Quantum Algorithm for SAT Problem and Quantum Mutual Entropy

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Abstract

It is von Neumann who opened the window for today’s Information epoch. He defined quantum entropy including Shannon’s information more than 20 years ahead of Shannon, and he introduced a concept what computation means mathematically.

In this paper I will report two works that we have recently done, one of which is on quantum algorithm in generalized sense solving the SAT problem (one of NP complete problems) and another is on quantum mutual entropy properly describing quantum communication processes.

1 Introduction

This paper consists of two parts, one (Sec.2 and 3) of which is about quantum algorithm solving the SAT problem based on a series of the papers [30, 2, 34, 31, 4] and another (Sec.4) is about quantum mutual entropy applying quantum communication processes based on the papers [27, 28, 35].

Although the ability of computer is highly progressed, there are several problems which may not be solved effectively, namely, in polynomial time. Among such problems, NP problem and NP complete problem are fundamental. It is known that all NP complete (NPC for short) problems are equivalent and have been studied for decades, for which all known algorithms have an exponential running time in the length of the input so far. An essential question to be asked for more than 30 years is whether there exists an algorithm to solve an NP complete problem in polynomial time. We found two different algorithms solving the NPC problems in polynomial time [30, 32, 33, 3]. In first two sections of the present paper we report the essence of these algorithms.

After von Neumann introduced quantum entropy [23] of density operators, many studies on various quantum entropies have appeared [28, 31], among which quantum mutual entropy plays an important role. That is, the mutual entropy
expresses the amount of information sending from input to output, so that it will be a basic quantity measuring the ability of a communication channel. I defined the quantum mutual entropy for density operators in 1983 [27] by using Umegaki’s relative entropy [40] and extended it to general C*-dynamical systems by means of Araki’s or Uhlmann’s relative entropy [6, 39, 31]. Recently several quantum mutual type entropies have appeared [38, 8, 10], and they are used to discuss communication processes. In Section 4 of this paper, we compare these mutual type entropies from the views of information communication based on the paper [35].

2 Quantum Chaos Algorithm of SAT

Let us remind what the P-problem and the NP-problem are [16, 13]: Let \( n \) be the size of input.

1. A P-problem is a problem whose time needed for solving the problem is at worst of polynomial time of \( n \). Equivalently, it is a problem which can be recognized in a polynomial time of \( n \) by deterministic Turing machine.

2. An NP-problem is a problem that can be solved in polynomial time by a nondeterministic Turing machine. This can be understood as follows: Let consider a problem to find a solution of \( f(x) = 0 \). We can check in polynomial time of \( n \) whether \( x_0 \) is a solution of \( f(x) = 0 \), but we do not know whether we can find the solution of \( f(x) = 0 \) in polynomial time of \( n \).

3. An NP-complete problem is a problem polynomially transformed NP-problem. We take the SAT (satisfiable) problem, one of the NP-complete problems, to study whether there exists an algorithm showing NPC becomes P. It is known that the SAT problem is equivalent to any other NPC problems.

Let \( X = \{x_1, \ldots, x_n\} \) be a set. Then \( x_k \) and its negation \( \bar{x}_k \) \((k = 1, 2, \ldots, n)\) are called literals and the set of all such literals is denoted by \( \overline{X} = \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\} \). The set of all subsets of \( X' \) is denoted by \( F(\overline{X}) \) and an element \( C \in F(\overline{X}) \) is called a clause. We take a truth assignment to all Boolean variables \( x_k \). If we can assign the truth value to at least one element of \( C \), then \( C \) is called satisfiable. When \( C \) is satisfiable, the truth value \( t(C) \) of \( C \) is regarded as true, otherwise, that of \( C \) is false. Take the truth values as “true ↔ 1, false ↔ 0”. Then \( C \) is satisfiable iff \( t(C) = 1 \).

Let \( L = \{0, 1\} \) be a Boolean lattice with usual join \( \lor \) and meet \( \land \), and \( t(x) \) be the truth value of a literal \( x \) in \( X \). Then the truth value of a clause \( C \) is written as \( t(C) = \lor_{x \in C} t(x) \).

Moreover the set \( C \) of all clauses \( C_j \) \((j = 1, 2, \ldots, m)\) is called satisfiable iff the meet of all truth values of \( C_j \) is 1; \( t(C) = \land_{j=1}^{m} t(C_j) = 1 \). Thus the SAT problem is written as follows:

**Definition 1** SAT Problem: Given a Boolean set \( X = \{x_1, \ldots, x_n\} \) and a set...
$\mathcal{C} = \{C_1, \cdots, C_m\}$ of clauses, determine whether $\mathcal{C}$ is satisfiable or not.

That is, this problem is to ask whether there exists a truth assignment to make $\mathcal{C}$ satisfiable. It is known in usual algorithm that it is polynomial time to check the satisfiability only when a specific truth assignment is given, but we can not determine the satisfiability in polynomial time when an assignment is not specified.

In [30] we discussed the quantum algorithm of the SAT problem, which was rewritten in [7] with showing that the OM SAT-algorithm is combinatic. Ohya and Masuda pointed out [30] that the SAT problem, hence all other NP problems, can be solved in polynomial time by quantum computer if the superposition of two orthogonal vectors $|0\rangle$ and $|1\rangle$ is physically detected. However this detection is considered not to be possible in the present technology. The problem to be overcome is how to distinguish the pure vector $|0\rangle$ from the superposed one $\alpha |0\rangle + \beta |1\rangle$, obtained by the OM SAT-quantum algorithm, if $\beta$ is not zero but very small. If such a distinction is possible, then we can solve the NPC problem in the polynomial time.

In [32, 33] it is shown that it can be possible by combining nonlinear chaos amplifier with the quantum algorithm, which implies the existence of a mathematical algorithm solving NP=P. The algorithm of Ohya and Volovich is going beyond usual (unitary) quantum Turing algorithm. So the next question is (1) whether there exists more general Turing machine scheme combining the unitary quantum algorithm with chaos dynamics, or (2) whether there exists another method to achieve the above distinction of two vectors by a suitable unitary evolution. In the paper [4], we discussed that the stochastic limit, recently extensively studied by Accardi and coworkers [2], can be used to find another method of (2).

In this paper, we review mathematical frame of quantum algorithm in Section 2 and the OV-chaos algorithm. In Section 3, based on the idea of quantum adaptive dynamics [1, 25, 4], we discuss how it can be used to solve the problem NP=P.

### 2.1 Quantum algorithm

The quantum algorithms discussed so far are rather idealized because computation is represented by unitary operations. A unitary operation is rather difficult to realize in physical processes, more realistic operation is one allowing some dissipation like semigroup dynamics. For such a realization, we have to generalize the concept of quantum Turing machine so that the generalized one contains non-unitary operations. This work has been done in the papers [4, 21, 5], about which we will not discussed here. We will, in this paper, explain the algorithms solving the SAT problem in polynomial time.

First we remind the procedure of usual quantum algorithm which is needed the computation of the truth value $t(\mathcal{C})$ of the SAT.

Let $\mathcal{H}$ be a Hilbert space describing input, computation and output (result). As usual, the Hilbert space is $\mathcal{H} = \bigotimes_1^n \mathbb{C}^2$, and let the basis of $\mathcal{H} =$
$N \otimes C^2$ be: $e_0 (= |0\rangle) = |0\rangle \otimes \cdots \otimes |0\rangle \otimes |0\rangle, e_1 (= |1\rangle) = |0\rangle \otimes \cdots \otimes |0\rangle \otimes |1\rangle, \ldots, e_{2N-1} (= |2N-1\rangle) = |1\rangle \otimes \cdots \otimes |1\rangle \otimes |1\rangle$.

Any number $t (0, \ldots, 2^N - 1)$ can be expressed by $t = \sum_{k=1}^{N} a_t^{(k)} 2^{k-1}, a_t^{(k)} = 0$ or 1, so that the associated vector is written by $|t\rangle (= e_t) = \otimes_{k=1}^{N} |a_t^{(k)}\rangle$.

And applying $n$-tuples of Hadamard matrix $H \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to the vacuum vector $|0\rangle$, we get $H |0\rangle (= \xi (0)) \equiv \otimes_{j=1}^{N} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$. Put $W (t) \equiv \otimes_{j=1}^{N} \begin{pmatrix} 1 & 0 \\ 0 & \exp \left( \frac{2\pi i t}{2^N} \right) \end{pmatrix}$.

Then we have

$$\xi (t) \equiv W (t) \xi (0) = \frac{1}{\sqrt{2^N}} \sum_{k=0}^{2^N-1} \exp \left( \frac{2\pi i tk}{2^N} \right) |k\rangle,$$

which is called Discrete Fourier Transformation. Thus altogether of the above operations, it follows a unitary operator $U_F (t) \equiv W (t) H$ and the vector $\xi (t) = U_F (t) |0\rangle$.

All conventional unitary algorithms can be written as the following three steps by means of certain channels on the state space in $\mathcal{H}$ (i.e., a channel is a map sending a state to another state):

1. Preparation of state: Take a state $\rho$ (e.g., $\rho = |0\rangle \langle 0|$) applying the unitary channel defined by the above $U_F (t): \Lambda^*_F \equiv Ad_{U_F (t)}$

   $$\Lambda^*_F = Ad_{U_F} \Rightarrow \Lambda^*_F \rho = U_F \rho U_F^*$$

2. Computation: Let $U$ a unitary operator on $\mathcal{H}$ representing the computation followed by a suitable programming of a certain problem, then the computation is described by a channel $\Lambda^*_U \equiv Ad_{U}$ (unitary channel). After the computation, the final state $\rho_f$ will be

   $$\rho_f = \Lambda^*_U \Lambda^*_F \rho.$$

3. Register and Measurement: For registration of the computed result and its measurement we might need an additional system $K$ (e.g., register), so that the lifting $\mathcal{E}^*_m$ from $\mathcal{S} (\mathcal{H})$ to $\mathcal{S} (\mathcal{H} \otimes K)$ in the sense of $\mathbb{K}$ is useful to describe this stage. Thus the whole process is written as

   $$\rho_f = \mathcal{E}^*_m (\Lambda^*_U \Lambda^*_F \rho).$$

Finally we measure the state in $K$: For instance, let $\{P_k; k \in J\}$ be a projection valued measure (PVM) on $K$

   $$\Lambda^*_m \rho_f = \sum_{k \in J} I \otimes P_k \rho_f I \otimes P_k,$$
after which we can get a desired result by observations in finite times if the size of the set $J$ is small.

**Remark 2** When dissipation is involved the above three steps have to be generalized so that dissipative nature is involved. Such a generalization can be expressed by means of suitable channel, not necessarily unitary. (1) Preparation of state: We may be use the same channel $\Lambda^*_p = \text{Ad}_U$ in this first step, but if the number of qubits $N$ is large so that it will not be built physically, then $\Lambda^*_p$ should be modified, and let denote it by $\Lambda^*_p$. (2) Computation: This stage is certainly modified to a channel $\Lambda^*_c$, reflecting the physical device for computer. (3) Registering and Measurement: This stage will be remained as above. Thus the whole process is written as

$$\rho_f = E_m(\Lambda^*_c \Lambda^*_p \rho).$$

### 2.2 Quantum algorithm of SAT

We explain the algorithm of the SAT problem which has been introduced by Ohya-Masuda [30] and developed by Accardi-Sabbadini [7]. This quantum algorithm is described by a combination of the unitary operators discussed in the previous section on a Hilbert space $\mathcal{H}$. The detail of this section is given in the papers [30, 7, 33], so we will discuss just the essence of the OM algorithm. Throughout this subsection, let $n$ be the total number of Boolean variables used in the SAT problem.

Let 0 and 1 of the Boolean lattice $L$ be denoted by the vectors $|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the Hilbert space $\mathbb{C}^2$, respectively. That is, the vector $|0\rangle$ corresponds to falseness and $|1\rangle$ does to truth.

As we explained in the previous section, an element $x \in X$ can be denoted by 0 or 1, so by $|0\rangle$ or $|1\rangle$. In order to describe a clause $C$ with at most $n$ length by a quantum state, we need the n-tuple tensor product Hilbert space $\mathcal{H} \equiv \otimes^n \mathbb{C}^2$. For instance, in the case of $n = 2$, given $C = \{x_1, x_2\}$ with an assignment $x_1 = 0$ and $x_2 = 1$, then the corresponding quantum state vector is $|0\rangle \otimes |1\rangle$, so that the quantum state vector describing $C$ is generally written by $|C\rangle = |x_1\rangle \otimes |x_2\rangle \in \mathcal{H}$ with $x_k = 0$ or 1 ($k=1,2$).

Once $X \equiv \{x_1, \ldots , x_n\}$ and $C = \{C_1, C_2, \ldots , C_m\}$ are given, the SAT is to find the vector

$$|t(C)\rangle \equiv \bigwedge_{j=1}^m \forall x \in C_j \ t(x),$$

where $t(x)$ is $|0\rangle$ or $|1\rangle$ when $x = 0$ or 1, respectively, and $t(x) \land t(y) \equiv t(x \land y)$, $t(x) \lor t(y) \equiv t(x \lor y)$.

For any two qubits $|x\rangle$ and $|y\rangle$, $|x,y\rangle$ and $|x^N\rangle$ is defined as $|x\rangle \otimes |y\rangle$ and $|x\rangle \otimes \cdots \otimes |x\rangle$, respectively. The usual (unitary) quantum computation can
be formulated mathematically as the multiplication by unitary operators. Let $U_{\mathrm{NOT}}, U_{\mathrm{CN}}$ and $U_{\mathrm{CCN}}$ be the three unitary operators defined as

$$U_{\mathrm{NOT}} \equiv |1\rangle \langle 0| + |0\rangle \langle 1|,$$

$$U_{\mathrm{CN}} \equiv |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes U_{\mathrm{NOT}},$$

$$U_{\mathrm{CCN}} \equiv |0\rangle \langle 0| \otimes I \otimes I + |1\rangle \langle 1| \otimes |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes |0\rangle \langle 1| \otimes U_{\mathrm{NOT}}.$$  

$U_{\mathrm{NOT}}, U_{\mathrm{CN}}$ and $U_{\mathrm{CCN}}$ are often called NOT-gate, Controlled-NOT gate and Controlled-Controlled-NOT gate, respectively. For any $k \in \mathbb{N}$, $U_H^{(N)}(k)$ denotes the $k$-tuple Hadamard transformation on $(\mathbb{C}^2)^{\otimes N}$ defined as

$$U_H^{(N)}(k) |0^N\rangle = \frac{1}{2^{k/2}} (|0\rangle + |1\rangle)^{\otimes k} |0^{N-k}\rangle = \frac{1}{2^{k/2}} \sum_{i=0}^{2^{k-1}} |e_i\rangle \otimes |0^{N-k}\rangle.$$

The above unitary operators can be extended to the unitary operators on $(\mathbb{C}^2)^{\otimes N}$:

$$U_{\mathrm{NOT}}^{(N)}(u) \equiv I^{\otimes u-1} \otimes (|0\rangle \langle 1| + |1\rangle \langle 0|) I^{\otimes N-u-1},$$

$$U_{\mathrm{CN}}^{(N)}(u,v) \equiv I^{\otimes u-1} \otimes |0\rangle \langle 0| \otimes I^{\otimes N-u-1} + I^{\otimes u-1} \otimes |1\rangle \langle 1| \otimes I^{\otimes N-v-1} \otimes U_{\mathrm{NOT}} \otimes I^{\otimes N-v-1},$$

$$U_{\mathrm{CCN}}^{(N)}(u,v,w) = I^{\otimes u-1} \otimes |0\rangle \langle 0| \otimes I^{\otimes N-u-1} + I^{\otimes u-1} \otimes |1\rangle \langle 1| \otimes I^{\otimes v-1} \otimes |0\rangle \langle 0| \otimes I^{\otimes N-v-1} + I^{\otimes u-1} \otimes |1\rangle \langle 1| \otimes I^{\otimes w-1} \otimes U_{\mathrm{NOT}} \otimes I^{\otimes N-w-1},$$

where $u, v$ and $w$ be positive integers satisfying $1 \leq u < v < w \leq N$.

Furthermore we have the following three unitary operators $U_{\mathrm{AND}}, U_{\mathrm{OR}}$ and $U_{\mathrm{COPY}}$, called the logical gates; (see [7])

$$U_{\mathrm{AND}} \equiv \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} \{|\varepsilon_1, \varepsilon_2, \varepsilon_1 \land \varepsilon_2\rangle \langle \varepsilon_1, \varepsilon_2, 0| + |\varepsilon_1, \varepsilon_2, 1 - \varepsilon_1 \land \varepsilon_2\rangle \langle \varepsilon_1, \varepsilon_2, 1|\}$$

$$= |0,0,0\rangle \langle 0,0,0| + |0,0,1\rangle \langle 0,0,1| + |1,0,0\rangle \langle 1,0,0| + |1,0,1\rangle \langle 1,0,1| + |0,1,0\rangle \langle 0,1,0| + |0,1,1\rangle \langle 0,1,1| + |1,1,0\rangle \langle 1,1,0| + |1,1,1\rangle \langle 1,1,1|.$$  

$$U_{\mathrm{OR}} \equiv \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} \{|\varepsilon_1, \varepsilon_2, \varepsilon_1 \lor \varepsilon_2\rangle \langle \varepsilon_1, \varepsilon_2, 0| + |\varepsilon_1, \varepsilon_2, 1 - \varepsilon_1 \lor \varepsilon_2\rangle \langle \varepsilon_1, \varepsilon_2, 1|\}$$

$$= |0,0,0\rangle \langle 0,0,0| + |0,0,1\rangle \langle 0,0,1| + |1,0,0\rangle \langle 1,0,0| + |1,0,1\rangle \langle 1,0,1| + |0,1,0\rangle \langle 0,1,0| + |0,1,1\rangle \langle 0,1,1| + |1,1,0\rangle \langle 1,1,0| + |1,1,1\rangle \langle 1,1,1|.$$
\[ U_{COPY} \equiv \sum_{\varepsilon_1, \varepsilon_2 \in \{0, 1\}} \{ |\varepsilon_1, \varepsilon_1 \rangle \langle \varepsilon_1, 0 | + |\varepsilon_1, 1 - \varepsilon_1 \rangle \langle \varepsilon_1, 1 | \}
\]
\[ = |0, 0 \rangle \langle 0, 0 | + |0, 1 \rangle \langle 0, 1 | + |1, 1 \rangle \langle 1, 0 | + |1, 0 \rangle \langle 1, 1 |. \]

Here \( \varepsilon_1 \) and \( \varepsilon_2 \) take the value 0 or 1. We call \( U_{AND}, U_{OR} \) and \( U_{COPY} \), AND gate, OR gate and COPY gate, respectively, whose extensions to \( (\mathbb{C}^2)^{\otimes N} \) are denoted by \( U_{AND}^{(N)}, U_{OR}^{(N)} \) and \( U_{COPY}^{(N)} \), which are expressed as

\[ U_{AND}^{(N)}(u, v, w) = \sum_{\varepsilon_1, \varepsilon_2 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1 \rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2 \rangle \langle \varepsilon_2|
\]
\[ = I^{\otimes w-v-u-1} \otimes |\varepsilon_1 \wedge \varepsilon_2 \rangle \langle 0 | I^{\otimes N-w-v-u} + I^{\otimes u-1} \otimes |\varepsilon_1 \rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2 \rangle \langle \varepsilon_2|
\]
\[ = I^{\otimes w-v-u-1} \otimes |1 - \varepsilon_1 \wedge \varepsilon_2 \rangle \langle 1 | I^{\otimes N-w-v-u}.
\]

\[ U_{OR}^{(N)}(u, v, w) = \sum_{\varepsilon_1, \varepsilon_2 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1 \rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2 \rangle \langle \varepsilon_2|
\]
\[ = I^{\otimes w-v-u-1} \otimes |\varepsilon_1 \vee \varepsilon_2 \rangle \langle 0 | I^{\otimes N-w-v-u} + I^{\otimes u-1} \otimes |\varepsilon_1 \rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2 \rangle \langle \varepsilon_2|
\]
\[ = I^{\otimes w-v-u-1} \otimes |1 - \varepsilon_1 \vee \varepsilon_2 \rangle \langle 1 | I^{\otimes N-w-v-u}.
\]

\[ U_{COPY}^{(N)}(u, v) = \sum_{\varepsilon_1 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1 \rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_1 \rangle \langle \varepsilon_1|
\]
\[ + I^{\otimes u-1} \otimes |\varepsilon_1 \rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |1 - \varepsilon_1 \rangle \langle 1 | I^{\otimes N-w-v-u}.
\]

where \( u, v \) and \( w \) are positive integers satisfying \( 1 \leq u < v < w \leq N \). These operators can be written, in terms of elementary gates, as

\[ U_{OR}^{(N)}(u, v, w) = U_{CN}^{(N)}(u, w) \cdot U_{CN}^{(N)}(v, w) \cdot U_{CCN}^{(N)}(u, v, w),
\]
\[ U_{AND}^{(N)}(u, v, w) = U_{CCN}^{(N)}(u, v, w),
\]
\[ U_{COPY}^{(N)}(u, v) = U_{CN}^{(N)}(u, v).
\]

Let \( \mathcal{C} \) be a set of clauses whose cardinality is equal to \( m \). Let \( \mathcal{H} = (\mathbb{C}^2)^{\otimes n+\mu+1} \) be a Hilbert space and \( |v_0\rangle \) be the initial state \( |v_0\rangle = |0^\mu, 0^\nu, 0\rangle \), where \( \mu \) is the number of dust qubits (the details are seen in [19]). Let \( U_{C}^{(n)} \) be a unitary operator for the computation of the SAT:

\[ U_{C}^{(n)} |v_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |\varepsilon_i, x^\mu, t, (\mathcal{C})\rangle \equiv |v_f\rangle
\]

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where \( x^\mu \) denotes a \( \mu \) strings of binary symbols and \( t_{e_i} (\mathcal{C}) \) is a truth value of \( \mathcal{C} \) with \( e_i \).

Let \( \{ s_k; k = 1, \ldots, m \} \) be the sequence defined as

\[
\begin{align*}
  s_1 & = n + 1, \\
  s_2 & = s_1 + \text{card} (C_1) + \delta_{1, \text{card}(C_1)} - 1, \\
  s_i & = s_{i-1} + \text{card} (C_{i-1}) + \delta_{1, \text{card}(C_{i-1})}, \quad 3 \leq i \leq m,
\end{align*}
\]

where \( \text{card} (C_i) \) means the cardinality of a clause \( C_i \). Take a value \( s \) as

\[
s = s_m - 1 + \text{card} (C_m) + \delta_{1, \text{card}(C_m)}.
\]

Note that the number \( m \) of the clause is at most 2\( n \). Then we have \([19]\): The total number of dust qubits \( \mu \) is

\[
\mu = s - 1 - n = \sum_{k=1}^{m} \text{card} (C_k) + \delta_{1, \text{card}(C_k)} - 2
\]

for \( m \geq 2 \). In order to construct \( U_{\mathcal{C}}^{(n)} \) concretely, we use the following unitary gates for this concrete expression \([30, 2]\):

\[
U_{\text{AND}}^{(x)} (k) = \begin{cases} 
  U_{\text{AND}}^{(x)} (s_{k+1} - 1, s_{k+2} - 2, s_{k+2} - 1), & 1 \leq k \leq m - 2, \\
  U_{\text{AND}}^{(x)} (s_m - 1, s_{f} - 1, s_{f}), & k = m - 1,
\end{cases}
\]

\[
U_{\text{OR}}^{(x)} (k) = U_{\text{OR}}^{(x)} (l, s_k - \text{card} (C_k) - 1, s_k - \text{card} (C_k) - 2) \cdots U_{\text{OR}}^{(x)} (l, s_k, s_k + 1) \bar{U}_{\text{OR}}^{(x)} (l_1, l_2, s_k),
\]

\[
\bar{U}_{\text{OR}}^{(x)} (u, v, w) = \begin{cases} 
  U_{\text{OR}}^{(x)} (u, v, w), & x_u \in C_k \\
  U_{\text{NOT}}^{(x)} (u) \cdot U_{\text{OR}}^{(x)} (u, v, w) \cdot U_{\text{NOT}}^{(x)} (u), & x_u \in C_k
\end{cases}, \quad \bar{x}_{u, \bar{x}_v} \in C_k
\]

where \( l_1, l_2, l_3, l_4 \) are positive integers such that \( x_z \in C_k \) or \( \bar{x}_z \in C_k, (z = l_1, \ldots, l_4) \).

**Theorem 3** The unitary operator \( U_{\mathcal{C}}^{(n)} \), is represented as

\[
U_{\mathcal{C}}^{(n)} = U_{\text{AND}}^{(n+\mu+1)} (m, \ldots, 1) \cdot U_{\text{AND}}^{(n+\mu+1)} (m, \ldots, 1) \cdots U_{\text{AND}}^{(n+\mu+1)} (2) \cdot U_{\text{AND}}^{(n+\mu+1)} (1) \cdot U_{\text{AND}}^{(n+\mu+1)} (m) \cdot U_{\text{OR}}^{(n+\mu+1)} (m, \ldots, 1) \cdots U_{\text{OR}}^{(n+\mu+1)} (1) \cdot U_{\text{OR}}^{(n+\mu+1)} (n).
\]

Applying the above unitary operator to the initial state, we obtain the final state \( \rho \). The result of the computation is registered as \( |t (\mathcal{C})\rangle \) in the last section of the final vector, which will be taken out by a projection \( P_{\mathcal{C}, \mu, 1} \equiv I^{\otimes n+\mu} \otimes |1\rangle \langle 1| \) onto the subspace of \( \mathcal{H} \) spanned by the vectors \( |x^\mu, \varepsilon^\mu, 1\rangle \).

The following theorem is easily seen.
Theorem 4  

$C$ is SAT if and only if

$$P_{n+\mu,1}U_C^{(\mu)}|v_0\rangle \neq 0$$

According to the standard theory of quantum measurement, after a measurement of the event $P_{n+\mu,1}$, the state $\rho = |v_f\rangle \langle v_f|$ becomes

$$\rho \rightarrow \frac{P_{n+\mu,1}\rho P_{n+\mu,1}}{Tr\rho P_{n+\mu,1}} =: \rho'$$

Thus the solvability of the SAT problem is reduced to check that $\rho' \neq 0$. The difficulty is that the probability

$$Tr\rho P_{n+\mu,1} = ||P_{n+\mu,1}|v_f\rangle||^2 = \frac{|T(C_0)|}{2^n}$$

is very small in some cases, where $|T(C_0)|$ is the cardinality of the set $T(C_0)$, of all the truth functions $t$ such that $t(C_0) = 1$.

We put $q \equiv \sqrt{\frac{r}{2^n}}$ with $r = |T(C_0)|$. Then if $r$ is suitably large to detect it, then the SAT problem is solved in polynomial time. However, for small $r$, the probability is very small so that we in fact do not get an information about the existence of the solution of the equation $t(C_0) = 1$, hence in such a case we need further deliberation.

Let go back to the SAT algorithm. After the quantum computation, the quantum computer will be in the state

$$|v_f\rangle = \sqrt{1 - q^2} |\varphi_0\rangle \otimes |0\rangle + q |\varphi_1\rangle \otimes |1\rangle$$

where $|\varphi_1\rangle$ and $|\varphi_0\rangle$ are normalized $n (=n + \mu)$ qubit states and $q = \sqrt{r/2^n}$. Effectively our problem is reduced to the following 1 qubit problem: The above state $|v_f\rangle$ is reduced to the state

$$|\psi\rangle = \sqrt{1 - q^2} |0\rangle + q |1\rangle,$$

and we want to distinguish between the cases $q = 0$ and $q > 0$(small positive number).

It will be not possible to amplify, by a unitary transformation, the above small positive $q$ into suitable large one to be detected, e.g., $q > 1/2$,with staying $q = 0$ as it is. The amplification would be not possible if we use the standard model of quantum computations with a unitary evolution. What we did in \cite{32,33} is to propose to use the output $|\psi\rangle$ of the quantum computer as an input for another device involving chaotic dynamics. That is, it is proposed to combine quantum computer with a chaotic dynamics amplifier in \cite{32,33}. Such a quantum chaos computer is a new model of computations and we could demonstrate that the amplification is possible in the polynomial time.
2.3 Chaos algorithm of SAT

Here we will argue that chaos can play a constructive role in computations (see [32, 33] for the details). Chaotic behavior in a classical system usually is considered as an exponential sensitivity to initial conditions. It is this sensitivity we would like to use to distinguish between the cases $q = 0$ and $q > 0$ mentioned in the previous section.

Consider the so called logistic map

$$x_{n+1} = ax_n(1 - x_n) \equiv g(x), \quad x_n \in [0, 1].$$

The properties of the map depend on the parameter $a$. If we take, for example, $a = 3.71$, then the Lyapunov exponent is positive, the trajectory is very sensitive to the initial value and one has the chaotic behavior [26]. It is important to notice that if the initial value $x_0 = 0$, then $x_n = 0$ for all $n$.

It is known [14] that any classical algorithm can be implemented on quantum computer. Our quantum chaos computer will be consisting from two blocks. One block is the ordinary quantum computer performing computations with the output $|\psi\rangle = \sqrt{1 - q^2} |0\rangle + q |1\rangle$. The second block is a computer performing computations of the classical logistic map. This two blocks should be connected in such a way that the state $|\psi\rangle$ first be transformed into the density matrix of the form

$$\rho = q^2 P_1 + (1 - q^2) P_0$$

where $P_1$ and $P_0$ are projectors to the state vectors $|1\rangle$ and $|0\rangle$. This connection is in fact nontrivial and actually it should be considered as the third block. One has to notice that $P_1$ and $P_0$ generate an Abelian algebra which can be considered as a classical system. In the second block the density matrix $\rho$ above is interpreted as the initial data $\rho_0$, and we apply the logistic map as

$$\rho_m = \frac{(I + g^m(\rho_0)\sigma_3)}{2}$$

where $I$ is the identity matrix and $\sigma_3$ is the z-component of Pauli matrix on $\mathbb{C}^2$. To find a proper value $m$ we finally measure the value of $\sigma_3$ in the state $\rho_m$ such that

$$M_m \equiv tr \rho_m \sigma_3.$$ 

We obtain

**Theorem 5**

$$\rho_m = \frac{(I + g^m(q^2)\sigma_3)}{2}, \text{ and } M_m = g^m(q^2).$$

Thus the question is whether we can find such a $m$ in polynomial steps of $n$ satisfying the inequality $M_m \geq \frac{1}{2}$ for very small but non-zero $q^2$. Here we have to remark that if one has $q = 0$ then $\rho_0 = P_0$ and we obtain $M_m = 0$ for all $m$. If $q \neq 0$, the stochastic dynamics leads to the amplification of the small magnitude
q in such a way that it can be detected as is explained below. The transition from \( \rho_0 \) to \( \rho_m \) is nonlinear and can be considered as a classical evolution because our algebra generated by \( P_0 \) and \( P_1 \) is abelian. The amplification can be done within at most \( 2n \) steps due to the following propositions. Since \( g^m(q^2) \) is \( x_m \) of the logistic map \( x_{m+1} = g(x_m) \) with \( x_0 = q^2 \), we use the notation \( x_m \) in the logistic map for simplicity.

**Theorem 6** For the logistic map \( x_{n+1} = ax_n(1 - x_n) \) with \( a \in [0, 4] \) and \( x_0 \in [0, 1] \), let \( x_0 = \frac{1}{2} \) and a set \( J \) be \{0, 1, 2, \ldots, n, \ldots 2n\}. If \( a \) is 3.71, then there exists an integer \( m \) in \( J \) satisfying \( x_m > \frac{1}{2} \).

**Theorem 7** Let \( a \) and \( n \) be the same in the above proposition. If there exists \( m_0 \) in \( J \) such that \( x_{m_0} > \frac{1}{2} \), then \( m_0 > \frac{n-1}{\log_2 3.71 - 1} \).

According to these theorems, it is enough to check the value \( x_m \) (\( M_m \)) around the above \( m_0 \) when \( q \) is \( \frac{1}{2} \) for a large \( n \). More generally, when \( q = \frac{k}{2^n} \) with some integer \( k \), it is similarly checked that the value \( x_m \) (\( M_m \)) becomes over \( \frac{1}{2} \) within at most \( 2n \) steps.

The complexity of the quantum algorithm for the SAT problem was discussed in Section 3 to be in polynomial time. We have only to consider the number of steps in the classical algorithm for the logistic map performed on quantum computer. It is the probabilistic part of the construction and one has to repeat computations several times to be able to distinguish the cases \( q = 0 \) and \( q > 0 \). Thus it seems that the quantum chaos computer can solve the SAT problem in polynomial time.

In conclusion of [33], the quantum chaos computer combines the ordinary quantum computer with quantum chaotic dynamics amplifier. It may go beyond the usual quantum Turing algorithm, but such a device can be powerful enough to solve the \( \text{NP} \)-complete problems in the polynomial time. The detail estimation of the complexity of the SAT algorithm is discussed in [19].

In the next two sections we will discuss the SAT problem in a different view, that is, we will show that the same amplification is possible by unitary dynamics defined in the stochastic limit.

## 3 Quantum Adaptive Algorithm of SAT

The idea to develop a mathematical approach to adaptive systems, i.e. those systems whose properties are in part determined as responses to an environment [1, 25], were born in connection with some problems of quantum measurement theory and chaos dynamics.

The mathematical definition of adaptive system is in terms of observables, namely: *an adaptive system is a composite system whose interaction depends on a fixed observable (typically in a measurement process, this observable is the observable one wants to measure). Such systems may be called observable-adaptive.*
In the paper [4] we extended this point of view by introducing another natural class of adaptive systems which, in a certain sense, is the dual to the above defined one, namely the class of state–adaptive systems. These are defined as follows: a state–adaptive system is a composite system whose interaction depends on the state of at least one of the sub–systems at the instant in which the interaction is switched on. We applied the state-adaptivity to quantum computation.

The difference between state–adaptive systems and nonlinear dynamical systems should be emphasized:

(i) in nonlinear dynamical systems (such as those whose evolution is described by the Boltzmann equation, or nonlinear Schrödinger equation, ..., ) the interaction Hamiltonian depends on the state at each time \( t \):

\[
H_I = H_I(\rho_t) \quad \forall t.
\]

(ii) in state–adaptive dynamical systems (such as those considered in the present paper) the interaction Hamiltonian depends on the state only at time \( t = 0 \):

\[
H_I = H_I(\rho_0).
\]

Now from the general theory of stochastic limit [2] one knows that, under general ergodicity conditions, an interaction with an environment drives the system to a dynamical (but not necessarily thermodynamical) equilibrium state which depends on the initial state of the environment and on the interaction Hamiltonian.

Therefore, if one is able to realize experimentally these state dependent Hamiltonians, one would be able to drive the system \( S \) to a pre–assigned dynamical equilibrium state depending on the input state \( \psi_0 \).

In the following subsection we will substantiate the general scheme described above with an application to the SAT problem described in the previous sections.

### 3.1 Stochastic Limit and SAT Problem

We illustrate the general scheme described in the previous section in the simplest case when the state space of the system is \( \mathcal{H}_S \equiv \mathbb{C}^2 \). We fix an orthonormal basis of \( \mathcal{H}_S \) as \( \{ e_0, e_1 \} \).

The unknown state (vector) of the system at time \( t = 0 \)

\[
\psi := \sum_{\varepsilon \in \{0,1\}} \alpha_{\varepsilon} e_{\varepsilon} = \alpha_0 e_0 + \alpha_1 e_1 \quad \|\psi\| = 1.
\]

In Sec. 3, \( \alpha_1 \) corresponds to \( q \) and \( e_j \) does to \( |j\rangle \) \( (j = 0, 1) \). This vector is taken as input and defines the interaction Hamiltonian in an external field

\[
H_I = \lambda |\psi\rangle\langle\psi| \otimes (A_g^+ + A_g)
\]

\[
= \sum \lambda \alpha_{\varepsilon} \pi_{\varepsilon} |e_{\varepsilon}\rangle\langle e_{\varepsilon}'| \otimes (A_g^+ + A_g)
\]

where \( \lambda \) is a small coupling constant. Here and in the following summation over repeated indices is understood.
The free system Hamiltonian is taken to be diagonal in the $e_ε$–basis

$$H_S := \sum_{ε\in\{0,1\}} E_ε|e_ε⟩⟨e_ε| = E_0|e_0⟩⟨e_0| + E_1|e_1⟩⟨e_1|$$

and the energy levels are ordered so that $E_0 < E_1$. Thus there is a single Bohr frequency $ω_0 := E_1 - E_0 > 0$. The 1–particle field Hamiltonian is

$$S_Ig(k) = e^{itω(k)}g(k)$$

where $ω(k)$ is a function satisfying the basic analytical assumption of the stochastic limit. Its second quantization is the free field evolution

$$e^{itH_0}A_0e^{-itH_0} = AS_Ig$$

We can distinguish two cases as below, whose cases correspond to two cases of Sec. 3, i.e., $q > 0$ and $q = 0$.

**Case (1).** If $α_0, α_1 \neq 0$, then, according to the general theory of stochastic limit (i.e., $t → t/λ^2$) \[2\], the interaction Hamiltonian $H_I$ is in the same universality class as

$$\tilde{H}_I = D \otimes A^+_0 + D^+ \otimes A_g$$

where $D := |e_0⟩⟨e_1|$. The interaction Hamiltonian at time $t$ is then

$$\tilde{H}_I(t) = e^{-itω_0}D \otimes A^+_0 + \text{ h.c.} = D \otimes A^+(e^{it(ω(p)−ω_0)}g) + \text{ h.c.}$$

and the white noise ($\{b_t\}$) Hamiltonian equation associated, via the stochastic golden rule, to this interaction Hamiltonian is

$$∂_t U_t = i(Db_t^+ + D^+b_t)U_t$$

Its causally normal ordered form is equivalent to the stochastic differential equation

$$dU_t = (iDdb_t^+ + iD^+ dB_t - γ_- D^+ Ddt)U_t,$$

where $dB_t := b_t dt$.

The causally ordered inner Langevin equation is

$$dj_t(x) = dU^*_t xU_t + U^*_t x dB_t + dB^*_t x dU_t$$

where $j_t(x) := U^*_t xU_t$. Therefore the master equation is

$$\frac{d}{dt} P^t(x) = (Imγ)_t[D^+ D, P^t(x)] - (Reγ_-)[D^+ D, P^t(x)] + (Reγ_-)D^+ P^t(x)D$$

13
where $D^+ D = |e_1⟩⟨e_1|$ and $D^+ XD = ⟨e_0, x e_0| e_1⟩⟨e_1|$. 

The dual Markovian evolution $P^*_t$ acts on density matrices and its generator is

$$L_p = (\text{Im} \gamma_1) i [\rho, D^+ D] - (\text{Re} \gamma_1) \{ \rho, D^+ D \} + (\text{Re} \gamma_1) D \rho D^+$$

Thus, if $\rho_0 = |e_0⟩⟨e_0|$ one has

$$L_p \rho_0 = 0$$

so $\rho_0$ is an invariant measure. From the Frigerio–Fagnola–Rebolledo criteria, it is the unique invariant measure and the semigroup $\exp(tL_p)$ converges exponentially to it.

**Case (2).** If $\alpha_1 = 0$, then the interaction Hamiltonian $H_I$ is

$$H_I = \lambda |e_0⟩⟨e_0| \otimes (A_g^+ + A_g)$$

and, according to the general theory of stochastic limit, the reduced evolution has no damping and corresponds to the pure Hamiltonian

$$H_S + |e_0⟩⟨e_0| = (E_0 + 1) |e_0⟩⟨e_0| + E_1 |e_1⟩⟨e_1|$$

therefore, if we choose the eigenvalues $E_1, E_0$ to be integers (in appropriate units), then the evolution will be periodic.

Since the eigenvalues $E_1, E_0$ can be chosen a priori, by fixing the system Hamiltonian $H_S$, it follows that the period of the evolution can be known a priori. This gives a simple criterion for the solvability of the SAT problem because, by waiting a sufficiently long time one can experimentally detect the difference between a damping and an oscillating behavior.

A precise estimate of this time can be achieved either by theoretical methods or by computer simulation. Both methods will be analyzed in the full paper [5].

**Conclusion 8** We pointed out that it is possible to distinguish two different states, $\sqrt{1 - q^2} |0⟩ + q |1⟩$ ($q \neq 0$) and $|0⟩$ by means of the adaptive dynamics with the stochastic limit.

**Conclusion 9** Finally we remark that our algorithm can be described by a deterministic generalized quantum Turing machine [21, 5].

### 4 Comparison of Various Quantum Mutual Type Entropies

There exist several different types of quantum mutual entropy. The classical mutual entropy was introduced by Shannon to discuss the information transmission from an input system to an output system [20]. Then Kolmogorov [24], Gelfand and Yaglom [17] gave a measure theoretic expression of the mutual entropy by means of the relative entropy defined by Kullback and Leibler. The Shannon’s expression of the mutual entropy is generalized to one for finite dimensional quantum (matrix) case by Holevo [18, 22]. Ohya took the measure
theoretic expression by KGY and defined quantum mutual entropy by means of quantum relative entropy \[27, 28\]. Recently Shor \[38\] and Bennett et al. \[10\] took the coherent information and defined the mutual type entropy to discuss a Shannon’s coding theorem. In this section, we compare these mutual types entropies.

The most general form of the quantum mutual entropy defined by Ohya, generalizing the KGY measure theoretic mutual entropy, is given as

\[
I_1 (\varphi; \Lambda) = \sup \left\{ \int_{S} S^{AU} (\Lambda \omega, \Lambda \varphi) \, d\mu; \; \mu \in M_{\varphi} (S) \right\}.
\]

Here \(S\) is the set of all states in a certain C*-algebra (or von Neumann algebra) describing a quantum system, \(S^{AU} (\cdot, \cdot)\) is the relative entropy of Araki \[6\] or Uhlmann \[39\] and \(\mu\) is a measure decomposing the state \(\varphi\) into extremal orthogonal states, i.e., \(\varphi = \int_{x \in S} \omega d\mu, \) in \(S\), whose set is denoted by \(M_{\varphi} (S)\).

In the case that the C*-algebra is \(B (\mathcal{H})\) and \(S\) is the set of all density operators, the above mutual entropy goes to

\[
I_1 (\rho; \Lambda) = \sup \left\{ \sum_{n} S^{U} (\Lambda E_n, \Lambda \rho); \; \rho = \sum_{n} \lambda_n E_n \right\},
\]

where \(\rho\) is a density operator (state), \(S^{U} (\cdot, \cdot)\) is Umegaki’s relative entropy and \(\rho = \sum_{n} \lambda_n E_n\) is a Schatten-von Neumann (one dimensional spectral) decomposition. The SN decomposition is not always unique unless \(S\) is Choque simplex, so we take the supremum over all possible decompositions. It is easily shown that we can take orthogonal decomposition instead of the SN decomposition \[29\].

These quantum mutual entropy are completely quantum, namely, they describe the information transmission from a quantum input to a quantum output. When the input system is classical, the state \(\rho\) is a probability distribution and the Schatten-von Neumann decomposition is unique with delta measures \(\delta_n\) such that \(\rho = \sum_{n} \lambda_n \delta_n\). In this case we need to code the classical state \(\rho\) by a quantum state, whose process is a quantum coding described by a channel \(\Gamma\) such that \(\Gamma \delta_n = \sigma_n\) (quantum state) and \(\sigma \equiv \Gamma \rho = \sum_{n} \lambda_n \sigma_n\). Then the quantum mutual entropy \(I_1 (\rho; \Lambda)\) becomes Holevo’s one, that is,

\[
I_1 (\rho; \Lambda \Gamma) = S (\Lambda \sigma) - \sum_{n} \lambda_n S (\Lambda \sigma_n)
\]

when \(\sum_{n} \lambda_n S (\Lambda \sigma_n)\) is finite.

Let us discuss the entropy exchange \[8\]. For a state \(\rho\), a channel \(\Lambda\) is defined by an operator valued measure \(\{A_j\}\) such as \(\Lambda (\cdot) \equiv \sum_{j} A_j^* \cdot A_j\). Then define a matrix \(W = (W_{ij})\) with \(W_{ij} = \frac{tr A_i^* \rho A_j}{tr \rho}\), by which the entropy exchange is defined by

\[
S_e (\rho, \Lambda) = -tr W \log W.
\]
Using the above entropy exchange, two mutual type entropies are defined as below and they are applied to the study of quantum version of Shannon’ coding theorem [8, 38, 10]. The first one is called the coherent information $I_2 (\rho; \Lambda)$ and the second one is $I_3 (\rho, \Lambda)$, which are defined by

\[
I_2 (\rho, \Lambda) \equiv S (\Lambda \rho) - S_e (\rho, \Lambda), \\
I_3 (\rho, \Lambda) \equiv S (\rho) + S (\Lambda \rho) - S_e (\rho, \Lambda).
\]

By comparing these mutual entropies for information communication processes, we have the following theorem [25]:

**Theorem 10** When $\{A_j\}$ is a projection valued measure and $\dim(\text{ran} A_j) = 1$, for arbitrary state $\rho$ we have (1) $I_1 (\rho, \Lambda) \leq \min \{S (\rho), S (\Lambda \rho)\}$, (2) $I_2 (\rho, \Lambda) = 0$, (3) $I_3 (\rho, \Lambda) = S (\rho)$.

From this theorem, the entropy $I_1 (\rho, \Lambda)$ only satisfies the inequality held in classical systems, so that only this entropy can be a candidate as quantum extension of the classical mutual entropy. Other two entropies can describe a sort of entanglement between input and output, such a correlation can be also described by quasi-mutual entropy, a slight generalization of $I_1 (\rho, \Lambda)$, discussed in [29, 9].

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**References**

[1] L. Accardi and K. Imafuku: Control of Quantum States by Decoherence, to appear in Open Systems and Information Dynamics, 2003

[2] L.Accardi, Y.G. Lu, I.Volovich: Quantum Theory and its Stochastic Limit. Springer Verlag 2002; Japanese translation, Tokyo–Springer 2003.

[3] L.Accardi and M.Ohya, Compound channels, transition expectations, and liftings, Appl. Math. Optim., Vol.39, 33-59, 1999.

[4] L.Accardi and M.Ohya, A stochastic limit approach to the SAT problem, to appear.

[5] L.Accardi and M.Ohya, Generalized quantum Turing machine and stochastic limit for the SAT problem, in preparation.

[6] H.Araki: Relative entropy of states of von Neumann Algebras, Publ.RIMS, Kyoto Univ.Vol.11, 809-833, (1976); Relative entropy for states of von Neumann algebras II, Publ.RIMS, Kyoto Univ., 13, pp.173–192, (1977)
[7] L. Accardi and Ruben Sabbadini, On the Ohya–Masuda quantum SAT Algorithm, Proceedings Intern. Conf. "Unconventional Models of Computations", I. Antoniou, C.S. Calude, M. Dinneen (eds.) Springer 2001 ; Preprint Volterra, N. 432, 2000

[8] H. Barnum, M.A. Nielsen and B.W. Schumacher, Information transmission through a noisy quantum channel, Physical Review A, Vol. 57, No. 6, 4153-4175, 1998.

[9] V.P. Belavkin and M. Ohya, Quantum entropy and information in discrete entangled states, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. 4, No. 2, 137-160 (2001); Quantum entanglements and entangled mutual entropy, Proc. R. Soc. Lond. A. 458, 209-231 (2002)

[10] C.H. Bennett, P.W. Shor, J.A. Smolin, and A.V. Thapliyalz, Entanglement-Assisted Capacity of a Quantum Channel and the Reverse Shannon Theorem, quant-ph/0106052

[11] E. Bernstein and U. Vazirani, Quantum complexity theory, Proc. of the 25th Annual ACM Symposium on Theory of Computing, ACM, New York, pp.11-22 (1993), SIAM Journal on Computing 26, 1411 (1997)

[12] C.H. Bennett, E. Bernstein, G. Brassard and U. Vazirani, Strengths and Weaknesses of Quantum Computing, quant-ph/9701001

[13] R. Cleve, An Introduction to Quantum Complexity Theory, quant-ph/9906111

[14] D. Deutsch, Quantum theory, the Church-Turing principle and the universal quantum computer, Proc. of Royal Society of London series A, 400, pp. 97-117, 1985.

[15] A. Ekert and R. Jozsa, Quantum computation and Shor’s factoring algorithm, Reviews of Modern Physics, 68 No. 3, pp. 733-753, 1996.

[16] M. Garey and D. Johnson, Computers and Intractability - a guide to the theory of NP-completeness, Freeman, 1979.

[17] I.M. Gelfand and A.M. Yaglom, Calculation of the amount of information about a random function contained in another such function, Amer. Math. Soc. Transl., 12, pp.199-246, (1959)

[18] A.S. Holevo, Some estimates for the amount of information transmittable by a quantum communication channel (in Russian), Problemy Peredachi Informacii, 9, pp. 3-11, (1973)

[19] S. Iriyama and S. Akashi, Complexity of Ohya-Masuda-Volovich algorithm, to appear

[20] R.S. Ingarden, A. Kossakowski and M. Ohya, Information Dynamics and Open Systems, Kluwer, (1997)
[21] S. Iriyama, M. Ohya and I. Volovich, Generalized quantum Turing machine and its application to the SAT chaos algorithm, TUS(Tokyo University of Science) preprint, 2003.

[22] R.S.Ingarden, Quantum information theory, Rep. Math. Phys., 10, pp.43-73, 1976.

[23] J.von Neumann, Die Mathematischen Grundlagen der Quantenmechanik, Springer-Berlin, 1932.

[24] A.N.Kolmogorov, Theory of transmission of information, Amer. Math. Soc. Translation, Ser.2, 33, pp.291–321, 1963.

[25] A.Kossakowski, M.Ohya and Y.Togawa, How can we observe and describe chaos?., Open System and Information Dynamics 10(3): 221-233, 2003

[26] M. Ohya, Complexities and Their Applications to Characterization of Chaos, Int. Journ. of Theort. Phy., 37, 495, 1998.

[27] M. Ohya, On compound state and mutual information in quantum information theory, IEEE Trans.Information Theory, 29, pp.770–777 (1983)

[28] M. Ohya, Some aspects of quantum information theory and their applications to irreversible processes, Rep.Math.Phys., Vol.27, 19-47, (1989)

[29] M.Ohya, Fundamentals of quantum mutual entropy and capacity, Open Systems and Information Dynamics, 6. No.1, 69-78, 1999.

[30] M.Ohya and N.Masuda, NP problem in Quantum Algorithm, Open Systems and Information Dynamics, 7 No.1 33-39, 2000.

[31] M. Ohya and D.Petz, Quantum Entropy and its Use, Springer-Verlag, (1993)

[32] M.Ohya and I.Volovich, Quantum computing, NP-complete problems and chaotic dynamics, Quantum Information II, eds. T.Hida and K.Saito, World Sc. 2000; quant-ph/9912100 and J.Opt.B, 5,No.6 639-642, 2003

[33] M. Ohya and I. Volovich, New quantum algorithm for studying NP-complete problems, Rep.Math.Phys.,52, No.1,25-33, 2003

[34] M.Ohya and I. Volovich, Quantum Information, Computation, Cryptography and Teleportation, Springer-Verlag (to appear).

[35] M. Ohya and N.Watanabe, Remarks on quantum mutual entropy, TUS preprint.

[36] D. Petz and M. Mosonyi, Stationary quantum source coding, Journal of Mathematical Physics, Vol.42, 4857-4864, 2001.

[37] P. Shor, Algorithm for quantum computation, Discrete logarithm and factoring algorithm, Proceedings of the 35th Annual IEEE Symposium on Foundation of Computer Science, pp.124-134, 1994.
[38] P. Shor, The quantum channel capacity and coherent information, Lecture Notes, MSRI Workshop on Quantum Computation, 2002.

[39] A. Uhlmann, Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in interpolation theory, Commun. Math. Phys., Vol. 54, 21-32, 1977.

[40] H. Umegaki, Conditional expectations in an operator algebra IV (entropy and information), Kodai Math. Sem. Rep., Vol. 14, 59-85, (1962)