Optimized Sparse Projections for Compressive Sensing

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Abstract—We consider designing a sparse sensing matrix which contains few non-zero entries per row for compressive sensing (CS) systems. By unifying the previous approaches for optimizing sensing matrices based on minimizing the mutual coherence, we propose a general framework for designing a sparse sensing matrix that minimizes the mutual coherence of the equivalent dictionary and is robust to sparse representation error. An alternating minimization-based algorithm is proposed for designing sparse sensing matrices. Experiments with real images show that the obtained sparse sensing matrix (even each row is extremely sparse) significantly outperforms a random dense sensing matrix in terms of the signal recovery accuracy.

Index Terms—Sparse sensing matrix, structured sensing matrix, mutual coherence, compressive sensing

I. INTRODUCTION

Compressive sensing (CS)—that aims to break through the Shannon-Nyquist limit for sampling signals—has attracted great attention since it was introduced a little more than a decade ago [1]–[3]. The performance of a CS system strongly depends on the properties of two matrices: a sensing matrix \( \Phi \in \mathbb{R}^{M \times N} \) (where \( M \ll N \)) which compresses the signal and a dictionary \( \Psi \in \mathbb{R}^{N \times L} \) (where \( L \geq N \)) that captures the concise structure of the signal. In particular, we say \( x \in \mathbb{R}^N \) is sparse (in \( \Psi \)) if \( x \approx \Psi s \) with \( \|s\|_0 \leq K \), where \( K \ll L \) and \( \| \cdot \|_0 \) counts the number of nonzero elements.

The choice of dictionary \( \Psi \) depends on the signal model and traditionally we attempt to concisely capture the structure contained in the signals of interest by a well-designed dictionary. Typical examples include the Fourier matrix for frequency-sparse signals, and a multiband modulated Discrete Prolate Spheroidal Sequences (DPSS’s) dictionary for sampled multiband signals [5]. Dictionary learning is another popular approach that learns the dictionary by giving a set of representative signals (called training data) [6]–[8].

Another important factor in CS is to choose an appropriate sensing matrix \( \Phi \) preserving the useful information contained in the signal \( x \) such that it is possible to recover the signal from its low dimensional measurement \( y = \Phi x \). It has been shown that if the equivalent dictionary \( \Phi \Psi \) satisfies the restricted isometry property (RIP), a sparse signal can be exactly reconstructed from its linear measurement \( y \) [1], [9]. Despite the fact that a random matrix satisfies the RIP with high probability [9], testing whether a matrix satisfies the RIP is NP-hard [10]. Therefore, mutual coherence—another property of the sensing matrices that is much easier to verify—has been alternatively utilized to design the sensing matrix [11]–[17].

In hardware implementation [18] and applications like electrocardiography (ECG) compression [19] and data stream computing [20], it is useful that the sensing matrix \( \Phi \) is a sparse matrix, i.e., it contains very few non-zero elements per row. Sparsity implies that it is faster to apply the resulting sensing matrix for efficiently acquiring a signal. We note that a set of iterative methods for sparse linear systems also require performing the sensing matrix to a vector (which is usually called residual) in each iteration. We refer the reader to [20] for more applications where a sparse sensing matrix is useful and also crucial. To the best of our knowledge, this is the first work to design a sparse sensing matrix via enhancing the mutual coherence property of the equivalent dictionary. The main contributions and the paper organization are given as follows.

By unifying the previous approaches for optimizing sensing matrices based on mutual coherence [11]–[16], the first contribution is to propose a unifying framework for designing a sparse sensing matrix based on the mutual coherence of the equivalent dictionary in Section II. The designed sparse sensing matrix is also robust to sparse representation error (SRE), which is generally not negligible in practical applications like image processing [15]. Our proposed framework also covers the design of structured sparse sensing matrices which consist of a sparse matrix and a structured matrix (like a discrete cosine transform (DCT) matrix) that can be very efficiently applied to a vector.

The second contribution is to provide an alternating minimization-based algorithm for designing sparse sensing matrices in Section III. We utilize the projected gradient descent to update the sensing matrix. We demonstrate the performance of the obtained sparse sensing matrix with experiments on real images in Section IV.

II. A FRAMEWORK FOR DESIGNING SPARSE SENSING MATRIX

A. Review: A uniform framework for sensing matrix design

To begin, we briefly review the concept of mutual coherence to CS. The mutual coherence of \( Q \in \mathbb{R}^{M \times L} \) is defined as

\[
\mu(Q) \triangleq \max_{1 \leq i \neq j \leq L} \frac{|q_i^T q_j|}{\|q_i\|_2 \|q_j\|_2} \geq \mu \triangleq \sqrt{\frac{L - M}{M(L - 1)}}
\]

(1)
where \( q_i \) is the \( i \)-th column of \( Q \) and \( \mu \) is the lower bound of the mutual coherence of \( Q \) \cite{[1]}.

With the projected measurements \( y \) of the form

\[
y = \Phi x
\]

and the prior information that \( x \) is sparse in \( \Psi \), we can recover the signal as \( \hat{x} = \Psi \hat{s} \) where

\[
\hat{s} = \arg \min_{s} \|y - \Phi \Psi s\|_2^2 \\
\text{subject to } \|s\|_0 \leq K
\]

which can be exactly or approximately solved via convex methods \cite{[1], [22], [23]} or greedy algorithm \cite{[24]}, like the orthogonal matching pursuit (OMP). It is shown in \cite{[24]} that OMP can exactly solve (3) (and hence obtain accurate estimation of \( x \)) if

\[
K < \frac{1}{2} \left[ 1 + \frac{1}{\mu(D)} \right]
\]

where \( D = \Phi \Psi \) is referred to as the equivalent dictionary of the CS system. This is a simple motivation of designing a sensing matrix with small mutual coherence \( \mu(D) \) as it enables a large set of sparse signals that can be recovered from the projected measurement. We refer the reader to \cite{[11], [14]} and the references therein for more discussions.

We now unify the approaches for designing sensing matrix proposed in \cite{[11]–[17]} as follows:

\[
\min_{\Phi, G \in \mathcal{G}_t} \|G - \Psi^T \Phi^T \Phi \Psi\|_F^2 + \lambda \|\Phi E\|_F^2
\]

where \( \mathcal{G}_t \) is the set of targeted Gram matrices that we will discuss soon, the second term \( \|\Phi E\|_F^2 \) is utilized to make the sensing matrix robust to the sparse representation error (SRE) and will be detailed soon, and \( \lambda \geq 0 \) is the trade-off parameter to balance the mutual coherence of the equivalent dictionary and robustness of the sensing matrix to the SRE.

The sensing matrices in \cite{[11], [13], [14]} are obtained via solving (5) with \( \lambda = 0 \) and \( \mathcal{G}_t \) being the following set of relaxed Equiangular Tight Frame (ETF) Gram matrices

\[
\mathcal{G}_t^{\text{ETF}} = \left\{ G \in \mathbb{R}^{L \times L} : G(k, k) = 1, \forall k, \max_{i \neq j} |G(i, j)| \leq \xi \right\}
\]

where \( G(i, j) \) is the \( (i,j) \)th entry of \( G \) and \( \xi \in \{0, 1\} \) is a pre-set threshold and is usually chosen as \( \mu \).

The equivalent dictionary with the optimized sensing matrix has small mutual coherence and the corresponding CS system has better performance than the one with a random sensing matrix for the exactly sparse signals. However, it was recently realized that such a sensing matrix is not robust to SRE (for example it exists when we represent the real images with a learned dictionary) and thus the corresponding CS system yields poor performance, especially in image processing \cite{[15]}. Thus, in \cite{[15]}, the SRE matrix \( E = X - \Psi S \) (where \( X \in \mathbb{R}^{N \times J} \) is the training data and \( S \) consists of the sparse coefficients of \( X \) in \( \Psi \)) is incorporated into the framework (5) to design a sensing matrix that is also robust to SRE. Simulations have shown that the designed sensing matrices (with \( E = \tilde{E} \) and either \( \mathcal{G}_t = \{I\} \) or \( \mathcal{G}_t = \mathcal{G}_t^{\text{ETF}} \)) achieve state-of-the-art performance for CS-based image compression \cite{[15]}. To get rid of the dependence of the training data \( X \) and SRE matrix \( E \), it is proposed to set \( \|E\|_F = 1 \) in (5). The designed sensing matrix in \cite{[17]} has similar or slightly better performance than the one obtained in \cite{[15]}.

B. A framework for designing sparse sensing matrices

As we explained before, in applications like ECG compression \cite{[19]}, data stream computing \cite{[20]} and hardware implementation \cite{[18]}, it is useful to have a sparse sensing matrix as it can be applied very efficiently for acquiring measurements. To further explain the computational complexity of a CS system for acquiring measurements as in (2), we note that the overall computational complexity of \( \Phi x \) for a general dense sensing matrix is \( O(MN) \), while it reduces to \( O(M\kappa) \) for a sparse sensing matrix \( \Phi \) with \( \kappa \) non-zero elements in each row. Aside from a sparse sensing matrix, a structured sparse sensing matrix of the form \( \Phi = \Phi A \) can also be utilized to reduce the computational complexity of acquiring measurements. Here, \( \Phi \in \mathbb{R}^{M \times N} \) is a sparse matrix and \( A \in \mathbb{R}^{N \times N} \) is a structured matrix that can be efficiently applied to a vector. Similar structure appears in double sparsity dictionary \cite{[25]}.

Typical choices of \( A \) include the DCT matrix and Walsh-Hadamard matrix both of which can be applied to a length-\( N \) vector in \( O(N \log N) \).

By adopting the widely utilized framework (5) for designing general sensing matrices, we now propose a general framework for optimizing a structured sparse sensing matrix such that it is robust to SRE and the resulting equivalent dictionary also has small mutual coherence:

\[
\tilde{\Phi} = \arg \min_{\Phi, G \in \mathcal{G}_t} \|G - \Psi^T \Phi^T \Phi \Psi\|_F^2 + \lambda \|\Phi E\|_F^2
\]

subject to

\[
\|\Phi(m,:)\|_0 \leq \kappa, \forall m
\]

and the obtained sensing matrix is \( \Phi = \tilde{\Phi} A \). Here \( \Phi(m,:) \) represents the \( m \)-th row of \( \Phi \). A few remarks are as follows.

Remark 2.1:

- As we stated before, we usually choose \( A \in \mathbb{R}^{N \times N} \) as the one that can be applied to a vector with complexity \( O(N \log N) \). Thus, the total computational complexity of \( \Phi x = \Phi A x \) is \( O(N \log N + M\kappa) \). When \( A \) is the identity matrix, then \( \Phi = \tilde{\Phi} \) and the corresponding complexity reduces to \( O(M\kappa) \). This is a significant reduction of computational complexity compared with a dense sensing matrix which requires \( O(MN) \) computational complexity to sense a signal.

- The choice of the structure matrix \( A \) depends on specific applications. For example, the DCT matrix has been demonstrated to be useful in image processing \cite{[25]}. We also show the DCT matrix can improve the signal reconstruction accuracy when the sensing matrix is very sparse in Section IV.

- Similar to \cite{[17]}, we also suggest to replace \( E \) with the identity matrix (i.e., get rid of the SRE matrix) in (6)\footnote{If we model each column of \( E \) as an i.i.d. Gaussian random vector of zero mean and covariance \( \sigma^2 I \), then \( \|\Phi E\|_F^2 \) converges in probability and almost surely to \( \sigma^2 \|\Phi\|_F^2 \), when the number of training samples \( J \) approaches to \( \infty \).}.
for most applications. As we argued before, \( \| \Phi A \tilde{E} \|_F^2 \) approaches to \( \| \Phi A \|_F^2 \) when the SRE is modeled as Gaussian noise.

- Aside from the fact that \( \Phi \) is parameterized by \( \tilde{\Phi} A \), \( \tilde{\Phi} \) differs from \( \Phi \) in that the former has a sparse constraints on the rows of the sensing matrix \( \Phi \). However, this sparsity constraint makes \( \tilde{\Phi} \) highly nonconvex. We will propose an algorithm to address this constraint and solve \( \tilde{\Phi} \) in next section.

### III. Algorithms for Designing Sparse Sensing Matrix

In this section, we provide an alternating minimization-based algorithm for solving the general sparse sensing matrix design problem \( \tilde{\Phi} \) but with slightly simplified notations:

\[
\begin{align*}
\min_{\tilde{\Phi}, G} \rho(\tilde{\Phi}, G) &= \| G - \Psi^T \tilde{\Phi}^T \Phi \|_F^2 + \lambda \| \tilde{E} \|_F^2 \\
\text{subject to} & \| \Phi(m,:) \|_0 \leq \kappa, \forall m, G \in \mathcal{G}_t
\end{align*}
\]

(7)

where \( \Psi = A \Psi \) and \( \tilde{E} = A E \). Let \( \mathcal{G}_t : \mathbb{R}^{L \times L} \rightarrow \mathbb{R}^{L \times L} \) denote an orthogonal projector onto the set \( \mathcal{G}_t \). First note that the solution of minimizing \( \rho \) in terms of \( G \) when \( \Phi \) is fixed is given by

\[
\tilde{G} = \arg\min_{G \in \mathcal{G}_t} \rho(\Phi, G) = \mathcal{P}_{\mathcal{G}_t}(\Psi^T \Phi^T \tilde{\Phi} \Psi)
\]

(8)

In particular, when \( \mathcal{G}_t \) is the set of relaxed ETF Gram matrices \( \mathcal{G}_{\text{ETF}} \), the orthogonal projector is given by

\[
(\mathcal{P}_{\mathcal{G}_{\text{ETF}}}(G))(i,j) = \begin{cases} 1, & i = j \\ \text{sign}(G(i,j)) \min\{|G(i,j)|, \xi\}, & i \neq j \end{cases}
\]

Now we consider solving (7) in terms of \( \Phi \) when \( G \) is fixed:

\[
\min_{\Phi} \rho(\tilde{\Phi}, G) \text{ subject to } \| \tilde{\Phi}(m,:) \|_0 \leq \kappa, \forall m
\]

(9)

Without the sparsity constraint, the recent work \(^{[25]}\) has shown that a number of iterative algorithms (including gradient descent) can provably solve \( \min_{\Phi} \rho(\tilde{\Phi}, G) \)\(^{[2]}\). Thus, with the sparsity constraint in (9), we utilize the projected gradient descent (PGD) to solve (9). The gradient of \( \rho(\tilde{\Phi}, G) \) in terms of \( \tilde{\Phi} \) is given as follows:

\[
\nabla_{\tilde{\Phi}} \rho(\tilde{\Phi}, G) = 2\lambda \tilde{\Phi} \tilde{E} \tilde{E}^T - 4 \tilde{\Phi} \Psi G \Psi^T + 4 \tilde{\Phi} \Psi \Psi^T \tilde{\Phi} \tilde{\Psi} \Psi^T
\]

For convenience, we let \( \mathcal{S}_\kappa \) denote the set of matrices whose rows have at most \( \kappa \) non-zero elements:

\[
\mathcal{S}_\kappa := \{ Z \in \mathbb{R}^{M \times N} : \| Z(m,:) \|_0 \leq \kappa, \forall m \}
\]

We denote \( \mathcal{P}_{\mathcal{S}_\kappa} : \mathbb{R}^{M \times N} \rightarrow \mathbb{R}^{M \times N} \) as an orthogonal projector onto the set of \( \mathcal{S}_\kappa \). The pseudocode of PGD for solving (9) is given in Algorithm 2. We now summarize the alternating minimization-based algorithm for solving (7) in Algorithm 1:

In terms of the convergence of PGD for solving (9), we extend \(^{[27]}\) Theorem 3.1 from sparse vectors to sparse constraints of matrices in the following result which establishes the convergence of PGD for solving a general sparsity-constrained problem \( \min_{Z \in \mathcal{S}_\kappa} f(Z) \), where \( f \) is a general function. We omit the details of the derivation here.

**Theorem 1.** Suppose \( f : \mathbb{R}^{M \times N} \rightarrow \mathbb{R} \) is lower bounded and its gradient is Lipschitz with constant \( L_c \):

\[
\| \nabla f(Z) - \nabla f(Y) \| \leq L_c \| Z - Y \|, \forall Z, Y \in \mathbb{R}^{M \times N}.
\]

Let \( \{ Z^k \}_{k \geq 0} \) be the sequence generated by the PGD method with a constant stepsize \( \eta < \frac{1}{L_c} \):

\[
Z_{k+1} = \mathcal{P}_{\mathcal{S}_\kappa}(Z_k - \eta \nabla f(Z_k))
\]

Then

- \( f(Z^k) - f(Z^{k+1}) \geq \frac{\eta}{2} L_c \| Z^k - Z^{k+1} \| \)
- \( \| Z^k - Z^{k+1} \|_F \rightarrow 0 \)
- the sequence \( \{ f(Z^k) \}_{k \geq 0} \) converges.
- any accumulation point \( Z^* \) of \( \{ Z^k \}_{k \geq 0} \) is an \( \eta \)-Stationary point:

\[
Z^* = \mathcal{P}_{\mathcal{S}_\kappa}(Z^* - \eta \nabla f(Z^*))
\]

**Algorithm 1 Proposed Algorithm**

**Initialization:**

- Initial value \( \tilde{\Phi}_0 \), the number of outer iterations \( \text{Iter}_{\text{max}} \), the sparsity level \( \kappa \) and the given trade-off parameter \( \lambda \).

**Output:**

- Sparse sensing matrix \( \tilde{\Phi} \).

1: \( i \leftarrow 1 \)
2: \( \text{while } i \leq \text{Iter}_{\text{max}} \text{ do} \)
3: \( G_i = \mathcal{P}_{\mathcal{G}_t}(\tilde{\Phi}_i \tilde{\Psi}^T \tilde{\Phi}_i \tilde{\Psi}) \)
4: \( \tilde{\Phi}_i = \text{argmin}_{\tilde{\Phi}} \| \tilde{\Phi}(m,:) \|_0 \leq \kappa \rho(\tilde{\Phi}, G_i) \text{ with Algorithm 2} \)
5: \( i \leftarrow i + 1 \)
6: \( \text{end while} \)
7: \( \text{return } \tilde{\Phi}_{\text{Iter}_{\text{max}}} \)

**Algorithm 2 Projected Gradient Descent for solving (9)**

**Initialization:**

- Initial value \( \tilde{\Phi}^0 \), step-size \( \eta \), the number of inner iterations \( \text{Inneriter}_{\text{max}} \).

**Output:**

- \( \tilde{\Phi} \) that solves (9).

1: \( \text{while } k \leq \text{Inneriter}_{\text{max}} \text{ do} \)
2: \( \tilde{\Phi}^{k+1} = \mathcal{P}_{\mathcal{S}_\kappa}(\tilde{\Phi}^k - \eta \nabla \rho(\tilde{\Phi}^k, G)) \)
3: \( k \leftarrow k + 1 \)
4: \( \text{end while} \)
5: \( \text{return } \tilde{\Phi}_{\text{Inneriter}_{\text{max}}} \)

**Remark 3.1:**

- When \( \mathcal{G}_t \) consists of a single element (like \( \mathcal{G}_t = \{ 1 \} \)), the problem (7) reduces to (9) and hence the sparse sensing matrix can be simply obtained with Algorithm 2.
- We note that (9) is a sparsity-constrained problem and thus we can solve it by many other algorithms developed for CS, like OMP and convex methods (by replacing the sparsity with \( \ell_1 \) norm).
IV. SIMULATIONS

In this section, we conduct a set of experiments on real images to illustrate the performance of the obtained sparse sensing matrices. In these experiments, we set \( \mathcal{G}_k = \{I\} \).

Similar to [17], we obtain a high dimensional dictionary \( \Psi \in \mathbb{R}^{N \times L} \) as follows: 1) we first randomly obtain \( 10^6 \) non-overlapping \( \sqrt{N} \times \sqrt{N} \) patches from LabelMe training dataset \( \mathbb{D} \) and then arrange each patch as a length-\( N \) vector; 2) with the obtained \( N \times 10^6 \) training data, we then learn the dictionary \( \Psi \) using online dictionary learning algorithm in [29].

Combining the dictionary \( \Psi \) with different sensing matrices leads to different CS systems (Dictionary + Sensing matrix):

| \( \kappa \) | Elaine | Couple | Barbara | Child | Plane | Man |
|---|---|---|---|---|---|---|
| \( \kappa = 10 \) | \( 25.99 \) | \( 29.59 \) | \( 27.40 \) | \( 27.40 \) | \( 22.76 \) | \( 22.76 \) |
| \( \kappa = 30 \) | \( 31.99 \) | \( 31.99 \) | \( 31.99 \) | \( 31.99 \) | \( 29.59 \) | \( 29.59 \) |
| \( \kappa = 50 \) | \( 36.60 \) | \( 36.60 \) | \( 36.60 \) | \( 36.60 \) | \( 33.34 \) | \( 33.34 \) |
| \( \kappa = 10 \) | \( 32.92 \) | \( 32.92 \) | \( 32.92 \) | \( 32.92 \) | \( 30.95 \) | \( 30.95 \) |
| \( \kappa = 30 \) | \( 35.36 \) | \( 35.36 \) | \( 35.36 \) | \( 35.36 \) | \( 31.22 \) | \( 31.22 \) |
| \( \kappa = 50 \) | \( 39.04 \) | \( 39.04 \) | \( 39.04 \) | \( 39.04 \) | \( 31.05 \) | \( 31.05 \) |

The performance for different CS systems is evaluated by peak signal to noise ratio (PSNR), defined by

\[
\sigma_{\text{psnr}} \triangleq 10 \times \log_{10} \left( \frac{(2^r - 1)^2}{\sigma_{\text{mse}}} \right) \text{ (dB)}
\]

where, \( r = 8 \) bits per pixel and \( \sigma_{\text{mse}} \triangleq \frac{1}{J} \| H - \hat{H} \|^2_F \), with \( H, J \) being the original image and the number of pixels, \( \hat{H} \) being the recovered image by the different CS systems.

In the sequel, we always set the parameters \( M = 80, N = 256, L = 800, K = 16 \) and \( \lambda = 0.5 \). By setting \( \kappa = 30 \) and \( A = I \), Fig.1 shows the convergence (with respect to the objective function and the iterates) of Algorithm 2 for solving (9) with a constant stepsize \( \eta = 0.06 \) and an initialization as a closed-form solution of (9) without the sparsity constraints (see [15] Theorem 2). Similar convergence result is also observed when \( A \) is the DCT matrix. We observe from Fig.1 that the objective value sequence is convergent and the change of iterates \( \| \Phi^{k+1} - \Phi^k \|_F \) converges to 0, coinciding with the theoretical convergence guarantee in Theorem 1.

We now compare \( \text{CS}_{\text{sparse}} \), \( \text{CS}_{\text{sparse}} \) with \( \text{CS}_{\text{bispar}} \) and \( \text{CS}_{\text{MT}} \) in terms of PNSR for real images. Fig.2 displays the \( \sigma_{\text{psnr}} \) versus the sparsity \( \kappa \) when the CS systems applied to the image “Lena”. Table I provides \( \sigma_{\text{psnr}} \) for these CS systems when applied on other 6 images.

![Figure 1](image1.png)

Figure 1. Convergence of the proposed algorithm: (Left) Evolution of objective value \( \rho(\Phi, I) \). (Right) The iterates change defined as \( \| \Phi^{k+1} - \Phi^k \|_F \). Here, \( M = 80, N = 256, L = 800, K = 16, \lambda = 0.5, \kappa = 30, A = I \).

![Figure 2](image2.png)

Figure 2. \( \sigma_{\text{psnr}} \) (dB) versus sparsity \( \kappa \) of sensing matrix. \( \text{CS}_{\text{MT}} \) and \( \text{CS}_{\text{randu}} \) are dense matrix. We plot it as a straight line for more convenient visual comparison. Here, \( M = 80, N = 256, L = 800, K = 16, \lambda = 0.5 \).

Remark 4.1:

- As seen from Fig.2 and as also expected, \( \text{CS}_{\text{sparse}} \) and \( \text{CS}_{\text{sparse}} \) have better performance when increase the sparsity \( \kappa \) (the number of non-zero elements in each row). However, it is surprising that even when the sparsity is very low, for example, \( \kappa = 10 \), \( \text{CS}_{\text{sparse}} \) is only 0.53dB inferior than \( \text{CS}_{\text{MT}} \), but still has more than 3dB better than \( \text{CS}_{\text{randu}} \) (which has a dense sensing matrix). We note that the gap between \( \text{CS}_{\text{sparse}} \) and \( \text{CS}_{\text{MT}} \) is almost negligible (with 0.15 dB) when \( \kappa \geq 30 \). This demonstrates our main argument that we can design a sparse sensing matrix that has similar performance as a dense matrix, but can significantly reduce the computational cost for sensing signals.

- As seen from Fig.2 and Table 1, when \( \kappa \) is small, \( \text{CS}_{\text{sparse}} \) has better performance than \( \text{CS}_{\text{sparse}} \) because of \( A \). However, we note that the existence of \( A \) also increases the sensing cost in \( \text{CS}_{\text{sparse}} \) (up to \( O(N \log N) \)) than the one in \( \text{CS}_{\text{sparse}} \).

- Similar results are also observed for different settings of \( M, N, L, K \). We finally note that we only compare with \( \text{CS}_{\text{MT}} \) because as shown in [17] (and we also observed this), \( \text{CS}_{\text{MT}} \) has better performance than the ones in [11].

For CS-based image compression.

As we stated before, compared to the sensing matrix obtained by minimizing (5) with \( \mathcal{G}_k = \{I\} \), the one obtained by minimizing (5) with \( \mathcal{G}_k = \mathcal{G}^{*M} \) yields better performance for signals that are exactly sparse, but has worse performance for signals (like images) that are not exactly sparse under the dictionary [14, 15]. Due to limited room, we only show the results on real images.
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