Entanglement detection and lower bound of the convex-roof extension of the negativity

Ming Li¹, Tong-Jiang Yan¹,² and Shao-Ming Fei³,⁴

¹ College of Science, China University of Petroleum, 266555 Qingdao, People’s Republic of China
² State Key Laboratory of Information Security (Institute of Software, Chinese Academy of Sciences), 100049 Beijing, People’s Republic of China
³ School of Mathematical Sciences, Capital Normal University, 100048 Beijing, People’s Republic of China
⁴ Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

E-mail: liming@upc.edu.cn.

Received 31 August 2011, in final form 15 November 2011
Published 19 December 2011
Online at stacks.iop.org/JPhysA/45/035301

Abstract
We present a set of inequalities based on mean values of quantum mechanical observables similar to nonlinear entanglement witnesses for bipartite quantum systems. These inequalities give rise to sufficient and necessary conditions for separability of all bipartite pure states and even some mixed states. In terms of these mean values of quantum mechanical observables, a measurable lower bound of the convex-roof extension of the negativity is derived.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

Entanglement is not only the characteristic trait of quantum mechanics but also a vital resource for many aspects of quantum information processing such as quantum computation, quantum metrology and quantum communication [1]. One of the fundamental problems in quantum entanglement theory is to determine which states are entangled and which are not, either theoretically or experimentally. The entanglement witness [2, 3] is the most useful approach to characterize quantum entanglement experimentally. In recent years, there has been considerable effort in constructing and analyzing the structure of entanglement witness (see [4–8] and references therein). Generally the Bell inequalities [9–14] can be recast as entanglement witnesses. Better entanglement witnesses can be also constructed from more effective Bell-type inequalities.

On the other hand, quantifying quantum entanglement is also a significant problem in quantum information theory. A number of entanglement measures such as the entanglement of formation and distillation [15–17], negativity [18] and relative entropy [17, 19] have been proposed for bipartite systems [16, 19–24]. The negativity was derived from the positive partial transposition (PPT) [25]. It bounds two relevant quantities characterizing the entanglement of mixed states: the channel capacity and the distillable entanglement. The convex-roof extension
of the negativity (CREN) [26] gives a better characterization of entanglement, which is nonzero for PPT entangled quantum states.

In this paper, similar to the nonlinear entanglement witnesses and Bell-type inequalities, we present a set of inequalities based on mean values of quantum mechanical observables, which can serve as necessary and sufficient conditions for the separability of bipartite pure quantum states and the isotropic states. These inequalities are also closely related to the measure of the quantum entanglement. According to the violation of these inequalities, we derive an experimentally measurable lower bound for the CREN.

We first give a brief review of the 3-setting nonlinear entanglement witnesses enforced by the indeterminacy relation of complementary local observables for two-qubit systems [7]. For a two-level system, there are three mutually complementary observables \( A_i = \vec{\alpha}_i \cdot \vec{\sigma} \), where \( \vec{\alpha}_i \), \( i = 1, 2, 3 \), are three normalized vectors that are orthogonal to each other, and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices. \( \mu_A = -iA_1A_2A_3 \) is the so-called orientation of \( A_s \). \( \mu_A \) takes values \( \pm 1 \).

Similarly, one can define three mutually complementary observables \( B_i = \vec{\beta}_i \cdot \vec{\sigma} \) for \( i = 1, 2, 3 \) with the corresponding orientations \( \mu_B \). It has been shown that [7] (i) a two-qubit state is separable if and only if the following inequality holds for all sets of observables \( \{A_i, B_j\}_{i=1,2,3} \) with the same orientation:

\[
\sqrt{(A_1B_1 + A_2B_2)^2 + (A_3 + B_3)^2} - \langle A_3B_3 \rangle \rho \leq 1; \tag{1}
\]

(ii) for a given entangled state, the maximal violation of the above inequality is \( 1 - 4\lambda_{\text{min}} \), with \( \lambda_{\text{min}} \) being the minimal eigenvalue of the partially transposed density matrix. The maximal possible violation for all states is 3, which is attainable by the maximal entangled states.

For qubit–qutrit systems, a similar inequality has been presented in [8], which detects the quantum entanglement also necessarily and sufficiently. However, the approaches in [7] and [8] cannot be directly generalized to higher dimensional systems, since it is based on the PPT criterion that is both necessary and sufficient only for separability of two-qubit and qubit–qutrit states. For general higher dimensional \( M \times N \) bipartite quantum systems, a new approach has been employed in [14]. Let \( \rho \in \mathcal{H}_{AB} \) be any pure quantum state in the vector space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \) with dimensions \( \dim \mathcal{H}_A = M \) and \( \dim \mathcal{H}_B = N \). Assume that \( L^A_\alpha \) and \( L^B_\beta \) are the generators of special orthogonal groups \( SO(M) \) and \( SO(N) \), respectively. The \( (M-1)/2 \) generators \( L^A_\alpha \) are given by \( (|i\rangle \langle k| - |k\rangle \langle i|) \), \( 1 \leq j < k \leq M \), where \( |i\rangle \) is the usual canonical basis of \( \mathcal{H}_A \), a column vector with the \( i \)-th row 1 and the rest zeros. \( L^B_\beta \) can be similarly defined. The matrix operators \( L^A_\alpha \) (resp. \( L^B_\beta \)) have \( M-2 \) (resp. \( N-2 \)) rows and \( M-2 \) (resp. \( N-2 \)) columns that are identically zero. We define the operators \( A_\alpha^j \) (resp. \( B_\beta^j \)) from \( L^A_\alpha \) (resp. \( L^B_\beta \)) by replacing the four entries in the positions of the two nonzero rows and two nonzero columns of \( L^A_\alpha \) (resp. \( L^B_\beta \)) with the corresponding four entries of the matrix \( \vec{\alpha}_i \cdot \vec{\sigma} \) (resp. \( \vec{\beta}_i \cdot \vec{\sigma} \)), and keeping the other entries of \( A_\alpha^j \) (resp. \( B_\beta^j \)) zero.

By using \( L^A_\alpha \) and \( L^B_\beta \), the pure state \( \rho \) can be projected to ‘two-qubit’ ones [14]:

\[
\rho_{\alpha\beta} = \frac{L^A_\alpha \otimes L^B_\beta \rho (L^A_\alpha)^\dagger \otimes (L^B_\beta)^\dagger}{\text{Tr} \left[ (L^A_\alpha \otimes L^B_\beta \rho (L^A_\alpha)^\dagger \otimes (L^B_\beta)^\dagger) \right]}, \tag{2}
\]

where \( \alpha = 1, 2, \ldots, \frac{M(M-1)}{2}; \beta = 1, 2, \ldots, \frac{N(N-1)}{2} \). As the matrix \( L^A_\alpha \otimes L^B_\beta \) has \( MN - 4 \) rows and \( MN - 4 \) columns that are identically zero, one can directly verify that there are at most \( 4 \times 4 = 16 \) nonzero elements in each matrix \( \rho_{\alpha\beta} \). For every pure state \( \rho_{\alpha\beta} \), the corresponding Bell operators are defined by

\[
B_{\alpha\beta} = \tilde{A}_1^\alpha \otimes \tilde{B}_1^\beta + \tilde{A}_2^\alpha \otimes \tilde{B}_2^\beta + \tilde{A}_3^\alpha \otimes \tilde{B}_3^\beta - \tilde{A}_4^\alpha \otimes \tilde{B}_4^\beta, \tag{3}
\]
where $\hat{A}_\alpha^\dagger = L_{\alpha}^A \hat{A}_\alpha^\dagger (L_{\alpha}^A)^\dagger$ and $\hat{B}_\beta^\dagger = L_{\beta}^B \hat{B}_\beta^\dagger (L_{\beta}^B)^\dagger$ are the Hermitian operators. It has been shown that any bipartite pure quantum state is entangled if and only if at least one of the following inequalities is violated [14]:

$$|\langle B_{\alpha\beta} \rangle| \leq 2. \quad (4)$$

Inequalities (4) work only for general high-dimensional bipartite pure states. Combining the approaches in [7] and [14], we now define the mean value of the nonlinear operators $\tilde{B}_{\alpha\beta}$,

$$\langle \tilde{B}_{\alpha\beta}\rangle = \sqrt{\langle \hat{A}_\alpha^\dagger \hat{B}_\beta^\dagger \rangle^2 + \langle \hat{A}_\alpha^\dagger + \hat{B}_\beta^\dagger \rangle^2 - \langle \hat{A}_\alpha^\dagger \hat{B}_\beta^\dagger \rangle^2}, \quad (5)$$

for high-dimensional bipartite mixed states.

**Theorem 1.** Any bipartite quantum state $\rho \in \mathcal{H}_{AB}$ is entangled if any one of the following inequalities is violated:

$$\frac{1}{\text{Tr}(L_{\alpha} \otimes L_{\beta} \rho I_{\text{A}} L_{\alpha}^A \otimes L_{\beta}^B)} |\langle \tilde{B}_{\alpha\beta} \rangle| \leq 1, \quad \text{where } \alpha = 1, 2, \ldots, \frac{M(M-1)}{2} \text{ and } \beta = 1, 2, \ldots, \frac{N(N-1)}{2}. \quad (6)$$

**Proof.** Assume that $\rho$ is a separable (not entangled) quantum state. Since the separability of a state does not change under the local operation $L_{\alpha}^A \otimes L_{\beta}^B$, one has that for any $\alpha$ and $\beta$, $\rho_{\alpha\beta} = \frac{1}{M^2N^2} \rho I_{\text{A}} \otimes I_{\text{B}}$, which can be treated as a two-qubit state, must also be separable. According to the analysis in [7], a two-qubit state $\rho$ is separable if and only if (1) holds, which contradicts condition (6). Thus, we have that if any one of inequalities (6) is violated, $\rho$ must be an entangled quantum state.

It is obvious that inequalities (6) must not be weaker than the Bell inequalities given in [14] for detecting the entanglement of mixed quantum states, since (6) supplies a sufficient and necessary condition for separability of two-qubit (mixed) quantum states, while violating the CHSH inequality is just a sufficient condition for the two-qubit entanglement. Actually, (6) is strictly stronger, as seen from the following examples.

**Example 1.** We consider a $(3 \times 3)$-dimensional state introduced in [27] by Bennett et al. Set $|\xi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle)$, $|\xi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle)$, $|\xi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle)$, $|\xi_3\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle)$, $|\xi_4\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle)$, $|\xi_5\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle)$. Let

$$\rho = \frac{1}{4} \left( I_3 - \sum_{i=0}^{4} |\xi_i\rangle \langle \xi_i| \right).$$

This state is entangled according to the realignment criterion [28]. We consider the mixture of $\rho$ and the maximal entangled singlet $P_\pi = |\psi_+\rangle \langle \psi_+|$, where $|\psi_+\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{4} |ii\rangle$:

$$\rho_p = (1-p)\rho + pP_\pi. \quad (7)$$

By straightforward computation, the bell inequalities (4) detect the entanglement for $0.57602 \leq p \leq 1$, while (6) detect the entanglement for $0.18221 \leq p \leq 1$.

**Example 2.** Consider the state

$$\rho_p(a) = (1-p)\rho(a) + pP_+ \quad (8)$$

J. Phys. A: Math. Theor. 45 (2012) 035301 M Li et al
Figure 1. The differences $D(p)$ between the right and the left sides of inequalities (6) (solid line) and the Bell inequalities (4) (dotted line).

where

$$\rho(a) = \frac{1}{8a+1} \begin{pmatrix}
a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2}
\end{pmatrix}$$

is the weakly inseparable state given in [29], $0 < a < 1$.

Take $a = 0.236$, which is the case that $\rho(a)$ violates the realignment criterion [28] maximally. From figure 1 we see that the bell inequalities (4) detect the entanglement for $0.26 \leq p \leq 1$, while (6) detect the entanglement for the whole region of $0 < p \leq 1$.

**Example 3.** Isotropic states [30] with dimensions $M = N = d$ can be written as the mixtures of the maximally mixed state and the maximally entangled state

$$|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle,$$

$$\rho = \frac{1-x}{d^2} I_d \otimes I_d + x|\psi_+\rangle\langle\psi_+|.$$

Inequalities (6) can detect the entanglement for $x \leq \frac{1}{d^2 + 1}$, which agrees with the result in [30]. Thus, (6) serves as a sufficient and necessary condition of separability for isotropic states.

Inequalities (6) can not only be used to detect the entanglement, but also have some direct relations with the negativity. The negativity of a bipartite quantum states $\rho$ with dimensions $d(H_A) = M$ and $d(H_B) = N$ ($M \leq N$) is defined by [31]

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_A}\| - 1}{M - 1},$$

where $\rho^{T_A}$ is the partial transpose of $\rho$ and $\|R\| = \text{Tr}\sqrt{RR^\dagger}$ stands for the trace norm of the matrix $R$. The negativity is defined based on the positive partial transpose (PPT) criterion [25] which cannot detect the PPT bound entanglement. Thus, it is not sufficient for the negativity to be a good measure of entanglement. Lee et al in [26] introduced the CREN $\mathcal{N}_m(\rho)$. For the
pure bipartite quantum states $|\psi\rangle$, $\mathcal{N}_m(|\psi\rangle)$ is exactly the negativity $\mathcal{N}(|\psi\rangle)$ defined in (10). For a mixed bipartite quantum state $\rho$, the CREN is defined by

$$
\mathcal{N}_m(\rho) = \min_k \sum p_k \mathcal{N}_m(|\psi_k\rangle),
$$

(11)

where the minimum is taken over all the ensemble decompositions of $\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|$. The CREN can detect the PPT bound entanglement, since it is zero if and only if the corresponding quantum state is separable. Lee et al also show that $\mathcal{N}_m(\rho)$ does not increase under local quantum operations and classical communication. However, generally it is very difficult to calculate CREN analytically. Here we present an experimentally measurable tight lower bound of CREN for arbitrary bipartite quantum states, in terms of the violation of inequalities (6).

**Theorem 2.** For any bipartite quantum states $\rho \in \mathcal{H}_{AB}$,

$$
\mathcal{N}_m(\rho) \geq \frac{1}{M-1} \sum_{\alpha\beta} |C_{\alpha\beta}| \left( \frac{X(\rho_{\alpha\beta})}{2} + 1 \right) - (M - 1),
$$

(12)

where $C_{\alpha\beta} = \text{Tr}(L_\alpha \otimes L_\beta \rho^T L_\alpha \otimes L_\beta)$, $X(\rho_{\alpha\beta}) = \min\{0, d(\rho_{\alpha\beta})\}$ and $d(\rho_{\alpha\beta}) = \frac{\|\rho_{\alpha\beta}\|_{\max}}{\|\rho_{\alpha\beta}\|} - 1$ stands for the difference of the left and right sides of inequalities (6).

**Proof.** Let $|\psi\rangle = \sum_i \sqrt{\mu_i} |ii\rangle$ be a bipartite pure state in Schmidt form. One has

$$
\mathcal{N}_m(|\psi\rangle) = \frac{2}{M-1} \sum_{i<j} \sqrt{\mu_i \mu_j},
$$

(13)

Note that $\sum_i \mu_i = 1$. By calculating the trace norm of $L_\alpha \otimes L_\beta \langle |\psi\rangle \langle \psi|^{T_\alpha} L_\alpha \otimes L_\beta$ for each $\alpha$ and $\beta$, we derive that

$$
\sum_{\alpha\beta} \|C_{\alpha\beta}^{i\psi} |\psi\rangle_{\alpha\beta} \langle \psi|^{T_\alpha}\| = (M - 1)^2 + 2 \sum_{i<j} \sqrt{\mu_i \mu_j},
$$

(14)

where $|\psi\rangle_{\alpha\beta} = \frac{L_{\alpha} \otimes L_{\beta} |\psi\rangle}{\sqrt{C_{\alpha\beta}}}$. Let $\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|$ be the optimal decomposition which fulfills that $\mathcal{N}_m(\rho)$ attains its minimum. In terms of (13) and (14), we obtain

$$
\mathcal{N}_m(\rho) = \sum_k p_k \mathcal{N}_m(\rho_k)
$$

$$
= \frac{1}{M-1} \sum_k p_k \sum_{\alpha\beta} \|C_{\alpha\beta}^{k} \rho_{\alpha\beta}^{k} \langle \rho_{\alpha\beta}^{k} \rangle^{T_\alpha}\| - (M - 1)
$$

$$
\geq \frac{1}{M-1} \sum_{\alpha\beta} \left( \sum_k p_k \|C_{\alpha\beta}^{k} \| \| \rho_{\alpha\beta}^{k} \|^T \right) - (M - 1)
$$

$$
= \frac{1}{M-1} \sum_{\alpha\beta} \left( \sum_k p_k L_\alpha \otimes L_\beta \rho_{\alpha\beta}^{k} \langle \rho_{\alpha\beta}^{k} \rangle^{T_\alpha} L_\alpha \otimes L_\beta \right) - (M - 1)
$$

$$
= \frac{1}{M-1} \sum_{\alpha\beta} \|L_\alpha \otimes L_\beta \rho_{\alpha\beta}^{T_\alpha} L_\alpha \otimes L_\beta\| - (M - 1)
$$

$$
= \frac{1}{M-1} \sum_{\alpha\beta} |C_{\alpha\beta}| \| \rho_{\alpha\beta}^{T_\alpha} \| - (M - 1)
$$

$$
= \frac{1}{M-1} \sum_{\alpha\beta} |C_{\alpha\beta}| \left( \frac{X(\rho_{\alpha\beta})}{2} + 1 \right) - (M - 1),
$$

(15)
where we have used that $\|\rho_{TA}\|$ has at most one negative eigenvalue (see [32]) in deriving the last equation.

□

Remark. For the isotropic states (9), our lower bound (12) shows that $\mathcal{N}_m(\rho) \geq 4x^{-1}$, which matches with the formula derived in [26]. Thus, in this case the lower bound is exact for the CREN. Moreover, our lower bound is experimentally measurable, in the sense that $C_{\alpha\beta} = \text{Tr}(L_{\alpha} \otimes L_{\beta} \rho L_{\alpha} \otimes L_{\beta})$ is the mean value of the Hermitian operator $L_{\alpha} L_{\alpha}^\dagger \otimes L_{\beta} L_{\beta}^\dagger$ and $X(\rho_{\alpha\beta}) = \min\{0, d(\rho_{\alpha\beta})\}$ is determined by the mean value of the operator $\mathcal{B}_0^\dagger$. On the other hand, according to the proof of the theorem, the lower bound (12) for pure bipartite quantum states is also exact. Thus, based on the continuity of the CREN, for the weakly mixed quantum state $\rho$ with $\text{Tr}\{\rho^2\} \approx 1$, (12) supplies a good estimation of the CREN.

In conclusion, we have derived a set of inequalities that can detect the better entanglement of quantum mixed states. These inequalities serve as sufficient and necessary conditions for separability for all bipartite pure states and the isotropic states. Nevertheless, generally bound entangled states cannot be detected by these inequalities. We also find that these inequalities have close relations with the CREN. A measurable lower bound for the CREN has been obtained.

Acknowledgments

This work was supported by the NSFC 10875081, NSFC 11105226, KZ200810028013, PHR201007107, the open fund of the State Key Laboratory of Information Security (Graduate University of Chinese Academy of Sciences) and the Natural Science Fund of Shandong Province (no ZR2010FM017).

References

[1] Nielsen M and Chuang I 2000 Quantum Information and Computation (Cambridge: Cambridge University Press)
[2] Horodecki M, Horodecki P and Horodecki R 1996 Phys. Lett. A 223 1
[3] Terhal B M 2000 Phys. Lett. A 271 319
[4] Terhal B M 2002 Theor. Comput. Sci. 287 313
[5] Tith G and Gline O 2005 Phys. Rev. Lett. 94 060501
[6] Chruscinski D and Pytel J 2010 Phys. Rev. A 82 052310
[7] Yu S X, Pan J W, Chen Z B and Zhang Y D 2003 Phys. Rev. Lett. 91 217903
[8] Zhao M J, Ma T, Fei S M and Wang Z X 2011 Phys. Rev. A 83 052120
[9] Bell J S 1964 Physics I 195
[10] Gisin N 1991 Phys. Lett. A 154 201
[11] Gisin N and Peres A 1992 Phys. Lett. A 162 15
[12] Clauser J F, Horne M A, Shimony A and Holt R A 1969 Phys. Rev. Lett. 23 880
[13] Mermin N D 1990 Phys. Rev. Lett. 65 1838
[14] Ardehali M 1992 Phys. Rev. A 46 5375
[15] Belinskii A V and Klyshko D N 1993 Phys. Usp. 36 653
[16] Li M and Fei S M 2010 Phys. Rev. Lett. 104 240502
[17] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[18] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 Phys. Rev. A 53 2046
[19] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 Phys. Rev. Lett. 78 2275
[20] Vedral V, Plenio M B, Jacobs K and Knight P L 1997 Phys. Rev. A 56 4452
[21] Vedral V and Plenio M B 1998 Phys. Rev. A 57 1619
[22] Źyczkowski K and Horodecki P 1998 Phys. Rev. A 58 883
[19] Schumacher B and Westmoreland M D 2000 Relative entropy in quantum information theory (arXiv:quant-ph/0004045)
[20] Horodecki M, Horodecki P and Horodecki R 1998 Phys. Rev. Lett. 80 5239
[21] Rains E M 2001 IEEE Trans. Inform. Theory 47 2921
[22] Werner R F and Wolf M M 2000 Phys. Rev. A 61 062102
[23] Terhal B M, Gerd K and Vollbrecht K G H 2000 Phys. Rev. Lett. 85 2625
[24] Hill S and Wootters W K 1997 Phys. Rev. Lett. 78 5022
Wootters W K 1998 Phys. Rev. Lett. 80 2245
[25] Peres A 1996 Phys. Rev. Lett. 76 1413
[26] Lee S, Chi D P, Oh S D and Kim J 2003 Phys. Rev. A 68 062304
[27] Bennett C H, DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 1999 Phys. Rev. Lett. 82 5385
[28] Chen K and Wu L A 2003 Quantum Inform. Comput. 3 193
[29] Horodecki P 1997 Phys. Lett. A 232 333
[30] Horodecki M and Horodecki P 1999 Phys. Rev. A 59 4206
[31] Vidal G and Werner R F 2002 Phys. Rev. A 65 032314
[32] Verstraete F, Audenaert K, Dehaene J and Moor B D 2001 J. Phys. A: Math. Gen. 34 10327