T-Duality and Lie bialgebroid structures

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Abstract. We show that the geometric notion of duality behind T-duality, between two string theories on different manifolds $E, \hat{E}$ in the sense of [3][4], is precisely that of Lie bialgebroids due to Mackenzie and Xu [11].

1. Introduction

Lie bialgebroids are infinitesimal counterparts of Poisson groupoids, they were introduced by Kirill Mackenzie and Ping Xu in [11] and generalize both the double structure of Poisson manifolds (i.e. pairs $(TM, T^*M)$, where $M$ is a Poisson manifold) and that of Lie bialgebras (in the sense of V.G. Drinfel’d, see [13]). Lie bialgebroids encode the most natural definition of duality for Lie algebroids, these are pairs of dual vector bundles over a base manifold $M$ defining Lie algebroids $(A, A^*)$ satisfying, in addition, the compatibility condition that the coboundary operator $d_{A^*} : A \to \bigwedge^2 A$, associated to $A^*$, be a derivation of the graded Lie bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$. The main goal of this paper is to show that the geometric notion of duality behind the so-called topological T-duality between two string theories on different manifolds $E, \hat{E}$, in the sense of [3] [4], is precisely that of Lie bialgebroids. In this setup the manifolds $E$ and $\hat{E}$ are principal $T$-bundles, where $T = S^1 \times \cdots \times S^1$ is a torus, over the same base manifold $M$.

T-duality is an important tool in string theory. Among other features, it gives rise to an equivalence between string theories which in their low-energy limit might look different, but exhibit a one-to-one correspondence between fields, states and other defining ingredients of the corresponding theories. In the context of principal bundles, the smooth version of T-duality is a duality of torus bundles induced by closed invariant 3-forms as follows [3] [4] (see also [7]). Let us denote by $E$ and
\( \hat{E} \) two principal \( T \)-bundles, where \( T \) denotes a \( k \)-torus with Lie algebra \( t \), over the same base manifold \( M \). The source of the duality will be a non-degenerate pairing \( \Phi_o : t \times t^* \to \mathbb{R} \), where \( t^* \) denotes the dual of the Lie algebra \( t \), and it will be realized on the so-called correspondence space \( E \times_M \hat{E} \) (the fiber product) associated to the space-time manifolds \( E \) and \( \hat{E} \). Together with the principal \( T \)-bundles \( E \xrightarrow{\pi} M \) and \( \hat{E} \xrightarrow{\hat{\pi}} M \) we will consider closed \( T \)-invariant 3-forms \( H \) and \( \hat{H} \), with integral periods on \( E \) and \( \hat{E} \), respectively. The correspondence space defines principal fibrations over \( E \) and \( \hat{E} \), and the pull-back of the \( T \)-invariant 3-forms \( H \) and \( \hat{H} \) gives rise to an element \( \Phi \in \Omega^2(E \times_M \hat{E}) \) which plays a double role in the construction: geometrically it will define the double structure encoding the \( T \)-duality, topologically it intertwines between the twisted cohomologies (and twisted K-theories) of \( (E, H) \) and \( (\hat{E}, \hat{H}) \) (see, e.g. [3] [4] [5]).

**Definition 1.1.** We say that the pairs \( (E, H) \) and \( (\hat{E}, \hat{H}) \) are \( T \)-dual if

(i) \( p^*H - \hat{p}^*\hat{H} = d\Phi \) in the correspondence space \( E \times_M \hat{E} \), for a \( T \times \hat{T} \cong T^2 \)-invariant form \( \Phi \),

(ii) The form \( \Phi \) restricted to the fibers of \( E \times_M \hat{E} \) gives rise to the non-degenerate pairing \( \Phi_o : t \times t^* \to \mathbb{R} \).

\[
\begin{array}{c}
(E, H) \\
p \downarrow \quad \hat{p} \\
\hat{E} \\
\hat{\pi} \downarrow \quad \hat{\pi} \\
M.
\end{array}
\]

Given a principal \( T \)-bundle \( E \xrightarrow{\pi} M \) and a closed invariant 3-form \( H \) on \( E \), with integral periods, a pair \( (E, H) \) is called \( T \)-dualizable if there exists a closed 2-form \( \hat{F} \in \Omega^2(M, t^*) \) with integral periods such that, for all \( X \in t \),

\[
i_X H = \hat{F}(X),
\]

where \( \hat{F}(X) \) means the natural pairing (with values in \( \Omega^2(M) \)) between \( t \) and \( \Omega^2(M, t^*) \). It has been shown that, given a \( T \)-dualizable pair \( (E, H) \), then there exists a \( T \)-dualizable pair \( (\hat{E}, \hat{H}) \) that is \( T \)-dual to \( (E, H) \). As a matter of fact, in the case of circle bundles, the 2-forms defined by (9) corresponding to \( T \)-dual pairs have a natural geometric interpretation as curvatures of the principal \( S^1 \)-bundles \( E \) and \( \hat{E} \). Namely, there exist curvature forms \( F \) and \( \hat{F} \) on \( E \) and \( \hat{E} \), respectively, such that at the level of cohomology

\[
\int_{S^1} H = \hat{F} \quad \text{and} \quad \int_{\hat{S}^1} \hat{H} = F.
\]
In that case $\Phi = -A \wedge \hat{A}$, where $A$ and $\hat{A}$ denote connections on the $S^1$-bundles $E$ and $\hat{E}$ satisfying $F = dA$ and $\hat{F} = d\hat{A}$, respectively (see [3] [4] and references therein).

Recall that, given a principal $G$-bundle $P \xrightarrow{\pi} M$, the sub-bundle induced by the right-invariant vector fields on $P$, denoted by $X_R(P)$, is integrable and can be identified with the space of sections of the bundle $TP/G$. Since the projection map $TP \rightarrow P$ is $G$-equivariant we obtain a map $TP/G \rightarrow P/G \simeq M$ so that we can see $TP/G$ as a vector bundle over $M$. If we consider $a = d\pi : TP/G \rightarrow TM$ and the bracket as the restriction of the Lie bracket from $TP$, we induce a Lie algebroid structure on $TP/G$, called the Atiyah algebroid of $P$, and denoted by $A(P)$. Thus, we obtain the following short exact sequence of vector bundles over $M$:

$$0 \rightarrow \text{Ad}(P) \rightarrow A(P) \rightarrow TM \rightarrow 0,$$

splittings of which correspond with connections on $P$ (see [1][10] and references therein). Given any Lie algebroid $A$, a differential $d_A$ is defined on the graded algebra of sections of the exterior algebra of the dual vector bundle, $\Gamma(\Lambda^* A^*)$, called the Lie algebroid differential of $A$. This differential generalizes the Chevalley–Eilenberg cohomology operator on $\Lambda^* g^*$ (when $A$ is a Lie algebra $g$), the usual de Rham differential on forms (when $A = TM$) and the Lichnerowicz–Poisson differential $[\Pi, \cdot]$ on multi-vector fields on $M$ (when $A = T^* M$ is the cotangent bundle of a Poisson manifold $(M, \Pi)$), among others.

The total spaces of circle bundles in $T$-dual pairs $(E, H)$ and $(\hat{E}, \hat{H})$ are compact contact manifolds, and both $F$ and $\hat{F}$ define symplectic forms on the base space manifold $M$ (see [2]). We will use the associated Poisson structure to prove that the Atiyah algebroids $L = A(E)$ and $\hat{L} = A(\hat{E})$ over $M$ define a Lie bi-algebroid, i.e. they are dual (with anchor $a$ and bracket $[,]$, $a_*$ and bracket $[,]_*$, respectively) and $d_L$ is a derivation of the Schouten bracket of $\hat{L}$, so that for all $\alpha, \beta \in \Gamma(L)$:

$$d_L[\alpha, \beta]_* = [d_L \alpha, \beta]_* + [\alpha, d_L \beta]_*.$$  

(4)

The duality pairing giving the identification of $\hat{L}$ with $L^*$ is the defined by

$$\langle X, Y \rangle := \Phi(\hat{X}^v, \hat{Y}^v) + F(X, Y),$$  

(5)

where $\hat{X}$ and $\hat{Y}$ denote horizontal lifts to the correspondence space $E \times_M \hat{E}$ of the vector fields on each fibration.

In section 2 we recall the basic facts and definitions on Lie bialgebroids, and the way they can be used to define Poisson algebras of functions on smooth manifolds. In section 3 we show that $T$-duality of circle bundles can be naturally interpreted as a Lie bialgebroid induced by the Atiyah algebroids associated to the $T$-dual pairs, and we will address some of the implications of this results.
2. Lie bialgebroids and Poisson algebras

Consider a Lie algebroid $A$ over a smooth manifold $M$, with anchor $a : \Gamma(A) \to \mathcal{X}(M)$ and bracket $[\cdot, \cdot]$, whose dual $A^*$ is also equipped with a Lie algebroid structure, say $a^* : \Gamma(A^*) \to \mathcal{X}(M)$ and bracket $[\cdot, \cdot]^*_s$:

\[ A \xrightarrow{a} TM \xleftarrow{a^*} A^* \]

If the coboundary operator associated to $A^*$,

\[ d_\ast : \Gamma(A) \to \bigwedge^2 \Gamma(A), \]

satisfies the cocycle condition (on $\Gamma(A)$)

\[ d_\ast [x, y] = [d_\ast x, y] + [x, d_\ast y], \quad (6) \]

i.e. if $d_\ast$ is a derivation of the Schouten algebra ($\Gamma(\bigwedge^\bullet A^*), [\cdot, \cdot]^*_s$), we say that the pair $(A, A^*)$ is a Lie bialgebroid. Here we use the coboundary operator $d_\ast : \Gamma(\bigwedge^k A) \to \Gamma(\bigwedge^{k+1} A)$ and the Schouten bracket given by

\[ d\alpha(x_1, x_2, \ldots, x_{k+1}) = \sum_{i=1} (-1)^{i-1} a(x_i)\alpha(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1}) \]

\[ + \sum_{1<i<j<k+1} (-1)^{i+j} \alpha([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1}). \]

and

\[ [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] = \sum_{i,j} (-1)^{i+j} [x_i, y_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_j \wedge \cdots \wedge y_m, \]

for $x_1 \wedge \cdots \wedge x_k \in \Gamma(\bigwedge^k L), y_1 \wedge \cdots \wedge y_m \in \Gamma(\bigwedge^m L)$, and $[x, f] = -[f, x] = a(x)f$ for $f \in \mathcal{C}^\infty(M)$, respectively.

It has been shown [8][10][11] that pairs defining Lie bialgebroids are naturally self-dual: $(A, A^*)$ is a Lie bialgebroid if and only if $(A^*, A)$ is a Lie bialgebroid, and those conditions are equivalent to the coboundary operator associated to $A$ to satisfy the cocycle condition (on $\Gamma(A^*)$), i.e. to $d : \Gamma(A^*) \to \bigwedge^2 \Gamma(A^*)$ to be derivation of the Schouten algebra ($\Gamma(\bigwedge^\bullet A^*), [\cdot, \cdot]^*_s$) associated to $A^*$. Moreover, the bialgebroid condition can also be written in terms of Lie algebroid
morphisms and Poisson maps, namely it is equivalent to the Poisson bundle map
\( \pi_\# : T^*A \rightarrow TA \) to induce a Lie algebroid morphism with respect to the Lie
algebroid structure on \( T^*A \rightarrow A^* \), see theorem 6.2 in [11].

Using the coboundary operators \( d \) and \( d^* \), associated to a dual pair \((A, A^*)\)
defining a Lie bialgebroid, it is possible to associate to the space of smooth maps
on the base manifold \( M \) a Poisson structure [8]:

**Proposition 2.1.** Let \((A, A^*)\) be a Lie bialgebroid over a smooth manifold \( M \).
Then, for any \( f, g \in C^\infty(M) = \Gamma(\Lambda^0 A) = \Gamma(\Lambda^0 A^*) \),
\[
\{f, g\}_{(A,A^*)} = \langle df, d^* g \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) denotes a duality pairing between \( A \) and \( A^* \), defines a Poisson bracket structure satisfying
\[
d\{f, g\}_{(A,A^*)} = [df, dg]^*.
\]

**Example 2.1.** In the case of a Poisson manifold \((M, \Pi)\), both \( TM \) and \( T^*M \)
define Lie algebroids and, moreover, they are in duality. The (graded) Jacobi identity makes of the coboundary operator \( d_* = [\Pi, \cdot] \) a derivation, so that
\((TM, T^*M)\) is a Lie bialgebroid over \( M \), sometimes called the standard Lie bialgebroid of \((M, \Pi)\). The Poisson bracket associated to \( f, g \in C^\infty(M) \) is, as expected,
\[
\{f, g\}_{(TM, T^*M)} = \Pi(df, dg).
\]

In general, given a Lie bialgebroid \((A, A^*)\) over a smooth manifold \( M \) and
a function \( f \in C^\infty(M) \), a *hamiltonian field* will be any element \( X_f \in \Gamma(A) \)
satisfying
\[
X_f + d_* f = 0.
\]
In our last example, this corresponds exactly to the usual Hamiltonian vector fields of Poisson geometry.

3. T-dual pairs and Lie bialgebroids

Given \( T \)-dual pairs \((E, H) \) and \((\hat{E}, \hat{H})\) (of principal circle bundles with integral \( H \)-fluxes), we will denote by \( L = A(E) \) and \( \hat{L} = A(\hat{E}) \), respectively, their associated
Atiyah algebroids. Recall that, in the topological setting, for \((E, H)\) there exists
a closed 2-form \( \hat{F} \in \Omega^2(M, t^*) \) with integral periods such that, for all \( X \in t \),
\[
i_X H = \hat{F}(X),
\]
where \( \hat{F}(X) \) means the natural pairing (with values in \( \Omega^2(M) \)) between \( t \) and
\( \Omega^2(M, t^*) \). The 2-forms defined by (9) corresponding to \( T \)-dual pairs have a nat-
ural geometric interpretation as curvatures of the principal \( S^1 \)-bundles \( E \) and \( \hat{E} \)
and \( \Phi = -A \wedge \hat{A} \), where \( A \) and \( \hat{A} \) denote connections on the \( S^1 \)-bundles \( E \) and
\( \hat{E} \), satisfying \( F = dA \) and \( \hat{F} = d\hat{A} \), respectively (see [3] [4] and references therein).

The \( \mathbb{S}^1 \)-bundles \( E \) and \( \hat{E} \), which we assume are compact manifolds, define contact structures on the total spaces, and \( F \) and \( \hat{F} \) define symplectic forms on the base space \( M \) (see [2]), we will prove that the Atiyah algebroids \( L \) and \( \hat{L} \) over \( M \) define a Lie bi-algebroid, i.e. they are dual with anchor \( a \) and bracket \([\cdot, \cdot]_a\), \( a_\ast \) and bracket \([\cdot, \cdot]_{a_\ast}\), respectively, and \( d_L \) is a derivation of the Schouten bracket of \( \hat{L} \): \( d_L[a, \beta]_a = [d_L, \alpha, \beta]_a + [\alpha, d_L, \beta]_a \) for all \( \alpha, \beta \in \Gamma(\hat{L}) \). The duality pairing giving the identification of \( \hat{L} \) with \( L^\ast \) is the defined by

\[
\langle X, Y \rangle := \Phi(\hat{X}^v, \hat{Y}^v) + F(X, Y), \tag{10}
\]

where \( \hat{X} \) and \( \hat{Y} \) denote horizontal lifts to the correspondence space \( E \times_M \hat{E} \) of the vector fields on each fibration. This pairing is non-degenerate, as follows from the fact that \( F \) is non-degenerate and acts only on the horizontal part of vector fields and since, on the other hand, \( (E, H) \) and \( (\hat{E}, \hat{H}) \) are \( T \)-dual so that the form \( \Phi \) is non-degenerate on the vertical part of the corresponding extensions, by definition.

On the Atiyah algebroid \( \hat{L} = A(\hat{E}) \) over \( M \) we will consider the local frame \( \{\partial_\phi, X_{x_1}, \ldots, X_{x_n}\} \), where \( (x_1, \ldots, x_n) \) denotes local coordinates on a point in the base and the vector fields \( X_{x_j} \), for \( j = 1, \ldots, n \), are the Hamiltonian vector fields associated to the local coordinates with respect to the symplectic structure \( F \) over \( M \) (for \( L = A(E) \) we will denote the corresponding frame by \( \{\partial_\theta, X_{x_1}, \ldots, X_{x_n}\} \)).

Using the pairing (10) we can easily compute the differential \( d_L \) (of a smooth function on \( M \) or a section of \( L^\ast \)) as the next lemma shows.

**Lemma 3.1.** Let \( f \in C^\infty(M) \), then

(a) \( d_L f = X_f \) and \( d_L X_f = 0 \).

(b) \( d_L \partial_\phi = 0 \).

**Proof.** For (a), let \( X = \partial_\theta \in \Gamma(L) \). Then,

\[ d_L f(\partial_\theta) = 0 = \langle X_f, \partial_\theta \rangle, \]

since \( a(\partial_\theta) = 0 \). Also, if \( X = X_{x_j} \) for \( j = 1, \ldots, n \),

\[ d_L f(X_{x_j}) = a(X_{x_j}) f = df(X_{x_j}) = F(X_f, X_{x_j}) = \langle X_f, X_{x_j} \rangle \]

for all \( f \in C^\infty(M) \). Thus, for an arbitrary section \( X \in \Gamma(L) \), the result follows. The second part is just the fact that \( d_L^2 = 0 \).

To prove (b) let \( X_1, X_2 \in \mathfrak{X}(M) \), then

\[ d_L \partial_\phi(X_1, X_2) = X_1(\langle \partial_\phi, X_2 \rangle) - X_2(\langle \partial_\phi, X_1 \rangle) - \langle \partial_\phi, ([X_1, X_2]) \rangle = 0 \]
and
\[
d_L\phi(f \partial \theta, X_1) = -X_1((\partial \phi, f \partial \theta)) - \{\partial \phi, [f \partial \theta, X_1]\}
\]
\[
= -X_1(f) - \{\partial \phi, -X_1(\partial f)\} = 0.
\]

\[
\]

Using the preceding lemma it follows that (6) is true for the vector fields of the frame described before. More explicitly, the following equations hold for \(j, k = 1, \ldots, n\):

\[
d_L[X_{xj}, X_{xk}] = [d_LX_{xj}, X_{xk}] + [X_{xj}, d_LX_{xk}],
\]

(11)

\[
d_L[\partial \phi, X_{xj}] = [d_L\partial \phi, X_{xj}] + [\partial \phi, d_LX_{xj}],
\]

(12)

**Theorem 3.1.** The pair \((A(E), \hat{A}(\hat{E}))\) defines a Lie bi-algebroid.

**Proof.** In order to show that the pair \((L = A(E), \hat{L} = A(\hat{E}))\) defines a Lie bialgebroid, we need to show that (6) holds for arbitrary sections of \(L^*\). By linearity we only need to show that they remain true if we multiply the \(X_{xj}'s\) and \(\partial \phi's\) in the equations above by smooth function in \(C^\infty(M)\). To this end, we will use that the Poisson structure \(\pi\) defined by \(F\), satisfy the well-known relations

\[
X_{fg} = fX_g + gX_f, \quad X_f(g) = \{f, g\}
\]

and we will write \(\{x^j, x^k\} = \pi^{jk}\). Thus, on the left hand side of (11) we have

\[
d_L[fX_{xj}, gX_{xk}] = d_L((fX_{xj})g)X_{xk} - (gX_{xj}f)X_{xk} + f[gX_{xj}, X_{xk}]
\]

\[
= d_L((fX_{xj}g)X_{xk} - (gX_{xj}f)X_{xk} + f[gX_{xj}, X_{xk}])
\]

\[
= X_{fX_{xj}}(g) \wedge X_{xk} - X_{gX_{xk}(f)} \wedge X_{xj} + X_{fg} \wedge X_{\pi^{jk}},
\]

while on the right hand side we obtain

\[
[d_LfX_{xj}, gX_{xk}] + [fX_{xj}, d_LgX_{xk}]
\]

\[
= [X_f \wedge X_{xj}, gX_{xk}] + [fX_{xj}, X_g \wedge X_{xk}]
\]

\[
= [X_f, gX_{xk}] \wedge X_{xj} - [X_{xj}, gX_{xk}] \wedge X_f
\]

\[
+ [fX_{xj}, X_g] \wedge X_{xk} - [fX_{xj}, X_{xk}] \wedge X_g
\]

\[
= \{f, g\}X_{xk} \wedge X_{xj} + gX_{\{f, x_j\}} \wedge X_{xk} - \{x^j, g\}X_{xk} \wedge X_f
\]

\[
- gX_{\pi^{jk}} \wedge X_f - \{f, g\}X_{xj} \wedge X_{xk} + fX_{\{x_j, g\}} \wedge X_{xk}
\]

\[
+ \{x^k, f\}X_{xj} \wedge X_g - fX_{\pi^{jk}} \wedge X_g
\]

\[
= \{x^j, g\}X_f \wedge X_{xk} + fX_{\{x_j, g\}} \wedge X_{xk} - gX_{\{x_k, f\}} \wedge X_{xj}
\]

\[
- \{x^k, f\}X_g \wedge X_{xj} + gX_f \wedge X_{\pi^{jk}} + fX_g \wedge X_{\pi^{jk}}.
\]
showing that the equality remains true. For the case of (12) the computation on the left hand side yields
\[ d_L[f\partial_\phi, gX_{x^j}] = d_L(-gX_{x^j}(f)\partial_\phi) = -X_gX_{x^j}(f)\wedge\partial_\phi, \]
while on the right-hand side we have
\[ [d_L(f\partial_\phi), gX_{x^j}] + [f\partial_\phi, d_L(gX_{x^j})] = \{X_gX_{x^j}, f\}X_{x^j} - \{f, g\}X_{x^j} \]
\[ = -X_g\{X_{x^j}, f\}X_{x^j} - \{f, g\}X_{x^j} \]
\[ = \{f, g\}X_{x^j} - \{g, f\}X_{x^j} \]
\[ = \{f, g\}X_{x^j}. \]

This result gives us new examples of Lie bi-algebroids and, therefore, of Courant algebroids. Indeed, as shown in [9], any Lie bialgebroid \((A, A^*)\) gives rise to a canonical Courant algebroid over the same base manifold, namely \(E = A \oplus A^*\). The Courant algebroid structure of \(E\) comes from the following natural combination of the operations defined on each Lie algebroid: for \(x + \alpha, y + \beta \in \Gamma(E)\), the pairing
\[ \langle x + \alpha, y + \beta \rangle = \beta(x) + \alpha(y), \]
the anchor
\[ a(x + \alpha) = a(x) + a_*(\alpha) \]
and the bracket
\[ [x + \alpha, y + \beta] = [x, y] + \mathcal{L}_x\beta - i_\beta d_L\alpha \]
on sections of \(E\), with the natural notations.

**Corollary 3.1.** The vector bundle \(\mathcal{A} = A(E) \oplus A(\hat{E})\) admits a structure of Courant algebroid over \(M\).

**Example 3.1.** Over \(S^2\) consider the Lie algebroids \(TS^2 \oplus \mathfrak{t}\) and \(A(S^3)\), i.e. the Atiyah algebroids of the \(T\)-dual pairs \((S^2 \times S^1, H)\) and \((S^3, 0)\) respectively. Then they define a Lie bi-algebroid:
Example 3.2. Let $M$ be a compact manifold and let $E$ be a principal $S^1$-bundle over $M$, with $F$ a representative for $c_1(E)$. Consider the pair $(E, H)$, where $H \in \Omega^3(M)$. Then, for any $X \in \mathfrak{t}$ we obtain $i_X H = 0$ and, hence, if we take $\hat{F} = 0$ we obtain as $T$-dual $\hat{E}$ the trivial bundle $S^1 \times M$. Now endow such bundle with the connection $\hat{A} = d\varphi \otimes \partial \varphi$ and observe that, with the notations used in the introduction, $\Omega = -\varphi$ and in consequence

$$
\hat{H} = F \square \hat{A} + H = F \wedge d\varphi + H = d\varphi \wedge F + H.
$$

Hence, the pair $(E, H)$ is $T$-dual to $(S^1 \times M, d\varphi \wedge F + H)$ and in particular, $(E, 0)$ has as $T$-dual $(S^1 \times M, d\varphi \wedge F)$. Thus, considering the associated Atiyah algebroids, $(A(E), A(S^1 \times M))$ defines a Lie bialgebroid.

Remark 3.1. In [7] it was shown that $T$-duality, in this context, can be understood as an isomorphism of Courant algebroids $A^{(H)}_E$ and $\hat{A}^{(H)}_E$ which, locally, can be identified with the Courant algebroid $T M \oplus T^* M \oplus \mathfrak{t} \oplus \mathfrak{t}^*$. Here $A^{(H)}_E$ denotes the Courant algebroid $A(E) \oplus A(E)^*$ over $M$ with bracket twisted by the 3-form $H$ and $\hat{A}^{(H)}_E$ denotes the Courant algebroid $A(\hat{E}) \oplus A(\hat{E})^*$ and bracket twisted by $\hat{H}$. It is shown there that, as expected, $T$-duality is just a permutation of the $\mathfrak{t}$ and $\mathfrak{t}^*$ factors.

Remark 3.2. The Poisson structure used in the proofs in section 3 is equivalent to the one associated to the contact structure in the principal bundles $E$ and $\hat{E}$ (see [6]), and it coincides also with the one associated to the bialgebroid $(L, \hat{L})$ defined in proposition 2.1, as can be verified by a direct computation.

Remark 3.3. The interpretation of $T$-duality of circle bundles with 3-form flux done here can be generalized to the case of classical $T$-dual torus bundles of higher rank, as defined in [12]. In the general case the $T$-dual space associated to a torus bundle can be understood as a noncommutative torus, and no longer as the total space of a principal fibration.

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