SUSY transformations between digonalizable and non-diagonalizable Hamiltonians

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Abstract. Recently (see quant-ph/0503040) an explicit example has been given of a $\mathcal{PT}$-symmetric non-diagonalizable Hamiltonian. In this paper we show that such Hamiltonians appear as supersymmetric (SUSY) partners of Hermitian (hence diagonalizable) Hamiltonians and they can be turned back to diagonalizable forms by appropriate SUSY transformations.

1. It is well-known that there exist non-Hermitian Hamiltonians which cannot be reduced to a diagonal form by the change of the basis (so called non-diagonalizable Hamiltonians, see e.g. [1]). To illustrate better our ideas we will consider here only regular Sturm-Liouville problems. The set of eigenfunctions of a non-diagonalizable Hamiltonian is not complete in corresponding Hilbert space [2, 3]. The characteristic determinant has multiple roots. Together with any eigenfunction with a simple eigenvalue coinciding with a multiple root of the characteristic determinant there exists a set of associated functions [2, 3]. The linear hull of the eigenfunction and corresponding set of the associated functions forms for the given value of the energy the root subspace (see e.g. [3]). Recently an explicit example of an exactly solvable $\mathcal{PT}$-symmetric non-diagonalizable Hamiltonian was given [4].

We have discovered that supersymmetry (SUSY) transformations may convert an Hermitian (hence digonalizable) Hamiltonian to a non-diagonalizable Hamiltonian, which in particular can possess the $\mathcal{PT}$ symmetry, and vice versa. The possibility which does not appear in the linear algebra. It is related with the possibility to “create” more than one “bound state” at a given non-degenerate value of the energy. Since the energy level is non-degenerate the other state cannot be an eigenfunction of the Hamiltonian but it can be an associated function. In the opposite process, when we are “deleting” an eigenfunction having a non-zero associated function, the latter is transformed to a “real eigenfunction”. This looks like it “emerges from the background” and, therefore it may be called “background eigenfunction”.

2. We have found that the possibility described above appears if second order SUSY transformations or higher are used.

Let us consider two ordinary second order differential equations

$$ (h_0 - E)\psi_E(x) = 0 \quad h_0 = -\partial_x^2 + V_0(x) \quad x \in [a, b] \quad (1) $$

$$ (h_1 - E)\varphi_E(x) = 0 \quad h_1 = -\partial_x^2 + V_1(x) \quad x \in [a, b] \quad (2) $$
with $a$ and $b$ being finite numbers. We say that the Hamiltonian $h_1$ is related with $h_0$ by a second order SUSY transformation if

1. $V_1 = V_0 - 2[\log W(u_1, u_2)]''$ 

\[ W(u_1, u_2) = u_1^2 u_2^2 - u_1^2 u_2 \neq 0 \quad \forall x \in (a, b) \]

2. $(h_0 - \alpha_{1,2})u_{1,2}(x) = 0$. 

It is known (see e.g. [6]) that in this case $\varphi_E$ is related with $\psi_E$ as follows:

\[ \varphi_E = L\psi_E = W(u_1, u_2, \psi_E)/W(u_1, u_2) \quad E \neq \alpha_1, \alpha_2 \]

\[ \varphi_{\alpha_{1,2}} = u_{2,1}/W(u_1, u_2). \]

We are using the symbol $W$ to denote Wronskians and will everywhere suppose that $W(u_1, u_2) \neq 0 \forall x \in (a, b)$. Equation (5) holds for any $\psi_E$ from the two dimensional space $\ker(h_0 - E)$. The operator $L$ intertwines the Hamiltonians $h_0$ and $h_1$, $Lh_0 = h_1L$.

Let us suppose that $V_0(x)$ is a real-valued and sufficiently smooth function for $x \in [a, b]$. Consider two boundary value problems, that we will denote (I) and (II) respectively, defined by the equations (1) and (2) and the boundary conditions

\[ \psi_E(a) = \psi_E(b) = 0 \]

\[ \varphi_E(a) = \varphi_E(b) = 0. \]

It is well-known (see e.g. [3]) that the problem (I) has only discrete, simple and real spectrum of eigenvalues $E = E_n$, $n = 0, 1, 2, \ldots$.

We will now formulate conditions for $u_1$ and $u_2$ leading to a complex-valued $V_1(x)$ given by (3) with a real and simple spectrum coinciding with the spectrum of $V_0$ except for one level and the Hamiltonian $h_1$ is non-diagonalizable.

It follows from (3) and (11) that

\[ \varphi_E = \frac{1}{W(u_1, u_2)} [(W(u_1, u_2)E + \alpha_2 u_1^2 u_2 - \alpha_1 u_1 u_2')\psi_E + (\alpha_1 - \alpha_2)u_1 u_2 \psi_E']. \]

It is clear from here that if both $u_1$ and $\psi_E$, $E \neq \alpha_1, \alpha_2$, satisfy the boundary conditions (7) then $\varphi_E$ given by (11) satisfies the boundary conditions (8). The only possibility for $u_1$ to satisfy the zero boundary conditions is to be an eigenfunction of $h_0$, $u_1 = \psi_{E_k}$, so that it is (up to an inessential phase factor) real and $\alpha_1 = E_k$, which we shall suppose to be the case. This means that the Hamiltonian $h_1$ has the same spectrum as $h_0$ except maybe for the values $\alpha_1$ and $\alpha_2$ but since $u_1$ is supposed to satisfy the boundary conditions (7), the function $\varphi_{\alpha_2}$ given in (6) is an eigenfunction of $h_1$ and $E = \alpha_2$ is the spectral point for $h_1$. Remembering that we want to keep the real character of the spectrum of $h_1$ we have to choose $\alpha_2$ real also. So, we choose both $\alpha_1$ and $\alpha_2$ to be real and the function $u_1$ is fixed to be real but we want to get a complex potential difference defined by equation (3). This is possible if $u_2$ is a complex linear combination of two real linearly independent solutions of equation (11). Let $\alpha_2(\neq \alpha_1)$
also coincides with a spectral point \( E_l \) of \( h_0 \), \( \alpha_2 = E_l \), and \( u_2 = \psi_{E_1} + ic\psi_{E_1}^{(2)}, \ c \in \mathbb{R} \) where \( \psi_{E_1} \) satisfies the boundary conditions \((7)\) and \( \psi_{E_1}^{(2)} \) is any real solution of \( \text{Eq. } (1) \) at \( E = E_l \) linearly independent with \( \psi_{E_1} \). We notice that \( u_2(x) \neq 0 \ \forall x \in [a,b] \).

We claim that with \( u_1 \) and \( u_2 \) being chosen as it is described above the potential \( V_1 \) given in \( (3) \) has the spectrum coinciding with the spectrum of the initial \( V_0 \) except for the point \( E = \alpha_1 = E_k \) which is absent. At the energy \( E = \alpha_2 = E_l \) except for an eigenfunction of \( h_1 \) there exists an associated function (see e.g. \([2, 3]\) and also \([4]\)) which we will also call “background eigenfunction”. It satisfies the inhomogeneous equation

\[
(h_1 - E_l)\chi_{E_1} = \varphi_{E_1} \quad \chi_{E_1}(a) = \chi_{E_1}(b) = 0
\]

and also the homogeneous one with the squared Hamiltonian

\[
(h_1 - E_l)^2\chi_{E_1} = 0 \quad \chi_{E_1}(a) = \chi_{E_1}(b) = 0 \quad \chi_{E_1} \neq \varphi_{E_1}.
\]

We would like to stress that the set \( \{\varphi_n\}, \ n = 0, 1, 2, \ldots; \ n \neq k \) (\( E_k \) is “deleted”) is not complete in \( L^2(a,b) \). To have a complete set one has to add to this set the function \( \chi_{E_1} \) \([2, 3]\).

As it was already pointed out all spectral points \( E_n, \ n \neq k, l \) of \( h_0 \) are spectral points of \( h_1 \) also. So, to prove our claim it remains to analyze only the points \( E = \alpha_1 = E_k \) and \( E = \alpha_2 = E_l \).

One of the solutions \( \varphi_{E_1}^{(1)} = \varphi_{\alpha_1} \) of the Schrödinger equation with \( E = \alpha_1 = E_k \) is given by \([6]\) from which it follows that \( \varphi_{E_1}^{(1)}(a) \neq 0 \) and \( \varphi_{E_1}^{(1)}(b) \neq 0 \). A solution vanishing at one of the bounds, for instance at \( x = a \)

\[
\varphi_{E_k}^{(2)}(x) = \varphi_{E_k}^{(1)}(x) \int_a^x \frac{1}{[\varphi_{E_k}^{(1)}(y)]^2} dy
\]

does not vanish at the other bound. This means that \( E = \alpha_1 = E_k \) is not a spectral point of \( h_1 \).

To get a solution of the Schrödinger equation at \( E = \alpha_2 = E_l \) one can use formula \([3]\) with \( \psi_E = \psi_{E_1} \) which gives us the function \( \varphi_{E_1} \) satisfying the zero boundary conditions meaning as it was already mentioned that \( E = E_l \) is the spectral point for \( h_1 \). Moreover, since \( u_1(a) = 0 \), from \([3]\) one gets \( \varphi_E(a) = (E - \alpha_2)\psi_{E_1}(a) \). Now if \( \psi_{E_1}(a) \to \psi_{E_1}(a) \) when \( E \to E_l \), remembering that \( \psi_{E_1}(a) \) is an analytic function of \( E \) having a simple zero at \( E = E_l \) (see e.g. \([5]\)) we conclude that the function \( \varphi_E(a) \) is also an analytic function of \( E \) but it has a double zero at \( E = \alpha_2 = E_l \). In such a case together with the function \( \varphi_{E_1} \) there exists an associated function \( \chi_{E_1} = \partial\varphi_E/\partial E \). It is evident that \( \chi_{E_1}(a) = \chi_{E_1}(b) = 0 \) and the equation \([10]\) it satisfies can be obtained by taking the derivative of equation \([2]\) with respect to \( E \). Since \( \partial \partial E_{L\psi_E} \) and \( L \) is independent of \( E \) one has \( \chi_{E_1} = L\psi_{E_1}, \ \tilde{\psi}_{E_1} = (\partial \varphi_{E_1}/\partial E)_{E = E_l} \). The function \( \tilde{\psi}_{E_1} \) satisfies the equation \((h_0 - E_1)\tilde{\psi}_{E_1} = \psi_{E_1} \) but it does not satisfy the zero boundary conditions which agrees with the fact that \( h_0 \) is a diagonalizable Hamiltonian. Operator \( L \) \([6]\) turns \( \psi_{E_1} \) into a solution of the equation \((10)\) satisfying the zero boundary conditions thus transforming it into a “background eigenfunction” of \( h_1 \).

In contrast to the usual SUSY scheme the opposite process, the “deletion” of the level \( E = E_l \) does not actually delete this level. If we take the Hamiltonian \( h_1 \) as the initial Hamiltonian

\[
\psi_{E_l}(x) = \psi_{E_1} + ic\psi_{E_1}^{(2)}, \ c \in \mathbb{R} \text{ where } \psi_{E_1}
\]
for the next second order transformation, leading to the Hamiltonian \( h_2 \), and choose one of the transformation functions defining the transformation operator \( L^{(2)} \) of the next step to be equal to \( \varphi_{E_i} \), actual eigenfunction at \( E = E_i \) is deleted but the associated function \( \chi_{E_i} \) “comes out of the background” and becomes a true eigenfunction of \( h_2 \) at \( E = E_i \). This statement is readily seen if one acts by \( L^{(2)} \), which is constructed in a similar way as \( L = L^{(1)} \) given in (13) and intertwines now \( h_1 \) and \( h_2 \), on both sides of Eq. (10), takes into account the intertwining relation \( L^{(2)} h_1 = h_2 L^{(2)} \) and the property \( L^{(2)} \varphi_{E_i} = 0 \). If a non-diagonalizable Hamiltonian has only one associated function it is transformed in this way into a diagonalizable Hamiltonian.

3. The simplest example illustrating the possibilities described above is the boundary value problem with the zero initial potential \( V_0(x) = 0 \). We will choose \( a = -\pi \) and \( b = \pi \). The solutions of the boundary value problem (1), (7) is well-known, for instance, its discrete spectrum is \( E = E_n = \frac{1}{4} n^2 \), \( n = 1, 2, \ldots \).

Let us choose \( u_1 = \sin(Ax) \) and \( u_2 = \exp(-iBx) \), \( A, B \in \mathbb{R} \). Formula (3) gives us the following \( PT \)-symmetric Hamiltonian:

\[
V_1 = \frac{2A^2(A^2 - B^2)}{[\cos(Ax) - iB \sin(Ax)]^2}.
\]

For \( A = 1 \) the function \( u_1 \) coincides with the first excited state of \( h_0 \) and for \( B \neq n/2 \) this potential is diagonalizable with the spectrum \( E = E_n = \frac{1}{4} n^2 \), \( n = 1, 3, 4, 5, \ldots \) and \( E_{\alpha_2} = B^2 \). For \( B = 2 \) the function \( u_2 \) is a complex linear combination of the fourth excited state and another solution of Eq. (1) with \( V_0(x) = 0 \) at the same energy and the level \( E_{\alpha_2} \) merges with the existing level \( E = 4 \) which “goes to the background”. The potential (13) becomes non-diagonalizable with the discrete spectrum \( E = E_n = \frac{1}{4} n^2 \), \( n = 1, 3, 4, 5, \ldots \) studied in detail in (4).

Now we would like to illustrate the possibility to transform the non-diagonalizable potential (13) at \( A = 1 \) and \( B = 2 \) into a diagonalizable one. We choose \( V_1 \) as the initial potential and take \( u_1 = \varphi_4 \) and \( u_2 = \varphi_{\text{left}} \) where \( \varphi_{\text{left}} \) is such that \( \varphi_{\text{left}}(-\pi) = 0 \). This yields us the following potential:

\[
V_2 = \frac{(\kappa^2 - 1)[\kappa^2 - 1 - \kappa^2 \cos(2x) + \cos(2\kappa x + 2\kappa \pi)]}{[\kappa \cos(\kappa x + \kappa \pi) \sin x - \sin(\kappa x + \kappa \pi) \cos x]^2} \quad \kappa \neq 1
\]

where we denoted \( \alpha_2 = \kappa^2 \). It is regular \( \forall x \in (-\pi, \pi) \) provided \( 0.5 \leq \kappa \leq 1.5 \), \( \kappa \neq 1 \) and has the spectrum \( E = E_n = \frac{n^2}{4} \), \( n = 1, 3, 4 \ldots \) and \( E = \alpha_2 = \kappa^2 \). For \( \kappa = 1 \) \( \varphi_{\text{left}} = L\psi_{\text{left}} = 0 \) and to realize the transformation with \( \alpha_2 = 1 \) one has to use the solution obtained with the help of formula (3). This corresponds to the backward transformation from \( V_1 \) to \( V_0 = 0 \) and hence one gets \( V_2 = 0 \). For a real \( \kappa \) the potential (14) is real and corresponds to the Hermitian (hence diagonalizable) Hamiltonian \( h_2 = -\frac{d^2}{dx^2} + V_2 \). So, we have transformed the non-diagonalizable Hamiltonian \( h_1 \) to the diagonalizable \( h_2 \). One can also transform \( h_1 \) into a non-Hermitian diagonalizable \( h_2 \) by choosing a complex linear combination of two linearly independent solutions of equation (2) corresponding to the same value of \( E = \alpha_2 \) as transformation function \( u_2 \).

Our last example is related with the possibility to enlarge the root subspace corresponding to \( E = 4 \) of the potential (13) at \( A = 1 \) and \( B = 2 \) from the dimension two till the dimension
For this aim we take \( u_1 = \varphi_{E_1} \) and \( u_2 = [9 - \exp(-2ix)]/[1 - 3\exp(2ix)] \) which yields the potential
\[
V_2 = 6 \frac{25e^{ix} + 324e^{2ix} + 1350e^{3ix} + 2500e^{4ix} + 2025e^{5ix}}{(3 + 25e^{ix} + 81e^{2ix} + 75e^{3ix})^2}.
\]
It has the spectrum \( E = \frac{n^2}{4}, \ n = 3, 4, 5, \ldots \).

We hope that the possibility to transform non-diagonalizable \( \mathcal{PT} \)-symmetric Hamiltonians to diagonalizable forms may find application in complex quantum mechanics which is currently under developments.

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