AVERAGING PRINCIPLE FOR STOCHASTIC REAL
GINZBURG-LANDAU EQUATION DRIVEN BY \( \alpha \)-STABLE PROCESS

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Abstract. In this paper, we study a system of stochastic partial differential equations with
slow and fast time-scales, where the slow component is a stochastic real Ginzburg-Landau
equation and the fast component is a stochastic reaction-diffusion equation, the system
is driven by \( \alpha \)-stable process with \( \alpha \in (1,2) \). Using the classical Khasminskii approach
based on time discretization and the techniques of stopping times, we show that the slow
component strong converges to the solution of the corresponding averaged equation under
some suitable conditions.

1. Introduction

In this paper, we are interested in studying the averaging principle for stochastic real
Ginzburg-Landau equation driven by \( \alpha \)-stable process, i.e., considering the following sto-
chastic slow-fast system on torus \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \):

\[
\begin{align*}
&dX^\varepsilon_t(\xi) = \left[ \frac{\partial^2}{\partial \xi^2}X^\varepsilon_t(\xi) + X^\varepsilon_t(\xi) - (X^\varepsilon_t(\xi))^3 + f(X^\varepsilon_t(\xi), Y^\varepsilon_t(\xi)) \right] dt + dL_t, \quad X^\varepsilon_0 = x, \\
&dY^\varepsilon_t(\xi) = \frac{1}{\varepsilon} \left[ \frac{\partial^2}{\partial \xi^2}Y^\varepsilon_t(\xi) + g(X^\varepsilon_t(\xi), Y^\varepsilon_t(\xi)) \right] dt + \frac{1}{\varepsilon^{1/\alpha}} dZ_t, \quad Y^\varepsilon_0 = y,
\end{align*}
\]

where \( \varepsilon > 0 \) is a small parameter describing the ratio of time scales between the slow
component \( X^\varepsilon_t \) and fast component \( Y^\varepsilon_t \). The coefficients \( f \) and \( g \) satisfy some suitable
conditions. \( \{L_t\}_{t \geq 0} \) and \( \{Z_t\}_{t \geq 0} \) are mutually independent cylindrical \( \alpha \)-stable process,
\( \alpha \in (1,2) \). Under some assumptions, we aim to prove \( X^\varepsilon \) convergent to \( \bar{X} \) in the strong
sense, i.e.,

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|_{L^2(\mathbb{T})}^\kappa \right) = 0
\]

holds for any \( 0 < \kappa < 1 \), where \( \bar{X} \) is the solution of the corresponding averaged equation
(see Eq. (2.13) below).

The theory of averaging principle has a long and rich history. Bogoliubov and Mitropolsky
[2] first studied the averaging principle for the deterministic systems. Later on, the theory
of averaging principle for stochastic differential equations was firstly proved by Khasminskii
[21], see, e.g., [14, 15, 16, 18, 19, 28] for further generalization. Recently, averaging principle
for stochastic reaction-diffusion systems has become an active research area which attracted
much attention (see for instance [3, 4, 5, 6, 10, 11, 12, 22, 24, 25]).

To the best of our knowledge, there are rarely results on the averaging principle for stochastic
partial differential equations with nonlinear term on this topic. The averaging principle for

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one dimensional stochastic Burgers equation and two dimensional stochastic Navier-Stokes equation have been studied in [7] and [17] respectively. Averaging principle for stochastic Kuramoto-Sivashinsky equation with a fast oscillation was studied by Gao in [13]. But the noise considered in above references are Wiener noise.

This paper focus on studying the averaging principle for stochastic real Ginzburg-Landau equation driven by $\alpha$-stable process. Bao et al. [1] study the averaging principle for two-time scale stochastic partial differential equations driven by $\alpha$-stable noise without the nonlinear term. Xu et al. [26] study the strong convergence of the averaging principle for slow-fast SPDEs driven by Poisson random measures. However, $\alpha$-stable noise does not have second moment, so we can’t take $p$-th moment of the solution ($p \geq 2$), hence some methods developed in [7, 17, 26] do not work in this situation. So, the most challenge here is how to deal with the nonlinear term and $\alpha$-stable noise. To overcome these difficulties, we shall deal with the nonlinear term and the stable noise more carefully. The techniques of stopping times will be used frequently.

The proof of our main result is divided into several steps. Firstly, we follow the skills in [27] to give a priori estimate of the solution $(X_\varepsilon^t, Y_\varepsilon^t)$, which is very important to constrict some stopping times later. Meanwhile, we prove an estimate of $|X_\varepsilon^t - X_s^\varepsilon|$ when $s, t$ before the stopping time. Secondly, based on the Khasminskii discretization introduced in [21], we split the interval $[0, T]$ into some subintervals of size $\delta > 0$ which depends on $\varepsilon$, and on each interval $[k\delta, (k + 1)\delta)$, $k \geq 0$, we construct an auxiliary process $(\hat{X}_\varepsilon^t, \hat{Y}_\varepsilon^t)$ which associate with the system (1.1). Finally, by controlling the difference processes $X_\varepsilon^t - \hat{X}_\varepsilon^t$ and $\hat{X}_\varepsilon^t - \bar{X}_t$ respectively, we obtain (1.2) when time before the stopping time. Moreover, we use the priori estimates of the $X_\varepsilon^t$ and $\bar{X}_t$ to get a control the term of time after stopping time.

The paper is organized as follows. In the next section, we introduce some notation and assumptions that we use throughout the paper, and give out the main result. The section 3 is devoted to prove the strong convergence. The final section is the appendix, where we show the detailed proof of existence and uniqueness of solution, and the corresponding Galerkin approximation.

Along the paper, $C, C_p, C_T$ and $C_{p,R,T}$ denote positive constants which may change from line to line, where $C_p$ depends on $p$, $C_T$ depends on $T$, and $C_{p,R,T}$ depends on $p, R, T$.

### 2. Notations and main results

For $p \geq 1$, let $L^p(\mathbb{T})$ be the space of $p$-th power integrable $\mathbb{R}$-valued functions on torus $\mathbb{T}$ and $| \cdot |_{L^p}$ be the usual norm. For $k \in \mathbb{N}$, $W^{k,2}(\mathbb{T})$ is the Sobolev space of all functions in $L^2(\mathbb{T})$ whose differentials belong to $L^2(\mathbb{T})$ up to the order $k$. Let

$$H := \{ x \in L^2(\mathbb{T}) : \int_{\mathbb{T}} x(\xi) d\xi = 0 \}$$

be a separable real Hilbert space with inner product

$$\langle x, y \rangle := \int_{\mathbb{T}} x(\xi)y(\xi) d\xi, \quad x, y \in H$$

and norm

$$\| x \| := \left( \int_{\mathbb{T}} x^2(\xi) d\xi \right)^{1/2}, \quad x \in H.$$
Denote Laplacian operator $\Delta$ by

$$Ax := \Delta x := \frac{\partial^2}{\partial x^2} x, \quad x \in W^{k,2}(\mathbb{T}) \cap H.$$  

The eigenfunctions of $\Delta$ are given by

$$\{e_i : e_i = \sqrt{2} \cos(2\pi i \xi)\}_{i \in \{1,2,\ldots\}} \cup \{e_j : e_j = \sqrt{2} \sin(2\pi j \xi)\}_{j \in \{-1,-2,\ldots\}},$$

which is an orthonormal basis of $H$. For any $k \in \mathbb{Z}_+ := \mathbb{Z} \setminus \{0\}$,

$$Ae_k = -\lambda_k e_k \quad \text{with} \quad \lambda_k = 4\pi^2 |k|^2.$$  

For any $s \in \mathbb{R}$, we define

$$H^s := \mathcal{D}((A)^{s/2}) := \left\{ u = \sum_k u_k e_k : u_k = \langle u, e_k \rangle \in \mathbb{R}, \sum_k \lambda_k^s u_k^2 < \infty \right\},$$

and

$$(-A)^{s/2} u := \sum_k \lambda_k^{s/2} u_k e_k, \quad u \in \mathcal{D}((-A)^{s/2}),$$

with the associated norm

$$\|u\|_s := \|(-A)^{s/2} u\| = \sqrt{\sum_k \lambda_k^s u_k^2}.$$  

It is easy to see $H^0 = H$ and $H^{-s}$ be the dual space of $H^s$. Notice that the dual action is also denoted by $\langle \cdot , \cdot \rangle$ without confusion. Denote $V := H^1$, which is densely and compactly embedded in $H$. It is well known that $A$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}\}_{t \geq 0}$.

Define nonlinear operator,

$$N(x)(\xi) = x(\xi) - [x(\xi)]^3, \quad x \in H.$$  

We shall often use the following inequalities:

\begin{align}
\|x\|_{\sigma_1} &\leq C_{\sigma_1,\sigma_2} \|x\|_{\sigma_2}, \quad x \in H^{\sigma_2}, \sigma_1 \leq \sigma_2; \quad (2.1) \\
\|e^{tA} x - x\| &\leq C_{\sigma, t^2} \|x\|_{\sigma}, \quad x \in H^{\sigma}, \sigma \geq 0, t > 0; \quad (2.2) \\
\|e^{tA} x\| &\leq e^{-\lambda t} \|x\|, \quad x \in H, t \geq 0; \quad (2.3) \\
\|e^{tA} x\|_{\sigma_2} &\leq C_{\sigma_1,\sigma_2} t^{-\sigma_2+1}\|x\|_{\sigma_1}, \quad x \in H^{\sigma_2}, \sigma_1 \leq \sigma_2, t > 0; \quad (2.4) \\
\|N(x)\|_{-\sigma} &\leq C_{\sigma} (1 + \|x\|^3_{1/2}), \quad x \in H^{1/2}, \sigma \in [0, 1/2); \quad (2.5) \\
\|N(x) - N(y)\| &\leq C (1 + \|x\|^2_{1/2} + \|y\|^2_{1/2}) \|x - y\|, \quad x, y \in H^{1/2}; \quad (2.6) \\
\|N(x) - N(y)\| &\leq C (1 + \|x\|^2_{2\sigma} + \|y\|^2_{2\sigma}) \|x - y\|_{2\sigma}, \quad x, y \in H^{2\sigma}, \sigma \geq 1/6; \quad (2.7) \\
\langle x, N(x) \rangle &\leq \frac{1}{4}, \quad x \in H. \quad (2.8)
\end{align}

The proof of (2.6)-(2.8) can be founded in [27, Appendix] and we will show (2.5) in the Appendix.

With the above notations, the system (1.1) can be rewritten as:

\begin{align}
\left\{ \begin{array}{l}
\frac{dX_t^\varepsilon}{\varepsilon} = [AX_t^\varepsilon + N(X_t^\varepsilon) + f(X_t^\varepsilon, Y_t^\varepsilon)] dt + dL_t, \quad X_0^\varepsilon = x, \\
\frac{dY_t^\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} [AY_t^\varepsilon + g(X_t^\varepsilon, Y_t^\varepsilon)] dt + \frac{1}{\varepsilon^{1/\sigma}} dZ_t, \quad Y_0^\varepsilon = y,
\end{array} \right. \quad (2.9)
\end{align}
where \( \{L_t\}_{t \geq 0} \) and \( \{Z_t\}_{t \geq 0} \) are mutually independent cylindrical \( \alpha \)-stable process given by
\[
L_t = \sum_{k \in \mathbb{Z}_+} \beta_k L_t^k e_k, \quad Z_t = \sum_{k \in \mathbb{Z}_+} \gamma_k Z_t^k e_k, \quad t \geq 0,
\]
where \( \alpha \in (1, 2) \), \( \{\beta_k\}_{k \in \mathbb{Z}_+} \) and \( \{\gamma_k\}_{k \in \mathbb{Z}_+} \) are two given sequence of positive numbers and \( \{L_t^k\}_{k \in \mathbb{Z}_+} \) and \( \{Z_t^k\}_{k \in \mathbb{Z}_+} \) are independent one dimensional \( \alpha \)-stable processes satisfying for any \( k \in \mathbb{Z}_+ \) and \( t \geq 0 \),
\[
\mathbb{E}[e^{iL_t^k}] = \mathbb{E}[e^{iZ_t^k}] = e^{-t|h|^\alpha}, \quad h \in \mathbb{R}.
\]
We impose the conditions on the functions \( f, g : H \times H \to H \).

**A1.** \( f \) and \( g \) are Lipschitz continuous, i.e., there exist constants \( C > 0 \) and \( L_f, L_g > 0 \) such that for any \( x_1, x_2, y_1, y_2 \in H \),
\[
\|f(x_1, y_1) - f(x_2, y_2)\| \leq L_f (\|x_1 - x_2\| + \|y_1 - y_2\|)
\]
and
\[
\|g(x_1, y_1) - g(x_2, y_2)\| \leq C (\|x_1 - x_2\| + L_g \|y_1 - y_2\|).
\]

**A2.** There exist constants \( C_1, C_2 > 0 \) such that
\[
C_1 \lambda_k^{-\beta} \leq \beta_k \leq C_2 \lambda_k^{-\beta}, \quad \text{with} \quad \beta > \frac{1}{2} + \frac{1}{2\alpha}
\]
and
\[
\sum_{k \in \mathbb{Z}_+} \frac{\gamma_k}{\lambda_k} < \infty. \tag{2.10}
\]

**A3.** \( f \) is uniformly bounded, i.e., there exists \( C > 0 \) such that
\[
\sup_{x, y \in H} \|f(x, y)\| \leq C.
\]

**A4.** The smallest eigenvalue \( \lambda_1 \) of \( -A \) and the Lipschitz constant \( L_g \) satisfy
\[
\lambda_1 - L_g > 0.
\]

**Remark 2.1.** Under the condition **A1-A3**, for any given initial value \( x, y \in H, \varepsilon > 0 \), system \( (2.9) \) exists a unique mild solution \( X^\varepsilon \in \mathcal{D}([0, \infty); H) \cap \mathcal{D}((0, \infty); V) \), \( Y^\varepsilon \in H \) (see Appendix below). By [20], in general, \( (2.10) \) in condition **A2** does not imply that \( Y^\varepsilon \in \mathcal{D}([0, \infty); H) \), only imply \( Y^\varepsilon \in H \), but it is enough for us to prove our main result. Condition **A4** is called the dissipative condition, which used to give the uniform estimate of \( Y^\varepsilon \) with respect to \( \varepsilon \) and the exponential ergodicity of frozen equation (see Proposition 3.7 below).

**Remark 2.2.** Define
\[
L_A(t) := \int_0^t e^{(t-s)A}dL_s, \quad Z_A(t) := \int_0^t e^{(t-s)A}dZ_s.
\]
Refer to [27, Lemma 3.1] and [23, (4.12)], if condition **A2** holds, then for any \( T > 0, 0 \leq \theta < \beta - \frac{1}{2\alpha} \) and \( 0 < p < \alpha \),
\[
\mathbb{E} \sup_{0 \leq t < T} \|L_A(t)\|^p_{2\theta} \leq C_1 T^{p/\alpha}, \quad \sup_{t \geq 0} \mathbb{E}\|Z_A(t)\|^p \leq C_2,
\]
where \( C_1 \) depends on \( \alpha, \theta, \beta, p \) and \( C_2 \) depends on \( \alpha, p \).
Based on the Banach fixed point theorem, we have the following existence and uniqueness of the mild solution of system (2.9), whose proof is given in the appendix.

**Theorem 2.3.** Assume the conditions A1-A3 hold. Then for every \( \varepsilon > 0 \), \( x \in H \), \( y \in H \), system (2.9) admits a unique mild solution \( X^\varepsilon(\omega) \in D([0, \infty); H) \cap D((0, \infty); V) \) and \( Y^\varepsilon(\omega) \in H \), \( t \geq 0 \), \( \mathbb{P}\text{-a.s.} \). Moreover, for any \( t \geq 0 \)

\[
\begin{aligned}
X^\varepsilon_t &= e^{tA}x + \int_0^t e^{(t-s)A}N(X^\varepsilon_s)ds + \int_0^t e^{(t-s)A}f(X^\varepsilon_s, Y^\varepsilon_s)ds + \int_0^t e^{(t-s)A}dL_s, \\
Y^\varepsilon_t &= e^{tA/\varepsilon}y + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}g(X^\varepsilon_s, Y^\varepsilon_s)ds + \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s.
\end{aligned}
\]  

(2.11)

The main result of this paper is the following theorem.

**Theorem 2.4.** Assume the conditions A1-A4 hold. Then for any \( x \in H^0 \) with \( \theta \in (1/2, 1] \), \( y \in H \), \( T > 0 \) and \( 0 < \kappa < 1 \),

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \tilde{X}_t\|^\kappa \right) = 0,
\]

(2.12)

where \( \tilde{X}_t \) is the solution of the corresponding averaged equation:

\[
\begin{aligned}
\begin{cases}
\displaystyle d\tilde{X}_t = [A\tilde{X}_t + N(\tilde{X}_t) + \tilde{f}(\tilde{X}_t)] \, dt + dL_t, \\
\tilde{X}_0 = x,
\end{cases}
\end{aligned}
\]

(2.13)

with the average \( \tilde{f}(x) = \int_H f(x, y)\mu^x(dy) \). Here \( \mu^x \) is the unique invariant measure of the frozen equation

\[
\begin{cases}
\displaystyle dY_t = [AY_t + g(x, Y_t)] \, dt + d\tilde{Z}_t, \\
Y_0 = y,
\end{cases}
\]

\( \tilde{Z}_t \) is a version of \( Z_t \) and independent of \( L_t \) and \( Z_t \).

**Remark 2.5.** Since we only have the priori estimate \( \mathbb{E} \left( \sup_{0 \leq t \leq T} \|X^\varepsilon_t\| \right) < \infty \) (see Lemma 3.1 below), the main result (2.12) holds only for \( 0 < \kappa < 1 \).

3. **Proof of Theorem 2.4**

In this section, we are devoted to proving Theorem 2.4. The proof consists of the following several steps. In the first step, we give a priori estimate of the solution \( (X^\varepsilon_t, Y^\varepsilon_t) \) in Lemma 3.1, which is used to construct a stopping time \( \tau^\varepsilon_R \). Then Lemma 3.2 gives a uniform estimate of \( \|X^\varepsilon_t\|_\theta \) when \( t \leq T \land \tau^\varepsilon_R \) for \( \theta \in (1/2, 1] \), which is used to obtain an estimate of the expectation of \( X^\varepsilon_s - X^\varepsilon_t \) when \( 0 \leq s \leq t \leq T \land \tau^\varepsilon_R \) in Lemma 3.3. In the second step, following the idea inspired by Khasminskii in [21], we introduce an auxiliary process \( (\hat{X}^\varepsilon_t, \hat{Y}^\varepsilon_t) \in H \times H \) and also give the uniform bounds, see Lemma 3.4. Meanwhile, we introduce a new stopping time \( \hat{\tau}^\varepsilon_R \leq \tau^\varepsilon_R \). Then Lemma 3.5 is used to deduce an estimate of the difference process \( X^\varepsilon_t - \hat{X}^\varepsilon_t \) when \( t \leq T \land \hat{\tau}^\varepsilon_R \), which will be stated in Lemma 3.6. In the third step, we study the frozen equation and average equation. After defining another stopping time \( \hat{\tau}^\varepsilon_R \leq \hat{\tau}^\varepsilon_R \), we give a control of \( \hat{X}^\varepsilon_t - \hat{X}_t \) when \( t \leq T \land \hat{\tau}^\varepsilon_R \) in Lemma 3.10. Finally, in order to prove the main result, it is sufficient to control the term of time after the stopping \( \hat{\tau}^\varepsilon_R \), which will be done by the priori estimates of the \( X^\varepsilon_t, \hat{X}_t \) (see Lemma 3.9).
3.1. Some priori estimates of \((X_t^\varepsilon, Y_t^\varepsilon)\). We first prove the uniform bounds for \(p\)-moment of the solutions \(X_t^\varepsilon\) and \(Y_t^\varepsilon\) for the system (2.9), with respect to \(\varepsilon \in (0, 1)\) and \(t \in [0, T]\). The main proof follows the techniques in [8], [9] and [27], where the authors deal with the 2D stochastic Navier-Stokes equation, 1D stochastic Burgers’ equation and stochastic real Ginzburg-Landau equation driven by \(\alpha\)-stable noise, respectively.

Inspired by the above references, we first have a fast review about the purely jump Lévy process as following. \(L_t^k\) are independent one dimensional \(\alpha\)-stable processes, so they are purely jump Lévy processes and have the same characteristic function, i.e.,

\[
\mathbb{E}e^{i\xi L_t^k} = e^{\psi(\xi)}, \quad \forall t > 0, k \in \mathbb{Z}_+,
\]

\(\psi(\xi)\) is a complex valued function called Lévy symbol given by

\[
\psi(\xi) = \int_{\mathbb{R}\setminus\{0\}} (e^{i\xi y} - 1 - i\xi y 1_{|y| \leq 1}) \nu(dy),
\]

\(\nu(dx) = \frac{c}{|x|^\alpha} dx\) is the Lévy measure with constant \(c > 0\) and satisfies

\[
\int_{\mathbb{R}\setminus\{0\}} 1 \wedge |y|^2 \nu(dy) < \infty.
\]

For \(t > 0\) and \(\Gamma \in \mathcal{B}(\mathbb{R} \setminus \{0\})\), the Poisson random measure associated with \(L_t^k\) is defined by

\[
N^k(t, \Gamma) = \sum_{s \leq t} 1_{\Gamma}(L^k_s - L^k_{s-}),
\]

and the corresponding compensated Poisson measure is given by

\[
\tilde{N}^k(t, \Gamma) = N^k(t, \Gamma) - t\nu(\Gamma).
\]

By Lévy-Itô’s decomposition, one has

\[
L_t^k = \int_{|x| \leq 1} x\tilde{N}^k(t, dx) + \int_{|x| > 1} xN^k(t, dx).
\]

**Lemma 3.1.** Under conditions \(A1\)-\(A4\), for any \(x, y \in H\) and \(T > 0\), there exists a constant \(C_T > 0\) such that for all \(\varepsilon \in (0, 1)\),

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^2 + \mathbb{E}\int_0^T \|X_t^\varepsilon\|^2 dt \right) \leq C_T (1 + \|x\|), \tag{3.1}
\]

\[
\sup_{0 \leq t \leq T} \mathbb{E}\|Y_t^\varepsilon\|^2 \leq C_T (1 + \|x\| + \|y\|). \tag{3.2}
\]

**Proof.** For \(m \in \mathbb{N}_+\), put \(H_m = \text{span}\{e_k, |k| \leq m\}\) and let \(\pi_m\) be the projection from \(H\) to \(H_m\). Consider the Galerkin approximation of system (2.9):

\[
\begin{cases}
  dX_t^{m,\varepsilon} = [AX_t^{m,\varepsilon} + N^m(X_t^{m,\varepsilon}) + f^m(X_t^{m,\varepsilon}, Y_t^{m,\varepsilon})]dt + d\tilde{L}_t^m, \quad X_0^{m,\varepsilon} = x^m \in H_m, \\
  dY_t^{m,\varepsilon} = \frac{1}{\varepsilon}[AY_t^{m,\varepsilon} + g^m(X_t^{m,\varepsilon}, Y_t^{m,\varepsilon})]dt + \frac{1}{\varepsilon}Z_t^m, \quad Y_0^{m,\varepsilon} = y^m \in H_m,
\end{cases}
\tag{3.3}
\]

where \(X_t^{m,\varepsilon} = \pi_mX_t^{\varepsilon}, N^m(X_t^{m,\varepsilon}) = \pi_m(N(X_t^{m,\varepsilon}))\), \(f^m(X_t^{m,\varepsilon}, Y_t^{m,\varepsilon}) = \pi_m(f(X_t^{m,\varepsilon}, Y_t^{m,\varepsilon}))\), \(g^m(X_t^{m,\varepsilon}, Y_t^{m,\varepsilon}) = \pi_m(g(X_t^{m,\varepsilon}, Y_t^{m,\varepsilon}))\), \(\tilde{L}_t^m = \sum_{|k| \leq m} \beta_k L_t^{k} e_k\) and \(Z_t^m = \sum_{|k| \leq m} \gamma_k Z_t^{k} e_k\).

Now, define a smooth function \(U\) on \(H_m\) by

\[
U(x) = (\|x\|^2 + 1)^{1/2}, \quad x \in H_m.
\]

Then for any \(x, y \in H_m\),

\[
|U(x) - U(y)| \leq |\|x\|^2 + 1)^{1/2} - (\|y\|^2 + 1)^{1/2}| \leq \|x - y\|. \tag{3.4}
\]
By Itô’s formula, we get

\[
U(X^{m,\varepsilon}_t) = U(x^m) + \int_0^t \left\{ \frac{-\|X^{m,\varepsilon}_s\|^2}{(\|X^{m,\varepsilon}_s\|^2 + 1)^{1/2}} + \frac{(N_m(X^{m,\varepsilon}_s, X^{m,\varepsilon}_s))}{(\|X^{m,\varepsilon}_s\|^2 + 1)^{1/2}} \right\} ds
+ \sum_{|k| \leq m} \int_0^t \int_{|x| \leq 1} \left[ U(X^{m,\varepsilon}_{s-} + x\beta_k e_k) - U(X^{m,\varepsilon}_{s-}) \right] \tilde{N}_k(ds, dx)
+ \sum_{|k| \leq m} \int_0^t \int_{|x| > 1} \left[ U(X^{m,\varepsilon}_{s-} + x\beta_k e_k) - U(X^{m,\varepsilon}_{s-}) \right] N_k(ds, dx)
\]

:= U(x^m) + I^m_1(t) + I^m_2(t) + I^m_3(t) + I^m_4(t). \tag{3.5}

For \(I^m_1(t)\), By (2.8) and condition A3, there exists \(C > 0\) such that

\[
I^m_1(t) \leq C t + \int_0^t \frac{-C\|X^{m,\varepsilon}_s\|^2}{(\|X^{m,\varepsilon}_s\|^2 + 1)^{1/2}} ds. \tag{3.6}
\]

For \(I^m_2(t)\), by the Burkholder-Davis-Gundy inequality and (3.4), we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|I^m_2(t)\| \right] \leq C \sum_{|k| \leq m} \mathbb{E} \left( \int_0^T \int_{|x| \leq 1} |U(X^{m,\varepsilon}_{s} + x\beta_k e_k) - U(X^{m,\varepsilon}_{s})|^2 \nu(dx) ds \right)^{1/2}
\leq C T^{1/2} \sum_{|k| \leq m} \beta_k \left( \int_{|x| \leq 1} |x|^2 \nu(dx) \right)^{1/2} \leq C T^{1/2} \sum_{k \in \mathbb{Z}_+} \beta_k. \tag{3.7}
\]

For \(I^m_3(t)\), the Taylor’s expansion follows

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|I^m_3(t)\| \right] \leq C \sum_{|k| \leq m} \beta_k^2 \int_0^T \int_{|x| \leq 1} |x|^2 \nu(dx) ds
\leq C T \sum_{k \in \mathbb{Z}_+} \beta_k \int_{|x| \leq 1} |x|^2 \nu(dx) \leq C T \sum_{k \in \mathbb{Z}_+} \beta_k. \tag{3.8}
\]

For \(I^m_4(t)\), by (3.4) again, we obtain

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|I^m_4(t)\| \right] \leq C \sum_{|k| \leq m} \mathbb{E} \left( \int_0^T \int_{|x| \leq 1} |U(X^{m,\varepsilon}_{s} + x\beta_k e_k) - U(X^{m,\varepsilon}_{s})| \nu(ds, dx) \right)
\leq C T \sum_{k \in \mathbb{Z}_+} \beta_k \int_{|x| > 1} |x| \nu(dx) \leq C T \sum_{k \in \mathbb{Z}_+} \beta_k. \tag{3.9}
\]

Notice that condition A2 implies \(\sum_{k \in \mathbb{Z}_+} \beta_k < \infty\). Then combining (3.5)-(3.9), we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \|X^{m,\varepsilon}_t\|^2 + 1 \right)^{1/2} \right] + \mathbb{E} \int_0^T \frac{\|X^{m,\varepsilon}_t\|^2}{(\|X^{m,\varepsilon}_t\|^2 + 1)^{1/2}} dt \leq C_T (1 + \|x\|), \tag{3.10}
\]

where the constant \(C_T\) depends on \(T\). By Theorem 4.2 in the appendix below, for any \(t > 0\), when \(W = H\) or \(V\),

\[
\lim_{m \to \infty} \|X^{m,\varepsilon}_t\|_W = \|X^\varepsilon_t\|_W, \quad \mathbb{P} - a.s.\]
Hence by Fatou’s Lemma in (3.10) implies
\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} \| X_t^\varepsilon \| \right) + \mathbb{E} \int_0^T \frac{\| X_t^\varepsilon \|^2}{\| X_t^\varepsilon \|^2 + 1} dt \leq C_T (1 + \| x \|). \]  
(3.11)

Notice that
\[ Y_t^\varepsilon = e^{tA/\varepsilon} y + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon} g(X_s^\varepsilon, Y_s^\varepsilon) ds + \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} dZ_s. \]

Then for any \( t \geq 0 \), by property (2.3), we have
\[ \| Y_t^\varepsilon \| \leq \| y \| + \frac{1}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} (C + L_g \| X_s^\varepsilon \| + L_g \| Y_s^\varepsilon \|) ds + \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} dZ_s. \]

Define \( \tilde{Z}_t := \frac{1}{\varepsilon^{1/\alpha}} Z_{t\varepsilon} \), which is also a cylindrical \( \alpha \)-stable process. Then by [23, (4.12)],
\[ \mathbb{E} \left\| \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} dZ_s \right\| = \mathbb{E} \left\| \int_0^{t/\varepsilon} e^{(t-s)A/\varepsilon} d\tilde{Z}_s \right\| \]
\[ \leq C \left( \sum_k \gamma_k^\alpha \frac{1 - e^{-\alpha \lambda_k t/\varepsilon}}{\alpha \lambda_k} \right)^{1/\alpha} \]
\[ \leq C \left( \sum_k \frac{\gamma_k^\alpha}{\alpha \lambda_k} \right)^{1/\alpha}. \]

This and (3.11), we have for any \( t \leq T \),
\[ \mathbb{E} \| Y_t^\varepsilon \| \leq \| y \| + \frac{C}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} ds + \frac{L_g}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} \mathbb{E} \| X_s^\varepsilon \| ds \]
\[ + \frac{L_g}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} L_g \mathbb{E} \| Y_s^\varepsilon \| ds + \frac{1}{\varepsilon^{1/\alpha}} \mathbb{E} \left\| \int_0^t e^{(t-s)A/\varepsilon} dZ_s \right\| \]
\[ \leq C_T (1 + \| x \| + \| y \|) + \frac{L_g}{\lambda_1} \sup_{0 \leq t \leq T} \mathbb{E} \| Y_t^\varepsilon \|. \]

Hence (3.2) holds due to \( L_g < \lambda_1 \) in condition A4. The proof is complete. \[ \square \]

In order to study the high regularity of the slow component \( X_t^\varepsilon \), we need to construct the following stopping time, i.e., for any \( \varepsilon \in (0, 1) \), \( R > 0 \),
\[ \tau_R^\varepsilon := \inf \{ t > 0, \| X_t^\varepsilon \| \geq R \}. \]

**Lemma 3.2.** Under the conditions A1-A3, for any \( x \in H^\theta \) with \( \theta \in (1/2, 1] \), \( y \in H \), \( T > 0 \), \( 1 \leq p < \alpha \) and \( R > 0 \), there exists a constant \( C_{p,R,T} > 0 \) such that for all \( \varepsilon \in (0, 1) \),
\[ \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \tau_R^\varepsilon} \| X_t^\varepsilon \|_p^p \right) \leq C_{p,R,T} (\| x \|_\theta^p + 1). \]  
(3.12)

**Proof.** Recall that
\[ X_t^\varepsilon = e^{tA} x + \int_0^t e^{(t-s)A} N(X_s^\varepsilon) ds + \int_0^t e^{(t-s)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t e^{(t-s)A} dL_s. \]
According to properties (2.4) and (2.5), for any $\theta \in (1/2, 1]$, we have

$$\|X^\varepsilon_t\|_\theta \leq \|e^{tA}x\|_\theta + \left\| \int_0^t e^{(t-s)A} N(X^\varepsilon_s)ds \right\|_\theta + \left\| \int_0^t e^{(t-s)A}(f(X^\varepsilon_s, Y^{\varepsilon}_s))ds \right\|_\theta + \left\| \int_0^t e^{(t-s)A}dL_s \right\|_\theta$$

$$\leq \|x\|_\theta + \int_0^t (t-s)^{-1/2}\|N(X^\varepsilon_s)\|_{-(1-\theta)}ds + C\int_0^t (t-s)^{-\theta/2}ds + \|L(t)\|_\theta$$

$$\leq \|x\|_\theta + \|L(t)\|_\theta + C + C_R\int_0^t (t-s)^{-1/2}\|X^\varepsilon_s\|_{\theta}ds,$$

Using the interpolation inequality,

$$\|X^\varepsilon_s\|_{\theta/3} \leq C\|X^\varepsilon_s\|_{2/3}^{1/3}\|X^\varepsilon_s\|_{\theta}^{1/3}.$$  

Then for any $t \leq T \land \tau^\varepsilon_R$

$$\|X^\varepsilon_t\|_\theta \leq \|x\|_\theta + \|L(t)\|_\theta + C_T + \int_0^t (t-s)^{-1/2}\|X^\varepsilon_s\|_{\theta}^{1/3}\|X^\varepsilon_s\|_{\theta}^{2/3}ds$$

$$\leq \|x\|_\theta + \sup_{0 \leq s \leq t} \|L(t)\|_\theta + C_T + C_R\int_0^t (t-s)^{-1/2}\|X^\varepsilon_s\|_{\theta}ds,$$

which implies

$$\sup_{0 \leq t \leq T \land \tau^\varepsilon_R} \|X^\varepsilon_t\|_\theta^3 \leq C_T \left( \|x\|_\theta + \sup_{0 \leq t \leq T} \|L(t)\|_\theta + 1 \right)^3 + C_{R,T}\int_0^{T \land \tau^\varepsilon_R} \|X^\varepsilon_s\|_{\theta}^3 ds.$$  

Then by the Gronwall’s lemma, we obtain

$$\sup_{0 \leq t \leq T \land \tau^\varepsilon_R} \|X^\varepsilon_t\|_\theta \leq C_{R,T}(\|x\|_\theta + \sup_{0 \leq t \leq T} \|L(t)\|_\theta + 1).$$

By Remark 2.2, for any $1 \leq p < \alpha$, we have

$$\mathbb{E}\left( \sup_{0 \leq t \leq T \land \tau^\varepsilon_R} \|X^\varepsilon_t\|_{\theta}^p \right) \leq C_{p,R,T}(\|x\|_{\theta}^p + 1).$$  

The proof is complete.  

Because that we will use the approach based on time discretization later, we first give an estimate of $X^\varepsilon_{t+h} - X^\varepsilon_t$ when $0 \leq t \leq t+h \leq T \land \tau^\varepsilon_R$.

**Lemma 3.3.** Under the conditions A1-A3, for any $x \in H^\theta$ with $\theta \in (1/2, 1]$, $y \in H$, $T > 0$, $1 \leq p < \alpha$ and $R > 0$, there exists a constant $C_{p,R,T} > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\mathbb{E}\left[ \|X^\varepsilon_{t+h} - X^\varepsilon_t\|_{1[0 \leq t \leq t+h \leq T \land \tau^\varepsilon_R]}^p \right] \leq C_{p,R,T}(\|x\|_{\theta}^p + 1)h^{\frac{\alpha}{2}}.$$
Proof. After simple calculations, we have
\[
X^\varepsilon_{t+h} - X^\varepsilon_t = (e^{Ah} - I)X^\varepsilon_t + \int_t^{t+h} e^{(t+s)A} N(X^\varepsilon_s)ds + \int_t^{t+h} e^{(t+s)A} f(X^\varepsilon_s, Y^\varepsilon_s)ds + \int_t^{t+h} e^{(t+s)A} dL_s
\]
\[
:= I_1 + I_2 + I_3 + I_4.
\]
For $I_1$, by (2.2) and Lemma 3.2, for any $1 \leq p < \alpha$ we have
\[
\mathbb{E} \left[ \|I_1\|^p 1_{\{0 \leq t \leq t+h \leq T \land \tau_R^\varepsilon\}} \right] \leq C h^{\frac{p}{\alpha}} \mathbb{E} \left[ \|X^\varepsilon_t\|^p 1_{\{0 \leq t \leq T \land \tau_R^\varepsilon\}} \right] \leq C_{p,R,T}(\|x\|_\theta^p + 1) h^{\frac{p}{\alpha}}.
\] (3.14)
For $I_2$, by (2.5) and interpolation inequality, we get
\[
\|I_2\| 1_{\{0 \leq t \leq t+h \leq \tau_R^\varepsilon \land T\}} \leq C \left[ \int_t^{t+h} (t + h - s)^{-1 - \frac{1}{2}} \|N(X^\varepsilon_s)\|_{-(1-\theta)} ds \right] 1_{\{0 \leq t \leq t+h \leq T \land \tau_R^\varepsilon\}}
\leq C \left[ \int_t^{t+h} (t + h - s)^{-1 - \frac{1}{2}} (1 + \|X^\varepsilon_s\|^2 \|X^\varepsilon_s\|_{(1-\theta)}) ds \right] 1_{\{0 \leq t \leq t+h \leq T \land \tau_R^\varepsilon\}}
\leq C h^{1 + \frac{\theta}{2}} + C_R \sup_{0 \leq t \leq T \land \tau_R^\varepsilon} \|X^\varepsilon_t\|_{(1-\theta)} h^{\frac{1}{2}}.
\]
Then by Lemma 3.2, we have
\[
\mathbb{E} \left[ \|I_2\|^p 1_{\{0 \leq t \leq t+h \leq \tau_R^\varepsilon \land T\}} \right] \leq C_{p,R,T}(\|x\|_\theta^p + 1) h^{\frac{p}{\alpha}}.
\] (3.15)
For $I_3$, by condition A2, we obtain
\[
\mathbb{E} \|I_3\|^p \leq C h^p.
\] (3.16)
For $I_4$, Remark 2.2 implies
\[
\mathbb{E} \|I_4\|^p \leq C_p h^\frac{\alpha}{2}.
\] (3.17)
Putting (3.14)-(3.17) together, which complete the proof. \hfill \qed

3.2. Estimates of the auxiliary process $(\hat{X}^\varepsilon, \hat{Y}^\varepsilon)$. Following the idea inspired by Khasminskii [21], we introduce an auxiliary process $(\hat{X}^\varepsilon, \hat{Y}^\varepsilon) \in H \times H$. Specifically, we split the interval $[0, T]$ into some subintervals of size $\delta > 0$, where $\delta$ is a positive number depends on $\varepsilon$ and will be chosen later. With the initial value $\hat{Y}^\varepsilon_0 = Y^\varepsilon_0 = y$, we construct the process $\hat{Y}^\varepsilon$ as follows:
\[
d\hat{Y}^\varepsilon_t = \frac{1}{\varepsilon} \left[ A\hat{Y}^\varepsilon_t + g(X^\varepsilon_{t(\delta)}, \hat{Y}^\varepsilon_t) \right] dt + \frac{1}{\varepsilon^{1/\alpha}} dZ_t, \quad \hat{Y}^\varepsilon_0 = y,
\] which satisfying
\[
\hat{Y}^\varepsilon_t = e^{tA/\varepsilon} y + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon} g(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) ds + \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} dZ_s,
\] (3.18)
where $t(\delta) = \lfloor \frac{t}{\delta} \rfloor \delta$ is the nearest breakpoint proceeding $t$. Then we construct the process $\hat{X}^\varepsilon$ as follows:
\[
d\hat{X}^\varepsilon_t = \left[ A\hat{X}^\varepsilon_t + N(\hat{X}^\varepsilon_t) + f(X^\varepsilon_{t(\delta)}, \hat{Y}^\varepsilon_t) \right] dt + dL_t, \quad \hat{X}^\varepsilon_0 = x,
\] which satisfies
\[
\hat{X}^\varepsilon_t = e^{tA} x + \int_0^t e^{(t-s)A} N(\hat{X}^\varepsilon_s) ds + \int_0^t e^{(t-s)A} f(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) ds + \int_0^t e^{(t-s)A} dL_s.
\] (3.19)
The following Lemma gives a control of the auxiliary process \((\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)\). Since the proof almost follows the steps in the proof of Lemma 3.1, we omit the proof here.

**Lemma 3.4.** Under conditions **A1-A4**, for any \(x, y \in H\) and \(T > 0\), there exists a constant \(C_T > 0\) such that

\[
E \left( \sup_{0 \leq t \leq T} \|\hat{X}_t^\varepsilon\| \right) + E \int_0^T \frac{\|\hat{X}_t^\varepsilon\|^2}{(\|\hat{X}_t^\varepsilon\|^2 + 1)^{1/2}} dt \leq C_T (1 + \|x\|), \tag{3.20}
\]

\[
\sup_{0 \leq t \leq T} E \|Y_t^\varepsilon\| \leq C_T (1 + \|x\| + \|y\|).
\]

**Lemma 3.5.** Under the conditions **A1-A4**, for any \(x \in H^\theta\) with \(\theta \in (1/2, 1]\), \(y \in H\), \(1 \leq p < \alpha\), \(T > 0\) and \(R > 0\), there exists a constant \(C_{p,R,T} > 0\) such that

\[
E \left( \int_0^{T \land \tau^\varepsilon_R} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\| dt \right)^p \leq C_{p,R,T} (\|x\|^p_0 + 1) \delta^\varepsilon_R.
\]

Here \(\tau^\varepsilon_R\) comes from Lemma 3.2.

**Proof.** By the construction of \(Y_t^\varepsilon\) and \(\hat{Y}_t^\varepsilon\), we have

\[
Y_t^\varepsilon - \hat{Y}_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t e^{(t-s)/\varepsilon} \left[ g(X_s^\varepsilon, Y_s^\varepsilon) - g(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) \right] ds.
\]

Then for any \(t > 0\),

\[
\|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\| \leq \frac{1}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} L_g \|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\| ds + \frac{1}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} L_g \|Y_s^\varepsilon - \hat{Y}_s^\varepsilon\| ds.
\]

By Fubini’s theorem,

\[
\int_0^{T \land \tau^\varepsilon_R} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\| dt \leq \frac{1}{\varepsilon} \int_0^{T \land \tau^\varepsilon_R} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} L_g \|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\| ds dt + \frac{1}{\varepsilon} \int_0^{T \land \tau^\varepsilon_R} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} L_g \|Y_s^\varepsilon - \hat{Y}_s^\varepsilon\| ds dt
\]

\[
= \frac{L_g}{\varepsilon} \int_0^{T \land \tau^\varepsilon_R} \left( \int_s^{T \land \tau^\varepsilon_R} e^{-\lambda_1(t-s)/\varepsilon} dt \right) \|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\| ds
\]

\[
+ \frac{L_g}{\varepsilon} \int_0^{T \land \tau^\varepsilon_R} \left( \int_s^{T \land \tau^\varepsilon_R} e^{-\lambda_1(t-s)/\varepsilon} dt \right) \|Y_s^\varepsilon - \hat{Y}_s^\varepsilon\| ds
\]

\[
\leq C \int_0^{T \land \tau^\varepsilon_R} \|X_s^\varepsilon - X_{s(\delta)}^\varepsilon\| ds + \frac{L_g}{\lambda_1} \int_0^{T \land \tau^\varepsilon_R} \|Y_s^\varepsilon - \hat{Y}_s^\varepsilon\| ds.
\]

By Lemma 3.3 and \(L_g < \lambda_1\), we have

\[
E \left( \int_0^{T \land \tau^\varepsilon_R} \|Y_t^\varepsilon - \hat{Y}_t^\varepsilon\| dt \right)^p \leq C_{p,R,T} (\|x\|^p_0 + 1) \delta^\varepsilon_R.
\]

The proof is complete. \(\square\)
In the next lemma, we shall deal with the difference process \( X^\varepsilon_t - \hat{X}^\varepsilon_t \). To this end, we construct another stopping time, i.e., for any \( \varepsilon \in (0, 1) \), \( R > 0 \),

\[
\overline{\tau}_R^\varepsilon := \inf \left\{ t > 0 : \| X^\varepsilon_t \| + \| \hat{X}^\varepsilon_t \| + \int_0^t \frac{\| X^\varepsilon_s \|^2}{\| \hat{X}^\varepsilon_s \|^2 + 1} ds + \int_0^t \frac{\| \hat{X}^\varepsilon_s \|^2}{\| \hat{X}^\varepsilon_s \|^2 + 1} ds \geq R \right\}.
\]

**Lemma 3.6.** Under the conditions **A1-A4**, for any \( x \in H^\theta \) with \( \theta \in (1/2, 1] \), \( y \in H \), \( 1 \leq p < \alpha \), \( T > 0 \) and \( R > 0 \) there exists a constant \( C_{p,R,T} > 0 \) such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \overline{\tau}_R^\varepsilon} \| X^\varepsilon_t - \hat{X}^\varepsilon_t \|^p \right) \leq C_{p,R,T} (\| x \|^p + 1) \delta^{\frac{p}{2}}.
\]

**Proof.** In view of (3.19) and (2.11), we write

\[
X^\varepsilon_t - \hat{X}^\varepsilon_t = \int_0^t e^{(t-s)A} \left[ N(X^\varepsilon_s) - N(\hat{X}^\varepsilon_s) \right] ds + \int_0^t e^{(t-s)A} \left[ f(X^\varepsilon_s, Y^\varepsilon_s) - f(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) \right] ds.
\]

Using properties (2.1) and (2.6), condition **A1**, we get

\[
\| X^\varepsilon_t - \hat{X}^\varepsilon_t \| \leq \int_0^t \left\| N(X^\varepsilon_s) - N(\hat{X}^\varepsilon_s) \right\| ds + \int_0^t \left\| f(X^\varepsilon_s, Y^\varepsilon_s) - f(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s) \right\| ds
\]

\[
\leq C \int_0^t \left( 1 + \| X^\varepsilon_s \|^2 + \| \hat{X}^\varepsilon_s \|^2 \right) \| X^\varepsilon_s - \hat{X}^\varepsilon_s \| ds
\]

\[
+ C \int_0^t \left( \| X^\varepsilon_s - X^\varepsilon_{s(\delta)} \| + \| Y^\varepsilon_s - \hat{Y}^\varepsilon_s \| \right) ds
\]

\[
= C \int_0^t \left[ \| X^\varepsilon_s \|^2 (\| Y^\varepsilon_s \|^2 + 1)^{1/2} + \| \hat{X}^\varepsilon_s \|^2 (\| \hat{X}^\varepsilon_s \|^2 + 1)^{1/2} + 1 \right] \| X^\varepsilon_s - \hat{X}^\varepsilon_s \| ds
\]

\[
+ C \int_0^t \left( \| X^\varepsilon_s - X^\varepsilon_{s(\delta)} \| + \| Y^\varepsilon_s - \hat{Y}^\varepsilon_s \| \right) ds.
\]

Then by the definition of \( \overline{\tau}_R^\varepsilon \), we have

\[
\sup_{0 \leq t \leq T \wedge \overline{\tau}_R^\varepsilon} \| X^\varepsilon_t - \hat{X}^\varepsilon_t \|
\]

\[
\leq C \int_0^T \left\{ (R + 1) \left[ \frac{\| X^\varepsilon_{s(\delta_s)} \|}{\| X^\varepsilon_{s(\delta_s)} \|^2 + 1} + \frac{\| \hat{X}^\varepsilon_{s(\delta_s)} \|}{\| \hat{X}^\varepsilon_{s(\delta_s)} \|^2 + 1} \right] + 1 \right\} \| X^\varepsilon_{s(\delta_s)} - \hat{X}^\varepsilon_{s(\delta_s)} \| ds
\]

\[
+ C \int_0^{T \wedge \overline{\tau}_R^\varepsilon} \left( \| X^\varepsilon_s - X^\varepsilon_{s(\delta_s)} \| + \| Y^\varepsilon_s - \hat{Y}^\varepsilon_s \| \right) ds.
\]

The Gronwall’s inequality implies

\[
\sup_{0 \leq t \leq T \wedge \overline{\tau}_R^\varepsilon} \| X^\varepsilon_t - \hat{X}^\varepsilon_t \|
\]

\[
\leq C_T \int_0^{T \wedge \overline{\tau}_R^\varepsilon} \left( \| X^\varepsilon_s - X^\varepsilon_{s(\delta_s)} \| + \| Y^\varepsilon_s - \hat{Y}^\varepsilon_s \| \right) ds \leq C_{R,T} \int_0^{T \wedge \overline{\tau}_R^\varepsilon} \left( \| X^\varepsilon_s - X^\varepsilon_{s(\delta_s)} \| + \| Y^\varepsilon_s - \hat{Y}^\varepsilon_s \| \right) ds.
\]
Notice that \( \tilde{\tau}_R \leq \tau_R \), then it follows from Lemmas 3.3 and 3.5, we have
\[
E \left[ \sup_{0 \leq t \leq T \wedge \tilde{\tau}_R} \left\| X_t^\varepsilon - \dot{X}_t^\varepsilon \right\|^p \right] \leq C_{p,R,T}(\| x \|^p_0 + 1)\delta^\alpha T.
\]
The proof is complete. □

3.3. The frozen and averaged equation. For any fixed \( x \in H \), we first consider the following frozen equation associated with the fast component:
\[
dY_t = [AY_t dt + g(x,Y_t)] dt + d\tilde{Z}_t, \quad Y_0 = y, \tag{3.21}
\]
where \( \tilde{Z}_t \) is a version of \( Z_t \) and independent of \( \{L_t\}_{t \geq 0} \) and \( \{Z_t\}_{t \geq 0} \). Since \( g(x,\cdot) \) is Lipschitz continuous, it is easy to prove that for any fixed \( x,y \in H \), the Eq. (3.21) has a unique mild solution denoted by \( Y_t^{x,y} \). For any \( x \in H \), let \( P_t^x \) be the transition semigroup of \( Y_t^{x,y} \), that is, for any bounded measurable function \( \varphi \) on \( H \) and \( t \geq 0 \),
\[
P_t^x \varphi(y) = E\varphi(Y_t^{x,y}), \quad y \in H.
\]
The asymptotic behavior of \( P_t^x \) has been studied in many literatures, the following result shows the existence and uniqueness of the invariant measure and gives the exponential convergence to the equilibrium (see [1, Lemma 3.3]).

Proposition 3.7. Assume that conditions A1, A2 and A4 hold. For any \( x,y \in H \), \( P_t^x \) admits a unique invariant measure \( \mu^x \). Moreover, there exists \( C > 0 \),
\[
\left\| E f(x,Y_t^{x,y}) - \int_H f(x,z)\mu^x(dz) \right\| \leq C(1 + \| x \| + \| y \|)e^{-(\lambda_1-L_0)t}L_f.
\]
The following lemma is used to prove the existence and uniqueness of the solution of corresponding averaged equation, we state it ahead.

Lemma 3.8. For any \( x_1,x_2 \in H \), \( y \in H \) and \( T > 0 \), there exists a constant \( C_T > 0 \) such that
\[
\sup_{0 \leq t \leq T} \left\| Y_t^{x_1,y} - Y_t^{x_2,y} \right\| \leq C_T \| x_1 - x_2 \|.
\]
Proof. Notice that
\[
Y_t^{x_1,y} - Y_t^{x_2,y} = \int_0^t e^{(t-s)A}g(x_1,Y_s^{x_1,y}) - g(x_2,Y_s^{x_2,y})ds.
\]
By the Lipschitz continuous of \( g \), we get
\[
\sup_{0 \leq t \leq T} \left\| Y_t^{x_1,y} - Y_t^{x_2,y} \right\| \leq C \int_0^T \left( \| x_1 - x_2 \| + \| Y_s^{x_1,y} - Y_s^{x_2,y} \| \right) ds
\]
\[
\leq C_T \| x_1 - x_2 \| + \int_0^T \| Y_s^{x_1,y} - Y_s^{x_2,y} \| ds.
\]
The Gronwall’s inequality implies
\[
\sup_{0 \leq t \leq T} \left\| Y_t^{x_1,y} - Y_t^{x_2,y} \right\| \leq C_T \| x_1 - x_2 \|.
\]
The proof is complete. □
Now, we introduce the averaged equation, which satisfies:

\[
\begin{align*}
\{ & d\bar{X}_t = [A\bar{X}_t + N(\bar{X}_t) + \bar{f}(\bar{X}_t)] \, dt + dL_t, \\
& \bar{X}_0 = x,
\end{align*}
\] (3.22)

where

\[\bar{f}(x) = \int_H f(x, y)\mu^x(dy), \quad x \in H.\]

The existence and uniqueness of the solution and its priori estimates of Eq. (3.22) is the following lemma.

**Lemma 3.9.** Under conditions **A1-A3**, Eq. (3.22) exits a unique mild solution \(\bar{X}_t\) satisfying

\[
\bar{X}_t = e^{tA}x + \int_0^t e^{(t-s)A}N(\bar{X}_s)ds + \int_0^t e^{(t-s)A}\bar{f}(\bar{X}_s)ds + \int_0^t e^{(t-s)A}dL_s.
\] (3.23)

Moreover, for any \(x \in H\) and \(T > 0\), there exists a constant \(C_T > 0\) such that

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \|\bar{X}_t\| \right) + \mathbb{E}\int_0^T \frac{\|\bar{X}_t\|}{\sqrt{1 + \|X_t\|}} \, ds \leq C_T (1 + \|x\|).
\] (3.24)

**Proof.** It is sufficient to check that the \(\bar{f}\) is Lipschitz continuous and bounded, then the results can be easily obtained by following the procedures in Theorem 2.3 and Lemma 3.1. Obviously, \(\bar{f}\) is bounded by the boundness of \(f\). It remain to show \(\bar{f}\) is Lipschitz.

In fact, for any \(x_1, x_2, y \in H\) and \(t > 0\), by Proposition 3.7 and Lemma 3.8, we have

\[
\|\bar{f}(x_1) - \bar{f}(x_2)\| \leq \left\| \int_H f(x_1, z)\mu^{x_1}(dz) - \int_H f(x_2, z)\mu^{x_2}(dz) \right\|
\]

\[
\leq \left\| \int_H f(x_1, z)\mu^{x_1}(dz) - \mathbb{E}f(x_1, Y_t^{x_1,y}) \right\| + \left\| \mathbb{E}f(x_2, Y_t^{x_2,y}) - \int_H f(x_2, z)\mu^{x_2}(dz) \right\|
\]

\[
+ \left\| \mathbb{E}f(x_1, Y_t^{x_1,y}) - \mathbb{E}f(x_2, Y_t^{x_2,y}) \right\|
\]

\[
\leq C(1 + \|x_1\| + \|x_2\| + \|y\|)e^{-\alpha(t - L_t)} + C(\|x_1 - x_2\| + \mathbb{E}\|Y_t^{x_1,y} - Y_t^{x_2,y}\|)
\]

\[
\leq C(1 + \|x_1\| + \|x_2\| + \|y\|)e^{-\alpha(t - L_t)} + C\|x_1 - x_2\|.
\]

Hence, the proof is completed by letting \(t \to \infty\). \(\square\)

Now, we intend to estimate the difference process \(X_t^\varepsilon - \bar{X}_t\). Similar as the argument in Lemma 3.6, we further construct a new stopping time, i.e., for any \(\varepsilon \in (0, 1)\), \(R > 0\),

\[
\tilde{\tau}_R^\varepsilon := \inf \left\{ t > 0 : \|X_t^\varepsilon\| + \|\bar{X}_t\| + \|\bar{X}_t\| + \int_0^t \frac{\|X_s^\varepsilon\|^2}{\|X_s^\varepsilon\|^2 + 1} \, ds \right\}
\]

\[
+ \int_0^t \frac{\|\bar{X}_s\|^2}{\|\bar{X}_s\|^2 + 1} \, ds + \int_0^t \frac{\|\bar{\bar{X}}_s\|^2}{\|\bar{\bar{X}}_s\|^2 + 1} \, ds \geq R\right\}.
\]

**Lemma 3.10.** Under the conditions **A1-A4**, for any \(x \in H^\theta\) with \(\theta \in (1/2, 1]\), \(y \in H\), \(1 \leq p < \alpha\), \(T > 0\) and \(R > 0\), there exists a constant \(C_{p,R,T} > 0\) such that for all \(\varepsilon \in (0, 1)\),

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \|\bar{X}_t^\varepsilon - \bar{X}_t\|^p \right) \leq C_{p,R,T} \left[ (\|x\|^p_\theta + 1)\delta^{\frac{\alpha p}{2}} + (1 + \|x\| + \|y\|)^{p/2}e^{\frac{p}{2}\alpha} \right].
\]
Proof. From (3.19) and (3.23), it is easy to see
\[
\hat{X}_t^\varepsilon - \bar{X}_t = \int_0^t e^{(t-s)A} \left[ N(\hat{X}_s^\varepsilon) - N(\bar{X}_s) \right] ds + \int_0^t e^{(t-s)A} \left[ f(X_s^\varepsilon, \bar{Y}_s) - \bar{f}(X_s) \right] ds \\
+ \int_0^t e^{(t-s)A} \left[ \tilde{f}(X_s^\varepsilon) - \bar{f}(\hat{X}_s) \right] ds + \int_0^t e^{(t-s)A} \left[ \tilde{f}(\hat{X}_s) - \bar{f}(\bar{X}_s) \right] ds \\
:= \sum_{k=1}^4 J_k(t). \tag{3.25}
\]

For \( J_1(t) \), according to (2.6), we have
\[
\sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| J_1(t) \| \leq C \int_0^{T^{\wedge}_R^\varepsilon} \left( 1 + \| \bar{X}_s^\varepsilon \|_1^2 + \| \bar{X}_s^\varepsilon \|_2^2 \right) \| \bar{X}_s^\varepsilon - \bar{X}_s \| ds \tag{3.26}
\]
\[
\leq C \int_0^{T^{\wedge}_R^\varepsilon} \left( R + 1 \right) \left\{ \left( \frac{\| \bar{X}_s^\varepsilon \|_1^2}{\| X_s^\varepsilon \|_2^2 + 1} \right)^{1/2} + \left( \frac{\| \bar{X}_s^\varepsilon \|_2^2}{\| X_s^\varepsilon \|_1^2 + 1} \right)^{1/2} \right\} \| \bar{X}_s^\varepsilon - \bar{X}_s \| ds.
\]

For \( J_3(t) \) and \( J_4(t) \), by the Lipschitz continuity of \( \tilde{f} \), we obtain
\[
\sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| J_3(t) \| \leq C \int_0^{T^{\wedge}_R^\varepsilon} \| X_s^\varepsilon - \hat{X}_s^\varepsilon \| ds \tag{3.27}
\]
and
\[
\sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| J_4(t) \| \leq C \int_0^{T^{\wedge}_R^\varepsilon} \| \hat{X}_s^\varepsilon - \bar{X}_s \| ds. \tag{3.28}
\]

Then by (3.25) to (3.28), we have
\[
\sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| X_t^\varepsilon - \hat{X}_t \| \\
\leq C \int_0^T \left\{ \left( R + 1 \right) \left[ \left( \frac{\| \bar{X}_s^\varepsilon \|_1^2}{\| X_s^\varepsilon \|_2^2 + 1} \right)^{1/2} + \left( \frac{\| \bar{X}_s^\varepsilon \|_2^2}{\| X_s^\varepsilon \|_1^2 + 1} \right)^{1/2} \right] + 1 \right\} \| \bar{X}_s^\varepsilon - \bar{X}_s \| ds \\
+ \sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| J_2(t) \| + \int_0^{T^{\wedge}_R^\varepsilon} \| X_s^\varepsilon - \hat{X}_s^\varepsilon \| ds.
\]

The Gronwall’s inequality and the definition of \( \hat{t}_R^\varepsilon \) imply
\[
\sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| \hat{X}_t^\varepsilon - \bar{X}_t \| \leq \left[ \sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| J_2(t) \| + \int_0^{T^{\wedge}_R^\varepsilon} \| X_s^\varepsilon - \hat{X}_s \| ds \right] \left( e^{C_{R,T}} \right).
\]

Notice that \( \hat{t}_R^\varepsilon \leq \tilde{t}_R^\varepsilon \), then by Lemma 3.6, we obtain
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| \hat{X}_t^\varepsilon - \bar{X}_t \| \right) \leq C_{p,R,T} \mathbb{E} \left( \sup_{0 \leq t \leq T^{\wedge}_R^\varepsilon} \| J_2(t) \| \right) + C_{p,R,T} \left( \mathbb{E} \left( \| X \|_p \right) + 1 \right) \delta^{-p}. \tag{3.29}
\]

Now, it is remain to estimate \( J_2(t) \). Set \( n_t = \left[ \frac{t}{\delta} \right] \), we write
\[
J_2(t) = J_{2,1}(t) + J_{2,2}(t) + J_{2,3}(t),
\]
where

\[ J_{2,1}(t) = \sum_{k=0}^{n_1-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left[ f(X^\varepsilon_{k\delta}, \hat{Y}_s^\varepsilon) - \hat{f}(X^\varepsilon_{k\delta}) \right] ds, \]

\[ J_{2,2}(t) = \sum_{k=0}^{n_1-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left[ \hat{f}(X^\varepsilon_{k\delta}) - \hat{f}(X^\varepsilon_s) \right] ds, \]

\[ J_{2,3}(t) = \int_{nt\delta}^t e^{(t-s)A} \left[ f(X^\varepsilon_{nt\delta}, \hat{Y}_s^\varepsilon) - \hat{f}(X^\varepsilon_s) \right] ds. \]

For \( J_{2,2}(t) \), we have

\[ \sup_{0 \leq t \leq T^{\varepsilon}_R} \| J_{2,2}(t) \| \leq C \int_0^{T^{\varepsilon}_R} \| X^\varepsilon_s - X^\varepsilon_t \| ds. \quad (3.30) \]

For \( J_{2,3}(t) \), it follows from the boundness of \( f \),

\[ \sup_{0 \leq t \leq T^{\varepsilon}_R} \| J_{2,3}(t) \| \leq \delta. \quad (3.31) \]

For \( J_{2,1}(t) \), from the construction of \( \hat{Y}_t^\varepsilon \), we obtain that, for any \( k \in \mathbb{N}_s \) and \( s \in [0, \delta) \),

\[ \hat{Y}_{s+k\delta}^\varepsilon = \hat{Y}_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_{k\delta}^{k\delta+s} A\hat{Y}_r^\varepsilon dr + \frac{1}{\varepsilon} \int_{k\delta}^{k\delta+s} g(X^\varepsilon_{k\delta}, \hat{Y}_r^\varepsilon) dr + \frac{1}{\varepsilon^{1/\alpha}} \int_{k\delta}^{k\delta+s} dZ_r \]

\[ = \hat{Y}_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_0^s A\hat{Y}_{r+k\delta}^\varepsilon dr + \frac{1}{\varepsilon} \int_0^s g(X^\varepsilon_{k\delta}, \hat{Y}_{r+k\delta}^\varepsilon) dr + \frac{1}{\varepsilon^{1/\alpha}} \int_0^s dZ_{k\delta}(r) \quad (3.32) \]

where \( Z_{k\delta}(t) := Z_{t+k\delta} - Z_{k\delta} \) is the shift version of \( Z_t \), which is also a cylindrical \( \alpha \)-stable process.

Recall that \( \hat{Z}_t \) be a cylindrical \( \alpha \)-stable process which is independent of \( (X^\varepsilon_{k\delta}, \hat{Y}_{k\delta}^\varepsilon) \). We construct a process \( Y^X_{\varepsilon,k\delta}, \hat{Y}^\varepsilon_{k\delta} \) by means of \( Y^x_{t,y} |_{(x,y)=(X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_{k\delta})} \), i.e.,

\[ Y^X_{\varepsilon,k\delta}, \hat{Y}^\varepsilon_{k\delta} = \hat{Y}^\varepsilon_{k\delta} + \int_0^s AY^X_{r,k\delta}, \hat{Y}^\varepsilon_{k\delta} dr + \int_0^s g(X^\varepsilon_{k\delta}, Y^X_{r,k\delta}, \hat{Y}^\varepsilon_{k\delta}) dr + \int_0^s d\hat{Z}_r \]

\[ = \hat{Y}^\varepsilon_{k\delta} + \frac{1}{\varepsilon} \int_0^s AY^X_{r,k\delta}, \hat{Y}^\varepsilon_{k\delta} dr + \frac{1}{\varepsilon} \int_0^s g(X^\varepsilon_{k\delta}, Y^X_{r,k\delta}, \hat{Y}^\varepsilon_{k\delta}) dr + \frac{1}{\varepsilon^{1/\alpha}} \int_0^s d\hat{Z}_r. \quad (3.33) \]

where \( \hat{Z}_t := \varepsilon^{1/\alpha} \hat{Z}^\varepsilon_t \) is again a cylindrical \( \alpha \)-stable process by self-similar property of stable Lévy processes. Then the uniqueness of the solution to Eq. (3.32) and Eq. (3.33) implies that the distribution of \( (X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_{s+k\delta})_{0 \leq s \leq \delta} \) coincides with the distribution of \( (X^\varepsilon_{k\delta}, Y^X_{\varepsilon,k\delta}, \hat{Y}^\varepsilon_{k\delta})_{0 \leq s \leq \delta} \).
Then we try to control \( \|J_{2,1}(t)\| \):

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|J_{2,1}(t)\|^2 \right) = \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{n_t-1} e^{(t-(k+1)\delta)A} \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s) - \bar{f}(X^\varepsilon_{k\delta}) \right] ds \right\|^2
\]

\[
\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^{n_t-1} \left\| \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s) - \bar{f}(X^\varepsilon_{k\delta}) \right] ds \right\|^2 \right\}
\]

\[
\leq \frac{T}{\delta} \max_{0 \leq k \leq \lfloor \frac{T}{\delta} \rfloor - 1} \mathbb{E} \left\| \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s) - \bar{f}(X^\varepsilon_{k\delta}) \right] ds \right\|^2
\]

\[
= C_T \varepsilon \max_{0 \leq k \leq \lfloor \frac{T}{\delta} \rfloor - 1} \mathbb{E} \int_0^{\frac{T}{\delta}} \int_0^{\frac{T}{\delta}} \Psi_k(s, r) ds dr,
\]

where

\[
\Psi_k(s, r) = \mathbb{E} \left\langle e^{(\delta-r)A} \left( f(X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s) - \bar{f}(X^\varepsilon_{k\delta}) \right), e^{(\delta-r)A} \left( f(X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s) - \bar{f}(X^\varepsilon_{k\delta}) \right) \right\rangle | (x, y) = (X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s + r\delta)
\]

Then for \( s > r \), by the Markov property, Proposition 3.7 and condition A3,

\[
\Psi_k(s, r) = \mathbb{E} \left\langle e^{(\delta-r)A} \left( f(x, Y^x_{s-r}) - \bar{f}(x) \right), e^{(\delta-r)A} \left( f(x, Y^x_{s-r}) - \bar{f}(x) \right) \right\rangle | (x, y) = (X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s + r\delta)
\]

\[
\leq C \mathbb{E} \left\langle e^{(\delta-r)A} \mathbb{E} \left[ f(x, Y^x_{s-r}) - \bar{f}(x) \right] | X^\varepsilon_{k\delta}, \hat{Y}^\varepsilon_s + r\delta \right\rangle e^{-(\lambda_1-L_0)(s-r)} \mathbb{E} \left\langle e^{(\lambda_1-L_0)(s-r)} \right\rangle |
\]

\[
\leq C_T \varepsilon \max_{0 \leq k \leq \lfloor \frac{T}{\delta} \rfloor - 1} \int_0^{\frac{T}{\delta}} \int_0^{\frac{T}{\delta}} \Psi_k(s, r) ds dr.
\]

In the last inequality, Lemmas 3.1 and 3.4 have been used. Hence

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|J_{2,1}(t)\|^2 \right) \leq C_T \varepsilon \int_0^{\frac{T}{\delta}} \int_0^{\frac{T}{\delta}} e^{-(\lambda_1-L_0)(s-r)} ds dr
\]

\[
\leq C_T \varepsilon (1 + \|x\| + \|y\|).
\]
This, together with (3.30), (3.31), (3.34) and Lemma 3.3, we get

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \| J_2(t) \|^p \right) \leq C_{p,R,T} \left[ (\| x \|_q^p + 1) \delta^{p/2} T + (1 + \| x \| + \| y \|)^{p/2} \frac{\varepsilon^{p/2}}{\delta^{p/2}} \right].
\]  

(3.35)

According to the estimates (3.29) and (3.35), we obtain

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \| \hat{X}_t^\varepsilon - \bar{X}_t \|^p \right) \leq C_{p,R,T} \left[ (\| x \|_q^p + 1) \delta^{p/2} + \delta^p + (1 + \| x \| + \| y \|)^{p/2} \frac{\varepsilon^{p/2}}{\delta^{p/2}} \right],
\]

which complete the proof. □

3.4. Proof of Theorem 2.4.

Proof. By Lemmas 3.1, 3.6, 3.9 and 3.10, we obtain, for any \( \kappa \in (0, 1) \),

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \| X_t^\varepsilon - \bar{X}_t \|^\kappa \right) \\
\leq \mathbb{E}\left( \sup_{0 \leq t \leq T} \| X_t^\varepsilon - \bar{X}_t \|^\kappa \right) + \mathbb{E}\left( \sup_{0 \leq t \leq T} \| X_t^\varepsilon - \bar{X}_t \|^\kappa 1_{\{T > \hat{\tau}_R^\varepsilon\}} \right) \\
\leq \mathbb{E}\left( \sup_{0 \leq t \leq T} \| X_t^\varepsilon - \hat{X}_t^\varepsilon \|^\kappa \right) + \mathbb{E}\left( \sup_{0 \leq t \leq T} \| \hat{X}_t^\varepsilon - \bar{X}_t \|^\kappa \right) \\
+ C \left[ \mathbb{E} \sup_{0 \leq t \leq T} (\| X_t^\varepsilon \| + \| \bar{X}_t \|) \right]^\kappa \mathbb{P}(T > \hat{\tau}_R^\varepsilon)^{1/q} \\
\leq C_{\kappa,R,T} \left[ (\| x \|_q + 1) \delta^{p/2} + \delta^\kappa + (1 + \| x \| + \| y \|)^{\kappa/2} \frac{\varepsilon^{\kappa/2}}{\delta^{\kappa/2}} \right] + \frac{C_{\kappa,T}}{R^{1/q}},
\]

(3.36)

where \( \kappa + \frac{1}{q} = 1 \) and the last inequality comes from chebyshev’s inequality,

\[
\mathbb{P}(T > \hat{\tau}_R^\varepsilon) \leq \mathbb{E} \sup_{0 \leq t \leq T} \| X_t^\varepsilon \| + \int_0^T \frac{\| X_s^\varepsilon \|^2}{(\| X_s^\varepsilon \|^2 + 1)^{1/2}} ds \] / R \\
+ \mathbb{E} \sup_{0 \leq t \leq T} \| \hat{X}_t^\varepsilon \| + \int_0^T \frac{\| \hat{X}_s^\varepsilon \|^2}{(\| \hat{X}_s^\varepsilon \|^2 + 1)^{1/2}} ds \] / R \\
+ \mathbb{E} \sup_{0 \leq t \leq T} \| \bar{X}_t \| + \int_0^T \frac{\| \bar{X}_s \|^2}{(\| \bar{X}_s \|^2 + 1)^{1/2}} ds \] / R \\
\leq \frac{C_T}{R},
\]

where we use the the priori estintas in Lemmas 3.1, 3.4 and 3.9.

Now, taking \( \delta = \varepsilon^{1/2} \) and letting \( \varepsilon \to 0 \) firstly, then \( R \to \infty \) in (3.36), we have

\[
\lim_{\varepsilon \to 0} \mathbb{E}\left( \sup_{0 \leq t \leq T} \| X_t^\varepsilon - \bar{X}_t \|^\kappa \right) = 0.
\]

(3.37)

The proof is complete. □
4. Appendix

4.1. Some estimates about the nonlinearity $N$. The proof of (2.6)-(2.8) can be founded in [27, Appendix] and we show (2.5) here.

By the Sobolev embedding theorem, for any $\sigma \in [0,1/2)$,

$$\|N(x)\|_{-\sigma} \leq C \left(1 + \|x\|_{\frac{3}{1-\sigma}}\right).$$

which implies

$$\|N(x)\|_{-\sigma} \leq C \left(1 + \|x\|_{\frac{3}{1-\sigma}}\right).$$

4.2. The existence and uniqueness of solution of system (2.9). Fix $\varepsilon > 0$, for all $\omega \in \Omega$, define

$$W^\varepsilon(\omega) := X^\varepsilon(\omega) - L_A(t,\omega), \quad V^\varepsilon(\omega) := Y^\varepsilon(\omega) - Z^\varepsilon(t,\omega),$$

where $L_A(t,\omega) = \int_0^t e^{(t-s)A}dL_s$ and $Z^\varepsilon(t) := \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}dZ_s$. Then

$$\begin{cases}
\partial_t W^\varepsilon = AW^\varepsilon + N(W^\varepsilon + L_A(t)) + f(W^\varepsilon + L_A(t), V^\varepsilon + Z_A(t)), & W^\varepsilon_0 = x \\
\partial_t V^\varepsilon = \frac{1}{\varepsilon} [AV^\varepsilon + g(W^\varepsilon + L_A(t), V^\varepsilon + Z_A(t))], & V^\varepsilon_0 = y.
\end{cases} \tag{4.1}$$

For each $T > 0$, define

$$K^\varepsilon_T(\omega) := \sup_{t \leq T} \|L_A(t)\|_1 + \int_0^T \|Z^\varepsilon_A(t)\|dt.$$ 

By Remark 2.2, for every $k \in \mathbb{N}$, there exists some set $N^\varepsilon_k$ such that $\mathbb{P}(N^\varepsilon_k) = 0$ and

$$K^\varepsilon_k(\omega) < \infty, \quad \omega \notin N^\varepsilon_k.$$ 

Define $N^\varepsilon = \cup_{k \geq 1} N^\varepsilon_k$, it is easy to see $\mathbb{P}(N^\varepsilon) = 0$ and that for all $T > 0$

$$K^\varepsilon_T(\omega) < \infty, \quad \omega \notin N^\varepsilon.$$ 

Lemma 4.1. Assume the conditions A1-A3 hold. Then the following statements hold.

(i) For every $\varepsilon > 0$, $x \in H$, $y \in H$ and $\omega \notin N^\varepsilon$, there exists some $0 < \hat{T}(\omega) < 1$, depending on $\|x\|$ and $K^\varepsilon_T(\omega)$, such that system (4.1) admits a unique solution $W^\varepsilon(\omega) \in C([0,\hat{T}); H)$ and $V^\varepsilon(\omega) \in C([0,\hat{T}); H)$ satisfying for all $\sigma \in [1/6, 1/2]$,

$$\|W^\varepsilon\|_{2\sigma} \leq C(t^{-\sigma} + 1),$$

(ii) Let $\sigma \in [1/6, 1/2]$. For every $\varepsilon > 0$, $x \in H^{2\sigma}$, $y \in H$ and $\omega \notin N^\varepsilon$, there exists some $0 < \hat{T}(\omega) < 1$, depending on $\|x\|$, $\sigma$ and $K^\varepsilon_T(\omega)$, such that system (4.1) admits a unique solution $W^\varepsilon(\omega) \in C([0,\hat{T}); H^{2\sigma})$ and $V^\varepsilon(\omega) \in C([0,\hat{T}); H)$ satisfying

$$\sup_{0 \leq t \leq \hat{T}} \|W^\varepsilon\|_{2\sigma} \leq 1 + \|x\|_{2\sigma}.$$
Proof. We shall apply the Banach fixed point theorem. Since these two statements will be proved by the same method, we only prove statement (i).

Let \(0 < T \leq 1\) and \(B > 0\) be some constants to be determined later. For \(\sigma = \frac{1}{6}\), define

\[
S := \left\{ u = (u_1, u_2) : u_i \in C([0, T]; H), i = 1, 2, u_1(0) = x, u_2(0) = y, \right. \\
\left. u_1(t) \in H^{2\sigma}, \forall t \in (0, T], \sup_{0 \leq t \leq T} [t^\sigma \|u_1(t)\|_{2\sigma}] + \sup_{0 \leq t \leq T} \|u_1(t)\| + \sup_{0 \leq t \leq T} \|u_2(t)\| \leq B \right\}.
\]

Given any \(u = (u_1, u_2), v = (v_1, v_2) \in S\), define

\[
d(u, v) = \sup_{0 \leq t \leq T} [t^\sigma \|u_1(t) - v_1(t)\|_{2\sigma}] + \sup_{0 \leq t \leq T} \|u_1(t) - v_1(t)\| + \sup_{0 \leq t \leq T} \|u_2(t) - v_2(t)\|.
\]

Then \((S, d)\) is a closed metric space. Further define a map \(F : S \to (C([0, T]; H), C([0, T]; H))\) as the following: for any \(u = (u_1, u_2) \in S\),

\[
F(u)(t) = (F(u)_1(t), F(u)_2(t)),
\]

where

\[
\begin{align*}
F(u)_1(t) &= e^{\frac{1}{6} - t} + \int_0^t e^{\frac{1}{6} - s} A N(u_1(s) + L_A(s)) ds + \int_0^t e^{\frac{1}{6} - s} \frac{1}{3} e^{\frac{1}{6} - s} g(u_1(s) + L_A(s), u_2(s) + Z^\varepsilon_A(s)) ds, \\
F(u)_2(t) &= e^{\frac{1}{6} - t} y + \frac{1}{3} \int_0^t e^{\frac{1}{6} - s} \frac{1}{3} e^{\frac{1}{6} - s} g(u_1(s) + L_A(s), u_2(s) + Z^\varepsilon_A(s)) ds.
\end{align*}
\] (4.2)

We shall prove that there exist \(T_0 > 0\) and \(B_0 > 0\) such that whenever \(T \in (0, T_0]\) and \(B > B_0\), the following two statements hold:

(a) \(F(u) \in S\) for \(u \in S\);
(b) \(d(F(u), F(v)) \leq \frac{1}{2}d(u, v)\) for \(u, v \in S\).

It is obvious that \(F(u)(0) = (x, y)\). By the condition A3, properties (2.4) and (2.5), we have

\[
\|F(u)_1(t)\|_{2\sigma} \leq C t^{-\sigma}\|x\| + C \int_0^t (t - s)^{-\sigma} \left(1 + \|u_1(s)\|_{3\sigma}^3 + \|L_A(s)\|_{3\sigma}^3\right) ds + C \int_0^t (t - s)^{-\sigma} ds
\]

\[
\leq C t^{-\sigma}\|x\| + C \int_0^t (t - s)^{-\sigma} \left[1 + (K_1^\varepsilon)^3 + \|u_1(s)\|_{3\sigma}^3\right] ds + C t^{1-\sigma},
\]

which implies

\[
\sup_{0 \leq t \leq T} \left[t^\sigma \|F(u)_1(t)\|_{2\sigma}\right] \leq C\|x\| + C \sup_{0 \leq t \leq T} \left[t^\sigma \int_0^t (t - s)^{-\sigma} \left(1 + (K_1^\varepsilon)^3 + s^{-3\sigma} B^3\right) ds\right]
\]

\[
\leq C\|x\| + CT^{1-3\sigma} \left[1 + (K_1^\varepsilon)^3 + B^3\right].
\] (4.3)

Meanwhile,

\[
\sup_{0 \leq t \leq T} \|F(u)_1(t)\| \leq \|x\| + \int_0^T \|N(u_1(s) + L_A(s))\| ds \\
+ \int_0^T \|f(u_1(s) + L_A(s), u_2(s) + Z^\varepsilon_A(s))\| ds
\]

\[
\leq \|x\| + C \int_0^T (1 + \|u_1(s)\|_{2\sigma}^3 + \|L_A(s)\|_{2\sigma}^3) ds
\]

\[
\leq \|x\| + CT^{1-3\sigma} \left[1 + (K_1^\varepsilon)^3 + B^3\right].
\] (4.4)
Furthermore,

\[
\sup_{0 \leq t \leq T} \| \mathcal{F}(u)_2(t) \| \leq \| y \| + \frac{1}{\varepsilon} \left[ \int_0^T \| g(u_1(s) + L_A(s), u_2(s) + Z_A^\varepsilon(s)) \| ds \right] \\
\leq \| y \| + \frac{C T}{\varepsilon} \left[ \sup_{0 \leq s \leq T} (1 + \| u_1(s) \| + \| u_2(s) \| + \| L_A(s) \|) \right] + \frac{C}{\varepsilon} \int_0^T \| Z_A^\varepsilon(s) \| ds \\
\leq \| y \| + \frac{C T}{\varepsilon} (1 + B + K_1^\varepsilon) + \frac{C}{\varepsilon} K_1^\varepsilon. \tag{4.5}
\]

It is easy to see the continuity of \( \mathcal{F}(u)_1(t) \) and \( \mathcal{F}(u)_2(t) \). As \( T > 0 \) is sufficiently small and \( B \) is large enough, statement (a) follows from (4.3)-(4.5).

Now, let’s prove statement (b). Given any \( u = (u_1, u_2), v = (v_1, v_2) \in S \), by (2.7)

\[
t^\sigma \| \mathcal{F}(u)_1(t) - \mathcal{F}(v)_1(t) \|_{2\sigma} \\
\leq C t^\sigma \int_0^t (t - s)^{-\sigma} \| N(u_1(s) + L_A(s)) - N(v_1(s) + L_A(s)) \| ds \\
+ C t^\sigma \int_0^t (t - s)^{-\sigma} \| f(u_1(s) + L_A(s), u_2(s) + Z_A^\varepsilon(s)) - f(v_1(s) + L_A(s), v_2(s) + Z_A^\varepsilon(s)) \| ds \\
\leq C t^\sigma \int_0^t (t - s)^{-\sigma} \left[ 1 + (K_1^\varepsilon)^2 + \| u_1(s) \|_{2\sigma}^2 + \| v_1(s) \|_{2\sigma}^2 \right] \| u_1(s) - v_1(s) \|_{2\sigma} ds \\
+ C t^\sigma \int_0^t (t - s)^{-\sigma} \| u_1(s) - v_1(s) \| + \| u_2(s) - v_2(s) \| ds
\]

Notice that \( \| u_1(s) \|_{2\sigma} \leq s^{-\sigma} B \) and \( \| v_1(s) \|_{2\sigma} \leq s^{-\sigma} B \), we have

\[
t^\sigma \| \mathcal{F}(u)_1(t) - \mathcal{F}(v)_1(t) \|_{2\sigma} \leq C \left[ 1 + (K_1^\varepsilon)^2 \right] t^\sigma \int_0^t (t - s)^{-\sigma} s^{-\sigma} \| u_1(s) - v_1(s) \|_{2\sigma} ds \\
+ C B^2 t^\sigma \int_0^t (t - s)^{-3\sigma} s^{-3\sigma} \| u_1(s) - v_1(s) \|_{2\sigma} ds \\
+ C t^\sigma \int_0^t (t - s)^{-\sigma} \| u_1(s) - v_1(s) \| + \| u_2(s) - v_2(s) \| ds \\
\leq C \left[ 1 + (K_1^\varepsilon)^2 \right] t^{1-\sigma} \sup_{0 \leq s \leq t} [s^\sigma \| u_1(s) - v_1(s) \|_{2\sigma}] \\
+ C B^2 t^{1-3\sigma} \sup_{0 \leq s \leq t} [s^\sigma \| u_1(s) - v_1(s) \|_{2\sigma}] \\
+ C t \sup_{0 \leq s \leq t} [\| u_1(s) - v_1(s) \| + \| u_2(s) - v_2(s) \|] \\
\leq C \left[ 1 + (K_1^\varepsilon)^2 + B^2 \right] t^{1-3\sigma} d(u, v).
\]

This implies

\[
\sup_{0 \leq t \leq T} \| t^\sigma \| \mathcal{F}(u)_1(t) - \mathcal{F}(v)_1(t) \|_{2\sigma} \leq C \left[ 1 + (K_1^\varepsilon)^2 + B^2 \right] T^{1-3\sigma} d(u, v). \tag{4.6}
\]
Meanwhile, by (2.7)
\[
\|F(u)_1(t) - F(v)_1(t)\| 
\leq \int_0^t \|N(u_1(s) + L_A(s)) - N(v_1(s) + L_A(s))\| ds 
+ \int_0^t \|f(u_1(s) + L_A(s), u_2(s) + Z^\xi_A(s)) - f(v_1(s) + L_A(s), v_2(s) + Z^\xi_A(s))\| ds 
\leq C \int_0^t \left[ 1 + (K_1^\xi)^2 + \|u_1(s)\|_2^2 + \|v_1(s)\|_2^2 \right] \|u_1(s) - v_1(s)\|_{2\sigma} ds 
+ C \int_0^t \|u_1(s) - v_1(s)\| + \|u_2(s) - v_2(s)\| ds,
\]
which implies
\[
\sup_{0 \leq t \leq T} \|F(u)_1(t) - F(v)_1(t)\| \leq CT^{1-\sigma} \left[ 1 + (K_1^\xi)^2 \right] \sup_{0 \leq t \leq T} \|s^\sigma\|u_1(t) - v_1(t)\|_{2\sigma} 
+ CB^2 T^{1-3\sigma} \sup_{0 \leq t \leq T} \|s^\sigma\|u_1(t) - v_1(t)\|_{2\sigma} 
+ CT \sup_{0 \leq t \leq T} \|\|u_1(t) - v_1(t)\| + \|u_2(t) - v_2(t)\| \|
\leq C \left[ 1 + (K_1^\xi)^2 + B^2 \right] T^{1-3\sigma} d(u, v). \tag{4.7}
\]
Furthermore,
\[
\sup_{0 \leq t \leq T} \|F(u)_2(t) - F(v)_2(t)\| 
\leq \frac{1}{\varepsilon} \int_0^T \|g(u_1(s) + L_A(s), u_2(s) + Z^\xi_A(s)) - g(v_1(s) + L_A(s), v_2(s) + Z^\xi_A(s))\| ds 
\leq \frac{CT}{\varepsilon} \sup_{0 \leq s \leq T} \|\|u_1(s) - v_1(s)\| + \|u_2(s) - v_2(s)\| \|
\leq \frac{CT}{\varepsilon} d(u, v). \tag{4.8}
\]
By (4.6) to (4.8) and choosing $T$ small enough, it is easy to see statement (b) holds. Finally, system (4.1) has a unique solution in $S$ by the Banach fixed point theorem.

Let $(W, V) \in S$ be the solution obtained by the above, for every $\sigma \in \left[\frac{1}{6}, \frac{1}{2}\right],$
\[
\|W^\xi_\sigma\|_{2\sigma} \leq Ct^{-\sigma} \|x\| + C \int_0^t (t - s)^{-\sigma} \|N(W^s_\sigma + L_A(s))\| ds + C \int_0^t (t - s)^{-\sigma} ds
\leq Ct^{-\sigma} \|x\| + C \int_0^t (t - s)^{-\sigma} \left[\|W^\xi_s\|_{1/3}^3 + (K_1^\xi)^3 + 1 \right] ds + Ct^{1-\sigma}
\leq Ct^{-\sigma} \|x\| + C \int_0^t (t - s)^{-\sigma} \left[s^{-1/2}B^3 + (K_1^\xi)^3 + 1 \right] ds + Ct^{1-\sigma}
\leq C(1 + t^{-\sigma}),
\]
where $C$ is some constant depending on $\|x\|$, $\sigma$ and $K_1^\xi(\omega)$. The proof is complete. \qed
Now, we give a position to prove Theorem 2.3.

The proof of Theorem 2.3: By Lemma 4.1, Eq. (4.1) admits a unique local solution \( W^\varepsilon \in C([0, T]; H) \cap C((0, T]; V), V^\varepsilon \in C([0, T]; H) \) for some \( T > 0 \), then follow the same steps in [27, Lemma 4.2], we can extend this solution to be \( W^\varepsilon \in C([0, \infty); H) \cap C((0, \infty); V), V^\varepsilon \in C([0, \infty); H) \).

We now prove that the uniqueness of the solution. Suppose there are two solutions \( W^\varepsilon \in C([0, T]; H) \cap C((0, T]; V), V^\varepsilon \in C([0, T]; H) \) and \( \tilde{W}^\varepsilon \in C([0, T]; H) \cap C((0, T]; V), \tilde{V}^\varepsilon \in C([0, T]; H) \). Thanks to the uniqueness of \([0, T] \), we have \( W^\varepsilon_T = \tilde{W}^\varepsilon_T \) and \( V^\varepsilon_T = \tilde{V}^\varepsilon_T \). For any \( T_0 > T \), it follows from the continuity that

\[
\sup_{T \leq t \leq T_0} \| W^\varepsilon_t - \tilde{W}^\varepsilon_t \|_1 \leq \tilde{C}, \quad \sup_{T \leq t \leq T_0} \| \tilde{W}^\varepsilon_t \|_1 \leq \tilde{C},
\]

where \( \tilde{C} > 0 \) depends on \( T_0, \omega \). Hence, for all \( t \in [T, T_0] \), we have

\[
\| W^\varepsilon_t - \tilde{W}^\varepsilon_t \|_1 \leq C \int_T^t (t - s)^{-1/2} \| N(W^\varepsilon_s + L_A(s)) - N(\tilde{W}^\varepsilon_s + L_A(s)) \| ds
\]

\[
+ C \int_T^t (t - s)^{-1/2} \| f(W^\varepsilon_s + L_A(s), V^\varepsilon_s + Z^\varepsilon_A(s)) - f(\tilde{W}^\varepsilon_s + L_A(s), \tilde{V}^\varepsilon_s + Z^\varepsilon_A(s)) \| ds
\]

\[
\leq C \int_T^t (t - s)^{-1/2} \left[ 1 + (K^\varepsilon_1)^2 + \| W^\varepsilon_s \|^2 + \| \tilde{W}^\varepsilon_s \|^2 \right] \| W^\varepsilon_t - \tilde{W}^\varepsilon_t \|_1 ds
\]

\[
+ C \int_T^t (t - s)^{-1/2} \left[ \| W^\varepsilon_s - \tilde{W}^\varepsilon_s \| + \| V^\varepsilon_s - \tilde{V}^\varepsilon_s \| \right] ds
\]

\[
\leq C K_1^\varepsilon \int_T^t (t - s)^{-1/2} \| W^\varepsilon_s - \tilde{W}^\varepsilon_s \|_1 ds + C \int_T^t (t - s)^{-1/2} \left[ \| W^\varepsilon_s - \tilde{W}^\varepsilon_s \| + \| V^\varepsilon_s - \tilde{V}^\varepsilon_s \| \right] ds,
\]

where \( K_1^\varepsilon = 1 + (K^\varepsilon_1)^2 + 2\tilde{C}^2 \). On the other hand,

\[
\| V^\varepsilon_t - \tilde{V}^\varepsilon_t \| \leq \frac{1}{\varepsilon} \int_T^t \| g(W^\varepsilon_s + L_A(s), V^\varepsilon_s + Z^\varepsilon_A(s)) - g(\tilde{W}^\varepsilon_s + L_A(s), \tilde{V}^\varepsilon_s + Z^\varepsilon_A(s)) \| ds
\]

\[
\leq \frac{C}{\varepsilon} \int_T^t \left[ \| W^\varepsilon_s - \tilde{W}^\varepsilon_s \| + \| V^\varepsilon_s - \tilde{V}^\varepsilon_s \| \right] ds.
\]

By (4.9) and (4.10), Gronwall’s inequality implies \( W^\varepsilon_t = \tilde{W}^\varepsilon_t \) and \( V^\varepsilon_t = \tilde{V}^\varepsilon_t \) for all \( t \in [T_0, T] \). Since \( T_0 \) is arbitrary, we get the uniqueness of the solution.

For any \( \varepsilon > 0 \) fixed, \( L_A \in D([0, \infty); V) \) and \( Z^\varepsilon_A(t) \in H \). By the discussion above, \( X^\varepsilon_t = W^\varepsilon_t + L_A(t), Y^\varepsilon_t = V^\varepsilon_t + Z^\varepsilon_A(t) \) is the unique solution to (2.9). □

### 4.3. Galerkin approximation

Recall the Galerkin approximation of system (2.9):

\[
\begin{align*}
\frac{dX^m_{t,\varepsilon}}{dt} &= [AX^m_{t,\varepsilon} + N^m(X^m_{t,\varepsilon}) + f^m(X^m_{t,\varepsilon}, Y^m_{t,\varepsilon})] dt + d\tilde{L}^m_t, \quad X^m_{0,\varepsilon} = x^m \in H_m
\end{align*}
\]

\[
\begin{align*}
\frac{dY^m_{t,\varepsilon}}{dt} &= \frac{1}{\varepsilon} [AY^m_{t,\varepsilon} + g^m(X^m_{t,\varepsilon}, Y^m_{t,\varepsilon})] dt + \frac{1}{\varepsilon} d\tilde{Z}^m_t, \quad Y^m_{0,\varepsilon} = y^m \in H_m.
\end{align*}
\]

**Theorem 4.2.** For every \( x \in H, y \in H, \) system (4.11) has a unique mild solution \( X^{m,\varepsilon}(\omega) \in D([0, \infty); H) \cap D((0, \infty); V) \) and \( Y^{m,\varepsilon}(\omega) \in H \), i.e.,

\[
\begin{align*}
X^{m,\varepsilon}_t &= e^{tA}x^m + \int_0^t e^{(s-t)A}N^m(X^{m,\varepsilon}_s) ds + \int_0^t e^{(t-s)A}f^m(X^{m,\varepsilon}_s, Y^{m,\varepsilon}_s) ds + \int_0^t e^{(t-s)A}d\tilde{L}^m_s,
\end{align*}
\]

\[
\begin{align*}
Y^{m,\varepsilon}_t &= e^{tA/\varepsilon}y^m + \frac{1}{\varepsilon} \int_0^t e^{(s-t)A/\varepsilon}g^m(X^{m,\varepsilon}_s, Y^{m,\varepsilon}_s) ds + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}d\tilde{Z}^m_s.
\end{align*}
\]

Moreover, \( \mathbb{P} \)-a.s.,

\[
\lim_{m \to \infty} \| X^{m,\varepsilon}_t - X^\varepsilon_t \| = 0, \quad t \geq 0,
\]

(4.12)
\[
\lim_{m \to \infty} \|X_t^{m, \epsilon} - X_t^\epsilon\|_1 = 0, \quad t > 0. \tag{4.13}
\]

**Proof.** Follow the same argument in Theorem 2.3, it is easy to system (4.11) has a unique mild solution \(X_t^{m, \epsilon}(\omega) \in D([0, \infty); H) \cap D((0, \infty); V)\) and \(Y_t^{m, \epsilon} \in H\). It remains to prove (4.12) and (4.13). Since when \(t = 0\), (4.12) holds obviously. We only show (4.13).

It is easy to see that for any \(T > 0\),

\[
\sup_{0 \leq t \leq T} [\|X_t^\epsilon\| + \|Y_t^\epsilon\|] \leq \tilde{B}
\]

where \(\tilde{B} > 0\) depends on \(\|x\|, \|y\|, T\) and \(K_t^\epsilon\). By property (2.5), for any \(\theta \in (1/2, 1], t \in (0, T]
\[
\|X_t^\epsilon\|_\theta \leq \|e^{tA}x\|_\theta + \| \int_0^t e^{(t-s)A}N(X_s^\epsilon)ds \|_\theta + \| \int_0^t e^{(t-s)A}f(X_s^\epsilon, Y_s^\epsilon)ds \|_\theta + \| \int_0^t e^{(t-s)A}dL_s \|_\theta \\
\leq t^{-\theta/2}\|x\| + \int_0^t (t-s)^{-1/2}\|N(X_s^\epsilon)\|_{(1-\theta)}ds + C \int_0^t (t-s)^{-\theta/2}ds + \|L_A(t)\|_\theta \\
\leq t^{-\theta/2}\|x\| + K_t^\epsilon + C_T + C \int_0^t (t-s)^{-1/2}(1 + \|X_s^\epsilon\|_\theta^3)ds \\
\leq t^{-\theta/2}\|x\| + K_t^\epsilon + C_T + \int_0^t (t-s)^{-1/2}\|X_s^\epsilon\|^2_\theta ds \\
\leq t^{-\theta/2}\|x\| + K_t^\epsilon + C_T + C \sup_{0 \leq s \leq t} \|X_s^\epsilon\|^2 \int_0^t (t-s)^{-1/2}\|X_s^\epsilon\|_\theta ds, \\
\leq t^{-\theta/2}\|x\| + K_t^\epsilon + C_T + C \tilde{B}^2 \int_0^t (t-s)^{-1/2}\|X_s^\epsilon\|_\theta ds. \\
\leq t^{-\theta/2}\|x\| + K_t^\epsilon + C_T + C \tilde{B}^2 \left[ \int_0^t (t-s)^{-p/2}ds \right]^{1/p} \left( \int_0^t \|X_s^\epsilon\|^q_\theta ds \right)^{1/q},
\]

where \(1/p + 1/q = 1\) with \(1 < p < 2\). Then we have

\[
\|X_t^\epsilon\|_\theta^q \leq C \left( t^{-\theta/2}\|x\| + K_t^\epsilon + C_T \right)^q + C_T B^2 q \int_0^t \|X_s^\epsilon\|^q_\theta ds.
\]

The Gronwall’s lemma implies

\[
\|X_t^\epsilon\|_\theta \leq \tilde{C}(t^{-\theta/2} + 1), \tag{4.14}
\]

where \(\tilde{C} > 0\) depends on \(\|x\|, \|y\|, T\) and \(K_t^\epsilon\). Observe

\[
X_t^{m, \epsilon} - X_t^\epsilon = e^{tA}(x_m - x) + \int_0^t e^{(t-s)A}(\pi_m - I)N(X_s^\epsilon)ds \\
+ \int_0^t e^{(t-s)A}(N_m(X_s^{m, \epsilon}) - N_m(X_s^\epsilon))ds + \int_0^t e^{(t-s)A}(\pi_m - I)f(X_s^\epsilon, Y_s^\epsilon)ds \\
+ \int_0^t e^{(t-s)A}(f_m(X_s^{m, \epsilon}, Y_s^{m, \epsilon}) - f_m(X_s^\epsilon, Y_s^\epsilon))ds + \left[ \tilde{L}^m_A(t) - L_A(t) \right] \\
:= \sum_{i=1}^6 I_i(t). \tag{4.15}
\]

It is clear that

\[
\lim_{m \to \infty} \|I_1(t)\|_1 = 0, \quad \lim_{m \to \infty} \|I_6(t)\|_1 = 0. \tag{4.16}
\]
For $I_4(t)$, notice that $f$ is bounded and by dominated convergence theorem,
$$
\|I_4(t)\|_1 \leq C \int_0^t (t-s)^{-1/2}\|\pi_m - I\|f(X_s^\varepsilon, Y_s^\varepsilon)\|ds \to 0, \quad m \to \infty. \quad (4.17)
$$
For $I_5(t)$,
$$
\|I_5(t)\|_1 \leq C \int_0^t (t-s)^{-1/2}\|X_s^{m,\varepsilon} - X_s^\varepsilon\| + \|Y_s^{m,\varepsilon} - Y_s^\varepsilon\| ds. \quad (4.18)
$$
For $I_2(t)$, the property (2.5) and (4.14) imply for any $0 < s < t$
$$
\lim_{m \to \infty} \|\pi_m - I\|N(X_s^\varepsilon) = 0.
$$
This and the dominated convergence theorem yield
$$
\|I_2(t)\|_1 \leq C \int_0^t (t-s)^{-1/2}\|\pi_m - I\|N(X_s^\varepsilon)\|ds \to 0, \quad m \to \infty. \quad (4.19)
$$
It remains to estimate $I_3(t)$. By (2.7), (4.14) and interpolation inequality,
$$
\|I_3(t)\|_1 \leq C \int_0^t (t-s)^{-1/2}\|N^m(X_s^{m,\varepsilon}) - N^m(X_s^\varepsilon)\|ds \\
\leq C \int_0^t (t-s)^{-1/2}\|N(X_s^{m,\varepsilon}) - N(X_s^\varepsilon)\|ds \\
\leq C \int_0^t (t-s)^{-1/2}(1 + \|X_s^{m,\varepsilon}\|_{1/3}^2 + \|X_s^\varepsilon\|_{1/3}^2)\|X_s - X_s^{m,\varepsilon}\|_1ds \\
\leq C \int_0^t (t-s)^{-1/2}(1 + \|X_s^\varepsilon\|_{1/3}^{2/3}\|X_s^\varepsilon\|_{1/3}^{1/3} + \|X_s^{m,\varepsilon}\|_{1/3}^{2/3}\|X_s^{m,\varepsilon}\|_{1/3}^{1/3})\|X_s^{m,\varepsilon} - X_s^\varepsilon\|_1ds \\
\leq C \int_0^t (t-s)^{-1/2}(1 + \bar{B}A^{-1/3}\bar{C}s^{-1/3})\|X_s^{m,\varepsilon} - X_s^\varepsilon\|_1ds. \quad (4.20)
$$
On the other hand,
$$
Y_t^{m,\varepsilon} - Y_t^\varepsilon = e^{tA/\varepsilon}(y^m - y) + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}(\pi_m - I)g(X_s^\varepsilon, Y_s^\varepsilon)ds \\
+\frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}(g^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) - g^m(X_s^\varepsilon, Y_s^\varepsilon))ds + [\tilde{Z}^m_{A}(t) - Z^\varepsilon_{A}(t)] \\
:= \sum_{i=1}^4 J_i(t). \quad (4.21)
$$
where $Z^m_{A}(t) := \frac{1}{\varepsilon^1/2} \int_0^t e^{(t-s)A/\varepsilon}d\tilde{Z}_s^m$. It is clear that
$$
\lim_{m \to \infty} \|J_1(t)\| = 0, \quad \lim_{m \to \infty} \|J_4(t)\| = 0. \quad (4.22)
$$
For $J_2(t)$, by dominated convergence theorem, we obtain
$$
\|J_2(t)\| \leq C \int_0^t \|\pi_m - I\|g(X_s^\varepsilon, Y_s^\varepsilon)\|ds \to 0, \quad m \to \infty. \quad (4.23)
$$
For $J_3(t)$, by the Lipschitz continuous of $g$,
$$
\|J_3(t)\| \leq C \int_0^t (\|X_s^{m,\varepsilon} - X_s^\varepsilon\| + \|Y_s^{m,\varepsilon} - Y_s^\varepsilon\|) ds. \quad (4.24)
$$
By (4.21)-(4.24), Fatou’s lemma and Gronwall’s Lemma, we have
$$
\limsup_{m \to \infty} \|Y_t^{m,\varepsilon} - Y_t^\varepsilon\| \leq C \int_0^t \limsup_{m \to \infty} \|X_s^{m,\varepsilon} - X_s^\varepsilon\| ds. \quad (4.25)
$$
Combing (4.15)-(4.20) and (4.25), then by Fatou’s lemma, we obtain
\[
\limsup_{m \to \infty} \|X^{m, \epsilon}_t - X^\epsilon_t\|_1 \leq C \int_0^t (t - s)^{-1/2} \left( \limsup_{m \to \infty} \|X^{m, \epsilon}_s - X^\epsilon_s\| + \int_0^s \limsup_{m \to \infty} \|X^{m, \epsilon}_r - X^\epsilon_r\| \, dr \right) \, ds
\]
\[
+ C \int_0^t (t - s)^{-1/2} \left( 1 + 2 \tilde{B}^{4/3} \tilde{C} s^{-1/3} \right) \limsup_{m \to \infty} \|X^\epsilon_s - X^{m, \epsilon}_s\|_1 \, ds
\]
\[
\leq C \int_0^t (t - s)^{-1/2} \left( 1 + 2 \tilde{B}^{4/3} \tilde{C} s^{-1/3} \right) \limsup_{m \to \infty} \|X^\epsilon_s - X^{m, \epsilon}_s\|_1 \, ds
\]
\[
\leq C \left[ \int_0^t (t - s)^{-p/2} \left( 1 + 2 \tilde{B}^{4/3} \tilde{C} s^{-1/3} \right)^p \, ds \right]^{1/p} \left[ \int_0^t \limsup_{m \to \infty} \|X^\epsilon_s - X^{m, \epsilon}_s\|^q_1 \, ds \right]^{1/q}.
\]
where \(\frac{1}{p} + \frac{1}{q} = 1\) with \(1 \leq p < \frac{6}{5}\). Then we have
\[
\limsup_{m \to \infty} \|X^{m, \epsilon}_t - X^\epsilon_t\|_1 \leq \hat{C} \int_0^t \limsup_{m \to \infty} \|X^\epsilon_s - X^{m, \epsilon}_s\|^q_1 \, ds,
\]
where \(\hat{C}\) depends on \(\|x\|, \|y\|, T\) and \(K^\epsilon_T\). Hence, the Grownwall’s Lemma implies
\[
\limsup_{m \to \infty} \|X^{m, \epsilon}_t - X^\epsilon_t\|_1 = 0.
\]
The proof is complete. \(\square\)

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