Struwe-like solutions for the Stochastic Harmonic Map Flow

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Abstract

We give a new result on the well-posedness of the two-dimensional Stochastic Harmonic Map flow, whose study is motivated by the Landau-Lifshitz-Gilbert model for thermal fluctuations in micromagnetics. We construct strong solutions that belong locally to the spaces $C([s, t]; H^1) \cap L^2([s, t]; H^2)$, $0 \leq s < t \leq T$. In that sense, these maps are a counterpart of the so-called “Struwe solutions” of the deterministic model. We also give a natural criterion of uniqueness that extends A. Freire’s Theorem to the stochastic case. Both results are obtained under the condition that the noise term has a trace-class covariance in space.

1 Introduction

1.1 Motivations

In this paper, we are interested in the existence, uniqueness and regularity of the parabolic stochastic differential equation

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = u|\nabla u|^2 + u \times \xi & \text{in } [0, T] \times \mathbb{T}^2 \\
u(0) = u_0 & \text{in } \mathbb{T}^2,
\end{cases}$$

(Stratonovitch sense) whose unknown $u : \Omega \times [0, T] \times \mathbb{T}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ takes values in the unit sphere $\mathbb{S}^2 \equiv \{x \in \mathbb{R}^3 : |x| = 1\}$, and where the initial data $u_0$ belongs to the critical space $H^1$. We also assume regularity in space for $\xi = \xi(\omega, t, x)$, which is the time derivative of a Wiener process with finite trace class covariance in $H^1$. We will first construct the counterpart of the so-called “Struwe solutions” in the presence of noise (Theorem 1). Then, a similar result as that of Freire’s uniqueness Theorem will be given, providing a natural criterion of uniqueness leading to the solution obtained above (Theorem 2).

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The deterministic equation corresponding to (1.1) has been first studied in the early sixties by J. Eells and J.H. Sampson [17], in order to build harmonic maps from a general manifold (which here is simply the two-dimensional torus $\mathbb{T}^2$) onto another, typically a unit sphere. Trying to find harmonic maps $u : M \to N$ between two manifolds provides an important example of a variational problem occurring within non-flat metrics. It echoes several physical models, such as liquid crystals [9], or W.F. Brown's theory for continuous micromagnetics [5]. Their common feature is the necessity for ground states to minimize the functional $E := \int_M |\nabla u|^2 \, dx$, under the pointwise constraint $u(x) \in N \, \text{a.e.}$

Whether there exists or not a harmonic map, within the homotopy class of a given smooth map $\varphi : M \to N$ is by itself an important topic for geometers. To answer that question, the approach initiated by J. Eells and J.H. Sampson consists in adding a time variable to the unknown, and then studying the Heat flow associated to $E$, namely

$$\frac{\partial u}{\partial t} - \Delta_M u = u|\nabla u|^2, \quad t \geq 0 , \quad u|_{t=0} = \varphi, \quad (\text{HMF})$$

where in view of the applications we let here $N := \mathbb{S}^2$. The next step is to show convergence of the solution as $t \to \infty$, towards an harmonic map. This follows from asymptotic estimates, yielding finally a solution to the problem, see e.g. [17, 15, 16].

**Struwe-like solutions** Unfortunalely, the latter method fails unless the target manifold has non-positive sectional curvature, a somewhat restrictive hypothesis. If $M$ denotes a surface, M. Struwe has shown in [43] that (HMF) admits a solution $u$ such that

- $u$ fulfills (HMF) in the sense of distributions;
- $u$ is a classical, smooth solution, with the exception of finitely many points $(t_i, x^k_i), k \leq K_i, i \leq I$.

The latter map is the one that we might refer to the "Struwe solution". As will be shown below, the Struwe solution has a natural counterpart in presence of noise. Note that existence of a weak martingale solution has been provided in [8], where the authors are able to deal with a three dimensional domain. This is done via finite-dimensional Galerkin approximations and uniform energy bounds on the corresponding family.

Our approach here is different, in the sense that we work at the level of a regularized stochastic PDE, but still infinite-dimensional. We will obtain strong solutions by taking sufficiently “nice” initial data, as well as a sufficiently correlated noise. A similar tightness argument as before will be used thanks to a priori estimates, which are justified by the fact that the approximations are sufficiently regular in space to apply the Itô formula. This is somewhat faithful to Struwe's original approach, with the difference that for stochastic PDEs, existence and uniqueness have more varied aspects. We will indeed see that this method yields strong solutions in the probabilistic sense.

Moreover, the justification of the Itô formula requires here a bootstrap argument. This method suffers the fact that, no matter which state space $X$ we choose for a solution $t \to u(t)$, we will always have $u \notin C^{1/2}(0, T; X)$, so that adapting the deterministic tools may be involved. We will circumvent this problem by using the ideas presented in [13]. However, an additional difficulty here is the polynomial nonlinearity $u|\nabla u|^2$ which, to the best of the author's knowwledge, has not been treated so far.
Criticality and uniqueness  It turns out that the Struwe solution is unique in the class of solutions depending continuously of the initial data \( \varphi \) in \( H^1 \), locally in time. That is: for some \( t_1 > 0 \) and for every \( t < t_1 \), then \( u_n(0) \to \varphi \) in \( H^1 \) implies \( u_n \to u \) in \( C([0, t]; H^1) \cap L^2([0, t]; H^2) \). At the singular points (more precisely when \( t \not\to t_i \) and \( x \to x_i^k \)), we observe peaks in the energy density \( x \mapsto |\nabla u(t, x)|^2 \) that are called in the literature "forward bubblings". An amount of energy is then released: although \( u \) stays in \( H^1 \) for all times, the inequality
\[
E(t_1) \leq \liminf_{t \to t_1} E(u(t))
\]
is strict. The key ingredient for the proof of that result is a sharp interpolation inequality that permits to control the nonlinear term through the energy bound – see Proposition 2 below. Such an estimate is of course, specific to the dimension two.

It has been widely observed (see e.g. [30] for an overview of the subject) that the harmonic map problem
\[
-\Delta_M u = u|\nabla u|^2, \quad \text{where } u = u(x),
\]
has specific features in dimension two, as for instance a theorem due to F. Helein [29] states that any 2D weakly harmonic map (that is in the sense of distributions) is actually harmonic in the classical sense. Concerning this time (HMF), the associated natural energy \( E \) fulfills the a priori bound:
\[
E_t - E_0 + \iint_{[0, t] \times M} |\Delta u + u|\nabla u|^2|^2 = 0,
\]
which barely fails to give well-posedness of the flow. Indeed, (1.3) yields that the nonlinearity \( u|\nabla u|^2 \) belongs at each time to \( L^1 \), which in dimension two “hardly differs from \( H^{-1} \)” in the sense that \( L^p \to H^{-1} \) is always true unless \( p = 1 \). This small difference turns out to be important: if for some reason we could obtain that \( u \in C(H^{-1}) \) locally in time, then standard results on heat equations would yield well-posedness. In the time-independent case, this criticality is outpassed in the proof of the latter Helein’s theorem, by slightly increasing regularity from the symmetries of the associated variational problem. More precisely, noticing that the nonlinearity has the particular form
\[
u^i|\nabla u|^2 = \langle A, \nabla u \rangle^i = \sum_{k, j} A^{i,j}_k \partial_k u^i, \; i \leq 3, \quad \text{with } \text{div } A = 0 \quad (\text{this latter conservation law stems from the fact that } u \text{ is harmonic}),
\]
then classical results on the decomposition of 2D vector fields, imply the existence of \( \beta \in (\mathbb{R}^3)^{\otimes 2} \) such that
\[
u^i|\nabla u|^2 = \frac{\partial \beta^i}{\partial x_1} \cdot \frac{\partial u}{\partial x_2} - \frac{\partial \beta^i}{\partial x_2} \cdot \frac{\partial u}{\partial x_1} := \{\beta, u\}^i, \quad i = 1, 2, 3.
\]
Now, Wente’s inequality [47], which can be seen as a two-dimensional analogue of the more celebrated 3D “div – curl Lemma” (see [39, 44]), states that the quantity \( \{\beta, u\} \) has the rather unexpected property of being continuous in \( H^{-1} \) with respect to the weak topology of \( H^1 \), although being nonlinear.

In the non-stationary setting we can still write the latter decomposition, with the difference that \( \text{div } A(t) \neq 0 \), and therefore we do not have (1.4). Nevertheless, it is still possible to treat apart some additional non-divergence-free term. This approach turns out to be essential in the proof of the following uniqueness result:
Theorem: uniqueness criterion for (HMF) in 2D ([20]). Any weak solution $u$ of (HMF) such that $E_t \equiv \frac{1}{2} \int_{|t|\times M} |\nabla u|^2$ is non-increasing with $t \geq 0$ is the Struwe solution.

Remark 1. The notation $\int_{|t|\times M}$ means that we integrate the trace of $|\nabla u|^2$ on $|t|\times M$, which is defined for every $t \in [0, T]$. Note that P. Topping has given an example where $E_t \leq E_s$ for a.e. $t \geq s$ and yet $u$ is not the Struwe solution (see [45]). In the following we will systematically assume that $E_t$ corresponds to the latter integral.

The proof of this theorem exploits the fact that although $\text{div} \ A(t) \neq 0$, we can write $A(t) = \nabla \alpha + \nabla^\perp \beta$, where the second term is a divergence-free tensor, so that by Wente’s theorem $I \equiv \nabla^\perp \beta : \nabla u$ can be written as the sum of a small $C(H^{-1})$ term, plus a regular remainder. The additional time-regularity is obtained as a consequence of the monotonicity of $E$. On the other hand, the first term $I \equiv \nabla \alpha : \nabla u$ is controlled as

$$\iint_{[0,T] \times M} |\Delta \alpha|^2 \lesssim \iint_{[0,T] \times M} |\Delta u + u|\nabla u|^2|^2,$$

where due to the a priori estimate (1.3), the r.h.s. in the latter bound is finite for every “reasonable” definition of a weak solution to (HMF). Denoting by $u_j$, $j = 1,2$ two solutions of the problem, uniqueness is provided by linearizing the equation for $u_1 - u_2$, around the solution that corresponds to that constructed in [43]. This proof has however the disadvantage of appealing to M. Struwe’s existence part. Here our uniqueness theorem, namely Thm. 2 uses a different approach, though related through technical aspects. This is new even in the deterministic setting, where the new proof can be computed simply by letting $W \rightarrow 0$.

The Harmonic Map Flow perturbed by Gaussian noise As already pointed out, the minimization problem associated to $E$ relates the theory of micromagnetism where admissible configurations of the magnetization of a ferromagnetic domain $M$, $1 \leq \dim M \leq 3$, are the minimizers of the Dirichlet energy. Out of equilibrium, the dynamics of the magnetization $u : [0, T] \times M \rightarrow \mathbb{S}^2$ is governed by Landau-Lifshitz-Gilbert equation [36, 21, 22]

$$\frac{\partial u}{\partial t} = \gamma u \times H_{\text{eff}} - u \times (u \times H_{\text{eff}}), \quad (1.5)$$

where $H_{\text{eff}} := -\nabla E(u)$, and $\gamma \in \mathbb{R}$ is the gyromagnetic ratio. The geometrical constraint on the magnetization, namely “$u(x) \in \mathbb{S}^2$, $\forall x$”, stems from the fact that below the so-called Curie point, the value of $|u(x)|$ depends on the temperature only. If we stay below this level, but at a sufficiently high temperature so that thermal effects are no longer negligible, then fluctuations of the effective field are such that

$$H_{\text{eff}} = \Delta u + \xi \quad (1.6)$$

where $\xi = \xi(t, x)$ denotes Gaussian white noise, see [6, 4]. In the framework of stochastic equations in infinite dimensions, this term is classically constructed as the formal sum

$$\xi(t, x) := \frac{dW}{dt}(t)(x) \equiv \sum_{\ell \in \mathbb{N}} \frac{dB_t}{dt}(t) \phi_\ell(x), \quad (1.7)$$
where: \((e_\ell)\) denotes an orthonormal system of \(L^2(\mathbb{T}^2;\mathbb{R}^3)\), \((B_\ell)\) denotes an i.i.d. family of real-valued Brownian motions, and \(\phi : L^2 \to L^2\) measures the spatial correlation of \(W\) through:

\[
\mathbb{E} \langle W(t), f \rangle \langle W(s), g \rangle = \langle \phi f, \phi g \rangle_{L^2}, \quad \forall f, g \in L^2(\mathbb{T}^2;\mathbb{R}^3).
\]

Let us mention that solvability of (1.9) in the case where \(\xi\) is white in time and space (that is when \(\phi = \text{id}\)) is not a problem that we address here. In dimension two, the cylindrical Wiener process is not better than \(\cap \epsilon > 0 H^{-1-\epsilon}\) in space, which matches the regularity of the nonlinearity. Hence (1.5) is critical in the sense given in [27], so that the theory of regularity structures does not apply. We will assume throughout the paper that \(\phi\) is Hilbert-Schmidt from \(L^2\) to \(H^1\), which is needed to make sense of the energy.

Now, because of the norm constraint we have the vectorial identity: \(-u \times (u \times \Delta u) = \Delta u + u |\nabla u|^2\), so that setting for simplicity

\[
\gamma = 0,
\]

while forgetting the contribution \(-u \times (u \times \circ dW)\) to keep \(u \times dW\) only, we end up with the Stratonovitch equation:

\[
\begin{align*}
\frac{d}{dt} u &= (\Delta u + u |\nabla u|^2) dt + u \times \circ dW, \quad &\text{on } \Omega \times [0, T] \times \mathbb{T}^2 \\
u|_{t=0} = u_0, \quad &\text{on } \Omega \times \mathbb{T}^2,
\end{align*}
\]

where \(\sum |\phi e_\ell|^2_{H^1} < \infty\), and \(u_0 \in H^1\).

The parallel between (HMF) and the (deterministic) (1.5) has provided interesting insights, see e.g. [2, 25, 28, 24]. Let us mention that the results above could be stated in presence of a gyromagnetic term in (1.5), provided however, that local smooth solutions exist for regular data \((u(0), \phi)\) (see [31]). Unfortunately, the method presented in sec. 2.4 below to obtain local solvability ceases to work for the case \(\gamma \neq 0\). We hope to successfully come back to this question in a forthcoming work. To simplify the presentation, we will restrict our attention to the case where \(M\) equals \(\mathbb{T}^2\), the two-dimensional torus. Nevertheless our results could be adapted to the case of a general surface with boundary, endowed with a Riemannian metric (see e.g. [34] for a treatment of the deterministic case).

Note that the a priori estimate on the energy writes this time:

\[
E_t - E_0 + \int_{\Omega \times [0, t] \times M} |\Delta u + u |\nabla u|^2|^2 = C_\phi t + \mathcal{M}(t),
\]

where \(\mathcal{M}(t)\) is a martingale, and \(C_\phi \equiv |\nabla \phi|^2_{L^2}\) (see the notations below). Although in this context the energy cannot decrease pathwisely, we see that regular solutions of (1.9) must be such that the quantity

\[
\mathcal{G}(t) := E_t - C_\phi t
\]

defines a supermartingale with respect to \((\mathcal{F}_t)\). This property turns out to be the “correct” stochastic counterpart as that of A. Freire’s criteria that the energy decreases. This is somehow reminiscent to the notion of “energy solution” given for the 3D Navier-Stokes equation in [19].
1.2 Notation and settings

For $d \in \mathbb{N}$, $\alpha, p \in \mathbb{R}$, the usual Lebesgue and Sobolev-Slobodeckij spaces $L^2(\mathbb{T}^2; \mathbb{R}^d), L^p(\mathbb{T}^2; \mathbb{R}^d)$, $W^{\alpha,p}(\mathbb{T}^2; \mathbb{R}^d), H^\alpha(\mathbb{T}^2; \mathbb{R}^d) \equiv W^{0,2}(\mathbb{T}^2; \mathbb{R}^d)$, etc. are occasionally abbreviated as $L^2$, $L^p$, $W^{\alpha,p}$, $H^\alpha$. We write the corresponding norms as $|\cdot|_{L^2}, |\cdot|_{L^p}, |\cdot|_{W^{\alpha,p}}, |\cdot|_{H^\alpha}$. The $L^2$ inner product will be denoted by $\langle \cdot, \cdot \rangle$, namely

$$\langle f, g \rangle := \int_{\mathbb{T}^2} f(x)g(x) \, dx, \quad f, g \in L^2.$$ \hfill (1.11)

When referring to space-time elements, we write $f \in L^q([0, T]; X)$ to say that $\int_0^T |f(t, x)|_X^q < \infty$, $X$ being any of the Banach spaces above. The associated norms are sometimes abbreviated as $\| \cdot \|_{L^q([0, T]; X)}$, or simply $\| \cdot \|_{L^q X}$, when it is clear from the context. When $\alpha p \geq 1$, we will denote by $W^{\alpha,p}_0([0, T]; X)$ the space corresponding to those $f \in W^{\alpha,p}([0, T]; X)$, such that $f(0) = 0$ in the sense of traces, and similarly for $H^\alpha_0([0, T]; X)$.

The notation

$$f \in L^{2}_{\text{loc}}([\tau_1, \tau_2]; H^2)$$ \hfill (1.12)

means that $\|f\|_{L^2(K; H^2)} < \infty$ for every compact interval $K \subset [\tau_1, \tau_2]$.

If $X, Y$ are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators. We denote by $\gamma(X)$ the space of $\gamma$-radonifying operators from the Hilbert space $L^2$ onto $X$, that is

$$T \in \gamma(X) \Leftrightarrow |T|^2_{\gamma(X)} \equiv \int_{\Omega} \left| \sum_{\ell \in \mathbb{N}} \gamma_{\ell}(\tilde{\omega}) Te_{\ell} \right|^2_{Y} \mathbb{P}(d\tilde{\omega}) < \infty,$$

for all orthonormal system $(e_{\ell}) \subset L^2$, and for $(\gamma_{\ell})$ an i.i.d. family of $\mathcal{N}(0, 1)$ random variables defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. When $X$ is a Hilbert space, $\gamma(X)$ corresponds to the class of Hilbert-Schmidt operators from $L^2$ onto $X$ and it will be denoted by

$$\mathbb{L}_2(X), \quad \text{or simply} \quad \mathbb{L}_2 \quad \text{if} \quad X = L^2.$$ 

For $s \in \mathbb{R}$ we also use the abbreviation

$$\mathbb{L}^2_s := \mathbb{L}_2(H^s).$$

In the whole paper, we fix a stochastic basis $\mathcal{F}_t = (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, W)$, that is a filtered probability space, together with an $L^2(\mathbb{T}^2; \mathbb{R}^3)$-valued Wiener process $W$ with respect to the filtration $\mathcal{F}_t$. We assume that $W$ has spatial covariance $\phi \phi^*$, where to simplify the presentation, we make the assumption that the correlation is “isotropic”, namely there exists an Hilbert-Schmidt operator $\phi : L^2(\mathbb{T}^2; \mathbb{R}) \rightarrow L^2(\mathbb{T}^2; \mathbb{R})$ such that $\phi : L^2(\mathbb{T}^2; \mathbb{R}^3) \rightarrow H^1(\mathbb{T}^2; \mathbb{R}^3)$, is the operator given by

$$\phi f := (\phi f_1, \phi f_2, \phi f_3).$$ \hfill (1.13)

We assume in addition that $W$ is given by the sum

$$W(t) := \sum_{\ell \in \mathbb{N}} B_{\ell}(t) \phi e_{\ell},$$ \hfill (1.14)

where $e_{\ell}$ and $B_{\ell}, \ell \in \mathbb{N}$ are as in (1.7).
Assuming that \( u \) is solution to (1.9) and that \( \Psi \in C^1(L^2;\mathbb{L}_2) \), the Stratonovitch product is given by the rule: \( \Psi(u) \circ dW = \Psi(u) dW + 1/2 \sum_{\ell \in \mathbb{N}} \Psi'(u)[u \times \phi_{\ell}](\phi_{\ell} \, dt) \), provided the right hand side is convergent. For \( \phi \) as in (1.13), this gives a correction \( F_{\phi} \, u \, dt \) where

\[
F_{\phi}(x) := \sum_{\ell \in \mathbb{N}} - (\phi_{\ell}(x))^2, \quad x \in \mathbb{T}^2.
\]  

### 1.3 Notion of solution and main Results

We will make use of two different notions of solution for (1.9). The “strong solutions” are both strong in PDE and Probabilistic sense, and yield the correct framework to locally characterise the so-called Struwe solution constructed in the theorem below.

**Definition 1** (local strong solutions). Assume that a stochastic basis \( \mathbb{F} \) is given, that \( 0 \leq \tau_1 < \tau_2 \leq T \) are stopping times, and that \( u_1 \in L^2(\Omega;H^1) \) is \( \mathcal{F}_{\tau_1} \)-measurable. Given a progressively measurable process \( u: \Omega \times [0,T) \to H^1 \), we say that \((u,\tau_1;\tau_2)\), is a local strong solution of (1.9) on \([\tau_1,\tau_2]\), with initial data \( u_1 \) if the following conditions are fulfilled:

(i) for \( \mathbb{P} \otimes dt \otimes dx \) a.e. \((\omega, t, x)\) with \( \tau_1(\omega) \leq t < \tau_2(\omega) \), there holds

\[
|u(\omega, t, x)|_{\mathbb{R}^2} = 1 ;
\]

(ii) the process \( u \) has paths in \( C([\tau_1,\tau_2];H^1) \cap L^2_{\text{loc}}([\tau_1,\tau_2];H^2) \mathbb{P} - \text{a.s.} \);

(iii) \( \mathbb{P} - \text{a.s.}, \) for \( t \in [\tau_1,\tau_2] \):

\[
 u(t) - u_1 = \int_{\tau_1}^t (\Delta u + u|\nabla u|^2 + F_{\phi} u) \, dt + \int_{\tau_1}^t u \times dW \text{ in the sense of Bochner, respectively Itô integral in } L^2 .
\]

Our main results are stated in the following two theorems. Here we let \( H^1(\mathbb{T}^2;\mathbb{R}^3) := H^1(\mathbb{T}^2;\mathbb{R}^3) \cap \{v: v(x) \in \mathbb{S}^2 \text{ a.e.} \} \).

**Theorem 1.** Let \( W \) be as in (1.14). For all \( T > 0 \), and \( u_0 \in H^1 \) taking values in \( \mathbb{S}^2 \), there exists \( u \in L^2(\Omega;L^\infty(0,T;H^1)) \) and a finite sequence of stopping times \( \theta^0 \equiv 0 < \theta^1 < \theta^2 < \cdots < \theta^J(\omega) \), with

\[
\mathbb{P} \left( \theta^J = T \right) = 1,
\]

such that for each \( 0 \leq j \leq J-1 \), \( (u|_{[\theta^j,\theta^{j+1})};\theta^j,\theta^{j+1}) \) is a local strong solution of (1.9) with initial data \( f^j \) at \( t = \theta^j \), where \( f^j \) is uniquely determined by

\[
u(\xi_k) \to f^j, \quad \mathbb{P} - \text{a.s. weakly in } H^1(\mathbb{T}^2;\mathbb{R}^3), \quad \text{for every sequence of stopping times } \xi_k \not\equiv \theta^j .
\]

Moreover, \( u \) has support in \( C([0,T];L^2) \), and provided \( \omega \in \{J > 1\} \), we have for \( 0 \leq j \leq J-2 \):

\[
\forall \varrho > 0, \quad \exists \xi_{\varrho} \in [\theta^j,\theta^{j+1}], \quad \sup_{x \in \mathbb{T}^2} \int_{B(x,\varrho)} |\nabla u(\xi_{\varrho},y)|^2 \, dy \geq \varepsilon_1 .
\]

up to a null-set.
Remark 2. Using (3.69) together with (2.2), the proof below shows that the solutions constructed in Theorem 1 are unique in their class. Namely, if we are given a martingale solution \( v \), fulfilling the property:

there exists a positive stopping time \( \tau \) such that

\[
\inf_{\mathcal{Y} > 0} \sup_{0 \leq t \leq \tau} \sup_{x \in \mathbb{T}^2} \int_{B(x, \varphi)} |\nabla v(t, y)|^2 \, dy = 0'',
\]

then we have \( v|_{[0, \tau]} = u|_{[0, \tau]} \).

Remark 3. Concerning the Dirichlet problem, the existence of finite-time blowing-up solutions has been provided in [32], but further degeneracy assumptions on the noise have to be made.

In the general case, it remains an open problem to build such singular solutions. Moreover, instability results on the deterministic equation suggest a possible “regularization by noise” phenomenon. It is indeed expected that the sequence \( \vartheta_j : 0 \leq j \leq J \) is always the set \( \{0, T\} \), see [38] and the closing remarks in [32].

The second notion of solution that we need to introduce corresponds to that of [10, chap. 8]. It is also motivated by the results obtained in [8, 3, 31].

Definition 2 (weak martingale solution). A weak martingale solution for (1.9) is a couple \( (\mathcal{F}', u') \), where \( \mathcal{F}' = (\Omega', \mathcal{F}', \mathbb{P}', (\mathcal{F}'_t)_{t \in [0, T]}, W') \), \( W' \) has covariance \( \phi \phi^* \), and \( u' : \Omega' \times [0, T] \rightarrow H^1 \), a \( \mathbb{F}'_t \)-progressively measurable process satisfies the following assumptions:

(i') \( |u'|_{\mathbb{R}^3} = 1 \), \( \mathbb{P}' \otimes dt \otimes dx \)-a.e.;

(ii') \( u' \in C([0, T]; L^2), \mathbb{P}' \)-a.s. and

\[
\mathbb{E}' \left[ \sup_{t \in [0, T]} |\nabla u'(t)|^2_{L^2} + \int_0^T |\Delta u' + u'|\nabla u'|^2_{L^2} \, dt \right] < \infty;
\]

(iii') \( u' \) satisfies \( u'(t) = u_0 + \int_0^t (\Delta u' + u'|\nabla u'|^2 + F_{\varphi} u') \, ds + \int_0^t u'(s) \times dW' \) for all \( t \in [0, T] \), \( \mathbb{P}' \)-a.s., where the first integral is the Bochner integral, and the second is the \( \text{Itô} \) integral, in the space \( L^2 \).

Remark 4. By “gluing together” the local solutions constructed in Theorem 1, straightforward computations shows that the resulting map is in fact a (martingale) solution over the whole interval \( [0, T] \). This solution will be referred to as “the Struwe solution”.

Theorem 2 (Uniqueness criterion). Let \( u \) denote a martingale solution of (1.9) in the sense given in definition 2, with an associated \( \phi \in \mathbb{L}^1_2 \).

Assume that \( u \) is such that \( \mathcal{G}' \), its energy corrected for the mean injection rate from \( W \), namely the process

\[
\mathcal{G}'(t) := \frac{1}{2} |\nabla u(t)|^2_{L^2} - |\nabla \varphi|^2_{L^2} t, \quad t \in [0, T]
\]

(here \( u(t) \) denotes the trace onto the slices \( \{t\} \times \mathbb{T}^2 \)), is a supermartingale with respect to \( (\mathcal{F}_t) \). Then \( u \) is the Struwe solution (see remark 4).
Outline of the paper. Section 2 is devoted to preliminary results that will be used throughout the proofs of the main Theorems. Of particular interest are the interpolation inequalities, namely Propositions 1 and 2. We will also recall well-known results on parabolic equations. They will be used for proving both results, especially Theorem 1 where a bootstrap argument is needed.

We will prove Theorem 1 in Section 3, which is divided into successive steps. We first collect uniform a priori estimates, namely Proposition 5 and Corollary 1, which will yield tightness of a sequence of stopped processes. Corollary 1 also provides a bootstrap for the solution, which will play a key role in the proof that the existence time is uniform with respect to a compact set of initial data. These arguments will ensure convergence towards a weak martingale solution. Noticing that the solution obtained has enough regularity to apply a basic Grönwall estimate, we will then make use of the celebrated Gyöngy-Krylov argument to yield convergence towards a local strong solution. By reiterating the process, we will be able to construct the “Struwe solution” on the whole interval $[0, T]$. Theorem 2 will be treated in section 4. Writing “Helein’s decomposition” of the nonlinearity $u|\nabla u|^2$, we show that the gradient part is controlled by the energy bound. This was already remarked in [20], but the new insight here is that we can define a process $\tilde{u}$, whose singular part has been removed. The equation on $v := u - \tilde{u}$ can then be linearized around the latter “renormalized map”, and the supermartingale property will be used at this stage to obtain more regularity on the singular part, yielding then $v \in L^4W^{1,4}$, from which the uniqueness follows. We point out that our method has the advantage of not referring to the Struwe solution, and therefore the proofs of Theorem 1 and 2 (up to slight modification of the conclusion) are independent.

2 Preliminaries

2.1 Interpolation inequalities

The following multiplicative inequality corresponds to a particular case of of [35, II Thm. 2.2].

Proposition 1. There exists a constant $C_0 > 0$, such that for every $f \in H^1$ with $\int_{T^2} f = 0$, there holds

$$\int_{T^2} |f|^4 \leq C_0 \left( \int_{T^2} |f|^2 \right) \left( \int_{T^2} |\nabla f|^2 \right).$$

(2.1)

As a byproduct, the following Lemma is obtained in [43, Lemma 3.1]. It will play a central role in the proof of Theorem 1.

Proposition 2. For any $T > 0$, there exists a constant $C_1 > 0$, such that for all $v \in C([0, T]; H^2)$, for all $\varrho > 0$:

$$\iint_{[0, T] \times T^2} |\nabla v|^4 \, dt \leq C_1 \sup_{t \in [0, T]} \int_{y \in T^2} |\nabla v(t)|^2$$

$$\times \left( \iint_{[0, T] \times T^2} |\nabla^2 v|^2 \, dt + \iint_{[0, T] \times T^2} \frac{|\nabla v|^2}{\varrho^2} \right).$$

(2.2)
2.2 Parabolic estimates for deterministic PDEs

We recall regularity results associated to the deterministic cauchy problem with unknown \( \varphi \):

\[
\begin{aligned}
\partial_t \varphi - \Delta \varphi &= f(t, x), \quad \text{in } [0, T] \times \mathbb{T}^2, \\
\varphi(0, \cdot) &= 0, \quad \text{for } x \text{ in } \mathbb{T}^2.
\end{aligned}
\] (2.3)

The following parabolic estimates are well known.

**Proposition 3.**

(i) Let \( p \in (1, \infty) \) and \( \alpha > 0 \). For \( f \in C^\alpha(0, T; L^p) \), Problem (2.3) has a unique solution in \( C^1(0, T; L^p) \cap C(0, T; W^{2,p}) \).

(ii) For \( f \in L^2(0, T; H^{-1}) \), Problem (2.3) has a unique solution in \( H^1_0(0, T; H^{-1}) \cap L^2(0, T; H^1) \).

(iii) Let \( 1 < p, q < \infty \). For \( f \in L^q(0, T; L^p) \), Problem (2.3) has a unique solution in \( W^{1,q}_0(0, T; L^p) \cap L^q(0, T; W^{2,p}) \).

Moreover, every solution above depends continuously on \( f \) within the corresponding Banach spaces.

**Proof.** The first statement can be found in [10]. The second and third statements can be found respectively in [37] and [23]. \( \blacksquare \)

2.3 Parabolic estimates: stochastic case

We recall existence, uniqueness and regularity for weak solutions of the parabolic equation with multiplicative noise:

\[
\begin{aligned}
\mathrm{d}Z - \Delta Z \mathrm{d}t &= \Psi(t) \mathrm{d}\hat{W}, \quad \text{in } \Omega \times (0, T) \times \mathbb{T}^2, \\
Z(\cdot, 0) &= 0, \quad \text{in } \Omega \times \mathbb{T}^2,
\end{aligned}
\] (2.4)

(Itô sense) where this time \( \hat{W}(t) = \sum_{k \in \mathbb{N}} B_k(t) e_k \) is a cylindrical Wiener process. Under suitable hypotheses on \( \Psi \) (see the Lemma below) a weak solution \( Z \) of (2.4) exists and is unique. It is given by the stochastic convolution, namely:

\[
Z(t) = \int_0^t S(t - s) \Psi(s) \mathrm{d}\hat{W}, \quad t \in [0, T].
\] (2.5)

**Proposition 4.** Let \( \alpha \geq 0, \ p \in [2, \infty), \ r \geq 1, \) and let \( \Psi \) be a progressively measurable process in \( L^r(\Omega; L^r(0, T; \gamma(L^2, W^{\alpha,r}))) \).

(i) For \( p > 2 \), for every \( \delta \in (0, 1 - 2/r) \) and \( \lambda \in [0, 1/2 - 1/r - \delta/2) \), then \( Z \in L^\lambda(\Omega; C^\lambda(0, T; W^{\alpha+\delta,p})) \).

Moreover

\[
\mathbb{E}\|Z\|_{C^\lambda(0, T; W^{\alpha+\delta,p})}^\lambda \leq C \mathbb{E}\|\Psi\|_{L^r(0, T; \gamma(W^{\alpha,p}))}^r.
\]

(ii) For any \( p \geq 2, \ \delta \in (0, 1) \), we have \( Z \in L^\lambda(\Omega; C^\lambda(0, T; \gamma(W^{\alpha,r}))) \), and

\[
\mathbb{E}\|Z\|_{L^\lambda(0, T; W^{\alpha+\delta,p})}^\lambda \leq C \mathbb{E}\|\Psi\|_{L^r(0, T; \gamma(W^{\alpha,p}))}^r.
\]

**Proof.** The first statement, the proof of which relies on the factorization method, is a particular case of [7, Corollary 3.5]. The second point can be found in [33]. \( \blacksquare \)
2.4 Local solvability

We first need to investigate local solvability of the Itô equation
\[ d\nu = (\Delta \nu + v|\nabla \nu|^2 + F_\phi \nu) \, dt + v \times dW, \quad \nu(0) = \nu_0, \]
for regular data \( \nu_0 \) and \( \phi \). To this aim we will switch to the mild formulation
\[ \nu(t) - S(t)\nu_0 = \int_0^t S(t-s)(v|\nabla \nu|^2 + F_\phi \nu) \, ds + \int_0^t S(t-s)[v \times dW], \quad t \in [0, T], \]
\( t \in [0, T) (\tau \) sufficiently small), where \( (S(t))_{t \in [0, T]} \) denotes the Heat semigroup \( e^{t\Delta} \).

For convenience we now define an extended state space \( \Omega^1, p := W^{1, p} \cup \{ \Delta \} \), where the terminal state \( \Delta \) is an isolated point.

**Theorem 3.** Let \( p \in (2, \infty), \alpha > 0, q > 2/\alpha \), and let \( \phi \in \gamma(W^{a, p}) \). Then, for every \( \nu_0 \in W^{1, p} \) with \( |\nu_0| = 1 \) a.e., there exist a unique \((\nu, \tau)\) where \( \nu \in L^q(\Omega; C(0, T; \Omega^1, p)) \) denotes a progressively measurable process, whereas \( \tau \) is a positive stopping time, such that:

(i) \((\nu \cdot \wedge \tau, \tau)\) is a local solution of (2.6);

(ii) on \((\tau < T)\), we have \( \limsup_{t \to -\tau} |\nu(t)|_{W^{1, p}} = \infty \).

In addition, if \( \nu_0 \in W^{3, p} \) and \( \phi \in \gamma(W^{2+\alpha, p}) \) where \( 2 \leq \beta < 4 \), then we have \( \nu(\cdot \wedge \tau_3) \in C(0, T; W^{\beta, p}) \) for some \( 0 < \tau_3 \leq \tau \).

**Remark 5.** The solution above fulfills the norm constraint (1.16), provided \( t \in [0, \tau) \) is such that \( ||\nabla u(t)||_{L^\infty} < \infty \). This can be shown by an application of Itô Formula to the functional \( F(u) := |1 - |u|^2|_{L^2}^2 \), together with Grönwall (see [31] for details). Hence, the embedding \( W^{2, p} \hookrightarrow W^{1, \infty} \) for \( p > 2 \) shows that any \( W^{2, p} \) mild solution takes values in the sphere.

**Proof.** The proof is based on a classical fixed point argument whose details can be found in [31]. Let us give a short picture. Since the noise term cannot be estimated pathwise, we cannot operate a fixed point in \( L^2(\Omega; B_R) \), where \( B_R \) is some ball in \( C([0, T]; W^{1, p}) \). This leads us to truncate the nonlinearities by the use of a cut-off function \( \theta : \mathbb{R}^+ \to [0, 1] \), which has the following properties:
\[ \theta \in C^\infty(0, 2), \quad \theta(x) = 1, \text{ for all } 0 \leq x \leq 1. \quad (2.7) \]

For \( R > 0 \), and \( x \in \mathbb{R}^+ \), we denote by \( \theta_R(x) = \theta(x/R) \). We first solve the fixed point problem \( u(t) = \psi^R(u) \), where for a fixed \( R > 0 \), we define the map \( \psi^R \) on \( L^2(\Omega; C([0, T]; W^{1, p})) \) by the formula:
\[ \psi^R(v) = S(t)u_0 + \int_0^t S(t-s)[\theta_R(||v(t)||_{W^{1, p}}) v(s)|\nabla v(s)|^2] \, ds \]
\[ + \int_0^t S(t-s)[F_\phi v(s)] \, ds + \int_0^t S(t-s)[v(s) \times dW(s)], \quad (2.8) \]
for \( t \in [0, T] \). We show that provided \( T \) is sufficiently small, depending on \( R \) and \( u_0 \), then:
(i) for $q > 2/\alpha$, $\psi^R$ maps the space $L^q(\Omega; C([0, T]; W^{1,p}))$ onto itself;

(ii) $\psi^R|_{L^q(\Omega; C([0, T]; W^{1,p}))}$ is a contraction.

Then, Picard Theorem yields existence and uniqueness of a local solution $(u_R, \tau_R)$.

The proof of (i) and (ii) is standard, and we will omit the details here. For example, let us treat (i) on the stochastic convolution $Z := \int_0^\tau S(\cdot - s) \nu \times dW$, forgetting the other contributions. Letting $\epsilon > 0$ such that

$$\alpha > \epsilon > (p - 2)/2,$$

and using Proposition 4 we obtain

$$E\|Z\|^q_{C[0, T; W^{1,p}]} \leq CT^q E\|u \times \phi\|^q_{C[0, T; W^{r,p}]} \leq CT^q \|\phi\|^q_{Y(W^{2,r})} E\|u\|^q_{C[0, T; W^{r,p}}$$

where

$$r = \frac{2p}{2 - (1 - \epsilon)p} \quad \text{and} \quad r' = \frac{2}{1 - \epsilon}.$$

Using the two-dimensional Sobolev embedding Theorem we have therefore:

$$E\|Z\|^q_{C[0, T; W^{1,p}]} \leq CT^q \|\phi\|^q_{Y(W^{a,p})} E\|u\|^q_{C[0, T; W^{1,p}]}.$$

The treatment of the other terms is similar, as well as the the second statement (ii).

Using a localization procedure (see also [12, Theorem 4.1]) we can build a maximal solution as follows: we define the stopping times

$$\tau_m = \inf\{t \in [0, T], \ |u_m|_{W^{1,p}} \geq m\}, \quad (2.9)$$

and show that the sequence $(\tau_m)$ is non-decreasing and that $u_{m+1}(t) = u_m(t)$ for $t \in [0, \tau_m]$ a.s. The local solution $(u; \tau \equiv \sup_{m \geq 0} \tau_m)$ is then defined in an obvious way.

Higher regularity is obtained by writing the mild equation on $v := \Delta u$, and again using regularizing properties of $S(\cdot)$.

**Remark 6.** The reason why we chose the Banach space $W^{1,p}$ with $p > 2$ is motivated by the treatment of the nonlinearity $|v|\nabla v|^2$ in (i) and (ii). We have the sharp inequality

$$\left| \int_0^t S(\cdot - s) |v\nabla v|^2 \right|_{W^{1,p}} \leq C \int_0^t \frac{|v|\nabla v|^2_{L^{p/2}}}{(t-s)^{1/2+1/p}}$$

$$\leq CT^{1/2-1/p} \sup_{s \leq t} |v|_{L^\infty} |\nabla v|^2_{L^p} \leq CT^{1/2-1/p} \sup_{s \leq t} |v|^3_{W^{1,p}} \quad (2.10)$$

where the bound $|S(r)|_{L^{p/2}} W^{1,p}) \leq r^{-1/2-1/p}, r > 0$, is well-known, see e.g. [41, p. 25].

### 3 Proof of Theorem 1

#### 3.1 Step 1: a priori estimates

If $u$ denotes a solution of (1.9) up to a stopping time $\zeta \leq T$, an important role is played by the “tension”, defined for every $t \in [0, \zeta), \mathbb{P}$-a.s. by:

$$\mathcal{F}_u(t) := \Delta u(t) + u(t)|\nabla u(t)|^2. \quad (3.1)$$
Take \( \rho > 0 \) and \( \varepsilon_1 > 0 \). According to our definitions 1 and 2, this quantity defines an element of \( L^2(\mathbb{T}^2) \), \( dt \otimes \mathbb{P} \)– almost everywhere. Moreover, thanks to the identity \( \Delta |u|^2/2 = \Delta u \cdot u + |\nabla u|^2 \), it fulfills the geometrical property that

\[
\mathcal{F}_u = \Delta u -(u \cdot \Delta u) u,
\]

(3.2)

namely it is pointwisely equal to the orthogonal projection of the laplacian onto the plane \( \langle u \rangle \), up to a \( \mathbb{P} \otimes dt \otimes dx \)– null set.

For any solution that is continuous in time with values in \( H^1 \), it makes sense to define the stopping time \( \zeta(u, \varrho; \varepsilon_1) \leq T \) as follows:

\[
\zeta(u, \varrho; \varepsilon_1) := \inf \{ 0 \leq t < T, \sup_{x \in \mathbb{T}^2} \int_{B(x, \varrho)} |\nabla u(t,y)|^2 \, dy \geq \varepsilon_1 \}.
\]

(3.3)

We will also denote by \( E_t := \frac{1}{2} |\nabla u(t)|^2 \).

**Proposition 5 (a priori estimates).** Fix \( \varrho \in \mathbb{L}^1_{\mathbb{T}^2} \), and assume that \( (u, T) \) denotes a local strong solution of \((1.9)\), where for convenience we suppose that \( T > 0 \) is deterministic. Then

\[
E_t - E_0 + \int_0^t |\mathcal{F}_u|_{L^2}^2 \, ds = t|\nabla \phi|_{L^2}^2 + \int_0^t \langle \nabla u, u \times \nabla \mathcal{W} \rangle, \quad \text{a.s. for } t \leq T.
\]

(3.4)

Moreover, letting \( r \geq 1 \), we have:

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} (E_t)^r + \left( \int_0^T |\mathcal{F}_u|_{L^2}^2 \, dt \right)^r \right] \leq C(r, E_0, |\phi|_{L^2}),
\]

(3.5)

and there exists \( \varepsilon_1^* > 0 \) such that for \( \varepsilon_1 \in (0, \varepsilon_1^*) \), \( \varrho > 0 \):

\[
\mathbb{E} \left[ \left( \int_0^{\zeta(u, \varrho; \varepsilon_1)} |\Delta u|_{L^2}^2 \, dt \right)^r \right] \leq C(\varrho, E_0, |\phi|_{L^2}),
\]

(3.6)

The constants above depend on the indicated quantities, but not on \( u \).

**Remark 7.** The “optimal value” of \( \varepsilon_1^* \) corresponds to the inverse of \( C_1 \), namely the constant in the interpolation inequality given in Proposition 2 (in particular, it is independent of \( T \) and \( \varrho \)).

**Proof of Proposition 5.**

**Proof of (3.4).** The solution \( u \) has enough regularity to apply the Itô Formula given in [10] (letting \( H := H^1 \) and \( F(u) := |\nabla u|^2/2 \), so that

\[
E_t - E_0 = \int_0^t \langle -\Delta u, \Delta u + u|\nabla u|^2 \rangle \, ds - t|\nabla \phi|_{L^2}^2 + \int_0^t \langle \nabla u, u \times \nabla \mathcal{W} \rangle.
\]

(3.7)

Moreover by our assumption that \( |\phi|_{L^1} < \infty \), the Stratonovitch integral makes sense as \( \int_0^t \langle \nabla u, u \times \nabla \mathcal{W} \rangle = 1/2 \int_0^t \sum_{\ell \in \mathbb{N}} T_\ell(s) \, ds + \int_0^t \langle \nabla u, u \times \nabla \mathcal{W} \rangle \). Use now an adapted basis \( (\varepsilon \ell)_{\ell \in \{1,2,3\} \times \mathbb{N}} \), built over an orthonormal system \( (f_\ell)_{\ell \in \mathbb{N}} \) of \( L^2(\mathbb{T}^2; \mathbb{R}) \) in the following
way: for \( \ell \in \mathbb{N} \), we set \( e_{1,\ell} = (f_{\ell}, 0, 0) \), \( e_{2,\ell} = (0, f_{\ell}, 0) \), and \( e_{3,\ell} := (0, 0, f_{\ell}) \). Denoting by \( \Phi_{\ell} := \Phi e_{\ell} \), we have for each \( \ell \equiv (k, \ell) \in \{1, 2, 3\} \times \mathbb{N} \):

\[
T_{\ell} = \langle \nabla u \times \Phi_{\ell}, u \times \nabla \Phi_{\ell} \rangle + |u \times \nabla \Phi_{\ell}|_{L^2}^2 + \langle \nabla u, (u \times \Phi_{\ell}) \times \nabla \Phi_{\ell} \rangle =: A_{\ell} + B_{\ell} + C_{\ell}.
\]

By (1.13) and \( |u| = 1 \), we have on the one hand:

\[
\sum_{1 \leq k \leq 3} B_{k,\ell} = 2|\nabla \Phi_{\ell}|_{L^2}^2.
\]

On the other hand, using coordinates, there holds

\[
\sum_{1 \leq k \leq 3} A_{k,\ell} = -\int_{\mathbb{T}^2} \sum_{j} \frac{\partial}{\partial x_j} u \cdot (u \times \partial_j \Phi_{k,\ell}) \times \Phi_{k,\ell} = \sum_{j} \int_{\mathbb{T}^2} (\partial_j u \cdot u) \Phi_{k,\ell} \partial_j (\Phi_{k,\ell}) = 0, \quad (3.8)
\]

where we have used \( \partial_j u \cdot u = 0 \), for \( 1 \leq j \leq 2 \). Similarly, we have \( \sum_{k \leq 3} C_{k,\ell} = 0 \), whence the Itô correction is given by:

\[
\frac{1}{2} \int_{0}^{t} T_{\ell}(s) \, ds = t|\nabla \Phi_{\ell}|_{L^2}^2.
\]

By (3.2), we have also \( \langle -\Delta u, \bar{\Delta} u + u |\nabla u|^2 \rangle_{\mathbb{R}^3} = -|\bar{\mathcal{F}}_{u}|_{\mathbb{R}^3}^2 \rangle \), hence (3.4) follows.

Proof of (3.5). Denoting the martingale term in (3.4) by \( X(t) := \int_{0}^{t} \langle \nabla u, u \times dW \rangle \), Burkholder-Davies-Gundy inequality writes for any \( r \in [1, \infty) \):

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} X(t)^r \right] \leq C(r) \mathbb{E} \left( \int_{0}^{T} |\Phi^* \text{div}(u \times \nabla u)|_{L^2}^2 \, ds \right)^{r/2}. \quad (3.9)
\]

Since \( L_2(L^2, H^1) \subset \mathcal{L}(L^2, H^1) \), we observe that

\[
\mathbb{E} \left( \int_{0}^{T} |\Phi^* \text{div}(u \times \nabla u)|_{L^2}^2 \, ds \right) \leq C(|\Phi|_{L^2}) \mathbb{E} \int_{0}^{T} |u \times \nabla u|_{L^2}^2 \, ds, \quad (3.10)
\]

and therefore the r.h.s. in (3.9) is bounded by \( C(r, |\Phi|_{L^2}) \mathbb{E} \left( \int_{0}^{T} E_s \, ds \right)^{r/2} \). Going back to (3.4), taking the power \( r \geq 1 \), the supremum, and the expectation, there comes

\[
\mathbb{E} \left[ \sup_{t \leq s \leq t} (E_s)^r \right] \leq C(E_0, r, |\Phi|_{L^2}) \left( 1 + \int_{0}^{T} \mathbb{E} \left[ \sup_{t \leq s \leq t} (E_s)^r \right] \, ds \right), \quad (3.11)
\]

a.s. for \( t \in [0, T] \). Hence, the claimed bound follows obtained by Grönwall Lemma. Reusing (3.4), (3.9), and injecting the latter bound gives the bound on \( \mathbb{E}(\| \bar{\mathcal{F}}_{u} \|_{L^2 L^2})^r \).

Proof of (3.6). For \( t \in [0, T] \) we have a.s.

\[
\int_{0}^{t} E_s \, ds = \int_{0}^{t} \langle \Delta u, \Delta u + u |\nabla u|^2 \rangle \, ds.
\]

Expanding this term in (3.4), and still denoting by \( X(t) := \int_{0}^{t} \langle \nabla u, u \times dW \rangle \), then for \( t \in [0, T] \):

\[
E_t - E_0 + \int_{0}^{t} |\Delta u|^2 \, ds - X(t) - t|\nabla \Phi|^2 \leq \int_{0}^{1} \langle -\Delta u, u |\nabla u|^2 \rangle \, ds \leq \frac{1}{2} \int_{0}^{t} |\Delta u|^2 \, ds \geq \frac{1}{2} \int_{0}^{t} |\nabla u|^4 \, ds,
\]

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(since $|u| = 1$ a.e.). Using now Proposition 2 it follows that
\[
\int_0^{\zeta(u, \varphi; \epsilon_1)} |\nabla u|_{L^2}^4 \, ds \leq C_1 \epsilon_1 \int_0^{\zeta(u, \varphi; \epsilon_1)} \left( \frac{C}{\epsilon^2} |\nabla u|_{L^2}^2 + |\Delta u|_{L^2}^2 \right) \, ds,
\]
and this finally yields the estimate:
\[
\frac{1}{2}(1 - C_1 \epsilon_1) \int_0^{\zeta(u, \varphi; \epsilon_1)} |\Delta u|_{L^2}^2 \, ds \leq \frac{C}{\epsilon^2} \int_0^T |\nabla u|_{L^2}^2 \, ds + E_0 + \sup_{t \leq T} X(t) + T|\nabla \phi|_{L^2}^2. \tag{3.12}
\]
Provided $\epsilon_1$ is chosen to be sufficiently small, namely $< C_1^{-1}$, then (3.6) follows. \hfill \blacksquare

Remark 8. Fixing $\epsilon_1 < \epsilon_1^*$, the above result can be improved to yield the exponential bounds:
\[
\mathbb{E} \exp \left( m \sup_{r \leq \zeta} |\nabla u(r)|_{L^2}^2 \right) \leq C(m, \varphi) \tag{3.13}
\]
\[
\mathbb{E} \exp \left( m \int_0^{\zeta} |\Delta u(r)|_{L^2}^2 \, dr \right) \leq C(m, \varphi, T, E_0, |\phi|_{L^2}) \tag{3.14}
\]
\[
\mathbb{E} \exp \left( m \int_0^{\zeta} |\nabla u|_{L^2}^4 \, dr \right) \leq C(m, \varphi, T, E_0, |\phi|_{L^2}). \tag{3.15}
\]
for all $m \geq 1$.

Proof. The first bound is obtained as a consequence of the definition of $\zeta = \zeta(\epsilon_1, \varphi)$, namely: write that for each $t \in [0, T]$, a.s.
\[
\sup_{t \leq \zeta} E_t \leq \sum_{1 \leq k \leq N_0} \int_{B(x_k, \varphi)} |\nabla u(t, y)|^2 \, dy \leq N_0 \epsilon_1 \frac{1}{2}, \tag{3.16}
\]
where $\{x_1, \ldots, x_{N_0}\} \subset \mathbb{T}^2$ denotes a finite set such that $\cup_{1 \leq k \leq N_0} B(x_k, \varphi) = \mathbb{T}^2$. Hence, $\exp(\sup_{t \leq \zeta} m E_t)$ is integrable, and his norm is bounded by a constant depending on $m, \varphi, \epsilon_1$. This proves (3.13).

For the second bound, starting from (3.4), and writing for simplicity $X(t) := \int_0^{\zeta(u, \varphi; \epsilon_1)} (\nabla u, u \times dW)$, we have for $t \in [0, \zeta]$:
\[
m \int_0^t |\Delta u|_{L^2}^2 \, dt \leq C(1 + X(t)),
\]
where the constant above depends on the quantities $m, \varphi, E_0, T, |\phi|_{L^2}$, so that by the inequality $e^{a+b} \leq e^{2a} + e^{2b}$, it holds that:
\[
\mathbb{E} \exp \left( m \int_0^{\zeta} |\Delta u|_{L^2}^2 \, dt \right) \leq \exp(2C) + \mathbb{E} \sup_{t \leq T} \exp(2C X(t)), \tag{3.17}
\]
provided one can prove that the r.h.s. above is finite. However, applying Itô Formula to $Y(t) := \exp(m X(t))$ yields for every $t \leq \zeta$:
\[
Y(t) - Y(0) = m \int_0^t Y(r) \, dW + m^2 \int_0^t Y(r) |\phi|^2 \text{div}(u \times \nabla u)|_{L^2}^2 \, dr
\leq \int_0^t m Y(r) \, dW + C(|\phi|_{L^2}, \varphi, \epsilon_1, m) \int_0^t Y(r) \, dr, \tag{3.18}
\]
by (3.16). Taking the expectation in (3.18) and applying Grönwall, we end up with the bound

\[ \sup_{t \leq T} \mathbb{E} \exp(mX(t)) \leq C(m, \rho, T, E_0, |\phi|_{L^1}) . \]

Now, from Doob’s Inequality for submartingales, we also have

\[ \mathbb{E} \sup_{t \leq T} \exp(CX(t)) \leq C' \sup_{t \leq T} \mathbb{E} \exp X(t) , \]

for another such constant. This, together with (3.17), yields our second estimate (3.14).

The bound (3.15) follows by combining (3.14) with Proposition 1 and the estimate (3.16).

\[ \square \]

**Corollary 1 (bootstrap).** Consider \( \phi \in L^2_T \) and \( \rho > 0 \), \( \zeta \equiv \zeta(\phi) \) as in Proposition 5. Let \( u \) be a solution of (1.9) supported in \( C([0, T]; H^2) \cap L^2([0, T]; H^3) \). Then for \( m \geq 1 \):

\[ \mathbb{E} \left( \sup_{0 \leq t \leq \zeta} |\Delta u(t)|^{2m}_{L^2} \right) \leq C(m, \rho, T, |u_0|_{H^2}, |\phi|_{L^2}). \tag{3.19} \]

Moreover the following bootstrap principle holds: suppose

\[ \phi \in \bigcap_{k \in \mathbb{N}} L^2(H^k) . \]

and assume in addition that the above solution verifies \( u(0) \in C^\infty(\mathbb{T}^2) \). Then for each \( m \in \mathbb{N} \), we have

\[ \zeta < \tau_{H^k}(u) \tag{3.20} \]

where \( \tau_{H^k}(u) \) denotes the maximal existence time in \( C(H^m) \).

To prove this corollary, we need a refined version of Grönwall Lemma. We recall that a superadditive function \( \varepsilon \) on the simplex \( \Delta_T := \{(s, t) \in [0, T]^2 : s < t \} \) is by definition a function of two parameters such that

\[ \varepsilon(s, u) + \varepsilon(u, t) \leq \varepsilon(s, t), \]

for each \( 0 \leq s \leq u \leq t \leq T \). Moreover, a control function is a superadditive map that is positive.

The following lemma is proved in [14]. Note here that we allow for a \( \varepsilon_2 \) which has no sign, however it is straightforward to check that the proof remains identical.

**Lemma 1.** Fix \( T > 0 \) and consider \( G : [0, T] \to [0, \infty) \), continuous. Let \( \varepsilon_1 : \Delta_T \to [0, \infty) \) denote a control function, and \( \varepsilon_2 : \Delta_T \to \mathbb{R} \) be superadditive. Assume that there exists \( \kappa \) such that for each \( t, s \in [0, T] \):

\[ G_t - G_s \leq \left( \sup_{r \in [s, t]} G_r \right) \varepsilon_1(s, t)^{1/\kappa} + \varepsilon_2(s, t) . \]

Then, there exists a constant \( C_\kappa > 0 \) such that:

\[ \sup_{t \leq T} G_t \leq 2 \exp(\max[1, C_\kappa \varepsilon_1(0, T)]) \left[ G_0 + \sup_{t \leq T} |\varepsilon_2(0, t)| \right] . \tag{3.21} \]
Proof of Corollary 1.

Step 1: stochastic estimates. To prove the bound (3.19), we first apply Itô Formula to \(1/2|\mathcal{T}_u|^2\). There comes

\[
\frac{1}{2} |\mathcal{T}_u(t)|^2 - \frac{1}{2} |\mathcal{T}_u(s)|^2 + \int_s^t |\nabla \mathcal{T}_u(r)|^2 \, \mathrm{d}r = \int_s^t \langle \mathcal{T}_u, \nabla u \rangle^2 + u \nabla u \cdot \nabla \mathcal{T}_u \, \mathrm{d}r + M(s, t)
\]

(3.22)

for all \(0 \leq s \leq t \leq \zeta\), a.s., where \(M(s, t) \equiv M(t) - M(s)\) denotes the increment of the semi-martingale

\[
M(t) := \int_0^t \langle \mathcal{T}_u, \Delta(u \circ \mathrm{d}W) + |\nabla u|^2 u \circ \mathrm{d}W + u \nabla(u \circ \mathrm{d}W) \cdot \nabla u \rangle.
\]

(3.23)

To estimate \(\sup_{t \leq T} M(t)\), first expand the term \(\Delta(u \circ \mathrm{d}W)\) so that the latter semi-martingale rewrites as

\[
M(t) = \int_0^t \langle \mathcal{T}_u, \mathcal{T}_u \circ \mathrm{d}W \rangle + \int_0^t \langle \mathcal{T}_u, u \circ \Delta \circ \mathrm{d}W \rangle + \int_0^t \langle \mathcal{T}_u, \nabla u \rangle^2 + \int_0^t \langle \mathcal{T}_u, \nabla(u \circ \mathrm{d}W) \cdot \nabla u \rangle,
\]

(3.24)

where for a tensor \((f^\ell_i)_i \leq 3, j \leq 2\) we denote by

\[
(\nabla u \wedge f)^\ell := (\sum_j \partial_j u^{\ell+1} f^{\ell+2}_j - \partial_j u^{\ell+2} f^{\ell+1}_j)
\]

(here the index \(\ell\) runs over \(\mathbb{Z}/3\mathbb{Z}\)). We now evaluate each term of (3.24) separately. Noting that \(\mathcal{T}_u \perp \mathcal{T}_u \times W\), it is clear that

\[
A_t = 0.
\]

(3.25)

Similarly, by the fact that \(u \perp \mathcal{T}_u\), we have for the last term:

\[
D_t = 0.
\]

Concerning the second and the third terms, it is more convenient to use coordinates, write for instance

\[
C_t \equiv \sum_{\ell, j} \int_0^t \langle \partial_j u^{\ell+1} \circ \partial_j \mathcal{T}_u^{\ell+2} - \partial_j u^{\ell+2} \circ \partial_j \mathcal{T}_u^{\ell+1}, \mathcal{T}_u^\ell \rangle
\]

\[
= - \sum_{\ell, j} \int_0^t \left\{ \langle u^{\ell+1} \circ \partial_j \mathcal{T}_u^{\ell+2} - u^{\ell+2} \circ \partial_j \mathcal{T}_u^{\ell+1}, \partial_j \mathcal{T}_u^\ell \rangle + \langle u^{\ell+1} \partial_j \mathcal{T}_u^{\ell+2} - u^{\ell+2} \partial_j \mathcal{T}_u^{\ell+1}, \mathcal{T}_u^\ell \rangle \right\},
\]

(3.26)
so that \( B_t + C_t = \int_0^t \langle \nabla \mathcal{T}_u, u \times \nabla \circ dW \rangle \). Using the Itô form of the latter Stratonovich integral, we have

\[
M(t) := \hat{M}(t) + \int_0^t \sum_{\ell \in \mathbb{N}} \frac{1}{2} T_\ell(s) \, ds, \tag{3.26}
\]

where \( \hat{M}(t) \) is the corresponding Itô integral and can be estimated as follows, using Burkholder-Davies-Gundy inequality:

\[
\mathbb{E} \sup_{r \leq t} |\hat{M}_r|^m \leq C(m) \mathbb{E} \left( \int_0^t |\phi^* \text{div}(u \times \nabla \mathcal{T}_u)|^2 \, ds \right)^{m/2},
\]

for any \( t \in [0, T] \) and \( m > 1 \). Appealing to a similar argument as for (3.10), we end up with

\[
\mathbb{E} \sup_{r \leq t} |\hat{M}_r|^m \leq C(m, |\phi|_{L^2}) \mathbb{E} \left( \int_0^t |\nabla \mathcal{T}_u|^2 \, ds \right)^{m/2}. \tag{3.27}
\]

It remains to estimate the trace term in (3.26). We have

\[
T_\ell := \langle \text{div}(u \times \nabla \phi_\ell), \mathcal{T}_u \times \phi_\ell \rangle + \langle \text{div}(u \times \nabla \phi_\ell), 2 \nabla u \wedge \nabla \phi_\ell \rangle + \langle \text{div}(u \times \nabla \phi_\ell), 2u(u \times \nabla \phi_\ell) \cdot \nabla u \rangle + \langle \nabla \mathcal{T}_u, (u \times \phi_\ell) \times \partial_j \phi_\ell \rangle =: \sum_{k=1}^5 T^k_\ell
\]

where we have denoted by \( \phi_\ell := \phi e_\ell \), and also by \( \langle \nabla \mathcal{T}_u, (u \times \phi_\ell) \times \nabla \phi_\ell \rangle = \sum_{j=1,2} \langle \partial_j \mathcal{T}_u, (u \times \phi_\ell) \times \partial_j \phi_\ell \rangle \). Straightforward but cumbersome computations show that we have a bound

\[
\mathbb{E} \left( \int_0^T \sum_{\ell \in \mathbb{N}} \frac{1}{2} T_\ell(s) \, ds \right)^m \leq C(|\phi|_{L^\infty}, m, \phi, T, E_0). \tag{3.28}
\]

For instance:

\[
T^1_\ell \leq |u|_{L^4} |\nabla \phi_\ell|_{L^4} |\mathcal{T}_u|_{L^2} |\phi_\ell|_{L^\infty} \leq C |\phi_\ell|_{H^2}^2 (|\nabla u|_{L^2}^2 + |\Delta u|_{L^2}^2 + |\mathcal{T}_u|_{L^2}^2)
\]

using again the interpolation inequalities. Again, we have

\[
T^2_\ell \leq C |\nabla \phi_\ell|_{L^2} (|\Delta u|_{L^2} |\nabla \phi_\ell|_{L^\infty} + |\nabla u|_{L^2} |\Delta \phi_\ell|_{L^2}) \leq C |\phi_\ell|_{H^4}^2 (|\Delta u|_{L^2}^2 + |\nabla u|_{L^2}^2 + |\phi_\ell|_{H^2}^2),
\]

and the remaining terms are estimated in the same way. Summing over \( \ell \in \mathbb{N} \), integrating in time and using the energy estimates, we end up with (3.28).

**Step 2: bound on the Laplacian.** From (3.22) and \( \mathcal{T}_u \perp u \) we have

\[
\frac{1}{2} |\mathcal{T}_u(t)|_{L^2}^2 - \frac{1}{2} |\mathcal{T}_u(s)|_{L^2}^2 + \int_s^t |\nabla \mathcal{T}_u(r)|_{L^2}^2 \, dr = \int_s^t (|\mathcal{T}_u|^2 |\nabla u|^2) \, dr + M(s, t).
\]

Using Hölder and (2.1), there comes:

\[
\frac{1}{2} \left( |\mathcal{T}_u(t)|_{L^2}^2 - |\mathcal{T}_u(s)|_{L^2}^2 + \int_s^t |\nabla \mathcal{T}_u(r)|_{L^2}^2 \, dr \right) \leq \left( \int_s^t |\nabla u|^4 \, dr \int_s^t |\mathcal{T}_u|^4 \, dr \right)^{1/2} + M(s, t) \tag{3.29}
\]

\[
\leq \sqrt{20} \left( \int_s^t |\nabla u|^4 \, dr \right)^{1/2} \left( \sup_{r \in [s, t]} |\mathcal{T}_u(r)|_{L^2}^2 + \int_s^t |\nabla \mathcal{T}_u|^2 \, dr \right) + M(s, t),
\]

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Hence, we can apply Lemma 1 with \( \varepsilon_1(s, t) := C_0 / 4 \int_0^T |\nabla u(r)|^4 \, dr \), \( \varepsilon_2(s, t) := M(s, t) \) and 
\( G_t := |\mathcal{T}_u(t)|^2_{L^2} + |\mathcal{T}_u|_{L^2}^2 \). This yields the pathwise estimate

\[
\sup_{t \in [0, \varpi]} G_t \leq C \left( 1 + \exp \int_0^T |\nabla u|_{L^2}^4 \, dr \right) \left( |\mathcal{T}_u(0)|^2_{L^2} + \sup_{t \in [0, \varpi]} |M(t)| \right),
\]

for some universal constant \( C > 0 \). Using now the exponential bound (3.15), (3.27) and (2.8), we end up with

\[
\mathbb{E} \left[ \sup_{t \in [0, \varpi]} |\mathcal{T}_u(t)|^2_{L^2} + \int_0^\varpi |\nabla \mathcal{T}_u(r)|^2_{L^2} \, dr \right] \leq C \left( 1 + \delta(t) \mathbb{E} \left[ \int_0^T |\nabla \mathcal{T}_u|^2_{L^2} \right]^{1/2} \right),
\]

where we denote by

\[
\delta(t) := \mathbb{E} \left[ \exp 2 \int_0^t |\nabla u|^4_{L^2} \, dr \right]^{1/2}
\]

and where the constant \( C \) depends on the quantities \( |\mathcal{T}_u(0)|, \varrho, |\phi|_{L^2}, T, \varepsilon_1 \). Writing \( \delta(t) \mathbb{E}[\int_0^t |\nabla \mathcal{T}_u|^2_{L^2}] \leq C\delta(t)^{2/2} + (2C)^{-1} \mathbb{E}[\int_0^t |\nabla \mathcal{T}_u|^2_{L^2}] \), and then absorbing to the left in (3.31), we obtain

\[
\mathbb{E} \left[ \sup_{t \in [0, \varpi]} |\mathcal{T}_u(t)|^2_{L^2} + \int_0^\varpi |\nabla \mathcal{T}_u(r)|^2_{L^2} \, dr \right] \leq C(|\mathcal{T}_u(0)|, \varrho, |\phi|_{L^2}, T, \varepsilon_1).
\]

Now, applying 2 to the constant function \( v = u(t, \cdot) \), we have for all \( t \in [0, \varpi] \):

\[
|\Delta u(t)|^2_{L^2} \leq 2|\mathcal{T}_u(t)|^2_{L^2} + 2|\nabla u(t)|^4_{L^4},
\]

\[
\leq 2|\mathcal{T}_u(t)|^2_{L^2} + 2C_1 \varepsilon_1 \left( |\Delta u(t)|^2_{L^2} + \frac{C}{\varphi^2} |\nabla u(t)|^2_{L^2} \right),
\]

and taking \( \varepsilon_1 \) smaller if necessary, we end up with (3.19).

**Step 3: increasing the regularity of the stochastic convolution.** We appeal here to the same arguments as that of [13]: define the stochastic convolution:

\[
Z(t) := \int_0^t S(t - s) u \times dW, \quad t \in [0, \varpi],
\]

and for simplicity denote by \( L^m L^2 := L^m(0, \varpi; L^2) \), \( CL^2 := C(0, \varpi; L^2) \) and so on. Using (3.19), Proposition 4 yields that for every \( 4 < m < \infty \), fixing for instance \( \delta := 1/2 < 1 - 2/m \) we have with \( \lambda = 0 \):

\[
\mathbb{E} \left[ \|Z\|^m_{C^{H^{5/2}}} \right] \leq C \mathbb{E} \| u \times \phi \|^m_{L^m L^2} \leq C' (|\phi|_{L^4}^p, T) \left( 1 + \mathbb{E} \| \Delta u \|^m_{C^{L^2}} \right),
\]

where the second inequality comes from the fact that for any \( k \in \mathbb{N} \):

\[
|u \times \phi e_k|_{H^2} \leq |u|_{H^2} |\phi e_k|_{W^{2,\infty}},
\]

together with the embedding \( H^4 \hookrightarrow \ W^{2,\infty} \).
Step 4: Increasing the regularity of the solution. Observe that $y := u - Z$ is a solution of the following PDE with random coefficients:

$$\partial_t y - \Delta y = u|\nabla u|^2.$$  \hfill (3.36)

However, from (3.19) and the Sobolev embedding theorem in dimension two, we can deduce that:

$$f \equiv u|\nabla u|^2 \in CL^p,$$  \hfill (3.37)

for any $p \in [1, \infty)$, a.s., so that using Proposition 3, we have in particular $y \in CW^{2,4}$ and

$$\mathbb{E}\|y\|^m_{CW^{2,4}} \leq C_p(1 + \mathbb{E}\|\nabla u\|^{2m}_{CL^8}) \leq C_p \left(1 + \mathbb{E}\|\Delta u\|^{2m}_{CL^2}\right),$$  \hfill (3.38)

for every $m \geq 1$. Observe that by (3.35) and $H^{5/2} \hookrightarrow W^{2,4}$, there holds:

$$\mathbb{E}\|Z\|^m_{CW^{2,4}} \leq C(1 + \mathbb{E}\|\Delta u\|^m_{L^mL^2}),$$  \hfill (3.39)

for $u = y + Z$, namely

$$\mathbb{E}\|u\|^m_{CW^{2,4}} \leq C(|\phi|_{L^2})(1 + \mathbb{E}\|\Delta u\|^m_{L^mL^2}).$$

We have now $\nabla f = \nabla u|\nabla u|^2 + 2u\nabla^2 u\nabla u \in CL^2$ whence $f \in CH^1$ and

$$\mathbb{E}\|y\|^m_{CH^3} \leq C\mathbb{E}\|\nabla u\|^2_{CH^1} \leq C(1 + \mathbb{E}\|\nabla u\|_{CL^2})^2 \mathbb{E}\|\nabla u\|_{CL^4} + 2\mathbb{E}\|\nabla^2 u\|_{CL^4}\mathbb{E}\|\nabla u\|_{CL^4}^m \leq \mathbb{E}P_m(\|\Delta u\|_{CL^2}),$$  \hfill (3.40)

where $P_m$ is a polynomial.

We can now repeat Step 3 to obtain

$$\mathbb{E}\|Z\|^m_{CH^3} \leq \mathbb{E}P(\|\Delta u\|_{CL^2}),$$  \hfill (3.41)

and finally

$$\mathbb{E}\|u\|^m_{CH^3} \leq \mathbb{E}P(\|\Delta u\|_{CL^2}) < \infty \quad m \geq 1,$$  \hfill (3.42)

for another such polynomial, depending on $m, |\phi|_{L^2}$. Reiterating the argument above, a straightforward induction shows that provided $\phi \in \cap_{m \in \mathbb{N}}L^2(H^m)$, then

$$u(\cdot \land \zeta) \in \cap_{m \in \mathbb{N}}L^2(\Omega; CH^m).$$

This finishes the proof of Corollary 1.

Remark 9. The reason why a bootstrap argument is needed will be seen in (3.66) and (3.67). Whenever

$$t = \zeta(\rho)$$

for some $\rho > 0$, the bootstrap ensures the possibility to extend the solution during a positive time after $t$, in a space where the Itô formulas (3.7) and (3.22) can make sense.

However the regularity “$u \in C(H^3)$” turns out to be sufficient to make them rigorous, and hence to prove Theorem 1.
3.2 Step 2: Tightness

We now define a sequence \( \{W_n, n \in \mathbb{N}\} \) of Wiener processes in \( L^2(\mathbb{T}^2; \mathbb{R}^3) \) where for each \( n \in \mathbb{N} \), \( W_n \) is given by the sum

\[
W_n := \sum_{\ell \in \mathbb{N}} B_{\ell}(\cdot) \phi_n e_\ell,
\]

(3.43)

for \((e_\ell), (B_\ell)\) as above, and \( \phi_n \equiv (\phi_n^1, \phi_n^2, \phi_n^3) \) denotes a sequence of Hilbert-Schmidt operators. Consider the regularized problem:

\[
dv_n = (\Delta v_n + v_n|\nabla v_n|^2 + F_{\phi_n} v_n) \, dt + v_n \times dW_n.
\]

(3.44)

We make the following assumptions:

(A1) For all \( n \in \mathbb{N} \), we have \( v_n(0) \in C^\infty(\mathbb{T}^2) \), moreover: \( v_n(0) \to u_0 \) in \( H^1 \) and \( \frac{1}{2}|\nabla v_n(0)|_{L^2}^2 \leq CE_0 \equiv C_0 |u_0|^2_{L^2}; \)

(A2) For all \( n \in \mathbb{N} \), we have \( \phi_n \in \cap_{k \in \mathbb{N}} L^2_k \), and \( \phi_n \to \phi \) in \( L_2(2; H^1); \)

Note that (A2) is possible by considering e.g. the sequence of finite rank operators \( \phi_n := \sum_{k \in \mathbb{N}} \Phi_\ell e_k \langle e_k, \cdot \rangle. \) Furthermore thanks to Theorem 3 and (and also Remark 6), the assumptions (A1) and (A2) ensure that

(A3) For every \( n \in \mathbb{N} \), there exists a unique maximal strong solution \((v_n; \tau_n)\) to (3.44), having continuous paths with values in \( H^3 \). We have the property

\[
\tau_n = T \quad \text{or} \quad \limsup_{t \to \tau} |u(t)|_{H^3} = \infty.
\]

Now, fix \( \epsilon_1 \in (0, \epsilon_1^*), \) choose a sequence \( \rho_k \to 0 \), define the following stopping times:

\[
\zeta_{n,k} := \inf \left\{ 0 \leq t < \tau_n, \sup_{x \in \mathbb{T}^2} \int_{B(x, \rho)} |\nabla u(t, y)|^2 \, dy \geq \epsilon_1 \right\},
\]

(3.45)

for \( n, k \in \mathbb{N}, \) and denote by \( u_{n,k}, k \in \mathbb{N}, \) the “mildly stopped process”:

\[
u_n(t) \quad \text{if} \quad 0 \leq t \leq \zeta_{n,k}, \]

(3.46)

\[
e^{-\left(t-\zeta_{n,k}\right)\Delta} v_n(\zeta_{n,k}) \quad \text{if} \quad \zeta_{n,k} < t \leq T,
\]

(the reason for this definition will become clearer in the proof of Claim 1).

We will also denote by

\[
U_n := \{u_{n,k}; k \in \mathbb{N}\}, \quad n \in \mathbb{N}.
\]

(3.47)

Claim 1. For every \( \delta < 1 \), the sequence \( \{U_n, n \in \mathbb{N}\} \) is tight in \( E := \Pi_{k \in \mathbb{N}} L^2([0, T]; H^{1+\delta}) \cap C([0, T]; H^\delta). \)

Proof of Claim 1. The proof is rather similar than that of [8, Lemma 4.2]. It uses the a priori estimates, together with the following classical compactness result (“Aubin-Lions Lemma”): If \( B_0 \subset B \subset B_1 \) are Banach spaces, such that \( B_0, B_1 \) are reflexive, and the embedding of \( B_0 \) in \( B \) is compact, and if \( (\beta, p, q) \in (0, 1) \times (1, \infty) \times (1, \infty) \) with \( \beta p > 1 \) then \( L^q(0, T; B_0) \cap W^{\beta,p}(0, T; B_1) \hookrightarrow L^q(0, T; B) \) and \( C(0, T; B_0) \cap W^{\beta,p}(0, T; B_1) \hookrightarrow C(0, T; B) \) (with compact embeddings).
Regularity in time. We need uniform estimates in some space $W^{\beta,p}(0,T;B_1)$, where $B_1$ can be any reflexive Banach space containing $L^2$, and $\beta p > 1$. These bounds essentially follow from the equation on $v_{n,k}$.

As in the proof of Lemma 4.1 in [8], we can write, using (3.44):

$$v_n(t) - v_n(0) = \int_0^t (\Delta v_n + v_n|\nabla v_n|^2)\,ds + \int_0^t F_{\phi_n} v_n\,ds + \int_0^t v_n \times dW_n(s)$$

for all $n \in \mathbb{N}$ and $t \in [0,\tau_n)$, a.s. Recall that this equation holds in the sense of Bochner, and Itô integrals in $L^2$.

Now, the bound $\mathbb{E}\|f_n^{1}\|_{W^{1,2}(0,\tau_n;L^2)^2} \leq C(\|\phi\|_{L^2(H^1)})$ is a consequence of the uniform estimate (3.5), and by the definition of the correcting term $F_{\phi_n} v_n$ and (A3), we obtain $\mathbb{E}\|f_n^{2}\|_{W^{1,2}(0,\tau_n;L^2)^2} \leq C(\|\phi\|_{L^2(H^1)})$. Lastly, using Lemma 2.1 from [18], for any $\beta \in (0,\frac{1}{2})$, $\alpha > p \geq 2$ there exists a constant depending only on $\beta, p, \|\phi\|_{L^2}$ such that: $\mathbb{E}\|f_n^{2}\|_{W^{\beta,p}(0,\tau_n;L^2)^2} \leq C(\beta, p, \|\phi\|_{L^2})$. Putting these bounds together, we have for each $n, k \in \mathbb{N}$:

$$\mathbb{E}\|u_{n,k}\|_{W^{\alpha,q}(0,T;L^2)} \leq C(\alpha, \|\phi\|_{L^2(H^1)}), \quad (3.48)$$

for some $1 \geq \alpha > 0$ and $q \geq 1$ with $\alpha q > 1$, depending on $\beta, p$.

Bounds on the whole time interval and conclusion. Applying Proposition 5, we have for all $n, k \in \mathbb{N}$:

$$\mathbb{E}\sup_{0 \leq t \leq T} |\nabla u_{n,k}|_{L^2}^2 + \mathbb{E}\int_0^T |\Delta u_{n,k}|_{L^2}^2 \,ds \leq C(k, E_0, \|\phi\|_{L^2}). \quad (3.49)$$

The fact that this uniform bound holds on the whole interval $[0, T]$ (and not only on $[0, \tilde{\zeta}_{n,k}]$) is however not clear. This is precisely the reason why we extend $u_{n,k}$ after $\tilde{\zeta}_{n,k}$ by the solution of a linear parabolic equation involving the bilaplacian, see (3.46). This technical tool allows to “forget” the value of $|\Delta u(\zeta_{n,k})|_{L^2}$ (on which we have no control when $\phi_n$ is not bounded in $L^3_\beta$, see (3.19)). Indeed: for the sectorial operator $A := \Delta^2$, $D(A) := H^4$, we have the classical inequality

$$|e^{-tA}f|_{D(A^{1/2})} \leq C \frac{|f|_{D(A^{1/4})}}{t^{1/4}}, \quad \text{for } t > 0, \text{ and } f \in H^1.$$  

Therefore, by the definition (3.46) we have

$$\mathbb{E}\int_0^T |\Delta u_{n,k}(t)|_{L^2}^2 \,dt \leq C \mathbb{E}\int_0^T \frac{|\nabla u(\zeta_{n,k})|^2_{L^2}}{(t-\tilde{\zeta}_{n,k})^{1/2}} \,dt \text{,} \quad (3.50)$$

which is bounded by a constant $C(E_0, T, \phi)$, using (3.5).

Using Proposition 5, we have for all $n, k \in \mathbb{N}$:

$$\mathbb{E}\sup_{0 \leq t \leq T} |\nabla u_{n,k}|_{L^2}^2 + \mathbb{E}\int_0^T |\Delta u_{n,k}|_{L^2}^2 \,ds \leq C(k, E_0, T, \|\phi\|_{L^2}). \quad (3.51)$$

The tightness follows: for $\delta < 1$, set first $B_0 = H^1, B = H^\delta, B_1 = L^2$, and then $q = 2, B_0 = H^2, B = H^{1+\delta}, B_1 = L^2$, so that the embedding

$$C([0, T]; H^1) \cap W^{\beta,p}([0, T]; L^2) \cap L^2([0, T]; H^2) \hookrightarrow C(0, T; H^\delta) \cap L^2([0, T]; H^{1+\delta})$$
is compact by Aubin-Lions Lemma. We conclude using the estimates (3.48)–(3.51), together with Tychonov Theorem, Markov inequality. We refer the reader to [31] for details.

By classical properties of Wiener processes the sequence \( \{ (U_n, Z_n, W_n) , n \in \mathbb{N} \} \) is also tight in \( E \times \prod_{k \in \mathbb{N}} [0, T] \times C^\alpha ([0, T]; H^1) \) for some \( \alpha \in (0, \frac{1}{2}) \), where we let for \( n \in \mathbb{N} \):

\[
Z_n := \{ \xi_{n,k} , k \in \mathbb{N} \}.
\]

Therefore, by Prokhorov Theorem there exists an extraction \( n_\ell, \ell \in \mathbb{N} \), and a law \( \mu \) supported in \( \prod_{k \in \mathbb{N}} (L^2 ([0, T]; H^2) \cap C([0, T]; H^1)) \times \prod_{k \in \mathbb{N}} [0, T] \times C^\alpha ([0, T]; H^1) \) such that \( \mathcal{L} (U_{n_\ell}, Z_{n_\ell}, W_{n_\ell}) \rightarrow \mu \) weakly. By a standard application of Skorohod theorem, we however obtain a little more.

**Corollary 2.** There exist

- a stochastic basis \( \Omega' = (\Omega', \mathcal{F}', \mathbb{P}', (\mathcal{F}_t')_{t \in [0, T]}, W') \), where \( W' \) is a Wiener process in \( L^2 \) with covariance \( \phi \theta' \);

- a sequence of r.v. \( \{ \{ u'_\ell,k, k \in \mathbb{N} \}, \{ \xi'_\ell,k, k \in \mathbb{N} \}, W'_\ell \} , \ell \in \mathbb{N} \} \), where for each \( \ell, k \in \mathbb{N} \), \( u'_\ell,k : \Omega' \rightarrow C([0, T]; H^1) \cap L^2 ([0, T]; H^2) \) denotes a predictable process, and \( \xi'_\ell,k \) is a positive stopping time, whereas \( W'_\ell \) is an \( L^2 \)-valued Wiener process with respect to \( (\mathcal{F}_t') \) of which the square-root of the covariance is \( \psi'_\ell \equiv (\psi'_\ell, \psi'_\ell, \psi'_\ell) : = \phi_{n_\ell} \);

- limits \( u'_k(\omega') \in \cap_{\delta < 1} C(0, T; H^\delta) \cap L^2 ([0, T]; H^{1+\delta}) \), and \( \xi'_k(\omega') \in [0, T] \), for every \( k \in \mathbb{N} \),

such that the following convergences hold for each \( k \in \mathbb{N} \):

\[
u'_\ell k \rightarrow u'_k \quad \mathbb{P}' - a.s. \quad (3.53)
\]

in every \( C(0, T; H^\delta) \cap L^2 ([0, T]; H^{1+\delta}) \) for \( \delta < 1 \);

\[
\xi'_\ell,k \rightarrow \xi'_k \quad \mathbb{P}' - a.s. \quad (3.54)
\]

\[
W'_\ell \rightarrow W' \quad \mathbb{P}' - a.s. \quad \text{in every } C^\alpha ([0, T]; H^1) , \quad \text{for } \alpha < \frac{1}{2} \quad (3.55)
\]

\[
\mathbb{E} \langle \int_0^\cdot u'_\ell k \times dW'_\ell, X \rangle \rightarrow \mathbb{E} \left\langle \int_0^\cdot u'_k \times dW', X \right\rangle ,
\]

for every predictable process \( X \) in \( L^2 (\Omega' \times [0, T] \times \mathbb{T}^2) \).

**Proof.** The proof is standard. These properties are a consequence of Skorohod embedding Theorem (see [46]), and classical properties of Wiener processes: we write that

\[
\mathbb{E}' \left[ \left( M_{\ell,k}(t) - M_{\ell,k}(s) \right) \varphi \left( (u'_\ell W'_\ell) |_{[0,n]} \right) \right] = 0 ,
\]

\[
\mathbb{E}' \left[ \left[ \langle M_{\ell,k}(t), a \rangle_{L^2} - \langle M_{\ell,k}(s), b \rangle_{L^2} - \int_s^t \langle u'_\ell \times \psi'_\ell a, u'_\ell \times \psi'_\ell b \rangle_{L^2} ds \right] \times \varphi \left( (u'_\ell W'_\ell) |_{[0,n]} \right) \right] = 0,
\]

for any \( \ell, k \in \mathbb{N}, 0 \leq s \leq t \leq T, a, b \in L^2 \) and \( \varphi \) bounded continuous, where \( M_{\ell,k}(t) = u'_\ell k(t) - u'_\ell k(0) - \int_0^t \langle \Delta u'_\ell k, u'_\ell k \rangle_{L^2} = F_{\psi'_\ell} u'_\ell k \) ds. We can then take the limits in (3.57) and (3.58), and apply the Martingale Representation Theorem (see [10]). Details of this argument can be found in the monograph [42], see also [1].
3.3 **Step 3: below estimates for** \( \lim_{n \to \infty} \zeta'_{k,n} \)

Uniform bounds from below for the stopping time \( \zeta'_{n,k} \) will guarantee the existence of the “Struwe solution” during a positive time, and therefore the present section can be considered, together with the justification of the bootstrap (namely Corollary 1), as the core of the argument. By strong convergence of \( u'_{t,k}(0) \) towards \( u'_{k}(0) \) in \( H^1 \), and the fact that \( \rho_k \to 0 \), we can assume without restriction that for some \( \lambda \geq 2 \), and for all \( k \in \mathbb{N} \):

\[
\sup_{x \in \mathbb{T}^2} \int_{B(x, \lambda \rho_k)} |\nabla u'_{t,k}(0)|^2 \leq \frac{\epsilon_1}{2}, \quad \text{uniformly in } \ell \in \mathbb{N},
\]

(3.59)

(note that we also use compactness of \( \mathbb{T}^2 \) here). We will always assume (3.59) in the following.

**Claim 2.** For each \( k \in \mathbb{N} \), the limit point \( \zeta'_{k} \) of the sequence \( \{\zeta'_{t,k}, \ell \in \mathbb{N}\} \) (see Corollary 2) verifies

\[
P'(\zeta'_{k} > 0) = 1.
\]

To prove the claim, we need the following local dissipation estimate.

**Lemma 2.** Let \( \eta \in C^\infty_0(\mathbb{T}^2; \mathbb{R}) \), \( \rho > 0 \), and \( x \in \mathbb{T}^2 \), such that \( \text{spt}(\eta) \subset B(x, \rho) \) and \( |\nabla \eta|_{L^\infty} \leq \frac{\lambda}{2} \). Then, for every local solution \( (u; \tau) \) which is supported in \( C([0, \tau]; H^1) \cap L^2([0, \tau]; H^2) \) for some stopping time \( \tau > 0 \), there holds a.s. for \( t \in [0, \tau] \):

\[
\frac{|\eta \nabla u(t)|^2}{2} - \frac{|\eta \nabla u(0)|^2}{2} - \int_0^t \left\langle \eta \nabla u, \eta u \times \nabla dW \right\rangle = \int_t^\tau \left\langle \eta^2 \nabla u, \nabla (\mathcal{T}_u) \right\rangle ds,
\]

where we denote by \( |\nabla \phi|^2 = \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{T}^2} \eta(x)^2 |\nabla \phi(x)|^2, dx \).

**Proof of Lemma 2.** Itô Formula writes for \( \frac{1}{2} |\eta \nabla u|^2 \) :

\[
\frac{|\eta \nabla u(t)|^2}{2} - \frac{|\eta \nabla u(0)|^2}{2} - \int_0^t \left\langle \eta \nabla u, \eta u \times \nabla dW \right\rangle = \int_t^\tau \left\langle \eta^2 \nabla u, \nabla (\mathcal{T}_u) \right\rangle ds,
\]

Moreover, we have the identity \( \int_0^t \left\langle \eta \nabla u, \eta u \times \nabla dW \right\rangle = \iint \sum_{\ell \in \mathbb{Z}} \eta^2 |\nabla \phi|^2 + \int_0^t \left\langle \eta \nabla u, \eta u \times \nabla dW \right\rangle \) (the computations are identical as that of (3.4), replacing \( \langle \cdot , \cdot \rangle \) by \( \langle \cdot , \eta \cdot \rangle \)). We obtain:

\[
\frac{|\eta \nabla u(t)|^2}{2} - \frac{|\eta \nabla u(0)|^2}{2} - \int_0^t |\eta \nabla \phi|^2 ds - \int_0^t \left\langle \eta^2 \nabla u, u \times \nabla dW \right\rangle
\]

\[
= \int_0^t -\left\langle 2 \eta \nabla \eta \nabla u + \eta^2 \Delta u, \mathcal{T}_u \right\rangle ds
\]

\[
\leq \int_0^t \left|\eta \mathcal{T}_u \right|^2 + \left|\eta^2 \mathcal{T}_u \right|^2 + \frac{C}{\rho^2} |\nabla u|^2 \right| ds,
\]

a.s., where we have used that \( \mathcal{T}_u \cdot \Delta u = |\mathcal{T}_u|^2 \). This proves (3.60).

**Proof of Claim 2.** We first prove that for all \( \ell, k \in \mathbb{N} \), then \( P'(\zeta'_{t,k} > 0) = 1 \). Fix \( \ell, k \in \mathbb{N} \). We observe that:

\[
P'(\zeta'_{t,k} > 0) = 1 - P'(\zeta'_{t,k} = 0)
\]

\[
= 1 - \lim_{N \to \infty} P'(\zeta'_{t,k} \leq 1 / N).
\]

(3.61)
To show that $\mathbb{P}(\zeta_{\ell,k} \leq 1/N) \to 0$ as $N \to \infty$, we need to circumvent the presence of a supremum in the definition of $\zeta$, which is not well adapted for martingale inequalities. This can be done via a discretization method, which relies on the following geometrical fact.

**Covering argument.** There exist constants $C = C(\mathbb{T}^2) > 0$, $\lambda = \lambda(\mathbb{T}^2) \in (1,3]$, and a sequence of integers $\{N_k, k \in \mathbb{N}\}$ with $\limsup_{k \to \infty}(\Theta_k)^2N_k \leq C$, such that for all $k \in \mathbb{N}$, there are points $\{x^1_k, x^2_k, \ldots, x^{N_k}_k\} \subset \mathbb{T}^2$ fulfilling the property:

\[
\text{"For all } x \in \mathbb{T}^2 \text{ there exists } i \in \{1,\ldots,N_k\} \text{ with } B(x, \Theta_k) \subset B(x^i_k, \lambda \Theta_k)."
\]

**Proof.** It suffices to take $\lambda = 2$, and to consider any finite cover $\mathbb{T}^2 = \cap_{i \leq N_k} B(x^i_k, \Theta_k)$. Then we have also $\mathbb{T}^2 = \cap_{i \leq N_k} B(x^i_k, 2\Theta_k)$, and this cover fulfills the required property since any ball $B(x, \Theta_k)$ is included in $B(x, 2\Theta_k)$ for $|x - x^i_k| < \Theta_k$. This proves the covering argument.

Now, for each $k \in \mathbb{N}$, and each $x^i_k$, consider $\eta = \eta_{\lambda \Theta_k, i} \in C^\infty_0(\mathbb{T}^2)$ with spt $\eta \subset B(x^i_k, 2\lambda \Theta_k)$ with

\[
\mathbb{1}_{B(\lambda \Theta_k, x^i_k)} \leq \eta, \quad \sup_{x \in B(x^i_k, 2\lambda \Theta_k)} |\nabla \eta(x)| \leq \frac{C}{\Theta_k},
\]

for some $C > 0$ independent of $i, k$. To lighten the notations, denote by

\[
\mathcal{C}(k) := \{\eta \equiv \eta_{\lambda \Theta_k, i}, 1 \leq i \leq N_k\}
\]

where the functions $\eta_{\lambda \Theta_k, i}$ are as above, so that in particular $\# \mathcal{C}(k) = N_k$ is finite. Using the bound on the local dissipation, namely (3.60), we have for all $\eta \in \mathcal{C}(k)$:

\[
\frac{1}{2} \left( |\nabla u_{\ell,k}(t)|_{L^2}^2 - |\nabla u_{\ell,k}(t)|_{L^2}^2 \right) \leq V^\eta_{\ell,k}(t), \quad \text{for } t \in [0, \zeta_{\ell,k}],
\]

where we denote by: $V^\eta_{\ell,k}(t) := t|\nabla \psi|^2_{L^2} + C/\Theta_k \int_0^t |\nabla u_{\ell,k}(s)|_{L^2}^2 \, ds + \int_0^t \langle \eta \nabla u_{\ell,k}, \eta u_{\ell,k} \times \nabla W(t) \rangle$ (we recall that $\psi$ denotes $\phi_{n_x}$).

Moreover, the Burkholder-Davies-Gundy inequality for $\int_0^t \langle \eta \nabla u_{\ell,k}, \eta u_{\ell,k} \times \nabla W(t) \rangle$ gives

\[
\mathbb{E}' \sup_{0 \leq t \leq 1/N} V^\eta_{\ell,k}(t) \leq \frac{|\nabla \psi|^2_{L^2}}{N} + \mathbb{E}' \int_0^{1/N} |\nabla u_{\ell,k}(s)|_{L^2}^2 \, ds + \mathbb{E}' \int_0^{1/N} \left( \int_0^s |\nabla u_{\ell,k}(r)|_{L^2}^2 \, dr \right)^{1/2} \left( \int_0^s |\nabla u_{\ell,k}(r)|_{L^2}^2 \, dr \right)^{1/2} + C(\phi_{1,1}) \left( \mathbb{E}' \int_0^{1/N} |\nabla u_{\ell,k}(s)|_{L^2}^2 \, ds \right)^{1/2}
\]

On the other hand, according to the definition (3.45), we have

\[
\left\{ \zeta_{n,k} \leq \frac{1}{N} \right\} \subset \left\{ \zeta_{n,k} < \tau_n \text{ and } \zeta_{n,k} \leq \frac{1}{N} \right\} \cup \left\{ \zeta_{n,k} = \tau_n \text{ and } \tau_n \leq \frac{1}{N} \right\} =: \Omega_1 \cup \Omega_2
\]

but thanks to the bootstrap argument, namely Corollary 1, we know that

\[
\mathbb{P}(\Omega_2) = 0.
\]

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Therefore, by (3.64), (3.59) and (3.65), we see that \( \{ |\eta \nabla u'_{\ell,k} |_{L^2}^2 \geq \varepsilon_1 \} \subset \{ V_{\ell,k}^\eta \geq \varepsilon_1 / 4 \} \) so that using on the other hand (3.62), and Markov inequality, we obtain

\[
\mathbb{P}' \left( \xi'_{\ell,k} \leq \frac{1}{N} \right) = \mathbb{P}' \left( \sup_{t \in [0,1/N]} \sup_{x \in \mathbb{T}^2} \int_{B(x,\varepsilon_k)} |\nabla u'_{\ell,k}(t)|^2 \geq \varepsilon_1 \right) \\
\leq \sum_{\eta \in \mathcal{G}(k)} \mathbb{P}' \left( \sup_{t \in [0,1/N]} V_{\ell,k}^\eta(t) \geq \frac{\varepsilon_1}{4} \right) \\
\leq 4 \varepsilon_1 \sum_{\eta \in \mathcal{G}(k)} \left\{ \frac{|\eta \nabla \phi_{\ell,k} |_{L^2}^2}{N} + \frac{C}{\varrho_k^2} \mathbb{E}' \int_0^1 |\nabla u'_{\ell,k}(s)|_{L^2}^2 \, ds \right. \\
+ \left. C(\| \phi_{1/2} \|_{L^2}) \left( \mathbb{E}' \int_0^1 |\eta \nabla u'_{\ell,k}(s)|_{L^2}^2 \, ds \right)^{1/2} \right\}.
\]

(3.68)

By the previous paragraph, the right hand side of (3.68) converges to 0 as \( N \to \infty \), and the convergence holds uniformly in \( \ell \in \mathbb{N} \).

**Conclusion.** Writing that for each \( \ell \in \mathbb{N} \):

\[
\left\{ \xi'_{\ell,k} \leq \frac{1}{N} \right\} \subset \left\{ |\xi'_{\ell,k} - |\xi'_{\ell-1,k} - \xi'_{\ell,k}| | \leq \frac{1}{N} \right\},
\]

so that:

\[
\mathbb{P}' \left( \xi'_{\ell,k} \leq \frac{1}{N} \right) \leq \mathbb{P}' \left( \xi'_{\ell,k} \leq \frac{2}{N} \right) + \mathbb{P}' \left( |\xi'_{\ell,k} - \xi'_{\ell-1,k} | | \leq \frac{1}{N} \right).
\]

The conclusion follows by \( |\xi'_{\ell,k} - \xi'_{\ell-1,k} | \mathbb{P}' \to 0 \) as \( \ell \to \infty \), and the uniform convergence of \( \mathbb{P}'(\xi'_{\ell,k} \leq 1/N) \) as \( N \to \infty \).

### 3.4 Step 4: uniqueness and the end of the Proof

We start by showing a useful Grönwall estimate on the difference of two martingale solutions \( u, v \) of (1.9) that are defined on a common stochastic basis \( \mathcal{F}' \equiv (\Omega', \mathcal{F}', \mathbb{P}', (\mathcal{F}')_{t \in [0,T]}, W') \), and both supported in \( C([0,\zeta]; H^1) \cap L^2([0,\zeta]; H^2) \) for some \( \zeta > 0 \). Namely, denoting by \( f := u - v \), we have

\[
\frac{1}{2} |f(t)|_{L^2}^2 \leq C \int_0^t (|\nabla u|^4_{L^4} + |\nabla v|^4_{L^4} + 1) |f(s)|_{L^2}^2 \, ds, \quad \mathbb{P}' - \text{a.s. for } t \in [0,\zeta].
\]

(3.69)

**Proof of (3.69).** We have \( f \in C([0,\zeta]; H^1) \), \( f(0) = 0 \) and

\[
df = (\Delta f + u |\nabla u|^2 - v |\nabla v|^2) \, dt + f \times \circ dW', \quad \text{on } \Omega' \times [0,\zeta] \times \mathbb{T}^2.
\]

(3.70)

Itô formula on \( \frac{1}{2} |f|^2_{L^2} \) gives a.s.

\[
d\left( \frac{|f|^2_{L^2}}{2} \right) = (f, f \times \circ dW') + (\Delta f + u |\nabla u|^2 - v |\nabla v|^2)) \, dt \\
= (- |\nabla f|^2_{L^2} + (f, u |\nabla u|^2 - v |\nabla v|^2)) \, dt.
\]

(3.71)
Using Hölder Inequality, the second term in the right hand side of (3.71) is estimated as
\[
\int_0^t \langle f, u|\nabla u|^2 - v|\nabla v|^2 \rangle \, ds \leq \int_0^t (|f|_{L^4}^2 |\nabla u|_{L^4}^2 + |f|_{L^2} |\nabla u + \nabla v|_{L^4} |\nabla f|_{L^2} ) \, ds
\]
\[
\leq C \int_0^t (|\nabla u|_{L^4}^2 + |\nabla v|_{L^4}^2) |f|_{L^2}^2 \, ds + \frac{1}{2} \int_0^t |\nabla f|^2_{L^2} \, ds,
\]
a.s. for \( t \in [0, \zeta] \). Since by Proposition 1 \( |f|_{L^4}^2 \leq C (|\nabla f|_{L^2} + |f|_{L^2}) |f|_{L^2} \), using again \( ab \leq a^2/2 + b^2/2 \) yields:
\[
\int_0^t \langle f, u|\nabla u|^2 - v|\nabla v|^2 \rangle \, ds \leq C \int_0^t (|\nabla u|^4_{L^4} + |\nabla v|^4_{L^4} + 1) |f|_{L^2}^2 \, ds + \int_0^t |\nabla f|^2_{L^2} \, ds. \tag{3.72}
\]
Putting together (3.71) and (3.72), we obtain (3.69). Note that all computations above make sense since \( u, v \in L^2([0, \zeta]; H^2) \hookrightarrow L^4([0, \zeta]; W^{1,4}) \), by (2.1).

**Corollary 3.** For any \( u_0 \in H^1 \), and \( \phi \in L^1_{\text{loc}} \), there exists a local strong solution \( (u_*, \zeta_*) \) for (1.9).

**Proof.** We use the famous Gyöngy and Krylov argument [26] (see also [48] and [40] for related results). If we consider another extraction \( \{m_\ell, \ell \in \mathbb{N}\} \), then it is straightforward that the sequence
\[
\left\{ \left\{ U_{m_\ell}, Z_{m_\ell}, W_{m_\ell}; U_{m_\ell}, Z_{m_\ell}, W_{m_\ell} \right\}, \ell \in \mathbb{N} \right\}, \tag{3.73}
\]
is tight in \( \mathcal{X} := \left(E \times [0, T]^{\mathbb{N}} \times C^\alpha(0, T; H^1) \right)^2 \), so that it is not restrictive to assume the existence of \( \{u^{m_\ell}_k, \zeta^{m_\ell}_k, W^{m_\ell}_k\}, \ell \geq 0 \), and \( \{u_k, \zeta_k, W_k\} \), random variables on \( \Omega' \), such that the conclusions of Corollary 2 hold with \( u'' \) instead of \( u' \).

Fixing \( k \in \mathbb{N} \), by (3.53)–(3.56) it is straightforward to show that the limits \( u^{m_\ell}_k, u''_k \) are martingale solutions on \([0, \zeta'_k] \), resp. \([0, \zeta''_k] \). Moreover, they are both supported in \( C([0, \kappa_k]; H^1) \cap L^2([0, \kappa_k]; H^2) \) where \( \kappa_k := \min(c''_k, \zeta''_k) > 0 \). It follows from relation (3.69) that \( u^{m_\ell}_k|_{[0, \kappa_k]} = u''_k|_{[0, \kappa_k]} \), and by reiteration we have also \( \zeta^{m_\ell}_k = \zeta''_k \), so that the weak limit of the sequence defined in (3.73) is supported in the diagonal of \( \mathcal{X} \). This gives in particular the convergence of the whole sequence \( (u_{n,k}, \zeta_{n,k})_{n \in \mathbb{N}} \) towards a strong solution \( u_k : \Omega \times [0, \zeta_k] \rightarrow L^2 \).

**Definition of \((u_*, \zeta_*)\).** The definition (3.3) implies that \( \zeta_{n,k} \leq \zeta_{n,k+1} \), \( \mathbb{P} \)-a.s., \( \forall n, k \in \mathbb{N} \). We can take the limit as \( n \rightarrow \infty \), so that for each \( k \):
\[
\zeta_k \leq \zeta_{k+1} \quad \mathbb{P} \text{-a.s.,} \tag{3.74}
\]
and the following definition is not ambiguous:
\[
(\text{u}_*(u_0))(t) := \begin{cases} u_k(t) & \text{if } t \in [0, \zeta_k) \text{ for some } k \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.75}
\]

This defines a local strong solution \( (u_*(u_0), \zeta_*(u_0)) \), where we let
\[
\zeta_*(u_0) := \sup_{k \in \mathbb{N}} \zeta_k. \tag{3.76}
\]
End of the Proof of Theorem 1. There remains to show (1.17), (1.18) and (1.19). We progressively state these properties through successive steps.

Step 1. Proof of (1.18). We show existence and uniqueness for the limit of \( \{ f_k := u_k(\zeta_k), k \in \mathbb{N} \} \) in \( L^2(\Omega; H^1) \)-weak. For \( k, p \in \mathbb{N} \), using the equation on \( u_k \) and \( u_{k+p} \) gives:

\[
\mathbb{E}[f_{k+p} - f_k]_{L^2}^2 \leq C \int_{\zeta_k}^{\zeta_{k+p}} \left| \Delta u_{k+p} + u_{k+p} |\nabla u_{k+p}|^2 + F_{\phi} u_{k+p} \right|_{L^2}^2 \, dt
+ C(|\phi|_{L^2}^2(\Omega)) \mathbb{E} \int_{\zeta_k}^{\zeta_{k+p}} |u_{k+p}|_{L^\infty}^2 \, dt \quad (3.77)
\]

Since the sequence \( \{ \zeta_k \} \) is bounded, by (3.74) we have \( \text{a.s.}-\lim_{k \to \infty} |\zeta_{k+p} - \zeta_k| = 0 \). Therefore, using (3.5), \( |u_{n,k}| = 1 \) a.e. and (3.77) gives that \( \{ f_k \}_{k \in \mathbb{N}} \) is a Cauchy sequence in \( L^2(\Omega \times \mathbb{T}^2) \). Its limit \( f = f(\omega, x) \) is in \( L^2(\Omega; H^1) \) by Prop. 5.

To prove (1.17) and (1.19), we first need to establish the fact that the singular points are finite, \( \mathbb{P}- \text{a.s.} \). We show in addition that during blow-up the solution releases a quantum of energy. This will be used in the proof of (1.17).

Step 2. Finiteness of the singular set. Denote by \( u := u_*(u_0) \) and by

\[
\text{Sg}(f) = \left\{ x \in \mathbb{T}^2, \exists x_k \to x, |\nabla u(\zeta_k)|_{L^2(B(x_k, \rho_k))} \geq \varepsilon_1 \text{ for all } k \right\}. \quad (3.78)
\]

Using the definition of \( \zeta_k \), for every proper family \( \{ x^i \}_{i \in I} \) there exist \( x_k^i \to x^i \) with \( \int_{B(x_k^i, \rho_k)} |\nabla u(\zeta_k)|^2 \geq \varepsilon_1 \). By semicontinuity of the norm with respect to weak convergence, for any fixed \( k \in \mathbb{N} \) we have:

\[
|\nabla f|_{L^2(\mathbb{T}^2)}^2 \leq \liminf_{p \to \infty} |\nabla f_p|_{L^2(\mathbb{T}^2 \cup \{ x^i_k, \rho_k \})}^2
\leq |\nabla f_k|_{L^2(\mathbb{T}^2 \cup B(x^i_k, \rho_k))}^2 \leq |\nabla f_k|_{L^2(\mathbb{T}^2)}^2 - \sum_{i \in I} |\nabla f_k|_{L^2(\mathbb{T}^2 \cup B(x^i_k, \rho_k))}^2 \quad (3.79)
\]

(we can assume without restriction that the balls \( B(x^i_k, \rho_k) \) are disjoint since \( x^i \neq x^j \) for \( i \neq j \)). The right hand side in (3.79) is bounded by \( |u(\zeta_k)|_{L^2(\mathbb{T}^2)} - (\# I) \varepsilon_1 \), and this holds for any \( k \in \mathbb{N} \). Taking the limit in (3.79) gives then

\[
|\nabla f|_{L^2(\mathbb{T}^2)} \leq \liminf_{k \to \infty} |\nabla f_k|_{L^2(\mathbb{T}^2 \cup B(x^i_k, \rho_k))}^2 \leq \liminf_{k \in \mathbb{N}} |\nabla f_k|_{L^2(\mathbb{T}^2)}^2 - (\# I) \varepsilon_1. \quad (3.80)
\]

(this implies in particular \( \# \text{Sg}(f) < \infty \)).

Step 3. Definition of the maximal solution. For \( m \in \mathbb{N}^* \), define a measurable process \( u : \Omega \times [0, T] \to H^1 \), and a stopping time \( \theta^m \) recursively by \( \{ u(0, \theta^1), \theta^1 := (u_*(u_0), \zeta_*(u_0)) \} \) the solution defined by (3.75)-(3.76) and \( \mathbb{P}- \text{a.s.} \). For \( m \geq 1 \):

\[
\begin{cases}
    u(\theta^m) := \lim_{t \to \theta^{m-1}} u_*(u^{m-1})(t) \text{ in } L^2(\Omega; H^1) \text{ weak}, \\
    \theta^{m+1} := \theta^m + \zeta_*(u(\theta^m)), \\
    u(t|_{[\theta^m, \theta^{m+1})}) := u_*(u(\theta^m))(t), \quad t \in [\theta^m, \theta^{m+1}).
\end{cases} \quad (3.81)
\]

This procedure can be repeated by the fact that the limit \( f \) is in \( L^2(\Omega; H^1) \) and is measurable with respect to \( \mathcal{F}_{\zeta_*} \).
Step 4. Proof of (1.17). To prove that the solution constructed above is global, we define the \( \mathbb{N} \cup \{ \infty \} \)-valued process

\[
N_t := \begin{cases} \# \{(x, s) \in \mathbb{T}^2 \times [0, t), \inf_{e \in \partial \{B(x, \eta)\}} |\nabla u(s - e, y)|^2 \, dy > 0\}, & \text{if } t \leq \sup_{m \in \mathbb{N}} \theta^m, \\ \infty & \text{if } t \in (\sup_{m \in \mathbb{N}} \theta^m, T] \end{cases}
\]  
(3.82)

By (3.80), we have

\[\mathbb{P}(\forall m \in \mathbb{N}, \ \theta^m < T) \leq \mathbb{P}(N_T = \infty).\]  
(3.83)

By (3.80), and Proposition 5 we see that

\[\mathbb{E}[|\nabla u(\theta^1)|^2] \leq \lim_{k \to \infty} |\nabla u(\zeta_k)|^2 - \epsilon_1 \mathbb{E}[N_1] \leq |\nabla u(0)|^2 + C(|\phi|_{L^2}) - \epsilon_1 \mathbb{E}[N_1],\]

and a straightforward induction implies that for each \( t \in [\theta^m, \theta^{m+1}) \): \( \mathbb{E}[|\nabla u(\theta^m)|^2 \leq |\nabla u(0)|^2 + C(|\phi|_{L^2}) t - \epsilon_1 \mathbb{E}[N_t] \), which finally gives the bound:

\[\mathbb{E}[N_T] \leq (\epsilon_1)^{-1} (|\nabla u(0)|^2 + C(\phi) T).\]  
(3.84)

The conclusion now follows from (3.83) and (3.84): we have \( \mathbb{P}(\forall m \in \mathbb{N}, \ \theta^m < T) = 0 \), and thus \( \mathbb{P}(\exists m \in \mathbb{N}, \ \theta^m = T) = 1 \).

This finishes the proof of Theorem 1.

\[\Box\]

4 Proof of Theorem 2

4.1 Treatment of the regular part of the solution

Let \((u, \mathfrak{B})\), denote a martingale solution in the sense of Definition 2. In order to prove theorem 2, we aim to decompose \( u \) into \( \hat{u} + v \), where \( v \) is the “singular part”. We first need to isolate the term in \( u|\nabla u|^2 \) that corresponds to possible degeneracies. Using that \( u \cdot \nabla u = 0 \), Heinlein’s decomposition writes for \( i = 1, 2, 3 \):

\[
u^i|\nabla u|^2 = \sum_{1 \leq j \leq 3, 1 \leq k \leq 2} (u^i \partial_k u^j - u^j \partial_k u^i) \partial_k u^j = \sum_{1 \leq j \leq 3, 1 \leq k \leq 2} \Lambda^{i,j}_k \partial_k u^j \equiv A: \nabla u^i,\]  
(4.1)

where from now on the double dots \( X:f \) will be used to denote the “collapse of the \((k, j)\) indices” of two tensors \((X^i_{k,j}) \in (\mathbb{R}^3)^{\otimes 2} \otimes \mathbb{R}^2\) and \((f^i_j) \in (\mathbb{R}^3)^{\otimes 2}\), namely

\[
X:f := \sum_{1 \leq j \leq 3, 1 \leq k \leq 2} X^i_{k,j} f^i_j \text{1}_{1 \leq i \leq 3}.
\]

We recall the following classical theorem for the decomposition of two-dimensional vector fields. The following version can be found in [11], as a consequence of Prop. 1 p. 215, and Prop. 3 p. 222.
Theorem 4 (Helmholtz). We have the orthogonal decomposition:

\[
L^2(\mathbb{T}^2; (\mathbb{R}^3)^{\otimes 2} \otimes \mathbb{R}^2) = \nabla H^1(\mathbb{T}^2; (\mathbb{R}^3)^{\otimes 2}) \oplus \nabla^\perp H^1(\mathbb{T}^2; (\mathbb{R}^3)^{\otimes 2}).
\] (4.2)

The corresponding projections are continuous in \( L^2 \).

Applying Theorem 4, we write for each \( t \in [0, T] \):

\[
A(t) = \nabla \alpha(t) + \nabla^\perp \beta(t),
\]

where \( A(t) \) is defined by (4.1) with \( u \) replaced by \( u(t) \equiv \) the trace of \( u \) onto \( \{t\} \times \mathbb{T}^2 \).

Taking the divergence, we obtain for each \( 1 \leq i, j \leq 3 \):

\[
\text{div} A^{i,j} = u^i \Delta u^j - u^j \Delta u^i,
\] (4.3)

and since \( |u^i \Delta u - \Delta u^i|_{L^2(\mathbb{R}^3)^{\otimes 2}}^2 = |\mathcal{T}u^i|_{L^2(\mathbb{R}^3)}^2 \), we have \( \| \text{div} A \|_{L^2([0, T]; L^2)} \leq C \| \mathcal{T}u \|_{L^2([0, T]; L^2)} \).

On the other hand since \( \text{div} A = \Delta \alpha \), we obtain that

\[
\| \Delta \alpha \|_{L^2([0, T] \times \mathbb{T}^2)} \leq C \| \mathcal{T}u \|_{L^2([0, T]; L^2)},
\] (4.4)

\( \mathbb{P} \)-a.s. Consider now the equation (with unknown \( \hat{u} \)):

\[
\begin{cases}
    d\hat{u} - \Delta \hat{u} \ dt = \nabla \alpha \nabla u \ dt + u \times \circ \ dW, & \text{on } \Omega \times [0, T] \times \mathbb{T}^2, \\
    \hat{u}|_{t=0} = u_0, & \text{on } \Omega \times \mathbb{T}^2.
\end{cases}
\] (4.5)

Note that \( \hat{u} \) solves (4.5) in the sense of distributions if and only if

\[
\hat{u} = u^\partial + u^\dagger + Z,
\] (4.6)

where respectively

\[
\begin{align*}
\partial_t u^\partial - \Delta u^\partial &= 0, \quad u^\partial(0) = u_0, \quad (4.7) \\
\partial_t u^\dagger - \Delta u^\dagger &= \nabla \alpha \nabla u, \quad u^\dagger(0) = 0, \quad (4.8) \\
dZ = \Delta Z \ dt + u \times \circ \ dW, \quad Z(0) = 0. \quad (4.9)
\end{align*}
\]

From (4.5), we can now deduce better regularity for \( \hat{u} \), namely:

Claim 3. With probability one, there exists a unique solution \( \hat{u} \) of (4.5) in \( L^4([0, T]; W^{1,4}) \).

Proof. By Proposition 4, there exists a unique weak solution \( Z \) to (4.9), given by \( Z(t) = \int_0^t S(t-s)F_\phi u \ ds + \int_0^t S(t-s) u \times \circ \ dW \) moreover we have:

\[
\mathbb{E} \| Z \|_{L^4([0, T]; W^{1,4})}^4 \leq C \mathbb{E} \| Z \|_{L^4([0, T]; H^{3/2})}^4 \leq C' T^4 \sup_{t \leq T} \left( \| u \times \phi \|_{L^1_2}^4 \right) \leq C'' T^4 (1 + \| \nabla \phi \|_{L^2}^4),
\]

using the cancellations in (3.8). Moreover, by Proposition 3 and \( L^{4/3} \hookrightarrow W^{-1,4} \):

\[
\| \int_0^t S(t-s)F_\phi u \|_{L^4([0, T]; W^{1,4})} \leq \| F_\phi u \|_{L^4([0, T]; L^{4/3})} \leq \sum_{\ell \in \mathbb{N}} \| \phi \ell \|_{L^{8/3}}^2 \leq C \| \phi \|_{L^2}^2.
\]

Therefore

\[
Z(\omega) \in L^4([0, T]; W^{1,4}), \quad \text{a.s.} \quad (4.10)
\]
On the other hand, Hölder Inequality, (2.1) and (4.4) yield
\[ \|\nabla \alpha \nabla u\|_{L^4 L^4} \leq \|\nabla \alpha\|_{L^4 L^4} \|\nabla u\|_{L^\infty L^2} \leq \|\mathcal{F}_u\|_{L^2 L^2} \|\nabla u\|_{L^\infty L^2}. \] (4.11)

By the embedding \( W^{1,4/3} \rightarrow L^4 \) we also have that \( \langle \nabla \alpha, \nabla u \rangle \in W^{-1,4} \), so that by Proposition 3 there exists a unique \( u^i \equiv \mathcal{Y}(\nabla \alpha, \nabla u) \in L^4([0, T]; W^{1,4}) \) such that (4.8) holds.

Finally, Proposition 3 yields existence and uniqueness of \( u^i \in C([0, T]; H^1) \cap L^2([0, T]; H^2) \) solving (4.7), which by (2.1) also belongs to \( L^4([0, T]; W^{1,4}) \).

This proves Claim 3.  

\[ \Box \]

### 4.2 Decomposition of “\( \nabla^\perp \beta \)”.

The previous paragraph shows that the symmetric part of \( A^{i,j} \equiv u^i \nabla u^j - u^j \nabla u^i \) is controlled by the bound on the tension \( \mathcal{F}_u \). This yields \( L^4 W^{1,4} \)-regularity for the renormalized solution \( \hat{u} \). Oppositely, the antisymmetric part \( \nabla^\perp \beta \nabla u \) can be singular, at least without further assumptions on \( u \). However, using that \( |A|^2 = 2|\nabla u|^2 \) we can write:

\[ \mathcal{G}(t) - \mathcal{G}(s) + C_\phi(t - s) = \frac{1}{4} \left( \|\nabla(\alpha(t) - \alpha(s))\|_{L^2}^2 + \|\nabla^\perp(\beta(t) - \beta(s))\|_{L^2}^2 \right) \] (4.12)

so that additional regularity can be provided by (local) continuity for \( t \mapsto \mathcal{G}(t) \). This can be obtained from the supermartingale property.

**Claim 4.** With full probability, \( t \in [0, T] \mapsto |\nabla^\perp \beta(t)|_{L^2}^2 \) is right continuous.

**Proof of Claim 4.** Let \( s \in [0, T] \). Define for \( p, n \in \mathbb{N} \), the set \( \mathcal{V}(p, n) = \{ \omega \in \Omega : \exists t_n(\omega) \in [s, s + (n + 1)^{-1}], |\mathcal{G}(t_n) - \mathcal{G}(s)| > (p + 1)^{-1} \} \). It is convenient to write that

\[ \{ \omega : t \mapsto \mathcal{G}(t) \text{ is not right-continuous at } t = s \} = \bigcup_{p \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \mathcal{V}(p, n) \] (4.13)

**Right continuity of \( \mathcal{G} \).** Reasoning by contradiction, assume that there exists \( p \in \mathbb{N} \) such that \( \Omega_p \equiv \bigcap_{n \in \mathbb{N}} \mathcal{V}(p, n) \) has positive probability. The Optional Sampling Theorem (see e.g. [46, Chap. I-6], implies

\[ \mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t_n) - \mathcal{G}(s)] \leq 0 \text{ a.s.} \] (4.14)

Moreover, classical facts on supermartingales (see e.g. [46, Thm. 6.8]) imply the existence of \( \overset{\rightarrow}{\mathcal{G}}(s) := \text{a.s.-lim}_{n \to \infty} \mathcal{G}(t_n) \). Note that by the right continuity assumption on \( (\mathcal{F}_t) \), the set \( \Omega_p \) is \( \mathcal{F}_s \) measurable. On the one’s hand, there holds

\[ \mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t_n) - \mathcal{G}(s)]|_{\Omega_p} = \mathbb{E}[\mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t_n) - \mathcal{G}(s)]|_{\Omega_p}] \]
\[ \leq \mathbb{E}[\mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t_n) - \mathcal{G}(s)]|_{\Omega_p}] \]
\[ \leq 0 \text{, by (4.14).} \] (4.15)

On the other hand \( \mathbb{P}(u \in C([0, T]; L^2)) = 1 \), therefore \( \nabla u(t_n) \rightharpoonup \nabla u(s) \text{ a.s. in } H^{-1} \), and since \( \nabla u(s) \in L^2(\Omega \times T^2) \), we have in fact

\[ \nabla u(t_n) \rightharpoonup \nabla u(s) \text{ weakly in } L^2(T^2), \text{ a.s.} \] (4.16)
By lower semicontinuity of the $L^2$--norm, we have $\mathcal{G}(s) \leq \bar{\mathcal{G}}(s) = \lim \mathcal{G}(t_n)$, a.s. On the other hand, since on $\Omega_p$ we have $|\mathcal{G}(t_n) - \mathcal{G}(s)| > 1/(p + 1)$ for all $n \geq 0$, it follows that

$$\mathds{1}_{\Omega_p} (\mathcal{G}(s) - \mathcal{G}(t_n)) = \frac{1}{p + 1}. $$

This lower bound, together with (4.15) and $\mathbb{P} (\Omega_p) > 0$, leads to a contradiction.

The right continuity of $\beta$ follows by (4.12).

### 4.3 Conclusion

To end the proof of Theorem 2, analogous arguments as that of the proof given in [20] will be used, although the important difference here is that we do not refer to the Struwe solution. This gives in addition a new proof of A. Frei's Theorem.

Denote by

$$v := u - \hat{u}, \quad (4.17)$$

and note that $v$ is a weak solution of

$$\partial_t v - \Delta v = \nabla^\perp \beta : \nabla \hat{u} + \nabla^\perp \beta^t : \nabla v, \quad v(0) = 0. \quad (4.18)$$

Let us point out that because of the cancellation that occurs in (4.17), we make no use of any stochastic argument here. Therefore in the next computations we will simply omit the sample, assuming $\omega \in \Omega \setminus \mathcal{N}, \mathcal{N}$ being a $\mathbb{P}$--null set such that $\hat{u}(\omega) \in L^4([0, T]; \mathbb{W}^{1,4})$ for $\omega \notin \mathcal{N}$.

By density of $C_{t,x}^\infty$ in $C((0, T]; H^1)$, Claim 4 yields that for all $\varepsilon > 0$ there exists $0 < \tau(\varepsilon) \leq T$ and $\beta_\varepsilon \in L^\infty((0, \tau]; H^1)$, $\beta^t_\varepsilon \in C^\infty((0, \tau] \times \mathbb{T}^2)$ with:

$$|\beta|_{[0, \tau]} = |\beta_\varepsilon| + |\beta^t_\varepsilon| \quad \text{and} \quad |\beta_\varepsilon|_{L^\infty([0, \tau]; H^1)} \leq \varepsilon. \quad (4.19)$$

Choosing $\varepsilon, \tau > 0$ as in (4.19), we let $g_\varepsilon := (\nabla^\perp \beta : \nabla \hat{u} + \nabla^\perp \beta^t_\varepsilon : \nabla v) |_{[0, \tau]}$. Using the abbreviation $L^p L^q := L^p([0, \tau]; L^q)$ for notational sake, immediate computations yield that $g_\varepsilon \in L^4 W^{-1,4}$, since $L^{4/3} \hookrightarrow W^{-1,4}$ and

$$\|g_\varepsilon\|_{L^4 W^{4/3}} \leq \|\nabla^\perp \beta\|_{L^\infty L^2} \|\nabla \hat{u}\|_{L^4 L^4} + C \|\nabla v\|_{L^\infty L^2} < \infty, \quad (4.20)$$

where the bound on $\hat{u}$ is justified by Claim 3.

In the sequel, we will denote by $\mathcal{Y}$ and $\mathcal{Y}'$ the bounded isomorphisms given by

$$\mathcal{Y} := \mathcal{Y}(\nabla^\perp \beta : \nabla \hat{u}, \nabla^\perp \beta^t_\varepsilon : \nabla v) \quad \text{and} \quad \mathcal{Y}' := \nabla^\perp \beta : \nabla \hat{u} + \nabla^\perp \beta^t_\varepsilon : \nabla v.$$

In (4.18), we replace now $v$ by the unknown $\Phi$ and write the corresponding equation first with the help of $\mathcal{Y}'$ as:

$$\Phi = \mathcal{Y}'(\nabla^\perp \beta : \nabla \Phi) = \mathcal{Y}' g_\varepsilon. \quad (4.21)$$

Letting $T_\varepsilon \mathcal{Y} := \mathcal{Y}'(\nabla^\perp \beta_\varepsilon : \nabla \Phi)$, the parabolic estimates, and the continuous embedding $W^{1,4/3} \hookrightarrow L^4$ give

$$\|T_\varepsilon \mathcal{Y}\|_{L^4 W^{4/3}} \lesssim \|\nabla^\perp \beta_\varepsilon : \nabla \Phi\|_{L^4 W^{-1,4}} \lesssim \|\nabla^\perp \beta_\varepsilon : \nabla \Phi\|_{L^4 L^{4/3}}. \quad (4.22)$$

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Hölder Inequality gives then \( \| T_\epsilon \Phi \|_{L^4 W^{1,4}} \leq C(\tau) \| \nabla_\perp \beta_\epsilon \|_{L^\infty L^2} \| \nabla \Phi \|_{L^4 L^4} \leq C(\tau) \epsilon \| \Phi \|_{L^4 W^{1,4}} \), with a constant depending on \( \tau \) but not on \( \tau \) (because the operator norm of \( \mathcal{V} \) increases with \( \tau \)). Taking \( \epsilon < C(\tau)^{-1} \) we have \( \| T_\epsilon \|_{L^4 W^{1,4}} < 1 \) yielding the convergence of the Neumann Series \( \sum_{n \leq N} (T_\epsilon)^n \rightarrow (\text{id} - T_\epsilon)^{-1} \). This gives the existence of a (unique) \( \Phi \in L^4 W^{1,4} \), solving (4.18). We do not know however at this stage whether \( \Phi \) equals \( \nu \).

Since the bound (4.20) yields also \( g_\epsilon \in L^2 H^{-1} \), the same reasoning as above, but with \( U : L^2 H^{-1} \rightarrow L^2 H^1 \cap H^1_0 H^{-1} \) instead of \( \mathcal{V} \), leads to the equation

\[
\Psi - \mathcal{V} (\nabla_\perp \beta_\epsilon; \nabla \Psi) = \mathcal{V} g_\epsilon,
\]

with unknown \( \Psi \) in \( L^2 H^1 \). We will now make use of the following “regularity by compensation” result.

**Theorem 5** ([47]). For \( a, b \in H^1(\mathbb{T}^2; \mathbb{R}) \), let \( \varphi \) be the unique solution of

\[
\varphi + \Delta \varphi = \{a, b\}
\]

on \( \mathbb{T}^2 \) where \( \{a, b\} \) denotes the Poisson bracket \( \partial_1 a \partial_2 b - \partial_2 a \partial_1 b \). Then \( \varphi \in C(\mathbb{T}^2; \mathbb{R}) \cap H^1(\mathbb{T}^2; \mathbb{R}) \), and

\[
|\varphi|_{L^\infty} + |\nabla \varphi|_{L^2} \leq C |\nabla a|_{L^2} |\nabla_\perp b|_{L^2},
\]

for a constant independent of \( \varphi \).

Denoting by \( \tilde{T}_\epsilon \Psi := \mathcal{V} (\nabla_\perp \beta_\epsilon; \nabla \Psi) \), the parabolic estimates, Theorem 5 and (4.19) give that for all \( \Psi \in L^2 H^1 \):

\[
\| \tilde{T}_\epsilon \Psi \|_{L^2 H^1} \leq C \| \beta_\epsilon, \Psi \|_{L^2 H^{-1}} \leq C(\tau) \epsilon \| \Psi \|_{L^2 H^1}.
\]

Assuming in addition \( \epsilon < \min(C(\tau), C'(\tau)) \), the same argument as above yields uniqueness of \( \Psi \) within the class \( L^2 H^1 \), solving (4.21). Since we already know that \( \nu \) belongs to this class (because \( u \) does), and since \( L^4 W^{1,4} \rightarrow L^2 H^1 \), we obtain that

\[
\nu = \Psi = \Phi
\]

and therefore

\[
u = \Psi = \Phi \]

\( \mathbb{P} \)-a.e.

Finally, the computations of paragraph 3.4 (see (3.69)) show that \( u \) is necessarily the Struwe solution constructed in Theorem 1. Theorem 2 is now proved. 

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