FREQUENCY THEOREM FOR THE REGULATOR PROBLEM WITH UNBOUNDED COST FUNCTIONAL AND ITS APPLICATIONS TO NONLINEAR DELAY EQUATIONS

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Abstract. We study the quadratic regulator problem with an unbounded cost functional of general type. The motivation comes from delay equations, which has the feedback part with discrete delays (or, in other words, delta-like measurements, which are unbounded in $L^2$). We treat the problem in an abstract context of a certain Hilbert space, which is rigged by a Banach space. We obtain a version of the non-singular frequency theorem, which guarantees the existence of a unique optimal process, starting in the Banach space. We show that the optimal cost (that is the value of the quadratic functional on the optimal process) is given by the “quadratic form” of a bounded linear operator from the Banach space to its dual and this form can be used as a Lyapunov-like functional. For a large class of non-autonomous nonlinear delay equations in feedback form we obtain an analog of the circle criterion, which is a natural extension of the corresponding criterion for ODEs.

1. Introduction

We start from a precise statement of the main theorem and then present a discussion. In what follows we assume that all the vector spaces are complex unless otherwise is specified. Let $A$ be the generator of a $\mathcal{C}_0$-semigroup $G(t)$, where $t \geq 0$, acting in a Hilbert space $\mathbb{H}$. Let $\Xi$ be another Hilbert space and $B \in L(\Xi; \mathbb{H})$ be bounded. The equation

$$\dot{u} = Au + B\xi$$

is called a control system. For every $T > 0$, $u_0 \in \mathbb{H}$ and $\xi = \xi(\cdot) \in L_2(0, T; \Xi)$ there exists a unique mild solution $u(t) = u(t, u_0, \xi)$, where $u(0) = u_0$ and $t \in [0, T]$, to (1.1), which is a continuous $\mathbb{H}$-valued function and given by

$$u(t) = G(t)u_0 + \int_0^t G(t-s)B\xi(s)ds.$$ 

For any operator $C \in L(\mathbb{H}, \Xi)$ the operator $A + BC$ is the generator of a $\mathcal{C}_0$-semigroup $G_C(t)$ (see, for example, Theorem 7.5 in [1]). The pair $(A, B)$ is called

1. exponentially stabilizable, if there are $C \in L(\mathbb{H}, \Xi)$ and constants $M, \varepsilon > 0$ such that $\|G_C(t)\| \leq Me^{-\varepsilon t}$ for all $t \geq 0$;

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Throughout this paper $L(E; F)$ denotes the space of bounded linear operators from $E$ to $F$. If $E = F$ we usually write $L(E)$.
For every $M$ the elements of $E$ are equivalent. As a by-product of the frequency theorem \cite{23,24}, it turns out that the above three properties are equivalent. Here we, however, make the use of slightly different notions.

Let $\mathbb{E}$ be a Banach space, which is continuously and densely embedded in $\mathbb{H}$. We identify the elements of $\mathbb{E}$ and their images in $\mathbb{H}$ under the embedding. We require also that $\mathbb{E} \subset \mathcal{D}(A)$.

Identifying the Hilbert space $\mathbb{H}$ with its dual, we obtain the rigging $\mathbb{E} \subset \mathbb{H} \subset \mathbb{E}^*$.

By $\langle u, f \rangle$ we denote the dual pairing between $u \in \mathbb{E}$ and $f \in \mathbb{E}^*$. In virtue of the embedding, we have $\langle u, v \rangle_{\mathbb{H}} = \langle u, v \rangle_{\mathbb{H}}$ provided that $v \in \mathbb{H}$.

Let $\mathcal{F}(u, \xi)$, where $u \in \mathbb{E}$ and $\xi \in \mathbb{E}$ be an unbounded in $\mathbb{H} \times \mathbb{E}$ quadratic form such as

$$\mathcal{F}(u, \xi) = (\mathcal{F}_1 u, u)_{\mathbb{H}} + 2 \text{Re}(\mathcal{F}_2 u, \xi) + (\mathcal{F}_3 \xi, \xi)_{\mathbb{E}} + (\mathcal{F}_4 u, u)_{\mathbb{E}}, \quad (1.4)$$

where $\mathcal{F}_1 = \mathcal{F}_1 \in \mathcal{L}(\mathbb{H})$, $\mathcal{F}_2, \mathcal{F}_4 \in \mathcal{L}(\mathbb{E}; \mathbb{E})$, $\mathcal{F}_3 = \mathcal{F}_3 \in \mathcal{L}(\mathbb{E})$ and $\mathcal{F}_5 = \mathcal{F}_5 \in \mathcal{L}(\mathbb{E})$.

Let us consider the spaces $\mathcal{Z}_1 := L_2(0, +\infty; \mathbb{H})$, $\mathcal{Z}_0 := L_2(0, +\infty; \mathbb{H}) \cap \mathcal{C}([0, +\infty]; \mathbb{E})$ and $\mathcal{Z}_2 := L_2(0, +\infty; \mathbb{E})$. Note that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are Hilbert spaces with the usual inner product and we do not consider any norm on $\mathcal{Z}_0$ until the next section. We consider the unbounded quadratic functional $\mathcal{J}_\mathcal{F}(u(\cdot), \xi(\cdot))$ in the space $\mathcal{Z} := \mathcal{Z}_1 \times \mathcal{Z}_2$ given by

$$\mathcal{J}_\mathcal{F}(u(\cdot), \xi(\cdot)) := \int_0^\infty \mathcal{F}(u(t), \xi(t))dt. \quad (1.5)$$

We suppose that

- **(QF)** There are constants $C_2, C_4 > 0$ such that for all $u(\cdot) \in \mathcal{Z}_0^1$ we have

$$\int_0^\infty \|\mathcal{F}_j u(t)\|^2_{\mathbb{E}} dt \leq C_j (\|u(0)\|^2_{\mathbb{H}} + \|u\|^2_{\mathcal{Z}_1}), \quad (1.6)$$

where $j \in \{2, 4\}$.

Clearly, under **(QF)** the quadratic functional $\mathcal{J}_\mathcal{F}$ is well-defined on $\mathcal{Z}_0^1 \times \mathcal{Z}_2 \subset \mathcal{Z}$.

- **(REG)** For every $u_0 \in \mathbb{E}$ and $\xi(\cdot) \in L_2(0, T; \mathbb{E})$ for the solution $u(t) = u(t, u_0, \xi)$ of (1.1) we have $u(\cdot) \in \mathcal{C}([0, T]; \mathbb{E})$.

The above two assumptions are natural for delay equations and smoothes the dealing with $\mathcal{J}_\mathcal{F}$ in the presence of unboundedness.

We say that the pair $(A, B)$ is $(L_2, \mathbb{H})$-controllable in $\mathbb{E}$ if for every $u_0 \in \mathbb{E}$ there exists $\xi(\cdot) \in L_2(0, +\infty; \mathbb{E})$ such that $u(\cdot) = u(\cdot, u_0, \xi) \in L_2(0, +\infty; \mathbb{E})$. If, in addition, we have $u(\cdot) = u(\cdot, u_0, \xi) \in L_2(0, +\infty; \mathbb{H})$, we say that $(A, B)$ is $(L_2, \mathbb{E})$-controllable in $\mathbb{E}$.

As in \cite{23}, let $\mathcal{M}_{u_0}$ be the set of all $(u(\cdot), \xi(\cdot)) \in \mathcal{Z}$ such that $u(\cdot) = u(\cdot, u_0, \xi)$. Every such pair is called a process through $u_0$. A process through $u_0 \in \mathbb{E}$ is called optimal if it is a minimum point of $\mathcal{J}_\mathcal{F}$ on $\mathcal{M}_{u_0}$. The assumption of $(L_2, \mathbb{H})$-controllability of $\mathbb{E}$ for the pair $(A, B)$ is equivalent to the property that $\mathcal{M}_{u_0}$ is non-empty for all $u_0 \in \mathbb{E}$ that makes the problem of minimization for the quadratic functional \cite{1,5} meaningful. Below, as in \cite{23}, we will show that $\mathcal{M}_{u_0}$ is a closed affine subspace of $\mathcal{Z}$ given by a proper translate of $\mathcal{M}_0$. For the further investigations the key consideration is that under **(REG)** any process through

\footnote{It is clear that (1) implies (2) and (2) implies (3). The only nontrivial case is that (3) implies (1).}
Moreover, if the spectrum of $A$ axis, we consider the value $\omega(u)$.

Theorem 1. We have

\[ \langle u(\cdot), B \xi(\cdot) \rangle \in \mathbb{M}_{u_0}, \]

where the infimum is taken over all $\omega \in \mathbb{R}$, $u \in \mathbb{E}$ and $\xi \in \Xi$ such that $i \omega u = Au + B \xi$.

Moreover, if the spectrum of $A$ does not intersect with a neighbourhood of the imaginary axis, we consider the value

\[ \alpha_3 := \inf_{\omega \in \mathbb{R}} \inf_{\xi \in \Xi} \frac{\mathcal{F}(\lambda A - i \omega I)^{-1} B \xi(\cdot), \xi(\cdot)}{\| \xi(\cdot) \|^2_\Xi}. \]  

The main result of the present paper is the following theorem.

Theorem 1. Let the pair $(A, B)$ be $(L_2, \mathbb{H})$-controllable in $\mathbb{E}$ and $(QF)$, $(REG)$ be satisfied. We have the following:

1. If $\alpha_1 > 0$, then for every $u_0 \in \mathbb{E}$ the quadratic functional \([u(\cdot), u_0, \xi^0(\cdot), \xi(\cdot), u_0, \xi(\cdot)]\) has a unique minimum $\langle u(\cdot), u_0, \xi^0(\cdot), \xi(\cdot), u_0, \xi(\cdot) \rangle \in \mathbb{Z}$.

2. For $V(u) := \langle u, Pu \rangle$ and any $T \geq 0$ we have

\[ V(u(T)) - V(u_0) + \int_0^T \mathcal{F}(u(t), \xi(t))dt \geq 0, \]

where $u(t) = u(t, u_0, \xi)$ is the solution to \([u(\cdot), u_0, \xi^0(\cdot), \xi(\cdot), u_0, \xi(\cdot)]\) with arbitrary $u(0) = u_0 \in \mathbb{E}$ and $\xi(\cdot) \in L_2[0, T; \Xi]$.

Moreover, $\alpha_1 > 0$ is equivalent to $\alpha_2 > 0$ and $\alpha_2 > 0$ is equivalent to $\alpha_3 > 0$ if the latter is well-defined in the above given sense.

In Section 4 we present applications of Theorem 1 concerned with a natural extension of the circle criterion for general class of nonlinear non-autonomous delay equations in feedback form. Moreover, Theorem 1 possibly solves the problem concerned with the unification of Smith’s results, which was posed by the author in [2, 1] in the case of delay systems with discrete delays (see also the discussion after Theorem 4 below).

We note that in the case $\mathbb{E} \neq \mathbb{H}$ any generalizations of the Ricatti equations for $P$, which were useful in the continuous case (see [23, 22, 21, 26]), may not make any sense. Indeed, the expression

\[ 2 \text{Re}(Au + B \xi, Pu) \]

make sense only when $Au + B \xi \in \mathbb{E}$. In the case of delay equations the image of $B$ avoid the space $\mathbb{E}$ (which in this case is the space of continuous functions) and, therefore, $u \in \mathcal{D}(A)$ and $\xi \in \Xi$ in \([1.13]\) cannot be arbitrary as it is usually required. This is also reflected in the fact that the map $t \mapsto u(t) \in \mathbb{E}$ is not differentiable for such equations even for “nice” initial conditions. It is also interesting, whether it is possible to express the optimal control
\( \xi_0(t, u_0) \) in the closed form as \( \xi_0(t, u_0) = Hu_0(t, u_0) \), where \( H \in \mathcal{L}(\mathbb{E}; \Xi) \) is independent of \( u_0 \), analogously to the results in [23, 22, 24]. It seems that ideas from [3] may help in this direction.

It has been a while since it became clear that delay equations can be posed in a proper Hilbert space setting. This is very natural for linear equations [6] and some of these aspects are reflected in the classical studies [18], where they were used implicitly. Well-posedness of general non-linear delay equations in Hilbert spaces was studied by G. F. Webb [31], G. F. Webb and M. Badii [32], M. Faheem and M. R. M. Rao [16]. However, these studies (which are based on the theory of accretive operators) have strong limitations and in many situations it is easier to act in a more concrete way to obtain the well-posedness (for example, if the delay part in the nonlinear term is given by a bounded in \( L_2 \) operator, one can use a standard fixed-point argument). The first reason why one may should be interested in the use of Hilbert spaces corresponding to delay equations is the effective dimension estimates of attractors. For delay systems this is, for example, done by J. Mallet-Paret [25] and by J. W-H. So and J. Wu [30]. The second reason (and this is the topic of the present paper) is the construction of Lyapunov functionals from a general viewpoint using the Hilbert space geometry [14, 13, 23, 22, 24]. Such functionals may serve not only to determine stability or dissipativity properties as they are usually used. It is shown by the author [1, 2], the consideration of such functionals lead to a unification of several papers of R. A. Smith on autonomous and nonautonomous ODEs, delay and parabolic equations, where analogs of the Poincaré-Bendixson theory (for autonomous systems) and convergence theorems (for periodic system) were obtained [29, 28]. Although such a unification leads to a possibility of wider applications (for example, to parabolic equations with boundary controls and their “delayed” versions), some obstacles were encountered in the case of delay systems with discrete delays and this led to the present study. The consideration of delay equations in a proper Hilbert space is not much used in the papers on dynamics of delay equations and usually they are treated in the space of continuous functions (see, for example, the recent monographs on dynamical systems [10, 12]). As to the topic of the present study, there are many papers on delay equations in Lur’e (feedback) form (see, for example, works of P. A. Bliman [7, 8], H. Wenzhang [33], H. Yong and M. Wu [36]), where the existence of very special Lyapunov functionals for very special delay equations was investigated and none of the papers even tried to consider the problem from a general viewpoint. It become known since the papers of V. A. Yakubovich [34] and V. A. Yakubovich and A. L. Likhitarmnikov [23, 22] that the existence of such functionals is linked with the optimization of certain quadratic functionals on affine subspaces of a Hilbert space (in our case this is exactly the functional \( J_F \) in (1.5)). Apparently, their results concerned with mathematical problems of engineering systems are hard to understand for most of both engineers and mathematicians (due to different reasons for each of them) and this is probably the reason that such a strong tool as the frequency theorem for infinite-dimensional systems did not get the attention it deserves, although its finite-dimensional version known as the Kalman-Yakubovich-Popov lemma is widely spread due to its great success in the study of non-linear systems (see, for example, the monographs of A. Kh. Gelig, G. A. Leonov and V. A. Yakubovich [17] or N. V. Kuznetsov and V. Reitmann [20] for wider applications). The advantage of the frequency-domain methods is that they provide conditions (for the existence of certain operators), which can be effectively verified in practice. At least with the use of computer computations.

It seems interesting that for a general linear system corresponding to delay equations the exponential stability can be established using a bounded in \( H \) operator as it is shown
by R. Datko [14, 13]. But it is reasonable that one cannot expect this in the general non-linear case. However, this is the case when all the delays in the non-linear part are given by bounded in $L_2$ operators. It reflects in the boundedness of the form $\mathcal{F}$ (from [1, 4]), which can be used to determine the stability, in $H \times Z$ and this case is covered by the Likhtarnikov-Yakubovich frequency theorem for $C_0$-semigroups [22] or our Theorem [1] with $E := H$.

It is well-known obstacle for the stability theory of infinite-dimensional dynamical systems that some Lyapunov functionals, say $V(u) = \langle Pu, u \rangle$, are not necessarily coercive (in fact, for parabolic problems the operator $P$ obtained from the frequency theorem or certain Lyapunov inequality is compact [3]) and thus abstract stability results cannot be directly applied using this kind of Lyapunov functionals. However, for the case of systems in Lur'e form and the operator $P$ obtained from the frequency theorem, this sometimes can be avoided by using the finiteness of the energy integral to prove boundedness of solutions and then use “smoothing” properties of the corresponding system to prove convergence (see Section [4]).

In [29] R. A. Smith studied the existence of a special class of functionals $(u, Pu)$ for delay equations, extending his results for ODEs, which were close to the ones can be obtained via the Kalman-Yakubovich-Popov lemma. However, the obtained conditions of existence were not satisfactory and this forced him to abandon this approach in his subsequent works [28].

In Section 2 we present a proof of Theorem 1. In Section 3 we establish several corollaries useful for applications. They are, in particular, concerned with the realizability of $P$ and the construction of $j$-dimensional cones in $E$. In Section 4 we derive the circle criterion for general nonlinear non-autonomous delay equations in feedback form and consider its applications to a concrete example of the Goodwin delay equations in $\mathbb{R}^3$.

2. Proofs

Below, we always suppose that the pair $(A, B)$ is $(L_2, \mathbb{H})$-controllable in $E$. From [1, 2] it is clear that $\mathcal{M}_0$ is a closed subspace of $Z = L_2(0, +\infty; \mathbb{H}) \times L_2(0, +\infty; Z)$. Let us show that there exists operator $D \in \mathcal{L}(E; Z)$ such that $\mathcal{M}_{u_0} = \mathcal{M}_0 + Du_0$. Indeed, let $u_0 \in E$ and let $z = z \in \mathcal{M}_0$ be any process through $u_0$. We define $Du_0 := z - \Pi_{\mathcal{M}_0} z$, where $\Pi_{\mathcal{M}_0} : Z \to \mathcal{M}_0$ is the orthogonal projector onto $\mathcal{M}_0$. Note that $z_1 - z_2 \in \mathcal{M}_0$ for any $z_1, z_2 \in \mathcal{M}_{u_0}$ and, therefore, the definition of $Du_0$ is independent on the choice of $z \in \mathcal{M}_{u_0}$. From [1, 1] it is clear that $D$ is closed. Therefore, by the closed graph theorem we have $D \in \mathcal{L}(E; Z)$.

Let us endow the space $\mathcal{Z}_0 := Z_0 \times Z_2$ with the norm

$$\|(u(\cdot), \xi(\cdot))\|_0 := \|(u(\cdot), \xi(\cdot))\|_Z + \|w(0)\|_E.$$ (2.1)
Note that the space $Z_0$ with the above defined norm is not Banach, but it will not cause any problems for us. Note also that since $Du_0 \in \mathcal{M}_{u_0}$, we have in fact that $D\mathbb{E} \subset Z_0$ and the map $D: \mathbb{E} \to Z_0$ is continuous.

**Lemma 1.** Suppose $\alpha_1 > 0$ and let $(QF)$, $(REG)$ be satisfied. Then for every $u_0 \in \mathbb{E}$ there exists a unique minimum $(u^0(\cdot), u^0(\cdot), \xi^0(\cdot), u^0(\cdot))$ of $\mathcal{J}_F$ on $\mathcal{M}_{u_0}$. Moreover, there exists $T \in L(\mathbb{E}; Z)$ with its image in $Z_0$ such that $(u^0(\cdot), u^0(\cdot), \xi^0(\cdot), u^0(\cdot)) = T u_0$ for all $u_0 \in \mathbb{E}$.

**Proof.** Let $u_0 \in \mathbb{E}$ and consider two processes $(u_1(\cdot), \xi_1(\cdot)), (u_2(\cdot), \xi_2(\cdot)) \in \mathcal{M}_{u_0}$. Put $h_u(\cdot) := u_2(\cdot) - u_1(\cdot)$ and $h_\xi := \xi_2(\cdot) - \xi_1(\cdot)$. Note that $(h_u, h_\xi) \in \mathcal{M}_0$. We have

$$\mathcal{J}_F(u_1(\cdot) + h_u(\cdot), \xi_1(\cdot) + h_\xi(\cdot)) - \mathcal{J}_F(u_1(\cdot), \xi_1(\cdot)) =$$

$$= \int_0^\infty 2 \text{Re} \, L(h_u(t), h_\xi(t); u_1(t), \xi_1(t))dt + \int_0^\infty Q(h_u(t), h_\xi(t))dt,$$

where

$$L(h_u, h_\xi; u_1, \xi_1) = (h_u, F_1 u_1)_\mathbb{E} + (h_\xi, F_2 u_1)_\mathbb{E} + (F_3 h_u, \xi_1)_\mathbb{E} + (F_4 h_\xi, \xi_1)_\mathbb{E}$$

and

$$Q(h_u, h_\xi) = (F_1 h_u, h_u)_\mathbb{E} + 2 \text{Re}(F_2 h_u, h_\xi)_\mathbb{E} + (F_3 h_\xi, h_\xi)_\mathbb{E} + (F_4 h_u, F_4 h_\xi)_\mathbb{E}.$$  \hspace{1cm} (2.2)

This in particular shows that $\mathcal{J}_F$ is continuous on $\mathcal{M}_{u_0}$ since $h_u(0) = 0$. Standard argumentation shows that the necessary and sufficient condition for $\mathcal{J}_F$ to attain a minimum at $(u^1(\cdot), \xi^1(\cdot))$ is that

$$\int_0^\infty L(h_u, h_\xi; u_1, \xi_1(t))dt = 0 \text{ for all } (h_u, h_\xi) \in \mathcal{M}_0$$

and

$$\int_0^\infty Q(h_u(t), h_\xi(t))dt \geq 0 \text{ for all } (h_u, h_\xi) \in \mathcal{M}_0.$$  \hspace{1cm} (2.3)

Now let us act in an abstract context. We put $h(\cdot) := (h_u(\cdot), h_\xi(\cdot)) \in \mathcal{M}_0$. For every $z(\cdot) = (u_1(\cdot), \xi_1(\cdot)) \in Z_0$ the left-hand side of (2.5) defines a continuous linear functional on $\mathcal{M}_0$. Therefore, by the Riesz representation theorem there exists an element $Qz \in \mathcal{M}_0$ such that

$$\int_0^\infty L(h(t); z(t))dt = (h, Qz)_Z \text{ for all } h \in \mathcal{M}_0.$$  \hspace{1cm} (2.4)

Clearly, $Q$ is a linear unbounded operator $Z_0 \subset Z \to \mathcal{M}_0$. Since $\mathcal{M}_{u_0} = \mathcal{M}_0 + Du_0$, any $z \in \mathcal{M}_{u_0}$ can be written as $z = z_0 + Du_0$ for some $z_0 \in \mathcal{M}_0$. Therefore, for such $z$ the “orthogonality condition” (2.5) can be written as

$$(h, Qz_0 + QDu_0) = 0 \text{ for all } h \in \mathcal{M}_0,$$  \hspace{1cm} (2.5)

or, equivalently,

$$Qz_0 = -QDu_0.$$  \hspace{1cm} (2.6)

Since the functional in the left-hand side of (2.5) for $(u_1(\cdot), \xi_1(\cdot))$ changed to $z_0$ depend continuously on $z_0 \in \mathcal{M}_0$, the operator $Q_0 = Q|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{M}_0$ is bounded. From (2.7) it is clear that

$$\inf_{z \in \mathcal{M}_0} \frac{(z, Qz)_Z}{(z, z)_Z} = \alpha_1 > 0.$$  \hspace{1cm} (2.8)

In particular, the form of $Q_0$ is real-valued and, consequently, $Q_0$ is a bounded self-adjoint operator having a coercive form. Therefore, the Lax-Milgram theorem guarantees that $Q_0^{-1}$ is bounded and (2.9) has a unique solution $z_0 = -Q_0^{-1} QDu_0$. Since (2.6) is also satisfied,
the optimal process is now given by \( z = z(u_0) = -Q_0^{-1}QD u_0 + Du_0 =: Tu_0 \). Clearly, \( T \) is a linear operator \( \mathbb{E} \to \mathcal{Z} \) (in fact, its image lies in \( \mathcal{Z}_0 = \mathcal{Z}_1^1 \times \mathcal{Z}_2 \)). To show that \( T \) is continuous it is sufficient to show that \( D: \mathbb{E} \to \mathcal{Z}_0 \) and \( Q: \mathcal{Z}_0 \to \mathcal{M}_0 \) are continuous when \( \mathcal{Z}_0 \) is endowed with the norm \((2.4)\).

The operator \( Q: \mathcal{Z}_0 \to \mathcal{M}_0 \) is continuous since the left-hand side of \((2.7)\) is continuous in \( z \in \mathcal{Z}_0 \) uniformly in \( \|h\|_z \leq 1 \). Indeed, let \( z_n \to z \) in \( \mathcal{Z}_0 \) as \( n \to \infty \) then from \((2.7)\) we have
\[
(h, Qz_n - Qz)_z \to 0 \text{ uniformly in } \|h\|_z \leq 1. \tag{2.11}
\]

By the DiniBanach theorem for every \( n \) there exists \( h_n \in \mathcal{M}_0 \) of norm 1 such that \((h_n, Qz_n - Qz) = \|Qz_n - Qz\|_z \). Now the continuity of \( Q \) follows from \((2.11)\) with \( h = h_n \). For the operator \( D \) the continuity is more obvious since it is continuous as an operator \( \mathbb{E} \to \mathcal{Z} \) and, clearly, if \( Du_0 = (u(\cdot), \xi(\cdot)) \) then \( u(0) = u_0 \) that shows the required continuity. Thus the proof is finished. \( \square \)

**Lemma 2.** Let the assumptions of Lemma 1 hold. Then there exists \( P \in \mathcal{L}(\mathbb{E}; \mathbb{E}^*) \) such that for all \( u_0 \in \mathbb{E} \) we have
\[
\langle u_0, Pu_0 \rangle = J_F(u^0(\cdot, u_0), \xi^0(\cdot, u_0)), \tag{2.12}
\]
where \((u^0(\cdot, u_0), \xi^0(\cdot, u_0)) \in \mathcal{M}_{u_0}\) is the optimal process through \( u_0 \).

**Proof.** Let \( z = z(\cdot) \in \mathcal{Z}_0 \) be fixed. Consider the map
\[
\mathbb{E} \ni u_0 \mapsto \int_0^\infty L((Tu_0)(t); z(t))dt, \tag{2.13}
\]
where \( T \) and \( L \) are defined in Lemma 1. Clearly, \((2.13)\) defines a continuous linear functional on \( \mathbb{E} \), which we denote by \( \bar{P}z \). Clearly, \( \bar{P} \) is a linear operator \( \mathcal{Z}_0 \to \mathbb{E}^* \) and as in Lemma 1 one can show that it is continuous in the norm of \( \mathcal{Z}_0 \). Thus for \( z = Tu_0 \) we have
\[
\langle u_0, \bar{P}Tu_0 \rangle = \int_0^\infty L((Tu_0)(t); (Tu_0)(t))dt = J_F(u^0(\cdot, u_0), \xi^0(\cdot, u_0)). \tag{2.14}
\]
Therefore, \( P := \bar{P}T \) satisfies the required property. \( \square \)

The following lemma can be proved in the same way as Lemma 6 in [34].

**Lemma 3.** Let the assumptions of Lemma 1 hold. Then the optimal process \((u^0(\cdot, \cdot), \xi^0(\cdot, \cdot)) \) satisfies the semigroup property, i.e. for all \( u_0 \in \mathbb{E}\) and \( t, s \geq 0 \) we have
\[
u^0(t + s, u_0) = u^0(t, u^0(s, u_0)) \text{ and } \xi^0(t + s, u_0) = \xi^0(t, u^0(s, u_0)). \tag{2.15}
\]

The following lemma is an analog of Lemma 2 from [23]. The essential part of its proof of the second statement is based on application of the Parseval’s identity, which requires only the continuity of the quadratic form (the form of \( Q_0 \) obtained in the proof of Lemma 1) on \( \mathcal{M}_0 \). We omit its proof here.

**Lemma 4.** 1. Suppose that \( \alpha_3 \) from \((1.9)\) is well-defined (i.e. the operator does not have spectrum in a neighborhood of the imaginary axis). If \( \alpha_3 > 0 \) then \( \alpha_2 > 0 \).

2. If \( \alpha_2 > 0 \) then \( \alpha_1 > 0 \).

Now we can finish the proof of Theorem 1...
Proof of Theorem 2. The first part of the theorem follows from Lemma 1 and Lemma 2.

Let \( u(t) = u(t, u_0, \xi) \), where \( t \in [0, T] \), be the solution to (1.1) with \( u(0) = u_0 \in \mathbb{E} \) and \( \xi(\cdot) \in L_2(0, T; \mathbb{H}) \). Consider the process \( (\bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{M}_{u_0}, \) where
\[
\bar{u}(t) = \begin{cases} u(t), & \text{if } t \in [0, T), \\ u^0(t - T, u(T)), & \text{if } t \geq T. \end{cases}
\]
and
\[
\bar{\xi}(t) = \begin{cases} \xi(t), & \text{if } t \in [0, T), \\ \xi^0(t - T, u(T)), & \text{if } t \geq T. \end{cases}
\]
From \( \mathcal{J}_\mathcal{F}(u^0(\cdot, u_0), \xi^0(\cdot, u_0)) \leq \mathcal{J}_\mathcal{F}(\bar{u}(\cdot), \bar{\xi}(\cdot)), \) Lemma 2 and Lemma 3 we have
\[
V(u(T)) - V(u_0) + \int_0^T F(u(t), \xi(t))dt \geq 0.
\]
Thus the second part of the theorem is proved. The remainder follows from Lemma 4. \( \square \)

3. Realification of the Operator \( P \) and Its Consequences

Here we suppose that all the spaces \( \mathbb{H}, \mathbb{E} \) and \( \mathbb{E} \) and corresponding operators \( A, B \) are real. Suppose the form \( F \) is given. We may consider its extension to the complexifications \( \mathbb{H}^C, \mathbb{E}^C \) and \( \mathbb{E}^C \) given by \( \mathcal{F}^C(u_1 + iu_2, \xi_1 + i\xi_2) := F(u_1, \xi_2) + F(u_2, \xi_2) \). We suppose that \( F^C \) has the form as in (1.4).

**Theorem 2.** Let the pair \((A, B)\) be \((L_2, \mathbb{H})\)-controllable in \( \mathbb{E} \) and \((QF), (REG)\) be satisfied (in the real context). Suppose that for \( \alpha_1 \) given by (1.7) with the form \( \mathcal{F}^C \) we have \( \alpha_1 > 0 \). Then there exists \( P \in \mathcal{L}(\mathbb{E}; \mathbb{E}^*) \) such that for \( V(u_0) := \langle u_0, Pu_0 \rangle \) and all \( u_0 \in \mathbb{E} \) we have
\[
V(u(T)) - V(u_0) + \int_0^T F(u(t), \xi(t))dt \geq 0,
\]
where \( \xi(\cdot) \in L_2(0, T; \mathbb{H}) \) is arbitrary and \( u(t) = u(t, u_0, \xi) \).

**Proof.** The conditions of the theorem allows us to apply Theorem 1 to the operators \( A^C \) and \( B^C \), spaces \( \mathbb{H}^C, \mathbb{E}^C, \mathbb{E}^C \) and the quadratic functional corresponding to the form \( \mathcal{F}^C \). Thus, there exists \( P \in \mathcal{L}(\mathbb{E}; \mathbb{E}^*) \) such that
\[
\langle u_1(t) + iu_2(t), \bar{P}(u_1(T) + iu_2(T)) \rangle - \langle u_{0,1} + iu_{0,2}, \bar{P}(u_{0,1} + iu_{0,2}) \rangle + \int_0^T F(u_1(t), \xi_1(t))dt + \int_0^T F(u_2(t), \xi_2(t))dt \geq 0,
\]
where \( u_1(t) = u(t, u_{0,1}, \xi_1) \) and \( u_2(t) = u(t, u_{0,2}, \xi_2) \) is the solutions in real spaces. Note that \( \bar{P} \) can be represented as
\[
\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix},
\]
where \( \bar{P}_{ij} \in \mathcal{L}(\mathbb{E}, \mathbb{E}^*) \) for \( i, j \in \{1, 2\} \) and, moreover, \( \langle u, \bar{P}_{12}u \rangle = \langle u, \bar{P}_{21}u \rangle = 0 \) for all \( u \in \mathbb{E} \) since the form \( \langle u, Pu \rangle \) is real-valued. Putting \( u_2 \equiv 0 \) and \( \xi_2 \equiv 0 \) in (3.2), we get (3.1) for \( \bar{P} := \bar{P}_{11} \). \( \square \)

In applications, it is more convenient to use the frequency condition \( \alpha_3 > 0 \) and consider (3.1) with reversed inequality as well as add a small perturbation to the form \( F \), which do not disturb the frequency condition. We collect this in the following theorem.
Theorem 3. Let the pair \((A, B)\) be \((L_2, \mathbb{H})\)-controllable in \(E\) and \((QF)\), \((REG)\) be satisfied (in the real context). Assume that \(A\) does not have spectrum in a neighborhood of the imaginary axis. Let the frequency-domain condition be satisfied
\[
\sup_{\omega \in \mathbb{R}} \sup_{\xi \in \Xi} \mathcal{F}C(-A - pI)^{-1} B\xi, \xi) := \bar{\alpha}_3 < 0. \tag{3.4}
\]
Then there exists \(P \in \mathcal{L}(E; E^*)\) and a number \(\delta > 0\) such that for \(V(u_0) := \langle u_0, Pu_0 \rangle\) and all \(u_0 \in E\) we have
\[
V(u(T)) - V(u_0) + \int_0^T \mathcal{F}(u(t), \xi(t))dt \leq -\delta \int_0^T (|u(t)|^2_{\mathbb{H}} + |\xi(t)|^2_{\Xi}) dt, \tag{3.5}
\]
where \(\xi(\cdot) \in L_2(0, T; \Xi)\) is arbitrary and \(u(t) = u(t, u_0, \xi)\).
Proof. The result follows after applying Theorem 2 to the form \(\mathcal{F}C + \delta(|u|^2_{\mathbb{H}} + |\xi|^2_{\Xi})\) with any \(\delta < -\bar{\alpha}_3\).

Now suppose that there exists a decomposition of \(E\) into the direct sum of two subspaces \(E^s\) and \(E^n\), i. e. \(E = E^s \oplus E^n\), such that \(u_0 \in E^s\) we have \(\|G(t)u_0\|_{\mathbb{H}} \to 0\) as \(t \to +\infty\) and any \(u_0 \in E^n\) admits a unique extension backwards such that \(G(t)u_0\) is well-defined for \(t \leq 0\) and \(G(t)u_0 \to 0\) as \(t \to -\infty\).

Theorem 4. Suppose that \(A\) is the generator of a \(C_0\)-semigroup in \(\mathbb{H}\) and the space \(E\), continuous embedded in \(\mathbb{H}\), admits the decomposition \(E = E^s \oplus E^n\) as above and \(\dim E^n =: j < \infty\). Consider the operator \(P \in \mathcal{L}(E; E^*)\), the functional \(V(u) := \langle u, Pu \rangle\) and the set \(K := \{u \in E \mid V(u) \leq 0\}\). Let the inequality
\[
V(u(T)) - V(u(0)) \leq -\delta \int_0^T |u(t)|^2_{\mathbb{H}} dt \tag{3.6}
\]
hold for all \(T > 0\), \(u_0 \in E\) and \(u(t) = G(t)u_0\).
Then the set \(K\) is a \(j\)-dimensional cone in \(E\) in the sense that
1) \(K\) is closed;
2) \(\alpha u \in K\) for all \(u \in K\) and \(\alpha \geq 0\);
3) \(u \in K\) if and only if \(-u \in K\);
4) We have
\[
j = \max_{\mathcal{F}} \dim \mathcal{F} =: d(K), \tag{3.7}
\]
where the infimum is taken over all linear subspaces \(\mathcal{F}\) such that \(\mathcal{F} \subset K\).
Proof. Taking \(u_0 \in E^s\) and taking it to the limit as \(T \to +\infty\) in (3.6) we have
\[
V(u_0) \geq \delta \int_0^{+\infty} |u(t)|^2_{\mathbb{H}} dt. \tag{3.8}
\]
Analogously, for \(u_0 \in E^n\) we have
\[
V(u_0) \leq -\delta \int_{-\infty}^0 |u(t)|^2_{\mathbb{H}} dt. \tag{3.9}
\]
Therefore, \(V(u) > 0\) for all \(u \in E^s, u \neq 0\), and \(V(u) < 0\) for all \(u \in E^n, u \neq 0\).
The properties 1), 2), 3) are obvious. Let a subspace \(\mathcal{F} \subset E\) such that \(\mathcal{F} \subset K\) be given. We fix \(k > j\) vectors \(e_1, \ldots, e_k \in E\). Since \(E = E^s \oplus E^n\), for all \(i = 1, \ldots, k\) there exists a unique decomposition
\[
e_i = e_i^s + e_i^n, \tag{3.10}
\]
where \( e_i^s \in \mathbb{E}^s \) and \( e_i^u \in \mathbb{E}^u \). Since \( k > \dim \mathbb{E}^u \) there are constants \( c_i \) such that
\[
\sum_{i=1}^{k} c_i e_i^u = 0. \tag{3.11}
\]
From this we have
\[
\sum_{i=1}^{k} c_i e_i = \sum_{i=1}^{k} c_i e_i^s + \sum_{i=1}^{k} c_i e_i^u = \sum_{i=1}^{k} c_i e_i^s. \tag{3.12}
\]
From (3.12) and since \( \mathbb{F} \subset \mathcal{K} \), we must have
\[
0 \geq V \left( \sum_{i=1}^{k} c_i e_i \right) = V \left( \sum_{i=1}^{k} c_i e_i^s \right) \geq 0. \tag{3.13}
\]
Thus, \( V \left( \sum_{i=1}^{k} c_i e_i^s \right) = 0 \) or, in virtue of (3.8), \( \sum_{i=1}^{k} c_i e_i^s = 0 \). But from (3.12) it follows that \( e_1, \ldots, e_k \) are linearly dependent. Since this holds for any \( k > j \) vectors in \( \mathbb{F} \), we have \( \dim \mathbb{F} \leq j \). Clearly, \( \mathbb{F} = \mathbb{E}^u \) lies in \( \mathcal{K} \) and has dimension \( j \). Thus, \( d(\mathcal{K}) = j \). \( \square \)

Note that if \( j = 1 \) then the cone \( \mathcal{K} \) is convex (and it defines a partial order on \( \mathbb{E} \)). For \( j > 1 \) this no longer true and one considers usually a pseudo-order given by \( \mathcal{K} \). The papers of R. A. Smith also motivated the study of systems (semi-flows), which are monotone w. r. t. such high-rank cones \( \mathcal{K} \). Instead of inequalities like (3.5) in this abstract theory the monotonicity or strict monotonicity of the semi-flow w. r. t. the pseudo-order given by the cone \( \mathcal{K} \) is considered (in applications, the motonicity follows from the inequality [1, 2]).

For ODEs such theory was initiated by L. A. Sanchez [27] and recently L. Feng, Yi Wang and J. Wu [15] extended his ideas to the infinite-dimensional context. It turns out that this abstract monotonicity still leads to some analogs of the Poincaré-Bendixson theory. However, some topological information such as the existence of inertial manifolds and, especially, the existence of orbitally stable orbits, which can be obtained if some inequality like (3.5) is used [1, 2], seems unreachable for this abstract theory.

4. Nonlinear delay equations

Let us demonstrate possible applications to the following delay equation in \( \mathbb{R}^n \):
\[
\dot{x}(t) = \tilde{A}x_t + bf \left( t, \sum_{j=1}^{n} x_j(t - \tau_j) \right) \tag{4.1}
\]
here \( \tau > 0 \) is a positive constant and \( \tau_j \in [0, \tau] \); \( x \in \mathbb{R}^n \), \( x_t = x(t + \cdot) \in C([-\tau, 0]; \mathbb{R}^n) \); \( b \) is a constant \( n \)-vector and \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is a scalar-valued continuous function. We do not put any restrictions on the linear bounded operator \( \tilde{A} : C([-\tau, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n \). In fact, it is given by a \( n \times n \) matrix-valued function of bounded variation \( \tilde{A}(s) \), \( s \in [-\tau, 0] \), as
\[
\tilde{A}\phi = \int_{-\tau}^{0} d\tilde{A}(s)\phi(s). \tag{4.2}
\]
Also there is a \( 1 \times n \)-matrix valued function of bounded variation \( \tilde{C}(s) \) such that
\[
C\phi := \sum_{j=1}^{n} \phi_j(-\tau_j) = \int_{-\tau}^{0} d\tilde{C}(s)\phi(s). \tag{4.3}
\]

For convenience, we consider two types of nonlinearities \( f \). Namely,
(N1) \( f(t,0) = 0 \) for all \( t \in \mathbb{R} \) and for some \( \kappa_1 \leq \kappa_2 \) satisfy the sector condition
\[
\kappa_1 \sigma \leq f(t, \sigma) \leq \kappa_2 \sigma \quad \text{for all} \ t, \sigma \in \mathbb{R}.
\] (4.4)

(N2) There are \( \kappa_1 \leq \kappa_2 \) such that \( f \) satisfy the monotone sector condition
\[
\kappa_1 \leq \frac{f(t, \sigma_1) - f(t, \sigma_2)}{\sigma_1 - \sigma_2} \leq \kappa_2 \quad \text{for all} \ t, \sigma_1, \sigma_2 \in \mathbb{R} \text{ with } \sigma_1 \neq \sigma_2.
\] (4.5)

From the classical theory [18] it follows that for every \( \phi_0 \in C([\tau, 0]; \mathbb{R}^n) \) and \( t_0 \in \mathbb{R} \) there exists a unique continuous function \( x(\cdot) = x(\cdot, t_0, \phi_0): [\tau + t_0, +\infty) \to \mathbb{R}^n \), which is continuously differentiable for \( t \geq t_0 \) (in the sense that \( x(\cdot) \in C^1([t_0, +\infty); \mathbb{R}^n) \)), satisfies (4.4) for \( t \geq t_0 \) and starts from \( \phi_0 \), i.e. \( x(s) = \phi_0(s) \) for \( s \in [\tau + t_0, t_0] \). Our aim is to provide conditions under which all the solutions approach zero as \( t \to +\infty \) in the case of (N1) or converge to a unique bounded solution in the case of (N2). In fact these conditions will include some kind of uniform stability of the entire system. This gives an extension of results of V. A. Yakubovich from [35], where finite-dimensional systems (however, with a possible discontinuous nonlinearity \( f \)) are studied. In particular, in the case of (N2) the unique bounded solution is almost periodic provided that \( f \) is so (and in fact it has the same frequencies). Here we do not discuss this case in details and refer to the classical book of B. M. Levitan and V. V. Zhikov [21] and also to D. N. Cheban [11] for recent developments in the general theory of oscillations. Note that within the conditions of the Yakubovich result it is possible to estimate the fractal dimension of the attracting almost periodic solution [4] and this seems hard to achieve in our case since the Lyapunov function \( V(u) = \langle u, Pu \rangle \) in our case is not coercive. Our methods can be extended for a larger class of systems, which includes several nonlinearities.

Now put \( \mathbb{H} := \mathbb{R} \times L_2(-\tau, 0; \mathbb{R}^n), \mathbb{Z} := \mathbb{R} \). Consider \( A: D(A) \subset \mathbb{H} \to \mathbb{H} \) defined as
\[
\begin{bmatrix} x \\ \phi \end{bmatrix} \mapsto \begin{bmatrix} \tilde{A} \phi \\ \frac{d}{dt} \phi \end{bmatrix}
\] (4.6)

for \( (x, \phi) \in D(A) := \{(x, \phi) \in \mathbb{H} \mid \phi(0) = x, \phi \in W^{1,2}(-\tau, 0; \mathbb{R}^n) \} \). It can be shown that \( A \) is a closed operator, which generates a \( C_0 \)-semigroup \( G(t), t \geq 0, \) in \( \mathbb{H} \) (see Theorem 3.23 from [3]). Define the operator \( B \in \mathcal{L}(\mathbb{Z}; \mathbb{H}) \) as \( B \xi := (b_\xi, 0) \). With (4.1) we associate the control system
\[
\dot{u} = Au + B \xi.
\] (4.7)

Let us consider \( \mathbb{E} := C([-\tau, 0]; \mathbb{R}^n) \) and its embedding in \( \mathbb{H} \) as \( \phi \mapsto \langle \phi(0), \phi \rangle \). By Theorem 1 in Section 6 from [?] we have (REG) satisfied. Consider the operator \( C: \mathbb{E} \to \mathbb{R} \) defined in (4.3) and the quadratic form \( F(u, \xi) \), where \( u \in \mathbb{E}, \xi \in \mathbb{Z} \), defined as
\[
F(u, \xi) := (\xi - \kappa_1 Cu)(\kappa_2 Cu - \xi)
\] (4.8)

Its extension for \( u \in \mathbb{E}^C, \xi \in \mathbb{Z}^C \) is given by
\[
F^C(u, \xi) := \text{Re} \left[ (\xi - \kappa_1 Cu)^*(\kappa_2 Cu - \xi) \right].
\] (4.9)

Clearly, \( F^C \) has the form (1.4) with \( F_1 \equiv 0, F_2 u := \frac{1}{4}(x_2 Cu - x_1 Cu), F_3 \xi := \xi, F_4 u = Cu \) and \( F_5 := -x_1 x_2 I \). Now if \( u(\cdot) = (x(\cdot), \phi(\cdot)) \in Z^I = L_2(0, +\infty; \mathbb{H}) \cap C([0, +\infty); \mathbb{E}) \), we
Lemma 5. Let $M$ have for some constant $M_0 > 0$

$$\int_0^\infty \left| (Cu)(t) \right|^2 dt = \int_0^\infty \left| \sum_{j=1}^n x_j(t-\tau_j) \right|^2 dt \leq n \sum_{j=1}^n \int_0^{\tau_j} |x_j(t-\tau_j)|^2 dt + n \sum_{j=1}^n \int_0^\infty |x_j(t)|^2 dt \leq M_0 \left( \|u(\cdot)\|^2_{L^2_\tau(0,+\infty;\mathbb{R})} + \|u(\cdot)\|^2_{L^2_\tau(0,+\infty;\mathbb{R})} \right).$$

(4.10)

that verifies (QF). Now we consider the transfer function of the triple $(A^C, B^C, C^C)$, which is given for $p \in \mathbb{C} \setminus \sigma(A)$ by

$$W(p) = C^C(A^C - pI)^{-1} B^C.$$  

(4.11)

Note that since $C^C(A^C - pI)^{-1} B^C$ is a linear operator from $\mathbb{E}^C = \mathbb{C}$ to $\mathbb{C}$, it may be identified with a complex number. This is how one should understand the equality in (4.11).

The spectrum of the operator $A$ is determined by the solutions $p \in \mathbb{C}$ to the characteristic equation

$$\det \left( pI - \int_{-\tau}^0 d\bar{A}(s)e^{ps} \right) = 0.$$  

(4.12)

We are also interested in the characteristic equation corresponding to the linear system obtained from (4.1) with $f(t, \sigma) = \kappa_0 \sigma$ for some fixed $\kappa_0 \in [\kappa_1, \kappa_2]$:

$$\det \left( pI - \int_{-\tau}^0 d\bar{A}(s)e^{ps} - \kappa_0 b \int_{-\tau}^0 d\bar{C}(s)e^{ps} \right) = 0.$$  

(4.13)

Now we can formulate two circle criteria for (4.1).

**Theorem 5.** Suppose that there exists $\kappa_0 \in [\kappa_1, \kappa_2]$ such that equation (4.13) as well as (4.12) do not have roots with $\Re p \geq 0$. Let the frequency-domain condition

$$\Re \left[ (1 + \kappa_1 W(i\omega))^*(1 + \kappa_2 W(i\omega)) \right] > 0 \text{ for all } \omega \in [-\infty, +\infty].$$  

(4.14)

be satisfied. Then

1. If (N1) holds, then every solution of (4.1) approach zero as $t \to +\infty$ and the zero is uniformly Lyapunov stale.
2. If (N2) holds, then the entire system is positively uniformly Lyapunov stable. If at least one solution is bounded on $\mathbb{R}$, then every solution of (4.1) approach a unique bounded solution as $t \to +\infty$. This bounded solution is stationary, periodic or almost periodic provided that $f$ is so.

Before giving a proof, we have to establish several lemmas. The following lemma is a consequence of the fact that the operator $\frac{d}{ds}$ generates a $C_0$-semigroup of left shifts in $L_2(-\tau, 0; \mathbb{R}^n)$ and has the domain $W^{1,2}(-\tau, 0; \mathbb{R}^n)$.

**Lemma 5.** Let $x: [-\tau, T] \to \mathbb{R}^n$ be a classical solution to

$$\dot{x} = f(t, x_t),$$  

(4.15)

where $f: \mathbb{R} \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is continuous. Suppose that $x(s) = \phi_0(s)$ for $s \in [-\tau, 0]$ and $\phi_0 \in W^{1,2}(-\tau, 0; \mathbb{R}^n)$. Then $x_t \in W^{1,2}(-\tau, 0; \mathbb{R}^n)$, $t \mapsto x_t$ is in $C^1(0, T; L_2(-\tau, 0; \mathbb{R}^n))$ and for all $t \in (0, T)$ we have

$$\frac{d}{dt} x_t = \frac{d}{ds} x_t \text{ in } L_2(-\tau, 0; \mathbb{R}^n).$$  

(4.16)
We say that \( u \colon [t_0, T] \to \mathbb{H} \) corresponds to a classical solution \( x(\cdot) = x(\cdot, t_0, \phi_0) : [-\tau + t_0, T] \to \mathbb{R}^n \) of (4.1) if \( u(t) = (x(t), x_t) \) for all \( t \in [t_0, T] \). If in this case we have \( \phi_0 \in W^{1,2}(-\tau, 0; \mathbb{R}^n) \), then Lemma 5 guarantees that for all \( t \in (t_0, T) \) the equation

\[
\dot{u}(t) = Au(t) + Bf(Cu(t))
\]

(4.17)
is satisfied.

**Lemma 6.** Let the base hypotheses of Theorem 5 be satisfied. Then there exists an operator \( P \in \mathcal{L}(\mathbb{E}; \mathbb{E}^*) \) such that \( V(u) = \langle u, Pu \rangle > 0 \) for all \( u \in \mathbb{E}, u \neq 0 \), and the inequalities

\[
\text{Case of (N1): } V(u(T)) - V(u(t_0)) \leq -\delta \int_{t_0}^{T} |u(t)|_E^2 dt
\]

(4.18)

\[
\text{Case of (N2): } V(u_1(T) - u_2(T)) - V(u_1(t_0) - u_2(t_0)) \leq -\delta \int_{t_0}^{T} |u_1(t) - u_2(t)|_E^2 dt
\]

are satisfied for all \( t_0 < T \) and all \( u, u_1, u_2 : [t_0, T] \to \mathbb{H} \) corresponding to classical solutions of (4.1).

**Proof.** For simplicity of notation we consider only the case \( t_0 = 0 \). Let us apply Theorem 3 for the pair \((A, B)\) and the form \( \mathcal{F} \). The checking of \( L_2 \)-controllability is not required since \( A \) generates an exponentially table \( C_0 \)-semigroup due to our assumptions. Note that condition (4.14) in is exactly the same as (3.1). Thus, there is operator \( P \in \mathcal{L}(\mathbb{E}; \mathbb{E}^*) \) such that for \( V(u) := \langle u, Pu \rangle \) we have

\[
V(u(T)) - V(u(0)) + \int_{0}^{T} \mathcal{F}(u(t), \xi(t)) dt \leq -\delta \int_{0}^{T} (|u(t)|_E^2 + |\xi(t)|^2) dt
\]

(4.19)

for all \( \xi(\cdot) \in L_2(0, T; \Xi) \) and \( u(t) = u(t, u(0), \xi) \), which is a solution to (4.7).

Putting \( \xi(t) = \kappa_0 u(t) \) in (4.19) (this is possible since we can take the solutions \( u(t) \) corresponding to the operator \( A + \kappa_0 BC \)), we have

\[
V(u(T)) - V(u(0)) \leq -\delta \int_{0}^{T} |u(t)|_E^2 dt.
\]

(4.20)

for any solution \( \dot{u}(t) = (A + \kappa_0 BC)u(t) \) with \( u(0) \in \mathbb{E} \). Since we assumed that the operator \( A + \kappa_0 BC \) generates an exponentially stable semigroup, we must have \( V(u(T)) \to +\infty \) as \( T \to +\infty \). This implies that

\[
V(u_0) \geq \delta \int_{0}^{+\infty} |u(t)|_E^2 dt
\]

(4.21)

and, consequently, the functional \( V \) is positive.

Let \( x : [-\tau, +\infty) \to \mathbb{R}^n \) be a classical solution of (4.1) with initial condition \( \phi_0 \in W^{1,2}(-\tau, 0; \mathbb{R}^n) \). Consider \( u(t) := (x(t), x(t + \cdot)) \in \mathbb{R} \). By Lemma 3 we have the equality

\[
\dot{u}(t) = Au(t) + Bf(Cu(t))
\]

(4.22)

If we put \( \xi(t) := f(Cu(t)) \) then we get that \( u(t) = u(t, u(0), \xi), \) i.e. \( u(t), \xi(t) \) satisfy (4.19). From this we deduce in the case of (N1) the inequality

\[
V(u(T)) - V(u(0)) \leq -\delta \int_{0}^{T} |u(t)|_E^2 dt
\]

(4.23)
for all \( u(\cdot) \), which corresponds to classical solutions of (4.1). In the case of (N2) we put
\[
\xi(t) := f(Cu_1(t)) - f(Cu_2(t))
\]
and \( u(t) := u_1(t) - u_2(t) \) to get
\[
V(u_1(T) - u_2(T)) - V(u_1(0) - u_2(0)) \leq -\delta \int_0^T |u_1(t) - u_2(t)|^2 dt
\]
(4.24)
for \( u_1(\cdot), u_2(\cdot) \), which corresponds to any classical solutions of (4.1).

\[\text{Lemma 7.}\]
Let the base hypotheses of Theorem 2 be satisfied. There exists a constant \( M > 0 \) such that any classical solutions \( x(\cdot) = x(\cdot, t_0, \phi_0), x_1(\cdot) = x_1(\cdot, t_0, \phi_1, 0), x_2(\cdot) = x_2(\cdot, t_0, \phi_2, 0) \)
for all \( t \geq t_0 \) satisfy
\[
\begin{align*}
\text{Case of (N1)}: & \quad \|x\|_{\mathbb{H}} \leq M\|\phi_0\|_{\mathbb{H}}, \\
\text{Case of (N2)}: & \quad \|x_{1,t} - x_{2,t}\|_{\mathbb{H}} \leq M\|\phi_{1,0} - \phi_{2,0}\|_{\mathbb{H}}.
\end{align*}
\]
(4.25)

\[\text{Proof.}\]
Indeed, from Lemma 6 in the case of (N1) we have
\[
\int_{t_0}^{+\infty} |u(t)|^2 dt \leq \delta^{-1} V(u(t_0)).
\]
(4.26)
where \( u(\cdot) \) corresponds to \( x(\cdot) \). By Lemma 5 we have that
\[
u(t) = G(t)u_0 + \int_{t_0}^{t} G(t-s)Bf(C(u(s)))ds.
\]
(4.27)
Put \( \xi(t) := f(Cu(t)) \) for \( t \geq t_0 \). From (4.10) and (4.26) we have for \( \nu := \max |x_1|, |x_2| \)
\[
\int_{t_0}^{+\infty} |\xi(t)|^2 dt \leq \nu^2 \int_{t_0}^{+\infty} |Cu(t)|^2 dt \leq \nu^2 M_0\|\phi_0\|_{\mathbb{H}} + \delta^{-1} V(u(t_0)) \leq M_1\|\phi_0\|_{\mathbb{H}},
\]
(4.28)
where \( M_1 > 0 \) is a proper constant. Since the semigroup \( G(t) \) is exponentially stable, there
are constant \( M_2, \varepsilon > 0 \) such that \( |G(t)v|_{\mathbb{H}} \leq M_2e^{-\varepsilon t}\|v\|_{\mathbb{H}} \) for all \( v \in \mathbb{H} \) and all \( t \geq 0 \). Now
from (4.27) one can easily deduce the first inequality in (4.25). The second inequality can
be proved analogously.

\[\text{Proof of Theorem 4.}\]
In the case of (N1) we have the uniform Lyapunov stability of the zero solution, which is given by Lemma 4.25. Let \( x(\cdot) = x(\cdot, t_0, \phi_0) \) be any classical solution. Again by Lemma 4.25 the function \( x(\cdot) \) is bounded. Due to (4.26) there exists a sequence
\( t_k \to +\infty \) as \( k \to +\infty \) such that \( x(t_k) \to 0 \). From this we in fact may assume that
\( \|x_{1,t}\|_{\mathbb{H}} \to 0 \) as \( k \to +\infty \). But this along with the Lyapunov stability gives the convergence
\( x(t) \to 0 \) as \( t \to +\infty \).

In the case of (N2) the positive uniform Lyapunov stability by Lemma 4.25. Let \( x_1(\cdot) = x(\cdot, 0, \phi_{1,0}) \) and \( x_2(\cdot) = x(\cdot, 0, \phi_{2,0}) \) be two bounded on \( \mathbb{R} \) classical solutions. Since for the corresponding functions \( u_1(t) = (x_1(t), x_{1,t}), u_2(t) = (x_2(t), x_{2,t}) \) the right-hand side
\[
\int_{t_0}^{+\infty} |u_1(t) - u_2(t)|^2 dt \leq \delta^{-1} V(u_1(t_0) - u_2(t_0))
\]
(4.29)
is bounded in \( t_0 \in \mathbb{R} \), there must exist a sequence \( t_k \to +\infty \) such that \( x_1(t_k) - x_2(t_k) \to 0 \)
and in fact \( \|x_{1,t} - x_{2,t}\|_{\mathbb{H}} \to 0 \) as \( k \to +\infty \). But in virtue of the uniform positive Lyapunov stability this means that \( x_1 \equiv x_2 \). Therefore there is a unique bounded solution \( x^*(\cdot) = x(\cdot, 0, \phi_0) \). As above we can show that any other classical solution \( x(t) \) is attracted by \( x^* \) such that \( \|x^* - x(\cdot)\|_{\mathbb{H}} \to 0 \) as \( t \to +\infty \). Its almost periodicity (if \( f \) is almost periodic) can be
proved in a standard way and follows from the first theorem of Favard [21]. If \( f \) is periodic
with period \( T \) then \( x^*(\cdot + T) \) is also a classical solution bounded on \( \mathbb{R} \). By the established
uniqueness, $x^*(\cdot + T) = x^*(\cdot)$, i. e. $x^*$ is periodic with the same period as $f$. The proof is finished. □

Let us consider a more concrete example, which requires different representations of the system in Lur’e form. Namely, we will study the following delay equations in $\mathbb{R}^3$ called Goodwin delay equations:

$$\begin{align*}
\dot{x}_1(t) &= -\lambda x_1(t) + g(x_3(t-\tau)), \\
\dot{x}_2(t) &= -\lambda x_2(t), \\
\dot{x}_3(t) &= -\lambda x_3(t),
\end{align*}$$

(4.30)

where $\lambda, \tau > 0$ are parameters. Equations (4.30) serves as a model for certain chemical reactions concerned with the protein synthesis. Thus, $x_j$’s represent chemical concentrations, which are non-negative, and we are interested in the dynamics of (4.30) in the cone of non-negative functions $C^+ := \{ \phi \in C([-\tau, 0]; \mathbb{R}^3) : \phi_j \geq 0, j = 1, 2, 3 \}$, which is clearly invariant w. r. t. the solutions of (4.30). Let us fix $g : \mathbb{R} \to \mathbb{R}$ as

$$g(\eta) = \frac{1}{1 + |\eta|^3}. \quad (4.31)$$

In [28] R. A. Smith obtained for (4.30) the conditions of existence of an orbitally stable periodic solution (see also [1, 2], where Smith’s approach is treaten by the present author from a general viewpoint and its links with various versions of the frequency theorem are established). These conditions, in particular, require that the unique stationary point must be Lyapunov unstable. For a more general region in the space of parameters, which includes the case of Lyapunov stable stationary point, he obtained the Poincaré-Bendixson dichotomy (i. e. the convergence of solutions either to the stationary point or to a periodic orbit). However, in the stability region the existence of periodic orbits is not guaranteed. It is interesting to obtain conditions under which there are indeed no periodic orbits and all the solutions converge to the stationary point.

It is straightforward to verify that $-2\sqrt{2}/3 \leq g'(\sigma) \leq 0$ for $\sigma \geq 0$. Let $\hat{g}$ be a function, which coincides with $g$ on $\mathbb{R}_+$ and it is smoothly extended to $\mathbb{R}$ with preserving the inequality $-2\sqrt{2}/3 \leq \hat{g}(\sigma) \leq 0$ for all $\sigma \in \mathbb{R}$.

Let $\rho \geq 0$. Along with (4.30) we also consider the systems

$$\begin{align*}
\dot{x}_1(t) &= -\lambda x_1(t) - \rho x_3(t-\tau) + (\hat{g}(x_3(t-\tau)) + \rho x_3(t-\tau)), \\
\dot{x}_2(t) &= -\lambda x_2(t), \\
\dot{x}_3(t) &= -\lambda x_3(t),
\end{align*}$$

(4.32)

where the nonlinearity $g_\rho(\sigma) = \hat{g}(\sigma) + \rho \sigma$ satisfies (N2) as

$$-2\sqrt{2}/3 + \rho =: \kappa_2 \leq g_\rho'(\sigma) \leq \kappa_1 := \rho. \quad (4.33)$$

Let $A_\rho : \mathcal{D}(A_\rho) \to \mathbb{H}$ and $B$ be the linear operator corresponding to the linear part of (4.32). Its spectrum is given by the roots of equation

$$\lambda^3 + \rho e^{-\tau \lambda} = 0. \quad (4.34)$$

Let $\theta = \theta(\tau, \lambda)$ be the unique solution to $\tau \lambda \tan(\theta) = \pi - 3\theta$. In [28] (see also [1]) it is shown that for $0 \leq \rho < (\lambda \sec(\theta(\lambda, \tau)))^3$ the roots of (4.34) have negative real parts. The transfer
Figure 1. A numerically obtained region in the space of parameters $(\tau, \lambda)$ of system (4.30) with $g(\sigma) = (1 + |\sigma|^3)^{-1}$, for which the conditions of Theorem 6 are satisfied and therefore the global asymptotic stability of the unique stationary point holds. Compare with Fig. 2 from [28] and Fig. 2 from [1].

function is given by

$$W_\rho(p) = -\frac{1}{(\lambda + p)^3 e^{\tau p} + \rho}. \quad (4.35)$$

Theorem 6. Suppose there is $\rho \in [0, 2\sqrt{2}/3]$ such that $0 \leq \rho < (\lambda \sec(\theta(\lambda, \tau)))^3$ and

$$\text{Re} \left[ (1 + (-2\sqrt{2}/3 + \rho)W_\rho(i\omega)^* (1 + \rho W_\rho(i\omega)) \right] > 0 \text{ for all } \omega \in [-\infty, +\infty]. \quad (4.36)$$

is satisfied. Then any solution to (4.30), which starts in $C^+$, approaches zero as $t \to +\infty$ and the zero is Lyapunov stable w. r. t. $C^+$.

Proof. We apply Theorem 5 to (4.32), the pair $(A_\rho, B)$, the form $F$ as in (4.8) and $\kappa_0 = 0 \in [\kappa_1, \kappa_2] = [-2\sqrt{2}/3 + \rho; \rho]$. The nonlinearity $g_\rho$ satisfies (N2) and the stationary point in $C^+$ plays the role of a unique bounded solution. Clearly, dynamics of (4.32) and (4.30) coincide on $C^+$. Thus, the conclusion follows from Theorem 5.

On Fig. 1 we present a numerically obtained region in the spaces of parameters, which is given by Theorem 6 and therefore gives the uniform stability and convergence to the unique stationary solution. It indeed contains a large part of the region displayed at Fig. 2 from [1], for which the Poincaré-Bendixson dichotomy was established in [28], as well as reveals a new region with uniform stability and convergence, which was not discovered later. Thus, for these parameters we strengthened the result.

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