We discuss the status of the next-to-next-to-leading order calculation in scattering processes, describing briefly the challenges that have been overcome and what challenges are still to be resolved.

1 Introduction

The Standard Model (SM), the quantum theory of the strong (QCD) and electro-weak interactions, is in very good shape. In the last twenty years, we have accumulated a wide body of experimental evidence of the viability of the SM. LEP and SLC, with a well understood initial state, have proven to be ideal experiments for testing the strong and weak sector of the theory. Systematic studies of large classes of jet shape variables have been done to an extent that eliminates any doubt that perturbative QCD is at work in $e^+e^-$ annihilation. HERA data on $ep$ collisions have shown clearly the scaling violation and have given the best precise measurements of structure functions. The highest energy scales currently available to test the SM are reached in the hadron colliders. These machines go beyond the task of testing: they are discovery machines. In fact, $W$ and $Z$ vector bosons and $t\bar{t}$ quark cross sections were computed in a perturbative framework of the SM, and represent remarkable examples where the model has allowed physicists to predict cross sections for, at the time, unknown particles.

Even if the SM seems to work so well, there is an entire sector, the Higgs boson sector, that has not yet been discovered. We hope that LHC will not only discover the Higgs boson, but will provide useful information about the mass, couplings, charge, spin and color properties of the Higgs boson. In addition to this not yet discovered sector, the SM has some “disturbing” features that lead us to think that it is not the final theory: questions exist as to why there are three families of fermions and why there are so many ad hoc parameters. In addition, more profound problems such as the strong CP violation, the cosmological constant and the hierarchy problem have not been completely understood. Finally, a quantum theory of gravity has not yet
been formulated, so that the SM looks more like an effective theory, that should be integrated or embraced by a more general theory.

Any extension of the SM has to face the fact that the SM has predicted with high degree of accuracy the results from the experiments at high-energy accelerators. This means that any signal of new physics must be very small, and reaching a higher level of precision in our understanding of hard production phenomena becomes a fundamental issue in searching for new physics. This goal can be achieved in two ways:
- with better detectors and experimental analysis;
- by refining our theoretical calculations.

One way (and surely not the only one) to refine our theoretical calculations is to go from next-to-leading order (NLO) to next-to-next-to-leading order (NNLO). There are several reasons for why this step is vital in reducing the theoretical uncertainties:
- the dependence from the unphysical renormalization and factorization scales is going to be reduced;
- the presence of an additional hard parton gives rise to better matching between the partonic and the hadronic final state;
- double radiation from one of the incoming partons or single radiation from both of the two incoming partons creates more complicated transverse-momentum patterns for the final state partons, and may provide a better and more theoretically motivated description of the data, without the need of an intrinsic $k_T$ for the parton in the incoming hadron;
- the need of power correction contributions to event shape variables is going to be reduced, since part of the $1/Q$ contributions will be taken into account by the NNLO term.

2 The perfect mayonnaise

As it is well known among the Italian chefs, in order to make mayonnaise, one must have all the ingredients and then must follow the recipe. But, even in this case, one is not sure that all the ingredients will amalgamate correctly to produce mayonnaise and will not instead create simple scrambled eggs.

2.1 Ingredients

Not all the ingredients are currently available. The matrix elements at NNLO order that are needed for $2 \rightarrow 2$ scattering ($e^+e^- \rightarrow e^+e^-, gg \rightarrow gg, qq' \rightarrow qq', q\bar{q} \rightarrow g\gamma \ldots$) are the following:
1. the square of $2 \rightarrow 4$ tree amplitude,
2. the interference of the $2 \rightarrow 3$ one-loop amplitude with the tree-level one,
3. the square of the $2 \rightarrow 2$ one-loop amplitude,
4. the interference of the $2 \rightarrow 2$ two-loop amplitude with the tree-level one,

where all the particles are taken massless and the external legs are light-like.

*Mutatis mutandis*, these are the same ingredients that are needed in $e^+e^- \rightarrow Z, \gamma \rightarrow 3$ jets, at LEP or at a future linear collider. In fact, in this case, one of the incoming particles becomes a final one, and the remaining incoming particle is time-like. With a space-like incoming particle, the same ingredients can describe deep-inelastic scattering in $ep$ colliders.

All these matrix elements are now available, mainly due to the removal of a major stumbling block in the calculation of two-loop diagrams.

The first calculation of a two-loop four-point scattering amplitude was performed in the case of the maximal-helicity-violating gluon-gluon scattering. Subsequently, generic $2 \rightarrow 2$ scattering matrix elements at two loops have become tractable for massless particle exchanges in the loops and with light-like external legs. Analytic expansion in $\epsilon = (4 - D)/2$, where $D$ is the space-time
dimension, have been computed and, at the same time, algorithms were developed for the tensor reduction to master integrals of all relevant two-loop topologies.

This technology has already been applied to a wide range of physically interesting processes. The interference of tree and two-loop graphs (together with the simpler self-interference of one-loop diagrams) for various processes have now been computed, including Bhabha scattering \((e^+e^- \to e^+e^-)\) in the massless electron limit, and all the QCD \(2 \to 2\) parton-parton scattering processes \((gg \to gg, gg \to q\bar{q}\) and \(q\bar{q} \to q\bar{q}\)). Two-loop helicity amplitudes have also been derived for gluon fusion into photons \((gg \to \gamma\gamma)\), light-by-light scattering \((\gamma\gamma \to \gamma\gamma)\) and gluon-gluon scattering \((gg \to gg)\).

The case where the internal propagators are massless but one external leg is off-shell has also been intensively studied, leading to the evaluation of all associated planar and non-planar master integrals needed in the NNLO computation of the decay of an off-shell photon to three partons \((\gamma^* \to q\bar{q}g)\), relevant for three jet production in electron-positron annihilation.

2.2 A missing ingredient

In hadron-hadron collisions, factorization of the collinear singularities from the incoming partons requires the evolution of the parton density functions to be known to an accuracy matching the hard scattering matrix element. This entails knowledge of the three-loop splitting functions. An approximate expression based upon the calculations of some moments in Mellin space, the knowledge of the most singular \(\log(1/x)\) behaviour at small \(x\) and some exactly known terms has been computed by van Neerven & Vogt. Very recently a set of NNLO parton-distribution functions, which was obtained by fitting the new precise data for deep-inelastic scattering from HERA and for inclusive jet production at the Tevatron, has been provided by the MRST collaboration.

2.3 The recipe

The infrared singularities present in the two-loop \(2 \to 2\) contributions must cancel against the contributions from the one-loop \(2 \to 3\) processes when one particle is unresolved, and the contribution from the tree-level \(2 \to 4\) processes when two particles are unresolved. Unresolved particles are either soft or collinear with one of the other partons in the event, and both of these configurations have the appearance of a \(2 \to 2\) scattering. While the soft and collinear limits of these amplitudes has been vastly investigated, a systematic procedure for analytically carrying through the integration of the squared amplitudes over the soft and collinear phase space and the cancellation of the \(\epsilon\) poles has not yet been established.

2.4 The final mixing

A lot of thinking and work remains to be done. In fact, even when all the ingredients will be available and the entire recipe for the cancellation of the infrared poles will be formulated, we still have the task of assembling all the finite pieces in a fast and reliable partonic Monte Carlo program.

This task should not be underestimated, since speed and numerical accuracy are main goals in this part of the process. This will be the most time consuming part in the entire project, since a lot of effort should be put in speeding up the program, helping the Monte Carlo integration with importance sampling and phase-space remapping.

This is analogous to the whipping part of the mayonnaise-making process. In fact, even if one puts together all the ingredients and follows the recipe, mixing may not work. It would be a pity if the mayonnaise would not reach a fluffy and light consistency.
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