AN IMPROVEMENT OF ANDŐ DILATIONS

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Abstract. One of the most important results in operator theory is Andő’s generalization of von Neumann’s inequality to two commuting contractions on a Hilbert space. More explicitly, Andő proved that every pair of commuting contractions $(T_1, T_2)$ can be dilated to a pair of commuting isometries $(V_1, V_2)$.

We improve Andő’s dilation by reducing the space on which it acts substantially and by constructing an Andő’s dilation with nice structure. Indeed, for a pair of commuting contractions $(T_1, T_2)$ on a Hilbert space, we show:

(B) There exists a Hilbert space $F$, a commuting pair $(V_1, V_2)$ of isometries on $\mathcal{H} \oplus H_2(F)$ such that $(V_1^*, V_2^*)|_\mathcal{H} = (T_1^*, T_2^*)$ and most importantly $(V_1, V_2)|_{H_2(F)} = (M_\varphi, M_\psi)$, where $\varphi$ and $\psi$ are inner functions of the form $\varphi(z) = P_\perp U + zP_\perp$ and $\psi(z) = U^*P + zU^*P_\perp$ for some unitary $U$ and projection $P$ in $\mathcal{B}(F)$;

(C) There exists an isometry $\Pi : \mathcal{H} \oplus H_2(D_T) \to \mathcal{H} \oplus H_2(F)$ such that $\Pi^*V_1V_2\Pi = V_T$ and $\Pi^*(V_1, V_2)\Pi = (M_\Phi, M_\Psi)$, where $\Phi(z) = \Lambda^*(PU + zP_\perp U)\Lambda$, $\Psi(z) = \Lambda^*(U^*P_\perp + zU^*P)\Lambda$ for some isometry $\Lambda : D_T \to F$ and $V_T$ is the minimal isometric dilation of $T = T_1T_2$ constructed by Schäffer.

This result, in particular, describes precisely when a given contraction can be factorized as the product of two contractions. Our approach is based on the dilation theory of the tetrablock $E = \{(a_{11}, a_{22}, \det A) : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $||A|| < 1\}$ largely developed in [7].

In the special case when $T = T_1T_2$ is pure, i.e., if $T^*n \to 0$ strongly, results like above have been obtained recently in [10]. We derive the results in this special case as a simple application of the model theory of the tetrablock developed in [14].

There was no uniqueness on the triple $(\Lambda, U, P)$ showed in [10]. The beauty of the model theory of the tetrablock adorns this paper with the following uniqueness result: Let $(T_1, T_2)$ be a commuting pair of contractions such that their product is pure, then the set

$\{\Lambda^*PU\Lambda, \Lambda^*U^*P_\perp \Lambda, \Theta_T\}$

is a complete set of unitary invariance for $(T_1, T_2)$, where $\Theta_T$ is the characteristic function of $T = T_1T_2$.

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1. Notations and terminologies

- For a Hilbert space \( \mathcal{E} \), \( \mathcal{B}(\mathcal{E}) \) denotes the algebra of bounded operators. All the operators in this paper have been assumed to be bounded;
- For a contraction \( T \) on a Hilbert space \( \mathcal{H} \), we denote
  \[
  D_T = (I - T^*T)^{1/2} \quad \text{and} \quad \mathcal{D}_T = \overline{\text{Ran}D_T};
  \]
- For two operators \( A \) and \( B \), the notation \([A, B]\) will stand for the commutator of \( A \) and \( B \), i.e., \([A, B] = AB - BA\);
- For a Hilbert space \( \mathcal{E} \), \( H^2(\mathcal{E}) \) denotes the Hilbert space of those \( \mathcal{E} \)-valued analytic functions on the unit disk \( D \) for which the coefficients (belong to \( \mathcal{E} \)) of Taylor series expansion around the origin, are square summable. Note that \( H^2 \otimes \mathcal{E} \) is another realization of \( H^2(\mathcal{E}) \), where \( H^2 \) is the Hardy space over the unit disk;
- For \( \varphi \in H^\infty(\mathcal{B}(\mathcal{E})) \), the algebra of \( \mathcal{B}(\mathcal{E}) \)-valued bounded analytic functions on \( \mathbb{D} \), \( \mathcal{M}_\varphi \) denotes the bounded operator on \( H^2(\mathcal{E}) \) defined by
  \[
  \mathcal{M}_\varphi f(z) = \varphi(z)f(z), \quad \text{for all } f \in H^2(\mathcal{E}) \text{ and } z \in \mathbb{D};
  \]
- For an operator \( F \) on a Hilbert space \( \mathcal{H} \), \( w(F) \) denotes the numerical radius of \( F \) which is the following real number:
  \[
  w(F) = \sup \{ |\langle Fh, h \rangle| : h \in \mathcal{H} \}.
  \]
- For an \( n \)-tuple of operators \( T = (T_1, T_2, \ldots, T_n) \) and \( m = (m_1, m_2, \ldots, m_n) \) in \( \mathbb{Z}_+^n \), the operator \( T^m \) denotes \( T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n} \).

2. Introduction

One of the most important results in operator theory is that for every contraction \( T \) on a Hilbert space \( \mathcal{H} \), there exists an isometry \( V \) on a bigger Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) such that \( V \) is a co-extension of \( T \). This in turn gives a naive proof of the famous von-Neumann inequality [16]:

\[
\|f(T)\| \leq \|f\|_{\infty, \mathbb{D}},
\]

for every polynomial \( f \) in one variable. Note that the space \( \mathcal{K} \) can always be chosen to be

\[
\mathcal{K} = \overline{\text{span}}\{V^n h : n \geq 0 \text{ and } h \in \mathcal{H}\}.
\]

The operator \( V \), when restricted to this \( \mathcal{K} \), is called the minimal co-extension of \( T \) for natural reason. It is a purely one variable phenomenon that all the minimal isometric co-extensions of a contraction are unitarily equivalent. Schäffer in [1] gave an explicit construction of a minimal isometric co-extension. He showed that if \( T \) is a contraction on a Hilbert space \( \mathcal{H} \), then \( V_T : \mathcal{H} \oplus H^2(D_T) \rightarrow \mathcal{H} \oplus H^2(D_T) \) given by

\[
V_T = \begin{pmatrix} T & 0 \\ D_T & M_z \end{pmatrix},
\]

is a minimal isometric co-extension of \( T \). T. Andô in his remarkable paper [3] has extended this one dimensional phenomenon to two variables. Indeed, he showed that if \( (T_1, T_2) \) is a commuting pair of contractions on a Hilbert space \( \mathcal{H} \), then there exists a commuting pair of isometries \( (V_1, V_2) \) on a bigger Hilbert space \( \mathcal{K}_A \) containing \( \mathcal{H} \).
such that \((V_1, V_2)\) is a co-extension of \((T_1, T_2)\). This in turn proves an analogue of the von-Neumann inequality for \((T_1, T_2)\):

\[
\|f(T_1, T_2)\| \leq \|f\|_{\infty, \mathbb{C}^2},
\]

for every polynomial \(f\) in two variables. Andô’s theorem turns out to be equivalent to the commutant lifting theorem, another eminent result in operator theory. Andô’s construction of \((V_1, V_2)\) indeed reveals that \((V_1, V_2)\) is a co-extension of \((T_1, T_2)\). However, \(i\) the space \(\mathcal{K}_A\) is very large, in fact it is \(\mathcal{K}_A = \mathcal{H} \oplus \mathcal{I} \oplus \mathcal{I} \cdots\), where \(\mathcal{I}\) is the direct sum of four copies of \(\mathcal{H}\), and \(ii\) his construction fails to reveal any explicit feature of \((V_1, V_2)\). We show that the space \(\mathcal{K}_A\) can be chosen to be substantially small, viz., \(\mathcal{K}_A = \mathcal{H} \oplus \mathcal{H}^2(F)\), where

- \(\mathcal{F} = \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}\), if \(\dim(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}) < \infty\) and
- \(\mathcal{F} = (l^2 \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2}\), if \(\dim(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}) = \infty\).

We have also been able to give an explicit construction of \((V_1, V_2)\). We show that \(\mathcal{H}\) is co-invariant under \((V_1, V_2)\) (as in the case of Andô) and interestingly \(\mathcal{K}_A = \mathcal{H} \oplus \mathcal{H}^2(F)\), where

\[
(V_1, V_2)|_{\mathcal{H}^\perp} = (M_\varphi, M_\psi)
\]

for some inner one degree polynomials \(\varphi\) and \(\psi\) with coefficients in \(\mathcal{B}(\mathcal{F})\) such that \(M_\varphi M_\psi = M_z\) on \(\mathcal{H}^2(F)\).

Moreover, we find an isometry \(\Pi : \mathcal{H} \oplus \mathcal{H}^2(\mathcal{D}_T) \to \mathcal{H} \oplus \mathcal{H}^2(\mathcal{F})\) such that

\[
\Pi^*V_1V_2\Pi = V_T, \quad (2.2)
\]

where \(V_T\) is the minimal isometric co-extension of \(T = T_1T_2\) as in \(\text{(2.1)}\). This is the content of Theorem 6.1—the first main result of this paper, which in particular answers the following question:

**When can a given contraction be factorized as the product of two contractions?**

Note that the above question has a very simple answer:

\(T = T_1T_2\) if and only if there exists a commuting pair \((V_1, V_2)\) of isometry such that \((V_1, V_2, V_1V_2)\) is a co-extension of \((T_1, T_2, T)\).

The proof is a simple application of Andô’s theorem to the pair \((T_1, T_2)\). Our success lies in finding an explicit Andô’s dilation \((V_1, V_2)\) and showing that their product is minimal, in the sense of \(\text{(2.2)}\).

Pure contractions are always special as they are precisely the compressions of \(M_z\) on some vector-valued Hardy space to co-invariant subspaces. In this special case, the above question has been answered very beautifully, recently in [10]. They proved:

**Theorem 2.1** (Das-Sarkar-Sarkar, [10]). Let \(T\) be a pure contraction on a Hilbert space \(\mathcal{H}\). Then the following are equivalent:

(i) \(T = T_1T_2\) for some commuting contractions \(T_1\) and \(T_2\);

(ii) there exist \(\mathcal{B}(\mathcal{D}_{T^*})\)-valued polynomials \(\varphi, \psi\) of degree less or equal to one and a joint co-invariant subspace \(\mathcal{Q}\) of \(\mathcal{H}^2(\mathcal{D}_{T^*})\) such that \((T_1, T_2, T)\) is unitarily equivalent to \(P_\mathcal{Q}(M_\varphi, M_\psi, M_z)|_\mathcal{Q}\) and \(P_\mathcal{Q}M_z|_\mathcal{Q} = M_\varphi|_\mathcal{Q} = M_\psi|_\mathcal{Q}\).
Moreover, $Q$ can be chosen to be as in the Sz.-Nagy and Foias representation of the pure contraction $T$ and

$$
\phi(z) = \Lambda^*(P^U + zP)\Lambda, \quad \psi(z) = \Lambda^*(U^*P + zU^*P^U)\Lambda
$$

where $U$ is a unitary, $P$ is a projection in $\mathcal{B}(\mathcal{F})$ and $\Lambda: \mathcal{D}_{T^*} \to \mathcal{F}$ is an isometry, for some Hilbert space $\mathcal{F}$.

Their motivation came from a celebrated paper of Berger-Coburn and Lebow [6] where they proved a version of Theorem 2.1 above for isometries. They used the Sz.-Nagy and Foias dilation theory for pure tetrablock contractions [15] and the transfer function realization formula for functions in $H^\infty(\mathcal{B}(\mathcal{D}_{T^*}))$ [5]. Our approach is very different than that in [6] and [10] and based on the analysis of the operator theory [7] in the tetrablock, which is the following non-convex but polynomially convex domain in $\mathbb{C}^3$:

$$
E = \left\{(x_{11}, x_{22}, \det X) : X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \text{ with } \|X\| < 1 \right\}.
$$

This domain arose in connection with the $\mu$ synthesis problem that arises in control engineering and was first studied in [1] and [2] for its geometric properties. There are several characterizations of a member of $E$, which can be found in both of [1] and [2].

Let $A(E)$ be the algebra of functions holomorphic in $E$ and continuous in $\overline{E}$. The distinguished boundary of $E$ (denoted by $bE$), i.e., the shilov boundary with respect to $A(E)$, is found in [1] and [2] to be the set

$$
bE = \left\{(x_{11}, x_{22}, \det X) : X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \text{ whenever } X \text{ is unitary} \right\}.
$$

The operator theory on the tetrablock was first developed in [7].

**Definition 2.2.** A triple $(A, B, T)$ of commuting bounded operators on a Hilbert space $\mathcal{H}$ is called a tetrablock contraction if $E$ is a spectral set for $(A, B, T)$, i.e., the Taylor joint spectrum of $(A, B, T)$ is contained in $E$ and

$$
||f(A, B, T)|| \leq ||f||_{\infty, E} = \sup\{|f(x_1, x_2, x_3)| : (x_1, x_2, x_3) \in E\}
$$

for any polynomial $f$ in three variables.

It turns out that in case the set is polynomially convex, as in the case of the tetrablock, the condition of the Taylor joint spectrum being inside the set, is redundant, see Lemma 3.3 in [7]. There are $E$-analogues of unitaries and isometries.

**Definition 2.3.** A tetrablock unitary is a commuting triple of normal operators such that its Taylor joint spectrum is contained in $bE$.

A tetrablock isometry is the restriction of a tetrablock unitary to a joint invariant subspace.

There are several characterizations of a tetrablock unitary and a tetrablock isometry which can be found in [7]. We shall need characterizations of tetrablock isometries.

**Theorem 2.4** (Bhattacharyya, [7]). Let $\underline{V} = (V_1, V_2, V_3)$ be a commuting triple of bounded operators. Then the following are equivalent:

1. $\underline{V}$ is a tetrablock isometry;
(2) $V_3$ is an isometry, $V_2$ is a contraction and $V_1 = V_2^* V_3$;
(3) $V_3^* V_3 = I$, spectral radii of $V_1$ and $V_2$ are no greater than one and $V_1 = V_2^* V_3$;
(4) $V_3$ is an isometry and $\Lambda$ is a tetrablock contraction.

Therefore the family of tetrablock isometries is vast. We proceed with a concrete example of a family of tetrablock isometries which will be useful in the sequel.

**Example 2.5.** Let $F_1$ and $F_2$ be two operators on a Hilbert space $\mathcal{E}$ such that $w(F_i) \leq 1$ for $i = 1, 2$ and $[F_1, F_2] = 0$ and $[F_1, F_1^*] = [F_2, F_2^*]$. Then $(M_{F_1+zF_2}, M_{F_2+zF_1^*}, M_z)$ is a tetrablock isometry on $H^2(\mathcal{E})$.

That the triple in the above example is indeed a tetrablock isometry, can be checked by part (2) of Theorem 2.4, see [7] for a detailed proof of this.

Under a certain condition, Bhattacharyya showed in [7] that for every tetrablock contraction $(A, B, T)$ on $\mathcal{H}$, there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a tetrablock isometry $(V_1, V_2, V_3)$ on $\mathcal{K}$ such that

$$(V_1^*, V_2^*, V_3^*)|_{\mathcal{H}} = (A^*, B^*, T^*).$$

In other words, under a certain condition every tetrablock contraction has a dilation, a terminology that will be defined in the next section. We show, by means of an example, that this condition is only sufficient and not necessary in general.

A pure tetrablock contraction is a tetrablock contraction with the last entry pure. A model theory for pure tetrablock contractions has been developed in [14]. Using the model theory we have been able to give a very brief proof of Theorem 2.1 above.

Another important tool in the dilation theory is the following. For a contraction $T$, the characteristic function $\Theta_T$ is defined by

$$\Theta_T(z) = [-T + zD_T^*(I_{\mathcal{H}} - zT^*)^{-1}D_T]|_{D_T}, \text{ for all } z \in \mathbb{D}. \quad (2.3)$$

Because of the relation $T D_T = T^* D_{T^*}$ see [15], one easily realizes that for each $z$ in $\mathbb{D}$, $\Theta_T(z)$ is in $\mathcal{B}(D_T, D_{T^*})$. This seemingly hideous expression of the characteristic function becomes an obvious generalization of the Möbius transformations preserving $\mathbb{D}$, when one considers the scalar contractions.

The beauty of the model theory of the tetrablock adorns this article with the following uniqueness theorem—the second main result of this paper:

**Theorem 2.6.** Let $T$ be a pure contraction on a Hilbert space $\mathcal{H}$ such that $T = T_1 T_2$ for some commuting contractions $T_1, T_2$ on $\mathcal{H}$. Let the triple $(P, U, \Lambda)$ be associated to $(T_1, T_2, T)$ by Theorem 2.1. Then

$$\{\Lambda^* P U \Lambda, \Lambda^* U^* P^\perp \Lambda, \Theta_T\}$$

is a set of complete unitarily invariance for the pair $(T_1, T_2)$.

This paper is organized as follows: Section 3 gathers previous results that will be used in this paper. Section 4 and 5 produce a family of tetrablock contractions and their dilation. Section 6 proves the first main result of the paper. Section 7 shows how the model theory of pure tetrablock contractions can be used to derive Theorem 2.1. Section 8 proves Theorem 2.6, the second main result of this paper.
3. Previous results

Let $T = (T_1, T_2, \ldots, T_n)$ be a commuting $n$-tuple of operators on a Hilbert space $\mathcal{H}$. An $n$-tuple of commuting operators $V = (V_1, V_2, \ldots, V_n)$ on $\mathcal{K}$ is called dilation of $T$ if $V$ is a co-extension of $T$. Moreover the dilation $V$ is said to be minimal if

$$\mathcal{K} = \text{span}\{V^m h : m \in \mathbb{Z}_+^n, h \in \mathcal{H}\}.$$

3.1. Pair of commuting contractions. Let $(T_1, T_2)$ be a pair of commuting contractions on a Hilbert space $\mathcal{H}$. Denote $T = T_1T_2$. The following technique due to Andô has been very useful in operator theory:

$$D_T^2 = I - T_2^* T_2 + T_2^* T_1 T_1 T_2 = D_{T_2}^2 + T_2^* D_{T_1}^2 T_1.$$

This shows that the operator $\Lambda : D_T \to D_{T_1} \oplus D_{T_2}$ defined by

$$\Lambda D_T h = D_{T_1} T_2 h \oplus D_{T_2} h \text{ for all } h \in \mathcal{H},$$

is an isometry and $\text{Ran} \Lambda$ is the space

$$\mathcal{E} = \bigvee\{D_{T_1} T_2 h \oplus D_{T_2} h : h \in \mathcal{H}\} \subseteq D_{T_1} \oplus D_{T_2}.$$ (3.1)

as its range. Also it is easy to see that

$$D_{T_2}^2 + T_2^* D_{T_1}^2 T_2 = D_{T_2}^2 + T_2^* D_{T_1}^2 T_1,$$

which shows that the operator $U : \mathcal{E} \to D_{T_1} \oplus D_{T_2}$ defined by

$$U(D_{T_1} T_2 h \oplus D_{T_2} h) = D_{T_1} h \oplus D_{T_2} T_1 h \text{ for all } h \in \mathcal{H},$$ (3.2)

is an isometry. One can always add an infinite dimensional Hilbert space to $D_{T_1} \oplus D_{T_2}$ if necessary, to extend $U$ as a unitary.

We shall denote the extended unitary operator by $U$ itself and the resulting space when we add an infinite dimensional Hilbert space to $D_{T_1} \oplus D_{T_2}$ by $\mathcal{F}$. (3.3)

3.2. Tetrablock contraction. Note that if $(A, B, P)$ is a tetrablock contraction, then $A$, $B$ and $T$ are contractions. It was proved in [14] that for every tetrablock contraction $(A, B, T)$ on a Hilbert space $\mathcal{H}$, there exist two operators $F_1$ and $F_2$ in $\mathcal{B}(D_T)$ with $w(F_i) \leq 1$ for $i = 1, 2$ such that

$$A - B^* T = D_T F_1 D_T \text{ and } B - A^* T = D_T F_2 D_T.$$

(3.4)

It is easy to see that any two bounded operators $F_1$, $F_2$ on $\mathcal{D}_T$ satisfying (3.4) are unique. These unique operators are called the fundamental operators of the tetrablock contraction $(A, B, T)$.

Fundamental operators ever since its invention [8] have been proved to be extremely important in the dilation theory. They exist, apart from the tetrablock, in the symmetrized polydisk also, see [11]. Fundamental operators yet again, as this paper will show, play an important role in finding the explicit Andô’s dilation.

Let $(A, B, T)$ be a tetrablock contraction on a Hilbert space $\mathcal{H}$ with $F_1$ and $F_2$ as its fundamental operators. Define two operators $\mathcal{A}, \mathcal{B} : \mathcal{H} \oplus H^2(\mathcal{D}_T) \to \mathcal{H} \oplus H^2(\mathcal{D}_T)$ by

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ F_2^* D_T & M_{F_1 + z F_2} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} B & 0 \\ F_1^* D_T & M_{F_2 + z F_1^*} \end{pmatrix}.$$ (3.5)
With the help of the fundamental operators, Bhattacharyya showed that a tetrablock contraction has a normal boundary dilation under a certain condition.

**Theorem 3.1.** [Bhattacharyya, 7] Let \((A, B, P)\) be a tetrablock contraction on a Hilbert space \(\mathcal{H}\) with \(F_1\) and \(F_2\) as its fundamental operators. Then

1. the triple \((A, B, V_T)\) as defined in (3.3) is a minimal tetrablock isometric dilation of \((A, B, T)\), if
   \[
   [F_1, F_2] = 0 \text{ and } [F_1, F_1^*] = [F_2, F_2^*] \text{ holds.}
   \]

2. if there is another tetrablock isometric dilation \((W_1, W_2, W_3)\) on \(\mathcal{K}'\) of \((A, B, T)\) such that \(W_3\) is the minimal isometric dilation of \(T\), then \((W_1, W_2, W_3)\) is unitarily equivalent to \((A, B, V_T)\).

A remark on the condition (3.6) of the above theorem is in order.

**Remark 3.2.** Let \(F_1\) and \(F_2\) be two bounded operators on a Hilbert space \(L\). Then it can be checked by a fairly simple computation that the pair \((M_{F_1+zF_1^*}, M_{F_2+zF_2^*})\) on \(H^2(L)\) is commuting if and only if \(F_1\) and \(F_2\) satisfy (3.6). Therefore for the triple \((A, B, V_T)\) in Theorem 3.1 to be commuting \(F_1\) and \(F_2\) must have to satisfy (3.6). In fact, this is the only reason why such an assumption is needed. Later in [9], a normal boundary dilation of tetrablock contractions have been explicitly constructed under the same condition.

We show in Section 5 that this condition is only sufficient and not necessary, in general.

### 3.3. Pure tetrablock contractions

We quickly recall the model theory for pure tetrablock contractions established in [14], from which we shall derive Theorem 2.1 proved in [10].

For a contraction \(T\) on a Hilbert space \(\mathcal{H}\), define \(W: \mathcal{H} \rightarrow H^2(D_{T^*})\) by

\[
W(h) = \sum_{n=0}^{\infty} z^n \otimes D_{T^*} T^n h, \text{ for all } h \in \mathcal{H}.
\]  

(3.7)

This operator is significant when \(T\) is pure, known from the time of Arveson [4]. It is a straightforward computation to show that \(W\) has the following properties:

\[
\|Wh\|^2 = \|h\|^2 - \lim_{n} \|T^n h\|^2 \text{ for all } h \in \mathcal{H} \text{ and } WT^* = M_z^* W.
\]

Consequently, when \(T\) is a pure contraction, \(W\) is an isometry and \(T\) is unitarily equivalent to \(P_Q M_z|_Q\), where \(Q = RanW\). This is the Sz.-Nagy and Foias representation of \(T\).

Now let \((A, B, T)\) be a pure tetrablock contraction on a Hilbert space \(\mathcal{H}\) and \(G_1\) and \(G_2\) be the fundamental operators of \((A^*, B^*, T^*)\). With \(W\) as defined in (3.7), the following two equalities were obtained in [14]:

\[
WA^* = M_{G_1^* + zG_2} W \text{ and } WB^* = M_{G_2^* + zG_1} W.
\]

Consequently, the following model for a pure tetrablock contraction is obtained.
Theorem 3.3 (Sau[14]). Let \((A, B, T)\) be a pure tetrablock contraction on a Hilbert space \(\mathcal{H}\). Then \((A, B, T)\) is unitarily equivalent to the triple

\[
(T_\mathcal{Q}M_{G_1^* + zG_2}, P_\mathcal{Q}M_{G_2 + zG_1}, P_\mathcal{Q}M_2)
\]

via the unitary \(W : \mathcal{H} \to \mathcal{Q}\) defined in (3.7), where \(G_1, G_2\) are the fundamental operators of \((A^*, B^*, T^*)\).

Moreover if \(G_1\) and \(G_2\) satisfy \([G_1, G_2] = 0\) and \([G_1, G_1^*] = [G_2, G_2^*]\), then as observed before, the triple \((M_{G_1^* + zG_2}, M_{G_2 + zG_1}, M_z)\) is a tetrablock isometry and indeed by Theorem 3.3 the triple \((M_{G_1^* + zG_2}, M_{G_2 + zG_1}, M_z)\) is a tetrablock isometric dilation of \((A, B, T)\).

Given two contractions \(T\) and \(T'\) on Hilbert spaces \(\mathcal{H}\) and \(\mathcal{H}'\) respectively, we say that the characteristic functions of \(T\) and \(T'\) coincide if there are unitary operators \(u : \mathcal{D}_T \to \mathcal{D}_{T'}\) and \(u_* : \mathcal{D}_{T^*} \to \mathcal{D}_{{T'}^*}\) such that the following diagram commutes for all \(z \in \mathbb{D}\),

\[
\begin{array}{ccc}
\mathcal{D}_T & \xrightarrow{\Theta_T(z)} & \mathcal{D}_{T^*} \\
\downarrow u & & \downarrow u_* \\
\mathcal{D}_{T'} & \xrightarrow{\Theta_{{T'}^*(z)}} & \mathcal{D}_{{T'}^*}
\end{array}
\]

Sz.-Nagy and Foias proved that the characteristic function of a completely-non-unitary contraction is a complete unitary invariant [15].

We end this section with the following result which will play the main role in the proof of Theorem 2.6

Theorem 3.4. Let \((A, B, T)\) and \((A', B', T')\) be two pure tetrablock contractions defined on \(\mathcal{H}\) and \(\mathcal{H}'\) respectively. Suppose \((G_1, G_2)\) and \((G_1', G_2')\) are fundamental operators of \((A^*, B^*, T^*)\) and \((A'^*, B'^*, T'^*)\) respectively. Then \((A, B, T)\) is unitarily equivalent to \((A', B', T')\) if and only if the characteristic functions of \(T\) and \(T'\) coincide and \((G_1, G_2)\) is unitarily equivalent to \((G_1', G_2')\) by the same unitary that is involved in the coincidence of the characteristic functions of \(T\) and \(T'\).

4. A FAMILY OF TETRABLOCK CONTRACTIONS AND THEIR FUNDAMENTAL OPERATORS

We start with observing the following, which is the root of this paper.

Lemma 4.1. Let \((T_1, T_2)\) be a commuting pair of contractions on a Hilbert space \(\mathcal{H}\) and \(T = T_1 T_2\). The triple \((T_1, T_2, T)\) is a tetrablock contraction.

Proof. The proof is a simple application of Andô’s Theorem [3]. Define the map \(\pi : \mathbb{D} \times \mathbb{D} \to \mathbb{C}^3\) by \(\pi(z_1, z_2) = (z_1, z_2, z_1 z_2)\). Then by definition of the set tetrablock, it follows that \(\text{Ran}(\pi) \subset \mathbb{E}\). Now let \(f\) be any polynomial in three variables. By Andô’s theorem,

\[
\|f \circ \pi(T_1, T_2)\| \leq \|f \circ \pi\|_{\infty, \bar{\mathbb{D}}^2} \leq \|f\|_{\infty, \mathbb{E}},
\]

which proves the lemma. \(\square\)
4.1. The fundamental operators of \((T_1, T_2, T)\). It is usually very difficult to find the fundamental operators of a given tetrablock contraction. On the other hand, the dilation theory of the tetrablock is heavily dependent on the fundamental operators, and so is the proof of our main result. Therefore we first find the fundamental operators of the tetrablock contraction \((T_1, T_2, T)\).

Let \(E_1\) and \(E_2\) on \(\mathcal{F}\) be the operators defined by

\[
E_1 = P^\perp U \quad \text{and} \quad E_2^* = PU
\]

where \(P\) is the orthogonal projection in \(\mathcal{B}(\mathcal{F})\) onto the first component. Note that \(E_1\) and \(E_2\) have the following properties:

\[
E_1(D_T T_2 h \oplus D_T^2 h) = 0 \oplus D_T T_1 h \quad \text{and} \quad E_2^*(D_T T_2 h \oplus D_T^2 h) = D_T^* h \oplus 0, \quad \text{for all} \ h \in \mathcal{H}.
\]

Proposition 4.2. Let \(E_1\) and \(E_2\) be as defined in (4.1). Then the operators \(F_1\) and \(F_2\) on \(D_T\) defined by

\[
F_i = \Lambda^* E_i \Lambda, \quad \text{for} \ i = 1, 2
\]

are the fundamental operators of the tetrablock contraction \((T_1, T_2, T_{1T2})\).

Proof. Note that for every \(h, h' \in \mathcal{H}\) and \(i = 1, 2\),

\[
\langle D_T F_i D_T h, h' \rangle = \langle E_i \Lambda D_T h, \Lambda D_T h' \rangle \]

\[
= \langle E_i (D_T T_2 h \oplus D_T^2 h), (D_T T_2 h \oplus D_T^2 h) \rangle \]

\[
= \left\{ \begin{array}{ll}
\langle (0 \oplus D_T T_2^* h), (D_T T_2 h \oplus D_T^2 h) \rangle = \langle (T_1 - T_2^* T) h, h' \rangle & \text{if} \ i = 1 \\
\langle (D_T T_2 h \oplus D_T^2 h), (D_T^* h \oplus 0) \rangle = \langle (T_2 - T_1^* T) h, h' \rangle & \text{if} \ i = 2.
\end{array} \right.
\]

This proves the claim. \(\square\)

Lemma 4.3. The operators \(E_1\) and \(E_2\) on \(\mathcal{F}\) as defined in (4.1) satisfy

\[E_1 E_2 = E_2 E_1 = 0 \quad \text{and} \quad E_1 E_1^* + E_2^* E_2 = E_1^* E_1 + E_2 E_2^* = I_{\mathcal{F}}.\]

Proof. This is obvious from the definitions of the operators in concern. \(\square\)

Now the following lemma follows from what we observed in Example 2.5.

Lemma 4.4. The triple \((M_{E_1+zE_2}, M_{E_2+zE_1}, M_z)\) is a tetrablock isometry on \(H^2(\mathcal{F})\).

5. A dilation of \((T_1, T_2, T)\)

It should be noted that the fundamental operators \(F_1\) and \(F_2\) of \((T_1, T_2, T)\) need not satisfy

\[\lbrack F_1, F_2 \rbrack = 0 \quad \text{and} \quad \lbrack F_1, F_1^* \rbrack = \lbrack F_2, F_2^* \rbrack.\]

Therefore Theorem [3.1] fails to produce a dilation of \((T_1, T_2, T)\). However we construct a tetrablock isometric dilation of \((T_1, T_2, T)\). This in particular proves that the condition [3.0] is only a sufficient one and not necessary, in general.

Theorem 5.1. Let \((T_1, T_2)\) be a commuting pair of contractions on a Hilbert space \(\mathcal{H}\) and \(T = T_1 T_2\). Define three operators \(V_1, V_2, V : \mathcal{H} \oplus H^2(\mathcal{F}) \to \mathcal{H} \oplus H^2(\mathcal{F})\) by

\[
V_1 = \begin{pmatrix} T_1 & 0 \\ E_2^* \Lambda D_T & M_{E_1+zE_2} \end{pmatrix}, \quad V_2 = \begin{pmatrix} T_2 & 0 \\ E_1^* \Lambda D_T & M_{E_2+zE_1} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} T & 0 \\ \Lambda D_T & M_z \end{pmatrix} \quad \text{(5.1)}
\]
The triple \((V_1, V_2, V)\) is commuting and a tetrablock isometric dilation of \((T_1, T_2, T)\).

**Proof.** The proof is mere bare hand matrix computation. Therefore the proof will be terse. We show that the \((21)\)-entry in the matrix representation of \(V_1V_2\) and \(V_2V_1\) are \(VD_T\). This would prove that \(V = V_1V_2 = V_2V_1\). Therefore we need to show

\[
E_2^*\Lambda_D T_2 + M_{E_1 + zE_2^*}E_1^*\Lambda_D T = E_1^*\Lambda_D T_1 + M_{E_2 + zE_1^*}E_2^*\Lambda_D T. \tag{5.2}
\]

In the following computation for all \(h \in \mathcal{H}\) we use \(E_1E_2 = 0\):

\[
(E_2^*\Lambda_D T_2 + M_{E_1 + zE_2^*}E_1^*\Lambda_D T)h = E_2^*\Lambda_D T_2h + E_1^*\Lambda_D T_1h = PU\Lambda_D T_2h + P\Lambda_D T_1h = \Lambda_D T_1h = U^*(D_{T_1}h \oplus 0) + U^*(0 \oplus D_{T_2}T_1h) = U^*PU\Lambda_D T_2h + U^*P\Lambda_D T_1h = E_2E_2^*\Lambda_D T_2h + E_1E_1^*\Lambda_D T_1h = (E_1^*\Lambda_D T_1 + M_{E_2 + zE_1^*}E_2^*\Lambda_D T)h.
\]

Next thing we show is that \(V_1\) and \(V\) are isometries, that will prove \(V_2\) an isometry too. The proof that \(V\) is an isometry, is easier and can be computed by means of a simple matrix computation. Therefore it only remains to show that \(V_1\) is an isometry, which is equivalent to the following equalities being true:

\[
T_1^*T_1 + D_T\Lambda^*E_2E_2^*\Lambda_D T = I \quad \text{and} \quad D_T\Lambda^*E_2M_{E_1 + zE_2^*} = 0. \tag{5.3}
\]

The first equality is true because for \(h \in \mathcal{H}\),

\[
\langle E_2^*\Lambda_D T_2h, E_2^*\Lambda_D T_2h \rangle = \langle D_{T_1}^2h, h \rangle,
\]

and the second equality is true because \(E_1E_2 = 0\) and for every \(h \in \mathcal{H}\), \(\zeta \in \mathcal{F}\), \(n \geq 0\),

\[
\langle D_T\Lambda^*E_2M_{E_1 + zE_2^*}(z^n\zeta), E_2^*\Lambda_D T_2h \rangle = \langle z^{n+1}D_T\Lambda^*(\zeta), E_2^*\Lambda_D T_2h \rangle = 0.
\]

This completes the proof. \(\square\)

We end this section with a couple of remarks.

**Remark 5.2.** It should be noted that \(V = V_1V_2\) is not the minimal isometric dilation of \(T = T_1T_2\). Hence part (2) of Theorem 3.4 doesn’t apply.

**Remark 5.3.** Observe that if \(\Pi : \mathcal{H} \oplus H^2(\mathcal{D}_T) \to \mathcal{H} \oplus H^2(\mathcal{F})\) is the isometry defined by

\[
\Pi = \begin{pmatrix} I & 0 \\ 0 & I \otimes \Lambda \end{pmatrix}
\]

then \(\Pi^*V_1V_2\Pi = V_T\), the minimal isometric dilation of \(T = T_1T_2\) as in 2.1 and

\[
(\Pi^*V_1\Pi, \Pi^*V_2\Pi) = \left( \begin{pmatrix} T_1 & 0 \\ F_2^*D_T & M_{F_1 + zF_2}^* \end{pmatrix}, \begin{pmatrix} T_2 \\ F_1^*D_T & M_{F_2 + zF_1}^* \end{pmatrix} \right),
\]

where \(F_1, F_2\) are the fundamental operators of \((T_1, T_2, T_1T_2)\). This shows that the fundamental operators have always been there in the scene.
6. The representation

This section proves the following theorem – the first main result of this paper.

**Theorem 6.1.** Let $T$ be a contraction on a Hilbert space $\mathcal{H}$ and $V_T$ on $\mathcal{H} \oplus H^2(\mathcal{D}_T)$ be the minimal isometric dilation of $T$ constructed by Schäffer. Then the following are equivalent:

(A) $T = T_1T_2$ for some commuting contractions $T_1$ and $T_2$ on $\mathcal{H};$

(B) There exists a Hilbert space $\mathcal{F}$, a commuting pair $(V_1, V_2)$ of isometries on $\mathcal{H} \oplus H^2(\mathcal{F})$ and an isometry $\Pi : \mathcal{H} \oplus H^2(\mathcal{D}_T) \to \mathcal{H} \oplus H^2(\mathcal{F})$ such that

$$(V_1^*, V_2^*)|_\mathcal{H} = (T_1^*, T_2^*), \quad \Pi^*V_1V_2\Pi = V_T \quad \text{and} \quad (V_1, V_2)|_{H^2(\mathcal{F})} = (M_\phi, M_\psi),$$

where $\phi(z) = (P^\perp U + zPU)\Lambda$ and $\psi(z) = U^*P + zU^*P^\perp$ are inner functions for some unitary $U$ and projections $P$ in $\mathcal{B}(\mathcal{F});$

(C) There exists a pair of contractions $(W_1, W_2)$ on $\mathcal{H} \oplus H^2(\mathcal{D}_T)$ such that

$$(W_1^*, W_2^*)|_\mathcal{H} = (T_1^*, T_2^*), \quad V_T|_\mathcal{H} = W_1^*W_2^*|_\mathcal{H} \quad \text{and} \quad (W_1, W_2)|_{H^2(\mathcal{D}_T)} = (M_\phi, M_\psi),$$

where $\Phi(z) = \Lambda^* (PU + zP^\perp U)\Lambda$ and $\Psi(z) = \Lambda^*(U^*P^\perp + zU^*P)\Lambda$ for some unitary $U$ and projections $P$ in $\mathcal{B}(\mathcal{F})$ and isometry $\Lambda : \mathcal{D}_T \to \mathcal{F}.$

**Proof.** We prove (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (A). Theorem 5.1 proves (A) $\Rightarrow$ (B) with $V_1, V_2$ as defined in (5.1) and $\Pi : \mathcal{H} \oplus H^2(\mathcal{D}_T) \to \mathcal{H} \oplus H^2(\mathcal{F})$ is

$$\Pi = \begin{pmatrix} I & 0 \\ 0 & I \otimes \Lambda \end{pmatrix},$$

where $\Lambda : \mathcal{D}_T \to \mathcal{F}$ is the isometry defined by (3.1). (B) $\Rightarrow$ (C) follows when one chooses

$$(W_1, W_2) = \Pi^*(V_1, V_2)\Pi$$

where $V_1, V_2$ are as defined in (5.1). And (C) $\Rightarrow$ (A) is obvious. \hfill $\square$

7. The pure and isometry cases

In [6], Berger, coburn and Lebow showed that a pure isometry $V$ is a product of two isometries $V_1$ and $V_2$ if and only if there exists a Hilbert space $\mathcal{F}$, a unitary $U$ and a projection $P$ in $\mathcal{B}(\mathcal{F})$ such that

$$(V_1, V_2) \text{ is unitarily equivalent to } (M_\phi, M_\psi),$$

where

$$\Phi(z) = P^\perp U + zPU \quad \text{and} \quad \Psi(z) = U^*P + zU^*P^\perp \quad (7.1)$$

In an attempt to generalize this result, Das, Sarkar and Sarkar proved a version of this for pure contractions, Theorem 2.1. In this section we show how the results in both the cases follow from the model theory explored in [14].
7.1. **Proof of Theorem 2.1.** Theorem 3.3 and Lemma 4.1 say that if \( T_1, T_2 \) are two commuting contractions on a Hilbert space \( \mathcal{H} \) and \( T = T_1T_2 \), then

\[
(T_1, T_2, T_1T_2)
\]

is unitarily equivalent to \( P_Q(M_{G_1} \oplus zG_2, M_{G_2 + zG_1}, M_z)|Q, \)

where \( G_1 \) and \( G_2 \) are the fundamental operators of \((T_1^*, T_2^*, T_1^*T_2^*)\) and \( Q = \text{Ran} W \).

In section 4, we have calculated the fundamental operators of \((T_1, T_2, T_1T_2)\). One can easily find the fundamental operators of \((T_1^*, T_2^*, T_1^*T_2^*)\) by following the same technique as in Section 4 and by replacing \((T_1, T_2)\) by \((T_1^*, T_2^*)\). And these are

\[
(G_1, G_2) = (\Lambda_s H_1 \Lambda_s, \Lambda^*_s H_2 \Lambda_s)
\]

(7.2)

where \( \Lambda_s : D_T^* \rightarrow D_{T_1^*} \oplus D_{T_2^*} \) is the isometry defined by

\[
\Lambda_s D_T^* h = D_{T_1^*} T_1^* h \oplus D_{T_2^*} T_2^* h \text{ for all } h \in \mathcal{H},
\]

and \( (H_1, H_2) = (P^* U_s, U^*_s P) \) where \( P \) is the projection of \( \mathcal{F}_s \) onto the first component and \( U_s \) is the unitary on \( \mathcal{F}_s \) that satisfies

\[
U_s(D_{T_1^*} T_1^* h \oplus D_{T_2^*} T_2^* h) = D_{T_1^*} T_1^* h \oplus D_{T_2^*} T_2^* h \text{ for all } h \in \mathcal{H}.
\]

Note that \( \mathcal{F}_s \) is the space after possible addition of an infinite dimensional Hilbert space to \( D_{T_1^*} \oplus D_{T_2^*} \). This proves Theorem 2.1.

7.2. **The isometry case.** The following simple things are to be noted in the case when \((T_1, T_2, T_1T_2)\) is replaced by \((V_1, V_2, V_1V_2)\) for commuting pair of isometries \( V_1 \) and \( V_2 \):

(a) The triple \((V_1, V_2, V_1V_2)\) is a tetrablock isometry;
(b) The operator \( \Delta : D_{V^*} \rightarrow D_{V_1^*} \oplus D_{V_2^*} \) defined by

\[
\Delta D_{V^*} h = D_{V_1^*} V_2^* h \oplus D_{V_2^*} V_1^* h \text{ for all } h \in \mathcal{H},
\]

is actually a unitary;
(c) The operator

\[
\Gamma (D_{V_1^*} V_2^* h \oplus D_{V_2^*} V_1^* h) = D_{V_1^*} h \oplus D_{V_2^*} V_1^* h \text{ for all } h \in \mathcal{H}
\]

is a unitary from \( D_{V_1^*} \oplus D_{V_2^*} \) onto itself. Therefore there is no need of a bigger space.

Now apply these observations to the following theorem which can be found in [14].

**Theorem 7.1.** [Sau, 14] Let \((A, B, P)\) be a pure tetrablock isometry. Then \((A, B, P)\) is unitarily equivalent to \((M_{G_1} + zG_2, M_{G_2 + zG_1}, M_z)\), where \( G_1 \) and \( G_2 \) are the fundamental operators of \((A^*, B^*, P^*)\).

Note that the fundamental operators of \((V_1^*, V_2^*, V_1^*V_2^*)\) are given by

\[
G_1 = \Delta^* P^* \Gamma \Delta \text{ and } G_2 = \Delta^* \Gamma^* P \Delta
\]

where \( P \) is the orthogonal projection of \( D_{V_1^*} \oplus D_{V_2^*} \) onto the first component. This proves the result of Berger, Coburn and Lebow.
8. Uniqueness

This section proves the second main result of this paper, i.e., Theorem 2.6.

So let $T$ be a pure contraction such that $T = T_1T_2$ for two commuting contractions $T_1$ and $T_2$. We observed in Section 7 that the fundamental operators $G_1, G_2$ of $(T_1^*, T_2^*, T_1^*T_2^*)$ are given by (7.2). Hence by Theorem 3.4 it follows that

$$(\Lambda_s H_1 \Lambda_s, \Lambda_s^* H_2 \Lambda_s, \Theta_{T_1T_2})$$

is a complete set of unitary invariance for the pair $(T_1, T_2)$.

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