On dendrites, generated by polyhedral systems and their ramification points.

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Abstract

The paper considers systems of contraction similarities in \( \mathbb{R}^d \) sending a given polyhedron \( P \) to polyhedra \( P_i \subset P \), whose non-empty intersections are singletons and contain the common vertices of those polyhedra, while the intersection hypergraph of the system is acyclic. It is proved that the attractor \( K \) of such system is a dendrite in \( \mathbb{R}^d \).

The ramification points of such dendrite have finite order whose upper bound depends only on the polyhedron \( P \), and the set of the cut points of the dendrite \( K \) is equal to the dimension of the whole \( K \) iff \( K \) is a Jordan arc.

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1 Introduction

Though the study of topological properties of dendrites from the viewpoint of general topology proceed for more than three quarters of a century \[5,11,12\], the attempts to study the geometrical properties of self-similar dendrites are rather fragmentary.

In 1985 M. Hata \[8\] studied the connectedness properties of self-similar sets and proved that if a dendrite is an attractor of a system of weak contractions in a complete metric space, then the set of its endpoints is infinite.

In 1990 Ch. Bandt showed in his unpublished paper \[2\] that the Jordan

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arcs connecting pairs of points of a post-critically finite self-similar dendrite are self-similar, and the set of possible values for dimensions of such arcs is finite. Jun Kigami in his work [9] applied the methods of harmonic calculus on fractals to dendrites; on a way to this he developed effective approaches to the study of structure of self-similar dendrites. D. Croydon in his thesis [6] obtained heat kernel estimates for continuum random tree and for certain family of p.c.f. random dendrites on the plane. D. Dumitru and A. Mihail [7] made an attempt to get a sufficient condition for a self-similar set to be a dendrite in terms of sequences of intersection graphs for the refinements of the system $S$.

There are many papers [4, 3, 15] discussing examples of self-similar dendrites, but systematic approach to the study of self-similar requires to find the answer to the following questions: What kind of topological restrictions characterise the class of dendrites generated by systems of similarities in $\mathbb{R}^d$? What are the explicit construction algorithms for self-similar dendrites? What are the metric and analytic properties of morphisms of self-similar structures on dendrites?

To approach these questions, we start from simplest and most obvious settings, which were used by many authors [2, 13]. We consider systems $S$ of contraction similarities in $\mathbb{R}^d$ defined by some polyhedron $P \subset \mathbb{R}^d$, which we call contractible $P$-polyhedral systems.

We prove that the attractor of such system $S$ is a dendrite $K$ in $\mathbb{R}^d$ (Theorem 14), and there is a dense subset of $K$ such that punctured neighbourhoods of its points split to a finite disjoint union of subsets of solid angles $\Omega_l$, equal to the solid angles of $P$ (Theorem 10); we show that the orders of points $x \in K$ have an upper bound, depending only on $P$ (Theorem 20); and that Hausdorff dimension of the set $CP(K)$ of the cut points of $K$ is strictly smaller than the dimension of the set $EP(K)$ of its end points unless $K$ is a Jordan arc (Theorem 21).

1.1 Preliminaries

Dendrites. A dendrite is a locally connected continuum containing no simple closed curve.

In the case of dendrites the order $Ord(p, X)$ of the point $p$ with respect to $X$ is equal to the number of components of the set $X \setminus \{p\}$. the points of order 1 are called end points in $X$, and cut points are called usual points if $Ord(p, X) = 2$ and ramification points, if $Ord(p, X) \geq 3$.

We will use the following statements selected from [3, Theorem 1.1]:

**Theorem 1** For a continuum $X$ the following conditions are equivalent:
(a) $X$ is dendrite;
(b) every two distinct points of $X$ are separated by a third point;
(c) each point of $X$ is either a cut point or an end point of $X$;
(d) each nondegenerate subcontinuum of $X$ contains uncountably many cut points of $X$.
(e) for each point $p \in X$ the number of components of the set $X \setminus \{p\} = \text{ord}(p,X)$ whenever either of these is finite;
(f) the intersection of every two connected subsets of $X$ is connected;
(g) $X$ is locally connected and uniquely arcwise connected.

**Self-similar sets.** Let $(X, d)$ be a complete metric space. A mapping $F : X \to X$ is a contraction if $\text{Lip} F < 1$. The mapping $S : X \to X$ is called a similarity if
\[ d(S(x), S(y)) = rd(x,y) \]
for all $x, y \in X$ and some fixed $r$.

**Definition 2** Let $S = \{S_1, S_2, \ldots, S_m\}$ be a system of (injective) contraction maps on the complete metric space $(X, d)$. A nonempty compact set $K \subset X$ is said to be invariant with respect to $S$, if $K = \bigcup_{i=1}^{m} S_i(K)$.

We also call the subset $K \subset X$ self-similar with respect to $S$.

Throughout the whole paper, the maps $S_i \in S$ are supposed to be similarities and the set $X$ to be $\mathbb{R}^d$.

**Notation.** $I = \{1, 2, \ldots, m\}$ is the set of indices, $I^* = \bigcup_{n=1}^{\infty} I^n$ - is the set of all finite $I$-tuples, or multiindices $j = j_1 j_2 \ldots j_n$, where $ij$ is the concatenation of the corresponding multiindices;
we say $i \sqsubset j$, if $i = i_1 \ldots i_n$ is the initial segment in $j = j_1 \ldots j_{n+k}$ or $j = ik$ for some $k \in I^*$; if $i \not\sqsubset j$ and $j \not\sqsubset i$, $i$ and $j$ are incomparable;
we write $S_j = S_{j_1 j_2 \ldots j_n} = S_{j_1} S_{j_2} \ldots S_{j_n}$ and for the set $A \subset X$ we denote $S_j(A)$ by $A_j$; we also denote by $G_S = \{S_j, j \in I^*\}$ the semigroup, generated by $S$;
$I^\infty = \{\alpha = \alpha_1 \alpha_2 \ldots, \ \alpha_i \in I\}$ - index space; and $\pi : I^\infty \to K$ is the index map, which sends $\alpha$ to the point $\bigcap_{n=1}^{\infty} K_{\alpha_1 \ldots \alpha_n}$.

**Definition 3** The system $S$ satisfies the open set condition (OSC) if there exists a non-empty open set $O \subset X$ such that $S_i(O), \{1 \leq i \leq m\}$ are pairwise disjoint and all contained in $O$.
We say the self-similar set $K$ defined by the system $S$ satisfies the one-point intersection property if for any $i \neq j$, $S_i(K) \cap S_j(K)$ is not more than one point.

We use the following convenient criterion of connectedness of the attractor of a system $S$ [8, 10].

**Theorem 4** Let $K$ be the attractor of a system of contractions $S$ in a complete metric space $(X,d)$. Then the following statements are equivalent:

1) for any $i, j \in I$ there are $\{i_0, i_1, \ldots, i_n\} \subset I$ such that $i_0 = i, i_n = j$ and $S_{i_k}(K) \cap S_{i_{k+1}}(K) \neq \emptyset$ for any $k = 0, 1, \ldots, n - 1$.

2) $K$ is arcwise connected.

3) $K$ is connected.

**Proposition 5** If a self-similar set $K$ is connected, it is locally connected.

**Zippers and multizippers.** The simplest way to construct a self-similar curve is to take a polygonal line and then replace each of its segments by a smaller copy of the same polygonal line; this construction is called zipper and was studied by Aseev, Tetenov and Kravchenko [1].

**Definition 6** Let $X$ be a complete metric space. A system $S = \{S_1, \ldots, S_m\}$ of contraction mappings of $X$ to itself is called a zipper with vertices $\{z_0, \ldots, z_m\}$ and signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m), \varepsilon_i \in \{0, 1\}$, if for $i = 1 \ldots m$,

$$S_i(z_i) = z_{i-1} + \varepsilon_i, \quad S_i(z_{m-i}) = z_{m-i-1},$$

More general approach for building self-similar curves and continua is provided by a graph-directed version of zipper construction [14]:

**Definition 7** Let $\{X_u, u \in V\}$ be a system of spaces, all isomorphic to $\mathbb{R}^d$. For each $X_u$ let a finite array of points be given $\{x^{(u)}_0, \ldots, x^{(u)}_{m_u}\}$. Suppose for each $u \in V$ and $0 \leq k \leq m_u$ we have some $v(u,k) \in V$ and $\varepsilon(u,k) \in \{0, 1\}$ and a map $S^{(u)}_k : X_u \to X_u$ such that $S^{(u)}_k(x^{(v)}_0) = x^{(u)}_{k-1}$ or $x^{(u)}_k$ and $S^{(u)}_k(x^{(v)}_{m_v}) = x^{(u)}_{k}$ or $x^{(u)}_{k-1}$, depending on the signature $\varepsilon(u,k)$.

The graph directed iterated function system (IFS) defined by the maps $S^{(u)}_k$ is called a multizipper $Z$.

The attractor of multizipper $Z$ is a system of connected and arcwise connected compact sets $K_u \subset X_u$ satisfying the system of equations

$$K_u = \bigcup_{k=1}^{m_u} S^{(u)}_k(K_{v(u,k)}), \quad u \in V.$$
We call the sets $K_u$ the components of the attractor of $\mathcal{Z}$.

The components $K_u$ of the attractor of $\mathcal{Z}$ are Jordan arcs if the following conditions are satisfied:

**Theorem 8** Let $\mathcal{Z}_0 = \{S_k^{(u)}\}$ be a multizipper with node points $x_k^{(u)}$ and a signature $\varepsilon = \{(v(u,k),\varepsilon(u,k)), u \in V, k = 1, \ldots, m_u\}$. If for any $u \in V$ and any $i, j \in \{1, 2, \ldots, m_u\}$, the set $K_{(u,i)} \cap K_{(u,j)} = \emptyset$ if $|i - j| > 1$ and is a singleton if $|i - j| = 1$, then any linear parametrization $\{f_u : I_u \to K_u\}$ is a homeomorphism and each $K_u$ is a Jordan arc with endpoints $x_0^{(u)}$, $x_m^{(u)}$.

2 Contractible polyhedral systems.

Let $P \subset \mathbb{R}^d$ be a finite polyhedron homeomorphic to a $d$-dimensional ball and let $V_P = \{A_1, \ldots, A_{n_P}\}$ be the set of its vertices, and $\Omega(P, A_i)$ be the solid angles at the vertices of $P$.

Consider a system of similarities $S = \{S_1, \ldots, S_m\}$, which define polyhedra $P_i = S_i(P)$ and satisfy the following conditions:

(D1) For any $i \in I$, $P_i \subset P$;

(D2) For any $i \neq j$, $i, j \in I$, the intersection $P_i \cap P_j$ is either empty, or is a common vertex of $P_i$ and $P_j$;

(D3) $V_P \subset \bigcup_{i \in I} S_i(V_P)$;

(D4) The set $\tilde{P} = \bigcup_{i=1}^m P_i$ is contractible.

**Definition 9** A system $S$, satisfying D1-D4, is called $P$-polyhedral system of similarities.

The similarities $S_i \in S$ are contractions, therefore the system $S$ has the attractor $K$; the system $S$ generates the semigroup $G_S = \{S_j, j \in I^*\}$ and therefore defines the set of polyhedra $G_S(P) = \{P_j, j \in I^*\}$. The properties of this system of refining polyhedra define the geometric properties of the invariant set $K$. First we focus on those properties, which follow from D1—D3 only, which corresponds to a class of point connected self-similar sets, as they were defined by R.Strichartz [13]. The relative position of solid angles of polyhedra $P_j$ will be our special interest:

**Theorem 10** Let $S$ be a $P$-polyhedral system of similarities.

(a) The system $S$ satisfies (OSC).

(b) $P_j \subset P_i$ iff $j \supset i$. 

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(c) If $i \subset j$, then $S_i(V_P) \cap P_j \subset S_j(V_P)$.
(d) For any incomparable $i, j \in I^*$, $\#(P_i \cap P_j) \leq 1$ and $P_i \cap P_j = S_i(V_P) \cap S_j(V_P)$.
(e) The set $G_S(V_P)$ of vertices of polyhedra $P_j$ is contained in $K$.
(f) If $x \in K\setminus G_S(V_P)$, then $\#\pi^{-1}(x) = 1$.

Proof: (a) It follows from D1, D2 that the interior of $P$ is the desired open set for (OSC); (b) follows from (OSC):

(c) Notice that D2, D3 imply the condition (D3a): for any $i \in I$, $P_i \cap V_P \subset S_i(V_P)$:

Indeed, if $x \in P \setminus V_P$ and $S_i(x) = A \in V_P$, then, since there is $j \in I$, such that $A \in S_j(V_P)$, $P_i \cap P_j \notin S_i(V_P)$, which contradicts D3.

Using induction, we derive from D3a, that for any $k \in I^*$, $P_k \cap V_P \subset S_k(V_P)$:

Let now $j = ik$ and $A \in S_i(V_P) \cap S_k(P_k)$. It means that $S_i^{-1}(A) \in V_P \cap P_k$, and therefore $S_i^{-1}(A) \subset S_k(V_P)$, or $A \in S_i(V_P)$.

(d) Represent a pair of incomparable multiindices as $ik, kj$, where $i_1 \neq j_1$.

Since $P_{ki} \cap P_{kj} \neq \emptyset$, $P_i \cap P_j \neq \emptyset$. But $P_i \cap P_j \subset P_{ki} \cap P_{kj}$. The last set is nonempty and therefore it consists solely of a common vertex of $P_i$ and $P_j$; by (c), this point is also a common vertex of $P_i$ and $P_j$; therefore $P_{ki} \cap P_{kj} = S_{ki}(V_P) \cap S_{kj}(V_P)$.

(e) For any vertex $A \in V_P$ there are $A_1 \in V_P$ and $\alpha_1 \in I$ such that $S_{\alpha_1}(A_1) = A$. By induction we get that for any $n$ there are such $A_n \in V_P$ and $\alpha_1 \ldots \alpha_n \in I^n$, that $S_{\alpha_1 \ldots \alpha_n}(A_n) = A$. In this case, $\bigcap_{n=1}^\infty S_{\alpha_1 \ldots \alpha_n}(P) = \{A\}$ and $A \in K$. Thus, $V_P \subset K$, and therefore $G_S(V_P) \subset K$.

(f) If $\pi^{-1}(x)$ contains non-equal $\alpha, \beta \in I^\infty$, then for some $n, \alpha_1 \ldots \alpha_n$ and $\beta_1 \ldots \beta_n$ are incomparable; therefore $x \in P_{\alpha_1 \ldots \alpha_n} \cap P_{\beta_1 \ldots \beta_n}$, so $x \in G_S(V_P)$.

(g) First let $\alpha \in I^\infty$ and $\pi(\alpha) = A \in V_P$. As in (e), for any $n$, $S_{\alpha_1 \ldots \alpha_n}(A_n) = A$ and $S_{\alpha_1 \ldots \alpha_n}(\Omega(P, A_n)) \subset \Omega(P, A)$. Moreover, the solid angles $S_{\alpha_1 \ldots \alpha_n}(\Omega(P, A_n))$ form a nested sequence. Since the set $\{\Omega(P, B), B \in V_P\}$ is finite, there is a solid angle $\Omega_\alpha$ and a number $N \in \mathbb{N}$, such that if $n > N$, then $S_{\alpha_1 \ldots \alpha_n}(\Omega(P, A_n)) = \Omega_\alpha$. At the same time, $S_{\alpha_1 \ldots \alpha_n}(P) \subset \Omega_\alpha$. If for some $\beta \in I^\infty, \beta \neq \alpha, \pi(\beta) = A$, then, according to (d), $\Omega_\alpha \cap \Omega_\beta = \{A\}$. Thus, the set $\pi^{-1}(A)$ can be mapped bijectively to the family of disjoint solid angles $\Omega_k$ with common vertex $A$. 

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The measure $\theta(\Omega_k)$ is greater or equal to $\theta_{\min} = \min\{\theta(\Omega(P,A)), A \in V_P\}$, therefore the set of different $\alpha \in I^\infty$ such that $\pi(\alpha) = A$, does not exceed $\theta(\Omega(P,A))/\theta_{\min}$, if $A \in V_P$, and $\theta_F/\theta_{\min}$, if $A \in \tilde{P}$, where $\theta_F$ is the measure of complete solid angle in $\mathbb{R}^d$. \hfill \blacksquare

Now we discuss some properties of $S$ which follow from the condition D4.

Applying Hutchinson operator $T(A) = \bigcup_{i \in I} S_i(A)$ of the system $S$ to the polyhedron $P$, we get the set $\tilde{P} = \bigcup_{i \in I} P_i$. We define $\tilde{P}^{(1)} = T(P)$, $\tilde{P}^{(n+1)} = T(\tilde{P}^{(n)})$. Thus we get a nested family of sets $\tilde{P}^{(1)} \supset \tilde{P}^{(2)} \supset \ldots \supset \tilde{P}^{(n)} \supset \ldots$, whose intersection is $K$.

The composition of two contractible $P$-polyhedral systems is also of the same type:

**Lemma 11** Let $S$ and $S'$ be contractible $P$-polyhedral systems of similarities. Then $S'' = \{ S_i \circ S'_j, S_i \in S, S_j \in S' \}$ is also contractible $P$-polyhedral system of similarities.

**Proof:** (D1) is obvious since $S_i \circ S'_j(P) \subset S_i(P) \subset P$.

(D2) Let $Q_1 = S_{i_1} \circ S'_{j_1}(P)$ and $Q_2 = S_{i_2} \circ S'_{j_2}(P)$ be two polyhedra for the system $S''$; consider their intersection:
if $i_1 \neq i_2$, then $Q_1 \cap Q_2 \subset P_{i_1} \cap P_{i_2}$, where the right part is either empty, or for some $A_1, A_2 \in V_P$, $P_{i_1} \cap P_{i_2} = \{ S_{i_1}(A_1) \} = \{ S_{i_2}(A_2) \}$. Since $A_1 \in S'_{j_1}(V_P)$, $A_2 \in S'_{j_2}(V_P)$, $Q_1 \cap Q_2 = S_{i_1} \circ S'_{j_1}(V_P) \cap S_{i_2} \circ S'_{j_2}(V_P)$; if $i_1 = i_2$, then $Q_1 \cap Q_2 = S_{i_2}(P_{j_1} \cap P_{j_2})$ where the right part is either empty or a singleton contained in $S'_{j_1}(V_P) \cap S'_{j_2}(V_P)$.

(D3) holds, because for any vertex $A \in V_P$, there are $A_1 \in V_P$ and $S_{i_1} \in S$ such that $S_{i_1}(A_1) = A$; also there are $S'_{i_2} \in S'$ and $A_2 \in V_P$ such that $S'_{i_2}(A_2) = A_1$; therefore $S_{i_1} \circ S'_{i_2}(A_2) = A$. If $x \in P$ and $S_{i_1} \circ S'_{i_2}(x) = A$, then $S'_{i_2}(x) \in V_P$, therefore $x \in V_P$.

(D4) The sets $\tilde{P} = \bigcup_{i=1}^{m} P_i$ and $\tilde{P}' = \bigcup_{i=1}^{m'} P'_i$ are strong deformation retracts of $P$, containing the set $V_P$. Let $\varphi'(X,t) : P \times [0,1] \to P$ be the deformation retraction from $P$ to $\bigcup_{i=1}^{m'} P'_i$. The map $\varphi'$ satisfies the following conditions: $\varphi'(x,0) = Id$, $\varphi'(x,1)(P) = \tilde{P}'$ and for any $t \in [0,1]$, $\varphi'(x,t)|_{\tilde{P}'} = Id_{\tilde{P}'}$.

Define the map $\varphi'_i : P_i \times [0,1] \to P_i$ by a formula $\varphi'_i(x,t) = S_i \circ \varphi'(S_i^{-1}(x),t)$. 

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Each map $\varphi'_i$ is a deformation retraction from $P_i$ to $S_i(P')$.
Observe that the map $\varphi'_i$ keeps all the vertices $S_i(A_k)$ of the polyhedron $P_i$ fixed. Therefore we can define a deformation retraction $\tilde{\varphi}(x,t) : \tilde{P} \times [0,1] \to \bigcup_{i=1}^m S_i(P') = \tilde{P}$ by a formula

$$\tilde{\varphi}(x,t) = \varphi'_i(x,t), \quad \text{if } x \in P_i$$

The map $\tilde{\varphi}$ is well-defined and continuous because if $P_i \cap P_j = \{S_i(A_k)\} = \{S_j(A_l)\}$ for some $k$ and $l$, then $\varphi'_i(S_i(A_k),t) \equiv \varphi'_j(S_j(A_l),t) \equiv S_i(A_k)$.
Moreover, $\tilde{\varphi}(x,0) = x$ on $\tilde{P}$, and $\tilde{\varphi}(\tilde{P},1) \equiv \bigcup_{i=1}^m S_i(P')$ and $\tilde{\varphi}(x,t)|_{P''} \equiv Id$.
So $\tilde{\varphi}(x,t)$ is a deformation retraction from $\tilde{P}$ to $P''$.
Therefore, the set $P'' = \bigcup S_i \circ S'_j(P)$ is contractible.\[\blacksquare\]

**Corollary 12** If $S$ is a contractible $P$-polyhedral system, the same is true for $S^{(n)} = \{S_j, j \in P^n\}$. \[\blacksquare\]

The contractibility of the set $\tilde{P}$ and the condition **D2** imply, that any simple closed curve in $\tilde{P}$ lies in one of the polyhedra $P_i$; this can be derived from the following Lemma:

**Lemma 13** Let $B_i, i = 1, \ldots, n$ be a finite family of topological balls, such that for any $i, j$ the intersection $B_i \cap B_j$ contains no more than one point and the set $X = \bigcup_{i=1}^n B_i$ is simply-connected. Then any simple closed curve in $X$ lies in some $B_i$.

**Proof:** Choose in each of $B_i$ a point $O_i \in \dot{B}_i$ and for each $\{p_{ij}\} = B_i \cap B_j$ take a Jordan arc $\gamma_{ij}$ with endpoints $O_i$ and $p_{ij}$ so that $\gamma_{ij} \cap \gamma_{ij'} = \{O_i\}$ if $j' \neq j$. Let $\Gamma$ be a topological graph with vertices $O_i, i = 1, \ldots, n$ and $p_{ij}$ whose edges are $\gamma_{ij}$. Since for any $i$ the union $\bigcup_{j} \gamma_{ij}$ is a strong deformation retract of the ball $B_i$, $\Gamma$ is a strong deformation retract of the set $X$ and therefore the graph $\Gamma$ is a tree.
Let $l$ be some Jordan arc in $X$. Suppose $l$ is in general position in the sense that $p_{ij} \in l$ if $l \cap \dot{B}_i \neq \varnothing$ and $l \cap \dot{B}_j \neq \varnothing$. Each point $p_{ij}$ splits $X$ to no less than 2 components. Therefore if $l \ni p_{ij}$, the arc $l$ is not closed. Thus, any simple closed curve in $X$ lies completely in one of the balls $B_i$. \[\blacksquare\]

**Theorem 14** The attractor $K$ of contractible $P$-polyhedral system of similarities $S$ is a dendrite.
Proof: By Corollary 12 the sets $\tilde{P}^{(n)}$ are contractible, compact and satisfy $\tilde{P}^{(1)} \supset \tilde{P}^{(2)} \supset \tilde{P}^{(3)} \ldots$. The diameter of components of the interior of any of $\tilde{P}^{(n)}$ does not exceed $\text{diam} P \cdot q^n$, where $q = \max \text{Lip}(S_i)$. Thus the set $K = \bigcap \tilde{P}^{(n)}$ is connected and has empty interior. Since $K$ is connected, it is locally connected and arcwise connected [10], Theorem 1.6.2, Proposition 1.6.4.

Let $l$ be some Jordan arc in $K$. For any $n \in \mathbb{N}$, $l \subset \tilde{P}^{(n)}$, so it follows from Lemma 13 that if $l$ has non-zero diameter, it is not closed. Therefore $K$ is a dendrite. ■

The dendrite $K$ is contained in the polyhedron $P$; in general, its intersection with the boundary of $P$ may be uncountable or it can contain even some whole edges of $P$. The same is also true for the intersection of the dendrite $K$ with each polyhedron $S_j(P_j), j \in I^*$. Nevertheless it follows from D2 that a subcontinuum $L \subset K$ can "penetrate" to a polyhedron $S_j(P_j)$ only through its vertices, namely:

**Proposition 15** Let $j \in I^*$ be a multiindex. For any continuum $L \subset K$, whose intersection with both $P_j$ and its exterior $\tilde{C}P_j$ is nonempty, the set $L \setminus \tilde{P}_j \cap P_j \subset S_j(V_P)$.

**Proof:** Observe that for any polyhedron $P_j, j \in I^k$ the set $\tilde{P}^{(k)} \setminus S_j(V_P)$ is disconnected, and $P_j \setminus S_j(V_P)$ is its connected component, whose intersection with $K$ is equal to $S_j(K \setminus S_j(V_P))$. Therefore $L \setminus S_j(V_P)$ is also disconnected. ■

### 2.1 The main tree and ramification points

Since $K$ is a dendrite, by Theorem 1 for any vertices $A_i, A_j \in V_P$ there is unique Jordan arc $\gamma_{ij} \subset K$ connecting $A_i, A_j$. As it was proved by C. Bandt [2], these arcs are the components of the attractor of a graph-directed system of similarities. We show that this system is a Jordan multizipper [14]:

**Theorem 16** The arcs $\gamma_{ij}$ are the components of the attractor of some Jordan multizipper $\mathcal{Z}$.

**Proof:** We say, that polyhedra $P_{i_1}, \ldots, P_{i_s}, i_k \in I$ form a chain, connecting $x$ and $y$, if $x \in P_{i_1}, y \in P_{i_s}$ and the intersection $P_{i_k} \cap P_{i_l}$ is empty, if $|l - k| > 1$ and is a common vertex of $P_k$ and $P_l$, if $|l - k| = 1$.

For the vertices $A_i, A_j$, there is unique chain of polyhedra in the $P$-polyhedral system $\mathcal{S}$, which consists of those $P_k$, for which $\#P_k \cap \gamma_{ij} \geq 2$; we denote the polyhedra forming the chain and corresponding maps as $P'_{ij} = S'_{ij}(P), k = 1, \ldots, m_{ij}$, keeping in mind that all $S'_{ijk} \in \mathcal{S}$. 

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Let \( u(i, j, k) \) \( v(i, j, k) \) be the indices of vertices \( P \), for which
\[
S'_{ijk}(A_u) = P'_{ij(k-1)} \cap P'_{ij(k+1)} = z_{ijk}, \quad \text{if } 1 < k < m_{ij},
\]
and if \( k = 1 \) or \( k = m_{ij} \), \( u(i, j, 1) = A_i = z_{ij0} \) and \( v(i, j, m_{ij}) = A_j = z_{ijm_{ij}} \).

Thus for any triple \((i, j, k), 1 \leq k \leq m_{ij}\), such \( u, v \in \{1, \ldots, n_P\} \) are specified, that \( S'_{ijk}(z_{uv0}) = z_{ij(k-1)} \) and \( S'_{ijk}(z_{uvm_{uv}}) = z_{ijk} \).

Therefore the system \( \{S'_{ijk}\} \) is a multizipper \( Z \) with node points \( z_{ijk} \).

Since the relations:
\[
\gamma_{ij} = \bigcup_{i=1}^{m_{ij}} S'_{ijk}(\gamma_{u(i,j,k), v(i,j,k)}) = \bigcup_{i=1}^{m_{ij}} \gamma_{ijk}
\]
are satisfied, the arcs \( \gamma_{ij} \) form a complete set of the components of the attractor of the multizipper \( Z \).

Since each arc \( \gamma_{ijk} \) lies in \( P_{ijk} \),
\[
\gamma_{ijk} \cap \gamma_{ijl} = \emptyset,
\]
if \( |k - l| > 1 \) and
\[
\gamma_{ijk} \cap \gamma_{ijl} = \{z_{ijk}\},
\]
and \( l = k \pm 1 \). Therefore \( Z \) satisfies the conditions of the Theorem \( \S \) and is a Jordan multizipper. \( \blacksquare \)

The set \( \hat{\gamma} = \bigcup_{i \neq j} \gamma_{ij} \) is a subcontinuum of the dendrite \( K \) and therefore is a dendrite. Since all the end points of \( \hat{\gamma} \) are contained in \( V_P \), \( \hat{\gamma} \) is a finite dendrite or topological tree [5, A.17]. Let \( n_E \) be the number of end points of \( \hat{\gamma} \). As it was pointed out by Kigami [9], \( \hat{\gamma} \) may be represented as union of at most \((n_E - 1)\) Jordan arcs having disjoint interiors.

**Definition 17** The union \( \hat{\gamma} = \bigcup_{i \neq j} \gamma_{ij} \) is called the main tree of the dendrite \( K \). The ramification points of \( \hat{\gamma} \) are called main ramification points of the dendrite \( K \).

The following statement establishes the relations between the sets of vertices \( V_P \), end points \( EP(\hat{\gamma}) \) and cut points \( CP(\hat{\gamma}) \) of the main tree \( \hat{\gamma} \):

**Proposition 18** Let \( x \in K \).

(a) \( \hat{\gamma} \subset \bigcup_{A_j \in V_P} \gamma_{A_j x} \); besides, if \( \hat{\gamma} = \bigcup_{A_j \in V_P} \gamma_{A_j x} \), then \( x \in \hat{\gamma} \);
(b) \( EP(\hat{\gamma}) = V_P \setminus CP(\hat{\gamma}) \);
(c) \( x \in CP(\hat{\gamma}) \) iff there are vertices \( A_i, A_j \), belonging to different components of \( K \setminus \{x\} \);
(d) for \( x \in CP(K) \), \( Ord(x, K) = Ord(x, \hat{\gamma}) \) iff for any component \( C_i \) of the set \( K \setminus \{x\} \), \( C_i \cap V_P \neq \emptyset \).

**Proof:** For any \( A_i, A_j \in V_P \), \( \gamma_{A_iA_j} \subset \gamma_{A_i} \cup \gamma_{A_j} \), which gives (a). To get (b), notice that if \( x \in \hat{\gamma} \) is not a vertex then \( x \) is the inner point of some arc \( \gamma_{A_iA_j} \), therefore it is a cut point of \( \hat{\gamma} \) and therefore \( x \notin EP(\hat{\gamma}) \).

(c): Since \( \gamma_{A_i} \cap \gamma_{A_j} = \{x\} \), we have \( \gamma_{A_i} \cup \gamma_{A_j} = \gamma_{A_iA_j} \). So \( x \) is a cut point of \( \gamma_{A_iA_j} \), and therefore of \( \hat{\gamma} \).

(d): Necessity is obvious, so we prove sufficiency. By (c), \( x \in CP(\hat{\gamma}) \). The number of components of \( K \setminus \{x\} \) is no greater than \( n_P \), so \( Ord(x, K) \) is finite. Let \( C_i, l = 1, \ldots, k, k = Ord(x, K) \) be the components of \( K \setminus \{x\} \). It also follows from (c) that \( x \in \hat{\gamma} \) and that the two vertices \( A_i \) and \( A_j \) lie in the same component \( C_i \) iff \( x \notin \gamma_{A_iA_j} \). Therefore all the vertices of \( P \), belonging to the same component \( C_i \) of the set \( K \setminus \{x\} \), lie also in the same component of \( \hat{\gamma} \setminus \{x\} \), which implies \( Ord(x, \hat{\gamma}) = Ord(x, K) \). \( \blacksquare \)

To evaluate the order \( Ord(x, K) \) of the points \( x \in K \), first we have to evaluate the order of the vertices \( A \in V_P \), which is related to the number of preimages \( n_A = \#\pi^{-1}(A) \) of the point \( A \) in \( I^\infty \), and we evaluate it using measures \( \theta_A \) of solid angles at the vertices of \( P \).

Let \( \theta_A = \theta(\Omega(P, A)) \) be the \((d - 1)\)-dimensional measure of solid angle of \( P \) at \( A \), \( \theta_{\max} = \max\{\theta_A, A \in V_P\} \), and \( \theta_{\min} = \min\{\theta_A, A \in V_P\} \).

For \( t \in \mathbb{R} \), \([t]\) means \( \text{Ceil}(t) \), i.e. minimal integer, less or equal to \( t \).

**Proposition 19** Let \( A \in V_P \).

(a) If \( \#\pi^{-1}(A) = 1 \), then there are \( i \in I^* \), \( A' \in V_P \), such that \( A = S_i(A') \) and \( Ord(A, K) = Ord(A', \hat{\gamma}) \); then \( Ord(A, K) \leq n_P - 1 \);
(b) If \( n_A = \#\pi^{-1}(A) > 1 \), then there are \( i_k \in I^* \), \( A'_k \in V_P \), where \( k = 1, \ldots, n_A \), such that \( A_k = S_{i_k}(A'_k) \) and \( Ord(A, K) = \sum_{k=1}^{n_A} Ord(A'_k, \hat{\gamma}) \); then

\[
Ord(A, K) \leq (n_P - 1) \left( \left\lceil \frac{\theta_A}{\theta_{\min}} \right\rceil - 1 \right) \leq (n_P - 1) \left( \left\lceil \frac{\theta_{\max}}{\theta_{\min}} \right\rceil - 1 \right) \quad (1)
\]

**Proof:** Let \( \#\pi^{-1}(A) = 1 \) and \( \{C_l, l = 1, \ldots, k\} \) be some finite set of components of \( K \setminus \{A\} \). Since \( \{A\} \) is the intersection of unique sequence of polyhedra \( P_{j_1} \supset P_{j_1j_2} \supset \cdots \supset P_{j_1 \cdots j_s} \cdots \), there is such \( s \), that \( \text{diam} P_{j_1 \cdots j_s} < \text{diam} C_l \) for
any \( l = 1, \ldots, k \). Then, by Proposition \([15]\) each component \( C_l \) contains a vertex of a polyhedron \( P_{j_1\ldots j_s} \), different from \( A \). Therefore \( k \leq n_P - 1 \), and \( \text{Ord}(A, K) \leq n_P - 1 \).

Since \( \text{Ord}(A, K) \) is finite, we have the right to suppose that \( k = \text{Ord}(A, K) \), and \( \{C_1, \ldots, C_k\} \) is a complete set of components of \( K \setminus \{A\} \).

Let \( j = j_1\ldots j_s \) and \( A = S_j(A') \). The sets \( C_l \cap P_j, l = 1, \ldots, k \) are the components of \( K \setminus \{A\} \). Since \( (K \cap P_j) \setminus \{A\} = S_j(K \setminus \{A'\}) \), the set \( K \setminus \{A'\} \) consists of \( k \) components \( C_l' \), such that \( S_j(C_l') = C_l \cap P_j \). Since each component \( C_l' \) contains vertices of \( P \), by Proposition \([15]\)(d), \( \text{Ord}(A', \hat{\gamma}) = \text{Ord}(A', K) = \text{Ord}(A, K) \leq n_P - 1 \).

Suppose that \( n_A = \#\pi^{-1}(A) > 1 \). By Theorem \([10]\)(g) there is a family \( \{\Omega_1, \ldots, \Omega_{n_A}\} \) of disjoint solid angles with the same vertex \( A \), and of respective polyhedra \( P_{j} \ni A \), such that \( P_{j} \subset \Omega_k \) and \( \Omega(P_{j_k}, A) = \Omega_k \).

Let \( A_k \in V_P \) and \( S_{j_k}(A_k) = A \). Keeping in mind that \( \#\pi^{-1}(A) = 1 \) and following the argument of the part (a) we can choose such \( j_k \) and \( A_k' \) that \( \text{Ord}(A', K) = \text{Ord}(A_k, \hat{\gamma}) \); therefore \( \text{Ord}(A, K_{j_k}) = \text{Ord}(A_k, K) \leq n_P - 1 \) and \( \text{Ord}(A, K) \leq n_A(n_P - 1) \). Taking into account the inequality \( n_A \leq \left| \frac{\theta_A}{\theta_{\min}} \right| - 1 \leq \left| \frac{\theta_{\max}}{\theta_{\min}} \right| - 1 \), we get the inequality (1). \[ \square \]

**Theorem 20**  
(i) \( CP(K) \subset G_S(\hat{\gamma}) \);
(ii) If \( y \notin G_S(V_P) \), then there are \( j \in I^* \), \( x \in CP(\hat{\gamma}) \), such that \( y = S_j(x) \) and \( \text{Ord}(y, K) = \text{Ord}(x, \hat{\gamma}) \leq n_P \).
(iii) If \( y \in G_S(V_P) \), then there are multiindices \( j_k, k = 1, \ldots, s \) and vertices \( A_1', \ldots, A_s' \), such that for any \( k \), \( S_{j_k}(A_k') = y \), and any \( l \neq k \), \( S_{j_k}(P) \cap S_{j_l}(P) = \{y\} \);

in this case, \( \text{Ord}(y, K) = \sum_{k=1}^{s} \text{Ord}(A_k', \hat{\gamma}) \leq (n_P - 1) \left( \left| \frac{\theta_F}{\theta_{\min}} \right| - 1 \right) \), where \( \theta_F \) is the measure of full angle in \( \mathbb{R}^d \).

**Proof.** (ii) Let \( \{C_1, \ldots, C_k\} \) be some set of components of \( K \setminus \{y\} \), and \( \rho = \min_{l=1, \ldots, k} \text{diam}(C_l) \). Suppose \( j \in I^* \) is such that \( y \in P_j \) and \( \text{diam}(P_j) < \rho \).

By Proposition \([15]\) for any \( l \), \( C_l \cap S_j(V_P) \neq \emptyset \), therefore \( k \leq n_P \). Thus, \( \text{Ord}(y, K) \leq n_P \).

So we can suppose that \( k = \text{Ord}(y, K) \) and \( \{C_1, \ldots, C_k\} \) is the set of all components of \( K \setminus \{y\} \).

Let \( x = S_j^{-1}(y) \). Then the sets \( C_l' = S_j^{-1}(C_l \cap P_j), l = 1, \ldots, k \), form a full set of components of \( K \setminus \{x\} \), while for any \( l \), \( C_l' \cap V_P \neq \emptyset \). Then, by Proposition \([15]\) \( \text{Ord}(x, \hat{\gamma}) = \text{Ord}(x, K) = \text{Ord}(y, K) \leq n_P \).
Let \( n_y = \# \pi^{-1}(y) \). By Theorem 10(g) there is a family \( \{\Omega_1, \ldots, \Omega_{n_y}\} \) of disjoint solid angles with vertex \( y \), and of respective polyhedra \( P_{j_k} \ni y \), such that \( P_{j_k} \subset \Omega_k \). Using the argument of Proposition 19(b), we obtain that \( \text{Ord}(y, K) \leq n_y(n_P - 1) \) and therefore, choosing the polyhedra \( P_{j_k} \) of sufficiently small diameter, we obtain that for any \( k, y \in S_{j_k}(\hat{\gamma}) \), \( \text{Ord}(y, K_{j_k}) = \text{Ord}(y, S_{j_k}(\hat{\gamma})) \).

This gives the estimate

\[
\text{Ord}(y, K) \leq (n_P - 1) \left( \left\lceil \frac{\theta_F}{\theta_{\text{min}}} \right\rceil - 1 \right)
\]

(i) follows from (ii) and (iii).

Theorem 21 Let \( (P, S) \) be a contractible \( P \)-polyhedral system and \( K \) be its attractor. (i) \( \dim_H(CP(K)) = \dim_H(\hat{\gamma}) \leq \dim_H EP(K) = \dim_H(K) \); (ii) \( \dim_H(CP(K)) = \dim_H(K) \) iff \( K \) is a Jordan arc.

Proof: Since \( CP(K) = G_\delta(\hat{\gamma}) \), \( \dim_H(CP(K)) = \dim_H(\hat{\gamma}) \). If \( K \) is not a Jordan arc, the set \( EP(K) \) is infinite \([3]\) Theorem 5.2] and contains a point \( x \notin \hat{\gamma} \). Let \( \varepsilon < d(x, \hat{\gamma})/2 \). Take such \( n \) that for any \( j \in I^n \), \( \text{diam}(P_j) < \varepsilon \). Then the set \( J = \{ j \in I^n : P_j \cap \hat{\gamma} \neq \emptyset \} \) is a proper subset of \( I^n \), because \( x \notin P_j \) for any \( j \in J \). Let \( S' = \{ P_j, j \in J \} \) and \( K' \) be the attractor of the system \( S' \). Since the sets \( \{ S_j, j \in J \} \) cover \( \hat{\gamma}, K' \supset \hat{\gamma} \). At the same time, the similarity dimension \( \dim_s(S') \) of the system \( S' \) is strictly smaller than that of \( S^{(n)} \) which is equal to \( \dim_s(S) = \dim_H(K) \) in its turn. Therefore, \( \dim_H(\hat{\gamma}) \leq \dim_H(K') < \dim_H(K) \). Since \( EP(K) = K \setminus CP(K) \), \( \dim_H(EP(K)) = \dim_H(K) \).

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