A regularity theory for stochastic generalized Burgers’ equation driven by a multiplicative space-time white noise

Beom-Seok Han

Abstract
We introduce the uniqueness, existence, $L_p$-regularity, and maximal Hölder regularity of the solution to semilinear stochastic partial differential equation driven by a multiplicative space-time white noise:

$$u_t = au_{xx} + bu_x + cu + \tilde{b}|u|^\lambda u_x + \sigma(u) \dot{W}, \quad (t, x) \in (0, \infty) \times \mathbb{R}; \quad u(0, \cdot) = u_0,$$

where $\lambda > 0$. The function $\sigma(u)$ is either bounded Lipschitz or super-linear in $u$. The noise $\dot{W}$ is a space-time white noise. The coefficients $a, b, c$ depend on $(\omega, t, x)$, and $\tilde{b}$ depends on $(\omega, t)$. The coefficients $a, b, c, \tilde{b}$ are uniformly bounded, and $a$ satisfies ellipticity condition. The random initial data $u_0 = u_0(\omega, x)$ is nonnegative.

To establish the $L_p$-regularity theory, we impose an algebraic condition on $\lambda$ depending on the nonlinearity of the diffusion coefficient $\sigma(u)$. For example, if $\sigma(u)$ has Lipschitz continuity, linear growth, and boundedness in $u$, for $T < \infty$ and $\varepsilon > 0$,

$$u \in C^{1/4-\varepsilon, 1/2-\varepsilon}_{t,x}([0, T] \times \mathbb{R}) \quad (a.s.).$$

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2021R1C1C2007792) and the BK21 FOUR (Fostering Outstanding Universities for Research) funded by the Ministry of Education (MOE, Korea) and National Research Foundation of Korea (NRF).
On the other hand, if \( \lambda \in (0, 1) \) and \( \sigma(u) = |u|^{1+\lambda_0} \) with \( \lambda_0 \in [0, 1/2) \), for \( T < \infty \) and \( \varepsilon > 0 \),
\[
 u \in C_{t,x}^{1/2-(\lambda-1/2)\varepsilon,1/2-(\lambda-1/2)\varepsilon-\varepsilon}([0, T] \times \mathbb{R}) \quad (a.s.) .
\]
It should be noted that if \( \sigma(u) \) is bounded Lipschitz in \( u \), the Hölder regularity of the solution is independent of \( \lambda \). However, if \( \sigma(u) \) is super-linear in \( u \), the Hölder regularities of the solution are affected by nonlinearities, \( \lambda \) and \( \lambda_0 \).

**Keywords**  Stochastic partial differential equation · Nonlinear · Super-linear · Stochastic generalized Burger’s equation · Space-time white noise · Hölder regularity

**Mathematics Subject Classification**  60H15 · 35R60

### 1 Introduction

One of the most well-known example of semilinear stochastic partial differential equation (SPDE) is a stochastic generalized Burgers’ equation driven by space-time white noise \( \dot{W} \):

\[
 u_t = Lu + f(u) + (g(u))_x + \sigma(u)\dot{W}_t, \quad t > 0; \quad u(0, \cdot) = u_0(\cdot), \quad (1.1)
\]

where \( L \) is a second order operator, \( f(u) \) and \( g(u) \) are nonlinear functions, and \( \sigma(u) \) is a bounded function. Many studies have been conducted for Eq. (1.1); see [1–16]. For example, if \( f(u) = 0 \) and \( g(u) = u^2 \), the existence and properties of a solution to (1.1) with additive noise are introduced in [1, 2]. Also, in [3], similar results are obtained for equation (1.1) with multiplicative noise. In [7, 10], the regularity and moment estimate of solutions are achieved. Besides, the unique solvability of stochastic Burger’s equation with random coefficients is contained in [8]. Furthermore, the unique solvability of (1.1) with more general \( f \) and \( g \) is considered in [4–6]. Especially in [4], the unique solvability of (1.1) is provided under the assumption that \( f \) is linear and \( g \) has quadratic growth. For more information, see [11] and references therein.

One can notice that most of the above results are obtained under the condition that \( \sigma(u) \) is bounded. Thus, as another example of a semilinear equation, we suggest an equation having an unbounded diffusion coefficient. Consider a stochastic partial differential equation with nonlinear diffusion coefficient \( |u|^{\gamma} \):

\[
 u_t = Lu + \xi |u|^\gamma \dot{W}, \quad t > 0; \quad u(0, \cdot) = u_0(\cdot), \quad (1.2)
\]

where \( L \) is a second order differential operator, \( \gamma > 0 \), and \( \dot{W} \) is a space-time white noise; see [17–28]. Particularly, [18–21, 27, 28] describe the solvability of (1.2) with \( \gamma > 1 \). In [21], the long-time existence of a mild solution to equation (1.2) is proved if \( L = \Delta, \xi = 1, \mathcal{I} = (0, 1) \), and \( u(0) = u_0 \) is deterministic, continuous, and nonnegative. On the other hand, in [18, Section 8.4], the existence of a solution in
\(L_p(\mathbb{R})\) spaces is obtained with a second-order differential operator \(L = aD^2 + bD + c\), where the bounded coefficients \(a, b, c, \xi\) depending on \((\omega, t, x)\). Also, in \([28]\), on a unit interval, interior regularity and boundary behavior of a solution are obtained with \(L = aD^2 + bD + c\), random and space and time-dependent coefficients \(a, b, c, \xi\), and random nonnegative initial data \(u(0) = u_0\). In the case of \(\gamma > 3/2\), \([29]\) shows that if \(L = \Delta, \xi = 1\), and nonnegative initial data \(u_0\) is nontrivial and vanishing at endpoints, then there is a positive probability that a solution blows up in finite time.

This paper aims to obtain the uniqueness, existence, \(L_p\) regularity, and maximal Hölder regularity of the solution to a semilinear stochastic partial differential equation driven by a multiplicative space-time white noise

\[
u_t = au_{xx} + bu_x + cu + |u|^\lambda u_x + \sigma(u)\dot{W}_t, \quad t > 0; \quad u(0, \cdot) = u_0(\cdot), \tag{1.3}
\]

where \(\lambda > 0\), and \(\dot{W}\) is a space-time white noise. The function \(u_0\) is a nonnegative random initial data. The coefficient \(a(\omega, t, x), b(\omega, t, x), c(\omega, t, x)\) are \(\mathcal{P} \times \mathcal{B}(\mathbb{R})\)-measurable, and \(\dot{b}(\omega, t)\) is \(\mathcal{P}\)-measurable. The coefficients \(a, b, c, \) and \(\dot{b}\) are uniformly bounded, and the leading coefficient \(a\) satisfies the ellipticity condition. The diffusion coefficient \(\sigma(\omega, t, x, u)\) is \(\mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R})\)-measurable. According to the conditions on \(\lambda\) and \(\sigma(u)\), we separate two cases:

(i) \(\lambda \in (0, 1]\) and \(\sigma(u)\) has bounded Lipschitz continuity and linear growth in \(u\),

(ii) \(\lambda \in (0, 1)\) and \(\sigma(u) = \mu|u|^{1+\lambda_0}\) with \(\lambda_0 \in [0, 1/2]\) (i.e., \(\sigma(u)\) is super-linear).

It should be noted that different condition on \(\lambda\) is assumed depending on the types of diffusion coefficient \(\sigma(u)\). In other words, if \(\sigma(u)\) is bounded in \(u\), \(\lambda\) is assumed to be less than or equal to 1; \(\lambda \in (0, 1]\). On the other hand, if \(\sigma(u) = |u|^{1+\lambda_0}\) with \([0, 1/2]\), \(\lambda\) is considered to be less than 1; \(\lambda \in (0, 1)\).

Our result has several advantages. The first novelty of our work is that all the results are obtained with random initial data \(u_0(\omega, x)\) and random coefficients \(a(\omega, t, x), b(\omega, t, x), c(\omega, t, x), \dot{b}(\omega, t),\) and \(\sigma(\omega, t, x, u)\). This accomplishment is one of the benefit of the Krylov’s \(L_p\)-theory, compared with the researches that the coefficients \(a, b, c, \) and \(\dot{b}\) are assumed to be constants.

Secondly, the regularity of the solution is presented, and the dependence of the regularity of the solution is proposed. For example, if \(\lambda \in (0, 1]\), \(\sigma(u)\) has Lipschitz continuity, linear growth, and boundedness in \(u\), and nonnegative initial data \(u_0 \in U_p^{1/2} \cap L_1(\Omega; L_1(\mathbb{R}))\) for any \(p > 2\), then Eq. \((1.3)\) has a solution \(u \in \mathcal{H}_{p,loc}^{1/2}\) (see Definition 2.5) such that for any \(T < \infty\) and small \(\varepsilon > 0\), almost surely

\[
u \in C_{t,x}^{1/2-\varepsilon,1/4-\varepsilon}([0, T] \times \mathbb{R}).
\]

On the other hand, if \(\lambda \in (0, 1)\), \(\sigma(u) = |u|^{1+\lambda_0}\) with \(\lambda_0 \in [0, 1/2]\), and nonnegative initial data \(u_0 \in U_p^{1/2-(\lambda-1/2)\vee \lambda_0} \cap L_1(\Omega; L_1(\mathbb{R}))\) for any \(p > 2\), then equation \((1.3)\) has a solution \(u \in \mathcal{H}_{p,loc}^{1/2-(\lambda-1/2)\vee \lambda_0}\) such that for any \(T < \infty\) and small \(\varepsilon > 0\), almost surely

\[
u \in C_{t,x}^{1/2-(\lambda-1/2)\vee \lambda_0-\varepsilon,1/2-(\lambda-1/2)\vee \lambda_0-\varepsilon}([0, T] \times \mathbb{R}).
\]
It should be noted that if the diffusion coefficient $\sigma(u)$ is bounded, the regularities of the solution is independent of $\lambda$. However, if the diffusion coefficient $\sigma(u) = |u|^{1+\lambda_0}$, $\lambda$ and $\lambda_0$ affect the solution regularities.

Lastly, we suggest a sufficient condition for the unique solvability in $L_p$ spaces. For example, the coefficient $\hat{b}$ is considered to be independent of $x$ to handle the nonlinear term $\hat{b}|u|^\lambda u_x$. Also, we assume algebraic conditions on $\lambda$ (and $\lambda_0$ if $\sigma(u) = |u|^{1+\lambda_0}$). Specifically, if $\sigma(u)$ has Lipschitz continuity, linear growth, and boundedness in $u$, $\lambda$ is taken to be less than or equal to 1; $\lambda \in (0, 1)$. On the other hand, if $\sigma(u) = |u|^{1+\lambda_0}$, $\lambda$ and $\lambda_0$ are expected to satisfy $\lambda \in (0, 1)$ and $\lambda_0 \in [0, 1/2)$. Indeed, if $\sigma(u)$ has boundedness in $u$, we separate the solution $u$ into the noise-related part and the nonlinear-related part. Then, to control the nonlinear-related part, the chain rule and fundamental theorem of calculus are employed with the assumption that $\lambda \in (0, 1)$; see Lemma 4.4. However, if $\sigma(u) = |u|^{1+\lambda_0}$, the nonlinear terms $|u|^\lambda u_x$ and $|u|^{1+\lambda_0}$ should be controlled simultaneously. Since the uniform $L_1$ bound of the solution is obtained and the nonlinear terms are interpreted as

$$u^\lambda u_x = \frac{1}{1+\lambda} \left(|u|^{1+\lambda}\right)_x = \frac{1}{1+\lambda} \left(|u|^\lambda \cdot |u|\right)_x$$

the surplus parts $|u|^\lambda$ and $|u|^{\lambda_0}$ should be dominated by the uniform $L_1$ bound of the solution in the estimate. Therefore, $|u|^\lambda$ and $|u|^{\lambda_0}$ need to be summable to power $s = 1/\lambda$ and $2s_0 = 1/\lambda_0$ with $s, s_0 > 1$; see Sect. 5. Thus, we assume $\lambda \in (0, 1)$ and $\lambda_0 \in [0, 1/2)$.

This paper is organized as follows. In Sect. 2, preliminary definitions and properties are introduced. Sect. 3 provides the existence, uniqueness, $L_p$-regularity, and maximal Hölder regularity of a solution to equation (1.3). Sections 4 and 5 contain proof of the main results.

We finish introduction with the notations. Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and natural numbers, respectively. We use $:= \cdot$ to denote definition. For a real-valued function $f$, we define

$$f^+ := \frac{f + |f|}{2}, \quad f^- = -\frac{f - |f|}{2}$$

For a normed space $F$, a measure space $(X, \mathcal{M}, \mu)$, and $p \in [1, \infty)$, a space $L_p(X, \mathcal{M}, \mu; F)$ is a set of $F$-valued $\mathcal{M}$-measurable function satisfying

$$\|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left(\int_X \|u(x)\|^p_F \mu(dx)\right)^{1/p} < \infty. \quad (1.4)$$

A set $\mathcal{M}$ is the completion of $\mathcal{M}$ with respect to the measure $\mu$. For $\alpha \in (0, 1]$ and $T > 0$, a set $C^\alpha([0, T]; F)$ is the set of $F$-valued continuous functions $u$ such that

$$|u|_{C^\alpha([0, T]; F)} := \sup_{t \in [0, T]} \|u(t)\|_F + \sup_{s, t \in [0, T], s \neq t} \frac{|u(t) - u(s)|_F}{|t - s|^\alpha} < \infty.$$
For $a, b \in \mathbb{R}$, set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. Let $S = S(\mathbb{R}^d)$ denote the set of Schwartz functions on $\mathbb{R}^d$. The Einstein’s summation convention with respect to $i, j, k$ is assumed throughout the article. A generic constant is denoted as $N$, which varies from line to line, and $N = N(a, b, \ldots)$ denotes that the constant $N$ depends only on $a, b, \ldots$. For functions depending on $\omega, t, x$, the argument $\omega \in \Omega$ is usually omitted.

2 Preliminaries

This section is devoted to reviewing the definitions and properties of the Brownian sheet $W(A)$ and stochastic Banach spaces $\mathcal{H}^{\gamma+2}_p(\tau)$. The Brownian sheet $W(A)$ induces space-time white noise $\dot{W}$, and it is used to interpret Eq. (1.3). The stochastic Banach space $\mathcal{H}^{\gamma+2}_p(\tau)$ is employed as a solution space. For more detail, see [18, 30, 31].

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is assumed to satisfy the usual conditions. A set $\mathcal{P}$ denotes the predictable $\sigma$-field related to $\{\mathcal{F}_t\}_{t \geq 0}$.

Definition 2.1 (The Brownian sheet) The Brownian sheet
\[ W(A) = \int_A W(ds, dx) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \int_0^\infty 1_A(s, x) \eta_k(x) dxdw_k^s, \] (2.1)
where $\{w^k_t : k \in \mathbb{N}\}$ is a set of one-dimensional independent Wiener processes and $\{\eta_k : k \in \mathbb{N}\}$ is a set of orthonormal basis in $L^2(\mathbb{R})$. Without loss of generality, we may assume that $\eta_k$ is bounded for each $k$; e.g. see [18, Section 8.3].

Remark 2.1 To interpret the noise part of Eq. (1.3), we employ Walsh’s stochastic integral with respect to Gaussian random measure $W(ds, dx)$; see [25]. It should be mentioned that the stochastic integral with respect to the martingale measure can be written as a series of Itô’s stochastic integral. For example, if $A$ is a bounded Borel measurable subset in $(0, \infty) \times \mathbb{R}$, we have

Definition 2.2 (Bessel potential space) Let $p > 1$ and $\gamma \in \mathbb{R}$. The space $H^\gamma_p = H^\gamma_p(\mathbb{R})$ is the set of all tempered distributions $u$ on $\mathbb{R}$ such that
\[ \|u\|_{H^\gamma_p} := \|(1 - \Delta)^{\gamma/2} u\|_{L^p} = \left\| \mathcal{F}^{-1} \left[ \left(1 + |\xi|^2\right)^{\gamma/2} \mathcal{F}(u)(\xi) \right] \right\|_{L^p} < \infty. \]
Similarly, \( H^\gamma_p(\ell_2) = H^\gamma_p(\mathbb{R}; \ell_2) \) is a space of \( \ell_2 \)-valued functions \( g = (g_1, g_2, \cdots) \) such that

\[
\|g\|_{H^\gamma_p(\ell_2)} := \left\| \left( 1 - \Delta \right)^{\gamma / 2} g \right\|_{\ell_2} = \left\| \mathcal{F}^{-1} \left[ \left( 1 + |\xi|^2 \right)^{\gamma / 2} \mathcal{F}(g)(\xi) \right] \right\|_{\ell_2} < \infty.
\]

When \( \gamma = 0 \), we write \( L^p := H^0_p \) and \( L^p(\ell_2) := H^0_p(\ell_2) \).

**Remark 2.2** Note that for \( \gamma \in (0, \infty) \) and \( u \in S \), the operator \( (1 - \Delta)^{-\gamma / 2} \) has a representation.

\[
(1 - \Delta)^{-\gamma / 2} u(x) = \int_{\mathbb{R}} R_\gamma(x - y) u(y) dy,
\]

where

\[
|R_\gamma(x)| \leq N(\gamma) \left( e^{-\frac{|x|}{2}} 1_{|x| \geq 2} + A_\gamma(x) 1_{|x| < 2} \right),
\]

and

\[
A_\gamma(x) := \begin{cases} |x|^{\gamma - 1} + 1 + O(|x|^{\gamma + 1}) & \text{if } 0 < \gamma < 1, \\
\log(2/|x|) + 1 + O(|x|^2) & \text{if } \gamma = 1, \\
1 + O(|x|^{\gamma - 1}) & \text{if } \gamma > 1. 
\end{cases}
\]

For more detail, see [30, Proposition 1.2.5.].

We introduce the space of point-wise multipliers in \( H^\gamma_p \).

**Definition 2.3** Fix \( \gamma \in \mathbb{R} \) and \( \alpha \in [0, 1) \) such that \( \alpha = 0 \) if \( \gamma \in \mathbb{Z} \) and \( \alpha > 0 \) if \( |\gamma| + \alpha \) is not an integer. Define

\[
B^{\gamma + \alpha} = \begin{cases} B(\mathbb{R}) & \text{if } \gamma = 0, \\
C^{\gamma - 1, 1}(\mathbb{R}) & \text{if } \gamma \text{ is a nonzero integer}, \\
C^{\gamma + \alpha}(\mathbb{R}) & \text{otherwise}; \end{cases}
\]

\[
B^{\gamma + \alpha}(\ell_2) = \begin{cases} B(\mathbb{R}, \ell_2) & \text{if } \gamma = 0, \\
C^{\gamma - 1, 1}(\mathbb{R}, \ell_2) & \text{if } \gamma \text{ is a nonzero integer}, \\
C^{\gamma + \alpha}(\mathbb{R}, \ell_2) & \text{otherwise}, \end{cases}
\]

where \( B(\mathbb{R}) \) is the space of bounded Borel functions on \( \mathbb{R} \), \( C^{\gamma - 1, 1}(\mathbb{R}) \) is the space of \( |\gamma| - 1 \) times continuous differentiable functions whose derivatives of \((|\gamma| - 1)\)-th order derivative are Lipschitz continuous, and \( C^{\gamma + \alpha} \) is the real-valued Hölder spaces. The space \( B(\ell_2) \) denotes a function space with \( \ell_2 \)-valued functions, instead of real-valued function spaces.

Below we gather the properties of \( H^\gamma_p \).
Lemma 2.1  Let $p > 1$ and $\gamma \in \mathbb{R}$.

(i) The space $C^\infty_c(\mathbb{R})$ is dense in $H^\gamma_p$.

(ii) Let $\gamma - 1/p = n + \nu$ for some $n = 0, 1, \ldots$ and $\nu \in (0, 1]$. Then, for any $i \in \{0, 1, \ldots, n\}$, we have

$$\left| D^i u \right|_{C(\mathbb{R})} + \left[ D^n u \right]_{C^\nu(\mathbb{R})} \leq N \| u \|_{H^\gamma_p}, \quad (2.2)$$

where $N = N(p, \gamma)$ and $C^\nu$ is a Zygmund space.

(iii) The operator $D_i : H^\gamma_p \rightarrow H^{\gamma + 1}_p$ is bounded. Moreover, for any $u \in H^{\gamma + 1}_p$,

$$\left\| D^i u \right\|_{H^\gamma_p} \leq N \| u \|_{H^{\gamma + 1}_p},$$

where $N = N(p, \gamma)$.

(iv) Let $\mu \leq \gamma$ and $u \in H^\gamma_p$. Then $u \in H^\mu_p$ and

$$\| u \|_{H^\mu_p} \leq \| u \|_{H^\gamma_p}.$$

(v) (Isometry). For any $\mu, \gamma \in \mathbb{R}$, the operator $(1 - \Delta)^{\mu/2} : H^\gamma_p \rightarrow H^{\gamma - \mu}_p$ is an isometry.

(vi) (Multiplicative inequality). Let

$$\varepsilon \in [0, 1], \quad p_i \in (1, \infty), \quad \gamma_i \in \mathbb{R}, \quad i = 0, 1,$$

$$\gamma = \varepsilon \gamma_0 + (1 - \varepsilon) \gamma_1, \quad 1/p = \varepsilon/p_0 + (1 - \varepsilon)/p_1.$$

Then, we have

$$\| u \|_{H^\gamma_p} \leq \| u \|^{\varepsilon}_{H^{\gamma_0}_{p_0}} \| u \|^{1-\varepsilon}_{H^{\gamma_1}_{p_1}}.$$

(vii) Let $u \in H^\gamma_p$. Then, we have

$$\| au \|_{H^\gamma_p} \leq N \| a \|_{B^{\gamma\gamma_1+\alpha}} \| u \|_{H^\gamma_p} \quad \text{and} \quad \| bu \|_{H^\gamma_p(\ell_2)} \leq N \| b \|_{B^{\gamma\gamma_1+\alpha}(\ell_2)} \| u \|_{H^\gamma_p},$$

where $N = N(\gamma, p)$ and $B^{\gamma\gamma_1+\alpha}, B^{\gamma\gamma_1+\alpha}(\ell_2)$ are introduced in Definition 2.3.

Proof  The above results are well-known; for instance, for (i)-(vi), see Theorem 13.3.7 (i), Theorem 13.8.1, Theorem 13.3.10, Corollary 13.3.9, Theorem 13.3.7 (ii), Exercise 13.3.20 of [31], respectively. For (vii), see [18, Lemma 5.2]. \qed

Now stochastic Banach spaces and solution spaces are provided. For more detail, see Section 3 of [18].

Definition 2.4  (Stochastic Banach spaces) Let $\tau \leq T$ be a bounded stopping time, $p > 1$ and $\gamma \in \mathbb{R}$. Define
Remark 2.3 Let \( \gamma \leq \tau \) be understood in the sense of distribution. For example, for a bounded stopping time \( \tau \) converges uniformly in \( (\omega, t) \) with \( 0 < t \leq \tau(\omega) \) to \( \xi(\omega) \) in \( L_p(\Omega, \mathcal{F}_\infty, dP) \) for each \( \omega \) almost surely.

Definition 2.5 (Solution spaces) Let \( \tau \leq T \) be a bounded stopping time and \( p \geq 2 \).

(i) For \( u \in \mathbb{H}^{\gamma+2}_p(\tau) \), we write \( u \in \mathcal{H}^{\gamma+2}_p(\tau) \) if there exists \( u_0 \in U_{\gamma+2}^p(\tau) \) and \( (f, g) \in \mathbb{H}_p^\gamma(\tau) \times \mathbb{H}_p^{\gamma+1}(\tau, \ell_2) \) such that

\[
\begin{align*}
  du &= f dt + \sum_{k=1}^{\infty} g^k d w_i^k, \
  t \in (0, \tau]; \
  u(0, \cdot) &= u_0
\end{align*}
\]

in the sense of distributions. In other words, for any \( \phi \in \mathcal{S} \), the equality

\[
(u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s, \cdot), \phi) dw^k_s
\]  
(2.3)

holds for all \( t \in [0, \tau] \) almost surely.

(ii) The norm of the function space \( \mathcal{H}^{\gamma+2}_p(\tau) \) is defined as

\[
\|u\|_{\mathcal{H}^{\gamma+2}_p(\tau)} := \|u\|_{\mathbb{H}^{\gamma+2}_p(\tau)} + \|f\|_{\mathbb{H}^\gamma_p(\tau)} + \|g\|_{\mathbb{H}^{\gamma+1}_p(\tau, \ell_2)} + \|u_0\|_{U^{\gamma+2}_p}.
\]  
(2.4)

(iii) When \( \gamma + 2 = 0 \), we write \( L_p(\tau) := \mathcal{H}^0_p(\tau) \).

(iv) For a stopping time \( \tau \in [0, \infty] \), we say \( u \in \mathcal{H}^{\gamma+2}_{p,loc}(\tau) \) if there exists a sequence of bounded stopping times \( \{\tau_n : n \in \mathbb{N}\} \) such that \( \tau_n \uparrow \tau \) (a.s.) as \( n \to \infty \) and \( u \in \mathcal{H}^{\gamma+2}_p(\tau_n) \) for each \( n \). The stopping time \( \tau \) is omitted if \( \tau = \infty \). We write \( u = v \) in \( \mathcal{H}^{\gamma+2}_{p,loc}(\tau) \) if there exists a sequence of bounded stopping times \( \{\tau_n : n \in \mathbb{N}\} \) such that \( \tau_n \uparrow \tau \) (a.s.) as \( n \to \infty \) and \( u = v \) in \( \mathcal{H}^{\gamma+2}_p(\tau_n) \) for each \( n \).

Remark 2.3 Let \( p \geq 2 \) and \( \gamma \in \mathbb{R} \). For any \( g \in \mathbb{H}^{\gamma+1}_p(\tau, \ell_2) \), the series of stochastic integral in (2.3) converges uniformly in \( t \) in probability on \( [0, \tau \wedge T] \) for any \( T \). Therefore, \( (u(t, \cdot), \phi) \) is continuous in \( t \) (See, e.g., [18, Remark 3.2]).

Remark 2.4 A stochastic partial differential equation driven by space-time white noise is understood in the sense of distribution. For example, for a bounded stopping time \( \tau \leq T \), consider

\[
u_t = u_{xx} + u \tilde{W}_t, \quad (t, x) \in (0, \tau) \times \mathbb{R}; \quad u(0, \cdot) = u_0.
\]  
(2.5)
We interpret Eq. (2.5) as follows: for any \( \phi \in \mathcal{S} \), equality
\[
(u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t \int_{\mathbb{R}} u(s, x) \phi_{xx}(x) \, dx \, ds + \int_0^t \int_{\mathbb{R}} u(s, x) \phi(x) W(ds, dx)
\]
holds for all \( t \leq \tau \) almost surely. By the way, due to Remark 2.1, the Walsh’s integral with respect to \( W(ds, dx) \) can be written as the series of Itô stochastic integral:
\[
\int_0^t \int_{\mathbb{R}} u(s, x) \phi(x) W(ds, dx) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}} u(s, x) \phi(x) \eta_k(x) \, dx \, dw^k_s
\]
holds for all \( t \leq \tau \) almost surely. Thus, equation (2.5) means for any \( \phi \in \mathcal{S} \), the equality
\[
(u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t \int_{\mathbb{R}} u(s, x) \phi_{xx}(x) \, dx \, ds + \int_0^t \int_{\mathbb{R}} u(s, x) \phi(x) \eta_k(x) \, dx \, dw^k_s
\]
holds for all \( t \in [0, \tau] \) almost surely. Therefore, we consider
\[
du = u_{xx} \, dt + u \eta_k \, dw^k_t, \quad (t, x) \in (0, \tau) \times \mathbb{R}; \quad u(0, \cdot) = u_0.
\]

Next the properties of the solution space \( \mathcal{H}^{\gamma+2}_p(\tau) \) are introduced.

**Theorem 2.1** Let \( \tau \leq T \) be a bounded stopping time.

(i) For any \( p \geq 2, \gamma \in \mathbb{R}, \mathcal{H}^{\gamma+2}_p(\tau) \) is a Banach space with the norm \( \| \cdot \|_{\mathcal{H}^{\gamma+2}_p(\tau)} \).

(ii) If \( p > 2, \gamma \in \mathbb{R}, \) and \( 1/p < \alpha < \beta < 1/2 \), then for any \( u \in \mathcal{H}^{\gamma+2}_p(\tau) \), we have
\[
\| u \|_{\mathcal{C}^{\alpha-1/p}(0, \tau]; \mathcal{H}^{\gamma+2-2\beta}_p(\tau) (a.s.)} \]

\[
\mathbb{E}\| u \|_{\mathcal{C}^{\alpha-1/p}(0, \tau]; \mathcal{H}^{\gamma+2-2\beta}_p(\tau) (a.s.)} \leq N(d, p, \gamma, \alpha, \beta, T) \| u \|_{\mathcal{H}^{\gamma+2}_p(\tau)}, \quad (2.6)
\]

(iii) Let \( p > 2, \gamma \in \mathbb{R}, \) and \( u \in \mathcal{H}^{\gamma+2}_p(\tau) \). If there exists \( \gamma_0 < \gamma \) such that
\[
\| u \|_{\mathcal{H}^{\gamma+2}_p(\tau \wedge t)} \leq N_0 + N_1 \| u \|_{\mathcal{H}^{\gamma+2}_{p0}(\tau \wedge t)} \]

\[
\| u \|_{\mathcal{H}^{\gamma+2}_p(\tau \wedge T)} \leq N_0 N. \quad (2.7)
\]

for all \( t \in (0, T) \), then we have

\[
\| u \|_{\mathcal{H}^{\gamma+2}_p(\tau \wedge T)} \leq N_0 N. \quad (2.8)
\]

where \( N = N(N_1, p, \gamma, T) \).
Proof For (i) and (ii), we refer the reader [18, Theorem 3.7] and [18, Theorem 7.2]. To show (iii), apply Lemma 2.1 (vi) to (2.7). Then, we have

\[ \|u\|_{H_p^{\gamma+2}(\tau \wedge t)}^p \leq N_0 + N_1 \|u\|_{H_p^{\gamma+2}(\tau \wedge t)}^p \]

\[ \leq N_0 + \frac{1}{2} \mathbb{E} \int_0^{\tau \wedge t} \|u(s, \cdot)\|_{H_p^{\gamma+2}}^p ds + N \mathbb{E} \int_0^{\tau \wedge t} \|u(s, \cdot)\|_{H_p^\gamma}^p ds \]

\[ \leq N_0 + \frac{1}{2} \mathbb{E} \int_0^{\tau \wedge t} \|u(s, \cdot)\|_{H_p^{\gamma+2}}^p ds + N \mathbb{E} \sup_{r \leq \tau \wedge s} \|u(r, \cdot)\|_{H_p^\gamma}^p ds, \]

where \( N = N(N_1, p, \gamma) \). By removing \( \frac{1}{2} \|u\|_{H_p^{\gamma+2}(\tau \wedge t)}^p \) both sides and applying (2.6), we have

\[ \|u\|_{H_p^{\gamma+2}(\tau \wedge t)}^p \leq 2N_0 + 2N \int_0^t \|u\|_{H_p^{\gamma+2}(\tau \wedge s)}^p ds, \]

where \( N = N(N_1, p, \gamma, T) \). By the Grönwall’s inequality, we have (2.8). The theorem is proved.

Corollary 2.1 Let \( \tau \leq T \) be a bounded stopping time and \( \kappa \in (0, 1/2) \). Suppose \( p \in (2, \infty) \), \( \alpha, \beta \in (0, \infty) \) satisfy

\[ \frac{1}{p} < \alpha < \beta < \frac{1}{2} \left( \frac{1}{2} - \kappa - \frac{1}{p} \right). \]

Then, for any \( \delta \in \left[ 0, \frac{1}{2} - \kappa - 2\beta - \frac{1}{p} \right) \), we have \( u \in C^{\alpha-1/p}([0, \tau]; C^{1/2-\kappa-2\beta-1/p-\delta}) \) (a.s.) and

\[ \mathbb{E}|u|_{C^{\alpha-1/p}([0, \tau]; C^{1/2-\kappa-2\beta-1/p-\delta})}^p \leq N \|u\|_{H_p^{1/2-\kappa}(\tau)}^p, \]

where \( N = N(p, \alpha, \beta, \kappa, T) \).

Proof Set \( \gamma = 1/2 - \kappa - 2\beta - \delta \). Then, by Lemma 2.1 (ii), we have

\[ |u(t, \cdot)|_{C^{1/2-\kappa-2\beta-1/p-\delta}} \leq N \|u(t, \cdot)\|_{H_p^{1/2-\kappa-2\beta-\delta}} \leq N \|u(t, \cdot)\|_{H_p^{1/2-\kappa-2\beta}} \]

for all \( t \in [0, \tau] \) almost surely. By (2.11) and (2.6), we have (2.10). The corollary is proved.

Springer
3 Main results

This section provides the uniqueness, existence, $L_p$ regularity, and maximal Hölder regularity of the solution to semilinear equation

$$du = (au_{xx} + bu_x + cu + \tilde{b}|u|^\lambda u_x)\,dt + \sigma(u)\eta_k\,dw_k^x; \quad u(0, \cdot) = u_0(\cdot) \quad (3.1)$$
on $(t, x) \in (0, \infty) \times \mathbb{R}$ with $\lambda > 0$. The coefficients $a, b, c$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable, $\tilde{b}$ is $\mathcal{P}$-measurable, and $a, b, c, \tilde{b}$ are uniformly bounded; see Assumption 3.1. Notice that $\tilde{b}$ is assumed to be independent of $x$ to control the nonlinear term $\tilde{b}|u|^\lambda u_x$ in the estimate; see Remarks 3.1 and 3.5. The set of bounded functions $\{\eta_k : k \in \mathbb{N}\}$ is an orthonormal $L_2(\mathbb{R})$ basis introduced in Remark 2.1.

Based on the conditions of nonlinear terms $\tilde{b}|u|^\lambda u_x$ and $\sigma(u)$, we separate two cases. Each case is discussed in Sects. 3.1 and 3.2;

Case (1) $\lambda \in (0, 1]$ and $\sigma(u)$ has Lipschitz continuity, linear growth, and boundedness in $u$,

Case (2) $\lambda \in (0, 1)$ and $\sigma(u) = |u|^{1+\lambda_0}$ with $\lambda_0 \in [0, 1/2]$.

Notice that the condition on $\lambda$ is changed according to the types of diffusion coefficient $\sigma(u)$. In other words, if $\sigma(u)$ is bounded (Case 1), we consider $\lambda$ is less than or equal to 1; $\lambda \in (0, 1]$. On the other hand, if $\sigma(u)$ is unbounded (Case 2), $\lambda$ is assumed to be less than 1; $\lambda \in (0, 1)$.

Besides, the regularity of the solution varies in each case. For example, if $\sigma(u)$ is bounded (Case 1) and $u_0 \in \cap_{p>2} U^{1/2}_p \cap L_1(\Omega; L_1)$, then solution $u$ introduced in Corollary 3.1 satisfies for $T < \infty$ and small $\varepsilon > 0$,

$$\sup_{t \leq T} |u(t, \cdot)|_{C^{1/2-\varepsilon}(\mathbb{R})} + \sup_{x \in \mathbb{R}} |u(\cdot, x)|_{C^{1/4-\varepsilon}([0, T])} < \infty \quad (a.s.).$$

On the other hand, if $\sigma(u)$ is unbounded (Case 2) and $u_0 \in \cap_{p>2} U^{1/2-(\lambda-1/2)\vee\lambda_0}_p \cap L_1(\Omega; L_1)$, solution $u$ introduced in Corollary 3.2 satisfies for $T < \infty$ and small $\varepsilon > 0$,

$$\sup_{t \leq T} |u(t, \cdot)|_{C^{1/2-(\lambda-1/2)\vee\lambda_0-\varepsilon}(\mathbb{R})} + \sup_{x \in \mathbb{R}} |u(\cdot, x)|_{C^{1/2-(\lambda-1/2)\vee\lambda_0-\varepsilon}^{1/2-(\lambda-1/2)\vee\lambda_0-\varepsilon}([0, T])} < \infty \quad (a.s.).$$

Notice that the solution regularity independent of $\lambda$ if $\sigma(u)$ is bounded. On the contrary, the regularity of solution depends on $\lambda$ and $\lambda_0$ if $\sigma(u)$ is super-linear.

3.1 The first case: generalized Burgers’ equation with the bounded Lipschitz diffusion coefficient $\sigma(u)$

This section contains results of Eq. (3.1) with $\lambda \in (0, 1]$. The diffusion coefficient $\sigma(u)$ has Lipschitz continuity, linear growth, and boundedness in $u$; see Assumption 3.2. The $L_p$-solvability of Eq. (3.1) is provided in Theorem 3.3. Also, we obtain the Hölder regularity of the solution by applying the Hölder embedding theorem for
Furthermore, by employing the uniqueness of the solution in $p$ (Theorem 3.4), we achieve the maximal Hölder regularity of the solution; see Corollary 3.1.

The following are assumptions on coefficients to Eq. (3.1). Note that $\sigma(u)$ is bounded in Assumption 3.2.

**Assumption 3.1**

(i) The coefficients $a = a(t, x)$, $b = b(t, x)$, and $c = c(t, x)$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable.

(ii) The coefficient $\bar{b} = \bar{b}(t)$ is predictable.

(iii) There exists $K > 0$ such that

$$K^{-1} \leq a(t, x) \leq K$$

for all $(\omega, t, x) \in \Omega \times [0, \infty) \times \mathbb{R}$.

$$|a(t, \cdot)|_{C^2(\mathbb{R})} + |b(t, \cdot)|_{C^2(\mathbb{R})} + |c(t, \cdot)|_{C^2(\mathbb{R})} + |\bar{b}(t)| \leq K$$

for all $(\omega, t) \in \Omega \times [0, \infty)$.

**Assumption 3.2**

(i) The function $\sigma(t, x, u)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable.

(ii) There exists $K > 0$ such that for $\omega \in \Omega$, $t > 0$, $x, u, v \in \mathbb{R}$,

$$|\sigma(t, x, u) - \sigma(t, x, v)| \leq K|u - v| \quad \text{and} \quad |\sigma(t, x, u)| \leq K(|u| \wedge 1).$$

Now we provide the $L_p$-solvability of equation (3.1).

**Theorem 3.3** Let $\lambda \in (0, 1]$. Assume $p > 2$ and $\kappa \in (0, 1/2)$ satisfy

$$p - 1 + \lambda \in 2\mathbb{N} \quad \text{and} \quad p > \frac{6}{1 - 2\kappa}.$$

Suppose Assumptions 3.1 and 3.2 hold. For the nonnegative initial data $u_0 \in U_p^{1/2 - \kappa} \cap L_1(\Omega; L_1)$, equation (3.1) has a unique nonnegative solution $u$ in $H_{p, \text{loc}}^{1/2 - \kappa}$. Furthermore, if $\alpha$ and $\beta$ satisfy (2.9), then for any $T \in (0, \infty)$ and small $\varepsilon > 0$, almost surely

$$\|u\|_{C^{\alpha - 1/p, \frac{p}{2} - 1/p}(0, T; C^{1/2 - \kappa - 2\beta - 1/p}(\mathbb{R}))} < \infty.$$

**Proof** See Proof of Theorem 3.3 in Sect. 4.

**Remark 3.1** The coefficient $\bar{b}$ is assumed to be independent of $x$, and it is a sufficient condition for the existence of a global solution. In fact, we control the nonlinear term $|u|^\lambda u_x$ with the bound of $\sigma(u)$ and the initial data $u_0$ to show the local existence time blows up. Thus, in the $L_p$ estimate, the chain rule and fundamental theorem of calculus are employed to apply Grönwall’s inequality. Here, the coefficient $\bar{b}$ should be taken out of the integral with respect to $x$ to use the fundamental theorem of calculus. Thus, $\bar{b}$ is assumed to be independent of $x$; see Lemma 4.4.
Remark 3.2

(i) There exist $\alpha$ and $\beta$ satisfying (2.9) since $p > \frac{6}{1-2\kappa}$.

(ii) The summability $p$ is assumed to satisfy $p - 1 + \lambda \in 2\mathbb{N}$ to employ the fundamental theorem of calculus; see (4.30) of Lemma 4.4.

(iii) The condition on the initial data $u_0 \in L_1(\Omega; L_1)$ is assumed to obtain $L_1$ bound of the solution.

Remark 3.3

The regularity of the solution is independent of $\lambda$. Indeed, the nonlinear term $|u|^\lambda u_x$ does not affect the regularity of the solution when we extend the local solutions to a global one. More precisely, we prove that the regularity of the local solution is independent of $\lambda$ and the local solution is non-explosive. To prove non-explosive property, it suffices to show that there is a uniform $L_p$ bounded of the local solutions since we employ Krylov’s $L_p$ theory; see [18, Section 8]. During the proof, we separate the local solution into two parts to obtain the uniform $L_p$ bound of the local solution; noise-related part and nonlinear-related part. In Lemma 4.3, we show that the regularity of the noise-related part is mainly affected by space-time white noise and independent of $\lambda$. Also, it turns out that for any $\lambda \in (0, 1]$, the $L_p$ bound of nonlinear-related part is dominated by noise-related part; see Lemma 4.4. Thus, we achieve a uniform $L_p$ bound of the local solutions independent of $\lambda$.

Remark 3.4

The Hölder regularity of the solution introduced in Theorem 3.3 depends on $\alpha$ and $\beta$. For example, for small $\varepsilon > 0$, set $\alpha = \frac{1}{p} + \frac{\varepsilon}{4}$ and $\beta = \frac{1}{p} + \frac{\varepsilon}{2}$. Then, we have

$$\sup_{t \leq T} |u(t, \cdot)|_{C^{\alpha}} < \infty \quad (a.s.).$$

Similarly, for small $\varepsilon > 0$, let $\alpha = \frac{1}{2} \left( \frac{1}{2} - \kappa - \frac{1}{p} \right) - \varepsilon$ and $\beta = \frac{1}{2} \left( \frac{1}{2} - \kappa - \frac{1}{p} \right) - \varepsilon$. Then, we have

$$\sup_{x \in \mathbb{R}} |u(\cdot, x)|_{C^{\beta}} < \infty \quad (a.s.).$$

To obtain the maximal Hölder regularity of the solution, the parameter $p$ should be large. The following theorem provides the uniqueness of the solution in $p$.

Theorem 3.4

Assume that all the conditions of Theorem 3.3 holds. Let $u \in \mathcal{H}_{p,\text{loc}}^{1/2-\kappa}$ be the solution of equation (3.1) introduced in Theorem 3.3. If $q > p$ and $u_0 \in U_q^{1/2-\kappa} \cap L_1(\Omega; L_1)$, then $u \in \mathcal{H}_{q,\text{loc}}^{1/2-\kappa}$.

Proof

See Proof of Theorem 3.4 in Section 4. 

By combining Theorems 3.3 and 3.4, we obtain the maximal Hölder regularity of the solution.

Corollary 3.1

Suppose $u_0 \in U_p^{1/2} \cap L_1(\Omega; L_1)$ for all $p > 2$. Then, for $T > 0$ and small $\varepsilon > 0$, we have

$$\sup_{t \leq T} |u(t, \cdot)|_{C^{1/2-\varepsilon}(\mathbb{R})} + \sup_{x \in \mathbb{R}} |u(\cdot, x)|_{C^{1/4-\varepsilon}([0, T])} < \infty$$
almost surely.

Proof Let $T \in (0, \infty)$ and small $\varepsilon > 0$. For each $p > \frac{6}{1-\varepsilon/2}$, Theorem 3.3 with $\kappa = \varepsilon/4$ yields that there exists a unique solution $u = u_p \in \mathcal{H}_{p}^{1/2-\varepsilon/4}(T)$ to equation (3.1). Since $u_0 \in U_{p}^{1/2-\varepsilon/4}$ for all $p > 2$, Theorem 3.4 implies that all the solutions $u = u_p$ coincides. Then, by letting $\delta = 0, \kappa = \frac{\varepsilon}{4}, p = \frac{12}{\varepsilon}, \alpha = \frac{1}{p} + \frac{\varepsilon}{8},$ and $\beta = \frac{1}{p} + \frac{\varepsilon}{4}$ in Corollary 2.1, almost surely

$$\sup_{t \leq T} |u(t, \cdot)|^p_{C^{2-\varepsilon}(\mathbb{R})} \leq |u|^p_{C^{\alpha-\frac{1}{p}}([0,T]; C^{\frac{1}{2}-\varepsilon-2\beta - \frac{1}{p}}(\mathbb{R}))} < \infty.$$ 

On the other hand, set $\delta = 0, \kappa = \frac{\varepsilon}{4}, p = \frac{12}{\varepsilon}, \alpha = \frac{1}{2} (1 - \kappa - \frac{1}{p}) - \frac{\varepsilon}{2},$ and $\beta = \frac{1}{2} \left(1 - \kappa - \frac{1}{p}\right) - \frac{\varepsilon}{2}$ in Corollary 2.1. Then, almost surely

$$\sup_{x \in \mathbb{R}} |u(\cdot, x)|^p_{C^{2-\varepsilon}(0,T]} \leq |u|^p_{C^{\alpha-\frac{1}{p}}([0,T]; C^{\frac{1}{2}-\varepsilon-2\beta - \frac{1}{p}}(\mathbb{R}))} < \infty.$$ 

The corollary is proved.

3.2 The second case: modified Burgers’ equation with the super-linear diffusion coefficient $\sigma(u)$

In this section, we consider

$$du = (au_{xx} + bu_x + cu + b|u|^\lambda u_x) dt + \mu |u|^{1+\lambda_0} \eta_k dw_k^k; \quad u(0, \cdot) = u_0(\cdot), \quad (3.6)$$

on $(t, x) \in (0, \infty) \times \mathbb{R}$ with $\lambda \in (0, 1)$ and $\lambda_0 \in [0, 1/2)$. Theorem 3.6 contains the uniqueness, existence, and regularity of the solution to equation (3.6). Besides, the Hölder regularity of the solution follows from the Hölder embedding theorem for $\mathcal{H}_{p}^{1/2+\varepsilon}(\tau)$ (Corollary 2.1). Similarly to Sect. 3.1, by considering large $p$, we obtain the maximal Hölder regularity of the solution; see Corollary 3.2.

Below is the assumption for the coefficient $\mu$ of (3.6).

Assumption 3.5 (i) The coefficient $\mu(t, x)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable.
(ii) There exists $K > 0$ such that for $\omega \in \Omega, t > 0, x \in \mathbb{R},$

$$|\mu(t, \cdot)|_{C(\mathbb{R})} \leq K. \quad (3.7)$$

We introduce the $L_p$-solvability results on Eq. (3.6).

Theorem 3.6 Let $\lambda \in (0, 1), \lambda_0 \in [0, 1/2), 1/2 > \kappa > (\lambda - 1/2) \vee \lambda_0,$ and $p > \frac{6}{2-2\kappa}$. Suppose Assumptions 3.1 and 3.5 hold. For nonnegative initial data $u_0 \in U_{p}^{1/2-\kappa} \cap L_1(\Omega; L_1)$, Eq. (3.6) has a unique nonnegative solution $u$ in $\mathcal{H}_{p, loc}^{1/2-\kappa}$. Furthermore, if
\( \alpha \) and \( \beta \) satisfy (2.9), then for any \( T \in (0, \infty) \) and small \( \varepsilon > 0 \), almost surely
\[
\|u\|_{C^{\alpha-1/\rho}([0, T]; C^{1/2-\kappa-2\beta-1/\rho}(\mathbb{R}))}^\rho < \infty. \tag{3.8}
\]

**Proof** See **Proof of Theorem 3.6** in Sect. 5. \( \square \)

**Remark 3.5** The coefficient \( \bar{b} \) is assumed to be independent of \( x \) (Assumption 3.1), and it is a sufficient condition for the existence of a global solution. Indeed, since a uniform \( L_1 \) bound of the solution is employed to extend existence time, we show that all the paths of \( \|u(t, \cdot)\|_{L_1} \) are bounded. Intuitively, if we take integration and expectation to (3.6), the nonlinear term \( \bar{b}|u|^\lambda u_x \) is removed since the nonlinear term is interpreted as
\[
\bar{b}|u|^\lambda u_x = \bar{b} \frac{1}{1 + \lambda} (|u|^{1+\lambda})_x = \bar{b} \frac{1}{1 + \lambda} (|u|^\lambda \cdot |u|)_x \tag{3.9}
\]
and \( \bar{b} \) is independent of \( x \). Thus, it turns out that \( \|u(t, \cdot)\|_{L_1} \) is a local martingale, and its trajectories are bounded almost surely; see Lemma 5.2.

**Remark 3.6** (i) Note that \( \lambda \) is assumed to be less than 1. Indeed, due to the unbounded diffusion coefficient, we only obtain the uniform \( L_1 \) bound of the solution instead of \( L_p \) bound \( (p > 1) \). Since the surplus part \( |u|^\lambda \) in (3.9) should be summable for some \( s = 1/\lambda > 1 \), \( \lambda \) is less than 1.
(ii) Because \( p > \frac{6}{1-2\kappa} \), there are \( \alpha \) and \( \beta \) satisfying (2.9).
(iii) The condition on the initial data \( u_0 \in L_1(\Omega; L_1) \) is employed to obtain the uniform \( L_1 \) bound of the solution.

**Remark 3.7** The regularity of the solution depends on \( \lambda \) and \( \lambda_0 \). Indeed, the uniform \( L_1 \) bound of solutions is achieved, and the bound is employed to prove the local solution is non-explosive. Then, restrictions on solution regularity are required to control the nonlinear terms \( |u|^\lambda u_x \) and \( |u|^\lambda_{0+1} \) since the surplus parts \( |u|^\lambda \) and \( |u|^\lambda_0 \) should be summable to power \( s = 1/\lambda \) and \( 2s_0 = 1/\lambda_0 \), respectively; see Lemma 5.3.

**Remark 3.8** The Hölder regularity of the solution introduced in Theorem 3.6 depends on \( \alpha \) and \( \beta \). For example, for small \( \varepsilon > 0 \), let \( \alpha = \frac{1}{p} + \frac{\varepsilon}{4} \) and \( \beta = \frac{1}{p} + \frac{\varepsilon}{2} \). Then, we have
\[
\sup_{t \leq T} |u(t, \cdot)|_{C^{\alpha-\kappa-\frac{1}{2p} - \varepsilon}(\mathbb{R})} < \infty,
\]
almost surely. Similarly, for small \( \varepsilon > 0 \), set \( \alpha = \frac{1}{2} \left( \frac{1}{2} - \kappa - \frac{1}{p} \right) - \varepsilon \) and \( \beta = \frac{1}{2} \left( \frac{1}{2} - \kappa - \frac{1}{p} \right) - \frac{\varepsilon}{2} \). Then, we have
\[
\sup_{x \in \mathbb{R}} |u(\cdot, x)|_{C^{\frac{1}{2} - \kappa - \frac{3}{2p} - \varepsilon}(0, T]} < \infty
\]
almost surely.
Next, we provide the uniqueness of the solution in \( p \) to achieve the maximal Hölder regularity of the solution.

**Theorem 3.7** Assume that all the conditions of Theorem 3.6 holds. Let \( u \) be the solution of Eq. (3.6) introduced in Theorem 3.6. If \( q > p \) and \( u_0 \in U_q^{1/2-\kappa} \cap L_1(\Omega; L_1) \), then \( u \in H_{q,loc}^{1/2-\kappa} \).

**Proof** See **Proof of Theorem 3.7** in Sect. 5. \( \square \)

By Theorems 3.6 and 3.7, the maximal Hölder regularity of the solution follows.

**Corollary 3.2** Suppose \( u_0 \in U_p^{1/2-(\lambda-1/2)\vee \lambda_0} \cap L_1(\Omega; L_1) \) for all \( p > 2 \). Then, for \( T > 0 \) and small \( \varepsilon > 0 \), we have

\[
\sup_{t \leq T} |u(t, \cdot)|_{C^{1/2-(\lambda-1/2)\vee \lambda_0-\varepsilon}(\mathbb{R})} + \sup_{x \in \mathbb{R}} |u(\cdot, x)|_{C^{1/2-(\lambda-1/2)\vee \lambda_0-\varepsilon}([0,T])} < \infty
\]

almost surely.

**Proof** Even though the proof is similar to the proof of Corollary 3.1, we contain the proof for the completeness of the article. Let \( T \in (0, \infty) \) and \( \kappa = \left( \lambda - \frac{1}{2} \right) \vee \lambda_0 + \frac{\varepsilon}{4} \) for small \( \varepsilon > 0 \). Note that for each \( p > \frac{6}{1-2\kappa} \), there exists a unique solution \( u = u_p \in H_p^{1/2-\kappa}(T) \) to equation (3.6) by Theorem 3.6. Since \( u_0 \in U_p^{1/2-(\lambda-1/2)\vee \lambda_0} \) for all \( p > 2 \), Theorem 3.7 implies that all the solutions \( u = u_p \) coincides. Then, by considering \( p = \frac{12}{\varepsilon}, \alpha = \frac{1}{p} + \frac{\varepsilon}{8}, \) and \( \beta = \frac{1}{p} + \frac{\varepsilon}{4} \) in Corollary 2.1, almost surely

\[
\sup_{t \leq T} |u(t, \cdot)|_{C^{1/2-(\lambda-1/2)\vee \lambda_0-\varepsilon}(\mathbb{R})} \leq |u|_{C^{a-1/p}([0,T]; C^{1/2-\kappa-2\beta-1/p}(\mathbb{R}))} < \infty.
\]

On the other hand, if we consider \( p = \frac{4}{\varepsilon}, \alpha = \frac{1}{2} \left( \frac{1}{2} - \kappa - \frac{1}{p} \right) - \frac{\varepsilon}{2}, \) and \( \beta = \frac{1}{2} \left( \frac{1}{2} - \kappa - \frac{1}{p} \right) - \frac{\varepsilon}{4} \) in Corollary 2.1, almost surely

\[
\sup_{x \in \mathbb{R}} |u(\cdot, x)|_{C^{\frac{1}{2}-(\lambda-1/2)-\varepsilon}([0,T])} \leq |u|_{C^{a-1/p}([0,T]; C^{1/2-\kappa-2\beta-1/p}(\mathbb{R}))} < \infty.
\]

The corollary is proved. \( \square \)

### 4 Proof of the first case: generalized Burgers’ equation with the bounded Lipschitz diffusion coefficient \( \sigma(u) \)

This section suggests proof of Theorems 3.3 and 3.4. In the proof of Theorem 3.3, the uniqueness of a global solution follows from the uniqueness of the local ones; see Lemma 4.1. Also, we construct a global solution candidate by pasting local ones to show the existence of a global solution; see Remark 4.1. To check the global solution
candidate $u$ is non-explosive, we show that for any $T < \infty$,

$$P \left( \left\{ \omega \in \Omega : \sup_{t \leq T, x \in \mathbb{R}} |u(t, x)| > R \right\} \right) \to 0 \quad (4.1)$$

as $R \to \infty$. Then, to prove (4.1), we separate the local solution $u_m$ in two parts; noise-related part $v$ and nonlinear-related part $w_m := u_m - v$. In Lemma 4.4, we control the noise-related part $v$ with the initial data $u_0$ and bound of $\sigma(u)$. On the other hand, in Lemma 4.4, the other part $w_m$ is dominated by $v$. In the proof of Lemma 4.4, we employ the fundamental theorem of calculus with $p - 1 + \lambda \in 2\mathbb{N}$.

First of all, we introduce a theorem describing the uniqueness, existence, and $L_p$-regularity of a solution to semilinear equation

$$du = (au_{xx} + bu_x + cu + f(u))dt + g^k(u)dw^k\cdot \quad t > 0; \quad u(0, \cdot) = u_0(\cdot). \quad (4.2)$$

where $f(u)$ and $g(u)$ satisfy Assumption 4.1. This result is used to construct local solutions.

**Assumption 4.1 ($\tau$)**

(i) The function $f(t, x, u)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable and $f(t, x, 0) \in \mathbb{H}^\gamma_p(\tau)$.

(ii) The function $g^k(t, x, u)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable and $g(t, x, 0) = (g^1(t, x, 0), g^2(t, x, 0), \ldots) \in \mathbb{H}^\gamma_{p+1}(\tau, \ell_2)$.

(iii) For any $\varepsilon > 0$, there exists a constant $N_\varepsilon$ such that

$$\|f(u) - f(v)\|_{\mathbb{H}^\gamma_p(\tau)} + \|g(u) - g(v)\|_{\mathbb{H}^\gamma_{p+1}(\tau, \ell_2)} \leq \varepsilon \|u - v\|_{\mathbb{H}^\gamma_{p+2}(\tau)} + N_\varepsilon \|u - v\|_{\mathbb{H}^\gamma_{p+1}(\tau)}$$

for any $u, v \in \mathbb{H}^\gamma_{p+2}(\tau)$.

**Theorem 4.2** Let $\tau \leq T$ be a bounded stopping time, $\gamma \in [-2, -1]$, and $p \geq 2$. Suppose Assumptions 3.1 and 4.1 ($\tau$) hold. Then, for any $u_0 \in \mathcal{U}_\gamma^{p+2}$, equation (4.2) has a unique solution $u \in \mathcal{H}^{\gamma+2}_p(\tau)$ such that

$$\|u\|_{\mathcal{H}^{\gamma+2}_p(\tau)} \leq N \left( \|f(0)\|_{\mathcal{H}^\gamma_{p}(\tau)} + \|g(0)\|_{\mathcal{H}^\gamma_{p+1}(\tau, \ell_2)} + \|u_0\|_{\mathcal{U}_{\gamma}^{p+2}} \right), \quad (4.3)$$

where $N$ depends on constants $d, p, \gamma, K, T$ and the function $N_\varepsilon$ ($\varepsilon > 0$).

**Proof** We only consider case $\tau \leq T$ because case $\tau = T$ follows from [18, Theorem 5.1].

**Step 1. (Existence).** Set

$$\tilde{f}(t, u) := 1_{t \leq \tau} f(t, u) \quad \text{and} \quad \tilde{g}(t, u) := 1_{t \leq \tau} g(t, u).$$

Note that $\tilde{f}(u)$ and $\tilde{g}(u)$ satisfy Assumption 4.1 ($T$). Therefore, by [18, Theorem 5.1], there exists a unique solution $u \in \mathcal{H}^{\gamma+2}_p(T)$ such that $u$ satisfies equation (4.2) with
\[ f, \tilde{g}, \text{instead of} \ f, g. \] Since \( \tau \leq T \), we have \( u \in H^{\gamma+2}_p(\tau) \) and \( u \) satisfies equation (4.2) and estimate (4.3) with \( f, g \).

**Step 2. (Uniqueness).** Suppose \( v \in H^{\gamma+2}_p(\tau) \) is another solution to equation (4.2). Then, [18, Theorem 5.1] yields that there exists a unique solution \( \tilde{v} \in H^{\gamma+2}_p(T) \) satisfying

\[
d\tilde{v} = \left( a^{ij}(x) \tilde{v}_{x^i x^j} + b^i(x) \tilde{v}_{x^i} + c(x) \tilde{v} + \bar{f}(v) \right) dt + \sum_{k=1}^{\infty} \tilde{g}^k(v) dw^k_t, \quad 0 < t < T \tag{4.4}
\]

with the initial data \( \tilde{v}(0, \cdot) = u_0 \). Notice that \( \bar{f}(v) \) and \( \bar{g}(v) \) are used in (4.4), instead of \( f(\tilde{v}) \) and \( g(\tilde{v}) \). Let \( \tilde{v} := v - \tilde{v} \). Then, for each \( \omega \in \Omega \),

\[
d\tilde{v} = \left( a^{ij}(x) \tilde{v}_{x^i x^j} + b^i(x) \tilde{v}_{x^i} + c(x) \tilde{v} \right) dt, \quad 0 < t < \tau; \quad \tilde{v}(0, \cdot) = 0.
\]

Thus, [18, Theorem 5.1] implies \( v(t, \cdot) = \tilde{v}(t, \cdot) \) in \( H^{\gamma+2}_p \) for all \( t \leq \tau \) almost surely. Therefore, we can replace \( f(\tilde{v}), g(\tilde{v}) \) with \( \bar{f}(v), \bar{g}(\tilde{v}) \) in equation (4.4). Similarly, there exists \( \tilde{u} \in H^{\gamma+2}_p(T) \) satisfying (4.2) and \( u(t, \cdot) = \tilde{u}(t, \cdot) \) in \( H^{\gamma+2}_p \) for all \( t \leq \tau \) almost surely. Since the uniqueness result in \( H^{\gamma+2}_p(T) \) yields \( \tilde{u} = \tilde{v} \) in \( H^{\gamma+2}_p(T) \), we have \( u = v \) in \( H^{\gamma+2}_p \) for almost every \((\omega, t) \in (0, \tau] \). Thus, we have \( u = v \) in \( H^{\gamma+2}_p(\tau) \). The theorem is proved.

Let \( h(\cdot) \in C_c^\infty(\mathbb{R}) \) be a nonnegative function such that \( h(z) = 1 \) on \( |z| \leq 1 \) and \( h(z) = 0 \) on \( |z| \geq 2 \). Then, for \( m \in \mathbb{N} \), define

\[
h_m(z) := h(z/m). \tag{4.5}
\]

Below lemma provides the existence and uniqueness of local solutions \( u_m \).

**Lemma 4.1** Let \( \lambda \in (0, \infty) \) \( T \in (0, \infty) \), \( \kappa \in (0, 1/2) \), and \( p \geq 2 \). Suppose Assumptions 3.1 and 3.2 hold. For a bounded stopping time \( \tau \leq T \), \( m \in \mathbb{N} \), and nonnegative initial data \( u_0 \in U^{1/2-\kappa}_p \), there exists \( u_m \in H^{1/2-\kappa}_p(\tau) \) satisfying equation

\[
du = \left( au_{xx} + bu_x + cu + \bar{b} \left( u^{1+\lambda}_+ h_m(u) \right)_x \right) dt + \sigma(u) \eta_t dw^k_t; \quad u(0, \cdot) = u_0(\cdot), \tag{4.6}
\]

where \((t, x) \in (0, \tau) \times \mathbb{R} \). Furthermore, \( u_m \geq 0 \).

**Proof** Notice that for \( u, v \in \mathbb{R} \), we have

\[
\left| u^{1+\lambda}_+ h_m(u) - v^{1+\lambda}_+ h_m(v) \right| \leq N_m |u - v|. \tag{4.7}
\]
Indeed,

\[
|u^{1+\lambda}_+ h_m(u) - v^{1+\lambda}_+ h_m(v)| \leq \left\{ \begin{array}{ll}
|u^{1+\lambda}_+ h_m(u) - v^{1+\lambda}_+ h_m(v)| & \leq N_m |u - v| \\
|u^{1+\lambda}_+ h_m(u) - (2m)^\lambda u| & \leq (2m)^\lambda (u - v) \leq N_m |u - v| & \text{if } u, v \geq 0, \\
v^{1+\lambda}_+ h_m(v) & \leq (2m)^\lambda v \leq (2m)^\lambda (v - u) \leq N_m |u - v| & \text{if } u < 0, v \geq 0, \\
0 & \leq N_m |u - v| & \text{if } u, v < 0.
\end{array} \right.
\]

(4.8)

Thus, Lemma 2.1 (iii), Remark 2.2, and Minkowski’s inequality implies that for \( u, v \in \mathbb{H}^{1/2-\kappa}_p (\tau), \)

\[
\left\| \left( u_+(t, \cdot) \right)^{1+\lambda}_+ h_m(u(t, \cdot)) \right\|_x - \left( \left( v_+(t, \cdot) \right)^{1+\lambda}_+ h_m(v(t, \cdot)) \right\|_x \leq N_m \left\| (u_+(t, \cdot))^{1+\lambda}_+ h_m(u(t, \cdot)) - (v_+(t, \cdot))^{1+\lambda}_+ h_m(v(t, \cdot)) \right\|_{H_p^{-3/2-\kappa}}\]

\[
\leq N \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(y)| \right) \left( \int_{\mathbb{R}} |(u^{1+\lambda}_+ h_m(u) - v^{1+\lambda}_+ h_m(v))(t, x - y)| dy \right)^p dx \right)
\]

and

\[
\left\| \sigma (u) \eta - \sigma (v) \eta \right\|_{H_p^{-1/2-\kappa}(\ell_2)}^p \leq \int_{\mathbb{R}} \left( \sum_k \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)| ((\sigma(u) - \sigma(v))(s, y) \eta_k(y)) dy \right)^2 \right)^{p/2} dx \\
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)|^2 (\sigma(s, y, u(s, y)) - \sigma(s, y, v(s, y)))^2 dy \right)^{p/2} dx \\
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(y)|^2 (u(s, x - y) - v(s, x - y))^2 dy \right)^{p/2} dx \\
\leq \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(y)|^2 dy \right)^{p/2} \int_{\mathbb{R}} |u(s, x) - v(s, x)|^p dx
\]

(4.10)

on \((\omega, t) \in (0, \tau].\) Since Remark 2.2 implies

\[
\int_{\mathbb{R}} |R_{1/2+\kappa}(y)| dy + \int_{\mathbb{R}} |R_{1/2+\kappa}(y)|^2 dy < \infty,
\]

\(
\text{Springer}
\)
by taking integration with respect to \((\omega, t)\) and applying Lemma 2.1 (vi), we have

\[
\| (u_{n_+}^{1+\lambda} h_m(u))_x - (v_{n_+}^{1+\lambda} h_m(v))_x \|_{\mathbb{H}^{1/2-\kappa}(\tau)}^p + \| \sigma(u) \eta - \sigma(v) \eta \|_{\mathbb{H}^{1/2-\kappa}(\tau, \ell_2)}^p \\
\leq N_m \| u - v \|_{L^p(\tau)}^p \\
\leq \varepsilon \| u - v \|_{\mathbb{H}^{1/2-\kappa}(\tau)}^p + N_\varepsilon \| u - v \|_{\mathbb{H}^{1/2-\kappa}(\tau)}^p
\]

for any \(\varepsilon > 0\). Therefore, Theorem 4.2 yields that there exists \(u_m \in \mathcal{H}^{1/2-\kappa}_p(\tau)\) satisfying equation (4.6).

To prove \(u_m \geq 0\), take a sequence of functions \(\{u^n_0 \in U^1_p : u^n_0 \geq 0, n \in \mathbb{N}\}\) such that \(u^n_0 \to u_0\) in \(U^{1/2-\kappa}_p\). By Theorem 4.2, there exists a unique solution \(u^n_m \in \mathcal{H}^1_p(\tau)\) satisfying equation

\[
du^n_m = \left( au^n_{m,xx} + bu^n_{mx} + cu^n_m + b \left( (u_{n_+}^{1+\lambda} h_m(u^n_m))_x \right) \right) dt + \sum_{k=1}^n \sigma(u^n_m) \eta_k dw^k_t
\]

on \((t, x) \in (0, \tau) \times \mathbb{R}\) with initial data \(u^n_m(0, \cdot) = u^n_0(\cdot)\). Again, by Theorem 4.2,

\[
\| u_m - u^n_m \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p - N \| u_0 - u^n_0 \|_{U^{1/2-\kappa}_p}\]

\[
\leq N \| b \left( (u_{n_+}^{1+\lambda} h_m(u^n_m))_x \right) - b \left( (u_{n_+}^{1+\lambda} h_m(u^n_m))_x \right) \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p \\
+ N \| \sigma(u_m) \eta - \sigma(u^n_m) \eta \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p \\
\leq N \| u_m - u^n_m \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p + N \| \sigma(u_m) - \sigma(u^n_m) \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p \]

\[
\leq N \| u_m - u^n_m \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p + N \| \sigma(u_m) \eta 1_{k>n} \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p
\]

where \(N = N(m, d, p, \kappa, K, T)\). Then, Theorem 2.1 (iii) implies

\[
\| u_m - u^n_m \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p \leq N \| u_0 - u^n_0 \|_{U^{1/2-\kappa}_p}^p + N \| \sigma(u_m) \eta 1_{k>n} \|_{\mathcal{H}^{1/2-\kappa}(\tau, \ell_2)}^p
\]

where \(N = N(m, d, p, \kappa, K, T)\). Note that

\[
\| \sigma(u_m) \eta 1_{k>n} \|_{\mathcal{H}^{1/2-\kappa}(\ell_2)}^p \leq N \int_{\mathbb{R}} \left( \sum_{k>n} \left[ \int_{\mathbb{R}} R_{1/2+\kappa}^2(x-y) \sigma(t, y, u_m(t, y)) \eta_k(y) dy \right]^2 \right)^{p/2} dx.
\]
Also,

\[
\sum_{k>n} \left[ \int_{\mathbb{R}} R_{1/2+k}(x-y) \sigma(t, y, u_m(t, y)) \eta_k(y) dy \right]^2 \\
\leq \sum_{k} \left[ \int_{\mathbb{R}} R_{1/2+k}(x-y) \sigma(t, y, u_m(t, y)) \eta_k(y) dy \right]^2 \\
\leq \int_{\mathbb{R}} R_{1/2+k}(x-y) \left| \sigma(t, y, u_m(t, y)) \right|^2 dy \\
\leq K^2 \int_{\mathbb{R}} R_{1/2+k}(y) \left| u_m(t, x-y) \right|^2 dy.
\]

Thus, by the dominated convergence theorem, we have

\[
\| \sigma(t, \cdot, (u_m(t, \cdot)) \eta 1_{k>n}) \|_{H_p^{-1/2-\kappa}(\ell_2)}^p \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]

for almost all \((\omega, t)\). Then, by using the dominated convergence theorem again, we have

\[
\| \sigma(t, \cdot, (u_m(t, \cdot)) \eta 1_{k>n}) \|_{\mathbb{E}H_p^{-1/2-\kappa}(\tau, \ell_2)}^p \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Thus, it suffices to show that \(u_m^n \geq 0\). Since \(u_m^n \in \mathcal{H}_p^1(\tau)\), By applying [32, Theorem 2.5] with \(f = \bar{b}((u_m^n)^{1+\lambda}h_m(u_m^n))\), we have \(u_m^n \geq 0\). The lemma is proved. \(\square\)

The following lemma provides \(L_1\) bound of local solution \(u_m\) on \(\Omega \times (0, T) \times \mathbb{R}\). This bound is used to control the noise-related part of the local solution; see (4.18). Recall that \(h(\cdot) \in C_c^\infty(\mathbb{R})\) is a nonnegative function satisfying \(h(z) = 1\) on \(|z| \leq 1\) and \(h(z) = 0\) on \(|z| \geq 2\). Also, \(h_m\) is defined as \(h_m(z) := h(z/m)\).

**Lemma 4.2** Suppose all the assumptions of Lemma 4.1 hold for \(\tau = T\) and we assume that \(u_0 \in U_p^{1/2-\kappa} \cap L_1(\Omega; L_1)\). Let \(u_m \in \mathcal{H}_p^{1/2-\kappa}(T)\) be the solution to equation (4.6) introduced in Lemma 4.1. Then, we have

\[
\mathbb{E} \int_0^T \| u_m(t, \cdot) \|_{L_1} dt \leq N \| u_0 \|_{L_1(\Omega; L_1)}
\]

where \(N = N(K, T)\).

**Proof** Note that \(h_k(x) = h(x/k) \in C_c^\infty(\mathbb{R})\) and \(h_k(x) \rightarrow 1\) as \(k \rightarrow \infty\). For a bounded stopping time \(\tau \leq T\) and \(k \in \mathbb{N}\), by employing Itô’s formula, expectation,
and integration by parts, we have

\[
\mathbb{E} e^{-4K\tau} (u_m(\tau, \cdot), h_k)
= \mathbb{E}(u_0, h_k) + k^{-2} \mathbb{E} \int_0^\tau \int_{\mathbb{R}} u_m(s, x) a(s, x) h_{xx}(x/k)e^{-4Ks} dx ds
+ k^{-1} \mathbb{E} \int_0^\tau \int_{\mathbb{R}} u_m(s, x) \left(2a_x - b - \bar{b}(u_m)^\gamma h_m(u_m)\right)(s, x) h_x(x/k) e^{-4Ks} dx ds
+ \mathbb{E} \int_0^\tau \int_{\mathbb{R}} u_m(s, x) \left(a_{xx}(s, x) - b_x(s, x) + c(s, x) - 4K\right) h(x/k)e^{-4Ks} dx ds.
\]

(4.12)

Since \(u_0 \in L_1(\Omega; L_1)\) and (3.3), we have

\[
e^{-4KT} \mathbb{E}(u_m(\tau, \cdot), h_k)
\leq \mathbb{E} \int_{\mathbb{R}} u_0(x) dx + k^{-2} N \mathbb{E} \int_0^\tau \int_{\mathbb{R}} |u_m(s, x)||h_{xx}(x/k)| dx ds
+ k^{-1} N \mathbb{E} \int_0^\tau \int_{\mathbb{R}} |u_m(s, x)||h_x(x/k)| dx ds.
\]

(4.13)

Let \(q := \frac{p}{p-1}\). By Hölder’s inequality and changing of variable, we have

\[
k^{-2} \mathbb{E} \int_0^\tau \int_{\mathbb{R}} |u_m(s, x)||h_{xx}(x/k)| dx ds
\leq k^{-2} ||u_m||_{L_p(\tau)} \left( \int_{\mathbb{R}} |h_{xx}(x/k)|^q dx \right)^{1/q}
\leq k^{-2+1/q} N_m
\]

and

\[
k^{-1} \mathbb{E} \int_0^\tau \int_{\mathbb{R}} |u_m(s, x)||h_x(x/k)| dx ds
\leq k^{-1} ||u_m||_{L_p(\tau)} ||h_x(\cdot/k)||_{L_q}
\leq k^{-1+1/q} N_m ||h_x||_{L_q}
\leq k^{-1+1/q} N_m.
\]

Therefore, we have

\[
\mathbb{E}(u_m(\tau, \cdot), h_k) \leq \mathbb{E}\|u_0\|_{L_1(\Omega \times \mathbb{R})} e^{4KT} + k^{-1+1/q} N_m e^{4KT},
\]

(4.14)

where \(N_M\) is independent of \(k\). By letting \(\tau = t\) and taking integration on \(t \in (0, T)\), we have

\(\oplus\) Springer
\[ \mathbb{E} \int_0^T (u_m(t, \cdot), h_k) dt \leq T \mathbb{E} \int_{\mathbb{R}} u_0(x) dx e^{4KT} + k^{-1+1/q} N_m T e^{4KT}, \]

where \( N_m \) is independent of \( k \). By letting \( k \to \infty \), we have

\[ \mathbb{E} \int_0^T \| u_m(s, \cdot) \|_{L^1} ds \leq N(K, T) \mathbb{E} \| u_0 \|_{L^1}. \]

The lemma is proved. \( \square \)

**Remark 4.1** Now we introduce a global solution candidate. Let \( T \in \mathbb{N}, \kappa \in (0, 1/2), p > \frac{6}{1-2\kappa} \) and \( m \in \mathbb{N} \). Suppose \( u_m \in \mathcal{H}^{1/2-\kappa}_p (T) \) is the solution introduced in Lemma 4.1. By Corollary 2.1, we have \( u_m \in C([0, T]; C(\mathbb{R})) \) (a.s.). Thus, for \( R \in \{1, 2, \ldots, m\} \), we can set \( \tau^R_m \)

\[ \tau^R_m := \inf \left\{ t \in [0, T] : \sup_{x \in \mathbb{R}} |u_m(t, x)| \geq R \right\}. \quad (4.15) \]

Note that

\[ \tau^R_m \leq \tau^m_m. \quad (4.16) \]

Indeed, if \( R = m \), (4.16) is obvious. If \( R < m \), we have \( u_m \wedge m = u_m \wedge m \wedge R = u_m \wedge R \) for \( t \leq \tau^R_m \). Therefore, \( u_m \) and \( u_R \) satisfy

\[ du = \left( a u_{xx} + bu_x + cu + \tilde{b} \left( u^{1+\kappa} h_R(u) \right)_x \right) dt + \sigma(u) \eta_k dw^k_t, \quad 0 < t \leq \tau^R_m \]

with the initial data \( u(0, \cdot) = u_0 \). On the other hand, \( u_R \wedge R = u_R \wedge R \wedge m = u_R \wedge m \) for \( t \leq \tau^R_R \). Thus, \( u_m \) and \( u_R \) satisfy

\[ du = \left( a u_{xx} + bu_x + cu + \tilde{b} \left( u^{1+\kappa} h_m(u) \right)_x \right) dt + \sigma(u) \eta_k dw^k_t, \quad 0 < t \leq \tau^R_R \]

with the initial data \( u(0, \cdot) = u_0 \). Notice that the uniqueness result in Lemma 4.1 yields that \( u_m = u_R \) for all \( t \leq (\tau^R_m \vee \tau^R_R) \wedge T \), for any positive integer \( T \). Also, note that \( \tau^R_R = \tau^R_m \) (a.s.). Indeed, for \( t < \tau^R_m \),

\[ \sup_{s \leq t} \sup_{x \in \mathbb{R}} |u_R(s, x)| = \sup_{s \leq t} \sup_{x \in \mathbb{R}} |u_m(s, x)| \leq R, \]

and this implies \( \tau^R_m \leq \tau^R_R \). Similarly, \( \tau^R_m \geq \tau^R_R \). Besides, we have \( \tau^R_m \leq \tau^m_m \) since \( m > R \). Therefore, \( \tau^R_R \leq \tau^m_m \).

Define

\[ u := u_m \mathbb{1}_{[0, \tau^m_m \wedge T]}(t) \]
and set \( \tau_\infty := \lim \sup_{m \to \infty} \lim \sup_{T \to \infty} \tau_m^m \wedge T \). It should be remarked that \( u(t, x) \) is well-defined on \( \Omega \times [0, \infty) \times \mathbb{R} \) and the nontrivial domain of \( u \) is \( \Omega \times [0, \tau_\infty) \times \mathbb{R} \).

The following lemma introduces a noise dominating part of the solution and the estimate of it.

**Lemma 4.3** Let \( T \in (0, \infty) \), \( \kappa \in (0, 1/2) \), and \( p > \frac{6}{1 - 2\kappa} \). If \( u_0 \in U^1_{p, \kappa}(T) \), there exists \( v \in H^{1/2-\kappa}_p(T) \) such that

\[
dv = (av_{xx} + bv_x + cv)\,dt + \sigma(u)\eta d\omega^k \quad (t, x) \in (0, T) \times \mathbb{R}; \quad v(0, \cdot) = u_0.
\]

Furthermore, we have

\[
\mathbb{E} \sup_{t \leq T, x \in \mathbb{R}} |v(t, x)|^p \leq N(p, \kappa, K, T) \left( \|u_0\|_{U^1_{p, \kappa}}^p + \mathbb{E}\|u_0\|_{L^1} \right) \tag{4.18}
\]

and

\[
\mathbb{E} \sup_{t \leq T} \|v(t, \cdot)\|_{L^1}^{1/2} \leq N(K, T)\mathbb{E}\|u_0\|_{L^1}^{1/2}. \tag{4.19}
\]

**Proof** Let \( T \in (0, \infty) \). First, we show that there exists \( v \in H^{1/2-\kappa}_p(T) \) satisfying (4.17) and (4.18). To employ Theorem 4.2, we need to show

\[
\|\sigma(u)\eta\|_{H^{1/2-\kappa}_p(T, \ell_2)}^p < \infty.
\]

Set \( \tau_m^m \) as in (4.15). Then, Remark 2.2, Remark 4.1, and Fatou’s lemma yield

\[
\|\sigma(u)\eta\|_{H^{1/2-\kappa}_p(T, \ell_2)}^p = \mathbb{E} \int_0^T \left\| (1 - \Delta)^{1/2+\kappa} (\sigma(s, u(s))\eta) (\cdot) \right\|_{L^p}^p \, ds
\]

\[
\leq \mathbb{E} \int_0^{\tau_\infty \wedge T} \int_{\mathbb{R}} \left( \sum_k \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)\sigma(s, y, u(s, y))\eta(y)dy\right)^2 \right)^{p/2} \, dx ds
\]

\[
= \mathbb{E} \int_0^{\tau_\infty \wedge T} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)\sigma(s, y, u(s, y))\eta(y)dy\right)^2 \, dx ds
\]

\[
\leq \liminf_{m \to \infty} \mathbb{E} \int_0^{\tau_m^m \wedge T} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)\sigma(s, y, u_m(s, y))\eta(y)dy\right)^2 \, dx ds
\]

\[
\leq \liminf_{m \to \infty} \mathbb{E} \int_0^{\tau_m^m \wedge T} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)\sigma(s, y, u_m(s, y))\eta(y)dy\right)^2 \, dx ds
\]

\[
\leq \liminf_{m \to \infty} \mathbb{E} \int_0^{\tau_m^m \wedge T} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)\sigma(s, y, u_m(s, y))\eta(y)dy\right)^2 \, dx ds
\]

\[
\leq \liminf_{m \to \infty} \mathbb{E} \int_0^{\tau_m^m \wedge T} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x - y)\sigma(s, y, u_m(s, y))\eta(y)dy\right)^2 \, dx ds
\]
Also, by Remark 2.2, (3.4), Minkowski’s inequality, and (4.11), we have
\[
\mathbb{E} \int_0^{\tau_m} \left( \int_{\mathbb{R}} \left| R_{1/2+\kappa}(x-y) \right|^2 (\sigma(s, y, u_m(s, y)))^2 dy \right)^{p/2} dx ds \\
\leq K^p \mathbb{E} \int_0^{\tau_m} \left( \int_{\mathbb{R}} \left| R_{1/2+\kappa}(y) \right|^2 (u_m(s, x-y))^2 dy \right)^{p/2} dx ds \tag{4.21}
\]
\[
\leq K^p \left( \int_{\mathbb{R}} \left| R_{1/2+\kappa}(y) \right|^2 dy \right)^{p/2} \mathbb{E} \int_0^{\tau_m} \int_{\mathbb{R}} |u_m(s, x)| dx ds \\
\leq N \|u_0\|_{L_1(\Omega; L_1)},
\]
where \( N = N(p, \kappa, K, T) \). Since \( N \) is independent of \( m \), (4.20) and (4.21) imply
\[
\|\sigma(u)\eta\|_{\mathbb{E}^{p-1/2-\kappa}(T, \ell_2)}^p \leq N \|u_0\|_{L_1(\Omega; L_1)}, \tag{4.22}
\]
where \( N = N(p, \kappa, K, T) \). Therefore, by Theorem 4.2, there exists \( v \in \mathcal{H}_{p}^{1/2-\kappa}(T) \) satisfying (4.17) and
\[
\|v\|_{\mathcal{H}_{p}^{1/2-\kappa}(T)} \leq N \|u_0\|_{U_p^{1/2-\kappa}} + N \|\sigma(u)\eta\|_{\mathbb{E}^{p-1/2-\kappa}(T, \ell_2)},
\]
where \( N = N(p, \kappa, K, T) \). Thus, by (4.22) and Theorem 2.1 (ii), we have (4.18).

To show (4.19), follow the proof of Lemma 4.2. Then, instead of (4.14), for any bounded stopping time \( \tau \leq T \),
\[
\mathbb{E}(v(\tau, \cdot), h_k) \leq T \mathbb{E} \int_{\mathbb{R}} u_0(x) dx e^{AKT} + k^{-1+1/q} N e^{AKT},
\]
where \( N = N(m, p, K, T) \). Then, (e.g. [33, Theorem III.6.8])
\[
\mathbb{E} \sup_{t \leq T} \|v(t, \cdot)\|_{L_1}^{1/2} \leq N(K, T) \mathbb{E} \|u_0\|_{L_1}^{1/2}.
\]

The lemma is proved. \( \square \)

The following lemma provides an estimate of the nonlinear part of the local solution;
\( u_m := u_m - v \).

**Lemma 4.4** Suppose all the assumptions on Theorem 3.3 hold. Assume that \( u_m \in \mathcal{H}_{p}^{1/2-\kappa}(T) \) and \( v \in \mathcal{H}_{p}^{1/2-\kappa}(T) \) are solutions introduced in Lemmas 4.1 and 4.3, respectively. Let \( \tau_m \) be a stopping time defined as in (4.15). Then, for any stopping time \( \tau \leq \tau_m \),
\[
\int_{\mathbb{R}} |u_m(\tau \land t, x) - v(\tau \land t, x)|^p dx \\
\leq N(p, K, T) \left( \sup_{t \leq \tau, x \in \mathbb{R}} |v(t, x)|^{p+\lambda-2 \over 2-\kappa} + \sup_{t \leq \tau, x \in \mathbb{R}} |v(t, x)|^{p(1+\lambda)-2 \over 1} \right) \sup_{t \leq \tau} \|v(t, \cdot)\|_{L_1}, \tag{4.23}
\]
\( \odot \) Springer
for all \( t \leq \tau \) almost surely.

**Proof** Let \( \tau \leq \tau_m^n \) and set \( w_m := u_m - v \). Then, for any \( \phi \in S \),

\[
(w_m(t, \cdot), \phi) = \int_0^t (w_m(s, \cdot), (a(s, \cdot)\phi)_{xx} - (b(s, \cdot)\phi)_x + c(s, \cdot)\phi)ds \tag{4.24}
\]

for all \( t \leq \tau \) almost surely. Choose a nonnegative function \( \zeta \in C_c^\infty(\mathbb{R}) \) such that \( \int_\mathbb{R} \zeta(x)dx = 1 \). For \( x \in \mathbb{R} \) and \( \varepsilon > 0 \), by plugging in \( \phi(\cdot) := \frac{1}{\varepsilon} \zeta\left(\frac{\cdot - x}{\varepsilon}\right) \) to (4.24), we have

\[
w_m^{(\varepsilon)}(t, x) = \int_0^t (aw_m^{(\varepsilon)}_{xx}(s, x) - 2(a_x w_m^{(\varepsilon)}(s, x) + (a_{xx} w_m^{(\varepsilon)}(s, x)ds
\]

\[
- \int_0^t (bw_m^{(\varepsilon)}_x(s, x) + (b_x w_m^{(\varepsilon)}(s, x)ds
\]

\[
+ \int_0^t (cw_m^{(\varepsilon)}(s, x) + \tilde{b}(s) \left((w_m(s, \cdot) + v(s, \cdot))^{1+\lambda}\right)(s, x)ds
\]

for all \( t \leq \tau \) almost surely. Let \( M \in \mathbb{N} \) be specified later. Then, chain rule, integration with respect to \( x \), and integration by parts imply

\[
e^{-M\tau \Lambda_1} \int_{\mathbb{R}} \left| w_m^{(\varepsilon)}(\tau \wedge t, x) \right|^p dx
\]

\[
= -p \int_0^{\tau \wedge t} \int_{\mathbb{R}} \left( \left| w_m^{(\varepsilon)}(s, x) \right|^{p-1} \right)_x (aw_m^{(\varepsilon)}(s, x)e^{-Ms}dxds
\]

\[
- p \int_0^{\tau \wedge t} \int_{\mathbb{R}} \left( \left| w_m^{(\varepsilon)}(s, x) \right|^{p-1} \right)_x ((2a_x - b) w_m^{(\varepsilon)}(s, x)e^{-Ms}dxds
\]

\[
+ p \int_0^{\tau \wedge t} \int_{\mathbb{R}} \left| w_m^{(\varepsilon)}(s, x) \right|^{p-1} ((a_{xx} - b_x + c) w_m^{(\varepsilon)}(s, x)e^{-Ms}dxds
\]

\[
- \frac{1}{2} p \int_0^{\tau \wedge t} \int_{\mathbb{R}} \left( \left| w_m^{(\varepsilon)}(s, x) \right|^{p-1} \right)_x ((w_m(s, \cdot) + v(s, \cdot))^{1+\lambda})^{(\varepsilon)}(s, x)\tilde{b}(s)e^{-Ms}dxds
\]

\[
- M \int_0^{\tau \wedge t} \int_{\mathbb{R}} \left| w_m^{(\varepsilon)}(s, x) \right|^p e^{-Ms}dxds
\]

for all \( t > 0 \) almost surely. Notice that (3.3) and (3.2) imply

\[
\left| (aw_m)^{(\varepsilon)}_x(s, x) - a(s, x)w_m^{(\varepsilon)}_x(s, x) \right|
\]

\[
= \varepsilon^{-1} \int_{\mathbb{R}} (a(s, x - \varepsilon y) - a(s, x))w_m(s, x - \varepsilon y)\zeta'(y)dy \tag{4.26}
\]

\[
\leq N(K) \int_{\mathbb{R}} |w_m(s, x - \varepsilon y)||y\zeta'(y)|dy
\]
and
\[ - \int_{\mathbb{R}} a(s, x) \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx \leq -K^{-1} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx. \]  
(4.27)

Thus, by (4.26), (4.27), and Young’s inequality,
\[ - \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx \leq - \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx \]
\[ = - \int_{\mathbb{R}} a(s, x) \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx \]
\[ \leq N(K) \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx \int_{\mathbb{R}} \left| w_m(s, x - \varepsilon y) |y \zeta'(y)|dy \right|^2 dx \]
\[ - K^{-1} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx \]
\[ \leq N \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left( \int_{\mathbb{R}} \left| w_m(s, x - \varepsilon y) |y \zeta'(y)|dy \right|^2 \right) dx \]
\[ - \frac{1}{2} K^{-1} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 dx. \]  
(4.28)

Besides, we have
\[ \left| (2a_x - b) w_m^{(e)}(s, x) \right| = \int_{\mathbb{R}} (2a_y(s, y) - b(s, y)) w_m(s, y) \zeta_e(x - y)dy \]
\[ \leq K \int_{\mathbb{R}} |w_m(s, y)| \zeta_e(x - y)dy \]
\[ = K(|w_m(s, \cdot)|)^{(e)}(x) \]
and \[ \left| (\alpha_{xx} - b_x + c) w_m^{(e)}(s, x) \right| \leq K(|w_m(s, \cdot)|)^{(e)}(x). \]
Thus, by Young’s inequality, we have
\[ - p(p - 1) \int_{0}^{\tau t} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 e^{-Ms} dx ds \]
\[ + p \int_{0}^{\tau t} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-1} \left( \alpha_{xx} - b_x + c \right) w_m^{(e)}(s, x) e^{-Ms} dx ds \]
\[ \leq \frac{1}{8} K^{-1} p(p - 1) \int_{0}^{\tau t} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left| w_{mx}^{(e)}(s, x) \right|^2 e^{-Ms} dx ds \]  
(4.29)

\[ + N \int_{0}^{\tau t} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-2} \left( |w_m(s, \cdot)| \right)^{(e)}(x) e^{-Ms} dx ds \]
\[ + N \int_{0}^{\tau t} \int_{\mathbb{R}} \left| w_m^{(e)}(s, x) \right|^{p-1} \left( |w_m(s, \cdot)| \right)^{(e)}(x) e^{-Ms} dx ds. \]
Furthermore, since \( p - 1 + \lambda \in 2\mathbb{N} \), we have
\[
\int_\mathbb{R} |u^{(e)}_m(s, x)|^{p-1+\lambda} u^{(e)}_{mx}(s, x) dx = \int_\mathbb{R} \left( w^{(e)}_m(s, x) \right)^{p-1+\lambda} u^{(e)}_{mx}(s, x) dx \\
= \frac{1}{p + \lambda} \int_\mathbb{R} \left( \left( w^{(e)}_m(s, x) \right)^{p+\lambda} \right) dx \\
= 0
\]
for \( s \leq \tau \). Thus, Young’s inequality yields that
\[
\int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-2} u^{(e)}_{mx}(s, x) \left( (w_m(s, \cdot) + v(s, \cdot))^{1+\lambda} \right)^{(e)}(x) dx \\
= - \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-2} u^{(e)}_{mx}(s, x) \Phi^{(e)}(w_m, v, s, x) dx \\
\leq \frac{1}{4K} \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-2} \left| u^{(e)}_{mx}(s, x) \right|^2 dx + N \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-2} \Phi^2_e(w_m, v, s, x) dx, \tag{4.30}
\]
where \( \Phi_e(w_m, v, s, x) = \left( (w_m(s, \cdot) + v(s, \cdot))^{1+\lambda} \right)^{(e)}(x) - \left| u^{(e)}_m(s, x) \right|^{1+\lambda} \). Therefore, applying (4.28), (4.29), and (4.30) to (4.25), we have
\[
e^{-M\tau} \int_\mathbb{R} \left| u^{(e)}_m(t \wedge t, x) \right|^p dx \\
\leq N \int_0^{\tau \wedge t} \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-2} \left( \int_\mathbb{R} \left| w_m(s, x - \varepsilon y) \right| |y\xi'(y)| dy \right)^2 dx ds \\
+ N \int_0^{\tau \wedge t} \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-2} \left| \left| w_m(s, \cdot) \right|^{(e)}(x) \right|^2 e^{-M\varepsilon} dx ds \\
+ N \int_0^{\tau \wedge t} \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-1} \left| \left| w_m(s, \cdot) \right|^{(e)}(x) \right| e^{-M\varepsilon} dx ds \\
+ N \int_0^{\tau \wedge t} \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^{p-2} \Phi^2_e(w_m, v, s, x) e^{-M\varepsilon} dx ds \\
- M \int_0^{\tau \wedge t} \int_\mathbb{R} \left| u^{(e)}_m(s, x) \right|^p e^{-M\varepsilon} dx ds.
\]
Thus, by letting \( \varepsilon \downarrow 0 \), we have
\[
e^{-M\tau} \int_\mathbb{R} \left| w_m(t \wedge t, x) \right|^p dx \\
\leq (N - M) \int_0^{\tau \wedge t} \int_\mathbb{R} \left| w_m(s, x) \right|^p e^{-M\varepsilon} dx ds \\
+ N \int_0^{\tau \wedge t} \int_\mathbb{R} \left| w_m(s, x) \right|^{p-2} \left( (w_m(s, x) + v(s, x))^{1+\lambda} - (w_m(s, x))^{1+\lambda} \right)^2 e^{-M\varepsilon} dx ds
\]
\( \odot \) Springer
Choose $M = 2N$. Since $|a^{1+\lambda} - b^{1+\lambda}| \leq N(\lambda)|a - b|(a^{\lambda} + b^\lambda)$ for $\lambda, a, b \in [0, \infty)$, Young’s inequality implies that

$$e^{-M\tau \wedge t} \int_\mathbb{R} |w_m(\tau \wedge t, x)|^p \, dx$$

$$\leq N \int_0^{\tau \wedge t} \int_\mathbb{R} |w_m(s, x)|^{p-2} \left| (w_m(s, x) + v(s, x))^{1+\lambda} - (w_m(s, x))^{1+\lambda} \right|^2 \, dx \, ds$$

$$e^{-Ms} \int_\mathbb{R} w_m(s, x) \, dx$$

$$\leq N \int_0^{\tau \wedge t} \int_\mathbb{R} |w_m(s, x)|^{p-2} (2|w_m(s, x)|^{\lambda} + |v(s, x)|^{\lambda}) |v(s, x)| e^{-Ms} \, dx \, ds$$

$$\leq N \int_0^{\tau \wedge t} \int_\mathbb{R} |w_m(s, x)|^p e^{-M\tau \wedge s} \, dx \, ds$$

$$+ N \int_0^{\tau \wedge t} \int_\mathbb{R} (|v(s, x)|^{\frac{p}{\lambda}} + |v(s, x)|^{\frac{p(1+\lambda)}{2}}) e^{-Ms} \, dx \, ds$$

$$\leq N \int_0^{\tau \wedge t} \int_\mathbb{R} |w_m(\tau \wedge s, x)|^p e^{-M\tau \wedge s} \, dx \, ds$$

$$+ N \left( \sup_{t \leq \tau, x \in \mathbb{R}} |v(t, x)|^{\frac{p-2+\lambda}{\lambda}} + \sup_{t \leq \tau, x \in \mathbb{R}} |v(t, x)|^{\frac{p(1+\lambda)-2}{2}} \right)$$

$$\times \int_0^{\tau \wedge t} \int_\mathbb{R} |v(s, x)| e^{-Ms} \, dx \, ds$$

for all $t > 0$ almost surely. Recall that $w_m := u_m - v \in \mathcal{H}_p^{1/2-\kappa}(\tau_m^m)$. By Theorem 2.1 (ii), we have $w_m \in C([0, \tau_m^m]; L_p)$ (a.s.). For almost sure $\omega$, by Grönwall’s inequality, we have (4.23). The lemma is proved. \hfill \Box

By combining Lemmas 4.3 and 4.4, we show that solution candidate $u$ is non-explosive.

**Lemma 4.5** All the conditions of Theorem 3.3 hold. Let $u$ be the function introduced in Remark 4.1. Then, for any $T < \infty$, we have

$$\lim_{R \to \infty} P \left( \left\{ \omega \in \Omega : \sup_{t \leq T, x \in \mathbb{R}} |u(t, x)| > R \right\} \right) = 0. \quad (4.31)$$

**Proof** Let $T < \infty$. Suppose $v$ is the solution to (4.17) introduced in Lemma 4.3. For $m, S > 0$, define

$$\tau^1(S) := \inf \left\{ t \geq 0 : \|v(t, \cdot)\|_{L_1} \geq S \right\}, \quad \tau^2(S) := \inf \left\{ t \geq 0 : \sup_{x \in \mathbb{R}} |v(t, x)| \geq S \right\},$$

and

$$\tau := \tau(m, T) := \tau_m^m \wedge \tau^1(S) \wedge \tau^2(S) \wedge T,$$
where \( \tau_m^m \) is defined as in (4.15). By Lemma 4.3, \( \tau^1(S) \) and \( \tau^2(S) \) are well-defined stopping times and thus \( \tau \) is a stopping time. Then, by Theorem 4.2, Hölder’s inequality, and Minkowski’s inequality, we have

\[
\|u_m\|_{L^p(\tau)}^{p/(2-\kappa)} - N \|u_0\|_{L^p(\tau)}^{p/(2-\kappa)} \\
\leq N \left( \int_0^\tau \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x)|^2 dx \right)^{p/2} \int_{\mathbb{R}} |u_m(s, x)|^p dx ds \right)^{1+\lambda} \\
\leq N \left( \|R_1\|_{L^p(\tau)}^{p/(2-\kappa)} \int_{\mathbb{R}} |u_m(s, x)|^p dx ds \right)^{1+\lambda} \\
\leq N \left( \|R_1\|_{L^p(\tau)}^{p/(2-\kappa)} \int_{\mathbb{R}} |u_m(s, x)|^p dx ds \right)^{1+\lambda} \\
+ N \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x)|^2 dx \right)^{p/2} \int_{\mathbb{R}} |u_m(s, x)|^p dx ds,
\]

(4.32)

where \( N = N(p, \kappa, K, T) \). Since \( \frac{p}{p-\kappa} \left( \frac{1}{2} - \kappa \right) < 1 \) and \( 2 \left( \frac{1}{2} - \kappa \right) < 1 \), by Remark 2.2, we have

\[
\left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x)|^\frac{p}{p-\kappa} dy \right)^{p-\lambda} + \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x)|^2 dx \right)^{p/2} < \infty.
\]

(4.33)

Besides, since \( s \leq \tau \) and \( \tau \leq \tau_m^m \), Lemma 4.4 yields

\[
\int_{\mathbb{R}} |u_m(s, x)|^p dx \leq N(p) \left( \int_{\mathbb{R}} |u_m(s, x) - v(s, x)|^p dx + \int_{\mathbb{R}} |v(s, x)|^p dx \right) \\
\leq N(p, K, T) \Theta(v) \sup_{s \leq \tau} \|v(s, \cdot)\|_{L^1} \\
\leq N(S, p, \lambda, K, T),
\]

for all \( s \leq \tau \) almost surely, where

\[
\Theta(v) = \sup_{s \leq \tau, x \in \mathbb{R}} |v(s, x)|^{\frac{p+\lambda-2}{2-\kappa}} + \sup_{s \leq \tau, x \in \mathbb{R}} |v(s, x)|^{\frac{p(1+\lambda)-2}{2}} + \sup_{s \leq \tau, x \in \mathbb{R}} |v(s, x)|^{p-1}.
\]
Therefore, by (4.32), (4.33), and (4.34), we have

\[
\|u_m\|_{H^{1/2-\kappa}\tau}^p \leq N(p, \kappa, K, T)\|u_0\|_{H^{1/2-\kappa}\tau}^p + N(S, p, \kappa, \lambda, K, T).
\]

(4.35)

It should be remarked that \( N \) is independent of \( m \). Therefore, by Corollary 2.1, \( \mathbb{E} \sup_{t \leq \tau, x} |u_m(t, x)|^p \) is bounded by the right hand side of (4.35). Thus, for any \( R > 0 \), by Chebyshev’s inequality and (4.35), we have

\[
P\left( \sup_{t \leq \tau, x \in \mathbb{R}} |u(t, x)| > R \right) \leq \frac{1}{R^p} \mathbb{E} \sup_{t \leq \tau, x \in \mathbb{R}} |u(t, x)|^p
\]

\[
= \frac{1}{R^p} \mathbb{E} \sup_{t \leq \tau, x \in \mathbb{R}} |u_m(t, x)|^p
\]

\[
\leq \frac{1}{R^p} \|u_m\|_{H^{1/2-\kappa}\tau}^p
\]

\[
\leq \frac{1}{R^p} N(u_0, S, p, \kappa, \lambda, K, T).
\]

(4.36)

On the other hand, by Chebyshev’s inequality and Lemma 4.3, we have

\[
P\left( \tau^1(S) < T \right) + P\left( \tau^2(S) < T \right)
\]

\[
\leq P\left( \sup_{t \leq T} \|v(t, \cdot)\|_{L^1} > S \right) + P\left( \sup_{t \leq T, x \in \mathbb{R}} |v(t, x)| > S \right)
\]

\[
\leq \frac{1}{\sqrt{S}} \mathbb{E} \sup_{t \leq T} \|v\|_{L^1}^{1/2} + \frac{1}{Sp} \mathbb{E} \sup_{t \leq T, x \in \mathbb{R}} |v(t, x)|^p
\]

\[
\leq \frac{1}{\sqrt{S}} N(u_0, p, \kappa, K, T).
\]

(4.37)

Thus, Chebyshev’s inequality, Fatou’s lemma, (4.36), and (4.37) yield

\[
P\left( \sup_{t \leq \tau, x \in \mathbb{R}} |u(t, x)| > R \right) = P\left( \sup_{t \leq \tau_\infty \wedge T, x \in \mathbb{R}} |u(t, x)| > R \right)
\]

\[
\leq P\left( \sup_{t \leq \tau_\infty \wedge \tau^1(S) \wedge \tau^2(S), x \in \mathbb{R}} |u(t, x)| > R \right) + P\left( \tau^1(S) < T \right)
\]

\[
+ P\left( \tau^2(S) < T \right)
\]

\[
\leq \liminf_{m \to \infty} P\left( \sup_{t \leq \tau(m, S, T), x \in \mathbb{R}} |u(t, x)| > R \right) + P\left( \tau^1(S) < T \right)
\]

\[
+ P\left( \tau^2(S) < T \right)
\]

\[
\leq \frac{N_1}{R^p} + \frac{N_2}{\sqrt{S}}.
\]
where \( N_1 = N_1(u_0, S, p, \kappa, \lambda, K, T) \) and \( N_2 = N_2(u_0, p, \kappa, K, T) \). Notice that \( N_1 \) is independent of \( R \), and \( N_2 \) is independent of \( R \) and \( S \). By letting \( R \to \infty \), and \( S \to \infty \) in order, we get (4.31). The lemma is proved. \( \square \)

**Proof of Theorem 3.3** Step 1. (Uniqueness). Suppose \( u, \tilde{u} \in \mathcal{H}_p^{1/2-\kappa} \) are nonnegative solutions of equation (3.1). By Definition 2.5 (iv), there are bounded stopping times \( \tau_n (n = 1, 2, \cdots) \) such that

\[
\tau_n \uparrow \infty \quad \text{and} \quad u, \tilde{u} \in \mathcal{H}_p^{1/2-\kappa}(\tau_n).
\]

Fix \( n \in \mathbb{N} \). Since \( p > \frac{6}{1-2\kappa} \), by Corollary 2.1, we have \( u, \tilde{u} \in C([0, \tau_n]; C(\mathbb{R})) \) (a.s.) and

\[
\mathbb{E} \sup_{t \leq \tau_n} \sup_{x \in \mathbb{R}} |u(t, x)|^p + \mathbb{E} \sup_{t \leq \tau_n} \sup_{x \in \mathbb{R}} |\tilde{u}(t, x)|^p < \infty. \tag{4.38}
\]

For \( m \in \mathbb{N} \), define

\[
\tau_{m,n}^1 := \inf \left\{ t \geq 0 : \sup_{x \in \mathbb{R}} |u(t, x)| \geq m \right\} \land \tau_n, \\
\tau_{m,n}^2 := \inf \left\{ t \geq 0 : \sup_{x \in \mathbb{R}} |\tilde{u}(t, x)| \geq m \right\} \land \tau_n,
\]

and

\[
\tau_{m,n} := \tau_{m,n}^1 \land \tau_{m,n}^2. \tag{4.39}
\]

Due to (4.38), \( \tau_{m,n}^1 \) and \( \tau_{m,n}^2 \) are well-defined stopping times, and thus \( \tau_{m,n} \) is a stopping time. Observe that \( u, \tilde{u} \in \mathcal{H}_p^{1/2-\kappa}(\tau_{m,n}) \) and \( \tau_{m,n} \uparrow \tau_n \) as \( m \to \infty \) almost surely. Fix \( m \in \mathbb{N} \). Notice that \( u, \tilde{u} \in \mathcal{H}_p^{1/2-\kappa}(\tau_{m,n}) \) are solutions to equation

\[
dv = \left( av_{xx} + bv_x + cv + \tilde{b} \left( v^{1+\lambda} h_m(v) \right) \right) dt + \sigma^k(v) \eta_k dw^k_t, \quad 0 < t \leq \tau_{m,n}
\]

with the initial data \( v(0, \cdot) = u_0 \). By the uniqueness result in Lemma 4.1, we conclude that \( u = \tilde{u} \) in \( \mathcal{H}_p^{1/2-\kappa}(\tau_{m,n}) \) for each \( m \in \mathbb{N} \). The monotone convergence theorem yields \( u = \tilde{u} \) in \( \mathcal{H}_p^{1/2-\kappa}(\tau_n) \) and this implies \( u = \tilde{u} \) in \( \mathcal{H}_p^{1/2-\kappa}_{loc} \).

Step 2. (Existence). For \( m \in \mathbb{N} \), define a stopping time \( \tau_{m}^m \) and \( u \) as in Remark 4.1. Let \( T < \infty \). Observe that

\[
\left\{ \omega \in \Omega : \tau_{m}^m < T \right\} \subset \left\{ \omega \in \Omega : \sup_{t \leq \tau_{m}^m} |u(t, x)| \leq \sup_{t \leq T, x \in \mathbb{R}} |u(t, x)| \right\}.
\]
Since \( \sup_{t \leq \tau_m^m, x \in \mathbb{R}} |u(t, x)| = \sup_{t \leq \tau_m^m, x \in \mathbb{R}} |u_m(t, x)| = m \) (a.s.), by Lemma 4.5,

\[
\lim_{m \to \infty} \sup_{m} P \left( \tau_m^m < T \right) \leq \lim_{m \to \infty} P \left( \sup_{t \leq T, x \in \mathbb{R}} |u(t, x)| \geq m \right) \to 0.
\]

Since \( T < \infty \) is arbitrary, this implies \( \tau_m^m \to \infty \) in probability. Since \( \tau_m^m \) is increasing in \( m \), we conclude that \( \tau_m^m \uparrow \infty \) (a.s.).

Lastly, Set \( \tau_m := \tau_m^m \wedge m \). Recall that

\[
u(t, x) := u_m(t, x) \text{ for } t \in [0, \tau_m^m].
\]

Observe that \( |u_m(t)| \leq m \) for \( t \in [0, \tau_m^m] \cap [0, \infty) \) and thus \( u_m \) satisfies (3.1) for all \( t \in [0, \tau_m^m] \cap [0, \infty) \) (a.s.). Since \( u = u_m \) for \( t \in [0, \tau_m^m] \cap [0, \infty) \) and \( u_m \in H_p^{1/2-\kappa}(T) \) for any \( T < \infty \), it follows that \( u \in H_p^{1/2-\kappa}(\tau_m) \) and \( u \) satisfies (3.1) for all \( t \leq \tau_m \) (a.s.). Since \( \tau_m \uparrow \infty \) (a.s.) as \( m \to \infty \), we have \( u \in H_p^{1/2-\kappa} \).

**Step 3. (Hölder regularity).** Let \( T < \infty \). Since \( u \in H_p^{1/2-\kappa}(T) \), by employing Corollary 2.1, we have (3.5). The theorem is proved.

**Proof of Theorem 3.4** The proof of Theorem 3.4 is motivated by [18, Corollary 5.11]. Let \( q > p \). By Theorem 3.3, there exists a unique solution \( \bar{u} \in H_{q, loc}^{1/2-\kappa} \) satisfying equation (3.1). By Definition 2.5 (iv), there exists \( \tau_n \) such that \( \tau_n \to \infty \) (a.s.) as \( n \to \infty \), \( u \in H_{p, loc}^{1/2-\kappa}(\tau_n) \) and \( \bar{u} \in H_{q, loc}^{1/2-\kappa}(\tau_n) \). Fix \( n \in \mathbb{N} \). Since \( \frac{6}{1-2\kappa} < p < q \), we can define \( \tau_{m,n} \) (\( m \in \mathbb{N} \)) as in (4.39). For any \( p_0 > p \), we have

\[
u \in L_{p_0}(\tau_{m,n})
\]

since

\[
E \int_0^{\tau_{m,n}} \int_{\mathbb{R}} |\nu(t, x)|^{p_0} dx dt \leq m^{p_0-p} E \int_0^{\tau_{m,n}} \int_{\mathbb{R}} |\nu(t, x)|^p dx dt < \infty.
\]

Observe that \( \bar{b} \left( u^{1+\lambda} \right)_x \in H_{q}^{-1}(\tau_{m,n}) \). Indeed, by (3.3), Lemma 2.1 (iii), (vii), we have

\[
E \int_0^{\tau_{m,n}} \| \bar{b}(s) \left( (u(s, \cdot))^{1+\lambda} \right)_x \|_{H_{q}^{-1}}^q ds \leq N E \int_0^{\tau_{m,n}} \int_{\mathbb{R}} |u(s, x)|^{q(1+\lambda)} dx ds < \infty.
\]

Also, by Lemma 2.1 (vii) again, we have

\[
a u_{xx} \in H_{q}^{-2}(\tau_{m,n}), \quad b u_x \in H_{q}^{-1}(\tau_{m,n}), \quad \text{and} \quad c u \in L_{q}(\tau_{m,n}).
\]

Therefore, since \( L_{q}(\tau_{m,n}) \subset H_{q}^{-1}(\tau_{m,n}) \subset H_{q}^{-2}(\tau_{m,n}) \), Lemma 2.1 (iv) yields

\[
a u_{xx} + b u_x + c u + b \left( u^{1+\lambda} \right)_x \in H_{q}^{-2}(\tau_{m,n}). \tag{4.40}
\]
By (3.4),

$$
\|\sigma(u)\|_{L^q(\tau_{m,n}, \ell_2)}^q = \mathbb{E} \int_0^{\tau_{m,n}} \int_{\mathbb{R}^d} \left( \sum_k |\sigma^k(s, x, u(s, x))|^2 \right)^{q/2} dx ds \\
\leq N \|u\|_{L^q(\tau_{m,n})}^q < \infty.
$$

(4.41)

Thus, we have

$$
\sigma(u) \in L_q(\tau_{m,n}, \ell_2) \subset H^{-1}_q(\tau_{m,n}, \ell_2).
$$

(4.42)

Due to (4.40) and (4.42), the right hand side of equation (3.1) is $H^{-2}_p$-valued continuous function on $[0, \tau_{m,n}]$. Thus, $u$ is in $L_q(\tau_{m,n})$ and $u$ satisfies (3.1) for all $t \leq \tau_{m,n}$ almost surely with $u(0, \cdot) = u_0(\cdot)$. On the other hand, since $\tilde{b}(u^{1+\lambda}) \in H^{-3/2-\kappa}_q(\tau_{m,n})$ and $\sigma(u) \in H^{-1/2-\kappa}_q(\tau_{m,n}, \ell_2)$, Theorem 4.2 implies that there exists $v \in H^{1/2-\kappa}_q(\tau_{m,n})$ satisfying

$$
dv = \left( av_{xx} + bv_x + cv + \tilde{b}(u^{1+\lambda}) \right) dt + \sigma^k(u) \eta_k dw^k_t, \quad 0 < t \leq \tau_{m,n}
$$

(4.43)

with the initial data $v(0, \cdot) = u_0$. In (4.43), $\tilde{b}(u^{1+\lambda})$ and $\sigma^k(u)$ are used instead of $b(v^{1+\lambda})$ and $\sigma^k(v)$. Since $u \in L_q(\tau_{m,n})$ satisfies equation (4.43), $w := u - v \in L_q(\tau_{m,n})$ satisfies

$$
dw = (aw_{xx} + bw_x + cw) dt, \quad 0 < t \leq \tau_{m,n}; \quad w(0, \cdot) = 0.
$$

By Theorem 4.2, we have $w = 0$ and thus $u = v$ in $L_q(\tau_{m,n})$. Therefore, $u$ is in $H^{1/2-\kappa}_q(\tau_{m,n})$. Note that $\tilde{u} \in H^{1/2-\kappa}_q(\tau_{m,n})$ satisfies equation (3.1). By Theorem 4.1, we have $u = \tilde{u}$ in $H^{1/2-\kappa}_q(\tau_{m,n})$. The theorem is proved.

\qed

5 Proof of the second case: modified Burgers’ equation with the super-linear diffusion coefficient $\sigma(u)$

This section proposes proof of Theorems 3.6 and 3.7. The main idea of the proof for Theorem 3.6 is similar to the proof of Theorem 3.3. However, we deal with the nonlinear terms $|u|^1 u_x$ and $|u|^{1+\lambda_0}$ simultaneously. Intuitively, we understand equation (3.6) as

$$
du = \left( a^{ij} u_{x_i x_j} + b^i u_x^i + cu + \tilde{b}^i (\xi u)_x^i \right) dt + \mu \xi_0 u \eta_k dw^k_t, \quad t > 0; \quad u(0) = u_0,
$$

(5.1)
where $\xi = |u|^{1+\lambda}/u$ and $\xi_0 = |u|^{1+\lambda_0}/u \ (0/0 := 0)$. Then, Theorem 4.2 and (4.32) yield

$$
\|u\|_{H_p^{1/2-\epsilon}(\tau)} \leq N \left( \|u_0\|_{U_p^{1/2-\epsilon}} + \|\tilde{b}\xi u\|_{H_p^{-3/2-\epsilon}} + \|\mu \xi_0 u\|_{H_p^{-1/2-\epsilon}}(\ell_2) \right) 
\leq N \|u_0\|_{U_p^{1/2-\epsilon}} + N \left( \|\xi\|_{L_s}^1 + \|\xi_0\|_{L_{s_0}}^{1/2+\epsilon_0} \right) \|u\|_{L_p}
$$

where $s > 1$ and $s_0 > 1$. Note that the coefficients $\xi \in L_s$ and $\xi_0 \in L_{s_0}$ should be controlled to extend the local solution to a global one. Since we have the uniform $L_1$ bound of $u_m$ (Lemma 5.2), the case $s = 1/\lambda$ and $s_0 = 1/(2\lambda_0)$ is considered. Therefore, the conditions on $\lambda \in (0, 1)$ and $\lambda_0 \in (0, 1/2)$ are required. With the assumptions on $\lambda$ and $\lambda_0$, the non-explosive property of local solutions is proved; see Lemma 5.3.

Recall that $h(\cdot) \in C^\infty_c(\mathbb{R})$ satisfies $h \geq 0$, $h(z) = 1$ on $|z| \leq 1$, and $h(z) = 0$ on $|z| \geq 2$. Besides,

$$
h_m(z) := h(z/m).
$$

The following lemma provides the unique existence of the local solution.

**Lemma 5.1** Let $\lambda, \lambda_0 \in (0, \infty)$, $T \in (0, \infty)$, $\kappa \in (0, 1/2)$, and $p > \frac{6}{1-2\kappa}$. Suppose Assumptions 3.1 and 3.5 hold. Then, for a bounded stopping time $\tau \leq T$, $m \in \mathbb{N}$, and nonnegative initial data $u_0 \in U_p^{1/2-\kappa}$, there exists a unique $u_m \in H_p^{1/2-\kappa}(\tau)$ such that $u_m$ satisfies equation

$$
du = \left( au_{xx} + bu_x + cu + \tilde{b}(u_+^{1+\lambda}h_m(u))_x \right) dt + \mu u_+^{1+\lambda_0}h_m(u)\eta_k dw_k, \quad (5.2)
$$
on $(t, x) \in (0, \tau) \times \mathbb{R}$ with the initial data $u(0, \cdot) = u_0(\cdot)$. Furthermore, $u_m \geq 0$.

**Proof** Follow the proof of Lemma 4.1. However, instead of (4.10), we apply

$$
\left\| \mu u_+^{1+\lambda_0}h_m(u)\eta - \mu v_+^{1+\lambda_0}h_m(v)\eta \right\|_{H_p^{-1/2-\epsilon}(\ell_2)}^p
\leq \int_{\mathbb{R}} \left( \sum_k \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x-y)| \mu(s, y)\Phi_{m,\lambda_0}(u, v, s, y)\eta_k(y)dy \right)^2 \right)^{p/2} dx
\leq N_m \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(y)|^2 (u(s, x-y) - v(s, x-y))^2 dy \right)^{p/2} dx
\leq N_m \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(y)|^2 dy \right)^{p/2} \int_{\mathbb{R}} |u(s, x) - v(s, x)|^p dx,
$$

where $\Phi_{m,\lambda_0}(u, v, s, y) = \left( u_+^{1+\lambda_0}(s, y)h_m(u(s, y)) - v_+^{1+\lambda_0}(s, y)h_m(v(s, y)) \right)$. The lemma is proved. $\square$
**Remark 5.1** Suppose (5.2) holds for all \( t \), and \( u_m \) is the solution Lemma 5.1. Then, for any \( \phi \in S \),

\[
M_t := \sum_{k=1}^{\infty} \int_{0}^{t} \int_{\mathbb{R}} \mu u_{1+\lambda_0} h_{m}(u) \eta_k \phi dx dM^k_s, \quad t < \infty
\]

is a square integrable martingale. Indeed,

\[
\sum_{k=1}^{\infty} \int_{0}^{t} \left( \int_{\mathbb{R}} \mu(s, x) u_{1+\lambda_0} h_{m}(u(s, x)) \eta_k(x) \phi(x) dx \right)^2 ds \\
\leq N \sum_{k=1}^{\infty} \int_{0}^{t} \left( \int_{\mathbb{R}} \eta_k(x) \phi(x) dx \right)^2 ds \\
\leq N \| \phi \|^2_{L_2} \\
< \infty
\]

almost surely.

**Lemma 5.2** Suppose all the conditions of Lemma 5.1 hold for \( \tau = T \), and we assume that \( u_0 \in U^{1/2-k}_{p} \cap L_1(\Omega; L_1) \). Let \( u_m \) be the solution to equation (5.2) introduced in Lemma 5.1. Then

\[
\mathbb{E} \sup_{t \leq T} \| u_m(t, \cdot) \|_{L_1}^{1/2} \leq N \| u_0 \|_{L_1(\Omega \times \mathbb{R})}^{1/2}.
\]

**Proof** Follow the proof of Lemma 4.2. When we obtain (4.14), however, employ [33, Theorem III.6.8] instead of integration in \( t \). Then, we have

\[
\mathbb{E} \sup_{t \leq T} \left( \int_{\mathbb{R}} u_m(t, x) h_k(x) dx \right)^{1/2} \leq 3e^{2KT} \mathbb{E} \| u_0 \|_{L_1}^{1/2} + k^{-1/2+1/(2q)} N e^{2KT},
\]

where \( N = N(m, p, K, T) \). By letting \( k \to \infty \), the monotone convergence theorem implies that

\[
\mathbb{E} \sup_{t \leq T} \| u_m(t, \cdot) \|_{L_1}^{1/2} \leq 3e^{2KT} \mathbb{E} \| u_0 \|_{L_1}^{1/2}.
\]

The lemma is proved.

**Remark 5.2** To construct a global solution from the local ones, we need to show the local solution \( u_m \) is non-explosive. Thus, we prove \( \tau_m := \tau_m^m \wedge m \to \infty \) (a.s.) as \( m \to \infty \), where \( \tau_m^R \) is a stopping time introduced in Remark 4.1. Then, the uniqueness of \( u_m \) (Lemma 5.1) implies that a global solution

\[
u(t, x) := u_m(t, x) \quad \text{for} \quad t \leq \tau_m
\]
is well-defined. Note that we employ Lemma 5.2 to show

\[
\tau_m \to \infty \quad \text{as} \quad m \to \infty \quad (a.s.).
\]  \tag{5.5}

Since \(L_1\) norm of \(u_m\) is uniformly bounded, a stopping time \(\tau_m(S) := \inf\{t \geq 0 : \|u_m(t, \cdot)\|_{L_1} \geq S\}\) is well-defined and \(\tau_m(S) \to \infty\) in probability as \(S \to \infty\). Then, we obtain (5.5) by considering the local solution \(u_m\) on \([0, \tau_m(S)]\); see Lemma 5.3.

**Lemma 5.3** Suppose all the conditions of Theorem 3.6 hold, and \(u_m\) is the solution to equation (5.2) introduced in Lemma 5.1. Then, we have

\[
\lim_{R \to \infty} \sup_{m} P \left( \left\{ \omega \in \Omega : \sup_{t \leq \tau, x \in \mathbb{R}} |u_m(t, x)| \geq R \right\} \right) = 0. \quad \tag{5.6}
\]

**Proof** Let \(T \in (0, \infty)\). For \(m, S \in \mathbb{N}\), set

\[
\tau_m(S) := \inf\{t \geq 0 : \|u_m(t, \cdot)\|_{L_1} \geq S\} \land T.
\]

With the help of Lemma 5.2, the random time \(\tau_m(S)\) is a well-defined stopping time. Let \(t \in (0, T)\). Then, Theorem 4.2, Remark 2.2, Hölder inequality, and Minkowski’s inequality imply

\[
\begin{align*}
\|u_m\|_{H^{1/2-\kappa}(\tau_m(S) \land T)}^p & \leq N \left( \int_{\mathbb{R}} R_{1/2+\kappa}(x-y) |u_m(s, y)|^{1+\lambda} dy \right)^p dx ds \\
& + N \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x-y)|^2 |u_m(s, y)|^{2+2\lambda\kappa} dy \right)^{p/2} dx ds
\end{align*}
\]

Observe that

\[
\begin{align*}
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} R_{1/2+\kappa}(x-y) |u_m(s, y)|^{1+\lambda} dy \right)^p dx \\
& \leq \|u_m(s, \cdot)\|_{L^1}^p \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x-y)|^{\frac{1}{1-\kappa}} |u_m(s, y)|^{\frac{1}{1-\kappa}} dy \right)^{p(1-\lambda)} dx \quad \tag{5.8}
\end{align*}
\]

\[
\begin{align*}
& \leq S^{p\lambda} \|R_{1/2+\kappa}\|_{L^{\frac{1}{1-\kappa}}}^p \left( \int_{\mathbb{R}} |u_m(s, x)|^p dx \right)
\end{align*}
\]
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x-y)|^2 |u_m(s, y)|^{2+2\lambda_0} dy \right)^{p/2} dx \\
\leq \|u_m(s, \cdot)\|_{L_1^p}^{p\lambda_0} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |R_{1/2+\kappa}(x-y)|^{2-2\lambda_0} |u_m(s, y)|^{2-2\lambda_0} dy \right)^{p(1-\lambda_0)} dx \\
\leq S^{p\lambda_0} \|R_{1/2+\kappa}\|_{L_2^{2,2\lambda_0}(\mathbb{R})} \int_{\mathbb{R}} |u_m(s, x)|^p dx
\]

for all \(s \in (0, \tau_m(S) \wedge t)\) almost surely. Therefore, by applying (5.8) and (5.9) to (5.7), we have

\[
\|u_m\|_{H^{1/2-\kappa}(\tau_m(S) \wedge t)}^p - N \|u_0\|_{U_p^{1/2-\kappa}}^p \\
\leq N \left( S^{p\lambda} \|R_{1/2+\kappa}\|_{L_1^{1,\tau}(\mathbb{R})} + S^{p\lambda_0} \|R_{1/2+\kappa}\|_{L_2^{2,2\lambda_0}(\mathbb{R})} \right) \|u_m\|_{L_p^{p}(\tau_m(S) \wedge t)},
\]

where \(N = N(p, \kappa, K, T)\). Note that \(\|R_{1/2+\kappa}\|_{L_1^{1,\tau}(\mathbb{R})} + \|R_{1/2+\kappa}\|_{L_2^{2,2\lambda_0}(\mathbb{R})} < \infty\) since \(\kappa > (\lambda - 1/2) \vee \lambda_0\). Thus,

\[
\|u_m\|_{H^{1/2-\kappa}(\tau_m(S) \wedge t)}^p \leq N_1 \|u_0\|_{U_p^{1-\kappa}}^p + N_2 \|u_m\|_{L_p^{p}(\tau_m(S) \wedge t)}^p,
\]

where \(N_1 = N_1(p, \kappa, K, T)\) and \(N_2 = N_2(S, \lambda, \lambda_0, p, \kappa, K, T)\). By Corollary 2.1 and Theorem 2.1 (iii), we have

\[
\mathbb{E} \sup_{t \leq \tau_m(S) \wedge T, x \in \mathbb{R}} |u_m(s, x)|^p \leq N \|u_m\|_{H^{1/2-\kappa}(\tau_m(S) \wedge T)}^p \leq N \|u_0\|_{U_p^{1-\kappa}},
\]

where \(N = N(S, \lambda, \lambda_0, p, \kappa, K, T)\). By the way, by (5.3) and Chebyshev’s inequality, we have

\[
P \left( \sup_{t \leq T} \|u_m(t, \cdot)\|_{L_1} \geq S \right) \leq \frac{1}{\sqrt{S}} \mathbb{E} \sup_{t \leq T} \|u_m(t, \cdot)\|_{L_1}^{1/2} \leq \frac{N}{\sqrt{S}},
\]

where \(N = N(K, T)\). Therefore, by Chebyshev’s inequality twice, we have

\[
P \left( \sup_{t \leq T, x \in \mathbb{R}} |u_m(t, x)| > R \right) \\
\leq P \left( \sup_{t \leq \tau_m(S) \wedge T, x \in \mathbb{R}} |u_m(t, x)| > R \right) + P(\tau_m(S) < T)
\]
\[ \leq P \left( \sup_{t \leq \tau_m(S) \wedge T, x \in \mathbb{R}} |u_m(t, x)| > R \right) + P \left( \sup_{t \leq T} \|u_m(t, \cdot)\|_{L^1} \geq S \right) \]

where \( N_1 = N_1(S, \lambda, \lambda_0, p, \kappa, K, T) \) and \( N_2 = N_2(K, T) \). By taking the supremum with respect to \( m \), letting \( R \to \infty \), \( S \to \infty \) in order, we get (5.6). The lemma is proved. \( \square \)

Now we introduce the proof of Theorem 3.6. The motivation of the proof follows from [18, Section 8.4] and Proof of Theorem 2.11 in [28].

**Proof of Theorem 3.6**

**Step 1. (Uniqueness).** Follow the Step 1 of Proof of Theorem 3.3. The only difference is \( \kappa \in (0, (\lambda - 1/2) \vee \lambda_0, 1/2) \) instead of \( \kappa \in (0, 1/2) \).

**Step 2. (Existence).** Let \( \kappa \in (0, (\lambda - 1/2) \vee \lambda_0, 1/2) \) and \( T \in (0, \infty) \). By Lemma 5.1, there exists nonnegative \( u_m \in H^{1/2-\kappa}_p(T) \) satisfying equation (3.6). By Corollary 2.1, we have \( u_m \in C([0, T]; C(\mathbb{R})) \) (a.s.). Thus, we can define \( \tau^m_R \leq \tau^m_m \). Therefore, Lemma 5.3 implies

\[ \limsup_{m \to \infty} P \left( \tau^m_m < T \right) = \limsup_{m \to \infty} P \left( \sup_{t \leq T, x \in \mathbb{R}} |u_m(t, x)| \geq m \right) \]

\[ \leq \limsup_{m \to \infty} \sup_{n} P \left( \sup_{t \leq T, x \in \mathbb{R}} |u_n(t, x)| \geq m \right) \to 0, \]

which implies \( \tau^m_m \to \infty \) in probability. Since \( \tau^m_m \) is increasing, we conclude that \( \tau^m_m \uparrow \infty \) (a.s.). Define \( \tau_m := \tau^m_m \wedge m \) and

\[ u(t, x) := u_m(t, x) \quad \text{for} \quad t \in [0, \tau_m]. \]

Note that \( u \) satisfies (3.6) for all \( t \leq \tau_m \), because \( |u(t)| = |u_m(t)| \leq m \) for \( t \leq \tau_m \). Since \( u = u_m \in H^{1-\kappa}_p(\tau_m) \) for any \( m \), we have \( u \in H^{1-\kappa}_p,loc \).

**Step 3. (Hölder regularity).** Let \( T < \infty \). Since \( u \in H^{1/2-\kappa}_p(T) \), by employing Corollary 2.1, we have (3.8). The theorem is proved. \( \square \)

**Proof of Theorem 3.7** Follow the proof of Proof of Theorem 3.4. It should be mentioned that we assume \( \kappa \in (0, (\lambda - 1/2) \vee \lambda_0, 1/2) \). \( \square \)

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**References**

1. Bertini, L., Cancrini, N., Jona-Lasinio, G.: The stochastic Burgers’ equation. Commun. Math. Phys. 165(2), 211–232 (1994)
2. Da Prato, G., Debussche, A., Temam, R.: Stochastic Burgers’ equation. Nonlinear Differ. Equ. Appl. 1(4), 389–402 (1994)
3. Da Prato, G., Gatarek, D.: Stochastic Burgers’ equation with correlated noise. Stochs. Int. J. Prob. Stoch. Process. 52(1–2), 29–41 (1995)
4. Gyöngy, I.: Existence and uniqueness results for semilinear stochastic partial differential equations. Stoch. Process. Appl. 73(2), 271–299 (1998)
5. Gyöngy, I., Rovira, C.: On stochastic partial differential equations with polynomial nonlinearities. Stochs. Int. J. Prob. Stoch. Process. 67(1–2), 123–146 (1999)
6. Gyöngy, I., Rovira, C.: On $L_p$-solutions of semilinear stochastic partial differential equations. Stoch. Process. Appl. 90(1), 83–108 (2000)
7. León, J.A., Nualart, D., Pettersson, R.: The stochastic Burgers’ equation: finite moments and smoothness of the density. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3(03), 363–385 (2000)
8. Englezos, N., Frangos, N.E., Kartala, X.I., Yannacopoulos, A.N.: Stochastic Burgers’ pdes with random coefficients and a generalization of the Cole-Hopf transformation. Stoch. Process. Appl. 123(8), 3239–3272 (2013)
9. Catuogno, P., Olivera, C.: Strong solution of the stochastic Burgers equation. Appl. Anal. 93(3), 646–652 (2014)
10. Lewis, P., Nualart, D.: Stochastic Burgers’ equation on the real line: regularity and moment estimates. Stochastics 90(7), 1053–1086 (2018)
11. Röckner, M., Sobol, Z.: Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations. Ann. Probab. 34(2), 663–727 (2006)
12. Crighton, D.G., Dowling, A.P., Ffowcs-Williams, J., Heckl, M., Leppington, F., Bartram, J.F.: Modern methods in analytical acoustics lecture notes (1992)
13. Ladyzhenskaya, O.A. Solonnikov, V.A., Ural’tseva, N.N.: Linear and quasi-linear equations of parabolic type, Vol. 23, American Mathematical Soc., (1968)
14. Dix, D.B.: Nonuniqueness and uniqueness in the initial-value problem for Burgers’ equation. SIAM J. Math. Anal. 27(3), 708–724 (1996)
15. Bekiranov, D.: The initial-value problem for the generalized Burgers equation. Differ. Integral Equ. 9(6), 1253–1265 (1996)
16. Tersenov, A.S.: On the generalized Burgers equation. Nonlinear Differ. Equ. Appl. 17(4), 437–452 (2010)
17. Gomez, A., Lee, K., Mueller, C., Wei, A., Xiong, J.: Strong uniqueness for an spde via backward doubly stochastic differential equations. Stat. Probab. Lett. 83(10), 2186–2190 (2013)
18. Krylov, N.V.: An analytic approach to spdes. Stoch. Partial Differ. Equ Six Perspect. 64, 185–242 (1999)
19. Krylov, N.V.: On a result of C. Mueller and E. Perkins, Probab. Theory Relat. Fields 108(4) (1997) 543–557
20. Krylov, N.V.: On SPDE’s and superdiffusions, Ann. Probab. (1997) 1789–1809
21. Mueller, C.: Long time existence for the heat equation with a noise term. Probab. Theory Relat. Fields 90(4), 505–517 (1991)
22. Burdzy, K., Mueller, C., Perkins, E.: Nonuniqueness for nonnegative solutions of parabolic stochastic partial differential equations. Illinois J. Math. 54(4), 1481–1507 (2010)
23. Mytnik, L.: Weak uniqueness for the heat equation with noise, Ann. Probab. (1998) 968–984
24. Mytnik, L., Perkins, E.: Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: the white noise case. Probab. Theory Relat. Fields 149(1–2), 1–96 (2011)
25. Walsh, J.B.: An introduction to stochastic partial differential equations, in: École d’Été de Probabilités de Saint Flour XIV-1984, Springer, (1986), 265–439
26. Xiong, J.: Super-Brownian motion as the unique strong solution to an SPDE. Ann. Probab. 41(2), 1030–1054 (2013)
27. Choi, J., Han, B.: A regularity theory for stochastic partial differential equations with a super-linear diffusion coefficient and a spatially homogeneous colored noise. Stoch. Process. Appl. 135, 1–30 (2021)
28. Han, B., Kim, K.: Boundary behavior and interior Hölder regularity of solution to nonlinear stochastic partial differential equations driven by space-time white noise. J. Differ. Equ. 269, 9904–9935 (2020)
29. Mueller, C.: The critical parameter for the heat equation with a noise term to blow up in finite time, Ann. Probab. (2000) 1735–1746
30. Grafakos, L.: Modern Fourier analysis, Vol. 250, Springer, (2009)
31. Krylov, N.V.: Lectures on elliptic and parabolic equations in Sobolev spaces, Vol. 96, American Mathematical Soc., (2008)
32. Krylov, N.V.: Maximum principle for spdes and its applications, stochastic differential equations: theory and applications: a Volume in Honor of Professor Boris L Rozovskii, World Scientific, (2007), 311–338
33. Krylov, N.: Introduction to the theory of diffusion processes, Providence, (1995)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.