Bases in Linear Algebra

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Abstract

In this paper we define the $S$-bases for the spaces of tempered distributions. These new bases are the analogous of Hilbert bases of separable Hilbert spaces for the continuous case (they are indexed by $m$-dimensional Euclidean spaces) and enjoy properties similar to those shown by algebraic bases in the finite dimensional case. The $S$-bases are one possible rigorous and extremely manageable mathematical model for the “physical” bases used in Quantum Mechanics.

1 $S$-Linear independence

A finite family $v$ of vectors of a vector space is said linearly independent if any zero linear combination of $v$ is given by a zero system of coefficients. This is exactly the definition we generalize in the following.

Definition (of $S$-linear independence). Let $v \in S(\mathbb{R}^m, S'_n)$ be an $S$-family of tempered distributions in the space $S'_n$ indexed by the Euclidean space $\mathbb{R}^m$. The family $v$ is said to be $S$-linearly independent, if any coefficient distribution $a \in S'_m$ such that

$$\int_{\mathbb{R}^m} av = 0_{S'_n}$$

must be the zero distribution $0_{S'_m}$. In other terms the family $v$ is said $S$-linearly independent if and only if any zero $S$-linear combination of the family $v$ has necessarily a zero coefficient system.

We shall denote the superposition of a family of distribution $v$ with respect to a distributional system of coefficients $a$, also by the multiplication symbol $a.v$. 

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Example (The Dirac family). The Dirac family in $\mathcal{S}_n'$ is $\mathcal{S}$ linearly independent. In fact, we have
\[ \int_{\mathbb{R}^n} u \delta = u, \]
for all $u \in \mathcal{S}_n'$, and then the relation $u \cdot \delta = 0_{\mathcal{S}_n}$ is equivalent to $u = 0_{\mathcal{S}_n}$.

Example (the Fourier families). Let $a, b$ be two real non-zero numbers, recall that the $(a, b)$-Fourier family in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ is the family
\[ \left( a^{-n} \left[ e^{-ib(p \cdot \cdot)} \right] \right)_{p \in \mathbb{R}^n}, \]
of smooth and bounded regular tempered distributions (recall that $[f]$ denote the distribution canonically associated to a locally summable function $f$). In the particular case $a = 1$ and $b = -1/\hbar$ (with $\hbar$ the reduced Planck constant) we obtain what we call the De Broglie family, i.e. the family
\[ \left( e^{(i/\hbar)(p \cdot \cdot)} \right)_{p \in \mathbb{R}^n}. \]
All the Fourier families are $\mathcal{S}$ linearly independent. In fact, let $\varphi$ be the $(a, b)$-Fourier family, and let
\[ \int_{\mathbb{R}^n} u \varphi = 0_{\mathcal{S}_n'(\mathbb{C})}. \]
For every test function $\phi$ in $\mathcal{S}_n(\mathbb{C})$, we have
\[ 0 = \left( \int_{\mathbb{R}^n} u \varphi \right) (\phi) = u (\hat{\varphi} (\phi)) = u \left( \mathcal{S}_{(a,b)} (\phi) \right) = F_{(a,b)} (u) (\phi), \]
where $\mathcal{S}_{(a,b)}$ is the Fourier-Schwartz transformation on the space $\mathcal{S}_n(\mathbb{C})$, so that the Fourier transform of the distribution $u$ must be zero, i.e.
\[ F_{(a,b)} (u) = 0_{\mathcal{S}_n'(\mathbb{C})}, \]
and hence $u = 0_{\mathcal{S}_n'(\mathbb{C})}$, being the Fourier transformation $F_{(a,b)}$ an injective operator.

2 Linear and $\mathcal{S}$ linear independence

A first elementary connection between the classic linear independence and the new $\mathcal{S}$ linear independence is given by the following theorem, which states that
the $S$-linear independence is a stronger requirement for an $S$-family than the requirement of the simple linear independence.

Recall that an infinite family $v$ of vectors of a vector space is said to be linearly independent if any finite subfamily of $v$ is linearly independent.

**Theorem.** Let $v \in S(\mathbb{R}^m, S'_n)$ be an $S$-linearly independent family. Then, the family $v$ is linearly independent. Consequently, the $S$-linear hull $S\text{span}(v)$ is an infinite dimensional subspace of $S'_n$.

**Proof.** By contradiction, assume the family $v$ linearly dependent. Then there exists a linearly dependent finite subfamily of $v$. More precisely, there are a positive integer $k \in \mathbb{N}$ and a $k$-sequence $\alpha \in (\mathbb{R}^m)_k$ of points belonging to the $m$-dimensional Euclidean space $\mathbb{R}^m$ such that the $k$-sequence of distributions $v_\alpha = (v_{\alpha_i})_{i=1}^k$, extracted from the family $v$ by means of the index selection $\alpha$ is a linearly dependent system of $S'_n$. Then there exists a non-zero scalar $k$-tuple $a \in K^k$ such that the finite linear combination $\sum a_i v_{\alpha_i}$ is zero, that is such that

$$\sum_{i=1}^k a_i v_{\alpha_i} = 0_{S'_n}.$$ 

Consider the tempered distribution $d = \sum_{i=1}^k a_i \delta_{\alpha_i}$ as a coefficient distribution, we have

$$\int_{\mathbb{R}^m} dv = \int_{\mathbb{R}^m} \left( \sum_{i=1}^k a_i \delta_{\alpha_i} \right) v =$$

$$= \sum_{i=1}^k a_i \int_{\mathbb{R}^m} \delta_{\alpha_i} v =$$

$$= \sum_{i=1}^k a_i v_{\alpha_i} =$$

$$= 0_{S'_n}.$$ 

Now, since the distribution $d$ is different from the zero distribution $0_{S'_m}$, the preceding equality contradicts the $S$-linear independence of the family $v$, against our assumptions. ■

### 3 Topology and $S$-linear independence

The last theorem of the above section shows that, for what concerns the $S$-families, the $S$-linear independence implies the usual linear independence. Actually, the
Linear independence is more restrictive than the linear independence, as we shall see later by a simple notable example of family which is linearly independent but not \( S \)-linearly independent. On the contrary the \( S \)-linear independence is less restrictive than the \( \beta(S'_n) \)-topological independence, as it is shown below.

**Topological independence.** We recall that a system of vectors \( v = (v_i)_{i \in I} \) in the space \( S'_n \), indexed by a non-void index set \( I \), is said \( \beta(S'_n) \)-topologically free (respectively, \( \sigma(S'_n) \)-topologically free) if and only if there exists a family \( L = (L_i)_{i \in I} \) of \( \beta(S'_n) \)-continuous (respectively, \( \sigma(S'_n) \)-continuous) linear forms on \( S'_n \) such that

\[
L_i(v_k) = \delta_{ik},
\]

for any pair \((i, k) \in I^2\), where the family \( \delta = (\delta_{ik})_{(i, k) \in I^2} \) is the Kronecker family on the square \( I^2 \). Note that the above relation can be written as \( L \otimes v = \delta \), where \( \delta \) is the Kronecker family.

If the family \( v \) is not topologically free it is said topologically bound. If the family \( v \) is topologically free, any family \( L \) of continuous linear forms, satisfying the above relations is said a dual family of the family \( v \). So, to say that a family \( v \) is topologically free is equivalent to say that \( v \) has a dual family of linear continuous forms.

Recalling that (by reflexivity) any continuous linear functional on the space \( S'_n \) is canonically and univocally representable by a test function in \( S_n \), to say that the family \( v \) is topologically free is equivalent to say that the bi-orthonormality condition

\[
\langle g_i, v_k \rangle = \delta_{ik},
\]

is true, for any pair \((i, k) \in I^2\), where \( g \) is a suitable family of test functions.

**Theorem.** Every \( S \)-family in the space \( S'_n \) is \( \beta(S'_n) \)-topologically bound and, thus, \( \sigma(S'_n) \)-topologically bound. Consequently, no \( S \)-family has a dual family of test functions.

**Proof.** Let \( v \) be any \( S \)-family in the space \( S'_n \) indexed by \( \mathbb{R}^m \). And let \( L \) be an arbitrary family in the dual \( S''_n \) indexed by the same index set. Being the Schwartz space \( (S_n) \) reflexive, for every \( i \), there is a test function \( g_i \) in \( S_n \) canonically generating the functional \( L_i \), that is such that

\[
L_i = \langle \cdot, g_i \rangle.
\]

In other terms, the test function \( g_i \) is such that \( L_i(u) = u(g_i) \), for every tempered distribution \( u \) in \( S'_n \). Assume the existence of an index \( i \) such that \( L_i(v_i) = 1 \), then we deduce

\[
1 = L_i(v_i) = v_i(g_i) = v(g_i)(i),
\]

being \( v \) an \( S \)-family, the function \( v(g_i) \) is continuous, then there is a neighborhood \( U \) of the point \( i \) in which the function \( v(g_i) \) is strictly positive. Then, for every point \( k \) in the neighborhood \( U \), we have

\[
L_i(v_k) = v_k(g_i) = v(g_i)(k) > 0,
\]
and then $L$ cannot verify the condition

$$L_i(v_k) = \delta_{ik},$$

for any pair $(i, k) \in I^2$. So we cannot find a dual family of functionals for $v$ and, consequently, $v$ cannot be topologically independent. ■

**Note.** By the same proof, it is possible to prove that *every* $C^0$-family of tempered distributions is strongly topologically bound. Consequently every smooth family of tempered distributions is strongly topologically bound.

## 4 Multiplicity of representations

It’s simple to prove the following property that characterizes the $S$-linear dependence of a family of distributions by explicit multiplicity of representations of some member of the family itself.

**Property.** An $S$ family $v$ in $S_n$ indexed by $\mathbb{R}^m$ is $S$ linearly dependent if and only if there is a point index $p$ in $\mathbb{R}^m$ and a tempered distribution $a$ in $S_n'$ different from the Dirac delta distribution $\delta_p$ such that the $p$-th member of the family is representable also by

$$v_p = \int_{\mathbb{R}^m} av.$$

**Proof.** **Necessity.** Indeed, if $v_p$ fulfills that property we have that the $S$-linear combination $(a - \delta_p)v$ is zero with a non-zero coefficient distribution, so that the family $v$ is $S$-linearly dependent. **Sufficiency.** Vice versa, let, for every index-point $p$, the term $v_p$ of the family be representable in a unique way as the superposition $v_p = \delta_p v$. Assume $v$ $S$-linearly dependent, then there is a coefficient system $a$ different from zero such that $a.v = 0$, hence

$$v_p = \int_{\mathbb{R}^m} \delta_p v - 0 = \int_{\mathbb{R}^m} \delta_p v - \int_{\mathbb{R}^m} av = \int_{\mathbb{R}^m} (\delta_p - a)v,$$

since $a$ is a non-zero distribution, the distribution $\delta_p - a$ is different from the Dirac delta $\delta_p$; and so the member $v_p$ is representable in another (different) way, against the assumption. ■
5 Characterizations of \( S \)-linear independence

By the Dieudonné-Schwartz theorem we immediately deduce two characterizations.

We say an \( S \)-family \( v \) to be *topologically exhaustive* (with respect to the weak* topology or the strong topology) if its \( S \)-linear hull \( \bar{S} \text{span} (v) \) is \( \sigma(S'_n) \)-closed (or strongly closed, which is the same).

**Theorem.** Let \( v \in S(\mathbb{R}^m, S'_n) \) be a topologically exhaustive family, that is an \( S \)-family whose \( S \)-linear hull \( \bar{S} \text{span} (v) \) is \( \sigma(S'_n) \)-closed. Then the following assertions are equivalent

- 1) the family \( v \) is \( S \)-linearly independent;
- 2) the superposition operator \( \int_{\mathbb{R}^m} (\cdot, v) \) is an injective topological homomorphism for the weak* topologies \( \sigma(S'_m) \) and \( \sigma(S'_n) \);
- 3) the superposition operator \( \int_{\mathbb{R}^m} (\cdot, v) \) is an injective topological homomorphism for the strong topologies \( \beta(S'_m) \) and \( \beta(S'_n) \);
- 4) the operator \( \hat{\nu} \) is a surjective topological homomorphism for the weak topologies \( \sigma(S_n) \) and \( \sigma(S_m) \);
- 5) the operator \( \hat{\nu} \) is a surjective topological homomorphism of the topological vector space \( (S_n) \) onto the space \( (S_m) \).

**Remark (on the coordinate operator).** In the conditions of the above theorem, if the family \( v \) is \( S \)-linearly independent, we can consider the algebraic isomorphism from the space \( S'_m \) onto the \( S \)-linear hull \( \bar{S} \text{span} (v) \) sending every
tempered distribution \( a \in S'_m \) into the superposition \( a.v \), that is the restriction of the injection

\[
\int_{\mathbb{R}^m} (\cdot, v) : S'_m \to S'_n : (a, v) \mapsto a.v
\]

to the pair of sets \((S'_m, S\text{span}(v))\). We shall denote the inverse of this isomorphism by the symbol \([\cdot]|v\). It is an important consequence of the preceding theorem that

- **Theorem.** The operator

  \[
  [\cdot]|v : S\text{span}(v) \to S'_m
  \]

  is a topological isomorphism, with respect to the topology induced by the weak* topology \(\sigma(S'_n)\) on the \(S\) linear hull \(S\text{span}(v)\) and to the weak* topology \(\sigma(S'_m)\), if and only if the \(S\) linear hull \(S\text{span}(v)\) is \(\sigma(S'_n)\)-closed, that is if the family \(v\) is topologically exhaustive.

6 \(S\) **Bases**

**Definition (of \( S\) basis).** Let \( v \in S(\mathbb{R}^m, S'_n) \) be an \( S\) family in \( S'_n \), and let \( V \) be a subspace of the space \( S'_n \). The family \( v \) is said an \( S\) **basis** of the subspace \( V \) if it is \( S\) linearly independent and it \( S\) generates \( V \), that is

\[
S\text{span}(v) = V.
\]

In other terms, the family \( v \) is said an \( S\) basis of the subspace \( V \) if and only if the superposition operator of the family \( v \) is injective and its range coincides with the subspace \( V \).

The Dirac family \( \delta \) of \( S'_n \) is an \( S\) basis of the whole \( S'_n \) (indeed its superposition operator is the identity on \( S'_n \)). We call the Dirac family \( \delta \) the **canonical \( S\) basis** of \( S'_n \) (because of reasons which will appear clear soon, and that can be summarized into the relation \( u.\delta = u \), valid for any tempered distribution \( u \)) or the **Dirac basis** of \( S'_n \).

Moreover, the following complete version of the Fourier expansion-theorem, allow us to call the Fourier families of \( S'(\mathbb{R}^n, \mathbb{C}) \) by the name of **Fourier bases** of \( S'(\mathbb{R}^n, \mathbb{C}) \).

**Theorem (geometric form of the Fourier expansion theorem).** In the space of complex tempered distributions \( S'_n(\mathbb{C}) \) the Fourier families are \( S\) bases (of the entire space \( S'_n(\mathbb{C}) \)).

**Proof.** Indeed, the superposition operators of the Fourier families in \( S'_n \) are the Fourier transforms upon \( S'_n \) which are bijective. ■
7 Algebraic characterizations of $^S$bases

The following is an elementary but meaningful generalization of the Fourier expansion theorem.

**Theorem (characterization of an $^S$basis).** Let $v \in S(\mathbb{R}^m, S'_n)$ be an $^S$family. Then,

- 1) the family $v$ $^S$generates the space $S'_n$ if and only if the superposition operator $^t(\hat{v})$ is surjective;
- 2) the family $v$ is $^S$linearly independent if and only if the superposition operator $^t(\hat{v})$ is injective;
- 3) the family $v$ is an $^S$basis of the space $S'_n$ if and only if the superposition operator $^t(\hat{v})$ is bijective.

**Proof.** The proof follows immediately from the definitions, however we see it. First of all the superposition operator $^t(\hat{v})$ is well defined because $v$ is an $^S$family. Moreover, it is obvious, by the very definitions, that the family $v$ $^S$generates the space $S'_n$ if and only if the superposition operator $^t(\hat{v})$ is surjective, and that $v$ is $^S$linearly independent if and only if the superposition operator $^t(\hat{v})$ is injective. ■

7.1 Example

The following point gives us an example of linearly independent family which is not $^S$linearly independent and also an example of a linearly independent system of $^S$generators which is not an $^S$basis.

**Example (a system of linearly independent $^S$generators that is not an $^S$basis).** Let $v = (\delta'_x)_{x \in \mathbb{R}}$ be the family in $S'_1$ of the first derivatives of the Dirac distributions. The family $v$ is of class $S$, in fact

$$v(\phi)(x) = v_x(\phi) = \delta'_x(\phi) = -\phi'(x),$$

and the the derivative $\phi'$ is an $^S$function. Consequently, the operator associated with the family $v$ is the derivation in the test function space $S_1$ up to the sign and, then, the superposition operator $^t(\hat{v})$ is the derivation in the space $S'_1$. This last superposition operator is a surjective operator (every tempered distribution has a primitive) but it is not injective (every tempered distribution has many primitives), then the family $v$ is a system of $^S$generators for the space $S'_1$, but it is not $^S$linearly independent. Moreover, note that, however, the family $v$ is
linearly independent. In fact, let \( P \) be a finite subset of the real line \( \mathbb{R} \), and let, for every point \( p_0 \) in \( P \), \( f_{p_0} \) be a test function in \( S_1 \) such that

\[
f'_{p_0}(p) = \delta_{p_0,p},
\]

for every index \( p \) in \( P \), here \( \delta(\ldots) \) is the Kronecker delta upon the square \( P^2 \). Now, if \( a = (a_p)_{p \in P} \) is a finite family of scalars such that

\[
\sum_{p \in P} a_p v_p = 0_{S'_1},
\]

then

\[
0 = \left( \sum_{p \in P} a_p v_p \right) (f_{p_0}) = \sum_{p \in P} a_p \delta_{p_0,p} = a_{p_0},
\]

for every index \( p_0 \) in \( P \). And so, any finite linear combination of members of \( v \) is zero only with respect to a zero-system of coefficients.

8 Totality of \( S \) bases

We give another way to characterize an \( S \) basis.

**Theorem (characterization of an \( S \) basis).** Let \( v \in \mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n) \) be an \( S \) family. Then:

1) the family \( v \) is a system of \( S \) generators of the entire space \( \mathcal{S}'_n \) if and only if the family \( v \) is total in the space \( \mathcal{S}_n \) (in the sense that if \( v_p(g) = 0 \), for every index-point \( p \), then \( g = 0 \)) and the linear hull of the family \( v \) is weakly* closed;

2) the family \( v \) is \( S \) linearly independent if and only if it is total in the distribution space \( \mathcal{S}'_m \), in the sense that if \( a.v = 0 \) then \( a = 0 \);

3) a family \( v \) is an \( S \) basis of the space \( \mathcal{S}'_n \) if and only if the family \( v \) is total both in the function space \( \mathcal{S}_n \) and distribution space \( \mathcal{S}'_m \).

**Proof.** 1) Indeed, the condition means that the operator generated by the family \( v \) is injective and this (by the Dieudonné-Schwartz theorem) implies the surjectivity of the superposition operator of \( v \), since the image of the superposition operator of \( v \) is closed. 2) This is exactly the definition of linear independence. 3) Follows from the two before. ■
9 Topological characterizations of $\mathcal{S}$ bases

By the Dieudonné-Schwartz theorem we immediately take a characterization.

**Theorem.** Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'^n)$ be a family of tempered distributions. Then the following assertions are equivalent

- 1) the family $v$ is an $\mathcal{S}$ basis of the space $\mathcal{S}'^n$;
- 2) the superposition operator $\int_{\mathbb{R}^m} (\cdot, v)$ is a topological isomorphism for the weak* topologies $\sigma(\mathcal{S}'^m)$ and $\sigma(\mathcal{S}'^n)$;
- 3) the superposition operator $\int_{\mathbb{R}^m} (\cdot, v)$ is a topological isomorphism for the strong topologies $\beta(\mathcal{S}'^m)$ and $\beta(\mathcal{S}'^n)$;
- 4) the operator $\hat{v}$ is a topological isomorphism for the weak topologies $\sigma(\mathcal{S}^n)$ and $\sigma(\mathcal{S}^m)$;
- 5) the operator $\hat{v}$ is a topological isomorphism of the topological vector space $(\mathcal{S}^n)$ onto $(\mathcal{S}^m)$.

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