BIG COHEN-MACaulAY MODULES,
MORPHISMS OF PERFECT COMPLEXES, AND
INTERSECTION THEOREMS IN LOCAL ALGEBRA

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Abstract. There is a well known link from the first topic in the title to the third one. In this paper we thread that link through the second topic. The central result is a criterion for the tensor nilpotence of morphisms of perfect complexes over commutative noetherian rings, in terms of a numerical invariant of the complexes known as their level. Applications to local rings include a strengthening of the Improved New Intersection Theorem, short direct proofs of several results equivalent to it, and lower bounds on the ranks of the modules in every finite free complex that admits a structure of differential graded module over the Koszul complex on some system of parameters.

1. Introduction

A big Cohen-Macaulay module over a commutative noetherian local ring $R$ is a (not necessarily finitely generated) $R$-module $C$ such that some system of parameters of $R$ forms a $C$-regular sequence. In [16] Hochster showed that the existence of such modules implies several fundamental homological properties of finitely generated $R$-modules. In [17], published in [18], he proved that big Cohen-Macaulay modules exist for algebras over fields, and conjectured their existence in the case of mixed characteristic. This was recently proved by Y. André in [2]: as a major consequence many “Homological Conjectures” in local algebra are now theorems.

A perfect $R$-complex is a bounded complex of finite projective $R$-modules. Its level with respect to $R$, introduced in [6] and defined in 2.3, measures the minimal number of mapping cones needed to assemble a quasi-isomorphic complex from bounded complexes of finite projective modules with differentials equal to zero.

The main result of this paper, which appears as Theorem 3.3, is the following

**Tensor Nilpotence Theorem.** Let $f: G \to F$ be a morphism of perfect complexes over a commutative noetherian ring $R$.

If $f$ factors through a complex whose homology is $I$-torsion for some ideal $I$ of $R$ with height $I \geq \text{level} R$, then the induced morphism

$$\otimes_R^n f: \otimes_R^n G \to \otimes_R^n F$$

is homotopic to zero for some non-negative integer $n$. 

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Big Cohen-Macaulay modules play an essential, if discreet role in the proof, as a tool for constructing special morphisms in the derived category of \( R \); see Proposition 3.7.

In applications to commutative algebra it is convenient to use another property of morphisms of perfect complexes: \( f \) is fiberwise zero if \( H(k(p) \otimes_R f) = 0 \) holds for every \( p \) in \( \text{Spec } R \). Hopkins [21] and Neeman [25] have shown that this is equivalent to tensor nilpotence; this is a key tool for the classification of the thick subcategories of perfect \( R \)-complexes.

It is easy to see that the level of a complex does not exceed its span, defined in [21]. Due to these remarks, the Tensor Nilpotence Theorem is equivalent to the Morphic Intersection Theorem.

If \( f \) is not fiberwise zero and factors through a complex with \( I \)-torsion homology for an ideal \( I \) of \( R \), then there are inequalities:

\[
\text{span } F + \text{span } G - 1 \geq \text{level}^R \text{Hom}_R(G, F) \geq \text{height } I + 1.
\]

In Section 4 we use this result to prove directly, and sometimes to generalize and sharpen, several basic theorems in commutative algebra. These include the Improved New Intersection Theorem, the Monomial Theorem and several versions of the Canonical Element Theorem. All of them are equivalent, but we do not know if they imply the Morphic Intersection Theorem; a potentially significant obstruction to that is that the difference \( \text{span } F - \text{level } F \) can be arbitrarily big.

2. Perfect complexes

Throughout this paper \( R \) will be a commutative noetherian ring.

This section is a recap on the various notions and construction, mainly concerning perfect complexes, needed in this work. Pertinent references include [6, 26].

2.1. Complexes. In this work, an \( R \)-complex (a shorthand for ‘a complex of \( R \)-modules’) is a sequence of homomorphisms of \( R \)-modules:

\[
X := \cdots \rightarrow X_n \xrightarrow{\partial^n_X} X_{n-1} \xrightarrow{\partial^{n-1}_X} X_{n-2} \rightarrow \cdots
\]

such that \( \partial^n X \partial^X = 0 \). We write \( X^g \) for the graded \( R \)-module underlying \( X \). The \( i \)th suspension of \( X \) is the \( R \)-complex \( \Sigma^i X \) with \( (\Sigma^i X)_n = X_{n-i} \) and \( \partial^X_{n+i} = (-1)^i \partial^X_{n-i} \) for each \( n \). The span of \( X \) is the number

\[
\text{span } X := \sup \{i \mid X_i \neq 0\} - \inf \{i \mid X_i \neq 0\} + 1
\]

Thus span \( X = -\infty \) if and only if \( X = 0 \), and span \( X = \infty \) if and only if \( X_i \neq 0 \) for infinitely many \( i \geq 0 \). The span of \( X \) is finite if and only if span \( X \) is a natural number. When span \( X \) is finite we say that \( X \) is bounded.

Complexes of \( R \)-modules are objects of two distinct categories.

In the category of complexes \( \mathcal{C}(R) \) a morphism \( f : Y \rightarrow X \) of \( R \)-complexes is a family \((f_i : Y_i \rightarrow X_i)_{i \in \mathbb{Z}}\) or \( R \)-linear maps satisfying \( \partial^n_X f_i = f_{i-1} \partial^n_Y \). It is a quasi-isomorphism if \( H(f) \), the induced map in homology, is bijective. Complexes that can be linked by a string of quasi-isomorphisms are said to be quasi-isomorphic.
The derived category $\mathcal{D}(R)$ is obtained from $\mathcal{C}(R)$ by inverting all quasi-isomorphisms. For constructions of the localization functor $\mathcal{C}(R) \to \mathcal{D}(R)$ and of the derived functors $\otimes_R^L$, see e.g. [13, 31, 24]. When $P$ is a complex of projectives with $P_i = 0$ for $i \ll 0$, the functors $P \otimes_R^L$ and $\text{RHom}_R(P, ?)$ are represented by $P \otimes_R ?$ and $\text{Hom}_R(P, ?)$, respectively. In particular, the localization functor induces for each $n$ a natural isomorphism of abelian groups

\begin{equation}
H_{-n}(\text{RHom}_R(P, X)) \cong \text{Hom}_{\mathcal{D}(R)}(P, \Sigma^n X).
\end{equation}

2.2. Perfect complexes. In $\mathcal{C}(R)$, a perfect $R$-complex is a bounded complex of finitely generated projective $R$-modules. When $P$ is perfect, the $R$-complex $P^* := \text{Hom}_R(P, R)$ is perfect and the natural biduality map $P^* \to P^{**} = \text{Hom}_R(\text{Hom}_R(P, R), R)$ is an isomorphism. Moreover for any $R$-complex $X$ the natural map $P^* \otimes_R X \to \text{Hom}_R(P, X)$ is an isomorphism. In the sequel these properties are used without comment.

2.3. Levels. A length $l$ semiprojective filtration of an $R$-complex $P$ is a sequence of $R$-subcomplexes of finitely generated projective modules

$$0 = P(0) \subseteq P(1) \subseteq \cdots \subseteq P(l) = P$$

such that $P(i - 1)^\perp$ is a direct summand of $P(i)^\perp$ and the differential of $P(i)/P(i-1)$ is equal to zero, for $i = 1, \ldots, l$. For every $R$-complex $F$, we set

$$\text{level}^R F = \inf \left\{ l \in \mathbb{N} \mid F \text{ is a retract of some } R\text{-complex that has a semiprojective filtration of length } l \right\}.$$

By [6, 2.4], this number is equal to the level $F$ with respect to $R$, defined in [6, 2.1]. In particular, level$^R F$ is finite if and only if $F$ is quasi-isomorphic to some perfect complex. When $F$ is quasi-isomorphic to a perfect complex $P$, one has

\begin{equation}
\text{level}^R F \leq \text{span} P.
\end{equation}

Indeed, if $P := 0 \to P_b \to \cdots \to P_a \to 0$, then consider the filtration by subcomplexes $P(n) := P_{<n+a}$. The inequality can be strict; see [2.4] below.

When $R$ is regular, any $R$-complex $F$ with $\text{H}(F)$ finitely generated satisfies

\begin{equation}
\text{level}^R F \leq \text{dim} R + 1.
\end{equation}

For $R$-complexes $X$ and $Y$ one has

\begin{equation}
\text{level}^R (\Sigma^i X) = \text{level}^R X \quad \text{for every integer } i, \text{ and}
\end{equation}

\begin{equation}
\text{level}^R (X \oplus Y) = \max \left\{ \text{level}^R X, \text{level}^R Y \right\}.
\end{equation}

These equalities follow easily from the definitions.

**Lemma 2.4.** The following statements hold for every perfect $R$-complex $P$.

1. $\text{level}^R (P^*) = \text{level}^R P$.
2. For each perfect $R$-complex $Q$ there are inequalities

$$\text{level}^R (P \otimes_R Q) \leq \text{level}^R P + \text{level}^R Q - 1,$$

$$\text{level}^R \text{Hom}_R(P, Q) \leq \text{level}^R P + \text{level}^R Q - 1.$$
Proof. (1) If $P$ is a retract of $P'$, then $\text{level}^R P \leq \text{level}^R P'$ and $P^*$ is a retract of $(P')^*$. Thus, we can assume $P$ itself has a finite semiprojective filtration $\{P(n)\}_{i=0}^m$. The inclusions $P(l - i) \subseteq P(l - i + 1) \subseteq P$ define subcomplexes

$$P^*(i) := \text{Ker}(P^* \to P(l - i)^*) \subseteq \text{Ker}(P^* \to P(l - i + 1)) =: P^*(i + 1)$$

of finitely generated projective modules. They form a length $l$ semiprojective filtration of $P^*$, as $P^*(i - 1)^S$ is a direct summand of $P^*(i)^S$ and there are isomorphisms

$$\frac{P^*(n)}{P^*(n - 1)} \cong \left( \frac{P(l - n + 1)}{P(l - n)} \right)^*.$$

This gives $\text{level}^R P^* \leq \text{level}^R P$, and the reverse inequality follows from $P \cong P^*$.

(2) Assume first that $P$ has a semiprojective filtration $\{P(n)\}_{i=0}^m$ and $Q$ has a semiprojective filtration $\{Q(n)\}_{j=0}^m$. For all $h, i$, we identify $P(h) \otimes_R Q(i)$ with a subcomplex of $P \otimes_R Q$. For each non-negative integer $n \geq 0$ form the subcomplex

$$C(n) := \sum_{j \geq 0} P(j + 1) \otimes_R Q(n - j)$$

of $P \otimes_R Q$. A direct computation yields an isomorphism of $R$-complexes

$$\frac{C(n)}{C(n - 1)} \cong \sum_{j \geq 0} \frac{P(j + 1)}{P(j)} \otimes_R \frac{Q(n - j)}{Q(n - j - 1)}.$$

Thus $\{C(n)\}_{i=0}^{l+m-1}$ is a semiprojective filtration of $P \otimes_R Q$.

The second inequality in (2) follows from the first one, given (1) and the isomorphism $\text{Hom}_R(P, Q) \cong P^* \otimes_R Q$. Next we verify the first inequality. There is nothing to prove unless the levels of $P$ and $Q$ are finite. Thus we may assume that $P$ is a retract of a complex $P'$ with a semiprojective filtration of length $l = \text{level}^R P$ and $Q$ is a retract of a complex $Q'$ with a semiprojective filtration of length $m = \text{level}^R Q$. Then $P \otimes_R Q$ is a retract of $P' \otimes_R Q'$, and—by what we have just seen—this complex has a semiprojective filtration of length $l + m - 1$, as desired. □

2.5. Ghost maps. A ghost is a morphism $g \colon X \to Y$ in $\mathcal{D}(R)$ such that $\text{H}(g) = 0$; see [11, §8]. Evidently a composition of morphisms one of which is ghost is a ghost.

The next result is a version of the "Ghost Lemma"; cf. [11, Theorem 8.3], [29, Lemma 4.11], and [5, Proposition 2.9].

Lemma 2.6. Let $F$ be an $R$-complex and $c$ an integer with $c \geq \text{level}^R F$.

When $g \colon X \to Y$ is a composition of $c$ ghosts the following morphisms are ghosts

$$F \otimes_R^L g \colon F \otimes_R^L X \to F \otimes_R^L Y$$

and

$$\text{RHom}_R(F, g) \colon \text{RHom}_R(F, X) \to \text{RHom}_R(F, Y)$$

Proof. For every $R$-complex $W$ there is a canonical isomorphism

$$\text{RHom}_R(F, R) \otimes_R^L W \xrightarrow{\cong} \text{RHom}_R(F, W),$$

so it suffices to prove the first assertion. For that, we may assume that $F$ has a semiprojective filtration $\{F(n)\}_{i=0}^m$, where $l = \text{level}^R F$. By hypothesis, $g = h \circ f$ where $f \colon X \to W$ is a $(c - 1)$-fold composition of ghosts and $h \colon W \to Y$ is a ghost. Tensoring these maps with the exact sequence of $R$-complexes

$$0 \to F(1) \xrightarrow{i} F \xrightarrow{\pi} G \to 0$$
TENSOR NILPOTENT MORPHISMS

where $G := F/F(1)$, yields a commutative diagram of graded $R$-modules

\[
\begin{array}{c}
\text{H}(F(1) \otimes_R X) \rightarrow \text{H}(F \otimes_R X) \rightarrow \text{H}(G \otimes_R X) \\
\text{H}(F(1) \otimes f) \downarrow \text{H}(F \otimes f) \downarrow \text{H}(G \otimes f) \\
\text{H}(F(1) \otimes_R W) \rightarrow \text{H}(F \otimes_R W) \rightarrow \text{H}(G \otimes_R W) \\
\text{H}(F(1) \otimes h) \downarrow \text{H}(F \otimes h) \downarrow \\
\text{H}(F(1) \otimes_R Y) \rightarrow \text{H}(F \otimes_R Y) \rightarrow \text{H}(G \otimes_R Y)
\end{array}
\]

where the rows are exact. Since $\text{level}^R G \leq l - 1 \leq c - 1$, the induction hypothesis implies $G \otimes f$ is a ghost; that is to say, $\text{H}(G \otimes f) = 0$. The commutativity of the diagram above and the exactness of the middle row implies that

\[
\text{Im} \text{H}(F \otimes f) \subseteq \text{Im} \text{H}(i \otimes W)
\]

This entails the inclusion below.

\[
\text{Im} \text{H}(F \otimes g) = \text{H}(F \otimes h)(\text{Im} \text{H}(F \otimes f)) \\
\subseteq \text{H}(F \otimes h)(\text{Im} \text{H}(i \otimes W)) \\
\subseteq \text{Im} \text{H}(i \otimes Y) \text{H}(F(1) \otimes h)) = 0
\]

The second equality comes from the commutativity of the diagram. The last one holds because $F(1)$ is graded-projective and $\text{H}(h) = 0$ imply $\text{H}(F(1) \otimes h) = 0$. \qed

2.7. Koszul complexes. Let $x := x_1, \ldots, x_n$ be elements in $R$.

We write $K(x)$ for the Koszul complex on $x$. Thus $K(x)^2$ is the exterior algebra on a free $R$-module $K(x)_1$ with basis $\{\bar{x}_1, \ldots, \bar{x}_n\}$, and $\partial^K$ is the unique $R$-linear map that satisfies the Leibniz rule and has $\partial^K(\bar{x}_i) = x_i$ for $i = 1, \ldots, n$. In particular, $K(x)$ is a DG (differential graded) algebra, and so its homology $\text{H}(K(x))$ is a graded algebra with $\text{H}_0(K(x)) = R/(x)$. This implies $\text{H}(K(x)) = 0$.

Evidently $K(x)$ is a perfect $R$-complex; it is indecomposable when $R$ is local; see [1, 4.7]. As $K(x)_i$ is non-zero precisely for $0, \ldots, n$, from (2.3.1) one gets

\[
\text{level}^R K(x) \leq \text{span} K(x) = n + 1.
\]

Equality holds if $R$ is local and $x$ is a system of parameters; see Theorem 4.2 below. However, span $K(x) - \text{level}^R K(x)$ can be arbitrarily large; see [1, Section 3].

For any Koszul complex $K$ on $n$ elements, there are isomorphisms of $R$-complexes

\[
K^* \cong \Sigma^{-n} K \quad \text{and} \quad K \otimes_R K \cong \bigoplus_{i=0}^n \Sigma^i K^{(n)}.
\]

See [8] Propositions 1.6.10 and 1.6.21. It thus follows from (2.3.3) that

\[
(2.7.1) \quad \text{level}^R \text{Hom}_R(K, K) = \text{level}^R (K \otimes_R K) = \text{level}^R K.
\]

In particular, the inequalities in Lemma 2.4(2) can be strict.

3. Tensor nilpotent morphisms

In this section we prove the Tensor Nilpotence Theorem announced in the introduction. We start by reviewing the properties of interest.
3.1. **Tensor nilpotence.** Let \( f : Y \to X \) be a morphism in \( \mathcal{D}(R) \).

The morphism \( f \) is said to be **tensor nilpotent** if for some \( n \in \mathbb{N} \) the morphism

\[
\underbrace{f \otimes_R L_R \cdots \otimes_R L_R}_n : Y \otimes_R L_R \cdots \otimes_R L_R Y \to X \otimes_R L_R \cdots \otimes_R L_R X
\]

is equal to zero in \( \mathcal{D}(R) \); when the \( R \)-complexes \( X, Y \) are perfect this means that the morphism \( \otimes^n f : \otimes^n_R Y \to \otimes^n_R X \) is homotopic to zero. When \( X \) is perfect and \( f : X \to \Sigma^l X \) is a morphism with \( \otimes^n f \) homotopic to zero the \( n \)-fold composition

\[
X \to \Sigma^l X \xrightarrow{\Sigma^l f} \Sigma^{2n} X \xrightarrow{\Sigma^{2l} f} \cdots \xrightarrow{\Sigma^{nl} f} \Sigma^{nl} X
\]

is also homotopic to zero. The converse does not hold, even when \( R \) is a field for in that case tensor nilpotent morphisms are zero.

3.2. **Fiberwise zero morphisms.** A morphism \( f : Y \to X \) that satisfies

\[
k(p) \otimes_R^L f = 0 \quad \text{in} \quad \mathcal{D}(k(p)) \quad \text{for every} \quad p \in \text{Spec} \ R
\]

is said to be **fiberwise zero**. This is equivalent to requiring \( k \otimes_R^L f = 0 \) in \( \mathcal{D}(k) \) for every homomorphism \( R \to k \) with \( k \) a field. In \( \mathcal{D}(k) \) a morphism is zero if (and only if) it is a ghost, so the latter condition is equivalent to \( H(k \otimes_R^L f) = 0 \).

In \( \mathcal{D}(k) \), a morphism is tensor nilpotent exactly when it is zero. Thus if \( f \) is tensor nilpotent, it is fiberwise zero. There is a partial converse: **If a morphism** \( f : G \to F \) **of perfect** \( R \)-**complexes is fiberwise zero, then it is tensor nilpotent.** This was proved by Hopkins [21, Theorem 10] and Neeman [25, Theorem 1.1].

The next result is the Tensor Nilpotence Theorem from the Introduction. Recall

that an \( R \)-module is said to be \( I \)-torsion if each one of its elements is annihilated by some power of \( I \).

**Theorem 3.3.** Let \( R \) be a commutative noetherian ring and \( f : G \to F \) a morphism of perfect \( R \)-complexes. If for some ideal \( I \) of \( R \) the following conditions hold

1. \( f \) factors through some complex with \( I \)-torsion homology, and
2. \( \text{level}^R \text{Hom}_R(G, F) \leq \text{height} I \),

then \( f \) is fiberwise zero. In particular, \( f \) is tensor nilpotent.

The proof of the theorem is given after Proposition 3.7.

**Remark 3.4.** Lemma 2.4 shows that the inequality (2) is implied by

\[
\text{level}^R F + \text{level}^R G \leq \text{height} I + 1
\]

the converse does not hold; see (2.7.1).

On the other hand, condition (2) cannot be weakened: Let \((R, \mathfrak{m}, k)\) be a local ring and \( G \) the Koszul complex on some system of parameters of \( R \) and let

\[
f : G \to (G/\mathfrak{m}^{d-1} G) \cong \Sigma^d R
\]

be the canonical surjection with \( d = \dim R \). Then \( G \) is an \( \mathfrak{m} \)-torsion complex and \( \text{level}^R G = d + 1 \); see 2.7. Evidently \( H(k \otimes_R f) \neq 0 \), so \( f \) is not fiberwise zero.

In the proof of Theorem 3.3 we exploit the functorial nature of \( I \)-torsion.
3.5. Torsion complexes. The derived $I$-torsion functor assigns to every $X$ in $D(R)$ an $R$-complex $\Gamma_I X$; when $X$ is a module it computes its local cohomology: $H^j_I(X) = H_{-n}(\Gamma^j_I X)$ holds for each integer $n$. There is a natural morphism $t : \Gamma^1_I X \to X$ in $D(R)$ that has the following universal property: Every morphism $Y \to X$ such that $H(Y)$ is $I$-torsion factors uniquely through $t$; see Lipman [24 Section 1]. It is easy to verify that the following conditions are equivalent.

(1) $H(X)$ is $I$-torsion.
(2) $H(X)_p = 0$ for each prime ideal $p \not\supset I$.
(3) The natural morphism $t : \Gamma^1_I X \to X$ is a quasi-isomorphism.

When they hold, we say that $X$ is $I$-torsion. Note a couple of properties:

(3.5.1) If $X$ is $I$-torsion, then $X \otimes^L_R Y$ is $I$-torsion for any $R$-complex $Y$.
(3.5.2) There is a natural isomorphism $\Gamma^1_I(X \otimes^L_R Y) \cong \Gamma^1_I(X) \otimes^L_R Y$.

Indeed, $H(X_p) \cong H(X)_p = 0$ holds for each $p \not\supset I$, giving $X_p = 0$ in $D(R)$. Thus

$$(X \otimes^L_R Y)_p \cong X_p \otimes^L_R Y \cong 0$$

holds in $D(R)$. It yields $H(X_p) \cong H(X)_p = 0$ holds for any $p \not\supset I$, giving $X_p = 0$ in $D(R)$. Thus

A proof of the isomorphism in (3.5.2) can be found in [24, 3.3.1].

3.6. Big Cohen-Macaulay modules. Let $(R, m, k)$ be a local ring.

A (not necessarily finitely generated) $R$-module $C$ is big Cohen-Macaulay if every system of parameters for $R$ is a $C$-regular sequence, in the sense of [8, Definition 1.1]. In the literature the name is sometimes used for $R$-modules $C$ that satisfy the property for some system of parameters for $R$; however, the $m$-adic completion of $C$ is then big Cohen-Macaulay in the sense above; see [8, Corollary 8.5.3].

The existence of big Cohen-Macaulay was proved by Hochster [16, 17] in case when $R$ contains a field as a subring, and by André [2] when it does not; for the latter case, see also Heitmann and Ma [15].

In this paper, big Cohen-Macaulay modules are visible only in the next result.

Proposition 3.7. Let $I$ be an ideal in $R$ and set $c := \text{height } I$.

When $C$ is a big Cohen-Macaulay $R$-module the following assertions hold.

(1) The canonical morphism $t : \Gamma^1_I C \to C$ from the $I$-torsion complex $\Gamma^1_I C$ (see 3.5) is a composition of $c$ ghosts.
(2) If a morphism $g : G \to C$ of $R$-complexes with level$^R G \leq c$ factors through some $I$-torsion complex, then $g = 0$.

Proof. (1) We may assume $I = (x)$, where $x = \{x_1, \ldots, x_c\}$ is part of a system of parameters for $R$; see [8, Theorem A.2]. The morphism $t$ factors as

$$\Gamma^1_{(x_1, \ldots, x_c)}(C) \to \Gamma^1_{(x_1, \ldots, x_{c-1})}(C) \to \cdots \to \Gamma^1_{(x_1)}(C) \to C.$$ Since the sequence $x_1, \ldots, x_c$ is $C$-regular, we have $H_i(\Gamma^1_{(x_1, \ldots, x_j)}(C)) = 0$ for $i \neq -j$; see [8, (3.5.6) and (1.6.16)]. Thus every one of the arrows above is a ghost, so that $t$ is a composition of $c$ ghosts, as desired.

(2) Suppose $g$ factors as $G \to X \to C$ with $X$ an $I$-torsion $R$-complex. As noted in [63, the morphism $X \to C$ factors through $t$, so $g$ factors as

$$G \xrightarrow{g} X \xrightarrow{g''} \Gamma^1_I C \xrightarrow{t} C.$$
In view of the hypothesis \( \text{level}^R G \leq c \) and part (1), Lemma 2.6 shows that 

\[
\text{RHom}_R(G, t) : \text{RHom}_R(G, RΓ F) \rightarrow \text{RHom}_R(G, C)
\]

is a ghost. Using brackets to denote cohomology classes, we get 

\[
[g] = [tg''g'] = H_0(\text{RHom}_R(G, t))(\langle g''g' \rangle) = 0 .
\]

Due to the isomorphism (2.1.1), this means that \( g \) is zero in \( D(R) \). \( \square \)

**Lemma 3.8.** Let \( f : G \rightarrow F \) be a morphism of perfect \( R \)-complexes, where \( G \) is finite free with \( G_i = 0 \) for \( i \ll 0 \) and \( F \) is perfect. Let \( f' : F^* \otimes_R G \rightarrow R \) denote the composed morphism in the next display, where \( e \) is the evaluation map:

\[
F^* \otimes_R G \xrightarrow{F^* \otimes_R f} F^* \otimes_R F \xrightarrow{e} R.
\]

If \( f \) factors through some \( I \)-torsion complex, then so does \( f' \). The morphism \( f' \) is fiberwise zero if and only if so is \( f \).

**Proof.** For the first assertion, note that if \( f \) factors through an \( I \)-torsion complex \( X \), then \( F^* \otimes_R f \) factors through \( F^* \otimes_R X \), and the latter is \( I \)-torsion.

For the second assertion, let \( k \) be a field and \( R \rightarrow k \) be a homomorphism of rings. Let \( (\_)^\vee \) and \( (\_)^\gamma \) stand for \( k \otimes_R (\_ \otimes_R (-)) \) and \( \text{Hom}_k(\_ \otimes_k (-), k) \), respectively. The goal is to prove that \( f = 0 \) is equivalent to \( f' = 0 \).

Since \( F \) is perfect, there are canonical isomorphisms

\[
F^* \otimes k \cong \text{Hom}_R(F, k) \cong \text{Hom}_k(k \otimes_R F, k) = (F)^\vee .
\]

Given this, it follows that \( f' \) can be realized as the composition of morphisms

\[
(F)^\vee \otimes_k C \xrightarrow{(F)^\vee \otimes_k f} (F)^\vee \otimes_k F \xrightarrow{e} k .
\]

If \( F = 0 \), then \( f = 0 \) and \( f' = 0 \) hold. When \( F \) is nonzero, it is easy to verify that \( f' \neq 0 \) is equivalent to \( f' \neq 0 \), as desired. \( \square \)

**Proof of Theorem 3.3.** Given morphisms of \( R \)-complexes \( G \rightarrow X \rightarrow F \) such that \( F \) and \( G \) are perfect and \( X \) is \( I \)-torsion for an ideal \( I \) with

\[
\text{level}^R \text{Hom}_R(G, F) \leq \text{height } I ,
\]

we need to prove that \( f \) is fiberwise zero. This implies the tensor nilpotence of \( f \), as recalled in 3.3.

By Lemma 3.8 the morphism \( f' : F^* \otimes_R G \rightarrow R \) factors through an \( I \)-torsion complex, and if \( f' \) is fiberwise zero, so if \( f \). The isomorphisms of \( R \)-complexes

\[
(F^* \otimes_R G)^* \cong G^* \otimes_R F \cong \text{Hom}_R(G, F)
\]

and Lemma 2.4 yield \( \text{level}^R (F^* \otimes_R G) = \text{level}^R \text{Hom}_R(G, F) \). Thus, replacing \( f \) by \( f' \), it suffices to prove that if \( f : G \rightarrow R \) is a morphism that factors through an \( I \)-torsion complex and satisfies \( \text{level}^R G \leq \text{height } I \), then \( f \) is fiberwise zero.

Fix \( p \in \text{Spec } R \). When \( p \nmid I \), we have \( X_p = 0 \), by 3.5(2). For \( p \mid I \) we have

\[
\text{level}_p G_p \leq \text{level}^R G \leq \text{height } I \leq \text{height } I_p ,
\]

where the first inequality follows directly from the definitions; see [6, Proposition 3.7]. It is easy to verify that \( X_p = I_p \)-torsion. Thus, localizing at \( p \), we may further assume \( (R, \mathfrak{m}, k) \) is a local ring, and we have to prove that \( \text{H}(k \otimes_R f) = 0 \) holds.

Let \( C \) be a big Cohen-Macaulay \( R \)-module. It satisfies \( \mathfrak{m}C \neq C \), so the canonical map \( \pi : R \rightarrow k \) factors as \( R \xrightarrow{\gamma} C \xrightarrow{\gamma} k \). The composition \( G \xrightarrow{f} R \xrightarrow{\pi} C \) is zero in

\[
\text{level}^R G \leq \text{height } I \leq \text{height } I_p .
\]
D(R), by Proposition 3.7. We get \( \pi f = \varepsilon \gamma f = 0 \), whence \( H(k \otimes_R \pi) H(k \otimes_R f) = 0 \). Since \( H(k \otimes_R \pi) \) is bijective, this implies \( H(k \otimes_R f) = 0 \), as desired. \( \square \)

The following consequence of Theorem 3.3 is often helpful.

**Corollary 3.9.** Let \((R, m, k)\) be a local ring, \( F \) a perfect \( R \)-complex, and \( G \) an \( R \)-complex of finitely generated free modules. If a morphism of \( R \)-complexes \( f: G \to F \) satisfies the conditions

1. \( f \) factors through some \( m \)-torsion complex,
2. \( \sup F^a - \inf G^a \leq \dim R - 1 \),

then \( H(k \otimes_R f) = 0 \).

**Proof.** An \( m \)-torsion complex \( X \) satisfies \( k(p) \otimes_R^L X = 0 \) for any \( p \in \text{Spec} \ R \setminus \{m\} \). Thus a morphism, \( g \), of \( R \)-complexes that factors through \( X \) is fiberwise zero if and only if \( k \otimes_R^L g = 0 \). This remark will be used in what follows.

Condition (2) implies \( G_n = 0 \) for \( n \ll 0 \). Let \( f' \) denote the composition \( F^* \otimes_R G \xrightarrow{F^* \otimes_R f} F^* \otimes_R F \xrightarrow{e} R \), where \( e \) is the evaluation map. Since \( \inf (F^* \otimes_R G)^a = - \sup F^a + \inf G^a \), Lemma 3.8 shows that it suffices to prove the corollary for morphisms \( f: G \to R \).

As \( f \) factors through some \( m \)-torsion complex, so does the composite morphism \( G_{\leq 0} \subseteq G \xrightarrow{f} R \).

It is easy to check that if the induced map \( H(k \otimes_R G_{\leq 0}) \to H(k \otimes_R R) = k \) is zero, then so is \( H(k \otimes_R f) \). Thus we may assume \( G_n = 0 \) for \( n \not\in [-d + 1, 0] \), where \( d = \dim R \). This implies \( \text{level}^R G \leq d \), so Theorem 3.3 yields the desideratum. \( \square \)

For some applications the next statement, with weaker hypothesis but also weaker conclusion, suffices. The example in Remark 3.4 shows that the result cannot be strengthened to conclude that \( f \) is fiberwise zero.

**Theorem 3.10.** Let \( R \) be a local ring and \( f: G \to F \) a morphism of \( R \)-complexes. If there exists an ideal \( I \) of \( R \) such that

1. \( f \) factors through an \( I \)-torsion complex, and
2. \( \text{level}^R F \leq \text{height} I \),

then \( H(C \otimes_R^L f) = 0 \) for every big Cohen-Macaulay module \( C \).

**Proof.** Set \( c := \text{height} I \) and let \( t: R \Gamma_I C \to C \) be the canonical morphism. It follows from (3.5.1) that \( C \otimes_R^L f \) also factors through an \( I \)-torsion \( R \)-complex. Then the quasi-isomorphism (3.5.2) and the universal property of the derived \( I \)-torsion functor, see (3.5) implies that \( C \otimes_R^L f \) factors as a composition of the morphisms:

\[
C \otimes_R^L G \longrightarrow (R \Gamma_I C) \otimes_R^L F \xrightarrow{t \otimes_R^L F} C \otimes_R^L F
\]

By Proposition 3.7(1) the morphism \( t \) is a composition of \( c \) ghosts. Thus condition (2) and Lemma 2.6 imply \( t \otimes_R^L F \) is a ghost, and hence so is \( C \otimes_R^L f \). \( \square \)
4. Applications to local algebra

In this section we record applications the Tensor Nilpotence Theorem to local algebra. To that end it is expedient to reformulate it as the Morphic Intersection Theorem from the Introduction, restated below.

**Theorem 4.1.** Let \( R \) be a commutative noetherian ring and \( f \colon G \to F \) a morphism of perfect \( R \)-complexes.

If \( f \) is not fiberwise zero and factors through a complex with \( I \)-torsion homology for an ideal \( I \) of \( R \), then there are inequalities:

\[
\text{span} \ F + \text{span} \ G - 1 \geq \text{level} R(\text{Hom}_R(G,F)) \geq \text{height} I + 1.
\]

**Proof.** The inequality on the left comes from Lemma 2.4 and (2.3.1). The one on the right is the contrapositive of Theorem 3.3. \( \square \)

Here is one consequence.

**Theorem 4.2.** Let \( R \) be a local ring and \( F \) a complex of finite free \( R \)-modules:

\[
F := 0 \to F_d \to F_{d-1} \to \cdots \to F_0 \to 0
\]

For each ideal \( I \) such that \( I \cdot H_i(F) = 0 \) for \( i \geq 1 \) and \( I \cdot z = 0 \) for some element \( z \) in \( H_0(F) \setminus m H_0(F) \), where \( m \) is the maximal ideal of \( R \), one has

\[
d + 1 \geq \text{span} \ F \geq \text{level} R F \geq \text{height} I + 1.
\]

**Proof.** Indeed, the first two inequalities are clear from definitions. As to third one, pick \( \tilde{z} \in F_0 \) representing \( z \) in \( H_0(F) \) and consider the morphism of complexes \( f : R \to F \) given by \( r \mapsto r \tilde{z} \). Since \( z \) is not in \( m H_0(F) \), one has

\[
H_0(k \otimes_R f) = k \otimes_R H_0(f) \neq 0
\]

for \( k = R/m \). In particular, \( k \otimes_R f \) is nonzero. On the other hand, \( f \) factors through the inclusion \( X \subseteq F \), where \( X \) is the subcomplex defined by

\[
X_i = \begin{cases} 
F_i & i \geq 1 \\
R \tilde{z} + d(F_1) & i = 0
\end{cases}
\]

By construction, we have \( H_i(X) = H_i(F) \) for \( i \geq 1 \) and \( I H_0(X) = 0 \), so \( H(X) \) is \( I \)-torsion. The desired inequality now follows from Theorem 4.1 applied to \( f \). \( \square \)

The preceding result is a stronger form of the Improved New Intersection Theorem of Evans and Griffith [12]; see also [19, §2]. First, the latter is in terms of spans of perfect complexes whereas the one above is in terms of levels with respect to \( R \); second, the hypothesis on the homology of \( F \) is weaker. Theorem 4.2 also subsumes prior extensions of the New Intersection Theorem to statements involving levels, namely [6, Theorem 5.1], where it was assumed that \( I \cdot H_0(F) = 0 \) holds, and [1] Theorem 3.1 which requires \( H_i(F) \) to have finite length for \( i \geq 1 \).

In the influential paper [18], Hochster identified certain \textit{canonical elements} in the local cohomology of local rings, conjectured that they are never zero, and proved that statement in the equal characteristic case. He also gave several reformulations that do not involve local cohomology. Detailed discussions of the relations between these statements and the histories of their proofs are presented in [28] and [20].

\[1\]This and the other statements in this section were conjectures prior to the appearance of [2].
Some of those statements concern properties of morphisms from the Koszul complex on some system of parameters to resolutions of various $R$-modules. This makes them particularly amenable to approaches from the Morphic Intersection Theorem.

In the rest of this section we uncover direct paths to various forms of the Canonical Element Theorem and related results.

We first prove a version of [15, 2.3]. The conclusion there is that $f_d$ is not zero, but the remarks in [15, 2.2(6)] show that it is equivalent to the following statement.

**Theorem 4.3.** Let $(R, m, k)$ be a local ring, $x$ a system of parameters for $R$, and $K$ the Koszul complex on $x$, and $F$ a complex of free $R$-modules.

If $f : K \to F$ is a morphism of $R$-complexes with $H_0(k \otimes_R f) \neq 0$, then one has

$$H_d(S \otimes_R f) \neq 0 \quad \text{for} \quad S = R/(x) \quad \text{and} \quad d = \dim R.$$ 

**Proof.** Recall from [2.7] that $\partial(K)$ lies in $(x)K$, that $K_1$ has a basis $\bar{x}_1, \ldots, \bar{x}_d$ and $K^2$ is the exterior algebra on $K_1$. Thus $K_d$ is a free $R$-module with basis $x = \bar{x}_1 \cdots \bar{x}_d$ and $H_d(S \otimes_R K) = S(1 \otimes x)$, so we need to prove $f(K_d) \not\subseteq (x)F_d + \partial(F_{d+1}).$

Arguing by contradiction, we suppose the contrary. This means

$$f(x) = x_1 y_1 + \cdots + x_d y_d + \partial^F(y)$$

with $y_1, \ldots, y_d \in F_d$ and $y \in F_{d+1}$. For $i = 1, \ldots, d$ set $x_i^* := \bar{x}_1 \cdots \bar{x}_{i-1} \bar{x}_{i+1} \cdots \bar{x}_d$; thus $\{x_1^*, \ldots, x_d^*\}$ is a basis of the $R$-module $K_{d-1}$. Define $R$-linear maps

$$h_{d-1} : K_{d-1} \to F_d \quad \text{by} \quad h_{d-1}(x_i^*) = (-1)^{i-1} y_i \quad \text{for} \quad i = 1, \ldots, d.$$ 

$$h_d : K_d \to F_{d+1} \quad \text{by} \quad h_d(x) = y.$$ 

Extend them to a degree one map $h : K \to F$ with $h_i = 0$ for $i \neq d-1, d$. The map

$$g := f - \partial^F h - h\partial^K : K \to F$$

is easily seen to be a morphism of complexes that is homotopic to $f$ and satisfies $g_d = 0$. This last condition implies that $g$ factors as a composition of morphisms

$$K \xrightarrow{g'} F_{<d} \subseteq F.$$ 

The complex $K$ is $m$-torsion; see [2.7] Thus Corollary [3.9] applied to $g'$, yields $H(k \otimes_R g') = 0$. This gives the second equality below:

$$H_0(k \otimes_R f) = H_0(k \otimes_R g) = H_0(k \otimes_R g') = 0.$$ 

The first one holds because $f$ and $g$ are homotopic, and the last one because $g_0 = g_0'$. The result of the last computation contradicts the hypotheses on $H_0(k \otimes_R f)$.

A first specialization is the Canonical Element Theorem.

**Corollary 4.4.** Let $I$ be an ideal in $R$ containing a system of parameters $x_1, \ldots, x_d$.

With $K$ the Koszul complex on $x$ and $F$ a free resolution of $R/I$, any morphism $f : K \to F$ of $R$-complexes lifting the surjection $R/(x) \to R/I$ has $f_d(K) \neq 0$. □

As usual, when $A$ is a matrix, $I_d(A)$ denotes the ideal of its minors of size $d$.

**Corollary 4.5.** Let $R$ be a local ring, $x$ a system of parameters for $R$, and $y$ a finite subset of $R$ with $(y) \supseteq (x)$.

If $A$ is a matrix such that $Ay = x$, then $I_d(A) \not\subseteq (x)$ for $d = \dim R$. □
Theorem 4.8. Let $\alpha$ be where $\mu$ the maps $\mu$. The equality follows from Corollary 4.6.

Proof. Let $K$ and $F$ be the Koszul complexes on $x$ and $y$, respectively. The matrix $A$ defines a unique morphism of DG $R$-algebras $f: K \to F$. Evidently, $H_0(k \otimes_R f)$ is the identity map on $k$, and hence is not zero. Since $f_d$ can be represented by a column matrix whose entries are the various $d \times d$ minors of $A$, the desired statement is a direct consequence of Theorem 4.3.

A special case of the preceding result yields the Monomial Theorem.

Corollary 4.6. When $y_1, \ldots, y_d$ is a system of parameters for local ring, one has $(y_1 \cdots y_d)^n \notin (y_1^{n+1}, \ldots, y_d^{n+1})$ for every integer $n \geq 1$.

Proof. Apply Corollary 4.5 to the inclusion $(y_1^{n+1}, \ldots, y_d^{n+1}) \subseteq (y_1, \ldots, y_d)$ and

$$
A := \begin{bmatrix}
y_1^n & 0 & \cdots & 0 \\
0 & y_2^n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & y_d^n
\end{bmatrix}.
$$

We also deduce from Theorem 4.3 another form of the Canonical Element Theorem. Roberts [27] proposed the statement and proved that it is equivalent to the Canonical Element Theorem; a different proof appears in Huneke and Koh [22].

Recall that for any pair $(S, T)$ of $R$-algebras the graded module $\text{Tor}_i^R(S, T)$ carries a natural structure of graded-commutative $R$-algebra, given by the $\wedge$-product of Cartan and Eilenberg [10, Chapter XI.4].

Lemma 4.7. Let $R$ be a commutative ring, $I$ an ideal of $R$, and set $S := R/I$. Let $G \to S$ be some $R$-free resolution, $K$ be the Koszul complex on some generating set of $I$, and $g: K \to G$ a morphisms of $R$-complexes lifting the identity of $S$.

For every surjective homomorphism $\psi: S \to T$ of of commutative rings there is a commutative diagram of strictly graded-commutative $S$-algebras

$$
\begin{array}{c}
S \otimes_R K = \bigwedge_S H_1(S \otimes_R K) \xrightarrow{\alpha} \bigwedge_S \text{Tor}_1^R(S, S) \xrightarrow{\mu^S} \text{Tor}_1^R(S, S) \\
\xrightarrow{\wedge \psi \text{Tor}_1^R(S, \psi)} \bigwedge_T \text{Tor}_1^R(S, T) \xrightarrow{\mu^T} \text{Tor}_1^R(S, T)
\end{array}
$$

where $\alpha_1 = H_1(S \otimes_R g)$, the map $\alpha$ is defined by the functoriality of exterior algebras, and the maps $\mu^S$ are defined by the universal property of exterior algebras.

Proof. The equality follows from $\partial^K(K) \subseteq IK$ and $K^2 = \bigwedge_R K_1$. The resolution $G$ can be chosen to have $G_{\leq 1} = K_{\leq 1}$; this makes $\alpha$ surjective, and the surjectivity of $\alpha$ follows. The map $\text{Tor}_1^R(S, \psi)$ is surjective because it can be identified with the natural map $I/I^2 \to I/IJ$, where $J = \text{Ker}(R \to T)$; the surjectivity of $\bigwedge \psi \text{Tor}_1^R(S, \psi)$ follows. The square commutes by the naturality of $\wedge$-products.

Theorem 4.8. Let $(R, m, k)$ be a local ring, $I$ a parameter ideal, and $S := R/I$.

For each surjective homomorphism $S \to T$ the morphism of graded $T$-algebras

$$
\mu^T: \bigwedge_T \text{Tor}_1^R(S, T) \to \text{Tor}_1^R(S, T)
$$

has the property that $\mu^T \otimes_T k$ is injective. In particular, $\mu^k$ is injective.
Theorem 4.3, with \( f \) the second equality in the string respectively. We have \( \mu \)

The first equality comes from Lemma 4.7, with Remark 4.9 homomorphism of graded \( \mathbb{Q} \)

Tor \( \psi \) and the canonical surjection of finite free \( \mathbb{R} \)

Let Theorem 5.1. Element Theorem, in the form of Theorem 4.3 above, is used in the proof.

for cyclic modules in [9, 1.4]; background is reviewed after the proof. The Canonical

R-resolution of some

The desired inequality is vacuous when \( d = 0 \), so suppose \( d \geq 1 \). Let \( x \) be

the said system of parameters of \( R \) and \( K \) the Koszul complex on \( x \). Since \( F \) is a

direct system of parameters. Under the additional assumptions that \( R \) is a domain and \( F \) is a resolution of some \( R \)-module, the theorem below was proved in [3, 6.4.1], and earlier for cyclic modules in [9, 1.4]; background is reviewed after the proof. The Canonical Element Theorem, in the form of Theorem 4.3 above, is used in the proof.

\[
\begin{align*}
\bigwedge \alpha \quad \text{Tor}^R(S,T) & \xrightarrow{\mu_T} \text{Tor}^R(S,T) \\
\bigwedge \alpha \quad \text{Tor}^R(S,\pi) & \\
\bigwedge k \quad \text{Tor}^R(S,k) & \xrightarrow{\mu_k} \text{Tor}^R(S,k)
\end{align*}
\]

It is easy to verify that \( \pi \) induces a bijective map

\[
\text{Tor}^R_1(S,\pi) \otimes_T k: \text{Tor}^R_1(S,T) \otimes_T k \to \text{Tor}^R_1(S,k),
\]

so \((\pi, \text{Tor}^R(S,\pi)) \otimes_T k\) is an isomorphism. Thus it suffices to show \( \mu^k \) is injective.

Let \( K \) be the Koszul complex on a minimal generating set of \( I \). Let \( G \xrightarrow{g} S \) and \( F \xrightarrow{h} k \) be \( R \)-free resolutions of \( S \) and \( k \), respectively. Lift the identity map of \( S \) and the canonical surjection \( \psi: S \to k \) to morphisms \( g: K \to G \) and \( h: G \to F \), respectively. We have \( \mu^S \alpha = H(S \otimes_R g) \) and \( \text{Tor}^R(S,\psi) = H(S \otimes_R h) \). This implies the second equality in the string

\[
\mu_d^k \text{Tor}^R_d(S,\pi) \alpha_d = \text{Tor}^R_d(S,\psi) \mu_d \alpha_d = H_d(S \otimes_R h g) \neq 0.
\]

The first equality comes from Lemma 4.7 with \( T = k \), and the non-equality from Theorem 4.3 with \( f = h g \). In particular, we get \( \mu^k_0 \neq 0 \). We have an isomorphism \( \text{Tor}^R(S,k) \cong \bigwedge k^d \) of graded \( k \)-algebras, so \( \mu^k \) is injective by the next remark. □

Remark 4.9. If \( Q \) is a field, \( d \) is a non-negative integer, and \( \lambda: \bigwedge Q^d \to B \) is a homomorphism of graded \( Q \)-algebras with \( \lambda_d \neq 0 \), then \( \lambda \) is injective.

Indeed, the graded subspace \( \bigwedge Q^d \) of exterior algebra \( \bigwedge Q^d \) is contained in every every non-zero ideal and has rank one, so \( \lambda_d \neq 0 \) implies \( \text{Ker}(\lambda) = 0 \).

5. Ranks in finite free complexes

This section is concerned with DG modules over Koszul complexes on sequences of parameters. Under the additional assumptions that \( R \) is a domain and \( F \) is a resolution of some \( R \)-module, the theorem below was proved in [3, 6.4.1], and earlier for cyclic modules in [9, 1.4]; background is reviewed after the proof. The Canonical Element Theorem, in the form of Theorem 4.3 above, is used in the proof.

Theorem 5.1. Let \((R, m, k)\) be a local ring, set \( d = \dim R \), and let \( F \) be a complex of finite free \( R \)-modules with \( \text{H}_0(F) \neq 0 \) and \( F_i = 0 \) for \( i < 0 \).

If \( F \) admits a structure of DG module over the Koszul complex on some system of parameters of \( R \), then there is an inequality

\[
\text{rank}_R(F_n) \geq \binom{d}{n} \quad \text{for each} \quad n \in \mathbb{Z}.
\]

Proof. The desired inequality is vacuous when \( d = 0 \), so suppose \( d \geq 1 \). Let \( x \) be the said system of parameters of \( R \) and \( K \) the Koszul complex on \( x \). Since \( F \) is a DG \( K \)-module, each \( \text{H}_i(F) \) is an \( R/(x) \)-module, and hence of finite length.

First we reduce to the case when \( R \) is a domain. To that end, let \( \mathfrak{p} \) be a prime ideal of \( R \) such that \( \dim(R/\mathfrak{p}) = d \). Evidently, the image of \( x \) in \( R \) is a system of
parameters for $R/p$. By base change, $(R/p) \otimes_R F$ is a DG module over $(R/p) \otimes_R K$, the Koszul complex on $\mathfrak{x}$ with coefficients in $R/p$, with

$$H_0((R/p) \otimes_R F) \cong R/p \otimes H_0(F) \neq 0.$$  

Moreover, the rank of $F^\natural$ as an $R$-module equals the rank of $(R/p) \otimes_R F^\natural$ as an $R/p$-module. Thus, after base change to $R/p$ we can assume $R$ is a domain.

Choose a cycle $z \in F_0$ that maps to a minimal generator of the $R$-module $H_0(F)$. Since $F$ is a DG $K$-module, this yields a morphism of DG $K$-modules

$$f: K \to F \quad \text{with} \quad f(a) = az.$$  

This is, in particular, a morphism of complexes. Since $k \otimes_R H_0(F) \neq 0$, by the choice of $z$, Theorem 4.3 applies, and yields that $f(K_d) \neq 0$. Since $R$ is a domain, this implies $f(Q \otimes_R K_d)$ is non-zero, where $Q$ is the field of fractions of $R$.

Set $\Lambda := (Q \otimes_R K)^\natural$ and consider the homomorphism of graded $\Lambda$-modules

$$\lambda := Q \otimes_R f^\natural: \Lambda \to Q \otimes_R F^\natural.$$  

As $\Lambda$ is isomorphic to $\bigwedge Q^d$, Remark 4.2 gives the inequality in the display

$$\text{rank}_R(F_n) = \text{rank}_Q(Q \otimes_R F_n) \geq \text{rank}_Q(\Lambda_n) = \binom{d}{n}.$$  

Both equalities are clear. \hfill \Box

The inequalities (5.1.1) are related to a major topic of research in commutative algebra. We discuss it for a local ring $(R, \mathfrak{m}, k)$ and a bounded $R$-complex $F$ of finite free modules with $F_{<0} = 0$, homology of finite length, and $H_0(F) \neq 0$.

5.2. **Ranks of syzygies.** The celebrated and still open Rank Conjecture of Buchsbaum and Eisenbud [9, Proposition 1.4], and Horrocks [14, Problem 24] predicts that (5.1.1) holds whenever $F$ is a resolution of some module of finite length.

That conjecture is known for $d \leq 4$. Its validity would imply $\sum_n \text{rank}_R F_n \geq 2^d$. For $d = 5$ and equicharacteristic $R$, this was proved in [4, Proposition 1] by using Evans and Griffith’s Syzygy Theorem [12]; in view of [2], it holds for all $R$.

In a breakthrough, M. Walker [30] used methods from K-theory to prove that $\sum_n \text{rank}_R F_n \geq 2^d$ holds when $R$ contains $\frac{1}{2}$ and is complete intersection (in particular, regular), and when $R$ is an algebra over some field of positive characteristic.

5.3. **Obstructions for DG module structures.** Theorem 5.1 provides a series of obstruction for the existence of any DG module structure on $F$. In particular, it implies that if $\text{rank}_R F < 2^d$ holds with $d = \dim R$, then $F$ supports no DG module structure over $K(\mathfrak{x})$ for any system of parameters $\mathfrak{x}$. Complexes satisfying the restriction on ranks were recently constructed in [23, 4.1]. These complexes have nonzero homology in degrees 0 and 1, so they are not resolutions of $R$-modules.

5.4. **DG module structures on resolutions.** Let $F$ be a minimal resolution of an $R$-module $M$ of nonzero finite length and $\mathfrak{x}$ a parameter set for $R$ with $\mathfrak{x}M = 0$.

When $F$ admits a DG module structure over $K(\mathfrak{x})$ the Rank Conjecture holds, by Theorem 5.1. It was conjectured in [9, 1.2] that such a structure exists for all $F$ and $\mathfrak{x}$. An obstruction to its existence was found in [3, 1.2], and examples when that obstruction is not zero were produced in [3, 2.4.2]. On the other hand, by [3, 1.8] the obstruction vanishes when $\mathfrak{x}$ lies in $\text{ann}_R(M)$.
It is not known if $F$ supports some DG $K(\mathfrak{x})$-module structure for special choices of $\mathfrak{x}$; in particular, for high powers of systems of parameters contained in $\text{ann}_R(M)$.

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