A Quasi-Monte Carlo solution for the mutual inductance of misaligned circular coils

L Babić¹, M Dadić²,
¹ Končar – Power Plant and Electric Traction Engineering, Fallerovo šetalište 22, Zagreb, Croatia
² University of Zagreb, Faculty of Electrical Engineering and Computing, Department of Fundamentals of Electrical Engineering and Measurements, Unska 3, Zagreb, Croatia

¹ lucija.babic@koncar-ket.hr
² martin.dadic@fer.hr

Abstract. This paper presents a quasi-Monte Carlo method for calculating the mutual inductance of misaligned circular coils which can be useful in the areas of designing and producing the devices based on the mutual inductance (biomedical applications, sensor modelling, wireless battery chargers, etc.). Validation of the method is presented through the examples. The results obtained highly match previously published data that uses other methods which is shown graphically. Advantages of the quasi-Monte Carlo method are the availability of a deterministic bound for error (unlike the probabilistic in the Monte-Carlo method), a faster convergence rate and the high flexibility.

1. Introduction
Growing needs for developing and designing of new devices based on the mutual inductance with better performances demand new solutions of the calculations. With the powerful and general numerical methods such as finite element method (FEM) and boundary element method (BEM) it is possible to calculate accurately the mutual inductance of almost any 3-D arrangement of conductors. But in some circumstances there is still necessity to address the problem using the other analytical and semi-analytical methods. The quasi-Monte Carlo method based on Van der Corput sequence, presented in this paper, considerably simplifies the mathematical procedures and reduces the computational effort. It also gives a deterministic bound for error. This computation technique can be useful in the area of wireless battery chargers for general purpose and biomedical applications, sensor modeling, coreless printed circuit board transformers, superconducting magnetic levitation, etc.

This paper will deal with the calculation of mutual inductance of coils in perfect alignment, lateral misalignment and angular misalignment. For simplicity, it assumes the case of one-turn coils which are placed in the free-space.

2. Neumann’s formula
Neumann’s formula for the mutual inductance [1] is expressed by
2.1. Perfect alignment case

According to the figure 1, table 1 and [1] mutual inductance is expressed by

\[ M_i = \mu_0 \sqrt{ab} \left( \frac{2}{k} - k \right) K(k) - \frac{2}{r} E(k) \]  

(2)

where

\[ k \equiv \left( \frac{4ab}{(a+b)^2 + d^2} \right)^{1/2} \]  

(3)

and K(k) and E(k) are the complete elliptic integrals of the first and second kind.

The bracketed expression from the equation (2) will be denoted by

\[ G(r) \equiv \left( \frac{2}{r} - r \right) K(r) - \frac{2}{r} E(r). \]  

(4)

Now equation (2) can be written as

\[ M_i = \mu_0 \sqrt{ab} G(k). \]  

(5)

This exact expression will be used as the basis for comparing the mutual inductances in the nonideal cases to follow.

Table 1. Coil and configuration parameters.

| Parameter | Meaning                                      |
|-----------|----------------------------------------------|
| a, b      | Coil radii                                   |
| d         | Coil spacing                                 |
| \( \Delta \) | Lateral misalignment                        |
| \( \alpha \) | Angular misalignment                        |
| \( M_i \) | Mutual inductance – ideal case               |
| \( M_L \) | Mutual inductance - Lateral misalignment     |
| \( M_A \) | Mutual inductance - Angular misalignment     |
| \( \mu_0 \) | Free-space permeability \((4\pi \times 10^{-7} \text{ H/m})\) |

2.2. Lateral misalignment case

Lateral misalignment configuration can be observed in figure 2. By simplifying and adjusting the Neuman’s equation, the expression for the mutual inductance in this case can be written as

\[ M_L = \frac{\mu_0 ab}{4\pi} \frac{\cos \beta}{\sqrt{ab_L}} G(r) \, d\phi \]  

(6)

where

\[ b_L \equiv \sqrt{b^2 + \Delta^2 + 2\Delta b \cos \phi}, \quad \tan \beta \equiv \frac{\Delta \sin \phi}{b + \Delta \cos \phi}, \quad r \equiv \left( 4a \frac{b_L}{(a+b_L)^2 + d^2} \right)^{1/2}. \]  

(7)
When \( \Delta = 0 \), or perfectly aligned, \( M_L \) reduces to \( M_i \). When the \( \Delta \neq 0 \) it is impossible to integrate equation (6) exactly. This paper will introduce a new possible way to calculate the equation (6).

2.3. Angular misalignment case

Coils are in an angular misalignment but the plane perpendicular to coil 1 is passing through the both coil centers as can be seen from the figure 3. Expression for calculating this mutual inductance is derived from the adjustment of equation (1) \[1\]

\[
M_A = \frac{\mu_0 a b}{\pi \sqrt{\cos \alpha}} \int_0^{\pi} \left( \frac{\cos \lambda}{\cos \phi} \right)^{3/2} G(r) d\phi
\]

(8)

where the \( G(r) \) is given in equation (4).

It is worth of mentioning that the \( \lambda \) is independent of \( \theta \) and for the perfect alignment case, when the \( \alpha = 0^\circ \), the mutual inductance \( M_A \) is reduced to the \( M_i \).

3. Monte-Carlo integration

Monte-Carlo integration is a powerful technique for numerical integration using random numbers. According to the Monte-Carlo integration, numerical calculation of \( \int_0^1 f(x) dx \) can be performed by using the generator of \( n \) pseudo numbers \( x_i \), \( i \in N \) and by using the numerical method for calculating \( \frac{1}{n} \sum_{i=1}^{n} f(x_i) \) where \( (x_1, x_2, x_3, \ldots) \) is a sequence of independently random variables distributed on \([0,1]\) interval which gives us the approximation to the integral. \[2\] Larger values of \( n \) will produce more accurate approximations.

Equations (6) and (10) can now be written in a form of Monte-Carlo integration

\[
M_L = \frac{\mu_0 a b}{\sqrt{\cos \alpha \ n}} G(r) \frac{1}{n} \sum_{i=1}^{n} \cos \beta
\]

(9)

\[
M_A = \frac{\mu_0 a b}{\sqrt{\cos \alpha \ n}} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\cos \lambda}{\cos \phi} \right)^{3/2} G(r)
\]

(10)

4. Quasi-Monte Carlo integration using Van der Corput sequence and Error bound

The simple Monte Carlo method can be upgraded by a method that uses a better distributed sequence of numbers unlike the generation of independent uniform random variables. The more these sequences
are uniformly distributed and the less is the discrepancy. One of sequences that can be used is Van der Corput sequence. Figure 4 shows the graphical comparison of Van der Corput sequence and the random variables distribution over the [0,1] interval.

![Comparison of Van der Corput sequence and RandomReal function from the WolframMathematica® distribution on the interval [0,1] with the number of variables n=10 000.](image)

**Figure 4.** Comparison of the Van der Corput sequence and RandomReal function from the WolframMathematica® distribution on the interval [0,1] with the number of variables n=10 000.

### 4.1. Van der Corput sequence

It is a low discrepancy sequence over the unit interval whose terms are obtained as follows: to compute the \(i\)th point \(x_i\) of the Van Der Corput sequence of base \(b\), the integer number \(i\) is at first expanded on base \(b\). Then \(x_n\) is obtained by reversing the digits after the decimal point. [3]

### 4.2. Error Bounding andBounding of the Discrepancy

Other main issue with the generation of independent uniform random variables is that you cannot be drastically sure of the precision of the result because of the probabilistic nature of the error bounding. Thus it is of a great value to be able to determine the error bounding while choosing the different number of variables over the chosen interval. A useful equation that provides an upper bound to discrepancy \(D_n(x)\) for a base 2 Van der Corput sequence is [3]

\[
n \cdot D_n \leq \frac{\log_{10}(n)}{3 \times \log_{10}(2)} + 1 \quad \forall n \geq 1
\]  

(11)

And so called Koksma-Hlawka inequality will provide an upper bound for the error in quasi-Monte Carlo approximation [3]

\[
\left| \sum_{i=1}^{n} f(x_i) - \int_{0}^{1} f(t) dt \right| \leq V(f) \cdot D_n
\]  

(12)

with \(D_n(f)\) the discrepancy of the function \(f\) and \(V(f)\) its variation. The total variation of function \(f\) on [0,1] is

\[
V(f) = \sup \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|
\]  

(13)
5. Results

Results presented in this paper are obtained by using the Wolfram Mathematica® software and are graphically compared with the results from [1].

5.1. Lateral misalignment example

Dimensions of circular coils and lateral misalignment:
\( a = 1.0 \text{ cm}, \quad b = 1.0 \text{ cm} \)
\( d = 0.5 \text{ cm}, \quad \Delta = \{0, 0.1, 0.2, 0.3, 0.4, 0.5\} \text{ cm} \)

Results obtained with \( n = 10 \, 000 \).

Table 2. Lateral misalignment example.

| Lateral misalignment \( \Delta \) (cm) | \( M_L \) (H) | Error Bounding percentage (%) |
|--------------------------------------|--------------|-------------------------------|
| 0                                   | 1.11261\times10^{-8} | 0.0542924                     |
| 0.1                                 | 1.0868\times10^{-8}  | 0.0544287                     |
| 0.2                                 | 1.02401\times10^{-8} | 0.054845                      |
| 0.3                                 | 9.37911\times10^{-9} | 0.0555645                     |
| 0.4                                 | 8.41474\times10^{-9} | 0.0566305                     |
| 0.5                                 | 7.43747\times10^{-9} | 0.0581155                     |

\( V(f) \) and the error bounding for this example are

\[
V(f_{\text{ext}}) = \frac{\mu_0 a b}{\sqrt{a^2 b^2} / \cos \alpha} \cdot V(f)
\]

\[
\text{Error bounding} \leq V(f_{\text{ext}}) \cdot D_n .
\]

5.2. Angular misalignment example

Dimensions of circular coils and angular misalignment:
\( a = 1.0 \text{ cm}, \quad b = 0.9 \text{ cm} \)
\( d = 0.5 \text{ cm}, \quad \alpha = \{0, 6, 10, 14, 16, 20\} \degree \)

Results obtained with \( n = 10 \, 000 \).

Table 3. Angular misalignment example.

| Angular misalignment \( \alpha \) (°) | \( M_A \) (H) | Error Bounding percentage (%) |
|--------------------------------------|--------------|-------------------------------|
| 0                                    | 9.85367\times10^{-9} | 0.0542924                     |
| 6                                    | 9.9207\times10^{-9}  | 0.0645385                     |
| 10                                   | 1.00368\times10^{-8} | 0.072094                      |
| 14                                   | 1.02015\times10^{-8} | 0.0800614                     |
| 16                                   | 1.02962\times10^{-8} | 0.0840651                     |
| 20                                   | 1.04848\times10^{-8} | 0.0915331                     |

\( V(f) \) and error bounding for this example are

\[
V(f_{\text{extA}}) = \frac{\mu_0 a b}{\sqrt{a^2 b^2} / \cos \alpha} \cdot V(f)
\]

\[
\text{Error bounding} \leq V(f_{\text{extA}}) \cdot D_n .
\]
5.3. Graphical comparison

Figure 5. Comparison of mutual inductance under lateral misalignments $M_L$ (normalized by $4\pi \text{nH}$) with the data from [1].

Figure 6. Comparison of mutual inductance under angular misalignments $M_A$ (normalized by $4\pi \text{nH}$) with the data from [1].

6. Conclusion

This paper presents a new method for the calculation of the mutual inductance of the circular coils. This method offers an alternative to FEM and BEM. It is characterized by the speed of the calculation as well as the possibility to calculate the specific equation within the determined percentage of error bounding. It can be useful in the process of designing and developing new devices when mutual inductance of coils is set and their lateral or angular misalignment should be calculated.

References
[1] Soma M, Galbraith D C and White R L 1987 Radio-frequency coils in implantable devices: misalignment analysis and design procedure IEEE Trans. Biomed. Eng. 34 276-282
[2] Sarapa D N 1992 Teorija vjerojatnosti (Zagreb: Školska knjiga)
[3] Legrand X, Xémard A, Fleury G, Auriol P and Nucci C A 2008 A quasi-Monte Carlo integration method applied to the computation of the Pollaczek integral IEEE Trans Pow. Del. 23 1527-34