New exact and analytic solutions in Weyl integrable cosmology from Noether symmetry analysis

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Abstract

We consider a cosmological model in a Friedmann–Lemaître–Robertson–Walker background space with an ideal gas defined in Weyl Integrable gravity. In the Weyl–Einstein a scalar field is introduced in a geometric way. Furthermore, the scalar field and the ideal gas interact in the gravitational Action Integral. Furthermore, we introduce a potential term for the scalar field potential and we show that the field equations admit a minisuperspace description. Noether’s theorem is applied for the constraint of the potential function and the corresponding conservation laws are constructed. Finally, we solve the Hamilton-Jacobi equation for the cosmological model and we derive a family of new analytic solutions in Weyl Integrable cosmology. Some closed-form expressions for the Hubble function are presented.

1. Introduction

Noether symmetry analysis is a powerful method for the construction of analytic solutions for dynamical system which follow from a variational principle. Noether symmetries have been widely applied in gravitational physics and cosmology for the investigation of the integrability properties for the field equations [1–7], also Noether symmetries have been used as a geometric selection rule [8] for the determination of the unknown parameters and functions in the gravitational Action Integral, which have been introduced in order to explain the cosmological observations [9–12].

Indeed, in large scales, the Universe is observed to be in an acceleration phase, which it is attributed to the dark energy. For the origin of the dark energy various approaches have been proposed the last decades in the literature, see for instance [13–24]. In the majority of the dark energy models, the new degrees of freedom are introduced in the gravitational Action integral such that the cosmological dynamics to be evolved in a way such that the physical parameters to describe the observable universe.

In this piece of work, we are interested on the construction of exact and analytic solutions for the cosmological field equations in Weyl Integrable Gravity (WIG) also known as Weyl Integrable Spacetime (WIST) [25–30]. In Einstein theory of gravity the fundamental geometric object of the theory is the metric tensor. On the contrary, in Weyl-Einstein theory of gravity (in WIG) the fundamental geometric objects are the metric tensor and a scalar field. The novelty of the WIG is that the new degrees of freedom are introduced by the geometry of the physical space. Moreover, because of the geometric characteristics of the theory, the scalar field of WIG interacts in the Action Integral with the rest of the matter source of the Universe. Cosmological models with interaction in the dark sector of the Universe have been studied before theoretically and phenomenologically with very interesting results, see for instance [31–34] and references therein.

In our analysis we will show that when the matter source is described by an ideal gas, for instance by a radiation or a dust fluid source, the cosmological field equations in WIG admit a minisuperspace description and there exists a Lagrangian function which produces the field equations under variation. The Noether symmetry analysis is applied for the determination of the scalar field potential and the derivation of invariant functions and conservation laws. The plan of the paper is as follows.
In section 2 we present the basic properties and definitions for the theory of variational symmetries, in particular we present the two theorems of E. Noether for the invariant transformations of the Action Integral. In section 3 we define our model which is that of a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe with an ideal gas as a matter source defined in WIG. Moreover we determine the minisuperspace for the cosmological model of our consideration and we write the point-like Lagrangian which describes the dynamical system.

Furthermore, in section 4, we apply the Noether symmetry condition in order to constrain the free function of the cosmological model. We find that for a specific family of the exponential potential the field equations form a Hamiltonian integrable system where the Hamilton-Jacobi equation can be solved in a closed-form expression. Furthermore, exact and analytic closed-form expressions which solve the field equations are presented. Finally, in section 5 we summarize our results.

2. Noether symmetry analysis

The symmetry analysis is a systematic mathematical method for the determination of invariant functions, similarity transformations and conservation laws, for the study of differential equations. Symmetry analysis was established by S. Lie at the end of the 19th century in a series of books [35–37]. As it was shown by S. Lie, the transformation group which leaves invariant a given differential equation, or a system of differential equations, can be used to simplify the differential equation.

E. Noether in 1918 published a pioneer work [38] for the determination of conservation laws of differential equations which follow from a variational principle. Inspired by the spirit of the work of S. Lie, Noether proved two novel theorems. The first theorem treats the invariance of the Action Integral under an infinitesimal transformation, while the second theorem provides a one to one correspondence for the symmetries of the Action Integral with conservation laws for the differential equations. Some similar studies on finite groups before the work of Noether, are that of Hamel [39, 40], Herglotz [41], Kneser [42] and Klein [43], For a modern discussion on Noether’s work we refer the reader to the review [44].

We continue our discussion of the case of Lagrangian functions which describe second-order differential equations and invariant transformations with point symmetries as generators.

Consider the infinitesimal transformation

\[ t' = t + \varepsilon \xi(t, x^k), \]

\[ x'^i = x^i + \varepsilon \eta^i(t, x^k), \]

with generator the vector field

\[ X = \frac{\partial t'}{\partial \varepsilon} \partial_t + \frac{\partial x'^i}{\partial \varepsilon} \partial_i. \]

A dot indicates derivative with respect to the variable \( t \). For the dynamical system of second-order ordinary differential equations \( H(t, x^k, \dot{x}^k, \ddot{x}^k) \) which follows from the variation of the Lagrangian function \( \mathcal{L} = \mathcal{L}(t, x^k, \dot{x}^k) \), the vector field \( X \) is a variational symmetry, i.e. Noether symmetry, if there exists a function \( f \) such that the following condition is held

\[ X^{(1)} \mathcal{L} + \mathcal{L} \dot{\xi} - \dot{f} = 0. \]

Hence, the Euler-Lagrange equations, that is the dynamical system \( H(t, x^k, \dot{x}^k, \ddot{x}^k) \) remain invariant under the action of the point transformation (1), (2). The symmetry condition (3) is known as Noether’s first theorem.

For a given Lagrangian function, equation (3) gives a monomial expression which is identical zero if the coefficients of the independent monomials are zero. From the later a system of linear partial differential equations is defined.

The second Noether’s theorem relates the existence of Noether symmetries to that of conservation laws. Indeed, if \( X \) is the generator of the infinitesimal transformation (1), (2) which satisfies the symmetry condition (3) for a specific function \( f \), then the function

\[ I(t, x^k, \dot{\xi}^k) = \xi \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \dot{x}^k - \mathcal{L} \right) - \eta \frac{\partial \mathcal{L}}{\partial x^i} + \dot{f}, \]

is a conservation law for the dynamical system \( H(t, x^k, \dot{x}^k, \ddot{x}^k) \), that is, \( I(t, x^k, \dot{\xi}^k) = 0 \).

For a recent review on the application of Noether symmetries in gravitational theories we refer the reader to [45] while other applications of symmetries in gravitation physics can be found for instance in [46–49].
3. Weyl integrable Gravity

In WIG the geometry is defined by the metric tensor $g_{\mu\nu}$ and the scalar field $\phi$, in which $\tilde{g}_{\mu\nu} = \phi g_{\mu\nu}$. The gravitational Action Integral is considered to be the

$$S_W = \int dx^4 \sqrt{-\tilde{g}} \left( \tilde{R} + \xi (\nabla_\mu \phi \nabla^\mu \phi) \right) g^{\mu\nu} - V(\phi) + L_m, \quad (5)$$

in which $\tilde{R}$ is the Ricci scalar of the metric tensor $\tilde{g}_{\mu\nu}$, $\xi$ is the coupling constant of the scalar field $\phi$, $V(\phi)$ is the scalar field potential and $L_m$ is the Lagrangian component which attributes the matter source. In the following we shall assume that $L_m$ describes a perfect fluid.

By definition, the covariant derivative $\nabla_\mu$ is defined according to the Christoffel symbols of the metric tensor $\tilde{g}_{\mu\nu}$.

The field equation (9), with the use of (7) and (8) can be expressed in the equivalent form

$$\tilde{G}_{\mu\nu} + \tilde{\nabla}_\lambda (\nabla^\lambda g) - (2\xi - 1)(\nabla^\mu \phi) (\nabla^\nu \phi) + \xi g_{\mu\nu} g^{\alpha\lambda} (\nabla^\alpha \phi) (\nabla^\lambda \phi) - V(\phi) g_{\mu\nu} = e^{-\frac{2}{\lambda}} T_{\mu\nu}^{(m)}, \quad (9)$$

where $\tilde{G}_{\mu\nu}$ is the Weyl-Einstein tensor and $T_{\mu\nu}^{(m)} = \left( \rho_m + p_m \right) u_\mu u_\nu + \rho_m g_{\mu\nu}$ is the energy momentum tensor which describes the matter source, $\rho_m, p_m$ are the energy density and pressure components respectively, while $u^\mu$ is the observer.

The field equation (9), with the use of (7) and (8) can be expressed in the equivalent form

$$G_{\mu\nu} = \lambda \left( \nabla^\alpha (\nabla_\alpha \phi) (\nabla^\beta \phi) - \frac{1}{2} g_{\mu\nu} g^{\beta\lambda} (\nabla_\beta \phi) (\nabla_\lambda \phi) \right) - V(\phi) g_{\mu\nu} = e^{-\frac{2}{\lambda}} T_{\mu\nu}^{(m)}, \quad (10)$$

in which the new parameter $\lambda$ is defined as $2\lambda = 4\xi - 3$. When $\lambda > 0$, that is $\xi > \frac{3}{2}$, the scalar field is a real field, while when $\lambda < 0$, i.e. $\xi < \frac{3}{2}$, the scalar field $\phi$ is a phantom field because its energy density can be negative.

The equations of motion for the scalar field and the fluid source are

$$-g^{\mu\nu} \nabla_\mu \phi + V(\phi) + \frac{1}{2\lambda} e^{-\frac{2}{\lambda}} \rho_m = 0 \quad (11)$$

$$(\nabla_\mu e^{-\phi} p_m) u^\mu + e^{-\phi} \nabla_\mu u^\mu (\rho_m + p_m) = 0. \quad (12)$$

In the following we assume the perfect fluid to be an ideal gas, that is $p_m = \omega_m \rho_m$ in which $\omega_m$ is a constant parameter. For $w_m = 0$, the matter source $\rho_m$ describes a pressureless fluid source known as a dust fluid, while for $w_m = \frac{1}{3}$, the matter source $\rho_m$ describes a radiation fluid. Furthermore, for this analysis $w_m$ is constraint as $w_m \in (-1, 1)$.

3.1. FLRW spacetime

According to the cosmological principle in large scales the Universe is assumed to be isotropic and homogeneous described by the FLRW geometry. In addition, from the cosmological observations the spatial curvature is very small, thus the Universe is described by the spatially flat FLRW spacetime with line element

$$ds^2 = -N^2 dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \quad (13)$$

We assume the comoving observer $u_\mu = \frac{1}{N} \delta^\mu_0$, and $H = \frac{4\pi}{3N a}$ is the Hubble constant. For the scalar field $\phi$ we assume that it inherits the symmetries of the background space, such that $\phi = \phi(t)$. Hence, the gravitational field equations are

$$3H^2 - \frac{\lambda}{2N^2} \phi^2 - V(\phi) - e^{-\frac{2}{\lambda}} \rho_m = 0, \quad (14)$$
\[
\dot{H} + H^2 + \frac{1}{6} e^{-2\phi} (\rho_m + 3p_m) + \frac{1}{3} \left( \frac{\lambda}{N^2} \phi^2 - V(\phi) \right) = 0, \tag{15}
\]
\[
\dot{\phi} + 3H\dot{\phi} + V(\phi) + \frac{1}{2\lambda} e^{-2\phi} \rho_m = 0, \tag{16}
\]
\[
\dot{\rho}_m + 3NH (\rho_m + p_m) - \rho_m \dot{\phi} = 0. \tag{17}
\]

For an ideal gas, i.e. \( p_m = w_m \rho_m \), equation (17) gives \( \rho_m = \rho_{m0} a^{-3(\omega \rho_{m0} + 1)} e^{\phi} \), where \( \rho_{m0} \) is an integration constant which describes the energy density of the matter source at the present time.

If we replace \( \rho_m \) in the field equations (14)-(16) it is easy to see that the rest of the field equations can be reproduced from the variation of the singular Lagrangian function
\[
\mathcal{L}(N, a, \dot{a}, \phi, \dot{\phi}) = \frac{1}{N} \left( -3a\dot{a}^2 + \frac{\lambda}{2} a^2 \dot{\phi}^2 \right) - N(a^3V(\phi) + \rho_{m0} e^{\phi} a^{m0}). \tag{18}
\]

Lagrangian function (18) is a singular Lagrangian since \( \frac{\partial \mathcal{L}}{\partial \dot{a}} = 0 \). The second-order differential equations (15), (16) are the Euler–Lagrange equations with respect to the variables \( a \) and \( \phi \), i.e. Equations \( \frac{\partial \mathcal{L}}{\partial a} - \frac{\partial \mathcal{L}}{\partial \dot{a}} = 0 \); \( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \), respectively. Furthermore, equation (14) follows from the variation of (18) with respect to the lapse function \( N \), that is, \( \frac{\partial \mathcal{L}}{\partial N} = 0 \). Equation (14) is nothing else than a constraint equation which can be seen as the Hamiltonian for the autonomous dynamical system (15), (16) if without loss of generality we consider \( N(t) = N(a(t), \phi(t)) \).

In the latter case we apply the theory of symmetries of differential equations for the determination of conservation laws for the field equations. Specifically, we consider the application of Noether’s first theorem for the constraint of the scalar field potential \( V(\phi) \), such that, the dynamical system described by the point-like Lagrangian (18) admits Noether point symmetries. Moreover, with the use of Noether’s second theorem conservation laws can be constructed.

**4. Exact and analytic solutions from symmetry analysis**

Without loss of generality we consider \( N(t) = 1 \). Hence for the infinitesimal generator
\[
X = \xi(t, a, \phi) \partial_t + \eta^a(t, a, \phi) \partial_a + \eta^\phi(t, a, \phi) \partial_\phi,
\]
and the point-like Lagrangian (18); the symmetry condition (3) gives that for nonzero potential function \( V(\phi) \), the Noether point symmetries for the cosmological model of our consideration WIG are, the vector field \( X_1 = \partial_t \) for arbitrary potential, while for
\[
V(\phi) = V_0 \exp \left( -\frac{\phi}{w_m - 1} \right), \tag{20}
\]
there exists the additional Noether symmetry
\[
X_2 = 2t \partial_t + \frac{2}{3} a \partial_a + 4(w_m - 1) \partial_\phi. \tag{21}
\]

From Noether’s second theorem we are able to construct the conservation laws. For the vector field \( X_1 \) the conservation laws is the Hamiltonian function \( \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{a}} + \frac{\partial \mathcal{L}}{\partial \phi} \dot{\phi} - \mathcal{L} \), where from (14) it follows \( \mathcal{H} = 0 \). Moreover, for the vector field \( X_2 \) we find the conservation law \( I_0 = 2t \mathcal{H} - 4a^2 \dot{a} + 4\lambda a^3(w_m - 1) \dot{\phi}, \) that is,
\[
I_0 = -4a^2 \dot{a} + 4\lambda a^3(w_m - 1) \dot{\phi}. \tag{22}
\]

We proceed with the application of the symmetry vector and of the conservation laws for the determination of exact and analytic solutions for the field equations.

**4.1. Exact solution**

For the cosmological model with the scalar field potential (20), from the vector field \( X_2 \) we define the invariant functions \( U = at^{-2}, V = \phi - 2(w_m - 1) \ln t \). We assume that the invariant functions are constant, that is,
\[
a(t) = a_0 t^{2}, \quad \phi(t) = 2(w_m - 1) \ln t. \tag{23}
\]

By replacing in the field equations (14)–(17) it follows that (23) is an exact solution if and only if
\[
V_0 = \frac{1}{3} \frac{w_m - 1}{w_m + 1} (6\lambda - 1 - 12w_m \lambda + 6\lambda w_m^2), \tag{24}
\]
\[ \rho_{mn} = \frac{a_0^{3(1+w_m)}}{3(1+w_m)} (6\lambda - 1 + 12w_m\lambda + 6\lambda w_m^2). \] (25)

However, exact solution (25) describes a universe by a stiff fluid. The equation of the state parameter for the effective fluid is \( w_{ef} = -1 \).

We continue with the determination of the analytic solution for the given model.

### 4.2. Hamilton-Jacobi equation

In order to solve the field equations we apply the Hamilton-Jacobi approach. We define the new variable \( \psi = \phi - 6(w_m - 1)\ln a \), such that we write the field equations on the normal coordinates.

In the new variables the point-like Lagrangian (18) reads,

\[ \mathcal{L}(a, \dot{a}, \psi, \dot{\psi}) = (1 - 6(w_m - 1)^2 a^2 - 6(w_m - 1)\lambda a^2 \dot{a} \dot{\psi} - \frac{1}{2} \lambda a^2 \dot{\psi}^2 + a^{-3}(\rho_{mn} a^2 + V_0 e^{-\frac{\psi}{m_{rad}}}). \] (26)

We define the momentum

\[ p_a = 6(1 - 6(w_m - 1)^2)\lambda a \dot{a} - 6(w_m - 1)\lambda a^2 \dot{\psi}, \] (27)

\[ p_\psi = -6(w_m - 1)\lambda a^2 \dot{a} - \lambda a^3 \dot{\psi}, \] (28)

from where it follows the Hamiltonian

\[ \mathcal{H}(a, \psi, p_a, p_\psi) = \frac{1}{12a} p_a^2 - \frac{w_m - 1}{a^2} p_a p_\psi + \frac{6(w_m - 1)^2 (w_m - 1)}{2\lambda a^3} p_\psi^2 - a^{-3}(\rho_{mn} a^2 + V_0 e^{-\frac{\psi}{m_{rad}}}) = 0. \] (29)

In the new variables the conservation law \( I_0 \) reads

\[ I_0 = \frac{2}{3} a p_a. \] (30)

In expression (29), we replace \( p_a = \frac{\partial}{\partial a} S(a, \psi) \), \( p_\psi = \frac{\partial}{\partial \psi} S(a, \psi) \), from where we derive the Hamilton-Jacobi equation

\[ 0 = \frac{1}{12a} \left( \frac{\partial S(a, \psi)}{\partial a} \right)^2 - \frac{w_m - 1}{a^2} \left( \frac{\partial S(a, \psi)}{\partial \psi} \right) \left( \frac{\partial S(a, \psi)}{\partial a} \right) + \frac{(6(w_m - 1)^2 (w_m - 1))}{2\lambda a^3} \left( \frac{\partial S(a, \psi)}{\partial \psi} \right)^2 - a^{-3}(\rho_{mn} a^2 + V_0 e^{-\frac{\psi}{m_{rad}}}). \] (31)

Moreover, from the conservation law (30) we write the constraint equation

\[ I_0 = \frac{2}{3} a \frac{\partial}{\partial a} S(a, \psi). \] (32)

By solving the Hamilton-Jacobi equation we determine the functional form for the Action \( S(a, \psi) \), which can be used to write the field equations into an equivalent system of two-first order ordinary differential equations

\[ \dot{a} = \frac{1 - w_m}{a^2} \left( \frac{\partial S(a, \psi)}{\partial a} \right) + \frac{1}{6a} \left( \frac{\partial S(a, \psi)}{\partial a} \right), \] (33)

\[ \dot{\psi} = \frac{(6(w_m - 1)^2 (w_m - 1))}{\lambda a^3} \left( \frac{\partial S(a, \psi)}{\partial a} \right) + \frac{1 - w_m}{a^2} \left( \frac{\partial S(a, \psi)}{\partial a} \right). \] (34)

For a specific expression for the Action \( S(a, \psi) \), the dynamical system (33), (34) is called the analytic solution for the field equations.

Consequently, from \( I_0 \), the Action is given by the closed-form expression

\[ S(a, \psi) = F(\psi) \] (35)

with

\[ \left( \frac{\partial F(\psi)}{\partial \psi} \right)^2 = \frac{2\lambda (V_0 + \rho_{mn} e^{\frac{\psi}{m_{rad}}})}{e^{\frac{-\psi}{m_{rad}}}(6(w_m - 1)^2 - 1)}. \] (36)

On the other hand for \( I_0 \neq 0 \) the Action is

\[ S(a, \psi) = \frac{3}{2} I_0 \ln a + F(\psi), \] (37)
with

\[
\left( \frac{\partial F(\psi)}{\partial \psi} \right) = \frac{6I_0 e^{-\psi} (w_m - 1) \lambda \pm \sqrt{\kappa(\psi)}}{4e^{-\psi}(6\lambda(w_m - 1)^2 - 1)},
\]

where

\[
\kappa(\psi) = 2e^{-\psi} (3e^{-\psi} I_0^2 + 16V_0(6\lambda(w_m - 1)^2 - 1) + 16e^{-i\pi/4}(6\lambda(w_m - 1)^2 - 1)\rho_{ab}).
\]

Let us focus on the case with \( I_0 = 0 \) and for simplicity let us assume \( \rho_{ab} = 0 \). The Action \( S(a, \psi) \) reads

\[
\left( \frac{\partial F(\psi)}{\partial \psi} \right)^2 = \frac{2\lambda(V_0 e^\psi + \rho_{m0} e^{2\psi})}{6\lambda - 1}.
\]

Thus, the reduced system is

\[
a^2 \dot{a} = \pm \frac{2\lambda(V_0 e^\psi + \rho_{m0} e^{2\psi})}{6\lambda - 1},
\]

\[
\dot{\psi} = \pm \frac{1}{a^2} \frac{2(6\lambda - 1)}{\lambda}(V_0 e^\psi + \rho_{m0} e^{2\psi}).
\]

Therefore, \( \frac{d\psi}{da} = \frac{1}{a} \frac{(6\lambda - 1)}{\lambda} \), or \( \psi(a) = \frac{6\lambda - 1}{\lambda} \ln a \). From the latter, it follows that the Hubble function is expressed by the closed-form expression

\[
(H(a))^2 = \frac{2\lambda}{6\lambda - 1} (V_0 a^{-\frac{\lambda}{2}} + \rho_{m0} a^{-3 - \frac{\lambda}{2}}).
\]

This is an analytic solution which describes an equivalent system of two ideal gases non interacting with constant equation of state parameters.

For the arbitrary parameter \( \rho_{m0} \) it follows \( \frac{d\psi}{da} = \frac{1}{a} \frac{(6\lambda - 1)}{\lambda} \), in which the Hubble function is determined

\[
\frac{H(a)^2}{H_0^2} = \frac{\dot{\Omega}_1 \dot{\Omega}_2}{\Omega_1 \Omega_2},
\]

with \( \dot{\Omega}_1 \equiv \dot{\Omega}_1(\lambda, \omega_m, V_0), \dot{\Omega}_2 \equiv \dot{\Omega}_2(\lambda, \omega_m, \rho_{m0}) \) and indices

\( p_1 = -6\omega_m + \frac{1}{\omega_m - 1} \lambda, p_2 = -3 + 3\omega_m(\omega_m - 2) - \frac{1}{2\lambda}. \)

5. Conclusions

In this study we applied the symmetry analysis of differential equations; specifically, the Noether symmetry conditions, for the constrain of the unknown parameters of a cosmological model with an ideal gas in WIG. A scalar field is introduced in the gravitational Action Integral as a result of Weyl geometry. The scalar field is minimally coupled with gravity but interacts with the matter source. The gravitational field equations admit a minisuperspace description. There exists a point-like Lagrangian function which produces the field equations under variation.

From Noether’s two theorems we were able to constrain the unknown potential function for the scalar field and we found that when it is exponential as it is given by expression (20) the field equations admit a second conservation law. Consequently the field equations form an integrable dynamical system and the Hamilton-Jacobi equation was explicitly solved. For a special value of the second conservation law we were able to write the Hubble function in a closed-form expression, where the cosmological solution consists of two ideal gases.

In the case of a general exponential potential \( V(\phi) = V_0 \exp \left(-\frac{\phi}{\kappa}\right) \). The closed-form solution follows

\[
a(t) = a_0 e^{\frac{\phi(t)}{\kappa}}, \quad \phi(t) = 2\kappa \ln t,
\]

with constraints for the free parameters of the model

\[
V_0 = -\frac{\kappa}{3(1 + \omega_m)^2}(6\kappa\lambda(\omega_m^2 - 1) - 2 - \kappa),
\]

\[
\rho_{m0} = -\frac{4\kappa}{3(1 + \omega_m)^2}(6\kappa\lambda(\omega_m^2 + 1) - 2 - \kappa).
\]

This solution is not the analytic solution of the model, because it has fewer free parameters than the degrees of freedom of the theory. However, it is clear that cosmology in WIG in terms of the background space provides cosmological solutions of special interests.
At this point it is important to mention that the cosmological perturbations for a model with background equations similar to that of this study were investigated in [50]. Thus similar results are expected and for this study.

From such analysis we conclude that the Noether symmetry analysis is a geometric criterion which provides physically accepted solutions in cosmological studies.

Data availability statement

No new data were created or analysed in this study.

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