THE SHARP WEIGHTED BOUND FOR MULTILINEAR MAXIMAL FUNCTIONS AND CALDERÓN-ZYGMUND OPERATORS

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Abstract. We investigate the weighted bounds for multilinear maximal functions and Calderón-Zygmund operators from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\vec{w})$, where $1 < p_1, \ldots, p_m < \infty$ with $1/p_1 + \cdots + 1/p_m = 1/p$ and $\vec{w}$ is a multiple $A_\beta$ weight. We prove the sharp bound for the multilinear maximal function for all such $p_1, \ldots, p_m$ and prove the sharp bound for $m$-linear Calderón-Zygmund operators when $p \geq 1$.

1. Introduction and Main Results

A weight is a non-negative locally integrable function. Given $p$, $1 < p < \infty$, an $A_p$ weight is one that satisfies the following

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty. $$

It is well known that the Hardy-Littlewood maximal operator and Calderón-Zygmund operators are bounded on $L^p(w)$ when $w \in A_p$. The sharp dependence for the Hardy-Littlewood maximal function is given by

$$\|M\|_{L^p(w) \to L^p(w)} \leq C_{n,p} [w]_{A_p}^{p'}. $$

Inequality (1.1) was first proven by Buckley [1]. (We refer the reader to [11] for a beautiful proof of this inequality and a summary of the history.) Later is was proven by Hytönen [8] that if $T$ is a Calderón-Zygmund operator then

$$\|T\|_{L^p(w) \to L^p(w)} \leq C_{n,p,T} [w]_{A_p}^{\max(1, \frac{p'}{p})}. $$

Again, we refer the reader to [9, 12] for the background material and further references. In this article we prove the the multilinear analogs of inequalities (1.1) and (1.2). We begin with a few definitions.

First, let us define multiple $A_\beta$ weights. In [13], Lerner, Ombrosi, Pérez, Torres and Trujillo-González introduced the theory of multiple $A_\beta$ weights.

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Then the following results hold:

\[ \|M_{1} \|_{L^{p}(w)} \leq C \prod_{i=1}^{m} \|f_{i}\|_{L^{p_{i}}(w_{i})}. \]  

We say that \( w \) satisfies the multilinear \( A_{\vec{P}} \) condition if

\[ [\vec{w}]_{A_{\vec{P}}} := \sup_{Q} \left( \frac{1}{|Q|} \int_{Q} v_{\vec{w}} \right)^{p/p'} \prod_{i=1}^{m} \left( \frac{1}{|Q|} \int_{Q} w_{i}^{1-p_{i}'}/p_{i}' \right)^{p_{i}'}, \]

where \( [\vec{w}]_{A_{\vec{P}}} \) is called the \( A_{\vec{P}} \) constant of \( \vec{w} \). When \( p_{i} = 1, \) \( (\frac{1}{|Q|} \int_{Q} w_{i}^{1-p_{i}'}/p_{i}' \right) < \infty, \)

It is easy to see that in the linear case (that is, if \( m = 1 \) \( [w]_{A_{\vec{P}}} = [w]_{A_{p}} \) is the usual \( A_{p} \) constant. In \( [13] \), it was shown that for \( 1 < p_{1}, \cdots, p_{m} < \infty, \)

\[ \vec{w} \in A_{\vec{P}} \] if and only if \( w_{i}^{1-p_{i}'}/p_{i}' \in A_{p_{i}'} \) and \( v_{\vec{w}} \in A_{mp_{i}}. \)

Given \( \vec{f} = (f_{1}, \cdots, f_{m}) \), we define the multilinear maximal function by

\[ \mathcal{M}(\vec{f}) = \sup_{Q \ni x} \prod_{i=1}^{m} \left| \frac{1}{|Q|} \int_{Q} f_{i} \right|. \]

In \( [13] \), the authors proved that \( \vec{w} \in A_{\vec{P}} \) if and only if

\[ \|\mathcal{M}(\vec{f})\|_{L^{p}(w_{\vec{w}})} \leq C \prod_{i=1}^{m} \|f_{i}\|_{L^{p_{i}}(w_{i})}. \]

Recall that inequality \( (1.1) \) is sharp in the sense that the exponent on \( [w]_{A_{p}} \) cannot be improved. The analogous question for the operator \( \mathcal{M} \) has remained open. In \( [5] \), by mixed estimates involving \( A_{\infty} \), Damián, Lerner and Pérez proved the following result.

**Theorem A.** \( [5] \) Theorem 1.2] Let \( 1 < p_{i} < \infty, i = 1, \cdots, m \) and \( 1/p = 1/p_{1} + \cdots + 1/p_{m} \). Denote by \( \alpha = \alpha(p_{1}, \cdots, p_{m}) \) the best possible power in

\[ \|\mathcal{M}(\vec{f})\|_{L^{p}(w_{\vec{w}})} \leq C_{m,n,\vec{P}}[\vec{w}]_{A_{\vec{P}}}^{\alpha} \prod_{i=1}^{m} \|f_{i}\|_{L^{p_{i}}(w_{i})}. \]

Then the following results hold:

1. For all \( 1 < p_{1}, \cdots, p_{m} < \infty, \frac{m}{mp_{m} - 1} \leq \alpha \leq \frac{1}{p} \left( 1 + \sum_{i=1}^{m} \frac{1}{p_{i} - 1} \right). \)
2. If \( p_{1} = p_{2} = \cdots = p_{m} = r > 1, \) then \( \alpha = \frac{m}{r - 1}. \)

Interestingly, the mixed estimates involving \( A_{\infty} \) do not yield the sharp dependence on the constant \( [\vec{w}]_{A_{\vec{P}}} \). The sharp bound along the diagonal is obtained using similar methods to those found in \( [11] \) and these techniques only seem to work along the diagonal. In this paper we find the optimal power on \( [\vec{w}]_{A_{\vec{P}}} \) for the full range of exponents, \( 1 < p_{1}, \cdots, p_{m} < \infty. \)
Theorem 1.2. Suppose \(1 < p_1, \ldots, p_m < \infty\), \(1/p = 1/p_1 + \cdots + 1/p_m\), and \(\vec{w} \in A_{\vec{p}}\). Then

\[
\|M(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C_{m,n,\vec{p}}[\vec{w}] A_{\vec{p}} \max(\frac{p'_1}{p}, \ldots, \frac{p'_m}{p}) \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}. \tag{1.4}
\]

Moreover the exponent \([\vec{w}] A_{\vec{p}}\) is the best possible.

We emphasize that our bounds (1.4) not only improves those in Theorem A, but is also the best possible. See Figure 1 for a visualization of the bilinear case.

Next we turn to study weighted bounds of multilinear Calderón-Zygmund operators. The theory of multilinear Calderón-Zygmund operators originated in the works of Coifman and Meyer [3, 4] and was later developed by Christ and Journé [2], Kenig and Stein [10], and Grafakos and Torres [7]. The last work provides a comprehensive account of general multilinear Calderón-Zygmund operators which we follow in this paper.

Definition 1.3. Let \(T\) be a multilinear operator initially defined on the \(m\)-fold product of Schwartz spaces and taking values into the space of tempered distributions,

\[T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).\]

we say that \(T\) is an \(m\)-linear Calderón-Zygmund operator if, for some \(1 \leq q_i < \infty\), it extends to a bounded multilinear operator from \(L^{q_1} \times \cdots \times L^{q_m}\) to \(L^q\), where \(1/q_1 + \cdots + 1/q_m = 1/q\), and if there exists a function \(K\), defined off the diagonal \(x = y_1 = \cdots = y_m\) in \((\mathbb{R}^n)^{m+1}\), satisfying

\[T(f_1, \ldots, f_m) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m)f_1(y_1) \cdots f_m(y_m)dy_1 \cdots dy_m\]
for all \(x \not\in \bigcap_{j=1}^{m} \text{supp } f_j\):
\[
|K(y_0, y_1, \cdots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn}}
\]
and
\[
|K(y_0, \cdots, y_i, \cdots, y_m) - K(y_0, \cdots, y_i', \cdots, y_m)| \leq \frac{A|y_i - y_i'|^{\varepsilon}}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn+\varepsilon}}
\]
for some \(A, \varepsilon > 0\) and all \(0 \leq i \leq m\), whenever \(|y_i - y_i'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_i - y_k|\).

It was shown in [7] that if \(1/r_1 + \cdots + 1/r_m = 1/r\) when \(1 < r_i < \infty\) for all \(i = 1, \cdots, m\); and \(T\) is bounded from \(L^{r_1} \times \cdots \times L^{r_m}\) to \(L^{r,\infty}\) when at least one \(r_i = 1\). In particular, \(T\) is bounded from \(L^1 \times \cdots \times L^1\) to \(L^{1/m,\infty}\).

A weighted theory for \(m\)-linear Calderón-Zygmund operators was developed in [13], where it was shown that such operators are bounded from \(L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)\) to \(L^p(\bar{w})\) when \(\bar{w} \in A_\bar{P}\). In [5], the authors proved a multilinear version of the \(A_2\) conjecture. Specifically, for \(p_1 = \cdots = p_m = m + 1\), it was shown that
\[
\|T(f)\|_{L^p(\bar{w})} \lesssim [\bar{w}]_{A_\bar{P}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}
\]
where the estimate for the power of \([\bar{w}]_{A_\bar{P}}\) is the best possible.

Due to the lack of an appropriate extrapolation theorem for multilinear operators with multiple weights, let alone a version with good constants, the sharp estimate is unknown for any other choices of \(p_i\). In this paper, we give a sharp estimate for the case of \(p \geq 1\). Specifically, we prove the following theorem, again we refer the reader to Figure 2 for a visualization of the bilinear case.

**Theorem 1.4.** Let \(T\) be a multilinear Calderón-Zygmund operator, \(\bar{P} = (p_1, \cdots, p_m)\) with \(1 < p_1, \cdots, p_m < \infty\) and \(1/p_1 + \cdots + 1/p_m = 1/p \leq 1\). If \(\bar{w} = (w_1, \cdots, w_m) \in A_{\bar{P}}\), then
\[
\|T(f)\|_{L^p(\bar{w})} \leq C_{n,m, \bar{P}, T, \bar{A}} [\bar{w}]_{A_{\bar{P}}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}
\]
Moreover, for certain multilinear operators the exponent on \([\bar{w}]_{A_{\bar{P}}}\) is the best possible.

In order to prove Theorem 1.4 we will approximate multilinear Calderón-Zygmund operators by positive dyadic operators. The result of Damián, Lerner and Pérez [5] states the following (see Section 2 for pertinent definition).
Theorem 1.5. [5] Theorem 1.4] Let $T$ be a multilinear Calderón-Zygmund operator and let $X$ be a Banach function space over $\mathbb{R}^n$ equipped with Lebesgue measure. Then, for any appropriate $\vec{f}$,

$$
\|T(\vec{f})\|_X \leq C_{T,m,n} \sup_{\mathcal{D}, \mathcal{S}} \|A_{\mathcal{D}, \mathcal{S}}(|\vec{f}|)\|_X.
$$

When $p \geq 1$, $X = L^p(v)$ is a Banach space. However, for $0 < p < 1$ it is not. Since the $m$-linear operators map into $L^p$ for $p > 1/m$ we are are unable to obtain the full range. It is an interesting question as to whether the same decomposition can be obtained for non Banach spaces such as $L^p$ for $0 < p < 1$. Moreover, we believe that inequality (1.5) should hold for all $1 < p_1, \ldots, p_m < \infty$.

We will actually prove the estimate in Theorem 1.4 for the sparse operators $A_{\mathcal{D}, \mathcal{S}}$. For these operators, the main techniques are an extension of those found in [15], in which the second author proved the sharp weighted bound for linear Calderón-Zygmund operators without extrapolation.

The rest of this article is devoted to the following. In Section 2 we state some brief preliminary material. In Section 3 we will prove the main estimates in Theorems 1.2 and 1.4 and in Section 4 we will provide examples to show that our results are sharp.

2. Preliminaries

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$
2^{-k}(0, 1)^n + j, \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.
$$

Denote the standard grid by $\mathcal{D}$.

By a general dyadic grid $\mathcal{D}$ we mean a collection of cubes with the following properties: (i) for any $Q \in \mathcal{D}$ its sidelength, $l_Q$, is of the form $2^k$, $k \in \mathbb{Z}$; (ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathcal{D}$; (iii) the cubes of a fixed sidelength $2^k$ form a partition of $\mathbb{R}^n$. 

Figure 2. The sharp exponents on $[\vec{w}]_{A_{\vec{p}}}$ for bilinear Calderón-Zygmund operators when $p \geq 1$.
Now we define the dyadic maximal function with respect to arbitrary weight

\[ M^D_w f(x) = \sup_{Q \ni x, Q \in D} \frac{1}{w(Q)} \int_Q |f|w. \]

It is well-known that

\[ (2.1) \quad \|M^D_w f\|_{L^p(w)} \leq p' \|f\|_{L^p(w)}, \quad 1 < p < \infty, \]

we refer the readers to [15] for a proof.

We will also need the notion of a sparse family of cubes. Given a dyadic grid \( D \) we say that a family \( S \) is sparse if there are disjoint majorizing subsets, that is, for each \( Q \in S \) there exists \( E_Q \subset Q \) such that \( \{E_Q\}_{Q \in S} \) is pairwise disjoint and \( |E_Q| \geq \frac{1}{2}|Q| \). Sparse families have long played a role in Calderón-Zygmund theory, our definition can be found in [9]. Finally, in [5], the multilinear sparse operators,

\[ A_{\mathcal{D}, S}(\vec{f}) = \sum_{Q \in S} \left( \prod_{i=1}^m \frac{1}{|Q|} \int_Q f_i \right) \chi_Q, \]

were defined and used to approximate multilinear Calderón-Zygmund operators (see Theorem 1.5).

### 3. Proof of Theorems 1.2 and 1.4

First, we give a proof for Theorem 1.2.

**Proof of Theorem 1.2.** In [5], the authors proved that there exists \( 2^n \) families of dyadic grids \( \mathcal{D}_\beta \) such that

\[ \mathcal{M}(\vec{f})(x) \leq 6^{mn} \sum_{\beta=1}^{2^n} \mathcal{M}^{\mathcal{D}_\beta}(\vec{f})(x), \]

where

\[ \mathcal{M}^{\mathcal{D}_\beta}(\vec{f})(x) = \sup_{Q \ni x, Q \in \mathcal{D}_\beta} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i|. \]

Without loss of generality, it suffices to prove that

\[ \|M^D_w (\vec{f})\|_{L^p(vw)} \leq C_{m,n,\beta}[w]_{A_\beta} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}. \]

for a general dyadic grid \( \mathcal{D} \), and \( M^D_w (\vec{f}) = M^D_w (f_1 \sigma_1, \ldots, f_m \sigma_m) \). Moreover, it was shown in [5] Lemma 2.2 that there exists a sparse subset \( S \subset \mathcal{D} \) such that

\[ \mathcal{M}^{\mathcal{D}}(\vec{f}) \lesssim \sum_{Q \in S} \prod_{i=1}^m \left( \frac{1}{|Q|} \int_Q |f_i| \sigma_i \right) \chi_{E_Q}. \]
Without loss of generality, assume that \( p_1 = \min\{p_1, \ldots, p_m\} \). We have

\[
\int_{\mathbb{R}^n} M_\sigma^g(\vec{f})^p v_\vec{w} \lesssim \sum_{Q \in S} \prod_{i=1}^m \left( \frac{1}{|Q|} \int_Q |f_i| \sigma_i \right)^p v_\vec{w}(Q)
\]

\[
= \sum_{Q \in S} v_\vec{w}(Q)^{p_1'} \prod_{i=1}^m \left[ \frac{\sigma_i(Q)^{p_1'/p_i'}}{|Q|^{mp_1'}} \left( \prod_{i=1}^m \int_Q |f_i| \sigma_i \right)^p \right] v_\vec{w}(Q)^{p_1'}
\]

\[
\leq |\vec{w}|_{A_{p_1'}}^{p_1'} \sum_{Q \in S} \frac{2^{mp_1'-1}|E_Q|^{mp_1'-1}}{\prod_{i=1}^m \sigma_i(Q)^{p_1'/p_i'}} \left( \prod_{i=1}^m \int_Q |f_i| \sigma_i \right)^p.
\]

By Hölder’s inequality, we have

\[
|E_Q| = \int_{E_Q} v_\vec{w} \sigma_1^{mp_1'} \cdots \sigma_m^{mp_m'} \leq v_\vec{w}(E_Q)^{1/mp_1'} \sigma_1(E_Q)^{1/mp_1'} \cdots \sigma_m(E_Q)^{1/mp_m'}.
\]

Therefore,

\[
|E_Q|^{mp_1'-1} \leq v_\vec{w}(E_Q)^{p_1'-1} \sigma_1(E_Q)^{p_1'/p_1} \cdots \sigma_m(E_Q)^{p_1'/p_m}
\]

and

\[
\frac{p(p_1'-1)}{p_1'} - \frac{p}{p_i} = \frac{pp_1'}{p_1'} - p \geq 0.
\]

Since \( E_Q \subset Q \), we have

\[
v_\vec{w}(E_Q)^{p_1'-1} \leq v_\vec{w}(Q)^{p_1'-1}
\]

and hence

\[
\sigma_i(E_Q)^{p_1'/p_1} \leq \sigma_i(Q)^{p_1'/p_1 - p}, \quad i = 1, \ldots, m.
\]

It follows that

\[
\sum_{Q \in S} \frac{|E_Q|^{mp_1'-1}}{v_\vec{w}(Q)^{p_1'-1} \prod_{i=1}^m \sigma_i(Q)^{p_1'/p_i}} \left( \prod_{i=1}^m \int_Q |f_i| \sigma_i \right)^p \leq \sum_{Q \in S} \prod_{i=1}^m \left( \frac{1}{\sigma_i(Q)} \int_Q |f_i| \sigma_i \right)^p \sigma_i(E_Q)^{p/p_i}
\]

\[
\leq \prod_{i=1}^m \left( \sum_{Q \in S} \left( \frac{1}{\sigma_i(Q)} \int_Q |f_i| \sigma_i \right)^{p_i} \sigma_i(E_Q)^{p/p_i} \right)
\]

\[
\leq \prod_{i=1}^m \|M_\sigma^g(f_i)\|_{L^{p_i}(\sigma_i)}^p.
\]
\[
\|M_D(\vec{f})\|_{L^p(v\vec{w})} \leq C_{m,n,\vec{P}} \max_{\vec{w}} (\vec{P}) \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.
\]

This completes the proof. \qed

We now turn our attention to the proof of Theorem 1.4. First, we note the following symmetry of \(A_{\vec{P}}\) weights.

**Lemma 3.1.** Suppose that \(\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}\) and that \(1 < p_1, \ldots, p_m < \infty\) with \(1/p_1 + \cdots + 1/p_m = 1/p\). Then \(\vec{w}' := (w_1, \ldots, w_{i-1}, v_1^{1-p'}, w_{i+1}, \ldots, w_m) \in A_{\vec{P}}\) with \(\vec{P}' = (p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_m)\) and

\[
[\vec{w}']_{A_{\vec{P}}} = [\vec{w}]_{A_{\vec{P}}}^{p'/p}.
\]

**Proof.** We will prove the conclusion for \(i = 1\); the other cases are similar. Notice that

\[
1/p' + 1/p_2 + \cdots + 1/p_m = 1/p_1
\]

and

\[
v_1^{(1-p')p'/p'} \cdot v_2^{p_2/p_2} \cdots v_m^{p_m/p_m} = w_1^{1-p'}.
\]

By the definition of multiple \(A_{\vec{P}}\) constant, we have

\[
[\vec{w}]_{A_{\vec{P}}} = \sup_Q \left( \frac{1}{|Q|} \int_Q w_1^{1-p'} \right) \cdot \left( \frac{1}{|Q|} \int_Q (v_1^{1-p'})^{1-p} \right)^{p_1/p}
\]

\[
\times \prod_{i=2}^m \left( \frac{1}{|Q|} \int_Q v_i^{1-p'} \right)^{p_i/p_i'}
\]

\[
= [\vec{w}]_{A_{\vec{P}}}^{p'/p}.
\]

\qed

By Theorem 1.5 we reduce our problem to consider the behavior of the operator \(A_{\vec{P},S}\). For these operators we have the following Theorem, which holds for all \(1 < p_1, \ldots, p_m < \infty\).

**Theorem 3.2.** Suppose that \(1 < p_1, \ldots, p_m < \infty\) with \(1/p_1 + \cdots + 1/p_m = 1/p\) and \(\vec{w} \in A_{\vec{P}}\). Then

\[
\|A_{\vec{P},S}(\vec{f})\|_{L^p(v\vec{w})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max(1, p_1/p, \ldots, p_m/p)} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.
\]

**Proof.** We first consider the case when \(1/m < p \leq 1\). In this case

\[
\int_{\mathbb{R}^n} A_{\vec{P},S}(\vec{f})^p v_{\vec{w}} \leq \sum_{Q \in S} \left( \prod_{i=1}^m \frac{1}{|Q|} \int_Q f_i \right)^p v_{\vec{w}}(Q),
\]
which can be handled in exactly the same manner as the estimates in proof of Theorem 1.2.

Now consider the case \( p \geq \max_i p_i' \). It is sufficient to prove that

\[
\| A_{\varphi, S}(f^\sigma) \|_{L^p(v^\alpha)} \lesssim [\tilde{w}] A_{\varphi} \prod_{i=1}^{m} \| f_i \|_{L^{p_i}(\sigma_i)},
\]

where \( \sigma_i = w_{i}^{1-p_i'} \), \( A_{\varphi, S}(f^\sigma) = A_{\varphi, S}(f_{1, \sigma_1}, \ldots, f_{m, \sigma_m}) \), and \( f_i \geq 0 \). By duality, it suffices to estimate the \((m+1)\)-linear form

\[
\int_{\mathbb{R}^n} A_{\varphi, S}(f^\sigma) g v^\tilde{w} = \sum_{Q \in S} \int_{Q} g v^\tilde{w} \cdot \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} f_i \sigma_i
\]

for \( g \geq 0 \) belonging to \( L^p(v^\alpha) \). We have

\[
\sum_{Q \in S} \int_{Q} g v^\tilde{w} \cdot \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} f_i \sigma_i
\]

\[
= \sum_{Q \in S} v^\tilde{w}(Q) \prod_{i=1}^{m} \sigma_i(Q)^{\frac{1}{p_i}} \cdot \int_{Q} g v^\tilde{w} \cdot \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} f_i \sigma_i
\]

\[
\leq [\tilde{w}] A_{\varphi} \sum_{Q \in S} \frac{|Q|^{m(p-1)}}{v^\tilde{w}(Q) \prod_{i=1}^{m} \sigma_i(Q)^{\frac{1}{p_i}}} \cdot \int_{Q} g v^\tilde{w} \cdot \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} f_i \sigma_i
\]

\[
\leq [\tilde{w}] A_{\varphi} \sum_{Q \in S} \frac{2^{m(p-1)}|E_Q|^{m(p-1)}}{v^\tilde{w}(Q) \prod_{i=1}^{m} \sigma_i(Q)^{\frac{1}{p_i}}} \cdot \int_{Q} g v^\tilde{w} \cdot \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} f_i \sigma_i.
\]

By (3.3),

\[
|E_Q| \leq v^\tilde{w}(E_Q)^{\frac{1}{mp_1}} \sigma_1(E_Q)^{\frac{1}{mp_1}} \cdots \sigma_m(E_Q)^{\frac{1}{mp_m}}.
\]

Since \( p \geq \max_i \{ p_i' \} \) and \( E_Q \subset Q \), we have \( \sigma_i(Q)^{1-\frac{p_i}{p_i'}} \leq \sigma_i(E_Q)^{1-\frac{p_i}{p_i'}} \) for any \( i = 1, \ldots, m \). Therefore,

\[
\sum_{Q \in S} \int_{Q} g v^\tilde{w} \cdot \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} f_i \sigma_i
\]

\[
\leq 2^{m(p-1)}[\tilde{w}] A_{\varphi} \sum_{j,k} v^\tilde{w}(E_Q)^{\frac{1}{mp}} \prod_{i=1}^{m} \sigma_i(E_Q)^{\frac{p_i-1}{p_i'}} \sigma_i(Q)^{1-\frac{p_i}{p_i'}}
\]

\[
\cdot \int_{Q} g v^\tilde{w} \cdot \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} f_i \sigma_i
\]

\[
\leq 2^{m(p-1)}[\tilde{w}] A_{\varphi} \sum_{Q \in S} v^\tilde{w}(E_Q)^{\frac{1}{mp}} \prod_{i=1}^{m} \sigma_i(E_Q)^{\frac{1}{p_i'}} \frac{1}{v^\tilde{w}(Q)} \int_{Q} g v^\tilde{w}.
\]
∥M∥

\[ \sum_{Q \in S} \left( \frac{1}{\sigma_i(Q)} \int_Q v_i w_i(E_Q) \right)^{1/p_i} \]

\[ \prod_{i=1}^m \left( \sum_{Q \in S} \left( \frac{1}{\sigma_i(Q)} \int_Q f_i \sigma_i E(Q) \right) \right)^{1/p_i} \]

2^{m(p-1)} [\tilde{w}] \|M_{\mathcal{F}_w}^g\|_{L^p(v_\tilde{w})} \prod_{i=1}^m \|M_{\sigma_i(f_i)}^g\|_{L^{p_i}(\sigma_i)}

2^{m(p-1)} [\tilde{w}] \|g\|_{L^p(v_\tilde{w})} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}

\[
\int_{\mathbb{R}^n} A_{\mathcal{G},\mathcal{S}}(f_1, \ldots, f_m)g = \int_{\mathbb{R}^n} A_{\mathcal{G},\mathcal{S}}(f_1, \ldots, f_i-1, g, f_i+1, \ldots, f_m)f_i.
\]

Without loss of generality suppose \( p'_1 \geq \max(p, p'_2, \ldots, p'_m) \). Hence, by duality and self adjointness we have

\[
\|A_{\mathcal{G},\mathcal{S}}\|_{L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(v_\tilde{w})} = [w^1]_{\tilde{p}_1} = [\tilde{w}]_{A_{\mathcal{G}}}.
\]

4. Examples

Finally, we end with some examples to show that our bounds are sharp.

First we show that Theorem 1.2 is sharp. Consider the case \( m = 2 \) (we leave it to the reader to modify the example for \( m > 2 \)) and suppose that we had a better exponent than the one in inequality 1.4 that is, suppose

\[
\|M\|_{L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(v_\tilde{w})} \lesssim [\tilde{w}]_{A_{\mathcal{G}}}^{\max(p_1, p_2)}
\]

for some \( r < 1 \). Further suppose that \( p'_1 \geq p'_2 \). For \( 0 < \varepsilon < 1 \), let

\[
f_1(x) = |x|^{-n} \chi_{B(0,1)}(x), \quad f_2(x) = |x|^{-\frac{n}{p_2}} \chi_{B(0,1)}(x), \quad w_1(x) = |x|^{(n-\varepsilon)(p_1-1)}
\]

and \( w_2(x) = 1 \). Calculations show that

\[
\|f_1\|_{L^{p_1}(w_1)} \approx \varepsilon^{-1/p_1}, \quad \|f_2\|_{L^{p_2}(w_2)} \approx \varepsilon^{-1/p_2}, \quad v_\tilde{w}(x) = |x|^{(n-\varepsilon)p_1}
\]

and

\[
[\tilde{w}]_{A_{\mathcal{G}}} \approx \varepsilon^{-p/p'_1}.
\]
For $x \in B(0,1)$ we have
\[
\mathcal{M}(f_1, f_2)(x) \gtrsim \frac{1}{|x|^n} \int_{B(0,|x|)} |y|^\varepsilon - n \, dy_1 \cdot \frac{1}{|x|^n} \int_{B(0,|x|)} |y_2|^\varepsilon - \frac{n}{r_2} \, dy_2
\]
\[
\gtrsim \frac{f_1(x)f_2(x)}{\varepsilon \cdot \left(\frac{r_2}{n} + n\right)} \gtrsim \frac{f_1(x)f_2(x)}{\varepsilon}.
\]
Hence,
\[
\|\mathcal{M}(f_1, f_2)\|_{L^p(v_d)} \gtrsim \frac{1}{\varepsilon} \left( \int_{B(0,1)} |x|^{(\varepsilon - n)(p + \frac{n}{r_2} - \frac{p_1}{p_2})} \, dx \right)^{1/p}
\]
\[
\approx \frac{1}{\varepsilon} \left( \int_0^1 x^{\varepsilon - 1} \, dx \right)^{1/p}
\]
\[
= \frac{1}{\varepsilon} \left( \frac{1}{1} \right)^{1/p}.
\]
Combining this with inequality (4.1) we see for some $r < 1$,
\[
\left( \frac{1}{\varepsilon} \right)^{1+\frac{1}{p}} \lesssim \left( \frac{1}{\varepsilon} \right)^{r+\frac{1}{p}},
\]
which is impossible as $\varepsilon \to 0$.

Next we show that Theorem 1.4 is sharp. Recall that for $i = 1, \ldots, n$, the $m$-linear $i$th Riesz transform is defined by
\[
R_i(\vec{f})(x) = p.v. \int_{\mathbb{R}^n} \frac{\sum_{j=1}^m (x_i - (y_j)_i)}{(\sum_{j=1}^m |x - y_j|^2)^{(n+1)/2}} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,
\]
where $(y_j)_i$ denotes the $i$th coordinate of $y_j$.

Suppose that $m = 2$, $p'_1 \geq p'_2$ and $p'_1 \geq p$. Let
\[
U = \{ x \in \mathbb{R}^n : |x| \leq 1, 0 < x_i \leq x_1, i = 2, \ldots, n \},
\]
\[
V = \{ x \in \mathbb{R}^n : |x| \leq 1, x_i \leq 0, i = 1, \ldots, n \}.
\]
For $0 < \varepsilon < 1$, let $f_1(x) = |x|^{\varepsilon - n} \chi_V(x)$, $f_2(x) = |x|^{\varepsilon - \frac{n}{r_2}} \chi_V(x)$, $w_1(x) = |x|^{(n-\varepsilon)(p_1-1)}$ and $w_2(x) = 1$. For $x \in U$ and $y_j \in V$ with $|y_j| \leq |x|$, we have
\[
\frac{\sum_{j=1}^2 (x_1 - (y_j)_1)}{(\sum_{j=1}^2 |x - y_j|^2)^{1/2}} \geq \frac{2 |x|^{1/n}}{4 |x|} \gtrsim 1.
\]
Therefore,
\[
\frac{\sum_{j=1}^2 (x_1 - (y_j)_1)}{(\sum_{j=1}^2 |x - y_j|^2)^{(2n+1)/2}} \gtrsim \frac{1}{|x|^{2n}}.
\]
It follows that
\[
R_1(\vec{f})(x) = p.v. \int_{\mathbb{R}^n} \frac{\sum_{j=1}^2 (x_i - (y_j)_i)}{(\sum_{j=1}^2 |x - y_j|^2)^{(2n+1)/2}} f_1(y_1) f_2(y_2) dy_1 dy_2
\]
\[
\gtrsim \int_{|y_1| \leq |x|} \int_{|y_2| \leq |x|} \frac{1}{|x|^{2n}} |y_1|^{\varepsilon - n} : |y_2|^{\varepsilon - \frac{n}{r_2}} dy_1 dy_2
\]
\[ \geq \frac{1}{\varepsilon} f_1(x)f_2(x). \]

Hence
\[
\| R_1(\tilde{f}) \|_{L^p(v_\omega)} \geq \frac{1}{\varepsilon} \left( \int_U |x|^{(\varepsilon-n)(p+p_2-p/p')} \, dx \right)^{1/p} \\
= \frac{1}{\varepsilon} \left( \int_U |x|^{\varepsilon-n} \, dx \right)^{1/p} \\
\geq \frac{1}{\varepsilon} \left( \int_{\{|x| \leq 1\}} |x|^{\varepsilon-n} \, dx \right)^{1/p} \quad \text{(by symmetry)} \\
\geq \left( \frac{1}{\varepsilon} \right)^{1+1/p}.
\]

Then by similar arguments as the above we can show that the exponent is sharp when \( \max(p'_1, p'_2) \geq p \geq 1 \). When \( p \geq \max(p'_1, p'_2) \). Again, suppose that \( p'_1 \geq p'_2 \). We consider the adjoint in the first variable, \( (R_1)^{1,*} \). Notice that
\[
(R_1)^{1,*}(f_1, f_2)(x) = \int_{(\mathbb{R}^n)^2} \frac{2(y_1)_1 - x_1 - (y_2)_1}{|x - y_1|^2 + |y_1 - y_2|^2(2n+1)/2}f(y_1)f(y_2)dy_1dy_2.
\]

Let
\[
U_1 = \{ x \in \mathbb{R}^n : |x| \leq 1, x_1 \leq x_i < 0, i = 2, \cdots, n \}, \\
V_1 = \{ x \in \mathbb{R}^n : |x| \leq 1, x_i \geq 0, i = 1, \cdots, n \}.
\]

For \( 0 < \varepsilon < 1 \), let \( f_1(x) = |x|^{\varepsilon-n}\chi_{U_1}(x), f_2(x) = |x|^{\varepsilon-n}\chi_{V_1}(x), w_1(x) = |x|^{(\varepsilon-n)p_1/p} \) and \( w_2(x) = 1 \). Then \( v_\omega(x) = |x|^{\varepsilon-n}, v_\omega^{1-p'_i} = |x|^{(n-\varepsilon)(p'_i-1)} \) and \( w_1^{1-p'_i} = |x|^{(n-\varepsilon)p'_i/p} \). Similar arguments as the above show that
\[
\| (R_1)^{1,*} \|_{L^{p'_i}(v_\omega^{1-p'_i}) \times L^{p_2}(w_2) \rightarrow L^{p'_i}(w_1^{1-p'_i})} \geq \frac{1}{\varepsilon} = \left[ \bar{w}^{-1/p'} \right]_{A_{p_1}} = \left[ \bar{w} \right]_{A_{p'_i}}.
\]

Therefore,
\[
\| R_1 \|_{L^{p_1}(w_1) \times L^{p_2}(w_2) \rightarrow L^{p}(v_\omega)} = \| (R_1)^{1,*} \|_{L^{p'_i}(v_\omega^{1-p'_i}) \times L^{p_2}(w_2) \rightarrow L^{p'_i}(w_1^{1-p'_i})} \geq \left[ \bar{w} \right]_{A_{p'_i}}.
\]

This shows the sharpness of the exponent when \( p \geq \max\{p'_i\} \), which completes the proof.

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