ON CHOW STABILITY FOR ALGEBRAIC CURVES

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ABSTRACT. In the last decades there have been introduced different concepts of stability for projective varieties. In this paper we give a natural and intrinsic criterion of the Chow, and Hilbert, stability for irreducible complete reduced curves $C$, with at most ordinary nodes and cusps as singularities, in a projective space $\mathbb{P}^n$. Namely, if the restriction $T_{\mathbb{P}^n}|_C$ of the tangent bundle of $\mathbb{P}^n$ to $C$ is stable then $C \subset \mathbb{P}^n$ is Chow stable, and hence Hilbert stable. We apply this criterion to describe a smooth open set of the irreducible component $\text{Hilb}_s^{Ch}$ of the Hilbert scheme of $\mathbb{P}^n$ containing the generic smooth Chow-stable curve of genus $g$ and degree $d > g + n - \left\lfloor \frac{g + 1}{n + 1} \right\rfloor$. Moreover, we describe the quotient stack of such curves. Similar results are obtained for the locus of Hilbert stable curves.

1. Introduction

In [33] Mumford introduced the GIT notion of Chow stability (see Definition 2.1) giving projective moduli spaces of projective varieties. However, in general there is no simple way to know when a variety is Chow stable, mainly, because the Hilbert-Mumford criterion has not been successfully simplified or interpreted for varieties of higher dimension. Many authors have turned to other methods and have introduced different concepts of stability for producing moduli of varieties (see e.g. [2], [27], [40]). Some of them were defined with the aim to understand the relation of the algebro-geometric stability and the existence of special metrics. It was R. Berman who, in [7], proved that a Fano manifold admitting a Kähler-Einstein metric is $K$-polystable. The breakthrough result has been achieved recently by Xiuf-Xiong Chen, Simon Donaldson and Song Sun in [41]. They showed that if a Fano manifold is $K$-stable then it admits a Kähler-Einstein metric. For more information in this direction see [41] and [34] and the bibliography therein.

In this paper, for irreducible curves, we prove in Section 2

**Theorem 1.1.** Let $C \subset \mathbb{P}^n$ be a complex irreducible complete reduced curve with at most ordinary nodes and cusps as singularities. If the restriction $T_{\mathbb{P}^n}|_C$ of the tangent bundle of $\mathbb{P}^n$ to $C$ is stable (respectively semistable) then $C \subset \mathbb{P}^n$ is Chow stable (respectively Chow semistable).

Another way of stating Theorem 1.1 for smooth curves, via the Hitchin-Kobayashi correspondence, is:
Let $C \subset \mathbb{P}^n$ be a complex smooth irreducible curve. If $T_{|C}$ admits an Hermitian-
Einstein metric then $C \subset \mathbb{P}^n$ is Chow poly-stable, and Chow stable if $T_{|C}$ is irreducible.

Our theorem provides a sufficient condition for the Chow stability for irreducible curves. The proof of the theorem is not complicated, mainly we use \[33, \text{Theorem 4.12}], Proposition \[2.7] and the relation between the tangent bundle and the syzygy bundle. But our
statement is not in the literature and our main contribution is the interpretation of Chow
(and Hilbert) stability via the stability of the restriction of the tangent bundle of the
projective space to $C$. One may conjecture that Theorem 1.1 holds also for varieties of
higher dimension, that is,

Conjecture 1.2. Let $X \subset \mathbb{P}^n$ be a complex smooth irreducible variety. If the restriction
$T_{|X}$ of the tangent bundle of $\mathbb{P}^n$ to $X$ is $O_X(1)$-stable (respectively semistable) then
$X \subset \mathbb{P}^n$ is Chow stable (respectively Chow semistable).

The theorem is still true if we drop the assumption that the base field is $\mathbb{C}$. The proof
works for any field $K = K$ but we will stay with $\mathbb{C}$ since some of our applications work
only for curves in $\mathbb{C}$.

Our viewpoint also sheds some new light on the singularities of $C \subset \mathbb{P}^n$. Suppose that
$C \subset \mathbb{P}^n$ is a complex irreducible curve and $p \in C$ is a singularity of multiplicity $\mu_p$. In
Proposition 2.4 we prove

$If T_{|C}$ is stable then $\mu_p < \frac{\deg(O_C(1))}{n}$ for any $p \in C$.

It is well known (see \[33, \text{Theorem 4.15}]), that if $C \subset \mathbb{P}^n$ is a smooth irreducible curve
of genus $g \geq 1$ embedded by a complete linear system of degree $d > 2g$ then $C$ is Chow
stable. We use Theorem 1.1 to extend such result to curves with at most ordinary nodes
and cusps as singularities (see Theorem 3.1).

In particular, if $\phi_{K_C} : C \to \mathbb{P}(H^0(C, K_C)^*)$ is the natural morphism induced by the
canonical bundle $K_C$ of $C$ then we have (see Theorem 3.1)

Theorem 1.3. Let $C$ be a complex irreducible complete reduced curve with at most ordinary
nodes and cusps as singularities of genus $g \geq 2$. Then $\phi_{K_C}(C) \subset \mathbb{P}(H^0(C, K_C)^*)$ is
Chow-semistable and Chow-stable if $C$ is non-hyperelliptic.

For non complete linear systems and lower degrees the existence of Chow stable/semistable
curves is established by our next theorem (see Theorem 3.2). Recall that for a general
curve of genus $g > 0$, a sharp lower bound for the existence of generated linear series
$(L, V)$ of type $(d, n + 1)$ is $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$. That is, the Brill-Noether number
$p(g, d, n + 1) := g - (n + 1)(n - d + g)$ is non-negative (see Section 3 for the definition of
a Petri curve).

Theorem 1.4. Let $C \subset \mathbb{P}^n$ be an irreducible smooth curve of genus $g \geq 1$ and degree $d$
embedded by the linear series $(L, V)$ of type $(d, n + 1)$. If $C$ and $(L, V)$ are general and
ρ(g, d, n + 1) ≥ 0 then $C \subset \mathbb{P}^n$ is Chow semistable. Moreover, $C \subset \mathbb{P}^n$ is Chow stable if one of the following conditions

1. $C$ is general of genus $g \geq 2$ and $\gcd(d, n) = 1$;
2. $C$ is a curve of genus $g = 1, d \geq n + 1$ and $\gcd(d, n) = 1$;
3. $C$ is a curve of genus $g = 2, d \geq n + 2$ with $d \neq 2n$;
4. $C$ is a Petri curve of genus $g \geq 3$ and $n \leq 4$;
5. $C$ is a Petri curve of genus $g \geq 3, n \geq 5$ and $g \geq 2(n - 2)$;

is satisfied.

Mainly, Theorems 1.4 and 1.3 summarizes the known results on the stability of the syzygy bundle. The breakthrough result in this direction has been achieved recently in [9], where Butler’s Conjecture (see [14, Conjecture 2]) was proved for line bundles on smooth curves. For singular curves with at most ordinary nodes and cusps as singularities the stability of the syzygy bundle was consider in [10].

Let $\text{Hilb}^{P(t)}_{\mathbb{P}^n}$ be the Hilbert scheme of $\mathbb{P}^n$ with Hilbert polynomial $P(t) = dt + (1 - g)$. The problem of describing the Hilbert scheme parameterizing embedded curves seems very natural and interesting too. In general, the Hilbert schemes $\text{Hilb}^{P(t)}_{\mathbb{P}^n}$ can be quite pathological. The principal significance of Theorem 1.4 is that it allows one to describe an open smooth subscheme of an irreducible component of a suitable Hilbert scheme $\text{Hilb}^{P(t)}_{\mathbb{P}^n}$.

In order to state our results we recall from [11, Chapter XXI] that given a family $p : C \to S$ of smooth curves of genus $g > 1$ there exists a relative linear series $\mathcal{G}^n_d$ over the moduli space $M_g$ of curves of genus $g \geq 2$. We use $\mathcal{G}^n_d$ to give a structure to the set of triplets

$$\mathcal{B}^n_d = \{ z := (C, (L, V), \alpha) \},$$

where

1. $C$ is a Petri curve;
2. $(L, V)$ is a linear series of type $(d, n + 1)$ and the natural morphism
   $$\phi_{L,V} : C \to \mathbb{P}(V^*)$$
   induced by $(L, V)$ is an embedding;
3. $\alpha : \mathbb{P}(V^*) \to \mathbb{P}^n$ is an isomorphism.

Actually, $\mathcal{B}^n_d$ is a $\text{PGL}(n + 1)$–principal bundle over the open irreducible subscheme $\mathcal{P}^n_d \subset \mathcal{G}^n_d$ consisting of pairs $(C, (L, V))$ where $C$ is a Petri curve and $(L, V)$ is a generated linear series of type $(d, n + 1)$ with $\phi_{L,V}$ an embedding.

There is a natural algebraic morphism

$$\Gamma : \mathcal{B}^n_d \to \text{Hilb}^{P(t)}_{\mathbb{P}^n}$$

$$(C, (L, V), \alpha) \mapsto [\alpha(\phi_{L,V}(C)) \subset \mathbb{P}^n].$$
Denote by $\text{Hilb}_{\text{Ch}}^s$ (respectively $\text{Hilb}_{\text{Ch}}^{ss}$) the irreducible component of the Hilbert scheme $\text{Hilb}_P^{P(t)}$ containing the generic Chow stable (respectively Chow semi-stable) curve. Recall that for $d = m(2g - 2)$ with $m > 5$, Mumford uses the $m$-canonical embedding $C \subset \mathbb{P}(H^0(C, K^m_C)^*)$ to construct from $\text{Hilb}_{\text{Ch}}^s$ the moduli space $M_g$. Note that in our case we consider different embedding of the same curve and the embedding can be by non-complete linear systems and of degree $< m(2g - 2)$ with $m > 0$.

We can now formulate our main results (see Theorems 4.4 and 4.5 and Corollary 4.6).

**Theorem 1.5.** Let $g, d$ and $n$ be positive integers with $g \geq 3$. For any $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$, $\text{Hilb}_{\text{Ch}}^{ss} \neq \emptyset$. Moreover, if one of the conditions in Theorem 1.4 is satisfied then

1. $\text{Hilb}_{\text{Ch}}^s \neq \emptyset$.
2. $\Gamma(\mathcal{B}^n_d) \subset \text{Hilb}_{\text{Ch}}^s$.
3. $\dim \text{Hilb}_{\text{Ch}}^s = 3g - 3 + \rho(g, d, n + 1) + n(2g - 2)$.
4. $\text{Hilb}_{\text{Ch}}^s$ is smooth at $\Gamma(z)$ for any $z \in \mathcal{B}^n_d$ and $\mathcal{P}^n_d$.
5. $\dim \text{Hilb}_{\text{Ch}}^s/\text{SL}(n + 1) = \dim \mathcal{P}^n_d = 3g - 3 + \rho(g, d, n + 1)$.
6. $\Gamma : \mathcal{B}^n_d \rightarrow \text{Hilb}_{\text{Ch}}^s$ is an open immersion.

Moreover, the quotient stack $[\text{Hilb}_{\text{Ch}}^s/\text{SL}(n + 1)]$ is a smooth irreducible Deligne-Mumford stack of dimension $3g - 3 + \rho(g, d, n + 1)$.

The theorem gains in interest if we recall that the goal of the so called Hassett-Keel program is to find the minimal model of the moduli space $M_g$ of curves via the successive constructions of modular birational models of $M_g$. For a recent account of the theory we refer the reader to [3] and [24]. From the above theorem we have that for low degrees, up to $d \geq g + n - \left\lfloor \frac{g}{n+1} \right\rfloor$, the Petri curves are in the image of the natural morphism $p : \text{Hilb}_{\text{Ch}}^s/\text{SL}(n + 1) \rightarrow M_g$, and the bound is sharp. It may be possible that a compactification of $p(\text{Hilb}_{\text{Ch}}^s/\text{SL}(n + 1))$ could be easier to describe rather than the loci of $m$-canonically embedded curves, but we will not develop this point here.

In [22] Gieseker introduced the concept of Hilbert stability for projective curves and Morrison in [32, Corollary 3.5 (i)], proved that Chow stability implies Hilbert stability. Therefore, we can reformulate the above results in terms of Hilbert stability (see Section 5).

In particular, for complex irreducible complete reduced curves with at most ordinary nodes and cusps as singularities, from the above results we conclude that

$\phi_{K_C}(C) \subset \mathbb{P}^{g-1}$ is Hilbert semistable, and Hilbert stable if $C$ is non-hyperelliptic.

Moreover, any irreducible curve $C \subset \mathbb{P}^{g-1}$, of degree $2g - 2$, $g \geq 2$ with $T\mathbb{P}^{\text{dim}_C}$ stable, the multiplicity $\mu_p < 2$, for any $p \in C$.

This article is organized as follows. In Section 2, we review some of the standard facts on Chow stability and establish the relation between Chow stability and stability of the restriction of the tangent bundle. In Section 3 we summarize the relevant material on
the stability of $T\mathbb{P}^n_C$ and prove Theorem 1.4. The main results, Theorems 4.3 and 4.5 and Corollary 4.6 are proved in the fourth section. In Section 5 we extend the results of Chow stability to Hilbert stability.

Notation: Given a vector bundle $E$ over $C$ we denote by $d_E$ the degree, by $n_E$ the rank and by $\mu(E) := \frac{d_E}{n_E}$ the slope of $E$. For abbreviation, we write $H^i(E)$ instead of $H^i(C, E)$, whenever it is convenient.

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2. Chow stability

In this section we recall from [33] the definition of Chow stability and linear stability for projective varieties $X \subset \mathbb{P}^n$ and prove Theorem 1.1.

Let $X \subset \mathbb{P}^n$ be a non-degenerate irreducible complex projective variety of dimension $r \geq 1$ and degree $d \geq 2$. The Chow form $F_X$ associated to $X \subset \mathbb{P}^n$ is defined as follows (see [33]).

Consider the locus $Y_X \subset \mathbb{G}(n-r-1, \mathbb{P}^n)$ defined by

$$Y_X := \{ H \in \mathbb{G}(n-r-1, \mathbb{P}^n) : H \cap X \neq \emptyset \}.$$

It is well known that $Y_X$ is an irreducible divisor in $\mathbb{G}(n-r-1, \mathbb{P}^n)$ of degree $d$ (in the Plücker coordinates). Moreover, $Y_X$ is the zero set of a section $F_X \in H^0(\mathbb{G}(n-r-1, \mathbb{P}^n), \mathcal{O}(d))$ and $F_X$ is determined up to multiplicative constants. Therefore, it defines a point

$$[F_X] \in \mathbb{P}(H^0(\mathbb{G}(n-r-1, \mathbb{P}^n), \mathcal{O}(d))^*) .$$

We call $[F_X]$ the Chow form of $X \subset \mathbb{P}^n$. There is a natural action of $SL(n+1)$ on $\mathbb{P}(H^0(\mathbb{G}(n-r-1, \mathbb{P}^n), \mathcal{O}(d))^*)$ and a GIT concept of $SL(n+1)$-stability (semistability and polystability) for the Chow forms. Denote by $\mathcal{U}^s$ (respectively $\mathcal{U}^{ss}$) the set of $SL(n+1)$-stable (respectively $SL(n+1)$-semistable) points in $\mathbb{P}(H^0(\mathbb{G}(n-r-1, \mathbb{P}^n), \mathcal{O}(d))^*)$.

Definition 2.1. A projective irreducible variety $X \subset \mathbb{P}^n$ is Chow stable if the Chow form $[F_X]$ is $SL(n+1)$-stable. Similarly we define Chow semistability and Chow polystability.

A non-degenerate complex projective variety $X \subset \mathbb{P}^n$ defines a point $[X \subset \mathbb{P}^n]$ in a Hilbert scheme $Hilb_{\mathbb{P}^n_C}^{P(t)}$ of $\mathbb{P}^n$, with a suitable Hilbert polynomial $P(t)$. Denote by $Hilb_{\mathbb{P}^n_C}^s$ (respectively $Hilb_{\mathbb{P}^n_C}^{ss}$) the irreducible component of the Hilbert scheme $Hilb_{\mathbb{P}^n_C}^{P(t)}$ containing the generic Chow stable (respectively Chow semistable) variety. Recall that the set of
stable Chow forms is irreducible and open in \( \mathbb{P}(H^0(\mathbb{G}(n-r-1, \mathbb{P}^n), \mathcal{O}(d))^*) \). Our aim is to describe an open set of \( \text{Hilb}_{C_h}^L \) when \( P(t) = dt + 1 - g \).

Let \( X \subset \mathbb{P}^n \) be as above. Let \( (\mathcal{L}, V) \) be the generated linear series on \( X \) defining the embedding. That is, \( \mathcal{L} = \mathcal{O}_X(1) \) is a line bundle on \( X \) and
\[
V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \subseteq H^0(X, \mathcal{L})
\]
is a linear subspace of sections of dimension \( n + 1 \) which generates \( \mathcal{L} \) and induces the closed immersion
\[
\phi_{\mathcal{L}, V} : X \to \mathbb{P}(V^*) = \mathbb{P}^n.
\]
Actually, \( \mathcal{L} = \phi_{\mathcal{L}, V}^*(\mathcal{O}_{\mathbb{P}^n}(1)) \). Note that the embedding need not be a canonical embedding, neither \( (\mathcal{L}, V) \) a complete linear system.

We now introduce the notion of linear stability, following Mumford [33]. Given a linear space \( B \subset \mathbb{P}^n \) of dimension \( n - m - 1 \) denote by \( \pi_B : \mathbb{P}^n - B \to \mathbb{P}^m \) the canonical projection.

**Definition 2.2.** [33, Definition 2.16]. If \( X \subset \mathbb{P}^n \) is embedded by the linear series \( (\mathcal{L}, V) \), we say that \( X \subset \mathbb{P}^n \) (or \( (\mathcal{L}, V) \)) is linearly stable (respectively linearly semistable) if for all linear subspaces \( W \subset V \)
\[
\deg(\pi_{\mathbb{P}(W)}(X)) \cdot \dim W - r > \deg X (n + 1 - r) \quad (\text{respectively } \geq)
\]
where \( \pi_{\mathbb{P}(W)}(X) \) is the image cycle of \( X \) under \( \pi_{\mathbb{P}(W)} \) and \( \dim \pi_{\mathbb{P}(W)}(X) = r \).

**Remark 2.3.** Recall from [33, Proposition 2.5] that if we project from a point \( p \in X \) then
\[
\deg \pi_p(X) = \deg X - \mu_p,
\]
where \( \mu_p \) is the multiplicity of \( p \in X \).

Here is an elementary property of these concepts.

**Proposition 2.4.** Let \( X \subset \mathbb{P}^n \) be a projective variety of dimension \( r \geq 1 \) and \( p \) a point in \( X \). If \( X \subset \mathbb{P}^n \) is linearly stable then \( \mu_p < \frac{\deg X}{n + 1 - r} \). In particular, if \( X \) is a curve then \( \mu_p < \frac{\deg X}{n} \).

**Proof.** Suppose that \( V \) be a vector space of dimension \( n + 1 \) and \( \mathbb{P}^n = \mathbb{P}^n(V) \). Assume that \( X \subset \mathbb{P}^n(V) \) is linearly stable and let \( W \subset V \) be a subspace of dimension \( n \). The following inequality follows from (2.1) and Remark 2.3
\[
\deg X - \mu_p > \frac{\deg X}{n + 1 - r}.
\]
Hence, from (2.2), \( \mu_p < \frac{\deg X}{n + 1 - r} \), which is the desired conclusion. \( \square \)

The relation between Chow stability and linear stability for curves is established by the next theorem, which goes back as far as [33].
Theorem 2.5. [33, Theorem 4.12] Let $C \subset \mathbb{P}^n$ be a curve. If $C \subset \mathbb{P}^n$ is linearly stable (respectively semistable) then $C \subset \mathbb{P}^n$ is Chow stable (respectively Chow semistable).

Remark 2.6. Mumford in [33] uses a complete linear system to define Chow stability and in the proof of [33, Theorem 4.12]. However, the same definitions and proofs work for non complete linear systems. Since our aim is far from giving, and repeating, all the technical details of this assertion we will leave it to the reader to verify that we can use linear series instead of complete linear systems.

In the remainder of this section we assume $X$ to be an irreducible curve $C$.

Let $(L, V)$ be a generated linear series of type $(d, n+1)$ over $C$, that is, the degree of $L$ is $d$ and $\dim V = n+1$. Recall that the pull-back, by $\phi_{L,V}$, of the dual of the Euler sequence tensored by $O_{\mathbb{P}^n}(1)$ induces the following exact sequence

\[
0 \to M_{L,V} \to V \otimes O_C \to L \to 0
\]

of vector bundles over $C$. The kernel $M_{L,V}$ of the evaluation map $V \otimes O_C \to L$ is called the (1st-)syzygy bundle of $(L, V)$ (sometimes also a Lazarsfeld or Lazarsfeld-Mukai bundle). If $V = H^0(C, L)$, we denote $M_{L,V}$ by $M_L$ and $\phi_{L,V}$ by $\phi_L$.

In general, the syzygy bundles can be defined for any variety $X$ and arise in a variety of algebraic and geometric problems. For curves, the stability of $M^*_L$ has been related to problems like: Green’s Conjecture, the Minimal Resolution Conjecture, computing Koszul cohomology groups, theta-divisors, Brill-Noether theory and coherent system theory. In some sense, they have the information on how complicated $X$ sits in $\mathbb{P}^n$.

In the next proposition\(^1\), we relate the stability of the syzygy bundle with linear stability. This result will prove extremely useful in the following sections.

Proposition 2.7. Let $C \subset \mathbb{P}^n$ be a curve of degree $d$ embedded by the linear series $(L, V)$. If the syzygy bundle $M_{L,V}$ is stable (respectively semistable) then

1. $C \subset \mathbb{P}^n$ is linearly stable (respectively semistable);
2. for any $p \in C$, $\mu_p < \frac{d}{n}$ (respectively $\leq$);
3. $C \subset \mathbb{P}^n$ is Chow stable (respectively Chow semistable).

Proof. Recall that the stability of $M_{L,V}$ gives the following inequality

\[
\frac{\deg(F)}{rk F} < \frac{\deg(M_{L,V})}{rk M_{L,V}},
\]

\(^1\)The first part of the proposition was also proved recently in [10].
for every proper subbundle \( F \subset M_{L,V} \). In particular, for those subbundles that fit into the following diagram

\[
\begin{array}{c}
0 \rightarrow M_{L',W} \rightarrow W \otimes O_C \rightarrow L' \rightarrow 0,
\end{array}
\]

(2.5)

\[
\begin{array}{c}
0 \rightarrow M_{L,V} \rightarrow V \otimes O_C \rightarrow L \rightarrow 0,
\end{array}
\]

where \( W \subset V \) a linear subspace that defines the line bundle \( L' \). Applying (2.4) we deduce that

\[
\frac{-\deg(L)}{\dim W - 1} < \frac{-\deg(L)}{n},
\]

(2.6)

Since \( \deg C = \deg(L) \) and \( \deg(p_{F(W)}(C)) = \deg(L') \), (2.6) shows that \( C \) is linearly stable, by (2.1), which completes the proof of Part (1). The proof for semistability is similar.

Part (2) follows from Proposition 2.4. Part (3) follows from (1) and Theorem 2.5.

\[\Box\]

**Remark 2.8.** The implications: Chow stability implies linear stability or linear stability implies stability of \( M_{L,V} \) are not true in general. In [31] was proved the equivalence of linear stability and stability of \( M_{L,V} \) for certain bounds given by the Clifford index of the curve \( C \), and in [39] the second author prove that, for general curves, linear stability implies stability of \( M_{L,V} \) if \( \text{codim} V(n - 1) < g \).

We can now prove Theorem 1.1

**Proof of Theorem 1.1** The proof is based on the following observation. For complex irreducible complete reduced curve with at most ordinary nodes and cusps as singularities the stability of \( TP^n |_C \) is equivalent to the stability of the syzygy bundle \( M_{L,V} \) since

\[
TP^n |_C = M_{L,V}^* \otimes L.
\]

(2.7)

By assumption, \( TP^n |_C \) is stable. We conclude from (2.7), that \( M_{L,V} \) is stable, hence that \( C \subset \mathbb{P}^n \) is linearly stable, and finally that \( C \subset \mathbb{P}^n \) is Chow stable, and this is precisely the assertion of the theorem. Similarly, we obtain Chow semistability.

\[\Box\]

In the next section we summarize the known cases were \( TP^n |_C \) is stable.

**3. Stability of \( TP^n |_C \)**

Let \( C \) be a complex irreducible curves with at most ordinary nodes and cusps as singularities of genus \( g \geq 1 \). In this section we summarize without proofs the relevant material in the stability of the syzygy bundles over \( C \). For the proofs we refer the reader to e.g. [28], [11], [8], [10] and [9].
It is well known ([18, Proposition 3.2]) that for smooth irreducible curves and complete linear system, $M_L$ is stable if $d > 2g$ and semistable if $d = 2g$. Moreover, in [35] Paranjape and Ramanan consider the case $(K_C, H^0(K_C))$, showing in particular that $M_{K_C}$ is always semistable, and is indeed stable if $C$ is non-hyperelliptic. In [10] the above results were proved for curves with at most ordinary nodes and cusps as singularities. Another way of stating these results using Theorem 1.1 is as follows:

**Theorem 3.1.** Let $C \subset \mathbb{P}^n$ be a complex irreducible curves with at most ordinary nodes and cusps as singularities embedded by a complete linear system. Then

(1) $C \subset \mathbb{P}^{d-g}$ is Chow semistable if $d \geq 2g$ and Chow stable if $d > 2g$.

(2) $\phi_{K_C}(C) \subset \mathbb{P}^{g-1}$ is Chow semistable, and Chow stable if $C$ is non-hyperelliptic.

The reference [15] includes an example to show that these nice results for any curve do not extend beyond $d \geq 2g$. For general smooth irreducible curves and lower degrees, $M_L$ is always semistable (see [37]) and conditions for stability are given in [13, Theorem 2] (see also [11] and [8]). The stability of $M_L$ was proved also for

(1) special curves ([12]),

(2) curves computing the Clifford dimension ([28]),

(3) linear systems computing the Clifford index ([15]), and

(4) bounded with the Clifford index ([30] and [31]).

A breakthrough result has been achieved recently in [9], where the semistability of the syzygy bundle for non complete linear series was proved and stability under some conditions for general and Petri curves. Recall that a smooth curve is called Petri if for every line bundle $L$ on $C$, the cup product map

$$\mu : H^0(C, L) \otimes H^0(C, L^* \otimes K_C) \to H^0(C, K_C)$$

is injective.

The next theorem is a reformulation of the results on syzygy bundles in terms of Chow stability for smooth irreducible curves.

**Theorem 3.2.** Let $(L, V)$ be a generated linear series of type $(d, n+1)$ over an irreducible smooth curve $C$ of genus $g \geq 1$. Assume $C$ and $(L, V)$ are general. Then $\phi_{L,V}(C) \subset \mathbb{P}^n$ is Chow semistable. Moreover, $\phi_{L,V}(C) \subset \mathbb{P}^n$ is Chow stable if one of the following conditions

(1) $C$ is general curve of genus $g \geq 2$ and $\gcd(d, n) = 1$;

(2) $C$ is a curve of genus $g = 1$, $d \geq n + 1$ and $\gcd(d, n) = 1$;

(3) $C$ is a curve of genus $g = 2$, $d \geq n + 2$ with $d \neq 2n$;

(4) $C$ is a Petri curve of genus $g \geq 3$ and $n \leq 4$;

(5) $C$ is a Petri curve of genus $g \geq 3$, $n \geq 5$ and $g \geq 2(n - 2)$;

is satisfied.
Proof. From Theorem 1.1 we only need to show that, under the above hypothesis, the syzygy bundle $M_{L,V}$ is semistable.

The semistability of $M_{L,V}$ was proved in [9, Theorem 5.1] for a general $C$ and $(L,V)$. (see also [37] for complete linear systems). Case (1) follows immediately from the equality $\gcd(d,n) = 1$.

For (2), (3), the stability of $M_{L,V}$ follows from [28], if $C$ is a curve of genus $g = 1$, $d \geq n + 1$ and $\gcd(d,n) = 1$, and from [11, Proposition 6.5] and [8, Theorem 8.2], if $C$ is a curve of genus $g = 2$, $d \geq n + 2$ with $d \neq 2n$.

Let $C$ be a Petri curve of genus $g \geq 3$. The stability of $M_{L,V}$ was proved in [8, §7] when $n \leq 4$ and in [9, Theorem 6.1.] when $n \geq 5$ and $g \geq 2(n-2)$.

From what has already been proved (see Theorem 1.1) it follows that $\phi_{L,V}(C) \subset \mathbb{P}^n$ is Chow stable, which is the desired conclusion. \hfill \qed

Remark 3.3. Note that in the above cases we can have $C \subset \mathbb{P}^n$ with $n \neq d - g$.

4. The Hilbert scheme

Let $\text{Hilb}^P_{\mathbb{P}^n}$ be the Hilbert scheme of $\mathbb{P}^n$ with Hilbert polynomial $P(t) = dt + 1 - g$. Recall that $\text{Hilb}^P_{\mathbb{P}^n}$ is the irreducible component of $\text{Hilb}^P_{\mathbb{P}^n}$ containing the points $[C \subset \mathbb{P}^n] \in \text{Hilb}^P_{\mathbb{P}^n}$ such that the Chow form $[F_C]$ is Chow stable and generic. In this section we are interested in describing a smooth open set of $\text{Hilb}^P_{\mathbb{P}^n}$ for $d - g > n - \left\lfloor \frac{g}{n+1} \right\rfloor$. For a treatment of the case $n = d - g$ and $d >> 0$ we refer the reader to [24].

Denote by $M_g$ the moduli space of smooth curves. An important open sublocus $P_g$ of $M_g$ is the one whose points represent Petri curves. The basic varieties of the Brill-Noether theory for moving curves were defined in [1, Chapter XXI]. In particular, for a family $p : C \to S$ of smooth curves of genus $g > 1$, there exists (see [1, Theorem 3.13]) an $S$-scheme $G^g_d$ representing the functor

$$G^g_d : \text{Sch}/S \longrightarrow \text{Sets} \quad T \longmapsto \left\{ \text{equivalence classes of families of } g^*_d \text{'s on } p : C \to S \text{ parametrized by } T \right\}.$$

Moreover, there is a morphism $\Phi : G^g_d \to M_g$ with fibre at $C$ the variety $G^g_d(C)$ of linear series $(L,V)$ of type $(d,n+1)$ on $C$.

Let us denote by $P^g_d \subset G^g_d$ the open subscheme of generated linear series over Petri curves with $\phi_{L,V}$ an embedding. That is, $w = (C, (L,V)) \in P^g_d$ if and only if $C$ is a Petri curve and $(L,V)$ a generated linear series of type $(d,n+1)$ with $\phi_{L,V}$ an embedding.

Remark 4.1. It is well known that for Petri curves $C$, $G^g_d(C)$ is empty if the Brill-Noether number $\rho(g,d,n+1) := g - (n+1)(n-d+g)$ is negative and is irreducible and smooth of dimension $\rho(g,d,n+1)$ if $\rho(g,d,n+1) > 0$. 
The following Proposition may be proved in much the same way as the above results.

**Proposition 4.2.** (Proposition 5.26 and Corollary 5.30) $\mathcal{G}_d^n$ is empty if the Brill-Noether number $\rho(g, d, n + 1) = g - (n + 1)(n - d + g)$ is negative and is irreducible and smooth of dimension $3g - 3 + \rho$ if $\rho(g, d, n + 1) > 0$.

**Corollary 4.3.** If $d - g > n - \left\lfloor \frac{g}{n+1} \right\rfloor$ then $\mathcal{P}_d^n$ is open subscheme of $\mathcal{G}_d^n$. Moreover, $\mathcal{P}_d^n$ is irreducible and smooth of dimension $3g - 3 + \rho$.

**Proof.** The proof is straightforward. □

Fix $\mathbb{P}^n$. For any $w = (C, (L, V)) \in \mathcal{P}_d^n$, let

$$\Lambda_w := \{ \alpha : \mathbb{P}(V^*) \to \mathbb{P}^n : \alpha \text{ is an isomorphism} \}.$$  

This define a $PGL(n + 1)$--principal bundle $\mathcal{B}_d^n \to \mathcal{P}_d^n$ over $\mathcal{P}_d^n$ with fibre $\Lambda_w$ at $w$. Since $\mathcal{G}_d^n$ represents the functor $\mathcal{G}_d^n$, we have a natural injective morphism

$$\Gamma : \mathcal{B}_d^n \to \text{Hilb}_{\mathbb{P}^n}^{P(t)},$$

defined by

$$z := (C, (L, V), \alpha : \mathbb{P}(V^*) \to \mathbb{P}^n) \mapsto [\alpha(\phi_{L, V}(C)) \subset \mathbb{P}^n].$$

The relation between $\mathcal{B}_d^n$ and $\text{Hilb}_{\mathbb{C}^n}$ is established by our next Theorem.

**Theorem 4.4.** If $g, d$ and $n$ are positive integers such that $g \geq 3$ and $d - g \geq n - \left\lfloor \frac{g}{n+1} \right\rfloor$ then $\text{Hilb}_{\mathbb{C}^n}^g \neq \emptyset$. Moreover, $\text{Hilb}_{\mathbb{C}^n}^g \neq \emptyset$ if one of the conditions in Theorem 3.2 is satisfied. In particular, $\Gamma(\mathcal{B}_d^n) \subset \text{Hilb}_{\mathbb{C}^n}$.

**Proof.** Under the conditions stated above, there exist generated linear series $(L, V)$ of type $(d, n + 1)$, with $\phi_{L, V}$ an embedding, over a general or Petri curve $C$. Theorem 3.2 shows that, for general $(L, V)$, $\phi_{L, V}(C) \subset \mathbb{P}^n$ is Chow semistable, hence $\text{Hilb}_{\mathbb{C}^n}^g \neq \emptyset$, which is precisely the first assertion of the theorem. Similarly, $\text{Hilb}_{\mathbb{C}^n}^g \neq \emptyset$ under the conditions of Theorem 3.2 and $\Gamma(\mathcal{B}_d^n) \subset \text{Hilb}_{\mathbb{C}^n}$ as claimed. □

**Theorem 4.5.** Under the conditions in Theorem 4.4

1. $\dim \text{Hilb}_{\mathbb{C}^n}^g = 3g - 3 + \rho(g, d, n + 1) + n(n + 2)$.
2. $\text{Hilb}_{\mathbb{C}^n}^g$ is smooth at $\Gamma(z)$, for any $z \in \mathcal{B}_d^n$ and
3. $\dim \text{Hilb}_{\mathbb{C}^n}^g/\text{SL}(n + 1) = \dim \mathcal{P}_d^n = 3g - 3 + \rho(g, d, n + 1)$.

Moreover, $\Gamma : \mathcal{B}_d^n \to \text{Hilb}_{\mathbb{C}^n}^g$ is an open immersion when $g \geq 3$.

**Proof.** The morphism $\Gamma : \mathcal{B}_d^n \to \text{Hilb}_{\mathbb{C}^n}^g$ is injective and from Corollary 4.3 $\mathcal{B}_d^n$ is smooth irreducible of dimension $3g - 3 + \rho(g, d, n + 1) + n(n + 2)$.

We claim that $\text{Hilb}_{\mathbb{C}^n}^g$ is smooth at $\Gamma(z) = [\alpha(\phi_{L, V}(C)) \subset \mathbb{P}^n]$ of dimension $\chi(N_{C/\mathbb{P}^n})$, where $N_{C/\mathbb{P}^n}$ is the normal bundle of $C$ in $\mathbb{P}^n$. To see this, it suffices to show that
$h^0(C, N_{C/P^n}) = \chi(N_{C/P^n})$ or, equivalently, that $h^1(C, N_{C/P^n}) = 0$. Recall that the normal bundle fits into the following exact sequence
\begin{equation}
0 \to TC \to T\mathbb{P}^n|_C \to N_{C/P^n} \to 0
\end{equation}
of vector bundles over $C$. Let
\begin{equation}
0 \to H^0(TC) \to H^0(T\mathbb{P}^n|_C) \to H^0(N_{C/P^n}) \to H^1(TC) \to H^1(T\mathbb{P}^n|_C) \to H^1(N_{C/P^n}) \to 0
\end{equation}
be the cohomology sequence of (4.1).

Let us first prove that $h^1(C, T\mathbb{P}^n|_C) = 0$. From (2.7) we have $T\mathbb{P}^n|_C = M_{L,V}^* \otimes L$. This gives
\begin{equation}
(T\mathbb{P}^n|_C)^* \otimes K_C = M_{L,V} \otimes L^* \otimes K_C.
\end{equation}
Tensor the exact sequence (2.3) with $L^* \otimes K_C$ to get the exact sequence
\begin{equation}
0 \to M_{L,V} \otimes L^* \otimes K_C \to V \otimes \mathcal{O}_C \otimes L^* \otimes K_C \to K_C \to 0.
\end{equation}
Since $C$ is a Petri curve, (3.1) shows that $h^0(C, M_{L,V} \otimes L^* \otimes K_C) = 0$.

Therefore, from (4.3), $h^0(C, (T\mathbb{P}^n|_C)^* \otimes K_C) = 0$; and by Serre duality, $h^1(C, T\mathbb{P}^n|_C) = 0$. Hence, from this and Riemann-Roch formula we deduce that
\begin{equation}
h^0(C, T\mathbb{P}^n|_C) = d(n + 1) + n(1 - g) = \rho(g, d, n + 1) + n(n + 2).
\end{equation}

Let us now compute the cohomology of the normal bundle. From what has already been proved and the sequence (4.2) we deduce that $H^1(C, N_{C/P^n}) = 0$, and hence $\chi(N_{C/P^n}) = h^0(C, N_{C/P^n})$, as claimed.

As $g > 2$, we have $H^0(C, TC) = 0$ and $h^1(C, TC) = 3g - 3$. We conclude from (4.2) and (4.5) that
\begin{equation}
h^0(C, N_{C/P^n}) = 3g - 3 + \rho(g, d, n + 1) + n(n + 2),
\end{equation}
hence that
\begin{equation}
\dim \text{Hilb}_{c^h} = h^0(C, N_{C/P^n}) = 3g - 3 + \rho(g, d, n + 1) + n(n + 2) = \dim \mathcal{B}_d^3
\end{equation}
and finally that $\text{Hilb}_{c^h}$ is smooth at $\Gamma(z)$. This clearly forces
\begin{equation}
\dim \text{Hilb}_{c^h}/SL(n + 1) = 3g - 3 + \rho(g, d, n + 1).
\end{equation}
From [23 Corollaire (4.4.9)] we conclude that $\Gamma : \mathcal{B}_d^3 \to \text{Hilb}_{c^h}$ is an open immersion, which proves the theorem.

\begin{corollary}
The quotient stack $[\text{Hilb}_{c^h}/SL(n + 1)]$ is a smooth irreducible Deligne-Mumford stack of dimension $3g - 3 + \rho(g, d, n + 1)$.
\end{corollary}

\begin{proof}
It is clear that $\text{Hilb}_{c^h}$ is an atlas and, according to the results of the previous theorem, it is a smooth and irreducible scheme, and this is precisely the assertion of the corollary.
\end{proof}
5. Hilbert Stability

The moduli space $M_g$ of smooth curves was constructed also by Gieseker (see [22]) as a GIT quotient of a locally closed subset of a suitable Hilbert scheme parametrizing $m$-canonically embedded curves, for $m$ sufficiently large. He uses a concept of Hilbert stability, which we will recall now.

Let $C \subset \mathbb{P}^n$ be a a projective curve over $\mathbb{C}$ and $(L, V)$ the generated linear series on $C$ defining the embedding. By Serre’s vanishing theorem there exists an integer $m_0$ such that for all integers $m > m_0$, $H^1(C, L^m) = 0$ and the restriction

$$\lambda_m : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(C, L^m)$$

is surjective. Taking the $N = h^0(X, L^m)$ exterior powers we see that

$$\bigwedge^N \lambda_m : \bigwedge^N H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to \bigwedge^N H^0(C, L^m) = \mathbb{C},$$

defines a point $H_M(X) \in \bigwedge^N H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))^*$ and $H_m(X)$ is determined up to multiplicative constants. Therefore, it defines a point $[H_m(C)] \in \mathbb{P}(\bigwedge^N H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))^*)$, called $[H_m(C)]$ the $m$th-Hilbert point of $X \subset \mathbb{P}^n$. There is a natural action of $SL(n+1)$ on $\mathbb{P}(\bigwedge^N H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))^*)$ and a GIT concept of stability (semistability and polystability).

**Definition 5.1.** A projective irreducible curve $X \subset \mathbb{P}^n$ is Hilbert stable if the $m$th-Hilbert point $[H_m(X)]$ is $SL(n+1)$-stable for infinite values of $m$. Similarly we define Hilbert semistability and Hilbert polystability.

Denote by $Hilb^s$ the irreducible component of the Hilbert scheme $Hilb^{P(t)}$ containing the generic Hilbert stable curve.

Morrison in [32 Corollary 3.5 (i)], proved that Chow stability implies Hilbert stability. From what has already been proved, for complex irreducible complete reduced curve $C \subset \mathbb{P}^n$ with at most ordinary nodes and cusps as singularities we deduce that

- if $T|_C$ is stable then $C \subset \mathbb{P}^n$ is Hilbert stable.
- If $T|_C$ is irreducible and admits an Hermitian-Einstein metric then $C \subset \mathbb{P}^n$ is Hilbert stable.
- Under the conditions of Theorem 3.2
  1. $Hilb^s_Hilb \neq \emptyset$.
  2. $\Gamma(B^r_d) \subset Hilb^s_Hilb$.
  3. $Hilb^s_Hilb$ has dimension $3g - 3 + \rho(g, d, n + 1) + n(n + 2)$ and is smooth at $\Gamma(z)$ for any $z \in P^r_d$.
  4. $\dim Hilb^s_Hilb/SL(n + 1) = \dim P^r_d = 3g - 3 + \rho(g, d, n + 1)$.

In particular, we have

**Theorem 5.2.** If $C$ is a complex irreducible complete reduced curves with at most ordinary nodes and cusps as singularities then $\phi_{K_C}(C) \subset \mathbb{P}^{g-1}$ is Hilbert semistable, and Hilbert
stable if \( C \) is non-hyperelliptic. Moreover, any irreducible curve \( C \subset \mathbb{P}^{g-1} \), of degree \( 2g-2 \) with \( g \geq 2 \), has multiplicity \( \mu_p < 2 \), for any \( p \in C \), if \( T^n|_C \) is stable.

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