Explicit Cosmological Coarse Graining via Spatial Averaging

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March 21, 2022

Abstract

The present matter density of the Universe, while highly inhomogeneous on small scales, displays approximate homogeneity on large scales. We propose that whereas it is justified to use the Friedmann-Lemaître-Robertson-Walker (FLRW) line element (which describes an exactly homogeneous and isotropic universe) as a template to construct luminosity distances in order to compare observations with theory, the evolution of the scale factor in such a construction must be governed not by the standard Einstein equations for the FLRW metric, but by the modified Friedmann equations derived by Buchert [7, 8] in the context of spatial averaging in Cosmology. Furthermore, we argue that this scale factor, defined in the spatially averaged cosmology, will correspond to the effective FLRW metric provided the size of the averaging domain coincides with the scale at which cosmological homogeneity arises. This allows us, in principle, to compare predictions of a spatially averaged cosmology with observations, in the standard manner, for instance by computing the luminosity distance versus red-shift relation. The predictions of the spatially averaged cosmology would in general differ from standard FLRW cosmology, because the scale-factor now obeys the modified FLRW equations. This could help determine, by comparing with observations, whether or not cosmological inhomogeneities are an alternative explanation for the observed cosmic acceleration.

1 Introduction

The goal of Cosmology is to describe the evolution history of the Universe as a whole, tracing it as far back as possible from the present epoch. This seemingly daunting task is made tractable by the fact that the observed matter density in the Universe appears to be approximately homogeneous when viewed on the largest scales. Further, observations of the Cosmic Microwave Background (CMB) radiation reveal it to be isotropic (after subtracting a dipole contribution due to our local motion) down to the level of one part in $10^5$. The twin conditions of homogeneity and isotropy, if valid on all length scales, would immensely simplify the form of the metric and matter tensors used in the General Relativistic description of Cosmology. It can be shown [1] that on purely geometric grounds, the most general metric describing a homogeneous and isotropic universe takes on the form of the Friedmann-Lemaître-Robertson-Walker (FLRW) line

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element. The standard method of proceeding is to assume that the geometry of the universe can be described by a background metric with small perturbations, where the background is exactly homogeneous and isotropic and is hence of the FLRW form. (For a detailed discussion of the assumptions involved in the standard Cosmology, see Ref. [2].)

This approach is adopted not only for the early universe, for example in describing Big Bang nucleosynthesis and the dynamics of the CMB anisotropies, but is also assumed to be valid at recent epochs. In particular, the luminosity distances constructed to compare data from Type Ia Supernovae (SNe) with theory, assume a metric of the FLRW form with its dynamics given by the Einstein equations for a homogeneous and isotropic universe. We know, however, that the matter content of the Universe today is not homogeneous on small scales - one finds inhomogeneity at the scale of individual galaxies for example; and there is evidence [3] that voids of order $30h^{-1}\text{Mpc}$ in diameter account for 40-50% of the observed Universe. There is also evidence for voids 3-5 times this size [4], as well as local voids on smaller scales [5]. The evidence for homogeneity of the matter distribution in the Universe only appears at scales larger than about $100h^{-1}\text{Mpc}$ or so [6]. It is then reasonable to assume that the metric describing our Universe must also reflect the inhomogeneity of the matter on the corresponding scales. One may now ask whether it is at all justified to use the luminosity distances constructed using the FLRW metric with its associated dynamics to compare theory with observational data from SNe. In this paper we argue that the use of such luminosity distances is indeed justified, provided we are only interested in using such constructions in probing length scales larger than a certain minimum scale which is fixed by requiring that the assumption of homogeneity be valid (a lower bound on such a scale could be $100h^{-1}\text{Mpc}$ for example). More importantly, we propose that the dynamics of this (effective or template) macroscopic metric must be governed not by the standard Einstein equations applied to the FLRW metric, but by the effective equations obtained after spatial averaging on constant time slices, as developed by Buchert [7, 8].

An outstanding open issue has been the following: how does one compare the predictions of the inhomogeneous, spatially averaged cosmology with observations, and should one be attempting an interpretation in terms of spatially averaged variables in the first place? For instance, one could study light propagation and determine the luminosity distance versus redshift ($D_L(z)$) relation which best fits the observations, within an inhomogeneous model of the Universe. Following the important and interesting early work of Dyer and Roeder, where the angular-size and luminosity distances were obtained for beams propagating in "lumpy" FLRW universes [9], detailed investigations have been carried out in [10]. Such a program has also received considerable attention in the context of the spherically symmetric inhomogeneous Lemaitre-Tolman-Bondi (LTB) models [11, 12]. However, it is not clear to us that this is necessarily the best way to proceed, at least as far as distant SNe are concerned. A realistic computation of light propagation in an inhomogeneous geometry seems difficult at best, and while the use of simplified toy models such as the LTB solutions in this endeavor may be justified on the grounds that we may inhabit a locally underdense region [4], it is not clear that the assumptions justifying the use of the toy model remain valid at large scales (see also Ref. [12] for a recent review of the LTB approach). And after all, the concordance model infers the presence of a dark energy by comparing the predictions of the standard homogeneous and isotropic FLRW cosmology against observations [13]. If one decides to interpret the FLRW cosmology as a template against which observations may be compared, then in order to decide whether

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Here $h$ is the conventional parameter which appears in the definition of the Hubble constant $H_0 = 100h \text{km s}^{-1}\text{Mpc}^{-1}$. 
cosmic inhomogeneities could be an alternative to the concordance model, one must construct a homogeneous and isotropic cosmology, after a suitable smoothing procedure, and compare its predictions with observations such as the $D_L(z)$ relation for Type Ia SNe.

One should note that the effects of averaging within the standard perturbed FLRW framework of Cosmology, have been studied by several authors (see, e.g., Refs. [14, 15]), and it has been argued that these effects must necessarily be small. The papers in Refs. [14, 15] however, do not necessarily address the entire problem. For example, Ref. [14] deals with only superhorizon fluctuations, whereas work done by others (see, e.g., Ref. [16]) indicates that there may be nontrivial contributions due to averaging from subhorizon fluctuations where the matter perturbations have gone nonlinear. And while the authors of Ref. [15] construct a cosmological model with negligible effects of averaging, there do exist toy models of inhomogeneous spacetimes in which these effects are large (see, e.g., Ref. [17]).

Furthermore, the possibility of nonperturbative effects of averaging has not, to our knowledge, been satisfactorily addressed, much less ruled out. It is important to realize that current $N$-body simulations for example, which are based on a Newtonian approximation of gravity, consistently miss possible relativistic effects of averaging. This is because the modifications due to averaging in a Newtonian setting can be shown to reduce to boundary terms in general [18], and to vanish in the case of spherical symmetry [19]. These terms therefore do not contribute in simulations which either set periodic boundary conditions on the simulation box, or use a spherically symmetric simulation region. As we will see, the “backreaction” effects in Buchert’s framework (i.e., the modifications to the Cosmological equations due to averaging) are coupled to the average of the spatial curvature, this being a completely relativistic effect. Nontrivial consequences of this coupling deserve more attention than they have received in the context of $N$-body simulations (see Ref. [19]).

We also note that an argument for describing the Universe using FLRW plus Newtonian perturbations (e.g., as given in Ref. [20]), is based on the fact that it is Newtonian perturbations and a negative pressure component, which describe the Universe very well. In other words, one argues that the effect of averaging inhomogeneities must be small, but at the cost of adding an ill-understood Dark Energy component in the Universe. On the other hand one should allow for the following possibility (as of now neither confirmed nor ruled out): at an initial epoch, say at the end of inflation, the perturbations in the homogeneous metric would give a small “backreaction” as a result of averaging. As a result of its coupling with the curvature, this backreaction would grow and become significant around the time of structure formation, when matter inhomogeneities on small scales become nonlinear. The backreaction would also nontrivially affect the growth of the metric perturbations. Such a possibility could only be tested within a fully nonperturbative treatment of the evolution of inhomogeneities.

The goal of this paper is modest. We argue that the scale factor, defined in the spatially averaged cosmology via the volume of a domain, should correspond to an effective FLRW metric, provided the size of the averaging domain coincides with the scale at which cosmological homogeneity arises. This allows us, in principle, to compare predictions of a spatially averaged cosmology with observations, in the standard manner, for instance by computing the $D_L(z)$ relation. (However, see Ref. [21] for an alternative approach to matching the spatially averaged dynamics with observations.) The predictions of the spatially averaged cosmology would in general differ from standard FLRW cosmology, because the scale-factor now obeys modified Einstein equations. We will demonstrate that consistent models of spatially averaged FLRW templates exist which can, at least in principle, explain observations such as those of the $D_L(z)$
relation for Supernovae.

In a nutshell, our new results in this paper are that we propose a template metric which takes into account the kinematical effects of inhomogeneities in the Universe. To implement this idea, we first suggest that in Buchert’s spatial averaging scheme, the size of the averaging domain should coincide with the cosmological scale at which homogeneity sets in. We then argue that, as a first step towards predicting the relation between inhomogeneities and observations, the expansion factor $a_D(t)$ defined by Buchert should be identified with the scale-factor $a(t)$ in standard FLRW cosmology. The template metric is then the FLRW metric, with $a_D(t)$ playing the role of $a(t)$. We then show, with the help of a few toy examples, that the luminosity distance - redshift relation in a model Universe with an accelerating $a_D(t)$ can be computed in the standard manner. To the best of our knowledge, such a template metric has not been proposed in earlier work by other researchers.

2 The Coarse Grained Picture and Cosmology

At the length scales of cosmological interest, individual galaxies are small enough to be treated as point-like objects. Given the large number ($\sim 10^{11}$) of galaxies estimated to exist in the observed Universe, it is also assumed that the matter content of the Universe can be described to a good approximation as a continuum fluid. At the level of the matter distribution, this picture is comparable to that of a gas composed of a large number of molecules, wherein the internal structure of the molecules can be ignored when dealing with the macroscopic properties of the gas which is treated as a continuum fluid. Further, the complex intermolecular interactions in the gas at the microscopic level can be dealt with in a mean field approximation by considering volume averages of the quantities of interest.

This picture is made precise by introducing the concept of physically infinitesimal volumes or coarse-graining cells. The fluid approximation is assumed to be valid provided the (effectively) infinitesimal volumes used in defining continuum quantities such as the matter density of the fluid, correspond to physical volumes which are large enough to contain a very large number of the particles constituting the fluid. At the same time, since these volumes are to be effectively infinitesimal, their physical size must also be much smaller than the scale of the fluid as a whole. We can then say that the length scales $L$ being probed by these physically infinitesimal volumes must lie in a certain range $L_1 < L < L_2$ which is determined by the details of the fluid (see also [23]). Clearly we must satisfy $L_2 \ll L_{syst}$ where $L_{syst}$ is the size of the system. In the cosmological context, $L_{syst}$ would be the Hubble scale, and in order to deal with homogeneous and isotropic effective descriptions we would also have to satisfy $L_1 \gtrsim L_{hom}$ where the homogeneity scale $L_{hom}$ as mentioned earlier can be as large as $100h^{-1}$ Mpc. The physical quantities we wish to deal with, such as the matter density, pressure, fluid velocity field, etc. are defined by suitably averaging or coarse graining over the physically infinitesimal volumes (hence the terminology of coarse-graining cells (CGCs)). For example, the matter density of a given CGC is defined as the mass contained in it divided by its physical volume. The (effectively) continuum matter density is built up by repeating this for every CGC in the fluid.

Since we deal with the notion of a continuum fluid in Cosmology, clearly we need a working description of coarse graining appropriate in this context. However, the underlying theory

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2 In the cosmological context, these length scales will be probed by the metric. Since we will soon specialize to a metric written in comoving coordinates, these length scales must be understood to be comoving scales.
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\[ \mathbf{g}_{\text{mic}} \rightarrow \mathbf{g}_{\text{mac}} \]
\[ \mathbf{G}_{\text{mic}} = \mathbf{T}_{\text{mic}} \rightarrow \mathbf{G}_{\text{mac}} = \mathbf{T}_{\text{mac}} \]

Figure 1: Symbolic depiction of the modifications to Einstein’s equations caused by an explicit smoothing. Adapted from Ellis [24], fig. 7. See text for details.

being General Relativity, there are problems. The crucial issue is that along with the matter distribution (which is now tensorial), the metric for the spacetime also needs to be coarse grained. As discussed by Ellis [24], in defining a smoothing operation to go from the scale of stars (Scale 1 in Ellis’ terminology, microscopic in ours) to cosmological scales where homogeneity can be assumed to have set in (Scale 5 for Ellis, macroscopic for us), one would employ two logically different smoothing operators, one each on the metric and the matter tensor. We denote the smoothing operator acting on the metric as \( S_{\text{met}} \) and the one acting on the matter tensor as \( S_{\text{matt}} \) (our notation differs from that used in Ref. [24]). We assume that Einstein’s equations are valid on the microscopic scale of stars, which is reasonable since it is at these scales that Solar System tests and binary pulsar studies probe General Relativity. Symbolically then (see fig. 1), the metric \( \mathbf{g}_{\text{mic}} \) at this scale is used to construct the Einstein tensor \( \mathbf{G}_{\text{mic}} \) which is equated to the matter tensor \( \mathbf{T}_{\text{mic}} \), giving Einstein’s equations \( \mathbf{G}_{\text{mic}} = \mathbf{T}_{\text{mic}} \). To obtain the corresponding description at the macroscopic scale, we use the smoothing operators to get a new metric tensor \( \mathbf{g}_{\text{mac}} = S_{\text{met}}(\mathbf{g}_{\text{mic}}) \) and a matter tensor \( \mathbf{T}_{\text{mac}} = S_{\text{matt}}(\mathbf{T}_{\text{mic}}) \). The macroscopic metric \( \mathbf{g}_{\text{mac}} \) is used to construct the Einstein tensor \( \mathbf{G}_{\text{mac}} \) at the macroscopic scale. Formally, one can think of the map \( S_{\text{met}} \) as inducing a map \( S_{\text{Einstein}} \) which acts on the microscopic Einstein tensor \( \mathbf{G}_{\text{mic}} \) to yield \( \mathbf{G}_{\text{mac}} = S_{\text{Einstein}}(\mathbf{G}_{\text{mic}}) \). Since the operators \( S_{\text{Einstein}} \) and \( S_{\text{matt}} \) are in general different, we have \( \mathbf{G}_{\text{mac}} \neq \mathbf{T}_{\text{mac}} \). Hence, if the Einstein equations are valid on a given length scale, then they are in general not valid on some other length scale connected to the first by appropriately defined smoothing operators. If we can re-express the new Einstein tensor \( \mathbf{G}_{\text{mac}} \) in terms of quantities obtained by acting the operator \( S_{\text{matt}} \) on both sides of the Einstein equations at the microscopic scale, then it should be possible to write \( \mathbf{G}_{\text{mac}} = \mathbf{T}_{\text{mac}} + \mathbf{P}_{\text{mac}} \) where \( \mathbf{P}_{\text{mac}} \) represents corrections to the Einstein equations arising from an explicit smoothing. In the standard approach, it is assumed that these corrections are small enough to be neglected. (It should be emphasized that given the recent evidence for voids in the Universe, one or more additional scales may eventually have to be included in the averaging scheme.)

Added to all this is the complication that smoothing or averaging operations are notoriously difficult to define for tensor quantities in a gauge covariant manner. Some examples of averaging techniques employed in General Relativity (and in Cosmology in particular) can be found in Ref. [23]. Here we restrict ourselves to the formalism developed by Buchert [7, 8] for volume
averages of scalar quantities in a given choice of time-slicing. In the following sections we recall this formalism and argue that it is naturally adapted to the notion of the smoothing operator for the matter sector. We will reserve the term “averaging” for the matter smoothing defined in this way. As for the smoothing of the metric (which we will refer to as “coarse graining”), we will take recourse to cosmological observations and argue that the result of such an operation on the inhomogeneous metric must yield the FLRW metric on a suitable coarse graining scale.

3 Averaging in a Dust Dominated Spacetime

For simplicity we restrict our attention to a pressure-less fluid or “dust”, although a generalization to fluids with non-zero pressure is also possible (see Ref. [8]). For a general spacetime containing irrotational dust, the metric can be written in synchronous and comoving gauge:

$$ds^2 = -dt^2 + h_{ij}(\vec{x}, t) dx^i dx^j.$$ (1)

The expansion tensor $\Theta^i_j$ is given by $\Theta^i_j \equiv (1/2)h^{ik}h_{kj}$ where the dot refers to a derivative with respect to (proper) time $t$. The traceless symmetric shear tensor is defined as $\sigma^i_j \equiv \Theta^i_j - (\Theta/3)\delta^i_j$ where $\Theta = \Theta^i_i$ is the expansion scalar. The Einstein equations can be split [7] into a set of scalar equations and a set of vector and traceless tensor equations. The scalar equations are the Hamiltonian constraint (2a) and the evolution equation for $\Theta$ (2b),

$$(3) \mathcal{R} + 2/3 \Theta^2 - 2\sigma^2 = 16\pi G \rho, \quad (2a)$$

$$(3) \mathcal{R} + \dot{\Theta} + \Theta^2 = 12\pi G \rho, \quad (2b)$$

where $(3)\mathcal{R}$ is the Ricci scalar of the 3-dimensional hypersurface of constant $t$, $\sigma^2$ is the rate of shear defined by $\sigma^2 \equiv (1/2)\sigma^i_j \sigma^j_i$ and $\rho$ is the matter density of the dust. Note that all quantities in Eqns. (2) generically depend on both position $\vec{x}$ and time $t$. Eqns. (2a) and (2b) can be combined to give Raychaudhuri’s equation

$$\dot{\Theta} + 1/3 \Theta^2 + 2\sigma^2 + 4\pi G \rho = 0.$$ (3)

The continuity equation $\dot{\rho} = -\Theta \rho$ which gives the evolution of $\rho$, is consistent with Eqns. (2a), (2b). We only consider the scalar equations, since the spatial average of a scalar quantity can be defined in a gauge covariant manner within a given foliation of space-time. For the space-time described by (1), the spatial average of a scalar $\Psi(\vec{x}, t)$ over a comoving domain $D$ at time $t$ is defined by

$$\langle \Psi \rangle _D = \frac{1}{V_D} \int_D d^3 x \sqrt{h} \Psi,$$ (4)

where $h$ is the determinant of the 3-metric $h_{ij}$ and $V_D$ is the volume of the comoving domain given by $V_D = \int_D d^3 x \sqrt{h}$. Spatial averaging is, by definition, not generally covariant. Thus the choice of foliation is relevant, and should be motivated on physical grounds. In the context of cosmology, averaging over freely-falling observers is a natural choice, especially when one intends to compare the results with standard FLRW cosmology. Following the definition (4) the following commutation relation then holds [7]

$$\langle \Psi \rangle _D - \langle \dot{\Psi} \rangle _D = \langle \Psi \Theta \rangle _D - \langle \Psi \rangle _D \langle \Theta \rangle _D,$$ (5)

Latin indices take values 1..3, Greek indices take values 0..3. We set $c = 1$. 

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which yields for the expansion scalar $\Theta$

$$\langle \Theta \rangle_D - \langle \dot{\Theta} \rangle_D = \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 .$$  \hspace{1cm} (6)

Introducing the dimensionless scale factor $a_D \equiv (V_D/V_{D_0})^{1/3}$ normalized by the volume of the domain $D$ at some initial time $t_i$, we can average the scalar Einstein equations (2a), (2b) and the continuity equation to obtain 

$$\langle \rho \rangle_D \cdot D = -\langle \dot{\Theta} \rangle_D \langle \rho \rangle_D ; \hspace{0.5cm} \langle \Theta \rangle_D = \frac{3}{a_D} \frac{\dot{a}_D}{a_D} ,$$  \hspace{1cm} (7a)

$$3 \left( \frac{\dot{a}_D}{a_D} \right)^2 - 8\pi G \langle \rho \rangle_D + \frac{1}{2} \langle R \rangle_D = -\frac{Q_D}{2} ;$$  \hspace{1cm} (7b)

$$3 \left( \frac{\ddot{a}_D}{a_D} \right) + 4\pi G \langle \rho \rangle_D = Q_D .$$  \hspace{1cm} (7c)

Here $\langle R \rangle_D$, the average of the spatial Ricci scalar $^3R$, is a domain dependent spatial constant. The ‘kinematical backreaction’ $Q_D$ is given by

$$Q_D \equiv \frac{2}{3} \left( \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 \right) - 2\langle \sigma^2 \rangle_D$$  \hspace{1cm} (8)

and is also a spatial constant over the domain $D$. A necessary condition for (7c) to integrate to (7b) takes the form of the following differential equation involving $Q_D$ and $\langle R \rangle_D$,

$$\dot{Q}_D + 6 \frac{\dot{a}_D}{a_D} Q_D + \langle R \rangle_D + 2 \frac{\ddot{a}_D}{a_D} \langle R \rangle_D = 0 .$$  \hspace{1cm} (9)

4 Smoothing Operators and Modified Field Equations

The averager defined in Eqn. (4) is a natural candidate for the smoothing operator $S_{\text{mat}}$ which must act on the matter tensor although we have not yet specified the domain size. Note that this averaging is defined on an inhomogeneous underlying geometry, i.e. it does not smoothen out the geometry itself. It remains to determine the operator $S_{\text{met}}$ that will coarse grain the metric. Although we do not have an explicit definition for this operator, we can define a notional smoothing operator acting on the metric, by using physical inputs from the Universe we observe. Namely, we propose that since the Universe has a matter content which is homogeneous on large scales, and since the CMB is highly isotropic around us, this notional operator must reflect these observations by acting on the general metric of Eqn. (1) to yield a FLRW metric.

We begin by making precise the notion of “homogeneity on large scales”, by requiring that in the range $R \equiv (L_1, L_2)$ of CGC sizes allowed by the fluid approximation validity criterion,
there exist a subset \( R_{\text{hom}} \equiv \subset R \) satisfying the following condition: When the microscopic matter density \( \rho(\vec{x}, t) \) is averaged over a length scale \( L \in R_{\text{hom}} \), the resulting smoothed density must lose all information about the particular CGC over which the smoothing was performed. In other words, for \( L \in R_{\text{hom}} \), at any given \( t \), the smoothed matter density must have the same value in all CGCs. It is in this sense that the smoothed matter density is homogeneous (see however, the discussion at the end of this section concerning an explicit smoothing of the metric). This definition can be related to the concept of statistical homogeneity if we assume that the total volume of the observable Universe is large enough that each CGC effectively samples one of an ensemble of configurations of matter distribution (see Ref. [26] for a careful discussion of this point). It is possible that homogeneity sets in at a scale too large to be considered as the size of the CGCs, namely that the range \( R_{\text{hom}} \) is not contained in the range \( R \) and that \( L'_2 \leq L_{\text{Hubble}} \) is not satisfied. We will assume that this does not happen (for a more detailed discussion on this assumption, see Sec. 2.1 of Ref. [2]). We also make the crucial assumption that \( R_{\text{hom}} \) does not change with time. This will later allow us to consistently analyze light propagation using the effective FLRW metric. This assumption needs some justification and we will return to this issue in the discussion.

Now define a coarse-graining cell to be an averaging domain \( D \) as used in Buchert’s formalism, with a characteristic comoving size \( L \in R_{\text{hom}} \). The averaged density is then \( \langle \rho \rangle_D(t) \), independent of the location of the particular CGC being considered. It also follows that the averaged expansion scalar \( \langle \Theta \rangle_D = 3(\dot{a}_D/a_D) = (V_D/V_D) \) is homogeneous in the above sense (using Eqn. (7a)) and describes the evolution of the physical volume of any given CGC. Thus, \( \langle \Theta \rangle_D \) plays the role of the expansion scalar for a geometry whose infinitesimal spatial volume elements probe length scales of the size of our CGCs (see however the discussion at the end of this section). It is therefore natural to demand that \( \langle \Theta \rangle_D \) be the expansion scalar for our notionally smoothed out macroscopic metric. This is equivalent to demanding that the scale factor of this homogeneous and isotropic effective metric be identified with the functional \( a_D(t) \) (up to a constant factor which we choose to be unity by an appropriate choice of units for the effective comoving coordinates). We are thus left with an effective metric whose line element can be written as

\[
ds_{\text{eff}}^2 = -dt^2 + a_D(t)^2 \left( \frac{dr^2}{1 - k_D r^2} + r^2 d\Omega^2 \right),
\]

which probes comoving length scales of size \( L \) as chosen above. The 3-Ricci scalar of this effective metric is given by \( R_{\text{eff}} = 6k_D/a_D^2 \) and should be thought of as arising from the coarse graining of the geometry. Following Ellis’ ideas from Sec. 2 then, \( R_{\text{eff}} = S_{\text{Einstein}}(\langle R \rangle_D) \) is in general different from the averaged 3-Ricci scalar \( \langle R \rangle_D \) which arises from applying the matter averager to the inhomogeneous scalar curvature \( \langle R \rangle_D = S_{\text{matt}}(\langle R \rangle_D) \). The constant \( k_D \) has the usual interpretation as a parameter characterizing all possible homogeneous and isotropic 3-metrics. We can write \( \langle R \rangle_D \) in terms of \( R_{\text{eff}} \) by defining \( \Delta R \equiv \langle R \rangle_D - 6k_D/a_D^2 \). In doing so we are merely shifting focus from (the a priori unknown) \( \langle R \rangle_D \) to another unknown \( \Delta R \). As the notation suggests, we think of \( \Delta R \) as being a correction (a part of \( P_{\text{mac}} \) in the notation of Sec. 2) see Eqns. (111) below) to the 3-Ricci scalar of the FLRW metric, although this correction need not be small compared to \( R_{\text{eff}} \) and need not evolve proportional to \( a_D^{-2} \). We choose not to normalize the magnitude of the constant \( k_D \), retaining instead the normalization of the functional \( a_D(t) \) at the initial time \( t_i \). The evolution equations appropriate at this scale are Buchert’s equations (7b) and (7c) together with the integrability condition (9). For completeness, and in the spirit of Ellis’ arguments we can rewrite equations (7b) and (7c) as
follows

\[
\left( \frac{\dot{a}_D}{a_D} \right)^2 + \frac{k_D}{a_D^2} = \frac{8\pi G}{3} \langle \rho \rangle_D - \frac{1}{6} (\Delta R + Q_D) \tag{11a}
\]

\[
\frac{\dot{a}_D}{a_D} = -\frac{4\pi G}{3} \langle \rho \rangle_D + \frac{1}{3} Q_D \tag{11b}
\]

where we have symbolically denoted the specific components of the general modified equations to which the various terms of Buchert’s equations belong. The integrability condition ([9]) which supplants these equations is purely a consequence of having non-zero corrections \( P_{mac} \) to the standard FLRW equations. As a quick check, we note that requiring the corrections \( P_{mac} \) to vanish implies both \( Q_D = 0 \) (from Eqn. (11b)) and \( \Delta R = 0 \), i.e. \( \langle R \rangle_D = 6k_D/a_D^2 \) (from Eqn. (11a)); the integrability condition ([9]) is identically satisfied and we recover the usual FLRW solution.

In the arguments leading to Eqns. (11), we have ignored the effects of a virtual change in the averaging length scale \( L \). Considerable effort has been spent by Buchert and Carfora [27] in studying such effects. In particular, in the spirit of the real space Renormalization Group formalism, these authors derive a novel curvature backreaction effect analogous to the kinematical backreaction \( Q_D \), together with a volume scaling effect. The inclusion of these effects would change the evolution equations satisfied by the scale factor \( a_D \). We also note that Buchert and Carfora’s techniques (involving Ricci flow of the inhomogeneous geometry into a smooth geometry) form one candidate for an explicit realization of the metric coarse graining operator \( S_{met} \). Seen in this light, it is apparent that by accounting for the above effects, we would also have to appropriately modify our notion of large scale homogeneity (to include the volume scaling effect and the residual information from the curvature fluctuations). In the arguments leading to the effective metric (10), the scale factor \( a_D(t) \) would be replaced by an appropriately “dressed” quantity characterizing the evolution of effectively infinitesimal volumes in the smoothed geometry. However, the main arguments leading to an effective metric which is FLRW, will not be altered given the existence of the range \( R_{hom} \subset R \). For simplicity, we ignore these additional effects in this paper, and hope to investigate them in future work.

It should be noted that a notion of coarse-graining is always implicit in standard FLRW cosmology. It is assumed that a coarse-graining (while satisfying the conditions of the fluid approximation) is possible on a length scale which coincides with the scale at which homogeneity is achieved (in other words, the existence of \( R_{hom} \subset R \) is assumed). Standard cosmology however ignores the possible modifications to FLRW cosmology brought about by the process of explicitly smoothing over the inhomogeneities - an issue which Buchert’s approach addresses. Our key proposal in this paper is that the size of the averaging domain should be taken to coincide with the scale of homogeneity, and that (up to some corrections) the quantity \( a_D(t) \) corresponds to the scale factor of a template FLRW metric.

We now illustrate how our proposal can facilitate the comparison of the predictions of the spatially averaged cosmology with observations, for example by computing the \( D_L(z) \) relation using the scale factor \( a_D(t) \), which has a modified time evolution compared to the corresponding scale factor in standard cosmology.
5 Luminosity Distance using the Effective Metric

In our approach, we take the effective or template metric to be FLRW, and since we are holding the size $L$ of the averaging domain fixed, there is nothing conceptually new to be done while studying its null geodesics. The important difference, of course, is that the evolution of the scale factor is now given by the modified Friedmann equations (11) (subject to the caveats listed at the end of the previous section). We note that the use of the FLRW metric is, strictly speaking “unrealistic” in the sense that in the real Universe, light travels mostly in vacuum which has a non-zero Weyl tensor whereas in the FLRW cosmologies which are conformally flat, light travels through a geometry with vanishing Weyl tensor (see also Refs. [9] and [2]). This issue is clearly relevant in the standard approach as well. While we have not addressed this issue in the context of averaging, we emphasize that our approach must be thought of as a first step in gauging the effects of inhomogeneities, by using an explicitly averaged construction of the template FLRW model to be used. The arguments relating the redshift of a source to its comoving distance are exactly the same as in the standard approach. In particular, we have for the redshift $z$ of a source that emits light at time $t$ which is received by an observer at time $t_0$,

$$1 + z = \frac{a_D(t_0)}{a_D(t)} ,$$

and the comoving distance $r$ to the source is determined by solving the equation

$$\int_0^r \frac{dr}{\sqrt{1 - k_D r^2}} = \int_{t_0}^t \frac{dt'}{a_D(t')} .$$

A caveat to the above relations is that the “step-size” used in computing the integrals involved must be understood to be no smaller than the size $L$ of the CGCs. Further, the entire construction must be interpreted as a “fitting template” that allows us to compare observations with theoretical predictions in a straightforward manner. With our choice of conventions, the magnitude of $k_D$ is a dynamically determined parameter of the model and determines the form of the luminosity distance $D_L(z)$ as follows. Solving Eqn. (13) for $r = r_{em}(z)$ gives

$$r_{em}(z) = S_{k_D}(\alpha(z)) ; \quad \alpha(z) = \int_{a_D(z)}^{a_D(t_0)} \frac{da_D}{a_D H_D} ,$$

$$D_L(z) = a_D(t_0) (1 + z) r_{em}(z) ,$$

where $H_D = \dot{a_D}/a_D$ is the Hubble parameter and the function $S_{k_D}$ is defined as

$$S_{k_D}(x) = \begin{cases} \frac{1}{\sqrt{k_D}} \sinh \left( \frac{\sqrt{k_D} x}{2} \right) , & k_D > 0 \\ \frac{1}{\sqrt{-k_D}} \sinh \left( \frac{\sqrt{-k_D} x}{2} \right) , & k_D < 0 \end{cases} .$$

In order to complete the picture we need the functional dependence of the Hubble parameter $H_D$ on the scale factor $a_D$. This requirement is complicated by the fact that the system of equations (11), (12), (13) and (14) is only consistent, it does not close. In order to obtain the

---

6The subscript 0 will indicate the value of the quantity at the present epoch $t_0$, and the subscript $i$, the value at the initial epoch $t_i$. 

---
required relation then, it is necessary to make certain assumptions about the evolution of the kinematical backreaction $Q_D$ and the averaged 3-Ricci scalar $\langle R \rangle_D$. Buchert et al. [21] have analyzed a class of scaling solutions of the form

$$Q_D = Q_D a_D^n ; \quad \langle R \rangle_D = \mathcal{R}_D a_D^n.$$  \quad \text{(16)}$$

The case $n = -6$, $p = -2$ is the only scaling solution with $n \neq p$. For the cases $n = p$, the backreaction is proportional to the averaged 3-curvature $\mathcal{R}$.

$$Q_D = r \mathcal{R}_D ; \quad r = \frac{n + 2}{n + 6} ; \quad n = -\frac{1 + 3r}{1 + r}.$$ \quad \text{(17)}$$

The forms of the integrability condition (9) and the solutions (16) indicate that the scaling solutions can be superposed to obtain new solutions. We will use two such superpositions to construct models which contain an accelerating scale factor $a_D$ and which also yield analytic expressions for the luminosity distance $D_L(z)$. The solutions we consider are of the form

$$\langle R \rangle_D = \frac{6k_D}{a_D^2} + \frac{\beta}{r} a_D^n ; \quad Q_D = \beta a_D^n,$$ \quad \text{(18)}$$

in keeping with our earlier discussion of the quantity $\Delta R$. Namely, we model $\Delta R$ by a single scaling solution. Here $r$ and $n$ are related as in Eqn. (17), and $k_D$ and $\beta$ are constants with dimensions of $(\text{length})^{-2}$. We are interested in models which admit an accelerating scale factor $a_D$, and will hence assume $\beta > 0$ and $k_D < 0$ (the reason for which will be apparent shortly).

Inserting the relations (18) in Eqns. (11a) and (11b) we find

$$\frac{\dot{a}_D^2}{a_D^2} + \frac{k_D}{a_D^2} = \frac{8\pi G}{3} \rho_i a_D^n - \frac{2}{3} \frac{\beta}{n + 2} a_D^n,$$ \quad \text{(19a)}$$

$$\frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3} \rho_i a_D^n + \frac{\beta}{3} a_D^n,$$ \quad \text{(19b)}$$

where we have defined $\rho_i \equiv \langle \rho \rangle_D(t_{in})$ and hence written $\langle \rho \rangle_D = \rho_i / a_D^3$ using the continuity equation (7a). The next two examples which we will consider are pathological, in that the luminosity distances $D_L(z)$ constructed for these models are not defined for all $z > 0$. Nevertheless, since these models yield analytical results, we shall display the expressions for $D_L(z)$ that we obtain. In general the computation of $D_L(z)$ expressions would have to be performed numerically.

Case 1 : $n = -3$. The backreaction $Q_D$ in this model decays at the same rate as the averaged matter density $\langle \rho \rangle_D$. We choose initial conditions such that $\beta > 4\pi G \rho_i$, which ensures that the model accelerates (cf. Eqn. (19b)). This is consistent with Eqn. (19a) due to the presence of the term $k_D a_D^2 < 0$, which allows the right hand side to be negative. Defining $A \equiv (2/3)(\beta - 4\pi G \rho_i) > 0$ we find

$$a_D^2 H_D = a_D \dot{a}_D = \sqrt{a_D} \left[ a_D^2 - A \right]^{1/2}.$$ \quad \text{(20)}$$

We have retained the notation of Ref. [21]. The proportionality constant $r$ should not be confused with the comoving coordinate in the effective metric.
Defining the “volume deceleration” \( q_D \equiv -\ddot{a}/(a \dot{H}_D^2) \) and evaluating Eqns. (19a) and (19b) at the present epoch \( t_0 \), we find

\[
q_{D_0} = \frac{\Omega_{m0}}{2} - \frac{Q_{D_0}}{3H_{D_0}^2} = -\frac{1}{2} \frac{A}{a_{D_0}^3 H_{D_0}^2} < 0;
\]

\[
k_{D} = (a_{D_0}H_{D_0})^2 \left( \Omega_{m0} - 1 - \frac{2Q_{D_0}}{3H_{D_0}^2} \right) = -(a_{D_0}H_{D_0})^2 (1 - 2q_{D_0}) < 0,
\]

where \( Q_{D_0} = \beta/a_{D_0}^3 \) is the present value of the backreaction, and we have defined the present matter density parameter \( \Omega_{m0} = (8\pi G \rho_0)/(3H_{D_0}^2) = (8\pi G \rho_i)/(3a_{D_0}^3 H_{D_0}^2) \) in the usual way. The function \( \alpha(z) \) defined in Eqn. (21) which determines the luminosity distance \( D_L(z) \) reduces to

\[
\alpha(z) = \frac{2}{\sqrt{-k_D}} \ln \left[ \frac{1 + \sqrt{1 - 2q_{D_0}}}{\sqrt{1 - 2q_{D_0}} + \sqrt{1 + 2q_{D_0}z}} \right].
\]

Since \( q_{D_0} < 0 \), the redshift is constrained to take values \( 0 \leq z \leq 1/(2q_{D_0}) \).

Case 2 : \( n = -5/2 \). Here the backreaction \( Q_D \) decays slower than the matter density, and if we assume \( 0 < \beta < 4\pi G \rho_i \), then we have a situation wherein the effective scale factor initially decelerates, and then starts accelerating after a certain epoch which is determined by the values of the parameters \( \beta \) and \( \rho_i \). Using Eqn. (19a) we have

\[
a_D\dot{a}_D = \sqrt{a_D} \left( \frac{8\pi G}{3\rho_i} - \frac{4\beta}{3\sqrt{a_D}} k_D a_D \right)^{1/2}.
\]

Now we have \( q_{D_0} = (\Omega_{m0}/2) - Q_{D_0}/(3H_{D_0}^2) \) as before with \( Q_{D_0} = \beta/a_{D_0}^3 \), and

\[
-k_D = (a_{D_0}H_{D_0})^2 (1 + \Omega_{m0} - 4q_{D_0}) \tag{24}
\]

The volume deceleration \( q_{D_0} \) will be negative provided \( a_{D_0} \) corresponds to a sufficiently late epoch, namely if \( a_{D_0} > (4\pi G \rho_i/3\beta)^2 \) as can be seen from Eqn. (19b), and provided \( k_D < 0 \) as seen from Eqn. (21). Assuming these conditions are met, we find for \( \alpha(z) \)

\[
\alpha(z) = \frac{2}{\sqrt{-k_D}} \left\{ \ln \left[ (C + 1 - 2q_{D_0}) \sqrt{1 + z} \right] \right.
\]

\[
- \ln \left[ C^2 - (\Omega_{m0} - 2q_{D_0}) \sqrt{1 + z} + C \left\{ 1 - 2(\Omega_{m0} - 2q_{D_0}) (\sqrt{1 + z} - 1) + \Omega_{m0}z \right\}^{1/2} \right] \right\}
\]

\[
C \equiv \sqrt{1 + \Omega_{m0} - 4q_{D_0}}.
\]

The range of allowed values for \( z \) is constrained in this case as well. For example, the quantity within the smaller braces in the previous equation must be positive, and this can be shown to imply that

\[
(z - z_+)(z - z_-) > 0,
\]

where \( z_\pm \) are the roots of the quadratic polynomial

\[
\mathcal{P}(z) \equiv M_2 z^2 + (2M_1 M_2 - 1)z + M_1^2 - 1; \quad M_1 = 1 + \frac{1}{2(\Omega_{m0} - 2q_{D_0})}; \quad M_2 = \frac{\Omega_{m0}}{2(\Omega_{m0} - 2q_{D_0})}.
\]

\[
(27)
\]
An analysis of this condition reveals that unless $\Omega_{m0} > 4q_{D0}^2$, the range $0 < z_- < z < z_+$ must be excluded from consideration when $q_{D0} < 0$. Requiring the arguments of the logarithms in Eqn. (25) to be positive would also impose some condition on the allowed values for $z$. In any case, since there is no a priori reason to expect that $\Omega_{m0} > 4q_{D0}^2$ holds, we see that this model also contains a pathology in general.

Although both models considered above must be regarded as unphysical (since the luminosity distance of a source is a measurable quantity), consistent models also exist. For example, the case $n = 0$, $r = -1/3$ corresponding to a constant backreaction, mimics the cosmological constant of the standard cosmology. Further, it may be argued that current observations seem observationally relevant in any case. To show that the formalism does admit consistent (accelerating) models with $k_D = 0$ as well, we now give an example of such a model.

Case 3: $n = -1$, $k_D = 0$. In this case, although the effective spatial curvature $R_{eff}$ is zero, the average of the physical spatial curvature $\langle R \rangle_D$ does not vanish, and is given by (see Eqn. (18)) $\langle R \rangle_D = -5\beta/a_D$. If we take $\beta > 0$, which guarantees that the scale factor will accelerate for $a_D > (4\pi G\rho_i/\beta)^{1/2}$ (see Eqn. (19a)), then we have $\langle R \rangle_D < 0$. The quantity $\alpha(z)$ is now given by

$$\alpha(z) = \int_{a_D(z)}^{a_{D0}} \frac{da_D}{(C_1 a_D + C_2 a_D^3)^{1/2}} ; \quad C_1 = \frac{8\pi G}{3} \rho_i , \quad C_2 = \frac{2\beta}{3} .$$

(28)

Since $C_1$ and $C_2$ are positive, this integral is well behaved for all $z$. Further, since $k_D = 0$, we have $D_L(z) = a_{D0}(1 + z)\alpha(z)$. Defining $\Omega_{Q0} \equiv (2/3)(Q_{D0}/H^2_{D0})$ and noting that $C_1 = \Omega_{m0}H^2_{D0}a_{D0}^3$ and $C_2 = (2/3)Q_{D0}a_{D0} = \Omega_{Q0}H^2_{D0}a_{D0}$, we get

$$D_L(z) = \frac{1 + z}{H_{D0}} \int_0^z \frac{d\tilde{z}}{[\Omega_{m0}(1 + \tilde{z})^3 + \Omega_{Q0}(1 + \tilde{z})]^{1/2}} .$$

(29)

In this notation, which is reminiscent of that in the standard LCDM model, if we use Eqn. (19a) evaluated at present time we also see that $\Omega_{m0} + \Omega_{Q0} = 1$, and that acceleration begins at a redshift given by

$$1 + z_{acc} = a_{D0} \left( \frac{\beta}{4\pi G\rho_i} \right)^{1/2} = \left( \frac{\beta/a_{D0}}{4\pi G\rho_0} \right)^{1/2} = \left( \frac{\Omega_{Q0}}{\Omega_{m0}} \right)^{1/2} .$$

(30)

As an example, in fig. 2 we have plotted the behaviour of $D_L(z)$ in units of $cH_{D0}^{-1}$ in this model (for redshifts between $z = 0$ and $z = 1$), setting $\Omega_{m0} = 0.3$, $\Omega_{Q0} = 0.7$. In this case the acceleration redshift is $z_{acc} \simeq 0.53$. For comparison we also show the $D_L(z)$ curve in the standard LCDM model with $\Omega_m = 0.3$ and $\Omega_{\Lambda} = 0.7$, for which the acceleration redshift is $z_{acc} = (2\Omega_{\Lambda}/\Omega_m)^{1/3} - 1 \simeq 0.67$. [Interestingly, a model in which the backreaction $Q_D$ behaves as $a_D^{-1}$ at early times, has recently been studied by Li and Schwarz [8] in the context of spatial averaging of a perturbed FLRW Cosmological model.]

We have therefore demonstrated the existence of models of the backreaction which can admit an accelerating scale factor. It is important to keep in mind though, that the models...
Figure 2: The luminosity distance (in units of $cH_0^{-1}$) versus redshift for the backreaction model with $n = -1$ defined in Eqns. (28) and (29) (solid line) and for the standard LCDM model with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ (dotted line).

considered above were not physically well-motivated, they served only to demonstrate the construction of luminosity distances using the modified dynamics. A more realistic scenario would probably contain a backreaction which becomes significant only at late times, when structure formation has advanced sufficiently far. It will be an interesting exercise to numerically construct luminosity distances for such models and compare them with those of the concordance model.

6 Discussion

We have argued that for the purposes of Cosmological study, an explicit coarse graining of the matter density in the Universe must be performed using the formalism of spatial averaging developed by Buchert. Due to the observed large scale homogeneity of the matter density and the isotropy of the CMB radiation, the notionally smoothed out template metric obtained after this coarse graining must be described by a FLRW line element, with the scale factor being given by Buchert’s functional $a_D(t)$. The procedure for constructing luminosity distances is therefore identical to that in the standard approach, except that the evolution of the effective scale factor must be given by the modified FLRW equations (7) supplemented by the integrability condition (9). (The scale factor and its evolution will be further modified by including perfect fluid sources with non-zero pressure [8] and the curvature backreaction and volume effects [27].) Such a construction of luminosity distances must be interpreted as a convenient way of comparing observations with theoretical models, while retaining the essence of the physical inhomogeneous geometry. It will be an interesting project to compare such luminosity distances in realistic backreaction scenarios with current Supernovae observations in an attempt to constrain the
magnitude and evolution of the backreaction.

We end by discussing some additional issues concerning an explicit smoothing of the metric which we avoided in the present treatment, and also some potential problems with our construction. Firstly, we return to the issue of holding the comoving range \(R_{\text{hom}}\), and in particular the smoothing scale \(L\), constant. This is justified only so long as the condition \(L \ll L_{\text{Hubble}}(z)\) is satisfied, where \(L_{\text{Hubble}}(z)\) is the appropriate comoving scale associated with the observable Universe at redshift \(z\). For example, if we choose a scale \(L\) which satisfies this condition at the present epoch, the same scale will not in general be valid at the last scattering epoch of the CMB. However, it seems reasonable that for a small enough redshift range, the condition \(L \ll L_{\text{Hubble}}(z)\) can be satisfied with a single constant \(L\). For epochs sufficiently far back in the past, the scale \(L\) would have to be reduced, and the construction of the \(D_L(z)\) relation would no longer be valid.

Further, a change in the size of the averaging domain would in general affect both the scale factor \(a_D\) and the "constant" \(k_D\) in a non-trivial way. In particular, any change in \(k_D\) would render the effective FLRW metric \((-14)\) meaningless. Also, a change in \(k_D\) need not necessarily be caused by a change in the averaging domain size. It seems likely \((29)\) that an explicit smoothing of the geometry using Buchert and Carfora's technique will yield a 3-space of different spatially constant curvatures at different times, this difference being independent of any change in the averaging domain size with time and being possibly incommensurate with our assumption of \(\mathcal{R}_{\text{eff}} \propto a_D^{-2}\). This could be interpreted in terms of a change in \(k_D\) with time, and in such a case, one would be dealing with a different FLRW template at different times. The effects of such a construction are not clear at present. Clearly this issue needs more careful study, and we hope to address this in the future. Of course, if \(k_D\) is made a function of time, care must be taken that the matter content is now so chosen that Einstein's equations continue to be satisfied, as pointed out by Räsänen \((31)\) in response to \((30)\).

Also, as pointed out by Räsänen \((2)\), the use of a homogeneous and isotropic FLRW template will necessarily ignore physical effects such as a non-trivial shear in the underlying geometry. A potential problem with our template construction is that effects of the non-trivial scalar curvatures \((3)\mathcal{R}(\vec{x}, t)\) on light propagation are also simplified to a proportionality to \(a_D^{-2}\). Although the full scalar curvature will show up in the evolution of \(a_D(t)\) via its spatial average \(<\mathcal{R}>_D\), it is possible that this construction may inaccurately model the real Universe. A study of the evolution of the effective spatial curvature as mentioned in the preceding paragraph may help resolve this issue.

There is also the related issue of degeneracy, namely that several underlying inhomogeneous models may reproduce the same backreaction effects when averaged. For example, it was shown \((17)\) in the context of the LTB solutions, that models with a vanishing spatial curvature (the so called marginally bound models) in the inhomogeneous geometry also have a vanishing backreaction \(Q_D\) after averaging. One is now faced with the situation wherein interpreting say luminosity distance observations within the inhomogeneous geometry may distinguish two inhomogeneous models, but interpreting the data after averaging would not make this distinction and would hence perhaps rule out both models. This issue has been recently highlighted by Enqvist and Mattsson (the last paper in Ref. \((11)\)) who show that a marginally bound LTB model can fit the data from Type Ia SNe without any spatial averaging. Since the standard FLRW matter dominated model is also (a special, homogeneous case of) a marginal LTB model which doesn't fit the SNe data \((13)\), the formalism after spatial averaging would rule out both models by demanding a non-trivial backreaction instead. However, since the LTB (and in fact any in-
homogeneous) models themselves face a similar degeneracy issue (for example several different LTB models can currently claim to fit the SNe data \[11\]), this matter cannot be considered as settled, and also deserves further attention.

Lastly, we note that Buchert’s averaging scheme itself has been criticized in the literature (see, e.g., Ref. \[20\]) on the grounds that the effects of averaging in a noncovariant manner as done in this scheme, are likely to be gauge artifacts. It should be noted that within second-order perturbation theory, gauge invariance of the averaged equations has been demonstrated in \[16\] and \[28\]. It is also possible to approach the problem of determining modifications to the standard Cosmological equations via a covariant method such as Zalaletdinov’s Macroscopic Gravity \[32\] for example. Such an attempt has in fact been made recently and has led to effective Cosmological equations very similar to those of Buchert (see Ref. \[33\]). The proposal in the present paper however, is attractive to us due to its simplicity in implementation, and may be treated as a phenomenological model of the more rigorous equations derived in Ref. \[33\].

Acknowledgments

We have greatly benefited from our correspondence with Thomas Buchert and Syksy Räsänen, and it is a pleasure to acknowledge their insightful comments and criticism of an earlier draft. AP also thanks Rakesh Tibrewala for useful and stimulating discussions.

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