On-demand thermoelectric generation of equal-spin Cooper pairs

Felix Keidel,1 Sun-Yong Hwang,2 Björn Trauzettel,1,3 Björn Sothmann,2 and Pablo Burset4

1 Institute for Theoretical Physics and Astrophysics, University of Würzburg, D-97074 Würzburg, Germany
2 Theoretische Physik, Universität Duisburg-Essen and CENIDE, D-47048 Duisburg, Germany
3 Würzburg-Dresden Cluster of Excellence ct.qmat, Germany
4 Department of Applied Physics, Aalto University, 00076 Aalto, Finland

(Dated: July 9, 2019)

Superconducting spintronics is based on the creation of spin-triplet Cooper pairs in ferromagnet-superconductor (F-S) hybrid junctions. Previous proposals to manipulate spin-polarized supercurrents on-demand typically require the ability to carefully control magnetic materials. We, instead, propose a quantum heat engine that drives equal-spin supercurrents on-demand without the need for manipulating magnetic components. We consider a S-F-S junction, connecting two leads at different temperatures, on top of the helical edge of a two-dimensional topological insulator. The heat and charge currents generated by the thermal bias are caused by different transport processes, where electron cotunneling is responsible for the heat flow to the cold lead and, strikingly, only crossed Andreev reflections contribute to the charge current. Such a purely nonlocal Andreev thermoelectric effect generates a spin-polarized supercurrent between the superconductors that can be switched on/off by tuning their relative phase. We further demonstrate that the detection of the spin-triplet supercurrent is facilitated by rather low fluctuations of the thermoelectric current for temperature gradients comparable to the superconducting gap.

Introduction. — The new field of superconducting spintronics has emerged since the creation of spin-triplet Cooper pairs in experiments [1–3]. The development of spintronics had already benefited from the use of superconducting materials, resulting in longer spin lifetimes and energy-efficient components [4, 5]. Now, triplet supercurrents formed by spin-polarized Cooper pairs add the possibility of transporting a net spin component at zero resistance and thus pave the way for spintronic devices that are less liable to overheat [6–16]. The key challenge in the field is the nonequilibrium and on-demand generation of equal-spin Cooper pairs in a viable fashion [17–21], desirably avoiding the complicated manipulation of magnetic components.

In this Letter, we propose a thermoelectric engine that produces a spin-polarized supercurrent on demand from a temperature gradient. We consider a superconductor–ferromagnetic-insulator–superconductor (S-F-S) junction on top of the helical edge state of a quantum spin Hall insulator (QSHI) [22–28] connecting a hot and a cold bath, cf. Fig. 1(a). Only two microscopic transport processes couple the baths: quantum tunneling of electrons, known as electron cotunneling (EC), and crossed Andreev reflection (CAR). The QSHI edge states comprise one-dimensional Dirac fermions characterized by spin-momentum locking [29, 30]. Therefore, while EC amounts to a spin-polarized normal current, the peculiar transport properties of the helical edge states guarantee that CAR always converts electrons into holes with the same spin, creating equal-spin Cooper pairs at the superconductors [31–33]. Our key finding is that the heat-to-supercurrent conversion of this engine is near-perfect, with almost complete suppression of the normal particle current. This is only possible due to a unique interference effect for CAR processes in our setup. As sketched in Fig. 1(b), CAR requires a spin-flip process at the central ferromagnet and an Andreev reflection at either the left or the right superconductor. In an asymmetric junction, the different phases acquired in each path constitute interference, making CAR transmission strongly asymmetric in energy and thus creating an Andreev-dominated thermoelectric current, cf. Fig. 1(c,d).

Harvesting waste heat by quantum thermoelectric effects has become essential in modern nanoscale de-

![Figure 1](https://example.com/figure1.png)

**FIG. 1.** Quantum heat engine generating equal-spin Cooper pairs. (a) SFS junction on the helical edge of a QSHI connecting hot (L) and cold (R) baths. (b) Lowest order contributions to equal-spin CAR. (c) Transmission probabilities for electrons (T^ec_RL) and holes (T^he_RL), and difference in Fermi distributions δf for a temperature gradient θ. (d) Unequal distance between F and SL (dNL) and SR (dNR) results in an Andreev-dominated thermoelectric current I_R(θ).
proximity-induced superconductivity, and the de Gennes Hamiltonian in the Nambu basis \( \Psi \).

Conducting and ferromagnetic order by a Bogoliubov–de Gennes edge states of a QSHI in proximity to superconducting iron clusters has recently been reported in Ref. [59]. The observation of helical edge states \([26–28]\), hybrid structures like the emergence of Majorana bound states \([47–50]\) or exotic odd-frequency superconducting pairing \([32, 33, 51, 52]\).

Given recent advances in the experimental realization of helical edge states \([26–28]\), hybrid structures like the one sketched in Fig. 1(a) are within reach: superconductors \([53–56]\) have been successfully coupled to QSHIs \([57, 58]\), and monolayer QSHIs provide a new promising platform to induce ferromagnetic order \([28, 56]\). The observation of Majorana modes in helical hinge states of Bi(111) films under the influence of superconductivity and magnetic iron clusters has recently been reported in Ref. [59].

We theoretically describe the one-dimensional helical edge states of a QSHI in proximity to superconducting and ferromagnetic order by a Bogoliubov–de Gennes Hamiltonian in the Nambu basis \( \Psi(x) = (\psi_1, \psi_1^\dagger, \psi_2^\dagger, -\psi_2^\dagger) \) of the form \( (\hbar = v_F = 1) \)

\[
H_{\text{BdG}} = H_0 + H_S + H_F, \tag{1}
\]

with \( H_0 = \hat{p}_x \hat{\tau}_3 \hat{\sigma}_3 - \mu \hat{\tau}_3 \hat{\sigma}_0 \) the Hamiltonian of the free helical edge, \( H_S = \Delta(x) \cos \phi(x) \hat{\tau}_1 + \Delta(x) \sin \phi(x) \hat{\tau}_2 \hat{\sigma}_0 \) the proximity-induced superconductivity, and \( H_F = \hat{\tau}_0 \hat{\mathbf{m}}(x) \cdot \hat{\sigma} \equiv \hat{\tau}_0 (m_1 \cos \lambda \hat{\sigma}_1 + m_1 \sin \lambda \hat{\sigma}_2 + m_2 \hat{\sigma}_3) \) describing the effect of the ferromagnetic barrier. Here, \( \hat{p}_x = -i \hbar \partial_x \) and \( \hat{\sigma}_i (\hat{\tau}_i) \) are Pauli matrices acting in spin (Nambu) space.

We consider a system with two S regions (named SL and SR) separated by two normal regions (NL and NR) surrounding one ferromagnetic insulator (F): their respective widths are \( d_S \) for \( X \in \{\text{SL, NL, F, NR, SR}\} \). The pair potential is assumed equal for both superconductors and constant, \( \Delta(x) = \Delta_0 \), a valid approximation as long as the Fermi wavelength in each superconductor is much smaller than the proximity-induced coherence length. For simplicity, we take the phase of the pair potential \( \phi(x) = \phi \) in SR and zero otherwise. The F region is modeled by constant \( m_\parallel(x) = m_0 \) within F, and we choose \( m_\perp = 0 \) since its effect can be absorbed in the phase difference \( \phi \) between the superconductors \([33, 48, 60]\). Without loss of generality the angle \( \lambda \) is set to zero. Finally, we assume that all regions reside at the same chemical potential, \( \mu(x) = 0 \) everywhere.

In the following, we consider that all leads except L are at the same temperature \([61] (T_{\text{SL}} = T_{\text{SR}} = T_R = T_0) \) and set \( T_L = 0 + \theta \), introducing the temperature difference \( \theta \).

The electric current in the right lead after a temperature bias is applied to the left lead is given by \( I_R = I_{R}^{\text{ee}} + I_{R}^{\text{ec}} \), where \([62]\)

\[
I_{R}^{\text{ee}} = 2I_0 \int_{-\infty}^{\infty} \frac{dE}{\Delta_0} T_{R\ell}^{\text{ec}}(E) \delta f(E), \tag{2a}
\]

\[
I_{R}^{\text{ec}} = -2I_0 \int_{-\infty}^{\infty} \frac{dE}{\Delta_0} T_{R\ell}^{\text{ee}}(E) \delta f(E), \tag{2b}
\]

with \( T_{R\ell}^{\text{ec}} \) the CAR probability, \( T_{R\ell}^{\text{ee}} \) the EC probability, \( \theta = -\Delta_0 / \hbar, \delta f(E) = f[E, 0, \theta] - f(E, 0, 0) \), and \( f(E, \mu, \tau) = \{1 + \exp[(E - \mu) / \tau]\}^{-1} \) the Fermi distribution function. The probabilities are obtained by solving the scattering problem defined by the solutions of Eq. (1) in every region \([32, 33, 48, 63–68]\).

Helicity determines that particles arriving to the right lead have the same spin polarization as the injected particles on the left lead. While this does not restrict the quantum tunneling of electrons through the junction, \( i.e., \) EC processes, CAR processes are only possible if injected electrons and transmitted holes have the same spin \([31–33]\).

By breaking time-reversal symmetry, the F region facilitates equal-spin CAR processes. As sketched in Fig. 1(b), incident electrons can be transmitted as holes through the junction if at least one spin-flip process takes place at the F region and one Andreev reflection occurs at either superconductor. Crucially, scattering events involving an Andreev reflection at the right superconductor will acquire an extra phase \( \phi \) and a phase shift \( \theta \) compared to the ones where the reflection takes place at SL, which are only shifted by \( \theta \) (note that we set \( \hbar = v_F = 1 \)). The interference between these two processes is a unique property of CAR, not present in EC, resulting in an unusually strong asymmetry of the transmission probability with the energy, \( \phi \) Fig. 1(c).

Furthermore, since \( \delta f(E) \) is odd in \( E \), only the antisymmetric part of the transmissions will contribute to the charge current.

Resonant scattering at each S-F hybrid junctions always gives rise to zero-energy Majorana (quasi-)bound states, with additional finite energy Andreev states depending on the cavity’s width \([32, 33, 47, 48, 50]\). The hybridization between the bound states at each S-F cavity is controlled by the phase difference between the superconductors \([33, 60]\). This, in turn, allows for the control of the electric current through the junction.
As we describe in detail below, the interference of CAR processes leads to a particular thermoelectric effect, where the current can be completely dominated by equal-spin Andreev processes. At the same time, the energy current is given by 

\[ I_{\text{ec}}(E) = \frac{\gamma(E)}{2E} \sin[2E(d_{\text{NR}}-d_{\text{NL}})] \sin \phi. \]  

The sinusoidal behaviour of the current with \( \phi \) is shown in the inset of Fig. 2(a), revealing the phase difference \( \phi \) as an ideal knob to tune the thermoelectric effect. 

Increasing the temperature gradient drives larger thermoelectric currents [see Fig. 2 (a)], but also potentially larger fluctuations [71]. It is thus essential for the characterization of the proposed heat engine to identify a parameter regime where the fluctuations are the smallest with respect to the average current. That is, where the Fano factor \( F = S_{\text{RR}}/|2eI_R| \), with \( S_{\text{RR}} \) the current fluctuations in the right lead, is minimal. The zero-frequency fluctuation of \( I_R \) is given by 

\[ S_{\text{RR}} = eI_0 \int_{-\infty}^{\infty} \frac{dE}{\Delta_0} \sum_{\alpha,\beta} \text{sgn}(\alpha) \text{sgn}(\beta) \times A_{k\gamma,i\delta}(R\alpha,E)A_{l\delta,k\gamma}(R\beta,E) f_{k\gamma}(E) \left[ 1 - f_{l\delta}(E) \right] \]  

with 

\[ A_{k\gamma,i\delta}(i\alpha,E) = \delta_{i,k} \delta_{\delta,\gamma} \delta_{\alpha,\beta} - [s^\alpha_{ik}(E)]^* s^{\delta\beta}_{il}(E). \]
where Greek letters label Nambu indices, with $\text{sgn}(\alpha) = \pm 1$ for $\alpha = e, h$, Latin symbols represent reservoirs \{L, R, SL, SR\}, $s_{ik}^\gamma$ denotes the amplitude for a particle of type $\gamma$ in reservoir $k$ to be scattered into reservoir $i$ as a particle of type $\alpha$, and $f_{j\beta}(E) = f(E, \text{sgn}(\beta)\mu_j, k_BT_j)$ is the Fermi distribution for particles $\beta$ in reservoir $j$ [72].

After identifying the phase difference $\phi$ as the tuning parameter to control the thermoelectric current, it is striking to see that the fluctuations are almost independent of it, cf. Fig. 3(a). This indicates that they are mostly caused by thermal noise. Because of the carrier-selective heat and charge transfer in this setup, thermal noise is due to normal processes that do not experience interference and $S_{RR}$ increases steadily with the temperature bias $\theta$. By contrast, the Andreev-dominated current appears to saturate for higher bias, see Fig. 3(b). Importantly, when the current is maximum, the Fano factor becomes minimum, see Fig. 3(c), thus demonstrating that the current is enhanced over its fluctuations. Moreover, as the inset of Fig. 3(c) indicates, the minimum of the Fano factor is rather stable for temperature biases $\theta \geq T_c$. Note that in order to find an Andreev-dominated current we need $T_0$ to be sufficiently large ($T_0 \approx T_c/2$), which always leads to rather large Fano factors. We provide more details on the $T_0$-dependence of the noise in the supplemental material [62]. Recently, the electronic noise due to temperature differences in mesoscopic conductors, different than thermal or shot noise, was measured and proposed as an accurate temperature probe [73].

Finally, an Andreev-dominated thermoelectric current in R fulfills $I_L = -I_R$, with $I_L$ caused by local Andreev processes. Importantly, even though current conservation demands that the currents on the superconducting leads must be balanced, both $I_{SL}$ and $I_{SR}$ are nonzero for finite $\theta$ and fulfill $I_{SL}(\phi) = I_{SR}(-\phi)$, cf. inset in Fig. 3(b). In our setup, CAR processes due to a temperature gradient are only possible by the simultaneous creation of equal-spin Cooper pairs on one superconductor and their annihilation on the other. The temperature bias thus creates opposite sign supercurrents in the superconductors that could be measured as a finite thermophase in a setup similar to the one depicted in Fig. 1(a) [45]. Moreover, for a bias $\theta \sim T_c$ close to the minimum of fluctuations, the magnitude of the temperature-induced supercurrent is comparable to $I_0$, the zero-temperature maximum Josephson current with a typical value of $\sim 1 \mu A$. The sizeable spin-polarized thermoelectric current proposed here is thus within experimental reach and its detection and control should be accessible for temperature biases comparable to the superconducting gap.

**Summary.** — We propose a quantum heat engine that can be electrically controlled to drive spin-polarized supercurrents from a temperature bias on demand. Our proposal is based on a unique transport mechanism taking place at a S-F-S junction on the helical edge of a QSHI. Nonlocal Andreev processes through the junction experience an interference effect between the contributions from each superconductor. This interference is not present for normal processes, resulting in carrier-selective heat and charge currents where normal processes transfer heat and Andreev processes transfer charge. Due to the strong spin-orbit coupling at the helical edge state, the thermoelectric current is completely dominated by equal-spin Andreev processes. We discussed how the proposed spin-triplet thermoelectric effect could be measured as a thermophase appearing between the superconductors. The measurement is further facilitated by the low fluctuations of the spin-polarized nonlocal current.

The authors are grateful to M. Moskalets for valuable discussions. We acknowledge support from the DFG (SPP 1666 and SFB 1170, Project-ID 258499086), the Cluster of Excellence EXC 2147 (Project-ID 39085490), the Ministry of Innovation NRW via the “Programm zur Förderung der Rückkehr des hochqualifizierten Forschungsnachwuchses aus dem Ausland”, the Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant No. 743884 and the Academy of Finland (project 312299).
[1] Matthias Eschrig, “Spin-polarized supercurrents for spintronics,” Phys. Today 64, 43 (2010).
[2] Jacob Linder and Jason W.A. Robinson, “Superconducting spintronics,” Nat. Phys. 11, 307–315 (2015).
[3] Matthias Eschrig, “Spin-polarized supercurrents for spintronics: a review of current progress,” Reports on Progress in Physics 78, 104501 (2015).
[4] Hyunsoo Yang, See-Hun Yang, Saburo Takahashi, Sadamichi Maekawa, and Stuart S. P. Parkin, “Extremely long quasiparticle spin lifetimes in superconducting aluminium using nmg tunnel spin injectors,” Nature Materials 9, 586 (2010).
[5] F. Hübner, M. J. Wolf, D. Beckmann, and H. v. Löhneysen, “Long-range spin-polarized quasiparticle transport in mesoscopic al superconductors with a zeeman splitting,” Phys. Rev. Lett. 109, 207001 (2012).
[6] I. Sosnin, H. Cho, V. T. Petrashev, and A. F. Volkov, “Superconducting Phase Coherent Electron Transport in Proximity Conical Ferromagnets,” Phys. Rev. Lett. 96, 157002 (2006).
[7] R S Keizer, S T B Goennenwein, T M Klapwijk, G Miao, G Xiao, and A Gupta, “A spin triplet supercurrent through the half-metallic ferromagnet CrO2,” Nature 439, 825–827 (2006).
[8] J W A Robinson, J D S Witt, and M G Blamire, “Controlled Injection of Spin-Triplet Supercurrents into a Strong Ferromagnet,” Science 329, 59–61 (2010).
[9] J. W. A. Robinson, Gábor B. Halász, A. I. Buzdin, and M. G. Blamire, “Enhanced supercurrents in josephson junctions containing nonparallel ferromagnetic domains,” Phys. Rev. Lett. 104, 207001 (2010).
[10] Trupti S. Khaire, Mazin A. Khawasneh, W. P. Pratt, and Norman O. Birge, “Observation of Spin-Triplet Superconductivity in Co-Based Josephson Junctions,” Phys. Rev. Lett. 104, 137002 (2010).
[11] Carolin Klose, Trupti S. Khaire, Yixing Wang, W. P. Pratt, Norman O. Birge, B. J. McMorran, T. P. Ginley, J. A. Borchers, B. J. Kirby, B. B. Maranville, and J. Unguris, “Optimization of Spin-Triplet Supercurrent in Ferromagnetic Josephson Junctions,” Phys. Rev. Lett. 108, 127002 (2012).
[12] J. W. A. Robinson, F. Chiodi, M. Egilmez, Gábor B. Halász, and M. G. Blamire, “Supercurrent enhancement in Bloch domain walls,” Sci. Rep. 2, 699 (2012).
[13] J W A Robinson, N Banerjee, and M G Blamire, “Triplet pair correlations and nonmonotonic supercurrent decay with Cr thickness in Nb/Cr/Fe/Nb Josephson devices,” Phys. Rev. B 89, 104505 (2014).
[14] A. Srivastava, L. A. B. Olde Olthof, A. Di Bernardo, S. Komori, M. Amado, C. Palomares-Garcia, M. Allodoust, K. Halterman, M. G. Blamire, and J. W. A. Robinson, “Magnetization Control and Transfer of Spin-Polarized Cooper Pairs into a Half-Metal Manganite,” Phys. Rev. Appl. 8, 044008 (2017).
[15] Simon Diesch, Peter Machon, Michael Wolz, Christoph Sürgers, Detlef Beckmann, Wolfgang Belzig, and Elke Scheer, “Creation of equal-spin triplet superconductivity at the Al/EuS interface,” Nat. Commun. 9, 5248 (2018).
[16] Kun-rok Jeon, Chiara Ciccarelli, Andrew J Ferguson, Hidekazu Kurebayashi, Lesley F Cohen, Xavier Montiel, Matthias Eschrig, Jason W A Robinson, and Mark G Blamire, “Enhanced spin pumping into superconductors provides evidence for superconducting pure spin currents,” Nat. Mater. 17, 499–503 (2018).
[17] N Banerjee, J.W.A. Robinson, and M G Blamire, “Reversible control of spin-polarized supercurrents in ferromagnetic Josephson junctions,” Nat. Commun. 5, 4771 (2014).
[18] Marianne Etzelmüller Batthen and Jacob Linder, “Spin Seebeck effect and thermoelectric phenomena in superconducting hybrids with magnetic textures or spin-orbit coupling,” Sci. Rep. 7, 41409 (2017).
[19] Daniel Breunig, Pablo Burset, and Björn Trauzettel, “Creation of spin-triplet cooper pairs in the absence of magnetic ordering,” Phys. Rev. Lett. 120, 037701 (2018).
[20] Jabir Ali Ouassou, Jason W. A. Robinson, and Jacob Linder, “Controlling spin supercurrents via nonequilibrium spin injection,” (2018), arXiv:1810.08623.
[21] James Jun He, Kanta Hiroki, Keita Hamamoto, and Naoto Nagaosa, “Spin supercurrent in two-dimensional superconductors with Rashba spin-orbit interaction,” (2018), arXiv:1811.09057.
[22] Markus König, Steffen Wiedmann, Christoph Brüne, Andreas Roth, Hartmut Buhmann, Laurens W. Molenkamp, Xiao-Liang Qi, and Shou-Cheng Zhang, “Quantum Spin Hall Insulator State in HgTe Quantum Wells,” Science 318, 76–770 (2007).
[23] C. Brüne, A. Roth, E. G. Novik, M. Konig, H. Buhmann, E. M. Hankiewicz, W. Hanke, J. Sinova, and L. W. Molenkamp, “Evidence for the ballistic intrinsic spin Hall effect in HgTe nanostuctures,” Nature Physics 6, 448–454 (2010).
[24] Ivan Knez, Rui-Rui Du, and Gerard Sullivan, “Evidence for Helical Edge Modes in Inverted InAs/GaSb Quantum Wells,” Phys. Rev. Lett. 107, 136603 (2011).
[25] Christoph Brüne, Andreas Roth, Hartmut Buhmann, Evelina M. Hankiewicz, Laurens W. Molenkamp, Joseph Maciejko, Xiao-Liang Qi, and Shou-Cheng Zhang, “Spin polarization of the quantum spin Hall edge states,” Nat Phys 8, 485–490 (2012).
[26] F. Reis, G. Li, L. Dudy, M. Bauer, E. Glass, W. Hanke, R. Thomale, J. Schäfer, and R. Claessen, “Bismuthene on a SiC substrate: A candidate for a high-temperature quantum spin Hall material,” Science 357, 287–290 (2017).
[27] J. Kammhuber, M C. Cassidy, F. Pei, M. P. Nowak, A. Vuiik, Ö. Gül, D. Car, S. R. Plissard, E. P. A. M. Bakkers, M. Wimmer, and L. P. Kouwenhoven, “Conductance through a helical state in an Indium antimonide nanowire,” Nat. Commun. 8, 478 (2017).
[28] Sanfeng Wu, Valla Fatemi, Quinn D Gibson, Kenji Watanabe, Takashi Taniguchi, Robert J Cava, and Pablo Jarillo-Herrero, “Observation of the Quantum Spin Hall Effect up to 100 Kelvin in a Monolayer,” Science 359, 76–79 (2018).
[29] Congjun Wu, B Andrei Bernevig, and Shou-cheng Zhang, “Helical Liquid and the Edge of Quantum Spin Hall Systems,” Phys. Rev. Lett. 96, 106401 (2006).
[30] Cenke Xu and J. E. Moore, “Stability of the quantum spin Hall effect: Effects of interactions, disorder, and Z2 topology,” Phys. Rev. B 73, 045322 (2006).
[31] P. Adroguer, C. Grenier, D. Carpentier, J. Cayssol, P. Degiovanni, and E. Orignac, “Probing the helical edge states of a topological insulator by Cooper-pair in-
François Crépin, Pablo Burset, and Björn Trauzettel, “Odd-frequency triplet superconductivity at the helical edge of a topological insulator,” Phys. Rev. B 92, 100507 (2015).

Félix Keidel, Pablo Burset, and Björn Trauzettel, “Tunable hybridization of Majorana bound states at the quantum spin Hall edge,” Phys. Rev. B 97, 075408 (2018).

B Roche, P Roulleau, T Jullien, Y Jompol, I Farrer, D.A. Ritchie, and D.C. Glattli, “Harvesting dissipated energy with a mesoscopic ratchet,” Nat. Commun. 6, 6738 (2015).

P. Machon, M. Eschrig, and W. Belzig, “Nonlocal ThermoElectric Effects and Nonlocal Onsager relations in a Three-Terminal Proximity-Coupled Superconductor-Ferromagnet Device,” Phys. Rev. Lett. 110, 047002 (2013).

Mikhail S. Kalenkov and Andrei D. Zaikin, “Electron-hole imbalance and large thermoelectric effect in superconducting hybrids with spin-active interfaces,” Phys. Rev. B 90, 134502 (2014).

Mikhail S. Kalenkov and Andrei D. Zaikin, “Enhancement of thermoelectric effect in diffusive bilayers with magnetic interfaces,” Phys. Rev. B 91, 064504 (2015).

A. Ozaeta, P. Virtanen, F. S. Bergeret, and T. T. Heikkilä, “Predicted very large thermoelectric effect in ferromagnet-superconductor junctions in the presence of a spin-splitting magnetic field,” Phys. Rev. Lett. 112, 057001 (2014).

Sun-Yong Hwang, Rosa López, and David Sánchez, “Large thermoelectric power and figure of merit in a ferromagnetic-quantum dot-superconducting device,” Phys. Rev. B 94, 054506 (2016).

S. Kolenda, M. J. Wolf, and D. Beckmann, “Observation of Thermoelectric Currents in High-Field Superconductor-Ferromagnet Tunnel Junctions,” Phys. Rev. Lett. 116, 097001 (2016).

Razieh Beiranvand and Hossein Hamzehpour, “Spin-dependent thermoelectric effects in graphene-based superconductor junctions,” J. Appl. Phys. 121, 063903 (2017).

Zhan Cao, Tie-Feng Fang, Lin Li, and Hong-Gang Luo, “Thermoelectric-induced unitary Cooper pair splitting efficiency,” Applied Physics Letters 107, 212601 (2015).

Sun-Yong Hwang, Pablo Burset, and Björn Söthmann, “Odd-frequency superconductivity revealed by thermopower,” Phys. Rev. B 98, 161408(R) (2018).

J. Wiedenmann, E. Bocquillon, R. S. Deacon, S. Hartinger, O. Herrmann, T. M. Klapwijk, L. Maier, C. M. Armitage, C. Bré, J. J. Yoshida, K. Nakagawa, K. Ishibashi, T. Taniguchi, and N. Nagaosa, “Signature of Majorana bound states in the quantum spin Hall insulator HgTe,” Nature Mater. 7, 476–480 (2018).

T. Heikkilä, “Colloquium: Nonequilibrium Effects and Nonlocal Onsager relations in a Three-Terminal Proximity-Coupled Superconductor-Ferromagnet Device,” Phys. Rev. Lett. 110, 047002 (2013).

F. Giazotto, T. T. Heikkilä, and F. S. Bergeret, “Very large thermopower in ferromagnetic josephson junctions,” Phys. Rev. Lett. 114, 067001 (2015).

F. Giazotto, P. Solinas, A. Braggio, and F. S. Bergeret, “Ferromagnetic-Insulator-Based Superconducting Junctions as Sensitive Electron Thermometers,” Phys. Rev. Appl. 4, 044016 (2015).

Liang Fu and C. L. Kane, “Josephson current and noise at a superconductor/quantum-spin-hall-insulator/superconductor junction,” Phys. Rev. B 79, 161408(R) (2009).

François Crépin, Björn Trauzettel, and Fabrizio Dolcini, “Signature of Majorana bound states in transport properties of hybrid structures based on helical liquids,” Phys. Rev. B 89, 205115 (2014).

François Crépin and Björn Trauzettel, “Parity Measurement in Topological Josephson Junctions,” Phys. Rev. Lett. 112, 077002 (2014).

Christoph Fleckenstein, Felix Keidel, Björn Trauzettel, and Niccolo Traverso Ziani, “The invisible Majorana bound state at the helical edge,” Eur. Phys. J. Spec. Top. 227, 1377–1386 (2018).

Jorge Cayao and Annica M. Black-Schaffer, “Odd-frequency superconducting pairing and subgap density of states at the edge of a two-dimensional topological insulator without magnetism,” Phys. Rev. B 96, 155426 (2017).

C. Fleckenstein, N. Traverso Ziani, and B. Trauzettel, “Conductance signatures of odd-frequency superconductivity in quantum spin Hall systems using a quantum point contact,” Phys. Rev. B 97, 134523 (2018).

Ivan Knez, Rui-Rui Du, and Gerard Sullivan, “Andreev Reflection of Helical Edge Modes in InAs/Gr/Sb Quantum Spin Hall Insulator,” Phys. Rev. Lett. 109, 186603 (2012).

O. Herrmann, T. M. Klapwijk, L. Maier, C. M. Armitage, C. Bré, J. J. Yoshida, K. Nakagawa, K. Ishibashi, T. Taniguchi, and N. Nagaosa, “Signature of Majorana bound states in the quantum spin Hall insulator HgTe,” Nature Mater. 7, 476–480 (2018).

Valia Fatemi, Sanfeng Wu, Yuan Cao, Landry Bretheau, Quinn D. Gibson, Kenji Watanabe, Takashi Taniguchi, Robert J. Cava, and Pablo Jarillo-Herrero, “Electrically tunable low-density superconductivity in a monolayer topological insulator,” Science 362, 922–925 (2018).

J. Wiedenmann, E. Bocquillon, R. S. Deacon, S. Hartinger, O. Herrmann, T. M. Klapwijk, L. Maier, C. M. Armitage, C. Bré, J. J. Yoshida, K. Nakagawa, K. Ishibashi, T. Taniguchi, and N. Nagaosa, “Signature of Majorana bound states in the quantum spin Hall insulator HgTe,” Nature Nanotech 12, 137 (2017).

Berthold Jack, Yonglong Xie, Jian Li, Sangjun Jeon, B. Andrei Bernevig, and Ali Yazdani, “Observation of a Majorana zero mode in a topologically protected edge channel,” Science 1444, eaax1444 (2019).

Johan Nilsson, A. R. Akhmerov, and C. W. J. Beenakker, “Splitting of a Cooper Pair by a Pair of Majorana Bound States,” Phys. Rev. Lett. 101, 120403 (2008).

To account for finite temperatures in the superconductors, the gap is approximated as $\Delta(T) \approx \Delta_0 \tanh(1.74\sqrt{T_c/T - 1})$, with $T_c$ the critical temperature and $T$ the superconductor’s temperature.
[62] See Supplemental Material, which includes References [66, 73–77], at [URL will be inserted by publisher] for details of the derivation of the nonlocal current, the analysis of the CAR amplitude, and more information on the current fluctuations in the right lead.

[63] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, “Transition from metallic to tunneling regimes in superconducting microconstrictions: Excess current, charge imbalance, and supercurrent conversion,” Phys. Rev. B 25, 4515–4532 (1982).

[64] M. Büttiker, “Scattering theory of current and intensity noise correlations in conductors and wave guides,” Phys. Rev. B 46, 12485–12507 (1992).

[65] Yositake Takane and Hiromichi Ebisawa, “Conductance Formula for Mesoscopic Systems with a Superconducting Segment,” J. Phys. Soc. Japan 61, 1685–1690 (1992).

[66] M. P. Anantram and S. Datta, “Current fluctuations in mesoscopic systems with Andreev scattering,” Phys. Rev. B 53, 16390–16402 (1996).

[67] Yuli V. Nazarov and Yaroslav M. Blanter, Quantum Transport (Cambridge University Press, Cambridge, 2009).

[68] Pier A. Mello and Narendra Kumar, Quantum Transport in Mesoscopic Systems (Oxford University Press, 2004).

[69] Note that the location of the extremes of $\delta f$, and thus the width of the integration window in Eq. (2), are mainly determined by the base temperature $T_0$.

[70] Here, we discuss the case of a ferromagnetic insulator, as opposed to another recent proposal to use the edge states of a quantum spin Hall insulator for efficient thermoelectricity, Daniel Gresta, Mariano Real, and Liliana Ararachea, “Optimal thermoelectricity with quantum spin-Hall edge states,” arXiv:1904.12688 (2019).

[71] Sara Kheradsoud, Nastaran Dashti, Maciej Misiorny, Patrick P Potts, Janine Splettstoesser, and Peter Samuelsson, “Power, Efficiency and Fluctuations in a Quantum Point Contact as Steady-State Thermoelectric Heat Engine,” arXiv:1904.03912 (2019).

[72] Since we neglect quasiparticle injection in the superconductors, the sum over the reservoirs in Eq. (5) is effectively restricted to L and R.

[73] Ofir Shein Lumbroso, Lena Simine, Abraham Nitzan, Dvira Segal, and Oren Tal, “Electronic noise due to temperature differences in atomic-scale junctions,” Nature 562 (2018), 10.1038/s41586-018-0592-2.

[74] U. Sivan and Y. Imry, “Multichannel Landauer formula for thermoelectric transport with application to thermopower near the mobility edge,” Phys. Rev. B 33, 551–558 (1986).

[75] A. Bardas and D. Averin, “Peltier effect in normal-metal–superconductor microcontacts,” Phys. Rev. B 52, 12873–12877 (1995).

[76] Jacob Linder and Marianne Etzelmüller Bathen, “Spin caloritronics with superconductors: Enhanced thermoelectric effects, generalized onsager response-matrix, and thermal spin currents,” Phys. Rev. B 93, 224509 (2016).

[77] P. N. Butcher, “Thermal and electrical transport formalism for electronic microstructures with many terminals,” Journal of Physics: Condensed Matter 2, 4869 (1990).
In this supplemental material, we provide more details regarding the derivation of the expression for the current in the right lead and the crossed Andreev reflection amplitude. Furthermore, we show the behavior of the current fluctuations in the right lead as a function of the base temperature.

**Current in right lead**

In order to derive Eq. (2) of the main text, we follow the formalism developed in Refs. [74–77]. As a starting point, by virtue of the standard approach in mesoscopic physics [66] and to recapitulate the main text, the charge current on the right side of the setup \( I_R \) is given by

\[
I_R = \frac{e}{h} \int_{-\infty}^{\infty} dE \sum_{\alpha,\beta,j} \text{sgn}(\alpha) \left[ \delta_{R,j} \delta_{\alpha,\beta} - T_{R,j}^{\alpha\beta}(E) \right] f_{j\beta}(E),
\]

where Greek summation indices \( \alpha, \beta \in \{e, h, \} \) run over the electron/hole degree of freedom with \( \text{sgn}(\alpha) = \pm 1 \) for \( \alpha = e/h \), the Latin index \( j \in \{L, R, SL, SR\} \) runs over all reservoirs, \( T_{R,j}^{\alpha\beta} = |s_{R,j}^{\alpha\beta}|^2 \), with \( s_{R,j}^{\alpha\beta} \) the scattering amplitude for a particle of type \( \alpha \) in reservoir \( j \) to be scattered into reservoir \( R \) as a particle of type \( \beta \), and \( f_{j\beta}(E) = f(E, \text{sgn}(\beta)\mu_j, k_BT_j) \), with \( f(E, \mu, \tau) = \{1 + \exp[(E - \mu)/\tau]\}^{-1} \), is the Fermi distribution for particles \( \beta \) in reservoir \( j \).

The lengthy expression arising from performing the sum in Eq. (S1) can be simplified substantially. Using our assumption of grounded superconductors, \( \mu_{SR} = \mu_{SL} = 0 \), and equal superconductor temperatures, \( T_{SR} = T_{SL} = T_0 \), the Fermi functions in the superconductors coincide and \( f_{SL,e}(E) = f_{SL,h}(E) = f_{SR,e}(E) = f_{SR,h}(E) = f_0(E) = f(E, 0, k_BT_0) \) follows.

Furthermore, by employing unitarity of the scattering matrix and conservation of quasiparticle current, which implies

\[
\sum_{j,\beta} T_{ij}^{\alpha\beta} = \sum_{i,\alpha} T_{ij}^{\alpha\beta} = 1,
\]

we can eliminate the coefficients involving SL, SR and arrive at

\[
I_R = \frac{e}{h} \int_{-\infty}^{\infty} dE \left\{ \left[1 - T_{RR}^{ee}(E) + T_{RR}^{eh}(E)\right] [f_{Re}(E) - f_0(E)] - \left[1 - T_{RH}^{eh}(E) + T_{RR}^{ch}(E)\right] [f_{Rh}(E) - f_0(E)] 
+ \left[T_{RL}^{he}(E) - T_{RL}^{eh}(E)\right] [f_{Le}(E) - f_0(E)] - \left[T_{RL}^{ch}(E) - T_{RL}^{hh}(E)\right] [f_{Lh}(E) - f_0(E)] \right\}.
\]

Next, by recognizing that \( f_{he}(E) - f_0(E) = f_0(-E) - f_{eh}(-E) \) and that particle-hole symmetry enforces \( T_{ij}^{\alpha\beta}(E) = T_{ij}^{\beta\alpha}(-E) \) where \( \alpha = h, e \) if \( \alpha = e, h \), the terms in Eq. (S3) corresponding to the injection of holes can be folded back onto their charge conjugated counterparts, yielding

\[
I_R = \frac{2e}{h} \int_{-\infty}^{\infty} dE \left\{ \left[1 - T_{RR}^{ee}(E) + T_{RR}^{he}(E)\right] [f_{Re}(E) - f_0(E)] + \left[T_{RL}^{he}(E) - T_{RL}^{eh}(E)\right] [f_{Le}(E) - f_0(E)] \right\}.
\]

The first term in Eq. (S4) corresponds to local current contributions in the right lead resulting from injection from the right reservoir, while the second term in Eq. (S4) describes the charge current in the right lead rooting in CAR and EC processes of particles injected from the left reservoir.

We stress that for our choice of chemical potentials and temperatures Eq. (S4) is general for a four terminal setup with two superconducting leads, irrespective of the specific scattering problem at hand. Eq. (S4) holds even when including quasiparticle injection from and into the superconductors.

Importantly, since we assume equilibrium between the right reservoir and the superconductors, \( \mu_{SR} = T_0 \), the local current contribution vanishes identically. Therefore, the current in the right lead is a purely nonlocal effect and solely given by \( I_R = I_{R}^{he} + I_{R}^{ee} \), with

\[
I_{R}^{he} = 2I_0 \int_{-\infty}^{\infty} \frac{dE}{\Delta_0} T_{RL}^{he}(E) [f_{Le}(E) - f_0(E)],
\]

\[
I_{R}^{ee} = 2I_0 \int_{-\infty}^{\infty} \frac{dE}{\Delta_0} T_{RL}^{ee}(E) [f_{Le}(E) - f_0(E)].
\]
FIG. S 1. Sketch of the scattering region. Full (dashed) lines correspond to electrons (holes), whereas the color distinguishes spin up (green) from spin down (brown) modes. Additionally, the arrows indicate the direction of propagation. The shaded rectangles indicate the barriers, and the interfaces are numbered for clarity.

9

and

\[
I_{RL}^{ee} = -2I_0 \int_{-\infty}^{\infty} \frac{dE}{\Delta_0} T_{RL}^{ee}(E) \left[ f_{Le}(E) - f_0(E) \right],
\]

(S 5b)

where \( I_0 = e\Delta_0/h \). Upon identifying \( f_{Le}(E) - f_0(E) \equiv \delta f(E) \), Eqs. (S 5a) and (S 5b) correspond to Eqs. (2a) and (2b) of the main text.

The helicity of the QSHI edge states profoundly affects the resulting current. By convention, incoming and right-moving particles from the left and outgoing right-moving particles and holes on the right of the system must have spin \( \uparrow \). Consequently, the edge states act as a spin filter for nonlocally driven current.

Analysis of the crossed Andreev reflection amplitude

In this section of the supplementary material, we discuss the CAR amplitude in more detail and provide a derivation of Eqs. (3) and (4) of the main text. To that end, we decompose the full scattering problem into simpler pieces, namely the superconducting barriers SL and SR, the ferromagnetic region F, and the intermediate normal domains NL and NR. We follow the same approximations as stated in the main text. For the sake of readability we slightly change notation compared to the main text and denote transmissions and reflections as \( t_{ij}^{\alpha\beta} \) with \( i \neq j \) and \( r_{ii}^{\alpha\beta} \), respectively.

In order to write the S-matrix elements of the full system in terms of the scattering coefficients of the single constituents, we proceed as follows. Globally, the amplitudes of incoming and outgoing modes at the outmost interfaces (1) and (6) (see Fig. S 1) are related by the full scattering matrix \( S \) according to

\[
\begin{pmatrix}
    b_{e}^{(1)} \\
    b_{h}^{(1)} \\
    b_{e}^{(6)} \\
    b_{h}^{(6)}
\end{pmatrix}
= S
\begin{pmatrix}
    a_{e}^{(1)} \\
    a_{h}^{(1)} \\
    a_{e}^{(6)} \\
    a_{h}^{(6)}
\end{pmatrix},
\]

(S 6)

with

\[
S = \begin{pmatrix}
    R & T' \\
    T & R'
\end{pmatrix}
\quad \text{and} \quad
R = \begin{pmatrix}
    r_{ee}^{LL} & r_{eh}^{LL} \\
    r_{he}^{LL} & r_{hh}^{LL}
\end{pmatrix},
R' = \begin{pmatrix}
    r_{ee}^{RR} & r_{eh}^{RR} \\
    r_{he}^{RR} & r_{hh}^{RR}
\end{pmatrix},
T = \begin{pmatrix}
    t_{ee}^{RL} & t_{eh}^{RL} \\
    t_{he}^{RL} & t_{hh}^{RL}
\end{pmatrix},
T' = \begin{pmatrix}
    t_{ee}^{LR} & t_{eh}^{LR} \\
    t_{he}^{LR} & t_{hh}^{LR}
\end{pmatrix}.
\]

(S 7)

Here, \( a_{i}^{(i)} \) (\( b_{i}^{(i)} \)) with \( i = 1, 6 \) denotes the amplitude of an incoming (outgoing) mode at interface \( (i) \) of particle/hole type \( \alpha \). For the scattering coefficients of the total system we choose the convention that the right (left) sub- and superscript refers to the incoming (outgoing) particle, e.g. \( t_{he}^{RL} \) is the amplitude for an electron to be crossed Andreev reflected from left to right.
Within the system the amplitudes between positions \((i)\) and \((i+1)\) are related by scattering matrices associated with single S and F barriers or the intermediate NL and NR regions. We denote the amplitude of rightmovers (leftmovers) of type \(\alpha\) at interface \((i)\) with \(p_{\alpha}^{(i)}(m_{\alpha}^{(i)})\). Specifically, they are related by

\[
\begin{pmatrix}
    p_{\alpha}^{(1)} \\
    m_{\alpha}^{(2)} \\
    p_{\alpha}^{(2)} \\
    m_{\alpha}^{(3)}
\end{pmatrix}
= S_{\text{SL}}
\begin{pmatrix}
    a_{\alpha}^{(1)} \\
    a_{\alpha}^{(2)} \\
    m_{\alpha}^{(2)} \\
    m_{\alpha}^{(2)}
\end{pmatrix}
\quad \text{with} \quad S_{\text{SL}} =
\begin{pmatrix}
    0 & r_{\text{SL}}^{\text{eh}}(11) & t_{\text{SR}}^{\text{ee}}(21) & 0 \\
    r_{\text{SL}}^{\text{he}}(11) & 0 & t_{\text{SL}}^{\text{eh}}(21) & 0 \\
    t_{\text{SL}}^{\text{he}}(21) & 0 & 0 & r_{\text{SL}}^{\text{eh}}(22) \\
    r_{\text{SL}}^{\text{he}}(12) & 0 & t_{\text{SL}}^{\text{eh}}(22) & 0
\end{pmatrix},
\tag{S 8a}
\]

\[
\begin{pmatrix}
    m_{\alpha}^{(2)} \\
    a_{\alpha}^{(2)} \\
    p_{\alpha}^{(2)} \\
    m_{\alpha}^{(3)}
\end{pmatrix}
= S_{\text{NL}}
\begin{pmatrix}
    a_{\alpha}^{(1)} \\
    a_{\alpha}^{(2)} \\
    p_{\alpha}^{(2)} \\
    p_{\alpha}^{(3)}
\end{pmatrix}
\quad \text{with} \quad S_{\text{NL}} =
\begin{pmatrix}
    0 & 0 & t_{\text{NL}}^{\text{ee}}(23) & 0 \\
    0 & 0 & 0 & t_{\text{NL}}^{\text{eh}}(23) \\
    t_{\text{NL}}^{\text{ee}}(32) & 0 & 0 & 0 \\
    0 & 0 & t_{\text{NL}}^{\text{eh}}(32) & 0
\end{pmatrix},
\tag{S 8b}
\]

\[
\begin{pmatrix}
    m_{\alpha}^{(3)} \\
    a_{\alpha}^{(3)} \\
    p_{\alpha}^{(3)} \\
    p_{\alpha}^{(4)}
\end{pmatrix}
= S_{\text{F}}
\begin{pmatrix}
    a_{\alpha}^{(1)} \\
    a_{\alpha}^{(2)} \\
    p_{\alpha}^{(2)} \\
    p_{\alpha}^{(4)}
\end{pmatrix}
\quad \text{with} \quad S_{\text{F}} =
\begin{pmatrix}
    r_{\text{F}}^{\text{ee}}(33) & 0 & t_{\text{F}}^{\text{ee}}(34) & 0 \\
    0 & r_{\text{F}}^{\text{ee}}(34) & 0 & t_{\text{F}}^{\text{ee}}(34) \\
    r_{\text{F}}^{\text{eh}}(43) & 0 & r_{\text{F}}^{\text{eh}}(44) & 0 \\
    t_{\text{F}}^{\text{eh}}(43) & 0 & r_{\text{F}}^{\text{eh}}(44) & 0
\end{pmatrix},
\tag{S 8c}
\]

\[
\begin{pmatrix}
    a_{\alpha}^{(4)} \\
    m_{\alpha}^{(5)} \\
    r_{\alpha}^{(5)} \\
    m_{\alpha}^{(6)}
\end{pmatrix}
= S_{\text{NR}}
\begin{pmatrix}
    a_{\alpha}^{(1)} \\
    m_{\alpha}^{(2)} \\
    p_{\alpha}^{(4)} \\
    p_{\alpha}^{(5)}
\end{pmatrix}
\quad \text{with} \quad S_{\text{NR}} =
\begin{pmatrix}
    0 & 0 & t_{\text{NR}}^{\text{ee}}(45) & 0 \\
    0 & 0 & 0 & t_{\text{NR}}^{\text{eh}}(45) \\
    t_{\text{NR}}^{\text{ee}}(54) & 0 & 0 & 0 \\
    0 & t_{\text{NR}}^{\text{eh}}(54) & 0 & 0
\end{pmatrix},
\tag{S 8d}
\]

\[
\begin{pmatrix}
    p_{\alpha}^{(6)} \\
    m_{\alpha}^{(5)} \\
    r_{\alpha}^{(6)} \\
    a_{\alpha}^{(6)}
\end{pmatrix}
= S_{\text{SR}}
\begin{pmatrix}
    a_{\alpha}^{(1)} \\
    m_{\alpha}^{(2)} \\
    r_{\alpha}^{(5)} \\
    a_{\alpha}^{(6)}
\end{pmatrix}
\quad \text{with} \quad S_{\text{SR}} =
\begin{pmatrix}
    0 & r_{\text{SR}}^{\text{eh}}(55) & t_{\text{SR}}^{\text{ee}}(56) & 0 \\
    r_{\text{SR}}^{\text{he}}(55) & 0 & t_{\text{SR}}^{\text{eh}}(56) & 0 \\
    t_{\text{SR}}^{\text{ee}}(65) & 0 & 0 & r_{\text{SR}}^{\text{eh}}(66) \\
    r_{\text{SR}}^{\text{he}}(56) & 0 & r_{\text{SR}}^{\text{he}}(65) & 0
\end{pmatrix}.
\tag{S 8e}
\]

In Eqs. (S 8a) to (S 8e), \(t_{X(iii)}^{\alpha\beta}\) corresponds to a reflection process of a particle of type \(\beta\) into type \(\alpha\) at interface \((ii)\) of region X, whereas \(t_{X(ij)}^{\alpha\beta}\) represents a transmission of particle \(\alpha\) from interface \(j\) to \(i\) through region X with \(X \in \{\text{SL, NL, F, NR, SR}\}\).

The scattering problems need to be set up such that the scattering coefficients alone capture the phase shifts picked up due to propagation. Specifically, there are four solutions of the Bogoliubov–de Gennes Hamiltonian at each interface \((i)\) given by (note that we set \(h = v_F = 1\))

\[
\phi_+^\alpha(x) = e^{iE_x x} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \phi_-^\alpha(x) = e^{-iE_x x} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \phi_+^h(x) = e^{-iE_x x} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \phi_-^h(x) = e^{iE_x x} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\tag{S 9}
\]

corresponding to right- (+) and leftmoving (−) electrons and holes (\(e\) and \(h\), respectively). The scattering matrix relating interfaces \((i)\) and \((i+1)\) is then obtained by constructing scattering states out of the solutions \(\phi_\pm^\alpha(x - x_\langle i\rangle)\) and \(\phi_\pm^\alpha(x - x_\langle i+1\rangle)\), respectively, where \(x_\langle j\rangle\) denotes the location of interface \((j)\).

One can now use all 16 subequations of Eqs. (S 8a) to (S 8e) not involving the outgoing amplitudes \(b_{\alpha}^{(i)}\) in order to write the 16 coefficients \(p_{\alpha}^{(i)}, m_{\alpha}^{(i)}\) in terms of the ingoing amplitudes \(a_{\alpha}^{(i)}\). Subsequently, the remaining 4 linear equations can be used to relate the outgoing and ingoing amplitudes \(a_{\alpha}^{(i)}\) and \(a_{\alpha}^{(i)}\) in the form of Eq. (S 6) and we can read off \(t_{RL}^{b\alpha}\).
We obtain

$$t_{RL}^{he} = t_1 + t_2 = (t_1 + t_2) \sum_{n=0}^{\infty} \Sigma^n,$$

(S 10)

where we define self-energies as

$$\Sigma = \Sigma^L_1 + \Sigma^R_1 - \Sigma^L_1 \Sigma^R_1 + \Sigma^L_2 + \Sigma^R_2 - \Sigma^L_2 \Sigma^R_2 + \Sigma^S + \Sigma^h.$$

(S 11)

Before we provide the lengthy expressions for $t_1$, $t_2$ and $\Sigma$ in terms of the elements of the scattering matrices defined in Eq. (S 8), we first give a graphical explanation of Eq. (S 10) (see Fig. S 2). The CAR coefficient $t_{RL}^{he}$ is given by the sum of the two lowest order processes $t_1$ and $t_2$ necessary to convert an electron incoming from the left into a hole leaving the heterostructure to the right as also shown in the main text, augmented by the insertion of all possible closed loops, denoted $\Sigma^L_1$, $\Sigma^R_1$ and $\Sigma^L_3$.

The schematic paths in Fig. S 2 correspond to the expressions

$$t_1 = t_{ee}^{SL(21)}t_{ee}^{NL(32)}t_{SL(22)}^{he}t_{ee}^{NL(23)}t_{F(33)}^{he}t_{F(43)}^{RH}t_{F(54)}^{RH}t_{SR(65)}^{RH},$$

$$t_2 = t_{ee}^{SL(21)}t_{ee}^{NL(32)}t_{F(43)}^{he}t_{SR(55)}^{Rh}t_{F(44)}^{Rh}t_{F(54)}^{Rh}t_{SR(65)}^{Rh},$$

(S 12)

for the two lowest order paths for CAR, i.e., (i) transmission through SL, reflection at F, local Andreev reflection at SL under electron-hole conversion, and then transmission all the way through to the right lead ($t_1$); (ii) transmission through SL and F, local Andreev reflection at SR under electron-hole conversion, reflection at F followed by tunneling to the right lead ($t_2$),

$$\Sigma^L_1 = t_{ee}^{NL(32)}t_{F(33)}^{he}t_{SL(22)}^{he}t_{ee}^{NL(23)}t_{F(33)}^{he}t_{F(43)}^{Rh}t_{F(54)}^{Rh}t_{SR(65)}^{Rh},$$

$$\Sigma^R_1 = t_{ee}^{NL(23)}t_{F(33)}^{he}t_{F(54)}^{Rh}t_{SR(55)}^{Rh}t_{F(44)}^{Rh}t_{F(54)}^{Rh}t_{SR(65)}^{Rh},$$

(S 13)
for the two closed loops in the left (L) and right (R) S-F cavities,
\[ \Sigma^L_1 = i t_{NL(32)}^{ee} t_{F(43)}^{ee} t_{NR(54)}^{hh} t_{SL(22)}^{ee} t_{F(34)}^{ee} t_{SL(22)}^{hh}, \]
\[ \Sigma^L_2 = i t_{NL(32)}^{hh} t_{F(43)}^{hh} t_{NR(54)}^{hh} t_{SL(22)}^{ee} t_{F(34)}^{ee} t_{SL(22)}^{hh}. \]

(S14)

for the two closed loops between the SL and SR barriers - i.e., one involving spin-up electrons and spin-down holes (denoted with superscript ↑), and the other one built from spin-down electrons and spin-up holes (↓) - and finally
\[ \Sigma^L_3 = i t_{NL(32)}^{ee} t_{F(43)}^{ee} t_{SR(55)}^{hh} t_{NL(23)}^{hh} t_{SL(22)}^{ee} t_{SL(22)}^{hh}, \]
\[ \Sigma^R_3 = i t_{NL(32)}^{hh} t_{F(43)}^{hh} t_{SR(55)}^{hh} t_{NL(23)}^{hh} t_{SL(22)}^{ee} t_{SL(22)}^{hh}. \]

(S15)

for the loops between the S regions with an additional detour in each of the cavities (see Fig. S2). The superscript refers to the particle type along the long paths connecting the S regions.

In order to derive Eq. (3) of the main text, one can now analyze the energy dependence of \( \Sigma \) by making use of particle hole symmetry. The terms \( \Sigma^L_1 \) and \( \Sigma^R_1 \) involve all four modes and all possible reflections and thus are invariant under charge conjugation, i.e.,
\[ \Sigma^L/R(E) = (\Sigma^L/R(-E))^*, \]  
(S16)

which implies that the real and imaginary parts fulfill
\[ \text{Re} \left[ \Sigma^L/R(E) \right] = \text{Re} \left[ \Sigma^L/R(-E) \right], \quad \text{Im} \left[ \Sigma^L/R(E) \right] = -\text{Im} \left[ \Sigma^L/R(-E) \right]. \]
(S17)

By contrast, \( \Sigma^{h/l}_2 \) and \( \Sigma^{e/h}_3 \) are charge conjugated partners of one another, such that
\[ \Sigma^L_2(E) = \left( \Sigma^R_2(-E) \right)^*, \quad \Sigma^R_3(E) = \left( \Sigma^L_3(-E) \right)^*. \]
(S18)

and hence
\[ \text{Re} \left[ \Sigma^L_2(E) \right] = \text{Re} \left[ \Sigma^R_2(-E) \right], \quad \text{Im} \left[ \Sigma^L_2(E) \right] = -\text{Im} \left[ \Sigma^R_2(-E) \right], \]
\[ \text{Re} \left[ \Sigma^R_3(E) \right] = \text{Re} \left[ \Sigma^L_3(-E) \right], \quad \text{Im} \left[ \Sigma^R_3(E) \right] = -\text{Im} \left[ \Sigma^L_3(-E) \right]. \]
(S19)

Next, we consider the complex valued functions \( u(E), w(E), z(E) \) defined as
\[ u(E) = \Sigma^L_1(E) + \Sigma^R_1(E) - \Sigma^L_1(E) \Sigma^R_1(E), \]
\[ w(E) = \Sigma^L_2(E) + \Sigma^R_2(E) - \Sigma^L_2(E) \Sigma^R_2(E), \]
\[ z(E) = \Sigma^R_3(E) + \Sigma^L_3(E), \]
(S20)

such that \( u + w + z = \Sigma \). Importantly,
\[ u^*(E) = u(-E), \quad w^*(E) = w(-E), \quad z^*(E) = z(-E), \]
(S21)

because of Eqs. (S16) and (S18). Eq. (S21) immediately leads to the crucial result
\[ [\Sigma(E)]^* = \Sigma(-E), \]
(S22)

demonstrating that \( |\Sigma| \) is indeed even in \( E \). Going back to the full crossed Andreev reflection coefficient, we can rewrite Eq. (S10) as
\[ t_{RL}^{he} = \frac{1 - \Sigma^*}{1 + |\Sigma|^2 - \Sigma - \Sigma^*}, \]
(S23)

where the denominator \( d(E) = 1 + |\Sigma(E)|^2 - \Sigma(E) - (\Sigma(E))^* \) is real and obeys
\[ d^*(E) = d(-E) \quad \Rightarrow \quad |d(E)|^2 = |d(-E)|^2, \]
(S24)
while finally the numerator of the second term in Eq. (S 23) is \( n(E) = 1 - (\Sigma(E))^* \) with the property

\[
n^*(E) = n(-E) \quad \Rightarrow \quad |n(E)|^2 = |n(-E)|^2.
\] (S 25)

In conclusion, the modulus of the contribution of all higher order corrections given by \( 1/(1 - \Sigma) = d/n \) is indeed even in energy.

As the final step, we turn to the first term in Eq. (S 23) responsible for the interference effect. From Eq. (S 12), we first write

\[
t_1 + t_2 = t_{ee}^{SL(21)} t_{ee}^{NL(32)} t_{hh}^{NR(54)} t_{hh}^{SR(65)} \left( t_{ee}^{F(33)} t_{ee}^{he} t_{SL(22)}^{hh} t_{SL(22)}^{hh} t_{F(43)}^{he} t_{NR(54)}^{hh} t_{SR(55)}^{hh} t_{F(44)}^{hh} \right).
\] (S 26)

Now, using the explicit form of the coefficients we provide in Eq. (S 32) below, we have

\[
\begin{align*}
t_{ee}^{SL(21)} &= t_{hh}^{F(43)}; \quad r_{ee}^{F(43)} = t_{hh}^{F(44)}, \quad r_{he}^{S(55)} = e^{i\theta} r_{ee}^{S(22)}; \\
t_{ee}^{NL(32)} &= t_{hh}^{NL(32)} = e^{i\phi}; \quad t_{ee}^{NR(54)} = t_{hh}^{NR(54)} = e^{i\phi}.
\end{align*}
\] (S 27)

Thus, Eq. (S 26) simplifies to

\[
t_1 + t_2 = t_{ee}^{SL(21)} t_{ee}^{NL(32)} t_{hh}^{NR(54)} t_{hh}^{SL(22)} t_{ee}^{F(33)} t_{SL(22)}^{he} t_{F(43)}^{he} t_{SL(22)}^{ee} (e^{2i\phi} + e^{2i\phi}) ,
\] (S 28)

which is of the form

\[
t_1 + t_2 = 2|t(E)| e^{i\pi(E)} \cos(\Delta \phi/2).
\] (S 29)

Here, we have introduced the phase difference between the two paths \( \Delta \phi = \phi + 2(d_{NR} - d_{SL}) E \), an unimportant global phase \( \tau \), and the modulus of a product of scattering coefficients fulfilling \( |t(E)| = |t(-E)| \) (see below). Combining Eqs. (S 23) to (S 25) and (S 29), we thus show that the absolute squared of the CAR coefficient can be written in the form

\[
|t_{RL}^{he}|^2 = T_{RL}^{he}(E) = \gamma(E, \phi) \cos^2 [\phi/2 + (d_{NR} - d_{SL}) E],
\] (S 30)

where \( \gamma(E, \phi) \) is even in \( E \) and given by (we have restored the dependence on the phase difference \( \phi \))

\[
\gamma(E, \phi) = \frac{4|t(E, \phi)|^2 |n(E, \phi)|^2}{|d(E, \phi)|^2},
\] (S 31)
as stated in Eq. (3) in the main text.

In order to show that \( |t(E)| \) is indeed even in \( E \), we define \( A_S(E) = \text{arccosh}(E/\Delta) \), \( A_F(E) = \text{arccosh}(E/m_\parallel) \) and

\[
\begin{align*}
\Omega_S(E) &= \begin{cases} 
\sqrt{E^2 - \Delta^2} & \text{for } E > \Delta \\
-i\sqrt{E^2 - \Delta^2} & \text{for } -\Delta < E < \Delta \\
-\sqrt{E^2 - \Delta^2} & \text{for } E < -\Delta 
\end{cases} \\
\Omega_F(E) &= \begin{cases} 
\sqrt{E^2 - m_\parallel^2} & \text{for } E > m_\parallel \\
i\sqrt{m_\parallel^2 - E^2} & \text{for } -m_\parallel < E < m_\parallel \\
-\sqrt{E^2 - m_\parallel^2} & \text{for } E < -m_\parallel 
\end{cases}.
\end{align*}
\]

The coefficients are then found to be (together with Eq. (S 27))

\[
\begin{align*}
t_{ee}^{SL(21)}(E) &= \frac{\sinh [A_S(E)]}{\sinh [A_S(E) - i\phi] \Omega_S(E)}, \quad r_{he}^{SL(22)}(E) = -i \frac{\sinh [A_S(E)]}{\sinh [A_S(E) - i\phi] \Omega_S(E)}, \\
t_{hh}^{SL(65)}(E) &= \frac{\sinh [A_S(E)]}{\sinh [A_S(E) - i\phi] \Omega_S(E)}, \quad r_{he}^{F(33)}(E) = -i \frac{\sinh [A_F(E)]}{\sinh [A_F(E) - i\phi] \Omega_F(E)}; \\
t_{ee}^{F(43)}(E) &= \frac{\sinh [A_F(E)]}{\sinh [A_F(E) - i\phi] \Omega_F(E)}, \quad t_{ee}^{NL(32)}(E) = e^{i\phi} E, \quad t_{hh}^{NR(54)}(E) = e^{i\phi} E.
\end{align*}
\] (S 32)

From Eq. (S 32) one can check that the modulus of all coefficients is an even function of \( E \), and thus \( |t(E)| \) must be even as well.
Dependence of the noise in the right lead on the base temperature

In the main text, we focus on the behavior of the current fluctuations and the Fano factor as a function of the temperature difference $\theta$ at fixed base temperature $T_0$. Here, we provide some information about the dependence on $T_0$. We fix the phase difference between the superconductors to be at the optimal operating value $\phi = \pi/2$. The temperature difference is set to be $\theta = 2T_0$ (plotted in cyan) and $\theta = T_c$ (orange). Note that at $T_0 = T_c/2$, both cases coincide and we also recover the results of the main text.

Fig. S3 (a) shows the noise in the right lead $S_{RR}$. For all choices for the temperature gradient $\theta$ the fluctuations do not grow monotonically up to $T_0 = T_c$, but reach a maximum below that value. This is connected to the fact that the scattering problem and thus the conductance change dramatically when the two superconducting barriers are removed. Note that there are several contributions to the noise in setups like this, namely an equilibrium contribution due to finite temperatures, a non-equilibrium contribution due to a temperature gradient and, additionally, a contribution due to the thermoelectrically created nonlocal charge current. The interplay of these sources of fluctuations has been carefully studied in [73]. In Fig. S3 (b), we plot the total current in the right lead produced by the temperature gradient. For smaller values of $T_0$, the current for $\theta = T_c$ is of course much larger than for $\theta = 2T_0$, but for both cases it is not dominated by Andreev contributions below $T_0 \approx T_c/2$. The quite sharp drop of the $\theta = T_c$ case for $T_0$ smaller than $T_c/2$ is mostly due to the vanishing of the normal contribution. Finally, Fig. S3 (c) displays the Fano factor $F = S_{RR}/|2eI_R|$ resulting from noise and average current in Fig. S3 (a) and (b). For small base temperatures, Fano factors below 10 are possible. However, note that a dominating Andreev contribution to the current requires $T_0$ to be of the order of $T_c/2$, which increases the Fano factor at the optimal working point of our device.