QCD Factorization for Quarkonium Production in Hadron Collisions at Low Transverse Momentum

J.P. Ma\(^1\), C. Wang\(^3\)

\(^1\) Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735, Beijing 100190, China
\(^2\) Center for High-Energy Physics, Peking University, Beijing 100871, China
\(^3\) School of Physics, Peking University, Beijing 100871, China

Abstract

Inclusive production of a quarkonium \(\eta_{c,b}\) in hadron collisions at low transverse momentum can be used to extract various Transverse-Momentum-Dependent (TMD) gluon distributions of hadrons, provided the TMD factorization for the process holds. The factorization involving unpolarized TMD gluon distributions of unpolarized hadrons has been examined with on-shell gluons at one-loop level. In this work we study the factorization at one-loop level with diagram approach in the most general case, where all TMD gluon distributions at leading twist are involved. We find that the factorization holds and the perturbative effects are represented by one perturbative coefficient. Since the initial gluons from hadrons are off-shell in general, there exists the so-called super-leading region found recently. We find that the contributions from this region can come from individual diagrams at one-loop level, but they are cancelled in the sum. Our factorized result for the differential cross-section is explicitly gauge-invariant.

1. Introduction

Theoretical predictions of the inclusive quarkonium production at large transverse momentum in hadron collisions can be made by using the standard collinear factorization of QCD. Using the predictions one can extract from experimental results the gluon distribution functions of initial hadrons. These distribution functions are important for making predictions of other processes and for providing the information about inner structure of initial hadrons. However the information is limited because the extracted gluon distributions are one-dimensional. It is possible to extract three-dimensional gluon distributions, called as Transverse-Momentum-Dependent (TMD) gluon distributions, by using processes involving small transverse momenta. For this purpose, one needs to establish TMD factorizations.

In TMD factorization nonperturbative- and perturbative effects in a process are consistently separated. TMD parton distributions are defined with QCD operators and represent the separated nonperturbative effects. TMD quark distributions can be extracted from processes of Drell-Yan and Semi-Inclusive DIS. For these two processes, TMD factorizations have been established in [1, 2] and in [3, 4]. It is noted that TMD factorization has been first established for inclusive \(e^+e^-\)-annihilations into two nearly back-to-back hadrons in the seminal work in [5], where only TMD parton fragmentation functions are involved. For extracting TMD gluon distributions it is suggested to use processes in hadron collisions like Higgs-production [6, 7], quarkonium production [8], two-photon production [9], the production of a quarkonium combined with a photon [10] and double-quarkonium production [11]. In [12, 13] the TMD distribution of linearly polarized gluons inside a nucleus and its phenomenology have been studied.
TMD factorization for the suggested processes has been derived only at tree-level, except the process of Higgs- and quarkonium production. At one-loop level, TMD factorization has been examined for Higgs-production in [6] and for the production of $\eta_c, \eta_b$ in [14]. In these studies one takes the incoming gluons from the initial hadrons as on-shell and spin-averaged. The corresponding perturbative coefficient is determined at one-loop. Since only on-shell gluons with spin averaged are considered, one examines in fact the TMD factorization for the contribution from the scattering of unpolarized gluons coming from unpolarized hadrons. By taking the transverse momenta of incoming gluons into account, the gluon as a parton from an unpolarized hadron can be linearly polarized according to [15]. Certainly one can use on-shell-gluon scattering as in the studies of [6, 14] to study the factorization of the contribution from linearly polarized gluons at one-loop. But this is difficult because one needs to study the on-shell gluon scattering at three-loop, as discussed in [14]. The factorization for the processes with polarized hadrons has not been examined beyond tree-level.

We will use the subtractive approach to examine TMD factorization for the inclusive production of a $^1S_0$-quarkonium $\eta_Q$, which is $\eta_c$ or $\eta_b$. The approach is based on diagram expansion at hadron level and explained in [16, 17, 18]. In this approach the nonperturbative effects can be systematically subtracted in terms of diagrams. Using the approach TMD factorization for Drell-Yan processes has been examined [19], where TMD quark distributions are involved. In the case with TMD gluon distributions the situation is rather subtle, e.g., there can be super-leading-power contributions related to gluons as found in [16], and TMD factorization can be violated in certain processes as shown in [20, 21, 22]. We notice here that gluons from initial hadrons participating in a hard scattering are in general off-shell, i.e., the gluons have momenta slightly off-shell and unphysical polarizations. This brings up complications in examining the factorization and for obtaining gauge invariant results. The complications do not appear in [6, 14]. In the subtractive approach the complications can be correctly addressed.

In our work, we take the initial hadrons as arbitrarily polarized. We can show that at one-loop level the factorization holds and is explicitly gauge invariant. The contributions from the super-leading region do not appear at tree-level in our case, but they do appear at one-loop level. The nonzero contributions come from two different sets of diagrams. They are canceled in the sum. Our result shows that there is only one perturbative coefficient for all various contributions involving different TMD gluon distributions. The coefficient is determined at one-loop.

The above discussions are mainly relevant to the initial hadrons. For the nonperturbative effects related to the quarkonium in the final state, one can employ the factorization with nonrelativistic QCD(NRQCD) proposed in [23]. A quarkonium can be taken as a bound state of a heavy quark $Q$ and a heavy anti-quark $\bar{Q}$. The heavy quark or heavy anti-quark moves with a small velocity $v$ in the rest frame of the quarkonium. For the production of $\eta_Q$, the production rate at the leading order of $v$ can be written as a product of the production rate of a $Q\bar{Q}$ pair in color- and spin singlet with a NRQCD matrix element. The NRQCD matrix element characterizes the transmission of the produced $Q\bar{Q}$ into $\eta_Q$. In this work we will also add the correction at the next-to-leading order of $v$ in NRQCD factorization at tree-level, since the correction and the one-loop correction can be of the same importance. We notice here that in NRQCD factorization for a $P$-wave quarkonium one needs to consider not only the color-singlet $Q\bar{Q}$ pair, but also the color-octet $Q\bar{Q}$ pair [23]. Taking the color-octet $Q\bar{Q}$ pair into account, TMD factorization for the production of a $P$-wave quarkonium is violated [24]. It is noted that NRQCD factorization with the color-octet $Q\bar{Q}$ pair is also violated at two-loop level. But this violation can be avoided by adding gauge links in NRQCD color-octet matrix elements [25].

Our work is organized as in the following: In Sect.2. we study TMD factorization at tree-level. Notations are introduced. In Sect.3. we study TMD factorization at one-loop level. We will show that one needs to introduce a soft factor to complete TMD factorization. The mentioned complications from
unphysical incoming gluons will be explained in detail. In Sect.4. we will give our main result of TMD factorization for the differential cross-section with arbitrary hadrons in the initial state. The differential cross-section is given in detail in the case that one of the initial hadrons is unpolarized and another one is of spin-1/2. Sect.5. is our summary.

2. TMD Factorization at Tree-Level

In this section we first introduce notations and TMD gluon density matrix or distributions in Subsection 2.1. In Subsection 2.2. we consider the tree-level contribution with one-gluon exchange. The contribution from two-gluon exchange is studied in Subsection 2.3.

2.1. Notations and TMD Gluon Distributions

For our purpose it is convenient to use the light-cone coordinate system, in which a vector $a^\mu$ is expressed as $a^\mu = (a^+, a^-, \bar{a}_\perp) = ((a^0 + \bar{a}^3)/\sqrt{2}, (a^0 - \bar{a}^3)/\sqrt{2}, a^1, a^2)$ and $a^2_\perp = (a^1)^2 + (a^2)^2$. We introduce two light-cone vectors: $n$ and $l$. With these light-cone vectors one can build two tensors in the transverse space. The vectors and tensors are:

$$ n^\mu = (0, 1, 0, 0), \quad l^\mu = (1, 0, 0, 0), \quad g_{\mu\nu}^{\perp} = g^{\mu\nu} - n^\mu l^\nu - n^\nu l^\mu, \quad \epsilon_{\perp}^{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} l_\mu n_\nu, \quad (1) $$

with $\epsilon_{\perp}^{12} = -\epsilon_{\perp}^{21} = 1$.

We consider the process

$$ h_A(P_A) + h_B(P_B) \rightarrow \eta_Q(q) + X, \quad (2) $$

where $\eta_Q$ stands for $\eta_c$ or $\eta_b$. It is a $1S_0$-quarkonium consisting of a heavy $QQ$ pair with $Q = c$ or $Q = b$. The momenta in Eq.(2) are given by:

$$ P^\mu_A \approx (P^+_{\perp A}, 0, 0, 0), \quad P^\mu_B \approx (0, P^+_{\perp B}, 0, 0), \quad q^\mu = (q^+, q^-, \bar{q}_\perp) = (xP^+_{\perp A}, yP^+_{\perp B}, \bar{q}_\perp), \quad (3) $$

where we have neglected the masses of hadrons, i.e., $P^+_{\perp A} \approx 0$ and $P^+_{\perp B} \approx 0$. The mass of the quarkonium is $M_Q$ and the invariant mass of the initial hadrons is $s \approx 2P^+_{\perp A}P^+_{\perp B}$. At the leading order of the velocity expansion one has $M_Q \approx 2M_Q$. We will also use the notation $q^2 = Q^2 = M_Q^2$. We are interested in the kinematical region of $q_\perp/Q \ll 1$. In this region, one can establish TMD factorization to express the differential cross-section in terms of TMD gluon distributions.

TMD gluon distributions are defined with QCD operators. To define them for $h_A$ we introduce the gauge link along the direction $u^\mu = (u^+, u^-, 0, 0)$ with $u^+ \ll u^-$:

$$ \mathcal{L}_u(z) = P \exp \left( -ig_s \int_{-\infty}^{0} d\lambda u \cdot A(\lambda u + z) \right), \quad (4) $$

where the gluon field $A^\mu$ is in the adjoint representation. The gluonic density matrix of $h_A$ is defined as:

$$ \Gamma_A^{\mu\nu}(k) = \frac{1}{xP^+_{\perp A}} \int \frac{d\xi - d^2\xi_{\perp}}{(2\pi)^3} e^{-i\xi^k} \langle h_A | \left( G^{+\mu}(\xi)\mathcal{L}_u(\xi) \right)^a \left( \mathcal{L}_u^+(0)G^{+\nu}(0) \right)^a | h_A \rangle, \quad (5) $$

with $\xi^\mu = (0, \xi^-, \xi_{\perp})$ and $k^\mu = (xP^+_{\perp A}, 0, \bar{k}_{\perp})$. $G^{\mu\nu}$ is the field strength tensor. The definition is gauge-invariant. The gauge links in Eq.(5) are taken off light-cone. This is to avoid light-cone singularities if we take $u^+ = 0$. In this work we will use Feynman gauge. In this non-singular gauge fields at infinite space-time are zero. If one works with a singular gauge, one needs to implement gauge links along the transverse direction at $\xi^- = -\infty$ to make the definition gauge-invariant [26, 27].
The density matrix is defined for hadrons with an arbitrary spin. In general one can decompose the
density matrix with scalar functions. These functions are TMD gluon distributions. Here we briefly
discuss the case for $h_A$ with spin-1/2. We assume that the spin of $h_A$ is described by the helicity $S_L$ and
the transverse spin $s_{\perp}^{\mu}$. At leading power or leading twist the indices $\mu$ and $\nu$ of the density matrix are
transverse. The decomposition reads[15]:

$$
\Gamma_{\mu\nu}^{(c)}(k) = \begin{pmatrix}
\frac{1}{2}g_{\mu\nu} f_g(x, k_{\perp}) + \frac{1}{2M_A^2} \left( k_{\perp}^{\mu} k_{\perp}^{\nu} + \frac{1}{2}g_{\perp}^{\mu\nu} k_{\perp}^2 \right) H_{\perp}^{(c)}(x, k_{\perp}) + \frac{1}{2} S_L \left[ -i \epsilon_{\perp}^{\mu\nu} \Delta G_{\perp}(x, k_{\perp}) \right] \\
- \frac{1}{M_A^2} \tilde{k}_{\perp}^{\mu} \Delta H_{\perp}^{(c)}(x, k_{\perp}) \\
- \frac{1}{L} \left( \tilde{k}_{\perp}^{\mu} s_{\perp}^{\nu} + \tilde{s}_{\perp}^{\mu} k_{\perp}^{\nu} \right) \left( \Delta H_{\perp}^{(c)}(x, k_{\perp}) - \frac{k_{\perp}^2}{2M_A^2} \Delta H_{\perp}^{(c)}(x, k_{\perp}) \right) \\
+ \frac{1}{M_A^2} \tilde{k}_{\perp}^{\mu} \tilde{k}_{\perp}^{\nu} k_{\perp} \cdot s_{\perp} \Delta H_{\perp}^{(c)}(x, k_{\perp})
\end{pmatrix},
$$

(6)

with $\tilde{k}_{\perp}^{\mu} = \epsilon_{\perp}^{\mu\nu} k_{\perp}^{\nu}$ and $\tilde{s}_{\perp}^{\mu} = \epsilon_{\perp}^{\mu\nu} s_{\perp}^{\nu}$. The TMD gluon distribution $f_g$ corresponds to the standard gluon
distribution function of an unpolarized hadron if we integrate out the transverse momentum formally.

By taking the transverse momentum into account, the gluon as a parton in an unpolarized hadron can
be linearly polarized indicated by $H_{\perp}$ in Eq.(6).

Similarly, one can define the gluon density matrix $\Gamma_{\mu\nu}^{(B)}(k)$ or TMD gluon distributions of $h_B$, where
one uses instead of $L_u$ the gauge link $L_v$ along the direction $v$ with $v^- \ll v^+$. In the following subsections
and Sect.3 we will set $u = n$ and $v = l$ for convenience to study the factorization. It is well-known that
there will be light-cone singularities in TMD gluon density matrices with the setting. But this will not
affect our analysis, one can always make in each step the substitution $n \rightarrow u$ and $l \rightarrow v$ for subtractions.
We will make the substitution in our final result.

Figure 1: (a): Tree-level diagram of one-gluon exchange. (b): The bubbles in the middle of Fig.1a stand
for the amplitude of $g^* g^* \rightarrow \eta_Q$.

2.2. TMD Factorization of One-Gluon-Exchange Contributions

At tree-level the contribution to the differential cross-section of Eq.(2) are from diagrams represented
by Fig.1a. It can be written as:

$$
\frac{d\sigma}{d^4q} = \frac{1}{2s(2\pi)^3} \int d^4k_A d^4k_B \delta(q^2 - 4m_Q^2)(2\pi)^4 \delta^4(k_A + k_B - q) M_{\alpha\beta}^{ab}(k_A, k_B) \left( \mathcal{M}^{cd}_{\delta\nu}(k_A, k_B) \right)^\dagger
$$
\begin{align}
\cdot \left[ \int \frac{d^4 \xi}{(2\pi)^4} e^{i \xi \cdot k_A} \langle h_A | A^{\alpha \beta}(0) A^{a,\alpha}(\xi) | h_A \rangle \right] \cdot \left[ \int \frac{d^4 \eta}{(2\pi)^4} e^{i \eta \cdot k_B} \langle h_B | A^{d,\nu}(0) A^{b,\nu}(\eta) | h_B \rangle \right].
\end{align}

In Eq. (7), the correlation function of gluon fields in the first \([\cdots]\) of the second line is represented by the lower bubble in Fig. 1a, the correlation function in the second \([\cdots]\) is represented by the upper bubble. The bubble in the left-middle part of Fig. 1a is the amplitude given by Fig. 1b, i.e., \(\mathcal{M}_{ab}(k_A, k_B)\) is the amplitude of \(g^*(k_A) g^*(k_B) \rightarrow \eta_Q\). At amplitude level, there is only one gluon exchanged between bubbles. There can be more exchanged gluons. The case of two-gluon exchange will be studied in the next subsection.

We will take the leading order of the small velocity expansion in NRQCD [23]. At the leading order, the quark \(Q\) and \(\bar{Q}\) carry the same momentum \(q/2\) in Fig. 1a. The broken line there is the cut. The cut cutting the lines of the heavy quark pair means also that we take the projection for the pair into the color-singlet \(1S_0\)-state. The projection is standard and can be found, e.g., in [20]. Under the above approximation, one easily finds the amplitude given in Fig. 1b:

\begin{align}
\mathcal{M}^{ab,\alpha\mu}(k_A, k_B) &= g_s^2 \delta^{ab} e^{\alpha\mu\sigma} k_{A\sigma} k_{B\rho} f_L(k_A, k_B), \\
f_L(0, 0) &= -\frac{\psi^*(0)}{m_Q^2 \sqrt{2N_c m_Q}},
\end{align}

where we introduced the form factor \(f_L\) for two-gluon fusion into \(\eta_Q\). The two gluons are in general off-shell. The explicit result of \(f_L(0, 0)\) for on-shell gluons is given. Here we use the wave function at the origin for the projection. In the final result it will be replaced with NRQCD matrix element.

In the kinematical region under our consideration the produced \(\eta_Q\) carries a small transverse momentum \(q_\perp \sim \lambda Q\) with \(\lambda \ll 1\). The leading power contributions arise when upper- and lower bubbles are jet-like functions. The power counting for the momenta carried by the gluons leaving the bubbles are:

\[ k_A^\mu \sim (1, \lambda^2, \lambda), \quad k_B^\mu \sim (\lambda^2, 1, \lambda, \lambda). \]

The power counting for the gauge vector field \(A^\mu\) in the correlation functions represented by the lower- or upper bubble is the same as for the momenta given in the above, respectively. Taking the tree-level result in Eq. (8) we have the leading order contribution in \(\lambda\) for the combination appearing in Eq. (7):

\begin{align}
\mathcal{M}_{ab}(k_A, k_B) A^{a,\alpha}(\xi) A^{b,\mu}(\eta) &= g_s^2 f_L(0, 0) \delta^{ab} \delta^{\alpha\mu} \left( k_{A\perp} A_{a,\alpha}(\xi) - k_{A\perp} A^{a,+,\alpha}(\xi) \right) \\
&\cdot \left( k_{B\perp} A_{b,\mu}(\eta) - k_{B\perp} A^{b,-,\mu}(\eta) \right) + \mathcal{O}(\lambda^3),
\end{align}

the leading order is at \(\lambda^2\). In the expansion one also needs to expand \(f_L(k_A^2, k_B^2)\) in \(\lambda\). This gives the factor \(f_L(0, 0)\) in the above. From the power counting, the so-called super-leading region can appear here in the amplitude if there is a contribution proportional to \(A^{a,+,\alpha}(\xi) A^{b,-,\alpha}(\eta)\). Such a contribution can be at order of \(\lambda^0\) or \(\lambda^1\). Usually, if we take the gluons from \(h_{A,B}\) as on-shell, as the explicit calculation in [14], one will not meet the super-leading regions, because the on-shell gluons are always transversely polarized. In Eq. (10) the contribution proportional to \(A^{a,+,\alpha}(\xi) A^{b,-,\mu}(\eta)\) is combined with two transverse momenta. Therefore, the super-leading region gives no contribution here. We also note that the introduced form factor \(f_L(k_A^2, k_B^2)\) in general is not gauge-invariant, but \(f_L(0, 0)\) is gauge-invariant obviously.

Since we are only interested in the leading order of \(\lambda\), the \(-\)-components of gauge fields in the correlation function of \(h_A\) can always be neglected. The \(-\)-components of momenta carried by these fields can always be neglected except in the correlation function. The latter results in that one can perform the integration over those \(-\)-components of momenta and the corresponding \(+\)-components of the space-time coordinate vectors in the correlation function trivially. Therefore, we introduce the
The amplitude introduced before Eq.(11) is:

\[ \frac{d\sigma}{d^4q} = \frac{\pi\delta(xys - Q^2)}{s} g_A^2 f_L^2(0,0) \int d^2k_A d^2k_B d^2k_{\perp} \delta^2(k_{A\perp} + k_{B\perp} - q_{\perp}) \delta^{ab} \delta^{cd} \epsilon_{\perp \alpha \mu} \epsilon_{\perp \beta \nu} \cdot \left[ \int \frac{d^3\xi}{(2\pi)^3} e^{i\xi \cdot k_A} \langle h_A | \hat{G}^{c*} \delta^{+} + \beta(0) \hat{G}^{a*} \alpha(\xi) | h_A \rangle \cdot \int \frac{d^3\nu}{(2\pi)^3} e^{i\nu \cdot k_B} \langle h_B | \hat{G}^{d*} \delta^{-\mu} + \nu(0) \hat{G}^{b*} \delta^{-\mu} (\nu) | h_B \rangle \right], \quad (11) \]

with \( k_A^- = 0 \) and \( k_B^+ = 0 \). Approximately one may write the two correlation functions in the second line as the gluon density matrix \( \Gamma_A \) and \( \Gamma_B \), respectively. However, in these correlation functions, \( \hat{G}^{\mu\nu} \) given by \( \hat{G}^{\mu\nu} = \partial^\mu G^{\nu} - \partial^\nu G^{\mu} \) is not exactly the field strength tensor operator. To obtain it, one needs to consider the contributions from the exchange of two- or more gluons.

Figure 2: (a) The diagram with two-gluon change. The middle bubble in the left part is the sum of diagrams given in (b) and (c). (b) The tree amplitude for \( g^* g^* \rightarrow \eta_Q \), in which all gluons are attached to the quark line. (c) The tree amplitude for \( g^* g^* \rightarrow \eta_Q \), in which a three-gluon vertex is involved.

2.3. TMD Factorization of Two-Gluon-Exchange Contributions

Now we consider the contribution in which two gluons come from \( h_A \) as in those diagrams given in Fig.2a. The contribution after neglecting the unimportant components of momenta and with the notation introduced before Eq.(11) is:

\[ \frac{d\sigma}{d^4q} = \frac{\pi}{s} \int d^3k_1 d^3k_2 d^3k_B \delta(q^2 - 4m_Q^2) \delta^4(k_1 + k_2 + k_B - q) \mathcal{M}_{a_1a_2b}(k_1, k_2, k_B) \cdot \left( \frac{1}{2} \right) \left[ \int \frac{d^3\xi_1 d^3\xi_2}{(2\pi)^6} e^{i\xi_1 \cdot k_1 + i\xi_2 \cdot k_2} \langle h_A | A^{c* \beta}(0) A^{a_1 \alpha_1}(\xi_1) A^{b* \alpha_2}(\xi_2) | h_A \rangle \right] \cdot \left( \mathcal{M}_{b_1b_2}(k_1 + k_2, k_B) \right) \left[ \int \frac{d^3\nu}{(2\pi)^3} e^{i\nu \cdot k_B} \langle h_B | A^{d* \mu}(0) A^{b* \mu}(\nu) | h_B \rangle \right], \quad (12) \]

The amplitude \( \mathcal{M}_{a_1a_2b}(k_1, k_2, k_B) \) is the sum of contributions from Fig.2b and Fig.2c. It is the amplitude for three-gluon fusion into \( \eta_Q \) with the exclusion of those diagrams in which the gluon 1 and gluon 2
combine into one gluon with the three-gluon vertex. Those diagrams are in fact included in the lower bubble. A factor $1/2$ has to be implemented to avoid double-counting.

The three-gluon amplitude given by Fig.2b and Fig.2c will be used extensively in our work. Hence, we give the detailed result for the expansion with the power counting in $\lambda$. In the expansion one should keep the $i\epsilon$ factors in propagators, because these factors have a physical meaning as we will see. The calculation of diagrams are straightforward, although it is tedious. But after the expansion the result takes a rather simple form for combinations with gauge fields appearing in Eq. (12). We have from Fig.2b and Fig.2c the following:

$$
\mathcal{M}_{a_1a_2b}(k_1, k_2, k_B) \approx -i g_s^3 f a_1a_2b f_L(0, 0) \int \frac{d^3\tilde{\xi}_1 d^3\tilde{\xi}_2 d^3\tilde{\eta}}{(2\pi)^9} e^{+i\tilde{\xi}_1 \cdot k_1 + i\tilde{\xi}_2 \cdot k_2 + i\tilde{\eta} \cdot k_B} A^{a_1, a_1}(\hat{\xi}_1) A^{a_2, a_2}(\hat{\xi}_2) A^{b, \mu}(\hat{\eta})
$$

where the terms come from the quark propagators in Fig.2b and Fig.2c. Those diagrams are in fact included in the lower bubble. The three-gluon amplitude given by Fig.2b and Fig.2c will be used extensively in our work. Hence, we give the detailed result for the expansion with the power counting in $\lambda$. In the expansion one should keep the $i\epsilon$ factors in propagators, because these factors have a physical meaning as we will see. The calculation of diagrams are straightforward, although it is tedious. But after the expansion the result takes a rather simple form for combinations with gauge fields appearing in Eq. (12). We have from Fig.2b and Fig.2c the following:

$$
\mathcal{M}_{a_1a_2b}(k_1, k_2, k_B) \approx -i g_s^3 f a_1a_2b f_L(0, 0) \int \frac{d^3\tilde{\xi}_1 d^3\tilde{\xi}_2 d^3\tilde{\eta}}{(2\pi)^9} e^{+i\tilde{\xi}_1 \cdot k_1 + i\tilde{\xi}_2 \cdot k_2 + i\tilde{\eta} \cdot k_B} A^{a_1, a_1}(\hat{\xi}_1) A^{a_2, a_2}(\hat{\xi}_2) A^{b, \mu}(\hat{\eta})
$$

where the $(B^- - \text{term})$ is proportional to $A^{b, -}(\hat{\eta})$. It is exactly cancelled in the sum of Fig.2b and Fig.2c. The above results are symmetric in exchange of the gluon 1 and 2. Taking the sum we obtain the two-gluon-exchange contribution from Fig.2a at the leading order of $\lambda$:

$$
\frac{d\sigma}{d^3q} = \frac{\pi \delta(xys - Q^2)}{s} g_5 f_3 f_L(0, 0) \int d^3k_1 d^3k_2 d^3k_B \delta^4(k_1 + k_2 + q - \eta) \epsilon_{\lambda\alpha\mu} \left( -\delta^{cd} \epsilon_{\beta\nu} \right) \left[ \int \frac{d^3\tilde{\eta}}{(2\pi)^3} e^{i\tilde{\eta} \cdot k_B} \langle h_B| \hat{G}^{d, -\nu}(0)|\hat{G}^{b, \mu}(\hat{\eta})|h_B \rangle \right] \left[ \int \frac{d^3\tilde{\xi}_1}{(2\pi)^3} e^{i\tilde{\xi}_1 \cdot k_1} \langle h_A| \hat{G}^{c, +\beta}(0)|f a_1a_2b \rangle \right]
$$

$$
\left[ A^{a_1, +}(\hat{\xi}_1) A^{a_2, -}(\hat{\xi}_2) - i \int d^3k_2 \frac{d^3\tilde{\xi}_2}{(2\pi)^3} e^{i\tilde{\xi}_2 \cdot k_2} \frac{1}{k_2^+ + i\epsilon} A^{a_2, +}(\hat{\xi}_2) \hat{G}^{a_2, +\alpha}(\hat{\xi}_1) \right] \frac{d\sigma}{d^3q}.
$$

We note here that the $\pm i\epsilon$'s in the denominators come from different places. The factor $+i\epsilon$ comes from the gluon propagators in Fig.2c and indicates that the interactions through the exchange of the collinear gluon are of the initial-state, while the factor $-i\epsilon$ comes from the quark propagators in Fig.2b and Fig.2c and indicates final-state interactions. In the sum the effects of final-state interactions are completely canceled and there is no term with the eikonal propagator $1/(k_{1,2}^+ - i\epsilon)$. The sum only contains those eikonal propagators $1/(k_{1,2}^+ + i\epsilon)$ representing initial-state interactions. This result is important here for the factorization. If the effects of initial-state- and final-state interactions exist simultaneously, they can be potential sources for breaking the factorization, as discussed in [20, 21], or one needs to introduce non-universal TMD gluon distributions for the factorization [28]. Then, the prediction power is lost.
Now we consider the product in the gluon density matrix \( \Gamma^{\mu \nu}_A \) defined in Eq.(5):

\[
\left( L_n^\dagger(\tilde{\xi}) G^{+\mu}(\tilde{\xi}) \right)^a = \hat{G}_a^{\mu} + i g_s f^{abc} A^b,+,\mu(\tilde{\xi}) A^c,\mu(\tilde{\xi}) - i g_s f^{abc} \int \frac{d^3 \tilde{\xi}_2}{(2\pi)^3} e^{i k \cdot (\tilde{\xi}_2 - \tilde{\xi})} A^{b,+,\mu}(\tilde{\xi}_2) \hat{G}_c^{+,\mu}(\tilde{\xi}) + O(g_s^2). \tag{15}
\]

Comparing this expression with Eq.(14), one can realize that the first term in Eq.(14) is the missing part for the field strength tensor \( G^{+\alpha} \) in the one-gluon-exchange contribution, and the second term forms a part of the gauge link. One may consider the exchange of more gluons to obtain full gauge links. Therefore, we can write the result of TMD factorization at tree-level as:

\[
\frac{d\sigma}{d^4 q} = \frac{xy \pi \delta(x y s - Q^2)}{2(N_c^2 - 1)} g_s^4 f_L^2(0,0) \int d^2 k_A \perp d^2 k_B \perp \frac{1}{2} \sigma^2 (k_A \perp + k_B \perp - q_\perp) \epsilon_\perp \alpha \mu \epsilon_\perp \beta \nu \Gamma^{\alpha \beta}_A (k_A) \Gamma^{\mu \nu}_B (k_B) \left( 1 + O(\alpha_s) + O(\lambda) \right) \tag{16}
\]

with \( k_A^\mu = (x P^+_A, 0, \vec{k}_A \perp) \) and \( k_B^\mu = (0, y P^-_B, \vec{k}_B \perp) \). In the above, the gluon density matrices are defined gauge-invariantly. \( f_L(0,0) \) is gauge invariant as discussed before. Hence, the result in Eq.(16) is gauge invariant. This result can be still represented by Fig.1a, where the lower bubble represents \( \Gamma^{\beta \alpha}_A \), the upper bubble represents \( \Gamma^{\mu \nu}_B \). The obtained result in Eq.(16) will be corrected beyond tree-level.

3. TMD Factorization at One-Loop-Level

One-loop correction comes from diagrams in which one has one-gluon-exchange in the middle part of Fig.1. The one-loop correction can be divided into two parts. One part is the real part, in which an additional gluon is exchanged in the middle part of Fig.1a crossing the cut. Another part is the virtual part, in which the exchanged gluon does not cross the cut. We will consider the two parts in the following subsections separately.

Figure 3: Diagrams of the real part of one-loop correction.

3.1. The Real Part

The real part is given by the diagrams given in Fig.3 where the bubbles in the middle represent the three-gluon- and two-gluon amplitudes explained in the last section. For the bubbles attached with three
gluons, we always identify the two gluons coming from the below to the bubbles in the middle part are the gluon 1 and the gluon 2, as specified in Fig.2c. With this identification, The total contribution of the real part is:

\[
\left(\frac{d\sigma}{dq}\right)_R = \frac{d\sigma}{dq} + \left(\frac{d\sigma}{dq}\right)_{h.c.} + \frac{d\sigma}{dq} \cdot \cdot \cdot .
\]

(17)

In the kinematical region of \( q_\perp \sim Q\lambda \) with \( \lambda \ll 1 \) the contributions at the leading order of \( \lambda \) come from the region of the gluon momentum which is collinear to \( P_A \), or to \( P_B \), and soft, with the standard power counting. We first consider the case that the exchanged gluon is collinear to \( P_A \) in each diagram.

The leading contribution from Eq.3a can be easily derived by using the leading result of the two-gluon amplitude in Eq.(10) and by taking the pattern of the momentum \( k \) of the exchanged gluon as \( k^\mu \sim (1, \lambda^2, \lambda, \lambda) \). The leading contribution from Eq.3a is:

\[
\left(\frac{d\sigma}{dq}\right)_3 = \frac{\pi\delta(xys-Q^2)}{s(N_c^2-1)} g_s^4 f_L^2 (0,0) \int d^3k_1 d^3k_B d^4k \cdot \frac{\delta^4(k_1+k_B-q-k)}{(2\pi)^4} \cdot \left(\frac{\Gamma_{\mu\nu}^{\alpha\beta}}{k_B}\right) \cdot \left(\begin{array}{c}
(k_1-k)^+ g_{\alpha\alpha_1} - n_{\alpha_1}(k_1-k)^\perp \\
(k_1-k)^+ g_{\beta_1} - n_{\beta_1}(k_1-k)^\perp
\end{array}\right) \cdot \left(\begin{array}{c}
k_B^\rho \\
-\frac{1}{(k_1-k)^2} g_{\rho\beta_1}
\end{array}\right) \\
\left(\begin{array}{c}
g_{\alpha_1}(k_1-k)^\perp \\
g_{\beta_1}(k_1-k)^\perp
\end{array}\right) \cdot \left(\begin{array}{c}
-2k_1-k)^\rho g^{\beta_1} + (k_1-k)^\beta g^{\rho\beta_1}
\end{array}\right) \cdot \left(\begin{array}{c}
\epsilon_\perp \cdot \epsilon_\perp \cdot \epsilon_{\cdot \cdot \cdot}
\end{array}\right) \cdot \cdot \cdot .
\]

(18)

In the above we have in fact included those contributions from exchange of all possible gluons between the upper bubble and the bubbles in the middle of Fig.3a. Therefore we have now in the above \( \Gamma_{\mu\nu}^\mu \) representing the upper bubble. This will be implicitly implied in the whole analysis. The \cdot \cdot \cdot stand for power-suppressed contributions which are neglected.

The power of \( g_s \) indicates that the contribution in Eq.(18) can be a part of the \( \mathcal{O}(\alpha_s) \)-correction in the tree-level factorization in Eq.(16). However, this needs to be examined. We note here that the \( \Gamma_{\mu\nu}^\mu \) represented by the lower bubble in Fig.1a is a jet-like correlation function, which is the sum of all diagrams in which parton lines carry the momenta collinear to \( P_A \). Because the exchanged gluon in Fig.3a is collinear to \( P_A \), the contribution from the exchanged gluon can be already included in the lower bubble \( \Gamma_{\mu\nu}^\mu \) entirely or partly. Therefore, the correct contribution to the \( \mathcal{O}(\alpha_s) \)-correction is only obtained after subtracting the corresponding contribution from \( \Gamma_{\mu\nu}^\mu \). If one simply takes the contribution from Fig.3a as the \( \mathcal{O}(\alpha_s) \)-correction in Eq.(16) without the subtraction, a double-counting happens.

Figure 4: Diagrams of the gluon TMD distribution for the subtraction
Beyond tree-level, the TMD gluon density matrix receives the contributions from diagrams in Fig. 4. In these diagrams, the double lines represent the gauge links. The contribution from Fig. 4 is:

\[
\Gamma_{A}(k_A) = \frac{1}{xP^+} \int \frac{d^4k^2d^4k_1}{(2\pi)^4} (-2\pi\delta(k^2)) \delta^3(k_A - k_1 + k)(i(k_A \cdot n g^{\alpha_1} - (k_1 - k)^\nu n_\nu) \\
-\frac{i}{(k_1 - k)^2 + i\varepsilon}(-g_{s}f_{ab1c}((-k - k_1)^{\alpha_1} g_\rho + (2k_1 - k)_\rho g^{\alpha_1} + (2k - k_1)^{\alpha_1} g_\rho) \\
-\frac{i}{(k_1 - k)^2 - i\varepsilon}g_s f^{ac1c}(k + k_1)^{\beta_1} g^{\beta_3} + (-2k + k_1)^{\rho} g^{\beta_3} + (k_1 - k)^{\beta_1}) \\
\int \frac{d^2\xi}{(2\pi)^3} e^{i\xi \cdot k_1} \langle h_A | A_{\beta_1}^a (0) A_{\alpha_1}^b (\xi) | h_A \rangle.
\]

(19)

Comparing Eq. (18) with Eq. (19), we find that the contribution from Fig. 4 takes a factorized form:

\[
\frac{d\sigma}{d^4q} = \frac{\pi\delta(xys - Q^2)}{s} \int d^3k_1 d^3k_B \frac{d^4k}{(2\pi)^4} \delta^4(k_1 + k_B + q - k)(-2\pi\delta(k^2)) \\
M^{a_1b_2}(k_1, -k, k_B) \left[ \int \frac{d^3\xi}{(2\pi)^3} e^{i\xi \cdot k_1} \langle h_A | A_{\beta_1}^a (0) A_{\alpha_1}^b (\xi) | h_A \rangle \right] \left( -ig_{s}f_{L}(0, 0) \delta^a c^1 \right) f^{a_2c_1} \\
\left\{ \left( \epsilon_{\perp \beta_1 \nu}(k_1 - k)^+ - n_{\beta_1} \epsilon_{\perp \alpha_1 \nu}(k_1 - k)^\sigma \right) \right\} \left( (-k_1 - k)^{\beta_1} g^{\alpha_2 \beta_1} + (2k_1 - k)^{\beta_1} g^{\alpha_2 \beta_1} + (2k_1 - k)^{\alpha_2} g^{\beta_1} \right) \\
\left\{ \frac{i}{(k_1 - k)^2 - i\varepsilon} + \cdots \right\}
\]

(20)

This result indicates that the contribution from Fig. 4 is already included in the gluon density matrix $\Gamma_A$ in Eq. (16). Therefore, it should be subtracted from the $O(\alpha_s)$-correction. This leads to that the contribution from Fig. 4 will not contribute to the $O(\alpha_s)$-correction.

Now we consider the contribution from Fig. 5. Taking the exchanged gluon as collinear to $P_A$ and the leading result for the two-gluon amplitude represented by the right bubble in the middle, the contribution reads:

\[
\frac{d\sigma}{d^4q} = \frac{\pi\delta(xys - Q^2)}{s} \int d^3k_1 d^3k_B \frac{d^4k}{(2\pi)^4} \delta^4(k_1 + k_B - q - k)(-2\pi\delta(k^2)) \\
M^{a_1b_2}(k_1, -k, k_B) \left[ \int \frac{d^3\xi}{(2\pi)^3} e^{i\xi \cdot k_1} \langle h_A | A_{\beta_1}^a (0) A_{\alpha_1}^b (\xi) | h_A \rangle \right] \left( -ig_{s}f_{L}(0, 0) \delta^a c^1 \right) f^{a_2c_1} \\
\left\{ \left( \epsilon_{\perp \beta_1 \nu}(k_1 - k)^+ - n_{\beta_1} \epsilon_{\perp \alpha_1 \nu}(k_1 - k)^\sigma \right) \right\} \left( (-k_1 - k)^{\beta_1} g^{\alpha_2 \beta_1} + (2k_1 - k)^{\beta_1} g^{\alpha_2 \beta_1} + (2k_1 - k)^{\alpha_2} g^{\beta_1} \right) \\
\left\{ \frac{i}{(k_1 - k)^2 - i\varepsilon} + \cdots \right\}
\]

(21)

In the above there are still some contributions at higher order of $\lambda$. These contributions need to be separated and neglected as those represented by $\cdots$. We note here that among diagrams in Fig. 5, the contribution from Fig. 5 is the most difficult to analyze.

To find the leading contribution we can use the leading order result of the three-gluon amplitude in Eq. (14). We expand the relevant combination in $\lambda$:

\[
M^{a_1b_2}(k_1, -k, k_B) \int \frac{d^3\xi}{(2\pi)^3} e^{i\xi \cdot k_1} A_{\beta_1}^a (\xi) A_{\alpha_1}^b (\xi) \\
= -g_{s}f_{a_1b_2} \frac{2A_0}{m_Q^2} \frac{1}{(2\pi)^6} \epsilon^{\xi \cdot k_1 + \bar{\xi} \cdot k_B} \xi_{\perp} G_{\rho}^{\sigma_\rho} \left( \frac{1}{k^+ + i\varepsilon} n_{a_2} \left( (k_1^+ - k^+) A_{\alpha_1}^a (\xi) - k_{\perp \sigma} A_{\alpha_1}^a (\xi) \right) \\
+ A_{\alpha_2} A_{\beta_1}^a (\xi) \right) + l_{a_2} (O(\lambda^3)) + O(\lambda^2)
\]

(22)
where the gluon field $A_{\mu}^{a}(\vec{q})$ is from the correlation function of $h_B$, and $A_{\alpha_1,\alpha_2}(\vec{x})$ is from that of $h_A$. The combination is a vector with the index $\alpha_2$ which will be contracted with the remaining terms in Eq. (21). This index is carried by the exchanged gluon. It is noted that the power for different $\alpha_2$ is different. The terms proportional to $n_{\alpha_2}$ in (21) are at order of $\lambda$, while the term with $\alpha_2=\perp$ is at order of $\lambda^0$. At first look one may neglect the terms proportional to $n_{\alpha_2}$. But one can not neglect them, because all terms contracted with the remaining terms in Eq. (21) will give the leading order contribution to the differential cross-section, except the terms proportional to $t_{\alpha_2}$.

It will be lengthy to give the full result for the contribution from Fig 3b. However, one can rather easily find the factorized form of the contribution. We consider the contributions from Fig 4b and Fig 4c to the TMD gluon density matrix:

\[
\Gamma_{\mu\nu}^{\alpha}(k_A) = \frac{1}{xP^+} \int \frac{d^4k d^3k_1}{(2\pi)^4} (-2\pi\delta(k^2)) \delta^3(k_A - k_1 + k)(i)(k_A \cdot ng^{\mu\alpha} - k_1^\mu n^\alpha)
\]

\[
\frac{-i}{n \cdot k - i\varepsilon} (g_{s \rho} f_{\rho a c}^{\alpha c}) (-i)(k_A \cdot ng^{\mu\beta_1} - (k_1 - k)^\mu n_{\beta_1})
\]

\[
\frac{i}{(k_1 - k)^2 - i\varepsilon} g_{s f a c}^{\alpha c} \left( (k_1 + k)^{\beta_1} g^{\rho\beta} + (-2k_1 + k)^\rho g^{\beta_1} + (k_1 - 2k)^\beta g^{\rho\beta_1} \right)
\]

\[
\int \frac{d^3\xi}{(2\pi)^3} e^{i\xi \cdot k_1} \langle h_A | A^{\alpha}_{\beta}(0) A^{\beta}_{\alpha}(\xi) | h_A \rangle.
\]

\[
\Gamma_{\mu\nu}^{\alpha}(k_A) = \frac{1}{xP^+} \int \frac{d^4k d^3k_1}{(2\pi)^4} (-2\pi\delta(k^2)) \delta^3(k_A - k_1 + k)(i)(k_A \cdot ng^{\nu\rho} + k^\nu n_\rho)
\]

\[
\frac{i}{n \cdot k + i\varepsilon} (g_{s n}^{\alpha} f_{\rho a c}^{\alpha c}) (-i)(k_A \cdot ng^{\mu\beta_1} - (k_1 - k)^\mu n_{\beta_1})
\]

\[
\frac{i}{(k_1 - k)^2 - i\varepsilon} g_{s f a c}^{\alpha c} \left( (k_1 + k)^{\beta_1} g^{\rho\beta} + (-2k_1 + k)^\rho g^{\beta_1} + (k_1 - 2k)^\beta g^{\rho\beta_1} \right)
\]

\[
\int \frac{d^3\xi}{(2\pi)^3} e^{i\xi \cdot k_1} \langle h_A | A^{\alpha}_{\beta}(0) A^{\beta}_{\alpha}(\xi) | h_A \rangle.
\]

With these expressions and result in Eq. (22), we find that the contribution from Fig 3b takes the factorized form:

\[
\frac{d\sigma}{d^4q} = \frac{xy\pi\delta(xys - Q^2)}{2(N_c^2 - 1)} g_A^4 f_L^2(0,0) \int d^2k_{A\perp} d^2k_{B\perp} \delta^2(k_{A\perp} + k_{B\perp} - q_{\perp})
\]

\[
\epsilon_{\perp\alpha_2} \epsilon_{\perp\beta_2} \Gamma_{\mu\nu}^{\alpha}(k_B) \left( \Gamma_{\alpha_2}^{\beta_\alpha}(k_A) \right) + \Gamma_{\alpha_2}^{\beta_\alpha}(k_A) \right).
\]

In the above the contribution factorized with Fig 4b is from these terms proportional to $1/(k^+ - i\varepsilon)$ in the (21) in Eq. (22), while the contribution factorized with Fig 4c is from these terms proportional to $1/(k_1^+ + i\varepsilon)$. Again, the contribution from Fig 3b is totally subtracted. It does not contribute to the $O(\alpha_s)$-correction.

The remaining diagram which needs to be studied is Fig 3a. The contribution involves the three-gluon amplitudes only. The leading order result can be obtained in a straightforward way by using the result in Eq. (22). The leading contribution comes from the term with the index $\alpha_2 = \perp$. We obtain the contribution at the leading order of $\lambda$:

\[
\frac{d\sigma}{d^4q} = \frac{\pi\delta(xys - Q^2)}{s(N_c^2 - 1)} g_A^4 f_L^2(0,0) \int d^2k_1 d^2k_B \frac{d^4k}{(2\pi)^4} \delta^4(k_1 + k_B - q - k)(-2\pi\delta(k^2))\epsilon_{\perp\alpha_2} \epsilon_{\perp\beta_2} k^+.
\]
This indicates that the contribution from Fig. 3c will not contribute to the $O(\alpha_s)$-correction in Eq. (16). Performing a similar analysis one will also find that the leading contributions from the exchange of the gluon collinear to $P_A$ are already included in the gluon density matrix $\Gamma_{\mu}^{\mu}(k_A)$ in the factorized form in Eq. (16) at tree-level. Therefore, these leading contributions do not contribute to the $O(\alpha_s)$-correction in Eq. (16). Performing a similar analysis one will also find that the leading contributions from the exchange of the gluon collinear to $P_B$ are already included in the gluon density matrix $\Gamma_{\mu}^{\mu}(k_B)$. We conclude here that the leading contributions of the real part from the exchange of a collinear gluon will not contribute to the $O(\alpha_s)$-correction. They are correctly factorized into TMD gluon density matrices. Therefore, the possible $O(\alpha_s)$-correction to Eq. (16) can only come from the real part subtracted with the collinear contribution, i.e., from the difference:

$$
\frac{d\sigma}{d^4q}_{R,s} = \frac{d\sigma}{d^4q}_{R} - \frac{xy \pi \delta(xy - Q^2)}{2(N_c^2 - 1)} g_A^d f_L^2(0,0) \int d^2k_{A\perp} d^2k_{B\perp} \delta^2(k_{A\perp} + k_{B\perp} - q_{\perp}) \epsilon_{\perp \alpha \mu} \epsilon_{\perp \beta \nu} \Gamma_{\mu}^{\mu}(k_B) \Gamma_{\beta \alpha}(k_A)_{\text{Fig. 4}} + \text{h.c.}. \tag{28}
$$

The contribution from the above expression at the leading order of $\lambda$ can only come from the momentum region where the exchanged gluon is soft. We will study the soft-gluon contribution in the next subsection.

Before turning to the soft-gluon contribution, it is interesting to compare the results here with the calculation done in [14] by replacing the initial hadrons with on-shell gluons. In [14] one can only examine the factorization of a part of the differential cross-section, in which only $f_\beta$ in Eq. (6) is involved. i.e., the contribution from unpolarized gluons in unpolarized hadrons. With the employed approach here, we are able to examine the factorization with the entire gluon density matrices. In this approach the initial gluons are in general off-shell. This brings up some additional complications in comparison with the study in [14]. The complications are indicated by the fact that we need to consider additional contributions represented by those contributions to gluon density matrices given by Fig. 4a and Fig. 4b.

### 3.2 Soft-Gluon Contributions and the Soft Factor

In the kinematical region of $q_{\perp}/Q \sim \lambda \ll 1$, dominant contributions can come from the momentum region in which the exchanged gluon in Fig. 3 is soft. The soft gluon carries the momentum $k$ at the order:

$$
k^\mu \sim (\lambda, \lambda, \lambda, \lambda). \tag{29}$$
In analyzing the collinear contributions, e.g., the three-gluon amplitude contracted with gauge fields in Eq. (13), there are no contributions from the super-leading region in the set of diagrams given by Fig. 2b or Fig. 2c. However, we will have the contributions from the super-leading region in analyzing the contributions of soft gluons. We first consider the three-gluon contribution contracted with gauge fields as that in Eq. (22).

For the case of the soft gluon we have from Fig. 2b and Fig. 2c the contributions at the leading order of \( \lambda \):

\[
\mathcal{M}_{\alpha_1 \alpha_2 \mu}(k_1, -k, k_B) \left|_{2c} \right. \int \frac{d^3 \xi_1 d^3 \eta}{(2\pi)^6} e^{+i\xi_1 \cdot k_1 + i\eta \cdot k_B} A^{a_1, a_2}(\tilde{\xi}_1) A^{b, \mu}(\eta)
\]

\[
= -g_s f^{a_1 a_2 b} A_0 \int \frac{d^3 \xi_1 d^3 \eta}{(2\pi)^6} e^{+i\xi_1 \cdot k_1 + i\eta \cdot k_B} \frac{4i k_1^\beta}{q \cdot k - i\varepsilon} \epsilon_{\lambda a_2 \beta} A^{a_1, +}(\tilde{\xi}_1) A^{b, -}(\eta) \left(1 + \mathcal{O}(\lambda)\right),
\]

\[
\mathcal{M}_{\alpha_1 \alpha_2 \mu}(k_1, -k, k_B) \left|_{2b} \right. \int \frac{d^3 \xi_1 d^3 \eta}{(2\pi)^6} e^{+i\xi_1 \cdot k_1 + i\eta \cdot k_B} A^{a_1, a_2}(\tilde{\xi}_1) A^{b, \mu}(\eta)
\]

\[
= g_s f^{a_1 a_2 b} A_0 \int \frac{d^3 \xi_1 d^3 \eta}{(2\pi)^6} e^{+i\xi_1 \cdot k_1 + i\eta \cdot k_B} \frac{4i k_1^\beta}{q \cdot k - i\varepsilon} \epsilon_{\lambda a_2 \beta} A^{a_1, +}(\tilde{\xi}_1) A^{b, -}(\eta) \left(1 + \mathcal{O}(\lambda)\right). \tag{30}
\]

If we calculate the soft-gluon contributions to the differential cross-section by using the result from the set of diagrams in Fig. 2b or Fig. 2c, one will find that each contribution will be at the order of \( \lambda \) lower than that of the tree-level result given in Eq. (10). This is the contribution from the super-leading region as discussed in [16]. However, the contributions from the super-leading region are canceled in the sum of Fig. 2b and Fig. 2c from Eq. (30).

To find the contributions from the leading region, one has to expand the contribution from Fig. 2b and Fig. 2c at the next-to-leading order of \( \lambda \). The contribution from each set of diagrams is very lengthy, but the sum takes the simple form:

\[
\mathcal{M}_{\alpha_1 \alpha_2 \mu}(k_1, -k, k_B) \left|_{2c} \right. \int \frac{d^3 \xi_1 d^3 \eta}{(2\pi)^6} e^{+i\xi_1 \cdot k_1 + i\eta \cdot k_B} A^{a_1, a_2}(\tilde{\xi}_1) A^{b, \mu}(\eta)
\]

\[
= -g_s f^{a_1 a_2 b} A_0 \int \frac{d^3 \xi_1 d^3 \eta}{(2\pi)^6} e^{+i\xi_1 \cdot k_1 + i\eta \cdot k_B} \frac{2}{m_Q^2 (k^+ - i\varepsilon)} \epsilon^a_\rho \hat{G}^{b, \rho}(\eta) \left[ g_{\sigma \rho} k^+ A^{a_1, +}(\tilde{\xi}) + \eta_{a_2} \left( k_1^+ A^{a_1, +}(\tilde{\xi}) - k_1 A^{a_1, +}(\tilde{\xi}) \right) \right] \left(1 + \mathcal{O}(\lambda)\right). \tag{31}
\]

Using this result we can calculate the contribution from the soft-gluon exchange of Fig. 3b at the leading power:

\[
\frac{d\sigma}{d^2 q} \bigg|_{3,s} = \pi \delta(xys - Q^2) \frac{g_s f^2_L(0, 0)}{s} \int d^3 k_A d^3 k_B \int \frac{d^4 k}{(2\pi)^4} \delta^4(k_A + k_B - q - k_\perp) \epsilon_{\perp \sigma \rho} \epsilon_{\perp \mu \nu} \Gamma_{B}^{\nu \rho}(k_B) \Gamma_A^{\sigma \mu}(k_A) \frac{k_B^+ k_A^-}{N_c^2 - 1} \left[ \frac{1}{N_c^2 - 1} 2\pi\delta(k^2) f_{abc} f_{abc} \right] \left(1 + \mathcal{O}(\lambda)\right), \tag{32}
\]

the factor in \([\ldots]\) is a part of the soft factor introduced in the below.

We note that there are nonzero soft-gluon contributions in the subtracted collinear contributions in Eq. (25). The contribution from the exchanged gluon collinear to \( P_A \) from Fig. 3b, given in Eq. (24), has the same soft-gluon contribution as given in Eq. (32). After analyzing all contributions we have the difference in Eq. (25) at the leading power:

\[
\frac{d\sigma}{d^2 q} \bigg|_{R,s} \approx -\pi \delta(xys - Q^2) \frac{g_s f^2_L(0, 0)}{s} \int d^3 k_A d^3 k_B \int \frac{d^4 k}{(2\pi)^4} \delta^4(k_A + k_B - q - k_\perp) \epsilon_{\perp \sigma \rho} \epsilon_{\perp \mu \nu}
\]
\[
\Gamma_B^\mu(k_B) \Gamma_A^\nu(k_A) \frac{k_B^+ k_A^-}{N_c^2 - 1} \left[ \frac{1}{N_c^2 - 1} 2\pi \delta(k^2) f^{abc} f^{abc} \frac{1}{k^+ - i\varepsilon} \frac{1}{k^- + i\varepsilon} \right] + h.c. \tag{33}
\]

The difference is nonzero at the leading order of \(\lambda\). If we take the factorization as given in Eq.(16), then the difference should be taken as the \(O(\alpha_s)\)-correction. However, the difference is the effect of the soft gluon. It should be taken as a nonperturbative effect which needs to be factorized. For this one needs to implement a soft factor in Eq.(16).

The needed soft factor is defined as:
\[
\tilde{S}(\vec{\ell}_\perp) = \int \frac{d^2b_\perp}{(2\pi)^2} e^{i\vec{b}_\perp \cdot \vec{\ell}_\perp} S^{-1}(\vec{b}_\perp) \tag{34}
\]

with
\[
S(\vec{b}_\perp) = \frac{1}{N_c^2 - 1} \langle 0 | \text{Tr} \left[ \mathcal{L}_u(\vec{b}_\perp, -\infty) \mathcal{L}_u(\vec{b}_\perp, -\infty) \mathcal{L}_u^\dagger(\vec{0}, -\infty) \mathcal{L}_v(\vec{0}, -\infty) \right] | 0 \rangle. \tag{35}
\]

This soft factor has been introduced in the TMD factorization of Higgs production in [6]. At leading order one has
\[
\tilde{S}(\vec{\ell}_\perp) = \delta^2(\vec{\ell}_\perp) + O(\alpha_s). \tag{36}
\]

Figure 5: One-loop corrections for the soft factor. The two diagrams with their conjugated diagrams give the one-loop correction for the soft factor with \(u^+ = 0\) and \(v^- = 0\).

For the convenience we take \(u^+ = 0\) and \(v^- = 0\) here. There are corrections at one-loop. One can divide the corrections into a virtual- and a real part. The real part is given by Fig.5a and the virtual part is given by Fig.5b. For \(u^+ \neq 0\) and \(v^- \neq 0\) there are more diagrams in which one gluon is exchanged between gauge links along the same direction. We will come back later to the contributions from those diagrams. The sum of Fig.5a and its conjugated diagram reads:
\[
\tilde{S}(\vec{\ell}_\perp) \biggr|_{5a+h.c.} = -g_s^2 \int \frac{d^4k}{(2\pi)^4} \delta^2(\vec{\ell}_\perp - \vec{k}_\perp) \left[ \frac{1}{N_c^2 - 1} 2\pi \delta(k^2) f^{abc} f^{abc} \frac{1}{k^+ - i\varepsilon} \frac{1}{k^- + i\varepsilon} \right] + h.c. \tag{37}
\]

We note that the terms in \([\cdots]\) are exactly those in \([\cdots]\) of Eq.(33). Now we modify the tree-level factorization in Eq.(16) as:
\[
\frac{d\sigma}{d^4q} = \frac{xy\delta(xy s - Q^2)}{2(N_c^2 - 1)} g_s A_{\parallel} \int_0 d^2k_{A\perp} d^2k_{B\perp} \delta^2(\vec{k}_{A\perp} + \vec{k}_{B\perp} + \vec{\ell}_\perp - q_{\perp})
\]
\[
\epsilon_{\perp\alpha\mu} \epsilon_{\perp\beta\nu} \Gamma_A^{\alpha\beta}(k_A) \Gamma_B^{\mu\nu}(k_B) \tilde{S}(\vec{\ell}_\perp) \left( 1 + O(\alpha_s) + O(\lambda) \right). \tag{38}
\]
With this modification, one can see that the soft-gluon contribution in Eq. (33) is now in the soft factor. Combining the results in the last subsection we conclude that the real part of one-loop correction at the leading order of $\lambda$ are included in TMD gluon density matrices and the soft factor. The real part does not give the $O(\alpha_s)$-correction. Only the virtual part can give the correction.

![Figure 6: Virtual corrections.](image)

### 3.3. The Virtual Part

The virtual part receives contributions from two sets of diagrams. One set consists of diagrams, in which an additional gluon is exchanged between quark lines in diagrams given in Fig.1b. The contributions from this set of diagrams contain Coulomb- and I.R. divergences. These divergences are correctly factorized with NRQCD factorization into the NRQCD matrix element. After NRQCD factorization, the contributions from this set will give a part of the $O(\alpha_s)$-correction. Another set of diagrams consists of those in which an additional gluon is exchanged between a quark- and a gluon line or between gluon lines. This set of diagrams is given by Fig.6a and Fig.6b with the three-gluon amplitude defined before. The virtual part can be written as the sum:

$$
\frac{d\sigma}{d^4q} = \left( \frac{d\sigma}{d^4q}_{6a} + \frac{d\sigma}{d^4q}_{6b} - \frac{d\sigma}{d^4q}_{G} + \frac{d\sigma}{d^4q}_{Q} \right) + h.c.,
$$

(39)

where the last term in $(\cdots)$ is the contribution from the set of diagrams where an additional gluon is exchanged between quark lines. In the sum of first two terms, the contribution from the gluon-exchange between gluon lines is double-counted. This is corrected by the third term which stands for the contribution from the gluon-exchange between gluon lines. As discussed, only the first three terms are relevant to the TMD factorization.

The contribution from Fig.6a is given as:

$$
\frac{d\sigma}{d^4q}_{6a} = \frac{\pi\delta(xys - Q^2)}{s} \int d^3k_A d^3k_B \delta^4(k_A + k_B - q) \int \frac{d^4k_2}{(2\pi)^4} \frac{-i}{k_2^2 + i\varepsilon} \frac{-i}{k_2^2 + i\varepsilon}
$$

$$
M_{\alpha_1\alpha_2\beta}(k_1, k_2, k_B) \left[ \int \frac{d^3\xi}{(2\pi)^3} e^{i\xi \cdot k_A} \langle h_A| A^{\alpha_1}_\beta(0) A^{\alpha_2}_\alpha(\xi)| h_B \rangle \right] \left( \frac{1}{2} \right)
$$

$$
\langle M_{\beta\delta}^a(k_A, k_B) \rangle^+ \left[ \int \frac{d^3\eta}{(2\pi)^3} e^{i\eta \cdot k_B} \langle h_B| A^{d,\mu}(0) A^{b,\mu}(\eta)| h_B \rangle \right]
$$

$$
\left( - g_s f^{a_1a_2a} \right) \left( (-k_1 + k_2)^\alpha g^{a_1a_2} + (-k_2 - k_A)^a g^{a_2\alpha} + (k_A + k_1)^{\alpha_2} g^{\alpha_1} \right)
$$

(40)
with \( k_1 + k_2 = k_A \). A factor 1/2 should be added to avoid a double counting. The contribution is essentially a part of one-loop correction to the introduced form factor \( f_L(k_A^2, k_B^2) \). It contains collinear- and infrared divergences. One may think that these divergences can be regularized by the off-shellnesses \( k_A^2 \) and \( k_B^2 \). However, in order to find the leading power contribution one has to expand the form factor in \( k_A^2 \) and \( k_B^2 \). Only the contribution with \( k_A^2 = k_B^2 = 0 \) is at the leading power, because of that the \( \delta \)-function \( \delta^2(k_{A\perp} + k_{B\perp} - q_\perp) \) in Fig.6 in the order of \( \lambda^{-2} \). Therefore, the contribution from Fig.6 at the leading power of \( \lambda \) is a part of one-loop correction to the on-shell form factor \( f_L(0,0) \). Similar situation also appears in TMD factorization of Drell-Yan processes and a detailed discussion about virtual corrections can be found in [19]. The collinear- and infrared divergences can be regularized with dimensional regularization. These divergences need to correctly be factorized or subtracted.

Using the early result one can obtain the collinear contribution in which the gluon with \( k_2 \) is collinear to \( P_A \):

\[
\frac{d\sigma}{d^4q} \bigg|_{\delta_{\perp\mu}} = \frac{\pi \delta(xys - Q^2)}{s} g_s f_L^2(0,0) \int d^3k_A d^3k_B \delta^4(k_A + k_B - q) \int d^4k_2 \frac{-i}{(2\pi)^4 k_2^2 + i\varepsilon k_2^2 + i\varepsilon} f^{a_1a_2b}\epsilon_{\perp\rho\mu}^{\nu\delta} M_2(n_{a_2} - k_0^\mu n^{a_1}) \left[ \int \frac{d^4\xi}{(2\pi)^3} \epsilon^{\xi \nu \lambda} k_\lambda \delta(\xi) \right] \left[ \int \frac{d^4\eta}{(2\pi)^3} \delta(\eta) \right] \left[ -g_s f^{a_1a_2b}\epsilon_{\perp\rho\mu}^{\nu\delta} M_2(n_{a_2} - k_0^\mu n^{a_1}) \right].
\]

The corresponding contribution to the gluon TMD distribution from Fig.6 is:

\[
\Gamma^{\mu\nu}_{\perp}(k_A) \bigg|_{\perp} = \frac{1}{x\rho^+} \int \frac{d^4k}{(2\pi)^4 k^2 + i\varepsilon} \left( \frac{-i}{(k_A - k)^2 + i\varepsilon} g_s f^{dab}((k_A + k)^\rho g_\delta^\alpha + (2k_A - k)^\rho g_\delta^\alpha + (-k_A + 2k)^\alpha g_{\perp\delta}) \right)\right)
\]

where \( k \) is the momentum of the gluon attached to the gauge link and is flowing into the gauge link. Comparing Eq.\( (41) \) with Eq.\( (42) \), one finds:

\[
\frac{d\sigma}{d^4q} \bigg|_{\delta_{\perp\mu}} = \frac{xy\pi\delta(xys - Q^2)}{2(N_c^2 - 1)} g_s f_L^2(0,0) \int d^3k_A d^3k_B \delta^4(k_A + k_B - q) \delta_{\perp\mu\perp\nu} \Gamma^{\mu\nu}_{\perp}(k_B) \left( \Gamma^{\beta\alpha}_{\perp}(k_A) \bigg|_{\perp} \right),
\]

therefore, the collinear contribution is already included in the TMD gluon density matrix. Performing the analysis for the case that the gluon is collinear to \( P_B \), one obtains the similar result. We then have the difference

\[
\frac{d\sigma}{d^4q} \bigg|_{V,s} = \frac{d\sigma}{d^4q} \bigg|_{V} - \frac{xy\pi\delta(xys - Q^2)}{2(N_c^2 - 1)} g_s f_L^2(0,0) \int d^2k_{A\perp} d^2k_{B\perp} \delta^4(k_{A\perp} + k_{B\perp} - q_\perp) \epsilon_{\perp\alpha\beta} \epsilon_{\perp\alpha\beta} \left( \Gamma^{\mu\nu}_{\perp}(k_B) \left( \Gamma^{\beta\alpha}_{\perp}(k_A) \bigg|_{\perp} \right) + \Gamma^{\beta\alpha}_{\perp}(k_A) \left( \Gamma^{\mu\nu}_{\perp}(k_B) \bigg|_{\perp} \right) \right) + h.c.,
\]

\[
(44)
\]
which does not contain any collinear divergence. But, the difference contains infrared divergences from the soft-gluon exchange.

For the soft-gluon exchange we consider the case that $k_2^\mu$ is soft, i.e., $k_2^\mu \sim (\lambda, \lambda, \lambda, \lambda)$. In this case, $k_1$ is collinear. Therefore, in analyzing the soft contribution from Fig. 6a, the factor $1/2$ should be replaced with 1 to obtain the correct result. Using the result in the last subsection for the three-gluon amplitude, we have the soft-gluon contribution:

$$\frac{d\sigma}{d^4q} \bigg|_{\text{soft}} = \frac{xy\delta(xys - Q^2)}{2(N_c^2 - 1)} g_A^4 f_L^2(0, 0) \int d^3k_A d^3k_B \delta^4(k_A + k_B - q)\epsilon_\perp_{\rho\mu}\epsilon_\perp_{\beta\nu} \Gamma_B^{\nu\mu}(k_B) \Gamma_A^{\beta\rho}(k_A) \left[ -\frac{g_s^2}{N_c^2 - 1} f_{a_1a_2} f_{a_1a_2} \int \frac{d^4k_2}{(2\pi)^4} \frac{-i}{k_2^2 + i\epsilon} \cdot \frac{i}{k_2^\perp - i\epsilon} \cdot \frac{i}{k_2^+ + i\epsilon} \right].$$ (45)

Again, there is also a soft-gluon contribution in the gluon TMD distribution from Fig. 6c with the same factor in $[\cdots]$ in Eq. (45). At the end one finds the soft-gluon contribution for the difference in Eq. (44) which is similar to the case of real corrections:

$$\frac{d\sigma}{d^4q} \bigg|_{\text{soft}} = -\frac{xy\delta(xys - Q^2)}{2(N_c^2 - 1)} g_A^4 f_L^2(0, 0) \int d^3k_A d^3k_B \delta^4(k_A + k_B - q)\epsilon_\perp_{\rho\mu}\epsilon_\perp_{\beta\nu}\Gamma_B^{\nu\mu}(k_B) \Gamma_A^{\beta\rho}(k_A) \left[ -\frac{g_s^2}{N_c^2 - 1} f_{a_1a_2} f_{a_1a_2} \int \frac{d^4k_2}{(2\pi)^4} \frac{-i}{k_2^2 + i\epsilon} \cdot \frac{i}{k_2^\perp - i\epsilon} \cdot \frac{i}{k_2^+ + i\epsilon} \right] + \text{h.c.},$$ (46)

This soft-gluon contribution is in fact included in the soft factor. The soft factor receives the contribution from Fig. 5b and its conjugated diagram. It is:

$$\tilde{S}(\ell_\perp, \mu, \rho) \bigg|_{\text{soft}, \beta} = \delta^2(\ell_\perp) \frac{g_s^2}{N_c^2 - 1} f_{a_1a_2} f_{a_1a_2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} \cdot \frac{i}{k^\perp - i\epsilon} \cdot \frac{i}{k^+ + i\epsilon} + \text{h.c.}.$$ (47)

If we take the factorized form as in Eq. (48), then the soft-gluon contribution is included in the soft factor. Based on the results in this subsection and previous ones, we find that at the leading power of $q_\perp \sim \lambda Q$ with $\lambda \ll 1$, the real part of the one-loop correction are correctly factorized into TMD gluon densities and the introduced soft factor. It will not contribute to the $O(\alpha_s)$-correction. The virtual part gives contributions to the correction, but the collinear- and infrared divergences are subtracted into TMD gluon density matrices and the soft factor, respectively. The correction is finite.

4. The Final Result

In the study of the previous sections, we have set $u = n$ and $v = l$ for convenience. This setting will generate light-cone singularities in the subtraction. To present the final result, we undo the setting for TMD gluon density matrices and the soft factor. For $u^+ \neq 0$ and $v^- \neq 0$ the TMD density matrix $\Gamma_A^{\mu\nu}$ or $\Gamma_B^{\mu\nu}$ will depend on an extra parameter $\zeta_u$ or $\zeta_v$, respectively. The soft factor contains the parameter $\rho$. These parameters are defined as:

$$\zeta_u^2 = \frac{2u^-}{u^+} \left(P_A^+ \right)^2, \quad \zeta_v^2 = \frac{2v^+}{v^-} \left(P_B^- \right)^2, \quad \rho^2 = \frac{u^-v^+}{u^+v^-}.$$ (48)

Our final factorized result can be written as:

$$\frac{d\sigma}{dxdydq_\perp} = \frac{2\pi\delta(xys - Q^2)}{Q^2} \sigma_0 \int d^2k_{A\perp} d^2k_{B\perp} d^2\ell_\perp \delta^2(k_{A\perp} + k_{B\perp} + \ell_\perp - q_\perp)\epsilon_\perp_{\alpha\mu}\epsilon_\perp_{\beta\nu} \mathcal{H}(\zeta_u, \zeta_v, \rho) \Gamma_A^{\beta\alpha}(k_A, \zeta_u) \Gamma_B^{\nu\mu}(k_B, \zeta_v) \tilde{S}(\ell_\perp, \rho),$$ (49)
where the dependence on the renormalization scale $\mu$ in each term in the last line is suppressed. $\sigma_0$ is given by
\[
\sigma_0 = \frac{(4\pi \alpha_s)^2}{N_c(N_c^2 - 1)m_Q} |\psi(0)|^2 = \frac{(4\pi \alpha_s)^2}{2N_c^2(N_c^2 - 1)m_Q} \langle O^{(1)} S_0^{[1]} \rangle.
\]
(50)

Here, we have expressed the quantity $|\psi(0)|^2$ with the corresponding NRQCD matrix element $\langle O^{(1)} S_0^{[1]} \rangle$.

As mentioned in the subsection 3.2 there are more diagrams for the one-loop correction of the soft-factor for the case of $u^+ \neq 0$ and $v^- \neq 0$. In these diagrams there is one-gluon exchange between gauge links along the same direction. In this case the TMD gluon density matrices also receive one-loop contributions from one-gluon exchange between gauge links along the same direction. These contributions are exactly canceled in Eq.(38) or Eq.(49) by those from the soft factor.

In Eq.(49) $\mathcal{H}$ is the perturbative coefficient which starts at the order of $\alpha_s^0$. Because of that all one-loop real corrections are subtracted into TMD gluon density matrices and the soft factor from our analysis in Sect. 3.2., the coefficient is determined by the virtual correction. It is determined by the form factor of the fusion of two on-shell gluons into $\eta_Q$ after the subtraction of collinear- and infrared divergences with TMD gluon density matrices and the soft factor as shown in Sect. 3.3. Since there is only one form factor for the fusion, we have then correspondingly in Eq.(49) only one perturbative coefficient. It is noted that the form factor of the fusion with two on-shell gluons and the subtraction are gauge-invariant, $\mathcal{H}$ and Eq.(49) are hence also gauge-invariant.

At the leading order of $\alpha_s$ $\mathcal{H}$ is 1. Beyond the leading order $\mathcal{H}$ will depend on $\zeta_u$, $\zeta_v$ and $\rho$ because of the subtraction. The dependences will be canceled by those of TMD gluon density matrices and the soft factor. $\mathcal{H}$ is obtained in [24]. But there are several typos and errors in constant terms. We will give here the corrected one. For NRQCD factorization we have made the expansion in the small velocity $v$, as discussed in the section of Introduction. We have only taken the leading order $v^0$. However, the correction from the next-to-leading order of $v$, i.e., the relativistic correction, is at the same level of the importance as the $O(\alpha_s)$-correction, as discussed in [23]. Here we also include the relativistic correction. This correction can be extracted from the results in [29]. We have:

\[
\mathcal{H}(\zeta_u, \zeta_v, \rho) = 1 - \frac{4}{3} \frac{\langle P^{(1)} S_0^{[1]} \rangle}{m_Q^3 \langle O^{(1)} S_0^{[1]} \rangle} + \frac{\alpha_s N_c}{4\pi} \ln^2 \frac{\zeta_s^2}{Q^2} + \ln^2 \zeta_u^2 - \ln \rho^2 \left( 1 + 2 \ln \frac{\mu^2}{Q^2} \right) + 2 \ln \frac{\mu^2}{Q^2} \\
+ \frac{7}{2} \pi^2 + \frac{2}{N_c} \left( 5 - \frac{1}{4} \pi^2 \right) + O(\alpha_s^2) + O(v^4)
\]
(51)

with $Q^2 = 4m_Q^2$. In the second term in Eq.(51) there is a ratio of two NRQCD matrix elements defined in [23]. This term is at order of $v^2$. The neglected correction from the expansion of the small velocity $v$ is now at order of $v^4$.

After giving our main result in Eq.(49) it is worthy to discuss the problem of the so-called scheme-dependence in TMD factorization. The scheme-dependence arises because one can define different TMD parton distributions. In the case of TMD quark distributions one can define subtracted quark distributions to absorb the corresponding soft factor as suggested in [30]. One may also use the definition from the soft-collinear effective theory given in [31]. With different definitions one obtains the similar factorized result with different perturbative coefficients. The difference can be calculated perturbatively as discussed in [32] [33]. Similarly, one can also work with different definitions of TMD gluon distributions. In this work, we only give our result in Eq.(49) with the unsubtracted TMD gluon distributions defined in Eq.(5). Hence, the soft factor $\tilde{S}$ appears explicitly.
The TMD gluon density matrix of \( h_A \) or \( h_B \) depends on the parameter \( \zeta_u \) or \( \zeta_v \), respectively. The dependence is determined by Collins-Soper equation. This equation can be used to resume terms of the large logarithms of \( q_\perp/Q \) in perturbative coefficient functions of collinear factorization. In this way, one obtains the standard Collins-Soper-Sterman(CSS) resummation formalism in [1]. However, in CSS formalism there can be certain freedom to re-define perturbative coefficient functions to make them process-independent or universal. This has been noticed in [34]. An application by using the freedom is given in [35] for Higgs production in hadron collisions. Following the work in [34], the impact of the scheme-dependence in TMD factorization on the correspondingly derived resummation formalism has been studied in [36]. The difference between different schemes can be determined perturbatively. It is noted that our scheme of TMD factorization here and hence the corresponding resummation formalism are referred as Ji-Ma-Yuan scheme according to [33, 36].

From our result one can derive the resummation formula of \( \ln(q_\perp/Q) \) for the production of \( \eta_c, \eta_b \). The resummation for the production of \( J/\psi \) or \( \Upsilon \) has been studied in [37]. In these resummations the quantum numbers of the produced heavy quark pair are fixed. The resummation for the production of a heavy quark pair in general case is studied in [38, 39].

In the factorization formula in Eq. (19) the physical effects from initial hadrons in the processes only appear in TMD gluon density matrices. They take different parametrization forms for different hadrons, e.g., different spins of hadrons. With the formula one can derive the angular distribution for given hadrons in the initial state. But, the results can be very lengthy. Here, we consider a realistic case in which \( h_B \) is of spin-0 or unpolarized, \( h_A \) is of spin-1/2. The classification of the gluon TMD distributions of a spin-1/2 hadron is given in Eq. (8). For \( h_B \) we have:

\[
\Gamma_B^{\mu\nu}(k_B, \zeta_B^2) = -\frac{1}{2} g_1^{\mu\nu} \tilde{f}_g(y, k_{B\perp}) + \frac{1}{2M_B^2} \left( k_{B\perp}^\mu k_{B\perp}^\nu + \frac{1}{2} g_1^{\mu\nu} k_{B\perp}^2 \right) \tilde{H}^\perp(y, k_{B\perp}).
\]

(52)

With the factorization formula we can derive the result of the differential cross-section in the considered case as:

\[
\frac{d\sigma(x, y, q_\perp)}{dx dy dq_\perp} = \frac{2\pi\delta(xys - Q^2)}{Q^2} \sigma_0 \mathcal{H} \left[ A(x, y, q_\perp) + \tilde{s}_\perp \cdot q_\perp B(x, y, q_\perp) \right],
\]

(53)

where \( \tilde{s}_\perp \) is the transverse-spin vector and \( \tilde{s}_\perp^{\mu} = \epsilon_\perp^{\mu\nu} s_\perp^{\nu} \). The two coefficient functions are expressed with gluon TMD distributions as:

\[
A(x, y, q_\perp) = \frac{1}{2} \int d^2k_{A\perp} d^2k_{B\perp} d^2\ell_\perp \delta^2(k_{A\perp} + k_{B\perp} + \ell_\perp - q_\perp) \tilde{S}(\ell_\perp, \rho) \left[ f_g(x, k_{A\perp}) \tilde{f}_g(y, k_{B\perp}) - \frac{1}{4M_A^2 M_B^2} (2(k_{A\perp} \cdot k_{B\perp})^2 - k_{A\perp}^2 k_{B\perp}^2) H^\perp(x, k_{A\perp}) \tilde{H}^\perp(y, k_{B\perp}) \right],
\]

\[
B(x, y, q_\perp) = \frac{-1}{2M_A q_\perp^2} \int d^2k_{A\perp} d^2k_{B\perp} d^2\ell_\perp \delta^2(k_{A\perp} + k_{B\perp} + \ell_\perp - q_\perp) \tilde{S}(\ell_\perp, \rho) \left\{ k_{A\perp} \cdot q_\perp \tilde{f}_g(y, k_{B\perp}) G_T(x, k_{A\perp}) + \frac{1}{8M_B^2} \left[ (2k_{A\perp} \cdot k_{B\perp} q_\perp \cdot k_{B\perp} + k_{B\perp}^2 k_{A\perp} \cdot q_\perp) \left( \Delta H^\perp_T(x, k_{A\perp}) - \frac{k_{A\perp}^2}{2M_A^2} \Delta H^\perp(x, k_{A\perp}) \right) 
\right.
\]

\[
+ 2k_{A\perp} \cdot k_{B\perp} \left( q_\perp \cdot k_{A\perp} k_{A\perp} \cdot k_{B\perp} + k_{A\perp}^2 k_{B\perp} \cdot q_\perp \right) \Delta H^\perp_T(x, k_{A\perp}) \tilde{H}^\perp(y, k_{B\perp}) \right\}. \]

(54)

At tree-level, i.e., with \( \mathcal{H} = 1 \) in Eq. (53), parts of the above expression has been derived before. The function \( A \) is given in [8] and the first term in \( B \) is derived in [40]. In [14] the factorization of the
contribution with $f_g \tilde{f}_g$ is examined at one-loop level. In this work, we have examined the factorization of all contributions at one-loop with the general result given in Eq. (49).

The obtained differential cross-section does not depend on the helicity of the polarized hadron. But it depends on the transverse spin. This dependence will give an Single transverse-Spin Asymmetry (SSA). The same SSA has been studied with the twist-3 collinear factorization in [40]. In the kinematical region $\Lambda_{QCD} \ll q_\perp \ll Q$, both the factorizations apply, as shown for SSA in Drell-Yan processes in [41], where SSA is only generated by Sivers quark distribution in TMD factorization. In our case SSA is not only generated by the gluonic Sivers function $G_T$, but also by other two T-odd TMD gluons distributions. Therefore, the relation between the two factorizations for SSA needs to carefully examined.

5. Summary

We have studied TMD factorization for $\eta_Q$-production in hadron collisions at low transverse momenta. If the factorization holds, one can use the production process to extract TMD gluon distributions from experiments. We have explicitly shown that the factorization holds at one-loop level, in which all nonperturbative effects are factorized into TMD gluon density matrices and a soft factor defined as the vacuum expectation value of product of gauge links. There is only one perturbative coefficient standing for all perturbative effects. This coefficient is determined at one-loop level and implemented with the relativistic correction of $\eta_Q$. With the result here, all TMD gluon distribution functions at leading power can be safely extracted from experimental data.

In general the initial gluons from the initial hadrons are off-shell, and they can be with nonphysical polarizations. This makes the study of the TMD factorization more complicated than that in the case of on-shell gluons. However, with the complication we can still show that the factorization holds and is gauge-invariant. It is interesting to note that at one-loop level there exist contributions from the super-leading region from different sets of diagrams. But, they are cancelled in the sum. With our factorized result at one-loop, it is possible to show the factorization beyond one-loop for the studied process. One can also use the approach employed here for examining TMD factorization of other processes involving TMD gluon distributions mentioned in the Introduction.

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