Minimal determinantal representations of bivariate polynomials

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Abstract

For a square-free bivariate polynomial \( p \) of degree \( n \) we introduce a simple and fast numerical algorithm for the construction of \( n \times n \) matrices \( A, B, \) and \( C \) such that \( \det(A + xB + yC) = p(x, y) \). This is the minimal size needed to represent a bivariate polynomial of degree \( n \). Combined with a square-free factorization one can now compute \( n \times n \) matrices for any bivariate polynomial of degree \( n \). The existence of such symmetric matrices was established by Dixon in 1902, but, up to now, no simple numerical construction has been found, even if the matrices can be nonsymmetric. Such representations may be used to efficiently numerically solve a system of two bivariate polynomials of small degree via the eigenvalues of a two-parameter eigenvalue problem. The new representation speeds up the computation considerably.

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1. Introduction

Let

\[
p(x, y) := \sum_{i=0}^{n} \sum_{j=0}^{n-i} p_{ij} x^i y^j,
\]

where \( p_{ij} \in \mathbb{C} \) for all \( i, j \), be a bivariate polynomial of degree \( n \), where we assume that \( p_{ij} \neq 0 \) for at least one index such that \( i + j = n \). We say that matrices \( A, B, C \in \mathbb{C}^{m \times m} \), where \( m \geq n \), form a determinantal representation of order \( m \) of the polynomial \( p \) if

\[
\det(A + xB + yC) = p(x, y).
\]

It is known since Dixon’s 1902 paper \cite{5} that every bivariate polynomial of degree \( n \) admits a determinantal representation with symmetric matrices of order \( n \). However, the construction of such matrices is far from trivial and up to now there have been no efficient numerical
algorithms, even if we do not insist on matrices being symmetric. We introduce the first efficient numerical construction of determinantal representations that returns \( n \times n \) matrices for a square-free bivariate polynomial of degree \( n \), which, with the exception of the symmetry, agrees with Dixon’s result. For non square-free polynomials one can combine it with a square-free factorization to obtain a representation of order \( n \).

Our motivation comes from the following approach for finding roots of systems of bivariate polynomials, proposed by Plestenjak and Hochstenbach in [16]. Suppose that we have a system of two bivariate polynomials

\[
\begin{align*}
p(x, y) &= \sum_{i=0}^{n_1} \sum_{j=0}^{n_1-i} p_{ij} x^i y^j = 0, \\
q(x, y) &= \sum_{i=0}^{n_2} \sum_{j=0}^{n_2-i} q_{ij} x^i y^j = 0.
\end{align*}
\]  

(3)

The idea is to construct matrices \( A_1, B_1, C_1 \) of size \( m_1 \times m_1 \) and matrices \( A_2, B_2, C_2 \) of size \( m_2 \times m_2 \) such that

\[
\begin{align*}
\det(A_1 + xB_1 + yC_1) &= p(x, y), \\
\det(A_2 + xB_2 + yC_2) &= q(x, y)
\end{align*}
\]  

(4)

and then numerically solve the equivalent two-parameter eigenvalue problem

\[
\begin{align*}
(A_1 + xB_1 + yC_1) u_1 &= 0, \\
(A_2 + xB_2 + yC_2) u_2 &= 0.
\end{align*}
\]  

(5)

Here it does not matter whether the matrices are symmetric. Except in some special cases, we get \( m_1 m_2 > n_1 n_2 \) and thus (5) is a singular two-parameter eigenvalue problem. Its finite regular eigenvalues, which can be computed numerically with a staircase type algorithm [9], are the roots of (3). This is a generalization of a well-known approach to numerically compute the roots of a univariate polynomial as eigenvalues of its companion matrix.

In our approach to the computation of roots of a system of bivariate polynomials we must find the right balance. As we can not exploit the symmetry in (4), it is not necessary that the
matrices have this property. If we want a representation with small matrices this might take too much time in the first phase, while a representation with large matrices slows down the second phase. By using the most efficient representations at the time with matrices of order $\frac{1}{6}n^2 + \mathcal{O}(n)$, a numerical method was implemented in Matlab [14] that is competitive to some existing methods for polynomials up to degree 9, see [16] for details.

The computational complexity of the new construction of determinantal representations that we give in the paper is just slightly above that of the construction used in [16], yet the matrices are much smaller. While the matrices in [16] are of order $\frac{1}{6}n^2 + \mathcal{O}(n)$, the new construction gives matrices of the minimal possible order $n$ for a square-free bivariate polynomial of degree $n$. This decreases the overall asymptotic complexity of solving the system (3) via determinantal representations from $\mathcal{O}(n^{12})$ to $\mathcal{O}(n^6)$ when $n = n_1 = n_2$. Moreover, if both polynomials $p_1$ and $p_2$ are square-free, then $m_1m_2 = n_1n_2$ and (5) is a nonsingular two-parameter eigenvalue problem, which is much easier to solve numerically than a singular one.

The rest of the paper is organized as follows. In Section 2 we give a short overview of existing determinantal representations. In Section 3 we show that a square-free polynomial can always be transformed into a form required by the algorithm in Section 4, where we give a determinantal representation of order $n$ for a square-free bivariate polynomial of degree $n$. In Section 5 we extend the representation of order $n$ to non square-free polynomials and discuss other options for such polynomials. We end with some numerical experiments in Section 6 and conclusions in Section 7.

2. Overview of existing determinantal representations

In the semidefinite programming (SDP) there is a large interest in symmetric determinantal representations of the real zero polynomials, a particular subset of polynomials related to the linear matrix inequality (LMI) constraints. For an overview, see, e.g., [10, 20]. A real bivariate polynomial $p$ satisfies the real zero condition with respect to $(x_0, y_0) \in \mathbb{R}^2$ if for all $(x, y) \in \mathbb{R}^2$ the univariate polynomial $p_{(x,y)}(t) := p(x_0 + tx, y_0 + ty)$ has only real zeros. A two-dimensional LMI set is defined as $\{(x, y) \in \mathbb{R}^2 : A + XB + yC \succeq 0\}$, where $A$, $B$, and $C$ are symmetric matrices of size $m \times m$ and $\succeq 0$ stands for positive semidefinite. For this particular subset of polynomials there do exist some procedures that involve slow symbolic computation and other expensive steps, see, e.g., [12, 13]. However, besides being too slow these algorithms are limited to the real zero polynomials only.

While in SDP and LMI the matrices have to be symmetric or Hermitian, this is not important in our case. We are looking for a simple and fast numerical construction of matrices as small as possible that satisfy (2).

Here is a list of some available determinantal representations for generic bivariate polynomials. The first group of determinantal representations has the property that the elements of matrices $A$, $B$, and $C$ depend affine-linearly on the coefficients of the polynomial $p$. Such determinantal representations are named uniform in [3]. The first such representation of order $n^2$ is given by Khazanov in [7] as a special case of a linearization of a multiparameter polynomial matrix. This is improved to a representation or order $\frac{1}{6}n(n+1)$ by Muhić and Plestenjak in [9, Appendix]. Quarez [17] gives symmetric representations of multivariate polynomials,
which results in a representation of order \( \frac{1}{4}n^2 + \mathcal{O}(n) \) for a bivariate polynomial. A smaller nonsymmetric uniform representation of the same asymptotic order is described in [16]. Recently, a uniform representation of order \( 2n - 1 \) was presented in [3], which is the first uniform representation such that the order of matrices grows linearly and not quadratically with \( n \). All uniform representations do not require any computation, the construction is very simple and fast as one just puts the coefficients of \( p \) on prescribed places in the matrices \( A, B, \) and \( C \).

If we allow computations, we can obtain smaller representations. In [16], a representation of order \( \frac{1}{n} n^2 + \mathcal{O}(n) \) is given. This representation is used in [14] as a part of a numerical method for the roots of a system of bivariate polynomials that is competitive to the existing numerical methods for polynomials of small degree. In the following sections we upgrade the approach from [16] to obtain minimal determinantal representations of order \( n \) while maintaining approximately the same complexity of computations involved in the construction.

3. Preliminary transformations

We can homogenize (1) into

\[
p_h(x, y, z) := z^n p \left( \frac{x}{z}, \frac{y}{z} \right) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} p_{ij} x^i y^j z^{n-i-j},
\]

where \((x, y, z)\) are points in the projective plane. It is easy to see that if \( n \times n \) matrices \( A, B, C \) are such that \( \det(A + xB + yC) = p(x, y) \), then

\[
\det(zA + xB + yC) = p_h(x, y, z).
\]

This also works in the opposite direction. If we construct a determinantal representation (7) with matrices of order \( n \) of the homogeneous polynomial \( p_h \), then we get a determinantal representation of \( p(x, y) \) by simply setting \( z = 1 \).

The homogeneous form gives us more freedom in the following sense. We can apply a linear change of variables

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = T \begin{bmatrix}
\tilde{x} \\
\tilde{y} \\
\tilde{z}
\end{bmatrix},
\]

where \( T \) is a nonsingular \( 3 \times 3 \) matrix, and transform \( p_h(x, y, z) \) into a homogeneous polynomial \( \tilde{p}_h(\tilde{x}, \tilde{y}, \tilde{z}) \). If we find matrices \( \tilde{A}, \tilde{B}, \) and \( \tilde{C} \) for \( \tilde{p}_h \) such that \( \det(\tilde{z}A + \tilde{x}B + \tilde{y}C) = \tilde{p}_h(\tilde{x}, \tilde{y}, \tilde{z}) \), this gives a determinantal representation of \( p \) after we substitute \( \tilde{x}, \tilde{y}, \tilde{z} \) back to \( x, y, z \) and set \( z = 1 \).

We say that a polynomial (1) is square-free if it is not a multiple of a square of a non-constant polynomial. The following result that relies on Bézout’s theorem and Bertini’s theorem (see, e.g., [18]), ensures that our construction can be applied to any square-free polynomial.
Theorem 1. Let \( p \) be a bivariate non-constant polynomial of degree \( n \). A generic line \( \mathcal{L} \) of the form \( ax + by + cz = 0 \) in the projective plane intersects the curve \( p_h(x, y, z) = 0 \), where \( p_h \) is the homogenized polynomial \( p \), in \( n \) distinct points if and only if \( p \) is square-free.

Proof. Let \( \mathcal{C} \) be the zero set of \( p_h \). When \( \mathcal{C} \) is smooth, it follows from Bertini’s theorem that a generic line \( \mathcal{L} \) intersects \( \mathcal{C} \) in distinct points. This can be easily extended to the case when \( \mathcal{C} \) has finitely many singular points, which is true for a square-free polynomial. Namely, in the Zariski topology the lines containing a singular point form a proper closed subset of all possible lines. As \( p \) has degree \( n \), there are \( n \) points in \( \mathcal{L} \cap \mathcal{C} \) by Bézout’s theorem.

On the other hand, if \( p \) is not square-free then \( p(x, y) = q(x, y)^k r(x, y) \), where \( k \geq 2 \) and \( q \) is a non-constant polynomial. Clearly, each intersection of an arbitrary line \( \mathcal{L} \) with the zero set of \( q_h \), the homogenization of \( q \), appears with multiplicity at least \( k \) in \( \mathcal{L} \cap \mathcal{C} \). \( \square \)

Let \( (1) \) be a bivariate polynomial \( p \) of degree \( n \) that we want to linearize. We can assume that \( p \) has the following properties:

(a) \( p_{n0} \neq 0 \),
(b) all zeros \( \xi_1, \ldots, \xi_n \) of the polynomial

\[
\alpha := p_h(\xi, 1, 0) = p_{n0} \xi^n + p_{n-1,1} \xi^{n-1} + \cdots + p_{0n} = 0 \quad (9)
\]

are simple.

The above holds for a generic square-free polynomial \( p \). If not, one can apply a random linear substitution \((8)\). In particular, an equivalent formulation of (b) is that in the projective plane the line \( z = 0 \) and the curve \( p_h(x, y, z) = 0 \) intersect in \( n \) distinct points. If follows from Theorem \( 1 \) that this holds after a random linear substitution \((8)\).

In a preliminary step of the construction we apply a linear substitution of the form

\[
x = \tilde{x} + s \tilde{y} + t \tilde{z}, \quad y = \tilde{y}, \quad z = \tilde{z},
\]

where we set \( s \) and \( t \) so that \( \tilde{p}_{n0} = \tilde{p}_{0,n-1} = 0 \) in the transformed polynomial \( \tilde{p} \) while properties (a) and (b) still hold. Indeed, \( \tilde{p}_{n0} = \tilde{p}_h(1, 0, 0) = p_h(1, 0, 0) = p_{n0} \neq 0 \) and it follows from \( \tilde{p}_h(\xi, 1, 0) = p_h(\xi + s, 1, 0) \) that all roots of \((9)\) shift for \( s \) and thus remain simple. It is easy to see that

\[
\tilde{p}_{n0} = \tilde{p}_h(0, 1, 0) = p_h(s, 1, 0) = h(s)
\]

and

\[
\tilde{p}_{0,n-1} = \frac{d}{d\tilde{z}} \tilde{p}_h(0, 1, 0) = t \frac{d}{dx} p_h(s, 1, 0) + \frac{d}{d\tilde{z}} p_h(s, 1, 0) = th'(s) + \frac{d}{d\tilde{z}} p_h(s, 1, 0).
\]

Therefore, if we select \( s \) as one of the roots of \((9)\) and

\[
t = \frac{-p_{n-1,0}s^{n-1} + p_{n-2,1}s^{n-2} + \cdots + p_{1,n-2}s + p_{0,n-1}}{h'(s)},
\]

then coefficients \( \tilde{p}_{n0} \) and \( \tilde{p}_{0,n-1} \) are both zero. Note that \( t \) is well defined because all roots of \((9)\) are simple and thus \( h'(s) \neq 0 \).
4. Determinantal representation for a square-free polynomial

Let (11) be a bivariate square-free polynomial of degree \( n \) that we want to linearize. After the preliminary linear transformations from the previous section we can assume that:

(a) \( p_{n0} \neq 0 \),
(b) all zeros \( \xi_1, \ldots, \xi_{n-1} \) of the polynomial

\[ v(\xi) := p_{n0} \xi^{n-1} + p_{n-1,1} \xi^{n-2} + \cdots + p_{1,n-1} \]

are simple and nonzero,
(c) \( p_{0n} = p_{0,n-1} = 0 \).

The construction is based on bivariate polynomials \( q_0, \ldots, q_{n-1} \), constructed recursively as

\[
\begin{align*}
q_0(x, y) &:= 1, \\
q_1(x, y) &:= f_{11} q_0(x, y), \\
q_2(x, y) &:= f_{21} q_1(x, y) + f_{22} q_0(x, y), \\
&\vdots \\
q_{n-1}(x, y) &:= f_{n-1,1} q_{n-2}(x, y) + f_{n-1,2} q_{n-3}(x, y) + \cdots + f_{n-1,n-1} q_0(x, y),
\end{align*}
\]

where each coefficient \( f_{ij} \) has a linear form \( f_{ij} = \alpha_{ij} x + \beta_{ij} y \) with \( \alpha_{ij} = 1 \) for \( i < n-1 \). It follows from the construction (11) that \( q_j \) is a bivariate polynomial of degree \( j \) for \( j = 0, \ldots, n-1 \). The ansatz for a determinantal representation of \( p \) is an \( n \times n \) bivariate pencil

\[
A + xB + yC = \begin{bmatrix}
\gamma_{00} + \gamma_{10} x & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-2} & p_{n0} x \\
-f_{11} & 1 & 0 & \cdots & 0 & 0 \\
-f_{22} & -f_{21} & 1 & \cdots & 0 & 0 \\
-f_{33} & -f_{32} & -f_{31} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
-f_{n-1,n-1} & -f_{n-1,n-2} & \cdots & \cdots & -f_{n-1,2} & -f_{n-1,1}
\end{bmatrix}.
\]

One can see from (11) and (12) that

\[
\begin{bmatrix}
1 \\
q_1(x, y) \\
q_2(x, y) \\
\vdots \\
q_{n-1}(x, y)
\end{bmatrix}
= \begin{bmatrix}
d(x, y) & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-2} & p_{n0} x \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & -f_{21} & 1 & 0 & \cdots & 0 \\
0 & -f_{32} & -f_{31} & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & -f_{n-1,n-2} & \cdots & \cdots & -f_{n-1,2} & -f_{n-1,1}
\end{bmatrix},
\]

where

\[
d(x, y) = \gamma_{00} + \gamma_{10} x + \gamma_1 q_1(x, y) + \cdots + \gamma_{n-2} q_{n-2}(x, y) + p_{n0} x q_{n-1}(x, y).
\]
It follows that \( \det(A + xB + yC) = d(x, y) \) and we will show how to set the values of \( \alpha_{ij}, \beta_{ij}, \) and \( \gamma_{00}, \gamma_{10}, \gamma_{11}, \ldots, \gamma_{n-2} \) so that \( \det(A + xB + yC) = p(x, y) \).

For a better understanding we first give a quick overview of the algorithm. Then we explain the details and give the complete procedure in Algorithm 3, followed by an example. In the quick overview we display the structure of a polynomial by a diagram, where dots in the \( j \)-th row stand for zero or nonzero coefficients at \( x^{j-1}, x^{j-2}y, \ldots, y^{j-1} \), respectively. The following diagram is for the case \( n = 5 \). Notice the two white dots representing \( p_{04} = p_{05} = 0 \).

\[
p = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

We build the representation (12) in a loop by adding subdiagonals from top to bottom and elements in the first row from right to left. We start by taking \( f_1 = x - \xi_i y \), for \( i = 1, \ldots, n - 1 \). Because of (10) \( p_{00}x(x - \xi_1 y) \cdots (x - \xi_{n-1} y) \) agrees with the part of \( p \) of degree \( n \) and we get a residual of degree \( n - 1 \) with a zero coefficient at \( y^{n-1} \). In case \( n = 5 \) we get

\[
r^{(1)}(x, y) := p(x, y) - \det \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-f_{11} & 1 & -f_{21} & 0 & 0 \\
-f_{22} & 1 & -f_{31} & 0 & 0 \\
-f_{33} & 1 & -f_{41} & 0 & 0 \\
-f_{44} & 1 & -f_{51} & 0 & 0 \\
\end{bmatrix} \bigg|_{x=0} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array},
\]

where \( r^{(1)} \) is a polynomial of degree 4 with a zero coefficient at \( y^4 \). Now we add \( f_{22}, \ldots, f_{n-1,2} \) in the second subdiagonal to annihilate the part of degree \( n - 1 \). In case \( n = 5 \) we get

\[
s^{(2)}(x, y) := p(x, y) - \det \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-f_{11} & 1 & -f_{21} & 0 & 0 \\
-f_{22} & 1 & -f_{31} & 0 & 0 \\
-f_{33} & 1 & -f_{41} & 0 & 0 \\
-f_{44} & 1 & -f_{51} & 0 & 0 \\
\end{bmatrix} \bigg|_{x=0} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}.
\]

There are \( n - 1 \) free parameters \( \beta_{22}, \ldots, \beta_{n-1,2} \) and \( \alpha_{n-1,2} \) in \( f_{22}, \ldots, f_{n-1,2} \) to zero \( n - 1 \) coefficients in the residual. We show later that this can be done by applying a direct formula for \( \alpha_{n-1,2} \) and solving a nonsingular triangular system of linear equations for \( \beta_{22}, \ldots, \beta_{n-1,2} \). Notice that the introduction of \( f_{22}, \ldots, f_{n-1,2} \) does not affect the part of the determinant of degree \( n \). Next we add \( \gamma_{n-2} \) to make the coefficient at \( y^{n-2} \) zero. In case \( n = 5 \) we get

\[
r^{(2)}(x, y) := p(x, y) - \det \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-f_{11} & 1 & -f_{21} & 0 & 0 \\
-f_{22} & 1 & -f_{31} & 0 & 0 \\
-f_{33} & 1 & -f_{41} & 0 & 0 \\
-f_{44} & 1 & -f_{51} & 0 & 0 \\
\end{bmatrix} \bigg|_{x=0} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \bigg|_{y=0}.
\]
By comparing (15) to (14) we see that we reduced the degree of the residual by one and preserved the form where the coefficient at $y^d$, where $d$ is the degree of the residual, is zero. In a similar way we fix the remaining parameters. We start the step with the residual $r^{(k-1)}$, which has degree $n-k+1$ and a zero coefficient at $y^{n-k+1}$. In the first part we set $f_{kk}, \ldots, f_{n-1,k}$ so that the new residual $s^{(k)}$ has degree $n-k$. Then we set $\gamma_{n-k}$ to annihilate the term $y^{n-k}$ in $s^{(k)}$ and obtain the residual $r^{(k)}$ for the next step. We continue in the same manner until the final residual $r^{(n-1)}$ has the form $\gamma_0 + \gamma_{10}x$. In our example, after two additional steps, we get

$$
\begin{bmatrix}
0 & \gamma_1 & \gamma_2 & \gamma_3 & p_{50}x \\
-f_{11} & 1 & & & \\
-f_{22} & -f_{21} & 1 & & \\
-f_{33} & -f_{32} & -f_{31} & 1 & \\
-f_{44} & -f_{43} & -f_{42} & -f_{41} & 1
\end{bmatrix}
= r^{(4)}(x, y) := p(x, y) - \det
$$

where $r^{(4)}(x, y) = \gamma_0 + \gamma_{10}x$. We obtain the final determinantal representation by putting $\gamma_0 + \gamma_{10}x$ at position $(1, 1)$ in (12).

Let us show in more details the step from the residual $r^{(k-1)}$ to the residual $r^{(k)}$. Let $q_j^{(k-1)}$ for $j = 0, \ldots, n-1$ be the polynomials (11) in step $k-1$, i.e., $q_0^{(k-1)}(x, y) = 1$ and

$$
q_j^{(k-1)}(x, y) = \sum_{\ell=1}^{\min(j, k-1)} f_{j\ell} q_{j-\ell}^{(k-1)}(x, y), \quad j = 1, \ldots, n-1. \tag{16}
$$

It is clear from (16) that $q_j^{(k-1)}$ and $q_j^{(k)}$ are equal for $j < k$ and differ only in terms of degree $j-k+1$ or less for $j \geq k$. We take

\begin{equation}
r^{(k-1)}(x, y) := p(x, y) - p_{n0}x q_{n-1}^{(k-1)}(x, y) - \gamma_{n-2} q_{n-2}^{(k-1)}(x, y) - \cdots - \gamma_{n-k+1} q_{n-k+1}^{(k-1)}(x, y) \\
= r_0^{(k-1)} + r_{10}^{(k-1)}x + r_{11}^{(k-1)}y + \cdots + r_{n-k+1,0}^{(k-1)}x^{n-k+1} + \cdots + r_{1, n-k}^{(k-1)}x y^{n-k}, \tag{17}
\end{equation}

select the terms of degree $n-k+1$ and form

$$
u_{n-k+1}(x, y) := r_{n-k+1,0}^{(k-1)}x^{n-k+1} + \cdots + r_{1, n-k}^{(k-1)}x y^{n-k}.
$$

Notice that $r_{0, n-k+1}^{(k-1)} = 0$, therefore we can write

$$
u_{n-k+1}(x, y) := p_{n0}x h_{n-k}(x, y), \tag{18}
$$

where $h_{n-k}$ is a polynomial of degree $n-k$. It follows from (17) and (18) that in order to zero all terms of degree $n-k+1$ in $r^{(k-1)}$ the part of degree $n-k$ in $q_n^{(k)} - q_{n-1}^{(k-1)}$ has to agree with $h_{n-k}$. By comparing $q_n^{(k)}$ to $q_{n-1}^{(k-1)}$ we see that we have to set the parameters $\beta_{kk}, \ldots, \beta_{n-1,k}$ and $\alpha_{n-1,k}$ so that

$$
h_{n-k}(x, y) = \sum_{l=k}^{n-1} \sum_{i=1}^{l-k} \sum_{j=1}^{l-1} f_{l-k} f_{l} f_{i+1} f_{j+1} f_{k+1,1} \cdots f_{n-1,1} + f_{11} f_{k+1,2} \cdots f_{n-1,1} + \cdots + f_{11} \cdots f_{n-k-1,1} f_{n-k}. \tag{19}
$$
This is equivalent to finding $\beta_{kk}, \ldots, \beta_{n-1,k}$, and $\alpha_{n-1,k}$ such that

$$h_{n-k}(t, 1) = \sum_{\ell=k}^{n-2} (t + \beta_{\ell k}) \prod_{i=1}^{\ell-k} (t - \xi_i) \prod_{j=\ell+1}^{n-1} (t - \xi_j) + (\alpha_{n-1,k} t + \beta_{n-1,k}) \prod_{i=1}^{n-k-1} (t - \xi_i). \quad (19)$$

By inspecting the coefficients at $t^{n-k}$ in (19) we get

$$\alpha_{n-1,k} = h_{n-k}(1, 0) - \sum_{\ell=k}^{n-1} (t - \xi_1) \prod_{j=\ell+1}^{n-1} (t - \xi_j). \quad (20)$$

For the remaining parameters $\beta_{kk}, \ldots, \beta_{n-1,k}$ it follows from (19) that

$$g_{n-k-1}(t) := h_{n-k}(t, 1) - t \sum_{\ell=k}^{n-2} \prod_{i=1}^{\ell-k} (t - \xi_i) \prod_{j=\ell+1}^{n-1} (t - \xi_j) - \alpha_{n-1,k} t \prod_{i=1}^{n-k-1} (t - \xi_i)$$

$$= \sum_{\ell=k}^{n-2} \beta_{\ell k} \prod_{i=1}^{\ell-k} (t - \xi_i) \prod_{j=\ell+1}^{n-1} (t - \xi_j) + \beta_{n-1,k} \prod_{i=1}^{n-k-1} (t - \xi_i), \quad (21)$$

where $g_{n-k-1}$ is a polynomial of degree $n-k-1$. We need $\beta_{kk}, \ldots, \beta_{n-1,k}$ such that $g_{n-k-1}$ is a linear combination of the polynomials

$$w_{\ell}(t) := \prod_{i=1}^{\ell-1} (t - \xi_i) \prod_{j=\ell+k}^{n-1} (t - \xi_j), \quad \ell = 1, \ldots, n-k. \quad (23)$$

The following lemma shows that such $\beta_{kk}, \ldots, \beta_{n-1,k}$ do exist because the above polynomials form a basis for the set of polynomials of degree less than or equal to $n-k-1$.

**Lemma 2.** Polynomials $w_1, \ldots, w_{n-k}$ defined in (23) form a basis for all polynomials of degree less than or equal to $n-k-1$ for $k = 1, \ldots, n-2$.

**Proof.** We have $n-k$ polynomials $w_1, \ldots, w_{n-k}$ of degree $n-k-1$. If we look at the matrix of values of these polynomials in points $\xi_1, \ldots, \xi_{n-k}$,

$$
\begin{bmatrix}
  w_1(\xi_1) & w_2(\xi_1) & \cdots & w_{n-k}(\xi_1) \\
  w_1(\xi_2) & w_2(\xi_2) & \cdots & w_{n-k}(\xi_2) \\
  \vdots & \vdots & \ddots & \vdots \\
  w_1(\xi_{n-k}) & w_2(\xi_{n-k}) & \cdots & w_{n-k}(\xi_{n-k})
\end{bmatrix},
$$

we see that the matrix is lower triangular with nonzero elements on the diagonal, because it follows from (23) that $w_{\ell}(\xi_j) = 0$ for $j < \ell$ and $w_\ell(\xi_j) \neq 0$ for $\ell = 1, \ldots, n-k$. The matrix is therefore nonsingular and the polynomials $w_1, \ldots, w_{n-k}$ satisfy the Haar condition. As a result $w_1, \ldots, w_{n-k}$ form a basis for all polynomials of degree less than or equal to $n-k-1$. □
It is well known that two polynomials of degree \( n - k - 1 \) or less are equal if they have the same values at \( n - k \) distinct points. We use this to write down a system of linear equations for \( \beta_{kk}, \ldots, \beta_{n-1,k} \) from (21) and (22) by choosing \( \xi_1, \ldots, \xi_{n-k} \) as \( n - k \) distinct points. This gives

\[
\begin{bmatrix}
w_1(\xi_1) & \beta_{kk} \\
w_1(\xi_2) & \beta_{k+1,k} \\
\vdots & \vdots \\
w_1(\xi_{n-k}) & \beta_{n-1,k}
\end{bmatrix} \begin{bmatrix}
w_2(\xi_2) \\
\vdots \\
w_2(\xi_{n-k}) \\
\vdots \\
w_{n-k}(\xi_{n-k})
\end{bmatrix} = \begin{bmatrix}g_{n-k-1}(\xi_1) \\
g_{n-k-1}(\xi_2) \\
\vdots \\
g_{n-k-1}(\xi_{n-k})
\end{bmatrix}
\]

(24)

and we know from Lemma 2 that the system is nonsingular. In addition to being lower triangular the system is also banded, and thus can be solved efficiently with the forward substitution.

Once we set the coefficients \( f_1, \ldots, f_{n-1} \) the updated polynomials \( q_0^{(k)}, \ldots, q_{n-1}^{(k)} \) give the intermediate residual

\[
s^{(k)}(x, y) := p(x, y) - p_{n0}x q_{n-1}^{(k)}(x, y) - \gamma_{n-2} q_{n-2}^{(k)}(x, y) - \cdots - \gamma_{n-k+1} q_{n-k+1}^{(k)}(x, y)
\]

\[
= s_0^{(k)} + s_1^{(k)} x + s_2^{(k)} y + \cdots + s_{n-k}^{(k)} x^{n-k} + s_{0,n-k}^{(k)} y^{n-k}
\]

We now insert \( \gamma_{n-k} q_{n-k}^{(k)} \) to zero the coefficient at \( y^{n-k} \). A simple computation shows that we have to choose

\[
\gamma_{n-k} = (-1)^{n-k} \frac{s_{0,n-k}^{(k)}}{\xi_1 \cdots \xi_{n-k}},
\]

(25)

which is well defined because \( \xi_1, \ldots, \xi_{n-2} \neq 0 \). The new residual is

\[
r^{(k)}(x, y) := p(x, y) - p_{n0}x q_{n-1}^{(k)}(x, y) - \gamma_{n-2} q_{n-2}^{(k)}(x, y) - \cdots - \gamma_{n-k} q_{n-k}^{(k)}(x, y)
\]

and compared to (17) the degree of the residual is reduced by one. We repeat the procedure and after \( n - 1 \) steps we get the final residual \( r^{(n-1)}(x, y) = r_0^{(n-1)} + r_1^{(n-1)} x \). The overall algorithm is presented in Algorithm 3.

**Algorithm 3.** Given a bivariate polynomial of degree \( n \)

\[
p(x, y) = p_0 + p_{10} x + p_{01} y + \cdots + p_{n0} x^n + p_{n-1,1} x^{n-1} y + \cdots + p_{1,n-1} x y^{n-1}
\]

such that \( p_{n0} \neq 0, p_{0n} = p_{0,n-1} = 0 \), and all roots of \( p_{n0} \xi^{n-1} + p_{n-1,1} \xi^{n-2} + \cdots + p_{1,n-1} = 0 \) are simple and nonzero, the output are \( n \times n \) matrices \( A, B, \) and \( C \) such that \( \det(A + xB + yC) = p(x, y) \).

1. Compute the roots \( \xi_1, \ldots, \xi_{n-1} \) of \( p_{n0} \xi^{n-1} + p_{n-1,1} \xi^{n-2} + \cdots + p_{1,n-1} = 0 \).
2. \( q_0(x, y) = 1 \)
3. for \( j = 1, \ldots, n - 1 \)
4. \( q_j(x, y) = (x - \xi_j) q_{j-1}(x, y) \)
5. \( r(x, y) = p(x, y) - p_{n0}xq_{n-1}(x, y) \)

6. for \( k = 2, \ldots, n - 1 \)

7. \( h(t) = (r_{n-k+1,0}t^{n-k} + r_{n-k,1}t^{n-k-1} + \cdots + r_{1,n-k})/p_{n0} \)

8. \( \alpha_{n-1,k} = r_{n-k+1,0}/p_{n0} - n + k + 1 \)

9. \( g(t) = h(t) - \sum_{l=k}^{n-2} (t - \xi_l) \prod_{j=l+1}^{n-1} (t - \xi_j) - \alpha_{n-1,k} t \prod_{l=1}^{n-k-1} (t - \xi_l) \)

10. for \( m = 1, \ldots, n - k \)

11. for \( \ell = 1, \ldots, m \)

12. \( w_{ml} = \prod_{i=1}^{\ell-1} (\xi_m - \xi_i) \prod_{j=\ell+1}^{n-1} (\xi_m - \xi_j) \)

13. Solve

\[
\begin{bmatrix}
    w_{11} & w_{21} & w_{22} \\
    \vdots & \ddots & \ddots \\
    w_{n-k,1} & w_{n-k,2} & \cdots & w_{n-k,n-k}
\end{bmatrix}
\begin{bmatrix}
    \beta_{kk} \\
    \beta_{k+1,k} \\
    \vdots \\
    \beta_{n-1,k}
\end{bmatrix}
= 
\begin{bmatrix}
    g(\xi_1) \\
    g(\xi_2) \\
    \vdots \\
    g(\xi_{n-k})
\end{bmatrix}
\]

14. \( q_k(x, y) = q_k(x, y) + (x + \beta_{kk} y)q_0(x, y) \)

15. for \( j = k + 1, \ldots, n - 2 \)

16. \( q_j(x, y) = (x - \xi_j y)q_{j-1}(x, y) + \sum_{l=2}^{k} (x + \beta_{jl} y)q_{j-l}(x, y) \)

17. \( q_{n-1}(x, y) = (x - \xi_{n-1} y)q_{n-2}(x, y) + \sum_{l=2}^{k} (x + \beta_{n,l} y)q_{n-l}(x, y) \)

18. \( s(x, y) = p(x, y) - p_{n0}xq_{n-1}(x, y) - \sum_{l=2}^{k} \gamma_{n-l} q_{n-l}(x, y) \)

19. \( \gamma_{n-k} = (-1)^{n-k} q_{0,n-k}/(\xi_1 \cdots \xi_{n-k}) \)

20. \( r(x, y) = s(x, y) - \gamma_{n-k} q_{n-k}(x, y) \)

Return

\[
A = \begin{bmatrix}
    r_{00} & \gamma_1 & \cdots & \gamma_{n-2} & 0 \\
    1 & \ddots & \ddots & \ddots & \ddots \\
    & 1 & \ddots & \ddots & \ddots \\
    & & 1 & \ddots & \ddots \\
    & & & 1 & \ddots \\
    & & & & 1
\end{bmatrix}, 
B = \begin{bmatrix}
    r_{10} & 0 & \cdots & 0 & p_{n0} \\
    -1 & 0 & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    -1 & \cdots & -1 & 0 & \ddots \\
    -\alpha_{n-1,n-1} & \cdots & -\alpha_{n-1,2} & -1 & 0
\end{bmatrix}, 
C = \begin{bmatrix}
    0 \\
    -\beta_{11} & 0 \\
    -\beta_{22} & -\beta_{21} & 0 \\
    \vdots & \ddots & \ddots & \ddots \\
    -\beta_{n-1,n-1} & \cdots & -\beta_{n-1,n-2} & -\beta_{n-1,1} & 0
\end{bmatrix}.
\]

**Remark 4.** In Algorithm 3 we require that all roots of (10) are simple and nonzero. The nonzero condition is related to expression (25) for \( \gamma_{n-k} \) that involves only the roots \( \xi_1, \ldots, \xi_{n-2} \). Thus it is sufficient that the roots of (10) are simple. If one of the roots is zero we order them so that \( \xi_{n-1} = 0 \) and then everything works fine.

To further clarify the algorithm, we give Example 5 with most of the details for the construction of a determinantal representation of a bivariate polynomial of degree 5.
Example 5. We would like to linearize the bivariate polynomial

\[ p(x, y) := 1 - x - 3y + 3x^2 - 7xy - 6y^2 + 10x^3 + 9x^2y - 14xy^2 - 4y^3 + 8x^4 + 7x^3y - 8x^2y^2 - 4xy^3 + 2x^5 - 10x^3y^2 + 8xy^4 \]

that satisfies the conditions \( p_{50} \neq 0 \) and \( p_{05} = p_{04} = 0 \).

In the initial part before the main loop we compute the roots \( \xi_1, \ldots, \xi_4 \) of the polynomial

\[ p_{50}\xi^4 + p_{41}\xi^3 + p_{32}\xi^2 + p_{23}\xi + p_{14} = 2\xi^4 - 10\xi^2 + 8. \]

The roots, which are all simple and nonzero, are

\[ \xi_1 = -2, \quad \xi_2 = -1, \quad \xi_3 = 1, \quad \xi_4 = 2. \]

This gives the coefficients \( f_{11}, \ldots, f_{41} \) in the first subdiagonal of (12):

\[ f_{11} = x + 2y, \quad f_{21} = x + y, \quad f_{31} = x - y, \quad f_{41} = x - 2y. \]

The coefficient at \( q_4 \) in the first row of (12) is \( 2x \). The corresponding residual is

\[ r^{(1)}(x, y) = 1 - x - 3y + 3x^2 - 7xy - 6y^2 + 10x^3 + 9x^2y - 14xy^2 - 4y^3 + 8x^4 + 7x^3y - 8x^2y^2 - 4xy^3. \]

Now we enter the main loop.

- \((k = 2)\) From the terms of \( r^{(1)} \) of degree 4 we define
  \[ s_4(x, y) = 8x^4 + 7x^3y - 8x^2y^2 - 4xy^3 \]
  and divide \( s_4(t, 1) \) by \( 2t \) to obtain \( h_3(t) = 4t^3 + \frac{7}{2}t^2 - 4t - 4 \). Now we have to find the coefficients \( \beta_{22}, \beta_{32}, \beta_{42}, \) and \( \alpha_{42} \) such that
  \[ h_3(t) = (t + \beta_{22})(t - \xi_3)(t - \xi_4) + (t + \beta_{32})(t - \xi_1)(t - \xi_4) + (t + \beta_{42})(t - \xi_1)(t - \xi_2). \]

We get \( \alpha_{42} \) from (20) as \( \alpha_{42} = 4 - 2 = 2 \). This gives

\[ g_2(t) = h_3(t) - t(t - \xi_3)(t - \xi_4) - t(t - \xi_1)(t - \xi_4) - \alpha_{42}t(t - \xi_1)(t - \xi_2) = \frac{1}{2}t^2 - 6t - 2. \]

For \( \beta_{22}, \beta_{32}, \) and \( \beta_{42} \) we set the linear system

\[
\begin{bmatrix}
(\xi_1 - \xi_3)(\xi_1 - \xi_4) & 0 & 0 \\
(\xi_2 - \xi_3)(\xi_2 - \xi_4) & (\xi_2 - \xi_1)(\xi_2 - \xi_4) & 0 \\
0 & (\xi_3 - \xi_1)(\xi_3 - \xi_4) & (\xi_3 - \xi_1)(\xi_3 - \xi_2)
\end{bmatrix}
\begin{bmatrix}
\beta_{22} \\
\beta_{32} \\
\beta_{42}
\end{bmatrix}
= \begin{bmatrix}
g_2(\xi_1) \\
g_2(\xi_2) \\
g_2(\xi_3)
\end{bmatrix}.
\]

When we insert the values we get

\[
\begin{bmatrix}
12 & 0 & 0 \\
6 & -3 & 0 \\
0 & -3 & 6
\end{bmatrix}
\begin{bmatrix}
\beta_{22} \\
\beta_{32} \\
\beta_{42}
\end{bmatrix}
= \begin{bmatrix}
12 \\
\frac{7}{2} \\
-\frac{15}{2}
\end{bmatrix}.
\]
and the solution is \( \beta_{22} = 1, \beta_{32} = \frac{1}{2}, \) and \( \beta_{42} = -1, \) which gives

\[
s^{(2)}(x, y) = 1 - x - 3y + 3x^2 - 7xy - 6y^2 + 6x^3 + 7x^2y - 12xy^2 + 4y^3.
\]

We compute \( \gamma_3 \) from (25) to annihilate the term \( y^3 \) in \( s^{(2)} \). We get \( \gamma_3 = -4/(\xi_1 \xi_2 \xi_3) = 2. \)

The residual at the end of step \( k = 2 \) is

\[
r^{(2)}(x, y) = 1 - x - 3y - x^2 - 12xy - 6y^2 + 4x^3 + 3x^2y - 10xy^2.
\]

- \( (k = 3) \) From \( r^{(2)} \) we get \( h_2(t) = 2t^2 + \frac{3}{2}t - 5. \) Now we need \( \beta_{33}, \beta_{43}, \) and \( \alpha_{43} \) such that

\[
h_2(t) = (t + \beta_{33})(t - \xi_4) + (\alpha_{43} t + \beta_{43})(t - \xi_1).
\]

We get \( \alpha_{43} \) from (20) as \( \alpha_{43} = 2 - 1 = 1. \) This gives

\[
g_1(t) = h_2(t) - t(t - \xi_4) - \alpha_{43} t(t - \xi_1) = \frac{3}{2}t - 5.
\]

For \( \beta_{33} \) and \( \beta_{43} \) we set the linear system

\[
\begin{bmatrix}
\xi_1 - \xi_4 & 0 \\
\xi_2 - \xi_4 & \xi_2 - \xi_1
\end{bmatrix}
\begin{bmatrix}
\beta_{33} \\
\beta_{43}
\end{bmatrix}
= 
\begin{bmatrix}
g_1(\xi_1) \\
g_1(\xi_2)
\end{bmatrix}.
\]

When we insert the values we get

\[
\begin{bmatrix}
-4 & 0 \\
-3 & 1
\end{bmatrix}
\begin{bmatrix}
\beta_{33} \\
\beta_{43}
\end{bmatrix}
= 
\begin{bmatrix}
-8 \\
-13/2
\end{bmatrix}
\]

and the solution is \( \beta_{33} = 2 \) and \( \beta_{43} = -\frac{1}{2}. \) This gives

\[
s^{(3)}(x, y) = 1 - 3x - 7y - x^2 - 12xy - 6y^2.
\]

We compute \( \gamma_2 = -6/(\xi_1 \xi_2) = -3 \) from (25) to annihilate the term \( y^2 \) in \( s^{(3)} \). The residual after step \( k = 3 \) is

\[
r^{(3)}(x, y) = 1 - 4y + 2x^2 - 3xy.
\]

- \( (k = 4) \) From \( r^{(3)} \) we get \( h_1(t) = t - \frac{3}{2}. \) Now we have to find coefficients \( \beta_{44} \) and \( \alpha_{44} \) such that \( h_1(t) = \alpha_{44} t + \beta_{44}. \) Clearly, the answer is \( \alpha_{44} = 1 \) and \( \beta_{44} = -\frac{3}{2}. \) The new residual is

\[
s^{(4)}(x, y) = 1 - 4y.
\]

From (25) we compute \( \gamma_1 = 4/\xi_1 = -2. \) The final residual after the main loop is

\[
r^{(4)}(x, y) = 1 + 2x.
\]
The final $5 \times 5$ determinantal representation is
\[
A + xB + yC = \begin{bmatrix}
1 + 2x & -2 & 3 & 2 & 2x \\
-2 & -y & 1 & & \\
-x & -y & 1 & & \\
-x & -y & 1 & & \\
-x & -y & 1 & & \\
\end{bmatrix}.
\]

Let us remark that the polynomial in Example 5 was constructed in such way that the roots $\xi_1, \ldots, \xi_{n-1}$ are all real, which results in real matrices in the determinantal representation. In a generic case, even if the polynomial is real, the roots can be complex and the representation has complex matrices.

5. Determinantal representation for a non square-free polynomial

If a bivariate polynomial $p$ is not square-free, then it cannot be transformed into a polynomial that satisfies the conditions of Algorithm 3. In such case we have several options.

First, for a polynomial of degree $n \leq 5$ we can apply the algorithm in [4] that returns a determinantal representation of order $n$ for a non square-free polynomial as well. For a polynomial of degree $n > 5$ we can use one of the available symbolic or numerical tools, e.g., NAClab [22], and factorize $p$ into a product $p(x, y) = p_1(x, y)p_2(x, y) \ldots p_k(x, y)$, where $p_i$ is a square-free polynomial of degree $d_i$ for $i = 1, \ldots, k$ and $d_1 + \cdots + d_k = n$. Now we apply Algorithm 3 to obtain $d_i \times d_i$ matrices $A_i, B_i,$ and $C_i$ such that $p_i(x, y) = \det(A_i + xB_i + yC_i)$ for $i = 1, \ldots, k$. We arrange them in block diagonal matrices $A = \text{diag}(A_1, \ldots, A_k)$, $B = \text{diag}(B_1, \ldots, B_k)$, and $C = \text{diag}(C_1, \ldots, C_k)$ and get $n \times n$ matrices such that $\det(A + xB + yC) = p(x, y)$. So, combined with a square-free factorization, one can find a determinantal representation of order $n$ for each bivariate polynomial of degree $n$.

As the square-free factorization is more complex than Algorithm 3, it takes most of the computational time in the above procedure. In our case, where we use representations to compute roots of a system of two bivariate polynomials, it is more efficient for polynomials of small degree to use larger representations that can be constructed faster. For instance, for each polynomial of degree $n$ there exists a uniform determinantal representation of order $2n - 1$ [3]. As an example, a uniform representation of order 7 for the polynomial (11) of degree $n = 4$ is
\[
A + xB + yC = \begin{bmatrix}
-x & 1 & -x & 1 & -x & 1 & -x \\
1 & -x & 1 & -x & 1 & -x & 1 \\
p_{00} & p_{10} & p_{20} & p_{30} + p_{40}x & -y & 1 & -y \\
p_{01} & p_{11} & p_{21} + p_{31}x & 1 & -y & 1 & -y \\
p_{02} + p_{03}y & p_{12} + c_{22}x & p_{22} & 1 & -y & 1 & -y \\
p_{13}x + p_{04}y & p_{23} & p_{33} & 1 & -y & \end{bmatrix}.
\]
The construction of a uniform representation of order $2n - 1$ is immediate, but we need more time in the second phase to numerically solve a larger associated two-parameter eigenvalue problem, which is singular in addition. Still, for polynomials of small degree this might altogether be faster than using a representation of order $n$ that requires a square-free factorization.

6. Numerical examples

A new numerical approach for computing roots of systems of bivariate polynomials by constructing determinantal representations of the polynomials and solving the obtained two-parameter eigenvalue problem was proposed in [16]. Using the representation of order $\frac{1}{2}n^2 + O(n)$ for a bivariate polynomial of degree $n$ (we refer to this method as Lin2) it was shown that the approach is competitive for polynomials of degree 9 or less. The method was compared to several numerical methods for polynomial systems: NSolve in Mathematica 9 [21], BertiniLab 1.4 [11] running Bertini 1.5 [2], NAClab 3.0 [22], and PHCLab 1.04 [6] running PHCpack 2.3.84, which was the fastest of these methods.

Using the uniform representation of a smaller order $2n - 1$ for a bivariate polynomial of degree $n$ (we refer to this method as MinUnif), computational times were significantly improved in [3] and the largest degree such that the approach is competitive with the existing numerical methods for polynomial systems was raised to 15.

In a numerical experiment we compare the new representation of order $n$ from Section 4 (we refer to it as MinRep) to Lin2, MinUnif, and PHCLab on a similar set of random polynomials as in [3] and [16]. We take systems of full bivariate polynomials of the same degree, whose coefficients are random real numbers uniformly distributed on $[0, 1]$ or random complex numbers, such that real and imaginary parts are both uniformly distributed on $[0, 1]$.

In order to overcome some difficulties that we noticed while testing MinRep, some heuristics were applied in the implementation. In practice, even if the polynomial satisfies the initial conditions of Algorithm 3 i.e., the roots of (10) are nonzero and simple, the obtained matrices $A$, $B$, and $C$ can be such that the error between $p(x, y)$ and $\text{det}(A + xB + yC)$, caused by the numerical computation, is too large. This happens for instance when the roots of (10) are ill-conditioned. Also, when the roots of (10) are close to each other, then linear systems (24) can be ill-conditioned. A usual remedy for this is to apply a random linear transformation (8).

To get out of such troubles, we compute the determinantal representation and check its quality by computing

$$\nu := \max_{i=1, \ldots, k} \frac{|p(x_i, y_i) - \text{det}(A + x_iB + y_iC)|}{|p(x_i, y_i)|} + \epsilon$$

on a set of $k$ random points $(x_i, y_i), i = 1, \ldots, k$. If $\nu \cdot \max(\|A\|_\infty, \|B\|_\infty, \|C\|_\infty) > \delta$ for a given $\delta$ (in our experiments we use $\epsilon = 10^{-4}, k = 200$, and $\delta = 10^{-8}$), then we first compute a new representation of the polynomial, where we exchange the roles of $x$ and $y$, and, if this does not help, then we apply a random change of variables (5). We noticed that this heuristics does not improve the situation for polynomials of degree 11 or more. It seems that for a generic bivariate polynomial the largest safe degree, when we can expect that the method works, is 10. This does not mean that the method cannot fail for polynomials of smaller degree. Similar
to other methods based on determinantal representations, this can happen for some systems that appear to be to difficult for this approach and require computation in higher precision.

Table 1: Average computational times in milliseconds for MinRep, MinUnif, and PHCLab for random bivariate polynomial systems of degree 3 to 10. For MinUnif separate results are included for real ($\mathbb{R}$) and complex polynomials ($\mathbb{C}$).

| d  | MinRep | MinUnif ($\mathbb{R}$) | MinUnif ($\mathbb{C}$) | Lin2 | PHCLab |
|----|--------|------------------------|------------------------|------|--------|
| 3  | 6      | 6                      | 6                      | 7    | 210    |
| 4  | 8      | 9                      | 11                     | 11   | 247    |
| 5  | 11     | 15                     | 18                     | 18   | 289    |
| 6  | 15     | 25                     | 32                     | 32   | 344    |
| 7  | 20     | 40                     | 55                     | 70   | 409    |
| 8  | 29     | 70                     | 98                     | 191  | 499    |
| 9  | 46     | 112                    | 172                    | 439  | 607    |
| 10 | 64     | 184                    | 301                    | 1111 | 739    |

The results in Table 1 show that the new representation has a big potential as it is much faster than the previous ones. The results were obtained on a 64-bit Windows version of MATLAB R2015b running on an Intel Core i5-6200U 2.30 GHz processor with 8 GB of RAM. For each $n$ we apply the methods to the same set of 50 real and 50 complex random polynomial systems of degree $n$ and measure the average time. The accuracy of all methods on this random set of polynomials is comparable. For MinUnif, where determinantal representations have real matrices for real polynomials, we report separate results for polynomials with real and complex coefficients. Although MinRep is still the fastest method for $11 \leq n \leq 15$, the computed roots are in most cases useless and, based on the results from [3], we rather suggest that MinUnif is applied to polynomials of such degree. As in MinUnif no computation is involved in the construction of determinantal representations, this phase is not affected by numerical errors. On the other hand, the matrices in MinUnif are approximately double size compared to the matrices in MinRep and to extract the final solution one has to compute the finite eigenvalues of a pair of singular pencils, which is more delicate than solving a regular pencil with the QZ algorithm in MinRep. A future research might give more insight into the cases when MinRep does not perform so well and further improve the method.

We performed a limited number of tests comparing MinUnif to the square-free factorization approach from Section 5. We observed that the square-free factorization approach is faster for polynomials of degree $n \geq 8$. Although MinUnif is faster for polynomials of degree $n = 6$ and $n = 7$, it is also less accurate because of the multiple roots. We therefore recommend to use the square-free factorization for all non square-free polynomials of degree $n \geq 5$.

7. Conclusions

We presented a numerical construction for the determinantal representation of a square-free bivariate polynomial of degree $n$ with matrices of order $n$. The computation requires only routines for roots of univariate polynomials of degree $n$ and solutions of linear systems.
of order less than \( n \), both should be available in any numerical package. When combined with a square-free factorization, we believe that this is the first construction that reaches the theoretical lower order from Dixon's theorem for all bivariate polynomials. We do not get symmetric matrices, but this is not really important in our application.

The new representation can be used to numerically compute the roots of a system of bivariate polynomials. Compared to the previous results, the new representation speeds up the computation considerably.

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**References**

[1] F.V. Atkinson, Multiparameter Eigenvalue Problems, Academic Press, New York, 1972.

[2] D.J. Bates, J.H. Husenstei n, A.J. Sommese, C.W. Wampler, Bertini: Software for Numerical Algebraic Geometry, available at bertini.nd.edu; doi: dx.doi.org/10.7274/R0H41PB5.

[3] A. Boralevi, J. van Doornmalen, J. Draisma, M.E. Hochstenbach, B. Plestenjak, Uniform determinantal representations, arXiv:1607.04873 (2016), to appear in SIAM Journal on Applied Algebra and Geometry.

[4] A. Buckley, B. Plestenjak, Simple determinantal representations of up to quintic bivariate polynomials, arXiv.1609.00498 (2016).

[5] A. Dixon. Note on the reduction of a ternary quartic to a symmetrical determinant. Proc. Camb. Phil. Soc. 11 (1902) 350–351.

[6] Y. Guan, J. Verschelde, PHClab: A MATLAB/Octave interface to PHCpack. In: M. Stillman, J. Verschelde, N. Takayama (eds), Software for Algebraic Geometry, volume 148 of The IMA Volumes in Mathematics and its Applications, Springer, New York, (2008) 15–32.

[7] V.B. Khazanov, To solving spectral problems for multiparameter polynomial matrices, J. Math. Sci. 141 (2007) 1690–1700.

[8] Y. Guan, J. Verschelde, PHClab: A MATLAB/Octave interface to PHCpack. In: M. Stillman, J. Verschelde, N. Takayama (eds), Software for Algebraic Geometry, volume 148 of The IMA Volumes in Mathematics and its Applications, Springer, New York, (2008) 15–32.

[9] A. Muhić, B. Plestenjak, On the quadratic two-parameter eigenvalue problem and its linearization, Linear Algebra Appl. 432 (2010) 2529–2542.

[10] T. Netzer, A. Thom, Polynomials with and without determinantal representations, Linear Algebra Appl. 437 (2012) 1579–1595.

[11] A. Newell, BertiniLab: toolbox for solving polynomial systems, MATLAB Central File Exchange, www.mathworks.com/matlabcentral/fileexchange/48536-bertinilab

[12] D. Plaumann, R. Sinn, D.E. Speyer, C. Vinzant, Computing Hermitian determinantal representations of hyperbolic curves, Internat. J. Algebra Comput. 25 (2015) 1327–1336.

[13] D. Plaumann, B. Sturmfels, C. Vinzant, Computing Linear Matrix Representations of Helton-Vinnikov Curves, Operator Theory: Advances and Applications 222 (2012) 259–277.

[14] B. Plestenjak, BiRoots, MATLAB Central File Exchange, http://www.mathworks.com/matlabcentral/fileexchange/54159-biroots

[15] B. Plestenjak, MultiParEig: toolbox for multiparameter eigenvalue problems, MATLAB Central File Exchange, www.mathworks.com/matlabcentral/fileexchange/47844-multipareig
[16] B. Plestenjak, M.E. Hochstenbach, Roots of bivariate polynomial systems via determinantal representations, SIAM J. Sci. Comput. 38 (2016) A765–A788.
[17] R. Quarez, Symmetric determinantal representation of polynomials, Linear Algebra Appl. 436 (2012) 3642–3660.
[18] K.E. Smith, L. Kahanpää, P. Kekäläinen, W. Traves, An Invitation to Algebraic Geometry, Springer, New York, 2000.
[19] J. Verschelde, Algorithm 795: PHCpack: a general-purpose solver for polynomial systems by homotopy continuation, ACM Trans. Math. Softw., 25 (1999) 251–276.
[20] V. Vinnikov, LMI representations of convex semialgebraic sets and determinantal representations of algebraic hypersurfaces: past, present, and future, in H. Dym, M.C. de Oliveira, M. Putinar (eds.), Mathematical Methods in Systems, Optimization, and Control: Festschrift in Honor of J. William Helton. Operator Theory: Advances and Applications 222, Birkhäuser (2012) 325–349.
[21] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, Illinois, 2012.
[22] Z. Zeng, T.-Y. Li, NAClab: a Matlab toolbox for numerical algebraic computation, ACM Commun. Comput. Algebra 47 (2013) 170–173.