Computads and slices of operads.

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Abstract

For a given \(\omega\)-operad \(A\) on globular sets we introduce a sequence of symmetric operads on \(\mathbb{Set}\) called slices of \(A\) and show how the connected limit preserving properties of slices are related to the property of the category of \(n\)-computads of \(A\) being a presheaf topos.

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1 Introduction.

Computads were invented by Street [13] as a tool for the presentation of strict \(n\)-categories. They attracted a new wave of interest in recent years due to the development of the theory of weak higher categories. It also became evident that we often need some more general types of computads than Street’s computads. For example, the theory of surface diagrams in 3D-space naturally leads to the use of so-called Gray-computads [12]. In our paper [4] computads for magma-type globular theories were used.

In our paper [3] we construct a general theory of computads for finitary monads on globular sets. An important class of such monads consists of so-called analytic monads [5] which can be identified with higher operads in \(\mathbb{Span}\) in the sense of [1]. The examples in the previous paragraph all belong to this class of monads.

In [3] some properties of computads for analytic monads were established. In particular, it was claimed that computads form a presheaf topos. This statement in the case of Street’s 2-computads was proved by Shanuel and then reproved by Carboni and Johnstone [6]. Unfortunately, the proof we gave in [3] and [4]

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turned out to be incorrect. In [11] Makkai and Zawadowski observed that the category of Street’s 3-computads can not be a presheaf topos.

In this paper we study this question more carefully. We find a sufficient condition when computads for a given analytic monad on globular sets do form a presheaf category. The condition is given in terms of a sequence of symmetric operads in the category of sets which we can construct from the analytic monad. We call this sequence the sequence of *slices of the operad*. We also show that if the slices are *normalised* then the condition is even necessary.

We also give examples of monads for which this condition is satisfied. A surprising result is that $n$-computads for weak $n$-categories do form a presheaf category for any $n$. This result is also true for 3-computads for Gray-categories.

It seems to us that the slices of operads are closely related to the coherence problem for weak $n$-categories and we suggest a couple of conjectures about it in section 3.

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## 2 Computads.

By an $n$-globular (globular if $n = \omega$) set we mean a sequence (infinite if $n = \omega$) of sets

$$X_0, X_1, \ldots, X_k, \ldots, X_n$$

together with source and target maps

$$s_{r-1}, t_{r-1} : X_r \rightarrow X_{r-1}$$

satisfying the equation:

$$s_{r-1} \cdot s_r = s_{r-1} \cdot t_r , \ t_{r-1} \cdot s_r = t_{r-1} \cdot t_r.$$  

The set $X_r$ is called the set of $r$-cells of $X$. Sometimes we will use also notation $(X)_r$ for this set.

Every $(n-1)$-globular set can be considered as an $n$-globular set with empty set of $n$-cells. So we have a chain of inclusion functors

$$Set = \text{Glob}_0 \subseteq \text{Glob}_1 \subseteq \ldots \subseteq \text{Glob}_k \subseteq \text{Glob}_{k+1} \ldots \subseteq \text{Glob}$$

and each of the inclusion functors

$$L_k : \text{Glob}_k \rightarrow \text{Glob}_{k+1}$$
has a right adjoint
\[ tr_k : \text{Glob}_n \rightarrow \text{Glob}_k. \]

Let \( A = (A, \mu, \varepsilon) \) be a finitary monad on \( \text{Glob} \). We denote by \( A_n \) the \( n \)-truncation of \( A \), i.e. the restriction of \( A \) to the category \( \text{Glob}_n \) of \( n \)-globular sets. The category of algebras of \( A_n \) will be denoted by \( \text{Alg}_n \) and the corresponding forgetful functor will be denoted by
\[ W_n : \text{Alg}_n \longrightarrow \text{Glob}_n. \]

We now make the following inductive definition [3]:
The category \( \text{Comp}_0 \) of \( A_0 \)-computads is \( \text{Glob}_0 \). The functors
\[ W_0 = W_0 : \text{Alg}_0 \rightarrow \text{Comp}_0 \]
\[ F_0 = F_0 : \text{Comp}_0 \rightarrow \text{Alg}_0 \]
are the forgetful and free \( A_0 \)-algebra functors, respectively.

Let us suppose now that the category \( \text{Comp}_{n-1} \) of \( A_{n-1} \)-computads is already defined together with two functors:
\[ W_{n-1} : \text{Alg}_{n-1} \rightarrow \text{Comp}_{n-1} \]
\[ F_{n-1} : \text{Comp}_{n-1} \rightarrow \text{Alg}_{n-1} \]
such that \( F_{n-1} \) is left adjoint to \( W_{n-1} \).

**Definition 2.1** An \( A_n \)-computad \( C \) is a triple \( (C, \phi, C') \) consisting of an \( n \)-globular set \( C \), an \( A_{n-1} \)-computad \( C' \) and an isomorphism
\[ \phi : W_{n-1}(F_{n-1}C') \rightarrow tr_{n-1}C \]
in \( \text{Glob}_{n-1} \).

Let \( G \) be an object of \( \text{Alg}_n \). The counit of the adjunction \( F_{n-1} \dashv W_{n-1} \) gives a morphism
\[ r_{n-1} : F_{n-1}W_{n-1}tr_{n-1}G \rightarrow tr_{n-1}G. \]

Define an \( n \)-globular set \( G \) in the following way. The \((n-1)\)-skeleton of \( G \) coincides with \( W_{n-1}F_{n-1}W_{n-1}tr_{n-1}G \) and
\[ G_n = \{ (\xi, a, \eta) \in G_{n-1} \times G_n \times G_{n-1} \mid s_{n-2}\xi = s_{n-2}\eta, \ t_{n-2}\xi = t_{n-2}\eta, \ s_{n-1}a = r_{n-1}(\xi), \ t_{n-1}a = r_{n-1}(\eta) \}. \]

Define
\[ s_{n-1}(\xi, a, \eta) = \xi, \ t_{n-1}(\xi, a, \eta) = \eta. \]
Then put

\[ W_nG = (G, id, W_{n-1}tr_{n-1}G) \]

For an \( A_n \)-computad \( C = (C, \phi, C') \), define

\[ V_n(C) = C \]

and \( V_0 = id \) for \( n = 0 \).

Define a natural transformation

\[ \Theta_n : V_nW_n \to W_n \]

to be the morphism of \( n \)-globular sets which coincides with

\[ W_{n-1}tr_{n-1} : W_{n-1}\mathcal{F}_{n-1}W_{n-1}tr_{n-1}G \to W_{n-1}tr_{n-1}G \]

up to dimension \( n - 1 \) and has

\[ \Theta_n(\xi, a, \eta) = a \]

in dimension \( n \).

Let us define a new monad \( I_A \) on globular sets by means of the following pushout.

\[
\begin{array}{ccc}
L_{n-1}tr_{n-1}X & \longrightarrow & L_{n-1}tr_{n-1}AX \\
\downarrow & & \downarrow \\
X & \longrightarrow & IAX \\
\end{array}
\]

The algebras of \( I_A \) are globular sets together with an \( A_{n-1} \)-algebra structure on its \((n-1)\)-truncation. Notice that the categories of \( A_n \)-computads and \((I_A)_{n-1} \)-computads are canonically isomorphic. Moreover, the functor \( V \) together with the \( A_{n-1} \)-algebra structure on \( tr_{n-1}VC \simeq W_{n-1}(\mathcal{F}_{n-1}C') \) is left adjoint to the forgetful functor from the category of \( I_A \)-algebras to \( A_n \)-computads and \( \Theta_n \) is the counit of this adjunction. So, the functor \( \mathcal{F}_n \) is canonically isomorphic to a composite of \( V \) and \( \Gamma \) which is left adjoint to the restriction functor

\[ l^* : Alg_n \longrightarrow Alg_{I_A} \]

induced by an obvious morphism of monads

\[ l : I_A \to A_n. \]

This left adjoint exists due to the finitary assumption [10].

We also can talk about \( \omega \)-computads. Recall [3] that the \( n \)-truncation of an \((n+1)\)-computad \((C, \phi, C)\) is the \( n \)-computad \( C \).
Definition 2.2 Let $A$ be a finitary monad on $Glob$. An $\omega$-computad for $A$ is a sequence $C_n$ of $n$-computads for $A$ together with a sequence of isomorphisms $c_n : tr_n(C_{n+1}) \rightarrow C_n$. A morphism of $\omega$-computads is a sequence of morphisms of $n$-computads which commutes in the obvious sense with the structure isomorphisms.

We use the techniques of [10] for an explicit construction of the left adjoint $\Gamma$ into the category of $A_n$-algebras.

Let $X = M_0$ be an $I_A$-algebra and let $M_1$ be the following coequalizer in $Glob_n$

$$
\begin{array}{c}
M_1 \\
\downarrow \pi_1 \\
A_nM_0
\end{array}
\begin{array}{c}
\downarrow Ak \\
\downarrow \eta
\end{array}
\begin{array}{c}
A_nI_AM_0
\end{array}
$$

where $k$ is the $I_A$-algebra structure morphism for $X$ and $\eta$ is the composite $\mu \cdot A_n(l)$. Notice, that $k$ is an identity in dimension $n$.

Suppose that the globular set $M_r$, together with the morphism $\pi_r : A_nM_{r-1} \rightarrow M_r$, are already constructed. Then define $M_{r+1}$ to be the following coequalizer.

$$
\begin{array}{c}
M_{r+1} \\
\downarrow \pi_{r+1} \\
A_nM_r
\end{array}
\begin{array}{c}
\downarrow A_n\pi_r \\
\downarrow id
\end{array}
\begin{array}{c}
A_n^2M_{r-1}
\end{array}
\begin{array}{c}
\downarrow \epsilon_n \\
\downarrow \mu_n
\end{array}
\begin{array}{c}
A_n^2M_{r-1}
\end{array}
$$

Then we have the following sequence of morphisms

$$
M_0 \xrightarrow{\epsilon_n} A_nM_0 \xrightarrow{\pi_1} M_1 \xrightarrow{\epsilon_n} A_nM_1 \xrightarrow{\pi_2} \ldots
$$

We denote the colimit of it by $M_\infty X$. According to [10] $M_\infty X$ has a natural $A_n$-algebra structure given by $\pi_\infty = \text{colim} \pi_r$, and this is indeed the free $A_n$-algebra generated by $X$.

3 Suspensions and slices of globular operads.

Every strict $\omega$-category has an underlying globular set. This functor has a left adjoint

$$
D : Glob \rightarrow \omega\text{-Cat}.
$$
We will also denote by \((D, \mu, \epsilon)\) the monad generated by this adjunction (notice, that in [1] this monad was denoted by \(D_s\)). In [1] a description of \(D\) in terms of plain trees was presented.

Recall [14] that a natural transformation \(p : R \to Q\) between two functors is called \textit{cartesian} if for every morphism \(f : X \to Y\) the naturality square

\[
\begin{array}{ccc}
R(X) & \xrightarrow{R(f)} & R(Y) \\
\downarrow p & & \downarrow p \\
Q(X) & \xrightarrow{Q(f)} & Q(Y)
\end{array}
\]

is a pullback. Recall also that an endofunctor \(A\) on \(Glob\) is called \textit{analytic} if it is equipped with a cartesian natural transformation (augmentation) \(p : A \to D\). Such an endofunctor is determined up to isomorphism by a collection

\[p(1) : A(1) \to D(1),\]

where 1 is the terminal globular set and it is connected limits preserving. A monad on \(Glob\) is called analytic if its functor part is analytic and unit and multiplication are cartesian natural transformation. The category of analytic monads is equivalent to the category of \(\omega\)-operads in \(\text{Span}\).

The following definition is due to Joyal [9]. An endofunctor \(a\) on \(Set\) is called \textit{analytic} if it can be represented as a ‘Taylor series’

\[
a(X) = \sum_{n \geq 0} A[n] \times \Sigma_n X^n,
\]

where \(A[n], n \geq 0\), is a symmetric collection, i.e. a family of sets equipped with an action of the symmetric group \(\Sigma_n\) on \(A[n]\). The analytic functors are closed under composition and the monoids in this monoidal categories are called \textit{symmetric operads}.

Symmetric operads are a special case of algebraic theories in \(Set\). Another special case of algebraic theories called \textit{strongly regular} theories was considered by Carboni and Johnstone in [6]. These are theories which can be given by equations without permutations and repetitions of symbols. For example, the theory of monoids is such a theory, but the theory of commutative monoids is not. In [6] a characterisation of strongly regular theories is established. They are exactly the theories given by \textit{nonsymmetric operads} in \(Set\). The last are monoids with respect to composition in the monoidal category of endofunctors of the form

\[
a(X) = \sum_{n \geq 0} A[n] \times X^n,
\]

where \(A[n], n \geq 0\) is a nonsymmetric collection, i.e. just a sequence of sets. We will call the functors of the form \((\ast)\) \textit{strongly analytic}. It was also proved in [6] that strongly analytic functors preserve connected limits.
Definition 3.1 An \( n \)-globular set \( X \) is called \( k \)-terminal if its \( k \)-th truncation is a terminal \( k \)-globular set. An algebra of a monad on \( n \)-globular sets is called \( k \)-terminal if its underlying globular set is \( k \)-terminal.

We denote by \( \text{Glob}_n^{(k)} \) the category of \( k \)-terminal \( n \)-globular sets. Clearly, \( \text{Glob}_n^{(k)} \) is isomorphic to \( \text{Glob}_{n-k-1} \). For a monad \( A \) on \( \text{Glob} \) we denote by \( \text{Alg}_n^{(k)} \) the category of \( k \)-terminal algebras of \( A_n \). We have a restriction of the forgetful functor \( W \)

\[ W^{(k)} : \text{Alg}_n^{(k-1)} \to \text{Glob}_n^{(k-1)}, \quad k \geq 1. \]

It is not hard to prove that this functor is monadic at least for a finitary monad \( A \) [15]. Hence, we have a monad \( S^k A_n \) on \( \text{Glob}_{n-k} \) such that its category of algebras is equivalent to \( \text{Alg}_n^{(k-1)} \). We also put \( S^0 A = A \).

Definition 3.2 [15] \( S^k A_n \) is called the \( k \)-fold suspension of \( A_n \)

Now if \( k = n \) then \( S^k A_k \) is a monad on \( \text{Glob}_0 = \text{Set} \). The proposition 2.1 and the theorem 10.2 from [2] assert that this monad is actually a symmetric operad on \( \text{Set} \).

Definition 3.3 The symmetric operad \( S^k A_k \) will be called the \( k \)-th slice of \( A \). We will denote this operad by \( \mathcal{P}_k(A) \).

Example 3.1 For any operad \( A \) its 0-slice is given by a symmetric operad which underlying collection consists of a monoid \( A[U_0] \) in dimension 1 and empty sets in other dimensions. The tree \( U_0 \) is the only tree of height 0.

Example 3.2 The first slice of the terminal operad \( D \) is free monoid operad. All the higher slices are the free commutative monoid operad.

It is proved in [2, Theorem 10.1] that the first slice of an operad is always a free symmetric operad on some nonsymmetric operad [2] and, hence, is always a strongly regular theory.

Example 3.3 For the bicategory operad, its first slice is the nonsymmetric operad freely generated by a pointed collection which has exactly one operation in dimensions 0, 1, 2. The second slice is the free commutative monoid operad.

Example 3.4 For the Gray operad \( G \) [1] the first slice is the free monoid operad, the second slice is the double-monoid-with-common-unit operad i.e. a set with two independent monoid structures and common unit. So \( \mathcal{P}_2(G) \) is a strongly regular theory. The third slice is the free commutative monoid operad.

Example 3.5 For a free operad on a globular collection, the slices are free symmetric operads on some nonsymmetric collections and are, therefore, strongly
regular theories. The category of $\omega$-computads for such operads were used in [4].

**Example 3.6** For the universal contractible $\omega$-operad $K$ from [1] the slices are free symmetric operads on nonsymmetric collections. This can be easily seen from the construction of $K$ given in [1]. Hence, all the slices of $K$ are strongly regular theories. Recall that the algebras of $K$ are by definition weak $\omega$-categories.

**Example 3.7** For the universal contractible $n$-operad its slices up to dimension $n - 1$ are free symmetric operads on some nonsymmetric collections but its $n$-th slice is the free commutative monoid operad. The algebras of this operad are weak $n$-categories.

In the theory of symmetric operads a very important condition is freeness of the action of the symmetric groups. For example, $E_\infty$-operads are exactly those operads which are contractible and have free action of the symmetric groups. If the action is not free it usually means that the corresponding algebras have some homotopy degeneracy like the vanishing of some Whithead products or Postnikov invariants.

From the examples above we see that slices carry with them some information about the homotopy behaviour of the higher operads. It seems to us that the condition for slices to be regular theories is the correct analogue of the condition of freeness of action of the symmetric groups. So our conjecture is

**Conjecture 3.1** Suppose that an $n$-operad $A$ is contractible, contains a system of binary compositions [1], and all its slices up to dimension $n - 1$ are strongly regular theories. Then every weak $n$-category is weakly equivalent to an $A$-algebra.

At the time of writing it is not completely clear what the right notion of ‘semistrict’ $n$-category should be. The desirable properties are:

- every weak $n$-category must be equivalent to a semistrict one;
- the notion of ‘semistrict’ $n$-category is ‘minimal’ with the above property.

In dimension 2 this is just the notion of strict 2-category. In dimension 3 it is the notion of Gray-category [8]. Crans has a candidate for dimension 4 and some ideas about higher dimensions [7]. Here we risk to suggesting a conjecture.

**Conjecture 3.2** There is a unique contractible $n$-operad $G_n$ with the property that $\mathcal{P}_k(G_n)$, $0 \leq k \leq n - 1$, is the free $k$-fold monoid operad. A semistrict $n$-category is an algebra for this operad.
4 Weak limits and coequalisers

This section has a technical character and contains some elementary facts about weak pullbacks and coequalisers we will need in next section.

Definition 4.1 Let $F : \Lambda \to C$ be a functor between two categories and let $W \xrightarrow{p_\lambda} F(c_\lambda)$ be a cone over $F$. It is called a weak limit of $F$ if for any other cone $V \xrightarrow{q_\lambda} F(c_\lambda)$ there exists a morphism $r : V \to W$ such that $q_\lambda = p_\lambda \cdot r$.

Remark 4.1 It is obvious that if limit of a functor $F$ exists then it is a retract of any weak limit of $F$. Moreover, in order to prove that $W$ is a weak limit it is enough to construct a section of the canonical morphism from $W$ to the limit of $F$ which makes some obvious diagrams commutative. We will use this simple observation extensively.

Following [9] and [15] we call a natural transformation between two functors weak cartesian provided every naturality square is a weak pullback.

Lemma 4.1 Suppose

$$
\begin{array}{ccc}
C & \xrightarrow{p} & A \\
\downarrow{C(f)} & & \downarrow{\chi} \\
C(Y) & \xrightarrow{p_Y} & A(Y)
\end{array}
$$

is a coequaliser of two weakly cartesian transformations between functors $A, B : \Lambda \to \text{Set}$. Then $p$ is weakly cartesian.

Proof. Let $f : X \to Y$ be a map of sets and let $P$ be the pullback of $C(f)$ and $p_Y$ i.e.

$$
P = \{(c,a) | C(f)(c) = p_Y(a)\}.
$$

We have to prove that there is a section $s$ of the canonical map $A(X) \to P$ which makes the following diagram commutative

$$
\begin{array}{ccc}
C(X) & \xrightarrow{p_X} & A(X) \\
\downarrow{C(f)} & & \downarrow{s} \\
C(Y) & \xrightarrow{p_Y} & A(Y)
\end{array}
$$

Let us take $(c,a) \in P$ and let $a' \in A(X)$ be such that $p_X(a') = c$. Put $y = A(f)(a')$. Then $p_Y(y) = p_Y(a)$. The last equality means $x$ and $a$ are equivalent with respect to the equivalence relation generated by $\chi$ and $\zeta$. Without loss of generality we can assume that there is a finite sequence $b_1, \ldots, b_k$ of elements of $B(Y)$ such that

$$
y = \chi(b_1), \quad a = \zeta(b_k), \quad \zeta(b_i) = \chi(b_{i+1}).
$$
Since $\chi$ is weakly cartesian we can find a $b'_1 \in B(X)$ such that $B(f)(b'_1) = b_1$ and $\chi(b'_1) = a'$. Then consider the element $\zeta(b'_1)$. We have $p_X(\zeta(b'_1)) = c$ and

$$A(f)(\zeta(b'_1)) = \zeta(B(f)(b'_1)) = \zeta(b_1) = \chi(b_2).$$

Therefore, we can find $b'_2$ such that $B(f)(b'_2) = b_2$ and $\chi(b'_2) = \zeta(b'_1)$. Then again $p_X(\zeta(b'_2)) = c$ and

$$A(f)(\zeta(b'_2)) = \zeta(B(f)(b'_2)) = \zeta(b_2) = \chi(b_3).$$

We can continue this process and finally we get

$$p_X(\zeta(b'_k)) = c$$

and

$$A(f)(\zeta(b'_k)) = \zeta(b_k) = a.$$ 

Hence, we can put $s(c,a) = \zeta(b'_k)$. The lemma is therefore proved.

\[\square\]

**Lemma 4.2** Sequential colimits in \textit{Set} preserve weak pullbacks.

**Proof.** It is well known that sequential colimits in \textit{Set} preserve pullbacks. So it is enough to prove that in a sequential colimit of weak pullbacks we can choose the sections of the retractions from pullbacks to weak pullbacks naturally.

Let us fix a section $q_i : P_i \rightarrow W_i$ of the canonical retraction $W_i \rightarrow P_i$ for every $i \geq 0$. We will construct a new section $s_i$ inductively.

We take $s_0 = q_0$. Now suppose the retraction $s_i$ in the $i$-th weak pullback

$$\begin{array}{ccc}
A_i & \xrightarrow{s_i} & W_i \\
| & & | \\
C_i & \xleftarrow{P_i} & B_i
\end{array}$$

is already constructed. Then we can construct $s_{i+1}$ in the following way. Let $a \in P_{i+1}$ belong to the image of the limiting map $\lambda_i : P_i \rightarrow P_{i+1}$ and let us choose a $b \in P_i$ such that $\lambda_i(b) = a$. Then we put $s_{i+1}(a) = w_i(s_i(b))$, where $w_i : W_i \rightarrow W_{i+1}$. If $a$ does not belong to $im(\lambda_i)$ then we put $s_{i+1}(a) = q_{i+1}(a)$.

The sections $s_i$ obviously induce a section

$$\text{colim} P_i \rightarrow \text{colim} W_i$$

of the canonical map $\text{colim} W_i \rightarrow \text{colim} P_i$ which completes the proof.

\[\square\]

By a similar diagram-chase method one can easily prove the following lemma.
Lemma 4.3 Suppose that in a commutative diagram of coequalisers

\[
\begin{array}{ccc}
C_2 & \xleftarrow{p_2} & A_2 \\
\downarrow{\phi} & & \downarrow{\zeta} \\
C_1 & \xleftarrow{p_1} & A_1
\end{array}
\begin{array}{ccc}
& & \xrightarrow{\psi} \\
& & \downarrow{} \\
& & B_2 \\
& & \xleftarrow{} \\
& & B_1
\end{array}
\]

both right commutative squares are weak pullbacks and \(\psi\) and \(\zeta\) are monomorphisms, then colimiting map \(\phi\) is a monomorphism.

\[\square\]

The following lemma is obvious.

Lemma 4.4 If a commutative square is weakly cartesian and one of the limiting maps is a monomorphism then the square is cartesian.

\[\square\]

Lemma 4.5 Analytic functors on \(\text{Glob}\) preserve connected weak limits.

Proof. Let \(A\) be an analytic functor on \(\text{Glob}\) and let \(C\) be a weak connected limit of a diagram of globular sets \(F : \Lambda \to \text{Glob}\). Then there is a retraction

\[r : C \to \lim_{\Lambda} F.\]

Hence, we have a retraction

\[A(r) : A(C) \to A(\lim_{\Lambda} F) \simeq \lim_{\Lambda} A(F)\]

which proves the lemma.

\[\square\]

Lemma 4.6 Let \(\phi : A \to B\) be a natural transformation in \(\text{Glob}_n\) such that \(\text{tr}_{n-1}\phi\) is cartesian and \((\phi)_n : (A)_n \to (B)_n\) is weakly cartesian in \(\text{Set}\). Then \(\phi\) is weakly cartesian.

Proof. Let \(f : X \to Y\) be a morphism of globular sets and let \(P\) be a pullback of \(\phi\) and \(B(f)\). Then we can assume that \(\text{tr}_{n-1}P = \text{tr}_{n-1}A\). Let \(\psi : (P)_n \to (A(X))_n\) be a section of the canonical retraction \((r)_n : (A(X))_n \to (P)_n\) which exists due to the weak cartesianness of \((\phi)_n\). We have to prove that \(\psi\) respects source and target operators.

Indeed, consider a map \(\alpha = s_{n-1}(\psi) : (P)_n \to (A(X))_{n-1}\). Then we have

\[s_{n-1}(pA)_n = s_{n-1}(pA)_n((r)_n(\psi)) = s_{n-1}((Af)_n(\psi)) = (Af)_{n-1}(\alpha).\]
Analogously

\[ s_{n-1}(p_B)_n = (\phi)_{n-1}(\alpha), \]

where \( p_A, p_B \) are canonical projections from the pullback \( P \). Since \((Af)_{n-1}\)
and \((\phi)_{n-1}\) are also projections of a pullback we have by its universal property
that \( \alpha \) must coincide with \( s_{n-1} : (P)_n \to (A(X))_{n-1} \). So \( \psi \) commutes with the
source operator. Analogously it commutes with target operator.

\textbf{Lemma 4.7} Suppose \( \phi : A \to B \) is a weakly cartesian transformation between
two strongly analytic functors in \( \text{Set} \), then it is cartesian.

\textbf{Proof.} By [6] and a theorem of Joyal [9, 15] we can assume that \( A \) and \( B \) both
are given by free symmetric collections \( A[n] = \alpha[n] \times \Sigma_n, B[n] = \beta[n] \times \Sigma_n \) and
\( \phi \) is given by equivariant maps of symmetric collections

\[ \phi[n] : \alpha[n] \times \Sigma_n \to \beta[n] \times \Sigma_n, \ n \geq 0. \]

The map \( \phi[n] \) is determined in its turn by a map of nonsymmetric collections

\[ \psi[n] : \alpha[n] \to \beta[n] \times \Sigma_n. \]

Then the natural transformation \( \phi \) is the coproduct over \( n \) of the composites

\[ \begin{array}{ccc}
\alpha[n] \times \Sigma_n \times X^n & \cong & \alpha[n] \times X^n \\
\psi[n] \times 1 & \to & \beta[n] \times \Sigma_n \times X^n \\
1 \times k & \to & \beta[n] \times X^n \to (\beta[n] \times \Sigma_n) \times \Sigma_n X^n,
\end{array} \]

where \( k \) is the action of \( \Sigma_n \) on \( X^n \). Then for the unique map \( X \to 1 \) we have the following commutative naturality diagram

\[ \begin{array}{ccc}
A[n] \times X^n & \xrightarrow{\psi[n] \times 1} & B[n] \times \Sigma_n \times X^n & \xrightarrow{1 \times k} & B[n] \times X^n \\
A[n] & \xrightarrow{\psi[n]} & B[n] \times \Sigma_n & \to & B[n]
\end{array} \]

In this diagram both left and right squares are obviously pullbacks; hence, so is
the big square. This is enough to imply the transformation \( \phi \) is cartesian.

\textbf{Lemma 4.8} Let \( A \) be an analytic functor on \( \text{Glob} \) and let \( f : X \to Y \) be a map
of globular sets such that for a fixed \( n \geq 0 \) the map \( (f)_n \) is a monomorphism.
Then \( (A(f))_n \) is a monomorphism.

\textbf{Proof.} Since \( A \) is strongly analytic it is sufficient to prove the lemma for the
case \( A = D \). Then it is obvious from the construction of \( D \) given in [1].
5  Computads and slices of operads.

Let

\[ k_n : A_n(F_n) \rightarrow F_n \]

be a natural transformation which is given on a computad \( C \) by the structure map of the algebra \( F_n(C) \).

**Theorem 5.1** Suppose for an \( n \)-operad \( A \) all the slices \( \mathcal{P}_k(A) \), \( 0 \leq k \leq n \), are strongly regular theories, then

- \( k_n \) is a cartesian natural transformation;
- \( F_n \) preserves connected limits.

**Proof.** We will prove the theorem by induction. If \( n = 0 \) the proposition is obvious because the 0-operads are just monoids and 0-computads are sets.

We assume, therefore, that the natural transformation

\[ tr_{n-1}k_n = k_{n-1} : A_{n-1}F_{n-1} \rightarrow F_{n-1} \]

is cartesian and \( F_{n-1} \) preserves connected limits.

Now we can use Kelly’s method to construct the left adjoint to the restriction functor

\[ Alg^{(n-1)} \rightarrow Alg^{(n-1)}_{I_A} \]

First of all observe that for the operad \( I_A \) the natural transformation

\[ \kappa : I_A(-) \rightarrow (-) \]

is cartesian on the category of \((n-1)\)-terminal \( I_A \)-algebras because \( tr_{n-1}\kappa \) is the constant map

\[ 1 : A_{n-1}(1) \rightarrow 1 \]

and \( \kappa \) is an identity in dimension \( n \). Hence, \( A(\kappa) \) is cartesian.

By lemma 4.1 the resulting colimit map

\[ \pi_1 : A_nM_0 \rightarrow M_1 \]

is weakly cartesian in dimension \( n \) and, therefore, by lemma 4.6 \( \pi_1 \) is weakly cartesian because \( tr_{n-1}\pi_1 = 1 \).

Analogously we have that in Kelly’s construction all \( \pi_r \) are weakly cartesian transformations and all \( M_r \) preserve connected limits.

The last sequential colimit of Kelly’s construction

\[ \text{colim} \pi_r : A_nM_\infty \rightarrow M_\infty \]

is weakly cartesian by lemmas 4.6 and 4.2. This map is obviously the map

\[ j : A_n(\mathcal{P}_n) \rightarrow \mathcal{P}_n \]
given on $X$ by the structure morphism of the algebra $\mathcal{P}_n(X)$. Since $\mathcal{P}_n$ is strongly regular theory the functor $(A_n(P_n))_n$ is strongly analytic as well. Hence, by lemma 4.7 $j$ is even cartesian.

Now let

$\widehat{(-)}: \text{Alg}_{I_A} \rightarrow \text{Alg}_{I_A}^{(n-1)}$

be a functor which assigns to an $I_A$-algebra $X$ the $(n-1)$-terminal $I_A$-algebra $\hat{X}$ with $(\hat{X})_n = (X)_n$. We obviously have a natural morphism of $I_A$-algebras $X \rightarrow \hat{X}$.

For a computad $C$ we therefore have a coequalisers diagram

\[
\begin{array}{ccc}
M_1 & \rightarrow & A_n(VC) \\
\downarrow & & \downarrow \\
N_1 & \rightarrow & A_n(V\hat{C})
\end{array}
\]

\[
\begin{array}{ccc}
A_n(VC) & \leftarrow & A_nI_A(VC) \\
\downarrow & & \downarrow \\
A_n(V\hat{C}) & \leftarrow & A_nI_A(V\hat{C})
\end{array}
\]

In this diagram the two right vertical morphisms are monomorphisms in dimension $n$ by lemma 4.8 and, therefore, the colimiting map is a monomorphism in dimension $n$ by lemma 4.3. In addition, the left square is a weak pullback by lemma 4.1.

What we have here is a map of the first stages of the Kelly machine for $VC$ and $V\hat{C}$ respectively. Continuing this process we have as an output of the Kelly machine in dimension $n$, a weak pullback

\[
\begin{array}{ccc}
(F_nC)_n & \leftarrow & (A_n(F_nC))_n \\
\downarrow & & \downarrow \\
(P_nV\hat{C})_n & \leftarrow & (A_n(P_nV\hat{C}))_n
\end{array}
\]

with vertical morphisms being monomorphisms. So it is a pullback. By a similar argument, the natural transformation

$$ (F_nC)_n \rightarrow (A_n(P_nV\hat{C}))_n $$

is cartesian.

For a computad morphism $f : C \rightarrow C'$ we have now the following commutative cube.
In this diagram the front and rear vertical squares are pullbacks. The bottom horizontal square is a pullback because $j_n$ is cartesian. Hence, we have that the top horizontal square is a pullback in dimension $n$. It is also a pullback after truncation by our inductive assumption. So we have proved that $k_n$ is cartesian.

Finally, we have to prove that $F_n$ preserves connected limits. To do this it is sufficient to prove this result in dimension $n$.

Let $C$ be a connected limit of computads $C_\lambda$ and let $c_\lambda : C \rightarrow C_\lambda$ be the canonical projection. So we have a cartesian square

$$
(F_nC_\lambda)_n \xleftarrow{(F_nC)_n} (F_nC)_n \\
(P_n\hat{V}C_\lambda)_n \xleftarrow{(P_n\hat{V}C)_n} (P_n\hat{V}C)_n
$$

But $(P_n\hat{V}C)_n$ is naturally isomorphic to $(P_n\lim(\hat{V}C_\lambda))_n$ because $V$ obviously preserves limits in dimension $n$. So, after the limit we have a pullback

$$
\lim((F_nC_\lambda))_n \xleftarrow{(F_nC)_n} (F_nC)_n \\
\lim((P_n\hat{V}C_\lambda))_n \xleftarrow{(P_n\hat{V}C)_n} (P_n\hat{V}C)_n
$$

where the bottom arrow is an isomorphism because $P_n$ preserves connected limits. So the top arrow is, and we completed the proof of the theorem.

**Theorem 5.2** Suppose that for an operad $A$ the slices $P_k(A), 0 \leq k \leq n-1$ are strongly regular theories. Then the category of $n$-computads of $A$ is a presheaf topos.
Proof. The proof generalizes example 3.6 from [6]. We use induction on $n$. If $n = 0$ the statement is true by definition.

Suppose we know that the category $\text{Comp}_{n-1}$ is a presheaf topos. Consider the functor

$$T_{n-1} : \text{Comp}_{n-1} \longrightarrow \text{Set},$$

which assigns to a computad $C$ the set of parallel pairs of $(n-1)$-cells from $W_{n-1}F_{n-1}C$. Then we have the equivalence of categories

$$\text{Comp}_n \sim \text{Set} \downarrow T_{n-1}.$$

Now we prove that $T_{n-1}$ preserves connected limits. Notice that $T_{n-1}$ is isomorphic to the following composite

$$\text{Comp}_{n-1} \xrightarrow{F_{n-1}} \text{Alg}_{n-1} \xrightarrow{\text{Alg}_{n-1}(A_{n-1}S^{n-1}, -)} \text{Set}$$

where $S^{n-1}$ is the $n-1$-globular set which has two elements $-$ and $+$ in every dimension and

$$s(-) = s(+) = - ,\ t(-) = t(+) = +.$$ 

By our assumption, $F_{n-1}$ preserves pullbacks (wide pullbacks), so $T_{n-1}$ does. According to the results of [6] this is sufficient for $\text{Set} \downarrow T_{n-1}$ to be a preasheaf topos.

\[\square\]

Corollary 5.2.1 The following categories of computads are presheaf toposes:

- the category of Street 2-computads (Shanuel, Carboni-Johnstone [6]);
- the category of Gray-computads [12] and the category of 3-computads for Gray-categories;
- the category of $k$-computads for weak $n$-categories for all $0 \leq k \leq n$;
- the category of $k$-computads for $P$-magmas [4].

Proof. See examples in section 3.

The following theorem extends the example of Makkai-Zawadowski.

Theorem 5.3 Let $A$ be an operad such that its slices $P_k(A), 1 \leq k \leq n-2$, are normalised in the sense that $P_k(A)[0] = 1$. Then the category of $n$-computads is a presheaf topos if and only if all the slices $P_k(A), 0 \leq k \leq n-1$, are strongly regular theories.
Proof. We only need to prove the only if part of the theorem. For this we will show that if there exists $P_k(A)$ which is not strongly regular then the category of $k$-computads is not a presheaf topos; this implies that the category of $n$-computads is not a presheaf topos either. So without loss of generality we can assume that the category of $(n-1)$-computads is a presheaf topos but $P_{n-1}(A)$ is not strongly regular, in particular, it is not connected limits preserving.

Let $\vartheta_k$, $0 \leq k \leq n-2$ be a $k$-computad defined by induction

$$\vartheta_k = (O_{k-1}, id, \vartheta_{k-1})$$

and $\vartheta_0 = 1$, where $O_{k-1}$ is a $(k-1)$-terminal $k$-globular set with empty set of cells of dimension $k$. For this definition to be valuable, we have to prove that $\mathcal{F}_k \vartheta_k = 1$.

If $k = 0$ it follows from $A_0(1) = 1$. Suppose we have proved it up to dimension $k-1$. Then by applying the Kelly machine we see that the calculation of $\mathcal{F}_k \vartheta_k$ amounts to the calculation of a free $A_k$-algebra on the $I_{A_k}$-algebra $O_{k-1}$. So the algebra $\mathcal{F}_k \vartheta_k$ is isomorphic to $P_k(A)(\emptyset) = 1$.

Let us consider the full subcategory of $Comp_{n-1}$ consisting of computads $C$ with $tr_{n-2}C = \vartheta_{n-2}$. Obviously, this subcategory is isomorphic to the category of sets. By the above argument, the restriction of $\mathcal{F}_{n-1}$ to this subcategory is isomorphic to $P_{n-1}(A)$ and, hence, is not connected limit preserving. So the functor $T_{n-1}$ is not connected limit preserving either and hence $Comp_n \sim Set \downarrow T_{n-1}$ cannot be a presheaf topos by a theorem from [6] again.

Corollary 5.3.1 (Makkai-Zawadowski [11]) The category of Street $n$-computads for $n \geq 3$ is not a presheaf topos.

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