The estimation of the length of a convex curve in two-dimensional Alexandrov space

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It is proved the generalization of Toponogov theorem about the length of the curve in two-dimensional Riemannian manifolds in the case of two-dimensional Alexandrov spaces.

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MSC: 53C44, 52A40

Let $R$ be an Alexandrov space of curvature $\geq c$ homeomorphic to a disc (see [1, p. 308]). Suppose $G$ is a domain in $R$ that is bounded by a rectifiable curve $\gamma$. Denote by $\tau(\gamma_1)$ the integral geodesic curvature (the swerve) of a subarc $\gamma_1$ of $\gamma$ [1, p. 309]. A curve $\gamma$ is called $\lambda$-convex with $\lambda > 0$ if any subarc $\gamma_1$ of $\gamma$ satisfies

$$\frac{\tau(\gamma_1)}{s(\gamma_1)} \geq \lambda > 0,$$

where $s(\gamma_1)$ is the length of $\gamma_1$. For regular curves in a two-dimensional Riemannian manifold this condition is equivalent to the assumption that the geodesic curvature at each point of this curve is $\geq \lambda > 0$. In the general case the condition (0.1) allows $\gamma$ to have a corner points.

We prove the following theorem.

**Theorem 1.** Let $G$ be a domain homeomorphic to a disc and $G$ lies in a two-dimensional Alexandrov space of curvature $\geq c$ (in the sense of Alexandrov).

I. If the boundary curve $\gamma$ of $G$ is $\lambda$-convex and $c + \lambda^2 > 0$, then the length $s(\gamma)$ of $\gamma$ satisfies

1. for $c = 0$

$$s(\gamma) \leq \frac{2\pi}{\lambda};$$

2. for $c > 0$

$$s(\gamma) \leq \frac{2\pi\sqrt{c}}{\sqrt{c + \lambda^2}};$$

3. for $c < 0$

$$s(\gamma) \leq \frac{2\pi\sqrt{-c}}{\sqrt{-c + \lambda^2}}.$$

II. All these inequalities attain equalities if and only if the domain $G$ is a disc on the plane of constant curvature $c$.

This theorem is the generalization of Toponogov theorem [2] about the length of the curve in two-dimensional Riemannian manifolds. We need the following statements to prove Theorem [1].
Theorem A. (A. D. Alexandrov [1] p. 269) A metric space with inner metric of curvature $\geq c$ homeomorphic to a sphere is isometric to a closed convex surface in a simply connected space of constant curvature $c$.

Theorem B. (A. V. Pogorelov [3] pp. 119-167, 267, 320-321, [4]) Closed isometric convex surfaces in three-dimensional Euclidean and spherical spaces are equal up to a rigid motion.

Theorem C. (A. D. Milka [5]) Closed isometric convex surfaces in three-dimensional Lobachevsky space are equal up to a rigid motion.

Theorem D. (W. Blaschke [6]) Let $\gamma$ be a closed embedded $C^2$ regular curve in Euclidean plane.

I. If the curvature $k$ of $\gamma$ at each its point $P$ satisfies

$$k \geq \lambda > 0,$$

then the curve belongs to the disc that is bounded by the circle of the radius $R = 1/\lambda$ tangent to the curve at point $P$.

II. If the curvature $k$ of $\gamma$ at each its point $P$ satisfies

$$0 \leq k \leq \lambda,$$

then the circle of the radius $R = 1/\lambda$ tangent to the curve at the point $P$ belongs to the domain $G$ that is bounded by the curve $\gamma$.

The Theorem D is true if the condition for the curve’s curvature $k$ is substituted with the same condition for the specific curvature $\tau(\gamma_1)/s(\gamma_1)$ for any arc.

Lemma 1. Let $\gamma$ be a closed embedded rectifiable curve in Euclidean plane.

I. If for any subarc $\gamma_1$ of $\gamma$ the specific curvature $\frac{\tau(\gamma_1)}{s(\gamma_1)}$ satisfies

$$\frac{\tau(\gamma_1)}{s(\gamma_1)} \geq \lambda > 0,$$

then the curve $\gamma$ belongs to the disc that is bounded by the circle of the radius $R = 1/\lambda$ tangent to the support straight line at a point $P \in \gamma$.

II. If for any subarc $\gamma_1$ of $\gamma$ the specific curvature $\frac{\tau(\gamma_1)}{s(\gamma_1)}$ satisfies

$$0 \leq \frac{\tau(\gamma_1)}{s(\gamma_1)} \leq \lambda,$$

then the circle of the radius $R = 1/\lambda$ tangent to the curve at a point $P$ belongs to the domain $G$ that is bounded by the curve $\gamma$.

Proof. I. In this case the support function $h(\phi)$, $0 \leq \phi \leq 2\pi$ of the curve $\gamma$ is $C^{1,1}$ regular and a.e. it satisfies the equation

$$h + h'' = R, \quad 0 \leq R \leq \frac{1}{\lambda},$$

where $R$ is a radius of curvature for $\gamma$. Therefore

$$h(\phi) = \int_0^\phi R(\sigma) \sin(\phi - \sigma) \, d\sigma$$
and the proof coincides with Blaschke proof [6].

II. The radius-vector \( r(s) \) of the curve \( \gamma \) is \( C^{1,1} \) regular vector function. Fix the initial point \( P_0 \) on \( \gamma \) and denote by \( e_1 \) the unit tangent vector of \( \gamma \) at \( P_0 \) and by \( e_2 \) the unit normal vector of \( \gamma \) at \( P_0 \). Let \( P(s) \) be the point on \( \gamma \) such that the length of the arc \( \gamma(s) = P_0P(s) \) equals to \( s \). The function \( \tau(s) = \tau(\gamma(s)) \) is the integral geodesic curvature of the arc \( \gamma(s) \) and \( \tau(s) \leq \lambda s \). Therefore

\[
r'(s) = \cos \tau(s) e_1 + \sin \tau(s) e_2.
\] (0.2)

If we compare (0.2) with the equation for the circle of radius \( \frac{1}{4} \), we obtain the proof. \( \square \)

H. Karcher proved the generalization of Blaschke theorem in spherical space \( S^2 \) and in Lobachevsky space \( \mathbb{H}^2 \) for regular curves [7]. We formulate Lemma 2 for the case when the curvature \( S \) equals to 1 and the curvature \( \mathbb{H}^2 \) is equal to \(-1\). The Lemma 2 is true for the planes of any constant curvature \( c \) and the proof is the same.

**Lemma 2.** Let \( \gamma \) be a closed embedded rectifiable curve in \( \mathbb{H}^2 \) or \( S^2 \).

I. If the specific curvature satisfies

\[
\frac{\tau(\gamma_1)}{s(\gamma_1)} \geq \coth R_0 = \lambda,
\]

for any subarc \( \gamma_1 \) of \( \gamma \) in \( \mathbb{H}^2 \), then the curve \( \gamma \) belongs to the disc that is bounded by the circle of radius \( R_0 \) tangent to the support straight line of \( \gamma \) at a point \( P \in \gamma \).

II. If the specific curvature satisfies

\[
\frac{\tau(\gamma_1)}{s(\gamma_1)} \geq \cot R_0 = \lambda,
\]

for any subarc \( \gamma_1 \) of \( \gamma \) in \( S^2 \), then the curve \( \gamma \) belongs to the disc that is bounded by the circle of radius \( R_0 \) tangent to the support straight line of \( \gamma \) at a point \( P \in \gamma \).

**Proof.** The curve \( \gamma \) is a closed convex curve. At any point \( P \) of \( \gamma \) there exists a support straight line (geodesic map in the plane of constant curvature).

I. \( \gamma \in \mathbb{H}^2 \). Let \( S \) be a circle of radius \( R_0 \) tangent to the support straight line of \( \gamma \) from the side containing \( \gamma \). Assume that the center of the circle \( S \) is the origin of the coordinate system in the Cayley-Klein model of Lobachevsky plane and also it is the origin for support function \( h \) of curve \( \gamma \). The support function \( h \) is \( C^{1,1} \) regular and a.e. the radius of curvature \( R \) of \( \gamma \) equals

\[
R = \frac{g + g''}{\left(1 - \frac{(g')^2}{1 + g^2}\right)^{3/2}},
\] (0.3)

where \( g(h) = \tanh h \) is the support function for the curve \( \tilde{\gamma} \), and \( \tilde{\gamma} \) is the image of \( \gamma \) under the geodesic map \( \mathbb{H}^2 \) into \( \mathbb{E}^2 \) [9]. The radius of curvature \( \tilde{R} \) of \( \tilde{\gamma} \) is a.e. equal to

\[
\tilde{R} = R \left(1 - \frac{(g')^2}{1 + g^2}\right)^{3/2}, \quad 0 \leq \tilde{R} \leq R.
\] (0.4)

The image of the circle \( S \) under the geodesic map is the circle \( \tilde{S} \) in Euclidean plane \( \mathbb{E}^2 \) with the center in the origin of Cartesian orthogonal coordinate system. The curvature of \( \tilde{S} \) equals to \( \coth R_0 \). From Lemma II (I.) it follows that \( \tilde{\gamma} \) belongs to the disc being bounded by the circle \( \tilde{S} \). Applying the inverse geodesic transformation, we obtain that the curve \( \gamma \) belongs to the disc that is bounded by the circle \( S \) in Lobachevsky plane \( \mathbb{H}^2 \).
II. $\gamma \in \mathbb{S}^2$. Let $\gamma$ be the polar to $\gamma$ curve in $\mathbb{S}^2$. The radius vector of $\gamma$ is $C^{1,1}$ regular and its curvature is $\leq \tan R_0$ a.e. Let $P_0$ be a point on $\gamma$ and $S$ be a circle of the radius $\pi/2 - R_0$ tangent to $\gamma$ at the point $P_0$. The curvature of this circle is equal to $\tan R_0$. The center $\mathcal{O}$ of $S$ is the south pole of the sphere. Consider the geodesic map of the sphere $\mathbb{S}^2$ into the plane tangent to $\mathbb{S}^2$ at the point $\mathcal{O}$. The curve $\gamma$ is mapped to the curve $\tilde{\gamma} \in \mathbb{E}^2$ and the circle $S$ is mapped into the circle $\tilde{S}$ of the curvature $\tan R_0$. The curvature $k(\tilde{\gamma})$ of the circle $\tilde{S}$ is $\leq \tan R_0$. From Lemma [II] (II) it follows that the circle $\tilde{S}$ belongs to the domain that is bounded by the curve $\tilde{\gamma}$. Applying the inverse geodesic transformation, we obtain that the circle $\tilde{S}$ belongs to the domain bounded by $\gamma$ and the polar curve $\gamma$ belongs to the disc bounded by the polar circle $S$ of the radius $R_0$. 

Proof of the Theorem [I] Let $G_1$ and $G_2$ be two copies of the domain $G$. Let us glue the domains $G_1$ and $G_2$ along their boundary curves $\gamma_1$ and $\gamma_2$ by isometry between these curves. We obtain a manifold $F$ homeomorphic to the two-dimensional sphere with the intrinsic metric. Since the sum of the integral geodesic curvatures of any two identified arcs of the boundary curves is non-negative, from the Alexandrov gluing theorem [I, p. 318] it follows that $F$ is Alexandrov space of curvature $\geq c$. By Theorem A this manifold can be isometrically embedded as a closed convex surface in the simply-connected space $M^3(c)$ of constant curvature $c$. From Theorem B and C it follows that up to the rigid motion this surface is unique.

By plane domains we will understand domains on totally-geodesic two-dimensional surfaces in spaces of constant curvature; similarly we will call geodesic lines in these spaces as lines.

Perform the reflection of the surface $F_1$ with respect to a plane $\pi$ passing through three points on $\gamma$ that do not belong to a line. We will get the mirrored surface $F_2$. The domains $G_1$ and $G_2$ are mapped to the domains $\tilde{G}_1$ and $\tilde{G}_2$ on $F_2$; the curve $\gamma$ is mapped to $\tilde{\gamma}$. But $G_1$ is isometric to $G_2$ and $\tilde{G}_2$ is isometric to $\tilde{G}_1$. Let us reverse the orientation of the domains $\tilde{G}_1$, $\tilde{G}_2$. Then the surface $F_2$ will be isometric to $F_1$ and they will have the same orientation. By Theorems B and C the surface $F_1$ can be mapped into the surface $F_2$ by a rigid motion of the ambient space. But the three points of the curve $\gamma$ are fixed under this rigid motion. Thus it follows that this motion is the identity mapping and, moreover, the curve $\gamma$ coincides with the curve $\tilde{\gamma}$. Such situation is possible only when the curve $\gamma$ is a plane curve and it is the boundary of a convex cup isometric to the domain $G$. Recall that the convex cup is a convex surface with a plane boundary $\gamma$ such that the surface is a graph over a plane domain $\mathcal{G}$ enclose by $\gamma$. Note that, since $\gamma$ is a convex curve on the plane, then the integral geodesic curvature of any arc of the curve $\gamma$ is non-negative viewed both as a curve on the cup and as a curve on a plane $[10]$.

Let us show that the integral geodesic curvature of any arc of $\gamma$ calculated on $G$ is not less than the corresponding integral geodesic curvature of it that is calculated on the cup $G$. This means that $\gamma$ as a boundary curve of $\mathcal{G}$ is also $\lambda$-convex.

Recall that the intrinsic curvature $\omega(D)$ of a Borel set $D$ on a convex surface in a space of constant curvature $c$ is

$$\omega(D) = \psi(D) + cF(D),$$

where $\psi(D)$ is the extrinsic curvature, $F(D)$ is the area of $D$ [I, p. 397]. Consider a closed convex surface $M$ bounded by $G$ and the plane domain $\mathcal{G}$, and a surface $\mathcal{M}$ made up from the double-covered domain $\mathcal{G}$.

The intrinsic curvature concentrated on $\gamma$ equals

$$\omega(\gamma) = \tau_\gamma(G) + \tau_\gamma(\mathcal{G}),$$

where $\tau_\gamma(G)$, $\tau_\gamma(\mathcal{G})$ are the integral geodesic curvatures of $\gamma$ computed in $G$ and $\mathcal{G}$ respectively. Since $F(\gamma) = 0$, we have

$$\psi_\mathcal{M}(\gamma) = \tau_\gamma(G) + \tau_\gamma(\mathcal{G}),$$
\( \psi_M(\gamma) = 2\tau_\gamma(G) \).

From the definition of the extrinsic curvature [1, p. 398] it follows that \( \psi_M(\gamma) \geq \psi_M(\gamma) \) because each plane supporting to \( M \) at a point of \( \gamma \) is also supporting to \( M \). Thus we obtain \( \tau_\gamma(G) \geq \tau_\gamma(G) \). Moreover, this inequality holds for any subarc of \( \gamma \) as well.

I. The curve \( \gamma \) is a \( \lambda \)-convex curve lying in the plane of constant curvature \( c \). From Lemmas \textbf{1} and \textbf{2} it follows that the curve \( \gamma \) belongs to the disc bounded by the circle of radius \( R_0 \). The curvature and the length \( s \) of these circle equals

1. for \( c = 0 \), \( \lambda = \frac{1}{R_0} \), \( s = 2\pi R_0 \);
2. for \( c > 0 \), \( \lambda = \sqrt{c} \cot \sqrt{c}R_0 \), \( s = 2\pi \sin \sqrt{c}R_0 \);
3. for \( c < 0 \), \( \lambda = \sqrt{-c} \coth \sqrt{-c}R_0 \), \( s = 2\pi \sinh \sqrt{-c}R_0 \).

The curve \( \gamma \) on the plane of constant curvature \( c \) bounds the convex domain \( G \). It follows that the length of \( \gamma \) satisfies

\[
s(\gamma) \leq \begin{cases} 
\frac{2\pi}{\lambda} & \text{if } c = 0 \\
\frac{2\pi \sqrt{c}}{\sqrt{c} + \lambda} & \text{if } c > 0 \\
\frac{2\pi \sqrt{-c}}{\sqrt{-c} + \lambda} & \text{if } c < 0
\end{cases} \quad (0.5)
\]

II. Suppose that there is equality in (0.5). Then the domain \( G \) is a disc bounded by the circle \( \gamma \). Furthermore, \( \tau_\gamma(G) = \tau_\gamma(G) \) and the intrinsic curvature of \( \gamma \) satisfies \( \omega_M(\gamma) = \omega_M(\gamma) = 2\tau_\gamma(G) \) and the extrinsic curvature for any subarc \( \gamma_1 \) of \( \gamma \) satisfies

\[
\psi_M(\gamma) = \psi_M(\gamma) \quad (0.6)
\]

It follows that the surface \( M \) and \( M \) coincide, \( M \) is the double-covered disk and then \( G \) is a disk. If \( M \) doesn’t coincide with \( M \) then there exists the set of a positive measure of supporting planes to \( M \) along \( \gamma \), that are not planes of support to \( M \). It follows that the extrinsic curvatures of \( M \) and \( M \) along \( \gamma \) don’t coincides. This contradicts the equality (0.6). The Theorem 1 is proved.

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