The derivative of the conjugacy for the pair of tent-like maps from an interval into itself

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Abstract

We consider in this article the properties of the topological conjugacy of the piecewise linear unimodal maps \( g : [0, 1] \to [0, 1] \), all whose kinks belong to the complete pre-image of 0. We call such maps firm carcass maps. We prove that every firm carcass maps \( g_1 \) and \( g_2 \) are topologically conjugated. For the conjugacy \( h \) such that \( h \circ g_1 = g_2 \circ h \) we denote \( \{h_n, n \geq 1\} \) the piecewise linear approximations of \( h \), whose graphs connect the points \( \{(x, h(x)), g_1^n(x) = 0\} \). For any \( x \in [0, 1] \) we reduce the question about the value of \( h'(x) \) to the properties of the sequence \( \{h'_n(x), n \geq 1\} \). We prove that each conjugacy of firm carcass maps either has the length 2, or is piecewise linear

1 Introduction

The topological conjugation is a powerful tool for the investigation of properties of one-dimensional dynamical systems. The classical example in the pedagogic of one-dimensional dynamical systems is the conjugacy of the maps

\[ x \mapsto 2x - \left|1 - 2x\right| \]  

and \( f : x \mapsto 4x(1 - x) \), which was stated at first in [1]. Due to the form of the graph, the map (1.1) is called tent map. The mentioned example inspirits the desire to describe all the functions, which are topologically conjugated to the tent map. This description is also given in [1]. By the definition of the topological conjugation, for the increasing conjugacy \( h : [0, 1] \to [0, 1] \), the map \( g = h \circ f \circ h^{-1} \) appears to be of the form

\[ g(x) = \begin{cases} g_l(x), & 0 \leq x \leq v, \\ g_r(x), & v \leq x \leq 1, \end{cases} \]  

where \( v \in (0, 1) \), the function \( g_l \) increase, the function \( g_r \) decrease, and

\[ g(0) = g(1) = 1 - g(v) = 0. \]

We will call unimodal map the mentioned \( g \).

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Theorem 1. [1, p. 53] The tent map (1.1) is topologically conjugated to the unimodal map $g$ if and only if the complete pre-image of 0 under the action of $g$ is dense in $[0, 1]$.

Remind, that the set $g^{-\infty}(a) = \bigcup_{n \geq 1} g^{-n}(a)$, where $g^{-n}(a) = \{ x \in [0, 1] : g^n(x) = a \}$ for all $n \geq 1$, is called the complete pre-image of $a$ (under the action of the map $g$).

In spite of the elegance of the statement of Theorem 1 it may appear to be quite difficult to resolve for a given unimodal map $g$, whether or not $g^{-\infty}(0)$ is dense in $[0, 1]$. Thus, it is natural to restrict the attention to some subclasses of the unimodal maps.

The simplest partial case of the unimodal maps is when the graphs of both $g_l$ and $g_r$ in (1.2) are segments of lines, i.e. the map (1.2) is of the form

$$f_v(x) = \begin{cases} \frac{x}{v}, & \text{if } 0 \leq x \leq v, \\ \frac{1-x}{1-v}, & \text{if } v \leq x \leq 1, \end{cases}$$

(1.3)

where $v \in (0, 1)$ is a parameter. Due to [2] and [3], we will call the map (1.3) a skew tent map.

Remark that $f_{1/2}$ is the tent map (1.1). The existence and the uniqueness of the conjugacy of $f_{v_1}$ and $f_{v_2}$ of the form (1.3) for all distinct $v_1, v_2 \in (0, 1)$ was stated in [2]. In other words, there exists the unique continuous invertible solution $h : [0, 1] \to [0, 1]$ of the functional equation

$$h \circ f_{v_1} = f_{v_2} \circ h.$$

(1.4)

We have independently proved in [4] the conjugateness of the tent map and $f_v$ for all $v \in (0, 1)$. Precisely, we have proved the following

**Proposition 1.** [4, Lema 10] For every $v \in (0, 1)$ the set $f_v^{-\infty}(0)$ is dense in $[0, 1]$, where $f_v$ is skew tent map.

The next property of the conjugacy from (1.4) is obtained in [2].

**Proposition 2.** [2, Prop. 2] If the derivative of the continuous invertible solution $h$ of (1.4) is finite at some $x \in [0, 1]$, then $h'(x) = 0$.

This result is especially interesting, because Lebesgue’s Theorem (see [6] or [7, p. 15]) claims that each nondecreasing function has a finite derivative at every point with the possible exception of the points of a set of measure zero. The conjugacy $h$ of the maps $f_{v_1}$ and $f_{v_2}$ is also studied in [3].

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3In fact, Proposition 2 of [2] says that $h'(x) = 0$ everywhere, where it exists. Nevertheless, it follows from the proof of Proposition 2, that the authors of [2] assume that the derivative can be only finite. In fact, they take an arbitrary point $x \in (0,1)$ and construct a sequence $k_n$ such that $x \in I_n = \left[ \frac{k_n}{2^n}, \frac{k_n+1}{2^n} \right]$ and $p_n = h \left( \frac{k_n}{2^n} \right) - h \left( \frac{k_n+1}{2^n} \right)$. After this they claim that if $h'(x)$ exists and is non-zero, then $\frac{p_{n+1}}{p_n} \to \frac{1}{2}$. But this is true only in the case $h'(x) < \infty$. Following [5, Sect 92, 101], we will assume, that the derivative is a limit, which can be also infinite and in this case so is the value of the derivative.
Proposition 3. [Prop. 1] For any distinct \( v_1, v_2 \in (0, 1) \) the length of the graph of the conjugacy \( h \) of maps \( f_{v_1} \) and \( f_{v_2} \) of the form (1.3) equals 2.

Remark that the length of the graph, mentioned in Proposition 3, is the maximum possible length of a monotone \([0, 1] \rightarrow [0, 1]\) function.

We will call the map defined on the set \( A \subseteq \mathbb{R} \) with values in \( \mathbb{R} \), linear, if its graph is a line (or a line segment). We will call the map piecewise linear, if its domain can be divided into finitely many intervals, such that on each of them the map is linear. A point, where the piecewise linear map is not differentiable, will be called a kink.

The piecewise linear unimodal map will be called carcass map. A carcass map, all whose kinks belong to the complete pre-image of 0, will be called a firm carcass map.

Let \( g_1, g_2 : [0, 1] \rightarrow [0, 1] \) be firm carcass maps. For every \( n \geq 1 \) denote by \( h_n : [0, 1] \rightarrow [0, 1] \) the increasing piecewise linear map such that \( h_n(g_1^{-n}(0)) = g_2^{-n}(0) \) and all the kinks of \( h_n \) belong to \( g_1^{-n}(0) \). We will prove that these \( g_1 \) and \( g_2 \) are topologically conjugated, moreover, each of them is conjugated to the tent map. We will prove that \( h = \lim_{n \rightarrow \infty} h_n \), where the limit is considered point-wise, exists and this limit function is the homeomorphism, which satisfies the functional equation

\[
    h \circ g_1 = g_2 \circ h. \tag{1.5}
\]

In fact, the idea of this construction appeared in the original proof of Theorem 1, given in [1]. We will prove the following generalization our Proposition 1.

Theorem 2. The complete pre-image of 0 under the action of every firm carcass map is dense in \([0, 1]\).

Notice that Theorems 1 and 2 imply that every firm carcass map is topologically conjugated to the tent-map.

For any \( x \in [0, 1] \) denote

\[
    L(x) = \begin{cases}
        \lim_{n \rightarrow \infty} h_n'(x-) & \text{if } x > 0 \\
        \lim_{n \rightarrow \infty} h_n'(x+) & \text{otherwise}
    \end{cases} \tag{1.6}
\]

and

\[
    R(x) = \begin{cases}
        \lim_{n \rightarrow \infty} h_n'(x+) & \text{if } x < 1 \\
        \lim_{n \rightarrow \infty} h_n'(x-) & \text{otherwise}
    \end{cases} \tag{1.7}
\]

We will prove the following theorem.

Theorem 3. Let \( g_1 \) and \( g_2 \) be firm carcass maps and let \( h \) be the conjugacy, which satisfies (1.5).
1. If for at least one \( x \in [0, 1] \) the derivative \( h'(x) \) exists, is positive and finite, then \( h \) is piecewise linear.

2. If \( h \) is not piecewise linear, then \( h'(x) \) exists if and only if there exists \( L(x) \), \( R(x) \) and, moreover, \( L(x) = R(x) \). In this case \( h'(x) = L(x) \).

For every \( a, b \in \{0, 1\} \) denote

\[
\varphi_v(a, b) = \begin{cases} 
  v & \text{if } a = b, \\
  1 - v & \text{if } a \neq b
\end{cases}
\]  

(1.8)

Theorem 3 can be specified in the case of the conjugation of the tent map with a skew tent map.

**Theorem 4.** For any \( v \in (0, 1) \) let \( h \) be the conjugacy of the tent map \( f \) and the map \( f_v \) of the form (1.3), i.e.

\[
f \circ h = h \circ f_v.
\]

The derivative \( h'(x) \) exists if and only if there exists the limit \( \varphi_\infty(x) = \prod_{k=2}^{\infty} (2 \varphi_v(x_k, x_{k-1})) \), where \( 0.x_1x_2\ldots \) is the binary expansion of \( x \). Moreover, in this case \( h'(x) = \varphi_\infty(x) \).

We have obtained the partial cases of Theorem 4 in our previous works. We have proved Theorem 4 for the binary finite numbers in [8] and we also have proved Theorem 4 for the rational numbers in [9].

Notice, that Proposition 2 is a simple corollary of Theorem 4.

Proposition 3 can be generalized for the firm carcass maps as follows.

**Theorem 5.** Let \( h \) be the conjugacy of any firm carcass maps. If \( h \) is not piecewise linear, then the length of the graph of \( h \) equals 2.

Our work consists of eight sections except introduction. In section 2 we present some basic facts about topological conjugacy and the tent map. Section 3 is devoted to the construction of the linear approximations of the conjugacy of unimodal functions, which was obtained at first in the original proof of Theorem 1 and we generalize this Theorem. We use this approximation in our further reasonings. In section 4 we construct the generalization of the binary expansion of a number, precisely for any \( x \in [0, 1] \) and the map \( g \) of the form (1.2) we construct an infinite sequence \( \{x_i, i \geq 1, x_i \in \{0; 1\}\} \), which determines the number \( x \). We also specify the properties of the obtained sequence for firm carcass maps and prove Theorem 2. In Section 5 we study the derivatives of the conjugacy of firm carcass maps, precisely we prove Theorem 3 there. In Section 6 we prove Theorem 5. In We devote Section 7 to the topological conjugacy of the tent map and a skew tent map. We derive Theorem 3 from Theorem 3 there give an alternative proof of Theorem 5. We state some hypothesis for the further research in Section 8.
2 Basic facts and properties

2.1 Basic facts about tent map

We will state some, quite clear, properties of tent maps in this section. These properties are almost evident, whenever they are already formulated from one hand and, maybe, all of them are mentioned in some text books on the Theory of Dynamical Systems as illustrations, or examples. Nevertheless, we want to state these properties explicitly, because we will generalize them later in Section 4 for an arbitrary unimodal map.

Remark 2.1. Notice that we can rewrite the formula (1.1) as

\[ f(x) = \begin{cases} 
2x & \text{if } 0 \leq x \leq 1/2, \\
2 - 2x & \text{if } 1/2 \leq x \leq 1.
\end{cases} \]

Remark 2.1 provides the rule to construct the binary expansion of \( f(x) \) by the binary expansion of \( x \in [0,1] \).

Notation 2.2. Denote \( R(t) = 1 - t \) for \( t \in \{0; 1\} \).

Remark 2.3. Let \( f \) be the tent map and

\[ x = 0.x_1x_2 \ldots x_n \ldots \] (2.1)

be the binary expansion of an arbitrary \( x \in [0,1] \). Then the binary expansion of \( f(x) \) is

\[ f(x) = \begin{cases} 
0.x_2x_3 \ldots x_n \ldots , & \text{if } x_1 = 0, \\
0.R(x_2)R(x_3) \ldots R(x_n) \ldots , & \text{if } x_1 = 1.
\end{cases} \]

Notice, that we understand (2.1) as

\[ x = \sum_{i=1}^{\infty} x_i 2^{-i}, \] (2.2)

precisely we will understand (2.1) and (2.2) as two different forms to write the same fact. We will use (2.1) because we think that this form of representation is more visual than (2.2).

Remark 2.3 provides the description of the set \( f^{-n}(0) \) for all \( n \geq 1 \).

Remark 2.4. For every \( n > 1 \) the set \( \{ x < 1 : f^n(x) = 0 \} \) consists of all the \( x \in [0,1] \) with the binary expansion

\[ x = 0.x_1x_2 \ldots x_{n-1}. \]

Next, Remark 2.4 can be rewritten as follows.
Remark 2.5. For every $n \geq 1$ we have that

$$f^{-n}(0) = \left\{ \frac{k}{2^{n-1}}, 0 \leq k \leq 2^{n-1} \right\}.$$ 

Remark 2.6. The graph of the $n$th iteration $f^n$ has the following properties:

1. $f^n(0) = 0$, i.e. the graph passes through origin.
2. The graph consists of $2^n$ line segments, whose tangents are either $2^n$, or $-2^n$.
3. Each maximal part of monotonicity of $f^n$ connects the line $y = 0$ and $y = 1$.
4. Let $x_1, x_2$ be such that $\{f^n(x_1); f^n(x_2)\} = \{0; 1\}$ and $f^n$ is monotone on $[x_1, x_2]$. Then $f^{n+1}(x_1) = f^{n+1}(x_2) = 0$. Precisely, for $x_3 = \frac{x_1 + x_2}{2}$ we have that $f^{n+1}(x_3) = 1$ and, moreover, $f^{n+1}$ increase on $[x_1, x_3]$ and decrease on $[x_3, x_2]$, being linear at each of these intervals.

2.2 Properties of topological conjugation

The change of coordinates is one of the classical illustration (explanation) of what topological conjugation is. Thus, it preserves a lot of properties of maps and points. From another hand, if topologically conjugated maps are “similar” in some sense, then local properties of the conjugacy can be globalized. Moreover, this globalization still holds in the case, when we are talking about semi conjugation.

Lemma 2.7. Let $g_1$ and $g_2$ be unimodal maps and let a solution $h$ of (1.5) be continuous. Then:

1. For any fixed point $x$ of $g_1$, the point $h(x)$ is a fixed point of $g_2$.
2. If $h$ is invertible then for any periodical point $x$ of $g_1$ of period $n$, the point $h(x)$ is periodical point of $g_2$ of period $n$.

The next fact us quite technical, but is important for the further corollaries.

Lemma 2.8. Suppose that $h$ is a continuous solution of (1.5). Let $a, b \in [0, 1]$ be such that $g_1$ and $g_2$ are linear on $[a, b]$ and $[h(a), h(b)]$ respectively. Then for every $c \in (a, b)$

$$\frac{h(c) - h(a)}{c - a} \cdot \frac{b - a}{h(b) - h(a)} = \frac{(h \circ g_1)(c) - (h \circ g_1)(a)}{g_1(c) - g_1(a)} \cdot \frac{g_1(b) - g_1(a)}{(h \circ g_1)(b) - (h \circ g_1)(a)}$$

holds.

Proof. It follows from (1.5) that $(h \circ g_1)(c) - (h \circ g_1)(a) = (g_2 \circ h)(c) - (g_2 \circ h)(a)$, and, by linearity of $g_2$ on $[h(a), h(b)]$, $(g_2 \circ h)(c) - (g_2 \circ h)(a) = g'_2(c) \cdot (h(c) - h(a))$, whence

$$(h \circ g_1)(c) - (h \circ g_1)(a) = g'_2(c) \cdot (h(c) - h(a)).$$
Analogously, 
\[(h \circ g_1)(b) - (h \circ g_1)(a) = g'_2(c) \cdot (h(b) - h(a)).\]

By linearity of \(g_1\) on \([a, b]\) obtain 
\[g_1(c) - g_1(a) = g'_1(c) \cdot (c - a)\]
and 
\[g_1(b) - g_1(a) = g'_1(c) \cdot (b - a).\]

Now, 
\[
\frac{(h \circ g_1)(c) - (h \circ g_1)(a)}{g_1(c) - g_1(a)} \cdot \frac{g_1(b) - g_1(a)}{(h \circ g_1)(b) - (h \circ g_1)(a)} = \\
= \frac{g'_2(c) \cdot (h(c) - h(a))}{g'_1(c) \cdot (c - a)} \cdot \frac{g'_1(c) \cdot (b - a)}{g'_2(c) \cdot (h(b) - h(a))}
\]
and we are done. \(\square\)

The next lemma follows from Lemma \(2.8\)

**Lemma 2.9.** Let \(g_1\) and \(g_2\) be carcass maps and let solution \(h\) of (1.5) be continuous. Then:

1. If \(h\) is constant on some interval, then \(h\) is constant in the entire \([0, 1]\).
2. If \(h\) is linear on some interval, then \(h\) is piecewise linear in the entire \([0, 1]\).
3. If \(h\) is differentiable on some interval, then \(h\) is piecewise differentiable on the entire \([0, 1]\) (i.e. \(h\) is differentiable everywhere on \([0, 1]\) except, possibly, finitely many points).

### 3 The Stanislaw Ulam’s construction

We will generalize Theorem [1] in this section. We will also introduce some notations during the proof, which are necessary for our further reasonings. The construction of the sequence \(\{h_n, n \geq 1\}\) below is the simple generalization of the proof of Theorem [1] given in [1].

The following remark is the generalization of Remark \(2.6\)

**Remark 3.1.** The graph of the \(n\)th iteration \(g^n\) of an arbitrary unimodal function \(g\) has the following properties:

1. The graph consists of \(2^n\) monotone curves.
2. Each maximal part of monotonicity of \(g^n\) connects the line \(y = 0\) and \(y = 1\).
3. If \(x_1, x_2\) are such that \(\{g^n(x_1); g^n(x_2)\} = \{0; 1\}\) and \(g^n\) is monotone on \([x_1, x_2]\), then \(g^{n+1}(x_1) = g^{n+1}(x_2) = 0\) and there is \(x_3 \in (x_1, x_2)\) such that \(g^{n+1}(x_3) = 1\). Moreover in this case \(g^{n+1}\) increase on \([x_1, x_3]\) and decrease on \([x_3, x_2]\).
Since, by Remark 3.1, the set \( g^{-n}(0) \) consists of \( 2^{n-1} + 1 \) points, then the notation follows.

**Notation 3.2.** For every unimodal map \( g : [0, 1] \to [0, 1] \) and for every \( n \geq 1 \) denote \( \{ \mu_{n,k}(g), 0 \leq k \leq 2^{n-1} \} \) such that \( g^n(\mu_{n,k}(g)) = 0 \) and \( \mu_{n,k}(g) < \mu_{n,k+1}(g) \) for all \( k \).

**Remark 3.3.** Notice that \( \mu_{n,k}(g) = \mu_{n+1,2k}(g) \) for all \( 0 \leq k \leq 2^{n-1} \).

**Calculation 3.4.** For each unimodal map \( g \), every \( n \geq 2 \) and \( k, 0 \leq k \leq 2^{n-2} \) the equalities

\[
g(\mu_{n,k}(g)) = \mu_{n-1,k}(g) \tag{3.1}
\]

and

\[
g(\mu_{n,k}(g)) = g(\mu_{n,2^{n-1}-k}(g)) \tag{3.2}
\]

hold.

For every two unimodal maps \( g_1, g_2 : [0, 1] \to [0, 1] \) and every \( n \in \mathbb{N} \) define the map \( \hat{h}_n : g_1^{-n}(0) \to g_2^{-n}(0) \) by

\[
\hat{h}_n(\mu_{n,k}(g_1)) = \mu_{n,k}(g_2) \tag{3.3}
\]

for all \( 0 \leq k \leq 2^{n-1} \).

**Lemma 3.5.** For every maps \( g_1, g_2 : [0, 1] \to [0, 1] \) the map \( \hat{h}_n : g_1^{-n}(0) \to g_2^{-n}(0) \), defined by (3.3), satisfies the equation

\[
\hat{h}_n \circ g_1 = g_2 \circ \hat{h}_n. \tag{3.4}
\]

**Proof.** By Calculation 3.4 for every \( k, 0 \leq k \leq 2^{n-2} \) it follows from (3.1) that

\[
\begin{cases}
\hat{h}_n(\mu_{n,k}(g_1)) = \hat{h}_n(\mu_{n-1,2k}(g_1)) = \mu_{n-1,2k}(g_2) \\
g_2(\hat{h}_n(\mu_{n,k}(g_1))) = g_2(\mu_{n,k}(g_2)) = \mu_{n-1,2k}(g_2),
\end{cases} \tag{3.5}
\]

and it follows from (3.2) that

\[
\begin{cases}
\hat{h}_n(\mu_{n,2^{n-1}-k}(g_1)) = \hat{h}_n(\mu_{n-1,2k}(g_1)) = \mu_{n-1,2k}(g_2) \\
g_2(\hat{h}_n(\mu_{n,2^{n-1}-k}(g_1))) = g_2(\mu_{n,k}(g_2)) = \mu_{n-1,2k}(g_2). \tag{3.6}
\end{cases}
\]

Now (3.5) and (3.6) imply (3.4).

**Lemma 3.6.** Suppose that unimodal maps \( g_1, g_2 : [0, 1] \to [0, 1] \) are topologically conjugated and \( h : [0, 1] \to [0, 1] \) is the conjugacy such that (1.5) holds. Then

\[
h(\mu_{n,k}(g_1)) = \mu_{n,k}(g_2) \tag{3.7}
\]

for all \( n \geq 1 \) and \( k, 0 \leq k \leq 2^{n-1} \).
Proof. By (1.3), the number \( h(0) \) is a fixed point of \( g_2 \). Since \( h(0) \in \{0; 1\} \), then \( h(0) = 0 \).

Thus, \( h \) increase. It follows by induction on \( n \) from (1.3) that \( h \circ g_1^n = g_2^n \circ h \) for all \( n \geq 1 \). The obtained equality and \( h(0) = 0 \) imply that \( h(g_1^{-n}(0)) = g_2^{-n}(0) \), whence (3.7) follows.

The next theorem is the generalization of Theorem 1.

**Theorem 6 (Ulam’s Theorem).** Let \( g_1, g_2 : [0, 1] \rightarrow [0, 1] \) be unimodal maps and suppose that \( g_1^{-\infty}(0) \) is dense in \([0, 1]\). Then \( g_1 \) and \( g_2 \) are topologically conjugated if and only if \( g_2^{-\infty}(0) \) is dense in \([0, 1]\). Moreover, in this case the conjugacy is unique.

**Proof.** For every \( n \geq 1 \) define \( \hat{h}_n : g_1^{-n}(0) \rightarrow g_2^{-n}(0) \) by (3.3).

Suppose that \( h \) is the topological conjugacy of \( g_1 \) and \( g_2 \). By Lemma 3.6 the map \( h \) coincides with \( \hat{h}_n \) on the domain of \( \hat{h}_n \), thence the density of \( g_2^{-\infty}(0) \) follows from the continuity of \( h \).

Now assume that \( g_2^{-\infty}(0) \) is dense in \([0, 1]\). Define \( h : [0, 1] \rightarrow [0, 1] \) by \( h = \lim_{n \rightarrow \infty} h_n \), where the limit is considered point-wise. The function \( h \) is well-defined due to the density of \( g_1^{-\infty}(0) \) in \([0, 1]\) and it is continuous due to the density of \( g_2^{-\infty}(0) \) in \([0, 1]\). Also denote \( \hat{g}_n = h_n \circ g_1 \circ h_n^{-1} \).

By Lemma 3.5 obtain that \( \hat{g}_n(x) = g_2(x) \) for all \( x \in g_2^{-n}(0) \) and (1.5) follows.

**Remark 3.7.** It follows from (3.7) that the conjugacy of maps \( g_1, g_2 \) of the form (1.2) increase.

Remark, that the sequence \( \{h_n, n \geq 1\} \) for the approximation of the conjugacy of maps \( f_v \) of the form (1.3) also appeared in [3]. These maps are denoted by \( T_c \) in [3], where \( c \in (0, 1) \) means the same as \( v \in (0, 1) \) in our notations. The solution \( \varphi \) of the functional equation \( \varphi \circ T_{c_1} = T_{c_2} \circ \varphi \) is found in [3] as the limit of the sequence \( \{\varphi_n, n \geq 0\} \), where \( \varphi_0(x) = x \) for all \( x \in [0, 1] \) and

\[
\varphi_{n+1}(x) = \begin{cases} 
c_2 \varphi_n \left( \frac{x}{c_1} \right) & \text{if } 0 \leq x \leq c_1, \\
(c_2 - 1) \varphi_n \left( \frac{x-1}{c_1-1} \right) + 1 & \text{if } c_1 < x \leq 1. 
\end{cases}
\] (3.8)

Notice, that the sequence of functions \( \{h_n, n \geq 1\} \) satisfies (3.8), whence \( \varphi_n = h_{n+1} \) for all \( n \geq 0 \).

The sequence (3.8) is considered in [2] too for the unimodal (not necessary piecewise linear) maps \( g_1 \) and \( g_2 \), where, additionally, the peak of \( g_1 \) is 1/2. By [2, Lema 3] for any such \( g_1 \) and \( g_2 \) there is a unique fixed element \( h \) of (3.8), which is bounded in the Banach space with the norm \( |h| = \sup_{x \in [0,1]} |h(x)| \). This fact is independent on wether of not \( g_1 \) and \( g_2 \) are topologically conjugated. Clearly, if \( g_1 \) and \( g_2 \) are not topologically conjugated, then the fixed element \( h \) of (3.8) is not a homeomorphism.
4 The map-expansion of a number

4.1 General properties

The construction below is the generalization of the binary expansion of a number. By any given $x \in [0, 1]$ and by a unimodal map $g$ we will construct an infinite sequence $\{x_i, i \geq 1\}$ with $x_i \in \{0, 1\}$ for all $i \geq 1$, and denote

$$k_n = \sum_{i=0}^{n} x_i 2^{n-i} \quad (4.1)$$

for all $n \geq 1$.

Lemma 4.1. For every $x \in (0, 1) \setminus g^{-\infty}(0)$ there exists the infinite sequence $\{x_i, i \geq 1\}$ with the following properties:

1. $x_i \in \{0, 1\}$ for all $i \geq 1$,
2. $x \in (\mu_{n+1,k_n}, \mu_{n+1,k_n+1})$, where $k_n$ is defined by (4.1).

Proof. The sequence $\{k_n, n \geq 1\}$ such that $x \in (\mu_{n+1,k_n}, \mu_{n+1,k_n+1})$ exists, because $g^n(x) \neq 0$ for all $n$. For any $n \geq 1$ assume that (4.1) holds. Then it follows from Remark 3.3 that $x \in (\mu_{n+2,2k_n}, \mu_{n+2,2k_n+2})$, whence either $k_{n+1} = 2k_n$, or $k_{n+1} = 2k_n + 1$. Thus, there exists $x_n \in \{0, 1\}$ such that $k_{n+1} = 2 \cdot k_n + x_n$, whence $k_{n+1} = x_n + 2 \cdot \sum_{i=0}^{n} x_i 2^{n-i} = \sum_{i=0}^{n+1} x_i 2^{n+1-i}$ and we are done. \qed

Notation 4.2. For every $x \in [0, 1]$ construct the infinite sequence $\{x_k, k \geq 0\}$ as follows:

1. If $x = 1$, then set $x_k = 1$ for all $k \geq 1$.
2. If $x < 1$, but $x \in g^{-n}(x) = 0$ for some $n \geq 1$, then set $x_1, \ldots, x_{n-2}$ to be such that the number $k$ from the equality $x = \mu_{n+1,k}$, has the binary expansion $k = \sum_{i=1}^{n} x_i 2^{n-i}$; and set $x_k = 0$ for all $k > n$.
3. If $g^n(x) \neq 0$ for all $n \geq 1$, then denote $\{x_k, k \geq 1\}$ the sequence from Lemma 4.1.

It will be convenient for us to add $x_0 = 0$ to the left of the sequences from Notation 4.2. Thus, call the constructed sequence the $g$-expansion of $x$ and write

$$x \xrightarrow{g} x_0x_1 \ldots x_n \ldots \quad (4.2)$$

Remark 4.3. The $g$-expansion of any $x \in [0,1]$ is unique if and only if $g^{-\infty}(0)$ is dense in $[0,1]$. Remind that, by Ulam’s Theorem 7, in this case the map $g$ is topologically conjugated to the tent map.
Thus, if $g^{-\infty}(0)$ is dense in $[0,1]$, then write

$$x \xrightarrow{g} x_0x_1 \ldots x_n \ldots$$ (4.3)

for the $g$-expansion of $x$. Say that $x$ is $g$-finite, if there is $k$ such that $x_i = 0$ for all $i > k$. Otherwise say that $x$ is $g$-infinite. Analogously to rational numbers, assume that the $g$-expansion of a $g$-finite number is finite (i.e. does not contain the infinite series of zeros).

**Remark 4.4.** Remark, that if $g$ is the tent map (1.1), then $g$-expansion is the classical binary expansion of a number.

**Remark 4.5.** Notice, that for numbers $x, y \in [0, 1]$ the inequality $x \leq y$ holds if and only if the $g$-expansion of $x$ is $\leq$ than $g$-expansion of $y$ in the natural lexicographical order.

In spite of non-uniqueness of $g$-expansion in general, the analogue of Remark 2.3 holds for $g$-expansion.

**Remark 4.6.** Let (4.2) be a $g$-expansion of $x \in [0, 1]$. Then

$$g(x) \xrightarrow{g} \begin{cases} 0.x_2x_3 \ldots x_n \ldots, & \text{if } x_1 = 0, \\ 0.R(x_2)R(x_3) \ldots R(x_n) \ldots, & \text{if } x_1 = 1, \end{cases}$$ (4.4)

where $R$ is defined in Notation 2.2. Moreover, if (4.3) holds, then “$\xrightarrow{g}$” is “$\xleftarrow{g}$” in (4.4).

### 4.2 Specification for firm carcass map

Let $g$ be a firm carcass map, which will be fixed till the end of this section. Denote by $n_0$ the minimal natural number such that $g^{n_0}(x) = 0$ for each kink $x$ of $g$.

**Notation 4.7.** For any $n \geq 1$ and $k$, $0 \leq k < 2^{n-1}$ denote $I_{n,k} = (\mu_{n,k}, \mu_{n,k+1})$ and $\#I_{n,k} = \mu_{n,k+1} - \mu_{n,k}$.

**Remark 4.8.** For every $k$, $0 \leq k < 2^{n_0-1}$ the graph of the function $g$ on $I_{n_0,k}$ is a segment of a line.

**Notation 4.9.** For any $n \geq 1$ and $k$, $0 \leq k < 2^n$ denote

$$\delta_{n,k} = \frac{\mu_{n+1,k+1} - \mu_{n,k}}{\mu_{n,k+1} - \mu_{n,k}}$$

**Remark 4.10.** Suppose that $I_{n+2,p} \subseteq I_{n+1,k}$, where (4.1) is the binary expansion of $k$. By Remark 3.3 there there exists $x_{n+1} \in \{0; 1\}$ such that $p = 2k + x_{n+1}$. Thus, by Notation 4.7

$$\#I_{n+2,p} = \#I_{n+1,k} \cdot R^{x_{n+1}}(\delta_{n+1,k})$$.
Remark 4.11. For any \( n \geq n_0 - 1 \) and \( k, 0 \leq k < 2^{n-1} \) denote \( I_{n-1,k^*} = g(I_{n,k}) \). Then
\[
\delta_{n-1,k^*} = \begin{cases} 
\delta_{n,k} & \text{if } I_{n,k} \subset I_{2,0}, \\
1 - \delta_{n,k} & \text{otherwise.}
\end{cases}
\]

Using Notation 2.2 we can rewrite Remark 4.11 as follows.

Remark 4.12. For any \( n \geq n_0 \) and \( k, 0 \leq k < 2^n \) with the binary expansion (4.1) denote \( I_{n,k^*} = g(I_{n+1,k}) \). Then
\[
\delta_{n+1,k} = R^n(\delta_{n,k^*}).
\]

Notation 4.13. Denote \( v_k = \delta_{n_0-1,k} \) for every \( k, 0 \leq k < 2^{n_0-1} \).

Notation 4.14. Denote \( V = \{ v_k, 0 \leq k < 2^{n_0-1} \} \), where \( v_k \) are defined in Notation 4.13, and \( v_- = \min_{v \in V} \{ v, 1-v \} \) and \( v^+ = \max_{v \in V} \{ v, 1-v \} \).

The following remark follows from Remark 4.12.

Remark 4.15. For any \( n \geq n_0 \) and any \( p \) and \( k \) such that \( I_{n,k} \subset I_{n_0-1,p} \) the restriction
\[
#I_{n_0-1,p} \cdot (v_-)^{n-n_0+1} \leq #I_{n,k} \leq #I_{n_0-1,p} \cdot (v^+)^{n-n_0+1}
\]
holds.

Remark 4.16. Let \( n \geq 1 \) and \( k, 0 \leq k < 2^n \) has binomial expansion (4.1). Then
\[
g(I_{n+1,k}) = I_{n,p},
\]
where the binomial expansion of \( p \) is
\[
p = \begin{cases} 
x_2 \ldots x_n & \text{if } x_1 = 0, \\
R(x_2) \ldots R(x_n) & \text{if } x_1 = 1.
\end{cases}
\]

The next remark follows from Remarks 4.12 and 4.16 by induction.

Remark 4.17. Let \( n \geq n_0 \) and \( k, 0 \leq k < 2^{n-1} \) has the binary expansion (4.1). Then
\[
\delta_{n+1,k} = R^n(\delta_{n_0,p}), \text{ where } p = \sum_{i=n-n_0+2}^{n} R^a(x_i)2^{n-i},
\]
\[
\alpha_{n-n_0+1} = \sum_{i=1}^{n-n_0+1} \left| x_i - x_{i-1} \right| \quad (4.5)
\]
and \( x_0 = 0 \).

Remark 4.18. Let (4.1) be the binary expansion of a natural number \( k \) and let \( \alpha_n \) be defined by (4.5). Then \((-1)^{\alpha(k)} = (-1)^x_n \).
Proof. For every \( i < n \) such that \( x_i = 1 \) denote \( p \) the number, which is obtained from \( k \) by the change of \( x_i \) to 0. Then
\[
\alpha_n(k) - \alpha_n(p) = (1 - x_{i-1}) + (1 - x_{i+1}) - x_{i-1} - x_{i+1} = 2(1 - x_{i-1} - x_{i+1}),
\]
which is an even number, whence \((-1)^{\alpha(k)} = (-1)^{\alpha(p)}\) and the necessary equality follows. \( \square \)

By Remark 4.18 we can rewrite Remark 4.17 as

**Remark 4.19.** Let \( n \geq n_0 \) and \( k, 0 \leq k < 2^{n_1} \) has the binary expansion (4.1). Then \( \delta_{n+1,k} = \mathcal{R}^{x_{n-n_0}+1}(\delta_{n_0,p_{i-1}}) \), where
\[
p_{i-1} = \sum_{j=i-n_0+2}^{i} \mathcal{R}^{x_{i-n_0+1}}(x_j)2^{i-j} \tag{4.6}
\]
for all \( i, n_0 + 1 \leq i \leq n \).

**Lemma 4.20.** For any \( n < n_0 \) and \( k, 0 \leq k < 2^{n_1} \),
\[
\#I_{n+1,k_n} = \#I_{n_0,k_{n_0-1}} \cdot \prod_{i=n_0+1}^{n} \mathcal{R}^{x_{i-n_0}+1}(\delta_{n_0,p_{i-1}}) \tag{4.7}
\]
holds, where \( p_{i-1} \) is defined by (4.6) for all \( i, n_0 + 1 \leq i \leq n \).

Proof. By Remark 4.10,
\[
\#I_{n+1,k_n} = \#I_{n,k_{n-1}} \cdot \mathcal{R}^{x_n}(\delta_{n,k_{n-1}}). 
\]
Now, by Remark 4.19 we can continue
\[
\#I_{n,k_{n-1}} \cdot \mathcal{R}^{x_n}(\delta_{n,k_{n-1}}) = \#I_{n,k_{n-1}} \cdot \mathcal{R}^{x_{n-n_0}+1}(\delta_{n_0,p_{i-1}}),
\]
where \( p_{n-1} \) is defined in (4.6). Now (4.7) follows by induction. \( \square \)

We are now ready to prove Theorem 2.

**Proof of Theorem 2** Denote \( d = \max_{k, 0 \leq k < 2^{n_0}-1} \#I_{n_0-1,k} \). By Remark 4.15,
\[
\#I_{n,k} \leq d \cdot (v^+)^{n-n_0+1}
\]
for any \( n \geq n_0 \) and \( k, 0 \leq k \leq 2^{n_1} \).

Since \( v^+ < 1 \), then
\[
\lim_{n \to \infty} \left( \max_{k, 0 \leq k < 2^{n_1-1}} \#I_{n,k} \right) \leq \lim_{n \to \infty} d \cdot (v^+)^{n-n_0+1} = 0
\]
and theorem follows. \( \square \)
5 The derivative of the conjugation

We will start this section with some technical calculations, which we will use for the further reasonings.

5.1 Technical calculations

Notation 5.1. Let \( g \) be a firm carcass map, which will be fixed up to the end of this section.

Let \( x \in [0, 1] \) be fixed till the end of the section, \( \{ x_n, n \geq 0 \} \) be its \( g \)-expansion and \( k_n \) be defined by (4.1). Denote \( \hat{x}_n = \mu_{n+1,k_n}, \hat{x}_n^+ = \mu_{n+1,k_n+1}, \hat{x}_n^\pm = \mu_{n+2,2k_n+1}, \hat{x}_n^- = \mu_{n+1,k_n-1}, \hat{x}_n^+ = \mu_{n+2,2k_n-1} \).

Remark 5.2. Notice, that, by Notation 4.7,
\( I_{n+1, k_n} = (\hat{x}_n, \hat{x}_n^+) \) and
\( I_{n+2, 2k_n} = (\hat{x}_n^- , \hat{x}_n^+) \).

Remark 5.3. Notice that
\( \hat{x}_n^- \leq \hat{x}_n^\pm \leq x \leq \hat{x}_n^\pm \leq \hat{x}_n^+ \)
for every \( n \geq 1 \). Moreover, all “\( \leq \)” are “\( < \)”, whenever \( x \not\in \{0, 1\} \).

Suppose that \( x_{n+1} = 1 \). Then we can specify the \( g \)-expansion (4.3) of \( x \) as
\[ x \xleftarrow{g} 0x_1x_2 \ldots x_n 1 \underbrace{0 \ldots 0}_{t \text{ zeros}} 1 x_{n+t+3}x_{n+t+4} \ldots \] (5.1)

Remark 5.4. If (5.1) is the \( g \)-expansion of \( x \in [0, 1] \), then
\[ x_{n+1} = x_{n+t+2} = 1, \]
\[ x_{n+2} = x_{n+3} = \ldots = x_{n+t+1} = 0, \]
\[ \hat{x}_{n+1} = \hat{x}_{n+2} = \ldots = \hat{x}_{n+t+1}. \]

Denote
\[ b_i = \hat{x}_{n+i+1}^- , 0 \leq n \leq t. \] (5.2)

Calculation 5.5. If \( x \in (0, 1) \) has \( g \)-expansion (5.1) and \( \{ b_i, i \geq 0 \} \) is given by (5.2), then
\[ b_0 = \hat{x}_n \] and
\[ b_i = \hat{x}_{n+i}^+ \] (5.3)
for all \( i, 1 \leq i \leq t. \)
Proof. Since $x_{n+1} = 1$, then $\hat{x}_{n+1} = \hat{x}_n$, whence $b_0 = \hat{x}_n$. Next,

$$\hat{x}_{n+1} = \hat{x}_{n+2} = \ldots \hat{x}_{n+t} = \hat{x}_{n+t+1} \neq \hat{x}_{n+t+2}$$

implies $\hat{x}_{n+i+1} = \hat{x}_{n+i}$ for $i = 1, \ldots, t$, whence (5.3) follows. 

\[\square\]

Lemma 5.6. Suppose that $n \geq n_0 - 1$. Then the following hold:

1. For any $i$, $0 \leq i \leq n_0 + 1$ write

$$\#([\hat{x}_n, \hat{x}_n^+] \cdot (v_-)^{i+1} \leq \#([b_i, \hat{x}_{n+i}]) = \#([\hat{x}_n, \hat{x}_n^+] \cdot (v^+)^{i+1}. \quad (5.4)$$

2. For every $i, n_0 + 2 \leq i \leq t$ we have that

$$\#([\hat{x}_n, \hat{x}_n^+] \cdot (v_-)^{n_0+2} \cdot v_i^{i-n_0-1} \leq \#([b_i, \hat{x}_{n+i}]) \leq \#([\hat{x}_n, \hat{x}_n^+] \cdot (v^+)^{n_0+2} \cdot v_i^{i-n_0-1}. \quad (5.5)$$

3. For any $i$, $1 \leq i \leq n_0$ the restriction

$$\#[\hat{x}_n, \hat{x}_n^+] \cdot (v_-)^i \leq \#[\hat{x}_{n+1}, \hat{x}_{n+i}] \leq \#[\hat{x}_n, \hat{x}_n^+] \cdot (v^+)^i \quad (5.6)$$

holds.

4. For every $i, n_0 \leq i \leq t - n_0 + 1$ we have that

$$\#[\hat{x}_{n+1}, \hat{x}_{n+i}] = \#[\hat{x}_{n+1}, \hat{x}_{n+n_0}] \cdot v_i^{i-n_0} \quad (5.7)$$

and

$$\#[\hat{x}_n, \hat{x}_n^+] \cdot (v_-)^{n_0} \cdot v_i^i - n_0 \leq \#[\hat{x}_{n+1}, \hat{x}_{n+i}] \leq \#[\hat{x}_n, \hat{x}_n^+] \cdot (v^+)^{n_0} \cdot v_i^{i-n_0}. \quad (5.8)$$

5. For every $i, t - n_0 + 2 \leq i \leq t$ we have

$$\#[\hat{x}_n, \hat{x}_n^+] \cdot v_i^{i-2n_0+1} \cdot (v_-)^{i-t+2n_0-1} \leq \#[\hat{x}_{n+1}, \hat{x}_{n+i}] \leq \#[\hat{x}_n, \hat{x}_n^+] \cdot v_i^{i-2n_0+1} \cdot (v^+)^{i-t+2n_0-1}. \quad (5.9)$$

Proof. The restriction (5.4) follows from Remark 4.10.

By Calculation 5.5 the $g$-expansion of $\hat{x}_{n+1}$ is

$$\hat{x}_{n+1} \leftarrow g \rightarrow x_0 x_1 \ldots x_n 1,$$

the $g$-expansion of $b_i$ is

$$b_i \leftarrow g \rightarrow x_0 x_1 \ldots x_n 0 \underbrace{1 \ldots 1}_{i \text{ ones}},$$

and $b_i = \mu_{n+i+1, p_i}$, where the binary expansion of $p_i$ is

$$p_i = x_1 \ldots x_n \underbrace{0 1 \ldots 1}_{i \text{ ones}} .$$
Thus,

\((b_i, \hat{x}_{n+1}) = I_{n+2+i,p_i}\).

By Remark 4.19 for any \(i, 0 \leq i \leq t\) denote \(q_i\) the number with the binary expansion

\[ q_i = \mathcal{R}^{x_{n+3+i-n_0}}(x_{n+4+i-n_0}) \cdots \mathcal{R}^{x_{n+3+i-n_0}}(x_{n+1+i}) \]

and write

\[ \delta_{n+2+i,p_i} = \mathcal{R}^{x_{n+3+i-n_0}}(\delta_{n_0-1, q_i}). \]

If \(n_0 < i \leq t\), then

\[ \delta_{n+2+i,p_i} = \mathcal{R}^{1}(\delta_{n_0-1, \mathcal{R}^1(q_i)}) = 1 - \delta_{n_0-1,0} = 1 - v_0. \] (5.10)

If \(i \geq n_0 + 2\), then (5.10) implies

\[ \#([b_i, \hat{x}_{n+1}]) = \#([b_{i-1}, \hat{x}_{n+1}]) \cdot \mathcal{R}^{1}(\delta_{n+1+i, p_{i-1}}) = \#([b_{i-1}, \hat{x}_{n+1}]) \cdot v_0, \]

whence

\[ \#([b_i, \hat{x}_{n+1}]) = \#([b_{n_0+1}, \hat{x}_{n+1}]) \cdot v_{i-n_0 - 1} \]

and (5.5) follows.

The restriction (5.6) follows from Remark 4.15.

Remind that \([\hat{x}_{n+1}, \hat{x}_{n+1}^+] = [\hat{x}_{n+i}, \hat{x}_{n+i}^+] = I_{n+i+1,k_{n+i}}\).

Notice, that for every \(i, 1 \leq i < t\) the last \(i - 1\) binary digits of \(k_{n+i}\) are zero and the last binary digit of \(k_{n+i}\) is 1. Thus, if \(n_0 \leq i \leq t - n_0 + 1\), then the last \(n_0 - 1\) binary digits of \(k_{n+i}\) are zero and, by Remark 4.19,

\[ \delta_{n+i+1,k_{n+i}} = \delta_{n_0-1,0}. \]

Now, by Remark 4.12

\[ \#([\hat{x}_{n+1}, \hat{x}_{n+1}^+]) = \#([\hat{x}_{n+1}, \hat{x}_{n+i-1}^+]) \cdot v_0, \]

whence (5.7) follows by induction.

Restrictions (5.8) and (5.9) follow from (5.7) by Remark 4.15.

**5.2 The case of firm carcass maps**

Let \(g_1\) and \(g_2\) be firm carcass maps, which will be fixed to the end of the section. Denote by \(n_0\) the minimal natural number such that \(g_i^{-n_0}(0)\) contains all the kinks of \(g_i\), where \(i \in \{1, 2\}\).

For every \(n \geq 1\) even \(k\), \(0 \leq k \leq 2^{n-1}\) and \(i \in \{1, 2\}\) define \(\mu_{n,k}(g_i), I_{n,k}(g_i), \delta_{n,k}(g_i), v(g_i), \mathcal{V}(g_i), v_-(g_i)\) and \(v^+(g_i)\) as in Notations 4.7, 4.9 and 4.14.
Lemma 5.7. For every \( x \in [0, 1] \) the \( g_2 \)-expansion of \( h(x) \) coincides with the \( g_1 \)-expansion of \( x \).

Notation 5.8. Denote by \( h \) be the unique continuous invertible solution of the functional equation (1.5). Let a number \( x \in [0, 1] \) be fixed till the end of the section and let (4.3) be its \( g_1 \)-expansion.

Remark 5.9. If \( x \notin g_1^{-n}(0) \) for some \( n \geq 1 \) then
\[
h'_{n+1}(x) = \frac{h(\hat{x}_n^+) - h(\hat{x}_n)}{\hat{x}_n^+ - \hat{x}_n}.
\]

Remark 5.10. If \( x \notin g_1^{-n}(0) \) for some \( n > 1 \) then there exist \( v \in \mathcal{V}(g_1) \) and \( w \in \mathcal{V}(g_2) \) such that
\[
h'_n(x) = h'_{n-1}(x) \cdot \frac{v}{w}.
\]

Remark 5.11. If for some \( x \in [0, 1] \) the derivative \( h'(x) \) exists, then
1. If \( x > 0 \), then there exists \( \lim_{n \to \infty} h'_n(x-) \), and \( h'(x) = \lim_{n \to \infty} h'_n(x-) \);
2. If \( x < 1 \), then there exists \( \lim_{n \to \infty} h'_n(x+) \), and \( h'(x) = \lim_{n \to \infty} h'_n(x+) \).

Suppose that for some \( n \geq 1 \) the \( g_1 \)-expansion (4.3) can be specified as (5.1). The next fact follows from (5.7), Lemma 5.7 and Remark 5.9.

Remark 5.12. For every \( i \geq 1 \)
\[
\left( \frac{v_-(g_2)}{v^+(g_1)} \right)^i \cdot h'_{n+1}(x) \leq h'_{n+i}(x) \leq \left( \frac{v^+(g_2)}{v_-(g_1)} \right)^i \cdot h'_{n+1}(x)
\]

Remark 5.13. For every \( i, n_0 \leq i \leq t - n_0 + 1 \)
\[
h'_n(x) = h'_{n_0+n}(x) \cdot \left( \frac{v_0(g_2)}{v_0(g_1)} \right)^{i-n_0}
\]

Lemma 5.14. Suppose that \( g_1^{n+1}(x) = 0 \) and \( s \in [\hat{x}_n^-, \hat{x}_n^+] \) for \( n > n_0 \). Then
\[
v_-(g_2) \cdot h'_{n+1}(\hat{x}_n^-) \leq \frac{h(\hat{x}_n) - h(s)}{\hat{x}_n - s} \leq h'_{n+1}(\hat{x}_n^-) \cdot \frac{1}{v(g_1)}.
\]

Proof. Since \( g_1^{n+1}(x) = 0 \), then \( x = \hat{x}_n \). Denote \( A(\hat{x}_n^-, h(\hat{x}_n^-)) \), \( S(s, h(s)) \), \( X(x, h(x)) \), \( S_-(\hat{x}_n^-, h(\hat{x}_n^-)) \) and \( S^+(\hat{x}_n^+, h(\hat{x}_n^-)) \) (see Fig. 1a).

Also let \( k_{S,X} \) be the tangent of \( S_- X \), let \( k_{SX} \) be \( k_{S,X} \) and let \( k_{S+X} \) be the tangent of \( S^+ X \). Then
\[
k_{S,X} \leq k_{SX} \leq k_{S+X},
\]
because \( s \in [\hat{x}_n^-, \hat{x}_n^+] \) and \( h \) increase (by Remark 3.7).
By Remark 4.15 and Lemma 5.7

\[ k_{S-X} \geq \frac{(h(x) - h(\hat{x}_n^-)) \cdot v_-(g_2)}{x - \hat{x}_n^-} \]

and

\[ k_{S+X} \leq \frac{h(x) - h(\hat{x}_n^-)}{(x - x_n^-) \cdot v_-(g_1)}. \]

Now lemma follows from Remark 5.9.

\[ \square \]

Lemma 5.15. Suppose that \( x \) is \( g_1 \)-infinite and \( s \in [\bar{x}_n, \hat{x}_n^+] \) for \( n > n_0 \). There exist \( k_- \) and \( k_+ \), independent on \( x \), and \( i \geq 1 \) such that

\[
k_- \cdot h'_{n+i}(\bar{x}_n^-) \leq \frac{h(\hat{x}_n) - h(s)}{\hat{x}_n - s} \leq k_+ \cdot h'_{n+i}(\bar{x}_n^-).
\]

Proof. Define \( \{b_i, 0 \leq i \leq t\} \) by (5.2).

Notice, that for any \( s_-, s^+, x_- \) and \( x^+ \) such that

\[
\begin{cases}
    s_- \leq s \leq s^+ < x^- \leq x \leq x^+ \\
    s_- < s^+ < x_- < x^+
\end{cases}
\]

if follows from Remark 3.7 that

\[
k_{S-X} \leq k_{SX} \leq k_{S+X-},
\]

where:

1. \( k_{SX} \) is the tangent of the line, which connects points \((s, h(s))\) and \((x, h(x))\);
2. \( k_{S-X} \) is the tangent of the line, which connects points \((s-, h(s_-))\) and \((x^-, h(x_-))\) and,

finally

3. \( k_{S+X-} \) is the tangent of the line, which connects points \((s^+, h(s_-))\) and \((x^-, h(x^+))\).

By definitions,

\[
k_{SX} = \frac{h(x) - h(s)}{x - s},
\]

\[
k_{S-X} = \frac{h(x_-) - h(s^+)}{x^+ - s_-},
\]

and

\[
k_{S+X-} = \frac{h(x^+) - h(s_-)}{x_- - s^+}.
\]

If \( s \in [b_i, b_{i+1}) \) for some \( i, 0 \leq i < t \) then take

\[
s_- = b_i, \quad s^+ = b_{i+1},
\]

\[
x_- = \hat{x}_{n+1}, \quad x^+ = \hat{x}_{n+i+2}.
\]

(5.11)
a) Construction of $S, S^+, S^-$ and $X$

b) Case $t = 0$ in Lemma 5.15

Figure 1: Proof of Lemmas 5.14 and 5.15

a) $s \in [\hat{x}_n, \hat{x}_{n+1}]$

b) $s \in [\hat{x}_{n+1}, \hat{x}_{n+1})$

Figure 2: Case $t = 1$ in Lemma 5.14

And if $s \in [b_i, \hat{x}_{n+1})$ then take

$$s_- = b_i, \quad s^+ = \hat{x}_{n+1}, \quad x_- = \hat{x}_{n+t+2}, \quad x^+ = \hat{x}_{n+t+1}.$$  \hspace{1cm} (5.12)

The case $t = 0$ is presented at Figure 1b.

The case $t = 1$ is illustrated at Figure 2.

Suppose that $s \in [b_i, b_{i+1})$ for some $i$, $0 \leq i < t$, and that the numbers $s_-, s^+, x_-$ and $x^+$ are defined by (5.11).

Since $x_- - s^+ = \hat{x}_{n+1} - b_{i+1}$ we can use Lemma 5.6 directly to obtain the restrictions for $x_- - s^+$. In order to use the same Lemma for the restriction of $x^+ - s_- = \hat{x}_{n+i+2} - b_i$, we will write

$$x^+ - s_- = (\hat{x}_{n+i+2} - \hat{x}_{n+1}) + (\hat{x}_{n+1} - b_i)$$

and then use Lemma 5.6 to obtain the restriction of each summand.

For $0 \leq i \leq n_0$ by (5.4) of Lemma 5.6

$$x_- - s^+ \leq \#[\hat{x}_n, \hat{x}^+_n]$$  \hspace{1cm} (5.13)

and

$$\#[\hat{x}_n, \hat{x}^+_n] \cdot (v_- (g_1))^{n_0+1} \leq \hat{x}_{n+1} - s_-.$$  \hspace{1cm} (5.14)
If \( n_0 + 1 \leq i < t \), then by (5.5) of Lemma 5.6,
\[
x_+ - s^+ \leq \#[\hat{x}_n, \hat{x}_n^+] \cdot (v^+ (g_1))^{n_0 + 2} \cdot (v_0 (g_1))^{t - n_0}
\] (5.15)
and
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot (v_-)_0^{n_0 + 2} \cdot v_0^{t - n_0 - 1} \leq \hat{x}_{n + 1} - s_-
\] (5.16)

For \( i, 0 \leq i \leq n_0 - 2 \) it follows from (5.6) of Lemma 5.6 that
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot (v_- (g_1))^{n_0} \leq x^+ - \hat{x}_{n + 1}.
\] (5.17)

If \( n_0 - 1 \leq i \leq t - n_0 - 1 \) then it follows from (5.8) of Lemma 5.6 that
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot (v_- (g_1))^{n_0} \cdot (v_0 (g_1))^{i + 2 - n_0} \leq x^+ - \hat{x}_{n + 1}.
\] (5.18)

If \( t - n_0 \leq i < t \), then it follows from (5.9) of Lemma 5.6 that
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot v_{t - 2n_0 + 1} \cdot (v_-)^{2n_0} \leq x^+ - \hat{x}_{n + 1}.
\] (5.19)

If \( 0 \leq i \leq n_0 - 2 \) then if follows from (5.14) and (5.17) then
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot 2 \cdot (v_-)^{n_0 + 1} \leq x^+ - s_-
\] (5.20)

If \( n_0 - 1 \leq i \leq n_0 \) then it follows from (5.14) and (5.18) that
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot 2 \cdot (v_-)^{n_0 + 1} \cdot v_0^{i + 2 - n_0} \leq x^+ - s_-
\] (5.21)

If \( n_0 + 1 \leq i \leq t - n_0 - 1 \), then, by (5.16) and (5.18) obtain
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot 2 \cdot (v_-)^{n_0 + 2} \cdot v_0^{t - n_0} \leq x^+ - s_-
\] (5.22)

If \( t - n_0 \leq i < t - 1 \), then, by (5.16) and (5.19) obtain
\[
\#([\hat{x}_n, \hat{x}_n^+]) \cdot 2 \cdot (v_-)^{2n_0} \cdot v_{t - n_0} \leq x^+ - s_-. 
\] (5.23)

\[
K_{S, x^+} \geq \min(x^+ - s_-) \over \max(x_- - s^+)
\]
and
\[
K_{S, x_-} \leq \max(x_- - s^+) \over \min(x^+ - s_-)
\]

If \( 0 \leq i \leq n_0 - 2 \), then, by (5.13) and (5.20)
\[
K_{S, x^+} \geq \#([\hat{y}_n, \hat{y}_n^+]) \cdot 2 \cdot (v_- (g_2))^{n_0 + 1} \over \#([\hat{x}_n, \hat{x}_n^+])
\] (5.24)
and
\[
K_{S^{+}X_{-}} \leq \frac{\# [\hat{y}_{n}, \hat{y}_{n}^{+}]}{\# [\bar{x}_{n}, \bar{x}_{n}^{+}]} \cdot 2 \cdot (v_{-}(g_{1}))^{n_{0}+1}.
\] (5.25)

If \(n_{0} - 1 \leq i \leq n_{0}\) then by (5.13) and (5.21)
\[
K_{S^{-}X^{+}} \geq \frac{\# [\hat{y}_{n}, \hat{y}_{n}^{+}]}{\# [\bar{x}_{n}, \bar{x}_{n}^{+}]} \cdot 2 \cdot (v_{-}(g_{2}))^{i+2-n_{0}} \cdot (v_{0}(g_{2}))^{i+2-n_{0}}
\] (5.26)

and
\[
K_{S^{+}X^{-}} \leq \frac{\# [\hat{y}_{n}, \hat{y}_{n}^{+}]}{\# [\bar{x}_{n}, \bar{x}_{n}^{+}]} \cdot 2 \cdot (v_{-}(g_{1}))^{n_{0}+1} \cdot (v_{0}(g_{1}))^{i+2-n_{0}}
\] (5.27)

If \(n_{0} + 1 \leq i \leq t - n_{0} - 1\) then by (5.15) and (5.22)
\[
K_{S^{-}X^{+}} \geq \frac{\# [\hat{y}_{n}, \hat{y}_{n}^{+}]}{\# [\bar{x}_{n}, \bar{x}_{n}^{+}]} \cdot 2 \cdot (v_{-}(g_{2}))^{n_{0}+2} \cdot (v_{0}(g_{2}))^{i+2-n_{0}} \cdot (v_{0}(g_{1}))^{i+2-n_{0}}
\] (5.28)

and
\[
K_{S^{+}X^{-}} \leq \frac{\# [\hat{y}_{n}, \hat{y}_{n}^{+}]}{\# [\bar{x}_{n}, \bar{x}_{n}^{+}]} \cdot (v_{+}(g_{2}))^{n_{0}+2} \cdot (v_{0}(g_{2}))^{i-n_{0}} \cdot (v_{0}(g_{1}))^{i-n_{0}}
\] (5.29)

If \(t - n_{0} \leq i \leq t - 1\) then by (5.15) and (5.23)
\[
K_{S^{-}X^{+}} \geq \frac{\# [\hat{y}_{n}, \hat{y}_{n}^{+}]}{\# [\bar{x}_{n}, \bar{x}_{n}^{+}]} \cdot 2 \cdot (v_{-}(g_{2}))^{2n_{0}} \cdot (v_{0}(g_{2}))^{i-n_{0}} \cdot (v_{0}(g_{1}))^{i-n_{0}}
\] (5.30)

and
\[
K_{S^{+}X^{-}} \leq \frac{\# [\hat{y}_{n}, \hat{y}_{n}^{+}]}{\# [\bar{x}_{n}, \bar{x}_{n}^{+}]} \cdot (v_{+}(g_{2}))^{n_{0}+2} \cdot (v_{0}(g_{2}))^{i-n_{0}} \cdot (v_{0}(g_{1}))^{i-n_{0}}
\] (5.31)

Due to Remark 5.9 it follows from (5.24), (5.25), (5.26) and (5.27) that for any \(i, 0 \leq i \leq n_{0}\)
\[
K_{S^{-}X^{+}} \geq h'_{n+1}(x) \cdot 2 \cdot (v_{-}(g_{2}))^{n_{0}+1} \cdot (v_{0}(g_{2}))^{2}
\] and
\[
K_{S^{+}X^{-}} \leq \frac{h'_{n+1}(x)}{2 \cdot (v_{-}(g_{1}))^{n_{0}+1} \cdot (v_{0}(g_{1}))^{2}}.
\]

By Remark 5.12
\[
\left(\frac{v_{-}(g_{2})}{v_{+}(g_{1})}\right)^{n_{0}} \cdot h'_{n+1}(x) \leq h'_{n+1}(x) \leq \left(\frac{v_{+}(g_{2})}{v_{-}(g_{1})}\right)^{n_{0}} \cdot h'_{n+1}(x).
\] (5.32)

If \(n_{0} + 1 \leq i \leq t - n_{0} - 1\), then it follows from (5.28) and (5.29), Remarks 5.9 and 5.13 and from just obtained (5.32) that
\[
K_{S^{-}X^{+}} \geq h'_{n+1}(x) \cdot \frac{2 \cdot (v_{-}(g_{2}))^{n_{0}+2} \cdot (v_{0}(g_{2}))^{i+2-n_{0}}}{(v_{+}(g_{1}))^{n_{0}+2} \cdot (v_{0}(g_{1}))^{i-n_{0}}} \geq
\]
\[
\geq h'_{n+1}(x) \cdot \left(\frac{v_{+}(g_{2})}{v_{-}(g_{1})}\right)^{n_{0}} \cdot \frac{2 \cdot (v_{-}(g_{2}))^{n_{0}+2} \cdot (v_{0}(g_{2}))^{i+2-n_{0}}}{(v_{+}(g_{1}))^{n_{0}+2} \cdot (v_{0}(g_{1}))^{i-n_{0}}} \cdot \left(\frac{v_{-}(g_{1})}{v_{+}(g_{2})}\right)^{n_{0}} \geq
\]
\[
\geq h'_{n_{0}+n}(x) \cdot \frac{(v_{0}(g_{2}))^{i-n_{0}}}{(v_{0}(g_{1}))^{2}} \cdot \frac{2 \cdot (v_{-}(g_{2}))^{2} \cdot (v_{0}(g_{2}))^{2}}{(v_{+}(g_{1}))^{2}} \cdot \left(\frac{v_{-}(g_{1})}{v_{+}(g_{2})}\right)^{n_{0}} \geq
\]

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\[ h_{n+1}(x) \cdot \frac{2 \cdot (v_-(g_2))^2 \cdot (v_0(g_2))^2 \cdot (v_-(g_1) \cdot v_-(g_2))^{n_0}}{(v_+(g_1))^2} \]

and

\[ K_{S+X_-} \leq h'_{n+1}(x) \cdot \frac{(v_+(g_2))^{n_0+2} \cdot (v_0(g_2))^{i-n_0}}{2 \cdot (v_-(g_1))^{n_0+2} \cdot (v_0(g_1))^{i+2-n_0}} \leq h'_{n+1}(x) \cdot \left( \frac{v_-(g_2)}{v_+(g_1)} \right)^{n_0} \cdot \frac{(v_+(g_2))^{n_0+2} \cdot (v_0(g_2))^{i-n_0}}{2 \cdot (v_-(g_1))^{n_0+2} \cdot (v_0(g_1))^2} \cdot \left( \frac{v_+(g_1)}{v_-(g_2)} \right)^{n_0} \]

\[ \leq h'_{n+1}(x) \cdot \frac{(v_+(g_2))^2}{2 \cdot (v_0(g_1))^2} \cdot \left( \frac{v_+(g_1) \cdot v_+(g_2)}{v_-(g_1) \cdot v_-(g_2)} \right)^{n_0} \cdot \left( \frac{v_+(g_1)}{v_-(g_2)} \right)^{n_0} \]

If \( t - n_0 \leq i \leq t - 1 \), then it follows from (5.30), (5.31), (5.32) and Remarks 5.9 and 5.13 that

\[ K_{S_-X^+} \geq h'_{n+1}(x) \cdot \frac{2 \cdot (v_-(g_2))^{2n_0} \cdot (v_0(g_2))^{i-n_0}}{(v_+(g_1))^{n_0+2} \cdot (v_0(g_1))^{i-n_0} \cdot (v_+(g_2))^{n_0}} \geq h'_{n+1}(x) \cdot \frac{2 \cdot (v_-(g_2))^{2n_0} \cdot (v_0(g_2))^{i-n_0} \cdot (v_-(g_1))^{n_0}}{(v_+(g_1))^{n_0+2} \cdot (v_0(g_1))^{i-n_0} \cdot (v_+(g_2))^{n_0}} \geq h'_{n+1}(x) \cdot \frac{2 \cdot (v_-(g_2))^{2n_0} \cdot (v_0(g_2))^{i-n_0} \cdot (v_-(g_1))^{n_0}}{(v_+(g_1))^{n_0+2} \cdot (v_0(g_1))^{i-n_0} \cdot (v_+(g_2))^{n_0}} = h'_{n+1}(x) \cdot \frac{2 \cdot (v_-(g_2))^{2n_0} \cdot (v_0(g_2))^{i-n_0} \cdot (v_-(g_1))^{n_0}}{(v_+(g_1))^{n_0+2} \cdot (v_0(g_1))^{i-n_0} \cdot (v_+(g_2))^{n_0}} \]

This proves lemma for the case (5.11).

The case (5.12) can be considered analogously.

In the same manner as in Lemmas 5.14 and 5.15 we can prove the following lemma.

**Lemma 5.16.** Suppose that \( x < 1 \). There exist \( k_- \) and \( k^+ \), independent on \( x \), and \( i \geq 1 \) such that for any \( n \geq n_0 \) and \( s \in \left( \hat{x}_n, \hat{x}_n^+ \right] \) there exits \( i \geq 1 \) such that

\[ k_- \cdot h'_{n+i}(\hat{x}_n^+) \leq h(\hat{x}_n) - h(s) \leq k^+ \cdot h'_{n+i}(\hat{x}_n^+). \]
Suppose that $x$ is $g$-finite and $\tilde{n}$ is such that $g_1^{\tilde{n}}(x) = 0$.

Denote $I_{n,k} = (\mu_{n,k}(g_1), \mu_{n,k+1}(g_1))$. For any $t \in I_{n,k}$ denote

$$\Delta_L(I_{n,k}, t) = \left| \frac{h(t) - \mu_{n,k}(g_2)}{h'_n(t) \cdot (t - \mu_{n,k}(g_1))} - 1 \right|$$

and

$$\Delta_R(I_{n,k}, t) = \left| \frac{\mu_{n,k+1}(g_2) - h(t)}{h'_n(t) \cdot (\mu_{n,k+1} - t)} - 1 \right| .$$

Notice that the following conditions are equivalent:

1. $\Delta_L(I_{n,k}, t) = 0$;
2. $\Delta_R(I_{n,k}, t) = 0$;
3. The point $(t, h(t))$ belongs to the graph of $h_n$.

The next lemma follows from Lemma 2.8.

**Lemma 5.17.** For every $n > n_0$, every $k$, $0 \leq k < 2^{n-1}$ and $t \in I_{n,k}$ we have that

1. If $g_1$ increase on $I_{n,k}$, then
   $$\Delta_L(I_{n,k}, t) = \Delta_L(g_1(I_{n,k}), g_1(t))$$

   and

   $$\Delta_R(I_{n,k}, t) = \Delta_R(g_1(I_{n,k}), g_1(t)) .$$

2. If $g_1$ decrease on $I_{n,k}$, then
   $$\Delta_L(I_{n,k}, t) = \Delta_R(g_1(I_{n,k}), g_1(t))$$

   and

   $$\Delta_R(I_{n,k}, t) = \Delta_L(g_1(I_{n,k}), g_1(t)) .$$

**Proof.** Lemma follows from the direct calculations by (1.5). \qed

If $x \notin g_1^{-\infty}(0)$, then for any $n > n_0$ denote

$$\Delta_n(x) = \{ \Delta_L([\tilde{x}_n, \tilde{x}_n^+], \Delta_R([\tilde{x}_n, \tilde{x}_n^+])) \} .$$

If $g_1^{\tilde{n}}(x) = 0$, then for every $n > \max\{n_0, \tilde{n}\}$ denote

$$\Delta_n(x) = \{ \Delta_R([\tilde{x}_n^-, \tilde{x}_n], \Delta_L([\tilde{x}_n, \tilde{x}_n^+])) \} .$$

The next lemma follows from Lemma 5.17.

**Lemma 5.18.** For any $n \geq n_0$, we have that $\Delta_n(x) = \Delta_{n_0-1}(x)$.
We are now ready to prove Theorem 3.

**Proof of Theorem 3.** Suppose that the derivative \( h'(x) \) exists, is positive and finite.

Since the derivative \( h'(x) \) exists, is positive and finite, then \( \lim_{n \to \infty} \Delta_n(x) = 0 \), whence it follows from Lemma 5.18 that \( \Delta_{n_0-1}(x) = 0 \). Thus, there exists \( k, 0 \leq k < 2^{n_0-2} \) such that

\[
\sup_{t \in I_{n_0-1,k}} \Delta_R(I_{n_0-1,k}, t) = \sup_{t \in I_{n_0-1,k}} \Delta_L(I_{n_0-1,k}, t) = 0,
\]

which means that \( h \) is linear on \( I_{n_0-1,k} \). Now piecewise linearity of \( h \) on the whole \([0, 1]\) follows from part 2 of Lemma 2.9.

Part 2 of Theorem 3 follows from Lemmas 5.15, 5.16 and 5.18.

### 6 Length of the graph of the conjugacy

We will prove Theorem 3 in this section. Let \( h \) be the conjugacy of firm carcass maps \( g_1 \) and \( g_2 \).

**Lemma 6.1.** Either \( h \) is piecewise linear, or \( h \) is differentiable at all \( g_1 \)-finite points.

**Proof.** Remind that, by the construction of \( h_n \), we have that if \( x \in I_{n+1,k_n} \), then

\[
h_n'(x) = \frac{\#I_{n+1,k_n}(g_2)}{\#I_{n+1,k_n}(g_1)}
\]

for all \( n \geq n_0 \), whence, by Lemma 1.20,

\[
h_n'(x) = \frac{\#I_{n_0,k_{n_0-1}}(g_2)}{\#I_{n_0,k_{n_0-1}}(g_1)} \cdot \prod_{i=n_0+1}^{n} \frac{\mathcal{R}^{x_i+x_{i-n_0+1}}(\delta_{n_0,p_{i-1}}(g_2))}{\mathcal{R}^{x_i+x_{i-n_0+1}}(\delta_{n_0,p_{i-1}}(g_1))} \tag{6.1}
\]

Notice that if \( x < 1 \) then (6.1) also determines \( h_n'(x+) \). Suppose that \( x \) is \( g_1 \)-finite. Then there is \( m > n_0 \) such that for all \( i > m \) we have that

\[
\mathcal{R}^{x_i+x_i-n_0+1}(\delta_{n_0,p_{i-1}}) = \delta_{n_0,0},
\]

whence the multipliers in the infinite product

\[
R(x) = \frac{\#I_{n_0,k_{n_0-1}}(g_2)}{\#I_{n_0,k_{n_0-1}}(g_1)} \cdot \prod_{i=n_0+1}^{\infty} \frac{\mathcal{R}^{x_i+x_i-n_0+1}(\delta_{n_0,p_{i-1}}(g_2))}{\mathcal{R}^{x_i+x_i-n_0+1}(\delta_{n_0,p_{i-1}}(g_1))} \tag{6.2}
\]

stabilize on

\[
P = \frac{\delta_{n_0,p_{i-1}}(g_2)}{\delta_{n_0,p_{i-1}}(g_1)} \tag{6.3}
\]

and the limit (6.2) exists.
The left derivative $L(x)$ in the case $x$ is $g_1$-finite can be found as

$$L(x) = \frac{\# I_{x_0^+,x_{n_0-1}}(g_2)}{\# I_{x_0^+,x_{n_0-1}}(g_1)} \cdot \prod_{i=n_0+1}^{\infty} \frac{\mathcal{R}^{y_i+n_i-1}(\delta_{x_0,g_{i-1}}(g_2))}{\mathcal{R}^{y_i+n_i-1}(\delta_{x_0,g_{i-1}}(g_1))},$$

where

$$q_i-1 = \sum_{j=i-n_0+1}^{i-1} \mathcal{R}^{y_i-n_i}(y_j)2^{i-j}$$

and the sequence $\{y_i, i \geq 1\}$ can be constructed as follows:

1. $y_i = x_i$ for all $i$ before the beginning of the infinite series of zeros in the $g_1$-expansion of $x$.
2. For those $i$, where $x_i = 0$ is an element of infinite series of zeros take $y_i = 1$.

Thus, for all $i > m$ we have that $q_i-1 = \sum_{j=i-n_0+1}^{i-1} \mathcal{R}(1)2^{i-j}$ in (6.5), i.e. $q_i-1 = 0$. Also the infinite product (6.4) for $L(x)$ stabilizes on $\mathcal{P}$, which is defined by (6.3).

Now, by Theorem 5 either $h$ is piecewise linear, or the infinite products $L(x)$ and $R(x)$ exist.

We are now ready to prove Theorem 5.

**Proof of Theorem 5** For any $g_1$-finite $x \in [0,1]$ define

$$d_n = \frac{h(\hat{x}_n^+) - h(\hat{x}_n)}{\hat{x}_n^+ - \hat{x}_n}.$$

Since $\lim_{n \to \infty} d_n = \lim_{n \to \infty} h'_n(x)$, then, by Theorem 3 and Lemma 6.1 either $\lim_{n \to \infty} d_n = 0$, or $\lim_{n \to \infty} d_n = +\infty$. In each of these cases we have that

$$\lim_{n \to \infty} \sqrt{\frac{(h(\hat{x}_n^+) - h(\hat{x}_n))^2 + (\hat{x}_n^+ - \hat{x}_n)^2}{(\hat{x}_n^+ - \hat{x}_n) + (h(\hat{x}_n^+) - h(\hat{x}_n))}} =$$

$$= \lim_{n \to \infty} \left[ \frac{1}{1 + \frac{h(\hat{x}_n^+) - h(\hat{x}_n)}{\hat{x}_n^+ - \hat{x}_n} + \frac{\hat{x}_n^+ - \hat{x}_n}{2(h(\hat{x}_n^+) - h(\hat{x}_n))}} \right] =$$

$$= \sqrt{\frac{1}{1 + \frac{1}{0+\infty}}} = 1.$$

Now, for every $\varepsilon > 0$ and every $g_1$-finite $x \in (0,1)$ there is an interval $I(x,\varepsilon) = (\hat{x}_n,\hat{x}_n^+)$ such that

$$1 - \frac{\sqrt{(h(\hat{x}_n^+) - h(\hat{x}_n))^2 + (\hat{x}_n^+ - \hat{x}_n)^2}}{(\hat{x}_n^+ - \hat{x}_n) + (h(\hat{x}_n^+) - h(\hat{x}_n))} < \varepsilon.$$  \hspace{1cm} (6.6)

Analogously for $x = 0$ and $n$ such that (6.6) holds denote $I(0,\varepsilon) = [0,x_n^+]$. For $x = 1$ and $n$ such that (6.6) holds denote $I(1,\varepsilon) = (\hat{x}_n,1]$. Notice that every $I(x,\varepsilon)$ is open in $[0,1]$ and $\bigcup_x I(x,\varepsilon) = [0,1]$, where the union is taken for all $g_1$-finite $x \in [0,1]$.

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Notice, that by construction for all \( x_1, x_2 \) the closed intervals \( I(x_1, \varepsilon) \) and \( I(x_2, \varepsilon) \) either do not intersect, or their intersection consists of one point, or one of these sets is a subset of other.

It follows from the compactness of \([0, 1]\) that there exist \( x_1, \ldots, x_m \) such that

\[
\bigcup_{i=1}^{m} I(x_i, \varepsilon) = [0, 1],
\]

(6.7)

Now Theorem 5 follows from (6.6) and (6.7). \( \square \)

7 Skew tent maps

The results about the conjugation of the carcase maps, which are obtained in Sections 5 and 6. Precisely, we will reduce Theorem 4 from Theorem 3.

7.1 The derivative of the conjugation

The following specification of Remark 4.19 is clear for the case of skew tent maps.

**Remark 7.1.** Let \( g \) be the skew tent map with \( g(v) = 1 \) and \( x \in [0, 1] \) has \( g \)-representation \( \{(x_n, k_n), n \geq 1\} \). Then

\[
\#I_{n+1,k_n} = \#I_{n,k_{n-1}} \cdot \mathcal{R}^{x_n+x_{n+1}}(v).
\]

for any \( n \geq 1 \)

**Proof.** Since \( I_{2,0} = [0, 1/2] \) and \( I_{2,1} = [1/2, 1] \), denote \( I_1 = [0, 1] \).

By Remark 4.12

\[
\delta_{2,k} = \mathcal{R}^k(\delta_{1,0}) = \mathcal{R}^k(v)
\]

for any \( k \in \{0; 1\} \)

Thus, by Remark 4.19 for every \( n \geq 2 \) we have

\[
\delta_{n+1,k_n} = \mathcal{R}^{x_{n-2}}(\delta_{2,\mathcal{R}^{x_{n-2}}(x_{n-1})}) = \mathcal{R}^{x_{n-1}}(v).
\]

Now it follows from Remark 4.10 that

\[
\#I_{n+1,k_n} = \#I_{n,k} \cdot \mathcal{R}^{x_n}(\delta_{n,k}) = \#I_{n,k} \cdot \mathcal{R}^{x_n+x_{n+1}}(v).
\]

\( \square \)

The same result as in Remark 7.1 can be obtained directly from the basic notions (such as iteration of a function). We show these reasonings in lemma below.
Lemma 7.2. For any \( v \in (0, 1) \) and all \( n \geq 1 \) equalities

\[
\mu_{n+1,4k+1}(f_v) = \mu_{n,2k}(f_v) + v(\mu_{n,2k+1}(f_v) - \mu_{n,2k}(f_v)),
\]

and

\[
\mu_{n+1,4k+3}(f_v) = \mu_{n,2k+1}(f_v) + (1 - v)(\mu_{n,2k+2}(f_v) - \mu_{n,2k+1}(f_v))
\]

hold for all \( k, 0 \leq k < 2^{n-2} \).

Proof. For every \( m \geq 1 \) and \( t, 0 \leq t \leq 2^{m-1} \) we will write \( \mu_{m,t} \) instead of \( \mu_{m,t}(f_v) \).

Notice, that

\[
\mu_{n+1,4k} < \mu_{n+1,4k+1} < \mu_{n+1,4k+2} < \mu_{n+1,4k+3} < \mu_{n+1,4k+4}
\]

are consequent zeros of \( f_v^{n+1} \),

\[
\mu_{n+1,4k} < \mu_{n+1,4k+2} < \mu_{n+1,4k+4}
\]

are the consequent zeros of \( f_v^n \), and

\[
\mu_{n+1,4k} < \mu_{n+1,4k+4}
\]

are the consequent zeros of \( f_v^{n-1} \).

Since \( \mu_{n+1,4k+2} \) is zero of \( f_v^n \), but is not zero of \( f_v^{n-1} \), then \( f_v^{n-1}(\mu_{n+1,4k+2}) = 1 \), whence \( M(\mu_{n+1,4k}, 0), Q(\mu_{n+1,4k+2}, 1) \) and \( N(\mu_{n+1,4k+4}, 0) \) are the consequent kinks of the graph of \( f^{n-1} \) (see Fig. 3a.). Thus, the explicit formulas of \( f_v^{n-1}(x) \) for \( x \in [\mu_{n+1,4k}, \mu_{n+1,4k+4}] \) are

\[
f_v^{n-1}(x) = \begin{cases} 
\frac{x - \mu_{n+1,4k}}{\mu_{n+1,4k+2} - \mu_{n+1,4k}} & \text{if } x \in [\mu_{n+1,4k}, \mu_{n+1,4k+2}), \\
1 - \frac{x - \mu_{n+1,4k+2}}{\mu_{n+1,4k+4} - \mu_{n+1,4k+2}} & \text{if } x \in (\mu_{n+1,4k+2}, \mu_{n+1,4k+4}].
\end{cases}
\] (7.1)

Figure 3: Parts of the graph of iterations of \( f_v \)
Numbers \( \mu_{n+1,4k+1} \) and \( \mu_{n+1,4k+3} \) are solutions of equations \( f_v^{n+1}(\mu) = 0 \), \( f_v^n(\mu) = 1 \) and \( f_v^{n-1}(\mu) = v \). The graph of \( f_v^n \) is given at Fig. 3b., where \( S(\mu_{n+1,4k+1}, 1), K(\mu_{n+1,4k+2}, 0) \) and \( T(\mu_{n+1,4k+3}, 1) \).

Thus, by (7.1), \( \mu_{n+1,4k+1} \) and \( \mu_{n+1,4k+3} \) can be found from equations

\[
\frac{\mu_{n+1,4k+1} - \mu_{n+1,4k}}{\mu_{n+1,4k+2} - \mu_{n+1,4k}} = v
\]

and

\[
1 - \frac{\mu_{n+1,4k+3} - \mu_{n+1,4k+2}}{\mu_{n+1,4k+4} - \mu_{n+1,4k+2}} = v,
\]

whence lemma follows from Remark 3.3.

The next result follows from Remark 7.1 (which is the same as Lemma 7.2).

**Remark 7.3.**

\[
\hat{x}_{n+1} - \hat{x}_n = (\hat{x}_n - \hat{x}_n) \cdot \alpha(x_n, x_{n+1}).
\]

The next result follows from remarks 5.9 and 7.3.

**Remark 7.4.** For every \( x \notin g_1^{-\infty}(0) \) we have that

\[
h_n'(x) = \prod_{k=2}^{n} \left( 2 \alpha_v(x_k, x_{k-1}) \right),
\]

precisely

\[
L(x) = R(x) = \prod_{k=2}^{\infty} \left( 2 \alpha_v(x_k, x_{k-1}) \right),
\]

Where \( L(x), R(x) \) and \( \alpha_v \) are defined by (1.6), (1.7) and (1.8).

Now Theorem 4 follows from Theorem 3 and Remark 7.4.

### 7.2 Length of the graph of the conjugacy

Now we will find the explicit formula for the length of the graph of \( h_{n+1} \), which approximates the conjugation \( h \) of the tent map \( f \) and skew tent map \( f_v \). The graph of \( h_{n+1} \) divides \([0, 1]\) into \( 2^n \) equal parts, where it is linear. At each of these parts the derivative of \( h_{n+1} \) equals \( \prod_{i=2}^{n+1} \alpha_i \). Each of these \( n \) multipliers can be either \( 2v \), of \( 2(1 - v) \).

Notice, that all the values \( \alpha_i \in \{2v, 2-2v\} \) are possible in the sequence \( \alpha_2, \ldots, \alpha_{n+1} \) in (7.2) for \( h_{n+1}'(x) \). Indeed, the map \( h_{n+1} \) has \( 2^n \) intervals of linearity and there are exactly \( 2^n \) choices for the independent values of \( \alpha_2, \ldots, \alpha_{n+1} \). Thus, for every \( k, 0 \leq k \leq n \) there are \( C_n^k \) intervals of linearity of \( h_{n+1} \), where the derivative of \( h_{n+1} \) equals \( (2v)^k(2 - 2v)^{n-k} \).
Let $t$ be a tangent of the graph of $h_{n+1}$ on some of its interval of linearity and $\alpha$ be the derivative of $h_{n+1}$ on its interval. Clearly, $\tan \alpha = t$. Then $\cos \alpha = \sqrt{\frac{1}{1+t^2}}$ and the length of the graph on the interval is $\frac{1}{2^n \cos \alpha} = \frac{1}{2^n} \sqrt{1+t^2}$.

We can now express the length of the graph of $h_{n+1}$ on the entire $[0, 1]$ as

$$l_{n+1}(v) = \frac{1}{2^n} \cdot \sum_{k=0}^{n} C_n^k \cdot \sqrt{1 + \frac{2^n v^{2k}(1-v)^{2(n-k)}}{}}.$$  \hspace{1cm} (7.3)

The following combinatorial fact follows from Theorem 5.

**Lemma 7.5.** For every $v \in (0, 1) \setminus \{0.5\}$ the limit $\lim_{n \to \infty} l_n(v) = 2$ holds, where $l_n(v)$ are defined by (7.3).

Notice, that expression (7.3) has sense also for $v \in \{0; 0.5; 1\}$. Obviously, $l_n(0) = l_n(1) = 1$ and $l_n(0.5) = \sqrt{2}$. Moreover, the case $l_n(0.5)$ corresponds to the trivial conjugation $y = x$ of the mapping $f$ with itself. We have presented in [10, Ch. 13.1] the numerical calculations of $l_n(v)$, given by (7.3), for different $v \in (0, 1)$. We have calculated $l_n(v)$ up to so huge values of $n$, that $l_n(v) > 1.97$.

Georgiy Shevchenko, professor of Taras Shevchenko National University of Kyiv (Ukraine), noticed us that Lemma 7.5 can be simply proven with the use of reasonings, which are simple for the specialists in probability theory.

**Proof of Lemma 7.5 by G. Shevchenko.** Clearly,

$$l_n = \int_0^1 \sqrt{1 + (h_n'(x))^2} dx \leq \int_0^1 \left(1 + h_n'(x)\right) dx = 2,$$

so it suffices to prove that $\lim \inf_{n \to \infty} l_n \geq 2$.

Let us consider the probability space $(\Omega, \mathcal{F}, P) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$, where $\lambda$ is the Lebesgue measure. For $\omega \in [0, 1]$, define by $X_k(\omega)$, $k \geq 1$, the $k$th digit in the binary representation of $\omega$, so that

$$\omega = \sum_{k=1}^{\infty} 2^{-k} X_k(\omega).$$

Considered as random variables on $(\Omega, \mathcal{F}, P)$, these digits are independent and have the Bernoulli distribution $B(1, \frac{1}{2})$, i.e. $P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$.

By Remark 7.4, $h'_{n+1}(\omega) = \prod_{k=2}^{n+1} \alpha_k(\omega)$ for almost all $\omega \in \Omega$, where $\alpha_k(\omega) = 2v$ if $X_k(\omega) = X_{k-1}(\omega)$ and $\alpha_k(\omega) = 2(1-v)$ otherwise; $X_0(\omega) = 0$. Then we can write

$$h'_n(\omega) = 2^n v^{S_n(\omega)} (1-v)^{n-S_n(\omega)},$$

where $S_n(\omega) = \sum_{k=1}^{n} 1_{X_k(\omega) = X_{k-1}(\omega)}$. Observe that the random variables $Y_k = 1_{X_k(\omega) = X_{k-1}(\omega)}$ are independent and have the Bernoulli distribution $B(1, \frac{1}{2})$. 
Then
\[
    l_n = \mathbb{E} \left[ \sqrt{1 + (h'_n(\omega))^2} \right] = \mathbb{E} \left[ \sqrt{1 + \prod_{k=2}^{n} \alpha_k(\omega)^2} \right]
\]
\[
    = \mathbb{E} \left[ \sqrt{1 + 2^{2n}n(1 - v)^{2n-2S_n(\omega)}} \right],
\]
where \( \mathbb{E} [\cdot] \) denotes the expectation on \((\Omega, \mathcal{F}, \mathbb{P})\).

Now assume without loss of generality that \( v > 1/2 \) and take any number \( b \in (1/2, v) \). Estimate
\[
l_n \geq \mathbb{P}(S_n \leq bn) + 2^n \mathbb{E} \left[ v^{S_n(\omega)}(1 - v)^{n-S_n(\omega)} \mathbb{1}_{S_n > bn} \right]. \tag{7.4}
\]
Since the random variables \( Y_k, k \geq 1 \), are independent and identically distributed, by the firm law of large numbers, \( S_n/n \to \mathbb{E}[Y_1] = 1/2, n \to \infty \), almost surely. In particular, \( \mathbb{P}(S_n \leq bn) \to 1, n \to \infty \).

Further, define a measure \( \mathbb{P} \ll \mathbb{P} \) on \((\Omega, \mathcal{F})\) by
\[
    \frac{d\mathbb{P}_n}{d\mathbb{P}}(\omega) = h'_n(\omega) = 2^n \prod_{k=1}^{n} v^{Y_k}(1 - v)^{1-Y_k}.
\]
Since
\[
    \mathbb{E} \left[ \frac{d\mathbb{P}_n}{d\mathbb{P}}(\omega) \right] = \mathbb{E} [h'_n(\omega)] = \int_0^1 h'_n(x)dx = 1,
\]
\( \mathbb{P}_n \) is a probability measure. Moreover, it is easy to see that under \( \mathbb{P}_n \) the random variables \( Y_k, k = 1, \ldots, n \) are independent and identically distributed and have the Bernoulli distribution \( B(1, v) \). Now
\[
    2^n \mathbb{E} \left[ v^{S_n(\omega)}(1 - v)^{n-S_n(\omega)} \mathbb{1}_{S_n > bn} \right] = 2^n \mathbb{E}_n \left[ \mathbb{1}_{S_n > bn} \right] = \mathbb{P}_n(S_n > bn).
\]
Appealing to the firm law of large numbers once more, \( S_n/n \to v, n \to \infty \). In particular, since \( b < v \), \( \mathbb{P}_n(S_n > bn) \to 1, n \to \infty \). Consequently, in view of (7.4), we get \( \lim \inf_{n \to \infty} l_n \geq 2. \)

8 Hypothesis

Theorems 4 and 5 can be also probed in the case, when we change there a firm carcass map by a carcass map \( g \), all whose kinks are \( g \)-rational, i.e. the \( g \)-expansion of every kink of \( g \) is either \( g \)-finite, or is a periodical sequence of numbers from the set \( \{0; 1\} \).

We think that Theorems 4 and 5 are not true for carcass maps in general, i.e. some additional assumptions about the kinks of maps are necessary.
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