SLOPES OF FIBERED SURFACES WITH A FINITE CYCLIC AUTOMORPHISM

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Abstract. We study slopes of finite cyclic covering fibrations of a fibered surface. We give a best possible lower bound of the slope of these fibrations. We also give the slope equality of finite cyclic covering fibrations of a ruled surface and observe the local concentration of the global signature of these surfaces on a finite number of fiber germs. We also give an upper bound of the slope of finite cyclic covering fibrations of a ruled surface.

1. Introduction

Let $S$ be a complex projective smooth surface and $f: S \rightarrow B$ a surjective morphism from $S$ to a smooth complete curve $B$ with connected fibers. The datum $(S, f, B)$ or simply $f$ is called a fibered surface or a fibration. The genus $g$ of the general fiber of $f$ is called the genus of the fibered surface $f$. We call a fibration $f$ relatively minimal if the fibers contain no $(-1)$-curves (i.e. a smooth rational curve with self-intersection number $-1$). Put $K_f = K_S - f^* K_B$ and call it the relative canonical divisor. We consider the following relative invariants of $f$:

\begin{align*}
\chi_f & := \chi(O_S) - (g - 1)(b - 1), \\
K_f^2 & := K_S^2 - 8(g - 1)(b - 1), \\
e_f & := e(S) - 4(g - 1)(b - 1),
\end{align*}

where $b$ is the genus of the base curve $B$ and $e(S)$ is the topological Euler characteristic of $S$. They satisfy the following well-known results:

(1) (Noether) $12\chi_f = K_f^2 + e_f$.
(2) (Arakelov) If $g \geq 2$, then $f$ is relatively minimal if and only if $K_f$ is nef.
(3) (Ueno) If $g \geq 2$ and $f$ is relatively minimal, then $\chi_f \geq 0$, and $\chi_f = 0$ if and only if $f$ is locally trivial.
(4) If $g \geq 2$, then $e_f \geq 0$, and $e_f = 0$ if and only if $f$ is smooth.

When $f$ is not locally trivial, we define the slope of $f$ as

$$\lambda_f = \frac{K_f^2}{\chi_f}.$$ 

It follows $0 < \lambda_f \leq 12$ from the Noether’s formula. The slope of a fibration turns out to be sensible to a lot of geometric properties, both of the fibers of $f$ and of the surface $S$ itself.

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(cf. [1]). If $f$ is relatively minimal and $g \geq 2$, the slope $\lambda_f$ satisfies the following inequality:

$$\lambda_f \geq 4 - \frac{4}{g},$$

called the slope inequality. It was proven by Horikawa and Persson for hyperelliptic fibrations, and by Xiao for general fibrations (cf. [5]). The equality holds only when $f$ is hyperelliptic. Given the family of fibered surface satisfying a certain property, a sharper slope inequality holds. In [3], Cornalba and Stoppino show the sharp bound for the slope of double cover fibrations and construct a fibration which attains the bound. Our first aim is to generalize the lower bound of the slope of double cover fibrations to of finite classical cyclic covering ones. In §2, we will prove

Theorem 1.1. Let $f: S \to B$ be a genus $g \geq 2$ fibration which is the relatively minimal model of a classical $n$-cyclic covering of a genus $h$ fibered surface. Assume $h \geq 1$ and $g \geq (2n-1)(2hn+n-1)/(n+1)$. Then, we have

$$K_f^2 \geq \frac{24(g-1)(n-1)}{2(2n-1)(g-1) - n(n+1)(h-1)} \chi_f.$$

Moreover, we will construct an example showing that the inequality is sharp. The essential idea of the proof is much similar to that in [3], but the computation is more complicated. When $g \geq n(n-1)/2$, the same inequality is also true for $h = 0$. We will treat the case where $h = 0$ and show the so-called slope equality in §3. Namely, we will show that

Theorem 1.2. Let $f: S \to B$ be as above and assume $h = 0$. Then, there exists a $\mathbb{Q}$-valued function $\text{Ind}(F_p)$ on $B$ such that $\text{Ind}(F_p) = 0$ for a general fiber $F_p$ and satisfies the following equality:

$$K_f^2 = \frac{24(g-1)(n-1)}{2(2n-1)(g-1) + n(n+1)} \chi_f + \sum_{p \in B} \text{Ind}(F_p).$$

Moreover, $\text{Ind}(F_p)$ is non-negative when $g \geq n(n-1)/2$.

This theorem is a generalization of the hyperelliptic case in [6]. The function $\text{Ind}(F_p)$ is called the Horikawa index of $f$. A general discussion for slope equalities and Horikawa indices can be found in [1]. Finally, we will examine the upper bound of the slope. More precisely, we will show that

Theorem 1.3. Let $f: S \to B$ be a fibered surface as above and assume $h = 0$ and $n \geq 4$. Then,

$$K_f^2 \leq \left(12 - \frac{48n(n-1)(r-1)}{n(n+1)r^2 - 8(2n-1)r + 24n} \right) \chi_f$$

where $r = 2g/(n-1) + 2$ is the number of branch points for the $n$-cyclic covering of general fibers.

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2. Bounds on the slope of fibered surfaces with a finite cyclic automorphism

Let $n$ be a positive integer. Let $X$ be a complex projective smooth surface and $R$ an effective divisor on $X$ linearly equivalent to $n\delta$ for some divisor $\delta$. We define a graded $\mathcal{O}_X$-algebra structure on $\mathcal{A} = \bigoplus_{j=0}^{n-1} \mathcal{O}_X(-j\delta)$ by multiplying the section of $\mathcal{O}_X(n\delta)$ defining $R$ and put $Y = \text{Spec}_X \mathcal{A}$. The natural map $\varphi: Y \to X$ is called a classical $n$-cyclic covering branched over $R$ (cf. [2]).

Locally, $Y$ is defined by $z^n = r(x, y)$, where $r(x, y)$ is the local equation of $R$. From this representation, one sees that $R$ is smooth if and only if $Y$ is smooth, and $R$ is reduced if and only if $Y$ is normal. Moreover, if $Y$ is smooth, we have

\begin{align}
\varphi^*R &= nR_0, \\
K_Y &= \varphi^*K_X + (n-1)R_0, \\
\text{Aut}(Y/X) &\cong \mathbb{Z}/n\mathbb{Z},
\end{align}

where $R_0$ is the effective divisor defined by $z = 0$ locally. In general, an $n$ sheeted Galois covering $\varphi: Y \to X$ is said to be totally ramified if the inverse image of any branch point consists of one point, that is, it satisfies (2.1) for some $R_0$, and to be an $n$-cyclic covering if it satisfies (2.2).

Remark 2.1. If $R$ is smooth but not irreducible and $n \geq 3$, a totally ramified $n$-cyclic covering $\varphi: Y \to X$ branched over $R$ is not necessarily a classical $n$-cyclic covering defined above. Indeed, let $\pi: \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism of degree $n$ defined by $z \mapsto z^n$ and set $X = Y = \mathbb{P}^1 \times \mathbb{P}^1$ and $\varphi = \text{Id} \times \pi: Y \to X$. Then, $\varphi$ is a totally ramified $n$-cyclic covering branched over $\mathbb{P}^1 \times \{0, \infty\}$ but not the classical $n$-cyclic covering since $\mathbb{P}^1 \times \{0, \infty\}$ is not linearly equivalent to $n\delta$ for any divisor $\delta$.

In this paper, an $n$-cyclic covering means a classical $n$-cyclic covering constructed above unless otherwise noted.

Here we show the following elementary lemma for the later use.

Lemma 2.2. Let $n \geq 4$ be a positive integer and $a, b$ integers such that $\text{gcd}(a, b, n) = 1$. Then, it follows that $a + 2b \notin n\mathbb{Z}$ or $2a + b \notin n\mathbb{Z}$.

Proof. Suppose $a + 2b \in n\mathbb{Z}$ and $2a + b \in n\mathbb{Z}$. Then, we have $3(a + b) \in n\mathbb{Z}$ and $a - b \in n\mathbb{Z}$. If $n \notin 3\mathbb{Z}$, we have $a + b \in n\mathbb{Z}$ and it follows $a \in n\mathbb{Z}$ and $b \in n\mathbb{Z}$, which contradicts $\text{gcd}(a, b, n) = 1$. Thus, we have $n = 3k$ for some integer $k \geq 2$. Since $a - b \in n\mathbb{Z}$, we may write $a = b + 3lk$ for some $l$. Then $2a + b = 3(b + 2lk)$. Since $2a + b \in n\mathbb{Z} = 3k\mathbb{Z}$, we have $b \in k\mathbb{Z}$. Then, $a \in k\mathbb{Z}$, contradicting $\text{gcd}(a, b, n) = 1$.

This lemma is apparently false when $n = 3$ as the case $a = b = 1$ shows.

Let $X$ be a smooth projective surface with an automorphism $\sigma$ of order $n$. We take a fixed point $x \in \text{Fix}(\sigma)$ and a local coordinate $(U; z_1, z_2)$ centered at $x$ such that $\sigma(U) = U$. We write $\sigma(z) = (\sigma_1(z_1, z_2), \sigma_2(z_1, z_2))$ and expand $\sigma_i(z_1, z_2) = a_{i,1}z_1 + a_{i,2}z_2 + \cdots$ near $x$. The
Jacobian matrix at $x$ is

$$(J\sigma)_x = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$ 

Since $\sigma^n = \text{Id}$, we may assume

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \zeta^{k_1} & 0 \\ 0 & \zeta^{k_2} \end{pmatrix}$$

where $\zeta = e^{2\pi i/n}$ and $0 \leq k_1 \leq k_2 \leq n - 1$. Since $\sigma \neq \text{Id}$, we have $k_2 > 0$. Clearly, $x$ is a smooth point of a 1-dimensional fixed part if and only if $k_1 = 0$, and $x$ is an isolated fixed point if and only if $k_1 > 0$. Such a point $x$ is called a fixed point of type $(k_1, k_2)$. Since $\sigma$ is of order $n$, we have gcd$(k_1, k_2, n)$ = 1. We consider the blow-up $\rho: \tilde{X} \to X$ of a fixed point $x$ of type $(k_1, k_2)$. Let $E$ be the exceptional curve of $\rho$ and $\sigma$ the automorphism of $\tilde{X}$ of order $n$ induced by $\sigma$. By easy calculations, one sees that $E$ is fixed by $\sigma$ if $k_1 = k_2$ and that there are exactly two isolated fixed points 0, $\infty$ on $E$ of types $(k_1, k_2 - k_1)$ and $(k_1 - k_2, k_2)$, respectively if otherwise. Applying Lemma 2.2 to $(a, b) = (k_1, -k_2)$ repeatedly, we have

**Lemma 2.3.** Let $X$ be a smooth projective surface with an automorphism $\sigma$ of order $n$. Then, there exists a birational morphism $\rho: \tilde{X} \to X$ such that the automorphism $\sigma$ on $\tilde{X}$ induced by $\sigma$ has no isolated fixed points if and only if $n \leq 3$ or any isolated fixed point of $\sigma$ is of type $(k, k)$ for some $k$.

Let $f: S \to B$ be a genus $g \geq 2$ relatively minimal fibered surface. In this section, suppose the following ($\ast$):

($\ast$) There exists a genus $h$ fibered surface $\bar{\phi}: \tilde{W} \to B$ which is not necessarily relatively minimal and an $n$-cyclic covering $\bar{\theta}: \tilde{S} \to \tilde{W}$ branched over a smooth divisor $\tilde{R} \in |n\tilde{\delta}|$ for some $n \geq 2$ and $\tilde{\delta} \in \text{Pic}(\tilde{W})$ such that $f: S \to B$ is the relatively minimal model of $\tilde{f} = \bar{\phi} \circ \bar{\theta}: \tilde{S} \to B$.

Let $\tilde{F}$ and $\tilde{\Gamma}$ be general fibers of $\tilde{f}$ and of $\bar{\phi}$, respectively. The restriction map $\bar{\theta}|_{\tilde{F}}: \tilde{F} \to \tilde{\Gamma}$ is a finite morphism branched over $\tilde{R} \cap \tilde{\Gamma}$. By the Hurwitz formula, it follows that

$$(2.3) \quad r = \frac{2(g - 1 - n(h - 1))}{n - 1},$$

where $r := \tilde{R}\tilde{\Gamma}$.

Let $\tilde{\sigma}$ be a generator of $\text{Aut}(\tilde{S}/\tilde{W}) \simeq \mathbb{Z}/n\mathbb{Z}$ and $\rho: \tilde{S} \to S$ a natural birational morphism. Put $\text{Fix}(\tilde{\sigma}) = \{x \in \tilde{S} | \tilde{\sigma}(x) = x\}$. It is easy to see that $\text{Fix}(\tilde{\sigma})$ is a disjoint union of smooth curves and $\bar{\theta}(\text{Fix}(\tilde{\sigma})) = \tilde{R}$. Let $\varphi: W \to B$ be a relatively minimal model of $\bar{\phi}$ and $\tilde{\psi}: \tilde{W} \to W$ the natural birational morphism. We may decompose $\tilde{\psi}$ into a series of blow-ups. Let $\psi_i: W_i \to W_{i-1}$ be a blow-up at $x_i \in W_{i-1}$ such that $\tilde{\psi} = \psi_1 \circ \cdots \circ \psi_N$, where $W_N = \tilde{W}$ and $W_0 = W$. We define $R_i$ as $R_N = \tilde{R}$, $R_{i-1} = (\psi_i)_*R_i$ inductively. Set $R = R_0$, $E_i = \psi_i^{-1}(x_i)$ and $m_i = \text{mult}_{x_i}(R_{i-1})$. 


Lemma 2.4. (1) $m_i \in n\mathbb{Z}$ or $n\mathbb{Z}+1$ for any $i = 1, \ldots, N$. Moreover, $m_i \in n\mathbb{Z}$ if and only if $E_i$ is not contained in $R_i$.

(2) We have

$$R_i = (\psi_i)^*R_{i-1} - n\left\lfloor \frac{m_i}{n} \right\rfloor E_i,$$

where $[x]$ is the greatest integer not greater than $x$.

Proof. Since $\tilde{R}$ is reduced, every $R_i$ is reduced. Set $\delta_N = \tilde{\delta}$. Since

$$\text{Pic}(\tilde{W}) = (\psi_N)^*\text{Pic}(W_{N-1}) \oplus \mathbb{Z}[E_N],$$

there exist $\delta_{N-1} \in \text{Pic}(W_{N-1})$ and $d_N \in \mathbb{Z}$ such that $\delta_N = (\psi_N)^*\delta_{N-1} - d_NE_N$. Inductively, we may take $\delta_{i-1} \in \text{Pic}(W_{i-1})$ and $d_i \in \mathbb{Z}$ such that $\delta_i = (\psi_i)^*\delta_{i-1} - d_iE_i$. Since $\tilde{R} \sim n\tilde{\delta} = (\psi_N)^*n\delta_{N-1} - nd NE_N$ and $R_{N-1} = (\psi_N)^*R_N$, it follows $R_{N-1} \sim n\delta_{N-1}$ and inductively $R_i \sim ndi$. If $E_i$ is not contained in $R_i$, then $R_i$ is the proper transform of $R_{i-1}$ and so $m_i = nd_i$. If $E_i$ is contained in $R_i$, then $R_i - E_i$ is the proper transform of $R_{i-1}$ and so $m_i = nd_i + 1$ since $R_i$ is reduced. Hence $d_i = [m_i/n]$ in both cases.

Lemma 2.5. Let $E$ be a $(-1)$-curve on a fiber of $\tilde{f}$ and $L = \tilde{\theta}(E)$. Then, the following hold:

(1) If $E$ is contained in $\text{Fix}(\tilde{\sigma})$, then $L$ is a vertical $(-n)$-curve in $\tilde{R}$. Conversely, for any vertical $(-n)$-curve $L$ in $\tilde{R}$, there exists a $(-1)$-curve $E$ on a fiber of $\tilde{f}$ contained in $\text{Fix}(\tilde{\sigma})$ such that $\tilde{\theta}^*L = nE$.

(2) If $E$ is not contained in $\text{Fix}(\tilde{\sigma})$, then $L$ is a vertical $(-1)$-curve and there exist $(-1)$-curves $E_2, \ldots, E_n$ on a fiber of $\tilde{f}$ such that $\tilde{\theta}^*L = E_1 + E_2 + \cdots + E_n$ and $E_1, E_2, \ldots, E_n$ are disjoint, where $E_1 = E$. Moreover, let $\overline{\varphi}: \overline{S} \to \overline{S}$ be a contraction of $E_1, E_2, \ldots, E_n$ and $\overline{\psi}: \overline{W} \to \overline{W}$ a contraction of $L$, then there exists a natural $n$-cyclic covering $\overline{\vartheta}: \overline{S} \to \overline{W}$ branched over $\overline{R} = \overline{\psi}_*\overline{R}$ such that $\overline{\vartheta} \circ \overline{\varphi} = \overline{\psi} \circ \overline{\vartheta}$.

Proof. Suppose that $E$ is contained in $\text{Fix}(\tilde{\sigma})$. Since $L$ is contained in $\tilde{R}$, it follows $\tilde{\theta}^*L = nE$. Hence we have $nL^2 = (\tilde{\theta}^*L)^2 = n^2E^2 = -n^2$. Thus, we get $L^2 = -n$. Clearly, the restriction map $\tilde{\theta}|_E: E \to L$ is an isomorphism and then $L$ is a $(-n)$-curve. Since $E$ is contracted by $\rho$, $L$ is vertical with respect to $\tilde{\sigma}$. The rest assertion of (1) is obvious.

Suppose that $E$ is not contained in $\text{Fix}(\tilde{\sigma})$. If $\tilde{\theta}^*L = E$, we have that $L^2 = -1/n$ and then it is a contradiction. Hence there exist $(n-1)$ curves $E_2, \ldots, E_n$ such that $\tilde{\theta}^*L = E_1 + E_2 + \cdots + E_n$, where $E_1 = E$. Every $E_i$ is a $(-1)$-curve since $E$ is a $(-1)$-curve and $E$ is translated into $\tilde{E}_i$ by some composite of $\tilde{\sigma}$. All $\tilde{E}_i$’s are contracted by $\rho$ since the relatively minimal model of $\tilde{f}: \tilde{S} \to B$ is unique. Hence $E_i$’s are disjoint and equivalently $L$ and $\tilde{R}$ is disjoint. Thus, $\overline{R} := \overline{\psi}_*\overline{R}$ is smooth and we have $\overline{R} = \overline{\psi} \overline{R}$. Moreover, there exist $\delta \in \text{Pic}(\overline{W})$ such that we have $\overline{R} \sim n\delta$ and $\tilde{\delta} = \overline{\psi} \delta$. Hence, we can construct the $n$-cyclic covering $\overline{\vartheta}: \overline{S} \to \overline{W}$ branched over $\overline{R}$ where $\overline{S} = \text{Spec}(\bigoplus_{j=0}^{n-1} \mathcal{O}_{\overline{W}}(-j\delta))$ and then $\overline{S}$ is isomorphic to $\tilde{S}$ naturally.

□
From this lemma, we may assume that all vertical $(-1)$-curves are contained in $\text{Fix}(\tilde{\sigma})$. Then, it follows that $\tilde{\sigma}$ induces the automorphism $\sigma$ of $S$ over $B$ of order $n$ from the next easy lemma.

**Lemma 2.6.** We write $\rho = \rho_1 \circ \cdots \circ \rho_k$ as a composite of blow-ups $\rho_i : \tilde{S}_i \rightarrow \tilde{S}_{i-1}$. Then, $\tilde{\sigma}$ induces an isomorphism $\bar{\sigma}_i$ of order $n$ on $\tilde{S}_i$ and $\rho_i$ is a blow-up at a fixed point of $\bar{\sigma}_{i-1}$.

**Proof.** Let $E_i$ be the exceptional curve of $\rho_i$. By the above assumption, $E_k$ is contained in $\text{Fix}(\tilde{\sigma})$ and then $\tilde{\sigma}$ induces an isomorphism $\bar{\sigma}_{k-1}$ on $\tilde{S}_{k-1}$. If $E_{k-1}$ is translated into another curve by $\bar{\sigma}_{k-1}$, then $E_{k-1}$ and $\text{Fix}(\bar{\sigma}_{k-1})$ are disjoint. Thus $E_{k-1}$ is also a $(-1)$-curve on $\tilde{S}$ and then this contradicts the assumption. Hence $\bar{\sigma}_{k-1}(E_{k-1}) = E_{k-1}$ and then $\bar{\sigma}_{k-1}$ induces an isomorphism $\bar{\sigma}_{k-2}$ on $\tilde{S}_{k-2}$. The assertion follows inductively.

From Lemma 2.6, it follows that $n \leq 3$ or any isolated fixed point of $\sigma$ is of type $(k,k)$ for some $k$. Consider the case where $n = 3$.

**Lemma 2.7.** If $n = 3$, any isolated fixed point $x$ of $\sigma$ is of type $(1,1)$ or type $(2,2)$.

**Proof.** Let $x$ be an isolated fixed point of type $(1,2)$. Then, $x$ is blown up exactly three times. Let $E$ be the proper transform of the exceptional curve of the first blow-up and $E_1$, $E_2$ the exceptional curves of the blow-up at the isolated fixed point of types $(1,1)$ and $(2,2)$, respectively. Clearly, $E$ is a $(-3)$-curve not contained in $\text{Fix}(\tilde{\sigma})$ while $E_1$, $E_2$ are contained in it. Let $L$, $L_1$, $L_2$ be the image of $E$, $E_1$, $E_2$ by $\tilde{\sigma}$ respectively. One sees easily that $L$ is a $(-1)$-curve not contained in $\tilde{R}$, and $L_1$, $L_2$ are $(-3)$-curves in $\tilde{R}$ and these three curves are contained in a fiber of $\tilde{\varphi}$. Hence $L$ is contracted by $\tilde{\psi}$ and then the images of $L_1$ and $L_2$ intersect in one point transversely. On the other hand, the restriction map $\theta|_E : E \rightarrow L$ is a cyclic covering branched over $L \cap \tilde{R}$, and $L \cap \tilde{R}$ contains two points $L \cap L_1$ and $L \cap L_2$. By the Hurwitz formula, $L \cap \tilde{R}$ consists of exactly two points. Thus, $L$ intersects $L_1$ and $L_2$ only in components of $\tilde{R}$ and then the multiplicity of the point to which $L$ is contracted is 2. It contradicts Lemma 2.4.

From these lemmas, the next lemma follows.

**Lemma 2.8.** Any curve contracted by $\rho$ is a $(-1)$-curve on a fiber of $\tilde{f}$ contained in $\text{Fix}(\tilde{\sigma})$.

**Proof.** We write $\rho = \rho_1 \circ \cdots \circ \rho_k$ as above. It is sufficient to show that any $\rho_i$ is a blow-up of an isolated fixed point. If $\rho_i$ is a blow-up of a smooth point of a 1-dimensional fixed part (i.e. a fixed point of type $(0,k)$ for some $k$), one isolated fixed point of type $(n-k,k)$ appears on the exceptional curve $E_i$. Then, it contradicts Lemmas 2.6 and 2.7.

From this lemma, we can reconstruct $\tilde{f} : \tilde{S} \rightarrow B$ by blowing up isolated fixed points of $\sigma$.

**Definition 2.9.** A fibered surface $f : S \rightarrow B$ with a $B$-automorphism $\sigma$ of $S$ of order $n$ is called a *fibration with an automorphism $\sigma$ of type $(g,h,n)$* if $f : S \rightarrow B$ satisfies $(\ast)$ and $\sigma$ is of order $n$ and induced by a generator of $\text{Aut}(\tilde{S}/\tilde{W})$.

Our aim in this section is to prove the following theorem:
Theorem 2.10. Let \( f : S \to B \) be a fibration with an automorphism of type \((g, h, n)\). Assume \( h \geq 1 \) and \( g \geq (2n - 1)(2hn + n - 1)/(n + 1) \). Then we have

\[
K_f^2 \geq \lambda_{g,h,n} \chi_f,
\]

where

\[
\lambda_{g,h,n} = \frac{24(g - 1)(n - 1)}{2(2n - 1)(g - 1) - n(n + 1)(h - 1)}.
\]

Remark 2.11. From (2.3), the hypothesis \( g \geq (2n - 1)(2hn + n - 1)/(n + 1) \) is equivalent to \( r \geq 3g/(2n - 1) + 3 \), and we have

\[
\lambda_{g,h,n} = 8 \cdot \frac{1 + r(n + 1)/6(g - 1)}{n}.
\]

We obtain an \( n \)-cyclic cover \( \theta_i : S_i \to W_i \) branched over \( R_i \) by setting

\[
S_i = \text{Spec}( \bigoplus_{j=0}^{j=n-1} \mathcal{O}_{W_i}(-j\delta_i)).
\]

Since \( R_i \) is reduced, \( S_i \) is a normal surface. There exists a natural birational morphism \( S_i \to S_{i-1} \). Set \( S' = S_0, \theta = \theta_0, \delta = \delta_0 \) and \( f' = \varphi \circ \theta \).

\[
\tilde{S} = S_N \xrightarrow{\tilde{\theta}} S_{N-1} \to \cdots \to S_0 = S' \xrightarrow{\theta} S
\]

\[
\tilde{W} = W_N \xrightarrow{\tilde{\psi}_N} W_{N-1} \xrightarrow{\tilde{\psi}_{N-1}} \cdots \xrightarrow{\tilde{\psi}_1} W_0 = W \xrightarrow{\varphi} B
\]

Then, we have

\[
K_{\tilde{\varphi}} = \tilde{\psi}^* K_{\varphi} + \sum_{i=1}^{N} E_i,
\]

(2.4)

\[
\tilde{\delta} = \tilde{\psi}^* \delta - \sum_{i=1}^{N} \left[ \frac{m_i}{n} \right] E_i,
\]

(2.5)

where \( E_i \) is the total transform of \( E_i \). Since

\[
K_{\tilde{S}} = \tilde{\theta}^*(K_{\tilde{W}} + (n - 1)\tilde{\delta})
\]

and

\[
\chi(\mathcal{O}_{\tilde{S}}) = n\chi(\mathcal{O}_{\tilde{W}}) + \frac{1}{2} \sum_{j=1}^{n-1} j\tilde{\delta}(j\tilde{\delta} + K_{\tilde{W}}),
\]
we get

\[(2.6) \quad K_f^2 = n(K_\varphi^2 + 2(n-1)K_\varphi \delta + (n-1)^2 \delta^2), \]

\[(2.7) \quad \chi_f = n\chi_\varphi + \frac{1}{2} \sum_{j=1}^{n-1} j\delta(j\delta + K_\varphi). \]

Similarly, we have

\[(2.8) \quad \omega_f^2 = n(K_\varphi^2 + 2(n-1)K_\varphi \delta + (n-1)^2 \delta^2) \]

\[(2.9) \quad \chi_f = n\chi_\varphi + \frac{1}{2} \sum_{j=1}^{n-1} j\delta(j\delta + K_\varphi). \]

Hence, we obtain

\[(2.10) \quad \omega_f^2 - K_f^2 = n \sum_{i=1}^{N} \left( (n-1) \left( \left\lfloor \frac{m_i}{n} \right\rfloor - 1 \right)^2, \right. \]

\[(2.11) \quad \chi_f - \chi_f = \frac{1}{12} n(n-1) \sum_{i=1}^{N} \left( (n-1) \left( \left\lfloor \frac{m_i}{n} \right\rfloor - 3 \right). \right. \]

**Lemma 2.12.** Assume

\[ g \geq 2n-1 \frac{(2hn + n - 1)}{n+1}. \]

Then, we have

\[(2.12) \quad \omega_f^2 \geq \lambda_{g,h,n} \chi_f, \]

where

\[ \lambda_{g,h,n} = \frac{24(g-1)(n-1)}{2(n-1)(g-1) - n(n+1)(h-1)}. \]

**Proof.** Suppose that \( h = 0 \). Then, \( R \) is numerically equivalent to \( -rK_\varphi/2 + M\Gamma \) for some \( M \in \frac{1}{2}\mathbb{Z} \) since \( \varphi : W \to B \) is a \( \mathbb{P}^1 \)-bundle and \( K_W\Gamma = -2, R\Gamma = \bar{R}\bar{\Gamma} = r \). Moreover, \( K_\varphi^2 = 0 \) and \( \chi_\varphi = 0 \). Hence it follows

\[ \omega_f^2 = \frac{4(g-1)(n-1)}{n} M, \]

\[ \chi_f = \frac{2(2n-1)(g-1) + n(n+1)}{6n} M \]

from \((2.3),(2.8)\) and \((2.9)\). Thus, we get \( \omega_f^2 = \lambda_{g,0,n} \chi_f \).

If \( h = 1 \), we have \( K_\varphi^2 = 0 \) and \( \chi_\varphi = \chi(O_W) \) since \( \varphi \) is a relatively minimal elliptic surface. Then, we have

\[ \omega_f^2 - \lambda_{g,1,n} \chi_f = n \left( K_\varphi^2 - \frac{12(n-1)}{2n-1} K_\varphi \right) + \frac{n(n-1)(n+1)}{2n-1} K_\varphi \delta \]

\[(2.13) \quad = \frac{n(n-1)}{2n-1} ((n+1)K_\varphi \delta - 12\chi_\varphi). \]
By the canonical bundle formula, $K\varphi$ is numerically equivalent to $\chi(\mathcal{O}_W)\Gamma + \sum_{i=1}^l (1 - 1/k_i)\Gamma$ where $\Gamma_i = k_i D_i, i = 1, \ldots, l$ are all multiple fibers of $\varphi$, $D_i$ the fundamental cycle of $\Gamma_i$ and $k_i \in \mathbb{Z}$. Hence, we have

$$\text{By the canonical bundle formula, } K\varphi \delta \geq \chi(\mathcal{O}_W)\Gamma \delta = \frac{2(g - 1)}{n(n - 1)} \chi(\mathcal{O}_W).$$

Thus,

$$\frac{n(n - 1)}{2n - 1}((n + 1)K\varphi \delta - 12\chi\varphi) \geq \frac{n(n - 1)}{2n - 1} \left( \frac{2(n + 1)(g - 1)}{n(n - 1)} - 12 \right) \chi(\mathcal{O}_W)$$

(2.14)

$$= \frac{2}{2n - 1}((n + 1)g - (2n - 1)(3n - 1))\chi(\mathcal{O}_W).$$

By (2.13), (2.14) and hypothesis $g \geq (2n - 1)(3n - 1)/(n + 1)$, we have (2.12).

Consider the case where $h \geq 2$. We compute $\omega_{\mathcal{F}'}^2 - \lambda_{g,h,n}\chi_{\mathcal{F}'}$ by (2.8), (2.9) as follows:

$$\omega_{\mathcal{F}'}^2 - \lambda_{g,h,n}\chi_{\mathcal{F}'} = n(K\varphi^2 + 2(n - 1)K\varphi \delta + (n - 1)^2 \delta^2) - \lambda_{g,h,n}\left( n\chi\varphi + \frac{1}{4}n(n - 1)K\varphi \delta + \frac{1}{12}n(n - 1)(2n - 1)\delta^2 \right)$$

$$= n(K\varphi^2 - \lambda_{g,h,n}\chi\varphi) + \frac{n(n - 1)}{4}(8 - \lambda_{g,h,n})K\varphi \delta$$

+ $$\frac{n(n - 1)}{12}(12(n - 1) - (2n - 1)\lambda_{g,h,n})\delta^2.$$  (2.15)

Since the slope inequality of $\varphi$ gives us

$$K\varphi^2 \geq \frac{4(h - 1)}{h} \chi\varphi;$$

we have

$$K\varphi^2 - \lambda_{g,h,n}\chi\varphi = \frac{h\lambda_{g,h,n}}{4(h - 1)} \left( K\varphi^2 - \frac{4(h - 1)}{h} \chi\varphi \right) + \left( 1 - \frac{h\lambda_{g,h,n}}{4(h - 1)} \right) K\varphi^2$$

(2.16)

$$\geq \left( 1 - \frac{h\lambda_{g,h,n}}{4(h - 1)} \right) K\varphi^2.$$  

We consider the intersection matrix of $\{K\varphi, \delta, \Gamma\}$

$$\left( \begin{array}{ccc}
K\varphi & K\varphi \delta & K\varphi \Gamma \\
K\varphi \delta & \delta^2 & \delta \Gamma \\
K\varphi \Gamma & \delta \Gamma & 0
\end{array} \right).$$  (2.17)

By Arakerov theorem, we have $K\varphi^2 \geq 0$ and then the matrix is not negative definite. Hence the determinant of (2.17) is non-negative, that is, we have

$$2(K\varphi \delta)(\delta \Gamma)(K\varphi \Gamma) - \delta^2(K\varphi \Gamma)^2 - (\delta \Gamma)^2 K\varphi^2 \geq 0$$  (2.18)
by the Hodge index theorem. Since
\[
\delta \Gamma = \frac{r}{n} = \frac{2(g-1-n(h-1))}{n(n-1)}
\]
and
\[
K_{\varphi} \Gamma = 2(h-1),
\]
the inequality (2.18) is equivalent to
\[
(2.19) \quad 2(g-1-n(h-1))K_{\varphi} \delta - n(n-1)(h-1)\delta^2 \geq \frac{1}{n(n-1)(h-1)}(g-1-n(h-1))^2 K_{\varphi}^2.
\]
On the other hand, by definition of \( \lambda_{g,h,n} \), we have
\[
\frac{n(n-1)\lambda_{g,h,n}}{4} (8 - \lambda_{g,h,n}) = \frac{n(n-1)(n+1)}{2(2n-1)(g-1) - n(n+1)(h-1)} 2(g-1-n(h-1))
\]
and
\[
\frac{n(n-1)}{12} (12(n-1) - (2n-1)\lambda_{g,h,n}) = \frac{n(n-1)(n+1)}{2(2n-1)(g-1) - n(n+1)(h-1)} (-n(n-1)(h-1)).
\]
Hence, by (2.15), (2.16), (2.19), (2.20) and (2.21), we get
\[
\omega_{f'}^2 - \lambda_{g,h,n} \chi_{f'} \geq n \left( 1 - \frac{h\lambda_{g,h,n}}{4(h-1)} \right) K_{\varphi}^2 
+ \frac{(n+1)(g-1-n(h-1))^2}{(h-1)(2(2n-1)(g-1) - n(n+1)(h-1))} K_{\varphi}^2.
\]
By definition of \( \lambda_{g,h,n} \), we have
\[
1 - \frac{h\lambda_{g,h,n}}{4(h-1)} = \frac{(g-1)(-hn+2h-2n+1) - n(n+1)(h-1)}{(h-1)(2(2n-1)(g-1) - n(n+1)(h-1))}.
\]
Thus, the right hand side of the inequality (2.22) is
\[
(2.23) \quad \frac{(g-1)((n+1)g-(2hn+n-1)(2n-1))}{(h-1)(2(2n-1)(g-1) - n(n+1)(h-1))} K_{\varphi}^2.
\]
By assumption \( g \geq (2n-1)(2hn+n-1)/(n+1) \), (2.23) is non-negative. Hence we completes the proof. \( \square \)

**Proof of the Theorem.** Let \( \varepsilon \) be the number of blow-ups of \( \rho: \tilde{S} \to S \). Then, we have
\[
(2.24) \quad K_{f'}^2 = K_{f}^2 - \varepsilon.
\]
Using (2.10), (2.11), (2.20) and (2.21), we can calculate $K_f^2 - \lambda_{g,h,n} \chi_f$ as follows:

$$K_f^2 - \lambda_{g,h,n} \chi_f = K_f^2 - \lambda_{g,h,n} \chi_f + \varepsilon$$

$$= \omega_f^2 - n \sum_{i=1}^{N} \left( (n-1) \left\lfloor \frac{m_i}{n} \right\rfloor - 1 \right)^2 - \lambda_{g,h,n} \chi_f'$$

$$+ \lambda_{g,h,n} \frac{1}{12} n(n-1) \sum_{i=1}^{N} \left\lfloor \frac{m_i}{n} \right\rfloor \left( (2n-1) \left\lfloor \frac{m_i}{n} \right\rfloor - 3 \right) + \varepsilon$$

$$= \omega_f^2 - \lambda_{g,h,n} \chi_f' + \frac{1}{2} n(n-1)((2n-1)\lambda_{g,h,n} - 12(n-1)) \sum_{i=1}^{N} \left\lfloor \frac{m_i}{n} \right\rfloor \sum_{i=1}^{N} \left\lfloor \frac{m_i}{n} \right\rfloor - nN + \varepsilon$$

$$= \omega_f^2 - \lambda_{g,h,n} \chi_f' + \frac{n^2(n-1)^2(n+1)(h-1)}{2(2n-1)(g-1) - n(n+1)(h-1)} \sum_{i=1}^{N} \left\lfloor \frac{m_i}{n} \right\rfloor - nN + \varepsilon$$

(2.25)

When $h \geq 1$, the right hand side of (2.25) increases monotonically with respect to the multiplicity $m_i$. So we may assume that $[m_i/n] = 1$ for all $i = 1, \ldots, N$. Then,

$$\frac{n^2(n-1)^2(n+1)(h-1)}{2(2n-1)(g-1) - n(n+1)(h-1)} \sum_{i=1}^{N} \left\lfloor \frac{m_i}{n} \right\rfloor \sum_{i=1}^{N} \left\lfloor \frac{m_i}{n} \right\rfloor - nN$$

$$= \frac{n^2(n-2)N}{2(2n-1)(g-1) - n(n+1)(h-1)}((n+1)(n-2)(h-1) + 2(g-1))$$

$$\geq 0.$$ 

Combining them with Lemma 2.12 we conclude the proof. \qed

**Example 2.13.** Let $B$ and $\Gamma$ be smooth curves of genus $b$ and $h$, respectively. Let $\delta_1$ and $\delta_2$ be divisors on $B$ and $\Gamma$ of degree $N$ and $M$, respectively. For $N$ and $M$ sufficiently large, the divisor $\delta := p_1^* \delta_1 + p_2^* \delta_2$ on $B \times \Gamma$ gives us a base point free linear system, where $p_1$ and $p_2$ are the natural projections from $B \times \Gamma$ to $B$ and to $\Gamma$, respectively. Thus, we can take a smooth divisor $R \in |n\delta|$ for $n > 0$ by Bertini’s theorem. Hence we may construct an $n$-cyclic covering $\theta: S \rightarrow B \times \Gamma$ branched over $R$. Let $f: S \rightarrow B$ be the composite of $p_1$ and $\theta$. We will compute $K_f^2$ and $\chi_f$. Let $F$ be a general fiber of $f$ and $a$ the genus of the fibration $f$.
Applying the Hurwitz formula to $f|_F: F \to \Gamma = \{t\} \times \Gamma$, we have
\[
2g - 2 = n(2h - 2) + (n - 1)n(p^*_1\delta_1 + p^*_2\delta_2)p^*_1t
= n(2h - 2) + (n - 1)nM.
\]
Hence
\[
M = 2(2g - 2 - n(h - 1)) / n(n - 1).
\]
Since $K_{p_1} = p^*_2K_{\Gamma}$ and $\chi_{p_1} = 0$, we obtain
\[
K^2_f = (\theta^*(K_{p_1} + (n - 1)\delta))^2
= n(p^*_1(n - 1)\delta_1 + p^*_2((n - 1)\delta_2 + K_{\Gamma}))^2
= 2n(n - 1)N((n - 1)M + 2(h - 1)),
\]
and
\[
\chi_f = n\chi_{p_1} + \frac{1}{2} \sum_{j=1}^{n-1} j\delta(j\delta + K_{p_1})
= \frac{1}{4} n(n - 1)\delta K_{p_1} + \frac{1}{12} n(n - 1)(2n - 1)\delta^2
= \frac{1}{4} n(n - 1)N(2h - 2) + \frac{1}{12} n(n - 1)(2n - 1)2NM
= \frac{1}{6} n(n - 1)N(3(h - 1) + (2n - 1)M).
\]
Thus, we get
\[
\frac{K^2_f}{\chi_f} = \frac{12(2h - 1) + (n - 1)M}{3(h - 1) + (2n - 1)M}
= \frac{2(2n - 1)(g - 1)}{n(n - 1)(g - 1) - n(n + 1)(h - 1)}
= \lambda_{g,h,n}.
\]
This example implies that the bound of Theorem 2.10 on the slope is sharp when $h$ and $n$ are fixed.

3. **Slope equalities for cyclic covering fibrations of a ruled surface**

In this section, we consider a fibration $f: S \to B$ with an automorphism $\sigma$ of type $(g, 0, n)$, i.e., there exist a ruled surface $\tilde{P}$ over $B$ and an $n$-cyclic covering $\tilde{\theta}: \tilde{S} \to \tilde{P}$ branched over a smooth divisor $\tilde{R}$, and $f$ is the relatively minimal model of the genus $g$ fibration $\tilde{f}: \tilde{S} \to B$. Moreover, we assume that $n \geq 3$. The case where $n = 2$ is extensively studied in [6]. The ruling $\tilde{\varphi}: \tilde{P} \to B$ has infinitely many relatively minimal models.
Lemma 3.1. There exists a relatively minimal model $\varphi: P \to B$ of $\tilde{\varphi}$ satisfying that
\[
\text{mult}_x(R_h) \leq \frac{r}{2} = \frac{g}{n-1} + 1
\]
for all $x \in R_h$, where $R_h$ is the sum of horizontal components of $R$ with respect to $\varphi$.

Proof. We take a relatively minimal model $\varphi: P \to B$ of $\tilde{\varphi}$ arbitrarily. Suppose that $x$ is a point of $R$ at which $R$ has the multiplicity greater than $r/2$. Let $\Gamma$ be the fiber of $\varphi$ through $x$. We will perform the elementary transformation at $x$. Let $\psi_1: P_1 \to P$ be the blow-up at $x$, $E_1$ the exceptional curve for $\psi_1$ and $E'_1$ the proper transform of $\Gamma$. Then, $E'_1$ is also a $(-1)$-curve. Let $\psi'_1: P_1 \to P'$ be the contraction of $E'_1$ and $\varphi': P'_1 \to B$ the induced fibration. Set $x' = \psi'_1(E'_1)$ and $\Gamma' = \psi'_1(\Gamma_1)$. Then, $\Gamma'$ is the fiber of $\varphi'$ over $t = \varphi(\Gamma)$. Let $m$ be the multiplicity of $R$ at $x$. Put $R_1 = \psi_1^* R - n[m/n]E_1$ and $R' = (\psi'_1)^* R_1$. Let $m'$ be the multiplicity of $R'$ at $x'$. Clearly, it follows $R_1 = (\psi'_1)^* R' - n[m'/n]E'_1$. Since $R \Gamma = R_1 \psi_1^* \Gamma = R_1 (E_1 + E'_1)$, we have
\[
\left[ \frac{m}{n} \right] + \left[ \frac{m'}{n} \right] = \frac{r}{n}.
\]
Moreover, since $m > r/2$, we get
\[
\left[ \frac{m'}{n} \right] \leq \frac{r}{2n} \leq \left[ \frac{m}{n} \right].
\]

Hence we have $m' \leq r/2 + 1$ by Lemma 2.4. If $m' = r/2 + 1$, then we have $m' \in n\mathbb{Z} + 1$ and hence $E'_1$ is contained in $R_1$. In particular, $\Gamma$ is contained in $R$. Since $r/2 \in n\mathbb{Z}$, we have $m = r/2 + 1$. Hence the multiplicity of $R_h$ at $x$ is $r/2$. If $m' \leq r/2$, we replace the relatively minimal model $P$ with $P'$. Since the number of singularities of $R$ are finite, we obtain a relatively minimal model satisfying the condition inductively. \qed

We take a relatively minimal model $\varphi: P \to B$ of $\tilde{\varphi}$ satisfying this lemma. In the previous section, we have seen that vertical $(-1)$-curves in $\text{Fix}(\tilde{\varphi})$ and vertical $(-n)$-curves in $\tilde{R}$ are in one-to-one correspondence via $\tilde{\varphi}$. We will examine how vertical $(-n)$-curves in $\tilde{R}$ appear. Let $L$ be a vertical irreducible curve in $\tilde{R}$. Since $\tilde{\varphi}^* L = nD$ for some $D \simeq \mathbb{P}^1$ in $\text{Fix}(\tilde{\varphi})$, $L$ is a $(-an)$-curve for some positive integer $a$. The image of $L$ by the natural birational morphism $\tilde{\psi}: \tilde{P} \to P$ is a point or a fiber of $\varphi$. If $\tilde{\psi}(L)$ is a point, then $L$ is the proper transform of an exceptional curve $E$ appeared during the process $\tilde{\psi}$ of blowing-ups. Moreover, $E$ is blown up $an - 1$ times since $L^2 = -an$ and $E^2 = -1$. Then, set $C = E$ and $c = an - 1$. If $\tilde{\psi}(L)$ is a fiber $\Gamma$ of $\varphi$, then $L$ is the proper transform of $\Gamma$ and it is blown up $an$ times since $L^2 = -an$ and $\Gamma^2 = 0$. Then, set $C = \Gamma$ and $c = an$. When $E = E_j$ is the blow-up $\tilde{\psi}_j$ at $x = x_j$ in the previous notation $\tilde{\psi} = \psi_1 \circ \cdots \circ \psi_N$, we have $m = m_j \in n\mathbb{Z} + 1$ since $E$ is contained in $R_j$. We drop the index and set $R = R_i$ for simplicity. Let $x_1, x_2, \ldots, x_l$ be all singularities of $R$ on $C$ and $m_i$ the multiplicity of $R$ at $x_i$. Clearly, we have $1 \leq l \leq c$. We consider a local analytic branch $D$ of $R - C$ which has the multiplicity $m \geq 2$ at $x_i$ (i.e. $D$ has a cusp $x_i$). There are two cases as follows:
(i) the case where $D$ is not tangent to $C$ at $x_i$. Then, the proper transform of $D$ does not meet that of $C$ after blowing up at $x_i$. Hence, the local intersection number $(DC)_{x_i}$ of $D$ and $C$ is $m$.

(ii) the case where $D$ is tangent to $C$ at $x_i$. Then, there are three cases after blowing up at $x_i$.

(1) the case where the proper transform of $D$ is tangent to neither that of $C$ nor the exceptional curve.

(2) the case where the proper transform of $D$ is tangent to that of $C$.

(3) the case where the proper transform of $D$ is tangent to the exceptional curve. Then, the multiplicity $m'$ of the singularity of the proper transform of $D$ is less than $m$. It follows $(DC)_{x_i} = m + m'$.

From these observation, we obtain $(D\Gamma)_{x_i} = (k + 1)m$ if $D$ becomes type (1) as above when we blow up at $x_i$ $k$ times, and $(D\Gamma)_{x_i} = km + m'$ if $D$ becomes type (3) as above when blow up $k$ times. We define $s_{i,k}$ to be the number of local analytic branches of $R - C$ meeting $C$ at $x_i$ which have the local intersection number $k$, where we regard a branch $D$ of the multiplicity $m \geq 2$ at $x_i$ satisfying $(D\Gamma)_{x_i} = (k + 1)m$ as $m$ branches $D_1, \ldots, D_m$ satisfying $(D_i\Gamma)_{x_i} = k + 1$ for all $i = 1, \ldots, m$, and a branch $D$ of the multiplicity $m \geq 2$ at $x_i$ satisfying $(D\Gamma)_{x_i} = km + m'$ as $m - m'$ branches $D_1, \ldots, D_{m - m'}$ satisfying $(D_j\Gamma)_{x_i} = k$ for all $j = 1, \ldots, m - m'$ and $m'$ branches $D_{m - m' + 1}, \ldots, D_m$ satisfying $(D_j\Gamma)_{x_i} = k + 1$ for all $j = m - m' + 1, \ldots, m$. Let $i_{\text{max}}$ be the maximal integer $k$ such that $s_{i,k} \neq 0$. We may assume
that \( i_{\text{max}} \geq (i + 1)_{\text{max}} \) for any \( i \). Set \( x_{i,1} = x_i \) and \( m_{i,1} = m_i \). Let \( \psi_{i,1} : P_{i,1} \to P \) be the blow-up at \( x_{i,1} \), \( E_{i,1} \) the exceptional curve of \( \psi_{i,1} \) and \( R_{i,1} = \psi_{i,1}^* R - n[m_{i,1}/n]E_{i,1} \). Inductively, We define \( x_{i,j}, m_{i,j}, \psi_{i,j}, P_{i,j}, E_{i,j} \) and \( R_{i,j} \) to be the intersection point of \( R_{i,j-1} \) and \( E_{i,j-1} \), the multiplicity of \( R_{i,j-1} \) at \( x_{i,j} \), the blow-up \( \psi_{i,j} : P_{i,j} \to P_{i,j-1} \) at \( x_{i,j} \), the exceptional curve of \( \psi_{i,j} \) and \( \psi_{i,j}^* R_{i,j-1} - n[m_{i,j}/n]E_{i,j} \), respectively.

**Lemma 3.2.** The following hold:

1. \( m_{i,1} = \sum_{k=1}^{i_{\text{max}}} s_{i,k} + 1 \) and \( m_{i,i_{\text{max}}} \in n\mathbb{Z} \).
2. \( m_{i,j} \in n\mathbb{Z} \) (resp. \( n\mathbb{Z} + 1 \)) if and only if \( m_{i,j+1} = \sum_{k=j+1}^{i_{\text{max}}} s_{i,k} + 1 \) (resp. \( \sum_{k=j+1}^{i_{\text{max}}} s_{i,k} + 2 \)).
3. \( ((R - C)C)_{x_i} = \sum_{k=1}^{i_{\text{max}}} ks_{i,k} \).
4. \( c = \sum_{i=1}^{l} i_{\text{max}} \).

**Proof.** Since \( m_{i,j} \) is the number of branches of \( R_{i,j-1} \) through \( x_{i,j} \) and Lemma 2.4, we have \( m_{i,1} = \sum_{k=1}^{i_{\text{max}}} s_{i,k} + 1 \) and (2). If \( m_{i,i_{\text{max}}} \in n\mathbb{Z} + 1 \), then \( x_{i,i_{\text{max}}+1} \) is the singular point of the multiplicity 2. Since \( n \geq 3 \), this contradicts Lemma 2.4. Thus, we have (1). (3) is clear from the definition of \( s_{i,k} \). Since every \( x_i \) on \( C \) is blown up exactly \( i_{\text{max}} \) times, we obtain (4). \( \square \)

Let \( t \) be the intersection number of \( R - C \) and \( C \). This is the number of branch point \( r \) if \( \tilde{\psi}(L) \) is a fiber of \( \varphi \), or the multiplicity of the point \( x \) to which \( C \) is contracted if \( \tilde{\psi}(L) \) is a point. From Lemma 3.2 (3), we get

\[
t = \sum_{i=1}^{l} \sum_{k=1}^{i_{\text{max}}} ks_{i,k}.
\]

Let \( c_i \) be the number of \( m_{i,j} \) belonging to \( n\mathbb{Z} + 1 \). Clearly, we have \( 0 \leq c_i \leq i_{\text{max}} - 1 \). Set \( d_{i,j} = [m_{i,j}/n] \).

**Proposition 3.3.** The following equalities hold:

\[
t + c + \sum_{i=1}^{l} c_i = \sum_{i=1}^{l} \sum_{j=1}^{i_{\text{max}}} m_{i,j},
\]

\[
\frac{t + c}{n} = \sum_{i=1}^{l} \sum_{j=1}^{i_{\text{max}}} d_{i,j}.
\]

**Proof.** From Lemma 3.2 we have

\[
t = \sum_{i=1}^{l} \sum_{k=1}^{i_{\text{max}}} ks_{i,k}
\]

\[
= \sum_{i=1}^{l} \left( \sum_{j=1}^{i_{\text{max}}} m_{i,j} - c - c_i \right).
\]

The other equality is clear. \( \square \)

We will examine the property of \( m_{i,j} \).
Lemma 3.4. The following hold:

1. \( m_{i,j} \geq m_{i,j+1} \), and \( m_{i,j} = m_{i,j+1} \) if and only if \( s_{i,j} = 0 \) when \( m_{i,j} \in n\mathbb{Z} \), or \( s_{i,j} = 1 \) when \( m_{i,j} \notin n\mathbb{Z} + 1 \).

2. If \( m_{i,j} \in n\mathbb{Z} + 1 \) and \( m_{i,j} \in n\mathbb{Z} \), then \( m_{i,j} > m_{i,j+1} \).

3. If \( m_{i,j} = nd_{i,j} + 1 \in n\mathbb{Z} + 1 \), then \( d_{i,j} - d_{i,j+1} \geq n - 3 \).

Proof. If \( m_{i,j} < m_{i,j+1} \), we have \( s_{i,j} = 0 \) and \( m_{i,j} + 1 = m_{i,j+1} \) since \( m_{i,j} - m_{i,j+1} = s_{i,j} - 1, s_{i,j}, \) or \( s_{i,j} + 1 \). Then, we have \( m_{i,j} \in n\mathbb{Z} + 1 \) by Lemma 3.2 (2) and then \( m_{i,j+1} \in n\mathbb{Z} + 2 \). This contradicts Lemma 2.4 and \( n \geq 3 \). Hence, \( m_{i,j} \geq m_{i,j+1} \). the rest of (1) follows from Lemma 3.2 (2). If \( m_{i,j} \in n\mathbb{Z} + 1 \) and \( m_{i,j} \in n\mathbb{Z} \), then \( m_{i,j} = \sum_{k=j}^{i_{\max}} s_{i,k} + 2 \) and \( m_{i,j+1} = \sum_{k=j}^{i_{\max}} s_{i,k} + 1 \) by Lemma 3.2 (2). Then, \( m_{i,j} - m_{i,j+1} = s_{i,j} + 1 > 0 \) and hence (2) follows. Suppose that \( m_{i,j} = nd_{i,j} + 1 \in n\mathbb{Z} + 1 \). Let \( C' \) be the exceptional curve \( E_{i,j} \) and define \( x', m'_i, d'_{i,j}, c' \) etc. on \( C' \) similarly to \( C \). Since \( C' \) become a \((a'-c')\)-curve by blowing up for some \( a' \geq 1 \), we have \( c' = a' - 1 \). We may assume \( m_{i,j+1} = m'_{i,1} \). Then we have

\[
\frac{m_{i,j} + c'}{n} = \sum_{p=1}^{i_{\max}} \sum_{q=1}^{v_{p}} d'_{i,j+1} + c' - 1.
\]

Hence we get

\[
d_{i,j} - d_{i,j+1} \geq a'(n - 1) - 2 \geq n - 3.
\]

and thus (3) follows.

Proposition 3.5.

1. If \( g < (n - 1) \left( \frac{an(n - 1)}{2} - 1 \right) \), then there are no vertical \((-an)\)-curves in \( \tilde{R} \).

2. If \( (n - 1) \left( \frac{an(n - 1)}{2} - 1 \right) \leq g < (n - 1)(an^2 - (a + 1)n - 1) \), then any vertical \((-an)\)-curve in \( \tilde{R} \) is the proper transform of a fiber of \( \varphi \).

3. Let \( x \) be a singular point with multiplicity \( m \in n\mathbb{Z} + 1 \). If the proper transform \( L \) of the exceptional curve obtained by blowing up at \( x \) is a \((-an)\)-curve, then we have

\[
\left\lfloor \frac{m}{n} \right\rfloor \geq a(n - 1) - 1.
\]

Proof. Let \( L \) be a vertical \((-an)\)-curve in \( \tilde{R} \). By Proposition 3.3 we have

\[
\frac{t + c}{n} = \sum_{i=1}^{l} \sum_{j=1}^{i_{\max}} d_{i,j} \geq c.
\]

Hence, we get

\[
t \geq (n - 1)c.
\]
If $L$ is the proper transform of a fiber of $\varphi$, then the above inequality is
\[ r \geq an(n - 1). \]
Hence, we have
\[(3.1) \quad g \geq (n - 1) \left( \frac{an(n - 1)}{2} - 1 \right). \]
If $L$ is the proper transform of an exceptional curve $E$ of $\tilde{\psi}$, then we have
\[ m \geq (an - 1)(n - 1), \]
where $m$ is the multiplicity of the point obtained by contracting $E$. Hence (3) follows. Moreover, we get
\[ m \leq \frac{r}{2} + 1 \]
by Lemmas 3.1 and 3.4 (1). Thus, we get
\[(3.2) \quad g \geq (n - 1)(an^2 - (a + 1)n - 1) \]
from the above two inequalities. By an easy computation, we have
\[ (n - 1)(an^2 - (a + 1)n - 1) \geq (n - 1) \left( \frac{an(n - 1)}{2} - 1 \right) \]
(with equality holding if and only if $a = 1$ and $n = 3$). Hence, we get (1) and (2) by (3.1) and (3.2).

When $a = 1$, we have

**Corollary 3.6.** (1) If
\[ g < \frac{(n - 2)(n - 1)(n + 1)}{2}, \]
then any irreducible components of $\tilde{R}$ is horizontal.
(2) If
\[ \frac{(n - 2)(n - 1)(n + 1)}{2} \leq g < (n - 1)(n^2 - 2n - 1), \]
then any vertical $(-n)$-curve in $\tilde{R}$ is the proper transform of a fiber of $\varphi$.
(3) The multiplicity of any singular point of type $n\mathbb{Z} + 1$ is greater than or equal to $(n - 1)^2$.

**Definition 3.7.** By using the datum $\{d_{i,j}\}$, one can construct a diagram as in Table II. We call it the $(d_{i,j})$-diagram. Similarly, we define the $(m_{i,j})$-diagram.
Example 3.8. We consider $n = 3$ and $g = 4$. It follows $r = 6$. Let $L$ be a $(-n)$-curve in $\tilde{R}$.

(1) Suppose that $\tilde{\psi}(L)$ is a fiber $\Gamma$ of $\varphi$. Then, we have $d_{i,j} = 1$ and $m_{i,j} = 3$ or $4$ for all $i,j$. From Lemma 3.4, all the possible $(d_{i,j})$ and $(m_{i,j})$-diagrams on $\Gamma$ are as follows:

**Table 2.** $(d_{i,j})$-diagrams on $\Gamma$

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |

(2) Suppose that $\tilde{\psi}(L)$ is a point. Since $d_{i,j} \leq r/2n = 1$, we have $d_{i,j} = 1$ and $m_{i,j} = 3$ or $4$ for all $i,j$. From Lemma 3.4, all the possible $(d_{i,j})$ and $(m_{i,j})$-diagrams on the exceptional curve $E$ which has the proper transform $L$ are as follows:

**Table 4.** $(d_{i,j})$-diagrams on $E$

|   |   |
|---|---|
| 1 | 1 |

|   |   |
|---|---|
| 1 | 1 |

**Table 5.** $(m_{i,j})$-diagrams on $E$

|   |   |   |
|---|---|---|
| 3 | 3 | 3 |

|   |   |   |
|---|---|---|
| 3 | 3 | 3 |

|   |   |   |
|---|---|---|
| 3 | 4 | 3 |

|   |   |   |
|---|---|---|
| 3 | 3 | 4 |

|   |   |   |
|---|---|---|
| 3 | 3 | 4 |
(i) when $i_{\text{max}} = 1$ and $m_{i,1} = 3$, it follows $s_{i,1} = 2$. Thus, the branch curve near $x_i$ is as in Figure 1.

**Figure 1.** $s_{i,1} = 2$

![Figure 1](image1)

(ii) when $i_{\text{max}} = 2$ and $m_{i,1} = 3, m_{i,2} = 3$, it follows $s_{i,1} = 0, s_{i,2} = 2$. Thus, the branch curve near $x_i$ is as in Figure 2.

**Figure 2.** $s_{i,1} = 0, s_{i,2} = 2$

![Figure 2](image2)

(iii) when $i_{\text{max}} = 2$ and $m_{i,1} = 4, m_{i,2} = 3$, it follows $s_{i,1} = 2, s_{i,2} = 1$. Thus, the branch curve near $x_i$ is as in Figure 3.

**Figure 3.** $s_{i,1} = 2, s_{i,2} = 1$

![Figure 3](image3)

(iv) when $i_{\text{max}} = 3$ and $m_{i,1} = 3, m_{i,2} = 3, m_{i,3} = 3$, it follows $s_{i,1} = 0, s_{i,2} = 0, s_{i,3} = 2$. Thus, the branch curve near $x_i$ is as in Figure 4.

**Figure 4.** $s_{i,1} = 0, s_{i,2} = 0, s_{i,3} = 2$

![Figure 4](image4)
(v) when $i_{\text{max}} = 3$ and $m_{i,1} = 3$, $m_{i,2} = 4$, $m_{i,3} = 4$, it follows $s_{i,1} = 1$, $s_{i,2} = 1$, $s_{i,3} = 1$. Thus, the branch curve near $x_i$ is as in Figure 5.

**Figure 5.** $s_{i,1} = 1$, $s_{i,2} = 1$, $s_{i,3} = 1$

---

**Example 3.9.** Suppose $n = 4$. From (3.6), there are no $(-n)$-curves in $\tilde{R}$ for $g \leq 15$. We consider the case $g = 21$. Then $r = 16$. Let $L$ be a $(-n)$-curve in $\tilde{R}$.

1) Suppose that $\tilde{\psi}(L)$ is a fiber $\Gamma$ of $\phi$. Then, $c = 4$ and $5 = \sum_{i,j} d_{i,j}$. Hence, all the possible $(d_{i,j})$ and $(m_{i,j})$-diagrams on $\Gamma$ are as follows:

| Table 6. $(d_{i,j})$-diagrams on $\Gamma$ |
|-----------------------------------------|
| $2 | 1 | 1 | 1$ | $1 | 1 | 1 | 2 | $1 | 2 | 1 | 1 | $1 | 1 | 1$ | 2 | $1 | 1 | 1$ | 2 |

| Table 7. $(m_{i,j})$-diagrams on $\Gamma$ |
|-----------------------------------------|
| $8 | 4 | 4 | 4 | $4 | 8 | 4 | 4 | $4 | 8 | 4 | 4 | $4 | 8 | 4 | 4 | $4 | 8 | 4 | 4 |

(2) Suppose that $\tilde{\psi}(L)$ is a point. Then we have $c = 3$. By $r/2 + 1 = 9$ and (3.6), it follows that the multiplicity of any singularity of type $n\mathbb{Z} + 1$ is 9. Thus, we get $t = 9$ and $3 = \sum_{i,j} d_{i,j}$. Hence $d_{i,j} = 1$ for any $i$, $j$. From Lemma 3.4, all $(d_{i,j})$ and $(m_{i,j})$-diagrams on the exceptional curve $E$ which has the proper transform $L$ are as follows:
Table 8. \((d_{i,j})\)-diagrams on \(E\)

\[
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 1 \\
\hline
\end{array}
\]

Table 9. \((m_{i,j})\)-diagrams on \(E\)

\[
\begin{array}{|c|c|c|}
\hline
4 & 4 & 4 \\
\hline
4 & 4 & 4 \\
\hline
\end{array}
\]

Definition 3.10. For \(k \geq 1\), we define a function \(\alpha_k(F)\) on \(B\) (or equivalently on fibers of \(f\)) as follows:
For \(p \in B\), \(\alpha_k(F_p)\) equals the number of singularities of multiplicity \(kn\) or \(kn + 1\) of \(R\) over the point \(p\).
We call \(\alpha_k(F_p)\) the \(k\)-th singularity index of \(F_p\) and \(\alpha_k(F) = \sum_{p \in B} \alpha_k(F_p)\) the \(k\)-th singularity index of \(f\). Clearly, \(\alpha_k(F_p) = 0\) if \(F_p\) is a general fiber.
We define the 0-th singularity index \(\alpha_0(F_p)\) of \(F_p\) as follows:
Let \(D_1\) be the sum of all vertical \((-n)\)-curves contained in \(\tilde{R}\) and \(\tilde{R} = \tilde{R}_0 + D_1\) the decomposition of \(\tilde{R}\). Then, \(\alpha_0(F_p)\) equals the ramification index of the restriction map \(\tilde{\varphi}|_{\tilde{R}_0}: \tilde{R}_0 \to B\) over \(p\), that is, the ramification index of the restriction map \(\tilde{\varphi}|_{(\tilde{R}_0)_h}: (\tilde{R}_0)_h \to B\) over \(p\) minus the sum of the topological Euler characteristic of irreducible components of \((\tilde{R}_0)_v\) over \(p\).
By definition, we have
\[
\alpha_0 := \sum_{p \in B} \alpha_0(F_p) = (K_{\tilde{\varphi}} + \tilde{R}_0)\tilde{R}_0
\]
and call it the 0-th singularity index of \(f\). If \(F_p\) is a general fiber, then \(\alpha_0(F_p) = 0\) since there exist no irreducible components of \((\tilde{R}_0)_v\) over \(p\) and \(\tilde{\varphi}|_{(\tilde{R}_0)_h}: (\tilde{R}_0)_h \to B\) is not ramified over \(p\).

Remark 3.11. The singularity indices defined above are somewhat different from these in \([6]\) because all singularities are essential if \(n \geq 3\). One can check the value of these singularity indices is independent on a relatively minimal model \(P\) of \(\hat{P}\) satisfying 3.1 by the same proof for \(n = 2\) in \([6]\).

Definition 3.12. Given positive integers \(m, l, \{i_{\text{max}}\}\) and \(\{d_{i,j}\}\) such that \(m = kn + 1 \in n\mathbb{Z} + 1, 1 \leq l \leq n - 1, i_{\text{max}} \geq (i + 1)_{\text{max}}, \sum_{i=1}^l i_{\text{max}} = n - 1, d_{i,j} \geq d_{i,j+1}, \sum_{i,j} d_{i,j} = (m - 1)/n + 1, d_{i,j} \leq [r/2n]\). We define the singularity index \(\beta_{(k,(d_{i,j}))}(F_p)\) of type \((k,(d_{i,j}))\) of \(F_p\) by the number of singularities over \(p \in B\) at which the exceptional curve appeared
by blowing up has the \((d_{i,j})\)-diagram. Clearly, \(\beta_{(k,(d_{i,j}))}(F_p) = 0\) if \(F_p\) is a general fiber. Set 
\[ \beta_{(k,(d_{i,j}))} = \sum_{p \in B} \beta_{(k,(d_{i,j}))}(F_p) \]
and is called the singularity index of type \((k,(d_{i,j}))\) of \(f\).

Given positive integers \(n\) of type \((d_{i,j})\), we have \(\sum_{i,j} d_{i,j} = r/n + 1\), \(d_{i,j} \leq [r/2n]\). We define the singularity index \(\gamma_{(d_{i,j})}(F_p)\) of type \((d_{i,j})\) of \(F_p\) as follows:

If \(\Gamma_p\) has the \((d_{i,j})\)-diagram, \(\gamma_{(d_{i,j})}(F_p)\) equals 1, otherwise, \(\gamma_{(d_{i,j})}(F_p)\) equals 0.

We call \(\gamma_{(d_{i,j})} = \sum_{p \in B} \gamma_{(d_{i,j})}(F_p)\) the singularity index of type \((d_{i,j})\) of \(f\).

From Lemma 2.8 it follows easily

**Lemma 3.13.** Let \(\varepsilon\) be the number of blow-ups of \(\rho\). Then, we have

\[ \varepsilon = \sum_{k,d_{i,j}} \beta_{(k,(d_{i,j}))} + \sum_{d_{i,j}} \gamma_{(d_{i,j})}. \]

We have seen that \(R\) is numerically equivalent to \(-rK_\varphi/2 + MT\) for some half-integer \(M\).

Then we can represent \(M\) in the singularity indices by two calculations of \((K_\varphi + \tilde{R})\tilde{R}\) as follows:

From (2.4) and (2.5), we have

\[
(K_\varphi + \tilde{R})\tilde{R} = \left(\tilde{\psi}^* (K_\varphi + R) + \sum_{i=1}^N \left(1 - n \left[\frac{m_i}{n}\right]\right) E_i\right) \left(\tilde{\psi}^* R - n \left[\frac{m_i}{n}\right] E_i\right) \\
= (K_\varphi + R)R - \sum_{i=1}^N n \left[\frac{m_i}{n}\right] \left(n \left[\frac{m_i}{n}\right] - 1\right) \\
= \left(1 - \frac{r}{2}\right) K_\varphi + MT - \left(-\frac{r}{2} K_\varphi + MT\right) - \sum_{k=1}^{\left[\frac{r}{2}\right]} nk(nk - 1)\alpha_k \\
= 2(r - 1)M - n \sum_{k=1}^{\left[\frac{r}{2}\right]} k(nk - 1)\alpha_k.
\]

On the other hand, we have

\[
(K_\varphi + \tilde{R})\tilde{R} = (K_\varphi + \tilde{R}_0)\tilde{R}_0 + D_1(K_\varphi + D_1) \\
= \alpha_0 - 2\varepsilon.
\]

Hence we obtain

\[
M = \frac{1}{2(r - 1)} \left(\alpha_0 + n \sum_{k=1}^{\left[\frac{r}{2}\right]} k(nk - 1)\alpha_k - 2 \left(\sum_{k,d_{i,j}} \beta_{(k,(d_{i,j}))} + \sum_{d_{i,j}} \gamma_{(d_{i,j})}\right)\right)
\]

22
Next, we will compute $K_\tilde{\phi}$ and $\chi_\tilde{\phi}$. We have

$$\tilde{\delta}^2 = \delta^2 - \sum_{i=1}^{N} \left( \frac{m_i}{n} \right)^2 = \frac{2rM}{n^2} - \sum_{k=1}^{\left[ \frac{r}{2n} \right]} k^2 \alpha_k,$$

and

$$\tilde{\delta}K_\tilde{\phi} = \delta K_\phi + \sum_{i=1}^{N} \left[ \frac{m_i}{n} \right] = -\frac{2M}{n} + \sum_{k=1}^{\left[ \frac{r}{2n} \right]} k\alpha_k.$$

Further,

$$K^2_\tilde{\phi} = K^2_\phi - N = -\sum_{k=1}^{\left[ \frac{r}{2n} \right]} \alpha_k.$$

Thus, we get

$$K^2_\tilde{\phi} = -n \sum_{k=1}^{\left[ \frac{r}{2n} \right]} \alpha_k + 2n(n-1) \left( -\frac{2M}{n} + \sum_{k=1}^{\left[ \frac{r}{2n} \right]} k\alpha_k \right) + n(n-1)^2 \left( -\frac{2rM}{n^2} + \sum_{k=1}^{\left[ \frac{r}{2n} \right]} k^2 \alpha_k \right)$$

and

$$\chi_\tilde{\phi} = \frac{1}{12} n(n-1)(2n-1) \left( \frac{2rM}{n^2} - \sum_{k=1}^{\left[ \frac{r}{2n} \right]} k^2 \alpha_k \right) + \frac{1}{4} n(n-1) \left( -\frac{2M}{n} + \sum_{k=1}^{\left[ \frac{r}{2n} \right]} k\alpha_k \right)$$

by (2.6) and (2.7). Hence we obtain

$$K^2_\tilde{\phi} = \frac{2(n-1)((n-1)r-2n)}{n} M - n \sum_{k=1}^{\left[ \frac{r}{2n} \right]} ((n-1)k-1)^2 \alpha_k + \varepsilon$$

and

$$\chi_\tilde{\phi} = \frac{n-1}{6n} (r(2n-1) - 3n) M - \frac{n-1}{12} \sum_{k=1}^{\left[ \frac{r}{2n} \right]} ((2n-1)k^2 - 3k) \alpha_k$$

by (2.6) and (2.7). Hence we obtain

$$K^2_\tilde{\phi} = \frac{2(n-1)((n-1)r-2n)}{n} M - n \sum_{k=1}^{\left[ \frac{r}{2n} \right]} ((n-1)k-1)^2 \alpha_k + \varepsilon$$

and

$$\chi_\tilde{\phi} = \frac{n-1}{r-1} \left( \frac{(n-1)r-2n}{n} (\alpha_0 - 2\varepsilon) + (n+1) \sum_{k=1}^{\left[ \frac{r}{2n} \right]} (-nk^2 + rk) \alpha_k \right) - n \sum_{k=1}^{\left[ \frac{r}{2n} \right]} \alpha_k + \varepsilon$$
and

\[
\chi_f = \frac{n-1}{6n}(r(2n-1) - 3n)M - \frac{n(n-1)}{12} \sum_{k=1}^{\lfloor \frac{n}{2n} \rfloor} ((2n-1)k^2 - 3k)\alpha_k
\]

\[
= \frac{n-1}{12(r-1)} \left( \frac{(2n-1)r - 3n}{n} (\alpha_0 - 2\varepsilon) + (n + 1) \sum_{k=1}^{\lfloor \frac{n}{2n} \rfloor} (-nk^2 + rk)\alpha_k \right).
\]

Moreover, we obtain

\[
e_f = 12\chi_f - K_f^2
\]

(3.6)

\[
= (n - 1)\alpha_0 + n \sum_{k=1}^{\lfloor \frac{n}{2n} \rfloor} \alpha_k - (2n - 1)\varepsilon
\]

by Noether's formula. From these, it follows

\[
K_f^2 - \lambda_{g,0,n}\chi_f = \frac{n}{(2n-1)r - 3n} \sum_{k=1}^{\lfloor \frac{n}{2n} \rfloor} ((n+1)(n-1)(-nk^2 + rk) - (2n-1)r + 3n)\alpha_k + \varepsilon
\]

and the right hand side is non-negative if \( r \geq n + 2 \). Indeed, the polynomial \((n+1)(n-1)(-nk^2 + rk) - (2n-1)r + 3n\) in \( k \) is monotonically increasing and we have \((n+1)(n-1)(-nk^2 + rk) - (2n-1)r + 3n = n(n-2)(r-n-2)\) when \( k = 1 \). Hence, we obtain the following theorem:

**Theorem 3.14.** Let \( f : S \to B \) be a fibration with an automorphism of type \((g,0,n)\). Then, we have the following equality

\[
K_f^2 = \lambda_{g,0,n}\chi_f + \sum_{p \in B} \text{Ind}(F_p),
\]

where \( \text{Ind}(F_p) : B \to \mathbb{Q} \) is the function defined by

\[
\text{Ind}(F_p) = \frac{n}{(2n-1)r - 3n} \sum_{k=1}^{\lfloor \frac{n}{2n} \rfloor} ((n+1)(n-1)(-nk^2 + rk) - (2n-1)r + 3n)\alpha_k(F_p)
\]

\[
+ \sum_{k,d_{i,j}} \beta(k, (d_{i,j})) (F_p) + \sum_{d_{i,j}} \gamma(d_{i,j}) (F_p)
\]

and non-negative for \( g \geq n(n-1)/2 \) (or \( r \geq n + 2 \) equivalently) and takes 0 for a general fiber \( F_p \).

For an oriented compact real 4-dimensional manifold \( X \), the signature \( \text{Sign}(X) \) is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form on \( H^2(X) \). From the slope equality, we observe the local concentration of \( \text{Sign}(S) \) on a finite number of fiber germs.
Corollary 3.15. (cf. [1]) Let $f : S \to B$ as above. Then, we have
\[
\text{Sign}(S) = \sum_{p \in B} \sigma(F_p),
\]
where $\sigma(F) : B \to \mathbb{Q}$ is defined by
\[
\sigma(F_p) = \frac{-(n-1)(n+1)r}{3n(r-1)} \alpha_0(F_p) + \frac{[\frac{r}{n}]}{3n(r-1)} \left( \frac{(n-1)(n+1)(-nk^2 + rk)}{3(r-1)} - n \right) \alpha_k(F_p)
\]
\[
+ \frac{1}{3n(r-1)} \left( (n+2)(2n-1)r - 3n \right) \left( \sum_{k,d_i,j} \beta_{(k,d_i,j)}(F_p) + \sum_{d_i,j} \gamma_{(d_i,j)}(F_p) \right).
\]

**Proof.** By the index theorem ([4] p.126), it follows
\[
\text{Sign}(S) = \sum_{p+q \equiv 0(\text{mod}2)} h^{p,q}(S) = K_f^2 - 8\chi_f.
\]
Since $K_f^2 = \lambda \chi_f + \text{Ind}$ and $12\chi_f = K_f^2 + \epsilon_f$, we have
\[
K_f^2 = \frac{12}{12 - \lambda} \text{Ind} + \frac{\lambda}{12 - \lambda} \epsilon_f,
\]
\[
\chi_f = \frac{1}{12 - \lambda} \text{Ind} + \frac{1}{12 - \lambda} \epsilon_f,
\]
where $\lambda = \lambda_{g,0,n}$ and $\text{Ind} = \sum_{p \in B} \text{Ind}(F_p)$. Then, we obtain
\[
\text{Sign}(S) = \frac{4}{12 - \lambda} \text{Ind} - \frac{8 - \lambda}{12 - \lambda} \epsilon_f.
\]
Hence, by the definition of Ind and (3.6), we obtain
\[
\text{Sign}(S) = \frac{-(n-1)(n+1)r}{3n(r-1)} \alpha_0 + \frac{[\frac{r}{n}]}{3n(r-1)} \left( \frac{(n-1)(n+1)(-nk^2 + rk)}{3(r-1)} - n \right) \alpha_k
\]
\[
+ \frac{1}{3n(r-1)} \left( (n+2)(2n-1)r - 3n \right) \left( \sum_{k,d_i,j} \beta_{(k,d_i,j)} + \sum_{d_i,j} \gamma_{(d_i,j)} \right).
\]

We will examine the upper bound of the slope of $f$. Let $D_a$ be the sum of all vertical ($-an$)-curves in $\tilde{R}$. Clearly, we have $\tilde{R}_a = \sum_{a=1}^{\tilde{c}} D_a$ and $(\tilde{R}_0)_v = \sum_{a=2}^{\tilde{c}} D_a$. For a divisor $T$ of vertical curves and $p \in B$, we denote $T(p)$ to be the maximal divisor of $T$ which consists of vertical curves over $p$. We can write $T = \sum_{p \in B} T(p)$. We consider a family $\{L^i\}_i$ of vertical irreducible curves in $\tilde{R}$ over $p$ satisfying the following:

(i) $L^1$ is the proper transform of the fiber $\Gamma_p$ or an exceptional curve $E^1$. 
(ii) For $i \geq 2$, $L^i$ is the proper transform of an exceptional curve $E^i$ which contracts to a point $x^i$ on $C^k$ or a proper transform of $C^k$ for some $k < i$, where $C^1 = E^1$ or $\Gamma_p$ of which $L^1$ is the proper transform, and $C^j = E^j$ for $j < i$.

(iii) $\{L^i\}_i$ is maximum in the families satisfying (i) and (ii).

All vertical irreducible curves in $\tilde{R}$ over $p$ are decomposed to the disjoint union of such families uniquely. We denote this decomposition of $\tilde{R}_v(p)$ by $D^1(p) + \cdots + D^{\nu p}(p)$. Let $D^i_a(p)$ be the sum of all vertical $(-an)$-curves in $D^i(p)$. Then, it follows easily that $\tilde{R}_v(p) = D^1(p) + \cdots + D^{\nu p}(p)$ and $D_a(p) = \sum_{i=1}^{\nu p} D^i_a(p)$ and $D^i(p) = \sum_{a=1}^{\xi} D^i_a(p)$. Put $j_a(p) = \#D_a(p)$, $j^i_a(p) = \#D^i(p)$, $j^i_a(p) = \#D^i_a(p)$ and $j(p) = \#\tilde{R}_v(p)$, where $\#T$ is the number of irreducible components of an effective divisor $T$. Firstly, we will examine the relation of them and the singularity indices. Let $\alpha^+_0(F_p)$ be the ramification index of $\hat{\varphi}: \tilde{R}_h \to B$ over $p$ and $\alpha^-_0(F_p) = \alpha_0(F_p) - \alpha^+_0(F_p)$.

**Lemma 3.16.** Let $L$ be a vertical curve in $\tilde{R}$ over $p$ and $C$, $l$, $i_{\max}$, $E_{i,j}$ as before. Then, $L$ contributes at least $\sum_{i=1}^{\nu p} \sum_{j=1}^{i_{\max}} (j - 1) \tilde{R}_h \hat{E}_{i,j}$ to $\alpha^+_0(F_p)$, where $\hat{E}_{i,j}$ is the proper transform of $E_{i,j}$.

**Proof.** When $\Gamma_p = mG + \cdots$ and $\tilde{R}_h G = d$, it follows that the ramification index of $\hat{\varphi}: \tilde{R}_h \to B$ over $p$ is greater than or equal to $(m - 1)d$ since $\alpha^+_0(F_p) = r - \#(\text{Supp}(\Gamma_p) \cap \text{Supp}(\tilde{R}_h))$. Moreover, we can check easily that the multiplicity of $\hat{E}_{i,j}$ in $\Gamma_p$ is greater than or equal to $j$. From these, the assertion follows. 

**Lemma 3.17.** The following hold:

1. $\alpha^+_0(F_p) \geq (n - 2)(j(p) - \eta_p)$.
2. $\alpha^-_0(F_p) = -2 \sum_{a=2}^{\xi} j_a(p)$.
3. $\varepsilon(F_p) = j_1(p)$, where $\varepsilon(F_p) = \sum_{k,d_{i,j}} \beta((k,d_{i,j}))(F_p) + \sum_{d_{i,j}} \gamma((d_{i,j}))(F_p)$.
4. $\sum_{k=1}^{\lfloor r/2 \rfloor} \alpha_k(F_p) \geq \sum_{a=1}^{\xi} (an - 2) j_a(p) + 2\eta_p - \max\{j(p) - 2\eta_p, 0\}$.

**Proof.** (2), (3) are obvious. Let $D^i(p) = L^1 + \cdots + L^{j^i(p)}$ be the irreducible decomposition as above. For $L^k$, let $C^k$, $l^k$, $i^k_{\max}$, $x_{i,j}^k$, $E_{i,j}^k$ be as before. When $L^k$ is a $(-an)$-curve, $L^k$ is obtained by $\sum_{i=1}^{i^k_{\max}} = an - 1$ blow-ups (or $an$ blow-ups when $k = 1$ and $C^1 = \Gamma_p$) as the proper transform of $C^k$. Thus, counting in disregard of overlaps, $D^i(p)$ is obtained by blowing up $\sum_{k=1}^{i^k_{\max}} + \sum_{i=1}^{i^k_{\max}} \alpha_k(F_p) \geq \sum_{a=1}^{\xi} (an - 2) j_a(p) + 2\eta_p - \sum_{t=1}^{\eta_p} \kappa^t(p)$.
Combining this and \(0 \leq \sum_{t=1}^{\eta_p} t^t(p) \leq \max\{j(p) - 2\eta_p, 0\}\), (4) follows. Since \(x^2, \ldots, x^j(p)\) are of type \(n^Z + 1\) and Lemma \(3.2\), it follows that

\[
\sum_{k=1}^{j^t(p)} \sum_{i=1}^{t^k} i^k_{\text{max}} \geq j^t(p) - 1 + \sum_{k=1}^{j^t(p)} l^k
\]

and equivalently

\[(3.7)\]

\[
\sum_{k=1}^{j^t(p)} \sum_{i=1}^{t^k} (i^k_{\text{max}} - 1) \geq j^t(p) - 1.
\]

From Lemma \(3.16\), \(L^k\) contributes at least \(\sum_{i=1}^{t^k} \sum_{j=1}^{\eta_{i,j}^k} (j - 1)\tilde{R}_{i,j}^{k}E_{i,j}^k\) to \(\alpha_0^+(F_p)\). Let us estimate the part \(\sum_{i=1}^{t^k} \sum_{j=1}^{\eta_{i,j}^k} (j - 1)\tilde{R}_{i,j}^{k}E_{i,j}^k\) of it. Since \(m_i^{i,j} \in \mathbb{N}\), it follows \(\tilde{E}_{i,j}^{k}\) is not contained in \(\tilde{R}\) and then \(R\tilde{E}_{i,j}^{k} \geq 0\). Moreover, \(R\tilde{E}_{i,j}^{k} \geq n\) since \(L^k\) intersects with \(\tilde{E}_{i,j}^{k}\) and \(\tilde{R} \sim n\tilde{\Delta}\). We consider singularities of \(R\) on \(\tilde{E}^k_{i,j}\).

(i) the case that there exists a singularity of type \(n^Z\) on \(\tilde{E}_{i,j}^{k}\). Let \(E'\) be the exceptional curve of the blow-up at this point. Then, the proper transform \(\tilde{E}'\) of \(E'\) is not contained in \(\tilde{R}\). If there exists a singularity of type \(n^Z\) on \(E'\), we can repeat this operation. Thus, we obtain an exceptional curve \(E''\) over \(\tilde{x}_{i,j}^{k}\) such that there exist no singularities on \(E''\) or singularities of type \(n^Z + 1\) only on \(E''\). The latter case is treated in (iii). We consider the former case. If \(E''\) intersects with a vertical curve \(L\) in \(\tilde{R}\), then the image of \(L\) intersects with \(L^k\) after blowing down of \(E'\). Hence, it follows that \(L = \tilde{E}_{i,j}^{k}\) since two vertical irreducible curve over \(p\) intersects transversely. Thus, we get \(\tilde{R}_{i,j}^{k}E'' \leq 1\). Since \(E''\) intersects with \(\tilde{R}\), we get \(\tilde{R}_{i,j}^{k}E'' \geq n - 1\). Since the multiplicity of \(\tilde{E}''\) in \(\tilde{\Gamma}_p\) is equal to that of \(\tilde{E}_{i,j}^{k}\), at least \((i_{\text{max}}^k - 1)(n - 1)\) is contributed to \(\alpha_0^+(F_p)\).

(ii) the case that there exist no singularities on \(\tilde{E}_{i,j}^{k}\).

If \(\tilde{E}_{i,j}^{k}\) is contained in \(\tilde{R}\), then \(\tilde{R}_{i,j}^{k}E_{i,j}^k = 2\). If \(\tilde{E}_{i,j}^{k}\) is not contained in \(\tilde{R}\), then \(\tilde{R}_{i,j}^{k}E_{i,j}^k = 1\). Thus, we get \(\tilde{R}_{i,j}^{k}E_{i,j}^k \geq n - 2\).

(iii) the case that there exist singularities of type \(n^Z + 1\) only on \(\tilde{E}_{i,j}^{k}\).

The proper transform of the exceptional curve for any singularity on \(\tilde{E}_{i,j}^{k}\) belongs to other \(D^u(p)\) and become \(L^1\) in \(D^u(p)\). Since the multiplicity of it on \(\tilde{\Gamma}_p\) is greater than or equal to \(i_{\text{max}}^k\), we may assume \(\tilde{E}_{i,j}^{k}\) contributes \((i_{\text{max}}^k - 1)(n - 2)\) to \(\alpha_0^+(F_p)\).
From (i), (ii), (iii) and (3.7), we have
\[
\eta_p \sum_{t=1}^{j(t)p} \sum_{k=1}^{l(k)} \sum_{i=1}^{(i_{\max}^k - 1)} (\bar{R}_k \bar{E}_i^k) \geq \sum_{t=1}^{\eta_p} \sum_{k=1}^{j(t)p} \sum_{i=1}^{l(k)} (i_{\max}^k - 1)(n-2)
\]
\[
\geq \sum_{t=1}^{\eta_p} (j(t)p - 1)(n-2)
\]
\[
= (j(p) - \eta_p)(n-2).
\]
Hence, (1) follows.

Using Lemma 3.17, we will give an upper bound of the slope of \(f\) for \(n \geq 4\).

Theorem 3.18. Let \(f: S \to B\) be a fibered surface with an automorphism of type \((g, 0, n)\), \(n \geq 4\). Then,
\[
K_f^2 \leq \left(12 - \frac{48n(n-1)(r-1)}{n(n+1)r^2 - 8(2n-1)r + 24n}\right) \chi_f.
\]

Proof. Put \(\mu = 48n(n-1)(r-1)/(n(n+1)r^2 - 8(2n-1)r + 24n)\), \(\mu' = (n-1)\mu/12(r-1)\) and \(A = \max\{j(p) - 2\eta_p, 0\}\). Then, we have
\[
(12 - \mu)\chi_f - K_f^2
\]
\[
= e_f - \mu\chi_f
\]
\[
= (n-1)\alpha_0 + n \sum_{k=1}^{\lceil \frac{r+1}{2n} \rceil} \alpha_k - (2n-1)\varepsilon
\]
\[
- \mu' \left(\frac{r(2n-1) - 3n}{n}\alpha_0 + (n+1) \sum_{k=1}^{\lceil \frac{r+1}{2n} \rceil} (-nk^2 + rk)\alpha_k - \frac{2(r(2n-1) - 3n)}{n}\varepsilon\right)
\]
\[
= \left((n-1) - \frac{r(2n-1) - 3n}{n}\mu'\right)\alpha_0 - \left((2n-1) - \frac{2(r(2n-1) - 3n)}{n}\mu'\right)\varepsilon
\]
\[
+ \sum_{k=1}^{\lceil \frac{r+1}{2n} \rceil} \left\{\mu' (n+1)(k - \frac{r}{2n})^2 + n - \frac{(n+1)r^2}{4n}\mu'\right\}\alpha_k
\]
\[
\geq \left((n-1) - \frac{r(2n-1) - 3n}{n}\mu'\right)\alpha_0 - \left((2n-1) - \frac{2(r(2n-1) - 3n)}{n}\mu'\right)\varepsilon
\]
\[
+ \sum_{k=1}^{\lceil \frac{r+1}{2n} \rceil} \left(n - \frac{(n+1)r^2}{4n}\mu'\right)\alpha_k.
\]
Set \(A_n = n - 1 - ((r(2n-1) - 3n)/n)\mu'\) and \(B_n = n - ((n+1)r^2/4n)\mu'\). From the definition of \(\mu\), it follows that \(A_n\) and \(B_n\) are positive and \(-2A_n + nB_n - 1 = 0\). From Lemma 3.17.
we have

\[
A_n\alpha_0(F_p) + B_n \sum_{k=1}^{[\frac{r}{2n}]} \alpha_k(F_p) - (2A_n + 1)\varepsilon(F_p)
\]

\[
\geq ((n - 4)A_n + (n - 2)B_n - 1)j_1(p) + \sum_{a=2}^{\xi} ((n - 4)A_n + (an - 2)B_n)j_a(p)
\]

\[
+(-n - 2)A_n + 2B_n)\eta_p - AB_n
\]

\[
= \begin{cases} 
((n - 4)A_n + (n - 2)B_n - 1)j_1(p) + \sum_{a=2}^{\xi} ((n - 4)A_n + (an - 2)B_n)j_a(p) \\
+(-n - 2)A_n + 2B_n)\eta_p & (j(p) \leq 2\eta_p). \\
((n - 4)A_n + (n - 3)B_n - 1)j_1(p) + \sum_{a=2}^{\xi} ((n - 4)A_n + (an - 3)B_n)j_a(p) \\
+(-n - 2)A_n + 4B_n)\eta_p & (j(p) \geq 2\eta_p).
\end{cases}
\]

Note that we may assume \( r \geq n(n - 1) \) by Corollary 3.6. If \( n \geq 4 \), one can see

\[
-(n - 2)A_n + 2B_n < 0,
\]

and then

\[
((n - 4)A_n + (n - 2)B_n - 1)j_1(p) + \sum_{a=2}^{\xi} ((n - 4)A_n + (an - 2)B_n)j_a(p)
\]

\[
+(-n - 2)A_n + 2B_n)\eta_p
\]

\[
\geq (-2A_n + nB_n - 1)j_1(p) + \sum_{a=2}^{\xi} (-2A_n + anB_n)j_a(p)
\]

\[
= \sum_{a=2}^{\xi} ((a - 1)nB_n + 1)j_a(p)
\]

\[
\geq 0
\]

If \( n \geq 5 \), it follows that \(-(n - 2)A_n + 4B_n\) is negative. Hence, the assertion follows if \( n \geq 5 \).

Suppose that \( n = 4 \). Then, one can see \( B_4 > 1 \) and \(-2A_4 + 4B_4 = 1\). Hence, we have

\[
(B_4 - 1)j_1(p) + \sum_{a=2}^{\xi} ((4a - 3)B_4)j_a(p) + (-2A_4 + 4B_4)\eta_p \geq 0.
\]

Thus, the assertion follows when \( n = 4 \).

\[\square\]

**Remark 3.19.** If \( f \) is not locally trivial and \( K_f^2 = (12 - \mu)\chi_f \), then \( \alpha_k = 0 \) for any \( k \neq r/2n \) and all singularities of type \( n\mathbb{Z} \) are involved in giving vertical curves in \( \tilde{R} \). However, one can prove easily that it does not happen. Hence, the upper bound in Theorem 3.18 is not sharp.
Appendix

Let \( f : S \rightarrow B \) be a genus \( g \geq 2 \) fibered surface. Let \( e_f(F_p) = e(F_p) - e(F) = e(F_p) - 2 + 2g \) for any fiber \( F_p \), where \( F \) is a general fiber. It is well known that \( e_f(F_p) \geq 0 \) and the equality holds if and only if \( F_p \) is smooth, and \( e_f = \sum_{p \in B} e_f(F_p) \). On the other hand, if \( f \) is a fibered surface with an automorphism of type \((g, 0, n)\), we have obtained another local concentration of \( e_f \) as \( e_f = (n - 1)\alpha_0 + n \sum_{k=1}^{[r/2n]} \alpha_k - (2n - 1)\varepsilon \). In fact, these two representation coincide. Namely, the following assertion holds:

**Proposition 3.20.** For any \( p \in B \), it follows that

\[
e_f(F_p) = (n - 1)\alpha_0(F_p) + n \sum_{k=1}^{[r/2n]} \alpha_k(F_p) - (2n - 1)\varepsilon(F_p).
\]

**Proof.** It is sufficient to show that

\[
e_f(\tilde{F}_p) = (n - 1)\alpha_0^+(F_p) + n \sum_{k=1}^{[r/2n]} \alpha_k(F_p) - 2(n - 1)j(p).
\]

Let \( N = \sum_{k=1}^{[r/2n]} \alpha_k(F_p) \) be the number of blow-ups on \( \Gamma_p \) and \( \tilde{\Gamma}_p = \sum_{i=0}^{N} m_i \Gamma_i \) the irreducible decomposition. We may assume \( \Gamma_i \) and \( \tilde{R}_h \) are transverse. Put \( r_i = \Gamma_i\tilde{R} \) and \( F_i = \theta^*\Gamma_i \). Then,

\[
F_p = \sum_{i=0}^{N} m_i F_i = \sum_{r_i>0} m_i F_i + \sum_{r_i=0} m_i(F_{i,1} + \cdots + F_{i,n}) + \sum_{\Gamma_i \subset \tilde{R}} m_i n F_i',
\]

where \( F_{i,j} \), \( F_i' \) are smooth rational curves. For \( r_i > 0 \), the restriction map \( F_i \rightarrow \Gamma_i \) is \( n \)-cyclic covering. From the Hurwitz formula, we have \( 2g(F_i) = 2(2n) - 2 = 2n + (n - 1)r_i \). Let \( N_1 \), \( N_2 \) and \( N_3 \) be the number of intersection points of two \( \Gamma_i \) and \( \Gamma_j \) which is contained in \( \tilde{R}_h \), not contained in \( \tilde{R}_h \) and that one \( \Gamma_i \) is contained in \( \tilde{R} \), respectively. Clearly, it follows \( N = N_1 + N_2 + N_3 \). Let \( J = j(p) \) and \( K \) the number of \( \Gamma_i \) such that \( r_i = 0 \). Then, we have

\[
e(\tilde{F}_p) = \sum_{r_i>0} e(F_i) + 2nK + 2J - N_1 - nN_2 - N_3
\]

\[
= \sum_{r_i>0} (2n - (n - 1)r_i) + 2nK + 2J - N_1 - nN_2 - N_3
\]

\[
= 2n(N + 1) - 2(n - 1)J - (n - 1) \sum_{r_i>0} r_i - N - (n - 1)N_2.
\]

Since \( e(\tilde{F}) = 2n - (n - 1)r \), we have

\[
e_f(\tilde{F}_p) = (2n - 1)N - 2(n - 1)J + (n - 1) \left( r - \sum_{r_i>0} r_i \right) - (n - 1)N_2.
\]
On the other hand, we have
\[
\alpha_0^+ (F_p) = r - \# (\text{Supp}(\tilde{\Gamma}_p) \cap \text{Supp}(\tilde{R})) = r - \sum_{r_i > 0} r_i + N_1 + N_3.
\]

Combing these, the assertion follows. □

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