Comment on “Nonlinear band structure in Bose-Einstein condensates: Nonlinear Schödinger equation with a Kronig Penney potential”

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In their recent paper [Phys. Rev. A 71, 033622], B. T. Seaman et al. studied Bloch states of the condensate wave function in a Kronig-Penney potential and calculated the band structure. They argued that the effective mass is always positive when a swallow-tail energy loop is present in the band structure. In this comment, we reexamine their argument by actually calculating the effective mass. It is found that there exists a region where the effective mass is negative even when a swallow-tail is present. Based on this fact, we discuss the interpretation of swallow-tails in terms of superfluidity.

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The energy of the condensate per site is given by

\[ E = \frac{1}{2m} \frac{d^2}{dx^2} + V(x) + \frac{g}{2} |\Psi_0|^2 \]

The condensate wave function can be written as

\[ \Psi_0(x) = \sqrt{n_0} A(x) e^{iS(x)} \],

where \( A(x) \) and \( S(x) \) mean the amplitude and phase of the condensate. \( A(x) \) is normalized by the density at the center of each site \( n_0 \equiv |\Psi_0((j+1/2)a)|^2 \). Thus, Eq. (1) is reduced to

\[ -\frac{1}{2m} \frac{d^2}{dx^2} A(x) + \frac{Q^2}{2m} A^{-3} + V(x) A(x) - \mu A + gn_0 A^3 = 0 \]

\[ A^2 \frac{dS}{dx} = Q. \] (6)

Equation (6) is the equation of continuity and \( Q \) describes the superfluid momentum.

Assuming that the condensate sits in the first Bloch band, one obtains the solution of Eq. (5) in the region \((j-1/2)a < x < (j+1/2)a\),

\[ A(x)^2 = (1 - \beta_-) \times \text{sn}^2 \left( \frac{\sqrt{\beta_- - \beta_+} (x - ja) + x_0}{\xi_0}, \sqrt{\frac{1 - \beta_-}{\beta_+ - \beta_-}} + \beta_+ \right) \]

where

\[ \beta_\pm = \frac{\mu}{gn_0} + \frac{1}{2} \pm \sqrt{\left( \frac{2\mu}{gn_0 - 1} \right)^2 - 4(Q\xi_0)^2} \]. (8)

FIG. 1: First Bloch band of the energy of the condensate \( E(K) \) for \( V_0 = 1gn_{av}\xi_{av} \) and \( a = 5\xi_{av} \), where \( n_{av} = N_C/a \) and \( \xi_{av} = (mgn_{av})^{-1/2} \).
The phase $S$ where $S_0 = B. T. Seaman, L. D. Carr, and M. J. Holland, Phys. Rev. A 71, 033622 (2005).

Substituting Eqs. (7) and (11) into Eq. (4), one obtains the first Bloch band of the energy of the condensate $E(K)$ as shown in Fig. 1. A swallow-tail energy loop is present at the edge of the first Brillouin zone.

The group velocity $v_g(K)$ and the effective mass $m^*(K)$ are given by

$$v_g(K) = \frac{\partial E}{\partial K},$$

$$m^*(K) = \left( \frac{\partial^2 E}{\partial K^2} \right)^{-1}.$$

$v_g(K)$ and $m^*(K)$ are shown in Figs. 2 and 3, respectively. Reflecting the presence of the swallow-tail, $v_g(K)$ has a reentrant structure. In the lower portion of the swallow-tail, $v_g(K)$ increases as $K$ increases until $K_M$, which gives the maximum value of the group velocity. As $K$ increases further from $K_M$ to the swallow-tail edge $K_E$, $v_g(K)$ decreases. In the upper portion of the swallow-tail, $v_g(K)$ increases as $K$ increases from $\frac{\pi}{a}$ to $K_E$. It is obvious in Fig. 3 that the effective mass is negative in the region of the lower portion between $K_M$ and $K_E$. This fact suggests the dynamical instability of the condensate in the region, which has been shown in Ref. 3 by the linear stability analysis when the lattice constant is much larger than the healing length. The region where the effective mass is negative is present also in the case of a sinusoidal periodic potential, because $v_g(K)$ exhibits the same reentrant behavior. It is worth mentioning that the region of the negative effective mass exists for any values of $V_0$ and $a$ except in the limit of $V_0 \to 0$.

In Refs. 1, 3, the presence of swallow-tails in the band structure was interpreted as a manifestation of superfluidity of condensates. This interpretation is inconsistent with the negative effective mass in the lower portion of the swallow-tail for the following reason. It is assumed in the interpretation that the Bloch states in the lower portion correspond to local minima in the energy landscape. However, they do not actually correspond to local minima, but to saddle points when the effective mass is negative. Moreover, it was insisted in Refs. 1, 3 that as a superfluid, the condensate can screen out the periodic potential and the system follows the energy spectrum in the absence of the periodic potential until the superfluid momentum reaches the swallow-tail edge. This argument is not valid, since the superfluidity breaks down due to the dynamical instability when the effective mass is negative.

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