ON THE EQUALITY OF DE RHAM DEPTH AND FORMAL GRADE IN CHARACTERISTIC ZERO

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ABSTRACT. Let \( Y \subset \mathbb{P}^n_k \) be a nonsingular proper closed subset of projective \( n \)-space over a field \( k \) of characteristic zero and let \( I \subset R = k[x_0, \ldots, x_n] \) be the homogeneous defining ideal of \( Y \). We show that in this case, the de Rham depth of \( Y \) is the same as the so-called formal grade of \( I \) in \( R \).

1. INTRODUCTION

Let \( Y \subset \mathbb{P}^n_k \) be a proper closed subset and let \( I \subset R = k[x_0, \ldots, x_n] \) be the homogeneous defining ideal of \( Y \) over a field \( k \) with homogeneous maximal ideal \( m = (x_0, \ldots, x_n) \).

In [3] the author has shown that in the case of an \( F \)-finite regular local ring \( A \) of positive characteristic the formal grade of \( I \) in \( A \), \( \text{fgrade}(I, A) := \min\{i \mid \lim \leftarrow H^i_m(R/I^i) \neq 0 \} \) is the same as \( F\text{-depth}\ Y = \text{Frobenius depth of } Y = \text{Spec}(A/I) \). In this note we try to consider its analogue in characteristic zero case and show that the so called de Rham depth of \( Y \), \( \text{DR-depth} \ Y = \text{fgrade}(I, R) \). To be more precise we show the following:

**Theorem 1.1.** Let \( Y \subset \mathbb{P}^n_k \) be a nonsingular proper closed subset of projective \( n \)-space over a field \( k \) of equicharacteristic zero and let \( I \subset R = k[x_0, \ldots, x_n] \) be the homogeneous defining ideal of \( Y \). Then \( \text{DR-depth} \ Y = \text{fgrade}(I, R) \).

2. PRELIMINARIES

Let \( Y \) be a scheme of finite type over a field \( k \) of characteristic 0 which admits an embedding as a closed subscheme of a smooth scheme \( X \). We denote by \( \Omega^p_X \) the sheaf of \( p \)-differential forms on \( X \) over \( k \). With these notations in mind we define the de Rham complex of \( X \) as the complex of sheaves of differential forms

\[
\Omega_X^* : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \ldots
\]

De Rham cohomolgy of \( Y \) is defined as hypercohomology of the formal completion \( \hat{\Omega}^*_{\hat{X}} \) of the de Rham complex of \( X \) along \( Y \):

\[
H^i_{\text{dR}}(Y/k) = H^i(\hat{X}, \hat{\Omega}^*_{\hat{X}}),
\]

see [4] and [5] for more details. If there is no ambiguity on the field \( k \) we use \( H^i_{\text{dR}}(Y) \) instead of \( H^i_{\text{dR}}(Y/k) \).

\textbf{2010 Mathematics Subject Classification.} 13D45, 14F40.

\textbf{Key words and phrases.} formal grade, de Rham depth.

The author is supported in part by a grant from Tafresh University.
Definition 2.1. Let $A$ be a complete local ring with coefficient field $k$ of characteristic zero. Let 
$\pi: R \to A$ be a surjection of $k$-algebras where $R = \mathbb{k}[[x_1, \ldots, x_n]]$ for some $n$ and let $Y \to X$ (where $Y = \text{Spec } A, X = \text{Spec } R$) be the corresponding closed immersion. Let $P \in Y$ be the closed point. The (local) de Rham cohomology of $Y$ is defined by

$$H_{p, dR}^i(Y) = H_{p}^{i}(X, \Omega_{X}^{*}), \quad \text{for all } i.$$ 

by [5] III. Proposition 3.1] we have

$$H_{p, dR}^{i}(Y) = H_{p}^{i}(\hat{X}, \hat{\Omega}_{X}^{*}) \simeq H_{p, dR}^{i}(\text{Spec } \hat{\mathcal{O}}_{X, P})$$

as $k$-spaces for all $i$.

It is noteworthy to mention that $H_{p, dR}^{i}(Y)$ and $H_{p, dR}^{i}(Y)$ are finite dimensional $k$-vector spaces for all $i$.

Definition 2.2. Let $Y$ be a Noetherian scheme of equicharacteristic zero. The "de Rham depth of $Y$" which is abbreviated by $\text{DR-depth } Y$ is defined as

$$\text{DR-depth } Y = \max \{ d \in \mathbb{Z} \mid H_{p, dR}^{i}(Y) \neq 0, i < d - \dim \{y\}, \quad \text{where } d \text{ is an integer and } \overline{\{y\}} \text{ denotes the closure of } \{y\}. \}

Here, and by our assumption on the scheme $Y \subset \mathbb{P}_k^n$ we may suppose that $y$ is a closed point of $Y$. See [6] pp. 340].

We conclude this section with some review on the concept of formal grade which is the index of the minimal nonvanishing formal cohomology module, $\lim H_{m}^{i}(R/I^i)$, where $R$ is a local ring with unique maximal ideal $m$ and $I$ is an ideal of $R$. Recall that $H_{m}^{i}(\cdot)$ is the local cohomology functor with support in $V(m)$. Notice that local cohomology module of $R$ with support in $V(I)$ is defined as $H_{I}^{i}(R) = \lim H_{m}^{i}(R/I^i, R)$. For the regular local ring $R$, one can interpret the last non vanishing index of $H_{I}^{i}(R)$, which is known as cohomological dimension of $I$ in $R$, by using $\text{fgrade}(I, R)$. We denote the cohomological dimension of $I$ in $R$ with $\text{cd}(R, I)$, see Remark 3.1 below. For more information on this topic we refer the reader to [2].

3. **Proof of Theorem 1.1**

Remark 3.1. Let $R = \mathbb{k}[x_0, \ldots, x_n]$ be a polynomial ring over a field $k$ and $m = (x_0, \ldots, x_n)$ be its homogeneous maximal ideal. Note that $k = R/m$. Using a suitable gonflement of $R$ (cf. [11] Chapter IX, Appendix 2]) one can take a faithfully flat local homomorphism $f: R \to S$ with $f(m) = n$ from $(R, m)$ to a regular local ring $(S, n)$ such that $S/n$ is the algebraic closure of $k$. Because of faithfully flatness, $S$ contains a field. Now note that by local duality [2] 11.2.5]

$$\lim_{\leftarrow} H_{m}^{i}(S/(IS)^i) \simeq \lim_{\rightarrow} \text{Hom}_{S}(\text{Ext}_{S}^{\dim S - i}(S/(IS)^i, S), E_S)$$

and

$$\text{Hom}_{S}(\lim_{\rightarrow} \text{Ext}_{S}^{\dim S - i}(S/(IS)^i, S), E_S) \simeq \text{Hom}_{S}(\lim_{\leftarrow} H_{IS}^{i}(S/(IS)^i, S), E_S).$$

This shows that $\text{fgrade}(IS, S) = \dim S - \text{cd}(S, IS)$. On the other hand, by flat base change theorem [2], $\text{cd}(S, IS) = \text{cd}(R, I)$. Hence, we conclude that $\text{fgrade}(I, R) = \text{fgrade}(IS, S)$.
Remark 3.2. Suppose that $Q \subseteq k_0 \subseteq k$, where $k$ is a field extension of a field $k_0$. Let $X_0$ be a smooth scheme defined over $k_0$. Let $X := X_0 \times_{k_0} k$, by [5, III, Section 5], it implies the natural isomorphism between algebraic de Rham cohomology groups:

$$H^i_{dR}(X/k) \simeq H^i_{dR}(X_0/k_0) \otimes k_0 k, \ i \in \mathbb{Z},$$

as $k$ is faithfully flat over $k_0$.

Proposition 3.3. Let $Y \subset \mathbb{P}^n_k$ be a proper closed subset of projective $n$-space over a field $k$ of characteristic zero and let $I \subset R = k[x_0, \ldots, x_n]$ be the homogeneous defining ideal of $Y$. Then DR-depth $Y \geq \text{fgrade}(I, R)$.

Proof. By virtue of Remark 3.1 and 3.2 we may reduce the question to the case $k = C$. Put $\text{fgrade}(I, R) = u$. Thus, $\lim_{m} (R/I)^m = 0$ for all $j < u$. By using local duality [2, 11.2.5] and taking account that the inverse limit commutes with direct limit in the first place of $\text{Hom}_R(\cdot, \cdot)$ we observe that $H^i(C(I)) = 0$ for all $i > \dim R - u$. As $\dim R - u > 0$ then $H^i(\mathbb{A}^n_C - C(Y), F) = 0$ for all $i \geq \dim R - u$ and all quasi-coherent sheaf $F$ on $\mathbb{A}^n_C - C(Y)$, where $\mathbb{A}^n_C$ and $C(Y)$ are, respectively, the affine cones over $\mathbb{P}^n_C$ and $Y$. Since the natural morphism $\mathbb{A}^n_C - C(Y) \to \mathbb{P}^n_C - Y$ is affine, it follows that $H^i(\mathbb{P}^n_C - Y, F) = 0$ for all $i \geq \dim R - u$ and all quasi-coherent sheaf $F$ on $\mathbb{P}^n_C - Y$. By virtue of [6, Theorem 4.4] we have DR-depth $Y \geq t$, as required.

Proof of Theorem [1].

The Proposition 3.3 ensures that DR-depth $Y \geq \text{fgrade}(I, R)$. To prove converse direction, note that by [6] Theorem 2.8, it is enough to show that $\text{Supp} H^i(R) \subseteq \{m\}$ for all $i > \text{ht}(I)$. If $p$ is a prime ideal not containing $I$, it is clear that $(H^i(R))_p = 0$. Hence, suppose that $p \neq m$ is a prime ideal containing $I$. As $Y$ is nonsingular, the ideal $IR_p$ is a complete intersection in $R_p$. Then, it follows that $(H^i(R))_p \cong H^i_{IR_p}(R_p) = 0$ for all $i > \text{ht}(I)$.

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