Bi-stable objects that are pushed between states by an external field are often used as a simple model to study memory formation in disordered materials. Such systems, called hysterons, are typically treated quasistatically. Here, we generalize hysterons to explore the effect of dynamics in a simple spring system with tunable bistability and study how the system chooses a minimum. Changing the timescale of the forcing allows the system to transition between a situation where its fate is determined by following the local energy minimum to one where it is trapped in a shallow well determined by the path taken through configuration space. Oscillatory forcing can lead to transients lasting many cycles, a behavior not possible for a single quasistatic hysteron.

I. INTRODUCTION

The ability of a physical system to store information about how it was prepared — memory — is now recognized as being crucial for the behavior in a large variety of disordered materials [1]. Jammed packings of soft spheres subjected to repeated cycles of shear, cyclically crumpled sheets of paper, and interacting spins in an oscillating magnetic field all form memories of how they were trained [2–7]. Memory in such systems hinges on the ability to learn a pathway between metastable states of the energy landscape. It has been likened to the memory seen in a collection of bi-stable elements, called hysterons, which flip between states when an external field is raised above or below a critical value as shown in Fig. 1(a) [8–10]. Although an enormous simplification from the original materials, ensembles of hysterons are able to capture some features of the memory formation seen in complex systems surprisingly well [1, 10, 11].

However, such hysterons fail to capture certain features of real systems [10, 12–14]. As long as the hysterons are independent, for example, the configuration produced at the end of the first cycle is guaranteed to be the same as that found after subsequent cycles of the same amplitude (since each hysteron separately has this property). By contrast, cyclically sheared packings can take many cycles to train, and can even exhibit a multi-period response [14] in which the periodicity of the response is an integer multiple of the driving period as first demonstrated in systems with friction [16]. Recent work has shown that generalizing the simple idea of a hysteron as an independent two-state object by adding interactions can result in long training times and multi-period responses [12, 13].

Here, we generalize the behavior of hysterons in a different way: by studying the effect of dynamics. Starting from a two-spring configuration that gives rise to a symmetric double-well potential, we add features one at a time to uncover the criteria for landing in one basin or the other. When a symmetry-breaking third spring is added, the behavior is determined by a competition between the timescale of applied forcing and the timescale of inherent system dynamics that relaxes the system to lower energy. There is a crossover, depending on forcing velocity, between energy-dominated and path-dominated selection criteria. Following from this, for oscillatory driving we find a critical frequency which separates the two regimes. Finally, we characterize the effect of allowing the system to age by slowly evolving the spring stiffnesses.

II. UNBUCKLED-TO-BUCKLED TRANSITION FOR TWO SPRINGS

In two dimensions, two identical harmonic springs with rest lengths $\ell_0$ and stiffnesses $k$ can be connected by a single node to produce a bistable system as shown in Fig. 1(b,c). The central-node location is the only variable since the positions of the two outer nodes are explicitly controlled. We restrict the motion of those outer nodes to be symmetric about the $x$-axis so that the middle node moves only in one dimension along $x$. The distance of a given outer node from its position when the two outer nodes are exactly $2\ell_0$ apart is given by $\epsilon$. The energy is the sum of the spring energies:

$$E = k(\sqrt{x^2 + (\ell_0 - \epsilon)^2} - \ell_0)^2$$

which to first order in $\epsilon$ and fourth order in $x$ is:

$$E \approx k\left(\frac{1}{4\ell_0^2} x^4 - \frac{\epsilon}{\ell_0} x^2\right).$$

The energy is symmetric around $x = 0$, the center of symmetry of the energy landscape. When $\epsilon > 0$, the quartic and quadratic terms are of opposite sign and the
energy is given by a bistable (double-well) potential. This corresponds to a buckled configuration. For \( \epsilon < 0 \), the springs are stretched and there is only a single minimum.

\[
2\ell_0 x = 0
\]

In addition to the spring forces, an applied strain, \( \gamma \), the landscape changes: one well disappears at low strain and the other disappears at high strain. For large \( \epsilon \), the two wells are deep. For small \( \epsilon \), the wells become smaller and the minima are closer together; they coalesce when \( \epsilon \rightarrow 0 \). For \( \epsilon \leq 0 \), there is only one minimum.

\( E_{v_x} \approx k \left( \frac{1}{4\ell_0^4} x^4 - \frac{\epsilon}{\ell_0^2} x^2 \right) + \beta v_{o,x}^* x. \)  \hspace{1cm} (2)

We probe the transition from one to two minima by bringing the two outer nodes together in the \( y \)-direction at a fixed velocity \( v_y \). Starting from the stretched state with one minimum, \( \epsilon \) changes from negative to positive as the two nodes approach one another, causing the initial minimum to separate into two distinct minima. If the motion is only along the \( y \)-direction (co-linear motion of the two outer nodes), the system chooses a minimum randomly (if there is any noise) or remains stuck in the unstable equilibrium position \( x = 0 \).

However, if the outer nodes move in the \( x \)- as well as the \( y \)-direction, the path of the outer nodes dictates into which minimum the system will eventually come to rest. One can see already that this two-state system is different from the conventional hysteron; the \( \beta \) of applied boundary motion, not just the resulting energy landscape, determines the configuration of the system.

\section{III. ADDITION OF A WEAK THIRD SPRING}

We study the behavior of this system with overdamped dynamics. The \( x \)-velocity of the middle node is given by \( v_{m} = f/\beta \) where \( f \) is the \( x \)-component of the total force from all springs and \( \beta \) is a damping coefficient. In addition to the spring forces, an \( x \)-velocity of the outer nodes, \( v_{o,x} \), will also influence the motion of the middle node. We choose a reference frame in which the outer nodes are stationary in the \( x \)-direction. By its definition the position \( x = 0 \) then also remains stationary so that the middle node moves with respect to \( x = 0 \) with an additional velocity: \( \Delta v_m = -v_{o,x} \). This movement, due solely to this motion of the origin, is in addition to the velocity caused by the springs themselves. With overdamped dynamics, this can be treated as if there were an additional effective force in the \( x \)-direction \( f_{\text{eff}} = -\beta v_{o,x} \):

\[
E_{v_x} \approx k \left( \frac{1}{4\ell_0^4} x^4 - \frac{\epsilon}{\ell_0^2} x^2 \right) - f_{\text{eff}} x. \hspace{1cm} (3)
\]

where \( f_{\text{eff}} \) is the small force due to the weak spring. The form of this equation is identical to that of Eq. 2 so we can include any boundary-node motion in the \( x \)-direction as an effective weak spring force: \( f_{\text{eff}} = f_w - \beta v_{o,x} \).

This simple model of a hysteron has two distinct mechanisms that can compete to determine the effective force. If the effective force is dominated by the weak spring, then the behavior will be “energy dominated,” with the landscape changing slowly enough that the energetics determine which well is chosen. If, on the other hand, the velocity of the boundary nodes dominates, then the behavior will be “path dominated”: the energy landscape will change too quickly for the system to keep up with the local minimum and so it will become trapped in a state determined by the path of the boundaries.

We can make the crossover between energy-dominated and path-dominated outcomes explicit by finding the critical \( x \)-velocity of the boundary nodes that leads to zero effective force: \( f_{\text{eff}} = f_w - \beta v_{o,x} = 0 \). If we start with \( \epsilon < 0 \) at \( x = 0 \) and bring the nodes together at a
constant velocity, we find that for \( v_{a,x} < f_w/\beta \), the system is energy-dominated and ends up in the global minimum; for \( v_{a,x} > f_w/\beta \), the system is path-dominated and becomes trapped in the shallower minimum.

**IV. OSCILLATORY FORCING**

Now that we have understood the basics of these three-spring systems, we can bring them closer to the hysteron picture by modulating the weak force with time so that the bi-stable system is subjected to oscillatory forcing:

\[
f_w(t) = f_0 + A \sin(\omega t).
\]

Recall that independent quasi-static hysterons are completely trained in one cycle. However, in this realization of a hysteron with dynamics, it can take many cycles before the system reaches its steady-state behavior.

We analyze the case where there are two stable minima at time \( t = 0 \) and the middle node begins in the energetically unfavorable well. If both wells remain stable at all times, the middle node will never leave its initial well. However, if for some fraction of the cycle, the higher-energy minimum disappears, the middle node may escape to the deeper minimum.

We simulate this behavior by calculating the net force on (and hence velocity of) the middle node and updating its position accordingly. Length and time are measured in units of \( \ell_0 \) and \( \beta/k \) respectively. The node position versus time is shown in Fig. 2a. At low frequency \( \omega \), the system initially remains in the metastable minimum but after several cycles escapes to the global minimum. As \( \omega \) increases, it takes longer for the node to reach that global minimum. This time diverges at a critical frequency \( \omega_c \), above which the system never falls into the deeper well.

We find the crossover time at which the middle node falls into the lower energy well. The critical frequency, \( \omega_c \), is determined by the limit when the crossover time versus \( A \) diverges. Unless otherwise noted, we have chosen \( \beta = 0.10, \epsilon = 0.01, A = -10^{-4} \) and \( f_0 = -0.00108 \), and have set the initial middle-node position to be \( x = 0.12 \). As shown in Fig. 2a, \( \omega \propto A^6 \) with \( \alpha = 1.001 \pm 0.004 \).

To calculate the scaling of \( \omega_c \) with \( A \), we need to find the frequency at which the motion in one half of the cycle is exactly undone by the motion in the other half. Any longer of a period and the system will start to creep progressively toward the deeper well; any shorter, and the system will be stuck eternally.

We can calculate \( \omega_c \) analytically under a set of simplifying assumptions. (i) We replace the sine function in Eq. 4 with a square wave so that \( f_w \) takes on only two discrete values \( f_0 + A \) and \( f_0 - A \), shown schematically in Fig. 3. (ii) We define a force \( f_c \) to be the value at which a second minimum just forms and assume \( 0 < f_c - f_0 \ll f_c \) so that the time-averaged potential is just barely stable and \( f_c - f_0 \ll A \) so that the potential is relatively flat (i.e., variation in the force is small compared to its magnitude in the vicinity of the critical position \( x_c \) where the force from the strong springs is a maximum). (iii) We assume small oscillation amplitudes: \( A \ll f_c \). Fig. 3 provides a graphical interpretation of these statements.

With the assumption of flatness in (ii), we can approximate the displacement of the middle node as the time spent in half of the cycle \( T/2 \) times a characteristic velocity of the middle node \( v \approx \langle f \rangle/\beta \), where \( \langle f \rangle \) is the force averaged over time. The time average can be well approximated as a position average for low spatial variation in \( f \), which is the case we are interested in here. Finally, we Taylor expand the force as a quadratic around \( x_c \):

\[
f(x) \approx (f_0 \pm A) - f_c - \frac{3k}{\ell_0^2} (x - x_c)^2 x_c.
\]

We estimate the critical frequency by determining the size of a limit cycle centered at \( x_c \) and then finding the
This is consistent with the exponent $\alpha$ obtained in the simulations. In fact, we see from Fig. 2b that $\alpha \sim 1$ even when $A$ is extended to values larger than $f_0$, well outside the regime of the calculated exponent.

V. DIRECTED AGING

Existence of a critical frequency indicates that the behavior of a dynamic hysterons is determined by a competition between the timescale of external driving, set by $\omega$, and the time it takes to travel a given distance, set by $\beta$ and the net force from the springs at any given moment. Any process that alters one of these timescales can therefore change the behavior of the system.

One example of such a process is directed aging in which local properties of a spring network, such as the spring constants or bond lengths, evolve in response to the stresses imposed on each bond [17-19]. This leads to changes in the global elastic response of the system.

In the dynamic hysterons described above, each spring, $\ell_i$, undergoes a strain $\delta\ell_i$ when the system is driven. We evolve the spring constants at an aging rate $q$ according to the energy stored in each bond:

$$k_i(t + 1) = k_i(t) - q k_i(t) \delta\ell_i^3. \quad (7)$$

Fig. 4. Trajectories of the three systems shown in Fig. 2a, after the system has been aged. Now only the black curve falls into the global minimum because the critical frequency has decreased to $\omega_c \approx 0.088$.

Fig. 4 shows the trajectories for the same system at the same frequencies depicted in Fig. 2a, after the system is aged. Starting with $k = 1$ in the strong spring and $k_w = 10^{-3}$ in the weak spring, we update the stiffness values according to Eq. (7) with $q = 5 \times 10^{-5}$. Note that all three springs are aged under the strain they undergo when $\epsilon = 0.01$ and $x = 0.12$. For the weak spring, aging strain is fixed at $|f_0 + A|/10^{-3}$. After 50 iterations, we fix the spring constants at the aged values and drive the system as in Fig. 2a. Because the critical frequency is shifted to a lower value, the middle node can only fall into the global minimum for the lowest frequency shown. It is also possible to increase the critical frequency by aging the system in ways that slow down the dynamics.
VI. CONCLUSION

We have introduced a model that generalizes the notion of hysterons to a dynamical unit that exhibits a variety of distinct behaviors in response to driving. Starting with two springs connected to an overdamped node, we show that the system’s path is the only factor that determines which well it will choose. Adding a weak spring breaks the symmetry and brings in the energy landscape as another way to determine which minimum is chosen. Thus there is a competition between energy and path for choosing the minimum. For cyclic driving, the system can require multiple cycles to reach its global minimum.

The model presented is suitable for studying the effect of dynamics in cyclically driven experimental systems where a frequency-dependent behavior is observed. In addition, the sensitivity of the model to directed aging shows its versatility and opens the door to a large variety of training possibilities. One natural generalization of this model would be to include interactions between pairs of dynamical hysterons. In the case of ordinary hysterons, the presence of interactions leads to memories that include multiple cycles [12, 13]. It would be interesting to see if unexpected behavior emerges by allowing interactions between pairs of dynamical hysterons.

Hysterons with more than two wells can also be constructed in a network of springs. As an example, in Fig. 5 we demonstrate a three-well potential with one free node connected to four springs. If the wells are designated A, B, and C, by prescribing the motion of the boundary nodes one can create a reversible boundary motion that causes the middle node to transition from $A \rightarrow B \rightarrow C \rightarrow A$ over the course of a single full cycle (see SI). In this case, the topology of the node’s trajectory is nontrivial, a feature that may be important in more complex systems like jammed packings that are sheared repeatedly and where the phase-space trajectory is typically complex (topologically nontrivial) but cyclic.

We can conceive of using dynamical hysterons in designing mechanical metamaterials. As an example, consider a machine with buckling mechanisms in the form of robotic arms that can switch between two states depending on the frequency. In this case, the dynamical hysterons allow changing the structure by controlling the frequency of an external driving force.

The system presented here is simple enough to be studied analytically, yet the resulting dynamics are rich with complexity. The examples above illustrate that it is easily adaptable to a variety of systems, i.e., where interactions between hysterons play a role, or more complex scenarios, as in the case of a three-well system. This dynamical hysteron model can thus provide insight into the effect of new timescales on a wide variety of complex systems.

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![FIG. 5. A node (green circle) connected by four springs (dashed lines) to the four boundary nodes (black dots). Underlaid is the associated energy landscape for the free node. Orange dots mark the location of the three energy minima. By a suitable reversible manipulation of the boundary nodes, for example as described in the SI, this system behaves as a three-state hysterons that visits the minima sequentially over the course of one cycle: $A \rightarrow B \rightarrow C \rightarrow A$.](image-url)
VII. SUPPLEMENTAL INFORMATION

A. Chiral motion in three-well systems

Using a series of fairly simple motions of the three-well boundary nodes, we can lead the system quasistatically through the cycle $A \rightarrow B \rightarrow C \rightarrow A$. We will use two types of moves: one in which two nodes are pulled away from the center, which causes the two wells opposite those nodes to disappear, and one in which one node is brought toward the center and another is moved azimuthally, which causes one well (e.g. $C$) to remain unaffected and the others to flow (e.g. $A \rightarrow B$).

Let the top node be node $T$ and the bottom left and right nodes be nodes $L$ and $R$, respectively.

Video 1 shows the following set of steps:

1. Nodes $T$ and $R$ move outward. Starting from well $A$, this does nothing

2. Node $T$ moves toward the center and node $R$ moves azimuthally toward node $L$. This moves us from well $A$ to well $B$

3. Nodes $L$ and $R$ move outward. This moves us from well $B$ to well $C$

Now these steps are repeated in reverse order. Note that each step had individually reversible boundary conditions, so repeating them in reverse order takes the boundaries back along the same path to the starting point.

1. Nodes $L$ and $R$ move outward. We remain in well $C$

2. Node $T$ moves toward the center and node $R$ moves azimuthally toward node $L$. We remain in well $C$

3. Nodes $T$ and $R$ move outward. We move from well $C$ back to well $A$

We have therefore moved the boundaries “out” and “back” along the same path just as you might when shearing a solid out to some maximum strain and back to its original box shape. The result, unlike anything obtainable from a two-well system, is chiral in nature.