ONE CONSTRUCTION OF A K3 SURFACE WITH THE DENSE SET OF RATIONAL POINTS

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ABSTRACT. We prove that there exists a number field $K$ and a smooth K3 surface $S_{22}$ over $K$ such that the geometric Picard number of $S_{22}$ equals 1, $S_{22}$ is of genus 12, and the set of $K$-points is Zariski dense in $S_{22}$.

1. INTRODUCTION

In the present paper, we study the density problem for the set of rational points on algebraic varieties over a number field, and consider the following

Question 1.1. Are there a number field $K$ and a smooth K3 surface $S$ over $K$ such that the geometric Picard number of $S$ equals one and the set of $K$-points is Zariski dense in $\mathbb{P}^3$?  

It is expected that the answer to Question 1.1, which is a version of a modified Weak Lang Conjecture (see [7, Conjecture 1.3]), is positive, and the corresponding statement holds true for all K3 surfaces $S$ (after taking an appropriate $K$ for each of these surfaces). However, no evidence for the existence of any answer to Question 1.1 has been known so far (cf. [3], [5], [9], [29]). The present paper aims to eliminate this defect:

Theorem 1.2. There exists a number field $K$ and a smooth K3 surface $S_{22}$ over $K$ such that the geometric Picard number of $S_{22}$ equals one, $S_{22}$ is of genus 12, and the set of $K$-points is Zariski dense in $S_{22}$.

Recall that the genus equals 12 for the surface $S_{22}$ from Theorem 1.2 means that $S_{22}$ can be embedded into the projective space $\mathbb{P}^{12}$ as a subvariety of degree 22 (cf. Section 2). Let us describe the approach towards the proof of Theorem 1.2. First of all, there exists a K3 surface $S$ of type R (see Proposition 3.8 and Definition 3.9). Namely, the Picard lattice $\text{Pic}(S)$ is generated by a very ample divisor $H$ and a $(-2)$-curve $C_0$ such that $(H^2) = 70$ and $H \cdot C_0 = 2$. In particular, $(S, H)$ is a polarized K3 surface of genus 36. Furthermore, the divisors $H - kC_0$ also provide polarizations on $S$ for $1 \leq k \leq 4$ (see Lemma 3.10 and Remark 3.11). Moreover, the surface $(S, H - kC_0)$ is BN general for all $0 \leq k \leq 4$ (see Definition 3.2, Lemma 3.12 and Remark 3.13). In particular, due to S. Mukai, there exists a rigid (stable) vector bundle $E_3$ of rank 3 on $S$ such that the first Chern class of $E_3$ equals $H - 4C_0$ and $\dim H^0(S, E_3) = 7$ (see Theorem-definition 3.7 and Remark 3.8). Furthermore, such $E_3$ is unique for these properties and gives a morphism $\Phi_{E_3} : S \to G(3, 7)$ into the Grassmanian $G(3, 7) \subset \mathbb{P}(\Lambda^3 C^7)$, considered with respect to the Plücker embedding (see Theorem-definition 3.6 and Remark 3.8). Moreover, $\Phi_{E_3}$ coincides with the embedding $S \hookrightarrow \mathbb{P}^{12}$ given by the linear system $|H - 4C_0|$, and the surface $S \subset G(3, 7) \cap \mathbb{P}^{12}$, when identified with $\Phi_{E_3}(S)$, can be given by explicit equations in $G(3, 7)$ (see Theorem 3.14). Note that one can run these arguments in the same way for any general polarized K3 surface $(S_{22}, L_{22})$ of genus 12 (see Remark 3.15). Furthermore, all the preceding constructions can be carried over an appropriate number field $K$, which makes one able to apply the geometric description of the surfaces $S$ and $S_{22}$ (now defined over $K$) to study their arithmetic. For instance, the set of $K$-points on $S$ turns out to be Zariski dense (see Proposition 3.17).

Remark 1.3. The surface $S$ is one of the examples of smooth K3 surfaces (of genus 12, which, as one can see, is the least possible value for all genera of polarizations on $S$) which contain a smooth rational curve and have geometric Picard number 2. Other examples (over $K$) are the minimal resolution of a double cover of $\mathbb{P}^2$ with ramification at a general sextic having an ordinary double point (genus 2 case), and a general quartic surface in $\mathbb{P}^3$ with a line (genus 3 case). Surprisingly, the latter two types of surfaces, in addition to $S$, also possess the density property for the set of $K$-points (see [3, Proposition 3.1] and [7, Theorem 1.5]), which leaves one with the question on what is beyond this phenomenon. In fact, this observation was one of the motivations for the result of Theorem 1.2 and

1)See [1, 9, 28] and references therein for the excellent surveys on the density problem.
we believe that modifying the arguments below, one might also come up with the positive answer to Question 1.1
in the case of genus 2 and 3 polarized K3 surfaces. On the other hand, we should mention that there are examples
of smooth K3 surfaces over $K$, which have geometric Picard number 2, Zariski dense set of $K$-points, but do not contain any $(-2)$-curves (see [27]). Thus, one faces the extremely important problems of constructing new examples
of surfaces of types we mentioned, and understanding relations between them and the role these surfaces (might)
play in constructions of examples similar to that in Theorem 1.2 but of other genera.

Further, the main technical tool in the proof of Theorem 1.2, namely, Proposition 1.9 was inspired by the following

**Theorem 1.4** (see [10, Theorem 1]). Let $B$ be a complex curve and $F := \mathbb{C}(B)$ its function field. There exist
non-isotrivial K3 surfaces over $F$ of any genus between 2 and 10, having geometric Picard number 1 and Zariski
dense set of $F$-points.

Yet our arguments are different from that in paper [10] (for instance, the proof of Proposition 1.9 is a direct
parameter count), the idea of constructing the example we need over a function field first was of great value for us.
More specifically, applying Proposition 1.9 and the functional version of the Mordell Conjecture (see Theorem 3.18)
together with the previously stated density property for $S$, we construct a K3 surface (the surface $S_2$ in Section 5)
of the kind similar to that in Theorem 1.4 which additionally has genus 12 and for which $F$ can be taken to be the
function field $K(\mathbb{P}^1)$. Playing with this surface and its specialization (at infinity), we arrive at the surface from
Theorem 1.2 (see the arguments at the end of Section 4).

**Remark 1.5**. In the proof of Theorem 1.2 we came up with an interesting fact about the group of automorphisms
$\text{Aut}(C_{22})$ of a general codimension 2 linear section $C_{22}$ of a general Fano threefold of the first species, having
anticanonical degree 22 (see [18], [21], [23] and Section 5). Namely, such sections are smooth canonical curves of
genus 12, and they form an analytic subset $\mathcal{Z}$ in the moduli space $\mathcal{M}_{12}$ of all canonical curves of genus 12. One
can show that $\dim \mathcal{Z} = 31$, which is less than $\dim \mathcal{M}_{12} = 33$, and hence it is not a priori clear that the group
$\text{Aut}(C_{22})$ is trivial. We prove its triviality in Section 5 (see Proposition 5.2 and Corollary 5.16). We believe,
however, that there should be a simpler proof of this fact, definitely known to specialists, but we could not find a
reference. Moreover, one can modify the proof of Theorem 1.2 in a way to avoid the usage of this result on triviality
(cf. Remark 4.17). Finally, let us note that one could try construct examples, similar to that in Theorem 1.2 but
of other genera, by combining the result of Theorem 1.4 together with the arguments in the proof of Theorem 1.2
applied to the examples from Remark 1.3.

2. Notation and conventions

All algebraic varieties, if the other is not specified, are assumed to be defined over an algebraically closed field
of characteristic 0. Throughout the paper we use standard facts, notions and notation from [6], [8], [2], [15], [11].
However, let us introduce some more:

- For a field $k$, we denote by $\bar{k}$ the algebraic closure of $k$. We also put $k^* := k \setminus \{0\}$. For an algebraic variety $V$
defined over $k$, we denote by $V(k)$ the set of $k$-points on $V$.
- We denote by $\text{Sing}(V)$ the singular locus of an algebraic variety $V$.
- We denote by $Z_1 \cdot \ldots \cdot Z_k$ the intersection of algebraic cycles $Z_1, \ldots, Z_k$, $k \in \mathbb{N}$, in the Chow ring of a
normal algebraic variety $V$.
- $D_1 \sim D_2$ stands for the the linear equivalence of two Weil divisors $D_1, D_2$ on a normal algebraic variety $V$.
  We denote by $\text{Cl}(V)$ (respectively, $\text{Pic}(V)$) the group of Weil (respectively, Cartier) divisors on $V$ modulo
linear equivalence.
- For a Weil divisor $D$ on a normal algebraic variety $V$, we denote by $\mathcal{O}_V(D)$ the corresponding divisorial sheaf on $V$ (sometimes we denote both by $\mathcal{O}_V(D)$ (or $D$)).

\[2\] At the same time, in the case of Fano threefolds of lower anticanonical degrees, and hence their codimension 2 linear sections
of lower genera, namely, between 3 and 9, this result on triviality is known to be true (see [19], [20]).
For a (coherent) sheaf $F$ on a projective normal variety $V$, we denote by $H^i(V, F)$ the $i$-th cohomology group of $F$. We put $h^i(V, F) := \dim H^i(V, F)$ and $\chi(V, F) := \sum_{i=1}^{\dim V} (-1)^i h^i(V, F)$. We also denote by $c_i(F)$ the $i$-th Chern class of $F$.

For a vector bundle $E$ on a smooth projective variety $V$, we put $\det E := c_1(E)$. For any point $x \in V$, we denote by $E_x$ the fiber of $E$ over $x$. We frequently identify $E$ with its sheaf of sections. We denote by $\text{rank}(E)$ the rank of $E$.

We call a vector bundle $E$ on a smooth projective variety $V$ stable if $\det E$ is ample and $\chi(V, F \otimes (\det E)^n)/\text{rank}(F) < \chi(V, E \otimes (\det E)^n)/\text{rank}(E)$ for $n \gg 0$ and every coherent subsheaf $F$ in $E$ with $0 < \text{rank}(F) < \text{rank}(E)$.

For a Cartier divisor $M$ on a projective normal variety $V$, we denote by $|M|$ the corresponding complete linear system on $V$. For an algebraic cycle $Z$ on $V$, we denote by $|M - Z|$ the linear subsystem in $|M|$ consisting of all the divisors passing through $Z$. For a linear system $\mathcal{M}$ on $V$, we denote by $\text{Bs}(\mathcal{M})$ the base locus of $\mathcal{M}$. If $\mathcal{M}$ on $V$ is movable (i.e., $\dim \text{Bs}(\mathcal{M}) < \dim V - 1$), we denote by $\Phi_{\mathcal{M}}$ the corresponding rational map.

We denote by $G(m, n)$ (or $G(m, \mathbb{k}^n)$) the Grassmanian of $m$-dimensional linear subspaces in $\mathbb{k}^n$, or, equivalently, $(m - 1)$-dimensional linear subspaces in $\mathbb{P}^{n-1}$. We always identify $G(m, n)$ with its image under the Plücker embedding $G(m, n) \hookrightarrow \mathbb{P}((\Lambda^m \mathbb{k}^n)^*)$.

All K3 surfaces are assumed to be smooth and projective. We write $(\mathcal{S}, L)$ for a K3 surface $\mathcal{S}$ with an ample divisor $L$. We call $L$ polarization on $\mathcal{S}$, and the pair $(\mathcal{S}, L)$ – polarized K3 surface $\mathcal{S}$. We also call the integer $(L^2)/2 + 1$ genus of $(\mathcal{S}, L)$. All polarizations are assumed to primitive, i.e., the class of $L$ in the lattice $H^2(\mathcal{S}, \mathbb{Z})$ is a primitive vector. We denote by $k_g$ the moduli space of all polarized K3 surfaces of genus $g$.

For algebraic varieties $V'$ and $V''$, we denote by $A(V'[V'')]$ the sheaf of germs of local holomorphic maps from $V'$ to the group $\text{Aut}(V'')$ of automorphisms of $V''$.

For a linear space $V$, we denote by $V^\vee$ the dual linear space. We denote by $\mathbb{P}(V)$ projective space associated with $V$.

We denote by $\langle d \rangle$ the cyclic group generated by an element $d$. For a set $V$ and a group $\langle d \rangle$ acting on $V$, we denote by $V^d$ the subset of $\langle d \rangle$-fixed points in $V$. We also denote by $\#V$ the cardinality of $V$.

By general, when referring to an algebraic variety $V$, we usually mean, if the other is not specified, a point in a Zariski open subset in a Hilbert scheme corresponding to $V$.

### 3. Preliminaries

#### 3.1. Preliminaries

Let us formulate several auxiliary notions and results which we will use in the proof of Theorem \[12\]. We start with the geometry first:

**Definition 3.2** (see [21] Definition 3.8). A polarized K3 surface $(\mathcal{S}, L)$ of genus $g$ is called BN general if $h^0(\mathcal{S}, L_1)h^0(\mathcal{S}, L_2) < g + 1$ for all non-trivial $L_1, L_2 \in \text{Pic}(\mathcal{S})$ such that $L = L_1 + L_2$.

**Example 3.3.** A general K3 surface in the moduli space $\mathcal{K}_g$ is BN general. All BN general K3 surfaces of genus $g$ form a (non-empty) Zariski open subset in $\mathcal{K}_g$.

**Definition 3.4.** Let $V$ be a smooth projective variety and $E$ a vector bundle on $V$. Then $E$ is said to be generated by global sections if the natural homomorphism $ev_E : H^0(V, E) \otimes \mathcal{O}_V \rightarrow E$ of $\mathcal{O}_V$-modules is surjective.

**Remark 3.5.** (see [18] Section 2). In the notation of Definition 3.4 if $E$ is generated by global sections, then one gets a natural morphism $\Phi_E : V \rightarrow G(r, N)$, where $r := \text{rank}(E)$, $N := h^0(V, E)$, which sends $x \in V$ to the subspace $E_x'$ of $H^0(V, E)^\vee$.

One also gets the equality $E = \Phi_E^*(\mathcal{E})$ for the universal vector bundle $\mathcal{E}$ on $G(r, N)$ so that $H^0(V, E) = H^0(G(r, N), \mathcal{E})$. Furthermore, if the natural homomorphism $\bigwedge^r H^0(V, E) \rightarrow H^0(V, \bigwedge^r E)$, induced by the $r$-th exterior power of $ev_E$, is surjective, then $\Phi_E$ coincides with the embedding $\Phi_{|\det E|} : V \hookrightarrow \mathbb{P} := \mathbb{P}(H^0(V, L)^\vee)$ given by the linear system $|\det E|$. More precisely, the diagram

$$
\Phi_E : V \rightarrow G(r, N) \cap \mathbb{P} \subset G(r, N)
\cap \mathbb{P} \rightarrow \mathbb{P}(\bigwedge^r H^0(V, E)^\vee)
$$
commutes.

**Theorem-definition 3.6** (see [21], [18], [22], [24]). Let \((\mathcal{S}, L)\) be a polarized K3 surface of genus \(g\). Assume that \((\mathcal{S}, L)\) is BN general. Then for every pair of integers \((r, s)\) with \(g = rs\), there exists a stable vector bundle \(E_r\) on \(\mathcal{S}\) of rank \(r\) such that the following holds:

1. \(\det E_r = L\);
2. \(h^i(\mathcal{S}, E_r) = 0\) for all \(i > 0\) and \(h^0(\mathcal{S}, E_r) = r + s\);
3. \(E_r\) is generated by global sections and the natural homomorphism \(\bigwedge^r H^0(\mathcal{S}, E_r) \rightarrow H^0(\mathcal{S}, \bigwedge^r E_r) = \bigwedge^r H^0(\mathcal{S}, L)\) is surjective (cf. Remark 3.9);
4. every stable vector bundle on \(\mathcal{S}\), which satisfies (1) and (2), is isomorphic to \(E_r\).

**Proposition 3.8** (see [14] Corollary 1.5). General element \(S \in | -K_X|\) is a K3 surface such that the lattice \(\text{Pic}(S)\) is generated by a very ample divisor \(H \sim -K_X|_S\) and the \((-2)\)-curve \(C_0 := \hat{E}|_S\). Furthermore, \((H^2) = 70\) and \(H \cdot C_0 = 2\).

**Definition 3.9.** The surface \(S\) from Proposition 3.8 is called K3 surface of type R.

For a K3 surface \(S\) of type R, the divisor \(H\) provides a primitive polarization on \(S\). It turns out that \(S\) admits several more primitive polarizations:

**Lemma 3.10.** For a K3 surface \(S\) of type R, the divisor \(H - 4C_0\) is ample on \(S\).

**Proof.** Let \(Z\) be an irreducible curve on \(S\) such that \((H - 4C_0) \cdot Z \leq 0\). Then we have \(Z \neq C_0\). Write \(Z = aH + bC_0\) in \(\text{Pic}(S)\) for some \(a, b \in \mathbb{Z}\). Note that \(a > 0\) because the linear system \(|m(H + C_0)|\) is basepoint-free for \(m \gg 0\) (it provides the contraction of the \((-2)\)-curve \(C_0\) and \((H + C_0) \cdot Z = 72a\). On the other hand, we have

\[
0 \geq (H - 4C_0) \cdot Z = 62a + 10b,
\]

which implies that \(b < -6a\). But then we get

\[
(Z^2) = 70a^2 + 4ab - 2b^2 \leq -26a^2 < -2,
\]

which is impossible. Thus, \((H - 4C_0) \cdot Z > 0\) for every curve \(Z \subset S\), and hence \(H - 4C_0\) is ample by the Nakai–Moishezon criterion, since \((H - 4C_0)^2 = 22\).

**Remark 3.11.** Using exactly the same arguments as in the proof of Lemma 3.10, one can prove that \(H - kC_0\) are ample divisors on \(S\) for \(k = 1, 2, 3\), which provide polarizations on \(S\) of genera 33, 28, 21, respectively. This can be also seen via geometric arguments. Indeed, let \(\pi : \mathbb{P}^{37} \dashrightarrow \mathbb{P}^{34}\) be the linear projection from the plane \(\Pi\) passing through the conic \(C_0\). The blow up \(f_1 : Y_1 \rightarrow X\) of \(C_0\) resolves the indeterminacy of \(\pi\) on \(X\) and gives a morphism \(g_1 : Y_1 \rightarrow X_1 := \pi(X)\). It can be easily seen that \(Y_1\) is a weak Fano threefold and \(X_1 \subset \mathbb{P}^{34}\) is an anticanonically embedded Fano threefold of genus 33 (cf. the proof of Proposition 6.15 in [12]). Moreover, we get \(\text{Pic}(Y_1) = \mathbb{Z} \cdot K_{Y_1} + \mathbb{Z} \cdot E_{f_1}\), where \(E_{f_1} \simeq \mathbb{F}_4\) is the \(f_1\)-exceptional divisor, and the morphism \(g_1\) contracts the surface \(f_1^{-1}(E)\) to the point. In particular, the locus \(\text{Sing}(X_1)\) consists of a unique point, \(\text{Pic}(X_1) = \mathbb{Z} \cdot K_{X_1} + \mathbb{Z} \cdot E(1)\), where \(E(1) := g_{1*}(E_{f_1})\). One can prove that \(E(1)\) is a cone over a rational normal curve of degree 4 such that \(E(1) = X_1 \cap \mathbb{P}^5\). In particular, there exists a rational normal curve \(C_1 \subset X_1 \setminus \text{Sing}(X_1)\) of degree 4 with \(C_1 = X_1 \cap \Pi_1\) for some linear space \(\Pi_1 \simeq \mathbb{P}^4\). Proceeding with \(X_1, \Pi_1, \text{etc.}\) in the same way as with \(X, \Pi, \text{etc.}\) above, we obtain three other anticanonically embedded Fano threefolds \(X_2 \subset \mathbb{P}^{29}, X_3 \subset \mathbb{P}^{22}, X_4 \subset \mathbb{P}^{13}\) of genera 28, 21, 12, respectively, such that \(\text{Sing}(X_k)\) consists of a unique point, \(\text{Pic}(X_k) = \mathbb{Z} \cdot K_{X_k}\) and \(\text{Cl}(X_k) = \mathbb{Z} \cdot K_{X_k} + \mathbb{Z} \cdot E(k)\) for all \(k\), where \(E(k)\) is a cone over a rational normal curve of degree \(2 + 2k\). By construction, general surface \(S_k \in |-K_{X_k}|\) is isomorphic to \(S\), \(1 \leq k \leq 4\). Furthermore, identifying \(S\) with \(S_k\), we
Lemma 3.12. For a K3 surface \( S \) of type \( R \), the polarized K3 surface \( (S, H - 4C_0) \) (of genus 12) is BN general.

Proof. Let

\[
H - 4C_0 = L_1 + L_2
\]

for some \( L_1, L_2 \in \text{Pic}(S) \). We may assume that both \( h^0(S, L_1), h^0(S, L_2) > 0 \). Write

\[
L_i = a_i H + b_i C_0
\]

in \( \text{Pic}(S) \) for some \( a_i, b_i \in \mathbb{Z} \). Note that \( a_i \geq 0 \) (cf. the proof of Lemma 3.10). Thus, we get that, say \( a_1 = 1 \) and \( a_2 = 0 \). Then, in particular, we have \( b_2 \neq 0 \).

Now, if \( b_2 < 0 \), then \( h^0(S, L_2) = 0 \), and we are done. Further, if \( b_2 > 0 \), then \( b_1 < -5 \), and hence we get

\[
h^0(S, L_1) h^0(S, L_2) = h^0(S, H + b_1 C_0) < h^0(S, H - 4C_0) = 13,
\]
since \( h^0(S, L_2) = h^0(S, b_2 C_0) = 1 \). \( \square \)

Remark 3.13. Using exactly the same arguments as in the proof of Lemma 3.12 one can prove that the polarized K3 surfaces \( (S, H - kC) \) (cf. Remark 3.11) are also BN general for all \( 0 \leq k \leq 3 \).

Lemmas 3.10, 3.12 and Theorem-definition 3.6 imply that there exists a rigid vector bundle \( E_3 \) on \( S \) of rank 3 such that \( \det E_3 = H - 4C_0 \) and \( h^0(S, E_3) = 7 \). From Theorem-definition 3.6 and Remark 3.3 we obtain the morphism \( \Phi_{E_3} : S \to G(3, 7) \cap \mathbb{P}^{12} \subset \mathbb{P}(\mathbb{A}^3 \otimes \mathbb{C}^7) \), which coincides with the embedding \( \Phi_{H-4C_0} : S \to \mathbb{P}^{12} \). We also have \( E_3 = \Phi_{E_3}(\mathcal{E}_7) \) for the universal vector bundle \( \mathcal{E}_7 \) on \( G(3, 7) \). Let us recall the explicit description of the image \( \Phi_{E_3}(S) \):

Theorem 3.14 (see [21, Theorem 5.5]). The surface \( S = \Phi_{E_3}(S) \) coincides with the locus

\[
G(3, 7) \cap (\Lambda = 0) \cap (\Sigma_0 = 0)
\]

for some global sections

\[
\Lambda \in \bigwedge^3 H^0(S, E_3) = H^0(G(3, 7), \bigwedge^3 \mathcal{E}_7) \simeq \mathbb{C}^7
\]

and

\[
\Sigma_0 := (\sigma_1, \sigma_2, \sigma_3) \in \bigwedge^2 H^0(S, E_3) \otimes \mathbb{C}^3 = H^0(G(3, 7), \bigwedge^2 \mathcal{E}_7) \simeq \mathbb{C}^7.
\]

Remark 3.15 (see [18, 21, 23]). One can repeat the previous arguments literally in the case of any BN general polarized K3 surface \( (S_{22}, L_{22}) \) of genus 12. Namely, \( S_{22} \) can be embedded into \( G(3, 7) \cap \mathbb{P}^{12} \), where it coincides with a locus \( G(3, 7) \cap (\Lambda = 0) \cap (\tau_1 = \tau_2 = \tau_3 = 0) \) for some \( \Lambda \in \bigwedge^3 \mathbb{C}^7 \), \( \tau_1, \tau_2, \tau_3 \in \bigwedge^2 \mathbb{C}^7 \), so that \( \mathcal{O}_{S_{22}}(L_{22}) \simeq \mathcal{O}_{G(3,7)}(1)|_{S_{22}} \) (cf. Theorem 3.14). Conversely, any such locus (for general \( \Lambda \) and \( \tau_i \)) defines a BN general polarized K3 surface of genus 12 (cf. Example 3.3). Moreover, one can prove that \( (S_{22}, L_{22}) \) is uniquely determined up to an isomorphism by the \( GL_7(\mathbb{C}) \)-orbits of \( \Lambda \) and \( \tau_i \). From this it is easy to construct a birational map between \( K_{12} \) and a \( \mathbb{P}^{13} \)-bundle over the orbit space \( M_{3} := G(3, \mathbb{A}^2 \mathbb{C}^7)/PGL_7(\mathbb{C}) \). Furthermore, the triple \( (\tau_1, \tau_2, \tau_3) \) corresponds to a general point in \( M_3 \), while \( \Sigma_0 \) corresponds to a general point in a codimension 1 (irreducible) locus in \( M_3 \) (see [14, Corollary 1.5]).

3.16. Let us now turn to the arithmetic. Recall that the above Fano threefold \( X \) is constructed as follows: one takes the weighted projective space \( \mathbb{P} := \mathbb{P}(1, 1, 4, 6) \), embeds \( \mathbb{P} \) into \( \mathbb{P}^{18} \) anticanonically, picks up a general point \( O \in \text{Sing}(\mathbb{P}) \), and projects \( \mathbb{P} \) from \( O \). Then the image of \( \mathbb{P} \) under this projection is precisely the threefold \( X \). Definitely, this construction of \( X \) can be run over a number field \( K \), say \( K = \mathbb{Q}(\sqrt{-1}) \). Moreover, the above construction of the surface \( S \) and embedding of \( S \) into \( \mathbb{P}^{12} \) can also be carried over an appropriate \( K \), which makes it possible to assume that \( S = \Phi_{E_3}(S) \) from Theorem 3.14 is defined over \( K \) (the same holds, of course, for \( S_{22} \) from Remark 3.15 with the same \( K \)). In this setting, we get the following

Proposition 3.17. The set \( S(K) \), after possibly replacing \( K \) with its finite extension, is Zariski dense in \( S \).
4.1. We use the notation and conventions of Section 3. Consider \(K\) \(\mathbb{C}\) Put (see Theorem 3.14 and Remark 3.15). It follows from Remark 3.15 that we can assume for some (general) \(\bar{\xi}\) the result follows from [3, Theorem 1.1]. □

Finally, the following result will play a crucial role in the proof of Theorem 1.2.

**Theorem 3.15** (see [16]). Let \(\mathcal{C}\) be a smooth curve of genus \(\geq 2\) over the function field \(\bar{K}(\mathbb{P}^1)\) such that the set \(\mathcal{C}(\bar{K}(\mathbb{P}^1))\) is infinite. Then there exists a curve \(\mathcal{C}_0\) over \(\bar{K}\) such that \(\mathcal{C}\) and \(\mathcal{C}_0 \times \mathbb{P}^1\) are birationally isomorphic over \(\bar{K}(\mathbb{P}^1)\).

4. PROOF OF THEOREM 1.2

4.1. We use the notation and conventions of Section 3. Consider \(G(3,7) \times \mathbb{P}^1 \times \mathbb{P}^1\) as the Grassmanian \(G(3,7)\) over \(K(t)\). Consider also the locus \(V_{22}^\xi \subset G(3,7) \times \mathbb{P}^1\) given by the equation \(\Sigma_0 + t\Sigma_{22} = 0\), where \(\Sigma_{22} := (\tau_1, \tau_2, \tau_3)\). Take \(\Lambda \in \mathbb{P}^{12}(K) = \wedge^3 H^0(G(3,7), \mathcal{E}_t)(K)\) such that \(V_{22}^\xi \cap (\Lambda = 0) \cap (t = 0) = S\) and \(V_{22}^\xi \cap (\Lambda = 0) \cap (t = \infty) = S_{22}\) (see Theorem 3.14 and Remark 3.15). It follows from Remark 3.15 that we can assume

\[
(\tau_1, \tau_2, \tau_3) = (\tau, \sigma_2, \sigma_3)
\]

for some (general) \(\tau \in \wedge^2 K^7 = \wedge^2 H^0(G(3,7), \mathcal{E}_t)(K)\).

4.3. Put \(S_\xi := V_{22}^\xi \cap (\Lambda = 0)\) and \(Z := \mathbb{P}^1\). Consider the projection \(p : G(3,7) \times Z \to Z\) on the second factor and restrict it to \(\bar{S}_\xi \subset G(3,7) \times Z\). We obtain a family \(f := p|_{S_\xi} : S_\xi \to Z\) of surfaces such that \(f^{-1}(0) = S\) and \(f^{-1}(\infty) = S_{22}\). Finally, for a general \(\lambda \in \mathbb{P}^{12}(K)\) we put \(C_\xi := S_\xi \cap (\lambda = 0)\), and refer to \(S_\xi\) and \(C_\xi\) as a surface and a curve over \(K(t)\), respectively.

4.4. Put \(C := S \cdot C_\xi\). Note that the curve \(C\) is a general hyperplane section of the surface \(S\) (over \(K\)). Pick up a point \(O \in C\). Let us find a point \(O_\xi \in C_\xi(\bar{K}(t))\) such that \(O\) is the specialization of \(O_\xi\) at \(t = 0\), i.e., \(O = S \cdot O_\xi\) on \(S_\xi\).

4.5. Let \(U\) be the space of all \((3 \times 7)\)-matrices of rank 3 over \(\bar{K}\). The group \(G := GL_3(\bar{K})\) acts on \(U\) in the natural way so that \(G(3,7) = U/G\) is the geometric quotient (cf. [17, 8.1]). In particular, any point in \(G(3,7)\) is the \(G\)-orbit of some matrix \(M \in U\), having the entries, say \(m_{i,j}, 1 \leq i \leq 3, 1 \leq j \leq 7\). Thus, any section of the morphism \(p\) (or, equivalently, any point in \(G(3,7)/(\bar{K}(t))\)) is represented by such \(M\) for which \(m_{i,j} = p_{i,j}(t), 1 \leq i \leq 3, 1 \leq j \leq 7\), where \(p_{i,j}(t)\) are polynomials in \(t\). On this way, note also that the equations \(\Sigma_0 + t\Sigma_{22} = 0, \Lambda = 0, \lambda = 0\) on \(G(3,7) \times Z\) lift to some \(G\)-invariant polynomial relations between the entries \(m_{i,j}, 1 \leq i \leq 3, 1 \leq j \leq 7\), of all matrices \(M \in U\).

4.6. Identify the point \(O \in C \subset G(3,7)\) with a matrix in \(U\). We can assume that

\[
O = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0
\end{pmatrix}.
\]

Let us also identify \(U\) with the quotient \(U/\bar{K}^*\) by the normal subgroup \(\bar{K}^* \subset G\). Consider a point \(M \in G(3,7)(\bar{K}(t))\) such that the corresponding matrix has the entries

\[
\begin{pmatrix}
Y_{i,j} + X_{i,j}t
\end{pmatrix}
\]

for some \(X_{i,j}, Y_{i,j} \in \bar{K}, 1 \leq i \leq 3, 1 \leq j \leq 7\), where we additionally put \(Y_{i,j} := 0\) for all \(i + j > 4\). Regard \(X_{i,j}, Y_{i,j}\) as projective coordinates on \(\mathbb{P}^{29}\) and identify \(M\) with the corresponding point in \(\mathbb{P}^{29}\). In this setting, we have \(G = PGL_3(\bar{K})\), and the \(G\)-action on \(U\) descends to \(\mathbb{P}^{29}\).
4.8. Let us introduce one more piece of notation. Let \( X \) (respectively, \( Y \)) be the \((3 \times 7)\)-matrix with the entries \( X_{i,j} \) (respectively, \( Y_{i,j} \), \( 1 \leq i \leq 3, 1 \leq j \leq 7 \), and \( Y_{i,j} = 0 \) for all \( i + j > 4 \)). Let also \( X_j \) (respectively, \( Y_j \)), \( 1 \leq j \leq 7 \), be the \( j \)-column of \( X \) (respectively, \( Y \)). Then we can write

\[
\tau(X + Y) = \tau(Y) + \sum_{i=1}^{3} \left( \sum_{j=1}^{7} \beta_{j}^{(i)} X_{j} \right) \wedge Y_i + \tau(X)
\]

for some \( \beta_{j}^{(i)} \in K \). Let us denote by \( Q_{1,\alpha}(X, X) \) (respectively, \( Q_{1,\alpha}(Y, Y) \)) the \( \alpha \)-th component of the vector \( \tau(X) \) (respectively, \( \tau(Y) \)), and by \( Q_{1,\alpha}(X, Y) \) the \( \alpha \)-th component of the vector \( \sum_{i=1}^{3} ( \sum_{j=1}^{7} \beta_{j}^{(i)} X_{j} ) \wedge Y_i, 1 \leq \alpha \leq 3 \). Similarly, we define \( Q_{i+1,\alpha}(X, X), Q_{i+1,\alpha}(Y, Y), Q_{i+1,\alpha}(X, Y) \) for \( \sigma_i, 1 \leq i \leq 3 \). Finally, we can write

\[
\Lambda(X + Y) = \Lambda(Y) + \sum_{i,k=1}^{3} \left( \sum_{j=1}^{7} \beta_{j,k}^{(i)} X_{j} \right) \wedge Y_i \wedge Y_k + \sum_{i=1}^{3} \left( \sum_{j,k=1}^{7} \beta_{j,k}^{(i)} X_{j} \wedge X_{k} \right) \wedge Y_i + \Lambda(X)
\]

for some \( \beta_{j,k}^{(i)} \in K \). Similarly, we define \( \beta_{j,k}^{(i)} \) for \( \lambda \).

**Proposition 4.9.** The set \( C_{\xi}(\tilde{K}(t)) \) contains a point \( O_{\xi} \) of the form \((4.7)\) such that \( O \) is the specialization of \( O_{\xi} \) at \( t = 0 \).

*Proof.* In the above notation, every \((2 \times 2)\)-minor of \( M \) is a quadratic polynomial in \( t \), with homogeneous polynomials in \( X_{i,j}, Y_{i,j} \) as coefficients. On this way, substitute \( p_{i,j}(t), 1 \leq i \leq 3, 1 \leq j \leq 7 \), from \((4.7)\) into the equation \( \Sigma_0 + t \Sigma_{22} = 0 \) and equate all the coefficients of the resulting 9 cubic polynomials in \( t \) to zero. Then, since \((4.2)\) holds and \( O \in C \) (cf. \((4.7)\)), we get a closed \( G \)-invariant scheme in \( \mathbb{P}^{29} \), given by the equations

\[
\begin{cases}
Q_{1,\alpha}(X, X) = Q_{1,\alpha}(X, Y) + Q_{2,\alpha}(X, X) = Q_{1,\alpha}(Y, Y) + Q_{2,\alpha}(X, Y) = 0, \\
Q_{l,\alpha}(X, X) = Q_{l,\alpha}(Y, Y) = 0,
\end{cases}
\]

where \( 1 \leq \alpha \leq 3, l \in \{3, 4\} \).

Further, every \((3 \times 3)\)-minor of \( M \) is a cubic polynomial in \( t \), with homogeneous polynomials in \( X_{i,j}, Y_{i,j} \) as coefficients. Then, after substituting \( p_{i,j}(t) \) from \((4.7)\) into the equations \( \Lambda = \lambda = 0 \) and equating all the coefficients of the resulting 2 cubic polynomials in \( t \) to zero, we arrive at a closed \( G \)-invariant scheme in \( \mathbb{P}^{29} \), given by the equations (again we use that \( O \in C \))

\[
\begin{align*}
\sum_{i,k=1}^{3} \left( \sum_{j=1}^{7} \beta_{j,k,\gamma}^{(i)} X_{j} \right) \wedge Y_i \wedge Y_k &= \sum_{i=1}^{3} \left( \sum_{j,k=1}^{7} \beta_{j,k,\gamma}^{(i)} X_{j} \wedge X_{k} \right) \wedge Y_i = R_{\gamma}(X, X) = 0, \\
&\quad \gamma \in \{1, 2\}, \text{ where } R_{\gamma}(X, X) \text{ is a linear combination of } (3 \times 3)\text{-minors of the matrix } X.
\end{align*}
\]

Consider a rank 1 matrix

\[
\begin{pmatrix}
W_{l_1} \\
W_{l_2} \\
\vdots \\
W_{l_7}
\end{pmatrix}
\begin{pmatrix}
W_{l_1} \\
W_{l_2} \\
\vdots \\
W_{l_7}
\end{pmatrix}
\begin{pmatrix}
W_{l_1} \\
W_{l_2} \\
\vdots \\
W_{l_7}
\end{pmatrix}
\end{equation}
\]

for some \( W_l, W_j \in \tilde{K} \) and general linear forms \( l_i := l_i(Z_1, \ldots, Z_N) \in H^0(\mathbb{P}^{N-1}, O(1)) \) in \( Z_1, \ldots, Z_N \in \tilde{K} \) with \( N \gg 0 \). Put \( X \) of the form \((4.12)\) and \( Y := O \) in \((4.10), (4.11)\). Let \( V \) be the linear space spanned by \( x_{1,i} := W_{l_i}, x_{2,i} := W_{l_1}, x_{3,i} := W_{l_2}, 1 \leq i \leq 7 \). Then \( Q_{l,\alpha}, 2 \leq l \leq 4, 1 \leq \alpha \leq 3 \), turns into an element \( H_{l,\alpha} \) in the linear subspace \( V_{\alpha} \) of \( V \), given by equations

\[
\begin{align*}
x_{1,1} = \ldots = x_{3,3} = x_{i(\alpha),j} = 0,
4 \leq j \leq 7, \text{ for some fixed } i(\alpha) (\{i(1), i(2), i(3)\} = \{1, 2, 3\}). \text{ In the same way, we get the elements } H_{1,\alpha} \in V, 1 \leq \alpha \leq 3, \text{ and } H_{\alpha} \in \bigoplus_{\alpha} V_{\alpha}, \text{ corresponding to } Q_{1,\alpha} \text{ and } \lambda, \text{ respectively. Moreover, by generality of } \tau, \sigma_2, \sigma_3, \lambda \text{ and } \Lambda, \text{ we can assume that } H_{l,\alpha}, 2 \leq l \leq 4, H_{1,\alpha}, H_{\alpha} \text{ and } H_{\lambda} \text{ are general points in } V_{\alpha}, V \text{ and } \bigoplus_{\alpha} V_{\alpha}, \text{ respectively. On this way, after compactifying via } W, \text{ from } (4.10), (4.11) \text{ we get a closed scheme } \Gamma \text{ in } \mathbb{P}^{N+2} \text{ with coordinates } Z_l, W_l, W_j, \text{ given by equations}
\end{align*}
\]

\[
\begin{align*}
&W^2 - H_1(W_{l_1}, \ldots, W_{l_7}) = H_2(W_{l_1}, \ldots, W_{l_7}) = \ldots = H_{11}(W_{l_1}, \ldots, W_{l_7}) = 0,
&F_l(W_{l_1}, \ldots, W_{l_7}) = 0,
\end{align*}
\]
1 ≤ l ≤ 3, for some linearly independent forms \( H_j, F_i \in H^0(\mathbb{P}^{20}, \mathcal{O}(1)) \).

**Lemma 4.13.** \( W \neq 0 \) identically on \( \Gamma \).

**Proof.** Assume the contrary. Then \( \Gamma \) coincides with the (reduced) scheme

\[
\bigcap_{1 \leq t \leq 3} \bigcap_{1 \leq l \leq 11} \{ W = H_f(W_1, \ldots, W_2 l_t) = F_i(W_1, \ldots, W_2 l_t) = 0 \}.
\]

In particular, the linear subsystem \( \mathcal{L} \subseteq |\mathcal{O}(2) - \Gamma| \), spanned by the quadrics \( W^2 - H_1(W_1, \ldots, W_2 l_t), H_j(W_1, \ldots, W_2 l_t), F_i(W_1, \ldots, W_2 l_t), 2 \leq j \leq 11, 1 \leq l \leq 3 \), is movable with \( \text{Bs}(\mathcal{L}) = \Gamma \). From this we deduce that the image \( \Phi|_{\mathcal{O}(2) - \Gamma} : (\mathbb{P}^{N + 2}) \) is at most 13-dimensional. On the other hand, the rational map \( \Psi : \mathbb{P}^{N + 2} \rightarrow \mathbb{P}^{N + 2} \), given by the quadrics \( H_1(W_1, \ldots, W_2 l_t) + WF \) for all \( F \in H^0(\mathbb{P}^{N + 2}, \mathcal{O}(1)) \), factors through \( \Phi|_{\mathcal{O}(2) - \Gamma} \). Then, restricting \( \Psi \) to the hypersurface \( (H_1(W_1, \ldots, W_2 l_t) = 0) \), we get a birational map, hence \( \dim \Phi|_{\mathcal{O}(2) - \Gamma}(\mathbb{P}^{N + 2}) \geq N + 1 \), a contradiction. \( \square \)

It follows from Lemma 4.13 that there exists a matrix \( X_0 \) of the form (4.12) such that the corresponding point in \( \mathbb{P}^{N + 2} \) belongs to \( \Gamma \cap \{ W = 1 \} \). Then the pair \( (X, Y) := (X_0, O) \) provides a common solution for (4.10) and (4.11). The latter determines a point \( O_\xi \in G(3, 7)(\bar{K}(t)) \) of the form (4.7) because for \( (X, Y) = (X_0, O) \) one gets a rank 3 matrix in (4.7) whenever \( |t| \ll 1 \). Thus, we have \( O_\xi \in C_\xi(\bar{K}(t)) \) by construction, with \( O \in C \) being the specialization of \( O_\xi \) at \( t = 0 \).

Proposition 4.9 is completely proved. \( \square \)

**Corollary 4.14.** The set \( C_\xi(\bar{K}(t)) \) is infinite.

**Proof.** This follows immediately from Proposition 4.9 because \( O \in C \) was chosen arbitrary. \( \square \)

**Remark 4.15.** It follows from the proof of Proposition 4.9 that all \( \lambda \) in \( \mathbb{P}^{12} \) with \( \lambda(O_\xi) = 0 \) form a codimension \( \leq 2 \) linear subspace \( \mathcal{L} \). Indeed, the first linear constraint is \( \lambda(O) = 0 \) and the second one is obtained from (4.11), with \( \gamma = 2, Y = O \) and \( X = X_0 \). Furthermore, for any \( \lambda' \in \mathbb{P}^{12} \) with \( \lambda'(O) = 0 \) and every \( \lambda \in \mathcal{L} \), running the arguments in the proof of Proposition 4.9 we get a point \( O_\xi' \in G(3, 7)(\bar{K}(t)) \) such that \( \lambda'(O_\xi') = \lambda(O_\xi ') = 0 \). Finally, all the preceding arguments apply literally if one replaces \( O \in C = S \cap (\lambda = 0) \) with \( O \in S_{22} \cap (\lambda = 0) \).

In the above notation, consider the (smooth) curve \( C_\xi \). Since the genus of \( C_\xi \) equals 12, Corollary 4.14 and Theorem 3.18 imply that \( C_\xi \), when considered as a smooth surface over \( K \), together with the morphism \( f|_{C_\xi} : C_\xi \rightarrow Z \), is birationally isomorphic over \( Z \) to \( \Gamma \times Z \) for some smooth curve \( \Gamma \) over \( \bar{K} \). In particular, a general fiber of \( f|_{C_\xi} \) is isomorphic to \( \Gamma \). Further, after possibly a coordinate change on \( Z \), we can take \( U \subset Z \), a Zariski open subset such that \( 0, \infty \in U \). \( f|_{C_\xi} \) is smooth over \( U \), and \( (f|_{C_\xi})^{-1}(t') \simeq \Gamma \) for every \( t' \in U \setminus \{0\} \). Put also \( C_{U, \xi} := f^{-1}(U) \cap C_\xi \).

**Lemma 4.16.** The surfaces \( C_{U, \xi} \) and \( C \times U \) are isomorphic over \( U \), i.e., the family \( f|_{C_{U, \xi}} : C_{U, \xi} \rightarrow U \) is trivial.

**Proof.** For every \( t' \in U \setminus \{0\} \), we have \( (f|_{C_\xi})^{-1}(t') \simeq \Gamma \) by definition of \( U \). Take a small disk \( \Delta \subseteq U \) around 0 and consider the period map \( \zeta : \Delta \rightarrow \mathfrak{S}_{12} \) associated with \( f|_{C_\xi} \), into the Ziegel upper half space \( \mathfrak{S}_{12} \). The map \( \zeta \) is a holomorphic morphism, constant on \( \Delta \setminus \{0\} \) and hence on \( \Delta \), i.e., we have

\[
C = (f|_{C_\xi})^{-1}(0) \simeq (f|_{C_\xi})^{-1}(t') \simeq \Gamma.
\]

Thus, all fibers of the morphism \( f|_{C_{U, \xi}} \) are isomorphic to the curve \( C \), which implies that the family \( f|_{C_{U, \xi}} : C_{U, \xi} \rightarrow U \) is locally trivial. But \( \# \text{Aut}(C) = 1 \) by Proposition 5.2 below. Thus, we get \( H^1(U, \mathcal{A}_U[C]) = 0 \), which implies that the family \( f|_{C_{U, \xi}} : C_{U, \xi} \rightarrow U \) is trivial. \( \square \)

**Remark 4.17.** Alternatively, in the proof of Lemma 4.16, instead of vanishing \( H^1(U, \mathcal{A}_U[C]) = 0 \) one may use the fact that through every point on \( C \) there is a section of \( f|_{C_{U, \xi}} \). In this case, it is enough to assume only that \( C \) is smooth.

3) Note that \( W^2 \) is algebraically independent of \( W_1, \ldots, W_2 l_t \), hence of \( H_1, \ldots, F_3 \), in \( \bar{K}(\mathbb{P}^{N + 2}) \).
Lemma 4.18. Through every point in \( C_{U_1} \subseteq C_\xi \) there is at most one section of \( f|_{C_\xi} \).

Proof. Indeed, otherwise there exists a rational dominant map \( \mathbb{P}^1 \rightarrow C \), induced by the projection \( C_{U_1} \simeq C \times U \rightarrow C \) on the first factor (see Lemma 4.16), which is impossible. \( \square \)

Lemma 4.19. Let \( O_\xi \) be a point in \( C_\xi(K(t)) \) such that its specialization \( O := S \cdot O_\xi \) at \( t = 0 \) is a point in \( S(K) \). Then \( O_\xi \in C_\xi(K(t)) \).

Proof. The point \( O \) is \( \text{Gal}(\bar{K}/K) \)-fixed, which implies that the section \( O_\xi \) is \( \text{Gal}(\bar{K}/K) \)-invariant, since otherwise we get a contradiction with Lemma 4.18. Hence \( O_\xi \in C_\xi(K(t)) \). \( \square \)

Remark 4.20. Consider a curve \( \hat{C}_\xi \subset S_\xi \) over \( K(t) \), given by the equation \( \hat{\lambda} = 0 \) for some \( \hat{\lambda} \in \mathbb{P}^{12}(K) \). Then, in the previous notation, the family \( f|_{\hat{C}_{U_1,\xi}} : \hat{C}_{U_1,\xi} \rightarrow U \), where \( \hat{C}_{U_1,\xi} := \hat{C}_\xi \cap f^{-1}(U) \), is smooth for every \( \hat{\lambda} \) in some Zariski open subset \( \mathcal{L} \subset \mathbb{P}^{12}(K) \) (with \( \mathcal{L} \ni \lambda \)). Moreover, the same arguments as in the proof of Lemma 4.18 imply that the family \( f|_{\hat{C}_{U_1,\xi}} : \hat{C}_{U_1,\xi} \rightarrow U \) is trivial for every such \( \hat{\lambda} \), and hence the statements of Lemmas 4.18 and 4.19 hold also for \( \hat{C}_{U_1,\xi} \) and \( \hat{C}_\xi \), respectively.

In the above notation, let \( \mathcal{U} \) be the set of all points in \( S_\xi(K(t)) \) as in Proposition 4.19 when \( O \) runs through \( S(K) \) and \( \lambda \) runs through \( \mathcal{L} \). Then Proposition 4.17 and Lemma 4.19 (cf. Remark 4.20) imply that \( \mathcal{U} \subset S_\xi(K(t)) \) is a Zariski dense set in \( S_\xi \). Thus, the surface \( S_\xi \) provides an example of a (non-isotrivial) K3 surface over the function field \( K(t) \) such that \( S_\xi \) is of genus 12, geometric Picard number of \( S_\xi \) is 1, and the set of \( K(t) \)-points is Zariski dense in \( S_\xi \) (cf. Theorem 1.1). We can, however, extract more from the construction of \( S_\xi \):

4.21. Firstly, for any point \( O_\xi \in \mathcal{U} \) we have \( S_{22} \cdot O_\xi \in S_{22}(K) \), since both \( O_\xi \) and \( S_{22} \) are defined over \( K(t) \), hence both are \( \text{Gal}(\bar{K}/K) \)-invariant. We can define the map \( h : \mathcal{U} \rightarrow S_{22}(K) \) via \( O_\xi \mapsto S_{22} \cdot O_\xi \). Put \( \mathcal{U}' := h(\mathcal{U}) \). Consider \( \mathcal{U} \) and \( \mathcal{U}' \) as topological subspaces in \( S_\xi(K(t)) \) and \( S_{22}(K) \), respectively, with the induced Zariski topologies.

Lemma 4.22. \( \dim \mathcal{U}' > 0 \).

Proof. Suppose that \( \dim \mathcal{U}' = 0 \). We can assume that \( \mathcal{U}' \) is a point. But then \( \mathcal{L} \) must be of codimension 1 in \( \mathbb{P}^{12}(K) \), a contradiction. \( \square \)

Further, shrinking \( \mathcal{U} \) if necessary, we get the following

Lemma 4.23. The map \( h \) is 1-to-1.

Proof. Take a general point \( O \in \mathcal{U}' \). In the notation of Remark 4.15 consider the linear space \( \mathcal{L}_{O_\xi} \) for some \( O_\xi \in \mathcal{U} \) with \( h(O_\xi) = S_{22} \cdot O_\xi = O \) (we have switched the roles between \( S \) and \( S_{22} \)). Lemma 4.22 implies that we can take \( \lambda \in \mathcal{L}_{O_\xi} \cap \mathcal{L} \). Furthermore, for a general \( \lambda' \in \mathbb{P}^{12}(K) \) with \( \lambda'(O) = 0 \), we have \( \lambda' \in \mathcal{L} \) and \( \lambda'(O_\xi) = \lambda(S_{22}) = 0 \) for some \( O_\xi \in \mathcal{U} \) with \( h(O_\xi) = O \) (see Remark 4.15 and Lemma 4.19). Then Lemma 4.18 implies that \( O_\xi = O_\xi \). Hence there is a unique point \( O_\xi \in \mathcal{U} \) with \( h(O_\xi) = O \). \( \square \)

It follows from Lemma 4.22 that \( h^{-1} : \mathcal{U}' \rightarrow \mathcal{U} \) is a continuous map because for any Zariski closed subset \( R \subseteq S_\xi \) the set \( h(R) := S_{22} \cap R \subseteq S_{22} \) is just the specialization of \( R \) at \( t = \infty \). Then we get \( \dim \mathcal{U}' \geq \dim \mathcal{U} \) by the definition of topological dimension (note that \( h^{-1} \) is a continuous bijection). On the other hand, \( \dim \mathcal{U} \geq 2 \) because \( \mathcal{U} \) is dense in \( S_\xi \). Thus, we get \( 2 \leq \dim \mathcal{U}' \leq \dim S_{22} = 2 \), which implies that \( \mathcal{U}' \), hence also \( S_{22}(K) \), is dense in \( S_{22} \). This proves Theorem 1.2.

5. Appendix on the automorphisms of curves

5.1. We use the notation and conventions of Section 3. Let \( C \) be a general hyperplane section of the surface \( S = \Phi_{E_5}(S) \subset \mathbb{P}^{12} \) (see Theorem 3.14). \( C \) is a smooth canonical curve of genus 12, having \( K_C \sim H'|_{\mathcal{C}} \), where \( H' \sim H - 4C_0 \) is a genus 12 polarization on \( S \). The main purpose of the present section is the proof of the following

Proposition 5.2. The group \( \text{Aut}(C) \) is trivial.

Before proving Proposition 5.2, let us introduce several auxiliary objects and make some conventions:

4) We can always choose \( C \) to be defined over \( K \).
5.3. Consider the morphism \( f := \Phi_{|H-5\ell|} : S \rightarrow \mathbb{P}^1 \), whose general fiber is a smooth elliptic curve (see the proof of Proposition 5.17). Note that all fibers of \( f \) are reduced.

**Lemma 5.4.** The elliptic fibration \( f : S \rightarrow \mathbb{P}^1 =: B \) is non-isotrivial.

**Proof.** Suppose that \( f \) is isotrivial. Let \( E_0 \) be a smooth fiber of \( f \), with \( E_0 \cong E_\xi \) for the general fiber \( E_\xi \) of \( f \).

Take a Galois extension \( F \supset \overline{K}(B) \) with the Galois group \( G \) such that \( S' := S \times_B B' \cong E_0 \times B' \) (cf. the proof of Lemma 5.16) for a smooth curve \( B' \) having \( \overline{K}(B') = F \) and \( B'/G = B \). The group \( G \) acts on the surface \( S' \) in the natural way so that \( S = S'/G \). In particular, we get that either \( S \cong S' \) or \( f \) has a multiple fiber, a contradiction. \( \square \)

Let \( P_1, \ldots, P_r \) be all ramification points of the morphism \( \phi := f|_C : C \rightarrow B \) (cf. Lemma 5.4). Then, taking \( P_1, \ldots, P_r \) general, we may assume that

- the fibers \( F_i := \phi^{-1}(\phi(P_i)) \) are smooth for all \( 1 \leq i \leq r \).

**Lemma 5.5.** The morphism \( \phi \) has only simple ramifications and the monodromy group of \( \phi \) acts on the linear space \( H^1(C, \overline{K}) \) by transpositions.

**Proof.** It follows from Lemma 5.4 that locally-analytically near each \( P_j \) the surface \( S \) is given by the equation \( y^2 = x + z \) in local coordinates \( x, y, z \) so that \( f \) is the restriction of projection on the \( z \)-coordinate. Then \( C \) is given by the equation \( z = y^m \) on \( S \) for some \( m \in \mathbb{N} \). In particular, \( \phi \) has only simple ramifications, which implies that near \( P_j \) the curve \( C \) is glued of two charts \( \{(z, h(z)) : z \in \Delta \} \) and \( \{(z, -h(z)) : z \in \Delta \} \) over \( \Delta \), where \( \Delta \subset B \) is a small disk and \( h : \Delta \rightarrow F_j \) is a holomorphic map. In particular, the monodromy of \( \phi \) acts on \( H^1(C, \overline{K}) \) by transpositions. \( \square \)

5.6. Fix a point \( Q_j := \phi(P_j) \in \mathbb{P}^1 \) and take a small disk \( U_j \subset \mathbb{P}^1 \) around \( Q_j \). Then the preimage \( \phi^{-1}(U_j) \) is a disjoint union of open connected subsets \( V_{j,i} \cong U_j \subset C \):

\[
\phi^{-1}(U_j) = \bigcup_{i=1}^{N_j} V_{j,i},
\]

where \( 1 \leq i \leq N_j \), for some \( N_j \in \mathbb{N} \). Furthermore, since \( \phi \) has only simple ramifications, \( \phi|_{V_{j,i}} \) coincides with the morphism \( x \mapsto y := x^a \) in appropriate coordinates \( x \) and \( y \) on \( V_{j,i} \) and \( U_j \), respectively, where \( a \in \{1, 2\} \). Put \( V_{j,i} := U_j^{(1)} \) if \( \phi|_{V_{j,i}} \) has \( a = 1 \), and \( V_{j,i} := U_j^{(2)} \) if \( \phi|_{V_{j,i}} \) has \( a = 2 \). In particular, the monodromy action on \( V_{j,i}^{(1)} \) is trivial, while it coincides with a \( \mathbb{Z}/2\mathbb{Z} \)-action on \( V_{j,i}^{(2)} \). Let also \( E_{j,i} \), \( 1 \leq i \leq N_j \), be a circle on \( V_{j,i}^{(2)} \) such that \( \phi(E_{j,i}) = \partial(U_j) \cong S^1 \). Varying \( 1 \leq j \leq r \) in these arguments, we arrive at the set of generators

\[
B := \bigcup_{j=1}^r \{ E_{j,1}, \ldots, E_{j,N_j} \}
\]

of \( H^1(C, \overline{K}) \) (over \( \overline{K} \)).

**Proof of Proposition 5.2** Suppose that the group \( \text{Aut}(C) \) is non-trivial. Then there exists an element \( \tau \in \text{Aut}(C) \) of a prime order \( \ell \). Put \( G := \langle \tau \rangle \). Take the product \( W := C \times \ldots \times C \) of \( \ell \) copies of \( C \) and consider the algebraic subset

\[
C := \{ (x_1, \ldots, x_\ell) \in W \mid x_1 \in C \text{ and } x_i := \tau^{i-1}(x_1) \text{ for all } 2 \leq i \leq \ell \} \subset W.
\]

Note that \( C \cong C \). Furthermore, the group \( G \) acts non-trivially on \( C \) via the cyclic permutation of the coordinates of any point \( (x_1, \tau(x_1), \ldots, \tau^{\ell-1}(x_1)) \in C \), and the morphism

\[
\tilde{\phi} := (f, \ldots, f)|_C : C \rightarrow \mathbb{P}^1 \cong \tilde{\phi}(C) \subset B \times \ldots \times B
\]

to the product of \( \ell \) copies of \( B = \mathbb{P}^1 \) is equivariant with respect to this \( G \)-action. Finally, the morphism \( \tilde{\phi} \) possesses all the preceding properties of the morphism \( \phi \), and hence we may additionally assume that

- the morphism \( \phi : C \rightarrow \mathbb{P}^1 \) is \( G \)-equivariant. In this setting, we may also assume that the set \( B \) in (5.7) is \( G \)-invariant.

\[5\text{Since } S \text{ is locally glued of two charts } (z, y) \text{ and } (z, -y).\]
Further, since $\tau$ is of order $\ell$, every point in the locus $C^\tau$ is non-degenerate, and hence we have
\[
\#C^\tau = 2 \sum_{k=0}^{2} (-1)^k \text{Trace} \left[ \tau^* : H^k(C, \bar{K}) \to H^k(C, \bar{K}) \right] = 2 - \text{Trace} \left[ \tau^* : H^1(C, \bar{K}) \to H^1(C, \bar{K}) \right] := 2 - T
\]
by the Lefschetz formula. Let $M$ be a matrix associated with the $\tau^*$-action on $H^1(C, \bar{K})$.

**Lemma 5.8.** The matrix $M$ has entries only from the set $\{0, 1\}$ and consists of blocks, which correspond to the orbits of the $G$-action on the set $\mathcal{B}$ in (5.7). In particular, we have $T = \text{Trace}(M) \geq 0$, or, equivalently, $\#C^\tau = 2 - T \leq 2$.

**Proof.** Recall that the monodromy of $\phi$ acts on $H^1(C, \bar{K})$ by transpositions (see Lemma 5.5). Then, since $\phi$ is $G$-equivariant and $\mathcal{B}$ is $G$-invariant, we get that $\tau^*$ acts on $\mathcal{B}$ by permutations, hence the result. \qed

**Lemma 5.9.** The equality $\text{deg}(\phi) = 12$ holds. In particular, every fiber $F$ of the morphism $\phi$ consists of $12 - 2N$ distinct points, where $N$ is the number of ramification points in $F$.

**Proof.** This follows from the fact that $\text{deg}(\phi) = (H - 4C_0) \cdot (H - 5C_0) = 12$ and $\phi$ has only simple ramifications (see Lemma 5.6). \qed

**Lemma 5.10.** Let $C^\tau = \emptyset$. Then $\ell \in \{2, 3\}$.

**Proof.** Indeed, since $C^\tau = \emptyset$, none of the sets $V_{j,i}$ from the construction of the set $\mathcal{B}$ can be $G$-invariant, $1 \leq j \leq r$, $1 \leq i \leq N_j$. Then, in particular, $G$ acts freely on $\mathcal{B}$, and hence $\ell = \#G$ divides $24 = \dim H^1(C, \bar{K})$, i.e., $\ell \in \{2, 3\}$. \qed

**Lemma 5.11.** $C^\tau \neq \emptyset$.

**Proof.** Suppose that $C^\tau = \emptyset$, i.e., $G$ acts freely on $C$. Consider the quotient $C' := C/G$. This is a smooth curve of genus $g'$ such that the quotient morphism $C \to C'$ is étale of degree $\ell = \#G$. Then the Hurwitz formula implies that
\[
22 = 2\ell(g' - 1),
\]
which is impossible, since $\ell \in \{2, 3\}$ by Lemma 5.10. \qed

**Lemma 5.12.** Let $C^\tau \cap \phi^{-1}(\phi(P_i)) = \emptyset$ for all $1 \leq i \leq r$. Then $\ell = 2$.

**Proof.** Note that the condition $C^\tau \cap \phi^{-1}(\phi(P_i)) = \emptyset$ for all $1 \leq i \leq r$ implies that $G$ acts freely on the set $\mathcal{B}$ (cf. the proof of Lemma 5.10). Then we obtain that $\ell \in \{2, 3\}$ exactly as in the proof of Lemma 5.10.

Further, pick up a point $P \in C^\tau$ (see Lemma 5.11) such that $P$ is not ramified at all points in $F_P := \phi^{-1}(\phi(P))$, which is possible by our assumption. Then, since the fiber $F_P$ is $G$-invariant, $\#F_P = 12$ (see Lemma 5.9) and $\ell \in \{2, 3\}$, there exists another point $Q \in F_P \cap C^\tau \setminus \{P\}$. In particular, we have $\{P, Q\} \subseteq C^\tau$, and hence $2 - T = \#C^\tau \geq 2$. Moreover, the latter implies that $\#C^\tau = 2$ by Lemma 5.8. Thus, we get $C^\tau = \{P, Q\}$, and hence $\ell \mid 10$, i.e., $\ell = 2$. \qed

**Lemma 5.13.** Let $P_j \in C^\tau$ for some $1 \leq j \leq r$. Then $\ell = 2$.

**Proof.** It follows from our assumption that there exists a $G$-fixed element $E_{j,i} \in H^1(C, \bar{K})$ (cf. the proof of Lemma 5.10). Then the definition of the matrix $M$ from Lemma 5.8 implies that $T \geq 1$. Moreover, since $2 - T = \#C^\tau \neq 0$ by Lemma 5.11, we obtain that $T = 1$, i.e., $C^\tau = \{P_j\}$. Then, since $\text{deg}(\phi) = 12$ (see Lemma 5.9), the fiber $F_j := \phi^{-1}(\phi(P_j))$ is $G$-invariant, and $\phi$ has only simple ramifications, we get that $G$ acts freely on a set consisting of $\#F_j - 2 = 10$ elements. In particular, $\ell = \#G$ divides $10$, i.e., $\ell \in \{2, 5\}$.

Further, since $\phi$ is $G$-equivariant, there exists another $G$-invariant fiber $F'$ of $\phi$, with $F' \neq F_j$. Note that $G$ acts freely on $F'$. In particular, either $\phi$ is not ramified at all points of $F'$, or $F'$ contains at least two ramification points of $\phi$. Then Lemma 5.9 implies that either $\#F' = 12$ or $\#F' \leq 8$. Thus, since $\#F'$ is even (see Lemma 5.9) and $\ell \in \{2, 5\}$, we get that $\ell = 2$. \qed

**Lemma 5.14.** Let $C^\tau \cap \phi^{-1}(\phi(P_j)) \neq \emptyset$ for some $1 \leq j \leq r$. Then $\ell = 2$. 

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Proof. Pick up a point $P \in C^r \cap \phi^{-1}(\phi(P_j))$. Note that $\ell = 2$ provided $B^G \neq \emptyset$ by the similar arguments as in the proof of Lemma 5.13. Note also that $\ell = 2$ provided $\phi$ ramifies at $P$ (see Lemma 5.10).

Suppose now that $G$ acts freely on the sets $B$ and $\{P_1, \ldots, P_r\}$. In this case, we have $\ell \in \{2, 3\}$ (cf. the proof of Lemma 5.10). Furthermore, by assumption $\phi$ must have at least two ramification points, say $q_1$ and $q_2$, in the fiber $F_j := \phi^{-1}(\phi(P_j))$. This implies that $\#F_j \leq 6$ (see Lemma 5.9). Moreover, if $q_1, q_2$ are the only ramification points in $F_j$, then, since $\ell \in \{2, 3\}$ and $G$ acts freely on the set $\{q_1, q_2\}$, we get $\ell = 2$. Hence we may assume that $\#F_j \leq 6$ (cf. Lemma 5.9).

Further, by the similar arguments as in the proof of Lemma 5.12, we have $C^r = \{P, Q\}$, with $Q \in F_j \setminus \{P\}$. Thus, since $\#F_j \leq 6$ is even (see Lemma 5.9), we obtain that $G$ acts freely on either a 2- or 4-element subset in $F_j$, and hence $\ell = 2$.

Since $\ell = 2$ (see Lemmas 5.12, 5.14) and $\phi$ is $G$-equivariant, all the sets $V_{j,i}$ must be $G$-invariant, $1 \leq i \leq N_j$, $1 \leq j \leq r$. In particular, since $\deg(\phi) = 12$ (see Lemma 5.9) and $\phi$ has only simple ramifications, we get that $\#C^r \geq r = 46$ by the Hurwitz formula. On the other hand, we have $\#C^r \leq 2$ by Lemma 5.8 a contradiction.

Proposition 5.2 is completely proved.

5.15. Recall that the locus $G(3, 7) \cap (r_1 = r_2 = r_3 = 0)$ (see Remark 3.15) is a general (smooth) anticanonically embedded Fano threefold $V_{22} \subset \mathbb{P}^{11}$ (see [18], [21], [23]). Let $C_{22} := V_{22} \cap \mathbb{P}^{11}$ be a general codimension 2 linear section of $V_{22}$. Then $C_{22}$ is a smooth canonical curve of genus 12, having $K_{C_{22}} \sim -K_{V_{22}}|_{C_{22}}$.

Corollary 5.16. The group $\text{Aut}(C_{22})$ is trivial.

Proof. Let $M_{12}$ be the moduli space of all smooth canonical curves of genus 12. Theorem 3.14 Remark 3.15 and the construction of the curves $C$ and $C_{22}$ imply that the points $[C]$ and $[C_{22}]$ in $M_{12}$, corresponding to $C$ and $C_{22}$, respectively, belong to an irreducible analytic subset $Z \subset M_{12}$. On the other hand, all smooth canonical curves of genus 12 and non-trivial group of automorphisms are parameterized by an analytic subset $\mathfrak{A}_{12} \subset M_{12}$. Hence, since $[C] \not\in \mathfrak{A}_{12}$ (see Proposition 5.2) and $[C], [C_{22}] \in Z$, we get that also $[C_{22}] \not\in \mathfrak{A}_{12}$.

Further, taking the curve $C_{22}$ defined over $K$, we get the following

Corollary 5.17. $C_{22}$ has no non-trivial forms over $K$.

Proof. This follows from Corollary 5.3 (cf. [26 Ch. III]).

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