Information Scrambling over Bipartitions: Equilibration, Entropy Production, and Typicality

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In recent years, the out-of-time-ordered correlator (OTOC) has emerged as a diagnostic tool for many-body quantum chaos and information scrambling. Here, we provide exact analytical results for the long-time averages of the OTOC for a typical pair of random local operators supported over two regions of a bipartition, thereby revealing its connection with eigenstate entanglement. We uncover a hierarchy of constraints over the structure of the spectrum of Hamiltonian systems, for instance integrable models, and elucidate how they affect the equilibration value of the OTOC. We provide operational significance to this “bipartite OTOC,” by unraveling intimate connections with operator entanglement, average entropy production, and scrambling of information at the level of quantum channels.

Introduction.— A characteristic feature of chaotic quantum systems is their ability to quickly spread “localized” information over subsystems, thereby making it inaccessible to local observables. Although unitary evolution retains all information, this local inaccessibility manifests itself as equilibration in closed systems, and has been termed “information scrambling” [1–5].

For Hamiltonian quantum dynamics, scrambling can be probed by examining the overlap of a time-evolved local operator \( V(t) := U_t^\dagger V U_t \) with a second static operator \( W \). This overlap is commonly quantified via the strength of the commutator

\[
C_{V,W}(t) := \frac{1}{2} \text{Tr} \left( [V(t),W]^\dagger [V(t),W] \rho_\beta \right)
\]  

(1)

where \( \rho_\beta \) denotes the thermal state at inverse-temperature \( \beta \). From the perspective of information spreading, \( C_{V,W}(t) \) is a natural quantity to consider since it constitutes a state-dependent variant of the Lieb-Robinson scheme; the latter enforces a fundamental restriction on the speed of correlations spreading in non-relativistic quantum systems [6–9]. In Eq. (1), it is convenient to consider pairs of operators \( V,W \) which at \( t = 0 \) act nontrivially on different subsystems, thus commute; we follow this convention here.

The commutator \( C_{V,W}(t) \) is intimately linked to the out-of-time-order correlator (OTOC) [10, 11] which is a 4-point function with an unconventional time-ordering

\[
F_{V,W}(t) := \text{Tr} \left( V(t)^\dagger W(t) V(t) W(t) \rho_\beta \right).
\]  

(2)

The connection between the two arises when \( V,W \) are unitary; Eq. (1) then immediately reduces to \( C_{V,W}(t) = 1 - \text{Re} \left[ F_{V,W}(t) \right] \). In this paper we focus on the infinite temperature, \( \beta = 0 \) case.

The OTOC has been extensively utilized to study chaos in quantum systems [12–15]. Scrambling is a characteristic signature of the latter, and the OTOC can successfully diagnose the transition to chaoticity [16–24], for instance, via its initial decay rate.

Per se, the OTOC’s ability to probe dynamical features such as chaoticity clearly depend on the choice of operators \( V,W \). However, it is desirable to be able to capture these features as independently as possible from the specific choice of operators. This insensitivity can be achieved by averaging over a set of operators, a strategy also considered in Refs. [23, 25–29]. It is crucial to remark that for the averaged OTOC to faithfully capture information spreading, the averaging process must preserve the initial locality of the system, i.e., which subsystems \( V,W \) initially act upon — an observation that was quintessential in revealing the correct behavior of the OTOC and its connection with Loschmidt echo [29].

Given a bipartition of a finite-dimensional Hilbert space \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \), we will henceforth focus on averaging \( C_{V_A,W_B}(t) \) over the (independent) unitary operators \( V_A,W_B \), whose support is over subsystems \( A \) and \( B \), respectively. The resulting quantity

\[
G(t) := 1 - \frac{1}{d} \text{Re} \int dV dW \text{Tr} \left( V_A^\dagger(t) W_B(t) V_A(t) W_B \right),
\]  

(3)

depends only on the dynamics and the Hilbert space cut, where we denote \( V_A = V \otimes I_B, \ W_B = I_A \otimes W \) and the averaging is performed according to the Haar measure [30]. We will refer to \( G(t) \) for brevity as the bipartite OTOC, and analyzing its properties will be the focus of the present paper.

It was recently shown in Ref. [29], where \( G(t) \) was first introduced, that under the assumptions of (i) weak coupling between \( A \) and \( B \), and (ii) Markovianity, that \( G(t) \) exhibits a close connection with the Loschmidt echo [31]; the latter has been widely employed to characterize chaos [32]. Here, we first show, without any of the previous assumptions, that \( G(t) \) is, in fact, amenable to exact analytical treatment, and we uncover its direct relation with entropy production, information spreading,
and entanglement. We also rigorously prove that the average case is also the typical one, hence justifying the averaging process. All proofs of the claims appearing in the text can be found in Appendix A.

The bipartite OTOC. — We begin by bringing Eq. (4) in a more explicit form which will be the starting point for a sequence of results. This can be achieved by working on the doubled space $\mathcal{H} \otimes \mathcal{H}'$, where $\mathcal{H}' = \mathcal{H}_A' \otimes \mathcal{H}_{B'}$ is a replica of the original Hilbert space.

**Proposition 1.** Let $S_{AA'}$ be the operator over $\mathcal{H} \otimes \mathcal{H}'$ that swaps $A$ with its replica $A'$ and $d = \text{dim}(\mathcal{H})$. Then

$$G(t) = 1 - \frac{1}{d^2} \text{Tr} \left( S_{AA'} U_t^{(2)} S_{AA'} U_0^{(2)} \right).$$

(4)

The analogous expression for $BB'$ also holds.

The above formula immediately exposes a connection between the bipartite OTOC and the operator entanglement of the evolution $E_p(U_t)$, as defined in Ref. [33]; the two quantities, remarkably, coincide exactly. This observation also allows one to express the entangling power [34] $e_p(U_t)$ as a function of the bipartite OTOC for the symmetric case $d_A = d_B$. The former quantifies the average entanglement produced by the evolution and has been established as an indicator of global chaos in few-body systems [35–38].

**Proposition 2.** Let $G_U$ denote the bipartite OTOC for the evolution $U$. Then, (i) $E_p(U_t) = G_U$, and (ii) for a symmetric bipartition $d_A = d_B$,

$$e_p(U_t) = \frac{d}{(\sqrt{d} + 1)^2} \left( G_{U_t} + G_{U_t S_{AB}} - G_{S_{AB}} \right).$$

(5)

How informative is the average $G(t)$? — Usually, one is interested in behavior of the OTOC for a typical choice of random unitary operators. Due to measure concentration [39], we prove the two essentially coincide, i.e., the probability that a random instance deviates significantly from the mean is exponentially suppressed as the dimension of either of the subsystems $A$ and $B$ grows large.

**Proposition 3.** Let $P(\epsilon)$ be the probability that a random instance of $C_{V,W}(t)$ deviates from its Haar average $G(t)$ more than $\epsilon$. Then,

$$P(\epsilon) \leq 2 \exp \left( -\frac{\epsilon^2 d_{\text{max}}}{64} \right),$$

(6)

where $d_{\text{max}} = \max\{d_A, d_B\}$.

In the definition of the bipartite OTOC and to obtain the replica formula Eq. (4), we have so far considered averaging over the uniform (Haar) ensemble which continuously extends over the whole unitary group. Although natural from a mathematical viewpoint, this choice can turn out to be rather complicated on physical and numerical grounds [40]. Nonetheless, we show in Appendix B that Haar averaging can be replaced by any unitary ensemble that forms a 1-design [41–44] without altering $G(t)$. Such ensembles mimic the Haar randomness only up to the first moment, which is the depth of randomness that the OTOC can probe [23]. The latter assumption is thus much weaker than Haar randomness. For instance, consider the case of a spin-1/2 many-body system split into two parts, $A$ and $B$. Instead of averaging over Haar random unitaries $V_A$ and $W_B$, that typically do not factor, the 1-design (equivalent) picture prescribes to instead consider only fully factorized unitaries with support over $A$ and $B$, e.g., products of local Pauli matrices.

**Time-averaging the bipartite OTOC.** — In finite dimensional quantum systems, nontrivial quantum expectation values or quantities such as $C_{V,W}(t)$ do not converge to a limit for $t \to \infty$. Instead, after a long time they typically oscillate around an equilibrium value [45–50] which can be extracted by time-averaging $\overline{X}(t) := \int_0^T dt X(t)$. We now turn to examine this long-time behavior $\overline{G}(t)$ of the bipartite OTOC as a function of the Hamiltonian and the Hilbert space cut.

Let us begin with the case of a chaotic dynamics, which entails level repulsion statistics [15] and an “incommensurable” relation among the energy levels. As such, chaotic Hamiltonians satisfy (either exactly or to very good approximation) the no-resonance condition (NRC). The energy levels and energy gaps feature nondegeneracy. This has important implications for the long-time behavior of their bipartite OTOC, as we will see soon.

Let us spectrally decompose $H = \sum_k E_k |\phi_k\rangle \langle \phi_k|$ and use $\rho^{(x)}_k := \text{Tr}_X (|\phi_k\rangle \langle \phi_k|)$ to denote the reduced density operator over $X = A, B$ corresponding to the $k$th Hamiltonian eigenstate ($X$ corresponds to the complement). Below, $\langle X, Y \rangle := \text{Tr}(X^\dagger Y)$ denotes the Hilbert-Schmidt inner product [51], which gives rise to the operator 2-norm $\|X\|_2 := \sqrt{\langle X, X \rangle}$.

**Proposition 4.** Consider a Hamiltonian satisfying the NRC. Then

$$\overline{G(t)}^{\text{NRC}} = 1 - \frac{1}{d^2} \sum_{\chi \in \{A,B\}} \left( \| R(\chi) \|_2^2 - \frac{1}{2} \| R_D^{(\chi)} \|_2^2 \right)$$

(7)

where $R^{(x)}$ is the Gram matrix of the reduced Hamiltonian eigenstates $\{\rho^{(x)}_k\}_{k=1}^d$, i.e.,

$$R_k^{(x)} := \langle \rho^{(x)}_k, \rho^{(x)}_l \rangle$$

(8)

while $(R_D^{(\chi)})_{kl} = R_k^{(\chi)} \delta_{kl}$.

Let us first point out some basic, yet important properties of the above formula. The matrix $R^{(x)}$ is real and symmetric, while $R_D^{(\chi)}$ is positive-semidefinite and diagonal. Moreover, the completeness of the Hamiltonian eigenvectors imposes $\sum_k \rho_k^{(x)} = I$, thus the rescaled $\tilde{R}^{(\chi)} := R^{(x)}/d_X$ are doubly stochastic, i.e., $\sum_i \tilde{R}^{(x)}_{ij} = \sum_j \tilde{R}^{(x)}_{ij} = 1 \forall j$. As $\tilde{R}^{(x)}$ is a (rescaled) Gram matrix, its
A time-averaging over any Hamiltonian evolutions. Its estimate depends only on the eigenprojectors of the Hamiltonian and can be found in Appendix (A). The value of the Haar average can be performed exactly, with result

\[
\overline{G}(t) = \frac{(d_A^2 - 1)(d_B^2 - 1)}{d^2 - 1}.
\]

The following ordering holds.

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2 Here chaoticity concretely means that the Hamiltonian spectrum satisfies the NRC and that the entanglement of the typical eigenvectors in the bulk, which determine the equilibration value, resembles that of Haar random vectors \([62, 63]\), i.e., \(\bar{\text{Tr}}(\rho_k^2) \approx (d_A + d_B)/(d + 1)\) thus \(\epsilon = \mathcal{O}(1/d_{\text{min}})\).
Proposition 6. For any given Hamiltonian, the corresponding estimates are related with the exact time-average $G(t)$ as

$$G_{\text{Haar}} \geq G(t)_{\text{NRC}} \geq G(t)_{\text{NRC}^+} \geq G(t).$$

(11)

The above constitutes a proof that coincidences in the spectrum of a Hamiltonian up to the “gaps of gaps” (i.e., degeneracy over the energy levels and their gaps) always reduces the equilibration value of the bipartite OTOC.

Proposition 8. 

$G(t) = \frac{d_\chi}{d_\chi} + 1 \int d\chi S_{\text{lin}}[\Lambda^A_t(|\psi_U\rangle\langle\psi_U|)]$  

(13)

where $\chi = A, B$ and $|\psi_U\rangle := U|\psi_0\rangle$ corresponds to Haar random pure states over $H_\chi$.

The analogous expression for $BB'$ also holds.

The quantum map $\Lambda^A_t(x)$ is unital, i.e., the maximally mixed state is a fixed point. As such, the transformation $\rho_\chi \mapsto \Lambda^A_t(\rho_\chi)$ results always in an output state whose spectrum is more disordered than the input one [64]. As a result, when $\rho_\chi$ is pure, the effect of the reduced time-dynamics is to scramble and hence produce entropy. Let us now turn to examine this connection more closely.

Bipartite OTOC as entropy production. — We now show that the bipartite OTOC $G(t)$ is nothing but a measure of the average entropy production over pure states, with the latter quantified by linear entropy $S_{\text{lin}}$. 

Proposition 8. 

$G(t) = \frac{d_\chi}{d_\chi} + 1 \int d\chi S_{\text{lin}}[\Lambda^A_t(|\psi_U\rangle\langle\psi_U|)]$  

(13)

where $\chi = A, B$ and $|\psi_U\rangle := U|\psi_0\rangle$ corresponds to Haar random pure states over $H_\chi$.

In this manner, the bipartite OTOC can be fully characterized by linear entropy measurements over any of the $A, B$ subsystems. To obtain a satisfactory estimate of the mean in the RHS of Eq. (13), one does not, in practice, need to sample over the full Haar ensemble. An adequate estimate can be obtained with a rapidly decreasing number of necessary samples, as the dimension $d_\chi$ grows. More precisely, let $P(\epsilon)$ be the probability of the entropy $S_{\text{lin}}[\Lambda^A_t(|\psi\rangle\langle\psi|)]$ deviating from $\frac{d_\chi}{d_\chi} G(t)$ by more than $\epsilon$ for an instance of a random state. We show in Appendix A that

$$P(\epsilon) \leq \exp\left(-\frac{d_\chi \epsilon^2}{64}\right).$$

(14)

The linear entropy, although, per se, a nonlinear functional, can be turned into an ordinary expectation value if two (uncorrelated) copies of the quantum state are simultaneously available, $1 - S_{\text{lin}} = \text{Tr}(S^2)$ for $S = S_{AA'}, S_{BB'}$. This considerably simplifies its experimental accessibility as opposed to other entanglement measures, an important simplification for certain experimental setups [65–68]. As a result, Proposition 8 and the typicality result Eq. (14) suggest that the bipartite OTOC is, in turn, tractable via linear entropy measurements. We provide more details in Appendix C.

From Eq. (13) one can also infer the upper bound $G(t) \leq 1/2d_\chi := G_{\text{max}}^{(x)}$ announced earlier that follows from the range of the linear entropy function. The bound is thus achievable only when $\Lambda^A_t(x)$ is equal to the completely depolarizing map $T(x)(\cdot) := \text{Tr}(\cdot)\frac{I_\chi}{d_\chi}$.
Bipartite OTOC and information spreading.— The bipartite OTOC measures the average ability of the reduced time-evolution to erase information, as captured by the entropy production over a random pure state. This naturally raises the question as to whether $G(t)$ can also be understood as a measure of distance between $A_i^{(x)}$ and the depolarizing map $\mathcal{T}^{(x)}$, that is, in the space of quantum channels (i.e., Completely Positive and Trace Preserving (CPTP) maps [69]).

A straightforward answer can be obtained by resorting to the duality between quantum states and operatations [69]. Let $\rho_\mathcal{E} := \mathcal{E} \otimes \mathcal{T}(|\phi^+\rangle\langle\phi^+|)$ denote the (Choi) state corresponding to the CPTP map $\mathcal{E}$, where $|\phi^+\rangle := d^{-1/2}\sum_{i=1}^d |ii\rangle$ is a maximally entangled state.

Proposition 9. The bipartite OTOC is a measure of the distance between the reduced time-evolution and the depolarizing map:

$$G(t) = G_{\text{max}}^{(x)} - \left\| \rho_{A_i^{(x)}} - \rho_{\mathcal{T}^{(x)}} \right\|_2^2. \quad (15)$$

As an application, the proposition above can be utilized to bound the distance $\|A_i^{(x)} - \mathcal{T}^{(x)}\|_\infty$ given by the diamond norm [70, 71]; the latter is a well-established measure of distance between quantum channels since it admits an operational interpretation in terms of discrimination on the level of quantum processes [72]. The distinguishability of the two operations satisfies $\|A_i^{(x)} - \mathcal{T}^{(x)}\|_\infty \leq d_\chi^{3/2}G_{\text{max}}^{(x)} - G(t)$ (see Appendix A), therefore if $G_{\text{max}}^{(x)} - G(t)$ decays faster than $d_\chi^{-3}$, then asymptotically the two channels are essentially indistinguishable.

Summary.— We showed that the bipartite OTOC is amenable to exact analytical treatment and, quite remarkably, is equal to the operator entanglement of the dynamics. This identity allows one to establish a rigorous quantitative connection between the OTOC and the notion of entangling power, a well-established quantifier of few-body chaos. We then studied the late-time averages of the bipartite OTOC and provide a hierarchy of estimates for systems that violate the conditions of a “generic spectrum”. Finally, we unravel the operational significance of the OTOC by establishing intimate connections with entropy production and information scrambling at the level of quantum channels. Possible future directions include applying further these theoretical tools to concrete many-body systems and uncovering relations with thermalization, localization, and other many-body phenomena.

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APPENDICES

Appendix A: Proofs

Here we restate the Propositions, as well as other mathematical claims appearing in the main text, and give their proof.

Proposition 1

**Proposition 1.** Let $S_{AA'}$ be the operator over $\mathcal{H} \otimes \mathcal{H}'$ that swaps $A$ with its replica $A'$ and $d = \dim(\mathcal{H})$. Then

$$G(t) = 1 - \frac{1}{d^2} \text{Tr} \left( S_{AA'} U_t \otimes S_{AA'} U_t^{\dagger} \right).$$

(4)

The analogous expression for $BB'$ also holds.

**Proof.** Let $S$ be the operator over $\mathcal{H} \otimes \mathcal{H}'$ that swaps $H$ with its replica $H'$. Then for any operators $X, Y$ acting over $\mathcal{H}$ it holds that

$$\text{Tr} (XY) = \text{Tr} \left[ S (X \otimes Y) \right],$$

(A1)

as it can be easily verified by expressing both sides in a basis. Notice that in our case, where $\mathcal{H}$ carries a bipartition, one can further decompose $S = S_{AA'} S_{BB'}$.

Using the above identity the OTOC averaging in Eq. (3) can be written as

$$G(t) = 1 - \frac{1}{d} \text{Re} \int dV dW \text{Tr} \left( S V_A^\dagger(t) W_B \otimes V_A(t) W_B \right)$$

$$= 1 - \frac{1}{d} \text{Re} \int dV dW \text{Tr} \left( S U_t \otimes (V_A \otimes V_A) U_t^{\dagger} (W_B \otimes W_B) \right)$$

$$= 1 - \frac{1}{d} \text{Re} \text{Tr} \left[ S U_t \otimes (\int dV V_A^\dagger \otimes V_A) U_t^{\dagger} \left( \int dW W_B^\dagger \otimes W_B \right) \right].$$

Now the two independent averages can be easily performed since for unitary operators over $\mathcal{H} \cong \mathbb{C}^d$ the corresponding Haar integrals evaluate to

$$\int dU U \otimes U^\dagger = \frac{S}{d}$$

(A2)

where $S$ is again the swap operator over the doubled space.

A quick way to prove the well-known identity (A2) is by using Eq. (A1) to write

$$UXU^\dagger = \text{Tr}_H \left[ (U \otimes U^\dagger) (X \otimes I) S \right]$$

and then using the fact that

$$\int dUUXU^\dagger = \frac{\text{Tr}(X)}{d}$$

(A3)

which follows directly from the left/right invariance of the Haar measure [30].

Using Eq. (A2) twice, we get

$$G(t) = 1 - \frac{1}{d} \text{Re} \text{Tr} \left( S U_t \otimes S_{AA'} U_t^{\dagger} \right)$$

$$= 1 - \frac{1}{d^2} \text{Tr} \left( S_{AA'} U_t \otimes S_{AA'} U_t^{\dagger} \right).$$

Since $[S, X \otimes Z] = 0$ for all operators $X, Z$, the analogous expression for $BB'$ holds, i.e.,

$$G(t) = 1 - \frac{1}{d^2} \text{Tr} \left( S_{BB'} U_t \otimes S_{BB'} U_t^{\dagger} \right).$$

(A4)

$\blacksquare$

Notice that the symmetry of the Haar measure forces the bipartite OTOC to be time-reversal invariant, i.e., $G(t) = G(-t)$. 

Proposition 2

**Proposition 2.** Let $G_U$ denote the bipartite OTOC for the evolution $U$. Then, (i) $E_{op}(U_t) = G_{U_t}$, and (ii) for a symmetric bipartition $d_A = d_B$, 

$$e_p(U_t) = \frac{d}{(\sqrt{d} + 1)^2} (G_{U_t} + G_{U_t} s_{AB} - G_{S_{AB}}).$$  \hspace{1cm} (5)

*Proof.* (i) The key observation here is that the bipartite OTOC $G_{U_t}$, in the form of Eq. (4), coincides with the operator entanglement $E(U_t)$ as defined in Ref. [33] (see Eq. (6) therein). Let us for completeness recall the definition from [33] and briefly reproduce the argument.

The main idea behind operator entanglement in [33] is to first express the unitary evolution $U$ (over the bipartite Hilbert space $\mathcal{H}_{AB}$) as a state in the doubled space $\mathcal{H}_{AB} \otimes \mathcal{H}_{A'B'}$ via

$$|U\rangle = U \otimes I_{A'B'} |\phi^+\rangle$$  \hspace{1cm} (A5)

for the maximally entangled state $|\phi^+\rangle$ and then evaluate the linear entropy of the state $\sigma_U = \text{Tr}_{B'B'}(|U\rangle\langle U|)$, i.e.,

$$E_{op}(U) := S_{lin}(\sigma_U) = 1 - \text{Tr}(\sigma_U^2).$$  \hspace{1cm} (A6)

Evaluating the above expression, as in the proof of Proposition 1, one obtains exactly Eq. (4), hence $E_{op}(U_t) = G_{U_t}$.

(ii) For the symmetric case $d_A = d_B$, the result follow by combining the first part of the current Proposition and Eq. (12) of Ref. [33].

Finally, we note that by direct substitution, one has $G_{S_{AB}} = 1 - 1/d$.  \hspace{1cm} ■

Proposition 3

**Proposition 3.** Let $P(\epsilon)$ be the probability that a random instance of $C_{V_A, W_B}(t)$ deviates from its Haar average $G(t)$ more than $\epsilon$. Then,

$$P(\epsilon) \leq 2 \exp \left( -\frac{\epsilon^2 d_{\text{max}}}{64} \right),$$  \hspace{1cm} (6)

where $d_{\text{max}} = \max\{d_A, d_B\}$.

The proof relies on measure concentration and, in particular, Levy’s lemma which we shall recall shortly (see, e.g., [73]). Below we are also going use various operator (Schatten) $k$-norms [51]; the latter are defined as $\|X\|_k := (\sum_i s_i^k)^{1/k}$ where $\{s_i\}$ are the singular values of $X$. The case $\|X\|_{\infty} := \max_i \{s_i\}$ corresponds to the usual operator norm. For $k \geq 1$, one always has $\|X\|_k \leq \|X\|_1$.

We also remind the reader that a function $f : U(d) \to \mathbb{R}$ is said to be Lipschitz continuous with constant $K$ if it satisfies

$$|f(V) - f(W)| \leq K \|V - W\|_2$$  \hspace{1cm} (A7)

for all $V, W \in U(d)$. For brevity, in this section we denote the Haar averages as $\langle \langle \cdot \rangle \rangle_U$ and also occasionally drop the explicit time dependence.

**Theorem** (Levy’s lemma). Let $U \in U(d)$ be distributed according to the Haar measure and $f : U(d) \to \mathbb{R}$ be a Lipschitz continuous function. Then for any $\epsilon > 0$

$$\text{Prob}\{|f(U) - \langle f(U) \rangle_U| \geq \epsilon\} \leq \exp \left( -\frac{d\epsilon^2}{4K^2} \right),$$  \hspace{1cm} (A8)

where $K$ is a Lipschitz constant.

During the course of the proof of the Proposition, the following two continuity results will come in handy.

**Lemma 1.** (i) The function $f_W(V) : U(d_A) \to \mathbb{R}$ with $f_W(V) := C_{V_A, W_B}(t)$ is Lipschitz continuous with constant $K_f = 2$ for all $t \in \mathbb{R}$ and $W \in U(d_B)$. 


(ii) The function $g(W) : U(d_B) \rightarrow \mathbb{R}$ with $g(W) := \langle C_{V_A,W_B}(t) \rangle_V$ is Lipschitz continuous with constant $K_g = 2/d_A$ for all $t \in \mathbb{R}$.

Proof of lemma. (i) Let $X, Y \in U(d_A)$. We need to show that

$$|f_W(X) - f_W(Y)| \leq K_f \|X - Y\|_2.$$ 

Following the proof of Proposition 1, we can express

$$f_W(V) = 1 - \frac{1}{d} \text{Re Tr} \left[ SU_t \otimes^2 (V^\dagger_A \otimes V_A) U_t \otimes^2 (W_B \otimes W_B) \right]$$

therefore

$$|f_W(X) - f_W(Y)| \leq \frac{1}{d} \left| \text{Tr} \left[ U_t \otimes^2 (W_B \otimes W_B) SU_t \otimes^2 (X_A \otimes Y_A) \right] \right|$$

$$\leq \frac{1}{d} \|X_A \otimes X_A - Y_A \otimes Y_A\|_1,$$

where in the last step we used the inequality $\|\text{Tr} (AB)\| \leq \|A\|_1 \|B\|_\infty$ and the fact that $\|U_t \otimes^2 (W_B \otimes W_B) SU_t \otimes^2 \|_\infty = 1$ since the operator within the norm is unitary.

In order to express the last norm as a function of the difference $X_A - Y_A$, we first add and subtract $Y_A \otimes X_A$ and then use the triangle inequality. This results in

$$\frac{1}{d} \|X_A \otimes X_A - Y_A \otimes Y_A\|_1 \leq \frac{1}{d} \left( \|X_A \otimes X_A - Y_A \otimes Y_A\|_1 + \|Y_A \otimes (X_A - Y_A)\|_1 \right)$$

$$\leq \frac{1}{d} \left( \|X_A \otimes Y_A\|_\infty \|I \otimes X_A\|_1 + \|X_A - Y_A\|_\infty \|Y_A \otimes I\|_1 \right)$$

where for the last step we utilized the inequality $\|AB\|_1 \leq \|A\|_1 \|B\|_\infty$. Now notice that $\|I \otimes X_A\|_1 = d$ since $X_A$ is unitary, and similarly for $\|Y_A \otimes I\|_1$. Therefore we can bound

$$|f_W(X) - f_W(Y)| \leq \|X_A - Y_A\|_\infty + \|X_A - Y_A\|_\infty \leq 2 \|X_A - Y_A\|_\infty = 2 \|X - Y\|_\infty \leq 2 \|X - Y\|_2,$$

from which clearly one can take $K_f = 2$.

(ii) First notice that the Haar average over $V_A = V \otimes I_B$ can be performed, as was done in the proof of Proposition 1. The result is

$$g(W) = 1 - \frac{1}{d} \text{Re Tr} \left[ SU_t \otimes^2 S_A^A \frac{S_A}{d_A} U_t \otimes^2 W_B \otimes W_B \right]$$

$$= 1 - \frac{1}{d} \text{Re Tr} \left[ U_t \otimes^2 S_{BB}^B \frac{S_B}{d_A} U_t \otimes^2 W_B \otimes W_B \right].$$

Considering the relevant difference, we can bound

$$|g(X) - g(Y)| \leq \frac{1}{d_A} \left| \text{Tr} \left[ U_t \otimes^2 S_{BB}^B U_t \otimes^2 (X_B \otimes X_B - Y_B \otimes Y_B) \right] \right|$$

$$\leq \frac{1}{d_A} \left( \|X_B \otimes X_B - Y_B \otimes Y_B\|_1 \right).$$

Now one can follow the exact same steps as in part (i); the result is identical except of the extra factor $1/d_A$ that carries through, which originates from the averaging. This results in

$$|g(X) - g(Y)| \leq \frac{2}{d_A} \|X - Y\|_2$$

from which one can take $K_g = 2/d_A$.

Everything is now in place to give the proof of Proposition 3.
Proof. Let $\epsilon > 0$. We want to show that, for $V \in U(d_A)$ and $W \in U(d_B)$ distributed independently according to the Haar measure, it holds

$$\Prob(\gamma \geq \epsilon) \leq \exp\left(-\frac{\epsilon^2 d_{\max}}{64}\right)$$

where $\gamma := |C_{V_A, W_B} - G|$ and by definition $G = \langle C_{V_A, W_B} \rangle_{V,W}$.

Let us consider any pair $V_A, W_B$ that satisfies $\epsilon \leq \gamma$. Then, from the triangle inequality also

$$\epsilon \leq \alpha + \beta,$$

where we set $\alpha := |C_{V_A, W_B} - \langle C_{V_A, W_B} \rangle_V|$ and $\beta := |\langle C_{V_A, W_B} \rangle_V - G|$. Hence we have for the corresponding probabilities

$$\Prob\{\gamma \geq \epsilon\} \leq \Prob\{\alpha + \beta \geq \epsilon\}.$$

However, if $\alpha + \beta \geq \epsilon$ then necessarily $\alpha \geq \epsilon/2$ or $\beta \geq \epsilon/2$, therefore we also have

$$\Prob\{\alpha + \beta \geq \epsilon\} \leq \Prob\{\{\alpha \geq \epsilon/2\} \cup \{\beta \geq \epsilon/2\}\}.$$

Using the standard union bound over the last expression results in

$$\Prob\{\gamma \geq \epsilon\} \leq \Prob\{\alpha \geq \epsilon/2\} + \Prob\{\beta \geq \epsilon/2\}. \quad (A9)$$

The two Probabilities in Eq. (A9) can be bounded using Levy’s lemma. For that, let us first define the auxiliary functions $f_W(V)$ and $g(W)$ as in Lemma 1. Combining the Lipschitz continuity result from there with Levy’s lemma, one gets measure concentration bounds

$$\Prob_V\{|C_{V_A, W_B} - \langle C_{V_A, W_B} \rangle_V| \geq \epsilon/2\} \leq \exp\left(-\frac{d_A c^2}{64}\right) \quad \forall W \quad (A10a)$$

$$\Prob\{|\langle C_{V_A, W_B} \rangle_V - G| \geq \epsilon/2\} \leq \exp\left(-\frac{d_B c^2}{64}\right) \quad (A10b)$$

We are almost done; it suffices to notice that the bound (A10a) is uniform in $W$, hence it is also applicable to $\Prob\{\alpha \geq \epsilon/2\}$. Therefore we arrive at

$$\Prob\{|C_{V_A, W_B}(t) - G(t)| \geq \epsilon\} \leq \exp\left(-\frac{d_A c^2}{64}\right) + \exp\left(-\frac{d_B c^2}{64}\right) \leq 2\exp\left(-\frac{d_A c^2}{64}\right). \quad (A11)$$

Notice the resulting bound is independent of the dynamics, as long as the latter is unitary. Finally, one can obtain the analogous bound for $A \leftrightarrow B$ by inverting the roles of $V$ and $W$ in the proof. Therefore we obtain Eq. (6).  

**Proposition 4**

**Proposition 4.** Consider a Hamiltonian satisfying the NRC. Then

$$\overline{G^\text{NRC}(t)} = 1 - \frac{1}{d^2} \sum_{\chi \in \{A, B\}} \left(\|R^{(\chi)}\|^2 - \frac{1}{2} \|R_D^{(\chi)}\|^2\right) \quad (7)$$

where $R^{(\chi)}$ is the Gram matrix of the reduced Hamiltonian eigenstates $\{|\rho^{(\chi)}\rangle\}_{k=1}^d$, i.e.,

$$R^{(\chi)}_{kl} := \langle \rho^{(\chi)}_k, \rho^{(\chi)}_l \rangle \quad (8)$$

while $(R_D^{(\chi)})_{kl} := R_D^{(\chi)} \delta_{kl}$.

Here we give a straightforward proof assuming the NRC holds exactly. For a more detailed discussion, see also the section of the proof of Proposition 6.
Proof. Our starting point is Eq. (4), which we need to time-average. Since the Hamiltonian is by assumption nondegenerate, we can spectrally decompose \( H = \sum_{k=1}^{d} E_k P_k \), where \( P_k := |\phi_k\rangle \langle \phi_k| \). We then have

\[
G(t) = 1 - \frac{1}{d^2} \sum_{klmn} \exp \left[ i(E_k + E_l - E_m - E_n)t \right] \text{Tr} \left[ S_{AA'}(P_k \otimes P_l) S_{AA'}(P_m \otimes P_n) \right].
\]

Time-averaging the exponential results in

\[
\exp \left[ i(E_k + E_l - E_m - E_n)t \right] \Rightarrow \delta_{E_k+E_l-E_m-E_n,0} \text{NRC} \quad \delta_{k,m}\delta_{l,n} + \delta_{k,n}\delta_{l,m} - \delta_{k,l}\delta_{m,n}
\]

where in the last step we used the fact that energy gaps are nondegenerate. Thus

\[
G(t) = 1 - \frac{1}{d^2} \left( \sum_{kl} \text{Tr} \left[ S_{AA'}(P_k \otimes P_l) S_{AA'}(P_k \otimes P_l) \right] + \sum_{kl} \text{Tr} \left[ S_{AA'}(P_k \otimes P_l) S_{AA'}(P_l \otimes P_k) \right] - \sum_{k} \text{Tr} \left[ S_{AA'}(P_k \otimes P_k) S_{AA'}(P_k \otimes P_k) \right] \right)
\]

\[
= 1 - \frac{1}{d^2} \left( \sum_{kl} \left| \text{Tr} \left[ (P_k \otimes P_l) S_{AA'} \right] \right|^2 + \sum_{kl} \left| \text{Tr} \left[ (P_k \otimes P_l) S_{BB'} \right] \right|^2 - \sum_{k} \left| \text{Tr} \left[ (P_k \otimes P_k) S_{AA'} \right] \right|^2 \right),
\]

where for the second term we used that \( P_l \otimes P_k = S(P_k \otimes P_l)S \) and \( S = S_{AA'} S_{BB'} \).

Now, notice that partial traces can be formally performed, giving

\[
\text{Tr}_{AA'BB'} [(P_k \otimes P_l) S_{AA'}] = \text{Tr}_{AA'} \left[ \text{Tr}_{BB'} (P_k \otimes P_l) S_{AA'} \right] = \text{Tr}_{AA'} \left[ (\rho^{(A)}_k \otimes \rho^{(A)}_l) S_{AA'} \right] = \text{Tr} \left( \rho^{(A)}_k \rho^{(A)}_l \right) = R_{kl}^{(A)},
\]

and similarly

\[
\text{Tr}_{AA'BB'} [(P_k \otimes P_l) S_{BB'}] = R_{kl}^{(B)},
\]

\[
\text{Tr}_{AA'BB'} [(P_k \otimes P_k) S_{AA'}] = \text{Tr}_{AA'BB'} [(P_k \otimes P_k) S_{BB'}] = R_{kk}^{(A)} = R_{kk}^{(B)}
\]

where in the last line we used the fact that the spectra of \( \rho^{(A)}_k \) and \( \rho^{(B)}_k \) are equal, up to (irrelevant for the trace) zeroes. The result follows by expressing the matrix 2-norm as \( \|X\|_2^2 = \sum_{ij} |X_{ij}|^2 \).

\[\Box\]

**Proposition 5.** If the entanglement of the Hamiltonian eigenstates deviates up to \( \epsilon \) from \( E_{\text{max}} \) with respect to the \( A-B \) cut, i.e., \( E_{\text{max}} - E(|\phi_k\rangle) \leq \epsilon \) for all \( k \), then

\[
\left| G_{\text{ME}}(t) - G(t) \right|_{\text{NRC}} \leq \frac{6\epsilon}{d_{\text{min}}} + \frac{5\epsilon^2}{2d_{\text{max}}} + \frac{2\lambda^2}{d_{\text{max}}}
\]

where \( \lambda = d_{\text{max}}/d_{\text{min}} \).

Proof. To simplify the notation, we assume \( d_A \leq d_B \). First of all, notice that one can express the difference \( E_{\text{max}} - E(|\psi_{AB}\rangle) \) as the distance

\[
E_{\text{max}} - E(|\psi_{AB}\rangle) = \text{Tr} (\rho_B^2) - 1/d_B = \|\rho_B - I/d_B\|_2^2 \geq \|\rho_A - I/d_A\|_2^2 = \text{Tr} (\rho_A^2) - 1/d_A.
\]

Setting for brevity \( \Delta^{(x)}_k := \rho^{(x)}_k - I/d_x \), we have by assumption \( E_{\text{max}} - E(|\phi_k\rangle) = \|\Delta^{(B)}_k\|_2 \leq \epsilon \) and hence also \( \|\Delta^{(A)}_k\|_2 \leq \epsilon \) for all \( k \). Moreover, we will shortly need

\[
\|\rho_k^{(x)} - I/d_A\|_2 \leq \epsilon \quad \forall k
\]

\[\Box\]
Let’s start from Eq. (7). Using the fact that \( \| R_D^{(A)} \|^2 = \| R_D^{(B)} \|^2 \) and recalling \( G_{\text{ME}}(t)_{\text{NRC}} = (1 - 1/d)^2 \) we get by the triangle inequality

\[
| G_{\text{ME}}(t)_{\text{NRC}} - G(t)_{\text{NRC}} | \leq \left| \frac{1}{d^2} \| R^{(A)} \|^2 - \frac{1}{d} \right| + \left| \frac{1}{d^2} \| R^{(B)} \|^2 - \frac{1}{d} \right| + \frac{1}{d^2} \| R^{(A)} \|^2 - 1.
\]

We can bound the first term as

\[
\left| \frac{1}{d^2} \| R^{(A)} \|^2 - \frac{1}{d} \right| = \frac{1}{d^2} \left| \sum_{k,l} \{ \langle \rho_k^{(A)}, \rho_l^{(A)} \rangle \}^2 - \frac{1}{d} \right| \leq \frac{1}{d^2} - \frac{1}{d} + \frac{2}{d} \epsilon + \epsilon^2
\]

where we used Eq. (A12) and the Cauchy-Schwartz inequality

\[
| \langle \Delta_k^{(x)}, \Delta_l^{(x)} \rangle | \leq \| \Delta_k^{(x)} \|_2 \| \Delta_l^{(x)} \|_2 \leq \epsilon.
\]

Analogously for the second term,

\[
\left| \frac{1}{d^2} \| R^{(B)} \|^2 - \frac{1}{d} \right| \leq \frac{1}{d^2} - \frac{1}{d^2} + \frac{2}{d} \epsilon + \epsilon^2
\]

For the third one, we have

\[
\| R_D^{(A)} \|^2 = \sum_k \{ \langle \rho_k^{(A)}, \rho_l^{(A)} \rangle \}^2 = \frac{d_B}{d_A} + \frac{2}{d_A} \sum_k \langle \Delta_k^{(A)}, \Delta_k^{(A)} \rangle + \sum_k \langle \Delta_k^{(A)}, \Delta_k^{(A)} \rangle^2 \leq d_B \left( \frac{1}{d_A} + 2 \epsilon + d_A \epsilon^2 \right)
\]

Thus, under the convention \( d_A \leq d_B \),

\[
\| R_D^{(A)} \|^2 - 1 \leq \frac{d_B}{d_A} - 1 + d_B \epsilon (2 + d_A \epsilon).
\]

Putting the inequalities together, we have

\[
| G_{\text{ME}}(t)_{\text{NRC}} - G(t)_{\text{NRC}} | \leq 2 \epsilon \left( \frac{1}{d^2} + \frac{1}{d^2} + \frac{1}{d^2} d_B \right) + \epsilon^2 \left( 2 + \frac{1}{d} \right) + \frac{\lambda - 1}{d^2} + \frac{\lambda^2 - 1}{d^2} \]

(A13)

which can be relaxed to give Eq. (9) by use of \( \frac{\lambda^2 - 1}{d^2} \geq \frac{\lambda - 1}{d^2} \).

**Proposition 6**

*Proposition 6.* For any given Hamiltonian, the corresponding estimates are related with the exact time-average \( \overline{G}(t) \) as

\[
\overline{G}^{\text{Haar}} \geq G(t)_{\text{NRC}} \geq G(t)_{\text{NRC}^+} \geq \overline{G}(t).
\]

Before giving the proof of the Proposition, we first briefly discuss some general facts regarding infinite time-averages, their connection with the NRC and the NRC\(^+\), and how they give rise to the corresponding estimates.

Let us consider unitary quantum dynamics \( U_t(\cdot) = U_t(\cdot)U_t^\dagger \) generated by a Hamiltonian \( H = \sum_k \hat{E}_k \Pi_k \), where \( \Pi_k \) denotes the projector onto the \( k \)-th eigenspace. As a warm-up, let us calculate the time-average of the superoperator \( U_t \). The latter can be easily performed by noticing that \( \exp \left[ -i(\hat{E}_k - \hat{E})t \right] = \delta_{kt} \). It results to

\[
\mathcal{P}_H := \overline{U}_t = \sum_k \Pi_k(\cdot) \Pi_k
\]

(A14)

which is the (Hilbert-Schmidt orthogonal) projector onto the commutant of the algebra generated by \( \{ \Pi_k \} \), i.e., the projector whose range is the space of operators commuting with \( H \).

The object of interest for us is, in fact, \( \overline{U}_t^{\text{NRC}} \) since

\[
\overline{G}(t) = 1 - \frac{1}{d^2} \langle S_{AA'}, \overline{U}_t^{\text{NRC}}(S_{AA'}) \rangle.
\]

(A15)
Reasoning as above, it follows that the resulting superoperator is again a projector, whose range is the space of operators over the replicated Hilbert space $\mathcal{H}^{\otimes 2}$ that commute with $H^{(2)} := H \otimes I + I \otimes H$. The projector can be explicitly expressed as

\begin{equation}
\mathcal{P}_{H^{(2)}} := \overline{\mathcal{U}}^{\otimes 2} = \sum_{kln} \delta_{E_k - E_m, E_l - E_n} \Pi_k \otimes \Pi_{l(\cdot)} \Pi_{m(\cdot)} \otimes \Pi_n
\end{equation}

To evaluate the above sum, let us for a moment examine what happens when the energy gaps $\{\tilde{E}_k - \tilde{E}_l\}_{kl}$ are nondegenerate. i.e.,

\begin{equation}
\text{NRC}^+ : \quad \tilde{E}_k + \tilde{E}_l = \tilde{E}_m + \tilde{E}_n \iff (k = m \land l = n) \lor (k = n \land l = m).
\end{equation}

We will refer to this condition over the spectrum as NRC$^+$, since it constitutes a relaxed version of the NRC. Without any assumption over the spectrum, one can always separate two contributions

\begin{equation}
\mathcal{P}_{H^{(2)}} = \mathcal{P}_{\text{NRC}^+} + \mathcal{P}_{\text{NRC}^-}
\end{equation}

where

\begin{equation}
\mathcal{P}_{\text{NRC}^+} := \sum_{kl} \Pi_k \otimes \Pi_{l(\cdot)} \Pi_{k(\cdot)} \otimes \Pi_l + \sum_{kl} \Pi_k \otimes \Pi_{l(\cdot)} \Pi_{k(\cdot)} \otimes \Pi_l - \sum_k \Pi_k \otimes \Pi_{k(\cdot)} \Pi_{k(\cdot)} \otimes \Pi_k
\end{equation}

and $\mathcal{P}_{\text{NRC}^-}$ is any possibly remaining piece, which vanishes if and only if the Hamiltonian does indeed satisfy NRC$^+$. Disregarding $\mathcal{P}_{\text{NRC}^-}$, one gets the estimate

\begin{equation}
\overline{G(t)}_{\text{NRC}^+} := 1 - \frac{1}{d^2} \text{Tr} [S_{AA'} \mathcal{P}_{\text{NRC}^+} (S_{AA'})]
\end{equation}

\begin{equation}
= 1 - \frac{1}{d^2} \left( \sum_{kl} \text{Tr} [S_{AA'} (\Pi_k \otimes \Pi_l) S_{AA'} (\Pi_k \otimes \Pi_l)] + \sum_k \text{Tr} [S_{AA'} (\Pi_k \otimes \Pi_l) S_{AA'} (\Pi_l \otimes \Pi_k)]
\right.
\end{equation}

\begin{equation}
\left. - \sum_k \text{Tr} [S_{AA'} (\Pi_k \otimes \Pi_k) S_{AA'} (\Pi_k \otimes \Pi_k)] \right).
\end{equation}

where the second equation follows from the proof of Proposition 4. Clearly, if all projectors $\{\Pi_k\}$ are rank-1, then Eq. (A21) collapses to the corresponding one for NRC, Eq. (7). Notice that one can evaluate $\overline{G(t)}_{\text{NRC}^+}$ regardless of whether the Hamiltonian spectrum actually satisfies NRC$^+$, and obtain the NRC$^+$ estimate mentioned in the main text.

Evidently, one can also express the NRC time-average, Eq. (7), in terms of the corresponding projector

\begin{equation}
\overline{G(t)}_{\text{NRC}} = 1 - \frac{1}{d^2} \text{Tr} [S_{AA'} \mathcal{P}_{\text{NRC}} (S_{AA'})].
\end{equation}

If the Hamiltonian does not satisfy NRC, performing a (possibly nonunique) decomposition $H = \sum_k E_k |\phi_k\rangle \langle \phi_k|$ and evaluating Eq. (7) gives rise to the corresponding NRC estimate.

Finally, for the case of Haar random unitaries, one has the corresponding projector $\overline{\mathcal{U}}^{\otimes 2}_{\text{Haar}} := \mathcal{P}_{\text{Haar}}$ whose range is given by the algebra generated by $\{I, S\}$ [74]. We evaluate its explicit expression in the next section.

We are now ready to give the proof of Proposition 6.

\textit{Proof.} The key observation here is that, by construction, the range of each projector satisfies

\begin{equation}
\text{Ran} (\mathcal{P}_{H^{(2)}}) \supseteq \text{Ran} (\mathcal{P}_{\text{NRC}^+}) \supseteq \text{Ran} (\mathcal{P}_{\text{NRC}}) \supseteq \text{Ran} (\mathcal{P}_{\text{Haar}}).
\end{equation}

Since all of the above are Hilbert-Schmidt orthogonal projectors, it also follows that

\begin{equation}
\mathcal{P}_{H^{(2)}} \supseteq \mathcal{P}_{\text{NRC}^+} \supseteq \mathcal{P}_{\text{NRC}} \supseteq \mathcal{P}_{\text{Haar}}.
\end{equation}

As a result,

\begin{equation}
\langle S_{AA'}, \mathcal{P}_{H^{(2)}} (S_{AA'}) \rangle \geq \langle S_{AA'}, \mathcal{P}_{\text{NRC}^+} (S_{AA'}) \rangle \geq \langle S_{AA'}, \mathcal{P}_{\text{NRC}} (S_{AA'}) \rangle \geq \langle S_{AA'}, \mathcal{P}_{\text{Haar}} (S_{AA'}) \rangle,
\end{equation}

from which Eq. (11) follows immediately. 

\hfill \blacksquare
Proof of Eq. (10)

The Haar average

\[ G_{\text{Haar}} = \frac{(d_A^2 - 1)(d_B^2 - 1)}{d^2 - 1} \]

can be derived using fact that \( U^2 \) is the CPTP orthogonal projector over the algebra generated by \( \{ I, S \} \) \cite{74}, i.e.,

\[ P_{\text{Haar}}(X) := U^2(X) = \frac{1}{2} \sum_{\alpha = \pm 1} (I + \alpha S)(I + \alpha S, X), \quad (A26) \]

where \( S \) swaps \( H \) and its duplicate \( H' \), as usual. Plugging the above into Eq. (4), one gets

\[ G_{\text{Haar}} = 1 - \frac{1}{2d^2} \sum_{\alpha = \pm 1} |\langle I + \alpha S, S_{AA'} \rangle|^2 \]

which, after some simple algebra, simplifies to the announced result.

Proposition 8

Proposition 8.

\[ G(t) = \frac{d_\chi + 1}{d_\chi} \int dU \ S_{\text{lin}} \left[ \Lambda_t^{(\chi)}(\langle \psi_U \rangle \langle \psi_U |) \right] \quad (13) \]

where \( \chi = A, B \) and \( |\psi_U \rangle := U |\psi_0 \rangle \) corresponds to Haar random pure states over \( \mathcal{H}_\chi \).

Proof. Let us do the \( \chi = A \) case. The result relies on the observation that one can express \( S_{AA'} \) in Eq. (12) through the Haar average \cite{74}

\[ \int dU (\langle \psi_U \rangle \langle \psi_U |)^{\otimes 2} = \frac{1}{d_A(d_A + 1)} (I_{AA'} + S_{AA'}). \quad (A27) \]

Performing the substitution results in

\[ G(t) = 1 + \frac{1}{d_A^2} \text{Tr} (S_{AA'}) - \frac{d_A + 1}{d_A} \int dU \text{Tr} \left( S_{AA'} \left[ \Lambda_t^{(A)}(\langle \psi_U \rangle \langle \psi_U |) \right]^{\otimes 2} \right) \]

\[ = \frac{d_A + 1}{d_A} \left( 1 - \int dU \text{Tr} \left[ \Lambda_t^{(A)}(\langle \psi_U \rangle \langle \psi_U |)^2 \right] \right) \]

\[ = \frac{d_A + 1}{d_A} \int dU \ S_{\text{lin}} \left[ \Lambda_t^{(A)}(\langle \psi_U \rangle \langle \psi_U |) \right] \]

where we used the fact that \( \Lambda_t^{(A)}(I) = I \) and the identity of Eq. (A1).

The \( \chi = B \) case follows similarly.

\[ \blacksquare \]

Proof of Eq. (14)

We need to prove that

\[ \text{Prob} \left\{ \left| S_{\text{lin}} \left[ \Lambda_t^{(\chi)}(\langle \psi \rangle \langle \psi |) \right] - \frac{d_\chi}{d_\chi + 1} G(t) \right| \geq \epsilon \right\} \leq \exp \left( -\frac{d_\chi^2 \epsilon^2}{64} \right) \quad (A28) \]

where \( |\psi \rangle \) is a Haar random pure state. We will make use of the concentration of measure machinery, briefly presented before the proof of Proposition 3.
The result follows by the use of Levy’s lemma and Proposition 8, if one shows that the function $f : U(d_x) \to \mathbb{R}$ with $f(V) := S_{\text{lin}}[\Lambda_t^{(x)}(\psi_V \psi_V)]$ is Lipschitz continuous with $K = 4$. As before, we denote $\psi_V := V \psi_0$ for some (irrelevant) reference state $|\psi_0\rangle$.

Indeed, let us show the Lipschitz continuity. We have

$$\left| f(V) - f(W) \right| = \left| \left| \Lambda_t^{(x)}(\psi_V \psi_V) \right\rangle - \left| \Lambda_t^{(x)}(\psi_W \psi_W) \right\rangle \right|_2^2 = \left( \left| \Lambda_t^{(x)}(\psi_V \psi_V) \right\rangle \right|_2^2 - \left| \Lambda_t^{(x)}(\psi_W \psi_W) \right\rangle \right|_2^2 \leq 2 \left\| \Lambda_t^{(x)}(\psi_V \psi_V) - \Lambda_t^{(x)}(\psi_W \psi_W) \right\|_1 \leq 2 \left\| \mathcal{U}_t \left( \left| \psi_V \right\rangle \left\langle \psi_V \right| \otimes I_x \right) - \mathcal{U}_t \left( \left| \psi_W \right\rangle \left\langle \psi_W \right| \otimes I_x \right) \right\|_1$$

where in the second to last line we used the monotonicity of the 1-norm under the partial trace and in the last line that it is unitarily invariant. Utilizing the inequality $\left\| X \right\|_1 \leq \sqrt{\text{Rank}(X)} \left\| X \right\|_2$, we have

$$\left| f(V) - f(W) \right| \leq 2 \sqrt{2} \left\| \left| \psi_V \right\rangle \left\langle \psi_V \right| - \left| \psi_W \right\rangle \left\langle \psi_W \right| \right\|_2 = 4 \sqrt{1 - \left| \left\langle \psi_V | \psi_W \right\rangle \right|^2} \leq 4 \sqrt{2(1 - \left| \left\langle \psi_V | \psi_W \right\rangle \right|)} \leq 4 \left\| \psi_V \right\| - \left\| \psi_W \right\| \leq 4 \left\| V - W \right\| \leq 4 \left\| V - W \right\|_2$$

hence one can take $K = 4$.

**Proposition 9**

**Proposition 9.** The bipartite OTOC is a measure of the distance between the reduced time-evolution and the depolarizing map:

$$G(t) = G_{\text{max}}^{(x)} - \left\| \rho^{(x)} - \rho_T^{(x)} \right\|_2^2.$$

**Proof.** Let us first express the Choi states explicitly as

$$\rho^{(x)}_t = (\Lambda^{(x)}_t \otimes I) |\phi^{+}\rangle \langle \phi^{+}| = \frac{1}{d_x} \sum_{i,j} \Lambda^{(x)}_t (|i\rangle \langle j|) \otimes |i\rangle \langle j|$$

$$\rho_T^{(x)} = (T^{(x)} \otimes I) |\phi^{+}\rangle \langle \phi^{+}| = \left( I_x \frac{d_x}{d_x^2} \right)^{\otimes 2}.$$

Writing $S_{XX'} = \sum_{i,j=1}^{d_x} |i\rangle \langle j| \otimes |i\rangle \langle j|$ one also has from Eq. (12)

$$G(t) = 1 - \frac{1}{d_x^2} \sum_{i,j} \left| \Lambda_t^{(x)} (|i\rangle \langle j|) \right|_2^2.$$

Thus, expanding the Choi state distance,

$$\left\| \rho^{(x)}_t - \rho_T^{(x)} \right\|_2^2 = \left( \rho^{(x)}_t - \rho_T^{(x)} \right) \left( \rho_T^{(x)} - \rho^{(x)}_t \right) = \left( \rho^{(x)}_t - \rho_T^{(x)} \right) - 2 \left( \rho_T^{(x)} \right)^2 + \left( \rho_T^{(x)} \right)^2 = \left\| \rho^{(x)}_t \right\|_2^2 - \frac{1}{d_x^2} \sum_{i,j} \left| \Lambda^{(x)}_t (|i\rangle \langle j|) \right|_2^2 - \frac{1}{d_x^2}$$

$$= 1 - G(t) + \frac{1}{d_x^2}$$

which is what we wanted. \(\blacksquare\)
Proof of \( \|\Lambda_t^{(x)} - T^{(x)}\|_\Theta \leq d^{3/2}_\chi \sqrt{G_{\max}^{(x)} - G(t)} \) and an application on information spreading

We will prove that

\[
\sqrt{G_{\max}^{(x)} - G(t)} \leq \|\Lambda_t^{(x)} - T^{(x)}\|_\Theta \leq d^{3/2}_\chi \sqrt{G_{\max}^{(x)} - G(t)} .
\]

Proof. The result follows easily by utilizing the inequalities

\[
\|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_1 \leq \|\mathcal{E}_1 - \mathcal{E}_2\|_1 \leq d \|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_1
\]

that hold for any pair of CPTP maps. The inequality was reported by John Watrous in [75]. The result follows by use of the inequality \( \|X\|_1 \leq \sqrt{d} \|X\|_2 \) and Proposition 9.

As an additional application of Eq. (A29), we can utilize it to bound from above the fraction of time such that \( \|\Lambda_t^{(x)} - T^{(x)}\|_\Theta \geq \epsilon \) holds true. This can be done by combining Eq. (A29) with our earlier time-averages. The result

\[
\text{Prob} \{ t \mid \|\Lambda_t^{(x)} - T^{(x)}\|_\Theta \geq \epsilon \} \leq \frac{2d^{3/2}_\chi}{d\epsilon} \kappa ,
\]

where \( \kappa := \sqrt{1 + \frac{d^2_\chi}{2} (G_{\Haar} - G(t))} \), demonstrates in yet another way that if \( d_T \gg d_\chi \) and \( \kappa = O(1) \) (i.e., the equilibration is sufficiently close to the Haar estimate), then the reduced evolution is necessarily close to the maximally mixing one for a large fraction of time.

Proof. Our starting point will be inequality (A29), \( \|\Lambda_t^{(x)} - T^{(x)}\|_\Theta \leq d^{3/2}_\chi \sqrt{G_{\max}^{(x)} - G(t)} \). By taking the time-average of both sides, and then using the concavity of the square root, we obtain

\[
\|\Lambda_t^{(x)} - T^{(x)}\|_\Theta \leq d^{3/2}_\chi \sqrt{G_{\max}^{(x)} - G(t)} \leq d^{3/2}_\chi \sqrt{(G_{\max}^{(x)} - G_{\Haar}) + (G_{\Haar} - G(t))} \leq 2 \frac{d^{3/2}_\chi}{d_T} \kappa ,
\]

where we approximated the difference

\[
G_{\max}^{(x)} - G_{\Haar} = \frac{(d^2_\chi - 1)^2}{d^2_\chi (d^2 - 1)} \leq 2 \frac{d^4_\chi}{d_T^2} .
\]

Finally, Eq. (A30) follows by the use of Markov’s inequality.

**Appendix B: Haar measure, unitary k-designs and the bipartite OTOC**

Here we discuss in more details how the Haar measure in the definition of the bipartite OTOC, Eq. (3), can be replaced by other possible averaging choices, in a way that Eq. (4) (and everything that stems from it) remains valid.

Let us first recall the definition of a (unitary) \( k \)-design [23, 41–44]. Consider an ensemble of unitary operators \( \Lambda = \{(p_i, U_i)\}_i \) and define the family of CPTP maps

\[
\mathcal{E}_\Lambda^{(k)} := \sum_i p_i U_i^{\otimes k}(\cdot) U_i^{\dagger\otimes k} \quad \text{(B1)}
\]

\[
\mathcal{E}_{\Haar}^{(k)} := \int dU U^{\otimes k}(\cdot) U^{\dagger\otimes k} \quad \text{(B2)}
\]

for \( k \in \mathbb{N} \). The ensemble \( \Lambda \) forms a \( k \)-design if \( \mathcal{E}_\Lambda^{(k)} = \mathcal{E}_{\Haar}^{(k)} \). In words, a \( k \)-design emulates Haar averaging up to (at least) the \( k^{th} \) moment.

Now, let us investigate what is the freedom over the possible probability measures of \( V_\Lambda \) and \( W_B \) in Eq. (3), such that Eq. (4) holds true without modification. It is easy to see, by the proof of Proposition 1, that we are in fact looking for a unitary ensemble \( \Lambda \) retaining the validity of Eq. (A2). In turn, the latter is just a vectorized form of the 1-design condition \( \mathcal{E}_\Lambda^{(1)} = \mathcal{E}_{\Haar}^{(1)} \). One can therefore substitute the Haar measure over \( U(d_\Lambda) \) and \( U(d_B) \) with 1-designs over the corresponding spaces; the full Haar randomness is not probed by the OTOC [23].
Moreover, 1-designs factorize, i.e., if $\Lambda_1 = \{(p_i^{(1)}, U_i^{(1)})\}_i$ and $\Lambda_2 = \{(p_j^{(2)}, U_j^{(2)})\}_j$ are 1-designs over $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, then $\Lambda_1 \otimes \Lambda_2 := \{(p_i^{(1)}, p_j^{(2)}, U_i^{(1)} \otimes U_j^{(2)})\}_{ij}$ is a 1-design over $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. This follows just by the 1-design condition in the form of Eq. (A2) and the fact that the swap operator over the duplicated space $\mathcal{H} \otimes \mathcal{H}'$ factorizes $S_{AB;A'B'} = S_{AA'}S_{BB'}$.

This last fact has an important implication for the physically relevant case of many-body systems. Consider the case where $\mathcal{H}_\chi = \bigotimes_i \mathcal{H}_\chi^{(i)}$ for $\chi = A, B$, i.e., when $A$ and $B$ are made up of (not necessarily identical) individual subsystems. Then the OTOC of Eq. (3) remains unchanged if the averages $\int dV_A$ and $\int dW_B$ are replaced by the unitary ensemble $\bigotimes_i \Lambda_\chi^{(i)}$, where each $\Lambda_\chi^{(i)}$ is a 1-design on $\mathcal{H}_\chi^{(i)}$. In other words, it is always enough to average over unitary operators that factorize completely. For instance, in the case of a spin-1/2 many-body system $\mathcal{H}_\chi^{(i)} \cong \mathbb{C}^2$ such an example is given by the Pauli 1-design $\Lambda_{\chi;\text{Pauli}}^{(i)} := \{1/4, \sigma_k\}_{k=0}^{3}$ [76].

**Appendix C: Estimating the bipartite OTOC via linear entropy measurements of random pure states**

Here we present a simple protocol for the estimation of the bipartite OTOC via repeated measurements of a single expectation value.

\[
\begin{align*}
|\psi\rangle \otimes |\psi\rangle & \quad A \quad U_t \quad B \quad S_{AA'} \\
I_B/d_B & \quad |\psi\rangle \otimes |\psi\rangle & \quad A' \quad U_t \quad B' \\
\end{align*}
\]

\[\text{FIG. 2. Protocol to for the estimation of the purity } 1 - S_{\text{lin}} \left[ \Lambda_t^{(A)}(|\psi\rangle\langle\psi|) \right] \text{ according to Eq. (13). The resulting purity constitutes also an estimate of the bipartite OTOC, up to a simple proportionality factor. The final measurement of the swap operator can be realized, for instance, by measuring the expectation value of } A \text{ and } A' \text{ over any preferred product basis } \{|i\rangle \otimes |j\rangle\}_{i,j=1}^{d_A}, \text{ without the need for coherences.}\]

The main idea relies on the simple fact that, as pointed out in the main text, the linear entropy of a state can be expressed as an expectation value, $1 - S_{\text{lin}}(\rho) = \text{Tr}(S\rho^\otimes^2)$ at the expense of requiring two copies of the state $\rho$, though uncorrelated. Combining Proposition 8 with the above observation, one can realize a protocol for estimating the bipartite OTOC via measuring the expectation value of the swap operator over pairs of randomly generated states $|\psi\rangle \in \mathcal{H}_A$. We schematically draw the protocol in Figure 2.

Averaging the resulting expectation value over Haar random pure states $|\psi\rangle$ converges to the exact value of the bipartite OTOC. In light of Eq. (14), the expected number of sample for this convergence to a given accuracy drops fast as $d_A$ increases. Clearly, the corresponding protocol with the roles of $A$ and $B$ interchanged is formally equivalent.