ALGORITHMS FOR DETERMINING INTEGER COMPLEXITY

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Introduction.

The complexity $\|n\|$ of a natural number $n$ is defined as the least number of 1’s needed to express the positive integer $n$ using only the two operations $+$ and $\times$ and parentheses. At present there is a growing literature about this topic. For details we refer to [9], [4], [10], [2], [1], [6].

We present three algorithms to compute the complexity $\|n\|$ of all natural numbers $n \leq N$. The first of them is a brute force algorithm, computing all these complexities in time $O(N^2)$ and space $O(N \log^2 N)$. The main problem of this algorithm is the time needed for the computation. In 2008 there appeared three independent solutions to this problem: [11], [7], [3]. All three are very similar. Only [11] gives an estimation of the performance of its algorithm, proving that the algorithm computes the complexities in time $O(N^{1+\beta})$, where $1 + \beta = \log 3/\log 2 \approx 1.584963$. The other two algorithms, presented in [7] and [3], were very similar but both superior to the one in [11]. In Section 2 we present a version of these algorithms and in Section 4 it is shown that they run in time $O(N^\alpha)$ and space $O(N \log \log N)$. (Here $\alpha = 1.230175$).

The authors of the present paper, at the moment of sending [3] to a journal (Oct. 2008) saw Martin N. Fuller’s program [7] posted in the OEIS. Recognizing the similarity of our own approach in [3] we contacted Fuller and proposed to publish our results jointly. This started a very active collaboration, improving the results in [7] and [3]. The main problem with the algorithm presented in Section 2 concerned the space requirement. In [7] Fuller has given the idea of how to reduce the space requirements. He then programmed this improved algorithm and used it to compute the complexities for all $n \leq 10^{11}$.

For some reason (unknown to us) there was an interruption of this collaboration after which we could not contact Fuller any more. The present paper is the result of the above collaboration and we both think that Fuller should have been the main author of this paper. Of course he is not responsible for any errors contained in it.

In Section 2 we present the algorithm of [7] and [3]. The main advantage of this algorithm with respect to that in [11] is the definition of kMax in Section 2.7. This explains the difference in performance from $O(N^{1+\beta})$ to $O(N^\alpha)$.

In Section 3 we present a detailed description of this space-improved algorithm of Fuller and in Section 5 we prove that it runs in time $O(N^\alpha)$ and space $O(N^{(1+\beta)/2} \log \log N)$, where $\alpha = 1.230175$ and $(1 + \beta)/2 \approx 0.792481$. 
We have delayed the publication of the present paper in the hope of again getting in contact with M. Fuller. In the mean time it has been published [6] where the computation of the complexities of all $n \leq 10^{12}$ is announced. In [6] it is said that they have used an algorithm inspired by the description in [7] of the space-improved algorithm of Fuller, but do not give a detailed description of their algorithm. The results proved in this paper about the performance of the algorithms are not trivial and are still missing.

There are many known results about the complexity of natural numbers. We will use the inequality $\|n\| \leq 3 \log_2 n$ for $n > 1$, which appeared first in [4].

Since we try to explain two algorithms, we have tried to concentrate on explaining things to human beings how to proceed instead of trying to explain to a computer. This is the definition of *Literate programming*. We follow the recommendations of Knuth in [8].

The full power of these programs will be apparent when implemented in a language such as C or Pascal, but to explain the algorithm we have preferred Python as a language. In this way we can concentrate on the computations needed and not on the manipulation of data.

To one not used to Python it is a very simple language, the main difficulty possibly being that the scopes of the for, while, etc. loops are indicated by indentations. We also note that for integers $a$ and $b$ Python defines $a/b$ as $\lfloor a/b \rfloor$.

1. Brute force algorithm.

This is the algorithm used in [2] to compute the values of $\|n\|$ for $1 \leq n \leq n_{\text{Max}}$ with $n_{\text{Max}} = 200,000$.

It is based on the fact that $\|1\| = 1$, and (also see Section 2.1)

\[
\|n\| = \min_{a,b < n \in \mathbb{N}} \|a\| + \|b\|.
\]

```
<1 Brute force algorithm> ≡
nMax = 20000
Compl() =
s = range(nMax+1)
divisors = [[1] for n in s]
for k in range(2, nMax+1):
    j=1
    n = k
    while n <= nMax:
        divisors[n].append(k)
        m = n+k
        Compl[1]=1
for n in range(2, nMax+1):
    S=nMax+1
    for k in range(1,1+n/2):
        a = Compl[k]+Compl[n-k]
        if a < S:
            S=a
    div_n = divisors[n]
tau_n = len(div_n)
```
\begin{verbatim}
P = nMax + 1
for k in range(1, tau_n - 1):
    d = div_n[k]
    a = Compl[d] + Compl[n/d]
    if a < P:
        P = a
Compl[n] = min(S, P)
\end{verbatim}

We will not explain this simple minded algorithm. It is not very useful. At the end Compl[n] will give us the value of \( \|n\| \) for \( 1 \leq n \leq nMax \). Observe that for each \( n \) we compute

\[
S := \min_{a,b \in \mathbb{N}} \|a\| + \|b\| \quad \text{and} \quad P := \min_{a,b \in \mathbb{N}} \|ab\| = \min_{a+b = n} \|a\| + \|b\|.
\]

The complexity of \( n \) is then the minimum of these two quantities: \( \|n\| = \min(S, P) \).

It is easy to prove the following

**Proposition 1.** The above brute force algorithm computes the values of \( \|n\| \) for \( 1 \leq n \leq N \) in time \( \mathcal{O}(N^2) \) and space \( \mathcal{O}(N \log^2 N) \).

To store the complexities we need only \( \mathcal{O}(N \log \log N) \) space. But our program stores the divisors of all \( 1 \leq n \leq N \) requiring \( \mathcal{O}(N \log^2 N) \) space.

## 2. Time Improved Algorithm.

### 2.1. Idea of the algorithm.

The main disadvantage of the brute force algorithm is its running time \( \mathcal{O}(N^2) \). This time is spent mainly on checking all the instances of \( \|n - k\| + \|k\| \) for \( 1 \leq k \leq n/2 \). We will improve the time by finding a suitable kMax such that

\[
S = \min_{1 \leq k \leq n/2} \|l\| + \|n - k\| = \min_{1 \leq k \leq \text{kMax}} \|k\| + \|n - k\|.
\]

In this new program we will define a function: ‘complexity(·)’. Given an integer nMax, the output of \( \mathcal{B} = \text{complexity}(\text{nMax}) \) will be a list \( \mathcal{B} \) containing the values of the complexities of the numbers \( 1 \leq n \leq \text{nMax} \). More precisely we will have \( \mathcal{B}[n] = \|n\| \).

### 2.2. Plan of the second algorithm.

After some initializations the program will consist of only one main loop. In this loop we will compute successively the complexities of the natural numbers \( 2 \leq n \leq \text{nMax} \). The first part of this computation will consist of the computation of an adequate kMax for the corresponding \( n \). The computation of kMax uses an auxiliary function \( \mathcal{E} \).

We import the module \texttt{math} because we are using the mathematical function \texttt{log} in our program.

\textbf{[2.2 Time improved algorithm ]}
2.3. **Definition of $E$.** For each natural number $k$ let $E(k)$ be the largest number with complexity $k$ (see [1]). Then $E(1) = 1$, and for $j \geq 1$, $E(3j) = 3^j$, $E(3j + 1) = 4 \cdot 3^{j-1}$, $E(3j - 1) = 2 \cdot 3^{j-1}$. We extend this definition by $E(0) := 1$. In this way $E$ coincides with the sequence A000792 in the OEIS [13]. We have

$$\text{(3)} \quad 3^{(n-1)/3} \leq E(n) \leq 3^{n/3} \quad n \geq 0.$$ 

Note that $E$ is a non decreasing function. The following algorithm computes $E$:

```python
<2.3 definition of E>
from math import *
def complexity(nMax):
    <2.4 Initializing Compl>
    <2.6 Main Loop>
    return Compl

For later application we state the following easily proved fact.

**Proposition 2.** The cost of computing $E(n)$ is $O(n)$ operations and $O(n)$ space.

**Proof.** Observe that to write $3^{n/3}$ we need $O(n)$ space. \hfill \Box

2.4. **Initializing Compl.** The complexity function outputs a list Compl[·]. We first fill this list with a value which is an upper bound for all the true complexities. This upper bound of Compl[·] will be denoted by $c_{\text{Max}}$. Since $\|n\| \leq \frac{3}{\log 2} \log n$, we may take this bound equal to $\lfloor 3\log n_{\text{Max}} / \log 2 \rfloor$. We take a unit more because in case $n_{\text{Max}}$ is a power of 2 we are near the singularity of the $\lfloor \cdot \rfloor$ function.

We also initialize Compl[1] = 1.

```python
<2.4 Initializing Compl>
cMax = int(3*log(nMax)/log(2))+1
Compl = [cMax for n in range(0,nMax+1)]
Compl[1] = 1
```

2.5. **Main Loop.** Now we have the correct value of Compl[1]. Dynamically we compute Compl[n] for $2 \leq n \leq n_{\text{Max}}$. The pertinent loop consists of the following parts:

```python
<2.5 Main Loop>
for n in range(2,nMax+1):
    <2.6 usual best value>
    <2.7 computing kMax>
    <2.8 testing the sums>
    <2.9 testing the products>
```
2.6. **Usual best value.** When computing \( \|n\| \) we already have \( \text{Compl}[j] = \|j\| \) for all \( 1 \leq j \leq n - 1 \), and for \( k \geq n \) we have \( \text{Compl}[k] \geq \|k\| \). We will proceed to check all sums and products to replace \( \text{Compl}[k] \) by better values, so that eventually we will obtain the correct values.

We call \( \|n-1\|+1 \) the *usual best value* of \( \|n\| \). At this stage we simply replace \( \text{Compl}[n] \) by this bound if needed.

```python
<2.6 Usual best value>

a = \text{Compl}[n-1]+1
if a < \text{Compl}[n]:
    \text{Compl}[n] = a
```

2.7. **Computing kMax.** To compute kMax we use a bound \( s \) with \( \|n\| \leq s \). Here we always take \( s = \|n-1\|+1 \).

Here is our procedure to compute kMax. Subsequently we will prove that it has the desired properties.

```python
<2.7 Computing kMax>

target = \text{Compl}[n-1]
t = target/2
while E(t)+E(target-t) < n:
    t = t-1
kMax = E(t)
```

**Proposition 3.** For \( n \geq 3 \) the above procedure stops in finite time giving a value

\[
1 \leq \text{kMax} \leq \frac{n}{2} \left(1 - \sqrt{1 - \frac{4}{n^2}3\|n-1\|/3}\right).
\]

**Proof.** Observe that in the second line we obtain \( t = \|n-1\|/2 \) (by the convention of Python \( m/2 \) returns the integer part of the fraction \( m/2 \)). Therefore, for \( n \geq 3 \) after the second line \( t \) will be a natural number or 0. In each while-loop of the while \( t \) is replaced by \( t - 1 \), but if we arrive at \( t = 0 \) we will reach the condition of the while loop

\[
E(0)+E(\|n-1\|-0) \geq 1 + (n-1) = n
\]

and the program will stop with \( t \geq 0 \). Hence at finite time.

For each natural number \( t \) we will have

\[
E(t)+E(\|n-1\|-t) \leq 3^{t/3} + 3^{(\|n-1\|-t)/3} < n
\]

whenever \( x^2 - nx + 3^{|n-1|/3} < 0 \) with \( x = 3^{t/3} \). For \( n \geq 29 \), the roots \( x_1, x_2 \) of this polynomial in \( x \) are real, because its discriminant is \( n^2 - 4 \cdot 3^{|n-1|/3} > 0 \). In fact

\[
2 \cdot 3^{|n-1|/6} \leq 2 \exp\left(\frac{\log 3}{6} - \frac{3}{\log 2} \log n\right) < n, \quad n \geq 29.
\]

We also have \( 0 < x_1 < x_2 \) because \( x_1x_2 > 0 \), so that \( x_1 \) and \( x_2 \) have the same sign and since \( x_1 + x_2 = n \) they are positive.
The initial value of $t = \lfloor (n-1)/2 \rfloor$, so that \( t = (n-1)/2 - \varepsilon \) with \( \varepsilon = 0 \) or \( 1/2 \). For \( n \geq 31 \), we will have

\[
3^{t/3} + 3^{(\|n-1\| - t)/3} = 3^{\|n-1\|/6 - \varepsilon/3} + 3^{\|n-1\|/6 + \varepsilon/3} \leq (3^{-\varepsilon/3} + 3^{\varepsilon/3}) \exp\left(\frac{\log 3}{\log 4} \log n\right) < n.
\]

It follows that this initial value of \( t \) satisfies \( x_1 < 3^{t/3} < x_2 \) when \( n \geq 31 \). While \( 3^{t/3} > x_1 \) we will get \( E(t) + E(\|n-1\| - t) < n \), so that we end the while loop with a value \( t = t_\varepsilon \) such that \( 3^{t_\varepsilon} \leq x_1 \). Then

\[
(5) \quad k_{\text{Max}} = E(t_\varepsilon) \leq 3^{t_\varepsilon/3} \leq x_1 = \frac{n}{2} \left( 1 - \sqrt{1 - 4 \cdot 3^{\|n-1\|/3} / n^2} \right).
\]

While computing the needed complexities one may check that (4) is also true for \( 3 \leq n \leq 31 \).

\[\square\]

**Corollary 4.** For \( n \geq 2 \) we have \( k_{\text{Max}} \leq 2n^\beta \) where \( \beta = \frac{\log 3}{\log 2} - 1 \approx 0.584963 \).

**Proof.** We have \( 3^{\|n-1\|/3} < 3^{|\log n|/\log 2} = n^{1+\beta} \) and \( 4n^{\beta-1} < 1 \) for \( n \geq 29 \), so that in this case

\[
\sqrt{1 - \frac{4}{n^2} 3^{\|n-1\|/3}} > \sqrt{1 - 4n^{\beta-1}} > 1 - 4n^{\beta-1}.
\]

Therefore, by Proposition 3 we get

\[
k_{\text{Max}} \leq \frac{n}{2} \left( 1 - \sqrt{1 - 4 n^2 3^{\|n-1\|/3}} \right) < 2n^\beta.
\]

For \( 2 \leq n \leq 29 \) we may check directly that \( k_{\text{Max}} < 2n^\beta \) (for \( 2 \leq n \leq 50 \) we have \( k_{\text{Max}} = 1 \) except for \( n = 24 \) and \( n = 48 \) for which \( k_{\text{Max}} = 2 \)).

We have the following main result about \( k_{\text{Max}} \).

**Proposition 5.** For \( n \geq 2 \) we have

\[
S := \min_{1 \leq k \leq n/2} \|k\| + \|n-k\| = \min_{1 \leq k \leq k_{\text{Max}}} \|k\| + \|n-k\|.
\]

**Proof.** If not there would be a \( k_0 \) with \( 1 \leq k_0 \leq n/2 \) and

\[
(6) \quad \|k_0\| + \|n-k_0\| < \min_{1 \leq k \leq k_{\text{Max}}} \|k\| + \|n-k\|.
\]

In particular \( \|k_0\| + \|n-k_0\| < 1 + \|n-1\| \), so that \( \|k_0\| + \|n-k_0\| \leq \|n-1\| \). One of the two numbers \( k_0 \) and \( n-k_0 \), let us call it \( u \), would satisfy \( \|u\| \leq \|n-1\|/2 \). Therefore, \( \|u\| \leq t_0 \), the initial value of \( t \) in our procedure to get \( k_{\text{Max}} \), and

\[
\|u\| + \|n-u\| \leq \|n-1\|.
\]

We will also have \( \|u\| \leq t_\varepsilon \) the final value of \( t \) so that, by definition \( k_{\text{Max}} = E(t_\varepsilon) \). In the other case we would have \( t_\varepsilon < \|u\| \leq t_0 \) so that \( t' := \|u\| \) would be one of the values of \( t \) in the procedure (not the last) and we would therefore have

\[
E(t') + E(\|n-1\| - t') < n.
\]

From this we get a contradiction

\[
n = u + (n-u) \leq E(\|u\|) + E(\|n-u\|) \leq E(t') + E(\|n-1\| - t') < n.
\]
Therefore, \( u \leq E(\|u\|) \leq E(t_e) = \text{kMax} \), so that by (6) we get the contradiction

\[ \|u\| + \|n - u\| = \|k_0\| + \|n - k_0\| < \min_{1 \leq k \leq \text{kMax}} \|k\| + \|n - k\| \leq \|u\| + \|n - u\|. \]

\[ \square \]

2.8. Testing the sums. By Proposition 5 we only have to check \( \|k\| + \|n - k\| < \text{Compl}\[n]\) for \(1 \leq k \leq \text{kMax}. \) In 2.6 Usual best value we have checked \( k = 1. \) We need not check \( k = 2, k = 3, \ldots, k = 5. \) The first value we will check is \( k = 6. \) We explain: In [1] it is shown that the least value \( b \) such that

\[ S := \inf_{1 \leq k \leq n} \|k\| + \|n - k\| = \|b\| + \|n - b\| \]

is a solid number, where \( b \) is a solid number if \( b = u + v \) implies \( \|b\| < \|u\| + \|v\|. \)

It follows that we only need to test \( \|k\| + \|n - k\| < \text{Compl}\[n]\) when \( k \) is a solid number. But the sequence of solid numbers starts with 1, 6, 8, \ldots. Hence we arrive at the following algorithm to check the sums:

\[<2.8 Testing the sums>\]

```python
for m in range(6, kMax + 1):
    sumvalue = Compl[m]+ Compl[n-m]
    if sumvalue < Compl[n]:
        Compl[n] = sumvalue
```

2.9. Testing the products. Here we consider all the multiples \( k \cdot n \) of \( n \) and substitute the value of \( \text{Compl}\[k \cdot n]\) if necessary. Therefore, when we arrive at the case \( n \) all its divisors will have been checked, so that we will have the correct value of \( \text{Compl}\[n]\).

\[<2.9 Testing the products>\]

```python
for k in range(2, min(n, nMax/n)+1):
    prodvalue = Compl[k]+ Compl[n]
    if prodvalue < Compl[k*n]:
        Compl[k*n] = prodvalue
```

3. Fuller algorithm.

In practice the main limitation of the algorithm presented in Section 1 is the space requirement \( O(N \log \log N) \). We present here an idea of Fuller, partially expressed in [7] to overcome this difficulty.

The output of the new program is different from that of Section 1. Now we will compute successively the complexities of all numbers \( n \leq N \) but we do not store all these values. So the output of the program will consist of several statistics about the complexities of the numbers \( 1 \leq n \leq N. \) These statistics, which may change from run to run, will be computed simultaneously with the complexities.
3.1. The basic idea. The algorithm will store the minimal data needed to calculate \( \|n\| \):

- \( \|a\| \) for \( a \mid n \).
- \( \|k\| \) for \( k \leq \text{kMax} \).
- \( \|n - k\| \) for \( k \leq \text{kMax} \).

Small divisors and small summands of \( n \) will be stored in a fixed block, containing \( \|n\| \) up to some fixed upper limit \( H \). Recent calculations will be stored in a running block around \( n \) which is used to retrieve \( \|n - k\| \). Large divisors \( \|n/2\|, \|n/3\|, \ldots \) will be calculated and stored in the same way as \( n \), using additional running blocks and sharing the same fixed block.

The program will have several parameters: \( N \) (or \( \text{nMax} \) in the program) will be the limit to which we compute the complexities. We will have a fixed block where we will store the complexities for \( 1 \leq n \leq H \). Thus \( H \) is the length of the fixed block. We will have a number of running blocks \( B_j \) all of them of length \( \ell = 2^\ell \). \( B_1 \) will be the main running block, where we will store the last computed complexities. The other running blocks \( B_2, B_3, \ldots \) will contain the complexities of the large divisors \( n/2, n/3, \ldots \) of the numbers immediately following \( B_1 \).

All running blocks \( B_j \) (for \( 1 \leq j \leq N/H \)), each of length \( L \) will contain \( \|n\| \) for the range \( H_j - L < n \leq H_j \), where \( H_j \) will start at \( H \) and then move in steps of size \( \ell \) to finish at or just above \( N/j \).

For the computation to run smoothly we need that these parameters satisfy

\[
(7) \quad \text{kMax} < \ell, \quad N \leq \ell H, \quad \ell \mid H, \quad \ell \mid N, \quad H \geq \sqrt{L N}.
\]

The algorithm was used by Fuller in 2009 to compute \( n \) up to \( N = 10^{11} \) using just over \( 10^9 \) bytes of memory. The running time was 106 hours using 1 processor of a 2.8 GHz dual core PC. The parameters used for this run were \( N = 10^{11}, H = 10^9, L = 10^6 \). Sieving was used to find potential factors \( 2 \leq a \leq b, H_j - \ell < ab \leq H_j \). Thus there was a performance gain by using a large \( L \). The program runs roughly \( 1/3 \) of the speed when using \( L = 10^5 \) instead of \( L = 10^6 \).

3.2. General description. Roughly the program executed the following steps:

1. Calculate \( \|n\| \) for the fixed block \( B_0 \), using the earlier algorithm of Section 2.
2. Copy from \( B_0 \) into \( B_1 \) for the range \( H - L < n \leq H \). Set \( H_1 = H \).
3. Iterate the following steps while \( H_1 < N \):
   a. Set \( h = H_1 + \ell \).
   b. In descending order \( \left\lceil \frac{h}{\ell} \right\rceil > j \geq 1 \):
      i. If \( B_j \) has not been initialized, copy it from \( B_0 \) and set \( H_j = H \).
ii. If $H_i < \ell \lceil h/j \ell \rceil$, increase it by $\ell$. Fill the bottom half of the new range with the values of the top half of the old range. Calculate new values as shown in Section 3.8.

Here we have the scheme of Fuller’s program

3.3. Setting parameters. As we have said we fix the three parameters $N$ or $nMax$, $H$ the length of the fixed block, and $\ell$ or step being the size of the unit shift for the running blocks. This is also half of the length $L$ of the running blocks. We use a function to check that the parameters satisfy the conditions (7). The program prints the result of this check. Bad parameters may cause an index error.

```python
3.3 Setting parameters ≡
3.3.1 def of checkparameters
nMax = 1000000
H = 200000
step = 10000
L = 2*step
test = checkparameters(nMax,H,step)
if test:
   print 'GOOD PARAMETERS'
else:
   print 'NOT GOOD PARAMETERS'
```

3.3.1. Checking parameters. The function ‘checkparameters( )’ is defined. It checks whether our parameters satisfy the inequalities (7). We apply Corollary 4 to check whether $kMax \leq \ell$.

```python
3.3.1 def of checkparameters ≡
def checkparameters(nMax,H,step):
test = True
beta = 0.584962501
if step < 2*nMax**beta:
test = False
if nMax > step*H:
test = False
if step*(H/step) != H:
test = False
if step*(nMax/step) != nMax:
test = False
if H < sqrt(L*nMax):
test = False
return test
```
3.4. **Initialize** $B_0$. The fixed block $B_0$ and the running blocks $B_j$ are members of a list $B$. Here we simply compute the complexities contained in the fixed block (using the function defined in Section 1) and put them as the first term in the list. The successive terms of this list will be the running blocks. Hence we will have $B[0][n] = \|n\|$ for $1 \leq n \leq H$.

We also initialize Heads. This will be a list with $(H_j)^r_{r=0}$ as elements, where the fixed block $B_0$ will contain the complexities of $n$ for $1 \leq n \leq H = H_0$, and $B_j$, when initialized, will be the list formed by the complexities of $n$ for $H_j - L < n \leq H_j$. Hence we will have $B[j][n] = \|n + H_j - L + 1\|$ for $0 \leq n \leq L - 1$.

3.5. **Creating the first running block.** All running blocks $B_j$ with $j \geq 1$ are created with $H_j = H$. Later they will be shifted by steps of length $\ell$ to finish at or just above $N/j$.

3.5.1. **Definition of CreateRunningBlock.** The definition of this function is very simple. We simply copy the last half of the fixed block and put it as running block. We also have to update the Heads list.

3.6. **Definition of the function Shift.** The running block $B_j$ contains the complexities of $n$ for $H_j - 2\ell < n \leq H_j$. While running the program this block is shifted $\ell$ units to the right. This is done in two steps, first the function Shift simply puts $H_j = H_j + \ell$, and copies the second half of the block $B_j$ into the first half of this block and initializes the new values to any upper bound for $\|n\|$. Subsequently the function ‘ComputeRunningBlock’ will compute the new complexities $\|n\|$ for $H_j < n \leq H_j + \ell$.

The most delicate point of the program will be to show that the function ComputeRunningBlock can complete its task. But we may assume that in some way this has been done and analyze the Main Loop. This allows us to have a picture of which running blocks have been initialized at a given point in the program and which values the $H_j$ will have.
We have initialized the new values of $B_j$ to 255. We need that this value is an upper bound of $\|n\|$ for all $n \leq n_{\text{Max}}$. Since $\|n\| \leq \frac{3}{\log_2 \log n}$ we may safely take 255 for $n_{\text{Max}} \leq 3.86 \times 10^{25}$. The value 255 is useful because in this way we may take $B_j$ as a byte array.

3.7. Main Loop. The main computation is that of the first running block $B_1$, while $H_1$ will indicate where we are in the task of computing all the complexities up to $n_{\text{Max}}$. Each run of the main loop consists essentially of a shift of this running block from $H_1$ to $H_1 + \ell$. Hence we have to compute $\|n\|$ for $H_1 < n \leq H_1 + \ell$. For $n = ab$ we must read $\|a\|$ and $\|b\|$. This is the purpose of the other running blocks and the fixed block. For small $a$, the running block $B_a$ must contain the values of $\|b\|$. The value of $\|a\|$ will be read from the fixed block. For $n = a + b$ with $a < k_{\text{Max}}$ we read $\|a\|$ from the fixed block and $\|b\|$ from the running block $B_1$.

It follows that we have to shift $B_a$, if needed, before computing $B_1$. To update $B_a$ we follow the same procedure as to update $B_1$. Hence we proceed to update the $B_a$ in descending order of $a$. We also have to initialize the needed running blocks.

Here is the algorithm for the main loop. We will show that the necessary readings can easily be performed.

\begin{verbatim}
<3.7 Main Loop> ≡
while Heads[1] < nMax:
    h = min(Heads[1]+step,nMax)
    for j in range(ceiling(h/H)-1,0,-1):
        newH = step*ceiling(h,(j*step))
        if notInitialized(j) and (newH > H):
            CreateRunningBlock(B,Heads)
            if Heads[j] < newH:
                Shift(B,j)
                CalculateRunningBlock(j,newH)
\end{verbatim}

Proposition 6. At the start of each run of the main loop, except the first, we will have initialized the running blocks $B_j$ for $1 \leq j < \lceil H_1/H \rceil$ and $H_j = \ell \lceil H_1/j\ell \rceil$.

Proof. At the start we have initialized only $B_1$ and $H_1 = H$. Then we put $h = H_1 + \ell = H + \ell$. Since $1 < \ell < H$ we would have $\lceil h/H \rceil = 2$. The index $j$ in this first run will take only the value $j = 1$. Since $B_1$ is initialized we do not initialize any running block in this run. Since $H < \ell \lceil (H + \ell)/\ell \rceil = H + \ell$, the block $B_1$ will be shifted and $H_1$ will be put equal to $H + \ell$.

Therefore, at the end of the first run of the loop we have $H_1 = H + \ell$, so that $\lceil H_1/H \rceil = 2$, and the only running block initialized is $B_1$, and $H_1 = \ell \lceil H_1/\ell \rceil = H + \ell$. Therefore, the Proposition is true in this case.

Now by induction we may show that if our claim is true when a run starts, then it will be true at the end of this run.

\footnote{Since $\ell \mid H$, we have $\ell \leq H$. If $\ell = H$, we have $\ell = H \geq \sqrt{LN} = \sqrt{2^N}$, so that $\ell \geq 2N$, contradicting $\ell \mid N$.}
At the start of a run we will have initialized the running block $B_j$ for $1 \leq j < \lceil H_1/H \rceil$, and $H_j = \ell[H_1/j\ell]$.

We will initialize in the next loop those $B_j$ not yet initialized, i.e., such that $\lceil H_1/H \rceil \leq j$, and such that new$H > H$, i.e., such that $H < \ell[(H_1 + \ell)/j\ell]$. Given natural numbers $a$, $b$ and $c$, the relation $a < \lfloor b/c \rfloor$ is equivalent to $ac < b$. Therefore, since $\ell | H$, the condition $H < \ell[(H_1 + \ell)/j\ell]$ is equivalent to $H_j < H_1 + \ell$ and this is equivalent to $j < \lceil (H_1 + \ell)/H \rceil$. Thus at the end of the loop all blocks $B_j$ with $1 \leq j < \lceil (H_1 + \ell)/H \rceil$ will be initialized. Since the new $H_1$ will then be equal to $H_1 + \ell$ we get half of our assertion.

Now we have to show that at the end of the loop $H_j$ will be equal to $H_j = \ell[(H_1 + \ell)/j\ell]$. For one of those blocks $B_j$ that were initialized at the start of the loop we have $H_j = \ell[H_1/j\ell]$. If we have $H_j < \ell[(H_1 + \ell)/j\ell]$ then this block is shifted in this loop. If not, then we will have $H_j \geq \ell[(H_1 + \ell)/j\ell]$.

In the first case $H_j/\ell = \lceil H_1/j\ell \rceil < \lceil (H_1 + \ell)/j\ell \rceil$. This is an instance of $\lceil x \rceil < \lceil x + y \rceil$ with $0 < y \leq 1$, so that we will have $\lceil x + y \rceil = \lceil x \rceil + 1$. Therefore, $\lceil (H_1 + \ell)/j\ell \rceil = H_j/\ell + 1$. It follows that $H_j + \ell = \ell[(H_1 + \ell)/j\ell]$ and after the shift we will have what we want.

In the second case, the block has not been shifted, and $\ell[(H_1 + \ell)/j\ell] \leq H_j = \ell[H_1/j\ell]$. It follows that $H_j = \ell[(H_1 + \ell)/j\ell]$. Again what we want.

Now let $B_j$ be one of the running blocks that have just been initialized. So $j \geq \lceil H_1/H \rceil$ and $H < \ell[(H_1 + \ell)/j\ell]$. After the initialization we have $H_j = H$, so that this block will be shifted. After the shift $H_j$ is changed to $H + \ell$. Therefore, we must show that $H + \ell = \ell[(H_1 + \ell)/j\ell]$. First notice that $j \geq \lceil H_1/H \rceil$ is equivalent to $H \geq \ell[H_1/j\ell]$. Therefore

$$\frac{H_1}{j\ell} \leq \frac{H}{\ell} < \frac{H_1}{j\ell} + \frac{1}{j}.$$

But since $0 < 1/j \leq 1$ this implies

$$\left\lfloor \frac{H_1}{j\ell} \right\rfloor = \frac{H}{\ell} \leq \left\lfloor \frac{H_1}{j\ell} + \frac{1}{j} \right\rfloor = \frac{H}{\ell} + 1$$

from which we get $H + \ell = \ell[(H_1 + \ell)/j\ell]$.

3.8. **Definition of the function CalculateRunningBlock.** After each application of the function $\text{Shift}(B, j)$, we must apply $\text{CalculateRunningBlock}(j, \text{new}H)$. This computes the improved values of the complexity for $H_j - \ell < n \leq H_j$. Recall that the values of $B_j$ were initialized to the upper bound 255.

Calculating new values for $B_j$. The steps are:

1. For each $2 \leq a \leq b$, $H_j - \ell < ab \leq H_j$:
   a. Read $\|a\|$ from $B_0$. Read $\|b\|$ from $B_0$ if $b \leq H$, otherwise from $B_{ab}$.
   b. Set $\|a\| + \|b\|$ as the array value of $\|ab\|$ if it is lower than the current value.
2. For each $H_j - \ell < n \leq H_j$ in ascending order:
a. Read $\|n - 1\| + 1$ from $B_j$ and set this as the array value of $\|n\|$ if it is lower than the current value.

b. Calculate $k_{Max}$ for $n$ using the current array value for $\|n\|$.

c. For each $6 \leq k \leq k_{Max}$:

* Read $\|k\|$ from $B_0$ and $\|n - k\|$ from $B_j$. Set $\|n - k\| + \|k\|$ as the array value for $\|n\|$ if it is lower than the current value.

Proposition 7. The procedure $\text{Products}(j, \text{newH})$ is correct, i.e. the needed complexities $\|a\|$ and $\|b\|$ are contained in the indicated blocks.

Proof. 1. Proof that $a \leq H$ so that $\|a\|$ can always be read from the fixed block.

Since $a^2 \leq ab \leq \text{newH}$ we have $a \leq \sqrt{\text{newH}}$. On the other hand we have $\text{newH} \leq N + \ell$, because we only compute the complexities up to $N$. Then by (7) $a \leq \sqrt{\text{newH}} \leq \sqrt{N + \ell} < \sqrt{NL} \leq H$. 

2. Proof that \( \frac{\text{new}H}{a} \leq H \) or that \( B_{aj} \) has been initialized.

Assume that \( B_{aj} \) is not initialized. Since we are shifting the block \( B_j \), we are in step \( j \) of the Main Loop 3.8. The first not initialized running block will be \( B_u \) with \( u = \lceil h/H \rceil \), so that \( j < u \leq aj \), since \( B_j \) is initialized and \( B_{aj} \) is not. From \( \lceil h/H \rceil \leq aj \) we get \( h/H \leq aj \), so that \( h/j \leq aH \).

Since \( B_j \) has already been shifted if needed, we have by Proposition 6, that \( H_j = \text{new}H = \ell \lceil h/j \ell \rceil \). Because \( \ell \mid H \) it follows that

\[
\text{new}H = \ell \left\lceil \frac{h}{j \ell} \right\rceil \leq \ell \left\lceil \frac{aH}{\ell} \right\rceil = aH.
\]

Therefore, \( b \leq \frac{\text{new}H}{a} \leq H \). Hence if \( B_{aj} \) is not initialized, then \( \frac{\text{new}H}{a} \leq H \) and we may read \( \|b\| \) from the fixed block \( B_0 \). But the program chooses Block\( B = B_0 \) when \( \frac{\text{new}H}{a} \leq H \).

3. If \( B_{aj} \) is initialized. By Proposition 6, we have in this case \( aj < \lceil h/H \rceil \) and \( \text{new}H = \ell \lceil h/j \ell \rceil \). This implies respectively that \( aj < h/H \), and (since \( \ell \mid \text{new}H \)) \( h/j \ell \leq \text{new}H/\ell \). It follows that

\[
aH < \frac{h}{j} \leq \text{new}H \quad \text{so that} \quad H < \frac{\text{new}H}{a}.
\]

Hence in this case the program puts Block\( B = B_{aj} \). In fact we can read \( \|b\| \) from \( B_{aj} \) because

\[
H_{aj} - \ell \leq \frac{\text{new}H - \ell}{a} < b \leq \frac{\text{new}H}{a} \leq H_{aj}.
\]

The two intermediate inequalities are true because we have by hypothesis \( \text{new}H - \ell < ab \leq \text{new}H \).

To prove the first inequality put \( h = j\ell p - r \) with \( 0 \leq r < j\ell \) and \( p = aq - s \) with \( 0 \leq s < a \). Then \( h = aj\ell q - (sj\ell + r) \) and \( 0 \leq sj\ell + r < (s + 1)j\ell \leq aj\ell \), so that

\[
H_{aj} - \ell = \ell \left( \left\lceil \frac{h}{aj\ell} \right\rceil - 1 \right) = \ell(q - 1) = \ell \frac{aq - s - (a - s)}{a} \leq \frac{\ell}{a} (p - 1) = \frac{\ell}{a} \left( \left\lceil \frac{h}{j\ell} \right\rceil - 1 \right) = \frac{\text{new}H - \ell}{a}.
\]

With the same notation

\[
\frac{\text{new}H}{a} = \frac{\ell}{a} \left\lceil \frac{h}{j\ell} \right\rceil = \frac{\ell}{a} p = \frac{\ell}{a} (aq - s) \leq \ell q = \ell \left\lceil \frac{h}{aj\ell} \right\rceil = H_{aj}.
\]

Hence if \( B_{aj} \) is initialized then \( \|b\| \) can be read from Block\( B \) which in this case will be \( B_{aj} \), since in this case, \( H < \frac{\text{new}H}{a} \). \( \square \)
3.8.2. Definition of Sums. This is simpler than the case of the products. The definition of the function \( \text{kMaxfor}(n, s) \) should be clear after our explanation in Section 2.7. The definition of the function \( E(\cdot) \) is contained in Section 2.3.

As explained in Section 2.8 we only have to test the values of \( k \leq \text{kMax} \) which are solid numbers. We precompute the first few of these numbers and put them in a file in the form \( \text{solid} = [1, 6, 8, 9, 12, \ldots] \). If this file is not large enough an index-error will appear. In practice this will not be a problem because \( \text{kMax} \) is relatively small and the sequence of solid numbers is easy to compute.

```python
<3.8.2 definition of Sums> ≡
from solid import *
def kMaxfor(n, s):
target = s
t = target/2
while E(t)+E(target-t) < n:
t = t-1
return E(t)
def Sums(j, newH):
    for n in range(newH-step+1, newH+1):
        Block = B[j]
        shift = newH-L+1
        s = Block[n-shift]+1
        n = n
        kMax = kMaxfor(n, s)
        Block0 = B[0]
        r = 0
        b = solid[r]
        while b <= kMax:
            if Block[n-shift] > Block[n-b-shift]+Block0[b]:
                Block[n-shift] = Block[n-b-shift]+Block0[b]
            r = r+1
            b = solid[r]
```

Since we have taken \( \ell > \text{kMax} \) for all values of \( n \) in \( 1 \leq n \leq N \) it is clear that \( \|n - k\| \) can always be read from the current running block. Indeed, since the complexities \( \|n\| \) for \( \text{newH} - \ell < n \leq \text{newH} \) are computed in increasing order and the values in the first half of the running block (which has length \( \ell \)) are correct from the start. Also the value of \( \|k\| \) can always be read from the fixed block since \( k \leq \text{kMax} \leq \ell \leq H \).

3.9. Some definitions. There are two simple functions in the main loop that we have not yet defined. The ceiling function is used for certain ranges in the Main Loop. To check whether a running block is initialized we use the length of the Heads list. Each time a running block is initialized we put a new element in this list.

```python
<3.8.2 Some definitions> ≡
def ceiling(n, n):
    return n/n*(n%1==0)
def notInitialized(j):
    return len(Heads) < j+1
```

4. Upper bounds for \( \|n\| \).

To estimate the running time of our algorithms we need upper bounds for \( \|n\| \). We get some useful upper bounds by expressing \( n \) in a base \( b \) and using Horner’s algorithm. For
example \( n = r_0 r_1 r_2 \ldots r_k \) where \( 0 \leq r_j < b \) and \( r_0 \geq 1 \). This is equivalent to
\[
(8) \quad n = r_k + r_{k-1} b + r_{k-2} b^2 + \cdots + r_0 b^k = r_k + b(r_{k-1} + b(r_{k-2} + \cdots + b(r_1 + br_0) \cdots )).
\]
For each digit \( 0 \leq r < b \) we write \( D(b, r) \) for the complexity of multiplying by \( b \) and adding \( r \). E. g. \( D(6, 4) \leq 7 \) because \( \|6n + 4\| = \|3(2n + 1) + 1\| \leq \|n\| + 3 + 2 + 1 + 1 = \|n\| + 7 \). In general we define \( D(b, r) \) as the least number satisfying
\[
(9) \quad \|r + bn\| \leq \|n\| + D(b, r)
\]
for all \( n \geq 1 \).

We will always have \( D(b, r) \leq \|b\| + \|r\| \). This inequality may be strict. For example we have just seen that \( D(6, 4) \leq 7 \) and \( \|6\| + \|4\| = 9 \).

**Proposition 8.** If the expansion of \( n \) in base \( b \) is given by \( n = r_0 r_1 r_2 \ldots r_k \), then
\[
(10) \quad \|n\| \leq \|r_0\| + \sum_{j=1}^{k} D(b, r_j).
\]

**Proof.** For numbers with one digit this is trivially true. In the other case we have \( n = r_k + mb \) with \( m = r_0 r_1 r_2 \ldots r_{k-1} \) so that by induction we get
\[
\|n\| \leq \|m\| + D(b, r_k) = \|r_0\| + \sum_{j=1}^{k-1} D(b, r_j) + D(b, r_k).
\]

\[\square\]

By means of the following proposition we may easily obtain upper bounds for \( D(b, r) \).

**Proposition 9.** Let \( d \mid b \) with \( 1 < d < b \), where \( b = da \). For \( 1 \leq r \leq b \) put \( r = qd + s \). Then
\[
(11) \quad D(b, r) = D(da, qd + s) \leq D(d, s) + D(a, q).
\]

**Proof.** We have
\[
\|nb + r\| = \|(na + q)d + s\| = D(d, s) + \|na + q\| \leq D(d, s) + D(a, q) + \|n\|.
\]

\[\square\]

We define a function \( D_0(b, r) \) for \( b \geq 2 \) and \( 0 \leq r < b \) by induction on \( b \geq 2 \).

First when \( b = p \) is prime
\[
(12) \quad D_0(p, r) := \|p\| + \|r\|
\]
where we take \( \|0\| := 0 \), and in general
\[
(13) \quad D_0(b, r) := \min \{\|b\| + \|r\|, \frac{\min_{1 \leq d < b, d \mid b} D_0(d, \mod(r, d)) + D_0(b/d, \lfloor r/d \rfloor)}{b} \}.
\]
By induction we will then find that
\begin{equation}
D(b, r) \leq D_0(b, r).
\end{equation}

\textbf{Remark 10.} The best bounds are usually obtained with bases of the form \( b = 2^n3^m \). Notice that to compute \( D(b, r) \) we need to precompute only the values of \( D(b', r) \) for all proper divisors \( b' | b \).

4.1. \textbf{Average bound of the complexities.} We will give here an application to the general theory of the complexity of natural numbers. We may define several constants associated with the bounds on the complexity.

\textbf{Definition 11.} Let \( C_{\text{max}} \) denote the \( \limsup_{n \to \infty} \frac{\| n \|}{\log n} \). Let \( C_{\text{avg}} \) be the infimum of all \( C \) such that
\[ \| n \| \leq C \log n \]
for a set of natural numbers of density 1.

We have
\[ \frac{3}{\log 3} \leq C_{\text{avg}} \leq C_{\text{max}} \leq \frac{3}{\log 2}. \]

In [4] it is said that Isbell has shown using the expression of \( n \) in basis 24 that \( C_{\text{avg}} \leq 3.475 \). In [12] Steinerberger considered also a related problem obtaining for a slightly different constant the bound \( C'_{\text{avg}} \leq 3.332 \).

\textbf{Proposition 12.} For any basis \( b \geq 2 \) we have
\begin{equation}
C_{\text{avg}} \leq \frac{1}{b \log b} \sum_{r=0}^{b-1} D(b, r).
\end{equation}

In particular we get
\[ C_{\text{avg}} = \frac{41747875}{2^73^8\log(2^93^8)} = 3.3080772123153688960 \ldots \leq 3.309. \]

\textit{Proof.} Given a number in base \( b \) as \( n = r_0 r_1 \ldots r_k \) with \( r_0 \neq 0 \), then \( n \geq b^k \) so that by (10) we have
\[ \frac{\| n \|}{\log n} \leq \frac{\| r_0 \|}{k \log b} + \frac{1}{k \log b} \sum_{j=1}^{k} D(b, r_j). \]

The first term tends to 0 as \( n \) tends to \( \infty \). By Chernoff’s Theorem, for almost all numbers the difference
\[ \frac{1}{k} \sum_{j=1}^{k} D(b, r_j) - \frac{1}{b} \sum_{r=0}^{b-1} D(b, r) \]
is small. This proves (15).
We may easily compute the numbers $\frac{1}{b} \sum_{r=0}^{b-1} D(b, r)$, for increasing bases. The smaller values are obtained for bases of the form $b = 2^n3^m$. Having computed all numbers $D(b, r)$ for all bases $2^n3^m < 3359232 = 2^93^8$, we get the best value for $b = 2^93^8$ for which

$$\frac{1}{b \log b} \sum_{r=0}^{b-1} D(b, r) = 3.3080772123153688960 \ldots .$$

\[\square\]

5. Performance of the time-improved algorithm.

**Theorem 13.** The time-improved algorithm presented in Section 2 computes all $\|n\|$ for $n \leq N$ in time $\mathcal{O}(N^{1.230175})$ and space $\mathcal{O}(N \log \log N)$.

**Proof.** Clearly we need space for the array Compl. At the start of the algorithm we initialize this array by a common value $c_{\text{Max}}$ of the order $\log N$ (observe that to simplify the notation we have put $N = n_{\text{Max}}$), and we need $\log \log N$ bits to store this value. During the calculation each element of the array is decreased, so that we need $\mathcal{O}(N \log \log N)$ bits of space. The space needed for the rest of the computation is only $\mathcal{O}(\log N)$, needed to store some numbers $\leq N$ or to compute $E(t)$ for $t$ of the order of $\log N$.

The test for the products takes a number of operations of the order

$$\sum_{n=1}^N \min(n, N/n) = \sum_{n=1}^{\sqrt{N}} n + \sum_{n=\sqrt{N}}^N \frac{N}{n} = \mathcal{O}(N) + \mathcal{O}(N \log N) = \mathcal{O}(N \log N).$$

The cost of computing $k_{\text{Max}}$ for a value of $n$ is $\mathcal{O}(\log^2 n)$ operations. In fact the starting value of $k$ is of the order $\|n - 1\|$, so of order $\log n$. In the procedure $k$ will be changed at most $k$ times and for each of these we have to compute two values of $E$ for numbers of size $\log n$ at a cost of $\mathcal{O}(\log n)$ operations and $\mathcal{O}(\log n)$ space.

In the main loop for each $n \leq n_{\text{Max}}$ we have precomputed $\|n - 1\|$. We compute $k_{\text{Max}}$ for such an $n$ and then in the check of sums we run the variable $m$ from 6 to $k_{\text{Max}}$ requiring a fixed number of operations for each value of $m$. The cost of all this is of the order of

$$C = \sum_{n=1}^N (\log^2 n + k_{\text{Max}}(n)) = \mathcal{O}(N \log^2 N) + \sum_{n=1}^N n \left(1 - \sqrt{1 - \frac{4}{n^2} 3^{\|n - 1\|/3}}\right).$$

Since

$$\frac{4^{3^{\|n - 1\|/3}}}{n^2} \leq 4 \frac{\exp\left(\frac{\log 3}{3} - \frac{3}{\log 2} \log n\right)}{n^2} \leq 4 n^{-0.415637},$$

tends to 0 when $n \to \infty$ we may bound the cost by

$$C \leq \mathcal{O}(N \log^2 N) + \mathcal{O}\left(\sum_{n=1}^N \frac{3^{\|n - 1\|/3}}{n}\right).$$
Now choose a base \( b \) and apply the bound (10). Taking a number \( a \) such that \( b^{a-1} \leq N < b^a \) we will have

\[
\mathcal{O}\left(\sum_{n=1}^{N} \frac{3^{|n-1|/3}}{n}\right) \leq \mathcal{O}\left(\sum_{\ell=1}^{a} \sum_{b^{\ell-1} \leq n < b^{\ell}} \frac{3^{|n|/3}}{n}\right) = \mathcal{O}\left(\sum_{\ell=1}^{a} b^{-\ell} \sum_{b^{\ell-1} \leq n < b^{\ell}} 3^{|n|/3}\right).
\]

Now in the inner sum \( n \) runs through all the numbers that in base \( b \) have \( \ell \) digits. By (10) we will have

\[
\sum_{b^{\ell-1} \leq n < b^\ell} 3^{|n|/3} \leq C_b \sum_{b^{\ell-1} \leq n < b^\ell} 3^{1/3} \sum_{j=1}^{b^{\ell-1}} D(b,b_j) \leq C_b \sum_{d=0}^{b^{\ell-1}-1} 3^{1/3} D(b,d)^{\ell} = C_b A_b^\ell.
\]

Hence

\[
C \leq \mathcal{O}(N \log^2 N) + \mathcal{O}_b\left(\sum_{\ell=1}^{a} (A_b/b)^\ell\right) = \mathcal{O}(N \log^2 N) + \mathcal{O}_b((A_b/b)^a).
\]

Since \( N \sim b^a \) with a constant only depending on \( b \)

\[
(A_b/b)^a = \exp\left(a \log(A_b/b)\right) = b^a \frac{\log(A_b/b)}{\log b} = \mathcal{O}(N^\alpha),
\]

where

\[
\alpha = \frac{\log(A_b/b)}{\log b} = -1 + \frac{1}{\log b} \log \left(\sum_{d=0}^{b^{\ell-1}-1} 3^{1/3} D(b,d)\right).
\]

We have computed \( \alpha \) for all bases \( b = 2^n 3^m \leq 3188246 \). For \( b = 2239488 = 2^{10} 3^7 \) we found the smallest value

\[
\alpha = \frac{\log(3^6 2^{-10} (30357189 + 21079056 \cdot 3^{1/3} + 14571397 \cdot 3^{2/3}))}{\log(2^{10} 3^7)} = 1.230174997215298061586 \cdot \ldots < 1.230175.
\]

\( \square \)

6. Performance of Fuller's algorithm.

**Proposition 14.** The space-improved algorithm presented in Section 3 computes \( \|n\| \) for all \( n \leq N \) in time \( \mathcal{O}(N^\alpha) \) using \( \mathcal{O}(N^{1+\beta/2} \log \log N) \) bits of storage. ( \( \alpha = 1.230175 \) and \( (1 + \beta)/2 \approx 0.792481 \).)

**Proof.** In the last run of the Main Loop we have \( H_1 \leq N + \ell \) so that the total number of initialized running blocks will be \( \leq \left\lceil (N + \ell)/H \right\rceil \leq N/H + 2 \).

Hence we need a fixed block of length \( H \) and \( N/H \) running blocks of length \( L \). Each entry in the blocks must contain a value of the complexity, each of \( \log \log N \) bytes. So the required space will be

\[
\leq \left(H + \frac{N}{H} L\right) \log \log N.
\]
The value of $L$ is limited by our conditions $k\text{Max} \leq \ell$ and $L = 2\ell$. Therefore, an $L$ of the order $O(N^\beta)$ or larger would be adequate by Corollary 4. Given $N$ and $L$ the best choice of $H$ (the one requiring less space) is $H = \sqrt{LN}$, and this will give a space requirement of $\sqrt{NL}\log\log N$. So the best choice will be to take $L = O(N^\beta)$, and $H = O(N^{(1+\beta)/2})$. It is easy to see that increasing $H$ and $N$ a little (if needed) we may also satisfy the conditions $\ell | H$ and $\ell | N$.

So the space requirement for the algorithm is $O(N^{(1+\beta)/2}\log\log N)$, and this choice will satisfy all the conditions in (7).

The time and the number of operations needed for the computation is as follows

1. Computing the fixed block takes $O(H^\alpha)$ operations.

2. Copying from $B_0$ into $B_1$ takes $O(L)$ operations.

The running block $B_j$ starts at $H$ and ends at $\approx N/j$ in steps of size $\ell$. Hence $B_j$ must be adapted about $N/j\ell$ times.

There are $N/H$ running blocks each of length $L$. Initializing all these requires

$$\leq C \sum_{j \leq N/H} \frac{N}{j} = O(N \log N/H)$$

operations.

To compute the new values for the block $B_j$ (for the products) requires a fixed number of operations for each $2 \leq a \leq b$ with $H - \ell < ab \leq N/j$. The total cost of the products is therefore

$$\leq C \sum_{j \leq N/H} \sum_{a \leq \sqrt{N/j}} \frac{N}{aj} \leq C \sum_{j \leq N/H} \frac{N}{j} \log \sqrt{N/j} \leq CN \log N \log(N/H) \leq O(N \log^2 N).$$

To compute the new values for the block $B_j$ (for the sums) we have to perform the same number of operations as in the time-improved algorithm of Section 2. This cost was $O(N^\alpha)$. Hence the cost for the sums in the block $B_j$ is at most $C(N/j)^\alpha$. It follows that all the sums for the block cost

$$\leq C \sum_{j \leq N/H} \left(\frac{N}{j}\right)^\alpha \leq C\zeta(\alpha)N^\alpha = O(N^\alpha).$$

So the total cost of the algorithm is

$$O(N^\alpha) + O(N \log^2 N) + O(N \log(N/H)) = O(N^\alpha).$$

□

7. Acknowledgement

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