Baryons and the Chern-Simons Term in Holographic QCD with Three Flavors

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We study dynamical baryons in the holographic QCD model of Sakai and Sugimoto in the case of three flavors, with special interest in the construction of the Chern-Simons (CS) term. The baryon classical solution in this model is given by the BPST instanton, and we carry out the collective coordinate quantization of this solution. The CS term should give rise to a first-class constraint which selects baryon states with the correct spins. However, the original CS term written in terms of the CS 5-form does not do so. Considering this fact, we propose a new CS term that is gauge invariant and takes the form of an integral over a six-dimensional space whose boundary is the original five-dimensional spacetime of the holographic model. Collective coordinate quantization using our new CS term leads to the correct baryon states and their mass formula.

§1. Introduction

Among the various approaches to the holographic dual of large \(N_c\) QCD, the model proposed by Sakai and Sugimoto\(^1\),\(^2\) is, at present, one of the most successful, both theoretically and phenomenologically. This model with \(N_f\) massless quarks is constructed using the brane configuration of \(N_c\) D4-branes and \(N_f\) D8-branes in type IIA superstring theory. They analyzed the effective theory of D8-branes on the D4-brane background, which is a \(U(N_f)\) Yang-Mills (YM) theory with the Chern-Simons (CS) term on a curved five-dimensional background. They found that this model has a massless pion, corresponding to the Nambu-Goldstone boson of chiral symmetry breaking, and an infinite number of massive (axial-)vector mesons. It accurately reproduces various phenomenologically important parameters, such as the masses and the couplings of the mesons. Moreover, if we truncate all the massive modes, then the effective theory is the Skyrme model,\(^3\) with the Wess-Zumino-Witten (WZW) term,\(^4\),\(^5\) which is known as the effective theory of massless mesons.

The Sakai-Sugimoto model (SS-model) can also describe the baryon degrees of freedom. It has been argued that in the AdS/CFT correspondence, a baryon is identified as a D-brane wrapped around a sphere.\(^6\) In the SS-model, this D-brane, which is a D4-brane wrapped on \(S^4\) in the color D4-brane background, is realized as a soliton in the effective theory of the D8-brane, namely, the five-dimensional YM+CS theory. Therefore, when we quantize the collective coordinates of the instanton, the baryon spectra are expected to appear as in the case of the Skyrme model.\(^7\)

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In Ref. 8), explicit construction of the baryon solution in the YM+CS theory and its collective coordinate quantization were carried out in the \( N_f = 2 \) case with an approximation assuming a large 't Hooft coupling, \( \lambda \gg 1 \) (see also Refs. 9) and 10) for the construction of the solution). The baryon solution at a fixed time was found to be the BPST instanton solution\(^{11}\) whose size is of order \( \lambda^{-1/2} \), which is determined by energy balance: The curved color D4-brane background tends to shrink the instanton size, while the Coulomb self-energy from the CS term favors a larger instanton. Quantization of the collective coordinates, including the size of the instanton, leads to a baryon spectra that agree fairly well with experiments if we employ a suitable Kaluza-Klein mass scale \( M_{\text{KK}} \) of the theory.

The purpose of this paper is to extend the study of Ref. 8) to the case of three flavors, \( N_f = 3 \). In fact, this is not a simple problem. First, all the quarks are massless in the original SS-model, and we have to modify the model so that at least the strange quark possesses a mass. This is absolutely necessary for comparison of the model with experiments. Though there have appeared a number of proposals to generate quark and meson masses in the SS-model,\(^{12)–15)}\) actual calculations seem difficult at present. In this paper, we focus on another problem regarding the \( N_f = 3 \) SS-model, namely, the problem associated with the CS term. As mentioned above, the \( U(1) \) part of the CS term plays an important role in giving the instanton a non-vanishing size, even in the \( N_f = 2 \) case. Contrastingly, the \( SU(N_f) \) part of the CS term vanishes identically in the \( N_f = 2 \) case, and \( N_f = 3 \) is the first nontrivial case in which the non-abelian part of the CS term enters the analysis of the theory.

To explain the problem concerning the CS term in the \( N_f = 3 \) SS-model, let us recall the role of the WZW term in the quantization of the collective coordinates of the \( SU(3) \) rotation of the baryon solution in the \( N_f = 3 \) Skyrme model.\(^{16)–20)}\) (Note that the WZW term vanishes identically in the \( N_f = 2 \) case.) In this case, there arises a first-class constraint

\[
J_8 = \frac{N_c}{2\sqrt{3}}, \quad (1.1)
\]

where \( J_8 \) is the eighth-component of the charge of \( SU(3)_J \), whose first three components, \( J_1, J_2 \) and \( J_3 \) constitute the \( SU(2) \) group corresponding to spatial rotation, and the quantity on the RHS, \( N_c/(2\sqrt{3}) \), is from the WZW term. The constraint (1.1) selects the correct baryon states with spin 1/2 for the flavor octet and those with spin 3/2 for the decuplet from the \( SU(3)_J \) octet and decuplet, respectively, which also contain other states with incorrect spins.

In the SS-model, the CS term should play the role of the WZW term in the Skyrme model. (Recall that the WZW term reduces to the CS term in the low energy limit.\(^1)\)) However, in the collective coordinate quantization of the baryon solution in the SS-model with \( N_f = 3 \), the CS term originally proposed in Refs. 1) and 2) [given in (2.4) of §2] vanishes identically, as we see in §3. This implies the absurd result that the constraint in the SS-model is \( J_8 = 0 \) instead of (1.1).

To overcome this difficulty, we propose a new CS term for the SS-model [see Eq. (4.4)]. Our new CS term is strictly gauge invariant, in contrast to the original CS term given in Refs. 1) and 2), which is not invariant under “large” gauge transformations. However, to define our CS term, we need a fictitious sixth coordinate, just
as the WZW term requires a fifth coordinate. With our new CS term, we can carry out the collective coordinate quantization of the baryon solution and get the desired constraint (1.1). The two CS terms, (2.4) and (4.4), are naively the same if we use the relation tr $\mathcal{F}^3 = d\omega_5(A)$. The reason why the two CS terms lead to different results is that the BPST instanton solution requires two patches in order to describe it in the entire four-dimensional space, including both the origin and the infinity, and hence the space of the integration for (2.4) is not the only boundary of that for (4.4) (see Appendix C for details). In this paper, we introduce the sixth dimension for our CS term by hand. It would be interesting if the origin of this extra dimension was in the ten dimensions of IIA superstring theory, though this seems difficult to show as we discuss in §6.

This paper is organized as follows. In §2, we present our model, five-dimensional $U(N_f)$ YM+CS theory in a curved background, and we obtain the classical solution representing a baryon. We keep $N_f$ arbitrary in this section, and set $N_f = 3$ in §3 and after. In §3, we introduce the collective coordinates into the baryon solution and obtain their lagrangian for the case $N_f = 3$. There, we find that the original CS term does not have the desired effect. We also find that the WZW term obtained from this CS term in the low energy limit cannot reproduce the constraint (1.1). In §4, we propose a new CS term and show that it leads to the constraint (1.1). Then, in §5, we complete the collective coordinate quantization using our new CS term and obtain the baryon mass formula. We also make a brief comparison of this formula with experimental data, though we have to introduce a strange quark mass for more serious analyses. The final section (§6) is devoted to a summary and discussion. The appendices contain various technical details. In particular, in Appendices C and D, we present details concerning our new CS term.

§2. SS-model with $N_f$ flavors and a baryon solution

In this section, we present the action of the SS-model with $N_f$ flavors and obtain its classical solution representing a baryon. Although in this paper we are particularly interested in the case of three flavors, $N_f = 3$, we keep $N_f$ arbitrary in this section.

2.1. The action of the SS-model

We consider the effective theory of $N_f$ probe D8-branes in a background of $N_c$ D4-branes. Discarding the dependence on the $S^4$ around which the D8-branes are wrapped, this effective theory is a $U(N_f)$ gauge theory in the five-dimensional subspace of the world volume of the D8-branes. The $U(N_f)$ gauge field $A$, which is hermitian and corresponds to an open string with both ends attached to the D8-branes, is given by

$$ A = A_\mu dx^\mu + A_z dz,$$

where $\mu, \nu = 0, 1, 2, 3$ are four-dimensional Lorentz indices and $z$ is the coordinate of the fifth-dimension. The action of the theory consists of the Yang-Mills (YM) part,
$S_{\text{YM}}$, and the Chern-Simons (CS) part, $S_{\text{CS}}$,

$$ S = S_{\text{YM}} + S_{\text{CS}} , $$

with

$$ S_{\text{YM}}[A] = -\kappa \int d^4x dz \, \text{tr} \left[ \frac{1}{2} h(z) F_{\mu \nu}^2 + k(z) F_{\mu \nu}^2 \right] , $$

$$ S_{\text{CS}}[A] = \frac{N_c}{24\pi^2} \int_{M_5} \omega_5^{U(N_f)}(A) , $$

where $F = dA + iA^2$ is the field strength, and $\omega_5^{U(N_f)}(A)$ is the CS 5-form defined by

$$ \omega_5^{U(N_f)}(A) = \text{tr} \left( A F^2 - \frac{i}{2} A^3 F - \frac{1}{10} A^5 \right) . $$

In $S_{\text{YM}}$ (2.3), $\kappa$ is written in terms of the 't Hooft coupling $\lambda$ and the number of colors $N_c$ as

$$ \kappa = a \lambda N_c , \quad \left( a = \frac{1}{216\pi^3} \right) $$

and $h(z)$ and $k(z)$ are the warp factors, given by

$$ h(z) = (1 + z^2)^{-1/3} , \quad k(z) = 1 + z^2 . $$

The space of the integration in (2.4) [and also in (2.3)] is $M_5 = \mathbb{R} \times M_4$, where $\mathbb{R}$ corresponds to the time $t$ and $M_4$ to the coordinates $(x, z)$. Here, we adopt the original CS term, (2.4), considered in Refs. 1) and 2). Although we need to refine the definition of the CS term for proper quantization around the baryon solution, the present one, (2.4), is sufficient for obtaining classical solutions, because the equations of motion (EOM) are not affected by the redefinition of the CS term.

Let us decompose the $U(N_f)$ gauge field $A$ into the $SU(N_f)$ part $A$ and the $U(1)$ part $\hat{A}$ as

$$ A = A + \frac{1}{\sqrt{2N_f}} \hat{A} = A^a t_a + \frac{1}{\sqrt{2N_f}} \hat{A} , $$

where $t_a$ ($a = 1, 2, \cdots, N_f^2 - 1$) are the hermitian generators of $SU(N_f)$, normalized as

$$ \text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab} . $$

Using $A$ and $\hat{A}$, the actions $S_{\text{YM}}$ and $S_{\text{CS}}$ read

$$ S_{\text{YM}} = -\kappa \int d^4x dz \, \text{tr} \left[ \frac{1}{2} h(z) \hat{F}_{\mu \nu}^2 + k(z) \hat{F}_{\mu \nu}^2 \right] - \frac{1}{2} \kappa \int d^4x dz \, \text{tr} \left[ \frac{1}{2} h(z) F_{\mu \nu}^2 + k(z) F_{\mu \nu}^2 \right] , $$

$$ S_{\text{CS}} = \frac{N_c}{24\pi^2} \int \left[ \omega_5^{SU(N_f)}(A) + \frac{1}{\sqrt{2N_f}} \left( 3\hat{A} \text{tr} F^2 + \frac{1}{2} \hat{A} \hat{F}^2 \right) \right] . $$
with $M,N = 1,2,3,z$ and $\epsilon_{123z} = +1$. The genuine non-abelian part, $\omega^\text{SU}(N_f)_5(A)$, is missing in the $N_f = 2$ case.

2.2. Classical solution representing a baryon

In this subsection, we obtain the classical solution of the SS-model representing a baryon in the $1/\lambda$ expansion by assuming that the 't Hooft coupling $\lambda$ is sufficiently large. The number of flavors $N_f$ is kept arbitrary, i.e., not restricted to the $N_f = 3$ case. Our solution is an extension of the $N_f = 2$ solution given in Ref. 8) to generic $N_f$,\(^{\text{a)}}\) and it essentially represents the embedding of the $SU(2)$ BPST instanton solution into $SU(N_f)$. A nontrivial point in the $N_f \geq 3$ case is the appearance of the time component $A_0$ of the $SU(N_f)$ part of the gauge field, which is absent in the $N_f = 2$ case.

In order to carry out a systematic $1/\lambda$ expansion, we follow Ref. 8), rescaling the coordinates $x^M = (x,z)$ and the gauge field $A$ as

$$x^M \rightarrow \lambda^{-1/2}x^M, \quad x^0 \rightarrow x^0, \quad A_0 \rightarrow A_0, \quad A_M \rightarrow \lambda^{1/2}A_M,$$

$$F_{MN} \rightarrow \lambda F_{MN}, \quad F_{0M} \rightarrow \lambda^{1/2}F_{0M}.$$  \hspace{1cm} (2.12)

Note that $S_{CS}$ is invariant under this rescaling, while $S_{YM}$ is expanded as

$$S_{YM} = -aN_c \int d^4 x dz \left[ \frac{\lambda}{2} \hat{F}_{MN}^2 + \left( -\frac{z^2}{6} \hat{F}_{ij}^2 + z^2 \hat{F}_{iz}^2 - F_{0M}^2 \right) + O(\lambda^{-1}) \right]$$

$$- a \frac{N_c}{2} \int d^4 x dz \left[ \frac{\lambda}{2} \hat{F}_{MN}^2 + \left( -\frac{z^2}{6} \hat{F}_{ij}^2 + z^2 \hat{F}_{iz}^2 - \hat{F}_{0M}^2 \right) + O(\lambda^{-1}) \right],$$  \hspace{1cm} (2.13)

with $i,j = 1,2,3$. Here, we have used (2.6) for $\kappa$. From this action, the EOM is obtained as follows:

$$D_M F_{0M} + \frac{1}{64\pi^2 a} \sqrt{\frac{2}{N_f}} \epsilon_{MNPK} \hat{F}_{MN} F_{PQ}$$

$$+ \frac{1}{64\pi^2 a} \epsilon_{MNPK} \left\{ F_{MN} F_{PQ} - \frac{1}{N_f} \tr(F_{MN} F_{PQ}) \right\} + O(\lambda^{-1}) = 0,$$  \hspace{1cm} (2.14)

\(^{\text{a)}}\) The baryon solution in the $N_f = 2$ case is also analyzed in 9). Moreover, the energy (2.26) and the size (2.27) of the solution for a generic $N_f$ are obtained in 10).
\[ D_N F_{MN} + \mathcal{O}(\lambda^{-1}) = 0 , \]  
\[ \partial_M \hat{F}_{0M} + \frac{1}{64\pi^2 a} \sqrt{2} \epsilon_{MNPQ} \left\{ \text{tr}(F_{MN} F_{PQ}) + \frac{1}{2} \hat{F}_{MN} \hat{F}_{PQ} \right\} + \mathcal{O}(\lambda^{-1}) = 0 , \]  
\[ \partial_N \hat{F}_{MN} + \mathcal{O}(\lambda^{-1}) = 0 , \]

where (2.14) and (2.15) are the EOM for the SU\((N_f)\) part, and (2.16) and (2.17) are those for the U\((1)\) part.

Let us obtain the static soliton solution of the EOM (2.14)–(2.17) corresponding to a baryon. In this paper, we wish to construct a solution whose energy is correctly obtained to the next-to-leading order in the \(1/\lambda\) expansion. First, let us solve (2.15). For the purpose of the present paper, it is sufficient to consider the leading part, \(D_N F_{MN} = 0\), and the solution carrying a unit baryon number is obtained by embedding the SU\((2)\) BPST instanton solution \(^{11)}\) in the flat four-dimensional space into SU\((N_f)\):

\[ A_{\text{cl}}^M(x) = -i f(\xi) g(x) \partial_M g(x)^{-1} , \]

where \(f(\xi)\) and \(g(x)\) are given by\(^\ast\)

\[ f(\xi) = \frac{\xi^2}{\xi^2 + \rho^2} , \quad \xi = \sqrt{(x^M - X^M)^2} , \]

\[ g(x) = \begin{pmatrix} g^{SU(2)}(x) & 0 \\ 0 & 1_{N_f-2} \end{pmatrix} , \quad g^{SU(2)}(x) = \frac{1}{\xi} ((z - Z)1_2 + i(x^i - X^i)\tau_i) . \]

Here, \(1_N\) denotes the \(N \times N\) identity matrix, and \(\tau_i\) \((i = 1, 2, 3)\) are the Pauli matrices. The constants \(X^M = (X, Z)\) and \(\rho\) represent the position and the size of the instanton, respectively. Note that these constants are also rescaled as in (2.12). The field strengths of this solution are given by

\[ F_{\text{cl}}^{ij} = \frac{4\rho^2}{(\xi^2 + \rho^2)^2} \epsilon_{ijkl} t_k , \quad F_{\text{cl}}^{iz} = \frac{4\rho^2}{(\xi^2 + \rho^2)^2} t_i , \]

where \(t_i\) is the SU\((N_f)\) embedding of \(\tau_i\): \(t_i = \frac{1}{2} \begin{pmatrix} \tau_i & 0 \\ 0 & 0 \end{pmatrix} \).

The solutions to the U\((1)\) part of the EOM, (2.17) and (2.16), are the same as in the \(N_f = 2\) case.\(^8\) We have

\[ \hat{A}_{\text{cl}}^M = 0 , \]

and

\[ \hat{A}_{\text{cl}}^0 = -\sqrt{\frac{2}{N_f}} \frac{1}{8\pi^2 a} \frac{1}{\xi^2} \left( 1 - \frac{\rho^4}{(\xi^2 + \rho^2)^2} \right) . \]

This \(\hat{A}_{\text{cl}}^0\) has been chosen to be regular at the origin, \(\xi = 0\), and vanish at the infinity, i.e., for \(\xi \to \infty\).

\(^\ast\) We have chosen \(g^{SU(2)}(x)\) in (2.20) as the hermitian conjugate of \(g(x)\) appearing in Ref. 8) in order to make the corresponding \(A_{\text{cl}}^M\) (2.18) carry a unit baryon number, \(N_B = +1\) [see (3.29)].
Finally, let us solve (2.14) to obtain $A_0$. In the $N_f = 2$ case, the third term in (2.14) is missing, and the solution vanishing at $\xi = \infty$ is simply given by $A_0 = 0$. For arbitrary $N_f$, substituting (2.18) and (2.22) into (2.14), we have

$$D_M^2 A_0 - \frac{3}{2 \pi^2 a} \frac{\rho^4}{(\xi^2 + \rho^2)^2} \left( P_2 - \frac{2}{N_f} 1_{N_f} \right) = 0 ,$$

(2.24)

where the matrix $P_2$ is given by $P_2 = \text{diag}(1,1,0,\ldots,0)$. Equation (2.24) leads to the following nontrivial regular solution, which commutes with $A_0$ (2.18), vanishes at the infinity, and has the same $\xi$-dependence as (2.23):

$$A_0^{\text{cl}} = -\frac{1}{16\pi^2 a} \frac{1}{\xi^2} \left( 1 - \frac{\rho^4}{(\xi^2 + \rho^2)^2} \right) \left( P_2 - \frac{2}{N_f} 1_{N_f} \right) .$$

(2.25)

The mass $M$ of our static soliton solution is obtained by using the relation $S = -\int dt M$. Substituting the above solution into (2.13) and (2.11), we get

$$M = \kappa \int d^3 x d\tau \left[ \frac{1}{2} (F_{MN}^{\text{cl}})^2 - \lambda^{-1} \left( \frac{z^2}{6} (F_{ij}^{\text{cl}})^2 + z^2 (F_{12}^{\text{cl}})^2 - (F_{0M}^{\text{cl}})^2 \right) \right]
- \frac{\kappa}{2} \lambda^{-1} \int d^3 x d\tau (F_{0M}^{\text{cl}})^2
- \frac{\kappa}{24\pi^2 a} \lambda^{-1} \epsilon_{MNPOQ} \int d^3 x d\tau \left[ \sqrt{\frac{2}{N_f}} \frac{3}{8} \tilde{A}_0^{\text{cl}} \text{tr}(F_{MN}^{\text{cl}} F_{PQ}^{\text{cl}}) + \frac{3}{4} \text{tr}(A_0^{\text{cl}} F_{MN}^{\text{cl}} F_{PQ}^{\text{cl}}) \right]
+ \mathcal{O}(\lambda^{-1})
= 8\pi^2 \kappa \left[ 1 + \lambda^{-1} \left( \frac{\rho^2}{6} + \frac{1}{320\pi^4 a^2} \frac{1}{\rho^2} + \frac{Z^2}{3} \right) \right] + \mathcal{O}(\lambda^{-2}) .$$

(2.26)

The contributions from the two terms $\text{tr}(F_{0M}^{\text{cl}})^2$ and $\text{tr}(A_0^{\text{cl}} F_{MN}^{\text{cl}} F_{PQ}^{\text{cl}})$ are absent in the $N_f = 2$ case.\(^8\) It is interesting that the mass formula (2.26) is nonetheless independent of the number of flavors, $N_f$. The values of $\rho$ and $Z$ for the stable solution are determined by minimizing $M$:\(^9\)

$$\rho^2 = \frac{1}{8\pi^2 a} \sqrt{\frac{6}{5}} , \quad Z = 0 .$$

(2.27)

Note that the size of the instanton is also independent of $N_f$. If we express this in terms of the original variable [see (2.12)], $\rho^2$ is rescaled back as $\rho^2 \rightarrow \lambda \rho^2$. This implies that the size of our solution is of order $\lambda^{-1/2}$. Inserting (2.27) into (2.26), the mass of the soliton is found to be

$$M = 8\pi^2 \kappa + \sqrt{\frac{2}{15} N_c} .$$

(2.28)

The very small size, of order $\lambda^{-1/2}$, of the baryon solution implies that the higher-order derivative terms in the D-brane effective action, which have been omitted in

\(^8\) This is equivalent to solving the sub-leading part of the EOM (2.15) and (2.17) projected on to the subspace of deformations of the solution in the $\rho$ and $Z$ directions.
(2.3), might have important contributions, as mentioned in Ref. 8). However, we leave this issue for future study and continue analysis based on the YM action (2.3) in the rest of this paper.

§3. Necessity of modifying the CS term

Having constructed the baryon classical solution in §2, our next task is to carry out the quantization of the collective coordinates of this solution. However, as mentioned in the Introduction, there arises the problem that, in the \( N_f = 3 \) case, the constraint (1.1), necessary for selecting the baryon states with the correct spins, cannot be obtained from the CS term (2.4) of Refs. 1) and 2).

In this section, we first introduce the collective coordinates into our baryon classical solution (§3.1), obtain the lagrangian of the collective coordinates (§3.2), and then explain how the CS term (2.4) fails to give the constraint (1.1) (§3.3). We also show that the WZW term obtained as the low-energy limit of the CS term (2.4) cannot reproduce the constraint (1.1) either (§3.4). In the rest of the paper, we restrict ourselves to the three flavor case, \( N_f = 3 \).

3.1. Introducing the collective coordinates

We take the following moduli of the classical solution as the collective coordinates for quantization:

- \( SU(3) \) orientation \( W \in SU(3) \)
- Size of the instanton \( \rho \)
- Position of the instanton \( X^M = (X, Z) \)

More specifically, we analyze the quantum mechanical system consisting of the above three kinds of moduli promoted to time-dependent variables \( (W(t), X^M(t), \rho(t)) \). Note that \( \rho \) and \( Z \) are not genuine moduli, as seen from the fact that the mass (2.26) of the solution depends on them. However, as in the \( N_f = 2 \) case, the masses of the modes \( \rho \) and \( Z \) are much smaller than those of the other massive modes for large \( \lambda \). Therefore, we regard \( \rho \) and \( Z \) as collective coordinates, in addition to \( W \) and \( X \).

In order to derive the lagrangian of these collective modes, we approximate the slowly moving soliton by the static solution obtained in the last section with \( X^\alpha = (X^M, \rho) \) and the \( SU(3) \) orientation \( W \) made time dependent. Thus, the \( SU(3) \) gauge field is assumed to be of the form*)

\[
A_M(t, x) = W(t)A^\text{cl}_M(x; X^\alpha(t))W(t)^{-1}, \\
A_0(t, x) = W(t)A^\text{cl}_0(x; X^\alpha(t))W(t)^{-1} + \Delta A_0(t, x),
\]

(3.1)

where \( A^\text{cl}_M(x; X^\alpha(t)) \) is the BPST instanton solution (2.18) with time-dependent \( X^\alpha \). The \( U(1) \) parts of the gauge field, \( \tilde{A}_M(x, t) \) and \( \tilde{A}_0(x, t) \), are given simply by (2.22)

*) Here, we adopt a manner of introducing the collective coordinates of the \( SU(3) \) rotation that differs from that of Ref. 8). The gauge field (3.1) used in this paper and the corresponding one in 8), Eq. (4.2) there, (extended to the \( N_f = 3 \) case) are related through the gauge transformation in terms of \( Y(t, x) \), defined by \(-iY^{-1}\dot{Y} = \Delta A_0 \). The variable \( V \) in Ref. 8) and \( W \) in this paper are related by \( V(t, x) = Y(t, x)W(t) \).
and (2.23), respectively, with $X^\alpha$ made time dependent:

$$\hat{A}_M(x,t) = 0, \quad \hat{A}_0(x,t) = \hat{A}_0^\text{cl}(x; X^\alpha(t)) \quad (3.2)$$

The extra term $\Delta A_0(x,t)$ in (3.1) for $A_0$ is introduced so that the EOM of $A_0$, namely, the Gauss law constraint (2.14), is satisfied for the present gauge field with time-dependent moduli.\(^{1}\) Let us see how $\Delta A_0$ is determined. For $A(x,t)$ of (3.1), we find

$$F_{MN} = W(t)F_{MN}^\text{cl} W(t)^{-1} \quad (3.3)$$

$$F_{0M} = W(t) \left( \hat{X}^\alpha \frac{\partial}{\partial X^\alpha} A_0^\text{cl} \right) W(t)^{-1} \quad (3.4)$$

where $\Phi(t,x)$ is defined by

$$\Phi(t,x) = W(t)^{-1} \Delta A_0 W(t) - iW(t)^{-1}\hat{W}(t) \quad (3.5)$$

Then, (2.14) implies

$$D_M^\text{cl} \left( \hat{X}^N \frac{\partial}{\partial X^N} A_M^\text{cl} + \rho \frac{\partial}{\partial \rho} A_M^\text{cl} - D_M^\text{cl} \Phi \right) = 0 \quad (3.6)$$

and thus the problem of determining $\Delta A_0$ has been reduced to that of solving (3.6) for $\Phi$.

The solution to (3.6) is given by the sum of three terms, $\Phi = \Phi_X + \Phi_\rho + \Phi_{SU(3)}$, each of which depends on the time derivative of the corresponding collective coordinates. The determination of the solution $\Phi$ is explained in Appendix A of Ref. 8) in the case $N_f = 2$. In the present, $N_f = 3$ case, $\Phi_X$ and $\Phi_\rho$ are the same as in the $N_f = 2$ case, $\Phi_X = -\hat{X}^N A_N^\text{cl}$ and $\Phi_\rho = 0$, and we have only to solve $D_M^\text{cl} D_M^\text{cl} \Phi_{SU(3)} = 0$. The derivation of $\Phi_{SU(3)}$ is given in Appendix A, and there we find that $\Phi$ in the $N_f = 3$ case is

$$\Phi(t,x) = -\hat{X}^N(t) A_N^\text{cl}(x; X^\alpha(t)) + \chi^a(t) \Phi_a(x; X^\alpha(t)) \quad (3.7)$$

where the functions $\Phi_a(x; X^\alpha(t))$ ($a = 1, \ldots, 8$) are given by (A.14) in terms of $u^a(\xi)$ of (A.12), and the quantities $\chi^a(t)$ are arbitrary. In order to relate $\chi^a(t)$ to $W(t)$, we impose the condition\(^{**}\)

$$\Delta A_0(t,x) \to 0 \quad \text{as} \quad z \to +\infty. \quad (3.8)$$

\(^{1}\) The general principle for introducing time-dependent collective coordinates into a classical solution is to make sure that the EOM of the collective coordinates implies the field theory EOM. In gauge theories, this requirement is automatically satisfied, except for the EOM of $A_0$. For $A_0$, we have to add an extra term by hand to ensure its EOM. We thank S. Sugimoto, T. Sakai and S. Yamato for discussions on this matter.

\(^{**}\) The condition (3.8) with $z \to +\infty$ may seem strange. In fact, $\Delta A_0(t,x)$ of (3.1), and hence $A_0(t,x)$ itself, does not tend to zero in the limit $z \to -\infty$, since $g(x) \to \text{diag}(-1,-1,1) \neq \mathbf{1}_3$ in this limit. Equation (3.8) should be regarded as a consequence of the condition $\mathfrak{A}_0(t,x) \to 0$ ($\xi \to \infty$), requiring that the gauge field $\mathfrak{A}_0$ in the patch containing the infinity $\xi = \infty$ be regular there. (See Appendix C.1.)
Then, because we have $\Phi_a(x) \to t_a$ and $A^{cl}_M(x) \to 0$ as $z \to +\infty$, we obtain

$$\chi^a(t) = -2i \text{tr}(t_a W(t)^{-1} \dot{W}(t)) \quad (3.9)$$

Summarizing, we find that $F_{0M}$ is given by

$$F_{0M} = W(t) \left( \dot{X}^N F^{cl}_{MN} + \dot{\rho} \frac{\partial}{\partial \rho} A^{cl}_M - \chi^a D^{cl}_M \Phi_a - D^{cl}_M A^{cl}_0 \right) W(t)^{-1}, \quad (3.10)$$

where we have used $\frac{\partial}{\partial X^N} A^{cl}_M = -\partial_N A^{cl}_M$. The $SU(3)$ part of the gauge field 1-form $A(t, x)$, given in (3.1), is concisely expressed as

$$A(t, x) = (A^{cl}(x; X^\alpha(t)) + \Phi(t, x) dt)^{W(t)}, \quad (3.11)$$

where $A^V$ is the gauge transform of $A$ in terms of $V(t, x) \in SU(3)$:

$$A^V = V(A - id) V^{-1}. \quad (3.12)$$

Because the $U(1)$ part of $A(x, t)$, is simply given by (3.2), the formula (3.11) is extended to the entire $A = A + \hat{A}$ as

$$A(t, x) = (A^{cl}(x; X^\alpha(t)) + \Phi(t, x) dt)^{W(t)}. \quad (3.13)$$

3.2. Lagrangian of the collective coordinates

The lagrangian $L$ of the collective coordinates $X^\alpha(t) = (X(t), Z(t), \rho(t))$ and $W(t)$ is obtained as $S_{YM} + S_{CS} = \int dt L$ by substituting (3.3) and (3.10) into $S_{YM}$ (2.13):*

$$L = -M + aN_c \int d^3 x dz \text{tr} \left( F^{2}_{0M} - (F^{cl}_{0M})^2 \right) + L_{CS}$$

$$= -M + aN_c \int d^3 x dz \text{tr} \left( \dot{X}^N F^{cl}_{MN} + \dot{\rho} \frac{\partial}{\partial \rho} A^{cl}_M - \chi^a D^{cl}_M \Phi_a \right)^2 + L_{CS}, \quad (3.14)$$

where $L_{CS}$ is defined by

$$S_{CS}[A] - S_{CS}[A^{cl}] = \int dt L_{CS}. \quad (3.15)$$

Then, performing the integrations over $(x, z)$, we obtain

$$L = -M_0 + \frac{m_X}{2} \dot{X}^2 + L_Z + L_\rho + L_\rho W + L_{CS}, \quad (3.16)$$

where $L_Z$, $L_\rho$ and $L_\rho W$ are given by

$$L_Z = \frac{m_Z}{2} \left( \dot{Z}^2 - \omega^2 Z^2 \right), \quad (3.17)$$

*) In obtaining the last expression in (3.14), we have carried out integration-by-parts for the term $\text{tr}(O_M D^{cl}_M A^{cl}_0)$ with $O_M = X^\alpha (\partial / \partial X^\alpha) A^{cl}_M - D^{cl}_M \Phi$, changing it to $- \text{tr}(A^{cl}_0 D^{cl}_M O_M)$, which vanishes due to (3.6). The surface term can be dropped, because we have $A^{cl}_0 \sim 1/\xi^2$ and $O_M \sim 1/\xi^3$ as $\xi \to \infty$. 

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\[ L_\rho = \frac{m_\rho}{2} (\dot{\rho}^2 - \omega_\rho^2 \rho^2) - \frac{K}{m_\rho \rho^2}, \quad (3.18) \]

\[ L_{\rho W} = m_\rho \rho^2 \left( \frac{1}{8} \sum_{a=1}^{3} (\chi^a)^2 + \frac{1}{16} \sum_{a=4}^{7} (\chi^a)^2 \right), \]

\[ = 2 T_1(\rho) \sum_{a=1}^{3} \left[ \text{tr}(-iW^{-1} \dot{W} t_a) \right]^2 + 2 T_2(\rho) \sum_{a=4}^{7} \left[ \text{tr}(-iW^{-1} \dot{W} t_a) \right]^2, \quad (3.19) \]

with the various quantities defined as follows:

\[ M_0 = 8 \pi^2 \kappa, \quad (3.20) \]

\[ m_X = m_Z = \frac{m_\rho}{2} = 8 \pi^2 \kappa \lambda^{-1} = 8 \pi^2 a N_c, \quad (3.21) \]

\[ \omega_Z^2 = \frac{2}{3}, \quad \omega_\rho^2 = \frac{1}{6}, \quad (3.22) \]

\[ K = \frac{N_c m_\rho}{40 \pi^2 a} = \frac{2}{5} N_c^2, \quad (3.23) \]

\[ T_1(\rho) = \frac{1}{4} m_\rho \rho^2, \quad T_2(\rho) = \frac{1}{8} m_\rho \rho^2. \quad (3.24) \]

The expressions for \( M_0, m_X, m, \omega_Z, \omega_\rho, \) and \( Q \equiv K/m_\rho \) are the same as in the \( SU(2) \) case.\(^8\) The ratio of the moments of inertia, \( T_2(\rho)/T_1(\rho) = 1/2, \) follows from the powers 1 and 1/2 of \( f(\xi) \) in \( u^a(\xi) \) (A.12) for \( a = 1, 2, 3 \) and \( a = 4, \cdots, 7, \) respectively.

### 3.3. The CS term (2.4)

Let us evaluate the CS term (2.4) for the configuration (3.13) to determine the dependence on the collective coordinates \( W(t) \) and \( X^\alpha(t) \). Using the formulas for \( \omega_5 \) which are listed in Appendix B, we obtain (with the superscript \( U(3) \) on \( \omega_5 \) omitted for simplicity)

\[ \omega_5(\mathcal{A}) = \omega_5((\mathcal{A}^c + \Phi dt)^W) \]

\[ = \omega_5(\mathcal{A}^c + \Phi dt) + \frac{1}{10} \text{tr}(-iW^{-1} \dot{W} dt)^5 + d\alpha_4(-iW^{-1} \dot{W} dt, \mathcal{A}^c + \Phi dt) \]

\[ = \omega_5(\mathcal{A}^c) + 3 \text{tr}(\Phi dt (F^c)^2) + d\beta(\Phi dt, \mathcal{A}^c) + d\alpha_4(-iW^{-1} \dot{W} dt, \mathcal{A}^c) \]

\[ = \omega_5(\mathcal{A}^c) + 3 \text{tr}(\Phi dt (F^c)^2) + d\beta(\Phi dt, \mathcal{A}^c) + d\alpha_4(-iW^{-1} \dot{W} dt, \mathcal{A}^c), \quad (3.25) \]

where \( \beta \) and \( \alpha_4 \) are given in (B.4) and (B.2). In obtaining the last expression, we have used the fact that \( \tilde{\mathcal{A}}^c_M(x; \tilde{X}^\alpha(t)) = \hat{F}^c_M(x; \tilde{X}^\alpha(t)) = 0. \)

As mentioned in the Introduction, the dependences of the CS term (2.4) on the collective coordinates, and in particular on \( W(t) \), cancel out among the last three terms of (3.25). This is seen as follows. First, note that, in the term \( 3 \text{tr}(\Phi dt (F^c)^2) \), we have \( (F^c)^2 = \frac{1}{2} \mathcal{P}_2 \text{tr}(F^c)^2 \) and

\[ \text{tr}(\Phi \mathcal{P}_2) = \frac{1}{\sqrt{3}} \chi^8(t), \quad (3.26) \]
where we have used (3.7), (A.14) and
\[
P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{2}{\sqrt{3}} t_8 + \frac{2}{3} 1_3, \quad t_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\] (3.27)

Therefore, we obtain
\[
\frac{N_c}{24\pi^2} \int_{M_5=\mathbb{R} \times M_4} 3 \text{tr}(\Phi dt (F^{cl})^2) = \frac{N_c}{24\pi^2} \frac{\sqrt{3}}{2} \int dt \chi^8(t) \int_{M_4} \text{tr}(F^{cl})^2 = \frac{N_c}{2\sqrt{3}} \int dt \chi^8(t),
\] (3.28)

where we have used the fact that our classical solution has unit baryon number (instanton number):
\[
N_B = \frac{1}{8\pi^2} \int_{M_4} \text{tr}(F^{cl})^2 = 1.
\] (3.29)

The evaluations of \( \int d\beta \) and \( \int d\alpha_4 \) are similar and simpler. We have, using \((A^{cl})^3 \rightarrow (-igdg^{-1})^3 \propto P_2 \) and \( F^{cl}(x) \sim 1/\xi^4 \) as \( \xi \to \infty \),
\[
\frac{N_c}{24\pi^2} \int_{\mathbb{R} \times M_4} d\beta(\Phi dt, A^{cl}) = \frac{N_c}{24\pi^2} \int_{\mathbb{R} \times M_4} d\alpha_4(-iW^{-1}\bar{W} dt, A^{cl})
\]
\[
= \frac{N_c}{24\pi^2} \frac{i}{4\sqrt{3}} \int dt \chi^8(t) \int_{\partial M_4} \text{tr}(-igdg^{-1})^3 = -\frac{N_c}{4\sqrt{3}} \int dt \chi^8(t),
\] (3.30)

where we have used another expression of (3.29), namely,
\[
-\frac{i}{24\pi^2} \int_{S^3} (-igdg^{-1})^3 = \frac{1}{2} P_2.
\] (3.31)

From (3.28) and (3.30), we find that the sum of the contributions of the three terms in (3.25) cancels out as stated above:
\[
S_{CS}[A] = S_{CS}[A^{cl}],
\] (3.32)

Namely, \( L_{CS} (3.15) \) vanishes:
\[
L_{CS} = 0.
\] (3.33)

In the above calculation of \( L_{CS} \), the collective coordinate \( W(t) \) of the \( SU(3) \) rotation has been introduced as it appears in (3.1) and (3.13). However, because the original CS term (2.4) is not strictly a gauge invariant quantity [see (B.1)], it in fact depends on the manner in which the collective coordinates are introduced. For example, if we adopt the manner of Ref. 8) in terms of \( V(t, x) \), the CS term (2.4) becomes more involved. In this case, we can show that the terms linear in \( \chi^a(t) \) are missing from (2.4). On the other hand, if we adopt the configuration \( A(t, x) = A^{cl}(x; X^a(t)) + \Phi(t, x)dt \), which is gauge equivalent to (3.13), the \( d\alpha_4 \) term in (3.25) does not appear, and we get a non-vanishing \( L_{CS} \): \( L_{CS} = \frac{N_c}{4\sqrt{3}} \chi^8(t) \). However, this is half of the desired quantity, giving the constraint (1.1). We do not know whether there exists a gauge that reproduces the constraint (1.1). In any case, this gauge non-invariance is also an undesirable feature of the original CS term (2.4).
3.4. WZW term

In Ref. 1), it is shown that the Skyrme action including the WZW term is correctly reproduced as the low energy limit of the action (2.2) of holographic QCD. In particular, the WZW term comes from the CS term of (2.4) and is given by

\[
S_{WZW} = \frac{N_c}{240 \pi^2} \int_{\mathbb{R} \times M_4} \text{tr} L^5, \tag{3.34}
\]

where the left-current 1-form \( L \) is defined by

\[
L = -i U(t, x, z) dU(t, x, z)^{-1}, \tag{3.35}
\]

with \( U(t, x, z) = \text{P exp} \left( i \int_{-\infty}^{z} dz' A_z(t, x, z') \right) \).

In this WZW term, the coordinate \( z \) plays the role of the fifth dimension, with \( z = \infty \) corresponding to the real four-dimensional space-time, \((t, x)\).

In this subsection, we show that the WZW term (3.34) cannot give the desired constraint (1.1) either. This is, of course, consistent with the result of the last subsection.

Inserting (3.1) into (3.36), we have

\[
U(t, x, z) = W(t)U_{cl}(x, z)W(t)^{-1}, \tag{3.37}
\]

where \( U_{cl} \) is given, with \( A_z^{cl}(x) = \mathbf{x} \cdot \mathbf{\tau} / (\xi^2 + \rho^2) \), by

\[
U_{cl}(x, z) = \text{P exp} \left( i \int_{-\infty}^{z} dz' A_z^{cl}(x, z') \right) = \exp \left( i H(r, z) \mathbf{\hat{r}} \cdot \mathbf{\tau} \right), \tag{3.38}
\]

with

\[
H(r, z) = \frac{r}{\sqrt{r^2 + \rho^2}} \left( \arctan \frac{z}{\sqrt{r^2 + \rho^2}} + \frac{\pi}{2} \right). \tag{3.39}
\]

We omit the collective coordinates other than \( W(t) \) here for simplicity.

For \( U \) given in (3.37), we have

\[
L_0 = W \left[ U_{cl}(-iW^{-1}\dot{W})U_{cl}^{-1} + iW^{-1}\dot{W} \right] W^{-1},
\]

\[
L_M = W L_M^{cl} W^{-1}, \tag{3.40}
\]

and accordingly,

\[
\text{tr} L^5 = 5 \text{tr} \left( -iW^{-1}\dot{W} dt \left[ (R_M^{cl} dx^M) \right]^4 - (L_M^{cl} dx^M) \right). \tag{3.41}
\]

Here, \( L^{cl} \) and \( R^{cl} \) are given by (3.35), with \( U \) replaced by \( U_{cl} \) and \( U_{cl}^{-1} \), respectively. We can show generically that, for \( U_{cl} \) of the spherically symmetric form (3.38) with an arbitrary \( H(r, z) \) not restricted to (3.39), we have

\[
(R_M^{cl} dx^M)^4 = (L_M^{cl} dx^M)^4 = 0, \tag{3.42}
\]

and hence the WZW term of (3.34) vanishes.
§4. New CS term

As we saw in the last section, the CS term (2.4) cannot reproduce the constraint (1.1) necessary for selecting baryon states with the correct spins. Another potential problem concerning the CS term (2.4) is that it is not strictly a gauge-invariant quantity. Indeed, it is not invariant under “large” gauge transformations [see (B.1)]. Therefore, the physics resulting from its use can depend on the choice of the gauge.

To overcome these problems, here we propose another CS term for the holographic QCD (2.2). Its construction is largely parallel with that of the WZW term in the Skyrme model.\cite{4,5}

We introduce a new, fictitious sixth coordinate $s$, which takes values in the interval $[0,1]$, and consider a six-dimensional spacetime $M_6$ with the coordinates $(t, x^M, s) = (t, x, z, s)$ (see Fig. 1). The subspace corresponding to $s = 0$ is the boundary of $M_6$, and it is the original five-dimensional spacetime $M_5 = \mathbb{R} \times M_4$, where the YM action $S_{YM}$ (2.3) is defined. Accordingly, the gauge field on $M_6$ has an $s$ component, and it is now a function of the coordinates $(t, x, s) = (t, x, z, s)$:

$$A(t, x, s) = A_0(t, x, s)dt + A_M(t, x, s)dx^M + A_s(t, x, s)ds.$$  \hspace{1cm} (4.1)

Also, it must satisfy the condition

$$A(t, x, s = 0) = A(t, x). \quad \text{(except for the s-component, } A_s)$$  \hspace{1cm} (4.2)

Following the case of the WZW term,\cite{19} we take as the space $M_6$ in the baryon sector the direct product $M_6 = D_2 \times M_4$. Here, $D_2$ is the two-dimensional disc for $(t, s)$, and $M_4$ is for $(x, z)$. On $D_2$, $t$ is the angular coordinate and $s$ the radial coordinate, with $s = 0$ and $s = 1$ corresponding to the boundary and the center, respectively (see Fig. 2). More precisely, we regard the space of $t$ as $S^1$ by identifying $t = +\infty$ and $t = -\infty$. In this case, the gauge field on $M_6$ must respect the fact that $s = 1$ is a point on $D_2$ and satisfy conditions including

$$A(t, x, s = 1) = t\text{-indep.}$$  \hspace{1cm} (4.3)
With the above extension to the six-dimensional spacetime $M_6$, our new CS term is given by

$$S_{\text{CS}}^{\text{new}} = \frac{N_c}{24\pi^2} \int_{M_6} \text{tr} \, \mathcal{F}^3,$$

(4.4)

where $\mathcal{F}(A) = dA + iA^2$ is the field strength on $M_6$ also having the $s$ component. The ambiguity in the six-dimensional extension (4.4) is an integer times $2\pi$ and hence does not affect $\exp iS_{\text{CS}}^{\text{new}}$, as in the case of the WZW term.

Because we have

$$\text{tr} \, \mathcal{F}^3 = d\omega_5(A),$$

(4.5)

and $\partial M_6 = M_5$, our new CS term (4.4) may seem to be merely an equivalent rewriting of the original one, (2.4). This is indeed the case in the topologically trivial sector without baryons. In the baryon sector, however, due to the fact that we need two patches to express the BPST instanton on $M_4(\simeq S^4)$, $M_5$ is not the only boundary of $M_6$ for gauge non-invariant quantities, such as $\omega_5$. For this reason, our new CS term can differ from the original one in the sector with baryons. (The baryon configuration on $M_6$ given in this section is for the patch containing the origin, $\xi = 0$. See Appendix C for the construction of baryon configurations in both patches.)

For the collective coordinate quantization of baryons using our new CS term, we extend the gauge field (3.13) defined on $M_5$ to $M_6$ as

$$A(t, x, s) = (A^{\text{cl}}(x, s; X^\alpha(t, s)) + \Phi(t, x, s) dt + \Psi(t, x, s) ds) W(t, s).$$

(4.6)

Compared with (3.13), the various quantities, including $A^{\text{cl}}$ and the collective co-ordinates $(W, X^\alpha)$, are extended so as to depend also on $s$, and a new term, $\Psi ds$, has been added. These extensions should be carried out in such a manner that the conditions (4.2) and (4.3) are satisfied. The details of the extensions are described in Appendix C, and here we explain only the part necessary for the arguments in this section. First, the $s$ dependence of $A^{\text{cl}}(x, s)$ should be introduced only into the time component $A^{\text{cl}}_0$ in such a way that the following conditions are satisfied:

$$A^{\text{cl}}_0(x, s = 0) = A^{\text{cl}}_0(x), \quad A^{\text{cl}}_0(x, s = 1) = 0, \quad [A^{\text{cl}}_0(x, s), g(x)] = 0.$$

(4.7)

The $s$ dependence of $A^{\text{cl}}_0(x, s)$ can be quite arbitrary, as long as these conditions are satisfied (there is no EOM for $s \neq 0$), and the other components $A^{\text{cl}}_M(x)$ on $M_6$ should not have explicit $s$ dependence and should be the same as on $M_5$. The second condition in (4.7) ensures that the the Coulomb self-energy of the baryon solution given by $S_{\text{CS}}^{\text{new}}[A^{\text{cl}}]$ is the same as that from the original CS term, $S_{\text{CS}}[A^{\text{cl}}]$. (The term $S_{\text{CS}}^{\text{new}}[A^{\text{cl}}]$ reduces to the difference between $S_{\text{CS}}[A^{\text{cl}}]$ at $s = 0$ and $s = 1$, and the latter vanishes, due to the second condition of (4.7)). The third condition in (4.7), stating that $A^{\text{cl}}_0(x, s)$ is spanned by $1_3$ and $t_8$, will become necessary when we consider the two patches in Appendix C. Other $s$-dependent quantities appearing in (4.6) should, of course, coincide with the original ones on $M_5$ at $s = 0$:

$$W(t, s = 0) = W(t), \quad X^\alpha(t, s = 0) = X^\alpha(t), \quad \Phi(t, x, s = 0) = \Phi(t, x).$$

(4.8)
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We have to introduce the $\Psi ds$ term in (4.6) in order to make the $s$-component of the gauge field in the patch containing the infinity, $\xi = \infty$, regular and vanish there (see the end of Appendix C.2).

Let us calculate the new CS term, (4.4), for the baryon configuration with the collective coordinates given by (4.6). We find that the result is just what is necessary for reproducing the constraint (1.1). For this purpose, we first consider $F$ for $A + \delta A$ with $\delta A = \Phi dt + \Psi ds$, and expand it in powers of $\delta A$:

$$F(A + \delta A) = F(A) + D\delta A + i(\delta A)^2,$$

with $D\delta A = d\delta A + i(A\delta A + \delta AA)$. This leads to

$$\text{tr} F(A + \delta A)^3 = \text{tr} F(A)^3 + 3 d\text{tr} \left( A F(A)^2 + \delta A (D\delta A) F(A) \right),$$

where we have used the fact that $(\delta A)^3 = (D\delta A)(\delta A)^2 = DF = 0$ and $D^2 \delta A = i [F, \delta A]$. Using the gauge invariance of $S_{\text{CS}}^{\text{new}}$ and the formula (4.10), $S_{\text{CS}}^{\text{new}}$ for (4.6) is evaluated as follows:

$$S_{\text{CS}}^{\text{new}} [A] = (A^{\text{cl}} + \Phi dt + \Psi ds)^W - S_{\text{CS}}^{\text{new}} [A^{\text{cl}}]$$

$$= \frac{N_c}{24\pi^2} \int_{M_6} 3 d\text{tr} \left( A F^{\text{cl}})^2 + \delta A (D^{\text{cl}} \delta A) F^{\text{cl}} \right)$$

$$= \frac{N_c}{8\pi^2} \int_{M_5} \text{tr} (\Phi dt (F^{\text{cl}})^2) = \frac{N_c}{2\sqrt{3}} \int dt \chi^8(t),$$

where we have used the fact that, on $M_5$ with $s = 0$, we have $\delta A = \Phi dt$, $\delta A (D^{\text{cl}} \delta A) = 0$ and $\hat{F}^{\text{cl}}_{MN} = 0$. The last equality is merely (3.28). Equation (4.11) implies that $L_{\text{CS}}$ (3.15) for our new CS term (4.4) is

$$L_{\text{CS}} = \frac{N_c}{2\sqrt{3}} \chi^8(t) = \frac{N_c}{\sqrt{3}} \text{tr} (-iW(t)^{-1}W(t) t_8).$$

This $L_{\text{CS}}$ is the same as that appearing in the collective coordinate quantization of the $SU(3)$ Skyrme model,\cite{16,17,18,19} and it leads to the desired condition (1.1) (see the next section).

We should add a comment on the derivation of (4.11). We mentioned above that $M_5$ is not the only boundary of $M_6$ for gauge non-invariant quantities. Fortunately, $\text{tr}(\delta A (F^{\text{cl}})^2 + \delta A (D^{\text{cl}} \delta A) F^{\text{cl}})$ is gauge invariant, since every constituent, $\delta A$, $D^{\text{cl}} \delta A$ and $F^{\text{cl}}$, transforms covariantly under the gauge transformation. Therefore, we do not need to consider two patches for describing the instanton. By contrast, if we repeat the calculation of (4.11) by first using the formula (4.5), we do need two patches, since $\omega_5$ is not gauge invariant, and we obtain the same result as (4.11). Details of the calculation are given in Appendix C.

§5. Quantization of the collective coordinates

In §§3.1 and 3.2, we introduced the collective coordinates into the baryon solution and obtained their lagrangian, (3.16), except the last term $L_{\text{CS}}$, given in (3.15),
from the CS term. In this section, by adopting the new CS term, (4.4), and hence $L_{CS}$ given by (4.12), we complete the collective coordinate quantization, thereby obtaining the baryon spectra in the three-flavor model of holographic QCD.

5.1. Hamiltonian

Let us start with the lagrangian of the collective coordinates (3.16), with $L_{CS}$ given by (4.12). This lagrangian differs from the standard collective coordinate lagrangian of the $SU(3)$ Skyrme model in that there are $L_Z$ and $L_\rho$ terms and the moments of inertia, $I_1(\rho)$ and $I_2(\rho)$, depend on the dynamical variable $\rho$. However, the quantization is straightforward, and we obtain the following Hamiltonian of the system [omitting the center-of-mass coordinate, $X(t)$]:

$$H = M_0 + H_Z + H_\rho + H_\rho W \ ,$$

with

$$H_Z = -\frac{1}{2m_Z} \partial_\rho^2 + \frac{1}{2} m_Z \omega_\rho^2 Z^2 \ ,$$

$$H_\rho = -\frac{1}{2m_\rho} \frac{1}{\rho} \partial_\rho (\rho^n \partial_\rho) + \frac{1}{2} m_\rho \omega_\rho^2 \rho^2 + \frac{K}{m_\rho \rho^2} \ ,$$

$$H_\rho W = \frac{1}{2I_1(\rho)} \sum_{a=1}^{3} (J_a)^2 + \frac{1}{2I_2(\rho)} \sum_{a=4}^{7} (J_a)^2 \ .$$

This system must be supplemented with the constraint (1.1), as seen from the fact that $\chi^8$ appears only in $L_{CS}$, (4.12), in the lagrangian (3.16). Here, we have chosen the representation in which $Z$ and $\rho$ are diagonalized. In (5.4), $J_a$ is the charge of the right $SU(3)_I$ transformation applied to $W$:

$$[J_a, W] = W t_a, \quad [J_a, J_b] = i f_{abc} J_c .$$

The present system is invariant only under the $SU(2)$ subgroup of $SU(3)_I$, which is the group of rotations in the $x$-space spanned by $(J_1, J_2, J_3)$. In addition, our system is fully invariant under the $SU(3)_I$ flavor transformation. The charge $I_a$ of $SU(3)_I$ satisfies

$$[I_a, W] = -t_a W, \quad [I_a, I_b] = i f_{abc} I_c, \quad [I_a, J_b] = 0 \ .$$

Because the relation $I = -WJW^{-1}$ holds for $I = I_a t_a$ and $J = J_a t_a$, we have $\text{tr} I^2 = \text{tr} J^2$ and $\text{tr} I^3 = -\text{tr} J^3$.

The first term in (5.3) is chosen so that it is hermitian with respect to the inner product, $(f, g) = \int_0^\infty d\rho \rho^n f^*(\rho) g(\rho)$. In the $N_f = 2$ case studied in Ref. 8), we used $\eta = 3$, since we identified $\rho$ and $W$ as the radial coordinate and the orientation, respectively, of the part of the instanton moduli space $\mathbb{R}^4/\mathbb{Z}_2$ with line element $(\delta s)^2 = \rho^2 \frac{1}{2} \text{tr}(-iW^{-1} \delta W)^2 + (\delta \rho)^2$. In the $N_f = 3$ case considered presently, it is natural to set $\eta = 8$. However, we leave $\eta$ arbitrary until we compare our result for the baryon spectra with experimental data.
5.2. Baryon mass formula

Let us solve the Schrödinger equation for our collective coordinate system to obtain the spectra. First, we consider the hamiltonian \( H_{\rho} + H_{\rho W} \) by taking the \((p,q)\) representation for \( SU(3)_I \). For a state in this representation with spin \( j \), we have

\[
\sum_{a=1}^{8}(J_a)^2 = \sum_{a=1}^{8}(I_a)^2 = \frac{1}{3}(p^2 + q^2 + pq + 3(p + q)), \tag{5.7}
\]

\[
\sum_{a=1}^{3}(J_a)^2 = j(j + 1), \tag{5.8}
\]

and the \( \rho \) part of the hamiltonian, \( H_{\rho} + H_{\rho W} \), is

\[
H_{\rho}^{\text{tot}} = -\frac{1}{2m_{\rho}} \frac{1}{\rho^n} \partial_\rho (\rho^n \partial_\rho) + \frac{1}{2} m_{\rho} \omega_{\rho}^2 \rho^2 + \frac{K'}{m_{\rho} \rho^2}, \tag{5.9}
\]

where \( K' \) is the sum of \( K \) and the contribution from \( H_{\rho W} \):

\[
K' = \frac{N_c^2}{15} + \frac{4}{3} (p^2 + q^2 + pq + 3(p + q)) - 2j(j + 1). \tag{5.10}
\]

The first term, \( \frac{N_c^2}{15} \), is the sum of \( K = (2/5)N_c^2 \) and \(-N_c^2/3\), coming from \(-(J_8)^2/(2I_2(\rho))\), with \( J_8 \) given by (1.1).

Now we consider solving the Schrödinger equation

\[
H_{\rho}^{\text{tot}} \phi(\rho) = E_{\rho W} \phi(\rho). \tag{5.11}
\]

This equation is reduced using the form

\[
\phi(\rho) = e^{-z^2/2} z^\beta v(z), \tag{5.12}
\]

with

\[
z = m_{\rho} \omega_{\rho}^2, \quad \beta = \frac{1}{4} \left( \sqrt{(\eta - 1)^2 + 8K'} - (\eta - 1) \right), \tag{5.13}
\]

to a confluent hypergeometric differential equation for \( v(z) \):

\[
\left\{ z \frac{d^2}{dz^2} + \left( 2\beta + \frac{\eta + 1}{2} - z \right) \frac{d}{dz} + \left( \frac{E_{\rho W}}{2\omega_{\rho}} - \beta - \frac{\eta + 1}{4} \right) \right\} v(z) = 0. \tag{5.14}
\]

A normalizable regular solution to (5.14) exists only when \( E_{\rho W}/(2\omega_{\rho}) - \beta - (\eta+1)/4 = n_{\rho} = 0,1,2,3,\cdots \). Thus, the energy eigenvalues are given by

\[
E_{\rho W} = \omega_{\rho} \left( 2n_{\rho} + \frac{1}{2} \sqrt{(\eta - 1)^2 + 8K'} + 1 \right). \tag{5.15}
\]

The eigenvalues of the \( Z \) part of the hamiltonian, \( H_Z \), appearing in (5.2) are simply those of the harmonic oscillator:

\[
E_Z = \omega_Z \left( n_Z + \frac{1}{2} \right). \quad (n_Z = 0,1,2,3,\cdots) \tag{5.16}
\]
Adding (5.15) and (5.16), the baryon mass formula in the present model is finally obtained as
\[ M = M_0 + \sqrt{\frac{(\eta - 1)^2}{24}} + \frac{K'}{3} + \sqrt{\frac{2}{3}}(n_\rho + n_Z + 1). \] (5.17)

In the above arguments, \( N_c \) was arbitrary and we did not impose the constraint (1.1) on the states specified by \((p, q)\) and \(j\). Setting \( N_c = 3 \), the constraint (1.1),
\[ J_8 = \frac{\sqrt{3}}{2}, \] (5.18)
implies that \((p, q)\) must satisfy
\[ p + 2q = 3 \times \text{(integer)}. \] (5.19)

The allowed states with smaller \((p, q)\) satisfying the constraints (5.19) and (5.18), along with their \(K'\) values, are as follows:
\[
\begin{align*}
(p, q) &= (1, 1), \quad j = \frac{1}{2}, \quad K' = \frac{111}{10}, \hspace{1cm} \text{(octet)} \\
(p, q) &= (3, 0), \quad j = \frac{3}{2}, \quad K' = \frac{171}{10}, \hspace{1cm} \text{(decuplet)} \\
(p, q) &= (0, 3), \quad j = \frac{1}{2}, \quad K' = \frac{231}{10}, \hspace{1cm} \text{(anti-decuplet)} 
\end{align*}
\] (5.20)

5.3. Comparison with experimental data

The presently studied three-flavor holographic QCD model is not realistic, as all the quarks are massless. Thus, it does not make much sense to seriously compare the obtained baryon spectrum (5.17) with experimental data unless we at least add a strange quark mass to break the \(SU(3)_I\) symmetry. [See Refs. 12–15 for attempts to introduce quark masses in the SS-model.] Below, we carry out a comparison of our baryon mass formula (5.17) with the observed spectra of baryons. However, we keep our analysis very short for this reason.

From (5.17) with \( \eta = 8 \), we find that the difference between the masses of the octet and the decuplet baryons with the same \((n_\rho, n_Z)\), and that between the octet and the anti-decuplet are given in units of \( M_{KK} \) as follows:
\[
\begin{align*}
M_{10} - M_8 &= 0.386, \\
M_{10}^* - M_8 &= 0.725. 
\end{align*}
\] (5.21)

The value of \( M_{10} - M_8 \) is much smaller (by nearly 64%) than the corresponding value \( (M_{l=3} - M_{l=1} = 0.600) \) in the \( N_f = 2 \) case.\(^8\) Therefore, the value of \( M_{KK} \) favored by the experimentally observed value, \( M_{10}^{\text{exp}} - M_8^{\text{exp}} = (1232 - 940) \text{ MeV} = 292 \text{ MeV} \), of low-lying non-strange baryons is
\[ M_{KK} = 756 \text{ MeV}. \] (5.23)

This is smaller than the value \( M_{KK} = 949 \text{ MeV}, \) determined from the \( \rho \) meson mass,\(^1,2\) but it is larger than \( M_{KK} \simeq 500 \text{ MeV}, \) obtained in the \( N_f = 2 \) case.\(^8\) The
dependence of the mass formula (5.17) on \((n_\rho, n_Z)\) is the same as in the \(N_f = 2\) case [see Eq. (5.31) of Ref. 8)]. Therefore, (5.17) with \(M_{KK}\) given by (5.23) predicts larger masses for the excited baryon states than in Ref. 8), though the agreement with the experimental data is not so bad. Finally, adopting the value (5.23) for \(M_{KK}\), Eq. (5.22) for the anti-decuplet predicts
\[
M_{10^*} - M_8 = 548 \text{ MeV}.
\] (5.24)
This is close to the experimental value, \(M_{10^{*\exp}} - M_8^{\exp} = (1530 - 940) \text{ MeV} = 590 \text{ MeV},\) obtained using the reported \(\Theta^+\) mass of 1530 MeV.\(^{21}\) Of course, we cannot take this result seriously, due to the lack of a strange quark mass in our model.

§6. Summary and discussion

In this paper, we studied baryons in the SS-model with three flavors. The baryon solution is given by an \(SU(3)\) embedding of the BPST instanton solution with a small size, of order \(\lambda^{-1/2}\), and we carried out the collective coordinate quantization of the baryon solution. Although our analysis is largely parallel to that previously given for the two flavor case,\(^8\) the three flavor case is the first nontrivial case in which the non-abelian part of the CS term should play a critical role of giving a constraint that selects baryons with the correct spins. We found that the original CS term, (2.4), given in terms of the CS 5-form does not have the necessary effect, and we proposed another CS term, (4.4), by introducing a fictitious sixth coordinate, \(s\). These two CS terms naively appear to be equivalent, but they are different in the baryon sector, which cannot be described only by one patch. In fact, we found that our new CS term leads to the desired constraint. Using our new CS term, we completed the collective coordinate quantization and obtained the baryon mass formula (5.17). We found that the \(N-\Delta\) mass difference favors a value of \(M_{KK}\) that is larger than that in the \(SU(2)\) case\(^8\) but smaller than that determined from the \(\rho\) meson mass.\(^1,2\) Of course, serious comparison of our mass formula with experimental data is meaningless, since all the quarks are massless in the present model.

We finish this paper by discussing some remaining problems regarding the three-flavor SS-model, especially concerning the CS term. First is the origin of the sixth coordinate, \(s\), used for expressing our CS term, given in (4.4). In this paper, the coordinate \(s\) was introduced by hand, like the fifth coordinate in the WZW term. However, recall that the original CS term, (2.4), was obtained from the coupling
\[
S_{CS}^{D8} = \frac{1}{48\pi^3} \int_{D8} C_3 \text{ tr } \mathcal{F}^3 ,
\] (6.1)
where the integration is over the D8-brane, and \(C_3\) is the RR 3-form of the D4-brane background. The integral in Eq. (6.1) vanishes identically if we consider only \(A_0\) and \(A_M\) \((M = 1, 2, 3, z)\) on D8, depending only on \((t, x^M)\). For this reason, the authors of Ref. 1) adopted (2.4), which is obtained from (6.1) by carrying out integration-by-parts, using the formula (4.5), discarding the surface term, and then using \(1/(2\pi) \int_{S^4} dC_3 = N_c\). It would be interesting if we could directly relate our
CS term (4.4) with (6.1) and find a “physical origin” of the sixth coordinate \( s \). We cannot, however, adopt (6.1) itself instead of our (4.4) for a number of reasons. For example, if we allow a gauge field component other than \( A_0 \) and \( A_M \) for (6.1), it must also be contained in the YM action \( S_{YM} \).

The second problem regards the reproducibility of the chiral anomaly in QCD in the presence of the background gauge field defined by \( A_{L/R}(t, x) = \lim_{z \to +\infty/-\infty} A(t, x, z) \). The chiral anomaly is correctly reproduced from the original CS term, (2.4), using the gauge transformation property (B.1) of \( \omega_5(A) \).\(^1\) By contrast, if we adopt our new CS term \( S_{CS}^{\text{new}} \), given in (4.4), it seems that no anomaly arises, since (4.4) is strictly gauge invariant. A quick remedy to this problem is to add to \( S_{CS}^{\text{new}} \) the boundary term

\[
\Delta S_{CS} = -\frac{N_c}{24\pi^2} \left( \int_{Z_+} - \int_{Z_-} \right) \omega_5(A) ,
\]

where the integration region \( Z_{\pm} \) is the \( z = \pm \infty \) boundary of \( M_6 \). Note that \( \Delta S_{CS} \) vanishes in the absence of the background gauge fields \( A_{L/R} \), because we have \( Z_+ = Z_- \) in this case. The modified CS term \( S_{CS}^{\text{new}} + \Delta S_{CS} \) reproduces the chiral anomaly and the WZW term with the background gauge fields \( A_{L/R} \), at least in the sector without baryons. It is desirable to find a more concise definition of the CS term that gives both the constraint (1.1) and the anomaly.

Finally, for serious comparison of our results, in particular, the baryon mass formula (5.17), with experiments, we have to redo the analysis by introducing a strange quark mass. This is the most important task concerning the three flavor SS-model.

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**Appendix A**

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**Determination of \( \Phi(t, x) \)**

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In this appendix, we solve (3.6) to obtain \( \Phi(3.7) \) in the \( SU(3) \) case. We essentially follow Appendix A of Ref. 8). Let us decompose \( \Phi \) into three parts, each of which depends on the time derivative of one of the three kinds of collective coordinates:

\[
\Phi = \Phi_X + \Phi_\rho + \Phi_{SU(3)} .
\]
Then, (3-6) reduces to the following three equations:

\[ D_M^{\text{cl}} \left( \hat{X}^N \frac{\partial}{\partial X^N} A_M^{\text{cl}} - D_M^{\text{cl}} \Phi_X \right) = 0, \]  
\[ D_M^{\text{cl}} \left( \hat{\rho} \frac{\partial}{\partial \rho} A_M^{\text{cl}} - D_M^{\text{cl}} \Phi_\rho \right) = 0, \]  
\[ D_M^{\text{cl}} D_M^{\text{cl}} \Phi_{SU(3)} = 0. \]

The solutions to Eqs. (A-2) and (A-3) are the same as in the $SU(2)$ case of Ref. 8):

\[ \Phi_X = -\hat{X}^M A_M^{\text{cl}}, \quad \Phi_\rho = 0. \]

To solve (A-4), it is convenient to work in the singular gauge, namely, the gauge in which the BPST solution is singular at the origin but regular at the infinity. Let us represent each quantity in the singular gauge by the symbol used for the corresponding one in the regular gauge, with an overline. The BPST solution in the singular gauge is related to (2.18) in the regular gauge through the gauge transformation in terms of $g(x)^{-1}$:

\[ \overline{A}_M^{\text{cl}}(x) = g(x)^{-1} \left( A_M^{\text{cl}}(x) - i \partial_M \right) g(x) = -i \left( 1 - f(\xi) \right) g(x)^{-1} \partial_M g(x). \]

Because $\Phi_{SU(3)}$ transforms covariantly under the gauge transformation, we have

\[ \overline{\Phi}_{SU(3)}(t, x) = g(x; X(t))^{-1} \Phi_{SU(3)}(t, x) g(x; X(t)), \]

and Eq. (A-4) in the singular gauge is

\[ D_M^{\text{cl}} D_M^{\text{cl}} \Phi_{SU(3)} = 0. \]

This equation can be reduced by assuming the form

\[ \overline{\Phi}_{SU(3)} = u^a(\xi) \xi \]

and using the properties $\partial_M \overline{A}_M^{\text{cl}} = 0$ and $(x - X)^M \overline{A}_M^{\text{cl}} = 0$, to the following differential equation for each $u^a(\xi)$:

\[ \frac{1}{\xi^3} \frac{d}{d\xi} \left( \xi^3 \frac{d}{d\xi} u^a(\xi) \right) = C_a \frac{(1 - f(\xi))^2}{\xi^2} u^a(\xi). \]

Here, $C_a$ is defined in terms of the structure constant $f_{abc}$ of $SU(3)$ by $4 \sum_{c=1}^3 \sum_{d=1}^8 f_{acd} f_{bcd} = \delta_{ab} C_a$, and it is given explicitly by

\[ C_a = \begin{cases} 8, & (a = 1, 2, 3) \\ 3, & (a = 4, 5, 6, 7) \\ 0, & (a = 8) \end{cases} \]
The solution to (A.10) regular at $\xi = 0$ is

$$u^a(\xi) = \begin{cases} 
  f(\xi), & (a = 1, 2, 3) \\
  f(\xi)^{1/2}, & (a = 4, 5, 6, 7) \\
  1, & (a = 8)
\end{cases}$$

(A.12)

to a multiplicative constant for each $u^a$. Returning to the regular gauge, we find that the general solution to (A.4) is

$$\Phi_{SU(3)}(t, x) = \chi^a(t) \Phi_a(x; X^\alpha(t)),$$

(A.13)

with $\Phi_a$ given by

$$\Phi_a(x; X^\alpha(t)) = u^a(\xi) g(x; X(t)) t^a g(x; X(t))^{-1},$$

(A.14)

and the quantities $\chi^a(t)$ being arbitrary functions of $t$ only. Note that $X^\alpha = (X^M, \rho)$ in $u^a(\xi)$ is also made time dependent.

If we had solved (A.4) in the regular gauge by assuming (A.9) for $\Phi_{SU(3)}$, we would have obtained (A.10) with $1 - f$ replaced by $f$. However, its solutions are divergent either at $\xi = 0$ or at $\xi = \infty$.

**Appendix B**

**Formulas Involving $\omega_5$**

Here, we present formulas involving $\omega_5(A)$, given in (2.5) (where the gauge group can be arbitrary). First, under the gauge transformation, $A \rightarrow A^V = V(A - id) V^{-1}$, we have

$$\omega_5(A^V) = \omega_5(A) + \frac{1}{10} \text{tr} L^5 + d\alpha_4(L, A),$$

(B.1)

with $\alpha(L, A)$ defined by

$$\alpha_4(L, A) = \frac{1}{2} \text{tr} \left[ L (AF + FA - iA^3) + \frac{i}{2} LAL - iL^3 A \right]. \quad (L = -iV^{-1}dV)$$

(B.2)

Second, the change of $\omega_5(A)$ under an arbitrary infinitesimal deformation, $A \rightarrow A + \delta A$, is

$$\omega_5(A + \delta A) = \omega_5(A) + 3 \text{tr}(\delta A F^2) + d\beta(\delta A, A) + O((\delta A)^2),$$

(B.3)

where $\beta(\delta A, A)$ is

$$\beta(\delta A, A) = \text{tr} \left[ \delta A \left( FA + A F - \frac{i}{2} A^3 \right) \right].$$

(B.4)

**Appendix C**

**Another Derivation of (4.11)**

In this appendix, we present another way of deriving the result given in (4.11). Here, we reduce (4.4) to surface integrations by using (4.5), taking into account the
fact that $M_5$ is not the unique boundary of $M_6$ for $\omega_5(\mathcal{A})$. For this purpose, we first define the gauge fields on the two patches in $M_5$ (see Appendix C.1) and in $M_6$ (see Appendix C.2). Another derivation of (4.11) is given in Appendix C.3.

C.1. Baryon configurations on the two patches in $M_5$

First, we need two patches for describing the baryon solution (BPST solution) in the entire $M_4 (\simeq S^4)$ including both the origin $\xi = 0$ and the infinity $\xi = \infty$. Let $M_4^{(0)}$ and $M_4^{(\infty)}$ be the patches containing the origin and the infinity, respectively, separated by the boundary $B$: $M_4 = M_4^{(0)} + M_4^{(\infty)}$ and $\partial M_4^{(0)} = -\partial M_4^{(\infty)} = B$ (see Fig. 3).

In the patch $M_4^{(0)}$, we adopt the BPST solution $A_{A_M}^{(0)} (2.18)$, while in the other patch $M_4^{(\infty)}$ we use $\tilde{\Phi}_{A_M}^{(0)}$ (A-6) in the singular gauge. These two are related via the $SU(2)$ gauge transformation in terms of $g(x)^{-1}$. The time components of the solution, $\tilde{A}_{A_M}^{(0)} (2.23)$ and $A_{A_M}^{(0)} (2.25)$, are the same for the two patches, since they are $SU(2)$ invariant, and they are indeed regular both at the origin and at the infinity. Summarizing, the $U(3)$ classical solutions, $\mathcal{A}^{(0)}(x)$ in $M_4^{(0)}$ and $\tilde{\mathcal{A}}^{(0)}(x)$ in $M_4^{(\infty)}$, are related as a whole via the gauge transformation in terms of $g(x)^{-1}$:

$$\tilde{\mathcal{A}}^{(0)}(x) = (\mathcal{A}^{(0)}) g(x)^{-1} (x) = g(x)^{-1} (\mathcal{A}^{(0)}(x) - id) g(x) \ , \quad (C.1)$$

The baryon configuration $\mathcal{A}(t, x)$ containing the collective coordinates and defined on $M_5^{(0)} = \mathbb{R} \times M_4^{(0)}$ is given by (3.13). As seen from the arguments in Appendix A, the corresponding one, $\tilde{\mathcal{A}}(t, x)$, on $M_5^{(\infty)} = \mathbb{R} \times M_4^{(\infty)}$ is given by\(^{\star}\)

$$\tilde{\mathcal{A}}(t, x) = A^{W(t)} g(x; X(t))^{-1} (t, x) = (\tilde{\mathcal{A}}^{(0)}(x; X^\alpha(t)) + \tilde{\Phi}(t, x) dt)^{W(t)} \ , \quad (C.2)$$

with

$$\tilde{\Phi}(t, x) = g(x; X(t))^{-1} (\Phi(t, x) - i \partial_0) g(x; X(t))$$

$$= -\dot{X} (t) \tilde{\mathcal{A}}^{(0)}(x; X^\alpha(t)) + \sum_{a=1}^{8} \chi^a(t) u^a(\xi) t_a \ . \quad (C.3)$$

\(^{\star}\) Note that the gauge transformation $\mathcal{A}^V = V (A - id) V^{-1}$ on $\mathcal{A}$ has the property $\mathcal{A}^{V_1 V_2} = (\mathcal{A}^{V_2})^{V_1}$. 

---

Fig. 3. The space $M_4 (\simeq S^4)$ for $x^M = (x, z)$ in the baryon sector consists of two patches, $M_4^{(0)}$ and $M_4^{(\infty)}$, which are separated by the boundary $B$. 

$\xi = 0$

$\xi = \infty$
Note that all the components of $\mathbf{A}(t, x)$ vanish sufficiently fast as $\xi \to \infty$. In particular, we have $A_0(t, x) = O(1/\xi^2)$ as $\xi \to \infty$ due to the fact that $u^a(\xi) = 1 + O(1/\xi^2)$ for all $a$. Therefore, our $\mathbf{A}(t, x)$ is indeed well-defined on $M_4^{(\infty)}$ containing the infinity.

C.2. Baryon configurations on $M_6$

Next, we must extend the baryon configurations on $M_5 = M_5^{(0)} + M_5^{(\infty)}$ to $M_6 = D_2 \times M_4 = M_6^{(0)} + M_6^{(\infty)}$, with $M_6^{(0/\infty)} = D_2 \times M_4^{(0/\infty)}$, for our new CS term (4.4). Recall that $D_2$ is the disc with angular coordinate $t$ and radial coordinate $s$, and $s = 0$ and $s = 1$ correspond to the circumference and the center of the disc, respectively (Fig. 2).

The baryon configuration $\mathcal{A}(t, x, s)$ (4.1) on $M_6^{(0)} = D_2 \times M_4^{(0)}$ must satisfy the condition (4.2) at $s = 0$. In addition, it must respect the fact that $s = 1$ is a point on $D_2$ and satisfy several conditions, including (4.3). Explicitly, $\mathcal{A}(t, x, s)$ is given by (4.4) in terms of the $s$-dependent collective coordinates $(W(t, x), X^\alpha(t, s))$, as well as $\Phi(t, x, s)$ and $\Psi(t, x, s)$, which satisfy the conditions (4.8) at $s = 0$ and the following ones at $s = 1$, necessary for $s = 1$ to be a point on $D_2$:

$$O(t, x, s = 1) = t\text{-indep.}, \quad \partial_s O(t, x, s)|_{s=1} = 0. \quad (O = \mathcal{W}, \mathcal{X}^\alpha, \Phi \Psi) \quad \text{(C.4)}$$

As explained below (4.6), the classical configuration $\mathcal{A}^{\text{cl}}$ in (4.1) is given by $\mathcal{A}^{\text{cl}}(x, s) = A_0^{\text{cl}}(x, s)dt + A_M^{\text{cl}}(x)dx^M$, with the $s$-dependent $A_0^{\text{cl}}(x, s)$ satisfying the condition (4.7).

The baryon configuration $\mathbf{A}(t, x, s)$ on the other patch, $M_6^{(\infty)} = D_2 \times M_4^{(\infty)}$, which is an extension of (C.2), is given by

$$\mathbf{A}(t, x, s) = \mathbf{A}^{W(t,s)g(x;X(t,s))^{-1}W(t,s)^{-1}}(t, x, s) = (\mathbf{A}^{\text{cl}}(x, s; X^\alpha(t, s))) + \Phi(t, x, s)dt + \Psi(t, x, s)ds \mathcal{W}(t, s). \quad \text{(C.5)}$$

This extension should satisfy

$$\mathbf{A}(t, x, s = 0) = \mathbf{A}(t, x), \quad \mathbf{A}(t, x, s = 1) = t\text{-indep.}, \quad \mathbf{A}(t, x, s) \to 0 \quad \text{as} \quad \xi \to \infty. \quad \text{(C.6)}$$

The precise meaning of the third condition is that $\mathbf{A}$ tends to zero faster than $O(1/\xi)$. The classical configuration $\mathbf{A}^{\text{cl}}$ in (C.5) is given by $\mathbf{A}^{\text{cl}}(x, s) = A_0^{\text{cl}}(x, s)dt + \mathbf{A}_M^{\text{cl}}(x)dx^M$, in terms of the same $A_0^{\text{cl}}(x, s)$ as in $\mathcal{A}^{\text{cl}}(x, s)$ on $M_6^{(0)}$. Owing to the third condition in (4.7), Eq. (C.1) continues to hold on $M_6$:

$$\mathbf{A}^{\text{cl}}(x, s) = (\mathcal{A}^{\text{cl}})^{g(x)^{-1}}(x, s) = g(x)(\mathcal{A}^{\text{cl}}(x, s) - id)g(x)^{-1}. \quad \text{(C.7)}$$

Note that the following relations hold:

$$\Phi(t, x, s) = g(x; X(t, s))^{-1}(\Phi(t, x, s) - i\partial_0)g(x; X(t, s)),$$
$$\Psi(t, x, s) = g(x; X(t, s))^{-1}(\Psi(t, x, s) - i\partial_s)g(x; X(t, s)). \quad \text{(C.8)}$$

Our $S_{\text{CS}}^{\text{new}}$ is independent of the details of the manner in which we extend the various quantities into $M_6$. In particular, $\Phi(t, x, s)$ and $\Psi(t, x, s)$ for $s \neq 0$ are subject...
to no restrictions of the Gauss law constraint, and hence they are not uniquely determined. An example of \( \Phi(t, x, s) \) and \( \Psi(t, x, s) \) is

\[
\Phi(t, x, s) = -\dot{X}^N(t, s) A_N^c(x; X^\alpha(t, s)) \\
- 2i \sum_{a=1}^{8} u^a(\xi) \text{tr} \left[ t_a W(t, s)^{-1} \partial_0 W(t, s) \right] g(x; X(t, s)) t_a g(x; X(t, s))^{-1},
\]

(C.9)

\[
\Psi(t, x, s) = -\partial_s X^N(t, s) A_N^c(x; X^\alpha(t, s)) \\
- 2i \sum_{a=1}^{8} u^a(\xi) \text{tr} \left[ t_a W(t, s)^{-1} \partial_s W(t, s) \right] g(x; X(t, s)) t_a g(x; X(t, s))^{-1}.
\]

(C.10)

The corresponding \( \overline{\Phi}(t, x, s) \) and \( \overline{\Psi}(t, x, s) \) are obtained from (C.9) and (C.10), respectively, by replacing \( A_N^c \) with \( \overline{A}_N^c \) and removing \( g(x; X(t, s)) \). Note that we have \( \overline{A}_s(t, x, s) = O(1/\xi^2) \) as \( \xi \to \infty \) for the present \( \overline{\Psi} \).

C.3. Rederivation of (4.11)

Having finished the preparation, let us turn to the evaluation of \( S_{\text{CS}}^{\text{new}} \) given in (4.4) by reducing it to surface integrations. Taking \( \mathcal{A}(t, x, s) \) (4.6) and \( \overline{\mathcal{A}}(t, x, s) \) (C.5) as the gauge field on \( M_6^{(0)} \) and \( M_6^{(\infty)} \), respectively, and using the relations \( \partial M_6^{(0)} = M_5^{(0)} + D_2 \times B \) and \( \partial M_6^{(\infty)} = M_5^{(\infty)} - D_2 \times B \), we obtain

\[
\int_{M_6} \text{tr} \left( \mathcal{F}^3 - (\mathcal{F}^c)^3 \right) = \int_{M_5^{(0)}} \left( \omega_5(\mathcal{A}) - \omega_5(\mathcal{A}^c) \right) + \int_{M_5^{(\infty)}} \left( \omega_5(\overline{\mathcal{A}}) - \omega_5(\overline{\mathcal{A}}^c) \right) \\
+ \int_{D_2 \times B} \left[ \left( \omega_5(\mathcal{A}) - \omega_5(\overline{\mathcal{A}}) \right) - \left( \omega_5(\mathcal{A}^c) - \omega_5(\overline{\mathcal{A}}^c) \right) \right].
\]

(C.11)

Then, recall (3.32), which states that the original CS term (2.4) does not depend on the collective coordinates. In quite the same manner, calculation in the singular gauge leads to

\[
\int_{M_5 = M_5^{(0)} + M_5^{(\infty)}} \left( \omega_5(\overline{\mathcal{A}}) - \omega_5(\overline{\mathcal{A}}^c) \right) = 0.
\]

(C.12)

Using this and the formula (B.1) with \( V = W g^{-1} W^{-1} \) relating \( \omega_5(\overline{\mathcal{A}}) = \omega_5(\mathcal{A} W g^{-1} W^{-1}) \) and \( \omega_5(\mathcal{A}) \), Eq. (C.11) is rewritten as

\[
\int_{M_6} \text{tr} \left( \mathcal{F}^3 - (\mathcal{F}^c)^3 \right) = \int_{\partial M_6^{(0)} = M_5^{(0)} + D_2 \times B} \left[ \left( \omega_5(\mathcal{A}) - \omega_5(\overline{\mathcal{A}}) \right) - \left( \omega_5(\mathcal{A}^c) - \omega_5(\overline{\mathcal{A}}^c) \right) \right] \\
= - \int_{\partial M_6^{(0)} = M_5^{(0)} + D_2 \times B} \left\{ \frac{1}{10} \text{tr} [L^5 - (-igdg^{-1})^5] \right\}
\]
where $L$ is given by

$$ L = -iWgW^{-1}d(Wg^{-1}W) $$

$$ = -iW \left[g(W^{-1}dW)g^{-1} - W^{-1}dW + gdg^{-1}\right]W^{-1}. \quad (C.14) $$

More precisely, $g = g(x; X^M)$ explicitly given in (C.13) and that appearing in $L$ given in (C.14) are different: The former is from the classical solution and has a constant and arbitrary instanton position $X^M$, while the latter has the $(t, s)$-dependent position $X^M(t, s)$. However, we do not need to distinguish the two, since the instanton position can be absorbed through a shift of $x^M$. (Note that the origin $\xi = 0$, the infinity $\xi = \infty$, and the boundary $B$ are defined in terms of the relative coordinate $(x - X^M)$.)

First, let us confirm that we can safely discard the exact term $d\left[\alpha_4(L, A) - \alpha_4(-igdg^{-1}, A^{cl})\right]$ in (C.13). A possible dangerous term at the origin $\xi = 0$ on $M_5^{(0)}$ is $\text{tr} L^3 A$ with $L \Rightarrow W(-ig\partial M g^{-1}dx^M)W^{-1} \sim 1/\xi$ and $A \Rightarrow A_0 dt \sim \xi^0$. Taking this into account and adding the boundary $\partial M_4^{(0)}$ of infinitesimal radius $\xi = \epsilon$, we have

$$ \int_{M_5^{(0)}+D_2 \times B} d\left[\alpha_4(L, A) - \alpha_4(-igdg^{-1}, A^{cl})\right] $$

$$ = \int dt \int_{\partial M_4^{(0)}} \text{tr} \left\{ (W(-ig\partial M g^{-1}dx^M)W^{-1})^3 A_0 - (-ig\partial M g^{-1}dx^M)^3 A_0^{cl} \right\} $$

$$ = \int dt \int d\Omega_3 \text{tr} \left[ t_8 \left( \Phi + iW^{-1}W \right) \right] = 0, \quad (C.15) $$

where we have used $W^{-1}A_0 W = A_0^{cl} + \Phi + iW^{-1}W$ obtained from (3.13), and $(-ig\partial_M g^{-1}dx^M)^3 \sim (1/\xi)^3 P_2 \xi^3 d\Omega_3$, with $P_2$ given by (3.27). The last equality, leading to zero, is due to (3.7) and the fact that $g(x)$ commutes with $t_8$.

Thus, we are left with the first term of (C.13). The integrand can in fact be rewritten into the exact form

$$ -\frac{1}{10} \text{tr} \left[ L^5 - (-igdg^{-1})^5 \right] = d \text{tr} (\mathcal{O}_A + \mathcal{O}_B), \quad (C.16) $$

with $\mathcal{O}_A$ and $\mathcal{O}_B$ given, respectively, by

$$ \mathcal{O}_A = -\frac{i}{2}(-iW^{-1}dW) \left[ (-igdg^{-1})^3 - (-ig^{-1}dg)^3 \right], \quad (C.17) $$

$$ \mathcal{O}_B = -\frac{i}{2}(-iW^{-1}dW) \left[ g(-iW^{-1}dW)g^{-1}(-igdg^{-1})^2 - g^{-1}(-iW^{-1}dW)g(-ig^{-1}dg)^2 \right. $$

$$ \left. - \frac{1}{2}(-igdg^{-1})(-iW^{-1}dW)(-ig^{-1}dg) + \frac{1}{2}(-ig^{-1}dg)(-iW^{-1}dW)(-ig^{-1}dg) \right]. \quad (C.18) $$

The $\mathcal{O}_B$ term containing only two $-igdg^{-1}$ can be safely dropped. However, the $\mathcal{O}_A$ term with $(-igdg^{-1})^3 \sim 1/\xi^3$ near the origin $\xi = 0$ needs careful treatment, like
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(C.15). Another way to evaluate \( \int_{M_5^{(0)} + D_2 \times B} d\mathcal{O}_A \) is to note that \( d\mathcal{O}_A = 0 \) holds on \( M_5^{(0)} \), since we have

\[
(-i g \partial_M g^{-1} dx^M)^4 = (-i g^{-1} \partial_M g dx^M)^4 = 0 \quad \text{on} \ M_4 , \tag{C.19}
\]

for a spherically symmetric \( g(x) \) of (2.20) [c.f., (3.42)], and \((-i W^{-1} dW)^2 = 0 \) on \( M_5 \), which is a surface with \( s = 0 \). Using this fact, we obtain

\[
\int_{M_5^{(0)} + D_2 \times B} d tr \mathcal{O}_A = \int_{\{s=0\} \times B} tr \mathcal{O}_A \\
= tr \left\{ \int dt (-i W(t)^{-1} \dot{W}(t)) (-i) \int_{B=S^3} (-i g dg^{-1})^3 \right\} \\
= \frac{24 \pi^2}{\sqrt{3}} \int dt \ tr \left[ t_8 (-i W(t)^{-1} \dot{W}(t)) \right] , \tag{C.20}
\]

where we have used (3.31) and (3.27). This implies our previous result, (4.11).

### Appendix D

**WZW Term from \( S_{\text{CS}}^{\text{new}} \)**

In this appendix, we show how the WZW term is correctly reproduced from our new CS term, (4.4), in the low energy limit. We start with a configuration \( A(t, x, s) \) in \( M_6 \) which vanishes at the infinity, \( \xi = \infty \), and therefore at \( z = \pm \infty \). (This configuration is not necessarily a baryon configuration.) For carrying out the expansion in terms of the modes in \( z \)-space, we change to the \( A_z = 0 \) gauge via the gauge transformation in terms of

\[
V(t, x, z, s) = P \exp \left( i \int_{-\infty}^{z} dz' A_z(t, x, z', s) \right) . \tag{D.1}
\]

The gauge field \( A_\alpha \) (\( \alpha = 0, 1, 2, 3, s \)) in the \( A_z = 0 \) gauge satisfies the boundary condition (using the same symbol \( A \) for the gauge field in the \( A_z = 0 \) gauge)

\[
A_\alpha(t, x, s) \rightarrow \begin{cases} 
-i U(t, x, s) \partial_\alpha U(t, x, s)^{-1}, & (z \to +\infty) \\
0, & (z \to -\infty)
\end{cases} \tag{D.2}
\]

with \( U(t, x, s) \) given by

\[
U(t, x, s) = P \exp \left( i \int_{-\infty}^{z} dz A_z(t, x, z, s) \right) . \tag{D.3}
\]

Therefore, we can mode expand \( A_\alpha \) as

\[
A_\alpha(t, x, s) = -i U(t, x, s) \partial_\alpha U(t, x, s)^{-1} \times \psi_+(z) + (\text{massive modes}) , \tag{D.4}
\]

where \( \psi_+(z) \) is the zero-mode, given in Ref. 1):

\[
\psi_+(z) = \frac{1}{2} + \frac{1}{\pi} \arctan z \rightarrow \begin{cases} 
1, & (z \to +\infty) \\
0, & (z \to -\infty)
\end{cases} . \tag{D.5}
\]
Then, let us calculate our CS term, (4.4), for the gauge field (D.4) by discarding the contribution from the massive modes. First, the field strengths are given by

\[ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i [A_\alpha, A_\beta] = i \left[ -i U \partial_\alpha U^{-1}, -i U \partial_\beta U^{-1} \right] \psi_+(\psi_+ - 1) + \text{(massive modes)}, \]

\[ F_{z\alpha} = \partial_z A_\alpha = -i U \partial_\alpha U^{-1} \times \frac{d}{dz} \psi_+(z) + \text{(massive modes)}. \]  

Using this, we obtain

\[ \text{tr} F^3 = \frac{6}{23} \text{tr} (F_{\alpha\beta} F_{\gamma\delta} F_{\zeta\kappa}) \epsilon^{\alpha\beta\gamma\delta\kappa} d^6 x \]

\[ = 3 \text{tr} (-i UdU^{-1})^5 \times [\psi_+(z)(\psi_+(z) - 1)]^2 \frac{d\psi_+(z)}{dz} dz + \text{(massive modes)}. \]  

(D-7)

The \( z \)-integration of (D-7) is trivially carried out, and we finally get the desired result:

\[ S_{\text{CS}}^{\text{new}} = \frac{N_c}{240 \pi^2} \int \text{tr} (-i U(t, x, s) dU(t, x, s)^{-1})^5 + \text{(contribution from massive modes)}. \]  

(D-8)

Note that this WZW term is different from the WZW term given in Ref. 1), (3.34) with (3.35) and (3.36), with regard to the definition of the Skyrme field \( U \) in terms of \( A_z \) and the fact that the extra fifth coordinate is \( s \) in the present WZW term, while it is \( z \) in (3.34). However, in the topologically trivial sector without baryons, these two WZW terms are equivalent, because they are determined by the Skyrme field at the boundary, namely, by P \( \exp \left( i \int_\infty^{-\infty} dz A_z(t, x, z) \right) \).

Let us consider the Skyrme field (D.3) for our WZW term in the baryon sector. In the baryon sector, the gauge field \( \mathcal{A}(t, x, s) \) (C.5) in the singular gauge satisfies the condition \( \mathcal{A} \to 0 (\xi \to \infty) \).\(^{1)}\) Adopting

\[ \mathcal{A}_z(t, x, s) = W(t, s)\mathcal{A}_z^\text{cl}(x)W(t, s)^{-1}, \]  

(D-9)

as \( A_z \) in (D-3) (ignoring collective coordinates other than \( W \)), we have

\[ U(t, x, s) = W(t, s)U^\text{cl}(x)W(t, s)^{-1}, \]  

(D-10)

with \( U^\text{cl} \) defined by (c.f., Ref. 23))

\[ U^\text{cl}(x) = P \exp \left( i \int_\infty^{-\infty} dz \mathcal{A}_z^\text{cl}(x, z) \right). \]  

(D-11)

Plugging (D-10) with \( s \)-dependent \( U^\text{cl} \) into (D-8) leads to the same result, Eq. (4.11), as of course it should. Explicit expressions for the various quantities are

\[ \mathcal{A}_z^\text{cl}(x) = \left( \frac{1}{\xi^2 + \rho^2 - \frac{1}{\xi^2}} \right) (x \cdot \tau), \]  

(D-12)

\(^{1)}\) Contrastingly, \( \mathcal{A}(t, x, s) \) in the regular gauge does not vanish at \( \xi = \infty \), since \( \Delta A_0(t, x) \to 0 \) as \( z \to -\infty \). [See the footnote above Eq. (3.8).]
and
\[ U^{\text{cl}}(x) = \exp\left(-i\mathcal{H}(r) \hat{x} \cdot \tau \right), \tag{D-13} \]
with
\[ \mathcal{H}(r) = \pi \left(1 - \frac{r}{\sqrt{r^2 + \rho^2}}\right). \tag{D-14} \]

Note that \( \mathcal{H}(r) \) has the same behavior as the corresponding function of the Hedgehog solution in the Skyrme model:\textsuperscript{23)} \( \mathcal{H}(r = 0) = \pi \) and \( \mathcal{H}(r \to \infty) = \mathcal{O}(1/r^2) \). In any case, what is important for reproducing (4.11) is that the collective coordinate of the \( SU(3) \) rotation, \( W \), depends on the extra coordinate of the WZW term, as well as on \( t \). This is not satisfied in (3.37), where the extra coordinate is \( z \).

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