On the quantization of the Bateman system of a continuum

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Abstract
Within the scheme of quantization proposed in the author’s previous works, Bateman’s dual system is extended to a field theory of a continuum that can be used to model the dynamics of a viscoelastic medium. The model includes the wavevector as a controllable parameter. The level-crossing at the critical wavevectors is shown to be related to the system’s ‘geometrical phase’ that will brings about an observable effect in one dimension. The energy eigenstates of the quantized system entail condensation of bosons near critical wavevectors. The autocorrelation function vanishes at the critical damping limit. The dispersion relation is not strongly affected by the condensation. Plastic deformation is understood as a critical condensation of bosons.

1. Introduction
The Bateman system [1] is a model of damped motions of a particle under harmonic and dissipative forces. Since an action exists, the model can be subjected to canonical formulation. The Bateman system is also known as Bateman’s dual system (BDS), which features an additional auxiliary variable that undergoes amplifying motion. If the two variables x and y are naively interpreted as the coordinates of two particles, canonical quantization of BDS leads to unacceptable consequences that no stable ground state exists [2, 3] and that Heisenberg’s uncertainty relation is not satisfied [4, 5]. The cause of these pathologies can be traced back to the bilinear structure of the Lagrangian where saddle-point instability is already inherent.

The above obstacles in quantization are bypassed if the auxiliary variable is regarded not as a coordinate of a particle but as a Lagrange multiplier that has been introduced to derive, through invoking the variational principle, a correct dissipative equation of motion of a real particle. Interestingly, quantizing the system renders the role of the auxiliary variable clear: it exists to maintain correct quantum correlations among dynamical variables. The dynamical variables must be expressed as a linear combination of observable and auxiliary variables. This feature of the BDS was first pointed out for the massless case [6]. Then, the arguments were straightforwardly extended to the massive BDS [7] with a slight modification of the quantization procedure. It involves

(i) Finding two independent solutions to the equations of motion of the massive BDS.
(ii) Dividing the system of kinematical variables (x, y) into two subsystems (x₁, y₁) and (x₂, y₂) that correspond to two independent solutions respectively.
(iii) Imposing quantization conditions that are consistent with the two solutions on (x₁, y₁) and (x₂, y₂).
(iv) Forming the ‘z-representation’ of the coordinates by linearly combining (x₁) and y, i = 1, 2.
(v) Reconstructing the original variables by x = x₁ + x₂ and y = y₁ + y₂, which automatically fulfill the canonical quantization condition on x and y.

The Hamiltonian of the original system is given by adding the subHamiltonians for the subsystems together. The resultant symmetry is U(1) × U(1).
The next step of our study will be to explore whether the quantized BDS can model physical reality. In particular, it remains an intriguing question whether the theoretical discontinuity between the under- and overdamped regions of the model parameters has any physical correspondence to observable phenomena. However, it may not necessarily be easy to vary continuously the parameters of the original BDS, i.e., the mass, the friction constant and the elastic constant over sufficiently wide ranges. The primary concern of the present paper is to construct and explore a model which experiences the above two regions by changing not the model parameters but a kinematical variable. In such a model, it can be expected that the effects of the theoretical discontinuity between the two regions on various physical quantities are subjects of direct observation.

The model that we will investigate is of a one-component continuum that experiences elastic and dissipative forces simultaneously. In other words, we are interested in the quantum theory of a viscoelastic medium with spatial variance. In such a system, the kinematical variable is a wavevector whose value can be varied throughout the under- and overdamped regions. Then the procedure of quantization is straightforwardly applied to the problem.

One of the remarkable features of the BDS is that, at the critical point that separates the above two regions, the eigenvalues of the Hamiltonian are degenerate and the wavefunctions are divergent. In the present paper, we will ask and attempt to answer the question of what is the relation between these aspects of the model and Berry’s theorem on the ‘geometrical phase’ [8–11].

For the model to be comparable with observation, we require an energy operator by which the ground state and excited states are constructed. Unfortunately, a naive interpretation of the Hamiltonian as energy does not hold in the BDS. By definition, the Hamiltonian is a constant of motion, whereas the kinematic energy of the BDS exponentially varies. Therefore, we have to define the energy operator separately from the Hamiltonian. For this purpose, we shall resort to a correspondence to classical mechanics. The physical ground state will be defined in this way.

This paper is organized as follows. The model of a continuum is defined in section 2. The procedures for quantization in the over- and underdamped regions are discussed in sections 3 and 4, respectively. In section 5, a comment on the level-crossing and the ‘geometrical phase’ is given. In section 6, the energy operator is defined and its eigenstates with boson condensation are obtained. In section 7, the autocorrelation function is examined in order to evaluate the effect of boson condensation on the dispersion relation. The conclusions are presented in section 8.

2. BDS of a continuum

The quantum theory of the lowest dimensional BDS within the broken $O(2, 1)$ or $SU(1, 1)$ representation was introduced by Feshbach and Tikochinsky [2] and was later extended to infinite degrees of freedom by Celeghini et al [3]. The non-dissipative limit of their modeling does not afford a conservative harmonic system. Instead, we shall adopt the representation based on exact $U(1) \times U(1)$ symmetry [7], which is free from the pathology inherent in the $O(2, 1)$ or $SU(1, 1)$ representation [3]. Our model reduces to a free harmonic oscillator model in the non-dissipative limit and will be shown to incorporate the standard model of viscoelasticity.

We consider a Lagrangian

$$L = \int dr \left( m\dot{\xi}^2 + \gamma (\nabla \eta \cdot \nabla \xi - \nabla \eta \cdot \nabla \dot{\xi})/2 - \kappa \nabla \eta \cdot \nabla \xi \right),$$  \hspace{1cm} (2.1)

where $\xi$ stands for some characteristic quantity associated with the deformation of a continuum body. $\eta$ is the Lagrange multiplier introduced to derive the equation of motion for $\xi$. $m$ is the mass density, $\gamma$ is the frictional constant and $\kappa$ is the elastic constant. The spatial gradients are due to that the force on a volume element is proportional to the net difference of deformations of adjacent volume elements. The dissipation rate is assumed to be proportional to the difference of deformation velocity between the adjacent elements with a proportional constant $\gamma$. The Lagrangian (2.1) is a natural modification of the standard model of the elastic continuum (see, e.g., [12]) to the BDS type. The corresponding Hamiltonian is given by

$$H = \int dr \left( \frac{1}{m} \left( p_\xi - \frac{\gamma}{2} \nabla^2 \eta \right) p_\eta + \frac{\kappa}{2} \nabla \eta \cdot \nabla \xi \right),$$  \hspace{1cm} (2.2)

where

$$p_\xi = m\dot{\xi} + \gamma \nabla^2 \eta/2, \quad p_\eta = m\dot{\eta} - \gamma \nabla^2 \xi/2$$  \hspace{1cm} (2.3)

are momenta conjugate to $\xi$ and $\eta$, respectively. The effects of external force will be incorporated by non-derivative terms, if necessary.

The equations of motion that follow from (2.1) are

$$m\ddot{\xi}(r, t) - \gamma \nabla^2 \xi(r, t) - \kappa \nabla \eta(r, t) = 0,$$  \hspace{1cm} (2.4a)
\begin{equation}
m\dddot{x}(r, t) + \gamma \nabla^2 \dot{x}(r, t) - \kappa \nabla^2 \dot{x}(r, t) = 0.
\end{equation}

By decomposing \( \xi(r, t) \) and \( \eta(r, t) \) in Fourier series as

\begin{equation}
\xi(r, t) = \frac{1}{\sqrt{V}} \sum_k \xi_k(t) e^{ikr}, \quad \eta(r, t) = \frac{1}{\sqrt{V}} \sum_k \eta_k(t) e^{-ikr}
\end{equation}

we obtain the equations for the Fourier coefficients with \( \mathbf{k} \neq 0 \)

\begin{align}
m\dddot{\xi}_k + \gamma_k \dddot{\xi}_k + \kappa_k \xi_k &= 0, \\
m\dddot{\eta}_k - \gamma_k \dddot{\eta}_k + \kappa_k \eta_k &= 0,
\end{align}

where \( \gamma_k \equiv \gamma \mathbf{k}^2 \) and \( \kappa_k \equiv \kappa \mathbf{k}^2 \) are nothing but the dissipation rate and the elastic constant, respectively, in the BDS of a point particle. Hence, the arguments on the BDS presented in the literature are straightforwardly applicable to our model. It should be commented that, if it were not for dissipation, the quantization of \( \xi \) governed by (2.4a) or (2.6a) would yield a spectrum of an acoustic phonon, i.e., \( \omega_k = \sqrt{\kappa/m |\mathbf{k}|} \) for the angular frequency \( \omega_k \).

The canonical Hamiltonian of the BDS is given by

\begin{equation}
H_0[\xi, \eta] = \sum_k (m\dddot{\xi}_k + \kappa_k \xi_k) \equiv \sum_k H_0[\xi_k, \eta_k],
\end{equation}

where, from (2.3), \( \xi_k \) and \( \eta_k \) are related to the conjugate momenta by

\begin{equation}
p_{\xi_k} = m\dddot{\eta}_k - \gamma_k \eta_k /2, \quad p_{\eta_k} = m\dddot{\xi}_k + \gamma_k \xi_k /2.
\end{equation}

Writing the equations that are obeyed by independent solutions of (2.6) in the following form (Step i in the Introduction):

\begin{equation}
\dddot{\xi}_{i,k} + \lambda_{i,k} \xi_{i,k} = 0, \quad \dddot{\eta}_{i,k} - \lambda_{i,k} \eta_{i,k} = 0, \quad (i = 1, 2),
\end{equation}

it is easy to show that \( H_0[\xi, \eta] \) is split into two subHamiltonians [4, 7, 13]

\begin{equation}
H_0[\xi, \eta] = H_0[\xi_1, \eta_1] + H_0[\xi_2, \eta_2]
\end{equation}

which is equivalent to a decomposition into Dedene’s Hamiltonian [13]. The eigenvalues \( \lambda_{i,k} \) are given by

\begin{equation}
\lambda_{1,k} = \frac{\gamma_k}{2m} \left( 1 + \sqrt{1 - \frac{4m \kappa_k}{\gamma_k^2}} \right), \quad \lambda_{2,k} = \frac{\gamma_k}{2m} \left( 1 - \sqrt{1 - \frac{4m \kappa_k}{\gamma_k^2}} \right).
\end{equation}

Equation (2.9) imply that each subsystem is equivalent to the massless BDS and that quantization of our continuum system can be performed in accordance with the method devised in previous works [6, 7].

It is customary to classify the sets of parameters into four cases: the massless region for \( w_k \equiv 4m \kappa_k / \gamma_k^2 = 0 \), the overdamped region for \( 0 < w_k < 1 \), the critical damping for \( w_k = 1 \), and the underdamped region for \( w_k > 1 \). Note that (2.10) can express only one of the two independent solutions at the critical damping in the BDS and the decomposition of the Hamiltonian in accordance with (2.9) and (2.10) is not applicable to this case of the BDS. We will not consider the critically damped case in this paper, unless otherwise explicitly mentioned.

The original variables \( \xi_k \) and \( \eta_k \) are reconstructed by (Step ii)

\begin{equation}
\xi_k = \xi_{1,k} + \xi_{2,k}, \quad \eta_k = \eta_{1,k} + \eta_{2,k}.
\end{equation}

\( \xi_{1,k} \) and \( \eta_{1,k} \) are inversely expressed in terms of \( \xi_k \) and \( \eta_k \) by using (2.9) as

\begin{align}
\xi_{1,k} &= -\xi_k / \Delta, \quad \eta_{1,k} = \eta_k / \Delta, \\
\xi_{2,k} &= \xi_k / \Delta, \quad \eta_{2,k} = -\eta_k / \Delta,
\end{align}

where \( \Delta \equiv \lambda_{+,k} - \lambda_{-,k} \).

The equations of motion (2.6) are homogeneous and two sets of independent solutions are easily found. In subsequent sections, we will analyze them for over- and underdamped regions.

Before going into the details of quantization, we will comment briefly on the relation between our model and the Maxwell model of viscoelasticity. The Maxwell model, in which a spring and a dashpot are serially connected, assumes that the total deformation \( \varepsilon \) is given by the sum of spring’s and dashpot’s deformations, \( \varepsilon_1 \) and \( \varepsilon_2 \), as

\begin{align}
\varepsilon_1 &= \frac{1}{E} \sigma \varepsilon, \quad \varepsilon_2 = \frac{1}{E \tau} \sigma \varepsilon, \\
\varepsilon &= \varepsilon_1 + \varepsilon_2,
\end{align}

where \( \sigma \) is the stress and \( E \) and \( \tau \) are constants. As (2.14) is analogous to the set of (2.13) and (2.12), let us seek a condition in which the system (2.13) is really equivalent to the above Maxwell model. By identifying \( \varepsilon \) with \( \xi_{i,k} \), we have
\[ \sigma = E e_{1,k} = E \hat{e}_{1,k}. \]

Substitute (2.13a) and (2.13b) to (2.15) to obtain
\[ -\left( \frac{\lambda_{+}}{\tau} \dot{\xi}_k + \frac{\lambda_{-}}{\tau} \ddot{\xi}_k \right) = \frac{\tau}{\gamma} \left( \frac{\lambda_{+}}{\tau} \xi_k + \frac{\lambda_{-}}{\tau} \dot{\xi}_k \right) \]

This equation coincides with (2.6a) if \( \gamma_k / m = \lambda_{+} / \tau \) and \( \kappa_k / m = \lambda_{-} / \tau \) hold. This condition is satisfied by either
\[ \frac{4m \kappa}{\gamma} \xi_k^2 \rightarrow 1, \quad \tau = \frac{\gamma}{2\kappa}, \]

or
\[ \kappa = 0, \quad \tau = \frac{m}{\gamma k^2}. \]

Our model defined by (2.9)–(2.11) reduces to the Maxwell model when the parameters are related by (2.17) or (2.18). The former and the latter are the conditions for the critically damped BDS and the massless BDS, respectively. For detailed discussions of viscoelasticity, see, e.g., [14] or [15].

### 3. Quantization in the overdamped region

In the overdamped region, the wavevector is constrained as \( \vec{k} \equiv |k| / k_0 > 1 \) where \( k_0 \equiv 2 \sqrt{m \kappa / \gamma} \). Introducing the amplitudes \( \xi_{1,k} \) and \( \xi_{2,k} \), the coordinates are expressed as
\[ (\xi_{1,k}, \eta_{1,k}) = \left( \frac{h}{\sqrt{\gamma_k \delta_k}} e^{-\lambda_{+} u \delta_{t,k}} \chi, \ i e^{\lambda_{-} u \delta_{t,k}} \chi \right), \]
\[ (\xi_{2,k}, \eta_{2,k}) = \left( \frac{h}{\sqrt{\gamma_k \delta_k}} (i e^{-\lambda_{+} u \delta_{t,k}} - e^{\lambda_{-} u \delta_{t,k}}) \right), \]
\[ \delta_k^{(w)} = \sqrt{1 - w_k} = \sqrt{k^2 - 1 / \vec{k}}, \]

where \( w_k = 4m \kappa / \gamma k^2 < 1 \), \( \lambda_{i,k} \), the decay rates of \( \xi_{i,k} \), have been given by (2.11). For a later convenience, we here give other expressions:
\[ \lambda_{1,k} = \frac{\tau}{2m} \left( 1 + \delta_k^{(w)} \right), \]
\[ \lambda_{2,k} = \frac{\tau}{2m} \left( 1 - \delta_k^{(w)} \right). \]

For a given \( \vec{k} \), there exist two branches, which merge to \( 2K / \gamma \) at \( \vec{k} = 1 \); \( \xi_{1,k} \) are decaying, while \( \eta_{1,k} \) are amplifying. The solutions for \( i = 1 \) and 2 respectively constitute the subsystems equivalent to Dede’s Hamiltonian [16] or the massless BDS [6]. The original coordinates are given by (2.12). The Hamiltonian for the \( k \)th component is decomposed into two subHamiltonians \( H_B[\xi_{1,k}, \eta_{1,k}] \) and \( H_B[\xi_{2,k}, \eta_{2,k}] \) as was given in (2.10):
\[ H_B[\xi_{i,k}, \eta_{i,k}] = H_0[\xi_{i,k}, \eta_{i,k}] + H_0[\xi_{2,k}, \eta_{2,k}], \]

By construction, these subHamiltonians are mutually independent. The ‘subLagrangian’ for each subsystem is given by the one for the massless BDS [6].

As was shown in [6], quantization is not possible in the representation that diagonalizes either the coordinate \( \xi \) or \( \eta \). Instead, we have to adopt a representation in which some linear combinations of the damping and amplifying coordinates are employed [4] to construct physical wavefunctions, as is explained below.

Following the prescription given by Takahashi [6, 7], we require the coordinate variables to obey the commutation relations (Step iii)
\[ [\xi_{1,k}, \Pi_{1,k}^{(j)}] = [\eta_{1,k}, \Pi_{1,k}^{(j)}] = i \frac{\hbar}{2} \delta_{ij}, \]
\[ [\xi_{1,k}, \eta_{1,k}] = [\xi_{2,k}, \eta_{2,k}] = i \frac{\hbar}{\gamma k \xi_k}. \]

where \( \Pi_{1,k}^{(j)} \) and \( \Pi_{1,k}^{(j)} \) are the momenta conjugate to \( \xi_{1,k} \) and \( \eta_{1,k} \) respectively (see (3.5)). These relations follow from the fact that \( (\xi_{1,k}, \eta_{1,k}) \) constitutes the coordinates of the effectively massless subsystem constructed as (2.13). The factor \( 1/2 \) on the rhs of (3.4a) implies that the above commutation relations, which have been uniquely derived in order to be consistent with the original equations of motion, are for the subsystems defined
by $H_0[\xi_{l,k}, \eta_{l,k}]$ and $H_0[\zeta_{l,k}, \eta_{l,k}]$\cite{7}. The quantization conditions (3.4) are distinctive in that they require non-commutability among the variables that were treated as independent in prevailing approaches [2–5, 17]. Notwithstanding, the condition (3.4) is compatible with the canonical quantization condition [7].

The conjugate momenta $\Pi^\xi_{l,k}$ and $\Pi^\eta_{l,k}$ are defined with recourse to the corresponding sub-Lagrangians by

$$\Pi^\xi_{l,k} = \frac{\gamma_k \delta_k^{(0)}}{2} \eta_{l,k}, \quad \Pi^\eta_{l,k} = -\frac{\gamma_k \delta_k^{(0)}}{2} \xi_{l,k},$$

$$\Pi^\zeta_{l,k} = -\frac{\gamma_k \delta_k^{(0)}}{2} \eta_{l,k}, \quad \Pi^\eta_{l,k} = \frac{\gamma_k \delta_k^{(0)}}{2} \zeta_{l,k}.$$ \hspace{1cm} (3.5)

Accordingly, the representation to be adopted, which we may call the ‘$\zeta$-representation’ in this paper, is given by (Step iv)

$$\zeta_{1,k}(t) = \frac{\xi_{1,k} - i \eta_{1,k}}{\sqrt{2}} = \frac{\hbar}{2 \gamma_k \delta_k^{(0)}} (e^{\lambda_k \alpha_{1,k}(0)} + e^{-\lambda_k \alpha_{1,k}(0)}),$$

$$\zeta_{2,k}(t) = \frac{\xi_{2,k} + i \eta_{2,k}}{\sqrt{2}} = i \frac{\hbar}{2 \gamma_k \delta_k^{(0)}} (e^{-\lambda_k \alpha_{2,k}(0)} - e^{\lambda_k \alpha_{2,k}(0)}),$$ \hspace{1cm} (3.6a)

for the coordinates and

$$\Pi_{1,k}(t) = \frac{\Pi^\xi_{1,k} + i \Pi^\eta_{1,k}}{\sqrt{2}} = \frac{\hbar}{2 \gamma_k \delta_k^{(0)}} (e^{\lambda_k \alpha_{1,k}(0)} - e^{-\lambda_k \alpha_{1,k}(0)}),$$

$$\Pi_{2,k}(t) = \frac{\Pi^\xi_{2,k} - i \Pi^\eta_{2,k}}{\sqrt{2}} = \frac{\hbar}{2 \gamma_k \delta_k^{(0)}} (e^{\lambda_k \alpha_{2,k}(0)} + e^{-\lambda_k \alpha_{2,k}(0)}),$$ \hspace{1cm} (3.6b)

for the conjugate momenta. In the above definition of the coordinates and the conjugate momenta, the translational invariance of the model has been taken into account. The quantization conditions for each subsystem in the $\zeta$-representation read

$$[\zeta_{1,k}(t), \Pi_{1,k}(t)] = \frac{1}{2} i \hbar \delta_{kl} \delta_{h,k'},$$ \hspace{1cm} (3.7a)

$$[\zeta_{1,k}(t), \zeta_{j,k'}(t)] = [\Pi_{1,k}(t), \Pi_{j,k'}(t)] = 0.$$ \hspace{1cm} (3.7b)

The operators in the $\zeta$-representation for the original variables are constructed by (Step v)

$$\zeta_k(t) = \zeta_{1,k}(t) + \zeta_{2,k}(t),$$ \hspace{1cm} (3.8a)

and

$$\Pi_k(t) = \Pi_{1,k}(t) + \Pi_{2,k}(t).$$ \hspace{1cm} (3.8b)

These operators satisfy the canonical commutation relation

$$[\zeta_k, \Pi_k] = i \hbar \delta_{kl} \delta_{h,k'}.$$ \hspace{1cm} (3.9)

The form of the linear combinations as given by (3.6) that satisfy (3.7) and (3.9) is not unique. (One may multiply an arbitrary number $s$ to the rhs of (3.8a) and $s^{-1}$ to the rhs of (3.8b). Here we choose $s = 1$.) Once we fixed the form of $\zeta_k$ and $\Pi_k$ in our $\zeta$-representation, we employ any operators in this representation throughout the calculations.

In terms of $\zeta_{1,k}$ and $\Pi_{1,k}$, the operators $a_{l,k}$ and $\bar{a}_{l,k}$ are written by utilizing (3.6) as

$$a_{l,k}(0) = \frac{1}{\sqrt{\hbar}} \left( \frac{\gamma_k \delta_k^{(0)}}{2} \zeta_{1,k} + \frac{2}{\gamma_k \delta_k^{(0)}} \Pi_{1,k} \right),$$

$$\bar{a}_{l,k}(0) = \frac{1}{\sqrt{\hbar}} \left( \frac{\gamma_k \delta_k^{(0)}}{2} \zeta_{1,k} - \frac{2}{\gamma_k \delta_k^{(0)}} \Pi_{1,k} \right),$$ \hspace{1cm} (3.10a)

$$a_{2,k}(0) = \frac{1}{\sqrt{\hbar}} \left( -\frac{\gamma_k \delta_k^{(0)}}{2} \zeta_{2,k} + \frac{2}{\gamma_k \delta_k^{(0)}} \Pi_{2,k} \right),$$

$$\bar{a}_{2,k}(0) = \frac{1}{\sqrt{\hbar}} \left( \frac{\gamma_k \delta_k^{(0)}}{2} \zeta_{2,k} + \frac{2}{\gamma_k \delta_k^{(0)}} \Pi_{2,k} \right).$$ \hspace{1cm} (3.10b)
where
\[ \gamma_k \delta_k^{(i)} = \frac{4\pi k}{\gamma}. \]

In (3.10), \( \zeta_{i,k} \) and \( \Pi_{i,k} \) are operators evaluated at \( t = 0 \). Obviously, \( a_{i,k} \) and \( \bar{a}_{i,k} \), which are generally not Hermitian conjugate of each other, obey the commutation relations
\[ [a_{i,k}(0), \bar{a}_{i,k}(0)] = \delta_k^0 \delta_{i,k}. \]

Then, each subHamiltonian in (2.10) takes the form
\[ \hat{H}_{i,k} \equiv \hat{H}_0[\zeta_{i,k}, \eta_{i,k}] = -i\hbar \sum_k \lambda_{i,k} \bar{a}_{i,k} a_{i,k}. \]

This is invariant under the global \( U(1) \times U(1) \) transformation \( \zeta_{i,k} \rightarrow e^{i\phi} \zeta_{i,k}, \eta_{i,k} \rightarrow e^{-i\phi} \eta_{i,k} \) (or \( a_{i,k} \rightarrow e^{i\phi} a_{i,k}, \bar{a}_{i,k} \rightarrow e^{-i\phi} \bar{a}_{i,k} \)).

We define the bare vacuum \( \ket{0} \) by \( a_{i,k}\ket{0} = \ket{0} \bar{a}_{i,k} = 0 \), where \( \ket{0} \) is the time reversal conjugate of \( \ket{0} \). The time-reversal conjugate of any state is defined by replacements \( \bar{a}_{i,k}(t) \rightarrow a_{i,k}(t) \) together with taking complex conjugate of \( c \)-numbers in remaining parts which include components of the vector in the Hilbert space spanned by the eigenstates \( \bar{a}_{i,k}^\dagger \ket{0} / \sqrt{n!} \) and \( \ket{0} a_{i,k}^\dagger / \sqrt{n!} \). Their eigenvalues \( -i\hbar \lambda_{i,k} \) are pure imaginary.

The eigenfunctions of \( \hat{H}_{i,k} \) and their time reversal conjugate are given by
\[ \psi_{n,i}(\zeta_{i,k}, t) = (n!)^{-1/2} e^{-n\lambda_{i,k}^2 t} \bar{H}_n^{1/2} \zeta_{i,k}^n \psi_0(\zeta_{i,k}), \]
\[ \bar{\psi}_{n,i}(\zeta_{i,k}, t) = (n!)^{-1/2} e^{-n\lambda_{i,k}^2 t} \bar{H}_n^{1/2} \zeta_{i,k}^n \bar{\psi}_0(\bar{\zeta}_{i,k}), \]

where \( H_n \) is the Hermite polynomial. The vacuum state is given by
\[ \psi_0(z) = A_0^{-1/2} e^{-z^2/(2\hbar^2)}, \bar{\psi}_0(z) = \bar{\psi}_0(z)^*, \]
\[ \int_C \bar{\psi}_0(\bar{\zeta}_{i,k}) \psi_0(\zeta_{i,k}) d\zeta_{i,k} = 1. \]

The integration path \( C \) on the complex \( \zeta_{i,k} \)-plane in (3.15b) determines the absolute value of the normalization constant \( A_0 \). For simplicity, we chose \( C \) so as for the variable \( (2\gamma_k \delta_k^{(i)} / \hbar)^{1/2} \zeta_{i,k} \) to be real. This means that, when \( \gamma_k \delta_k^{(i)} \) is real and positive, \( C \) is the real axis (i.e., \( \eta_{i,k} = 0 \)) and gives
\[ |A_0| = (\pi \hbar / \gamma_k \delta_k^{(i)})^{1/2}. \]

If \( \gamma_k \delta_k^{(i)} \) were a complex number with a phase \( \theta \), the path \( C \) would be a line that intersects with the real axis at the origin with an angle \( -\theta/2 \). The change of \( C \) of this kind will take place when analytic continuation on the real axis of the complex parameter plane is considered. The inner product is taken between a ket vector and a time-reversal conjugate vector.

The total Hamiltonian is given by
\[ \hat{H}_B = -i\hbar \sum_{i,k} \lambda_{i,k} n_{i,k}, n_{i,k} = \bar{a}_{i,k}(0) a_{i,k}(0). \]

Accordingly, the full wavefunction is given by a product
\[ \psi_{mn}(\zeta_{i,k}, \zeta_{j,k}, t) = \psi_{mn}(\zeta_{i,k}, t) \psi_{2,n}(\bar{\zeta}_{i,k}, t) \]

In the Hilbert space defined previously, \( \bar{a}_{i,k}(0) \) is the hermitian conjugate of \( a_{i,k}(0) \). Like the Feshbach - Tikochinsky Hamiltonian [2], the \( O(2) \) symmetry of \( \hat{H}_B \) is broken by the \( \delta_k^{(i)} \) in \( \lambda_{i,k} \), although this does not cause any instability of the system because \( \hat{H}_B \) is not the system’s energy.

The Heisenberg equations of motion
\[ \hbar \dot{a}_{i,k} = -i[a_{i,k}, \hat{H}_B], \hbar \dot{\bar{a}}_{i,k} = -i[\bar{a}_{i,k}, \hat{H}_B] \]

and their solutions
\[ a_{i,k}(t) = e^{-\lambda_{i,k}^2 t} a_{i,k}(0), \quad \bar{a}_{i,k}(t) = e^{\lambda_{i,k}^2 t} \bar{a}_{i,k}(0) \]

are of course consistent with the classical solutions (3.1).

Green’s function \( G_k(t) \) is evaluated as
\[ G_k(t) \equiv -\frac{i}{\hbar} \langle 0 | T \zeta_{i,k}(t) \zeta_{i,k}(0) | 0 \rangle \]
\[ = \frac{1}{2\gamma_k} \left[ \theta(t)(e^{-\lambda_{i,k}^2 t} + e^{-\lambda_{i,k}^2 t}) + \theta(-t)(e^{\lambda_{i,k}^2 t} + e^{\lambda_{i,k}^2 t}) \right], \]
The system couples with a source $J(t)$ after $t = 0$, the particular solution is given by

$$
\zeta_k(t) = \zeta_k^{(0)}(t) + \int_0^\infty dt' G_k(t - t') J(t').
$$

(3.22)

$\zeta_k^{(0)}(t)$ is the solution of homogeneous equation.

The Fourier transform of $G_k(t)$ provides us the knowledge of the level structure. Let us define

$$
\tilde{g}_k^{(0)}(\omega) \equiv \int_0^\infty g_k^{(0)}(t) \cos \omega t dt,
$$

(3.23a)

$$
\tilde{g}_k^{(0)}(\omega) \equiv \frac{e^{-\lambda_k t} + e^{-\lambda_k t'}}{2},
$$

(3.23b)

for $k > 1$. Since $\lambda_{i,k}$ are real, $\tilde{g}_k^{(0)}(\omega)$ has a peak at $\omega = 0$ and is monotonically decreasing. It exhibits no physically interesting structure. Notice that (3.23) is mathematically meaningful also at $k = 1$ that corresponds to the Maxwell model of viscoelasticity.

4. Quantization in the underdamped region and the dispersion relation

In the underdamped region, the wavevector is constrained as $k = |k|/k_0 < 1$. The decay rates of $\zeta_{i,k}$ in the $i$th subsystem are complex:

$$
\lambda_{i,k} = \frac{\gamma_k}{2m} \left( 1 \pm i \delta_k^{(0)}(\omega) \right) = \frac{2\kappa}{\gamma} \left( F_1 \pm i \sqrt{1 - F_2} \right), \quad (i = 1, 2),
$$

(4.1)

where $\delta_k^{(0)}(\omega) \equiv \sqrt{w_k - 1}$ with $w_k = 4m\kappa/\gamma^2 > 1$. In the literature, $(\gamma_k/2m)\delta_k^{(0)}$ is sometimes called the reduced frequency. Here and below, $i = 1$ (2) for upper (lower) sign in (4.1). The canonical momenta are expressed as

$$
\Pi_{1,k} = \pm i \frac{\gamma_k \delta_k^{(0)}}{2} \eta_{1,k}, \quad \Pi_{2,k} = \mp i \frac{\gamma_k \delta_k^{(0)}}{2} \zeta_{1,k},
$$

(4.2a)

$$
\gamma_k \delta_k^{(0)} = \frac{4m\kappa}{\gamma} \sqrt{1 - F_2}.
$$

(4.2b)

Their temporal behaviors are given by

$$
(\zeta_{1,k}, \eta_{1,k}) = \sqrt{\frac{h}{\gamma_k \delta_k^{(0)}}} (e^{-\lambda_k t} a_{1,k}(0), e^{i\lambda_k t} \tilde{a}_{1,k}(0)),
$$

(4.3a)

$$
(\Pi_{1,k}, \Pi_{2,k}) = i \sqrt{\frac{h\gamma_k \delta_k^{(0)}}{2}} (e^{i\lambda_k t} \tilde{a}_{1,k}(0), -e^{-\lambda_k t} a_{1,k}(0)),
$$

(4.3b)

$$
(\zeta_{2,k}, \eta_{2,k}) = \sqrt{\frac{h}{\gamma_k \delta_k^{(0)}}} (e^{-\lambda_k t} d_{2,k}(0), e^{i\lambda_k t} \tilde{d}_{2,k}(0)),
$$

(4.3c)

$$
(\Pi_{1,k}, \Pi_{2,k}) = i \sqrt{\frac{h\gamma_k \delta_k^{(0)}}{2}} (e^{-\lambda_k t} \tilde{d}_{2,k}(0), -e^{i\lambda_k t} d_{2,k}(0)).
$$

(4.3d)

The Hamiltonian takes the same form as (3.13) with $\lambda_{i,k}$ now being given by (4.1).

Note that $\lambda_{i,k} \rightarrow \pm i \sqrt{\kappa/m} |k|$ and $\gamma_k \delta_k^{(0)} \rightarrow 2 \sqrt{\kappa m} |\kappa|$ for $\gamma \rightarrow 0$. This ensures that the operators and therefore the Hamiltonian in $U(1) \times U(1)$ representation are well-defined and take the forms of the genuinely elastic system in the conservative limit.

As in the overdamped region, we transfer to the $\zeta$-representation by

$$
\zeta_{1,k} = \frac{\zeta_{1,k} + \eta_{1,k}}{\sqrt{2}} = \sqrt{\frac{h}{2\gamma_k \delta_k^{(0)}}} (e^{-\lambda_k t} a_{1,k}(0) + e^{i\lambda_k t} \tilde{a}_{1,k}(0)),
$$

$$
\zeta_{2,k} = \frac{\zeta_{2,k} - \eta_{2,k}}{\sqrt{2}} = i \sqrt{\frac{h}{2\gamma_k \delta_k^{(0)}}} (e^{-\lambda_k t} d_{2,k}(0) - e^{i\lambda_k t} \tilde{d}_{2,k}(0)).
$$

(4.4a)
for coordinates and by
\[
\begin{align*}
\Pi_{1,k} &= \frac{\Pi_{1,k}^c + \Pi_{1,k}^o}{\sqrt{2}} = \frac{i}{2} \sqrt{\frac{2\gamma k}{\sum\omega^2}} \left( e^{\lambda_k \omega^2} \hat{a}_{1,k}(0) - e^{-\lambda_k \omega^2} \hat{a}_{1,k}(0) \right), \\
\Pi_{2,k} &= \frac{\Pi_{2,k}^c - \Pi_{2,k}^o}{\sqrt{2}} = \frac{1}{2} \sqrt{\frac{2\gamma k}{\sum\omega^2}} \left( e^{\lambda_k \omega^2} \hat{a}_{2,k}(0) + e^{-\lambda_k \omega^2} \hat{a}_{2,k}(0) \right).
\end{align*}
\] (4.4b)

for conjugate momenta. Comparing these with (3.6), we see that the coordinate and momentum operators in the underdamped region are obtained by replacing \( \delta_k^{(o)} \) for the overdamped region with \( \delta_k^{(u)} \). These operators also satisfy the commutation relations (3.7). (More generally, the replacement \( \delta_k^{(o)} \rightarrow e^{i\beta} \delta_k^{(u)} \) with arbitrary phase \( \beta \) does not alter the commutation relation and is also allowed.) As in the overdamped case, the Hamiltonian is given by (3.17) with \( \lambda_{1,k} \) being given by (4.1). The \( O(2) \) symmetry is broken by the \( \delta_k^{(u)} \).

As for the Fourier transform of the autocorrelation function, we have
\[
\tilde{g}_k^{(u)}(\omega) = \frac{1}{2} \int_0^\infty (e^{\lambda_k \omega^2} + e^{-\lambda_k \omega^2}) \cos \omega t \, dt
\]
\[
= \gamma \frac{k^2}{(\omega - k)^2 + k^2} + \frac{k^2}{(\omega + k)^2 + k^2},
\] (4.5)

where \( \omega \equiv (\gamma/2\kappa) \omega \). This \( \tilde{g}_k^{(u)}(\omega) \) appears expresses the existence of an excitation with the ‘bare’ dispersion relation
\[
\omega_b(k) = k \sqrt{1 - k^2},
\] (4.6)

and a half-value width \( \bar{k} \) (the excitation at \( -k \sqrt{1 - k^2} \) is a time-reversal image of the former one). However, we have to note that the position of the peak \( \tilde{g}_0(k) \) in \( \tilde{g}_k^{(u)}(\omega) \) is determined by \( \frac{\partial \tilde{g}_k^{(u)}(\omega)}{\partial \omega} = 0 \). This condition is solved as
\[
\omega_b(k) = k \sqrt{2 \omega / \omega^2 - 1}, \quad 0 \leq k \leq \sqrt{3}/2.
\] (4.7)

The \( \bar{k} \) dependences of \( \omega_0(k) \) and \( \omega_b(k) \) are depicted in figure 1.

\( \omega_0(k) \) and \( \omega_b(k) \) almost coincide with each other for \( \bar{k} < 0.5 \). Above \( \bar{k} \approx 0.6 \), \( \omega_0(k) \) begins to decreases faster than \( \omega_b(k) \) and becomes 0 at \( \bar{k} = \bar{k}_0 \equiv \sqrt{3}/2 = 0.8660 \cdots \). This downward bending of the curve is interpreted to signify the partition of the energy into lower energies that are shared among condensed \( \alpha \)-quanta. The dominance of the zero-mode at shorter wavelengths may be an indication of dislocation or rupture of material in small scales.

One measure of the peak broadening is \( r_p \equiv \left| |\tilde{g}_0^{(u)}(0) / \tilde{g}_k^{(u)}(\omega_0(k))| \right| \). The gross tendency of the increase of \( r_p \) with \( \bar{k} \) is also shown in figure 1. Above \( \bar{k} \), \( \tilde{g}_k^{(u)}(\omega) \) peaks at \( \omega = 0 \). This does not mean that vibration does not occur: the pole still has a real part above \( \bar{k}_0 \). Actually, the spectrum of excitations becomes continuous because of the large width. Above \( \bar{k} = 1 \), the real part of the pole vanishes and the motion becomes utterly dissipative.
5. Unified expression for the wavefunctions and the ‘geometrical phase’

The under- and overdamped wavefunctions are brought into a single expression as

\[
\psi_{\alpha}(\zeta_{i,k}^1, t) = (n!)^{-1/2} e^{-i\zeta_{i,k}^1 H_{n}((2R_k/\hbar)\zeta_{i,k}^1)} \psi_{0}(\zeta_{i,k}^1),
\]

(5.1a)

\[
\psi_{0}(\zeta_{i,k}^1) = (R_k/(\pi\hbar))^{1/4} e^{-R_k\zeta_{i,k}^1/4}\gamma,
\]

(5.1b)

\[
R_k \equiv \gamma_k \delta_k = \frac{4\pi s}{\gamma} (\kappa^2 - 1)^{1/2},
\]

(5.1c)

\[
\lambda_{i,k} = \frac{2\kappa}{\gamma} (\kappa^2 - \kappa^2)^{1/2}.
\]

(5.1d)

Then, the restrictions on \( \zeta \) that have been posed so far separately for the over- and underdamped regions can be removed. Remember however that in our problem we have adopted the constraint of \( R_k^{1/2} \zeta_{i,k}^1 \) being kept real in evaluating inner product.

The variable \( \zeta_{i,k} \) as well as the parameter \( R_k \) in the Hermite polynomial \( H_n \) in (5.1a) can be complex, i.e., the \( H_n \) has generally been extended holomorphically to complex one. For mathematical details of holomorphic Hermite polynomials on a plane, see, e.g., [18], although the spatial dimension of our system is one and the norm of the wavefunction in (5.1) is defined over a line.

Because of (5.1b) and (5.1c), the wavefunctions possess a common factor \((K - 1)^{1/8}\) and do not exist at the critical point \( K = 1 \), where the ‘level crossing’ of higher states takes place. The level-crossing is generally connected to a non-trivial ‘geometrical phase’ due to anholonomy in accordance with the rule found and formulated by Pancharatnam [8], Longuet-Higgins [9], Berry [10] and many other researchers. See [11] for other references.

The point is that the temporal evolution of a quantum system is governed by the Hamiltonian that experiences adiabatic changes of parameters. We can examine whether the ‘geometrical phase’ occurs in our model, too, by following the argument presented by Berry [10], because the Hamiltonian (3.17), irrespective of the values of parameters, is a generator of time translation.

We take, for example, the bare ground state \((\sigma = 0 \text{ in (5.1)})\). The wavefunction with the normalization factor \((R_k/(\pi\hbar))^{1/2}\) depends on the parameters through \( R_k \). For a given \( k \), we calculate

\[
\Gamma = i \oint \int \prod_i \frac{\psi_{0}(\zeta_{i,k}^1) \partial}{\partial R_k} \frac{\psi_{0}(\zeta_{i,k}^1)}{\partial R_k} d\zeta_{i,k}^1 dR_k,
\]

(5.2)

where \( \zeta_{i,k}^1 \)-integrations are performed first with the value of \( R_k \) fixed and then \( R_k \)-integration is carried out along a path that avoids the branch point

\[
R_k = 0.
\]

The corresponding path in the complex parameter space is closed and encircles the point \( K = 1 \). On account of (5.1c), we see that, when a physical parameter vary its phase from 0 to \( 2\pi \), the phase of \( R_k \) can be chosen to vary from 0 to \( \pi \). We therefore parametrize as \( R_k = e^{i\theta/2}, 0 \leq \theta < 2\pi \), with a constant \( \varepsilon \) and identify \( dR_k \) with \((iR_k/2)d\theta \). Recalling that, as was argued in section 3, \( R_k \zeta_{i,k}^1 \) is kept real and thus the \( \theta \)-dependence emerges only from the full amplitude \((R_k/(\pi\hbar))^{1/2}\), \( \Gamma \) is evaluated as

\[
\Gamma = -\frac{1}{8} \oint \frac{d\theta}{\pi} = -\frac{\pi}{4}.
\]

(5.4)

The above result is a quarter of the value obtained by Blasone and Jizba [19] for the ‘Berry–Anandan phase’ of the BDS in \( O(2, 1) \) or \( SU(1, 1) \) representation [2, 3].

In a continuum system, there are multiple \( k \)'s that satisfy (5.3). In one dimension, if \( k \) satisfies (5.3), there exists another wavevector \( -k \) that satisfies (5.3). Since the number of relevant degrees of freedom is doubled, the result for \( \Gamma \) is also doubled:

\[
\Gamma = \frac{\pi}{2}, \quad \text{(one dimension)}.
\]

(5.5)

If the vibrational propagation had a polarization, the above result would give rise to an observable effect in one dimensional medium with torsion [20]. By the same reasoning, for higher dimensions, \( \Gamma \) is an integer multiple of \(-2\pi \). We conclude that the ‘geometrical phase’ in continuum system emerges in a way closely related to the level crossing. Only in one dimension, the ‘geometrical phase’ will be non-trivial.
6. Definition of energy operator

In the previous sections, we saw that, for a given wavevector \( \vec{k} = 1 \), the Hamiltonian \( \hat{H}_B \) consists of two branches of modes with distinctive eigenvalues. They are pure imaginary in the overdamped region, while, in the underdamped region, they are complex with the imaginary part of the eigenvalue, i.e., the reduced frequency, being positive or negative, i.e.,

\[
\operatorname{Im}(\lambda_{i,k}) = \pm \frac{1}{2m} \sqrt{4m \kappa_k - \gamma_k^2} (\vec{k} < 1).
\]

(Then, if \( \hat{H}_B \) were energy, the vacuum would not be the ground state). The two eigenvalues in each region coalesce into a single value \( \lambda_k \) at \( \vec{k} = 1 \).

We recall that \( \hat{H}_B \) is the generator of time translation but not the system’s kinematic energy \([6, 13]\). It is then meaningful to ask what the energy is. Actually, there is no canonical procedure in the BDS to unambiguously define the energy. In order to proceed further, therefore, we have to refer to classical mechanics.

The kinematic energy that corresponds to the classical energy will be constructed by noting that the observable motion is described by the coordinate \( \xi_{\vec{k}} \). Then, together with the equation of motion (2.6a), we may consider a quantity

\[
\hat{E}_k[\vec{p}_{\vec{k}}, \xi_{\vec{k}}] = \frac{1}{2} \sum_k \left( \frac{p^2_k}{m} + \kappa_k \xi_{\vec{k} - \vec{k}} \right)
\]

where \( p^2_k = m \kappa_k \) and the definition (2.13) of \( \xi_{\vec{k}} \) are used to eliminate the temporal derivatives. Obviously, \( \hat{E}_k[\vec{p}_{\vec{k}}, \xi_{\vec{k}}] \) is time dependent and monotonically decays with time toward 0. After quantization, the operator \( \hat{E}_k[\vec{p}_{\vec{k}}, \xi_{\vec{k}}] \) has coherent states \( \hat{A}_{\vec{k}}[n_{\vec{k}}, \vec{k}] \) with \( n_{\vec{k}} \) being arbitrary c-numbers as eigenstates with unbounded continuous eigenvalues. The zero point energy does not exist. Therefore, \( \hat{E}_k[\vec{p}_{\vec{k}}, \xi_{\vec{k}}] \) cannot be the energy operator. In this section, we examine the ‘kinematic energy’ defined in the \( \zeta \)-representation.

By the analogy with classical mechanics, let us take an operator given by a sum of the kinetic and potential energies:

\[
\hat{K}(t) \equiv \sum_k \left( \frac{1}{2m} \Pi_k(t) \Pi_k(t) + \frac{\kappa_k}{2} \dot{\zeta}_k(t) \dot{\zeta}_k(t) \right)
\]

\[
= \sum_k \hbar \frac{\kappa_k}{m} |k| \left( \hat{A}_k(t) A_k(t) + \frac{1}{2} \right),
\]

(6.1)

where

\[
A_k(t) = \frac{1}{\sqrt{2\hbar}} ((m \kappa_k)^{-1/4} \Pi_k(t) - i(m \kappa_k)^{1/4} \dot{\zeta}_k(t)),
\]

\[
\dot{A}_k(t) = \frac{1}{\sqrt{2\hbar}} ((m \kappa_k)^{-1/4} \Pi_k(t) + i(m \kappa_k)^{1/4} \dot{\zeta}_k(t)).
\]

(6.2)

The form of \( \hat{K}(t) \) is same as the one that has been adopted as the Hamiltonian in the quantum theory of elastic continuum. See, e.g., [12]. The non-vanishing equal-time commutator is given by

\[
[A_k(t), \dot{A}_k'(t)] = \delta_{k,k'}.
\]

(6.3)

\( \hat{K} \) has a dimension of energy and is time-dependent.

6.1. Energy in the overdamped region

In this subsection, we investigate the overdamped case, i.e., \( \omega_k < 1 \) or \( \vec{k} > 1 \). We introduce the time-dependent ground state \( |\vec{0}; t\rangle \) by

\[
A_k(t)|\vec{0}; t\rangle = 0 \text{ for all } k.
\]

(6.4)

The eigenstate of \( \hat{K} \) is then constructed as

\[
|\{n_k, k\rangle = \prod_k \hat{A}_k(t)^{n_k}/\sqrt{n_k!} |\vec{0}; t\rangle
\]

(6.5a)

\[
\hat{K}|\{n_k, k\rangle = \hbar \sum_k n_k \omega_k |\{n_k, k\rangle,
\]

(6.5b)

\[
\omega_k \equiv \sqrt{\kappa/m} |k|.
\]

(6.5c)
From (6.4) and (6.2), $|\mathbf{0}; t\rangle$ in the $\zeta$-representation is given by the product over all possible $k$ of

$$
\psi_{\alpha,k} = \left( \sqrt{\frac{m_s}{\hbar}} \right)^{3/2} \exp \left( -\sqrt{\frac{m_s}{\hbar}} \zeta^\dagger \right).
$$

(6.6)

The energies as eigenvalues of $\hat{K}$ are time-independent, while the eigenstates are time dependent. The spectrum coincides with that of the acoustic phonon in the conservative part of the original system. This fact is reconcileable with the system’s dissipative nature if the system is supposed to be somehow in an equilibrium state in which the kinetic energy is partitioned to phonon modes. Such a viewpoint has been presented previously by Feshbach and Tikochinsky [2] and Celeghini et al [3], according to which the auxiliary field in the BDS plays a role of heat bath or sink of energy flow. Our $K$ looks as if it substantiates their views. However, $K$ itself does not tell what the environment is because the Bateman Hamiltonian $H_B[\zeta, \eta]$ consists of variables of the system that is subject to observation.

The environment or the heat bath will be incorporated into the model by introducing some physical elements that as a whole have very large degrees of freedom and are randomly interacting with $\zeta$ and/or $\Pi_k$. There are several ways of introducing such elements that lead to the Langevin equations [21, 22]. Interactions with environments can also be represented by introducing certain nonlinear terms into the energy operator [23, 24]. Extending $K$ along these lines is out of the scope of the present paper.

From (3.8) and (3.6), $A_k$ and $\tilde{A}_k$ in (6.2) are expressed in terms of $\hat{a}_{i,k}$ and $\hat{a}_{i,k}^*$ as

$$
A_k(t) = \frac{1}{2}(D^o_{+,k}\alpha_k(t) + D^o_{-,k}\tilde{\alpha}_k(t)),
$$

$$
\tilde{A}_k(t) = \frac{1}{2}(D^o_{+,k}\tilde{\alpha}_k(t) + D^o_{-,k}\alpha_k(t)),
$$

(6.7)

where

$$
\alpha_k(t) = \frac{1}{\sqrt{2}}(-i\hat{a}_{1,k}(t) + a_{2,k}(t)),
$$

$$
\tilde{\alpha}_k(t) = \frac{1}{\sqrt{2}}(i\hat{a}_{1,k}(t) + a_{2,k}(t)),
$$

(6.8a)

$$
[\alpha_k(t), \tilde{\alpha}_k(t)] = i\delta_{k,0},
$$

$$
[\alpha_k(t), \alpha_l(t)] = [\tilde{\alpha}_k(t), \tilde{\alpha}_l(t)] = 0,
$$

(6.8b)

$$
D^o_{\pm,k} = \left( \frac{1 - wk}{w_k} \right)^{1/4} \pm \left( \frac{w_k}{1 - wk} \right)^{1/4}
$$

$$
= (K^2 - 1)^{1/4} \pm (K^2 - 1)^{-1/4}.
$$

(6.8c)

$D^o_{\pm,k}$ is real in the overdamped region. Obviously, $\alpha_k(t)$ annihilates the vacuum $|0\rangle$.

By virtue of (6.8b), the state $\exp[-R_k\tilde{\alpha}_k(t)\tilde{\alpha}_{-k}(t)]|0\rangle$ is annihilated by $A_k(t)$, so that the time-dependent ground state of $K(t)$ is constructed as the direct product of those states:

$$
|\mathbf{0}; t\rangle = \prod_k |\psi_k^{1/2}\exp\left[ -\frac{1}{2}R_k\tilde{\alpha}_k(t)\tilde{\alpha}_{-k}(t) \right]|0\rangle,
$$

$$
|\mathbf{0}; t\rangle = \prod_k |\psi_k^{1/2}\exp\left[ -\frac{1}{2}R_k\alpha_k(t)\alpha_{-k}(t) \right]|0\rangle,
$$

(6.9)

where $\gamma_k$ is the normalization factor. $R_k$ is given by

$$
R_k = \frac{D^o_{+,k}}{D^o_{-,k} + D^o_{+,k}} = \left( K^2 - 1 \right)^{1/2} - \frac{1}{(K^2 - 1)^{1/2} + 1}.
$$

(6.10)

The product in (6.9) involves $k$ and $-k$ distinctively. At first glance, a factor like $h(-i\hat{a}_{1,k}(t) + a_{2,k}(t))$ with an arbitrary function $h(x)$ seems to be generally allowed since it commutes with $A_k$. However, the combination $-i\hat{a}_{1,k}(t) + a_{2,k}(t)$ belongs to other representation than our $\zeta$-representation that should not be taken into account.

With use of a formula $|0\rangle \exp(\alpha_{-k}\alpha_{-k}) \exp(x\tilde{\alpha}_k\tilde{\alpha}_{-k})|0\rangle = (1 - x^2)^{-1}$ for a $c$-number $x$, $\gamma_k$ in (6.9) is given by

$$
\gamma_k = 1 - R_k^2,
$$

(6.11a)

$$
0 < \gamma_k \leq 1, \gamma_k \to 4/K \text{ for } |k| \to \infty.
$$

(6.11b)

The ground state is time-dependent but the normalization is temporally invariant. The portion the bare vacuum $|0\rangle$ occupies in $|\mathbf{0}; t\rangle$ is given by

$$
\gamma_k = 1 - R_k^2,
$$

(6.11a)

$$
0 < \gamma_k \leq 1, \gamma_k \to 4/K \text{ for } |k| \to \infty.
$$

(6.11b)
provided that the system’s volume $V$ is finite and a ultraviolet cut-off is introduced in the integration. \( \langle \alpha; t \mid 0 \rangle \) vanishes if the integration is extended to infinity.

The expectation value of the number of $\alpha_k$-quanta is equal to that of $\alpha_{-k}$-quanta. They are evaluated as

\[
\langle \alpha; t \mid \hat{a}_k(t)\hat{a}_k(t)\mid \alpha; t \rangle = R_k^2 \langle \alpha; t \mid \alpha_{-k}(t)\hat{a}_{-k}(t)\mid \alpha; t \rangle
\]

from which we have

\[
\langle n_i(k) \rangle = \langle \alpha; t \mid \hat{a}_k(t)\hat{a}_k(t)\mid \alpha; t \rangle = \langle \alpha; t \mid \hat{\alpha}_{-k}(t)\alpha_{-k}(t)\mid \alpha; t \rangle
\]

Similarly, the number of $a$-quanta is evaluated on account of

\[
a_{i,k}(t)\mid \alpha; 0 \rangle = -\left(i\sqrt{\frac{\omega}{k}}\right) R_k e^{-\frac{i\omega t}{k}} \hat{a}_{-k}(t)\mid \alpha; 0 \rangle,
\]

\[
a_{i,k}(t)\mid \alpha; t \rangle = -\left(i\sqrt{\frac{\omega}{k}}\right) R_k e^{-\frac{i\omega t}{k}} \hat{a}_{-k}(t)\mid \alpha; 0 \rangle,
\]

as

\[
\langle \alpha; t \mid n_{i,k}\mid \alpha; t \rangle = \frac{R_k^2}{2} \langle \alpha; t \mid \alpha_{-k}(t)\hat{a}_{-k}(t)\mid \alpha; t \rangle = \frac{R_k^2}{2(1 - R_k^2)}.
\]

$D_{\alpha k}^{(o)}$ diverge at $\bar{k} \to 1$ or $\bar{k} \to \infty$ and take minimum values at $\bar{k} = \sqrt{2}$, where $D_{\alpha k}^{(o)}$ vanishes. Accordingly, the number of $\alpha$-quanta varies from infinity to 0 as $\bar{k}$ varies from 1, (near the critical point) to $\sqrt{2}$, and then increases again to infinity as $\bar{k}$ increases to $\infty$ (dissipation dominance or short wavelength limit described by the massless BDS). Considering that $|k| = \infty$ is unphysical, $\bar{k} = \infty$ means two possibilities: the absence of inertia or elastic force. In either case, the motion is genuinely dissipative and phonons cannot be created. Therefore, the divergence of $\langle \alpha; t \mid n_{i,k}\mid \alpha; t \rangle$ at $\bar{k} = \infty$ indicates that the generation of $\alpha$-quanta is associated with dissipation. Actually, in order to avoid the divergence of the number density, a cut-off in the wavevector integration must be introduced.

By contrast, $\bar{k} = 1$ is the critical point realizable by the wavevectors $|k| = k_0 = 2\sqrt{m\hbar k}/\gamma$, about which the boson condensation takes place. However, $\langle \alpha; t \mid n_{i,k}\mid \alpha; t \rangle$ is not a direct measure of dissipation because $\langle \alpha; t \mid n_{i,k}\mid \alpha; t \rangle$ vanishes at $\bar{k} = \sqrt{2}$ where dissipation is also taking place. Physical meaning of this phenomenon associated with our ground state is unknown.

At the critical damping ($\bar{k} = 1$), the BDS allows two independent motions

\[
\xi_k(t) = e^{-\frac{\gamma k}{2m}}t, \quad \text{te}^{-\frac{\gamma k}{2m}t}.
\]

The first motion in (6.16) is the one generated by the equation of motion

\[
\dot{\xi}_k + \frac{\gamma k}{2m} \xi_k = 0.
\]

This is equivalent to the solution to the equation of massless BDS. Thus, we again observe the correspondence of the divergence of $\langle \alpha; t \mid n_{i,k}\mid \alpha; t \rangle$ at $\bar{k} = 1$ with the involvement of genuinely dissipative motion.

We regard $\bar{k}$ as the mechanical energy operator that acts on the Fock space spanned by $\{|\eta_k\rangle = \prod_k \hat{a}_{\eta k}^{(h)}(t)/\sqrt{nk} \mid \alpha; t \rangle$. Thermal properties of the system will be deduced from $\bar{k}$ by assigning the Bose distribution to the quanta. On the other hand, in order to take a closer look at the dynamics including dissipation, autocorrelation functions are suitable. We will explore this aspect of the system in the next section.

6.2. Energy in the underdamped region

We saw that the effect of dissipation emerges most eminently as infinite condensations of $a$- or $\alpha$-quanta at the massless or near the critical point. Namely, the energy per boson near these points becomes infinitesimal. Our interest in this subsection is in what is happening in the underdamped region, $\bar{k} < 1$.

The kinematic energy operator to be adopted has the same expression as (6.1). $\lambda_{i,k}$ for the underdamped region has been defined by (4.1). This means that any physical quantities are gained from the ones in the overdamped region by a replacement $\sqrt{\bar{E}^2 - 1} \to i\sqrt{1 - \bar{E}^2}$ in the arguments in section 3. But now the definitions for $D_{\alpha k}^{(o)}$ are changed. By substituting the coordinate operators (4.4) to the energy operator (6.1), it is readily confirmed that $D_{\alpha k}^{(o)}$ in the underdamped case is given by a replacement $1 - w_k \to w_k - 1$ in $D_{\alpha k}^{(o)}$ for the overdamped case as
The ground state is given by (6.9) with replacing $D^{(a)}_{\omega,k}$ by $D^{(a)}_{\omega,k}$ defined by (6.18). $|D^{(a)}_{\omega,k}|$ monotonically increases from minimum value to infinity as $k$ increases from 0 to 1. The number of $\alpha$-quanta, which is now given by

$$n^{(a)}_\alpha(k) = \frac{D^{(a)}_{\omega,k}}{4}$$

in accordance with (6.13), diverges as $k \to 1$, which signifies the boson condensation. Since $k_0 \equiv 2\sqrt{m\ell}/\gamma$, $k \to 1$ in the conservative limit $\gamma \to 0$ means $|k| \to \infty$. Namely, the bosons that condense in physical region is associated with the dissipation.

In the limit $k \to 0$, $D^{(a)}_{\omega,k}$ and the number of $\alpha$-quanta vanish. $\langle n^{(a)}_\alpha(k) \rangle$ as a function of $k$ is depicted in figure 1. The total number density within $0 \leq k < 1$ is given for $d$-dimension by

$$n^{(a)}_\alpha \equiv \frac{k_0^d}{(2\pi)^d} \int_k \tilde{\kappa}^d \langle n^{(a)}_\alpha(k) \rangle = C_d k_0^d,$$  

$$C_1 = \frac{1}{8\pi} \left( \frac{3\pi}{4} - 2 \right), \quad C_2 = \frac{1}{24\pi}, \quad C_3 = \frac{1}{8\pi^2} \left( \frac{5\pi}{16} - \frac{2}{3} \right).$$

As in the overdamped case, the divergence of the number of $\alpha$-quanta occurs at the boundary of the over- and underdamped regions. Nevertheless, 'physical' quantities constructed by $A_k$ and $A_k$ are well-defined everywhere.

7. Autocorrelation

7.1. Overdamped region

Since the eigenvalues of the energy operator $\hat{K}(t)$ do not depend on time, a question arises: where is the dissipation? The answer will be found by inspecting the property of the ground state.

The inner product is easily calculated by expanding $|\alpha; t \rangle$ and $|\alpha; t \rangle$ in a power series of $R_k$ and by noting the commutation relation derivable from (3.20) and (6.8a)

$$[\alpha_k(t), \tilde{\alpha}_k(0)] = g^{\alpha}_k(t),$$

as

$$\langle \alpha; t | \alpha; 0 \rangle = \prod_{k} \left[ \frac{1}{\tilde{\kappa}^d} \sum_{m=0}^\infty \frac{(-R_k)^{n+m}}{n!m!} \delta_{n,m} g^{\alpha}_k(t)^{2n} \right] = \prod_{k} \frac{g^{\alpha}_k(t)^2}{1 - R_k g^{\alpha}_k(t)^2},$$

where $g^{\alpha}_k(t)$ is defined by (3.23b) and the $(k)$ under the product symbol denotes that $-k$ is excluded when $k$ is included. In deriving (7.2), a use has been made of a fact that the terms of the same power of $-R_k$ and $\delta_{k}(t)\tilde{\delta}_{-k}(t)$ appearing in the power series expansion of $|\alpha; 0 \rangle$ and $|\alpha; t \rangle$ in $R_k$ contribute to the inner product.

$(\alpha; t | \alpha; 0)$ does not vanish for $t \to \infty$ but becomes equal to

$$\langle \alpha; \infty | \alpha; 0 \rangle = | \langle \alpha; t | \alpha; 0 \rangle |^2 = \prod_{k} \frac{g^{\alpha}_k(t)^2}{1 - R_k g^{\alpha}_k(t)^2}.$$

Namely, the ground state $|\alpha; t \rangle$ does not completely lose its information at $t = 0$. This implies that the component that remains persistent is $|0 \rangle$, the vacuum of $\alpha$-quanta. This feature is analogous to the relaxation process observed in viscoelasticity: after the external load is removed, the initial strain decays with time and approaches a smaller constant value.

We define the dimensionless autocorrelation function for the coordinate operators in the ground state by

$$G^{\alpha}_k(t) = \frac{\langle \gamma_k \delta^{\alpha}_k / \hbar | \alpha_k(t) \tilde{\zeta}_k(0) | \alpha; 0 \rangle \langle \alpha; t | \alpha; 0 \rangle^{-1}}{t > 0},$$

$$G^{\alpha}_k(t) = \frac{\langle \gamma_k \delta^{\alpha}_k / \hbar | \alpha_k(t) \tilde{\zeta}_k(0) | \alpha; 0 \rangle \langle \alpha; t | \alpha; 0 \rangle^{-1}}{t < 0}.$$  

From the viewpoint of the viscoelasticity, $G^{\alpha}_k(t)$ is the contribution from a quantum excitation of mode $k$ to the correlation of strains at different times. Using the expressions (3.6a) for $\zeta_k(t)$, together with (6.8), we rewrite $\zeta_k(t) = \zeta_{k,t} + \tilde{\zeta}_{k,t}$ as

$$\zeta_k(t) = \sqrt{\frac{\hbar}{\gamma_k \delta^{\alpha}_k}} (\alpha_k(t) - \tilde{\alpha}_{-k}(t)).$$

Substituting (7.5) into (7.4), $G^{\alpha}_k(t)$ with $t > 0$ is written as

$$G^{\alpha}_k(t) = -\langle \alpha; t | (\alpha_{-k}(t) - \tilde{\alpha}_{k}(t)) (\alpha_k(0) - \tilde{\alpha}_{-k}(0)) | \alpha; 0 \rangle \langle \alpha; t | \alpha; 0 \rangle^{-1}.$$  

(7.6)
To calculate (7.6), we use the following formulæ

$$\langle \mathbf{0}; t \mid \hat{\alpha}_k(t) \hat{\alpha}_k(0) \mid \mathbf{0}; 0 \rangle = \langle \mathbf{0}; t \mid \mathbf{0}; 0 \rangle \frac{R_{kk}^{(0)}(t)}{1 - R_{kk}^{(0)}(t)^2},$$  \hspace{1cm} (7.7a)

$$\langle \mathbf{0}; t \mid \alpha_k(t) \hat{\alpha}_k(0) \mid \mathbf{0}; 0 \rangle = \langle \mathbf{0}; t \mid \mathbf{0}; 0 \rangle \frac{\mathcal{G}_k^{(0)}(t)}{1 - R_{kk}^{(0)}(t)^2},$$  \hspace{1cm} (7.7b)

$$\langle \mathbf{0}; t \mid \alpha_k(t) \alpha_{-k}(0) \mid \mathbf{0}; 0 \rangle = \langle \mathbf{0}; t \mid \mathbf{0}; 0 \rangle - \langle \mathbf{0}; t \mid \mathbf{0}; 0 \rangle \frac{R_{kk}^{(0)}(t)}{1 - R_{kk}^{(0)}(t)^2}.$$  \hspace{1cm} (7.7c)

Derivations of (7.7a)–(7.7c) are presented in the appendix. From (7.6) and (7.7), we readily obtain

$$G_k^{(0)}(t > 0) = \frac{(1 + R_k)^2 \mathcal{G}_k^{(0)}(t)}{1 - R_k^{(0)}(t)^2},$$  \hspace{1cm} (7.8)

where \(\mathcal{G}_k^{(0)}(t)\), which is essentially the contribution from the bare vacuum, is given by (3.23b). The expression for \(G_k^{(0)}(t < 0)\) is obtained by the replacement \(t \to -t\) on the rhs of (7.8). \(R_k\) in the overdamped region takes the form

$$R_k = - \frac{\bar{\mathcal{K}}^2 - 2}{\bar{\mathcal{K}}^2 + 2 \sqrt{\mathcal{K}^2 - 1}},$$  \hspace{1cm} (7.9)

\(R_k \to -1\) for \(\bar{\mathcal{K}} \to 1\) or \(\mathcal{K} \to 1\), so that \(G_k^{(0)}(t)\) vanishes in this limit. This implies that the condensed vacuum contribution completely cancels the bare vacuum contribution when the critical point is approached.

\(G_k^{(0)}(t)\) decays exponentially without oscillations. Accordingly, its Fourier transform

$$\tilde{G}_k^{(0)}(\omega) \equiv \int_0^\infty G_k^{(0)}(t) \cos(\omega t) dt$$  \hspace{1cm} (7.10)

has a distribution centered at \(\omega = 0\).

### 7.2. Underdamped region

The dimensionless autocorrelation function takes the same form as (7.6). The expressions for \(D_{\omega k}^{(0)}\) are presented in (6.17). For the full autocorrelation function, we thus obtain for the Fourier transform of \(G_k^{(0)}(t)\):

$$G_k^{(0)}(t > 0) = \frac{(1 + R_k)^2 \mathcal{G}_k^{(0)}(t)}{1 - R_k^{(0)}(t)^2},$$  \hspace{1cm} (7.11a)

$$\tilde{G}_k^{(0)}(\omega) = \tilde{G}_k^{(0)}(\omega) \big|_{\omega \to -\omega} = \int_0^\infty G_k^{(0)}(t) \cos(\omega t) dt.$$  \hspace{1cm} (7.11b)

The underdamped counterparts of the physical quantities in the overdamped case are obtained by the replacement \(\sqrt{\bar{\mathcal{K}}^2 - 1} \to i \sqrt{1 - \bar{\mathcal{K}}^2}\) in \(\lambda_{\omega k}\). At the same time, from the arguments in section 5, \(R_k\) for the underdamped case takes the form

$$R_k = - \frac{1 - \sqrt{1 - \mathcal{K}^2}}{1 + \sqrt{1 - \mathcal{K}^2}}.$$  \hspace{1cm} (7.12)

Accordingly, the Fourier transform \(\tilde{G}_k^{(0)}(\omega)\) is given by (4.5). As for the functional behavior of \(\tilde{G}_k^{(0)}(\omega)\), we have to resort to numerical calculations.

The time-dependence of \(G_k^{(0)}(t)\) is shown in figure 2(a) for several \(\bar{\mathcal{K}}\). The oscillatory behaviors are due to the imaginary part of the decay rate:

$$\text{Im} \lambda_{\omega k} = \pm (2\kappa/\gamma) \bar{\mathcal{K}} \sqrt{1 - \mathcal{K}^2}.$$  \hspace{1cm} (7.13)

The dispersion relation is expected to be governed by this imaginary part. \(G_k^{(0)}(t)\) above \(\bar{\mathcal{K}} \approx 0.8\), although not depicted here, comes to lose the feature of oscillation.

The \(\bar{\mathcal{K}}\)-dependences of \(G_k^{(0)}(t)\) are shown in figure 2(b). For small \(\bar{\mathcal{K}}, G_k^{(0)}(t)\) monotonically decreases. Oscillatory behavior emerges for \(\bar{\mathcal{K}} > 2.9\). \(G_k^{(0)}(t)\) vanishes at \(\bar{\mathcal{K}} = 1\) and is continuously connected to \(G_k^{(0)}(t)\) at this point. As in the overdamped case, the contributions from the bare vacuum and the condensation completely cancels each other at \(\bar{\mathcal{K}} = 1\). In the region of small \(\bar{\mathcal{K}}, |R_k|\) is also small, such that the bare vacuum contribution \(\mathcal{G}_k^{(0)}(\omega)\) dominates over the condensation effect.

The functional behavior of the dispersion relation will be obtained by finding the location of the maximum of \(\tilde{G}_k^{(0)}(\omega)\). For this purpose, we performed the numerical integrations in (7.11b) to see the \(\omega\)-dependences of \(\tilde{G}_k^{(0)}(\omega)\). Some results are shown in figure 3.
Each curve of $\tilde{G}_k^{(\omega)}(t)$ shown in figure 3 has a maximum in the region $0 < \tilde{\omega} < 1$. The peak is sharp for small $\tilde{K}$ and becomes broader for larger $\tilde{K}$. The dispersion relation $\tilde{\omega}(\tilde{K})$ is constructed by finding a correspondence between the value of $\tilde{K}$ and the peak position. The result is shown in figure 1.

Two aspects of the effect of dissipation emerge. First, the one to one correspondence of the frequency to the wavenumber becomes subtle in the shorter wavelength region owing to the broadening of the width. Second, the
curve of \( \pi(\vec{k}) \) bends downward at smaller \( \vec{k} \) than the ‘bare’ dispersion relation. This feature is due to the existence of the damping factor \( \exp(-\gamma/2m)t \) in the mode operators, which gives rise to the condensation of \( \alpha \)-quanta in the energy ground state.

### 8. Conclusion

The purpose of the present paper was to examine how the BDS manifests its quantum properties in observable ways. BDS, when extended to a one component continuum system, is regarded as the simplest field theory of viscoelastic material. With the wavevector being a variable parameter, the model covers the massless, underdamping, and overdamping motions. We quantized the Bose field variable in accordance with the canonical procedure [6, 7]. Two kinds of bosons, which we called \( a \)-quanta, emerge as eigenmodes associated with two branches of the quantized Hamiltonian for the damping (harmonic) motions.

The two branches mutually intersect at the critical point where the wavefunction is singular. We confirmed these two characteristics to be consistent with each other in that the wavefunction generally develops a non-trivial ‘geometrical phase’ factor when the model parameters are varied on a closed loop that avoids the singular point, as is suggested by Berry’s theorem [10]. Anholonomy of this kind is expected to manifest itself in one dimension.

We then explored the consequence of quantization by constructing the kinematic energy operator by analogy with classical mechanics. The energy operator was diagonalized in terms of annihilation and creation operators of an \( \alpha \)-quantum, which is a superposition of \( a \)-quanta. The ground state was formed as a state of an exponential condensation of pairs of \( \alpha \)-quanta.

In the underdamped region, the spectra of the eigenstates and eigenvalues of the energy operator are nothing but those of the acoustic phonon. There exist critical wavevectors near which the condensation of \( \alpha \)-quanta is extremely enhanced in the energy eigenstates. The dispersion relation was deduced from the autocorrelation function and was shown to coincide with that of the acoustic phonon. There exist critical wavevectors near which the condensation of pairs of \( \alpha \)-quanta is exponential.

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In the underdamped region, the spectra of the eigenstates and eigenvalues of the energy operator are nothing but those of the acoustic phonon. There exist critical wavevectors near which the condensation of \( \alpha \)-quanta is extremely enhanced in the energy eigenstates. The dispersion relation was deduced from the autocorrelation function and was shown to coincide with that of the acoustic phonon at low and intermediate wave numbers, but exhibited marked deviations at larger wave numbers. The dispersion relation is cutoff at \( \sqrt{3}/2 \) times the critical wavevector. The eigenenergy and the peak width of the autocorrelation are in geometrical phase.

We then explored the consequence of quantizing the simplest Bateman model of a continuum with the aim of finding a criterion for the relevance of a quantized Bateman system to the real physics, although some refinement of the model will be necessary for quantitative comparison with experimental observations.

### Appendix

**Derivation of (7.7a)–(7.7c)**

We separate the non-trivial contributions to \( \langle \sigma; t | \hat{a}_k(t) \alpha_k(0) | \sigma; 0 \rangle \) by writing

\[
\langle \sigma; t | \hat{a}_k(t) \alpha_k(0) | \sigma; 0 \rangle = \mathcal{E}_{k}^2 \exp[-R_k \alpha_k(t) \alpha_{-k}(t)] \langle \hat{a}_k(t) \alpha_k(0) \rangle \\
\times \exp[-R_k \hat{a}_k(0) \hat{a}_{-k}(0)] \prod_{k', k''} \left[ \frac{\mathcal{E}_{k'}^2 \mathcal{E}_{k''}^2}{\mathcal{E}_{k}^2 \mathcal{E}_{k''}^2(k- \lambda k''(t))} \right] \\
\times \exp \left[ \frac{-1}{2} R_k \hat{a}_k(0) \hat{a}_{-k}(0) \right] \\
\langle \sigma; 0 | \hat{a}_k(t) \alpha_{-k}(0) | \sigma; 0 \rangle \right],
\]

(A1)

The last two lines are the transition amplitude \( \langle \sigma; t | \sigma; 0 \rangle \) of the ground state with the contributions of \( k \) and \( -k \) modes being excluded. Here, we use the formula

\[
\langle 0 | \exp[-R_k \alpha_k(t) \alpha_{-k}(t)] \exp[-R_k \hat{a}_k(0) \hat{a}_{-k}(0)] | 0 \rangle^{-1} = 1 - R_k^2 \mathcal{E}_k^2(t)^2,
\]

(A2)

where, after expanding the exponential factors in power series of \( R_k \), use has been made of the commutation relation (7.1). The remaining factor on the rhs of (A1) is calculated in a similar way.
\[ \mathcal{G}_k^2(0) \exp \left[ - R_k \alpha_k(t) \alpha_{-k}(t) \right] \alpha_k(t) \alpha_k(0) \exp \left[ - R_k \alpha_k(0) \alpha_{-k}(0) \right] = \mathcal{G}_k^2 \sum_{l,m} (-1)^{l+m} \frac{R_k^l R_k^m}{l! m!} \left( \alpha_k(t) \alpha_{-k}(t) \alpha_k(0) \alpha_{-k}(0) \right)^{l+m} \]

\[ = \mathcal{G}_k^2 \sum_{m} R_k^{2m} m g_m^{(0)}(t)^{2m-1} = \mathcal{G}_k^2 \frac{R_k^2(t) g_0^{(0)}(t)}{(1 - R_k^2(t) g_0^{(0)}(t)^2)^2}. \]  

(A3)

Combining (A2) and (A3), we have (7.7a).

Equation (7.7b) is readily obtained from (7.7a) by noting that

\[ \langle 0 | \exp \left[ - R_k \alpha_k(t) \alpha_{-k}(t) \right] \alpha_k(t) \alpha_k(0) \exp \left[ - R_k \alpha_k(0) \alpha_{-k}(0) \right] | 0 \rangle = R_k^{-2} \langle 0 | \exp \left[ - R_k \alpha_k(t) \alpha_{-k}(t) \right] \alpha_k(t) \alpha_k(0) \exp \left[ - R_k \alpha_k(0) \alpha_{-k}(0) \right] | 0 \rangle. \]

Equation (7.7c) are also derived in a similar way.

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