Quantization of string theory for $c \leq 1$

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**Abstract**

We consider the canonical quantization scheme for $c \leq 1$ ($(p,q)$-) string theories and compare it with what is known from matrix model approach. We derive explicitly a trivial (≡ topological) solution. We discuss a “dressing” operator which in principle allows one to obtain a non-trivial solution, but an explicit computation runs into a problem of analytic continuation of the formal expressions for $\tau$-functions. We discuss also the application of proposed scheme to the case of discrete matrix model and consider some parallels with mirror symmetry and background independence in string theory.

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1 Introduction

Recent development of string theory brought us to a to a more or less sensible progress in explicit description of the exact solutions to the simplest string models. Fortunately enough, it turns out that the correlation functions in such theories can be determined through their generation functions which not only solve the well-known integrable equations of the KP (Toda lattice) hierarchy but also appear to be a particular solution having simple integral representation form. This allows to find a kind of (rather degenerate) field theory representations for these solutions and try to interpret these representations as a second-quantized or string field theory.

From a general outlook we have here an example of a new phenomenon in the study of quantum systems – their relation to the solution of classical integrable equations. Among other examples one can mention the cases when correlation functions in the quantum integrable systems are solutions to the classical integrable equations. In string theory we have slightly different statement: generating function for correlators in some (exotic) physical system is a $\tau$-function of KP or Toda-lattice hierarchy.

From mathematical point of view this ”integrable structure” is related to the underlying module space structures whose topological characteristics can be computed as correlation functions in a corresponding topological theory. Moreover, almost all what is known about higher-dimensional (topological) theories relies upon corresponding structures of module spaces of the target-space theory. However, it is not still clear how to derive integrable structure directly from the properties of corresponding module spaces. Below we will review “integrable” approach to the case of simplest 2d gravity theories: $(p,q)$ models interacting with 2d gravity and try to clarify their features which should allow one to find parallels with the less trivial higher-dimensional cases.

It is necessary to point out that we are still dealing with two different problems in string theory. Starting from the end, the second one is related with the properties of integrable systems describing (or hypothetically describing) string theory. The success in $c \leq 1$ case here is mostly related to the fact that one needs to work with the simplest hierarchy of integral equations (one-component KP or Toda-lattice) where lots of useful facts were known in advance. The less trivial cases (multicomponent hierarchies, hierarchies with non-commuting flows etc) are either much less known or even not considered at all in the literature. However, the first problem is even more fundamental - what is the sense of the integrable equations from the first principles of string theory. At the moment, there exists a lot of different approaches (or languages) more or less useful when understanding this or that group of facts but neither of them gives complete understanding of the sense of the equations in the space of coupling constants.

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1 which are usually called topological 2d gravity plus topological matter, though this is not too informative and complete definition

2 and looking for in more complicated cases

3 or Baker-Akhiezer function if normalized (see below)
2 String equation and Heisenberg algebra

First, we will review the problem in the form related to the search of solutions to non-perturbative gravity interacting with the \((p,q)\)-matter, which still exists in a not satisfactory form except for trivial \((p,1)\) “topological” case (see [9, 10, 11]) where it can be represented in terms of matrix model in external matrix field.

It is well-known [3], that it can be reduced to a problem of description of a particular representation of the Heisenberg algebra. Indeed, consider the representation of the Heisenberg operators, satisfying the string equation \([\hat{P}, \hat{Q}] = 1\) in the “momentum” space

\[
\hat{P} = \lambda \\
\hat{Q} = \frac{\partial}{\partial \lambda} + Q(\lambda)
\]  

(1)

From the point of view of the KP hierarchy, we will also add some additional requirements on the “spectral parameter” implying that

\[
\lambda = \mu^p
\]  

(2)

then \((p, q)\) models correspond to the case where \(Q(\lambda)\) should be a polynomial of \(\mu\) of degree \(q\) [5], (while the corresponding wave functions should have specific asymptotics when \(\mu \to \infty\)).

Wave functions of this problem appear to be the Baker-Akhiezer functions of the corresponding integrable system and when acting on wave functions conditions (1) get the form of the Kac-Schwarz equations [6, 7]:

\[
\lambda \varphi_i(\mu) = \sum_j W_{ij} \varphi_j(\mu) \\
\hat{A} \varphi_i(\mu) = \sum_j A_{ij} \varphi_j(\mu)
\]  

(3)

where

\[
\lambda = W(\mu) \sim \mu^p \equiv W^{(W,Q)}(\mu) \\
A^{(W,Q)}(\mu) = N^{(W,Q)}(\mu) \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} [N^{(W,Q)}(\mu)]^{-1} = \\
= \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} - \frac{1}{2 W'(\mu)^2} W''(\mu) + Q(\mu)
\]  

(4)

The standard way to construct wave functions of the theory is to define the Fock vacuum by

\[
\hat{A} \Psi_0 = 0
\]  

(5)

with an obvious solution
\[ \Psi_0 = \sqrt{W'(\mu)} \exp \int QdW \]  

and the corresponding \( \tau \)-function is a determinant projection of higher states

\[ \Psi_n \sim W^n \Psi_0 \]  

(7)

to the states with a canonical asymptotics

\[ \varphi_i(\mu) \to \mu \to \infty \mu_i - 1 \]  

(8)

forming the conventional basis in the space of wave functions - the point of infinite-dimensional Grassmannian.

The only simple case arises when the Kac-Schwarz equations (3) have trivial solution, i.e. when \( p = 1 \).

Starting from normalization \( \varphi_1(\mu) = 1 \) (corresponding to \( \Psi_0 = \exp \int Qd\mu \)), and using first of eqs.(3) one can always get \( \Psi_n = \mu^n \exp \int Qd\mu \to \varphi_i(\mu) = \mu_i - 1 \) exactly. Then the second condition of (3) is fulfilled automatically for any \( Q(\mu) \).

However, one can see that the corresponding solutions are not “physically” trivial in the sense that they are related to more meaningful solutions by a kind of Fourier transformation. Indeed, it has been observed \[10, 11\] that the system of equations (3) possesses a duality symmetry which relates \((p,q)\) to \((q,p)\) solution. The duality transformation for the Baker-Akhiezer functions looks like

\[ \psi^{(P,Q)}(z) = [P'(z)]^{1/2} \int dQ \; e^{P(z)Q(x)} \psi^{(Q,P)}(x)[Q'(x)]^{-1/2} \]  

(9)

and it can be also written for the basis vectors in the Grassmannian

\[ \phi_i(\mu) = [W'(\mu)]^{1/2} \exp(-S_{W,Q}\Big|_{x=\mu}) \int dM_Q(x) f_i(x) \exp S_{W,Q}(x,\mu) \]  

(10)

with

\[ dM_Q(x) = dx \sqrt{Q'(x)} \]

\[ S_{W,Q}(x,\mu) = - \int_x^\mu WdQ + W(\mu)Q(x) \]  

(11)

and for the partition functions

\[ \tau^{(W,Q)} [M] = C[V,M] \int DX \tau^{(Q,W)} [X] \exp \left\{ Tr[1/2 \log Q'(X) + \int_X^X W(z)dQ(z) + W(M)Q(X)] \right\} \]  

(12)

(here, better to consider normalized partition function \( \tau^{(W,Q)} \to Z^{(W,Q)} \to \Psi_{BA}^{(W,Q)}(t_k - \frac{1}{k} Tr M^{-k}) \)). It makes possible to obtain solutions for nontrivial models – topological \((p,1)\) models \[4\] and their Landau-Ginzburg deformations \[3\].
\[ \varphi_i(\mu) = (p\mu^{p-1})^{1/2} \exp\left(-\sum t_k \mu^k\right) \int dx \ x^{i-1} \exp(-V(x) + x\mu^p) \] (13)

which are dual to \((p,1)\) model in the above sense.

Here, we immediately run into a puzzle: how to interpret this from the point of view of quantization theory. Indeed, the duality transformation \((10)\) is nothing but a transformation from \(\hat{p}\) to \(\hat{q}\) quantization procedure or from one to another representation of quantum algebra and as it is well-known the quantization should be independent of this. From this point of view \((p,1)\) and \((1,p)\) or trivial theory should be equivalent. On the other hand we know that the partition functions for \((p,1)\) theories are nontrivial and correspond to some well-known topological theories (twisted \(N = 2\) Landau-Ginzburg theories) interacting with topological gravity \((2,1)\) model corresponds to pure topological gravity and generates intersection indices on module spaces of Riemann surfaces with punctures). Thus, there should be a way to extract all this “topological” information from a “dual” partition function \(\tau^{(1,p)} \equiv 1\). We will return to this problem below.

3 Flows in the space of solutions

Now let us discuss briefly how one can try to construct explicitly less trivial solutions. First, let us mention that the corresponding projection

\[ Pr : \Psi_{n-1} \to \varphi_n \] (14)

will be highly nontrivial (singular) in this case, though the corresponding basis vectors \(\Psi_n\) span the whole space.

Another option is to start from above integral formulas (see also [11]). One can try to construct more or less explicit representation for complicated solutions starting from known ones and using a sort of “dressing” operator. This way is certainly not very useful from technical point of view and does not lead us to a final result, but it demonstrates some valuable properties of these solutions and in particular gives some observations close to that of [12].

The main formula can be derived from a simple fact that some sort of “dressing transformations” just maps one solution of \((3)\) to another.

\[ \varphi_i(z|t) = \exp \left( \sum \frac{p}{p+1} \delta_{k,p+1} \right) \exp \left( \sum C_{ij} z^i \hat{A}^j \right) \]

\[ = (pz^{p-1})^{1/2} \exp\left(-\sum t_k z^k\right) \exp \left( \sum C_{ij} \lambda^i \left( \frac{\partial}{\partial \lambda} \right)^j \int dx \ x^{i-1} \exp\left(-\frac{x^{p+1}}{p+1} + x\lambda\right) \] (15)

This formula has a simple quantum mechanical interpretation. From the point of view of the Heisenberg algebra representation it can be considered as a quantum mechanical matrix element of the following form

One can easily recognize in this formula an element of \(W_\infty\) in the sense of [8].
\[ \langle q_1 | \exp H(p, q) | q_0 \rangle \] (16)

with

\[ H(p, q) = \sum C_{ij} p^i q^j \] (17)

Thus, we have obtained a sort of path integral representation for a nontrivial solution (instead of a trivial solution in the topological case). An interesting question is if it can be computed via localization or other technique and does it have a solution for finite-dimensional matrices \( C_{ij} \).

### 3.1 Perturbation expansion

Now we are going to present few explicit examples of the formula (15), computed in perturbation theory. First, let us start with “expansion” of the matrix \( C_{ij} \) into different pieces which would have different physical meaning

\[ \sum C_{ij} x^j \equiv \Omega_i(x) \] (18)

We are going to consider the contribution of the two first terms in the sum (18), taking them in the following form

\[ \Omega_0(x) = -V(x) + \frac{x^{p+1}}{p+1} = -\sum_{j=1}^{p} v_j x^j \]

\[ \Omega_1(x) = \frac{1}{F'(x)} = cx^q \] (19)

One can demonstrate that the first one corresponds to the Landau-Ginzburg deformation of the potential (13) while the second one brings to a reparameterization in the space of fields and gives a nontrivial co-ordinate \( Q(X) \) in the sense of [12].

Indeed

\[ \exp \left( \frac{\partial}{\partial \lambda} \right)_j \int dxx^{i-1} \exp \left( -\frac{x^{p+1}}{p+1} + \lambda x \right) = \]

\[ = \int dxx^{i-1} \exp \left( -\frac{x^{p+1}}{p+1} + \Omega_0(x) + \lambda x \right) = \int dxx^{i-1} \exp \left( -V(x) + \lambda x \right) \] (20)

what corresponds to (13), and

\[ \exp \left( \frac{\partial}{\partial \lambda} \right) + \lambda \frac{\partial}{\partial \lambda} \right) \int dxx^{i-1} \exp \left( -\frac{x^{p+1}}{p+1} + \lambda x \right) = \]

\[ \int dxx^{i-1} \exp \left( -\frac{x^{p+1}}{p+1} \right) \exp \left( \Omega_0(x) + \Omega_1(x) \frac{\partial}{\partial x} \right) \exp \lambda x \] (21)

Then, using that
\[
\exp \left( \Omega_0(x) + \Omega_1(x) \frac{\partial}{\partial x} \right) = \exp B \exp \Omega_1 \partial
\]

with

\[
B = \int_x^{F^{-1}(1+F(x))} \frac{d\xi \Omega_0(\xi)}{\Omega_1(\xi)}
\]

\[
F'(x) = \frac{1}{\Omega_1(x)}
\]

and choosing \( F(x) = \frac{x^{1-q}}{\epsilon(1-q)} \), \( \Omega_0 \to \Omega_0 + \frac{\epsilon}{2}x^{q-1} \), one gets

\[
\exp \left( \Omega_0(x) + \Omega_1(x) \frac{\partial}{\partial x} \right) \exp \lambda x = \exp B \exp \lambda Q(x) \sim \sqrt{1 + \epsilon x^{q-1}} \exp \lambda Q(x)
\]

where

\[
Q(x) = (x^{1-q} + \epsilon(1-q))^{1/\epsilon} = x + \epsilon x^q + ...
\]

We see that this is indeed an infinitesimal reparameterization of “spectral co-ordinate” \( X \to Q(X) = X + \epsilon X^q + ... \).

### 3.2 Analytic continuation. Example of discrete matrix model

So, we see that the formula (15) is consistent with what we know about the exact solutions perturbatively. It means that one can consider non-trivial solutions as perturbations over trivial topological ones. However, the problems runs into the difficulties of analytic continuation and the most trivial example to demonstrate this is a discrete matrix model [13].

Indeed, when computing the integral

\[
\int DH \exp \left( -Tr \sum t_k H^k \right)
\]

there exists an essential difference between \( t_k = \frac{1}{2} \delta_{k,2} \) and “double-scaling” cases, considered in [14] (see also references therein).

In the first case, due to special properties of the Hermite polynomials (with measure \( \frac{1}{\sqrt{2\pi}} H^2 \)) which have rather simple integral representation there exists a GKM-like determinant formula which means that this case is actually quite similar to the topological \((p,1)\) models. The matrix integral computed in such a way gives a generation function for the correlators only in the Gaussian discrete matrix model, and that means that the Gaussian matrix model is an example of the simplest topological theory.

In contrast, considered “double-scaling” limits can be obtained from [24] only as a highly-nontrivial analytic continuation. For example, formulae from [13] look like
\[ \sum_{n>2, i_k>0} C_{i_3...i_n}(t_2)t_3^{i_3}...t_n^{i_n} \]  

where \( C_{i_3...i_n}(t_2) \) are in general non-analytic functions of \( t_2 \), like

\[ \int DH e^{-t_2 \text{Tr} H^2} = \int \prod_{i,j} dH_{ij} e^{-t_2 \sum_{i,j} H_{ij} H_{ji}} \sim t_2^{-\frac{N^2}{2}} \]  

The partitions functions for generic 2d gravity models will be in contrast non-analytic functions of higher times, for example \( t_4, t_6 \) etc.

### 3.3 BA-function computation

Let us compute the quasiclassical expansion for the Baker-Akhiezer function. We start with the \((p,1)\) case.

\[
\Psi(z,t) = \sqrt{p} z^{p-1} \int dx \exp (-V(x) + x z^p) = \\
= \sqrt{p} z^{p-1} \exp (-V(\mu) + \mu z^p) \int dx \exp \left( -\frac{1}{2} V''(\mu) (x-\mu)^2 - \frac{1}{3!} V'''(\mu) (x-\mu)^3 - ... \right) = \\
= \sqrt{p} z^{p-1} \exp (-V(\mu) + \mu z^p) \int d\xi \exp \left( -\frac{1}{2} \xi^2 - \frac{1}{3!} \frac{W''(\mu)}{W'(\mu)^{3/2}} \xi^3 - ... - \frac{1}{n!} \frac{W^{(n-1)}(\mu)}{W'(\mu)^{n/2}} \xi^n - ... \right)
\]

The integral results in

\[
\langle \exp \sum y_k \xi^k \rangle = \langle \sum P_m(y) \xi^m \rangle = \sum P_{2m}(y) \langle \xi^{2m} \rangle
\]

with

\[
y_1 = y_2 = 0 \\
y_n = - \frac{1}{n!} \frac{W^{(n-1)}(\mu)}{W'(\mu)^{n/2}}, \quad n > 2
\]

\( \langle ... \rangle \equiv \int d\xi \exp \left( -\frac{1}{2} \xi^2 \right) ... \)  

Computing gaussian integrals one gets

\[
\langle \xi^{2m} \rangle = (2m - 1)!!
\]  

and using that

\[
V(\mu) - \mu W(\mu) = - \sum_{-\infty}^{p+1} t_k z^k \\
\mu = \frac{1}{p} \sum_{-\infty}^{p+1} k t_k z^{k-p}
\]
\[
\frac{1}{W'(\mu)} = \frac{1}{p^2} \sum_{-\infty}^{p+1} k(k-p) t_k z^{k-2p}
\]

we get

\[
\Psi(z,t) = \sqrt{\frac{p^2}{p^2-1}} W'(\mu) \exp \left( -\sum t_k z^k \right) \left[ \frac{5}{12} W'' W - \frac{1}{8} W''' W^2 + ... \right]
\]

The terms in the square brackets behave like \( \frac{1}{z^{p+1}} \) and they are dependent on the first \( p \) terms given by the expansion of the pre-factor, giving the only interesting contribution

\[
\Psi(z,t) = \sqrt{\frac{p^2}{p^2-1}} \sum_{k=-\infty}^{p+1} \frac{k(k-p)}{p^2} t_k z^{k-p-1} \exp \left( -\sum t_k z^k \right) + ...
\]

The Baker-Akhiezer functions relation:

\[
\Psi(\tilde{t},\mu) = \exp \sum_{\tilde{t}} \mu \tilde{t} = \exp \sum t_k z^k
\]

where first sum is finite while the second one - infinite. The set of times \( \{\tilde{t}\} \) is simply related with the coefficients of the “superpotential” \( W(\mu) = \sum_{j=0}^{p} v_j \mu^j \)

\[
\tilde{t}_k = -\frac{1}{k(1-k)} \text{Res} \mu^{1-k} dW(\mu) = \frac{1}{k} v_j \delta_{j,k+1}
\]

4 Mirror symmetry and pq-duality

Now let us briefly make some comments on relation between the quantization scheme we advocated above for \( c \leq 1 \) theories and a popular question of mirror symmetry (see for example [17, 18] and references therein). One may hope that the parallels we will discuss below can shed light on possible appearance of the integrable hierarchies in the “higher-dimensional” theories.

First, the simple observation is that the quasiclassics distinguishes \( p \) and \( q \) from a symmetric formulation - indeed the classical limit depends on what we call an area operator in the theory (\( \beta_+ = \sqrt{\frac{2p}{q}} \) or \( \beta_- = \sqrt{\frac{2q}{p}} \)). In such case we have a sort of mirror map between \( (p,q) \) and \( (q,p) \) theory similar to \( R \rightarrow \frac{1}{R} \) in the case of \( c = 1 \) theories. In general, if we have two mirror manifolds only quantum theories are equivalent (not classical ones which depend on configuration space) and two different classical limits correspond to different spaces. It is well known already when studying classical limit of corresponding 2d conformal field theories, one has to fix either \( p > q \) or \( p < q \) in the \( (p,q) \) model and only one “screening” survives in the classical limit.

In the classical limit instead of \( [\hat{P}, \hat{Q}] = 1 \) we have the Poisson bracket

\[
\{W, Q\} = 1
\]

\(^5\)The most simple and illustrative example is the same effect in the WZNW model where only one screening has nice classical interpretation \([6]\)
which is actually generated by

\[ \{ z, t_1 \} = 1 \]
\[ \{ \tilde{z}, \tilde{t}_1 \} = 1 \] (39)

(where \( z^p = W(\mu) \) and \( \tilde{z}^q = Q(\mu) \), see [13, 16] etc. For trivial \((1, p)\) topological theories \( \tilde{z} = \mu \).

From this point of view what we consider is a quantization of a symplectic manifold

\[ \omega = \delta W \land \delta Q \] (40)

(in the simplest case)

\[ \omega = \delta z \land \delta t_1 \] (41)

and we can consider it along the lines [18].

The corresponding connection is “action” [11]

\[ S = \int WdQ + S_0 \]
\[ ds = \delta W \land \delta Q \] (42)

and \( S_0 \) parameterizes an “initial point”. Now, it is obvious that in the proposed quantization scheme the set of coupling constants depends on the way of quantization, so does the solutions (potentials) of the hierarchy, \( \tau \)- or the BA function etc. This can be easily seen already on the example of (35) and (36).

5 Conclusion

In these notes we have tried to present some ideas how the results of formulation of non-perturbative string theory in terms of hierarchies of integrable equations can appear through “canonical” way of quantization. One can hope, that this way will bring us to more understanding of the problem what is second-quantized string theory and why does this quantum theory possess a structure of hierarchies of classical integrable equations.

However, this “canonical” way of quantization is still very far from being completed. First, even in the simplest case of \( c \leq 1 \) theories we do not have detailed explanation of the geometry underlying the Heisenberg string equation. The fact that a generation function for correlators in string theory appears as a wave function in the second-quantized theory gives an analogy with a similar effect in the Chern-Simons theory.

The main problem is still how to generalize the language of integrable hierarchies to the higher-dimensional theories(like topological strings in the Calabi-Yau backgrounds, critical strings etc). The basic moment for the integrable hierarchies is the appearance of spectral curve via Miwa transformation. The dependence of the partition function of “Miwa times” should be identified with the dependence of
the partition function of higher-dimensional theory on the “homology co-ordinate”, or put differently on
the co-ordinate in the tangent bundle to the module space. Then, the “classical” component of the time
variables - an element of the finite-dimensional small phase space should be identified with the co-ordinate
on the module space itself.

Such way of thinking immediately leads us to an idea that the second-quantized string theory should
be based on the quantum module space. This object naturally appears in the frames of the Chern-Simons
theory (see [22] and references therein). This problem certainly deserves further investigation.

Let us finally add few comments about holomorphic anomaly. The “quasiclassical” \( \tau \)-function obeys
a homogeneous relation

\[
\sum t_j \frac{\partial}{\partial t_j} \log \tau_0 = 2 \log \tau_0
\]

spoilt by the contribution of the one-loop correction, having the form, for example, for the \((2, 1)\) theory

\[
\sum t_j \frac{\partial}{\partial t_j} \log \tau - 2 \log \tau = -\frac{1}{24}
\]

The similar expressions appear when one considers the logarithm of the partition function for the higher-
dimensional theories [21] and this should mean that the expression (44) should have a similar nature.

There is certainly a lot of other open questions. We are going to return to them in a separate
publication.

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