A solvable algebra for massless fermions

Stefan Groote and Rein Saar
Loodus- ja täppisteaduste valdkond, Füüsika Instituut,
Tartu Ülikool, W. Ostwaldi 1, 59411 Tartu, Estonia

Abstract

We derive the stabiliser group of the four-vector, also known as Wigner’s little group, in case of massless particle states, as the maximal solvable subgroup of the proper orthochronous Lorentz group of dimension four, known as the Borel subgroup. In the absence of mass, particle states are disentangled into left- and right-handed chiral states, governed by the maximal solvable subgroups $\text{sol}_{\pm}^2$ of order two. Induced Lorentz transformations are constructed and applied to general representations of particle states. Finally, in our conclusions it is argued how the spin-flip contribution might be closely related to the occurrence of nonphysical spin operators.

Keywords: solvable Lie group; Borel subgroup; massless particle states; chirality states

PACS: 02.20.Qs; 03.65.Ge
1 Introduction

As neutrino oscillations are observed in experiments, it seems obvious that all fermions carry a mass. Even though the mass spectrum reaches from small fractions of eV for neutrinos up to 175 GeV for the top quark, a hierarchy waiting still for an explanation, the fact that a fermion carries a mass allows to go to the rest frame of the particle and to observe both left-handed and right-handed states.

Therefore, the concept of massless fermions, moving with the speed of light, has to be considered as an approximation. This approximation holds true if some of the masses of fermions interacting in a perturbative calculation can be neglected compared to other, larger fermion masses. However, while assuming a fermion to be massless, one not only obtains an essential simplification of the calculation but also different symmetries which are not given for fermions with small but finite mass. As an example, the breakdown of these symmetries can cause spin-flip effects where the result in the mass-zero limit differs from the result for massless fermions \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\]. This effect can be understood as a discontinuity in freezing the spin of the fermion. However, to the best of our knowledge, a deeper understanding of these effects is still missing.

In this paper, we analyse the structure of Wigner’s little group for massless particles by adding a small but essential degree of freedom, given by the fact that the momentum vector of a massless particle defines a projective space. In doing so, we come to the conclusion that the stabiliser subgroup is not given by a semisimple group as for massive particles but by a solvable group. In Sec. 2 we give details on the Borel subgroup as the maximal solvable subgroup describing the stabiliser. In Sec. 3 we deal with the representation space in terms of common eigenvectors which, in a natural way, leads to the split-off of the representation space into a left- and right-handed part, described as Kronecker sum on Sec. 4 The two-dimensional subspaces are governed by the solvable groups \(\text{sol}_2^-\) and \(\text{sol}_2^+\) which are expressed in terms of the Chevalley basis in Sec. 5. Finally, in Sec. 6 we give our conclusions and present an outlook on how the Weyl equations for these massless states can be combined to a Dirac equation for fermions with mass.
1.1 Analysis of Wigner’s little group

In his paper “Sur la dynamique de l’électron” from July 1905 [12], Henri Poincaré formulates the “Principle of Relativity”, introduces the concepts of Lorentz transformation and Lorentz group, postulating the covariance of the laws of nature under Lorentz transformations. The full Lorentz group is a six-dimensional, noncompact and non-abelian real Lie group which is not connected. The four connected components of this group are related to each other via discrete transformations (parity and time reversal). None of these components is simply connected. In describing physics, one usually considers the component connected to the identity, called the proper orthochronous Lorentz group Lor(1, 3).

An important subgroup of Lor(1, 3) that preserves a given four-vector $p$ is Wigner’s little group. For $p$ describing the momentum of a massive particle, the condition $\Lambda_p p = p$ for the elements $\Lambda_p$ of the little group can be solved in the rest frame of the particle where the normalised momentum vector is given by $\hat{p} = (1; 0, 0, 0)^T$, leading to the block structure

$$\hat{\Lambda}_p = \begin{pmatrix} 1 & \bar{0}^T \\ 0 & D \end{pmatrix} = R_p,$$

where $DD^T = 1_3 = D^T D$. Therefore, the little group of a massive particle is isomorphic to SO(3). However, for a massless particle, the momentum vectors $p = (1; 0, 0, \varepsilon)$ with $\varepsilon = \pm 1$ for a movement along the z axis are projective vectors. Therefore, in solving the generalized equations $\Lambda_p = \lambda p$ and $\Lambda^T \eta p = \lambda^{-1} \eta p$ for $p = (1; 0, 0, \varepsilon)$ with a general value of $\varepsilon$ and the Minkowskian metric $\eta = \text{diag}(1; -1, -1, -1)$ via the block ansatz

$$\Lambda = \begin{pmatrix} A & \bar{B}^T \\ \bar{C} & D \end{pmatrix}$$

leads to $\varepsilon B_3 = \lambda - A$, $\varepsilon C_3 = A - \lambda^{-1}$, $\varepsilon D_{33} = \varepsilon \lambda - C_3$ and $\varepsilon D_{33} = B_3 + \varepsilon \lambda^{-1}$. The two last conditions are in agreement if and only if

$$\varepsilon^2 \lambda - A + \lambda^{-1} = \lambda - A + \varepsilon^2 \lambda^{-1} \quad \Leftrightarrow \quad (1 - \varepsilon^2)(\lambda - \lambda^{-1}) = 0.$$  

This equation marks the point where two different paths are possible to follow: for $\lambda = \lambda^{-1} = 1$ ($\lambda > 0$ for the proper orthochronous Lorentz group) one ends up again with
Wigner’s little group SO(3). For massless particles, however, one has $\varepsilon^2 = 1$ and, therefore, one can keep $\lambda > 0$ arbitrary, ending up with the Borel subgroup explained in the following.

### 1.2 Justification of the extension

The introduction of an extension of Wigner’s little group needs justification. Wigner introduces the little group as a stabiliser group with respect to the momentum vector $p$. However, because the four-length of the momentum vector for a massless particle is zero and, therefore, the multiplication of this vector with an arbitrary scale does not change the physics of this particle, the physical situation is better described by a projective space. The existence of an invariant subspace is guaranteed by the Lie–Kolchin theorem,

**Lie–Kolchin theorem**

*If $G$ is a connected and solvable linear algebraic group defined over an algebraically closed field and $\rho : G \rightarrow \text{GL}(V)$ is a representation on a nonzero finite-dimensional vector space $V$, then there exists a one-dimensional linear subspace $L$ of $V$ such that*

$$\rho(G)(L) = L.$$  \hspace{1cm} (4)

In 1956, Armand Borel generalised the Lie–Kolchin theorem as a fixed-point theorem for algebraic varieties \[13\] and, therefore, also for the projective space,

**Borel fixed-point theorem**

*If $G$ is a connected, solvable, algebraic group acting regularly on a non-empty, complete algebraic variety $V$ over an algebraically closed field, then there exists a fixed-point of $V$.*

As expressed by Eq. (3), the projectivity of the fixed-point is broken if $\varepsilon^2 < 1$, i.e. if the particle gains mass. In this case we are falling back to Wigner’s little group. The extension can be understood also on the level of Lie algebras, as for massless particles the interchange of space and time components of the momentum vector is an additional symmetry which
is absent for massive particles. Note in this context that also \( E(2) \) as the little group for massless particles proposed by Wigner is a solvable group, though not maximal.

2 The Borel subgroup \( \text{Bor}(1, 3; p) \)

From now on we use \( \varepsilon \) only as the sign of the momentum 3-component. The fact that the momentum vector \( p \sim (1; 0, 0, \varepsilon) \) for a massless particle is symmetric (up to the sign \( \varepsilon = \pm 1 \)) under the interchange of the first and the last component gives an additional element of the algebra which is missing so far in Wigner’s little group. In order to see this, one can find solutions for the character problem (summation over repeated indices is implied)

\[
\Lambda^\mu_{\nu}(p)p^\nu = \lambda(\Lambda)p^\mu. \tag{5}
\]

Solving this problem for the Lorentz matrix \( \Lambda_p = (\Lambda^\mu_{\nu}(p)) \) with \( \Lambda_p^{\top} \eta \Lambda_p = \eta \) one obtains

\[
\Lambda_p = \begin{pmatrix}
\cosh t + \frac{1}{2} e^{-t}(u^2 + v^2) & \varepsilon u & \varepsilon v & \varepsilon (\sinh t + \frac{1}{2} e^{-t}(u^2 + v^2)) \\
\varepsilon (u \cos w - \varepsilon \sin w) & \cos w & -\sin w & u \cos w - \varepsilon \sin w \\
\varepsilon (u \sin w + v \cos w) & \sin w & \cos w & u \sin w + v \cos w \\
\varepsilon (\sinh t - \frac{1}{2} e^{-t}(u^2 + v^2)) & -u & -v & \cosh t - \frac{1}{2} e^{-t}(u^2 + v^2)
\end{pmatrix}^{\top},
\tag{6}
\]

where we have chosen \( \lambda = e^t \) and introduced three additional parameters \( u, v \) and \( w \).

Expanding in these parameters one obtains \( \Lambda_p \approx \mathbb{1}_4 + T t + U u + V v + W w \), where

\[
T = \left. \frac{\partial \Lambda}{\partial t} \right|_0 = \begin{pmatrix}
0 & 0 & 0 & \varepsilon \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\varepsilon & 0 & 0 & 0
\end{pmatrix}, \quad U = \left. \frac{\partial \Lambda}{\partial u} \right|_0 = \begin{pmatrix}
0 & \varepsilon & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
V = \left. \frac{\partial \Lambda}{\partial v} \right|_0 = \begin{pmatrix}
0 & 0 & \varepsilon & 0 \\
0 & 0 & 0 & 0 \\
\varepsilon & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad W = \left. \frac{\partial \Lambda}{\partial w} \right|_0 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{7}
\]

\(^{1}\)The matrix is transposed (indicated by the upper index \( T \)) for reasons of visualisation only.
and the lower index “0” symbolises the initial value \( t = u = v = w = 0 \).  
\( T, U, V \) and \( W \) are generators of the maximal solvable Lie subgroup of \( \text{Lor}(1,3) \), i.e. the Borel subgroup \( \text{Bor}(1,3; p) \subset \text{Lor}(1,3) \). For the corresponding Lie algebra \( g = \text{span}_\mathbb{R}\{T,U,V,W\} = \text{bor}(1,3; p) \) one easily obtains

\[
[T, U] = U, \quad [T, V] = V, \quad [W, U] = -V, \quad [W, V] = U, \tag{8}
\]

with all other commutators being zero. Accordingly, one has \([g, g] = \text{span}_\mathbb{R}\{U, V\}\) and \([[[g, g], [g, g]] = 0\), so that \( g \) is solvable. Note that the element (6) of the Borel subgroup \( \text{Bor}(1,3; p) \) is given by a polar decomposition, i.e. it can be restored by calculating

\[
\Lambda_p = \exp(Uu + Vv) \exp(Tt + Ww). \tag{9}
\]

Because of \([T, W] = 0\), one has \( \exp(Tt) \exp(Ww) = \exp(Tt + Ww) = \exp(Ww) \exp(Tt) \).

The two parts of the second exponential factor commutate with each other. They constitute the maximal torus \( \text{Tor}(1,1; p) \) describing the transformations that leave the direction of the momentum vector \( p \) invariant: a boost directed along the \( z \) axis described by \( \exp(Tt) \), and a rotation about the \( z \) axis described by \( \exp(Ww) \). However, these two factors do not commute with the first exponential factor \( \Lambda_{u,v} := \exp(Uu + Vv) \) which constitutes the physically nontrivial part \( \mathcal{T}(2; p) \) of the Borel subgroup (translations),

\[
\Lambda_{u,v} = \exp \begin{pmatrix}
0 & \varepsilon u & \varepsilon v & 0 \\
\varepsilon u & 0 & 0 & -u \\
\varepsilon v & 0 & 0 & -v \\
0 & u & v & 0
\end{pmatrix} = \begin{pmatrix}
1 + \frac{1}{2}(u^2 + v^2) & \varepsilon u & \varepsilon v & -\frac{1}{2}\varepsilon(u^2 + v^2) \\
\varepsilon u & 1 & 0 & -u \\
\varepsilon v & 0 & 1 & -v \\
\frac{1}{2}\varepsilon(u^2 + v^2) & u & v & 1 - \frac{1}{2}(u^2 + v^2)
\end{pmatrix}. \tag{10}
\]

Note that due to the solvability, the series expansion breaks at the second order. Together, these two parts of the polar decomposition of \( \Lambda_p \) represent the Borel subgroup as a semidirect product,

\[
\text{Bor}(1,3; p) = \mathcal{T}(2; p) \rtimes \text{Tor}(1,1; p). \tag{11}
\]
2.1 A bridge from massive to massless

Even though the main emphasis of this paper is laid on an independent treatment of the little group of massless particles as the maximal noncompact solvable subgroup of the proper orthochronous Lorentz group, there is still a way to find a bridge connecting this part of the Lorentz group to the maximal compact simple subgroup which is quite remarkable. Starting with a massive particle, in the rest frame of this particle a proper orthochronous Lorentz transformation $\hat{\Lambda}_p = B_{r,s,t}R_{u,v,w}$ can be written as polar decomposition of the Wigner rotation matrix $R_{u,v,w}$ followed by a boost $B_{r,s,t}$, where

$$B_{r,s,t} = \exp \begin{pmatrix} 0 & r & s & t \\ r & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ t & 0 & 0 & 0 \end{pmatrix}, \quad R_{u,v,w} = \exp \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & w & -u \\ 0 & -w & 0 & -v \\ 0 & u & v & 0 \end{pmatrix}. \tag{12}$$

The transformation to the laboratory frame where the momentum vector of the particle is given by $p$ is performed with the help of the boost matrix $B_p = B_{0,0,\varepsilon \xi_p}$ parametrised by the momentum vector $p$,

$$B_p = \exp \begin{pmatrix} 0 & 0 & 0 & \varepsilon \xi_p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon \xi_p & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_p & 0 & 0 & \varepsilon s_p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \varepsilon s_p & 0 & 0 & c_p \end{pmatrix}, \tag{13}$$

where $c_p = \cosh \xi_p$ and $s_p = \sinh \xi_p$ with rapidity $\xi_p$. Accordingly, the proper orthochronous Lorentz transformation in the laboratory frame is given by

$$\Lambda_p = B_p B_{r,s,t} R_{u,v,w} B_p^{-1} = B_p B_{r,s,t} B_p^{-1} B_p R_{u,v,w} B_p^{-1}. \tag{14}$$

For the generic Lie algebra element generating the boost $B_{r,s,t}$ one obtains

$$B_p \begin{pmatrix} 0 & r & s & t \\ r & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ t & 0 & 0 & 0 \end{pmatrix} B_p^{-1} = \begin{pmatrix} 0 & c_p r & c_p s & t \\ c_p r & 0 & 0 & -\varepsilon s_p r \\ c_p s & 0 & 0 & -\varepsilon s_p s \\ t & \varepsilon s_p r & \varepsilon s_p s & 0 \end{pmatrix}. \tag{15}$$
Because $c_p, s_p \to \infty$ in the massless limit, $r$ and $s$ (but not $t$) have to be renormalized in order to obtain a finite matrix $B_p B_{r,s,t} B_p^{-1}$. This can be done by replacing $r$ by $x r$ and $s$ by $x s$ where $x c_p = x s_p \to 1$ in the massless limit $x \to 0$. Raising the generic element in Eq. (15) to the exponent, one obtains $B_{x r, x s, t} = B_{x r} B_{x s} B_t$, where the exponential factors $B_{x r}$ and $B_{x s}$ factorise and commute with each other and with the remaining factor $B_t$ due to the smallness of the renormalised parameters $x r$ and $x s$. The factor $B_t$ describes a boost along the $z$ axis. Compared to the boost $B_p$ in the same direction, the former is negligible in the massless limit. Therefore, one can replace $B_p$ with $B_p B_t^{-1} = B_t^{-1} B_p$ and obtain
\[
\Lambda_p = B_p B_{x r, x s} B^{-1}_p B_p R_{u,v,w} B^{-1}_p B_t \to \Lambda_{\varepsilon r, \varepsilon s} B_p R_{u,v,w} B^{-1}_p B_t.
\] (16)

Because of the renormalisation, $B_p B_{x r} B_{x s} B_p^{-1}$ is finite in the massless limit and gives $\Lambda_{\varepsilon r, \varepsilon s}$ which can be seen by comparing the result of the exponentiation with Eq. (10).

Looking at the second main factor in $\Lambda_p$, a similar consideration can be made for $B_p R_{u,v,w} B^{-1}_p B_t$. Starting from
\[
B_p \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & w & -u \\ 0 & -w & 0 & -v \\ 0 & u & v & 0 \end{pmatrix} B_p^{-1} \begin{pmatrix} 0 & \varepsilon s_p u & \varepsilon s_p v & 0 \\ \varepsilon s_p u & 0 & w & -c_p u \\ \varepsilon s_p v & -w & 0 & -c_p v \\ 0 & c_p u & c_p v & 0 \end{pmatrix},
\] (17)

$u$ and $v$ (but not $w$) have to be renormalized using again $x$ with $x c_p = x s_p \to 1$. Raising the generic element in Eq. (17) to the exponent, one obtains $R_{x u, x v, w} = R_{x u} R_{x v} R_w$ where again all three factors commute with each other. As $R_w$ commutes with $B_p^{-1}$ as well, this factor can be pulled out, and the remaining product $B_p R_{x u} R_{x v} B_p^{-1}$ gives $\Lambda_{u,v}$ in the massless limit. Therefore, in this limit, $\Lambda_p$ will decay into
\[
\Lambda_p \to \Lambda_{\varepsilon r, \varepsilon s} \Lambda_{u,v} R_w B_t, \quad R_w = \exp(W w), \quad B_t = \exp(T t).
\] (18)

In this product, $\Lambda_{u,v} R_w B_t$ constitutes the generic element of the Borel subgroup $\text{Bor}(1, 3; p)$ and $\Lambda_{\varepsilon r, \varepsilon s}$ constitutes the rest class $\text{Lor}(1, 3)/\text{Bor}(1, 3; p)$. To conclude, the little groups of massive and massless particles are connected by a singular transformation, induced by an infinitesimal boost, interpreted as contraction in the sense of Inoue and Wigner [14].
3 Common (pseudo)eigenvectors

The exponential representation (9) is a special case of the representation

$$\Lambda(\omega) = \exp\left(-\frac{1}{2} \omega_{\alpha\beta} e^{\alpha\beta}\right)$$

of the full Lorentz group where

$$(e^{\alpha\beta})_{\mu\nu} = \eta^{\alpha\nu} \eta^{\beta\mu} - \eta^{\alpha\mu} \eta^{\beta\nu}, \quad \eta = \text{diag}(1,-1,-1,-1).$$

Of course then, the generators $T$, $U$, $V$ and $W$ can be expressed in terms of the $e^{\alpha\beta}$,

$$T = -\varepsilon e^{03}, \quad U = e^{31} - \varepsilon e^{01}, \quad V = e^{32} - \varepsilon e^{02}, \quad W = e^{12}$$

with the non-vanishing parameters $\varepsilon \omega_{03} = \varepsilon \omega_{30} = t, \varepsilon \omega_{01} = \varepsilon \omega_{10} = \omega_{13} = -\omega_{31} = u, \varepsilon \omega_{02} = \varepsilon \omega_{20} = \omega_{23} = -\omega_{32} = v$ and $\omega_{21} = -\omega_{12} = w$. For technical reasons, instead of $\{T, U, V, W\}$ we may use the notation $\{T_0^\varepsilon, T_1^\varepsilon, T_2^\varepsilon, T_3^\varepsilon\} = \{T_i^\varepsilon\}_{i=0}^3$ in the following. The upper index $\varepsilon$ indicates the dependence on $\varepsilon$, where $T_0^\varepsilon = -T_0^\varepsilon$ and $T_3^{3-\varepsilon} = T_3^\varepsilon$. Because $T_3^\varepsilon = W$ does not depend on $\varepsilon$, one can skip the index in this case.

According to Lie’s theorem, a solvable algebra has a single common eigenvector. Solving the equations $T_i^\varepsilon \ell_0 = \lambda_0^{(i)} \ell_0$ ($i = 0, 1, 2, 3$), one obtains

$$\ell_0 = (1; 0, 0, \varepsilon)^T/\sqrt{2}, \quad \lambda_0^{(0)} = +1, \quad \lambda_1^{(0)} = 0, \quad \lambda_2^{(0)} = 0, \quad \lambda_3^{(0)} = 0.$$  \hspace{1cm} (21)

Not very surprisingly, the common eigenvector is just given by $p$. In order to specify the defective matrices $\hat{T}_i^\varepsilon$ of the solvable algebra, the equations

$$T_i^\varepsilon \ell_1 = \lambda_1^{(i)} \ell_1 + \gamma_1^{0(i)} \ell_0$$

$$T_i^\varepsilon \ell_2 = \lambda_2^{(i)} \ell_2 + \gamma_2^{0(i)} \ell_1 + \gamma_2^{0(i)} \ell_0$$

$$T_i^\varepsilon \ell_3 = \lambda_3^{(i)} \ell_3 + \gamma_3^{0(i)} \ell_2 + \gamma_3^{0(i)} \ell_1 + \gamma_3^{0(i)} \ell_0$$

are solved step-wise to obtain a system of pseudo-eigenvectors and -eigenvalues. Collecting all these equations in a single one, after some normalisation one obtains

$$T_i^\varepsilon P = P \begin{pmatrix} \lambda_i^{(0)} & \gamma_i^{0(1)} & \gamma_i^{0(2)} & \gamma_i^{0(3)} \\ 0 & \lambda_i^{(1)} & \gamma_i^{1(2)} & \gamma_i^{1(3)} \\ 0 & 0 & \lambda_i^{(2)} & \gamma_i^{2(3)} \\ 0 & 0 & 0 & \lambda_i^{(3)} \end{pmatrix}, \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -\varepsilon \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ \varepsilon & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (23)
where $P = (\ell_0, \ell_1, \ell_2, \ell_3)$ is rearranged in order to be unitary, $P^{-1} = P^\dagger$. Turning back to the original notation, one obtains $TP = P\hat{T}$, $UP = P\hat{U}$, $VP = P\hat{V}$ and $WP = P\hat{W}$, where

\[
\hat{T} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \hat{U} = \begin{pmatrix}
0 & \varepsilon & i\varepsilon & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{V} = \begin{pmatrix}
0 & i\varepsilon & \varepsilon & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{W} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

are upper triangular forms of the four generators.

### 3.1 Generating the (pseudo)eigenvectors

Even though the four generators have only a single common eigenvector, this is not the case for the generic element $\Lambda_p \in \text{Bor}(1,3;p)$ in Eq. (6). Solving the fourth-order equation $\det(\Lambda_p - \lambda I) = 0$ for $\lambda \in \{e^t, e^{iw}, e^{-iw}, e^{-t}\}$. The corresponding system of eigenvectors can be calculated. However, here we give a more elegant method to calculate this system of eigenvectors. Using the exponential representation (9) and

\[
(Uu + Vv)P = P(\hat{U}u + \hat{V}v), \quad (Tt + Ww)P = P(\hat{T}t + \hat{W}w),
\]

one obtains $\Lambda_p = PK_{u,v}K_{t,w}P^{-1}$ with unipotent $K_{u,v} = \exp(\hat{U}u + \hat{V}v)$ and semisimple $K_{t,w} = \exp(\hat{T}t + \hat{W}w)$,

\[
K_{u,v} = \begin{pmatrix}
1 & \varepsilon(u + iv) & \varepsilon(v + iu) & -\varepsilon(u^2 + v^2) \\
0 & 1 & 0 & -u + iv \\
0 & 0 & 1 & -v + iu \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad K_{t,w} = \begin{pmatrix}
e^t & 0 & 0 & 0 \\
0 & e^{iw} & 0 & 0 \\
0 & 0 & e^{-iw} & 0 \\
0 & 0 & 0 & e^{-t}
\end{pmatrix}.
\]
Because $K_{t,w}$ is a diagonal matrix containing the four eigenvectors, the system of eigenvectors is given by the matrix $Q$ obeying $\Lambda_p Q = Q K_{t,w}$. Inserting $\Lambda_p = P K_{u,v} K_{t,w} P^{-1}$ into this eigenvalue equation, after some rearrangements one obtains

$$P^{-1}Q = K_{u,v} K_{t,w} P^{-1} Q K_{t,w}^{-1}.$$  \((27)\)

This equation for the unknown quantity $P^{-1}Q$ can be solved iteratively, starting with $P^{-1}Q = 1$, i.e. $Q = P$. The iterative solution can be shown to converge to

$$P^{-1}Q = \begin{pmatrix}
1 & \varepsilon(u + iv) & \varepsilon(v + iu) & -\varepsilon(u^2 + v^2) \\
1 - e^{-iw} & 1 - e^{i+iu} & (1 - e^{-iw})(1 - e^{i+iu}) & 0 \\
0 & 1 & 0 & -u + iv \\
0 & 0 & 1 & -v + iu \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \((28)\)

Multiplying with $P$ from the left, one finally obtains the system of eigenvectors

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & \varepsilon(u + iv) & \varepsilon(v + iu) & -\varepsilon(u^2 + v^2) \\
1 - e^{-iw} & 1 - e^{i+iu} & (1 - e^{-iw})(1 - e^{i+iu}) & 0 \\
0 & 1 & i & -u + iv \\
0 & i & 1 & -v + iu \\
\varepsilon & u + iv & v + iu & 1 - e^{-iw} \\
1 - e^{-iw} & 1 - e^{i+iu} & (1 - e^{-iw})(1 - e^{i+iu}) & 1
\end{pmatrix}.$$  \((29)\)

Expressed in a slightly philosophically manner, one can say that starting from the very sparse boundary of four defective matrices, the Lie algebra (in this case, the Borel subalgebra) knits the sweater $Q$ for the Lie group in a straightforward, iterative way.

### 4 Kronecker sum of solvable algebras

Although we were able to analyse the solvable algebra $\text{bor}(1,3;p)$, the representation in terms of the generators $T, U, V$ and $W$ is not the best one to see the structure of this
algebra. Therefore, we use a second one, namely

\[ J_3^\varepsilon = \frac{1}{2}(-T - iW), \quad J_-^\varepsilon = \frac{1}{2}(-U + iV), \]

\[ K_3^\varepsilon = \frac{1}{2}(T - iW), \quad K_+^\varepsilon = \frac{1}{2}(U + iV), \] (30)

obeying

\[ [J_3^\varepsilon, J_3^\varepsilon] = 0, \quad [J_3^\varepsilon, J_-^\varepsilon] = -J_-^\varepsilon, \quad [J_-^\varepsilon, J_-^\varepsilon] = 0; \]

\[ [K_3^\varepsilon, K_3^\varepsilon] = 0, \quad [K_3^\varepsilon, K_+^\varepsilon] = K_+^\varepsilon, \quad [K_-^\varepsilon, K_-^\varepsilon] = 0 \] (31)

and \([J_3^\varepsilon, K_3^\varepsilon] = [J_3^\varepsilon, K_-^\varepsilon] = [J_-^\varepsilon, K_3^\varepsilon] = [J_-^\varepsilon, K_-^\varepsilon] = 0\). The first justification for the sign notations for \(J_-^\varepsilon\) and \(K_+^\varepsilon\) is given by the commutator relations (31). In terms of the pairs \(\{J_3^\varepsilon, J_-^\varepsilon\}\) and \(\{K_3^\varepsilon, K_+^\varepsilon\}\) of generators \(\text{bor}(1, 3; p)\) can be rewritten as a Kronecker sum \(\text{sol}_2^- \boxplus \text{sol}_2^+\) of two two-dimensional solvable algebras, as will be detailed in the following.

### 4.1 Weyl’s unitary trick

A deeper look at this change of representation unveils that this change is actually a composition of several steps. In order to illustrate these steps, one can start again with the proper orthochronous Lorentz group \(\text{Lor}(1, 3) \subset \text{SO}(1, 3)\), the elements of which are given by the exponential representation (19) where

\[ (e^{\alpha \beta})^\mu_{\nu} = \eta^\alpha_{\nu} \eta^{\beta \mu} - \eta^{\alpha \mu} \eta^\beta_{\nu}, \quad \eta = \text{diag}(1, -1, -1, -1). \]

This representation can be written in a different form as

\[ \Lambda_p = \exp(\vec{\tau} \cdot \vec{E} + \vec{\omega} \cdot \vec{B}), \] (32)

using an analogy to the electromagnetic field strength tensor \(F^{\mu \nu}\) to write

\[ -\frac{1}{2} \omega_{\alpha \beta} e^{\alpha \beta} = \begin{pmatrix}
0 & \omega_{01} & \omega_{02} & \omega_{03} \\
\omega_{01} & 0 & \omega_{12} & -\omega_{31} \\
\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\
\omega_{03} & \omega_{31} & -\omega_{23} & 0
\end{pmatrix} = \vec{\tau} \cdot \vec{E} + \vec{\omega} \cdot \vec{B}, \] (33)
where \( \vec{\tau} = (\omega_{01}, \omega_{02}, \omega_{03}) \), \( \vec{\omega} = (\omega_{23}, \omega_{31}, \omega_{12}) \) and \( \vec{E} = -(e^{01}, e^{02}, e^{03}) \), \( \vec{B} = -(e^{23}, e^{31}, e^{12}) \) or \( e^{0i} = -E_i \), \( e^{ij} = -\epsilon_{ijk} B_k \) with the convention of lower indices for \( \vec{E} \) and \( \vec{B} \) and related three-vectors, \( \epsilon_{123} = 1 \). The \( 3 + 3 \) generators of Lor(1, 3) obey the commutation relations

\[
[B_i, B_j] = \epsilon_{ijk} B_k, \quad [B_i, E_j] = \epsilon_{ijk} E_k, \quad [E_i, E_j] = -\epsilon_{ijk} B_k.
\]

(34)

Obviously, the algebra lor(1, 3) is a real algebra. It contains a compact subalgebra \( \mathfrak{k} \) related to the \( B^i \) which is isomorphic to the compact algebra so(3). Actually, lor(1, 3) is in the shape of a Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) characterized by the values \( \phi(\mathfrak{k}) = \mathfrak{k}, \phi(\mathfrak{p}) = -\mathfrak{p} \) of an involution \( \phi \). As vector spaces, \( \mathfrak{k} \) and \( \mathfrak{p} \) are orthogonal, because given a scalar product \( (\mathfrak{k}, \mathfrak{p}) \) invariant under this involution, one obtains

\[
(\mathfrak{k}, \mathfrak{p}) = (\phi(\mathfrak{k}), \phi(\mathfrak{p})) = (\mathfrak{k}, -\mathfrak{p}) = - (\mathfrak{k}, \mathfrak{p}) \Rightarrow (\mathfrak{k}, \mathfrak{p}) = 0
\]

(35)

However, \([\mathfrak{k}, \mathfrak{p}] \neq 0\). Therefore, we used the symbol \( \oplus \) instead of the symbol \( \oplus \) for the direct sum. The algebra \( \mathfrak{g} \) can be transformed to a compact form by using Weyl’s unitary trick. The result is an algebra \( \mathfrak{g}^* = \mathfrak{k} + i \mathfrak{p} \), where the implications for introducing an imaginary factor will be explained later. In case of lor(1, 3), the involution is given by

\[
\phi : e^{\mu\nu} \rightarrow \eta(e^{\mu\nu}) := \eta e^{\mu\nu} \eta = -e^{\mu\nu T}
\]

(matrix indices are suppressed) or \( \phi(\vec{B}) = \vec{B}, \phi(\vec{E}) = -\vec{E} \). Therefore, the compact form of lor(1, 3) is given by the generators \( B^i \) and \( iE^i \) obeying the commutation relations

\[
[B_i, B_j] = \epsilon_{ijk} B_k, \quad [B_i, (iE_j)] = \epsilon_{ijk} (iE_k), \quad [(iE_i), (iE_j)] = \epsilon_{ijk} B_k.
\]

(37)

Considered as a real algebra, this algebra is isomorphic to so(4). However, the generators are antihermitean, \( B_i^\dagger = -B_i \) and \( (iE_i)^\dagger = -iE_i \) and, therefore, the group is unitary. In general, Weyl’s unitary trick can be seen to lead always to unitary Lie groups.

### 4.2 Duplication and complexification

The addition of an imaginary factor \( i \) turns the real algebra into a complex algebra, at least for intermediate steps. In general, this process is called complexification and is denoted by
a lower index \( \mathbb{C} \) (or additional argument) to the algebra symbol. Given a real Lie algebra \( L \), the duplication of this algebra is given by \( \text{(15)} \)

\[
L + iL := \{x + iy \mid x, y \in L\}.
\]  
\[
(38)
\]

\( L + iL \) is still a real vector space. In defining the multiplication of an element \( x + iy \in L_C \) with a complex number \( \alpha = a + ib \in \mathbb{C} \) by \( (a + ib)(x + iy) := (ax - by) + i(bx + ay) \), and the commutator of two elements \( x + iy \) and \( x' + iy' \) by

\[
[x + iy, x' + iy'] := [x, x'] - [y, y'] + i([x, y'] + [y, x']),
\]

\[
(39)
\]

\( L + iL \) constitutes a complex form, denoted by \( L_C := \mathbb{C} \otimes_{\mathbb{R}} L \). This complex form again is a complex Lie algebra which is called the complexification of \( L \). Applied to the actual case, the complexification turns the real algebra \( \text{so}(1, 3) \) into the complex algebra \( \text{so}(4, \mathbb{C}) \).

However, it is obvious that the algebra given by the commutator relations \( \text{(37)} \) is real, not complex. The final algebra, therefore, is a real form of this complex algebra, defined as follows: a subalgebra \( K \) of the duplicated algebra \( L + iL \) is called real form if the complexification of this subalgebra is the same as the original algebra, \( K_C = L \). As the duplication is not unique \( \text{2} \), there are also different real forms to a given complex algebra.

### 4.3 Compactified and decompactified real forms

Most important real forms are the normal real form where the duplicates are again taken as separate elements, and the compact real form which exists for all (semi)simple complex Lie algebras. Because we will meet these forms in the lower-dimensional case, we postpone the discussion about the different real forms. In the actual case, one of the compact real forms is \( \text{so}(4) \). However, another one is given by

\[
A_i = \frac{1}{2}(B_i + iE_i), \quad \bar{A}_i = \frac{1}{2}(B_i - iE_i)
\]

\[
(40)
\]

with the commutation rules

\[
[A_i, A_j] = \epsilon_{ijk}A_k, \quad [A_i, \bar{A}_j] = 0, \quad [\bar{A}_i, A_j] = \epsilon_{ijk}A_k.
\]

\[
(41)
\]

\text{2For instance, a part of the basis elements can be duplicated with } i, \text{ another part with } -i.
Therefore, the algebra decomposes into two separate algebras which are isomorphic to\( su(2) \).\(^3\) Turning back to solvable groups, the decomposition into \( su(2) = \text{span}_R\{A_i\} \) and \( su(2) = \text{span}_R\{\bar{A}_i\} \) is not yet conform with the definitions given in Eq. (30). Looking at the definitions of \( T_i^\varepsilon \) on the one hand and the definitions of \( E_i \) and \( B_i \) on the other hand, one obtains

\[
J_3^\varepsilon = \frac{1}{2}(\varepsilon e^{03} - ie^{12}) = \frac{1}{2}(-\varepsilon E_3 + iB_3), \quad K_3^\varepsilon = \frac{1}{2}(-\varepsilon e^{03} - ie^{12}) = \frac{1}{2}(\varepsilon E_3 + iB_3)
\]

and generally \( J_i^\varepsilon = \frac{1}{2}(-\varepsilon E_i + iB_i) = i\bar{A}_i, \quad K_i^\varepsilon = \frac{1}{2}(\varepsilon E_i + iB_i) = iA_i. \) As \( A_i \) and \( \bar{A}_i \) are antihermitean, \( J_i^\varepsilon \) and \( K_i^\varepsilon \) are hermitean and, therefore, constitute decompactified subgroups generated by \( \exp(ij_iJ_i^\varepsilon) \) and \( \exp(ik_iK_i^\varepsilon) \). One obtains

\[
J_1^\varepsilon - iJ_2^\varepsilon = \frac{1}{2}(-\varepsilon E_1 + B_2 + i(\varepsilon E_2 + B_1)) = \frac{1}{2}(\varepsilon E_1 + B_2 - i(\varepsilon E_2 + B_1)) = J_1^\varepsilon, \quad K_1^\varepsilon + iK_2^\varepsilon = \frac{1}{2}(\varepsilon E_1 - B_2 + i(\varepsilon E_2 + B_1)) = \frac{1}{2}(\varepsilon E_1 - B_2 - i(\varepsilon E_2 + B_1)) = K_1^\varepsilon
\]

which is the other justification for the sign notations in \( J_\pm^\varepsilon \) and \( K_\pm^\varepsilon \). Actually, the algebra looks like \( \text{sl}(2, \mathbb{R}) \) with one generator missing \( (J_+^\varepsilon \) or \( K_-^\varepsilon \), respectively). In \( \text{lor}(1, 3) \), these missing generators exist. In \( \text{bor}(1, 3; p) \), however, the generators are found in the resp. other algebra with opposite sign \( \varepsilon \),

\[
J_+^\varepsilon = J_1^\varepsilon + iJ_2^\varepsilon = \frac{1}{2}(-\varepsilon E_1 + B_2 + i(\varepsilon E_2 + B_1)) = \frac{1}{2}(T_1^{-\varepsilon} + iT_2^{-\varepsilon}) = K_+^{-\varepsilon}, \quad K_-^\varepsilon = K_1^\varepsilon - iK_2^\varepsilon = \frac{1}{2}(\varepsilon E_1 + B_2 + i(\varepsilon E_2 + B_1)) = \frac{1}{2}(T_1^{-\varepsilon} - iT_2^{-\varepsilon}) = J_-^{-\varepsilon},
\]

while \( J_3^{\pm\varepsilon} = K_3^{\pm\varepsilon} \). Therefore, \( \text{bor}(1, 3; p) \) splits up into the subalgebras

\[
\text{sol}_2^- := \text{span}_R\{J_3^\varepsilon, J_1^\varepsilon\} = \text{span}_R\{K_3^{-\varepsilon}, K_-^{-\varepsilon}\} \quad \text{and} \quad \text{sol}_2^+ := \text{span}_R\{K_3^\varepsilon, K_1^\varepsilon\} = \text{span}_R\{J_3^-\varepsilon, J_1^-\varepsilon\}.
\]

As there is a homomorphism between the two algebras \( \text{sol}_2^+ \) and \( \text{sol}_2^- \), both solvable algebras are maximal and, therefore, are Borel subalgebras of the larger algebra \( \text{sl}(2, \mathbb{R}) \). For a free choice of \( \varepsilon \) one can represent the two Borel subalgebras as being generated by a solvable part

\(^3\text{su}(2) \) is preferred instead of \( \text{so}(3) \) because \( A_i \) and \( \bar{A}_i \) are antihermitean, leading to unitary groups.
of the set \( \{J_3^{\pm\varepsilon}, J_+^{\pm\varepsilon}, J_-^{\pm\varepsilon}\} \) of generators of \( \text{sl}(2, \mathbb{R}) \), thereby skipping the second (redundant) set \( \{K_3^{\pm\varepsilon}, K_+^{\pm\varepsilon}, K_-^{\pm\varepsilon}\} \). Alternatively, one can use the two sets and skip \( \varepsilon = +1 \). Though the first choice is more intriguing, for this paper we stay with the clearer second one.

### 4.4 In quest of left and right

Searching for eigenvectors of the set \( \{J_3, J_+, J_-\} \) one finds that these eigenvectors are disjoint, as known for semisimple algebras. The same holds for the set \( \{K_3, K_+, K_-\} \). However, for each of the solvable subalgebras \( \text{sol}_- \) and \( \text{sol}_+ \) one obtains only a single common eigenvector. In order to analyse the eigenvector structure, we return to the eigenvectors

\[
\ell_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \varepsilon \end{pmatrix}, \quad \ell_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \ell_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad \text{and} \quad \ell_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\varepsilon \end{pmatrix}
\]

of Sec. 3 to which we apply the algebra elements, obtaining

\[
\begin{align*}
J_3 \ell_0 &= -\frac{1}{2} \ell_0 & J_+ \ell_0 &= -\varepsilon \ell_1 & J_- \ell_0 &= 0 & K_3 \ell_0 &= +\frac{1}{2} \ell_0 & K_+ \ell_0 &= 0 & K_- \ell_0 &= \varepsilon \ell_1 \\
J_3 \ell_1 &= +\frac{1}{2} \ell_1 & J_+ \ell_1 &= 0 & J_- \ell_1 &= -\varepsilon \ell_0 & K_3 \ell_1 &= +\frac{1}{2} \ell_1 & K_+ \ell_1 &= 0 & K_- \ell_1 &= \varepsilon \ell_3 \\
J_3 \ell_2 &= -\frac{1}{2} \ell_2 & J_+ \ell_2 &= -\varepsilon \ell_3 & J_- \ell_2 &= 0 & K_3 \ell_2 &= -\frac{1}{2} \ell_2 & K_+ \ell_2 &= \varepsilon \ell_0 & K_- \ell_2 &= 0 \\
J_3 \ell_3 &= +\frac{1}{2} \ell_3 & J_+ \ell_3 &= 0 & J_- \ell_3 &= -\varepsilon \ell_2 & K_3 \ell_3 &= -\frac{1}{2} \ell_3 & K_+ \ell_3 &= \varepsilon \ell_1 & K_- \ell_3 &= 0
\end{align*}
\]  

(47)

For \( \{J_3, J_-\} \) the common eigenvector is given as a linear combination of \( \ell_0 \) and \( \ell_2 \) while for \( \{K_3, K_+\} \) the common eigenvector is given by the linear combination of \( \ell_0 \) and \( \ell_1 \). On the other hand, the common eigenvector for \( \{J_3, J_+\} \) is a linear combination of \( \ell_3 \) and \( \ell_1 \) while the common eigenvector of \( \{K_3, K_-\} \) is a linear combination of \( \ell_3 \) and \( \ell_2 \). While \( \ell_0 \) (\( \ell_3 \)) is proportional to the (space inverted) momentum four-vector \( p \), the interpretation of the eigenvectors \( \ell_1 \) and \( \ell_2 \) deserves more effort. For this one can take refuge to the circular polarisation [16, 17]. The representation

\[
\vec{E}(z, t) = E_0 \Re \left( (\vec{e}_x + i \vec{e}_y) e^{i k z - i \omega t} \right) = E_0 (\vec{e}_x \cos(kz - \omega t) - \vec{e}_y \sin(kz - \omega t))
\]

(48)
describes the right turn of the electric vector in the \((x, y)\) plane, as can be seen by comparing the solution for \(z = 0\) at \(t = 0\) and after a short time \(t = \Delta t\). Therefore, the vector \(\ell_1\) can be identified with the right turn. However, a turn can be identified with handedness or chirality only in combination with a direction of propagation, as in case of the circular polarisation by the argument \(kz - \omega t\). This direction is given by \(\ell_0\) (or \(\ell_3\)). Therefore, one can interpret (in case of \(\varepsilon = 1\))

\[
\begin{align*}
\{J_3, J_-\} & \quad \text{as forward-propagating left-handed}, \\
\{K_3, K_+\} & \quad \text{as forward-propagating right-handed}, \\
\{J_3, J_+\} & \quad \text{as backward-propagating left-handed, and} \\
\{K_3, K_-\} & \quad \text{as backward-propagating right-handed.} \\
\end{align*}
\]  

\[\text{(49)}\]

4.5 The irreducible representation

In terms of \(4 \times 4\) matrices the generators \(J_i\) and \(K_i\) \((i = 3, \pm)\) are, of course, not given in the irreducible representation. However, they can be related to irreducible representations in an easy way. In fact, there is a similarity transformation such that

\[
J_i \rightarrow S^{-1}J_iS =: J_i^\oplus, \quad K_i \rightarrow S^{-1}K_iS =: K_i^\oplus
\]

\[\text{(50)}\]

(a deeper understanding of the representation index \(\oplus\) will be given soon), where

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
-i & 0 & 0 & -i \\
0 & 1 & 1 & 0
\end{pmatrix}
\]

\[\text{(51)}\]

\[\text{In optics this solution is named left-polarised, as looked at in the direction the light comes from (passive direction). In our case, however, we consider the direction of propagation (active direction).}\]
and $S^{-1} = S^\dagger$. In detail, one obtains

\begin{align}
J_3^\Xi &= \frac{1}{2}(\sigma_3 \otimes 1), \\
J_\pm^\Xi &= \frac{1}{2}(\sigma_\pm \otimes 1), \\
K_3^\Xi &= \frac{1}{2}(1 \otimes \sigma_3), \\
K_\pm^\Xi &= \frac{1}{2}(1 \otimes \sigma_\pm),
\end{align}

(52)

where the outer product is defined by $(A \otimes B)_{(ik)(jl)} := A_{ij}B_{kl}$, i.e. the first matrix sets the frame for the second one. The matrices

\begin{align}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align}

(53)

are the usual Pauli matrices, and $\sigma_\pm = \sigma_1 \pm i\sigma_2$. The same similarity transformation via $S$ can be applied also to the generators $E_i$ and $B_i$ of the proper orthochronous Lorentz group Lor(1,3). One obtains

\begin{align}
E_i^\Xi := S^{-1}E_iS &= -\frac{1}{2}(\sigma_i \otimes 1 - 1 \otimes \sigma_i), \\
B_i^\Xi := S^{-1}B_iS &= -\frac{i}{2}(\sigma_i \otimes 1 + 1 \otimes \sigma_i).
\end{align}

(54)

These two results can be rewritten by employing the Kronecker sum

\begin{equation}
A \boxplus B := A \otimes 1 + 1 \otimes B.
\end{equation}

(55)

Using this notation, one obtains

\begin{align}
E_i^\Xi &= -\frac{1}{2}\sigma_i \boxplus \left(1 + \frac{1}{2}\sigma_i\right), \\
B_i^\Xi &= -\frac{i}{2}\sigma_i \boxplus \left(-\frac{i}{2}\sigma_i\right).
\end{align}

(56)

Therefore, the representation index $\boxplus$ indicates that in this representation obtained via the similarity transformation with $S$ the matrix can be written as a Kronecker sum. It is characteristic that

\begin{align}
J_i^\Xi &= \frac{1}{2}\sigma_i \boxplus \emptyset, \\
K_i^\Xi &= \emptyset \boxplus \frac{1}{2}\sigma_i
\end{align}

(57)

contribute only to the first or second component of the Kronecker sum, respectively. Following the argumentation of Sec. 4.4 one can conclude that the first component of the Kronecker sum (and thereby $J_i^\Xi$) is left-handed while the second component of the Kronecker sum (and thereby $K_i^\Xi$) is right-handed. Finally, we conclude that via the same similarity transformation $S$ the maximal solvable algebra bor(1,3; $p$) in the representation of this section can indeed be decomposed into the Kronecker sum $\text{sol}^- \boxplus \text{sol}^+$. 

18
5 The Chevalley basis

From Eqs. (57) it is obvious that sol$_-^2$ and sol$_+^2$ are isomorphic to Borel subalgebras of the real algebra sl(2, R), given in the Chevalley basis by the three generators

$$
\begin{align*}
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \sigma_+ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} & \text{and} & \sigma_- &= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \\
\end{align*}
$$

(58)

One can write sol$_-^2 +$ span$_R\{\sigma_3, \sigma_+\}$. The algebra sl(2, R) = span$_R\{\sigma_3, \sigma_+, \sigma_-\}$ can be complexified to obtain sl(2, C) = span$_C\{\sigma_3, \sigma_+, \sigma_-\}$. Therefore, sl(2, R) is a real form of sl(2, C). The compact real form of sl(2, C) is given by su(2) = span$_R\{\sigma_1, \sigma_2, \sigma_3\}$, while sl(2, R) can be called the decompactified real form of sl(2, C). In a similar way as the complexified version of lor(1, 3) is isomorphic to su(2) ⊕ su(2), the complexified version of the extended little algebra bor(1, 3; p) is isomorphic to sol$_-^2 +$ sol$_+^2$.

5.1 Common eigenvectors

The concept of common eigenvectors introduced in Sec. 4.4 pulls through to the very core, i.e. to the irreducible representation. The set of generators \{\sigma_3, \sigma_+\} of sol$_+^2$ have the common eigenvector (1, 0)$^T$ and the set \{1, 0\} of eigenvalues while for \{\sigma_3, \sigma_-\} (i.e. sol$_-^2$) the common eigenvector is (0, 1)$^T$ with eigenvalues \{-1, 0\}. Reintroducing the sign \varepsilon, the two non-trivial eigenvalue equations can be cast into the form

$$
\sigma_3 \psi_+ = \varepsilon \psi_+,
\psi_+ = \begin{pmatrix} (1 + \varepsilon) \psi^1 \\ (1 - \varepsilon) \psi^2 \end{pmatrix}.
$$

(59)

This is the first quantisation step. Indeed, introducing

$$
\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1
$$

(60)

one obtains the Weyl equation ((\sigma^\mu) := (\sigma_0; \sigma_1, \sigma_2, \sigma_3))

$$
0 = (\varepsilon \sigma_0 - \sigma_3) \psi_+ = \varepsilon p_\mu \sigma^\mu \psi_+ =: \varepsilon \sigma(p) \psi_+.
$$

(61)
However, this is not the only possible quantisation. Equivalently, one may write

$$\sigma_3 \psi_- = -\varepsilon \psi_-,$$

$$\psi_- = \begin{pmatrix} (1 - \varepsilon) \psi^1 \\ (1 + \varepsilon) \psi^2 \end{pmatrix}$$

or

$$0 = (\varepsilon \sigma_0 + \sigma_3) \psi_- = \varepsilon p_\mu \tilde{\sigma}^\mu \psi_- =: \varepsilon \tilde{\sigma}(p) \psi_-$$

$$((\tilde{\sigma}^\mu) = (\sigma_0; -\sigma_1, -\sigma_2, -\sigma_3))$$ which is the dual Weyl equation. In using the tilde notation for $\tilde{\sigma}$ one avoids the breakdown of the covariant notation. Using Weyl’s representation

$$\gamma^\mu_W = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5_W = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

of the Dirac matrices, for finite mass $m$ one ends up with the Dirac equation

$$(p_\mu \gamma^\mu_W - mc) \psi_W = 0, \quad \psi_W = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}.$$  

$\psi_+$ is the right-handed spinor and $\psi_-$ is the left-handed spinor. This is in agreement with the usual definition $\psi_R = \frac{1}{2}(1 + \gamma^5) \psi_W = (0, \psi_+)^T$ and $\psi_L = \frac{1}{2}(1 - \gamma^5) \psi_W = (\psi_-, 0)^T$.

### 5.2 Induced Lorentz transformations

The contractions of the momentum four-vector $p$ with $\sigma$ and $\tilde{\sigma}$ induces two (proper orthochronous) Lorentz transformations $A_\Lambda$ and $\tilde{A}_\Lambda$ which make the diagram

$$A_\Lambda: \quad \sigma(p) \quad \longrightarrow \quad \sigma(\Lambda p)$$

$$\pi \uparrow \quad \uparrow \sigma \quad \uparrow \sigma$$

$$\Lambda: \quad p \quad \longrightarrow \quad \Lambda p$$

$$\tilde{\pi} \downarrow \quad \downarrow \tilde{\sigma} \quad \downarrow \tilde{\sigma}$$

$$\tilde{A}_\Lambda: \quad \tilde{\sigma}(p) \quad \longrightarrow \quad \tilde{\sigma}(\Lambda p)$$

commutative. The induced Lorentz transformations are defined by

$$A_\Lambda \sigma(p) A_\Lambda^\dagger = \sigma(\Lambda p), \quad \tilde{A}_\Lambda \tilde{\sigma}(p) \tilde{A}_\Lambda^\dagger = \tilde{\sigma}(\Lambda p).$$
A long but straightforward calculation shows that

\[ A_\Lambda = \frac{\sigma^\mu \Lambda_{\mu\nu} \bar{\sigma}^\nu}{2 \text{tr}(A^\dag_\Lambda)}, \quad A^{-1}_\Lambda = \frac{\sigma^\mu \Lambda_{\mu\nu} \bar{\sigma}^\nu}{2 \text{tr}(A^\dag_\Lambda)}, \quad \bar{A}_\Lambda = \frac{\bar{\sigma}^\mu \Lambda_{\mu\nu} \sigma^\nu}{2 \text{tr}(A^\dag_\Lambda)}, \quad \bar{A}^{-1}_\Lambda = \frac{\bar{\sigma}^\mu \Lambda_{\mu\nu} \sigma^\nu}{2 \text{tr}(A^\dag_\Lambda)}. \]  

(67)

\( A_\Lambda \) and \( \bar{A}_\Lambda \) can be written in an exponential form similar to Eq. [19],

\[ A_\Lambda(\omega) = \exp \left( -\frac{1}{2} \omega_{\alpha\beta} a^{\alpha\beta} \right), \quad \bar{A}_\Lambda(\omega) = \exp \left( -\frac{1}{2} \omega_{\alpha\beta} \bar{a}^{\alpha\beta} \right). \]  

(68)

For the exponential coefficients of \( A_\Lambda \) one obtains

\[ a^{\alpha\beta} = \frac{1}{4}(e^{\alpha\beta})_{\mu\nu} \sigma^\mu \bar{\sigma}^\nu = -\frac{1}{2}(\sigma^\alpha \bar{\sigma}^\beta - \bar{\sigma}^\beta \sigma^\alpha) \]  

(69)

which can be detailed into \( a^{ij} = \frac{i}{2} \epsilon_{ijk} \sigma_k = -\epsilon_{ijk} b_k, \quad a^i = \frac{1}{2} \sigma_i = -e_i \)

with

\[ b_i = -\frac{i}{2} \sigma_i, \quad e_i = -\frac{1}{2} \sigma_i, \]  

(70)

where \( b_i \) and \( e_i \) obey the algebra lor(1,3),

\[ [b_i, b_j] = \epsilon_{ijk} b_k, \quad [b_i, e_j] = \epsilon_{ijk} e_k, \quad [e_i, e_j] = -\epsilon_{ijk} b_k. \]  

(71)

For the exponential coefficient of \( \bar{A}_\Lambda \) one obtains

\[ \bar{a}^{\alpha\beta} = \frac{1}{4}(e^{\alpha\beta})_{\mu\nu} \bar{\sigma}^\mu \sigma^\nu = -\frac{1}{2}(\sigma^\alpha \bar{\sigma}^\beta - \bar{\sigma}^\beta \sigma^\alpha) \]  

(72)

which gives \( \bar{a}^{ij} = -\frac{i}{2} \epsilon_{ijk} \sigma_k = :\epsilon_{ijk} \bar{b}_k, \quad \bar{a}^i = \frac{1}{2} \bar{\sigma}_i = :\bar{e}_i \). The generators

\[ \bar{b}_i = -\frac{i}{2} \bar{\sigma}_i, \quad \bar{e}_i = \frac{1}{2} \bar{\sigma}_i \]  

(73)

(note the sign changes compared to \( b_i, e_i \)) obey again the algebra lor(1,3),

\[ [\bar{b}_i, \bar{b}_j] = \epsilon_{ijk} \bar{b}_k, \quad [\bar{b}_i, \bar{e}_j] = \epsilon_{ijk} \bar{e}_k, \quad [\bar{e}_i, \bar{e}_j] = -\epsilon_{ijk} \bar{b}_k. \]  

(74)

Formally, the transitions to the induced Lorentz transformations can be considered as mappings \( \pi : \Lambda \rightarrow A_\Lambda \) with \( \pi(e^{\alpha\beta}) = a^{\alpha\beta} \) and \( \tilde{\pi} : \Lambda \rightarrow \bar{A}_\Lambda \) with \( \tilde{\pi}(e^{\alpha\beta}) = \bar{a}^{\alpha\beta} \). Under these mappings, the generators \( J^\varepsilon_i \) and \( K^\varepsilon_i \) of sol\(_2^\varepsilon \) are mapped onto the Chevalley basis. Under \( \pi \) one obtains

\[ J^\varepsilon_3 \rightarrow \frac{1}{4}(1 + \varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K^\varepsilon_3 \rightarrow \frac{1}{4}(1 - \varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

\[ J^\varepsilon_- \rightarrow \frac{1}{4}(1 + \varepsilon) \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad K^\varepsilon_- \rightarrow \frac{1}{4}(1 - \varepsilon) \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \]  

(75)
while under $\tilde{\pi}$ one obtains

$$
J_3^\varepsilon \rightarrow -\frac{1}{4}(1 - \varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_3^\varepsilon \rightarrow -\frac{1}{4}(1 + \varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

$$
J_-^\varepsilon \rightarrow -\frac{1}{4}(1 - \varepsilon) \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad K_+^\varepsilon \rightarrow -\frac{1}{4}(1 + \varepsilon) \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},
$$

(76)
i.e. the same result with $\varepsilon \leftrightarrow -\varepsilon$ and the total sign interchanged. Again, we are faced with the fact that half of the generators are mapped to zero. Taking into account the relations to $J_i^\varepsilon$ and $K_i^\varepsilon$, one can state that $\pi$ maps to the first component of the Kronecker sum while $\tilde{\pi}$ maps to the second component of the Kronecker sum. Due to Sec. 4, $\pi$ is the mapping to the left-handed sector, $\tilde{\pi}$ the mapping into the right-handed sector.

Using Eqs. (67) and performing a couple of simple conversions, one obtains the explicit shape for $A_{\Lambda}$ for $\Lambda$ of Eq. (6) with dependence on $\varepsilon$,

$$
A_{\Lambda}^\varepsilon = \begin{pmatrix} e^{(-\varepsilon t + i w)/2} & \frac{1}{2}(1 - \varepsilon)(u - iv)e^{(-\varepsilon t - i w)/2} \\ -\frac{1}{2}(1 + \varepsilon)(u + iv)e^{(\varepsilon t + i w)/2} & e^{(\varepsilon t - i w)/2} \end{pmatrix}
$$

(77)

($\tilde{A}_{\Lambda}^\varepsilon = A_{-\varepsilon}^\Lambda$), which can be rewritten as

$$
A_{\Lambda}^\varepsilon = R_{-\varepsilon}A_{u,v}^\varepsilon R_{i w},
$$

(78)

where

$$
R_x := \begin{pmatrix} e^{x/2} & 0 \\ 0 & e^{-x/2} \end{pmatrix}, \quad A_{u,v}^\varepsilon := \begin{pmatrix} 1 & \frac{1}{2}(1 - \varepsilon)(u - iv) \\ -\frac{1}{2}(1 + \varepsilon)(u + iv) & 1 \end{pmatrix}
$$

(79)

and $\det A_{u,v}^\varepsilon = \det R_x = 1$. Using $(A_{\Lambda}^{\varepsilon\dagger})^{-1} = (A_{\Lambda}^\varepsilon)^{-1\dagger} = A_{-\varepsilon}^\Lambda$ and $(\tilde{A}_{\Lambda}^{\varepsilon\dagger})^{-1} = A_{-\varepsilon}^\Lambda$ and Eq. (66), one obtains the Lorentz transformation $\psi_W(\Lambda p) = U(\Lambda)\psi_W(p)$ of the Weyl spinor where

$$
U(\Lambda) = \begin{pmatrix} A_{\Lambda}^\varepsilon & 0 \\ 0 & A_{-\varepsilon}^\Lambda \end{pmatrix}, \quad \psi_W(p) = \begin{pmatrix} \psi_-(p) \\ \psi_+(p) \end{pmatrix}.
$$

(80)
5.3 Representations of the proper orthochronous Lorentz group

Using the two mappings \( \pi : \text{Lor}(1, 3) \to \text{SL}(2, \mathbb{C}) \), \( \tilde{\pi} : \text{Lor}(1, 3) \to \text{SL}(2, \mathbb{C}) \) (\( \pi : \Lambda \mapsto A_\Lambda \) and \( \tilde{\pi} : \Lambda \mapsto \tilde{A}_\Lambda \)) and the Kronecker sum, one can define the representation \((1/2, 1/2)\) by

\[
\pi^{(1/2, 1/2)}(\Lambda) := (\pi \otimes \tilde{\pi})(\Lambda \boxplus \Lambda) = \pi(\Lambda) \boxplus \tilde{\pi}(\Lambda),
\]

for which (and for the choice \( \varepsilon = +1 \))

\[
\pi^{(1/2, 1/2)}(E_i) = -\frac{1}{2}\sigma_i \boxplus \left( +\frac{1}{2}\sigma_i \right) E_i^\boxplus, \quad \pi^{(1/2, 1/2)}(J_i) = \frac{1}{2}\sigma_i \boxplus 0 = J_i^\boxplus,
\]

\[
\pi^{(1/2, 1/2)}(B_i) = -\frac{i}{2}\sigma_i \boxplus \left( -\frac{i}{2}\sigma_i \right) B_i^\boxplus, \quad \pi^{(1/2, 1/2)}(K_i) = 0 \boxplus \frac{1}{2}\sigma_i = K_i^\boxplus.
\]

Therefore, the map \( \pi^{(1/2, 1/2)} : \text{Lor}(1, 3) \to \text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C}) \) may replace the similarity transformation via \( S \). The benefit of using this map instead of the similarity transformation is that such a construction can easily be generalized to a representation \( \pi^{(k,l)} \) of the proper orthochronous Lorentz group.

In proceeding to these general \((k, l)\) representations, the common eigenvectors \((1, 0)^T\) of the set \( \{\sigma_3, \sigma_+\} \) and \((0, 1)^T\) of the set \( \{\sigma_3, \sigma_-\} \) can be written as states \(|l; m\rangle = |1/2; 1/2\rangle \) and \(|l; m\rangle = |1/2; -1/2\rangle \), respectively, with

\[
\sigma_3|l; m\rangle = 2m|l; m\rangle, \quad \sigma_\pm|l; m\rangle = 2\rho(l; \pm m)|l; m \pm 1\rangle,
\]

where \( \rho(l; m) = \sqrt{(l-m)(l+m+1)} \). For the general \((k, l)\) representation the states are given by \(|k, l; m_k, m_l\rangle\), with

\[
\pi^{(k,l)}(J_3)|k, l; m_k, m_l\rangle = 2m_k|k, l; m_k, m_l\rangle,
\]

\[
\pi^{(k,l)}(J_\pm)|k, l; m_k, m_l\rangle = 2\rho(k; \pm m_k)|k, l; m_k \pm 1, m_l\rangle,
\]

\[
\pi^{(k,l)}(K_3)|k, l; m_k, m_l\rangle = 2m_l|k, l; m_k, m_l\rangle,
\]

\[
\pi^{(k,l)}(K_\pm)|k, l; m_k, m_l\rangle = 2\rho(l; \pm m_l)|k, l; m_k, m_l \pm 1\rangle.
\]

Of these \((2k + 1) \times (2l + 1)\) states, only those for \( \rho(k; -m_k) = 0 \) or \( \rho(l; m_l) = 0 \), i.e. for \( m_k = -k \) or \( m_l = l \) are common eigenstates of \( \text{sol}_2^- = \{J_3, J_-\} \) and \( \text{sol}_2^+ = \{K_3, K_+\} \), respectively, with \(|k, l; -k, l\rangle\) being the common eigenvector for both algebras [18].

5.4 Helicity

In order to define a helicity

\[ H(\vec{p}) = \hat{p} \cdot \vec{s}, \quad \hat{p} = \frac{\vec{p}}{|\vec{p}|}, \]  

(85)

one needs a spin vector \( \vec{s} \). This vector can be defined by \( s_i = i \hbar b_i \), because then the commutation relation \([b_i, b_j] = \epsilon_{ijk} b_k\) for the generators of \( A^\vee_1 \) leads to the usual commutation relation

\[ [s_i, s_j] = i\hbar \epsilon_{ijk} s_k \]  

(86)

of an angular momentum algebra. For the three-vector part \( \vec{p} = (0, 0, 1)^T \) of the momentum vector \( p \) generating the Borel subgroup \( \text{Bor}(1, 3; p) \), one obtains

\[ H(\vec{p}) = s_3 = i\hbar b_3 = \frac{\hbar}{2} \sigma_3 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(87)

Therefore, the common eigenvector \((1, 0)^T\) of \( \text{sol}_2^+ \) has helicity \( h = +\hbar/2 \) and the common eigenvector \((0, 1)^T\) of \( \text{sol}_2^- \) has helicity \( h = -\hbar/2 \), in agreement (for \( \varepsilon = +1 \)) with the previous understanding of left and right.

As \( b_i \) is the two-dimensional representation of \( B_i \), the concept of helicity can be generalised to representations \((k, l)\),

\[ H^{(k,l)}(\vec{p}) = \pi^{(k,l)}(i\hbar B_3) = \frac{\hbar}{2} \pi^{(k,l)} (\sigma_3 \boxplus \sigma_3), \]  

(88)

Applied to the state \(|k, l; m_k, m_l\rangle\) one obtains

\[ H^{(k,l)}(\vec{p})|k, l; m_k, m_l\rangle = \hbar (m_k + m_l)|k, l; m_k, m_l\rangle \]  

(89)

which means that the helicity of this state is \( h = \hbar (m_k + m_l) \).

6 Conclusions and Outlook

In this paper, we have calculated the stabiliser group of the proper orthochronous Lorentz group, which turns out to be the maximal solvable or Borel subgroup of dimension four.
We have explained the continuous transition between the stabiliser groups of massive and massless particles that describes the massless limit but fails for exactly massless states. We have dealt with the system of eigenvectors of the Borel subgroup and shown that the Borel subgroup can be described by a Kronecker sum of two two-dimensional solvable groups $\text{sol}_2^\pm$ representing right- and left-handed chirality states. Finally, in the Chevalley basis we have derived the Weyl and Dirac equations for massless and massive particles and have defined the helicity of the massless states. Note that without the generator $T$ such a Kronecker sum of chiral states would not emerge. The Borel subgroup as the maximal solvable subgroup of the proper orthochronous Lorentz group provides exactly four eigenvectors describing these two chiral states, of which the left handed state is populated by massless fermions, the right handed by antifermions. This is the physical content of our extension.

Even though the foundations for an explanation of the spin-flip effect are prepared by this, an exact formulation is not gained here but is aimed for a future publication. The effect is closely related to the concept of mass which we want to understand in more detail. In our argumentation we obtained unexpected help from a not yet published seminal work explaining in detail the construction of a spin operator by a linear combination of components of the Pauli–Lubanski pseudovector [19]. Not unexpectedly, the authors end up with two spin (tensor) operators and corresponding chirality states that are interchanged under parity transformation. Parity eigenstates can be constructed as particle or antiparticle compound states. Applying the Lorentz transformation to the massive states of Ref. [19], the parity eigenstates are shown to evolve to solutions of the Dirac equation.

In Ref. [19] it is emphasised that the two spin operators are neither axial nor Hermitian, and the same holds for the spin operators in the $(1/2, 0) \oplus (0, 1/2)$ representation. However, both properties are restored if applied to particle and antiparticle states. On the other hand, as both properties are essential for physical states, we can conclude that massless left- and right-handed states are physical only in the total absence of mass. This “gap of (un)physicalness” as an explanation for the spin-flip effect has to be investigated in detail.
Acknowledgments

The research was supported by the Estonian Research Council under Grant No. IUT2-27 and by the European Regional Development Fund under Grant No. TK133.

References

[1] T. D. Lee and M. Nauenberg, “Degenerate Systems and Mass Singularities,” Phys. Rev. 133 (1964), B1549-B1562

[2] S. Jadach and Z. Was, “QED $O(\alpha^3)$ radiative corrections to the reaction $e^+e^- \rightarrow \tau^+\tau^-$ including spin and mass effects,” Acta Phys. Polon. B 15 (1984), 1151 [erratum: Acta Phys. Polon. B 16 (1985), 483]

[3] R. Kleiss, “Hard Bremsstrahlung Amplitudes for $e^+e^-$ Collisions With Polarized Beams at LEP / SLC Energies,” Z. Phys. C 33 (1987), 433

[4] S. Jadach, J. H. Kühn, R. G. Stuart and Z. Was, “QCD and QED Corrections to the Longitudinal Polarization Asymmetry,” Z. Phys. C 38 (1988), 609 [erratum: Z. Phys. C 45 (1990), 528]

[5] A. V. Smilga, “Quasiparadoxes of massless QED,” Comments Nucl. Part. Phys. 20 (1991) no.1+2, 69-84

[6] H. F. Contopanagos and M. B. Einhorn, “Physical consequences of mass singularities,” Phys. Lett. B 277 (1992), 345-352

[7] B. Falk and L. M. Sehgal, “Helicity flip bremsstrahlung: An Equivalent particle description with applications,” Phys. Lett. B 325 (1994), 509-516

[8] J. G. Körner, A. Pilaftsis and M. M. Tung, “One Loop QCD Mass effects in the production of polarized bottom and top quarks,” Z. Phys. C 63 (1994), 575-579
[9] S. Groote, J. G. Körner and M. M. Tung, “Polar angle dependence of the alignment polarization of quarks produced in $e^+e^-$ annihilation,” Z. Phys. C 74 (1997), 615-629

[10] S. Dittmaier and A. Kaiser, “Photonic and QCD radiative corrections to Higgs boson production in $\mu^+\mu^- \rightarrow f\bar{f}$,” Phys. Rev. D 65 (2002), 113003

[11] S. Groote, J. G. Körner and J. A. Leyva, “$O(\alpha_s)$ corrections to the polar angle dependence of the longitudinal spin-spin correlation asymmetry in $e^+e^- \rightarrow q\bar{q}$,” Eur. Phys. J. C 63 (2009), 391-406 [erratum: Eur. Phys. J. C 74 (2014), 2789]

[12] H. Poincaré, “Sur la dynamique de l’électron”, Rendiconti del Circolo matematico di Palermo 21 (1906) 129–176 (sent to the editor on July 23th, 1905)

[13] A. Borel, “Groupes linéaires algébriques”, Annals Math. 64 (1956) 20

[14] E. Inonu, E. P. Wigner, “On the contraction of groups and their representations”, Proc. Nat. Acad. Sci. USA 39 (1953) 510–524

[15] G. G. A. Bäuerle and E. A. de Kerf, “Lie Algebras, Part 1: Finite and infinite dimensional Lie algebras and Applications in Physics”, North Holland, New York 1990

[16] J. D. Jackson, “Classical Electrodynamics”, John Wiley & Sons, New York 1962

[17] N. Ida, “Engineering Electrodynamics”, Springer Verlag, New York 2000

[18] R. Saar and S. Groote, “Mass, zero mass and … nophysics,” Adv. Appl. Clifford Algebras 27 (2017) no.3, 2739-2768

[19] T. Choi and S. Y. Cho, “Spin operators and representations of the Poincaré group,” [arXiv:1807.06425 [physics.gen-ph]]