CONTINUOUS FIELDS OF C*-ALGEBRAS OVER FINITE DIMENSIONAL SPACES

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Abstract. Let $X$ be a finite dimensional compact metrizable space. We study a technique which employs semiprojectivity as a tool to produce approximations of $C(X)$-algebras by $C(X)$-subalgebras with controlled complexity. The following applications are given. All unital separable continuous fields of C*-algebras over $X$ with fibers isomorphic to a fixed Cuntz algebra $C_n$, $n \in \{2, 3, ... \}$ are locally trivial. They are trivial if $n = 2$ or $n = \infty$. For $n \geq 3$ finite, such a field is trivial if and only if $(n-1)[1_A] = 0$ in $K_0(A)$, where $A$ is the C*-algebra of continuous sections of the field. We give a complete list of the Kirchberg algebras $D$ satisfying the UCT and having finitely generated K-theory groups for which every unital separable continuous field over $X$ with fibers isomorphic to $D$ is automatically locally trivial or trivial. In a more general context, we show that a separable unital continuous field over $X$ with fibers isomorphic to a $KK$-semiprojective Kirchberg C*-algebra is trivial if and only if it satisfies a K-theoretical Fell type condition.

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1. Introduction

Gelfand’s characterization of commutative C*-algebras has suggested the problem of representing non-commutative C*-algebras as sections of bundles. By a result of Fell [15], if the primitive spectrum $X$ of a separable C*-algebra $A$ is Hausdorff, then $A$ is isomorphic to the C*-algebra of continuous sections vanishing at infinity of a continuous field of simple C*-algebras over $X$. In particular $A$ is a continuous $C(X)$-algebra in the sense of Kasparov [18]. This description is very

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satisfactory, since as explained in [4], the continuous fields of C*-algebras are in natural correspondence with the bundles of C*-algebras in the sense of topology. Nevertheless, only a tiny fraction of the continuous fields of C*-algebras correspond to locally trivial bundles.

In this paper we prove automatic and conditional local/global trivialization results for continuous fields of Kirchberg algebras. By a Kirchberg algebra we mean a purely infinite simple nuclear separable C*-algebra [24]. Notable examples include the simple Cuntz-Krieger algebras [5]. The following theorem illustrates our results.

**Theorem 1.1.** A separable unital C(X)-algebra A over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to the same Cuntz algebra O_n, n \in \{2, 3, \ldots, \infty\}, is locally trivial. If n = 2 or n = \infty, then A \cong C(X) \otimes O_n. If 3 \leq n < \infty, then A is isomorphic to C(X) \otimes O_n if and only if \((n - 1)[1_A] = 0 \text{ in } K_0(A)\).

The case X = [0, 1] of Theorem 1.1 was proved in a joint paper with G. Elliott [10].

We parametrize the homotopy classes

\[ [X, \text{Aut}(O_n)] \cong \begin{cases} K_1(C(X) \otimes O_n) & \text{if } 3 \leq n < \infty, \\ \{e\} & \text{if } n = 2, \infty, \end{cases} \]

(see Theorem 23) and hence classify the unital separable C(SX)-algebras A with fiber O_n over the suspension SX of a finite dimensional metrizable Hausdorff space X.

To put our results in perspective, let us recall that none of the general basic properties of a continuous field implies any kind of local triviality. An example of a continuous field of Kirchberg algebras over [0, 1] which is not locally trivial at any point even though all of its fibers are mutually isomorphic is exhibited in [10] Ex. 8.4. Examples of nonexact continuous fields with similar properties were found by S. Wassermann [37].

A separable C*-algebra D is KK-semiprojective if the functor KK(D, -) is continuous, see Sec. 5. The class of KK-semiprojective C*-algebras includes the nuclear semiprojective C*-algebras and also the C*-algebras which satisfy the Universal Coefficient Theorem in KK-theory (abbreviated UCT [31]) and whose K-theory groups are finitely generated. It is very interesting that the only obstruction to local or global triviality for a continuous field of Kirchberg algebras is of purely K-theoretical nature.

**Theorem 1.2.** Let A be a separable C*-algebra whose primitive spectrum X is compact Hausdorff and of finite dimension. Suppose that each primitive quotient A(x) of A is nuclear, purely infinite and stable. Then A is isomorphic to C(X) \otimes D for some KK-semiprojective stable Kirchberg algebra D if and only if there is \(\sigma \in KK(D, A)\) such that \(\sigma_x \in KK(D, A(x))^{-1}\) for all \(x \in X\). For any such \(\sigma\) there is an isomorphism of C(X)-algebras \(\Phi : C(X) \otimes D \to A\) such that \(KK(\Phi|_D) = \sigma\).

We have an entirely similar result covering the unital case: Theorem 28. The required existence of \(\sigma\) is a KK-theoretical analog of the classical condition of Fell that appears in the trivialization theorem of Dixmier and Douady [12] of continuous fields with fibers isomorphic to the compact operators. An important feature of our condition is that it is a priori much weaker than the condition that A is KK_{C(X)}-equivalent to C(X) \otimes D. In particular, we do not need to worry at all about the potentially hard issue of constructing elements in KK_{C(X)}(A, C(X) \otimes D). To illustrate this point, let us note that it is almost trivial to verify that the local existence of \(\sigma\) is automatic
for unital $C(X)$-algebras with fiber $O_n$ and hence to derive Theorem 1.1. A $C^*$-algebra $D$ has the automatic local triviality property if any separable $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$ is locally trivial. A unital $C^*$-algebra $D$ has the automatic local triviality property in the unital sense if any separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$ is locally trivial. The automatic triviality property is defined similarly.

**Theorem 1.3.** (Automatic triviality) A separable continuous $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $O_2 \otimes K$ is isomorphic to $C(X) \otimes O_2 \otimes K$. The $C^*$-algebra $O_2 \otimes K$ is the only Kirchberg algebra satisfying the automatic local triviality property and hence the automatic triviality property.

**Theorem 1.4.** (Automatic local triviality in the unital sense) A unital $KK$-semiprojective Kirchberg algebra $D$ has the automatic local triviality property in the unital sense if and only if all unital $*$-endomorphisms of $D$ are $KK$-equivalences. In that case, if $A$ is a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$, then $A \cong C(X) \otimes D$ if and only if there is $\sigma \in KK(D, A)$ such that the induced homomorphism $K_0(\sigma) : K_0(D) \to K_0(A)$ maps $[1_D]$ to $[1_A]$.

It is natural to ask if there are other unital Kirchberg algebras besides the Cuntz algebras which have the automatic local triviality property in the unital sense. Consider the following list $\mathcal{G}$ of pointed abelian groups:

(a) $(\{0\}, 0)$;
(b) $(\mathbb{Z}, k)$ with $k > 0$;
(c) $(\mathbb{Z}/p^{s_1} \oplus \cdots \oplus \mathbb{Z}/p^{s_n}, p^{1} \oplus \cdots \oplus p^{n})$ where $p$ is a prime, $n \geq 1$, $0 \leq s_i < e_i$ for $1 \leq i \leq n$ and $0 < s_{i+1} - s_i < e_{i+1} - e_i$ for $1 \leq i < n$. If $n = 1$ then $1 \leq e_i \leq \cdots \leq e_n$ are given and then there exists integers $s_1, \ldots, s_n$ satisfying the conditions above if and only if $e_{i+1} - e_i > 2$ for each $1 \leq i \leq n$. If that is the case one can choose $s_i = i - 1$ for $1 \leq i \leq n$.

(d) $(G(p_1) \oplus \cdots \oplus G(p_m), g_1 \oplus \cdots \oplus g_m)$ where $p_1, \ldots, p_m$ are distinct primes and each $(G(p_j), g_j)$ is a pointed group as in (c).

(e) $(\mathbb{Z} \oplus G(p_1) \oplus \cdots \oplus G(p_m), k \oplus g_1 \oplus \cdots \oplus g_m)$ where $(G(p_j), g_j)$ are as in (d). Moreover we require that $k > 0$ is divisible by $p_1^{s_{n(j)}+1} \cdots p_m^{s_{n(j)}+1}$ where $s_{n(j)}$ is defined as in (c) corresponding to the prime $p_j$.

**Theorem 1.5.** (Automatic local triviality in the unital sense – the UCT case) Let $D$ be a unital Kirchberg algebra which satisfies the UCT and has finitely generated $K$-theory groups. (i) $D$ has the automatic triviality property in the unital sense if and only if $D$ is isomorphic to either $O_2$ or $O_\infty$. (ii) $D$ has the automatic local triviality property in the unital sense if and only if $K_1(D) = 0$ and $(K_0(D), [1_D])$ is isomorphic to one of the pointed groups from the list $\mathcal{G}$. (iii) If $D$ is as in (ii), then a separable unital $C(X)$-algebra $A$ over a finite dimensional compact Hausdorff space $X$ all of whose fibers are isomorphic to $D$ is trivial if and only if there exists a homomorphism of groups $K_0(D) \to K_0(A)$ which maps $[1_D]$ to $[1_A]$.

We use semiprojectivity (in various flavors) to approximate and represent continuous $C(X)$-algebras as inductive limits of fibered products of $n$ locally trivial $C(X)$-subalgebras where $n \leq \dim(X) < \infty$. This clarifies the local structure of many $C(X)$-algebras (see Theorem 1.2 and...
gives a new understanding of the K-theory of separable continuous $C(X)$-algebras with arbitrary nuclear fibers.

A remarkable isomorphism result for separable nuclear strongly purely infinite stable C*-algebras was announced (with an outline of the proof) by Kirchberg in [20]: two such C*-algebras $A$ and $B$ with the same primitive spectrum $X$ are isomorphic if and only if they are $KK_{C(X)}$-equivalent. This is always the case after tensoring with $O_2$. However the problem of recognizing when $A$ and $B$ are $KK_{C(X)}$-equivalent is open even for very simple spaces $X$ such as the unit interval or non-Hausdorff spaces with more than two points.

The proof of Theorem 1.6 (one of our main results) generalizes and refines a technique that was pioneered for fields over zero dimensional spaces in joint work with Pasnicu [11] and for fields over non-Hausdorff spaces with more than two points.

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2. $C(X)$-algebras

Let $X$ be a locally compact Hausdorff space. A $C(X)$-algebra is a C*-algebra $A$ endowed with a $*$-homomorphism $\theta$ from $C_0(X)$ to the center $ZM(A)$ of the multiplier algebra $M(A)$ of $A$ such that $C_0(X)A$ is dense in $A$; see [18], [3]. We write $fa$ rather than $\theta(f)a$ for $f \in C_0(X)$ and $a \in A$. If $Y \subseteq X$ is a closed set, we let $C_0(X,Y)$ denote the ideal of $C_0(X)$ consisting of functions vanishing on $Y$. Then $C_0(X,Y)A$ is a closed two-sided ideal of $A$ (by Cohen factorization). The quotient of $A$ by this ideal is a $C(X)$-algebra denoted by $A(Y)$ and is called the restriction of $A = A(X)$ to $Y$. The quotient map is denoted by $\pi_Y : A(X) \rightarrow A(Y)$. If $Z$ is a closed subset of $Y$ we have a natural restriction map $\pi_Z^Y : A(Y) \rightarrow A(Z)$ and $\pi_Z = \pi_Z^Y \circ \pi_Y$. If $Y$ reduces to a point $x$, we write $A(x)$ for $A(\{x\})$ and $\pi_x$ for $\pi(\{x\})$. The C*-algebra $A(x)$ is called the fiber of $A$ at $x$. The image $\pi_x(a) \in A(x)$ of $a \in A$ is denoted by $a(x)$. A morphism of $C(X)$-algebras $\eta : A \rightarrow B$ induces a morphism $\eta_Y : A(Y) \rightarrow B(Y)$. If $A(x) \neq 0$ for $x$ in a dense subset of $X$, then $\theta$ is injective. If $X$ is compact, then $\theta(1) = 1_{M(A)}$. Let $A$ be a C*-algebra, $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$. Throughout the paper we will assume that $X$ is a compact Hausdorff space unless stated otherwise. If $\varepsilon > 0$, we write $a \in_{\varepsilon} \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subseteq_{\varepsilon} \mathcal{G}$ if $a \in_{\varepsilon} \mathcal{G}$ for every $a \in \mathcal{F}$.

The following lemma collects some basic properties of $C(X)$-algebras.

**Lemma 2.1.** Let $A$ be a $C(X)$-algebra and let $B \subseteq A$ be a $C(X)$-subalgebra. Let $\alpha \in A$ and let $Y$ be a closed subset of $X$.

(i) The map $x \mapsto \|\pi_x(x)\|$ is upper semi-continuous.

(ii) $\|\pi_Y(a)\| = \max\{\|\pi_x(a)\| : x \in Y\}$

(iii) If $a(x) \in \pi_x(B)$ for all $x \in X$, then $a \in B$.

(iv) If $\delta > 0$ and $a(x) \in_{\delta} \pi_x(B)$ for all $x \in X$, then $a \in_{\delta} B$.

(v) The restriction of $\pi_x : A \rightarrow A(x)$ to $B$ induces an isomorphism $B(x) \cong \pi_x(B)$ for all $x \in X$. 

Proof. (i), (ii) are proved in [3] and (iii) follows from (iv). (iv): By assumption, for each \( x \in X \), there is \( b_x \in B \) such that \( \| \pi_x(a - b_x) \| < \delta \). Using (i) and (ii), we find a closed neighborhood \( U_x \) of \( x \) such that \( \| \pi_x(a - b_x) \| < \delta \). Since \( X \) is compact, there is a finite subcover \( \{ U_{x_i} \} \). Let \( (\alpha_i) \) be a partition of unity subordinated to this cover. Setting \( b = \sum_i \alpha_i b_{x_i} \in B \), one checks immediately that \( \| \pi_x(a - b) \| \leq \sum_i \alpha_i \| \pi_x(a - b_{x_i}) \| < \delta \), for all \( x \in X \). Thus \( \| a - b \| < \delta \) by (ii). (v): If \( \epsilon : B \to A \) is the inclusion map, then \( \pi_x(B) \) coincides with the image of \( \epsilon_{\times} : B/C(X,x)B \to A/C(X,x)A \). Thus it suffices to check that \( \epsilon_{\times} \) is injective. If \( \epsilon_{\times}(b + C(X,x)B) = \pi_x(b) = 0 \) for some \( b \in B \), then \( b = fa \) for some \( f \in C(X,x) \) and some \( a \in A \). If \( (f_{x}) \) is an approximate unit of \( C(X,x) \), then \( b = \lim_{n} f_{x}a = \lim_{n} f_{x}b \) and hence \( b \in C(X,x)B \). \( \square \)

A \( C(X) \)-algebra such that the map \( x \mapsto \| a(x) \| \) is continuous for all \( a \in A \) is called a continuous \( C(X) \)-algebra or a \( C^* \)-bundle \([3], [23], [4]\). A \( C^* \)-algebra \( A \) is a continuous \( C(X) \)-algebra if and only if \( A \) is the \( C^* \)-algebra of continuous sections of a continuous field of \( C^* \)-algebras over \( X \) in the sense of \([12\text{, Def. 10.3.1}]\), (see \([3], [4], [27]\)).

**Lemma 2.2.** Let \( A \) be a separable continuous \( C(X) \)-algebra over a locally compact Hausdorff space \( X \). If all the fibers of \( A \) are nonzero, then \( X \) has a countable basis of open sets. Thus the compact subspaces of \( X \) are metrizable.

**Proof.** Since \( A \) is separable, its primitive spectrum \( \text{Prim}(A) \) has a countable basis of open sets by \([12\text{, 3.3.4}]\). The continuous map \( \eta : \text{Prim}(A) \to X \) (induced by \( \theta : C_{0}(X) \to ZM(A) \cong C_{b}(\text{Prim}(A)) \)) is open since the \( C(X) \)-algebra \( A \) is continuous and surjective since \( A(x) \neq 0 \) for all \( x \in X \) (see \([3\text{, p. 388}]\) and \([27\text{, Prop. 2.1, Thm. 2.3}]\)). \( \square \)

**Lemma 2.3.** Let \( X \) be a compact metrizable space. A \( C(X) \)-algebra \( A \) all of whose fibers are nonzero and simple is continuous if and only if there is \( e \in A \) such that \( \| e(x) \| \geq 1 \) for all \( x \in X \).

**Proof.** By Lemma 2.2(i) it suffices to prove that \( \liminf_{n \to \infty} \| a(x_n) \| \geq \| a(x_0) \| \) for any \( a \in A \) and any sequence \( (x_n) \) converging to \( x_0 \) in \( X \). Set \( D = A(x_0) \) and let \( e \) be as in the statement. Let \( \psi : D \to A \) be a set-theoretical lifting of id\(_D\) such that \( \| \psi(d) \| = \| d \| \) for all \( d \in D \). Then \( \lim_{n \to \infty} \| \pi_x \psi(a(x_0)) - a(x_0) \| = 0 \) for all \( a \in A \), by Lemma 2.1(i). By applying this to \( e \), since \( \| e(x_n) \| \geq 1 \), we see that \( \lim_{n \to \infty} \| \pi_x \psi(e(x_0)) \| \geq 1 \). Since \( D \) is a simple \( C^* \)-algebra, if \( \varphi_n : D \to B_n \) is a sequence of contractive maps such that \( \lim_{n \to \infty} \| \varphi_n(\lambda c + d) - \lambda \varphi_n(c) - \varphi_n(d) \| = 0 \), \( \lim_{n \to \infty} \| \varphi_n(cd) - \varphi_n(c)\varphi_n(d) \| = 0 \), \( \lim_{n \to \infty} \| \varphi_n(c^*) - \varphi_n(c)^* \| = 0 \), for all \( c, d \in D, \lambda \in \mathbb{C}, \) and \( \lim_{n \to \infty} \| \varphi_n(c) \| > 0 \) for some \( c \in D \), then \( \lim_{n \to \infty} \| \varphi_n(c) \| = \| c \| \) for all \( c \in D \). In particular this observation applies to \( \varphi_n = \pi_x \psi \) by Lemma 2.1(i). Therefore

\[
\liminf_{n \to \infty} \| a(x_n) \| \geq \liminf_{n \to \infty} \| \pi_x \psi(a(x_0)) - \pi_x \psi(a(x_0)) \| = \| a(x_0) \| .
\]

Conversely, if \( A \) is continuous, take \( e \) to be a large multiple of some full element of \( A \). \( \square \)

Let \( \eta : B \to A \) and \( \psi : E \to A \) be *-homomorphisms. The pullback of these maps is

\[
B \oplus_{\eta, \psi} E = \{ (b, e) \in B \oplus E : \eta(b) = \psi(e) \}.
\]

We are going to use pullbacks in the context of \( C(X) \)-algebras. Let \( X \) be a compact space and let \( Y, Z \) be closed subsets of \( X \) such that \( Y = Y \cup Z \). The following result is proved in \([12\text{, Prop. 10.1.13}]\) for continuous \( C(X) \)-algebras.

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Lemma 2.4. If $A$ is a $C(X)$-algebra, then $A$ is isomorphic to $A(Y) \oplus_{\pi_Y} A(Z)$, the pullback of the restriction maps $\pi_{Y \cap Z}^Y : A(Y) \to A(Y \cap Z)$ and $\pi_{Y \cap Z}^Z : A(Z) \to A(Y \cap Z)$.

Proof. By the universal property of pullbacks, the maps $\pi_Y$ and $\pi_Z$ induce a map $\eta : A \to A(Y) \oplus_{\pi_Y} A(Z)$, $\eta(a) = (\pi_Y(a), \pi_Z(a))$, which is injective by Lemma 2.1(ii). Thus it suffices to show that the range of $\eta$ is dense. Let $b, c \in A$ be such that $\pi_{Y \cap Z}(b - c) = 0$ and let $\varepsilon > 0$. We shall find $a \in A$ such that $\|\eta(a) - (\pi_Y(b), \pi_Z(c))\| < \varepsilon$. By Lemma 2.1(i), there is an open neighborhood $V$ of $Y \cap Z$ such that $\|\pi_x(b - c)\| < \varepsilon$ for all $x \in V$. Let $\{\lambda, \mu\}$ be a partition of unity on $X$ subordinated to the open cover $\{Y \cup V, Z \cup V\}$. Then $a = \lambda b + \mu c$ is an element of $A$ which has the desired property.

Let $B \subset A(Y)$ and $E \subset A(Z)$ be $C(X)$-subalgebras such that $\pi_{Y \cap Z}^Y(E) \subseteq \pi_{Y \cap Z}^Z(B)$. As an immediate consequence of Lemma 2.4, we see that the pullback $B \oplus_{\pi_Y \cap Z, \pi_Z} E$ is isomorphic to the $C(X)$-subalgebra $B \oplus_{Y \cap Z} E$ of $A$ defined as $B \oplus_{Y \cap Z} E = \{a \in A : \pi_Y(a) \in B, \pi_Z(a) \in E\}$.

Lemma 2.5. The fibers of $B \oplus_{Y \cap Z} E$ are given by

$$\pi_x(B \oplus_{Y \cap Z} E) = \begin{cases} \pi_x(B), & \text{if } x \in X \setminus Z, \\ \pi_x(E), & \text{if } x \in Z, \end{cases}$$

and there is an exact sequence of $C^*$-algebras

$$0 \to \{b \in B : \pi_{Y \cap Z}(b) = 0\} \to B \oplus_{Y \cap Z} E \overset{\pi_Z}{\longrightarrow} E \to 0$$

Proof. Let $x \in X \setminus Z$. The inclusion $\pi_x(B \oplus_{Y \cap Z} E) \subseteq \pi_x(B)$ is obvious by definition. Given $b \in B$, let us choose $f \in C(X)$ vanishing on $Z$ and such that $f(x) = 1$. Then $a = (fb, 0)$ is an element of $A$ by Lemma 2.4. Moreover $a \in B \oplus_{Y \cap Z} E$ and $\pi_x(a) = \pi_x(b)$. We have $\pi_Z(B \oplus_{Y \cap Z} E) \subset E$, by definition. Conversely, given $e \in E$, let us observe that $\pi_{Y \cap Z}^Y(e) \in \pi_{Y \cap Z}^Y(B)$ (by assumption) and hence $\pi_{Y \cap Z}(e) = \pi_{Y \cap Z}(b)$ for some $b \in B$. Then $a = (b, e)$ is an element of $A$ by Lemma 2.4 and $\pi_Z(a) = e$. This completes the proof for the first part of the lemma and also it shows that the map $\pi_Z$ from the sequence (1) is surjective. Its kernel is identified using Lemma 2.1(iii).

Let $X, Y, Z$ and $A$ be as above. Let $\eta : B \to A(Y)$ be a $C(Y)$-linear $*$-monomorphism and let $\psi : E \to A(Z)$ be a $C(Z)$-linear $*$-monomorphism. Assume that

$$\pi_{Y \cap Z}^Y(\psi(E)) \subseteq \pi_{Y \cap Z}^Z(\eta(B)).$$

This gives a map $\gamma = \eta_{Y \cap Z}^{-1} \psi_{Y \cap Z} : E(Y \cap Z) \to B(Y \cap Z)$. To simplify notation we let $\pi$ stand for both $\pi_{Y \cap Z}^Y$ and $\pi_{Y \cap Z}^Z$ in the following lemma.

Lemma 2.6. (a) There are isomorphisms of $C(X)$-algebras

$$B \oplus_{\pi, \eta, \psi} E \cong B \oplus_{\pi_Y, \pi_Z} E \cong \eta(B) \oplus_{Y \cap Z} \psi(E),$$

where the second isomorphism is given by the map $\chi : B \oplus_{\pi_Y, \pi_Z} E \to A$ induced by the pair $(\eta, \psi)$. Its components $\chi_x$ can be identified with $\psi_x$ for $x \in Z$ and with $\pi_x$ for $x \in X \setminus Z$.

(b) Condition (2) is equivalent to $\psi(E) \subseteq \pi_Z(A \oplus_{Y \cap Z} \eta(B))$.

(c) If $F$ is a finite subset of $A$ such that $\pi_Y(F) \subset \eta(B)$ and $\pi_Z(F) \subset \psi(E)$, then $\pi_E(\eta(B) \oplus_{Y \cap Z} \psi(E)) = \chi(B \oplus_{\pi_Y, \pi_Z} E)$. 


Proof. This is an immediate corollary of Lemmas 2.1, 2.4, 2.5. For illustration, let us verify (c). By assumption \( \pi_x(F) \subset \pi_x(\eta) \) for all \( x \in X \setminus Z \) and \( \pi_x(F) \subset \psi^\ast(\eta) \) for all \( z \in Z \). We deduce from Lemma 2.5 that \( \pi_x(F) \subset \pi_x(\eta(B) \oplus_{Y \cap Z} \psi(E)) \) for all \( x \in X \). Therefore \( F \subset \eta(B) \oplus_{Y \cap Z} \psi(E) \) by Lemma 2.1(iv). \( \square \)

**Definition 2.7.** Let \( C \) be a class of C*-algebras. A \( C(Z) \)-algebra \( E \) is called \( C \)-elementary if there is a finite partition of \( Z \) into closed subsets \( Z_1, \ldots, Z_r \) (\( r \geq 1 \)) and there exist C*-algebras \( D_1, \ldots, D_r \) in \( C \) such that \( E \cong \bigoplus_{i=1}^r C(Z_i) \otimes D_i \). The notion of category of a \( C(X) \)-algebra with respect to a class \( C \) is defined inductively: if \( A \) is \( C \)-elementary then \( \text{cat}_C(A) = 0 \); \( \text{cat}_C(A) \leq n \) if there are closed subsets \( Y \) and \( Z \) of \( X \) with \( Y = Z \) and there exist \( C(Y) \)-algebra \( B \) such that \( \text{cat}_C(B) \leq n - 1 \), a \( C \)-elementary \( C(Z) \)-algebra \( E \) and an \( \ast \)-monomorphism of \( C(Y \cap Z) \)-algebras \( \gamma : E(Y \cap Z) \to B(Y \cap Z) \) such that \( A \) is isomorphic to \( B \oplus_{\pi, \gamma} E \) where \( \pi \) is isomorphic to \( \pi(Y \cap Z) \).

By definition \( \text{cat}_C(A) = n \) if \( n \) is the smallest number with the property that \( \text{cat}_C(A) \leq n \). If no such \( n \) exists, then \( \text{cat}_C(A) = \infty \).

**Definition 2.8.** Let \( C \) be a class of C*-algebras and let \( A \) be a \( C(X) \)-algebra. An \( n \)-fibered \( C \)-monomorphism \( (\psi_0, \ldots, \psi_n) \) into \( A \) consists of \( (n + 1) \ast \)-monomorphisms of \( C(X) \)-algebras \( \psi_i : E_i \to A(Y_i) \), where \( Y_0, \ldots, Y_n \) is a closed cover of \( X \), each \( E_i \) is a \( C \)-elementary \( C(Y_i) \)-algebra and \( \pi \) in Definition 2.7 consists of stable Kirchberg algebras. I f \( \text{cat}_C(B) \leq n - 1 \), a \( C \)-elementary \( C(Z) \)-algebra \( E \) and an \( \ast \)-monomorphism of \( C(Y \cap Z) \)-algebras \( \gamma : E(Y \cap Z) \to B(Y \cap Z) \) such that \( A \) is isomorphic to \( B \oplus_{\pi, \gamma} E \).

Given an \( n \)-fibered morphism into \( A \) we have an associated continuous \( C(X) \)-algebra defined as the fibered product (or pullback) of the \( \ast \)-monomorphisms \( \psi_i \):

\[
A(\psi_0, \ldots, \psi_n) = \{(d_0, \ldots, d_n) : d_i \in E_i, \pi_{Y_0 \cap Y_i} \psi_i(d_i) = \pi_{Y_0 \cap Y_j} \psi_j(d_j) \text{ for all } i, j\}
\]

and an induced \( C(X) \)-monomorphism (defined by using Lemma 2.4)

\[
\eta = \eta(\psi_0, \ldots, \psi_n) : A(\psi_0, \ldots, \psi_n) \to A \subset \bigoplus_{i=0}^n A(Y_i),
\]

\[
\eta(d_0, \ldots, d_n) = (\psi_0(d_0), \ldots, \psi_n(d_n)).
\]

There are natural coordinate maps \( p_i : A(\psi_0, \ldots, \psi_n) \to E_i, p_i(d_0, \ldots, d_n) = d_i \). Let us set \( X_k = Y_k \cup \cdots \cup Y_n \). Then (\( \psi_k, \ldots, \psi_n \)) is an \( (n - k) \)-fibered \( C \)-monomorphism into \( A(X_k) \). Let \( \eta_k : A(X_k)(\psi_k, \ldots, \psi_n) \to A(X_k) \) be the induced map and set \( B_k = A(X_k)(\psi_k, \ldots, \psi_n) \). Let us note that \( B_0 = A(\psi_0, \ldots, \psi_n) \) and that there are natural \( C(X_{k-1}) \)-isomorphisms

\[
B_{k-1} \cong B_k \oplus_{\pi \psi_k} E_{k-1} \cong B_k \oplus_{\pi, \gamma_k} E_{k-1}.
\]

where \( \pi \) stands for \( \pi_{X_k \cap Y_{k-1}} \) and \( \gamma_k : E_{k-1}(X_k \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1}) \) is defined by \( (\gamma_k)_x = (\eta_k)^{-1}(\psi_k)_x \), for all \( x \in X_k \cap Y_{k-1} \). In particular, this decomposition shows that \( \text{cat}_C(A(\psi_0, \ldots, \psi_n)) \leq n \).

**Lemma 2.9.** Suppose that the class \( C \) from Definition 2.7 consists of stable Kirchberg algebras. If \( A \) is a \( C(X) \)-algebra over a compact metrizable space \( X \) such that \( \text{cat}_C(A) < \infty \), then \( A \) contains a full properly infinite projection and \( A \cong A \otimes O_\infty \otimes K \).
Proof. We prove this by induction on \( n = \text{cat}_C(A) \). The case \( n = 0 \) is immediate since \( D \cong D \otimes O_\infty \) for any Kirchberg algebra \( D \) \cite{19}. Let \( A = B \otimes_{\pi, \gamma} E \) where \( B, E \) and \( \gamma \) are as in Definition 2.7 with \( \text{cat}_C(B) = n - 1 \) and \( \text{cat}_C(E) = 0 \). Let us consider the exact sequence \( 0 \to J \to A \to E \to 0 \), where \( J = \{ b \in B : \pi_{Y \cap Z}(b) = 0 \} \). Since \( J \) is an ideal of \( B \cong B \otimes O_\infty \otimes K \), \( J \) absorbs \( O_\infty \otimes K \) by \cite{22} Prop. 8.5. Since both \( E \) and \( J \) are stable and purely infinite, it follows that \( A \) is stable by \cite{30} Prop. 6.12 and purely infinite by \cite{22} Prop. 3.5. Since \( A \) has Hausdorff primitive spectrum, \( A \) is strongly purely infinite by \cite{3} Thm. 5.8]. It follows that \( A \cong A \otimes O_\infty \) by \cite{22} Thm. 9.1]. Finally \( A \) contains a full properly infinite projection since there is a full embedding of \( O_2 \) into \( A \) by \cite{3} Prop. 5.6]. \( \square \)

3. Semiprojectivity

In this section we study the notion of \( KK \)-semiprojectivity. The main result is Theorem 3.12. Let \( A \) and \( B \) be \( C^* \)-algebras. Two \( * \)-homomorphisms \( \varphi, \psi : A \to B \) are approximately unitarily equivalent, written \( \varphi \approx_u \psi \), if there is a sequence of unitaries \( (u_n) \) in the \( C^* \)-algebra \( B^+ = B + C1 \) obtained by adjoining a unit to \( B \), such that \( \lim_{n \to \infty} \| u_n\varphi(a)u_n^* - \psi(a) \| = 0 \) for all \( a \in A \). We say that \( \varphi \) and \( \psi \) are asymptotically unitarily equivalent, written \( \varphi \approx_{au} \psi \), if there is a norm continuous unitary valued map \( t \mapsto u_t \in B^+ \), \( t \in [0,1) \), such that \( \lim_{t \to 1} \| u_t\varphi(a)u_t^* - \psi(a) \| = 0 \) for all \( a \in A \). A \( * \)-homomorphism \( \varphi : D \to A \) is full if \( \varphi(d) \) is not contained in any proper two-sided closed ideal of \( A \) if \( d \in D \) is nonzero.

We shall use several times Kirchberg’s Theorem \cite{28} Thm. 8.3.3 and the following theorem of Phillips \cite{25}.

**Theorem 3.1.** Let \( A \) and \( B \) be separable \( C^* \)-algebras such that \( A \) is simple and nuclear, \( B \cong B \otimes O_\infty \), and there exist full projections \( p \in A \) and \( q \in B \). For any \( \sigma \in KK(A,B) \) there is a full \( * \)-homomorphism \( \varphi : A \to B \) such that \( KK(\varphi) = \sigma \). If \( K_0(\sigma)[p] = [q] \), then \( \varphi(p) = q \). If \( \psi : A \to B \) is another \( * \)-homomorphism such that \( KK(\psi) = KK(\varphi) \) and \( \psi(p) = q \), then \( \varphi \approx_{au} \psi \) via a path of unitaries \( t \mapsto u_t \in U(qBq) \).

Theorem 3.1 does not appear in this form in \cite{28} but it is an immediate consequence of \cite{28} Thm. 4.1.1. Since \( pAp \otimes K \cong A \otimes K \) and \( qBq \otimes K \cong B \otimes K \) by \cite{3}, and \( qBq \otimes O_\infty \cong qBq \) by \cite{22} Prop. 8.5, it suffices to discuss the case when \( p \) and \( q \) are the units of \( A \) and \( B \). If \( \sigma \) is given, \cite{28} Thm. 4.1.1 yields a full \( * \)-homomorphism \( \varphi : A \to B \otimes K \) such that \( KK(\varphi) = \sigma \). Let \( e \in K \) be a rank-one projection and suppose that \( \varphi(1_A) = [1_B \otimes e] \) in \( K_0(B) \). Since both \( \varphi(1_A) \) and \( 1_B \otimes e \) are full projections and \( B \cong B \otimes O_\infty \), it follows by \cite{28} Lemma 2.1.8] that \( u\varphi(1_A)u^* = 1_B \otimes e \) for some unitary in \( (B \otimes K)^+ \). Replacing \( u \) by \( u \varphi u^* \) we can arrange that \( KK(\varphi) = \sigma \) and \( \varphi \) is unital. For the second part of the theorem let us note that any unital \( * \)-homomorphism \( \varphi : A \to B \) is full and if two unital \( * \)-homomorphisms \( \varphi, \psi : A \to B \) are asymptotically unitarily equivalent when regarded as maps into \( B \otimes K \), then \( \varphi \approx_{au} \psi \) when regarded as maps into \( B \), by an argument from the proof of \cite{28} Thm. 4.1.4].

A separable nonzero \( C^* \)-algebra \( D \) is semiprojective \cite{11} if for any separable \( C^* \)-algebra \( A \) and any increasing sequence of two-sided closed ideals \( (J_n) \) of \( A \) with \( J = \bigcup_n J_n \), the natural map \( \lim_n \text{Hom}(D, A/J_n) \to \text{Hom}(D, A/J) \) (induced by \( \pi_n : A/J_n \to A/J \)) is surjective. If we weaken this condition and require only that the above map has dense range, where \( \text{Hom}(D, A/J) \) is given the point-norm topology, then \( D \) is called weakly semiprojective \cite{14}. These definitions do not
Examples 3.2. (Weakly semiprojective C*-algebras) Any finite dimensional C*-algebra is semiprojective. A Kirchberg algebra $D$ satisfying the UCT and having finitely generated K-theory groups is weakly semiprojective by work of Neubüser [29], H. Lin [24] and Spielberg [32]. This also follows from Theorem 3.12 and Proposition 3.14 below. If in addition $K_1(D)$ is torsion free, then $D$ is semiprojective as proved by Spielberg [33] who extended the foundational work of Blackadar [1] and Szymanski [34].

The following generalizations of two results of Loring [25] are used in section 5, see [10].

Proposition 3.3. Let $D$ be a separable semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective $*$-homomorphism, and let $\varphi : D \to B$ and $\gamma : D \to A$ be $*$-homomorphisms such that $\|\pi\gamma(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$. Then there is a $*$-homomorphism $\psi : D \to A$ such that $\pi\psi = \varphi$ and $\|\gamma(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.

Proposition 3.4. Let $D$ be a separable semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. For any two $*$-homomorphisms $\varphi, \psi : D \to B$ such that $\|\varphi(d) - \psi(d)\| < \delta$ for all $d \in \mathcal{G}$, there is a homotopy $\Phi \in \text{Hom}(D, C[0,1] \otimes B)$ such that $\Phi_0 = \varphi$ to $\Phi_1 = \psi$ and $\|\varphi(c) - \Phi(t)(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ and $t \in [0,1]$.

Definition 3.5. A separable C*-algebra $D$ is $KK$-stable if there is a finite set $\mathcal{G} \subset D$ and there is $\delta > 0$ with the property that for any two $*$-homomorphisms $\varphi, \psi : D \to A$ such that $\|\varphi(a) - \psi(a)\| < \delta$ for all $a \in \mathcal{G}$, one has $KK(\varphi) = KK(\psi)$.

Corollary 3.6. Any semiprojective C*-algebra is weakly semiprojective and $KK$-stable.

Proof. This follows from Proposition 3.3.

Proposition 3.7. Let $D$ be a separable weakly semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$ there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ such that for any C*-algebras $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. If in addition $D$ is $KK$-stable, then we can choose $\mathcal{G}$ and $\delta$ such that we also have $KK(\varphi) = KK(\psi)$.

Proof. This follows from [14 Thms. 3.1, 4.6]. Since the result is essential to us we include a short proof. Fix $\mathcal{F}$ and $\varepsilon$. Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of $D$ whose union is dense in $D$. If the statement is not true, then there are sequences of C*-algebras $C_n \subset A_n$ and $*$-homomorphisms $\varphi_n : D \to A_n$ satisfying $\varphi_n(\mathcal{G}_n) \subset_{1/n} C_n$ and with the property that for any $n \geq 1$ there is no $*$-homomorphism $\psi_n : D \to C_n$ such that $\|\varphi_n(c) - \psi_n(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Set $B_i = \prod_{n \geq i} A_n$ and $E_i = \prod_{n \geq i} C_n \subset B_i$. If $\nu_i : B_i \to B_{i+1}$ is the natural projection, then $\nu_i(E_i) = E_{i+1}$. Let us observe that if we define $\Phi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \ldots)$, then the image of $\Phi = \lim_{i} \Phi_i : D \to \lim_{i} (B_i, \nu_i)$ is contained in $\lim_{i} (E_i, \nu_i)$. Since $D$ is weakly semiprojective,
there is \( i \) and a \(*\)-homomorphism \( \Psi_i : D \to E_i \) of the form \( \Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \ldots) \) such that \( \|\Phi_i(c) - \Psi_i(c)\| < \varepsilon \) for all \( c \in F \). Therefore \( \|\varphi_i(c) - \psi_i(c)\| < \varepsilon \) for all \( c \in F \) which gives a contradiction. \( \square \)

It is useful to combine Propositions 3.7 and 3.3 in a single statement.

**Proposition 3.8.** Let \( D \) be a separable semiprojective C*-algebra. For any finite set \( F \subset D \) and any \( \varepsilon > 0 \), there exists a finite set \( G \subset D \) and \( \delta > 0 \) with the following property. Let \( \pi : A \to B \) be a surjective \(*\)-homomorphism which maps a C*-subalgebra \( A' \) of \( A \) onto a C*-subalgebra \( B' \) of \( B \). Let \( \varphi : D \to B' \) and \( \gamma : D \to A' \) be \(*\)-homomorphisms such that \( \gamma(G) \subset A' \) and \( \|\gamma(d) - \varphi(d)\| < \delta \) for all \( d \in G \). Then there is a \(*\)-homomorphism \( \psi : D \to A' \) such that \( \pi \psi = \varphi \) and \( \|\gamma(c) - \psi(c)\| < \varepsilon \) for all \( c \in F \).

**Proof.** Let \( G_L \) and \( \delta_L \) be given by Proposition 3.3 applied to the input data \( F \) and \( \varepsilon/2 \). We may assume that \( F \subset G_L \) and \( \varepsilon > \delta_L \). Next, let \( G_P \) and \( \delta_P \) be given by Proposition 3.7 applied to the input data \( G_L \) and \( \delta_L/2 \). We show now that \( G := G_L \cup G_P \) and \( \delta := \min\{\delta_P, \delta_L/2\} \) have the desired properties. We have \( \gamma(G_P) \subset A' \) since \( G_P \subset G \) and \( \delta \leq \delta_P \). By Proposition 3.7 there is a \(*\)-homomorphism \( \gamma' : D \to A' \) such that \( \|\gamma'(d) - \gamma(d)\| < \delta_L/2 \) for all \( d \in G_L \). Then, since \( G_L \subset G \) and \( \delta \leq \delta_L/2 \),

\[
\|\pi \gamma'(d) - \varphi(d)\| \leq \|\pi \gamma'(d) - \pi \gamma(d)\| + \|\pi \gamma(d) - \varphi(d)\| < \delta_L/2 + \delta \leq \delta_L
\]

for all \( d \in G_L \). Therefore we can invoke Proposition 3.3 to perturb \( \gamma' \) to a \(*\)-homomorphism \( \psi : D \to A' \) such that \( \pi \psi = \varphi \) and \( \|\gamma'(d) - \psi(d)\| < \varepsilon/2 \) for all \( d \in F \). Finally we observe that for \( d \in F \subset G_L \)

\[
\|\gamma(d) - \psi(d)\| \leq \|\gamma(d) - \gamma'(d)\| + \|\gamma'(d) - \psi(d)\| < \delta_L/2 + \varepsilon/2 < \varepsilon.
\]

\( \square \)

**Definition 3.9.** (a) A separable C*-algebra \( D \) is **KK-semiprojective** if for any separable C*-algebra \( A \) and any increasing sequence of two-sided closed ideals \( (J_n) \) of \( A \) with \( J = \bigcup_n J_n \), the natural map \( \lim KK(D, A/J_n) \to KK(D, A/J) \) is surjective.

(b) We say that the functor \( KK(D, -) \) is **continuous** if for any inductive system \( B_1 \to B_2 \to \ldots \) of separable C*-algebras, the induced map \( \lim KK(D, B_n) \to KK(D, \lim B_n) \) is bijective.

**Proposition 3.10.** Any separable KK-semiprojective C*-algebra is KK-stable.

**Proof.** We shall prove the statement by contradiction. Let \( D \) be separable KK-semiprojective C*-algebra. Let \( (G_n) \) be an increasing sequence of finite subsets of \( D \) whose union is dense in \( D \). If the statement is not true, then there are sequences of \(*\)-homomorphisms \( \varphi_n, \psi_n : D \to A_n \) such that \( \|\varphi_n(d) - \psi_n(d)\| < 1/n \) for all \( d \in G_n \) and yet \( KK(\varphi_n) \neq KK(\psi_n) \) for all \( n \geq 1 \). Set \( B_i \equiv \prod_{n \geq i} A_n \) and let \( \nu_i : B_i \to B_{i+1} \) be the natural projection. Let us define \( \Phi_i, \Psi_i : D \to B_i \) by \( \Phi_i(d) = (\varphi_1(d), \varphi_{i+1}(d), \ldots) \) and \( \Psi_i(d) = (\psi_1(d), \psi_{i+1}(d), \ldots) \), for all \( d \in D \). Let \( B_i' \) be the separable C*-subalgebra of \( B_i \) generated by the images of \( \Phi_i \) and \( \Psi_i \). Then \( \nu_i(B_i') = B_{i+1}' \) and one verifies immediately that \( \lim \Phi_i = \lim \Psi_i : D \to \lim (B_i', \nu_i) \). Since \( D \) is KK-semiprojective, we must have \( KK(\Phi_i) = KK(\Psi_i) \) for some \( i \) and hence \( KK(\varphi_n) = KK(\psi_n) \) for all \( n \geq i \). This gives a contradiction. \( \square \)
Proposition 3.11. A unital Kirchberg algebra $D$ is $KK$-stable if and only if $D \otimes K$ is $KK$-stable. $D$ is weakly semiprojective if and only if $D \otimes K$ is weakly semiprojective.

Proof. Since $KK(D, A) \cong KK(D, A \otimes K) \cong KK(D \otimes K, A \otimes K)$ the first part of the proposition is immediate. Suppose now that $D \otimes K$ is weakly semiprojective. Then $D$ is weakly semiprojective as shown in the proof of [32, Thm. 2.2]. Conversely, assume that $D$ is weakly semiprojective. It suffices to find $\alpha \in \text{Hom}(D \otimes K, D)$ and a sequence $(\beta_n)$ in $\text{Hom}(D, D \otimes K)$ such that $\beta_n \alpha$ converges to $id_{D \otimes K}$ in the point-norm topology. Let $s_i$ be the canonical generators of $O_\infty$. If $(e_{ij})$ is a system of matrix units for $K$, then $\lambda(e_{ij}) = s_i s_j^*$ defines a $*$-homomorphism $K \rightarrow O_\infty$ such that $KK(\lambda) \in KK(K, O_\infty)^{-1}$. Therefore, by composing $id_D \otimes \lambda$ with some isomorphism $D \otimes O_\infty \cong D$ (given by [29, Thm. 7.6.6]) we obtain a $*$-monomorphism $\alpha : D \otimes K \rightarrow D$ which induces a $KK$-equivalence. Let $\beta : D \rightarrow D \otimes K$ be defined by $\beta(d) = d \otimes e_{11}$. Then $\beta \alpha \in \text{End}(D \otimes K)$ induces a $KK$-equivalence and hence after replacing $\beta$ by $\theta \beta$ for some automorphism $\theta$ of $D \otimes K$, we may arrange that $KK(\beta \alpha) = KK(id_D)$. By Theorem 3.1, $\beta \alpha \approx_u id_{D \otimes K}$, so that there is a sequence of unitaries $u_n \in (D \otimes K)^+$ such that $u_n \beta \alpha (-) u_n^*$ converges to $id_{D \otimes K}$.

Theorem 3.12. For a separable C*-algebra $D$ consider the following properties:

(i) $D$ is $KK$-semiprojective.

(ii) The functor $KK(D, -)$ is continuous.

(iii) $D$ is weakly semiprojective and $KK$-stable.

Then (i) $\Leftrightarrow$ (ii). Moreover, (iii) $\Rightarrow$ (i) if $D$ is nuclear and (i) $\Rightarrow$ (iii) if $D$ is a Kirchberg algebra. Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) for any Kirchberg algebra $D$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. (i) $\Rightarrow$ (ii): Let $(B_n, \gamma_{n,m})$ be an inductive system with inductive limit $B$ and let $\gamma_n : B_n \rightarrow B$ be the canonical maps. We have an induced map $\beta : \varinjlim KK(D, B_n) \rightarrow KK(D, B)$. First we show that $\beta$ is surjective. The mapping telescope construction of L. G. Brown (as described in the proof of [1, Thm. 3.1]) produces an inductive system of C*-algebras $(T_n, \eta_{n,m})$ with inductive limit $B$ such that each $\eta_{n,n+1}$ is surjective, and each canonical map $\eta_n : T_n \rightarrow B$ is homotopic to $\gamma_n \alpha_n$ for some $*$-homomorphism $\alpha_n : T_n \rightarrow B_n$. In particular $KK(\eta_n) = KK(\gamma_n)KK(\alpha_n)$. Let $x \in KK(D, B)$. By (i) there are $n$ and $y \in KK(D, T_n)$ such that $KK(\eta_n)y = x$ and hence $KK(\gamma_n)KK(\alpha_n)y = x$. Thus $z = KK(\alpha_n)y \in KK(D, B_n)$ is a lifting of $x$. Let us show now that the map $\beta$ is injective. Let $x$ be an element in the kernel of the map $KK(D, B_n) \rightarrow KK(D, B)$. Consider the commutative diagram whose exact rows are portions of the Puppe sequence in KK-theory [2, Thm. 19.4.3] and with vertical maps induced by $\gamma_m : B_m \rightarrow B$, $m \geq n$.

$$
\begin{array}{ccc}
KK(D, C_{\gamma_n}) & \rightarrow & KK(D, B_n) \\
\downarrow & & \downarrow \\
KK(D, C_{\gamma_m}) & \rightarrow & KK(D, B_m)
\end{array}
$$

By exactness, $x$ is the image of some element $y \in KK(D, C_{\gamma_m})$. Since $C_{\gamma_m} = \varinjlim C_{\gamma_{m,n}}$, the map $\varinjlim KK(D, C_{\gamma_{m,n}}) \rightarrow KK(D, C_{\gamma_m})$ is surjective by the first part of the proof. Therefore there is $m \geq n$ such that $y$ lifts to some $z \in KK(D, C_{\gamma_m})$. The image of $z$ in $KK(D, B_m)$ equals $KK(\gamma_m)x$ and vanishes by exactness of the bottom row.
(iii) ⇒ (i): Let $A$, $(J_n)$ and $J$ be as in Definition 3.9. Using the five-lemma and the split exact sequence $0 \to KK(D, A) \to KK(D, A^\ast) \to KK(D, \mathbb{C}) \to 0$, we reduce the proof to the case when $A$ is unital. Let $x \in KK(D, A/J)$. Since the map $KK(D^+, A/J) \to KK(D, A/J)$ is surjective, $x$ lifts to some element $x^+ \in KK(D^+, A/J)$. By [29, Thm. 8.3.3], since $D^+$ is nuclear, there is a $*$-homomorphism $\Phi : D^+ \to A/J \otimes \mathcal{O}_\infty \otimes K$ such that $KK(\Phi) = x^+$ and hence if set $\varphi = \Phi|_D$, then $KK(\varphi) = x$. Since $D$ is weakly semiprojective, there are $n$ and a $*$-homomorphism $\psi : D \to A/J_n \otimes \mathcal{O}_\infty \otimes K$ such that $\|\pi_n \psi(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$, where $\mathcal{G}$ and $\delta$ are as in the definition of $KK$-stability. Therefore $KK(\pi_n \psi) = KK(\varphi)$ and hence $KK(\varphi)$ is a lifting of $x$ to $KK(D, A/J_n)$.

(i) ⇒ (iii): $D$ is $KK$-stable by Proposition 3.10. It remains to show that $D$ is weakly semi-projective. Since any nonunital Kirchberg algebra is isomorphic to the stabilization of a unital one (see [29, Prop. 4.1.3]) and since by Proposition 3.11 $D$ is $KK$-semiprojective if and only if $D \otimes K$ is $KK$-semiprojective, we may assume that $D$ is unital. Let $A$, $(J_n)$, $\pi_{m,n} : A/J_m \to A/J_n (m \leq n)$ and $\pi_n : A/J_n \to A/J$ be as in the definition of weak semiprojectivity. By [1, Cor. 2.15], we may assume that $A$ and the $*$-homomorphism $\varphi : D \to A$ (for which we want to construct an approximative lifting) are unital. In particular $\varphi$ is injective since $D$ is simple. Set $B = \varphi(D) < A/J$ and $B_n = \pi_n^{-1}(B) \subset A/J_n$. The corresponding maps $\pi_{m,n} : B_m \to B_n (m \leq n)$ and $\pi_n : B_n \to B$ are surjective and they induce an isomorphism $\lim\limits_{\to} (B_n, \pi_{n,n+1}) \cong B$.

Given $\varepsilon > 0$ and $\mathcal{F} \subset D$ (a finite set) we are going to produce an approximate lifting $\varphi_n : D \to B_n$ for $\varphi$. Since $1_B$ is a properly infinite projection, it follows by [1, Props. 2.18 and 2.23] that the unit $1_n$ of $B_n$ is a properly infinite projection, for all sufficiently large $n$. Since $D$ is $KK$-semiprojective, there exist $m$ and an element $h \in KK(D, B_m)$ which lifts $KK(\varphi)$ such that $K_0(h)[1_D] = [1_m]$. By [29, Thm. 8.3.3], there is a full $*$-homomorphism $\eta : D \to B_m \otimes K$ such that $KK(\eta) = h$. By [29, Prop. 4.1.4], since both $\eta(1_D)$ and $1_m$ are full and properly infinite projections in $B_m \otimes K$, there is a partial isometry $w \in B_m \otimes K$ such that $w^*w = \eta(1_D)$ and $ww^* = 1_m$. Replacing $\eta$ by $\eta(-)w^*$, we may assume that $\eta : D \to B_m$ is unital. Then $KK(\pi_m \eta) = KK(\pi_m)h = KK(\varphi)$. By Theorem 3.11 $\pi_m \eta \approx_{uh} \varphi$. Thus there is a unitary $u \in B$ such that $\|\pi_m \eta(d)u^* - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$. Since $C(\mathbb{T})$ is semiprojective, there is $n \geq m$ such that $u$ lifts to a unitary $u_n \in B_n$. Then $\varphi_n := u_n \pi_{m,n} \eta(-)u_n^*$ is a $*$-homomorphism from $D$ to $B_n$ such that $\|\pi_n \varphi_n(d) - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$.

**Corollary 3.13.** Any separable nuclear semi-projective $C^*$-algebra is $KK$-semi-projective.

**Proof.** This is very similar to the proof of the implication (iii) ⇒ (i) of Theorem 3.12. Alternatively, the statement follows from Corollary 3.11 and Theorem 3.12. □

Blackadar has shown that a semi-projective Kirchberg algebra satisfying the UCT has finitely generated K-theory groups [29, Prop. 8.4.15]. A similar argument gives the following:

**Proposition 3.14.** Let $D$ be a separable $C^*$-algebra satisfying the UCT. Then $D$ is $KK$-semi-projective if and only if $K_*(D)$ is finitely generated.

**Proof.** If $K_*(D)$ is finitely generated, then $D$ is $KK$-semi-projective by 3.11. Conversely, assume that $D$ is $KK$-semi-projective. Since $D$ satisfies the UCT, we infer that if $G = K_i(D)$ ($i = 0, 1$), then $G$ is semi-projective in the category of countable abelian groups, in the sense that if $H_1 \to H_2 \to \cdots$ is an inductive system of countable abelian groups with inductive limit $H$, then the natural map
\[ \text{lim} \text{ Hom}(G, H_n) \rightarrow \text{Hom}(G, H) \text{ is surjective. This implies that } G \text{ is finitely generated. Indeed, taking } H = G, \text{ we see that } \text{id}_G \text{ lifts to } \text{Hom}(G, H_n) \text{ for some finitely generated subgroup } H_n \text{ of } G \text{ and hence } G \text{ is a quotient of } H_n. \]

4. Approximation of \( C(X) \)-algebras

In this section we use weak semiprojectivity to approximate a continuous \( C(X) \)-algebra \( A \) by \( C(X) \)-subalgebras given by pullbacks of \( n \)-fibered monomorphisms into \( A \).

**Lemma 4.1.** Let \( D \) be a finite direct sum of simple \( C^* \)-algebras and let \( \varphi, \psi : D \rightarrow A \) be \( * \)-homomorphisms. Suppose that \( H \subset D \) contains a nonzero element from each simple direct summand of \( D \). If \( \| \psi(d) - \varphi(d) \| \leq \| d \| /2 \) for all \( d \in H \), then \( \varphi \) is injective if and only if \( \psi \) is injective.

**Proof.** Let \( \epsilon > 0 \) be given. By hypothesis there exist \( \{ a_1, \ldots, a_r \} \subset D \) and a \( * \)-monomorphism \( \iota : D \rightarrow A(x) \) such that \( \| \pi_x(a_i) - \iota(c_i) \| < \epsilon / 2 \) for all \( i = 1, \ldots, r \). Let \( U_n = \{ y \in X : d(x, y) \leq 1/n \} \). Choose a full element \( d_j \) in each direct summand of \( D \). Since \( D \) is weakly semiprojective, there is a \( * \)-homomorphism \( \varphi : D \rightarrow A(U_n) \) (for some \( n \)) such that \( \| \pi_x \sigma(c_i) - \iota(c_i) \| < \epsilon / 2 \) for all \( i = 1, \ldots, r \), and \( \| \pi_x \varphi(d_j) - \iota(d_j) \| \leq \| d_j \| /2 \) for all \( d_j \). Therefore

\[ \| \pi_x \varphi(c_i) - \pi_x(a_i) \| \leq \| \pi_x \varphi(c_i) - \iota(c_i) \| + \| \pi_x(a_i) - \iota(c_i) \| < \epsilon / 2 + \epsilon / 2 = \epsilon \]

and \( \varphi \) is injective by Lemma 4.1. If \( \varphi_U \varphi \), after increasing \( n \) and setting \( U = U_n \), we have

\[ \| \varphi_i(c_i) - \pi_U(a_i) \| = \| \pi_U \varphi_i(c_i) - a_i \| < \epsilon, \]

for all \( i = 1, \ldots, r \). This shows that \( \pi_U(F) \subset \varphi(D) \). If \( A \) is continuous, then after shrinking \( U \) we may arrange that \( \| \varphi \circ (d_j) \| \geq \| \varphi \circ (d_j) \| /2 \) for all \( d_j \) and all \( z \in U \). This implies that \( \varphi \) in injective for all \( z \in U \). \( \Box \)

**Lemma 4.2.** Let \( C \) be a class consisting of finite direct sums of separable simple weakly semiprojective \( C^* \)-algebras. Let \( X \) be a compact metrizable space and let \( A \) be a \( C(X) \)-algebra. Let \( F \subset A \) be a finite set, let \( \epsilon > 0 \) and suppose that \( A(x) \) admits an exhaustive sequence of \( C^* \)-algebras isomorphic to \( C^* \)-algebras in \( C \) for some \( x \in X \). Then there exist a compact neighborhood \( U \) of \( x \) and a \( * \)-homomorphism \( \varphi : D \rightarrow A(U) \) for some \( D \in C \) such that \( \pi_U(F) \subset \varphi(D) \). If \( A \) is a continuous \( C(X) \)-algebra, then we may arrange that \( \varphi \) is injective for all \( z \in U \).

**Proof.** Let \( F = \{ a_1, \ldots, a_r \} \) and \( \epsilon \) be given. By hypothesis there exist \( D \in C, \{ c_1, \ldots, c_r \} \subset D \) and a \( * \)-monomorphism \( \iota : D \rightarrow A(x) \) such that \( \| \pi_x(a_i) - \iota(c_i) \| < \epsilon / 2 \) for all \( i = 1, \ldots, r \). Set \( U_n = \{ y \in X : d(x, y) \leq 1/n \} \). Choose a full element \( d_j \) in each direct summand of \( D \). Since \( D \) is weakly semiprojective, there is a \( * \)-homomorphism \( \varphi : D \rightarrow A(U_n) \) (for some \( n \)) such that \( \| \pi_x \sigma(c_i) - \iota(c_i) \| < \epsilon / 2 \) for all \( i = 1, \ldots, r \), and \( \| \pi_x \varphi(d_j) - \iota(d_j) \| \leq \| d_j \| /2 \) for all \( d_j \). Therefore

\[ \| \pi_x \varphi(c_i) - \pi_x(a_i) \| \leq \| \pi_x \varphi(c_i) - \iota(c_i) \| + \| \pi_x(a_i) - \iota(c_i) \| < \epsilon / 2 + \epsilon / 2 = \epsilon \]

and \( \varphi \) is injective by Lemma 4.1. By Lemma 2.1(i), after increasing \( n \) and setting \( U = U_n \), we have

\[ \| \varphi(c_i) - \pi_U(a_i) \| = \| \pi_U \varphi(c_i) - a_i \| < \epsilon, \]

for all \( i = 1, \ldots, r \). This shows that \( \pi_U(F) \subset \varphi(D) \). If \( A \) is continuous, then after shrinking \( U \) we may arrange that \( \| \varphi \circ (d_j) \| \geq \| \varphi \circ (d_j) \| /2 \) for all \( d_j \) and all \( z \in U \). This implies that \( \varphi \) in injective for all \( z \in U \). \( \Box \)

**Lemma 4.3.** Let \( X \) be a compact metrizable space and let \( A \) be a separable continuous \( C(X) \)-algebra the fibers of which are stable Kirchberg algebras. Let \( F \subset A \) be a finite set and let \( \epsilon > 0 \). Suppose that there exist a KK-semiprojective stable Kirchberg algebra \( D \) and \( \sigma \in KK(D, A(x))^{-1} \) for some \( x \in X \). Then there exist a closed neighborhood \( U \) of \( x \) and a full \( * \)-homomorphism \( \psi : D \rightarrow A(U) \) such that \( KK(\psi) = \sigma_U \) and \( \pi_U(F) \subset \psi(D) \).
This shows that \( \pi \psi \). We extend \( f \) and hence such that \( \| \pi \psi(x) - \pi \psi_0(d) \| < \varepsilon \) for all \( d \in \mathcal{H} \) and \( KK(\pi \psi_0) = KK(\pi \psi) = \pi \sigma \). Since \( \lim_{m \to \infty} KK(D, A(U_m)) = KK(D, A(x)) \), we deduce that there is \( m \geq n \) such that \( KK(\pi U_m \psi_n) = \pi U_m \sigma U_m \). By increasing \( m \) we may arrange that \( \pi U_m (\mathcal{F}) \subset \pi \psi \psi_n(D) \) since we have seen that \( \pi U_m (\mathcal{F}) = \psi_0(\mathcal{H}) \subset \pi \psi \psi_n(D) \). We can arrange that \( \psi z \) is injective for all \( z \in U \) by reasoning as in the proof of Lemma 2.2. We conclude by setting \( U = U_m \) and \( \psi = \pi U_m \psi_n \).

The following lemma is useful for constructing fibered morphisms.

**Lemma 4.4.** Let \( (D_j)_{j \in J} \) be a finite family consisting of finite direct sums of weakly semiprojective simple C*-algebras. Let \( \varepsilon > 0 \) and for each \( j \in J \) let \( \mathcal{H}_j \subset D_j \) be a finite set such that for each direct summand of \( D_j \) there is an element of \( \mathcal{H}_j \) of norm \( \geq \varepsilon \) which is contained and is full in that summand. Let \( G_j \subset D_j \) and \( \delta_j > 0 \) be given by Proposition 3.7 applied to \( D_j \), \( \mathcal{H}_j \) and \( \varepsilon/2 \). Let \( X \) be a compact metrizable space, let \( (Z_j)_{j \in J} \) be disjoint nonempty closed subsets of \( X \) and let \( Y \) be a closed nonempty subset of \( X \) such that \( X = Y \cup ( \cup_j Z_j ) \). Let \( A \) be a continuous \( C(X) \)-algebra and let \( \mathcal{F} \) be a finite subset of \( A \). Let \( \eta : B(Y) \to A(Y) \) be a \(*\)-homomorphism of \( C(Y) \)-algebras and let \( \varphi_j : D_j \to A(Z_j) \) be \(*\)-homomorphisms such that \( \varphi_j x \) is injective for all \( x \in Z_j \) and \( j \in J \), and which satisfy the following conditions:

1. \( \pi Z_j(\mathcal{F}) \subset \varepsilon/2 \varphi_j(\mathcal{H}_j) \), for all \( j \in J \),
2. \( \pi(Y) \mathcal{F} \subset \varepsilon \eta(B) \),
3. \( \pi Y \cap Z_j \varphi_j(G_j) \subset \delta_j \pi Y \cap Z_j \eta(B) \), for all \( j \in J \).

Then, there are \( C(Z_j) \)-linear \(*\)-homomorphisms \( \psi_j : C(Z_j) \otimes D_j \to A(Z_j) \), satisfying

\[
\| \psi_j(c) - \psi_j(c) \| < \varepsilon/2, \quad \text{for all } c \in \mathcal{H}_j, \quad \text{and } j \in J,
\]

and such that if we set \( E = \bigoplus_j C(Z_j) \otimes D_j \), \( Z = \cup_j Z_j \), and \( \psi : E \to A(Z) = \bigoplus_j A(Z_j) \), \( \psi = \oplus_j \psi_j \), then \( \pi Z_j(\psi(E)) \subset \pi Y \cap Z_j \eta(B) \), \( \pi \mathcal{F} \subset \varepsilon \psi(E) \) and hence

\[
\mathcal{F} \subset \varepsilon \eta(B) \oplus Y \cap Z_j \psi(E) = \chi(B \oplus \pi \eta_0 \pi_0 E),
\]

where \( \chi \) is the isomorphism induced by the pair \( (\eta, \psi) \). If we assume that each \( D_j \) is KK-stable, then we also have \( KK(\varphi_j) = KK(\psi_j|D_j) \) for all \( j \in J \).

**Proof.** Let \( \mathcal{F} = \{ a_1, \ldots, a_r \} \subset A \) be as in the statement. By (i), for each \( j \in J \) we find \( \{ c_1^{(j)}, \ldots, c_r^{(j)} \} \subset \mathcal{H}_j \) such that \( \| \varphi_j(c_i^{(j)}) - \pi Z_j(a_i) \| < \varepsilon/2 \) for all \( i \). Consider the \( C(X) \)-algebra \( A \oplus \varepsilon \eta(B) \subset A \). From (iii), Lemma 2.1(iv) and Lemma 2.5 we obtain

\[
\varphi_j(G_j) \subset \delta_j \pi Z_j(A \oplus \varepsilon \eta(B)).
\]

Applying Proposition 3.7 we perturb \( \varphi_j \) to a \(*\)-homomorphism \( \psi_j : D_j \to \pi Z_j(A \oplus \varepsilon \eta(B)) \) satisfying (i), and hence such that \( \| \varphi_j(c^{(j)}_i) - \pi Z_j(a_i) \| < \varepsilon/2 \), for all \( i \). Therefore

\[
\| \psi_j(c_i^{(j)}) - \pi Z_j(a_i) \| < \| \psi_j(c_i^{(j)}) - \varphi_j(c_i^{(j)}) \| + \| \varphi_j(c_i^{(j)}) - \pi Z_j(a_i) \| < \varepsilon.
\]

This shows that \( \pi Z_j(\mathcal{F}) \subset \varepsilon \psi_j(D_j) \). From (i) and Lemma 4.4 we obtain that each \( \psi_j x \) is injective. We extend \( \psi_j \) to a \( C(Z_j) \)-linear \(*\)-homomorphism \( \psi_j : C(Z_j) \otimes D_j \to \pi Z_j(A \oplus \varepsilon \eta(B)) \) and then we
Finally, from (ii), (7) and Lemma 2.6 (c) we get
\[ \pi_Z(F) \subset \psi(E). \]
The property \( \psi(E) \subset (A \oplus Y \eta(B))(Z) \) is equivalent to \( \pi_{Y \cap Z}^Z(\psi(E)) \subseteq \pi_{Y \cap Z}^Y(\eta(B)) \) by Lemma 2.6(b).

Finally, from (ii), (7) and Lemma 2.6(c) we get \( F \subset \pi_Z(B) \oplus \pi_Z(E) \).

Let \( C \) be as in Lemma 4.2. Let \( A \) be a \( C(X) \)-algebra, let \( F \subset A \) be a finite set and let \( \varepsilon > 0 \).

An \((F, \varepsilon, C)\)-approximation of \( A \)

\[ \alpha = \{ F, \varepsilon, \{ U_i, \varphi_i : D_i \to A(U_i), \mathcal{H}_i, \mathcal{G}_i, \delta_i \}_{i \in I} \}, \]
is a collection with the following properties: \( \{ U_i \}_{i \in I} \) is a finite family of closed subsets of \( X \), whose interiors cover \( X \) and \( (D_i)_{i \in I} \) are \( C^* \)-algebras in \( C \); for each \( i \in I \), \( \varphi_i : D_i \to A(U_i) \) is a *-homomorphism such that \( (\varphi_i)_x \) is injective for all \( x \in U_i \); \( \mathcal{H}_i \subset D_i \) is a finite set such that \( \pi_{U_i}(F) \subset \varepsilon / 2, \varphi_i(\mathcal{H}_i) \) and such that for each direct summand of \( D_i \) there is an element of \( \mathcal{H}_i \) of norm \( \geq \varepsilon \) which is contained and is full in that summand; the finite set \( \mathcal{G}_i \subset D_i \) and \( \delta_i > 0 \) are given by Proposition 3.7 applied to the weakly semiprojective \( C^* \)-algebra \( D_i \) for the input data \( \mathcal{H}_i \) and \( \varepsilon / 2 \); if \( D_i \) is \( KK \)-stable, then \( \mathcal{G}_i \) and \( \delta_i \) are chosen such that the second part of Proposition 3.7 also applies.

**Lemma 4.5.** Let \( A \) and \( C \) be as in Lemma 4.2. Suppose that each fiber of \( A \) admits an exhaustive sequence of \( C^* \)-algebras isomorphic to \( C^* \)-algebras in \( C \). Then for any finite subset \( F \) of \( A \) and any \( \varepsilon > 0 \) there is an \((F, \varepsilon, C)\)-approximation of \( A \). Moreover, if \( A \), \( D \) and \( \sigma \) are as in Lemma 4.3 and \( \sigma_\varepsilon \in KK(D, A(x))^{-1} \) for all \( x \in X \), then there is an \((F, \varepsilon, C)\)-approximation of \( A \) such that \( C = \{ D \} \) and \( KK(\varphi_i) = \sigma_{U_i} \), for all \( i \in I \).

**Proof.** Since \( X \) is compact, this is an immediate consequence of Lemmas 4.2, 4.3 and Proposition 3.7. \( \square \)

It is useful to consider the following operation of restriction. Suppose that \( Y \) is a closed subspace of \( X \) and let \( (V_j)_{j \in J} \) be a finite family of closed subsets of \( Y \) which refines the family \( (Y \cap U_i)_{i \in I} \) and such that the interiors of the \( V_j \)'s form a cover of \( Y \). Let \( \iota : J \to I \) be a map such that \( V_j \subseteq Y \cap U_{\iota(j)} \). Define

\[ \iota^*(\alpha) = \{ \pi_Y(F), \varepsilon, \{ V_j, \pi_{V_j} \varphi_{\iota(j)} : D_{\iota(j)} \to A(V_j), \mathcal{H}_{\iota(j)}, \mathcal{G}_{\iota(j)}, \delta_{\iota(j)} \}_{j \in J} \}. \]

It is obvious that \( \iota^*(\alpha) \) is a \((\pi_Y(F), \varepsilon, C)\)-approximation of \( A(Y) \). The operation \( \alpha \mapsto \iota^*(\alpha) \) is useful even in the case \( X = Y \). Indeed, by applying this procedure we can refine the cover of \( X \) that appears in a given \((F, \varepsilon, C)\)-approximation of \( A \).

An \((F, \varepsilon, C)\)-approximation \( \alpha \) (as in (3)) is subordinated to an \((F', \varepsilon', C)\)-approximation, \( \alpha' = \{ F', \varepsilon', \{ U_{i'}, \varphi_{i'} : D_{i'} \to A(U_{i'}), \mathcal{H}_{i'}, \mathcal{G}_{i'}, \delta_{i'} \}_{i' \in I'} \}, \) written \( \alpha \prec \alpha' \), if

(i) \( F \subseteq F' \),

(ii) \( \varphi_{i}(\mathcal{G}_i) \subseteq \pi_{U_i}(F') \) for all \( i \in I \), and

(iii) \( \varepsilon' < \min \{ \{ \varepsilon_i \} \cup \{ \delta_i, i \in I \} \} \).

Let us note that, with notation as above, we have \( \iota^*(\alpha) \prec \iota^*(\alpha') \) whenever \( \alpha \prec \alpha' \).

The following theorem is the crucial technical result of our paper. It provides an approximation of continuous \( C(X) \)-algebras by subalgebras of category \( \leq \dim(X) \).
Theorem 4.6. Let $\mathcal{C}$ be a class consisting of finite direct sums of weakly semiprojective simple $C^*$-algebras. Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable continuous $C(X)$-algebra the fibers of which admit exhaustive sequences of $C^*$-algebras isomorphic to $C^*$-algebras in $\mathcal{C}$. For any finite set $F \subset \mathcal{X}$, any $\varepsilon > 0$ there exist $n \leq \dim(X)$ and an $n$-fibered $C$-monomorphism $(\psi_0, \ldots, \psi_n)$ into $A$ which induces a $*$-monomorphism $\eta : A(\psi_0, \ldots, \psi_n) \to A$ such that $F \subset_{\varepsilon} \eta(A(\psi_0, \ldots, \psi_n))$.

Proof. By Lemma 4.5, for any finite set $F \subset A$ and any $\varepsilon > 0$ there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$. Moreover, for any finite set $F \subset A$, any $\varepsilon > 0$ and any $n$, there is a sequence $\{\alpha_k : 0 \leq k \leq n\}$ of $(\mathcal{F}_k, \varepsilon, \mathcal{C})$-approximations of $A$ such that $(\mathcal{F}_0, \varepsilon_0) = (\mathcal{F}, \varepsilon)$ and $\alpha_k$ is subordinated to $\alpha_{k+1}$:

$$\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n.$$  

Indeed, assume that $\alpha_k$ was constructed. Let us choose a finite set $\mathcal{F}_{k+1}$ which contains $\mathcal{F}_k$ and liftings to $A$ of all the elements in $\bigcup_{i_k \in I_k} \varphi_{i_k}(\mathcal{G}_{i_k})$. This choice takes care of the above conditions (i) and (ii). Next we choose $\varepsilon_{k+1}$ sufficiently small such that (iii) is satisfied. Let $\alpha_{k+1}$ be an $(\mathcal{F}_{k+1}, \varepsilon_{k+1}, \mathcal{C})$-approximation of $A$ given by Lemma 4.5. Then obviously $\alpha_k \prec \alpha_{k+1}$. Fix a tower of approximations of $A$ as above where $n = \dim(X)$.

By [4, Lemma 3.2], for every open cover $\mathcal{V}$ of $X$ there is a finite open cover $\mathcal{U}$ which refines $\mathcal{V}$ and such that the set $\mathcal{U}$ can be partitioned into $n+1$ nonempty subsets consisting of elements with pairwise disjoint closures. Since we can refine simultaneously the covers that appear in a finite family $\{\alpha_k : 0 \leq k \leq n\}$ of approximations while preserving subordination, we may arrange not only that all $\alpha_k$ share the same cover $\{U_i\}_{i \in I}$, but moreover, that the cover $\{U_i\}_{i \in I}$ can be partitioned into $n+1$ subsets $\mathcal{U}_0, \ldots, \mathcal{U}_n$ consisting of mutually disjoint elements. For definiteness, let us write $\mathcal{U}_k = \{U_{i_k} : i_k \in I_k\}$. Now for each $k$ we consider the closed subset of $X$

$$Y_k = \bigcup_{i_k \in I_k} U_{i_k},$$

the map $i_k : I_k \to I$ and the $(\pi_{Y_k}(\mathcal{F}_k), \varepsilon_k, \mathcal{C})$-approximation of $A(Y_k)$, induced by $\alpha_k$, which is of the form

$$i_k^*(\alpha_k) = \{\pi_{Y_k}(\mathcal{F}_k), \varepsilon_k, \{U_{i_k}, \varphi_{i_k} : D_{i_k} \to A(U_{i_k}), \mathcal{H}_{i_k}, \mathcal{G}_{i_k}, \delta_{i_k}\}_{i_k \in I_k}\},$$

where each $U_{i_k}$ is nonempty. We have

$$\pi_{U_{i_k}}(\mathcal{F}_k) \subset_{\varepsilon_k/2} \varphi_{i_k}(\mathcal{H}_{i_k}),$$

by construction. Since $\alpha_k \prec \alpha_{k+1}$ we obtain

$$\mathcal{F}_k \leq \mathcal{F}_{k+1},$$

$$\varphi_{i_k}(\mathcal{G}_{i_k}) \leq \pi_{U_{i_k}}(\mathcal{F}_{k+1}), \text{ for all } i_k \in I_k,$$

$$\varepsilon_{k+1} > \min(\{\varepsilon_k\} \cup \{\delta_{i_k} : i_k \in I_k\}).$$

Set $X_k = Y_k \cup \cdots \cup Y_n$ and $E_k = \oplus_{i_k} C(U_{i_k}) \otimes D_{i_k}$ for $0 \leq k \leq n$. We shall construct a sequence of $C(Y_k)$-linear $*$-monomorphisms, $\psi_k : E_k \to A(Y_k)$, $k = n, \ldots, 0$, such that $(\psi_k, \ldots, \psi_n)$ is an $(n-k)$-fibered monomorphism into $A(X_k)$. Each map

$$\psi_k = \oplus_{i_k} \psi_{i_k} : E_k \to A(Y_k) = \oplus_{i_k} A(U_{i_k})$$

should
will have components $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ whose restrictions to $D_{i_k}$ will be perturbations of $\varphi_{i_k} : D_{i_k} \to A(U_{i_k})$, $i_k \in I_k$. We shall construct the maps $\psi_k$ by induction on decreasing $k$ such that if $B_k = A(X_k)(\psi_k, \ldots, \psi_n)$ and $\eta_k : B_k \to A(X_k)$ is the map induced by the $(n-k)$-fibered monomorphism $(\psi_k, \ldots, \psi_n)$, then

$$\pi_{X_{k+1} \cap U_{i_k}}(\psi_{i_k}(D_{i_k})) \subset \pi_{X_{k+1} \cap U_{i_k}}(\eta_{k+1}(B_{k+1})), \forall i_k \in I_k,$$

and

$$\pi_{X_k}(F_k) \subset \epsilon_k \eta_k(B_k).$$

Note that (13) is equivalent to

$$\pi_{X_{k+1} \cap \gamma_{I_k}}(\psi_k(E_k)) \subset \pi_{X_{k+1} \cap \gamma_{I_k}}(\eta_{k+1}(B_{k+1})).$$

For the first step of induction, $k = n$, we choose $\psi_n = \oplus_{i_k} \tilde{\varphi}_{i_k}$ where $\tilde{\varphi}_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ are $C(U_{i_k})$-linear extensions of the original $\varphi_{i_k}$. Then $B_n = E_n$ and $\eta_n = \psi_n$. Assume that $\psi_n, \ldots, \psi_{k+1}$ were constructed and that they have the desired properties. We shall construct now $\psi_k$. Condition (14) formulated for $k + 1$ becomes

$$\pi_{X_{k+1}}(F_{k+1}) \subset \epsilon_{k+1} \eta_{k+1}(B_{k+1}).$$

Since $\epsilon_{k+1} < \delta_{i_k}$, by using (11) and (16) we obtain

$$\pi_{X_{k+1} \cap U_{i_k}}(\varphi_{i_k}(G_{i_k})) \subset \delta_{i_k} \pi_{X_{k+1} \cap U_{i_k}}(\eta_{k+1}(B_{k+1})), \forall i_k \in I_k.$$

Since $F_k \subseteq F_{k+1}$ and $\epsilon_{k+1} < \epsilon_k$, condition (16) gives

$$\pi_{X_{k+1}}(F_k) \subset \epsilon_k \eta_{k+1}(B_{k+1}).$$

Conditions (9), (17) and (13) enable us to apply Lemma 4.4 and perturb $\tilde{\varphi}_{i_k}$ to a $*$-monomorphism $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ satisfying (13) and (14) and such that

$$KK(\psi_{i_k}|_{D_{i_k}}) = KK(\varphi_{i_k})$$

if the algebras in $C$ are assumed to be $KK$-stable. We set $\psi_k = \oplus_{i_k} \psi_{i_k}$ and this completes the construction of $(\psi_0, \ldots, \psi_n)$. Condition (14) for $k = 0$ gives $F \subset \epsilon \eta_0(B_0) = \eta(A(\psi_0, \ldots, \psi_n))$. Thus $(\psi_0, \ldots, \psi_n)$ satisfies the conclusion of the theorem. Finally let us note that it can happen that $X_k = X$ for some $k > 0$. In this case $F \subset \epsilon A(\psi_k, \ldots, \psi_n)$ and for this reason we write $n \leq \dim(X)$ in the statement of the theorem.

Proposition 4.7. Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable continuous $C(X)$-algebra of which are stable Kirchberg algebras. Let $D$ be a $KK$-semiprojective stable Kirchberg algebra and suppose that there exists $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(\tau(x))^{-1}$ for all $x \in X$. For any finite subset $F$ of $A$ and any $\epsilon > 0$ there is an $n$-fibered $C$-monomorphism $(\psi_0, \ldots, \psi_n)$ into $A$ such that $n \leq \dim(X)$, $C = \{D\}$, and each component $\psi_i : C(Y_i) \otimes D \to A(Y_i)$ satisfies $KK(\psi_i) = \sigma_{Y_i}$, $i = 0, \ldots, n$. Moreover, if $\eta : A(\psi_0, \ldots, \psi_n) \to A$ is the induced $*$-monomorphism, then $F \subset \epsilon \eta(A(\psi_0, \ldots, \psi_n))$ and $KK(\eta_x)$ is a $KK$-equivalence for each $x \in X$. □
Proof. We repeat the proof of Theorem 4.6 while using only \((\mathcal{F}_i, \varepsilon_i, \{D_i\})\)-approximations of \(A\) provided by the second part of Lemma 4.5. The outcome will be an \(n\)-fibered \((D)\)-monomorphism \((\psi_0, \ldots, \psi_n)\) into \(A\) such that \(\mathcal{F} \subset \varepsilon \eta(A(\psi_0, \ldots, \psi_n))\). Moreover we can arrange that \(KK(\psi_i) = \sigma Y_i\) for all \(i = 0, \ldots, n\), by (19), since \(KK(\psi_{i_k}) = \sigma U_{i_k}\) by Lemma 4.5. If \(x \in X\), and \(i = \min\{k : x \in Y_k\}\), then \(\eta_x \equiv (\psi_i)_x\), and hence \(KK(\eta_x)\) is a KK-equivalence. □

Remark 4.8. Let us point out that we can strengthen the conclusion of Theorem 4.6 and Proposition 4.7 as follows. Fix a metric \(d\) for the topology of \(X\). Then we may arrange that there is a closed cover \(\{Y'_0, \ldots, Y'_n\}\) of \(X\) and a number \(\ell > 0\) such that \(\{x : d(x, Y'_i) \leq \ell\} \subset Y_i\) for \(i = 0, \ldots, n\). Indeed, when we choose the finite closed cover \(U = (U_i)_{i \in I}\) of \(X\) in the proof of Theorem 4.6 which may be partitioned into \(n + 1\) subsets \(U_0, \ldots, U_n\) consisting of mutually disjoints elements, as given by [4, Lemma 3.2], and which refines all the covers \(U(\alpha_0), \ldots, U(\alpha_n)\) corresponding to \(\alpha_0, \ldots, \alpha_n\), we may assume that \(U\) also refines the covers given by the interiors of the elements of \(U(\alpha_0), \ldots, U(\alpha_n)\). Since each \(U_i\) is compact and \(I\) is finite, there is \(\ell > 0\) such that if \(V_i = \{x : d(x, U_i) \leq \ell\}\), then the cover \(\mathcal{V} = (V_i)_{i \in I}\) still refines all of \(U(\alpha_0), \ldots, U(\alpha_n)\) and for each \(k = 0, \ldots, n\), the elements of \(\mathcal{V}_k = \{V_i : U_i \in U_k\}\), are still mutually disjoint. We shall use the cover \(\mathcal{V}\) rather than \(U\) in the proof of the two theorems and observe that \(Y'_k \stackrel{\text{def}}{=} \bigcup_{i_k \in I_k} U_{i_k} \subset \bigcup_{i_k \in I_k} V_{i_k} = Y_k\) has the desired property. Finally let us note that if we define \(\psi'_i : E(Y'_i) \to A(Y'_i)\) by \(\psi'_i = \pi Y'_i \psi_i\), then \((\psi'_0, \ldots, \psi'_n)\) is an \(n\)-fibered \(C\)-monomorphism into \(A\) which satisfies the conclusion of Theorem 4.6 and Proposition 4.7 since \(\pi Y'_i(\mathcal{F}) \subset \varepsilon \psi'_i(E_t)\) for all \(i = 0, \ldots, n\) and \(X = \bigcup_{i=1}^n Y'_i\).

5. Representing \(C(X)\)-algebras as inductive limits

We have seen that Theorem 4.6 yields exhaustive sequences for certain \(C(X)\)-algebras. In this section we show how to pass from an exhaustive sequence to a nested exhaustive sequence using semiprojectivity. The remainder of the paper does not depend on this section.

Proposition 5.1. Let \(X, A\) and \(\mathcal{C}\) be as in Theorem 4.6. Let \((\psi_0, \ldots, \psi_n)\) be an \(n\)-fibered \(C\)-monomorphism into \(A\) with components \(\psi_i : E_i \to A(Y_i)\). Let \(F_i \subset E_i, F \subset A(\psi_0, \ldots, \psi_n)\) be finite sets and let \(\varepsilon > 0\). Then there are finite sets \(G_i \subset F_i\) and \(\delta_i > 0\), \(i = 0, \ldots, n\), such that for any \(C(X)\)-subalgebra \(A' \subset A\) which satisfies \(\psi_i(G_i) \subset A(Y_i)\), \(i = 0, \ldots, n\), there is an \(n\)-fibered \(C\)-monomorphism \((\psi'_0, \ldots, \psi'_n)\) into \(A'\), with \(\psi'_i : E_i \to A'(Y'_i)\) and such that \((i)\) \(\|\psi'_i(a) - \psi_i(a)\| < \varepsilon\) for all \(a \in F_i\) and all \(i \in \{0, \ldots, n\}\), \((ii)\) \((\psi'_j)^{-1}(\psi_i)_x = (\psi_j)^{-1}(\psi_i)_x\) for all \(x \in Y_i \cap Y_j\) and \(0 \leq i \leq j \leq n\). Moreover \(A(\psi_0, \ldots, \psi_n) = A'(\psi'_0, \ldots, \psi'_n)\) and the maps \(\eta : A(\psi_0, \ldots, \psi_n) \to A\) and \(\eta' : A'(\psi'_0, \ldots, \psi'_n) \to A'\) induced by \((\psi_0, \ldots, \psi_n)\) and \((\psi'_0, \ldots, \psi'_n)\) satisfy \((iii)\) \(\|\eta(a) - \eta'(a)\| < \varepsilon\) for all \(a \in F\).

Proof. Let us observe that if we prove (i) and (ii) then (iii) will follow by enlarging the sets \(F_i\) so that \(p_i(F) \subset F_i\), where \(p_i : A(\psi_0, \ldots, \psi_n) \to E_i\) are the coordinate maps. We proceed now with the proof of (i) and (ii) by making some simplifications. We may assume that \(E_0 = C(Y_0) \otimes D_0\) with \(D_0 \in \mathcal{C}\) since the perturbations corresponding to disjoint closed sets can be done independently of each other. Without any loss of generality, we may assume that \(F_0 \subset D_0\) since we are working with morphisms on \(E_0\) which are \(C(Y_0)\)-linear. We also enlarge \(F_0\) so that for each direct summand \(C\) of \(D_0, F_0\) contains an element \(c\) which is full in \(C\) and such that \(\|c\| \geq 2\varepsilon\).

The proof is by induction on \(n\). If \(n = 0\) the statement follows from Proposition 5.2 and Lemma 4.1. Assume now that the statement is true for \(n - 1\). Let \(E_i, \psi_i, A', F_i, 1 \leq i \leq n\)
and $\varepsilon$ be as in the statement. For $0 \leq i < j \leq n$ let $\eta_{j,i} : E_i(Y_i \cap Y_j) \to E_j(Y_i \cap Y_j)$ be the $*$-homomorphism of $(Y_i \cap Y_j)$-algebras defined fiberwise by $(\eta_{j,i})_x = (\psi_j)_x^{-1}(\psi_i)_x$.

Let $G_0$ and $\delta_0$ be given by Proposition 4.5 applied to the $C^*$-algebra $D_0$ for the input data $F_0$ and $\varepsilon$. For each $1 \leq j \leq n$ choose a finite subset $\mathcal{H}_j$ of $E_j$ whose restriction to $Y_j \cap Y_0$ contains $\eta_{j,0}(G_0)$. Consider the sets $F_j := F_j \cup \mathcal{H}_j$, $1 \leq j \leq n$ and the number $\varepsilon' = \min\{\delta_0, \varepsilon\}$. Let $G_1, \ldots, G_n$ and $\delta_1, \ldots, \delta_n$ be given by the inductive assumption for $n - 1$ applied to $A(X_1), A'(X_1), \psi_j, F_j$, $1 \leq j \leq n$ and $\varepsilon'$, where $X_1 = Y_1 \cup \cdots \cup Y_n$.

We need to show that $G_0, G_1, \ldots, G_n$ and $\delta_0, \delta_1, \ldots, \delta_n$ satisfy the statement. By the inductive step there exists an $(n - 1)$-fibered $\mathcal{C}$-monomorphism $(\psi'_1, \ldots, \psi'_n)$ into $A'(X_1)$ with components $\psi'_j : E_j \to A'(Y_j)$ such that

(a) $\|\psi_j(a) - \psi'_j(a)\| < \varepsilon'$ for all $a \in F_j \cup \mathcal{H}_j$ and all $1 \leq j \leq n$,

(b) $(\psi'_j)_x^{-1}(\psi_i)_x = (\psi_j)_x^{-1}(\psi'_j)_x$ for all $x \in Y_i \cap Y_j$ and $1 \leq i \leq j \leq n$.

The condition (b) enables to define a $*$-homomorphism $\varphi : E_0 \to A'(Y_0 \cap X_1)$ with fiber maps $\varphi_x = (\psi'_j)_x(\psi_j)_x^{-1}\psi_0)_x$ for $x \in Y_0 \cap Y_j$ and $1 \leq j \leq n$.

Let us observe that $\varphi_0 : E_0 \to A(Y_0)$ is an approximate lifting of $\varphi$. More precisely we have $\|\pi_{Y_0}(a) - \varphi(a)\| < \delta_0$ for all $a \in G_0$. Indeed, for $x \in Y_0 \cap Y_j$, $1 \leq j \leq n$ and $a \in G_0$ we have

$$\|(\psi'_j)_x(a(x)) - (\psi_j)_x(\pi_{Y_0}(a(x)))\| = \|(\psi'_j)_x(\psi_0)_x(a(x)) - (\psi_j)_x(\psi_0)_x(a(x))\| \leq \sup_{h \in \mathcal{H}_j} \|\psi_j(h) - \psi'_j(h)\| < \varepsilon' \leq \delta_0.$$

Since we also have $\psi_0(G_0) \subset A'(Y_0)$ by hypothesis, it follows from Proposition 4.8 that there exists $\varphi'_0 : D_0 \to A(Y_0)$ such that $\|\varphi'_0(a) - \varphi_0(a)\| < \varepsilon$ for all $a \in F_0$ and $\pi_{Y_0}^0 \circ \varphi'_0 = \varphi$. By Lemma 5.1 each $(\psi'_j)_x$ is injective since each $\psi_0)_x$ is injective. The $C(Y_0)$-linear extension of $\psi'_0$ to $E_0$ satisfies $(\psi'_j)_x^{-1}\psi_0)_x = (\psi'_j)_x^{-1}\psi'_0)_x$ for all $x \in Y_0 \cap Y_j$ and $1 \leq j \leq n$ and this completes the proof of (ii). Condition (i) follows from (b).

The following result gives an inductive limit representation for continuous $C(X)$-algebras whose fibers are inductive limits of finite direct sums of simple semiprojective $C^*$-algebras. For example the fibers can be arbitrary AF-algebras or Kirchberg algebras which satisfy the UCT and whose $K_1$-groups are torsion free. Indeed, by [29] Prop. 8.4.13, these algebras are isomorphic to inductive limits of sequences of Kirchberg algebras $(D_n)$ with finitely generated K-theory groups and torsion free $K_1$-groups. The algebras $D_n$ are semiprojective by [33].

**Theorem 5.2.** Let $\mathcal{C}$ be a class consisting of finite direct sums of semiprojective simple $C^*$-algebras.

Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable continuous $C(X)$-algebra such that all its fibers admit exhaustive sequences consisting of $C^*$-algebras isomorphic to $C^*$-algebras in $\mathcal{C}$. Then $A$ is isomorphic to the inductive limit of a sequence of continuous $C(X)$-algebras $A_k$ such that $\dim(A_k) \leq \dim(X)$.

**Proof.** By Theorem 4.10 and Proposition 5.1 we find a sequence $(\psi^{(k)}_0, \ldots, \psi^{(k)}_n)$ of $n$-fibered $\mathcal{C}$-monomorphisms into $A$ which induces $*$-monomorphisms $\eta^{(k)}_n : A_k = A(\psi^{(k)}_0, \ldots, \psi^{(k)}_n) \to A$ with the following properties. There is a sequence of finite sets $F_k \subset A_k$ and a sequence of $C(X)$-linear $*$-monomorphisms $\mu_k : A_k \to A_{k+1}$ such that

(i) $\|\eta^{(k+1)}_n(a) - \eta^{(k)}_n(a)\| < 2^{-k}$ for all $a \in F_k$ and all $k \geq 1$,

(ii) $\mu_k(F_k) \subset F_{k+1}$ for all $k \geq 1$,
Arguing as in the proof of [29, Prop. 2.3.2], one verifies that
\[
\varphi_k(a) = \lim_{j \to \infty} \eta^{(j)}(\mu_{j-1} \circ \cdots \circ \mu_k)(a)
\]
defines a sequence of \(\ast\)-monomorphisms \(\varphi_k : A_k \to A\) such that \(\varphi_{k+1}\mu_k = \varphi_k\) and the induced map \(\varphi : \lim_{\to k} (A_k, \mu_k) \to A\) is an isomorphism of \(C(X)\)-algebras. \(\Box\)

**Remark 5.3.** By similar arguments one proves a unital version of Theorem 5.2.

6. **When is a fibered product locally trivial**

For \(C^*\)-algebras \(A, B\) we endow the space \(\text{Hom}(A, B)\) of \(\ast\)-homomorphisms with the point-norm topology. If \(X\) is a compact Hausdorff space, then \(\text{Hom}(A, C(X) \otimes B)\) is homeomorphic to the space of continuous maps from \(X\) to \(\text{Hom}(A, B)\) endowed with the compact-open topology.

We shall identify a \(\ast\)-homomorphism \(\varphi \in \text{Hom}(A, C(X) \otimes B)\) with the corresponding continuous map \(X \to \text{Hom}(A, B), x \mapsto \varphi_x, \varphi_x(a) = \varphi(a)(x)\) for all \(x \in X\) and \(a \in A\). Let \(D\) be a \(C^*\)-algebra and let \(A\) be a \((C(X)\)-algebra. If \(\alpha : D \to A\) is a \(\ast\)-homomorphism, let us denote by \(\tilde{\alpha} : C(X) \otimes D \to A\) its (unique) \(C(X)\)-linear extension and write \(\tilde{\alpha} \in \text{Hom}_{C(X)}(C(X) \otimes D, A)\). For \(C^*\)-algebras \(D, B\) we shall make without further comment the following identifications

\[
\text{Hom}_{C(X)}(C(X) \otimes D, C(X) \otimes B) \equiv \text{Hom}(D, C(X) \otimes B) \equiv C(X, \text{Hom}(D, B)).
\]

For a \(C^*\)-algebra \(D\) we denote by \(\text{End}(D)\) the set of full (and unital if \(D\) is unital) \(\ast\)-endomorphisms of \(D\) and by \(\text{End}(D)^0\) the path component of \(\text{id}_D\) in \(\text{End}(D)\). Let us consider

\[
\text{End}(D)^* = \{ \gamma \in \text{End}(D) : KK(\gamma) \in KK(D, D)^{-1} \}.
\]

**Proposition 6.1.** Let \(X\) be a compact metrizable space and let \(D\) be a \(KK\)-semiprojective Kirchberg algebra. Let \(\alpha : D \to C(X) \otimes D\) be a full (and unital, if \(D\) is unital) \(\ast\)-homomorphism such that \(KK(\alpha_x) \in KK(D, D)^{-1}\) for all \(x \in X\). Then there is a full \(\ast\)-homomorphism \(\Phi : D \to C(X \times [0, 1]) \otimes D\) such that \(\Phi(x, 0) = \alpha_x\) and \(\Phi(x, t) \in \text{Aut}(D)\) for all \(x \in X\) and \(t \in (0, 1]\). Moreover, if \(\Phi_1 : D \to C(X) \otimes D\) is defined by \(\Phi_1(d)(x) = \Phi(x, 1)(d),\) for all \(d \in D\) and \(x \in X,\) then \(\alpha \approx_{uh} \Phi_1\).

**Proof.** Since \(X\) is a metrizable compact space, \(X\) is homeomorphic to the projective limit of a sequence of finite simplicial complexes \((X_i)\) by [13 Thm. 10.1, p.284]. Since \(D\) is \(KK\)-semiprojective, \(KK(D, \lim C(X_i) \otimes D) = KK(D, C(X) \otimes D)\) by Theorem 5.12. By Theorem 5.11 there is \(i\) and a full (and unital if \(D\) is unital) \(\ast\)-homomorphism \(\varphi : D \to C(X_i) \otimes D\) whose KK-class maps to \(KK(\alpha) \in KK(D, C(X) \otimes D)\). To summarize, we have found a finite simplicial complex \(Y,\) a continuous map \(h : X \to Y\) and a continuous map \(y \mapsto \varphi_y \in \text{End}(D),\) defined on \(Y,\) such that the full (and unital if \(D\) is unital) \(\ast\)-homomorphism \(h^* \varphi : D \to C(X) \otimes D\) corresponding to the continuous map \(x \mapsto \varphi_{h(x)}\) satisfies \(KK(h^* \varphi) = KK(\alpha)\). We may arrange that \(h(X)\) intersects all the path components of \(Y\) by dropping the path components which are not intersected. Since \(\alpha_x \in \text{End}(D)^*\) by hypothesis, and since \(KK(\alpha_x) = KK(\varphi_{h(x)})\), we infer that \(\varphi_y \in \text{End}(D)^*\) for all \(y \in Y\). We shall find a continuous map \(y \mapsto \psi_y \in \text{End}(D)^*\) defined on \(Y,\) such that the maps \(y \mapsto \psi_y \varphi_y\) and \(y \mapsto \varphi_y \psi_y\) are homotopic to the constant map \(i\) that takes \(Y\) to \(\text{id}_D\). It is clear that it suffices to deal separately with each path component of \(Y,\) so that for this part of the proof

we may assume that $Y$ is connected. Fix a point $z \in Y$. By [29] Thm. 8.4.1 there is $\nu \in \text{Aut}(D)$ such that $KK(\nu^{-1}) = KK(\varphi_2)$ and hence $KK(\nu \varphi_2) = KK(\text{id}_D)$. By Theorem 3.1 there is a unitary $u \in M(D)$ such that $w_\varphi z(-)u^*$ is homotopic to $\text{id}_D$. Let us set $\theta = w\nu(-)u^* \in \text{Aut}(D)$ and observe that $\theta \varphi_z \in \text{End}(D)^0$. Since $Y$ is path connected, it follows that the entire image of the map $y \mapsto \theta \varphi_y$ is contained in $\text{End}(D)^0$. Since $\text{End}(D)^0$ is a path connected H-space with unit element, it follows by [29] Thm. 2.4, p462 that the homotopy classes $[Y, \text{End}(D)^0]$ (with no condition on basepoints, since the action of the fundamental group $\pi_1(\text{End}(D)^0, \text{id}_D)$ is trivial by [29] 3.6, p166) form a group under the natural multiplication. Therefore we find $y \mapsto \psi_y^t \in \text{End}(D)^0$ such that $y \mapsto \psi_y^t \varphi_y$ and $y \mapsto \theta \varphi_y \psi_y^t$ are homotopic to $\iota$. It follows that $y \mapsto \psi_y \overset{\text{def.}}{=} \psi_y^t \theta$ is the homotopic inverse of $y \mapsto \varphi_y$ in $[Y, \text{End}(D)^*]$. Composing with $h$ we obtain that the maps $x \mapsto \varphi_{h(x)} \psi_{h(x)}$ and $x \mapsto \psi_{h(x)} \varphi_{h(x)}$ are homotopic to the constant map that takes $X$ to $\text{id}_D$. By the homotopy invariance of $KK$-theory we obtain that

$$KK(\tilde{h}^* \varphi \cdot h^* \psi) = KK(\tilde{h}^* \psi \cdot h^* \varphi) = KK(\iota_D),$$

where $\tilde{h}^* \varphi$ and $\tilde{h}^* \psi$ denote the $C(X)$-linear extensions of the corresponding maps and $\iota_D : D \to C(X) \otimes D$ is defined by $\iota_D(d) = 1_{C(X)} \otimes d$ for all $d \in D$. Let us recall that $KK(h^* \varphi) = KK(\alpha)$ and hence $KK(\tilde{h}^* \varphi) = KK(\tilde{\alpha})$. If we set $\tilde{\Psi} = h^* \psi$, then

$$KK(\tilde{\alpha} \tilde{\Psi}) = KK(\tilde{\Psi} \alpha) = KK(\iota_D).$$

By Theorem 3.1 $\tilde{\alpha} \tilde{\Psi} \overset{\text{def.}}{=} \iota_D$ and $\tilde{\Psi} \alpha \overset{\text{def.}}{=} \iota_D$, and hence $\tilde{\alpha} \tilde{\Psi} \overset{\text{def.}}{=} \text{id}_{C(X) \otimes D}$ and $\tilde{\Psi} \tilde{\alpha} \overset{\text{def.}}{=} \text{id}_{C(X) \otimes D}$. By [29] Cor. 2.3.4, there is an isomorphism $\Gamma : C(X) \otimes D \to C(X) \otimes D$ such that $\Gamma \overset{\text{def.}}{=} \tilde{\alpha}$. In particular $\Gamma$ is $C(X)$-linear and $\Gamma_x \in \text{Aut}(D)$ for all $x \in X$. Replacing $\Gamma$ by $u \Gamma(\cdot)u^*$ for some unitary $u \in M(C(X) \otimes D)$ we can arrange that $\Gamma|_D$ is arbitrarily close to $\alpha$. Therefore $KK(\Gamma|_D) = KK(\alpha)$ since $D$ is KK-stable. By Theorem 3.1 there is a continuous map $(0, 1] \to U(M(C(X) \otimes D))$, $t \mapsto u_t$, with the property that

$$\lim_{t \to 0} \|u_t \Gamma(a)u_t^* - \alpha(a)\| = 0, \text{ for all } a \in D.$$

Therefore the equation

$$d(x,t) = \begin{cases} \alpha_x, & \text{if } t = 0, \\ u_t(x) \Gamma_x u_t(x)^*, & \text{if } t \in (0, 1], \end{cases}$$

defines a continuous map $\Phi : X \times [0, 1] \to \text{End}(D)^*$ which extends $\alpha$ and such that $\Phi(X \times (0, 1]) \subset \text{Aut}(D)$. Since $\alpha$ is homotopic to $\Phi_1$, we have that $\alpha \overset{\text{ub.}}{=} \Phi_1$ by Theorem 3.1. $\square$

**Proposition 6.2.** Let $X$ be a compact metrizable space and let $D$ be a KK-semiprojective Kirchberg algebra. Let $Y$ be a closed subset of $X$. Assume that a map $\gamma : Y \to \text{End}(D)^*$ extends to a continuous map $\alpha : X \to \text{End}(D)^*$. Then there is a continuous extension $\eta : X \to \text{End}(D)^*$ of $\gamma$, such that $\eta(X \setminus Y) \subset \text{Aut}(D)$.

**Proof.** Since the map $x \mapsto \alpha_x$ takes values in $\text{End}(D)^*$, by Proposition 6.1 there exists a continuous map $\Phi : X \times [0, 1] \to \text{End}(D)^*$ which extends $\alpha$ and such that $\Phi(X \times (0, 1]) \subset \text{Aut}(D)$. Let $d$ be a metric for the topology of $X$ such that $\text{diam}(X) \leq 1$. The equation $\eta(x) = \Phi(x, d(x, Y))$ defines a map on $X$ that satisfies the conclusion of the proposition. $\square$
Lemma 6.3. Let $X$ be a compact metrizable space and let $D$ be a $KK$-semiprojective Kirchberg algebra. Let $Y$ be a closed subset of $X$. Let $\alpha : Y \times [0,1] \cup X \times \{0\} \to \End(D)$ be a continuous map such that $\alpha(x,0) \in \End(D)^*$ for all $x \in X$. Suppose that there is an open set $V$ in $X$ which contains $Y$ and such that $\alpha$ extends to a continuous map $\alpha_V : V \times [0,1] \cup X \times \{0\} \to \End(D)$. Then there is $\eta : X \times [0,1] \to \End(D)^*$ such that $\eta$ extends $\alpha$ and $\eta(x,t) \in \Aut(D)$ for all $x \in X \setminus Y$ and $t \in (0,1]$.

Proof. By Proposition 6.2 it suffices to find a continuous map $\tilde{\alpha} : X \times [0,1] \to \End(D)^*$ which extends $\alpha$. Fix a metric $d$ for the topology of $X$ and define $\lambda : X \to [0,1]$ by $\lambda(x) = d(x, X \setminus V)(d(x, X \setminus V) + d(x, Y))^{-1}$. Let us define $\tilde{\alpha} : X \times [0,1] \to \End(D)$ by $\tilde{\alpha}(x,t) = \alpha_V(x, \lambda(x)t)$ and observe that $\tilde{\alpha}$ extends $\alpha$. Finally, since $\tilde{\alpha}(x,t)$ is homotopic to $\tilde{\alpha}(x,0) = \alpha(x,0)$, we conclude that the image of $\tilde{\alpha}$ in contained in $\End(D)^*$.

Proposition 6.4. Let $X$ be a compact metrizable space and let $D$ be a $KK$-semiprojective stable Kirchberg algebra. Let $A$ be a separable $C(X)$-algebra which is locally isomorphic to $C(X) \otimes D$. Suppose that there is $\sigma \in KK(D,A)$ such that $\sigma_x \in KK(D,A(x))^{-1}$ for all $x \in X$. Then there is an isomorphism of $C(X)$-algebras $\psi : C(X) \otimes D \to A$ such that $KK(\psi|_D) = \sigma$.

Proof. Since $X$ is compact and $A$ is locally trivial it follows that $\text{cat}_D(A) < \infty$. By Lemma 2.3 $A \cong pAp \otimes \mathcal{O}_\infty \otimes K$ for some projection $p \in A$. By Theorem 5.1 there is a full $*$-homomorphism $\varphi : C(X) \to A$ such that $KK(\varphi) = \sigma$. We shall construct an isomorphism of $C(X)$-algebras $\psi : C(X) \otimes D \to A$ such that $\psi$ is homotopic to $\tilde{\varphi}$, the $C(X)$-linear extension of $\varphi$. Moreover the homotopy $(H_t)_{t \in [0,1]}$ will have the property that $H(x, t) : D \to A(x)$ is an isomorphism for all $x \in X$ and $t > 0$. We prove this by induction on numbers $n$ with the property that there are two closed covers of $X$, $W_1, ..., W_n$ and $Y_1, ..., Y_n$ such that $Y_i$ contained in the interior of $W_i$ and $A(W_i) \cong C(Y_i) \otimes D$ for $1 \leq i \leq n$. First we observe that the case $n = 1$ follows from Proposition 5.2. Let us now pass from $n - 1$ to $n$. Given two covers as above, there is yet another closed cover $V_1, ..., V_n$ of $X$ such that $V_i$ is a neighborhood of $Y_i$ and $W_i$ is a neighborhood of $V_i$ for all $1 \leq i \leq n$. Set $Y = \cup_{i=1}^{n-1} Y_i$, $V = \cup_{i=1}^{n-1} V_i$ and $W = \cup_{i=1}^{n-1} W_i$. By the inductive hypothesis applied to $A(V)$, and the covers $V_1, ..., V_{n-1}$ and $W_1 \cap V_1, ..., W_{n-1} \cap V$ there is a homotopy $h : D \to A(V) \otimes C[0,1]$ such that $h(x,0) = \varphi_x$ and $h(x,t) : D \to A(x)$ is an isomorphism for all $(x,t) \in V \setminus (0,1]$. Fix a trivialization $\nu : A(Y_{n+1}) \to C(Y_{n+1}) \otimes D$. Define a continuous map $\alpha : (V \cap Y_{n+1}) \times [0,1] \cup Y_{n+1} \times \{0\} \to \End(D)$ by setting $\alpha(x,t) = \nu_x h(x,t)$ if $(x,t) \in (V \cap Y_{n+1}) \times [0,1]$ and $\alpha(x,0) = \nu_x \varphi_x$ if $x \in Y_{n+1}$. Since $V \cap Y_{n+1}$ is a neighborhood of $Y \cap Y_{n+1}$ in $Y_{n+1}$ and since $\nu_x \varphi_x \in \End(D)^*$ for all $x \in Y_{n+1}$, by Lemma 6.3 there is a continuous map $\eta : Y_{n+1} \times [0,1] \to \End(D)^*$ which extends the restriction of $\alpha$ to $(Y \cap Y_{n+1}) \times [0,1] \cup Y_{n+1} \times \{0\}$. We conclude the construction of the desired homotopy by defining $H : D \to A(X) \otimes C[0,1]$ by $H(x,t) = h(x,t)$ for $(x,t) \in V \times [0,1]$ and $H(x,t) = \nu_x^{-1} \eta(x,t)$ for $(x,t) \in Y_{n+1} \times [0,1]$. 

Lemma 6.5. Let $D$ be a $KK$-semiprojective stable Kirchberg algebra. Let $X$ be a compact metrizable space and $Y, Z$ be closed subsets of $X$ such that $X = Y \cup Z$. Suppose that $\gamma : D \to C(Y \cap Z) \otimes D$ is a full $*$-homomorphism which admits a lifting to a full $*$-homomorphism $\alpha : D \to C(Y) \otimes D$ such that $\alpha_x \in \End(D)^*$ for all $x \in Y$. Then the pullback $C(Y) \otimes D \oplus_{\pi_Y \cap \pi_Z, \gamma \pi_Y \cap \pi_Z} C(Z) \otimes D$ is isomorphic to $C(X) \otimes D$. 

Proof. By Prop. 6.2 there is a \( \ast \)-homomorphism \( \eta : D \to C(Y) \otimes D \) such that \( \eta_x = \gamma_x \) for \( x \in Y \cap Z \) and such that \( \eta_x \in \text{Aut}(D) \) for \( x \in Y \setminus Z \). Using the short five lemma one checks immediately that the triplet \( (\tilde{\eta}, \tilde{\gamma}, \text{id}_{C(Z) \otimes D}) \) defines a \( C(X) \)-linear isomorphism:

\[
C(X) \otimes D = C(Y) \otimes D \oplus \pi_{y \cap Z} \pi_{y \cap Z} C(Z) \otimes D \rightarrow C(Y) \otimes D \oplus \pi_{y \cap Z} \tilde{\eta} \pi_{y \cap Z} C(Z) \otimes D.
\]

\( \square \)

Lemma 6.6. Let \( D \) be a KK-semiprojective stable Kirchberg algebra. Let \( Y, Z \) and \( Z' \) be closed subsets of a compact metrizable space \( X \) such that \( Z' \) is a neighborhood of \( Z \) and \( X = Y \cup Z \). Let \( B \) be a \( (Y) \)-algebra locally isomorphic to \( C(Y) \otimes D \) and let \( E \) be a \( (Z') \)-algebra locally isomorphic to \( C(Z') \otimes D \). Let \( \alpha : E(Y \cap Z') \rightarrow B(Y \cap Z') \) be a \( \ast \)-monomorphism of \( C(Y \cap Z') \)-algebras such that \( KK(\alpha_x) \in KK(E(x), B(x))^{-1} \) for all \( x \in Y \cap Z' \). If \( \gamma = \alpha_{Y \cap Z} \), then \( B(Y) \oplus \pi_{y \cap Z} \gamma \pi_{y \cap Z} E(Z) \) is locally isomorphic to \( C(X) \otimes D \).

Proof. Since we are dealing with a local property, we may assume that \( B = C(Y) \otimes D \) and \( E = C(Z') \otimes D \). To simplify notation we let \( \pi \) stand for both \( \pi_{Y \cap Z} \) and \( \pi_{Z' \cap Z} \) in the sequel. Let us denote by \( H \) the \( C(X) \)-algebra \( C(Y) \otimes D \oplus \pi_{Y \cap Z} C(Z) \otimes D \). We must show that \( H \) is locally trivial. Let \( x \in X \). If \( x \notin Z \), then there is a closed neighborhood \( V \) of \( x \) which does not intersect \( Z \), and hence the restriction of \( H \) to \( V \) is isomorphic to \( C(V) \otimes D \), as it follows immediately from the definition of \( H \). It remains to consider the case when \( x \in Z \). Now \( Z' \) is a closed neighborhood of \( x \) in \( X \). The restriction of \( H \) to \( Z' \) is isomorphic to \( C(Y \cap Z') \otimes D \oplus \pi_{Y \cap Z} C(Z) \otimes D \). Since \( \gamma : Y \cap Z \rightarrow End(D)^{\ast} \) admits a continuous extension \( \alpha : Y \cap Z' \rightarrow End(D)^{\ast} \), it follows that \( (H(Z')) \) is isomorphic to \( C(Z') \otimes D \) by Lemma 6.6. \( \square \)

Proposition 6.7. Let \( X, A, D \) and \( \sigma \) be as in Proposition 6.4. For any finite subset \( F \) of \( A \) and any \( \varepsilon > 0 \) there is a \( C(X) \)-algebra \( B \) which is locally isomorphic to \( C(X) \otimes D \) and there exists a \( C(X) \)-linear \( \ast \)-monomorphism \( \eta : B \rightarrow A \) such that \( F \subset \varepsilon(\eta(B)) \) and \( KK(\eta_x) \in KK(B(x), A(x))^{-1} \) for all \( x \in X \).

Proof. Let \( \psi_k : E_k = C(Y_k) \otimes D \rightarrow A(Y_k), k = 0, \ldots, n \) be as in the conclusion of Proposition 6.4. Strengthen as in Remark 1.8. Therefore we may assume that there is another \( n \)-fibered \( \{D\} \)-monomorphism \( (\psi_0', \ldots, \psi_n') \) into \( B \) such that \( \psi_k : C(Y_k') \otimes D \rightarrow A(Y_k'), Y_k' \) is a closed neighborhood of \( Y_k \), and \( \pi_{Y_{k}} \psi_k \) extends to a \( \ast \)-monomorphism of \( X_k \)-algebras \( B_k, \eta_k \) and \( \gamma_k \) be as in Definition 2.24. \( B_0 \) and \( \eta_0 \) satisfy the conclusion of the proposition, except that we need to prove that \( B_0 \) is locally isomorphic to \( C(X) \otimes D \). We prove by induction on decreasing \( k \) that the \( (X_k) \)-algebras \( B_k \) are locally trivial. Indeed \( B_n = C(X_n) \otimes D \) and assuming that \( B_k \) is locally trivial, it follows by Lemma 6.6 that \( B_{k-1} \) is locally trivial, since by 6.3

\[
B_{k-1} \cong B_k \oplus \pi_{\eta_k, \gamma_k^{-1}} E_{k-1} \cong B_k \oplus \pi_\pi \gamma_k E_{k-1}, \quad (\pi = \pi_{X_k \cap Y_{k-1}})
\]

and \( \gamma_k : E_{k-1}(X_k \cap Y_{k-1}) \rightarrow B_k(X_k \cap Y_{k-1}), (\gamma_k)_{x} = (\eta_k)^{-1}(\psi_{k-1})_{x} \), extends to a \( \ast \)-monomorphism \( \alpha : E_{k-1}(X_k \cap Y_{k-1}) \rightarrow B_k(X_k \cap Y_{k-1}), \alpha_x = (\eta_k)^{-1}(\psi_{k-1})_{x} \) and \( KK(\alpha_x) \) is a KK-equivalence since both \( KK((\eta_k)_{x}) \) and \( KK((\psi_{k-1})_{x}) \) are KK-equivalences. \( \square \)

7. When is a \( C(X) \)-algebra locally trivial

In this section we prove Theorems 1.1 - 1.5 and some of their consequences.

Proof of Theorem 1.2.
Proof. Let \( X \) denote the primitive spectrum of \( A \). Then \( A \) is a continuous \( C(X) \)-algebra and its fibers are stable Kirchberg algebras (see [5, 2.2.2]). Since \( A \) is separable, \( X \) is metrizable by Lemma 2.2. By Proposition 6.7, there is a sequence of \( C(X) \)-algebras \( (A_k)_{k=1}^{\infty} \) locally isomorphic to \( C(X) \otimes D \) and a sequence of \( C(X) \)-linear \( * \)-monomorphisms \((\eta_k : A_k \to A)_{k=1}^{\infty}\), such that \( KK(\eta_k)_x \) is a KK-equivalence for each \( x \in X \) and \((\eta_k(A_k))_{k=1}^{\infty} \) is an exhaustive sequence of \( C(X) \)-subalgebras of \( A \). Since \( D \) is weakly semiprojective and KK-stable, after passing to a subsequence of \((A_k)\) if necessary, we find a sequence \((\sigma_k)_{k=1}^{\infty} \), \( \sigma_k \in KK(D, A_k) \) such that \( KK(\eta_k)_x \sigma_k = \sigma \) for all \( k \geq 1 \). Since both \( KK(\eta_k)_x \) and \( \sigma_x \) are KK-equivalences, we deduce that \( (\sigma_k)_x \in KK(D, A_k(x))^{-1} \) for all \( x \in X \). By Proposition 6.4, for each \( k \geq 1 \) there is an isomorphism of \( C(X) \)-algebras \( \varphi_k : C(X) \otimes D \to A_k \) such that \( KK(\varphi_k) = \sigma_k \). Therefore if we set \( \theta_k = \eta_k \varphi_k \), then \( \theta_k \) is a \( C(X) \)-linear \( * \)-monomorphism from \( B \overset{def}{=} C(X) \otimes D \) to \( A \) such that \( KK(\theta_k) = \sigma \) and \((\theta_k(B))_{k=1}^{\infty} \) is an exhaustive sequence of \( C(X) \)-subalgebras of \( A \). Using again the weak semiprojectivity and the KK-stability of \( D \), and Lemma 4.1 after passing to a subsequence of \((\theta_k)_{k=1}^{\infty} \), we construct a sequence of finite sets \( F_k \subset B \) and a sequence of \( C(X) \)-linear \( * \)-monomorphisms \( \mu_k : B \to B \) such that

\[
\begin{align*}
(\text{i}) \quad & KK(\theta_{k+1} \mu_k) = KK(\theta_k) \quad \text{for all} \quad k \geq 1, \\
(\text{ii}) \quad & ||\theta_{k+1} \mu_k(a) - \theta_k(a)|| < 2^{-k} \quad \text{for all} \quad a \in F_k \quad \text{and} \quad k \geq 1, \\
(\text{iii}) \quad & \mu_k(F_k) \subset F_{k+1} \quad \text{for all} \quad k \geq 1, \\
(\text{iv}) \quad & \bigcup_{j=k+1}^{\infty} (\mu_{j-1} \circ \cdots \circ \mu_k)^{-1}(F_j) \quad \text{is dense in} \quad B \quad \text{and} \quad \bigcup_{j=k}^{\infty} \theta_j(F_j) \quad \text{is dense in} \quad A \quad \text{for all} \quad k \geq 1.
\end{align*}
\]

Arguing as in the proof of [28, Prop. 2.3.2], one verifies that

\[
\Delta_k(a) = \lim_{j \to \infty} \theta_j \circ (\mu_{j-1} \circ \cdots \circ \mu_k)(a)
\]

defines a sequence of \( * \)-monomorphisms \( \Delta_k : B \to A \) such that \( \Delta_{k+1} \mu_k = \Delta_k \) and the induced map \( \Delta : \lim_{\to k} (B, \mu_k) \to A \) is an isomorphism of \( C(X) \)-algebras. Let us show that \( \lim_{\to k} (B, \mu_k) \) is isomorphic to \( B \). To this purpose, in view of Elliott’s intertwining argument, it suffices to show that each map \( \mu_k \) is approximately unitarily equivalent to a \( C(X) \)-linear automorphism of \( B \). Since \( KK(\theta_k) = \sigma \), we deduce from (i) that \( KK((\mu_k)_x) = KK(\theta_k)_x \) for all \( x \in X \). By Proposition 6.4 this property implies that each map \( \mu_k \) is approximately unitarily equivalent to a \( C(X) \)-linear automorphism of \( B \). Therefore there is an isomorphism of \( C(X) \)-algebras \( \Delta : B \to A \). Let us show that we can arrange that \( KK(\Delta|_D) = \sigma \). By Theorem 5.1 there is a full \( * \)-homomorphism \( \alpha : D \to B \) such that \( KK(\alpha) = KK(\Delta)_x \sigma_x \). Since \( KK(\Delta)_x \sigma_x \in KK(D, D)^{-1} \), by Proposition 6.1 there is \( \Phi_i : D \to C(X) \otimes D \) such that \( \Phi_i \in Aut(C(X))(B) \) and \( KK(\Phi_i) = KK(\Delta)^{-1} \sigma_i \). Then \( \Phi = \Delta \Phi_1 : B \to A \) is an isomorphism such that \( KK(\Phi|_D) = KK(D, A(V))^{-1} \), for each \( v \in V \).

Dixmier and Douady [12] proved that a continuous field with fibers \( K \) over a finite dimensional locally compact Hausdorff space is locally trivial if and only if it verifies Fell’s condition, i.e. for each \( x_0 \in X \) there is a continuous section \( a \) of the field such that \( a(x) \) is a rank one projection for each \( x \) in a neighborhood of \( x_0 \). We have an analogous result:

**Corollary 7.1.** Let \( A \) be a separable \( C^* \)-algebra whose primitive spectrum \( X \) is Hausdorff and of finite dimension. Suppose that for each \( x \in X \), \( A(x) \) is KK-semiprojective, nuclear, purely infinite and stable. Then \( A \) is locally trivial if and only if for each \( x \in X \) there exist a closed neighborhood \( V \) of \( x \), a Kirchberg algebra \( D \) and \( \sigma \in KK(D, A(V)) \) such that \( \sigma_v \in KK(D, A(v)^{-1}) \) for each \( v \in V \).
Proof. One applies Theorem 1.2 for $D \otimes K$ and $A(V)$. \hfill \square

**Proposition 7.2.** Let $\psi$ be a full endomorphism of a Kirchberg algebra $D$. If $D$ is unital we assume that $\psi(1) = 1$ as well. Then the continuous $C[0, 1]$-algebra $E = \{ f \in C[0, 1] \otimes D : f(0) \in \psi(D) \}$ is locally trivial if and only if $\psi$ is homotopic to an automorphism of $D$.

Proof. Suppose that $E$ is trivial on some neighborhood of 0. Thus there is $s \in (0, 1]$ and an isomorphism $\theta : C[0, s] \otimes D \cong E[0, s]$. Since $E[0, s] \subset C[0, s] \otimes D$, there is a continuous path $\{(\theta_t)_{t \in [0, s]} : D \to E[0, s] \}$ such that $\theta_t \in \text{Aut}(D)$ for $0 < t \leq s$ and $\theta_0(D) = \psi(D)$. Set $\beta = \theta_0^{-1} \psi \in \text{Aut}(D)$. Then $\psi$ is homotopic to an automorphism via the path $\{(\theta_t)_{t \in [0, s]} : D \to E[0, s] \}$. Conversely, if $\psi$ is homotopic to an automorphism $\alpha$, then by Theorem 4.4 there is a continuous path $\{(u_t)_{t \in (0, 1]} : D \to C[0, 1] \}$ of unitaries in $D^+$ such that $\lim_{t \to 0} \| \psi(d) - u_t \alpha(d) u_t^* \| = 0$ for all $d \in D$. The path $\{(u_t)_{t \in [0, 1]} : D \to C[0, 1] \}$ defined by $\theta_0 = \psi$ and $\theta_t = u_t \alpha u_t^*$ for $t \in (0, 1]$ induces a $C[0, 1]$-linear $*$-endomorphism of $C[0, 1] \otimes D$ which maps injectively $C[0, 1] \otimes D$ onto $E$. \hfill \square

**Proof of Theorem 7.3.**

Proof. For the first part we apply Theorem 1.2 for $D = \mathcal{O}_2 \otimes K$ and $\sigma = 0$. For the second part we assert that if $D$ is a Kirchberg such that all continuous $C[0, 1]$-algebras with fibers isomorphic to $D$ are locally trivial then $D$ is stable and $KK(D, D) = 0$. Thus $D$ is KK-equivalent to $\mathcal{O}_2$ and hence that $D \cong \mathcal{O}_2 \otimes K$ by [29] Thm. 8.4.1. The Kirchberg algebra $D$ is either unital or stable [29] Prop. 4.1.3. Let $\psi : D \to D$ be a $*$-monomorphism such that $KK(D, \psi) = 0$ and such that $\psi(1_D) < 1_D$ if $D$ is unital. By Proposition 7.2 $\psi$ is homotopic to an automorphism of $\theta$ of $D$. Therefore $D$ must be nonunital (and hence stable), since otherwise $1_D$ would be homotopic to its proper subprojection $\psi(1_D)$. Moreover $KK(\theta) = KK(D, \psi) = 0$ and hence $KK(D, D) = 0$ since $\theta$ is an automorphism. \hfill \square

We turn now to unital $C(X)$-algebras.

**Theorem 7.3.** Let $A$ be a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space $X$. Suppose that each fiber $A(x)$ is nuclear simple and purely infinite. Then $A$ is isomorphic to $C(X) \otimes D$, for some $KK$-semiprojective unital Kirchberg algebra $D$, if and only if there is $\sigma \in KK(D, A)$ such that $K_0(\sigma)[1_D] = [1_A]$ and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For any such $\sigma$ there is an isomorphism of $C(X)$-algebras $\Phi : C(X) \otimes D \to A$ such that $KK(\Phi|_D) = \sigma$.

Proof. We verify the nontrivial implication. $X$ is metrizable by Lemma 2.2. $A$ is a continuous $C(X)$-algebra by Lemma 2.3. By Theorem 1.2 there is an isomorphism $\Phi : C(X) \otimes D \otimes K \to A \otimes K$ such that $KK(\Phi) = \sigma$. Since $K_0(\sigma)[1_D] = [1_A]$, and since $A \otimes K$ contains a full properly infinite projection, we may arrange that $\Phi(1_{C(X) \otimes D \otimes 1} \otimes 1_{11}) = 1_A \otimes \varepsilon_{11}$ after conjugating $\Phi$ by some unitary $u \in M(A \otimes K)$. Then $\varphi = \Phi|_{C(X) \otimes D \otimes 1}$ satisfies the conclusion of the theorem. \hfill \square

**Proof of Theorem 7.4.**

Proof. Let $D$ be a $KK$-semiprojective unital Kirchberg algebra $D$ such that every unital $*$-endomorphism of $D$ is a KK-equivalence. Suppose that $A$ is a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space the fibers of which are isomorphic to $D$. We shall prove that $A$ is locally trivial. By Theorem 7.3 it suffices to show that each point $x_0 \in X$ has a closed neighborhood
V for which there is \( \sigma \in KK(D, A(V)) \) such that \( K_0(\sigma)[1_D] = [1_{A(V)}] \) and \( \sigma_x \in KK(D, A(x))^{-1} \) for all \( x \in V \).

Let \( (V_n)_{n=1}^\infty \) be a decreasing sequence of closed neighborhoods of \( x_0 \) whose intersection is \( \{x_0\} \). Then \( A(x_0) \cong \lim_{n \to \infty} A(V_n) \). By assumption, there is an isomorphism \( \eta : D \to A(x_0) \). Since \( D \) is KK-semiprojective, there is \( m \geq 1 \) such that \( KK(\eta) \) lifts to some \( \sigma \in KK(D, A(V_m)) \) such that \( K_0(\sigma)[1_D] = [1_{A(V_m)}] \). Let \( x \in V_m \). By assumption, there is an isomorphism \( \phi : A(x) \to D \).

The \( K_0 \)-morphism induced by \( KK(\phi)\sigma_x \) maps \( [1_D] \) to itself. By Theorem 6.1 there is a unital \( \ast \)-homomorphism \( \psi : D \to A \) such that \( KK(\psi) = KK(\phi)\sigma_x \). By assumption we must have \( KK(\psi) \in KK(D, D)^{-1} \) and hence \( \sigma_x \in KK(D, A(x))^{-1} \) since \( \sigma \) is an isomorphism. Therefore \( A(V_m) \cong C(V_m) \otimes D \) by Theorem 7.4.

Conversely, let us assume that all separable unital continuous \( C[0,1] \)-algebras with fibers isomorphic to \( D \) are locally trivial. Let \( \psi \) be any unital \( \ast \)-endomorphism of \( D \). By Proposition 7.2 \( \psi \) is homotopic to an automorphism of \( D \) and hence \( KK(\psi) \) is invertible. \( \square \)

**Proof of Theorem 7.4**

**Proof.** Let \( A \) be as in Theorem 6.1 and let \( n \in \{2, 3, \ldots \} \cup \{\infty\} \). It is known that \( O_n \) satisfies the UCT. Moreover \( K_0(O_n) \) is generated by \( \{1_{O_n}\} \) and \( K_1(O_n) = 0 \). Therefore any unital \( \ast \)-endomorphism of \( O_n \) is a KK-equivalence. It follows that \( A \) is locally trivial by Theorem 1.4.

Suppose now that \( n = 2 \). Since \( KK(O_2, O_2) = KK(O_2, A) = 0 \), we may apply Theorem 1.4 with \( \sigma = 0 \) and obtain that \( A \cong C(X) \otimes O_2 \). Suppose now that \( n = \infty \). Let us define \( \theta : K_0(O_\infty) \to K_0(A) \) by \( \theta(\eta[1_{O_\infty}]) = k[1_A] \), \( k \in \mathbb{Z} \). Since \( O_\infty \) satisfies the UCT, \( \theta \) lifts to some element \( \sigma \in KK(O_\infty, A) \). By Theorem 1.4 it follows that \( A \cong C(X) \otimes O_\infty \). Finally let us consider the case \( n \in \{3, 4, \ldots \} \). Then \( K_0(O_n) = \mathbb{Z}/(n-1) \). Since \( O_n \) satisfies the UCT, the existence of an element \( \sigma \in KK(O_n, A) \) such that \( K_0(\sigma)[1_{O_n}] = [1_A] \) is equivalent to the existence of a morphism of groups \( \theta : \mathbb{Z}/(n-1) \to K_0(A) \) such that \( \theta(1) = [1_A] \). This is equivalent to requiring that \( (n-1)[1_A] = 0 \). \( \square \)

As a corollary of Theorem 1.4 we have that \( [X, Aut(O_\infty)] \) reduces to a point. The homotopy groups of the endomorphisms of the stable Cuntz-Krieger algebras were computed in [1]. Let \( v_1, \ldots, v_n \) be the canonical generators of \( O_n \), \( 2 \leq n < \infty \).

**Theorem 7.4.** For any compact metrizable space \( X \) there is a bijection \( [X, Aut(O_n)] \to K_1(C(X) \otimes O_n) \). The \( k \)-th homotopy group \( \pi_k(Aut(O_n)) \) is isomorphic to \( \mathbb{Z}/(n-1) \) if \( k \) is odd and it vanishes if \( k \) is even. In particular \( \pi_1(Aut(O_n)) \) is generated by the class of the canonical action of \( \mathbb{T} \) on \( O_n \), \( \lambda_z(v_i) = zv_i \).

**Proof.** Since \( O_n \) satisfies the UCT, we deduce that \( End(O_n)^* = End(O_n) \). An immediate application of Proposition 6.1 shows that the natural map \( Aut(O_n) \hookrightarrow End(O_n) \) induces an isomorphism of groups \( [X, Aut(O_n)] \cong [X, End(O_n)] \). Let \( \iota : O_n \hookrightarrow C(X) \otimes O_n \) be defined by \( \iota(v_i) = 1_{C(X)} \otimes v_i \), \( i = 1, \ldots, n \). The map \( \psi \mapsto u(\psi) = \psi(v_1)u(v_1)^* + \cdots + \psi(v_n)u(v_n)^* \) is known to be a homeomorphism from \( Hom(O_n, C(X) \otimes O_n) \) to the unitary group of \( C(X) \otimes O_n \). Its inverse maps a unitary \( W \) to the \( \ast \)-homomorphism \( \psi \) uniquely defined by \( \psi(v_i) = Wv_i \), \( i = 1, \ldots, n \). Therefore

\[
[X, Aut(O_n)] \cong [X, End(O_n)] \cong \pi_0(U(C(X) \otimes O_n)) \cong K_1(C(X) \otimes O_n).
\]
The last isomorphism holds since $\pi_0(U(B)) \cong K_1(B)$ if $B \cong B \otimes O_\infty$, by \cite{28} Lemma 2.1.7. One verifies easily that if $\varphi \in \text{Hom}(O_n, C(X) \otimes O_n)$, then $u(\psi \varphi) = \psi(\varphi)(\psi)$ holds for all $\psi \in \text{Hom}(O_n, C(X) \otimes O_n)$. Therefore the bijection $\chi : [X, \text{End}(D)] \rightarrow K_1(C(X) \otimes O_n)$ is an isomorphism of groups whenever $K_1(\tilde{\psi}) = \text{id}$ for all $\psi \in \text{Hom}(O_n, C(X) \otimes O_n)$. Using the $C(X)$-linearity of $\tilde{\psi}$ one observes that this holds if the $n-1$ torsion of $K_0(C(X))$ reduces to $\{0\}$, since in that case the map $K_1(C(X)) \rightarrow K_1(C(X) \otimes O_n)$ is surjective by the Künneth formula.

**Corollary 7.5.** Let $X$ be a finite dimensional compact metrizable space. The isomorphism classes of unital separable $C(SX)$-algebras with all fibers isomorphic to $O_n$ are parameterized by $K_1(C(X) \otimes O_n)$.

**Proof.** This follows from Theorems 11.1 and 11.3 since the locally trivial principal $H$-bundles over $SX = X \times [0,1]/X \times \{0,1\}$ are parameterized by the homotopy classes $[X, H]$ if $H$ is a path connected group \cite{17} Cor. 8.4]. Here we take $H = \text{Aut}(O_n)$. \hfill $\square$

Examples of nontrivial unital $C(X)$-algebras with fiber $O_n$ over a $2m$-sphere arising from vector bundles were exhibited in \cite{36}, see also \cite{33}.

We need some preparation for the proof of Theorem 11.5. Let $G$ be a group, let $g \in G$ and set $\text{End}(G, g) = \{\alpha \in \text{End}(G) : \alpha(g) = g\}$. The pair $(G, g)$ is called weakly rigid if $\text{End}(G, g) \subset \text{Aut}(G)$ and rigid if $\text{End}(G, g) = \{\text{id}_G\}$.

**Theorem 7.6.** If $G$ is a finitely generated abelian group, then $(G, g)$ is weakly rigid if and only if $(G, g)$ is isomorphic to one of the pointed groups from the list $G$ of Theorem 11.3.

**Proof.** First we make a number of remarks.

1. $(G, g)$ is weakly rigid if and only if $(G, \alpha(g))$ is weakly rigid for some (or any) $\alpha \in \text{Aut}(G)$. Indeed if $\beta \in \text{End}(G, g)$ then $\alpha \beta \alpha^{-1} \in \text{End}(G, \alpha(g))$.

2. By considering the zero endomorphism of $G$ we see that if $(G, g)$ is weakly rigid and $G \neq 0$ then $g \neq 0$.

3. If $(G \oplus H, g \oplus h)$ is weakly rigid, then so are $(G, g)$ and $(H, h)$.

4. Let us observe that $(\mathbb{Z}^2, g)$ is not weakly rigid for any $g$. Indeed, if $g = (a, b) \neq 0$, then the matrix $\begin{pmatrix} 1 + b^2 & -ab \\ -ab & 1 + a^2 \end{pmatrix}$ defines an endomorphism $\alpha$ of $\mathbb{Z}^2$ such that $\alpha(g) = g$, but $\alpha$ is not invertible since $\det(\alpha) = 1 + a^2 + b^2 > 1$.

5. Let $p$ be a prime and let $1 \leq e_1 \leq e_2$, $0 \leq s_1 < e_1$, $0 \leq s_2 < e_2$ be integers. If $(G, g) = (\mathbb{Z}/p^{e_1} \oplus \mathbb{Z}/p^{e_2}, p^{s_1} \oplus p^{s_2})$ is weakly rigid then $0 < s_2 - s_1 < e_2 - e_1$. Indeed if $s_1 \geq s_2$ then the matrix $\begin{pmatrix} 0 & p^{s_1-s_2} \\ p^{s_1-s_2} & 1 \end{pmatrix}$ induces a noninjective endomorphism of $(G, g)$. Also if $s_1 < s_2$ and $s_2 - s_1 \geq e_2 - e_1$ then $p^{e_1}b = 0$ in $\mathbb{Z}/p^{e_2}$, where $b = p^{s_2-s_1}$ and so the matrix $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ induces a well-defined noninjective endomorphism of $(G, g)$.

6. Let $p$ be a prime and let $1 \leq k$, $0 \leq s < e$ be integers. Suppose that $(\mathbb{Z} \oplus \mathbb{Z}/p^e, k \oplus p^s)$ is weakly rigid. Then $k$ is divisible by $p^{s+1}$. Indeed, seeking a contradiction suppose that $k$ can be written as $k = p^tc$ where $0 \leq t \leq s$ and $c$ are integers such that $c$ is not divisible by $p$. Let $d$ be
an integer such that $dc - 1$ is divisible by $p^r$. Then the matrix

\[
\begin{pmatrix}
1 & 0 \\
dp^s - t & 0
\end{pmatrix}
\]

induces a noninjective endomorphism of $(\mathbb{Z} \oplus \mathbb{Z}/p^r, k \oplus p^s)$.

Suppose now that $(G, g)$ is weakly rigid. We shall show that $(G, g)$ is isomorphic to one of the pointed groups from the list $\mathcal{G}$. Since $G$ is abelian and finitely generated it decomposes as a direct sum of its primary components

\[(20) \quad G \cong \mathbb{Z}^r \oplus G(p_1) \oplus \cdots \oplus G(p_m)\]

where $p_i$ are distinct prime numbers. Each primary component $G(p_i)$ is of the form

\[(21) \quad G(p_i) = \mathbb{Z}/p_i^{e_i+1} \oplus \cdots \oplus \mathbb{Z}/p_i^{e_i(n)}\]

where $1 \leq e_{i1} \leq \cdots \leq e_{in(i)}$ are positive integers. Corresponding to the decomposition we write the base point $g = g_0 \oplus g_1 \oplus \cdots \oplus g_m$ with $g_0 \in \mathbb{Z}^r$ and $g_i \in G(p_i)$ for $i \geq 1$. If $g_{ij}$ is the component of $g_i$ in $\mathbb{Z}/p^{e_i}$, then it follows from (1), (2) and (3) that we may assume that $g_{ij} = p^{s_{ij}}$ for some integer $0 \leq s_{ij} < e_{ij}$. Using (3) and (4) we deduce that $r = 1$ in (20) and that $g_0 = k \neq 0$ by (2). We may assume that $k \geq 1$ by (1). Then using (3) and (5) we deduce that for each $1 \leq i \leq m$, $0 < s_{ij+1} - s_{ij} < e_{ij+1} - e_{ij}$ for $1 \leq j < n(i)$. Finally, from (3) and (6) we see that $k$ is divisible by the product $p_1^{s_{1n(1)}} \cdots p_m^{s_{mn(m)}}$. Therefore $(G, g)$ is isomorphic to one of the pointed groups on the list $\mathcal{G}$.

Conversely, we shall prove that if $(G, g)$ belongs to the list $\mathcal{G}$ then $(G, g)$ is weakly rigid. This is obvious if $G$ is torsion free i.e. for $(\{0\}, 0)$ and $(\mathbb{Z}, k)$ with $k \geq 1$.

Let us consider the case when $G$ is a torsion group. Since

\[
\text{End}(G(p_1) \oplus \cdots \oplus G(p_m), g_1 \oplus \cdots \oplus g_m) \cong \bigoplus_{i=1}^m \text{End}(G(p_i), g_i)
\]

it suffices to assume that $G$ is a $p$-group,

\[(G, g) = (\mathbb{Z}/p^{e_1} \oplus \cdots \oplus \mathbb{Z}/p^{e_n}, p^s \oplus \cdots \oplus p^s)\]

with $0 \leq s_i < e_i$ for $i = 1, \ldots, n$ and $0 < s_{i+1} - s_i < e_{i+1} - e_i$ for $1 \leq i < n$. For each $0 \leq i, j \leq n$ set $e_{ij} = \max\{e_i - e_j, 0\}$. It follows immediately that $s_i < e_{ij} + s_i$ for all $i \neq j$. Let $\alpha \in \text{End}(G, g)$. It is well-known that $\alpha$ is induced by a square matrix $A = [a_{ij}] \in M_n(\mathbb{Z})$ with the property that each entry $a_{ij}$ is divisible by $p^{e_{ij}}$ and so $a_{ij} = p^{s_{ij}}b_{ij}$ for some $b_{ij} \in \mathbb{Z}$, see [10]. Since $\alpha(g) = g$, we have $\sum_{j=1}^n b_{ij}p^{e_{ij} + s_i} = p^s$ in $\mathbb{Z}/p^{e_i}$ for all $0 \leq i \leq n$. Since $e_{ij} + s_i > s_i$ for $i \neq j$ and $e_i > s_i$ we see that $b_{ii} - 1$ must be divisible by $p$ for all $1 \leq i \leq n$. Since $\det(A)$ is congruent to $b_{11} \cdots b_{nn}$ modulo $p$ it follows that $\det(A)$ is not divisible by $p$ and so $\alpha \in \text{Aut}(G)$ by [10].

Finally consider the case when $(G, g) = (\mathbb{Z} \oplus G(p_1) \oplus \cdots \oplus G(p_m), k \oplus g_1 \oplus \cdots \oplus g_m)$. If $\gamma \in \text{End}(G, g)$ then there exist $\alpha_i \in \text{End}(G(p_i), g_i)$ and $d_i \in G(p_i)$, $1 \leq i \leq n$, such that $\gamma(x_0 \oplus x_1 \oplus \cdots \oplus x_n) = x_0 \oplus (\alpha_1(x_1) + x_0d_1) \oplus \cdots \oplus (\alpha_m(x_m) + x_0d_m)$. Note that if each $\alpha_i$ is an automorphism then so is $\gamma$. Indeed, its inverse is $\gamma^{-1}(x_0 \oplus x_1 \oplus \cdots \oplus x_m) = x_0 \oplus (\alpha_1^{-1}(x_1) + x_0c_1) \oplus \cdots \oplus (\alpha_m(x_m)^{-1} + x_0c_m)$, where $c_i = -\alpha_i^{-1}(d_i)$. Therefore it suffices to consider the case $m = 1$, i.e.

\[(G, g) = (\mathbb{Z} \oplus \mathbb{Z}/p^{e_1} \oplus \cdots \oplus \mathbb{Z}/p^{e_n}, k \oplus p^{s_1} \oplus \cdots \oplus p^{s_n})\]

and $(G, g)$ is on the list $\mathcal{G}$ (e). In particular $k = p^{s_n + 1} \ell$ for some $\ell \in \mathbb{Z}$. Let $\gamma \in \text{End}(G, g)$. Then there exists $\alpha \in \text{End}(G(p))$ and $d \in G(p)$ such that $\gamma(x_0 \oplus x) = x_0 \oplus (\alpha(x) + x_0d)$. Just as above,
\( \alpha \) is induced by a square matrix \( A \in M_n(\mathbb{Z}) \) of the form \( A = [b_{ij} p^{s_{ij}}] \in M_n(\mathbb{Z}) \) with \( b_{ij} \in \mathbb{Z} \), \( e_{ij} = \max\{e_i - e_j, 0\} \). Since \( \gamma(g) = g \) we have that \( p^{s_{n+1} \ell d_i + \sum_{j=1}^n b_{ij} p^{s_{ij} + s_1} = p^{s_i} \) in \( \mathbb{Z}/p^s \) for all \( 0 \leq i \leq n \), where the \( d_i \) are the components of \( d \). By reasoning as in the case when \( G \) was a torsion group considered above, since \( s_{n+1} > s_i \) for all \( 1 \leq i \leq n \), \( e_i + s_j > s_i \) for all \( i \neq j \) and \( e_i > s_i \), it follows again that each \( b_{ii} - 1 \) is divisible by \( p \) and that the endomorphism \( \alpha \) of \( G(p) \) induced by the matrix \( A \) is an automorphism. We conclude that \( \gamma \) is an endomorphism.

**Proof of Theorem 1.5**

(i) By Theorem 1.4 both \( \mathcal{O}_2 \) and \( \mathcal{O}_\infty \) have the automatic triviality property. Conversely, suppose that \( D \) has the automatic triviality property, where \( D \) is a unital Kirchberg algebra satisfying the UCT and such that \( K_*(D) \) is finitely generated. We shall prove that \( D \) is isomorphic to either \( \mathcal{O}_2 \) or \( \mathcal{O}_\infty \).

Let \( Y \) be a finite connected CW-complex and let \( \iota : D \to C(Y) \otimes D \) be the map \( \iota(d) = 1 \otimes d \). Let \([D,C(Y) \otimes D]\) denote the homotopy classes of unital \(*\)-homomorphisms from \( D \) to \( C(Y) \otimes D \). By Theorem 3.1 the image of the map \( \Delta : [D,C(Y) \otimes D] \to KK(D,C(Y) \otimes D) \) defined by \([\varphi] \mapsto KK(\varphi) - KK(\iota) \) coincides with the kernel of the restriction morphism \( \rho : KK(D,C(Y) \otimes D) \to KK(C1_D,C(Y) \otimes D) \).

We claim that \( \ker \rho \) must vanish for all \( Y \). Let \( h \in \ker \rho \). Then there is a unital \(*\)-homomorphism \( \varphi : D \to C(Y) \otimes D \) such that \( \Delta[\varphi] = h \). By Theorem 1.4 each unital endomorphism of \( D \) induces a \( KK \)-equivalence. Therefore, by Proposition 3.1 there is a \(*\)-homomorphism \( \Phi : D \to (C(Y) \otimes D) \) such that \( \Phi_y \in \text{Aut}(D) \) for all \( y \in Y \) and \( KK(\Phi) = KK(\varphi) \). Therefore \( \Delta[\Phi] = KK(\Phi) - KK(\iota) = h \). By hypothesis, the Aut(\( D \))-principal bundle constructed over the suspension of \( Y \) with characteristic map \( y \mapsto \Phi_y \) is trivial. It follows then from [17, Thm. 8.2 p85] that this map is homotopic to the to the constant map \( Y \to \text{Aut}(D) \) which shrinks \( Y \) to id_\( D \). This implies that \( \Phi \) is homotopic to \( \iota \) and hence \( h = 0 \).

Let us now observe that \( \ker \rho \) contains subgroups isomorphic to \( \text{Hom}(K_1(D),K_1(D)) \) and \( \text{Ext}(K_0(D),K_0(D)) \) if \( Y = T \), since \( D \) satisfies the UCT. It follows that both these groups must vanish and so \( K_1(D) = 0 \) and \( K_0(D) \) is torsion free. On the other hand, \( (K_0(D),[1_D]) \) is weakly rigid by the first part of the proof. Since \( K_0(D) \) is torsion free we deduce from Theorem 7.1 that either \( K_0(D) = 0 \) in which case \( D \cong \mathcal{O}_2 \) or that \( (K_0(D),[1_D]) \cong (\mathbb{Z},k) \), \( k \geq 1 \), in which case \( D \cong M_k(\mathcal{O}_\infty) \) by the classification theorem of Kirchberg and Phillips.

To conclude the proof, it suffices to show that \( \ker \rho \neq 0 \) if \( D = M_k(\mathcal{O}_\infty), k \geq 2 \) and \( Y \) is the two-dimensional space obtained by attaching a disk to a circle by a degree-k map. Since \( K_0(C(Y) \otimes \mathcal{O}_\infty) \cong \mathbb{Z} \oplus \mathbb{Z}/k \) we can identify the map \( \rho \) with the map \( \mathbb{Z} \oplus \mathbb{Z}/k \to \mathbb{Z} \oplus \mathbb{Z}/k, x \mapsto kx \) and so \( \ker \rho \cong \mathbb{Z}/k \neq 0 \) if \( k \geq 2 \). \( \square \)
**Added in proof.** Some of the results from this paper are further developed in [9]. Theorem 1.2 was shown to hold for all stable Kirchberg algebras D. The assumption that X is finite dimensional is essential. Theorem 1.1. Theorem 1.5 (ii) extends as follows: $O_2$, $O_\infty$, and $B \otimes O_\infty$, where B is a unital UHF algebra of infinite type, are the only unital Kirchberg algebras which satisfy the UCT and have the automatic triviality property.

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