecture was proved by Rietsch [R98], [R99] and in a more explicit form by Marsh and Rietsch [MR].

For $v \leq w$ let $[v, w]$ be the subset of $\mathcal{X}_{\geq 0}$ corresponding to $\mathcal{B}_{0, v, w}$ under the bijection $\mathcal{B}_{\geq 0} \leftrightarrow \mathcal{X}_{\geq 0}$ in 1.5(d). We have $\mathcal{X}_{\geq 0} = \sqcup_{(v, w) \in W \times W; v \leq w} [v, w]$.

1.8. Let $v \leq w$ in $W$ and let $i = (i_1, i_2, \ldots, i_m) \in \mathcal{I}_w$. According to Marsh and Rietsch [MR], there is a unique sequence $t_1, t_2, \ldots, t_m$ with $t_k \in \{s_{i_k}, 1\}$ for $k \in [1, m]$, $t_1 t_2 \cdots t_m = v$ and such that $t_1 \leq t_1 t_2 \leq \cdots \leq t_1 t_2 \cdots t_m$ and $t_1 \leq t_1 s_{i_2}, t_1 t_2 \leq t_1 t_2 s_{i_2}, \ldots, t_1 t_2 \cdots t_{m-1} \leq t_1 t_2 \cdots t_{m-1} s_{i_m}$. Following [MR] we define a subset $\mathcal{Y}_{v, w, i}$ of $G$ to be the set of products $g_1 g_2 \cdots g_m$ in $G$ where $g_k = s_{i_k}$ if $t_k = s_{i_k}$ and $g_k = y_{i_k}(a_k)$ with $a_k \in \mathbf{R}_{\geq 0}$ if $t_k = 1$; according to [MR], the map $\mathcal{Y}_{v, w, i} \to \mathcal{B}$, $g \mapsto gB + g^{-1}$ is a homeomorphism

(a) $\mathcal{Y}_{v, w, i} \sim \mathcal{B}_{0, v, w}$.

Moreover we have a homeomorphism

(b) $\mathcal{Y}_{v, w, i} \sim \mathbf{R}_{\geq 0}^{[w] - [v]}$

to $g_1 g_2 \cdots g_m$ we associate the sequence consisting of the $a_k$ with $k$ such that $t_k = 1$). In particular, the factors $g_1, g_2, \ldots, g_m$ are uniquely determined by their product $g_1 g_2 \cdots g_m$. The composition of the inverse of (b) with (a) is a homeomorphism

(c) $\tau_1 : \mathbf{R}_{\geq 0}^{[w] - [v]} \sim \mathcal{B}_{0, v, w}$.

1.9. We preserve the setup of 1.8. Assume that $s_{i_1} w \leq w$. We show:
(a) We have $t_1 = 1$ if and only if $v \leq s_{i_1} w$.

If $t_1 = 1$ then from $v = t_1 \cdots t_m$ we deduce $v = t_2 t_3 \cdots t_m$; thus $v$ is equal to a product of a subsequence of $s_{i_2}, s_{i_3}, \ldots, s_{i_m}$ so that $v \leq s_{i_1} w$. Conversely, if $v \leq s_{i_1} w$ then by the results in 1.8 applied to $v, s_{i_1} w$ instead of $v, w$, we can find a sequence $t'_2, \ldots, t'_m$ with $t'_k \in \{s_{i_k}, 1\}$ for $k \in [2, m]$, $t'_2 \cdots t'_m = v$ and such that $t'_2 \leq t'_2 t'_3 \leq \cdots \leq t'_2 \cdots t'_m$ and $t'_2 \leq t'_2 s_{i_3}, \ldots, t'_2 t'_3 t'_4 \cdots t'_m = t'_2 \cdots t'_m s_{i_m}$. Taking $t'_1 = 1$ we have $t'_1 t'_2 \cdots t'_m = v$, $t'_1 \leq t'_1 t'_2 \leq \cdots \leq t'_1 t'_2 \cdots t'_m$ and $t'_1 \leq t'_1 s_{i_2}, t'_1 t'_2 \leq t'_1 t'_2 s_{i_3}, \ldots, t'_1 t'_2 \cdots t'_m = t'_1 t'_2 \cdots t'_m s_{i_m}$. By uniqueness we have $(t'_1, t'_2, \ldots, t'_m) = (t_1, t_2, \ldots, t_m)$. Thus $t_1 = 1$ and (a) is proved.
1.10. For any \( h \in G \) let \([h] : B \rightarrow B\) be the map \( B \mapsto hBh^{-1} \).

Let \( i \in I, a \in \mathbb{R}_{>0} \) and let \( v, w \) in \( W \) be such that \( v \leq w \). We show:

(a) If \( v \leq s_iw \leq w \), then \([y_i(a)] : B \rightarrow B\) restricts to a map \( B_{\geq 0,v,w} \rightarrow B_{\geq 0,v,w} \)
which is fixed point free.

(b) If \( s_iw \leq w, v \leq s_iw \), then \([y_i(a)] : B \rightarrow B\) restricts to the identity map
\( B_{\geq 0,v,w} \rightarrow B_{\geq 0,v,w} \).

(c) If \( w \leq s_iw \) then \([y_i(a)] : B \rightarrow B\) restricts to a map \( B_{\geq 0,v,w} \rightarrow B_{\geq 0,v,s_iw} \).

Assume first that \( s_iw \leq w \). We can find \( i = (i_1, i_2, \ldots, i_m) \in \mathcal{I}_w \) such that \( i_1 = i \). We use the notation of 1.8 relative to \( v, w, i \).

In case (a) we have \( t_1 = 1 \), see 1.9(a). In this case, left multiplication by \( y_i(a) \) restricts to a map \( \mathcal{Y}_{v,w,i} \rightarrow \mathcal{Y}_{v,w,i} \) given by \( g_1g_2 \ldots g_m \mapsto g'_1g'_2 \ldots g'_m \) where \( g_1 = y_i(a_1), g'_1 = y_i(1+a), g'_k = g_k \) for \( k > 1 \). To prove (a), it remains to use that \( a_1 \mapsto a_1 + a \) from \( \mathbb{R}_{>0} \) to \( \mathbb{R}_{>0} \) is fixed point free.

In case (b) we have \( t_1 = s_i \), see 1.9(a). In this case, for \( g_1g_2 \ldots g_m \in \mathcal{Y}_{v,w,i} \) we have \( g_1 = s_i \) and for some \( b \in B^+ \) we have \( y_i(a)g_1g_2 \ldots g_m = y_i(a)s_is_i\ldots g_m = s_is_i(-a)g_2 \ldots g_m = s_is_i(g_2 \ldots g_m) \) (the last equality follows by an argument in [MR, 11.9]). Hence (b) holds.

Assume that we are in case (c). Let \( B \in B_{\geq 0,v,w}, B' = y_i(a)By_i(a)^{-1} \). We have \( \text{pos}(B^+, B') = \text{pos}(y_i(a)^{-1}B^+y_i(a), B) \). This equals \( s_iw \) since

\[
\text{pos}(y_i(a)^{-1}B^+y_i(a), B^+) = s_i, \text{pos}(B^+, B) = w
\]

and \( s_iw \geq w \). We have

\[
\text{pos}(B^-, B') = \text{pos}(y_i(a)^{-1}B^-y_i(a), B) = \text{pos}(B^-, B) = v
\]
since \( y_i(a) \in B^- \). Thus \( B' \in B_{s_iw} \). By 1.5(e) we have \( B' \in B_{\geq 0} \) hence \( B' \in B_{\geq 0, s_iw} \). Hence (c) holds.

1.11. Let \( i \in I, a \in \mathbb{R}_{>0} \) and let \( v, w \) in \( W \) be such that \( v \leq w \). We show:

(a) If \( v \leq s_iv \leq w \), then \([x_i(a)] : B \rightarrow B\) restricts to a map \( B_{\geq 0,v,w} \rightarrow B_{\geq 0,v,w} \)
which is fixed point free.

(b) If \( v \leq s_iv, s_iv \not\leq w \), then \([x_i(a)] : B \rightarrow B\) restricts to the identity map
\( B_{\geq 0,v,w} \rightarrow B_{\geq 0,v,w} \).

(c) If \( s_iv \leq v \), then \([x_i(a)] : B \rightarrow B\) restricts to a map \( B_{\geq 0,v,w} \rightarrow B_{\geq 0,s_iv,w} \).

We apply 1.10(a)-(c) to \( wv_I, vw_I \) instead of \( v, w \) (note that \( vw_I \leq vw_I \)) and we apply the automorphism \( \phi \). We obtain the following statements.

(1) If \( vw_I \leq s_iwv_I \leq vw_I \), then \([x_i(a)] : B \rightarrow B\) restricts to a map

\[
\phi(B_{\geq 0,wv_I,vw_I}) \rightarrow \phi(B_{\geq 0,wv_I,vw_I})
\]

which is fixed point free.

(2) If \( s_iwv_I \leq vw_I, vw_I \not\leq s_iwv_I \) then \([x_i(a)] : B \rightarrow B\) restricts to the identity map
\( \phi(B_{\geq 0,wv_I,vw_I}) \rightarrow \phi(B_{\geq 0,wv_I,vw_I}) \).
If \( vw_l \leq s_i \cdot vw_l \), then \( [x_i(a)] : B \rightarrow B \) restricts to a map \( \phi(B_{\geq 0, vw_l, uw_l}) \rightarrow \phi(B_{\geq 0, uw_l, vw_l}) \).

It remains to note that

\[ \phi(B_{\geq 0, uw_l, vw_l}) = B_{\geq 0, v, w} \]

for any \( v \leq w \) in \( W \) (see 1.4(a) and 1.5(c)).

1.12. Let \( i \in I, a \in \mathbb{R}_{>0} \). From 1.10 we deduce:

(a) \( \{ B \in B_{\geq 0}; y_i(a) \in B \} = \sqcup_{(v, w) \in W \times W; v \leq w, s_i w \leq w, v \leq s_i w} B_{\geq 0, v, w} \).

From 1.11 we deduce:

(b) \( \{ B \in B_{\geq 0}; x_i(a) \in B \} = \sqcup_{(v, w) \in W \times W; v \leq w, v \leq s_i v, s_i w \leq w} B_{\geq 0, v, w} \).

1.13. Let \( (W \times W)_{\text{disj}} = \{(z, z') \in W \times W; \text{supp}(z) \cap \text{supp}(z') = \emptyset\} \). For \( (z, z') \in (W \times W)_{\text{disj}} \), let \( U_{\geq 0, z, z'} \) be the image of the (injective) map \( \mathbb{R}^m \times \mathbb{R}^m' \rightarrow G_{\geq 0} \) given by

\[
((a_1, a_2, \ldots, a_m), (a'_1, a'_2, \ldots, a'_m', i)) \mapsto y_{i_1}(a_1) y_{i_2}(a_2) \cdots y_{i_m}(a_m) x_{i'_1}(a'_1) x_{i'_2}(a'_2) \cdots x_{i'_m}(a'_m')
\]

where \( i = (i_1, \ldots, i_m) \in I_w, i' = (i'_1, \ldots, i'_m') \in I_{w'} \). This map is injective and its image is independent of the choice of \( i, i' \) (see [L94, 2.7, 2.9]). Note that \( U_{\geq 0, z, z'} \subset U_{\geq 0} \). More precisely, we have (see [L94, 6.6]):

\[ U_{\geq 0} = \sqcup_{(z, z') \in (W \times W)_{\text{disj}}} U_{\geq 0, z, z'} \]

We state the following result.

**Theorem 1.14.** Let \((z, z') \in (W \times W)_{\text{disj}}\) and let \( u \in U_{\geq 0, z, z'}, J = \text{supp}(z), J' = \text{supp}(z')\). We have

\[ B_{u, \geq 0} = \bigcap_{i \in J} B_{y_i(1), \geq 0} \cap \bigcap_{j \in J'} B_{x_j(1), \geq 0}. \]

Using 1.6(c) we see that the following implication implies the theorem.

(a) \[ V^u_+ = \bigcap_{i \in J} V^{y_i(1)}_+ \cap \bigcap_{j \in J'} V^{x_j(1)}_+ \]

Let \( m, m', i, i' \) be as in 1.13 and let \((a_1, a_2, \ldots, a_m), (a'_1, a'_2, \ldots, a'_m', i)\) be corresponding to \( u \) as in 1.13. We argue by induction on \( m + m' \). If \( m + m' = 0 \) the result is obvious. Assume now that \( m + m' \geq 1 \). If \( m \geq 1 \) let \( z_1 = s_{i_1} z \) and let \( u_1 = y_{i_1}(a_1)^{-1} u; \) we have \( |z_1| = m - 1 \) and \( u_1 \in U_{\geq 0, z_1, z'} \), \( u = y_{i_1}(a_1) u_1 \). If \( m' \geq 1 \) let \( z'_1 = s_{i'_1} z' \) and let \( u'_1 = x_{i'_1}(a'_1)^{-1} u; \) we have \( |z'_1| = m' - 1 \) and \( u'_1 \in U_{\geq 0, z'_1, z'} \), \( u = x_{i'_1}(a'_1) u'_1 \). From 1.6(a), (b), we have \( V^u_+ = V^{y_1(1)}_+ \cap V^{u_1}_+ \) if \( m \geq 1 \), \( V^u_+ = V^{x_1(1)}_+ \cap V^{u'_1}_+ \) if \( m' \geq 1 \). Since the induction hypothesis is applicable to \( V^{u_1}_+ \) (if \( m \geq 1 \)) and to \( V^{u'_1}_+ \) (if \( m' \geq 1 \)) we see that (a) is proved. This proves the theorem.
1.15. Let \( J \subset I, J' \subset I \) be such that \( J \cap J' = \emptyset \). We define
\[
Z_{J,J'} = \{(v, w) \in W \times W; v \leq w; s_i w \leq w, v \not\leq s_i w \quad \forall i \in J; \\
v \leq s_j v, s_j v \not\leq w \quad \forall j \in J'\}.
\]
Combining 1.14 with 1.12(a),(b) we obtain our main result.

**Corollary 1.16.** Let \((z, z') \in (W \times W)_{\text{disj}}\) and let \( u \in \mathcal{U}_{\geq 0, z, z'} \), \( J = \text{supp}(z), J' = \text{supp}(z') \). We have
\[
\mathcal{B}_{u, \geq 0} = \bigcup_{(v, w) \in Z_{J,J'}} \mathcal{B}_{\geq 0, v, w}.
\]

Thus \( \mathcal{B}_{u, \geq 0} \) admits a canonical, explicit, cell decomposition in which each cell is part of the canonical cell decomposition of \( \mathcal{B}_{\geq 0} \). The zero dimensional cells of \( \mathcal{B}_{u, \geq 0} \) are \( \mathcal{B}_{\geq 0, v, w} \), where \( w \in W \) is such that the set \( \{i \in I; s_i w \leq w\} \) contains \( J \) and is contained in \( I - J' \). (For example \( w_{J} \) is such a \( w \) since \( J \subset I - J' \).) In particular, \( Z_{J,J'} \neq \emptyset \), so that \( \mathcal{B}_{u, \geq 0} \neq \emptyset \). (This also follows from [L94, 8.11].)

1.17. In [L94, 2.11], \( G_{\geq 0} \) is partitioned into pieces indexed by \( W \times W \); by results in [L94] each piece is a cell. In [L91] the set of pieces of \( G_{\geq 0} \) is interpreted as a monoid \( G(\{1\}) \), the value of \( G \) at the semifield \( \{1\} \) with 1 element, so that pieces appear precisely as the fibres of a natural surjective map \( G_{\geq 0} \rightarrow G(\{1\}) = W \times W \) compatible with the monoid structures (the monoid structure on \( W \times W \) thus obtained is not the usual group structure). In particular, the product of two pieces of \( G_{\geq 0} \) is contained in a single piece of \( G_{\geq 0} \). Similarly, we can view the set of cells \( \mathcal{B}_{\geq 0, v, w} \) of \( \mathcal{B}_{\geq 0} \) (see 1.7) as \( G(\{1\}) \) or the value of \( \mathcal{B} \) at the semifield \( \{1\} \). From 1.10 and 1.11 we see that in the action of \( G_{\geq 0} \) on \( \mathcal{B}_{\geq 0} \), the result of applying a piece of \( G_{\geq 0} \) to a cell of \( \mathcal{B}_{\geq 0} \) is contained in a single cell of \( \mathcal{B}_{\geq 0} \). It follows that the action of \( G_{\geq 0} \) on \( \mathcal{B}_{\geq 0} \) induces an action of \( G(\{1\}) = W \times W \) on \( \mathcal{B}(\{1\}) \). We can identify \( \mathcal{B}(\{1\}) \) with \( \{(v, w) \in W \times W; v \leq w\} \). Then the action of \( W \times W \) becomes:
\[
(s_i, 1) : (v, w) \mapsto (v, s_i \ast w), (1, s_i) : (v, w) \mapsto (s_i \circ v, w)
\]
where for \( i \in I, v \in W, w \in W \) we define
\[
s_i \ast w = w \text{ if } s_i w \leq w, s_i \ast w = s_i w \text{ if } w \leq s_i w, \\
s_i \circ v = v \text{ if } v \leq s_i v, s_i \circ v = s_i v \text{ if } s_i v \leq v.
\]
Now let \( u \in \mathcal{U}_{\geq 0} \) and let \( z, z', J, J' \) be associated to \( u \) as in 1.16. Let \( \mathcal{B}(\{1\})_u \) be the set of cells of \( \mathcal{B}_{u, \geq 0} \) described in 1.16; this set is in bijection with \( Z_{J,J'} \) (see 1.16) and can be viewed as a subset of \( \mathcal{B}(\{1\}) \). Let \( \mathcal{B}(\{1\})_u \) be the set of all \( (w, w') \in W \times W = G(\{1\}) \) such that in the \( W \times W \)-action on \( \mathcal{B}(\{1\}) \) (as above), we have \( (w, w') \mathcal{B}(\{1\})_u \subset \mathcal{B}(\{1\})_u \). Clearly, \( \mathcal{B}(\{1\})_u \) is a submonoid of \( W \times W = G(\{1\}) \).

Let \( \mathcal{I}_u \) be the inverse image of \( \mathcal{I}(u)(\{1\}) \) under the canonical monoid homomorphism \( G_{\geq 0} \rightarrow G(\{1\}) \) (as above). Note that \( \mathcal{I}(u) \) is a submonoid of \( G_{\geq 0} \) which is related to the centralizer of \( u \) in \( G \) (although it is not in general contained in it). Clearly the \( G_{\geq 0} \)-action of \( \mathcal{B}_{\geq 0} \) restricts to a \( \mathcal{I}(u) \)-action on \( \mathcal{B}_{u, \geq 0} \).
1.18. Let $\tilde{B} = \{(u, B) \in U \times B; u \in B\}$. Let $\tilde{B}_{\geq 0} = \{(u, B) \in U_{\geq 0} \times B_{\geq 0}; u \in B\}$. From 1.16 we can deduce that $\tilde{B}_{\geq 0}$ is the disjoint union of the sets

(a) \[ \tilde{B}_{\geq 0, z, z', v, w} = U_{\geq 0, z, z'} \times B_{\geq 0, v, w} \]

where $(z, z')$ runs through $(W \times W)_{\text{disj}}$ and $(v, w)$ runs through $Z_{J,J'}$ where $J = \text{supp}(z)$, $J' = \text{supp}(z')$. The subset (a) is a cell of dimension $|z| + |z'| + |w| - |v|$. Note that the first projection $\tilde{B}_{\geq 0} \to U_{\geq 0}$ is a fibration over each $U_{\geq 0, z, z'}$.

Let $\tilde{B}(\{1\})$ be the indexing set for the set of cells of $\tilde{B}_{\geq 0}$. This is the set of all $(z, z', v, w) \in W^4$ such that $J = \text{supp}(z)$, $J' = \text{supp}(z')$ are disjoint and $(v, w) \in Z_{J,J'}$.

1.19. Let $B \in B_{\geq 0}$ and let $\underline{B} = B \cap U_{\geq 0}$. This is a closed subset of $B$ closed under multiplication. (If $u \in \underline{B}$, $u' \in \underline{B}$ then $u, u'$ are contained in the unipotent radical of $B$ hence $uu'$ is also contained in that unipotent radical, so that $uu'$ is unipotent. Since $u \in G_{\geq 0}, u' \in G_{\geq 0}$ we have also $uu' \in G_{\geq 0}$. Thus $uu' \in \underline{B}$.) We have $B \in B_{\geq 0, v, w}$ for well defined $v \leq w$ in $W$. We set

\[ \Xi_{v, w} = \{(z, z') \in (W \times W)_{\text{disj}}; s_i w \leq w, v \not\leq s_i w \quad \forall i \in \text{supp}(z); \]

\[ v \leq s_j v, s_j v \not\leq w \quad \forall j \in \text{supp}(z')\} \]

We show:

(a) \[ \underline{B} = \bigcup_{(z, z') \in \Xi_{v, w}} U_{\geq 0, z, z'} \]

Let $E$ be the right hand side of (a). Let $u \in \underline{B}$. Let $(z, z') \in (W \times W)_{\text{disj}}$ be such that $u \in U_{\geq 0, z, z'}$. Let $J = \text{supp}(z)$, $J' = \text{supp}(z')$. We have $B \in B_{u, \geq 0}$ hence by 1.16 we have $(v, w) \in Z_{J,J'}$ hence $(z, z') \in \Xi_{v, w}$ Thus, $u \in E$. We see that $\underline{B} \subset E$. Conversely, assume that $u \in E$. We can find $(z, z') \in \Xi_{v, w}$ such that $u \in U_{\geq 0, z, z'}$.

Let $J = \text{supp}(z)$, $J' = \text{supp}(z')$. We have $(v, w) \in Z_{J,J'}$. Since $B \in B_{\geq 0, v, w}$ from 1.16 we see that $B \in B_{u, \geq 0}$ so that $u \in B$ and $u \in \underline{B}$. We see that $E \subset \underline{B}$. This proves (a).

Let $H = \{i \in I; s_i w \leq w, v \not\leq s_i w\}$, $H' = \{j \in I; v \leq s_j v, s_j v \not\leq w\}$.

If $i \in H$ then $v \leq w, s_i w \leq w$ and this is known to imply $s_i v \leq w$. Thus $i \in H \implies i \not\in H'$ so that

(b) $\quad H \cap H' = \emptyset$.

We have $\Xi_{v, w} = \{(z, z') \in (W \times W)_{\text{disj}}; \text{supp}(z) \subset H, \text{supp}(z') \subset H'\}$. Using (b) we see that

(c) $\Xi_{v, w} = \{(z, z') \in W \times W; \text{supp}(z) \subset H, \text{supp}(z') \subset H'\}$.

Hence (a) becomes:

(d) \[ \underline{B} = \bigcup_{(z, z') \in W \times W; \text{supp}(z) \subset H, \text{supp}(z') \subset H'} U_{\geq 0, z, z'} \]

From (d) we see that $\underline{B}$ has a canonical cell decomposition with cells indexed by (c). One of these cells is $U_{\geq 0, w_H, w_{H'}}$ (of dimension $|w_H| + |w_{H'}|$); all other cells
have dimension strictly less than $|w_H| + |w_{H'}|$ and are contained in the closure of $U_{\geq 0, w_H, w_{H'}}$. We see that

(e) $\overline{B}$ is connected of dimension $|w_H| + |w_{H'}|$.

If $(v, w) = (1, w_I)$ then $H = H' = \emptyset$ and from (d) we see that $\overline{B} = U_{\geq 0, 1, 1} = \{1\}$. Recall that $B_{\geq 0, 1, w_I} = B > 0$. We see that:

(f) If $u \in U_{> 0}$ and $B \in B_u$ satisfies $B \in B_{> 0}$ then $u = 1$.

It is likely that, more generally, for any $B \in B_{> 0}$, $B \cap G_{\geq 0}$ consists of semisimple elements.

1.20. Let $g \in G_{\geq 0}$ and let $[g] : B \to B$ be as in 1.10. Let $B_{g, \geq 0}$ be as in 1.1. If $v \leq w$ in $W$ then, by 1.17, there are three possibilities:

(i) $[g]B_{\geq 0, v, w} = B_{\geq 0, v', w'}$ where $v' \leq w', (v', w') \neq (v, w)$;

(ii) $[g]B_{\geq 0, v, w} = B_{\geq 0, v, w}$ and $B \mapsto [g]B$, $B_{\geq 0, v, w} \to B_{\geq 0, v, w}$ has empty fixed point set;

(iii) $[g]B_{\geq 0, v, w} = B_{\geq 0, v, w}$ and $B \mapsto [g]B$, $B_{\geq 0, v, w} \to B_{\geq 0, v, w}$ has non-empty fixed point set (denoted by $B_{g, \geq 0, v, w}$).

It follows that $B_{g, \geq 0} = \sqcup_{v, w} B_{g, \geq 0, v, w}$ where $v, w$ is as in (iii). We conjecture that if $v, w$ is as in (iii), then $B_{g, \geq 0, v, w}$ is a cell. (When $g$ is unipotent this holds by 1.16.)

2. Examples

2.1. Assume that $G = SL_{n+1}(\mathbb{C})$ and $I = \{1, 2, \ldots, n\}$ with $n \geq 2$ and with $s_1s_2, s_2s_3, \ldots, s_{n-1}s_n$ of order 3. Let $J = \{1, 2, \ldots, n-1\}$. Let $-1$ be the unit element of $W$. We have $(w_J, -) \in (W \times W)_{\text{disj}}$ and $Z_{J, \emptyset}$ consists of

$$(w_J, w_J), (w_Js_n, w_Js_n), (w_J, w_Js_n), (w_Js_n, s_n, w_Js_n, s_n, \ldots, s_n, w_Js_n, s_n, \ldots, s_n),$$

(a) $\ldots, (w_Js_n, s_n, \ldots, s_2s_1, w_Js_n, s_n, \ldots, s_2s_1), (w_Js_n, s_n, \ldots, s_2, w_Js_n, s_n, \ldots, s_2s_1).$

Thus if $u \in U_{\geq 0, w_J, -}$ then $B_{u, \geq 0}$ is a union of $n + 1$ cells of dimension 0 and $n$ cells of dimension 1.

We have also $(w_J, s_n) \in (W \times W)_{\text{disj}}$ and $Z_{J, \{n\}}$ consists of the pairs in (a) other than the last two. Thus, if $u \in U_{\geq 0, z, s_n}$ then $B_{u, \geq 0}$ is a union of $n$ cells of dimension 0 and $n - 1$ cells of dimension 1.

Assume now that $n = 3$. We have $(s_1s_3, -) \in (W \times W)_{\text{disj}}$. Let $J = \{1, 3\}$. Then $Z_{J, \emptyset}$ consists of

$$(132, 132), (13, 132), (1321, 1321), (132, 1321), (1323, 1323), (132, 1323),$$

$$(1323, 1323),$$

(b) $$(1321, 13213), (1323, 13213), (132, 13213), (132312, 132312), (13231, 132132).$$

(We write $i_1i_2\ldots$ instead of $s_i s_{i_2} \ldots$) Thus, if $u \in U_{\geq 0, s_1s_3, -}$ then $B_{u, \geq 0}$ is a union of 6 cells of dimension 0, 6 cells of dimension 1 and one cell of dimension 2.
We have also \((s_1s_3,s_2) \in (W \times W)_{\text{disj}}\) and \(Z_{I,\{2\}}\) consists of the pairs in (b) other than the last two. Thus, if \(u \in U_{\geq 0,s_1s_3,s_2}\) then \(B_{u,\geq 0}\) is a union of 5 cells of dimension 0, 5 cells of dimension 1 and one cell of dimension 2.

In these examples \(B_{u,\geq 0}\) is contractible; but in the last two examples, \(B_{u,\geq 0}\) is not of pure dimension, unlike \(B_u\).

We have \((s_1, -) \in (W \times W)_{\text{disj}}\). Then \(Z_{\{1\},\emptyset}\) consists of

\[
(2, 2), (21, 21), (2, 21), (23, 23), (2, 23), (212, 212), (21, 212),
(232, 232), (23, 232), (213, 213), (21, 213), (23, 213), (2, 213), (2123, 2123),
(212, 2123), (213, 2123), (21, 2123), (2321, 2321), (232, 2321), (231, 2321),
(23, 2321), (2132, 2132), (213, 2132), (232, 2132), (212, 2132), (23, 2132),
(21232, 21232), (2123, 21232), (2132, 21322), (2122, 21232), (21, 21232),
(23212, 23212), (2321, 23212), (232, 23212), (231, 23212), (23, 23212),
(23, 23212), (213213, 213213), (32132, 213213), (12312, 213213), (2132, 213213).
\]

(We write \(i_1i_2\ldots\) instead of \(s_1, s_2\ldots\).) Thus if \(u \in U_{\geq 0,s_1, -}\) then \(B_{u,\geq 0}\) is a union of cells of dimension \(\leq 3\), two of which have dimension 3.

In each of the examples above, for any cell \(B_{\geq 0,v,w}\) of maximal dimension of \(B_{u,\geq 0}\), we have \(w = vw_H\) where \(H \subset I\) is such that \(u\) is conjugate in \(G\) to a regular unipotent element in the subgroup of \(G\) generated by \(\{y_i(a), x_i(a); i \in I - H, a \in \mathbb{C}\}\) and by \(T\). This is likely to be a general phenomenon.

For such \(u\) one can show that \(|w_H| \leq \dim B_u\) (complex dimension); this is compatible with \(|w_H| = \dim B_{u,\geq 0}\) (real dimension).

2.2. In this subsection we assume that \(G, I\) are as in 2.1 and \(n = 2\). In this case we can take \(V\) in 1.3 to be the adjoint representation of \(G\). The canonical basis \(\beta\) of \(V\) can be denoted by \(X_{-12}, X_{-1}, X_{-2}, t_1, t_2, X_1, X_2, X_{12}\) and the action of \(x_i(a), y_i(a), i \in I, a \in \mathbb{C}\) is as follows:

\[
\begin{align*}
x_i(a)X_{12} &= X_{12} \\
x_i(a)X_j &= X_j + aX_{12} \text{ if } i \neq j \in I \\
x_i(a)X_j &= X_j \text{ if } i = j \\
x_i(a)X_{-j} &= X_{-j} \text{ if } i \neq j \in I \\
x_i(a)X_{-j} &= X_{-j} + at_j + a^2X_j \text{ if } i = j \\
x_i(a)X_{-12} &= X_{-12} + aX_{-j} \text{ if } i \neq j \in I \\
x_i(a)t_j &= t_j + aX_i \text{ if } i \neq j \in I \\
x_i(a)t_j &= t_j + 2aX_i \text{ if } i = j \\
y_i(a)X_{12} &= X_{12} + aX_j \text{ if } i \neq j \in I \\
y_i(a)X_j &= X_j \text{ if } i \neq j \in I \\
y_i(a)X_j &= X_j + at_j + a^2X_{-j} \text{ if } i = j \\
y_i(a)X_{-j} &= X_{-j} + aX_{-12} \text{ if } i \neq j \in I \\
y_i(a)X_{-j} &= X_{-j} \text{ if } i = j \\
y_i(a)X_{-12} &= X_{-12}
\end{align*}
\]
modulo the homothety action of $R$
where $a_{-12}, a_{-1}, a_{-2}, c_1, c_2, a_1, a_2, a_{12}$ are in $R_{\geq 0}$ (not all 0) such that

\[ a_{-2}a_{-2} = c_2a_{-1}, a_1a_{-2} = c_1a_{-2}, a_{-1}a_{12} = c_1a_2, \]

\[ a_2a_{-12} = c_2a_{-1}, a_1a_{-12} = c_1a_{-2}, a_{-1}a_{12} = c_1a_2, \]

\[ a_{-2}a_{12} = c_2a_1, a_{12}(c_1 + c_2) = a_1a_2, a_{-12}(c_1 + c_2) = a_{-1}a_{-2}, \]

\[ c_1c_2 = a_{12}a_{-12}, c_1(c_1 + c_2) = a_1a_{-1}, c_2(c_1 + c_2) = a_2a_{-2} \]

modulo the homothety action of $R_{> 0}$.

The subsets $[v, w]$ of $X_{\geq 0}$ can be described as follows (the coefficients $a_{-12}, a_{-1}, a_{-2}, c_1, c_2, a_1, a_2, a_{12}$ are required to be in $R_{\geq 0}$ and are taken up to simultaneous multiplication by an element in $R_{> 0}$):

1. $[12, 12]: \{ a_{-12}X_{-12} \}$,
2. $[12, 21]: \{ a_{-2}X_{-2} \}$,
3. $[2, 2]: \{ a_1X_1 \}$,
4. $[1, 1]: \{ a_2X_2 \}$,
5. $[-, -]: \{ a_{12}X_{12} \}$,
6. $[21, 12]: \{ a_{-12}X_{-12} + a_{-2}X_{-2} \}$,
7. $[21, 21]: \{ a_{-12}X_{-12} + a_{-1}X_{-1} \}$,
8. $[12, 12]: \{ a_{-1}X_{-1} + a_2X_2 \}$,
9. $[2, 21]: \{ a_{-2}X_{-2} + a_1X_1 \}$,
10. $[-, 2]: \{ a_1X_1 + a_{12}X_{12} \}$,
11. $[-, 1]: \{ a_2X_2 + a_{12}X_{12} \}$,
12. $[2, 12]: \{ a_{-1}X_{-1} + c_1t_1 + a_1X_1; a_{-1}a_1 = c_1^2 \}$,
13. $[1, 21]: \{ a_{-2}X_{-2} + c_2t_2 + a_2X_2; a_{-2}a_2 = c_2^2 \}$,
14. $[2, 12]: \{ a_{-12}a_{-2}, a_{-1}a_{12} = c_1a_{-2} \}$,
15. $[1, 12]: \{ a_{-12}X_{-12} + a_{-1}X_{-1} + a_{-2}X_{-2} + c_1t_1 + a_1X_1, a_{-1}a_1 = c_1^2 \}$,
16. $c_1a_{-12} = a_{-1}a_{-2}, a_1a_{12} = c_1a_{-2}$,
17. $[1, 12]: \{ a_{-12}X_{-12} + a_{-1}X_{-1} + a_{-2}X_{-2} + c_2t_2 + a_2X_2; a_{-2}a_2 = c_2^2 \}$,
18. $c_2a_{-12} = a_{-1}a_{-2}, a_2a_{12} = c_2a_{-1}$,
19. $[-, 12]: \{ a_{-1}X_{-1} + c_1t_1 + a_1X_1 + a_2X_2 + a_{12}X_{12}; a_{-1}a_1 = c_1^2 \}$,
20. $c_1a_{12} = a_1a_2, a_{-1}a_{12} = c_1a_2$. 

TOTAL POSITIVITY IN SPRINGER FIBRES 11
We have \[ [-, 21] : \{a_{-2}X_{-2} + c_2 t_2 + a_2 X_2 + a_1 X_1 + a_{12} X_{12} ; a_{-2}a_2 = c_2^2, \]
\[ c_2 a_{12} = a_1 a_2, a_{-2} a_{12} = c_2 a_1 \}, \]
\[ [-, 121] : \]
\[ \{a_{-2}X_{-2} + a_{-1} X_{-1} + a_{-2}X_{-2} + c_1 t_1 + c_2 t_2 + a_1 X_1 + a_2 X_2 + a_{12} X_{12} \]
such that (a) holds.

2.3. We preserve the setup of 2.2.

(a) For any \( v \leq w \) in \( W \) there is a well defined subset \([v, w]\) of \( \beta \) such that \([v, w]\) consists of all lines in \( X_{\geq 0} \) which contain some vector spanned by an \( \mathbb{R}_{>0} \)-linear combination of vectors in \([v, w]\).

We have
\[ [[121, 121]] = \{X_{-12}\}, \]
\[ [[12, 12]] = \{X_{-1}\}, \]
\[ [[21, 21]] = \{X_{-2}\}, \]
\[ [[2, 2]] = \{X_1\}, \]
\[ [[1, 1]] = \{X_2\}, \]
\[ [[-, -]] = \{X_{12}\}, \]
\[ [[21, 121]] = \{X_{-12}, X_{-2}\}, \]
\[ [[12, 121]] = \{X_{-12}, X_{-1}\}, \]
\[ [[1, 12]] = \{X_{-1}, X_2\}, \]
\[ [[2, 21]] = \{X_{-2}, X_1\}, \]
\[ [[-, 2]] = \{X_1, X_{12}\}, \]
\[ [[-, 1]] = \{X_2, X_{12}\}, \]
\[ [[2, 12]] = \{X_{-1}, t_1, X_1\}, \]
\[ [[1, 21]] = \{X_{-2}, t_2, X_2\}, \]
\[ [[2, 121]] = \{X_{-12}, X_{-1}, X_{-2}, t_1, X_1\}, \]
\[ [[1, 121]] = \{X_{-12}, X_{-1}, X_{-2}, t_2, X_2\}, \]
\[ [[-, 12]] = \{X_{-1}, t_1, X_1, X_2, X_{12}\}, \]
\[ [[-, 21]] = \{X_{-2}, t_2, X_2, X_1, X_{12}\}, \]
\[ [[-, 121]] = \{X_{-12}, X_{-1}, X_{-2}, t_1, t_2, X_1, X_2, X_{12}\}. \]

We have

(b) \([v, w] = [[v, 121]] \cap [[-, w]]\) for any \( v \leq w \) in \( W \).

(c) There is a well defined partition \( \beta = \sqcup_{z \in W} \beta^-_z \) such that
\[ \{[-, w] = \sqcup_{z \in W, z \leq w} \beta^-_z \}
for any \( w \in W \). There is a well defined partition \( \beta = \sqcup_{z \in W} \beta^+_z \) such that
\[ [[v, 121]] = \sqcup_{z \in W, z \leq z \beta^+_z} \]

We have
\[ \beta^- = \{X_{12}\}, \beta^-_1 = \{X_2\}, \beta^-_2 = \{X_1\}, \beta^-_{12} = \{X_{-1}, t_1\}, \beta^-_{21} = \{X_{-2}, t_2\}, \]
\[ \beta^-_{121} = \{X_{-12}\}, \]
\[ \beta^+ = \{X_{12}\}, \beta^+_1 = \{t_2, X_2\}, \beta^+_2 = \{t_1, X_1\}, \beta^+_{21} = \{X_{-2}\}, \beta^+_{12} = \{X_{-1}\}, \]
\[ \beta^+_{121} = \{X_{-12}\}. \]
2.4. We return to the general case. Now 2.3(a),(b),(c) make sense in the general case (in (b),(c) we replace 121 by \( w_I \) and \(-\) by the unit element of \( W \)); we expect that these statements hold in the general case. In particular \( \beta^-_z \subset \beta \) and \( \beta^+_z \subset \beta \) are defined for \( z \in W \).

Let \( \eta^- \) be the unique vector in \( \beta \) such that the stabilizer of \( C \eta^- \) in \( G \) is \( B^- \). We can regard \( V \) naturally as a module over the universal enveloping algebra of the Lie algebra of \( G \) hence as a module over the universal enveloping algebra \( \mathfrak{U}^+ \) of the Lie algebra of \( U^+ \) and as a module over the universal enveloping algebra \( \mathfrak{U}^- \) of the Lie algebra of \( U^- \). Now \( \mathfrak{U}^+ \) (resp. \( \mathfrak{U}^- \)) has a canonical basis \( \hat{\beta}^+ \) (resp. \( \hat{\beta}^- \)), see [L90]) and the map \( c^+ : \hat{b} \mapsto \hat{b} \eta^- \) (resp. \( c^- : \hat{b} \mapsto \hat{b} \eta \)) from \( \hat{\beta}^+ \) (resp. \( \hat{\beta}^- \)) to \( V \) has image \( \beta \cup \{0\} \). From [L19, 10.2] we have a partition \( \hat{\beta}^+ = \bigsqcup_{w \in W} \hat{\beta}^+_w \); similarly we have a partition \( \hat{\beta}^- = \bigsqcup_{w \in W} \hat{\beta}^-_w \). We expect that for \( z \in W \), \( \beta^-_z \) is equal to \( c^-(\hat{\beta}^-_z) \) with 0 removed and that \( \beta^+_z \) is equal to \( c^+(\hat{\beta}^+_z) \) with 0 removed.

3. Partial flag manifolds

3.1. We fix \( H \subset I \). Let \( W^H \) be the set of all \( w \in W \) such that \( w \) has minimal length in \( wW_H \). Let \( P_H \) be the subgroup of \( G \) generated by \( \{x_i(a) ; i \in I, a \in C\} \), \( \{y_i(a) ; i \in H, a \in C\} \) and by \( T \) (a parabolic subgroup containing \( B^+ \)). Let \( \mathcal{P}_H \) be the variety whose points are the subgroups of \( G \) conjugate to \( P_H \). (We have \( P_\emptyset = B^+, P_\emptyset = B \).) Define \( \pi_H : \mathcal{B} \to \mathcal{P}_H \) by \( B \to P \) where \( P \in \mathcal{P}_H \) contains \( B \). As observed in [L98] to any \( P \in \mathcal{P}_H \) we can attach two Borel subgroups \( B',B'' \) of \( P \) such that \( \text{pos}(B^+,B') = b \in W^H, \text{pos}(B^-,B'') = a' \in W^H \); moreover \( B',B'' \) (hence \( a',b \)) are uniquely determined by \( P \). Let \( c = \text{pos}(B',B'') \in W^H \) and let \( a = wi \alpha \). We have \( a \leq bc \) (since \( B'' \in B_{a,b,c} \)) and \( ac^{-1} \leq b \) (since \( B' \in B_{a,c^{-1},b} \)). Conversely, if

(a) \((a,b,c) \in (w_I W^H) \times W^H \times W_H\) satisfy \( a \leq bc \) or equivalently \( ac^{-1} \leq b \), then the set of all \( P \in \mathcal{P}_H \) which give rise as above to \( a,b,c \) is non-empty; we denote this set by \( \mathcal{P}_{H,a,b,c} \). The subsets \( \mathcal{P}_{H,a,b,c} \) form a partition of \( \mathcal{P}_H \) indexed by the set \( \mathcal{P}_H(\{1\}) \) of triples \((a,b,c)\) as in (a).

Following [L98] we set \( \mathcal{P}_{H,\geq 0} = \pi_H(B_{\geq 0}) \), \( \mathcal{P}_{H,> 0} = \pi_H(B_{> 0}) \). For \((a,b,c) \in \mathcal{P}_H(\{1\})\) we set \( \mathcal{P}_{H,\geq 0,a,b,c} := \mathcal{P}_{H,a,b,c} \cap \mathcal{P}_{H,\geq 0} \)

so that we have

\[ \mathcal{P}_{H,\geq 0} = \bigsqcup_{(a,b,c) \in \mathcal{P}_H(\{1\})} \mathcal{P}_{H,\geq 0,a,b,c} \]

Let

\[ Z = \{(r,t) \in (w_I W^H) \times W ; r \leq t\}, \quad Z' = \{(r',t') \in W \times W^H ; r' \leq t'\} \]

We have bijections

\[ \alpha : \mathcal{P}_H(\{1\}) \to Z, \quad (a,b,c) \mapsto (a,bc), \quad \alpha' : \mathcal{P}_H(\{1\}) \to Z', \quad (a,b,c) \mapsto (ac^{-1},b) \]

We shall write \( r,t \mathcal{P}_{H,\geq 0} \) instead of \( \mathcal{P}_{H,\geq 0,a,b,c} \) where \((r,t) \in Z, (a,b,c) = \alpha^{-1}(r,t) \) and \( r',t' \mathcal{P}_{H,\geq 0} \) instead of \( \mathcal{P}_{H,\geq 0,a,b,c} \) where \((r',t') \in Z', (a,b,c) = \alpha'^{-1}(r',t') \). Thus we have
\[ \mathcal{P}_{H,\geq 0} = \bigcup_{(r,t) \in \mathbb{Z}_r, t} \mathcal{P}_{H,\geq 0} = \bigcup_{(r',t') \in \mathbb{Z}, t'} \mathcal{P}_{H,\geq 0} \]

and

\[ r.t \mathcal{P}_{H,\geq 0} = r'.t' \mathcal{P}_{H,\geq 0} \text{ if } (r', t') = \alpha' \alpha^{-1}(r, t). \]

In [R98, p.50,51] it is shown that for \((r, t) \in \mathbb{Z}, \pi_H\) restricts to a bijection

(b) \( \tilde{\alpha} : B_{\geq 0, r, t} \sim \rightarrow r.t \mathcal{P}_{H,\geq 0} \)

and that for \((r', t') \in \mathbb{Z}', \pi_H\) restricts to a bijection

(c) \( \tilde{\alpha}' : B_{\geq 0, r', t'} \sim \rightarrow r'.t' \mathcal{P}_{H,\geq 0} \).

This implies (by 1.7(a)) that \( \mathcal{P}_{H,\geq 0,a,b,c} \) is a cell for any \((a, b, c) \in \mathcal{P}_H(\{1\})\), so that the various \( \mathcal{P}_{H,\geq 0,a,b,c} \) form a cell decomposition of \( \mathcal{P}_{H,\geq 0} \); this justifies the notation \( \mathcal{P}_H(\{1\}) \).

3.2. Let \( H \) be as in 3.1. Let \( V_H \) be an irreducible rational \( G \)-module over \( \mathbb{C} \) such that \( \{g \in G; gL = L\} = \mathcal{P}_H \) for some (necessarily unique) line \( L \) in \( V \); let \( \eta \in L - \{0\} \). Let \( \beta \) be the basis of \( V_H \) (containing \( \eta \)) obtained by specializing at \( v = 1 \) the canonical basis [L90] of the corresponding module over the corresponding quantized enveloping algebra. Let \( X_H \) be the \( G \)-orbit of \( L \) in the set of lines in \( V_H \). We have a bijection

(a) \( \mathcal{P}_H \sim \rightarrow X_H \)

given by \( P \mapsto L_P \) where \( L_P \) is the unique \( P \)-stable line in \( V_H \). Let

(b) \( V_{H+} = \sum_{\beta \in \mathbb{R}_0} R_{\geq 0} g \subset V_H \).

From [L98] we see that \( V_H \) above can be chosen so that:

(c) under the bijection (a), \( \mathcal{P}_{H,\geq 0} \) corresponds to the set of lines in \( X_H \) which meet \( V_{H+} \neq \{0\} \).

In the sequel we assume that \( V_H \) has been chosen so that (c) holds.

3.3. For \( u \in U_{\geq 0} \) let \( \mathcal{P}_{H,u,u,0} = \{ P \in \mathcal{P}_{H,\geq 0}; u \in P \}, V_{H+}^u = \{ \xi \in V_{H+}; u\xi = \xi \} \). The proof of (a),(b),(c) below is identical to that of 1.6(a),(b),(c).

(a) Let \( u, u', u'' \) in \( U_{\geq 0} \) be such that \( u = u'u'' \). We have \( V_{H+}^u = V_{H+}^{u'} \cap V_{H+}^{u''} \).

(b) Let \( i \in I, a \in R_0 \). We have \( V_{H+}^{x_i(a)} = V_{H+}^{x_i(1)}, V_{H+}^{y_i(a)} = V_{H+}^{y_i(1)} \).

(c) Under the bijection 3.2(c), \( \mathcal{P}_{H,u,u,0} \) corresponds to the set of lines in \( X_H \) which meet \( V_{H+}^u \neq \{0\} \).

3.4. For any \( h \in G \) let \( [h]_H : \mathcal{P}_H \rightarrow \mathcal{P}_H \) be the map \( P \mapsto hPh^{-1} \).

Let \( i \in I, a \in R_0 \). Let \( (r, t) \in \mathbb{Z} \). Now (a),(b),(c) below follow immediately from 1.10(a),(b),(c) using 3.1(b).

(a) If \( r \leq s, t \leq t \), then \( [y_i(a)]_H : \mathcal{P}_H \rightarrow \mathcal{P}_H \) restricts to a map \( r.t \mathcal{P}_{H,\geq 0} \rightarrow r.t \mathcal{P}_{H,\geq 0} \) which is fixed point free.

(b) If \( s, t \leq t, r \neq s.t \), then \([y_i(a)]_H : \mathcal{P}_H \rightarrow \mathcal{P}_H \) restricts to the identity map \( r.t \mathcal{P}_{H,\geq 0} \rightarrow r.t \mathcal{P}_{H,\geq 0} \).

(c) If \( t \leq s.t \) then \([y_i(a)]_H : \mathcal{P}_H \rightarrow \mathcal{P}_H \) restricts to a map \( r.t \mathcal{P}_{H,\geq 0} \rightarrow r.s.t \mathcal{P}_{H,\geq 0} \); note that \( (r, s, t) \in \mathbb{Z} \).
3.5. Let \( i \in I, a \in \mathbb{R}_{> 0} \). Let \( (r', t') \in \mathbb{Z}' \). Now (a),(b),(c) below follow immediately from 1.11(a),(b),(c) using 3.1(c).

(a) If \( r' \leq s_i r' \leq t', \) then \( [x_i(a)]_H : \mathcal{P}_H \to \mathcal{P}_H \) restricts to a map \( r', t' \mathcal{P}_{H, \geq 0} \to r', t' \mathcal{P}_{H, \geq 0} \) which is fixed point free.

(b) If \( r' \leq s_i r', s_i r' \leq t', \) then \( [x_i(a)]_H : \mathcal{P}_H \to \mathcal{P}_H \) restricts to the identity map \( r', t' \mathcal{P}_{H, \geq 0} \to r', t' \mathcal{P}_{H, \geq 0} \).

(c) If \( s_i r' \leq r' \) then \( [x_i(a)]_H : \mathcal{P}_H \to \mathcal{P}_H \) restricts to a map \( r', t' \mathcal{P}_{H, \geq 0} \to s_i r', t' \mathcal{P}_{H, \geq 0} ; \) note that \( (s_i r', t') \in \mathbb{Z}' \).

3.6. Let \( i \in I, a \in \mathbb{R}_{> 0} \). From 3.4 we deduce:

(a) \[ \{ P \in \mathcal{P}_{H, \geq 0} ; y_i(a) \in P \} = \bigcup_{(r, t) \in \mathbb{Z} ; s_i t \leq r \leq s_i t} (r, t) \mathcal{P}_{H, \geq 0} . \]

From 3.5 we deduce:

(b) \[ \{ P \in \mathcal{P}_{H, \geq 0} ; x_i(a) \in P \} = \bigcup_{(r', t') \in \mathbb{Z}' ; r' \leq s_i r', s_i r' \leq t'} (r', t') \mathcal{P}_{H, \geq 0} . \]

The proof of the following result is entirely similar to that of 1.14, using 3.3(a),(b) instead of 1.6(a),(b).

**Theorem 3.7.** Let \( (z, z') \in (W \times W)_{\text{disj}} \) and let \( u \in U_{\geq 0, z, z'} , J = \text{supp} (z) , J' = \text{supp} (z') \). We have
\[
\mathcal{P}_{H, u, \geq 0} = \bigcap_{i \in I} \mathcal{P}_{H, y_i(1), \geq 0} \bigcap_{j \in J', P_{H, x_j(1), \geq 0}} .
\]

3.8. Let \( J \subset I, J' \subset I \) be such that \( J \cap J' = \emptyset \). We set
\[
Z_{H, J, J'} = \{(r, t), (r', t') \in \mathbb{Z} \times \mathbb{Z}' ;
(r', t') = a_i a \alpha^{-1} (r, t) , s_i t \leq r \leq s_i t \quad \forall i \in J, r' \leq s_j r', s_j r' \leq t' \quad \forall j \in J' \}.
\]

Combining 3.7 with 3.6(a),(b) we obtain the following result.

**Corollary 3.9.** Let \( (z, z') \in (W \times W)_{\text{disj}} \) and let \( u \in U_{\geq 0, z, z'} , J = \text{supp} (z) , J' = \text{supp} (z') \). We have
\[
\mathcal{P}_{H, u, \geq 0} = \bigcup_{(r, t), (r', t') \in Z_{H, J, J'}} (r, t) \mathcal{P}_{H, \geq 0} .
\]

Thus \( \mathcal{P}_{H, u, \geq 0} \) admits a canonical, explicit, cell decomposition in which each cell is part of the canonical cell decomposition of \( \mathcal{P}_{H, \geq 0} \). The zero dimensional cells of \( \mathcal{P}_{H, u, \geq 0} \) are indexed by
\[
\{(r, r), (r', r') ; r' \in W^H , r = r' w_H , s_i r \leq r \quad \forall i \in J, r' \leq s_j r' \quad \forall j \in J' \}.
\]

The last set contains for example \( ((r, r), (r', r')) \) where
\[ r' = w_j w_{J \cap H}, r = w_j w_{J \cap H} w_H . \]
3.10. From 3.4 and 3.5 we see that in the action of \(G_{\geq 0}\) on \(\mathcal{P}_{H,\geq 0}\), the result of applying a piece of \(G_{\geq 0}\) to a cell of \(\mathcal{P}_{H,\geq 0}\) is contained in a single cell of \(\mathcal{P}_{H,\geq 0}\). It follows that the action of \(G_{\geq 0}\) on \(\mathcal{P}_{H,\geq 0}\) induces an action of \(G\{1\} = W \times W\) on \(\mathcal{P}_H\{1\}\). We can identify \(\mathcal{P}_H\{1\}\) with
\[
\{(r, t), (r', t') \in Z \times Z'; (r', t') = \alpha' \alpha^{-1}(r, t)\}.
\]
Then the action of \(W \times W\) becomes:
\[
(s_i, 1): ((r, t), (r', t')) \mapsto ((r, s_i * t), \alpha' \alpha^{-1}(r, s_i * t)),
\]
\[
(1, s_i): ((r, t), (r', t')) \mapsto (\alpha \alpha^{-1}(s_i \circ r', t'), (s_i \circ r', t')),
\]
(notation of 1.17).

3.11. Let \(P \in \mathcal{P}_{H,\geq 0}\) and let \(\bar{P} = P \cap \mathcal{U}_{\geq 0}\). This is a closed subset of \(P\). We have \(P \in \mathcal{P}_{H,\geq 0} = \mathcal{P}_{H,\geq 0}\) for a well defined \(((r, t), (r', t')) \in Z \times Z'\) such that \((r', t') = \alpha' \alpha^{-1}(r, t)\). We set
\[
\Xi_{H,(r,t),(r',t')} = \{(z, z') \in (W \times W)_{\mathrm{disj}}; s_i t \leq t, r \not\leq s_i t \quad \forall i \in \mathrm{supp}(z); \quad r' \leq s_j r', s_j r' \not\leq t' \quad \forall j \in \mathrm{supp}(z')\}.
\]
We show:
\[
(a) \quad \bar{P} = \bigcup_{(z, z') \in \Xi_{H,(r,t),(r',t')}} \mathcal{U}_{\geq 0, z, z'}.
\]
Let \(E\) be the right hand side of (a). Let \(u \in \bar{P}\). Let \((z, z') \in (W \times W)_{\mathrm{disj}}\) be such that \(u \in \mathcal{U}_{\geq 0, z, z'}\). Let \(J = \mathrm{supp}(z)\), \(J' = \mathrm{supp}(z')\). We have \(P \in \mathcal{P}_{H,u,\geq 0}\) hence by 3.9 we have \(((r, t), (r', t')) \in Z_{H,J,J'}\) hence \((z, z') \in \Xi_{H,(r,t),(r',t')}\). Thus, \(u \in E\). We see that \(\bar{P} \subset E\). Conversely, assume that \(u \in E\). We can find \((z, z') \in \Xi_{H,(r,t),(r',t')}\) such that \(u \in \mathcal{U}_{\geq 0, z, z'}\). Let \(J = \mathrm{supp}(z), J' = \mathrm{supp}(z')\). We have \(((r, t), (r', t')) \in Z_{H,J,J'}\). Since \(P \in \mathcal{P}_{H,u,\geq 0}\), from 3.9 we see that \(P \in \mathcal{P}_{H,u,\geq 0}\) so that \(u \in P\) and \(u \in P\). We see that \(E \subset \bar{P}\). This proves (a).

From (a) we see that \(\bar{P}\) has a canonical cell decomposition with cells indexed by \(\Xi_{H,(r,t),(r',t')}\).

4. A conjecture and its consequences

4.1. An \(R_{>0}\)-positive structure on a set \(\mathfrak{X}\) is a finite collection \(f_e: R_{>0}^m \simto \mathfrak{X}\) of bijections \((e \in E)\) with \(m \geq 0\) fixed, such that for any \(e, e' \in E\), \(f_e^{-1} f_{e'}: R_{>0}^m \to R_{>0}^m\) is admissible in the sense of [L19, 1.2]. If \((\mathfrak{X}; f_e: R_{>0}^m \to \mathfrak{X}, e \in E)\), \((\mathfrak{X}; f_{\tilde{e}}: R_{>0}^m \to \mathfrak{X}, \tilde{e} \in \tilde{E})\) are two sets with \(R_{>0}\)-positive structure, a map \(\xi: \mathfrak{X} \to \mathfrak{X}\) is said to be a morphism if for some (or equivalently any) \(e \in E, \tilde{e} \in \tilde{E}\), the map \(f_{\tilde{e}}^{-1} \xi f_e: R_{>0}^m \to R_{>0}^m\) is admissible in the sense of [L19, 1.2]. We say that \(\xi\) is an isomorphism if it is a bijective morphism and \(\xi^{-1}\) is a morphism.

If \(\mathfrak{X}, \mathfrak{X}'\) have positive \(R_{>0}\)-structures, then \(\mathfrak{X} \times \mathfrak{X}'\) has a natural \(R_{>0}\)-positive structure.
4.2. Let $T_{>0} = T \cap G_{>0}$. For any basis $\chi_* = (\chi_1, \chi_2, \ldots, \chi_n)$ of $\text{Hom}(C^*, T)$ we have a bijection $R_{>0}^n \sim T_{>0}$ given by $(a_1, \ldots, a_n) \mapsto \chi_1(a_1) \chi_n(a_n)$. These bijections (for various $\chi_*$) define an $R_{>0}$-positive structure on $T_{>0}$.

Let $v \leq w$ in $W$. From [R08] we see that the bijections $\tau_1: R_{>0}^{[w]-[v]} \sim \mathcal{B}_{\geq 0, v, w}$ (see 1.8(c)) with $i \in I_w$ form an $R_{>0}$-positive structure on $\mathcal{B}_{\geq 0, v, w}$.

From 1.10 we see that for $i \in I$,

(a) the map $R_{>0} \times \mathcal{B}_{\geq 0, v, w} \rightarrow \mathcal{B}_{\geq 0, v, s_i w}$ given by $(a, B) \mapsto y_i(a) B y_i(a)^{-1}$ is a well defined morphism of sets with $R_{>0}$-positive structure.

From the definitions, for any $\chi \in \text{Hom}(C^*, T)$,

(b) the map $R_{>0} \times \mathcal{B}_{\geq 0, v, w} \rightarrow \mathcal{B}_{\geq 0, v, w}$ given by $(a, B) \mapsto \chi(a) B \chi(a)^{-1}$ is a well defined morphism of sets with $R_{>0}$-positive structure.

4.3. Replacing in 4.2 $v$, $w$ by $w_I w$, $w_I v$, we deduce that the bijections

$$\tau'_i: R_{>0}^{[w_I w]-[v]} = R_{>0}^{[w_I v]-[w_I w]} \sim \mathcal{B}_{\geq 0, w_I w, w_I v}$$

with $i' \in I_{w_I v}$ form an $R_{>0}$-positive structure on $\mathcal{B}_{\geq 0, w_I v, w_I v}$ and that for $i \in I$, the map

$$R_{>0} \times \mathcal{B}_{\geq 0, w_I w, w_I v} \rightarrow \mathcal{B}_{\geq 0, w_I w, s_i w_I v}$$

given by $(a, B) \mapsto y_i(a) B y_i(a)^{-1}$ is a well defined morphism of sets with $R_{>0}$-positive structure. We state:

**Conjecture 4.4.** The bijection $\mathcal{B}_{\geq 0, w_I w, w_I v} \sim \mathcal{B}_{\geq 0, v, w}$ defined by $\Phi$ (see 1.11(d)) is an isomorphism of $R_{>0}$-positive structures.

When $v = 1$, $w = w_I$ this can be deduced from [L97, §3].

In the remainder of this section we assume that this conjecture holds.

4.5. Let $v, w$ be as in 4.2. Using 4.4 we can reformulate the last statement in 4.3 as follows. For $i \in I$,

(a) the map $R_{>0} \times \mathcal{B}_{\geq 0, v, w} \rightarrow \mathcal{B}_{\geq 0, v, s_i v, w}$ given by $(a, B) \mapsto x_i(a) B x_i(a)^{-1}$ is a well defined morphism of sets with $R_{>0}$-positive structure.

4.6. We consider a sequence of admissible maps (see [L19, 1.2]):

$$\Phi_1: R_{>0} \times R_{>0}^{n_0} \rightarrow R_{>0}^{n_1}, \Phi_2: R_{>0} \times R_{>0}^{n_1} \rightarrow R_{>0}^{n_2},$$

$$\ldots, \Phi_\sigma: R_{>0} \times R_{>0}^{n_{\sigma-1}} \rightarrow R_{>0}^{n_\sigma}$$

where $n_0, n_1, \ldots, n_\sigma$ are in $N$. We define $\Phi: R_{>0}^{\sigma} \times R_{>0}^{n_0} \rightarrow R_{>0}^{n_\sigma}$ by

$$\Phi((a_1, a_2, \ldots, a_\sigma), b) = \ldots \Phi_3(a_3, \Phi_2(a_2, \Phi_1(a_1, b))) \ldots$$

where $b \in R_{>0}^{n_0}$. Clearly, $\Phi$ is admissible.
4.7. Consider the inverse image \( G_{r,-s} \) of \((r, s) \in W \times W\) under the map \( G_{\geq 0} \to W \times W\) in 1.17. Now \( G_{r,-s} \) has a natural \( \mathbb{R}_{>0} \)-positive structure (see \([L19]\)). Let \((i_1, i_2, \ldots, i_m) \in \mathcal{I}_r, (j_1, j_2, \ldots, j_l) \in \mathcal{I}_s\). Let \( v \leq w \) be elements of \( W \). Define \( v' \leq w' \) by \( v' = s_{i_l} \circ (\ldots s_{i_1} \circ (s_{i_m} \circ v)) \ldots, w' = s_{j_l} \ast (\ldots s_{j_1} \ast (s_{j_m} \circ w)) \ldots \) (notation of 1.17). The \( G_{\geq 0} \)-action on \( \mathcal{B}_{\geq 0} \) restricts to a map \( G_{r,-s} \times \mathcal{B}_{\geq 0,v,w} \to \mathcal{B}_{\geq 0,v',w'} \).

(a) This is a morphism of sets with \( \mathbb{R}_{>0} \)-positive structure.

Indeed, our map can be identified with a \( \Phi \) in 4.6 where each of \( \Phi_1, \ldots, \Phi_\sigma \) is a map as in 4.2(a),(b) or 4.5(a) hence is admissible.

4.8. We now fix a semifield \( K \). Let \((\mathfrak{X}; f_e : \mathbb{R}_{>0}^m \to \mathfrak{X}, e \in E)\) be a set with an \( \mathbb{R}_{>0} \)-positive structure. For any \( e, e' \in E \) the admissible bijection \( f_e^{-1} f_{e'} : \mathbb{R}_{>0}^m \to \mathbb{R}_{>0}^m \) induces a bijection \( f_{e,e'} : K^m \to K^m \). (This is obtained by replacing the indeterminates which appear in the formula for \( f_e^{-1} f_{e'} \) by elements of \( K \) instead of elements of \( \mathbb{R}_{>0} \).) There is a well defined set \( \mathfrak{X}(K) \) with bijections \( f_{e,e'} : K^m \to \mathfrak{X}(K) \) such that \( f_{e,e'} = f_{e',e}^{-1} f_{e,e'} \) for any \( e, e' \in E \). If \( \xi : \mathfrak{X} \to \mathfrak{X}' \) is a morphism of sets with \( \mathbb{R}_{>0} \)-positive structure then \( \xi \) induces a map of sets \( \xi(K) : \mathfrak{X}(K) \to \mathfrak{X}'(K) \).

4.9. In the setup of 4.7, the sets \( G_{r,-s}(K), \mathcal{B}_{\geq 0,v,w}(K), \mathcal{B}_{\geq 0,v',w'}(K) \) are defined as in 4.8 and by 4.8(a), the map \( G_{r,-s} \times \mathcal{B}_{\geq 0,v,w} \to \mathcal{B}_{\geq 0,v',w'} \) induces a map

(a) \( G_{r,-s}(K) \times \mathcal{B}_{\geq 0,v,w}(K) \to \mathcal{B}_{\geq 0,v',w'}(K) \).

We set \( G(K) = \bigsqcup_{(r,s) \in W \times W} G_{r,-s}(K), \mathcal{B}(K) = \bigsqcup_{(v,w) \in W \times W, v \leq w} \mathcal{B}_{\geq 0,v,w}(K) \). The maps \((a)\) define a map \( G(K) \times \mathcal{B}(K) \to \mathcal{B}(K) \). This is an action of \( G(K) \) (with the monoid structure induced from that of \( G_{\geq 0} \)) on the set \( \mathcal{B}(K) \).

4.10. Let \( H \subset I \). For any \((a, b, c) \in \mathcal{P}_H(\{1\})\), the cell \( \mathcal{P}_{H,\geq 0,a,b,c} \) in \( \mathcal{P}_{H,\geq 0} \) has a \( \mathbb{R}_{>0} \)-positive structure via a bijection as in 3.1(a) or as in 3.1(b) (these two bijections define the same \( \mathbb{R}_{>0} \)-positive structure, as we see easily from the definitions). Thus the set \( \mathcal{P}_{H,\geq 0,a,b,c}(K) \) is defined. We set

\[ \mathcal{P}_H(K) = \bigsqcup_{(a,b,c) \in \mathcal{P}_H(\{1\})} \mathcal{P}_{H,\geq 0,a,b,c}(K). \]

An argument similar to that in 4.9 shows that the \( G_{\geq 0} \)-action on \( \mathcal{P}_{H,\geq 0} \) (see 3.1(d)) induces an action of the monoid \( G(K) \) on \( \mathcal{P}_H(K) \).

4.11. Let \( u \in \mathcal{U}_{\geq 0} \). Let \( z, z', J, J' \) be as in 1.16. We set

\[ \mathcal{B}_u(K) = \bigsqcup_{(v,w) \in Z, J} \mathcal{B}_{\geq 0,v,w}(K), \]

\[ \mathcal{Z}(u)(K) = \bigsqcup_{(r,s) \in \mathcal{Z}_u(\{1\})} G_{r,-s}(K). \]

(notation of 1.15, 1.17). Then the action of \( G(K) \) on \( \mathcal{B}(K) \) restricts to an action of \( \mathcal{Z}(u)(K) \) (with the monoid structure induced from that of \( G(K) \)) on the set \( \mathcal{B}_u(K) \).
5. The map \( g \mapsto P_g \) from \( G_{\geq 0} \) to \( \mathcal{P}_{\geq 0} \)

5.1. In this section we give a new, simpler, definition of the map \( g \mapsto B_g \) from \( G_{\geq 0} \) to \( \mathcal{B}_{\geq 0} \) in [L94, 8.9(c)] and we extend it to a map \( g \mapsto P_g \) from \( G_{\geq 0} \) to \( \mathcal{P}_{\geq 0} \).

5.2. For a closed subgroup \( G' \) of \( G \) we denote by \( \mathfrak{L}G' \) the Lie algebra of \( G' \). Let \( g = \mathfrak{L}G \). Let \( g \in G_{\geq 0} \). We associate to \( g \) a parabolic subgroup \( P = P_g \) of \( G \) containing \( g \) as follows. By [L19, 9.1(a)], we have \( g = \oplus_{\alpha \in \mathfrak{R}_{>0}} \mathfrak{g}_\alpha \) where \( \mathfrak{g}_\alpha \) is the generalized \( \alpha \)-eigenspace of \( \text{Ad}(g) : g \rightarrow g \). It follows that we have \( g = \mathfrak{g}_{<1} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{>1} \) where \( \mathfrak{g}_{<1} = \oplus_{\alpha_1 < \alpha_1} \mathfrak{g}_\alpha \), \( \mathfrak{g}_{>1} = \oplus_{\alpha_1 > 1} \mathfrak{g}_\alpha \) and \( \mathfrak{g}_{1} = \mathfrak{g}_{<1} \oplus \mathfrak{g}_1 \). \( \mathfrak{g}_{>1} \) are opposed parabolic subalgebras of \( g \) with common Levi subalgebra \( \mathfrak{g}_1 \). Let \( P = P_g \) be the parabolic subgroup of \( G \) with \( \mathfrak{L}P = \mathfrak{g}_{>1} \). Let \( L = L_g \) be the Levi subgroup of \( P \) with \( \mathfrak{L}L = \mathfrak{g}_1 \). Note that \( L \) is the centralizer in \( G \) of the semisimple part \( g_s \) of \( g \). In particular we have \( g \in L \); more precisely, \( g \) is a central element of \( L \) times a unipotent element of \( L \). For example, if \( g \in G_{\geq 0} \) is unipotent then \( P_g = G \).

5.3. We now assume that \( g \in G_{\geq 0} \). We show:

(a) \( B_g = P_g \).

Let \( U_{>0}^+ \subset U^+, U_{>0}^- \subset U^- \) be as in [L94, 2.12]. Let \( T_{>0} \subset T \) be as in 4.7. By [L94, 8.10] we can find \( u \in U_{>0}^+, u' \in U_{>0}^-, t \in T_{>0} \) such that \( g = u'tuu'^{-1} \) and all eigenvalues of \( \text{Ad}(t) \) on \( \mathfrak{L}U^+ \) are \( > 1 \). Then all eigenvalues of \( \text{Ad}(t) \) on \( \mathfrak{L}B^+ \) are \( \geq 1 \). Now \( tu \) is \( U^+ \)-conjugate to \( t \) hence all eigenvalues of \( \text{Ad}(tu) \) on \( \mathfrak{L}B^+ \) are \( \geq 1 \). It follows that all eigenvalues of \( \text{Ad}(g) \) on \( \mathfrak{L}(u'B^+u'^{-1}) \) are \( \geq 1 \), so that \( \mathfrak{L}(u'B^+u'^{-1}) \subset \mathfrak{L}g \). Since \( t \) (and \( tu \)) is regular semisimple, \( L_g \) is a maximal torus of \( G \) so that \( P_g \in \mathcal{B} \); this implies that \( \mathfrak{L}(u'B^+u'^{-1}) = \mathfrak{L}g \). From the definition of \( \mathcal{B}_{>0} \) we have \( u'B^+u'^{-1} \in \mathcal{B}_{>0} \). Since \( g \in u'B^+u'^{-1} \), we have \( u'B^+u'^{-1} = B_g \). This proves (a).

5.4. Let \( g \in G_{\geq 0} \). We show:

(a) \( P_g \in \mathcal{P}_{\geq 0} \).

By [L94, 4.4] we can find a sequence \( g_1, g_2, g_3, \ldots \) in \( G_{>0} \) such that \( g = \lim_{n \to \infty} g_n \) in \( G \). Since \( B \) is compact, some subsequence of the sequence of Borel subgroups \( P_{g_n} \) converges in \( B \) to a Borel subgroup \( B \in \mathcal{B} \). We can assume that \( \lim_{n \to \infty} P_{g_n} = B \) in \( \mathcal{B} \). Since \( P_{g_n} \in \mathcal{B}_{>0} \subset \mathcal{B}_{\geq 0} \) and \( \mathcal{B}_{\geq 0} \) is closed in \( \mathcal{B} \), we have \( B \in \mathcal{B}_{\geq 0} \). Let \( p_n = \mathfrak{L}P_{g_n}, n = 1, 2, \ldots, p = \mathfrak{L}g \), and let \( b = \mathfrak{L}B \). We show:

(b) \( b \subset p \).

We have \( \lim_{n \to \infty} p_n = b \) in the Grassmannian of \( \text{dim} b \)-dimensional subspaces of \( g \). Since \( p_n \) is stable under \( \text{Ad}(g_n) \) for \( n = 1, 2, \ldots \), we see that \( b \) is stable under \( \text{Ad}(g) \). Moreover since all eigenvalues of \( \text{Ad}(g_n) : p_n \to p_n \) are \( \geq 1 \), we see that all eigenvalues of \( \text{Ad}(g) : b \to b \) are \( \geq 1 \). Hence (b) holds.

From (b) we deduce that \( B \subset P_g \). Since \( B \in \mathcal{B}_{\geq 0} \) we see that \( P_g \in \mathcal{P}_{\geq 0} \); this proves (a) and completes the verification of the statements in 5.1.

5.5. Let \( B \in \mathcal{B}_{>0} \). We show:

(a) \( B \cap G_{>0} \) is non-empty.

We have \( B = uB^+u^{-1} \) for some \( u \in U_{>0}^- \). Let \( \tilde{u} \in U_{>0}^+ \). By [L94, 7.2] we can
find $t \in T_{>0}$ such that $u\tilde{u}tu^{-1} \in G_{>0}$. We have $u\tilde{u}tu^{-1} \in uB^+u^{-1} = B$. Thus $u\tilde{u}tu^{-1} \in B \cap G_{>0}$ and (a) follows.

We conjecture that $B \cap G_{>0}$ is homeomorphic to $\mathbb{R}_{>0}^{\dim B}$. Consider for example the case where $G = SL_2(\mathbb{C})$. In this case there is a unique $z \in \mathbb{R}_{>0}$ such that $B$ equals $\left\{ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \mathbb{C} \mid ad = bc + 1, c + dz = z(a + bz) \right\}$. Then $B \cap G_{>0}$ can be identified with the set $\left\{ (a, b, c, d) \in \mathbb{R}_{>0}^4 \mid ad = bc + 1, c + dz = z(a + bz) \right\}$ or (via the substitution $d = (bc + 1)/a$) with the set $X_z = \left\{ (a, b, c) \in \mathbb{R}_{>0}^3 \mid c + \frac{bc + 1}{a} z = z(a + bz) \right\} = \left\{ (a, b, c) \in \mathbb{R}_{>0}^3 \mid az - c(bz + a) = z \right\}$. Setting $\epsilon = a - cz^{-1}$ we can identify $X_z$ with $X_z' = \left\{ (b, c, \epsilon) \in \mathbb{R}_{>0}^3 \mid \epsilon^2 + \epsilon(bz + cz^{-1}) - 1 = 0 \right\}$. The map $X_z' \to \mathbb{R}_{>0}^2$, $(b, c, \epsilon) \mapsto (b, c)$ is a homeomorphism: for any $(b, c) \in \mathbb{R}_{>0}^2$ there is a unique $\epsilon \in \mathbb{R}_{>0}$ such that $\epsilon^2 + \epsilon(bz + cz^{-1}) - 1 = 0$. This proves the conjecture in our case.

5.6. We show:

(a) If $g \in G_{>0}$ then $P_g \in B$; the image of the map $g \mapsto P_g$ from $G_{>0}$ to $B$ is exactly $B_{>0}$.

The fact that $P_g \in B_{>0}$ for $g \in G_{>0}$ follows from 5.3(a). Let $B \in B_{>0}$. Let $g \in B \cap G_{>0}$ (see 5.5(a)). From [L94, 8.9(a)] we see that $B = B_g$ hence by 5.3(a) we have $B = P_g$. This proves (a).

Now (a) provides a new definition of $B_{>0}$; it is the image of the map $g \mapsto P_g$ from $G_{>0}$ to $B$.

Let $G_{reg}$ be the set of regular elements in $G$ and let $G_{reg,>0} = G_{reg} \cap G_{>0}$. We conjecture that the map $g \mapsto P_g$ from $G_{>0}$ to $P_{>0}$ is surjective and that, moreover, its restriction $G_{reg,>0} \to P_{>0}$ is surjective.

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