Canonical transformations of the time for the Toda lattice and the Holt system.

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For the Toda lattice and the Holt system we consider properties of canonical transformations of the extended phase space, which preserve integrability. The separated variables are invariant under change of the time. On the other hand, mapping of the time induces transformations of the action-angles variables and a shift of the generating function of the Bäcklund transformation.

1 Introduction.

Let \( M \) be a \( 2n \)-dimensional symplectic manifold (phase space) with coordinates \( \{ p_j, q_j \}_{j=1}^n \). The Hamilton function \( H(p, q) \) defines the hamiltonian dynamical system on \( M \). Here \( p \) and \( q \) denote \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_n \), respectively.

By adding to \( M \) the time \( q_{n+1} = t \) and the Hamiltonian \( p_{n+1} = -H \) one gets \( 2n + 2 \)-dimensional extended phase space \( M_E \) of the given hamiltonian system \[9\]. Canonical functional \( S \) on \( M_E \) has the following completely symmetric form

\[
S = \int_{\tau_1}^{\tau_2} \sum_{i=1}^{n+1} p_i q'_i d\tau.
\]

On the extended phase space \( M_E \) the Jacobi, Euler-Lagrange and Hamilton variational principles \( \delta S = 0 \) are differed by an additional constraint

\[
\mathcal{H}(p_1, \ldots, p_{n+1}; q_1, \ldots, q_{n+1}) = 0. \tag{1.1}
\]

Here \( \mathcal{H} \) is called generalised Hamilton function \[9\], which determines the evolution.

By definition the Hamilton function \( H(p, q) \) is a function on \( M \). At the same time the Hamiltonian \( H \) and the time \( t \) are variables in \( M_E \), which are independent on the other variables \( (p, q) \). Of course, equation of the zero-valued energy surface \( \mathcal{H} = 0 \) may be rewritten as \( H(p, q) = H \) \[9\]. Thus, unless other wise indicated, \( H(p, q) \) denotes a function on \( M \), and \( H \) denotes independent variable in \( M_E \).

By definition canonical transformations of the extended phase space \( M_E \) preserve the Hamilton-Jacobi equation and differential form

\[
\alpha = \sum_{j=1}^{n} p_j dq_j - H dt.
\]

So, any canonical transformation of the time looks like

\[
d\bar{t} = v^{-1}(p, q) dt, \quad \bar{H} = v(p, q) H,
\]

where we used implicit transformation of the time \( t \) as in the general relativity.

Any canonical transformation of the initial phase space \( M \) maps any integrable system into the other integrable system. However, we have not a regular way to obtain canonical transformation of the extended phase space \( M_E \), which maps a given integrable system into the other integrable system. Nevertheless, we can try to construct such transformations by using different approaches developed for the integrable system.
In [16, 17] some canonical transformations of the time have been constructed for the Toda lattices and for the Stäckel systems by using transformations of the Lax matrices.

The aim of this letter is to study some properties of the such canonical transformations of the extended phase space. For the Toda lattice and the Henon-Heiles system we shall prove that separated variables are invariant by the change of the time. On the other hand, mapping of the time induces transformations of the action-angles variables and a shift of the generating function of the Bäcklund transformation.

2 The Toda lattice.

The periodical Toda lattice is described by the following Hamilton function

\[ H(p, q) = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + a_i e^{q_i - q_{i+1}}. \]  

(2.1)

Here \( \{p_i, q_i\} \) are canonical variables and the periodicity conventions \( q_{i+n} = q_i \) and \( p_{i+n} = p_i \) are always assumed for the indices of \( q_i \) and \( p_i \).

The \( n \times n \) Lax matrices [8, 3] for the Toda lattice are

\[ \mathcal{L}^{(n)}(\mu) = \sum_{i=1}^{n} p_i E_{i,i} + \sum_{i=1}^{n-1} (e^{q_i - q_{i+1}} E_{i+1,i} + E_{i,i+1}) + \mu e^{q_n - q_1} E_{1,n}, \]  

(2.2)

\[ \mathcal{A}^{(n)}(\mu, q) = \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} E_{i+1,i} + \mu e^{q_n - q_1} E_{1,n} \]

where \( E_{i,k} \) stands for the \( n \times n \) matrix with unity on the intersection of the \( i \)th row and the \( k \)th column as the only nonzero entry.

By abuse of notation we shall omit the superscript \( n \) of the Lax matrices (2.2) when it is not important. The exact solution of the equations of motion is due to existence of the Lax representation [8, 3]

\[ \{ H(p, q), \mathcal{L} \} = [\mathcal{L}, \mathcal{A}]. \]

Another \( 2 \times 2 \) Lax representation [1, 14] for the same Toda lattice is equal to

\[ T^{(1 \ldots n)}(\lambda) = L_1(\lambda) \cdots L_n(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \]  

(2.3)

where

\[ L_j = \begin{pmatrix} \lambda + p_j & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}, \quad A_j = \begin{pmatrix} \lambda & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}, \]  

(2.4)

such that

\[ \{ H(p, q), L_j \} = L_j A_j - A_{j-1} L_j, \quad \{ H(p, q), T^{(1 \ldots n)} \} = [T^{(1 \ldots n)}, A_n], \]

Sometimes, we shall omit the superscripts \( 1 \ldots n \) of the monodromy matrix \( T(\lambda) \) (2.3), too.

According to [17], canonical transformations of the extended phase space \( \mathcal{M}_E \)

\[ dt_j = e^{\theta_j - q_{j+1}} dt, \quad \tilde{H}_j = e^{\theta_{j+1} - q_j} (H + b), \quad b \in \mathbb{R} \]  

(2.5)

preserve integrability for any index \( 1 \leq j \leq n \).

Associated with the different indexes \( j \) canonical mappings (2.3) are related with each other by canonical transformations of the other variables \( (p, q) \). Hence, we shall omit subscript \( j \) for the new time \( \tilde{t} \) and the new Hamiltonian \( \tilde{H} \).

Any such change of the time gives rise to the following transformation of the Lax matrices

\[ \tilde{\mathcal{L}} = \mathcal{L} - \tilde{H} E_{j,j+1}, \quad \tilde{\mathcal{A}}(\mu, q) = v^{-1}(q) \mathcal{A}(\mu, q). \]  

(2.6)
and
\[ \overline{T}(1...n) = T(1...n) + T(1...j-1) \begin{pmatrix} H + b & 0 \\ 0 & 0 \end{pmatrix} T(j+2...n) \]
\[ = L_1 \cdots L_{j-1} \cdot \left[ L_j L_{j+1} + \begin{pmatrix} H + b & 0 \\ 0 & 0 \end{pmatrix} \right] \cdot L_{j+2} \cdots L_n, \]
\[ \widetilde{A}_n(\lambda, q) = v^{-1}(q) A_n(\lambda, q). \] (2.7)

Change of the time (2.5) maps the Toda lattice into the other integrable system. Coefficients of the polynomials
\[ P(\lambda) = \text{tr} \ T(\lambda), \quad \text{and} \quad \widetilde{P}(\lambda) = \text{tr} \ \overline{T}(\lambda) \]
are generating functions of the integrals of motion in the involution providing complete integrability of the systems.

The corresponding transformation of the spectral curves \( \det(\mathcal{L}(\mu) + \lambda I) = 0 \) or \( \det(T(\lambda) + \mu I) = 0 \) looks like
\[ C: \quad -\mu - \frac{1}{\mu} = \lambda^n + \lambda^{n-1} \rho + \lambda^{n-2} \left( \frac{p^2}{2} - H \right) + \ldots \]
\[ \overline{C}: \quad -\mu - \frac{1}{\mu} = \lambda^n + \lambda^{n-1} \rho + \lambda^{n-2} \left( \frac{p^2}{2} + b \right) + \ldots. \] (2.8)

Here \( p = \sum p_j \) is a total momentum, \( H \) and \( \overline{H} \) are the corresponding Hamilton functions.

The Poisson brackets relations for the \( n \times n \) Lax matrices can be expressed in the \( r \)-matrix form
\[ \{ \mathcal{L}(\mu), \mathcal{L}(\nu) \} = [ r_{12}(\mu, \nu), \mathcal{L}(\mu) ] + [ r_{21}(\mu, \nu) \overline{\mathcal{L}}(\nu) ]. \]

Here we used the standard notations
\[ \mathcal{L}(\mu) = \mathcal{L}(\mu) \otimes I, \quad \overline{\mathcal{L}}(\nu) = I \otimes \mathcal{L}(\nu), \]
\[ r_{21}(\mu, \nu) = -\Pi r_{12}(\nu, \mu) \Pi, \]
and \( \Pi \) is the permutation operator in \( \mathbb{C}^n \times \mathbb{C}^n \) [4]. Change of the time (2.3) maps the constant \( r \)-matrix for the Toda lattice
\[ r_{12}(\mu, \nu) = r_{12}^{\text{const}}(\mu, \nu) = \frac{1}{\mu - \nu} \left( \nu \sum_{m \geq i} + \mu \sum_{m < i} \right) E_{im} \otimes E_{mi} \]
into the following dynamical \( r \)-matrix
\[ r_{12}(\mu, \nu) = r_{12}^{\text{const}}(\mu, \nu) + r_{12}^{\text{dyn}}(\mu, \nu), \quad r_{12}^{\text{dyn}}(\mu, \nu) = \widetilde{A}(\nu, q) \otimes E_{j,j+1}, \]
where the second Lax matrix \( \widetilde{A}(\nu, q) \) and, therefore, dynamical \( r \)-matrix depend on coordinates only.

The \( 2 \times 2 \) monodromy matrix \( T(\lambda) \) (2.3) satisfies the following Sklyanin \( r \)-matrix relations
\[ \{ \mathcal{T}(\lambda), \mathcal{T}(\nu) \} = [ R(\lambda - \nu), \mathcal{T}(\lambda) \mathcal{T}(\nu) ], \quad R(\lambda - \nu) = \frac{\Pi}{\lambda - \nu}. \] (2.9)

Change of the time (2.5) transforms these quadratic relations into the following poly-linear ones
\[ \{ \mathcal{T}(\lambda), \mathcal{T}(\nu) \} = [ R(\lambda - \nu), \mathcal{T}(\lambda) \mathcal{T}(\nu) ] \]
\[ + [ r_{12}^{\text{dyn}}(\lambda, \nu), \mathcal{T}(\lambda) ] + [ r_{21}^{\text{dyn}}(\lambda, \nu), \mathcal{T}(\nu) ]. \]

The corresponding dynamical \( r \)-matrix is given by
\[ r_{12}^{\text{dyn}}(\lambda, \nu) = A_n(\lambda, q) \otimes \left( L_1 \cdots L_{j-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot L_{j+1} \otimes L_n \right). \]
The original symplectic form is written as

and by applying Arnold’s method \cite{1}, action variables have the form

From \( \det T^{(1\ldots n)} = T^{(1\ldots n)} + \left( \begin{array}{cc} H & 0 \\ 0 & 0 \end{array} \right) T^{(3\ldots n)} \), \( \widetilde{T}^{(1\ldots n)} \) is defined by (2.12). By using the standard form of the hyperelliptic curves \( C \) and \( \tilde{C} \) (2.8) and by applying Arnold’s method \cite{1}, action variables have the form

where \( \gamma \) is defined by (2.12). By using the standard form of the hyperelliptic curves \( C \) and \( \tilde{C} \) (2.8) and by applying Arnold’s method \cite{1}, action variables have the form

where \( \alpha_i \) and \( \tilde{\alpha}_i \) are \( \alpha \)-cycles of the Jacobi variety of the algebraic curves (2.8), respectively. In fact, polynomials \( P(\lambda) \), \( \tilde{P}(\lambda) \) and \( \alpha \)-cycles depend on the constants of motion, which are dropped in the notations. Thus, the Abel transformation linearizes equations of motion by using first kind abelian differentials on the corresponding spectral curves.
Finally let us consider the Bäcklund transformation $B_\nu$ for the Toda lattice $[11]$. As is well known $B_\nu$, transformation $B_\nu$ is canonical transformation $(p, q) \mapsto (P, Q)$ of the initial phase space $\mathcal{M}$ preserving all the integrals of motion (see $[15]$ for a more detailed list of properties of $T$ and $\mathcal{M}$).

For the Toda lattice canonical transformation $B_\nu$ can be described by the generating function

$$F_\nu(q | Q) = \sum_{i=1}^{n} \left( e^{q_i - Q_i} - e^{Q_i - q_{i+1} - \nu(q_i - Q_i)} \right),$$

(2.14)
such that

$$p_i = \frac{\partial F}{\partial q_i}, \quad P_i = -\frac{\partial F}{\partial Q_i}.$$  

(2.15)
The Bäcklund transformation $B_\nu$ has to preserve the spectral invariants of the Lax matrices $L(\mu)$ and $T(\lambda)$. As a consequence, to prove that $B_\nu$ preserves integrals of motion

$$I_k(p, q) = I_k(P, Q),$$

(2.16)
one verifies that $B_\nu$ preserves the spectrum of the Lax matrix $L(\mu)$

$$M(\mu, q, Q) L(\mu, p, q) = L(\mu, P, Q) M(\mu, q, Q),$$

(2.17)
where

$$M(\mu, q, Q) = \sum_{i=1}^{n-1} e^{Q_i - q_i} E_{i+1, i} + \mu e^{Q_n - q_i} E_{1, n}.$$  

(2.18)
(see $[10]$ for a detailed account of the theory of the Bäcklund transformation as gauge transformation).

Canonical transformation $[2.3]$ of the extended phase space $\mathcal{M}_E$ associated with arbitrary index $1 \leq j \leq n$ induces the following shift of the generating function

$$\tilde{F}_\nu(q | Q) = F_\nu(q, Q) + \Delta F.$$  

(2.19)
Note, here $\Delta F$ be independent variable of the extended phase space $\mathcal{M}_E$. It means that in $[2.15]$ all the partial derivatives $\partial_i$ with respect to any other coordinates of $\mathcal{M}_E$ are equal to zero. In this case the equality $(2.17)$ and the matrix $M(\mu, q, Q)$ $(2.18)$ are invariant with respect to the change of the time.

As above, the same Bäcklund transformations $B_\nu$ $(2.14)$ and $\tilde{B}_\nu$ $(2.19)$ are isospectral deformations of the corresponding $2 \times 2$ Lax matrices $T(\lambda)$ and $\tilde{T}(\lambda)$. For the Toda lattices the intertwining relations are equal to

$$M_i(\lambda, \nu) T_i(p, q) = T_i(P, Q) M_{i+1}(\lambda, \nu),$$

where

$$M_i(\lambda, \nu) = \begin{pmatrix} 1 & e^{Q_i - q_i} \\ -e^{-q_i} & \nu - \lambda - e^{Q_i - q_i} \end{pmatrix},$$

The same relations may be used after change of the time at $i \neq j, j + 1$. One additional non-factorized relation is given by

$$M_j(\lambda, \nu) = \begin{pmatrix} 1 & e^{Q_{j+1}} \\ -e^{-q_{j+1}} & \nu - \lambda - e^{Q_{j+1} - q_i} \end{pmatrix} = \begin{pmatrix} 1 & e^{Q_j} \\ -e^{-q_j} & \nu - \lambda - e^{Q_j - q_i} \end{pmatrix} = \begin{pmatrix} 1 & e^{Q_j} \\ -e^{-q_j} & \nu - \lambda - e^{Q_j - q_i} \end{pmatrix} M_{j+2}(\lambda, \nu).$$

The characteristic properties of the new Bäcklund transformation $\tilde{B}_\nu$ are verified following $[13]$. To prove the spectrality property we have to use one non-factorized relation as well.

Recall, the correspondence between the kernel of the corresponding quantum Baxter $Q$-operator and the function $F_\lambda(q | Q)$ is given by the semiclassical relation $[11]$. Note, that harmonic oscillator may be mapped into the Coulomb model by using canonical change of the time $[12]$. In quantum mechanics, such duality of the corresponding eigenvalue problems has been used by Fok, Schrödinger and many other. As an example, in the Birman-Schwinger formalism we can estimate spectrum of the one Hamiltonian $\tilde{H}$ by using known spectrum of the dual Hamiltonian $H$. So, it will be interesting to study such duality in framework of the quantum $Q$-operator theory.
3 The Henon-Heiles and Holt integrable systems.

The Holt system is defined by the Hamilton function
\[ \tilde{H}(p_x, p_y, x, y) = \frac{1}{2} (p_x^2 + p_y^2) + a x^{-2/3} \left( \frac{3b}{4} x^2 + y^2 + c \right). \] (3.1)

Only three integrable cases are known [13]
\[ (i) \, b = 1, \quad (ii) \, b = 6, \quad (iii) \, b = 16, \] (3.2)
while the remaining parameters \( a \) and \( c \) be an arbitrary constants. After canonical change of variables
\[ x = \frac{2}{3} x^{3/2}, \quad p_x = p_x \sqrt{x}, \quad y = -\frac{1}{2 \sqrt{3} a} p_y, \quad p_y = 2 \sqrt{3} a y. \]
and rescaling
\[ a \rightarrow 4 \left( \frac{3}{2} \right)^{1/3} a, \quad c \rightarrow \frac{c}{3a} \]
the Hamilton function (3.1) becomes
\[ \tilde{H}(p_x, p_y, x, y) = \frac{p_x^2 + p_y^2}{2x} + 2a \left( b x^2 + 3 y^2 \right) + \frac{2c}{x}. \] (3.3)

According to [13, 16], further canonical transformation of the extended phase space
\[ d\tilde{t} = x \, dt, \quad \tilde{H} \mapsto H = x \tilde{H}, \] (3.4)
preserves integrability and maps the Holt system into the Henon-Heiles system
\[ H(p_x, p_y, x, y) = \frac{p_x^2 + p_y^2}{2} + 2a x \left( b x^2 + 3 y^2 \right) + 2c. \] (3.5)

At \( b = 1 \) and at \( b = 16 \) the corresponding Lax matrices are \( 3 \times 3 \) matrices and the spectral curves are trigonal algebraic curves [5]. We shall consider these more complicated cases in the forthcoming publication.

At \( b = 6 \) the \( 2 \times 2 \) Lax matrices for the Henon-Heiles system is equal to [5, 16]
\[ \mathcal{L}(\lambda) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} = \begin{pmatrix} p_x^2/4 + p_y^2/4 & \lambda + x - y^2/4 \\ p_y^2/4 & -p_x^2/4 + p_y^2/4 \end{pmatrix} + 6a \left[ \lambda^2 - x \lambda + x^2 + y^2/4 \right] \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \] (3.6)
\[ \mathcal{A}(\lambda) = \begin{pmatrix} 0 & 1 \\ 6a (\lambda - 2x) & 0 \end{pmatrix}. \]

Change of the time (3.4) gives rise to the mapping of these Lax matrices
\[ \tilde{\mathcal{L}}(\lambda) = \mathcal{L}(\lambda) - \frac{1}{2} \tilde{H} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\mathcal{A}}(\lambda) = \frac{1}{x} \mathcal{A}(\lambda). \] (3.7)

and the following transformations of the corresponding hyperelliptic spectral curves
\[ \mu^2 = P(\lambda) = 6a \lambda^3 + \frac{1}{2} H - c + \frac{K}{\lambda}, \] (3.8)
\[ \mu^2 = \tilde{P}(\lambda) = 6a \lambda^3 - \frac{1}{2} \tilde{H} \lambda - \frac{c}{2} + \frac{\tilde{K}}{\lambda}. \]
The corresponding transformation of the $r$-matrix Poisson brackets has been considered in [16].

The separation variables $\{\lambda_1, \lambda_2\}$ for the both system are zeroes of the polynomial (see references in [16])

$$\mathbb{B}(\lambda) = \frac{(\lambda - \lambda_1) (\lambda - \lambda_2)}{\lambda} \quad (3.9)$$

and

$$\mu_i = \mathbb{A}(\lambda_i), \quad i = 1, 2.$$  

Notice that $\mu_i = \mu(\lambda_i)$ where $\mu(\lambda)$ is the eigenvalue of $L(\lambda)$.

According to (3.7), namely these entries of $T(\lambda)$ are invariant under change of the time (3.4).

These variables are the standard parabolic coordinates, which lie on the hyperelliptic curves (3.8), respectively. Applying Arnold’s method [1], action variables have the form

$$s_i = \oint_{\alpha_i} \sqrt{P(\lambda)} \, d\lambda, \quad \tilde{s}_i = \oint_{\tilde{\alpha}_i} \sqrt{\tilde{P}} \, d\lambda, \quad (3.10)$$

where $\alpha_i$ and $\tilde{\alpha}$ are $\alpha$-cycles of the Jacobi variety of the algebraic curves (3.8), respectively. Thus, the Abel transformation linearizes equations of motion by using first kind abelian differentials on the corresponding hyperelliptic spectral curves.

Change of the time (3.4) is related to ambiguity of the corresponding Abel map [16]. In fact, these integrable systems may be associated with the two different subsets of the differentials into the complete basis of first kind abelian differentials on the common hyperelliptic curve (3.8).

Now let us consider the known Bäcklund transformation $B_\nu$ for the Henon-Heiles system [18], which can be described by the generating function

$$F(x, y | X, Y) = \sqrt{6} a \nu y Y + \frac{2}{5} \sqrt{6} a (\nu - x - X) \left(2\lambda^2 - (x + X)\lambda + 2x^2 - xX + 2X^2 + \frac{5(y^2 + Y^2)}{4}\right). \quad (3.11)$$

The Bäcklund transformation $B_\nu$ preserves the spectrum of the Lax matrix $L(\lambda)$ (3.6) (for instance see [10, 7])

$$M(\lambda, \nu) L(\lambda, x, y) = L(\lambda, X, Y) M(\lambda, \nu), \quad (3.12)$$

where

$$M(\lambda, \nu) = \begin{pmatrix} -\sqrt{6} a (\nu - x - X) & 1 \\ \frac{6 a (\lambda - x - X)}{6 a (\nu - x - X)} & \sqrt{6} a (\nu - x - X) \end{pmatrix}. \quad (3.13)$$

Change of the time (3.4) gives rise to the shift of the generating function (3.11)

$$\tilde{F} = F - \sqrt{6} a (\nu - x - X) \tilde{H}$$

and factorization of the corresponding kernel of $Q$-operator in the semiclassical limit.

Here the Hamiltonian $\tilde{H}$ be independent variable of the extended phase space $M_E$. As for the Toda lattice, equality (3.12) and matrix $M(\mu, \nu)$ (3.13) are invariant under change of the time.

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