Double Reinforcement Learning for Efficient Off-Policy Evaluation in Markov Decision Processes

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Abstract

Off-policy evaluation (OPE) in reinforcement learning allows one to evaluate novel decision policies without needing to conduct exploration, which is often costly or otherwise infeasible. We consider for the first time the semiparametric efficiency limits of OPE in Markov decision processes (MDPs), where actions, rewards, and states are memoryless. We show existing OPE estimators may fail to be efficient in this setting. We develop a new estimator based on cross-fold estimation of q-functions and marginalized density ratios, which we term double reinforcement learning (DRL). We show that DRL is efficient when both components are estimated at fourth-root rates and is also doubly robust when only one component is consistent. We investigate these properties empirically and demonstrate the performance benefits due to harnessing memorylessness efficiently.

1 Introduction

Off-policy evaluation (OPE) is the problem of estimating mean rewards of a given policy (target policy) for a sequential decision-making problem using data generated by the log of another policy (behavior policy). OPE is a key problem in reinforcement learning (RL) (Jiang and Li, 2016; Li et al., 2015; Liu et al., 2018; Mahmood et al., 2014; Munos et al., 2016; Precup et al., 2000; Thomas and Brunskill, 2016) and it finds applications as varied as healthcare (Murphy, 2003) and education (Mandel et al., 2014). Because data can be scarce, it is crucial to use all available data efficiently, while at the same time using flexible, nonparametric estimators that avoid misspecification error.

In this paper, our goal is to obtain an estimator for policy value with minimal asymptotic mean squared error under nonparametric models for the sequential decision process and behavior policy, that is, achieving the semiparametric efficiency bound (Bickel et al.).

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Figure 1: $\mathcal{M}_1$: Non-Markov decision process (NMDP)

Figure 2: $\mathcal{M}_2$: Markov decision process (MDP)

\[
\text{EffBd}(\mathcal{M}_2) = \text{EffBd}(\mathcal{M}_{2b}) > \text{EffBd}(\mathcal{M}_{2q})
\]
\[
\wedge \quad \wedge
\]
\[
\text{EffBd}(\mathcal{M}_1) = \text{EffBd}(\mathcal{M}_{1b}) > \text{EffBd}(\mathcal{M}_{1q})
\]

Figure 3: Relationship between the semiparametric efficiency bounds in each model. $\mathcal{M}_1$, $\mathcal{M}_2$ are, respectively, NMDP and MDP with unknown behavior policy. $\mathcal{M}_{1b}$, $\mathcal{M}_{2b}$ are with known behavior policy. $\mathcal{M}_{1q}$, $\mathcal{M}_{2q}$ are with parametric assumptions on the $q$-functions. Inequalities are generically strict (see Theorem 3).

1998). Toward that end, we explore the efficiency bound and efficient influence function of the target policy value under two models: non-Markov decision processes (NMDP) and Markov decision processes (MDP). The two models are illustrated in Figs. 1 and 2 and defined precisely in Section 1.1. While much work has studied efficient estimation under $\mathcal{M}_1$ (Dudik et al., 2014; Jiang and Li, 2016; Kallus and Uehara, 2019; Thomas and Brunskill, 2016), work on $\mathcal{M}_2$ has been restricted to the parametric, finite-state-finite-action case (Jiang and Li, 2016) and no globally efficient estimators have been proposed. The two models are clearly nested and indeed we obtain that the efficiency bounds are generally strictly ordered (see Fig. 3). In other words, if we correctly leverage the Markov property, we can obtain OPE estimators that are more efficient than existing ones. This is quite important, given the practical difficulty of evaluation in long horizons (see, e.g., Gottesman et al., 2019) and given that many RL problems are Markovian.

We propose the Double Reinforcement Learning (DRL) estimator, which is given by by cross-fold estimation and plug-in of the $q$- and density ratio functions into the efficient influence function for each model, which we derive for the first time here. We show that DRL achieves the semiparametric efficiency bound globally even when these nuisances are
estimated at slow fourth-root rates and without restricting to Donsker or bounded entropy classes, enabling the use of machine learning method for the nuisance estimation in the spirit of Chernozhukov et al. (2018). Further, we show that DRL is consistent even if only some of the nuisances are consistently estimated, known as double robustness. To the best of our knowledge, this is the first proposed estimator shown to be globally efficient for OPE in MDPs.

The organization of the paper is as follows. In Section 1.1 we define the OPE problem and our models. In Section 1.2 we summarize semiparametric inference theory and in Section 1.3 we review the literature on OPE. In Section 2 we calculate the efficient influence functions and efficiency bounds in each of our models. In Section 3 we propose the DRL estimator and prove its efficiency and robustness in each model, while also reviewing the inefficiency of other estimators. In Section 4 we discuss how to estimate q-functions in an off-policy manner to be used in DRL as well as the efficiency bound under parametric assumptions on the q-function. In Section 5 we demonstrate the benefits of DRL empirically.

1.1 Problem Setup

A (potentially) non-Markov decision process (NMDP) is given by a sequence of state and action spaces \( S_t, A_t \) for \( t = 0, 1, \ldots, T \), an initial state distribution \( P_{s_0}(s_0) \), transition probabilities \( P_{s_t}(s_t \mid \mathcal{H}_{a_{t-1}}) \) for \( t = 1, \ldots, T \), and emission probabilities \( P_{r_t}(r_t \mid \mathcal{H}_{a_t}) \) for \( t = 0, \ldots, T \), where \( \mathcal{H}_{a_t} = (s_0, a_0, \ldots, s_t, a_t) \) is the state-action history up to \( a_t \). A (non-anticipatory) policy is a sequence of action probabilities \( \pi_t(a_t \mid \mathcal{H}_{s_t}) \), where \( \mathcal{H}_{s_t} = (s_0, a_0, \ldots, a_{t-1}, s_t) \) is the state-action history up to \( s_t \). Together, an NMDP and a policy define a joint distribution over trajectories \( \mathcal{H} = (s_0, a_0, r_0, s_1, a_1, r_1, \ldots, s_T, a_T, r_T) \), given by the product \( P_{s_0}(s_0)\pi_0(a_0 \mid \mathcal{H}_{s_0})P_{r_0}(r_0 \mid \mathcal{H}_{a_0})P_{s_1}(s_1 \mid \mathcal{H}_{s_0}) \cdots P_{r_T}(r_T \mid \mathcal{H}_{a_T}) \). The dependence structure of such a distribution is illustrated in Fig. 1. We denote this distribution by \( P_\pi \) and expectations in this distribution by \( E_\pi \) to highlight the dependence on \( \pi \).

A Markov decision process (MDP) is an NMDP where transitions and emissions only depend only on the recent state and action, \( P_{s_t}(s_t \mid \mathcal{H}_{a_{t-1}}) = P_{s_t}(s_t \mid s_{t-1}, a_{t-1}) \) and \( P_{r_t}(r_t \mid \mathcal{H}_{a_t}) = P_{r_t}(r_t \mid s_t, a_t) \), and where we restrict to policies that depend only on the recent state, \( \pi_t(a_t \mid \mathcal{H}_{s_t}) = \pi_t(a_t \mid s_t) \). MDPs have the important property that they are memoryless: given \( s_t \), the trajectory starting at \( s_t \) is independent of the past trajectory, so that \( s_t \) fully captures the current state of the system. This imposes a stricter dependence structure, which is illustrated in Fig. 2. In particular, connections between variables with different time indices occurs only via \( s_t \).

Our ultimate goal is to estimate the average cumulative reward of a policy,

\[
\rho_\pi = E_\pi \left[ \sum_{t=0}^{T} r_t \right].
\]

The quality and value functions (\( q \)- and \( v \)-functions) are defined as the following conditional averages of the cumulative reward to go, respectively:

\[
q_t(\mathcal{H}_{a_t}) = E_\pi \left[ \sum_{k=t}^{T} r_k \mid \mathcal{H}_{a_t} \right], \quad v_t(\mathcal{H}_{s_t}) = E_\pi \left[ \sum_{k=t}^{T} r_k \mid \mathcal{H}_{s_t} \right] = E_\pi \left[ q_t(\mathcal{H}_{a_t}) \mid \mathcal{H}_{s_t} \right].
\]
Note that the very last expectation is taken only over $a_t \sim \pi_t(a_t \mid \mathcal{H}_{s_t})$. For MDPs, we have $q_t(\mathcal{H}_{a_t}) = q_t(s_t, a_t)$ and $v_t(\mathcal{H}_{s_t}) = v_t(s_t) = \mathbb{E}_\pi[q_t(s_t, a_t) \mid s_t]$, where again the last expectation is taken only over $a_t \sim \pi_t(a_t \mid s_t)$. For brevity, we define the random variables $q_t = q_t(\mathcal{H}_{a_t})$, $v_t = v_t(\mathcal{H}_{s_t})$.

The off-policy evaluation (OPE) problem is to estimate the average cumulative reward of a given (known) target evaluation policy, $\pi^e$, using $n$ observations of trajectories $\mathcal{D} = \{\mathcal{H}^{(1)}, \ldots, \mathcal{H}^{(n)}\}$ independently generated by the distribution $P_{\pi^b}$ induced by using another policy, $\pi^b$, in the same decision process. This latter policy, $\pi^b$, is called the behavior policy and it may be known or unknown.

A model for the data generating process $P_\theta$ of $\mathcal{D}$ is given by the set of products $P_{s_0}(s_0)\pi^b_0(a_0 \mid \mathcal{H}_{s_0})P_{r_0}(r_0 \mid \mathcal{H}_{a_0})P_{s_1}(s_1 \mid \mathcal{H}_{a_0})\cdots P_{r_T}(r_T \mid \mathcal{H}_{a_T})$ over some possible values for each probability distribution in the product. We let $\mathcal{M}_1$ denote the nonparametric model where each distribution is unknown and free. We let $\mathcal{M}_{1b}$ denote the submodel of $\mathcal{M}_1$ where $\pi^b$ is known and fixed. We let $\mathcal{M}_{tb}$ denote any submodel of $\mathcal{M}_1$ where the functions $q_t(\mathcal{H}_{a_t})$ are restricted parametrically for $t = 0, \ldots, T$. We let $\mathcal{M}_2$, $\mathcal{M}_{2b}$, $\mathcal{M}_{2q}$ denote the corresponding models where both the decision process and the behavior policy are restricted to be Markovian. Since $\pi^e$ is given, the parameter of interest, $\rho^e$, is a function of just the part that specifies the decision process (initial state, transition, and emission probabilities).

To streamline notation, when no subscript is denoted, all expectations $E[\cdot]$ and variances $\text{var}[\cdot]$ are taken with respect to the behavior policy, $\pi^b$. At the same time, all $v$- and $q$-functions are for the target policy, $\pi^e$. The $L^p$-norm is defined as $\|g\|_p = \mathbb{E}[|g|^p]^{1/p}$. For any function of trajectories, we define its empirical average as

$$E_n[f(\mathcal{H})] = n^{-1} \sum_{i=1}^{n} f(\mathcal{H}^{(i)}).$$

We denote the density ratio at time $t$ between the target and behavior policy by

$$\eta_t(\mathcal{H}_{a_t}) = \frac{\pi^e_t(a_t \mid \mathcal{H}_{s_t})}{\pi^b_t(a_t \mid \mathcal{H}_{s_t})}.$$  

We denote the cumulative density ratio up to time $t$ and the marginal density ratio at time $t$ by, respectively,

$$\nu_t(\mathcal{H}_{a_t}) = \prod_{k=0}^{t} \eta_t(\mathcal{H}_{a_k}), \quad \mu_t(s_t, a_t) = \frac{p_t^e(s_t, a_t)}{p_t^b(s_t, a_t)}.$$  

where $p_t^e(s_t, a_t)$ denotes the marginal distribution of $s_t, a_t$ under $P_t$. Note that under $\mathcal{M}_2$, $\eta_t(\mathcal{H}_{a_t}) = \eta_t(a_t, s_t)$ and $\nu_t(\mathcal{H}_{a_t}) = \eta_t(s_t, a_t)$. Again, for brevity we define the variables $\eta_t = \eta_t(\mathcal{H}_{a_t})$, $\nu_t = \nu_t(\mathcal{H}_{a_t})$, $\mu_t = \mu_t(s_t, a_t)$.

We assume the following throughout this paper.

**Assumption 1** (Sequential overlap). The density ratio $\eta_t$ satisfies $0 \leq \eta_t \leq C$ for all $t = 0, \ldots, T$.

**Assumption 2** (Bounded rewards). The reward $r_t$ satisfies $0 \leq r_t \leq R_{\max}$ for all $t = 0, \ldots, T$. 

4
1.2 Summary of Semiparametric Inference

We briefly review semiparametric inference theory as it pertains to the relevance of our results. For a more general and complete presentation, we refer the reader to Bickel et al. (1998), Kosorok (2008), Tsiatis (2006), van der Vaart (1998). Suppose we have a model $M$ for the generating process of the iid data $H(1), \ldots, H(n)$, that is, a (potentially nonparametric) set of potential distributions for $H$ that also contains the true distribution $F \in M$ that generated the data. Consider a (scalar) parameter of interest $R: M \to \mathbb{R}$. That is, we want to estimate $R(F)$. Given an estimator $\hat{R}$ (or rather a sequence of estimators), its limiting law is the distribution limit of $\sqrt{n}(\hat{R} - R(F))$.

If $R$ can be shown to be appropriately pathwise differentiable at $F$ (see van der Vaart, 1998, Thm. 25.20 for precise conditions), then there will exist a square-integrable function $\phi(H)$, called the efficient influence function, such that $\text{EffBd}(M) = E[\phi^2(H)]$ bounds below the limiting variance of any regular estimator. Regular estimators are roughly those whose limiting law is invariant to vanishing local perturbations to the data generating process $F$, which is a desirable property else the estimator may be unreasonably sensitive to undetectable changes. Alternatively, $\text{EffBd}(M)$ also lower bounds the limiting variance of any estimator in any vanishing neighborhood of the true distribution $F \in M$ (van der Vaart, 1998, Thm. 25.21).

In these senses, $\text{EffBd}(M)$, known as the semiparametric efficiency bound, lower bounds the achievable mean-squared error (MSE) in estimating $R$ on the model $M$. If we can find an estimator whose limiting law has zero mean and variance $\text{EffBd}(M)$ then it must have the smallest-possible (asymptotic) MSE, and such estimators are known as (asymptotically) efficient. Under the same differentiability conditions on $R$, such regular estimators exist and, moreover, all efficient regular estimators must satisfy

$$\sqrt{n}(\hat{R} - R(F)) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi(H^{(i)}) + o_p(1),$$

that is, they are asymptotically linear with influence function $\phi$. This suggests an estimation strategy: try to approximate $\hat{\psi}(\mathcal{H}) \approx \phi(\mathcal{H}) + R(F)$ and use $\hat{R} = \frac{1}{n} \sum_{i=1}^{n} \hat{\psi}(H^{(i)})$. Done appropriately, this can provide an efficient estimate. Therefore, deriving the efficient influence function is important both for computing the semiparametric efficiency bound and for deriving good estimators.

When $M$ is a fully parametric model the semiparametric efficiency bound is the same as the Cramér-Rao bound. In fact, the semiparametric efficiency bound corresponds to the supremum of the Cramér-Rao bounds over all parametric submodels $F \in M_{\text{para}} \subset M$. Thus, it describes the best-achievable behavior by nonparametric estimators that work in every parametric submodel. Note that while the semiparametric efficiency bound depends on $R$, $F$, and $M$, our notation $\text{EffBd}(M)$ emphasizes in particular the dependence on the model $M$ because we always consider the same estimand (target policy value) but investigate how the bound changes as we further restrict the model (to be Markov).

1.3 Summary of Literature on OPE

OPE is a central problem in both RL and in closely related problems such as dynamic treatment regimes (DTR; Murphy et al., 2001). While the NMDP model $M_1$ is commonly...
the one assumed used in the causal inference literature in the context of marginal structural
model estimation (Robins, 2000; Robins et al., 2000) and DTRs (Chakraborty and Moodie,
2013; Murphy et al., 2001; Zhang et al., 2013), in RL one often assumes that the MDP
model \( M_2 \) holds. Nonetheless, with some exceptions that we review below, OPE methods
in RL have largely not leveraged the additional independence structure of \( M_2 \) to improve
estimation, and in particular, the effect of this structure on efficiency has not previously
been studied and no efficient evaluation method has been proposed.

Methods for OPE can be roughly categorized into three types. The first approach is
the direct method (DM), wherein we directly estimate the \( q \)-function and use it to directly
estimate the value of the target evaluation policy. As we review in greater detail, one can
estimate the \( q \)-function by a backward recursive regression. Once we have an estimate \( \hat{q}_0 \)
of the first \( q \)-function, the DM estimate is simply \( \hat{\rho}_{\pi e}^{DM} = \mathbb{E}_n \left[ \hat{\rho}_{\pi e}(s_0, a_0) \right] \), where the
inner expectation is simply over \( a_0 \sim \pi_e(\cdot | s_0) \) and is thus computable as a sum or integral
over a known measure and the outer expectation is simply an average over the \( n \) observations
of \( s_0 \). For DM, we can leverage the structure of \( M_2 \) by simply restricting \( q \)-functions to
be Markovian. However, DM can fail to be efficient even under \( M_1 \) unless \( q \)-functions are
parametric (and correctly specified) or extremely smooth (as shown by Hahn, 1998 but only
in the \( T = 0 \) case). DM is also not robust in that, if \( q \)-functions are inconsistently estimated,
the estimate will be inconsistent.

The second approach is importance sampling (IS), which averages the data weighted
by the density ratio of the evaluation and behavior policies. Given estimates \( \hat{\nu}_t \) of the cumulative density ratios (or, letting \( \hat{\nu}_t = \nu_t \) if the behavior policy is known), the IS
estimate is simply \( \hat{\rho}_{\pi e}^{IS} = \mathbb{E}_n \left[ \sum_{t=0}^T \hat{\nu}_t r_t \right] \). (An alternative but higher-variance IS estimator
is \( \mathbb{E}_n \left[ \sum_{t=0}^T \hat{\nu}_t \right] \).) When behavior policy is known, IS is unbiased and consistent, but
its variance tends to be large due to extreme weights. In particular, under \( M_1 \), IS with
\( \hat{\nu}_t = \nu_t \) is known to be inefficient (Hirano et al., 2003), which implies it must be inefficient
under \( M_2 \) as well. A common variant of IS is the self-normalized estimate \( \sum_{t=0}^T \frac{\hat{\nu}_t r_t}{\hat{\nu}_t} \) (Swaminathan and Joachims, 2015), which trades off some bias for variance but does not
make IS efficient.

The third approach is the doubly robust (DR) method, which combines DM and IS
and is given by adding the estimated \( q \)-function as a control variate (Dudik et al., 2014;
Jiang and Li, 2016; Scharfstein et al., 1999). The DR estimate has the form
\[
\hat{\rho}_{\pi e}^{DR} = \mathbb{E}_n \left[ \sum_{t=0}^T (\hat{\nu}_t (r_t - \hat{q}_t) + \hat{\nu}_{t-1} \mathbb{E}_{\pi e}[\hat{q}_t | s_t]) \right].
\]

DR is colloquially known to be efficient under \( M_1 \) but no precise result is available.
When state and action spaces are finite, the model \( M_1 \) is necessarily parametric, and, under
this parametric model, Jiang and Li (2016) study the Cramér-Rao lower bound and
observe that an infeasible DR estimator that uses oracle nuisance values instead of estimates,
\( \hat{q}_t = q_t \) and \( \hat{\nu}_t = \nu_t \), would achieve the bound. For completeness, we derive precisely the
more general semiparametric efficiency bound under \( M_1 \) (Theorem 1) and show that two
(feasible) variants of the standard DR estimate are semiparametrically efficient, either using
sample splitting with a rate condition (Theorem 4) or without sample splitting with a
Donsker condition (Theorem 6). Jiang and Li (2016) also study parametric Cramér-Rao
lower bounds under finite action and state space in the MDP model \( M_2 \), but no efficient
estimator, whether parametric or nonparametric, has been proposed. See also Remarks 2 and 4. There is a significant gap to deriving the semiparametric bound, which generalizes these results to more general action and state spaces and nonparametric models. More importantly, our derivation yields the efficient influence function, which provides a way to construct an efficient estimator under $\mathcal{M}_2$.

Many variations of DR have been proposed. Thomas and Brunskill (2016) propose both a self-normalized variant of DR and a variant blending DR with DM when density ratios are extreme. Farajtabar et al. (2018) propose to optimize the choice of $\hat{q}$ to minimize a variance estimate for DR rather than use a plug-in value. Kallus and Uehara (2019) propose a DR estimator that achieves local efficiency, has certain stability properties enjoyed by self-normalized IS, and at the same time is guaranteed to have asymptotic MSE that is never worse than both DR, IS, and self-normalized IS.

However, all of the aforementioned IS and DR estimators do not exploit MDP structure and, in particular, will fail to be efficient under $\mathcal{M}_2$. Recently, in the same finite-state-and-action-space setting studied by Jiang and Li (2016), Xie et al. (2019) studied an IS-type estimator that exploits MDP structure by replacing density ratios with marginalized density ratios, estimated by within-state-action averages since state and action spaces are assumed finite. However, this estimator is also not efficient, even in the finite setting. Remark 4 of Xie et al. (2019) points out the inefficiency of the estimator proposed therein. We compute the variance of such marginalized importance sampling estimators, including a nonparametric extension, which again shows they are inefficient (Theorems 11 to 13).

2 Semiparametric Inference for Off-Policy Evaluation

In this section, we derive the efficiency bounds and efficient influence functions for $\rho_{\pi^e}$ under the models $\mathcal{M}_1$, $\mathcal{M}_{1b}$, $\mathcal{M}_2$, and $\mathcal{M}_{2b}$. Recall that the former two models are NMDP and the latter two are MDP and that the second and fourth assume a known behavior policy.

2.1 Semiparametric Efficiency in Non-Markov Decision Processes

First, we consider the NMDP models $\mathcal{M}_1$ and $\mathcal{M}_{1b}$. We do this mostly for the sake of completeness since, while the influence function we derive below for the NMDP model has appeared in numerous papers on doubly-robust evaluation in RL (e.g., among others, Dudik et al., 2014; Farajtabar et al., 2018; Jiang and Li, 2016; Kallus and Uehara, 2019; Thomas and Brunskill, 2016), we are aware of no result showing rigorously that it in fact corresponds to the efficient influence function in the NMDP model or deriving the semiparametric efficiency bound for this model. (Note that, in contrast, the influence function we derive for the MDP model in the next section appears to be novel.)

**Theorem 1** (Efficiency bound under $\mathcal{M}_1$). The efficient influence function of $\rho_{\pi^e}$ under $\mathcal{M}_1$ is

$$
\phi^{\mathcal{M}_1}_{\text{eff}}(\mathcal{H}) = -\rho_{\pi^e} + \sum_{t=0}^{T} (\nu_t(r_t - q_t) + \nu_{t-1}v_t).
$$

(1)
The semiparametric efficiency bound under $\mathcal{M}_1$ is

$$\text{EffBd}(\mathcal{M}_1) = \text{var}(v_0) + \sum_{t=0}^{T} E \left[ \nu_t^2 \text{var} \left( r_t + v_{t+1} \mid \mathcal{H}_{a_t} \right) \right], \quad (2)$$

where $v_{T+1} = 0$.

Under $\mathcal{M}_{1b}$, the efficient influence function and bound are the same.

**Remark 1.** The efficient influence function and bound are both the same whether we know the behavior policy or not. Intuitively, this happens because the estimand $\rho^{\pi_e}$ does not in fact depend on behavior policy part of the data generating distribution, $P_{\pi_b}$, but only on the decision process parts (initial state, transition, and emission probabilities). This phenomenon mirrors the situation with knowledge of the propensity score in average treatment effect estimation in causal inference noted by Hahn (1998).

**Remark 2.** When the action and state spaces are discrete, $\mathcal{M}_1$ is necessarily a parametric model. In this discrete-space parametric model and with $r_t = 0$ for $t \leq T - 1$, Theorem 2 of Jiang and Li (2016) derives the Cramér-Rao lower bound for estimating $\rho^{\pi_e}$. Because the semiparametric efficiency bound is the same as the Cramér-Rao lower bound for parametric models, the bound coincides with ours in this special discrete setting. Our result is more general, establishing the limit on estimation in non-discrete, nonparametric settings and, moreover, establishes that the efficient influence function coincides with the structure of many doubly-robust OPE estimators used in RL (see references above).

**Remark 3.** The efficient influence function $\phi^{\mathcal{M}_1}_{\text{eff}}$ has the oft-noted doubly robust structure. Specifically,

$$\rho^{\pi_e} + E \left[ \phi^{\mathcal{M}_1}_{\text{eff}} (\mathcal{H}) \right] = E \left[ \sum_{t=0}^{T} \nu_t r_t \right] + E \left[ \sum_{t=0}^{T} (-\nu_t q_t + \nu_{t-1} v_t) \right] = 0 = E \left[ v_0 \right] + E \left[ \sum_{t=0}^{T} \nu_t (r_t - q_t + v_{t+1}) \right].$$

The first term in each line corresponds to IPW and direct method (DM) estimators, respectively. The second term in each line is a control variate, which remain mean zero even if we plug in different (i.e., wrong) $q$- and $v$-functions or density ratios, respectively. In this sense, it is sufficient to estimate only one part of these for consistent OPE. We will leverage this in Theorem 7 to achieve double robustness for DRL.

### 2.2 Semiparametric Efficiency in Markov Decision Processes

Next, we derive the efficiency bound and efficient influence function for $\rho^{\pi_e}$ under the models $\mathcal{M}_2$ and $\mathcal{M}_{2b}$, i.e., when restricting to MDP structure. To our knowledge, not only have these never before been derived, the influence function we derive has also not appeared in any existing OPE estimators in RL. We recall that under $\mathcal{M}_2$, we have $q_t = q_t(s_t, a_t)$ and $v_t = v_t(s_t)$. 

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8
Theorem 2 (Efficiency bound under $\mathcal{M}_2$). The efficient influence function of $\rho^{\pi^e}$ under $\mathcal{M}_2$ is

$$
\phi^{\mathcal{M}_2}_{\text{eff}}(H) = -\rho^{\pi^e} + \sum_{t=0}^{T} (\mu_t (r_t - q_t) + \mu_{t-1} v_t).
$$

(3)

The semiparametric efficiency bound under $\mathcal{M}_2$ is

$$
\text{EffBd}(\mathcal{M}_2) = \text{var}(v_0) + \sum_{t=0}^{T} \mathbb{E} \left[ \mu_t^2 \text{var} \left( r_t + v_{t+1} \mid s_t, a_t \right) \right],
$$

(4)

where $v_{T+1} = 0$.

Under $\mathcal{M}_2$, the efficient influence function and bound are the same.

Remark 4. Again, when the action and state spaces are discrete, $\mathcal{M}_2$ is necessarily a parametric model. In this discrete-space parametric model and with $r_t = 0$ for $t \leq T - 1$, Theorem 3 of Jiang and Li (2016) derives the Cramér-Rao lower bound, which must (and does) coincide with ours in this setting. Again, our result is more general, covering non-parametric models and estimators, and, importantly, derives the efficient influence function, which we will use to construct the first globally efficient estimator for $\rho^{\pi^e}$ under $\mathcal{M}_2$.

Remark 5. The difference between the efficient influence functions in the NMDP and MDP models, $\phi^{\mathcal{M}_1}_{\text{eff}}$ and $\phi^{\mathcal{M}_2}_{\text{eff}}$, is that the cumulative density ratio $\nu_t$ is replaced with the marginalized density ratio $\mu_t$ and that $q$- and $v$-functions only depend on recent state and action rather than full past trajectory.

Although the efficient influence function in Theorem 2 is derived de-novo in the proof, which is the most direct route to a rigorous derivation, we can also use the geometry of influence functions to understand the result relative to Theorem 1. The efficient influence function is always given by projecting the influence function of any regular asymptotic linear estimator onto the tangent space (Tsaiats, 2006, Thm. 4.3). Under $\mathcal{M}_2$, the function $\phi^{\mathcal{M}_1}_{\text{eff}}(H)$ from Theorem 1 can be shown to still be an influence function of some regular asymptotic linear estimator in $\mathcal{M}_2$. Projecting it onto the tangent space in $\mathcal{M}_2$, where we have imposed the independence of past and future trajectories given intermediate state, can be seen to exactly correspond to the above marginalization over the past trajectory, explaining this structure of $\phi^{\mathcal{M}_2}_{\text{eff}}(H)$.

Remark 6. The efficient influence function $\phi^{\mathcal{M}_2}_{\text{eff}}(H)$ also has a doubly robust structure. Specifically,

$$
\rho^{\pi^e} + \mathbb{E} \left[ \phi^{\mathcal{M}_2}_{\text{eff}}(H) \right] = \mathbb{E} \left[ \sum_{t=0}^{T} \mu_t r_t \right] + \mathbb{E} \left[ \sum_{t=0}^{T} (\mu_t q_t + \mu_{t-1} v_t) \right] = \mathbb{E} \left[ v_0 \right] + \mathbb{E} \left[ \sum_{t=0}^{T} \mu_t (r_t - q_t + v_{t+1}) \right].
$$

9
The first term on the first line corresponds to the marginalized IPW estimator of Xie et al. (2019). The first term on the second line corresponds to the DM estimator. The second term on each line corresponds to control variate terms. We will leverage this in Theorem 10 to achieve double robustness for DRL.

By comparing the efficiency bounds of Theorem 1 and Theorem 2 and using Jensen’s inequality, we can see that the Markov assumption reduces the efficiency bound, usually strictly so.

**Theorem 3.** If \( P_{\pi^b} \in \mathcal{M}_2 \) (i.e., the underlying distribution is an MDP), then

\[
\text{EffBd}(\mathcal{M}_2) \leq \text{EffBd}(\mathcal{M}_1).
\]

Moreover, the inequality is strict if there exists \( t \leq T \) such that both \( \nu_{t-1} \) and \( r_{t-1} + \nu_t \) are not constant given \( s_{t-1}, a_{t-1} \).

### 3 Efficient Estimation Using Double Reinforcement Learning

In this section, we construct the DRL estimator and then study its properties in the various models. In particular, we show that DRL is globally efficient under very mild assumptions. In the NMDP model, these assumptions are generally weaker than needed for efficiency of previous estimators. In the MDP model, this provides the first globally efficiency estimator for OPE. We further show that DRL enjoys certain double robustness properties when some nuisances are inconsistently estimated.

DRL is a meta-estimator; it takes in as input estimators for \( q \)-functions and density ratios and combines them in a particular manner that ensures efficiency even when the input estimators may not be well behaved. This is achieved by following the cross-fold sample-splitting strategy developed by Chernozhukov et al. (2018). We proceed by presenting DRL and its properties in each setting (NMDP and MDP). In the NMDP setting, DRL amounts to a cross-fold modification to the standard plug-in version of the RL OPE doubly robust estimator (Jiang and Li, 2016), which affords it some additional desirable properties. In the MDP setting, DRL is the first semiparametrically efficient and doubly robust estimator.

#### 3.1 Double Reinforcement Learning for NMDPs

Given a learning algorithm to estimate the \( q \)-function \( q(\mathcal{H}_{at}) \) and cumulative density ratio function \( \nu_t(\mathcal{H}_{at}) \), DRL for NMDPs proceeds as follows:

1. Split the data randomly into two halves, \( D_0 \) and \( D_1 \). Let \( J(\mathcal{H}^{(i)}) \in \{0,1\} \) be such that \( J(\mathcal{H}^{(i)}) \in D_j \).

2. For \( j = 0, 1 \), construct estimators \( \hat{\nu}_t^{(j)}(\mathcal{H}_{at}) \) and \( \hat{q}_t^{(j)}(\mathcal{H}_{at}) \) based on the training data \( D_j \).

3. Let

\[
\hat{\rho}_{\text{DRL}(\mathcal{M}_1)} = \mathbb{E}_n \left[ \sum_{t=0}^{T} \left( \hat{\nu}_t^{(1-J)} \left( r_t - \hat{q}_t^{(1-J)} \right) + \hat{\nu}_t^{(1-J)} \mathbb{E}_{\pi^*} \left[ \hat{q}_t^{(1-J)} | \mathcal{H}_{s_t} \right] \right) \right],
\]
Note that the inner expectation is only over $a_t \sim \pi^c(\cdot \mid \mathcal{H}_{s_t})$ and is computable as a sum or integral over a known measure.

In other words, we approximate the efficient influence function $\phi_{\text{ef}}^{M_1}(H) + \rho^{\pi^+}$ from Theorem 1 by replacing the unknown $q$- and density ratio functions with estimates thereof and we take empirical averages of this approximation, where for each data point we use $q$- and density ratio function estimates based only on the half-sample that does not contain it.

This estimator has several desirable properties. To state them, we assume the following conditions for the estimators, reflecting Assumptions 1 and 2:

**Assumption 3.** $0 \leq \hat{\nu}_t \leq C_t$, $0 \leq \hat{q}_t \leq (T + 1 - t)R_{\text{max}}$ for $0 \leq t \leq T$.

We first prove that DRL achieves the semiparametric efficiency bound, even if each nuisance estimator has a slow, nonparametric convergence rate ($\sqrt{n}$).

**Theorem 4 (Efficiency of $\hat{\rho}_{\text{DRL}(M_1)}$ under $M_1$).** Suppose $\|\hat{\nu}_t^{(j)} - \nu_t\|_2 = c_\nu (n^{-\alpha_1,1})$ and $\|\hat{q}_t^{(j)} - q_t\|_2 = c_q (n^{-\alpha_2,2})$ for $0 \leq t \leq T, j = 0, 1$, where $\alpha_{1,1} + \alpha_{2,2} \geq 1/2$, $\alpha_{1,1}, \alpha_{2,2} > 0$.

Then, the estimator $\hat{\rho}_{\text{DRL}(M_1)}$ achieves the semiparametric efficiency bound under $M_1$.

**Remark 7.** There are two important points to make about this result. First, we have not assumed a Donsker condition (van der Vaart, 1998) on the class of estimators $\hat{\nu}_t$ and $\hat{q}_t$. This is why this type of sample splitting estimator is called a double machine learning: the only required condition is a convergence rate condition at a nonparametric rate, allowing the use of complex machine learning estimators, for which one cannot verify the Donsker condition (Chernozhukov et al., 2018). Second, relative to the efficient influence function, which is defined in terms of the true $q$-function and cumulative density ratio, there is no inflation in DRL’s asymptotic variance due to plugging in estimated nuisance functions. This is due to the doubly robust structure of efficient influence function so that the estimation errors multiply and drop out of the first-order variance terms. This is in contrast to inefficient importance sampling estimators, as we will see in Theorem 13.

Often in RL, the behavior policy is known and need not be estimated. That is, we can let $\hat{\nu}_t^{(j)} = \nu$. In this case, as an immediate corollary, we have a much weaker condition for semiparametric efficiency: just that we estimate the $q$-function consistently, without a rate.

**Corollary 5 (Efficiency of $\hat{\rho}_{\text{DRL}(M_1)}$ under $M_{1b}$).** Suppose $\hat{\nu}_t = \nu_t$ and $\|\hat{q}_t^{(j)} - q_t\|_2 = c_q (1)$ for $0 \leq t \leq T, j = 0, 1$. Then, the estimator $\hat{\rho}_{\text{DRL}(M_1)}$ achieves the semiparametric efficiency bound under $M_{1b}$.

Without sample splitting, we have to assume a Donsker condition for the class of estimators in order to control a stochastic equicontinuity term (see, e.g., van der Vaart, 1998, Lemma 19.24). Although this is more restrictive, for completeness, we also include a theorem establishing the semiparametric efficiency of the standard plug-in doubly robust estimator for NMDPs (Jiang and Li, 2016) when assuming the Donsker condition for in-sample-estimated nuisance functions, since this result was never precisely established before.
Theorem 6 (Efficiency without sample splitting). Let \( \hat{\nu}_t, \hat{q}_t \) be estimators based on \( D \) and let \( \hat{\rho}_{DR}^c \) be estimators based on \( D \) and let \( \hat{\rho}_{DR}^c = \mathbb{E}_a \left[ \sum_{t=0}^T (\hat{\nu}_t (r_t - \hat{q}_t) - \hat{\nu}_t \hat{q}_t) \right] \). Suppose \( \| \hat{\nu}_t - \nu_t \|_2 = o_p(n^{-\alpha/2}) \), \( \| \hat{q}_t - q_t \|_2 = o_p(n^{-\alpha/2}) \), \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 > 0 \) for \( 0 \leq t \leq T \) and that \( \hat{\nu}_t, \hat{q}_t \) belong to a Donsker class. Then, the estimator \( \hat{\rho}_{DR}^c \) achieves the semiparametric efficiency bound under \( M_1 \).

Thus, in \( M_1 \), in comparison to the standard doubly robust estimator, DRL enjoys efficiency under milder conditions. To our knowledge, Theorems 4 and 5 are the first results precisely showing semiparametric efficiency for any OPE estimator.

In addition to efficiency, DRL enjoys a double robustness guarantee (as defined in Rotnitzky and Vansteelandt, [2014]). Specifically, if at least just one model is correctly specified, then the DRL is estimator is still consistent and asymptotically normal (CAN). specifically, if at least just one model is correctly specified then Theorem 7 ensures that the estimator is CAN.

Theorem 7 (Double robustness). Suppose \( \| \hat{\nu}_t^{(j)} - \nu_t^{(j)} \|_2 = o_p(n^{-\alpha/2}) \) and \( \| \hat{q}_t^{(j)} - q_t^{(j)} \|_2 = o_p(n^{-\alpha/2}) \) for \( 0 \leq t \leq T, j = 0, 1 \), where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 > 0 \). If, for each \( 0 \leq t \leq T \), either \( \nu_t^{(j)} = \nu_t \) and \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 > 0 \), then the estimator \( \hat{\rho}_{DRL(M_2)} \) is CAN around \( \rho^{\pi^b} \).

In particular, if the behavior policy is known so that \( \hat{\nu}_t^{(j)} = \nu_t \), we can always ensure the estimator is CAN (an example is the IS estimator, which has \( \hat{q}_t^{(j)} = q_t^{(j)} = 0 \)).

A remaining question is when can we get nonparametric estimators achieving the necessary rates for the \( q \)- and density ratio functions. We discuss estimating \( q \)-functions in Section 4. When the behavior policy is unknown, \( \nu_t \) can be estimated by estimating and plugging in \( \pi^b \), which can in turn be estimated by nonparametric regression. Specifically, we let \( \hat{\nu}_t^{(j)} = \prod_{0=t}^k \hat{\pi}_t^{(j)}/\hat{\pi}_t^{(j)}, \) where \( \hat{\pi}_t^{(j)} \) is a standard kernel regression estimator or sieve regression estimator [Newey and McFadden, 1994; Stone, 1994]. When \( \pi^b(a_t \mid \mathcal{H}_{st}) \) belongs to the Hölder class with smoothness parameter \( \alpha \) and the dimension of the space \( \mathcal{H}_{st} \) is \( d_{\mathcal{H}_{st}} \) and the dimension of the action space is 1, it can be shown (ibid.) that \( \| \hat{\pi}_t^{(j)} - \pi^b_t \|_2 = O_p(n^{-\alpha/(2\alpha + d_{\mathcal{H}_{st}})}) \). We therefore have the following result.

Lemma 8. Assume \( \hat{\pi}_t^{(j)} \) and \( \pi^b_t \) are uniformly bounded by some constant below and that \( \pi^b_t(a_t \mid \mathcal{H}_{st}) \) is Hölder with parameter \( \alpha \). Then, \( \| \hat{\nu}_t^{(j)} - \nu_t \|_2 = O_p(n^{-\alpha/(2\alpha + d_{\mathcal{H}_{st}})}) \).

Alternatively, parametric models can be used for \( q_t \) and (behavior policy is unknown) \( \nu_t \). Then, under standard regularity conditions, using MLE and other parametric regression estimators for behavior policy would yield \( \| \hat{\nu}_t^{(j)} - \nu_t^{(j)} \|_2 = O_p(n^{-1/2}) \), where \( \nu_t^{(j)} = \nu_t \) if the model is well-specified. Similarly, in Section 4, we discuss how using parametric \( q \)-models yields \( \| \hat{q}_t^{(j)} - q_t^{(j)} \|_2 = O_p(n^{-1/2}) \). If both models are correctly specified then Theorem 4 immediately implies DRL achieves the efficiency bound. When using parametric models, this is sometimes termed local efficiency (i.e., local to the specific parametric model). If only one model is correctly specified then Theorem 7 ensures the estimator is still CAN.

### 3.2 Double Reinforcement Learning for MDPs

Given a learning algorithm to estimate the \( q \)-function \( q_t(s_t, a_t) \) and marginal density ratio function \( \mu_t(s_t, a_t) \), DRL for MDPs proceeds as follows:

1. **Policy Evaluation:** Estimate the \( q \)-function \( q_t(s_t, a_t) \) using a parametric or nonparametric regression model, taking into account the policy \( \pi_t \).
2. **Policy Iteration:** Use the estimated \( q \)-function to improve the policy \( \pi_t \) using a policy improvement algorithm, such as value iteration or policy gradient.
3. **Convergence:** Repeat steps 1 and 2 until convergence is achieved, i.e., the value function \( q_t(s, a) \) or the policy \( \pi_t \) stops improving significantly.

This iterative process allows DRL to learn optimal policies for MDPs under varying environments and uncertainties, making it a powerful tool in reinforcement learning.
1. Split the data randomly into two halves, $D_0$ and $D_1$. Let $J(H^{(i)}) \in \{0, 1\}$ be such that $J(H^{(i)}) \in D_j$.

2. For $j = 0, 1$, construct estimators $\hat{\mu}_t^{(j)}(s_t, a_t)$ and $\hat{q}_t^{(j)}(s_t, a_t)$ based on the training data $D_j$.

3. Let

$$\hat{\rho}_{\text{DRL}}(M_2) = \mathbb{E}_n \left[ \sum_{t=0}^T \left( \hat{\mu}_t^{(1-j)} \left( r_t - \hat{q}_t^{(1-j)} \right) + \hat{\mu}_{t-1}^{(1-j)} \mathbb{E}_\pi \left[ \hat{q}_t^{(1-j)} | s_t \right] \right) \right],$$

Note that the inner expectation is only over $a_t \sim \pi^\epsilon(\cdot | s_t)$ and is computable as a sum or integral over a known measure.

Again, what we have done is approximate the efficient influence function $\phi_{\text{eff}}^M(H) + \rho^{\pi^\epsilon}$ from Theorem \textbf{2} and taken its empirical average, where for each data point we use $q$- and marginal density ratio function estimates based only on the half-sample that does not contain it. Again, to establish the properties of DRL for MDPs, we assume the following conditions for the estimators, reflecting Assumptions \textbf{1} and \textbf{2}.

**Assumption 4.** $0 \leq \hat{\mu}_t \leq \frac{C}{t}, 0 \leq \hat{q}_t \leq (T + 1 - t)R_{\text{max}}$ for $0 \leq t \leq T$.

The following result establishes that DRL is the first efficient OPE estimator for MDPs. In fact, it is efficient even if each nuisance estimator has a slow, nonparametric convergence rate ($\sqrt{n}$). Moreover, as before, we make no restrictive Donsker assumption; the only required condition is the convergence rate condition. This result leverages our novel derivation of the efficient influence function in Theorem \textbf{1} and the structure of the influence function, which ensures no variance inflation due to estimating the nuisance functions.

**Theorem 9** (Efficiency of $\hat{\rho}_{\text{DRL}}(M_2)$ under $M_2$). Suppose $\| \hat{\mu}_t^{(j)} - \mu_t \|_2 = c_p(n^{-\alpha_{t,1}})$ and $\| \hat{q}_t^{(j)} - q_t \|_2 = c_p(n^{-\alpha_{t,2}})$ for $0 \leq t \leq T, j = 0, 1$, where $\alpha_{t,1} + \alpha_{t,2} \geq 1/2, \alpha_{t,1}, \alpha_{t,2} > 0$. Then, the estimator $\hat{\rho}_{\text{DRL}}(M_2)$ achieves the semiparametric efficiency bound under $M_2$.

In addition to efficiency, DRL again enjoys a double robustness guarantee in $M_2$, as in $M_1$.

**Theorem 10** (Double robustness). Suppose $\| \hat{\mu}_t^{(j)} - \mu_t \|^2 = c_p(n^{-\alpha_{t,1}})$ and $\| \hat{q}_t^{(j)} - q_t \|^2 = c_p(n^{-\alpha_{t,2}})$ for $0 \leq t \leq T, j = 0, 1$, where $\alpha_{t,1} + \alpha_{t,2} \geq 1/2, \alpha_{t,1}, \alpha_{t,2} > 0$. If, for each $0 \leq t \leq T$, either $\mu_t^\dagger = \mu_t$ and $\alpha_{t,1} \geq 1/4$ or $q_t^\dagger = q_t$ and $\alpha_{t,2} \geq 1/4$, then the estimator $\hat{\rho}_{\text{DRL}}(M_2)$ is CAN around $\rho^{\pi^\epsilon}$.

**Remark 8.** When the behavior policy is known, the estimator $\hat{\rho}_{\text{DRL}}(M_1)$ is still CAN around $\rho^{\pi^\epsilon}$ under $M_2$ even without smoothness conditions on $\mu_t$ because $M_2$ is included in $M_1$ so that Theorem \textbf{7} applies. On the other hand, the estimator $\hat{\rho}_{\text{DRL}}(M_2)$ requires some smoothness conditions even if the behavior policy is known because $\mu_t$ must still be estimated. In this sense, when the behavior policy is known, $\hat{\rho}_{\text{DRL}}(M_1)$ is more robust than $\hat{\rho}_{\text{DRL}}(M_2)$ under $M_2$ but its asymptotic variance is bigger, and generally strictly so.
A remaining question is how to estimate the nuisances at the necessary rates. We discuss $q$-function estimation in Section 4. For estimating $\mu_b$, one can leverage the following relationship to reduce it to a regression problem:

$$
\mu_t(s_t, a_t) = \eta_t(s_t, a_t)w_t(s_t), \quad \text{where } w_t(s_t) = E[\nu_{t-1} | s_t].
$$

(5)

Thus, for example, when the behavior policy is known, we need only estimate $w_t$, which amounts to regressing $\nu_{t-1}$ on $s_t$. So, in particular, if $w_t(s_t)$ belongs to a Hölder class with smoothness $\alpha$ and $s_t$ has dimension $d_s$, estimating $w_t$ with a sieve-type estimator $\hat{w}_t$ based on the loss function $(\nu_{t-1} - w_t(s_t))^2$ and letting $\mu_t(j)(s_t, a_t) = \eta_t(s_t, a_t)\hat{w}_t(j)(s_t)$ will give a convergence rate $\|\hat{\mu}_t(j)(s_t, a_t) - \mu_t(s_t, a_t)\|_2 = O_p(n^{-\alpha/(\alpha + d_s)})$ (Chen, 2007). When the behavior policy is unknown, it can be first estimated to construct $\hat{\nu}_t$ and we can repeat the above replacing $\nu_t$ with $\hat{\nu}_t$. In particular, there will be no deterioration in rate if $\pi_t^i$ also belongs to a Hölder class with smoothness $\alpha$ and if we further split each $D_j$, estimate $\pi_t^i$ as in Lemma 8 on one half, and plug it in to estimate $w_t$ on the other half. Further strategies for estimating $\mu_t$ are discussed in Section 3.3 below.

In the special case where we use parametric models for $\mu_t$ and $q_t$, under some regularity conditions, parametric estimators will generally satisfy $\|\hat{\mu}_t - \mu_t^\dagger\|_2 = O_p(n^{-1/2})$ and $\|\hat{q}_t - q_t^\dagger\|_2 = O_p(n^{-1/2})$, where $q_t^\dagger = q_t$ and $\mu_t^\dagger = \mu_t$ if the models are well-specified. (See Section 3 regarding estimating the $q$-function.) Thus, if both models are correctly specified, then Theorem 9 yields local efficiency. If only one model is correctly specified, Theorem 10 yields double robustness.

### 3.3 Estimating Marginalized Density Ratios and the Inefficiency of Marginalized Importance Sampling

In this section we discuss strategies for estimating $\mu_t$ and also show that doing OPE estimation using only marginalized density ratios, as recently proposed, leads to inefficient evaluation in $M_2$. Specifically, we consider the Marginalized Importance Sampling (MIS) estimator with known behavior policy, given by DRL without sample splitting when we let $\hat{q}_t = 0$ and $\hat{\mu}_t = \eta_t\hat{w}_t$, where $\hat{w}_t$ is a regression estimate.

Specifically, the MIS estimator is:

$$
\hat{\rho}_{\text{MIS}}^c = \frac{1}{n} \sum_{t=0}^T \eta_t \hat{w}_t r_t.
$$

(6)

We focus on two cases: when $\hat{w}$ is estimated using a histogram by averaging $\nu_{t-1}$ by state over a finite state space and a nonparametric extension.

**Theorem 11** (Asymptotic variance of $\hat{\rho}_{\text{MIS}}^c$ with finite state space). Suppose $|S_t| < \infty$ for $0 \leq t \leq T$. Let

$$
\hat{w}_t(s_t) = \frac{\sum_{i=1}^n 1[s_t^{(i)} = s_t] \nu_{t-1}}{\sum_{i=1}^n 1[s_t^{(i)} = s_t]}.
$$

(7)
Then $\hat{\rho}_{\text{MIS}}^{\rho_e}$ is CAN around $\rho^{\rho_e}$ and its asymptotic MSE is

$$\text{var} \left[ \sum_{t=0}^{T} \mu_t r_t + (\nu_{t-1} - \omega_t) E_{\pi_e}[r_t | s_t] \right].$$  \hspace{1cm} (8)

For the proof of Theorem 11, we use an argument based on the theory of $U$-statistics \cite{van1998asymptotic, ch. 12} in order to rephrase the MIS estimator with $\hat{\rho}$ as in Eq. (7) in an asymptotically linear form: $\hat{\rho}_{\text{MIS}}^{\rho_e} = E_n[\sum_{t=0}^{T} \mu_t r_t + (\nu_{t-1} - \omega_t) E_{\pi_e}[r_t | s_t]] + o_p(n^{-1/2}).$

This influence function is different from the efficient influence function; therefore, $\hat{\rho}_{\text{MIS}}^{\rho_e}$ is not efficient (the efficient influence function is unique). In fact, we can confirm this fact by calculating and comparing the variances.

**Theorem 12.** If $P_{\pi_b} \in \mathcal{M}_2$ (i.e., the underlying distribution is an MDP), Eq. (8) is greater than or equal to $\text{EffBd}(\mathcal{M}_2)$. The difference is

$$\text{var} (\eta_t v_t(s_{t+1}) | s_t) + T - \sum_{t=0}^{T-1} E \left[ (w_t - \nu_{t-1})^2 \text{var} (\eta_t v_t(s_{t+1}) | s_t) \right].$$

We now turn to the nonparametric case, where we first consider a sieve-type extension of the $\hat{w}$ estimator.

**Theorem 13** (Asymptotic variance of $\hat{\rho}_{\text{MIS}}^{\rho_e}$ with nonparametric $w_t$ estimate). Suppose $E[(\nu_t - \mu_t)^q] < \infty$ for some $q > 1$. Let

$$\hat{w}_t(s_t) = \arg\min_{w_t(s_t) \in \Lambda_{\alpha,s_t}^d} E_n[(w_t(s_t) - \nu_{t-1})^2],$$  \hspace{1cm} (9)

where $\Lambda_{\alpha,s_t}^d$ is the space of Hölder functions with smoothness $\alpha$ and the dimension $d_{s_t}$. Assume $w_t \in \Lambda_{\alpha,s_t}^d$, $\alpha/(2\alpha + d_{s_t}) > 1/4$. Then the estimator $\hat{\rho}_{\text{MIS}}^{\rho_e}$ is CAN around $\rho^{\rho_e}$ and its asymptotic MSE is equal to Eq. (8).

**Remark 9.** The estimator Eq. (9) is over an infinite-dimensional function space. It can be replaced with a finite-dimensional approximation $\Lambda_{\alpha,s_t}^d$ such that $\Lambda_{\alpha,s_t}^d \rightarrow \Lambda_{\alpha,s_t}^d$. Following Example 1(b) in Section 8 \cite{shen1997nonparametric}, it can be shown that this will lead to the same asymptotic MSE as in Eq. (8) and not change the conclusion of Theorem 13.

**Remark 10.** When the action and sample space is continuous, the histogram estimator in Eq. (7) can also easily be extended to a kernel estimator:

$$\hat{w}_t(s_t) = \frac{\sum_{i=1}^{n} K_h(s_t^{(i)} - s_t) \nu_{t-1}}{\sum_{i=1}^{n} K_h(s_t^{(i)} - s_t)},$$  \hspace{1cm} (10)

where $K_h$ is a kernel with a bandwidth $h$.

The smoothness condition in Theorem 13 ensures we can estimate $w_t$ at fourth-root rates using Eq. (9). Following \cite{newey1994large} and utilizing a high-order kernel, we can obtain similar fourth-root rates for Eq. (10) and a similar variance result for MIS. Unlike Eq. (9), we cannot invoke a Donsker condition to prove a stochastic equicontinuity condition. However, it is still possible to show this directly based on a V-statistics theory (see Chapter 8 of \cite{newey1994large}).
Finally, we also consider estimating $\mu_t$ directly and nonparametrically using the relation

$$\mu_t(s_t, a_t) = E[\nu_t | s_t, a_t].$$

A sieve-type regression estimator for $\mu_t$ is then constructed as

$$\hat{\mu}_t(s_t, a_t) = \arg\min_{\mu_t(s_t, a_t) \in \Lambda_{d_{st} + d_{at}}^0} \text{Var}[\nu_t - \mu_t]^2].$$

**Theorem 14** (Asymptotic variance of $\hat{\rho}_{\text{DRL}}$ with nonparametric $\mu_t$ estimate). Suppose $E[(\nu_t - \mu_t)^q] < \infty$ for some $q > 1$. Let $\hat{\mu}_t$ be as in Eq. (12). Assume $\mu_t \in \Lambda_{d_{st} + d_{at}}^0$, $\alpha/(2\alpha + d_{st} + d_{at}) > 1/4$. Then the estimator $E_n \left[ \sum_{t=0}^T \hat{\mu}_t r_t \right]$ is CAN around $\rho^{\pi^e}$ and its asymptotic MSE is equal to

$$\text{var} \left[ \sum_{t=0}^T \mu_t r_t + (\nu_t - \mu_t)E[r_t | s_t, a_t] \right].$$

**Remark 11.** While both estimators for $\mu_t$ in Theorems 13 and 14 achieve fourth-root rates under the respective conditions, the resulting asymptotic variances in Eqs. (6) and (13) are different and generally incomparable. Both are inefficient, but which is larger is problem-dependent. Note that, in contrast, the asymptotic variance of DRL (Theorem 9) is the same (and is efficient) regardless of which way is used to estimate $\mu_t$ as long as we have the necessary rate. When the behavior policy is known, using Eq. (9) may be better than Eq. (12) when estimating $\mu_t$ nonparametrically because the smoothness condition is weaker and the convergence rate is faster (since $d_{st} < d_{at} + d_{st}$). However, when using parametric models, the rates are the same (under correct specification) and sometimes it is easier to model $\mu_t(s_t, a_t)$ rather than $w_t(s_t)$, as we do in Section 5.1.

### 4 Estimating the $q$-function and Efficiency Under $\mathcal{M}_{1q}$, $\mathcal{M}_{2q}$

In this section, we discuss the estimation of $q$-functions in an off-policy manner, parametrically or nonparametrically, which can be plugged into our estimators, $\hat{\rho}_{\text{DRL}, \mathcal{M}_1}$, $\hat{\rho}_{\text{DRL}, \mathcal{M}_2}$. On the way, we also derive the semiparametric efficiency bound when we impose parametric restrictions on $q$-functions, i.e., the models $\mathcal{M}_{1q}$, $\mathcal{M}_{2q}$.

To do this, we will leverage a recursive definition of the $q$-functions (Bertsekas, 2012). Under $\mathcal{M}_1$, the following recursion equation holds:

$$q_t = E \left[ r_t + E_{\pi^e} \left[ q_{t+1} | \mathcal{H}_{s_{t+1}} \right] | \mathcal{H}_{a_t} \right].$$

Under $\mathcal{M}_2$, we can further replace $\mathcal{H}_{s_{t+1}}$ with $s_{t+1}$ and $\mathcal{H}_{a_t}$ with $(s_t, a_t)$ in the above.

The recursion in Eq. (14) can equivalently be written as a set of conditional moment equations satisfied by the $q$-functions:

$$m_t(\mathcal{H}_{a_t}; \{q_1, \ldots, q_T\}) = 0 \quad \forall t \leq T,$$

where

$$m_t(\mathcal{H}_{a_t}; \{q'_{t_1}, \ldots, q'_T\}) = E \left[ r_t + E_{\pi^e} \left[ q'_{t+1}(\mathcal{H}_{a_{t+1}}) | \mathcal{H}_{s_{t+1}} \right] - q'_t(\mathcal{H}_{a_t}) | \mathcal{H}_{a_t} \right].$$

This formulation of the $q$-function in terms of conditional moment equations, along with the observation that $\rho^{\pi^e} = E[\pi^e(q_0(s_0, a_0) | s_0)]$ is determined by the $q$-function, allows us both to estimate the $q$-function efficiently, either parametrically and nonparametrically, and to characterize the efficiency bounds under $\mathcal{M}_{1q}$ and $\mathcal{M}_{2q}$. We start with the latter.
4.1 Efficiency Bounds Under $\mathcal{M}_{1q}$, $\mathcal{M}_{2q}$

In this section we consider the models where we restrict $q$-functions parametrically:

$$\mathcal{M}_{1q} = \{ P_{\pi^*} \in \mathcal{M}_1 : \exists \beta_t^* \in \Theta_{\beta_t}, q_t(\mathcal{H}_{a_t}) = q_t(\mathcal{H}_{a_t}; \beta_t^*) \ \forall t \leq T \} ,$$

$$\mathcal{M}_{2q} = \{ P_{\pi^*} \in \mathcal{M}_2 : \exists \beta_t^* \in \Theta_{\beta_t}, q_t(s_t, a_t) = q_t(s_t, a_t; \beta_t^*) \ \forall t \leq T \} ,$$

where $q_t(\mathcal{H}_{a_t}; \beta_t)$ or $q_t(s_t, a_t; \beta_t)$ is some parametric model for the $q$-function at time $t$ that is continuously differentiable with respect to the parameter $\beta_t$, $\Theta_{\beta_t}$ is some compact parameter space, and $\beta_t^*$ is the true parameter, which is assumed to lie in the interior of $\Theta_{\beta_t}$. For brevity we define $v_t(\mathcal{H}_{s_t}; \beta_t) = E_{\pi^*} [q_t(\mathcal{H}_{a_t}; \beta_t) | \mathcal{H}_{a_t}]$ and similarly $v_t(s_t; \beta_t)$.

Under $\mathcal{M}_{1q}$, $\mathcal{M}_{2q}$, Eq. (13) can be rephrased as a set of conditional moment restrictions on the parameter $\beta$ defined by $\beta = (\beta_1^T, \ldots, \beta_T^T)^T$. In particular, overloading notation and letting $m_t(\mathcal{H}_{a_t}; \beta) = m_t(\mathcal{H}_{a_t}; \{q_1(\cdot, \beta_1), \ldots, q_T(\cdot, \beta_T)\})$, we have that $\beta$ is defined by the set of conditional moment equations $m_t(\mathcal{H}_{a_t}; \beta) = 0 \ \forall t \leq T$. This observation is key in establishing the following result.

**Theorem 15** (Efficiency bound under $\mathcal{M}_{1q}$, $\mathcal{M}_{2q}$). Define $e_{q,t} = r_t + v_{t+1} - q_t$ and let

$$A_t = E \left[ \nabla_{\beta_t} q_t(\mathcal{H}_{a_t}; \beta_t^*) \text{var}(e_{q,t} | \mathcal{H}_{a_t})^{-1} \nabla_{\beta_t}^T q_t(\mathcal{H}_{a_t}; \beta_t^*) \right]^{-1} + B_t A_{t+1} B_t^T ,$$

$$B_t = E \left[ \nabla_{\beta_t} q_t(\mathcal{H}_{a_t}; \beta_t^*) \text{var}(e_{q,t} | \mathcal{H}_{a_t})^{-1} \nabla_{\beta_{t+1}}^T v_{t+1}(\mathcal{H}_{s_{t+1}}; \beta_{t+1}^*) \right] ,$$

$$A_T = E \left[ \nabla_{\beta_T} q_T(\mathcal{H}_{a_T}; \beta_T^*) \text{var}(e_{q,T} | \mathcal{H}_{a_T})^{-1} \nabla_{\beta_T}^T q_T(\mathcal{H}_{a_T}; \beta_T^*) \right]^{-1} ,$$

$$B_{-1} = E[\eta_0(s_0, a_0) \nabla_{\beta_0}^T q_0(s_0, a_0; \beta_0^*)] .$$

Then

$$\text{EffBd}(\mathcal{M}_{1q}) = \text{var}(v_0) + B_{-1} A_0 B_{-1}^T .$$

Moreover, the efficiency bound for estimating $\beta_t$ is $A_t$.

Finally, the corresponding efficiency bounds under $\mathcal{M}_{2q}$ are given by replacing $\mathcal{H}_{s_{t+1}}$ with $s_{t+1}$ and $\mathcal{H}_{a_t}$ with $(s_t, a_t)$ everywhere in the above.

**Remark 12.** When $T = 1$, EffBd($\mathcal{M}_{1q}$) above is equal to

$$\text{var}(v_0(s_0)) + B_{-1} A_0 B_{-1}^T ,$$

where

$$A_0 = E[\nabla_{\beta_0} q_0(s_0, a_0; \beta_0^*) \text{var}(r_0 | s_0, a_0)^{-1} \nabla_{\beta}^T q_0(s_0, a_0; \beta_0^*)]^{-1} ,$$

The Matrix Cauchy-Schwarz Inequality (Tripathi, 1999) immediately shows that this is upper bounded by EffBd($\mathcal{M}_{1q}$), as is also implied by $\mathcal{M}_{1q} \subset \mathcal{M}_1$ albeit less directly.

4.2 Parametric Estimation of $q$-functions

Next, we consider an estimation method for $\beta_t$ and $\rho^*$. Given the above observations, a natural way to estimate $\beta$ is by solving the following set of conditional moment equations given by $m_t(\mathcal{H}_{a_t}; \beta) = 0 \ \forall t \leq T$. For example, one approach when the $q$-model is a linear model specified as $\beta_T^T \phi_t(\mathcal{H}_{a_t})$ for some $d_{\phi_t}$-dimensional feature expansion $\phi_t$ is to choose $\hat{\beta}$
to minimize $\sum_{t=0}^{T} \sum_{i=1}^{d_{t}} (E_{n} (\{(r_{t} + v_{t+1}(H_{s_{t+1}}; \beta_{t+1}) - q_{t}(H_{a_{t}}; \beta_{t})\}) \phi_{t}(H_{a_{t}}))^{2}$, which corresponds exactly to backward-recursive ordinary least squares. That is, first $r_{t}$ is regressed on $\phi_{t}(H_{a_{t}})$ to obtain $\hat{\beta}_{t}$, then $q_{t}(H_{a_{t}}; \hat{\beta}_{t})$ is averaged over $\pi_{t}^{n}(a_{t} \mid H_{s_{t}})$ to obtain $\hat{v}_{t}$, then $r_{t-1} + \hat{v}_{t}$ is regressed on $\phi_{t}(H_{a_{t-1}})$ to obtain $\hat{\beta}_{t-1}$, and so on.

Although such an estimator can achieve the rate $O_{p}(n^{-1/2})$ under correct specification and standard conditions for M-estimators, it might not yield an efficient estimator for $\beta$ or for $\rho^{\ast}$ even when the $q$-model is linear as above, this can be easily solved by instead applying any efficient variant of the generalized method of moments (GMM), such as two-step GMM [Hansen 1982; Hansen et al. 1996], to the set of moment equations given by $m_{t}(H_{a_{t}}; \beta)\phi_{t}(H_{a_{t}}) = 0 \forall t \leq T, i \leq d_{a_{t}}$. This is almost the same as the above backward-recursive ordinary least squares but with an optimal weighting of the different moment conditions in the sum above.

When the $q$-model may be nonlinear, we can obtain an efficient estimator by instead applying the method of [Hahn 1997] to our set of conditional moment equations. Specifically, we can consider the set of $Tn_{m}$ moment equations $E [m_{t}(H_{a_{t}}; \beta)\phi_{t}(H_{a_{t}})] = 0 \forall t \leq T, i \leq m_{n}$, where $\phi_{1}(H_{a_{t}}), \phi_{2}(H_{a_{t}}), \ldots$ is a basis expansion of the $L^{2}$-space and $m_{n} \rightarrow \infty$ as $n \rightarrow \infty$. Then, applying any efficient variant of GMM to this set of moment conditions will yield an efficient estimator $\hat{\beta}$ of $\beta$.

In all of the above, replacing $H_{s_{t+1}}$ with $s_{t+1}$ and $H_{a_{t}}$ with $(s_{t}, a_{t})$, the same techniques can be applied in $M_{2}$. In either case, once we have an efficient estimate $\hat{\beta}$ of $\beta$, an efficient estimate for $\rho^{\ast}$, achieving the semiparametric efficiency bound in the appropriate model, is given by $\hat{\rho}_{DM} = E_{n} [v_{q}(s_{0}; \hat{\beta}_{0})]$.

### 4.3 Nonparametric Estimation of $q$-functions

The above observation in Eq. (13) that $q$-functions satisfy a set of conditional moment equations also lends itself to nonparametric estimation of the $q$-functions. In this section we briefly review how one approach to this, following the application of the method of [Ai and Chen 2012] to this set of conditional moment equation, can obtain the necessary fourth-root rates for use in DRL.

The estimator $\{{\hat{q}_{t}}\}_{t=0}^{T}$ is constructed as the following sieve minimum distance estimator:

$$\{{\hat{q}_{t}}\}_{t=0}^{T} = \arg\min_{q_{t} \in \Lambda_{t,n}} \sum_{t=0}^{T} E_{n} \left[ \hat{m}_{t}(H_{a_{t}}; q_{t}) \hat{\Sigma}_{t}^{-1} \hat{m}_{t}(H_{a_{t}}; q_{t}) \right],$$

where $\hat{m}_{t}(H_{a_{t}}; q_{t})$ is a nonparametric estimator for $m_{t}(H_{a_{t}}; q_{t})$, $\hat{\Sigma}_{t}$ is a nonparametric estimator for var $(e_{q_{t}, t} \mid H_{a_{t}})$, and $\Lambda_{t,n}$ is a sequence of approximation space whose union $\bigcup_{n=1}^{\infty} \Lambda_{t,n}$ is dense in some infinite dimensional space $\Lambda_{t}$. Alternatively, in $M_{2}$, we replace $H_{a_{t}}$ with $(s_{t}, a_{t})$ in the above.

[ Ai and Chen 2003] prove that applying the above with appropriate nonparametric estimators, under some smoothness conditions, we can obtain $\|{\hat{q}_{t} - q_{t}}\|_{F,t} = O_{p}(n^{-1/4})$, where $\| \cdot \|_{F,t}$ is the Fisher metric, which is defined as

$$\|g(H_{a_{t}})\|_{F,t}^{2} = E \left[ \text{var}(e_{q_{t}, t} \mid H_{a_{t}})g^{2} + \text{var}(e_{q_{t-1}, t-1} \mid H_{a_{t-1}})E_{n_{t}}[g(H_{a_{t}}) \mid H_{a_{t}}]^{2} \right].$$
Table 1: Experiment from Section 5.1: RMSE (and standard errors).

| Setting | n   | $\hat{\rho}_{IS}$ | $\hat{\rho}_{DRL(M_1)}$ | $\hat{\rho}_{DM}$ | $\hat{\rho}_{MIS}$ | $\hat{\rho}_{DRL(M_2)}$ |
|---------|-----|-------------------|-------------------------|------------------|------------------|-------------------------|
| (1)     | 1500| 42.4 (12.4)       | 36.1 (16.8)             | 0.70 (0.002)     | 40.8 (12.5)     | 0.70 (0.002)            |
|         | 3000| 20.4 (3.1)        | 7.8 (0.8)               | 0.50 (0.001)     | 20.8 (2.8)      | 0.50 (0.001)            |
|         | 4500| 20.2 (3.1)        | 6.6 (0.75)              | 0.43 (0.001)     | 21.5 (3.5)      | 0.43 (0.001)            |
| (2)     | 1500| 42.4 (12.4)       | 77.6 (29.1)             | 10.8 (0.002)     | 40.8 (12.5)     | 10.3 (3.5)              |
|         | 3000| 20.4 (2.5)        | 36.6 (6.9)              | 10.8 (0.001)     | 20.8 (2.8)      | 6.0 (0.6)               |
|         | 4500| 20.2 (3.1)        | 34.4 (9.6)              | 10.8 (0.001)     | 21.5 (3.5)      | 5.5 (2.0)               |
| (3)     | 1500| 42.4 (12.4)       | 36.1 (16.8)             | 0.70 (0.002)     | 87.7 (25.5)     | 0.73 (0.03)             |
|         | 3000| 20.4 (3.1)        | 7.8 (0.8)               | 0.50 (0.001)     | 37.3 (3.2)      | 0.51 (0.002)            |
|         | 4500| 20.2 (3.1)        | 6.6 (0.75)              | 0.43 (0.001)     | 53.5 (15.1)     | 0.44 (0.005)            |

We omit the details and refer the interested reader to Ai and Chen (2003). We only prove that this norm is in fact equivalent to the $L^2$-norm under mild conditions.

Lemma 16. Suppose $\text{var}[e_{q,t} | H_{at}]$ and $\text{var}[e_{q,t-1} | H_{a_{t-1}}]$ are bounded away from zero. Then, $\| \cdot \|_{F,k}$ and $\| \cdot \|_2$ are equivalent norms.

This means that, under the appropriate conditions, the estimator $\hat{q}$ obtains the rate $c_p(n^{-1/4})$ in terms of $L^2$-norm, as necessary for Theorems 4, 7, 9 and 10.

5 Experiments

We now turn to an empirical study of OPE and DRL. First, we construct a simulation to investigate the effect of using memorylessness on estimation variance as well as the effect of double robustness on model specification sensitivity. Then, we study comparative performance of different OPE estimators in two standard OpenAI Gym tasks.

Replication code for all experiments is available at http://github.com/CausalML/DoubleReinforcementLearningMDP.

5.1 The Effects of Leveraging Memorylessness and of Double Robustness

In this section we consider an MDP with a horizon of $T = 30$, binary actions, univariate continuous state, initial state distribution $p(s_0) \sim \mathcal{N}(0.5, 0.2)$, transition probabilities $P_t(s_{t+1} | s_t, a_t) \sim \mathcal{N}(0.02)$. The target and behavior policies we consider are, respectively,

$$\pi^e(a | s) \sim \text{Bernoulli}(p_e), \quad p_e = 0.2/(1 + \exp(-0.1s)) + 0.2U, \quad U \sim \text{Uniform}[0, 1]$$

$$\pi^b(a | s) \sim \text{Bernoulli}(p_b), \quad p_b = 0.9/(1 + \exp(-0.1s)) + 0.1U, \quad U \sim \text{Uniform}[0, 1].$$

We assume the behavior policy is known. Note that this setting is an MDP and belongs to $\mathcal{M}_2$.

We compare five estimators: $\hat{\rho}_{IS}$, $\hat{\rho}_{DRL(M_1)}$, $\hat{\rho}_{DM}$, $\hat{\rho}_{MIS}$, $\hat{\rho}_{DRL(M_2)}$ when nuisance functions $q_t(s, a)$ and $\mu_t(s)$ are estimated parametrically. We consider three settings:
1. Both models correct: \(q_t(s_t, a_t) = \beta_{1t} s_t + \beta_{2t} s_t a_t + \beta_{3t}, \mu_t(s_t, a_t) = \beta_{4t} s_t + \beta_{5t} s_t a_t + \beta_{6t}\).

2. Only \(\mu\)-model correct: \(q_t(s_t, a_t) = \beta_{1t} s_t^2 + \beta_{2t} s_t a_t + \beta_{3t}, \mu_t(s_t, a_t) = \beta_{4t} s_t + \beta_{5t} s_t a_t + \beta_{6t}\).

3. Only \(q\)-model correct: \(q_t(s_t, a_t) = \beta_{1t} s_t + \beta_{2t} s_t a_t + \beta_{3t}, \mu_t(s_t, a_t) = \beta_{4t} s_t^2 + \beta_{5t} s_t a_t + \beta_{6t}\).

Note that in the above, the “correct” models are in fact not exactly correct because \(E_{\pi^e}[a_t | s_t]\) is actually nonlinear in \(s_t\), but it is very nearly linear in the space of observed \(s_t\) values (for example, best linear fit for \(E_{\pi^e}[a_t | s_t]\) has an \(L^2\) distance \(3 \times 10^{-5}\) on [0, 1], which spans \(\pm 2.5\) standard deviations for \(s_0\)). We therefore treat them as correctly specified.

In all cases, to estimate \(q\)-models we use backward-recursive ordinary least squares. To estimate \(\mu\)-models we use backward-recursive ordinary least squares regression on each setting (and the standard error) in Table 1.

For each \(n = 1500, 3000, 4500\), we consider 50000 Monte Carlo replications. In each replication, we estimate the \(q\)- and \(\mu\)-models as above and compute, for each setting, each of \(\hat{\rho}_{IS}, \hat{\rho}_{DRL(M_2)}, \hat{\rho}_{DM}, \hat{\rho}_{MIS}, \hat{\rho}_{DRL(M_2)}\). We report the RMSE of each estimator in each setting (and the standard error) in Table 1.

Our first immediate observation is that \(\hat{\rho}_{DRL(M_2)}\) nearly dominates all other estimators, achieving similar or better performance in every setting and sample size. In particular, in settings (1) and (3), where the \(q\)-model is correct, it has performance similar to \(\hat{\rho}_{DM}\). Note that in settings (1) and (3), \(\hat{\rho}_{DM}\) is efficient for \(\mathcal{M}_{20}\) per Section 4.2 (or almost so; it would be efficient if we used efficient GMM instead of one-step GMM). In setting (1), \(\hat{\rho}_{DRL(M_2)}\) is locally efficient, while in setting (3), it is only doubly robust and performs almost imperceptibly worse than the efficient \(\hat{\rho}_{DM}\).

In setting (2), where the \(q\)-model is incorrect, \(\hat{\rho}_{DM}\) is inconsistent and \(\hat{\rho}_{DRL(M_2)}\) handily outperforms it. In the same setting (2), the consistent \(\hat{\rho}_{IS}\) and \(\hat{\rho}_{MIS}\) also outperform the inconsistent \(\hat{\rho}_{DM}\) but not by as much as \(\hat{\rho}_{DRL(M_2)}\). While \(\hat{\rho}_{DRL(M_1)}\) is doubly robust in setting (2) guaranteeing consistency, unlike the case of \(\hat{\rho}_{DRL(M_2)}\), the combination of large (unmarginalized) cumulative density ratios and a misspecified \(q\)-model leads to still worse performance in the sample sizes tested.

Generally, \(\hat{\rho}_{IS}, \hat{\rho}_{MIS}, \text{ and } \hat{\rho}_{DRL(M_1)}\) all have high RMSE due to the significant mismatch between the behavior and target policies so that cumulative density ratios are very large and only marginalizing them without also using a \(q\)-model helps only a little. In settings (1) and (2), where the \(\mu\)-model is correct, \(\hat{\rho}_{MIS}\) improves on \(\hat{\rho}_{IS}\) only slightly, while in setting (3), where \(\mu\)-model is incorrect, it performs significantly worse. This highlights the potential danger of misspecifying \(\mu\)-models compared to the robustness of importance sampling with known behavior policy (see also Remark 8).

While both \(\hat{\rho}_{IS}\) and \(\hat{\rho}_{DRL(M_1)}\) remain consistent throughout all settings, they are outperformed by the also-consistent \(\hat{\rho}_{DRL(M_2)}\), which leverages the MDP structure of \(\mathcal{M}_2\) and exhibits local efficiency in setting (1) and doubly robustness in settings (2) and (3).

### 5.2 Investigating Performance in RL Tasks: Cliff Walking and Mountain Car

We next compare the same OPE estimators using nonparametric nuisance estimation in two standard RL settings included in OpenAI Gym (Brockman et al., 2016): Cliff Walking and Mountain Car. For further detail on each setting, see Appendix C.
Table 2: Cliff Walking: RMSE (and standard errors)

| Size | $\hat{\rho}_{IS}$ | $\hat{\rho}_{DRL(M_1)}$ | $\hat{\rho}_{DM}$ | $\hat{\rho}_{MIS}$ | $\hat{\rho}_{DRL(M_2)}$ |
|------|---------------------|--------------------------|-------------------|-------------------|--------------------------|
| 500  | 18.8 (7.67)         | 3.78 (1.14)              | 2.63 (0.01)       | 12.8 (4.96)       | 1.44 (0.29)              |
| 1000 | 7.99 (0.89)         | 0.28 (0.026)             | 1.27 (0.002)      | 5.92 (0.78)       | 0.22 (0.34)              |
| 1500 | 7.64 (1.63)         | 0.098 (0.013)            | 1.01 (0.001)      | 5.55 (1.10)       | 0.075 (0.008)            |

Table 3: Mountain Car: RMSE (and standard errors)

| n    | $\hat{\rho}_{IS}$ | $\hat{\rho}_{DRL(M_1)}$ | $\hat{\rho}_{DM}$ | $\hat{\rho}_{MIS}$ | $\hat{\rho}_{DRL(M_2)}$ |
|------|---------------------|--------------------------|-------------------|-------------------|--------------------------|
| 500  | 6.85 (0.13)         | 3.72 (0.08)              | 4.30 (0.05)       | 6.82 (0.12)       | 3.53 (0.12)              |
| 1000 | 4.73 (0.07)         | 2.12 (0.04)              | 3.40 (0.008)      | 4.83 (0.06)       | 2.07 (0.04)              |
| 1500 | 3.41 (0.04)         | 1.82 (0.02)              | 3.30 (0.008)      | 3.40 (0.05)       | 1.69 (0.03)              |

First, we used $q$-learning to learn an optimal policy for the MDP and define it as $\pi^d$. Then we generate the dataset from the behavior policy $\pi^b = (1 - \alpha)\pi^d + \alpha\pi^u$ where $\pi^u$ is a uniform random policy and $\alpha = 0.8$. We define the target policy similarly but with $\alpha = 0.9$. Again, we assume the behavior policy is known. Note that this $\pi_d$ is fixed in each setting.

We estimate all $\mu$-functions by first estimating $w$-functions and using Eq. (5). For Cliff-Walking, we use a histogram estimator for $w$ as in Eq. (7). For Mountain Car, we use a kernel estimator for $w$ as in Eq. (10). We use the Epanechnikov kernel and choose an optimal bandwidth based on an $L^2$-risk criterion for $t = 1$; we then use this bandwidth for all other $t$ values as well for simplicity. For $q$-functions, we use backward-recursive regression. For Cliff-Walking, we use a histogram model, $q(s, a; \beta) = \sum_{s_j, a_k \in S, A} \beta_{jk} [s_j = s, a_k = a]$. For Mountain-Car, we use the mode $q(s, a; \beta) = \beta^\top \phi(s, a)$ where $\phi(s, a)$ is a 400-dimensional feature vector based on a radial basis function, generated using the RBFSampler method of `scikit-learn` based on Rahimi and Recht (2008).

We again compare $\hat{\rho}_{IS}$, $\hat{\rho}_{DRL(M_1)}$, $\hat{\rho}_{DM}$, $\hat{\rho}_{MIS}$, $\hat{\rho}_{DRL(M_2)}$. In each setting we consider varying evaluation dataset sizes and for each consider 1000 replications. We report the RMSE of each estimator in each setting (and the standard error) in Tables 2 and 3.

We again find that the performance of $\hat{\rho}_{DRL(M_2)}$ is superior to all other estimators in either setting. This is especially true in Cliff Walking. The estimator $\hat{\rho}_{DRL(M_2)}$ also improves upon $\hat{\rho}_{IS}$ and $\hat{\rho}_{DM}$ but not as much as $\hat{\rho}_{DRL(M_2)}$. The estimator $\hat{\rho}_{MIS}$ offers a slight improvement over $\hat{\rho}_{IS}$, but is still outperformed by $\hat{\rho}_{DRL(M_2)}$, $\hat{\rho}_{DRL(M_1)}$, and $\hat{\rho}_{DM}$.

That the improvement of $\hat{\rho}_{MIS}$ over $\hat{\rho}_{IS}$ and the overall improvements of $\hat{\rho}_{DRL(M_2)}$ is starker in Cliff Walking than in Mountain Car may be attributed to the difficulty of learning $w_t$ nonparametrically in a continuous state space.

6 Conclusions

We established the semiparametric efficiency bounds and efficient influence functions for OPE under either NMDP or MDP model, which quantify how fast one could hope to estimate policy value. While in the NMDP case, the influence function we derived has appeared frequently in OPE estimators, in the MDP case, the influence function is novel and has not
appeared in existing estimators. Our results also suggested how one could construct efficient estimators. We used this to develop DRL, which used our newly derived efficient influence function, with nuisances estimated in a cross-fold manner. This ensured efficiency under very weak and mostly agnostic conditions on the nuisance estimation method used. Notably, DRL is the first efficient OPE estimator for MDPs. In addition, DRL enjoyed double robustness properties. This efficiency and robustness translated to better performance in experiments.

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A Notation

We first summarize the notation we use in Table 4 and the abbreviations we use in Table 5. Notice in particular that, following empirical process theory literature, in the proofs we also use $\mathbb{P}$ to denote expectations (interchangeably and even simultaneously with $E$).

| Notation | Description |
| --- | --- |
| $\nabla_\beta$ | Differentiation with respect to $\beta$ |
| $r_t, s_t, a_t$ | Reward, state, action at $t$ |
| $J_{r_t}, J_{s_t}, J_{a_t}$ | History up to time $r_t, s_t, a_t$, including reward variables |
| $\mathcal{H}_{s_t}, \mathcal{H}_{a_t}$ | History up to time $s_t, a_t$, excluding reward variables |
| $\pi_t(a_t|\mathcal{H}_{s_t}), \pi_t(a_t|s_t)$ | Policy in NMDP and MDP case, respectively |
| $\pi_t^e, \pi_t^b$ | Target and behavior policies at $t$, respectively |
| $\rho_t$ | Policy value, $E[\sum_{t=0}^{\infty} r_t]$ |
| $v_t = v_t(\mathcal{H}_{s_t}), v_t(s_t)$ | Value function at $t$, in $M_1, M_2$ respectively |
| $q_t = q_t(\mathcal{H}_{a_t}), q_t(s_t, a_t)$ | $q$-function at $t$, in $M_1, M_2$ respectively |
| $\nu_t$ | Cumulative density ratio $\prod_{k=0}^{t} \pi_t^e / \pi_t^b$ |
| $\mu_t$ | Marginal density ratio $E[\nu_t | s_t, a_t]$ |
| $\eta_t$ | Instantaneous density ratio $\pi_t^e / \pi_t^b$ |
| $\Lambda$ | Tangent space |
| $\mathcal{M}$ | A model for the data generating distribution |
| $\mathcal{M}_1, \mathcal{M}_{1b}, \mathcal{M}_{1q}$ | NMDP model with unknown behavior policy, known behavior policy, and parametric $q$-function, respectively |
| $\mathcal{M}_2, \mathcal{M}_{2b}, \mathcal{M}_{2q}$ | MDP model with unknown behavior policy, known behavior policy, and parametric $q$-function, respectively |
| $C, R_{\text{max}}$ | Upper bound of density ratio and reward, respectively |
| $\prod(A|B)$ | Projection of $A$ onto $B$ |
| $\bigoplus$ | Direct sum |
| $\| \cdot \|_p$ | $L^p$-norm $E[|f|^p]^{1/p}$ |
| $\preceq^{\mathbb{P}}$ | Inequality up to constant |
| $E_\pi[\cdot], \mathbb{P}_\pi$ | Expectation with respect to a sample from a policy $\pi$ |
| $E[\cdot], \mathbb{P}$ | Same as above for $\pi = \pi^b$ |
| $E_n[\cdot], \mathbb{P}_n$ | Empirical expectation (based on sample from a behavior policy) |
| $n_j$ | The size of $\mathcal{D}_j$ |
| $E_{n_j}, \mathbb{P}_{n_j}$ | Empirical expectation on $\mathcal{D}_j$ |
| $\mathcal{G}_n$ | Empirical process $\sqrt{n}(\mathbb{P}_n - \mathbb{P})$ |
| $\text{Asmee}[\cdot], \text{var}[\cdot]$ | Asymptotic variance, variance |
| $\mathcal{N}(a, b)$ | Normal distribution with mean $a$ and variance $b$ |
| $\text{Uni}[a, b]$ | Uniform distribution on $[a, b]$ |
| $A_n = o_p(a_n)$ | The term $A_n/a_n$ converges to zero in probability |
| $A_n = O_p(a_n)$ | The term $A_n/a_n$ is bounded in probability |
| $A_d^\alpha$ | Hölder space with smoothness $\alpha$ with a dimension $d$ |
Table 5: Abbreviations

| Abbreviation | Description |
|--------------|-------------|
| NMDP         | Non-Markov Decision Process |
| MDP          | Markov Decision Process |
| RL           | Reinforcement Learning |
| CB           | Contextual Bandit |
| OPE          | Off policy Evaluation |
| MLE          | Maximum Likelihood Estimation |
| RAL          | Regular and Asymptotic Linear |
| CAN          | Consistent and Asymptotically Normal |
| MSE          | Mean Squared Error |

B Proofs

Proof of Theorem 1. The efficient influence function under $M_1$. The entire regular (regular model as defined in Chapter 7 \cite{vanderVaart1998}) parametric submodel under $M_1$ is

$$\{p_\theta(s_0)p_\theta(a_0|s_0)p_\theta(r_0|H_{a_0})p_\theta(s_1|H_{r_0})p_\theta(a_1|H_{s_1})p_\theta(r_1|H_{a_1}) \cdots p_\theta(r_T|H_{a_T})\},$$

where it matches with a true pdf at $\theta = 0$.

The score function of the model $M_1$ is decomposed as

$$g(J_s^T) = \sum_{k=0}^{T} g_{s_k|H_{a_{k-1}}} + \sum_{k=0}^{T} g_{a_k|H_{s_k}} + \sum_{k=0}^{T} g_{r_k|H_{a_k}}.$$ 

We first calculate an influence function for the target functional. Note that this influence function is not unique. We have

$$\nabla_\theta \mathbb{E}_{\pi^c} \left[ \sum_{t=0}^{T} r_t \right]$$

$$= \nabla_\theta \left[ \int \sum_{t=0}^{T} r_t \left\{ \prod_{k=0}^{T} p_\theta(s_k|H_{r_{k-1}})p_{\pi^c}(a_k|H_{s_k})p_\theta(r_k|H_{a_k}) \right\} d\mu(J_s^T) \right]$$

$$= \sum_{c=0}^{T} \{\mathbb{E}_{\pi^c}[\{\mathbb{E}_{\pi^c}(r_c|s_0) - \mathbb{E}_{\pi^c}(r_c)\}g_{s_0}] - \mathbb{E}_{\pi^c}[\{r_c - \mathbb{E}_{\pi^c}(r_c|H_{a_c})\}g_{r_c|H_{a_c}}] $$

$$- \mathbb{E}_{\pi^c} \left[ \mathbb{E}_{\pi^c} \left[ \sum_{t=c+1}^{T} r_t|H_{s_{c+1}} \right] - \mathbb{E}_{\pi^c} \left[ \sum_{t=c+1}^{T} r_t|H_{a_c} \right] \right] g_{s_{c+1}|H_{a_c}} \}$$

$$= \sum_{c=0}^{T} \{\mathbb{E}_{\pi^c}[\{\mathbb{E}_{\pi^c}(r_c|s_0) - \mathbb{E}_{\pi^c}(r_c)\}g(H_{s_{T+1}})] - \mathbb{E}_{\pi^c}[\{r_c - \mathbb{E}_{\pi^c}(r_c|H_{a_c})\}g(J_s^T)] $$

$$- \mathbb{E}_{\pi^c} \left[ \mathbb{E}_{\pi^c} \left[ \sum_{t=c+1}^{T} r_t|H_{s_{c+1}} \right] - \mathbb{E}_{\pi^c} \left[ \sum_{t=c+1}^{T} r_t|H_{a_c} \right] \right] g(J_s^T) \}.$$
\[
\begin{align*}
&= \mathbb{E} \left( -\rho^{\pi^e} + \sum_{c=0}^{T} \nu_c r_c - \left\{ \nu_c \sum_{t=c}^{T} \mathbb{E}_{\pi^e}(r_t|\mathcal{H}_{a_{t+c}}) - \nu_{c-1} \sum_{t=c}^{T} \mathbb{E}_{\pi^e}(r_t|\mathcal{H}_{a_{t+c}}) \right\} \right) g(\mathcal{J}_T).
\end{align*}
\]

This concludes that the following function is an influence function:

\[
\rho_{\text{eff}}^{M_1} = -\rho^{\pi^e} + \sum_{c=0}^{T} \nu_c r_c - \left\{ \nu_c \sum_{t=c}^{T} \mathbb{E}_{\pi^e}(r_t|\mathcal{H}_{a_{t+c}}) - \nu_{c-1} \sum_{t=c}^{T} \mathbb{E}_{\pi^e}(r_t|\mathcal{H}_{a_{t+c}}) \right\}.
\]

Next, we show that this influence function is the efficient influence function. In order to show this, we calculate the tangent space of model \( M_1 \). The (nuisance) tangent space of the model \( M_1 \) is the product space:

\[
\bigoplus_{0 \leq t \leq T} (A_t \bigoplus B_t \bigoplus C_t),
\]

\[
A_t = \{ q(s_t, \mathcal{H}_{a_{t-1}}) ; \mathbb{E}[q(s_t, \mathcal{H}_{a_{t-1}})|\mathcal{H}_{a_{t-1}}] = 0, q \in L^2 \},
\]

\[
B_t = \{ q(a_t, \mathcal{H}_{a_t}) ; \mathbb{E}[q(a_t, \mathcal{H}_{a_t})|\mathcal{H}_{a_t}] = 0, q \in L^2 \},
\]

\[
C_t = \{ q(r_t, \mathcal{H}_{a_t}) ; \mathbb{E}[q(r_t, \mathcal{H}_{a_t})|\mathcal{H}_{a_t}] = 0, q \in L^2 \}.
\]

The orthogonal space of the tangent space is the product of

\[
\bigoplus_{0 \leq t \leq T} (A'_t \bigoplus B'_t \bigoplus C'_t)
\]

such that

\[
A'_t \bigoplus A_t = A''_t, \quad A''_t = \{ q(J_{s_t}) ; \mathbb{E}[q(J_{s_t})|J_{r_{t-1}}] = 0, q \in L^2 \},
\]

\[
B'_t \bigoplus B_t = B''_t, \quad B''_t = \{ q(J_{a_t}) ; \mathbb{E}[q(J_{a_t})|J_{a_{t-1}}] = 0, q \in L^2 \},
\]

\[
C'_t \bigoplus C_t = C''_t, \quad C''_t = \{ q(J_{r_t}) ; \mathbb{E}[q(J_{r_t})|J_{a_t}] = 0, q \in L^2 \}.
\]

More specifically, we have the following lemma.

**Lemma 17.** The orthogonal tangent space is represented as

\[
A'_t = \{ q(J_{s_t}) - \mathbb{E}[q(J_{s_t})|\mathcal{H}_{s_t}] ; \mathbb{E}[q(J_{s_t})|J_{r_{t-1}}] = 0, q \in L^2 \},
\]

\[
B'_t = \{ q(J_{a_t}) - \mathbb{E}[q(J_{a_t})|\mathcal{H}_{a_t}] ; \mathbb{E}[q(J_{a_t})|J_{a_{t-1}}] = 0, q \in L^2 \},
\]

\[
C'_t = \{ q(J_{r_t}) - \mathbb{E}[q(J_{r_t})|\mathcal{H}_{a_t}, r_t] ; \mathbb{E}[q(J_{r_t})|J_{a_{t-1}}] = 0, q \in L^2 \}.
\]

**Proof.** We give a proof for \( A'_t \). Regarding the other cases, it is proved similarly. First, from the definition of the conditional expectation, \( A'_t \) and \( A_t \) are orthogonal. Thus, what we have to prove is \( \mathbb{E}[q(J_{s_t})|\mathcal{H}_{s_t}] \) is included in \( A_t \). This is proved as follows:

\[
\mathbb{E}[\mathbb{E}[q(J_{s_t})|\mathcal{H}_{s_t}]|J_{r_{t-1}}] = \mathbb{E}[q(J_{s_t})|\mathcal{H}_{a_{t-1}}] = \mathbb{E}[\mathbb{E}[q(J_{s_t})|J_{r_{t-1}}]|\mathcal{H}_{a_{t-1}}] = 0.
\]

If we can prove that the influence function Eq. (16) is orthogonal to the orthogonal tangent space Eq. (17), we can see that the above influence function is actually the efficient influence function. This fact is shown as follows.

28
**Lemma 18.** The derivative Eq. (16) is orthogonal to \( \{A'_t\}_{t=0}^{T+1}, \{B''_t\}_{t=0}^{T}, \{C'_t\}_{t=0}^{T} \).

**Proof.** The influence function is orthogonal to \( A'_k \): for \( t(J_{s_k}) \in A'_k \)

\[
E \left\{ \left\{ -\rho \pi^e + \sum_{c=0}^{T} \nu_c(H_{ac})r_c - \nu_c(H_{ac}) \sum_{t=c}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ac}] - \nu_{c-1}(H_{ac-1}) \sum_{t=c}^{T} \mathbb{E}_{\pi^e}[r_t|H_{sc}] \right\} \right\} t(J_{s_k})
\]

\[
= E \left\{ \sum_{c=k}^{T} \nu_c(H_{ac})r_c - \nu_{k-1} \sum_{t=k}^{T} \mathbb{E}_{\pi^e}[r_t|H_{sk}] \right\} t(J_{s_k})
\]

\[
= 0.
\]

The influence function is orthogonal to \( B''_k \): for \( t(J_{a_k}) \in B''_k \)

\[
E \left\{ \left\{ -\rho \pi^e + \sum_{c=0}^{T} \nu_c r_c - \nu_c \sum_{t=c}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ac}] - \nu_{c-1} \sum_{t=c}^{T} \mathbb{E}_{\pi^e}[r_t|H_{sc}] \right\} \right\} t(J_{a_k})
\]

\[
= E \left\{ \sum_{c=k}^{T} \nu_c r_c - \nu_k \sum_{t=k}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ak}] \right\} t(J_{a_k})
\]

\[
= E \left\{ \nu_k \sum_{t=k}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ak}] - \nu_k \sum_{t=k}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ak}] \right\} t(J_{a_k}) = 0.
\]

The influence function is orthogonal to \( C'_k \): for \( t(J_{r_k}) \in C'_k \)

\[
E \left\{ \left\{ -\rho \pi^e + \sum_{c=0}^{T} \nu_c(H_{ac})r_c - \nu_c(H_{ac}) \sum_{t=c}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ac}] - \nu_{c-1}(H_{ac-1}) \sum_{t=c}^{T} \mathbb{E}_{\pi^e}[r_t|H_{sc}] \right\} \right\} t(J_{r_k})
\]

\[
= E \left\{ \sum_{c=k}^{T} \nu_c(H_{ac})r_c \right\} t(J_{r_k})
\]

\[
= E \left\{ \nu_{k-1} \sum_{t=k}^{T} \mathbb{E}_{\pi^e}[r_t|J_{r_k}] \right\} t(J_{r_k}) = E \left\{ \nu_{k-1} \sum_{t=k}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ak}, r_k] \right\} t(J_{r_k})
\]

\[
= E \left\{ \nu_{k-1} \sum_{t=k}^{T} \mathbb{E}_{\pi^e}[r_t|H_{ak}, r_k] \right\} E[t(J_{r_k})|H_{ak}, r_k] = 0.
\]

This concludes the proof for \( \mathcal{M}_1 \).

**Efficient influence function under \( \mathcal{M}_{1b} \).** Next, we show that the efficiency bound is still the same even if we know the target policy. To show that, we derive an orthogonal space of the tangent space of the regular parametric submodel:

\[
\{p_\theta(s_0)p(a_0|s_0)p_\theta(r_0|H_{a_0})p_\theta(s_1|H_{r_0})p(a_1|H_{a_1})p_\theta(r_1|H_{a_1}) \cdots p_\theta(r_T|H_{a_T})\},
\]
where \( p(a_t | H_{s_t}) \) is fixed at \( \pi_t^b \). This is equal to

\[
\bigoplus_{0 \leq t \leq T} (A'_t \bigoplus B'_t \bigoplus C'_t)
\]  \hspace{1cm} (18)

This space Eq. (18) is orthogonal to the obtained efficient influence function under \( M_1 \). Therefore, the efficient influence function under \( M_{1b} \) is the same as the one under \( M_b \).

**Efficiency bound.** We use a law of total variance (Bowsher and Swain, 2012) to compute the variance of the efficient influence function.

\[
\begin{align*}
\text{var} & \left[ \sum_{t=0}^{T} (\nu_t r_t - (\nu_t q_t - \nu_{t-1} v_t)) \right] \\
& = \sum_{t=0}^{T+1} E \left[ \text{var} \left( E \left[ \nu_{t-1} r_{t-1} + \sum_{k=0}^{T} (\nu_k r_k - \nu_k q_k - \nu_{k-1} v_k) \right] | J_{a_t} \right) | J_{a_{t-1}} \right] \\
& = \sum_{t=0}^{T+1} E \left[ \text{var} \left( E \left[ \nu_{t-1} r_{t-1} + \sum_{k=t}^{T} (\nu_k r_k - \nu_k q_k - \nu_{k-1} v_k) \right] | J_{a_t} \right) | J_{a_{t-1}} \right] \\
& = \sum_{t=0}^{T+1} E \left[ \text{var} \left( E \left[ \nu_{t-1} r_{t-1} + \left( \sum_{k=t}^{T} \nu_k r_k \right) - \nu_t q_t - \nu_{t-1} v_t \right] | J_{a_t} \right) | J_{a_{t-1}} \right] \\
& = \sum_{t=0}^{T+1} E \left[ \nu_{t-1}^2 \text{var} (r_{t-1} + v_t (H_{s_t}) | H_{a_{t-1}}) \right].
\end{align*}
\]

Here, we used \( E[\sum_{k=t}^{T} \nu_k r_k | J_{a_k}] = \nu_k q_k \).

**Proof of Theorem 2**

**Efficient influence function under \( M_2 \).** The entire regular parametric submodel is

\[
\{ p_0(s_0)p_\theta(a_0 | s_0)p_\theta(r_0 | s_0, a_0)p_\theta(s_1 | s_0, a_0)p_\theta(a_1 | s_1)p_\theta(r_1 | s_1, a_1) \cdots p_\theta(r_T | s_T, a_T) \}.
\]

The score function of the parametric submodel is

\[
g(J_{s_T}) = \sum_{k=0}^{T} g_k | s_{k-1}, a_{k-1} + \sum_{k=0}^{T} g_{k+1} | s_k + \sum_{k=0}^{T} g_r | s_k, a_k.
\]

We first calculate the influence function of the target function. Note that this influence function is not only influence function. We have

\[
\begin{align*}
\nabla_\theta E_{\pi^t} \left[ \sum_{t=0}^{T} r_t \right] \\
= \nabla_\theta \int \sum_{t=0}^{T} r_t \left( \prod_{t=0}^{T} p_\theta(s_k | a_{k-1}, s_{k-1}) p_\pi^t(a_k | s_k)p_\theta(r_k | s_k, a_k) \right) d\mu(J_{s_T})
\end{align*}
\]
We will show this influence function is the efficient influence function:

\[ M = -\sum_{c=0}^{T} r_{t}|s_{c+1}| - \sum_{c=t+1}^{T} r_{t}|s_{c}, a_{c} \times g_{s_{c+1}|s_{c}, a_{c}} \]

\[ = -E_{\pi} \left[ \left( E_{\pi}[T] r_{t}|s_{c+1}|-E_{\pi}[T] r_{t}|s_{c}, a_{c} \right) g_{s_{c+1}|s_{c}, a_{c}} \right] \]

\[ = E \left[ \frac{p_{\pi}^{e}(s_{c}, a_{c})}{p_{\pi}^{b}(s_{c}, a_{c})} \left( E_{\pi}[T] r_{t}|s_{c+1}|-E_{\pi}[T] r_{t}|s_{c}, a_{c} \right) g_{s_{c+1}|s_{c}, a_{c}} \right] \]

\[ = E \left[ \left( -\rho_{e}^{e} - \sum_{c=0}^{T} \frac{p_{\pi}^{e}(s_{c}, a_{c})}{p_{\pi}^{b}(s_{c}, a_{c})} r_{t} - \frac{p_{\pi}^{e}(s_{c}, a_{c})}{p_{\pi}^{b}(s_{c}, a_{c})} \sum_{t=0}^{T} E_{\pi}[T] r_{t}|s_{c}, a_{c} |- \frac{p_{\pi}^{e}(s_{c-1}, a_{c-1})}{p_{\pi}^{b}(s_{c-1}, a_{c-1})} \sum_{t=0}^{T} E_{\pi}[T] r_{t}|s_{c}, a_{c} ] \right] \right] g(J_{S_{T}}) \]

Therefore, the following function is an influence function:

\[-\rho_{e}^{e} + \sum_{c=0}^{T} \frac{p_{\pi}^{e}(s_{c}, a_{c})}{p_{\pi}^{b}(s_{c}, a_{c})} r_{t} - \frac{p_{\pi}^{e}(s_{c}, a_{c})}{p_{\pi}^{b}(s_{c}, a_{c})} \sum_{t=0}^{T} E_{\pi}[T] r_{t}|s_{c}, a_{c} |- \frac{p_{\pi}^{e}(s_{c-1}, a_{c-1})}{p_{\pi}^{b}(s_{c-1}, a_{c-1})} \sum_{t=0}^{T} E_{\pi}[T] r_{t}|s_{c}, a_{c} ] \right] \right] g(J_{S_{T}}) \]

\[ (19) \]

We will show this influence function is the efficient influence function.

In order to show this, we calculate the tangent space of model \( M_{2} \). The tangent space of the model \( M_{2} \) is the product space:

\[ \bigoplus_{0 \leq t \leq T} (A_{t} \bigoplus B_{t} \bigoplus C_{t}) \]

\[ A_{t} = \{ q(s_{t}, s_{t-1}, a_{t-1}); E[q(s_{t}, s_{t-1}, a_{t-1})|s_{t-1}, a_{t-1}] = 0, q \in L^{2} \}, \]

\[ B_{t} = \{ q(a_{t}, s_{t}); E[q(a_{t}, s_{t})|s_{t}] = 0, q \in L^{2} \}, \]

\[ C_{t} = \{ q(r_{t}, s_{t}, a_{t}); E[q(r_{t}, s_{t}, a_{t})|s_{t}, a_{t}] = 0, q \in L^{2} \}. \]

The orthogonal space of the tangent space is the product of

\[ \bigoplus_{0 \leq t \leq T} (A'_{t} \bigoplus B'_{t} \bigoplus C'_{t}) \]

\[ (20) \]

such that

\[ A'_{t} \bigoplus A_{t} = A''_{t}, \quad A''_{t} = \{ q(J_{s_{t}}); E[q(J_{s_{t}})|J_{r_{t-1}}] = 0, q \in L^{2} \}, \]

\[ B'_{t} \bigoplus B_{t} = B''_{t}, \quad B''_{t} = \{ q(J_{a_{t}}); E[q(J_{a_{t}})|J_{s_{t}}] = 0, q \in L^{2} \}, \]

\[ C'_{t} \bigoplus C_{t} = C''_{t}, \quad C''_{t} = \{ q(J_{r_{t}}); E[q(J_{r_{t}})|J_{a_{t}}] = 0, q \in L^{2} \}. \]

More specifically, the orthogonal tangent space is represented as

\[ A'_{t} = \{ q(J_{s_{t}}) - E[q(J_{s_{t}})|s_{t}, a_{t-1}, s_{t-1}]; E[q(J_{s_{t}})|J_{r_{t-1}}] = 0, q \in L^{2} \}, \]

\[ 31 \]
\[ B'_t = \{ q(\mathcal{J}_{a_t}) - E[q(\mathcal{J}_{r_t})|s_t, a_t]; E[q(\mathcal{J}_{a_t})|\mathcal{J}_{s_t}] = 0, q \in L^2 \}, \]
\[ C'_t = \{ q(\mathcal{J}_{r_t}) - E[q(\mathcal{J}_{r_t})|r_t, s_t, a_t]; E[q(\mathcal{J}_{r_t})|\mathcal{J}_{a_t}] = 0, q \in L^2 \}. \]

If we can prove that the influence function Eq. (19) is orthogonal to the orthogonal tangent space Eq. (20), we can see that the above influence function is actually the efficient influence function. This fact is shown as follows.

**Lemma 19.** The derivative Eq. (19) is orthogonal to \( \{ A'_t \}_{t=0}^{T+1} \), \( \{ B''_t \}_{t=0}^{T} \), \( \{ C'_t \}_{t=0}^{T} \).

**Proof.** First, the influence function Eq. (19) is orthogonal to \( A'_k \); for \( t(\mathcal{J}_{s_k}) \in A'_k \)

\[
\begin{align*}
E \left[ \left\{ v_0 + \sum_{t=0}^{T} \mu_t(s_t, a_t)(r_t + v_{t+1} - q_t) \right\} t(\mathcal{J}_{s_k}) \right] \\
= E \left[ \left\{ \sum_{t=k-1}^{T} \mu_t(s_t, a_t)(r_t + v_{t+1} - q_t) \right\} t(\mathcal{J}_{s_k}) \right] \\
= E \left[ \mu_{k-1}(s_{k-1}, a_{k-1})(r_{k-1} + v_k - q_{k-1})t(\mathcal{J}_{s_k}) \right] \\
= E \left[ \mu_{k-1}(s_{k-1}, a_{k-1})v_k t(\mathcal{J}_{s_k}) \right] \\
= E \left[ \mu_{k-1}(s_{k-1}, a_{k-1})v_k E[t(\mathcal{J}_{s_k})|s_k, a_{k-1}, s_{k-1}] \right] = 0
\end{align*}
\]

Second, the influence function Eq. (19) is orthogonal to \( B''_k \); for \( t(\mathcal{J}_{a_k}) \in B''_k \)

\[
\begin{align*}
E \left[ \left\{ v_0 + \sum_{t=0}^{T} \mu_t(s_t, a_t)(r_t + v_{t+1} - q_t) \right\} t(\mathcal{J}_{a_k}) \right] \\
= E \left[ \left\{ \sum_{t=k}^{T} \mu_t(s_t, a_t)(r_t + v_{t+1} - q_t) \right\} t(\mathcal{J}_{a_k}) \right] = 0.
\end{align*}
\]

Third, the influence function Eq. (19) is orthogonal to \( C'_k \); for \( t(\mathcal{J}_{r_k}) \in C'_k \)

\[
\begin{align*}
E \left[ \left\{ v_0 + \sum_{t=0}^{T} \mu_t(s_t, a_t)(r_t + v_{t+1} - q_t) \right\} t(\mathcal{J}_{r_k}) \right] \\
= E \left[ \left\{ \sum_{t=k}^{T} \mu_t(s_t, a_t)(r_t + v_{t+1} - q_t) \right\} t(\mathcal{J}_{r_k}) \right] \\
= E \left[ \mu_k(s_k, a_k)(r_k + v_{k+1} - q_k) \right] t(\mathcal{J}_{r_k}) \\
= E \left[ \mu_k(s_k, a_k)(E[r_k + E[v_{k+1}|\mathcal{J}_{r,k}] - q_k]) \right] t(\mathcal{J}_{r_k}) \\
= E \left[ \mu_k(s_k, a_k)(r_k + E[v_{k+1}|s_k] - q_k) \right] E[t(\mathcal{J}_{r_k})|s_k, a_k, r_k] = 0.
\end{align*}
\]

**Efficient influence function under \( \mathcal{M}_{2b} \).** In Lemma 19, we check that the \( \phi_{\text{eff}}^{M_2} \) is orthogonal to \( B''_t \). This concludes the proof noting that the orthogonal tangent space of \( \mathcal{M}_{2b} \) is

\[
\bigoplus_{0 \leq t \leq T} (A'_t \bigoplus B''_t \bigoplus C'_t).
\]
Efficiency bound. To show an efficiency bound, we use a law of total variance \cite{Bowsher:2012}. Recall that we can also easily derive this variance form using another equivalent form of efficient influence function.

Proof of Theorem 4. Define

\begin{equation}
\frac{\sqrt{n}}{n_0} \mathbb{P}_{n_0} \phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) + \frac{n_1}{n} \mathbb{P}_{n_1} \phi(\{\hat{\nu}_k^{(0)}\}, \{\hat{q}_k^{(0)}\}),
\end{equation}

where \(\mathbb{P}_{n_0}\) is an empirical approximation based on a set of samples such that \(J = 0\), \(\mathbb{P}_{n_1}\) is an empirical approximation based on a set of samples such that \(J = 1\). Then we have

\begin{equation}
\sqrt{n/n_0} \mathbb{P}_{n_0} \phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \rho^{\pi_e} = \sqrt{n/n_0} \mathbb{P}_{n_0} \phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{\nu_k\}, \{q_k\})
\end{equation}

(21)
We analyze each term. To do that, we use the following relation:

\[ \phi(\{\hat{\nu}_k\}, \{\hat{q}_k\}) - \phi(\nu_k, q_k) = D_1 + D_2 + D_3, \quad \text{where} \]

\[ D_1 = \sum_{k=0}^{T} (\hat{\nu}_k - \nu_k)(-\hat{q}_k + q_k) + (\hat{\nu}_{k-1} - \nu_{k-1})(-\hat{\nu}_k + \nu_k), \]

\[ D_2 = \sum_{k=0}^{T} \nu_k(\hat{q}_k - q_k) + \nu_{k-1}(\hat{\nu}_k - \nu_k), \]

\[ D_3 = \sum_{k=0}^{T} (\hat{\nu}_k - \nu_k)(r_k - q_k + \nu_{k+1}). \]

First, we show the term Eq. (21) is \( O_p(1) \).

**Lemma 20.** The term Eq. (21) is \( O_p(1) \).

**Proof.** If we can show that for any \( \epsilon > 0, \)

\[ \lim_{n \to \infty} \sqrt{n}P[\mathbb{P}_{n_0}[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\nu_k, q_k)]
- E[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\nu_k, q_k)|\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] > \epsilon |D_1| = 0, \]

Then, by bounded convergence theorem, we would have

\[ \lim_{n \to \infty} \sqrt{n}P[\mathbb{P}_{n_0}[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\nu_k, q_k)]
- E[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\nu_k, q_k)|\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] > \epsilon] = 0, \]

yielding the statement.

To show Eq. (24), we show that the conditional mean is 0 and conditional variance is \( O_p(1) \). The conditional mean is

\[ E[\mathbb{P}_{n_0}[\phi(\{\nu_k^{(1)}\}, \{q_k^{(1)}\}) - \phi(\nu_k, q_k)|\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] \]

\[ - \mathbb{P}[\phi(\{\nu_k^{(1)}\}, \{q_k^{(1)}\}) - \phi(\nu_k, q_k)|D_1] = 0. \]

Here, we leveraged the sample splitting construction, that is, \( \hat{\nu}_k^{(1)} \) and \( \hat{q}_k^{(1)} \) only depend on \( D_1 \). The conditional variance is

\[ \text{var}[\sqrt{n}\mathbb{P}_{n_0}[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\nu_k, q_k)|D_1]]
= E[E[D_1^2 + D_2^2 + D_3^2 + 2D_1D_2 + 2D_2D_3 + 2D_2D_3|\{\hat{q}_k^{(1)}\}, \{\nu_k^{(1)}\}]|D_1]
= O_p(1). \]
Here, we used the convergence rate assumption and the relation \( \| \hat{\epsilon}_k^{(1)} - v_k \|_2 < \| \hat{q}_k^{(1)} - q_k \|_2 \) arising from the fact that the former is the marginalization of the latter over \( \pi_k^* \). Then, from Chebyshev's inequality:

\[
\sqrt{n_0}P[\mathbb{P}_{n_0}[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{v_k\}, \{q_k\})] - E[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\})] - \phi(\{v_k\}, \{q_k\})[\nu_k] > \epsilon |\mathcal{D}_1] \leq \frac{1}{\epsilon^2} \text{var}[\sqrt{n_0}\mathbb{P}_{n_0}[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{v_k\}, \{q_k\})][\mathcal{D}_1] = o_p(1). \]

\[\]

Lemma 21. The term Eq. (23) is \( o_p(1) \).

**Proof.**

\[
\sqrt{n}E[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\})] - E[\phi(\{v_k\}, \{q_k\})][\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] = \sqrt{n}E[\sum_{k=0}^{T} (\hat{\nu}_k^{(1)} - v_k)(-q_k^{(1)} + q_k) + (\hat{\nu}_k^{(1)} - v_k - q_k)(-\hat{\nu}_k^{(1)} + v_k) + \hat{q}_k^{(1)} - q_k)]
\]

\[
+ \sqrt{n}E[\sum_{k=0}^{T} \nu_k(\hat{q}_k^{(1)} - q_k) + \hat{v}_k^{(1)}(\hat{\nu}_k^{(1)} - v_k)]
\]

\[
+ \sqrt{n}E[\sum_{k=0}^{T} (\hat{\nu}_k^{(1)} - v_k)(r_k - q_k + v_k) + \hat{q}_k^{(1)}]
\]

\[
= \sqrt{n}E[\sum_{k=0}^{T} (\hat{\nu}_k^{(1)} - v_k)(-q_k^{(1)} + q_k) + (\hat{\nu}_k^{(1)} - v_k - q_k)(-\hat{\nu}_k^{(1)} + v_k) + \hat{q}_k^{(1)} - q_k)]
\]

\[
= \sqrt{n}E[\|\hat{\nu}_k^{(1)} - v_k\|_2]\|\hat{q}_k^{(1)} - q_k\|_2 = \sqrt{n} \sum_{k=0}^{T} o_p(n^{-\alpha_1})o_p(n^{-\alpha_2}) = o_p(1).
\]

Finally, we get

\[
\sqrt{n}(\mathbb{P}_{n_0}[\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \rho^{\pi^*}) = \sqrt{n/n_0}G_{n_0}[\phi(\{v_k\}, \{q_k\})] + o_p(1).
\]

Therefore,

\[
\sqrt{n}(\hat{\rho}^{\pi^*}_{\text{DRL}}(\mathcal{M}_1) - \rho^{\pi^*})
\]

\[
= n_0/n \sqrt{n}\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \rho^{\pi^*}) + n_1/n \sqrt{n}\mathbb{P}_{n_1}[\phi(\{\hat{v}_k^{(0)}\}, \{\hat{q}_k^{(0)}\}) - \rho^{\pi^*})
\]

\[
= \sqrt{n_0/n}G_{n_0}[\phi(\{v_k\}, \{q_k\})] + \sqrt{n_1/n}G_{n_1}[\phi(\{v_k\}, \{q_k\})] + o_p(1)
\]

\[
= G_{n_0}[\phi(\{v_k\}, \{q_k\})] + o_p(1),
\]

concluding the proof by showing the influence function of \( \hat{\rho}^{\pi^*}_{\text{DRL}}(\mathcal{M}_1) \) is the efficient one. \[\]

**Proof of Theorem.** We define \( \phi(\{\hat{\nu}_k\}, \{\hat{q}_k\}) \) as:

\[
\sum_{k=0}^{T} \hat{\nu}_k r_k - \hat{\nu}_{k-1} q_k - \mathbb{E}_{\pi^*}[\hat{q}_k(\mathcal{H}_{a_k})|\mathcal{H}_{s_k}] = \mathbb{E}_{\pi^*}[\hat{q}_k(\mathcal{H}_{a_k})|\mathcal{H}_{s_k}].
\]

35
The estimator $\hat{\rho}^{\pi^c}_{\text{DR}}$ is given by $\mathbb{P}_n\phi(\{\hat{\nu}_k\}, \{\hat{q}_k\})$. Then, we have

$$\sqrt{n}(\mathbb{P}_n\phi(\{\hat{\nu}_k\}, \{\hat{q}_k\}) - \rho^{\pi^c}) = \mathbb{G}_n[\phi(\{\hat{\nu}_k\}, \{\hat{q}_k\}) - \rho^{\pi^c}] + \mathbb{G}_n[\phi(\nu_k, q_k)] + \sqrt{n}(E[\phi(\{\hat{\nu}_k\}, \{\hat{q}_k\})|\nu_k, \{q_k\}] - \rho^{\pi^c}). \quad (25)$$

If we can prove that the term Eq. (25) is $o_p(1)$, the statement is concluded as in the proof of Theorem 4. We proceed to prove this.

First, we show $\phi(\{\hat{\nu}_k\}, \{\hat{q}_k\}) - \phi(\nu_k, q_k)$ belongs to a Donsker class. The transformation

$$(\nu_k, q_k) \rightarrow \sum_{k=0}^{T} \nu_k r_k - \nu_k q_k - \nu_{k-1} E^{\pi^c}[q_k|\mathcal{H}_{s_k}]$$

is a Lipschitz function. Therefore, by Example 19.20 in van der Vaart (1998), $\phi(\{\hat{\nu}_k\}, \{\hat{q}_k\}) - \phi(\nu_k, q_k)$ is an also Donsker class. In addition, we can also show that

$$\|\phi(\{\hat{\nu}_k\}, \{\hat{q}_k\}) - \phi(\nu_k, q_k)\|_2 = o_p(1),$$

as in Lemma 20. Therefore, from Lemma 19.24 in van der Vaart (1998), the term Eq. (25) is $o_p(1)$, concluding the proof.

**Proof of Theorem 7.** We use the following doubly robust structure

$$E \left[ \sum_{k=0}^{T} \nu_k r_k - \nu_k q_k - \nu_{k-1} E^{\pi^c}[q_k|\mathcal{H}_{s_k}] \right] = E\{E^{\pi^c}(q_0|s_0)\} + E \left[ \sum_{k=0}^{T} \nu_k r_k - \nu_k q_k + E^{\pi^c}(q_k|\mathcal{H}_{s_{k+1}}) \right] = \rho^{\pi^c}.$$ 

Then, as in the proof of Theorem 4

$$\sqrt{n}(\mathbb{P}_n\phi(\{\hat{\nu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \rho^{\pi^c})$$

$$= \sqrt{n/n_0\mathbb{G}_{n_0}}[\phi(\{\nu_k^{(1)}\}, \{q_k^{(1)}\}) - \phi(\nu_k^{(1)}, q_k^{(1)})] + \sqrt{n/n_0\mathbb{G}_{n_0}}[\phi(\nu_k^{(1)}, q_k^{(1)})] + \sqrt{n}(E[\phi(\nu_k^{(1)}, q_k^{(1)})] - \rho^{\pi^c})$$

$$= \sqrt{n/n_0\mathbb{G}_{n_0}}[\phi(\nu_k^{(1)}, q_k^{(1)})] + \sqrt{n}(E[\phi(\nu_k^{(1)}, q_k^{(1)})] - \rho^{\pi^c}) + o_p(1).$$

Here, we used

$$\sqrt{n/n_0\mathbb{G}_{n_0}}[\phi(\nu_k^{(1)}, q_k^{(1)})] - \phi(\nu_k^{(1)}, q_k^{(1)})] = o_p(1)$$

from Lemma 20 and

$$\sqrt{n/n_0}(E[\phi(\nu_k^{(1)}, q_k^{(1)})] - E[\phi(\nu_k^{(1)}, q_k^{(1)})]) = o_p(1),$$

which we prove below as in Lemma 21.
Lemma 22.\[\sqrt{n/n_0}(E[\phi(\{\nu_k^{(1)}\}, \{q_k^{(1)}\})|\nu_k^{(1)}, \{\hat{q}_k^{(1)}\}] - E[\phi(\{\nu_k\}, \{q_k\})]) = o_p(1).\]

Proof. First, consider the case where \(\nu_k = \nu_k^{\dagger}\).

\[\sqrt{n}E[\phi(\{\nu_k^{(1)}\}, \{q_k^{(1)}\})] - E[\phi(\{\nu_k\}, \{q_k\})]\]

\[= \sqrt{n}E\left(\sum_{k=0}^{T} (\nu_k^{(1)} - \nu_k) (-\hat{q}_k^{(1)} + q_k^{\dagger}) + (\nu_{k-1}^{(1)} - \nu_k) (-\hat{\nu}_k + v_k^{\dagger})\right)\]

\[= \sqrt{n}E\left(\sum_{k=0}^{T} \nu_k (\hat{q}_k^{(1)} - q_k^{\dagger}) + \nu_{k-1} (-\hat{\nu}_k + v_k^{\dagger})\right)\]

\[= \sqrt{n} \sum_{k=0}^{T} O(\|\nu_k^{(1)} - \nu_k\|_2\|q_k^{(1)} - q_k^{\dagger}\|_2 + \|\hat{\nu}_k^{(1)} - \nu_k\|_2^2)\]

\[= \sqrt{n} \sum_{k=0}^{T} \{c_p(n^{-\alpha_1})c_p(n^{-\alpha_2}) + c_p(n^{-2\alpha_1, 1})\} = c_p(1).\]

Next, consider the case where \(q_k = q_k^{\dagger}\):

\[\sqrt{n}E[\phi(\{\nu_k^{(1)}\}, \{q_k^{(1)}\})] - E[\phi(\{\nu_k^{\dagger}\}, \{q_k\})]\]

\[= \sqrt{n}E\left(\sum_{k=0}^{T} \nu_k (\hat{q}_k^{(1)} - q_k^{\dagger}) + \nu_{k-1} (\hat{\nu}_k - v_k^{\dagger})\right)\]

\[= \sqrt{n} \sum_{k=0}^{T} O(\|\nu_k^{(1)} - \nu_k\|_2\|q_k^{(1)} - q_k^{\dagger}\|_2 + \|\hat{\nu}_k^{(1)} - \nu_k\|_2^2)\]

\[= \sqrt{n} \sum_{k=0}^{T} \{c_p(n^{-\alpha_1})c_p(n^{-\alpha_2}) + c_p(n^{-2\alpha_1, 2})\} = c_p(1).\]

Using the above result, we prove the statement for each case below.
\( \nu\)-model is well-specified. First, consider the case when \( \nu_k^\dagger = \nu_k \):

\[
E[\phi(\{\nu_k\}, \{q_k^1\})] = E[\sum_{k=0}^{T} \nu_k r_k - \nu_k q_k^1 \langle H_{a_k} \rangle - \nu_{k-1} E_{\pi^e}[q_k^1 \langle H_{a_k} \rangle | s_k]]
\]

Then,

\[
\sqrt{n}(\mathbb{P}_{n_0} \phi(\{\hat{\nu}_k^1\}, \{\hat{q}_k^1\}) - \rho^{\pi^e}) = \sqrt{n/n_0 \mathbb{G}_{n_0} [\phi(\{\nu_k\}, \{q_k^1\})]} + o_p(1).
\]

Therefore,

\[
\sqrt{n}(\hat{\rho}_{\text{DRL}}^{\pi^e} - \rho^{\pi^e}) = \sqrt{n/n_0 \mathbb{G}_{n_0} [\phi(\{\nu_k\}, \{q_k^1\})]} + \sqrt{n_1/n \mathbb{G}_{n_1} [\phi(\{\nu_k\}, \{q_k^1\})]} + o_p(1)
\]

which shows \( \hat{\rho}_{\text{DRL}}^{\pi^e} \) is CAN around \( \rho^{\pi^e} \) when the model for the behavior policy is well-specified.

\( q\)-model is well-specified. Next, consider the case where \( q_k^1 = q_k \).

\[
E[\phi(\{\nu_k^1\}, \{q_k\})] = E \left[ E_{\pi^e}[q_0(\langle H_{a_0} \rangle) | s_0] + \sum_{k=0}^{T} \nu_k^1 r_k - q_k \langle H_{a_k} \rangle + E_{\pi^e}[q_k(\langle H_{a_k} \rangle) | s_{k+1}] \right]
\]

Then,

\[
\sqrt{n}(\mathbb{P}_{n_0} \phi(\{\hat{\nu}_k^1\}, \{\hat{q}_k^1\}) - \rho^{\pi^e}) = \sqrt{n/n_0 \mathbb{G}_{n_0} [\phi(\{\nu_k^1\}, \{q_k\})]} + o_p(1).
\]

Therefore,

\[
\sqrt{n}(\hat{\rho}_{\text{DRL}}^{\pi^e} - \rho^{\pi^e}) = \sqrt{n/n_0 \mathbb{G}_{n_0} [\phi(\{\nu_k^1\}, \{q_k\})]} + \sqrt{n_1/n \mathbb{G}_{n_1} [\phi(\{\nu_k^1\}, \{q_k\})]} + o_p(1)
\]

which shows \( \hat{\rho}_{\text{DRL}}^{\pi^e} \) is CAN around \( \rho^{\pi^e} \) when the model for the \( q\)-function is well-specified.

\( \square \)

**Proof of Lemma** We have

\[
\left\| \frac{1}{T} \sum_{t=0}^{T} \pi_t^{\pi^e} - \nu_k \right\| _2 \leq \left\| \sum_{i=0}^{T} \left( \prod_{t=0}^{i} \frac{\pi_t^{\pi^e}}{\pi_t^{\pi^e}} \prod_{t=i+1}^{T} \frac{\pi_t^{\pi^e}}{\prod_{t=0}^{T} \pi_t^{\pi^e}} \right) \right\| _2
\]

\[
\leq \sum_{t=0}^{k} \mathcal{O}(\| \pi_t^{\pi^e}/\pi_t^{\pi^e} - \eta_k \| _2)
\]

\[= \mathcal{O}_p(n^{-(\alpha/(\alpha+d_{\text{aug}}))}). \square \]
Proof of Theorem 6 Define $\phi(\{\hat{\mu}_k\}, \{\hat{q}_k\})$ as:

$$
\sum_{k=0}^{T} \hat{\mu}_k r_k - \hat{v}_{k-1} \{\hat{q}_k \hat{q}_k - \mathbb{E}_x[\hat{q}_k|\mathcal{H}_{a_k}|\mathcal{H}_{s_k}]\}.
$$

The estimator $\hat{\rho}_{\text{DRL}(M_2)}$ is given by

$$
\frac{n_0}{n} \mathbb{P}_{n_0} \phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}) + \frac{n_1}{n} \mathbb{P}_{n_1} \phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}).
$$

Then, we have

$$
\sqrt{n} (\mathbb{P}_{n_0} \phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}) - \rho^*) = \sqrt{n/n_0} \mathbb{P}_{n_0} \phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}) - \phi(\{\mu_k \}, \{q_k \}) \tag{28}
$$

$$
+ \sqrt{n/n_0} \mathbb{P}_{n_0} [\phi(\{\mu_k \}, \{q_k \})] + \sqrt{n} (\mathbb{E} [\phi(\{\hat{\mu}_k \}, \{\hat{q}_k \})[\{\hat{\mu}_k \}, \{\hat{q}_k \}] - \rho^*). \tag{29}
$$

We analyze each term. To do that, we use the following relation;

$$
\phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}) - \phi(\{\mu_k \}, \{q_k \}) = D_1 + D_2 + D_3, \quad \text{where}
$$

$$
D_1 = \sum_{k=0}^{T} (\hat{\mu}_k - \mu_k)(-\hat{q}_k + q_k) + (\hat{\mu}_k - \mu_k)(-\hat{v} + v_k),
$$

$$
D_2 = \sum_{k=0}^{T} \mu_k (\hat{q}_k - q_k) + \mu_k (\hat{v} - v_k),
$$

$$
D_3 = \sum_{k=0}^{T} (\hat{\mu}_k - \mu_k)(r_k - q_k + v_{k+1}).
$$

First, we show the term Eq. \ref{eq:28} is $\mathcal{O}_p(1)$.

Lemma 23. The term Eq. \ref{eq:28} is $\mathcal{O}_p(1)$.

Proof. If we can show that for any $\epsilon > 0$,

$$
\lim_{n \to \infty} \sqrt{n} \mathbb{P}_{n_0} \phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}) - \phi(\{\mu_k \}, \{q_k \}) - \mathbb{E} [\phi(\{\hat{\mu}_k \}, \{\hat{q}_k \})|\{\hat{\mu}_k \}, \{\hat{q}_k \}] > \epsilon |D_1| = 0. \tag{31}
$$

Then, by bounded convergence theorem, we have

$$
\lim_{n \to \infty} \sqrt{n} \mathbb{P}_{n_0} \phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}) - \phi(\{\mu_k \}, \{q_k \}) - \mathbb{E} [\phi(\{\hat{\mu}_k \}, \{\hat{q}_k \})|\{\hat{\mu}_k \}, \{\hat{q}_k \}] > \epsilon |D_1| = 0,
$$

yielding the statement.

To show Eq. \ref{eq:31}, we show that the conditional mean is 0 and conditional variance is $\mathcal{O}_p(1)$. The conditional mean is

$$
\mathbb{E} [\mathbb{P}_{n_0} \phi(\{\hat{\mu}_k \}, \{\hat{q}_k \}) - \phi(\{\mu_k \}, \{q_k \})|\{\hat{\mu}_k \}, \{\hat{q}_k \}] =
$$
\[ \mathbb{P}[\phi(\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{\mu_k\}, \{q_k\}) | D_1] = 0. \]

Here, we used a sample splitting construction, that is, \( \hat{\mu}_k^{(1)} \) and \( \hat{q}_k^{(1)} \) only depend on \( D_1 \). The conditional variance is

\[
\begin{align*}
\text{var}[\sqrt{n_0} \mathbb{P}_{n_0}[\phi(\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{\mu_k\}, \{q_k\}) | D_1] & = E[E[D_1^2 + D_2^2 + D_3 + 2D_1D_2 + 2D_2D_3 | \{\hat{q}_k^{(1)}\}, \{\mu_k^{(1)}\}] | D_1] \\
& = \sigma_p(1).
\end{align*}
\]

Here, we used the convergence rate assumption and the relation \( \| \hat{v}_k^{(1)} - v_k \|_2 \leq \| \hat{q}_k^{(1)} - q_k \|_2 \).

Then, from Chebyshev’s inequality,

\[
\sqrt{n_0} \mathbb{P}[\mathbb{P}_{n_0}[\phi(\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{\mu_k\}, \{q_k\})] - \phi(\{\mu_k\}, \{q_k\}) | D_1] > \epsilon | D_1] \leq \frac{1}{\epsilon^2} \text{var}[\sqrt{n_0} \mathbb{P}_{n_0}[\phi(\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{\mu_k\}, \{q_k\}) | D_1] = \sigma_p(1). \]

**Lemma 24.** The term Eq. (30) is \( \sigma_p(1) \).

**Proof.**

\[
\begin{align*}
\sqrt{n} E[\phi(\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \phi(\{\mu_k\}, \{q_k\}) | \{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] & = \sqrt{n} E[\sum_{k=0}^T (\hat{\mu}_k^{(1)} - \mu_k)(-\hat{q}_k^{(1)} + q_k) + (\hat{\mu}_k^{(1)} - \mu_k)(\hat{v}_k + v_k)|\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] \\
& + \sqrt{n} E[\sum_{k=0}^T \mu_k(\hat{q}_k^{(1)} - q_k) + \mu_k(-\hat{v}_k + v_k)|\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] \\
& + \sqrt{n} E[\sum_{k=0}^T (\hat{\mu}_k^{(1)} - \mu_k)(v_k + v_{k+1})|\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] \sum_{k=0}^T (\hat{\mu}_k^{(1)} - \mu_k)(\hat{q}_k^{(1)} + q_k) + (\hat{\mu}_k^{(1)} - \mu_k)(\hat{v}_k + v_k)|\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}] \\
& = \sqrt{n} \sum_{k=0}^T \mathcal{O}(\|\hat{\mu}_k^{(1)} - \mu_k\|_2 \|\hat{q}_k^{(1)} - q_k\|_2) = \sqrt{n} \sum_{t=0}^T \sigma_p(n^{-\alpha_1}) \sigma_p(n^{-\alpha_2}) = \sigma_p(1). \quad \square
\end{align*}
\]

Finally, we get

\[
\sqrt{n} (\mathbb{P}_{n_0} \phi(\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \rho^{\pi^e}) = \sqrt{n/n_0} \mathbb{G}_{n_0}[\phi(\{\mu_k\}, \{q_k\})] + \sigma_p(1).
\]

Therefore,

\[
\begin{align*}
\sqrt{n} (\hat{\rho}_{\text{DRL}(M_2)}^{\pi^e} - \rho^{\pi^e}) & = n_0/n\sqrt{n} \phi(\{\hat{\mu}_k^{(1)}\}, \{\hat{q}_k^{(1)}\}) - \rho^{\pi^e} + n_1/n\sqrt{n} \mathbb{P}_{n_1} \phi(\{\hat{\mu}_k^{(0)}\}, \{\hat{q}_k^{(0)}\}) - \rho^{\pi^e}
\end{align*}
\]

40
We proceed by considering each case. Therefore, which shows \( \hat{\rho} \) is well-specified. Then, concluding the proof by showing the influence function of \( \hat{\rho}^{\mu} \) is the efficient one. \( \square \)

Proof of Theorem 10. We use the following doubly robust structure

\[
\sqrt{n}(\mathbb{P}_{n_0} \phi(\{\mu_k^{(1)}\}, \{q_k^{(1)}\}) - \rho^{\mu}) = \sqrt{n/n_0} \mathbb{G}_{n_0} [\phi(\{\mu_k\}, \{q_k\})] + \sqrt{n/n_0} \mathbb{G}_{n_0} [\phi(\{\mu_k^+, \{q_k^\}])
\]

\[
+ \sqrt{n/n_0} \mathbb{E} [\phi(\{\mu_k\}, \{q_k\})]; \{\mu_k\}, \{q_k\}] - \mathbb{E} [\phi(\{\mu_k^{(1)}\}, \{q_k^{(1)}\})] + \sqrt{n}(\mathbb{E}[\phi(\{\mu_k\}, \{q_k\})] - \rho^{\mu})
\]

Then, as in the proof of Theorem 3

\[
\sqrt{n}(\mathbb{P}_{n_0} \phi(\{\mu_k^{(1)}\}, \{q_k^{(1)}\}) - \rho^{\mu}) = \sqrt{n/n_0} \mathbb{G}_{n_0} [\phi(\{\mu_k\}, \{q_k\})] + (\mathbb{E}[\phi(\{\mu_k\}, \{q_k\})] - \rho^{\mu} + c_p(1).
\]

We proceed by considering each case.

\( \mu \)-model is well-specified. First, consider the case when \( \mu_k^+ = \mu_k \):

\[
\mathbb{E}[\phi(\{\mu_k\}, \{q_k^+\})] = \mathbb{E}[\sum_{k=0}^T \mu_k r_k - \mu_k q_k - \mu_k q_k \mathbb{E}_T \phi(q_k, s_k, a_k)]
\]

Then,

\[
\sqrt{n}(\mathbb{P}_{n_0} \phi(\{\mu_k^{(1)}\}, \{q_k^{(1)}\}) - \rho^{\mu}) = \sqrt{n/n_0} \mathbb{G}_{n_0} [\phi(\{\mu_k\}, \{q_k^+\})] + c_p(1).
\]

Therefore,

\[
\sqrt{n}(\hat{\rho}^{\mu}_T - \rho^{\mu}) = \sqrt{n_0/n} \mathbb{G}_{n_0} [\phi(\{\mu_k\}, \{q_k^+\})] + \sqrt{n_1/n} \mathbb{G}_{n_1} [\phi(\{\mu_k\}, \{q_k^+\})] + c_p(1)
\]

\[
= \sqrt{n} \mathbb{G}_n [\phi(\{\mu_k\}, \{q_k^+\})] + c_p(1),
\]

which shows \( \hat{\rho}^{\mu}_T \) is CAN around \( \rho^{\mu} \) when the model for the \( \mu \)-function is well-specified.
\(q\)-model is well-specified. Next, consider the case where \(q_i^k = q_k\):

\[
\begin{align*}
E[\phi(\{\mu_k^i\}, \{q_k\})] &= E \left[ E_{\pi_e}[q(s_k, a_k)|s_0] + \sum_{k=0}^T \mu_k^i(r_k - q_k(s_k, a_k) + E_{\pi_e}[q_k(s_k, a_k)|s_{k+1}]) \right] \\
&= E \left[ E_{\pi_e}[q_0(s_0, a_0)|s_0] \right] = \rho^{\pi_e}.
\end{align*}
\]

We have

\[
\sqrt{n}(\mathbb{P}_n \phi(\{\mu_k^i\}, \{q_k\}) - \rho^{\pi_e}) = \sqrt{n/n_0}G_{n_0}[\phi(\{\mu_k^i\}, \{q_k\})] + o_p(1).
\]

Therefore,

\[
\sqrt{n}(\hat{\rho}_T^{M2} - \rho^{\pi_e}) = \sqrt{n/n_0}G_{n_0}[\phi(\{\mu_k^i\}, \{q_k\})] + \sqrt{n/n_1}G_{n_1}[\phi(\{\mu_k^i\}, \{q_k\})] + o_p(1)
\]

which shows \(\hat{\rho}_T^{\pi_e(x)}\) is CAN around \(\rho^{\pi_e}\) when the model for the \(q\)-function is well-specified.

**Proof of Theorem 11.** We first prove

\[
\mathbb{P}_n[\hat{\nu}_t(s_t)\eta_t r_t] = \mathbb{P}_n[w_t(s_t)\eta_t r_t] + \mathbb{P}_n[(\nu_{t-1} - w_t(s_t))E_{\pi_e}[r_t|s_t]] + o_p(n^{-1/2}).
\]

(32)

Noting

\[
\left\| \frac{1}{n} \sum_{i=1}^n \mathbb{I} \left[ s_t^{(i)} = s_t \right] \nu_{t-1} - p_{\pi_e}^1(s_t) \right\|_\infty = o_p(n^{-1/4}),
\]

\[
\left\| \frac{1}{n} \sum_{i=1}^n \mathbb{I} \left[ s_t^{(i)} = s_t \right] - p_{\pi_e}^1(s_t) \right\|_\infty = o_p(n^{-1/4}),
\]

\[
\hat{a}/\hat{b} = b^{-1}\{1 - \hat{b}^{-1}(\hat{b} - b)\}((\hat{a} - a) - a/b(\hat{b} - b)),
\]

we have

\[
\hat{w}_t(s_t) - w_t(s_t) + o_p(n^{-1/2})
\]

\[
= \frac{1}{p_{\pi_e}^1(s_t)} \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I} \left[ s_t^{(i)} = s_t \right] \nu_{t-1} - p_{\pi_e}^1(s_t) \right\} - w_t(s_t) \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I} \left[ s_t^{(i)} = s_t \right] - p_{\pi_e}^1(s_t) \right\}.
\]

Then,

\[
\mathbb{P}_n[\hat{\nu}_t(s_t)\eta_t r_t] = \frac{1}{n} \sum_{i=1}^n w_t(s_t^{(i)})\eta_t^{(i)} r_t^{(i)} + \frac{1}{n^2} \sum_{i=1}^n \eta_t^{(i)} r_t^{(i)} \left\{ \sum_{j=1}^n \mathbb{I} \left[ s_t^{(j)} = s_t^{(i)} \right] \nu_{t-1} - w_t(s_t^{(i)}) \right\} \sum_{j=1}^n \mathbb{I} \left[ s_t^{(j)} = s_t^{(i)} \right] \}
\]

\[
= \frac{2}{n(n-1)} \sum_{i<j} 0.5(a_{ij} + a_{ji}) + o_p(n^{-1/2}),
\]

42
where
\[ a_{ij} = \frac{\eta_i^{(i)} r_i^{(i)} I[s_t^{(j)} = s_t^{(i)}]}{p_{s_t^{(i)}}} \left( \nu_{t-1}^{(j)} - w_t(s_t^{(i)}) \right). \]

From U-statistics theory, by defining \( b_{ij}(\mathcal{H}^{(i)}, \mathcal{H}^{(j)}) = 0.5(a_{ij} + a_{ji}) \), we have
\[ \frac{2}{n(n-1)} \sum_{i<j} b_{ij}(\mathcal{H}^{(i)}, \mathcal{H}^{(j)}) = \frac{2}{n} \sum_{i=1}^{n} E[b_{ij}(\mathcal{H}^{(i)}, \mathcal{H}^{(j)})] + o_p(n^{-1/2}). \]

In addition,
\[ E[a_{ij} | \mathcal{H}^{(i)}] = \eta_i^{(i)} r_i^{(i)} \left\{ w_t(s_t^{(i)}) - w_t(s_t^{(i)}) \right\} = 0, \]
\[ E[a_{ji} | \mathcal{H}^{(i)}] = E \left[ \frac{\eta_i^{(i)} r_i^{(i)} I[s_t^{(j)} = s_t^{(i)}]}{p_{s_t^{(i)}}} \left( \nu_{t-1}^{(i)} - w_t(s_t^{(i)}) \right) | \mathcal{H}^{(i)} \right] \]
\[ = (\nu_{t-1}^{(i)} - w_t^{(i)}) E[\eta_i^{(i)} r_i^{(i)} | s_t^{(i)}] \]

Therefore, we have shown Eq. (32). Summing over \( t \) yields
\[ \mathbb{P}_n \left[ \sum_{t=0}^{T} w_t(s_t) \eta_t r_t \right] = \mathbb{P}_n \left[ \sum_{t=0}^{T} \left\{ w_t(s_t) \eta_t r_t + \nu_{t-1} E_{\pi_t} [r_t | s_t] - w_t E_{\pi_t} [r_t | s_t] \right\} \right] + o_p(n^{-1/2}), \]
which concludes the proof by establishing the influence function for \( \hat{\rho}_{\text{MIS}} \). \( \square \)

**Proof of Theorem 12** The difference of the influence functions belongs to the orthogonal tangent space. Therefore, the difference of variances is the variance of the difference of the influence functions. This is equal to
\[
\text{var} \left[ v_0 + \sum_{t=0}^{T} -\mu_t q_t + \mu_t v_{t+1} - \{ \nu_{t-1} - w_t(s_t) \} E_{\pi_t} [r_t | s_t] \right]
\]
\[
= \text{var}[v_0] + \sum_{t=1}^{T+1} E \left[ \text{var} \left( E \left[ \sum_{k=0}^{T} -\mu_k q_k + \mu_k v_{k+1} - \{ \nu_{k-1} - w_k(s_k) \} E_{\pi_k} [r_k | s_k] | \mathcal{J}_{s_{t-1}} \right] | \mathcal{J}_{s_{t-1}} \right) \right]
\]
\[
= \text{var}[v_0] + \sum_{t=1}^{T+1} E \left[ \text{var} \left( E \left[ \mu_{t-1} v_t + \sum_{k=t}^{T} -\mu_k r_k - \{ \nu_{k-1} - w_k(s_k) \} E_{\pi_k} [r_k | s_k] | \mathcal{J}_{s_{t-1}} \right] | \mathcal{J}_{s_{t-1}} \right) \right]
\]
\[
= \text{var}[v_0] + \sum_{t=1}^{T+1} E \left[ \text{var} \left( E \left[ \mu_{t-1} v_t - \sum_{k=t}^{T} \nu_{k-1} E_{\pi_k} [r_k | s_k] | \mathcal{J}_{s_{t-1}} \right] | \mathcal{J}_{s_{t-1}} \right) \right]
\]
\[
= \text{var}[v_0] + \sum_{t=1}^{T+1} E \left[ \text{var} \left( E \left[ \mu_{t-1} v_t - \nu_{t-1} v_t(s_t) | \mathcal{J}_{s_{t-1}} \right] | \mathcal{J}_{s_{t-1}} \right) \right]
\]
\[
= \text{var}[v_0] + \sum_{t=1}^{T+1} E \left[ \text{var} \left( \{ \mu_{t-1} - \nu_{t-1} \} v_t(s_t) | \mathcal{J}_{s_{t-1}} \right) \right]
\]
\[
\begin{align*}
\text{Proof of Theorem 13.} & \quad \text{We have} \\
\sqrt{n} & \left\{ \Pr \left[ \sum_{c=0}^{T} \hat{w}_c \eta_c (s_c, a_c) r_c | \hat{\mu}_c \right] - \rho^{\pi^e} \right\} \\
& = \mathbb{G}_n \left( \sum_{c=0}^{T} \hat{w}_c \eta_c r_c - \sum_{c=0}^{T} w_c \eta_c r_c \right) + \mathbb{G}_n \left( \sum_{c=0}^{T} w_c \eta_c r_c \right) \\
& + \sqrt{n} \left\{ \mathbb{E} \left[ \sum_{c=0}^{T} \hat{w}_c \eta_c r_c \mu_c \right] - \rho^{\pi^e} \right\} \\
& = \mathbb{G}_n \left( \sum_{c=0}^{T} w_c \eta_c r_c \right) + \mathbb{G}_n \left( \sum_{c=0}^{T} \left( \nu_{c-1} - w_c (s_c) \right) \right) \mathbb{E}_{\pi^e} [r_c | s_c].
\end{align*}
\]

From the second line to the third line, we used a fact that the stochastic equicontinuity term is \( o_p(1) \) as in the proof of Theorem 6 because \( \{ \hat{w}_c \eta_c r_c \} \) belongs to a Donsker class and the convergence rate condition holds. This fact is confirmed by the fact that a Hölder class with \( \alpha > d_{\mathcal{H}}/2 \) is a Donsker class (Example 19.9 in van der Vaart, 1998).

From the third line to the fourth line, we have used a result of Example 1(a) in Section 8 of Shen (1997). More specifically, the functional derivative of the loss function with respect to \( \hat{w}_c \) is
\[
g(s_c) \rightarrow \{ w_c (s_c) - \nu_{c-1} \} g(s_c),
\]
and the induced metric from the loss function is \( L^2 \)-metric with respect to \( p_{\pi^b}(s_c) \). The functional derivative of the target function with respect to \( \hat{\mu}_c \) is
\[
g(s_c) \rightarrow \mathbb{E}[g(s_c) \eta_c r_c | \hat{\mu}_c] = \mathbb{E}[g(s_c) \mathbb{E}_{\pi^e} [r_c | s_c]],
\]
and Riesz representation of the Hilbert space with the induced \( L^2 \)-metric with respect to \( p_{\pi^b}(s_c) \) is \( \mathbb{E}_{\pi^e} [r_c | s_c] \). Therefore, from Theorem 1 in Shen (1997),
\[
\mathbb{E} \left[ \sum_{c=0}^{T} \hat{w}_c \eta_c r_c \mu_c \right] - \rho^{\pi^e} = (\mathbb{P}_n - \mathbb{P}) \sum_{c=0}^{T} (\nu_{c-1} - w_c (s_c)) \mathbb{E}_{\pi^e} [r_c | s_c] + o_p(n^{-1/2}).
\]

Proof of Theorem 14. The proof is done as in the proof of Theorem 13. We study the following drift term:
\[
\mathbb{E} \left[ \sum_{c=0}^{T} \hat{\mu}_c r_c \mid \hat{\mu}_c \right].
\]
Here, the functional derivative of the loss function with respect to \( \hat{\mu}_c(s_c, a_c) \) is

\[
    g(s_c, a_c) \rightarrow E[g(s_c, a_c)(\mu_c - \nu_c)].
\]

and Riesz representation of the Hilbert space with the induced \( L^2 \)-metric with respect to \( p_{\pi_b}(s_c, a_c) \) is \( E[r_c|s_c, a_c] \). On the other hand, the functional derivative of the target function with respect to \( \hat{\mu}_c \) is

\[
    g(s_c, a_c) \rightarrow E[g(s_c, a_c)r_c] = E[g(s_c, a_c)E[r_c|s_c, a_c]].
\]

From Theorem 1 in [Shen (1997)]

\[
    E \left[ \sum_{c=0}^{T} \hat{\mu}_cr_c | \hat{\mu}_c \right] = P_n \left[ \sum_{c=0}^{T} (\nu_c - \mu_c)E[r_c|s_c, a_c] \right] + o_p(n^{-1/2}). \tag{□}
\]

**Proof of Theorem 15** We use the general framework developed in [Chamberlain (1992)] for establishing the efficiency bounds. For the current problem, noting that the orthogonal moment condition

\[
    E[e_{q,k+i}e_{q,k}] = E[E[e_{q,k+i}|H_{a_{k+i}}]e_{q,k}] = 0, \quad (0 \leq k < k+i \leq T)
\]

holds, the efficiency bound for \( \beta \) is represented as

\[
    \left\{ \sum_{k=0}^{T} \nabla_\beta m_k(H_{a_k}; \beta^*)\Sigma_k^{-1}(H_{a_k})\nabla_\beta m_k(H_{a_k}; \beta^*) \right\}^{-1},
\]

where \( \Sigma_k(H_{a_k}) = \text{var}[e_{q,k}|H_{a_k}] \). The statement of the theorem for the efficiency bound for \( \beta \) is arrived at by algebraic simplification of the above. The efficiency bound of \( \rho_{\pi_e} \) is calculated similarly. \( \tag{□} \)

**Proof of Lemma 16** Note \( \text{var}[e_{q,k}|H_{a_k}] \) and \( \text{var}[e_{q,k-1}|H_{a_{k-1}}] \) are upper and lower bounded by some constants by assumption. From Jensen’s inequality, we also know

\[
    E[g^2] = E[E[g^2|H_{a_k}]] \geq E[E[g|H_{a_k}]^2].
\]

This concludes that there exists some constant \( C_1, C_2 \) such that

\[
    C_1\|g\|_2 \leq \|g\|_{F,k} \leq C_2\|g\|_2. \tag{□}
\]

**C Additional Details from Section 5.2**

**Cliff Walking.** This RL task is detailed in Example 6.6 in [Sutton (2018)]. We consider a board of size \( 4 \times 12 \). The horizon was set to \( T = 400 \). Each time step incurs \(-1\) reward until the goal is reached, at which point it is 0, and stepping off the cliff incurs \(-100\) reward and a reset to the start.

45
Mountain Car. The RL task is as follows: a car is between two hills in the interval $[-0.7, 0.5]$ and the agent must move back and forth to gain enough power to reach the top of the right hill. The state space comprises position and velocity. There are three discrete actions: (1) forward, (2) backward, and (3) stay-still. The horizon was set to $T = 200$. The reward for each step is $-1$ until the position $0.5$ is reached, at which point it is $0$. The state space was continuous; thus, we obtained a 400-dimensional feature expansion using a radial basis function kernel as mentioned.

The Policy $\pi_d$. We construct the policy $\pi_d$ using standard $q$-learning (Sutton, 2018). For Cliff Walking, we use a $q$-learning in a tabular manner. Regarding a Mountain Car, we use $q$-learning based on the same feature expansion as above. We use 4000 sample to learn an optimal policy.