Approximate Message Passing with Unitary Transformation for Robust Bilinear Recovery

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Abstract—Recently, several promising approximate message passing (AMP) based algorithms have been developed for bilinear recovery with model $Y = \sum_{k=1}^{K} b_k A_k C + W$, where $\{b_k\}$ and $C$ are jointly recovered with known $A_k$ from the noisy measurements $Y$. The bilinear recover problem has many applications such as dictionary learning, self-calibration, compressive sensing with matrix uncertainty, etc. In this work, we propose a new bilinear recovery algorithm based on AMP with unitary transformation. It is shown that, compared to the state-of-the-art message passing based algorithms, the proposed algorithm is much more robust and faster, leading to remarkably better performance.

Index Terms—Approximate message passing, unitary transformation, bilinear recovery, compressive sensing, dictionary learning.

I. INTRODUCTION

In this work, we consider the following bilinear problem

$$Y = \sum_{k=1}^{K} b_k A_k C + W,$$

where $Y$ denotes measurements, matrices $\{A_k\}$ are known, $\{b_k\}$ and $C$ are to be recovered, and $W$ represents white Gaussian noise. When both $Y$ and $C$ are replaced with two vectors $y$ and $c$ respectively, the above multiple measurement vector (MMV) problem is reduced to a single measurement vector (SMV) problem. Model (1) covers a variety of problems, e.g., compressive sensing (CS) with matrix uncertainty [1], joint channel estimation and detection [2], self-calibration and blind deconvolution [3], and structured dictionary learning [4].

Recently, several approximate message passing (AMP) [5] based algorithms have been developed to solve the bilinear problem, which show promising performance, compared to existing non-message passing alternates [7]. The generalized AMP (GAMP) [8] was extended to solve a general bilinear problem, i.e., recover both $A$ and $X$ from observation $Y = AX + W$, and the algorithm is called bilinear GAMP (BiGAMP) [9]. The parametric BiGAMP (P-BiGAMP) is then proposed in [10], which works with the model (1) to recover $\{b_k\}$ and $C$. Lifted AMP was proposed in [11] by using the lifting approach [12], [13]. However, these AMP based algorithms are vulnerable to difficult $A$ matrices, e.g., ill-conditioned, correlated, rank-deficient or non-zero mean matrices as AMP can easily diverge in these cases [14].

It was discovered in [15] that the AMP algorithm can still perform well for difficult matrix $A$. Instead of working directly with the original model $y = Ax + w$, [15] proposed to apply AMP to a unitary transform of the original model, i.e., $U^H y = \Lambda V x + U^H w$ where the unitary matrix $U$ is obtained by the singular value decomposition (SVD) of matrix $A$, i.e., $A = U \Lambda V$. AMP with unitary transformation (called UTAMP for convenience) has been employed for sparse Bayesian learning (SBL) [16], and it shows outstanding performance even with difficult measurement matrices. This motivates the work in this paper to design robust message passing bilinear recovery algorithms with UTAMP. The vector AMP (VAMP) algorithm [17] was also proposed to address the divergence issue of AMP. To solve the bilinear recovery problem, building on VAMP, lifted VAMP was proposed in [18], and the bilinear adaptive VAMP (BAd-VAMP) was proposed in [7], which inherit the robustness of VAMP. It was shown in [7] that, BAd-VAMP is more robust and faster, and it can outperform lifted VAMP significantly.

In this work, a more robust and faster algorithm for bilinear recovery is designed based on UTAMP, which is called Bi-UTAMP for convenience. By using the lifting approach, the original bilinear problem is reformulated to a linear one. Then efficient hybrid message passing with UTAMP is designed to recover $\{b_k\}$ and $C$ simultaneously. It is shown in this paper that, Bi-UTAMP performs significantly better and is much faster than BAd-VAMP for difficult matrices.

The remainder of this paper is organized as follows. In Section II, AMP with unitary transformation is briefly introduced. Bi-UTAMP is designed for SMV problems and it is then extended for MMV problems in Section III. Numerical examples are provided in Section IV and conclusions are drawn in Section V.

Notations- Boldface lower-case and upper-case letters denote vectors and matrices, respectively, and superscripts $(\cdot)^H$ and $(\cdot)^T$ represent conjugate transpose and transpose, respectively. A Gaussian distribution of $x$ with mean $\tilde{x}$ and variance $\nu_x$ is represented by $\mathcal{N}(x; \tilde{x}, \nu_x)$. We also simply use $\mathcal{N}(m, v)$ to represent a Gaussian distribution with mean $m$ and variance $v$. Notation $\otimes$ represents the Kronecker product. The relation $f(x) = cg(x)$ for some positive constant $c$ is written as...
Algorithm 1 Vector Stepsize AMP

\begin{align*}
\text{Initialize } & \tau^{(0)} > 0 \text{ and } x^{(0)}. \text{ Set } s^{(-1)} = 0 \text{ and } t = 0. \\
\text{Repeat} & \\
1: & \tau_p = |A|^2 \tau^t_p \\
2: & p = Ax^t - \tau_p \cdot s^{t-1} \\
3: & \tau_s = 1 / (\tau_p + \beta^{-1} 1) \\
4: & s^t = s \cdot (y - p) \\
5: & 1 / \tau_q = \dot{A}^H \tau_s \\
6: & q = x^t + \tau_q \cdot \dot{A}^H s^t \\
7: & \tau_{x}^{t+1} = \tau_q \cdot g_{\tau_q}(q, \tau_q) \\
8: & x^{t+1} = g_s(q, \tau_q) \\
9: & t = t + 1 \\
\text{Until terminated} 
\end{align*}

Algorithm 2 UTAMP Version 1

Unitary transform: \(r = U^H y = \Phi x + \omega\), where \(\Phi = U^H A = \Lambda V\), and \(U\) is obtained from the SVD \(A = U \Lambda V\). Initialize \(\tau_x^{(0)} > 0\) and \(x^{(0)}\). Set \(s^{(-1)} = 0\) and \(t = 0\).

\begin{align*}
\text{Repeat} & \\
1: & \tau_p = \tau^t_x \lambda \\
2: & p = \Phi x^t - \tau_p \cdot s^{t-1} \\
3: & \tau_s = 1 / (\tau_p + \beta^{-1} 1) \\
4: & s^t = s \cdot (r - p) \\
5: & 1 / \tau_q = \dot{A}^H \tau_s \\
6: & q = x^t + \tau_q \cdot \dot{A}^H s^t \\
7: & \tau_{x}^{t+1} = \tau_q \cdot g_{\tau_q}(q, \tau_q) \\
8: & x^{t+1} = g_s(q, \tau_q) \\
9: & t = t + 1 \\
\text{Until terminated} 
\end{align*}

II. APPROXIMATE MESSAGE PASSING WITH UNITARY TRANSFORMATION

A. AMP Algorithms

The AMP algorithm [5] was developed based on the loopy belief propagation for compressive sensing with model

\[ y = Ax + w \tag{2} \]

where \(y\) is a measurement, \(A\) is a known \(M \times N\) measurement matrix, \(w\) is a white Gaussian noise vector with distribution \(\mathcal{N}(w; 0, \beta^{-1} I)\). AMP enjoys low complexity and its performance can be rigorously characterized by a scalar state evolution in the case of large i.i.d. (sub)Gaussian matrix \(A\) [19]. However, for a generic \(A\), the convergence of AMP cannot be guaranteed, e.g., AMP can easily diverge for non-zero mean, rank-deficient, correlated, or ill-conditioned matrix \(A\) [14, 15].

Inspired by [20], it was discovered in [15] that the AMP algorithm can still work well for difficult \(A\). In [15], instead of employing the original model [2], AMP is applied to a unitary transform of \(F\). As any matrix \(A\) can have an SVD \(A = U \Lambda V\), a unitary transformation with \(U^H\) can be performed, i.e.,

\[ r = \Phi x + \omega \tag{3} \]

where \(r = U^H y\), \(\Phi = U^H A = \Lambda V\), \(A\) is an \(M \times N\) rectangular diagonal matrix, and \(\omega = U^H w\) is still a zero-mean Gaussian noise vector with the same covariance matrix \(\beta^{-1} I\) as \(U^H\) is a unitary matrix. Then the vector stepsize AMP [8] shown in Algorithm [1] can be applied to model
yielding a version of the UTAMP shown in Algorithm 2. It is interesting that, with such simple pre-processing, the robustness of AMP is remarkably enhanced, enabling it to handle difficult A.

As in [13], applying the average operation to two vectors \( \tau_x \) in Line 7 and \( |\Phi^H|^2 \tau_x \) in Line 5 in Algorithm 2 leads to the second version of UTAMP Algorithm 3. Specifically, due to the average operation in Line 7 of Algorithm 2, \( \tau_x^t \) in Line 1 turns into a scaled all-one vector \( \tau_x^t \mathbf{1} \). With \( \Phi = AV \), it is not hard to show that

\[
\tau_p = |\Phi|^2(\tau_x^t \mathbf{1}) = \tau_x^t \Lambda,
\]

which is Line 1 of Algorithm 3. Performing the average operation to \( |\Phi^H|^2 \tau_x \), i.e.,

\[
\frac{1}{N} \mathbf{1}^H \Phi^H |\tau_x|^2 \Phi = \frac{1}{N} \Lambda^H \tau_x,
\]

leads to Line 5 of Algorithm 3. The two average operations result in lower complexity. Compared to Algorithm 1 and Algorithm 2, Line 1 and Line 5 of Algorithm 3 do not require matrix-vector product operations, i.e., the number of matrix-vector product operations is reduced from 4 to 2 per iteration, which is a significant reduction as the complexity of AMP-like algorithms is dominated by matrix-vector products.

In many cases, the noise precision \( \beta \) is unknown. The noise precision estimation can be incorporated into the algorithm as in [16], and UTAMP with noise precision estimation is summarized in Algorithm 4. It is noted that we rewrite \( v_h = 1/(1/\tau_p + \beta^{-1} - 1) \) in [16] as \( v_h = \tau_p/(1 + \beta^{-1} \tau_p) \), and rewrite \( h = v_h \cdot (\beta^{-1}r + p/\tau_p) \) in [16] as \( h = (\beta^{-1} \tau_p \cdot r + p)/(1 + \beta^{-1} \tau_p) \) to avoid numerical problem as \( \tau_p \) may contain zero elements.

In the above AMP algorithms, the function \( g_x(q, \tau_q) \) returns a column vector whose \( n \)-th element, denoted as \( [g_x(q, \tau_q)]_n \), is given by

\[
[g_x(q, \tau_q)]_n = \int x_n p(x_n) N(x_n; q_n, \tau_q) dx_n / \int p(x_n) N(x_n; q_n, \tau_q) dx_n,
\]

where \( p(x_n) \) is the prior of \( x_n \). Equation (6) can be interpreted as the minimum mean square error (MMSE) estimation of \( x_n \) based on the following model

\[
g_n = x_n + \omega
\]

where \( \omega \) is a Gaussian noise with mean zero and variance \( \tau_q \). The function \( g_x^t(q, \tau_q) \) returns a column vector and the \( n \)-th element is denoted by \( [g_x^t(q, \tau_q)]_n \), where the derivative is with respect to \( q_n \). Note that \( g_x(q, \tau_q) \) can also be changed for the MAP (maximum a posterior) estimation of \( x \).

B. State Evolution of UTAMP

The performance of UTAMP can be characterized by the following simple recursion (for a more general matrix \( A \) compared to AMP)

\[
\tau^t = \frac{1}{|\Lambda|^2} (\lambda / (v^t \lambda + \beta^{-1} - 1))
\]

\[
v_{x+1}^{t+1} = E \left[ |g_x(x + \sqrt{\tau} z) - x|^2 \right]
\]

III. BILINEAR UTAMP

In this section, we start with the case of SMV, and then extend it to the case of MMV.

A. Model Reformulation and Factor Graph Representation

Similar to the lifting approach, we define \( A \triangleq [A_1, \ldots, A_K]_{M \times NK} \) and \( b \triangleq [b_1, \ldots, b_K]^T \), then the original bilinear model can be reformulated as

\[
y = Ax + w
\]

with

\[
x = b \otimes c = \begin{pmatrix} b_1 c_1 \\ \vdots \\ b_K c \end{pmatrix}_{NK \times 1}
\]
where $x$ can be indexed as

$$x = [x_{1,1}, \ldots, x_{N,1}, \ldots, x_{n,k}, \ldots, x_{N,K}]^T$$  \quad (13)

with

$$x_{n,k} = c_n b_k.$$  \quad (14)

Assume priors $p(c)$ and $p(b)$ for $c$ and $b$. In addition, the precision of the noise denoted by $\beta$ is unknown. Our aim is to efficiently estimate $c$ and $b$. It is noted that, in some applications, $b_1 = 1$, so we do not need to estimate $b_1$, which can be treated separately as shown in the Bi-UTAMP algorithm.

Following UTAMP, we have SVD for matrix $A$, i.e.,

$$A = U \Lambda V$$

where $U$ and $V$ being unitary matrices and $\Lambda$ being a rectangular diagonal matrix. Performing unitary transformation, we have $r = \Phi x + \omega$, where $r = U^H y$, $\Phi = \Lambda V$ with size $M \times N K$, and $\omega = U^H w$, which is still white and Gaussian with the same precision $\beta$.

Define an auxiliary variable $z = \Phi x$. We have the following joint distribution and its factorization

$$p(x, c, b, \beta | r) \propto p(r | z, \beta) p(z | x) p(x | c, b) p(c) p(b) p(\beta)$$

$$f_z(z, \beta) f_x(x, z) f_c(x, c, b) f_b(b, \beta) f_(\beta).$$  \quad (15)

The probability functions and the corresponding factors (to facilitate the factor graph representation) are listed in Table 1. The factor graph representation of (15) is depicted in Fig. 2.

We note that the function $f_x(x, c, b)$ can be further factorized, i.e.,

$$f_x(x, c, b) = \prod_{k, n} f_{x_{n,k}}(b_k, \nu_{z_{k}}),$$  \quad (16)

and the factor $f_{x_{n,k}}(c_n, b_k)$ is shown in Fig. 3.

**Table I: Distributions and factors in (15)**

| Factor | Distribution | Function |
|--------|--------------|----------|
| $f_r$  | $p(r | x, \beta)$ | $N(z | r, \beta^{-1} I)$ |
| $f_z$  | $p(z | x)$ | $\delta(z - \Phi x)$ |
| $f_c$  | $p(x | c, b)$ | $\delta(x - b \circ c)$ |
| $f_{x_{n,k}}$ | $p(x_{n,k} | b_k, \nu_{z_{k}})$ | $\delta(x_{n,k} - b_k \circ c_{n,k})$ |
| $f_b$  | $p(b)$ | prior of $b$, e.g., prior promoting sparsity |
| $f_\beta$ | $p(\beta)$ | $\nu_{\beta}$ |

As shown in Fig. 2, we divide the graph into two parts: Part (i) and Part (ii), and elaborate the message updates in each part.

**B. Message Passing in Part (i)**

The derivation of message computations in this part mainly follows UTAMP. Note that the size of matrix $\Phi$ is $M \times NK$. We partition it into $K$ sub-matrices $\{\Phi_k, k = 1, \ldots, K\}$, each with a size of $M \times N$, i.e.,

$$\Phi = [\Phi_1, \ldots, \Phi_K].$$  \quad (17)

Accordingly, we also divide the length-$NK$ vector $x$ into $K$ length-$N$ vectors $\{x_k, k = 1, \ldots, K\}$, i.e.,

$$x = [x_1^T, \ldots, x_K^T]^T,$$  \quad (19)

where $x_0 = c$ if $b_0 = 1$. With the above definitions, we have the following model

$$r = \sum_{k=1}^{K} \Phi_k x_k + \omega.$$  \quad (20)

1) **Backward Message Passing:** Assume that the incoming message from Part (ii) are available, which are the mean and variance of $x_k$. Following UTAMP, we assume that the elements of $x_k$ have a common variance $\nu_{x_k}$, and the computation of $\nu_{x_k}$ will be detailed later. The mean of $x$ is denoted by $\hat{x}$. Then we calculate two vectors $\nu_p$ and $p$ as

$$\nu_p = \sum_{k=0}^{K} \Phi_k \nu_{x_k}$$  \quad (21)

$$p = \sum_{k=0}^{K} \Phi_k \hat{x}_k - \nu_p \cdot s,$$  \quad (22)

where $s$ is a vector, which is computed in the last iteration. Using the MF rule [21], [22] at the factor node $f_r$ in Fig. 2 as in [16], we can update the variance $\nu_z$ and mean $\hat{z}$ of $z$ as

$$\nu_z = 1 / (1 / \nu_p + \hat{\beta} M)$$  \quad (23)

$$\hat{z} = \nu_z \cdot (p / \nu_p + \hat{\beta} r)$$  \quad (24)

where $\hat{\beta}$ is the estimate of the noise precision $\beta$ in the last iteration. The estimate of the noise precision is then updated as

$$\hat{\beta} = \frac{M}{\| r - \hat{z} \|^2 + 1^T \nu_z},$$  \quad (25)

where we slightly abuse the use of the notation $\hat{\beta}$ as we do not distinguish it from the last iteration. The details for the derivation of (23) to (25) are omitted, which can be found in [16].
2) Forward Message Passing: With estimated noise precision in the backward message passing, we update the intermediate vectors $\nu_s$ and $s$ by

$$\nu_s = 1 / (\nu_p + \beta^{-1} 1)$$

$$s = \nu_{s^*} \cdot (r - p).$$

Then calculate vectors $\nu_{q_k}$ and $q_k$ for $k = 0, \ldots, K$ with

$$\nu_{q_k} = 1 / \langle |\Phi_k^H|^2 \nu_s \rangle$$

$$q_k = \bar{x}_k + \nu_{q_k} \Phi_k^H s.$$  

The messages $q_k$ and $\nu_{q_k}$ are the mean and variance of $x_k$, which are passed to Part (ii).

C. Message Passing in Part (ii)

We use the graph in Fig[3] to derive the forward and backward message computations.

1) Forward Message Passing: The $n$th entry of $q_k$ is denoted by $q_{n,k}$. With the incoming message from Part (i) $m_{x_{n,k} \rightarrow f_{x_{n,k}}}(x_{n,k}) = \mathcal{N}(x_{n,k}; q_{n,k}, \nu_{q_k})$, and the factor $f_{x_{n,k}} = \delta(x_{n,k} - b_k c_n)$, we can get an intermediate function node $f_{x_{n,k}}(c_n, b_k)$ according to the BP rule [23], [24], [25], i.e.,

$$\bar{f}_{x_{n,k}}(c_n, b_k) = \int_{x_{n,k}} f_{x_{n,k}}(x_{n,k} \rightarrow f_{x_{n,k}}(x_{n,k})) = \mathcal{N}(c_n b_k; q_{n,k}, \nu_{q_k}).$$

With the local function $\bar{f}_{x_{n,k}}(b_k, c_n)$, we can calculate the outgoing message from $f_{x_{n,k}}$ to $c_n$ with the MF rule, i.e.,

$$m_{f_{x_{n,k}} \rightarrow c_n}(c_n) = \exp \left\{ \int_{b_k} \log \bar{f}_{x_{n,k}}(b_k) \right\} = \mathcal{N}(c_n; \bar{c}_{n,k}, \bar{\nu}_{c_{n,k}})$$

where

$$\bar{c}_{n,k} = \frac{q_{n,k} b_k}{|b_k|^2 + \nu_{b_k}},$$

$$\bar{\nu}_{c_{n,k}} = \frac{\nu_{q_k}}{|b_k|^2 + \nu_{b_k}}.$$  

with $\bar{b}_k$ and $\nu_{b_k}$ being the mean and variance of the Gaussian belief of $b_k$, which are computed in [50] and [51]. It is noted that, in the case of $b_1 = 1$, we simply set $\bar{b}_1 = 1$ and $\nu_{b_1} = 0$. Then the message $m_{c_n \rightarrow f_{c_n}}(c_n)$ can be represented as

$$m_{c_n \rightarrow f_{c_n}}(c_n) = \mathcal{N}(c_n; \bar{c}_n, \bar{\nu}_{c_n})$$

with

$$\bar{\nu}_{c_n} = 1 / \sum_{k=1}^K \frac{1}{\bar{\nu}_{c_{n,k}}},$$

$$\bar{c}_n = \bar{\nu}_{c_n} \sum_{k=1}^K \bar{c}_{n,k}.$$  

So, the belief of $c_n$ ($n = 1, \ldots, N$) can be expressed as

$$b(c_n) = \int_{c_n} m_{c_n \rightarrow f_{c_n}}(c_n) f_c,$$

which is projected to be Gaussian, i.e.,

$$b'(c_n) = \mathcal{N}(c_n; \bar{c}_n, \nu_{c_n})$$

with

$$\bar{c}_n = \mathbb{E}\left[c_n | \nu_{c_n} \right]$$

$$\nu_{c_n} = \mathbb{V}\left[c_n | \nu_{c_n} \right],$$

which are the a posterior mean and variance of $c_n$ based on the prior $f_c$ and the following pseudo observation model

$$\bar{c}_n = c_n + w_1,$$

with $w_1$ denoting a Gaussian noise with mean 0 and variance $\nu_{c_n}$.

Similarly, we can calculate the outgoing message from $f_{x_{n,k}}$ to $b_k$ with the MF rule, i.e.,

$$m_{f_{x_{n,k}} \rightarrow b_k}(b_k) = \mathcal{N}(b_k; \bar{b}_{n,k}, \bar{\nu}_{b_{n,k}})$$

where

$$\bar{b}_{n,k} = \frac{q_{n,k} \bar{c}_{n,k}}{|\bar{c}_{n,k}|^2 + \nu_{\bar{c}_{n,k}}},$$

$$\bar{\nu}_{b_{n,k}} = \frac{\nu_{q_k}}{|\bar{c}_{n,k}|^2 + \nu_{\bar{c}_{n,k}}}.$$  

with $\bar{c}_n$ and $\nu_{c_n}$ being the mean and variance of the Gaussian belief of $c_n$, which are updated in [39] and [40]. Then the message $m_{b_k \rightarrow f_{b_k}}(b_k)$ can be represented as

$$m_{b_k \rightarrow f_{b_k}}(b_k) = \mathcal{N}(b_k; \bar{b}_k, \nu_{b_k})$$

with

$$\bar{b}_k = 1 / \sum_{n=1}^N \frac{1}{\bar{b}_{n,k}},$$

$$\nu_{b_k} = \nu_{b_k} \sum_{n=1}^N \frac{b_{n,k}}{\bar{b}_{n,k}}.$$  

Then we can compute the belief of each $b_k$,

$$b(b_k) = \int_{b_{v/b_k}} \prod_{b_{n,k}} m_{b_{n,k} \rightarrow f_{b_{n,k}}}(b_{n,k}) f_b,$$

which is projected to be Gaussian, i.e.,

$$b'(b_k) = \mathcal{N}(b_k; \hat{b}_k, \nu_{b_k})$$

with

$$\hat{b}_k = \mathbb{E}[b_k | \nu_{b_k}]$$

$$\nu_{b_k} = \mathbb{V}[b_k | \nu_{b_k}],$$

The a posteriori mean $\hat{b}_k$ and variance $\nu_{b_k}$ of $b_k$ are computed based on the prior $f_b$ and the following pseudo observation model

$$\bar{b}_k = b_k + w_k''$$

with $w_k''$ denoting a Gaussian noise with mean 0 and variance $\nu_{b_k}$. It is noted that, in the case of $b_1 = 1$, we simply set $\bar{b}_1 = 1$ and $\nu_{b_1} = 0.$
Algorithm 5 Bi-UTAMP for SMV

**Unitary transform:** \( r = U^H y = \Phi x + \omega, \) where \( A_{M \times NK} = U \Lambda V, \Phi = U^H A = \Lambda V, \) and \( x = b \otimes c \) with \( b = [b_1, ..., b_K]^T \) and \( c = [c_1, ..., c_N]^T. \)

Let \( \Phi = [\Phi_1, ..., \Phi_K], \Phi_k = [\Phi_k]^2 1_N, \) and \( x = [x_1, ..., x_K]^T, k = 1, ..., K \) and \( n = 1, ..., N. \)

**Initialize**

\( b_k, v_{b_k} = 1, \bar{x}_k = 0, s = 0 \) and \( \beta = 1. \)

**Repeat**

1. \( \nu_p = \sum_k \Phi_k \nu_{x_k} \)
2. \( p = \sum_k \Phi_k x_k - \nu_p \cdot s \)
3. \( \nu_z = \nu_{\nu_p} / (1 + \beta \nu_p) \)
4. \( \hat{z} = (\beta \nu_p \cdot r + p) / (1 + \beta \nu_p) \)
5. \( \lambda = M / (\| r - \hat{z} \|^2 + 1)^{1/2} \)
6. \( \nu_s = 1 / (\nu_p + \beta \lambda 1_M) \)
7. \( s = \nu_s \cdot (r - p) \)
8. \( \forall k: \nu_{q_k} = 1 / (\Phid \nu_s) \)
9. \( \forall k: q_k = \bar{x}_k + v_{q_k} \quad \Phid \hat{x}_k \)

(\( \text{In the case of } b_1 = 1, \text{ set } b_1 = 1 \) and \( \nu_{b_1} = 0. \))

10. \( \forall k: \nu_{c_k} = q_{b_k} / [(b_k^2 + \nu_{b_k})] \)
11. \( \forall k: \nu_{c_k} = 1_N \nu_{q_k} / [(b_k^2 + \nu_{b_k})] \)
12. \( \hat{c}_k = 1_N / (\sum_k 1_N \nu_{c_k}) \)
13. \( \nu_{c_k} = [\nu_{c_1}, ..., \nu_{c_N}]^T, \quad \hat{c} = [\hat{c}_1, ..., \hat{c}_N]^T \)
14. \( \forall n: \nu_{c_n} = \nu_{c_n} / [(\nu_{c_n} + \nu_{c_n})] \)
15. \( \forall n: \nu_{c_n} = \nu_{c_n} / [(\nu_{c_n} + \nu_{c_n})] \)
16. \( \nu_{c_k} = \nu_{c_k} / [(\nu_{c_k} + \nu_{c_k})] \)
17. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
18. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
19. \( \forall k: \nu_{b_k} = (1_N \nu_{q_k})^T \)
20. \( \forall k: \nu_{b_k} = (1_N \nu_{q_k})^T \)
21. \( \forall k: b_k = \mathbb{E}[\nu_{b_k} | \nu_{b_k}, \hat{b}_k, f_b] \)
22. \( \forall k: b_k = \mathbb{E}[\nu_{b_k} | \nu_{b_k}, \hat{b}_k, f_b] \)

(\( \text{In the case of } b_1 = 1, \text{ set } b_1 = 1 \) and \( \nu_{b_1} = 0. \))

23. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
24. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
25. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
26. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
27. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
28. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
29. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
30. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)
31. \( \forall k: \nu_{b_k} = \nu_{q_k} / [(\nu_{c_k} + \nu_{c_k})] \)

**Until terminated**

2) **Backward Message Passing:** The backward message \( m_{b \otimes f_x}(b) \) is Gaussian with mean \( \bar{b} \) and variance \( \hat{b} \), which can be calculated as

\[
\begin{align*}
\nu_b & = (1_N \otimes \nu_b) - 1_N / (\nu_b - 1_N) \\
\hat{b} & = \bar{b} \cdot (\nu_b / (1_N - \nu_b)) \cdot (\nu_b - 1_N) \cdot (\nu_b - 1_N)
\end{align*}
\]
be computed in parallel. The major difference lies in the computations of $b(b)$ and $b(\beta)$, where the messages from $f_{x_l}$ and $f_{r_l}$, ∀l, should be considered, i.e.,

$$b(b) \propto \prod_l m_{f_{x_l} \rightarrow b(b)} m_{f_{b} \rightarrow b(b)}$$

(67)

$$b(\beta) \propto \prod_l m_{f_{r_l} \rightarrow \beta(\beta)} m_{f_{x_l} \rightarrow \lambda(\beta)}$$

(68)

Similar to the SMV case, the message computation rules can be derived, which is summarized as Algorithm 5 (Bi-UTAMP for MMV).

E. Discussions and Complexity Analysis

We have the following remarks on Bi-UTAMP:

1: In some problems, $b_1$ is known, e.g., $b_1 = 1$. In this case, we can set $\tilde{b}_1 = 1$ and $\nu_1 = 0$ in Bi-UTAMP, which are indicated in Algorithm 5.

2: It can be shown that, when $b = b_1 = 1$, Bi-UTAMP is equivalent to UTAMP (Algorithm 5) exactly.

3: The robustness of Bi-UTAMP can be enhanced by simply damping $s$, i.e., Line 7 of the MMV Bi-UTAMP is changed as

$$s = (1 - \alpha)s + \alpha \nu_\nu \cdot (r - p)$$

(69)

with $\alpha \in (0, 1]$, where $\alpha$ is the damping factor and $\alpha = 1$ leads to the case without damping. Accordingly, Line 7 of the MMV Bi-UTAMP is changed as $s_l = (1 - \alpha)s_l + \alpha \nu_{\nu_l} \cdot (r - p)$.

4: The iterative process can be terminated based on some criterion, e.g., the normalized difference between the estimates of $b$ of two consecutive iterations is smaller than a threshold, i.e., $\|b^t - b^{t-1}\|^2/\|b(t)\|^2 < \epsilon$ where $b^t$ is the estimate of $b$ at the $t$th iteration and $\epsilon$ is a threshold.

5: As the bilinear problem has local minima, we can use the same strategy of restart as in [7] to mitigate the issue of being stuck at local minima. For each restart, we initialize $b$ with a different value.

6: Bi-UTAMP is attractive due to its low complexity compared to other alternates. Bi-UTAMP needs pre-processing, i.e., computing matrix $U$ from $A$, and performing unitary transform, and the complexity is $O(M^2NK)$. It is also worth mentioning that the pre-processing can be carried out offline (although we do not assume this in the simulations in Section IV). It can be seen from the iterative process of Bi-UTAMP that, there is no matrix inversion involved in Bi-UTAMP, and the complexity per iteration is $O(MNK)$ (in the case of SMV, $L = 1$), which linearly increases with $M$, $N$, $K$ and $L$. With lower complexity, Bi-UTAMP outperforms the state-of-the-art algorithm significantly.

IV. NUMERICAL EXAMPLES

In this section, we evaluate the performance of Bi-UTAMP and compare it with the state-of-the-art message passing based algorithm BAd-VAMP. Performance bounds are also included for reference.
the measurement matrix is modeled as \( A \). We aim to recover a sparse signal vector \( s \) sensing with matrix uncertainty [1].

![Fig. 6: Compressive sensing with correlated matrices: NMSE of \( b \) and \( c \) versus SNR with (a) \( \rho = 0.2 \) and (b) \( \rho = 0.3 \).](image)

**A. SMV Case**

For the SMV case, we take the example of compressive sensing with matrix uncertainty [1]. We aim to recover a sparse signal vector \( c \) from measurement \( y = A(b)c + w \), where the measurement matrix is modeled as \( A(b) = \sum_{k=1}^{K} b_k A_k \) with \( b_1 = 1 \), \( A_k \in \mathbb{R}^{M \times N} \) are known, and the uncertainty parameter vector \( b = [b_2, ..., b_K]^T \) is unknown. In addition the precision of the noise is unknown as well.

In the experiments, we set \( K = 11, N = 256, M = 150 \) and the number of nonzero elements in \( c \) is 10. The SNR is defined as \( SNR = 10 \log \frac{E[|A c|^2]}{E[|w|^2]} \). The uncertainty parameters \( \{b_2, ..., b_K\} \) are drawn from \( \mathcal{N}(0, 1) \) independently, and the nonzero elements of sparse vector \( c \) are drawn from \( \mathcal{N}(0, 1) \) independently, which are randomly located in \( c \). The performance of the methods are evaluated using NMSE(b) and NMSE(c)

![Fig. 5: Compressive sensing with correlated matrices: NMSE of \( b \) and \( c \) versus \( \rho \) at SNR = 40dB.](image)

![Fig. 7: Compressive sensing with non-zero mean matrix: NMSE of \( b \) and \( c \) versus \( \mu \) with SNR = 40dB.](image)

It is noted that, different from [2], we do not use median NMSEs, and the NMSEs are obtained by averaging the results from all trails. To demonstrate the robustness of Bi-UTAMP, we focus on tough measurement matrices, e.g., correlated matrix, non-zero mean matrix, and ill-conditioned matrix. Bi-UTAMP and BAd-VAMP use a same damping factor of 0.8 to enhance robustness.

1) **Correlated Measurement Matrix**: All matrices \( \{A_k\} \) are correlated, and \( A_k \) is constructed using \( A_k = C_L^{1/2} G_k C_R^{1/2} \), where \( G_k \) is an i. i. d. Gaussian matrix, and \( C_L \) is an \( M \times M \) matrix with the \((m, n)\)th element given by \( \rho^{m-n} \) where \( \rho \in [0, 1] \). Matrix \( C_R \) is generated in the same way but with a size of \( N \times N \). The parameter \( \rho \) controls the correlation of matrix \( A_k \). Fig 4 shows the NMSE performance of the algorithms versus SNR, where the correlation parameter \( \rho = 0.2 \) in (a) and \( \rho = 0.3 \) in (b). It can be seen that when \( \rho = 0.2 \), both Bi-UTAMP and BAd-VAMP perform very well and they deliver almost the same performance. However, when \( \rho = 0.3 \), Bi-UTAMP still works very well, and it significantly outperforms BAd-VAMP at relative high SNRs. We further evaluate the performance of Bi-UTAMP and BAd-VAMP with the parameter \( \rho \) at SNR = 40dB and the results are shown.
The coefficient matrix $C \in \mathbb{Y} = A$ is a goal of structured DL. The structured dictionary matrix is much faster than BAd-V AMP. The results are obtained using MATLAB (R2016b) on a computer. Matrices and different condition numbers for ill-conditioned matrices are used with the correlation parameter $\rho$. The results demonstrate the better performance of Bi-UTAMP. With SNR = 40 dB, the average run-time versus different $\rho$ is shown in Fig. 8, where the SNR = 40 dB. The results again demonstrate the better performance of Bi-UTAMP compared to BAd-V AMP especially when $\mu$ is relatively large.

2) Non-Zero Mean Measurement Matrix: The elements of matrix $A_k$ are independently drawn from a non-zero mean Gaussian distribution $\mathcal{N}(\mu, \nu)$. The mean $\mu$ measures the derivation from the i. d. zero-mean Gaussian matrix. In the simulations, for $\{A_k, k = 2 : K\}$, $v = 1$, and for $A_1$, $v = 20$. The NMSE performance of the algorithms versus $\mu$ is shown in Fig. 7, where the SNR = 40 dB. It can be seen from this figure that Bi-UTAMP can achieve much better performance compared to BAd-V AMP.

3) Ill-Conditioned Measurement Matrix: Each matrix $A_k$ is constructed based on the SVD $A_k = U_k \Lambda_k V_k$ where $\Lambda_k$ is a singular value matrix with $\Lambda_{k+1,k+1} = k^{1/(M-1)}$ (i.e., the condition number of the matrix is $k$). The NMSE performance of the algorithms versus the condition number is shown in Fig. 8 where the SNR = 40 dB. The results again demonstrate the better performance of Bi-UTAMP.

4) Run-Time Comparison: Fig. 9 compares the average run-time of Bi-UTAMP and BAd-UTAMP. In Fig. 9(a), correlated matrices are used with the correlation parameter $\rho = 0.3$. With SNR = 40 dB, the average run-time versus different $\rho$ for correlated matrices, different means for non-zero mean matrices, and different condition numbers for ill-conditioned matrices is given in Fig. 9(b), (c), and (d), respectively. The results are obtained using MATLAB (R2016b) on a computer with a 6-core Intel i7 processor. Fig. 9 shows that Bi-UTAMP is much faster than BAd-VAMP.

B. MMV Case

We take the structured dictionary learning (DL) as an example to demonstrate the performance of Bi-UTAMP. The goal of structured DL is to find a structured dictionary matrix $A = \sum_{k=1}^{K} b_k A_k \in \mathbb{R}^{M \times N}$ from the training samples $Y \in \mathbb{R}^{M \times L}$ with model $Y = AC + W$ for some sparse coefficient matrix $C \in \mathbb{R}^{N \times L}$. In the simulations, we assume square dictionary matrix $A$ with $M = N = 100$. The length of vector $b$, i.e., $K = 100$, and the number of non-zero elements set to be 20 in each column of $C$ and $L = 5$ for the training examples. Since the dictionary matrix $A$ has a structure, it can be learned with a small number of training samples. Bi-UTAMP is run for maximum 100 iterations and 10 restarts. In addition, to enhance the robustness, we use a damping factor 0.55 for Bi-UTAMP and BAd-V AMP. In addition, Lines 19-22 in Bi-UTAMP are executed once every two iterations. The performance is evaluated with NMSE of the estimates of $A$ and $C$. As the pair $(A, C)$ has a scalar ambiguity, the NMSE is calculated in the same way as in [7], i.e.,

$$\text{NMSE}(\hat{A}) \triangleq \frac{1}{J} \sum_{j=1}^{J} \min_d \frac{||A_j - d\hat{A}_j||^2}{||A_j||^2}$$

$$\text{NMSE}(\hat{C}) \triangleq \frac{1}{J} \sum_{j=1}^{J} \min_d \frac{||C_j - d\hat{C}_j||^2}{||C_j||^2}.$$ 

The NMSEs are obtained by averaging the results from all trails. To test the performance and robustness of the algorithms, correlated matrices $\{A_k\}$ generated in the same way as in the SMV case are used.
Figure 10 shows the NMSE performance NMSE(A) and NMSE(C) versus SNR with (a) \( \rho = 0 \) and (b) \( \rho = 0.1 \). It can be seen that when \( \rho = 0 \), i.e., \( \{ A_k \} \) are i.i.d. Gaussian, BAd-VAMP and Bi-UTAMP have similar performance. When \( \rho = 0.1 \), Bi-UTAMP can outperform BAd-UTAMP considerably. Figure 11 shows the NMSE versus \( \rho \) at SNR = 40dB, where we can see that Bi-UTAMP can achieve significantly better performance than BAd-VAMP. From these results, we conclude that Bi-UTAMP is more robust. Figure 12 shows the average run time versus (a) SNR and (b) \( \rho \). Again, Bi-UTAMP is much faster than BAd-VAMP.

V. CONCLUSIONS

In this paper, building on UTAMP, we have designed a new algorithm Bi-UTAMP for the bilinear recovery problem, where UTAMP is combined with hybrid message passing to achieve efficient recovery of the unknown variables. We have shown that Bi-UTAMP is much more robust and faster than the state-of-the-art algorithm, leading to significantly better performance.

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