Solving Nonsmooth Bi-Objective Environmental and Economic Dispatch Problem using Smoothing Techniques

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Abstract

The Environmental and Economic Dispatch problem (EEDP)[14, 13] is a non-linear Multi-objective Optimization Problem (MOP) which simultaneously satisfies multiple contradictory criteria, and it’s a nonsmooth problem when valvepoint effects, multi-fuel effects and prohibited operating zones have been considered. It is an important optimization task in fossil fuel fired power plant operation for allocating generation among the committed units such that fuel cost and pollution (emission level) are optimized simultaneously while satisfying all operational constraints. In this paper, we use smoothing functions with the gradient consistency property to approximate the nonsmooth multi-objective Optimization problem. Our approach is based on the smoothing method. In fact, we explain the convergence analysis of smoothing method by using approximate Karush-Kuhn-Tucker condition, which is necessary for a point to be a local weak efficient solution and is also sufficient under convexity assumptions. Finally, we give an application of our approach for solving the bi-objective EEDP.

Key words: Nonsmooth multi-objective optimization, locally Lipschitz function, Smoothing functions, approximate Karush-Kuhn-Tucker, economic and environmental dispatching problem.

1 Introduction

During the last decades the area of nonsmooth (nondifferentiable) Multiobjective Optimization Problems (MOP) has been extensively developed. The MOP refers to the process of simultaneously optimizing two or more real-valued objective functions. For nontrivial problems, no single point will minimize all given objective functions at once, and so the concept of optimality is to be replaced by the concept of Pareto optimality or efficiency. One should recall that a point is called Pareto optimal or efficient, if there is no different point with the same, or smaller, objective function values, such that there is a decrease in at least one objective function value. The nonsmooth MOP problem has applications in engineering [1], economics [2], mechanics [3] and other fields. For more details, see, for example, Miettinen [4].

In this paper, we concentrate on solving a classe of nonsmooth MOP that include \(\min\), \(\max\), absolute value functions or composition of the plus function with smooth functions. Which the approximations are constructed based on the smoothing function for the plus function. For this end we introduce the concept of approximate Karush-Kuhn-Tucker
AKKT condition for the approximate multiobjective problem inspired by Giorrg. G et al. [9] and we adapt it to prove the convergence analysis of the smoothing method, whose feasible set is defined by inequality constraints. Note that the AKKT condition has been widely used to define the stopping criteria of many practical contrained optimization algorithms [11, 12, 10]. The objective is to update the smoothing parameters to guarantee the convergence. We point out that Chen [8] has dealt with convergence analysis of smoothing method (in the scalar case) by using a gradient method. Finally, we give an application of our approach in solving bi-objective Economic and Environmental Dispatching Problem (EEDP) [14]. In fact, we transform the nonsmooth EEDP into a set of single-objective subproblems using the $\epsilon$-constraint method. The objective function of the subproblems is smoothed and the subproblems are solved by the interior point barrier method.

This paper is organized as follows. In Section 2, we state the problem under consideration and we recall some useful basic notations. In Section 3, we define a class of smoothing composite functions by using the plus function. In Section 4, to explain the convergence analysis of the smoothing method, we use sequential AKKT. Finally, we show a numerical application in Section 5.

2 Basic notations and properties

The following notations are used throughout this paper. By $\langle \cdot, \cdot \rangle$, we denote the usual inner product on $\mathbb{R}^n$, and by $\| \cdot \|$ we denote its corresponding norm. Let $\mathbb{R}^+_p = \{ x \in \mathbb{R}^p : x_i \geq 0, \ i = 1, \cdots, p \}$, $\mathbb{R}^-_p = \{ x \in \mathbb{R}^p : x_i \leq 0, \ i = 1, \cdots, p \}$, $\mathbb{R}^+_p = \{ x \in \mathbb{R}^p : x_i > 0, \ i = 1, \cdots, p \}$ and $\mathbb{R}^-_p = \{ x \in \mathbb{R}^p : x_i < 0, \ i = 1, \cdots, p \}$, we consider the partial orders $\succeq$ (respectively, $\preceq$) and $\succ$ (respectively, $\prec$), defined as $x \succeq y$ (respectively, $x \preceq y$) if and only if $x - y \in \mathbb{R}^+_p$ (respectively, $x - y \in \mathbb{R}^-_p$) and $x \succ y$ (respectively, $x \prec y$) if and only if $x - y \in \mathbb{R}^+_p$ (respectively, $x - y \in \mathbb{R}^-_p$). In this paper we consider the nonsmoothing multiobjective problem

$$P(1) : \begin{cases} \min F(x), \\
\text{subj to } x \in S. \end{cases}$$

where $S = \{ x \in \mathbb{R}^n / g_j(x) \leq 0, \ j = 1, \cdots, m \}$, the objective function $F : \mathbb{R}^n \to \mathbb{R}^p$ is given by $F(x) = (f_1(x), \cdots, f_p(x))$ nonsmooth, convex and locally lipschitz and $g_j : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, $j = 1, \cdots, m$, $S$ is a feasible set of $P(1)$. The set of active indexes at a point $x \in S$ is given by $J(x) = \{ j, \ g_j(x) = 0 \}$. A point $x^* \in S$ is called Pareto optimal point or (efficient solution) of problem $P(1)$ if there exists no other $x \in S$ with $f_i(x) \leq f_i(x^*), \ i = 1, \cdots, m$ and $f_j(x) < f_j(x^*)$ for at least one
index \(j\). If there exists no \(x \in S\) with \(f_i(x) < f_i(x^*)\) \(i = 1, \cdots, p\), then \(x^*\) is said to be a weak Pareto optimal point or (weak efficient solution) of problem \(P_{(1)}\).

**Definition 2.1.** [5] The upper Clarke directional derivative of a locally Lipschitz function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) at \(x\) in the direction \(d \in \mathbb{R}^n\) is

\[
f^o(x, d) = \limsup_{z \rightarrow x, t \downarrow 0} \frac{f(z + td) - f(y)}{t}
\]

and the Clarke subdifferential of \(f\) at \(x\) is given by

\[
\partial_c f(x) = \{ \lambda \in \mathbb{R}^n : \langle \lambda, d \rangle \leq f^o(x, d) \ \forall d \in \mathbb{R}^n \}
\]

When \(f\) is continuously differentiable, one has \(\partial_c f(x) = \{ \nabla f(x) \}\). Now, we recall some results which will be needed in our convergence analysis.

**Proposition 2.2.** [5]

Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be locally Lipschitz and \(h : \mathbb{R}^n \rightarrow \mathbb{R}\) continuously differentiable. Then

(i) \(\partial_c (f(x) + h(x)) = \partial_c f(x) + \nabla h(x).\)

(ii) If \(x^*\) is a local minimum of \(f\), then \(0 \in \partial_c f(x).\)

(iii) If \(f(x) = \max\{f_1(x), \cdots, f_p(x)\}\), where \(f_j : \mathbb{R}^n \rightarrow \mathbb{R}\) for all \(j \in \{1, \cdots, p\}\) are continuously differentiable, then

\[
\partial_c f(x) = \text{conv}\{\nabla f_j(x) : j = 1, \cdots, p \text{ such that } f_j(x) = f(x)\}
\]

(Here \(\text{conv}\) denotes the convex hull).

### 3 Smoothing function

Rockafellar and Wets have shown that for any locally Lipschitz function \(f\), we can construct a smoothing function by using the convolution

\[
f(x, \mu) = \int_{\mathbb{R}^n} f(x - y)\psi_\mu(y)dy = \int_{\mathbb{R}^n} f(y)\psi_\mu(x - y)dy
\]

where \(\psi_\mu : \mathbb{R}^n \rightarrow \mathbb{R}\) is a smooth kernel function, (see [6]). In this section we extend the smoothing method given by Chen [8] to solve nonsmooth MOP, for this, we start by considering a class of smoothing functions.
Definition 3.1. Let \( F : \mathbb{R}^n \to \mathbb{R}^p \) be a continuous function given by \( F(x) = (f_1(x), \cdots, f_p(x)) \), we define a smoothing function of \( F \) by \( \tilde{F} : \mathbb{R}^n \times \mathbb{R}^{p+} \to \mathbb{R}^p \) where \( \tilde{F}(x, \mu) = (\tilde{f}_1(x, \mu_1), \cdots, \tilde{f}_p(x, \mu_p)) \) such that for each \( i = 1, \cdots, p \) \( \tilde{f}_i(x, \mu_i) \) is continuously differentiable in \( \mathbb{R}^n \) for any fixed \( \mu_i \in \mathbb{R}^{p+} \), and for any \( x \in \mathbb{R}^n \)

\[
\lim_{y \to x, \mu_i \downarrow 0} \tilde{f}_i(y, \mu_i) = f_i(x)
\]

Now we can construct a smoothing method by using \( \tilde{F} \) and \( \nabla \tilde{F} \) as follows. The first step is to define a parametric smooth function \( \tilde{F}(x, \mu_k) \) to approximate \( F(x) \). The second step we find for a fixed \( \mu_k \in \mathbb{R}^{p+} \) an approximate solution of the smooth MOP

\[
P_{(\mu_k)} : \begin{cases} 
\min \tilde{F}(x, \mu_k), \\
\text{subj to } x \in S.
\end{cases}
\]

The last step, by updating \( \mu_k \), which guarantees the convergence of any accumulation point of a designated subsequence of the iteration sequence generated by the smoothing MOP algorithm is a AKKT point. So the Pareto optimal solutions (stationary points) of the approximate subproblems \( P_{(\mu_k)} \) converge to a Pareto optimal solution (stationary point) of the initial MOP \( P_{(1)} \). Note that the advantage of the smoothing method is to solve optimization problems with continuously differentiable functions which has a rich theory and powerful methods [7].

Many nonsmooth optimization problems can be reformulated by using the plus function \((h)^+\) for example \( \max(h, g) = h + (g - h)^+ \), \( \min(h, g) = h - (h - g)^+ \) and \( |h| = (h)^+ + (-h)^+ \). So that, in this paper, we present a class of smooth approximation for the plus function by convolution given by Chen [8].

Definition 3.2. [8] Let \( \rho : \mathbb{R} \to \mathbb{R}^+ \) be a piecewise continuous density function satisfying

\[
\rho(s) = \rho(-s) \quad \text{and} \quad \kappa := \int_{\mathbb{R}} |s| \rho(s) \, ds < \infty
\]

then the function \( \phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \) defined by

\[
\phi(h, \mu) := \int_{\mathbb{R}} (t - \mu s)^+ \rho(s) \, ds
\]

is a smoothing function of \((h)^+\).

Proposition 3.3. [8]
For any fixed $\mu > 0$, $\phi(\cdot, \mu)$ is continuously differentiable convex, strictly increasing, and satisfies
\[ 0 < \phi(h, \mu) - (h)_+ \leq \kappa \mu \] (2)
then for any $h \in \mathbb{R}$
\[ \lim_{h_k \rightarrow h \ \mu_k \downarrow 0} \phi(h_k, \mu_k) = (h)_+ \] (3)

Proposition 3.4. [8] Let $\partial(h)_+$ the Clarke subdifferential of $(h)_+$ and $G\phi(h)$ is the subdifferential associated with the smoothing function $\phi$ at $h$ given by
\[ G\phi(h) = \text{conv} \{ \tau / \nabla_t \phi(h_k, \mu_k) \rightarrow \tau , \ h_k \rightarrow h, \ \mu_k \downarrow 0 \} \] (4)
then
\[ G\phi(h) = \partial(h)_+ \]

Remark 3.5. The plus function $(h)_+$ is convex and globally Lipschitz continuous. Any smoothing function $\phi(h, \mu)$ of $(h)_+$ is also convex and globally Lipschitz. In addition, for any fixed $h$, the function $\phi$ is continuously differentiable, monotonically increasing and convex with respect $\mu > 0$ and satisfies
\[ 0 \leq \phi(t, \mu_2) - \phi(t, \mu_1) \leq \kappa (\mu_2 - \mu_1) \quad \text{for} \quad \mu_2 > \mu_1 \]

Now, we study properties of the smoothing function $\phi$. We assume that $F : \mathbb{R}^n \rightarrow \mathbb{R}^p$ given by $F(x) = (f_1(x), \ldots, f_p(x))$ is locally Lipschitz continuous. According to Rademacher’s theorem, $F$ is differentiable almost everywhere. For each $i = 1, \ldots, p$ the Clarke subdifferential of $f_i$ at a point $x$ is defined by
\[ \partial f_i(x) = \text{conv} \{ v / \nabla f_i(z) \rightarrow v, \ f_i \text{ is differentiable at } z, \ z \rightarrow x \} \]

For a locally Lipschitz function $f_i$, the gradient consistency
\[ \partial f_i(x) = \text{conv} \{ \lim_{x_k \rightarrow x} \nabla f_i(x_k, \mu_k^i) \} = Gf_i(x) \quad \forall x \in \mathbb{R}^n \]

between the Clarke subdifferential and subdifferential associated with the smoothing function of $f_i$ for each $i = 1, \ldots, p$. Note that the above result is important for the convergence of smoothing methods.

Throughout the rest of this paper we assume that the function $F$ is given by $F(x) = H((\varphi(x))_+)$ where $H(x)$ and $\varphi(x)$ are continuously differentiable, $H(x) = (h_1(x), \ldots, h_p(x))$ with components $h_i : \mathbb{R}^n \rightarrow \mathbb{R}, \ i = \{1, \ldots, p\}$ and $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ with $\varphi_j : \mathbb{R}^n \rightarrow \mathbb{R}, \ j = 1, \ldots, n$. Notice that
\((\varphi(x))_+ = ((\varphi_1(x))_+, \cdots, (\varphi_n(x))_+)\) and its smoothing function is 
\(\phi(\varphi(x), \mu) = (\phi(\varphi_1(x), \mu), \cdots, \phi(\varphi_n(x), \mu))^T\).

Now we show the gradient consistency of the smoothing composite functions using \(\phi\) in definition 3.2 for the plus function.

**Theorem 3.6.** Let \(F(x) = H((\varphi(x))_+), \) where \(\varphi : \mathbb{R}^n \to \mathbb{R}^n\) and \(H : \mathbb{R}^n \to \mathbb{R}^p\) are continuously differentiable, then for each \(i = 1, \cdots, p\), \(\tilde{f}_i(x, \mu_k) = h_i(\phi(\varphi(x)), \mu_k)\) is a smoothing function of \(f_i\) with the following properties.

(i) For any \(x \in \mathbb{R}^n\), \(\{\lim_{x_k \to x} \nabla \tilde{f}_i(x, \mu_k)\}\) is nonempty and bounded, and \(\partial f_i(x) = G \tilde{f}_i(x)\), for each \(i = \{1, \cdots, p\}\).

(ii) If \(H, \varphi_j\) are convex for each \(j \in \{1, \cdots, n\}\) and \(\varphi_j\) is monotonically nondecreasing, then for any fixed \(\mu_k \in \mathbb{R}^p_+\), \(\tilde{f}_i(\cdot, \mu_k)\) is convex.

**Proof 3.7.** For any fixed \(i \in \{1, \cdots, p\}\), we can derive this theorem by theorem 1 [8].

**Proposition 3.8.** Let \(\vartheta(t) = |t|, \vartheta_\mu(t) = \sin(\mu).\ln(\cosh(\frac{t}{\sin(\mu)}))\), \(0 < \mu < \frac{\pi}{2}\). Then

(i) \(0 \leq \vartheta(t) - \vartheta_\mu(t) \leq \sin(\mu) \ln(2)\).

(ii) \(\left| \frac{d\vartheta_\mu(t)}{dt} \right| < 1\), and \(\frac{d\vartheta_\mu(t)}{dt} \big|_{t=0} = 0\).

(iii) \(\vartheta_\mu(t)\) is convex.

**Proof.** (i) Let

\[
\vartheta_\mu(t) - \vartheta(t) = \sin(\mu).\ln(\cosh(\frac{t}{\sin(\mu)})) - |t|
= \sin(\mu).\ln(\frac{1}{2} \exp\left(\frac{-t}{\sin(\mu)}\right) + \frac{1}{2} \exp\left(\frac{t}{\sin(\mu)}\right)) - |t|
= \sin(\mu).\ln(\exp\left(\frac{-t - |t|}{\sin(\mu)}\right) + \exp\left(\frac{-t - |t|}{\sin(\mu)}\right)) + \sin(\mu).\ln(\frac{1}{2})
\]

Since \(1 < \exp\left(\frac{-t - |t|}{\sin(\mu)}\right) + \exp\left(\frac{-t - |t|}{\sin(\mu)}\right) \leq 2\). Thus,

\[
\sin(\mu).\ln(\frac{1}{2}) < \sin(\mu).\ln(\exp\left(\frac{-t - |t|}{\sin(\mu)}\right) + \exp\left(\frac{-t - |t|}{\sin(\mu)}\right)) + \sin(\mu).\ln(\frac{1}{2}) \leq \sin(\mu).\ln(2) + \sin(\mu).\ln(\frac{1}{2})
\]
then
\[ \sin(\mu) \ln\left(\frac{1}{2}\right) < \vartheta_\mu(t) - \vartheta(t) \leq 0. \]

Hence, we obtain
\[ 0 \leq \vartheta(t) - \vartheta_\mu(t) \leq \sin(\mu) \ln(2). \]

Therefore, \( \vartheta_\mu \) is a smoothing approximation function \( \vartheta \).

(ii) We have
\[
\frac{d\vartheta_\mu(t)}{dt} = \frac{\exp\left(\frac{t}{\sin \mu}\right) - \exp\left(-\frac{t}{\sin \mu}\right)}{\exp\left(\frac{t}{\sin \mu}\right) + \exp\left(-\frac{t}{\sin \mu}\right)} = \frac{\exp\left(\frac{2t}{\sin \mu}\right) - 1}{\exp\left(\frac{2t}{\sin \mu}\right) + 1}
\]

Since
\[ \left| \frac{\exp\left(\frac{2t}{\sin \mu}\right) - 1}{\exp\left(\frac{2t}{\sin \mu}\right) + 1} \right| < 1 \]

Then
\[ \left| \frac{d\vartheta_\mu(t)}{dt} \right| < 1 \]

(iii)
\[
\vartheta_\mu''(t) = \frac{\frac{d}{dt} \left[ \frac{d\vartheta_\mu(t)}{dt} \right]}{4 \frac{1}{\sin \mu} \exp\left(\frac{2t}{\sin \mu}\right)} \cdot \exp\left(\frac{2t}{\sin \mu}\right) + 1
\]

\[ \exp\left(\frac{-g_j(x)}{\sin \mu_j}\right) \right) \text{ for } j = \{1, \ldots, p\} \text{ and } 0 < \mu_j < \frac{\pi}{2}. \]

Thus \( \vartheta_\mu \) is convex.

\[ \square \]

**Proposition 3.9.** Let \( \vartheta(t) = |t| \), and a vector function \( g(x) = (g_1(x), \ldots, g_p(x))^T \) with components \( g_j : \mathbb{R}^n \to \mathbb{R} \), we denote \( \vartheta(g(x)) = |g(x)| = (|g_1(x)|, \ldots, |g_p(x)|) \) and \( \vartheta_\mu(g(x)) = (\vartheta_{\mu_1}(g_1(x)), \ldots, \vartheta_{\mu_p}(g_p(x)))^T \), with \( \vartheta_{\mu_j}(g_j(x)) = \sin(\mu_j) \ln\left(\frac{1}{2}\exp\left(\frac{g_j(x)}{\sin \mu_j}\right) + \frac{1}{2} \exp\left(-\frac{g_j(x)}{\sin \mu_j}\right) \right) \) for \( j = \{1, \ldots, p\} \) and \( 0 < \mu_j < \frac{\pi}{2} \). Then \( \vartheta_\mu(g(x)) \) is a smoothing approximation function of \( \vartheta(g(x)) \).

**Proof.** For any fixed \( j \in \{1, \ldots, p\} \), we can derive by proposition 3.9 that
\[ 0 < \vartheta(g_j(x)) - \vartheta_{\mu_j}(g_j(x)) \leq \sin(\mu_j) \ln(2). \]

Considering \( \kappa = \ln(2) \) and \( 0 \leq \sin(\mu_j) \leq \mu_j \forall \mu_j \in [0, \frac{\pi}{2}] \).

then,
\[ 0 < \vartheta(g_j(x)) - \vartheta_{\mu_j}(g_j(x)) \leq \kappa \mu \text{ for } j \in \{1, \ldots, p\}. \]
Therefore, $\vartheta_{\mu}(g(x))$ is a smoothing approximation function $\vartheta(g(x))$.

4  Smoothing Multiobjective Optimization Problem

In this section, we introduce AKKT condition for the multiobjective problem $P(\mu)$ inspired by Giorrg. G et al. [9]. Then we exploit it to prove convergence analysis of the smoothing method, whose feasible set is defined by inequality constraints. In fact, the solution of problem $P(1)$ is accomplished by solving a sequence of problems $P(\mu)$, where the value of $\mu$ is updated according to $\mu_{k+1} = \alpha \mu_k$ with $\alpha \in (0, 1)$ is the decreasing factor of $\mu$. We point out that Chen [8] is concerned with convergence analysis of the smoothing method (in the scalar case) by using a smoothing gradient method.

Definition 4.1. We say that the AKKT condition is satisfied for problem $P(\mu)$ at a feasible point $x^* \in S$ if there exists a sequence $(x_k) \subset \mathbb{R}^n$ and $(\lambda_k, \beta_k) \subset \mathbb{R}^p \times \mathbb{R}^m$ such that

$(C_0)$ $x_k \to x^*$

$(C_1)$ $\sum_{i=1}^p \lambda_k^i \nabla f_i(x_k, \mu_k^i) + \sum_{j=1}^m \beta_k^j \nabla g_j(x_k) \to 0$

$(C_2)$ $\sum_{i=1}^p \lambda_k^i = 1$

$(C_3)$ $g_j(x_k) < 0 \Rightarrow \beta_k^j = 0 \text{ for } j = \{1, \cdots, n\}$.

Remark 4.2. (i) A point satisfying the AKKT is called AKKT point.

(ii) The sequence of points $(x_k)$ is not required to be feasible.

(iii) Assuming $\beta_k \in \mathbb{R}^m_+$, condition $(C_3)$ is equivalent to

$\beta_k^j g_j(x_k) \leq 0 \text{ for sufficiently large } k, \forall j \notin J(x^*)$.

Each of these condition implies the condition

$\beta_k^j g(x_k) \to 0 \forall j \notin J(x^*)$

The following theorem establish necessary optimality conditions for problem $P(1)$.

Theorem 4.3. If $x^* \in S$ is a locally weakly efficient solution for Problem $P(1)$, then $x^*$ satisfies the AKKT condition with sequences $(x_k)$ and $(\lambda_k, \beta_k)$. In addition, for these sequence we have that $\beta_k = b_k(g(x_k))_+$ where $b_k > 0 \forall k$. 

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Proof 4.4. Its proof is based on Theorem 3.1 in [9], the gradient consistency Theorem 3.6 and Proposition 3.3.

In order to establish the sufficient condition we assume convexity assumption and the following condition.

**Assumption A:** We call a sum converging to zero, if
\[ \sum_{j=1}^{m} \beta_j^k g(x_k) \rightarrow 0. \]

**Theorem 4.5.** Assume that \( H, \varphi_j \) for \( j = \{1, \cdots, n\} \) and \( g_i \) for \( i = \{1, \cdots, m\} \) are convex and \( \varphi_j \) is monotonically nondecreasing. If \( x^* \in S \) satisfies the AKKT condition and assumption A is fulfilled then \( x^* \) is a global weak efficient solution of problem \( P_1 \).

**Proof 4.6.** By Theorem 3.2 in [9] and gradient consistency Theorem 3.6, we derive this Theorem.
5 Bi-Objective Nonsmooth Environmental and Economic Dispatch Problem

The Bi-objective Economic and Environmental Dispatch Problem (EEDP) is concerned with the minimization of generation costs and the emission of pollutants while representing systems operational constraints. Note that the two objectives are conflicting in nature and they both have to be considered simultaneously to find overall optimal dispatch. The EEDP is a multi-objective, nonlinear, and nonsmooth problem.

5.1 Notation

- \( C(P) \) function cost for all thermal units [\$/\text{unit}];
- \( E(P) \) total pollutant emission for all thermal units given in [kg/h];
- \( P \) vector of active power outputs for all thermal units [MW];
- \( P_i \) active power output of generating unit \( i \) [MW];
- \( E_i \) active emission output of generating unit \( i \) [MW];
- \( P_{i_{\text{max}}} \) maximum power output of generating unit \( i \) [MW];
- \( P_{i_{\text{min}}} \) minimum power output of generating unit \( i \) [MW];
- \( E_{i_{\text{max}}}(P) \) maximum pollutant emission given in [kg/h];
- \( E_{i_{\text{min}}}(P) \) minimum pollutant emission given in [kg/h].
5.2 Cost function

The growing costs of fuels and operations of power generating units require a development of optimization methods for Economic Dispatch (ED) problems. Standard optimization techniques such as direct search and gradient methods often fail to find global optimum solutions. The realistic operation of the ED problem considers the couple valve-point effects and multiple fuel options. The cost model integrates the valve-point loadings and the fuel changes in one frame. So, the nonsmooth cost function is given as [14]:

\[ C(P) = \sum_{i=1}^{n} C_i(P_i) = \sum_{i=1}^{n} a_i P_i^2 + b_i P_i + c_i + |g_i sin(h_i(P_i^{min} - P_i))| \]

such that: \( P_i^{min} \leq P_i \leq P_i^{max} \)

where \( g_i, h_i, a_i, b_i \) and \( c_i \) are the cost coefficients of generator \( i \).

5.3 Emission function

The emission function can be formulated as the sum of all types of emission considered, with convenient pricing or weighting on each emitted pollutant. In this paper, only one type of emission NOx is taken into account without loss of generality [13]. The volume of NOx emission is given as a function of generator output. That is, the sum of a quadratic and exponential function. The total amount of emission such as SO2 or NOx depends on the amount of power generated by unit. The NOx emission amount which is, the sum of a quadratic and exponential function can be realistically written as :

\[ E(P) = \sum_{i=1}^{n} E_i(P_i) = \sum_{i=1}^{n} \alpha_i P_i^2 + \beta_i P_i + \gamma_i \]

where, \( \alpha_i, \beta_i, \gamma_i, \eta_i \) and \( \delta_i \) are the coefficients of the \( i \)th generator emission characteristics.

Formulation of the Non-Smooth EEDP:

\[ P_{\text{NEEDP}} : \begin{cases} 
\min \{ C(P), E(P) \} \\
\text{subet to : } \sum_{i=1}^{n} P_i = P_d \\
P_i^{min} \leq P_i \leq P_i^{max}
\end{cases} \]

5.4 Application:

In order to solve EEDP with two generators we consider the following problem :
\[ \begin{align*}
P_{\text{NEEDP}} : & \quad \begin{cases} 
\min \{ C(P), E(P) \} \\
\text{subject to: } P_1 + P_2 = 650 \\
100 \leq P_1 \leq 600 \text{ and } 100 \leq P_2 \leq 400.
\end{cases}
\end{align*} \]

where
\[ \begin{align*}
C_1(P_1) &= 0.001562P_1^2 + 7.92P_1 + 561 + |300 \sin(0.0315(P_{\min} - P_1))|; \\
C_2(P_2) &= 0.00194P_2^2 + 7.85P_2 + 310 + |200 \sin(0.042(P_{\min} - P_2))|; \\
C(P) &= C_1(P_1) + C_2(P_2); \\
E_1(P_1) &= 0.0126P_1^2 + 1.355P_1 + 22.983; \\
E_2(P_2) &= 0.00765P_2^2 + 0.805P_2 + 363.70; \\
E(P) &= E_1(P_1) + E_2(P_2).
\end{align*} \]

**Step 1:** We apply the smoothing method to the nonsmooth objective function \( C(P) \) (see Propositions 3.9 and 3.8) to obtain a smooth objective function \( \tilde{C}(P, \mu) = \{ \tilde{C}_1(P_1, \mu), \tilde{C}_2(P_2, \mu) \} \) as follows:

\[ \begin{align*}
\tilde{C}_1(P_1, \mu) &= 0.001562P_1^2 + 7.92P_1 + 561 + \sin(\mu) \left[ \ln \left( \frac{1}{2} \exp \left( \frac{200 \sin(0.042(P_{\min} - P_2))}{\sin(\mu)} \right) \right) \right] \\
&\quad + \frac{1}{2} \exp \left( \frac{-200 \sin(0.042(P_{\min} - P_2))}{\sin(\mu)} \right) \\
&= \text{(5)}
\end{align*} \]

\[ \begin{align*}
\tilde{C}_2(P_2, \mu) &= 0.00194P_2^2 + 7.85P_2 + 310 + \sin(\mu) \left[ \ln \left( \frac{1}{2} \exp \left( \frac{300 \sin(0.0315(P_{\min} - P_1))}{\sin(\mu)} \right) \right) \right] \\
&\quad + \frac{1}{2} \exp \left( \frac{-300 \sin(0.0315(P_{\min} - P_1))}{\sin(\mu)} \right) \\
&= \text{(6)}
\end{align*} \]

**Step 2:** Each of \( P_{\text{NEEDP}}^\mu \) subproblems has the form

\[ \begin{align*}
P_{\text{NEEDP}}^\mu : & \quad \begin{cases} 
\min \{ \tilde{C}(P), E(P) \} \\
\text{subject to: } P_1 + P_2 = 650 \\
100 \leq P_1 \leq 600 \text{ and } 100 \leq P_2 \leq 400.
\end{cases}
\end{align*} \]
Step 3: The bi-objective subproblem $P_{\text{NEEDP}}^\mu$ is transformed into a set of single-objective subproblems using the $\epsilon$-constraint method. For both methods, the objective function of the subproblems are smoothed by the smoothing method and the subproblems are solved by the interior point barrier method [15].

$$
P_{\text{NEEDP}}^\mu, \epsilon_l : \begin{cases} 
\min \tilde{C}(P, \mu), \\
\text{subject to } P_1 + P_2 = 650, \\
E(P) < \epsilon_l; \\
100 \leq P_1 \leq 600, \quad 100 \leq P_2 \leq 400.
\end{cases}
$$

Step 4: To create constraint bound vector, consider the number of Pareto points $n = 70$; let $\tau = \frac{E_{\text{max}} - E_{\text{min}}}{n}$ and $\epsilon_{l+1} = \epsilon_l + \tau$, $l = \{1, \cdots, n - 1\}$ with $\epsilon_1 = E_{\text{min}}$, and solve each smoothed single-objective subproblem $P_{\text{NEEDP}}^\mu, \epsilon_l$ by the interior point barrier method.

Figure 2: Pareto Front using Smoothing-$\epsilon$-constraint - Interior point method for $\mu = 0.000001$.

Figure 3: Pareto Front using Smoothing-$\epsilon$-constraint - Interior point method for $\mu = 0.0001$.

In order to verify our approach performance, some simulations were performed and the results were compared with the PBC-HS-MLBIC method developed by Gonçalves et al. [16]. This work was chosen because we use the same parameters input. in the following table, $\tilde{C}(P)$ and $\tilde{E}(P)$ represent respectively, the total cost and total pollutant emission obtained by PBC-HS-MLBIC method [16] for two generators.

From the results presented in table 1, some remarks are rised. The first is the quality of our methodology, as our output results are significantly better compared to that of Gonçalves.
Table 1: Minimum fuel cost, minimum emission and comparison to output results of PBC-HS-MLBIC method.

| n  | $P_1$(MW) | $P_2$(MW) | $C(P)$ ($) | $E(P)$(kg/h) | $\hat{C}(P)$ ($) | $\hat{E}(P)$(kg/h) |
|----|-----------|-----------|-----------|-------------|-----------------|-----------------|
| 1  | 259,0835  | 390,7781  | 6390,6884 | 1735,0267   | 6698,45         | 1737,14         |
| 2  | 274,8133  | 375,1867  | 6382,8279 | 1740,72     | 6724,26         | 1747,20         |
| 3  | 282,1196  | 367,8804  | 6379,16   | 1746,44     | 6681,07         | 1752,92         |
|   |           |           |           |             |                 |                 |
|   |           |           |           |             |                 |                 |
|   |           |           |           |             |                 |                 |
| 35| 350,0203  | 299,9797  | 6362,9941 | 1903,0076   | 6848,11         | 1936,07         |
| 36| 350,0236  | 299,9764  | 6362,9941 | 1903,0198   | 6845,91         | 1941,79         |
| 37| 350,0243  | 299,9757  | 6362,9941 | 1903,0223   | 6842,53         | 1947,51         |
|   |           |           |           |             |                 |                 |
|   |           |           |           |             |                 |                 |
|   |           |           |           |             |                 |                 |
| 68| 350,0243  | 299,9757  | 6362,9941 | 1903,0225   | 6407,48         | 2124,93         |
| 69| 350,0243  | 299,9757  | 6362,9941 | 1903,0223   | 6389,74         | 2130,66         |
| 70| 350,0242  | 299,9758  | 6362,9941 | 1903,0222   | 6383,05         | 2135,34         |
Figure 4: Pareto Front using Smoothing-$\epsilon$-constraint - Interior point method for $\mu = 0.01$

Figure 5: Pareto Front using Smoothing-$\epsilon$-constraint - Interior point method for $\mu = 0.1$.

Figure 6: Pareto Front using Smoothing-$\epsilon$-constraint - Interior point method for $\mu = 0.5$.

Figure 7: Pareto Front using Smoothing-$\epsilon$-constraint - Interior point method for $\mu = 1$.

Conclusion:
In this paper, a class of nonsmooth multiobjective optimization problems that include $\min$, $\max$, absolute value functions or composition of the plus function $(t)^+$ with smooth functions is introduced, and some smoothing methods are presented. The algorithm is based on smoothing techniques to approximate the objective functions in all points where the function is nonsmooth. Numerical results show that the smoothing methods are promising for the nonsmooth MOP.
References

[1] Mistakidis E. S., Stavroulakis GE. Nonconvex optimization in mechanics. Smooth and nonsmooth algorithms, heuristics and engineering applications by the F.E.M. Dordrecht: Kluwer Academic Publisher; 1998.

[2] Outrata J., Kôcvara M. and Zowe J., Nonsmooth approach to optimization problems with equilibrium constraints. Theory, applications and numerical results. Dordrecht: Kluwer Academic Publishers; 1998.

[3] Moreau J. J., Panagiotopoulos P. D., Strang G.(editors), Topics in non-smooth mechanics. Basel: Birkhäuser; 1988

[4] Miettinen K., Nonlinear multiobjective optimization. Boston: Kluwer; 1999.

[5] Clarke F. H., Optimization and Nonsmooth Analysis. Wiley, New York; 1983.

[6] Rockafellar R. T., Wets R.J-B., Variational Analysis. Springer, New York; 1998.

[7] Nocedal J., Wright S. J., Numerical Optimization, 2nd edn. Springer, New York; 2006.

[8] Chen X., Smoothing methods for nonsmooth, nonconvex minimization. Math Program. 2012;134(1), pp. 71 -99.

[9] Giorgi G., Jiménez B., Novo V., Approximate Karush-Kuhn-Tucker condition in multiobjective optimization. J Optim Theory Appl. 2016; 171, pp.70-89

[10] Birgin E. G., Martínez J. M., Practical augmented Lagrangian methods for constrained optimization. Fundam Algorithms. 2014; vol 10. SIAM, Philadelphia.

[11] Qi L,Wei Z On the constant positive linear dependence condition and its application to SQP methods. SIAM J Optim 2000; 10:963-981.

[12] Chen L., Goldfarb D., Interior-point 2-penalty methods for nonlinear programming with strong global convergence properties. Math Program; 2006; 108,pp. 1-36

[13] Cheng Farag A. , Al-Baiyat S., T.C., "Economic load dispatch multiobjective optimization procedures using linear programming techniques", IEEE Transactions on Power Systems, 1995; vol. 10, no.2, pp: 731-738.
[14] Zhan J., Wu Q. H., Guo C., Zhou X., Economic dispatch with non-smooth objectives-part ii: dimensional steepest decline method. IEEE Trans Power Syst 2015;30(2), pp. 722-33.

[15] M. Schmidt, An Interior-Point Method for Nonlinear Optimization Problems with Locatable and Separable Nonsmoothness. EURO J. Comput. Optim. 2015; 3, 309-348.

[16] Gonçalves E, Balbo A, da D, Nepomuceno E, Baptista C, Soler E. M. Deterministic approach for solving multi-objective non-smooth environmental and economic dispatch problem. International Journal of Electrical Power and Energy Systems, vol. 2019;104, pp. 880-897.