A CHARACTERISTIC NUMBER OF BUNDLES DETERMINED BY MASS LINEAR PAIRS

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Abstract. Let $\Delta$ be a Delzant polytope in $\mathbb{R}^n$ and $b \in \mathbb{Z}^n$. Let $E$ denote the symplectic fibration over $S^2$ determined by the pair $(\Delta, b)$. We prove the equivalence between the fact that $(\Delta, b)$ is a mass linear pair (D. McDuff, S. Tolman, Polytopes with mass linear functions, part I. arXiv:0807.0900 [math.SG]) and the vanishing of a characteristic number of $E$ in the following cases: When $\Delta$ is a $\Delta_{n-1}$ bundle over $\Delta_1$; when $\Delta$ is the polytope associated with the one point blow up of $\mathbb{C}P^n$; and when $\Delta$ is the polytope associated with a Hirzebruch surface.

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1. Introduction

Let $T$ be the torus $(U(1))^n$ and $\Delta = \Delta(\mathbf{n}, k)$ the polytope in $t^*$ with $m$ facets defined by

\begin{equation}
\Delta(\mathbf{n}, k) = \bigcap_{j=1}^{m} \{ x \in t^* : \langle x, \mathbf{n}_j \rangle \leq k_j \},
\end{equation}

where $k_j \in \mathbb{R}$ and the $\mathbf{n}_j \in t$ are the outward conormals to the facets. The facet defined by the equation $\langle x, \mathbf{n}_j \rangle = k_j$ will be denoted $F_j$, and we put $\text{Cm}(\Delta)$ for the mass center of the polytope $\Delta$.

In [3] is defined the chamber $C_\Delta$ of $\Delta$ as the set of $k' \in \mathbb{R}^m$ such that the polytope $\Delta' := \Delta(\mathbf{n}, k')$ is analogous to $\Delta$; that is, the intersection $\cap_{j \in J} F_j$ is nonempty iff $\cap_{j \in J} F'_j \neq \emptyset$ for any $J \subset \{1, \ldots, m\}$. When we consider only polytopes which belong to the chamber of a fixed polytope we delete the $\mathbf{n}$ in the notation introduced in (1.1).

Further McDuff and Tolman introduced the concept of mass linear pair: Given the polytope $\Delta$ and $b \in t$, the pair $(\Delta, b)$ is mass linear if the map

$$k \in \mathbb{R}^m \mapsto \langle \text{Cm}(\Delta(k)), b \rangle \in \mathbb{R}$$

is linear on $C_\Delta$.

Let $(N, \Omega)$ be a closed connected symplectic $2n$-manifold. By $\text{Ham}(N, \Omega)$ we denote the Hamiltonian group of $(N, \Omega)$ [3]. If $\psi$ is a loop in $\text{Ham}(N, \Omega)$, then $\psi$ determines a Hamiltonian fibre bundle $E \to S^2$ with standard fibre $N$ via the clutching construction. In [3] various characteristic numbers for the fibre bundle $E$ ...
are defined. These numbers give rise to topological invariants of the loop $\psi$. In this note we will consider only the following characteristic number

$$I(\psi) := \int_E c_1(VTE) c^n,$$

where $VTE$ is the vertical tangent bundle of $E$ and $c \in H^2(E, \mathbb{R})$ is the coupling class of the fibration $E \to S^2$ [2, 3]. $I(\psi)$ depends only on the homotopy class of the loop $\psi$. Moreover the map

$$I : \psi \in \pi_1(\Ham(N)) \to I(\psi) \in \mathbb{R}$$

is an $\mathbb{R}$-valued group homomorphism [3].

Let us suppose that $\Delta$ is a Delzant polytope. We shall denote by $(M_{\Delta}, \omega_{\Delta}, \mu_{\Delta})$ the toric manifold determined by $\Delta$ ($\mu_{\Delta} : M \to t^*$ being the corresponding moment map). Given $\mathbf{b}$, an element in the integer lattice of $t$, we shall write $\psi_{\mathbf{b}}$ for the loop of Hamiltonian diffeomorphisms of $(M_{\Delta}, \omega_{\Delta})$ defined by $\mathbf{b}$. The bundle with fibre $M_{\Delta}$ determined by $\psi_{\mathbf{b}}$ will be denoted $E_{\Delta, \mathbf{b}}$, and we will let $I(\Delta; \mathbf{b})$ for the characteristic number $I(\psi_{\mathbf{b}})$. When we consider only polytopes in the chamber of a given polytope, we will write $I(k; \mathbf{b})$ instead of $I(\Delta(k); \mathbf{b})$ for $k$ in this chamber.

In Section 2 we study the characteristic number $I(k; \mathbf{b})$, when $(\Delta, \mathbf{b})$ is a linear pair and $k$ varies in the chamber of $\Delta$, and we prove that $I(k; \mathbf{b})$ is a homogeneous polynomial of ten $k_j$ (see Proposition 11).

In Section 3 we consider the case when the polytope $\Delta$ is a $\Delta_p$ bundle over $\Delta_1$ with $p > 1$ [5]. Then $M_{\Delta}$ is a $2(p + 1)$-dimensional manifold diffeomorphic to the total space of the fibre bundle $\mathbb{P}(L_1 \oplus \cdots \oplus L_p \oplus \mathbb{C}) \to \mathbb{C}P^1$, where each $L_j$ is a holomorphic line bundle over $\mathbb{C}P^1$. Given $\mathbf{b} \in \mathbb{Z}^{p+1}$, we prove that $I(k; \mathbf{b})$ is the product of two factors, $\mathcal{K}$ and $\mathcal{Z}(\mathbf{b})$, and that the first one is independent of $\mathbf{b}$ and the second one is independent of $k \in \mathcal{C}_{\Delta}$ (Theorem 10). We also prove the equivalence between the vanishing of $\mathcal{Z}(\mathbf{b})$ and the fact that $(\Delta, \mathbf{b})$ is a mass linear pair (Theorem 11). As a consequence we deduce that a necessary and sufficient condition for the vanishing of $I(k; \mathbf{b})$ on $\mathcal{C}_{\Delta}$ is that $(\Delta, \mathbf{b})$ be a mass linear pair (see Theorem 12). From this theorem we will deduce that $\psi_{\mathbf{b}}$ generates an infinite cyclic subgroup in $\pi_1(\Ham(M_{\Delta}))$, if the pair $(\Delta, \mathbf{b})$ is not mass linear (Proposition 13).

The case when the polytope $\Delta$ is the one associated with a Hirzebruch surface is considered in Subsection 1.1. Using calculations carried out in [7], we will prove the equivalence between the vanishing of $I(\Delta; \mathbf{b})$ and the fact that $(\Delta, \mathbf{b})$ is a mass linear pair (Theorem 14). In a Remark we give a general proof, based in general properties of the characteristic number $I$, of the implication: $(\Delta, \mathbf{b})$ is a mass linear pair $\implies I(k; \mathbf{b})$ vanishes on the chamber of $\Delta$. The arguments developed in this proof are applicable to other polytopes; for example to the polytope associated with the one point blow up of $\mathbb{C}P^n$ (see Proposition 20).

In Subsection 1.2 we consider the polytope $\Delta$ associated to the manifold one point blow up of $\mathbb{C}P^n$. We also prove the equivalence: $I(k; \mathbf{b}) = 0$ for all $k \in \mathcal{C}_{\Delta} \iff (\Delta, \mathbf{b})$ is a mass linear pair (see Theorem 22). As a consequence we deduce a simple sufficient condition for $\psi_{\mathbf{b}}$ to generate an infinite cyclic subgroup in $\pi_1(\Ham(M_{\Delta}))$ (Proposition 23).

In summary, we prove the equivalence between the vanishing of $I(k; \mathbf{b})$ for all $k \in \mathcal{C}_{\Delta}$ and the property of $(\Delta, \mathbf{b})$ being a mass linear pair, in the following cases: When $\Delta$ is a $\Delta_p$ bundle over $\Delta_1$, when $\Delta$ is the trapezoid associated to a Hirzebruch
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In the proof of the mentioned results plays a crucial role a formula for the characteristic number $I(\psi_b)$ obtained in [8]. This formula gives $I(\psi_b)$ in terms of the integrals, on the facets of the polytope, of the normalized Hamiltonian corresponding to the loop $\psi_b$ (see (2.1)).

Let $(\Delta, b)$ be a pair consisting of a Delzant polytope in $\mathbb{C}P^n$ and an element in the integer lattice of $t$. In view the above results one is tempted to conjecture the equivalence between the following statements

a) $I(k; b) = 0$ for all $k \in C_{\Delta}$.

b) $(\Delta, b)$ is a mass linear pair.

We think that a possible proof of this conjecture using formula (2.1) will probably involve general properties, valid for all Delzant polytopes, about $Cm(\Delta)$ and the mass center of the facets $F_j$.

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2. A CHARACTERISTIC NUMBER

Let us suppose that the polytope $\Delta$ defined in (1.1) is a Delzant polytope in $t^*$. If $b$ is in the integer lattice of $t$, an expression for the value of $I(\psi_b)$ in terms of integrals of the Hamiltonian function has been obtained in Section 4 of [8]

\begin{equation}
I(\Delta; b) := I(\psi_b) = -n \sum_{j=1}^{m} \int_{D_j} (\omega_{\Delta})^{n-1}, \tag{2.1}
\end{equation}

where $D_j := \mu_{\Delta}^{-1}(F_j)$ is oriented by the restriction of $\omega_{\Delta}$, and $f$ being the normalized Hamiltonian of the corresponding circle action; that is,

$$f = \langle \mu_{\Delta}, b \rangle + \text{constant} \quad \text{and} \quad \int_{M_{\Delta}} f (\omega_{\Delta})^n = 0.$$

That is,

\begin{equation}
I(\Delta; b) = n \sum_{j=1}^{m} \left( \langle Cm(\Delta), b \rangle \int_{D_j} (\omega_{\Delta})^{n-1} - \int_{D_j} \langle \mu_{\Delta}, b \rangle (\omega_{\Delta})^{n-1} \right), \tag{2.2}
\end{equation}

where

\begin{equation}
\langle Cm(\Delta), b \rangle = \frac{\int_{M_{\Delta}} \langle \mu_{\Delta}, b \rangle (\omega_{\Delta})^n}{\int_{M_{\Delta}} (\omega_{\Delta})^n}. \tag{2.3}
\end{equation}

If $\Delta = \Delta(n, k)$ we consider the polytope $\Delta' = \Delta(n, k')$ obtained from $\Delta$ by the translation defined by a vector $a$ of $t^*$. As we said, we write $I(k; b)$ and $I(k'; b)$ for the corresponding characteristic numbers. According to the construction of the respective toric manifolds (see [1]),

$$M_{\Delta'} = M_{\Delta}, \quad \omega_{\Delta'} = \omega_{\Delta}, \quad \mu_{\Delta'} = \mu_{\Delta} + a.$$

But the normalized Hamiltonians $f$ and $f'$ corresponding to the action of $b$ on $M_{\Delta}$ and $M_{\Delta'}$ are equal. Thus it follows from (2.1) that $I(k; b) = I(k'; b)$. More precisely, we have the evident proposition

**Proposition 1.** If $a$ is an arbitrary vector of $t^*$, then $I(k; b) = I(k'; b)$, for $k' = k_i + \langle a, n_i \rangle$, $i = 1, \ldots, m$. 
Following [1] we recall some points of the construction of \((M_\Delta, \omega_\Delta, \mu_\Delta)\) from the polytope \(\Delta\) defined by (1.1), in order to study the value of the integrals that appear in (2.2) and (2.3). Given \(\Delta\), we put \(r := m - n\) and \(\tilde{T} := (S^1)^r\). The \(n_i\) determine weights \(w_j \in \tilde{t}^\ast\), \(j = 1, \ldots, m\) for a \(\tilde{T}\)-action on \(C^m\). Then moment map for this action is

\[
J : z \in C^m \mapsto J(z) = \pi \sum_{j=1}^{m} |z_j|^2 w_j \in \tilde{t}^\ast.
\]

The \(k_i\) define a regular value \(\sigma\) for \(J\), and the manifold \(M\) is the following orbit space

\[
(2.4) \quad M_\Delta = \{ z \in C^m : \pi \sum_{j=1}^{m} |z_j|^2 w_j = \sigma \}/\tilde{T},
\]

where the relation defined by \(\tilde{T}\) is

\[
(2.5) \quad (z_j) \simeq (z'_j) \text{ iff there is } y \in \tilde{t} \text{ such that } z'_j = z_j e^{2\pi i (w_j, y)} \text{ for } j = 1, \ldots, m.
\]

Identifying \(\tilde{t}^\ast\) with \(R^r\), \(\sigma = (\sigma_1, \ldots, \sigma_r)\) and each \(\sigma_a\) is a linear combination of the \(k_j\)'s.

After a possible change in numeration of the facets, we can assume that \(F_1, \ldots, F_n\) intersect at a vertex of \(\Delta\). If we write \(z_j = \rho_j e^{i\theta_j}\), then the symplectic form can be written on \(\{ [z] \in M : z_i \neq 0, \forall i \}\)

\[
(2.6) \quad \omega_\Delta = (1/2) \sum_{i=1}^{n} d\rho_i^2 \wedge d\varphi_i,
\]

with \(\varphi_i\) an angular variables, linear combination of the \(\theta_j\)'s.

The action of \(T = (S^1)^n\) on \(M\)

\[
(\alpha_1, \ldots, \alpha_n)[z_1, \ldots, z_m] := [\alpha_1 z_1, \ldots, \alpha_n z_n, z_{n+1}, \ldots, z_m]
\]

gives \(M\) a structure of toric manifold. Identifying \(t^\ast\) with \(R^n\), the moment map \(\mu_\Delta : M_\Delta \rightarrow t = R^n\) is defined by

\[
\mu_\Delta([z]) = \pi(\rho_1^2, \ldots, \rho_n^2) + (d_1, \ldots, d_n),
\]

where the constants \(d_i\) are linear combinations of the \(k_j\)'s and

\[
(2.7) \quad \text{im } \mu_\Delta = \Delta.
\]

By Proposition [1] we can assume that all \(d_j\) are zero in the determination of \(I(k; b)\).

We write \(x_i := \pi \rho_i^2\), then

\[
\int_{M_\Delta} (\omega_\Delta)^n = n! \int_{\Delta} dx_1 \ldots dx_n, \quad \int_{M_\Delta} \langle \mu_\Delta, b \rangle (\omega_\Delta)^n = n! \int_{\Delta} \sum_{i=1}^{n} b_i x_i dx_1 \ldots dx_n.
\]

The following Lemma is useful to evaluate some integrals which will appear henceforth.

**Lemma 2.** If

\[
S_n(\tau) := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \left| \sum_{i=1}^{n} x_i \leq \tau, \quad 0 \leq x_j, \forall j \right. \right\},
\]
Thus, in the particular case that $\Delta = S$ then
\[
\text{degree } n (2.8) \text{ hence the coordinates of any vertex of } \Delta \text{ are linear combinations of the vertices are the solutions to subsets of } \Delta \text{ such that:}
\]
\[
\langle \alpha \rangle \text{ in an hyperplane } R \text{ of Euclidean motions in } \mathbb{R} \kappa \text{ is a linear combination of the form (2.9) with } \kappa \text{ linear combination of the } k_j.
\]

A hyperplane in $\mathbb{R}^n$ through a vertex $(x_1^0, \ldots, x_n^0)$ of $\Delta$ is given by an equation of the form
\[
\langle x, n \rangle = \langle x^0, n \rangle = : \kappa.
\]

So $\kappa$ is a linear combination of the $k_j$.

By drawing hyperplanes through vertices of $\Delta$ we can obtain a family $\{ \beta S \}$ of subsets of $\Delta$ such that:

a) Each $\beta S$ is the transformed of a simplex $S_\alpha(b, \tau)$ by an element of the group of Euclidean motions in $\mathbb{R}^n$.

b) For $\alpha \neq \beta$, $\alpha S \cap \beta S$ is a subset of the border of $\alpha S$.

c) $\bigcup_{\beta} \beta S = \Delta$.

So, by construction, each facet of $\beta S$ is of the form (2.9) with $\kappa$ linear combination of the $k_j$.

On the other hand the hyperplane $\pi$, $\langle x, n \rangle = \kappa$, is transformed by an element of $SO(n)$ in an hyperplane $\langle x, n' \rangle = \kappa$. If $T$ is a translation in $\mathbb{R}^n$ which applies $S_\alpha(b, \tau)$ onto $\beta S$, then this transformation maps $(0, \ldots, 0)$ in a vertex $a = (a_1, \ldots, a_n)$ of $\beta S$. So the translation $T$ transforms $\pi$ in $\langle x, n \rangle = \kappa + \langle a, n \rangle = : \kappa'$.

As the $a_i$ are linear combinations of the $k_j$, so is $\kappa'$. Hence any element of the group of Euclidean motions in $\mathbb{R}^n$ which maps $S_\alpha(b, \tau)$ onto $\beta S$ transforms the hyperplane $\pi$ through a vertex of $\Delta$ in an hyperplane $\langle x, n' \rangle = \kappa'$ with $\kappa'$ a linear combination of the $k_j$. In particular, $\tau$ is a linear combination of the $k_j$, and by (2.8)

\[
\int_{S_\alpha(b, \tau)} dx_1 \ldots dx_n = \int_{S_\alpha(b)} dx_1 \ldots dx_n
\]
is a monomial of degree $n$ of a linear combination of the $k_j$. Thus,

$$\int_M (\omega_\Delta)^n = \sum_{\beta} \int_\beta d\xi_1 \ldots d\xi_n,$$

is a homogeneous polynomial of degree $n$ of the $k_j$. 

Similarly

$$\int_{M_\Delta} \langle \mu_\Delta, b \rangle (\omega_\Delta)^n$$

is a homogeneous polynomial of degree $n + 1$ of the $k_j$. Analogous results hold for $\int_D (\omega_\Delta)^{n-1}$ and $\int_D \langle \mu_\Delta, b \rangle (\omega_\Delta)^{n-1}$.

It follows from (2.2), (2.3) together with the preceding argument the following proposition

**Proposition 3.** Given a Delzant polytope $\Delta$, if $b$ belongs to the integer lattice of $t$, then $I(k; b)$ is a rational function of the $k_i$ for $k \in C_\Delta$.

Analogously we have

**Proposition 4.** If $(\Delta, b)$ is mass linear pair, then $I(k; b)$ is a homogeneous polynomial in the $k_i$ of degree $n$, when $k \in C_\Delta$.

**Proposition 5.** If for the polytope $\Delta$ defined in (1.1) $n_i = -n_i$, with $i \neq a$, and $(\Delta, b)$ is a mass linear pair, then $I(k; b)$ is a polynomial divisible by $k_a - k_i$.

**Proof.** Given $k \in C_\Delta$, maintaining $k_a$ fixed we vary $k$ so that $k \in C_\Delta$ and $k_i \to k_a$.

In the limit $\Delta$ collapses in the facet $F_a$. As $(\Delta, b)$ is mass linear

$$\lim_{k_i \to k_a} \langle \text{Cm}(\Delta), b \rangle = \langle \text{Cm}(F_a), b \rangle,$$

where

$$\langle \text{Cm}(F_a), b \rangle = \frac{\int_{D_a} \langle \mu_\Delta, b \rangle (\omega_\Delta)^{n-1}}{\int_{D_a} (\omega_\Delta)^{n-1}}.$$

We write (2.2) as

$$I(k; b) = n \sum_{j=1}^m \mathcal{E}_j,$$

with

$$\mathcal{E}_j = \langle \text{Cm}(\Delta), b \rangle \int_{D_j} (\omega_\Delta)^{n-1} - \int_{D_j} \langle \mu_\Delta, b \rangle (\omega_\Delta)^{n-1}.$$

In this limit process the facet $F_a$ remains unchanged, so by (2.10) and (2.11)

$$\lim_{k_i \to k_a} \mathcal{E}_a = \langle \text{Cm}(F_a), b \rangle \int_{D_a} (\omega_\Delta)^{n-1} - \int_{D_a} \langle \mu_\Delta, b \rangle (\omega_\Delta)^{n-1} = 0.$$

On the other hand, the facets $F_j$, with $j \neq a$ give rise, in the limit $k_i \to k_a$, to a subdivision of $F_a$ (in Remark after Theorem 14 is detailed this subdivision in a particular case). Hence

$$\lim_{k_i \to k_a} \sum_{j \neq a} \int_{D_j} (\omega_\Delta)^{n-1} = \int_{D_a} (\omega_\Delta)^{n-1},$$

$$\lim_{k_i \to k_a} \sum_{j \neq a} \int_{D_j} \langle \mu_\Delta, b \rangle (\omega_\Delta)^{n-1} = \int_{D_a} \langle \mu_\Delta, b \rangle (\omega_\Delta)^{n-1}.$$
3. $\Delta_p$ Bundle over $\Delta_1$

Given the integer $p > 1$, following [5] we consider the following vectors in $\mathbb{R}^{p+1}$

\[ n_i = -e_i, \quad i = 1, \ldots, p, \quad n_{p+1} = \sum_{i=1}^{p} e_i, \quad n_{p+2} = -e_{p+1}, \quad n_{p+3} = e_{p+1} - \sum_{i=1}^{p} a_i e_i, \]

where $e_1, \ldots, e_{p+1}$ is the standard basis of $\mathbb{R}^{p+1}$ and $a_i \in \mathbb{Z}$. We write

\[ a := (a_1, \ldots, a_p) \in \mathbb{Z}^p, \quad A := \sum_{i=1}^{p} a_i. \]

Let $\lambda, \tau$ be real positive numbers with $\lambda + a_i > 0$, for $i = 1, \ldots, p$. We will consider the polytope $\Delta$ in $(\mathbb{R}^{p+1})^*$ defined by the above conormals $n_i$ and the following $k_i$

\[ k_1 = \cdots = k_p = k_{p+2} = 0, \quad k_{p+1} = \tau, \quad k_{p+3} = \lambda. \]

The polytope $\Delta$ is a $\Delta_p$ bundle on $\Delta_1$ (see [5]). When $p = 2$, $\Delta$ is the prism whose base is the triangle of vertices $(0, 0, 0)$, $(\tau, 0, 0)$ and $(0, \tau, 0)$ and whose ceiling is the triangle determined by $(0, 0, \lambda)$, $(\tau, 0, \lambda + a_1 \tau)$ and $(0, \tau, \lambda + a_2 \tau)$

We assume that the above polytope $\Delta$ is a Delzant polytope. The manifold (2.4) is in this case

\[ M_\Delta = \{ z \in \mathbb{C}^{p+3} : \sum_{i=1}^{p+3} |z_i|^2 = \tau/\pi, \quad -\sum_{j=1}^{p} a_j |z_j|^2 + |z_{p+2}|^2 + |z_{p+3}|^2 = \lambda/\pi \}/\sim, \]

where $(z_j) \simeq (z'_j)$ iff there are $\alpha, \beta \in U(1)$ such that

\[ z'_j = \alpha \beta^{-a_j} z_j, \quad j = 1, \ldots, p; \quad z'_{p+1} = \alpha z_{p+1}; \quad z'_{k} = \beta z_k, \quad k = p + 2, p + 3. \]

The symplectic form (2.6) is

\[ \omega_\Delta = (1/2)(\sigma_1 + \cdots + \sigma_p + \sigma_{p+2}), \]

where $\sigma_k = d\rho_k^2 \wedge d\varphi_k$.

And the moment map

\[ \mu_\Delta([z]) = \pi(\rho_1^2, \ldots, \rho_p^2, \rho_{p+2}^2). \]

Thus $M_\Delta$ is the total space of the fibre bundle $\mathbb{P}(L_1 \oplus \cdots \oplus L_p \oplus \mathbb{C}) \to \mathbb{C}P^1$, where $L_j$ is the holomorphic line bundle over $\mathbb{C}P^1$ with Chern number $a_j$.

Given $\hat{b} = (b_1, \ldots, b_p, 0) \in \mathbb{Z}^{p+1}$ we write

\[ B := \sum_{j=1}^{p} b_j, \quad a \cdot \hat{b} := \sum_{j=1}^{p} a_j b_j. \]
Proposition 6. Let \( \hat{b} = (b_1, \ldots, b_p, 0) \) be an element of \( \mathbb{Z}^{p+1} \), then

\[
\langle C_m(\Delta), \hat{b} \rangle = \frac{\tau}{p+2} \frac{\lambda(p+2)B + \tau(AB + a \cdot \hat{b})}{\lambda(p+1) + \tau A}.
\]

Proof. By Lemma 2

\[
\int_{M_\Delta} (\omega_\Delta)^{p+1} = (p+1)! \int_{\mathcal{S}_{p}(\tau)} (\lambda + \sum_{j=1}^{p} a_j x_j) = (p+1)! \left( \frac{\lambda \tau^p}{p!} + \frac{\tau^{p+1} A}{(p+1)!} \right).
\]

Similarly, for \( k = 1, \ldots, p \)

\[
\int_{M_\Delta} x_k (\omega_\Delta)^{p+1} = (p+1)! \left( \frac{\lambda \tau^p}{(p+1)!} + \frac{\tau^{p+2}}{(p+2)!} \sum_{j \neq k} a_j + \frac{2 \tau^{p+2} a_k}{(p+2)!} \right).
\]

So the \( k \)-th coordinate of \( C_m(\Delta), \bar{x}_k \), is

\[
\bar{x}_k = \frac{\tau}{p+2} \frac{\lambda(p+2) + \tau(A + a_k)}{\lambda(p+1) + \tau A}.
\]

So

\[
\langle C_m(\Delta), \hat{b} \rangle = \frac{\tau}{p+2} \frac{\lambda(p+2)B + \tau(AB + a \cdot \hat{b})}{\lambda(p+1) + \tau A}.
\]

\[\square\]

Theorem 7. Let \( \hat{b} = (b_1, \ldots, b_p, 0) \) be an element of \( \mathbb{Z}^{p+1} \). For all \( \tau > 0 \) and all \( \lambda \) with \( \lambda + a_i > 0 \),

\[
I(\lambda, \tau; \hat{b}) = K(\lambda, \tau) \hat{Z},
\]

where

\[
K(\lambda, \tau) = \frac{\tau^{p+1}}{p+2} \left( \frac{\tau}{\lambda(p+1) + \tau A} - 1 \right) \text{ and } \hat{Z} = 2((p+1)a \cdot \hat{b} - AB).
\]

Proof. We write

\[
\frac{1}{p+1} I(\lambda, \tau; \hat{b}) = \sum_{j=1}^{p+3} \langle (C_m(\Delta), \hat{b}) \Phi_j - \Phi_j' \rangle,
\]

where

\[
\Phi_j := \int_{z_j=0} (\omega_\Delta)^p, \quad \Phi_j' := \int_{z_j=0} (\mu_\Delta, b)(\omega_\Delta)^p.
\]

To calculate the values of the \( \Phi_i \), we will distinguish three cases, according to the value of \( i \):

a) \( i = 1, \ldots, p \)

b) \( i = p + 1 \)

c) \( i = p + 2, p + 3 \).

We will also respect this classification in the calculation of the \( \Phi_j' \).

a) We calculate the value of \( \Phi_1 \). On \( z_1 = 0 \)

\[
(\omega_\Delta)^p = \frac{p!}{2^p} (\sigma_2 \wedge \cdots \wedge \sigma_p \wedge \sigma_{p+2}).
\]

If we put \( x_i = \pi \rho_i^2 \), then

\[
\frac{1}{p!} \Phi_1 = \int_0^\tau dx_2 \int_0^{\tau-x_2} dx_3 \cdots \int_0^{\tau-\sum_{j=2}^{p-1} x_j} dx_p \int_0^{\lambda+\sum_{j=2}^{p} a_j x_j} dx_{p+2}.
\]
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It follows from Lemma 2
\[
\frac{1}{p!} \Phi_1 = \int_{S_{p-1}(\tau)} (\lambda + \sum_{j=2}^{p} a_j x_j) = \frac{\lambda \tau^{p-1}}{(p-1)!} + \frac{\tau^p}{p!} \sum_{j=2}^{p} a_j.
\]

For \( k = 1, \ldots, p \), a similar calculation gives
\[
\frac{1}{p!} \Phi_k = \int_{S_{p-1}(\tau)} \left( \lambda + \sum_{j=1}^{p-1} a_{jp} x_j + a_p \tau \right) = \frac{\lambda \tau^{p-1}}{(p-1)!} + \frac{\tau^p}{p!} \sum_{j=2}^{p} a_j.
\]

b) Next we consider \( \Phi_{p+1} \). Now \( x_{p+1} = 0 \). So \( x_p = \tau - \sum_{j=1}^{p-1} x_j \) and
\[
- \sum_{j=1}^{p-1} a_{jp} x_j - a_p \tau + x_{p+2} + x_{p+3} = \lambda, \quad \text{with } a_{jp} := a_j - a_p.
\]

Hence
\[
\frac{1}{p!} \Phi_{p+1} = \int_{S_{p-1}(\tau)} \left( \lambda + \sum_{j=1}^{p-1} a_{jp} x_j + a_p \tau \right) \left( \tau - \sum_{i=1}^{p} b_i x_i \right) = \frac{\lambda \tau^p}{(p-1)!} + \frac{\tau^p}{p!} A.
\]

\( \Phi_{p+2} = \Phi_{p+3} = \tau^p \).

Thus, it follows from (3.9) and (3.8)
\[
\Phi_{p+3} = (2 + pA) \tau^p + (p(p+1)) \lambda \tau^{p-1}
\]

Next we determine the values of the \( \Phi'_i \).

a') On \( z_k = 0 \), with \( k = 1, \ldots, p \), \( \langle \mu \Delta [z], \hat{b} \rangle = \sum_{i \neq k} b_i x_i \). Then
\[
\frac{1}{p!} \Phi'_k = \int_{S_{p-1}(\tau)} \left( \lambda + \sum_{j=1}^{p} a_j x_j \right) \sum_{i \neq k} b_i x_i.
\]

By Lemma 2
\[
\frac{1}{p!} \Phi'_k = \left( \sum_{i \neq k} b_i \right) \frac{\lambda \tau^p}{p!} + 2 \left( \sum_{i \neq k} a_i b_i \right) \frac{\tau^{p+1}}{(p+1)!} + \left( \sum_{i \neq j, j \neq k} a_i b_j \right) \frac{\tau^{p+1}}{(p+1)!}
\]

b') Using the notation introduced in b),
\[
\frac{1}{p!} \Phi'_{p+1} = \int_{S_{p-1}(\tau)} \left( \lambda + \sum_{j=1}^{p-1} a_{jp} x_j + a_p \tau \right) \left( \sum_{i=1}^{p-1} b_i x_i + b_p \tau \right),
\]

with \( b_{ip} := b_i - b_p \). Lemma 2 and a straightforward calculation give
\[
\Phi'_{p+1} = \lambda \tau^p B + (a \cdot \hat{b} + AB) \frac{\tau^{p+1}}{p+1}
\]

c')
\[
\Phi'_{p+2} = \Phi'_{p+3} = \frac{B \tau^{p+1}}{p+1}.
\]
From this last result together with (3.11) and (3.12) it follows

\[(3.13) \quad \sum_{j=1}^{p+3} \Phi'_j = B\lambda \tau^p + \left((p + 1)a \cdot \hat{b} + (p - 1)AB + 2B\right)\frac{\tau^{p+1}}{p + 1}\]

Taking into account (3.10), Proposition 6, (3.13) and (3.10), by means of an easy but tedious calculation one obtains

\[(3.14) \quad I(\Delta, \hat{b}) = \frac{2((p + 1)a \cdot b - AB)}{(p + 2)(\lambda(p + 1) + \tau A)} ((1 - A)\tau^{p+2} - (p + 1)\lambda\tau^{p+1}).\]

If we define \(\hat{Z}\) and \(\mathcal{K}(\tau, \lambda)\) as in the statement of theorem, then (3.14) can be written \(I(\Delta, \hat{b}) = \mathcal{K}(\tau, \lambda)\hat{Z}\).

\[\square\]

We write \(\hat{b}\) for the element \((0, \ldots, 0, b) \in \mathbb{Z}^{p+1}\).

**Proposition 8.** Given \(\hat{b} = (0, \ldots, 0, b) \in \mathbb{Z}^{p+1}\), then

\[(3.15) \quad \langle \text{Cm}(\Delta), \hat{b} \rangle = \frac{b}{2} \frac{(p + 1)(p + 2)\lambda^2 + 2(p + 2)A\lambda\tau + (a \cdot a + A^2)\tau^2}{(p + 2)((p + 1)\lambda + A\tau)},\]

where \(a \cdot a = \sum a_i^2\).

**Proof.** We need to calculate \(\int_M bx_{p+2}\omega^{p+1}\). By Lemma 2

\[(3.16) \quad \frac{1}{(p + 1)!} \int_M bx_{p+2}\omega^{p+1} = \frac{b}{2} \int_{S_p(\tau)} \left(\lambda + \sum_{j=1}^{p} a_j x_j\right)^2\]

\[= \frac{b}{2}\left(\frac{\lambda^2 \tau^p}{p!} + \frac{2A\lambda\tau^{p+1}}{(p + 1)!} + \frac{(a \cdot a + A^2)\tau^{p+2}}{(p + 2)!}\right).\]

Formula (3.15) is a consequence of (3.1) together with (3.16).

\[\square\]

**Theorem 9.** Let \(\hat{b} = (0, \ldots, 0, b)\) be an element of \(\mathbb{Z}^{p+1}\). For all for all \(\tau > 0\) and all \(\lambda\) with \(\lambda + a_i > 0\),

\[I(\lambda, \tau; \hat{b}) = \mathcal{K}(\lambda, \tau)\hat{Z},\]

where

\[\hat{Z} = b((p + 1)a \cdot a - A^2),\]

and \(\mathcal{K}(\lambda, \tau)\) is defined as in (3.3).

**Proof.** As in the preceding theorem

\[(3.17) \quad \frac{1}{p + 1} I(\lambda, \tau; \hat{b}) = \sum_{j=1}^{p+3} \langle \text{Cm}(\Delta), \hat{b} \rangle \Phi_j - \Phi'_j.\]

The \(\Phi_j\) in (3.17) as the same as in (3.6). But now \(\langle \mu, \hat{b} \rangle = bx_{p+2}\). We will follow the same steps as in the proof of Theorem 7 for calculating the \(\Phi'_j\).

\[a')\] The corresponding \(\Phi'_k = \int_M \langle \mu, \hat{b} \rangle \omega^p\) can be calculated as in Theorem 7. For \(k = 1, \ldots, p\) one has

\[\frac{1}{p!} \Phi'_k = \frac{b}{2}\left(\frac{\lambda^2 \tau^{p-1}}{(p - 1)!} + \frac{2\lambda \tau^p}{p!} \sum_{j \neq k} a_j + 2\frac{\tau^{p+1}}{(p + 1)!} \sum_{j \neq k} a_j^2 + \frac{\tau^{p+1}}{(p + 1)!} \sum_{j \neq i, j \neq k \neq i} a_ia_j\right).\]
b') Now
\[ \frac{1}{p!} \Phi'_p = \frac{b}{2} \left( \frac{\lambda^p - 1}{p!} + \frac{2\lambda A}{p!} + \frac{(a \cdot a + A^2)\tau}{p!} \right) + \frac{2\lambda^p A}{p!} + \frac{(a \cdot a + A^2)\tau}{p!} \]
\[ c') \text{In this case } \Phi'_{p+2} = 0, \quad \frac{1}{p!} \Phi'_{p+3} = b \left( \frac{\lambda^p}{p!} + \frac{A\tau + 1}{p!} \right). \]

If we insert in \( 3.17 \) the values for the \( \Phi_j \) obtained in the proof of Theorem 7, the above values of the \( \Phi'_j \) and \( 3.15 \) we arrive to \( I(\tau, \lambda; b) = K(\tau, \lambda)Z \).

Given \( b = (b_1, \ldots, b_p, b) \in \mathbb{Z}^{p+1} \), we write \( \hat{b} = (b_1, \ldots, b_p, 0) \) and \( \dot{b} := b - \hat{b} \).

We put \( Z(b) := (p + 1)(a \cdot (2\hat{b} + b\hat{a})) - A(2B + bA), \)
that is, \( Z = \hat{Z} + \hat{Z}. \) It follows from \( 2.2 \) that \( I(\tau, \lambda; b) \) is a group homomorphism with respect to the variable \( b \). By Theorem 7 and Theorem 9 one has

**Theorem 10.** If \( b = (b_1, \ldots, b_p, b) \in \mathbb{Z}^{p+1}, \) then
\[ I(\tau, \lambda; b) = K(\tau, \lambda)Z(b), \]
where \( Z(b) \) is given by \( 3.18 \).

This theorem expresses the value of \( I(\tau, \lambda; b) \) as the product of two factors. \( K \) is independent of the Hamiltonian loop, it depends only on \( \lambda, \tau \). On the contrary, the factor \( Z \) is constant on the chamber of the polytope.

Let \( b = \hat{b} + \dot{b} \) be as before, since \( \langle \text{Cm}(\Delta), b \rangle = \langle \text{Cm}(\Delta), \hat{b} \rangle + \langle \text{Cm}(\Delta), \dot{b} \rangle \), by \( 3.3 \) and \( 3.15 \)
\[ \lim_{\lambda \to 0} \langle \text{Cm}(\Delta), b \rangle = \frac{b}{2} \left( \frac{a \cdot a + A^2}{p + 2} \right) + \frac{(a \cdot \hat{b} + AB)}{(p + 2)} \]
\[ \lim_{\tau \to 0} \langle \text{Cm}(\Delta), b \rangle = \frac{b\lambda}{2}. \]

Therefore, \( (\Delta, b) \) is a mass iff
\[ 3.19 \quad \langle \text{Cm}(\Delta), b \rangle = \frac{b\lambda}{2} + \left( \frac{b}{2} \left( \frac{a \cdot a + A^2}{p + 2} \right) + \frac{(a \cdot \hat{b} + AB)}{(p + 2)} \right) \tau. \]

If we insert in the equation \( 3.19 \) the expressions for \( \langle \text{Cm}(\Delta), \hat{b} \rangle \) and \( \langle \text{Cm}(\Delta), \dot{b} \rangle \) given by \( 3.3 \) and \( 3.15 \) we obtain the following equivalent condition for the pair \( (\Delta, b) \) to be mass linear.

\[ 3.20 \quad B(p + 2) + bA(p + 2) = \frac{bA(p + 2)}{2} + \left( \frac{b(a \cdot a + A^2)}{2A} + \frac{b\cdot \hat{b} + AB}{A} \right)(p + 1). \]

That is,
\[ \frac{b(A^2 - (p + 1)a \cdot a)}{2} = (p + 1)a \cdot \hat{b} - AB. \]

Taking into account the definition of \( Z \) given in \( 3.18 \) we can state the following theorem

**Theorem 11.** If \( b \in \mathbb{Z}^{p+1}, \) then \( (\Delta, b) \) is a mass linear pair iff \( Z(b) = 0. \)
A consequence of Theorem 10 and Theorem 11 is the following result.

**Theorem 12.** Assume that the Delzant polytope $\Delta$ is a $\Delta_p$ bundle over $\Delta_1$. Given $b \in \mathbb{Z}^{p+1}$, then the pair $(\Delta, b)$ is mass linear iff the the characteristic number $I(k; b)$ of the Hamiltonian fibration $E_{\Delta(k), b} \to S^2$ is zero for all $k$ in the chamber of $\Delta$.

Given $\lambda$ and $\tau$, the map $b \in \mathbb{Z}^{p+1} \mapsto I(\lambda, \tau; b) \in \mathbb{R}$ is a group homomorphism. By Theorem 10 its kernel is $\mathcal{H} := \{ b \mid Z(b) = 0 \}$.

Taking into account (1.3), if $b \notin \mathcal{H}$ then $1 \neq [\psi_b] \in \pi_1(\text{Ham}(M_\Delta))$. So we have the following Proposition

**Proposition 13.** If $(\Delta, b)$ is not mass linear, then $\psi_b$ generates an infinite cyclic subgroup in $\pi_1(\text{Ham}(M_\Delta))$.

In particular $\hat{b} = (b_1, \ldots, b_p, 0)$ belongs to $\mathcal{H}$ iff

\begin{equation}
\label{eq:3.21}
\sum_{j=1}^{p} b_j( (p+1) a_j - \sum_{i=1}^{p} a_i ) = 0.
\end{equation}

We put $c_j := (p+1) a_j - \sum_{i=1}^{p} a_i$. If $(a_1, \ldots, a_p) \neq 0$, then $\hat{c} := (c_1, \ldots, c_p, 0)$ does not satisfy (3.21). Thus one has the following corollary

**Corollary 14.** If $(a_1, \ldots, a_p) \neq (0, \ldots, 0)$, then $[\psi_c]$ generates an infinite cyclic subgroup of $\pi_1(\text{Ham}(M_\Delta))$.

Analogously

**Corollary 15.** If $(a_1, \ldots, a_p) \neq (0, \ldots, 0)$ and $\hat{b} = (0, \ldots, 0, b) \neq 0$, then $[\psi_b]$ generates an infinite cyclic subgroup of $\pi_1(\text{Ham}(M_\Delta))$.

4. **One point blow up of $\mathbb{C}P^n$.**

4.1. **Hirzebruch surfaces.** Given $k \in \mathbb{Z}_{>0}$ and $\tau, \lambda \in \mathbb{R}_{>0}$ with $\sigma := \tau - k \lambda > 0$, we consider the Hirzebruch surface $M$ determined by these numbers. $M$ is the quotient

$$\{ z \in \mathbb{C}^4 : |z_1|^2 + k|z_2|^2 + |z_4|^2 = \tau / \pi, |z_2|^2 + |z_4|^2 = \lambda / \pi \} / \mathbb{T},$$

where the equivalence defined by $\mathbb{T} = (S^1)^2$ is given by

$$(a, b) \cdot (z_1, z_2, z_3, z_4) = (az_1, a^k b z_2, bz_3, az_4),$$

for $(a, b) \in (S^1)^2$. (The definition of Hirzebruch surface given in [7] can be obtained exchanging $z_1$ for $z_2$ in the above definition.)

The manifold $M$ equipped with the following $(U(1))^2$ action

$$(\xi_1, \xi_2) \cdot [z_j] = [\xi_1 z_1, \xi_2 z_2, z_3, z_4],$$

is a toric manifold. The corresponding moment polytope $\Delta$ is the trapezium in $\mathbb{R}^2$ with vertices $P_1 = (0, 0), P_2 = (0, \lambda), P_3 = (\tau, 0), P_4 = (\sigma, \lambda)$. The mass center of $\Delta$ is

\begin{equation}
\label{eq:4.1}
\text{Cm}(\Delta) = \left( \frac{3\sigma^2 - 3k\tau \lambda + k^2 \lambda^2}{3(2\tau - k\lambda)}, \frac{3\lambda \tau - 2k\lambda^2}{3(2\tau - k\lambda)} \right).
\end{equation}
A characteristic number of bundles determined by mass linear pairs

Given $b = (b_1, b_2) \in \mathbb{Z}^2$, the pair $(\Delta, b)$ is mass linear, iff there exist $A, B, C \in \mathbb{R}$ such that

$$\langle Cm(\Delta), b \rangle = A\tau + B\lambda + C,$$

can be expressed in terms of concepts introduced in [5]. The facets $P_1P_2$ and $P_3P_4$ of $\Delta$ are equivalent (according to Definition 1.11 of [5]). Thus, if $2b_2 = kb_1$, then $b$ is inessential (see Definition 1.13 in [5]).

We denote by $\phi$ the following isotopy of $M$

$$\phi_t[z] = [e^{2\pi it}z_1, z_2, z_3, z_4].$$

$\phi$ is a loop in the Hamiltonian group of $M$. By $\phi'$ we denote the Hamiltonian loop

$$\phi'_t[z] = [z_1, e^{2\pi it}z_2, z_3, z_4].$$

In Theorem 8 of [7] we proved that $I(\phi') = (-2/k)I(\phi)$. If $b = (b_1, b_2) \in \mathbb{Z}^2$, then

$$I(\psi_b) = b_1I(\phi) + b_2I(\phi') = (b_1 - (2/k)b_2)I(\phi).$$

From Proposition 16 one deduces the following theorem

**Theorem 17.** The pair $(\Delta, b)$ is mass linear iff $I(\psi_b) = 0$. (Equivalently $I(\tau, \lambda; b) = 0$ for all $(\tau, \lambda)$ in the chamber of $\Delta$.)

Remark. We will deduce the vanishing of $I(\tau, \lambda; b)$ on $C_\Delta$ when $(\Delta, b)$ is mass linear, by an indirect way; that is, without calculating the integrals of (2.2).

If $(\Delta, b)$ is a mass linear pair, $I(\tau, \lambda; b)$ is a homogeneous polynomial of degree 2 in $\tau, \lambda$, by Proposition 4. That is, (4.2)

$$I(\tau, \lambda; b) = C_1\lambda^2 + C_2\lambda\tau + C_3\tau^2.$$

Fixed $\tau$, if $\lambda \to 0$, then $\Delta$ converts into the segment $[0, \tau]$, and $\lim_{\lambda \to 0} Cm(\Delta) = (\tau/2, 0)$. In the limit, the facets of $\Delta$ give rise to the segments

$$F_1 = [0, \tau], F_2 = [0, \sigma], F_3 = [\sigma, \tau],$$

on the axis of abscissas. So $\lim_{\lambda \to 0} I(\tau, \lambda; b)$ is the sum of the contributions of $D_j = \mu^{-1}(F_j)$, $j = 1, 2, 3$ (see (2.2)). As $F_2, F_3$ is a decomposition of $F_1$, then

$$\int_{D_1} f_\omega = \sum_{j=2}^3 \int_{D_j} f_\omega,$$

and $\lim_{\lambda \to 0} I(\tau, \lambda; b) = -4 \int_{D_1} f_\omega$. On the other hand

$$\int_{D_1} f_\omega = \int_0^\tau x b_1 dx - \frac{\tau b_1}{2} = 0.$$

Hence $\lim_{\lambda \to 0} I(\tau, \lambda; b) = 0$, and $C_3$ in (4.2) is zero. This result can also be deduced from Proposition 4.

Now let us assume that $k = 1$. We denote by $\Delta'(\tau)$ the triangle of vertices $(0, 0)$, $(0, \tau)$, and $(\tau, 0)$; the corresponding toric manifold is $\mathbb{CP}^2$. As $\text{Ham}(\mathbb{CP}^2)$ has the homotopy type of $PU(3)$, the group homomorphism $I : \pi_1(\text{Ham}(\mathbb{CP}^2)) \to \mathbb{R}$ is zero.
For $\lambda >> 1$ and $0 < \sigma << 1$, the difference between the polytopes $\Delta(\tau, \lambda)$ and $\Delta'(\tau)$ is a triangle with small sides. Thus, by formula (2.2) for the characteristic class $I$, the expression $|I(\tau, \lambda; b) - I(\Delta'(\tau); b)|$ will be as small as we wish, if $\lambda$ is big enough and $\sigma$ is sufficiently small. As $I(\Delta'(\tau); b) = 0$, we conclude that

$$\lim_{\lambda \to \infty; \sigma \to 0} I(\lambda, \tau; b) = 0.$$ 

If we take $\sigma = a/\lambda$, with $a$ an arbitrary positive number, then

$$0 = \lim_{\lambda \to \infty} ((C_1 + C_2)\lambda^2 + aC_2).$$

That is, $C_i = 0$; and by (4.2), $I(\tau, \lambda; b) = 0$ on the chamber of $\Delta$.

### 4.2. One point blow up of $\mathbb{C}P^n$.

The mass center of the simplex $S_n(\tau)$, defined in Lemma 2, is the point $w$ with $\tau, \lambda \in \mathbb{R}_{> 0}$ and $\sigma := \tau - \lambda > 0$. Thus, by formula (2.2) for the characteristic class $I$, the expression $|I(\tau, \lambda; b)|$ will be as small as we wish, if $\lambda$ is big enough and $\sigma$ is sufficiently small. As $I(\Delta'(\tau); b) = 0$, we conclude that

$$\lim_{\lambda \to \infty; \sigma \to 0} I(\lambda, \tau; b) = 0.$$ 

If we take $\sigma = a/\lambda$, with $a$ an arbitrary positive number, then

$$0 = \lim_{\lambda \to \infty} ((C_1 + C_2)\lambda^2 + aC_2).$$

That is, $C_i = 0$; and by (4.2), $I(\tau, \lambda; b) = 0$ on the chamber of $\Delta$.

We have the following proposition

**Proposition 18.** The invariant $I(S_n(\tau); b)$ is zero for all $\tau$ and all $b \in \mathbb{Z}$.

In this subsection $\Delta$ will be

$$\Delta = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i \leq \tau, 0 \leq x_i, x_n \leq \lambda\},$$

where $\tau, \lambda \in \mathbb{R}_{> 0}$ and $\sigma := \tau - \lambda > 0$. That is, $\Delta$ is the polytope obtained truncating the simplex $S_n(\tau)$ by a “horizontal” hyperplane through the point $(0, \ldots, 0, \lambda)$.

As the volume of $S_n(\tau)$ is $\tau^n/n!$, it follows from (4.3)

$$(\tau^n - \sigma^n) \text{Cm}(\Delta) = \frac{\tau^n}{n+1} w - \sigma^n (\frac{\sigma}{n+1} w + \lambda \epsilon_n).$$

That is,

$$\text{Cm}(\Delta) = \frac{1}{\tau^n - \sigma^n} \left( (\frac{\tau^{n+1} - \sigma^{n+1}}{n+1}) w - \lambda \sigma^n \epsilon_n \right).$$

The pair $(\Delta, b = (b_1, \ldots, b_n))$ is mass linear iff there exist $A, B, C \in \mathbb{R}$ such that

$$\sum_{j=1}^{n-1} b_j \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} + b_n \left( \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} - (\tau - \sigma)\sigma^n \right) = (A\tau + B\sigma + C)(\tau^n - \sigma^n),$$

where
for all \( \tau, \sigma \) “admissible”. A straightforward calculation proves the following proposition

**Proposition 19.** The pair \((\Delta, b)\) is mass linear iff

\[
b_n = \frac{1}{n} \sum_{j=1}^{n-1} b_j.
\]

The manifold \(M_\Delta\) associated with \(\Delta\) is the one point blow up of \(\mathbb{CP}^n\). On the other hand, the general arguments showed in the Remark after Theorem 17 allow us to prove following Proposition

**Proposition 20.** Let \(\Delta\) be the polytope \((4.4)\), if \((\Delta, b)\) is a mass linear pair, then

\[
I(\tau, \lambda; b) = 0
\]

on \(\mathbb{C}^{\Delta}\).

**Proof.** By Proposition 11, \(I(\tau, \lambda; b) = \sum_{i=0}^{n} B_i \lambda^{n-i} \tau^i\). From (4.6) one obtains

\[
\lim_{\lambda \to 0} \text{Cm}(\Delta) = \frac{\tau}{n} \hat{w},
\]

with \(\hat{w} = (1, \ldots, 1, 0)\). Hence the contribution of the base \(F := \{x \in \Delta | x_n = 0\}\) to \(\lim_{\lambda \to 0} I(\tau, \lambda; b)\) is proportional to

\[
\left(\frac{n}{n} \sum_{j=1}^{n-1} b_j \int_{S_{n-1}(\tau)} 1 - \int_{S_{n-1}(\tau)} \sum_{i=1}^{n-1} b_i x_i\right) = 0.
\]

On the other hand, the other facets \(F_i\) of \(\Delta\) different from the base \(F\) give rise to a decomposition of \(F\) in the limit \(\lambda \to 0\). Thus, with the notation of (2.1),

\[
\lim_{\lambda \to 0} \sum_{D_i \neq D} \int_{D_i} f \omega^{n-1} = \lim_{\lambda \to 0} \int_{D} f \omega^{n-1},
\]

where \(D_i := \mu^{-1}(F_i)\) and \(D := \mu^{-1}(F)\). By (4.6) this limit vanishes. It follows from (2.2) that \(\lim_{\lambda \to 0} I(\tau, \lambda; b) = 0\). So

\[
I(\tau, \lambda; b) = \sum_{i=0}^{n-1} B_i \lambda^{n-i} \tau^i.
\]

For \(\lambda \gg 1\) and \(0 < \sigma << 1\), the difference between the polytopes \(\Delta(\tau, \lambda)\) and \(S_n(\tau)\) is a polytope with small edges. Thus, \(|I(\tau, \lambda; b) - I(S_n(\tau); b)|\) will be as small as we wish, when \(\lambda\) is big enough and \(\sigma\) is sufficiently small. By Proposition 18 \(I(S_n(\tau); b) = 0\) for all \(\tau\). Hence, \(\lim_{\lambda \to \infty; \sigma \to 0} I(\tau, \lambda; b) = 0\).

We take \(\sigma = \lambda^{-1/n}\); from

\[
0 = \lim_{\lambda \to \infty} \sum_{i=0}^{n-1} B_i \lambda^{n-i} (\lambda + \lambda^{-1/n})^i,
\]

one obtains a homogeneous system of \(n\) linearly independent equations

\[
B_0 + \cdots + B_{n-1} = 0, \quad B_1 + 2B_2 + \cdots + (n-1)B_{n-1} = 0, \quad \ldots, B_{n-1} = 0,
\]

for the \(n\) constants. So \(B_j = 0\), and \(I(\tau, \lambda; b) = 0\) on \(\mathbb{C}_\Delta\).

Next we will prove the reciprocal proposition of the preceding one. We denote by \(F_j\) the facet of \(\Delta\) defined by \(x_j = 0\), for \(j = 1, \ldots, n\); \(F_{n+1}\) will be the “ceiling”
\[ x_n = \lambda \text{ and } F_{n+2} \text{ the facet } x_1 + \cdots + x_n = \tau. \] We write as above

\[ \frac{1}{n} I(\tau, \lambda; b) = \langle Cm(\Delta), b \rangle \sum_{j=1}^{n+2} \Phi_j - \sum_{j=1}^{n+2} \Phi_j', \]

with \( \Phi_j = \int_{F_j} 1, \quad \Phi'_j = \int_{F_j} \sum_i b_ix_i. \) By Lemma 2

\[ \Phi_j = \tau^{n-1} - \sigma^{n-1}; \text{ for } j = 1, \ldots, n-1, n+2. \quad \Phi_n = \tau^{n-1}. \quad \Phi_{n+1} = \sigma^{n-1}. \]

Thus we have

\[ \sum_{j=1}^{n+2} \Phi_j = (n+1)\tau^{n-1} + (1-n)\sigma^{n-1}. \]

If \( b = \hat{b} = (b_1, \ldots, b_{n-1}, 0) \), by (4.5)

\[ \langle Cm(\Delta), \hat{b} \rangle = \frac{1}{n+1} \frac{\tau^{n+1} - \sigma^{n+1}}{\tau^n - \sigma^n} \sum_{j=1}^{n-1} b_j. \]

Similarly,

\[ \sum_{j=1}^{n+2} \Phi'_j = \frac{1}{n} (n\tau^n + (2-n)\sigma^n) \sum_{j=1}^{n-1} b_j. \]

If we put \( \epsilon := \sigma/\tau \), it follows from (4.7), (4.8) and (4.9)

\[ \frac{1}{n!} I(\tau, \lambda; \hat{b}) = -\frac{\tau^n}{n+1} \frac{\epsilon^{n-1}}{(n-2)!} \sum_{j=1}^{n-1} b_j + O(\epsilon^n). \]

Next we determine \( I(\tau, \lambda; \hat{b}) \), when \( \hat{b} = (0, \ldots, 0, b_n) \). Now

\[ \sum_{j=1}^{n+2} \Phi'_j = b_n \tau^n (1 + (1-n)\epsilon^{n-1} + (n-2)\epsilon^n) \]

and

\[ \langle Cm(\Delta), \hat{b} \rangle = b_n \frac{\tau^n}{n+1} \left( 1 - \frac{n\epsilon^n}{n} \right) + O(\epsilon^{n+1}). \]

Hence

\[ \frac{1}{n!} I(\tau, \lambda; \hat{b}) = b_n \frac{n\tau^n}{n+1} \frac{\epsilon^{n-1}}{(n-2)!} + O(\epsilon^n). \]

Now, if \( b = (b_1, \ldots, b_n) \), it follows from (4.10), (4.11)

\[ \frac{1}{n!} I(\tau, \lambda; b) = \left( n b_n - \sum_{j=1}^{n-1} b_j \right) \frac{\tau^n}{n+1} \frac{\epsilon^{n-1}}{(n-2)!} + O(\epsilon^n). \]

From (4.12) together with Proposition 19 we deduce the following proposition

**Proposition 21.** If \( \Delta \) is the polytope (4.4) and \( I(k; b) = 0 \) for all \( k \in C_\Delta \), then \((\Delta, b)\) is a mass linear pair.

Propositions 20 and 21 imply the following theorem

**Theorem 22.** If \( \Delta \) is the polytope (4.4), then \((\Delta, b)\) is a mass linear pair iff \( I(k; b) = 0 \) for all \( k \in C_\Delta \).
It follows from Theorem 22 together with Proposition 19 and the homomorphism (1.3) the following proposition

**Proposition 23.** If \( b = (b_1, \ldots, b_n) \in \mathbb{Z}^n \) and \( \sum_{j=1}^{n-1} b_j \neq nb_n \), then \( \psi_b \) generates an infinite cyclic subgroup in \( \pi_1(\text{Ham}(M_\Delta)) \).

**Remark.** When \( n = 3 \) the toric manifold \( M \) corresponding to \( \Delta \) is
\[
M = \left\{ z \in \mathbb{C}^5 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_5|^2 = \frac{\tau}{\pi}, \quad |z_3|^2 + |z_4|^2 = \frac{\lambda}{\pi} \right\}/T,
\]
where the action of \( T = (U(1))^2 \) is defined by
\[
(a, b)(z_1, z_2, z_3, z_4, z_5) = (az_1, az_2, abz_3, bz_4, az_5),
\]
for \( a, b \in U(1) \).

We consider the following loops in the Hamiltonian group of \( M \)
\[
\psi_t[z] = [z_1e^{2\pi it}, z_2, z_3, z_4, z_5], \quad \psi'[z] = [z_1, z_2e^{2\pi it}, z_3, z_4, z_5],
\]
\[
\tilde{\psi}_t[z] = [z_1, z_2, z_3e^{2\pi it}, z_4, z_5].
\]
In [8] (Remark in Section 4) we gave formulas that relate the characteristic numbers associated with these loops
\[
I(\psi) = I(\psi') = (1/3)I(\tilde{\psi}).
\]
So for \( b = (b_1, b_2, b_3) \in \mathbb{Z}^3 \),
\[
I(\psi_b) = (b_1 + b_2 - 3b_3)I(\psi).
\]
By Proposition 19 the vanishing of \( I(\psi_b) \) in (4.14) is equivalent to the fact that \( (\Delta, b) \) is a mass linear pair. This equivalence is a particular case of Theorem 22.

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