Phase sensitivity approaching quantum Cramér-Rao bound in a modified SU(1,1) interferometer

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SU(1,1) interferometers, based on the usage of nonlinear elements, are superior to passive interferometers in phase sensitivity. However, the SU(1,1) interferometer cannot make full use of photons carrying phase information as the second nonlinear element annihilates some of the photons inside. Here, we focus on improving phase sensitivity and propose a new protocol based on a modified SU(1,1) interferometer, where the second nonlinear element is replaced by a beam splitter. We utilize two coherent states as inputs and implement balanced homodyne measurement at the output. Our analysis suggests that the protocol we propose can achieve sub-shot-noise-limited phase sensitivity and is robust against photon loss and background noise. Our work is important for practical quantum metrology using SU(1,1) interferometers.

I. INTRODUCTION

Phase estimation is an important means of precision measurement, which provides a route to estimate many physical quantities that cannot be directly measured via conventional methods. Related to this, optical interferometers are an effective tool for phase estimation. In general, a conventional interferometer utilizes a coherent light source and intensity-based measurement; as a result, the phase sensitivity is bounded by the shot-noise limit \( \Delta \theta_{\text{SNL}} = 1/\sqrt{N} \) with \( N \) photons inside the interferometer on average [1]. To improve the phase sensitivity, two kinds of approaches are proposed. The first approach is to deploy quantum resources, including exotic nonclassical states [2–6] and photon-number-resolving measurements [7–9]; the second one is to change the evolution process through the use of nonlinear elements. A typical interferometer designed in terms of the second approach is usually known as an SU(1,1) interferometer, introduced by Yurke et al. [10], where two beam splitters in the conventional interferometer are replaced by two optical parametric amplifiers (OPAs).

In recent years, a large number of theoretical and experimental studies have focused on phase estimation using SU(1,1) interferometers. Plick et al. demonstrated that an SU(1,1) interferometer with two coherent states as inputs can achieve the sub-shot-noise-limited phase sensitivity by using conventional intensity measurement [11]. Li et al. proposed an SU(1,1) interferometer fed by coherent and squeezed vacuum states, the phase sensitivity can approach the Heisenberg limit with balanced homodyne measurement [12] and parity measurement [13]. An SU(1,1) interferometer using squeezed coherent state and coherent state was presented by Hu et al., they studied the phase sensitivity with balanced homodyne measurement and discussed the optimal squeezing fraction [14]. Meanwhile, there has been an interest in modified SU(1,1) interferometers, such as pumped-up and truncated SU(1,1) interferometers [15–17]. In addition, the phase sensitivity limit of an SU(1,1) interferometer was studied. Gong et al. analyzed the quantum Cramér-Rao bounds of various Gaussian inputs [18]. More recently, You et al. discussed the conclusive sensitivity limit for a protocol using single-mode inputs [19].

Indeed, the phase sensitivities of most protocols cannot achieve or approach the corresponding quantum Cramér-Rao bound, although they are capable of reaching the Heisenberg limit [18]. A major reason for this result originates from the drawback of SU(1,1) interferometers: many photons stimulated by the first OPA are absorbed by the second OPA. Therefore, only a part of photons carrying phase information contribute to the measurement, which limits the phase sensitivity. In this paper, we present a new protocol for phase estimation using a modified SU(1,1) interferometer; the second OPA in the conventional SU(1,1) interferometer is replaced by a beam splitter. Two coherent states are used as inputs, and balanced homodyne measurement is implemented at the output. The phase sensitivity of our protocol is compared with the previous SU(1,1)-type protocol and its corresponding quantum Cramér-Rao bound. For practical purposes, we also analyze the effects of photonic loss and background noise on phase sensitivity of our protocol.

The paper is organized as follows. Sec. II introduces model of the modified SU(1,1) interferometer and analyzes the phase sensitivity in an ideal scenario. In Sec. III, we compare the phase sensitivity of our protocol with that of the previous SU(1,1)-type protocol, and analyze the optimality of our protocol by calculating the quantum Cramér-Rao bound. The effects of photonic loss and background noise on phase sensitivity are discussed in Sec. IV. Finally, we summarize our results in Sec. V.

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II. MODEL AND PHASE SENSITIVITY OF MODIFIED SU(1,1) INTERFEROMETER

A. Model

The schematic of the modified SU(1,1) interferometer is shown in Fig. 1. Two coherent states, $|\alpha\rangle$ and $|\beta\rangle$, are injected into a pumped OPA. Then the generated states undergo phase shift of $\theta$ on one of the paths. After the unknown phase shift, these states are recombined by a beam splitter with transmittance of $\eta$. Finally, balanced homodyne measurement is implemented at one of the outputs.

![Fig. 1. Schematic of phase estimation protocol based on a modified SU(1,1) interferometer. A beam splitter, BS1, takes the place of the second OPA in a conventional SU(1,1) interferometer. One of the paths is then measured using balanced homodyne detection. Here the transmittance of BS1 and BS2 is given by $\eta$ and 1/2, respectively. The creation and annihilation operators for each stage are shown and given in the Appendix B. OPA, optical parametric amplifier; BS, beam splitter; LO, local oscillator; D, detector; S, subtracter.](image)

We assume that phase and gain parameter of the OPA are 0 and $g$, respectively; accordingly, the complex amplitudes of two coherent states can be expressed as $\alpha = |\alpha| e^{i\varphi}$ and $\beta = |\beta| e^{i\delta}$. Since the OPA changes the mean photon number of the input state, we use the mean photon number inside our interferometer as the benchmark. Thus the mean photon number is given by

$$N = \langle \hat{a}_1^\dagger \hat{a}_1 + \hat{b}_1^\dagger \hat{b}_1 \rangle = \langle |\alpha|^2 + |\beta|^2 \rangle \cosh (2g) + 2 \sinh^2 g + 2 |\alpha\beta| \sinh (2g) \cos (\varphi + \delta).$$

(1)

It is straightforward to show that when the total mean photon number of the input $N_{in} = |\alpha|^2 + |\beta|^2$ is constant, $|\alpha| = |\beta|$ and $\varphi + \delta = 0$ are the optimal conditions to obtain maximal mean photon number $N_{max}$ in our protocol. Throughout this paper, we assume such conditions are always met. That is, the mean photon number in our protocol is

$$N = 2|\alpha|^2 e^{2g} + 2\sinh^2 g. \hspace{1cm} (2)$$

B. Phase sensitivity

The measurement strategy in our protocol is balanced homodyne measurement, which can measure the quadrature of output by mixing with a strong coherent state. The phase therefore can be estimated from the difference between the two detectors D1 and D2. Here we consider amplitude quadrature $\hat{X}$ as measured observable, and the measurement operator can be written as

$$\hat{X} = \hat{a}_2^\dagger + \hat{a}_2. \hspace{1cm} (3)$$

The creation operator of the output modes is given by (See Appendix B)

$$\hat{a}_2^\dagger = \sqrt{\eta} e^{-i\delta} (\hat{a}_0^\dagger \cosh g + \hat{b}_0 \sinh g) - i \sqrt{1 - \eta} (\hat{b}_0^\dagger \cosh g + \hat{a}_0 \sinh g).$$

(4)

By using Eqs. (3) and (4), the expectation value of measurement operator is found to be

$$\langle \hat{X} \rangle = 2e^g \sqrt{\eta} |\alpha| \cos \theta. \hspace{1cm} (5)$$

Using the definition of visibility [20],

$$\mathcal{V} = \frac{\langle \hat{X} \rangle_{\text{max}} - \langle \hat{X} \rangle_{\text{min}}}{\langle \hat{X} \rangle_{\text{max}} + \langle \hat{X} \rangle_{\text{min}}}.$$  \hspace{1cm} (6)

we can find that the visibility of $\langle \hat{X} \rangle$ is 100% for any transmittance.

Furthermore, we can calculate that

$$\langle \hat{X}^2 \rangle = 2\eta |\alpha|^2 e^{2g} \left[ \cos (2\theta) + 1 \right] + \cosh (2g) - 2 \sinh (2g) \sqrt{\eta (1 - \eta)} \sin \theta.$$  \hspace{1cm} (7)

The phase sensitivity of our protocol can be calculated via error propagation formula, defined as [21]

$$\Delta \theta = \frac{\sqrt{\langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2}}{|\partial \langle \hat{X} \rangle / \partial \theta|}.$$  \hspace{1cm} (8)

Plugging Eqs. (5) and (7) into Eq. (8), we get

$$\Delta \theta = \frac{\sqrt{\cosh (2g) - 2 \sqrt{\eta (1 - \eta)} \sinh (2g) \sin \theta}}{2 \sqrt{\eta e^g |\alpha \sin \theta|}}.$$  \hspace{1cm} (9)

Clearly, the phase sensitivity depends on the phase shift $\theta$. Evidently, we obtain the optimal phase sensitivity at $\theta = \pi/2$.

Additionally, from Eq. (9), the phase sensitivity also depends on the value of transmittance $\eta$. We can obtain the optimal transmittance $\eta_{opt}$ by taking the derivative of Eq. (9) with respect to $\eta$. When a fixed gain parameter $g$, the optimal transmittance $\eta_{opt}$ is given by

$$\eta_{opt} = \frac{\cosh^2 (2g)}{\sinh^2 (2g) + \cosh^2 (2g)} = \frac{1 + \text{sech}(4g)}{2}.$$  \hspace{1cm} (10)
FIG. 2. The optimal transmittance \( \eta_{\text{opt}} \) of the beam splitter as a function of gain parameter \( g \).

This reveals that the optimal transmittance depends solely on the gain parameter \( g \), regardless of the mean photon number of the input. If the gain parameter is known, then the transmittance \( \eta \) can be chosen such that the phase sensitivity reaches its minimum. We note that this effect is pronounced at low gain parameters. In Fig. 2, we illustrate the relationship between the optimal transmittance \( \eta_{\text{opt}} \) and gain parameter \( g \). From Eq. (10) and Fig. 2, we conclude that the optimal transmittance approaches 0.5 when \( g \) is large. This suggests that a 50-50 beam splitter is nearly optimal for our protocol with a high gain parameter. Trivially, when \( g = 0 \), our protocol reduces to a classical interferometer, which is consistent with the result reported in Ref. [22].

In the following sections, the transmittance is fixed to be 1/2 for the sake of simplicity. Related to this, the optimal phase sensitivity turns out to be

\[
\Delta\theta_{\min} = \frac{1}{\sqrt{2e^{2g}|\alpha|}}.
\]  

According to the definition of mean photon number in Eq. (2), the shot-noise limit can be expressed as

\[
\Delta\theta_{\text{SNL}} = \frac{1}{\sqrt{2|\alpha|^2 e^{2g} + 2\sinh^2 g}}.
\]

In general, high-intensity coherent states (\( |\alpha|^2 \gg 1 \)) will be used as inputs, i.e., \( |\alpha|^2 e^{2g} \gg \sinh^2 g \). Then the shot-noise limit is approximately recast as

\[
\Delta\theta_{\text{SNL}} \simeq \frac{1}{\sqrt{2e^{2g}|\alpha|}}.
\]

By comparing Eqs. (11) and (13), one can find that the phase sensitivity of our protocol is superior to the shot-noise limit by a factor of \( e^g \).

III. COMPARISON WITH THE SU(1,1)-TYPE PROTOCOL AND THE QUANTUM CRAMÈR-RAO BOUND

In the previous section, we calculated the phase sensitivity of our protocol. In this section, we analyze the advantage of our protocol over the previous SU(1,1)-type protocol. Then we compare the phase sensitivity with the quantum Cramèr-Rao bound.

A. Comparison with the SU(1,1)-type protocol

In Ref. [13], Li et al. used different measurement strategies to analyze the phase sensitivity of an SU(1,1)-type protocol with two equal coherent states as inputs. Their results indicate that balanced homodyne measurement gives the optimal phase sensitivity, which has the following form

\[
\Delta\theta_{\text{SU}(1,1)} = \frac{1}{2e^g |\alpha| \sinh g}.
\]

By comparing Eqs. (11) with (14), one can find that the phase sensitivity of our protocol outperforms the previous SU(1,1)-type protocol. To quantify the improvement, we calculate the improvement factor \( K \), given by

\[
K = \frac{\sqrt{2e^g} e^g - e^{-g}}{e^g - e^{-g}}.
\]

For a high gain parameter, the phase sensitivity is improved by a factor of \( \sqrt{2} \). With the decrease of gain parameter, the improvement factor of phase sensitivity increases rapidly. That is, our protocol can give at least a factor of \( \sqrt{2} \) improvement in phase sensitivity in comparison with an SU(1,1)-type protocol.

In addition, noticing that the complexity of the measurement system is simplified. The modified SU(1,1) interferometer circumvents problems encountered by the SU(1,1) interferometer, such as pump phase-matching and gain-matching [12]. Hence, our protocol remains superior in terms of sensitivity and complexity when compared with an SU(1,1)-type protocol.

B. Comparison with the quantum Cramèr-Rao bound

Now we focus on the analysis of phase sensitivity limit of our protocol. The ultimate limit of phase sensitivity is given by the quantum Cramèr-Rao bound, \( \Delta\theta_{QCRB} = 1/\sqrt{\mathcal{F}} \) with \( \mathcal{F} \) being the QFI. In Ref. [18], Gong et al. demonstrated that three different phase configurations would give three different QFIs with two coherent states as inputs. This is due to the asymmetric gain of the OPA; in other words, we have \( \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \neq \langle \hat{b}_1^\dagger \hat{b}_1 \rangle \) for two unequal
coherent states ($|\alpha| \neq |\beta|$). For every phase configuration, the number of photons experiencing the estimated phase is different. Therefore, the QFIs calculated from different phase configurations are different.

In particular, when $|\alpha| = |\beta|$, all phase configurations give the same QFI, which is found to be

$$ F = 2|\alpha|^2 (e^{4\theta} + 1) + \sinh^2 (2g). $$

The corresponding quantum Cramér-Rao bound is given by

$$ \Delta \theta_{\text{QCRB}} = \frac{1}{\sqrt{2|\alpha|^2 (e^{4\theta} + 1) + \sinh^2 (2g)}}. $$

(17)

When $|\alpha|^2 \gg 1$ and $e^{4\theta} \gg 1$, we have

$$ \Delta \theta_{\text{QCRB}} \approx \frac{1}{\sqrt{2e^{2g}}} |\alpha|. $$

(18)

One can find that Eqs. (11) and (18) are the same, meaning that the phase sensitivity of our protocol approaches the quantum Cramér-Rao bound with the increase of $|\alpha|^2$ and $g$. That is, for high-intensity coherent states and high gain parameters, balanced homodyne measurement is approximately optimal strategy for our protocol.

In Fig. 3, we show the ratio between the ultimate phase sensitivity $\Delta \theta_{\text{QCRB}}$, given by quantum Cramér-Rao bound, and the optimal phase sensitivity $\Delta \theta_{\text{min}}$ as a function of gain parameter $g$ and mean photon number $|\alpha|^2$ in the coherent state. We can see that based on the current experimental techniques [23], it is feasible to achieve phase sensitivity close to the one governed by the quantum Cramér-Rao bound.

IV. ANALYSIS OF ROBUSTNESS REGARDING PHOTON LOSS AND BACKGROUND NOISE

In Sec. II, we analyzed the phase sensitivity in an ideal scenario. In practical applications, photon loss and background noise are inevitable and they will limit the attainable phase sensitivity. In this section, we discuss the effects of these two factors on phase sensitivity.

The process of photon loss can be modeled by adding two fictitious beam splitters with reflectivity of $L_1$ and $L_2$ in two paths [24]. Photons reflected by the fictitious beam splitters are regarded as photon loss, and the loss rate in each path is equal to the reflectivity of corresponding fictitious beam splitter. Background noise refers to photons in the surrounding environment, which enter the interferometer through another input port of the fictitious beam splitter.

In the optical regime, the mean photon number at room temperature is approximately $10^{-20}$ [25] and the environmental mode is treated as vacuum. However, when the measurement system is immersed in the environment with a small amount of stray light, the environmental mode is treated as a thermal state with $n_{th}$ photons on average. Generally, the mean photon number of background noise is much weaker than those of the input states. In this paper, $n_{th} = 1$ is used to simulate the intensity of background noise. Related to this, $n_{th} = 0$ indicates that there is only photon loss, and $n_{th} = 1$ indicates that photon loss and background noise exist simultaneously.

For a lossy scenario, the operator relationship between the input and output modes is rewritten as

$$ \hat{a}^\dagger_2 = \sqrt{\frac{1-L_1}{2}} e^{-i\theta} (\hat{a}^\dagger_0 \cosh g + \hat{b}^\dagger_0 \sinh g) - i \sqrt{\frac{L_1}{2}} \hat{c}^\dagger $$

$$ - i \sqrt{\frac{1-L_2}{2}} (\hat{b}^\dagger_0 \cosh g + \hat{a}^\dagger_0 \sinh g) - \sqrt{\frac{L_2}{2}} \hat{d}^\dagger, $$

(19)

where $\hat{c}^\dagger$ and $\hat{d}^\dagger$ represent the creation operators for environmental modes.

We can recalculate that

$$ \langle \hat{X}_L \rangle = \sqrt{2(1-L_1)} e^\theta |\alpha| \cos \theta, $$

(20)

and

$$ \langle \hat{X}_L^2 \rangle = 2|\alpha|^2 e^{2\theta} (1 - L_1) \cos^2 \theta + (L_1 + L_2) n_{th} + 1 $$

$$ - \sqrt{(1-L_1)(1-L_2)} \sinh (2g) \sin \theta $$

$$ + (2 - L_1 - L_2) \sin^2 g. $$

(21)

In the limit of $L_1 = L_2 = 1$, the inputs are completely lossy and the outputs are thermal states. Combining Eqs. (20) and (21), we have $\langle \hat{X}_L^2 \rangle - \langle \hat{X}_L \rangle^2 = 2n_{th} + 1$, which matches the variance of a thermal state without information on the estimated phase. In addition, Eq. (20) means that the visibility remains 100% regardless of lossy rates.
Since the lengths of two paths are close in a practical scenario, we consider \( L_1 = L_2 = L \) from now on. The phase sensitivity is given by

\[
\Delta \theta_L = \sqrt{(1 - L) \left[ 2 \sinh^2 g - \sinh (2g) \sin \theta \right] + 2L_{\text{th}} + 1} / \sqrt{2(1 - L)e^g |\alpha \sin \theta|}.
\]  

(22)

When the estimated phase sits at the optimal phase point \( \theta = \pi/2 \), the phase sensitivity reaches the minimum, which can be expressed as

\[
\Delta \theta_{L, \text{min}} = \frac{\sqrt{1 + \gamma}}{\sqrt{2e^{2g}|\alpha|}}
\]  

(23)

with

\[
\gamma = \frac{L}{1 - L} (2n_{\text{th}} + 1) e^{2g}.
\]  

(24)

It can be seen that \( \gamma \) is the increment of measurement variance induced by photon loss and background noise. As has been previously pointed out, phase sensitivity in an ideal scenario can outperform the shot-noise limit. Here we analyze the maximum value of lossy rate when the phase sensitivity drops below the shot-noise limit. By solving the equation with respect to Eqs. (12) and (23),

\[
\frac{1 - L_{\text{max}} + L_{\text{max}} (2n_{\text{th}} + 1) e^{2g}}{2 (1 - L_{\text{max}}) e^{4g} |\alpha|^2} = \frac{1}{2|\alpha|^2 e^{2g}}
\]  

(25)

one can calculate the maximal tolerable lossy rate \( L_{\text{max}} \).

We first consider the case of only photon loss, Fig. 4 gives the relationship between the maximal tolerable lossy rate and gain parameter with different mean photon numbers in coherent states. The mean number of photons we use is easy to engineer under the current experimental techniques; meanwhile, the condition \( |\alpha|^2 \gg 1 \) is satisfied. One can find that the effect of mean photon number on the maximal tolerable lossy rate is negligible. With the increase of gain parameter, the maximal tolerable lossy rate increases rapidly; after about \( g = 2 \), this rate tends to the limit, which is equal to 50%. This result indicates that increasing gain parameter can improve the robustness of our protocol.

In Fig. 5, we show the dependence of maximal tolerable lossy rate on gain parameter in the presence of both photon loss and background noise. Similarly, mean photon number has almost no effect on the maximal tolerable lossy rate. The maximal tolerable lossy rate tends to the limit after about \( g = 2 \), and the limit is 25%.

By comparing Fig. 4 with Fig. 5, it can be seen that the effect of background noise is significant. Although \( n_{\text{th}} \ll |\alpha|^2 \), the maximal tolerable lossy rate decreases to half of that without background noise. To determine the limit of the maximal tolerable lossy rate, we can calculate the combination of Eqs. (13) and (23) as follows

\[
\frac{1 - L_{\text{max}} + L_{\text{max}} (2n_{\text{th}} + 1) e^{2g}}{2 (1 - L_{\text{max}}) e^{4g} |\alpha|^2} = \frac{1}{2|\alpha|^2 e^{2g}}.
\]  

(26)

The approximate solution for Eq. (26) is found to be

\[
\lim_{g \gg 1, |\alpha| \gg 1} L_{\text{max}} = \frac{1}{2(n_{\text{th}} + 1)}.
\]  

(27)

Obviously, we have \( L_{\text{max}} = 0.5 \) and \( L_{\text{max}} = 0.25 \) for \( n_{\text{th}} = 0 \) and \( n_{\text{th}} = 1 \), respectively.

V. CONCLUSION

In conclusion, we propose a new protocol for phase estimation using a modified SU(1,1) interferometer. The second OPA in the conventional SU(1,1) interferometer is replaced by a 50-50 beam splitter. We use two coherent states as inputs and perform balanced homodyne measurement at the output. In an ideal scenario, the phase
sensitivity is superior to the shot-noise limit by a factor of
\(e^g\) with coherent inputs. Additionally, our protocol shows
the potential to surpass conventional SU(1,1) interferom-
eter in phase sensitivity. The improvement factor of the
phase sensitivity is \(\sqrt{2}\) at high gain parameters and in-
creases rapidly with the decrease of the gain parameter.
Finally, we analyze the effects of photon loss and back-
ground noise on phase sensitivity. The results indicate
that our protocol is robust against these two factors. For
high gain parameters, the phase sensitivity of our pro-
tocol can outperform the shot-noise limit even when the
lossy rate is relatively high. We believe that our protocol
is feasible to achieve phase sensitivity close to the quan-
tum Cramér-Rao bound based on current experimental
techniques.

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Appendix A: The exact expression for phase sensitivity calculated from classical Fisher information

For many measurement strategies, error propagation formula is a simple but approximate method to calculate
phase sensitivity. An exact expression for phase sensitivity equals the inverse of the square root of the classical Fisher
information, i.e., \(\Delta \theta = 1/\sqrt{\mathcal{F}_c}\). As for our protocol–Gaussian inputs and Gaussian measurement–the classical Fisher
information is described as

\[
\mathcal{F}_c = \frac{\langle \partial \langle \hat{X} \rangle / \partial \theta \rangle^2}{\langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2} + \frac{2\langle \partial \Delta \hat{X} / \partial \theta \rangle^2}{\langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2}
\]

with \(\Delta \hat{X} = \sqrt{\langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2}\).

One can find that the first term is related to error propagation formula. The second term at the optimal phase
point, \(\theta = \pi/2\), is 0 since the derivative in the numerator can be decomposed into the product of a prefactor and a
cosine function, \(\cos(\theta)\). This indicates that error propagation formula and the classical Fisher information give
the same result for the optimal phase sensitivity. For other phase points, the phase sensitivity calculated from error
propagation formula is greater than true value as the second term is non-negative. These results hold true for the
lossy scenario. Overall, it is sufficient to calculate the phase sensitivity via error propagation formula, for we merely
are interested in the optimal phase sensitivity of our protocol.

Appendix B: The expectation values of measurement operators with ideal and lossy scenarios

In an ideal scenario, the input-output field operator transformations of a beam splitter and those of an optical
parametric amplifier are given by

\[
\hat{a}_2 = \sqrt{\eta} a_1 e^{i\theta} + i\sqrt{1-\eta} b_1, \hspace{1cm} (B1)
\]

\[
\hat{b}_2 = i\sqrt{1-\eta} a_1 e^{i\theta} + \sqrt{\eta} b_1, \hspace{1cm} (B2)
\]

\[
\hat{a}_1 = \hat{a}_0 \cosh g + \hat{b}_0 \sinh g, \hspace{1cm} (B3)
\]

\[
\hat{b}_1 = \hat{b}_0 \cosh g + \hat{a}_0 \sinh g. \hspace{1cm} (B4)
\]

Regarding two equal coherent states \(\rho_A \otimes \rho_B\) (\(\rho_A = \rho_B = |\alpha\rangle \langle \alpha|\)), we have

\[
\langle \hat{a}_2 \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_2^\dagger \rangle = 2|\alpha|^2 e^{2g}\left[\eta \cos(2\theta) - 2\sqrt{\eta (1-\eta)} \sin \theta - (1-\eta) \right] - 2 \sinh(2g) \sqrt{\eta (1-\eta)} \sin \theta \hspace{1cm} (B5)
\]

and

\[
\langle \hat{a}_2 \hat{a}_1 \rangle = |\alpha|^2 e^{2g} \left[1 + 2\sqrt{\eta (1-\eta)} \sin \theta \right] + \sinh^2 g. \hspace{1cm} (B6)
\]

In a lossy scenario, the input-output field operator transformation of the whole interferometer is given by Eq. (19).
Related to this, the inputs are \(\rho_A \otimes \rho_B \otimes \rho_E \otimes \rho_E\), where \(\rho_E\) represents a thermal state with \(n_{th}\) photons on average.
Further, we have
\[ \langle \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \rangle = |\alpha|^2 e^{2g} [(1 - L_1) \cos (2\theta) - 2 \sqrt{(1 - L_1)(1 - L_2)} \sin \theta - (1 - L_2)] \]
\[ - \sqrt{(1 - L_1)(1 - L_2)} \sinh (2g) \sin \theta \] \hspace{1cm} (B7)
and
\[ \langle \hat{a}_2^\dagger \hat{a}_2 \rangle = \left( 1 - \frac{L_1 + L_2}{2} \right) \left( |\alpha|^2 e^{2g} + \sinh^2 g \right) + |\alpha|^2 e^{2g} \sqrt{(1 - L_1)(1 - L_2)} \sin \theta + \frac{L_1 + L_2}{2} n_{\text{th}}. \] \hspace{1cm} (B8)

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