Introduction to the Theory of Evolution Equations of Quantum Many-Particle Systems

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Abstract. In the paper we review some recent results of the theory of hierarchies of quantum evolution equations.

The evolution of infinite-particle quantum systems is described within the framework of the evolution of states by the quantum BBGKY hierarchy for marginal density operators or within the framework of an equivalent approach in terms of the marginal observables by the dual quantum BBGKY hierarchy. The nonperturbative solutions of the Cauchy problem of the dual quantum BBGKY hierarchy and the quantum BBGKY hierarchy are constructed.

The origin of the microscopic description of non-equilibrium quantum correlations is considered. It consists in one more approach to the description of the evolution of states of quantum many-particle systems by means of correlation operators. The correlation operators and the marginal correlation operators are governed by the von Neumann hierarchy and the quantum nonlinear BBGKY hierarchy respectively. The nonperturbative solutions of the Cauchy problem of both these hierarchies are constructed.

We develop also an approach to a description of the evolution of states by means of the quantum kinetic equations. For initial states which are specified in terms of a one-particle density operator the equivalence of the description of the evolution of quantum many-particle states by the Cauchy problem of the quantum BBGKY hierarchy and by the Cauchy problem of the generalized quantum kinetic equation together with a sequence of explicitly defined functionals of a solution of stated kinetic equation is established. The relationships of the specific quantum kinetic equations with the generalized quantum kinetic equation are discussed.

In conclusion the mean field asymptotic behavior of stated hierarchy solutions are established. The constructed asymptotics are governed by the quantum Vlasov hierarchy for limit states, the nonlinear quantum Vlasov hierarchy for limit correlations and the dual quantum Vlasov hierarchy for limit observables respectively.

Key words: dual BBGKY hierarchy; BBGKY hierarchy; nonlinear BBGKY hierarchy; von Neumann hierarchy; quantum kinetic equation; nonlinear Schrödinger equation; correlation operator; scaling limit; quantum many-particle system.

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1 Introduction: motivations and results

Experimental advances in the Bose condensation of atomic gases and in the strong correlated Fermi systems have stimulated interesting problems in the theory of evolution equations of quantum many-particle systems. Among them it is a description of collective behavior of interacting particles by quantum kinetic equations, i.e. the evolution equations for a one-particle marginal density operator \[12\]. The rigorous derivation of quantum kinetic equations is a very trendy subject nowadays, and the number of publications devoted to it has grown
last decade. In particular the considerable progress in the rigorous derivation of the nonlinear Schrödinger equation and the Gross-Pitaevskii equation for the Bose condensate \[8–19\] in mean field scaling limit as well as the quantum Boltzmann equation \[20, 21\] is observed. Owing to the intrinsic complexity and richness of this problem, first of all it is necessary to develop an adequate mathematical theory of evolution equations of quantum many-particle systems underlying of kinetic equations.

In this review we expound the foundation of the theory of quantum evolution equations of statistical mechanics. We construct nonperturbative solutions of the Cauchy problem of the quantum BBGKY hierarchy for marginal density operators and the dual quantum BBGKY hierarchy for marginal observables on appropriate Banach spaces. On basis of obtained results the origin of the microscopic description of non-equilibrium quantum correlations is considered. In particular, the nonperturbative solutions of the Cauchy problem of the von Neumann hierarchy for correlation operators and the nonlinear quantum BBGKY hierarchy for marginal correlation operators are constructed. We develop also one more approach to the description of the evolution of quantum many-particle systems by means of the quantum kinetic equations. For initial states which are specified in terms of a one-particle density operator the equivalence of the description of the evolution of quantum infinite-particle states by the Cauchy problem of the quantum BBGKY hierarchy and by the Cauchy problem of the generalized quantum kinetic equation is established. We also describe the mean field scaling asymptotic behavior of nonperturbative solutions of hierarchies of quantum evolution equations under consideration.

We outline the structure of the paper and the main results. In introductory section 2 we set forth the traditional approach to the description of the evolution of quantum many-particle systems. It is well known that a description of quantum many-particle systems is formulated in terms of two sets of objects: observables and states. The functional of the mean value of observables defines a duality between observables and states and as a consequence there exist two equivalent approaches to the description of the evolution, namely in terms of the evolution equations for observables (the Heisenberg equation) and for states (the von Neumann equation). We adduce some preliminary facts about dynamics of finitely many quantum particles described by these evolution equations within the framework of nonequilibrium grand canonical ensemble.

In section 3 one more an equivalent approach to the description of the evolution of states of quantum many-particle systems is given by means of operators which are interpreted as correlation operators. To justify the von Neumann hierarchy for correlation operators, at first we consider in detail the motivation of the introduction of correlation operators to the description of states or in other words, the origin of the microscopic description of correlations in quantum many-particle systems is considered.

In section 4 for the description of the evolution of infinite-particle quantum systems we introduce the hierarchies of evolution equations for the marginal observables, density operators and correlation operators. They are derived as the evolution equations for one more but an equivalent method of the description of states and observables of finitely many particles. The nonperturbative solutions of the Cauchy problem of the dual quantum BBGKY hierarchy for the marginal observables, the quantum BBGKY hierarchy for marginal density operators and the nonlinear quantum BBGKY hierarchy for marginal correlation operators are constructed as an expansion over particle clusters which evolution is governed by the corresponding-order cumulant (semi-invariant) of the groups of operators of the Heisenberg equations, the von Neumann equations and the von Neumann hierarchy of finitely many particles respectively.
In section 5 we develop an approach to a description of the evolution of states by means of the quantum kinetic equations. For initial states which are specified in terms of a one-particle density operator the equivalence of the description of the evolution of quantum many-particle states by the Cauchy problem of the quantum BBGKY hierarchy and by the Cauchy problem of the generalized quantum kinetic equation together with a sequence of explicitly defined correlation functionals of a solution of stated quantum kinetic equation is established. The relationship of the generalized quantum kinetic equation with the specific quantum kinetic equations is discussed.

In section 6 the mean field (self-consistent field) asymptotic behavior of constructed above solutions of the quantum hierarchies and the generalized quantum kinetic equation is established. In particular, the quantum Vlasov hierarchy for limit states, the nonlinear quantum Vlasov hierarchy for limit correlations and the dual quantum Vlasov hierarchy for limit observables are rigorously derived.

Finally in section 7 we conclude with some observations and perspectives for future research.

2 On evolution equations of quantum many-particle systems

It is well known that a description of quantum many-particle systems is formulated in terms of two sets of objects: observables and states. The functional of the mean value of observables defines a duality between observables and states and as a consequence there exist two approaches to the description of the evolution, namely in terms of the evolution equations for observables (the Heisenberg equation) and for states (the von Neumann equation). In this section we adduce some preliminary facts about dynamics of finitely many quantum particles described within the framework of nonequilibrium grand canonical ensemble.

2.1 Preliminaries

We consider a quantum system of a non-fixed, i.e. arbitrary but finite, number of identical (spinless) particles (nonequilibrium grand canonical ensemble) obeying the Maxwell-Boltzmann statistics in the space $\mathbb{R}^\nu$. We will use units where $\hbar = 2\pi\hbar = 1$ is a Planck constant, and $m = 1$ is the mass of particles.

Let $\mathcal{H}$ be a one-particle Hilbert space, then the $n$-particle space $\mathcal{H}_n$, is the tensor product of $n$ Hilbert spaces $\mathcal{H}$. We adopt the usual convention that $\mathcal{H}^\otimes 0 = \mathbb{C}$. We denote by $\mathbf{F}_\mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}_n$ the Fock space over the Hilbert space $\mathcal{H}$.

Let a sequence $g = (g_0, g_1, \ldots, g_n, \ldots)$ be an infinite sequence of self-adjoint bounded operators defined on the Fock space $\mathbf{F}_\mathcal{H}$ and $g_0 \in \mathbb{C}$. The operator $g_n$ defined on the $n$-particle Hilbert space $\mathcal{H}_n = \mathcal{H}^\otimes n$ will be denoted by $g_n(1, \ldots, n)$. Let the space $\mathcal{L}(\mathcal{F}_\mathcal{H})$ be the space of sequences $g = (g_0, g_1, \ldots, g_n, \ldots)$ of bounded operators $g_n$ defined on the Hilbert space $\mathcal{H}_n$ that satisfy symmetry condition: $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$, for arbitrary $(i_1, \ldots, i_n) \in (1, \ldots, n)$, equipped with the operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{H}_n)}$ [5]. We will also consider a more general space $\mathcal{L}_\gamma(\mathcal{F}_\mathcal{H})$ with a norm

$$
\|g\|_{\mathcal{L}_\gamma(\mathcal{F}_\mathcal{H})} = \max_{n \geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathcal{L}(\mathcal{H}_n)},
$$

where $\gamma > 0$. This norm defines a complete normed linear space which is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{L}_\gamma(\mathcal{F}_\mathcal{H})}$.
where $0 < \gamma < 1$. We denote by $\mathfrak{L}_{\gamma}(\mathcal{F}_H) \subset \mathfrak{L}_{\gamma}(\mathcal{F}_H)$ the everywhere dense set of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports. Observables of finitely many quantum particles are sequences of self-adjoint operators from the space $\mathfrak{L}_{\gamma}(\mathcal{F}_H)$ [5].

Let $\mathfrak{L}^1(\mathcal{F}_H) = \bigoplus_{n=0}^{\infty} \mathfrak{L}^1(\mathcal{H}_n)$ be the space of sequences $f = (f_0, f_1, \ldots, f_n, \ldots)$ of trace class operators $f_n = f_n(1, \ldots, n) \in \mathfrak{L}^1(\mathcal{H}_n)$ and $f_0 \in \mathbb{C}$, satisfying the mentioned above symmetry condition, equipped with the trace norm

$$
\| f \|_{\mathfrak{L}^1(\mathcal{F}_H)} = \sum_{n=0}^{\infty} \| f_n \|_{\mathfrak{L}^1(\mathcal{H}_n)} = \sum_{n=0}^{\infty} \text{Tr}_{1, \ldots, n} | f_n(1, \ldots, n) |,
$$

where $\text{Tr}_{1, \ldots, n}$ is the partial trace over $1, \ldots, n$ particles. The everywhere dense set of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports from the space $\mathfrak{L}^1(\mathcal{F}_H)$ we denote by $\mathfrak{L}_0^1(\mathcal{F}_H)$. The sequences of self-adjoint operators $f_n \in \mathfrak{L}^1(\mathcal{H}_n)$, $n \geq 1$, which kernels are known as density matrices, defined on the $n$-particle Hilbert space $\mathcal{H}_n = L^2(\mathbb{R}^{en})$, describe states of a quantum system of non-fixed number of particles.

The space $\mathfrak{L}(\mathcal{F}_H)$ is dual to the space $\mathfrak{L}^1(\mathcal{F}_H)$ with respect to the bilinear form

$$
(g, f) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, n} g_n f_n,
$$

where $g_n \in \mathfrak{L}(\mathcal{H}_n)$ and $f_n \in \mathfrak{L}^1(\mathcal{H}_n)$. The range of positive continuous linear functional $[1]$ on the space of observables $\mathfrak{L}(\mathcal{F}_H)$ is interpreted as the mean values (averages) of observables.

The Bose-Einstein and Fermi-Dirac statistics endow observables and states with additional symmetry properties [6] in comparison with the Maxwell-Boltzmann statistics. Let $\mathcal{H}$ be a one-particle Hilbert space, then the $n$-particle spaces $\mathcal{H}_n^{\pm}$, are correspondingly symmetric and antisymmetric tensor products of $n$ Hilbert spaces $\mathcal{H}$ that are associated with $n$-particle systems of bosons and fermions [6]. We denote by $\mathcal{F}_H^{\pm} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{\pm}$ the Bose and Fermi Fock spaces over the Hilbert space $\mathcal{H}$ respectively.

The symmetrization operator $S_n^+$ and the anti-symmetrization operator $S_n^-$ on $\mathcal{H}^{\otimes n}$ are defined by the formula

$$
S_n^\pm = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} (\pm 1)^{|\pi|} p_\pi,
$$

where the operator $p_\pi$ is a transposition operator of the permutation $\pi$ from the permutation group $\mathfrak{S}_n$ of the set $(1, \ldots, n)$ and $|\pi|$ denotes the number of transpositions in the permutation $\pi$. The operators $S_n^\pm$ are orthogonal projectors, i.e. $(S_n^+)^2 = S_n^+$, ranges of which are correspondingly the symmetric tensor product $\mathcal{H}_n^+$ and the antisymmetric tensor product $\mathcal{H}_n^-$ of $n$ Hilbert spaces $\mathcal{H}$.

### 2.2 The Heisenberg equation: the evolution of observables

The Hamiltonian $H_n$ of $n$-particle system is a self-adjoint operator with the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$

$$
H_n = \sum_{i=1}^{n} K(i) + \sum_{i_1 < i_2 = 1}^{n} \Phi(i_1, i_2),
$$

where $K(i)$ and $\Phi(i_1, i_2)$ are the kinetic and potential energy operators, respectively.
where $K(i)$ is the operator of a kinetic energy of the $i$ particle and $\Phi(i_1, i_2)$ is the operator of a two-body interaction potential. The operator $K(i)$ acts on functions $\psi_n$ that belong to the subspace $L^2_0(\mathbb{R}^{2n}) \subset D(H_n) \subset L^2(\mathbb{R}^{2n})$ of infinitely differentiable functions with compact supports according to the formula: $K(i)\psi_n = -\frac{1}{2}\Delta_{q_i}\psi_n$. Correspondingly we have: $\Phi(i_1, i_2)\psi_n = \Phi(q_{i_1}, q_{i_2})\psi_n$, and we assume that the function $\Phi(q_{i_1}, q_{i_2})$ is symmetric with respect to permutations of its arguments, translation-invariant and bounded function.

The evolution of observables $A(t) = (A_0, A_1(t, 1), \ldots, A_n(t, 1, \ldots, n), \ldots)$ is described by the initial-value problem of a sequence of the Heisenberg equations \[4, 5\]

$$
\frac{d}{dt} A(t) = \mathcal{N} A(t),
$$

$$
A(t)|_{t=0} = A(0),
$$

where $A(0) = (A_0, A_1(0)(1), \ldots, A_n(0)(1, \ldots, n), \ldots) \in \mathcal{L}(\mathcal{F}_\mathcal{H})$, if $g \in D(\mathcal{N}) \subset \mathcal{L}(\mathcal{F}_\mathcal{H})$, the generator $\mathcal{N} = \bigoplus_{n=0}^{\infty} \mathcal{N}_n$ is defined by the formula

$$
\mathcal{N}_n g_n \doteq -i(g_n H_n - H_n g_n),
$$

and $H_n$ is the Hamiltonian \[3\].

To determine a solution of the Cauchy problem \[1\]-\[5\] we introduce on the space $\mathcal{L}(\mathcal{F}_\mathcal{H})$ the following one-parameter family of operators $\mathcal{G}(t) = \bigoplus_{n=0}^{\infty} \mathcal{G}_n(t)$:

$$
\mathcal{G}_n(t) g_n \doteq e^{itH_n} g_n e^{-itH_n}.
$$

On the space $\mathcal{L}_\gamma(\mathcal{F}_\mathcal{H})$ the one-parameter mapping $\mathbb{R}^1 \ni t \mapsto \mathcal{G}(t) g$ defines an isometric $*$-weak continuous group of operators, i.e. it is a $C^*$-group \[5\]-\[6\]. The infinitesimal generator $\mathcal{N} = \bigoplus_{n=0}^{\infty} \mathcal{N}_n$ of this group of operators is a closed operator for the $*$-weak topology and on its domain of definition $D(\mathcal{N}_n) \subset \mathcal{L}(\mathcal{H}_n)$ which is everywhere dense for the $*$-weak topology, $\mathcal{N}_n$ is defined as follows in the sense of the $*$-weak convergence of the space $\mathcal{L}(\mathcal{H}_n)$

$$
w^* - \lim_{t \to 0} \frac{1}{t}(\mathcal{G}_n(t) g_n - g_n) = -i(g_n H_n - H_n g_n),
$$

where the operator $\mathcal{N}_n g_n = -i(g_n H_n - H_n g_n)$ is defined on the domain $D(H_n) \subset \mathcal{H}_n$.

The group of operators \[7\] preserves the self-adjointness of operators and the canonical commutation relations which are determined the physical meaning of operators characterizing particles and the structure of generator \[6\] of the Heisenberg equation \[4\].

On the space $\mathcal{L}_\gamma(\mathcal{F}_\mathcal{H})$ for initial-value problem \[1\]-\[5\] the following statement holds.

**Theorem 1.** A unique solution of initial-value problem \[1\]-\[5\] of the Heisenberg equation is determined by the formula

$$
A(t) = \mathcal{G}(t) A(0),
$$

where the one-parameter family $\{\mathcal{G}(t)\}_{t \in \mathbb{R}}$ of operators is defined by expression \[7\]. For $A(0) \in D(\mathcal{N}) \subset \mathcal{L}_\gamma(\mathcal{F}_\mathcal{H})$, operator \[8\] is a classical solution and for arbitrary initial data $A(0) \in \mathcal{L}_\gamma(\mathcal{F}_\mathcal{H})$, it is a generalized solution.
The average values of observables (mean values of observables) are defined by the positive continuous linear functional on the space $\mathcal{L}(\mathcal{F}_H)$

$$\langle A \rangle(t) = \langle A(t), D(0) \rangle = (I, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} A_n(t) D_n^0,$$

(10)

where $\text{Tr}_{1,\ldots,n}$ are the partial traces over $1, \ldots, n$ particles, $D(0) = (1, D_1^0, \ldots, D_n^0, \ldots)$ is a sequence of self-adjoint positive density operators defined on the Fock space $\mathcal{F}_H$ that describes the states of a quantum system of a non-fixed number of particles and $(I, D(0)) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} D_n^0$ is a normalizing factor (the grand canonical partition function). For $D(0) \in \mathcal{L}^1(\mathcal{F}_H)$ and $A(t) \in \mathcal{L}(\mathcal{F}_H)$ average value functional (10) exists and determines a duality between observables and states.

We note that in case of a system of fixed number $N$ of particles the observables and states are one-component sequences $A^{(N)}(t) = (0, \ldots, 0, A_N(t), 0, \ldots)$ and $D^{(N)}(0) = (0, \ldots, 0, D_N^0, 0, \ldots)$, respectively, and therefore, average value formula (10) reduces to the functional

$$\langle A^{(N)} \rangle(t) = \langle \text{Tr}_{1,\ldots,N} D_N^0 \rangle^{-1} \text{Tr}_{1,\ldots,N} A_N(t) D_N^0.$$

It is usually assumed that the normalizing condition: $\text{Tr}_{1,\ldots,N} D_N^0 = 1$, take place.

On the spaces $\mathcal{L}(\mathcal{H}_n^\pm)$ mapping (11) is defined and have the similar properties as in case of the Maxwell-Boltzmann statistics stated above.

Symmetrization and anti-symmetrization operators (2) are integrals of motion of the Heisenberg equation (4), and as a consequence it holds

$$G_n(t) S_n^\pm = S_n^\pm G_n(t),$$

and

$$N_n S_n^\pm = S_n^\pm N_n,$$

where the operator $N_n$ is defined by (6). Thus, the symmetry of observables is preserved in process of the evolution.

### 2.3 The von Neumann equation: the evolution of states

As a consequence of the validity for functional (10) of the following equality

$$(A(t), D(0)) = (I, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} G_n(t) A_n(t) D_n^0 =$$

$$(I, G(-t) D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} A_n^0(t) G_n(-t) D_n^0 \equiv (I, D(t))^{-1}(A(0), D(t)),$$

where $D(0) = (1, D_1^0(1), \ldots, D_n^0(1, \ldots, n), \ldots) \in \mathcal{L}^1(\mathcal{F}_H)$, it is possible to describe the evolution within the framework of the evolution of states. Indeed, the evolution of all possible states, i.e. the sequence $D(t) = (1, D_1(t, 1), \ldots, D_n(t, 1, \ldots, n), \ldots) \in \mathcal{L}^1(\mathcal{F}_H)$ of the density operators
$D_n(t), n \geq 1,$ is described by the initial-value problem of a sequence of the von Neumann equations (the quantum Liouville equations) \[3, 4\]

\[
\frac{d}{dt} D(t) = -\mathcal{N} D(t), \quad (11)
\]

\[
D(t)|_{t=0} = D(0). \quad (12)
\]

The generator \((-\mathcal{N}) = \bigoplus_{n=0}^{\infty} (-\mathcal{N}_n)\) of the von Neumann equation (11) is the adjoint operator to generator (6) of the Heisenberg equation (4) in the sense of functional (10), and for \(f \in \mathcal{L}_0^1(\mathcal{F}_H) \subset D(\mathcal{N}) \subset \mathcal{L}^1(\mathcal{F}_H)\) it is defined by the formula

\[
(-\mathcal{N}_n)f_n = -i(H_n f_n - f_n H_n), \quad (13)
\]

where the operator \(H_n\) is the Hamiltonian (3).

On the space of sequences of trace class operators \(\mathcal{L}^1(\mathcal{F}_H)\) for initial-value problem (11)-(12) the following statement is true.

**Theorem 2.** A unique solution of initial-value problem (11)-(12) of the von Neumann equation is determined by the formula

\[
D(t) = \mathcal{G}(-t) D(0), \quad (14)
\]

where the one-parameter family of operators \(\mathcal{G}(-t) = \bigoplus_{n=0}^{\infty} \mathcal{G}_n(-t)\), is defined by

\[
\mathcal{G}_n(-t)f_n = e^{-itH_n} f_n e^{itH_n}. \quad (15)
\]

If \(D(0) \in \mathcal{L}_0^1(\mathcal{F}_H) \subset \mathcal{L}^1(\mathcal{F}_H)\), operator (14) is a strong (classical) solution and for arbitrary \(D(0) \in \mathcal{L}^1(\mathcal{F}_H)\) it is a weak (generalized) solution.

On the space \(\mathcal{L}^1(\mathcal{H}_n)\) the mapping \(t \to \mathcal{G}_n(-t)f_n\) is an isometric strongly continuous group that preserves positivity and self-adjointness of operators. If \(f_n \in \mathcal{L}_0^1(\mathcal{H}_n) \subset D(\mathcal{N}_n)\), in the sense of the norm convergence of the space \(\mathcal{L}^1(\mathcal{H}_n)\) there exists the limit by which the infinitesimal generator of the group of evolution operators (15) is determined by

\[
\lim_{t \to 0} \frac{1}{t} (\mathcal{G}_n(-t)f_n - f_n) = -i(H_n f_n - f_n H_n),
\]

where \(H_n\) is the Hamiltonian (3) and operator (13) is defined on the domain \(D(H_n) \subset \mathcal{H}_n\).

On the spaces \(\mathcal{L}^1(\mathcal{H}_n^\pm)\) mapping (15) is defined and have the similar properties as in case of the Maxwell-Boltzmann statistics stated above.

Symmetrization and anti-symmetrization operators (2) are integrals of motion of the von Neumann equation (11), and as a consequence it holds

\[
\mathcal{G}_n(-t) S_n^\pm = S_n^\pm \mathcal{G}_n(-t),
\]

and

\[
(-\mathcal{N}_n) S_n^\pm = S_n^\pm (-\mathcal{N}_n),
\]

where the operator \((-\mathcal{N}_n)\) is defined by formula (13). Thus, the symmetry of states is preserved in process of the evolution.
3 The evolution equations for quantum correlations

One more an equivalent approach to the description of the evolution of states of quantum many-particle systems is given by means of operators which are interpreted as correlation operators. The correlation operators are governed by the von Neumann [32], [41]. To justify such hierarchy of quantum evolution equations, at first we consider in detail the motivation of the description of states within the framework of correlation operators or in other words, we consider the origin of the microscopic description of correlations in quantum many-particle systems.

3.1 Correlation operators

We introduce a sequence of operators $g(t) = (0, g_1(t, 1), \ldots, g_s(t, 1, \ldots, s), \ldots) \in \mathcal{L}^1(\mathcal{F}_H)$ defined by the cluster expansions of the density operators $D(t) = (1, D_1(t, 1), \ldots, D_s(t, 1, \ldots, s), \ldots) \in \mathcal{L}^1(\mathcal{F}_H)$

$$D_s(t, Y) = g_s(t, Y) + \sum_{P: Y = \bigcup_i X_i, |P| \geq 1} \prod_{X_i \subset P} g|_{X_i}|(t, X_i), \ s \geq 1, \quad (16)$$

where $\sum_{P: Y = \bigcup_i X_i, |P| > 1}$ is the sum over all possible partitions $P$ of the set $Y \equiv (1, \ldots, s)$ into $|P| > 1$ nonempty mutually disjoint subsets $X_i \subset Y$. The operators defined by recursion relations (16) are interpreted as the correlation operators of quantum many-particle systems, i.e. operators that characterize the correlations of particle states.

In order to construct a solution of recursion relations (16), i.e. to express the correlation operators in terms of the density operators, we introduce some notions. On sequences of operators $f, \tilde{f} \in \mathcal{L}^1(\mathcal{F}_H)$ we define the $*$-product

$$(f \ast \tilde{f})|_{Y}(Y) = \sum_{Z \subset Y} f|_{Z}|(Z) \tilde{f}|_{Y \setminus Z}|(Y \setminus Z), \quad (17)$$

where $\sum_{Z \subset Y}$ is the sum over all subsets $Z$ of the set $Y \equiv (1, \ldots, s)$. By means of definition (17) of the $*$-product we introduce the mapping $\text{Exp}_*$ and the inverse mapping $\text{Ln}_*$ on sequences $h = (0, h_1(1), \ldots, h_n(1, \ldots, n), \ldots)$ of operators $h_n \in \mathcal{L}^1(\mathcal{H}_n)$ by the expansions

$$\text{Exp}_*(h)|_{Y}(Y) = (I + \sum_{n=1}^{\infty} h_n|_{Y}|)_{Y}(Y) = I\delta|_{Y},0 + \sum_{P: Y = \bigcup_i X_i, X_i \subset P} \prod_{X_i \subset P} h|_{X_i}|(X_i), \quad (18)$$

where we use the notations accepted in (16), $\delta|_{Y},0$ is a Kronecker symbol, $I = (1, 0, \ldots, 0, \ldots)$, and respectively,

$$\text{Ln}_*(I + h)|_{Y}(Y) = \left(\sum_{n=1}^{\infty} (-1)^n h_n|_{Y}|\right)_{Y}(Y) = \sum_{P: Y = \bigcup_i X_i} (-1)^{|P| - 1}|P|! \prod_{X_i \subset P} h|_{X_i}|(X_i). \quad (19)$$
Hence in terms of sequences of operators recursion relations (16) are rewritten in the form

$$D(t) = \text{Exp}_* g(t),$$  \hspace{1cm} \text{(20)}$$

where $$D(t) = \mathbb{1} + (0, D_1(t, 1), \ldots, D_n(t, 1, \ldots, n), \ldots)$$. From this equation we obtain

$$g(t) = \ln_* D(t).$$

Thus, according to definition (17) of the $$\ast$$-product and mapping (19), in the component-wise form solutions of recursion relations (16) are represented by the expansions

$$g_s(t, Y) = D_s(t, Y) + \sum_{P: Y = \bigcup_i X_i, |P| > 1} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \subset P} D_{|X_i|}(t, X_i), \hspace{1cm} s \geq 1.$$  \hspace{1cm} \text{(21)}$$

The structure of expansions (21) means that the correlation operators have a sense of cumulants (semi-invariants) of the density operators governed by the Cauchy problem of the von Neumann equations (11)-(12). Therefore correlation operators (21) give an alternative approach to the description of the state evolution of quantum many-particle systems, namely within the framework of dynamics of correlations.

We emphasize that the possibility of the description of states within the framework of correlations arises naturally as a result of dividing of the series in expression (10) by the normalizing factor series, i.e. in consequence of redefining of functional (10).

To justify this proposition, i.e. to construct the mean-value functional in terms of correlation operators (21), we introduce necessary notions and formulate some equalities. For arbitrary $$f = (f_0, f_1, \ldots, f_n, \ldots) \in \mathcal{L}^1(\mathcal{F}_H)$$ and $$Y = (1, \ldots, s)$$ we define the linear mapping $$\partial_Y : f \to \partial_Y f$$, by the formula

$$(\partial_Y f)_n = f_{Y|+n}(Y, |Y| + 1, \ldots, |Y| + n), \hspace{1cm} n \geq 0.$$  \hspace{1cm} \text{(22)}$$

For the set $$\{Y\}$$ consisting of one element $$Y = (1, \ldots, s)$$ we have respectively

$$(\partial_{\{Y\}} f)_n = f_{1+n}(\{Y\}, s + 1, \ldots, s + n), \hspace{1cm} n \geq 0.$$  \hspace{1cm} \text{(23)}$$

On sequences $$\partial_Y f$$ and $$\partial_Y \tilde{f}$$ we introduce the $$\ast$$-product

$$(\partial_Y f \ast \partial_Y \tilde{f})|_X(X) = \sum_{Z \subset X} f|_{|Y|+|Z|}(Y, Z) \tilde{f}|_{X \setminus |Z|+|Y'|}(Y', X \setminus Z),$$  \hspace{1cm} \text{(24)}$$

where $$X, Y, Y'$$ are the sets, which elements characterize clusters of particles, and $$\sum_{Z \subset X}$$ is the sum over all subsets $$Z$$ of the set $$X$$. In particular case ($$Y = \emptyset, Y' = \emptyset$$) definition (24) reduces to (17).

For $$f = (0, f_1, \ldots, f_n, \ldots), f_n \in \mathcal{L}^1(\mathcal{H}_n)$$, according to definitions of mappings (18) and (23), the following equality holds

$$\partial_{\{Y\}} \text{Exp}_* f = \text{Exp}_* f \ast \partial_{\{Y\}} f,$$  \hspace{1cm} \text{(25)}$$

and for mapping (22) correspondingly,

$$\partial_Y \text{Exp}_* f = \text{Exp}_* f \ast \sum_{P: Y = \bigcup_i X_i} \partial_{X_1} f \ast \ldots \ast \partial_{X_{|P|}} f,$$  \hspace{1cm} \text{(26)}$$
where \( \sum_{P,Y=\bigcup_i X_i} \) is the sum over all possible partitions \( P \) of the set \( Y \equiv (1, \ldots, s) \) into \(|P|\) nonempty mutually disjoint subsets \( X_i \subset Y \).

According to the definition

\[
(I, f) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} f_n,
\]

for sequences \( f, \tilde{f} \in \mathfrak{L}^1(\mathcal{F}_H) \), the equality holds

\[
(I, f * \tilde{f}) = (I, f)(I, \tilde{f}).
\]  

(27)

In terms of mappings (22) and (23) the generalized cluster expansions of solution (14) of the von Neumann equation

\[
D_{s+n}(t, Y, X \setminus Y) = \sum_{P: \{Y\}, X \setminus Y = \bigcup_i X_i, X_i \subset P} \prod g(t, X_i), \quad s \geq 1,
\]  

(28)

where \( X \setminus Y \equiv (s + 1, \ldots, s + n) \), take the form

\[
\varnothing_Y D(t) = \varnothing_{\{Y\}} \text{Exp}_s g(t).
\]

(29)

We now deduce from the definition of functional (10) the functional for average values of observables in terms of correlation operators (21), for example, for \( s \)-ary observables \( A^{(s)} = (0, \ldots, 0, a_s(1, \ldots, s), \ldots, \sum_{i_1 < \ldots < i_s = 1} a_s(i_1, \ldots, i_s), \ldots) \), i.e.

\[
\langle A^{(s)} \rangle(t) = \frac{1}{s!} (I, D(t))^{-1} \text{Tr}_Y a_s(Y)(I, \varnothing_Y D(t)),
\]

where \( \text{Tr}_Y \equiv \text{Tr}_{1,\ldots,s} \). Using generalized cluster expansions (29) and as a consequence of equalities (25) and (27), we find

\[
(I, D(t))^{-1} (I, \varnothing_Y D(t)) = (I, D(t))^{-1} (I, \varnothing_{\{Y\}} \text{Exp}_s g(t)) = (I, D(t))^{-1} (I, \text{Exp}_s g(t)) (I, \varnothing_{\{Y\}} g(t)).
\]

According to generalized cluster expansions (29), as a final result we derive the following representation of the mean-value functional in case of \( s \)-ary observables

\[
\langle A^{(s)} \rangle(t) = \frac{1}{s!} \text{Tr}_Y a_s(Y)(I, \varnothing_{\{Y\}} g(t)),
\]

or in the componentwise form

\[
\langle A^{(s)} \rangle(t) = \frac{1}{s!} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,s+n} a_s(Y) g_{1+n}(t, \{Y\}, s + 1, \ldots, s + n),
\]

(30)

where the correlation operators of particle clusters \( g_{1+n}(t) \) are defined as solutions of generalized cluster expansions (29), namely

\[
g_{1+n}(t, \{Y\}, X \setminus Y) = \sum_{P: (Y), X \setminus Y = \bigcup_i X_i} (-1)^{|P|-1} (\prod_{i} |P| - 1)! \prod_{X_i \subset P} D(t, X_i), \quad s \geq 1,
\]

(31)

where the density operator \( D(t, X_i) \) is solution (14) of the von Neumann equation (11). For \( A^{(s)} \in \mathfrak{L}(\mathcal{F}_H) \) and \( g^{(s)} \in \mathfrak{L}^1(\mathcal{F}_H) \) functional (30) exists.

It should be emphasized that correlation operators that belong to the spaces \( \mathfrak{L}^1(\mathcal{F}_H) \) describe only finitely many particles, i.e. systems with finite average number of particles.
3.2 The von Neumann hierarchy

We consider quantum systems of particles obeying the Maxwell-Boltzmann statistics with the Hamiltonian

\[ H_n = \sum_{i=1}^{n} K(i) + \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k=1}^{n} \Phi^{(k)}(i_1, \ldots, i_k), \]

where \( \Phi^{(k)} \) is a \( k \)-body interaction potential. The evolution of all possible states is described in terms of the correlation operators \( g(t) = (0, g_1(t, 1), \ldots, g_s(t, 1, \ldots, s), \ldots) \in \mathfrak{L}^1(\mathcal{F}_H) \) governed by the following Cauchy problem

\[ \frac{d}{dt} g_s(t, Y) = -\mathcal{N}_s(Y) g_s(t, Y) + \]

\[ + \sum_{P: Y = \bigcup_i X_i, |P| > 1} \sum_{Z_1 \subset X_1, Z_1 \neq \emptyset} \ldots \sum_{Z_{|P|} \subset X_{|P|}, Z_{|P|} \neq \emptyset} \left( -\mathcal{N}_{\text{int}}^{(\sum_{r=1}^{|P|} |Z_r|)}(Z_1, \ldots, Z_{|P|}) \right) \prod_{X_i \subset P} g_{|X_i|}(t, X_i), \]

\[ g_s(t, Y)|_{t=0} = g_s^0(Y), \quad s \geq 1, \]

where the operator \( (-\mathcal{N}_{\text{int}}^{(n)}) \) is defined by

\[ (-\mathcal{N}_{\text{int}}^{(n)}) f_n \doteq -i(\Phi^{(n)} f_n - f_n \Phi^{(n)}), \]

\[ \sum_{P: Y = \bigcup_i X_i, |P| > 1} \quad \text{is the sum over all possible partitions } P \text{ of the set } Y \equiv (1, \ldots, s) \text{ into } |P| > 1 \text{ nonempty mutually disjoint subsets } X_i \subset Y \text{ and } \sum_{Z_j \subset X_j, Z_j \neq \emptyset} \quad \text{is a sum over nonempty subsets } Z_j \subset X_j. \]

We refer to the hierarchy of equations (33) as the von Neumann hierarchy [32].

It should be noted that the von Neumann hierarchy (33) is the evolution recurrence equations set. We cite an instance of the typical equations of hierarchy (33)

\[ \frac{d}{dt} g_1(t, 1) = -\mathcal{N}(1) g_1(t, 1), \]

\[ \frac{d}{dt} g_2(t, 1, 2) = -\mathcal{N}_2(1, 2) g_2(t, 1, 2) - \mathcal{N}_{\text{int}}^{(2)}(1, 2) g_1(t, 1) g_1(t, 2), \]

\[ \frac{d}{dt} g_3(t, 1, 2, 3) = -\mathcal{N}_3(1, 2, 3) g_3(t, 1, 2, 3) - \]

\[ - (\mathcal{N}_{\text{int}}^{(2)}(1, 2) + \mathcal{N}_{\text{int}}^{(2)}(1, 3) + \mathcal{N}_{\text{int}}^{(3)}(1, 2, 3)) g_1(t, 1) g_2(t, 2, 3) - \]

\[ - (\mathcal{N}_{\text{int}}^{(2)}(1, 2) + \mathcal{N}_{\text{int}}^{(2)}(2, 3) + \mathcal{N}_{\text{int}}^{(3)}(1, 2, 3)) g_1(t, 2) g_2(t, 1, 3) - \]

\[ - (\mathcal{N}_{\text{int}}^{(2)}(1, 3) + \mathcal{N}_{\text{int}}^{(2)}(2, 3) + \mathcal{N}_{\text{int}}^{(3)}(1, 2, 3)) g_1(t, 1) g_2(t, 1, 2) - \]

\[ - \mathcal{N}_{\text{int}}^{(3)}(1, 2, 3)(1, 2, 3) g_1(t, 1) g_1(t, 2) g_1(t, 3). \]

In case of a two-body interaction potential (3) the von Neumann hierarchy (33) reduces to the following form

\[ \frac{d}{dt} g_s(t, Y) = -\mathcal{N}_s(Y) g_s(t, Y) + \]

\[ + \sum_{P: Y = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2), \quad s \geq 1, \]
where \( \sum_{P : Y = X_1 \cup X_2 } \) is the sum over all possible partitions \( P \) of the set \( Y \equiv (1, \ldots, s) \) into two nonempty mutually disjoint subsets \( X_1 \subset Y \) and \( X_2 \subset Y \). In terms of kernels of correlation operators the first two equations of the von Neumann hierarchy (36) have the form

\[
i \frac{\partial}{\partial t} g_1(t, q_1; q'_1) = -\frac{1}{2} (\Delta q_1 - \Delta q'_1) g_1(t, q_1; q'_1),
\]

\[
i \frac{\partial}{\partial t} g_2(t, q_1, q_2; q'_1, q'_2) =
\]

\[
= ( -\frac{1}{2} \sum_{i=1}^{2} (\Delta q_i - \Delta q'_i) + (\Phi(q_1 - q_2) - \Phi(q'_1 - q'_2))) g_2(t, q_1, q_2; q'_1, q'_2) +
\]

\[
(\Phi(q_1 - q_2) - \Phi(q'_1 - q'_2)) g_1(t, q_1; q'_1) g_1(t, q_2; q'_2).
\]

We remark that the von Neumann hierarchy (33) (or (36)) is rigorously derived on basis of solutions (21) of cluster expansions (16) of density operators governed by the von Neumann equation (11).

To construct a solution of the Cauchy problem (33)-(34) we first consider its structure for physically motivated example of initial data, namely initial data satisfying a chaos property. A chaos property means the absence of state correlations in a system at the initial time. In this case the sequence of initial correlation operators is the following one-component sequence

\[
g(0) = (0, g^0_1(1), 0, \ldots).
\]

In fact, in terms of the sequence of the density operators this condition means that

\[
D(0) = (1, D^0_1(1), D^0_1(1)D^0_2(2), \ldots, \prod_{i=1}^{n} D^0_1(i), \ldots)\text{ in case of the Maxwell-Boltzmann statistics, then from (21) we derive (37).}
\]

On basis of representation (21) of the correlation operators in terms of the density operators and formula (14) we have

\[
g_s(t, Y) = \sum_{P : Y = \bigcup_i X_i} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \subset P} G_{|X_i|}(-t, X_i) \prod_{i=1}^{s} D^0_1(i), \quad s \ge 1.
\]

Taking into account the equality: \( D^0_1(i) = g^0_1(i), 1 \le i \le s \), we derive the formula of a solution of the Cauchy problem (33)-(34) for initial data (37)

\[
g_s(t, Y) = \mathfrak{A}_s(-t, Y) \prod_{i=1}^{s} g^0_1(i), \quad s \ge 1,
\]

where \( \mathfrak{A}_s(-t, Y) \) is the \( sth \)-order cumulant of the groups of operators (15) of the von Neumann equations (11)

\[
\mathfrak{A}_s(-t, Y) = \sum_{P : Y = \bigcup_i X_i} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \subset P} G_{|X_i|}(-t, X_i),
\]

and we use accepted above notations. It should be emphasized that the structure of expansions (39) of the evolution operators \( \mathfrak{A}_s(t, Y) \) means that they are determined by the cluster expansions of the groups of operators (15) of the von Neumann equation (11).
Thus, the cumulant nature of correlation operators induces the cumulant structure of a one-parametric mapping generated by solution (38). From (38) it is clear that in case of absence of correlations in a system at initial instant the correlations generated by the dynamics of a system are completely governed by cumulants (39) of groups (15).

Let us indicate some properties of a cumulant (semi-invariant) (39) of groups of operators (15). The corresponding-order cumulant is a solution of the cluster expansions of groups of operators

\[ G_s(-t) = \sum_{P:Y=\bigcup_{i}X_i} \prod_{X_i \subset P} \mathfrak{A}_{|X_i|}(-t,X_i), \quad s \geq 1, \]  

(40)

where \( \sum_P \) is the sum over all possible partitions \( P \) of the set \( Y \equiv (1, \ldots, s) \) into \( |P| \) nonempty mutually disjoint subsets \( X_i \subset Y \). The simplest examples of equations from recurrence relations (40) are given by the expressions

\[ G_1(-t,1) = \mathfrak{A}_1(-t,1), \]
\[ G_2(-t,1,2) = \mathfrak{A}_2(-t,1,2) + \mathfrak{A}_1(-t,1)\mathfrak{A}_1(-t,2). \]

In case of a quantum system of non-interacting particles we have: \( \mathfrak{A}_s(-t) = 0, \quad s \geq 2 \). Indeed, for non-interacting quantum particles it holds

\[ G_s(-t,1,\ldots,s) = \prod_{i=1}^{s} G_1(-t,i), \]

and hence,

\[ \mathfrak{A}_s(-t,Y) = \sum_{P:Y=\bigcup_{i}X_i} (-1)^{|P|-1}(|P|-1)! \prod_{X_i \subset P} \prod_{l_i=1}^{|X_i|} G_1(-t,l_i) = \]

\[ = \sum_{k=1}^{s} (-1)^{k-1}s(s,k)(k-1)! \prod_{i=1}^{s} G_1(-t,i) = 0. \]

Here \( s(s,k) \) are the Stirling numbers of the second kind and the following equality is used

\[ \sum_{P:Y=\bigcup_{i}X_i} (-1)^{|P|-1}(|P|-1)! = \sum_{k=1}^{s} (-1)^{k-1}s(s,k)(k-1)! = \delta_{s,1}, \]  

(41)

where \( \delta_{s,1} \) is a Kronecker symbol.

If \( s = 1 \), for \( f_1 \in \mathfrak{L}_1(\mathcal{H}) \subset \mathfrak{L}^1(\mathcal{H}) \) in the sense of the norm convergence of the space \( \mathfrak{L}^1(\mathcal{H}) \) the generator of first-order cumulant (39) is given by operator (13)

\[ \lim_{t \to 0} \frac{1}{t} (\mathfrak{A}_1(-t,1) - I) f_1 = -\mathcal{N}_1 f_1. \]

For \( s \geq 2 \) we obtain the following equality in the same sense

\[ \lim_{t \to 0} \frac{1}{t} \mathfrak{A}_s(-t,1,\ldots,s) f_s = -\mathcal{N}^{(s)}_{\text{int}}(1,\ldots,s) f_s, \]
where the operator \((-N_{\text{int}}^{(s)})\) is defined by formula (35).

If \(f_s \in \mathcal{L}^1(\mathcal{H}_s)\), then for \(sth\)-order cumulant (39) of groups of operators (15) the estimate is valid

\[
\|A_s(-t)f_s\|_{\mathcal{L}^1(\mathcal{H}_s)} \leq \sum_{P: Y = \bigcup_i X_i} (|P| - 1)! \|f_s\|_{\mathcal{L}^1(\mathcal{H}_s)} \leq \sum_{k=1}^s s(s, k)(k - 1)! \|f_s\|_{\mathcal{L}^1(\mathcal{H}_s)} \leq s! e^s \|f_s\|_{\mathcal{L}^1(\mathcal{H}_s)};
\]

where \(s(s, k)\) are the Stirling numbers of the second kind.

Hereafter we use the following notations: \((\{X_1\}, \ldots, \{X_{|P|}\})\) is a set, elements of which are \(|P|\) mutually disjoint subsets \(X_i \subset Y \equiv (1, \ldots, s)\) of the partition \(P: Y = \bigcup_{i=1}^{|P|} X_i\), i.e. \(|(\{X_1\}, \ldots, \{X_{|P|}\})| = |P|\). In view of these notations we state that \(Y\) is the set consisting of one element \(Y = (1, \ldots, s)\) of the partition \(P\) (\(|P| = 1\) and \(|\{Y\}| = 1\). We define the declusterization mapping \(\theta: (\{X_1\}, \ldots, \{X_{|P|}\}) \rightarrow Y\), by the formula

\[
\theta(\{X_1\}, \ldots, \{X_{|P|}\}) = Y.
\]

For arbitrary initial data a solution of the Cauchy problem (33)-(34) of the von Neumann hierarchy is given by the expansion (52)

\[
g_s(t, Y) = \sum_{P: Y = \bigcup_i X_i} \mathcal{A}_{|P|}(-t, \{X_1\}, \ldots, \{X_{|P|}\}) \prod_{X_i \subset P} g_0^{(1)}(X_i), \quad s \geq 1.
\]

Here \(\mathcal{A}_{|P|}(-t)\) is the \(|P|th\)-order cumulant of groups of operators (15) defined by the formula

\[
\mathcal{A}_{|P|}(-t, \{X_1\}, \ldots, \{X_{|P|}\}) = \sum_{P': (\{X_1\}, \ldots, (X_{|P|})) = \bigcup_k Z_k} (-1)^{|P'|-1}(|P'| - 1)! \prod_{Z_k \subset P'} \mathcal{G}_{\theta(Z_k)}(-t, \theta(Z_k)),
\]

where \(\sum_{P': (\{X_1\}, \ldots, (X_{|P|})) = \bigcup_k Z_k}\) is the sum over all possible partitions \(P'\) of the set \((\{X_1\}, \ldots, \{X_{|P|}\})\) into \(|P'|\) nonempty mutually disjoint subsets \(Z_k \subset (\{X_1\}, \ldots, \{X_{|P|}\})\). The simplest examples of correlation operators (44) are given by the expressions

\[
\begin{align*}
g_1(t, 1) &= \mathcal{A}_1(-t, 1)g_0^{(1)}(1), \\
g_2(t, 1, 2) &= \mathcal{A}_1(-t, \{1, 2\})g_0^{(1)}(1, 2) + \mathcal{A}_2(-t, 1, 2)g_1^{(1)}(1)g_1^{(2)}(2), \\
g_3(t, 1, 2, 3) &= \mathcal{A}_1(-t, \{1, 2, 3\})g_0^{(1)}(1, 2, 3) + \mathcal{A}_2(-t, \{2, 3\}, 1)g_0^{(1)}(1)g_0^{(2)}(2, 3) + \mathcal{A}_2(-t, \{1, 3\}, 2)g_0^{(1)}(2)g_0^{(1)}(1, 3) + \mathcal{A}_2(-t, \{1, 2\}, 3)g_0^{(1)}(3)g_0^{(1)}(1, 2) + \mathcal{A}_3(-t, 1, 2, 3)g_0^{(1)}(1)g_0^{(2)}(1)g_0^{(3)}(3).
\end{align*}
\]

The validity of solution expansion (44) can be verified by straightforward differentiation by time variable and also in the following way. Taking into account the fact that the von Neumann hierarchy (33) is the evolution recurrence equations set, we can construct a solution of initial-value problem (33)-(34) by integrating each equation of the hierarchy as the inhomogeneous
von Neumann equation. For example, using formula (15), as a result of the integration of the first two equations of hierarchy (33) we obtain the following equalities

\[ g_1(t, 1) = \mathcal{G}_1(-t, 1)g_0^0(1), \]
\[ g_2(t, 1, 2) = \mathcal{G}_2(-t, 1, 2)g_2^0(1, 2) + \]
\[ + \int_0^t dt_1 \mathcal{G}_2(-t + t_1, 1, 2)(-\mathcal{N}_{\text{int}}(1, 2))\mathcal{G}_1(-t_1, 1)\mathcal{G}_1(-t_1, 2)g_0^0(1)g_0^0(2). \]

Then for the second term on the right-hand side of this equation an analog of the Duhamel equation holds

\[ \int_0^t dt_1 \mathcal{G}_2(-t + t_1, 1, 2)(-\mathcal{N}_{\text{int}}(1, 2))\mathcal{G}_1(-t_1, 1)\mathcal{G}_1(-t_1, 2) = (46) \]
\[ = -\mathcal{G}_2(-t, 1, 2) - \mathcal{G}_1(-t, 1)\mathcal{G}_1(-t, 2) = \mathfrak{A}_2(-t, 1, 2), \]

where \( \mathfrak{A}_2(-t) \) is the second-order cumulant of groups of operators (15) defined by formula (39). For \( s > 2 \) a solution of the Cauchy problem (33)-(34) constructed by iterations is represented by expansions (44) as a consequence of transformations similar to an analog of the Duhamel equation (46).

We note, that in case of initial data (37) solution (38) of the Cauchy problem (33)-(34) of the von Neumann hierarchy may be rewritten in another representation. For \( n = 1 \), we have

\[ g_1(t, 1) = \mathfrak{A}_1(-t, 1)g_0^0(1). \]

Then, within the context of the definition of the first-order cumulant, \( \mathfrak{A}_1(-t) \), and the dual group of operators \( \mathfrak{A}_1(t) \), we express the correlation operators \( g_s(t) \), \( s \geq 2 \), in terms of the one-particle correlation operator \( g_1(t) \) using formula (38). Hence for \( s \geq 2 \) formula (38) is represented in the form of the functional with respect to one-particle correlation operators

\[ g_s(t, Y | g_1(t)) = \widehat{\mathfrak{A}}_s(t, Y) \prod_{i=1}^s g_1(t, i), \quad s \geq 2, \]

where \( \widehat{\mathfrak{A}}_s(t, Y) \) is \( s \)th-order cumulant (39) of the scattering operators

\[ \widehat{\mathcal{G}}_s(t, Y) \equiv g_s(-t, Y) \prod_{i=1}^s \mathcal{G}_1(t, i), \quad s \geq 1. \]

The generator of the scattering operator \( \widehat{\mathcal{G}}_s(Y) \) is determined by the operator

\[ \frac{d}{dt} \widehat{\mathcal{G}}_s(t, Y)|_{t=0} = \sum_{k=2}^s \sum_{i_1 < \ldots < i_k = 1}^s (-\mathcal{N}_{\text{int}}^{(k)}(i_1, \ldots, i_k)), \]
where the operator \((-\mathcal{N}_{\text{int}}^{(k)})\) acts on \(L^1_0(H_s) \subset L^1(H_s)\) according to formula (55).

On the space \(L^1(F_H)\) for initial-value problem (33)-(34) of the von Neumann hierarchy the following statement is true \[32, 41].

**Theorem 3.** For \(t \in \mathbb{R}\) a solution of the Cauchy problem (33)-(34) is given by the following expansion

\[
g_s(t, Y) = G(t; Y|g(0)) \equiv \sum_{P: Y = \bigcup_i X_i} \mathcal{A}_P(-t, \{X_1, \ldots, \{X_{|P|}\}) \prod_{X_i \in P} g_{|X_i|}^{0}(X_i), \quad s \geq 1,
\]

where \(\mathcal{A}_P(-t)\) is the \(|P|\)th-order cumulant (45) of the groups of operators. For \(g_0^n \in L^1_0(H_n) \subset L^1(H_n), n \geq 1\), expansion (48) is a strong (classical) solution and for arbitrary initial data \(g^n_0 \in L^1(H_n), n \geq 1\), it is a weak (generalized) solution.

The intrinsic properties of constructed solution (48) are generated by the properties of cumulants (45) of groups of operators of the von Neumann equations. We indicate some properties of the one-parameter mapping: \(t \to G(t|f)\), generated by solution (48).

For \(f_s \in L^1(H_s), s \geq 1\), the mapping \(G(t; Y|f)\) is defined and, according to the inequality

\[
\|\mathcal{A}_P(-t, \{X_1, \ldots, \{X_{|P|}\}) f_s\|_{L^1(H_s)} \leq |P|! e^{e_P} \|f_s\|_{L^1(H_s)},
\]

the following estimate is true

\[
\|G(t; Y|f)\|_{L^1(H_s)} \leq s! e^{2s} e^s,
\]

where \(e \equiv e^3 \max(1, \max_{P: Y = \bigcup_i X_i} \|f_{|X_i|}\|_{L^1(H_s)})\). The mapping \(G(t|f)\) has the group property, i.e. for \(f \in L^1(F_H)\) the equalities are fulfilled

\[
G(t_1 + t_2|f) = G(t_1|G(t_2|f)) = G(t_2|G(t_1|f)).
\]

On the subspaces \(L^1_0(H_s) \subset L^1(H_s), s \geq 1\), the infinitesimal generator \(\mathcal{N}(Y|f)\) of the group \(G(t; Y|f)\) is defined by the operator

\[
\mathcal{N}(Y|f) \equiv -\mathcal{N}_s(Y)f_s(Y) + \sum_{P: Y = \bigcup_i X_i, |P| \neq 1} \sum_{Z_1 \subset X_1, Z_1 \neq \emptyset} \ldots \sum_{Z_{|P|} \subset X_{|P|}, Z_{|P|} \neq \emptyset} (-\mathcal{N}_{\text{int}}^{(\sum_{Z_1} |Z_1|)}(Z_1, \ldots, Z_{|P|})) \prod_{X_i \in P} f_{|X_i|}(X_i), \quad s \geq 1,
\]

We remark that a particular solution of the steady von Neumann hierarchy is a sequence of the Ursell operators, for example, its first two elements have the form

\[
g_1(t, 1) = e^{-\beta K^{(1)}},
\]

\[
g_2(t, 1, 2) = e^{-\beta \sum_{i=1}^2 K^{(i)}}(e^{-\beta \Phi(1,2)} - I),
\]

where \(\beta\) is a parameter inversely proportional to temperature.
For the purpose of further application we introduce the notion of correlation operators of clusters of particles

\[ g_{1+n}(t, \{Y\}, X \setminus Y) = \mathcal{G}(t; \{Y\}, X \setminus Y | g(0)) \]

\[ = \sum_{P: \{Y\}, X \setminus Y = \bigcup_i X_i} \mathfrak{A}_{|P|}(-t, \{\theta(X_i)\}, \ldots, \{\theta(X_{|P|})\}) \prod_{X_i \in P} g^0_{|X_i|}(X_i), \]

where \( \mathfrak{A}_{|P|}(-t) \) is the \(|P|th\)-order cumulant defined by formula (15).

We remark that the relations between correlation operators of particle clusters \( \mathfrak{A}_{\{Y\}} g(t) \in \mathfrak{A}^I (\oplus_{n=0}^\infty \mathcal{H}_{s+n}) \) and correlation operators of particles (18) are given by the following equalities

\[ g_{1+n}(t, \{Y\}, X \setminus Y) = \sum_{P: \{Y\}, X \setminus Y = \bigcup_i X_i} (-1)^{|P|}(|P| - 1)! \prod_{X_i \in P} \sum_{P': \theta(X_i) = \bigcup_j Z_j, Z_j \in P'} \prod_{i=1}^{|P|} g_{|Z_j|}(t, Z_j). \]

In particular case \( n = 0 \), i.e. the correlation operator of a cluster of \(|Y|\) particles, these relations take the form

\[ g_{1+0}(t, \{Y\}) = \sum_{P: Y = \bigcup_i X_i, X_i \in \mathcal{P}} g_{|X_i|}(t, X_i). \]

Due to cluster expansions (16) it is possible to express many-particle correlation operators through the two-particle and one-particle correlation operators

\[ g_{1+n}(t, \{Y\}, X \setminus Y) = \sum_{Z \subset X \setminus Y, Z \neq \emptyset} g_2(t, \{Y\}, \{Z\}) \sum_{P: X \setminus (Y \cup Z) = \bigcup_i X_i} (-1)^{|P| - 1} ||P|! \prod_{i=1}^{|P|} g_1(t, \{X_i\}). \]

We note also that in case of many-particle systems obeying quantum statistics, i.e. many-particle systems of fermions or bosons, the von Neumann hierarchy has the form

\[ \frac{d}{dt} g_s(t, Y) = \mathcal{N}(Y | g(t)), \quad s \geq 1, \]

where the hierarchy generator is determined by

\[ \mathcal{N}(Y | g) \doteq -\mathcal{N}_s(Y) g_s(Y) + \]

\[ + \sum_{P: Y = \bigcup_i X_i, |P| \neq 1} \sum_{Z_1 \subset X_1, Z_1 \neq \emptyset} \ldots \sum_{Z_{|P|} \subset X_{|P|}, Z_{|P|} \neq \emptyset} (-\mathcal{N}^{\sum_{i=1}^{|P|} |Z_i|}_{\text{int}}(Z_1, \ldots, Z_{|P|})) \mathcal{S}^\pm_s \prod_{X_i \in P} g_{|X_i|}(X_i), \]

and the operators \( \mathcal{S}^\pm_s \) are defined by formula (2). A solution of the Cauchy problem of hierarchy (53) is given by the corresponding expansion

\[ g_s(t, Y) = \sum_{P: Y = \bigcup_i X_i} \mathfrak{A}_{|P|}(-t, \{X_1\}, \ldots, \{X_{|P|}\}) \mathcal{S}^\pm_s \prod_{X_i \in P} g^0_{|X_i|}(X_i), \quad s \geq 1. \]
4 The hierarchies of quantum evolution equations

For the description of the evolution of infinite-particle quantum systems the hierarchies of evolution equations for marginal observables, marginal density operators and marginal correlation operators are used \[1], \[2]. They are constructed as the evolution equations for one more but an equivalent method of the description of states and observables of finitely many particles.

Usually the evolution of infinite-particle quantum systems is described within the framework of the evolution of states by the quantum BBGKY hierarchy for marginal density operators. An alternative method of the description of state evolution is given in terms of the nonlinear quantum BBGKY hierarchy for marginal correlation operators. An equivalent approach to the description of the evolution of quantum systems is given within the framework of the marginal quantum BBGKY hierarchy \[31], \[40]. An alternative method of the description of states and observables of finitely many particles.

For a system of a finite average number of particles there exists an equivalent possibility to describe observables and states, namely, by sequences of marginal observables (the so-called s-particle observables) \( B(t) = (B_0, B_1(t, 1), \ldots, B_s(t, 1, \ldots, s), \ldots) \) and marginal states (s-particle density operators) \( F(0) = (1, F^0_1(1), \ldots, F^0_s(1, \ldots, s), \ldots) \) \[1], \[4]. These sequences are correspondingly introduced instead of sequences of observables \( A(t) \) and density operators \( D(0) \), in such way that mean value (10) does not change, i.e.

\[
\langle A \rangle(t) = (I, D(0))^{-1} A(t), D(0)) = (I, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} A_n(t) D_n^0 = \quad (54)
\]

\[
= (B(t), F(0)) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s} B_s(t, 1, \ldots, s) F_s^0(1, \ldots, s),
\]

where \( (I, D(0)) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} D_n^0 \) is a normalizing factor and \( I \) is the identity operator. Thus, the relationship of marginal observables and observables is determined by the formula

\[
B_s(t, Y) \doteq \sum_{n=0}^{s} (-1)^n \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n = 1} A_{s-n}(t, Y \setminus (j_1, \ldots, j_n)), \quad s \geq 1,
\]

\[
F_s^0(Y) \doteq (I, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,s+n} D_{s+n}^0(1, \ldots, s+n), \quad s \geq 1.
\]

As we can see from (55) one component sequences of marginal observables correspond to observables of certain structure, namely the marginal observable \( B^{(1)} = (0, a_1(1), 0, \ldots) \) corresponds to the additive-type observable \( A^{(1)} = (0, a_1(1), \ldots, \sum_{i=1}^{n} a_1(i), \ldots) \), and in the general case the \( k \)-ary-type marginal observable \( B^{(k)} = (0, \ldots, 0, a_{k}(1, \ldots, k), 0, \ldots) \) corresponds to the \( k \)-ary-type observable \( A^{(k)} = (0, \ldots, 0, a_{k}(1, \ldots, k), \ldots, \sum_{i_1 < \ldots < i_k = 1} a_{k}(i_1, \ldots, i_k), \ldots) \).

We emphasize that the evolution of marginal observables (55) of both finitely and infinitely many quantum particles is described by the initial-value problem of the dual BBGKY hierarchy. For finitely many particles the dual quantum BBGKY hierarchy is equivalent to the Heisenberg equations \[4\] (the dual equation to the Heisenberg equation \[4\] is the von Neumann equation \( [11] \)).
4.1 The dual quantum BBGKY hierarchy for marginal observables

The evolution of marginal observables is described by the initial-value problem of the dual quantum BBGKY hierarchy

\[ \frac{d}{dt} B_s(t, Y) = \left( \sum_{j=1}^{s} \mathcal{N}(j) + \sum_{j_1 < j_2=1}^{s} \mathcal{N}_{\text{int}}(j_1, j_2) \right) B_s(t, Y) + \]

\[ + \sum_{j_1 \neq j_2=1}^{s} \mathcal{N}_{\text{int}}(j_1, j_2) B_{s-1}(t, Y\backslash\{j_1\}), \]

\[ B_s(t) \mid_{t=0} = B_s^0, \quad s \geq 1, \] (57)

where on \( \mathcal{L}_0(\mathcal{H}_n) \subset \mathcal{L}(\mathcal{H}_n) \) the operators \( \mathcal{N}(j) \) and \( \mathcal{N}_{\text{int}}(j_1, j_2) \) are correspondingly defined by formulas

\[ \mathcal{N}(j) g_n = -i (g_n K(j) - K(j) g_n), \]

\[ \mathcal{N}_{\text{int}}(j_1, j_2) g_n = -i (g_n \Phi(j_1, j_2) - \Phi(j_1, j_2) g_n), \] (59)

We refer to recurrence evolution equations (57) as the dual quantum BBGKY hierarchy since it is the adjoint hierarchy of evolution equations to the quantum BBGKY hierarchy for the marginal density operators [3, 4] with respect to bilinear form (11). In case of the space \( \mathcal{H} = L^2(\mathbb{R}^\nu) \), evolution equations (57) for kernels of the operators \( B_s(t), \geq 1, \) are given by

\[ i \frac{\partial}{\partial t} B_1(t, q_1; q_1') = -\frac{1}{2} (-\Delta_{q_1} + \Delta_{q_1'}) B_1(t, q_1; q_1'), \]

\[ i \frac{\partial}{\partial t} B_s(t, q_1, \ldots, q_s; q_1', \ldots, q_s') = \left( -\frac{1}{2} \sum_{j=1}^{s} (-\Delta_{q_j} + \Delta_{q_j'}) + \right. \]

\[ + \sum_{1=j_1 < j_2}^{s} (\Phi(q_{j_1}' - q_{j_2}') - \Phi(q_{j_1} - q_{j_2})) B_s(t, q_1, \ldots, q_s; q_1', \ldots, q_s') + \]

\[ + \sum_{1=j_1 \neq j_2}^{s} (\Phi(q_{j_1}' - q_{j_2}') - \Phi(q_{j_1} - q_{j_2})) B_{s-1}(t, q_1, \ldots, q_{j_2-1}, q_{j_2+1}, \ldots, q_s; \]

\[ q_1', \ldots, q_{j_2-1}', q_{j_2+1}', \ldots, q_s'), \quad s \geq 2. \]

We note that in case of many-body interaction potentials (32) hierarchy (57) has the form (33)

\[ \frac{d}{dt} B_s(t, Y) = \mathcal{N}_s(Y) B_s(t, Y) + \]

\[ + \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_k=1}^{s} \mathcal{N}_{\text{int}}^{(k)}(j_1, \ldots, j_k) B_{s-n}(t, Y\backslash\{j_1, \ldots, j_n\}), \quad s \geq 1. \]

On space \( \mathcal{L}(\mathcal{H}_k) \) the operator \( \mathcal{N}_{\text{int}}^{(k)} \) is defined by

\[ \mathcal{N}_{\text{int}}^{(k)} g_k = -i (g_k \Phi^{(k)} - \Phi^{(k)} g_k). \] (60)
We note that for finitely many particles the dual quantum BBGKY hierarchy (57) (or (60)) is rigorously derived on basis of the Heisenberg equation (1) according to definition (55). Hence the structure of evolution equations for marginal observables is defined by the structure of expansion (55).

Let us introduce some abridged notations: \( Y \equiv (1, \ldots, s), X \equiv (j_1, \ldots, j_n) \subset Y \) and \( \{Y \setminus X\} \) is the set consisting of one element \( Y \setminus X = (1, \ldots, s) \setminus (j_1, \ldots, j_n) \), i.e. the set \( \{Y \setminus X\} \) is a connected subset of the set \( Y \).

To construct a solution of the Cauchy problem (57)-(58) we introduce the \((1 + n)th\)-order cumulant of groups of operators (7) as follows [30], [33]

\[
\mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) \equiv \sum_{P: (\{Y \setminus X\}, X) = \bigcup_i X_i} (-1)^{|P| - 1} (|P| - 1)! \prod_{X_i \subset P} \mathcal{G}_{[\theta(X_i)]}(t, \theta(X_i)), \quad n \geq 0,
\]

where \( \sum_P \) is the sum over all possible partitions \( P \) of the set \( \{\{Y \setminus X\}, j_1, \ldots, j_n\} \) into \( |P| \) nonempty mutually disjoint subsets \( X_i \subset \{(\{Y \setminus X\}, X)\} \).

For example,

\[
\mathfrak{A}_1(t, \{Y\}) = \mathcal{G}_s(t, Y),
\]

\[
\mathfrak{A}_2(t, \{Y \setminus (j)\}, j) = \mathcal{G}_s(t, Y) - \mathcal{G}_{s-1}(t, Y \setminus (j)) \mathcal{G}_1(t, j).
\]

Let us indicate some properties of cumulants (61). If \( g \in \mathfrak{L}_0(\mathcal{H}_s) \subset \mathfrak{L}(\mathcal{H}_s) \) in the sense of the \(*\)-weak convergence of the space \( \mathfrak{L}(\mathcal{H}_s) \) the generator of first-order cumulant (61) is given by operator (6)

\[
\lim_{t \to 0} \frac{1}{t}(\mathfrak{A}_1(t, \{Y\}) - I) g_s(Y) = \mathcal{N}_s g_s(Y).
\]

In case of \( n = 1 \) for \( g \in \mathfrak{L}_0(\mathcal{H}_s) \subset \mathfrak{L}(\mathcal{H}_s) \) we obtain the following equality in the sense of the \(*\)-weak convergence of the space \( \mathfrak{L}(\mathcal{H}_s) \)

\[
\lim_{t \to 0} \frac{1}{t} \mathfrak{A}_2(t, \{Y \setminus (j)\}, j) g_s(Y) = \sum_{i \in (Y \setminus (j))} \mathcal{N}_{int}(i, j) g_s(Y),
\]

where the operator \( \mathcal{N}_{int}(i, j) \) is defined by (59), and for \( n \geq 2 \) as a consequence of the fact that we consider a system of particles interacting by a two-body potential (3), it holds

\[
\lim_{t \to 0} \frac{1}{t} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) g_s(Y) = 0.
\]

Correspondingly in case of \( n \)-body interaction potential (32) it holds

\[
\lim_{t \to 0} \frac{1}{t} \mathfrak{A}_{n}(t, 1, \ldots, n) g_n = \mathcal{N}^{(n)}_{int} g_n,
\]

where the operator \( \mathcal{N}^{(n)}_{int} \) is defined by formula (60).

If \( g \in \mathfrak{L}(\mathcal{H}_s) \), then for \((1 + n)th\)-order cumulant (61) of groups of operators (7) the estimate is valid

\[
\| \mathfrak{A}_{1+n}(t) g_s \|_{\mathfrak{L}(\mathcal{H}_s)} \leq \sum_{P: (\{Y \setminus X\}, X) = \bigcup_i X_i} (|P| - 1)! \| g_s \|_{\mathfrak{L}(\mathcal{H}_s)} \leq \sum_{k=1}^{n+1} s(n + 1, k)(k - 1)! \| g_s \|_{\mathfrak{L}(\mathcal{H}_s)} \leq n! e^{n+2} \| g_s \|_{\mathfrak{L}(\mathcal{H}_s)},
\]

(62)
where \( s(n + 1, k) \) are the Stirling numbers of the second kind.

On the space \( \mathcal{L}_\gamma(\mathcal{F}_H) \) for abstract initial-value problem (57)-(58) the following statement is true \([10]\).

**Theorem 4.** If \( B(0) \in \mathcal{L}_\gamma(\mathcal{F}_H) \) and \( \gamma < e^{-1} \), then for \( t \in \mathbb{R} \) a unique solution of the Cauchy problem (57)-(58) of the dual quantum BBGKY hierarchy exists and it is determined by the expansion

\[
B_s(t, Y) = \sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n = 1} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) B^0_{s-n}(Y \setminus X), \quad s \geq 1, \tag{63}
\]

where the \((1 + n)\)th-order cumulant \( \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) \) is defined by formula (61). For \( B(0) \in \mathcal{L}_\gamma^0(\mathcal{F}_H) \subset \mathcal{L}_\gamma(\mathcal{F}_H) \) it is a classical solution and for arbitrary initial data \( B(0) \in \mathcal{L}_\gamma(\mathcal{F}_H) \) it is a generalized solution.

The simplest examples of marginal observables (63) are given by the expressions

\[
B_1(t, 1) = \mathfrak{A}_1(t, 1) B^0_1(1),
B_2(t, 1, 2) = \mathfrak{A}_1(t, \{1, 2\}) B^0_2(1, 2) + \mathfrak{A}_2(t, 1, 2)(B^0_1(1) + B^0_1(2)).
\]

We remark that at the initial time \( t = 0 \) solution (63) satisfies initial condition (58). Indeed, according to definition (7) and equality (11) for \( n \geq 0 \), we have

\[
\mathfrak{A}_{1+n}(0, \{Y \setminus X\}, X) = \sum_{\mathcal{P} \ni \{Y \setminus X\}, X = \bigcup_j X_j} (-1)^{|\mathcal{P}| - 1} (|\mathcal{P}| - 1)! I = I \delta_{n,0}.
\]

On the space \( \mathcal{L}_\gamma(\mathcal{F}_H) \) a solution of the initial-value problem of the dual BBGKY hierarchy defines a one-parametric mapping with the following properties \([33]\).

Owing to estimate (62) for cumulants (61), under the condition that \( \gamma < e^{-1} \) we have

\[
\|B(t)\|_{\mathcal{L}_\gamma(\mathcal{F}_H)} \leq e^2(1 - \gamma e)^{-1}\|B(0)\|_{\mathcal{L}_\gamma(\mathcal{F}_H)}.
\]

Then on the space \( g \in \mathcal{L}_\gamma(\mathcal{F}_H) \) the one-parametric mapping

\[
\mathbb{R}^1 \ni t \mapsto (U(t)g)_s(Y) \doteq \sum_{n=0}^{s} \frac{1}{(s-n)!} \sum_{j_1 \neq \ldots \neq j_{s-n} = 1} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) g_{s-n}(Y \setminus X), \quad s \geq 1, \tag{65}
\]

is a \( C^*_0 \)-group. The infinitesimal generator \( \mathcal{D} = \bigoplus_{s=0}^{\infty} \mathcal{D}_s \) of this group of operators is a closed operator for the \( * \)-weak topology and on the domain of the definition \( \mathcal{D}(\mathcal{D}) \subset \mathcal{L}_\gamma(\mathcal{F}_H) \) which is the everywhere dense set for the \( * \)-weak topology of the space \( \mathcal{L}_\gamma(\mathcal{F}_H) \) it is defined by the operator

\[
(Dg)_s(Y) \doteq \mathcal{N}_s(Y) g_s(Y) + \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_k = 1} \mathcal{N}^{(k)}_{\text{int}}(j_1, \ldots, j_k) g_{s-n}(Y \setminus \{j_1, \ldots, j_n\}),
\]
where the operator $N^{(k)}_{\text{int}}$ is given by formula (60).

In capacity of initial data we consider the additive-type observables, i.e. the one-component sequences $B^{(1)}(0) = (0, a_1(1), 0, \ldots)$ (the $k$-ary marginal observable is the sequence $B^{(k)}(0) = (0, \ldots, 0, a_k(1, \ldots, k), 0, \ldots)$ [29]). In this case solution expansion (63) attains the form

$$B^{(1)}_s(t, Y) = \mathfrak{A}_s(t, 1, \ldots, s) \sum_{j=1}^{s} a_1(j), \quad s \geq 1. \quad (66)$$

For such additive-type observable as number of particles, i.e. one-component sequence $N(0) = (0, I, 0, \ldots)$, according to definition of cumulants (61), solution expansion (66) get the following form

$$(N(t))_s(Y) = \mathfrak{A}_s(t, 1, \ldots, s) \sum_{j=1}^{s} I = s \delta_{s, 1} I, \quad s \geq 1,$$

where $I$ is the identity operator and $\delta_{s, 1}$ is a Kronecker symbol. Hence we have

$$|(N(t), F(0))| = |\text{Tr}_1 F_1^0(1)| \leq \|F_0^0\|_{L^1_H} < \infty.$$ 

Thus, the marginal density operators from the space $L^1_\alpha(F_H)$ describe quantum systems of finitely many particles.

We note that solution expansion (63), i.e. nonperturbative solution expansion, can be derived from solution (9) of the initial-value problem of the Heisenberg equation (4)-(5) on the basis of expansions (55). Since hierarchy (57) has the structure of recurrence equations, we also deduce that the solution can be constructed by successive integration of the inhomogeneous Heisenberg equations. Solution (63) is represented in the form of the perturbation (iteration) series as a result of applying of analogs of the Duhamel equation to cumulants (61) of groups of operators (7).

Cluster expansions (40) of group of operators can be put at the basis of all possible solution representations of the dual quantum BBGKY hierarchy (57). In fact, solving recurrence relations (40) with respect to the 1st-order cumulants for the separation terms, which are independent from the variable $Y \backslash X \equiv (j_1, \ldots, j_{s-n})$

$$\mathfrak{A}_{1+n}(t, \{Y \backslash X\}, X) = \sum_{Z \subset X} \mathfrak{A}_1(t, \{Y \backslash X \cup Z\}) \sum_{P : X \backslash Z = \bigcup_i X_i} (-1)^{|P|} |P|! \prod_{i=1}^{|P|} \mathfrak{A}_1(t, \{X_i\}),$$

where $\sum_{Z \subset X}$ is a sum over all subsets $Z \subset X$ of the set $X$ and taking into account the identity

$$\sum_{P : X \backslash Z = \bigcup_i X_i} (-1)^{|P|} |P|! \prod_{i=1}^{|P|} \mathfrak{A}_1(t, \{X_i\}) g_{s-n}(Y \backslash X) = \sum_{P : X \backslash Z = \bigcup_i X_i} (-1)^{|P|} |P|! g_{s-n}(Y \backslash X), \quad (67)$$

and the equality

$$\sum_{P : X \backslash Z = \bigcup_i X_i} (-1)^{|P|} |P|! = (-1)^{|X \backslash Z|}, \quad (68)$$
for expansion (63) we derive

\[ B_s(t, Y) = \sum_{n=0}^{s} \frac{1}{(s-n)!} \sum_{j_1 \neq \ldots \neq j_{s-n}=1}^{s} \sum_{Z \subset X} (-1)^{|X \setminus Z|} \mathcal{G}_{s-n+|Z|}(t, Y \setminus X \cup Z) B_{s-n}^0(Y \setminus X). \] (69)

Introducing the operator \( a^+ \) (an analog of the creation operator [40]):

\[ (a^+ g)_s(Y) = \sum_{j=1}^{s} g_{s-1}(Y \setminus \{j\}) \] (70)

defined on \( \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H}) \), as a result of the symmetry property of the Maxwell-Boltzmann statistics, expression (69) can be rewritten in the following compact form

\[ B(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!(n-k)!} (a^+)^{n-k} \mathcal{G}(t)(a^+)^k B^0 = e^{-a^+} \mathcal{G}(t)e^{a^+} B^0. \]

We can obtain one more representation for a solution of the initial-value problem of the dual quantum BBGKY hierarchy, if we express the cumulants \( \mathfrak{A}_{1+n}(t) \), \( n \geq 1 \), of groups of operators (7) with respect to the 1st-order and 2nd-order cumulants. In fact, it holds

\[ \mathfrak{A}_{1+n}(t, \{(Y \setminus X)\}, X) = \sum_{Z \subset X, \ Z \neq \emptyset} \mathfrak{A}_2(t, \{Y \setminus X\}, \{Z\}) \sum_{P: X \setminus Z = \bigcup_{i=1}^{P} X_i} (-1)^{|P|} |P|! \prod_{i=1}^{P} \mathfrak{A}_1(t, \{X_i\}), \]

where \( \sum_{Z \subset X, \ Z \neq \emptyset} \) is a sum over all nonempty subsets \( Z \subset X \) of the set \( X \). Then taking into account identity (67) and equality (68), we get the following representation for solution expansion (63) of the dual quantum BBGKY hierarchy

\[ B_s(t, Y) = \mathfrak{A}_1(t, Y) B^0_s(Y) + \]
\[ + \sum_{n=1}^{s} \frac{1}{(s-n)!} \sum_{j_1 \neq \ldots \neq j_{s-n}=1}^{s} \sum_{Z \subset X, \ Z \neq \emptyset} (-1)^{|X \setminus Z|} \mathfrak{A}_2(t, \{Y \setminus X\}, \{Z\}) B_{s-n}^0(Y \setminus X), \]

where \( Y \equiv (1, \ldots, s), \ X \equiv Y \setminus \{j_1, \ldots, j_{s-n}\}, \) i.e. \( Y \setminus X = (j_1, \ldots, j_{s-n}) \).

### 4.2 Marginal density operators

As stated above (54) the mean value of the marginal observable \( B(t) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H}) \) at \( t \in \mathbb{R} \) in the initial marginal state \( F(0) = (1, F^0_1(1), \ldots, F^0_s(1, \ldots, s), \ldots) \in \mathfrak{L}_\alpha(\mathcal{F}_\mathcal{H}) \) is defined by the functional

\[ (B(t), F(0)) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1, \ldots, s} B_s(t, 1, \ldots, s) F^0_s(1, \ldots, s). \] (71)

According to estimate (54), functional (71) exists under the condition that \( \gamma < e^{-1} \), and the following estimate holds

\[ \| (B(t), F(0)) \| \leq e^2 (1 - \gamma e)^{-1} \| B(0) \| \mathcal{L}_\gamma(\mathcal{F}_\mathcal{H}) \| F(0) \| \mathcal{L}_\gamma(\mathcal{F}_\mathcal{H}), \]
In consequence of the validity for functional \( \mathcal{I},D(0) \) of the following equalities
\[
(I, D(0))^{-1}(A(t), D(0)) = (I, D(t))^{-1}(A(0), D(t)) = (B(0), F(t)) = \\
= \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,...,s} B_s^0 (1,\ldots,s) F_s(t, 1,\ldots,s),
\]
where the marginal density operator \( F_s(t, 1,\ldots,s) \) is defined by means of density operators \( \mathcal{D} \) according to formula \( (14) \) and the marginal observable \( B_s^0 (1,\ldots,s) \) is defined by means of \( (5) \) according to formula \( (55) \) respectively, it is possible to describe the evolution within the framework of the marginal state evolution.

According to \( (30) \), marginal density operators \( (56) \) can be defined in terms of the correlation operators of clusters of particles \( (31) \) by the expansion
\[
F_s(t,Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,...,s+n} g_{1+n}(t, \{Y\}, s+1,\ldots,s+n), \quad s \geq 1,
\]
where the correlation operator \( g_{1+n}(t) \) is given by expansion \( (31) \). Starting from an alternative approach to the description of states by the von Neumann hierarchy, we define the marginal density operators by the use of solutions of the Cauchy problem of the von Neumann hierarchy for correlation operators of clusters of particles \( (51) \). As is obvious from what will be said the cumulant structure of the von Neumann hierarchy solution \( (51) \) induces the cumulant structure of solution expansion \( (97) \) of initial-value problem of the quantum BBGKY hierarchy for marginal density operators that are adopted for the description of infinite-particle systems, i.e. infinite-particle dynamics is generated by the dynamics of correlations.

### 4.3 The quantum BBGKY hierarchy

The evolution of states is described by the sequences \( F(t) = (1, F_1(t,1), \ldots, F_s(t,1,\ldots,s),\ldots) \) of the marginal density operators that satisfy the Cauchy problem of the quantum BBGKY hierarchy
\[
\frac{d}{dt} F_s(t,Y) = \left( \sum_{j=1}^{s} (-\mathcal{N}(j)) + \sum_{j_1<j_2=1}^{s} (-\mathcal{N}_{\text{int}}(j_1,j_2)) \right) F_s(t,Y) + \\
+ \sum_{j=1}^{s} \text{Tr}_{s+1} (-\mathcal{N}_{\text{int}}(j, s+1)) F_{s+1}(t,Y, s+1),
\]
where \( Y \equiv (1,\ldots,s) \) and on \( \mathcal{L}_0^1(\mathcal{H}_n) \subset \mathcal{L}_1^1(\mathcal{H}_n) \) the operators \( -\mathcal{N}(j) \) and \( -\mathcal{N}_{\text{int}}(j_1,j_2) \) are correspondingly defined by formulas
\[
(-\mathcal{N}(j)) f_n \doteq -i (K(j) f_n - f_n K(j)),
\]
\[
(-\mathcal{N}_{\text{int}}(j_1,j_2)) f_n \doteq -i (\Phi(j_1,j_2) f_n - f_n \Phi(j_1,j_2)).
\]
In case of the space $\mathcal{H} = L^2(\mathbb{R}^\nu)$, hierarchy of evolution equations \eqref{eq:73} for kernels of the operators $F_s(t)$, $s \geq 1$, (the marginal or $s$-particle density matrix) are given by

$$i\frac{\partial}{\partial t}F_s(t, q_1, \ldots, q_s; q'_1, \ldots, q'_s) = \left( -\frac{1}{2} \sum_{i=1}^{s} (\Delta_{q_i} - \Delta_{q'_i}) + \sum_{i<j=1}^{s} (\Phi(q_i - q_j) - \Phi(q'_i - q'_j)) \right) F_s(t, q_1, \ldots, q_s; q'_1, \ldots, q'_s) + \sum_{i=1}^{s} \int dq_{s+1} (\Phi(q_i - q_{s+1}) - \Phi(q'_i - q_{s+1})) F_{s+1}(t, q_1, \ldots, q_s, q_{s+1}; q'_1, \ldots, q'_{s+1}).$$

We note that in case of many-body interaction potentials \eqref{eq:32} hierarchy \eqref{eq:73} has the form \eqref{eq:33}

$$\frac{d}{dt}F_s(t, Y) = -\mathcal{N}_s(Y) F_s(t, Y) +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \sum_{Z \subset Y, Z \neq \emptyset} \left( -\mathcal{N}_\text{int}^{(Z+n)} \right)(Z, s+1, \ldots, s+n) F_{s+n}(t), \quad s \geq 1,$$

where the operator $\mathcal{N}_\text{int}^{(n)}$ is defined on $\mathfrak{L}_0^1(\mathcal{H}_n) \subset \mathfrak{L}^1(\mathcal{H}_n)$ by formula \eqref{eq:60}.

For finitely many particles the quantum BBGKY hierarchy \eqref{eq:73} is rigorously derived on basis of the von Neumann equation \eqref{eq:11} according to the definition of marginal density operators \eqref{eq:56}. The rigorous derivation of the quantum BBGKY hierarchy from the von Neumann hierarchy is given in \eqref{eq:16}. Another way of looking to the derivation of the quantum BBGKY hierarchy consists in the construction of the adjoint (dual) equations to the dual quantum BBGKY hierarchy \eqref{eq:57} with respect to bilinear form \eqref{eq:71}.

On the space $\mathfrak{L}_\alpha^1(\mathcal{F}_\mathcal{H})$ for abstract initial-value problem \eqref{eq:73}-\eqref{eq:74} the following statement holds \eqref{eq:31}.

**Theorem 5.** If $F(0) \in \mathfrak{L}_\alpha^1(\mathcal{F}_\mathcal{H})$ and $\alpha > e$, then for $t \in \mathbb{R}$ a unique solution of the Cauchy problem \eqref{eq:73}-\eqref{eq:74} of the quantum BBGKY hierarchy exists and is given by the expansion

$$F_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathfrak{A}_{1+n}(-t, \{Y\}, X \setminus \{Y\}) F_{s+n}^0(X), \quad s \geq 1,$$

where $Y \equiv (1, \ldots, s)$, $X \equiv (1, \ldots, s + n)$, and the evolution operator

$$\mathfrak{A}_{1+n}(-t, \{Y\}, X \setminus \{Y\}) = \sum_{P : \{Y\}, X \setminus \{Y\} = \bigcup X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \subset P} G_{\theta(X_i)}(-t, \theta(X_i))$$

is the $(1+n)$th-order cumulant of groups of operators \eqref{eq:15}, $\sum_P$ is the sum over all possible partitions $P$ of the set $\{Y\}, X \setminus \{Y\}$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset (\{Y\}, X \setminus Y)$. For initial data $F(0) \in \mathfrak{L}_{\alpha,0}^1 \subset \mathfrak{L}_\alpha^1(\mathcal{F}_\mathcal{H})$ it is a strong solution and for arbitrary initial data from the space $\mathfrak{L}_{\alpha,0}^1(\mathcal{F}_\mathcal{H})$ it is a weak solution.
Owing to estimate \([12]\), i.e.
\[
\|\mathfrak{A}_{1+n}(-t)f_{s+n}\|_{L^1(\mathcal{H}_{s+n})} \leq n!e^{n+2}\|f_{s+n}\|_{L^1(\mathcal{H}_{s+n})},
\]
for \(F^0 \in \mathfrak{H}_\alpha(\mathcal{F}_\mathcal{H})\) series \([77]\) converges on the norm of the space \(\mathfrak{H}_\alpha(\mathcal{F}_\mathcal{H})\) provided that \(\alpha > c\), and the inequality holds
\[
\|F(t)\|_{L^1(\mathcal{F}_\mathcal{H})} \leq c_{\alpha}\|F(0)\|_{L^1(\mathcal{F}_\mathcal{H})},
\]
where \(c_{\alpha} = e^2(1 - \frac{c}{\alpha})^{-1}\). The parameter \(\alpha\) can be interpreted as the value inverse to the average number of particles.

Let us indicate some properties of cumulants \([78]\). If \(n = 0\), for \(f_s \in \mathfrak{H}_0(\mathcal{H}_s) \subseteq \mathfrak{H}(\mathcal{H}_s)\) in the sense of the norm convergence of the space \(\mathfrak{H}_0(\mathcal{H}_s)\) the generator of first-order cumulant \([78]\) is given by operator \([73]\)
\[
\lim_{t \to 0} \frac{1}{t}(\mathfrak{A}_1(-t, \{Y\}) - I)f_s(Y) = -\mathcal{N}_sf_s(Y).
\]
In case of \(n = 1\) for \(f_{s+1} \in \mathfrak{H}_0(\mathcal{H}_{s+1}) \subseteq \mathfrak{H}(\mathcal{H}_{s+1})\) we obtain the following equality in the same sense
\[
\lim_{t \to 0} \frac{1}{t}\mathfrak{A}_2(-t, \{Y\}, s + 1)f_{s+1}(Y, s + 1) = \sum_{i=1}^s (-\mathcal{N}_{int}(i, s + 1))f_{s+1}(Y, s + 1), \quad (79)
\]
and for \(n \geq 2\) as a consequence of the fact that we consider a system of particles interacting by a two-body potential \([3]\), it holds
\[
\lim_{t \to 0} \frac{1}{t}\mathfrak{A}_{1+n}(-t, \{Y\}, X\backslash Y)f_{s+n}(X) = 0.
\]

With a view to generality we consider the case of many-body interaction potentials \([32]\). On the space \(\mathfrak{H}_\alpha(\mathcal{F}_\mathcal{H})\) a solution of the initial-value problem of the quantum BBGKY hierarchy is defined a one-parametric mapping \(\mathfrak{A}_{1+n}(-t, \{Y\}, X\backslash Y)f_{s+n}(X)\) the infinitesimal generator \(B\) in the sense of bilinear form \([11]\)
\[
\mathbb{R}^1 \ni t \mapsto (U(-t)f)_s(Y) \doteq \sum_{n=0}^{\infty} \frac{1}{n!}\text{Tr}_{s+1, \ldots, s+n} \mathfrak{A}_{1+n}(-t, \{Y\}, X\backslash Y)f_{s+n}(X)
\]
with the following properties \([33]\). If \(f \in \mathfrak{H}_\alpha(\mathcal{F}_\mathcal{H})\) and \(\alpha > c\), then one-parametric mapping \([80]\) is a \(C_0\)-group \([33]\). On the subspace \(\mathfrak{H}_{\alpha, 0} \subseteq \mathfrak{H}_\alpha(\mathcal{F}_\mathcal{H})\) the infinitesimal generator \(B = \bigoplus_{n=0}^{\infty} B_n\) of this group is defined by the operator
\[
(Bf)_s(Y) = -\mathcal{N}_sf_s(Y) + \sum_{k=1}^{s} \frac{1}{k!} \sum_{i_1 \neq \ldots \neq i_k} \sum_{n=1}^{\infty} \frac{1}{n!}\text{Tr}_{s+1, \ldots, s+n} (-\mathcal{N}_{int}^{(k+n)})(i_1, \ldots, i_k, X\backslash Y)f_{s+n}(X), \quad s \geq 1,
\]
where on \(\mathfrak{H}_{\alpha, 0}(\mathcal{H}_{s+n}) \subseteq \mathfrak{H}(\mathcal{H}_{s+n})\) the operator \((-\mathcal{N}_{int}^{(k+n)})\) is defined by formula \([85]\).
We indicate that nonperturbative solution (77) of the quantum BBGKY hierarchy is transformed to the form of the perturbation (iteration) series as a result of applying of analogs of the Duhamel equation (7) to cumulants (78) of groups of operators. To reduce expansion (77) to the form of the perturbation (iteration) series as a result of applying of analogs of the Duhamel equation [7] to cumulants (78) of groups of operators which act on the variables $Y$

$$A_{1+n}(-t, \{Y\}, X \setminus Y) = \sum_{Z \subseteq X \setminus Y} g_{|Y \cup Z|}(-t, Y \cup Z) \sum_{P: (X \setminus Y \cup Z) = \bigcup_{i} X_{i}} (-1)^{|P|} |P|! \prod_{X_{i} \subseteq P} g_{iX_{i}}(-t, X_{i}).$$

(80)

If $X_{i} \subseteq X \setminus Y$, then for the trace class operator $F_{|X|}^{0}$ and the unitary group of operator (15) the equality is valid

$$\text{Tr}_{s+1, \ldots, s+n} \prod_{X_{i} \subseteq P} g_{iX_{i}}(-t; X_{i}) F_{|X|}^{0}(X) = \text{Tr}_{s+1, \ldots, s+n} F_{|X|}^{0}(X).$$

Then, taking into account the equality

$$\sum_{P: (X \setminus Y \cup Z) = \bigcup_{i} X_{i}} (-1)^{|P|} |P|! = (-1)^{|X \setminus (Y \cup Z)|},$$

from expression (80) for solution expansion (77) we obtain

$$F_{s}(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} U_{1+n}(-t, \{Y\}, X \setminus Y) F_{|X|}^{0}(X),$$

(81)

where $U_{1+n}(-t)$ is the $(1+n)\text{th}$-order reduced cumulant of groups of operators (15)

$$U_{1+n}(-t, \{Y\}, X \setminus Y) = \sum_{Z \subseteq X \setminus Y} (-1)^{|X \setminus (Y \cup Z)|} g_{|Y \cup Z|}(-t, Y \cup Z).$$

Using the symmetry property of particles obeying the Maxwell-Boltzmann statistics, the equalities valid

$$\text{Tr}_{s+1, \ldots, s+n} \sum_{Z \subseteq X \setminus Y} (-1)^{|X \setminus (Y \cup Z)|} g_{|Y \cup Z|}(-t, Y \cup Z) F_{|X|}^{0}(X) =$$

$$= \text{Tr}_{s+1, \ldots, s+n} \sum_{k=0}^{n} (-1)^{k} \sum_{i_{1} < \ldots < i_{n-k} = s+1} g_{|Y|+n-k}(-t, Y, i_{1}, \ldots, i_{n-k}) F_{s+n}^{0}(X) =$$

$$= \text{Tr}_{s+1, \ldots, s+n} \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!(n-k)!} g_{|Y|+n-k}(-t, Y, s+1, \ldots, s+n-k) F_{s+n}^{0}(X).$$

Hence for the evolution operator $U_{1+n}(-t)$ it holds

$$U_{1+n}(-t, \{Y\}, X \setminus Y) = \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!(n-k)!} g_{s+n-k}(-t, Y, s+1, \ldots, s+n-k),$$

(82)
and consequently, we derive the representation which is written down in terms of the operator $a$ (an analog of the annihilation operator $[1]$)

$$(af)_n = \text{Tr}_{n+1} f_{n+1},$$

as the following expansion

$$F(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} a^{n-k} G(-t) a^k F^0.$$  \hfill (83)

We remark that this representation for the solution expansion for the first time is obtained in [4] by another method in the form

$$F(t) = e^a G(-t) e^{-a} F(0).$$

Finally, in view of the validity of the equality

$$\frac{d}{d\tau} G(-t + \tau) a G(-\tau) F(0) = G(-t + \tau) [N, a] G(-\tau) F(0),$$

where $[,]$ is the commutator of operators, namely in componentwise form

$$([N, a] f)_s(Y) = \sum_{j=1}^{s} \text{Tr}_{s+1} (-N_{\text{int}}(j, s + 1)) f_{s+1}(Y, s + 1),$$

expansion (84) is represented in the form of the perturbation series of the quantum BBGKY hierarchy (73)

$$F(t) = \sum_{n=0}^{\infty} \int_{0}^{t} \cdots \int_{0}^{t_{n-1}} dt_1 \cdots dt_n G(-t + t_1) [N, a] G(-t_1 + t_2) \cdots$$

$$\cdots G(-t_{n-1} + t_n) [N, a] G(-t_n) F(0),$$

or in componentwise form

$$F_s(t, Y) = \sum_{n=0}^{\infty} \int_{0}^{t} \cdots \int_{0}^{t_{n-1}} dt_1 \cdots dt_n \text{Tr}_{s+1, \ldots, s+n} G_s(-t + t_1) \sum_{j=1}^{s} (-N_{\text{int}}(j_1, s + 1)) \times$$

$$\times G_{s+1}(-t_1 + t_2) \cdots G_{s+n-1}(-t_{n-1} + t_n) \sum_{j_n=1}^{s+n-1} (-N_{\text{int}}(j_n, s + n)) G_{s+n}(-t_n) F^0_{s+n}(X).$$

Recurrence relations (40) underlie of the classification of possible solution representations of the Cauchy problem (73)-(74) of the quantum BBGKY hierarchy. In fact, using cluster expansion (15) of the group of operators (15) it is possible to construct other representations.
For example, solving the recurrence relations (40) with respect to the cumulants of first and second order, we have

\[ A_1(t, Y, X \setminus Y) = \sum_{Z \subset X \setminus Y, Z \neq \emptyset} \mathcal{A}_2(t, \{X\}, \{Z\}) \sum_{P: (X \setminus (Y \cup Z)) = \bigcup_{i} X_i} (-1)^{|P|} |P|! \prod_{X_i \subset P} A_1(t, \{X_i\}). \]

Summing up the relevant terms of expression (86) in expansion (78) similarly to the case of (81), we obtain the expansion representation of initial-value problem (73)-(74) through the second order cumulant

\[ F_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \sum_{Z \subset X \setminus Y, Z \neq \emptyset} (-1)^{|X \setminus (Y \cup Z)|} \mathcal{A}_2(t, \{X\}, \{Z\}) F_{s+n}^0(X). \]

For \( F(0) \in \mathcal{L}_1^1(\mathcal{F}_H) \) series (87) converges on the norm of the space \( \mathcal{L}_1^1(\mathcal{F}_H) \) and the estimate holds

\[ \|F(t)\|_{\mathcal{L}_1^1(\mathcal{F}_H)} \leq 2e^2 \|F(0)\|_{\mathcal{L}_1^1(\mathcal{F}_H)}. \]

The statement that for initial data from the space \( \mathcal{L}_1^1(\mathcal{F}_H) \) solution (87) of the Cauchy problem of the quantum BBGKY hierarchy (73)-(74) actually follows from the equivalence of different representations and the existence statement of solution (77).

4.4 The nonlinear quantum BBGKY hierarchy for marginal correlation operators

In view of the definition of mean-value functional (30), for example, the dispersion of an additive-type observable is determined by the functional

\[ \langle (A^{(1)} - \langle A^{(1)}(t) \rangle)^2 \rangle(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, 1+n} \left( a_1^2(1) - \langle A^{(1)} \rangle^2(t) \right) g_{1+n}(t, 1, \ldots, 1+n) + \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, 2+n} a_1(1) a_1(2) g_{2+n}(t, 1, \ldots, 2+n), \]

where \( \langle A^{(1)}(t) \rangle \) is defined by expression (30) and the operators \( g_{s+n}(t) \) are defined by expansions (21). For \( A^{(1)} \in \mathcal{L}(\mathcal{F}_H) \) and \( g \in \mathcal{L}^1(\mathcal{F}_H) \) functional (88) exists. Following to formula (88) we introduce the marginal correlation operators by the series

\[ G_s(t, 1, \ldots, s) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} g_{s+n}(t, 1, \ldots, s+n), \quad s \geq 1, \]

where the operator \( g_{s+n}(t, 1, \ldots, s+n) \) is defined by expansion (21) over solutions (14) of the von Neumann equations (11). According to estimate (49), series (89) exists and the estimate holds: \( \|G_s(t)\|_{\mathcal{L}_1^1(\mathcal{H}_s)} \leq s! (2e^2)^s c^s \sum_{n=0}^{\infty} (2e^2)^n c^n \).
Then the dispersion of an additive-type observable is defined within framework of the marginal correlation operators \(89\) as follows \[3\]

\[
\langle (A^{(1)} - \langle A^{(1)} \rangle(t))^2 \rangle(t) = \text{Tr}_1 \left( a_1^2(1) - \langle A^{(1)} \rangle^2(t) \right) G_1(t, 1) + \text{Tr}_{1,2} a_1(1) a_1(2) G_2(t, 1, 2).
\]

Thus, macroscopic characteristics of fluctuations of observable are determined by marginal correlation operators \(89\) on the microscopic level.

Assuming as a basis an alternative approach to the description of the state evolution within framework of the von Neumann hierarchy, we shall define the marginal correlation operators by the use of solutions of the Cauchy problem \(33\)-\(34\) of the von Neumann hierarchy for correlation operators by formula \(89\). We note that every term of marginal correlation operator expansion \(89\) is determined by the \((s + n)\)-particle correlation operator \((48)\) as contrasted to marginal density operator expansion \(72\) which is defined by the \((1 + n)\)-particle correlation operator \((51)\).

Traditionally marginal correlation operators are introduced by means of the cluster expansions of the marginal density operators \(72\) governed by the quantum BBGKY hierarchy \(73\)

\[
F_s(t, Y) = \sum_{P: Y = \bigcup_i X_i} \prod_{X_i \subset P} G_{|X_i|}(t, X_i), \quad s \geq 1,
\]

where \(\sum_{P: Y = \bigcup_i X_i}\) is the sum over all possible partitions \(P\) of the set \(Y = (1, \ldots, s)\) into \(|P|\) nonempty mutually disjoint subsets \(X_i \subset Y\). Hereupon solutions of cluster expansions \(90\)

\[
G_s(t, Y) = \sum_{P: Y = \bigcup_i X_i} (-1)^{|P|} |(P) - 1|! \prod_{X_i \subset P} F_{|X_i|}(t, X_i), \quad s \geq 1,
\]

are interpreted as the operators that describe correlations of many-particle systems. Thus, marginal correlation operators \(91\) are cumulants (semi-invariants) of the marginal density operators. It is obvious that definition \(89\) follows from \(91\) in consequence of definition \(72\) and relations \(52\) between correlation operators of particle clusters and correlation operators of particles.

The evolution of all possible states of quantum many-particle systems obeying the Maxwell-Boltzmann statistics with the Hamiltonian \(3\) can be described within the framework of marginal correlation operators governed by the nonlinear quantum BBGKY hierarchy

\[
\frac{d}{dt} G_s(t, Y) = \mathcal{N}(Y \mid G(t)) + \text{Tr}_{s+1} \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s + 1)) (G_{s+1}(t, Y, s + 1) + \sum_{P: Y = X_1 \cup X_2, \ i \in X_1; s + 1 \in X_2} G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2)),
\]

\[
G_s(t, Y) \Big|_{t=0} = G^0_s(Y), \quad s \geq 1.
\]

In equation \(92\) the operator \(\mathcal{N}(Y \mid G(t))\) is generator of the von Neumann hierarchy \(36\) defined by

\[
\mathcal{N}(Y \mid G(t)) \equiv -\mathcal{N}_s(Y) G_s(t, Y) + \sum_{P: Y = X_1 \cup X_2} \sum_{i_1 \in X_1, i_2 \in X_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2),
\]

where \(\mathcal{N}_s(Y)\) is generator of the von Neumann hierarchy \(36\) defined by

\[
\mathcal{N}_s(Y) \equiv \sum_{P: Y = \bigcup_i X_i} \prod_{X_i \subset P} \mathcal{N}_{|X_i|}(t, X_i).
\]
where $\sum_{P : Y = X_1 \cup X_2}$ is the sum over all possible partitions $P$ of the set $Y \equiv (1, \ldots, s)$ into two nonempty mutually disjoint subsets $X_1 \subset Y$ and $X_2 \subset Y$, and $\sum_{i \in X_1; s + 1 \in X_2}$ is the sum over all possible partitions of the set $(Y, s + 1)$ into two mutually disjoint subsets $X_1$ and $X_2$ such that $i$ particle belongs to the subset $X_1$ and $s + 1$ particle belongs to $X_2$.

We note that in case of many-body interaction potentials (32) hierarchy (92) has the form

$$\frac{d}{dt} G_s(t, Y) = \mathcal{N}(Y \mid G(t)) +$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{s} \sum_{j_1 < \ldots < j_k} \text{Tr}_{s+1, \ldots, s+1+n-k} (-\mathcal{N}_{\text{int}}^{(n+1)}(j_1, \ldots, j_k, s+1, \ldots, s+1+n-k)) \times$$

$$\times \sum_{P : (1, \ldots, s+1+n-k) = \bigcup_{i=1}^{s} X_i, X_i \subset P} \prod_{X_i \subset P} G_{|X_i|}(t, X_i),$$

where we use notations accepted above.

The nonlinear quantum BBGKY hierarchy (92) is derived on basis of the von Neumann hierarchy for correlation operators (33) according to definition (89) or on basis of the quantum BBGKY hierarchy for marginal density operators (73) according to cluster expansions (90). The evolution of marginal correlation operators of both finitely and infinitely many quantum particles is described by initial-value problem of the nonlinear quantum BBGKY hierarchy (92). For finitely many particles the nonlinear quantum BBGKY hierarchy is equivalent to the von Neumann hierarchy (33).

To construct a nonperturbative solution of the Cauchy problem (92)-(93) of the nonlinear quantum BBGKY hierarchy we first consider in capacity of initial data the marginal correlation operators satisfying a chaos property. In terms of marginal correlation operators (89) a chaos property means that

$$G_s^0(1, \ldots, s) = g_1^0(1) \delta_{s,1}, \quad s \geq 1,$$

where $\delta_{s,1}$ is a Kronecker symbol. Taking into account the structure of a solution (38) of the von Neumann hierarchy (33) in case of initial data (37), from expansion (89) we obtain

$$G_s(t, 1, \ldots, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathfrak{A}_{s+n}(-t, 1, \ldots, s+n) \prod_{i=1}^{s+n} g_1^0(i), \quad s \geq 1,$$

where $\mathfrak{A}_{s+n}(-t)$ is the $(s+n)$th-order cumulant (39) of groups of operators (15). In consequence of equality (96) we finally derive

$$G_s(t, 1, \ldots, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathfrak{A}_{s+n}(-t, 1, \ldots, s+n) \prod_{i=1}^{s+n} G_0^0(i).$$

Since estimate (42) holds, series (97) converges provided that $\|G_1(0)\|_{L^2(\mathcal{H})} \leq (2e)^{-1}$.

Thus, the cumulant structure of solution (44) of the von Neumann hierarchy (33) induces the cumulant structure of expansion solution (97) of the initial-value problem of the nonlinear quantum BBGKY hierarchy for marginal correlation operators.
We emphasize that in case of chaos initial data solution expansion (77) of the quantum BBGKY hierarchy (73) for marginal density operators differs from solution expansion (77) of the nonlinear quantum BBGKY hierarchy (92) for marginal correlation operators only by the order of the cumulants of the groups of operators of the von Neumann equations [28], [41].

\[ F_s(t, Y) = \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathfrak{A}_{1+n}(-t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_i^0(i), \quad s \geq 1, \]  

where the operator \( \mathfrak{A}_{1+n}(t) \) is the \((1+n)\)th-order cumulant (78). Series (98) converges under the condition that: \( \|F_i(0)\|_{L^1(\mathcal{H})} \leq e^{-1} \).

Let us construct a nonperturbative solution expansion of the Cauchy problem (92)-(93) in the case of general initial data. According to definition (89) of the marginal correlation operators, i.e.

\[ G(t) = e^a g(t), \]

where the operator \( e^a \) is defined by (83) and the sequence \( g(t) \) is a solution of the von Neumann hierarchy defined by group (48), i.e.

\[ g(t) = G(t \mid g(0)), \]

and the equality: \( g(0) = e^{-a} G(0) \), we finally derive

\[ G(t) = e^a G(t \mid e^{-a} G(0)). \]  

Thus, a solution of the Cauchy problem of the nonlinear quantum BBGKY hierarchy for marginal correlation operators is defined by the following one-parametric mapping

\[ \mathbb{R} \ni t \to \mathcal{U}(t \mid f) = e^a G(t \mid e^{-a} f), \]

which is defined on the space \( \mathcal{L}^1(\mathcal{H}) \) and has the group property. On the subspaces \( \mathcal{L}^1_s(\mathcal{H}_s) \subset \mathcal{L}^1(\mathcal{H}), s \geq 1 \), the infinitesimal generator \( \mathcal{B}(Y \mid f) \) of this group is defined by the operator

\[ \mathcal{B}(Y \mid f) \ni \mathcal{N}(Y \mid f) + \text{Tr}_{s+1} \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s + 1)) (f_{s+1}(t, Y, s + 1) + \sum_{P : (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \subset X_2} f_{|X_1|}(t, X_1) f_{|X_2|}(t, X_2)), \]

where the same notations as for formula (92) have been used.

To set down formula (100) in componentwise form, we observe that the following equality holds

\[ \prod_{X_i \subset \mathcal{P}} (e^{-a} G(0))_{|X_i|}(X_i) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \text{Tr}_{s+n+1,\ldots,s+n+k} \prod_{k_1=0}^{k} \frac{k!}{k_1!(k-k_1)!} \ldots \]

\[ \ldots \sum_{k_{|P|-2}=0}^{k_{|P|-2}} \frac{k_{|P|-2}!}{k_{|P|-1}!(k_{|P|-2}-k_{|P|-1})!} G^0_{|X_{|P|+k_{|P|-1}|}}(X_{|P|}, s + n + k - k_{|P|-1} + 1, \ldots, s + n + k). \]
Then in view of definitions (83) and (48), we have
\[ G_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \sum_{P: (Y, s+1, \ldots, s+n) = \bigcup_i X_i} \mathfrak{A}_{|P|}(-t, \{X_1\}, \ldots, \{X_{|P|}\}) \times \prod_{X_i \subset P} (e^{-sG(0)})|X_i|(X_i), \]
and as a result we derive a solution expansion of the nonlinear quantum BBGKY hierarchy [42]
\[ G_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} U_{1+n}(t; \{Y\}, s+1, \ldots, s+n \mid G(0)), \quad s \geq 1, \quad (101) \]
where the \((1 + n)th\)-order reduced cumulant \(U_{1+n}(t)\) of groups [18] has been introduced
\[ U_{1+n}(t; \{Y\}, s+1, \ldots, s+n \mid G(0)) \doteq \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \sum_{P: (1, \ldots, n+1-k) = \bigcup_i X_i} \mathfrak{A}_{|P|}(-t, \{X_1\}, \ldots, \{X_{|P|}\}) \times \prod_{k_{1}=0}^{k} \frac{k_{1}!}{(k_{1} - k_{1})!} \cdots \prod_{k_{|P|-1}=0}^{k_{|P|-2}} \frac{k_{|P|-2}!}{(k_{|P|-2} - k_{|P|-2})!} G_{|X_1|+k-k_1}(X_1, s+n-k_1+1, \ldots, s+n) \]
Let us indicate some properties of reduced cumulants [102] of groups of operators [18]. In case of \(n = 0\), first order reduced cumulant [102] has the form
\[ U_1(t; \{Y\} \mid f) = \sum_{P: Y = \bigcup_i X_i} \mathfrak{A}_{|P|}(-t, \{X_1\}, \ldots, \{X_{|P|}\}) \prod_{X_i \subset P} f_{|X_i|}(X_i), \]
i.e. it is the group of operators [18]. Its infinitesimal generator coincides with generator of the von Neumann hierarchy [36]
\[ \lim_{t \to 0} \frac{1}{t} (U_1(t; \{Y\} \mid f) - f_s(Y)) = \mathcal{N}(Y \mid f), \quad s \geq 1, \]
for \(f \in \mathfrak{L}^1(\mathfrak{F}_H)\) in the sense of the norm convergence of the space \(\mathfrak{L}^1(\mathcal{H}_s)\), where the operator \(\mathcal{N}(Y \mid f)\) is defined by formula [94].
In case of \(n = 1\) for second order reduced cumulant [102] in the same sense we obtain the following equality
\[ \text{Tr}_{s+1} \lim_{t \to 0} \frac{1}{t} U_2(t; \{Y\}, s+1 \mid f) = \sum_{i \in Y} \text{Tr}_{s+1}(N_{\text{int}}(i, s+1))(f_{s+1}(t, Y, s+1) + \sum_{P: (Y, s+1) = X_1 \cup X_2, i \in X_1; s+1 \in X_2} f_{|X_1|}(t, X_1)f_{|X_2|}(t, X_2)), \]
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where notations are used as above for hierarchy (92), and for \( n \geq 2 \) as a consequence of the fact that we consider a system of particles interacting by a two-body potential, it holds

\[
\text{Tr}_{s+1,\ldots,s+n} \lim_{t \to 0} \frac{1}{t} U_{1+n}(t; \{ Y \}, s + 1, \ldots, s + n \mid f) = 0.
\]

According to the estimate

\[
\| A \|_{\mathcal{P}} \left( -t, \{ X_1 \}, \ldots, \{ X_{|\mathcal{P}|} \} \right) f_n \leq |\mathcal{P}|^4 e^{4|\mathcal{P}|} \| f_n \|_{\mathcal{L}^1(\mathcal{H}_n)},
\]

on the space \( L^1(\mathcal{H}_s) \), series (101) converges provided that \( \max_{n \geq 1} \| G^0_n \|_{\mathcal{L}^1(\mathcal{H}_n)} < (2e^3)^{-1} \).

For abstract initial-value problem for hierarchy (92) on the space of sequences of trace-class operators \( L^1(\mathcal{F}_n) \) the following statement is true [42].

**Theorem 6.** If \( \max_{n \geq 1} \| G^0_n \|_{\mathcal{L}^1(\mathcal{H}_n)} < (2e^3)^{-1} \), then for \( t \in \mathbb{R} \) a solution of the initial-value problem (92) - (93) of the nonlinear quantum BBGKY hierarchy is determined by expansion (101). If \( G^0_n \in L^1_0(\mathcal{H}_n) \subset L^1(\mathcal{H}_n) \) it is a strong (classical) solution and for arbitrary initial data \( G^0_n \in L^1(\mathcal{H}_n) \) it is a weak (generalized) solution.

We note also that in case of many-particle systems obeying quantum statistics, i.e. many-particle systems of fermions or bosons the nonlinear quantum BBGKY hierarchy for marginal correlation operators has the form

\[
\frac{d}{dt} G_s(t,Y) = N(Y|G(t)) + \text{Tr}_{s+1} \sum_{i \in Y} (\mathcal{N}_{\text{int}}(i, s + 1))(G_{s+1}(t, Y, s + 1) +
\sum_{P: (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \in X_2} S^\pm_{s+1} G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2)), \quad s \geq 1,
\]

where \( N(Y|G(t)) \) is generator (54) of the von Neumann hierarchy of fermions or bosons and the operator \( S^\pm_{s+1} \) is defined by formula (2).

We emphasize that the evolution of marginal correlation operators of both finitely and infinitely many quantum particles is described by initial-value problem of the nonlinear quantum BBGKY hierarchy (92). For finitely many particles the nonlinear quantum BBGKY hierarchy is equivalent to the von Neumann hierarchy (33).

5 The origin of the quantum kinetic evolution

It is well known [1, 3] that in certain situations the collective behavior of many-particle systems can be adequately described by the kinetic equations, i.e. by evolution equations for a one-particle marginal density operator. In this section we discuss the problem of potentialities inherent in the description of the evolution of states of many-particle systems in terms of a one-particle density operator or more exactly the problem of an equivalence of the hierarchies of quantum evolution equations and quantum kinetic equations. We demonstrate that in fact, if initial data is completely defined by a one-particle marginal density operator, then all possible states of infinite-particle systems at arbitrary moment of time can be described within
the framework of a one-particle density operator and the marginal functionals of such states without any approximations \cite{36, 43}. By other words the state described in terms of the sequence $F(t) = (1, F_1(t), \ldots, F_s(t), \ldots)$ of marginal density operators can be described within the framework of the sequence $F(t | F_1(t)) = (1, F_1(t), F_2(t | F_1(t)), \ldots, F_s(t | F_1(t)), \ldots)$ of explicitly defined marginal functionals $F_s(t | F_1(t)), s \geq 2,$ of the solution $F_1(t)$ of the generalized quantum kinetic equation.

Thus, in this section the origin of the microscopic description of quantum many-particle systems by means of kinetic equations is considered.

### 5.1 The generalized quantum kinetic equation

We consider the Cauchy problem of the quantum BBGKY hierarchy \cite{73} with initial data $F(t)|_{t=0} = F^{(0)} \equiv (F_1^{(0)}(1), \ldots, \prod_{i=1}^{s} F_1^{(0)}(i), \ldots)$, which is intrinsic for the kinetic description of many-particle systems because in this case all possible states are characterized by means of a one-particle marginal density operator. Then we deal with initial-value problem of the quantum BBGKY hierarchy \cite{73}, which is not completely well-defined Cauchy problem, because the generic initial data $F^{(0)}$, is not independent for every unknown marginal density operator $F_s(t, 1, \ldots, s), s \geq 1,$ from the hierarchy of equations. Consequently such initial-value problem can be naturally reformulated as the new Cauchy problem for a one-particle density operator, that corresponds to the independent initial one-particle density operator and the sequence of explicitly defined marginal functionals of the state $F_s(t, 1, \ldots, s | F_1(t)), s \geq 2,$ of the solution $F_1(t)$ of such Cauchy problem \cite{36}.

Let us formulate the restated Cauchy problem and a sequence of the marginal functionals of the state which describe the evolution of all possible states of quantum many particles in an equivalent way as compared with the quantum BBGKY hierarchy.

The one-particle density operator $F_1(t)$ is governed by the following initial-value problem of the generalized quantum kinetic equation \cite{36}

$$
\frac{d}{dt} F_1(t, 1) = -\mathcal{N}(1)F_1(t, 1) +
+ \text{Tr}_2(\mathcal{N}_{\text{int}}(1, 2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3, \ldots, n+2} \mathfrak{V}_{1+n}(t, \{1, 2\}, 3, \ldots, n+2) \prod_{i=1}^{n+2} F_1(t, i),
$$

$$
F_1(t, 1)|_{t=0} = F^{(0)}_1(1),
$$

where the operator $\mathcal{N}_{\text{int}}(1, 2)$ is defined by formula \cite{75}, and the $(1+n)$th-order generated evolution operator $\mathfrak{V}_{1+n}(t), n \geq 0,$ is defined as follows (in case of $s = 2$ for $Y \equiv (1, \ldots, s)$ and $X \setminus Y \equiv (s+1, \ldots, s+n)$)

$$
\mathfrak{V}_{1+n}(t, \{Y\}, X \setminus Y) \doteq \sum_{k=0}^{n} (-1)^k \sum_{n_1=1}^{n} \ldots \sum_{n_k=1}^{n-n_1-\ldots-n_{k-1}} \frac{n!}{(n-n_1-\ldots-n_k)!} \times
$$

$$
\times \mathfrak{H}_{1+n-n_1-\ldots-n_k}(t, \{Y\}, s+1, \ldots, s+n-n_1-\ldots-n_k) \times
$$
The series of collision integral (103) converges under the condition: following expansions over products of the solution \( F_n \) where the \((1 + n)\text{-order}\) generated evolution operator 
\[
\times \prod_{j=1}^{k} D_j : \sum_{|D_j| \leq s + n - n_1 - \ldots - n_j} 1/|D_j|! \sum_{i_1 \neq \ldots \neq i_{|D_j|}} 1/|X_{i_j}!| \sum_{i_{i_j} \in D_j} \hat{\mathcal{A}}_{1+|X_{i_j}|}(t, i_{i_j}, X_{i_j}).
\]

In expansion (105) we denote by \( \sum_{D_j: Z_j = \bigcup_j X_{i_j}} \) is the sum over all possible dissections of the linearly ordered set \( Z_j \equiv (s + n - n_1 - \ldots - n_j + 1, \ldots, s + n - n_1 - \ldots - n_{j-1}) \) on no more than \( s + n - n_1 - \ldots - n_j \) linearly ordered subsets and \( \hat{\mathcal{A}}_{1+n}(t) \) is the \((1 + n)\text{-order}\) cumulant
\[
\hat{\mathcal{A}}_{1+n}(t, \{Y\}, X \setminus Y) = \sum_{P: \{Y\}, X \setminus Y = \bigcup_i X_i} (-1)^{|P| - 1} |P|! \prod_{X_i \subset P} \hat{\mathcal{G}}_{(\theta(X_i))}(t, \theta(X_i)),
\]
of the scattering operators (47) \( \hat{\mathcal{G}}_n(t) = \mathcal{G}_n(-t, 1, \ldots, n) \prod_{i=1}^{n} \mathcal{G}_1(t, i), \quad n \geq 1. \)

The series of collision integral (103) converges under the condition: \( \| F_1(t) \|_{\mathcal{L}(\mathcal{H})} < e^{-8} \) [39].

The marginal functionals of the state \( F_s(t, 1, \ldots, s | F_1(t)) \), \( s \geq 2 \), are represented by the following expansions over products of the solution \( F_1(t) \) of the Cauchy problem (103)-(104)
\[
F_s(t, Y | F_1(t)) \equiv \sum_{n=0}^{\infty} 1/n! \text{Tr}_{s+1, \ldots, s+n} \mathfrak{V}_{1+n}(t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_1(t, i),
\]
where the \((1 + n)\text{-order}\) generated evolution operator \( \mathfrak{V}_{1+n}(t), n \geq 0 \), is defined by expansion (105). The marginal functionals of the state are represented by converged series (106) under the condition that: \( \| F_1(t) \|_{\mathcal{L}(\mathcal{H})} < e^{-(3s+2)} \) [39].

We observe that the kinetic dynamics of states is described in terms of cumulants of scattering operators (47) in contrast to the evolution of states described by the quantum BBGKY hierarchy (73). We give a few examples of the generated evolution operators \( \mathfrak{V}_n, n \geq 1 \), of the lower orders
\[
\mathfrak{V}_1(t, \{Y\}) = \hat{\mathfrak{A}}_1(t, \{Y\}),
\]
\[
\mathfrak{V}_2(t, \{Y\}, s+1) = \hat{\mathfrak{A}}_2(t, \{Y\}, s+1) - \hat{\mathfrak{A}}_1(t, \{Y\}) \sum_{i=1}^{s} \hat{\mathfrak{A}}_2(t, i, s+1).
\]

It should be emphasized that in case under consideration, i.e. in case of the absence of correlations at initial time, the correlations generated by the dynamics of a system are completely governed by evolution operators (105).

Typical properties for the kinetic description of the evolution of constructed marginal functionals of the state (106) are induced by the properties of generated evolution operators (105). Let us indicate some intrinsic properties of the evolution operators \( \mathfrak{V}_{1+n}(t), n \geq 0 \), representative for cumulants (semi-invariants) of groups of operators.

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First of all we observe that they are solutions of the following recursive relations (the kinetic cluster expansions, which are in some sense analog of equilibrium virial expansions) \[36\]

\[\mathcal{A}_{1+n}(-t, \{Y\}, s+1, \ldots, s+n) = \]

\[= \sum_{n_1=0}^{n} \frac{n!}{(n-n_1)!} \mathfrak{g}_{1+n-n_1}(t, \{Y\}, s+1, \ldots, s+n-n_1) \sum_{D: Z = \bigcup_k X_k, |D| \leq s+n-n_1} \frac{1}{|D|!} \times \]

\[\times \prod_{i_1 \neq \ldots \neq i_{|D|}=1} \frac{1}{|X_k|!} \mathcal{A}_{1+|X_k|}(-t, i_k, X_k) \prod_{m=1, m \neq i_1, \ldots, i_{|D|}} \mathcal{A}_1(-t, m), \]

where \(\sum_{D: Z = \bigcup_k X_k, |D| \leq s+n-n_1}\) is the sum over all possible dissections \(D\) of the linearly ordered set \(Z \equiv (s + n - n_1 + 1, \ldots, s + n)\) on no more than \(s + n - n_1\) linearly ordered subsets.

Since in case of a system of non-interacting particles for scattering operators \(47\) the equality holds: \(\hat{g}_n(t) = I\), where \(I\) is a unit operator, then we have

\[\mathfrak{g}_{1+n}(t) = I \delta_{n,0},\]

where \(\delta_{n,0}\) is a Kronecker symbol. Similarly, at initial time \(t = 0\) it is true: \(\mathfrak{g}_{1+n}(0) = I \delta_{n,0}\).

The infinitesimal generator of the first-order generated evolution operator \(105\) is defined by the following limit in the sense of the norm convergence in the space \(L^1(\mathcal{H}_n)\)

\[\lim_{t \to 0} \frac{1}{t} (\mathfrak{g}_1(t, \{1, \ldots, n\}) - I) f_n = \sum_{i<j=1}^{n} (-\mathcal{N}_{\text{int}}(i, j)) f_n,\]

where the operator \((-\mathcal{N}_{\text{int}}(i, j))\) is defined by formula \[59\]. In general case, i.e. \(n \geq 2\), in the sense of the norm convergence on the space \(L^1(\mathcal{H}_n)\) for the \(n\)-order generated evolution operator \(105\) it holds

\[\lim_{t \to 0} \frac{1}{t} \mathfrak{g}_n(t, 1, \ldots, n) f_n = 0.\]

Before constructing a solution of initial-value problem \(103\)-\(104\) in the space \(L^1(\mathcal{H})\) we generalize kinetic equation \(103\) for particles interacting via many-body interaction potentials \(\Phi^{(n)}, n \geq 1\). In this case the generalized quantum kinetic equation has the form

\[\frac{d}{dt} F_1(t, 1) = -\mathcal{N}(1) F_1(t, 1) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{(n-k)!} \frac{1}{k!} \text{Tr}_{2, \ldots, n+1}(-\mathcal{N}^{(k+1)}_{\text{int}})(1, \ldots, k+1) \mathfrak{g}_{1+n-k}(t, \{1, \ldots, k+1\}, k+2, \ldots, n+1) \prod_{i=1}^{n+1} F_1(t, i),\]

where \(\mathfrak{g}_{1+n-k}(t)\), is the \((1+n-k)th\)-order generated evolution operator \(105\) and notations \(59, 60\) are used. The collision integral in the generalized quantum kinetic equation \(109\) is defined by the convergent series under condition that \(\|F_1(t)\|_{L^1(\mathcal{H})} < e^{-8} \[36\].
For the sake of a comparison of the structure of various collision integral components in (109) we give expressions of the collision integral term describing a two-body interaction and three particle correlations

\[ \text{Tr}_{2,3}(-\mathcal{N}^{(2)}_{\text{int}})(1, 2)\mathfrak{V}_2(t, \{1, 2\}, 3)F_1(t, 1)F_1(t, 2)F_1(t, 3), \]

and the collision integral term describing a three-body interaction

\[ \frac{1}{2!}\text{Tr}_{2,3}(-\mathcal{N}^{(3)}_{\text{int}})(1, 2, 3)\mathfrak{V}_1(t, \{1, 2, 3\})F_1(t, 1)F_1(t, 2)F_1(t, 3), \]

where the evolution operators \( \mathfrak{V}_2(t, \{1, 2\}, 3) \) and \( \mathfrak{V}_1(t, \{1, 2, 3\}) \) are defined by (105).

For the Cauchy problem (109)-(104) (or (103)-(104)) on the space \( L^1(\mathcal{H}) \) the following statement is true.

**Theorem 7.** The global in time solution of initial-value problem (109)-(104) is determined by the following expansion

\[ F_1(t, 1) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, 1+n} \mathfrak{A}_{1+n}(-t, 1, \ldots, n+1) \prod_{i=1}^{n+1} F_0^i(i), \quad (110) \]

where the cumulants \( \mathfrak{A}_{1+n}(-t) \), \( n \geq 0 \), are defined by formula (78). If \( \|F_0^1\|_{L^1(\mathcal{H})} < (e(1+e^9))^{-1} \), then for \( F_1^0 \in L^1(\mathcal{H}) \) it is a strong (classical) solution and for an arbitrary initial data \( F_1^0 \in L^1(\mathcal{H}) \) it is a weak (generalized) solution.

In section 4 the relationships of the evolution of observables and quantum states described in terms of marginal density operators have been considered in the general case. In case of initial states specified by a one-particle marginal density operator, the dual BBGKY hierarchy describes the dual picture of the evolution to the picture of the evolution of states governed by the generalized quantum kinetic equation and an infinite sequence of explicitly defined functionals of a solution of such evolution equation. In fact, the following equality is true

\[ (B(t), F^{(c)}) = (B(0), F(t | F_1(t))), \quad (111) \]

where the initial state \( F^{(c)} \) is given as above, the sequence \( B(t) \) defined by expansion (63) and \( F(t | F_1(t)) = (1, F_1(t), F_2(t | F_1(t)), \ldots, F_s(t | F_1(t)), \ldots) \) is a sequence of marginal functionals of the state (106) with the first element \( F_1(t) \) given by series (110).

To verify equality (111) we transform functional \( (B(t)|F^{(c)}) \) as follows

\[ (B(t), F^{(c)}) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1, \ldots, s} B_0^0(1, \ldots, s) \times \]

\[ \times \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathfrak{A}_{1+n}(-t, \{Y\}, s+1, \ldots, s+n) \prod_{i=1}^{s} F_1^0(i), \quad (112) \]

where the \((1+n)th\)-order cumulant \( \mathfrak{A}_{1+n}(-t, \{Y\}, s+1, \ldots, s+n) \) is defined by (78). For \( F_1^0 \in L^1(\mathcal{H}) \) and \( B_0^0 \in L^1(\mathcal{H}) \) obtained functional (112) exists under the condition \( \|F_1^0\|_{L^1(\mathcal{H})} < e^{-7} \). Then we expand the cumulants \( \mathfrak{A}_{1+n}(-t) \) over the new evolution operators \( \mathfrak{V}_{1+n}(t), n \geq 0, \)
into the kinetic cluster expansions (108). Representing series over the summation index \( n \) and the sum over the summation index \( n_1 \) in functional (112) as the two-fold series and identifying the series over the summation index \( n_1 \) with the products of one-particle density operators

\[
\sum_{n_1=0}^{\infty} \text{Tr} s_{s+n+1} \ldots s_{n+n_1} \sum_D Z = \bigcup_{|D| \leq s+n} X_k \sum_{i_1 < \ldots < |D| = 1} X_k \subset D \prod \frac{1}{|X_k|!} \times \\
\times \mathcal{A}_{1+1}(t, i_k, X_k) \prod_{l = 1, l \neq 1, \ldots, i_{|D|}}^{s+n} \mathcal{A}_1(-t, l) \prod_{j=1}^{n_1+n} F_1^0(j) = \prod_{i=1}^{s+n} F_1(t, i),
\]

we transform functional (112) to the form in terms of marginal functionals of the state (106). Thus, equality (111) holds.

In a particular case of the additive-type marginal observables \( B^{(1)}(t) \) given by (66) equality (111) is reduced to the form

\[
(B^{(1)}(t), F^{(c)}) = \text{Tr}_1 a_1(1) F_1(t, 1),
\]

where the one-particle marginal density operator \( F_1(t) \) is represented by the expansion

\[
F_1(t, 1) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s_{s+n+1}} \ldots \mathcal{A}_{1+n}(-t, 1, \ldots, n+1) \prod_{i=1}^{n+1} F_1^0(i),
\]

which coincides with solution (110) of the Cauchy problem (103)-(104). Hence for additive-type marginal observables the generalized quantum kinetic equation (103) is dual to the dual quantum BBGKY hierarchy (57) with respect to bilinear form (71).

Let us derive the evolution equation, which satisfies obtained expansion for the one-particle marginal density operator \( F_1(t) \). Taking into account equality (13) and observing the validity of equality (79) for cumulants of groups (15), we differentiate over the time variable in the sense of pointwise convergence on the space \( \mathcal{L}^1(\mathcal{H}) \) obtained expansion for \( F_1(t) \). As result it holds

\[
\frac{d}{dt} F_1(t, 1) = -\mathcal{N}_1(1) F_1(t, 1) + \\
+ \text{Tr}_2(-\mathcal{N}_{\text{int}}(1, 2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s_{s+n+2}} \mathcal{A}_{1+n}(t, \{1, 2\}, 3, \ldots, n+2) \prod_{i=1}^{n+2} F_1^0(i).
\]

In the second summand in the right-hand side of equality (114) we expand cumulants (78) of groups (15) into kinetic cluster expansions (108) and represent series over the summation index \( n \) and the sum over the summation index \( n_1 \) as the two-fold series. Then the following equalities take place

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s_{s+n+2}}(-\mathcal{N}_{\text{int}}(1, 2)) \mathcal{A}_{1+n}(t, \{1, 2\}, 3, \ldots, n+2) \prod_{i=1}^{n+2} F_1^0(i) = \\
= \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s_{s+n+2}}(-\mathcal{N}_{\text{int}}(1, 2)) \sum_{n_1=0}^{n} \frac{n!}{(n-n_1)!} \mathcal{A}_{1+n-n_1}(t, \{1, 2\}, 3, \ldots, n+2-n_1) \times
\]

\[
\times \prod_{i=1}^{n_1} F_1^0(i).\]
where \( Z \equiv (n + 3 - n_1, \ldots, n + 2) \) and \( Z' \equiv (n + 3, \ldots, n + 2 + n_1) \) are linearly ordered sets and the notations accepted above are used. Consequently, in case of \( s = 2 \) applying to the obtained expression formula (113), from equality (114) we derive

\[
\frac{d}{dt} F_1(t, 1) = -\mathcal{N}_1(1) F_1(t, 1) + \\
+ \text{Tr}_2(-\mathcal{N}_{\text{int}}(1, 2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3, \ldots, n+2} \mathfrak{A}_{1+n}(t, \{1, 2\}, 3, \ldots, n + 2) \prod_{i=1}^{n+2} F_1(t, i).
\]

where the operator \( S_s^\pm \) is defined by (2), and the marginal functionals of the states are represented by the series

\[
F_s(t, Y \mid F_1(t)) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathfrak{V}_{1+n}(t, \{Y\}, X \setminus Y) S_{s+n}^\pm \prod_{i=1}^{s+n} F_1(t, i),
\]

where the \((1+n)\text{th}\)-order generated evolution operator \( \mathfrak{V}_{1+n}(t) \), \( n \geq 0 \), is determined by expansion (105).
5.2 The marginal functionals of correlations

Within the framework of the description of states in terms of marginal correlation operators (91), the marginal correlation functionals $G_s(t, Y \mid F_1(t))$, $s \geq 2$, are represented by the expansions similar to (106), namely

$$G_s(t, Y \mid F_1(t)) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathfrak{V}_{1+n}(t, \theta(\{Y\}), s+1, \ldots, s+n) \prod_{i=1}^{s+n} F_1(t, i),$$

where it is used the notion of declusterization mapping (43). Hence in contrast to expansion (106) the $n$ term of expansions (115) of marginal correlation functionals $G_s(t, Y \mid F_1(t))$ is governed by $(1 + n)\text{th}$-order evolution operator (105) of the $(s + n)\text{th}$-order cumulants of the scattering operators, for example, as compared with (107) the lower order evolution operators $\mathfrak{V}_{1+n}(t, \theta(\{Y\}), s+1, \ldots, s+n)$, $n \geq 0$, have the form

$$\mathfrak{V}_1(t, \theta(\{Y\})) = \widehat{\mathfrak{A}}_s(-t, \theta(\{Y\})),$$

$$\mathfrak{V}_2(t, \theta(\{Y\}), s+1) =$$

$$= \widehat{\mathfrak{A}}_{s+1}(-t, \theta(\{Y\}), s+1) - \widehat{\mathfrak{A}}_s(-t, \theta(\{Y\})) \sum_{i=1}^{s} \widehat{\mathfrak{A}}_2(-t, i, s+1),$$

and in case of $s = 2$, it holds

$$\mathfrak{V}_1(t, \theta(\{1, 2\})) = \widehat{G}_2(t, 1, 2) - I,$$

where $\widehat{G}_2(t, 1, 2)$ is scattering operator (47).

We indicate that expansions (106) of marginal functionals of the state are nonequilibrium analog of the Mayer-Ursell expansions over powers of the density of equilibrium marginal density operators.

Within the framework of the description of states by marginal functionals of the state (106) the average value, for example, of the additive-type marginal observable $B^{(1)} = (0, a_1(1), 0, \ldots)$ is given by the functional

$$\langle B^{(1)} \rangle(t) = \text{Tr}_1 a_1(1) F_1(t, 1),$$

i.e. it is defined by a solution of the generalized quantum kinetic equation (103), or in general case of the $s$-ary marginal observable $B^{(s)} = (0, \ldots, 0, a_s(1, \ldots, s), 0, \ldots)$ by the functional

$$\langle B^{(s)} \rangle(t) = \frac{1}{s!} \text{Tr}_{1,\ldots,s} a_s(1, \ldots, s) F_s(t, 1, \ldots, s \mid F_1(t)), \quad s \geq 2,$$

where $F_s(t, 1, \ldots, s \mid F_1(t))$ is the marginal functional of the state (106). For $B^{(s)} \in \mathcal{L}(\mathcal{F}_H)$ and $F_1(t) \in \mathcal{L}(\mathcal{H})$ these functionals exist.

The dispersion of an additive-type observable is defined by a solution of the generalized quantum kinetic equation (103) and marginal correlation functionals (115) as follows

$$\langle (B^{(1)} - \langle B^{(1)} \rangle(t))^2 \rangle(t) =$$

$$= \text{Tr}_1 \left( a_1^2(1) - \langle B^{(1)} \rangle^2(t) \right) F_1(t, 1) + \text{Tr}_{1,2} a_1(1) a_1(2) G_2(t, 1, 2 \mid F_1(t)), $$
where the functional \( \langle B^{(1)} \rangle(t) \) is determined by expression (116).

We note that the dispersion of observables is minimal for states characterized by marginal correlation functionals (115) equals to zero, i.e. from macroscopic point of view the evolution of many-particle states with the minimal dispersion is the Markovian kinetic evolution. In fact functionals (115) or (106) characterize the correlations of states of quantum many-particle systems. We illustrate close links of functionals (115) and (106) in the following way

\[
F_2(t, 1, 2 | F_1(t)) = F_1(t, 1)F_1(t, 2) + G_2(t, 1, 2 | F_1(t)).
\]

Basically this equality gives the classification of all possible currently in use scaling limits. In the scaling limits it is assumed that chaos property of initial state preserves in time, i.e. the scaling limit means such limit of dimensionless parameters of a system in which the marginal correlation functional \( G_2(t, 1, 2 | F_1(t)) \) vanishes. According to definition (115), it is possible, if every finite particle cluster moves without collisions.

Finally we point out the relationship of generalized quantum kinetic equation (103) and the specific quantum kinetic equations. The last can be derived from the generalized quantum kinetic equation in the appropriate scaling limits [22] or as a result of certain approximations. Let us consider first two terms of expansion (106). If an interaction potential in (3) is a bounded operator and \( f_{s+1} \in L^1(H_{s+1}) \), then for the second-order cumulant of scattering operators (47) an analog of the Duhamel equation holds

\[
\hat{A}_2(t, \{Y\}, s + 1) f_{s+1} = \int_0^t d\tau G_s(-\tau, Y)G_1(-\tau, s + 1) \sum_{i_1=1}^s (-\mathcal{N}_{\text{int}}(i_1, s + 1)) \times
\]

\[
\times \hat{G}_{s+1}(\tau - t, Y, s + 1) \prod_{i_2=1}^{s+1} G_1(\tau, i_2) f_{s+1},
\]

and, consequently, for the second-order evolution operator \( \mathcal{W}_2(t, \{Y\}, s + 1) \) we have

\[
\mathcal{W}_2(t, \{Y\}, s + 1) f_{s+1} = (\hat{A}_2(t, \{Y\}, s + 1) - \hat{A}_1(t, \{Y\}) \sum_{i_1=1}^s \hat{A}_2(t, i_1, s + 1)) f_{s+1} =
\]

\[
= \int_0^t d\tau G_s(-\tau, Y)G_1(-\tau, s + 1) \left( \sum_{i_1=1}^s (-\mathcal{N}_{\text{int}}(i_1, s + 1)) \hat{G}_{s+1}(\tau - t, Y, s + 1) -
\]

\[
- \hat{G}_s(\tau - t, Y) \sum_{i_1=1}^s (-\mathcal{N}_{\text{int}}(i_1, s + 1)) \hat{G}_2(\tau - t, i_1, s + 1)) \prod_{i_2=1}^{s+1} G_1(\tau, i_2) f_{s+1}.
\]

Observing that in the kinetic (macroscopic) scale of the variation of variables [2] the groups of operators (7) of finitely many particles depend on microscopic time variable \( \varepsilon^{-1}t \), where \( \varepsilon \geq 0 \) is a scale parameter, the dimensionless marginal functionals of the state are represented in the form: \( F_s(\varepsilon^{-1}t, Y | F_1(t)) \). Then in the limit \( \varepsilon \to 0 \) the first two terms of the dimensionless marginal functional expansions (106)

\[
\hat{G}_s(\varepsilon^{-1}t, Y) \prod_{i=1}^s F_1(t, i) +
\]
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\[ + \int_0^{\varepsilon^{-1}t} d\tau \mathcal{G}_s(-\tau, Y) \text{Tr}_{s+1} \left( \sum_{i_1=1}^{s} (-\mathcal{N}_{\text{int}}(i_1, s + 1)) \mathcal{G}_s(\varepsilon^{-1}t, Y, s + 1) - \right. \]

\[ - \mathcal{G}_s(\varepsilon^{-1}t, Y) \sum_{i_1=1}^{s} (-\mathcal{N}_{\text{int}}(i_1, s + 1)) \mathcal{G}_2(\varepsilon^{-1}t, i_1, s + 1) \prod_{i_2=1}^{s+1} \mathcal{G}_1(\tau, i_2) F_1(t, i_2) \]

coincide with corresponding terms constructed by the perturbation method with the use of the weakening of correlation condition by Bogolyubov \[1\]. Thus, in the kinetic scale the collision integral of the generalized kinetic equation \[103\] takes the form of Bogolyubov’s collision integral \[1\] and we observe that in a space homogeneous case the collision integral of the first approximation has a more general form than the quantum Boltzmann collision integral.

### 5.3 On quantum kinetic equations in case of correlated initial data

We have proved that in case of initial data which is completely defined by a one-particle density operator, all possible states of infinite-particle systems of bosons or fermions at an arbitrary moment of time can be described within the framework of a one-particle density operator and explicitly defined functionals of such operator without any approximations. One of the advantages of this approach is the possibility to construct the kinetic equations in scaling limits in the presence of correlations of particle states at initial time, for instance, correlations characterizing the condensed states of interacting particles obeying Fermi-Dirac or Bose-Einstein statistics \[14\].

Let us consider initial data

\[ F(t)|_{t=0} = \left( F_1^0(1), F_2^0(1, 2), F_3^0(1, 2), \ldots, h_n(1, \ldots, n) \prod_{i=1}^{n} F_1^0(i), \ldots \right), \]

where the operators \( h_n \in \mathcal{L}^1(\mathcal{H}_n) \), \( n \geq 2 \), are specified initial correlations. Such initial data is typical for the condensed states of quantum gases, for example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition with the correlations which characterize the condensed state \[3\].

In the case under consideration the kinetic cluster expansions of cumulants of groups, i.e. recurrence relations \[108\], take the form

\[ \mathcal{A}_{1+n}(-t, \{Y\}, s + 1, \ldots, s + n) h_{1+n} (\{Y\}, s + 1, \ldots, s + n) \prod_{i=1}^{s+n} \mathcal{A}_1(t, i) = \]

\[ = \sum_{n_1=0}^{n} \frac{n!}{(n-n_1)!} \mathcal{G}_{1+n-n_1}(t, \{Y\}, s + 1, \ldots, s + n - n_1) \sum_{D_{s+n} \subseteq \{i_1 < i_2 < \ldots < i_{D_{s+n}}\} = 1} \prod_{k=1}^{[D_{s+n}]} \frac{1}{|X_k|!} \mathcal{A}_{1+|X_k|}(t, i_k, X_k) \prod_{k=1}^{[D_{s+n}]} h_{1+|X_k|}(i_k, X_k) \mathcal{A}_1(t, i_k) \prod_{j \in Z} \mathcal{A}_1(t, j), \]

where the notations of formula \[108\] are used. In terms of new evolution operators (the \((1+n)th\)-order scattering cumulants)

\[ \tilde{\mathcal{A}}_{1+n}(t, \{Y\}, X\setminus Y) = \mathcal{A}_{1+n}(-t, \{Y\}, X\setminus Y) h_{1+n} (\{Y\}, X\setminus Y) \prod_{i=1}^{s+n} \mathcal{A}_1(t, i), \]
the solutions $\mathcal{G}_{1+n}(t, \{Y\}, X \setminus Y)$, $n \geq 0$, of recurrence relations (117) are represented by the expansions

$$
\mathcal{G}_{1+n}(t, \{Y\}, X \setminus Y) = n! \sum_{k=0}^{n} (-1)^k \sum_{n_1=1}^{n} \ldots \sum_{n_k=1}^{n-n_1-\ldots-n_{k-1}} \frac{1}{(n-n_1-\ldots-n_k)!} \times (118)
$$

$$
\times \tilde{A}_{1+n-n_1-\ldots-n_k}(t, \{Y\}, s+1, \ldots, s+n-n_1-\ldots-n_k) \times 
\prod_{j=1}^{k} \frac{1}{|D_j|} \sum_{D_j: Z_j = \bigcup_{j} x_{i_j}, |D_j| \leq s+n-n_1-\ldots-n_j} \frac{1}{\prod_{i_1 \neq \ldots \neq i(D_j)=1} X_{i_j}!} \tilde{A}_{1+|X_{i_j}|}(t, i_{i_j}, X_{i_j}),
$$

where $\sum_{D_j: Z_j = \bigcup_{j} x_{i_j}}$ is the sum over all possible dissections of the linearly ordered set $Z_j \equiv (s+n-n_1-\ldots-n_j+1, \ldots, s+n-n_1-\ldots-n_{j-1})$ on no more than $s+n-n_1-\ldots-n_j$ linearly ordered subsets. For example,

$$
\mathcal{G}_1(t, \{Y\}) = \tilde{A}_1(t, \{Y\}) = A_1(-t, \{Y\})h_1(\{Y\}) \prod_{i=1}^{s} A_1(t, i).
$$

Therefore the one-particle density operator $F_1(t)$ is governed by the following generalized quantum kinetic equation

$$
\frac{d}{dt} F_1(t, 1) = -\mathcal{N}(1) F_1(t, 1) + (119)
$$

$$
+ \text{Tr}_2(-\mathcal{N}_{\text{int}}(1, 2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3, \ldots, n+2} \mathcal{G}_{1+n}(t, \{1, 2\}, 3, \ldots, n+2) \prod_{i=1}^{n+2} F_1(t, i),
$$

where the operator $\mathcal{N}_{\text{int}}(1, 2)$ is defined by formula (75), and the $(1+n)th$-order generated evolution operator $\mathcal{G}_{1+n}(t)$, $n \geq 0$, is defined by expansion (118). Correspondingly the marginal functionals of the state are represented by the following expansions

$$
F_s(t, Y | F_1(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathcal{G}_{1+n}(t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_1(t, i).
$$

Thus, the coefficients of generalized quantum kinetic equation (119) and generated evolution operators (118) of marginal functionals (120) are determined by the operators of initial correlations.

6 On scaling limits of hierarchy solutions

The current point of view on the problem of the derivation of quantum kinetic equations from underlying many-particle dynamics consists in the following. Since the evolution of states of infinitely many quantum particles is generally described by a sequence of marginal density operators governed by the quantum BBGKY hierarchy, then the evolution of states can be
effectively described by a one-particle density operator governed by the quantum kinetic equation in a suitable scaling limit \[22\,\text{[23]}\]. In this section the mean field (self-consistent field) scaling asymptotic behavior of stated above solutions is established. The constructed asymptotics are governed by the quantum Vlasov hierarchy for limit marginal density operators and limit solution of the generalized quantum kinetic equation \[34\], the nonlinear quantum Vlasov hierarchy for limit marginal correlation operators and the dual quantum Vlasov hierarchy for limit marginal observables, respectively \[43\]. In case of initial data satisfying a chaos property \[1\], which means the absence of correlations at initial time, the constructed asymptotics are governed by the Vlasov quantum kinetic equation. Moreover, within the framework of the description of the evolution by the dual quantum BBGKY hierarchy or by the generalized quantum kinetic equation it is possible to construct the kinetic equations in scaling limits in case of presence of correlations at initial time, for instance, correlations which characterize the condensate states \[38\].

6.1 A mean field limit of a solution of the dual quantum BBGKY hierarchy

We consider the \(n\)-particle system with the Hamiltonian

\[
H_n = \sum_{i=1}^{n} K(i) + \epsilon \sum_{i_1 < i_2 = 1}^{n} \Phi(i_1, i_2),
\]

(121)

where \(\epsilon > 0\) is a scaling parameter. At first we construct the mean field scaling limit of a solution of initial-value problem \([57]-[58]\) of the dual quantum BBGKY hierarchy.

Let for initial data \(B^0_s \in \mathcal{L}(\mathcal{H}_s)\) there exists the scaling limit \(b^0_s \in \mathcal{L}(\mathcal{H}_s)\), i.e.

\[
w^* - \lim_{\epsilon \to 0} (\epsilon^s B^0_s - b^0_s) = 0,
\]

(122)

then for arbitrary finite time interval there exists the mean field limit of solution \([63]\) of the Cauchy problem \([57]-[58]\) of the dual quantum BBGKY hierarchy in the sense of the \(~\)weak convergence of the space \(\mathcal{L}(\mathcal{H}_s)\)

\[
w^* - \lim_{\epsilon \to 0} (\epsilon^s B_s(t) - b_s(t)) = 0, \quad s \geq 1,
\]

(123)

which is defined by the expansion

\[
b_s(t, Y) = \sum_{n=0}^{s-1} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \mathcal{G}^0_s(t - t_1) \sum_{i_1 \neq j_1 = 1}^{s} \mathcal{N}_{\text{int}}(i_1, j_1) \mathcal{G}^0_{s-1}(t_1 - t_2) \ldots \mathcal{G}^0_{s-n+1}(t_{n-1} - t_n) \sum_{i_n \neq j_n \neq 1, \ldots, j_{n-1}}^{s} \mathcal{N}_{\text{int}}(i_n, j_n) \times \mathcal{G}^0_{s-n}(t_n) b^0_{s-n}(Y \setminus (j_1, \ldots, j_n)),
\]

(124)
where the following notation of the group of operators (7) of noninteracting particles is used

\[ G_{s-n+1}^0(t_{n-1} - t_n) = \prod_{j \in Y \setminus \{j_1, \ldots, j_{n-1}\}} G_1(t_{n-1} - t_n, j). \]

Before to prove this statement we give some comments. If \( b(0) \in \mathfrak{L}(\mathbb{F}_H) \), the sequence \( b(t) = (b_0, b_1(t), \ldots, b_s(t), \ldots) \) of limit marginal observables (124) is a generalized global solution of the initial-value problem of the dual quantum Vlasov hierarchy

\[
\frac{d}{dt} b_s(t, Y) = \sum_{i=1}^{s} \mathcal{N}(i) b_s(t, Y) + \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{N}_{\text{int}}(j_1, j_2) b_{s-1}(t, Y \setminus \{j_1\}), \quad (125)
\]

\[
b_s(t) \mid_{t=0} = b_s^0, \quad s \geq 1. \quad (126)
\]

This fact is proved similar to the case of an iteration series of the dual quantum BBGKY hierarchy [40]. It should be noted that equations set (125) has the structure of recurrence evolution equations. We make a few examples of the dual quantum Vlasov hierarchy (125) in terms of operator kernels of the limit marginal observables

\[
i \frac{\partial}{\partial t} b_1(t, q_1; q_1') = -\frac{1}{2} (-\Delta_{q_1} + \Delta_{q_1'}) b_1(t, q_1; q_1'),
\]

\[
i \frac{\partial}{\partial t} b_2(t, q_1, q_2; q_1', q_2') = -\frac{1}{2} \sum_{i=1}^{2} (-\Delta_{q_i} + \Delta_{q_i'}) b_2(t, q_1, q_2; q_1', q_2') + \left( \Phi(q_1' - q_2') - \Phi(q_1 - q_2) \right) \left( b_1(t, q_1; q_1') + b_1(t, q_2; q_2') \right).
\]

Let us consider a particular case of observables, namely the mean field limit of the additive-type marginal observables. In this case solution (63) of the dual quantum BBGKY hierarchy (67) has the form (66). If for the additive-type observables \( B^{(1)}(0) = (0, a_1(1), 0, \ldots) \) condition (122) is satisfied, i.e. it holds

\[
w^* - \lim_{\epsilon \to 0} (\epsilon^{-1} a_1(1) - b_1^0(1)) = 0,
\]

then according to statement (123), we have

\[
w^* - \lim_{\epsilon \to 0} (\epsilon^{-s} B_s^{(1)}(t) - b_s^{(1)}(t)) = 0,
\]

where the limit operator \( b_s^{(1)}(t) \) is defined by the expression

\[
b_s^{(1)}(t, Y) = \int_0^t dt_1 \ldots \int_0^{t_{s-2}} dt_{s-1} \mathcal{G}_s^0(t - t_1) \sum_{i_1 \neq j_1 = 1}^{s} \mathcal{N}_{\text{int}}(i_1, j_1) \mathcal{G}_{s-1}^0(t_1 - t_2) \ldots \mathcal{G}_2^0(t_{s-2} - t_{s-1}) \sum_{i_{s-1} \neq j_{s-1} = 1}^{s} \mathcal{N}_{\text{int}}(i_{s-1}, j_{s-1}) \times \mathcal{G}_1^0(t_{s-1}) b_1^0(Y \setminus \{j_1, \ldots, j_{s-1}\}), \quad s \geq 1,
\]
as a special case of expansion (124). We give examples of expressions (127)
\[ b^{(1)}_1(t, 1) = G_1(t, 1) b^0_1(1), \]
\[ b^{(1)}_2(t, 1, 2) = \int_0^t \prod_{i=1}^2 G_1(t - \tau, i) N_{\text{int}}(1, 2) \sum_{j=1}^2 G_1(\tau, j) b^0_1(j). \]

To establish the relationship of constructed mean field asymptotic behavior of marginal observables with asymptotic behavior of marginal states we consider initial data satisfying the factorization property or a chaos property \[ [1], \] which means the absence of correlations at initial time. For a system of identical particles, obeying the Maxwell-Boltzmann statistics, we have
\[ F(t)|_{t=0} = F^{(c)} \equiv (F^0_1(1), \ldots, \prod_{i=1}^s F^0_1(i), \ldots). \]
This assumption about initial data is intrinsic for the kinetic description of a gas, because in this case all possible states are characterized only by a one-particle marginal density operator. Let
\[ \lim_{\epsilon \to 0} \| \epsilon F^0_1 - f^0_1 \|_{L^1(\mathcal{H})} = 0, \]
then the limit of initial state satisfies a chaos property too
\[ f^{(c)} \equiv (f^0_1(1), \ldots, \prod_{i=1}^s f^0_1(i), \ldots). \]
For \( b(t) \in L_\gamma(F_{\mathcal{H}}) \) and \( f^0_1 \in L_1(\mathcal{H}) \), under the condition \( \| f^0_1 \|_{L^1(\mathcal{H})} < \gamma \), the mean field limit of mean value functional (71) exists and it is determined by the expansion
\[ (b(t), f^{(c)}) = \sum_{s=0}^\infty \frac{1}{s!} \text{Tr}_{1,\ldots,s} b_s(t, 1, \ldots, s) \prod_{i=1}^s f^0_1(i). \]
In consequence of the validity of the following equality for the limit additive-type marginal observables (127)
\[ (b^{(1)}_1(t), f^{(c)}) = \sum_{s=0}^\infty \frac{1}{s!} \text{Tr}_{1,\ldots,s} b^{(1)}_s(t, 1, \ldots, s) \prod_{i=1}^s f^0_1(i) = \]
\[ = \text{Tr}_1 b^0_1(1) f_1(t, 1), \]
where the operator \( b^{(1)}_s(t) \) is given by expansion (127) and the one-particle limiting density operator \( f_1(t, 1) \) is determined by the series
\[ f_1(t, 1) = \sum_{n=0}^\infty \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{i_1=1}^1 G_1(-t + t_1, i_1) \times \]
\[ \times (-N_{\text{int}}(1, 2)) \prod_{j_1=1}^2 G_1(-t_1 + t_2, j_1) \ldots \prod_{i_n=1}^n G_1(-t_n + t_{n+1}, t_n) \times \]
\[ \times \prod_{k_n=1}^s (-N_{\text{int}}(k_n, n + 1)) \prod_{j_n=1}^{n+1} G_1(-t_n + j_n) \prod_{i_1=1}^{n+1} f^0_1(i), \]
we establish that operator (131) is a solution of the initial-value problem of the Vlasov quantum kinetic equation
\[ \frac{d}{dt} f_1(t, 1) = -N(1)f_1(t, 1) + \text{Tr}_2(-N_{\text{int}}(1, 2))f_1(t, 1)f_1(t, 2), \] (132)
\[ f_1(t)|_{t=0} = f_1^0. \] (133)

Correspondingly, a chaos property in the Heisenberg picture of the evolution of quantum many-particle systems is fulfilled. This fact follows from the equality for the limit \( k \)-ary marginal observables, i.e. \( b^{(k)}(0) = (0, \ldots, b_k^0(1, \ldots, k), 0, \ldots) \)
\[ (b^{(k)}(t), f^{(c)}) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1, \ldots, s} b^{(k)}_s(t, 1, \ldots, s) \prod_{i=1}^{s} f_1^0(i) = \] (134)
\[ = \frac{1}{k!} \text{Tr}_{1, \ldots, k} b_k^0(1, \ldots, k) \prod_{i=1}^{k} f_1(t, i), \quad k \geq 2, \]
where the limit one-particle marginal density operator \( f_1(t, i) \) is defined by expansion (131) and therefore it is governed by the Cauchy problem (132)-(133).

Thus, in the mean field scaling limit an equivalent approach to the description of the kinetic evolution of quantum many-particle systems in terms of the Cauchy problem (132)-(133) of the Vlasov kinetic equation is given by the Cauchy problem (125)-(126) of the dual quantum Vlasov hierarchy for the additive-type marginal observables. In case of the \( k \)-ary marginal observables a solution of the dual quantum Vlasov hierarchy (125) is equivalent to the preservation of a chaos property for \( k \)-particle marginal density operators in the sense of equality (134).

6.2 A mean field limit of a solution of the quantum BBGKY hierarchy

Within the framework of the description of the evolution in terms of states the scaling mean field limit of a nonperturbative solution of the Cauchy problem (73)-(74) of the quantum BBGKY hierarchy in case of arbitrary initial data is stated as follows.

Let for initial data \( F_s^0, f_s^0 \in \mathcal{L}^1(\mathcal{H}_s) \) it holds
\[ \lim_{\epsilon \to 0} \| \epsilon^s F_s^0(1, \ldots, s) - f_s^0(1, \ldots, s) \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0, \] (135)
then for any finite time interval for solution (77) there exists the mean field limit
\[ \lim_{\epsilon \to 0} \| \epsilon^s F_s(t, 1, \ldots, s) - f_s(t, 1, \ldots, s) \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0, \] (136)
where the limiting operator \( f_s(t) \in \mathcal{L}^1(\mathcal{H}_s) \) is determined by the series
\[ f_s(t, 1, \ldots, s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \text{Tr}_{s+1, \ldots, s+n} \prod_{j_1=1}^{s} G_1(-t + t_1, j_1) \times \] (137)

which is norm convergent on the space \( L^1(F_\mathcal{H}) \) on finite time interval. Limiting marginal operators (137) are governed by the limiting BBGKY hierarchy known as the quantum Vlasov hierarchy.
In other words, under the condition on initial data $F_s^0 \in \mathcal{L}^1(\mathcal{H}_s)$

$$\lim_{\epsilon \to 0} \| \epsilon^s F_s^0(1, \ldots, s) - \prod_{j=1}^s f_1^0(j) \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0,$$

on finite time interval for solution (77) of the quantum BBGKY hierarchy it holds:

$$\lim_{\epsilon \to 0} \| \epsilon^s F_s(t, 1, \ldots, s) - \prod_{j=1}^s f_1(t, j) \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0,$$

where the one-particle limiting density operator $f_1(t, 1)$ is determined by series (131) which is a solution of the Cauchy problem (132)-(133) of the Vlasov quantum kinetic equation.

In case of pure states, i.e. if the limit one-particle density operator $f_1(t) = |\psi_t \rangle \langle \psi_t|$ is a one-dimensional projector onto a unit vector $|\psi_t \rangle$, we have

$$\lim_{\epsilon \to 0} \| \epsilon^s F_s(t) - |\psi_t \rangle \langle \psi_t|^{\otimes s} \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0,$$

where $|\psi_t \rangle$ is a solution of the Hartree equation, which in terms of the kernel $f_1(t, q, q') = \psi(t, q)\psi(t, q')$ of the marginal operator $f_1(t)$ has the form

$$i \frac{\partial}{\partial t} \psi(t, q) = -\frac{1}{2} \Delta q \psi(t, q) + \int dq' \Phi(q - q')|\psi(t, q')|^2 \psi(t, q).$$

If the kernel of the interaction potential is the Dirac measure $\Phi(q) = \delta(q)$, then we derive the cubic nonlinear Schrödinger equation

$$i \frac{\partial}{\partial t} \psi(t, q) = -\frac{1}{2} \Delta q \psi(t, q) + |\psi(t, q)|^2 \psi(t, q).$$

We observe that in the general case of many-body potentials with the scaled Hamiltonian

$$H_n = \sum_{i=1}^n K(i) + \sum_{k=2}^n \epsilon^{k-1} \sum_{i_1 < \ldots < i_k=1}^n \Phi(k)(i_1, \ldots, i_k),$$

the Hartree equation takes on form

$$i \frac{\partial}{\partial t} \psi(t, q_1) = -\frac{1}{2} \Delta_{q_1} \psi(t, q_1) +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \int dq_2 \ldots dq_{n+1} \Phi^{(n+1)}(q_1, \ldots, q_{n+1}) \prod_{i=2}^{n+1} |\psi(t, q_i)|^2 \psi(t, q_1).$$

We note also that in case of many-particle systems obeying quantum statistics statement (136) is true for operators from the corresponding spaces $\mathcal{L}^1(\mathcal{H}_s^\pm)$. For initial data satisfying a chaos property

$$f^{(c)} \equiv (f_1^0(1), \ldots, S_s^\pm \prod_{i=1}^s f_1^0(i), \ldots),$$

where the operator $S_s^\pm$ is defined by (2), the Vlasov quantum kinetic equation has the form

$$\frac{d}{dt} f_1(t, 1) = -\mathcal{N}(1)f_1(t, 1) + \text{Tr}_2(-\mathcal{N}_{\text{int}}(1, 2))S_s^\pm f_1(t, 1)f_1(t, 2),$$

and consequently for pure states of fermions we derive the Hartree-Fock equation.
6.3 On a mean field asymptotics of dynamics of correlations

We give some comments on the existence of mean field scaling limits of the constructed solution. The mean field limit $g_s(t,1,\ldots,s)$, $s \geq 1$, of marginal correlation operators $g_0$ exists

$$\lim_{\epsilon \to 0} \| \epsilon^s G_s(t) - g_s(t) \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0, \quad s \geq 1,$$

provided that

$$\lim_{\epsilon \to 0} \| \epsilon^s G^0_s - g^0_s \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0, \quad s \geq 1,$$

(143)

and it is governed by the nonlinear Vlasov quantum hierarchy

$$\frac{\partial}{\partial t} g_s(t,Y) = \sum_{i \in Y} (-N(i))g_s(t,Y) + \text{Tr}_{s+1} \sum_{i \in Y} (-N_{\text{int}}(i,s+1)) \times (g_{s+1}(t,Y,s+1) + \sum_{\mathcal{P} : (Y,s+1) = X_1 \cup X_2, i \in X_1; s+1 \in X_2} g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2)), $$

(144)

where we use the same notations as for hierarchy (92).

If initial data satisfies chaos property, then we establish

$$\lim_{\epsilon \to 0} \| \epsilon^s G_s(t) \|_{\mathcal{L}^1(\mathcal{H}_s)} = 0, \quad s \geq 2,$$

(145)

since solution expansions $g_0$ for marginal correlation operators are defined by the $(s + n)th$-order cumulants as contrasted to solution expansions $g_1$ for marginal density operators which defined by the $(1 + n)th$-order cumulants and in the same way as statement (140) the equality holds

$$\lim_{\epsilon \to 0} \| \frac{1}{\epsilon^n} \mathcal{A}_{s+n}(-t,1,\ldots,s+n)f_{s+n} \|_{\mathcal{L}^1(\mathcal{H}_{s+n})} = 0.$$

In the case of $s = 1$ provided that (143) we have

$$\lim_{\epsilon \to 0} \| \epsilon G_1(t) - g_1(t) \|_{\mathcal{L}^1(\mathcal{H})} = 0,$$

where for finite time interval the limiting one-particle marginal correlation operator $g_1(t,1)$ is given by the norm convergent on the space $\mathcal{L}^1(\mathcal{H})$ series

$$g_1(t,1) =$$

$$= \sum_{n=0}^{\infty} \left( \int_0^t dt_1 \ldots \int_0^{t_n-1} dt_n \text{Tr}_{2\ldots,n+1} \mathcal{G}_1(-t + t_1,1(-N_{\text{int}}(1,2)) \prod_{j_1=1}^{2} \mathcal{G}_1(-t_1 + t_2, j_1) \ldots \right.$$

$$\left. \ldots \prod_{i_n=1}^{n} \mathcal{G}_1(-t_n + t_{n_1}, i_n) \sum_{k_n=1}^{n} (-N_{\text{int}}(k_n, n + 1)) \prod_{j_n=1}^{n+1} \mathcal{G}_1(-t_n, j_n) \prod_{i=1}^{n+1} g^0_i(i) \right),$$

(146)

which obviously coincides with iteration series (131) of the Vlasov quantum kinetic equation. In view of the validity of limit (145) from the Vlasov nonlinear quantum hierarchy (144) we also conclude that limit one-particle marginal correlation operator (146) is governed by the Cauchy problem of the Vlasov quantum kinetic equation (132)-(133).

Therefore the Vlasov nonlinear quantum hierarchy (144) describes the evolution of initial correlations.
6.4 On scaling limits of the generalized quantum kinetic equation

In this section we consider the relationship of the specific quantum kinetic equations with the generalized quantum kinetic equation. First we construct the mean field (self-consistent field) asymptotics of solution (110) of initial-value problem of the generalized quantum kinetic equation for a system with the Hamiltonian (121), i.e.

\[
\frac{d}{dt} F_1(t,1) = -\mathcal{N}(1) F_1(t,1) + \varepsilon \text{Tr}_2(-\mathcal{N}_\text{int}(1,2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3,\ldots,n+2} \mathfrak{V}_{1+n}(t,\{1,2\},3,\ldots,n+2) \prod_{i=1}^{n+2} F_1(t,i),
\]

\[F_1(t,1)|_{t=0} = F_1^0(1),\]

and marginal correlation functionals (115).

If there exists the mean field limit \( f_1^0 \in L^1(\mathcal{H}) \) of initial data (148) in the following sense

\[
\lim_{\varepsilon \to 0} \| \epsilon F_1^0 - f_1^0 \|_{L^1(\mathcal{H})} = 0,
\]

then for arbitrary finite time interval there exists the limit \( f_1(t) \) of solution (110) of the generalized quantum kinetic equation (147)

\[
\lim_{\varepsilon \to 0} \| \epsilon F_1(t) - f_1(t) \|_{L^1(\mathcal{H}_1)} = 0,
\]

where the one-particle limiting density operator \( f_1(t,1) \) is determined by series (131) which is a strong solution of the Cauchy problem of the Vlasov quantum kinetic equation

\[
\frac{d}{dt} f_1(t,1) = -\mathcal{N}(1) f_1(t,1) + \text{Tr}_2(-\mathcal{N}_\text{int})(1,2) f_1(t,1) f_1(t,2),
\]

\[f_1(t)|_{t=0} = f_1^0.\]

Taking into account equality (149), for marginal functionals of the state (106) we establish

\[
\lim_{\varepsilon \to 0} \| \epsilon^s F_s(t, Y \mid F_1(t)) - \prod_{j=1}^{s} f_1(t,j) \|_{L^1(\mathcal{H}_s)} = 0,
\]

where \( f_1(t) \) is defined by series (131). This equality means that in the mean field scaling limit initial chaos property preserves in time.

The validity of this limit statement is the consequence of formulas (139) and (140) on an asymptotic perturbation of cumulants of groups and definition (105) of the generated evolution operators \( \mathfrak{V}_{1+n}(t), n \geq 0, \) of marginal functionals of the state (106). Indeed, if \( f_s \in L^1(\mathcal{H}_s), \) then the equality is correct

\[
\lim_{\varepsilon \to 0} \| (\mathcal{G}_s(t, Y) - I) f_s \|_{L^1(\mathcal{H}_s)} = 0.
\]

Further for the first-order generated evolution operator \( \mathfrak{V}_1(t, \{Y\}) \) the following equality holds

\[
\lim_{\varepsilon \to 0} \| (\mathfrak{V}_1(t, \{Y\}) - I) f_s \|_{L^1(\mathcal{H}_s)} = 0,
\]
for any finite time interval, and in the general case \( n \geq 1 \), we have

\[
\lim_{\epsilon \to 0} \left\| \frac{1}{\epsilon^n} \mathcal{G}_1 + n(t, \{Y\}, X \setminus Y) f_{s + n} \right\|_{\mathcal{L}^1(\mathcal{H}_{s + n})} = 0.
\]

According to this formula on an asymptotic perturbation of evolution operators \([105]\), we establish the mean field asymptotics of marginal correlation functionals \((115)\)

\[
\lim_{\epsilon \to 0} \left\| \epsilon^s G_s(t, Y | F_1(t)) \right\|_{\mathcal{L}^1(\mathcal{H}_s)} = 0, \quad s \geq 2.
\]

In case of initial states involving correlations for evolution operators \((118)\) in the mean field limit the equality holds

\[
\lim_{\epsilon \to 0} \left\| \mathcal{G}_1 + n(t, \{Y\}, s + 1, \ldots, s + n) f_{s + n} \right\|_{\mathcal{L}^1(\mathcal{H}_{s + n})} = 0, \quad n \geq 1,
\]

and in case of the first-order generated evolution operator \((118)\) we have respectively

\[
\lim_{\epsilon \to 0} \left\| G_1(t, \{Y\}) - \prod_{i_1=1}^s G_1(-t, i_1) h_1(\{Y\}) \prod_{i_2=1}^s G_1(t, i_2) f_s \right\|_{\mathcal{L}^1(\mathcal{H}_s)} = 0.
\]

Thus, the mean field asymptotics of marginal functionals \((115)\) has the form

\[
\lim_{\epsilon \to 0} \left\| \epsilon^s F_s(t, Y | F_1(t)) - \prod_{i_1=1}^s G_1(-t, i_1) h_1(\{Y\}) \prod_{i_2=1}^s G_1(t, i_2) \prod_{j=1}^s f_1(t, j) \right\|_{\mathcal{L}^1(\mathcal{H}_s)} = 0,
\]

that means the propagation of initial correlations in time in the mean field limit, and the limit one-particle density operator satisfies the modified Vlasov quantum kinetic equation

\[
\frac{d}{dt} f_1(t, 1) = -\mathcal{N}(1) f_1(t, 1) +
\]

\[
+ \text{Tr}_2(-\mathcal{N}_{\text{int}})(1, 2) \prod_{i_1=1}^2 G_1(-t, i_1) h_1(\{1, 2\}) \prod_{i_2=1}^2 G_1(t, i_2) f_1(t, 1) f_1(t, 2).
\]

If the kernel of the interaction potential is the Dirac measure \( \Phi(q) = \delta(q) \), then the sufficient equation for the description of pure state evolution governed by kinetic equation \((150)\) is the Gross-Pitaevskii-type equation

\[
i \frac{\partial}{\partial t} \psi(t, q) = -\frac{1}{2} \Delta_q \psi(t, q) + \int dq' dq'' b(t, q, q'; q'', q'') \psi(t, q'') \psi^*(t, q) \psi(t, q'),
\]

where the coupling ratio \( b(t, q, q'; q'', q''') \) is the kernel of the scattering amplitude operator \( \prod_{i_1=1}^2 G_1(-t, i_1) b_1(\{1, 2\}) \prod_{i_2=1}^2 G_1(t, i_2) \). Observing that in the kinetic (macroscopic) scale of the variation of variables the groups of operators \( (7) \) of finitely many particles depend on microscopic time variable \( \epsilon^{-1} t \), where \( \epsilon \geq 0 \) is a scale parameter, the dimensionless marginal functionals of the state are represented in the form: \( F_1(\epsilon^{-1} t, Y | F_1(t)) \). Then in the limit \( \epsilon \to 0 \) we obtain the Markovian kinetic evolution with the coefficient \( b(\infty, q, q; q', q'') \).
7 Conclusion and outlook

The concept of cumulants (61) of the groups of operators forms the basis of the nonperturbative solution expansions for hierarchies of quantum evolution equations, namely the dual quantum BBGKY hierarchy for marginal observables (57), the quantum BBGKY hierarchy for marginal density operators (73), the von Neumann hierarchy (33) for correlation operators and the nonlinear quantum BBGKY hierarchy for marginal correlation operators (92), and as well as it underlies of the kinetic evolution (103). The nonperturbative hierarchy solutions are represented in the form of an expansion over the particle clusters which evolution is governed by the corresponding order cumulant (61) of the groups of operators (7) of the Heisenberg equations (4) or the groups of operators (15) of the von Neumann equations (11).

We emphasize that intensional Banach spaces for the description of states of infinite-particle systems, which are suitable for the description of the kinetic evolution or equilibrium states, are different from the exploit spaces [1], [4]. Thus, marginal density operators or correlation operators from the space \( L^1(\mathcal{F}_H) \) describe finitely many quantum particles. In order to describe the evolution of infinitely many particles we have to construct solutions for initial data from more general Banach spaces than the space of sequences of trace class operators. For example, it can be the space of sequences of bounded translation invariant operators which contains the marginal density operators of equilibrium states [4]. In that case every term of the solution expansion (77) of the quantum BBGKY hierarchy or the nonlinear quantum BBGKY hierarchy (101) and correspondingly the generalized quantum kinetic equation (110) as well as mean-value functional (54) in case of the dual quantum BBGKY hierarchy (63) contains the divergent traces, which can be renormalized due to the cumulant structure of the solution expansions.

The origin of the microscopic description of non-equilibrium correlations of Bose and Fermi many-particle systems was considered in section 3. For the correlation operators that give an alternative description of the quantum state evolution of Bose and Fermi many-particle systems, the von Neumann hierarchy of nonlinear evolution equations (53) was introduced. In particular, it was established that in case of absence of correlations in the system at initial time, the correlations generated by the dynamics of a system (38) are completely determined by cumulants (39) of the groups of operators (15) of the von Neumann equations.

The links of constructed solution of the von Neumann hierarchy both with the solution of the quantum BBGKY hierarchy (72) and with the nonlinear quantum BBGKY hierarchy for marginal correlation operators (89) were discussed. The cumulant structure of solution (44) of the von Neumann hierarchy (33) induces the cumulant structure of solution expansions both the initial-value problem of the nonlinear quantum BBGKY hierarchy for marginal correlation operators (101) and the quantum BBGKY hierarchy for marginal density operators (77). Thus, the dynamics of infinite-particle systems is generated by the dynamics of correlations. We note that along with the definition within the framework of the non-equilibrium grand canonical ensemble the marginal density operators can be defined within the framework of dynamics of correlations (72) that allows to give the rigorous meaning to the states for more general classes of operators than trace-class operators.

We remark that the rigorous results on the evolution equations in functional derivatives for generating functionals of states and observables of classical many-particle systems, namely the BBGKY hierarchy and the dual BBGKY hierarchy in functional derivatives respectively, are presented in [45].
We developed also an approach to a description of the evolution of states by means of the quantum kinetic equations. It was demonstrated that in fact, if initial data is completely defined by a one-particle density operator, then all possible states of infinite-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator without any approximations. In other words, for mentioned states the evolution of states governed by the quantum BBGKY hierarchy \((73)\) can be completely described by the generalized quantum kinetic equation \((103)\). It should be emphasized that the kinetic evolution is an inherent property of infinite-particle systems. In spite of the fact that in terms of a one-particle marginal density operator from the space of trace-class operators can be described a system with the finite average number of particles, the generalized quantum kinetic equation has been derived on the basis of the formalism of nonequilibrium grand canonical ensemble since its framework is adopted to the description of infinite-particle systems in suitable Banach spaces as well.

We note that constructed marginal functionals of the state \((106)\) or \((115)\) characterize the correlations of states of quantum many-particle systems. Owing to that from macroscopic point of view the evolution of many-particle states with the minimal dispersion is the Markovian kinetic evolution, then from microscopic point of view such evolution is characterized by marginal correlation functionals \((115)\) which equal to zero.

An approach to the description of kinetic evolution of quantum many-particle systems in terms of the evolution of marginal observables is also developed. In the mean field limit the evolution of marginal observables is governed by the dual quantum Vlasov hierarchy \((125)\). One of the advantages of such approach as well as developed approach of the generalized quantum kinetic equation \((103)\) is the possibility to construct the kinetic equations in scaling limits in case of the presence of correlations at initial time \((151)\), for instance, correlations which characterize the condensate states of particles \([3]\).

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