The chromatic number of the convex segment disjointness graph

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Abstract

Let $P$ be a set of $n$ points in general and convex position in the plane. Let $D_n$ be the graph whose vertex set is the set of all line segments with endpoints in $P$, where disjoint segments are adjacent. The chromatic number of this graph was first studied by Araujo et al. [CGTA, 2005]. The previous best bounds are $\frac{2n}{3} \leq \chi(D_n) < n - \sqrt{\frac{n}{2}}$ (ignoring lower order terms). In this paper we improve the lower bound to $\chi(D_n) \geq n - \sqrt{2n}$, to conclude a near-tight bound on $\chi(D_n)$.

1 Introduction

Throughout this paper, $P$ is a set of $n > 3$ points in general and convex position in the plane. The convex segment disjointness graph, denoted by $D_n$, is the graph whose vertex set is the set of all line segments with endpoints in $P$, where two vertices are adjacent if the corresponding segments are disjoint. Obviously $D_n$ does not depend on the choice of $P$. This graph and other related graphs, were introduced by Araujo, Dumitrescu, Hurtado, Noy and Urrutia [1], who proved the following bounds on the chromatic number of $D_n$:

$$2 \left\lfloor \frac{1}{3} (n + 1) \right\rfloor - 1 \leq \chi(D_n) < n - \frac{1}{2} \left\lfloor \log n \right\rfloor.$$ 

Both bounds were improved by Dujmović and Wood [5] to

$$\frac{3}{4} (n - 2) \leq \chi(D_n) < n - \sqrt{\frac{3}{2} n} - \frac{1}{2} (\ln n) + 4.$$
In this paper we improve the lower bound to conclude near-tight bounds on $\chi(D_n)$.

**Theorem 1.**

$$n - \sqrt{2n + \frac{1}{2}} + \frac{1}{2} \leq \chi(D_n) < n - \sqrt{2n - \frac{1}{2} \ln n} + 4.$$

The proof of Theorem 1 is based on the observation that each colour class in a colouring of $D_n$ is a convex thrackle. We then prove that two maximal convex thrackles must share an edge in common. From this we prove a tight upper bound on the number of edges in the union of $k$ maximal convex thrackles. Theorem 1 quickly follows.

## 2 Convex thrackles

A **convex thrackle** on $P$ is a geometric graph with vertex set $P$ such that every pair of edges intersect; that is, they have a common endpoint or they cross. Observe that a geometric graph $H$ on $P$ is a convex thrackle if and only if $E(H)$ forms an independent set in $D_n$. A convex thrackle is **maximal** if it is edge-maximal. As illustrated in Figure 1(a), it is well known and easily proved that every maximal convex thrackle $T$ consists of an odd cycle $C(T)$ together with some degree 1 vertices adjacent to vertices of $C(T)$; see [2, 3, 4, 5, 6, 7, 8, 9]. In particular, $T$ has $n$ edges. For each vertex $v$ in $C(T)$, let $W_T(v)$ be the convex wedge with apex $v$, such that the boundary rays of $W_T(v)$ contain the neighbours of $v$ in $C(T)$. Every degree-1 vertex $u$ of $T$ lies in a unique wedge and the apex of this wedge is the only neighbour of $u$ in $T$.

![Figure 1: (a) maximal convex thrackle, (b) the intervals pairs $(I_u, J_u)$](image)


3 Convex thrackles and free $\mathbb{Z}_2$-actions of $S^1$

A $\mathbb{Z}_2$-action on the unit circle $S^1$ is a homeomorphism $f : S^1 \to S^1$ such that $f(f(x)) = x$ for all $x \in S^1$. We say that $f$ is free if $f(x) \neq x$ for all $x \in S^1$.

**Lemma 1.** If $f$ and $g$ are free $\mathbb{Z}_2$-actions of $S^1$, then $f(x) = g(x)$ for some point $x \in S^1$.

**Proof.** For points $x, y \in S^1$, let $\overrightarrow{xy}$ be the clockwise arc from $x$ to $y$ in $S^1$. Let $x_0 \in S^1$. If $f(x_0) = g(x_0)$ then we are done. Now assume that $f(x_0) \neq g(x_0)$. Without loss of generality, $x_0, g(x_0), f(x_0)$ appear in this clockwise order around $S^1$. Parametrise $x_0g(x_0)$ with a continuous injective function $p : [0, 1] \to x_0g(x_0)$, such that $p(0) = x_0$ and $p(1) = g(x_0)$. Assume that $g(p(t)) \neq f(p(t))$ for all $t \in [0, 1]$, otherwise we are done. Since $g$ is free, $p(t) \neq g(p(t))$ for all $t \in [0, 1]$. Thus $g(p([0, 1])) = g(p(0))g(p(1)) = g(x_0)x_0$. Also $f(p([0, 1])) = f(x_0)f(p(1))$, as otherwise $g(p(t)) = f(p(t))$ for some $t \in [0, 1]$. This implies that $p(t), g(p(t)), f(p(t))$ appear in this clockwise order around $S^1$. In particular, with $t = 1$, we have $f(p(1)) \in x_0g(x_0)$. Hence $x_0 \in f(x_0)f(p(1))$. Since $f$ is a $\mathbb{Z}_2$-action, $f(x_0) = p(t)$. This is a contradiction since $p(t) \in x_0g(x_0)$ but $f(x_0) \notin x_0g(x_0)$. \qed

Assume that $P$ lies on $S^1$. Let $T$ be a maximal convex thrackle on $P$. As illustrated in Figure 2(b), for each vertex $u$ in $C(T)$, let $(I_u, J_u)$ be a pair of closed intervals of $S^1$ defined as follows. Interval $I_u$ contains $u$ and bounded by the points of $S^1$ that are $1/3$ of the way towards the first points of $P$ in the clockwise and anticlockwise direction from $u$. Let $v$ and $w$ be the neighbours of $u$ in $C(T)$, so that $v$ is before $w$ in the clockwise direction from $u$. Let $p$ be the endpoint of $I_v$ in the clockwise direction from $v$. Let $q$ be the endpoint of $I_w$ in the anticlockwise direction from $w$. Then $J_u$ is the interval bounded by $p$ and $q$ and not containing $u$. Define $f_T : S^1 \to S^1$ as follows. For each $v \in C(T)$, map $w$ to the clockwise endpoint of $I_v$ to the anticlockwise endpoint of $J_v$, map the clockwise endpoint of $I_u$ to the clockwise endpoint of $J_v$, and extend $f_T$ linearly for the interior points of $I_v$ and $J_u$, such that $f_T(I_v) = J_v$ and $f_T(J_u) = I_v$. Since the intervals $I_v$ and $J_u$ are disjoint, $f_T$ is a free $\mathbb{Z}_2$-action of $S^1$.

**Lemma 2.** Let $T_1$ and $T_2$ be maximal convex thrackles on $P$, such that $C(T_1) \cap C(T_2) = \emptyset$. Then there is an edge in $T_1 \cap T_2$, with one endpoint in $C(T_1)$ and one endpoint in $C(T_2)$.

**Topological proof.** By Lemma 1, there exists $x \in S^1$ such that $f_{T_1}(x) = y = f_{T_2}(x)$. Let $u \in C(T_1)$ and $v \in C(T_2)$ so that $x \in I_u \cup J_u$ and $x \in I_v \cup J_v$, where $(I_u, J_u)$ and $(I_v, J_v)$ are defined with respect to $T_1$ and $T_2$ respectively. Since $C(T_1) \cap C(T_2) = \emptyset$, we have $u \neq v$ and $I_u \cap I_v = \emptyset$. Thus $x \notin I_u \cap I_v$. If $x \in J_u \cup J_v$ then $y \in I_u \cup I_v$, implying $u = v$. Thus $x \notin I_u \cap J_v$. Hence $x \in (I_u \cap J_v) \cup (J_u \cap I_v)$. Without loss of generality, $x \in I_u \cap J_v$. Thus $y \in J_u \cap I_v$. If $I_u \cap J_v = \{x\}$ then $x$ is an endpoint of both $I_u$ and $J_v$, implying $u \in C(T_2)$,
which is a contradiction. Thus $I_u \cap J_v$ contains points other than $x$. It follows that $I_u \subset J_v$ and $I_v \subset J_u$. Therefore the edge $uv$ is in both $T_1$ and $T_2$. Moreover one endpoint of $uv$ is in $C(T_1)$ and one endpoint is in $C(T_2)$. \qed \\

**Combinatorial Proof.** Let $H$ be the directed multigraph with vertex set $C(T_1) \cup C(T_2)$, where there is a blue arc $uv$ in $H$ if $u$ is in $W_{T_1}(v)$ and there is a red arc $uv$ in $H$ if $u$ is in $W_{T_2}(v)$. Since $C(T_1) \cap C(T_2) = \emptyset$, every vertex of $H$ has outdegree 1. Therefore $|E(H)| = |V(H)|$ and there is a cycle $\Gamma$ in the undirected multigraph underlying $H$. In fact, since every vertex has outdegree 1, $\Gamma$ is a directed cycle. By construction, vertices in $H$ are not incident to an incoming and an outgoing edge of the same color. Thus $\Gamma$ alternates between blue and red arcs. The red edges of $\Gamma$ form a matching as well as the blue edges, both of which are thrackles. However, there is only one matching thrackle on a set of points in convex position. Therefore $\Gamma$ is a 2-cycle and the result follows. \qed

4 Main Results

**Theorem 2.** For every set $P$ of $n$ points in convex and general position, the union of $k$ maximal convex thrackles on $P$ has at most $kn - \binom{k}{2}$ edges.

**Proof.** For a set $T$ of $k$ maximal convex thrackles on $P$, define

$$r(T) := |\{(v,T_i, T_j) : v \in C(T_i) \cap C(T_j), T_i, T_j \in T, T_i \neq T_j\}|.$$ 

The proof proceeds by induction on $r(T)$.

Suppose that $r(T) = 0$. Thus $C(T_i) \cap C(T_j) = \emptyset$ for all distinct $T_i, T_j \in T$. By Lemma 2, $T_i$ and $T_j$ have an edge in common, with one endpoint in $C(T_i)$ and one endpoint in $C(T_j)$. Hence distinct pairs of thrackles have distinct edges in common. Since every maximal convex thrackle has $n$ edges and we overcount at least one edge for every pair, the total number of edges is at most $kn - \binom{k}{2}$.

Now assume that $r(T) > 0$. Thus there is a vertex $v$ and a pair of thrackles $T_i$ and $T_j$, such that $v \in C(T_i) \cap C(T_j)$. As illustrated in Figure 2, replace $v$ by two consecutive vertices $v'$ and $v''$ on $P$, where $v'$ replaces $v$ in every thrackle except $T_j$, and $v''$ replaces $v$ in $T_j$. Add one edge to each thrackle so that it is maximal. Let $T'$ be the resulting set of thrackles. Observe that $r(T') = r(T) - 1$, and the number of edges in $T'$ equals the number of edges in $T$ plus $k$. By induction, $T'$ has at most $k(n+1) - \binom{k}{2}$ edges, implying $T$ has at most $kn - \binom{k}{2}$ edges. \qed \\

We now show that Theorem 2 is best possible for all $n \geq 2k$. Let $S$ be a set of $k$ vertices in $P$ with no two consecutive vertices in $S$. If $v \in S$ and $x, y$ are consecutive in this order in $P$, then $T_v := \{vw : w \in P \setminus \{v\}\} \cup \{xy\}$ is a maximal convex thrackle, and $\{T_v : v \in S\}$ has exactly $kn - \binom{k}{2}$ edges in total.

**Proof of Theorem 1.** If $\chi(D_n) = k$ then, there are $k$ convex thrackles whose union is the complete geometric graph on $P$. Possibly add edges to obtain $k$ maximal convex thrackles with $\binom{k}{2}$ edges in total. By Theorem 2, $\binom{k}{2} \leq kn - \binom{k}{2}$. The quadratic formula implies the result. \qed
Figure 2: Construction in the proof of Theorem 2.

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