ON TOEPLITZ MATRICES OVER GF(2)

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Abstract. For each positive integer \( n \) let \( A_n \) be a Toeplitz matrix over \( GF(2) \) and suppose for each \( n \) that \( A_n \) is the upper left corner of \( A_{n+1} \). We study the structure of the sequence \( \nu = \{ \nu_n : n \in \mathbb{N} \} \), where \( \nu_n = \text{null}(A_n) \) is the nullity of \( A_n \). As an application we recover a result of D. E. Daykin on the number of \( n \times n \) Toeplitz matrices over \( GF(2) \) of any specified rank.

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1. Introduction

Let \( F \) be any finite field. Corresponding to any sequence of elements \( \{a_0, a_1, \ldots \} \) in \( F \), with \( a_0 = 0 \), is a sequence of skew-symmetric Toeplitz matrices, \( A_n, n \in \mathbb{Z}^+ \), where \( A_n \) is

\[
\begin{bmatrix}
a_0 & a_1 & a_2 & \ldots & a_{n-1} & a_n \\
-a_1 & a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\
-a_2 & -a_1 & a_0 & \ldots & a_{n-3} & a_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 & a_0
\end{bmatrix}
\]

Sequences of matrices of this type were studied in [1]. For each such sequence let \( \nu = \{ \nu_n : n \in \mathbb{Z}^+ \} \) be the corresponding nullity sequence, where \( \nu_n = \text{null}(A_n) \) is the nullity of matrix \( A_n \). In [1] K. Culler and one of the authors showed that every nullity sequence \( \nu \) is the concatenation of infinitely many strings of the form \( 1, 2, \ldots, d-1, d, d-1, \ldots, 1 \) for some \( d \in \mathbb{N} \), or the concatenation of finitely many strings of this type followed by the sequence \( 1, 2, \ldots \). This result generalizes one obtained by R. T. Powers and one of the authors over \( F = GF(2) \), in [5]. Using this result it was proved in [1] that for \( n \) an odd integer and \( q \) the order of \( F \), the number of \( (n+1) \times (n+1) \) invertible skew-symmetric matrices over \( F \) is \( q^{n-1}(q-1) \) (there are no invertible \( n \times n \) matrices of this type, for \( n \) odd, since all such matrices have even rank, [4]). More generally, for any \( r \in \mathbb{Z}^+ \) the number of skew-symmetric \( n \times n \) Toeplitz matrices of rank \( r \) is determined in [1].

In this paper we consider sequences \( \{A_n\} \) of general \( (n+1) \times (n+1) \) Toeplitz matrices over \( GF(2) \). Such a sequence is uniquely determined by a pair of sequences

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The matrices sequence, taking values 1, 2, ..., \( a_0, a_1, \ldots, a_{n-1}, a_n \), \( b_0, b_1, a_0, \ldots, a_{n-2}, a_{n-1} \), ..., \( b_0, b_1, a_0, \ldots, a_{n-2}, a_{n-1} \)

As in [1] we ascertain the nullity sequences which can occur and show that \( \{v_n\} \) is a concatenation of strings of the form 0, 0, ..., 0 or 1, 2, ..., 1, where \( d \) can repeat any finite number of times; or the nullity sequence may consist of a concatenation of finitely many of strings of this type, followed by 1, 2, ... . We analyze the structure of the (right) kernels of matrices which satisfy these patterns and use this to determine the number of matrices \( A_1, \ldots, A_n \) which satisfy a specific nullity pattern, \( v_0, \ldots, v_{n-1} \). As a consequence we can determine the number of general \( n \times n \) Toeplitz matrices over \( GF(2) \) of any specified rank, a result obtained earlier and in more generality in [2].

2. Nullity Sequences

The main goal of this section is to determine the structure of any nullity sequence for a sequence of Toeplitz matrices over \( GF(2) \), defined as in the previous section. These results are assembled in Theorem 1 below, and are obtained by making an analysis of the sequence \( \{\ker(A_n)\} \) of kernels of the matrices. We shall show, for example, that for a finite section, or string, \( v_0, \ldots, v_{k+m} \) of the nullity sequence, taking values 1, 2, ..., 1, 2, ..., 1, 0, where \( d \) can repeat any finite number of times; or the nullity sequence may consist of a concatenation of finitely many of strings of this type, followed by 1, 2, ... . We analyze the structure of the (right) kernels of matrices which satisfy these patterns and use this to determine the number of matrices \( A_1, \ldots, A_n \) which satisfy a specific nullity pattern, \( v_0, \ldots, v_{n-1} \). As a consequence we can determine the number of general \( n \times n \) Toeplitz matrices over \( GF(2) \) of any specified rank, a result obtained earlier and in more generality in [2].

**Definition 1.** \( N(n, \nu) \) is the number of \( (n+1) \times (n+1) \) Toeplitz matrices \( A_n \) as in (1.1) with nullity \( \nu \).

For each \( n \in \mathbb{N} \) the matrix \( A_{n-1} \) occupies the top left \( n \times n \) corner of \( A_n \), and also the lower right corner of \( A_n \). From this we conclude that the sequence of ranks \( \{\rho_n = \operatorname{rank}(A_n) : n \in \mathbb{Z}^+\} \) is monotone increasing. This and the Rank-Nullity Theorem give the following result.

**Proposition 1.** For each \( n \in \mathbb{Z}^+ \), \( \rho_{n+1} \) is one of \( \rho_n \), \( 1 + \rho_n \), or \( 2 + \rho_n \) so \( \nu_{n+1} \) is \( \nu_n + 1 \), \( \nu_n \), or \( \nu_n - 1 \).

Let \( a_0, a_1, \ldots, a_n \) and \( b_1, b_2, \ldots \) be a pair of sequences in \( GF(2) \) and let \( \{A_n : n \in \mathbb{Z}^+\} \) be the sequence of Toeplitz matrices defined as above.

**Definition 2.** Given a fixed sequence of Toeplitz matrices defined from a pair of sequences in \( GF(2) \), the corresponding nullity sequence, \( \{\nu_n : n \in \mathbb{Z}^+\} \) is given by \( \nu_n = \dim(\ker(A_n)) = \dim(\ker(A_n)) \).

We will prove the following result on the possible forms of the nullity sequence (see Remark [1]).
Theorem 1. Every nullity sequence \( \nu = \{ \nu_n : n \in \mathbb{Z}^+ \} \) is given as follows.

1. \( \nu = \{ 1, 2, \ldots, d - 1, d, d, \ldots \} \), for some \( d \in \mathbb{N} \)
2. \( \nu = \{ 0, 1, 2, \ldots \} \)
3. \( \nu = \{ 1, 2, 3, \ldots \} \)
4. \( \nu \) is the concatenation of finite length strings of the following form.
   (a) an arbitrary number of zeroes
   (b) \( 1, 2, \ldots, d - 1, d, d - 1, \ldots, 2, 1, 0 \) for some positive integer \( d \geq 1 \)
   (c) \( 1, 2, \ldots, d - 1, d, d, \ldots, d - 1, \ldots, 2, 1, 0 \), where \( d \geq 1 \) may appear an arbitrary number of times
5. \( \nu \) is the concatenation of a finite number of finite length strings of the form above, followed by (1) or (3)

Let \( n \in \mathbb{N} \) and suppose \( A_{n-1} \) and \( A_n \) are given matrices with \( A_{n-1} \) embedded in \( A_n \) (in both the top left and lower right corners and satisfying \( \nu_{n-1} = 0 \), i.e. \( A_{n-1} \) is invertible, and \( \nu_n = 1 \). Then we will show that for any positive integer \( d \) there is one and only one string of matrices beginning with \( A_n \), written \( A_n, \ldots, A_{n+d} \) whose nullities \( \nu_n, \ldots, \nu_{n+d} \) are \( 1, 2, \ldots, d + 1 \), respectively, see Corollary \( \text{Corollary 8} \) First it will be helpful to introduce the following notation and terminology.

- \( \mathbb{F}^n \) will denote the vector space of column vectors of length \( n \) over \( \text{GF}(2) \).
- The superscript \( t \) denotes transpose so \( k = [k_0, \ldots, k_n]^t \) is a column vector.
- If \( k = [k_0, \ldots, k_n]^t \), then \( k' \) is the vector obtained from \( k \) by deleting its last entry.
- If \( k = [k_0, \ldots, k_n]^t \), then \( k' \) is the vector obtained from \( k \) by deleting its initial entry.
- \( \omega(k) = [k_0, \ldots, k_n, 0]^t \), i.e., \( \omega(k) \) appends a 0 to the end of vector \( k \).
- \( \sigma(k) = [0, k_0, \ldots, k_n]^t \), i.e., \( \sigma(k) \) places a 0 at the beginning of \( k \).
- We will say that \( A_n \) is embedded in \( A_{n+1} \) if \( A_n \) is the top left corner (also lower right corner) of \( A_{n+1} \), and we will denote this by \( A_n \rightarrow A_{n+1} \) or \( A_{n+1} \leftarrow A_n \).

Theorem 2. Let \( A_{n-1} \) and \( A_n \) be Toeplitz matrices with \( A_{n-1} \rightarrow A_n \), and nullity \( \nu_{n-1} = 0, \nu_n = 1 \), respectively. Then the column vector \( k = [k_0, \ldots, k_n]^t \) spanning \( \ker(A_n) \) satisfies \( k_0 = 1 \) and \( k_n = 1 \).

Proof. Let \( k \) be the vector spanning \( \ker(A_n) \). If \( k_0 = 0 \) then the vector \( k' \) is in the kernel of \( A_{n-1} \) in the lower right corner of \( A_n \), a contradiction unless \( k = 0 \). Then \( k_0 = 1 \). Next suppose that \( k_n = 0 \). Since \( \nu_n = 1 \) we have \( \rho_{n-1} = \rho_n \) so the last column of \( A_n \) is in the span of its preceding columns. By symmetry of \( A_n \) this implies that the first row of \( A_n \) is in the span of the succeeding rows. Therefore \( k' \) is in the kernel of the copy of \( A_{n-1} \) embedded in the top left of \( A_n \), a contradiction. Therefore \( k_n = 1 \), also. \( \square \)

Theorem 3. Let \( A_{n-1}, A_n, \ldots, A_{n+d} \) be matrices embedded in one another with nullities \( \nu_{n-1}, \nu_n, \ldots, \nu_{n+d} \) equal to \( 0, 1, 2, \ldots, d+1 \), respectively. Then for \( 1 \leq j \leq d-1 \), \( \ker(A_{n+j}) \) coincides with \( \text{span}(\omega(\ker(A_{n+j-1})), \sigma(\ker(A_{n+j-1}))) \).

Proof. As in the previous proof we have \( \rho_{n+j-1} = \rho_{n+j} \) and so we conclude as we did in the previous proof that the last row (respectively, the first row) of \( A_{n+j} \) is in the span of the preceding (resp., the succeeding rows) of \( A_{n+j} \). Since \( \omega(\ker(A_{n+j-1})) \) is in the kernel of the matrix obtained by deleting the last row of
and since the last row of \( A_{n+j} \) is in the span of the preceding rows, we conclude that \( \omega(\ker(A_{n+j-1})) \subset \ker(A_{n+j}) \). Similarly \( \sigma(\ker(A_{n+j-1})) \subset \ker(A_{n+j}) \).

From the results of the previous paragraph we see inductively that for \( 1 \leq j \leq d \), \( \ker(A_{n+j}) \) is spanned by \([k_0, \ldots, k_n, 0, \ldots, 0]^t = \omega^j(k), [0, k_0, \ldots, k_n, 0, \ldots, 0]^t = \sigma(\omega^{-1}(k)), \ldots, [0, 0, \ldots, k_0, k_n] = \sigma^d(k)\). The first \( n + j \) (out of \( n + j + 1 \)) of these vectors is in \( \omega(\ker(A_{n+j-1})) \) and the last \( n + j \) vectors are in \( \sigma(A_{n+j-1}) \). \( \square \)

**Corollary.** Assume the same hypotheses as in the previous theorem. Then for each \( j, 1 \leq j \leq d \) there is one and only one choice of a pair \( \{b_{n+j}, a_{n+j}\} \) such that \( \nu_{n+j} = \text{null}(A_{n+j}) = 1 + \nu_{n+j-1} = 1 + \text{null}(A_{n+j-1}) \).

**Proof.** From Theorem 3 the vector \( k = [k_0, \ldots, k_n]^t \in \ker(A_n) \) satisfies \( k_0 = 1 \) and \( k_n = 1 \). The vector \( \omega^j(k) \) is in the kernel of \( A_{n+j} \) and so it has dot product 0 with the last row of \( A_{n+j} \). Since the first entry of this vector is \( k_0 = 1 \) we conclude that \( b_{n+j} \) is uniquely determined by this fact. Similarly \( \sigma^j(k) \in \ker(A_{n+j}) \) and the vector ends in \( k_n = 1 \), so \( a_{n+j} \) is uniquely determined. Thus there is only one choice for the pair \( \{b_{n+j}, a_{n+j}\} \). \( \square \)

Our results imply that if \( \nu_{n-1}, \ldots, \nu_{n+d} \) is 0, 1, \ldots, \( d + 1 \) then there is one and only one choice of the pair \( \{b_{n+d+1}, a_{n+d+1}\} \) such that \( \nu_{n+d+1} = d + 2 \). We now show that there is one and only one choice of \( \{b_{n+d+1}, a_{n+d+1}\} \) so that \( \nu_{n+d+1} = d \). We also determine how to obtain \( \ker(A_{n+d+1}) \) from \( \ker(A_{n+d}) \).

**Theorem 4.** Suppose \( n \in \mathbb{N} \) is such that \( A_{n-1}, A_n, \ldots, A_{n+d} \) is an embedded string of Toeplitz matrices with nullities 0, 1, \ldots, \( d + 1 \). Then there is one and only one choice of \( \{b_{n+d+1}, a_{n+d+1}\} \) so that \( \nu_{n+d+1} = d \).

**Proof.** Consider the matrix

\[
A_{n+d+1} = \begin{bmatrix}
  a_0 & a_1 & a_2 & \ldots & a_{n+d+1} \\
  b_1 & a_0 & a_1 & \ldots & a_{n+d} \\
  b_2 & b_1 & a_0 & \ldots & a_{n+d-2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{n+d} & b_{n+d-1} & b_{n+d-2} & \ldots & a_0 \\
  b_{n+d+1} & b_{n+d} & b_{n+d-1} & \ldots & b_1 \\
\end{bmatrix}
\]

As usual we view the smaller matrix \( A_{n+d} \) positioned in both the top left and bottom right corner of \( A_{n+d+1} \).

From the proof of Theorem 3 the kernel of \( A_{n+d} \) has a basis \([k_0, \ldots, k_n, 0, \ldots, 0]^t\), \([0, k_0, \ldots, k_n, 0, \ldots, 0]^t\), \([0, 0, \ldots, 0, k_0, \ldots, k_n] \) in \( F^{n+d+1} \). Label these vectors \( v_0, v_1, \ldots, v_d \). Next append a 0 to the end of each of these vectors, obtaining the vectors \( \omega(v_0), \omega(v_1), \ldots, \omega(v_d) \). Note that each of these vectors has dot product 0 with the first \( n + d + 1 \) rows of \( A_{n+d+1} \), since \( v_0 \) through \( v_d \) form a basis for \( \ker(A_{n+d}) \). Note, however, that the vectors \( \omega(v_1), \ldots, \omega(v_d) \) all have dot product 0 with the last row of \( A_{n+d+1} \) because \( \omega(v_1), \ldots, \omega(v_d) \) coincide with the first \( d \) basis vectors \( v_0, \ldots, v_{d-1} \) and so have dot product 0 with the last row of \( A_{n+d} \).

So we have shown that there are at least \( d \) vectors in \( \ker(A_{n+d+1}) \) and these are obtained by deleting the first basis vector of \( A_{n+d} \) and appending a 0 to each of the remaining basis vectors of \( A_{n+d} \).

Note that for \( A_{n+d+1} \) to have nullity \( d \), neither the vector \([k_0, \ldots, k_n, 0, \ldots, 0]^t\) nor \([0, \ldots, 0, k_0, \ldots, k_n]^t \) (in \( F^{n+d+2} \)) is in the kernel of \( A_{n+d+1} \) and there is only one choice of the pair \( \{b_{n+d+1}, a_{n+d+1}\} \) which makes this so. \( \square \)
Corollary. Suppose Toeplitz matrices $A_{n-1} \rightarrow \cdots \rightarrow A_{n+d}$ are given with nullities $0, 1, \ldots, d + 1$, respectively. Then there are two choices of the pair of entries $\{b_{n+d+1}, a_{n+d+1}\}$ for which the corresponding matrix $A_{n+d+1}$ has nullity $\nu_{n+d+1} = d + 1$.

Proof. We know that the ranks of the matrices $A_n \rightarrow A_{n+1}$ are non-decreasing so that $\rho_{n+d} \leq \rho_{n+d+1} \leq \rho_{n+d} + 2$ so $\nu_{n+d+1}$ is $d + 1, d - 1$ or $d$. We have seen that there is only one choice of pair $\{b_{n+d+1}, a_{n+d+1}\}$ for which $A_{n+d+1}$ has nullity $d + 2$ and one for which $A_{n+d+1}$ has nullity $d$. Hence there must be two pairs for which $A_{n+d+1}$ has nullity $d + 1$.

We shall look more carefully below at $\ker(A_{n+d+1})$ when $\nu_{n+d+1} = \nu_{n+d}$, see Lemma 4.

Theorem 5. Suppose Toeplitz matrices $A_{n-1}$ through $A_{n+d}$ have nullities $0$ through $d + 1$ and $\nu_{n+d+1} = \nu(A_{n+d+1}) = d$. Then for any embedded Toeplitz matrices $A_{n+d+2}$ through $A_{n+2d+1}$ the nullities are $d - 1$ through $0$.

Proof. Since $\nu_{n+d} = d + 1$ and $\nu_{n+d+1} = d$ we conclude from Theorem 4 that the first and last entry of every vector in $\ker(A_{n+d+1})$ is 0. First suppose $\nu_{n+d+2}$ is not $d - 1$ then $\nu_{n+d+2}$ is either $d + 1$ or $d$, i.e. $\rho_{n+d+2}$ is either $\rho_{n+d+1}$ or $\rho_{n+d+1} + 1$.

Either case leads to the conclusion that $\omega(\ker(A_{n+d+1})) \subset \ker(A_{n+d+2})$ or $\sigma(\ker(A_{n+d+1})) \subset \ker(A_{n+d+2})$, or both. Suppose $\omega(\ker(A_{n+d+1})) \subset \ker(A_{n+d+2})$. Select a vector $v \in \ker(A_{n+d+1})$ whose first nonzero entry appears leftmost among all vectors in $\ker(A_{n+d+1})$. Then $\alpha(v) \in \ker(A_{n+d+2})$. Therefore, viewing $A_{n+d+1}$ in the lower right corner of $A_{n+d+2}$, we conclude that $\omega(v)$ is in the kernel of $A_{n+d+1}$, contradicting the leftmost position of the nonzero entry of $v$. Then by contradiction, $\sigma(\ker(A_{n+d+1}))$ is not a subspace of $\ker(A_{n+d+2})$ and by a similar proof, neither is $\sigma(\ker(A_{n+d+1}))$. Hence $\nu_{n+d+2} = d - 1$. Continuing this process inductively yields the result.

The proof of the theorem leads immediately to the following conclusion.

Corollary. Assume the same hypotheses as in the previous theorem. Then for each Toeplitz matrix $A_{n+d+j}$, for $j = 1, 2, \ldots, d$, and for any pair $\{b_{n+d+j+1}, a_{n+d+j+1}\}$ the corresponding matrix $A_{n+d+j+1} \rightarrow A_{n+d+j}$ has nullity one less than $A_{n+d+j}$.

Lemma 1. Let $d \geq 1$ and suppose for some $m \in \mathbb{N}$ that $A_m \rightarrow A_{m+1}$ satisfy $\nu_m = q = \nu_{m+1}$. Then $\ker(A_{m+1}) = \omega(\ker(A_m))$ or $\ker(A_{m+1}) = \sigma(\ker(A_m))$.

Proof. Since $\nu_m = \nu_{m+1}$ it follows that $\rho_{m+1} = \rho_m + 1$ so one and only one of the following occurs: either the last row of $A_{m+1}$ is in the span of the preceding rows, or the last column is in the span of the preceding columns. In the former case it is clear that $\omega(\ker(A_m)) \subset \ker(A_{m+1})$. In the latter case, it follows from the symmetry of $A_{m+1}$ that the first row is in the span of the succeeding rows and therefore $\sigma(\ker(A_m)) \subset \ker(A_{m+1})$. A dimension argument finishes the proof.

Theorem 6. Suppose for some $m \in \mathbb{N}$ that $A_m \rightarrow A_{m+1} \rightarrow A_{m+2}$ satisfy $0 < \nu_m = \nu_{m+1} = \nu_{m+2}$. Then either $\ker(A_{m+1}) = \omega(\ker(A_m))$ and $\ker(A_{m+2}) = \omega^2(\ker(A_m))$ or $\ker(A_{m+1}) = \sigma(\ker(A_m))$ and $\ker(A_{m+2}) = \sigma^2(\ker(A_m))$.

Proof. Suppose $\ker(A_{m+1}) = \omega(\ker(A_m))$ but $\ker(A_{m+2}) = \sigma(\ker(A_{m+1})) = \sigma(\omega(\ker(A_m)))$. Then $\ker(A_{m+2}) = \omega(\sigma(\ker(A_m)))$ and from this we have $\sigma(\ker(A_m)) \subset \ker(A_{m+1})$, a contradiction. Similarly the assumption $\ker(A_{m+1}) = \sigma(\ker(A_m))$
and \( \ker(A_{m+2}) = \omega(\ker(A_{m+1})) \) leads to a contradiction, so the conclusion of the theorem follows.

A proof similar to the above leads to the following result.

**Theorem 7.** Suppose for some \( m \in \mathbb{N} \) that \( A_m \rightarrow A_{m+1} \rightarrow A_{m+2} \rightarrow \cdots \rightarrow A_{m+r} \) satisfy \( 0 < \nu_m = \nu_{m+1} = \cdots = \nu_{m+r} \). If \( \ker(A_{m+1}) = \omega(\ker(A_m)) \) then \( \ker(A_{m+r}) = \omega^r(\ker(A_m)) \), and if \( \ker(A_{m+1}) = \sigma(\ker(A_m)) \) then \( \ker(A_{m+r}) = \sigma^r(A_m) \).

**Corollary.** If \( \nu_m = \nu_{m+1} = a > 0 \) then \( \nu_{m+2} \) is either \( a \) or \( a - 1 \). In the latter case the subsequent nullities are \( a - 2, \ldots, 1, 0 \).

**Proof.** By Proposition 1 the only other possibility is that \( \nu_{m+2} = a + 1 \), so suppose this is the case. We may assume this is the first time in the nullity sequence that this phenomenon has occurred, i.e. that there is an \( a > 0 \) and a triple \( \nu_m, \nu_{m+1}, \nu_{m+2} \) with nullities \( a, a, a + 1 \). From the preceding results we may assume that \( \nu_m, \nu_{m+1}, \nu_{m+2} \) is part of a nullity string \( \nu_{j-1}, \nu_j, \ldots, \nu_{m+2} \) of the form 0, 1, 2, \ldots, \( a-1 \), \( a \), \ldots, \( a+1 \), or, if \( j - 1 = 0 \), the string might be of the form 1, 2, \ldots, \( a-1 \), \( a \), \ldots, \( a+1 \). Since the argument for the latter is similar to the former we treat only the former case. Then by Theorem 2 there is a single nonzero vector \( k \in F^{j+2} \) such that \( k \) generates \( \ker(A_j) \) and \( k \) has initial entry \( k_0 = 1 \) and final entry \( k_j = 1 \). If \( \ell = j + a - 1 \), then by Theorem 3 \( \ker(A_j) \) is generated by the following vectors in \( F^{j+2} \): \([0, 0, \ldots, 0], [0, k, 0, \ldots, 0], \ldots, [0, \ldots, 0, k] \). Then by the previous theorem (with \( r = m - \ell \)) either \( \ker(A_j) = \omega^r(\ker(A_{j})) \) and \( \ker(A_{m+1}) = \omega^{r+1}(\ker(A_{j})) \) or \( \ker(A_j) = \sigma^{r+1}(\ker(A_{j})) \) and \( \ker(A_{m+1}) = \sigma^{r+1}(\ker(A_{j})) \). We assume the former case, the latter case being similar. From the form of \( k \) it follows that there is a vector \( v \in \ker(A_{m+1}) \) beginning with 1 and ending in 0. Since \( \nu_{m+2} = a + 1 \) it follows that \( \rho_{m+2} = \rho_{m+1} \). Thus the last row of \( A_{m+2} \) is in the span of the preceding rows, so \( \omega(v) \in \ker(A_{m+2}) \). Since the last column of \( A_{m+2} \) is in the span of the preceding columns there is a vector \( w \) which ends in 1. Then either \( w \) or \( \omega(v) + w \) is a vector that begins with 0 and ends in 1. By deleting the initial 0 from this vector and viewing \( A_{m+1} \) as being in the lower right corner of \( A_{m+2} \), we obtain a vector in \( \ker(A_{m+1}) \) that ends in 1, contradicting our assumption that \( \ker(A_{m+1}) = \omega(\ker(A_m)) \). The argument for the case \( \ker(A_{m+1}) = \sigma(\ker(A_m)) \) is similar.

So we have shown that the triple \( \nu_m, \nu_{m+1}, \nu_{m+2} \) is either \( a, a, a \) or \( a, a, a - 1 \). Assuming the latter (and that \( a - 1 > 0 \)) we will show that \( \nu_{m+3} = a - 2 \). As above, \( \ker(A_{m+1}) \) is either \( \omega^{r+1}(\ker(A_{j})) \) or \( \sigma^{r+1}(\ker(A_{j})) \). We assume the former case (with the proof of the latter case being similar). Then \( \ker(A_{m+1}) \) is generated by the vectors \( \omega^{r+1}([k, 0, \ldots, 0], \omega^{r+1}([0, k, 0, \ldots, 0], \ldots, \omega^{r+1}([0, \ldots, 0, k]) \). It can be verified that the image under \( \omega \) of the second through last of these vectors is in \( \ker(A_{m+2}) \) and therefore \( \omega^{r+2}([k, 0, \ldots, 0]) \) is not in \( \ker(A_{m+2}) \). It follows that every vector in \( \ker(A_{m+2}) \) begins and ends with a 0. Suppose then that \( \nu_{m+3} \geq a - 1 \). Then by an argument similar to the proof of Lemma 1 it follows that either \( \omega(A_{m+2}) \subset \ker(A_{m+3}) \), or \( \sigma(A_{m+2}) \subset \ker(A_{m+3}) \), and then by an argument similar to that in the last paragraph of Theorem 5 it follows that \( \nu_{m+3} = a - 2 \). Continuing in this way we verify that the nullities from \( \nu_{m+1} \) on are \( a, a - 1, a - 2, \ldots, 1, 0 \). 

□
Theorem 8. Suppose for some \( m \in \mathbb{N} \) that \( A_m \rightarrow A_{m+1} \) and \( 0 < a = \nu_m = \nu_{m+1} \). Then \( \nu_{m+2} \) is either \( a \) or \( a - 1 \). There are two matrices \( A_{m+2} \) for which \( A_{m+1} \rightarrow A_{m+2} \) and \( \nu_{m+2} = a \) and two for which \( \nu_{m+2} = a - 1 \).

Proof. From Theorem 5 and Corollary 7 it follows that there is an \( n \in \mathbb{N} \) such that, starting with \( n - 1 \) and ending with \( m + 1 \), the nullity string \( \nu_{n-1} \) through \( \nu_{m+1} \) is 0, 1, \ldots, \( a - 1 \), \( a \), \ldots, \( a \). (or if \( n = 1 \) it is possible that the nullity string is 1, \ldots, \( a - 1 \), a, \ldots, a). Let \( k \) span \( ker(A_n) \) then from Theorem 7 there is a positive integer \( s \) such that \( ker(A_{m+1}) = \omega^s( ker(A_{m+1-s}) \) or \( \sigma^s( ker(A_{m+1-s}) \) where \( ker(A_{m+1-s}) = \text{span}\{[k, 0, \ldots, 0], [0, k, 0, \ldots, 0], \ldots, [0, \ldots, 0, k]\} \). If \( ker(A_{m+1}) = \omega^s( ker(A_{m+1-s}) \) then since the first entry of \( k \) is 1, we may choose \( b_{m+2} \) so that \( \nu = [k, 0, \ldots, 0] \in F^{m+3} \) either is or is not in \( ker(A_{m+2}) \). If \( \nu \in ker(A_{m+2}) \) then the first column of \( A_{m+2} \) is in the span of the succeeding columns, hence the last row of \( A_{m+2} \) is in the span of the preceding rows, and it then follows that \( \omega( ker(A_{m+1}) \subset ker(A_{m+2}) \). Then by the preceding corollary \( A_{m+1} \) and \( A_{m+2} \) have the same nullity, so \( \omega( ker(A_{m+1}) = ker(A_{m+2}) \). Note that this holds regardless of the choice of \( a_{m+2} \). If \( \nu \notin ker(A_{m+2}) \) then since \( ker(A_{m+2}) \) is neither \( \omega( ker(A_{m+1}) \) nor \( \sigma( ker(A_{m+1}) \) it follows from the preceding corollary that \( \nu_{m+2} = a - 1 \). Since this is true regardless of the choice of \( a_{m+2} \) there are two choices of a matrix \( A_{m+2} \) satisfying these conditions.

A similar argument holds for the case where \( ker(A_{m+1}) = \sigma^s( ker(A_{m+1-s}) \).

Remark 1. Note that Theorem 4 follows by assembling Proposition 4, Corollary 5, Theorem 6 and its corollary, Theorem 7 and Theorem 8.

3. Additional Counting Results for Nullity Patterns

Theorem 9. If for some \( n \in \mathbb{Z}^+ \) we have \( \nu_n = 1 \) and \( \nu_{n+1} = 0 \), then there are two pairs of entries \( \{b_{n+2}, a_{n+2}\} \) for which the Toeplitz matrix \( A_{n+2} \) satisfies \( \nu_{n+2} = 0 \) and two for which \( \nu_{n+2} = 1 \).

Proof. First note that since \( \nu_n = 1 \) and \( \nu_{n+1} = 0 \), \( \rho_{n+1} = \rho_n + 2 \), and therefore \( \{a_{n+1}, a_{n+1}\}^t \) is not the zero vector. For if it were, then matrix \( A_n \) and the \( (n+1) \times (n+2) \) matrix obtained by appending \( \{a_{n+1}, a_{n+1}\}^t = [0, \ldots, 0]^t \) as the last column, both have the same rank. So then \( \rho(A_{n+1}) \leq \rho(A_n) + 1 \), a contradiction. Fix \( a_{n+2} \). Since \( A_{n+1} \) is invertible there is a vector \( \ell = [\ell_0, \ell_1, \ldots, \ell_{n+1}]^t \) satisfying \( A_{n+1} \cdot \ell = [a_{n+2}, a_{n+1}, \ldots, a_1]^t \). Note that \( \ell_0 = 1 \), for if \( \ell_0 = 0 \) it would follow (since \( A_n \) is embedded in the lower right corner of \( A_{n+1} \)) that \( \{a_{n+1}, a_{n+1}, \ldots, a_1\}^t \) is in the span of the column space of \( A_n \), and therefore the rank of \( A_{n+1} \) is at most one greater than the rank of \( A_n \). But the assumption that \( \nu_{n+1} = 0 \) and \( \nu_n = 1 \) means that \( \rho_{n+1} = 2 + \rho_n \). Setting \( L = [\ell, 1]^t = [\ell_0, \ell_1, \ldots, \ell_{n+1}, 1]^t \), then \( A_{n+2} \cdot L = [0, \ldots, 0, 1]^t \). We may choose \( b_{n+2} \) so that the last entry of \( A_{n+2} \cdot L \) is 1, so \( L \) is not in \( ker(A_{n+2}) \). Suppose there is a non-zero vector \( P \) in \( ker(A_{n+2}) \) then since the first \( n+2 \) columns of \( A_{n+2} \) are linearly independent (as a consequence of \( A_{n+1} \) lying in the top left corner of \( A_{n+2} \), it follows that the last entry of \( P \) is 1. Therefore \( A_{n+2} \cdot (L + P) = [0, \ldots, 0, 1]^t \). Since \( L + P \) has last entry 0, then by viewing \( A_{n+1} \) in the top left corner of \( A_{n+2} \), the last equation implies that the columns of \( A_{n+1} \) are dependent, a contradiction. Therefore, by contradiction, \( A_{n+2} \) is invertible.
Thus for each $a_{n+2}$ there is a unique entry $b_{n+2}$ such that the corresponding matrix $A_{n+2}$ is invertible. Thus the other two choices of the pair $\{b_{n+2},a_{n+2}\}$ yield matrices $A_{n+2}$ of nullity 1.

**Theorem 10.** Suppose $A_n \rightarrow A_{n+1}$ are Toeplitz matrices for some $n \in \mathbb{Z}^+$ with $\nu_n = 0 = \nu_{n+1}$. Then there are three choices of entries $\{b_{n+2},a_{n+2}\}$ for a matrix $A_{n+2} \leftarrow A_{n+1}$, such that $\nu_{n+2} = 0$, and one such that $\nu_{n+2} = 1$.

**Proof.** By Proposition 1 either $\nu_{n+2} = 0$ or $\nu_{n+2} = 1$. Suppose there are two Toeplitz matrices $A_{n+2}$ and $A'_{n+2}$ such that $A_n \rightarrow A_{n+1} \rightarrow A_{n+2}$ and $A_n \rightarrow A_{n+1} \rightarrow A'_{n+2}$, each with nullity 1. Then there are vectors $\ell$ and $L$ in $F^{n+3}$, with $l_0 = 1 = l_{n+2}$ and $L_0 = 1 = L_{n+2}$, by Theorem 2 such that $\ell \in \ker(A_{n+2})$ and $L \in \ker(A'_{n+2})$. Then the vector $v = \ell' + \ell' \in F^{n+1}$ satisfies $A_n v = 0$, a contradiction unless $\ell = L$, since $A_n$ is invertible. Since $\ell = L$ we conclude that there can be at most one Toeplitz matrix $A_{n+2}$ of nullity one such that $A_{n+1} \rightarrow A_{n+2}$.

To show that there is at least one Toeplitz matrix $A_{n+2}$ with nullity 1 and $A_{n+1} \rightarrow A_{n+2}$ choose $a_{n+2} \neq a'_{n+2}$. Then, since $A_{n+1}$ is invertible there exist vectors $v$ and $V$ in $F^{n+2}$ such that

(1) $A_{n+1} v = [a_{n+2}, a_{n+1}, a_1]^T$, and
(2) $A_{n+1} V = [a_{n+2}, a_{n+1}, a_1]^T$.

The sum of the initial entries, $v_0 + V_0$, is 1, otherwise $\langle v, v \rangle$ is in $\ker(A_n)$, a contradiction unless $v = V$, which is false. Suppose $v_0 = 1$ then we can choose $b_{n+2}$ such that $A_{n+2} [v_0, v_1, \ldots, v_{n+1}, 1]^T = 0$, so that $A_{n+2}$ has nontrivial kernel. Thus, by the comment made above, $A_{n+2}$ has nullity 1. □

**Remark 2.** In this remark we summarize the results of the counting arguments we have made thus far. The first two integers of each line indicate the nullities $\nu_{n-1}$ and $\nu_n$ of a pair of fixed Toeplitz matrices $A_{n-1}$ and $A_n$, with $A_{n-1} \rightarrow A_n$. The integer over the arrow represents the number of Toeplitz matrices $A_{n+1}$ satisfying $A_{n-1} \rightarrow A_n \rightarrow A_{n+1}$ and having nullity $\nu_{n+1}$, the third integer entry of the line.

1. $0, 0 \rightarrow 0$, Theorem 10
2. $0, 0 \rightarrow 1$, Theorem 10
3. $1, 0 \rightarrow 0$, Theorem 2
4. $1, 0 \rightarrow 1$, Theorem 2
5. For $d \geq 1$, $d - 1, d \rightarrow d + 1$,
6. For $d \geq 1$, $d - 1, d \rightarrow d$,
7. For $d \geq 1$, $d - 1, d \rightarrow d - 1$, Theorem 4
8. For $d \geq 1$, $d, d \rightarrow d$,
9. For $d \geq 1$, $d, d \rightarrow d - 1$,
10. For $d \geq 2$, $d, d - 1 \rightarrow d - 2$

**Definition 3.** For $\ell \in \mathbb{Z}^+$ let $A_\ell$ be a fixed Toeplitz matrix with nullity $\nu_\ell$ and let $\nu_\ell, \nu_{\ell+1}, \ldots, \nu_p$. Then the number of strings $A_\ell, A_{\ell+1}, \ldots, A_p$ with the prescribed nullities will be denoted by $C(\nu_\ell, \nu_{\ell+1}, \ldots, \nu_p)$.

**Lemma 2.** Let $m$ be a positive integer, then $C(1, 2, \ldots, m-1, m, m-1, \ldots, 2, 1, 0) = 2^{2m-2}$. 


Proof. For $0 < \nu_s$ and $\nu_{s+1} = \nu_s + 1$, by Corollary 5 there is, for a given matrix $A_s$ with nullity $\nu_s$, only one $A_{s+1}$ with $A_s \rightarrow A_{s+1}$ and with nullity $\nu_{s+1}$. If $A_s = m$ only one $A_{s+1}$ with $A_s \rightarrow A_{s+1}$ satisfies $\nu_{s+1} = m - 1$, by Theorem 3. By Theorem 5 if $m - 1 \geq \nu_s \geq 1$ then there are four choices of $A_{s+1}$ with nullity $\nu_{s+1} = \nu_s - 1$. □

Lemma 3. If $\nu_{n-1} = 0$ and 1 is repeated $u$ times, with $u > 1$, then $C(\nu_n, \ldots, \nu_{n+u-1}) = C(1, 1, \ldots, 1) = 2^{u-1}$. If $m > 1$ then $C(1, 2, \ldots, m-1, m, m, \ldots, m, m-1, \ldots, 2, 1, 0) = 2^{u-2}2^{m-2}$ if $m$ is repeated $u$ times.

Proof. The first statement follows from Theorem 8. For the second statement, as the nullity increases from 1 to $m$ there is only one choice of matrix at each stage, by Theorem 3. For each of the $u - 1$ occurrences of nullity $m$ after the first, there are two choices of matrix which fix the nullity at $m$, and then following the last occurrence of $m$ there are two choices of matrix that drop the nullity from $m$ to $m - 1$. Thereafter, as the nullity drops from $m - 1$ to 0 there are four choices of matrix at each stage. So we have a total of $2^{u-1} \cdot 2 \cdot 4^{m-1}$ choices □

The following result will be an essential part of the argument that half of the Toeplitz matrices of a given size are invertible. It will also be useful further on when we calculate the number of Toeplitz matrices of a given size of a specified rank.

Theorem 11. Let $\ell \in \mathbb{Z}^+$ and let $A_\ell$ be a Toeplitz matrix of nullity 1, where we assume that $\nu_{\ell-1} = 0$ if $\ell \geq 1$. Let $n$ be a positive integer. Then the number of matrices $A_{\ell+n}$ satisfying

1. $A_\ell \rightarrow A_{\ell+1} \rightarrow \ldots \rightarrow A_{\ell+n},$
2. $\nu_s > 0$ for $s = 1, 2, \ldots, n - 1,$ and
3. $\nu_{\ell+n} = 0$

is $n2^{n-1}$.

Proof. For $n = 1$ the only corresponding nullity string is $\nu_\ell, \nu_{\ell+1}$ which can only be 1, 0. By Theorem 3 there is only one choice for $A_{\ell+1}$ having nullity 0. For $n = 2$ the only nullity string is 1, 1, 0 and there are 4 matrices $A_{\ell+2}$ with this string. Suppose the statement is true for some even integer $n \geq 2$. Then for strings of matrices of length $n + 2$ satisfying the hypotheses, there are two types of nullity strings. First there are the nullity strings 1, 2, $\ldots$, $m - 1$, $m$, $\ldots$, $m$, $m - 1$, $\ldots$, 2, 1 obtained from the nullity strings of length $n$ by inserting $m$ twice, and there is the nullity string 1, 2, $\ldots$, $\frac{s}{2}$, $\frac{s+2}{2}$, $\frac{s+4}{2}$, $\ldots$, $\frac{s}{2}$, 2, 1, 0. By Lemma 8 there are $4 \cdot 4^s = 4 \cdot 2^n$ of the latter and by the induction assumption and Lemma 3 there are $4n2^{n-1}$ of the former. Adding gives $(n+2)2^{n+1}$, so the even case is established by induction. The proof of the odd case is similar. □

4. Counting Invertible Toeplitz Matrices

As an application of the counting arguments for nullity strings we will show that exactly half of the Toeplitz matrices $A_n$ are invertible, see Corollary 12. As a further application we will count the number of $A_n$’s of any specified rank, Corollary 15.

Definition 4. For $n \in \mathbb{N}$ let $\theta(n)$ be the number of $A_n$’s such that $\nu_{n-1} = 0 = \nu_n$. Define $\eta(n)$ to be the number of $A_n$’s such that $\nu_{n-1} = 1$ and $\nu_n = 0$.

Note that if $n \in \mathbb{N}$ and $A_n$ is invertible then $A_n$ falls either among those invertible matrices counted by $\theta(n)$ or by $\eta(n)$, and therefore we have the following.
Proposition 2. For $n \in \mathbb{N}$, the number of invertible $A_n$'s is $\theta(n) + \eta(n)$.

Remark 3. The formula in the theorem below computes the number of invertible matrices $A_n$ for $n \geq 2$. For $n = 0$ note that $A_0 = [1]$ is the only invertible $1 \times 1$ matrix. For $n = 1$ there are only two possible nullity strings $\nu_0, \nu_1$ ending in 0, namely 0,0 and 1,0. By Proposition 1 and Corollary 2, there are three matrices with the former and one with the latter nullity string. So $\theta(1) = 3$ and $\eta(1) = 1$, and there are $\theta(1) + \eta(1) = 4$ invertible $2 \times 2$ Toeplitz matrices.

Proposition 3. For $n \in \mathbb{N}$, $\theta(n+1) = 3\theta(n) + 2\eta(n)$.

Proof. It follows from Theorem 10 that if $A_{n-1}$ and $A_n$ both have nullity 0 then there are three choices of pairs $\{b_{n+1}, a_{n+1}\}$ for $A_{n+1}$ to have nullity 0. If $\nu_{n-1} = 1$ and $\nu_n = 0$ there are two choices for $\{b_{n+1}, a_{n+1}\}$ that make $A_{n+1}$ invertible, by Theorem 9.

Theorem 12. For $n \geq 2$ the number of invertible $(n+1) \times (n+1)$ Toeplitz matrices $A_n$ over $GF(2)$ is

$$n2^{n-1} + (n - 1)2^{n-2} + \sum_{j=1}^{n-2} [\theta(j) + 2\eta(j)]((n-1) - j)2^{n-2-j} + (3\theta(n-1) + 2\eta(n-1))$$

Proof. Each invertible Toeplitz matrix $A_n$ has a nullity string $\nu_0, ..., \nu_n$ which satisfies $\nu_n = 0$ and the rules of Theorem 11. We divide the collection of all of these invertible matrices into disjoint sets $S, S_0, ..., S_{n-1}$, depending on the structure of their nullity string. A matrix $A_n$ belongs to $S$ if $\nu_k > 0$ for all $k < n$, i.e., $\nu_n$ is the only 0 on the string. $A_n$ is in $S_1$ if $\nu_j = 0$ and $\nu_k > 0$ for $j < k < n - 1$.

By Theorem 11, there are $n2^{n-1}$ matrices $A_n$ in $S$. This is the first term of the formula in the statement of the proposition. The set $S_0$ consists of all matrices whose nullity string is $0, \nu_1, ..., \nu_{n-1}, \nu_n$, where $\nu_1, \nu_2, ..., \nu_{n-1}$ has no zeroes and $\nu_n = 0$. Then $\nu_1 = 1$, by Proposition 1 and there is only one choice for $A_1$, namely $[1, 1; 1, 1]$, so by Theorem 11, there are $(n-1)2^{n-2}$ matrices in $S_0$. This is the second term of the formula in the statement of the proposition. If a matrix is in $S_1$ its nullity sequence is either $0, 0, \nu_2, ..., \nu_n$ or $1, 0, \nu_2, ..., \nu_n$, with $\nu_n = 0$ and $\nu_2$ through $\nu_{n-1}$ non-zero. There are $\theta(1)$ strings $\nu_0 = 0, \nu_1 = 0$ so by Remark 2.2 and Theorem 11, there are $\theta(1)(n-2)2^{n-3}$ strings in $S_1$ beginning with $\nu_0 = 0$ and $\nu_1 = 0$. There are $\eta(1)$ strings $\nu_0 = 1, \nu_1 = 0$, so by Remark 2.4 and Theorem 11, there are $2\eta(1)(n-2)2^{n-3}$ strings in $S_1$ beginning with $\nu_0 = 1$ and $\nu_1 = 0$. This gives the first term in the summation. The other terms in the summation are obtained by a similar analysis of the properties of nullity sequences for the matrices in sets $S_2$ through $S_{n-2}$.

Now consider the set $S_{n-1}$, then the triple $\nu_{n-2}, \nu_{n-1}, \nu_n$ is either 0, 0, 0 or 1, 0, 0. By Remarks 2.1 and 2.3, the number of matrices in $S_{n-1}$ is $3\theta(n-1) + 2\eta(n-1)$.

Corollary. The number of invertible Toeplitz matrices $A_n$ is $2^{2n}$, half of the total number of $(n+1) \times (n+1)$ Toeplitz matrices.

Proof. The remark above shows that the conclusion holds for $n = 0$ and $n = 1$. For $n \geq 2$ we will prove the result using the formula in the previous theorem, along with induction on $n \in \mathbb{N}$ on the following formulas.
\[ \theta(n) = \frac{2^{2n+1} + 1}{3} \]

\[ \eta(n) = \frac{2^{2n} - 1}{3} \]

The two formulas are valid for \( n = 1 \) from the remark above. For \( n = 2 \) we see that \( \theta(2) \) is the count of all matrices with initial nullity string 0, 0, 0 or 1, 0, 0. There are nine of the former and two of the latter. For \( \eta(2) \) the relevant nullity strings are 0, 1, 0 and 1, 1, 0 for \( \nu_0, \nu_1, \nu_2 \). There is one of the former and there are four of the latter.

It is possible to show that the formula in the theorem, with \( n = 2 \), gives the answer 16. So now we assume that \( k \geq 2 \) the formulas (1), (2) are valid and also that the formula in the theorem is equal \( 2^{2k} \) for \( n = k \), i.e.

\[ k2^{k-1} + (k - 1)2^{k-2} \\
+ \sum_{j=1}^{k-2} [\theta(j) + 2\eta(j)] ((k - 1) - j)2^{k-2-j} \\
+ (3\theta(k - 1) + 2\eta(k - 1)) = 2^{2k} \]

We need to show that (12.3) is valid with \( k \) replaced by \( k + 1 \), i.e. that

\[ (k+1)2^k + k2^{k-1} + \sum_{j=1}^{k-1} [\theta(j) + 2\eta(j)] ((k - 1)2^{k-1-j} + (3\theta(k) + 2\eta(k)) = 2^{2k+2} \]
To show this it will suffice to if we subtract two times the left side of Eq (3) from the left side of Eq (4) we obtain $2^{2k+1}$. Taking this difference term by term gives

$$2^k + 2^{k-1} + \sum_{j=1}^{k-2} \left( \left[ \theta(j) + \eta(j) \right] 2^{k-1-j} \right) - 5\theta(k-1) - 2\eta(k-1) + 3\theta(k) + 2\eta(k)$$

$$= 2^k + 2^{k-1} + \sum_{j=1}^{k-2} \left( \frac{2^{2j+1} + 1}{3} + \frac{2^{2j} - 1}{3} \right) 2^{k-1-j}$$

$$= 2^k + 2^{k-1} + \sum_{j=1}^{k-2} \left( \frac{2^{2j+2} - 1}{3} \right) 2^{k-1-j}$$

$$= 2^k + 2^{k-1} + \sum_{j=1}^{k-2} \left( \frac{2^{k+j+1} - 2^{k-j-1}}{3} \right)$$

$$= 2^k + 2^{k-1} + \left( \frac{2^{k+2} + 2^{k+3} + \cdots + 2^{2k-1}}{3} - \frac{2^{k-2} + 2^{k-3} + \cdots + 2^{k-1}}{3} \right)$$

$$= 2^k + 2^{k-1} + \left( \frac{2^{2k} - 2^{k+2}}{3} \right) - \left( \frac{2^{k-1} - 2}{3} \right) - \left( \frac{2^{2k-2} - 1}{3} \right) + \frac{2^{2k+1} + 1}{3} + \frac{2^{2k-1} - 1}{3}$$

and a straightforward computation shows that this expression is $2^{2k+1}$.

To finish the induction we need to verify the formulas for $\theta(k+1)$ and $\eta(k+1)$. Since $\theta(k) + \eta(k) = 2^{2k}$ we have

$$\theta(k+1) = 3\theta(k) + 2\eta(k)$$

$$= 2(\theta(k) + \eta(k)) + \theta(k)$$

$$= 2^{2k+1} + \left( \frac{2^{2k+1} + 1}{3} \right)$$

$$= 3 \cdot 2^{2k+1} + 2^{2k+1} + 1$$

$$= \frac{2^{2k+3} + 1}{3}$$

and so

$$\eta(k+1) = 2^{2k+2} - \theta(k+1) = 2^{2k+2} - \left( \frac{2^{2k+3} + 1}{3} \right) = \frac{2^{2k+2} - 1}{3},$$

to finish the proof. □
Counting matrices of positive nullity

In this section we show that the theorem above can be applied to determine the number $N(n, \nu)$ of Toeplitz matrices $A_n$ with nullity $\nu$, for any $n \in \mathbb{Z}^+$. The first step is the following corollary to the theorem.

**Corollary.** Consider the set of all $(n+1) \times (n+1)$ Toeplitz matrices $A_n$ whose nullity string $\nu_0, \nu_1, \ldots, \nu_n$ is non-zero and satisfies $\nu_n = 1$. There is only one such matrix for $n = 0$, and for $n \geq 1$ there are $(n+3)2^{n-2}$ such matrices.

**Proof.** For $n = 0$, only $A_0 = [0]$ has nullity 1. For $n = 1$ the only nullity string that satisfies the conditions is 1,1, and there are two matrices $A_1$ with this string, namely, $[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}]$ and $[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$. So the formula holds for $n = 1$.

Suppose $n \geq 2$. By the previous theorem there are $(n+1)2^n$ matrices $A_{n+1}$ whose nullity string $\nu_0, \nu_1, \ldots, \nu_{n+1}$ satisfies the hypotheses of the previous theorem. Note that every string of this form is obtained by appending a 0 to the strings $\nu_0, \nu_1, \ldots, \nu_n$ satisfying the hypotheses of the corollary. One of these is 1,1,1,1 which, by the counting rules, is the string for $2^n$ Toeplitz matrices $A_n$ so that again by the counting rules 1,1,1,1,0 is the nullity string for $2^{n+1}$ matrices $A_{n+1}$. Now consider the other nullity strings satisfying the hypotheses of the corollary. Let $R$ be the number of matrices $A_n$ with this nullity string. Each of these ends in 2,1 so by the counting rules, if we append a 0 to the end there are four times as many matrices $A_{n+1}$ with the latter string as there are $A_n$’s with the former. So we have $(n+1)2^n = 2^{n+1} + 4R$, so $R = (n-1)2^{n-2}$ and the number of Toeplitz matrices satisfying the hypotheses of the corollary is $R + 2^n = (n+3)2^{n-2}$.

\[ \Box \]

**Theorem 13.** For non-negative integers $n$ and for $s \leq n+1$ the number of Toeplitz matrices $A_n$ with a positive nullity string $\nu_0, \nu_1, \ldots, \nu_n$ ending in $s$ agrees with the number of Toeplitz matrices $A_{n+1}$ with a positive nullity string ending in $s+1$.

**Proof.** We exhibit a one to one correspondence between non-zero nullity strings $\nu_0, \ldots, \nu_n$ of length $n+1$ ending in $s$ and nonzero nullity strings of length $n+2$ ending in $s+1$. First suppose $\nu_0, \ldots, \nu_n$ is 1,1,1,1, then we match this with the string 1,2,2,2 of length $n+2$. Otherwise, suppose $\nu_0, \nu_1, \ldots, \nu_n$ is some other non-zero nullity string and $\nu_n = s$. Then there is a $j > 1$ such that $j = \max(\nu_0, \nu_1, \ldots, \nu_n)$ and $\nu_0, \nu_1, \ldots, \nu_j - 1$ is 1,1,1,1. Consider the nullity string of length $n+2$ given by 1,2,2,2,1,1,1,1,1,1,1,1. It is not difficult that these matches exhibit a one to one correspondence between the non-zero nullity strings of length $n+1$ ending in $s$ and the non-zero nullity strings of length $n+2$ ending in $s+1$. Moreover it can also be verified using the counting rules that there are the same number of Toeplitz matrices $A_n$ corresponding to the string of length $n+1$ as there are matrices $A_{n+1}$ corresponding to the matched string. This establishes the proof.

\[ \Box \]

**Theorem 14.** For $n \in \mathbb{N}$ the number $N(n, 1)$ of Toeplitz matrices $A_n$ with nullity 1 is $2^{2n-2} + 2^{2n-1}$.

**Proof.** For $n = 1$ the possible nullity strings are 1,1 and 0,1. By Theorem 13 the number of Toeplitz matrices with nullity string 1,1 is 2. For $n = 1$ the number of Toeplitz matrices $A_1$ with nullity string 0,1 is 1 (as $A_0$ is [1] and $A_1$ must be $[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}]$) so the formula holds for $n = 1$, i.e. $N(1, 1) = 3 = 2^{2-1-2} + 2^{2-1-1}$.
For $n = 2$ we have nullity strings $1, 1, 1$ and $1, 2, 1$ with $5 = (2 + 3)2^{n-2}$ corresponding Toeplitz matrices $A_2$, we have nullity string $0, 1, 1$ with $2$ corresponding Toeplitz matrices, and we have $1, 0, 1$ and $0, 0, 1$ with $5 = \theta(1) + 2\eta(1)$ corresponding matrices. The total is then $N(2, 1) = 12 = 2^2 + 2^3$ matrices.

Let $n \geq 3$. From the Corollary above the number of Toeplitz matrices with nullity $1$ which satisfies $\nu_j > 0$ for $j = 0, 1, \ldots, n$ is $(n + 3)2^{n-2}$. Next we fix $j \in \{0, 1, \ldots, n - 1\}$ and count the number of Toeplitz matrices with nullity $1$ for which $\nu_j = 0$ and $\nu_k > 0$ for $j < k \leq n$. For the case $j = 0$ then the nullity string $\nu_0, \nu_1, \ldots, \nu_n$ satisfies $\nu_0 = 0, \nu_1 = 1, \ldots, \nu_n = 1$ and $\nu_k > 0$ for $1 \leq k \leq n$. By the previous Corollary and the comment in the first paragraph of this proof, the number of Toeplitz matrices satisfying these conditions is $(n + 2)2^{n-3}$. Now suppose $1 \leq j \leq n - 2$ then using Theorem 9 and Theorem 10 the number of Toeplitz matrices with nullity $1$ which satisfies

\[
(\theta(j) + 2\eta(j))(n - j + 2)2^{n-j-3} + (\theta(n - 1) + 2\eta(n - 1)).
\]

From the formulas for $\theta$ and $\eta$ in the first Corollary to Theorem 12 we have $\theta(j) + 2\eta(j) = (2^{j+2} - 1)/3$ so for $n = 3$ the expression above for $N(3, 1)$ is

\[
6 \cdot 2 + 5 + (\theta(1) + 2\eta(1)) \cdot \frac{4 \cdot 2^1 - 1}{3} + (\theta(2) + 2\eta(2)) = 12 + 5 + 5 \cdot 2 + 21 = 48 = 2^4 + 2^5.
\]

Now suppose for some $n \geq 3$ that $N(n, 1) = 2^{n-2} + 2^{n-1}$. Replacing $n$ with $n + 1$ in the expression for $N(n, 1)$ above, giving the expression for $N(n + 1, 1)$, we obtain the following expression for $N(n + 1, 1) - 2N(n, 1)$:

\[
N(n + 1, 1) - 2N(n, 1) = 2^{n-1} + 2^{n-2} + \sum_{j=1}^{n-2} (\theta(j) + 2\eta(j))2^{n-j-2} + (\theta(n) + 2\eta(n)),
\]

or

\[
N(n + 1, 1) - 2N(n, 1) = 2^{n-1} + 2^{n-2} + \sum_{j=1}^{n-2} \frac{2^{j+2} - 1}{3}2^{n-j-2} + \frac{2^{2n+2} - 1}{3}.
\]

A routine calculation shows that this expression is equal to $2^{2n-1} + 2^{2n}$. Therefore,

\[
N(n+1, 1) = 2N(n, 1) + 2^{2n-1} + 2^{2n} = 2(2^{2n-2} + 2^{2n-1}) = 4(2^{2n-2} + 2^{2n-1}) = 2^{2n+1} + 2^{2n+1}.
\]

This establishes the induction step and the result is proved.

\[\square\]

**Definition 5.** We use the notation $P(m, k)$ and $T(n, k)$ to denote the following.

1. For $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ let $P(m, k)$ denote the number of nullity strings $\nu_0, \nu_1, \ldots, \nu_m$ for which $\nu_j > 0$, for $j = 0, \ldots, m$, and $\nu_m = k$.
2. For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ let $T(n, k)$ denote the number of Toeplitz matrices $A_n$ for which $\nu_n = k$.

**Remark 4.** The following observations will be useful.
The only nullity string \( \nu_0, \ldots, \nu_{k-1} \) of length \( k \) ending in \( k \) is \( 1, 2, \ldots, k \), and there is only one Toeplitz matrix \( A_{k-1} \) with this string, so \( P(k-1, k) = 1 \). Clearly \( P(m, k) = 0 \) for \( m < k - 1 \).

(2) From Theorem 13 above we have \( P(m, 1) = P(m + k - 1, k) \) for all positive integers \( k \).

We now show, in fact, that \( T(n + k - 1, k) = T(n, 1) \) for all \( m \geq 0 \) and positive integers \( k \). Recall from Theorem 13 that \( T(n, 1) \) was calculated by dividing the set of all \( A_n \)'s with nullity 1 into disjoint sets \( S(n), S_{0}(n), \ldots, S_{n-1}(n) \) where \( S(n) \) is the set of \( A_n \)'s whose string \( \nu_0, \nu_1, \ldots, \nu_n \) has no zeroes, \( S_{0}(n) \) consists of those \( A_n \)'s for which \( \nu_0 = 0 \) and \( \nu_j > 0 \) for \( j = 1, 2, \ldots, n \), \( S_{1}(n) \) consists of those \( A_n \)'s for which \( \nu_1 = 0 \) and \( \nu_j > 0 \) for \( j = 2, \ldots, n \), and so on, up through \( S_{n-1}(n) \) the set of all of those \( A_n \)'s for which \( \nu_n-1 = 0 \) and \( \nu_n = 1 \). Then by the proof of Theorem 13 \( |S(n)| = P(n, 1), |S_{0}(n)| = P(n - 1, 1), |S_{1}(n)| = (\theta(j) + 2\eta(j))P(n - j - 1, 1) \) for \( j = 1, \ldots, n - 2 \) and \( |S_{n-1}(n)| = \theta(n - 1) + 2\eta(n - 1) \). Similarly the set of all Toeplitz matrices \( A_{n+k-1} \) with nullity \( k \) splits into disjoint sets \( S_{0}(n+k-1), S_{1}(n+k-1), \ldots, S_{n-1}(n+k-1) \) where \( S_{0}(n+k-1) \) consists of all those Toeplitz matrices \( A_{n+k-1} \) for which \( \nu_0, \ldots, \nu_{n+k-1} \) has no zeroes and \( \nu_{n+k-1} = k \), and for \( j = 0, 1, \ldots, n - 1 \), \( S_{j}(n+k-1) \) consists of those for which \( \nu_j = 0 \) while \( \nu_{j+1} \) through \( \nu_{n+k-1} \) are nonzero and \( \nu_{n+k-1} = k \). By Theorem 13 and arguing as in the proof of Theorem 14 we deduce that \(|S_{0}(n+k-1)| = P(n + k - 1, k), |S_{1}(n+k-1)| = P(n + k - 2, k), |S_{j}(n+k-1)| = (\theta(j) + 2\eta(j))P(n - j + k - 2, k) \) for \( j = 1, \ldots, n - 2 \), and \( |S_{n-1}(n+k-1)| = \theta(n - 1) + 2\eta(n - 1) \). Since \( P(r, 1) = P(r + k - 1, k) \) by the remark above, however, and \(|S(n)| + \sum_{j=0}^{n-1} |S_{j}(n)| \) is the number of \( A_n \)'s with nullity 1 (resp., \(|S_{0}(n+k-1)| + \sum_{j=0}^{n-1} |S_{j}(n+k-1)| \) is the number of matrices \( A_{n+k-1} \) with nullity \( k \), we have shown the following.

**Theorem 15.** For any positive integer \( k \) and any non-negative integer \( n \), \( T(n + k - 1, k) = T(n, 1) \).

We can combine the two preceding results to obtain the following.

**Corollary.** The number of \((n + 1) \times (n + 1)\) Toeplitz matrices \( A_n \) with nullity 1 is \( 2^{2n-2} + 2^{2n-2}, \) with nullity 2 is \( 2^{2n-4} + 2^{2n-3}, \) and so on. The number with nullity \( n \) is 3 and the number with nullity \( n + 1 \) is 1 (the zero matrix).

**Remark 5.** With this corollary and Corollary 12 we have recovered a result of D. Daykin [2] on the enumeration of Toeplitz matrices over \( GF(2) \) of specified rank. (In fact, Daykin has made these computations over any finite field, see also [3]).

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