Description of local multiplicity fluctuations and genuine multiparticle correlations

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Various parametrizations of the multiplicity distribution are studied using the recently published large statistics OPAL results on multidimensional local fluctuations and genuine correlations in $e^+e^- \rightarrow Z^0 \rightarrow$ hadrons. The measured normalized factorial and cumulant moments are compared to the predictions of the negative binomial distribution, the modified and generalized versions of it, the log-normal distribution and the model of the generalized birth process with immigration. This is the first study which uses the multiplicity distribution parametrizations to describe high-order genuine correlations. Although the parametrizations fit well the measured fluctuations and correlations for low orders, they do show certain deviations at high orders. We have shown that it is necessary to incorporate the multiparticle character of the correlations along with the property of self-similarity to attain a good description of the measurements.

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1 Introduction

The study of the multiplicity distribution of the produced hadrons along with the analysis of the correlations among them stands in the frontier of investigations in the area of multiparticle dynamics. The multiplicity distribution plays a fundamental role in extracting first information on the underlying particle production mechanism, while the correlations give details of the dynamics. Whereas the full-multiplicity distribution is a global characteristic and is influenced by conservation laws, the multiplicity distributions in restricted phase space domains contributing to the correlations are local characteristics and have an advantage of being much less affected by global conservation laws.

In the last decade, study of multiplicity distributions in limited regions (bins) has attracted high interest in view of search for local dynamical fluctuations of an underlying self-similar (fractal) structure, the so-called intermittency phenomenon. For review, see Refs. [1, 2]. This phenomenon is seen in various reactions; however, many questions about intermittency and, in particular, its origin are still open and further investigations are needed [2, 3].

Studying local fluctuations, one must remember that the fluctuation of a given number of particles, \( q \), is contributed by genuine lower-order, \( p < q \), correlations. To extract signals of these \( p \)-order correlations, one needs to use advanced statistical techniques such as normalized factorial cumulant moments (cumulants) [4–6]. However, this method requires measurements to be of high statistics (and of high precision), lack of which leads to a smearing out of high-order correlations [2, 5].

A real opportunity to extract genuine multiparticle correlations came with vast amount of multihadronic events collected now at LEP1. The statistics available allows one to decrease significantly the measurement errors and to reveal relatively small effects. The first reports representing the studies of genuine correlations in e\(^+\)e\(^-\) annihilation have just appeared [7, 8].

DELPHI [7] has analysed correlations in one- and two-dimensional angular bins of jet cones, while OPAL [8] has performed its study in three dimensional phase space using conventional kinematic variables such as rapidity, transverse momentum and azimuthal angle. DELPHI has shown an existence of the correlations up to third order, and OPAL with increased statistics has, for the first time in e\(^+\)e\(^-\) annihilation, calculated multidimensional cumulants and has established sensitive genuine correlations even at fifth order. Note that multidimensional analysis carried out in hadronic interactions shows that the cumulants are consistent with zero at \( q > 3 \) there, while in heavy-ion collisions non-zero cumulants have been observed at second order only [1, 2].

The genuine correlations measured in e\(^+\)e\(^-\) annihilations are found to exceed considerably the QCD analytic description and to indicate significant deviations from Monte Carlo (MC) predictions. A smallness of genuine correlations predicted by perturbative QCD is seen also in cumulant-to-factorial moment studies, even when higher orders of the analytical approximation have been searched for in hadron-hadron [9,10] and lepton-hadron [11] interactions. It is worth to note that except for problems arising due to various possibilities in defining of a proper topology of particles and a distance between them (see e.g. [2]), a rather important difficulty in interpreting results could come from a translation invariance breaking of the many-particle distributions [12]. This will lead to different results on moments/cumulants according to a variable used because different variables are sensitive to different hadroproduction mechanisms [12, 13]. For example, in the high-order genuine correlations obtained in Ref. [12] the study is performed in four-momentum difference squared, \( Q^2 \), dependent on Bose-Einstein correlations, whereas in the pseudorapidity analysis [10] no correlations higher than two-particle ones were found, since (pseudo)rapidity seems [13] to be more “natural” variable to search for jet formation than e.g. for Bose-Einstein correlations.
are used [14,15]. These findings along with the deviations obtained by L3 in its recent detailed MC analysis of local angular fluctuations [16] show particle bunching in small bins to be a sensitive tool to find differences between parton distributions treated by QCD vs. hadron ones detected in experiments. The study of bunchings seems to be critical to a choice of a more convenient basis for a suitable approach of multiple hadroproduction.

In this paper we compare the intermittency and correlation results from OPAL with predictions of various parametrizations, or regularities. We consider the negative binomial distribution, its modified and generalized versions, the log-normal distribution, and the pure-birth stochastic production mechanism. For the first time we examine these parametrizations with high-order genuine correlations. The incorporation of the multiplicity distribution in the study of correlations provides more advanced information by using various approximations and models. In particular, this study gives more understanding about the structure of multiparticle correlations, e.g. their relation to two-particle correlations.

All the above listed parametrizations are well-known in multiparticle high-energy physics for a long time and are used to describe the shape of the multiplicity distribution, either in full phase space or in its bins. Essentially all these parametrizations are sort of branching models and show an intermittent behavior. This feature becomes particularly important in hadroproduction studies in $e^+e^-$ annihilations where parton showers play a significant role and the hypothesis of local parton-hadron duality is applied to the hadronization process [3, 15].

### 2 Normalized factorial moments and cumulants from various parametrizations

There is a variety of models which describe particle production as a branching process, see e.g. [17]. The main prediction of these models is a suitable parametrization for the multiplicity distribution. Further, more details of the underlying dynamics come from investigations of the factorial moments and cumulants of the multiplicity distribution given [2].

One of the most popular parametrization used to describe the data for full and limited phase-space (basically, rapidity) bins is the negative binomial (NB) distribution, see Refs. [17–19] for review on the subject and its historical development.

The NB regularity depends on two parameters: $\langle n \rangle$ is the average multiplicity and $1/k$ is the so-called aggregation coefficient which influences the shape of the distribution. For the NB distribution, the normalized factorial moments and cumulants are governed by the parameter $1/k$ and can be derived [6, 20] from the following formulae,

\[
F_q = F_{q-1} \left( 1 + \frac{q-1}{k} \right)
\]

and

\[
K_q = (q - 1)! k^{1-q},
\]

respectively.

Fractal properties of the NB distribution have been studied elsewhere [6, 21–24]. To demonstrate the fractality of the NB distribution, it has been proposed to use its generalization [22] or to study individual sources obtained by consideration of different topologies, e.g. number of jets [24]. A good fit by the NB parametrization should yield $k$ independent of $q$-order, but
the limited statistics may also stabilize the $q$-dependence and hide the (unknown) real distribution \cite{21}. The bin-size dependence of the $k$ parameter, expected from the bin-size dependence of the factorial moments and cumulants, Eqs. (1) and (2), is a consequence of the unstable nature of the NB distribution, i.e. the convolution of the distributions in two neighbouring bins does not give the same type of distribution \cite{23}. In the other words, the bin-size dependence of $k$ reflects the fact that distribution in one bin depends on that in the other bin \cite{26}.

In $e^+e^-$ annihilations at the $Z^0$ peak \cite{27,29} and lower energies \cite{32}, the NB parametrization was found to fail consistently in describing the multiplicity distribution, either in rapidity bins or for the full phase space. Using $e^+e^-$ results on intermittency \cite{33}, it was also shown \cite{34} that this parametrization does not reproduce the large fluctuation patterns, while it is appropriate for phase space bins in which the fluctuations are sufficiently small. Such an effect has been observed in (pseudo)rapidity studies in other types of collisions \cite{24,35,38} too. The likely reason is the above noted instability of the NB distribution. Indeed, the NB law provides an acceptable description in the central regions of the multiplicity distribution, away from the tails \cite{17,30,32}. In this region, the distribution measured is mostly flat and, therefore, less sensitive to instability effects.

Another reason is that the NB regularity underestimates the high multiplicity tail \cite{34,36,38,40,41} which gives the main contribution to the fluctuations and which is influenced by instabilities. For high multiplicities the NB distribution transforms to the stable $\Gamma$-distribution \cite{25}. This type of distribution was found \cite{27} to be the most adequate to describe the multiplicity distribution of the OPAL data. This is in contradiction to the results obtained in nuclear collisions \cite{38}, where the $\Gamma$-distribution was found to be significantly inconsistent to reproduce the measurements: it underestimates the low-multiplicity parts of the experimental multiplicity distributions in different rapidity bins, while overestimates the high-multiplicity tails. Again in contrast to $e^+e^-$ data, the NB regularity is found to be the best one to describe small fluctuations in the multiplicity distribution in nuclear data, and large fluctuations are well reproduced by two-particle correlations \cite{38,40,43}.

Another popular choice for the parametrization of the multiplicity distribution is the log-normal (LN) or Gaussian distribution \cite{44,45}. This type of regularity of final state distribution can be obtained by assuming a scale invariant stochastic branching process to be the basis of the multiparticle production mechanism. The LN distribution is defined by two parameters, the average and dispersion. To describe the data a third parameter has been introduced \cite{45} to take into account an asymmetry in the shape of the full phase space distribution measured.

In $e^+e^-$ annihilation this model has been found to describe successfully the data for the full rapidity window \cite{27,30}. For restricted bins the best agreement between the LN parametrization and the data has been obtained for very small bins in the central rapidity region, while for the intermediate size bins the deviation observed is assumed to arise from perturbative (multi-jet) effects \cite{30}. Compared to the NB, the LN regularity is found to give much better description which leads to understanding the multiparticle production process as a scale invariant stochastic branching process.

The fluctuations have been studied in a model of this type in Ref. \cite{46}, within the so-called “$\alpha$-model” and it was found that in this particular case the normalized factorial moments obey

\footnote{Comparing the formation of dense groups of particles in $e^+e^-$ and in hadronic (or nuclear) interactions, one finds noticeable difference of the fluctuations structure being isotropic (self-similar) in the former case and anisotropic (self-affine) in the latter one \cite{39}. This is expected to reflect different dynamics of the hadroproduction process in these two types of collisions.}

\footnote{See preceding footnote.}
the recurrence relation,

$$\ln F_q = \frac{q(q - 1)}{2} \ln F_2.$$  

Strictly speaking, this formula connects the standard normalized LN moments $C_q = \langle n^q \rangle / \langle n \rangle^q$ rather than the factorial moments $F_q$. The difference between this two types of moments becomes negligible at large multiplicities which is not the case for small bins. Thus, the results based on the Eq. (3) with $F_2$ defined by data can deviate from the true LN regularity predictions, particularly at small bins [34].

Recently, the **modified negative binomial (MNB)** regularity has been introduced to correct deviation between the NB parametrization predictions and the $e^+e^-$ and $p\bar{p}$ data [17]. One finds [18] for the normalized factorial cumulants of the MNB distribution,

$$K_q^- = (q - 1)! k^{1-q} \frac{r^q - \Delta^q}{(r - \Delta)^q}.$$  

Here, $r = \Delta + \langle n \rangle / k$ and the superscript minus indicates that this law is applied for negatively charged particles. The MNB regularity reduces to the NB one if $\Delta = 0$, cf. Eq. (2).

The MNB parametrization has been found to give an accurate description of the full phase-space multiplicity distributions in $e^+e^-$ annihilation measured from a few GeV up to LEP2 energies [28, 17, 19, 52] and in lepton-nucleon scattering data in the wide energy range [53]. A similar energy dependence of the parameter $k$ has been obtained in these two types of collisions. Recently, a simple extension of the MNB law has been found to describe the charged particle multiplicity distributions in symmetric (pseudo)rapidity bins as well [54].

In contrast to the NB, the MNB parametrization is shown to reproduce fairly well the factorial moments and the cumulants of the full like-sign, e.g. negatively charged particle phase space from $e^+e^-$ data at the energies ranging from 14 to 91.2 GeV [53]. The fits of the multiplicity distributions, the moments and cumulants give rise to $k > 0$, $\Delta < 0$ and $0 < r < |\Delta| < 1$ [4] that, according to Eq. (4), leads to the cumulants being negative at even values of order $q$ and positive at odd ones. This fact has been utilized to explain the oscillations of the ratio of the cumulants to factorial moments as a function of $q$ [48].

To obtain the quantities for all charged particles one uses the fact that the number of charged particles produced in $e^+e^-$ collisions is twice that of negative ones. Then, the normalized factorial moments and cumulants are given [48] by

$$K_2 = K_2^- + \frac{1}{2k(r - \Delta)}, \quad K_3 = K_3^- + \frac{3K_2^-}{2k(r - \Delta)}, \quad \text{etc.}$$  

The stochastic nature of the NB and MNB laws allows to generalize them by introducing a stochastic equation of the **pure birth (PB)** process with immigration [58, 59]. Then, the NB and the MNB distributions can be derived from this birth process under the appropriate initial conditions, namely, the birth process with no particles in the initial stage leads to the

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4 Negative values of the parameter $\Delta$ are interpreted as the probability $-\Delta$ of intermediate neutral cluster to decay into the charged or neutral hadron pairs [19, 50]. Positive $\Delta$ values are also acceptable [17, 50], but in this case $k$ is the maximum number of sources at some initial stage of the cascading changing from one event to another, in contrast with the fixed $k$ value corresponding to the case of $\Delta < 0$ [19].

5 Nevertheless, at LEP1.5 energy, $\sqrt{s} \simeq 133$ GeV, an inverse inequality $|\Delta| < r \simeq 0.91$ can be found [51]. An increase of the parameter $r$ with increasing energy is allowed but this complicates the particle-production scenario leading to the MNB parametrization [57].
NB law, while the MNB distribution is resulted from the birth process with the initial binomial
distribution [56].

For the PB stochastic process one finds

\[ F_q = \Gamma(q)x^{1-q}L^{(1)}_{q-1}(-x) \]

\[ = 1 + \frac{q(q-1)}{x} + \frac{q(q-1)^2(q-2)}{2!x^2} + \frac{q(q-1)^2(q-2)^2(q-3)}{3!x^3} + \cdots \] (6)

\[ = 1 + \frac{q(q-1)}{x} + q \sum_{i=1}^{q-2} \frac{q-i-1}{(i+1)!x^{i+1}} \prod_{j=1}^{i}(q-j)^2 \]

for the normalized factorial moments [35, 58, 59], and

\[ K_q = q!x^{1-q} \] (7)

for the normalized cumulants. Here, \( L^{(1)}_{q}(a) \) is the associated Laguerre polynomial [60].

The PB stochastic model has been applied to describe the multiplicity distribution and its
moments in the entire (pseudo)rapidity range [58] and in its bins [59] in p\(\bar{p}\) collisions at c.m.
energy ranging from 11.5 to 900 GeV. Considering high order moments, it was shown [59] that
the results of this approach are close to the NB predictions revealing the stochastic nature of
particle production and, in particular, of the NB model. Further analysis [35], which includes
intermittency study extending from e\(^+\)e\(^-\) to nuclear collisions [6], have shown that the data is
well reproduced by this model. Note that the predictions of this model are systematically below
the NB calculations.

The last form of the multiplicity distribution, considered in this paper, is the recently intro-
duced **generalized negative binomial** distribution called the HNB distribution due to the
type of special H-function used to derive it [61,62]. This distribution represents an extension of
the NB regularity to the Poisson-transformed generalized \(\Gamma\)-distribution by incorporating some
perturbative QCD characteristics. Varying the shape parameter \(k > 0\), the scale parameter
\(\lambda > 0\) and the scaling exponent \(\mu \neq 0\), one receives special and limiting cases of the HNB
distribution. The Poisson and the LN distributions are special cases of the HNB distribution
in the limit of \(k \to \infty\) with \(\mu = 1\) and \(\mu \to 0\), respectively. The HNB regularity converges to
the NB distribution at \(\mu = 1\).

Applied to high-energy data [61–64], the HNB distribution has been found to agree with the
data, depending on the parameter \(\mu\) values being positive or negative [61] (\(|\mu| > 1\)) or approaching
zero, for different types of reactions. The HNB regularity with \(\mu > 1\) and \(k = 1\) (the Weibull
law) has been found to describe the data successfully in inelastic pp and p\(\bar{p}\) reactions up to
ISR energies and in deep-inelastic e\(^+\)p scattering at HERA energies in the entire rapidity phase
space [62] as well as in its restricted bins [63]. The multiplicity distribution from UA5 data of
non-diffractive p\(\bar{p}\) collisions at \(\sqrt{s} = 900\) GeV has been shown to be fitted reasonably well by
the LN (\(\mu \to 0\), \(k \to \infty\)) HNB limit in the full pseudorapidity window and in its symmetric
bins.

In e\(^+\)e\(^-\) annihilations, it was found that HNB describes the data below the top PETRA
energies for \(\mu < -1\) and \(k = 1\) [63], whereas at high energies the HNB description favour
\(\mu \approx 0\) and large \(k\) [61,62,64]. The latter, as well as the above mentioned success of the

\[ \text{See footnote 2.} \]

\[ \text{For } \mu \approx 0 \text{ the following reparametrization of the HNB density has been suggested [61,62]: } (k, \lambda, \mu) \leftrightarrow (p, \sigma, \alpha) \]

with \(p = 1/\sqrt{k}, \sigma = p/\mu\) and \(\alpha = \ln \lambda\). With these parameters, e.g. one gets the LN distribution when \(p = 0\).
LN limit HNB distribution to reproduce the non-diffractive UA5 data, is ascribed to the LN regularity of the multiplicity distribution obtained earlier \[27, 30, 45\] and recently explained by a renormalization group approach \[62\]. For the LEP1 data \(\mu\)-value lying between \(-1.2\) and \(-0.6\) and \(k \approx 20 \div 130\) have been obtained. Note that while the error range of the parameter \(\mu\) is found to be small, so that \(\mu\) is above \(-2.2\) and below zero, the errors for \(k\) allow it to vary between \(O(10)\) and \(O(10^4)\) \[61\]. The situation does not change when the energy increases up to higher than the \(Z^0\) peak or when the multiplicity distribution in rapidity bins instead that in full phase phase space is considered \[62\]. It is interesting that for central rapidity bins, \(\mu\) is obtained to be positive, \(0 < \mu < 1\), increasing with enlarging the bin size, while \(k\) decreases and is of \(O(10)\). Recent analysis \[64\] of the full phase space of uds-quark jet of the OPAL data \[67\] has shown the error bars for the \(k\)-parameter to lie between 20 and 420, with the central value at 54. The parameter \(\mu\) has been found to be about \(-0.5\).

One gets \[61\] for the HNB-defined normalized factorial moments,

\[
F_q = \frac{\Gamma(k + q/\mu)}{\Gamma(k)} \left[ \frac{\Gamma(k)}{\Gamma(k + 1/\mu)} \right]^q,
\]

approaching asymptotically for large \(q\) to the \(\Gamma\)-function of the rescaled rank, \(q/\mu\).

### 3 Comparison with OPAL measurements and discussion

In Figs. 1 and 2 we show, respectively, the normalized factorial moments and the normalized factorial cumulants, measured by OPAL \[8\] in \(e^+e^-\) annihilation and compared to a few parametrizations (lines) and to the MC \[68\] (shaded areas). The moments are represented in one-, two- and three dimensions of the phase space of rapidity, transverse momentum and azimuthal angle calculated with respect to the sphericity axis.

For the NB law we used the second-order factorial moments or cumulants to compute \(k\) and then the factorial moments and cumulants of order \(q \geq 3\), according to Eqs. (1) and (2). The resulting quantities are shown by the dashed lines.

From Fig. 1 one can see that in general the NB regularity underestimates the measured factorial moments. The deviation is more pronounced in one and two-dimensions for number of bins \(M > 5\) (in one projection) and \(q > 3\). Note that these our conclusions coincide with those from the analogous investigations \[34\] of earlier LEP1 results on intermittency \[33\]. The better agreement between the parametrization and the data we find for low-order \((q = 3)\) moments or for those in three dimensions, the cases when the NB predictions are within the MC results. Nevertheless, even in three dimensions, the NB values are below the data points at large \(M\) (small bin size) and high orders.

The situation becomes more clear when one addresses to the cumulants, namely the genuine correlations contributing to the fluctuations, Fig. 2. From one- and two-dimensional cumulants one can see that not only for high \(q\)’s, but even in the case of \(q = 3\) the correlations given by the NB model are weaker than those from the data. The same is observed for \(p_T\), \(y \times p_T\) and \(\Phi \times p_T\) projections (not shown). The NB cumulants lie far away from the measured ones in comparison with the MC predictions being much nearer to the data. Moreover, it is seen that

\[8\] It is worth to mention the remarkable different shapes of the multiplicity distribution in these two cases, namely, a heavy-tailed form with a narrow peak in pp collisions vs. a bell-like shape in e\(^+\)e\(^-\) annihilation. Nevertheless, recently it was found that the multiplicity distribution in both types of interactions show the same behaviour expected from the log-KNO scaling \[66\].
the discrepancy between the data and the NB results does not begin at some intermediate $M$, as in the case of the fluctuations, but is visible even at smaller $M$-values (larger bin sizes). This observation agrees with the inadequacy of the NB regularity to fit the full phase space multiplicity distribution, i.e. at $M = 1$. The NB parametrization describes the data reasonably well in three dimensions, although some deviations are seen for $M > 125.$

Using Eq. (2) we have estimated the parameter $k$ as a function of $M$ at different $q$. The parameter is found to decrease with increasing $M$ and to depend weakly on $q$. The values of $k$ lie between $\simeq 0.2 \div 2.8$ and $\simeq 5 \div 8$ at two extreme $M$ values and vary at fixed $M$ with a change in $q$ and with the dimensionality of subspaces. The higher is the dimension of the subspace, the smaller is the lower bound of the $k$-range. These lower bound values are almost independent of $q$. Conversely, the values of upper bound on $k$ show their $q$-dependence. They are about 8 at $q = 2$ and about 5 at $q = 3$ and 4 regardless of the subspace dimension. According to the expectations [21], the observed values of $k$ and its $q$-dependence do not seem to be related to the truncation effect but rather to small cascades. On the other hand, taking into account a long enough cascade at the $e^+e^-$ collision energies considered here, this conclusion confirms that NB encounters difficulties in a reasonable and consistent description of the measurements. It is interesting to note that the values of $k$ obtained are close to those found [54] in the MNB-type analysis of the multiplicity distributions in restricted rapidity intervals in $e^+e^-$ annihilations at the $Z^0$ peak [29,30].

Contrary to the NB, the LN regularity overestimates the data regardless of the dimensionality or the type of the variable used. The dotted line in Figs. 1 shows how the LN predictions compares the measurements. The cumulants are calculated from the factorial moments using their interrelations [2]. The smaller the bin size is, the larger the difference is. The LN parametrization describes the data quite well for order $q = 3$ only.

These findings are in agreement with the earlier studies of LN fits to factorial moments in $e^+e^-$ annihilation [34]. The LN distribution overestimates high multiplicities [4], which leads to an overestimate of the fluctuations and genuine correlations as shown here. In contrast to the studies of the full-multiplicity parametrization [27,30], the deviations are found for all bin sizes and not only for intermediate ones. This is in contrast with the above mentioned multi-jet (perturbative) effects [9] and indicates significant contribution of the non-perturbative stage dynamics, i.e. soft hadronization, to the formation of fluctuations and correlations. This agrees with recent theoretical studies [2]. Violation of the Gaussian law of Eq. (3) have already been observed in nuclear [37,41,42] and hadronic [43] interactions.

The predictions of another, the MNB regularity, are shown in Figs. 1 and 2 by the solid lines. The cumulants are calculated using Eqs. (4) and (5), while the factorial moments are derived from their relationships with cumulants [2]. The parameters $r$ and $\Delta$ have been fixed at the values $r = 0.91$ and $\Delta = -0.71$, the best values found to describe at least the third order cumulants. The only parameter depending on the bin size is $k$ extracted from $K_2$. The value of the $\Delta$ parameter is near to that found in multiplicity studies [28,47,50,51], while the parameter $r$ has the value $r \geq |\Delta|$, in contrast to the $r \leq |\Delta|$ inequality obtained from the analysis of the full multiplicity distributions. From the Figures, one can conclude that, in general, with the parameters obtained, the MNB regularity describes the data well, although underestimates the latter at small bins and the parameters $|\Delta|$ and $r$ have an inverse hierarchy.

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9 See, however, [21].

10 A better description of the multiplicity distribution in the case of smaller number of jets in $e^+e^-$ annihilation, namely for two-jet events, compared to the inclusive sample, has been also obtained by OPAL using the NB regularity [27].
The obtained exceeding of the value of $r$ over the $\Delta$-value is expected if one applies the MNB regularity not to negative-particle distributions but to all-charged-particle ones, see e.g. DELPHI publication \cite{28}. In this case, two effect are contributing: the number of sources increases with increasing the width of the phase space bin, while a neutral cluster decay gives 0, one or two particles hit in the given bin. \cite{57,58}. The latter effect is taken into account by implementing an additional parameter \cite{54}. This is not the case when corrected formulae \cite{3} are used, therefore the change of the inequality between $r$ and $|\Delta|$ has a physical meaning to be investigated.

The resulted factorial moments and cumulants of the multiplicity regularity given by the PB process, are represented in Figs. 1 and 2 by the dashed-dotted lines. The parameter $x$ is extracted from the second order cumulants $K_2$ and defines all the higher-order moments and cumulants, given by Eqs. (3) and (7), respectively. For both quantities the PB predictions are seen to lie lower than those from the above considered parametrizations. In all the cases the PB calculations underestimate the data. The difference between the PB predictions and the data is in contrast to earlier lower-energy $e^+e^−$ parametrization of the factorial moments in the rapidity subspace \cite{35}, where the PB process has been found to explain the data while underestimating higher-order moments.

No curves predicted by the HNB regularity are shown in the Figures, since we would like to estimate the regions of the parameters $\mu$ and $k$ of this approach. Combined analysis of the factorial moments and cumulants, the latter being calculated as combinations of the factorial moments \cite{4}, show that, assuming $k$ to be positive, the $\mu$-parameter is obtained to be either positive or negative. The factorial moments and cumulants are found to be sign-changing functions of $k$ for negative $\mu$, in particular at $−1 < \mu < 0$, so that one can find more than one $k$-region which satisfies the measurements\cite{36}. In this case we take the largest $k$-value, above which no sign-changing behaviour is seen and the calculation fits most of the data. For $\mu > 0$ the factorial moments and the cumulants decrease with increasing $k$.

At fixed $k$, the absolute value of the parameter $\mu$ decreases with the number $M$ of bins. The parameter is found to be limited in the interval $0.3 < |\mu| < 1.8$ when one considers the quantities under study of order $q = 2, 3$. For $q > 3$, one faces problems reaching the highest measured values of the moments and cumulants and, in the case of the highest $q = 5$ cumulants, fitting their lowest (negative) values. This narrows the interval of $\mu$ down to $0.6 < |\mu| < 1.6$. The values of the shape parameter $k$ depends on $\mu$, but are found to be $\lesssim 30$. The smallest $k$ is obtained to be 1.2 for $\mu < 0$ and 0.1 for $\mu > 0$. The larger the value of $|\mu|$ is, the smaller is the interval of $k$. For example, if at $|\mu| > 1.3$ the values of $k$ are only of a few units, $\simeq 2 ÷ 5$, then for $0.5 < |\mu| < 1.0$ these values lie between 2 and 20. The parameter $k$ slightly depends on the order $q$, increasing with $q$ for negative $\mu$ and decreasing for positive ones. This parameter depends also on the number of bins, being a decreasing function of $M$. Note that these two properties are similar to those observed here for the NB regularity.

Comparing the results obtained to the HNB studies \cite{61,62,65} of the LEP data and, particularly, to that of the uds-quark jets \cite{64} (see also Sect. 3), one can see similarities as well as

\footnote{See, however, footnote 5.}

\footnote{It is interesting to note that being limited to like-charged particles, a study of particle bunching is less dependent on correlations induced by charge conservation (and partly by resonance production), in addition to the above-mentioned advantage of such a study to be less affected by the energy-momentum constraints \cite{57}.}

\footnote{Our observation contradicts the property of the factorial moments and cumulants shown \cite{61,63} to oscillate around zero at $|\mu| > 1$ and not e.g. at $\mu > −1$. This disagreement can be assigned to different regions of the parameter $k$, found to be large ($k \rightarrow \infty$) in the case of the full-multiplicity distribution studies \cite{31,65} while having finite values in our investigation (\textit{vide infra}).}
important differences. Similar to those studies we find that the parameter $\mu$ can be positive as well as negative and does not exceed 2 in its absolute value. Moreover, this parameter shows the same behaviour with bin size, and $k$ tends to obtain large values \[62, 64\]. The main difference from the HNB studies is that the region $\mu \approx 0$ is excluded by our investigation, so that the LN law is not a suitable one. This conclusion agrees with the OPAL observation \[27\] and the above discrepancy between the data and the LN predictions. It is also worth to notice that (i) negative $\mu$'s can be also used to describe the multiplicity distributions in restricted bins, in addition to the positive ones found recently \[64\], and (ii) the values of the $k$-parameter are less than 30 and do not tend to infinity. All this shows that to describe the hadroproduction process correctly, one needs a more complicated scenario to be realized than those leading to the regularities discussed here, even generalized to the HNB case. One could not e.g., find $\mu$ to be only in the interval $0 < \mu < 1$ as it would expected due to our findings for the LN regularity ($\mu \approx 0$) to overestimate the measured fluctuations and correlations while the NB law ($\mu \rightarrow 1$) underestimates them.

4 Summary and Conclusions

To summarize, various regularities of the multiplicity distribution of charged particles are studied in restricted phase-space bins of $e^+e^-$ annihilation into hadrons at the $Z^0$ peak. The study is based on recent high-statistics results \[8\] on multidimensional local fluctuations and genuine correlations obtained with multihadronic OPAL sample by means of normalized factorial moments and cumulants. Such parametrizations as the log-normal and negative binomial distributions, modified and generalized versions of the negative binomial law, and the generalized stochastic birth process with immigration are considered. For the first time these parametrizations, being most common in the field of multiparticle production, are examined with genuine high-order correlations.

All the parametrizations are found to give a reasonably good description of low-order fluctuations while they do show deviations for the high-order fluctuations and correlations, especially at small resolutions. Some discrepancy can arise from a bin-size dependence of the measured multiplicity distribution and from its truncation. However, in our consideration, the influence of these effects is minimized since the models are based on the measured second-order factorial moments and cumulants, which carry most of the information given by the multiplicity distribution. Moreover, the effects mentioned arise mostly according to low statistics that is not the case for the data considered here, even at higher orders. To note is also that even with low statistics data, a simultaneous analysis of multiplicity distributions in different bin sizes and the corresponding factorial moments, carried out in $e^+e^-$ annihilation \[34\] and in nuclear reactions \[40\], does not show any sensitive influence of the finite statistics to the results.

From the study presented, one concludes that genuine high-order ($q > 3$) correlations have to be taken into account when the hadroproduction process is modelling, in particular by a multiplicative law for particle distributions. Indeed, since all the parametrizations used are essentially based on the average multiplicity and the two-particle correlations, the discrepancies between the predictions and the measurements indicate multiparticle character of bunching of hadrons. This could be considered as a reason why all the regularities give a good description of fluctuations and correlations at order $q = 3$. Our conclusion confirms the OPAL result from Ref. \[8\], on which we are based here and which shows the important contributions of many-particle correlations to the dynamical fluctuations by the decomposition of the factorial moments into lower-order cumulants. This our finding is also in agreement with the observation
of DELPHI shown the measured angular cumulants not to be reproduced by small genuine correlations given by perturbative QCD.

A self-similar nature of multihadron production is another issue of the above study. All the regularities result from the particle-production process of the stochastic nature. Therefore their capability to show the intermittent behaviour seen in the data can be attributed to their branching self-similar nature. The better agreement between the regularities and the measurements found in three dimensions, where QCD cascading is expected to be fully developed, stresses the essential self-similarity of the particle production mechanism. However, the discrepancies obtained show that a suitable hadroproduction model seems to be more sophisticated than those giving the parametrizations considered in the present paper.

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Figure 1: Factorial moments of order $q = 3$ to $5$ as a function of $M^{1/D}$, where $M$ is the number of bins of the $D$-dimensional subspaces of the phase space of rapidity, azimuthal angle, and transverse momentum, in comparison with the predictions of various multiplicity parametrizations and two Monte Carlo models. The data and the Monte Carlo predictions are taken from Ref. [8].
Figure 2: Cumulants of order $q = 3$ to $5$ as a function of $M^{1/D}$, where $M$ is the number of bins of the $D$-dimensional subspaces of the phase space of rapidity, azimuthal angle, and transverse momentum, in comparison with the predictions of various multiplicity parametrizations and two Monte Carlo models. The data and the Monte Carlo predictions are taken from Ref. [8].