Abstract

Inflation is known to be generically eternal to the future: the false vacuum is thermalized in some regions of space, while inflation continues in other regions. Here, we address the question of whether inflation can also be eternal to the past. We argue that such a steady-state picture is impossible and, therefore, that inflation must have had a beginning. First, it is shown that the old inflationary model is not past-eternal. Next, some necessary conditions are formulated for inflationary spacetimes to be past-eternal and future-eternal. It is then shown that these conditions cannot simultaneously hold in physically reasonable open universes.

1. Introduction

There are essentially two approaches to cosmology. The first, which may be called evolutionary, assumes that the universe was born a finite time ago. The task of cosmology is then to uncover how the universe evolved from its initial state to its present form. The second approach, which may be called the steady-state approach, assumes that the universe has always been the way it is now. The task of cosmology is then to understand the physical processes that sustain this steady state.

An attractive feature of the steady-state approach is that there is no need to impose any initial conditions on the universe or to deal with the problem of the initial singularity. This approach was extremely popular a few decades ago, but it fell out of favor in the 1960s under the pressure of observational evidence [1]. Recently, however, steady-state cosmology has made a comeback in the form of the eternal inflationary scenario.

Inflation is a state of rapid (quasi-exponential) expansion of the universe [2, 3, 4]. The inflationary expansion is driven by the potential energy of a scalar field \( \varphi \), while the field slowly “rolls down” its potential \( V(\varphi) \). When \( \varphi \) reaches the minimum of the potential, this vacuum energy thermalizes and inflation is followed by the usual radiation-dominated expansion.

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Soon after the inflationary scenario was proposed, it was realized that, once started, inflation never ends completely [5, 6, 7]. The evolution of the field $\phi$ is influenced by quantum fluctuations, and as a result thermalization does not occur simultaneously in different parts of the universe. Inflating regions constantly undergo thermalization, but the exponential expansion of the remaining regions more than compensates for the loss, and, at any time, there are parts of the universe that are still inflating.

A model in which the inflationary phase has no end and continually produces new islands of thermalization naturally leads to this question: can this model also be extended to the infinite past, avoiding in this way the problem of the initial singularity? The universe would then be in a steady state of eternal inflation without a beginning. We are going to argue here that the answer to this question is “no” [8, 9, 10].

Before analyzing the general case, we clarify the ideas using a simple model. In section 2 the possibility of eternal inflation is discussed in the context of the “old” inflationary scenario, in which the vacuum energy is strictly constant and vacuum decay occurs through bubble nucleation. Old inflation is known to be eternal to the future: bubbles cannot fill the entire universe since the space between them is expanding so fast [2, 3]. In the thermalized parts of the universe the distribution of matter produced by colliding bubbles walls is grossly inhomogeneous, making old inflation unsuitable as a realistic cosmological model. Here we disregard this aspect of the problem and only concern ourselves with the question of whether or not old inflation can be continued back to the infinite past. We review the arguments showing that it cannot.

Using this discussion as a guide, we formulate in section 3 two conditions that any spacetime describing an eternally inflating universe should satisfy. (In the Appendix we give an example of spacetimes where these conditions do not both hold, as well as an example where they do.) We then show in section 4 that, under very general assumptions, these conditions cannot be simultaneously satisfied.

We use the following conventions: the metric has signature $(+, -, -, -)$ and Einstein’s equation is $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$ (where $R_{\mu\nu}$ is the Ricci tensor associated with the metric $g_{\mu\nu}$, $R$ the scalar curvature and $T_{\mu\nu}$ the matter energy-momentum tensor). Our main result uses the Penrose-Hawking-Geroch “global techniques;” an overview of the background that we need is given in section 4. For further details and for the proofs of various standard global results, see, for example, Hawking and Ellis [11].

2. Old inflation

In the old inflationary scenario the false vacuum has the energy-momentum tensor

$$T_{\mu\nu} = \rho g_{\mu\nu},$$

where $\rho$ is constant. The homogeneous and isotropic solution of Einstein’s equation with $T_{\mu\nu}$ from (1) is de Sitter space, which can be represented in the form

$$ds^2 = dt^2 - e^{2Ht}d\vec{x}^2,$$

where

$$H^2 = \frac{8\pi G \rho}{3}. \quad (3)$$

This space has a horizon of radius $H^{-1}$; observers separated by a greater distance cannot communicate.

Bubbles nucleating in false vacuum expand, rapidly approaching the speed of light. In de Sitter space this corresponds to having asymptotically static boundaries in co-moving coordinates. The physical radius of a bubble formed at time $t_1$ is (for $H(t - t_1) \gg 1$)

$$r(t, t_1) \approx H^{-1} e^{H(t-t_1)}.$$

$$-2-$$
An expanding bubble can affect the geometry of the outside region only within a distance of approximately $H^{-1}$ from its boundary. Hence, although bubbles carve large volumes out of de Sitter space, the geometry of the remaining regions is practically unchanged. We shall first assume that inflation starts at some time $t_0$ and later consider the limit $t_0 \to -\infty$.

Bubble nucleation is a stochastic process with a constant probability $\lambda$ per unit spacetime volume. The probability for no bubbles to be formed in a 4-volume $\Omega$ is [2, 12]

$$P_{\Omega} = e^{-\lambda \Omega}. \tag{5}$$

Here is a quick derivation. Let $P(A)$ be the probability for no bubbles to nucleate in spacetime region $A$. Then, for non-overlapping regions $A$ and $B$, $P(A \cup B) = P(A)P(B)$, and for an infinitesimal volume $d\Omega$, $P \approx 1 - \lambda d\Omega$. The only function $P(\Omega)$ with these properties is (5).

The probability for a given spacetime point $x = (t, \vec{x})$ to be in the inflationary phase is given by (5) with $\Omega$ being the volume occupied by the false vacuum in the past light cone of $x$. For $H(t - t_0) \gg 1$, this volume is

$$\Omega = \frac{4\pi}{3H^2}(t - t_0). \tag{6}$$

(This equation is easily understood if we note that null geodesics continued to large negative values of $t_0$ asymptotically approach the horizon, which is a sphere of radius $H^{-1}$.) From (5), (6), the fraction of space that is still inflating at time $t$ is

$$f(t) \equiv \exp \left\{ -\frac{4\pi \lambda}{3H^3}(t - t_0) \right\}. \tag{7}$$

The function $f(t)$ decreases with time and vanishes as $t \to +\infty$. But the physical volume of the inflating regions, $V(t) \propto e^{3Ht}f(t)$, grows with time. The reason is very simple: for sufficiently small $\lambda$, the rate of expansion of the false vacuum regions is greater than their rate of decay.

![Figure 1](image.png)

**Figure 1:** A schematic snapshot of the old inflationary universe. The shaded regions represent bubbles and the white region represents the inflationary background in which they are embedded.

An argument similar to that in Ref. [6] shows that the inflating regions form a self-similar fractal of dimension $d < 3$. This fractal dimension can be found from

$$f(t) = \left( \frac{H^{-1}}{R} \right)^{3-d}, \tag{8}$$

- 3 -
where $H^{-1}$ is the size of the smallest bubbles and $R \approx H^{-1} \exp[H(t - t_0)]$ is the size of the largest bubbles (formed at $t \approx t_0$). A comparison of (7) and (8) gives

$$d = 3 - \frac{4\pi \lambda}{3H^3}.$$  \hspace{1cm} (9)

The meaning of the fractal dimension is easy to understand. Consider a sphere of radius $r$ centered on a point in the inflating region (see fig. 1). As $r$ is increased, the volume $V$ occupied by the false vacuum inside the sphere grows (on average) proportional to $r^4$. The deviation of $d$ from 3 can be attributed to the fact that, as the sphere becomes larger, it is likely to include larger and larger bubbles. Of course, the inflating regions have a fractal nature only on scales $H^{-1} < r < R$. For $r > R$, $V \propto r^3$.

Let us now ask what happens if we remove the beginning of inflation to the infinite past. As we said in the Introduction, we are not concerned here with whether or not this model is realistic (it is not). Rather, we are concerned with whether or not a consistent model of eternal inflation is obtained by letting $t_0 \to -\infty$.

As $t_0 \to -\infty$, the upper cutoff on the sizes of the bubbles is removed; i.e., $R \to \infty$. At the same time the probability (5) for a point to be in the inflationary phase and the fraction of space $f$ occupied by the false vacuum both vanish. Note, however, that for a point $(\vec{x}, t)$ in an inflating region, there is a finite probability that inflation will continue for any given time interval $\Delta t$. This probability is given by

$$P_{\Delta \Omega} = e^{-\lambda \Delta \Omega},$$  \hspace{1cm} (10)

where $\Delta \Omega$ is the 4-volume between the past light cones originating at $(\vec{x}, t)$ and $(\vec{x}, t + \Delta t)$; i.e.,

$$\Delta \Omega = \frac{4\pi}{3H^3} \Delta t.$$  \hspace{1cm} (11)

The vanishing of $f$ in (7) simply expresses the fact that an object of fractal dimension $d < 3$ cannot fill a 3-dimensional space; a randomly chosen point is most likely to be inside an infinitely large bubble. The physical volume occupied by the false vacuum is, however, still increasing with time, and it may appear that we have a model of eternal inflation.

The trouble with this model is that the metric (2) is geodesically incomplete [11]. To see what this means, consider a flat spacetime

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$  \hspace{1cm} (12)

and introduce a new time coordinate $\tau \equiv \ln t$. The metric may be written in terms of the new time coordinate as

$$ds^2 = e^{2\tau} d\tau^2 - dx^2 - dy^2 - dz^2$$  \hspace{1cm} (13)

As $\tau$ varies from $-\infty$ to $+\infty$, the Minkowski time $t$ changes from 0 to $\infty$, and thus the metric (13) covers only half of Minkowski spacetime.

How could this have been determined without the prior knowledge that (13) was obtained from (12)? One way to decide if there are such “missing regions” to the past is to calculate the proper time along past-directed timelike geodesics. For example, for the geodesic given by $\vec{x} = \text{const}$, starting at $\tau = 0$ and continuing back to $\tau \to -\infty$, we have

$$\int_{-\infty}^{0} e^\tau d\tau = 1.$$  \hspace{1cm} (14)

The finiteness of this proper time indicates that our spacetime has an “edge,” and can perhaps be continued beyond $\tau = -\infty$. (Note, however, that not all geodesically incomplete spacetimes
Figure 2: The full de Sitter spacetime. The time coordinate $\tilde{t}$ points upward and the spacelike cross sections orthogonal to $\tilde{t}$ are 3-spheres. The shaded portion of this spacetime lies to the past of the $t = -\infty$ surface and is not covered by the old coordinates.

The metric (2) which we used to describe the inflating universe is both timelike and null geodesically incomplete. All timelike geodesics, except the co-moving ones ($\vec{x} = \text{const}$), reach the surface $t = -\infty$ in a finite proper time and all null geodesics reach it within a finite lapse of their affine parameters. The full de Sitter space is covered by the metric [11]

$$ds^2 = d\tilde{t}^2 - H^{-2} \cosh^2(H\tilde{t}) d\Omega_3^2$$

(15)

where $d\Omega_3^2$ is the metric on a unit 3-sphere (see fig. 2). In this extended spacetime, the phase of the exponential expansion at $\tilde{t} > 0$ is preceded by a phase of exponential contraction at $\tilde{t} < 0$. Of course, the contracting phase does not describe an inflating universe. If such a contracting universe were filled by a false vacuum, the nucleating bubbles would rapidly fill the space. The whole universe would thermalize and collapse to a singularity without getting to the expanding phase.

In terms of the probability (10) for inflation to persist, a nonzero answer was obtained only because the volume $\Delta\Omega$ bounded by the two light cones was cut off at the surface $t = -\infty$. In the full de Sitter space (15), $\Delta\Omega = \infty$ and $P_{\Delta\Omega} = 0$.

3. Conditions for eternal inflation

The analysis in the previous section cannot be directly applied to realistic inflationary scenarios. In realistic models the false vacuum energy $\rho$ is replaced by the scalar field potential $V(\varphi)$ which can vary in space and time. The field $\varphi$ is usually assumed to be slowly-varying and the spacetime to be locally close to de Sitter, but the global structure of spacetime can be quite different. Moreover, quantum nucleation of bubbles is replaced by a quantum random walk of the field $\varphi$, which is followed by thermalization when $\varphi$ gets close to the bottom of the potential. One can still find the probability for inflation to persist at a given point for a specified period of time, but now this probability depends on the initial value of $\varphi$ at that point. As a result, the locations of thermalization regions are strongly correlated (unlike the bubble nucleation.
sites). The distribution of thermalized regions obtained in a numerical simulation [6] is shown in fig. 3. It can be shown that, as before, the inflating region is a fractal whose dimension is determined by the shape of the potential $V(\varphi)$. Finally, if the magnitude of $V(\varphi)$ gets near the Planck scale, the gravitational action may get significant quantum corrections, and Einstein’s equation can no longer be used.

See figure c in Ref. [6] for this picture.

**Figure 3:** A computer simulation of the inflationary universe [6]. The dark areas represent thermalized regions and the white ones the inflationary background.

In this section we shall formulate some necessary conditions that a spacetime should satisfy in order to describe an eternally inflating universe. We shall try to reduce to the minimum any assumptions about the dynamical laws that govern the evolution of geometry and of the scalar field $\varphi$. However, to make the discussion meaningful we will have to assume that, to a reasonable approximation, the spacetime can be treated as a classical Riemannian manifold. Although eternal inflation is sometimes described as occurring at the Planck scale, the description invariably relies on classical spacetime concepts such as “causality,” “beginning,” “end,” etc. It is also implicit in some of the discussion that follows that spacetime obeys the *causality condition*: i.e., it contains no closed null or timelike curves.

In the previous section we saw that the spacetime (2) describing an inflating universe is geodesically incomplete. If eternal inflation is possible, one should be able to construct a *complete* spacetime that has the necessary properties of (2). This leads us to the first of our conditions:

- **Condition 1:** Timelike and null geodesics are past-complete.

(We do not require future-completeness because that would preclude the existence of geodesics that encounter such things as black hole singularities.)

Now, it should be clear from the preceding discussion that the essential property required for inflation to be future-eternal is a non-zero probability for inflation to continue at a given point for a specified interval of time. In the old inflationary scenario this probability is given by (10) and its nonzero value is guaranteed by the finiteness of the 4-volume $\Delta \Omega$ in (11).
To formulate the corresponding requirement in the general case, we shall assume that
the boundaries of thermalized regions expand at a speed approaching the speed of light, like
the walls of bubbles expanding in a false vacuum. More precisely, it will be assumed that a
spacetime point $p$ can be in an inflating region only if its past, $I^{-}(p)$, intersects no thermalized
regions. In addition, it will be assumed that the probability of forming thermalized regions
does not vanish in the infinite past. Otherwise, it is possible for the false vacuum to survive
an infinitely long contraction phase. This possibility, however, is against the spirit of eternal
inflation which assumes a “steady-state” picture of the universe. With this assumption, the
probability of having no thermalized regions in an infinite volume vanishes, and we arrive at
the following condition:

- **Condition 2**: Let $p$ and $q$ be two points in an inflating region with $q$ to the future of $p$. Then the volume $\Delta \Omega$ of the difference of the pasts of $q$ and $p$ (i.e., the volume of $I^{-}(q) - I^{-}(p)$) is finite; i.e.,

$$\Delta \Omega < \infty.$$  (16)

The two conditions obtained here are very general. Neither rests on detailed assumptions about inflation. In particular, we make no assumptions about our spacetime being “locally close to de Sitter.” Such an assumption is often used to characterize inflationary scenarios. A more general approach might be to require that the Ricci tensor obeys $R_{\mu\nu}T^{\mu}T^{\nu} < 0$ for all timelike vectors (i.e., assuming Einstein’s equation, that the spacetime necessarily violates the strong energy condition). This, by itself, does not guarantee rapid expansion of the universe, but it is necessary if such expansion is to take place. To see this, observe that the geodesic focusing equation (the Landau-Raychaudhuri equation) for a congruence of rotation-free timelike geodesics may be written as [11]

$$\dot{\theta} \leq -\frac{1}{3} \theta^2 - R_{\mu\nu}T^{\mu}T^{\nu},$$  (17)

where the derivative is taken with respect to an affine parameter (the usual choice is the proper time), $\theta = D_{\mu}T^{\mu}$ is the divergence (or expansion) of the congruence, and $T^{\mu}$ is the tangent field to the geodesics with respect to the chosen parameter. If $R_{\mu\nu}T^{\mu}T^{\nu}$ is positive at some point, then initially parallel geodesics (i.e., $\theta = 0$) that pass through that point will start to converge (i.e., $\theta$ will become negative). This is not the sort of behavior one expects in an inflating spacetime.

For the particular results of this paper, we do not, however, need even such a general description of inflation. Indeed, it is possible to ask if conditions 1 and 2 are likely to hold in an arbitrary spacetime, whether it is inflating or not. In the Appendix we show that both conditions cannot hold in open Robertson-Walker spacetimes and we also construct an example in which they do hold. This example is not realistic, however, for in section 4 we show that a spacetime cannot simultaneously satisfy our two conditions, as long as it meets some mild (and physically reasonable) further requirements.

4. The general theorem

**Theorem**: A spacetime cannot simultaneously satisfy the following conditions:

A. It is past causally simple.
B. It is open.
C. Einstein’s equation holds, with a source that obeys the weak energy condition (i.e., the matter energy density is non-negative).
D. It is past null complete.
E. There is at least one point $p$ such that for some point $q$ to the future of $p$ the volume of the difference of the pasts of $q$ and $p$ is finite.
The proof of this theorem is supplied at the end of this section. We discuss the assumptions of the theorem in detail below, but here is a summary of what they mean: Assumptions A and B are made solely for mathematical convenience. The ultimate goal is to relax them (especially assumption B). Assumption C holds in standard inflationary spacetimes, and is physically quite reasonable. Assumption D is necessary for inflation to be past-eternal. Assumption E is necessary for inflation to be future-eternal.

Our result is similar in spirit to the standard singularity theorems of general relativity: in all the singularity results the existence of certain spacetime structures (trapped surfaces, reconverging null cones, compact spacelike hypersurfaces, etc.) is shown to be incompatible with geodesic completeness. None of the standard theorems, however, exactly fits the situation in which we are interested. For instance, several theorems (such as the multi-purpose Hawking-Penrose theorem [13]) assume the strong energy condition, known to be violated in inflationary scenarios. (In fact, as we have mentioned above, the violation of this energy condition is necessary for inflation to occur.) Others place much stronger restrictions on the global causal structure of spacetime than we do here (through assumption A). More significantly, assumption E is entirely new – as we have seen above, it captures a characteristic aspect of future-eternal inflationary spacetimes.

In order to discuss the assumptions in greater detail, we need certain notations and results from global general relativity. A summary is given below; the proofs of all our assertions may be found in Ref. [11].

Let $M$ be a spacetime. (We assume that $M$ is time-orientable, allowing a consistent global distinction between past and future.) A curve in $M$ is called causal if it is everywhere timelike or null (i.e., lightlike). Let $p \in M$. The causal and chronological pasts of $p$, denoted respectively by $J^{-}(p)$ and $\Gamma^{-}(p)$, are defined as follows:

$$J^{-}(p) = \{ q : \text{there is a future-directed causal curve from } q \text{ to } p \},$$

and

$$\Gamma^{-}(p) = \{ q : \text{there is a future-directed timelike curve from } q \text{ to } p \}.$$  

Thus $J^{-}(p)$ is the set of all points that can send signals to $p$ along timelike or null curves, and $\Gamma^{-}(p)$ is the set of all points that can send signals to $p$ along just timelike curves.

The past light cone of $p$ may then be defined as $E^{-}(p) = J^{-}(p) - \Gamma^{-}(p)$; i.e., $E^{-}(p)$ consists of all points that can send signals to $p$ along future-directed null curves, but not along timelike curves. It may be shown that the boundaries of the two kinds of pasts of $p$ are the same; i.e., $\bar{J^{-}}(p) = \bar{\Gamma^{-}}(p)$. Further, it may be shown that $E^{-}(p) \subset \bar{\Gamma^{-}}(p)$. In general, however, $E^{-}(p) \neq \bar{\Gamma^{-}}(p)$; i.e., the past light cone of $p$ (as we have defined it here) is a subset of the boundary of the past of $p$, but is not necessarily the full boundary of this past. This is illustrated in fig. 4.

A set is called achronal if no two points in it can be connected by a timelike curve; for instance, $\bar{\Gamma^{-}}(p)$ is achronal (if two points on it can be connected by a timelike curve, then the pastmost of the two points will lie inside $\bar{\Gamma^{-}}(p)$, not on its boundary).

With this background information, we return to our discussion of the assumptions of our theorem:

- **Assumption A:** A spacetime is past causally simple if $E^{-}(p) = \bar{\Gamma^{-}}(p) \neq \emptyset$ for all points $p$. By saying that $\bar{\Gamma^{-}}(p) \neq \emptyset$ we are ruling out causality violations, and by saying that $E^{-}(p) = \bar{\Gamma^{-}}(p)$ we are excluding scenarios such as the one in fig. 4.

  In a causally simple space, the boundary of the past of a point $p$ is “generated by null geodesics to $p$”; i.e., through each point $q \in \bar{\Gamma^{-}}(p)$ there passes a future directed null geodesic that has a future endpoint at $p$ [11].

- **Assumption B:** A universe is open if it contains no compact achronal edgeless hypersurfaces. This is an extension of the more common statement that a closed universe is one that contains a compact spacelike hypersurface.
Figure 4: An example of the causal complications that can arise in an unrestricted spacetime. Light rays travel along 45° lines in this diagram, and the two thick horizontal lines are identified. This allows the point q to send a signal to the point p along the dashed line, as shown, even though q lies outside what is usually considered the past light cone of p. The boundary of the past of p, \( \mathcal{I}^{-}(p) \), then consists of the past light cone of p, \( E^{-}(p) \), plus a further piece. Such a spacetime is not “causally simple.”

- **Assumption C:** An observer with four-velocity \( V^\mu \) will see a matter energy density of \( T_{\mu \nu} V^\mu V^\nu \). The weak energy condition is the requirement that \( T_{\mu \nu} V^\mu V^\nu \geq 0 \) for all timelike vectors \( V^\mu \).

  This is a reasonable restriction on the matter fields in spacetime. Indeed, the condition is known to be true for all classical matter. If the matter fields are quantum fields, however, it is possible for the weak energy condition to be violated by certain states of the fields. Violations of the standard energy conditions have been discussed by Tipler [14] who has pointed out that such pointwise conditions may be replaced by weaker integral, or averaged, conditions. An averaged weak energy condition has been discussed by Roman [15], and even weaker integral conditions have also been introduced [9].

  It follows by continuity from the weak energy condition that \( T_{\mu \nu} N^\mu N^\nu \geq 0 \) for all null vectors \( N^\mu \) and from Einstein’s equation that \( R_{\mu \nu} N^\mu N^\nu \geq 0 \). It is this final form, sometimes called the null convergence condition, that we will actually use. (Thus our results will remain true even in other theories of gravity, as long as the null convergence condition, or one of the weaker corresponding integral conditions, continues to hold.)

- **Assumptions D & E:** These have been discussed in detail in the previous section.

Here, now, is the proof of our main result:

**Proof:** Suppose that a spacetime obeys assumptions A–E. We show in two steps that a contradiction ensues.

1. A point \( p \) that satisfies assumption E has a finite past light cone. By a “finite past light cone” is meant a light cone \( E^{-}(p) \) such that every past-directed null geodesic that initially lies in the cone leaves it a finite affine parameter distance to the past of \( p \).

   Suppose, to the contrary, that a null geodesic \( \gamma \) lies in \( E^{-}(p) \) an infinite affine parameter distance to the past of \( p \). Let \( \nu \) be an affine parameter on \( \gamma \) chosen to increase to the past and to have the value 0 at \( p \). Consider a small ‘conical’ pencil of null geodesics in \( E^{-}(p) \) around \( \gamma \) and choose coordinates \( x^1 \) and \( x^2 \) on the spacelike cross sections of this pencil. Now vary this pencil along some timelike curve between \( p \) and \( q \). This sets up a null geodesic congruence; let \( N^\mu \) be the tangent vector field to these geodesics associated with the affine parameter \( \nu \). If \( u \) is a parameter along the timelike curve between \( p \) and \( q \), the surfaces of constant \( u \) will each consist of a pencil of null geodesics; choose \( u \) so that the pencil containing \( \gamma \) corresponds to \( u = 0 \). See fig. 5 for an illustration of this construction.
Figure 5: The geodesic congruence and the volume of interest. A small pencil of null geodesics emanating in the past direction from \( p \) is chosen around the geodesic of interest, \( \gamma \). The pencil is varied in the future direction till the vertex is at the point \( q \) (not shown above).

How do the null geodesics in this congruence move away from or towards \( \gamma \)? If one considers a “deviation vector” \( Z^\mu \) that connects points on \( \gamma \) to points on a nearby geodesic [11], a straightforward calculation yields the result that the only physically relevant variations in \( Z^\mu \) come from its components in the spacelike 2-space in \( E^- \) that is orthogonal to \( N^\mu \). This is most easily seen by introducing a pseudo-orthonormal basis \( \{ N^\mu, L^\nu, X^1, X^2 \} \) where \( L^\mu \) is a null vector such that \( N_\mu L^\mu = 1 \) and \( X_1^i \) and \( X_2^i \) are unit spacelike vectors orthogonal to \( N^\mu \) and \( L^\mu \). The deviation vector \( Z^\mu \) may be written as \( Z^\mu = nN^\mu + lL^\mu + x^1 X^1_1 + x^2 X^2_2 \). Now, it is possible to choose the affine parametrization so that \( n = 0 \). Further, \( l = Z^\mu N_\mu \) and the derivative of \( l \) vanishes in the direction of \( N^\mu \) (i.e., \( N^\nu D_\nu(Z^\mu N_\mu) = 0 \)).

The metric may be expressed in terms of the coordinates \((u, v, x^1, x^2)\) as [8]

\[
\begin{align*}
ds^2 &= g_{uu} du^2 + 2 du dv + 2 g_{ui} du dx^i + g_{ij} dx^i dx^j, \tag{18}
\end{align*}
\]

where \( i \) and \( j \) run from 1 to 2. The determinant of this metric is \( g = -(2)^g \), where \((2)^g\) is the determinant of \( g_{ij} \).

All of this means that the volume of the spacetime region occupied by the portion of the geodesic congruence between \( u = 0 \) and \( u = \Delta \) (where \( \Delta \) is infinitesimal) may be expressed as

\[
\Delta V = \Delta \int_0^\infty A(v) dv, \tag{19}
\]

where

\[
A(v) = \int \sqrt{(2)^g} d^2 x \tag{20}
\]

is the area of the spacelike cross section of the light cone orthogonal to \( N^\mu \) (see fig. 5). The region whose volume we are calculating is a subset of \( \bar{I}^- (q) - \bar{I}^- (p) \) (i.e., the closure of the difference of the pasts of \( q \) and \( p \)) and it must thus have a finite volume. In order for this to happen, \( A \) must decrease somewhere along \( \gamma \).

The propagation equation for \( A \) is [11]

\[
\dot{A} = \theta A \tag{21}
\]

where a dot represents a derivative with respect to \( v \) and \( \theta = D_\mu N^\mu \) is the divergence of the congruence. If \( A \) decreases it follows that \( \theta \) must become negative. But the propagation equation for \( \theta \) may be written as [11]

\[
\dot{\theta} \leq -\frac{1}{2} \theta^2 - R_{\mu\nu} N^\mu N^\nu \leq -\frac{1}{2} \theta^2 \tag{22}
\]

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(where we have used assumption C in the last step). If $\theta < 0$ somewhere, it follows that $\theta \to -\infty$ within a finite affine parameter distance.

The divergence of $\theta$ to $-\infty$ is a signal that the null geodesics from $p$ have refocused. It is a standard result in global general relativity [11] that points on such null geodesics beyond the focal point enter the interior of the past light cone (i.e., enter $I^- (p)$) and no longer lie in $E^- (p)$. This is illustrated in fig. 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{null_geodesics.png}
\caption{The past light cone of $p$, showing (in an exaggerated way) the focusing of some null geodesics. Points $r$ beyond the focal point lie in the interior of the past of $p$ (i.e., in $I^- (p)$) and no longer on the boundary of the past. The thrust of step 1 of the proof is to show that such focusing occurs along every past-directed null geodesic from $p$ that lies on the boundary (i.e., on $E^- (p)$), for a sufficient affine length.}
\end{figure}

Thus the null cone $E^- (p)$ must be finite in the sense defined above.

2. The result of step 1 contradicts assumption B. From causal simplicity it follows that $E^- (p)$ (being equal to the full boundary of the past of $p$, $I^- (p)$) is an edgeless surface. It is also achronal. And step 1 implies that $E^- (p)$ is compact. These three statements taken together contradict assumption B.

5. Discussion

The argument given above shows that inflation does not seem to avoid the problem of the initial singularity (although it does move it back into an indefinite past). In fact, our analysis of assumption E in section 3 suggests that almost all points in the inflating region will have a singularity somewhere in their pasts. In this sense, our result is stronger than most of the usual singularity results (which, in general, predict the existence of just one singularity, and in the case of the Hawking-Penrose theorem [13] fail to provide any information at all about the location of this singularity).

The only way to deal with the problem of the initial singularity is probably to treat the universe quantum mechanically and describe it by a wave function rather than by a classical spacetime.

The theorem that we have proved here is based on several assumptions which it would be desirable to further justify or relax. The principal relaxation that is necessary is in assumption B; i.e., closed universes must also be accommodated. It may appear at first sight that this will be difficult to achieve since assumption B entered into the proof above at a crucial place. However, all that we really need is to exclude situations where null geodesics recross after running around the whole universe (as they do in the static Einstein model). If the reconvergence of the null cone discussed above occurs on a scale smaller than the cosmological one, then essentially the same argument goes through. This approach to applying open universe
singularity theorems to closed universes has been outlined previously by Penrose [16] and it will be discussed in detail separately as will the relaxation of assumption A.

Assumption E, which plays a central role in our argument, requires further justification. It rests on the assumptions that (i) the boundaries of thermalized regions expand at speeds approaching the speed of light, and (ii) that the probability of finding no thermalized regions in an infinite spacetime volume vanishes. Both assumptions are plausible and, as we have seen in section 2, they are true in the original inflationary scenario. It would be interesting, however, to determine the exact conditions of validity for these assumptions and to investigate the possibility of relaxing them.

Appendix: Geodesic completeness and past volumes

Consider the open Robertson-Walker metrics; they may be expressed in the form

\[ ds^2 = a^2(\eta) [d\eta^2 - d\chi^2 - f^2(\chi) d\Omega^2] , \]  

(A1)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \) is the metric on a unit 2-sphere, \( \chi \geq 0 \), and

\[ f(\chi) = \begin{cases} \chi & \text{(spatially flat open universes)} \\ \sinh \chi & \text{(spatially curved open universes)} \end{cases} . \]  

(A2)

The range of the coordinate \( \eta \) depends on the behavior of the function \( a(\eta) \). As we shall see below, the range may or may not be infinite to the past.

The null geodesics that pass through \( \chi = 0 \) are given by

\[ a^2(\eta) \frac{d\eta}{dv} = \text{constant}, \]  

(A3)

and

\[ \chi(v) = \pm \eta(v) + \text{constant}, \]  

(A4)

where \( v \) is an affine parameter. If the geodesics are to be past complete, \( v \) must approach \(-\infty\) to the past (assuming that it is chosen to decrease in the past direction). The time coordinate \( \eta \) need not itself approach \(-\infty\); from (A3) it follows that all that is needed in order to guarantee past completeness for these null geodesics is that

\[ \int_{\eta_{\text{min}}}^{\eta} a^2(\hat{\eta}) \, d\hat{\eta} \]  

(A5)

diverge (where \( \eta_{\text{min}} \) represents the lower bound on \( \eta \)). If \( \eta_{\text{min}} \) is finite, the divergence of (A5) must arise from a blowing up of the function \( a^2(\eta) \) as \( \eta \to \eta_{\text{min}} \). In this case, the universe starts off "infinitely stretched" and, initially at least, contracts. Such a picture is inconsistent with inflation. But for the discussion below, it is not necessary to exclude this case and we make no assumptions here about whether or not \( \eta_{\text{min}} \) is finite.

If \( \eta \) is bounded below by \( \eta_{\text{min}} \), then \( \chi \) will be bounded above along each null geodesic by some value \( \chi_{\text{max}} \) (finite if \( \eta_{\text{min}} \) is, and infinite otherwise) when the geodesic is followed into the past.

Consider the volume \( \Delta \Omega \) between the past light cones of \( (\chi = 0, \eta = 0) \) and \( (\chi = 0, \eta = \Delta) \). Since \( \sqrt{-g} = a^4(\eta) f^2(\chi) \sin \theta \), we have

\[ \Delta \Omega = \int_0^{\chi_{\text{max}}} d\chi \int_{-\chi}^{-\chi+\Delta} d\eta \int_0^{\pi} d\theta \int_0^{2\pi} d\phi a^4(\eta) f^2(\chi) \sin \theta . \]  

(A6)
For small $\Delta$ this becomes
\[
\Delta \Omega \approx 4\pi \Delta \int_{0}^{\chi_{\text{max}}} d\chi \, a^4(-\chi) f^2(\chi)
\]  
(A7)

Now, the divergence of (A5), necessary for past null completeness, implies that
\[
\int_{0}^{\chi_{\text{max}}} d\chi \, a^2(-\chi)
\]  
(A8)
diverges. Thus $\int a^4 d\chi$ must also diverge. Further, $f^2$ is an increasing function. (Actually, all that we really need is that $f^2$ remain greater than some positive constant as $\chi \to \chi_{\text{max}}$.) The conclusion from these observations is that $\Delta \Omega$ in (A7) diverges to $+\infty$ in the open Robertson-Walker models if they are past null complete.

Thus past geodesic completeness is incompatible here with $\Delta \Omega < \infty$. It might seem that this must always be so, no matter what the metric. (I.e., it might seem that the spacetime volume between past-complete null cones must necessarily be infinite.) We now present an example showing that this is not the case. Consider the metric
\[
ds^2 = A^2(\eta)(d\eta^2 - d\chi^2) - B^2(\eta) \chi^2 d\Omega^2
\]  
(A9)

If the null geodesics that emanate from ($\eta = 0, \chi = 0$) are to be past complete, and the coordinate $\eta$ is to approach $-\infty$ when these geodesics are followed into the past, we must have
\[
\int_{-\infty}^{0} A^2(\eta) d\eta = \infty
\]  
(A10)
The past volume difference that we are interested in is now
\[
\Delta \Omega \approx 4\pi \Delta \int_{0}^{\infty} d\chi \chi^2 A^2(-\chi) B^2(-\chi)
\]
\[
= 4\pi \Delta \int_{-\infty}^{0} d\eta \eta^2 A^2(\eta) B^2(\eta)
\]  
(A11)
This integral will converge (and (A10) will diverge) if, for instance,
\[
A(\eta) = (1 - \eta)^{-\alpha}, \quad 0 \leq \alpha < \frac{1}{2}
\]  
(A12)
and
\[
B(\eta) = (1 - \eta)^{-\beta}, \quad \beta > 1
\]  
(A13)

This example escapes the clutches of the general theorem of section 4 because it violates the weak energy condition. It is instructive to examine how exactly the escape occurs. Let $p$ be a point on the $\chi = 0$ line with $\eta = \eta_0 < 1$. The divergence of the past-directed null geodesics from $p$ is
\[
\theta = 2 \frac{1}{A^2} \left( \frac{1}{\eta_0 - \eta} - \frac{1}{B} \frac{dB}{d\eta} \right)
\]  
(A14)
If $A$ and $B$ are given by (A12) and (A13), $\theta$ diverges to $+\infty$ at $p$ and it becomes negative for
\[
\eta < \frac{\beta \eta_0 - 1}{\beta - 1}
\]  
(A15)
This means that the null geodesics from $p$ have started to reconverge. This reconvergence is necessary for a finite past-volume, as is shown in the main argument in section 4. But the violation of the weak energy condition here allows the geodesics to avoid refocusing (an examination of (A14) reveals that it remains finite everywhere to the past of $p$), thereby providing an escape from our theorem.
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