LIMIT CYCLES BIFURCATING FROM A PERIODIC ANNULUS IN DISCONTINUOUS PLANAR PIECEWISE LINEAR HAMILTONIAN DIFFERENTIAL SYSTEM WITH THREE ZONES

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Abstract. In this paper, we study the number of limit cycles that can bifurcating from a periodic annulus in discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines. We prove that if the central subsystem, i.e. the system defined between the two parallel lines, has a real center and the others subsystems have centers or saddles, then we have at least three limit cycles that appear after perturbations of periodic annulus. For this, we study the number of zeros of a Melnikov function for piecewise Hamiltonian system and present a normal form for this system in order to simplify the computations.

1. INTRODUCTION AND MAIN RESULTS

The first works on piecewise differential systems appeared in the 1930s, see [1]. This class of systems have great applicability, mainly in mechanics, electrical circuits, control theory, etc (see for instance the book [5] and the papers [4, 7, 22, 23]). This subject has piqued the attention of researchers in qualitative theory of differential equations and numerous studies about this topic have arisen in the literature recently.

Piecewise differential systems with two zones are the most studied, either for their applications in modeling phenomena in general or for their apparent simplicity (see [10, 20]). As in the smooth case, the researches are mainly concentrated to the determination of the number and position of the limit cycles of these systems. In 1998, Freire, Ponce, Rodrigo and Torres in [9] proved that a continuous piecewise linear differential systems in the plane with two zones has at most one limit cycle. In the discontinuous case, the maximum number of limit cycles is not known, but important partial results about this problem have been obtained, see for example [2, 3, 11, 19].

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The problem becomes more complicated when we have more than two zones and there are a few works that deal with the discontinuous case (see \cite{6, 12, 17, 24}). However, when restrictive hypotheses such as symmetry and linearity are imposed, the issue of limit cycles is well explored. More precisely, for symmetric continuous piecewise linear differential systems with three zones, conditions for nonexistence and existence of one, two or three limit cycles have been obtained (see for instance the book \cite{15}). For the nonsymmetric case, examples with two limit cycles surrounding the only singular point at the origin was found in \cite{13, 16}.

Recently, some researchers have been trying to estimate the number of limit cycles in discontinuous piecewise Hamiltonian differential systems with three zones. In this direction, we have papers with one limit cycle, see \cite{8, 18} and more than two limit cycles, see \cite{26, 27, 28}. In this work, we contribute along these lines. Our goal is to study the number of limit cycles that can bifurcated from periodic annulus of families of discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines. We prove that if the central subsystem, i.e. the system between the two parallel lines, has a real center and the others subsystems have centers or saddles, then we have at least three limit cycles, visiting the three zones, that bifurcate from an periodic annulus. Our results are obtained by studying the number of zeros of the Melnikov function for piecewise Hamiltonian system, see the papers \cite{26, 27} for more details about the Melnikov function.

In order to set the problem, let \(h_i : \mathbb{R}^2 \to \mathbb{R}, \ i = L, R,\) be the functions \(h_L(x, y) = x + 1\) and \(h_R(x, y) = x - 1.\) Denote by \(\Sigma_L = h_L^{-1}(0)\) and \(\Sigma_R = h_R^{-1}(0)\) the switching curves. This straight lines decomposes the plane in three regions

\[
R_L = \{(x, y) \in \mathbb{R}^2 : x < -1\}, \quad R_C = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1\},
\]

and

\[
R_R = \{(x, y) \in \mathbb{R}^2 : x > 1\}.
\]

Consider the discontinuous planar piecewise linear near–Hamiltonian system with three zones, given by

\[
\begin{align*}
\dot{x} &= H_y(x, y) + \epsilon f(x, y), \\
\dot{y} &= -H_x(x, y) + \epsilon g(x, y),
\end{align*}
\]
with

\[ H(x, y) = \begin{cases} 
H^L(x, y) = \frac{b_L}{2} y^2 - \frac{c_L}{2} x^2 + a_L x y + \alpha_L y - \beta_L x, & x \leq -1, \\
H^C(x, y) = \frac{b_C}{2} y^2 - \frac{c_C}{2} x^2 + a_C x y + \alpha_C y - \beta_C x, & -1 \leq x \leq 1, \\
H^R(x, y) = \frac{b_R}{2} y^2 - \frac{c_R}{2} x^2 + a_R x y + \alpha_R y - \beta_R x, & x \geq 1,
\end{cases} \]

(2) \quad f(x, y) = \begin{cases} 
f_L(x, y) = p_L x + q_L y + r_L, & x \leq -1, \\
f_C(x, y) = p_C x + q_C y + r_C, & -1 \leq x \leq 1, \\
f_R(x, y) = p_R x + q_R y + r_R, & x \geq 1,
\end{cases} \]

(3) \quad g(x, y) = \begin{cases} 
g_L(x, y) = s_L x + u_L y + v_L, & x \leq -1, \\
g_C(x, y) = s_C x + u_C y + v_C, & -1 \leq x \leq 1, \\
g_R(x, y) = s_R x + u_R y + v_R, & x \geq 1,
\end{cases} \]

where the dot denotes the derivative with respect to the independent variable \( t \), here called the time, and \( 0 \leq \epsilon << 1 \). We call system (1) of left subsystem when \( x \leq -1 \), right subsystem when \( x \geq 1 \) and central subsystem when \( -1 \leq x \leq 1 \). Denote by \( X_L(x, y) \), \( X_C(x, y) \) and \( X_R(x, y) \) the planar piecewise linear vector fields associated with the left, central and right subsystem from (1)|\( \epsilon=0 \), respectively.

We will use the vector field \( X_L \) and the switching curve \( \Sigma_L \) in the next definitions. However, we can easily adapt the definitions to the vector fields \( X_C \) and \( X_R \) and the switching curve \( \Sigma_R \).

We say that the vector field \( X_L \) has a real equilibrium \( p \) if \( p \) is an equilibrium of \( X_L \) and \( p \in R_L \). Otherwise, we will say that \( X_L \) has a virtual equilibrium \( p \) if \( p \in (R_L)^c \), where \( (R_L)^c \) denotes the complementary of \( R_L \) in \( \mathbb{R}^2 \).

The derivative of function \( h_L \) in the direction of the vector field \( X_L \), i.e., the expression \( X_L h_L(p) = (X_L(p), \nabla h_L(p)) \), where \( \langle \cdot, \cdot \rangle \) is the usual inner product in \( \mathbb{R}^2 \), characterize the contact between the vector field \( X_L \) and the switching curve \( \Sigma_L \). When \( p \in \Sigma_L \) and \( X_L h_L(p) = 0 \) we say that \( p \) is a tangent point of \( X_L \). We distinguish the followings subsets of \( \Sigma_L \) (the same for \( \Sigma_R \)).

Crossing set:

\[ \Sigma^c_L = \{ p \in \Sigma_L : X_L h_L(p) \cdot X_C h_L(p) > 0 \}; \]

Sliding set:

\[ \Sigma^s_L = \{ p \in \Sigma_L : X_L h_L(p) > 0, X_C h_L(p) < 0 \}; \]

Escaping set:

\[ \Sigma^e_L = \{ p \in \Sigma_L : X_L h_L(p) < 0, X_C h_L(p) > 0 \}. \]
Suppose that system (1)\(|_{\epsilon=0}\) satisfies the following hypotheses:

(H1) The unperturbed central subsystem from (1)\(|_{\epsilon=0}\) has a real center and the others unperturbed subsystems from (1)\(|_{\epsilon=0}\) have centers or saddles.

(H2) The unperturbed system from (1)\(|_{\epsilon=0}\) has only crossing points on the straights lines \(x=\pm 1\), except by some tangent points.

(H3) The unperturbed system from (1)\(|_{\epsilon=0}\) has a periodic annulus consisting of a family of crossing periodic orbits around the origin such that each orbit of this family passes thought the three zones with clockwise orientation.

The main result in this paper is the follow.

**Theorem 1.** The number of limit cycles of system (1), satisfying hypothesis (Hi) for \(i=1,2,3\), which can bifurcate from the periodic annulus of the unperturbed system (1)\(|_{\epsilon=0}\) is at least three.

The paper is organized as follows. In Section 2 we introduce the first order Melnikov function associated to system (1). In Section 3 we obtain a normal form to system (1)\(|_{\epsilon=0}\) that simplifies the computations and in Section 4 we will prove Theorem 1.

## 2. Melnikov Function

In this section, we will present the first order Melnikov function associated to system (1) that we will use to prove the main result of this paper.

Suppose that (1)\(|_{\epsilon=0}\) satisfies the hypothesis (H3), i.e. there exists an open interval \(J=(\alpha, \beta)\) such that for each \(h \in J\) we have four points, \(A(h)=(1,h), A_1(h)=(1,a_1(h)) \in \Sigma_R\), with \(a_1(h) < h\), and \(A_2(h)=(-1,a_2(h)), A_3(h)=(-1,a_3(h)) \in \Sigma_L\), with \(a_2(h) < a_3(h)\), whose are determined by the following equations

\[
\begin{align*}
H^R(A(h)) &= H^R(A_1(h)), \\
H^C(A_1(h)) &= H^C(A_2(h)), \\
H^L(A_2(h)) &= H^L(A_3(h)), \\
H^C(A_3(h)) &= H^C(A(h)),
\end{align*}
\]

(4)

satisfying, for \(h \in J\),

\[
\begin{align*}
H^R_y(A(h)) H^R_y(A_1(h)) H^C_y(A_2(h)) H^C_y(A_3(h)) &\neq 0, \\
H^C_y(A(h)) H^C_y(A_1(h)) H^C_y(A_2(h)) H^C_y(A_3(h)) &\neq 0.
\end{align*}
\]
Moreover, system $[1]_{|\epsilon=0}$ has a crossing periodic orbit $L_h = L_h^R \cup L_h^C \cup L_h^L \cup L_h^L$ passing through these points (see Fig. 1), where

$$L_h^R = \{ (x, y) \in \mathbb{R}^2 : H^R(x, y) = H^R(A(h)) = \frac{b_R}{2} + (a_R + \alpha_R)h - \left(\frac{c}{2} + \beta_R\right), x > 1 \},$$

$$L_h^C = \{ (x, y) \in \mathbb{R}^2 : H^C(x, y) = H^C(A_1(h)), -1 \leq x \leq 1 \text{ and } y < 0 \},$$

$$L_h^L = \{ (x, y) \in \mathbb{R}^2 : H^L(x, y) = H^L(A_2(h)), x < 1 \},$$

$$L_h^L = \{ (x, y) \in \mathbb{R}^2 : H^C(x, y) = H^C(A_3(h)), -1 \leq x \leq 1 \text{ and } y > 0 \}.$$

Consider the solution of right subsystem from $[1]$ starting from point $A(h)$. Let $A_\epsilon(h) = (1, a_\epsilon(h))$ be the first intersection point of this orbit with straight line $x = 1$. Denote by $B_\epsilon(h) = (-1, b_\epsilon(h))$ the first intersection point of the orbit from central subsystem from $[1]$ starting at $A_\epsilon(h)$ with straight line $x = -1$, $C_\epsilon(h) = (-1, c_\epsilon(h))$ the first intersection point of the orbit from left subsystem from $[1]$ starting at $B_\epsilon(h)$ with straight line $x = -1$ and $D_\epsilon(h) = (1, d_\epsilon(h))$ the first intersection point of the orbit from central subsystem from $[1]$ starting at $C_\epsilon(h)$ with straight line $x = 1$ (see Fig. 2).

We define the Poincaré map of piecewise system $[1]$ as follows,

$$H^R(D_\epsilon(h)) - H^R(A(h)) = \epsilon M(h) + O(\epsilon^2),$$

where $M(h)$ is called the first order Melnikov function associated to piecewise system $[1]$. Then, using the same idea of Theorem 1.1 in [21], it is easy to obtain the following theorem.
Theorem 2. Consider system (3) with $0 \leq \epsilon << 1$ and suppose that the unperturbed system $(3)|_{\epsilon = 0}$ has a family of crossing periodic orbits around the origin. Then the first order Melnikov function can be expressed as

$$M(h) = \frac{H^R_y(A)}{H^C_y(A)} I_C + \frac{H^R_y(A)H^C_y(A_3)}{H^C_y(A)H^L_y(A_3)} I_L + \frac{H^R_y(A)H^C_y(A_3)H^L_y(A_2)}{H^C_y(A)H^L_y(A_3)H^C_y(A_2)} I_C + \frac{H^R_y(A)H^C_y(A_2)H^L_y(A_1)}{H^C_y(A)H^L_y(A_3)H^C_y(A_2)H^R_y(A_1)} I_R,$$

where

$$I_C = \int_{A_1A_2} g_C dx - f_C dy, \quad I_L = \int_{A_2A_3} g_L dx - f_L dy, \quad I_C = \int_{A_1A_2} g_C dx - f_C dy$$

and

$$I_R = \int_{A_1A_3} g_R dx - f_R dy.$$

Furthermore, if $M(h)$ has a simple zero at $h^*$, then for $0 < \epsilon << 1$, the system $(3)$ has a unique limit cycle near $L_{h^*}$.

3. Normal Form

In order to simplify the computations to prove Theorem 3 is convenient to do a continuous linear change of variables which transform
system \( \Pi |_{\epsilon=0} \) into a simple form. This change of variables is a homeomorphism with keeps invariant the straight lines \( x = \pm 1 \). Furthermore, this homeomorphism will be a topological equivalence between the systems. More precisely, we have the follow result.

**Proposition 3.** The discontinuous piecewise linear differential systems \( \Pi |_{\epsilon=0} \) satisfying assumption \((H_i)\), \( i = 1, 2, 3 \), after a change of variables can be written as

\[
\begin{align*}
\dot{x} &= H_y(x, y), \\
\dot{y} &= -H_x(x, y),
\end{align*}
\]

where

\[
H(x, y) = \begin{cases}
H^L(x, y) &= \frac{b_L}{2}y^2 - \frac{c_L}{2}x^2 + a_L xy + a_L y - \beta_L x, \quad x \leq -1, \\
H^C(x, y) &= \frac{1}{2}x^2 + \frac{1}{2}y^2, \quad -1 \leq x \leq 1, \\
H^R(x, y) &= \frac{b_R}{2}y^2 - \frac{c_R}{2}x^2 + a_R xy - a_R y - \beta_R x, \quad x \geq 1.
\end{cases}
\]

**Proof.** Through a translation, we can assume that the singularity of the central subsystem from \( \Pi |_{\epsilon=0} \) is the origin, i.e. \( \alpha_C = \beta_C = 0 \). By the hypotheses \((H1)\) and \((H3)\), we have that the central subsystem from \( \Pi |_{\epsilon=0} \) satisfy \( a_C^2 + b_C c_C < 0 \) and \( b_C > 0 \). Note that \( b_i \neq 0 \), for \( i = L, R \). In fact, if the singular points of the subsystems from \( \Pi |_{\epsilon=0} \) are centers, this is true due to the clockwise orientation of the orbits. Now, if the singular points are saddles and \( b_i = 0 \) we have a separatrices parallel to switching straight lines \( x = \pm 1 \). System \( \Pi |_{\epsilon=0} \) has four tangent points given by \( P_1 = (1, -a_C/b_C) \), \( P_2 = (1, -(a_R + \alpha_R)/b_R) \), \( P_3 = (-1, a_C/b_C) \) and \( P_4 = (-1, (a_L - \alpha_L)/b_L) \). By hypothesis \((H2)\), we have that the system \( \Pi |_{\epsilon=0} \) have only crossing points on the straight lines \( x = \pm 1 \), except in the tangent points. Hence, for all \( y \in \mathbb{R} \setminus \{ \pm a_C/b_C, -a_R + \alpha_R)/b_R \}, (a_L - \alpha_L)/b_L) \), we must have

\[
\langle X_L(-1, y), (1, 0) \rangle \langle X_C(-1, y), (1, 0) \rangle > 0
\]

and

\[
\langle X_R(1, y), (1, 0) \rangle \langle X_C(1, y), (1, 0) \rangle > 0.
\]

But this implies that \( b_L b_C > 0 \), \( b_R b_C > 0 \), \( P_1 = P_2 \) and \( P_3 = P_4 \). Therefore, as \( b_C > 0 \), we have that

\[
\alpha_L = \frac{a_L b_C - a_C b_L}{b_C}, \quad b_L > 0, \quad \alpha_R = \frac{-a_R b_C + a_C b_R}{b_C} \quad \text{and} \quad b_R > 0.
\]
Assuming the conditions (8), consider the change of variables 
\[
\begin{pmatrix}
 x \\
 y
\end{pmatrix} = \begin{pmatrix}
 1 & 0 \\
 -\frac{a_C}{b_C} & \frac{\sqrt{-a_C^2 - b_CCc}}{b_C}
\end{pmatrix} \begin{pmatrix}
 u \\
 v
\end{pmatrix}
\]
and rescaling the time by \( \tilde{t} = \sqrt{-a_C^2 - b_CCc} t \). Applying the change of variables and rescaling the time above and rewriting the parameters conveniently, system \((\Pi)|_{\epsilon=0}\) becomes system \((\mathcal{C})\).

In what follows, we will consider the discontinuous planar piecewise linear near–Hamiltonian system \((\Pi)\) with \(f(x,y), g(x,y)\) and \(H(x,y)\) given by \((2), (3)\) and \((7), \) respectively.

4. **Proof of Theorem 1**

The proof of Theorem 1 is a straightforward consequence of Corollaries \([11, 13, 17]\).

We can classify the systems that satisfy the hypothesis (H1) according to the configuration of their singular points. Thus, denoting the centers by the capital letter C and by S the saddles, in the case of three zones, we have the following three class of piecewise linear Hamiltonian systems: SCS, CCS and CCC. This is, CCC indicates that the singular points of the linear systems that define the piecewise differential system are centers and so on.

In order to computate the zeros of the first order Melnikov function, it is necessary to find the open interval \(J\), where it is define. For this, consider the follow proposition.

**Proposition 4.** Consider the system \((\Pi)\) with the hypotheses (Hi), \(i = 1, 2, 3\).

(a) If the system \((\Pi)|_{\epsilon=0}\) is of type SCS or CCS, then \(J = (0, \tau)\), where \(\tau = (a_R^2 - b_R\beta_R - \omega_{RS}^2)/b_R\omega_{RS} = \sqrt{a_R^2 + b_Rc_R}\), and the periodic annulus are equivalents to one of the figures of Fig. 3.

(b) If the system \((\Pi)|_{\epsilon=0}\) is of type CCC, then \(J = (0, \infty)\), and the periodic annulus are equivalents to one of the figures of Fig. 4.

**Proof.** Suppose that the system \((\Pi)|_{\epsilon=0}\) is of type SCS or CCS. Note that if the saddles are virtual or if they are under the straight lines \(x = \pm 1\), then we have not periodic orbits passing through the three zones. Denote by \(W^u_R\) and \(W^s_R\) (resp. \(W^u_L\) and \(W^s_L\)) the unstable and stable separatrices of the saddles of the right (resp. left) subsystems.
from \(\mathbb{I}\)\(_{\epsilon=0}\), respectively. Denote by \(P^i_L = W^i_L \cap \Sigma_L\) and \(P^i_R = W^i_R \cap \Sigma_R\), for \(i = u, s\). After some computate, is possible to show that

\[
P^u_L = \left( -1, \frac{a^2_L + b_L \beta_L - \omega^2_{LS}}{b_L \omega_{LS}} \right), \quad P^s_L = \left( -1, \frac{a^2_L + b_L \beta_L - \omega^2_{LS}}{b_L \omega_{LS}} \right),
\]

\[
P^u_R = \left( 1, \frac{a^2_R - b_R \beta_R - \omega^2_{RS}}{b_R \omega_{RS}} \right) \quad \text{and} \quad P^s_R = \left( 1, \frac{a^2_R - b_R \beta_R - \omega^2_{RS}}{b_R \omega_{RS}} \right),
\]

where \(\omega_{LS} = \sqrt{a^2_L + b_L c_L}\) and \(\omega_{RS} = \sqrt{a^2_R + b_R c_R}\). Note that we have a symmetry between the points \(P^u_L\) and \(P^s_L\) (resp. \(P^u_R\) and \(P^s_R\)) with respect to \(x\)-axis. Define by \(\tau\) the smallest ordinate value between the points \(P^u_R\) and \(P^u_L\), i.e. \(\tau = \min\{a^2_R - b_R \beta_R - \omega^2_{RS}\}/b_R \omega_{RS}, (a^2_L + b_L \beta_L - \omega^2_{LS})/b_L \omega_{LS}\}\}. Then, less than one reflection around the \(y\)-axis, we can assuming that \(\tau = (a^2_R - b_R \beta_R - \omega^2_{RS})/b_R \omega_{RS}\).

As the vector field \(X_C\) associated with the central subsystem from \(\mathbb{I}\)\(_{\epsilon=0}\) is \(X_C(x, y) = (y, -x)\), if system \(\mathbb{I}\)\(_{\epsilon=0}\) is of type SCS and the ordinates of the points \(P^s_R\) and \(P^u_L\) are distinct (see Fig. 3 (a)) or if system \(\mathbb{I}\)\(_{\epsilon=0}\) is of type CCS (see Fig. 3 (c) or (d)), then we have a homoclinic loop passing through the points \(P^s_R\) and \(P^u_L\). Otherwise, if system \(\mathbb{I}\)\(_{\epsilon=0}\) is of type SCS and the ordinates of points \(P^s_R\) and \(P^u_L\) are the same (see Fig. 3 (b)) then we have a heteroclinic orbit passing through the points \(P^s_R, P^u_R, P^s_L\) and \(P^u_L\). Moreover, the central subsystem from \(\mathbb{I}\)\(_{\epsilon=0}\) has a periodic orbit tangent to straight lines \(x = \pm 1\) in the points \(P_R = (1, 0)\) and \(P_L = (-1, 0)\). The Fig. 4 shows the possibles phase portraits of the system \(\mathbb{I}\)\(_{\epsilon=0}\) of type SCS and CCS.

Consider a initial point of form \(A(h) = (1, h)\), with \(h \in (0, \tau)\). By the hypothesis (H3), the system \(\mathbb{I}\)\(_{\epsilon=0}\) has a family of crossing periodic orbits that intersects the straight lines \(x = \pm 1\) at four points, \(A(h), A_1(h) = (1, a_1(h))\), with \(a_1(h) < h\), and \(A_2(h) = (-1, a_2(h))\), \(A_3(h) = (-1, a_3(h))\), with \(a_2(h) < a_3(h)\) satisfying

\[
H^R(A(h)) = H^R(A_1(h)), \quad H^C(A_1(h)) = H^C(A_2(h)), \quad H^L(A_2(h)) = H^L(A_3(h)), \quad H^C(A_3(h)) = H^C(A(h))
\]
where $H^r$, $H^c$ and $H^l$ are given by (7). More precisely, we have the equations

\[
\frac{b_R}{2}(h - a_1(h))(h + a_1(h)) = 0,
\]
\[
\frac{1}{2}(a_1(h) - a_2(h))(a_1(h) + a_2(h)) = 0,
\]
\[
\frac{b_L}{2}(a_2(h) - a_3(h))(a_2(h) + a_3(h)) = 0,
\]
\[
\frac{1}{2}(a_3(h) - h)(a_3(h) + h) = 0.
\]

As $a_1(h) < h$, $a_2(h) < a_3(h)$, $b_R > 0$ and $b_L > 0$, the only solution of system above is $a_1(h) = -h$, $a_2(h) = -h$ and $a_3(h) = h$, i.e. we have the four points given by $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$ and $A_3(h) = (-1, h)$. Moreover, system (11) has a periodic orbit $L_h$ passing through these points, for all $h \in (0, \tau)$. If $h \in [\tau, \infty)$ then the orbit of the system (11) with initial condition in $A(h)$ do not

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**Figure 3.** Phase portrait of system $\text{I}\{\epsilon = 0\}$ of type: (a) SCS with the ordinates of points $P_R^s$ and $P_L^r$ distinct; (b) SCS with the ordinates of points $P_R^s$ and $P_L^u$ equal; (c) CCS when left subsystem has a virtual center; (d) CCS when left subsystem has a real center.
return to straight line $x = 1$ to positive times, i.e. the system $(\|)|_{\epsilon=0}$ has no periodic orbit passing thought the point $A(h)$. Therefore, if $h \in (0, \tau)$ the system $(\|)|_{\epsilon=0}$ has a periodic annulus, formed by the periodic orbits $L_h$, limited by one (see Fig. 3 (a)–(c)) or two (see Fig. 3 (d)) periodic orbits tangent to the straight lines $x = \pm 1$, when $h = 0$, and a homoclinic loop (see Fig. 3 (a), (c) and (d)) or heteroclinic orbit (see Fig. 3 (b)), when $h = \tau$. Therefore, item (a) is proven.

To prove item (b), suppose that the system $(\|)|_{\epsilon=0}$ is of type CCC. The central subsystem from $(\|)|_{\epsilon=0}$ has a periodic orbit tangent to straight lines $x = \pm 1$ in the points $P_R = (1, 0)$ and $P_L = (-1, 0)$. Moreover, as in the previous case, for each $h \in (0, \infty)$ we have a periodic orbit $L_h$ passing through points $A(h) = (1, h), A_1(h) = (1, -h), A_2(h) = (-1, -h)$ and $A_3(h) = (-1, h)$. Therefore, the system $(\|)|_{\epsilon=0}$ has a continuum of periodic orbit formed by the periodic orbits $L_h$, with $h \in (0, \infty)$, and limited by one (see Fig. 4 (a)), two (see Fig. 4 (a)) or three (see Fig. 4 (c)) periodic orbits tangent to straight lines $x = \pm 1$, when $h = 0$. The Fig. 4 shows the possibles phase portraits of the system $(\|)|_{\epsilon=0}$ of type CCC.

The coefficients that multiply the integrals of first order Melnikov function (5) associated to system $(\|)$ can be easily calculated. More precisely, we have the immediate corollary.

**Corollary 5.** Let $J$ be the interval of definition of Melnikov function (5). For $h \in J$,

\[
\frac{H_y^R(A)}{H_y^C(A)} = b_R, \quad \frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^R(A_3)} = b_R, \quad \frac{H_y^R(A)H_y^C(A_3)H_y^R(A_2)}{H_y^C(A)H_y^R(A_3)H_y^C(A_2)} = b_R
\]

and

\[
\frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^R(A_3)H_y^C(A_2)H_y^L(A_1)} = 1.
\]

Then, the first order Melnikov function associated to system $(\|)$ can be written as

\[
M(h) = b_R \int_{A_3 A_3} g_c \, dx - f_c \, dy + \frac{b_R}{b_L} \int_{A_2 A_3} g_L \, dx - f_L \, dy
\]

\[
+ b_R \int_{A_1 A_2} g_c \, dx - f_c \, dy + \int_{A_1 A_1} g_R \, dx - f_R \, dy.
\]

In what follows, we will determinate the first order Melnikov function associated to system $(\|)$ when the system $(\|)|_{\epsilon=0}$ is of the type SCS, CCS and CCC. For this, we define the functions:
Figure 4. Phase portrait of system \( \|_{\epsilon=0} \) of type CCC when: (a) the left and right subsystems have virtual centers; (b) the left subsystem has a real center and right subsystem has a virtual center; (c) the left and right subsystems have real centers.

\[
\begin{align*}
\mathbf{f}_0(h) &= h, \quad h \in (0, \infty), \\
\mathbf{f}_C^R(h) &= (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right), \quad h \in (0, \infty), \\
\mathbf{f}_C^L(h) &= \left( (a_R^2 - b_R \beta_R)^2 + (2a_R^2 + b_R^2 h^2 - 2b_R \beta_R) \omega_{RC}^2 \right) F_R^C(h) \\
&\quad + \omega_{RC}^4 F_R^C(h), \quad h \in (0, \infty), \\
\mathbf{f}_S^R(h) &= (a_L^2 + b_L \beta_L)^2 + (2a_L^2 + b_L^2 h^2 + 2b_L \beta_L) \omega_{LC}^2 F_L^C(h) \\
&\quad + \omega_{LC}^4 F_L^C(h), \quad h \in (0, \infty), \\
\mathbf{f}_S^L(h) &= (a_R^2 - b_R \beta_R + b_R \omega_{RS} h - \omega_{RS}^2) \\
&\quad \times (-a_R^2 + b_R \beta_R + b_R \omega_{RS} h + \omega_{RS}^2) F_R^S(h), \quad h \in (0, \tau), \\
\mathbf{f}_S^R(h) &= (a_R^2 - b_R \beta_R + b_R \omega_{RS} h + \omega_{RS}^2) \\
&\quad \times (-a_R^2 + b_R \beta_R + b_R \omega_{LS} h + \omega_{LS}^2) F_R^S(h), \quad h \in (0, \tau),
\end{align*}
\]
with

\[ F^C_R(h) = \arccos \left( 1 - \frac{2\beta_R^2\omega_{RC}^2 h^2}{(a_R^2 - b_R\beta_R)^2 + (2a_R^2 + b_R^2 h^2 - 2b_R\beta_R)\omega_{RC}^2 + \omega_{RC}^4} \right), \]

\[ F^C_L(h) = \arccos \left( 1 - \frac{2b_L^2\omega_{LC}^2 h^2}{(a_L^2 + b_L\beta_L)^2 + (2a_L^2 + b_L^2 h^2 + 2b_L\beta_L)\omega_{LC}^2 + \omega_{LC}^4} \right), \]

\[ F^S_R(h) = \log \left( 1 - \frac{2b_R\omega_{RS} h}{a_R^2 + b_R\beta_R + b_R\omega_{RS} h + \omega_{RS}^2} \right), \]

\[ F^S_L(h) = \log \left( 1 + \frac{2b_L\omega_{LS} h}{a_L^2 + b_L\beta_L - b_L\omega_{LS} h - \omega_{LS}^2} \right), \]

where \( \omega_{is} = \sqrt{a_i^2 + b_ic_i} \) and \( \omega_{ic} = \sqrt{-a_i^2 - b_ic_i} \), for \( i = L, R \).

**Theorem 6.** Suppose that system \((1)_{\epsilon=0}\) is of the type SCS. Then the first order Melnikov function \( M(h) \) associated with system \((1)_{\epsilon=0}\) can be expressed as

\[ (11) \quad M(h) = k_0 f_0(h) + k^C_C f^C_C(h) + k^S_S f^S_S(h) + k^S_S f^S_S(h), \]

for \( h \in (0, \tau) \), where the functions \( f_0, f^C_C, f^S_S, f^S_S \) are the ones defined in \((10)\). Here the coefficients \( k_0 \) and \( k^i_j \), for \( i = L, C, R \) and \( j = C, S \), depend on the parameters of system \((1)\).

**Proof.** The orbit \((x_R(x, y), y_R(x, y))\) of the system \((1)_{\epsilon=0}\), such that \((x_R(0, 0), y_R(0, 0)) = (1, h)\), is given by

\[ x_R(t) = \frac{e^{-t\omega_{RS}}}{2\omega_{RS}^2} (b_R\beta_R - a_R^2 + \omega_{RS}^2 - b_R\omega_{RS} h) + \frac{1}{2\omega_{RS}^2} (2a_R^2 - 2b_R\beta_R) + \frac{e^{t\omega_{RS}}}{2\omega_{RS}^2} (b_R\beta_R - a_R^2 + b_R\omega_{RS} h + \omega_{RS}^2), \]

\[ y_R(t) = \frac{e^{-t\omega_{RS}}}{2b_R\omega_{RS}^2} (a_R^3 - a_R b_R \beta_R + a_R^2 \omega_{RS} + a_R b_R \omega_{RS} h + b_R \beta_R \omega_{RS}^2 - a_R^2 \omega_{RS}^2) + \frac{e^{t\omega_{RS}}}{2b_R\omega_{RS}^2} (b_R \omega_{RS}^2 h - \omega_{RS}^2) + \frac{1}{2b_R\omega_{RS}^2} (-2a_R^3 + 2a_R b_R \beta_R + 2a_R \omega_{RS}^2) \]

\[ + \frac{e^{t\omega_{RS}}}{2b_R\omega_{RS}^2} (a_R^3 - a_R b_R \beta_R - a_R^2 \omega_{RS} - a_R b_R \omega_{RS} h - b_R \beta_R \omega_{RS}^2) + \frac{e^{t\omega_{RS}}}{2b_R\omega_{RS}^2} (-a_R \omega_{RS}^2 + b_R \omega_{RS}^2 h + \omega_{RS}^2). \]
The fly time of the orbit \((x_R(x, y), y_R(x, y))\), from \(A(h) = (1, h)\) to \(A_1(h) = (1, -h)\), is

\[ t_R = \frac{1}{\omega_{RS}} \log \left( 1 - \frac{2b_R\omega_{RS}h}{-a_R^2 + b_R\beta_R + b_R\omega_{RS}h + \omega_{RS}^2} \right). \]

Now, for \(g_R\) and \(f_R\) defined in (2) and (3), respectively, we have

\[
\int_{AA_1}^{t_R} g_R dx - f_R dy = \\
= \int_0^{t_R} g_R(x_R(t), y_R(t)) \frac{dx_R(t)}{dt} - f_R(x_R(t), y_R(t)) \frac{dy_R(t)}{dt} \\
= \alpha_1 h + \alpha_2 f_R^s(h),
\]

with

\[
\alpha_1 = p_R + 2r_R - u_R + \frac{(p_R + u_R)(a_R^2 - b_R\beta_R)}{\omega_{RS}^2} \quad \text{and} \quad \alpha_2 = \frac{p_R + u_R}{2b_R\omega_{RS}^3}.
\]

The orbit \((x_{C1}(x, y), y_{C1}(x, y))\) of the system \([1]_{\epsilon=0}\), such that \((x_{C1}(0, 0), y_{C1}(0, 0)) = (1, -h)\), is given by

\[
x_{C1}(t) = \cos(t) - h\sin(t), \\
y_{C1}(t) = -h\cos(t) - \sin(t).
\]

The fly time of the orbit \((x_{C1}(x, y), y_{C1}(x, y))\), from \(A_1(h) = (1, -h)\) to \(A_2(h) = (-1, -h)\), is

\[ t_{C1} = \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right) \]

Now, for \(g_C\) and \(f_C\) defined in (2) and (3), respectively, we obtain

\[
\int_{A_1A_2}^{t_{C1}} g_C dx - f_C dy = \\
= \int_0^{t_{C1}} g_C(x_{C1}(t), y_{C1}(t)) \frac{dx_{C1}(t)}{dt} - f_C(x_{C1}(t), y_{C1}(t)) \frac{dy_{C1}(t)}{dt} \\
= -2v_C + \alpha_3 h + \alpha_4 f_C^s(h),
\]

with

\[
\alpha_3 = u_C - p_C \quad \text{and} \quad \alpha_4 = \frac{p_C + u_C}{2}.
\]
The orbit \((x_L(x, y), y_L(x, y))\) of the system \(1\) \(|\epsilon=0\), such that \((x_L(0, 0), y_L(0, 0)) = (-1, -h)\), is given by

\[
x_L(t) = -\frac{e^{-t\omega_L}}{2\omega_L^2}(-a_L^2 - b_L\beta_L - b_L\omega_{LS}h + \omega_{LS}^2) - \frac{1}{2\omega_L^2}(2a_L^2 + 2b_L\beta_L) \nonumber
\]

\[
- \frac{e^{-t\omega_L}}{2\omega_L^2}(-a_L^2 - b_L\beta_L + b_L\omega_{LS}h + \omega_{LS}^2), \nonumber
\]

\[
y_L(t) = -\frac{e^{-t\omega_L}}{2b_L\omega_{LS}^2}(a_L^3 + a_Lb_L\beta_L + a_L^2\omega_{LS} + a_Lb_L\omega_{LS}h + b_L\beta_L\omega_{LS} - a_L\omega_{LS}^2) \nonumber
\]

\[
- \frac{e^{-t\omega_L}}{2b_L\omega_{LS}^2}(b_L\omega_{LS}^2h - \omega_{LS}^3) - \frac{1}{2b_L\omega_{LS}^2}(-2a_L^3 - 2a_Lb_L\beta_L + 2a_L\omega_{LS}^2) \nonumber
\]

\[
- \frac{e^{t\omega_L}}{2b_L\omega_{LS}^2}(a_L^3 + a_Lb_L\beta_L - a_L^2\omega_{LS} - a_Lb_L\omega_{LS}h - b_L\beta_L\omega_{LS}) \nonumber
\]

\[
- \frac{e^{t\omega_L}}{2b_L\omega_{LS}^2}(-a_L\omega_{LS}^2 + b_L\omega_{LS}^2h + \omega_{LS}^3). \nonumber
\]

The fly time of the orbit \((x_L(x, y), y_L(x, y))\), from \(A_2(h) = (-1, -h)\) to \(A_3(h) = (-1, h)\), is

\[
t_L = \frac{1}{\omega_L} \log \left(1 + \frac{2b_L\omega_{LS}h}{a_L^2 + b_L\beta_L + b_L\omega_{LS}h - \omega_{LS}^2}\right). \nonumber
\]

Now, for \(g_L\) and \(f_L\) defined in \(2\) and \(3\), respectively, we obtain

\[
\int_{A_2A_3} g_L \, dx - f_L \, dy =
\]

\[
(14) \quad = \int_0^{t_L} \frac{d}{dt} x_L(t) - f_L(x_L(t), y_L(t)) \frac{d}{dt} y_L(t) \nonumber
\]

\[
= \alpha_5 h + \alpha_6 f_L^s(h), \nonumber
\]

with

\[
\alpha_5 = p_L - 2r_L - u_L + \frac{(p_L + u_L)(a_L^2 + b_L\beta_L)}{\omega_{LS}^2} \quad \text{and} \quad \alpha_6 = \frac{p_L + u_L}{2b_L\omega_{LS}^2}. \nonumber
\]

Finally, the orbit \((x_{c2}(x, y), y_{c2}(x, y))\) of the system \(1\) \(|\epsilon=0\), such that \((x_{c2}(0, 0), y_{c2}(0, 0)) = (-1, h)\), is given by

\[
x_{c2}(t) = -\cos(t) + h \sin(t), \nonumber
\]

\[
y_{c2}(t) = h \cos(t) + \sin(t). \nonumber
\]

The fly time of the orbit \((x_{c2}(x, y), y_{c2}(x, y))\), from \(A_3(h) = (-1, h)\) to \(A(h) = (1, h)\), is

\[
t_{c2} = \arccos \left(\frac{h^2 - 1}{h^2 + 1}\right) \nonumber
\]
Now, for \( g_C \) and \( f_C \) defined in (2) and (3), respectively, we obtain

\[
\begin{align*}
\int_{A_1} g_C dx - f_C dy = \\
= \int_{0}^{t_{c_2}} g_C(x_{c_2}(t), y_{c_2}(t)) \frac{d}{dt} x_{c_2}(t) - f_C(x_{c_2}(t), y_{c_2}(t)) \frac{d}{dt} y_{c_2}(t) \\
= 2v_C + \alpha_3 h + \alpha_4 f_C^C(h).
\end{align*}
\]

Therefore, replacing (12), (13), (14) and (15) in (9), we obtain

\[
M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h) + k_L^S f_L^S(h)
\]

with

\[
k_0 = \alpha_1 + 2b_R \alpha_3 + \frac{b_R}{b_L} \alpha_5, \quad k_C^C = 2b_R \alpha_4, \quad k_R^S = \alpha_2 \quad \text{and} \quad k_L^S = \frac{b_R}{b_L} \alpha_6.
\]

\[\square\]

**Remark 7.** Suppose that the system (11) is of type SCS and that the ordinates of the points \( P_u^L \) and \( P_s^R \) are equal, see Fig. 3 (b). Then the first order Melnikov function (11), is given by

\[
M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h),
\]

with

\[
k_0 = p_R - u_R + 2r_R + 2b_R \left( \frac{p_L - r_L}{b_L} + u_C - p_C \right) + (p_R + u_R) \left( \frac{a_R^2 - b_R \beta_R}{\omega_{RS}^2} \right) + (p_L + u_L) \left( \frac{a_R^2 - b_R \beta_R - \omega_{LS}^2}{\omega_{LS} \omega_{RS}} \right),
\]

\[
k_C^C = b_R (p_C + u_C) \quad \text{and} \quad k_R^S = \frac{\omega_{LS}(p_R + u_R) + \omega_{RS}(p_L + u_L)}{2b_R \omega_{LS} \omega_{RS}^3},
\]

where the functions \( f_0, f_C^C, f_R^S \) are the ones defined in (10). In fact, if the ordinates of the points \( P_u^L \) and \( P_s^R \) are equal, then

\[
\frac{a_L^2 + b_L \beta_L - \omega_{LS}^2}{b_L \omega_{LS}} = \frac{a_R^2 - b_R \beta_R - \omega_{RS}^2}{b_R \omega_{RS}}.
\]

Isolating the parameter \( \beta_L \) in the equality above, i.e.

\[
\beta_L = \frac{a_R^2 b_L \omega_{LS} - b_L b_R \beta_R \omega_{LS} - a_L^2 b_R \omega_{RS} + b_R \omega_{LS} \omega_{RS} - b_L \omega_{RS} \omega_{LS}^2}{b_L b_R \omega_{RS}},
\]

and replacing on function \( M(h) \) given by (11) we obtain the expression (16).

The next two theorems provide expressions for the Melnikov function in the cases CCS and CCC. The proof of these results is analogous to proof of Theorem 6.
Theorem 8. Suppose that systems $(1)|_{\epsilon=0}$ is of the type CCS. Then the first order Melnikov function $M(h)$ associated to system $(1)$ can be expressed as

$$M(h) = k_0 f_0(h) + k_C f_C(h) + k_R f_R(h) + k_L f_L(h),$$

for $h \in (0, \tau)$, where the functions $f_0, f_C, f_R, f_L$ are the ones defined in $(10)$. Here the coefficients $k_0$ and $k_i$, for $i = L, C, R$, depend on the parameters of system $(1)$.

Theorem 9. Suppose the systems $(1)|_{\epsilon=0}$ is of the type CCC. Then the first order Melnikov function $M(h)$ associated to system $(1)$ can be expressed as

$$M(h) = k_0 f_0(h) + k_C f_C(h) + k_R f_R(h) + k_L f_L(h),$$

for $h \in (0, \infty)$, where the functions $f_0, f_C, f_R, f_L$ are the ones defined in $(10)$. Here the coefficients $k_0$ and $k^C_i$, for $i = L, C, R$, depend on the parameters of system $(1)$.

The next corollaries provided lower bounds for the number of limit cycles of system $(1)$ that can bifurcate from a periodic annulus of system $(1)|_{\epsilon=0}$ in the cases SCS, CCS and CCC. But first, we will recall basic linear algebra results.

Let $\{f_0, f_1, \ldots, f_n\}$ a set of real functions defined on a proper interval $I \subset \mathbb{R}$. We say that $\{f_0, f_1, \ldots, f_n\}$ is linearly independent if the unique solution of the equation

$$\sum_{i=0}^{n} \alpha_i f_i(t) = 0,$$

for all $t \in I$, is $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$.

Proposition 10. If $\{f_0, f_1, \ldots, f_n\}$ is linearly independent then there exist $t_1, t_2, \ldots, t_n \in I$, with $t_i \neq t_j$ for $i \neq j$, and $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$, not all null, such that for every $j \in \{1, 2, \ldots, n\}$$$

$$\sum_{i=0}^{n} \alpha_i f_i(t_j) = 0.$$

For a proof of Proposition 10 see for instance [14].

Recall that if the Wronskian,

$$W(f_0, f_1, \ldots, f_n)(x) = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_n(x) \\ f'_0(x) & f'_1(x) & \cdots & f'_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n)}_0(x) & f^{(n)}_1(x) & \cdots & f^{(n)}_n(x) \end{vmatrix},$$
where \( \{f_0, f_1, \ldots, f_n\} \) is a set of functions with derivatives until order \( n \) on \( I \), is different of zero for some \( x \in I \), then \( \{f_0, f_1, \ldots, f_n\} \) is linearly independent on \( I \).

**Corollary 11** (Case SCS – Fig. 3 (a)). Consider the system (1) with \( a_L = b_L = b_R = c_R = 1, a_R = c_L = 0, \beta_L = 2 \) and \( \beta_R = -2 \). Then, for 
\( 0 < \epsilon << 1 \), the system (1) has at least three limit cycles.

**Proof.** For \( a_L = b_L = b_R = c_R = 1, a_R = c_L = 0, \beta_L = 2 \) and \( \beta_R = -2 \),

the eigenvalues of the linear part of the left, central and right subsystem from (1)|\( x=0 \) are \( \pm 1, \pm i \) and \( \pm 1 \), respectively, i.e. we have one center and two saddles. Moreover, the coordinates of the equilibrium points on the left and right subsystem from (1)|\( x=0 \) are \( (-3, 2) \) and \( (2, 0) \), respectively, and so the saddles are real. Note that \( P_L^* = (-1, 2) \) and \( P_R^* = (1, 1) \), i.e. the ordinates of this points are different and \( \tau = 1 \).

Therefore, the first order Melnikov function from Theorem 6 becomes

\[
M(h) = k_0 f_0(h) + k^C f^C_C(h) + k^R_k^R f^R_R(h) + k^L_k^L f^L_L(h), \quad \forall h \in (0, 1),
\]

with

\[
f_0(h) = h, \quad f^C_C(h) = (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right), \quad f^R_R(h) = (h^2 - 1) \log \left( -\frac{h + 1}{h - 1} \right),
\]

\[
f^L_L(h) = (h^2 - 4) \log \left( -\frac{h + 2}{h - 2} \right),
\]

\[
k_0 = 2(2p_L - p_C - r_L + r_R + u_C + u_L) + 3p_R + u_R, \quad k^C_C = p_C + u_C, \quad k^R_k^R = \frac{p_R + u_R}{2} \quad \text{and} \quad k^L_k^L = \frac{p_L + u_L}{2}.
\]

Consider the set of functions \( \mathcal{F}_{SCS} = \{f_0, f^C_C, f^R_R, f^L_L\} \). Using the algebraic manipulator Mathematica (see [25]), we compute the Wronskian \( W(f_0, f^C_C, f^R_R, f^L_L)(h) \) and evaluate in some point on the interval \( (0, 1) \).

More precisely, \( W(f_0, f^C_C, f^R_R, f^L_L)(0.4) = 9.16568 \). Then, set of functions \( \mathcal{F}_{SCS} \) is linearly independent on the interval \( (0, 1) \) and, by Proposition 10 there are \( h_i \in (0, 1) \), with \( i = 1, 2, 3 \), such that \( M(h_i) = 0 \). Therefore, for \( 0 < \epsilon << 1 \), the system (1) has a unique limit cycle near \( L_{h_i} \), for each \( i = 1, 2, 3 \), i.e. the system (1) has at least three limit cycles. \( \square \)

**Corollary 12** (Case SCS – Fig. 3 (b)). Consider the system (1) with \( a_L = b_L = b_R = c_R = \beta_L = 1, a_R = c_L = 0 \) and \( \beta_R = -2 \). Then, for 
\( 0 < \epsilon << 1 \), the system (1) has at least two limit cycles.
Proof. For \( a_L = b_L = b_R = c_R = \beta_L = 1, a_R = c_L = 0 \) and \( \beta_R = -2 \), the eigenvalues of the linear part of the left, central and right subsystem from (1) are \( \pm 1, \pm i \) and \( \pm 1 \), respectively, i.e. we have one center and two saddles. Moreover, the coordinates of the equilibrium points on the left and right subsystem from (1) are \((-2,1)\) and \((2,0)\), respectively, and so the saddles are real. Note that \( P_L = (-1,1) \) and \( P_R = (1,1) \), i.e. the ordinates of this points are the same and \( \tau = 1 \). Therefore, the first order Melnikov function from Theorem 6 becomes

\[
M(h) = k_0 f_0(h) + k^C_C f^C_C(h) + k^S_R f^S_R(h), \quad \forall \ h \in (0, 1),
\]

with

\[
f_0(h) = h,
\]

\[
f^C_C(h) = (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right),
\]

\[
f^S_R(h) = (h^2 - 1) \log \left( \frac{h + 1}{h - 1} \right),
\]

\[
k_0 = 2(u_C - p_C - r_L + r_R) + 3(p_R + p_L) + u_R + u_L, \quad k^C_C = p_C + u_C
\]

and

\[
k^S_R = \frac{p_R + p_L + u_R + u_L}{2}.
\]

Consider the set of functions \( \mathcal{F}_{SCS} = \{ f_0, f^C_C, f^S_R \} \). Using the algebraic manipulator Mathematica, we compute the Wronskian \( W(f_0, f^C_C, f^S_R)(h) \) and evaluate in some point on the interval \((0, 1)\). More precisely, \( W(f_0, f^C_C, f^S_R)(0.4) = -10.6955 \). Then, set of functions \( \mathcal{F}_{SCS} \) is linearly independent on the interval \((0, 1)\) and, by Proposition 11 there are \( h_i \in (0, 1) \), with \( i = 1, 2 \), such that \( M(h_i) = 0 \). Therefore, for \( 0 < \epsilon << 1 \), the system (1) has a unique limit cycle near \( L_{h_i} \), for each \( i = 1, 2 \), i.e. the system (1) has at least two limit cycles. \( \square \)

**Corollary 13** (Case CCS – Fig. 3 (c)). Consider the system (1) with \( a_L = \beta_L = b_R = c_R = 1, b_L = 2, c_L = -1, a_R = 0 \) and \( \beta_R = -2 \). Then for \( 0 < \epsilon << 1 \), the system (1) has at least three limit cycles.

Proof. For \( a_L = \beta_L = b_R = c_R = 1, b_L = 2, c_L = -1, a_R = 0 \) and \( \beta_R = -2 \), the eigenvalues of the linear part of left, central and right subsystem from (1) are \( \pm i, \pm i \) and \( \pm 1 \), respectively, i.e. we have two centers and one saddle. Moreover, the coordinates of the equilibrium point on the left and right subsystem from (1) are \((3, -2)\) and \((2, 0)\), respectively, and so we have a virtual center and a real saddle. Note that \( P_R = (1,1) \) and \( \tau = 1 \). Therefore, the first order Melnikov function from Theorem 8 becomes

\[
M(h) = k_0 f_0(h) + k^C_C f^C_C(h) + k^S_R f^S_R(h) + k^C_L f^C_L(h), \quad \forall \ h \in (0, 1),
\]
with
\[ f_0(h) = h, \]
\[ f_C^C(h) = (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right), \]
\[ f_S^R(h) = (h^2 - 1) \log \left( \frac{-h + 1}{h - 1} \right), \]
\[ f_C^L(h) = (h^2 + 4) \arccos \left( \frac{4 - h^2}{h^2 + 4} \right), \]
\[ k_0 = 2(u_C - p_C + r_R - u_L) - r_L - p_L + u_R + 3p_R, \quad k_C^C = p_C + u_C, \]
\[ k_R^S = \frac{p_R + u_R}{2} \quad \text{and} \quad k_L^S = \frac{p_L + u_L}{2}. \]
Consider the set of functions \( \mathcal{F}_{CCS} = \{ f_0, f_C^C, f_S^R, f_C^L \} \). Using the algebraic manipulator Mathematica, we compute the Wronskian \( W(f_0, f_C^C, f_S^R, f_C^L)(h) \) and evaluate in some point on the interval \((0, 1)\). More precisely, \( W(f_0, f_C^C, f_S^R, f_C^L)(0.4) = 13.25 \). Then, set of functions \( \mathcal{F}_{CCS} \) is linearly independent on the interval \((0, 1)\) and, by Proposition 10, there are \( h_i \in (0, 1) \), with \( i = 1, 2, 3 \), such that \( M(h_i) = 0 \). Therefore, for \( 0 < \epsilon << 1 \), the system (1) has a unique limit cycle near \( L_{h_i} \), for each \( i = 1, 2, 3 \), i.e. the system (1) has at least three limit cycles.

Corollary 14 (Case CCS – Fig. 3 (d)). Consider the system (1) with \( a_L = b_R = c_R = 1, b_L = 2, c_L = -1, a_R = 0 \) and \( \beta_R = \beta_L = -2 \). Then for \( 0 < \epsilon << 1 \), the system (1) has at least three limit cycles.

Proof. For \( a_L = b_R = c_R = 1, b_L = 2, c_L = -1, a_R = 0 \) and \( \beta_R = \beta_L = -2 \), the eigenvalues of the linear part of left, central and right subsystem from (1)\( |_{\epsilon=0} \) are \( \pm i, \pm i \) and \( \pm 1 \), respectively, i.e. we have two centers and one saddle. Moreover, the coordinates of the equilibrium point on the left and right subsystem from (1)\( |_{\epsilon=0} \) are \((-3, 1)\) and \((2, 0)\), respectively, and so we have a real center and a real saddle. Note that \( P^* = (1, 1) \) and \( \tau = 1 \). Therefore, the first order Melnikov function from Theorem 8 becomes
\[ M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h) + k_L^S f_L^S(h), \quad \forall h \in (0, 1), \]
with
\[ f_0(h) = h, \]
\[ f_C^C(h) = (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right), \]
\[ f_R^S(h) = (h^2 - 1) \log \left( \frac{-h + 1}{h - 1} \right), \]
\[ f_L^C(h) = (h^2 + 4) \arccos \left( \frac{4 - h^2}{h^2 + 4} \right). \]
Consider the set of functions $\mathcal{F}_{CCS} = \{f_0, f_C^c, f_R^c, f_L^c\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^c, f_R^c, f_L^c)(h)$ and evaluate in some point on the interval $(0, 1)$. More precisely, $W(f_0, f_C^c, f_R^c, f_L^c)(0.2) = -4.26846$. Then, set of functions $\mathcal{F}_{CCS}$ is linearly independent on the interval $(0, 1)$ and, by Proposition 10, there are $h_i \in (0, 1)$, with $i = 1, 2, 3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, the system (11) has at least three limit cycles. 

**Corollary 15** (Case CCC – Fig. 4(a)). Consider the system (11) with $a_L = b_r = \beta_L = 1$, $b_L = 2$, $c_L = c_R = -1$ and $a_r = \beta_R = 0$. Then, for $0 < \epsilon << 1$, the system (11) has at least three limit cycles.

**Proof.** For $a_L = b_r = \beta_L = 1$, $b_L = 2$, $c_L = c_R = -1$ and $a_r = \beta_R = 0$, the eigenvalues of the linear part of the left, central and right subsystem from (11)$_{i=0}$ are $\pm i$, $\pm i$ and $\pm i$, respectively, i.e. we have three centers. Moreover, the coordinates of the equilibrium points of the left and right subsystem from (11)$_{i=0}$ are $(3, -2)$ and $(0, 0)$, respectively, and so the centers are virtual. Therefore, the first order Melnikov function from Theorem 9 becomes

$$M(h) = k_0 f_0(h) + k_C^c f_C^c(h) + k_R^c f_R^c(h) + k_L^c f_L^c(h), \quad \forall h \in (0, \infty),$$

with

$$f_0(h) = h,$$

$$f_C^c(h) = (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right),$$

$$f_R^c(h) = (h^2 - 1) \arccos \left( \frac{1 - h^2}{h^2 + 1} \right),$$

$$f_L^c(h) = (h^2 + 4) \arccos \left( \frac{4 - h^2}{h^2 + 4} \right),$$

$$k_0 = 2(u_C - p_C + r_R - u_L) - r_L - p_L - u_R + p_R, \quad k_C^c = p_C + u_C,$$

$$k_R^c = \frac{p_R + u_R}{2} \quad \text{and} \quad k_L^c = \frac{p_L + u_L}{2}.$$

Consider the set of functions $\mathcal{F}_{CCC} = \{f_0, f_C^c, f_R^c, f_L^c\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^c, f_R^c, f_L^c)(h)$ and evaluate in some point on the interval $(0, \infty)$. More precisely, $W(f_0, f_C^c, f_R^c, f_L^c)(0.2) = -2.92151$. Then, set of functions $\mathcal{F}_{CCC}$ is linearly independent on the interval $(0, \infty)$ and, by Proposition 10, there are $h_i \in (0, \infty)$, with $i = 1, 2, 3$, such that $M(h_i) = 0$. Therefore,
for $0 < \epsilon << 1$, the system (1) has a unique limit cycle near $L_{h_i}$, for each $i = 1, 2, 3$, i.e. the system (1) has at least three limit cycles. \( \square \)

**Corollary 16** (Case CCC – Fig. 4(b)). Consider the system (1) with $a_L = b_r = 1$, $\beta_L = -3$, $b_L = 2$, $c_L = c_R = -1$ and $a_R = \beta_R = 0$. Then, for $0 < \epsilon << 1$, the system (1) has at least three limit cycles.

**Proof.** For $a_L = b_r = 1$, $\beta_L = -3$, $b_L = 2$, $c_L = c_R = -1$ and $a_R = \beta_R = 0$, the eigenvalues of the linear part of the left, central and right subsystem from (1) are $\pm i$, $\pm i$ and $\pm i$, respectively, i.e. we have three centers. Moreover, the coordinates of the equilibrium points of the left and right subsystem from (1) are $(5, 2)$ and $(0, 0)$, respectively, and so we have a real center and a virtual center. Therefore, the first order Melnikov function from Theorem 9 becomes

$$M(h) = k_0 f_0(h) + k_C C(f_C(h) + k_R R f_R(h) + k_L L f_L(h), \quad \forall h \in (0, \infty),$$

with

$$f_0(h) = h,$$

$$f_C(h) = (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right),$$

$$f_R(h) = (h^2 - 1) \arccos \left( \frac{1 - h^2}{h^2 + 1} \right),$$

$$f_L(h) = (h^2 + 4) \arccos \left( \frac{4 - h^2}{h^2 + 4} \right),$$

$$k_0 = 2(u_C - p_C + r_L + u_L) - r_L - u_L + p_R + 3p_L, \quad k_C = p_C + u_C,$$

$$k_R = \frac{p_R + u_R}{2} \quad \text{and} \quad k_L = \frac{p_L + u_L}{2}.$$

Consider the set of functions $F_{ccc} = \{f_0, f_C, f_R, f_L\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C, f_R, f_L)(h)$ and evaluate in some point on the interval $(0, \infty)$. More precisely, $W(f_0, f_C, f_R, f_L)(0.5) = 7.2124$. Then, set of functions $F_{ccc}$ is linearly independent on the interval $(0, \infty)$ and, by Proposition 10, there are $h_i \in (0, \infty)$, with $i = 1, 2, 3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, the system (1) has a unique limit cycle near $L_{h_i}$, for each $i = 1, 2, 3$, i.e. the system (1) has at least three limit cycles. \( \square \)

**Corollary 17** (Case CCC – Fig. 4(c)). Consider the system (1) with $a_L = b_r = 1$, $\beta_L = -3$, $b_L = \beta_R = 2$, $c_L = c_R = -1$ and $a_R = 0$. Then, for $0 < \epsilon << 1$, the system (1) has at least three limit cycles.

**Proof.** For $a_L = b_r = 1$, $\beta_L = -3$, $b_L = \beta_R = 2$, $c_L = c_R = -1$ and $a_R = 0$, the eigenvalues of the linear part of the left, central and right subsystem from (1) are $\pm i$, $\pm i$ and $\pm i$, respectively, i.e. we have
three centers. Moreover, the coordinates of the equilibrium points of the left and right subsystem from (1) are \((-5, 2)\) and \((2, 0)\), respectively, and so the centers are real. Therefore, the first order Melnikov function from Theorem 9 becomes

\[
M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^C f_R^C(h) + k_L^C f_L^C(h), \quad \forall h \in (0, \infty),
\]

with

\[
f_0(h) = h, \\
f_C^C(h) = (h^2 + 1) \arccos \left( \frac{h^2 - 1}{h^2 + 1} \right), \\
f_R^C(h) = (h^2 - 1) \arccos \left( \frac{1 - h^2}{h^2 + 1} \right), \\
f_L^C(h) = (h^2 + 4) \arccos \left( \frac{4 - h^2}{h^2 + 4} \right), \\
k_0 = 2(u_C - p_C + r_R + u_L) - r_L + u_R + 3(p_L + p_R), \\
k_C^C = p_C + u_C, \\
k_R^S = \frac{p_R + u_R}{2} \quad \text{and} \quad k_L^S = \frac{p_L + u_L}{2}.
\]

Consider the set of functions \(F_{ccc} = \{f_0, f_C^C, f_R^C, f_L^C\}\). Using the algebraic manipulator Mathematica, we compute the Wronskian \(W(f_0, f_C^C, f_R^C, f_L^C)(h)\) and evaluate in some point on the interval \((0, \infty)\). More precisely, \(W(f_0, f_C^C, f_R^C, f_L^C)(0.5) = 7.2124\). Then, set of functions \(F_{ccc}\) is linearly independent on the interval \((0, \infty)\) and, by Proposition 10, there are \(h_i \in (0, \infty)\), with \(i = 1, 2, 3\), such that \(M(h_i) = 0\). Therefore, for \(0 < \epsilon << 1\), the system (1) has a unique limit cycle near \(L_{h_i}\), for each \(i = 1, 2, 3\), i.e. the system (1) has at least three limit cycles. \(\square\)

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