Geometric Solutions to Algebraic Systems

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Abstract

A method to the explicit solutions of general systems of algebraic equations is presented via the metric form of affiliated Kähler manifolds. The solutions to these systems arise from sets of geodesic second order non-linear differential equations. Algebraic equations in various fields such as integers and rational numbers, as well as transcendental equations, are amenable. The case of Fermat’s set of equations is a subset.
The solution to systems of algebraic equations is important to many branches in mathematics. Furthermore, their solutions over specified fields such as integers and rationals are relevant to algebraic number theory. A primary point in this work is to bring together in a very precise sense the solutions in areas of algebra and differential geometry. The role of modular invariance and automorphic forms is explicit in the geometric solutions presented to the algebraic equations via the metric form of the Calabi-Yau manifolds.

A method is presented based on the geometry of Calabi-Yau manifolds, their toric limits and non-compact relatives, to obtain the solutions of polynomial and transcendental equations, in general. The means is obtained via the metric form on these manifolds and their geodesic flows, or line integrals, in these spaces. Given a set of equations, a complex manifold is constructed (e.g. toric variety, see for example [1]-[3]). An initial point on the manifold is chosen which represents a solution to the algebraic system; subsequently, solving a set of second order non-linear partial differential equations generates the general solution set to the algebraic system. The primary complication, not explicitly given here, is in solving these differential equations. A uniform theory towards the solution of algebraic systems is developed via the uniformization of these toric (and potentially more complicated) examples.

The sets of algebraic equations of primary interest are those of the form,

\[ P_c(z_i) = \sum_{j=1}^{m} a_{\sigma(j,i)} \prod_{l=1}^{n_{\sigma(j,i)}} z_{\tilde{\sigma}(j,l)}^{\sigma(j,l)} \]  

(1)

with \(c\) labeling the equation, and \(\sigma_c(j,l)\) labeling the \(l\)th term of the \(c\)th equation. The numbers \(a_{\sigma(j,i)}\) are coefficients multiplying the solved for variables \(z_j\). The permutations of the terms are labeled by \(\sigma\). An example set is,

\[ z_1^n + 2z_2z_4^n + 3z_3^n = 0 \]  

(2)

\[ 3z_2^m + 2z_3z_2^n + z_4^m = 0 \]  

(3)

The equations in (2) are general and contain, for example, the well known case pertaining to Fermat’s last theorem,

\[ P(z_i) = x^n + y^n - z^n \]  

(4)
Equations with \( m = \infty \), i.e. a power series, are also amenable by the following approach. These examples correspond to solutions to transcendental equations.

The interest in this work is to find the solution set to the Diophantine equations over a general field. The solutions may be restricted to integers and rational numbers, such as the case for the set in integers, and other examples.

Basically the methodology of the solution is given by the following steps: (1) modeling a Kähler space via the polynomials \( P_c(z_i) \) in terms of the holomorphic coordinates \( z_i \) (and the anti-holomorphic ones \( \bar{z}_i \) in \( P_c(\bar{z}_i) \)), (2) finding a simple sample solution of (1) using a restricted set of coordinates, and then (3) using a flow to other points in the manifold via a geodesic equation. Given the connectedness of the manifold all points are connected by a line from the sample point to the solution of interest.

Pertaining to Fermat’s set of equations, consider the point \( x^n = 1 \) for \( z = 1 \) in (1), which has solutions \( x = \exp(2\pi m/n) \) for \( m = 0 \) to \( m = n - 1 \). The standard metric to the six dimensional manifold modeled by the coordinates \( (x, y, z) \) and \( (\bar{x}, \bar{y}, \bar{z}) \) is constructed. Then flows along geodesics are used to find the general solution. The obstacle is solving the non-linear equations (although only second order) once the metric to the spaces are known. However, the solution is far more general than finding whether integer solutions exist.

The spaces used in the work are primarily (toric) Calabi-Yau. These manifolds have varieties defined by the equation set in (1), and may potentially be singular. The latter property is not irksome, as information is obtained from the singularities describing the existence of solutions, away from the locus of points in (1). Label the space pertinent to (1) as \( M_{P_c(z_i)} \) and its standard Riemannian metric as \( g_{\mu\nu} \). Its Kähler so that both \( g_{\mu\nu} = \partial_\mu \partial_\nu \ln \phi(z_i, \bar{z}_i) \) (in terms of \( z \) and \( \bar{z} \), \( g = g_{ij} \)) and its Christoffel connection is derived as \( \Gamma_{\rho,\mu\nu} = 1/2 \partial_\rho \partial_\mu \partial_\nu \ln \phi(z_i, \bar{z}_i) \) hold.

These polynomial sets of finite degree are modeled by a finite dimensional Calabi-Yau space, the equation sets of infinite degree (i.e. transcendental) are described by an infinite dimensional manifold. The infinite dimensional manifold (or a manifold of large dimension) actually comprises general solutions to the algebraic systems of lower dimension by tuning their coefficients \( a_i \) in (1). The Kähler geometrization of the algebra uniformizes to a large extent the solution of the equations.

After finding the original point A to the solution of \( P_c(z_i) = 0 \), a geodesic flow equation is given from point A to point B. The geodesic equation is, with the coordinates \( x = (z_i, \bar{z}_i) \),
\[
\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{5}
\]

with,

\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2} \partial^\rho \partial^\mu \partial^\nu \ln \phi(x_\mu, \bar{x}_\nu) \tag{6}
\]

The coordinates \( x \) contain both the holomorphic and anti-holomorphic components describing the geometry. Its complex form is

\[
\frac{d}{d\tau} \left\{ \frac{dz^i}{d\tau} + \frac{dz^j}{d\tau} \partial^i \partial^j \ln(\phi) \right\} - \frac{d^2 z^j}{d\tau^2} \partial^j \partial^i \ln(\phi) = 0 , \tag{7}
\]

or

\[
\frac{d}{d\tau} \left\{ \frac{dz^i}{d\tau} + \frac{dz^j}{d\tau} g^j, i \right\} - \frac{d^2 z^j}{d\tau^2} g^j, i = 0 . \tag{8}
\]

These equations are second order in derivatives, but non-linear because of the Kahler potential.

Given the initial coordinate \( A \), the possible solution at point \( B \) in (1) follows from the existence of the geodesic in (5) or (7). For example, the final point may belong to the integers, rationals, primes, reals, or any other set of interest. If the solutions to the geodesic equations are non-real, or singular, than the points are not allowed in the space \( M_{P_a(z_i)} \) spanning the original set of equations in (1). In the well known Fermat’s equations for \( n \geq 3 \) the non-integral solutions could not be reached by a geodesic and attempting to find one would result in a non-existent solution to the non-linear pdes; for example an imaginary component to the real coordinates \( x_\mu \) might develop, or there might be a singularity, signaling the failure of the integer solutions to exist on the manifold.

A complication is in solving the coupled set of geodesic differential equations. Solving these differential equations with the initial and final conditions, \( x^i \) and \( x^f \), is equivalent to solving for the algebraic solutions to \( P_c(z_i) = 0 \).

In the toric Calabi-Yau context these differential equations are uniformized in the sense that given the Kahler potential to the infinite dimensional Calabi-Yau variety and a systematic solution to the differential equations most algebraic systems could
Figure 1: An illustration of the path between two points A and B.
be solved for after finding the geodesics. This uniformization would correspond to the set of polynomials of infinite degree with coefficients that may be smoothly rotated to any value, hence spanning the basis of all of the sets of polynomials. This opens an avenue to finding relations between solution spaces of seemingly disparate sets of algebraic systems.

The upshot of the analysis is that given,

1) a metric on the variety parameterizing the set of polynomial equations
2) initial point A which is a simple solution

then the existence of a point B, a number in a specified field, may be found, or tested for, by solving a set of coupled second-order non-linear partial differential equations (the geodesic equations) subject to the boundary conditions, i.e. points A and B. In solving these differential equations, for example setting \( x^\mu_i \) and \( x^\mu \) to integers or rational numbers, one may systematically count and number the solutions.

We conclude with some discussion pertinent to the procedure of solving these algebraic systems.

The set of algebraic equations is connected to orbifolded torii on the geometric side via smooth interpolations of the potentially singular (toric) Calabi-Yau manifolds. These spaces have large volume limits to torii quotients. The sets of algebraic equations are thus connected to the former spaces in various dimensions via these deformations (including from one dimension to another by setting a coordinate to zero).

The topology of the Calabi-Yau manifolds, and its cohomology and homology, and curve type and numbering, should have an interesting interpretation in terms of the algebraic equations and their solutions in relevant (possibly non-trivial) fields. Indeed, the existence and counting of specific solutions to the algebraic equations, for example in the integer numbers, might be related to these topological properties in a general sense. There could be topological restrictions that specify classes of algebraic systems that do not have integer solutions.

In the geometric solutions produced here, totally geodesic submanifolds or closed geodesics parameterize sets of 'special' sets of numbers (sub-fields) along these curves. In a related point, the existence of these closed geodesics is often related to the countings in cohomological forms.

The algebraic systems are usually associated with Riemann surfaces of varying genus, together with countings of their solutions in specified fields. Modular invariance appears both in these associated Riemann surfaces and affiliated sums, for
example in the generalized L-series. A way of showing further automorphic properties
is by manifesting the metric properties of the algebraic Kähler variety in the geodesic
equation and its solution.

The metrics and the geodesic equation solutions are not evaluated in this paper.
A differential form, via an operation $O_{x_i} \cdot \ln \phi(z_i) = 0$, to find the Kähler potential,
based on the polytopic definition would be useful; in addition a classical solution to
the D-terms of the $\mathcal{N} = 2$ model with target the toric varieties would produce the
utilized metrics. The geodesic equations may be analyzed via a power series analysis
in the proper time. The number theoretic connection with the geometry is explicit
and computable.
References

[1] T. Hubsch, *Calabi-Yau manifolds, a bestiary for physicists*, World Scientific Publishing Co, Pte. Ltd. (1992).

[2] W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, Annals of Mathematics v.131, (1993).

[3] *Encyclopedic Dictionary of Mathematics*, Iwanamic Shoten Publishes, Tokyo, (1985), 3rd Ed., English Transl. MIT Press (1993).