A CLASS OF REDUCTIONS OF THE TWO-COMPONENT KP HIERARCHY AND THE HIROTA–OHTA SYSTEM

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We introduce a class of reductions of the two-component KP hierarchy that includes the Hirota–Ohta system hierarchy. The description of the reduced hierarchies is based on the Hirota bilinear identity and an extra bilinear relation characterizing the reduction. We derive the reduction conditions in terms of the Lax operator and higher linear operators of the hierarchy, as well as in terms of the basic two-component KP system of equations.

Keywords: two-component Kadomtsev–Petviashvili hierarchy, reduction, Hirota–Ohta system

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1. Introduction

In this paper, we introduce a class of reductions of the two-component KP hierarchy, which includes the Hirota–Ohta system hierarchy [1], [2] as the zero-order reduction. In the scalar case, a related class of reductions was introduced in [3], where it was demonstrated that the lowest-order reductions give rise to the CKP and BKP hierarchies. In the two-component case, a similar approach was developed in [4]. Our starting point is the \( \partial \)-dressing scheme, for which the definition of a class of reductions is rather transparent [5] and which can be used to construct a numerous explicit solutions. However, the algebraic description of the reduced hierarchies is based on the Hirota bilinear identity and an extra bilinear relation characterizing the reduction, and it does not necessarily require the dressing scheme. We derive the reduction conditions in terms of the Lax operator and higher linear operators of the hierarchy. The basic system of the two-component KP hierarchy with an additional symmetry constraint for the dynamics defining a special set of times is a closed system of equations with three independent variables \( x, y, t \) for six scalar functions. Each reduction of the class gives a set of three differential relations containing derivatives with respect to \( x \) and \( y \) for these functions, and a pair of different reductions produces a closed system of \((1+1)\)-dimensional equations.

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2. Nonlocal $\bar{\partial}$ problem and Hirota bilinear identity

We first recall a general setting to consider the multicomponent KP hierarchy in the framework of the $\bar{\partial}$-dressing method [6]. We start with a pair of adjoint canonically normalized matrix $\bar{\partial}$-problems

$$\frac{\partial}{\partial \lambda} \chi(\lambda, t) = \int_C d\nu \wedge d\bar{\nu} \chi(\nu)g(\nu, t)R(\nu, \lambda)g^{-1}(\lambda, t),$$

$$\frac{\partial}{\partial \lambda} \bar{\chi}(\lambda, t) = -\int_C d\nu \wedge d\bar{\nu} g(\lambda, t)R(\lambda, \nu)g^{-1}(\nu; t)\bar{\chi}(\nu, t).$$

We choose the following parameterization of the multicomponent loop group $\Gamma^{+N}$ defining the dynamics of the multicomponent KP hierarchy:

$$g(\lambda, t) = \exp\left(\sum_{n=1}^N \sum_{\alpha=1}^\infty P_\alpha \lambda^n t^{(\alpha)}\right).$$

Here, the projection matrices $P_\alpha$ form a basis of the commutative subalgebra of diagonal matrices,

$$(P_\alpha)_{\beta \gamma} = \delta_{\alpha \beta} \delta_{\beta \gamma}, \quad \alpha, \beta, \gamma = 1, \ldots, N,$$

and the hence have $N$ infinite series of dynamical variables $t^{(\alpha)}_n$.

The kernel $R(\lambda, \nu)$ is supposed to be equal to zero in some neighborhood of infinity for both variables $\lambda$ and $\mu$; we assume for simplicity that the support of the kernel belongs to the product of unit disks. The functions $\chi(\lambda, t)$, $\bar{\chi}(\lambda, t)$ are then analytic outside the unit disk, at infinity

$$\chi(\lambda, t) = I + \sum_{n=1}^\infty \chi_n(t)\lambda^{-n}, \quad \bar{\chi}(\lambda, t) = I + \sum_{n=1}^\infty \bar{\chi}_n(t)\lambda^{-n}.$$ Problems (1) imply the Hirota bilinear identity on the unit circle $S$:

$$\oint_S \chi(\nu; t)g(\nu, t)g^{-1}(\nu; t')\bar{\chi}(\nu; t')d\nu = 0. \quad (3)$$

In a more familiar form, for the Baker–Akheizer functions

$$\psi(\lambda; g) = \chi(\lambda)g(\lambda), \quad \bar{\psi}(\lambda; g) = g^{-1}(\lambda)\bar{\chi}(\lambda),$$

we have

$$\oint_S \psi(\nu; t)\bar{\psi}(\nu; t')d\nu = 0. \quad (4)$$

We also use the Cauchy–Baker–Akheizer (CBA) function (kernel) defined by the nonlocal $\bar{\partial}$-problems (1) with the pole normalization $(\lambda - \mu)^{-1}$ [6]:

$$\frac{\partial}{\partial \lambda} \chi(\lambda, \mu; t) = 2\pi i \delta(\lambda - \mu) + \int_C d\nu \wedge d\bar{\nu} \chi(\nu, \mu; t)g(\nu, t)R(\nu, \lambda)g^{-1}(\lambda; t),$$

$$\frac{\partial}{\partial \lambda} \bar{\chi}(\lambda, \mu; t) = 2\pi i \delta(\lambda - \mu) - \int_C d\nu \wedge d\bar{\nu} g(\lambda; t)R(\lambda, \nu)g^{-1}(\nu; t)\bar{\chi}(\nu, \mu; t).$$

After simple calculations, we obtain

$$\oint_S \chi(\nu, \lambda; t)g(\nu; t)g^{-1}(\nu; t')\bar{\chi}(\nu, \mu; t')d\nu = 0. \quad (6)$$

474
It follows from (6) with \( t = t' \) that outside the unit disk with respect to both variables, the function \( \chi(\lambda, \mu) \) is equal to \(-\tilde{\chi}(\mu, \lambda)\), and hence this identity should actually be written for a single function,
\[
\oint \chi(\nu, \lambda; t)g(\nu; t)g^{-1}(\nu; t')\chi(\mu, \nu; t') d\nu = 0.
\] (7)

By similar calculations, it follows that if both problems (5) are solvable, then \( \chi(\lambda, \mu) = -\tilde{\chi}(\lambda, \mu) \) for all \( \lambda, \mu \) where they are jointly defined.

Taking \( \lambda \to \infty \) and \( \mu \to \infty \), we reproduce identity (3) for \( \chi(\lambda; t) = \chi(\lambda, \infty; t) \), \( \tilde{\chi}(\lambda; t) = -\chi(\infty, \lambda; t) \).

In terms of the CBA function
\[
\Psi(\lambda, \mu; t) = g^{-1}(\mu, t)\chi(\lambda, \mu; t)g(\lambda, t),
\]
the Hirota bilinear identity becomes
\[
\oint \Psi(\nu, \lambda; t)\Psi(\mu, \nu; t') d\nu = 0.
\] (8)

3. A class of reductions of the two-component KP hierarchy

For the two-component KP hierarchy
\[
g(\lambda, t) = \exp\left(\sum_{n=1}^{\infty} \left( P_1 \lambda^n t_n^{(1)} + P_2 \lambda^n t_n^{(2)} \right) \right),
\] (9)
we consider a class of reductions
\[
R_T^T(-\lambda, -\mu)A^k = A^k J R(\mu, \lambda) J^{-1},
\] (10)
where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and the matrix \( A \) is equal to \( I \) or \( \sigma_3 \) and hence commutes or anticommutes with \( J \).

This class of reductions requires an involution
\[
g(-\lambda, t) = Jg(\lambda, t)^{-1} J^{-1},
\] (11)
and hence the reduction condition is compatible with the dynamics only if \( t_{2n+1}^{(1)} = t_{2n+1}^{(2)} \) (for odd times) or \( t_{2n}^{(1)} = -t_{2n}^{(2)} \) (for even times). We introduce a new set of times \( t_n \): \( t_{2k} = t_{2k}^{(1)} = -t_{2k}^{(2)} \) for even orders and \( t_{2k-1} = t_{2k-1}^{(1)} = t_{2k-1}^{(2)} \) for odd orders. The factor defining the dynamics of the kernel has the form
\[
g(\lambda, t) = \exp\left(\sum_{n=1}^{\infty} (I \lambda^{2n-1} t_{2n-1} + \sigma_3 \lambda^{2n} t_{2n}) \right).
\] (12)

For the first three times, we use the notation \( x = t_1, y = t_2, t = t_3 \).

In terms of the Baker–Akhiezer function, reduction (10) is characterized by an extra bilinear relation
\[
\oint \psi(\nu; t)JA^k \psi^T(-\nu; t') d\nu = 0,
\] (13)
and for the CBA function we have
\[
\oint \Psi(\nu, \lambda; t)JA^k \Psi^T(-\nu, -\mu; t') d\nu = 0.
\] (14)
The reduction condition corresponding to the Hirota–Ohta hierarchy is given by (10) with \( A = I, n = 0, \)
and
\[
R^T(-\lambda, -\mu) = JR(\mu, \lambda)J^{-1},
\]
and for the Baker–Akhiezer functions we have the condition
\[
\hat{\psi}(\lambda; t) = -J\psi^T(-\lambda; t)J,
\]
and identity (13) becomes
\[
\oint \psi(\nu; t)J\psi^T(-\nu; t')d\nu = 0.
\]
In terms of the CBA function, the reduction condition is
\[
\psi^T(-\lambda, -\mu) = J\psi(\mu, \lambda)J^{-1}.
\]

4. Two-component KP hierarchy

With the class of reductions (10) in mind, we first derive linear problems and equations for the two-component KP hierarchy with involution (11), and then we discuss the reduction conditions in terms of linear operators and equations.

We start with Hirota bilinear identity (6), (7) with the dependence on times defined in (12).

4.1. Linear operators. The action of operators \( \partial_{t_n} = \partial/\partial t_n \) on \( \psi \) and \( \psi^* = \hat{\psi}^T \) corresponds to the following (Manakov) operators acting on \( \chi \) and \( \chi^* \):

\[
D_{t_{2n-1}}\chi = \partial_{t_{2n-1}}\chi + \lambda^{2n-1}\chi, \quad D^*_{t_{2n-1}}\chi = \partial_{t_{2n-1}}\chi^* - \lambda^{2n-1}\chi^*,
\]
\[
D_{t_2}\chi = \partial_{t_2}\chi + \lambda^{2}\chi\sigma_3, \quad D^*_{t_2}\chi = \partial_{t_2}\chi^* - \lambda^{2}\chi^*\sigma_3.
\]

It follows from the Hirota bilinear identity that a differential operator \( \sum_{n,m} u_m^{(n)} D_n^m \) acting on \( \psi \) gives zero if and only if the result of acting on \( \chi \) with the respective Manakov operator has a zero projection to nonnegative powers of \( \lambda \):

\[
\left( \sum_{n,m} u_m^{(n)} D_n^m \chi \right) = 0 \quad \Leftrightarrow \quad \sum_{n,m} u_m^{(n)} D_n^m \chi = 0 \quad \Leftrightarrow \quad \sum_{n,m} u_m^{(n)} D_n^m \psi = 0. \tag{17}
\]

Using this observation, it is possible to construct linear operators of the hierarchy.

We start with the Lax operator. For the first three times \( x = t_1, y = t_2, \) and \( t = t_3 \), the Manakov operators are

\[
D_x\chi = \partial_x\chi + \lambda\chi, \quad D_y\chi = \partial_y\chi + \lambda^2\chi\sigma_3, \quad D_t\chi = \partial_t\chi + \lambda^3\chi,
\]
\[
D^*_x\chi^* = \partial_x\chi^* - \lambda\chi^*, \quad D^*_y\chi^* = \partial_y\chi^* - \lambda^2\chi^*\sigma_3, \quad D^*_t\chi^* = \partial_t\chi^* - \lambda^3\chi^*.
\]

Using (17), we obtain

\[
(D_y - \sigma_3 D_x^2)\chi = ([\chi_1, \sigma_3] D_x + ([\chi_2, \sigma_3] - 2\sigma_3 \chi_1 x - [\chi_1, \sigma_3] \chi_1))\chi,
\]
\[
(D_y + \sigma_3 D_x^2)\chi^* = ([\chi_1^*, \sigma_3] D_x^* - ([\chi_2^*, \sigma_3] + 2\sigma_3 \chi_1^* x - [\chi_1^*, \sigma_3] \chi_1^*))\chi^*.
\]
For the Baker–Akhiezer functions, we then have
\[ \partial_y \psi = \left( \sigma_3 \partial_x^2 + \frac{f}{g} \partial_x + U \right) \psi, \]
\[ \partial_y \psi^* = \left( -\sigma_3 \partial_x^2 + \frac{f^*}{g^*} \partial_x - U^* \right) \psi^*, \]
where
\[ \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} = [\chi_1, \sigma_3], \quad \begin{pmatrix} 0 & f^* \\ g^* & 0 \end{pmatrix} = -[\chi_1^*, \sigma_3], \]
\[ U = [\chi_2, \sigma_3] - 2\sigma_3 \chi_1 x - [\chi_1, \sigma_3] \chi_1, \]
\[ U^* = [\chi_2^*, \sigma_3] + 2\sigma_3 \chi_1^* x - [\chi_1^*, \sigma_3] \chi_1^*. \]

From the Hirota identity taken at equal times, we obtain
\[ \oint \chi(\nu; t) \chi^T(\nu; t) d\nu = 0, \]
whence \( \chi_1^* = -\chi_1^T \) and \( g = f^*, f = g^* \). Differentiating identity (3) with respect to \( x \) and taking it at equal times, we obtain
\[ \oint \chi^*(\nu; t)(\partial_x + \nu) \chi^T(\nu; t) d\nu = 0, \]
whence
\[ \chi_2^*(t) = -\chi_2^T - \chi_1^T x + \chi_1^T \chi_1^T. \]

Using this relation, it is easy to demonstrate that Lax operators (18) are (anti)adjoint (i.e., \( (V \partial_x)^* = -\partial_x V^T \)),
\[ \partial_y \psi = B_2 \psi, \quad \partial_y \psi^* = -B_2^* \psi^*, \]
where the operator \( B_2 \) is defined by (18).

Linear operators corresponding to the time \( t \) are
\[ \partial_t \psi = B_3 \psi, \quad \partial_t \psi^* = -B_3^* \psi^*, \]
where
\[ B_3 = \partial_x^3 + 3W \partial_x + W_1, \]
\[ W = -\chi_{1x}, \quad W_1 = 3\chi_{1x} \chi_1 - 3\chi_{2x} - 3\chi_{1xx}. \]

The higher linear operators can be written as
\[ \partial_t \psi = B_n \psi, \quad \partial_t \psi^* = -B_n^* \psi^*, \]
where
\[ B_{2m} = \sigma_3 \partial_x^{2m} + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \partial_x^{2m-1} + \sum_{k=0}^{2n-2} U_k^{(2m)} \partial_x^k, \]
\[ B_{2m+1} = \partial_x^{2m+1} + (2m + 1)W \partial_x^{2m-1} + \sum_{k=0}^{2n-2} W_k^{(2m+1)} \partial_x^k. \]
The coefficients of these operators can be expressed in terms of the coefficients of expansion of the function $\chi(\lambda; t)$. Lax operator (18) written in terms of this function gives the recursion formulas expressing the coefficients of the expansion of $\chi_n(t)$ via the coefficients of the Lax operator $f$, $g$ and $U$ (six scalar functions). Indeed,

$$\chi_y - \sigma_3 \chi_{xx} = -\lambda^2 [\chi, \sigma_3] + 2\lambda \sigma_3 \chi_x + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} (\chi_x + \lambda \chi) + U \chi,$$

and in terms of the expansion coefficients, we have

$$\chi_{ky} - \sigma_3 \chi_{kxx} = -[\chi_{k+2}, \sigma_3] + 2\sigma_3 \chi_{k+1} x + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} (\chi_{kx} + \chi_{k+1}) + U \chi_k.$$

To have a correct structure of recursion relations, we must split this equation into diagonal and antidiagonal parts, which gives

$$2\chi^a_{k+2} = -2\chi^a_{k+1} x + \begin{pmatrix} 0 & -f \\ g & 0 \end{pmatrix} (\chi^d_{k+1} + \chi^d_{kx}) - \sigma_3 (U \chi_k)^a + \sigma_3 \chi_{k}^a y - \chi_{kxx}^a,$$  \hfill (24)

$$2\chi^d_{k+1} x = \begin{pmatrix} 0 & -f \\ g & 0 \end{pmatrix} (\chi^a_{k+1} + \chi^a_{kx}) - \sigma_3 (U \chi_k)^d + \sigma_3 \chi_{k}^d y - \chi_{kxx}^d.$$  \hfill (25)

We write several terms of the recursion explicitly:

- the anti-diagonal part, $k = -1$,
  $$2\chi^a_1 = \begin{pmatrix} 0 & -f \\ g & 0 \end{pmatrix},$$

- the diagonal part, $k = 0$,
  $$2\chi^d_1 x = \frac{1}{2} f g I - \sigma_3 U^d,$$

- the anti-diagonal part, $k = 0$,
  $$2\chi^a_2 = -2\chi^a_{1x} + \begin{pmatrix} 0 & f \\ -g & 0 \end{pmatrix} \chi^d_1 - \sigma_3 U^a,$$

- the diagonal part, $k = 1$,
  $$2\chi^d_2 x = \begin{pmatrix} 0 & f \\ -g & 0 \end{pmatrix} (\chi^a_{2} + \chi^a_{1x}) - \sigma_3 (U \chi_1)^d + \sigma_3 \chi_{1}^d y - \chi_{1xx}^d.$$

4.2. Equations. The compatibility condition for linear equations (21) and (22) is given by the Zakharov–Shabat equation

$$\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial t} = [B_2, B_3].$$
which generates a closed system of equations for matrix coefficients of the operators $B_2$ and $B_3$:

$$U_t - W_{1y} - U_{xxx} + \sigma_3 W_{1xx} + \left( \begin{array}{cc} 0 & f \\ g & 0 \end{array} \right) W_{1x} + [U, W_1] - 3W U_x = 0,$$

$$\left( \begin{array}{cc} 0 & f_t \\ g_t & 0 \end{array} \right) + 3\sigma_3 W_{xx} - 3W \left( \begin{array}{cc} 0 & f_x \\ g_x & 0 \end{array} \right) + 3 \left( \begin{array}{cc} 0 & f \\ g & 0 \end{array} \right) W_x - 3[W, U] -$$

$$- \left[ W_1, \left( \begin{array}{cc} 0 & f \\ g & 0 \end{array} \right) \right] + 2\sigma_3 W_{1x} - 3W_y - 3U_{xx} = 0, \quad (26)$$

$$3U_x + 3 \left( \begin{array}{cc} 0 & f_{xx} \\ g_{xx} & 0 \end{array} \right) - 6\sigma_3 W_x + [W_1, \sigma_3] + 3 \left[ W, \left( \begin{array}{cc} 0 & f \\ g & 0 \end{array} \right) \right] = 0,$$

$$\left( \begin{array}{cc} 0 & f_x \\ g_x & 0 \end{array} \right) = [W, \sigma_3].$$

This system is a two-component KP system with the times $x, y, t$. Having recursion relations (24), (25) and the expressions for $W$ and $W_1$ in (22) in mind, we conclude that all the matrix functions in this system can be expressed in terms of $f$, $g$, and $U$, and the system should yield a closed system of equations for six scalar functions.

5. Reductions

5.1. Hirota–Ohta system hierarchy. The Hirota–Ohta system hierarchy is a reduction of the two-component KP hierarchy defined by condition (15), which is equivalent to

$$\psi^*(\lambda; t) = -J\psi(-\lambda; t)J.$$ For linear operators (23), we then have $B_n^* = JB_nJ$ (cf. [2]), and the reduced operator $B_2$ (21) has the form

$$B_2 = \sigma_3 \partial_x^2 + 2 \left( \begin{array}{cc} u & v \\ -\tilde{v} & -u \end{array} \right) \psi, \quad (27)$$

with $f = g = 0$ and $\chi_1 = -uI$. For the reduced operator $B_3$, $W = uI$. Reduced system (26) has the form

$$U_t - W_{1y} - U_{xxx} + \sigma_3 W_{1xx} + [U, W_1] - 3u U_x = 0,$$

$$3\sigma_3 u_{xx} + 2\sigma_3 W_{1x} - 3I u_y - 3U_{xx} = 0,$$

$$3U_x - 6\sigma_3 u_x + [W_1, \sigma_3] = 0,$$

where the second equation gives the expression for $W_{1x}$ in terms of $U$, and the third equation is implied by the second. From the first equation, we obtain an equation for one matrix $U$ of the form $U = u\sigma_3 + U^a$ (antidiagonal part), which, written in components, represents the Hirota–Ohta (coupled KP) system [1], [2]

$$4u_t - uu_x - 12u_{xxx} - 3u_{yy} + 12(\nu\tilde{v})_{xx} = 0,$$

$$2v_t + 6uv_x + v_{xxx} + 3uv_y + 6v\partial_x^{-1}u_y = 0,$$

$$2\tilde{v}_t + 6u\tilde{v}_x + \tilde{v}_{xxx} - 3\tilde{v}_{xy} - 6\tilde{v}\partial_x^{-1}u_y = 0. \quad (28)$$
5.2. Other reductions. We consider another zero-order reduction (10), (13) with \( A = \sigma_3 \) and \( k = 0 \).

Effectively, it amounts to replacing the matrix \( J \) with the matrix

\[
J' = \sigma_3 J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Then

\[
\psi^* (\lambda; t) = J' \psi (-\lambda; t) J',
\]

and we have \( B_n^* = -J' B_n J' \) for linear operators (23). The reduced Lax operator (18) has the form

\[
\partial_y \psi = \left( \sigma_3 \partial_x^2 + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \partial_x + uI + \frac{1}{2} \begin{pmatrix} 0 & f_x \\ g_x & 0 \end{pmatrix} \right) \psi.
\]

Thus, the Lax operator depends on the three functions \( f, g, \) and \( u \) instead of six functions in the general case of the two-component KP hierarchy, and matrix system (26) should give a closed system of equations for these three functions.

5.2.1. First-order reductions. We consider the reduction (13) with \( A = I \) and \( k = 1 \):

\[
\oint \psi (\nu; t) J \psi^T (-\nu; t') d\nu = 0.
\]

Taking this identity at equal times, we obtain

\[
\chi_2 J + J \chi_2^T - \chi_1 J \chi_1^T = 0.
\]

Recalling recursion relations (24) and (25), we obtain three scalar differential relations for six functions \( f, g, U \). These relations represent a reduction of system (26).

5.2.2. Higher reductions. Higher reductions can be considered similarly. In general, a reduction of an arbitrary order is a set of three scalar differential relations for six functions \( f, g, U \). A pair of reductions of different orders generates a closed \((1+1)\)-dimensional system for six functions, related to some stationary reductions of the hierarchy.

Another way to characterize the reduction is by the existence of an intertwining differential operator \( A_k \) of order \( k \), which defines a map from the wave functions of adjoint operators to the wave functions of the basic linear operators. A similar idea was used in [4] to construct differential reductions in the case of the two-dimensional Dirac operator. It is convenient to introduce a modified conjugation operation: for a matrix differential operator \( B \), we define \( B^\dagger = JB^* J^{-1} \). This operation has the standard properties \((B^\dagger)^\dagger = B\) and \((AB)^\dagger = B^\dagger A^\dagger \). We set \( \psi^\dagger (\lambda; t) = J \psi^* (\lambda; t) J^{-1} \). Then the reduction is characterized by the existence of a differential operator \( A_k \) of order \( k \) such that for any wave function \( \phi \),

\[
(\partial_y + B_2^\dagger) \phi = 0 \quad \Rightarrow \quad (\partial_y - B_2) A_k \phi = 0.
\]

Algebraically, this condition is equivalent to the operator equation

\[
(\partial_y - B_2) A_k = A_k (\partial_y + B_2^\dagger)
\]

(see [3] for the scalar case). Using this equation, it is possible to express the coefficients of \( A_k \) in terms of the coefficients of the Lax operator and obtain a reduction condition in terms of the coefficients of the Lax operator (or the solution of system (26)). This type of condition can also be used to define reductions in the context of the Lax–Sato equations, as we discuss in the Appendix (the scalar case is considered in [3]).
Appendix: Reductions in terms of the Lax–Sato equations

Here, we briefly describe the two-component KP hierarchy with times (12) in terms of the Lax–Sato equations and discuss the class of reductions corresponding to bilinear relation (13). In the scalar case, reductions of this type were described in [3].

The Lax–Sato equations define the dynamics of pseudodifferential operators

\[
L = \partial + U_1 \partial^{-1} + U_2 \partial^{-2} + \cdots,
\]
\[
M = \sigma_3 + V_1 \partial^{-1} + V_2 \partial^{-2} + \cdots,
\]

where \( U_n \) and \( V_n \) are \( 2 \times 2 \) matrices, \( \partial = \partial_x \), \( \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \), with the characteristic properties

\[
[L, M] = 0, \quad M^2 = 1.
\]

For odd times, we have

\[
\frac{\partial L}{\partial t_{2n+1}} = [(L^{2n+1})_+, L], \quad \frac{\partial M}{\partial t_{2n+1}} = [(L^{2n+1})_+, M],
\]

and for even times,

\[
\frac{\partial L}{\partial t_{2n}} = [(L^{2n}M)_+, L], \quad \frac{\partial M}{\partial t_{2n}} = [(L^{2n}M)_+, M].
\]

The Gelfand–Dickey reductions for this hierarchy are defined by the following conditions: for odd flows,

\[
(L^{2n+1})_- = 0, \quad (L^{2n+1})_+ = D^{(2n+1)},
\]

where \( D^{(2n+1)} \) is a differential operator of order \( 2n + 1 \) with matrix coefficients, and for even flows,

\[
(L^{2n}M)_- = 0, \quad (L^{2n}M)_+ = D^{(2n)}.
\]

Introducing a formal pseudodifferential dressing operator (related to the operator \((1 + \hat{K})[2]\))

\[
P = I + W_1 \partial^{-1} + W_2 \partial^{-2} + \cdots,
\]

allows expressing the operators \( L \) and \( M \) as

\[
L = P \partial P^{-1}, \quad M = P \sigma_3 P^{-1}.
\]

The operators \( L \) and \( M \) defined this way manifestly have the necessary characteristic properties. The dynamics of the dressing operator is governed by the Sato equations [7]

\[
\frac{\partial P}{\partial t_{2n+1}} = -(P \partial^{2n+1} P^{-1})_- P,
\]

\[
\frac{\partial P}{\partial t_{2n}} = -(P \partial^{2n} \sigma_3 P^{-1})_- P,
\]

which implies Eqs. (A.1) and (A.2). To find a dressing operator starting from \( L \) (or \( M \)), we must solve the factorization problem \( LP = P \partial \), \( MP = P \sigma_3 \). The reduction to the Hirota–Ohta system hierarchy is described by the conditions

\[
L^* = JLJ,
\]
\[
M^* = JMJ,
\]
\[
P^* = -JP^{-1}J.
\]
A class of reductions corresponding to bilinear relation (13) is defined by the following conditions: for \( A = I \),

\[
(P \partial^n JP^*)_- = 0,
\]

and for \( A = \sigma_3 \),

\[
(P \sigma_3 \partial^n P^*)__- = 0.
\]

Introducing the differential operators \( A_k = P \partial^k P^\dagger \) (for \( A = I \)) or \( A_k = P \sigma_3 \partial^k P^\dagger \) (for \( A = \sigma_3 \)), where we use the notation \( P^\dagger = JP^*J^{-1} \), we obtain the relations

\[
LA_k = A_k L^\dagger,
\]

\[
MA_k = A_k M^\dagger,
\]

and also relations of form (29).

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