Finding Integrals and Identities in the Newman–Penrose Formalism: a Comment

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Abstract—The goal of the present short letter is to prove, through a specific example (i.e., the class of Ricci-flat, Petrov type D geometries), that the original Newman-Penrose (NP) formalism, seen as an exterior differential system (EDS), suffices to provide the needed syzygies and integrals of the EDS under consideration—without (in principle) the aid of a computer algebra system (CAS).

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1. INTRODUCTION

In 1969 W. Kinnersley [1], using the NP formalism [2], found all the Petrov type D, Ricci flat, solutions to the Einstein’s Field Equations (EFEs). Yet, in doing so—as it seems—he neglected two fundamental identities (or constraints) on four NP variables and Cartan invariants as well, namely, \( \tau \tau - \pi \pi = 0 \) and \( \mu \bar{\mu} - \rho \bar{\rho} = 0 \). Since then, these identities have been constantly either overlooked or proven under special circumstances (e.g., for electrovac solutions).\(^1\) It was only until 2009, when B. S. Edgar and his collaborators [4] (see also references therein), by making an extended use of the Geroch–Held–Penrose (GHP) formalism [2], and of a CAS, succeeded in proving those identities in the general case (i.e., within Kinnersley’s assumptions). In that reference, it was—rather indirectly—implied that the results under consideration were provable only within the GHP formalism, and thus the latter is the optimal tool towards the invariant classification and study of classes of solutions to the EFEs.

In 2014 there was a kind of response to that paper by J.J. Ferrando and J.A. Saéz [5]. Using the tensorial algebra (of bivectors or 2-forms), and without any aid of a CAS, the authors proved the desired result, and they offered a much more refined and extended classification of the Petrov type D, Ricci flat, solutions. Nevertheless, when someone reads that third work, although beautiful and conceptually simple, one has the feeling that the authors know in advance what they want to prove; something which is not always the case.

2. THE NEWMAN–PENROSE FORMALISM AS AN EXTERIOR DIFFERENTIAL SYSTEM (EDS)

Let a pseudo-Riemannian space be described by the pair \((\mathcal{M}, g)\), where \(\mathcal{M}\) is a 4 dimensional, simply connected, Hausdorff, and \(C^\infty\) manifold, and \(g\) is a \(C^\infty\) metric tensor field on it that is a nondegenerate, covariant tensor field of order 2, with the property that at each point of \(\mathcal{M}\) one can choose a frame of 4 real vectors \(\{e_1, \ldots, e_4\}\), such that \(g(e_a, e_b) = \delta_{ab}\) where \(g\) (called the frame metric) is a constant symmetric matrix with prescribed signature.

The NP formalism [2] can be thought of as a torsionless local geometry described by a (special) class of bundles of quasi-orthonormal, semi-complex frames.

More precisely, a linear, complex combination of the four real frame vectors is chosen, such that it is

\[ e_a = (\Delta, \mathcal{D}, -\bar{\delta}, -\bar{\delta}), \quad \Delta = \bar{\Delta}, \quad \mathcal{D} = \bar{\mathcal{D}}, \quad (2.1) \]

where a bar over a symbol denotes complex conjugation, while

\[ g(e_a, e_b) = \delta_{ab} \equiv \eta_{ab} \]

\[ = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
\end{pmatrix}, \quad (2.2) \]

and the coframe is defined via

\[ e_a \downarrow \omega^b = \delta_a^b \]

\[ (2.3) \]

as

\[ \omega^a = (I, \mathbf{n}, \mathbf{m}, \bar{\mathbf{n}}), \quad \bar{I} = 1, \quad \bar{\mathbf{n}} = \mathbf{n}, \quad (2.4) \]
with the allocations
\[
\omega_0 = \tau l + \kappa n - \rho m - \sigma \hat{m},
\]
\[
\omega_1 = \gamma l + \epsilon n - \alpha m - \beta \hat{m},
\]
\[
\omega_2 = \nu l + \pi n - \lambda m - \mu \hat{m}.
\]

2. The Ricci equations
\[
d\omega^a_b + \omega^a_m \wedge \omega^m_b = \frac{1}{2} R^a_{bmn} \omega^m \wedge \omega^n \equiv \Omega^a_b
\]

with the allocations
\[
\Omega_0 = -\Psi_0 n \wedge \hat{m} - \Psi_1 (l \wedge n - m \wedge \hat{m})
\]
\[
- (\Psi_2 + 2\Lambda) m \wedge l - \Phi_{00} n \wedge m
\]
\[
- \Phi_{01} (l \wedge n + m \wedge \hat{m}) - \Phi_{02} \hat{m} \wedge l,
\]
\[
\Omega_1 = -\Psi_1 n \wedge \hat{m} - (\Psi_2 - \Lambda) (l \wedge n - m \wedge \hat{m})
\]
\[
- \Psi_3 m \wedge l - \Phi_{10} n \wedge m
\]
\[
- \Phi_{11} (l \wedge n + m \wedge \hat{m}) - \Phi_{12} \hat{m} \wedge l,
\]

3. FINDING INTEGRALS AND IDENTITIES IN THE NEWMAN PENROSE FORMALISM

In the present section, the focus will be on Kinnersley’s initial assumptions, i.e., all Petrov type D, Ricci-flat solutions to the EFEs.

The starting hypothesis implies [2] that there exists a family of frames such that \( \Lambda = 0 \), \( \Phi_{ij} = 0 \), and \( \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \), where the last equalities are obtained via the implementation of null rotations about the first \( l \) and second \( n \). Thus four of the six real degrees of freedom, corresponding to the use of the Lorentz group, have been used up. Still, there is a residual freedom; the system is invariant under spin-boost transformations as per
\[
(l, n, m, \hat{m}) \rightarrow \left( \zeta l, \frac{1}{\zeta} n, \zeta m, \frac{1}{\zeta} \hat{m} \right)
\]

This freedom is the reason for the existence of a whole family of frames. Based on the transformation laws for
the NP variables [6], it is deduced that the following set of NP quantities:

$$\{\mu \rho, \mu \bar{\rho}, \rho \mu, \bar{\tau} \pi, \tau \bar{\pi}, \pi \bar{\tau}, \Psi_2\}$$  \hspace{1cm} (3.2)

are left invariant under the action of the residual freedom. According to the Cartan-Kähler [7, 8] (CK) algorithm, all these quantities belong to (but without exhausting) the set of Cartan invariants.²

The Bianchi equations (2.13), along with the NP equations (2.6) imply:

$$\kappa = \lambda = \nu = \sigma = 0, \hspace{1cm} (3.3a)$$

$$d\Psi_2 = -3\mu \Psi_2 \mathbf{l} + 3\rho \Psi_2 \mathbf{n}$$

$$+ 3\pi \Psi_2 \mathbf{m} - 3\tau \Psi_2 \mathbf{m}. \hspace{1cm} (3.3b)$$

Now using

- the NP equations (2.6),
- the Ricci component equations [1, 4] and [2, 3] of (2.9),
- the fact that $d^2 \Psi_2 = 0$ (i.e., the post-Bianchi equation [4]),

there emerge four equations giving $d\mu, d\pi, d\rho, d\tau$ in terms of the spin connection coefficients:

$$d\mu = -(\gamma \mu + \mu^2 + \mu \gamma)\mathbf{l} + \mathbf{A} \mathbf{n}$$

$$+ (\alpha \mu + \mu \pi + \mu \bar{\beta} - \mu \bar{\rho}) \mathbf{m}$$

$$+(B + \beta \mu - \gamma \tau + \mu \bar{\alpha} + \tau \bar{\gamma}) \mathbf{m}, \hspace{1cm} (3.4a)$$

$$d\pi = -(\gamma \pi - \mu \pi + \pi \bar{\gamma} - \mu \bar{\tau})\mathbf{l} + \mathbf{F} \mathbf{n}$$

$$+ (\alpha \pi + \pi^2 - \pi \bar{\beta}) \mathbf{m} + (-A - \epsilon \mu + \beta \pi$$

$$- \pi \bar{\alpha} - \mu \bar{\epsilon} + \mu \bar{\rho} + \pi \bar{\pi} + \Psi_2) \mathbf{m}, \hspace{1cm} (3.4b)$$

$$d\rho = (\epsilon \rho + \rho^2 + \rho \bar{\epsilon}) \mathbf{n} + (-\beta \rho - \rho \tau - \rho \bar{\alpha} + \tau \bar{\beta}) \mathbf{m}$$

$$+ (A - \epsilon \mu + \gamma \rho + \rho \gamma - \mu \bar{\epsilon} - \bar{\tau} \bar{\bar{\pi}} + \pi \bar{\pi}) \mathbf{l}, \hspace{1cm} (3.4c)$$

$$d\tau = \mathbf{B} \mathbf{l} + (\epsilon \tau + \rho \tau - \tau \bar{\bar{\epsilon}} + \rho \bar{\pi}) \mathbf{n}$$

$$+ (-\beta \tau - \tau^2 + \tau \bar{\alpha}) \mathbf{m} + (A + \epsilon \mu - \alpha \tau$$

$$+ \tau \bar{\beta} + \mu \bar{\epsilon} - \rho \bar{\rho} - \pi \bar{\pi} - \Psi_2) \mathbf{m}. \hspace{1cm} (3.4d)$$

There are three unknown functions $(A, \beta, F)$, reflecting the fact that in each Ricci 2-form two directional derivatives, out of the four, are missing. The rest of the Ricci equations involve the operation of $d$ on $\alpha, \beta, \gamma, \epsilon$ and will not be needed.

Now one can search for $p$-forms which will be integrals of the NP–Cartan EDS. By construction, and by virtue of the CK algorithm [7], one can see that for a first, purely algebraic approach to the Petrov type D, Ricci–flat spaces, the Riemann tensor (i.e., $\Psi_2$) and its first covariant derivatives (i.e., $\mu, \pi, \rho, \tau$, —by virtue of (3.3b)), both being invariant under the residual freedom of spin-boosts (therefore the gauge fixing according to the CK algorithm can be performed in the second covariant derivative). For these reasons the $p$-form to be sought must be left-invariant under the residual freedom; thus it may not contain any of the gauge quantities $\alpha, \beta, \gamma, \epsilon$.

The first attempt will concern the Riemann tensor, for which a natural candidate is the spin-boost invariant component of the curvature 2-form

$$I_0 \equiv f(\Psi_2) \mathbf{l} \wedge \mathbf{n} + h(\Psi_2) \mathbf{m} \wedge \mathbf{m}, \hspace{1cm} (3.5)$$

and the property of being an integral reads

\[ \text{label} = \text{zeroth integral} \rightarrow I_0|_{\text{EDS}} = 0. \hspace{1cm} (3.6) \]

A simple calculation involving both the NP equations (2.6) and the Bianchi equations (3.3b) reveals two algebraic integrability conditions;

\[ (\tau \bar{\bar{\pi}} - \pi \bar{\bar{\tau}})(f(\Psi_2) + h(\Psi_2)) = 0, \hspace{1cm} (3.7a) \]

\[ (\mu \bar{\bar{\rho}} - \rho \bar{\bar{\mu}})(f(\Psi_2) + h(\Psi_2)) = 0. \hspace{1cm} (3.7b) \]

Thus,

\[ \left( (\tau \bar{\bar{\pi}} - \pi \bar{\bar{\tau}}) = 0, (\mu \bar{\bar{\rho}} - \rho \bar{\bar{\mu}}) = 0 \right) \]

\[ \vee \left( f(\Psi_2) + h(\Psi_2) = 0 \right), \hspace{1cm} (3.8) \]

i.e., either the desired identities will hold, or $h(\Psi_2) = -f(\Psi_2)$. For the time being let $h(\Psi_2) = -f(\Psi_2)$ only. Then, the substitution back to the condition (3.6) results in a simple ODE for the function $f$,

\[ 2f(\Psi_2) - 3\Psi_2 f'(\Psi_2) = 0, \hspace{1cm} (3.9) \]

the solution being

\[ I_0 = (\Psi_2)^{2/3}(1 \wedge \mathbf{n} - \mathbf{m} \wedge \mathbf{m}), \]

\[ dI_0|_{\text{EDS}} = 0. \hspace{1cm} (10.10) \]

(At this point it should be noted that the same ODE emerges when the identities hold.) This is a well known result—see, e.g., [11]. A note on topology is pertinent at this point. Indeed, applying the Gauss-Stokes theorem to $I_0$ for, e.g., the Kerr black hole, one can see that the corresponding integral assumes a null value if the area of integration does not contain the curvature singularity (at $r = 0$ and $\theta = \pi/2$) (since the integrand is left-invariant under continuous deformations of the integration area into a point), and it is a nonzero constant (related to the mass $M$) when the singularity is included—cf. the Morera-Cauchy theorems in complex analysis.

The second attempt will concern the first covariant derivative of the Riemann tensor, and a natural candidate is the Lorentz-invariant expression

\[ I_2 \equiv \frac{d\Psi_2}{3\Psi_2} \wedge \frac{d\Psi_2}{3\Psi_2}. \hspace{1cm} (11.1) \]
Then, by virtue of the Bianchi equations (2.13) in the form of (3.3b), the integral condition is already satisfied, i.e.,

$$dI_2|_{\text{EDS}} = 0$$  \hspace{1cm} (3.12)

since $I_2$ belongs to the ring of the EDS. Nevertheless, the quantities $I_0$, $I_2$ and $\bar{I}_2$ are functionally dependent, thus a syzygy emerges:

$$I_0 \wedge I_2 \wedge \bar{I}_2 = 0 \Rightarrow (\Psi_2)^{2/3}(\mu \bar{\rho} - \rho \bar{\mu} - \tau \bar{\tau} + \pi \bar{\pi})l \wedge n \wedge m \wedge \bar{m} = 0.$$  \hspace{1cm} (3.13)

Now, the real and imaginary parts of the previous expression result in

$$\tau \bar{\tau} - \pi \bar{\pi} = 0, \hspace{1cm} (3.14)$$

and

$$\rho \bar{\mu} - \mu \bar{\rho} = 0, \hspace{1cm} (3.15)$$

respectively.

4. DISCUSSION

In the present short paper, a naive method for finding identities (i.e., specific integrals) in an EDS—here, the NP formulation of the EFEs for the case of Ricci flat, Petrov Type D geometries—is presented. It is rather a proof of the concept that a closed EDS ideal, and it suffices to provide the needed syzygies without, in principle, resorting to either any modified formalism (like, e.g., the GHP formalism in this case) or the help of a CAS.

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