The applicability of causal dissipative hydrodynamics to relativistic heavy ion collisions

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We utilize nonequilibrium covariant transport theory to determine the region of validity of causal Israel-Stewart dissipative hydrodynamics (IS) and Navier-Stokes theory (NS) for relativistic heavy ion physics applications. A massless ideal gas with $2 \to 2$ interactions is considered in a $0+1$D Bjorken scenario, appropriate for the early longitudinal expansion stage of the collision. In the scale invariant case of a constant shear viscosity to entropy density ratio $\eta/s \approx \text{const}$, we find that Israel-Stewart theory is 10% accurate in calculating dissipative effects if initially the expansion timescale $\tau_0/\lambda_{tr,0} \gtrsim 2$. The same accuracy with Navier-Stokes requires three times larger $\tau_0/\lambda_{tr,0} \gtrsim 6$. For dynamics driven by a constant cross section, on the other hand, about 50% larger $\tau_0/\lambda_{tr,0} \gtrsim 3$ (IS) and 9 (NS) are needed. For typical applications at RHIC energies $\sqrt{s_{NN}} \sim 100 - 200$ GeV, these limits imply that even the Israel-Stewart approach becomes marginal when $\eta/s > \approx 0.15$. In addition, we find that the 'naive' approximation to Israel-Stewart theory, which neglects products of gradients and dissipative quantities, has an even smaller range of applicability than Navier-Stokes. We also obtain analytic Israel-Stewart and Navier-Stokes solutions in $0+1$D, and present further tests for numerical dissipative hydrodynamics codes in $1+1$, $2+1$, and $3+1$D based on generalized conservation laws.

I. INTRODUCTION

The realization that shear viscosity is likely nonzero in general[1–3], and therefore the perfect (Euler) fluid paradigm[4–7] of nuclear collisions at the Relativistic Heavy Ion Collider (RHIC) could have significant viscous corrections[8], has fuelled great interest in studying dissipative hydrodynamics[9–18]. Causality and stability problems[19] exhibited by standard first-order relativistic Navier-Stokes hydrodynamics[20, 21] steered most effort toward application of the second-order Israel-Stewart (IS) approach[22, 23].

However, unlike Navier-Stokes that comes from a rigorous expansion[24] in small gradients near equilibrium, the IS formulation is not a controlled expansion in some small parameter (see Section II). Moreover, though causality is restored in a region of hydrodynamic parameters, the stability of IS solutions is not necessarily guaranteed[25]. Therefore it is imperative to test the applicability of the IS approach against a stable, nonequilibrium theory.

In this work we perform such a test utilizing the fully stable and causal covariant transport approach[26–29]. We focus on the special case of $2 \to 2$ transport and a longitudinally boost invariant system[30] with transverse translational symmetry, i.e, $0+1$ dimensions. Follow-up studies in higher dimensions, such as our earlier comparison between transport and ideal hydrodynamics in $2+1$D[8], will be pursued in the future.

A similar study by Zhang and Gyulassy[27] compared kinetic theory and Navier-Stokes results. Here we compare to the causal IS solutions. In addition, we provide a series of tests and semi-analytic approximations that demonstrate the general behavior of IS solutions, which can be utilized to verify the accuracy of numerical IS solutions.

The paper is structured as follows. We start with reviewing the relationship between hydrodynamics and covariant transport (Sec. II), then proceed to discuss the Israel-Stewart equations (Sec. III). The basic observables studied here are introduced in Sec. IV, while the main results from the hydro-transport comparison are presented in Sec. V, together with implications for heavy-ion collisions. Many details are deferred to Appendices A-D. We highlight here the generalized conservation laws derived in App. B, and the detailed study of Israel-Stewart and Navier-Stokes solutions in App. C utilizing numerical and analytic methods.

II. HYDRODYNAMICS AND COVARIANT TRANSPORT

Hydrodynamics describes a system in terms of a few local, macroscopic variables[20], such as energy density $\varepsilon(x)$, pressure $p(x)$, charge density $n(x)$ and flow velocity $u^\mu(x)$. The equations of motion are energy-momentum and charge
conservation
\[ \partial_{\mu}T^{\mu\nu}(x) = 0 \quad , \quad \partial_{\mu}N^{\mu}(x) = 0 , \quad (1) \]
and the equation of state \( p(e, n) \). Ideal (Euler) hydrodynamics assumes local equilibrium in which case
\[ T^{\mu\nu}_{LR,id} = \text{diag}(\varepsilon, p, p, p) \quad , \quad N^{\mu}_{LR,id} = (n, 0) \quad [u_{LR} = (1, 0)] \quad (2) \]
in the fluid rest frame LR. Extension of the theory with additive corrections linear in flow and temperature gradients[20]
\[ \delta T^{\mu\nu}_{NS} = \eta_{\mu}(\nabla^{\mu}u^{\nu} + \nabla^{\nu}u^{\mu} - \frac{2}{3}\Delta^{\mu\nu}\partial^{\alpha}u_{\alpha}) + \zeta\Delta^{\mu\nu}\partial^{\alpha}u_{\alpha} \quad , \quad (3) \]
\[ \delta N^{\mu}_{NS} = \kappa_{q}\left(\frac{nT}{\varepsilon + p}\right)^{2}\nabla^{\mu}\left(\frac{\mu}{T}\right) \quad (\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu}u^{\nu} \quad , \quad \Delta^{\mu} \equiv \Delta^{\mu\nu}\partial_{\nu}) \quad (4) \]
leads via (1) to the relativistic Navier-Stokes (NS) equations. (We use the Landau frame convention \( u_{\mu}\delta T^{\mu\nu} \equiv 0 \) throughout this paper, i.e., the flow velocity is chosen such that momentum flow vanishes in the LR frame.) Here \( \eta_{\mu}(e, n) \) and \( \zeta(e, n) \) are the shear and bulk viscosities, while \( \kappa_{q}(e, n) \) is the heat conductivity of the matter. The most notable feature of NS theory relative to the ideal case is dissipation, i.e., entropy production. For consistency, the dissipative corrections (3)-(4) must be small, otherwise nonlinear terms and higher gradients should also be considered.

It is crucial that the above hydrodynamic equations can indeed be obtained from a general nonequilibrium theory, namely on-shell covariant transport[21, 27–29]. For a one-component system the covariant transport equation reads
\[ p^{\mu}\partial_{\mu}f(x, p) = S(x, p) + C[f, f](x, p) \quad (5) \]
where the source term \( S \) specifies the initial conditions and \( C \) is the collision term. Throughout this paper we consider the Boltzmann limit[43] with binary 2 \( \rightarrow \) 2 rates
\[ C[f, g](x, p) \equiv \iiint_{343}(f_{3}g_{4} - f_{1}g_{2})W_{12-34}\delta^{4}(p_{1}+p_{2}-p_{3}-p_{4}) \quad (6) \]
where \( f_{i} \equiv f(x, p_{i}) \) and \( f_{1} = \int d^{3}p_{1}/(2\pi) \). For dilute systems, \( f \) is the phasespace distribution of quasi-particles, while the transition probability \( W = (1/\pi)s(s - 4m^{2})\delta d/dt \) is given by the scattering matrix element[21]. Our interest here, on the other hand, is the theory near its hydrodynamic limit, \( W \rightarrow \infty \). In this case, “particles” and “interactions” do not necessarily have to be physical but could simply be mathematical constructs adjusted to reproduce the transport properties of the system near equilibrium[31]. The main advantage of transport theory is its ability to dynamically interpolate between the dilute and opaque limits.

The Euler and Navier-Stokes hydrodynamic equations follow from a rigorous expansion of (5) in small gradients near local equilibrium
\[ f(x, p) = f_{eq}(x, p)[1 + \phi(x, p)] \quad |\phi| \ll 1 \quad , \quad |p^{\mu}\partial_{\mu}\phi| \ll |p^{\mu}\partial_{\mu}f_{eq}|/f_{eq} , \quad (7) \]
and substitution of moments of the solutions
\[ N^{\mu}(x) = \int \frac{d^{3}p}{p_{0}}p^{\mu}f(x, p) \quad , \quad T^{\mu\nu}(x) = \int \frac{d^{3}p}{p_{0}}p^{\mu}p^{\nu}f(x, p) \quad , \quad (8) \]
into (1). The 0-th order \( \phi = 0 \) reproduces ideal hydrodynamics. The first order result is the solution to the linear integral equation
\[ p^{\mu}\partial_{\mu}f_{eq}(x, p) = 2C[f_{eq}, f_{eq}\phi_{NS}](x, p) \quad (9) \]
and leads to the Navier-Stokes equations.

Unfortunately the relativistic Navier-Stokes equations are parabolic and therefore acausal. A solution proposed by Mueller[32] and later extended by Israel and Stewart [22, 23] converts the NS equations into relaxation equations for the shear stress \( \pi^{\mu\nu} \), bulk pressure \( \Pi \), and heat flow \( q^{\mu} \). The dissipative corrections
\[ \delta T^{\mu\nu} \equiv \pi^{\mu\nu} - \Pi\Delta^{\mu\nu} \quad , \quad \delta N^{\mu} \equiv - \frac{n}{\varepsilon + p}q^{\mu} \quad (u_{\mu}q^{\mu} = 0 \quad , \quad u_{\mu}\pi^{\mu\nu} = u_{\mu}\pi^{\nu\mu} = 0) \quad (10) \]
dynamically relax on microscopic time scales \( \tau_{\pi}(e, n) \), \( \tau_{\Pi}(e, n) \), \( \tau_{q}(e, n) \) towards values dictated by gradients in flow and temperature. Causality is satisfied in a region of parameter space, however, stability is not guaranteed[25].
More importantly, unlike the Euler and NS equations, the Israel-Stewart approach is not a controlled approximation to the transport theory (5). Instead of an expansion in some small parameter, it corresponds to a quadratic ansatz [23, 33] for the deviation from local equilibrium

$$\phi_G(x, p) = D^\mu(x) \frac{p_\mu}{T} + C^\mu\nu(x) \frac{p_\mu p_\nu}{T^2} , \quad (u_\mu D^\mu = 0 = u_\mu C^\mu\nu u_\nu)$$  \hspace{1cm} (11)

where $D^\mu$ and $C^\mu\nu$ are determined by the local dissipative corrections $\pi^\mu\nu, \Pi$, and $q^\mu$[44]. In contrast, the Chapman-Enskog solution (9) contains all orders in momentum. An evident pathology of the quadratic form (11) is that, in general, $\phi_G$ is not bounded from below and thus the phase space density becomes negative at large momenta (cf. (7) and (62)). Furthermore, the two approaches give different results not only for the relaxation times [21, 23], e.g.,

$$\tau^\Pi_{NS} = 0 \quad , \quad \tau^\Pi_{IS} = \frac{3\eta_s}{2p}$$  \hspace{1cm} (12)

but also for the transport coefficients themselves. For an energy-independent isotropic cross section and ultrarelativistic particles ($T \gg m$) the difference is small [21], e.g.,

$$\eta^\Pi_{NS} \approx 0.8436 \frac{T}{\sigma_{tr}} \quad , \quad \eta^\Pi_{IS} = \frac{4T}{5\sigma_{tr}}$$  \hspace{1cm} (13)

but can be large for more complicated interactions. Here $\sigma_{tr} \equiv \int d\Omega_{cm} \sin^2 \theta_{cm} d\sigma / d\Omega_{cm}$ is the transport cross section (for isotropic, $\sigma_{tr} = 2\sigma_{TOT}/3$).

In the following Sections we analyze IS hydrodynamic solutions analytically and numerically, and test the accuracy of the IS approximation via comparison to solutions from full 2 + 2 transport theory.

III. ISRAEL-STEWART HYDRODYNAMICS AND BOOST INVARIANCE

A. Israel-Stewart equations

There seems to be some confusion regarding Israel-Stewart theory [22, 23] in the recent literature, therefore we start with reviewing the key ingredients. The starting point of Israel and Stewart (IS) is an entropy current that includes the IS approximation via comparison to solutions from full 2 + 2 transport theory.

$$S^\mu = u^\mu \left[ s_{eq} - \frac{1}{2T} \left( \beta_0 \Pi^2 - \beta_1 q_\mu q^\mu + \beta_2 \pi^\mu\nu \pi_{\mu\nu} \right) \right] + \frac{q^\mu}{T} \left( \frac{\mu n}{\varepsilon + p} + \alpha_0 \Pi \right) - \frac{\alpha_1 q_\mu \pi^\mu}{T}$$  \hspace{1cm} (14)

(we follow the Landau frame convention). Here $\mu$ is the chemical potential, and $s_{eq}$ is the entropy density in local equilibrium. The coefficients $\{\alpha_i(e, n)\}$ and $\{\beta_i(e, n)\}$ encode additional matter properties that complement the equation of state and the transport coefficients. Most importantly, $\{\beta_1\}$ control the relaxation times for dissipative quantities:

$$\tau_\Pi = \zeta \beta_0 \quad , \quad \tau_q = \kappa_T \beta_1 \quad , \quad \tau_\pi = 2\eta_s \beta_2 .$$  \hspace{1cm} (15)

The entropy current and relaxation times in Navier-Stokes theory are recovered when all the coefficients are set to zero $\beta_0 = \beta_1 = \beta_2 = 0 = \alpha_0 = \alpha_1$ (but as discussed previously, the IS and NS transport coefficients differ in general).

The requirement of entropy non-decrease ($\partial_\mu S^\mu \geq 0$), which IS satisfy via a positive semi-definite [46] quadratic ansatz

$$T \partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q_\mu q^\mu}{\kappa_T} + \frac{\pi^\mu\nu \pi_{\mu\nu}}{2\eta_s} \geq 0,$$  \hspace{1cm} (16)

leads to the identification of the dissipative currents:

$$\Pi = \zeta \left[ -\nabla_\mu u^\mu - \frac{1}{2} \Pi T \partial_\mu \left( \frac{\beta_0 u^\mu}{T} \right) - \beta_0 D\Pi + \alpha_0 \partial_\mu q^\mu - a_0 q^\mu D u_\mu \right]$$  \hspace{1cm} (17)

$$q^\mu = -\kappa_T T^\mu\nu T^{\eta/T} \left( \frac{\mu n}{\varepsilon + p} \right) + \frac{1}{2} \gamma_0 T \partial_\lambda \left( \frac{\beta_1 u^\lambda}{T} \right) + \beta_1 D q_\nu + \alpha_0 \nabla_\nu \Pi - \alpha_1 \partial_\nu \pi_{\lambda\nu} - a_0 \Pi D u_\nu + a_1 \pi_{\lambda\nu} D u^\lambda ,$$  \hspace{1cm} (18)

$$\pi^\mu\nu = 2\eta_s \left[ \nabla^{(\mu} u^{\nu)} - \frac{1}{2} \pi^\mu\nu T^\eta T_{\eta\lambda} \left( \frac{\beta_2 u^\lambda}{T} \right) - \beta_2 (D\pi^\mu\nu) - \alpha_1 \nabla^{(\mu} q^{\nu)} + a'_1 q^{(\mu} D u^{\nu)} \right]$$  \hspace{1cm} (19)

$$a'_i = \frac{\partial (\alpha_i/T)}{\partial (1/T)} \bigg|_{\mu/T = \text{const}} - \alpha_i .$$  \hspace{1cm} (20)
Here $D \equiv u^\mu \partial_\mu$ and the $\langle \rangle$ brackets denote traceless symmetrization and projection orthogonal to the flow

$$A^{(\mu \nu)} \equiv \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} (A_{\alpha \beta} + A_{\beta \alpha}) - \frac{1}{3} \Delta^{\mu \nu} \Delta_{\alpha \beta} A^{\alpha \beta}. \quad (21)$$

The new matter coefficients $\{a_i(e, n)\}$ are needed to describe how contributions from the $q^\mu$ and $q_\nu \pi^{\mu \nu}$ terms in (14) are split between the bulk pressure and heat flow, and the heat flow and shear stress evolution equations, respectively (in other words, a whole class of equations of motion generates the same amount of entropy - see Appendix A).

Notice that the time-derivatives of heat flow, $q^\mu$, and shear stress tensor, $\pi^{\mu \nu}$, are not expressed explicitly in (18)-(19) - instead, orthogonal projections to the flow velocity vector appear (cf. Eqs. (8a)-(8c) in [22]). Reordering the equations explicitly for the time derivatives gives rise to terms $-u^\mu q_\nu Du^\nu$ and $-(\pi^{\lambda \nu} u^\mu + \pi^{\lambda \nu} u^\mu) Du_\lambda$. There is therefore no need for a kinetic theory treatment [34] to derive these terms. They were missed in Ref. [12], but they are already present in standard IS theory as a trivial consequence of the product rule of differentiation and the orthogonality of the flow velocity and shear stress/heat flow.

As we saw above, the Israel-Stewart procedure only determines the equations of motion up to nonequilibrium terms that do not contribute to entropy production. In kinetic theory, further such terms arise [23] when the vorticity

$$\omega^{\mu \nu} \equiv \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} (\partial_\alpha u_\beta - \partial_\beta u_\alpha) \quad (22)$$

is nonzero. Including the vorticity terms, the complete set of evolution equations for the dissipative currents are:

$$D \Pi = -\frac{1}{\tau_\Pi} (\Pi + \zeta \nabla \cdot u)$$

$$+ \frac{\alpha_0}{\beta_0} \partial_\mu q^\mu - \frac{\alpha_1}{\beta_0} q^\mu Du^\mu$$

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$$D q^\mu = -\frac{1}{\tau_q} \left[ q^\mu + \frac{T^2}{T} \nabla \cdot \frac{p}{\rho} \nabla \mu (\frac{\mu}{T}) - u^\mu q_\nu Du^\nu \right]$$

$$- \frac{1}{\tau_q} q^\mu \nabla \cdot \frac{\lambda}{\lambda} + D \ln \frac{\beta_1}{\beta_1}$$

$$- \frac{\alpha_0}{\beta_1} \Pi + \frac{\alpha_1}{\beta_1} (\partial_\mu \pi^{\lambda \nu} + u^\mu \pi^{\lambda \nu} \partial_\lambda u_\nu) + \frac{\alpha_0}{\beta_1} - \frac{\alpha_1}{\beta_1} (\partial_\mu \pi^{\lambda \nu} + u^\mu \pi^{\lambda \nu} \partial_\lambda u_\nu)$$

$$D \pi^{\mu \nu} = -\frac{1}{\tau_{\pi}} \left( \pi^{\mu \nu} - 2 \eta \nabla (q^\mu u^\nu) \right)$$

$$- \frac{1}{\tau_{\pi}} \left( \nabla \lambda \nabla \lambda + D \ln \frac{\beta_2}{\beta_2} \right)$$

$$- \frac{\alpha_1}{\beta_2} q^\mu Du^\nu$$

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We will refer to these equations as “complete IS”. If we ignore their tensorial structure, the equations have the general form

$$\dot{X} = -\frac{1}{\tau_X} (X - X_{NS}) + X Y_X + Z_X \quad \text{(26)}$$

for each dissipative quantity $X$, where $X_{NS}$ is the value of $X$ in Navier-Stokes theory and $Y_X$, $Z_X$ are given by the ideal hydrodynamic fields and dissipative quantities other than $X$. Therefore, Israel-Stewart theory describes relaxation towards Navier-Stokes on a characteristic time $\tau_X$, provided $|Y_X| \tau_X \ll 1$ and $|Z_X| \tau_X \ll |X_{NS}|$.

In the last step of their derivation, Israel and Stewart neglect the first terms of the second lines, the ones with a factor 1/2 (this gives the Landau frame equivalent to (7.1a)-(7.1c) in [23]), because they expect to study astrophysical systems with small gradients $|\partial^\mu u^\nu + \partial^\nu u^\mu| / T \ll 1$, $|\partial^\mu e / (Te)| \ll 1$, $|\partial^\mu n / (Tn)| \ll 1$, near a global (possibly rotating) equilibrium state. The neglected terms are products of small gradients and the dissipative quantities. We will refer to this approximation as “naive IS” [47]. In heavy ion physics applications, on the other hand, gradients $\partial^\mu u^\nu / T$, $|\partial^\mu e / (Te)|$, $|\partial^\mu n / (Tn)|$ at early times $\tau \sim 1$ fm are large $\sim O(1)$, and therefore cannot be ignored. Nevertheless, hydrodynamics may still be applicable, provided the viscosities are unusually small $\eta / s_{eq} \sim 0.1$, $s / s_{eq} \sim 0.1$, where
$s_{eq}$ is the entropy density in local equilibrium. In this case, dissipative effects are still moderate, for example, pressure corrections from Navier-Stokes theory (3)

$$\frac{\delta T_{NS}^{\mu \nu}}{p} \approx 2 \frac{\eta_s}{s_{eq}} \frac{\nabla (\mu u^\nu)}{T} + \frac{\zeta}{s_{eq}} \frac{\nabla \alpha u^\alpha}{T} \frac{\varepsilon + p}{T} \sim O \left( \frac{8 \eta_s}{s_{eq}}, \frac{4 \zeta}{s_{eq}} \right).$$  (27)

Heat flow effects can also be estimated based on (4)

$$\frac{\delta N_{NS}^{\mu}}{n} \approx \kappa_q T \frac{n}{s_{eq}} \frac{\nabla (\mu/ T)}{T}.$$  (28)

For RHIC energies and above, at midrapidity, the correction is rather small even for large $\kappa_q$ because the baryon density and therefore $\mu_B/T$ is very low. For example, in a recent ideal fluid calculation at RHIC energy [36], these ratios were $n_B/s \approx 2.2 \cdot 10^{-3}$ and $\mu_B/T \approx 0.2$ in order to reproduce the observed net baryon spectra. These choices are also supported by thermal model analyses of particle ratios which lead to $\mu_B/T \approx 0.17$ [37].

**B. Viscous equations of motion for longitudinally boost-invariant 0+1D dynamics**

At this point we specialize the equations of motion to a viscous, longitudinally boost-invariant[48] system with transverse translation invariance and vanishing bulk viscosity:

$$\dot{n} + \frac{n}{\tau} = 0 \iff n(\tau) = \frac{\tau_0 n(\tau_0)}{\tau}$$  (29)

$$\dot{\pi} + \frac{e + p}{\tau} \pi = -\frac{\pi_L}{\tau}$$  (30)

$$\tau_s \dot{\pi}_L + \pi_L \left[ 1 + \frac{\tau_s}{2\tau} + \frac{n}{\eta_s T} \left( \frac{\tau_s}{\eta_s T} \right) \right] = -\frac{4\eta_s}{3\tau}$$  (31)

$$\pi_T = -\frac{\pi_L}{2}.$$  (32)

This special case is well known in the literature [9, 34, 38] as a useful approximation to the early longitudinal expansion stage of a heavy ion collision for observables near midrapidity $\eta \approx 0$. Here $\tau \equiv \sqrt{T^2 - z^2}$ is the Bjorken proper time, and the 'dot' denotes $d/d\tau$. $\pi_L$ and $\pi_T$ are the viscous corrections to the longitudinal and transverse pressure, i.e. the $\pi_{zz}$ and $\pi_{xx} = \pi_{yy}$ components of the shear stress tensor evaluated at local rest frame[49], respectively. All the other components of the stress tensor are zero due to symmetry. There is no equation for heat flow because the symmetries of the system—longitudinal boost-invariance, axial symmetry in the transverse plane and $\eta \rightarrow -\eta$ reflection symmetry—force the heat flow to be zero everywhere. We have chosen to ignore bulk viscosity since shear viscosity is expected to dominate at RHIC. In the following we also concentrate on a system of massless particles, where bulk viscosity is zero in general. It is worth noticing that these equations are identical in both Eckart and Landau frames, but in less restricted systems where heat flow is nonzero, Eckart and Landau frames differ.

To simplify the discussion and to facilitate comparison with transport results, from here on we concentrate on a system of massless particles with only $2 \rightarrow 2$ interactions. Particle number is then conserved and the equation of state is

$$e = 3p, \quad T = \frac{p}{n}.$$  (33)

Now the density equation decouples entirely and we end up with two coupled equations for the equilibrium pressure and the viscous correction $\dot{\pi}_L$. The shear stress relaxation time (12) and the shear viscosity (13) can be recast with the transport mean free path $\lambda_{tr} \equiv 1/(n\sigma_{tr})$ as

$$\eta_s = C n T \lambda_{tr}, \quad \tau_s = \frac{3C}{2} \lambda_{tr}, \quad C \approx \frac{4}{5},$$  (34)

and (30)-(31) can then be written as

$$\dot{\pi}_L + \frac{\pi_L}{\tau} \left( \frac{2 \kappa(\tau)}{3} + 4 + \frac{\pi_L}{3p} \right) = -\frac{8p}{9\tau}.$$  (35)
where

\[ \kappa(\tau) \equiv \frac{K(\tau)}{C} = \frac{nT\tau}{\eta_s}, \quad K(\tau) \equiv \frac{\tau}{\lambda_{tr}(\tau)}. \] (37)

The ratio of expansion and scattering timescales \( K \) controls how well ideal and/or dissipative hydrodynamics applies. This is essentially the inverse of the ratio of shear stress relaxation time to hydrodynamic timescales \( \tau_\pi/\tau = 3/(2\kappa) \). \( K \) is as well a generalization of the Knudsen number \( L/\lambda \), since the shortest spatial length scale is given by gradients in longitudinal direction \( L \sim 1/(\partial_z u_z) \sim \tau \). It is also a measure of the shear viscosity to entropy density ratio because for a system in chemical equilibrium \( s_{eq} = 4n \) and thus

\[ \frac{\eta_s}{s_{eq}} = \frac{T\tau}{4K} \] (38)

(see Sec. V E for the general case).

Similar treatment to relativistic Navier-Stokes theory leads to

\[ \pi_L = -\frac{4\eta_s}{3\tau} = -\frac{4p}{3\kappa} \] (39)

and the equation of motion

\[ \dot{p} + \frac{4p}{3\tau} = \frac{4}{9\kappa(\tau)} \frac{p}{\tau}. \] (40)

As discussed in the previous Section, the viscosities in NS and IS theories differ and, therefore, \( \kappa \) in (40) is not identical to the one in (36). We will ignore the difference because in our case it is only \( \approx 5\% \).

Finally, we note that in the “naive” Israel-Stewart approximation (36) changes to

\[ \dot{\pi}_L + \frac{2\kappa(\tau)\pi_L}{3\tau} = -\frac{8p}{9\tau}. \] (41)

### IV. Basic Observables

Here we introduce the basic observables investigated in this study, and discuss their evolution based on the analytic Israel-Stewart and Navier-Stokes solutions of Appendix C. It is important to emphasize that our observations will hold only during the longitudinal expansion stage of heavy ion collisions. After some time \( \tau \sim R/c_s \), transverse expansion sets in and hydrodynamics, whether Israel-Stewart or Navier-Stokes, eventually breaks down because for expansion in three dimensions \( \lambda_{tr} \sim \tau^3/\sigma \), i.e., \( \kappa \sim \sigma/\tau^2 \rightarrow 0 \) in the hadronic world where cross sections are bounded. It is interesting to note that \( \eta_s/s_{eq} \approx const \) would not decouple even for a three-dimensional expansion (because in that case \( T \propto 1/\tau \) and thus \( \lambda_{tr} \propto \eta/p \propto \tau \), while \( \tau_{exp} = 1/(\partial_\mu u^\mu) \propto \tau \), i.e., \( \kappa \sim const \)).

Throughout this section and the rest of the paper, the subscript ‘0’ refers to the value of quantities at the initial time \( \tau_0 \) (e.g., \( A_0 = A(\tau_0) \)). The most important parameters in the problem are the initial Knudsen number \( K_0 \), or the corresponding \( \kappa_0 \), and the initial shear stress to pressure ratio \( \xi_0 \equiv \pi_{L,0}/p_0 \).

#### A. Pressure anisotropy

The magnitude of dissipative corrections can be quantified through the ratio of viscous longitudinal shear and equilibrium pressure

\[ \xi \equiv \frac{\pi_L}{p}. \] (42)

A suitable equivalent measure is the pressure anisotropy coefficient

\[ R_p \equiv \frac{p_L}{p_T} = \frac{1 + \xi}{1 - \xi^2/2}, \] (43)

which is the ratio of the transverse and longitudinal pressures \( p_T \equiv p - \pi_L/2, p_L \equiv p + \pi_L \). In the ideal hydro limit the anisotropy is unity \( R_p \rightarrow 1 \).
The time-evolution of the anisotropy coefficient is given by the equations of motion (35) and (36):

\[ \dot{R}_p = -\frac{4}{3\tau} \frac{4 + 3\kappa \xi}{(2 - \xi)^2} . \]  

Thus, in IS theory the pressure anisotropy is a constant of motion when the viscous stress is equal to its Navier-Stokes value (39), or at asymptotically late times \( \tau \to \infty \). In contrast, from NS

\[ R_p^{NS} = \frac{3\kappa - 4}{3\kappa + 2} , \]

which is only constant for \( \kappa(\tau) = \text{const} \) (constant cross section), or in the ideal hydro limit \( \kappa \to \infty \) (in which case \( R_p \to 1 \)). From the above it also follows that in the special case of our boost invariant scenario, if the cross section is constant and the shear stress starts from its Navier-Stokes value, Navier-Stokes and Israel-Stewart theory coincide.

B. Longitudinal work

Dissipation also affects the evolution of the equilibrium (or average) pressure. From (35), for ideal hydro evolution the pressure drops as \( p(\tau) \propto \tau^{-4/3} \) due to longitudinal work. In the viscous case, the work done by the system is smaller because the viscous correction to the longitudinal pressure is usually negative \( \pi_L < 0 \). Therefore, pressure decreases slower than in ideal hydro, and deviations from the ideal evolution, such as the ratio

\[ \frac{p(\tau)}{p_{\text{ideal}}(\tau)} = \frac{T(\tau)}{T_{\text{ideal}}} \quad \text{(for conserved particle number)} \quad \text{(46)} \]

can be used to quantify dissipative effects.

Studies in the past\[27, 28\] have analyzed a closely related quantity, the transverse energy per unit rapidity, \( dE_T/d\eta \). This is simply a combination of the pressure anisotropy and deviation from ideal pressure

\[ \frac{dE_T}{d\eta} = \frac{3\pi T_0 dN}{4} \left( 1 - \frac{5}{16} \xi \right) = \frac{3\pi T_0 dN}{4} \left( \frac{\tau_0}{\tau} \right)^{-1/3} \frac{p(\tau)}{p_{\text{ideal}}(\tau)} \frac{3[7 + R_p(\tau)]}{8[2 + R_p(\tau)]} \]

provided the quadratic ansatz (11) is applicable (see Appendix D1).

We can make a few generic observations based on the analytic Israel-Stewart and Navier-Stokes results (C3)-(C4), (C8), (C21)-(C22), and (C29) from Appendix C. For a constant cross section, \( p/p_{\text{ideal}} \) grows without bound - dissipative corrections keep accumulating forever. The influence of the initial shear stress, or equivalently shear stress to pressure ratio \( \xi_0 \equiv \xi(\tau_0) \), is of \( O(\xi_0/\kappa_0) \) and thus vanishes in the large \( \kappa_0 \) limit. At late times \( \tau \gg \tau_0 \), for \( K_0 \gtrsim 2 \) and not too large initial shear stress to pressure ratio \( |\xi_0| \ll 2\kappa_0 \)

\[ \left( \frac{p}{p_{\text{ideal}}} \right)_{\sigma = \text{const}} \approx N \left( \frac{\tau}{\tau_0} \right)^\beta , \quad \beta \approx \frac{4}{9\kappa_0} \left( 1 - \frac{2}{3\kappa_0^2} \right) , \quad N \approx 1 + \frac{2}{3\kappa_0^2} + \frac{4}{3\kappa_0^4} - \frac{\xi_0}{2\kappa_0} , \quad (48) \]

i.e., for \( \tau \approx 10\tau_0 \) and \( K_0 = 2 \) the accumulated pressure increase is \( p/p_{\text{ideal}} \approx 1.3 \), while \( p/p_{\text{ideal}} \approx 1.15 \) if \( K_0 = 5 \). For a scale invariant system with \( \eta/s = \text{const} \), on the other hand, dissipative effects are more moderate for the same \( K_0 \) and at late times approach a finite upper bound

\[ \left( \frac{p}{p_{\text{ideal}}} \right)_{\eta/s = \text{const}} \approx \left[ 1 - \frac{2}{3\kappa_0} \left( \frac{\tau_0}{\tau} \right)^{2/3} \right] \left( 1 + \frac{2}{3\kappa_0} - \frac{\xi_0}{2\kappa_0} \right) \to 1 + \frac{2}{3\kappa_0} - \frac{\xi_0}{2\kappa_0} . \]

This is because scale invariant systems turn more and more ideal hydrodynamic as they evolve (as long as their expansion is only longitudinal). For the same \( K_0 = 2 \) and \( 5 \) with \( \xi_0 \approx 0 \), the bounds are modest, \( p/p_{\text{ideal}} \lesssim 1.25 \) and \( \lesssim 1.1 \), respectively.

C. Entropy

Another quantitative measure of the importance of dissipative effects is entropy production. Here we consider an ultra-relativistic system (thus \( \Pi = 0 \) and \( \beta_2 = 3/(4\rho) \)) with \( 2 \to 2 \) interactions, 1D Bjorken boost invariance, and
transverse translational, axial, and \( \eta \rightarrow -\eta \) reflective symmetries (imply \( q^\mu = 0 \)). Therefore, the entropy current (14) simplifies to

\[
S^\mu = \bar{s} u^\mu , \quad \bar{s} = s_{\text{eq}} - \frac{9 \pi^2}{16 \mu T} 
\]

(50)

where

\[
s_{\text{eq}} = n(4 - \chi) , \quad \chi \equiv \ln \frac{n}{n_{\text{eq}}(T)} = \frac{\mu}{T} 
\]

(51)

and

\[
n_{\text{eq}}(T) = \frac{g}{\pi^2} T^3 
\]

(52)

is the particle density in chemical equilibrium at temperature \( T \) for massless particles of degeneracy \( g \) in the Boltzmann limit. Dissipative contributions in the entropy density \( \bar{s} \) are negative, in accordance with the fundamental principle of maximal entropy in equilibrium.

The equations of motion (35)-(36) imply an entropy production rate

\[
\partial_\tau S^\mu = \frac{1}{\tau} \partial_\tau (\tau \bar{s}) = \frac{3 \kappa n}{4 \tau} \xi^2 \geq 0 . 
\]

(53)

Equivalently, the entropy per unit rapidity

\[
\frac{dS}{d\eta} = \tau A_T \bar{s} 
\]

(54)

never decreases

\[
\partial_\tau \left( \frac{dS}{d\eta} \right) = \frac{3 \kappa}{4 \tau} \frac{dN}{d\eta} \xi^2 \geq 0 . 
\]

(55)

Here \( A_T \) is the transverse area of the system, and in the last step we substituted the rapidity density \( dN/d\eta = \tau A_T n \).

Equation (54) is a special case of a generalized conservation law (B7) applied to the entropy current \( S^\mu \)

\[
\tau \int dx_T^2 \partial_\mu S^\mu = \partial_\tau \left( \tau \int dx_T^2 S_{0}^{LR} \right) - \partial_\eta \int dx_T^2 S_{3}^{LR} . 
\]

(56)

Analogous relations can be obtained for the energy, momentum, and charge density. In 0+1D these are quite trivial - they respectively reproduce (35), give identically zero, and \( dN/d\eta = \text{const} \). In higher dimensions, however, the generalized conservation laws present important constraints that any solution must satisfy at all times and, therefore, they are ideal for testing the accuracy of numerical solutions at each time step (see Appendix B).

Only the complete set of Israel-Stewart equations gives the correct rate of entropy production. The 'naive' approximation does not guarantee a monotonically increasing entropy

\[
(\partial_\mu S^\mu)^{\text{naive IS}} \overset{1S}{=} \frac{3 \kappa n}{4 \tau} \xi^2 \left( 1 - \frac{\xi + 4}{2 \kappa} \right) , 
\]

(57)

unless \( \kappa \) is sufficiently large and, away from equilibrium, it underpredicts for a given \( \xi \) the entropy production rate[50] (since \( \xi < -1 \) is unphysical). In contrast, the second law of thermodynamics does hold for Navier-Stokes

\[
(\partial_\mu S^\mu)^{\text{NS}} \overset{NS}{=} \frac{3 \kappa n}{4 \tau} \xi^2 \geq 0 . 
\]

(58)

The NS result is the same as (53) but with the shear stress restricted to its Navier-Stokes value. We note that in Israel-Stewart theory the naive entropy per unit rapidity, defined using the equilibrium entropy density

\[
\frac{dS'}{d\eta} = s_{\text{eq}} \tau A_T 
\]

(59)

does not increase monotonically. Rather, it increases (decreases) for negative (positive) \( \pi_L \).
Based on the analytic Israel-Stewart and Navier-Stokes results in Appendix C, we can outline general expectations for the entropy evolution. For a constant cross section by late times $\tau \gg \tau_0$ the entropy increase relative to the initial entropy is logarithmic with time

$$\left[ \frac{(dS/d\eta)}{(dS/d\eta)_0} \right]_{\eta = \text{const}} - 1 \approx \frac{1}{4 - \chi_0} \left( 3 \ln \frac{p_{\text{total}}}{p_0} - 9 \xi_0^2 \right) \approx \frac{1}{4 - \chi_0} \left( 3 \beta \ln \frac{\tau}{\tau_0} - \frac{3}{\kappa_0^2} + \frac{16}{3 \kappa_0^4} \right) \left( 2 - \frac{3 \xi_0}{4} \right)$$

(60)

where we considered initial conditions not too far from local equilibrium. E.g., by $\tau = 10 \tau_0$ with $K_0 = 2$ and chemical equilibrium initial conditions $\approx 20\%$ entropy is produced, while $\approx 10\%$ with $K_0 = 5$. For a scale invariant system with $\eta_s/s_{eq} = \text{const}$, on the other hand, entropy production is slower for the same $K_0$ and saturates at late times

$$\left[ \frac{(dS/d\eta)}{(dS/d\eta)_0} \right]_{\eta = \text{const}} - 1 \approx \frac{1}{4 - \chi_0} \left( 1 - \eta_s \right) \left( \eta_s \right) \frac{2}{\kappa_0} \left( \frac{16}{3 \kappa_0^4} \right)$$

(61)

For the same $K_0 = 2$ and 5 (and $\xi_0 \approx 0$), the entropy increase by $\tau = 10 \tau_0$ is smaller, $\approx 15$ and $\approx 6\%$, respectively. Based on this simple analytic formula for entropy production, we also confirm the results of Ref. [38], which considered IS hydrodynamics with a unique initial condition $\xi_0 \approx -16/(9 T_0 \tau_0) \times \eta_s/s_{eq}$ where $T_0 \approx 0.39 \text{ GeV} \times (0.14 \text{ fm}/\tau_0)^{1/3}$ and $\tau_0$ was varied between 0.5 and 1.5 fm.

V. REGION OF VALIDITY FOR DISSIPATIVE HYDRODYNAMICS

Here we determine the region of validity of dissipative hydrodynamics via comparison to full nonequilibrium twobody transport theory[26–29]. We consider two scenarios: Scenario I with a constant cross section, which is least favorable for hydrodynamics; and Scenario II with a growing $\sigma \propto \tau^{2/3}$, which is the most optimistic for applicability of hydrodynamics and is very close to $\eta_s/s_{eq} = \text{const}$ as we show in Appendix C. In the same Appendix we also study a scenario with $\sigma \propto 1/T^2$ that turns out to be close to Scenario II but with stronger dissipative effects, and discuss analytic Navier-Stokes and (approximate) Israel-Stewart solutions.

Due to scalings of the equations of motion, the results presented here are rather general. Equations (35)-(36) are invariant under rescaling of time, and/or joint rescaling of the pressures $p$ and $\pi_L$, provided the dimensionless $\kappa$ depends only on $p$, $\pi_L$, $\tau/\tau_0$ and no additional scales (all solutions studied here satisfy this condition). The same scalings are exhibited by the transport[28]. For a physically reasonable $p_0 > 0$, it is therefore convenient to consider dimensionless pressure variables $\tilde{p}(\tau) \equiv p(\tau)/p_0$ and $\pi_L(\tau)/p_0$, for which the solutions only depend on $\tilde{\tau} \equiv \tau/\tau_0$, $\kappa_0 \equiv K_0/C$ and the initial condition $\xi_0 \equiv \pi_L(0)/p_0$.

Unless stated otherwise, we initialize the transport based on the quadratic form (11). In our case of an ultrarelativistic system ($\epsilon = 3p$) in the Boltzmann limit with vanishing bulk pressure and heat flow

$$D^\mu = 0 , \quad C^{\mu\nu} = \frac{\pi^{\mu\nu}}{8p} \quad \Rightarrow \quad \phi_G(\eta = 0, p) = \frac{\xi}{16} \frac{2 p_\perp^2 - p_T^2}{T^2} ,$$

(62)

where $p_\perp \equiv \sqrt{p_x^2 + p_y^2}$ is the transverse momentum. We ensure nonnegativity of the phase space distribution via a $\Theta$-function

$$f(\eta = 0, p, \tau = \tau_0) = \frac{F(\xi)}{A_T \tau_0^3} \frac{dN}{d\eta} \frac{e^{-p/T}}{8 \pi T^3} \left[ 1 + \phi_G(\eta, p) \right] \Theta(1 + \phi_G(\eta, p))$$

(63)

where $A_T$ is the transverse area of the system (with the elimination of negative phase space contributions, a normalization factor $F(\xi) \leq 1$ is needed to set a given $dN/d\eta$). The cutoff does not affect the general scalings of transport solutions but does influence the initial pressure anisotropy (for example, values $R_p = 0.3$ and 1.75 set based on (62) change to $R_p \approx 0.476$ and 1.693 when the cutoff is applied). Therefore, we initialize hydrodynamics with a shear stress $\pi_L$ that gives the same initial pressure anisotropy as the transport.

The transport solutions were obtained using the MPC algorithm[39], which employs the particle subdivision technique to maintain covariance[26, 28]. Transverse translational invariance was maintained in the calculation through periodic boundary conditions in the two transverse directions. A longitudinal boost invariant system was initialized in a coordinate rapidity interval $-5 < \eta < 5$, and observables were calculated via averaging over $-2 < \eta < 2$ with proper Lorentz boosts of local quantities to $\eta = 0$. 
FIG. 1: Time evolution of pressure anisotropy $R_p \equiv p_L/p_T$ from covariant transport (solid lines with symbols) and Israel-Stewart dissipative hydrodynamics (solid lines) as a function of $K \equiv \tau / \lambda_0(\tau)$, from local equilibrium initial conditions $\pi_L(\tau_0) = 0$. Results for Navier-Stokes (dotted lines) and free streaming (dotted line with circles) are also shown. Left: $\sigma = \text{const}$ scenario, for which the curves are labeled by $K(\tau) = \text{const} = K_0 = 1, 2, 3, 6.67, \text{ and } 20$. For $K_0 = 1$, the Navier-Stokes result is negative and therefore not visible. Right: $\sigma \propto \tau^{2/3}$ scenario, for which $\eta_s/s_{eq} \approx \text{const}$ and the curves are labeled by the initial $K_0 = K(\tau_0) = 1, 2, 3$, and $6.67$.

A. Pressure anisotropy

Figure 1 shows the pressure anisotropy $p_L/p_T$ evolution as a function of the rescaled proper time $\tilde{\tau} = \tau/\tau_0$ from the transport (solid lines with symbols) and Israel-Stewart hydrodynamics (solid lines without symbols) with local equilibrium initial condition. The left panel shows calculations for the $\sigma = \text{const}$ scenario. For $K_0 = 1$, the anisotropy from IS hydro starts to fall rapidly below the transport above $\tau \gtrsim 2\tau_0$ and it is a factor $\sim 5$ smaller by late $\tau \sim 10\tau_0$. Clearly, the system cannot stay near equilibrium when the rate of scatterings equals the expansion rate. With increasing $K_0$, the undershoot becomes smaller and gradually vanishes as $K_0 \to \infty$. The difference is only $\sim 10\%$ already at $K_0 = 3$, and is rather small by $K_0 \approx 7$.

The right panel shows the same but for the growing cross section scenario with $\eta_s/s_{eq} \approx \text{const}$. The situation of course improves because in this case $K$ increases with time. For $K_0 = 1$, IS hydro undershoots the pressure anisotropy from the transport only by $\sim 20\%$ and the differences vanish at late times (since in this case both theories converge to $R_p = 1$ as $\tau \to \infty$). $\sim 10\%$ accuracy is achieved already for $K_0 = 2$, while for $K_0 = 3$, IS hydro is accurate to a few percent.

Moreover, the above findings hold for a wide range of the initial conditions, including large initial pressure anisotropies, as shown in Figures 2 and 3. These figures are for the same calculation but with $R_p(\tau_0) = 0.476$ and $1.693$, respectively (which correspond to $\xi_0 = -0.423$ and $0.375$). We emphasize that the results hold only if nonequilibrium corrections are close to the form (11) suggested by Grad. For such class of initial conditions, however, we find that Israel-Stewart hydrodynamics can well approximate the transport ($\sim 10\%$ accuracy) provided $K_0 \geq 3$, even for the most pessimistic constant cross section scenario. If $\eta_s/s_{eq} \approx \text{const}$, only $K_0 \geq 2$ is needed. We stress that in either case, there is no need for the initial conditions to be near the Navier-Stokes limit.

This is quite remarkable because from Figs. 1-3 it is clear that already the early evolution differs between IS
FIG. 2: Same as Fig. 1 but for an initial pressure anisotropy $R_p(\tau_0) = 0.476 (\xi_0 = -0.423)$. In the left plot, the Navier-Stokes curve for $K_0 = 1$ is negative and therefore not visible.

FIG. 3: Same as Fig. 1 but for an initial pressure anisotropy $R_p(\tau_0) = 1.693 (\xi_0 = 0.375)$. In the left plot, the Navier-Stokes curve for $K_0 = 1$ is negative and therefore not visible.
hydrodynamics and transport. E.g., for an equilibrium initial condition \((\xi(\tau_0) = 0)\), IS hydrodynamics (44) gives
\[
R_{p}^{\text{IS}}(\tau) = 1 - \frac{4(\tau - \tau_0)}{3\tau_0} + \mathcal{O}((\tau - \tau_0)^2) \tag{64}
\]
for any initial value and evolution scenario for \(\kappa\). From covariant transport, on the other hand (see Appendix D 2)
\[
R_{p}^{\text{transp}}(\tau) = 1 - \frac{8(\tau - \tau_0)}{5\tau_0} + \mathcal{O}((\tau - \tau_0)^2) . \tag{65}
\]
I.e., pressure anisotropy develops, universally, 20% faster from the transport than from IS hydrodynamics (if the evolution starts from equilibrium).

This illustrates a limitation of the hydrodynamic description of transport solutions. Similar discrepancies were observed in [8] in the early evolution of differential elliptic flow \(v_2(p_T)\). Remarkably, in our case, though the transport develops deviations from equilibrium faster, its rate of departure slows down quicker, which at intermediate times results in smaller accumulated dissipative corrections to the pressure anisotropy than from IS hydrodynamics. Eventually, the hydrodynamic evolution “catches up” to the transport, except for low \(K < 3\) in the \(\sigma = \text{const}\) scenario.

Figures 1-3 also show the Navier-Stokes approximation (dotted lines without symbols) for each of the Israel-Stewart results. By late times, the Navier-Stokes and Israel-Stewart solutions converge for both cross section scenarios, independently of the initial pressure anisotropy (for \(\sigma = \text{const}\) and \(K_0 = 1\), the NS anisotropy is negative and therefore not visible in the plots). However, the applicability of Navier-Stokes theory at early times depends, besides the value of \(K_0\), strongly on how far the initial shear stress is from its Navier-Stokes value (39). Navier-Stokes assumes that shear stress, and therefore the pressure anisotropy, relaxes immediately, but relaxation happens over a finite time. The approach toward the Navier-Stokes limit is governed by \(\tau_\pi = 3\tau/(2\kappa)\), therefore Navier-Stokes becomes applicable only after some time \(\Delta\tau \sim |R_0 - R_{NS}|/\tau_0/\kappa\). Note that the initial slope of the \(R(\tau)\) curves does not always reflect \(\tau_\pi\) directly because it is given by the initial derivative of \(\xi\)
\[
\dot{R}(\tau) \sim \frac{3}{2}\xi^2(\tau) = -\frac{3}{2\tau_\pi}(\xi - \xi_{NS}) + \mathcal{O}(1)\frac{\xi}{\tau} \tag{66}
\]
where we combined (26), (35), the observations that \(Y \sim \mathcal{O}(1)/\tau\) and \(Z = 0\), and assumed \(\xi\) is small. For local equilibrium initial conditions the slope of \(R(\tau)\) is therefore \(\sim \mathcal{O}(1)\xi_{NS}/\tau_\pi \sim \mathcal{O}(1)/\tau\), independently of \(K_0\) (cf. Figure 1 and also (64)). For initial shear stresses far away from the Navier-Stokes limit, on the other hand, the slope \(\sim \mathcal{O}(1)\xi/\tau_\pi \propto \kappa\) steepens with increasing \(K\) as seen in Figures 2 and 3.

The inaccurate description of early shear stress evolution in Navier-Stokes has a cumulative effect on the evolution of thermodynamic quantities, such as the pressure and the entropy, as we show in the next two Sections. Of course, the errors are proportional to ratio of the time the system spends away from the NS limit and the hydrodynamic timescale, i.e., \(\Delta\tau/\tau_0 \sim 1/\kappa\).

### B. Pressure evolution

Now we turn to the evolution of the (average) pressure. In ideal hydrodynamics \((K_0 \to \infty)\) the pressure drops rapidly with time \(p_{id} \propto \tau^{-4/3}\). Therefore it is more convenient to study dissipative effects relative to ideal hydrodynamics through the ratio \(p(\tau)/p_{id}(\tau)\).

Figure 4 shows the pressure relative to that in ideal hydrodynamics as a function of the rescaled proper time \(\hat{\tau} = \tau/\tau_0\) from the transport (solid lines with symbols) and Israel-Stewart hydrodynamics (solid lines without symbols) with local equilibrium initial condition. The left panel shows calculations for the \(\sigma = \text{const}\) scenario. For all \(K_0\) values, the evolution starts out the same between IS hydro and transport but then the hydro starts to accumulate deviations because it follows the shear stress evolution only approximately. For \(K_0 = 1\), IS hydro maintains 10% accuracy in the magnitude of dissipative corrections (i.e., \(p/p_{id} - 1\)) only up to \(\tau \approx 4\tau_0\). As \(K_0\) increases, the situation improves gradually, for \(K_0 = 3\), 10% accuracy holds up to \(\tau \approx 10\tau_0\), and for \(K_0 \approx 7\) the hydro stays within a few percent of the transport even until \(\tau = 20\tau_0\).

The right panel shows the same but for the growing cross section scenario with \(\eta_{s}/s_{eq} \approx \text{const}\). This scenario is more favorable for the hydrodynamic approximation because \(K \sim \tau^{2/3}\) grows with time. For \(K_0 = 1\), the error in the dissipative correction \((p/p_{id} - 1)\) is less than 10% up to \(\tau \approx 5\tau_0\), and already for \(K_0 = 2\) IS hydro is accurate to within better than 10% throughout the whole range \(\tau \leq 20\tau_0\) studied. The pressure evolution results therefore reinforce the regions of validity found in the previous Section \((K_0 \gtrsim 3\) for \(\sigma = \text{const}\), and \(K_0 \gtrsim 2\) for \(\eta_{s}/s_{eq} \approx \text{const}\)).

Clearly, the region of applicability for Navier-Stokes is more limited (Figure 4, dotted lines without symbols). For low \(K_0\), it overestimates the pressure corrections not only at late times but also at early \(\tau \sim f ew \times \tau_0\). \(K_0 \approx 7\) is
barely sufficient for 10% accuracy in viscous corrections for $\eta_s/s_{eq} \approx \text{const}$, but it is not enough in case of $\sigma = \text{const}$. Based on the trends with increasing $K_0$, we estimate that $K_0 \gtrsim 9 - 10$ is needed for Navier-Stokes with $\sigma = \text{const}$ to deviate less than 10% from the viscous effects calculated with the transport. Therefore, for local equilibrium initial conditions, Navier-Stokes theory becomes applicable at about three times shorter mean free paths, or equivalently three times larger longitudinal proper time $\tau$ (i.e., three times slower longitudinal expansion), than Israel-Stewart theory.

C. Entropy

Now we proceed with results on entropy production. In transport theory, the entropy current is defined as

$$S^\mu(x) = -\int \frac{d^3p}{p^0} p^\mu f(x, p) \left[ \ln \left( \frac{(2\pi)^3}{g} f(x, p) \right) - 1 \right]$$

(67)

where $g$ is the number of internal degrees of freedom. This nonlinear function of the phase-space density $f$ is cumbersome to evaluate with the MPC code, and therefore we here opt for an approximate result based on the truncated Israel-Stewart expression (50), evaluated using the pressure and shear stress from the transport. This includes dissipative corrections to the entropy up to quadratic order in $\phi$.

In the most dissipative $\sigma = \text{const}$ scenario with $K_0 = 1$, there is about 30% additional entropy produced by late times $\tau/\tau_0 \sim 10 - 20$ as can be seen in Figure 5 (left plot). For $\eta_s/s_{eq} \approx \text{const}$ (right plot), the same $K_0 = 1$ yields only about 20% extra entropy. With increasing $K_0$ entropy generation gradually weakens and by $K_0 \sim 7$ it is only 10 and 5 %, respectively.

The Israel-Stewart results are within 15% of the approximate transport results already for $K_0 = 1$, and about 10% accuracy in the calculated dissipative effect is achieved for $K_0 \gtrsim 3$ (for $\sigma = \text{const}$) and $K_0 \gtrsim 2$ (for $\eta_s/s_{eq} \approx \text{const}$). In contrast, the Navier-Stokes strongly overpredicts the entropy, unless $K_0$ exceeds about 6 for $\sigma = \text{const}$ or $\approx 3$ for $\eta_s/s_{eq} \approx \text{const}$. The bounds for 10% accuracy are in agreement with those found previously in Sec. VB.

D. Limitations of the 'naive' Israel-Stewart approximation

Now we discuss the applicability of the 'naive' Israel-Stewart equations. Figure 6 compares the pressure evolution in complete Israel-Stewart theory to that in the naive approximation, for local equilibrium initial conditions ($\xi_0 = 0$), as a function of the rescaled proper time $\tau/\tau_0$. Clearly, the naive result overshoots the pressure both for the constant cross section scenario and for $\eta_s/s_{eq} \approx \text{const}$, unless $K_0$ is large. This confirms expectations based on the analytic solutions in App. C. Though the 'naive' theory converges to the correct result at large enough $K_0 \sim 7 - 20$, comparison with Fig. 4 tells that it is even less accurate than Navier-Stokes theory.
FIG. 5: Same as Fig. 1 except for the time evolution of the entropy per unit rapidity, normalized by its initial value (note the linear time axis used this time). For the transport, entropy was calculated approximately using the Israel-Stewart entropy expression (50). Chemically equilibrated initial conditions (i.e., $\chi_0 = 0$) were assumed.

FIG. 6: Time evolution of the (average) pressure from complete Israel-Stewart theory (solid lines) and the ‘naive’ Israel-Stewart approximation (dotted) as a function of $K \equiv \tau / \lambda_{tr}(\tau)$, for local equilibrium initial conditions $\pi_L(\tau_0) = 0$. The pressure is plotted normalized to the pressure $p_{\text{ideal}}(\tau) = p_0(\tau_0/\tau)^{4/3}$ in ideal hydrodynamics. Left: $\sigma = \text{const}$ scenario, in which case $K(\tau) = \text{const} = K_0 = 2, 3, 6.67, 20$. Right: $\eta/s \approx \text{const}$ scenario, for which $\eta_s/s_{\text{eq}} \approx \text{const}$ and the curves are labeled by the initial $K_0 = K(\tau_0) = 1, 2, 3, 6.67$.

The reason for the large errors is that away from local equilibrium the ‘naive’ approach drives the shear stress more negative (compare (36) and (41), and note that typically $\pi_L < 0$). This is demonstrated in Fig. 7 where we plot the pressure anisotropy $R_p$, which is a monotonic function of $\xi = \pi_L/p$. For $\sigma = \text{const}$, we find that the naive approach saturates the anisotropy at a lower value than the complete theory, confirming analytic expectations in App. C1. For $\eta_s/s_{\text{eq}} \approx \text{const}$, the system does approach ideal hydrodynamic behavior eventually, however that occurs on a much longer timescale than from complete Israel-Stewart theory. This is in agreement with expectation based on the analytic solutions (C34)-(C36).

The pressure anisotropy results further reinforce our conclusion that the ‘naive’ Israel-Stewart approximation is poorer than Navier-Stokes (cf. Fig. 1). In heavy-ion collisions, gradients are large, at least initially, and therefore cannot be ignored even if dissipative corrections (e.g., $\pi_L/p$) are small.
E. Implications for heavy-ion physics

Having determined the region of validity (defined as 10% accuracy in dissipative effects) for Israel-Stewart and Navier-Stokes hydrodynamics in terms of the initial ratio of the expansion and scattering timescales $K_0 = \tau_0/\lambda_{tr,0}$

$$K_0^{IS} \gtrsim 3 \quad K_0^{NS} \gtrsim 9 \quad (\sigma = const)$$

$$K_0^{IS} \gtrsim 2 \quad K_0^{NS} \gtrsim 6 \quad (\eta_s/s_{eq} \approx const),$$

we now turn to implications for heavy-ion collisions. From (37), (51), and (52),

$$\kappa_0 = \frac{T_0\tau_0}{4 - \chi_0 \eta_s/\eta_{s,eq}} \approx 15.9 \times \frac{1}{1 - \chi_0/4} \left( \frac{T_0}{1 \text{ GeV}} \right) \left( \frac{\tau_0}{1 \text{ fm}} \right) \left( \frac{1/(4\pi)}{\eta_{s,eq}/s_0} \right), \quad K_0 \approx 0.8\kappa_0.$$  \hspace{1cm} (70)

Therefore, we can place an upper limit on the (initial) shear viscosity for which IS or NS reproduces with better than 10% accuracy the viscous corrections to basic observables such as pressure and entropy:

$$\frac{4\pi\eta_{s,0}}{s_{eq,0}} \bigg|_{IS} \lesssim 0.8T_0\tau_0 \quad \frac{4\pi\eta_{s,0}}{s_{eq,0}} \bigg|_{NS} \lesssim 0.25T_0\tau_0 \quad (\sigma = const)$$

$$\frac{4\pi\eta_{s}}{s_{eq}} \bigg|_{IS} \lesssim 1.2T_0\tau_0 \quad \frac{4\pi\eta_{s}}{s_{eq}} \bigg|_{NS} \lesssim 0.4T_0\tau_0 \quad (\eta_s/s_{eq} \approx const)$$

where we assumed chemical equilibrium initial conditions ($\chi_0 = 0$). If the shear viscosity of dense quark-gluon matter is bounded from below by $4\pi\eta_s/s_{eq} \gtrsim 1$, as has been conjectured recently, then the situation for Israel-Stewart is close to marginal. For $\eta_s/s_{eq} = 1/(4\pi)$, typical parton transport initial conditions ($T_0 = 0.7 \text{ GeV}, \tau_0 = 0.1 \text{ fm}$) translate into $K_0 \lesssim 1$, for which IS is not applicable for either of Scenario I or II, while for typical hydrodynamic initial conditions ($T_0 \sim 0.38 \text{ GeV}, \tau_0 = 0.6 \text{ fm}$) we have $K_0 \gtrsim 3$, sufficient for both scenarios (barely for $\sigma = const$).

On the other hand, Navier-Stokes may be marginally applicable only if $\eta_s/s_{eq} \lesssim 0.5/(4\pi)$ throughout the whole evolution, at least based on this 0+1D study, where acausal artifacts and instabilities do not arise. We emphasize that the bound quoted here is for initial conditions close to local equilibrium. The accuracy of the Navier-Stokes approximation strongly depends on how far the initial shear stress is from the Navier-Stokes value. If the evolution starts out near the Navier-Stokes limit, we expect Navier-Stokes to be accurate up to higher viscosities.

Within the region of applicability of Israel-Stewart, dissipative corrections to the average pressure and the entropy are modest and stay below $\sim 20\%$ even up to late times $\tau \lesssim 10\tau_0$. This may serve as a useful “rule of thumb” applicability condition for hydrodynamics: if dissipative corrections to average pressure and the entropy calculated from hydrodynamics are significantly larger than $20\%$, the validity of hydrodynamics is questionable.

The above findings reinforce a recent calculation\cite{18} in 2+1D that found good agreement between IS hydrodynamics and $2 \rightarrow 2$ transport, for conditions expected in $Au + Au$ at $\sqrt{s_{NN}} \sim 200 \text{ GeV/nucleon}$ at RHIC, in case of a small shear viscosity to entropy density ratio $\eta_s/s_{eq} \approx 1/(4\pi)$ (on average). The same study also found good agreement
between the two theories for a large constant transport cross section $\sigma_t \approx 13$ mb. That is also in line with our results here because it corresponds to $4\pi \eta_s/s_{eq}(\tau_0) \approx 0.25$ in the center of the collision zone, i.e., initially $\eta_s/s_{eq} \lesssim 1/(4\pi)$ in most of the system.

Finally we note that the applicability of the hydrodynamic approach on very short time and length scales is another important question. In typical real-life problems $T_0/\tau_0 \gg 1$ because the hydrodynamic expansion timescale $\tau$ is by orders of magnitude larger than the quantum (energy) timescale $1/T$. This also leaves ample room to make hydrodynamics applicable ($\kappa_0 \gg 1$) even for appreciable viscosities. In the heavy-ion case, however, the two timescales are comparable $T_0/\tau_0 \sim O(1)$, and therefore a macroscopic treatment may be marginal.

VI. CONCLUSIONS

Based on comparison to covariant transport theory, we explore the region of validity of Israel-Stewart and Navier-Stokes hydrodynamics in heavy-ion physics applications. We follow the evolution of the average pressure, pressure anisotropy, and entropy for a massless ideal gas in 0+1D longitudinally expanding Bjorken geometry. Binary $2 \rightarrow 2$ interactions are considered for two main scenarios, a fixed cross section $\sigma = \text{const}$ (Scenario I, pessimistic for hydrodynamics) and a scale invariant system with $\eta_s/s_{eq} \approx \text{const}$ (Scenario II, optimistic for hydrodynamics).

We find (Sec. V) that dissipative effects calculated from Israel-Stewart hydrodynamics reproduce those from the transport to within 10%, provided initially the expansion timescale is three (for Scenario I) or two (for Scenario II) times larger than the transport mean free path, i.e., the initial Knudsen number $K_0 = \tau_0/\lambda_{tr,0} \gtrsim 3$ or 2. When this criterion is fulfilled, Israel-Stewart is accurate even if initial pressure anisotropies are large $p_L/p_T \sim 0.4 - 1.7$ - there is no need to start near the Navier-Stokes limit. On the other hand, same accuracy from Navier-Stokes requires three times larger $K_0$, if the expansion starts from local thermal equilibrium (unlike for Israel-Stewart, the applicability of Navier-Stokes depends strongly on how far the initial shear stress is from its Navier-Stokes value). We emphasize that these findings apply only when initial viscous corrections are of the quadratic form suggested by Grad (11).

These results imply that (Sec. V E), for typical heavy ion initial conditions at RHIC energies, Israel-Stewart hydrodynamics is accurate up to $\eta_s/s_{eq} \lesssim 1.5/(4\pi)$, while for Navier-Stokes $\eta_s/s_{eq} \lesssim 0.5/(4\pi)$ is needed. This is supported by a recent 2+1D calculation[18] that finds good agreement between Israel-Stewart and Navier-Stokes for $\eta_s/s_{eq} \approx 1/(4\pi)$, and also for a large $\sigma_t \approx 13$ mb.

In addition, we test the accuracy of the naive Israel-Stewart approximation (Sec. V D) that neglects products of gradients and dissipative quantities in the equations of motion, and find that it has an even more limited applicability than Navier-Stokes.

We also compare in detail (App. C) Israel-Stewart and Navier-Stokes solutions in 0+1D for four scenarios, $\sigma = \text{const}$, $\sigma \propto 1/T^2$, $\sigma \propto \tau^{2/3}$ and $\eta_s/s_{eq} = \text{const}$, and find that results for the latter two are almost identical, even at low initial Knudsen numbers $K_0 \approx 1$. Moreover, we obtain analytic Israel-Stewart and Navier-Stokes solutions in 0+1D, which are useful for quick estimates (Secs. IV B and IV C) and to test numerical solution techniques. We also derive additional tests (App. B) based on generalized conservation laws for conserved currents, energy-momentum, and entropy, which can be utilized to verify the accuracy of numerical Israel-Stewart solvers in 1+1, 2+1, and 3+1 dimensions.

Finally we emphasize that the current study is limited to a massless ideal gas with particle number conserving interactions in 0+1D Bjorken geometry. The influence of the transverse expansion will be quantified in a future paper (requires at minimum a 1+1D approach). It will be also important to check how the results depend on the equation of state and the presence of particle non-conserving processes, such as radiative $2 \leftrightarrow 3$. For a nonconformal equation of state, bulk viscosity may become important[41, 42]. Ideally, one should also test the accuracy of the hydrodynamic approximation for nonequilibrium theories other than covariant transport.

Acknowledgments

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APPENDIX A: ORIGIN OF $a_0, a_1, a'_0, a'_1$ IN THE ISRAEL-STEWART EQUATIONS OF MOTION

The equations of motion (23)-(25) reproduce the entropy production rate (16) only approximately, up to typically small quartic and higher-order corrections in dissipative quantities. With $a_i \equiv 0 \equiv a'_i$, a contribution

$$\Pi q^\mu T \nabla_\mu (\alpha_0 / T) - q_\mu \pi^\mu T \nabla_\mu (\alpha_1 / T)$$

(A1)

would be missing from $T \partial_\mu S^\mu$ in (16). These terms are bilinear in the dissipative quantities and, therefore, can be split arbitrarily between the bulk pressure and heat, and heat and shear equations of motion. I.e., with

$$T \partial_\mu (\alpha_i / T) \equiv a_\mu^\nu + a'_{\mu \nu}$$

(A2)

(16) is identically satisfied but contributions to the equations of motion depend on $a^\nu_i \beta^\nu_0 D = (\ldots) + a^\nu_1 \beta^\nu_1 D a_0^\nu$, where we chose $1/T$ and $\mu/T$ as the two independent variables instead of $e$ and $n$. But the three terms are not independent - energy-momentum conservation (1) and the Gibbs-Duhem relation $s dT = dP - n d\mu$ provide one constraint

$$\frac{1}{T} \Delta^\nu_\alpha D u_\alpha + \nabla_\nu \frac{1}{T} = \frac{n}{\varepsilon + p} \nabla_\nu \mu$$

(A7)

and $\nabla_\nu (\mu/T)$ may be ignored, at least parametrically, because it is proportional to the heat flow (4) in the first-order (Navier-Stokes) theory. Therefore, to leading accuracy only one scalar function enters and we can write

$$a_\mu^\nu = -a_i(e,n) D u_\mu^\nu$$

(A8)

Analogous arguments give

$$T \nabla_\nu (\alpha_i / T) \approx T \frac{\partial (\alpha_i / T)}{\partial (1/T)} \nabla_\nu 1 = \frac{\partial (\alpha_i / T)}{\partial (1/T)} \nabla_\nu \alpha D u_\alpha$$

(A9)

from which (20) follows.

We plan to revisit the above approximations in a future study. In any case, they do not influence our 0+1D calculations here because the $a_i$ terms do not play a role (heat flow vanishes by symmetry).

APPENDIX B: GENERALIZED CONSERVATION LAWS

Here we present general relations of the form

$$\frac{d A(\tau)}{d\tau} = B(\tau)$$

(B1)

that can be used to test the accuracy of numerical dissipative hydrodynamics solutions in any dimensions. $A$ and $B$ only depend on the hydrodynamic fields at the given $\tau$. Evaluating them at each time step, one can either numerically differentiate $A(\tau)$ or integrate $B(\tau)$ and check how accurately the solutions satisfy (B1).
Consider a four-divergence $\partial_\mu A^\mu(x)$ (in regular Minkowski coordinates). Integration over a four-volume $V_4$ gives

$$ \int_{V_4} d^4x \, \partial_\mu A^\mu(x) = \int_{\sigma(V_4)} d\sigma_\mu(x) \, A^\mu(x) $$

(B2)

where $\sigma(V_4)$ is the 3D boundary (“surface”) of $V_4$. Now take the special case of a Bjorken “box” $V_4 = \Delta \tau \times \Delta \eta \times A_T$ with an infinite transverse area $A_T \to \infty$ but infinitesimal proper time and finite coordinate rapidity extensions $\Delta \tau \to 0$, $\Delta \eta = \eta_2 - \eta_1$. Assuming $A^\mu(x)$ drops faster than $1/x_T^2$ at large $|x_T|$, we can neglect surface terms at $|x_T| \to \infty$ and keep only contributions on $\tau = const$ and $\eta = const$ hypersurfaces:

$$ \int d\tau d\eta dT d x^2_T \partial_\mu A^\mu(x) = \sum_{\sigma(\tau+\Delta\tau)} \int_{\sigma(\tau)}^{\sigma(\tau+\Delta\tau)} \int_{\sigma(\eta_1)}^{\sigma(\eta_2)} \int_{\sigma(\eta)}^{\sigma(\eta+\Delta\eta)} A^\mu(x) $$

(B3)

where the surface normals are

$$ d\sigma_\mu = \tau dx^2_T d\eta u^\mu_B , \quad d\sigma^\mu = -d\tau dx^2_T u^\mu_3 , \quad \text{with } u^\mu_B \equiv (\gamma n, 0, \sinh \theta), \quad u^\mu_3 \equiv (0, 0, \cosh \theta) $$

(B4)

and we used $d^4x = d\tau d\eta dx^2_T$. Here, $u_B$ is the longitudinal Bjorken flow velocity, while $u_3$ is its orthonormal counterpart in the $t-z$ plane. Note that the actual flow velocity does not need to be $u_B$. Dividing by $\Delta \tau$ and taking the limit we arrive at

$$ \int d\eta dx^2_T \partial_\mu A^\mu = \partial_\tau \left( \int d\eta dx^2_T u^\mu_B A_\mu \right) - \int dx^2_T u^\mu_3 \left( A_\mu(\eta_1) - A_\mu(\eta_2) \right) , $$

(B5)

which is a generalized conservation law for the quantity

$$ A \equiv \tau \int d\eta dx^2_T u^\mu_B A_\mu . $$

(B6)

If $\partial_\mu A^\mu \equiv 0$, and the surface term $u^\mu_3(A_\mu(\eta_1) - A_\mu(\eta_2))$ vanishes, we have $A(\tau) = const$.

In a boost-invariant calculation the longitudinal extension of the system is formally infinite and thus a generalized conservation law for a quantity per unit rapidity is more practical. It can be obtained in a similar fashion if one divides by $\Delta \eta$ and takes the limit $\Delta \eta \to 0$. The result is

$$ \int dx^2_T \partial_\eta A^\eta = \partial_\tau \frac{dA}{d\eta} - \partial_\eta \int dx^2_T u^\mu_3 A_\mu , $$

(B7)

where

$$ \frac{dA}{d\eta} = \tau \int dx^2_T u^\mu_3 A_\mu . $$

(B8)

Again, if $\partial_\mu A^\mu \equiv 0$ and the $\eta$-derivative term vanishes, we have $dA/d\eta = const$.

1. Charge / Particle number

We first apply Eq. (B5) to a conserved current in Eckart frame: $N^\mu = n_\nu u^\mu$, where $u^\mu = \gamma (\cosh \theta, v e_B, \sinh \theta)$ is the flow four-velocity and $\theta$ is the flow rapidity. Now $u^\mu_B u_\mu = \gamma (\cosh (\eta - \theta))$ and $u^\mu_3 u_\mu = \gamma (\sinh (\eta - \theta))$. If the rapidity interval is so large that $N^\mu(\eta_1) = N^\mu(\eta_2) = 0$, or the system is boost invariant, $\eta \equiv \theta$, the surface terms are zero and we get a simple conservation law

$$ N = \tau \int d\eta dx^2_T \gamma n \cosh (\eta - \theta) = const . $$

(B9)

In a boost-invariant case, the coordinate rapidity integral is trivial and we get

$$ \frac{dN}{d\eta} = \tau \int dx^2_T \gamma n = const . $$

(B10)
2. Entropy

Second, we apply Eq. (B5) to the entropy current (14) and its divergence (16). If $S^\alpha(\eta_1) = S^\alpha(\eta_2) = 0$, we get

$$\partial_\tau S = \tau \int d\eta \, d^2x_T \left( \frac{\Pi^2}{\zeta T} - \frac{q_\mu q_\nu}{\kappa_q T^2} + \frac{\pi_{\mu\nu}\pi_{\mu\nu}}{2\eta_s T^4} \right) \geq 0,$$

where the entropy of the system is

$$S = \tau \int d\eta \, d^2x_T \, u_B^\mu S_\mu,$$

and the last inequality follows from the general properties $q^\mu q_\mu \leq 0$ and $\pi_{\mu\nu}\pi_{\mu\nu} \geq 0$.

For longitudinally boost invariant dynamics, it is more natural to follow entropy per unit rapidity:

$$\frac{dS}{d\eta} = \tau \int d^2x_T \, u_B^\mu S_\mu, \quad \partial_\tau \left( \frac{dS}{d\eta} \right) = \tau \int d^2x_T \left( \frac{\Pi^2}{\zeta T} - \frac{q_\mu q_\nu}{\kappa_q T^2} + \frac{\pi_{\mu\nu}\pi_{\mu\nu}}{2\eta_s T^4} \right) \geq 0.$$  \hspace{1cm} (B13)

3. Energy-momentum

Finally we derive the conservation equation corresponding to energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$. Contraction of the energy-momentum tensor with $u_B^\mu$ gives the conservation of energy. In a case where the entire system is within the interval $[\eta_1, \eta_2]$,

$$\partial_\tau E \equiv \partial_\tau \left( \tau \int d\eta \, d^2x_T \, u_B^\mu T_{\mu\nu} u_B^\nu \right) = 0.$$  \hspace{1cm} (B14)

Contraction with $u_R^\mu \equiv (0, e_R, 0)$ gives the change in transverse radial momentum. Substituting

$$\partial_\mu (T^{\mu\nu} u_R^\nu) = 0 + T^{\mu\nu} \partial_\mu u_R^\nu$$

into Eq. (B5) results in

$$\partial_\tau M_\tau \equiv \partial_\tau \left( \tau \int d\eta \, d^2x_T \, u_B^\mu T_{\mu\nu} u_R^\nu \right) = \tau \int d\eta \, d^2x_T \, u_R^\nu \partial_\nu u_R^\tau.$$  \hspace{1cm} (B16)

To be more specific, we also show as an example a boost-invariant, cylindrically symmetric case. In Landau frame

$$T^{\mu\nu} = (\varepsilon + p + \Pi)u^\mu u^\nu - (p + \Pi) g^{\mu\nu} + (-\hat{\pi}_2 - \hat{\pi}_3)u_1^\mu u_1^\nu + \hat{\pi}_2 u_1^\mu u_2^\nu + \hat{\pi}_3 u_3^\mu u_3^\nu,$$

where $u_3$ is the orthonormal counterpart of the flow velocity in the time-radial plane, while $u_2$ and $u_3$ are orthonormal counterparts of these in the axial and beam (rapidity) direction

$$u_1^\mu = \gamma(\eta, v \, e_R, s \, \eta) \quad u_1^\nu = \gamma(\eta, v \, e_R, s \, \eta) \quad u_1^\nu = u_R^\nu = 0, e_R, 0 \quad u_1^\nu = (s \, \eta, 0, \, s \, \eta).$$  \hspace{1cm} (B18)

These vectors are normalized to $u_1^2 = 1$, $u_2^2 = u_3^2 = -1$. The viscous pressure tensor components in the fluid rest frame are $\sigma_{LR}^{\mu\nu} = diag(0, -\hat{\pi}_2 - \hat{\pi}_3, \hat{\pi}_2, \hat{\pi}_3)$. It is important to notice that the surface terms in Eq. (B5) or the $\eta$-derivative term in Eq. (B7) are now nonzero. Contraction by $u_B^\mu$ as above and substitution into Eq. (B7) gives the evolution of the energy per unit rapidity:

$$\partial_\tau \left( \frac{dE}{d\eta} \right) \equiv \partial_\tau \left( \tau \int d^2x_T \, T^{00}(\eta=0) \right) = - \int d^2x_T \, (p + \Pi + \hat{\pi}_3).$$  \hspace{1cm} (B19)

Contraction by $u_R^\mu$ gives the evolution of transverse radial momentum per unit rapidity:

$$\partial_\tau \left( \frac{dM_\tau}{d\eta} \right) \equiv \partial_\tau \left( \tau \int d^2x_T \, T^{01}(\eta=0, \phi=0) \right) = \tau \int d^2x_T \, \frac{p + \Pi + \hat{\pi}_2}{R},$$  \hspace{1cm} (B20)

where we have used the relations

$$u_B^\mu T_{\mu\nu} u_R^\nu = -T^{01}(\eta=0, \phi=0) \quad \partial_\mu u_R^\mu = -\frac{1}{R} u_{2,a} u_{2,a}.$$  \hspace{1cm} (B21)

The above results reflect general expectations. Particle number, per unit rapidity $dN/d\eta$, is strictly conserved in both the ideal and the dissipative case. Entropy per unit rapidity $dS/d\eta$ is conserved for an ideal fluid but increases if there is dissipation. In both cases, the energy per unit rapidity $dE/d\eta$ decreases due to longitudinal work, while the radial momentum per unit rapidity $dM_\tau/d\eta$ increases due to build-up of radial flow, \textit{as long as the system stays near equilibrium} (i.e., the total pressure is dominated by the ideal part).
APPENDIX C: VISCOUS SOLUTIONS FOR VARIOUS CROSS-SECTION SCENARIOS

In this Section we analyze viscous Israel-Stewart and Navier-Stokes solutions for four different types of cross section: constant, \(\sigma \propto 1/T^2\), \(\sigma \propto \tau^{2/3}\), and \(\eta_s/s_{eq} = \text{const}\). For convenience, we will often use normalized quantities

\[
\hat{A}(\tau/\tau_0) = \frac{A(\tau)}{A(\tau_0)} .
\]

We will show that for typical observables of interest (average pressure, pressure anisotropy, entropy, shear viscosity to entropy ratio), \(\eta_s/s_{eq} = \text{const}\) dynamics is well approximated by \(\sigma \propto \tau^{2/3}\) already for \(K_0 = 1\).

In analytic considerations, it will be often convenient to drop the \(\pi^2_L\) term in the equations of motion (35)-(36), which is a good approximation for \(|\pi_L| \ll p\), i.e., the general region of validity of viscous hydrodynamics. This should not be confused with the “naive” Israel-Stewart approximation, which also ignores the 4/3 factor in (36). For the \(\sigma \propto \tau^{2/3}\) and \(\sigma = \text{const}\) scenarios we obtain this way accurate approximate analytic Israel-Stewart solutions. We also derive analytic Navier-Stokes solutions for \(\sigma = \text{const}\), \(\sigma \propto \tau^{2/3}\) and \(\sigma \propto 1/T^2\).

1. Solutions for ultra-relativistic gas with constant \(2 \rightarrow 2\) cross section

For a constant cross section,

\[
\lambda_{tr}(\tau) \propto \tau \quad \Rightarrow \quad K(\tau) = \frac{\tau_0}{\lambda_{tr}(\tau_0)} \equiv K_0 = \text{const} .
\]

If we ignore \(\pi^2_L\) term, the linear equations of motion (35)-(36) can be solved in a straightforward manner:

\[
\begin{align*}
\pi_L(\tilde{\tau}) &= \tilde{\tau}^{-\frac{4}{3} - \kappa_0} \left[ \frac{\pi_{L,0}}{2} T^2_{+}(\tilde{\tau}) - \frac{1}{2D} \left( \kappa_0 \pi_{L,0} + \frac{8p_0}{3} \right) T_{-}(\tilde{\tau}) \right] \quad \text{(C3)} \\
p(\tilde{\tau}) &= \tilde{\tau}^{-\frac{4}{3} - \kappa_0} \left[ \frac{p_0}{2} T^2_{+}(\tilde{\tau}) + \frac{1}{2D} (\kappa_0 p_0 - \pi_{L,0}) T_{-}(\tilde{\tau}) \right] \quad \text{(C4)}
\end{align*}
\]

where

\[
\kappa_0 \equiv \frac{K_0}{C} , \quad D \equiv \sqrt{\frac{8}{3} + \kappa_0^2} , \quad T_\pm(x) \equiv x^{D/3} \pm x^{-D/3} , \quad p(\tau_0) = p_0 , \quad \pi_L(\tau_0) = \pi_{L,0} . \quad \text{(C5)}
\]

For a practical approximate formula for the pressure evolution, see (48).

In the ideal hydrodynamic (\(\eta_s \rightarrow 0\), or equivalently \(\kappa_0 \rightarrow \infty\)) limit we recover

\[
\pi_L(\tau > \tau_0) = 0 , \quad p(\tau) = p_0 \left( \frac{\tau_0}{\tau} \right)^{4/3} .
\]

At late times the pressure anisotropy, irrespectively of its initial value \(R_{p,0}\), approaches a constant determined solely by the parameter \(\kappa_0\)

\[
R_\infty \equiv R_p(\tau \rightarrow \infty) = \frac{12\kappa_0 - 10}{9D + 3\kappa_0 + 14} < 1 . \quad \text{(C7)}
\]

For a finite \(\kappa_0\), the final anisotropy is below unity.

Therefore, with a constant cross section, the late-time behavior of the system does not become ideal hydrodynamic but instead the Navier-Stokes limit applies (cf. (44) and (45)). Indeed, for large \(\kappa_0\), (C3)-(C4) reproduce the NS solution

\[
p^{NS}(\tau) = p_0 \left( \frac{\tau_0}{\tau} \right)^{4/3 - 4/(9\kappa_0)} , \quad \pi^{NS}_L(\tau) = - \frac{4p^{NS}(\tau)}{3\kappa_0} ,
\]

and the final IS and NS anisotropies (C7) and (45) agree, \(R_\infty = 1 - 2/\kappa_0 + 4/(3\kappa_0^2) + \mathcal{O}(1/\kappa_0^3)\).

Because \(R_\infty\) is a monotonically increasing function of \(\kappa_0\), the final pressure anisotropy is a measure of the viscosity.

Inverting (C7),

\[
\kappa_0 = \frac{5 + 14R_\infty - R_\infty^2}{6 - 3R_\infty - 3R_\infty^2} . \quad \text{(C9)}
\]
where in the last step we approximated the temperature evolution using the leading NS term (C8). It is natural to measure viscosity relative to the density, which up to a factor \( (4 - \chi) \) is the same as \( \eta_s/s_\text{eq} \).

The exact analytic solutions to the 'naive' Israel-Stewart equations are analogous to (C4)-(C3) but involve different powers of \( T/\tau \)

\[
\eta_s(\tau)/n(\tau) = \frac{T \tau}{\kappa} \approx \frac{1 - R_{\infty}}{2} T_0 \tau_0 \left( \frac{\tau}{\tau_0} \right)^\gamma, \quad \gamma = \frac{2}{3} + \frac{4}{9} \eta_s(\tau_0) \frac{1}{T_0 \tau_0},
\]

(C10)

The late time behavior is governed by the exponent

\[
\delta^\text{naive} = -\frac{4}{3} + \frac{4}{9\kappa_0} + \frac{8}{9\kappa_0^3} + \mathcal{O}\left(\frac{1}{\kappa_0^3}\right),
\]

(C12)

which does incorporate correctly the ideal hydrodynamic limit \(-4/3\) and the Navier-Stokes correction \(4/(9\kappa_0)\) but is in general higher, the smaller the \(\kappa_0\), than the complete IS result \(\delta^\gamma = -4/3 + 4/(9\kappa_0) - 8/(27\kappa_0^3) + \mathcal{O}(1/\kappa_0^5)\). Therefore, the 'naive' approach overestimates the asymptotic pressure \(R^\text{naive}_\infty = 1 - 2/\kappa_0 - 8/(3\kappa_0^3) + \mathcal{O}(1/\kappa_0^5)\), and therefore, overpredicts the magnitude of the shear stress to pressure ratio \(\kappa\).

2. Solutions for ultra-relativistic gas with \(\sigma_2 \rightarrow 2 \propto 1/T^2\)

A constant cross section implies the existence of some external scale in the problem. For a scale-invariant system, however, the only scale available (in thermal and chemical equilibrium) is the temperature, and therefore the cross section behaves as \(\sigma \propto 1/T^2\). (37), (33) and (29) then give

\[
K(\tau) = K_0 \frac{T_0^2}{\tau^2} = \frac{K_0}{p^2\tau^2},
\]

(C13)

i.e., even without the \(\pi_L^2\) term, the equations of motion become nonlinear (but are easy to solve numerically).

For ideal hydrodynamic evolution, \(p \propto \tau^{-4/3}\) and thus, unlike for the case of a constant cross section,

\[
K(\tau) = K_0 \tau^{2/3}
\]

(C14)

increases with increasing \(\tau\). \(K(\tau)\) must grow in general in the viscous hydrodynamic case as well because dissipative corrections, namely the \(\pi_L/\tau\) term in (35), are assumed to be small (or else hydrodynamics is not applicable any longer). Consequently, the system gets closer and closer to ideal hydrodynamic behavior as time evolves (as long as the expansion is only one-dimensional). For example, the pressure anisotropy approaches unity at late times, for any \(\kappa_0 > 0\) and initial \(\pi_{L,0}/p_0\),

\[
R_p(\tau \rightarrow \infty) \rightarrow 1.
\]

(C15)

The exact Navier-Stokes solution

\[
p^{NS}(\tau) = \left(\frac{\tau_0}{\tau}\right)^{4/3} \frac{p_0^{4/3}}{\sqrt{1 + \frac{4}{3\kappa_0} \left(\frac{\tau_0}{\tau}\right)^{2/3} - 1}}
\]

behaves similarly. At late times \(p \propto \tau^{-4/3}\) as in the ideal case, therefore, \(\kappa(\tau \rightarrow \infty) = \kappa_0/(\tau^2\tau^2) \rightarrow \infty\), i.e., \(R_{\infty} = 1\).

The rate of approach to unity is controlled by the viscosity

\[
R_p^{NS}(\tau) = 1 - \frac{2}{\kappa_0} \left(\frac{\tau_0}{\tau}\right)^{2/3} \left[1 + \mathcal{O}(1/\kappa_0^2) + \mathcal{O}((\tau_0/\tau)^3)\right] \approx 1 - \frac{2}{T_0 \tau_0} \frac{\eta_s}{n} \frac{\tau_0}{\tau}^{2/3}.
\]

(C17)

Viscosity also increases the pressure relative to the ideal case

\[
\frac{p^{NS}(\tau \gg \tau_0)}{p_{\text{ideal}}} \rightarrow \frac{1}{\sqrt{1 - 4/3\kappa_0}} \approx 1 + \frac{2}{3T_0 \tau_0} \frac{\eta_s}{n}.
\]

(C18)
3. Solutions for ultra-relativistic gas with \( \sigma_{2\rightarrow 2} \propto \tau^{2/3} \)

Near the ideal hydro limit (i.e., for small viscosities and \( \pi_{L,0}/p_0 \)), one may substitute the approximate result (C14) in the equations of motion (35)-(36) directly. Provided we drop the \( \pi_L^2 \) term, these can be converted to a second-order linear differential equation, e.g., for \( p(\tau) \):

\[
\tau \ddot{p} + \frac{11}{3} \dot{p} + \frac{40}{27} \frac{p}{\tau} + 2 K(\tau) \left( \frac{4}{3} \frac{p}{\tau} \right) = 0 ,
\]

(C19)

with initial conditions

\[
p(\tau_0) = p_0 , \quad \dot{p}(\tau_0) = -\frac{4p_0 + \pi_{L,0}}{3\tau_0} .
\]

(C20)

The general solution with \( K(\tau) \) from (C14) is\(^{[51]}\)

\[
p(\tilde{\tau}) = \tilde{\tau}^{-4/3} \left[ C_- \tilde{\tau}^{-\frac{2\kappa_0}{3}} F_-(\kappa_0 \tilde{\tau}^{2/3}) + C_+ \tilde{\tau}^{\frac{2\kappa_0}{3}} F_+(\kappa_0 \tilde{\tau}^{2/3}) \right]
\]

(C21)

\[
\pi_L(\tilde{\tau}) = -3 \tilde{\tau}^{-1/3} \frac{d(\tilde{\tau}^{4/3} p(\tilde{\tau}))}{d\tilde{\tau}} ,
\]

(C22)

where

\[
F_{\pm}(x) \equiv _1F_1(\pm a, 1 \pm 2a; -x) , \quad a = \sqrt{\frac{2}{3}}
\]

(C23)

are shorthands for confluent hypergeometric functions of the first kind, while \( C_{\pm} \) are matched\(^{[52]}\) to the initial conditions (C20)

\[
C_{\pm} = \pm \frac{e^{\kappa_0}}{4a} \left[ p_0 G_{\mp}(\kappa_0) - \pi_{L,0} F_{\mp}(\kappa_0) \right]
\]

(C24)

\[
G_{\pm}(x) \equiv \pm 2a \left[ \frac{x}{1 \pm 2a} _1F_1(1 \pm a, 2 \pm 2a, -x) - _1F_1(\pm a, 1 \pm 2a, -x) \right] .
\]

(C25)

A very practical approximate formula for the pressure evolution is given by (49), which comes from the asymptotic forms (cf. (13.5.1) in \([40]\))

\[
_1F_1(a, b; -x) = \frac{\Gamma(b)}{\Gamma(b-a)} x^{-a} S(a, 1 + a - b, x) + \frac{\Gamma(b)}{\Gamma(a)} e^{-x} (-x)^{a-b} S(b-a, 1-a, -x) ,
\]

(C26)

where

\[
S(c, d, x) \equiv 1 + \frac{cd}{1!x} + \frac{c(c+1)d(d+1)}{2!x^2} + \frac{c(c+1)(c+2)d(d+1)(d+2)}{3!x^3} + \cdots
\]

(C27)

Note that the \( e^{-x} \) term in (C26) is crucial. For large \( \kappa_0 \), \( C_{\pm} \) are exponentially large, however, the \( e^{\kappa_0} \) factors drop out\(^{[53]}\) in linear combinations relevant for the pressure and shear stress.

At late times \( \tau \gg \tau_0/\kappa_0^{3/2} \) the IS solutions recover ideal hydrodynamics for any initial condition,

\[
p(\tau) \propto \left( \frac{\tau_0}{\tau} \right)^{4/3} , \quad \pi_L(\tau) \propto \left( \frac{\tau_0}{\tau} \right)^2 \Rightarrow \frac{\pi_L}{p(\tau)} \propto \left( \frac{\tau_0}{\tau} \right)^{2/3} \rightarrow 0 \quad \text{for} \quad \tau \gg \frac{\tau_0}{\kappa_0^{3/2}} ,
\]

(C28)

as can be inferred from (C26). The Navier-Stokes solution

\[
p^{NS}(\tau) = p_0 \left( \frac{\tau_0}{\tau} \right)^{4/3} \exp \left\{ \frac{2}{3\kappa_0} \left[ 1 - \left( \frac{\tau_0}{\tau} \right)^{2/3} \right] \right\}
\]

(C29)

exhibits the same features (as the Reader can easily verify). For the late-time evolution, this scenario gives smaller viscous corrections to the pressure and the pressure anisotropy than \( \sigma \propto 1/T^2 \). However, in the large \( \kappa_0 \) limit we recover the same results (C17) and (C18).
Analogous derivation gives the exact solutions in the 'naive' Israel-Stewart case:

\[ p(\tilde{\tau}) = C'_{\pm} \tilde{\tau}^{-2(1+a')/3} F'_{\pm}(\kappa_0 \tilde{\tau}^{2/3}) + C'_{\pm} \tilde{\tau}^{-2(1-a')/3} F'_{\pm}(\kappa_0 \tilde{\tau}^{2/3}) \, , \quad a' = \sqrt{\frac{5}{3}} \] (C30)

\[ \pi_L(\tilde{\tau}) = -3\tilde{\tau}^{-1/3} \frac{d[\tilde{\tau}^{4/3} p(\tilde{\tau})]}{d\tilde{\tau}} \] (C31)

where

\[ C'_{\pm} = \pm \frac{e^{-\kappa_0}}{4a'} \left[ p_0 C'_{\pm}(\kappa_0) - \xi_0 F'_{\pm}(\kappa_0) \right] \] (C32)

\[ F'_{\pm}(x) = \Gamma_1(1 \pm a', 1 \pm 2a', -x) \, , \quad G'_{\pm}(x) = 2x \frac{1 \pm a'}{1 \pm 2a'} F'_{\pm}(x) - 2(1 \pm a') F'_{\pm}(x) \] (C33)

With the help of (C26) it is straightforward (but somewhat lengthy) to determine the late-time behavior

\[ \frac{p}{p_{\text{ideal}}} = T(\tilde{\tau}) [P(\kappa_0) + \xi_0 X(\kappa_0)] \] (C34)

where in the 'naive' case

\[ T^{\text{naive}}(\tilde{\tau}) = 1 - \frac{2}{3\kappa_0 \tilde{\tau}^{2/3}} - \frac{7}{9\kappa_0^2 \tilde{\tau}^{4/3}} + O\left( \frac{1}{\kappa_0^3 \tilde{\tau}^2} \right) \] (C35)

\[ P^{\text{naive}}(\kappa_0) = 1 + \frac{2}{3\kappa_0} + \frac{5}{9\kappa_0^2} + O\left( \frac{1}{\kappa_0^3} \right) , \quad X^{\text{naive}}(\kappa_0) = -\frac{1}{2\kappa_0} - \frac{5}{6\kappa_0^2} + O\left( \frac{1}{\kappa_0^3} \right) . \] (C36)

Comparing to the 'complete' Israel-Stewart result (C21) (obtained in the small \( \xi \) limit)

\[ T^{\text{IS}}(\tilde{\tau}) \approx 1 - \frac{2}{3\kappa_0 \tilde{\tau}^{2/3}} - \frac{1}{9\kappa_0^2 \tilde{\tau}^{4/3}} + O\left( \frac{1}{\kappa_0^3 \tilde{\tau}^2} \right) \] (C37)

\[ P^{\text{IS}}(\kappa_0) \approx 1 + \frac{2}{3\kappa_0} - \frac{1}{9\kappa_0^2} + O\left( \frac{1}{\kappa_0^3} \right) , \quad X^{\text{IS}}(\kappa_0) \approx -\frac{1}{2\kappa_0} + \frac{1}{6\kappa_0^2} + O\left( \frac{1}{\kappa_0^3} \right) \] (C38)

we see that for the 'naive' approximation the evolution approaches ideal hydrodynamic \( p/p_{\text{ideal}} \sim \text{const} \) behavior later (deviation of \( T \) from unity is larger), and for near-equilibrium initial conditions (\( \xi_0 \approx 0 \)) the pressure saturates at a higher value (\( P \) is larger).

4. Solutions for ultra-relativistic gas with \( 2 \rightarrow 2 \) cross section and \( \eta_s/s_{eq} = \text{const} \)

The last scenario we consider is when the cross section is dynamically adjusted to maintain a constant shear viscosity to equilibrium entropy density ratio \( \eta_s/s_{eq} \), such as the conjectured lower bound of \( 1/(4\pi) \). From (29), (33), (51), and (52),

\[ \tilde{s}_{eq} = \frac{1}{\tilde{\tau}} \left( 1 + \frac{\ln [\tilde{\tau}^{4} \tilde{p}^3(\tilde{\tau})]}{4 - \chi_0} \right) \] (C39)

and thus

\[ \frac{\eta_s}{s_{eq}} = \frac{\eta_{s,0}}{s_{eq,0}} \frac{\tilde{p}(\tilde{\tau}) \tilde{\tau}^2}{K(\tilde{\tau})} \frac{4 - \chi_0}{4 - \chi_0 + \ln [\tilde{\tau}^{4} \tilde{p}^3(\tilde{\tau})]} , \] (C40)

where

\[ \frac{\eta_{s,0}}{s_{eq,0}} = \frac{T_0 \tau_0}{\kappa_0 (4 - \chi_0)} . \] (C41)

Therefore, \( \eta_s/s_{eq} = \text{const} \) requires

\[ K(\tilde{\tau}) = K_0 \tilde{p}(\tilde{\tau}) \tilde{\tau}^2 \frac{4 - \chi_0}{4 - \chi_0 + \ln [\tilde{\tau}^{4} \tilde{p}^3(\tilde{\tau})]} . \] (C42)
Within the generic region of validity for viscous hydrodynamics, $|\pi_L| \ll p$, this scenario also implies a growing $K(\tau) \sim \tau^{-2/3}$ and therefore convergence to the ideal limit at late times. Note that the double ratio $(\eta_s/s_{eq})/(\eta_{s,0}/s_{eq,0})$ as a function of $\tau/\tau_0$ depends only on $\pi_{L,0}/p_0$, $s_0$, the type of cross section (encoded in $\tilde{K}$), and $\chi_0$.

We now analyze the time evolution of $\eta_s/s_{eq}$ in the three earlier scenarios. Compared to the entropy density, $\eta_s/s_{eq}$ contains an additional multiplicative term that comes from the time evolution of the shear viscosity. Assume first, for simplicity, that we are very close to the ideal hydro limit, in which case $\eta_s/s_{eq} \propto \tau^{2/3}/K(\tau)$. For a constant cross section, this results in a growing $\eta_s/s_{eq} \propto \tau^{2/3}$; while for the other two cases, $\sigma \propto \tau^{2/3}$ or $\sigma \propto 1/T^2$ we obtain $\eta_s/s_{eq} \approx const$.

In reality, there are of course viscous effects. Because

$$\frac{\tilde{p}(\tilde{\tau}) \tilde{\tau}^2}{\tilde{K}(\tilde{\tau})} = \begin{cases} \tilde{p}(\tilde{\tau}) \tilde{\tau}^{4/3} \propto \tilde{\tau}^{2/3} & \text{for } \sigma = const \\ \tilde{p}(\tilde{\tau}) \tilde{\tau}^{4/3} \tilde{\tau}^{1/3} & \text{for } \sigma \propto 1/T^2 \\ \tilde{p}(\tilde{\tau}) \tilde{\tau}^{4/3} & \text{for } \sigma \propto \tau^{2/3} \end{cases} \quad (C43)$$

the relevant quantity that determines the evolution of $\eta_s/s_{eq}$ is $\tilde{p} \tilde{\tau}^{4/3}$. The last term in (C40) is only a logarithm. Therefore, the first term, (C43), dominates the behavior. Typically, $\pi_L < 0$ and thus dissipation generates an increasing $\tilde{p} \tilde{\tau}^{4/3}$. The increase in $\eta_s/s_{eq}$ is then fastest for the constant cross section case. The other two cases, $\sigma \propto 1/T^2$ and $\sigma \propto \tau^{2/3}$, are not equivalent when there is dissipation because for the latter the prefactor (C43) is only linear in $\tilde{p}(\tilde{\tau}) \tilde{\tau}^{4/3}$ and, therefore, $\eta_s/s_{eq}$ grows much slower.

5. Comparison of the various cross section scenarios

After exploring the general behavior, we compare numerical solutions for the four scenarios. Unless stated otherwise, for the $\eta_s/s_{eq} = const$ case we start the evolution from chemical equilibrium, i.e., take $\chi_0 = 0$. For the other three scenarios, the pressure and shear stress evolution does not depend on $\chi_0$. For simplicity, we start the evolution from $\pi_L(\tau_0) = 0$, and consider two extremes $K_0 = 1$, i.e., equal expansion and scattering timescales, and $K_0 = 6.67$, i.e., 6.67 times slower expansion than the timescale for scattering. On all figures, the dotted curves correspond to the approximation when the $\pi_L^2$ term in (36) is ignored.

Figure 8 shows the evolution of the pressure relative to the ideal hydrodynamic $p \sim \tau^{-4/3}$ result (for a comparison of the same observable between hydrodynamics and transport see Figure 4). Dissipation increases the pressure because it reduces the $pdV$ work. The effect is largest for the $\sigma = const$ scenario, while smallest for $\eta_s/s_{eq} = const$ and $\sigma \propto \tau^{2/3}$, which two give basically the same result. For $K_0 = 1$ the fourth scenario $\sigma \propto T^2$ is in between these limits but by $K_0 = 6.67$ it becomes equivalent to $\sigma \propto \tau^{2/3}$. Dropping $\pi_L^2$ terms in (36) (thin dotted lines) is a fair 10 – 15% approximation for $\sigma = const$ and $\sigma \propto 1/T^2$ at $K_0 = 1$, which improves to an essentially exact one by $K_0 = 6.67$. For the other two scenarios, $\eta_s/s_{eq} = const$ and $\sigma \propto \tau^{2/3}$, the nonlinear term can be safely ignored already for $K_0 = 1$. Note that for $K_0 = 6.67$, dissipative corrections to the pressure are still very modest 10 – 15% at late $\tau/\tau_0 \sim 20$ in all four cases studied.

Now we turn to the evolution of the viscous stress $\pi_L$ shown in Fig. 9. All four scenarios give very similar results for the early $\tau/\tau_0 \lesssim 1.5 - 2$ growth in magnitude but they differ in late-time relaxation. As inferred from the pressure evolution already, $\eta_s/s_{eq} = const$ and $\sigma \propto \tau^{2/3}$ are largely identical and relax quickly toward the ideal limit. $\sigma = const$ is the one that stays furthest away from equilibrium. For low $K_0 = 1$, the $\sigma \propto 1/T^2$ case lies in between but by $K_0 = 6.67$ it becomes identical to $\eta_s/s_{eq} = const$ and $\sigma \propto \tau^{2/3}$. The $\pi_L^2$ term in the equation of motion affects the pressure and the viscous stress similarly, and can be ignored for $K_0 = 6.67$ in all cases - for $\sigma \propto \tau^{2/3}$ and $\eta_s/s_{eq} = const$ even at $K_0 = 1$.

The same observations carry over to the pressure anisotropy $R_p = p_L/p_T$ shown in Figure 10. We plot this quantity because it is the same one shown in Figure 1 for the hydro-transport comparison in Sec. V (but note the logarithmic time axis there). These results further confirm that $\sigma \propto \tau^{2/3}$ is a very good approximation to $\eta_s/s_{eq} = const$ already for $K_0 = 1$.

Figure 11 shows entropy production $dS/d\eta$ as a function of proper time for the four scenarios, with local thermal ($\xi_0 = 0$) and chemical ($\chi_0 = 0$) equilibrium initial conditions. Due to scalings, only entropy relative to the initial one plays a role

$$\frac{(dS/d\eta)}{(dS_0/d\eta)} = \tilde{s} \tilde{s} = 1 + \frac{1}{4 - \chi_0} \left[ \ln(\tilde{\tau}^4 \tilde{p}^3) - \frac{9\xi_0^2(\tau)}{16} \right]. \quad (C44)$$

For $K_0 = 1$, a constant cross section generates about 35% extra entropy by late $\tau \sim 15 - 20\tau_0$. With $\sigma \propto 1/T^2$, the increase is only $\sim 30\%$, while $\sigma \propto \tau^{2/3}$ and $\eta_s/s_{eq} = const$ give the smallest increase of about 25%. For a
larger $K_0 \sim 7$, the system is much closer to ideal hydrodynamics and therefore entropy generation is slower - about 10% for $\sigma = \text{const}$, while only 5% for the other three cases. Note that these results also depend on $\chi_0$, but almost entirely through the explicit $1/(4-\chi_0)$ factor in (C44). Therefore, results for arbitrary $\chi_0 \neq 0$ can be obtained via straightforward rescaling. In the $\eta_s/s_{eq} = \text{const}$ case the shear stress and pressure evolution also depend on $\chi_0$ but only very weakly as we show later below (cf. Figure 13).

Figure 12 shows the evolution of the shear viscosity to equilibrium entropy density ratio $\eta_s/s_{eq}$, normalized by the initial value of the ratio. The entropy is calculated for a system starting from chemical equilibrium ($\chi_0 = 0$). The rough expectations that $\eta_s/s_{eq} \sim \tau^{2/3}$ for a constant cross section, while $\eta_s/s_{eq} \sim \text{const}$ for both $\sigma \propto 1/T^2$ and $\sigma \propto \tau^{2/3}$, hold within a factor of three already for $K_0 = 1$ and up to $\tau = 20\tau_0$ (note that the $\tau^{2/3}$ growth in the $\sigma = \text{const}$ case has been scaled out in the plots). Relative to this “zeroth order” behavior, for all three scenarios, $\eta_s/s_{eq}$ grows with time, reinforcing the general results in Sec. C4. The relative growth decreases with increasing $K_0$. The $K_0$ dependence is strongest for the constant cross section scenario: the factor of three gain by $\tau = 20\tau_0$ for $K_0 = 1$ is tamed to an about 25% increase for $K_0 \sim 7$. For the other two scenarios, $\sigma \propto 1/T^2$ and $\sigma \propto \tau^{2/3}$, the ratio stays nearly constant much more robustly. As expected (cf. end of Sec. C4), of all cases studied $\sigma \propto \tau^{2/3}$ approximates $\eta_s/s_{eq} = \text{const}$ the best, with only $\sim 10\%$ deviation accumulated by late $\tau = 20\tau_0$ even for a small $K_0 = 1$.

Finally, in Figure 13 we show that the results for $\eta_s/s_{eq} = \text{const}$ depend only weakly on the initial density, i.e., $\chi_0$. 

![Figure 8](image1.png)

**FIG. 8:** Pressure evolution from viscous hydrodynamics relative to the ideal hydrodynamic $p = p_0(\tau_0/\tau)^{-4/3}$ result in 0+1D Bjorken geometry for an ultrarelativistic gas with $2 \to 2$ interactions. Four scenarios are compared for $K_0 = 1$ (left) and 5 (right): $\sigma = \text{const}$ (dash-dot-dot), $\sigma \propto 1/T^2$ (long dash), $\sigma \propto \tau^{2/3}$ (short dash), and $\eta_s/s_{eq} = \text{const}$ (solid). Approximate results with dropping $\pi_L^2$ terms in the equation of motion are also shown (thin dotted lines).

![Figure 9](image2.png)

**FIG. 9:** Same as Fig. 8 but for the longitudinal viscous shear $\pi_L$ normalized by the initial pressure.
\[ \eta/s = \text{const} \]
\[ \sigma \propto \tau^{2/3} \]
\[ \sigma \propto 1/T^2 \]
\[ \sigma = \text{const} \]

\[ K_0 = 1 \]

\[ K_0 = 6.67 \]

\[ \eta/s = \text{const} \]

**FIG. 10:** Same as Fig. 8 but for the pressure anisotropy evolution.

\[ \eta/s = \text{const} \]
\[ \sigma \propto \tau^{2/3} \]
\[ \sigma \propto 1/T^2 \]
\[ \sigma = \text{const} \]

\[ K_0 = 1 \]

\[ K_0 = 6.67 \]

**FIG. 11:** Same as Fig. 8 but for the normalized entropy per unit rapidity \((dS/d\eta)/(dS_0/d\eta)\).

\[ \eta/s = \text{const} \]
\[ \sigma = \text{const} \]
\[ \sigma \propto 1/T^2 \]
\[ \sigma \propto \tau^{2/3} \]

\[ K_0 = 1 \]

\[ K_0 = 6.67 \]

**FIG. 12:** Same as Fig. 8 but for the shear viscosity to equilibrium entropy density ratio \(\eta_s/s_{eq}\). The results for \(\sigma = \text{const}\) are divided by \((\tau/\tau_0)^{2/3}\), otherwise they would quickly grow off the plot.
Density dependence in shear stress and pressure evolution arises in this case because the cross section is a function of the initial density (see (C42)). The dependence is weaker, the closer the system is to ideal hydrodynamics because in that case $p \propto \tau^{-4/3}$ and $\chi_0$ drops out from $K(\tau)$. But even for a pessimistic $K_0 = 1$, the pressure anisotropy (left plot), varies less than 10\% as we change the density by a factor of 4 around chemical equilibrium density ($\chi_0 = \pm \ln 4$). In fact a decrease in the density has a much weaker effect than an increase. The right plot shows the effect of the same initial density variation on entropy $dS/d\eta$ production normalized to the initial entropy. Most of the density dependence in the entropy change comes from the trivial $1/(4 - \chi_0)$ prefactor in (C44) which is there in any cross section scenario even if the shear stress and pressure evolution are independent of the density. To highlight dynamical density effects, we therefore plot, again for a pessimistic $K_0 = 1$, the normalized change in entropy

$$
\frac{4 - \chi_0}{4} \Delta (dS/d\eta) = \frac{4 - \chi_0}{4} \left( \frac{dS/d\eta}{dS_0/d\eta} - 1 \right)
$$

(C45)

(the scaling factor is chosen such that it has no effect for chemical equilibrium initial conditions $\chi_0 = 0$). The results show practically no density dependence, apart from few-per-cent changes, even for such a low $K_0$.

**APPENDIX D: USEFUL RELATIONS FROM COVARIANT TRANSPORT**

1. **Particle number and transverse energy**

The particle number and transverse energy distributions for particles crossing a 3D hypersurface $\sigma(x) = \text{const}$ are given by

$$
dN = dydp_T^2 \int p^\mu d\sigma_\mu(x)f(x,p) \quad (D1)
$$

$$
dE_T = dydp_T^2 \int p^\mu d\sigma_\mu(x)m_T f(x,p) \quad (D2)
$$

where $m_T \equiv \sqrt{p_T^2 + m^2}$, $p_T \equiv \sqrt{p_x^2 + p_y^2}$ is the transverse momentum, and $d\sigma_\mu(x)$ is the normal to the hypersurface at space time coordinate $x$. For our boost-invariant scenario it is natural to follow quantities per *unit coordinate*
rapidity as a function of the proper time $\tau$. For $\tau = const$ hypersurfaces $p^\mu d\sigma_\mu = m_T \tau c \omega d^2 x_T d\eta$, and in our 0+1D case, $f$ only depends on $sh\omega, p_\perp$, and $\tau$, where $\omega \equiv y - \eta$. Thus,

$$
\frac{dN(\tau)}{d\eta} = \tau A_T \int d^2 p_T d\omega m_T c \omega f(\tau, sh\omega, p_T) \quad (D3)
$$

$$
\frac{dE_T(\tau)}{d\eta} = \tau A_T \int d^2 p_T d\omega m^2_T c \omega f(\tau, sh\omega, p_T) . \quad (D4)
$$

$A_T$ is the transverse area of the system. With the local thermal equilibrium distribution for ultrarelativistic particles

$$
f(sh\omega, p_\perp) = N e^{-p_\perp c \omega / T} , \quad N = \frac{n}{8\pi T^3} \quad (D5)
$$

and the quadratic form (62), straightforward integration gives

$$
\frac{dN}{d\eta} = n \tau A_T = const \quad (D6)
$$

$$
\frac{dE_T(\tau)}{d\eta} = \frac{3\pi T}{4} \frac{dN}{d\eta} \left( 1 - \frac{5\xi}{16} \right) . \quad (D7)
$$

Clearly, dissipation slows the decrease of transverse energy (for typical $\pi_L < 0$), and $2 \rightarrow 2$ interactions of course conserve particle number.

Note that $dE_T(\eta)/(\tau A_T)$ is almost identical to the transverse pressure (D10), but has an extra $c \omega$ factor in the integrand.

2. Early pressure evolution

Here we evaluate the early transverse and longitudinal pressure evolution from the transport for a local equilibrium initial condition. The results hold for any interaction, not only $2 \rightarrow 2$.

In local equilibrium the collision term vanishes, thus in the vicinity of $\tau = \tau_0$ the evolution is governed by free streaming. In our 0+1D case, free streaming

$$
\left[ c \omega \frac{\partial}{\partial \tau} - \frac{sh\omega}{\tau} \frac{\partial}{\partial \omega} \right] f(sh\omega, p_\perp, \tau) = 0 \quad (\omega \equiv y - \eta) \quad (D8)
$$

implies

$$
f(sh\omega, p_\perp, \tau) = f(\tau sh\omega/\tau_0, p_\perp, \tau_0) . \quad (D9)
$$

Substituting a local thermal initial distribution for ultrarelativistic particles (D5), the definition of the energy-momentum tensor

$$
T^{\mu\nu}(\eta = 0, \tau) = \int \frac{d^3 p}{p_0} p^\mu p^\nu f = \int d^2 p_\perp dy p^\mu p^\nu f(shy, p_\perp, \tau) \quad (D10)
$$

gives the transverse pressure

$$
p_T(\tau) \equiv T^{xx}(\eta = 0, \tau) = N \int dp_\perp dp_\perp d\phi dy (p_\perp \cos \phi)^2 \exp\left[ -\frac{p_\perp}{T_0} \sqrt{1 + a^2 sh^2 y} \right] = \frac{3T_0 n}{2} \int_0^\infty \frac{dy}{(1 + a^2 sh^2 y)^2} \quad (D11)
$$

Here $a \equiv \tau/\tau_0$. Change of variables to $q = a shy$ leads to

$$
p_T(\tau) = \frac{3T_0 n}{2} \int_0^\infty \frac{dq}{(1 + q^2)^2 q^2 + a^2} = T_0 n \frac{3 \left[ \sqrt{a^2 - 1} + (a^2 - 2) a \cos \frac{1}{a} \right]}{4(a^2 - 1)^{3/2}} \quad (D12)
$$

Analogous calculation gives for the longitudinal pressure

$$
p_L(\tau) \equiv T^{zz}(\eta = 0, \tau) = 3T_0 n \int \frac{dy sh^2 y}{(1 + a^2 sh^2 y)^2} = T_0 n \frac{3}{2(a^2 - 1)} \left[ \frac{a \cos \frac{1}{a} - 1}{a^2} \right] . \quad (D13)
$$
Expanding near $a = 1$,

$$p_T(\tau) = T_0 n \left[ 1 - \frac{4(\tau - \tau_0)}{5\tau_0} + O((\tau - \tau_0)^2) \right] \quad (D14)$$

$$p_L(\tau) = T_0 n \left[ 1 - \frac{12(\tau - \tau_0)}{5\tau_0} + O((\tau - \tau_0)^2) \right] \quad (D15)$$

and thus (65) follows.

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[43] Bose (+) or Fermi (-) statistics can be included in a straightforward manner via substituting $f_1 g_2 \to f_1 g_2(1 \pm f_3)(1 \pm g_4)$ and $f_2 g_2 \to f_2 g_2(1 \pm f_3)(1 \pm g_4)$ in the collision term (6). The various hydrodynamic limits can then be derived analogously to the Boltzmann case, if one makes the convenient replacement $\phi \to (1 \pm f_{eq})\phi$ in (7).
[44] The alternative formulation based on transient thermodynamics[22, 23] also lacks a small expansion parameter.
[45] Unlike we here, Israel and Stewart choose $g^\mu v = \text{diag}(-1,1,1,1)$, $\Delta^{\mu v} = g^{\mu v} + u^\mu u^v$.
[46] Positive semi-definiteness follows from the general properties $q^\mu q_\mu \leq 0$ and $\pi^{\mu v} \pi_{\mu v} \geq 0$. [22, 23]
Note that in Ref. [35], the equivalent set of equations are called "full IS" and "simplified IS".

By a boost-invariant system we mean a system which obeys the scaling flow, $v = (0, 0, z/t)$, where all scalar quantities are independent of coordinate rapidity $\eta \equiv (1/2) \ln[(t+z)/(t-z)]$, and where all vector and tensor quantities can be obtained from their values at $\eta = 0$ by an appropriate Lorentz boost.

I.e., in the often employed curvilinear $\tau - \eta - x - y$ coordinates we have $\pi_{\eta \eta} = \tau^2 \pi_L$.

This however does not imply that the 'naive' IS equations always underpredict the total integrated entropy change over a finite time interval. The time evolution of $\xi(\tau)$ in the 'naive' approach differs in general from that in the complete theory.

First substitute $p(\tilde{\tau}) \equiv \bar{p}(\tilde{\tau}) \tilde{\tau}^{-4/3}$, then switch to a new variable $x \equiv -\kappa_0 \tilde{\tau}^{2/3}$, finally look for the solution in the form $\bar{p}(x) \equiv x^aq(x)$, and choose a suitable $a$.

Note that

$$\frac{d}{dx}F_1(a, b, x) \equiv \left[ \frac{a}{b} \right] F_1(a + 1, b + 1, x)$$

and from the Wronskian

$$G_-(x)F_+(x) - G_+(x)F_-(x) = 4ae^{-x}$$

(cf. $W\{1, 2\}$ in Eq. (13.1.20) in [40]).

For example,

$$\frac{\Gamma(1+2a)}{\Gamma(1+a)} F_- (\kappa_0) \kappa_0^{-a} - \frac{\Gamma(1-2a)}{\Gamma(1-a)} F_+ (\kappa_0) \kappa_0^{-a} = 2a \frac{e^{-\kappa_0}}{\kappa_0} \left[ 1 - \frac{1}{3\kappa_0} + O\left(\frac{1}{\kappa_0}\right) \right]$$

and

$$\frac{\Gamma(1+2a)}{\Gamma(1+a)} G_- (\kappa_0) \kappa_0^{-a} - \frac{\Gamma(1-2a)}{\Gamma(1-a)} G_+ (\kappa_0) \kappa_0^{-a} = 4ae^{-\kappa_0} \left[ 1 + \frac{2}{3\kappa_0} - \frac{1}{9\kappa_0^2} + O\left(\frac{1}{\kappa_0^3}\right) \right]$$

$(a = \sqrt{2/3})$. 
