CKM and PMNS mixing matrices from discrete subgroups of SU(2)

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Abstract. Remaining within the realm of the Standard Model (SM) local gauge group, this first principles derivation of both the PMNS and CKM matrices utilizes quaternion generators of the three discrete (i.e., finite) binary rotational subgroups of SU(2) called [3,3,2], [4,3,2], and [5,3,2] for three lepton families in \( \mathbb{R}^3 \) and four related discrete binary rotational subgroups [3,3,3], [4,3,3], [3,4,3], and [5,3,3] represented by four quark families in \( \mathbb{R}^4 \). The traditional 3x3 CKM matrix is extracted as a submatrix of the 4x4 CKM4 matrix. If these two additional quarks b’ and t’ of a 4th quark family exist, there is the possibility that the SM lagrangian may apply all the way down to the Planck scale. There are then numerous other important consequences. The Weinberg angle is derived using these same quaternion generators, and the triangle anomaly cancellation is satisfied even though there is an obvious mismatch of three lepton families to four quark families. In a discrete space, one can also use these generators to derive a unique connection from the electroweak local gauge group SU(2)\(_L\) x U(1)\(_Y\) acting in \( \mathbb{R}^4 \) to the discrete group Weyl E\(_8\) in \( \mathbb{R}^8 \). By considering Lorentz transformations in discrete (3,1)-D spacetime, one obtains another Weyl E\(_8\) discrete symmetry group in \( \mathbb{R}^8 \), so that the combined symmetry is Weyl E\(_8\) x Weyl E\(_8\) = “discrete” SO(9,1) in 10-D spacetime. This unique connection is in direct contrast to the \( 10^{500} \) possible connections for superstring theory!

1. Introduction

One of the greatest challenges in particle physics is to determine the first principles origin of the quark and lepton mixing matrices CKM and PMNS that relate the flavor states to the mass states. Within the realm of the Standard Model (SM) but considering a specific SU(2) discrete (i.e., finite) subgroup for each lepton and quark family, I determine that the true source of these mixings [1,2] is the mismatch of one Pauli generator for SU(2) with the corresponding generator for each proposed family subgroup. By using a linear combination of the generators for these specific discrete binary rotational subgroups of the electroweak (EW) gauge group SU(2)\(_L\) x U(1)\(_Y\), the mixing angles and matrices are determined directly.

One realizes immediately that the PMNS and CKM matrices are telling us that the very successful SM is an excellent approximation that requires but a small geometrical reconciliation in order to agree with the proposed discrete symmetry properties of the physical lepton and quark families. I repeat that these derivations are done within the realm of the SM and no alternative theoretical framework beyond the SM is required.

Many talks at this conference are devoted to deriving the neutrino PMNS mixing matrix and the quark CKM mixing matrix from various additional discrete horizontal symmetries. I suggest that they are missing the true origin of the mixing matrices: these matrices connect the lepton
and quark weak isospin $\pm \frac{1}{2}$ states representing discrete individual family symmetries acting collectively to the isospin $\pm \frac{1}{2}$ states of SU(2), the traditionally assumed continuous symmetry group having two flavor states per family.

The Standard Model (SM) is based upon symmetry properties of the local gauge group SU(2)$_L \times U(1)Y \times SU(3)C$, which defines an electroweak (EW) interaction part and a color interaction part acting in a continuous internal symmetry space. However, we know that the SM is incomplete, having 28 parameters. So, something important is absent in our traditional understanding. I strongly suspect that the SM gauge group may be the best one possible but requiring only the proposed correction in order to find agreement with the discrete symmetries represented by the leptons and quarks.

Perhaps the major assumption of the SM is that space is continuous. At present, no one knows whether space at the ultimate scale, the Planck scale of about $10^{-35}$ meters, is continuous or discrete. Although there is no evidence presently, one can suggest that space is actually discrete at the Planck scale and is described by an $C^2 = R^4$ lattice of mathematical nodes. The nodes themselves would possess no measurable physical properties, so the established physical properties of the fundamental leptons and quarks must "emerge" from the collection of nodes that as a unit exhibit the specific discrete rotational symmetries for each particle, a lepton or quark. In this manner, there is an end to the "Russian doll" hierarchy of particles within particles.

I point out that a discrete space at the Planck scale is not required absolutely for the mathematical and physical relationships and concepts that I discuss, but such a viewpoint does offer a reasonable avenue for trying to understand Nature at its ultimate scale.

2. Brief mathematical review

Each family of leptons and of quarks has two isospin flavor states that are the basis states in $C^2$ for SU(2). The subgroups of SU(2) for the leptons and quarks also have two orthogonal basis states per family.

In order to calculate the PMNS and CKM mixing matrix values in the following sections, I use unit quaternions instead of the more familiar and equivalent unitary 2x2 complex matrices of SU(2). The unit quaternion $q = a + bi + cj + dk$, where the coefficients a, b, c, d are real numbers for the one real and three imaginary axes. The unit quaternion generators are equivalent to the SU(2) generators, with the three Pauli generators $\sigma_x, \sigma_y, \sigma_z$, when multiplied by i, corresponding to the quaternion generators k, j, and i, respectively. These three quaternion generators are the keys to deriving the PMNS and CKM matrices from first principles [1,2].

The unit quaternion spans the space $R^4$ while the imaginary prime part spans the subspace $R^3$. With $i^2 = j^2 = k^2 = ijk = -1$, the quaternion $q$ can be expressed as an SU(2) matrix

$$\begin{bmatrix}
a + bi & c + di \\
-c + di & a - bi
\end{bmatrix}.$$ 

Both the quaternions and the SU(2) matrices operate in the unitary plane $C^2$ with its two orthogonal complex axes, so the quaternion can be written also as $q = u + vj$, with $u = a + bi$ and $v = c + di$.

In previous research [3,4] on possible mathematical lepton family origins I have determined that each lepton family represents its own unique discrete rotational subgroup of SU(2), i.e., a discrete quaternion group. That is, one requires three specific discrete (i.e., finite) binary rotational subgroups of the EW gauge group SU(2)$_L \times U(1)Y$, one group for each lepton family, thereby remaining within the realm of the SM lagrangian. The only finite (i.e., discrete) quaternion groups are [5,6]
with the 2 in front meaning binary (double) group, the double cover of the normal 3-D rotation group by SU(2) over SO(3).

The three lepton family groups are specifically the binary rotational groups called \([3,3,2]\), \([4,3,2]\), and \([5,3,2]\), (often labeled 2T, 2O, and 2I, respectively), which have discrete rotational symmetries in \(R^3\). Some properties of these groups are given in Table 1.

| Table 1. Lepton Family Group Assignments |
|------------------------------------------|
| Fam. Group | Order | \(N_i\) | Mass (\(\ell_i\)) (MeV) | Mass (\(\nu_i\)) (MeV) |
| \(\nu_e,e\) | [332] | 2T | 24 | 1 | 0.511 | \(~ 0\) |
| \(\nu_\mu,\mu\) | [432] | 2O | 48 | 108 | 105.7 | \(~ 0\) |
| \(\nu_\tau,\tau\) | [532] | 2I | 120 | 1728 | 1776.8 | \(~ 0\) |

My assignment of each particular binary rotational group to a specific lepton family in Table 1 is based upon the \(N_i\) value for each group. Here the integer \(N_i\) arises from the mathematical syzygy for each group relating the \(j\)-invariant of elliptic modular functions (and of the Monster Group) to functions of two complex variables that remain invariant under operations of the group. The origin of these different \(N_i\) values is derived in a famous book by F. Klein in 1884 titled "The Icosahedron and solutions of equations of the fifth degree", originally in German but available in English in a Dover edition.

One can see a remarkable similarity of the mass values in column 5 of the three charged leptons to the three \(N_i\) values associated with discrete groups \([332]\), \([432]\), and \([532]\). That was my first clue that these discrete binary rotational subgroups of SU(2) could be related to the fundamental leptons of the SM.

Each group has two degenerate basis states which must be taken in linear superposition to form the two actual orthogonal fermion flavor states in each family, i.e., \((\nu_e,e), (\nu_\mu,\mu), \text{ and } (\nu_\tau,\tau)\). That is, from basis states \(|1>\) and \(|2>\) having the same degenerate energy \(E_0\), one forms the new orthogonal flavor states \(|I>\) and \(|II>\) of different energies

\[
|I> = [ |1> - |2>]/\sqrt{2} \quad E_I = E_0 + A_i \quad (2)
\]

\[
|II> = [ |1> + |2>]/\sqrt{2} \quad E_{II} = E_0 - A_i \quad (3)
\]

where the quantity \(-A_i\) is associated with the quantum mechanical probability amplitude for state \(|1>\) to end up as state \(|2>\) and vice-versa. As a possibility, each of the different \(A_i\) values could be a function of the local spacetime environment. For an example, if the "up" flavor state \(|1>\) is a neutrino state \(|\nu>\), its passage through the material of the Sun or through the Earth may have measurable effects in different amounts for each of the families.

3. The PMNS matrix derivation
This section reviews the mathematical procedure used in my 2013 derivation [1] of the PMNS matrix from first principles. As you know, the three quaternions \(i, j, \text{ and } k\), can generate all
rotations in $\mathbb{R}^3$ about a chosen axis or, equivalently, all rotations in the plane perpendicular to this axis. For example, the quaternion $k$ is a binary rotation by $180^\circ$ in the $i$-$j$ plane.

One now must construct the three SU(2) generators, $U_1 = j$, $U_2 = k$, and $U_3 = i$, from the three quaternion generators from each of the discrete subgroups $[3,3,2]$, $[4,3,2]$, and $[5,3,2]$ for the three lepton families. The complete mathematical description [7] for the generators $R_s$ operating on the unit vector $x$ in $\mathbb{R}^3$ extending from the origin to the surface of the unit sphere $S^2$ is given by

$$ R_s = i x U_s $$

where $s = 1, 2, 3$ and

$$ U_1 = j, \quad U_2 = -icos\frac{\pi}{q} - jcos\frac{\pi}{p} + ksin\frac{\pi}{h}, \quad U_3 = i, $$

with $h = 4, 6, 10$ for the three lepton family groups $[p,q,2]$, respectively. Their $U_2$ generators are listed in Table 2, with the quantity $\phi = (\sqrt{5}+1)/2$, the golden ratio.

My three lepton family binary rotational groups, $[3,3,2]$, $[4,3,2]$, and $[5,3,2]$, all have the same SU(2) generators $U_1 = j$ and $U_3 = i$, but one sees immediately in Table 2 that each $U_2$ is a different quaternion generator operating in $\mathbb{R}^3$. One obtains the correct neutrino PMNS mixing angles from the linear superposition of their $U_2$’s by making the total $U_2 = k$, agreeing with the SU(2) generator. This particular combination of three discrete angle rotations is now equivalent to a rotation in the $i$-$j$ plane by the quaternion $k$.

### Table 2. Lepton Family Quaternion Generators $U_2$

| Fam. Grp. | Generator | Factor Angle$^\circ$ |
|----------|-----------|-----------------------|
| $\nu_e, \nu_e$ | 332 | $-\frac{1}{2}i - \frac{1}{2}j + \frac{1}{\sqrt{2}}k$ | -0.2645 105.337 |
| $\nu_\mu, \nu_\mu$ | 432 | $-\frac{1}{2}i - \frac{1}{\sqrt{2}}k + \frac{1}{2}j$ | 0.8012 36.755 |
| $\nu_\tau, \nu_\tau$ | 532 | $-\frac{1}{2}i - \frac{\phi}{2}j + \frac{\phi^{-1}}{2}k$ | -0.5367 122.459 |

The total contribution of all three $U_2$ generators should be $k$, so there are three equations for the three unknown factors, which are determined to be: $-5.537$, $16.773$, and $-11.236$, which become normalized to the values in the table. The resulting angles in Table 2 are the arccosines of these factors, i.e., their projections to the $k$-axis, but they are twice the rotation angles required in $\mathbb{R}^3$, a property of quaternion rotations.

Using one-half of these angles produces

$$ \theta_1 = 52.67^\circ, \quad \theta_2 = 18.38^\circ, \quad \theta_3 = 61.23^\circ, $$

then taking differences results in predicted mixing angles

$$ \theta_{12} = 34.29^\circ, \quad \theta_{13} = -8.56^\circ, \quad \theta_{23} = -42.85^\circ. $$

The absolute values of these mixing angles are all within the $1\sigma$ range of their values for the normal mass hierarchy [8,9] as determined from several experiments:

$$ \theta_{12} = \pm 34.47^\circ, \quad \theta_{13} = \pm 8.5^\circ, \quad \theta_{23} = \pm (38.39^\circ - 45.81^\circ). $$

The ± signs arise from the squares of the sines of the angles determined by the experiments.

For three lepton families, assuming that the charged lepton matrix is the identity, the neutrino flavor states $\nu_e$, $\nu_\mu$, $\nu_\tau$, and the mass states $\nu_1$, $\nu_2$, $\nu_3$, are related by the PMNS matrix $V_{ij}$.
\[
\begin{bmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{bmatrix} =
\begin{bmatrix}
V_{e1} & V_{e2} & V_{e3} \\
V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\
V_{\tau 1} & V_{\tau 2} & V_{\tau 3}
\end{bmatrix}
\begin{bmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{bmatrix}.
\]

The PMNS entries are the products of the sines and cosines of the predicted angles using the
standard parametrization of the matrix, producing:

\[
\begin{bmatrix}
0.817 & 0.557 & -0.149 e^{-i\delta} \\
-0.413 - 0.084 e^{i\delta} & 0.605 - 0.057 e^{i\delta} & -0.673 \\
-0.383 + 0.090 e^{i\delta} & 0.562 + 0.061 e^{i\delta} & 0.725
\end{bmatrix}.
\]

For direct comparison, the empirically estimated PMNS matrix for the normal hierarchy of
neutrino masses is

\[
\begin{bmatrix}
0.822 & 0.547 & -0.150 + 0.038 i \\
-0.356 + 0.0198 i & 0.704 + 0.0131 i & 0.614 \\
0.442 + 0.0248 i & -0.452 + 0.0166 i & 0.774
\end{bmatrix}.
\]

Comparing the \(V_{e3}\) elements from each, the phase angle \(\delta\) is confined to be \(0^\circ \leq \delta \leq \pm 14.8^\circ\),
an angle in agreement with the T2K collaboration value of \(\delta \approx 0\) but quite different from other proposed \(\delta \approx \pi\) values.

The derivation of the neutrino mixing angles from first principles means that deviations from
these values could indicate that the charged lepton mixing matrix is not the identity as assumed.

4. What has been learned?

At this point I pause to review what has been learned by the successful derivation of the neutrino
mixing angles and the PMNS matrix when each lepton family has its own specific discrete binary
rotational group in \(R^3\), namely, the discrete quaternion groups 2T, 2O, and 2I.

(i) The three neutrino mixing angles derive from the three \(U_2\) generator projections to the
k-axis in order to collectively act like the SU(2) quaternion generator \(k\).

(ii) Therefore, we have confirmation that the 3 lepton families represent the 3 discrete binary
rotational subgroups \([3,3,2]\), \([4,3,2]\), and \([5,3,2]\), of SU(2).

(iii) Leptons are 3-D entities existing in \(R^3\) as a subspace of \(R^4\) and \(C^2\).

(iv) There are no more lepton families because there are no more discrete binary rotational
subgroups in \(R^3\) that contain a volume, in agreement with the predictions from \(Z^0\) decays.

(v) Their mass ratios are related via syzygies to the j-invariant of elliptic modular functions,
which in turn are related to M"obius transformations and the Monster group.

(vi) With \(\theta_{23} = -42.85^\circ\) and the matrix agreement above, one can predict the normal mass
hierarchy for the neutrino mass states.

(vii) Space could be discrete at the Planck scale.

5. More background

In order to have a consistent geometrical approach toward understanding the SM, I have
proposed also \([3,4]\) that the quark families represent discrete binary rotational groups. However,
one must move up one spatial dimension from \(R^3\) to \(R^4\) and use the related four discrete binary
rotational subgroups \([3,3,3]\), \([4,3,3]\), \([3,4,3]\), and \([5,3,3]\), (or 5-cell, 16-cell, 24-cell, and 600-cell),
for the quarks, thereby dictating four quark families. Recall that both \(R^3\) and \(R^4\) are subspaces
of the unitary space \(C^2\).
Table 3. Lepton and quark families for the discrete binary rotational groups \([abc]\), their \(j\)-invariant constant \(N\), and the predicted mass values for the quarks based upon group-to-group \(N\) ratios (n.b. The mass values are the original published values from 1992).

| Leptons | Quarks |
|---------|--------|
| group   | family | N (MeV) | Mass | family | N (GeV) | Mass |
|         |        |         |      |        |         |      |
| [332]   | e\(^-\) | 1       | [1]  | 0.511  | s\(^{-1/3}\) | 1/4 | 0.011 |
|         | \(\nu_e\) | 0? | 0.0? |       | u\(^{+2/3}\) | 0.38 | 0.007 |
| [432]   | \(\mu^-\) | 108 | 108 | 103.5 |         |      |      |
|         | \(\nu_\mu\) | 0? | 0.0? |       | c\(^{+2/3}\) | [1.5] | 1.5 |
| [532]   | \(\tau^-\) | 1728 | 1728 | 1771.0 | b\(^{-1/3}\) | 108 | [5]  | 5.0  |
|         | \(\nu_\tau\) | 0? | 0.0? |       | t\(^{+2/3}\) | \(\sim\) 160 | 171.4 |
| [333]   |         |         |      |         |         |      |      |
|         | d\(^{-1/3}\) | 1/4 | 0.011 |         | c\(^{+2/3}\) | 0.38 | 0.007 |

Again, my assignment of a particular binary rotational group to a specific quark family in Table 3 is based upon the \(N_i\) value for each group. As for the leptons, the mass ratios of the fundamental quark families are determined by the ratios of the \(N_i\) values. For the quarks, the values in brackets, [1.5 GeV] and [5 GeV] in the right-hand mass columns, were used in 1992 to predict the top quark mass of \(\sim 160\) GeV and the mass values for the b’ and t’ quarks of the proposed 4th quark family.

These seven closely-related groups representing specific discrete rotational symmetries dictate the three known lepton families in \(R^3\) and four geometrically related quark families in \(R^4\). Consequently, being representatives of 3-D and 4-D entities, neither leptons nor quarks are to be considered as point objects. If my geometrical derivation of both the PMNS and CKM mixing matrices is based upon the correct reason for the mixing of flavor states to make the mass states, then one must reconcile the empirical data with the prediction of a fourth quark family. This fourth quark family of the t’ quark and the b’ quark has yet to be discovered.

My proposal that leptons are 3-D entities and that quarks are 4-D entities has several advantages. There is a clear distinction between leptons and quarks determined by inherent geometrical properties, with the specific consequence that leptons do not experience the color interaction via \(SU(3)_C\) because gluons and quarks are the only physical entities involving 4-D rotations associated with the three color charges that can be defined in \(R^4\). Also, one now has a geometrical reason for there being more than one family of leptons and of quarks, as well as there being a reason for an additional quark family to be discovered at the LHC.

However, a new problem arises with regard to there being 4 quark families and only 3 lepton families. The triangle anomalies do not cancel in the normal manner, i.e., across a generation with the lepton contributions being exactly the same value but opposite sign of the quark contributions. Remarkably, this triangle anomaly problem has a successful solution to be discussed in a later section, a solution closely related to the derivation of the PMNS and CKM matrices using the quaternion generators!
6. The CKM4 matrix derivation

The success of the above geometrical procedure for deriving the lepton PMNS matrix by using the quaternion generators from the 3 discrete binary rotation groups demands that the same approach should work for the quark families in \( \mathbb{R}^4 \) using the proposed 4 discrete binary rotation groups \([3,3,3],[4,3,3],[3,4,3], \) and \([5,3,3]\). If this procedure succeeds in deriving the CKM matrix elements as a 3x3 submatrix of CKM4, then a fourth sequential quark family, call its quark states \( b' \) and \( t' \), should exist in Nature.

These 4 discrete binary rotational groups for the quark families each have rotation subgroups of \( \text{SO}(4) = \text{SO}(3) \times \text{SO}(3) \), and they also have the double covering \( \text{SU}(2) \times \text{SU}(2) \). The \( \text{SO}(4) \) is the rotation group of the unit hypersphere \( S^3 \) in \( \mathbb{R}^4 \), with every 4-D rotation being simultaneous rotations in two orthogonal planes.

Mathematically, the 4 discrete binary groups for the quark families each can be identified \([5,6]\) as \((L/L_K; R/R_K)\) with the homomorphism \( L/L_K = R/R_K \). Here \( L \) and \( R \) are the specific discrete groups of quaternions listed above in section 2 and \( L_K \) and \( R_K \) are their kernels.

P. DuVal \([5]\) established that one only needs the cyclic groups \( 2C_n \) and \( 1C_n \) when considering the four discrete rotational symmetry groups, i.e., the ones I am using for the quark families. Essentially, vertices on the 4-D regular polytope can be projected to be a regular polygon on each of the two orthogonal planes in \( \mathbb{R}^4 \).

There will be 6 quaternion generators for each of the 4 groups, producing simultaneous rotations in two orthogonal planes. The two sets of Pauli matrices for producing continuous rotations can be identified with quaternion generators \( i, j, k \), and another \( i, j, k \), but they act on the two different \( S^2 \) spheres, i.e., in the two orthogonal planes. One considers this 4-D rotational transformation as the result of a bi-quatuerion operation \([10]\), or equivalently, a bi-spinor or Ivanchenko-Landau-Kähler spinor or Dirac-Kähler spinor operation.

For three quark families, one has the ”down” flavor states \( d', s', b' \), and their mass states \( d, s, b \), related by the CKM matrix. This quark mixing matrix for the left-handed components is defined in the standard way as

\[
V = U_L D_L^\dagger,
\]

but for four quark families in \( \mathbb{R}^4 \) the mathematics is a little different, for one must consider the bi-quatuerion case in which there will be Bogoliubov mixing \([10]\), producing two subfactors for each component, i.e.,

\[
U_L = W_{14,23}^u W_{12,34}^u, \quad D_L = W_{14,23}^d W_{12,34}^d
\]

with the \( W^u \) and \( W^d \) subfactor on the right mixing the 1st and 2nd generations and, separately, mixing the 3rd and 4th generations. The Bogoliubov mixing in the subfactor on the left mixes the 1st and 4th generations and, separately, the 2nd and 3rd generations. Therefore, the CKM4 matrix derives from

\[
V_{CKM4} = U_L D_L^\dagger = W_{14,23}^u W_{12,34}^u (W_{14,23}^d W_{12,34}^d)^\dagger.
\]

The product \( W_{12,34}^u W_{12,34}^{d\dagger} \) is given by

\[
W_{12,34}^u W_{12,34}^{d\dagger} = \begin{bmatrix}
    x_1 & y_1 & 0 & 0 \\
    z_1 & w_1 & 0 & 0 \\
    0 & 0 & x_2 & y_2 \\
    0 & 0 & z_2 & w_2
\end{bmatrix}.
\]

The upper left block is an \( \text{SU}(2) \) matrix that mixes generations 1 and 2 while the lower right block is an \( \text{SU}(2) \) matrix that mixes generations 3 and 4. Each 2x2 block relates the rotation angles and the phases via
\[
\begin{bmatrix}
  x & y \\
  z & w
\end{bmatrix} = \begin{bmatrix}
  \cos \theta e^{i \alpha} & -\sin \theta e^{i \beta} \\
  \sin \theta e^{i \gamma} & \cos \theta e^{i \delta}
\end{bmatrix}.
\]

The 4x4 matrix that achieves the Bogoliubov mixing has four possible forms for the four possible isospin cases obeying SU(2) x SU(2): (0, 0), (1/2, 0), (0, 1/2), and (1/2, 1/2). The (1/2, 1/2) is the one for equal, simultaneous, isospin 1/2 rotations in the two orthogonal planes for CKM4:

\[
W_{14,23}^{u,d} = \frac{1}{\sqrt{2}} \begin{bmatrix}
  1 & 0 & -1 & 0 \\
  0 & 1 & 0 & -1 \\
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1
\end{bmatrix}.
\]

Multiplying out these three 4x4 bi-quaternion mixing matrices,

\[
V_{CKM4} = \frac{1}{2} \begin{bmatrix}
  x_1 + x_2 & y_1 + y_2 & x_1 - x_2 & y_1 - y_2 \\
  z_1 + z_2 & w_1 + w_2 & z_1 - z_2 & w_1 - w_2 \\
  x_1 - x_2 & y_1 - y_2 & x_1 + x_2 & y_1 + y_2 \\
  z_1 - z_2 & w_1 - w_2 & z_1 + z_2 & w_1 + w_2
\end{bmatrix},
\]
in which the phases \(\alpha, \beta, \gamma, \delta\) have been ignored.

One determines the angles \(\theta_1\) and \(\theta_2\) from the quaternion generators of the 4 discrete binary rotation groups for the quark families. Projections of each of the four discrete symmetry 4-D entities onto the two orthogonal planes produces a regular polygon \([5,11]\) with the \(U_2\) generator \(i\exp[2\pi j/h]\), as given in Table 4, where the \(h\) values are 5, 8, 12, 30, for the \([3,3,3]\), \([4,3,3]\), \([3,4,3]\), and \([5,3,3]\), respectively. Again, \(U_1 = j\) and \(U_3 = i\), as for the leptons.

I need to determine the contribution from each \(U_2\) group generator that will make the sum add to 180°, i.e., make their collective action produce the rotation \(U_2 = k\). Expanding out the exponentials in terms of sines and cosines reveals four unknowns but only two equations. Alternately, because the four rotation angles sum to only 159°, we can use the same factor for each group, i.e., the ratio \(180^\circ/159^\circ = 1.132\).

| Fam. | Grp. | Generator | Angle° | Factor | Angle° |
|------|------|-----------|--------|--------|--------|
| u,d  | 333  | \(\exp[2\pi i/5]\) | 72     | 1.132  | 81.504 |
| c,s  | 433  | \(\exp[2\pi i/8]\) | 45     | 1.132  | 50.940 |
| t,b  | 343  | \(\exp[2\pi i/12]\) | 30     | 1.132  | 33.960 |
| t',b' | 533  | \(\exp[2\pi i/30]\) | 12     | 1.132  | 13.584 |

In the last column of Table 4 are the normalized angles which are twice the angle required. Therefore, taking the appropriate half-angle differences, i.e., between groups 1 and 2 and then between groups 3 and 4, produces the two required mixing angles

\[
\theta_1 = 15.282^\circ, \quad \theta_2 = 10.188^\circ.
\]
Substituting the cosines and sines of these two derived angles into the CKM4 matrix form above produces a mixing matrix symmetrical about the diagonal. Remember that I have ignored up to eight possible phases in the 2x2 blocks.

\[
V_{CKM4} = \begin{bmatrix}
0.9744 & 0.2203 & 0.0098 & 0.0433 \\
0.2203 & 0.9744 & 0.0433 & 0.0098 \\
0.0098 & 0.0433 & 0.9744 & 0.2203 \\
0.0433 & 0.0098 & 0.2203 & 0.9744
\end{bmatrix}
\]

One can compare the upper left 3x3 submatrix to the recent estimated absolute values [12]

\[
V_{CKM} = \begin{bmatrix}
0.9745 & 0.2246 & 0.0036 \\
0.2244 & 0.9736 & 0.0415 \\
0.0088 & 0.0407 & 0.9991
\end{bmatrix}
\]

Note that most of these estimated \(V_{CKM}\) values are probably good to within a few percent but some could have uncertainties as large as 10% or more.

Of some concern are my low values of 0.2203 for \(V_{us}\) and \(V_{cd}\). However, according to the Particle Data Group (2013) there are two possible values [12]: 0.2253 and 2204, the latter from tau decays. A recent report [13] further confirms the 0.2204 ± 0.0014 value from tau decays. Also, my derived symmetric CKM4 matrix \(V_{ub}\) value is high while the \(V_{td}\) value is reasonable, i.e., \(V_{td}\) at 0.0098 compares well with the estimated value of 0.0088.

The main diagonal \(V_{tb}\) element of CKM4 is 0.9744, quite a bit smaller than the suggested 0.9991 \(V_{tb}\) value for the 3x3 CKM matrix. However, if one imposes the unitarity condition on the rows and columns of the extracted 3x3 CKM submatrix, the new value for this \(V_{tb}\) matrix element would be 0.999, in agreement.

My final comment is that when I calculate CKM using only the first three quark groups \([3,3,3]\), \([4,3,3]\), and \([3,4,3]\), the resulting 3x3 CKM matrix will disagree significantly with the known CKM matrix. Therefore, one cannot eliminate a fourth quark family when discrete binary rotational groups are considered for the lepton and quark families.

7. What has been learned?

At this point I pause to review what has been learned by the successful derivation of the quark CKM4 mixing matrix and its CKM submatrix when each quark family has its own specific discrete binary rotational group in \(R^4\).

(i) The quark mixing angles derive from the four \(U_2\) generator projections to the k-axis in order to collectively act like the SU(2) quaternion generator \(k\). CKM is a submatrix of CKM4.

(ii) Therefore, we have confirmation that 4 quark families represent the 4 discrete binary rotational subgroups \([333]\), \([433]\), \([343]\), and \([532]\).

(iii) Quarks are 4-D entities defined in \(R^4\) and \(C^2\). (Unlike the 3-D leptons, they must combine to make a 3-D entity, as in QCD, as briefly explained in section 11.)

(iv) There are 4 quark families predicted, but where is the 4th quark family?

(v) The quark mass ratios are related to the \(j\)-invariant of elliptic modular functions, Möbius transformations, and the Monster group.

(vi) Space could be discrete at the Planck scale.

(vii) There would be an end to the ”Russian doll” hierarchy at the lattice nodes, which would possess no measurable physical parameters individually.
8. Discussion about the predicted b′ quark

The EW symmetry group SU(2)_L x U(1)_Y works extremely well, meaning that all its predictions agree with experiments so far. However, in this context there is no reason for Nature to have more than one fermion family, and certainly no reason for having 3 lepton families and at least 3 quark families. As far as I know, the traditional interpretation of the SM provides no answer that dictates the actual number of families, although the upper limit of 3 lepton families with low mass neutrinos is well established via Z^0 decays and via analysis of the CMB background.

My geometrical approach with discrete symmetries alters the default reliance upon SU(2) and its continuous symmetry transformations, for I utilize discrete binary rotational subgroups of SU(2) for the fundamental fermion states, a different subgroup for each lepton family and for each quark family. In this scenario one can surmise that the enormous success of the SM occurs because SU(2)_L x U(1)_Y is acting like a mathematical "cover group" for the actual underlying discrete rotations operating on the lepton states and quark states.

Assuming that my above matrix derivations are correct, the important question is: Where is the b′ quark of the predicted 4th quark family? In 1992 I predicted a top quark mass of about 160 GeV, a b′ quark mass of 65–80 GeV, and a t′ quark at a whopping 2600 GeV. These mass predictions were based upon the mass ratios being determined by the j-invariant function of elliptic modular functions and of fractional linear transformations, i.e., Möbius transformations. Note that all seven discrete groups I have for the fermions are related to the j-invariant and Möbius transformations, which have direct connections to practically all areas of fundamental mathematics.

With a predicted b′ mass that is much smaller than the top quark mass of 173.3 GeV and even smaller than the W mass at 80.4 GeV, one would have expected some production of the b′ at LEP, Fermilab, and the LHC. Yet, no clear indication of a b′ quark production and decay has appeared.

Perhaps the b′ quark has escaped detection at the LHC and lies hidden in the stored data from the runs at 7 TeV and 8 TeV. With a mass value below the W and Z masses, the b′ quark must decay via flavor changing neutral current (FCNC) decay channels such as b′ → b + γ and b′ → b + gluon. The b′ could have had an average lifetime too long for the colliders to have detected a reasonable number of its decays within the detector volumes and/or the energy cuts and angle cuts. However, the b′ quark and t′ quark would affect certain other decays that depend upon the heaviest "top" quark in a box or penguin diagram.

Another possibility is that a long lifetime might allow the formation of the quarkonium bound state b′-anti-b′, which has its own specific decay channels, to bb̅, gg, γγ, and WW* → ννℓℓ. Depending upon the actual quarkonium bound state, the spin and parity J^PC = 0^{++} or 0^{−+}. The relevant decay rates need to be calculated for comparison to the data.

And finally, as brought up in the question session of my presentation, there are two important theoretical considerations that need attention. First, associated with the mismatch of three lepton families to four quark families, e.g., the famous triangle anomalies do not cancel their infinities in the traditional manner. Second, EWSB involves the Weinberg angle θ_W, i.e., the weak mixing angle. What is the origin of θ_W in my geometrical approach? The solutions to these problems are discussed in the next sections, to be followed by the 'bigger picture'.

9. Triangle anomaly cancellation solution

The traditional cancellation procedure of matching each lepton family with a quark family "generation by generation" is known to produce the required triangle anomaly cancellation by summing the appropriate U(1)_Y, SU(2)_L, and SU(3)_C generators, producing the "generation" cancellation. But such an approach is not expected to work when there is the numerical mismatch of 3 lepton families to 4 quark families.

However, I now suspect that this "generation" idea is incomplete and actually incorrect.
Why? Because the derivation of the lepton and quark mixing matrices from the U2 generators of the discrete binary subgroups of SU(2) as shown above dictates that the 3 lepton families act as one collective lepton family for SU(2)\textsubscript{L} x U(1)\textsubscript{Y} and that the 4 quark families act as one collective quark family.

I have now created an effective single "generation" with one collective quark family matching one collective lepton family, so there is now the previously heralded "generation cancellation" of the triangle anomalies with the traditional summation of generator eigenvalues [14]. Furthermore, in the SU(3) representations one can show also that the quark and antiquark contributions cancel. Therefore, there are no SU(3)\times SU(3)\times U(1), SU(2)\times SU(2)\times U(1), U(1)\times U(1)\times U(1), or mixed U(1)-gravitational anomalies remaining.

There was always the suspicion that the traditional "generation" labeling was fortuitous because there was no specific reason for dictating the particular pairings of the lepton families to the quark families within the SM. That is, one could have paired the 1st lepton family with the 3rd quark family and still obtained the same cancellation result. Now, with the leptons and quarks representing the specific discrete binary rotational groups and acting collectively, a better understanding of how the families are related within the SM can be achieved.

10. Derivation of the Weinberg angle

Also considered in the question session was the statement that the Weinberg angle is determined by experiment only and not derivable from first principles. However, by using my quaternion generator approach that successfully derived the CKM4 and the PMNS mixing matrices, the Weinberg angle \(\theta_W\), i.e., the weak mixing angle, is derivable from first principles [14].

The four electroweak generators of the SM local gauge group SU(2)\textsubscript{L} x U(1)\textsubscript{Y} are typically labeled W\textsuperscript{+}, W\textsuperscript{0}, W\textsuperscript{−}, and B\textsuperscript{0}, but they can be defined equivalently as the quaternion generators i, j, k and b. But we do not require the full SU(2) to act upon the flavor states ±\(1\over 2\) for discrete rotations in the unitary plane C\(^2\) because the lepton and quark families represent specific discrete binary rotational symmetry subgroups of SU(2). That is, we require just a discrete subgroup of SU(2)\textsubscript{L} x U(1)\textsubscript{Y}.

One might suspect that the large 2I subgroup would be able to perform all the discrete symmetry rotations, but 2I omits some of the rotations in 2O. Instead, one finds that 2I x 2I’ works for the leptons and the quarks, where 2I’ provides the "reciprocal" rotations, i.e., the third generator U\textsuperscript{2} of 2I becomes the third generator U’\textsuperscript{2} for 2I’ by interchanging \(\phi\) and \(\phi^{-1}\):

\[
U_2 = -\frac{1}{2}i - \frac{\phi}{2}j + \frac{\phi^{-1}}{2}k, \quad U'_2 = -\frac{1}{2}i - \frac{\phi^{-1}}{2}j + \frac{\phi}{2}k.
\] (12)

Consider the three SU(2) generators i, j, k and their three simplest products: i x i = −1, j x j = −1, and k x k = −1. Now compare the three corresponding 2I x 2I’ discrete generator products: i x i = −1, j x j = −1, and

\[
U_2 U'_2 = -0.75 + 0.559i - 0.25j + 0.25k,
\] (13)
definitely not equal to −1. The reverse product U’\textsubscript{2}U\textsubscript{2} just interchanges signs on the i, j, k, terms.

One needs to multiply this product quaternion U\textsubscript{2}U’\textsubscript{2} by

\[
P = 0.75 + 0.559i - 0.25j + 0.25k
\] (14)
to make the result −1. Again, P’ has opposite signs for the i, j, k, terms only.

Given any unit quaternion q = Cos \(\theta\) + \(\hat{n}\) Sin \(\theta\), its power can be written as \(q^\alpha = \text{Cos } \alpha \theta + \hat{n}\text{ Sin } \alpha \theta\). Consider P to be a squared quaternion \(P = \text{Cos } 2\theta + \hat{n}\text{ Sin } 2\theta\) because we have the product of two quaternions U\textsubscript{2} and U’\textsubscript{2} (since we need the first term in P only). Therefore,
the quaternion square root of $P$ has $\cos \theta = \sqrt{0.75} = 0.866$, rotating the $U_2$ (and $U'_2$) in the unitary plane $C^2$ by the quaternion angle of $30^\circ$ so that each third generator becomes $k$. Thus the Weinberg angle, i.e., the weak mixing angle,

$$\theta_W = 30^\circ. \quad (15)$$

Therefore, the Weinberg angle derives from the mismatch of the third generator of $2I \times 2I'$ to the SU(2) third generator $k$.

The empirical value of $\theta_W$ ranges from $28.1^\circ$ to $28.8^\circ$, values less than the predicted $30^\circ$. The reason for the discrepancy is unknown (but see [15]), although one can surmise either (1) that in determining the Weinberg angle from the empirical data perhaps some contributions have been left out, or (2) the calculated $\theta_W$ is its value at the Planck scale at which the internal symmetry space and spacetime could be discrete instead of continuous.

11. Origin of quark color

With quark states being defined in $R^4$, there is a geometrical origin to color charge and gluon interactions. One must consider 4-D rotations in two orthogonal planes as the source of the color interaction and of color charge. Indeed, there are exactly 3 pairs of rotation planes: $[wx,yz]$, $[xy,zw]$, and $[yw,xz]$, which represent the red, green, and blue color charges. One sees immediately that they are an exact symmetry, in agreement with QCD. One can use the appropriate 4x4 real matrices to represent the gluon rotations from one color charge to another. In fact, eight specific 4x4 real matrices are equivalent to the SU(3) matrices traditionally used for the gluon interactions.

Therefore, the 3-D leptons do not experience the color interaction because only 4-D entities can have the 4-D color properties. In addition, particular geometrical combinations of the 4-D quarks can result in the 4x4 identity matrix that represents no 4-D rotation, i.e., one now has combined quarks (and anti-quarks) into hadrons which are 3-D entities in the $R^3$ subspace. Thereby, hadrons exist in our $(3,1)$-D spacetime.

We have therefore explained quark confinement as the inability of a 4-D quark state to exist in a 3-D space as a single entity. Instead, quarks must form the appropriate combinations called hadrons, again in agreement with QCD.

12. Why are there leptons?

At the DISCRETE ’08 conference in Valencia, Spain, Cecilia Jarlskog asked me to explain why there are leptons. She said that QCD is self-contained and could be a world by itself without the need for additional particles and interactions.

My answer is still the same six years later, that the mathematics called graph theory dictates my response. That is, the discrete groups that I have for the leptons and quarks are related to specific mathematical graphs of nodes and their connections. Kuratowski’s theorem states that a graph is planar (i.e., cannot remain more than 2-dimensional) if and only if the graph does not contain the $K_5$ or $K_{3,3}$ subgraph.

Fortunately for us all, my discrete binary group for the first quark family, the up and down quark states, is [3,3,3] with graph $\{3,3,3\}$, which is the $K_5$ graph! All other quark families must decay down to the first family representing [3,3,3] and $K_5$. But the color interaction changes color charge only and does not produce quark decays from one family to another. Therefore, one needs the weak interaction and additional particles called leptons.

The first lepton family, the electron and its neutrino state, represent the discrete binary group [3,3,2] with graph $\{3,3,2\}$, a subgraph of $K_5$ and also protected by Kuratowski’s theorem.
13. Some properties of the EW interaction

The above derivation of the Weinberg angle required the use of $2I \times 2I'$ in place of the traditional $SU(2)_L \times U(1)_Y$ local gauge group. We needed the substitute group because discrete rotations are required. Because the EW operators are quaternions and the lepton and quark states are represented also by quaternions, one has quaternions operating on quaternions, an action that mathematically produces left-handed doublets (LH) and right-handed (RH) singlets. Therefore, one has the origin of the maximal parity violation of the weak interaction.

In addition, because my geometrical approach to the SM occurs in $R^4$, there exist two equivalent 4-D real spaces: the particle space $R^4$ and the anti-particle space $R'$.

14. A brief look at the bigger picture!

Here I present a very brief outline of the bigger picture, suggesting how several mathematical connections to higher dimensional spaces and larger symmetry groups could dictate the SM behavior in $R^3$, $R^4$, and $C^2$.

We know from experiment and from the ability to derive the PMNS and CKM4 mixing matrices that the SM is an excellent approximation for understanding the behavior of leptons, quarks, and the interaction bosons in the lower energy region when the spatial resolution is not better than $10^{-24}$ meters. At smaller distance scales, perhaps one needs to consider a discrete space-time, for which the discrete binary rotation groups that I have suggested for the fundamental particles would reveal their true identity even more clearly. Quite possibly, with this emphasis on discrete subgroups of the local gauge group, the SM lagrangian will hold all the way down to the Planck scale.

If indeed the SM applies at the Planck scale, then one can show [15] that the Monster group dictates all of physics! One then must accept two surprising consequences: (1) The Universe is mathematics and is unique. (2) We humans are mathematics!

The second consequence is not easily appreciated but the argument would develop as follows: the fundamental leptons and quarks represent specific mathematical group properties → atoms are made of leptons and quarks → molecules are made of atoms → we humans are made of molecules → therefore we are mathematics!

The connection to the Monster Group is present already in determining the lepton and quark mass ratios, which are proportional to the j-invariant of elliptic modular functions, the same j-invariant that is the partition function for the Monster Group in a quantum field theory [17].

The mathematics of these discrete groups does even more for us, however, for there is a direct connection [16] from the lepton groups [3,3,2], [4,3,2], [5,3,2], and the quark groups [3,3,3], [4,3,3], [3,4,3], [5,3,3], in $R^3$ and $R^4$, respectively, and from the EW operations of $2I \times 2I'$ via special quaternions called iscosians to the discrete space $R^8$, thereby forming the $E_8$ lattice. Its discrete symmetry group is Weyl $E_8$, not the continuous symmetry group $E_8$ of M-theory.

One then separately considers the discrete (3,1)-D space-time of our existence and its discrete Lorentz transformations, which introduces another discrete $R^8$ lattice for relativistic space-time transformations via the same telescoping up from $R^4$ to $R^8$ via iscosians.

Mathematically, the two discrete $R^8$ spaces combine into a 10-D discrete space-time obeying the discrete symmetry transformations of "Weyl" $SO(9,1) = Weyl \ E_8 \times Weyl \ E_8$. This proposed unique connection to "Weyl" $SO(9,1)$ was a surprise to me because one has two 8-D spaces combining to make a 10-D space-time! Its direct and unique relationship back to the SM and
Lorentz transformations certainly is a welcome replacement to the $10^{500}$ ways possible for M-theory. One has eliminated the multiverse conjecture also.

Finally, one must not forget that having a fourth family of quarks provides a possible explanation for the baryon asymmetry of the Universe (BAU). From both the CKM and the PMNS matrices, one learns that their predicted CP violation (CPV) is at least 10 orders of magnitude too small to explain the BAU. That is, the important quantity called the Jarlskog value is much too small. But a 4th quark family resolves this issue [18] because substituting the fourth quark family mass values into the Jarlskog expression increases the CPV value by more than $10^{13}$! Voilà. One now has penguin diagrams distinguishing the particle and antiparticle decays with sufficient difference to have the particle dominated Universe we experience.

### Figure 1. Discrete symmetries for the lepton and quark families

| Invariant | Leptons | 3-D | Quarks | 4-D |
|-----------|---------|-----|--------|-----|
| $\{1/4\}$ | $\nu_e, e$ | $<3,3,3>$ | $u, d$ | |
| 1         | $\nu_\mu, \mu$ | $<4,3,2>$ | $c, s$ | $<4,3,3>$ |
| 108       | $\nu_\tau, \tau$ | $<5,3,2>$ | $t, b$ | $<3,4,3>$ |
| 1728      | $\nu_{\tau'}, \tau'$ | $<5,3,3>$ | |

The neutrino mixing angles and the PMNS mixing matrix have been derived successfully from the mismatch of the SU(2) generators to the corresponding generators for the discrete groups $[3,3,2]$, $[4,3,2]$, and $[5,3,2]$, one for each of the lepton families but now acting in combination. The quark mixing matrix CKM4 has been derived via the same procedure using the appropriate geometrical figures.

### 15. The geometrical representations

Figure 1 shows the table of the lepton and quark families representing their discrete geometrical entities. I displayed this same table at the DISCRETE '08 conference and these geometrical figures remain applicable for my presentation here at DISCRETE 2014. One can surmise that these discrete entities strongly suggest that space is discrete at the Planck scale; however, their symmetry properties can remain true in a continuous space also.

### 16. Summary

The neutrino mixing angles and the PMNS mixing matrix have been derived successfully from the mismatch of the SU(2) generators to the corresponding generators for the discrete groups $[3,3,2]$, $[4,3,2]$, and $[5,3,2]$, one for each of the lepton families but now acting in combination. The quark mixing matrix CKM4 has been derived via the same procedure using the appropriate geometrical representations.
generators from four specific related discrete binary rotational groups \([3,3,3], [4,3,3], [3,4,3], \) and \([5,3,3], \) corresponding to four quark families. The traditional CKM matrix is a submatrix of CKM4. However, neither quark of the 4th quark family has been detected yet. Their appearance could mean that the Standard Model lagrangian might be a good approximation to the ultimate lagrangian all the way down to the Planck scale, particularly if space-time is discrete.

In addition, now that the first principles origin of the lepton and quark mixings is understood in my geometrical approach to the SM, the Weinberg angle \(\theta_W\) was derived and found to be \(30^\circ\). Also, there was serious concern raised in the question session about the cancellation of the triangle anomaly because the traditional lepton-quark generational canceling no longer works with 3 lepton families and 4 quark families. However, the mixing angle derivation reveals that the 3 lepton families are acting in combination as one effective lepton family to cancel exactly the triangle anomaly contributions of 4 quark families acting as one effective quark family.

Finally, the important mathematical relationships originating from the SM acting in \(R^4\) can be telescoped up to \(R^8\) and higher dimensional spaces via icosians. I briefly introduced the fact that there exists a unique mathematical connection from the SM and Lorentz transformations to the discrete symmetry group in 10-D space-time called Weyl \(E_8\times E_8\) = ”Weyl” \(SO(9,1)\), in contrast to the \(10^{500}\) ways possible for M-theory.

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