Let $G$ be a connected algebraic reductive group over an algebraic closure of a prime field $\mathbb{F}_p$, defined over $\mathbb{F}_q$ thanks to a Frobenius $F$. Let $\ell$ be a prime different from $p$. Let $B$ be an $\ell$-block of the group of rational points $G^F$. Under mild restrictions on $\ell$, we show the existence of an algebraic reductive group $H$ defined over $\mathbb{F}_q$ via a Frobenius $F$, and of a unipotent $\ell$-block $b$ of $H^F$ such that:

- the respective defect groups of $b$ and $B$ are isomorphic,
- the associated Brauer categories are isomorphic and there is a height preserving one-to-one map from the set of irreducible representations of $b$ onto the set of irreducible representations of $B$.

**TOWARDS A JORDAN DECOMPOSITION OF BLOCKS OF FINITE REDUCTIVE GROUPS**

**MICHEL E. ENGUEHARD**

Foreword. In 2008 was published by Michel Enguehard in Journal of Algebra [0] “Vers une décomposition de Jordan des blocs des groupes réductifs finis”. So I really have to say some words on that [0]-paper and the present text, name it [∞]-paper. Several years after the publication of [0] I received, as the presumed author, several quite judicious questions on some results and proofs contained in it. I discovered in [0] inaccuracies, incomplete proofs, not to say more. To save my homonymous colleague reputation I decided to rewrite [0] in English, I hoped that bad English would be saved by better mathematics. In a sense this is a joint work from M. Enguehard and M. E. Enguehard.

Je continue en français, pour être mieux compris de l’auteur de [0] tout en espérant ne pas être lu par mes autres lecteurs, s’il y en a. Je l’imagine alternativement désinvolte et besogneux et l’ai maudit à la fois de sa légèreté et de son acharnement. J’ai craint de sombrer dans les même travers, ou dans la schizophrénie mathématique. Mais, qui sait? il se peut que toutes ces Propositions soient exactes.

Dedicated to Edmond Lavergne who gave me my middle name

ABSTRACT (that is English summary in [0]). Let $G$ be a reductive algebraic group over an algebraic closure of a prime field $\mathbb{F}_p$, defined over $\mathbb{F}_q$, with Frobenius endomorphism $F$. Let $G^F$ be the subgroup of rational points. The center of $G$ is not assumed to be connected.

Let $\ell$ be a prime number, different from $p$. If $(G^*, F)$ is in duality with $(G, F)$, then, by a theorem of M. Broué and J. Michel [9], for any $\ell$-block — further “block” means “$\ell$-block” — $B$ of $G^F$ there exists a unique $(G^*)^F$-conjugacy class $(s)$ of $\ell$-semi-simple elements such that at least one irreducible representation of $B$ belongs to the rational Lusztig’s series $\mathcal{E}(G^F, s)$ associated ([18], [22]) to the $G^F$-conjugacy class $(s)$. If $s = 1$, $B$ is said to be unipotent. If $G$ is not connected, with identity component $G^0$, define the “unipotent $\ell$-blocks of $G^F$” as the $\ell$-blocks that cover some unipotent $\ell$-block of $(G^0)^F$.

From $(G, F)$ and $(s)$, we construct a reductive algebraic group $(G(s), F)$ defined over $\mathbb{F}_q$ and, assuming $\ell$ good for $G$ and some other slight restrictions on $\ell$, see our Theorem 1.4,
a one-to-one map from the set of unipotent $\ell$-blocks of $G(s)^F$ onto the set of $\ell$-blocks of $G^F$ with associated class $(s)$ such that

if $B$ corresponds to $b$, then there is a height preserving one-to-one map from the set of irreducible representations $\text{Irr}(b)$ onto the set $\text{Irr}(B)$, the respective defect groups of $b$ and $B$ are isomorphic, the associated Brauer categories are isomorphic.

**Introduction**

Let $(G, F_q, F, G^*, \ell)$ as in the abstract above, and $s$ be a semi-simple element of $G^*F$ (the notation $G^*F$ has to be understood as $(G^*)^F$). When the center of $G$ is connected, more generally when the centralizer $C_{G^*}(s)$ is connected, there is a so called Jordan decomposition of irreducible representations which is defined by a one-to-one map between Lusztig series

$$\Psi_{G,s} : E(C_{G^*}(s)^F, 1) \rightarrow E(G^F, s)$$

with strong properties [26], see here section 1.3. A Jordan decomposition of blocks would associate to any block $B$ of $G^F$ a unipotent block $b$ of a related “finite reductive group”, with strong similarities between $B$ and $b$.

Let $s$ be an $\ell'$-semi-simple element in $G^*F$ such that $B$ acts trivially on some element of $E(G^F, s)$. By a theorem of Broué and Michel [9] the $G^*F$-conjugacy class of $s$ is well defined, $B$ will be said in series $(s)$.

Assume that $C_{G^*}(s)$ is a Levi subgroup of $G^*$ and let $L$ be a Levi subgroup of $G$ in the dual $G^F$-conjugacy class of the $G^*F$-conjugacy class of $C_{G^*}(s)$. Then $\Psi_{G,s}$ above may be defined from the Lusztig functor $R^G_L$ and there is a “perfect isometry” (see [8], [9],) between a unipotent block of $L^F$ and a block $B$ of $G^F$ with $\text{Irr}(B) = \Psi_{G,s}(\text{Irr}(b))$ : that is our Jordan decomposition.

A deeper result of Bonnafé and Rouquier (see [5] or [16], chapters 10, 11) say that if $C_{G^*}(s)$ is contained in an $F$-stable Levi subgroup $L^*$ of $G^*$ with dual $L$ in $G$, then $R^G_L$ induces a Morita equivalence between the sum of the blocks of $L$ in series $(s)$ and the sum of the blocks of $G^F$ in series $(s)$. If furthermore $L^* = C_{G^*}(s)$ there is a Morita equivalence between unipotent blocks of $L^F$ and blocks of $G^F$ associated to $(s)$. One may hope a similar result when the connected component $C_{G^*}(s)$ of $C_{G^*}(s)$ is a Levi subgroup of $G^*$.

In the more general case, $C_{G^*}(s)$ being a Levi subgroup of $G^*$ or not, one may construct a reductive group $G(s)$ defined over $F_q$, in duality with $C_{G^*}(s)$ — thus $G(s)$ is connected only if $C_{G^*}(s)$ is connected. Then, as said in the abstract above, there is a good one-to-one map between the set of unipotent blocks of $G(s)^F$ and the set of blocks of $G^F$ in series $(s)$. Some restrictions on $\ell$ and on the type of $G$ are required by our proof (Assumption 2.1.2 in Theorem 1.4). Thanks to the classification of blocks given in [15] by Cabanes and Enguehard the proof is of combinatorial type. Several of the properties we need for the final proof are proved inductively, reducing to minimal cases.

In Part 1 we first precise our notations on finite and algebraic groups, and morphisms. In section 1.2 are proved elementary properties of the centralizers in $G$ of finite $\ell$-subgroups. In section 1.3 we collect properties of the Jordan decomposition of irreducible representations of $G^F$, due to Lusztig [26], [27]. Our main Theorem is Theorem 1.4. In section 1.5 we present Jordan decomposition for 2-blocks of groups of classical type in odd characteristic, a case excluded in 1.4 and sections 2 to 4.

In Part 2 we mix the main results of Cabanes and Enguehard [13—15], where blocks are classified by so-called “cuspidal data”, see section 2.1, with Generalized Harish-Chandra theory for unipotent representations from Broué-Malle-Michel [10] and Jordan decomposition. We obtain a convenient description of the set $\text{Irr}(B)$ and a form of Generalized Harish-Chandra theory for representations in $E(G^F, s)$ when $C_{G^*}(s)$ is connected.
In Part 3, sections 3.1, 3.2, 3.3, the group $G(s)$, in duality with $C_{G^*}(s)$, is defined by a root datum. We prove the required properties to compute combinatorial parameters in Clifford theory for unipotent blocks between $G(s)^F$ and $G(s)^F$, where $G(s)^F$ is the connected component of 1 in $G(s)$. These properties are “Non-multiplicity conditions” (Proposition 3.1.1, (B), Proposition 3.1.2, (C)) and relations between defect groups and the quotient $G(s)^F/G(s)^{1}$ (Proposition 3.1.2, (A)-(B)). Using results of section 2.4, we obtain a one-to-one map sending a unipotent block $b$ of $G(s)^F$ to a block $B$ in series $(s)$ of $G^F$, with a height preserving one-to-one map $\text{Irr}(b) \to \text{Irr}(B)$, see Propositions 3.4.1, 3.4.2.

The part 4 is devoted to Brauer’s categories of the corresponding blocks $b$ and $B$. One may identify a defect group $D$ of $B$ with a defect group of $b$ (Proposition 4.1.2). Then for any subgroup $X$ of $D$, if $(X,b_X)$ and $(X,B_X)$ are Brauer subpairs of respectively $(1,b)$ in $G(s)^F$ and $(1,B)$ in $G^F$, the quotients $N_{G(s)^F}(X,b_X)/C_{G(s)^F}(X)$ and $N_{G^F}(X,B_X)/C_{G^F}(X)$ are isomorphic (Proposition 4.2.4).

In the Appendix, section 5.1, are collected results on Clifford theory for blocks. Section 5.2 is devoted to some useful remark on unipotent Generalized Harish-Chandra series, as computed in [10] (Broué-Malle-Michel). In section 5.3, assuming Mackey decomposition formula, we deduce for classical types with connected center a commutation formula between Jordan decomposition and Lusztig functor in any series from the similar one for unipotents. In section 5.4 we show that the description of $\text{Irr}(G^F)$ for $G^F = \text{SL}(\pm q)$ by Bonnafé [3] and Cabanes [12] imply the existence of Generalized $d$-Harish-Chandra series.

1. Backgrounds. A theorem

1.1. Notations and terminology

1.1.1. Finite groups

The cardinality of a finite set $X$, or the order of a group $X$, is denoted by $|X|$. The unit for multiplication in groups is denoted by 1 and if $X$ is a group such that $|X| = 1$, we may write $X = 1$. Group actions on sets or modules are on the left and conjugacy may be denoted exponentially : $^xy = x^y = x y y^{-1}$. The commutator of elements $x, y$ of a group is $[x, y] = x y y^{-1}$ and, for subsets $X, Y$ of a group $G$, $\langle X \rangle$ is the subgroup generated by $X$ and $[X, Y]$ is the subgroup generated by the set of commutators $[x, y]$ for $(x, y) \in X \times Y$.

If $X$ is a subset of a group acting on a set $E$, $E^X$ is the subset of fixed points. If $y \in E$ or $y \subseteq E$, and $X$ is a group acting on $E$, $X_y$ is the stabilizer of $y$ in $X : X_y = \{g \in X \mid g y = y\}$, with traditional exceptions if action is induced by conjugacy inside a group $X$ : if $Y \subseteq X$, $C_X(Y) = \cap_{y \in Y} X_y$ is the centralizer of $Y$ in $X$, $C_Y(Y) = Y' \cap C_X(Y)$, $Z(X) = C_X(X)$ is the center of $X$, $N_X(Y) = \{x \in X \mid x Y = Y\}$ is the normalizer of $Y$ in $X$. We may mix the two notations and write $N_X(Y, \lambda)$ for $N_X(Y, \lambda)$.

If $\pi$ is a set of primes, $\pi'$ is its complementary in the set of all primes, any integer $n$ is a product $n = n_\pi n_{\pi'}$, where any prime divisor of $n_\pi$ (resp. $n_{\pi'}$) belongs to $\pi$ (resp. $\pi'$). Let $X$ be a finite group, $X$ is said a $\pi$-group if $|X| = |X|_\pi$, an element $g$ of a group is said a $\pi$-element if there is an integer $n$ such that $g^n = 1$ and $n = n_\pi$; any $g \in X$ is a product $g = g_\pi g_{\pi'} = g_\pi g_{\pi'}$ where $g_\pi$ (resp. $g_{\pi'}$) is a $\pi$-element (resp. $\pi'$-element). We denote $X_\pi$ the set of $\pi$-elements of $X$, hence $X$ is a $\pi$-group if and only if $X = X_\pi$.

Let $X$ be a finite group. If $O$ is a commutative ring, $OX$ is the group algebra of $X$ on $O$. We denote by $\text{Irr}(X)$ the set of irreducible characters of $X$, i.e. the trace maps defined by simple $\mathbb{C}X$-modules. When $X$ is abelian the tensor product defines a product in $\text{Irr}(X)$, the group so obtained is denoted $X^\wedge$. Any
irreducible character of $X$ is a central function on $X$ and $\text{Irr}(X)$ is a basis of the space $\mathcal{CF}(X, \mathbb{C})$ of all central functions on $X$. If one consider a field $K$ of characteristic zero and containing a $|X|$th root of unity, one recover a set $\text{Irr}_K(X)$ which is in bijection with $\text{Irr}(X)$ (once a bijection between $|X|$th-roots of unity is given) and is a basis of $\mathcal{CF}(X, K)$. So we omit the subscript $K$ in that case. This applies to the field $K$ of any “$\ell$-modular splitting system” (see [28], §3.6) $(\mathcal{O}, K, k)$ of $X$. Such a triple allows to introduce $\ell$-blocks inside $\mathcal{O}X$ —as well in $kX$—, called blocks of $X$, and blocks idempotents, i.e. primitive idempotents in the center of $\mathcal{O}X$, see [28], §1.8. By extension from $\mathcal{O}$ to $K$ one obtains a partition

$$\text{Irr}(X) = \bigcup_B \text{Irr}(B)$$

where $B$ ranges over the set of blocks of $X$. Define, for $\xi \in \text{Irr}(X)$, the block $B_X(\xi)$ by

$$\xi \in \text{Irr}(B_X(\xi)).$$

On the space $\mathcal{CF}(X, K)$ is defined the usual scalar product

$$\langle \phi, \psi \rangle_X = |X|^{-1} \sum_{g \in X} \phi(g)\psi(g^{-1})$$

and, if $\phi, \psi \in \text{Irr}(X)$, then $\langle \phi, \psi \rangle_X = \delta_{\phi, \psi}$. The associated norm is denoted $||?||$, i.e. $||\phi||^2 = \langle \phi, \phi \rangle_X$ for $f \in \mathcal{CF}(X, K)$.

Given a morphism $\sigma: Y \to X$ (or simply an inclusion of groups $Y \subseteq X$), the restriction from $X$ to $Y$, applied to representations or to central functions, is denoted $\text{Res}_\sigma$ (or $\text{Res}_Y^X$). Induction from a subgroup $Y$ to a group $X$ is denoted $\text{Ind}_Y^X$. If $Y$ is a normal subgroup of $X$ and $\eta \in \text{Irr}(Y)$ one defines

$$\text{Irr}(X \mid \eta) = \{\chi \in \text{Irr}(X) \mid \langle \text{Res}_Y^X \chi, \eta \rangle_Y \neq 0\}.$$ 

Then we say that $\chi$ covers $\eta$. Similarly a block $b$ of $X$ covers a block $c$ of $y$ if there exist $\eta \in \text{Irr}(c)$ and $\chi \in \text{Irr}(b)$ such that $\chi$ covers $\eta$ [28], 5.1.

The notation “tensor product” of representations or of central functions is used to produce representations or central functions on a central product, as well with one fixed group : if $\chi \in \text{Irr}(X)$ and $\xi \in \text{Irr}(Y)$, and $Z$ injects in the centers of $X$ and $Y$, if furthermore $\chi/\chi(1)$ and $\xi/\xi(1)$ are equal on $Z$, $\chi \otimes \xi$ may be considered as an element of $\text{Irr}(X \times_Z Y)$. But when $X = Y$ and $\chi/\chi(1)$ and $\xi/\xi(1)$ are conjugate complexes on $Z$, $\chi \otimes \xi$ may be considered as an element of $\text{Irr}(X/Z)$ by restriction of the preceding one to the diagonal subgroup. We hope the good interpretation is given by the context.

1.1.2. Algebraic groups

All along $G$ is an algebraic group. The connected component of 1 is denoted $G^\circ$, but we prefer

1.1.2.0. Notations. $C_G^\circ(g)$ is $C_G(g)^\circ$, $Z^\circ(G)$ is $Z(G)^\circ$, $G^\circ F$ is $(G^\circ)^F$, $G^* F$ is $(G^*)^F$.

Let $p$ be a prime number, different from $\ell$. We consider first connected reductive groups $G$ on an algebraic closure $\overline{F}$ of a prime field $\mathbb{F}_p$, that are defined over a finite field $\mathbb{F}_q$ ($q$ a power of $p$) thanks to an endomorphism $F: G \to G$.

If $T$ is a maximal torus in $G$, then is defined a root datum

$$\mathcal{D}(G, T) := (X(T), Y(T), \Phi, \Phi^\vee)$$
(group of characters of $T$, group of one parameter subgroups of $T$, set of roots in $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$, set of coroots) and that root datum defines $G$ up to interior isomorphisms induced by $T$. To emphasize the choice of $T$ or/and to recall the algebraic group $G$, we may write $\Phi_G(T)$ instead of $\Phi$. The types of connected Dynkin diagrams of root system, or types of irreducible algebraic groups are denoted as usually $A_n, B_n, \ldots, E_6$. Assuming that $T$ is $F$-stable, $F$ acts on each of the 4 objects of the root datum, hence acts on the set of connected components of the Dynkin diagram of the root system $\Phi$. To every orbit of $F$ there corresponds an $F$-stable component of $[G,G]$, minimal as $F$-stable component, with a so-called rational type $(X,q^m)$, defined by one of the above types, twisted or not (so $2A_n, 2D_n, 3D_4, 2E_6$ appear) and an extension $F_{q^n}$ of $\mathbb{F}_q$. $G$ is said \textbf{rationally irreducible} if $[G,G]$ has only one such component. In reference to “Ennola’s conjecture” we write $(A_n,-q^m)$ instead of $(2A_n,q^m)$. 

A \textbf{dual root datum} of $\mathcal{D}(G,T)$ is isomorphic to $(Y(T),X(T),\Phi^\vee,\Phi)$. The pair $(Y(T),X(T))$ defines a so-called \textbf{dual torus} $T^*$, i.e. $X(T^*) = Y(T)$ and $Y(T^*) = X(T)$, and the dual root datum defines, up to some isomorphisms, an algebraic reductive group usually denoted $G^*$, said to be in duality with $G$ : so $T^*$ is a maximal torus in $G^*$ and $\mathcal{D}(G^*,T^*) \cong (Y(T),X(T),\Phi^\vee,\Phi)$. We may write then that $(T \subseteq G)$ and $(T^* \subseteq G^*)$ are in duality. Any maximal torus of $G^*$ is in duality with some maximal tori of $G$, more generally any Levi-subgroup of $G^*$ is in duality with some Levi subgroup of $G$. The action of $F$ on $\mathcal{D}(G,T)$ gives an action on $\mathcal{D}(G^*,T^*)$, hence some endomorphism of $G^*$, we denote $F$ for simplicity. Then a $G^*$-$F$-conjugacy class of $F$-stable Levi subgroups $L^*$ of $G^*$ corresponds to a $G^F$-conjugacy class of $F$-stable Levi subgroups $L$ of $G$ so that $L$ and $L^*$ may be defined by dual root data (the duality is not uniquely defined but is coherent with duality between $G$ and $G^*$, see formulas (1.3.2.1), (1.3.2.4), (1.3.2.5)). We say then that $(L,F)$ and $(L^*,F)$ (or $L$ and $L^*$) are in \textbf{dual conjugacy classes}. The Weyl group $W(G,T) := N_G(T)/T$ is a group of automorphisms of $\mathcal{D}(G,T)$, so is anti-isomorphic to $W(G^*,T^*)$. Given once for all an isomorphism $\mathbb{F}_q^\times \to (\mathbb{Q}/\mathbb{Z})_p$ and an imbedding $\mathbb{F}_q^\times \hookrightarrow \mathbb{Q}_p^\times$, one obtains isomorphisms $T^F \cong (T^*F)^\vee$, $(T^F)^\vee \cong T^*F$.

Let $\pi: G_{sc} \to [G,G]$ be a simply connected covering of the derived group of $G$. The kernel of $\pi$ is denoted $F(G)$. If $G$ is adjoint, $F(G)$ is the \textbf{fundamental group} common to groups of same type.

\section{1.1.3. Proposition.}  
(a) Let $G$ and $G^*$ be in duality. The finite abelian groups $F(G)$ and $Z(G^*)/Z^\circ(G^*)$ are in duality, hence are isomorphic.

(b) Let $L$ be a Levi subgroup of $G$. Then $Z(L)/Z^\circ(L)$ is isomorphic to a quotient of $Z(G)/Z^\circ(G)$ and $F(L)$ to a subgroup of $F(G)$.

\textbf{Proof.}  
(a) Let $\mathcal{D}(G,T) = (X,Y,\Phi,\Phi^\vee)$ and $\mathcal{D}(G^*,T^*)$ be root data in duality, defining the duality between $G$ and $G^*$. Let $\Omega$ be the group of weights, dual over $\mathbb{Z}$ of $\langle \Phi \rangle$. Then $F(G) = F([G,G])$ is isomorphic to the dual (as a finite abelian group ...) of the $p'$-torsion group of $\Omega/(X(T) \cap [G,G])$ hence isomorphic to the $p'$-torsion group of $Y(T \cap [G,G])/\mathcal{Z}\Phi^\vee$ (one has $\mathcal{Z}\Phi^\vee \subseteq Y(T \cap [G,G]) \subseteq Y$). One knows that the finite group $Z(G^*)/Z^\circ(G^*)$ is isomorphic to the dual of the $p'$-torsion of $Y/\mathcal{Z}\Phi^\vee$ ([17 4.5.8]. As $Y/Y(T \cap [G,G])$ has no torsion, the torsion groups of $Y/\mathcal{Z}\Phi^\vee$ and $Y(T \cap [G,G])/\mathcal{Z}\Phi^\vee$ are isomorphic.

Hence our claim.

(b) We may assume that $T \subseteq L$ and $L$ is defined by a subroot datum of $\mathcal{D}(G,T)$, so that the set $\Phi_L$ of roots of $L$ is contained in $\Phi$. As $\mathcal{Z}\Phi/\mathcal{Z}\Phi_L$ has no torsion, the torsion group of $X/\mathcal{Z}\Phi_L$ injects in the torsion group of $X/\mathcal{Z}\Phi$, hence $Z(L)/Z^\circ(L)$ is isomorphic to a quotient of $Z(G)/Z^\circ(G)$.}

\section{1.1.4. On morphisms}

(a) A morphism

$$\sigma: G \to H$$
between algebraic reductive groups is said to be \textit{isotypic} if its kernel is central in \( G \) and \([H,H] = \sigma([G,G])\).

If \( G, H \) and \( \sigma \) are defined on \( \mathbb{F}_q \) by some endomorphisms \( F \), we may write \( \sigma: (G,F) \to (H,F) \). Between groups \( G^*, H^* \) in duality with resp. \( G \) and \( H \), there exists “dual morphisms”

\[
\sigma^*: H^* \to G^*,
\]

where \( \sigma^* \) is isotypic and eventually defined over \( \mathbb{F}_q \).

Then \( F(G) \) (resp. \( F(H^*) \)) is isomorphic to a subgroup of \( F(H) \) (resp. \( F(G^*) \)) and \( Z(G^*)/Z^0(G^*) \) (resp. \( Z(H)/Z^0(H) \)) is isomorphic to a quotient of \( Z(H^*)/Z^0(H^*) \) (resp. \( Z(G)/Z^0(G) \)).

(b) An isotypic morphism \( \sigma: G \to H \) with kernel \( \{1\} \) will be called an \textbf{embedding}. Then \( F(G) \) is isomorphic to \( F(H) \). If \( \sigma^*: H^* \to G^* \) is a dual morphism of an embedding \( \sigma \), then \( \sigma^*(H^*) = G^* \) and \( Z(G^*)/Z^0(G^*) \) is isomorphic to \( Z(H^*)/Z^0(H^*) \).

If furthermore the center of \( H \) is connected, \( \sigma \) is called a \textbf{regular embedding}. An isotypic morphism \( \sigma: G \to H \) is a regular embedding if and only if one (and then every) dual morphism \( \sigma^*: H^* \to G^* \) satisfies : \( \sigma^*(H^*) = G^* \), the kernel of \( \sigma^* \) is a torus and \( F(H^*) = \{1\} \).

(c) For any \( (G,F) \) there exists a regular embedding \( \sigma: (G,F) \to (H,F) \). Given two regular embeddings \( \sigma_j: (G,F) \to (H_j,F) \) \((j = 1, 2)\) there exists a third one \( (G,F) \to (H,F) \) that factorizes through \( \sigma_1 \) and \( \sigma_2 \).

(d) For any isotypic morphism \( \sigma: G \to H \) (resp. \( (G,F) \to (H,F) \)) there exist regular embeddings \( G \to G_0 \), \( H \to H_0 \) (resp. \( (G,F) \to (G_0,F) \), \( (H,F) \to (H_0,F) \)) and an isotypic morphism \( \sigma_0: G_0 \to H_0 \) (resp. \( (G_0,F) \to (H_0,F) \)) whose kernel is a torus and extending \( \sigma \) (resp. \( (\sigma,F) \)).

On proofs. The existence of \( \sigma^* \) in (b) follows from the existence of a morphism of dual root data, so \( \sigma^* \) is not unique. The construction of regular embeddings is based on products as follows :

Let \( \sigma: (G,F) \to (H,F) \) be isotypic as in (d). Recall the center of \( G \) is contained in any maximal torus. Let \( T \) be an \( F \)-stable subtorus of \( G \) that contains the kernel \( K \) of \( \sigma \). Let

\[
G_1 = G \times_K T = G \times T/\{(k,k^{-1}) \mid k \in K\}
\]

The composed map \( G \to G \times T \to G_1 \) is an embedding (if \( K = Z(G) \), \( Z(G_1) \) is a torus hence that embedding is regular, a way to prove (c)). Then \( \sigma \) is the restriction of some \( \sigma_1: G_1 \to H \), where \( \text{Ker} \ \sigma_1 \) is the central torus \( T_1 := K \times_K T \), isomorphic to \( T \). A regular embedding \( G_1 \to G_0 \) defines a regular embedding \( H \to H_0 := G_0/T_1 \) such that the following diagram of isotypic morphisms is commutative

\[
\begin{array}{cccc}
K & T_1 & \downarrow & T_1 \\
\downarrow & \downarrow & & \downarrow \\
G & \leftrightarrow & G_1 & \leftrightarrow & G_0 \\
\downarrow & \sigma & & \downarrow \sigma_1 & & \downarrow \sigma_0 \\
H & \leftarrow & H & \leftarrow & H_0 \\
\end{array}
\]

1.1.5. \( \ell, q \) and \( d \). The decomposition \( G = G_aG_b \)

We have assumed once for all that \( \ell \) is a prime number, \( \ell \neq p \).

(1.1.5.1) If \( \ell \) is odd, let \( d = d_{q,\ell} \) be the order of \( q \) mod \( \ell \) and \( E = \{n \in \mathbb{N} \mid n_{\ell'} = d\} \). 

\[6\]
E is denoted $E_{q,\ell}$ in [16] Theorem 21.7.

For any positive integer $a$, let $\phi_a \in \mathbb{Z}[X]$ be the $a$-th cyclotomic polynomial. If $\ell > 2$ and $a \in \mathbb{N}^\times$, then $\ell$ divides $\phi_a(q)$ if and only if $a \in E$. Any algebraic group $(G, F)$ we consider, if defined on $\mathbb{F}_q$, has a polynomial order $P_{G,F}(X) \in \mathbb{Z}[X]$ such that $|G| = P_{G,F}(q)$ [16] Section 13.1. If $G$ is a torus its polynomial order is a product of cyclotomic polynomials. Let $A$ be a set of positive integers. A $\phi_A$-subgroup (or $\phi_a$-subgroup if $A = \{a\}$) $S$ of $(G, F)$ is an $F$-stable torus in $G$ such that $P_{S,F}$ is a product of $\phi_a$ where $a \in A$. One has “Sylow’s theorems” for $\phi_A$-subgroups in $G$. For any torus $T$, we denote $T_{\phi_A}$ its maximal $\phi_A$-subgroup. By definition of $E$ with respect to $\ell$ one has $T^F_{\ell} \subseteq T_{\phi_E}$. A Levi subgroup of $(G, F)$ is said to be $A$-split (or $a$-split if $A = \{a\}$) if it is the centralizer in $G$ of a $\phi_A$-subgroup. For more results on these notions see [10] or [16] Chapter 13.

1.1.5.2. Definition. [16] 22.4, 22.5. Let $(G, F)$ be defined on $\mathbb{F}_q$ as above and assume $\ell > 2$. Let $G_a$ be the product in $G$ of $Z^i(G)$ and all rationally irreducible components of $[G, G]$ of type $(\mathbb{A}_n, r)$ ($r = \pm q^m$) where $\ell$ divides $|r - 1|$. Let $G_b$ be the product in $G$ of all rationally irreducible components of $[G, G]$ which are not included in $G_a$.

One has $G = (G_aG_b)$ (central product), $Z(G_b)^F$ and $G^F/G^F_aG^F_b$ are commutative $\ell'$-groups.

The choice of components of $G_b$ is made so that $F(G_b)_F^F = 1 = (Z(G_b)/Z^0(G_b))^F$. For any central $F$-stable subgroup $A$ of $G$ one has $(G/A)_a = G_a/A \cap G_a$ and $(G/A)_b = G_b/A \cap G_b$. For any $F$-stable Levi subgroup $L$ of $G$, one has $L \cap G_a \subseteq L_a$ and $L_b \subseteq G_b$. Currently $L_b \neq L \cap G_b$ nevertheless one has $Z(L \cap G_b)_F^F = Z^0(L \cap G_b)_F^F$ by Proposition 1.1.3. In inductive proofs the following properties are frequently used [16] Proposition 22.5, Theorem 22.2:

1.1.5.3. Assume $\ell > 2$. If $Y$ is an $\ell$-subgroup of $G^F$ such that $Z(C_{G,F}(Y))_F \subseteq Z(G)G_a$, then $Y \subseteq G_a$.

If $G = G_b$ then any proper $E$-split Levi subgroup of $G$ is contained in a proper $d$-split Levi subgroup of $G$.

The last assertion is an immediate consequence of an easy to verify fact when $G = G_b$: if $L$ is an $F$-stable Levi subgroup of $G$ such that $Z^0(L)_{\phi_d} \subseteq Z^0(G)$, then $Z^0(L)_{\phi_E} \subseteq Z^0(G)$.

Let $(G^*, F)$ be in duality with $(G, F)$. Then one has isotypic morphisms

\[(1.1.5.4) \quad (G^*)_a \to (G_a)^*, \quad (G^*)_b \to (G_b)^*\]

1.2. Centralizers and connexity

For any subgroup $X$ of $G$ or $x \in G$ we denote

\[(1.2.0) \quad A_G(X) = C_G(X)/C_G^0(X), \quad A_G(x) = A_G(\langle x \rangle)\]

Proposition 1.1.3 (a) gives a relation between $A_G(G)$ and $F(G^*)$. Proposition 1.2.6 below relies $A_G(X)$ to $F(G)$ and $F(G^*)$ when $X$ is a finite $\ell$-subgroup of $G$.

We first note some elementary results for later use.

1.2.1. Lemma. Let $\rho: H \to K$ be a surjective morphism between groups, whose kernel $\text{Ker} \rho$ is central in $H$. Let $X$ be a subgroup of $H$ with finite exponent.

The exponent of $\rho^{-1}(C_K(\rho(X)))/C_H(X)$ divides the exponent of $X$ and the exponent of $\text{Ker} \rho \cap [H, H]$. If $\pi$ is a set of primes, $X$ is a finite $\pi$-group and $\text{Ker} \rho \cap [H, H]$ is a finite $\pi'$-group, then $\rho(C_H(X)) = C_K(\rho(X))$.

Proof. The last assertion is a direct consequence of the first one.
One defines a bi-morphism \((\rho^{-1}(C_K(\rho(X)))\cap C_H(X)) \times X \to \text{Ker} \rho \cap [H, H]\) by restriction of \((h,x) \mapsto [h,x] : \) indeed, when \([h,x] \in Z(H)\) — and this happens if \(\rho(h) \in C_K(\rho(X))\) and \(x \in X\) — one has, for \(h',y \in H, [h,x][h,y] = [h,xy]\) and \([h,x][h',x] = [hh',x].\) Let \(k\) be the exponent of \(X\), then \([h^k,x] = [h,x]^k = [h,x^k] = 1\) so that \(h^k \in C_H(X).\)
1.2.2. Proposition. Let $X$ be a subgroup of $G$.

(a) If $X.Z^o(G) = X'.Z^o(G)$ one has $A_G(X) = A_G(X')$. One has exact sequences of morphisms

$$1 \to Z^o(G)/(Z^o(G) \cap C_{[G,G]}^o(X)) \to C_G(X)/C_{[G,G]}^o(X) \to A_G(X) \to 1$$

$$1 \to C_{[G,G]}(X)/C_{[G,G]}^o(X) \to C_G(X)/C_{[G,G]}^o(X) \to Z^o(G)/(Z^o(G) \cap [G,G]) \to 1$$

hence $|C_{[G,G]}(X)/C_{[G,G]}^o(X)| = |A_G(X)|.|Z^o(G) \cap [G,G]/Z^o(G) \cap C_{[G,G]}^o(X)|$.

(b) If $H$ is an algebraic subgroup of $G$ and $X \subseteq H$, there exists an exact sequence of morphisms

$$1 \to C_G^o(X) \cap H/C_G^o(H) \to A_H(X) \to A_G(X) \to C_G(X)/C_G^o(X).C_H(X) \to 1$$

(c) If $X$ is a subgroup of $Y$, there exists an exact sequence of morphisms

$$1 \to C_G^o(X) \cap C_G(Y)/C_G^o(Y) \to A_G(Y) \to A_G(X) \to C_G(X)/C_G^o(X).C_G(Y) \to 1$$

Proof. The first assertion in (a) follows from the equality $G = Z^o(G).[G,G]$.

The four exact sequences are given by isomorphisms theorems, knowing that $C_{G}^o(X) \subseteq C_{G}^o(H)$ and $C_H(X) = H \cap C_G(X)$ in (b) and that $C_G^o(Y) \subseteq C_G^o(X)$ and $C_G^o(X).C_G(Y) \subseteq C_G(X)$ in (c).

1.2.3. Proposition. Let $X$ be a finite $p'$-subgroup of $G$.

(a) Let $Z$ be a central subgroup of $G$ and $\rho: G \to G/Z$ the quotient morphism. Then

$$\rho(C_G^o(X)) = C_{G/Z}^o(\rho(X))$$

so that $A_G(X)$ is isomorphic to a subgroup of $A_{p(G)}(\rho(X))$.

(b) Let $\sigma: G \to H$ be an embedding. Then $A_H(\sigma(X))$ is isomorphic to a quotient of $A_G(X)$.

(c) An isotypic morphism $\sigma: G \to H$ defines a morphism $A_G(X) \to A_H(\sigma(X))$.

Proof. (a) The property is well known in case $X$ is cyclic or more generally contained in a torus, $C_G^o(X)$ being described by a root datum [16] Proposition 13.13. Assume $X$ is generated by a finite set of semi-simple elements $x_j$, one has

$$C_G^o(X) \subseteq \cap_j C_G^o(x_j) \subseteq C_G(X)$$

and indices are finite. Applying $\rho$ we obtain, knowing that $Z \subseteq C_G(x_j)$,

$$\rho(C_G^o(X)) \subseteq \rho(\cap_j C_G^o(x_j)) \subseteq \cap_j \rho(C_G^o(x_j)) \subseteq \cap_j \rho(C_G(x_j)) = \rho(\cap_j C_G(x_j)) = \rho(C_G(X))$$

with finite indices, and, on $G/Z$-side

$$\cap_j \rho(C_G^o(x_j)) = \cap_j C_{G/Z}^o(\rho(x_j)) \subseteq \cap_j C_{G/Z}(\rho(x_j)) = C_{G/Z}(\rho(X))$$

with finite indices. But $\rho(C_G^o(X))$ is connected hence $\rho(C_G^o(X)) = C_{G/Z}(\rho(X))$.

(b) This is a special case of (a) in 1.2.2. Directly : one has $H = Z^o(H).\sigma(G)$, hence $C_H(\sigma(X)) = Z^o(H).C_{\sigma(G)}(\sigma(X))$ and $C_{\sigma(G)}(\sigma(X)) = Z^o(H).\sigma(C_G^o(X))$. By isomorphism theorems $A_H(\sigma(X))$ is a quotient of $A_{\sigma(G)}(\sigma(X)) \cong A_G(X)$. 


(c) Any isotypic morphism may be obtained by composition of an embedding and a quotient morphism.

As we are interested by centralizers of \(\ell\)-subgroups and \(\ell\) is different from the characteristic of the base field, the key-property is given by one of Steinberg’s theorems in [31], essentially that \(A_G(s)\) is 1 if \(s\) is semi-simple and \(G\) simply connected.

1.2.4. Proposition. (a) Let \(s\) be a semi-simple element of \(G\). The group \(A_G(s)\) is isomorphic to a subgroup of \(F(G)\) and its exponent divides the exponent of \(s\).

(b) If \(L\) is a Levi subgroup of \(G\) and \(s \in L\), \(A_L(s)\) is isomorphic to a subgroup of \(A_G(s)\).

(c) Let \(s\) and \(t\) be commuting semi-simple elements of \(G\) with coprime orders. One has

\[
C_G^s(s) \cap C_G^t(t) = C_G^t(st)
\]

hence \(A_G(st)\) is isomorphic to \(A_{C_G^s(s)}(t) \times A_{C_G^t(t)}(s)\).

(d) If \(C_G(s)\) is connected for any semi-simple \(\ell\)-element \(s\) in \(G\), then \(F(G)\) has order prime to \(\ell\).

(e) Assume that \(F(G)_{\ell} = \{1\}\). If the order of a semi-simple element \(s\) of \(G\) is prime to \(\ell\), \(|F(C_G^s(s))|\) is prime to \(\ell\).

Proof. (a) See [31] 9.5 and Proposition 1.2.3.

(b) is a special case of (b) in Proposition 1.2.2, with \((s, L)\) instead of \((X, H)\): one has \(L = C_G(Z^o(L))\) and \(L \cap C_G^s(s) = C_{C_G^s(s)}(Z^o(L))\).

(c) If \(F(G) = \{1\}\), then \(C_G^s(st) = C_G(st) = C_G(s) \cap C_G(t) = C_G^s(s) \cap C_G^t(t)\), thanks to (a).

When \(F(G) \neq \{1\}\) let \(\rho: H \to G\) be a covering such that \(F(H) = \{1\}\) (see Proposition 1.1.4.1). There exist semi-simple elements \(s’, t’\) in \(H\) with coprime orders such that \(\rho(s') = s\), \(\rho(t') = t\) so that \(\rho(C_H^o(s')) = C_G^s(s), \rho(C_H^o(t')) = C_G^t(t), \rho(C_H^o(s't')) = C_G^t(st)\). One obtains \(C_G^s(s) \cap C_G^t(t) = C_G^t(st)\). But \(C_G^s(s) \cap C_G(t) = C_{C_G^s(s)}(t)\) and \(C_G(s) \cap C_G^t(t) = C_{C_G^t(t)}(s)\). The isomorphism we claim follows.

(d) Let \(\rho: H \to [G, G]\) be a simply connected covering and \(G_1 = H/F(G)\). Then \(F(G_1)\) is isomorphic to \(F(G)_{\ell}\) and \(\rho\) factors through \(\rho_1: G_1 \to G\). By (a) if \(s_1 \in G_1\) is semi-simple and of order prime to \(\ell\), then \(G_1(s_1)\) is connected. If \(s_1 \in G_1\), then \(\rho_1(G_1(s_1)) \leq C_G(\rho_1(s_1)) = C_G^o(\rho_1(s_1)) = C_G^o(G_1(s_1)), \) hence \(G_1(s_1)\) is connected. Any semi-simple element \(t\) of \(G_1\) is a product \(t = t_\ell t_{\ell'}\) and, using (b) one has \(C_{G_1}(t) = C_{G_1}(t_\ell) \cap C_{G_1}(t_{\ell'}) = C_{G_1}(t_\ell) \cap C_{G_1}(t_{\ell'}) = C_G^o(t)\). By [31] 9.9 we have \(F(G_1) = \{1\}\).

(e) Let \(t \in C_G^o(s)\). By (a) \(C_G(t)\) is connected. We have \(C_{C_G^o(s)}(t) = C_G^o(s) \cap C_G^o(t) = C_G^o(st)\) by (b).

By (c) \(F(C_G^o(s))_{\ell} = \{1\}\).

When \(\ell\) is good for \(G\) (see [16] section 13.2) and \(s \in G_{\ell}\), then \(C_G^o(s)\) is a Levi subgroup of \(G\). If furthermore \(Z(G)_{\ell}^F = Z^o(G)_{\ell}^F\), then \(s \in Z^o(C_G^o(s))\) (see Proposition 1.1.3) hence \(C_G^o(z) = C_G(Z^o(C_G^o(s))_{\phi_E})\). Thus \(E\)-split Levi subgroup of \(G\) are good examples of (connected) centralizers of abelian \(\ell\)-subgroups of \(G^F\). In the following Proposition, (b) applies when \(G = Z^o(G)G_b\).

1.2.5. Proposition. [15] Propositions 2.4, 3.2, [16] Proposition 13.19. Assume the prime \(\ell\) is good for \(G\).

(a) If an \(F\)-stable Levi subgroup \(K\) of \(G\) satisfies \(K = C_K^o(Z(K)^F)\), then it is \(E\)-split.

(b) Assume that \(\ell\) does not divides \((Z(G)/Z^o(G))^F\). Let \(S\) be a \(\phi_E\)-subgroup of \(G\). Let \(K = C_G(S)\) be the associated \(E\)-split Levi subgroup of \(G\) (1.1.5). Then

\[
\begin{aligned}
& (b.1) Z^o(K)_{\ell}^F = Z(K)_{\ell}^F. \\
& (b.2) \ K = C_G^o(S^F) = C_K^o(Z(K)^F) \text{ and } \ K^F = C_G^o(Z(K)^F)_{\phi_E} = C_G^o(Z(K)_{\phi_E}^F).
\end{aligned}
\]
1.2.5.1. On Proposition 1.2.5. There exist examples of $E$-split Levi subgroup $K$ that does not satisfy the equality $K = C_G^o(Z(K))_\ell^F$ as in assertion (b.2) of Proposition 1.2.5, even with $\ell$ good, but they are deduced from one single extremal case, we want to describe here.

Indeed a good prime may divide the order of $Z(G)/Z^\circ(G)$ only in rational types $A_2$. Assume $K$ is an $E$-split Levi subgroup of $G$ such that $K \neq C_G^o(Z(K))_\ell^F$. Let $M = C_G^o(Z(K))_\ell^F$. Then $M$ is an $E$-split Levi subgroup of $G$ by assertion (a). Furthermore $K$ is an $E$-split Levi subgroup of $M$ and $Z(K)_\ell^F \subseteq Z(M)$. So we are reduced to check the $E$-split Levi subgroups $K$ of $G$ with $Z(K)_\ell^F \subseteq Z(G)$.

Let $K_1 = K \cap [G,G]$. Then $K_1$ is an $E$-split Levi subgroup of $[G,G]$ and $Z(K_1)_\ell^F \subseteq Z^\circ(G)Z(K_1)_\ell^F$, hence $C_G^o(Z(K_1)^F) = Z^\circ(G)C_G^o_{[G,G]}(Z(K_1)_\ell^F)$ so that $Z(K_1)_\ell^F \subseteq Z([G,G])$. The same situation occurs in $[G,G]$, we may assume $G$ semi-simple. In a simply connected covering $\pi: \hat{G} \to G$, $\hat{K} = \pi^{-1}(K)$ is an $E$-split Levi subgroup of $\hat{G}$ and $\pi(Z(\hat{K})^F) \subseteq Z(\hat{K})^F$ so that $G = C_{\hat{G}}^o(Z(\hat{K})^F) \subseteq \pi(C_{\hat{G}}(Z(\hat{K}))^F))$, hence $Z(\hat{K})_\ell^F \subseteq Z(\hat{G})$. Assume $G$ simply connected. $G$ is a direct product of rationally irreducible components and the inclusion occurs in each component. Assume $(G, F)$ of type $A_\ell(\text{eq})$, $\ell \in \{-1, 1\}$. The $E$-split Levi subgroups of such a group are easy to describe. Recall that the $E$-split Levi subgroups of $(G, F) = GL_n(F)$ where $G^F = GL_n(\text{eq})$ are direct products of the form $S \times A_{N(\text{eq})}$ where the polynomial degree of the torus $S$ may be written $\prod_i(X^{d_{eq}(i)} - 1)$ so that $|S^F| = \prod_i |(\text{eq})^{d_{eq}(i)} - 1|$ and $n = N + d. \sum_i d_{eq}(i)$, with the special case $N = 0$ and $|S^F| = \prod_i |(\text{eq})^{d_{eq}(i)} - 1|/(q - \epsilon) - 1$. One sees that, when $G^F = SL_n(\text{eq})$, the condition $Z(K)^F_\ell \subseteq Z(G)$ is satisfied only when $\ell$ divides $(q - \epsilon)$, $n$ divides $(q - \epsilon)\ell$ and $K$ is a so called “Coxeter torus”, that is a maximal $F$-stable torus $T$ such that $|T^F| = |(\text{eq})^n - 1|/\epsilon - 1$ (Coxeter tori are $G^F$-conjugate).

In the following Proposition on centralizers of $\ell$-subgroups, we assume $\ell$ good and use the decomposition $G = G_a.G_b$ in an inductive proof. But in view of a similar results for 2-groups in classical types, see Proposition 1.2.7, we notice that the hypothesis “$\ell$ good” is used only in parts (B) and (D) of the proof.

1.2.6. Proposition. Assume $\ell$ is good for $G$. Let $Y$ be a finite $\ell$-subgroup of $G$.

(a) $C_G^o(Y)$ is an algebraic reductive group.

(b) For any (eventually $F$-stable if $Y \subseteq G^F$) maximal torus $T_Y$ of $C_G^o(Y)$, there exists a maximal (eventually $F$-stable) torus $T$ of $G$ such that $T_Y \subseteq T$ and $Y \subseteq N_G(T)$. Thus $W(C_G^o(Y), T_Y)$ is isomorphic to a subgroup of $W_G(C_G(T_Y)) \cong N_G(C_G(T))/C_G(T_Y)$.

(c) Assume $Y \subseteq G^F$. Let $(T^*, Y^*)$ in duality with $(T, G)$. There exists a finite $\ell$-subgroup $Y'$ of $N_G(T^*)$ and a maximal $F$-stable torus $T_Y^*$ of $C_G^o(Y')$ such that $T_Y^* \subseteq T^*$ and the Levi subgroups $C_G(T_Y)$ of $G$ and $C_G(T_Y^*)$ of $G^*$ are in dual conjugacy classes. Moreover $(T_Y^*, Y')$ and $(T_Y, C_G^o(Y), F)$ are in duality. The groups $N_G(C_G^o(Y), T_Y)$ and $N_G(C_G^o(Y^*), T_Y^*)$ act on the root datum $\Xi(C_G^o(Y), T_Y)$ by contragredient actions.

(d) The group $Z(C_G^o(Y))/Z^\circ(C_G^o(Y))$ is isomorphic to a quotient of $Z(G)/Z^\circ(G)$ and $F(C_G^o(Y))$ is isomorphic to a subgroup of $F(G)$, with $F$-action when $Y \subseteq G^F$.

(e) $A_G(Y)$ is an $\ell$-group. If $F(G)\ell = \{1\}$ and $(Z(G)/Z^\circ(G))\ell = \{1\}$ then $A_G(Y) = \{1\}$. If $Y \subseteq G^F$, $F(G)\ell = \{1\}$ and $(Z(G)/Z^\circ(G))\ell = \{1\} —$ and that is the case when $G = Z^\circ(G)G_b —$ then $A_G(Y)^F = \{1\}$.

(f) If $L$ is a Levi subgroup of $G$ and $Y \subseteq L$, $A_L(Y)$ is isomorphic to a subgroup of $A_G(Y)$. If furthermore $Y \subseteq G^F$, $A_L(Y)^F$ is isomorphic to a subgroup of $A_G(Y)^F$.

Comments. In case $Y$ is abelian, specially contained in a torus, as $\ell$ is good $C_G^o(Y)$ is a Levi subgroup of $G$ and all these properties are well known (see [16] Proposition 13.16 and [13] 2.1). There are similar results for some automorphisms : if $\sigma$ is a quasi-simple automorphism and $(T_\alpha)$ is a maximal torus of $(G^\alpha)^o$, then
C_G(T_n) is a maximal and σ-stable torus of G (see [31] or [21] Theorem 1.8).

When ℓ is not good (c) and (d) may fail.

Proof of Proposition 1.2.6.

Note that any assertion referring to some Frobenius F in the Proposition applies to any finite ℓ-subgroup Y of G thanks to the fact that there exists a power F_i of F such that Y ⊆ G^{F_i}. Thus to prove (a), (b), (d) and (f) one may assume that Y ⊆ G^F. In (e) it is sufficient to prove that

\[ Y ⊆ G^F, \ F(Y)^{\ell} = \{1\} = (Z(G)/Z^G(G))^{\ell} \]  implies \( \Lambda_G(Y)^F = \{1\} \).

In (c) it is said that the morphism \( X(T) \to X(T_Y) \) given by the inclusion \( T_Y \subseteq T \) (see (b)) is direct, with a section \( Y(T_Y^*) \to Y(T^*) \) that is a \( \mathbb{N}_{W(G,T)}(W(C_G(T_Y),T)) \)-morphism. Note that if \( Y \subseteq N_C(T) \), then \( Y \) is a split extension of \( Y \cap T \) by \( Y/Y \cap T \), so \( Y \) is defined by the root datum \( \mathcal{D}(G,T) \). The torus \( T_Y \) is equally defined by \( \mathcal{D}(G,T) : \) one has \( T_Y = (T^Y)^{\circ} = T \cap C_G^o(Y) \).

On the connectivity of centralizers of finite nilpotent \( p' \)-subgroups (resp. ℓ-subgroups) assertion (e) says that \( G \) is a “good guy” (resp. “ℓ-good guy”) if \( Z(G) \) is connected and \( F(G) = 1 \) (resp. \( |Z(G)/Z^G(G)| \) and \( |F(G)| \) are prime to \( ℓ \)). As \( ℓ \) is good, if \( [G,G] \) has no component of type \( A \), \( G \) is an ℓ-good guy. It appears that Proposition 1.2.6 is easily proved using induction in the semi-simple rank for \( G \) an ℓ-good guy (see (D) in the proof below). Independently we verify all assertions when \( G \) is the good guy \( GL_n \) and access to any \( G \) of type \( A \) by a standard way. Thus some groups are considered twice, some partial results are proved twice.

(A) Some general implications:

All the properties in Proposition 1.2.6 go from two groups to their direct product.

(A.i) (a) implies (f).

Indeed \( L \cap C_G^o(Y) = C_{C_G^o(Y)}(Z^o(L)) \) is a Levi subgroup of the algebraic reductive group \( C_G^o(Y) \) so is connected: one has \( L \cap C_G^o(Y) \subseteq C_L^o(Y) \). As \( C_L(Y) = L \cap C_G(Y) \) and \( C_G(Y)/C_G^o(Y) \) is finite, one has \( L \cap C_G^o(Y) = C_L^o(Y) \) and obtains an injective map \( C_L(Y)/C_L^o(Y) \to C_G(Y)/C_G^o(Y) \). If \( Y \subseteq G^F \), that map commute with \( F \)-action.

(A.ii) The first assertion in (b) implies the second one.

One has \( T_Y = N_{C_G^o(Y)}(T_Y) \cap C_G(T_Y) \), hence by isomorphisms theorems \( W(C_G^o(Y),T_Y) = N_{C_G^o(Y)}(T_Y)/T_Y \cong N_{C_G(T_Y)}(T_Y).C_G(T_Y)/C_G(T_Y) \), a subgroup of \( N_C(C_G(T_Y))/C_G(T_Y) \).

(A.iii) The last assertion of (c) is a direct consequence of the preceding one.

By the anti-isomorphism \( W(G,T) \to W(G^*,T^*) \), \( N_C(C_G^o(Y),T_Y)/N_{C_G^o(Y)}(T_Y) \) maps onto \( N_{C_G^o(Y^*),(T_Y^*)}/N_{C_{G^o}(Y^*)}(T_Y^*) \).

(A.iv) (d) follows from

(d') If \( Z(G) = Z^o(G) \) and \( F(G) = \{1\} \), then \( Z(C_G^o(Y)) = Z^o(C_G^o(Y)) \) and \( F(C_G^o(Y)) = \{1\} \).

Given \( G \), there exists a commutative diagram of isotypic morphisms defined over \( \mathbb{F}_q \)

\[
\begin{array}{ccc}
(H,F) & \longrightarrow & (H_0,F) \\
\downarrow \pi & & \downarrow \pi_0 \\
(G,F) & \longrightarrow & (G_0,F)
\end{array}
\]

where horizontal maps are regular embeddings, \( \pi \) and \( \pi_0 \) are quotients by central tori and \( F(H) = F(H_0) = \{1\} \) (see Proposition 1.1.4).
Let $X$, $Y$ be $\ell$-subgroups of $G^F$, $H^F$ with $\pi(X) = Y$. (d') applies to $H_0$ : let $D_0 := C_{H_0}^o(X)$, we have $F(D_0) = \{1\}$ and $Z(D) = Z^*(D)$.

Let $C_0 = C_{G_0}^o(Y)$. By Proposition 1.2.3, $\pi_0(D_0) = C_0$, so that $\pi_0(Z(D_0)) = Z(C_0)$. As $Z(D_0)$ is connected, so is $Z(C_0)$. Let $C = C_0 \cap G$ and $C_0 = Z(G_0)C$, hence $Z(C_0) = Z(G_0)Z(C)$, $Z(C) = Z(C_0) \cap G$. As $Z(C_0)$ and $Z(G_0)$ are connected and $Z(G_0), Z^*(C)$ is connected with finite index in $Z(C_0)$, $Z(C_0) = Z(G_0).Z^*(C)$. Finally $Z(C) = Z(G).Z^*(C)$. By isomorphisms theorems $Z(C)/Z^*(C)$ is isomorphic to $Z(G)/Z^*(C) \cap Z(G)$. As $Z^*(G) \subseteq Z^*(C), Z(G)/Z^*(C) \cap Z(G)$ is a quotient of $Z(G)/Z^*(G)$. All morphisms commute with $F$ and $(Z(C)/Z^*(C))^F = Z(C)^F/Z^*(C)^F$ is a quotient of $Z(G)^F/Z^*(G)^F$.

Let $D = C^o_H(X)$. Clearly $D_0 = Z(H_0)D$, so that $[D, D] = [D_0, D_0]$ and $F(D_0) = \{1\}$ (by (d')) implies $F(D) = \{1\}$. As $\pi(X) = Y$, one has $\pi(D) = C, \pi([D, D]) = [C, C]$ and $[C, C]$ is isomorphic to $[D, D]/\ker \pi \cap [D, D]$. As $[D, D]$ is simply connected, $F(C) = \ker \pi \cap [D, D]$, a subgroup of $F(G) = \ker \pi \cap [H, H]$.

When (c) is satisfied, one sees that $F(C^o_H(Y))$ is isomorphic to $Z(C^o_H(Y'))/Z^o(C^o_H(Y'))$ by Proposition 1.1.3, hence a section of $Z(G^*)/Z^*(G^*)$, isomorphic to a section of $F(G)$.

(a.v) A short exact sequence :

\[(K) \quad 1 \rightarrow (Z^*(G) \cap [G, G])/Z^*(G) \cap C_{[G,G]}^o(Y) \rightarrow C_{[G,G]}^o(Y)/C_{[G,G]}^o(Y) \rightarrow A_G(Y) \rightarrow 1\]

If $Y$ is $F$-stable, the groups in (K) are $F$-stable and morphisms are $F$-morphisms.

We have $G = Z^*(G)\cap [G, G]$ hence $C_G(Y) = Z^*(G)C_{[G,G]}(Y)$ and $C_{G_0}^o(Y) = Z^*(G)C_{[G,G]}^o(Y)$. The two equalities imply that $A_G(Y)$ is a quotient of $C_{[G,G]}(Y)/C_{[G,G]}^o(Y)$. By isomorphism theorems kernels to $\ker \pi \cap [D, D]$, a subgroup of $F(G) = \ker \pi \cap [H, H]$. The property goes from $(G, F)$ to $(G^*, F)$ by (a) in Proposition 1.1.3.

We may assume $[G, G] \neq 1$ and $Y \nsubseteq Z^*(G)$, if not there is nothing to prove.

Let $z \in Z(Z^*(G)Y) \cap [G, G]$. As $\ell$ is good, $C_G^o(z)$ is a Levi subgroup of $G$, hence $F(C_G^o(z))^F = 1$ and $Z(C_G^o(z))^F = Z^*(C_G^o(z))^F$ (b) in Proposition 1.1.3). By Proposition 1.2.4 (a) $C_G^o(z)^F = C_G^o(z)^F$, hence $Y \subseteq C_G^o(z)$. The semi-simple rank of $C_G(z)$ is less than the one of $G$ and induction applies in $C_G(z)$. Let $T_Y$ be a maximal $F$-stable torus in $C_G(z)(Y)$. As $z \in T_Y, C_G(T_Y) \subseteq C_G(z)$, but $C_G(z)(Y) = C_G^o(Y)$ and $C_{C_G(z)}(Y) = C_G(Y)$. A maximal torus $T$ in $C_G(z)$ is maximal in $G$. Thus one obtains (a) and (b) for $Y$ in $G$ by (a) and (b) for $Y$ in $C_G(z)$, and $A_G(Y)^F = \{1\}$ by (c) in $C_G(z)$. Assertion (d) for $Y$ in $G$ is deduced from (d) for $Y$ in $C_G(z)$ and Proposition 1.1.3.)

To apply assertion (c) in $C_G(z)$, choose $L^* := C_G(z)^*$ as a Levi subgroup of $G^*$ : there exists an $\ell$-subgroup $Y'$ of $(L^*)^F$ and a maximal $F$-stable torus $T'$ of $L^*$ such that $Y' \subseteq N_{L^*}(T')$ and $T' = T^* \cap C_{L^*}(Y')$ maximal in $C_{L^*}(Y')$ with dualities between $(T \subseteq C_{G(z)}(T_Y))$ and $(T^* \subseteq C_{L^*}(T_Y^*))$ and between $(T_Y \subseteq C_{C_G(z)}(Y))$ and $(T_Y^* \subseteq C_{L^*}(Y'))$. As $T^*$ is a maximal torus in $G^*$, one has (c) for $Y'$. In $G$.

There is a “non rational version” of the preceding proof, with same steps, and one obtains:

Assume $Y \subseteq G_\ell$ and $F(G)_\ell = \{1\} = Z(G)_\ell/Z^*(G)_\ell$. One has (a), (b), (c), $F(C_G(Y))_\ell = \{1\}$, $Z(C_G(Y))_\ell = Z^*(C_G(Y))^F_\ell$ and $A_G(Y) = \{1\}$.

(C) Type $A$

(C.1) Assume $G = GL_n, Y \subseteq G^F$. 

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The representation of $Y$ on the space $\mathbb{F}^n$ is semi-simple, $\mathbb{F}^n = \oplus_E V_E$, where $E$ is a set of irreducible representations of $Y$, $V_E$ is isotypic with some multiplicity $m(E)$. With these notations one has an isomorphism $C_{G}(Y) \cong \prod_E \text{GL}_{m(E)}$. So $C_{G}(Y)$ is connected with a connected center, (a), (d) and (e) hold.

A maximal $F$-stable torus $T_Y$ of $C_{G}(Y)$ is defined by a decomposition of $\mathbb{F}^n$ as an $F$-stable direct sum $\oplus_{i \in I} V_i$ of irreducible $\mathbb{F} Y$-modules (Schur's lemma). A maximal torus of $G$ such that $T_Y \subseteq T$ and $Y \subseteq N_G(T)$ is defined by a choice in each $F$-space $V_i$ of an $Y$-stable family of generating and linearly independant lines, and $T$ is $F$-stable if the family is $F$-stable. Such a family exists because any representation of a nilpotent finite group is monomial. As $F$ acts by permutation on the set $I$ an $F$-stable family exists and defines $T$. Hence (b) holds.

One knows that $G = \text{GL}_n$ is isomorphic to $G^*$. Assertion (c) is clear in that case.

(C.2) Assume $Z(G)$ is connected.

There exists a regular covering $\pi: K := Z^*(G),H \rightarrow G$ where $H$ is a covering of $[G, G]$ and a direct product of linear groups, and $Z^*(G),H$ a central product. Furthermore the restriction of $\pi$, $K^F \rightarrow G^F$ is onto. There is some $\ell$-subgroup $X$ of $K^F$ such that $\pi(X) = Y$.

By (C.1) our Proposition is satisfied in $H$. As well it is satisfied in $K$. For any finite $\ell$-subgroup $Y$ of $K$, there exists a finite $\ell$-subgroup $Y_1$ of $H$ with $Y_1 Z^*(G) = Y Z^*(G)$. As for (a) (b) and (d) for $Y$ it follows from the equalities $K = Z^*(G),H$, $Z^*(K) = Z^*(G),Z^*(H)$ and (a) (b) and (d) for $Y_1$. As for (c), $K^*$ is a central product $Z^*(G),H^*$ and a couple of dual Levi subgroups in duality in $H$ and $H^*$ give a couple of dual Levi subgroups in $K$, $K^*$. The sequence $(K)$ in (A.v) gives $A_K(Y) = A_H(Y_1) = 1$.

A special case is $Z^*(G) = 1$, that is $G$ adjoint. By Proposition 1.2.3 $\pi(C^*_H(X)) = C^*_G(Y)$. By Lemma 1.2.1 $C_G(Y)/\pi(C_H(X))$ is an $\ell$-group. Thus $A_G(Y)$ is an $\ell$-group. The exact sequence $(K)$ implies that $C_{[H, H]}(X)/C^*_{[H, H]}(X)$ is an $\ell$-group for any finite $\ell$-subgroup $X$ of $H$. This apply to any morphism $G \rightarrow G_{ad}$ hence

in the sequence $(K)$ the group $C_{[G, G]}(X)/C^*_{[G, G]}(X)$ is an $\ell$-group. So is $A_G(Y)$ for any $G$ of type $A$.

Applying once more Proposition 1.2.3 and Lemma 1.2.1 to $\pi: K \rightarrow G$ with $\pi(X) = Y$ and to $K^F \rightarrow G^F$ if $Y \subseteq H^F$, one obtains (a), (b), (e) (recall that $\text{Ker} \pi \cap [K, K]$ is isomorphic to $F(G)$) and $Z(C^*_G(Y)) = Z^*(C^*_G(Y))$. On dual side one has a regular embedding $\pi^*: G^* \rightarrow K^*$. A dual $C^*_{K^*}(X')$ of $C^*_K(X)$ may be written $Z^*(K^*),C^*_{\pi^*(G^*)}(\pi^*(Y'))$ for some finite $\ell$-subgroup $Y' \subseteq G^*$ and one obtains easily (c) for $Y$ in $G$ from (c) for $X$ in $K : C^*_G(\pi^*(Y')) is isomorphic to $C^*_{\pi^*(G^*)}(\pi^*(Y')) in duality with $C^*_G(Y)$.

(C.3) End of the proof in type $A$.

Let us consider a regular imbedding $G \subseteq H$ defined over $\mathbb{F}_q$ with $F$-actions. One has $F(G) = F(H)$, $Z(H) = Z^*(H)$. By (C.2), (a) to (e) hold in $H$. For any $\ell$-subgroup $Y$ of $G$ one has $C^*_H(Y) = Z^*(H),C^*_G(Y)$, (a) and (b) for $Y$ in $H$ imply (a) and (b) for $Y$ in $G$, with $T_Y = Z^*(H),(T_Y \cap G), T_Y$ maximal in $C_H(Y)$, $T_Y \cap G$ maximal in $C^*_G(Y)$, $T = Z^*(H),(T \cap G)$.

As for (d), one knows that $Z^*(H),Z(C^*_G(Y)) = Z^*(C^*_H(Y)) = Z^*(H),Z(C^*_G(Y))$, hence $Z^*(C^*_G(Y)) \subseteq Z^*(H),Z(C^*_G(Y))$ and $Z(C^*_G(Y)) \subseteq Z^*(H) \cap Z^*(C^*_G(Y)) = Z^*(H) \cap Z(C^*_G(Y))$ are isomorphic. But $Z^*(H) \cap G = Z(G)$ and $Z^*(G) \subseteq C^*_G(Y)$, thus $(Z^*(H) \cap Z(C^*_G(Y)))/(Z^*(H) \cap Z^*(C^*_G(Y)))$ is isomorphic to a section of $Z(G)/Z^*(G)$.

By a covering dual of the embedding, say $\pi^*: G^* \rightarrow H^*$, then $\pi^*(T^*)$ is a dual of $T \cap G$. If a group $Y'$ is given by (c) for $Y$ in $H$, then the duality between $(T_Y \subseteq C^*_H(Y))$ and $(T^*_Y \subseteq C^*_H(Y'))$ induces a duality between $(T_Y \cap G \subseteq C^*_G(Y))$ and $(\pi^*(T_Y) \subseteq C^*_G(\pi^*(Y'))).$ If $C_H(T_Y) \cap C_G(T^*_Y)$ are dual Levi subgroups, so are $C_G(T_Y \cap G)$ and $(C_G(\pi^*(T^*_Y))$. That is (c) for $Y$ in $G$.

To compare $A_G(Y)$ and $A_H(Y)$ use the sequence $(K)$. The two groups are quotient of the same $\ell$-group
with included kernels: with $C^o = C^o_{[G,G]}(Y)$, $Z^o(G)\cap [G,G]/Z^o(G)\cap C^o$ injects in $Z^o(H)\cap [G,G]/Z^o(H)\cap C^o$, hence $A_H(Y)$ is a quotient of $A_G(Y)$ whose cokernel is $F$-isomorphic to a section of $Z([G,G])/Z^o(G)\cap C^o$, hence of $Z(G)/Z^o(G)$. If $Z(G)^F = Z^o(G)^F$ and $F(G)^F = 1$, then $A_H(Y)^F$ and $A_G(Y)^F$ are isomorphic and $A_H(Y)^F = 1$ by (C.2), that is the “rational part” of (e).

(D) End of the proof.

Assuming $Y \subseteq G^F$, we use the decomposition in central product defined in section 1.1.5: put $G'_b = Z^o(G).G_b$. Then $G = G_a.G'_b$ and $Z(G)/Z^o(G)$ is isomorphic to a direct product $Z(G_a)/Z^o(G) \times Z(G'_b)/Z^o(G)$.

Any $y \in G^F$ writes in a unique way in $(G_a)^F \times (G'_b)^F$. Let $Y_a$ and $Y_b$ be the projections of $Y$ on $(G_a)^F$ and $(G'_b)^F$. One has $C_G(Y) = C_{G_a}(Y_a).C_{G'_b}(Y_b)$, $C^o_G(Y) = C^o_{G_a}(Y_a).C^o_{G'_b}(Y_b)$, so that $A_G(Y)$ is isomorphic to $A_{G_a}(Y_a) \times A_{G'_b}(Y_b)$. By definition of $G'_b$, $Z(G)^F/Z(G)^F$ is isomorphic to $Z(G_a)^F/Z(G'_b)^F$ and $F(G)^F$ is isomorphic à $F(G_a)^F$. The assertions (a), (d) and (e) are true for $Y_a$ in $G_a$ and $Y_b$ in $G'_b$, there are true for $Y$ in $G$.

The properties required in (b) and (c) go up from $G_a$ and $G'_b$ to $G$ by straightforward constructions.

With clear notations $T_Y = (T_Y \cap G_a).(T_Y \cap G'_b) = T_{Y_a}T_{Y_b}$, $T = (T \cap G_a).(T \cap G'_b) = T_aT_b$ (central products on $Z^o(G)$), with $T_{Y_a} = (T^{Y_a})^o$, $T_{Y_b} = (T^{Y_b})^o$, hence $(T^Y)^o = T_Y$, $W(C^o_G(Y),T_Y) \cong W(C^o_{G_a}(Y_a),T_{Y_a}) \times W(C^o_{G'_b}(Y_b),T_{Y_b})$.

Dualities between $(T^{Y_a} \subseteq C^o_{G_a,*}(Y_a^*))$ and $(T_{Y_a} \subseteq C^o_{G_a,*}(Y_a^*))$, between $(T^{Y_b} \subseteq C^o_{G'_b,*}(Y_b^*))$ and $(T_{Y_b} \subseteq C^o_{G'_b,*}(Y_b^*))$ define, through the morphism $\rho: G^* \rightarrow G_a^* \times (G'_b)^*$, whose kernel is a central torus, a duality between $(T^Y \subseteq C^o_G(Y^*))$ and $(T^Y \subseteq C^o_G(Y^*))$ where $T^Y = \rho(Y_a^* \times Y_b^*)$ and $T^Y = \rho(T^{Y_a} \times T^{Y_b})$.

Similarly dualities between $(T_a \subseteq C_{G_a,*}(T_{Y_a}))$ and $(T_a^* \subseteq C_{G_a,*}(T_{Y_a}^*))$, between $(T_b \subseteq C_{G'_b,*}(T_{Y_b}))$ and $(T_b^* \subseteq C_{G'_b,*}(T_{Y_b}^*))$ give a duality between $(T \subseteq C_{G}(T_Y))$ and $(T^* \subseteq C_{G,*}(T^Y))$ where $T^* = \rho(T_a^* \times T_b^*)$. 

1.2.7. Proposition. Let $(G,F)$ be of classical type in odd characteristic, let $Y$ be a 2-subgroup de $G^F$.

(a) $C^o_G(Y)$ is reductive.

(b) For any (eventually $F$-stable if $Y \subseteq G^F$) maximal torus $T_Y$ of $C^o_G(Y)$, there exists a maximal (eventually $F$-stable) torus $T$ of $G$ such that $T_Y \supseteq T$ and $Y \subseteq N_G(T)$. Thus $W(C^o_G(Y),T_Y)$ is isomorphic to a subgroup of $W_G(C^o_G(T_Y)) \cong N_G(C^o_G(T_Y))/C^o_G(T_Y)$.

(c) If $G$ has no component of type B or C, then assertion (c) with $\ell = 2$ of Proposition 1.2.6 is true.

(d) The group $Z(C^o_G(Y))/Z^o(C^o_G(Y))$ is isomorphic to a quotient of $Z(G)/Z^o(G)$ and $F(C^o_G(Y))$ is isomorphic to a subgroup of $F(G)$, with $F$-action when $Y \subseteq G^F$.

(e) $A_G(Y)$ is an 2-group. If $F(G)^2 = \{1\}$ and $(Z(G)/Z^o(G))^2 = \{1\}$ then $A_G(Y) = \{1\}$. If $Y \subseteq G^F$, $F(G)^F = \{1\}$ and $(Z(G)/Z^o(G))^F = \{1\}$ then $A_G(Y)^F = \{1\}$.

(f) If $L$ is a Levi subgroup of $G$ and $Y \subseteq L$, $A_L(Y)$ is isomorphic to a subgroup of $A_G(Y)$. If furthermore $Y \subseteq G^F$, $A_L(Y)^F$ is isomorphic to a subgroup of $A_G(Y)^F$.

Proof. (a sketch of) We have seen in the proof of Proposition 1.2.6, using central products and regular isotypic morphisms, how to reduce to rationally irreducible types. Details are left to the reader.

In type A 2 is good, Proposition 1.2.6 applies.

Let $f$ be a non degenerate bilinear form on a space $V$ on $F$, defined on $F_q$, assume $G = SO(f)$. Thus $G^F$ is a symplectic or orthogonal group on $F_q$.

Under $Y$-action $V$ is a direct and orthogonal sum of isotypic $F[Y]$-modules

$$V = V_+ \oplus V_- \oplus (\oplus_{(E,F)}(V_E \oplus V_E'))$$
Here we have denoted $V_+$ (resp. $V_-$) the space of fixed points (resp. antifixed points: $y.v = -v$ for all $y \in Y$) and $\{E, E'\}$ belongs to a set of isomorphism classes of pairs of contragredient irreducible representations. Let $f_+$ (resp. $f_-$) be the restrictions of $f$ to $V_+$ (resp. $V_-$). Then $C_G(Y)$ is the intersection with $G$ (inside $\text{GL}(V)$) of the direct product

$$O(f_+) \times O(f_-) \times (\times_{\{E,E'\}} \text{GL}(m(E)))$$

Any factor $\text{GL}(m(E))$ is contained in $G$, so it acts on $V_E \oplus V_{E'}$ where $V_E$ and $V_{E'}$ are totally isotropic spaces with dual basis. One sees that $C_G(Y)$ is a 2-group. These properties are preserved by isotypic morphisms. There exists a group $(G_0, F)$ of same type such that $Z^{\circ}(G_0) = Z(G_0)$ and $\mathcal{F}(G_0) = 1$. One may verify that, for any 2-subgroup $Y_0$ of $G_0^F$, $C_{G_0}(Y_0)$ and $Z(C_{G_0}(Y_0))$ are connected, and $\mathcal{F}(C_{G_0}(Y_0)) = 1$. One deduce (d), (e) and (f) as in Proposition 1.2.6.

In types $\textbf{A}$ and $\textbf{D}$ there exists groups isomorphic to a dual, so (c) in Proposition 1.2.6 is true.

1.3. Jordan decomposition in $\text{Irr}(G^F)$

The properties we recall in that section are essentially due to G. Lusztig [26], [27]. In Proposition 1.3.2 we state what we need on Jordan decomposition of irreducible characters when the center of $G$ is connected.

As a consequence of “non-multiplicity condition” in an isotypic morphism $H \to G$ (Proposition 1.3.3), Propositions 1.3.6, 1.3.7 describe the link between Jordan decompositions in an isotypic morphism.

1.3.1. Some facts and notations.

Let $L$ be an $F$-stable Levi complement of the unipotent radical of a parabolic subgroup $P$ of $G$, then is defined a virtual $\mathcal{O}[G^F \times L^F]$-bimodule, where $\mathcal{O}$ may be the ring of an $\ell$-modular splitting system $(\mathcal{O}, K, k)$ (see 1.1.2), and, as a consequence. ” Deligne-Lusztig induction ”, which maps any $\mathcal{O}L^F$-module to a virtual $\mathcal{O}G^F$-module, see [20] chapter 11. By extension from $\mathcal{O}$ to $K$ and linearity is defined a linear map from $\mathcal{CF}(K, L^F)$ to $\mathcal{CF}(K, G^F)$, “ twisted induction” in [16] §8.3, where it is denoted $R_{L \subseteq P}^G$. Denote $R_{L \subseteq P}^G$ the linear map $\epsilon_G \epsilon_L R_{L \subseteq P}^G$ where $\epsilon_G = (-1)^{r(G)}$, and $r(G)$ is the semi-simple $F_q$-rank of $G$, we name it Lusztig induction, the dual map with respect to standard scalar products on spaces of central functions is called Lusztig restriction and denoted $^*R_{L \subseteq P}^G$. In an isotypic morphism $\sigma: (G, F) \to (H, F)$, as defined in 1.1.4, one has, by [6] Proposition 1.1 and with evident notations

$$(1.3.1.1) \quad \text{Res}_{\sigma: G^F \to H^F} \circ R_{M \subseteq Q}^H = R_{\sigma^{-1}(M) \subseteq \sigma^{-1}(Q)}^G \circ \text{Res}_{\sigma: \sigma^{-1}(M)^F \to M^F}$$

The character formula gives ([20] 12.17)

$$(1.3.1.2) \quad \forall \chi \in \mathcal{CF}(K, L^F) \quad (R_{L \subseteq P}^G \chi)(1) = \frac{|G^F|_{p'}}{|L^F|_{p'}} \chi(1)$$

Some properties, as formula (1.3.1.2), may be independant of the choice of $P$, given $L$, and the notation $R_{L \subseteq P}^G$ is frequently simplified in $R_L^G$. The map $R_{L \subseteq P}^G$ itself may be independant of the choice of $P$; it is the case if $L$ is a torus, see also Definition 2.2.1 and Proposition 2.2.4. The subspace of $\mathcal{CF}(K, G^F)$ generated by $\cup_T R_T^G(\mathcal{CF}(K, T^F))$ when $T$ ranges over $F$-stable maximal tori of $(G, F)$, is called the space of uniform functions on $G^F$. The orthogonal projection on the space of uniform functions of a central function $\chi$ on $G^F$ will be denoted $\pi_{\text{un}}^G(\chi)$; one has ([20] 12.12)

$$(1.3.1.3) \quad \pi_{\text{un}}^G(\chi) = \sum_{\{(T, \theta)\}/G^F} |W(G, T)^F|^{-1} \langle R_T^G \theta, \chi \rangle_{G^F} R_T^G \theta$$
By Mackey decomposition formula ([20] 11.13) and (1.3.1.3) \( \pi_{un} \) commute with Lusztig induction and restriction:

\[
(1.3.1.4) \quad \pi_{un}^L \circ R^G_L = R^G_L \circ \pi_{un}^G, \quad \pi_{un}^G \circ R^G_L = R^G_L \circ \pi_{un}^L
\]

Since the regular representation of \( G^F \) is a uniform function ([20] 12.14) \( \pi_{un} \) preserves the value on 1.

Let \((G, F)\) and \((G^*, F)\) be in duality and let \( s \in G^{*F} \) be semi-simple, the rational Lusztig series \( \mathcal{E}(G^F, s) \) is a subset of \( \text{Irr}(G^F) \). The set \( \mathcal{E}(G^F, s) \) is defined by the following property: for any couple of \( F \)-stable maximal torus \((T \subseteq G, T^* \subseteq G^*)\) in dual conjugacy classes, if \( \theta \in (T^F)^\wedge \) corresponds by duality to \( s \in T^* \), then \( R^G_T \theta \) writes in \( \mathcal{Z}(G^F, s) \). One has a partition \( \text{Irr}(G^F) = \bigcup_{(s)} \mathcal{E}(G^F, s) \) where \((s)\) ranges over \( G^{*F} \)-conjugacy class of semi-simple elements of \( G^{*F} \), see [20] 14.41, [16] § 8.4. Thus if, with evident notations, \((T_1, \theta_1)\) is not \( G^F \)-conjugate to \((T_2, \theta_2)\) — and that is equivalent to non \( G^{*F} \)-conjugacy between \((T^*_1, s_1)\) and \((T^*_2, s_2)\) see [20] 13.13 — then \( R^G_{T_1} \theta_1 \) and \( R^G_{T_2} \theta_2 \) have no common irreducible constituent.

Elements of \( \mathcal{E}(G^F, 1) \) are said to be unipotent. If \( \sigma : (G, F) \to (H, F) \) is an isotypic morphism, the restriction through \( \sigma \) restricts to a one-to-one map \( \mathcal{E}(H^F, 1) \to \mathcal{E}(G^F, 1) \) [20] 13.20.

One has \( \pi_{un}^G(K\mathcal{E}(G^F, s)) \subseteq K \mathcal{E}(G^F, s) \) and the orthogonal projection on the space \( K \mathcal{E}(G^F, s) \) commute with \( \pi_{un}^G \). Let us design by \( \pi_{un}^{G,s} \) the product of these projections:

\[
(1.3.1.5) \quad \pi_{un}^{G,s} : K \text{Irr}(G^F) \to \pi_{un}^G(K \mathcal{E}(G^F, s)) = \pi_{un}^G(K \text{Irr}(G^F)) \cap K \mathcal{E}(G^F, s)
\]

Lusztig [26] has defined an orthogonal basis \( \{ R^G_{T} \}_{f} \) of \( \pi_{un}^G(K \mathcal{E}(G^F, 1)) \) with indexation in some set of representations of \( W(F) \). There is a partition \( \mathfrak{F} \) of that set in so called families, defining a decomposition of \( \pi_{un}^G(K \mathcal{E}(G^F, 1)) \) in an orthogonal sum of subspaces \( \mathcal{F}(G, 1, f) = \sum_{f \in \mathfrak{F}} K R^G_f \), such that for any \( \chi \in \mathcal{E}(G^F, 1) \) there exists a unique family \( f \in \mathfrak{F} \) such that \( \pi_{un}^G(\chi) \in \mathcal{F}(G, 1, f) \):

\[
(1.3.1.6) \quad \pi_{un}^G(K \mathcal{E}(G^F, 1)) = \bigcup_{f \in \mathfrak{F}} \mathcal{F}(G, 1, f), \quad \mathcal{E}(G^F, 1) = \bigcup_{f \in \mathfrak{F}} \mathcal{E}(G^F, 1, f), \quad \pi_{un}^G(\mathcal{E}(G^F, 1, f)) \subseteq \mathcal{F}(G, 1, f)
\]

The decomposition of \( \pi_{un}^G(\chi) \) on the basis \( \{ R^G_{T} \}_{f} \) of \( \mathcal{F}(G, 1, f) \) is known, see [26], Chapter 4, for details.

When \( s \in (G^*)^F \) one defines \( \mathcal{E}_t(G^F, s) := \bigcup_{s \in C_{G^*}(s)} \mathcal{E}(G^F, st) \). We denote \( \text{Bl}(G^F; s) \) the set of blocks \( b \) of \( G^F \) such that \( \text{Irr}(b) \cap \mathcal{E}_t(G^F, s) \neq \emptyset \). By [9] (see [16] Theorem 9.12)

\[
(1.3.1.7) \quad b \in \text{Bl}(G^F; s) \text{ if and only if } \text{Irr}(b) \cap \mathcal{E}_t(G^F, s) \neq \emptyset \text{ if and only if } \text{Irr}(b) \cap \mathcal{E}(G^F, s) \neq \emptyset
\]

An element of \( \text{Bl}(G^F; 1) \) is said to be unipotent.

When \( G \) is not connected we define \( \mathcal{E}(G^F, 1) \) as the set of \( \chi \in \text{Irr}(G^F) \) that cover some \( \chi^s \in \mathcal{E}(G^s, 1) \). Elements of \( \mathcal{E}(G^F, 1) \) may be defined as the irreducible components of the \( R^G_T \), with suitable definitions of so called quasi-torus \( T \), and of Lusztig induction [21]. Similarly an \( \ell \)-bloc \( B \) of \( G^F \) is said to be unipotent if it covers a unipotent block of \( G^s, \) or equivalently if \( \text{Irr}(B) \cap \mathcal{E}(G^F, 1) \neq \emptyset \).

In the following Proposition we introduce one of our first important tool, the Jordan decomposition in \( \text{Irr}(G^F) \), see [26] 4.23, [20] 13.24. Recall that, when \( Z(G) \) is connected and \( s \) is semi-simple in \( G^* \), \( C_{G^*}(s) \) is connected (Propositions 1.1.3 and 1.2.4). By general conventions we assume that the duality between \( L \) and \( L^* \), \( G \) and \( G^* \) are defined around the same pair of dual maximal tori \((T \subseteq L, T^* \subseteq L^*)\) and so on for any “dual” sets \( \{ L_{ij} \}_{j} \), \( \{ L^*_{ij} \}_{j} \) of Levi subgroups of \( G, G^* \) with a common maximal torus \( T \subseteq L_j \), \( T^* \subseteq L_j^* \). Similar restrictions apply to dualities over \( \mathbb{F}_q \) between \( F \)-stable groups. It follows that the choice of duality between \( L \) and \( L^* \), hence the eventual different choices, if \( s \in L^* \), of \( \Psi_{L,s} \) do not affect \( R^G_L \circ \Psi_{L,s} \).
1.3.2. Proposition. For any connected algebraic reductive group \((G, F)\) defined on \(\mathbb{F}_q\), with connected center, and dual \((G^*, F^*)\), there exist one-to-one maps, named Jordan decompositions,

\[
\Psi_{L,s}: \mathcal{E}(C_{L^*}(s)^F, 1) \rightarrow \mathcal{E}(L^*, s)
\]

defined for any couple of \(F\)-stable Levi subgroups \((L \subseteq G, L^* \subseteq G^*)\) in dual conjugacy classes and any semi-simple element \(s\) in \(L^*F\) and that satisfy the following properties:

- Extend \(\Psi_{L,s}\) by linearity
  \[
  \Psi_{L,s}: K\mathcal{E}(C_{L^*}(s), 1) \rightarrow K\mathcal{E}(L^*, s)
  \]

  - (i) On orthogonal projections on the spaces of uniform functions:
    
    For any couple \((T \subseteq L, T^* \subseteq L^*)\) of \(F\)-stable maximal tori in dual conjugacy classes with \(s \in T^*F\) and any \(\lambda \in \mathcal{E}(C_{L^*}(s)^F, 1)\), one has
    
    \[
    (\Psi_{L,s}^T(\lambda), R^L_T (\Psi_{T,s}(1_{T^*F})))_{L^F} = (\lambda, R^C_{T,s}(1_{T^*F}))_{C_{L^*}(s)^F}
    \]

    
    \[
    R^L_T (\Psi_{T,s}(1_{T^*F})) = \Psi_{L,s}(R^C_{T,s}(1_{T^*F}))
    \]

    - (ii) Assume \(s\) is central in \(L^*\), the duality between \((L, F)\) and \((L^*, F)\) defines \(\hat{s} := \Psi_{L,s}(1_{L^F}) \in \text{Irr}(L^F)\) such that \(\hat{s}(1) = 1\). For any semi-simple \(t \in L^*F\), one has \(E(L^F, st) = \hat{s} \otimes E(L^F, t)\) and, for any \(\lambda \in \mathcal{E}(C_{L^*}(t)^F, 1)\),

    \[
    \Psi_{L, st}(\lambda) = \hat{s} \otimes \Psi_{L,t}(\lambda).
    \]

    - (iii) If \(C_{G^*}(s) \subseteq L^*\), then for any parabolic subgroup \(P\) of \(G\) with Levi complement \(L\) one has

    \[
    R^G_{L \subseteq P} \circ \Psi_{L,s} = \Psi_{G,s}.
    \]

    - (iv) Let \((\sigma, \sigma^*)\) be an isomorphism from \(((L, F), (L^*_1, F))\) to \(((L_1, F), (L^*_1, F))\) and \(s = \sigma^*(s_1)\). Then \((\sigma, \sigma^*)\) induces isomorphisms \(\tau: L^F \rightarrow L^F_1\) and \(\tau^*: C_{L^*_1}(s_1)^F \rightarrow C_{L^*}(s)^F\), and there is a commutative diagram with one-to-one maps

    \[
    \begin{array}{ccc}
    \mathcal{E}(C_{L^*_1}(s_1)^F, 1) & \xrightarrow{\text{Res}_{s_1}} & \mathcal{E}(C_{L^*}(s)^F, 1) \\
    \Psi_{L_1,s_1} & & \Psi_{L,s} \\
    \mathcal{E}(L^F_1, s_1) & \xrightarrow{\text{Res}} & \mathcal{E}(L^F, s)
    \end{array}
    \]

    - (v) \(\Psi_{L,s}\) is “functorial with respect to central products in \(L^*\).

    - (vi) For any \(\lambda \in \mathcal{E}(C_{L^*}(s)^F, 1)\), one has \(\Psi_{L,s}(\lambda)(1) = \frac{[L^F]_\lambda}{[L^*]_{s\lambda}}\lambda(1)\).

Comments.

The case \(s = 1\): \(\Psi_{G,1}: \mathcal{E}(G^*F, 1) \rightarrow \mathcal{E}(G^*F, 1)\) is a bijection between unipotent series of groups in duality. The hypothesis “\(Z(G)\) is connected” may be dropped (see Proposition 1.3.6 for a general statement in case \(Z(G)\) is not connected).

In case of tori \(T, T^*\) in dual conjugacy classes for \((L, L^*)\)—and so for \((G, G^*)\) — \(\Psi_{T,s}(1_{T^*F})\) is the image of \(s\) by an isomorphism \(T^*F \rightarrow (T^F)^\lambda\); the construction of it assume choices of isomorphisms \((\mathbb{Q}/\mathbb{Z})_{\mu'} \cong \mathbb{Q}^\times_{\mu'}\) and \((\mathbb{Q}/\mathbb{Z})_{\mu'} \cong \mathbb{F}^\times\), that are fixed one for all, and depends of the duality between \(T\) and \(T^*\). Thanks to [20] 13.13 \(R^L_T (\Psi_{T,s}(1_{T^*F}))\) is well defined as it is implicitly assumed in (J1), (J2). Recall the notation

\[
(1.3.2.1)\quad R^L_T (s) := R^L_T (\Psi_{T,s}(1_{T^*F}))
\]

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The relative Weyl groups \( W_G(T)^F \) and \( W_{C_G^*}(s)^F \) are isomorphic. By [20] 11.16 we have
\[
\langle R_G^T, s, R_T^G, s \rangle_{G^F} = \langle R_{C_G^*}^{C_G^*}(s) \rangle_{C_G^*(s)^F}.
\]
By [20] 13.12, 12.12 and (J1) and (J2) above \( \Psi_{G,s} \) send an orthogonal basis of \( \pi_{un}^{C_G^*}(s)(KE(C_G^*(s)^F), 1) \) to an orthogonal basis of \( \pi_{un}^{\alpha}(KE(G,F, s)) \), "Jordan decomposition commute with \( \pi_{un}^{\alpha} \) and restricts to an isomorphism of metric spaces

\[
\Psi_{G,s} \circ \pi_{un}^{C_G^*}(s) = \Psi_{G,s} \circ \pi_{un}^{\alpha}(s), \quad \Psi_{G,s} : \pi_{un}^{C_G^*}(s)(KE(C_G^*(s)^F), 1) \rightarrow \pi_{un}^{\alpha}(KE(G,F, s))
\]
(here we have used notation (1.3.1.5)). More precisely by \( \Psi_{G,s} \) and the decomposition (1.3.1.5) we may define families in \( E(G,F, s) : E(G,F, s)_f := \Psi_{G,s}(E(C_G^*(s)^F, 1)), \mathcal{FU}(G, s, f) := KE(G,F,s)_f \)

\[
\Psi_{G,s}(\mathcal{FU}(C_G^*(s), 1, f)) = \mathcal{FU}(G, s, f)
\]
In (1.3.2.3) the set \( \mathfrak{f} \) is defined from \( (W(C_G^*(s), F), \cdot) \).

Let \( M \) and \( M^* \) be Levi subgroups in duality of \( L \) and \( L^* \) respectively, such that \( s \in M^* \), then \( C_{M^*}(s) \) is a Levi subgroup of \( C_L^*(s) \). Let \( \alpha \in E(C_L^*(s)^F, 1) \), \( \lambda = \Psi_{L,s}(\alpha) \). Let be any couple of \( F \)-stable maximal tori \( (T \subseteq M, T^* \subseteq M^*) \) in dual conjugacy classes with \( s \in T^* \), put \( \theta = \Psi_{T,s}(1_{T,F}) \). We have \( R_{T,F}^L \theta = \Psi_{M,s}(R_{T,F}^{C_{M^*}(s)} 1_{T,F}) \). By transitivity of Lusztig induction and (J.1) we have

\[
\langle R_{M^*}^L \theta, R_{T,F}^L \theta \rangle_{M^*F} = \langle R_{L^*}^L \theta, 1_{T,F} \rangle_{T,F} = \Psi_{C_{M^*}(s)}(s, 1_{T,F})_{T,F} = \Psi_{C_{M^*}(s)}(s, 1_{T,F})_{T,F}
\]
Thus we enforce (1.3.2.2) in another commutation formula with Lusztig induction, we may combine with (1.3.1.4). The preceding equalities give

\[
\Psi_{L,s} \circ R_{C_{M^*}(s)}^{C_{M^*}(s)}(s) \circ \pi_{un}^{\alpha}(s)(\alpha) = (\pi_{un}^{M^*}(s) \circ R_{C_{M^*}(s)}^{\alpha}(s))_{T,F}(s) = (\pi_{un}^{\alpha}(s) \circ R_{C_{M^*}(s)}^{\alpha}(s))_{T,F}(s)
\]
and by adjunction

\[
\Psi_{L,s} \circ R_{C_{M^*}(s)}^{C_{M^*}(s)}(s) \circ \pi_{un}^{\alpha}(s)(\alpha) = (\pi_{un}^{M^*}(s) \circ R_{C_{M^*}(s)}^{\alpha}(s))_{T,F}(s) = (\pi_{un}^{\alpha}(s) \circ R_{C_{M^*}(s)}^{\alpha}(s))_{T,F}(s)
\]

The functoriality of the Jordan decomposition (assertion (iv)) apply in the following situation:

Let \( (L_j, \alpha) \) and \( (L_j^*, \alpha) \) be two pairs \( (j = 1, 2) \) of \( F \)-stable Levi subgroups of \( G \) and \( G^* \) respectively, \( L_j \) and \( L_j^* \) in dual classes. Assume that for some \( g \in G^F \) and \( s \in (L_1 \cap L_2^*)^F, (L_1, E(L,F,s)^g = (L_2, E(G,F,s)).
\]
The duality between \( L_j \) and \( L_j^* \) is defined around \( F \)-stable couples \( (T_1, T_1) \) with root data in duality. As \( g \) is defined modulo \( N_{G,F}(L, E(L,F,s)) \) we may assume that \( g \) sends such a couple \( (T_1, T_2) \) onto \( (T_1, T_2) \), assuming by choice \( s \in T_2^F \). Then there exists \( g^* \in G^F \) that sends \( (T_2 \subseteq B_2) \) onto \( (T_1 \subseteq B_1) \) so that \( g^* = s \). Then \( g \) and \( g^* \) induce by interior automorphisms dual morphisms \( L_1 \rightarrow L_2 \) and \( L_2 \rightarrow L_1 \)

As for (v) functoriality is as follows: assume \( G = G_1 \times G_2 \) a central product over a torus of \( F \)-stable groups \( G_j \). Let \( \pi_G : G_1 \times G_2 \rightarrow G \) so defined and let \( \pi^* : G^* \rightarrow G^*_1 \times G^*_2 \) a dual morphism. Put \( \pi^*(s) = (s_1, s_2) \), with \( s_i \in (G_i^*)^F \), so that \( \pi^*(C_{G_j^*}(s)) = C_{G_i^*}(s_1) \times C_{G_j^*}(s_2) \). Similarly \( L_1 = \pi(L_1 \times L_2) \) and \( \pi^*(L^*) = L_1^* \times L_2^* \). By what we saw in case of unipotent series one has a one-to-one map \( \sigma : E(L_1^*(s_1)^F, 1) \times E(L_2^*(s_2)^F, 1) \rightarrow E(L_{G_1^*}(s_1)^F, 1) \). As \( (s_1, s_2) = \pi^*(s) \), elements of \( E(L_1^*, s_1) \) and \( E(L_2^*, s_2) \) have equal restriction on the kernel of \( \pi \) that is the restriction of \( s_j \) in the notation of assertion (ii). Then, using the symbol \( \otimes \) in two different senses, one may write \( \Psi_{L,s}(\sigma(\alpha_1 \otimes \alpha_2)) = \Psi_{L_1,s_1}(\alpha_1) \otimes \Psi_{L_2,s_2}(\alpha_2) \).
1.3.3. Remarks. A generalization of (J.2) in the form

$$(J.3) \quad R_{L^P}^G \circ \Psi_{L,s} = \Psi_{G,s} \circ R_{C_L^L(s)}^{C_G^*}$$

with parabolic subgroups suitably defined or, better!, dropped, would be quite useful, as well as Mackey decomposition formula for Lusztig induction [20] chapter 11.

(a) In type A with connected center all central functions are uniform. In that case (J.2) implies (J.3). Asai and Shoji have shown that (J.3) is true in classical type with connected center for any $L$, see [29] and [24] Appendix, and unicity of Jordan’s decomposition follows in that case. In our Appendix, Proposition 5.3, we give an elementary proof of (J3) in classical types, assuming Mackey decomposition formula for $R_{L^P}^G \circ R_{L}^G$ and knowing Asai’s formulas that give $R_{C_L^L(s)}^{C_G^*} \alpha$ for any $\alpha \in \mathcal{E}(C_{L^L}(s), 1)$.

(b) If $s = 1$, (J.3) is true because $\mathcal{E}(G^F, 1)$ has a generic parametrization and Lusztig induction is “generic on unipotent functions” (see [10] 1.33). If $s$ is central, by assertion (ii) in Proposition 1.3.2, (J.3) goes from 1 to $s$.

More generally, if $C_{G^*}(s) \subseteq L^*$, a Levi subgroup of $G^*$ in the dual class of $L$, $R_{L}^G$ restricts to a one-to-one map $\mathcal{E}(L^F, s) \to \mathcal{E}(G^F, s)$ independantly of choices of parabolics. It follows that (J.3) is satisfied if $C_{G^*}(s)$ is a Levi subgroup of $G^*$:

Indeed let $G(s)$ be a Levi subgroup of $G$ in duality with $C_{G^*}(s)$, let $\hat{s} = \Psi_{L,s}(1_{L^F})$, and let $L(s)$ be a Levi subgroup of $G(s)$ in duality with $C_{L^L}(s)$. Let $\alpha \in \mathcal{E}(C_{G^*}(s)^F, 1)$ and let $\beta \in \mathcal{E}(C_{L^L}(s)^F, 1)$. By assertion (ii) we have

$$\Psi_{G,s}(\alpha) = R_{G(s)}^G (\hat{s} \otimes \Psi_{G(s),1}(\alpha)), \quad \Psi_{L,s}(\beta) = R_{L(s)}^L ((\text{Res}_{L(s)}^{G(s)} F ) \otimes \Psi_{L(s),1}(\beta))$$

hence

$$R_{L}^G (\Psi_{L,s}(\beta)) = R_{G(s)}^G ((\text{Res}_{L(s)}^{G(s)} F ) \otimes \Psi_{L(s),1}(\beta)) = R_{G(s)}^G (\hat{s} \otimes R_{L(s)}^G (\Psi_{L(s),1}(\beta)))$$

$$= R_{G(s)}^G (\hat{s} \otimes \Psi_{G(s),1}(R_{C_{G^*}(s)}^{C_{G^*}(s)} \beta)) = \Psi_{G,s}(R_{C_{G^*}(s)}^{C_{G^*}(s)} \beta).$$

There are deeper and stronger results in that hypothesis, existence of a perfect isometry [8] and, better, of a Morita equivalence (see [5], or [16] Chapters 10–12 for details).

Now let two semi-simple elements $s$, $t$ in $(G^*)^F$, with coprime order, such that $st = ts$. Assume $C_{G^*}(t)$ is a Levi subgroup of $G^*$, let $G(t)$ be a dual Levi subgroup in $G$. For any $\alpha \in \mathcal{E}(C_{G^*}(st)^F, 1)$ one has $\Psi_{G,st}(\alpha) = R_{G(t)}^G (\hat{t} \otimes \Psi_{G(t),s}(\alpha)).$

(c) If $L$ is split, $R_{L}^G$ is Harish-Chandra induction and (J.3) is satisfied. Indeed Jordan decomposition is defined such that Harish-Chandra series correspond. If $L_0$ is an $F$-stable Levi complement of an $F$-parabolic subgroup of $G$ (split Levi subgroup), and $\lambda \in \mathcal{E}(L_0^F, s)$ is cuspidal, that is $(L_0, \lambda)$ is a cuspidal datum in $(G, F)$, to $(L_0, \lambda)$ there corresponds a cuspidal datum $(L_0^s, \alpha)$ in $(C_{G^*}(s), F)$ where $L_0^s(s) = C_{L^L}(s)$ for some Levi subgroup $L^*$ of $G^*$ in duality with $L$, $\Psi_{L,s}(\alpha) = \lambda$ and the Hecke algebras one obtain as endomorphism algebras of Harish-Chandra modules are built on isomorphic relative rational Weyl groups $W := W_{C_{G^*}(s)}(L_0^s, \alpha) \cong W_{C_{G^*}}(L_0^s(s), \alpha)$ (see [26], especially theorems 8.6 and 4.23). Then for all $\beta \in \mathcal{E}(C_{G^*}(s)^F, 1)$, one has

$$\langle R_{L_0^s}^{C_{G^*}(s)} \alpha, \beta \rangle_{C_{G^*}(s)^F} = \langle R_{L_0^s}^G \lambda, \Psi_{G,s}(\beta) \rangle_{G^F}$$

because that integer is the degree of the same element of $\text{Irr}(W)$ that is associated to $\alpha$ and to $\Psi_{L,s}(\alpha)$ by the maps $\text{Irr}(W) \to \mathcal{E}(C_{G^*}(s)^F, 1)$ and $\text{Irr}(W) \to \mathcal{E}(G^F, s)$. Furthermore if one extends linearly the preceding
maps to $\eta_G: Z\text{Irr}(W) \to Z\mathcal{E}(G^F, s)$, $\eta_{G,s}: Z\text{Irr}(W) \to Z\mathcal{E}(C_{G^*}(s)^F, 1)$ and if $L$ is a split Levi-subgroup of $G$ with $L_0 \subseteq L$, one has $R^L_F \circ \eta_L = \eta_G \circ \text{Ind}_{W_L^F}(L_0, \zeta)$, and similarly $R^L_F \circ \eta_{L,s} = \eta_{G,s} \circ \text{Ind}_{W_L^F}(L_0, \alpha)$. But $\Psi_{G,s} \circ \eta_{G^*} = \eta_G$ and $\Psi_{L,s} \circ \eta_{L,s} = \eta_L$, (J.3) is satisfied.

From the construction of $\Psi_{G,s}$ with the Weyl group of $C_{G^*}(s)$, it follows that, if $[C_{L^*}(s), C_{L^*}(s)] = [C_{G^*}(s), C_{G^*}(s)]$, (J.3) is true. Under that assumption one has, when $\alpha \in \mathcal{E}(C_{G^*}(s)^F, 1)$,

$$R^L_F(\Psi_{L,s}(\text{Res}_{C_{L^*}(s)}\alpha)) = \Psi_{G,s}(\alpha)$$

(here to simplify the formula we have identify via restriction $\mathcal{E}(C_{G^*}(s), 1)$ with $\mathcal{E}([C_{G^*}(s), C_{G^*}(s)]^F, 1)$).

**1.3.4. Proposition.** Let $\sigma: (G, F) \to (H, F)$ be an isotypic morphism between connected reductive algebraic groups defined on $F_\eta$. Restriction from $G^F$ to $H^F$ has no multiplicity:

$$\forall (\chi, \xi) \in \text{Irr}(G^F) \times \text{Irr}(H^F) \quad (\text{Res}^{H^F}_{G^F} \xi, \chi)_{G^F} \in \{0, 1\}.$$  

**Proof.** Let $\eta: (K, F) \to (H, F)$ be a regular covering defined on $F_\eta$, $[K, K]$ is simply connected. The adjoint groups of $H^*, G^*, K^*$ are isomorphic and are in duality with $[K, K]$. Thus $[K, K] \to [H, H]$ factor through $[G, G]$. One has commutative diagrams

$$\begin{array}{ccc}
[K, K] & \longrightarrow & K \\
\downarrow & & \downarrow \eta \\
G & \stackrel{\sigma}{\longrightarrow} & H
\end{array} \quad \begin{array}{ccc}
[K, K]^F & \longrightarrow & K^F \\
\downarrow \eta & & \downarrow \eta \\
G^F & \stackrel{\sigma}{\longrightarrow} & H^F
\end{array}$$

where $K^F \to H^F$ is onto, as the kernel of $\eta$ is a torus, and $\sigma(G^F)$ is a normal subgroup of $H^F$. By a theorem of Lusztig, (see [16] Theorem 15.11, chapter 16), the restriction from $K^F$ to $[K, K]^F$ has no multiplicity. Thus the composed restriction from $H^F$ to $[K, K]^F$ has no multiplicity, hence there is no multiplicity between $H^F$ and $G^F$.  

**1.3.5. Embeddings and Jordan decomposition.**

Let $G \subseteq H$ be a regular embedding. What happens to Jordan decomposition in $\text{Irr}(H)$, as described by Proposition 1.3.2, by restriction from $H^F$ to $G^F$? An answer is given by Lusztig in [27].

In type $A$ Bonnafé [3] for non twisted type and Cabanes [12] for twisted type have define an explicit one-to-one map from $\mathcal{E}(C_{G^*}(s)^F, 1)$ onto $\mathcal{E}(G^F, s)$, where $\mathcal{E}(C_{G^*}(s)^F, 1)$ has to be understood in the extended definition for non connected reductive groups (see 1.3.1). See our section 5.4 for more details.

Let $\sigma^*: H^* \to G^*$ be a dual morphism of the inclusion of $G$ in $H$, it is a regular covering and by restriction of $\sigma^*$ the map $H^F \to G^F$ is onto. Let $t \in H^F$ and $\sigma^*(t) = s \in G^F$. We have seen in Proposition 1.2.4 that $A_{G^*}(s)$ is isomorphic to a subgroup of $F(G^*) = \text{Ker} \sigma^* \cap [H^*, H^*]$. By duality ($\text{Ker} \sigma^*)^F$ is isomorphic to $\text{Irr}(H^F/G^F)$ [16] (15.2). Therefore there is an injective map

$$\sigma_{H,s}: A_{G^*}(s)^F \longrightarrow \text{Irr}(H^F/G^F)$$

If $M$ is an $F$-stable Levi subgroup of $H$ and $L = M \cap G$ such that $s$ belongs to a dual Levi subgroup $L^*$ of $G^*$, then, through the injective morphism of $A_{L}(s)^F$ in $A_{G^*}(s)^F$ and isomorphism $H^F/G^F \cong M^F/L^F$, $\sigma_{M,s}$ is the restriction of $\sigma_{H,s}$. 


The existence of $\sigma_{H,s}$ allows us to define a subgroup $\tau_{H,s}(A')$ of $H^F$ for any subgroup $A'$ of $A_G^*(s)^F$ by the formula

$$\sigma_{H,s}(A') = \text{Irr}(H^F/\tau_{H,s}(A'))$$

One has $G^F \subseteq \tau_{H,s}(A')$ and $A_1 \subseteq A_2$ implies $\tau_{H,s}(A_1) \subseteq \tau_{H,s}(A_2)$.

By duality between an abelian group and its group of characters $\sigma_{H,s}$ gives an isomorphism

$$\sigma_{H,s}': H^F/\tau_{H,s}(A_G^*(s)^F) \longrightarrow (A_G^*(s)^F)^\wedge$$

The group $(H^F/G^F)^\wedge$ acts by tensor product on $\text{Irr}(H^F)$. It has been proved that $\sigma_{H,s}(A_G^*(s)^F)$ is the stabilizer of the subset $\mathcal{E}(H^F,t)$ of $\text{Irr}(H^F)$. So $A_G^*(s)^F$ acts on $\mathcal{E}(H^F,t)$. The same group $A_G^*(s)^F$, as quotient $C_{G^*}(s)^F/C_{G^F}^*(s)^F$, acts on $\mathcal{E}(C_{G^*}^*,(s)^F,1)$. The set $\mathcal{E}(C_{G^*}^*,(s)^F,1)$ identifies with $\mathcal{E}(C^t_{H^*}(t)^F,1)$ by restriction through $\sigma^*$. A fundamental result [27] is that $\Psi_{H,t}$ is a morphism for these two dual operations of $A_G^*(s)^F$:

$$\Psi_{H,t}(a) \otimes \sigma_{H,s}(a) = \Psi_{H,t}(a), \quad (a \in A_G^*(s)^F, a \in \mathcal{E}(C_{H^*}(t)^F,1))$$

For any $\chi \in \mathcal{E}(H^F,t)$, $\text{Res}_{H^F}^G \chi$ belongs to $\mathbb{Z}\mathcal{E}(G^F,s)$ [16] Proposition 15.6. Thus $(A_G^*(s)^F)^\wedge$ acts on $\mathcal{E}(G,s)$ through $(\sigma_{H,s}')^{-1}$ (1.3.5.3). That action is independent of the choice of $H$ in the regular embedding $G \subseteq H$ and of $t$ such that $\sigma^*(t) = s$. Using the non-multiplicity property in restriction from $H^F$ to $G^F$ (Proposition 1.3.4), one obtains [27] Proposition 8.1, [16] Corollary 15.14:

**1.3.6. Proposition.** Let $\sigma: (G,F) \rightarrow (H,F)$ be a regular embedding, let $\sigma^*: (H^*,F) \rightarrow (G^*,F)$ be a dual morphism, $t$ semi-simple in $H^*F$, $s = \sigma^*(t)$. Denote $A = A_G^*(s)^F$. There is a bijective map between sets of orbits

$$\mathcal{E}(G^F,s)/A^\wedge \leftrightarrow \mathcal{E}(C_{G^*}^*(s)^F,1)/A,$$

Let $\alpha \in \mathcal{E}(C_G^*(s)^F,1)$. To the orbit of $\alpha$ under $A$ in $\mathcal{E}(C_{G^*}^*(s)^F,1)$ there corresponds the set of irreducible components of $\text{Res}_{H^F}^G(\Psi_{H,t}(a))$, it is a regular orbit under $A^\wedge/(\Lambda_{\alpha})^\perp \cong (\Lambda_{\alpha})^\wedge$.

With notations introduced in Appendix, 5.1

$$\text{if } \Psi_{H,t}(\alpha) \in \text{Irr}(H^F | \chi), \text{then } \tau_{H,s}(A_G^*(s)^F)_\alpha = \text{Irr}_{H^F}^G(\Psi_{H,t}(\alpha)) = H^F_{\chi}. $$

**1.3.7. Proposition.** Let $\sigma: (G,F) \rightarrow (H,F)$ be an isotypic morphism between connected reductive algebraic groups defined on $\mathbb{F}_q$. Let $\sigma^*: H^* \rightarrow G^*$ be a dual morphism, $K^* = \text{Ker } \sigma^*$, and let $t$ be a semi-simple element of $H^*F$, $s = \sigma^*(t) \in G^*$, $\zeta \in \mathcal{E}(H^F,t)$.

(a) $\sigma^*$ defines by restriction from $H^*F$ to $G^*F$ a bijection $\mathcal{E}(C_{H^*}^t,(t)^F,1) \rightarrow \mathcal{E}(C_{G^*}^*,(s)^F,1)$ and an injective morphism $\tau: A_{H^*}(t) \rightarrow A_G^*(s)$ that transforms the action of $A_{H^*}(t)^F$ on $\mathcal{E}(C_{H^*}^t,(t)^F,1)$ in the action on $\mathcal{E}(C_{G^*}^*(s)^F,1)$ of its image in $A_G^*(s)^F$. The quotient $A_G^*(s)/\tau(A_{H^*}(t))$ is isomorphic to a subgroup of $K^* \cap [H^*,H^*]$.

(b) Let $\alpha \in \mathcal{E}(C_{H^*}^t,(t)^F,1)$ be in the orbit under $A_{H^*}(t)^F$ that is associated to the orbit of $\zeta \in \mathcal{E}(H^F,t)$ under $(A_{H^*}(t)^F)^\wedge$ by (1.3.6.1).

(b.1) If $Z(H)$ is connected, then $\Psi_{H,s}(a) = \zeta$. 

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(b.2) \( \text{Res}_{G^F \to H^F} \zeta \) is a sum of \( |A_G \cdot (s) F / \tau(A_{H^*}(t) F)| \) distinct elements of \( \mathcal{E}(G^F, s) \) and there are \( |A_G \cdot (s) F| / |A_{H^*}(t)(F)| \) elements of \( \mathcal{E}(H^F, t) \) with equal restriction \( \text{Res}_{G^F \to H^F} \zeta \).

**Proof.** We know by Proposition 1.2.3 that \( \sigma^*(C_{H^*}\cdot (t)), Z^\circ (G^*) = C_{G^0}(s) \) hence \( \sigma^* \) defines a map \( \tau : A_{H^*}(t) \to A_{G^*}(s) \) that commute with \( F \). By Lemma 2.1 the quotient \( A_{G^*}(s) / \tau(A_{H^*}(t)) \cong C_{G^*}(s) / \sigma^*(C_{H^*}(t)) \) is isomorphic to a subgroup of \( \text{Ker}\sigma \cap [H^*, H^*] \). The induced morphism \( C_{H^*}(t) \to C_{G^0}(s) \) is isotypic and identify the unipotent series of \( C_{H^*}(t) F \) and \( C_{G^0}(s) F \).

Consider regular embeddings defined over \( \mathbb{F}_q \) as in the proof of 1.1.4 (d), and dual morphisms:

\[
\begin{array}{cccc}
G & \to & G_0 & G^* \leftarrow G_0^* \\
\downarrow \sigma & & \downarrow \sigma_0 & \downarrow \sigma^* & \downarrow \sigma_0^* \\
H & \to & H_0 & H^* \leftarrow H_0^* \\
\end{array}
\]

(1.3.7.1)

Horizontal maps in the right diagram are coverings with central tori as kernels.

(b.1) follows directly from Propositions 1.2.3 and 1.3.6.

(b.2) follows from Proposition 1.3.6 and Clifford theory applied in the commutative diagram:

\[
\begin{array}{cccc}
G^F & \to & G_0^F \\
\downarrow \sigma & & \downarrow \sigma_0 \\
H^F & \to & H_0^F \\
\end{array}
\]

(1.3.7.2)

where all maps have invariant images and abelian cocernels.

Let \( t_0 \) be a semi-simple element in \( H_0^* F \) of image \( t \in H^* \) and put \( s_0 = \sigma_0^*(t_0) \in G_0^* \), so that \( s_0 \) maps on \( s \). By restriction in (1.3.7.1) we obtain one-to-one maps between \( \mathcal{E}(C_{H^*}(t)^F, 1), \mathcal{E}(C_{H^*}(t_0)^F, 1), \mathcal{E}(C_{G^0}(s_0) F, 1) \) and we identify these sets. The groups \( A_{H^*}(t)^F \), \( A_{H_0^*}(t_0)^F \), \( A_{G^*}(s)^F \), and \( A_{G^0}(s_0)^F \) act respectively on these unipotent series. As \( Z(G_0) \) and \( Z(H_0) \) are connected, \( A_{G_0}(s_0) = 1 = A_{H_0^*}(t_0) \). Furthermore \( \tau \) commute with \( F \) and actions on unipotent series.

Let \( \zeta_0 \in \text{Irr}(H_0^F \mid \zeta) \cap \mathcal{E}(H_0^F, t_0) \). Assume \( \zeta_0 = \Psi_{H_0^*, t_0}(\alpha) \) for a Jordan decomposition \( \Psi_{H_0^*, t_0} \) as in Proposition 1.3.5, where \( \alpha \in \mathcal{E}(C_{H_0^*}(t_0)^F, 1) \). We have by (1.3.6.2)

\[
\text{Res}_{H_0^*}^{H_0^F} (\zeta_0) = \sum_{h \in H_0^F / \tau_{H_0^*, t_0}(A_{H^*}(t)_F^0)} h \zeta
\]

hence \( \zeta_0(1) = \zeta(1)|A_{H^*}(t)_F^0| \). There are \( |A_{H^*}(t)^F / A_{H^*}(t)_F^0| \) elements in \( \mathcal{E}(H_0^F, t_0) \) with equal restriction to \( H^F \). We call “multiplicative factor of \( \text{Res}_{H_0^*}^{H_0^F} \) above \( \alpha \)” the quotient of Cardinals of the subsets of \( \mathcal{E}(H^F, t) \) and \( \mathcal{E}(H_0^F, t_0) \) that correspond to \( \alpha \), here it is \( |A_{H^*}(t)_F^0|^2 / |A_{H^*}(t)_F| \).

As the kernel of \( \sigma_0 \) is a torus, \( \sigma_0 \) restricts on a surjective morphism \( G_0^F \to H_0^F \), so that \( \text{Res}_{G_0^F \to H_0^F} \) sends \( \mathcal{E}(H_0^F, t_0) \) in \( \mathcal{E}(G_0^F, s_0) \). We may assume

\[
\chi_0 := \text{Res}_{G_0^F \to H_0^F} (\zeta_0) = \Psi_{G_0^*, s_0}(\alpha)
\]

for suitable Jordan decomposition \( \Psi_{G_0^*, s_0} \) and identification. Then we have, if \( \chi_0 \in \text{Irr}(G_0^F \mid \chi) \) :

\[
\text{Res}_{G_0^F}^{G_0^F} (\chi_0) = \sum_{g \in G_0^F / \tau_{G_0^*, s_0}(A_{G^*}(s)_F^0)} g \chi
\]

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By what we have seen in 1.3.5

\[ \tau_{G_0,s_0}(A_{G^*} (s)_{\alpha}) \subseteq \tau_{G_0,s_0} (\tau (A_{H^*} (t))_{\alpha}) = \sigma_0^{-1} (\tau_{H_0,t_0} (A_{H^*} (t)_{\alpha})) \]

with \( \chi_0(1) = \chi(1) |A_{G^*} (s)_{\alpha}| \) and the multiplicative factor of \( \text{Res}_{G^F}^{G_F} \) above \( \alpha \) is \( |A_{G^*} (s)_{\alpha}|^2 / |A_{G^*} (s)|^2 \).

In view of (1.3.7.2), (1.3.7.3), (1.3.7.4) and (1.3.7.5) we conclude by transitivity of restriction and that

\[ \chi(1) |A_{G^*} (s)_{\alpha}| = \chi(1) |A_{H^*} (t)_{\alpha}|. \]

The multiplicative factor of \( \text{Res}_{G^F}^{G_F} \) above \( \alpha \) is \( |A_{G^*} (s)_{\alpha}|^2 / |A_{G^*} (s)|^2 |A_{H^*} (t)_{\alpha}|^2 \). \( \blacksquare \)

1.4. Theorem. Let \((G, F)\) be a reductive algebraic group over an algebraic closure of a prime field \( \mathbb{F}_p \), defined over \( \mathbb{F}_q \), with Frobenius endomorphism \( F \), in duality over \( \mathbb{F}_q \) with \((G^*, F)\). Let \( G^F \) be the subgroup of rational points. Assume that Mackey decomposition formula holds for Lusztig induction inside closed \( F \)-stable subgroups of \( G \) and of \( G^* \).

Let \( \ell \) be an odd prime number, different from \( p \). Assume that \( \ell \geq 7 \) if \( G \) has a component of type \( E_8 \), \( \ell \geq 5 \) if \( G \) has a component of non classical type. Let \((G, F) \to (H, F)\) be a regular embedding with a dual morphism \( H^* \to G^* \). Let \( s \in G^*_F \), image of \( t \in H^*_F \).

There exist a reductive group defined over \( \mathbb{F}_q \), we denote \((G(s), F), \) such that

(A) \((G(s)^o, F)\) and \((C_{G^*}^o (s), F)\) are in duality, \( G(s)/C_{G^*}^o (s) \) is isomorphic to \( A_{G^*} (s) \) (see (1.2.0)), and so acts on \( G(s) \) coherently with the action of \( A_{G^*} (s) \) on the root datum of \( C_{G^*}^o (s) \).

(B) There exist a one-to-one map

\[ B_{G,s}: \text{Bl}(G(s)^F; 1) \to \text{Bl}(G^F; s) \]

from the set of unipotent \( \ell \)-blocks of \( G(s)^F \) onto the set of \( \ell \)-blocks of \( G^F \) in series \( s \) such that

(B.1) If \( G = H \), then Jordan decomposition defines a one-to-one map from \( \text{Irr}(b) \) onto \( \text{Irr}(B_{H,t}(b)) \) \( \) (where \( b \in \text{Bl}(H(t)^F; 1) \) : if \( h \in C_{H^*} (t)_{\ell}^F \), so that \( C_{H^*} (th) = C_{H(t)} (h) \), then

\[ (\Psi_{H,th} \circ \Psi_{H(t),h^{-1}}) (\text{Irr}(b) \cap \mathcal{E}(H(t)^F, h)) = \text{Irr}(B_{H,t}(b)) \cap \mathcal{E}(H^F, th) \]

(B.2) \( B_{G,s} \) and \( B_{H,t} \) respect Clifford theory in the following sense.

The regular embedding \( G \to H \) defines an embedding \( G(s)^o \to H(t) \). Let \( c \in \text{Bl}(H(t)^F; 1) \), \( b \in \text{Bl}(G(s)^F; 1) \), \( b_0 \in \text{Bl}(G(s)^F; 1) \) such that \( c \) restricts to \( b_0 \). Then the block \( B_{H,t}(c) \) covers \( B_{G,s}(b) \) if and only if \( b \) covers \( b_0 \).

(B.3) There is a one-to-one height preserving map \( \Psi_b \) from \( \text{Irr}(b) \) onto \( \text{Irr}(B_{G,s}(b)) \) such that

\[ \Psi_b (\zeta) (1) |G(s)^F|_{p'} = \zeta(1) |G^F|_{p'} \]

for all \( \zeta \in \text{Irr}(b) \)

(B.4) The defect groups of \( b \in \text{Bl}(G(s)^F; 1) \) and of \( B_{G,s}(b) \) are isomorphic. The Brauer categories of \( b \) and of \( B_{G,s}(b) \) are equivalent.

Proof. The all proof is contained in sections 2 to 5, we use here Propositions 2.1.4, 2.1.7, 2.1.10, 2.3.5, 2.3.6, 3.1.1, 3.4.1, 3.4.2, 4.1.2, 4.2.4.

(A) The group \( G(s) \) such that \( (A) \) holds is constructed in sections 3.1, 3.2, 3.3, see Proposition 3.1.1.

(B.1) By Proposition 2.1.7 a unipotent \( \ell \)-block \( b \) of \( H(t)^F \) is defined by a \( H(t)^F \)-conjugacy class of unipotent cuspidal data \((L(t), \alpha(t)) \) in \((H(t), F) \). See definitions 2.1.1 : \( L(t) \) is a \( d \)-split Levi subgroup of \( H \),

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The equality on degrees in (B.3) when $G$ is an immediate consequence of the definition of $\Psi_0$ and (2.3.5.3), (2.3.6.2), (2.3.6.3) we obtain a one-to-one map, restriction of $\Psi_{th,F}(t,F)$ in series $(t,F)$, $\Psi(B(t),t,F,0) \in B(F,F)$, $\Psi(B(t),t,F,0)$ and $\Psi_0(F,F)$ is defined by a $\Psi(b,F,0) \in B(b,F)$, $\Psi(b,F,0)$ and $\Psi_0$ is deduced from the degree formula in (B.3) for $G(s,F)$ and $G(s,F)$ is prime to $\ell$, if a block $b$ of $G(s,F)$ covers a block $b_0$ of $G(s,F)$, then a defect group of $b_0$ is a defect group of $b$. The defect groups of $b$ and $B_{G,s}(b)$ are computed in Proposition 4.1.2, they are isomorphic.

Then the degree formula in (B.3) imply that $\Psi_b$ preserves height.

The Brauer category of a block $B$ of a finite group $X$ is a small category whose objects are $\ell$-subpairs that contain $(\{1\},B)$ and morphisms are produced by restriction of interior automorphisms of $X$. By fusion theorems (see [32], §47 and (48.3)), once the defect groups $D$ of a block is given, the Brauer category is entirely defined by the groups of automorphisms

$$N_X(Y,b_Y)/Y.C_Y(X), \quad Y \subseteq D, \quad (\{1\},B) \subset (Y,b_Y) \subset (D,b_D)$$

Thus (B.4) follows from Proposition 4.2.4.
1.5. \( \ell = 2 \) for classical types in odd characteristic

In this section we obtain Jordan decomposition of blocks for \( \ell = 2 \) in classical types. We assume Mackey’s decomposition formula.

Two facts help us: there is only one unipotent 2-block and if \( s \) is a semi-simple element of odd order in \( G^*F \), then \( C^*_G(s) \) is a Levi subgroup of \( G^* \). Short of type \( A \), where \( 2 \) is good, \( C^*_G(s) \) is connected.

1.5.1. Hypotheses and notations. \( G \) is a reductive group defined over \( \mathbb{F}_q \), \( q \) odd. All components of \( G \) have classical type.

Let \( s \in G^*F \) be semi-simple of odd order. Let \( G(s)^0 \) be a Levi subgroup of \( G \) in the dual \( G^F \)-conjugacy class of the \( G^*F \)-class of \( C^*_G(s) \), duality around dual maximal tori \( T, T^* \). Let \( G(s) \subseteq N_G(G(s)^0) \) such that \( G(s)^0 \subseteq G(s) \), and \( G(s)/G(s)^0 \cong N_G(s)(T)/N_G(s)^0(T) \) has image \( A_G \) by the antimorphism between Weyl groups \( N_G(T)/T \) and \( N_G(T^*/T^*) \).

1.5.2. Proposition. Assume 1.5.1.

(a) \( G^F \) has only one unipotent 2-block, the principal block.

(b) If \( A_G = \{1\} \), and that is the case in types \( B, C \) and \( D \), \( G^F \) has only one 2-block \( b \) in series \( s \), such that \( \text{Irr}(b) = \mathcal{E}_2(G^F, s) \).

(c) A 2-Sylow subgroup of \( G(s)^0 \) is a defect group of any 2-block in series \( s \). Such a block has central defect group if and only if \( C^*_G(s) = T^* \) and \( T^F_2 \subseteq Z(G) \).

The condition \( T^F_2 \subseteq Z(G) \) is satisfied when \( G^F = \text{SL}_{2m}(q) \), \( m \) is odd, \( q \equiv 3 \mod 4 \) and \( T \) is a Coxeter torus in \( G \). There is a similar example in other classical types, \( G \) simply connected.

Proof. The notation \( \mathcal{E}_t(G^F, s) \) has been introduced in 1.3.1, see (1.3.1.7).

All properties reduce easily to the rationally irreducible case.

Assertion (a) is [16, Theorem 21.14].

(b) Assuming \( G(s) = G(s)^0 \), the virtual bimodule defining \( R^G_G(s) \) allows to construct a perfect isometry [8] and Morita equivalence [5] between the principal block \( b_1 \) of \( G(s)^0 \), unique in series \( 1 \) in \( G(s)^F \) by (a), and \( b \), therefore unique in series \( s \) in \( G^F \). The application \( \lambda \mapsto R^G_G(s) \Psi_G(s), 1 \otimes \lambda \) is a one-to-one map from \( \text{Irr}(b_1) = \mathcal{E}_2(G^F, s) \) onto \( \text{Irr}(b) = \mathcal{E}_2(G^F, s) \).

Furthermore a 2-Sylow subgroup of \( G(s)^0 \) is a common defect group of the two blocks and the map \( \lambda \mapsto R^G_G(s) \Psi_G(s), 1 \otimes \lambda \) preserve height by degrees formulas (1.3.1.2) and (vi) in Proposition 1.3.2.

Clearly (c) is true if \( A_G(s)^F = \{1\} \).

That happens if \( G \) has only types \( B, C, D \), because \( G(t) \) is then a 2-group (Proposition 1.2.6 (c)).

(c) We have to consider rational types \( A, 2A \).

Let \( G \to H \) be a regular embedding, let \( t \) semi-simple in \( H^F_2 \) of image \( s \) by a dual map \( \sigma : H^* \to G^* \). There exists a Levi subgroup \( H(t) \) of \( H \) such that \( G(s)^0 = H(t) \cap G \) and \( H(t) \) is in duality with \( C_H(t) \).

Assertion (b) applies to \( H \), \( t \). Let \( B(t) \) be the unique 2-block in series \( t \) of \( H^F \).

Any element of \( \mathcal{E}(H^F, t) \) (resp. \( \mathcal{E}_2(H^F, t) \)) restricts on \( G^F \) on a sum of elements of \( \mathcal{E}(G^F, s) \) (resp. \( \mathcal{E}_2(G^F, s) \)) and all \( \mathcal{E}(G^F, s) \) (resp. \( \mathcal{E}_2(G^F, s) \)) appear this way. Thus \( B(t) \) covers all 2-blocks in series \( s \) of \( G^F \) and these blocks are \( H^F \)-conjugate.

We have seen that a 2-Sylow subgroup \( E \) of \( H(t)^F \) is a defect group of \( B(t) \). One has \( G^F.H(t)^F = H^F \), \( G^F.E/G^F \) is a 2-Sylow subgroup of \( H^F/G^F \) and \( E \cap G^F \) is a 2-Sylow subgroup of \( G(s)^0 \). By [28] Chapter 5, 5.16, \( E \cap G^F \) is a defect group of any block of \( G^F \) covered by \( B(t) \).
We can now describe Brauer subpairs by standard arguments of local theory of blocks.

1.5.3. Proposition. Assume 1.5.1. Let $Y$ be a 2-subgroup of $G(s)^F$, $T_Y$ be a maximal $F$-stable torus of $C^s_G(Y)$, $T$ be a maximal $F$-stable torus of $G$ such that $Y \subseteq N_G(T)$ and $T_Y \subseteq T$. Let $T^*$ be a maximal torus in $C^s_G(s)$ in the $G^F$-conjugacy class of the $G^F$-class of $T$. Let $T^* \rightarrow T^*_Y$ be a dual morphism of the inclusion of $T_Y$ in $T$, it defines a Lusztig series $\mathcal{E}(C^s_G(Y)^F, s_Y)$.

If $b$ is a 2-block of $C_G(Y)^F$ that covers a block in series $(s_Y)$ of $C^s_G(Y)^F$ and $(\{1\}, B) \subset (Y, b)$ is an inclusion of subpairs in $G^F$, then $B$ is in series $(s)$.

Proof. We know that $C^s_G(Y)$ is reductive, the existence of $T$ has been proved in Proposition 1.2.7.

Given a group $(C^s_G(Y)^*, F)$ in duality with $C^s_G(Y)$ around torii $T_Y$, $T^*_Y$, $s_Y \in T^*_Y$ may be defined by $\Psi_{T_Y,s_Y}^{-1} = \psi_{T^*_Y}(\Psi_{T,s_Y}(1))$, so that $\mathcal{E}(C^s_G(Y)^F, s_Y)$ is defined. In types $\text{A} \text{ or } D$ (c) of Proposition 1.2.7 applies hence we may see $C^s_G(Y)^*$ as an algebraic subgroup $C^*$ of $G^*$ such that $s \in C^*$. Then we may assume $s_Y = s$.

(a) If $s = 1$, our claim is Brauer’s First Main Theorem, the inclusion between subpairs formed with principal blocks $B_1$ and $b$ : thanks to (a) in Proposition 1.5.2, $C_G(Y)^F/C^s_G(Y)^F$ is a 2-group, the principal block of $C_G(Y)^F$ is the unique block that covers the principal block $b_1$ of $C^s_G(Y)^F$.

If $s$ is central in $G$, one has blocks $B = \Psi_{G,s}(1) \otimes B_1$, $b_0 = \Psi_{C^s_G(Y),s_Y}(1) \otimes b_1$, $\Psi_{C^s_G(Y),s_Y}(1)$ extends to $C_G(Y)^F$, so is defined $b$. The inclusion of subpairs follows.

(b) Assume first $Z(G)$ connected and $Y$ cyclic, $Y = \langle y \rangle$.

By Proposition 1.5.2 is given a 2-block $B(s)$ of $G^F$. We have $T_Y = T$. Let $L_y := C_{G(s)}^s(y) = C^s_G(\langle y \rangle)$, $L_y$ a Levi subgroup of $C^s_G(\langle y \rangle)$. Let $\tilde{s} := \Psi_{G,s}(1)$, $\tilde{s}_y := \Psi_{L_y,s_Y}(1)$, hence $\tilde{s}_y = \text{Res}_{L_y}^{G(s)} \tilde{s}$. Let $\xi$ in $\mathcal{E}_2(G(s)^F, 1)$, then $\chi = R^G_L(\tilde{s} \otimes \xi) \in \text{Irr}(B(s))$. If $b(s)$ is the 2-block of $G(s)^F$ in series $(s)$ ($Z(G(s))$ is connected), $b(s)(\tilde{s} \otimes \xi) \in \text{Irr}(b(y))$. Let $\xi_y \in \mathcal{E}_2(L_y^F, 1)$, then $\chi_y := R^{C^s_G(y)}_{L_y}(\tilde{s}_y \otimes \xi_y) \in \text{Irr}(b(y))$ for some 2-block $b_y$ of $C^s_G(y)^F$ in series $(s_y)$.

Using the second Brauer Main Theorem, to prove an inclusion of subpairs $(1, B(s)) \subset (\langle y \rangle, b)$ in $G^F$, where $b$ covers $b_y$, we may consider only connected subpairs [16] 21.1, so it is sufficient to prove $b_y.d^B(\chi) \neq 0$. As $b_y.B_y = \chi_y$, consider $\langle b_y.d^B(\chi), \chi_y \rangle_{C^s_G(y)^F} = \langle d^B(\chi), \chi_y \rangle_{C^s_G(y)^F}$. By commutation formula [16], Theorem 21.4, adjunction and part (a) of the proof applied in in $G(s)$, we have

$$\langle d^B(\chi), \chi_y \rangle_{C^s_G(y)^F} = \langle d^B(R^G_{G(s)}(\tilde{s} \otimes \xi)), R^{C^s_G(y)}_{L_y}(\tilde{s}_y \otimes \xi_y) \rangle_{C^s_G(y)^F}$$

$$= \langle d^B(R^G_{G(s)}(R^G_{G(s)}(\tilde{s} \otimes \xi))), \tilde{s}_y \otimes \xi_y \rangle_{L_y^F}$$

$$= \langle d^B(b(s).R^G_{G(s)}(R^G_{G(s)}(\tilde{s} \otimes \xi))), \tilde{s}_y \otimes \xi_y \rangle_{L_y^F}$$

To compute $R^G_{G(s)}(R^G_{G(s)}(\tilde{s} \otimes \xi))$ we use Mackey formula. As $G(s)$ is in duality with $C_G^*(s)$, if $g \in G^F$ and $T^g \subseteq G(s)$, then $(T^g, (\text{Res}_{T^g}^{G(s)} \tilde{s}^g))$ is conjugate to $(T, \Psi_{T,s}(1))$ only if $g \in G(s)$. So

$$b(s).R^G_{G(s)}(R^G_{G(s)}(\tilde{s} \otimes \xi)) = \tilde{s} \otimes \xi$$

As $\tilde{s}_y$ is restriction of $\tilde{s}$ we obtain

$$\langle d^B(\chi), \chi_y \rangle_{C^s_G(y)^F} = \langle d^B(\xi), \xi_y \rangle_{L_y^F}$$

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Given $b_y$, the existence of $\xi_y \in \text{Irr}(b_y)$ and $\xi \in \mathcal{E}_2(G(s)^F, 1)$ such that $\langle d^y(\xi), \xi_y \rangle_{L_y^F} \neq 0$ is just our claim for unipotent blocks inside $G(s)$, we proved it in (a).

(c1) To easy induction, thanks to Proposition 1.2.7 — use freely — assume now $Z(G)$ connected and $[G, G]$ simply connected, with no restriction on the 2-subgroup $Y$ of $G^F$.

By (b) and induction we may assume that our claim is true for smaller groups $Y_1 (\|Y_1 \| < \|Y\|)$ and for groups $G_1$ with strictly smaller semi-simple rank.

We may assume $Z(G)_2^F = Z(C_G(Y))_2^F \subseteq Y$, with no change on $C_G(Y)$. Let $L_Y = C_{G(s)}(Y) = C_{C_G(Y)}(s)$, $L_Y$ is a Levi subgroup of $C_G(Y)$.

If $Z(Y) \neq Z(G)_2^F$, let $y \in Z(Y) \setminus Z(G)$. One has $Y \subseteq C_G(y)$. If $s_Y, s_y$ are given from $s$ as above, we have, by part (a) of the proof, three 2-blocks, $b(s_y), b(s_Y), b(s)$ of respectively $C_G(y)^F, C_G(Y)^F, G^F$.

By induction on the semi-simple rank of $G$ we have an inclusion of Brauer subpairs in $C_G(y)^F : (1, b(s_y)) \subseteq (Y, b(s_y))$. That inclusion may be seen in $G^F : (\langle y \rangle, b(s_y)) \subseteq (Y, b(s_y))$. By (b) we have an inclusion in $G^F : (1, b(s)) \subseteq (Y, b(s_Y))$. That imply by transitivity $(1, b(s)) \subseteq (Y, b(s_Y))$ in $G^F$.

It may happens that $Z(Y) = Z(G)_2^F$. If $Z(Y)$ is not a 2-Sylow subgroup of $C_G(s)(Y)^F$, let $x \in (T_Y)_2 \setminus Z(Y)$ ($x$ exists thanks to a good choice of $T_Y$). Consider $Y_1 = \langle Y, x \rangle$. We may take $T_Y = T_Y$, then $s_Y = s_Y$ and apply the preceding result on $Y_1$ to obtain an inclusion $(1, b(s)) \subseteq (Y_1, b(s_Y))$ of subpairs in $G^F$. Inductively we have an inclusion in $C_G(Y)^F : (1, b(s_1)) \subseteq (x, b(s_1))$ which may be read in $G^F$ as $(Y, b(s_Y)) \subseteq (Y_1, b(s_Y))$. By unicity in Brauer’s correspondence we have $(1, b(s)) \subseteq (Y, b(s_Y))$.

It may happens that $Z(Y) = Z(G)_2^F$ is a 2-Sylow subgroup of $C_G(s)(Y)^F$! Let $Y_1$ be a subgroup of $Y$ with index 2. If $C_G(Y) = C_G(Y_1)$ we have our claim by inductive hypothesis. One sees that if $C_G(Y_1) \neq C_G(Y)$, then $C_G(Y_1)$ is not rationally irreducible and that imply $Z(C_G(Y_1))_2^F \neq Z(G)_2^F$. Take $y \in Y \setminus Y_1$. We argue as above : we have an inclusion $(1, b(s)) \subseteq (Y_1, b(s_Y))$ in $G^F$, as well as $(1, b(s_Y)) \subseteq (\langle y \rangle, b(s_Y))$ in $C_G(Y_1)^F$.

(c2) Assuming only $Z(G)$ connected, there exists an isotypic epimorphism $H \to G$, with central kernel, such that $Z(H)$ is connected and $[H, H]$ simply connected (see Proposition 1.1.4 (d)). Then $G^F$ is a quotient of $H^F$ with central kernel. We consider $\text{Irr}(G^F)$ as a subset of $\text{Irr}(H^F)$. If $s$ has image $t \in H^*_F$ by a dual morphism, $t$ defines a 2-block $b(t)$ of $H^F$, and we have $\text{Irr}(b(s)) \subseteq \text{Irr}(b(t))$. Similarly, if $X$ is a 2-subgroup of $H^F$ of image $Y$ in $G^F$, $C_G(Y)$ is image of $C_H(X)$, $T_Y$ is image of some maximal torus $T_X$ in $C_H(X)$. There is an element of odd order $t_X$ in $T_X^F \subseteq C_H(X)^*$, defining a 2-block $b(t_X)$ of $C_Y(X)^F$. We may assume that $t_X$ is image of $s_Y$. Then by construction $\text{Irr}(b(s_Y)) \subseteq \text{Irr}(b(t_X))$ where $b(s_Y)$ is a 2-block of $C_G(Y)^F$. The inclusion of subpairs $(1, b(t)) \subseteq (X, b(t_X))$ in $H^F$ given by (c1) implies an inclusion of “connected subpairs” $(1, b(s)) \subseteq (Y, b(s_Y))$ in $G^F$, so an inclusion $(1, b(s)) \subseteq (Y, b_Y)$ in $G^F$ where $b_Y$ is the 2-block of $C_G(Y)^F$ that covers $b(s_Y)$.

(c3) To reach the more general case we have to consider a regular embedding $G \subseteq H$.

Let $H^* \to G^*$ be a dual map, $t$ semi-simple in $H^*_2^F$ of image $s, s_H(t)$ a Levi subgroup of $H$ in duality with $C_{H^*}(t)$ such that $G^*(t)^o = H(t) \cap G$. Let $B(t)$ be the unique 2-block idempotent of $H^F$ in series (t), $B(s)$ be the sum of 2-blocks idempotents of $G^F$ in series (s). As $H^F$ stabilizes $\mathcal{E}(G^F, s)$ and $\mathcal{E}_2(H^F, t)$ restricts in $\mathcal{E}_2(G^F, s), B(t)$ covers the 2-blocks in series (s) of $G^F$ and these are $H^F$-conjugate, components of $B(s)$. Let $X = G^F.H^*_2^F \subseteq H^F$, $H^F/X$ is an abelian 2-group. Thus $B(t)$ covers a unique 2-block $B'(t)$ of $X$ and $B'(t)$ covers the 2-blocks in series (s) of $G^F$. The embedding $C_G^o(Y) \subseteq C_H^o(Y)$ is regular and we may define $t_Y, s_Y$ semi-simple of odd order in $C_H(Y)^F$, $C_G^o(Y)^F$ respectively, such that $(1, B(t)) \subseteq (Y, B(t_Y))$, where $B(t_Y)$ covers the 2-blocks in series $(s_Y)$ of $C_G^o(Y)^F$. As $C_G(Y)^F/C_H^o(Y)^F$ is a 2-group, we may speak of “2-blocks in series $(s_Y)$ of $C_G(Y)^F$.” If $b$ is such a block, it is covered by $B(t_Y)$ in $C_H(Y)^F$, and by a well-defined 2-block $B'(t_Y)$ in $C_X(Y)$. 28
Proposition 5.1.4. (a) (i) applies with \((1, Y, GF, X)\) instead of \((U, V, Y, X)\). Assume \((1, B) \subset (Y, b) \subset GF\). As \(B'(t_Y)\) covers \(b\) and \((1, B'(t)) \subset (Y, B'(t_Y)) \subset X\), \(B'(t)\) covers \(B\), hence \(B\) is in series \((s)\).

1.5.4. Proposition. Assume 1.5.1. Let \(D\) be a 2-Sylow subgroup of \(G(s)^{GF}\), \(s_D \in C_G^e(D)^+\) as in Proposition 1.5.3. The set of 2-blocks in series \((s)\) of \(GF\) is a regular orbit under the action of \((AC_G^e(D), s_D)^*\).

Proof. In types \(B, C, D\), as \(s\) has odd order, we have \(A_{G,s} = 1\), as well as \(AC_G^e(D), s_D = 1\) (see Propositions 1.2.4, 1.1.3 and 1.2.7 (d)). There is only one 2-block in series \((s)\) by Proposition 1.5.2. So we may assume that \(G\) has type \(A\).

Then, as said at the beginning of the proof of Proposition 1.5.3, for any 2-subgroup \(Y\) of \(G(s)^{GF}\), we may see \(C_G^e(Y)^+\) as an \(F\)-stable reductive subgroup \(C^*\) of \(G^*\) with \(s \in C^*\) (Proposition 1.2.7 (c)). As \(C_G^e(s)\) is a Levi subgroup of \(G^*\) and \(C_G^e(s)\) is a Levi subgroup of \(C^*\), for some torus \(S\), \(C_G^e(s) = C_G^e(S)\), hence \(C_G^e(s) = C_G^e(S) = C^* \cap C_G^e(s)\). By Proposition 1.2.2 (b), with \((G^*, C^*)\) instead of \((G, X, A_{C^*} = 1)\) and our claim is clear.

So we consider as usual a regular embedding \(G \subseteq H\), with dual map \(H^* \rightarrow G^*\), \(t \in H^*\) semi-simple of odd order and of image \(\text{ts} = t, H(t) \subseteq H\), a Levi subgroup in duality with \(C_G^e(t) \subseteq H^*\) and such that \(G(s)^o = H(t) \cap G\). We have \(H = Z^o(H).G, tH = Z^o(H).G(s)^o, C_H^e(D) = Z^o(H).C_G^e(D)\), hence \(H^F = GF.H(t)^F = GF.C_{HF}(D), C_H^e(D)^F = Z^*(H)^F.C_G^e(D)^F\) and \(C_{HF}(D) = C_H^e(D)^F.C_{GF}(D)\).

The unique 2-block \(B(t)\) in series \((s)\) of \(GF\) covers any 2-block \(b\) in series \((s)\) of \(GF\). If \((D, b_D)\) is a maximal pair of \(b, b_D\) is covered by a block \(B_D\) of \(C_{HF}(D)\), such that \((1, B(t)) \subset (D, B_D)\) in \(H^F\). Any 2-block with central defect \(b_D\) of \(C_{GF}(D)\) covers one block with central defect \(b_D^*\) of \(C_{GF}(D)\). By conjugacy of maximal subpairs we see that \((H^F)b = GF.N_{HF}(D)b_D^*\).

The embedding \(C_G^e(D) \subset C_H^e(D)\) is regular and \(b_D^*\) has central defect. By Proposition 1.5.2 (c), \(C_G^e(D)\) is a torus, say \(S_D\). As well \(T_D := Z^o(H)S_D = C_{H(t)}^e(D)\) is a torus. There exists \(t_D \in T_D^*\), with image \(s_D\), related to \(t \in \left(Z(H)S\right)^o\) as \(s_D \in S^o\) is related to \(s \in S^*\). Then \(b_D^*\) is covered by the unique block \(B_D\) in series \((t_D)\) of \(C_{H}(D)^F\). The canonical character of \(B_D\) is \(\tau_D := \text{Res}_{C_{H}(D)^F}^{C_G^e(D)^F}(\Psi_{C_{H}(t)}^e(1))\). By Proposition 1.3.6, \(\text{Res}_{C_{H}(D)^F}^{C_G^e(D)^F}(\tau_D)\) is a sum of elements of \(E(C_G^e(D)^F, s_D)\), a regular orbit under \((AC_G^e(D), s_D)^*\).

Each one is the canonical character of one of the blocks of \(C_G^e(D)^F\) covered by \(B_D\). In other words, with notations of (1.3.5.2) \(H_\lambda = \tau_{H,s}(AC_G^e(D)^*, s_D)^F\) where \(AC_G^e(D)^*(s_D)\) is viewed as a subgroup of \(G\).

1.5.5. Proposition. Assume 1.5.1. There is a one-to-one map \(B_{G,s}\) from the set of unipotent 2-blocks of \(G(s)^F\) to the set of 2-blocks in series \((s)\) of \(GF\) such that, if \(b \in \Pi(G(s)^F)\),

\[
(1)\ b \text{ and } B_{G,s}(b) \text{ have a common defect group.}
\]

\[
(2)\ \text{There is a one-to-one map } \Psi_b \text{ from } \text{Irr}(b) \text{ onto } \text{Irr}(B_{G,s}(b)) \text{ that preserves height.}
\]

\[
(3)\ \text{The Brauer's category of } b \text{ and } B_{G,s}(b) \text{ are isomorphic.}
\]

On the proof.

When \(A_{G^*}(s)^F = 1\), as we have seen in the proof of Proposition 1.5.2, \(b\) is the principal and unique unipotent 2-block of \(G(s)^F\), \(\text{Irr}(b) = \mathcal{E}_2(G(s)^F, 1)\), \(B_{G,s}(b)\) is the unique 2-block in series \((s)\) of \(GF\), the map \(\Psi_b:\text{Irr}(b) \rightarrow \text{Irr}(B_{G,s}(b))\) is given using Lusztig induction \(\lambda \mapsto R_G^G(\Psi_{G,s}(1) \otimes \lambda)\). The map \(\Psi_b\) preserves height thanks to degree's formulas given in section 1.3.

Assuming \(A_{G^*}(s)^F \neq 1\), \(G\) has type \(A\). Consider now a regular embedding \((G, F) \subseteq (H, F)\) and \(t\) semi-simple of odd order in \(H^F\) with image \(s\) by a dual morphism. Using Clifford theory of irreducible
representations and of blocks and knowing that there is no multiplicity in restrictions of irreducible representations from $H^F$ to $G^F$ and from $G(s)^F$ to $G(s)^{o F}$, the crucial step is to verify the following combinatorial facts (see the proofs of Propositions 3.4.1 and 3.4.2 for more detailed arguments):

(a) If the unique 2-block in series $(t)$ of $H^F$ covers exactly $m$ blocks of $G^F$ (therefore in series $(s)$), then the unique unipotent 2-block of $G(s)^{o F}$ is covered by exactly $m$ blocks of $G(s)^F$. Here $m$ is given in Proposition 1.5.4.

(b) Let $\chi \in \mathcal{E}_2(H^F, t)$, hence $\chi = R_{H(t), t}^H(\Psi_{H(t), t}(1) \otimes \lambda)$, where $\lambda \in \mathcal{E}_2(H(t)^F, 1)$. Here we have $\lambda \in \mathcal{E}(H(t)^F, t_1)$, where $t_1 \in C_{H(t)}^F(1/2)$. Let $s_1$ be the image of $t_1$ in $C_{G(s)^{o F}}(s_1)^F$. There is a Levi subgroup $H(tt_1) \subseteq H(t)$ in the dual class of $C_{T^*}(tt_1)$ and $G(ss_1)^{o F} := H(tt_1) \cap G$ is a Levi subgroup of $G(s)^o$ in the dual class of $C_{G^o}(ss_1)$ (recall that $G$ has type $A$). Let $\alpha \in \mathcal{E}(H(tt_1)^F, 1)$ such that $\lambda = \Psi_{H(tt_1), t_1}(\alpha)$. The inclusion $G(ss_1)^o \subseteq H(tt_1)$ is a regular embedding. We may identify $\mathcal{E}(H(tt_1)^F, 1)$ and $\mathcal{E}(G(ss_1)^{o F}, 1)$, so that $\lambda$ defines $\mu := \Psi_{G(s)^{o F}, s_1}(\alpha) \in \mathcal{E}(G(s)^{o F}, s_1) \subseteq \text{Irr}(b(s))$.

We have to show that if $\chi$ covers exactly $n$ elements of $\mathcal{E}_2(G^F, s)$, then $\mu$ is covered by $n$ elements of $\text{Irr}(G(s)^F)$.

The set of irreducible components of $\text{Res}_{G^o}^H \chi$ is a regular orbit under $(A_{G^o}(ss_1)^F)^\wedge$ and $A_{G^o}(ss_1)$ is isomorphic to $A_{G(s)^o}(s_1)$. Thus $n = \left| A_{G^o}(ss_1)^F \right| = \left| A_{G(s)^o}(ss_1)^F \right|$ is the number of elements of $\text{Irr}(G(s)^F)$ that cover $\mu$.

The proof of (3) is left to the reader. It is similar and simpler that the proof of property (B.4) in Theorem 1.4, see section 4 below.
2. Cuspidal data, Generalized \(d\)-Harish-Chandra theory and blocks

In all this section are given \((G,F), (G^*,F), \ell > 2, d\) as in 1.1.2 and 1.1.5, see also Assumption 2.1.2.

2.1. Facts on cuspidal data and blocks

2.1.1. Definitions. An element \(\chi\) of \(\text{Irr}(G^F)\) is said to be \(d\)-cuspidal when for any proper \(d\)-split Levi subgroup \(L\) of \(G\) and any parabolic subgroup \(P\) of \(G\) admitting \(L\) as a Levi complement, one has \(\mathcal{R}_{L \subseteq P}^G \chi = 0\).

A \(d\)-cuspidal datum in \((G,F)\) is a couple \((L,\lambda)\), where \(L\) is a \(d\)-split Levi subgroup of \(G\) and \(\lambda\) is a \(d\)-cuspidal element of \(\text{Irr}(L^F)\). It is said in series (s), if \(\mathcal{R}_{F}^G \zeta\) writes in Lusztig series \(\mathcal{E}(G^F,s)\), \(s\) a semi-simple element of \(G^F\).

Let us denote in this hypothesis \(\mathcal{E}(G^F,(L,\lambda))\) the set of \(\chi \in \text{Irr}(G^F)\) such that \((\mathcal{R}_{L \subseteq P}^G \lambda,\chi)_{G^F} \neq 0\) for some \(P\).

Unipotent \(d\)-cuspidal data are described in [10]. Once \(d\)-split Levi subgroups are known, classification of \(d\)- cuspidal data in series (s) reduces to the existence of \(d\)-cuspidal elements in \(\mathcal{E}(G^F,s)\). Then one may rely \(d\)-cuspidality with Jordan decomposition. A first result is the following, to compare with Propositions 2.2.3, 2.1.4.

2.1.1.1. Assume \(C_{G^*}(s)\) is connected and that commutation formula \(\mathcal{R}_{L \subseteq P}^G \circ \Psi_{G,s} = \Psi_{G,s} \circ \mathcal{R}_{C_{G^*}(s)}^{C_{G^*}(s)}\) holds for any proper \(d\)-split Levi subgroup in \((G,F)\). Then \(\chi \in \mathcal{E}(G^F,s)\) is \(d\)-cuspidal if and only if

\[
 Z^s(C_{G^*}(s))_{\phi_4} = Z^s(G^*)_{\phi_4} \quad \text{and} \quad \chi = \Psi_{G,s}(\alpha), \quad \text{where} \quad \alpha \text{ is } d\text{-cuspidal in } \mathcal{E}(C_{G^*}(s)^F,1).
\]

Proof. By Propositions 1.3.2, 1.3.6. \(\Psi_{G,s}\) is well defined. The commutation formula is (J3), introduced in 1.3.3. Let \(\alpha \in \mathcal{E}(C_{G^*}(s)^F,1)\). Let \(L\) be any proper \(d\)-split Levi subgroup in \((G,F)\), and \(L^*\) in the dual \(G^*\)- conjugacy class such that \(s \in L^*\). As \(L^* \not\subseteq G^*\), \(Z^s(L^*)_{\phi_4} \neq Z^s(G^*)_{\phi_4}\).

If \(Z^s(C_{G^*}(s))_{\phi_4} = Z^s(G^*)_{\phi_4}\) and \(\alpha\) is \(d\)-cuspidal, then \(C_{L^*}(s)\) is a proper \(d\)-split Levi subgroup of \(C_{G^*}(s)\) so that \(\mathcal{R}_{C_{G^*}(s)}^{C_{G^*}(s)} \alpha = 0\). By commutation formula (J3) \(\mathcal{R}_{G^F} \Psi_{G,s}(\alpha) = 0 \circ \Psi_{G,s}(\alpha)\) is \(d\)-cuspidal.

If \(\Psi_{G,s}(\alpha)\) is \(d\)-cuspidal, by (J3) \(\mathcal{R}_{C_{G^*}(s)}^{C_{G^*}(s)} \alpha = 0\). That imply \(C_{L^*}(s) \neq C_{G^*}(s)\) for any proper \(d\)-split Levi subgroup of \(G^*\), hence \(C_{G^*}(Z^s(C_{G^*}(s))_{\phi_4}) = G^*\), equivalently \(Z^s(C_{G^*}(s))_{\phi_4} = Z^s(G^*)_{\phi_4}\). Then if \(L^*_s\) is any proper \(d\)-split Levi subgroup of \(C_{G^*}(s)\), \(L^*_s := C_{G^*}(Z^s(L^*_s))_{\phi_4}\) is a proper Levi subgroup of \(G^*\) and one has \(\mathcal{R}_{C_{G^*}(s)}^{C_{G^*}(s)} \alpha = 0\) by (J3): \(\alpha\) is \(d\)-cuspidal.

To use [15], we have to make some assumption on \(\ell\).

2.1.2. Assumption on \((G,F,\ell,d)\). \(G\) is defined on \(\mathbb{F}_q\) by \(F\), \(\ell\) is odd, \(d\) is the order of \(q\) modulo \(\ell\), “block” means “\(\ell\)-block”. If some component of \(G\) has exceptional type, or if some rational component of \(G\) has type \(3D_4\), then \(\ell \geq 5\). If some component of \(G\) has type \(E_8\), then \(\ell \geq 7\).

Assumption 2.1.2 on \((G,F,\ell,d)\) implies 2.1.2 on \((H,F,\ell,d)\) for any \(F\)-stable connected algebraic subgroup \(H\) of \(G\). Among consequences of Assumption 2.1.2, recall that \(\ell\) is good for \(G\) and that any \(d\)-cuspidal unipotent irreducible representation of \(G^F\) is the canonical irreducible representation of a block of \(G^F\) with central defect [16] Proposition 22.16.

From properties of Lusztig induction with respect to isotypic morphisms and of Jordan decomposition (see Proposition 1.3.2), transitivity of Lusztig restriction (\(\mathcal{R}_{F}^G \circ \mathcal{R}_{L \subseteq P}^G = \mathcal{R}_{F}^G\)) and the uniform criterion on \(d\)-cuspidal unipotent data [10] 3.13 one deduces the following equivalence.

2.1.3. Proposition. [15] Proposition 1.10, (i). Let \((G,F)\) and \((G^*,F)\) be in duality and \(s \in (G^*)^F\) semi-simple, \(\chi \in \mathcal{E}(G^F,s)\). Assume 2.1.2 on \((G,F,\ell,d)\). The following assertions (i) and (ii) are equivalent
(i) For every $F$-stable maximal torus $T$ of $G$ such that $T_{\phi_d} \nsubseteq Z(G)$, one has $R_{T,\phi_d}^G \chi = 0$.

(ii) $Z^e(C_{G^e}(s))_{\phi_d} \subseteq Z(G^e)$ and $\chi$ corresponds by Proposition 1.3.6 to a $C_{G^e}(s)^F$-orbit of $d$-cuspidal unipotent irreducible characters of $C_{G^e}(s)^F$ by Lusztig’s parametrization.

2.1.4. Proposition. [15] Theorem 4.2. Assume 2.1.2 on $(G,F,\ell,q)$. Let $s$ be a semi-simple $\ell'$-element of $G^F$. Any $d$-cuspidal datum in series $(s)$ of $(G,F)$ may be defined as follows

Let $(L^*_s,\alpha)$ be a unipotent $d$-cuspidal datum of $(C_{G^e}(s),F)$, let $L^* := C_{G^e}(Z^e(L^*_s)_{\phi_d})$, a d-split Levi subgroup of $G^*$, let $L$ be a Levi subgroup of $G$ in the dual $G^F$-conjugacy class of the $(G^*)^F$-class of $L^*$. To the orbit of $\alpha$ under $A_{L^*}(s)^F$ there corresponds, by (1.3.6.1), an orbit $\Lambda$ under $(A_{L^*}(s)^F)^\Lambda$ in $E(L^F,s)$. For any $\lambda \in \Lambda$, $(L,\lambda)$ is a $d$-cuspidal datum of $(G,F)$.

If $A_{G^e}(s)^F = \{1\}$, the map $(L^*_s,\lambda) \mapsto (L,\zeta)$ so defined induces a bijection from the set of $C_{G^e}(s)^F$-conjugacy classes of unipotent $d$-cuspidal data in $(C_{G^e}(s),F)$ to the set of $G^F$-conjugacy classes of $d$-cuspidal data in series $(s)$ of $(G,F)$.

Comments. We comment here the last assertion of Proposition 2.1.4.

When $A_{G^e}(s)^F = 1$ in the definition above one may write $\lambda = \Psi_{L,s}(\alpha)$. In this case $(L,\lambda)$ is defined up to $G^F$-conjugacy: $s$ is defined up to $G^{\ast F}$-conjugacy. $(L^*_s,\alpha)$ is defined up to $C_{G^e}(s)^F$-conjugacy, hence $(L^*,s)$ is defined up to $G^{\ast F}$-conjugacy, finally $(L,E(L^F,s))$ is defined up to $G^F$-conjugacy, so is $(L,\lambda)$ thanks to (iv) of Proposition 1.3.2.

Assume now that two unipotent $d$-cuspidal data $(L^*_{s,j},\alpha_j)$ ($j = 1,2$) define $G^F$-conjugate $(L_2,\lambda_2)$, $(L_1,\lambda_1)$. Thus $(L_j,E(L^F,s))$ ($j = 1,2$) are $G^F$-conjugate and, by our comments on (iv) of Proposition 1.3.2, there exists some $g^* \in C_{G^e}(s)^F$ inducing a dual morphism $L^*_1 = g^*L^*_2$, so that $\lambda_2 = \Psi_{L_2,s}(\alpha_2) = \lambda_1g = \Psi_{L_2,s}(g^*\alpha_1)$, hence $\alpha_2 = g^*\alpha_1$.

2.1.5. Proposition. Assume 2.1.2 on $(G,F,\ell,q)$.

(a) Let $\sigma:f(G,F) \to (H,F)$ be an isotypic morphism.

Let $M$ be a $d$-split Levi subgroup of $H$, $\mu \in E(M^F,t)$, $t \in (M^*)^F \subseteq H^*$, $L = \sigma^{-1}(M)$ and $\lambda$ be an irreducible component of $\text{Res}_{L^F \to M^F}\mu$. Then $(M,\mu)$ is a $d$-cuspidal datum in $(H,F)$ if and only if $(L,\lambda)$ is a $d$-cuspidal datum in $(G,F)$.

(b) Let $(G,F) = (G_1,F),(G_2,F)$ be a central product of connected reductive algebraic groups defined on $F^q$. Let $L = L_1,L_2$ be an $F$-stable Levi subgroup of $G$ where $L_j = L \cap G_j$, $\lambda_j \in \text{Irr}(L^F)$, $\lambda_j \in \text{Irr}(L^F_j)$ such that $\lambda \in \text{Irr}(L^F_j \mid \lambda_1 \otimes \lambda_2)$. Then $(L,\lambda)$ is a $d$-cuspidal datum in $(G,F)$ if and only if, for $j = 1,2$, $(L_j,\lambda_j)$ is a $d$-cuspidal datum in $(G_j,F)$.

(c) Assume $G = G_\alpha$ (see § 1.5.1). There is only one $G^F$-conjugacy class of $d$-cuspidal unipotent data in $(G,F)$, such as $(T,1_{T^F})$ where $T$ is a diagonal torus.

On proofs. In an isotypic morphism $\sigma:G \to H$, the sets of Levi subgroups $L$ of $G$ and $M$ of $H$ correspond bijectively by $L \mapsto M = \sigma(L)Z^e(H)$ and $M \mapsto L = \sigma^{-1}(M)$. Then $L$ is $d$-split if and only if $M$ is $d$-split. To verify (a) we may assume $L = H$ and $M = G$. If $\mu$ is unipotent, then $\text{Res}_{H^F}^G \mu \in E(G^F,1)$. By (1.3.1.1) $d$-cuspidality of $\mu$ is equivalent to $d$-cuspidality of $\text{Res}_{H^F}^G \mu$. If $\mu \in E(H^F,t)$ and $\sigma^*(t) = s$, $\lambda \in E(G^F,s)$ and there is some $\alpha \in E(C_{G^e},t)^F$, $1$ that corresponds to $\lambda$ and $\mu$ (once $E(C_{G^e},t)^F$, $1$ and $E(C_{G^e},s)^F$, $1$ are identified, see Proposition 1.3.7). By Proposition 2.1.4, $\mu$ is $d$-cuspidal if and only if $\alpha$ is $d$-cuspidal, if and only if $\lambda$ is $d$-cuspidal.

(b) is clear if the product is direct. The central quotient morphism is isotypic.
(c) If $G = G_\alpha$, the $d$-split Levi subgroups of $G$ are the diagonal Levi subgroups. For any diagonal Levi subgroups $M \subseteq L$ and $\chi \in \mathcal{E}(L^F, 1)$, $R_M^L \chi \neq 0$. If $(L, \alpha)$ is a unipotent $d$-cuspidal datum, $L$ is a diagonal torus, therefore $\alpha = 1_{L^F}$.

Our classification of blocks of $G^F$ is given by the two following propositions (see also [16] 21.7, Chapter 4, Exercise 4).

2.1.6. Proposition. [15] Theorem 2.5. Assume 2.1.2 on $(G, F, \ell, d)$. Let $K$ be an $E$-split Levi subgroup of $G$. Let $b_K$ be a block of $K^F$ in series $(s)$, where $s \in (K^*)^F_0 \subseteq G^*$. There exists an block $B$ of $G^F$ in series $(s)$, we denote $R_K^G b_K$, such that, for all $\xi \in \text{Irr}(b_K) \cap \mathcal{E}(K^F, s)$, any irreducible component of $R_{K^F}^G \xi$ belongs to $\text{Irr}(B)$, whatever be the parabolic $P$ with Levi complement $K$. Furthermore if $K^F = C_{G^F}(\mathbb{Z}(K)^F_0)$, then $(\{1\}, R_K^G b_K) \subseteq (\mathbb{Z}(K)^F_0, b_K)$, an inclusion of Brauer subpairs in $G^F$.

2.1.7. Proposition. [15] Theorem 4.1. Assume 2.1.2 on $(G, F, \ell, d)$. Let $s$ be a semi-simple element of $(G^*)^F_0$ and let $(L, \lambda)$ be a $d$-cuspidal datum in series $(s)$ in $(G, F)$. Then is defined an block $b_{G^F}(L, \lambda)$ of $G^F$ with the following properties :

(a) $\lambda \in \text{Irr}(b_{L^F}(L, \lambda))$.

(b) If $M$ is a $d$-split Levi subgroup of $G$ and $(L, \lambda)$ is a $d$-cuspidal datum in $(M, F)$, then, with notations of Proposition 2.1.6, $R_M^G (b_{MF}(L, \lambda)) = b_{G^F}(L, \lambda)$.

The map $(L, \lambda) \mapsto b_{G^F}(L, \lambda)$ is one-to-one from the set of $G^F$-conjugacy classes of $d$-cuspidal data in $\ell'$-series in $(G, F)$ onto the set of blocks of $G^F$.

Some comments on Proposition 2.1.7.

(a) For bad primes, see [23] who suggests that blocks correspond to $d$-cuspidal data $(L, \lambda)$ where $\lambda$ has central defect group, that is not always the case for all $d$-cuspidal data if $\ell$ is bad for $G$.

(b) The group $(G^F/[G, G]^F)^\wedge$ acts on the set of blocks of $G^F$ “by tensor product” that is by the equality (see Appendix, 5.1)

$$\theta \otimes b_{G^F}(L, \lambda) = b_{G^F}(L, (\text{Res}_{L^F}^{G^F} \theta) \otimes \lambda), \quad \theta \in (G^F/[G, G]^F)^\wedge$$

thanks to the equality $R_L^G ((\text{Res}_{L^F}^{G^F} \theta) \otimes \lambda) = R_L^G \theta \otimes \lambda$ [20] 12.6. In section 2.4 we compute the stabilizer of $b_{G^F}(L, \lambda)$ in terms of the unipotent $d$-cuspidal datum corresponding to $(L, \lambda)$ by Proposition 2.1.4.

(c) More generally in a central product $G = G_1.G_2$, by Proposition 2.1.5 a $d$-cuspidal datum $(L, \lambda)$ in $(G, F)$ covers a product of $d$-cuspidal data $(L_1 \times L_2, \lambda_1 \otimes \lambda_2)$ in $(G_1 \times G_2, F)$. Furthermore $G^F/G_1^F.G_2^F$ is isomorphic to $L^F/L_1^F.L_2^F$ by Lemma 1.2.1. By (1.3.1.1) applied to $\sigma: G_1 \times G_2 \rightarrow G$, $(G^F/[G, G]^F)^\wedge$ acts on the set of blocks of $G^F$ defined by $d$-cuspidal data with support $L_1.L_2$ as $(L^F/L_1^F.L_2^F)^\wedge$ acts on $d$-cuspidal elements in $\text{Irr}(L^F)$.

2.1.8. Corollary. Let $(G, F, \ell, d, s, L, \lambda)$ as in Proposition 2.1.7 and $\mu \in \text{Irr}(L^F)$. Then $R_{L^F}^G \lambda$ and $R_{L^F}^G \mu$ have a common irreducible component if and only if $\lambda$ and $\mu$ are conjugate under $N_{G^F}(L)^F$.

Proof. If $\lambda$ and $\mu$ are $N_{L}(G)^F$-conjugate, then $R_{L^F}^G \lambda = R_{L^F}^G \mu$.

Assume that $R_{L^F}^G \mu$ and $R_{L^F}^G \lambda$ have a common component $\chi \in \mathcal{E}(G^F, s)$. There exist some $d$-cuspidal datum $(L_0, \lambda_0)$ in $(L, F)$ such that $\mu \in \text{Irr}(b_{L^F}(L_0, \lambda_0))$. Then $\chi \in \text{Irr}(b_{G^F}(L_0, \lambda_0)) \cap \text{Irr}(b_{G^F}(L, \lambda))$ by Proposition 2.1.6. Hence $b_{G^F}(L, \lambda) = b_{G^F}(L_0, \lambda_0) \in (L, \lambda)$ and $(L_0, \lambda_0)$ are $G^F$-conjugate and $\mu = \lambda_0$.

In the following Proposition we precise the induction on blocks (Proposition 2.1.6) with respect to the parametrization by conjugacy classes of $d$-cuspidal data (Proposition 2.1.7).
2.1.9. Proposition Assume 2.1.2 on $(G, F, \ell, d)$. Let $K$ be an $E$-split Levi subgroup of $G$ and $(L_K, \lambda_K)$ be a $d$-cuspidal datum in $(K, F)$ in an $\ell'$-series, defining the block $b_{K^F}(L_K, \lambda_K)$ (Proposition 2.1.7). Let $L := C_G(Z^o(L_K)_{\phi_d})$. Then $R^L_{L_K} \lambda_K$ is $d$-cuspidal and one has, with notations of Proposition 2.1.6,

$$R^G_K(b_{K^F}(L_K, \lambda_K)) = b_{G^F}(L, R^L_{L_K} \lambda_K)$$

Proof. The important fact is that $R^L_{L_K} \lambda_K$ is $d$-cuspidal and we first prove it. By definition $L$ is the smallest $d$-split Levi subgroup of $G$ such that $L_K \subseteq L$.

If $G$ is a central product over $F$, $G = G_1.G_2$, then $K := K \cap G_i$ is $E$-split in $G_i$ and $K = K_1.K_2$. Let $L_{K,i} = L_K \cap G_i$, hence $L_K = L_{K,1}.L_{K,2}$, $L_{K,i}$ is $d$-split in $K_i$. By (b) in Proposition 2.1.5, $\text{Res}_{L_{K,i}}^{L}$ $\lambda_K$ writes $\lambda_{K,1} \otimes \lambda_{K,2}$ where each $\lambda_{K,i}$ is $d$-cuspidal. We have $L_i := L \cap G_i = C_{G_i}(Z^o(L_{K,i})_{\phi_d})$. If $R^L_{L_{K,i}} \lambda_{K,i}$ is $d$-cuspidal for $i = 1, 2$, as $\text{Res}_{L_{K,i}}^{L} (R^L_{L_K} \lambda_K) = R^{L_{i,1}}_{L_{K,i}} \lambda_{K,1} \otimes R^{L_{i,2}}_{L_{K,i}} \lambda_{K,2}$, by Proposition 2.1.5 again, $R^L_{L_K} \lambda_K$ is $d$-cuspidal.

So we may assume $G$ normally irreducible.

(a) In this first step we assume that $K$ is a maximal proper $E$-split Levi subgroup of $G$.

If $K$ is $d$-split, $L_K$ is $d$-split and $L = L_K$, we are done. If $G = Z^o(G).G_0$, then $K$ is $d$-split by 1.1.5.3.

Assume now that $G \neq Z^o(G).G_0$ and $K$ is not $d$-split. Let $A_{\alpha}((\alpha q)^{d_\alpha})$ be the rational type of $G$. Then $K$ has type $A_{\alpha}((\alpha q)^{d_\alpha}) \times A_m((\alpha q)^{d_\alpha})$, where $(n + 1) = (m + 1) \ell + n + 1$ and the type of $L_K$ has the form $(\times_j A_{\alpha_j}((\alpha q)^{d_\alpha})) \times (\times_i A_{\alpha_i}((\alpha q)^{d_\alpha}))$.

Then the rational type of $L$ is $(\times_j A_{\alpha_j}((\alpha q)^{d_\alpha})) \times (\times_i A_{\alpha_i}((\alpha q)^{d_\alpha}))$ and $L^*$ differs from $(L_K)^*$ only on the right side of that product.

We see that $L$ and $L_K$ have a common Coxeter torus. With a coherent choice of dualities $L^*$ and $(L_K)^*$ have a common Coxeter torus.

If $\mu_K \in \mathcal{E}(L_K^*, s)$, as $\mu_K$ is $d$-cuspidal, $C^o_{(L_K)^*}(s)$ is a Coxeter torus $T^*$ of $(L_K)^*$. By definition of $L^*$, $L^* = C_G((Z^o((L_K)^*)_{\phi_d})$ and we have $C_{L^*}(s) \cap (L_K)^* = C_{(L_K)^*}(s) = T^*$. For each value of the index $i$ in the decomposition above, the components of rational types $A_{\alpha_i}((\alpha q)^{d_{\alpha_i}})$ and $A_{\alpha_i}((\alpha q)^{d_{\alpha_i}})$ of $L^*$ and $(L_K)^*$ respectively are represented on the same vector space $V_i$ of dimension $d_{\alpha_i}(m_i + 1)$. As $C_{(L_K)^*}(s)$ is a Coxeter torus, product of Coxeter tori of each component, the semisimple element $s$ has $d_{\alpha_i}(m_i + 1)$ distinct eigenvalues with multiplicity one on $V_i$. Hence the $i$-component of $C_{L^*}(s)$ is a maximal torus of the $i$-component of $L^*$, that is $C_{L^*}(s) = T^*$, a Coxeter torus of $L^*$. If $T$ is a Coxeter torus of $L$, it is in the dual class of $T^*$. In case $Z^o(G)$ is connected, we have $\lambda_K = R^T_{L_K} (\Psi_{T,s}(1))$, hence $R^T_{L_K} \lambda_K = R^T_{F} (\Psi_{T,s}(1))$; it is a cuspidal element in $\mathcal{E}(L^*, t)$. In general case $\lambda_K$ corresponds to $1_T.F$ as well as $R^T_{L_K} \lambda_K$ by Propositions 2.1.4 and 1.3.6, and we have our claim.

(b) We use induction on the semi-simple rank of $G$.

If $K$ is not proper maximal as $E$-split Levi subgroup of $G$, let $K_1$ be a maximal proper $E$-split Levi subgroup of $G$ such that $K \subseteq K_1$. Denote $L_1 := C_{K_1}(Z^o(L_K)_{\phi_d})$, $L_1$ is an $E$-split Levi subgroup of $G$. We have $Z^o(L_1)_{\phi_d} = Z^o(L_K)_{\phi_d} = Z^o(L_{K_1})_{\phi_d}$, hence $L = C_G(Z^o(L_1)_{\phi_d})$. By induction hypothesis we know that $R^L_{L_K} \lambda_K$ is $d$-cuspidal. By transitivity of Lusztig induction and (a) $R^L_{L_K} (R^L_{L_1} \lambda_K) = R^L_{L_1} \lambda_K$ is $d$-cuspidal.

(c) On blocks:

By Proposition 2.1.7 any irreducible component $\xi$ of $R^L_{L_K} \lambda_K$ belongs to $\text{Irr}(b_{K^F}(L_K, \lambda_K))$. Then by Proposition 2.1.6 any irreducible component of $R^L_{L_K} \lambda_K$ belongs to $\text{Irr}(R^L_{L_K}(b_{K^F}(L_K, \lambda_K)))$. We see that any
irreducible component of \( R_{L_K}^G \lambda_K \) belongs to \( \text{Irr}(R_{L_K}^G (b_{K_F}(L_K, \lambda_K))) \). But \( R_{L_K}^G \lambda_K = R_L^G (R_{L_K}^L \lambda_K) \) writes in \( \mathbb{Z} \text{Irr}(b_{G^F}(L, R_{L_K}^L \lambda_K)) \) hence \( R_{L_K}^G (b_{K_F}(L_K, \lambda_K)) = b_{G^F}(L, R_{L_K}^L \lambda_K) \).

Propositions 2.1.4, 2.1.6, 2.1.7 suggest to compare blocks in series \( s \) of \( G^F \) and unipotent blocks of \( C_{G^*}(s)^{F} \) using \( d \)-cuspidal data. By Jordan decomposition, the relation is quite simple if the center of \( G \) is connected:

**2.1.10. Proposition.** Assume \( 2.1.2 \) on \( (G,F,\ell,q), Z(G) \) connected and \( s \) semi-simple in \( (G^*)_F^\ell \). By Proposition 2.1.4 there is a one-to-one map from the set of blocks in \( Z \text{Irr}(G^F, s) \) onto the set of unipotent \( d \)-cuspidal data. By Jordan decomposition, the relation is quite simple if the center of \( G \) is connected:

\[\text{as a consequence, by Proposition 2.1.7, there is a one-to-one map from the set of blocks } B \text{ of } G^F \text{ such that } \text{Irr}(B) \subseteq \mathcal{E}_\ell(G^F, s) \text{ onto the set of unipotent blocks of } C_{G^*}(s)^{F}.\]

**Proof.** In Proposition 2.1.4 the group \( L \) is defined up to \( G^F \)-conjugacy. The \( G^F \)-conjugacy class of \( (L, \mathcal{E}(L^F, s)) \) depends on the \( C_{G^*}(s)^{F} \)-conjugacy class of \( L^*_s \).

Assume now that two couples \((L_1, \mathcal{E}(L_1^F, s))\) and \((L_2, \mathcal{E}(L_2^F, s))\) are conjugate by some \( g \in G^F \). Then there exists \( h \in G^F \) such that \( (L_1^*, s) = (L_2^*, s)^h \). Indeed dualities are defined around couple of torii in duality \((T_i \subset L_i, T_i^* \subset L_i^*)\) \((i = 1, 2)\). Up to \( L_i^F \)-conjugacy, one may assume \( T_1 = T_2 \). Then there is \( h \in (G^*)^F \) such that \( hT_1^* = T_2^* \). By \((g,h)\)-conjugacy the root datum of \((L_1, L_1^*)\) with respect to \((T_1, T_1^*)\) is sent on the root datum of \((L_2, L_2^*)\) with respect to \((T_2, T_2^*)\). As \( g \) sends \( \mathcal{E}(L_1^F, s) \) on \( \mathcal{E}(L_2^F, s) \), up to \( (L_2^*)^F \)-conjugacy we may assume that \( h \in C_{G^*}(s)^F \). Therefore the \( d \)-split Levi subgroups \( L_1^*, s := C_{L_1^*}(s) \) and \( L_2^*, s := C_{L_2^*}(s) \) are \( C_{G^*}(s)^{F} \)-conjugate.

Denote \( S(L, s) \) the set of \( d \)-cuspidal \( \lambda \in \mathcal{E}(L^F, s) \).

Once \( s \) and the groups \( L, L^* \) are given, we are reduced to consider the set of orbits under \( N_G(L)^F \) on \( S(L, s) \). On the other side is the set of orbits under \( N_{C_{G^*}(s)^{F}}(L^*_s)^F \) on \( S(L^*_s, 1) \). One knows that \( N_G(L)^F \) acts on Lusztig series in \( L^F \) as \( W_G(L)^F = W_{G^*}(L^*)^F \) acts on classes of semi-simple elements of \( (L^*)^F \). We obtain a one-to-one map

\[\left( \bigcup_{n \in N_G(L)^F} n S(L, s) \right)/N_G(L)^F \cong S(L, s)/N_G(L, \mathcal{E}(L^F, s))^F\]

As \( Z(L) \) and \( Z(G) \) are connected \( \Psi_{G,s} \) et \( \Psi_{L,s} \) satisfy to Proposition 1.3.2, specially assertion (iv). The group \( N_G(L, \mathcal{E}(L^F, s))/L^F \) is isomorphic to the relative Weyl group \( W_{C_{G^*}(s)}(L^*_s)^F \), isomorphic to \( N_{W_{G^*}(s)}(W_L(s))/W_L(s)^F \) (see 2.2.5 below). It acts on the root datum of \( L \) with respect to \( T \) and on the root datum of \( L^*_s := C_{L^*}(s) \) with respect to \( T^* \). Via these isomorphisms and Jordan decomposition, the actions on \( \mathcal{E}(L^F, s) \) and on \( \mathcal{E}(L^*_s)^F \) are exchanged. Let \( \lambda \in S(L, s), \lambda = \Psi_{L,s}(s), \alpha \in S(L^*_s, 1) \) (Proposition 2.1.4). We have one-to-one maps on quotients

\[N_G(L, \mathcal{E}(L^F, s))/N_G(L, \lambda) \cong W_{C_{G^*}(s)}(L^*_s)^F / W_{C_{G^*}(s)}(L^*_s)^F \]

\[\cong N_{C_{G^*}(s)}(L^*_s)^F / N_{C_{G^*}(s)}(L^*_s)^F.\]

Let \( U = S(L^*_s, 1) \) be the set of \( d \)-cuspidal in \( \mathcal{E}(L^*_s)^F, 1 \). The one-to-one map \( S(L, s) \ihr U/N_{C_{G^*}(s)}(L^*_s)^F \) induces a one-to-one map \( S(L, s)/N_G(L, \mathcal{E}(L^F, s)) \ihr U/N_{C_{G^*}(s)}(L^*_s)^F \).

**2.1.11. Proposition.** On central defect. Assume 2.1.2 on \((G,F,\ell,d)\).

(a) Let \( \sigma: (G,F) \to (H,F) \) be an isotypic morphism defined on \( F_q \). Let \( \chi \in \text{Irr}(H^F) \) with central \( \ell \)-defect. Any irreducible component of \( \text{Res}_{G^F}^H \chi \) has central \( \ell \)-defect.
(b) Let $G = G_a.G_b$, the decomposition defined in 1.1.5.2. Let $\chi \in \text{Irr}(G^F | \chi_a \otimes \chi_b)$, where $\chi_a \in \text{Irr}(G^F_a)$ and $\chi_b \in \text{Irr}(G^F_b)$. Then $\chi$ has central $\ell$-defect if and only if $\chi_a$ and $\chi_b$ have central $\ell$-defect.

(c) Let $(L, \lambda)$ be a $d$-cuspidal datum in series $(s)$.

(i) Assume $G = G_b$. The block $b_{GF}(L, \lambda)$ has central defect group if and only if $Z(C_{G^F}(s))^F = Z(G^*)^F$ and $L = G$. Then $\lambda$ is the canonical character of $b_{GF}(L, \lambda)$.

(ii) Assume $G = G_a$. The block $b_{GF}(L, \lambda)$ has central defect group if and only if $C_{G^F_a}(s)$ is a torus with a dual $T$ in $G$ such that $T^F = Z(G^F)|_{\ell}$. Let $\theta = \Psi_{T, \lambda}(1, \gamma, r) \in (T^F)^\lambda$. Then $T$ is a Coxeter torus of $(G, F)$ and $L = T$. The canonical character of $b_{GF}(T, \lambda)$ is a component of $R^G_K(\theta)$.

Proof. (a) is a consequence of non-multiplicity in $\text{Res}_\sigma$ (Proposition 1.3.4). Let $\xi \in \text{Irr}(\sigma(G^F)) \subseteq \text{Irr}(G^F)$ such that $\chi \in \text{Irr}(H^F | \xi)$ and let $X$ be the stabilizer of $\xi$ in $H^F$. One has $\chi(1) = |H^F/X|_{\ell} \cdot \chi(1)|_{\ell}$ and the assumption on $\chi$ writes $\chi(1)|_{\ell} = |H^F/H(\xi^F)|_{\ell}$.

As $\sigma(G^F).Z(H^F) \subseteq X$ and by isomorphism theorem $\sigma(G^F).Z(H^F)/Z(H^F) \cong \sigma(G^F)/\sigma(G^F) \cap Z(H^F)$, we have

$$\xi(1)|_{\ell} = |X/\sigma(G^F).Z(H^F)|_{\ell} \cdot \sigma(G^F)/\sigma(G^F) \cap Z(H^F)|_{\ell}$$

But $Z(H^F) = Z(H^F)|_{\ell}$, hence $\sigma(G^F) \cap Z(H^F) \subseteq Z(\sigma(G))$. Moreover $\sigma(Z(G)) = Z(\sigma(G))$ and, since the kernel $K$ of $\sigma$ is contained in $Z(G)$, $\sigma(Z(G)) \cap \sigma(G^F) = \sigma(Z(G) \cap G^F.K) = \sigma(Z(G^F)) = \sigma(Z(G^F))$ so that $\sigma(G^F) \cap Z(H^F) = \sigma(Z(G^F))$ and by report in (2.1.11.1)

$$\xi(1)|_{\ell} = |X/\sigma(G^F).Z(H^F)|_{\ell} \cdot \sigma(G^F)/\sigma(Z(G^F))|_{\ell}$$

As the restriction of $\sigma$ to $G^F$ has kernel $K^F \subseteq Z(G^F)$, we have an isomorphism $G^F/Z(G^F) \cong \sigma(G^F)/\sigma(Z(G^F))$.

As $\xi(1)$ divides $|G^F/Z(G^F)|$, (2.1.11.2) shows that $\xi$ has central defect and the order of the group $X/\sigma(H^F).Z(H^F)$ is prime to $\ell$. Indeed the order of $s$ is prime to $\ell$.

(b) The morphism $G_a \times G_b \rightarrow G$ is isotypic, hence (a) gives half of the equivalence.

As $G_a \times G_b \rightarrow G$ is onto, $|G^F| = |G^F_a| |G^F_b|$. By definition of $G_b$, $Z(G_b^F)$ has order prime to $\ell$, hence the kernel and cokernel of $G^F_a \times G^F_b \rightarrow G^F$ are $\ell$-groups, by 1.1.5.2, Proposition 1.1.5.3 and Lemma 1.2.1. It follows that $Z(G_b^F)$ is isomorphic to $Z(G_a)^F$, and, by Clifford theory, that $\chi(1)|_{\ell} = \chi_a(1)|_{\ell} \cdot \chi_b(1)|_{\ell}$. Thus $\chi(1)|_{\ell} \cdot |Z(G^F)|_{\ell}$ is equivalent to $\chi(1)|_{\ell} \cdot |Z(G_a^F)|_{\ell} = |G_a^F|_{\ell}$ and $\chi_b(1)|_{\ell} \cdot |Z(G_b^F)|_{\ell} = |G_b^F|_{\ell}$.

(c) By [16], Proposition 22.16, if $\alpha \in \mathcal{E}(G^F, 1)$ is $d$-cuspidal, $b_{GF}(\alpha)$ has central defect. Reciprocally if $(L, \alpha)$ is a $d$-cuspidal unipotent datum, we have $L = C^o_G(Z(L)^F)$ and $Z(L)^F$ is contained in a defect group of $b_{GF}(L, \alpha)$ (Proposition 2.1.6). Hence $b_{GF}(L, \alpha)$ has central defect group if and only if $L = G$. When $G = G_a$ and $G$ is not a torus there is no unipotent with central defect group (Proposition 2.1.5).

When $s = s_P$, $Z(G)^F$ is contained in the kernel of any $\chi \in \mathcal{E}(G^F, s)$ because the characteristic function of $Z(G)^F$ is uniform [DM], 12.21. If $\chi$ is in a block of $G^F$ with central defect group, $\chi$ is the canonical character of its block. So let $\alpha \in \mathcal{E}(C_{G^F_a}(s)^F, 1)$ in the orbit under $A_{GF}(s)^F$ associated to $\chi$ by Proposition 1.3.6. We know that $A_{GF}(s)^F$ is an $\ell$-group, so that by (vi) in Proposition 1.3.2, Propositions 1.3.4, 1.3.6,

$$\chi(1)|_{\ell} = \alpha(1)|_{\ell} |(G^{*})^F/C_{G^F_a}(s)^F|_{\ell}$$

But $|(G^*)^F| = |G^F|$ hence $b_{GF}(\chi)$ has central defect group if and only if

$$|Z(G)^F|_{\ell} \cdot \alpha(1)|_{\ell} = |C_{G^F_a}(s)^F|_{\ell}$$

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(i) Assume $G_\alpha$ is a torus. We have $Z(G)^F = Z^*(G)^F$, as well as $G^*$, hence $|Z(G)^F|_\ell = |Z(G^*)^F|_\ell$. Clearly $Z(G^*) \subseteq Z(G_\alpha^*(s))$. We see that $b_{G^*}(\chi)$ has central defect only if $b_{G_\alpha^*(s)}(\alpha)$ has central defect and $Z(G^*)^F = Z(G_\alpha^*(s))^F$. By [13], Proposition 2.2, as $L^*$ is $d$-split in $G = G_\alpha$, one has $L^* = C_{G^*_T}(Z(L^*)^F)$ (In fact (ii) in [13], 2.2, is true under the hypothesis "$\ell \in \gamma(G^*, F)$", that is satisfied if $G = G_\alpha$). But $Z(L^*)^F \subseteq Z(G_\alpha^*(s))^F = Z(G^*)^F$. It follows that $L^* = G^*$, $L = G$, $\alpha$ is $d$-cuspidal.

(ii) Assume $G = G_\alpha$, non abelian. Let $T$ be a maximal $F$-stable torus in $L$ in duality with a diagonal torus $T^*_s$ of $(C_{G^*_T}(s), F)$. A defect group of $b_{G^*}(L, \lambda)$ is extension of $T^*_s$ by an $\ell$-Sylow subgroup of $W_{C^*_T}(F)$. It is central in $G^F$ if and only if $T^*_s \subseteq Z(G)$ and $W_{C^*_T}(F) = \{1\}$. Then $T$ is a Coxeter torus of $(G, F)$. Thus a diagonal torus $T^*_s$ of $(C_{G^*_T}(s), F)$ is a Coxeter torus of $(G^*, F)$. As a Coxeter torus it is a maximal $F$-stable proper Levi subgroup in $G^*$, hence $C_{G^*_T}(s) = T^*_s$ and $L = T$. In a regular embedding $G \rightarrow H$, with dual $H^* \rightarrow G^*$, $T$ is send in a Coxeter torus $S$ of $H$, $T^*$ is image of a Coxeter torus $S^*$ of $H^*$, $s$ is image of $t \in S^*F$ such that $C_{H^*}(s) = S^*$. We see that $R_{H, t}^\ell(\Psi_{H, t}(1_{S^*F}))$ is the unique element of $\mathcal{E}(H^*, H)$. We have $R_{H, t}^\ell = R_{H, t}^{1_{S^*F}}(R_{H, t}^{1_{S^*F}}(\Psi_{H, t}(1_{S^*F})))$ by (1.3.1.1). The Lusztig series $\mathcal{E}(G^F, s)$ is the set of irreducible components of $R_{H, t}^\ell$. Such a component has central defect if and only if $T^*_s \subseteq Z(G)$.

We note that (b) cannot be generalized to any isotropic morphism. Consider $G = SL_\ell \subseteq H := GL_\ell$ on $F_q$ and assume that $(q - 1)\ell = \ell$. The two groups $G^F$ and $H^F$ have isomorphic $\ell$-centers. If $T$ is a coxeter torus of $H$, $|T^*_s(\ell)| = \ell^2$ and $|(T \cap SL_\ell)^F(\ell)| = \ell$. By (c) (iii) in Proposition 2.1.11, with $T^* = C_{H^*}(s)$, $\chi := \Psi_{H, t}(1_{S^*F})$ has not central defect, but the components of $R_{H, t}^\ell$ have central defect.

2.2. Generalized Harish-Chandra theory and blocks

2.2.1. Definition. Let $d$ be an integer and $s$ a semi-simple element of $G^F$. We say that Generalized $d$-Harish-Chandra’s theory holds in $\mathcal{E}(G^F, s)$ (or shortly G.d-HC holds in $\mathcal{E}(G^F, s)$) if and only if:

for any $\chi \in \mathcal{E}(G^F, s)$ there exists a $d$-cuspidal datum $(L, \lambda)$ in $(G, F)$, uniquely defined up to $G^F$-conjugacy, and $\alpha \neq 0$, such that

$$\text{R}_{L \leq P} \chi = a.(\sum_{g \in G(L)^F/G(L)^{(\ell)}} \lambda^0),$$

independantly of the choice of the parabolic subgroup $P$.

Then $\mathcal{E}(G^F, s) = \cup_{(L, \lambda)} G^P \mathcal{E}(G^F, (L, \lambda))$ and it is a partition.

Note that if $(\chi, R_{L, F}^\ell \lambda)_{G^F} \neq 0$ and $\chi \in \mathcal{E}(G^F, s)$, then there is a $d$-split Levi subgroup $L^*$ in $G^*$, in the dual conjugacy class of $L$, such that $s \in L^*$ and $\lambda \in \mathcal{E}(L^*, s)$. That is why the property may be considered inside each rational series.

If $d = 1$, Generalized $d$-Harish-Chandra’s theory is just classical Harish-Chandra’s theory and holds in any type, any series [20] Chapter 6.

If $G = G_\alpha$, $d$-split Levi subgroups are diagonal Levi subgroups. That imply that G. $d$-HC theory reduces in non twisted type to classical Harish-Chandra’s theory. Then a decisive fact is that if $\chi \in \text{Irr}(G^F)$ and $L$ is a split Levi subgroup of $G$, then $\text{R}_{L}^\ell$ is an effective representation. In twisted type $^2A$ with $G = G_\alpha$ in view of $\text{End}(\alpha)$-duality we obtain a similar property at least when $Z(G)$ is connected. Then there exist a function

$$\epsilon^G : \text{Irr}(G^F) \rightarrow \{-1, 1\}$$

such that for any diagonal Levi subgroup $L$ and any $(\lambda, \chi) \in \text{Irr}(L^F) \times \text{Irr}(G^F)$ one has (see [20] 15.4 and reference)

$$\epsilon^G(\lambda)\epsilon^L(\chi)(\text{R}_{L}^\ell \lambda, \chi)_{G^F} \geq 0$$
Furthermore if \(\chi(1) = \chi'(1)\), then \(\epsilon^G(\chi) = \epsilon^G(\chi')\). In non twisted type we define \(\epsilon\) as a constant application with value 1. We may conjecture that (2.2.1.3) holds for \(G = G_a\) as well when \(Z(G)\) is not connected. Indeed under a conjecture relying Gelfand-Graev characters an Lusztig induction, one may extend Jordan decomposition for any series and (J3) [3], [12]. In that case, as we verify it in our Appendix, section 5.4, G.d-HC theory holds in \(\mathcal{E}(C_{G^*}(s)^F, 1)\) and transfers to \(\mathcal{E}(G^F, s)\).

If Proposition 2.1.7 applies then \(\cup_{(L, \lambda)} b_{G^F}(L, \lambda)\) is known to be a partition, hence

2.2.2. Proposition. Assume 2.1.2 on \((G, F, \ell, d)\) and \(s\) semi-simple in \((G^*)^F\). Generalized d-Harish-Chandra’s theory holds in \(\mathcal{E}(G^F, s)\), if and only if, for any \(d\)-cuspidal datum \((L, \lambda)\) in series \(s\) in \((G, F)\)

\[
\mathcal{E}(G^F, (L, \lambda)) = \text{Irr}(b_{G^F}(L, \lambda)) \cap \mathcal{E}(G^F, s).
\]

From the important paper of M. Broué, G. Malle J. Michel [10] it is known that Generalized d-Harish-Chandra theory holds in \(\mathcal{E}(G^F, 1)\). The fundamental theorem 3.2 in [10] gives precise results on degrees, value of scalar products, as \(a\) in (2.2.1.1), and relation with Lusztig’s map. We have retain the partition in “Generalized Harish-Chandra series” \(\mathcal{E}(G^F, (L, \lambda))\) and formula (2.2.1.1), a consequence of transitivity theorem [10] 3.11.

When \(Z(G)\) is connected and computation between Lusztig induction and Jordan decomposition (J3) in 1.3.3 is true, as it is the case in classical types (see [27], [24] Appendix), and if Proposition 2.1.4 applies, via \(\Psi_{G, s}\), G.d-HC holds in \(\mathcal{E}(G^F, s)\) because it holds in \(\mathcal{E}(C_{G^*}(s)^F, 1)\).

Let us consider no exceptional types. The proof of G.d-HC in unipotent series for exceptional types uses two properties:

(i) Mackey decomposition formula for restriction of Lusztig induction [20] 11.13;
(ii) orthogonal projection on the space of uniform functions (1.3.1.3).

From now on we enforce assumption 2.1.2 by Mackey decomposition formula. Mackey formula is proved for classical types and any \(q\) and for exceptional types if \(q > 2\) [4]. It gives us the norm of \(R^F_{L \leq P} \lambda\) for any \(d\)-cuspidal unipotent datum \((L, \lambda)\) in \((G, F)\), independantly of \(P\). To prove G.d-HC in \(\mathcal{E}(G^F, s)\) we need Mackey formula for and inside \(d\)-split Levi subgroups of \(G\) and of \(C_{G^*}(s)\).

2.2.3. Assumption. Assume 2.1.2 on \((G, F, \ell, d)\) and that Mackey decomposition formula for Lusztig induction holds inside closed \(F\)-stable subgroups of \(G\) or \(G^*\).

2.2.4. Proposition. Assume 2.2.3 on \((G, F, \ell, d)\) and \(Z(G)\) is connected. Let \(s\) be semi-simple in \((G^*)^F\). Then Generalized d-Harish-Chandra theory, as defined in 2.2.1, holds in \(\mathcal{E}(G^F, s)\).

Furthermore if \((L, \lambda)\) is a \(d\)-cuspidal datum in series \(s\) in \((G, F)\) associated to a \(d\)-cuspidal unipotent datum \((L^*_\alpha, \alpha)\) in \((C_{G^*}(s), F)\) by Proposition 2.1.4, there exist a one-to-one map

\[
\tilde{\Psi}_{G, s}(L, \lambda): \mathcal{E}(C_{G^*}(s)^F, (L^*_\alpha, \alpha)) \to \mathcal{E}(G^F, (L, \lambda))
\]

such that

\[
(2.2.4.1) \quad (\tilde{\Psi}_{G, s}(L, \lambda)(\beta))(1) = \frac{|G^F|}{|C_{G^*}(s)^F|} \beta(1)
\]

and

\[
(2.2.4.2) \quad R^F_L \lambda = \sum_{\beta \in \mathcal{E}(C_{G^*}(s)^F, (L^*_\alpha, \alpha))} (\beta, R^G_{L^*_\alpha}(\alpha)C_{G^*}(s)^F, \tilde{\Psi}_{G, s}(L, \lambda)(\beta))
\]
The proof of Proposition 2.2.4 is given in 2.2.8.

Clearly a candidate for \( \tilde{\Psi}_{G,s}(L, \lambda) \) is the restriction of \( \Psi_{G,s} \) to \( \mathcal{E}(C_{G^*}(s)^F, (L^*, \alpha)) \), but there is a doubt on the definition of \( \Psi_{G,s} \) in some exceptional cases, and we show only that

\[
\pi^G_{\text{un}} \circ \tilde{\Psi}_{G,s}(L, \lambda) = \pi^G_{\text{un}} \circ \Psi_{G,s}(L, \lambda)
\]

and that imply (2.2.4.1) by (vi) in Proposition 1.3.2.

To transfer \( G.d \)-HC from unipotent series to any Lusztig series, we have to use some classical properties on "relative Weyl groups" as the following

2.2.5. **W-argument.** Let \( L \) be a connected reductive \( F \)-stable subgroup of \( G \), and \( H \) a subgroup of \( G^F \) such that \( L^F \subseteq H \subseteq N_G(L) \). Let \( T \) be a maximally split torus of \( L \). Put \( W(HL, T) = N_{HL}(T)/T, T/T \), a subgroup of \( W(G, T) \).

There is a split short exact sequence

\[
1 \rightarrow W(L, T)^F \rightarrow W(HL, T)^F \rightarrow H/L^F \rightarrow 1
\]

**Proof.** By \( L^F \)-conjugacy of maximally split tori in \( L \) we have \( (HL)^F = H = N_{HL}(T)^F.L^F \) hence by an isomorphism theorem \( H/L^F \cong N_{HL}(T)^F/N_L(T)^F \). As \( T \) is connected one has \( W(HL, T)^F \cong N_{HL}(T)^F/T^F \) and \( W(L, T)^F \cong N_L(T)^F/T^F \), thus our claim. The extension is split : the stabilizer in \( H \) of an \( F \)-stable couple (torus \( \subseteq \) Borel) in \( L \) is a complement of \( N_L(T)^F \) in \( N_{HL}(T)^F \).

Note that Frattini’s argument on maximally split tori inside \( F \)-stable Borel subgroups may be used in type \(^2\mathbb{A}\) for diagonal tori and Borel subgroups.

Anti-isomorphisms between \( W(G, T) \) and \( W(G^*, T^*) \) for groups in duality extends easily to some relative Weyl groups.

As a first example assume that \( L \) is an \( F \)-stable Levi subgroup of \( G \), with \( T \subseteq L \subseteq G, F \) in duality with \( (T^* \subseteq L^* \subseteq G^*, F) \).

Then \( N_G(L)/L \cong N_{W(G,T)}(W(L,T))/W(L,T) = W_G(L) \cong W_G^*(L^*) \), \( W_G(L)^F \cong W_{G^*}(L^*)^F \) by 2.2.5. If \( s \in (T^*)^F \), put \( \theta = \Psi_{T,s}(1_{T^*}) \) and let \( N_{G^*}(L, \mathcal{E}(L^F, s)) \) be the stabilizer of \( \mathcal{E}(L^F, s) \) in \( N_G(L)^F \). Then, as \( (T, \theta) \) is defined by the series \( s \) modulo \( L^F \)-conjugacy, \( N_{G^*}(L, \mathcal{E}(L^F, s)) = N_{G^*}(L, T, \theta)L^F \), so that \( N_{G^*}(L, \mathcal{E}(L^F, s))/L^F \) is isomorphic to \( N_{G^*}(T, \theta)/N_{L^F}(T, \theta) \). One have an isomorphism

\[
(2.2.5.1) \quad N_{G^*}(L, \mathcal{E}(L^F, s))/L^F \cong (N_{G^*}(L^*) \cap C_{G^*}(s))/C_{L^F}(s)
\]

We need a restriction of that isomorphism in case of cuspidal data. To simplify notations, as \( T \) and \( T^* \) are fixed we write \( W(G) \) instead of \( W(G, T) \) and so on. Given \( s \in T^* \subseteq G^* \) and anti-isomorphism between \( W(G, T) \) and \( W(G^*, T^*) \) as above, we abbreviate the image of \( W(C_{G^*}(s), T^*) \) and of \( N_{G^*}(s)(T^*)/T^* \) in \( W_G^*(s) \) and \( W_G(s) \) respectively, two subgroups of \( W(G, T) \), so that the last isomorphism becomes

\[
(2.2.5.2) \quad N_{G^*}(L, \mathcal{E}(L^F, s))/L^F \cong N_{W_G(s)}(W_L(s))^F/W_L(s)^F
\]

With these notations we have

2.2.6. **Proposition.** Let \( (L, \lambda) \) be a \( d \)-cuspidal datum in series \( s \) of an \( F \)-stable Levi subgroup \( M \) of \( G \). Assuming Weyl groups are defined over dual \( F \)-stable torii \( T, T^* \) with \( T^* \) maximally split in \( C_{L^*}^\circ(s) \), one has an isomorphism

\[
[N_G(M) \cap N_{G^*}(L, \mathcal{E}(L^F, s))]/L^F \cong [N_{W(G)}(W(M)) \cap N_{W_G(s)^F}(W_L^*(s))]/W_L(s)^F.
\]
Proof. From the hypotheses we use the equalities \( L = M \cap C_G(Z(L)) \), \( L^* = M^* \cap C_G^-((Z^*))(ϕ_d) \) and \( Z^*(L^*)_{ϕ_d} = Z^*(C_L^*(s))_{ϕ_d} \) (Proposition 1.4.2). Therefore \( N_G(M)^F \cap N_G(L) = N_G(M)^F \cap N_G(Z(L))_{ϕ_d} \), \( N_G^-((M^*)^F \cap N_G^-((Z^*)_{ϕ_d})) \). As \( s \in L^* \), \( N_G^-((L^*)^F \cap C_G^-((s)) = N_G^-((s))((Z^*)_{ϕ_d})^F \). Now \( Z^*(L^*)_{ϕ_d} = Z^*(C_L^*(s))_{ϕ_d} \) implies \( N_G^-((M^*) \cap N_G^-((L^*) \cap C_G^-((s))^F = N_G^-((M^*) \cap NC_G^-((s))((C_L^*(s))^F \). Thus \( N_G^-((M^*) \cap N_G^-((L^*, s)) = (L^*)^F \cap NC_G^-((s))((C_L^*(s))^F \).

Using 2.2.5, (2.2.5.1) and (2.2.5.2) that last equality gives our claim.

Thanks to commutation of \( π_un \) with Lusztig induction (1.3.1.4) \( π^G_un \circ R^G_{L\subseteq P} \) is known and independent of \( P \). In [10] p.55 we read — translation in \( G^F \) of a generic result and using our notations —

It will turn out that there exists an essentially unique element \( γ \in ZE(G^F, 1) \) with \( π^G_un(γ) = π^G_un(R^G_L α) \) of minimal norm, and that this norm coincides with the norm of \( R^G_L α \) calculated from the Mackey formula.

2.2.7. Lemma. Assume 2.1.2 on \((G, F, ι, d)\) and \( Z(G) \) is connected. Let \( s \) be a semi-simple \( ι \)-element of \( G^* \). Let \((L^*, \alpha)\) be a \( d \)-cuspidal unipotent datum in \((C_G^-, s)\) and let \((L, \lambda)\) be an associated \( d \)-cuspidal datum in series \( (s) \) in \((G, F)\), hence \( \lambda = Ψ_{L,s}(α) \) (see Proposition 2.1.4).

(a) The relation

\[
R^G_L(Ψ_{L,s}(α)) = Ψ_{G,s}(R^G_{L^*}(s) α)
\]

is true when one of the following conditions is satisfied:

(i) \( C_G^-((s) \) is a Levi subgroup of \( G^* \).

(ii) \( L^* \) has type \( A \).

(iii) Mackey decomposition formula holds between \( E \)-split Levi subgroups in \( G \) and in \( C_G^-(s) \) and \( R^G_{L^*}(s) α \) is the unique element \( ξ \) of \( ZE(C_G^-(s))^F, 1) \) of minimal square norm such that

\[
π^G_un(s) = R^G_{L^*}(s) (π^G_un(α))
\]

When (iii) is satisfied \( R^G_L(γ) \) is the unique element \( χ \) of \( ZE(G^F, s) \) of minimal square norm such that 

\[π^G_un(γ) = R^G_L(π^G_un(Σ)).\]

(b) If \( Ψ_{G,s}(L, λ) \) exists as in Proposition 2.2.4 for any \( (L, λ) \) defined from a proper \( d \)-cuspidal unipotent datum \((L^*, α)\) in \((C_G-, s)\), one has the partition

\[
E(G^F, s) = \cup_{(L, Σ)/G^F} E(G^F, (L, Σ))
\]

Proof. Recall that \( L \) is in duality with \( L^* := C_G^-((Z^*)_{ϕ_d}). \) In formula (2.2.7.1) the parabolic subgroup is omitted because the result of induction is independent of it.

(i) A special case of (i) is \( s = 1 \). Then \( Ψ_{G,1} \) is a bijection \( E((G^*)^F, 1) \rightarrow E(G^F, 1) \) which commute with Lusztig induction, see [10].

If \( s \) is central in \( G^* \), then, by Proposition 1.3.2, (ii), tensor product by \( Ψ_{G,s}(1) \) induces a bijection \( E(G^F, 1) \rightarrow E(G^F, s) \) hence \( Ψ_{L,s}(α) = Ψ_{L,s}(1) \oplus Ψ_{L,1}(α) \). (2.2.7.1) follows from the cases \( s = 1 \) and equalities \( R^G_L(Ψ_{L,s}(α)) = Ψ_{G,s}(1) \oplus R^G_L(Ψ_{L,1}(α)) \), \( R^G_L(Ψ_{L,1}(α)) = Ψ_{G,1}(R^G_{L^*}(α)) \).
Assume now that $C_{G^*}(s)$ is a Levi subgroup of $G^*$ and use (iii) in Proposition 1.3.2: let $G(s)$ be a Levi subgroup of $G$ in duality with $C_{G^*}(s)$ such that $L^*_s$, a Levi subgroup of $G^*$, is in duality with $L(s) := G(s) \cap L$. Then $\Psi_{L,s}(\alpha) = R_{L(s)}^L (\Psi_{L(s),s}(1_{L(s)} \otimes \Psi_{L(s),1}(\alpha)))$. We have

$$R_G^G(\Psi_{L,s}(\alpha)) = R_{L(s)}^G (R_{L(s)}^G(\Psi_{L(s),s}(\alpha))), \quad R_{L(s)}^G(\Psi_{L(s),s}(\alpha)) = \Psi_{G(s),s}(R_{L^*_s}^{C_{G^*}}(\alpha))$$

(the case $s$ central above). Furthermore $R_{G(s)}^G$ restricts to a bijection from $E(G(s)^F, s)$ to $E(G^F, s)$ that exchanges $\Psi_{G(s),s}$ and $\Psi_{G,s} : R_{G(s)}^G(\Psi_{G(s),s}(R_{L^*_s}^{C_{G^*}}(\alpha))) = \Psi_{G,s}(R_{L^*_s}^{C_{G^*}}(\alpha))$, (2.2.7.1) follows.

(ii) If $L^*_s$ has type $A$, any central function on $L^*_s$ is uniform, hence $\Psi_{L,s}(\alpha) \in \pi_{un}^N(\mathcal{E}(G^F, s))$ by (1.3.2.2), and $R_{L(s)}^G(\Psi_{L(s),s}(\alpha))$ is uniform by (1.3.1.4). Then (1.3.2.5) applies and gives (2.2.7.1).

(iii) The square norms of $R_{L^*_s}^{C_{G^*}}(\alpha)$ and $R_{L^*_s}^L \lambda$ are given by Mackey decomposition formula:

$$||R_{L^*_s}^L \lambda||^2 = \langle R_{L^*_s}^L \lambda, R_{L^*_s}^L \lambda \rangle_{G^F} = \langle \lambda, \pi_{un}^N(R_{L^*_s}^L \lambda) \rangle_{G^F} \ldots$$

But as $\lambda$ is $d$-cuspidal and $L$ is $d$-split the formula for $R_{L(s)}^G(R_{L^*_s}^L \lambda)$ reduces to $R_{L(s)}^G(R_{L^*_s}^L \lambda) = \sum_{g \in N_{G,s}(L^*_s)^\prime} g \lambda$, so that $||R_{L(s)}^G(R_{L^*_s}^L \lambda)||^2 = |N_{G,s}(L^*_s)^\prime / L^*_s|^2$.

Similarly, $||R_{L^*_s}^{C_{G^*}}(\alpha)||^2 = |N_{G,s}(L^*_s)^\prime / L^*_s|^2$.

By Proposition 2.2.6 the quotients $N_{G^F}(L, \mathcal{E}(L^*_s, s)) / L^F$ and $N_{C_{G^*}(s)}(L^*_s)^F / L^*_s$ are isomorphic. By assertion (iv) of Proposition 1.3.2, $N_{C_{G^*}(s)}(L^*_s)^F / L^*_s$ is isomorphic to $N_{G}(L^*_s)^F / L^*_s$, so that

$$||R_{L^*_s}^G(\lambda)||^2 = ||R_{L^*_s}^{C_{G^*}}(\alpha)||^2.$$

By (1.3.2.5) and (1.3.1.4) $\pi_{un}^N(R_{L^*_s}^G(\lambda) = \Psi_{G,s}(\pi_{un}^N(R_{L^*_s}^{C_{G^*}}(\alpha))$.

More generally ”$\Psi$ commute with $\pi_{un}$” by formulas (1.3.2.2). Thus there exists a unique $\chi \in \mathcal{E}(G^F, s)$ such that $\pi_{un}^N(\chi) = \Psi_{G,s}(\pi_{un}^N(R_{L^*_s}^{C_{G^*}}(\alpha)))$ and $||\chi||^2 = ||R_{L^*_s}^{C_{G^*}}(\alpha)||^2$ and it is $\chi = \Psi_{G,s}(R_{L^*_s}^{C_{G^*}}(\alpha))$. We have $\chi = R_{L^*_s}^G(\lambda)$.

(b) Assume the existence of $\Psi_{G,s}(L, \lambda)$ for all proper unipotent $d$-cuspidal data $(L^*_s, \alpha)$ in $(C_{G^*}(s), F)$. Summing on the set of $C_{G^*}(s)^F$-conjugacy classes on such data denote $N := |\cup \mathcal{E}(C_{G^*}(s)^F, (L^*_s, \alpha))|$. Thanks to Propositions 2.1.7, 2.2.2 and hypothesis one has

$$(2.2.7.3) \quad N = \sum |\mathcal{E}(C_{G^*}(s)^F, (L^*_s, \alpha))| = \sum |\mathcal{E}(G^F, (L, \Psi_{L,s}(\alpha)))| \leq \sum |\text{Irr}(b_{G^F}(L, \Psi_{L,s}(\alpha))) \cap \mathcal{E}(G^F, s)|$$

where $L$ corresponds to $L^*_s$ as in Proposition 2.1.4 so that $L^*_s = C_{L,s}(s)$ and $(L, \Psi_{L,s}(\alpha))$ runs in the set of $G^F$-conjugacy classes of $d$-cuspidal data in series $(s)$ such that $C_{G^*}(s) \neq L^*_s$.

Let $N_0$ be the number of $d$-cuspidal elements of $\mathcal{E}(C_{G^*}(s)^F, 1)$. By G. d-HC in $\mathcal{E}(C_{G^*}(s)^F, 1)$, $N_0 + N$ is $|\mathcal{E}(C_{G^*}(s)^F, 1)|$. By the existence of $\Psi_{G,s}, |\mathcal{E}(G^F, s)| = |\mathcal{E}(C_{G^*}(s)^F, 1)|$ and $N_0$ is greater that the number of elements of the complement of $\cup_{L \neq L_0} \text{Irr}(b_{G^F}(L, \lambda)) \cap \mathcal{E}(G^F, s) \in \mathcal{E}(G^F, s)$ where $L_0$ is a Levi subgroup of $G$ in duality with $C_{G^*}(Z^K(C_{G^*}(s)))$. There are $N_0$ non $G^F$-conjugate $d$-cuspidal data in series $(s)$ of the form $(L_0, \Psi_{L_0,s}(\alpha))$ in $G^F$, in different $G^F$-conjugacy classes (see Proposition 2.2.6) and $L^*_s = C_{G^*}(s)$ implies $||R_{L_0}^L(\Psi_{L_0,s}(\alpha))||^2 = 1$, so that $\text{Irr}(b_{G^F}(L_0, \Psi_{L_0,s}(\alpha))) \cap \mathcal{E}(G^F, s) = \{R_{L_0}^L(\Psi_{L_0,s}(\alpha))\}$. In (2.2.7.3) above we have $N = \sum |\text{Irr}(b_{G^F}(L, \Psi_{L,s}(\alpha))) \cap \mathcal{E}(G^F, s)|$, hence the partition of $\mathcal{E}(G^F, s)$ by blocks is the partition by $d$-series $\mathcal{E}(G^F, (L, \Psi_{L,s}(\alpha)))$, that is (2.2.7.2).

2.2.8 Proof of Proposition 2.2.4.
Note that, once (2.2.7.2) is proved for all $G$ under assumption 2.2.3, then, for any $\chi \in E(G^F,(L,\lambda))$, (2.2.1.2), equivalently (2.2.4.2), is true, with $a = \langle \chi,R_L^g \lambda \rangle_{GF}$. Indeed let $\xi$ be some irreducible component of $R_L^g \chi$. By (2.2.7.2) applied to $(L,F)$, $\xi$ belongs to $E(L^F,(M,\mu))$ for some $d$-cuspidal datum $(M,\mu)$ in $(L,\lambda)$. Then $(M,\mu)$ is a $d$-cuspidal datum in $(G,F)$ and, by Propositions 2.1.7 and 2.1.6, we have
\[
\xi \in \text{Irr}(b_{L,F}(M,\mu)), \quad R_L^g (b_{L,F}(M,\mu)) = b_{GF}(M,\mu), \quad \chi \in \text{Irr}(b_{GF}(M,\mu)) \cap \text{Irr}(b_{GF}(L,\lambda))
\]
hence $(L,\lambda)$ and $(M,\mu)$ are $G^F$-conjugate.

Using Lemma 2.2.7 (b), we have to prove the existence of $\Psi_{G,s}(L,\lambda)$ for any proper unipotent $d$-cuspidal data $(L_s^\alpha,\alpha)$ in $C_{G^s}(s,F)$.

If $G$ is a central product over a torus $G_1, G_2$ of $F$-stable reductive subgroups with connected centers, $G^F = G_1^F . G_2^F$, $s$ has image $(s_1, s_2)$ in $G_1^* \times G_2^*$, the space $K\mathcal{E}(C_{G^s}(s,F),1)$ is an orthogonal product of $K\mathcal{E}(C_{G_1^s}(s_1,F),1)$ and $K\mathcal{E}(C_{G_2^s}(s_2,F),1)$. $\Psi_{G,s}$ is defined by $\Psi_{G_1,s_1} \times \Psi_{G_2,s_2}$. That’s why we may assume $(G,F)$ irreducible.

When (2.2.7.1) holds the restriction of $\Psi_{G,s}$ to $E(C_{G^s}(s,F),(L_s^\alpha,\alpha))$ as $\Psi_{G,s}(L,\lambda)$ is a good choice. In classical types (2.2.7.2) is a consequence of $G-d$-HC in $E(C_{G^s}(s,F),1)$, Proposition 2.1.10 and commutation between Lusztig induction and Jordan decomposition.

We have to consider groups $(G,F)$ of exceptional types when eventually Lemma 2.2.7 (a) don’t apply for some $d$-cuspidal datum.

Unfortunately there are unipotent $d$-cuspidal data which don’t satisfy any of the assumptions (i), (ii), (iii) in Lemma 2.2.7 on $(L_s^\alpha,\alpha)$. It happens when two $d$-cuspidal irreducible characters have equal projections on the space of uniform functions. The equality $\pi^\alpha_{un}(\alpha) = \pi^\alpha_{un}(\alpha')$ for distinct elements of $E(L^F,1)$ occurs only for algebraically conjugate representations in exceptional types.

We recall in Table 1 all these cases as described in [26] Chapter 4 and Appendix.

| Type | $\alpha$’s | $d$ | central defect |
|------|-------------|-----|---------------|
| $G_2$ | $\alpha(1) = \frac{1}{2} q \phi_1(q)^2 \phi_2(q)^2$ | 1, 2 | $\ell \geq 5$ |
| $F_4$ | $F_4[\theta_j], \ j = 1, 2$ | 1, 2, 4, 8 | $\ell \geq 3$ |
| | $F_4[\pm i]$ | 1, 2, 3, 6 | $\ell \geq 5$ |
| $E_6$ | $E_6[\theta_1], \ j = 1, 2$ | 1, 2, 4, 5, 8 | $\ell \geq 5$ |
| | $2E_6[\theta_1], \ j = 1, 2$ | 1, 2, 4, 8, 10 | $\ell \geq 5$ |
| $E_7$ | $E_7[\theta_1, \epsilon], \ j = 1, 2$ | 4, 5, 7, 8, 10, 14 | $\ell \geq 5$ |
| | $E_7[\theta_1, \epsilon'], \ j = 1, 2$ | 4, 5, 7, 8, 10, 14, 15, 20, 30 | $\ell \geq 5$ |
| | $E_7[\theta_1, \epsilon''], \ j = 1, 2$ | 4, 5, 7, 8, 10, 14, 15, 20, 30 | $\ell \geq 5$ |
| | $E_7[\theta_1, \epsilon'''], \ j = 1, 2$ | 2, 4, 5, 7, 8, 10, 14, 15, 18, 20, 24 | $\ell \geq 5$ |
| | $E_7[\theta_1, \epsilon''''], \ j = 1, 2$ | 2, 4, 5, 7, 8, 10, 14, 18, 20, 30 | $\ell \geq 5$ |
| | $E_8[\lambda_1], \ j = 1, 2, 3, 4$ | 1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 18, 24 | $\ell = 3, \ell \geq 7$ |
| | $E_8[-\theta_1], \ j = 1, 2$ | 1, 4, 5, 7, 8, 9, 10, 12, 14, 15, 20 | $\ell \geq 5$ |
| | $E_8[\theta_1], \ j = 1, 2$ | 1, 4, 5, 7, 8, 9, 10, 14, 20, 24, 30 | $\ell \geq 5$ |
| | $E_8[\pm i]$ | 1, 2, 3, 5, 6, 7, 9, 10, 14, 15, 18, 30 | $\ell \geq 3$ |

We have used notations of [26] to design elements of $E(L^F,1)$ in the second column. In the third one are the $d \in \mathbb{N}$ such that $\alpha$ is $d$-cuspidal and $\phi_d$ divides the polynomial order $P_{L,F}$ of $(L,F)$ (see 1.1.5; note
that if $\phi_d$ does not divides $P_{L,F}$ and $L$ is a $d$-split Levi subgroup of $G$, then $L = G$ and any $\chi \in \text{Irr}(G^F)$ is $d$-cuspidal). In the last column are the odd prime numbers $\ell$ such that $\alpha$ has central $\ell$-defect group, assuming that $d$ is the order of $q \mod \ell$, so that $d$ divides $(\ell - 1)$.

Inspection of the matrices of decomposition in [26] 4.14—4.16 shows

(2.2.8.2) Assume $G$ simple of any type. If $\alpha, \beta \in \mathcal{E}(G^F,1)$ and $\pi^G_{\text{un}}(\alpha), \pi^G_{\text{un}}(\beta)$ are proportional, then $\alpha = \beta$ short of the cases $\pi^G_{\text{un}}(\alpha) = \pi^G_{\text{un}}(\beta)$ listed in Table 1.

Assume that $C_{G^*}(s)$ is contained in a proper $d$-split Levi subgroup $M^*$ of $G^*$, with dual $M$ in $G$. By (iii) in Proposition 1.3.2, $R^G_L$ induces a bijection $\mathcal{E}(M^F,s) \rightarrow \mathcal{E}(G^F,s)$. $M^F$-conjugacy classes of $d$-cuspidal data in series $(s)$ and $G^F$-conjugacy classes of $d$-cuspidal data in series $(s)$ are in natural bijection. If our claim on $G.d$-HC holds in $M$, it holds in $G$. If (2.2.7.1) holds in $M$, it holds in $G$.

So we assume now that $s$ is isolated in $(G^*)^F$, that is $C_{G^*}(s)$ is not contained in any proper Levi subgroup of $G^*$. Assume further the existence in $(C_{G^*}(s), F)$ of a $d$-cuspidal unipotent datum $(L^*_s, \alpha)$ where $L^*_s$ is not of type $A$, so that (a) in Lemma 2.2.7 don’t apply. There are few cases to consider, we list in Table 2 in ascending rank, and use parametrizations of $\mathcal{E}(L^*_sF,1)$ given in [26], short of $\mathcal{E}(3D_4(q),1)$, notations of [10] Table 1.

| Case | Type of $G$ | Type of $C_{G^*}(s)$ | $d$ | $(L^*_s, \alpha)$ |
|------|-------------|----------------------|----|------------------|
| 1    | $F_4$       | $B_4$                | 2  | $(B_2, (0,2))$   |
| 2    |             |                      | 4  | $(B_2, (0,1))$   |
| 3    | $E_7$       | $A_1 \times C_3$    | 2  | $(C_2, (0,2))$   |
| 4    | $E_7$       | $D_6 \times A_1$    | 2  | $(D_4, (1,2))$   |
| 5    | $E_8$       | $D_8$                | 2  | $(D_4, (1,3))$   |
| 6    |             |                      | 3  | $2D_5$           |
| 7    |             |                      | 4  | $D_4$            |
| 8    |             |                      | 6  | $2D_5$           |
| 9    |             |                      | 8  | $2D_4$           |
| 10   | $E_7 \times A_1$ | 2  | $(E_7, [512a], [512''])$ |       |
| 11   |             |                      | 2  | $(2E_6, 2E_6[\theta^j])$, $j = 1, 2$ |       |
| 12   |             |                      | 2  | $(D_4, (1,2))$   |       |
| 13   |             |                      | 3  | $(3D_4 \times A_1, 3D_4[-1] \otimes \alpha')$ |       |
| 14   |             |                      | 6  | $(3D_4 \times A_1, \phi_{2,1} \otimes \alpha')$ |       |
| 15   | $E_6 \times A_2$ | 3  | $(3D_4, 3D_4[-1])$ |       |
| 16   | $2E_6 \times 2A_2$ | 2  | $(D_4, (1,3))$   |       |
| 17   |             |                      | 6  | $(3D_4, \phi_{2,1})$ |       |

(a) When $C_{G^*}(s)$ has a type $A$ component, the space $\pi^C_{\text{un}}(s)( K \mathcal{E}(C_{G^*}(s)^F,1))$ of uniform unipotent functions on $C_{G^*}(s)^F$ is an orthogonal product of the corresponding spaces for the two components and differ from all space of unipotent functions only on the other side, we have only to prove our claim on non type $A$ part. Thus the case 10 is clear.

(b) It happens, in cases 2, 3, 6, 8, 9, that $L^*_s$ is a maximal proper $d$-split Levi subgroup of $C_{G^*}(s)$ and $C_{G^*}(s)$ have a classical type. Then $R^C_{L^*_s}(s) \alpha$ is given by Asai’s $d$-hook formula ($d$ odd) [24], [10] (3.5) or a $d/2$-cohook formula [10] (3.9) in "one step". The formulas show that $R^C_{L^*_s}(s) \alpha$ is an algebraic sum of
irreducible \( \chi_j \) all in distinct families, families defined from \( W(C_G, (s), F) \). In fact (iii) of (a) in Lemma 2.2.7 is satisfied, hence the commutation formula (2.2.7.1).

(c) In some cases there exists between \( L^* \) and \( C_G^* (s) \) a maximal proper \( d \)-split Levi subgroup \( L^*_1 \) of \( C_G^* (s) \) and \( L^*_1 \) is also a \( d \)-split Levi subgroup of \( G^* \). Let \( L_1 \) be a \( (d \)-split) Levi subgroup of \( G \) in the dual class such that \( L \subseteq L_1 \). As \( s \) is central in \( L^*_1 \) (2.2.7.1) applies between \( L \) and \( L_1 : R^L_{L_1} (\Psi_{L,s}(\alpha)) = \Psi_{L_1,s}(R^L_{L_1} (\alpha)) \).

Assume first that \( C_G^* (s) \) has classical type. For any \( \beta \in E((L^*_1)^F), (L^*_1, \alpha) \) the hook or cokchoo formula gives, as in (a), \( \Psi_{G,s}(R^G_{L,s}(\beta)) = R^G_{L,s}(\Psi_{L,s}(\beta)) \). Finally we have (2.6.7.1) again. This applies to cases 1, 4, 5, 7 where \( L^*_1 = B_3(q),(q+1), D_5(q),(q+1)^2, 2D_7(q),[q+1], 3D_6(q),(q^2+1) \) respectively.

(d) In cases 12 to 17, \( L^*_1 \) is itself a Levi subgroup of \( G^* \), \( L^*_1 = D_4(q)[q+1]^4, [q^3-1], A_1(q), 3D_4(q),(q^2+1)^2, 4D_4(q),(q+1)^4, 3D_4(q),(q^2-q+1)^2 \), and \( R^C_{G^*}(s) \) may differ from the restriction of \( \Psi_{G,s} \).

In double case 11, with \( L^*_1 = L^*_1 \), \( L^*_1 \) is \( E_6(q),(q+1)^2 \). Instead of Asai’s formula we need the series \( 2E_6(q)[q+1] \) for different families, families defined from \( (\theta, s) \) in [15] Table 2.

2.3. On irreducible characters in a block of \( G^F, Z(G) \) connected

We want to describe \( \mathrm{Irr}(b_{GF}(L, \lambda)) \) for any \( d \)-cuspidal datum \( (L, \lambda) \) in an \( \ell' \)-series in \( (G, F) \). In [15] Theorem 2.8 it is shown, using induction on blocks (Proposition 2.1.6), \( Z(G) \) connected or not, how to recover \( \mathrm{Irr}(b) \cap E(G^F, s) \) knowing \( \mathrm{Irr}(b) \cap E(G^F, s_{\ell'}) \) for any bloc \( b \) of \( G^F \). To obtain our main theorem we refer to the case of connected center and rely the above connection to relation between unipotent \( d \)-cuspidal data in \( (C^*_{G^*}(s), F) \) and unipotent \( d \)-cuspidal data in \( (C^*_{G^*}(s_{\ell'}), F) \), in Proposition 2.3.5. A similar way is to refer to \( d \)-cuspidal data in any series, as in Proposition 2.3.6.

2.3.1. Notation. Let \( L \) and \( M \) be two \( F \)-stable connected reductive subgroups of \( (G, F) \), \( \lambda \in E(L^F, 1), \mu \in E(M^F, 1) \). Denote

\( (L, \lambda) \sim_{GF} (M, \mu) \)

the following equivalence relation

there exists \( g \in G^F \) such that \( [L, L] = [M, M]^g \) and \( \mathrm{Res}^{L^F}_{[L, L]^g} \lambda = (\mathrm{Res}^{M^F}_{[M, M]^g} \mu)^g \).

The relation \( \sim_{GF} \) is the extension by \( G^F \)-conjugacy of the relation \( \sim \) introduced in [15] (see Definition 23.1 and Proposition 23.2 in [16]).

Assertion (c) of Proposition 2.1.5 suggests that there are few \( d \)-cuspidal irreducible characters and few \( d \)-cuspidal data. The following Proposition is a consequence of that fact, and will be used to simplify the description of \( \mathrm{Irr}(b_{GF}(L, \lambda)) \cap E(G^F, st) \) (see [16] Theorem 23.2 in case \( s = 1 \)).
2.3.2. Proposition. Let $H$ be an $E$-split Levi subgroup of $G$.

(a) Let $(L_H, \lambda_H)$ be a unipotent $d$-cuspidal datum in $(H, F)$. There exists a unipotent $d$-cuspidal datum $(L, \lambda)$ in $(G, F)$ such that $[L, L] = [L_H, L_H], L_H = L \cap H$, and $\text{Res}^{L}_{[L,L]} F \lambda = \text{Res}^{L_H}_{[L_H,L_H]} F \lambda_H$.

(b) Let $(L, \lambda)$ and $(M, \mu)$ be two unipotent $d$-cuspidal data in $(G, F)$, and $(L_H, \lambda_H), (M_H, \mu_H)$ be two unipotent $d$-cuspidal data in $(H, F)$ such that $[L, L] = [L_H, L_H]$ and $[M, M] = [M_H, M_H]$. Then $L$ and $M$ are $G^F$-conjugate if and only if $L_H$ and $M_H$ are $H^F$-conjugate.

(c) The relation $(L, \lambda) \sim_{G^F} (L_H, \lambda_H)$ define a one-to-one map between the set of $G^F$-conjugacy classes of $d$-cuspidal unipotent data $(L, \lambda)$ in $(G, F)$ such that $[L, L] \subseteq g H g^{-1}$ for some $g \in G^F$ and the set of $H^F$-conjugacy classes of $d$-cuspidal unipotent data $(L_H, \lambda_H)$ in $(H, F)$.

Proof. Assertion (a) follows from (i) in [16] Proposition 23.3 where the relation

$$[L, L] = [L_H, L_H] \quad \text{and} \quad \text{Res}^{L}_{[L,L]} F \lambda = \text{Res}^{L_H}_{[L_H,L_H]} F \lambda_H$$

is denoted $(L, \lambda) \sim (L_H, \lambda_H)$. We have added the equality $L_H = H \cap L$.

Assume $[L, L] = [L_H, L_H]$. By [13] Proposition 1.7, $Z^o(L)$ is a maximal torus in $C^o_G([L, L])$ and $Z^o(L_H)$ is a maximal torus in $C^o_H([L_H, L_H]) = H \cap C^o_G([L, L])$. Then exist some $z \in C^o_G([L, L])$ such that $Z^o(L_H) \subseteq Z^o(L)^g$, hence $Z^o(L_H) = H \cap Z^o(L)^g$. Thus $L_H = Z^o(L_H), [L_H, L_H] = (H \cap Z^o(L)^g), [L, L] = H \cap L^g$. We may replace $(L, \lambda)$ by $(L^g, \lambda^g)$, preserving the above two equalities.

Clearly (b) implies (c).

Proof of (b):

If $G = G_a$, then $H = H_a$ and there is only one conjugacy class of $d$-cuspidal data in $G$ and in $H$. So we assume $G_b \neq \{1\}$ and $H \neq G$ and prove the Proposition inductively on the semi-simple rank of $G$.

The restriction along an isotypic morphism is a bijection on unipotent series and Proposition 2.1.5 allows us to assume $(G, F)$ rationally irreducible. By properties of scalar descent we assume that $G$ is irreducible.

(b.i) Assume that $H$ is $d$-split in $G$, so are $L_H$ and $M_H$.

Assume first that $L_H = (M_H)^h$ for some $h \in H^F$. Then $[L, L] = [M, M]^h$ hence, by [16] Proposition 22.8, $L$ and $M$ are $C^o_G([L, L])^F$-conjugate, so that $L$ and $M$ are $G^F$-conjugate.

Assume now that $L$ and $M$ are $G^F$-conjugate.

In classical types A, B, C, D with $d \geq 1$ [10, § 3], and in exceptional types with $d = 1$ [26] Appendix one sees on tables that

(i.a) $H = H_a$ is a torus or $H_b$ is rationally irreducible.

(i.b) If $L$ is not a torus, then $L_b$ is rationally irreducible.

(i.c) If $L_b$ and $M_b$ have same type, then $L$ and $M$ are $G^F$-conjugate.

From these facts $L_b = [L, L] = [L_H, L_H] = (L_H)_b$ (eventually $= 1$). In $(G, F)$ (resp. in $(H, F)$) the conjugacy class of $L$ (resp. $M$) is defined by $L_b$ (resp. $(L_H)_b$, hence our claim.

In exceptional type with $d > 1$ Table 1 in [10] show that (i.a) and (i.b) are no more true, but (i.c) is true for each $d$. One may verify, using properties (i.a), (i.b), (i.c) in classical irreducible type and (i.c) in exceptional irreducible type, that a proper maximal $d$-split Levi subgroup of $G$ cannot contain two non conjugate unipotent $d$-cuspidal data as $(L_H, \lambda_H)$ with $(L_H)_b$ of same type as $L_b$ and one conclude by induction.

(b.ii) In the general case with $G = G_b$ and $H$ a proper $E$-split Levi subgroup of $G$, let $K$ be a proper $d$-split Levi subgroup of $G$ that contains $H$ (1.1.5.3). By (a) there exist unipotent $d$-cuspidal data $(L_K, \lambda_K)$ and $(M_K, \mu_K)$ in $(M, F)$ such that $(L_H, \lambda_H) \sim_{K^F} (L_K, \lambda_K)$ and $(M_H, \mu_H) \sim_{K^F} (M_K, \mu_K)$. Induction applies in $K : L_K$ and $M_K$ are $K^F$-conjugate. By part (b.i) of our proof $L$ and $M$ are $G^F$-conjugate. 

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In the following proposition we make a connection between induction of blocks (Proposition 2.1.7) and cuspidal data, using [15] and [16].

2.3.3. Proposition. Assume 2.1.2 on \((G, F, \ell, d)\) and \(Z(G)\) is connected. Let \(H\) be an \(E\)-split Levi subgroup of \(G\) with a dual \(H^*\) in \(G^*\), such that \(s \in (H^*)^F_\ell\). Let \((L_H, \lambda_H)\) be a \(d\)-cuspidal datum in \((H, F)\) in series \((s)\), let \((L_{\ast,H}, \alpha_H)\) be a \(d\)-cuspidal unipotent datum in \((C_H^\ast, (s), F)\) such that \(L^\ast = C_H^\ast (Z^\ast (L_{\ast,H}^\ast, \alpha_H))\) belongs to the dual \((H^*)^F\)-conjugacy class of the \(H^F\)-conjugacy class of \(L_H, \lambda_H = \Psi_{L_H, s}(\alpha_H)\) (see Proposition 2.1.4).

Let \((L_{\ast,H}^\ast, \lambda)\) be a \(d\)-cuspidal unipotent datum in \((C_{G^*}^\ast, (s), F)\) such that \((L_{\ast,H}^\ast, \alpha_H) \sim_{C_{G^*}^\ast, (s), F} (L_{\ast,H}^\ast, \alpha)\). Then \((L_{\ast,H}^\ast, \lambda)\) defines a \(d\)-cuspidal datum \((L, \lambda) = (L, \Psi_{L_H, s}(\alpha_H))\) in series \((s)\) in \((G, F)\), (Proposition 2.1.4).

We have

\[ R^G_H (b_{HF} (L_H, \lambda_H)) = b_{GF} (L, \lambda) \]

and, if \(\xi \in \text{Irr}(b_{HF} (L_H, \lambda_H))\) and \(s_0 \in Z(H^*\ell^F)\), then \(R^G_H (\Psi_{L_H, s_0} (1) \otimes \xi) \in Z\text{Irr}(b_{GF} (L, \lambda))\).

Proof. We know by [16] Proposition 22.8, (iii) that \(Z^\ast (L_{\ast}^\ast (s))_{\phi_d}\) (resp. \(Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d}\)) is a maximal \(\phi_d\)-subgroup in \(C_{C_{G^*}^\ast, (s)} (L_{\ast,H}^\ast, s))_{\phi_d}\) (resp. \(C_{C_{G^*}^\ast, (s)} (L_{\ast,H}^\ast, s))_{\phi_d}\)). By Sylow’s theorem on \(\phi_d\)-subgroups ([16] Theorem 13.18) there exist \(c \in C_{C_{G^*}^\ast, (s)} (L_{\ast,H}^\ast, s))_{\phi_d}\) such that \(Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d} \subseteq Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d}\). Up to conjugacy by \(c\) we may assume \(Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d} \subseteq Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d}\). By construction and up to \(H^F\)-conjugacy we may assume \(Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d} \subseteq Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d}\) (Proposition 2.1.4).

By Proposition 2.1.6, for any \(\xi \in \text{Irr}(b_{HF} (L_H, \lambda_H))\cap \mathcal{E}(H^F, s)\), any suitable parabolic \(P, R^G_H \subseteq P\), \(\xi\) belongs to \(Z\text{Irr}(B)\) where \(B = R^G_H (b_{HF} (L_H, \lambda_H))\). We prove by induction on the rank of \(G\) that \(B = b_{GF} (L, \lambda)\): our claim for \(s_0 = 1\).

Let be \(G = G_a, G_b\) the decomposition in central product we defined in 1.1.5.2 and use the standard dichotomy whereas \(G_b \subseteq H\) or not (recall functorial properties of \(d\)-cuspidal data, Proposition 2.1.5).

If \(G = G_a\) there is only one block \(B_{GF} (s)\) and one \(G^F\)-conjugacy class of cuspidal data in \((L, \lambda)\) series \((s)\) : \((L, \lambda)\) is such that \(L_{\ast,H}^\ast\) is a diagonal torus in \(C_{G^*} (s)\) and \(\alpha = 1\) (Proposition 2.1.5, (c)). In that case \(H = H_a\) and the equality \(R^G_H (B_{GF} (s)) = B_{HF} (s)\) is evident. Assume more generally that \(G_b \subseteq H\). Then \(H = (H \cap G_a)G_b\) and there is a natural bijection from the set of \(\ell\)-blocks of \(H^F\) in series \((s)\) to the set of \(\ell\)-blocks of \(G_b^F\) in some series \((s_b)\), (or from the set of \(H^F\)-conjugacy classes of \(d\)-cuspidal data \((L_H, \lambda_H)\) in series \((s)\) and the set of \(G_b^F\)-conjugacy classes of \(d\)-cuspidal data in series \((s_b)\)), and as well to the set of \(G^F\)-conjugacy class of \(G^F\) in series \((s)\) : \((L_H, \lambda_H) \to (L, \lambda)\), with \(L_H \cap G_b = L \cap G_b\) and \(\text{Res}_{L_{HF}}^{L_{HF}} (L_H \cap G_b, \lambda_H) = \text{Res}_{L_{GF}}^{L_{GF}} (L_H \cap G_b, \lambda)\). Now \(R^G_H\) reduces to \(R^G_{G_a} \cap G_b\), so that \(R^G_H\) commutes with that bijection.

If \(G_b\) is not contained in \(H\), let \(M\) be a proper \(d\)-split Levi subgroup of \(G\) that contains \(H (1.1.5.3)\). We have \(Z^\ast (M)_{\phi_d} \subseteq Z^\ast (H)_{\phi_d} \subseteq Z^\ast (L_H)_{\phi_d} \subseteq Z^\ast (L_{\ast,H}^\ast, s))_{\phi_d}\) hence \(L \subseteq M\). In the dual class there exist a \(d\)-split Levi subgroup \(M^*\) of \(G^*\) such that \(L^* \subseteq M^*\). Inductive hypotheses says that \(R^M_H \xi \in Z\text{Irr}(b_{HF} (M, \lambda))\) for any \(\xi \in \text{Irr}(b_{HF} (L_H, \lambda_H))\cap \mathcal{E}(H^F, s)\). By Proposition 2.1.7, (ii) \(R^M_H \xi = R^M_H (R^M_H \xi) \in Z\text{Irr}(b_{GF} (L, \lambda))\), that is \(R^M_H (b_{HF} (L_H, \lambda_H)) = b_{GF} (L, \lambda)\).

Let \(s_0 \in Z(H^*\ell^F)\), \(C_{G^*} (s_0)\) is a Levi subgroup of \(G^*\). Let \(G(s_0)\) be a Levi subgroup of \(G\) in the dual class of \(C_{G^*} (s_0)\) and such that \(H \subseteq G(s_0)\). Denote \(b_0 = \Psi_{G(s_0), s_0}(1)\). As \(s_0\) is central in \((G(s_0))^F_\ell\), \(b_0(1) = 1\) (see Proposition 1.3.2, (ii)) and \(b_0\) has order a power of \(\ell\). Furthermore \(R^G_H (\Psi_{G(s_0), s_0}(1) \otimes \xi) = \theta_0 \otimes R^G_{G(s_0)} (\xi)\). By [15] Theorem 2.8, for any component \(\xi_0\) of \(R^G_H (\xi), R^G_{G(s_0)} (\xi_0)\) and \(b_0 \otimes R^G_{G(s_0)} (\xi_0)\) belongs to the same block of \(G^F\) that is \(b_{GF} (L, \lambda)\).

From [9] we know that if \((L, \lambda)\) is in series \((s)\) then \(\text{Irr}(b_{GF} (L, \lambda)) \subseteq \mathcal{E}(G^F, s)\). In Proposition 2.3.3 above appear some non \(\ell^t\)-series \((s, s_0)\). So it’s time to describe entirely \(\text{Irr}(b_{GF} (L, \lambda))\).
2.3.4. Notation. When \( s, s_0 \) are semi-simple in \((G^*)^F\) and \( s_0 = (ss_0)_\ell \), define \( \text{Bl}(G^F; s, s_0) \) as a set of blocks of \( G^F \) by

\[
B \in \text{Bl}(G^F; s, s_0) \text{ if and only if } \text{Irr}(B) \cap \mathcal{E}(G^F, ss_0) \neq \emptyset
\]

Thus \( \text{Bl}(G^F; s, 1) = \text{Bl}(G^F; s) \) and \( \text{Bl}(G^F; s, s_0) \subseteq \text{Bl}(G^F; s) \).

2.3.5. Proposition. Assume 2.2.3 on \((G, F, \ell, d)\). Let \((L, \lambda)\) be a d-cuspidal datum in series \( s \) in \((G, F)\) \((s \text{ semi-simple, } s \in (G^*)^F_\ell)\) and \((L_s, \alpha)\) an associated d-cuspidal unipotent datum in \((C_{G^*}(s), F)\) by Proposition 2.1.4. Assume \( Z(G) \) connected if \( s \neq 1 \). Let \( s_0 \in C_{G^*}(s)^F \). Then \( \text{Irr}(b_{G^F}(L, \lambda)) \cap \mathcal{E}(G^F, ss_0) \neq \emptyset \) if and only if there exist a d-split Levi subgroup \( L_{ss_0}^* \) of \( C_{G^*}(ss_0) \) such that \([L_{ss_0}^*, L_{ss_0}^*]\) is \( C_{G^*}(s)^F \)-conjugate to \([L_s^*, L_s^*]\). For such \( L_{ss_0}^* \) there exist a unique d-cuspidal \( \alpha_0 \in \mathcal{E}(L_{ss_0}^* F; 1) \) such that

\[
(L_{ss_0}^*, \alpha_0) \sim_{C_{G^*}(s)^F} (L_s^*, \alpha)
\]

When \((2.3.5.1)\) holds, let \( G(s_0) \) be a Levi subgroup of \( G \) in the dual \( G^F \)-conjugacy class of the \( G^* \)-conjugacy class of \( C_{G^*}(s_0) \), and let \((L_0, \lambda_0)\) be a d-cuspidal datum in series \( s \) in \((G(s_0), F)\) associated to \((L_{ss_0}^*, \alpha_0)\). One has

\[
(2.3.5.2) \quad \text{Irr}(b_{G^F}(L, \lambda)) \cap \mathcal{E}(G^F, ss_0) = \{ R_{G(s_0)}^G (\Psi_{G(s_0), s_0}(1) \otimes \xi) \mid \xi \in \mathcal{E}(G(s_0)^F, (L_0, \lambda_0)) \}
\]

Proof. Note that \((2.3.5.2)\) has been proved for \( s_0 = 1 \) (Propositions 2.2.2 and 2.2.4). If \( Z(G) \) is connected, \( Z(G(s_0)) \) is connected (Proposition 1.1.3 (b)), so this case gives also

\[
(2.3.5.3) \quad \mathcal{E}(G(s_0)^F, (L_0, \lambda_0)) = \Psi_{G(s_0), s_0}(\mathcal{E}(C_{G^*}(s_0)^F, (L_{ss_0}^*, \alpha_0))
\]

If \( s = 1 \), Proposition 2.3.5 is contained in [13] Theorem 4.4. So we assume \( Z(G) \) connected. Then \( G.d.HC \) holds in \( \mathcal{E}(G(s_0)^F, s) \) by Proposition 2.2.4.

Assuming \([L_{ss_0}^*, L_{ss_0}^*] = [L_s^*, L_s^*]\), the equality \( \text{Res}_{[L_s^*, L_s^*]^F}^{[L_{ss_0}^*, L_{ss_0}^*]^F} \alpha = \text{Res}_{[L_{ss_0}^*, L_{ss_0}^*]^F} \alpha_0 \) defines a unique \( \alpha_0 \) in \( \mathcal{E}(L_{ss_0}^* F; 1) \). Then if \( L_{ss_0}^* \) is d-split, \((L_{ss_0}^*, \alpha_0)\) is a d-cuspidal unipotent datum in \((C_{G^*}(ss_0), F)\) and we have \((2.3.5.1)\).

We may write \( G \) as a central product of rationally irreducible components \( G_j \) with connected center \( Z(G_j) = Z(G) \). Then by the usual process \( G^F \) is a central quotient of \( \times_j G_j^F \) and in \( \times_j G_j^F \), \( \mathcal{E}(G^F, ss_0) \) is in bijection with \( \times_j \mathcal{E}(G_j^F, s_j s_j) \). \((L, \lambda)\) is image of \( \times_j (L_j, \lambda_j) \), and \( \text{Irr}(b_{G^F}(L, \lambda)) \) is the set of elements of \( \times_j \text{Irr}(B_{G_j^F}(L_j, \lambda_j)) \) whose kernel contains the kernel of \( \times_j G_j^F \rightarrow G^F \), and so on...

So we assume \((G, F)\) rationally irreducible.

If \( G = G_s \), then there is only one conjugacy class of d-cuspidal data in series \( s \) in \((G, F)\) and in \((G(s_0), F)\) \((L_s^* \text{ and } L_{ss_0}^* \text{ are diagonal torii in } C_{G^*}(s) \text{ and } C_{G^*}(ss_0) \text{ respectively})\) and only one \( \ell \)-block of \( G^F \) such that \( \text{Irr}(b) \cap \mathcal{E}(G^F, ss_0) \neq \emptyset \). The condition \((2.3.5.1)\) is satisfied. One has \( \text{Irr}(b) = \cup_s \mathcal{E}(G^F, ss_0) \).

With notation of the Proposition, when \( \xi \in \text{runs in } \mathcal{E}(G(s_0)^F, s) \), \( R_{G(s_0)}^G (\Psi_{G(s_0), s_0}(1) \otimes \xi) \) runs in \( \mathcal{E}(G^F, ss_0) \) by Proposition 1.3.2, (ii) and (iii). That gives \((2.3.5.2)\).

If \( G = Z^*(G) G_0, C_{G^*}(s_0) \) is an \( E \)-split Levi subgroup of \( G^* : F^*(G)^F \) being prime to \( \ell \), by Proposition 1.1.3 \((Z(C_{G^*}(s_0))/Z^*(C_{G^*}(s_0)))^F \) is prime to \( \ell \), hence \( s_0 \in Z^*(C_{G^*}(s_0)) \), hence \( Z^*(C_{G^*}(s_0))_{\phi_E} \neq 1 \) by definition of \( E \). We may apply Proposition 2.3.3 with \( H = G(s_0) \). It gives us an inclusion

\[
R_{G(s_0)}^G (\Psi_{G(s_0), s_0}(1) \otimes \text{Irr}(b_{G(s_0)^F}(L_0, \lambda_0)) \cap \mathcal{E}(G(s_0)^F, s)) \subseteq \text{Irr}(b_{G^F}(L, \lambda)))
\]
Assume (2.3.5.1). By (2.3.5.2) for 1 in \( G(s_0) \) we know that
\[
\mathcal{E}(G(s_0)^F, s) = \bigcup_{(L_0, \lambda_0)} \text{Irr}(b_{G(s_0)^F}(L_0, \lambda_0)) \cap \mathcal{E}(G(s_0)^F, s)
\]
a disjoint union when \( d \)-cuspidal data \((L_0, \lambda_0)\) are considered modulo \( G(s_0)^F\)-conjugacy. Now \( G(s_0)^F\)-conjugacy on \( d \)-cuspidal data \((L_0, \lambda_0)\) in series \((s)\) in \((G(s_0), F)\) corresponds to \( C_{G^*}(s s_0)^F\) conjugacy of unipotent \( d \)-cuspidal data \((L_{s s_0}^*, \alpha_0)\) in \((C_{G^*}(s s_0), F)\) (Proposition 2.1.4), that corresponds, via (2.3.5.1) to \( C_{G^*}(s)^F\)-conjugacy of some unipotent \( d \)-cuspidal data \((L_s^*, \alpha_s)\) in \((C_{G^*}(s), F)\) (Proposition 2.3.2). Applying Proposition 2.1.4 again and Proposition 1.3.2, (ii) and (iii), we see that irreducible in series \((s)\) and in different blocks \( b_{G(s_0)^F}(L_0, \lambda_0) \) are sent by \( \xi \mapsto R^G_{G(s_0)}(\Psi_{G(s_0), s_0}(1) \otimes \xi) \) in series \((s s_0)\) and in different blocks of \( G^F\). Equality (2.3.5.2) is proved. \(\blacksquare\)

**Proposition 2.3.6.** Assume 2.2.3 on \((G, F, \ell, d)\) and \(Z(G)\) connected. Let \(s, s_0\) in \(G^* F\) such that \((s s_0)_\ell = s_0\).

Let \(G(s_0)\) be a Levi subgroup of \(d\)-cuspidal conjugacy class of the \(G^* F\)-conjugacy class of \(C_{G^*}(s_0)\).

(a) One defines a one-to-one map from the set of \(G(s_0)^F\)-conjugacy class of \(d\)-cuspidal data in series \((s)\) in \((G(s_0), F)\) onto the set of \(G^F\)-conjugacy class of \(d\)-cuspidal data in series \((s s_0)\) in \((G, F)\) as follows:

Let \(\alpha_0 \in E(C_{L_0^*(s)}F, 1)\), \(\lambda_0 = \Psi_{L_0, ss_0}(\alpha_0)\) such that \((L_0, \lambda_0)\) be a \(d\)-cuspidal datum in series \((s)\) in \((G(s_0), F)\). Here \(L_0^*\) belongs to the dual \(C_{G^*}(s_0)^F\)-conjugacy class of \(L_0\), \(s \in L_0^*\). Define \(M^*\) by

\[
M^* := C_{G^*}(Z^\alpha(L_0^*)_{\phi_d})
\]

Let \(M\) be in the dual \(G^F\)-conjugacy class of the \((G^*)^F\)-conjugacy class of \(M^*\) and \(\mu\) be defined by

\[
\mu = R^{M}_{L_0^*}(\Psi_{L_0, ss_0}(1) \otimes \lambda_0)
\]

Then \((M, \mu)\) is a \(d\)-cuspidal datum in series \((s s_0)\) in \((G, F)\).

Generalized \(d\)-Harish-Chandra theory holds in \(E(G^F, s s_0)\).

(b) Induction on blocks \(R^G_{G(s_0)}\), as defined in Proposition 2.1.6, restricts to a one-to-one map from \(\bl{Bl}(G(s_0)^F, s)\) onto \(\bl{Bl}(G^F, s s_0)\).

Let \(L_0^*, \alpha_0, (L_0, \lambda_0)\) and \((M, \mu)\) as in (a). Let \((L, \lambda)\) be a \(d\)-cuspidal datum in series \((s)\) in \((G, F)\), defined, thanks to Proposition 2.1.4, by some \(d\)-cuspidal unipotent datum \((L_s^*, \alpha_s)\) in \((C_{G^*}(s), F)\) such that

\[
(L_s^*, \alpha_s) \sim_{C_{G^*}(s)^F} (C_{L_0^*}(s), \alpha_0).
\]

One has

\[
R^G_{G(s_0)}(b_{G(s_0)^F}(L_0, \lambda_0)) = b_{G^F}(L, \lambda),
\]

\[
\text{Irr}(b_{G^F}(L, \lambda)) \cap E(G^F, s s_0) = E(G^F, (M, \mu))
\]

\[
R^G_{G(s_0)}(\Psi_{G(s_0), s_0}(1) \otimes E(G(s_0)^F, (L_0, \lambda_0))) = \Psi_{G, s s_0}(E(C_{G^*}(s s_0)^F, (C_{L_0^*}(s)^F, \alpha_0)))
\]

Proof. Let \(S := Z^\alpha(L_0^*)_{\phi_d}\). From (2.3.6.1) we have

\[
L_0^* = C_{M^*}(s_0), \quad S = Z^\alpha(M^*)_{\phi_d} = Z^\alpha(C_{L_0^*}(s))_{\phi_d}
\]
By Proposition 2.1.4 ($C_{L_1^*}(s), \alpha_0$) is a $d$-cuspidal unipotent datum in $(G_{L^*}(ss_0), F)$. As $s_0$ is central in the dual of $L_0$, we have (Proposition 1.3.2, (ii))

$$\Psi_{L_0,s_0}(1) \otimes \lambda_0 = \Psi_{L_0,ss_0}(\alpha_0)$$

By Proposition 1.3.2, (iii) which apply to $\Psi_{M,ss_0}$ with $C_{G^*}(ss_0) \subseteq L_0^*$ we have

$$\mu = \Psi_{M,ss_0}(\alpha_0)$$

Clearly $(M, \mu)$ is defined up to $G^F$-conjugacy.

To prove that $(M, \mu)$ is $d$-cuspidal, consider a $d$-split Levi subgroup $L_1$ in $G$ such that $L_1 \subseteq M$ and $\langle R_{M^*}^{L_1}(\lambda_1) \rangle_{L_0^F} \neq 0$ for some $\lambda_1 \in \text{Irr}(L_1^F)$. There is a $d$-split Levi subgroup $L_1^*$ in $G^*$, in duality with $L_1$, such that $ss_0 \in L_1^*$ and $L_1^* \subseteq M^*$. We may assume that $\lambda_1 \in \mathcal{E}(L_1^F, ss_0)$ by a good choice of $s_0$ (defined up to $L_1^F$-conjugacy). Define $\alpha_1$ by $\lambda_1 = \Psi_{L_1,ss_0}(\alpha_1)$. There is a Levi subgroup $L_{1,0}$ of $L_1 \cap L_0$ in duality with $C_{L_1^*}(s_0)$ so that (Proposition 1.3.2, (ii) again)

$$\Psi_{L_1,ss_0}(\alpha_1) = R_{L_{1,0}}^{L_1}(\Psi_{L_{1,0},ss_0}(\alpha_1)), \quad \langle R_{M}^{L_1}(\Psi_{L_1,ss_0}(\alpha_1)), \mu \rangle_{L_0} \neq 0$$

By Proposition 1.3.2, (ii), $\Psi_{L_{1,0},ss_0}(\alpha_1) = \Psi_{L_{1,0},s}(\alpha_1)$ hence

$$(2.3.6.5) \quad R_{L_{1,0}}^{M} \lambda_1 = R_{L_{1,0}}^{M}(\Psi_{L_{1,0},ss_0}(\alpha_1)) = R_{L_0}^{M}(\Psi_{L_{1,0},s}(\alpha_1)) = R_{L_{1,0}}^{L_0}(\Psi_{L_{1,0},s}(\alpha_1))$$

As $R_{L_{1,0}}^{M}$ is isometric on $\mathcal{E}(L_0^F, s_0)$ and on $\mathcal{E}(L_0^F, ss_0)$, using transitivity of Lusztig induction we have

$$(2.3.6.6) \quad \langle R_{L_1}^{M, \lambda_1, \mu} \rangle_{M^F} = \langle R_{L_{1,0}}^{L_0}(\Psi_{L_{1,0},ss_0}(\alpha_1)), \Psi_{L_{1,0},s}(\alpha_0) \rangle_{L_0^F} = \langle R_{L_{1,0}}^{L_0}(\Psi_{L_{1,0},s}(\alpha_1)), \Psi_{L_{1,0},s}(\alpha_0) \rangle_{L_0^F} = \langle R_{L_1}^{C_{M^*}(ss_0)}(\alpha_1, \alpha_0)C_{M^*}(ss_0) \rangle_{M^F} \neq 0$$

the second equality thanks to Proposition 1.3.2, (ii), the third one by Proposition 2.2.4, formula (2.2.4.2) applied to $L_{1,0} \subseteq L_0$. As $\alpha_0$ is assumed to be $d$-cuspidal, we obtain $C_{L_1^*}(ss_0) = C_{M^*}(ss_0)$ and $\alpha_0 = \alpha_1$. That imply $Z^F(L_1^*_\phi_d) = Z^F(C_{L_1^*}(ss_0))_{\phi_d} = Z^F(C_{L_1^*}(ss_0))_{\phi_d} = Z^F(M^*)_{\phi_d}$ by (2.3.6.4), hence $L_1^* = M^*$ and $(L_1, \lambda_1) = (M, \mu)$.

We have proved that our construction of $(M, \mu)$ from $(L_0, \lambda_0)$ in (a) provides a $d$-cuspidal datum in series $(ss_0)$ in $(G, F)$. By Proposition 2.2.4 we know that the set of $G\langle s_0 \rangle^{F}$-conjugacy classes of $d$-cuspidal datum in series $(s)$ in $(G\langle s_0 \rangle, F)$ define a partition of $\mathcal{E}(G\langle s_0 \rangle^F, s)$. Using the one-to-one map $(\xi \mapsto R_{G\langle s_0 \rangle}^{G\langle s_0 \rangle} (\Psi_{G\langle s_0 \rangle, ss_0}(1) \otimes \xi))$ from $\mathcal{E}(G\langle s_0 \rangle^F, s)$ onto $\mathcal{E}(G^F, ss_0)$ we obtain a partition of $\mathcal{E}(G^F, ss_0)$. By transitivity formula $R_{G\langle s_0 \rangle}^{G\langle s_0 \rangle} \circ R_{L_0}^{G\langle s_0 \rangle} \circ R_{G\langle s_0 \rangle}^{G\langle s_0 \rangle} = s_0$ is sent by $R_{G\langle s_0 \rangle}^{G\langle s_0 \rangle}$ on the partition of $\mathcal{E}(G^F, ss_0)$. Non $G\langle s_0 \rangle^F$-conjugate data $(L_0, \lambda_0)$ in series $(s)$ in $(G\langle s_0 \rangle, F)$ define by our construction in (a) non $G^F$-conjugate $d$-cuspidal data in series $(ss_0)$ in $(G, F)$. The definitions of $\mu$ and $\lambda_0$ from $\alpha_0$ imply

$$(2.3.6.7) \quad \mathcal{E}(G^F, (M, \mu)) = R_{G\langle s_0 \rangle}^{G\langle s_0 \rangle} (\Psi_{G\langle s_0 \rangle, ss_0}(1) \otimes \mathcal{E}(G\langle s_0 \rangle^F, (L_0, \lambda_0)))$$

Furthermore $G.d$-HC holds in $\mathcal{E}(G^F, ss_0)$ as it holds in $\mathcal{E}(G\langle s_0 \rangle^F, s)$. 
One can recover $(L_0, \lambda_0)$ from $(M, \mu)$ without using the partition (2.3.6.7) given by $G.d$-HC theory in $E(G(s_0)F, s)$ : consider $M^* \subseteq G^*$ in duality with $M$ and with $ss_0 \in M^*$, so that $\mu = \Psi_{M, ss_0}(\alpha), \alpha_0 \in E(M^*F, s_0), L$. Then $C_{M^*}(s_0)$ is $G$-split in $C_{G^*}(s_0)$. Take $L_0$ in the dual $G(s_0)F$-conjugacy class of $C_{M^*}(s_0), \lambda_0 = \Psi_{L_0, s}(\alpha_0)$, one has $\mu = \mathbb{R}L_0^G(\Psi_{L_0, s}(1) \otimes \lambda_0)$, so that by construction (a) $(L_0, \lambda_0)$ gives $(M, \mu).$ Using (2.3.6.6) one verifies easily that $\mu$ is $d$-cuspidal.

(b) By Proposition 1.2.5, $G(s_0)$ is an $E$-split Levi subgroup of $G$. As $\alpha_0$ is $d$-cuspidal, $(L_0, \Psi_{L_0, s}(\alpha_0))$ is a $d$-cuspidal datum in series $(s)$ in $(G(s_0), F)$. By Propositions 2.1.6, 2.1.9, $b_{G(s_0)}(L_0, \Psi_{L_0, s}(\alpha_0))$ and $R_{G(s_0)}^G(b_{G(s_0)}(L_0, \Psi_{L_0, s}(\alpha_0)))$ are well defined.

By Proposition 2.3.2, with $(C_{G^*}(s), C_{G^*}(ss_0), (C_{L^*_0}^L(s), \alpha_0))$ instead of $(G, H, (L_H, \lambda_H))$, we have $(L^*_0, \alpha) \sim C_{G^*}(s)^F(\alpha_0)$ for some $d$-cuspidal unipotent datum $(L^*_0, \alpha)$ in $(C_{G^*}(s), F)$ and may assume $C_{G^*}(s_0) \cap L^*_0 = C_{L^*_0}^L(s)$. Then $(L^*_0, \alpha)$ defines by Proposition 2.1.4 a $d$-cuspidal datum $(L, \lambda)$ in series $(s)$ in $(G, F)$, such that, for some dual $L^*$ of $L, s \in L^* \subseteq G^*, L^*_0 = L^* \cap C_{G^*}(s)$ and $\lambda = \Psi_{L_0, s}(\alpha)$.

By Propositions 2.3.3, 2.3.5, $G_{G(s_0)}^G(\xi) = \text{Irr}(b_G^F(\xi))$ and $G_{G(s_0)}^G(\Psi_{G(s_0), ss_0}(1) \otimes \xi) \in \text{Irr}(b_{G^*}^F(\xi))$ for any $\xi \in E(G(s_0)^F, (L_0, \lambda_0)), \text{ hence } b_{G^*}^F(\xi) \in \mathbf{Bl}(G^F; s, s_0)$. In other words and with notations of Proposition 2.1.7 we have (2.3.6.2)

\[(2.3.6.8) \quad G_{G(s_0)}^G(b_{G(s_0)}^F(L_0, \lambda_0)) = b_{G^*}^F(\xi) \in \mathbf{Bl}(G^F; s, s_0)\]

Given $(C_{L^*_0}(s), \alpha_0)$ above, $(L^*_0, \alpha)$ is defined up to $C_{G^*}(s)^F$-conjugacy, the map given in (2.3.6.8) maps $(G(s_0)^F; s) \rightarrow \mathbf{Bl}(G^F; s, s_0)$ is well defined. By Proposition 2.3.2, $C_{G^*}(s)^F$-conjugacy on $(L^*_0, \alpha)$ implies $C_{G^*}(ss_0)^F$-conjugacy on $(C_{L^*_0}^L(s), \alpha_0)$, hence $G_{G(s_0)}^G$ is one-to-one from $G_{G(s_0)}^G$ onto $\mathbf{Bl}(G^F; s, s_0)$.

From (2.3.6.2), (2.3.6.8) and Proposition 2.1.6 we deduce $E(G^F, (M, \mu)) \subseteq \text{Irr}(b_G^F(\xi))$. Now $G^F$-conjugacy implies $C_{G^*}(s)^F$-conjugacy on $(L^*_0, \alpha)$, hence $C_{G^*}(ss_0)^F$-conjugacy on $(C_{L^*_0}^L(s), \alpha_0)$, hence $G(s_0)^F$-conjugacy on $(L_0, \lambda_0)$, finally $G^F$-conjugacy on $(M, \mu)$, that is equality on sets $E(G^F, (M, \mu))$ inside $E(G^F, s_0)$. As we have seen, these last one form a partition of $E(G^F, ss_0)$. Therefore

\[\text{Irr}(b_{G^*}^F(\xi)) \cap E(G^F, ss_0) = E(G^F, (M, \mu))\]

hence the second equality of (2.3.6.3), thanks to (2.3.6.7).

By definition of $(L_0, \lambda_0), E(G(s_0)^F, (L_0, \lambda_0)) = \Psi_{G(s_0), s}(E(C_{G^*}(ss_0)^F, (C_{L^*_0}^L(s), \alpha_0))$. As $s_0$ is central in $G(s_0)^F, \Psi_{G(s_0), ss_0}(1) \otimes \Psi_{G(s_0), s} = \Psi_{G(s_0), ss_0}$. By Proposition 1.3.2 (iii), $G^G_{G(s_0)} \circ \Psi_{G(s_0), ss_0} = \Psi_{G, ss_0}$. The last equality of (2.3.6.3) follows.

2.4. On blocks and their irreducible characters in $G^F$ when $Z(G)$ is not connected

As usual we consider a regular embedding

\[(G, F) \rightarrow (H, F)\]

(see 1.1.4 (b)) and a dual morphism $H^* \rightarrow G^*$ to obtain by Clifford theory results on $d$-cuspidal data and blocks when the center of $G$ is not connected. Clearly assumption 2.2.3 on $(H, F, \ell, d)$ is equivalent to 2.2.3 on $(G, F, \ell, d)$.

We have only to consider types for which the fundamental group is not trivial. If the fundamental group of $G$ is trivial $Z([G, G]) = \{1\}$ and $H$ is a direct product $[G, G] \times Z^0(H)$. Any $G$ is a direct product $G = G_1 \times G_2$ where $G_2 = [G_2, G_2]$ and contains the components of type $E_8$ or $F_4$ or $G_2$, so that $Z([G_2, G_2]) = 1$. Therefore

50
we assume in that section that \( G = G_1 \). The proof 2.2.8 of Proposition 2.2.4 shows that the commutation formula (2.2.7.1) holds.

From Proposition 2.2.4, specially (2.2.4.1), (2.2.4.2), and section 5.2 in Appendix, we have

### 2.4.1. **Assume 2.2.3 on \((H, F, \ell, d)\) and \(Z(H)\) is connected.** Let \( (M, \mu) \) be a \( d \)-cuspidal datum in an \( \ell' \)-series in \((H, F)\). There exist \( \xi \in \text{Irr}(H^F) \) such that \((R^H_M \mu, \xi)_{H^F} \in \{-1, 1\}\) and, for any \( \chi \in \mathcal{E}(H^F, (M, \mu)) \) different from \( \xi \), \( \xi(1) \neq \chi(1) \).

In the proof of the following Proposition we use 2.4.1 as Geck and Bonnafé on 1-cuspidality in a regular embedding (see [3], 12.C).

**Proposition 2.4.2.** Let \((G, F) \to (H, F)\) be a regular embedding. Assume 2.2.3 on \((H, F, \ell, d)\). Let \((M, \mu)\) be a \( d \)-cuspidal datum in an \( \ell' \)-series in \((H, F)\). Let \( L := M \cap G, \nu := \text{Res}^M_L \mu, \lambda \) an irreducible component of \( \nu \). Then \((L, \lambda)\) is a \( d \)-cuspidal datum in \((G, F)\).

**Proof.** \((L, \lambda)\) is a \( d \)-cuspidal datum by Proposition 2.1.5.

(a) The restriction from \( M^F \) to \( L^F \) has no multiplicity, we have \( \nu = \sum_{g \in M^F/(M^F)_\lambda} \nu(g) \). From 2.4.1 we deduce

\[
(2.4.2.1) \quad \text{There exist some } \chi \in \text{Irr}(G^F) \text{ such that } |(R^G_L \nu, \chi)_{L^F}| = 1.
\]

The restriction from \( H^F \) to \( G^F \) has no multiplicity and \( H^F/G^F \) is abelian. Given two elements of \( \text{Irr}(H^F) \) their restrictions to \( G^F \) are disjoint or equal, if equal their degrees are equal. Let \( \xi \in \mathcal{E}(H^F, (L, \lambda)) \) as in 2.4.1. Then if \( \chi \in \mathcal{E}(H^F, (M, \mu)) \) and \( \chi \neq \xi \), \( (\text{Res}^H_{G^F} \xi, \text{Res}^H_{G^F} \chi)_{G^F} = 0 \). Thanks to the equality \( \text{Res}^H_{G^F}(R^H_M \mu) = R^G_L \nu \), we see that components of \( \text{Res}^H_{G^F} \xi \) occur with multiplicity \( \pm 1 \) in \( R^G_L \nu \), our claim (2.4.2.1).

The actions of \( M^F/L^F \) and \( N_{G^F}(L)/L^F \) on \( \text{Irr}(L^F) \) commute : as \([M^F, G^F] \subseteq L^F\), we see that for any \( \lambda' \in \text{Irr}(L^F) \) and any \( x \in M^F \), we have \( N_{G^F}(L, \lambda') = N_{G^F}(L, x\lambda') \), hence \( N_{G^F}(L, \lambda) \subseteq N_{G^F}(L, \nu) \).

The group \( N_{G^F}(L, \nu) \) acts on the set of irreducible components of \( \nu \), a regular orbit under \( M^F/(M^F)_\lambda \).

The set \( X \) of \( x \in M^F \) such that \( ^x\lambda = n\lambda \) for some \( n \in N_{G^F}(L, \nu) \) is a subgroup of \( M^F \) such that \( M_\chi \subseteq X \) and \( |X/(M^F)| = |N_{G^F}(L, \nu)/N_{G^F}(L, \lambda)| \).

We have \( \nu = \sum_{x \in X/(M^F)_\lambda} \sum_{h \in M^F/X} x_h \lambda \) and \( \langle R^G_L x_h \lambda, \chi \rangle_{G^F} = \langle R^G_L \nu, \chi \rangle_{G^F} \), hence

\[
\langle R^G_L \nu, \chi \rangle_{G^F} = |N_{G^F}(L, \nu)/N_{G^F}(L, \lambda)| \left( \sum_{h \in M^F/X} \langle R^G_L h \lambda, \chi \rangle_{G^F} \right)
\]

By assertion (2.4.2.1) \(|N_{G^F}(L, \nu)/N_{G^F}(L, \lambda)| = 1\).

(b) As \( \chi \in \mathcal{E}(G^F, (L, \lambda)) \), if \( h \in H^F \) and \( ^h\chi = \chi \), then \( \chi \in \text{Irr}(bG^F(L, \lambda)) \cap \text{Irr}(bG^F(hL, \lambda)) \) hence the \( d \)-cuspidal data \((L, \lambda)\) and \((hL, \lambda)\) are \( G^F \)-conjugate. We have \( H^F = G^F/M^F \) hence \( g \in G^F/(M^F)_\lambda \).  

### 2.4.2.2.** We may translate assertion (b) of Proposition 2.4.2 as follows.** If \( \chi \in \mathcal{E}(G^F, s) \) and \( \chi \) corresponds to the orbit under \( A_{G^F}(s)^F \) of \( \beta \in \mathcal{E}(C_{G^F}(s)^F, 1) \) by (1.3.6.1), then \( H_\chi = \tau_{H,s}(A_{G^F}(s)_\beta^F) \). Similarly, \( M_\nu = \tau_{M,s}(A_{L^F}(s)^F) \), where \( \alpha \) is unipotent \( d \)-cuspidal in \( \text{Irr}(C_{G^F}(s)^F) \) such that \( \beta \in \mathcal{E}(C_{G^F}(s)^F, (C_{L^F}(s), \alpha)) \). There are a one-to-one morphism \( \pi : A_{L^F}(s)^F \to A_{G^F}(s)^F \) (Proposition 1.2.6, (f)) and an isomorphism \( M^F/L^F \cong H^F/G^F \) so that \( G^F.\tau_{M,s}(A_{L^F}(s)^F) = \tau_{H,s}(\pi(A_{L^F}(s)^F)) \).

Assertion (b) writes \( \pi(A_{L^F}(s)_\alpha^F) \subseteq A_{G^F}(s)_\beta^F \).
By G.d-HC in $\mathcal{E}(C_{G^*}(s)^F, 1)$ we have $A_{G^*}(s)^F_{\beta} \cap \pi(A_{L^*}(s)^F) \subseteq \pi(A_{L^*}(s)^F_{\alpha})$. Finally

$$A_{G^*}(s)^F_{\beta} \cap \pi(A_{L^*}(s)^F) = \pi(A_{L^*}(s)^F_{\alpha}).$$

In the following, assuming a choice of suitable dual $F$-stable tori $T \subseteq M \subseteq H$, $T^* \subseteq M^* \subseteq H^*$, one has isomorphisms between Weyl groups $W(G, T \cap G) \cong W(H, T)$, $W(M \cap G, T \cap G) \cong W(M, T)$, ... to simplify notations we omit reference to the torii. Consider an action of $W_H(M)^F \times \operatorname{Irr}(H^F/G^F)$ on $\operatorname{Irr}(M^F)$ as follows:

$$W_{G^F \subset H^F}(M) = W_H(M)^F \times \operatorname{Irr}(H^F/G^F) : \operatorname{Irr}(M^F) \to \operatorname{Irr}(M^F)$$

$$(w, \theta) : \mu \mapsto w \mu \otimes (\operatorname{Res}^{H^F}_{M^F} \theta^{-1}).$$

**Proposition 2.4.3.** Let $G, H, F, \ell, d, M, \mu, L, \lambda, \nu$ as in Proposition 2.4.2. Define $M_G^F(\mu)$ by

$$M_G^F(\mu) = M^F \cap (\phi \ker \theta) \quad (\theta \in (H^F/G^F)^{\vee}, \ R_M^H \mu \otimes \theta = R_M^H \mu)$$

(a) Let $t \in M^*^F$ such that $\mu \in \mathcal{E}(M^F, t)$, and $\alpha \in \mathcal{E}(C_{M^*}(t)^F, 1)$ such that $\Psi_{M^*, t}(\alpha) = \mu$. Dualities being defined around maximal $F$-stable torii $T, T^*$ in $M, M^*$, with $t \in T^*$, let $s \in L^*^F$ be the image of $t$ and $A_{G^*}(s, L^*)^F$ be the image of $N_{W_G(s)}(W_L(s)^F)$ in $A_{G^*}(s)^F$. One has

$$G^F \cdot M_G^F(\mu) = \tau_{H^*, s}(A_{G^*}(s, L^*)^F_{\alpha}).$$

(b) The isomorphism of Proposition 2.2.6 sends $W_H(M)^F_\mu$ onto $W_{C_{H^*}(t)}(C_{M^*}(t))^F_\alpha$. One has three short exact sequences:

$$W_H(M)^F_\mu \to W_{G^F \subset H^F}(M)^F_\mu \to (M^F/M_G^F(\mu))^\wedge,$$

$$(M^F/\tau_{M^*, s}(A_{L^*}(s)^F_{\alpha}))^\wedge \to W_{G^F \subset H^F}(M)^F_\mu \to W_G(L)^F_\alpha,$$

$$W_H(M)^F_\mu \to W_G(L)^F_\alpha \to A_{G^*}(s, L^*)^F_\alpha/A_{L^*}(s)^F_\alpha.$$

(c) If $\chi \in \mathcal{E}(G^F, (L, \lambda))$, then $G^F \cdot M_G^F(\mu) \subseteq (H^F)_{\chi}$.

**Proof.** (a) The inclusion $M_G^F(\mu) \subseteq \ker \theta$ is equivalent to $G^F \cdot M_G^F(\mu) \subseteq \ker \theta$.

We have $A_{L^*}(s) \cong W_L(s)/W_L^*$. In Proposition 1.2.6 is defined an injective morphism from $A_{L^*}(s)$ to $A_{G^*}(s)$ by $wW_L^*(s) \mapsto wW_G^*(s)$ (notations of (2.2.5.2) and Proposition 2.2.6). With same definition that map extends to $N_{W_G(s)}(W_L^*(s))/W_L^*(s)$, so is defined $A_{G^*}(s, L^*)^F$. As well $A_{G^*}(s, L^*)^F$ may be defined as the stabilizer of the $C_{G^*}(s)^F$-conjugacy class of $L^*_s$ in $A_{G^*}(s)^F$.

(a.1) Let $\theta \in (H^F/G^F)^{\vee}$, we claim

$$(R_M^H \mu) \otimes \theta = R_M^H \mu \text{ if and only if there exist } w \in W_H(M)^F \text{ such that } (w, \theta) \in W_{G^F \subset H^F}(M)^F_\mu.$$ 

We have $(R_M^H \mu) \otimes \theta = R_M^H(\mu \otimes \operatorname{Res}_{M^F}^{H^F} \theta)$. If $(R_M^H \mu) \otimes \theta = R_M^H \mu$, $\theta$ stabilises the series $\mathcal{E}(H^F, t)$, hence $\tau_{H^*, s}(A_{G^*}(s)^F) \subseteq \ker \theta$ (see section 1.3.5) and the order of $\theta$ is prime to $\ell$. By Corollary 2.1.8, $\mu$ and $\mu \otimes (\operatorname{Res}_{M^F}^{H^F} \theta)$ are conjugate under $N_H(M)^F$. As $\mu$ is fixed under $W(M)^F$, the equality $(R_M^H \mu) \otimes \theta = R_M^H \mu$ is equivalent to “there exist $w \in W_H(M)^F$ such that $\mu \otimes \operatorname{Res}_{M^F}^{H^F} \theta = w \mu$”, that is $(w, \theta) \in W_{G^F \subset H^F}(M)^F_\mu$.

(a.2) Claim:

$$(w, \theta) \in W_{G^F \subset H^F}(M)^F_\mu \text{ imply } \theta \in \sigma_{H^*, s}(A_{G^*}(s, L^*)^F_{\alpha}).$$

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We have \( w(\text{Res}^H_M \mu) = \nu \in \mathcal{E}(L^F, s) \).

By Proposition 2.2.6, \( w \in W(M)^F \cap W_G \). As \( W(M^*) \cap W_G \) by anti-isomorphism \( W(H) \rightarrow W(H^*) \) and isomorphism theorem, \( w \) has image \( \tilde{w}^* \in N_{W_G}(W_L^*) \) \( F/W(M)^F \), hence in \( A_G \). (s, L^*)^F : \[
\begin{align*}
a := \tilde{w}^* W_G(s) \in A_G(s, L^*)^F
\end{align*}
\]

As seen in 1.3.5, (1.3.5.1), \( \sigma_{H,s}(a) \in (H^F/G^F)^{\wedge} \), \( \text{Res}^H_M \sigma_{H,s}(a) \in (M^F/L^F)^{\wedge} \). To \( \sigma_{H,s}(a) \) there corresponds by duality \( z \in Z(G^F)^{\wedge} \) such that \( a^* = tz \). By Proposition 1.3.2, (ii), \( \sigma_{H,s}(a) = \Psi_{H,z}(1) \) and \( \text{Res}^H_M \sigma_{H,s}(a) = \Psi_{M,z}(1) (\Psi_{H,z}(1) \) is a uniform function).

Any element of \( n \in N_T(T)^{\wedge} \) with image \( w \in W_H(W(M)) \) induces an automorphism of \( (M, F) \) that stabilizes \( \mathcal{E}(M, s) \). To \( nT \in W(H)^{\wedge} \) there corresponds \( n^*T^* \), where \( n^* \in N_H(T^*)^{\wedge} \) maps onto \( \tilde{w}^* \), and \( n, n^* \) induce dual automorphisms of \( (M, F), (M^*, F) \) respectively. Furthermore \( n^* \) acts on \( C_M(s) \) and on \( \mathcal{E}(C_M, (s, F), 1) \) as \( a \). By (iv) of Proposition 1.3.2,(4.3.2)
\[
\Psi_{M,tz}(a^*a) = w \mu \in \mathcal{E}(M^F, tz)
\]

By Proposition 1.3.2, (ii) again we obtain
\[
(4.3.3).
\[
\Psi_{M,t}(a^*a) = w \mu \otimes \text{Res}^H_M \sigma_{H,s}(a)^{-1}
\]

From the hypotheses \( \mu = \Psi_{M,t}(a) \) and \( \mu \otimes \text{Res}^H_M \theta = w \mu \) and (4.3.3) follows
\[
\Psi_{M,t}(a^*a) \otimes \text{Res}^H_M \theta = \Psi_{M,t}(a) \otimes \text{Res}^H_M \theta.
\]

By section 1.3.5, (1.3.5.4) \( \text{Res}^H_M \sigma_{H,s}(a) \otimes \theta^{-1} = \sigma_{M,s}(b) \) where \( b \in A_L(s)^F \) and \( b^* = a^*a \). Then \( \theta = \sigma_{H,s}(b) \) and \( ab^{-1} \) \( (s, L^*)^F \).

So (4.3.1) implies
\[
(4.3.4)
\]
\[
\theta = \sigma_{H,s}(c)^{-1} \text{ where } c \in A_G(s, L^*)^F.
\]

(a) Assume (4.3.4). Let \( z \in Z(G^F)^{\wedge} \) such that \( \sigma^* = tz \). Let \( \tilde{w}^* \in N_{W_G}(W_L^*) \) \( F/W(M)^F \) with image \( \tilde{w}^* W_G(s) \) \( A_G(s, L^*)^F \). As above to prove (2.4.3.4) we see that there exists \( \tilde{w} \) \( W(M)^F \cap W_G \) \( F/W(M)^F \subseteq W_H(M)^F \) with image \( \tilde{w}^* \) such that (iv) of Proposition 1.3.2 apply: \( \Psi_{M,tz}(a) = w \mu \) hence \( \Psi_{M,t}(a) = w \mu \otimes \text{Res}^H_M \sigma_{H,s}(c)^{-1} \). By definition of the action of \( W_{G^F \subseteq H^F}(M), (w, \theta) \in W_{G^F \subseteq H^F}(M) \).

We have proved that \( \text{R}_M^H \mu = R_M^H \mu \otimes \theta \) is equivalent to (4.3.4). By definition of \( \tau_{H,s} \) we have \( \cap_{a \in A_G(s, L^*)^F} \text{Ker}(\sigma_{H,s}(a)) = \tau_{H,s}(A_G(s, L^*)^F) \), hence \( G^F.M^F(\mu) = \tau_{H,s}(A_G(s, L^*)^F) \).

(b) The first assertion is a consequence of (iv) in Proposition 1.3.2.

(b1) We proved in (a2), (a3) that \( W_{G^F \subseteq H^F}(M) \mu \) maps onto \( (H^F/\tau_{H,s}(A_G(s, L^*)^F))^\wedge \) by projection on the right side. But \( H^F/\tau_{H,s}(A_G(s, L^*)^F) \cong M^F/\tau_{H,s}(A_G(s, L^*)^F) \cap M^F = M^F/M^F(\mu) \) that gives the first exact sequence.

(b2) Clifford theory shows that the image of \( W_{G^F \subseteq H^F}(M) \), by projection on left side is \( W_H(M)^F \). We have \( \Psi_{W_H}(M)^F = W_G(L)^F \). By (a) of Proposition 2.4.2, \( W_G(L)^F = W_G(L)^F \).

By (1.3.5.4) \( \mu = \mu \otimes \text{Res}^H_M \theta^{-1} = \sigma_{M,s}(a) \) where \( a \in A_L(s)^F \). The kernel of the left projection is \( \{1\} \times (H^F/G^F.\tau_{M,s}(A_L(s)^F))^\wedge \), isomorphic to \( (M^F/\tau_{M,s}(A_L(s)^F))^\wedge \).
(b.3) Clifford theory without multiplicities between $L^F$ and $M^F$ implies $(M^F/L^F)_{\mu} = (M^F/M^F)_{\mu}$ (see Appendix B), that is $\tau_{M,s}(A_{L^*}(s)^F) = M^F_{\mu}$. The projection of $W_{G^F \subseteq H^F}(M)_{\mu}$ on right side contains the kernel of the projection on the left side, that is $M^F_{\mu} \subseteq M^F_{\mu}$. As well $W_H(M)_{\mu} \subseteq W_G(L)_{\mu}$. The preceding exact sequences imply an isomorphism $W_G(L)_{\mu}^F/W_H(M)_{\mu}^F \cong M^F_{\mu}/M^F_{\mu}(\mu)$. The last quotient is in duality, via functions $\tau$, with $A_{G^*}(s, L^*)_{\alpha}^F/A_{L^*}(s)_{\alpha}^F$, we obtain the third exact sequence.

(c) Let $\chi \in E(G^F, (L, \lambda))$. Note that $H^F_{\chi} = \cap_{H^F \subseteq \ker} \theta$. So let $\theta \in (H^F/G^F)^* \cong \ker H^F_{\chi} \subseteq \ker \theta$. By (1.3.1) and Proposition 2.1.7 there exist $\xi \in \Irr(H \mid \chi) \cap E(H^F, (M, \mu))$. We know that $\xi \otimes \theta = \xi$, hence $R^G_{\mu} \mu = R^G_{\mu} \mu \otimes \theta$. By definition of $M^F_{\mu}(\mu)$, $M^F_{\mu}(\mu) \subseteq H^F_{\chi}$.

2.4.4. Proposition. Let $\sigma: (G, F) \to (H, F)$ be a regular embedding and $\sigma^*$ a dual morphism. Assume 2.2.3 on $(H, F, \ell, d)$. Let $(M, \mu)$ be a d-cuspidal datum in series $(t) \in (H, F)$, a dual $M^*$ of $M$, given as a Levi subgroup of $H^*$, $\mu = \Psi_{M, \alpha}(\alpha)$, $\alpha \in E(C^w_{M^*}(t)^F, 1)$. Let $L := \sigma^{-1}(M)$, $L^* = \sigma^*(t)$, $\lambda \in E(L^F, s)$ such that $\mu$ covers $\lambda$.

(a) The set of blocks $B$ of $H^F$ that cover $b_{G^F}(L, \lambda)$ and with $\Irr(B) \subseteq E_{\ell}(H^F, t)$ is a regular orbit under $A_{G^*}(s)^F / A_{G^*}(s, L^*)_{\alpha}^F$.

(b) The set of blocks of $G^F$ that are covered by $b_{H^F}(M, \mu)$ is a regular orbit under $H^F/\tau_{H^*}(A_{L^*}(s)_{\alpha}^F)$.

(c) If $C_{H^*}(t) = C_{H^*}(t)_{\alpha}$, hence if $G = G_{\alpha}$, then the blocks in series $(s)$ of $G^F$ are conjugate under $H^F$.

Proof. We know that $\Res_{H^F \to M^F} \mu$ is a sum of $M^F$-conjugate of $\lambda$, and that $\lambda$ is d-cuspidal. By Propositions 2.1.6, 2.1.7 and formula (1.3.1) any $\xi \in E(H^F, (M, \mu))$ covers some element of $E(G^F, (L, \mu))$ $(m \in M^F)$. Hence $b_{H^F}(M, \mu)$ covers $b_{G^F}(L, \lambda)$ and only these blocks of $G^F$ [28] Chapter 7, Lemmas 5.3, 5.7. From Proposition 2.1.7 again we have (see [15] Remark 2.7)

2.4.5. Domination between blocks of $H^F$ and $G^F$ is equivalent to domination between conjugacy classes of d-cuspidal data.

(a) Given a block $B = b_{H^F}(M, \mu)$ in series $(t)$ of $H^F$, modulo $(G^*)^F$-conjugacy on $s$, $(H^*)^F$-conjugacy on $t$ and $G^F$-conjugacy on $d$-cuspidal data in series $(s)$ in $(G, F)$, we may fix $t, s, M^*, L^*, M, L, \mu, \lambda$.

The blocks of $H^F$ that cover $b_{G^F}(L, \lambda)$ are the $b_{H^F}(M, \mu) \otimes \theta = b_{H^F}(M, \mu \otimes \Res_{H^F}^F) \theta$ where $\theta \in (H^F/G^F)^*$. These blocks are in series $(t)$ when $\theta \in \sigma_{H, s}(A_{G^*}(s)^F)$ (1.3.5, (1.3.5.1)). The number of such blocks is the number of orbits on the set of d-cuspidal $\mu \in E(M^F, t)$ under $W_H(M^F)$ that are fused under $W_s := W_H(M)^F \times \Res_{M^F}^F(\sigma_{H, s}(A_{G^*}(s)))$. By Proposition 2.4.3, (b), $W_s$ contains $W_{G^F} \subseteq H^F(M)_{\mu}$. As $W_H(M)^F \subseteq W_s$, the orbit of $W_s$ on the set of blocks of $H$ we consider is regular under $W_s/W_{G^F} \subseteq H^F(M)_{\mu}$ isomorphic to $A_{G^*}(s)^F / A_{G^*}(s, L^*)_{\alpha}^F$, that is (a) of the Proposition -- recall that the action of $A_{G^*}(s)^F$ on blocks of $H^F$ is by isomorphism with $(H^F/\tau_{H, s}(A_{G^*}(s))^F)^*$. Therefore

(b) On $G$-side, the set of $G^F$-conjugacy classes of d-cuspidal data in series $(s)$ with support conjugate to $L$ is in bijection with the $N := W_G(L)^F = N_G(L)^F/L^F$-orbits of d-cuspidal elements of $E(L^F, s)$. The $d$-cuspidal data that are covered by the $H^F$-class of $(M, \mu)$ are in the orbit of $\lambda$ under $W_H(M^F) \times (M^F/L^F)$, where $W_H(M)^F$ acts as $W_G(L)^F$. The stabilizer of the $W_G(L)^F$-orbit of $\lambda$ is just $W_G(L)^F \times M^F_{\chi}$. By (1.3.6.2), $M^F/M^F_{\chi} \cong (A_{L^*}(s)_{\alpha}^F)^*$, hence (b) is proved.

(c) We have $C_{H^*}(t) = C_{H^*}(t)_{\alpha}$ if and only if $C_{G^*}^s(s) = C_{G^*}^s(s)_{\alpha}$. In that case the only d-cuspidal unipotent data of $C_{H^*}(t, F)$ are the $(T^*, 1_{(T^*)^F})$ where $T^*$ is a diagonal torus of $C_{H^*}(t)$. Let $T_t$ (resp. $T_s := T_t \cap G$) be is in the dual $H^F$-conjugacy class of $T^*$ (resp. in the dual $G^F$-conjugacy class of $\sigma^*(T^*)$), and $M := C_H((T_t)_{\phi_\alpha})$ (resp. $L := C_G(T_t)_{\phi_\alpha} = M \cap G$). By Proposition 2.1.4 $(M, \mu)$ is, up to $H^F$-conjugacy the unique d-cuspidal datum in series $(t)$. Any element of $C_{G^*}(s)^F$ stabilizes the $C_{G^*}^s(s)^F$-conjugacy class
of diagonal tori. We have $A_G^*(s)^F = A_G^*(s, \sigma^*(T^*))_1^F$, a coherent result with the fact that there is only one block in series $(t)$ of $H^F$. The group $A_L^*(s)^F$ may be different from 1. 

\[ \square \]
3. The group \(G(s)\), “in duality with” \(C_{G^*}(s)\).

In this chapter we describe a good candidate to be named “a dual of \((C_{G^*}(s), F)\)” in case \(C_{G^*}(s)\) is not connected, and denote it as \((G(s), F)\), as announced in Theorem 1.4. The construction is based on dual root data with \(F\)-action, and we assume properties of functorial type, see Proposition 3.1.1, (A). Of course it may be generalized to construct a dual of a non-connected algebraic reductive group, eventually defined over \(\mathbb{F}_q\). We describe the set \(\text{Irr}(G(s)^F)\) and its partition in blocks. To obtain the set of blocks of \(G(s)^F\), we precise \(G(s)^F\)-conjugacy classes of maximal Brauer pairs in Proposition 3.1.2. In the two studies we have to verify a non-multiplicity property in a Clifford theory with quotient a subgroup of \(A_{G^*}(s)^F\) ((B) in Proposition 3.1.1, (C) in Proposition 3.1.2). All proofs in section 3.2 are inductive, the ”minimal case” being when \(G\) is irreducible and simply connected, and \(s\) is rationally quasi-isolated in \(G^*\).

Then, in section 3.3, we may rely “unipotent ”blocks of \(G(s)^F\) and their irreducible representations to blocks in series \((s)\) of \(G^F\) and their irreducible representations, the main result of this paper.

3.1. Propositions

3.1.1. Proposition. When \((G, F)\) and \((G^*, F)\) are dual algebraic reductive groups defined on \(\mathbb{F}_q\) and \(s \in G^*F\) is semi-simple, denote \(\mathcal{E}(G^*, s)\) the short exact sequence

\[
\mathcal{E}(G^*, s) \quad 1 \rightarrow C_{G^*}(s) \rightarrow G(s) \rightarrow A_{G^*}(s) \rightarrow 1
\]

(A) Let \((G(s)^\circ, F)\) be in duality with \((C_{G^*}(s), F)\) around a maximally split root datum of \((C_{G^*}(s), F)\).

There exists an extension

\[
\mathcal{E}(G, s) \quad 1 \rightarrow G(s)^\circ \rightarrow G(s) \rightarrow A_{G^*}(s) \rightarrow 1
\]

where \(A_{G^*}(s)\) and \(F\) act on the root datum of \(G(s)^\circ\) by transposition of their action on the dual one, and the action of \(F\) on \(G(s)^\circ\) and on \(A_{G^*}(s)\) extends in an action on \(G(s)\).

The various exact sequences \(\mathcal{E}(G, s)\) may be defined satisfying the following properties :

(A.1) If \((G, F)\) is a direct product \(\times_i (G_i, F_i)\), and \(s = (s_i)_i \in G^*\), \((s_i \in G_i^*)\), then \((\mathcal{E}(G, s), F)\) is isomorphic to the direct product of extensions \((\mathcal{E}(G_i, s_i), F)\).

(A.2) If \(C_{G^*}(s)\) is a Levi subgroup of \(G^*\), let \(L(s)\) be a Levi subgroup in the dual \(G^F\)-conjugacy class. Then \(G(s)\) is isomorphic to the subgroup \(N\) of \(N_G(L(s))\) such that \(N/L(s)\) is the image of \(C_{G^*}(s)\) by the isomorphism of relative Weyl groups \(W_{G^*}(C_{G^*}(s)) \cong W_G(L(s))\).

(A.3) If \(\sigma: (G, F) \rightarrow (H, F)\) is an isotypic morphism and \(\sigma^*: H^* \rightarrow G^*\) a dual one, assume \(s = \sigma^*(t)\) with \(t \in H^*F\). One has \(C_{G^*}(s) = \sigma^*(C_{H^*}(t))\) and \(\sigma^*(C_{H^*}(t)) \subseteq C_{G^*}(s)\) so is defined \(\alpha: A_{H^*}(t) \rightarrow A_{G^*}(s)\). Let \(G(t)\) be the inverse image of \(\alpha(A_{H^*}(t))\) in \(G(s)\) by \(\mathcal{E}(G, s)\), and let \(\mathcal{E}(G, t)\) be the restriction of \(\mathcal{E}(G, s)\) to \(\alpha(A_{H^*}(t))\). There is a morphism of short exact sequences

\[
\begin{align*}
\mathcal{E}(G, t) & \quad 1 \rightarrow G(s)^\circ \rightarrow G(t) \rightarrow \alpha(A_{H^*}(t)) \rightarrow 1 \\
\mathcal{E}(H, t) & \quad 1 \rightarrow H(t)^\circ \rightarrow H(t) \rightarrow A_{H^*}(t) \rightarrow 1
\end{align*}
\]

where \(\sigma_t\) is a dual isotypic morphism of the restriction of \(\sigma^*: C_{H^*}(t) \rightarrow C_{G^*}(s)\) and all maps commute with \(F\).

(B) ”Non-multiplicity condition” :
For any semi-simple $s_1 \in C_G^*(s)^F$ with order prime to the order of $s$,

$$\forall (\chi_0, \chi) \in \mathcal{E}(G(s)_{\sigma}^F, s_1) \times \text{Irr}(G(s)^F) \quad \langle \text{Res}_{G(s)}^{G(s)^F}(\chi), \chi_0\rangle_{G(s)_{\sigma}^F} \in \{0, 1\}.$$  

It is clear that, when $(G, G^*, F, s)$ is given, $\mathcal{E}(G, s)$ is defined up to an $F$-isomorphism. If $s$ is central in $G^*$, we may assume $G(s)^\circ = G = G(s)$.

The following Proposition and its Corollary describe Clifford theory of unipotent blocks between $G(s)^{\circ F}$ and $G(s)^F$. We fix a bijection $\Psi_{G(s)^{\circ}. F} : \mathcal{E}(G(s)^{\circ F}, 1) \to \mathcal{E}(C_G^*(s)^F, 1)$, $(\alpha \mapsto \alpha)$, and so for $d$-split Levi subgroups. By Proposition 1.3.1, these bijections preserves $d$-cuspidality and relative Weyl groups. Thus there is a bijection between the set of conjugacy classes of $d$-cuspidal unipotent data in $(G(s)^{\circ}, F)$ and the analogous set in $(C_{G^*}^*(s), F)$. Note that, as $G(s)^\circ$ is connected, the exact sequence $\mathcal{E}(G, s)$ gives by restriction an isomorphism

$$G(s)^F/G(s)^{\circ F} \cong A_{G^*}(s)^F.$$  

**3.1.2. Proposition.** Assumption 2.2.3 on $(G, F, \ell, d)$. Let $(G, F), (G^*, F), s \in G^*F$, $\mathcal{E}(G^*, s), \mathcal{E}(G, s)$ as in Proposition 3.1.1. Let

$$\rho : G(s)^F \to A_{G^*}(s)^F$$

be given by restriction to $F$-fixed points in $\mathcal{E}(G, s)$.

Let $(L(s), \alpha)$ be a $d$-cuspidal unipotent datum of $(G(s)^{\circ}, F)$ and $b := b_{G(s)^{\circ F}}(L(s), \alpha)$ the unipotent block so defined by Proposition 2.1.7. Let $L^*_s$ be a $d$-cuspidal Levi subgroup of $C_G^*(s)$ in the dual class of $L(s)$ and $L^* := C_{G^*}(Z(L^*_s)^\alpha_s)$. Let $D$ be a defect group of $b$, and $\alpha_D \in \text{Irr}(C_{G(s)^F}(D))$ be the canonical irreducible in a block $b_D$ such that $(1, b) \subseteq (D, b_D)$. Let $\beta_D \in \text{Irr}(C_{G(s)^F}(D) | \alpha_D)$. One has

(A) $\rho(C_{G(s)^F}(D)) = A_{L^*_s}(s)^F$ ;
(B) $N_{G(s)^F}(D)_{\alpha_D} \subseteq N_{G(s)^F}(D)_{\beta_D}$ ;
(C) "Non-multiplicity condition":

$$\forall (\chi, \chi_0) \in \text{Irr}(C_{G(s)^F}(D)) \times \text{Irr}(b_D), \quad \langle \text{Res}_{C_{G(s)^F}(D))}^{C_{G(s)^F}(D)}(\chi), \chi_0\rangle_{C_{G(s)^F}(D)} \in \{0, 1\}.$$  

**3.1.3. Corollary.** Notations and assumptions of Proposition 3.1.2 on $(G, F, \ell, d, \rho, (L(s), \alpha), b, L^*_s)$. Let $B$ be a block of $G(s)^F$ that covers $b$. One has

$$\rho(G(s)^F) = \rho(A_{G^*}(s), L^*_s)^F, \quad \rho(I_{G(s)^F}(B)) = A_{L^*_s}(s)^F.$$  

Proof of Corollary 3.1.3. Let $A := A_{G^*}(s)^F$. The notation $A_{G^*}(s), L^*_s)^F$ was introduced in Proposition 2.4.3. As $Z^0(L^*_s)^\alpha_s = Z^0(L^*_s)^\alpha_s$, $L^*_s = L^* \cap C_{G^*}^*(s)$ and $A_{G^*}(s), L^*_s)^F$ is the stabilizer of $(L^*_s, \alpha)$ in $A$.

The $d$-cuspidal unipotent data of $(G(s)^{\circ}, F)$ that define a fixed block $b$ of $G(s)^{\circ F}$ are conjugate hence $(G(s)^F)_b = G(s)^{\circ F}N_{G(s)^F}(L(s), \alpha)$. We have

$$\rho(N_{G(s)^F}(L(s))) = A_{L^*_s}, \quad A_b = A_{L^*_s, \alpha} = A_{G^*}(s), L^*_s)^F.$$
Let $C := C^G_{(s)^F}([L(s), L(s)])$. An $\ell$-Sylow subgroup of $C^F$ is a defect group of $b$ ([13] Theorem 4.4 or [16] Theorem 22.9). The normalizer $N_C(Z^e(L(s)))$ contains such an $\ell$-Sylow because $Z^e(L(s))$ is a maximal $F$-stable torus of $C$ and contains a maximal $\phi_q$-subgroup of $C$ [16] Proposition 22.8 or [13] Proposition 1.6. So we assume $D \subseteq N_{G(s)}(Z^e(L(s)))$ and $[L(s), L(s)]^D \subseteq C_{G(s)^F(D)} \subseteq L(s)$. We have $(1, b) \subseteq (D, b_D)$ for some block $b_D$ of $C_{G(s)^F(D)}$ with central defect and the canonical irreducible $\alpha_D$ in $b_D$ is the only element in $\text{Irr}(b_D)$ such that $\text{Res}_{C_{G(s)^F(D)}}^L(\alpha_D) = \alpha_D$ [16] Proposition 15.9, Lemma 22.18. Assertion (A) in Proposition 3.1.2 implies

\begin{equation}
(3.1.3.2) \quad \rho(C_{G(s)^F(D)}\alpha_D) = A_{L^\ast}(s)_{\alpha_D}
\end{equation}

By conjugacy of maximal Brauer subpairs in a fixed block, one has $G(s)^F_b = G(s)^F . N_{G(s)^F(D, \alpha_D)}$. Using (3.1.3.1),

\begin{equation}
(3.1.3.3) \quad \rho(N_{G(s)^F(D, \alpha_D)}) = A_{L^\ast, \alpha_D}.
\end{equation}

It is known that $D$ is a split extension of $Z := Z(L(s))^F$ by an $\ell$-Sylow subgroup of $W(C, Z^e(L(s)))^F$ and that $Z$ is a maximal normal abelian subgroup of $D$ [15] Lemma 4.16. As $L(s) = C^G_{G(s)^F(Z)}$ ([13] Proposition 3.3), $N_{G(s)}(D)^F \subseteq N_{G(s)}(Z)^F \subseteq N_{G(s)}(L(s))^F \subseteq N_{G(s)}(C)^F$.

That implies $\rho(N_{G(s)^F(D)}) \subseteq A_{L(s)} = A_{L^\ast}$.

By Frattini’s argument we have $N_{G(s)}(C)^F = C^F . N_{G(s)}(D)^F$, hence $\rho(N_{G(s)}(L(s))^F) \subseteq \rho(N_{G(s)}(D)^F)$.

We get

\begin{equation}
(3.1.3.4) \quad \rho(N_{G(s)^F(D)}) = A_{L^\ast}
\end{equation}

With notations of section 5.1 in Appendix, let $I(B) := I^{G(s)^F}_{G(s)^F}(B)$. We have $\rho(I(B)) \subseteq \rho(G(s)^F) = A_{L^\ast, \alpha}$ by (B.0) and (3.1.3.1). Thanks to Proposition 3.1.2, (C), Proposition 5.1.4 applies, with $(G(s)^F, G(s)^F)$ instead of $(G, H)$. By Proposition 3.1.2, (B), the last equality in (c) of Proposition 5.1.4 simplify in

\[ \rho(I^{G(s)^F}_{G(s)^F}(B)) = \rho(C_{G(s)^F(D)}\alpha_D) \]

The equality we claim follows from (3.1.3.2).

\[ \square \]

3.2. Proofs of Propositions 3.1.1 and 3.1.2. Minimal cases

3.2.1. Preliminary remarks

In 3.2 we assume $G$ irreducible, simply connected, $s$ ”rationally quasi-isolated” in $G^s$ and $A_{G^s}(s)^F \neq \{1\}$. By Proposition 1.2.4, the types $G_2$, $F_4$ and $E_8$ are excluded. A semi-simple element $s$ of $G^s$ is said quasi-isolated in $G^s$ if $C_{G^s}(s)$ is not contained in a Levi subgroup of a proper parabolic subgroup of $G^s$ or equivalently if $Z^e(C_{G^s}(s)) = Z^e(G^s)$. We say that a semi-simple element $s$ of $(G^s)^F$ is rationally quasi-isolated in $G^s$ if $C_{G^s}(s)^F$ is not contained in an $F$-stable Levi subgroup of a proper parabolic subgroup of $G^s$. A classification of quasi-isolated elements in reductive groups is given in [2]. If $s \in G^s F$, then $C_{G^s}(Z^e(C_{G^s}(s)))$ is an $F$-stable Levi subgroup of $G^s$ : a rationally quasi-isolated element of $G^s$ is quasi-isolated.

In types $B$, $C$ and $D$, there is no non central quasi-isolated semi-simple elements if $F$ has characteristic $2$ [2].

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Let $s$, $\rho$, $(L(s), \alpha)$, $L_\ast$, $L^\ast$, $L$, $D$, $\alpha_D$, $\beta_D$, $s_1$ be defined as in the Propositions to prove. Recall the properties to verify: two non-multiplicity conditions, (B) in Proposition 3.1.1 and (C) in Proposition 3.1.2, and, with notations of Proposition 3.1.2,

(A) $\rho(C_{G(s)}^s(D)) = A_{L^\ast}(s)^F$,
(B) $N_{G(s)}^s(D)_{\alpha_D} \subseteq N_{G(s)}^s(D)_{\mu_D}$.

Put

$$A := A_{G^\ast}(s)^F.$$

We note first several simple facts we use freely in section 3.2:

As $A$ is abelian, non-multiplicity condition is equivalent to maximal extensibility: $\chi_0 \in \mathcal{E}(G(s)^o, s_1)$ extends to its stabilizer in $G(F)^o$. If $A$ is cyclic then the maximal extensibility condition is satisfied.

If $L(s) = G(s)^o$, then (C) in Proposition 3.1.2 follows from (B) in Proposition 3.1.1; if furthermore $Z(G(s)^o)^F \subseteq Z(G(s))^F$ then (A) is satisfied.

If $A_{L^\ast}(s)^F = \{1\}$ then (A) implies (B).

If $|A|$ is a prime then (B) is satisfied.

Indeed, if $|A|$ is prime, then

- $\rho(C_{G(s)}^s(D)) = \{1\}$ and then $\beta_D = \alpha_D$ in (b), or
- $\rho(C_{G(s)}^s(D)) = A$ hence $N_{G(s)}^s(D) = N_{G(s)}^s(D).C_{G(s)}^s(D)$.

In any case (B) is true.

To verify (A) and (B) in types $A$, $D$ and $E$ we use the fact that $C_{G^\ast}^o(\cdot)$ and $G(s)^o$ have the same type. As $G^\ast$ is adjoint there exists an isotypic morphism $G(s)^o \to C_{G^\ast}^o(\cdot)$ that extends to

$$\pi_\ast : G(s) \to C_{G^\ast}^o(\cdot) \text{ with } \pi_\ast(G(s)^F) = C_{G^\ast}^o(\cdot)^F.$$

Then to prove (A) we shall verify that

$$\text{(3.2.1.1) If } A_{L^\ast}(s) \neq \{1\}, \text{ then } [L^\ast, L^\ast] \subseteq C_{G^\ast}^o(\pi_\ast(D)).$$

### 3.2.1.2. Lemma. Assume all roots of $G$ have same length. One has $\rho(C_{G(s)}^s(D)) \subseteq A_{L^\ast}(s)^F$. If furthermore (3.2.1.1) holds, then $\rho(C_{G(s)}^s(D)) = A_{L^\ast}(s)^F$.

**Proof.** When all roots have same length, $G$ and $G^\ast$ have same type. Consider, as in the proof of Corollary 3.1.3, a maximally split torus of $S$ of $L(s)$, in duality with a maximally split torus $S^\ast$ of $L_\ast$. The duality between $G(s)^o$ and $C_{G^\ast}(\cdot)$ may be defined by dual root data with respect to $(S, S^\ast)$. One know that $D$ is an $\ell$-Sylow subgroup of $C^F$ where $C := C_{G(s)}^o([L(s), L(s)])$ [16] Theorem 22.9. By [16] Proposition 22.7, as $Z^\circ(L(s))$ is a maximal torus in $C$ and contains a maximal $\phi_{\ell}$-subgroup of $C$, one may assume that $D \subseteq N_{G(s)}^s(Z^\circ(L(s)))$, so that $D$ is a split extension of $Z := Z(L(s))^F_\ell$ by an $\ell$-Sylow subgroup of $W(C,Z^\circ(L(s)))^F$. Furthermore, $Z$ is caracteristic in $D$ by [15] Lemma 4.16 and $L(s) = C_{G(s)}^o(Z)$ by [13] Proposition 3.3.

If $\ell$ divides the order of the kernel $Z_0$ of $G(s)^F \to C_{G^\ast}(s)^F$ then $d = 1$ in type $A$, $G(s)^o$ is a Levi subgroup of $G$, $Z_0$ is the kernel of $G^F \to G^\ast$, hence $G = G_\ast$, and $D$ is a Sylow subgroup of $G(s)^o F$, $L(s)$ is a diagonal torus of $G(s)^o$ and we’ll see in 3.2.2 (i) that $C_{G(s)}^o(D) \subseteq G(s)^o$ and $A_{L^\ast}(s) = \{1\}$.

Assume the kernel of the restriction of $\pi_\ast$ on groups of rational points is prime to $\ell$. Then $Z(L_\ast)^F = \pi_\ast(Z(L(s))^F)$ and $C_{G^\ast}(\pi_\ast(D)) = \pi_\ast(C_{G^\ast}(D))$. By Proposition 1.2.5, we have $L_\ast = C_{G^\ast}^o(\pi_\ast(Z(L_\ast)^F))$ and $L^\ast = C_{G^\ast}^o(Z(L^\ast)^F)$. As $L_\ast \subseteq L^\ast$, we have $Z(L^\ast)^F \subseteq Z(L^\ast)^F \subseteq C_{G^\ast}^o(Z(L^\ast)^F) \subseteq L^\ast$. 

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Hence $\pi_\ast(C_{G(s)}^\circ(D)) \subseteq C_{C_{G(s)}^\ast}(\pi_\ast(Z))^F \subseteq C_{L^\ast}(s)$. The last inclusion implies $\rho(C_{G(s)}^\circ(D)) \subseteq A_{L^\ast}(s)$.

We have $C_{L^\ast}(s) = Z^\ast(L^\ast).C_{[L^\ast,L^\ast]}(s)$ hence $C_{L^\ast}(s) = C_{\tilde{G}^\ast}(s).C_{[L^\ast,L^\ast]}(s)$. If (3.2.1.1) holds, then $C_{\tilde{G}^\ast}(\pi_\ast(D)).C_{L^\ast}(s)^F = C_{L^\ast}(s)^F$ so that $\rho(C_{G(s)}^\circ(D)) = A_{L^\ast}(s)^F$.

### 3.2.2. Type A

Here we assume that $G = \text{SL}_{n+1}$, a subgroup of $\text{GL}_{n+1}$, and $G^F = \text{SL}_{n+1}(r)$ where $r = \epsilon q$ (by convention $\text{SL}_n(-q)$ is $\text{SU}_n(q^2/q)$) so that $G^\ast = \text{PGL}_{n+1}$. Let $\tilde{G} = \text{GL}_{n+1}$ acting on $F^{n+1}$. To a natural $F$-epimorphism

$$\pi: \tilde{G} \to G^\ast$$

there corresponds $G \to \text{GL}_{n+1}$ between duals.

There exists $\tilde{s} \in \tilde{G}^F$ with the following properties:

(a) $\pi(\tilde{s}) = s$, $\tilde{s}$ is semi-simple and of order prime to $\ell$, so that $\pi(C_{\tilde{G}^\ast}(\tilde{s})) = C_{G^\ast}^\circ(s)$.

(b) Let $\Gamma$ be the set of eigenvalues of $\tilde{s}$ and, for $\gamma \in \Gamma$, let $V_\gamma$ be the corresponding eigenspace. All spaces $V_\gamma$ have the same dimension $m$, so that $n + 1 = |\Gamma| \cdot m$.

(c) The group $A_{G^\ast}(s)$ acts regularly on $\Gamma$ by translation in $F^m$ and on the set $\{V_\gamma\}_{\gamma \in \Gamma}$ via a morphism $v: A_{G^\ast}(s) \to F^m$ so that for any $\tilde{g} \in \tilde{G}$ with $\pi(\tilde{g}) \in G^\ast(s)$ and any $\gamma \in \Gamma$ then $\tilde{g}(V_\gamma) = V_{\gamma'}$ where $\gamma' = v(\rho(\pi(\tilde{g})))$. Let $\zeta$ be of order $|\pi(A_{G^\ast}(s))|$ in $F^m$, $\zeta$ is a generator of $v(A_{G^\ast}(s))$.

Let $c_0$ be the order of the orbit of $\zeta$ under the map $(\Phi:F \to F$, $\gamma \mapsto \gamma^\ast)$. Then $A := A_{G^\ast}(s)^F$ has order $c := |\Gamma|/c_0$ and $\zeta^c_0$ is a generator of $v(A)$. One sees easily that the orbit $\omega$ of $\gamma \in \Gamma$ under $\langle \zeta^c_0, \Phi \rangle$ acting on $\Gamma$ has order $c_0 \delta(\omega)$, where $\delta(\omega)$ is the order of the orbit of $\gamma$ under $\Phi$. To $\omega$ there corresponds a rational component of $C_{\tilde{G}^\ast}(s)$ of type $[\text{GL}_m(r^\delta(\omega))]^c$. One has $|\Gamma| = (\sum \delta(\omega))$, hence $c_0 = (\sum \delta(\omega))$.

If there is more than one orbit in $\Gamma$ under $\Phi$, there exists an $F$-stable proper Levi subgroup $K^\ast$ of $G^\ast$ such that $C_{G^\ast}(s)^F \subseteq K^\ast$, hence $A = A_{K^\ast}(s)^F$. Assuming $s$ rationally quasi-isolated, there is only one orbit $\omega$ and write $\delta = \delta(\omega) = c_0$. Then $C_{G^\ast}^\circ(s)$ has $c$ rational components, corresponding to a decomposition of $F^{n+1}$ in a direct sum $\bigoplus_{j \in [1,c]} V_j$ and $A$ acts on $[1,c]$.

Let $f := d/(d, \delta)$. A $d$-cuspidal unipotent datum of $(C_{G^\ast}^\circ(s), F)$ or of $(G(s)^\circ, F)$ is defined by a set of $c$ partitions without any $f$-hook, that is to say $c$ $f$-cores : for $j = 1, \ldots, c$, $\kappa(j)$ is a partition of $k(j)$, and $g(j) = (m - k(j))/f \in \mathbb{N}$. We have $L_\ast^c = \pi(\tilde{L}^\ast)$, where $\tilde{L}^\ast \cong \times_{j} ([\text{GL}_1(r^\delta)]^{g(j)} \times \text{GL}_{k(j)}(r^\delta))$ (if $f = 1$, then $k(j) = 0$ and $g(j) = m$ for all $j$). Then $L^\ast = \pi(M^\ast)$ where $(M^\ast)^F \cong \times_{j} ([\text{GL}_{f_j(d)}(r^\delta)]^{g(j)} \times \text{GL}_N(r))$ with $N = \delta(\sum j(k(j))$. One sees that

$$A_{L^\ast}^c(s)^F = \{a \in A \mid g \circ a = g\}, \quad A_{L^\ast}^c = \{a \in A \mid \kappa \circ a = \kappa\}.$$

Here we consider $A_{L^\ast}^c(s)^F$ as a subgroup of $A$. Indeed if $g(j) \neq 0$, any element of $\pi^{-1}((L^\ast)^F)$ stabilizes a non null subspace of $V_j$. By its component $\pi((\text{GL}_N(r))^F \cap C_{G^\ast}^\circ(s)^F, C_{L^\ast}(s)^F$ acts on the set of $j$ with $g(j) = 0$ as freely as $C_{G^\ast}^\circ(s)^F$. That gives $A_{L^\ast}^c(s)^F$. The three others equality are clear.

As $A$ is cyclic we have only to verify (A) and (B).

As said in condition (A2) in Proposition 3.1.3, we may assume that $G(s)^\circ$ is a Levi subgroup of $G$, with same rational type that $C_{G^\ast}^\circ(s)$. With evident notations, $L(s) = L_1, \ldots L_j, \ldots L_c$. A defect group $D$ of $b$ is an $\ell$-Sylow subgroup of $C_{G^\ast}(s)^\circ([L(s), L(s)])^F$ and $\pi^{-1}(C_{G^\ast}(s)^\circ([L(s), L(s)])^F) \cong \times_{j} ([\text{GL}_f(\delta)(r^\delta)] \times \text{GL}_k(j)(r^\delta))$.
D is a central product of the $\pi(D_j)$, where $D_j$ is an $\ell$-Sylow subgroup of a wreath product $GL_1(r^{f\delta}) \wr S_{\ell(j)}$ (because the torus with rational-points group $GL_1(r^{f\delta})$ is a minimal $d$-split Levi subgroup of the component $GL_{f(j)}(r^{f\delta})$).

(o) Let us consider the case $d = 1$, that is $G = G_\alpha$, to complete the proof of Lemma 3.2.1.2.

As $L(s)$ is a diagonal torus of $G(s)\alpha$ and $M = C_G(Z^r(L(s))_{\phi_1})$, $L(s)^F = SL_{m+1} \cap [GL_1(r^{f\delta})]^{mc}$ and $M^F = SL_{m+1} \cap [GL_3(r)]^{mc}$ and $L^r = \pi([GL_4(r)]^{mc})$. One sees that $\Lambda_{L^r}(s) = \{1\}$. One has $(r^{f\delta} - 1)_{\ell} = (r^d - 1)_{\ell}$ because $\delta$ is a divisor of $|A|$, prime to $\ell$. It follows that $Z(L(s))^F = Z(M)^F$ and, by Proposition 1.1.6, that $M^F = C_G(FZ(L(s))^F)$. Then $C_{G(s)^F}(D) \subseteq C_G(Z(L(s))^F) \subseteq M$.

(i) Assume that $m = 1$. Then $G(s)^G$ and $C_{G_\gamma}(s)$ are tori, $L(s) = G(s)^G$, $\alpha = 1_{L(s)^F}$, $D = G(s)^G$, $b_D$ is the principal block. In case $f > 1$, $D = \{1\}$ and $Z^r(C_{G_\gamma}(s))_{\phi_d} = \{1\}$. It follows that $L^* = G^*$ and $C_{G(s)^F}(D) = G(s)^G$. Hence (3.2.1.1), (A) and (B) are true in that case. In case $f = 1$, $d$ divides $\delta$. As $\delta$ is prime to $\ell$, $(r^{f\delta} - 1)_{\ell} = (r^d - 1)_{\ell}$. In other words $C_{G_\gamma}(s)^F \subseteq C_{G_\gamma}(s)_{\phi_d} = Z^0(L^*)_{\phi_d}$, so that $D \subseteq Z^0(M)_{\phi_d}$. This implies $\pi_s(D) \subseteq Z(L)^*$ hence (3.2.1.1) and (A) by Lemma 3.2.1.2. The condition (B) follows, $b_D$ being the principal block, $\alpha_D$ and $\beta_D$ are the unit characters.

(ii) Assume now that $g(j) = 0$ for any $j \in [1, c]$, but $m > 1$. Then $f > 1$ and again $D = \{1\}$. $A_L^r(s) = A_{G^*}(s)$, $C_{G(s)^F}(D) = N_{G(s)^F}(D) = G(s)^F$, (3.2.1.1), (A) and (B) are clearly true.

(iii) If $g(j) \neq 0$ for any $j$, by (3.2.1.1) $A_{L^r}(s)^F = \{1\}$. But $\pi(D_j) \neq \{1\}$ for any $j$ and $A$ acts regularly on $[1, c]$, hence $\rho(C_{G(s)^F}(D)) = \{1\}$, $\alpha_D = \beta_D$ and (A), (B) are true.

(iv) In the general case consider $J_0 = \{j \in [1, c] \mid g(j) = 0\}$ and $J_1 = [1, c] \setminus J_0$. That partition defines a decomposition $F^{n+1} = V_0 \oplus V_1$ and dual maximal proper $F$-stable Levi subgroups $H = H_0.H_1$, $H^* = H_0^*.H_1^*$ (central products, $H_j^*$ and $H_j$ are in duality "up to an isotypic morphism") of $G$ and $G^*$. One has $G(s)^G = G(s)^0.G(s)^0$ with $G(s)^0 = H_1 \cap G(s)^G$, $C_{G^*}(s) = (H_0^* \cap C_{G^*}(s)).(H_1^* \cap C_{G^*}(s))$. One may write $s = s_0.s_1 \in (H^0_0)^F.(H^1_1)^F$ and $A_{H_1^0}(s_0)^F \times A_{H_1^1}(s_1)^F$ may be viewed as a subgroup of $A_{G^*}(s)^F$ : it is the stabilizer in $A_{G^*}(s)$ of the decomposition $G(s)^0.G(s)^1$. By (3.2.1.1) $A_{L^*} \subseteq A_{H_0^0}(s_0)^F \times A_{H_1^1}(s_1)^F$. As seen in the extreme cases (ii) and (iii) above, where one $J_i = [1, c]$,

$$H^*_0 \subseteq L^*, \quad L(s) = (H_0 \cap G(s)^F).(L(s) \cap H_1), \quad D \subseteq H_1$$

hence (3.2.1.1). By (3.2.1.1) again, $A_L^r(s)^F = A_{H_0^0}(s_0)^F$. Now $C_{G(s)^F}(D) = G(s)^0.F.C_{G(s)^1}(D)$ and $\alpha_D = \alpha_0 \oplus \alpha_1$, where $\alpha_0$ has null defect. Furthermore $\rho(C_{G(s)^F}(D)) = A_{H^*_0}(s_0)^F$ (see (iii)) that is (A). Then $\beta_D = \beta_1 \oplus \alpha_1$. Similarly $N_{G(s)^F}(D)$ stabilizes the decomposition of $G(s)^G$ and, by (ii) and (iii),

$$\rho(N_{G(s)^F}(D)) = A_{H_0^0}(s_0)^F.\rho(N_{G(s)^1}(D)) \subseteq \rho(C_{G(s)^F}(D)).\rho(N_{G(s)^F}(D)).$$

This proves (B).

**Type B**

Let $\pi : Sp_{2n} \to G^* = PSp_{2n}$. Some $s \in \pi^{-1}(s)$ has order 2 or 4 and $s$ is conjugate to $-s$ under $A_{G^*}(s)$. In the standard action on $F^{2n}$, $s$ has eigenvalue 1 and $-1$ with multiplicity $2m$, and primitive 4-roots of 1 with multiplicity $(n - 2m)$. So $C_{G^*}(s)$ has type $C_m \times A_{n-2m-1} \times C_m$ with $0 \leq m \leq n/2$, $G(s)^G$ has type $B_m \times A_{n-2m-1} \times B_m$ (with usual conventions on $A_0, B_0, B_1$).

A generator of $A_{G^*}(s)$ exchange the factors of type $C_m$, if there are some, and acts as diagram automorphism on the factor of type $A$, if there is some. Then $L^*_s$ has type $C_{m_1} \times A_{m_1} \times C_{m_2}$ for some $m_i$ with $m_1, m_2 \leq m$, $m_3 \leq n - 2m - 1$ and $L^*$ has type $C_{m_1+m_2+m_3+1}$, short of the case $m_1 = m_2 = 0$, where $L^*$ has type $A_{m_3}$.
If \( L^*_r \neq C^r_{G^r}(s) \), that is \( L(s) \neq G(s)^o \), then \( A_{L^r}(s) = \{1\} \) and \( D \) has a non trivial intersection with one of the irreducible components of \( G(s)^o \), so that \( C_{G(s)^o}(D) \subseteq G(s)^o \).

If \( L^*_r = C^r_{G^r}(s) \), then the defect group \( D \) of \( b_{G(s)^o}^r(L(s),\alpha) \) is central in \( G(s)^o \) and contained in the type \( A \) factor. One has \( D = Z(G(s)^o)^F \). The polynomial order of \( Z^o(G(s)^o) \) has only cyclotomic factors \( \phi_1 \) and \( \phi_2 \) and \( \ell > 2 \), hence \( Z(G(s)^o)^F \subseteq Z(G(s)^o)^{\phi_d} \). But \( L^* = C_{G^r}(Z(C^r_{G^r}(s))^{\phi_d}) \) and \( A_{L^r}(s) \) acts trivially on \( Z(C^r_{G^r}(s))^{\phi_d} \). Dually \( A_{L^r}(s) \) acts trivially on \( Z(G(s)^o)^{\phi_d} \), therefore on \( D \), that is (A).

**Type C**

Take \( G = \text{Sp}_{2n} \), \( G^* = \text{SO}_{2n+1} \) and let \( \tilde{G} \) be the spin group of same type with \( \pi: \tilde{G} \to G^* \) a natural quotient. a semi-simple \( s \in G^* \) has non connected centralizer in \( G^* \) if and only if it has eigenvalues 1 and -1 in the standard action on \( \mathbb{F}_{2n+1} \) [16] Proposition 16.25. We may consider two cases for quasi-isolated semi-simple elements in \( G^* \):

**Case 1.** \( s \) is not isolated, has d’ordre 2, the fixed-point space of \( s \) in \( \mathbb{F}_{2n+1} \) has dimension \((2n-1)\), and \( C^r_{G^r}(s) \) is a maximal proper Levi subgroup of type \( B_{n-1} \).

\( G(s)^o \) is a Levi subgroup of \( G \) and has type \( C_{n-1} \). There exists an involution \( x \in \text{SO}_{2n+1}(q) \) acting on \( C^r_{G^r}(s) \) such that \( xy = x^{-1} \) if \( y \in Z^o(C^r_{G^r}(s)) \) and \( xy = y \) if \( y \in [C^r_{G^r}(s),C^r_{G^r}(s)] \). Then \( \rho(x) \) generates \( A \) and \( x \) acts on \( G(s)^o \) on the same way : \( xy = x^{-1} \) if \( y \in Z^o(G(s)^o) \) and \( xy = y \) if \( y \in [G(s)^o,G(s)^o] \).

If \( Z^o(G(s)^o)^F \neq \{1\} \) and \( L(s) \neq G(s)^o \), then \( \{1\} \neq D \subseteq [G(s)^o,G(s)^o] \), hence \( \rho(C_{G(s)^o}(D)) = A \). \( Z^o(C^r_{G^r}(s)) \subseteq L^* = C^r_{G^r}(Z^o(L^*_{s^r}))^{\phi_d} \) \( s \) is quasi-isolated in \( L^* \) so that \( A_{L^r}(s) = A^r_{G^r}(s) \).

If \( Z^o(G(s)^o)^F \neq \{1\} \), then \( d = 1 \), \( G(s)^o \) is a d-split Levi subgroup of \( G \), and \( G(s)^o = C^r_{G^r}(Z^o(G(s)^o)^F) \), hence \( M \subseteq G(s)^o \), \( L^* \subseteq C^r_{G^r}(s) \) and \( A_{L^r}(s) = \{1\} \). Furthermore \( Z^o(G(s)^o)^F \subseteq D \), \( G(s)^o = C^r_{G^r}(Z^o(G(s)^o)^F) \) [16] Lemma 13.17 and \( C^r_{G^r}(D) = C^r_{G}(D) \) (Proposition 1.2.4). Thus \( C_{G(s)^o}(D) \subseteq G(s)^o \) and (A) is true.

**Case 2.** \( s \) is isolated, has order 2, the fixed-point space of \( s \) in \( \mathbb{F}_{2n+1} \) has dimension \((2n+1)\) and \( C^r_{G^r}(s) \) has type \( D_m \times B_{n-m} \) \((1 < m \leq n)\).

Then \( C^r_{G^r}(s) \) is isomorphic to \( \text{SO}_{2m} \times \text{SO}_{2n-2m+1} \), \( s \) belongs to the first component and \( C^r_{G^r}(s) \) is isomorphic to \( (\text{O}_{2m} \times \text{O}_{2n-2m+1}) \cap \text{SO}_{2n+1} \). But \( \text{O}_{2k+1} = \text{SO}_{2k+1} \times (-\text{Id}) \). On \( G(s)^o = \text{SO}_{2m} \times \text{Sp}_{2n-2m} \) a generator of \( A \) acts by a diagonal automorphism on the first component and \( G(s)^o = \text{O}_{2m} \times \text{Sp}_{2n-2m} \).

If \( L(s) = G(s)^o \), then \( D = \{1\} \) and \( L^* = G^* \), hence (A).

In general, \( s \) is quasi-isolated in \( L^* \) with \( A_{L^r}(s) \neq \{1\} \) if and only if \( s \in Z(L^*) \). That condition is equivalent to \( \{[L^*_r,L^*_r]\} \) is not contained in the second component of \( C^r_{G^r}(s)^o \) or to \( \{[L(s),L(s)]\} \) is not contained in the component \( \text{Sp}_{2n-2m} \) of \( G(s)^o \). One sees easily that if \( X \) is an \( \ell \)-subgroup of \( \text{SO}_{2k}(q) \) with a non null space of fixed points on \( \mathbb{F}^{2k} \), then \( C_{O_{2k}(X)}.\text{SO}_{2k} = O_{2k} \). This applies to \( D \cap \text{SO}_{2m} \) in the first factor of \( G(s)^o \) when \( A_{L^r}(s) \neq \{1\} \) because \( D \) centralizes \( [L(s),L(s)] \).

If \( A_{L^r}(s) = \{1\} \), then \( d \) divides \( m \) and \( D \) contains an \( \ell \)-Sylow subgroup of \( \text{SO}_{2m} \). One can verify that if \( S \) is the minimal \( d \)-split Levi subgroup, it is a maximal torus in \( \text{SO}_{2m}(q) \) and \( C_{O_{2m}}(S^F) \subseteq \text{SO}_{2m} \). That shows that \( C_{G(s)^o}(D) \subseteq G(s)^o \).

### 3.2.4. Type D.

We assume that \( G \) is the spin group of a non degenerate symmetric bilinear form on a space of dimension \( 2n \) on \( \mathbb{F} \) and \( G^* = \text{PSO}_{2n} \), all defined on \( \mathbb{F}_q \) by \( F \). There are isotypic morphisms \( \pi: G \to G^*, G(s)^o \to C^r_{G^r}(s) \) and a restriction \( \pi_s: G(s)^o \to C^r_{G^r}(s)^F \) (see section 3.2.1).

(a) Assume \( s \) a semi-simple quasi-isolated element of \( G^* \) such that \( |A_{G^*}(s^F)| = 4 \).

Then \( s^4 = 1 \) and \( C^r_{G^r}(s) \) has type \( D_m \times A_{n-2m-1} \times D_m \) where \( 2 \leq m \leq n/2 \) (and with some conventional notations : if \( m = n/2 \), then the component of “type \( A_{-1} \)” is \( \{1\} \) and \( s \) is isolated) or \( C^r_{G^r}(s) \) has type
\(A_{n-3}\) (special case \(m = 1\) above). There exists \(s \in \text{SO}_{2n}\), which has image \(s\) in \(\text{PSO}_{2n}\) and four eigenvalues \(1, -1, i, -i\) \((i^2 = -1)\) with respective multiplicities \(m, m, n - 2m, n - 2m\) in the standard action of \(\text{SO}_{2n}\) on \(\mathbb{P}^{2n}\).

So \(C_{G^r}^o(s)\) is the image in \(\text{PSO}_{2n}\) of a subgroup of \(\text{SO}_{2n}\) isomorphic to \(\text{SO}_{2m} \times \text{GL}_{n-2m} \times \text{SO}_{2m}\), or to \(\text{GL}_1 \times \text{GL}_{2n-2} \times \text{GL}_1\) if \(m = 1\). The dual \(G(s)^\circ = G_1 . G_0 . G_2\) of \(C_{G^r}^o(s)\) maps dually in \(\text{SO}_{2m} \times \text{GL}_{n-2m} \times \text{SO}_{2m}\) (or if \(m = 1\), \(G(s)^\circ\) is a Levi subgroup of \(G_1\) and \(G_2\) are torii) and so is defined \(\tau\).

The groups of diagram automorphisms of \(C_{G^r}^o(s)\) and \(G(s)^\circ\) are isomorphic. Let \(\delta_1\) be a diagram automorphism of order 2 of the component \(G_1\) of type \(D_m\) of \(G(s)^\circ\) (if \(m = 1\), \(\delta_j\) acts by inversion on \(G_j\)) let \(\tau\) be a diagram automorphism defined on \(\mathbb{F}_q\) and of order 2 that exchange the two components \(G_1\) and \(G_2\), \(\delta_2 := \tau \delta_1\). If \(G_0 \neq \{1\}\) let \(\gamma\) be a diagram automorphism of order 2 and defined on \(\mathbb{F}_q\) of \(G_0\). If \(G_0 = \{1\}\) let \(\gamma = 1\). One has \(|\mathcal{F}(G(s)^\circ)| = |Z(C_{G^r}^o(s))/Z_o(C_{G^r}^o(s))| = 4\).

The group \(A := A_G^-(s)^F\) is non cyclic if and only if \(n\) is even. Then \(A\) is generated by diagram automorphisms: \(A = \langle \tau, \gamma, \delta_1 \delta_2 \rangle\). \(\delta_1 \delta_2\) is induced by the image in \(\text{PSO}_{2n}\) of \((\text{O}_{2m} \times \text{GL}_{n-2m} \times \text{O}_{2m})\) \(\cap \text{SO}_{2n}\). The extension \(C_{G^r}^o(s) \rightarrow C_{G^r}^o(G) \rightarrow A_G^-(s)\) is split. We may assume that \(\tau\) acts on \((\text{O}_{2m} \times \text{GL}_{n-2m} \times \text{O}_{2m})\) by exchange of the components \(O_{2m}\). Thus \(G(s)^F\) is a quotient by a finite central 2-group of a subgroup of the direct product \((\text{Spin}_{2n}(q), \langle \delta \rangle) \times \langle \tau \rangle \times \text{GL}_{n-2m}(q), \langle \gamma \rangle\).

Let \(s_1 \in C_{G^r}^o(s)^F\) with odd order and \(\xi \in \mathcal{E}(G(s)^\circ F, s_1)\). The kernel of \(\xi\) contains \(Z(G(s)^\circ)^F\). If \(G(s)^F\) fixes \(\xi\), \(\xi\) has the form \(\xi_1 \otimes \xi_0 \otimes \xi_2\) where \(\gamma(\xi_0) = \xi_0, \delta_j(\xi_j) = \xi_j\) and \(\tau(\xi_1) = \xi_2\). Then \(\xi\) is fixed under the all group \(\langle \delta, \tau, \gamma \rangle\). So \(\xi_1\) extends to \(\zeta_1 \in \text{Irr}(\text{Spin}_{2m}(q), \langle \delta \rangle)\), \(\xi_2\) extends to \(\tau(\zeta_1)\), \(\xi_0\) extends to \(\zeta_0 \in \text{Irr}(\text{GL}_{n-2m}(q), \langle \gamma \rangle)\). Then \(\zeta_1 \otimes \zeta_2\) extends to \(\text{Spin}_{2m}(q), \langle \delta \rangle) \times \langle \tau \rangle\). It follows that \(\xi\) extends to \(G(s)^\circ F, \langle \delta, \tau, \gamma \rangle\). The condition (B) in Proposition 3.1.1 is satisfied.

If \(n\) is odd, \(A\) is generated by \(\tau \delta_1 \gamma\).

One has \(L(s) = L_1 . L_0 . L_2\) a central product, and \(\alpha = \alpha_1 \otimes \alpha_0 \otimes \alpha_2\), where \((L_j, \alpha_j)\) is a d-cuspidal unipotent datum of \((G_j, F)\) eventually \(L_0 = \{1\}\) (Proposition 1.3.1(ii), if \(m = 1, \alpha_j = 1_G^F\) for \(j = 1, 2\)).

(a.1) If \(L(s) = G(s)^\circ\), then \(D = Z(G(s)^\circ)^F\).

If \(Z(G(s)^\circ)^F = \{1\}\) then \(D = \{1\}\) and \(L^* = G^*\) there is nothing to prove.

If \(Z(G(s)^\circ)^F \neq \{1\}\) then \(d = 1\).

If \(m = 1\), then \(C_{G^r}^o(s)\) is a d-split Levi subgroup of \(G^*\) and \(L(s)\) is a maximally split maximal torus in \(G(s)^\circ\), \(L^*\) is a d-split Levi subgroup of \(G^*\), \(L^* = L^*_s\), \(A_{L^*}(s) = \{1\}\), \(D\) is an \(\ell\)-Sylow subgroup of \(G(s)^F\), \(C_{G^r}(s)^F(D) \subseteq C_{G^r}(s)^F(Z(L(s))^F) = L(s)\) (Proposition 1.1.6), \(\alpha_D = \beta_D\), there is nothing to prove.

If \(m > 1\):

Then \(D \subseteq G_0, G_0\) is a torus of rank \(1, n = 2m + 1\), \(\gamma\) acts non trivially on \(D\) by \((x \mapsto x^{-1})\), hence \(\rho(C_{G^r}(s)^F(D)) = \langle \delta_1 \delta_2 \rangle\). As \(\ell\) is good and \(\mathcal{F}(G)\) is prime to \(\ell\), \(C_{G^r}(D) = C_{G^r}((G_0)_{\phi_D}) = M\) (Proposition 1.1.6); \(L^*\) is a maximal Levi subgroup of type \(D_{n-1}\) of \(G^*\) and \(A_{L^*}(s) = \langle \delta_1 \delta_2 \rangle\), that is (3.2.5.1) and (A).

By definition \(\beta_D\) is an extension of \(\alpha = \alpha_2 \otimes C_{G^r}(s)^F(D)\); \(\beta_D\) is also the restriction of some \(\beta_1 \otimes \alpha_0 \otimes \beta_2\) where \(\beta_j\) extends \(\alpha_j\) to \(O_{2m}(q)\) for \(j = 1, 2\). If \(\rho(N_{G^r}(s)^F(D)_{\alpha_2}) = A\), then \(\alpha_2 = \tau(\alpha_1)\) and \(\gamma(\alpha_0) = \alpha_0\).

There are four extensions of \(\alpha_1 \otimes \alpha_2\) to \(O_{2m}(q) \times O_{2m}(q)\) : \(\theta_1(\alpha_1) \otimes (\theta_2 \beta_2)\) where \(\theta_j\) is linear with square 1. Then \(\langle \tau \rangle\) has two fixed points on that set, such as \(\beta_1' \otimes \beta_2' \otimes \theta_1 \beta_1 \otimes \theta_2 \beta_2'\) where \(\theta_j \neq 1\) (and with a good choice of notations) and an orbit \(\theta_1(\beta_1') \otimes \beta_2', \beta_1' \otimes \theta_2 \beta_2'\). But the first two one’s have same restriction to \(O_{2m}(q) \times O_{2m}(q) \cap \text{SO}_{2m}(q)\), as well as the last one. In other words \(\tau\) fixes any extension of \(\alpha_1 \otimes \alpha_2\) to \(O_{2m}(q) \times O_{2m}(q) \cap \text{SO}_{2m}(q)\). This shows that \(G(s)^F\) stabilizes \(\beta_D\), that is (B).

(a.2) Assume \(L(s) \neq G(s)^\circ\).
If \( m > 1 \), then \( L(s) \) has type \( D_{m_1} \times A_{m_0} \times D_{m_2} \) \((m_0 > 0)\) or \( D_{m_1} \times D_{m_2} \) and \( M \) has type \( D_{m_1+m_0+m_2+1} \) or \( D_{m_1+m_2} \), with special cases where \( m_j = 1 \) for some \( j \in \{1, 2\} \), \( SO_2 \) being a torus. If \( m = 1 \), then \( L^* \) has type \( D_t \) with \( t \geq 2 \). Going back from \( G^* \) to \( SO_2 \), one sees that \([L^*_s, L^*]_s\) and \([L^*, L^*]_s\) have equal spaces of fixed points on \( \mathbb{P}^{2n} \). As \([L(s), L(s)]_s \subset C_{G(s)^\circ}(D), [L^*, L^*] \subset C_{G^*}(\pi_s(D))\), hence \( C_{L^*}(s) \subset C_{G^*}(\pi_s(D)) \). This implies (A).

The description of \( A \) above shows that there exists \( x \in Z^\circ(L^*) \) and \( a \in A \) such that \( a(x) \neq x \). Thus \( A_{L^*}(s) \neq A, [A_{L^*}(s)] \leq 2 \) and the non-multiplicity condition (C) of Proposition 3.1.2 is satisfied.

Assume \([A_{L^*}(s)] = 2 \) and \( \rho(N_{G(s)^\circ}(D)) = A \). The group \( A \) stabilizes the block \( b_{G(s)^\circ}(L(s), \alpha) \).

In case \( m > 1 \) this implies \( m_1 = m_2, \tau(\alpha_1, \alpha_2) = (\lambda_2, \alpha_1), \gamma(\alpha_0) = \alpha_0, \delta_j(\alpha_j) = \alpha_j \). As in case (a.1) the all group \( \langle \tau, \gamma, \delta \rangle \) acts on \( C_{G(s)^\circ}(D) \) and fixes \( \alpha_D \) and (B) follows as in (a.1).

If \( m = 1 \) then \( d > 1, \tau \in A_{L^*}(s) \) and \( A_{L^*}(s) \neq A \) is equivalent to \( L_0 \neq G_0 \). Then \( \rho(N_{G(s)^\circ}(D)) = A \) is equivalent to \( \gamma(\alpha_0) = \alpha_0 \). Once again the all group \( \langle \tau, \gamma, \delta \rangle \) acts on \( C_{G(s)^\circ}(D) \) and fixes \( \alpha_D \) : (B) follows as in (a.1).

(b) Assume \( s^2 = 1 \) and \( C_{G_0}^\circ(s) \) has type \( D_m \times D_{n-m}, (2 \leq m < n/2) \) or \( D_{n-1} \) (special case \( m = 1 \)).

Then \([A_{G_0}(s)] = 2 \) and \( A \) is generated by the image in \( G^* \) of an element of \( SO_{2n} \cap (O_{2m} \times O_{2(n-m)}) \).

When \( m = 1 \) \( C_{G_0}^\circ(s) \) and \( G(s)^\circ \) are Levi subgroups of \( G^* \) and \( G \) respectively.

If \( L(s) = G(s)^\circ \), then \( D = \{1\} \) or \( m = 1 \) and \( d = 1 \). In case \( m = 1 \), \( G(s)^\circ = C_{G_0}(D)^F = M^F \) by Proposition 1.1.6. So (A) is satisfied.

Assume \( L(s) \neq G(s)^\circ \). As in (a.2) above one sees that \([L^*, L^*] \subset C_{G^*}(\pi_s(D))\), hence (A).

(c) Assume \( s^2 = 1 \) and \( C_{G_0}^\circ(s) \) is a Levi subgroup of type \( A_{n-1} \).

Then \( n \) is even, \( s \) is the image of \( \hat{s} \in SO_{2n} \) such that \( \hat{s}^4 = 1 \) and \( \hat{s} \) has two eigenvalues with equal multiplicities \( n/2 \). The group \( A \) has order 2 and is generated by a diagram automorphism. Then \( D = \{1\} \) or \( A_{L^*}(s) = \{1\} \).

### 3.2.5. Type \( E_6 \).

In that type if \( s \) is a quasi isolated semi-simple element of \( G^* \) and \( \vert A_{G_0^*}(s) \vert > 1 \), then \( \vert A_{G_0}(s) \vert = 3 \).

\( s^3 = 1 \) and \( C_{G_0}^\circ(s) \) has type \( (A_2)^3 \) (s isolated) or \( D_4 \) or \( s^6 = 1 \) and \( C_{G_0}^\circ(s) \) has type \( (A_1)^4 \).

One has only to verify that \( A_{L^*}(s)^F \) \( \subset \rho(C_{G_0^G}(D)) \) and a sufficient condition is that \( L^* \subset C_{G^*}(\pi_s(D)) \) (see 3.2.1). One sees that \( \phi_{ad} \) does not divides the generic degree of \( C_{G_0}^\circ(s) \) and the order of the Weyl group of \( C_{G_0}^\circ(s) \) is prime to \( \ell \) (recall that \( \ell \geq 5 \) in our assumption). It follows that any \( \ell \)-subgroup of \( G(s)^\circ \) (resp. \( C_{G_0}^\circ(s) \)) is contained in a \( \phi_{ad} \)-subgroup of \( G(s)^\circ \) (resp. \( C_{G_0}^\circ(s)^F \)). Recall that \( Z^\circ(L(s))_{\phi_{ad}}^\circ \) is a maximal \( \phi_{ad} \)-subgroup of \( C_{G_0}^\circ(s)^\circ([L(s), L(s)]_s) \) and \( D \) an \( \ell \)-Sylow subgroup of \( C_{G_0}^\circ(s)^\circ([L(s), L(s)]_s) \). So \( D \subset Z^\circ(L(s))_{\phi_{ad}}^\circ \) and \( \pi_s(D) \subset Z^\circ(L^*_s)_{\phi_{ad}} \). Hence \( L^* = C_{G^*}(Z^\circ(L^*_s)_{\phi_{ad}}) \subset C_{G^*}(\pi_s(D)) \).

### 3.2.6. Type \( E_7 \).

In all cases we have to consider \( \vert A_{G_0^*}(s) \vert = 2 \):

- \( s^2 = 1 \), \( C_{G_0}^\circ(s) \) has type \( A_3 \times A_3 \times A_1 \) or \( s^4 = 1 \), \( C_{G_0}^\circ(s) \) has type \( D_4 \times A_1 \times A_1 \) or \( s^6 = 1 \), \( C_{G_0}^\circ(s) \) has type \( A_2 \times A_2 \times A_2 \) or \( s^2 = 1 \), \( C_{G_0}^\circ(s) \) has type \( E_6 \) or \( s^2 = 1 \), \( C_{G_0}^\circ(s) \) has type \( A_7 \).

The generic polynomial order of \( E_7(q) \) is, in symbolic notations, \( 0^{63}.1^{7}.2^{7}.3^{3}.4^{2}.5.6^{3}.7.8.9.10.12.14.18 \).

The proof we gave in 3.2.5 is available with a special attention in the last two cases when \( \phi_5, \phi_7 \) divide the polynomial order of \( C_{G_0^*}(s) \).

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Assume $C_{G^*}(s)$ has type $E_6$. $C_{G^*}(s)$ is a Levi subgroup of $G^*$. If $L(s) = G(s)^\circ$ and $Z(L(s))^F_\ell \neq \{1\}$, then $d = 1$, $Z^0(L^*_s) \supseteq Z^0(L^*_s)_{\phi_1}$ and $L^* = L^*_s = \text{CG}^*(Z(L^*_s)^F_\ell)$, so that $L^* \subseteq \text{CG}^*(\pi_s(D))$. That proves (A).

More generally one sees that for any $d$ and any prime $\ell \geq 5$, denoting $E := \{d\ell^b \mid b \in \mathbb{N}, b > 1\}$, if $L(s)$ is a $d$-split Levi subgroup of $G(s)^\circ$ then $Z^0(L(s))_{\phi_E} = Z^0(L(s))_{\phi_\ell}$. This imply $Z^0(L^*_s)_{\phi_E} = Z^0(L^*_s)_{\phi_\ell}$ and $L^* = \text{CG}^*(Z(L^*_s)^F_\ell)$, hence $L^* \subseteq \text{CG}^*(\pi_s(D))$.

3.3. From minimal cases to general one

We prove Propositions 3.1.1 and 3.1.2 by induction on the semi-simple rank of $G$ and begin with the more evident result:

3.3.1. Direct product

If $(G, F)$ is a direct product $\times_j (G_j, F)$, so is $(G^*, F)$, $s = (s_j)_j$ ($s_j \in G_j^*)$, $A_{G^*}(s) = \times_j A_{G^*_j}(s_j)$ and the condition (A.1) define the sequence $\mathcal{E}(G, s)$ as a direct product of the $\mathcal{E}(G_j, s_j)$. Any of the groups $L(s)$, $L$, $L^*$, $D$, $C_{G(s)^\circ}(D)$, $N_{G(s)^\circ}(D)$ decomposes in a direct product and $\alpha_D, \beta_D$ in a tensor product. Assertions (B) of Proposition 3.1.1 and (A), (B) and (C) of Proposition 3.1.2 are true for $(G, s)$ if and only if they are true for each direct component.

3.3.2. Isotypic morphisms

The non-contradiction between conditions (A.1), (A.2) and (A.3) of Proposition 3.1.1 are easily verified. We precise here condition (A.3).

Let $\sigma: (G, F) \to (H, F)$ be an isotypic morphism and $\sigma^*: H^* \to G^*$ a dual one, let $s = \sigma^*(t)$, as in (3.1.1.(A.3)). If $T^*$ is a maximally split torus in $C_{G^*}(s)$, $(\sigma^*)^{-1}(T^*)$ is a maximally split torus in $C_{H^*}(t)$. The restriction of $\sigma^*$ to $C_{H^*}(t)$ is isotypic and defines a morphism of sequences

\[
\begin{align*}
\mathcal{E}(H^*, t) & \quad \quad C_{H^*}(t) \quad \quad C_{H^*}(t) \quad \quad A_{H^*}(t) \\
\mathcal{E}(G^*, s) & \quad \quad C_{G^*}(s) \quad \quad C_{G^*}(s) \quad \quad A_{G^*}(s)
\end{align*}
\]

There are tori $T$ in $G$ and $\sigma(T)$ in $H$ in duality with respectively $T^*$ and $(\sigma^*)^{-1}(T^*)$. So are defined the connected reductive groups $(H(t)^\circ, F)$, $(G(s)^\circ, F)$ in duality with respectivement $(C_{H^*}(t), F)$ and $(C_{G^*}(s), F)$. There is a morphism

$$\sigma_t: (G(s)^\circ, F) \to (H(t)^\circ, F)$$

defined by duality from the restriction of $\sigma^*$, and such that $\sigma_t$ and $\sigma$ have equal restriction on $N_{G(s)^\circ}(T)$.

Consider a given sequence

$$\mathcal{E}(G, s) \quad 1 \to G(s)^\circ \to G(s) \to A_{G^*}(s) \to 1$$

The kernel $K$ of $\sigma_t$ is central and contained in $T$, so it is stable under the group $\alpha(A_{H^*}(t))$. Let $G(t)$ be the inverse image of $\alpha(A_{H^*}(t))$ in $G(s)$. By condition (3.1.1.(A.3)) $\mathcal{E}(H, t)$ is given, isomorphic to

$$1 \to H(t)^\circ = G(s)^\circ / K \to H(t) = G(t)/K \to \alpha(A_{H^*}(t)) \to 1$$

Note that $G(t)^F/G(s)^\circ F$ is isomorphic to $H(t)^F/H(t)^\circ F$. If conditions (3.1.1.(A.1)) and (3.1.1.(A.2)) are satisfied in the construction of $\mathcal{E}(G, s)$, they are as well satisfied for $\mathcal{E}(H, t)$.

Isotypic morphisms may be composed. Using functoriality of fiber product, one verify easily the compatibility of the constructions defined above. An interesting and special case is when $H^*$ is adjoint, the proof of the following is left to the reader.
Let $\sigma: (G, F) \to (H, F)$ be an isotypic morphism and $\sigma^*: H^* \to G^*$ a dual one, let $s = \sigma(t)$. Let $\bar{t}$ be the image of $t$ and $s$ in a common adjoint group of $G^*$ and $H^*$ with isomorphic simply connected duals $H_{sc}, G_{sc}$. One has a commutative diagram of sequences

$$
\begin{array}{c}
\mathcal{E}(G_{sc}, \bar{t}) \xrightarrow{\sigma^*} \mathcal{E}(H_{sc}, \bar{t}) \\
\downarrow \quad \downarrow \\
\mathcal{E}(G, s) \longrightarrow \mathcal{E}(H, t)
\end{array}
$$

3.3.3. Lemma. Assume $G$ quasi-simple and simply connected. Then (B) in Proposition 3.1.1 and (A), (B), (C) in Proposition 3.1.2 are satisfied.

Proof. Thanks to the studies in in section 3.2, we may assume that $s$ is not rationally quasi-isolated in $G^*$. Let $K^* = C_{G^*}(Z^*(C_{G^*}(s)))$. Let $K$ be a Levi subgroup of $G$ in the dual $G^F$-conjugacy class. One has $A_{K^*}(s)^F = A_{G^*}(s)^F$. From $(K, K^*, s)$ one define an exact short sequence $1 \to K(s)^o \to K(s) \to A_{K^*}(s) \to 1$ and the subsequence

$$
1 \to K(s)^o \to K(s)^o K(s)^F \to A_{K^*}(s)^F \to 1
$$

is isomorphic to

$$
1 \to G(s)^o \to G(s)^o G(s)^F \to A_{G^*}(s)^F \to 1
$$

For any $d$-split Levi subgroup $L_s^*$ of $C_{G^*}(s)$ one has $C_{G^*}(Z^*(L_s^*)_{\phi_d}) \cap C_{G^*}(s)^F = C_{K^*}(Z^*(L_s^*)_{\phi_d}) \cap C_{G^*}(s)^F$. The conditions (B) in Proposition 3.1.1 and (A), (B), (C) in Proposition 3.1.2 are equivalent for the two exact sequences and satisfied by 3.2.

3.3.4. Lemma. Assume $G$ rationally irreducible, $G = [G, G]$ and $F(G) = \{1\}$. Then (B) in Proposition 3.1.1 and (A), (B), (C) in Proposition 3.1.2 are satisfied.

Proof. $G$ is a direct product $\prod_{1 \leq j \leq k} G_j$ where $G_j$ is irreducible and $F(G_j) = G_{j+1}$ ($j < k$), $F(G_k) = G_1$. On dual side $G^* = \prod_{1 \leq j \leq k} G_j^*$ and $F(G_j^*) = G_{j+1}^*$, $F(G_k^*) = G_1^*$, $s = (s_j)_{1 \leq j \leq k}$ and $F(s_j) = s_{j+1}$, $F(s_k) = s_1$. It follows that the short exact sequence

$$
\mathcal{E}(G^*, s) \quad 1 \to C_{G^*}^o(s) \to C_{G^*}(s) \to A_{G^*}(s) \to 1
$$

is isomorphic to

$$
\prod_{1 \leq j \leq k} \mathcal{E}(G_j^*, s_j) \quad 1 \to \prod_{1 \leq j \leq k} C_{G_j^*}^o(s_j) \to \prod_{1 \leq j \leq k} C_{G_j^*}(s_j) \to \prod_{1 \leq j \leq k} A_{G_j^*}(s_j) \to 1
$$

and the short exact sequence

$$
\mathcal{E}(G, s) \quad 1 \to G(s)^o \to G(s) \to A_{G^*}(s) \to 1
$$

is isomorphic to (see (A.1) in Proposition 3.1.1)

$$
\prod_{1 \leq j \leq k} \mathcal{E}(G_j^*, s_j) \quad 1 \to \prod_{1 \leq j \leq k} G_j(s_j)^o \to \prod_{1 \leq j \leq k} G_j(s_j) \to \prod_{1 \leq j \leq k} A_{G_j^*}(s_j) \to 1
$$
About groups of rational points one have isomorphisms $G^F \cong G_1^F$, $G^s \cong G_1^F$, $\mathcal{A}_G^*(s)^F \cong \mathcal{A}_G^*(s_1)^F$ and isomorphic extensions

\[
[1 \to G(s)^F \to G(s) \to G_1^F(s)^F \to 1] \cong [1 \to G_1(s_1)^F \to G_1(s_1)^F \to \mathcal{A}_G^*(s_1)^F \to 1],
\]

\[
[1 \to C_{G_2}^o(s)^F \to C_{G_2}^o(s)^F \to A_{G_2}^*(s)^F \to 1] \cong [1 \to C_{G_1}^o(s_1)^F \to C_{G_1}^o(s_1)^F \to \mathcal{A}_G^*(s_1)^F \to 1].
\]

Let $d' := d/(k, d)$. Then $G(s)^o_F$-conjugacy classes of $d$-cuspidal data of $(G(s)^o, F)$ correspond to $G_1(s_1)^F$-conjugacy classes of $d'$-cuspidal data of $(G_1(s_1)^o, F^k)$. The assertions (B) in Proposition 3.1.1 and (A), (B), (C) relative to $(G, F, s)$ and to $(G_1, F^k, s_1)$ are satisfied.  

3.3.5. Lemma. Assume $\mathcal{F}(G) = \{1\}$ and $G$ rationally irreducible. Then (B) in Proposition 3.1.1 and (A), (B), (C) in Proposition 3.1.2 are satisfied.

Proof. (i) Assume $Z^o(G) \cap [G, G] = \{1\}$.

Then $G$ is a direct product $G = Z^o(G) \times [G, G]$ and $G^* = Z^o(G)^* \times [G, G]^*$. Any object we consider in Propositions 3.1.1 and 3.1.2 decomposes in a direct product. For a torus there is nothing to prove, the result follows from the preceding Lemma.

(ii) Assume $Z^o(G) \cap [G, G] \neq \{1\}$.

One has $Z(G) = Z^o(G)Z([G, G])$ hence $Z(G)/Z^o(G)$ is cyclic, so is $A_{G^o}(s)$ and the assertions (A) of Proposition 3.1.1 and (C) of Proposition 3.1.2 are satisfied. In types B, C and exceptionnel types, $Z(G) = Z^o(G)$ and $A_{G^o}(s) = \{1\}$, there is nothing to prove.

Denote $H = [G, G]$. A morphism in duality with the inclusion $H \subseteq G$ is $i^*: G^* \to (G^*)_{ad}$. Then $A_{G^o}(s)$ is a subgroup of $A_{H^o}(i^*(s))$. (A.3) in Proposition 3.1.1 describes the relation between $\mathcal{E}(G, s)$ and $\mathcal{E}(H, t)$:

\[
\begin{array}{ccccccc}
\mathcal{E}(H, s) & \to & H(t)^o & \to & H(s) & \to & \alpha(A_{G^o}(s)) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \cong \\
\mathcal{E}(G, s) & \to & G(s)^o & \to & G(s) & \to & A_{G^o}(s) & \to & 1
\end{array}
\]

As $C_{G^o}^o(s) \to C_{H^o}^o(t)$ is onto, $\sigma_s$ is injective. Let $t$, $\rho_H$, $(L_H(t), \alpha_H)$, $L_t^*$, $L_H^*$, $L_H$, ... be defined as $s$, $\rho$, $(L(s), \alpha)$, $L_s^*$, $L^*$, $L$, $D$, $\alpha_D$, $\beta_D$, $s_1$, one has

\[
G = Z^o(G).H, \quad L(s) = Z^o(G).\sigma_s(L_H(t)), \quad \alpha_H = \text{Res}_{L_H^o(t)}^{L_s^o} \alpha, \quad L_t^* = \iota^*(L_s^*), \quad L_H^* = \iota^*(L^*).
\]

As $D$ is an $\ell$-Sylow subgroup of $C_{G(s)}^o([L(s), L(s)])^F$, $D \subseteq Z^o(G).\sigma_s(D_H)$, $\sigma_s(D_H) = D \cap [G, G]$. With notations of section 3.2.1, $\pi_{t,H}^o = \iota^* \circ \pi_s \circ \sigma_s$.

From (3.2.1.1) $[L_H^*, L_s^*] \subseteq C_{o,H}^e(\pi_{s,H}(D_H))$ one deduces $[L^*, L^*] \subseteq C_{G,H}^e(\pi_s(D))$ (Proposition 1.2.3). That gives (A) for $(G, s)$ and $C_{G(s)}^o(F)(D) = \sigma_s(C_{H(s)}^o(F(D_H)))$ and we may assume that $\beta_D$ is the restriction of $\beta_D$. One has $N_{G(s)}(D) = Z^o(G).\sigma_s(N_{H(s)}(D_H))$ so that condition (B) for $H$ implies condition (B) for $G$.

3.3.6. Lemma Let $\sigma: (H, F) \to (G, F)$ be an isotypic morphism whose kernel is a central torus and such that $\sigma(H) = G$, let $\sigma^*: G^* \to H^*$ be a dual one, and let $t = \sigma^*(s)$. If Condition (B) in Proposition 3.1.1 and Proposition 3.1.2 are true with $(H, t)$ instead of $(G, s)$, then they are true for $(G, s)$.  

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Proposition 3.1.2. is true for $(B)$ of Proposition 3.1.2 for $(G,s)$ is satisfied for $(H,F,t,\sigma)$. By (3.3.6.2) and (3.3.6.3) the assertions (A) in Proposition 3.1.2 for $(G,F,s,s)$ is a direct product \( [h,E] \). Let \( \alpha \) be a semi-simple in \( C^0_{G,s}(s) \) and we may write

\[
\chi \in \text{Irr}(C^0_{G,s}(s)) \quad \text{and} \quad \xi \in \text{Irr}(E). 
\]

Thus any \( h \in C^0_{G,s}(s) \) is isomorphic to \( H^* \). The assertions (A) in Proposition 3.1.2 for $(G,F,s,s)$ is satisfied for $(H,F,t,\sigma)$. By (3.3.6.1) we have \( C^0_{G,s}(s) \) is a semi-simple in \( C^0_{G,s}(s) \) \( \subseteq \) \( C^0_{G,s}(s) \). As \( \alpha \) is induced by the restriction of \( \chi \) to an isomorphism \( \alpha \) of \( (M(t),\beta) \) in $(H(t),F)$. Here \( M(t) = \sigma^{-1}(L(s)) \), so that \( \sigma \) induces a bijection \( \mathcal{E}(M(t)^F,1) \to \mathcal{E}(L(s)^F,1) \) and we may write \( \alpha = \beta \).

A defect group \( D \) of \( b_{G(s)}^F(L(s),\alpha) \) is an \( \ell \)-Sylow subgroup of \( C^0_{G(s)}([L(s),L(s)])^F \). We have \( D = \sigma(E) \) where \( E \) is an \( \ell \)-Sylow subgroup of \( C^0_{H(t)}([L(t),L(t)])^F \), that is a defect group of \( b_{H(t)}^F(L(t),\alpha) \). Then \( \sigma(N_{H(t)}^F(E)) \subseteq N_{G(s)}^F(D) \) and \( \sigma(C_{H(t)}^F(E)) \subseteq C_{G(s)}^F(D) \), hence \( \rho_H(N_{H(t)}^F(E)) \subseteq \alpha(\rho_G(N_{G(s)}^F(D))) \) and \( \rho_H(C_{H(t)}^F(E)) \subseteq \alpha(\rho_G(C_{G(s)}^F(D))) \).

Let \( a \in \rho_G(N_{G(s)}^F(D)) \). There exists \( g \in N_{G(s)}^F(D) \) of image \( a \) and order prime to \( \ell \) (because the order of \( a \) is prime to \( \ell \)). Hence there exists \( g \in H(s)^F \) of order prime to \( \ell \) and such that \( \sigma(h) = g \), \( [h,E] \subseteq K^F, E \). But \( E \cap K^F = K^F \) and \( K^F \) is central in \( H(s)^F \), so \( [h,E] \subseteq E \). If \( a \in \rho_G(N_{G(s)}^F(D)) \) then we obtain \( [h,E] \subseteq K^F \). But \( E \) is an \( \ell \)-group and \( h \) an \( \ell \)-element, so this implies \( [h,E] = \{1\} \). We get

\[
(3.3.6.1) \quad N_{G(s)}^F(D) = N_{H(t)}^F(E)/K^F, \quad C_{G(s)}^F(D) = C_{H(t)}^F(E)/K^F
\]

\[
(3.3.6.2) \quad \alpha(\rho_G(N_{G(s)}^F(D))) = \rho_H(N_{H(t)}^F(E)), \quad \alpha(\rho_G(C_{G(s)}^F(D))) = \rho_H(C_{H(t)}^F(E))
\]

On the other hand if \( L' \) := \( C_{H^*}(Z(L'_{H^*})) \) and \( L''' \) := \( C_{G^*}(Z(L'''_{G^*})) \) the isomorphism \( \alpha \) gives

\[
(3.3.6.3) \quad \alpha(A_{L'''}(s)) = A_{L'^*}(t)
\]

By (3.3.6.2) and (3.3.6.3) the assertions (A) in Proposition 3.1.2 for $(G,s)$ and for $(H,t)$ are equivalent.

By (3.3.6.1) we have \( C_{G(s)}(D)^F = C_{H(t)}(E)^F/K^F \). As \( \alpha_D \) is defined by restriction from \( \alpha \), the canonical \( \beta_E \in \text{Irr}(C_{H(t)}^F(E)) \) is defined by \( \beta = \alpha \). This shows that \( \alpha_D \) is just the restriction via \( \sigma \) of \( \beta_E \). Thus any \( \chi_0 \in \text{Irr}(b_D) \) is restriction of some \( \xi_0 \in \text{Irr}(b_E) \) (evident notations). The assertions (B) of Proposition 3.1.2 for $(G,s)$ and for $(H,t)$ are equivalent. If the non-multiplicity condition (C) in Proposition 3.1.2. is true for $(H,t)$ it is true for $(G,s)$.

3.3.7. End of proof

Proof. (a) Assume \( \mathcal{F}(G) = \{1\} \). Then the decomposition of \( [G,G] \) in rationally irreducible components is a direct product \( [G,G] = \prod_j G_j \). Let \( H_j = Z^*G_j,G_j \) and \( H = \prod_j H_j \). There is a natural morphism \( \sigma:H \to G \) whose kernel is a torus and such that \( \sigma(H) = G \). One concludes by 3.3.1 and Lemmas 3.3.5, 3.3.6.
(b) Given $G$ without special assumption, let $\sigma: H \to G$ a dual morphism of a regular embedding $G^* \to H^*$. One has $\mathcal{F}(H) = \{1\}$ and the kernel of $\sigma$ is a torus. One concludes by (a) and Lemma 3.3.6.

3.4. Jordan decomposition on blocks.

In this section we prove (B.1) of our main theorem 1.4. We have defined in 1.3.1 $\text{Bl}(G_F^s)$ as the set of blocks $B$ of $G$ such that $\text{Irr}(b) \cap \mathcal{E}(G_F^s, s) \neq \emptyset$ ($G$ connected), with the special case $\text{Bl}(G_F^s; 1)$ when $G$ is not connected.

3.4.1. Proposition. Let $(G, F, \ell, d)$ with assumption 2.2.3, $\sigma: (G, F) \to (H, F)$ a regular embedding, $\sigma^*: H^* \to G^*$ a dual morphism. Let $t$ be a semi-simple element of $(H^*)_t^F$ and $s = \sigma^*(t)$. Let

$$\mathcal{E}(G, s) = \{1 \to G(s)^{\circ} \to G(s) \to A_G(s) \to 1\}$$

as constructed in Proposition 3.1.1.

By Propositions 2.1.4 and 2.1.7 is defined a one-to-one map

$$\mathcal{B}_{H, t}: \text{Bl}(H(t)^F; 1) \to \text{Bl}(H_F^s; t)$$

$\mathcal{B}_{H, 1}$ is identity.

By compatibility with Clifford theory of blocks there exist a one-to-one map

$$\mathcal{B}_{G, s}: \text{Bl}(G(s)^F; 1) \to \text{Bl}(G^F; s)$$

such that

(3.4.1.1) If a block $b$ of $G(s)^F$ covers a block $b_0$ of $G(s)^{\circ}F$ and $B$ is the unique unipotent block of $H(t)^F$ that covers $b_0$ through the morphism $G(s)^{\circ}F \to H(t)^F$, then $\mathcal{B}_{G, s}(b)$ is covered by $\mathcal{B}_{H, t}(B)$.

Proof. $\mathcal{B}_{H, t}$ is defined by composition of the following one-to-one maps

- $B = b_{H(t)^F}(M(t), \alpha_i)$, $(M(t), \alpha_i)$, a unipotent $d$-cuspidal datum in $(H(t), F)$
- $\to H(t)^F$-conjugacy class of $(M(t), \alpha_i)$ (Proposition 2.1.7)
- $\to C_{H^*}(t)^F$-conjugacy class of $(M^*_t, \alpha)$, a unipotent $d$-cuspidal datum in $(C_{H^*}(t), F)$, (Proposition 2.1.4, $\alpha_i = \Psi_{M(t), 1}(\alpha), M^*_t$ in duality with $M(t)$)
- $\to H^F$-conjugacy class of $d$-cuspidal data $(M, \mu)$ in series $(t)$ in $(H, F)$ (Proposition 2.1.4, $\mu = \Psi_{M(t), \ell}(\alpha)$, $M^* \cap C_{H^*}(t) = M^*_t$)
- $\to b_{H(t)}(M, \mu)$ (Proposition 2.1.7).

The isotypic morphism $\sigma_t: (G(s)^{\circ}, F) \to (H(t), F)$ is an embedding, because the kernel of the dual morphism $C_{H^*}(t) \to C_{G^*}^0(s)$ is a torus, and it induces by restriction a one-to-one map

$$\text{Res}_{\sigma_t}: \mathcal{E}(H(t)^F, 1) \to \mathcal{E}(G(s)^{\circ}F, 1)$$

hence a one-to-one map between conjugacy classes of $d$-cuspidal unipotent data (Proposition 2.1.2)

$$\langle M(t), \alpha_i \rangle \mapsto \langle \sigma^{-1}(M(t)), \text{Res}_{\sigma_t}(\alpha_i) \rangle$$

hence a bijection between unipotent blocks (Proposition 2.1.7) $b_0 \mapsto B$ such that

$$\text{Res}_{\sigma_t}(\text{Irr}(B) \cap \mathcal{E}(H(t)^F, 1)) = \text{Irr}(b_0) \cap \mathcal{E}(G(s)^{\circ}F, 1)$$
The existence of $\mathcal{B}_{G,s}$ for any $G$ under condition (3.4.1) follows from a combinatorial fact: given corresponding unipotent blocks $b_0$ and $B$ of $G(s)^o \cdot \sigma$ and $H(t)^F$ respectively, let

1. $m_0$ be the number of blocks of $G(s)^F$ that cover $b_0$,
2. $n_0$ be the number of $G(s)^F$-conjugate of $b_0$,
3. $m$ be the number of blocks in series $(t)$ of $H^F$ that are in the $\text{Irr}(H^F/G^F)$-orbit of $\mathcal{B}_{H,\lambda}(B)$,
4. $n$ be the number of blocks in series $(s)$ of $G^F$ that are covered by $\mathcal{B}_{H,\lambda}(B)$.

Indeed one has $m = n_0$ and $n = m_0$.

Assume $b_0 = b_{G(s)^o \cdot \sigma}(L(s), \alpha)$, hence, if $B = b_{H(t)^F}(M(t), \alpha)$ as above, $L(s)$ is the inverse image of $M(t)$ by the morphism $G(s)^o \to H(t)$ defined from $\sigma$. By Propositions 3.1.3 and 2.4.4 we have

$$m_0 = |A_{L,s}(s)_F| = n, \quad n_0 = |A_{G,s}(s)^F/A_{G,s}(s,L^*)_F| = m$$

To the orbit of $\alpha$ under $A_{G,s}(s)^F$ there correspond a set $\text{Bl}(H^F;t)_{(\alpha)}$ of $m$ blocks of $H^F$ in series $(t)$ covering $n$ blocks in series $(s)$ of $G^F$. By the first part of the proof, $\text{Bl}(H^F;t)_{(\alpha)} = B_{H,t}(\text{Bl}(H(t)^F;1)_{(\alpha)})$, where $\text{Bl}(H(s)^F;1)_{(\alpha)}$ is a set of $m$ unipotent blocks of $H(s)^F$, hence in one-to-one map with a set $\text{Bl}(G(s)^o \cdot \sigma; s)_{(\alpha)}$ of $m = n_0$ unipotent blocks of $G(s)^o \cdot \sigma$, covered by $n = m_0$ (unipotent) blocks of $G(s)^F$.

### 3.4.2. Proposition

Hypothesis and notations of Proposition 3.4.1 on $(G, F, \ell, d, \sigma, H, t, s)$. Let $t_0$ in $C_{H^*}(t)_F$ and $s^*(t_0) = s_0$ in $C_{G,s}(s)_F$. Let $B \in \text{Bl}(H(t)^F;1), b \in \text{Bl}(G(s)^F;1)$. Define $\mathcal{E}(G(s)^F, (s_0))$ by

$$\mathcal{E}(G(s)^F, (s_0)) = \{ \mu | \mu \in \text{Irr}(G(s)^F, s) \text{ for some } \lambda \in \mathcal{E}(G(s)^o \cdot \sigma, s_0) \}$$

There exist one-to-one maps $\Psi_{H,B}$ from $\text{Irr}(B)$ onto $\text{Irr}(\mathcal{B}_{H,t}(B))$ and $\Psi_{G,b}$ from $\text{Irr}(b)$ onto $\text{Irr}(\mathcal{B}_{G,s}(b))$ such that,

1. For any $t_0 \in C_{H^*}(t)_F$, $\Psi_{H,B}(\text{Irr}(B) \cap \mathcal{E}(H(t)^F, t_0)) = \text{Irr}(\mathcal{B}_{H,t}(B)) \cap \mathcal{E}(H^F, tt_0)$

2. $\text{Irr}(\mathcal{B}_{G,s}(b)) \cap \mathcal{E}(G^F, ss_0) \subseteq \Psi_{G,b}(\text{Irr}(b) \cap \mathcal{E}(G(s)^F, (s_0)))$

3. For any $\mu \in \text{Irr}(b)$, $\Psi_{G,b}(\mu)(1).|G(s)^F|_{p'} = \mu(1).|G^F|_{p'}$

**Proof.** There may be fusion under $G(s)^F$ of Lusztig series in $G(s)^o \cdot \sigma$, that’s why we use a different notation in (3.4.2), see part (B) of the proof.

Let be $b_0$, $b$, $B = b_{H(t)^F}(M(t), \alpha_t), \alpha_t = \Psi_{M(t),1}(\alpha)$ as in Proposition 3.4.1 and its proof.

1. We first consider the case $G = H$.

2. We have partitions [9], [16] Theorem 9.12

$$\text{Irr}(B) = \cup_{(s_0)}(\text{Irr}(B) \cap \mathcal{E}(H(t)^F, t_0)),$$

$$\text{Irr}(\mathcal{B}_{H,t}(B)) = \cup_{(h)}(\text{Irr}(\mathcal{B}_{H,t}(B)) \cap \mathcal{E}(H^F, h))$$

In the left equality $(s_0)$ runs on the set of $C_{H^*}(t)^F$-conjugacy classes of $\ell$-elements of $C_{H^*}(t)^F$. In the right one $(h)$ runs on the set of $H^*F$-conjugacy classes of semi-simple elements $h$ of $H^*F$ whose $\ell'$-component
$h_{ν}$ is $H^*F$-conjugate of $t$. The map $(t_0) ↦ (tt_0)$ is a one-to-one map between these two sets of conjugacy classes.

To prove (3.4.2.1) we define the restriction of $Ψ_{H,B}$

$$\text{Irr}(B) ∩ \mathcal{E}(H(t)^F, t_0) → \text{Irr}(\mathcal{B}_{H,t}(B)) ∩ \mathcal{E}(H^F, tt_0)$$

These two sets are described in Proposition 2.3.5. They are in bijection with sets of components of two Generalized d-H.C. series: replace in 2.3.5 $(G, s, a_0, L, λ, α, α_0)$ by $(H(t), 1, t_0, M(t), Ψ_{M(t_0), t}(α), α(t_0))$ and then, with $μ = Ψ_{M,t}(α)$, by $(H, t, t_0, M, μ, α, α(t_0))$.

On left side, unipotent block side, the source of $Ψ_{H,B}$, $\text{Irr}(B) ∩ \mathcal{E}(H(t)^F, t_0)$, if not empty, is in bijection with $\mathcal{E}(H(t)(t_0)^F, (M(t)(t_0), α(t_0)))$, where $H(t)(t_0)$ is a Levi subgroup of $H(t)$ in the dual $H^F$-conjugacy class of $C_{H^*}(t_0) = C_{H^*}(tt_0)$ and $(M(t)(t_0), α(t_0))$ is a $d$-cuspidal unipotent datum in $(H(t)(t_0), F)$, associated by duality to a $d$-cuspidal unipotent datum $(M^*_{t_0}, α(t_0))$ in $(C_{H^*}(t_0), F)$ such that, $M^*_t$ being in the dual $C_{G^*}(t)^F$-conjugacy class of the $H^F$-conjugacy class of $M(t)$, $(M^*_{t_0}, α(t_0)) ∼_{C_{H^*}(t)^F} (M^*_t, α)$. The map is $R^H_{H(t)^F}(Ψ_{H(t_0), t_0}(1) ⊗ -)$

(3.4.2.4) $ν ↦ ξ = R^H_{H(t)^F}(Ψ_{H(t_0), t_0}(1) ⊗ ν), \ ν(1).|C_{H^*}(t)^F|_{p'} = ξ(1).|C_{H^*}(t_0)^F|_{p'}$ the equality thanks to (1.3.1.2) and knowing that by duality $|H^F| = |C_{H^*}(t)^F|, |H(t)(t_0)^F| = |C_{H^*}(t_0)^F|$. On right side, $\text{Irr}(\mathcal{B}_{H,t}(B)) ∩ \mathcal{E}(H^F, tt_0)$, if not empty, is in bijection with $\mathcal{E}(H(t_0)^F, (M(t_0), μ(t_0)))$, where $H(t_0)$ is a Levi subgroup of $H$ in the dual conjugacy class of $C_{H^*}(t_0)$, $(M(t_0), μ(t_0))$ is a $d$-cuspidal datum in series $(t)$ in $(H(t_0), F)$ associated by Proposition 2.1.4 to a $d$-cuspidal unipotent datum $(M^*_{t_0}, α_0(t_0))$ in $(C_{H^*}(tt_0), F)$ such that $(M^*_{t_0}, α(t_0)) ∼_{C_{H^*}(t)^F} (M^*_t, α)$. Clearly, as $C_{C_{H^*}(t)^F}(tt_0) = C_{H^*}(tt_0)$, the existence of such a $d$-cuspidal datum $(M^*_{t_0}, α(t_0))$ is equivalent to the preceding one $(M^*_t, α(t_0))$ and we may assume $(M^*_{t_0}, α(t_0)) = (M^*_{t_0}, α_0(t_0))$. The map is $R^H_{H(t_0)^F}(Ψ_{H(t_0), t_0}(1) ⊗ -)$

(3.4.2.5) $ζ ↦ η = R^H_{H(t_0)^F}(Ψ_{H(t_0), t_0}(1) ⊗ ζ), \ ζ(1).|H^F|_{p'} = η(1).|C_{H^*}(t_0)^F|_{p'}$ by (1.3.1.2) and equality $|H(t_0)^F| = |C_{H^*}(t_0)^F|$. By Proposition 2.2.4 there exist one-to-one maps

$$\tilde{Ψ}_{H(t_0), t}(M(t)(t_0), μ(t_0)) : \mathcal{E}(C_{H^*}(tt_0)^F, (M^*_{t_0}, α(t_0)))) → \mathcal{E}(H(t)(t_0)^F, (M(t)(t_0), α(t_0)))$$

$$\tilde{Ψ}_{H(t_0), t}(M(t)(t_0), μ(t_0)) : \mathcal{E}(C_{H^*}(tt_0)^F, (M^*_{t_0}, α(t_0)))) → \mathcal{E}(H(t_0)^F, (M(t_0), μ(t_0)))$$

The restriction of $Ψ_{H,B}$ we are looking for is defined by the commutation formula

$$R^H_{H(t_0)}(Ψ_{H(t_0), t_0}(1) ⊗ -) \circ Ψ_{H(t_0), t}(M(t), μ(t_0))$$

$$= Ψ_{H,B} \circ R^H_{H(t_0)^F}(Ψ_{H(t_0), t_0}(1) ⊗ -) \circ Ψ_{H(t_0), t}(M(t)(t_0), α(t_0)).$$

From (2.2.4.1) we have, if $ν_0 ∈ Ε(C_{H^*}(tt_0)^F, (M^*_{t_0}, α(t_0))))$, $ν = Ψ_{H(t_0), t}(M(t)(t_0), α(t_0))(ν_0)$, $ν(1) = ν_0(1)$ (trivial case, $t$ is central in the dual of $H(t)(t_0)$) and if $ζ = Ψ_{H(t_0), t}(M(t)(t_0), μ(t_0))(ν_0)$,

$$ζ(1).|C_{H^*}(t_0)^F|_{p'} = ν(1).|C_{H^*}(t_0)^F|_{p'}.$$

With notations of (3.4.2.4) and (3.4.2.5) we obtain $η(1).|C_{H^*}(t)^F|_{p'} = ξ(1).|H^F|_{p'}$ whereas $Ψ_{H,B_0}(ξ) = η$ and $|C_{H^*}(t)^F| = |H(t)^F|$, that is (3.4.2.3) for $(H, t)$. 71
(B) Assuming now $Z(G)$ non-connected we may assume that type $E_8$ don’t appear in $G$ (see first lines of 2.4 or 3.2.1). Then by the proof of Proposition 2.2.4 the one-to-one map $\tilde{\Psi}_{H(t),t_0}(M(t),\alpha(t_0))$ (resp. $\tilde{\Psi}_{H(t),t_0}(M(t_0),\mu(t_0)))$ is just the restriction of Jordan decomposition $\Psi_{H(t),t_0}$ (resp. $\Psi_{H(t),t_0}$). Hence $\xi = \Psi_{H(t),t_0}$ for some $\beta \in \mathcal{E}(C_{H^*}(tt_0))^F$, $(M^*_t,\alpha(t_0)))$ and $\Psi_{H,B_0}(\xi_0) = \Psi_{H,B_0}(\beta)$. 

To prove the existence of $\Psi_{G,b}$ we proceed as in the proof of the preceding Proposition, using properties of the functions $\Psi$. We go from $(H(t)^F, H^F)$ to $(G(s)^F, G^F)$ in three steps through three restrictions via

$$G^F \to H^F, \quad G(s)^o F \to H(t)^F, \quad G(s)^o F \to G(s)^F$$

Thus we rely $\Psi_{G,b}$ to $\Psi_{H,B}$ by the condition

$$(3.4.2.6) \quad \text{If } \xi \in \text{Irr}(B) \text{ and } \mu \in \text{Irr}(b) \text{ cover } \lambda \in \text{Irr}(G(s)^o F), \text{ then } \Psi_{H,B}(\xi_0) \text{ covers } \Psi_{G,b}(\mu).$$

The three morphisms above have common properties: the kernel is $\{1\}$, the image is an invariant subgroup, the cokernel is abelian and there is no multiplicity in restrictions of irreducible representations (Propositions 1.3.4 and 3.1.1, (B)). In such a morphism $X \to Y$ there is a bijection between sets of orbits $\text{Irr}(Y)/\text{Irr}(X)$ indexed by $\eta \in \text{Irr}(Y | \chi)$, where $\chi \in \text{Irr}(X)$. Then $\chi$ is covered by $|Y/\chi|$ elements of $\text{Irr}(Y)$ and $\eta$ covers $|Y/\chi|$ elements of $\text{Irr}(X)$. We say that the “multiplicative factor” on the number of irreducible in corresponding orbits from $\text{Irr}(Y)$ to $\text{Irr}(X)$ is $|Y/\chi|/|Y/\chi|^2$.

The three maps restrict to series.

Consider first the stabilizer in $G(s)^F$ of $\mathcal{E}(G(s)^o F, s_0)$, or of the $C_{G^*}(s)^F$-conjugacy class of $s_0$ in $C_{G^*}(s)^F$. It has image $C_{G^*}(s)(s_0)^F.C_{G^*}(s)/C_{G^*}(s)$ in $A_{G^*}(s)^F$. We have $C_{G^*}(G^*(s_0)^F)$ is isomorphic to $C_{G^*}(s)^F \cap C_{G^*}(s_0)^F/C_{G^*}(s)^F \cap C_{G^*}(s_0)^F$ and that quotient is the component $A_{G^*,(s_0)}^F$, $A_{G^*}(s_0)^F$ we obtained in Proposition 1.2.4.

A block $b$ of $G(s)^F$ covers a block $b_0$ of $G(s)^o F$ if and only if some element of $\text{Irr}(b)$ covers an element of $\text{Irr}(b_0)$. When a block $b$ of $G(s)^F$ covers $b_0$ and a block $B$ of $H(t)^F$ covers $b_0$, then $\mathcal{B}_{H,t}(b)$ covers $\mathcal{B}_{G,s}(b)$ and there are similar partitions through series. The three restrictions above send series in series, so we consider separately sets of irreducible components

$$\eta \in \text{Irr}(\mathcal{B}_{H,t}(B)) \cap \mathcal{E}(H^F,tt_0) \mapsto \{ \chi \in \text{Irr}(\mathcal{B}_{G,s}(b)) \cap \mathcal{E}(G^F,s_0) \mid \eta \in \text{Irr}(H^F | \chi) \}$$

$$\xi \in \text{Irr}(B) \cap \mathcal{E}(H(t)^F,t_0) \mapsto \{ \lambda \in \text{Irr}(b_0) \cap \mathcal{E}(G(s)^o F,s_0) \mid \xi \in \text{Irr}(H(t)^F | \lambda) \}$$

$$\mu \in \text{Irr}(b) \cap \mathcal{E}(G(s)^F,s_0) \mapsto \{ \lambda \in \text{Irr}(b_0) \cap \mathcal{E}(G(s)^o F,s_0) \mid \mu \in \text{Irr}(G(s)^F | \lambda) \}$$

Thanks to the definition of $\Psi_{H,B}$, to any $\eta \in \text{Irr}(\mathcal{B}_{H,t}(B)) \cap \mathcal{E}(H^F,tt_0)$, or $\xi \in \text{Irr}(B) \cap \mathcal{E}(H(t)^F,t_0)$, we may associate some $\beta \in \mathcal{E}(C_{H^*}(tt_0)^F)$ such that $\xi = \Psi_{H,(t),t_0}(\beta)$ and $\Psi_{H,B}(\xi_0) = \Psi_{H,B}(\beta)$. Then, by Propositions 1.3.6, 1.3.7, any component of $\text{Res}_{G^*}^{H^F} \eta$ or of $\text{Res}_{G(s)^o F}^{H(t)^F} \xi$ is associated to an orbit of $\beta$ under $A_{G^*}(s_0)^F$. To compute the effect of the three restrictions on the number of elements in the sets of irreducible, and their degrees, we may choose the orbit of $\beta$ and apply 1.3.6, 1.3.7.

(i) From $H^F$ to $G^F$:

If $\eta = \Psi_{H,t_0}(\beta)$, $\chi \in \text{Irr}(G^F)$ and $\eta \in \text{Irr}(H^F | \chi)$, $|A_{G^*}(s_0)^F| \mid H^F$-conjugate of $\chi$ are covered by $|A_{G^*}(s_0)^F/A_{G^*}(s_0)^F| \mid$ elements of $\mathcal{E}(H^F,tt_0)$. Therefore

$$(3.4.2.7) \quad \chi(1), |A_{G^*}(s_0)^F| = \eta(1)$$

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and the “multiplicative factor of Res$^H_F$ above $\beta$” is

$$(3.4.2.8) \quad m_\beta(G^F \to H^F) = |A_{G^*}(ss_0)^F|/|A_{G^*}(ss_0)^F|$$

(ii) From $H(t)^F$ to $G(s)^oF$ :

By Proposition 1.2.4 $\cal F(C_H(t))$ is prime to $\ell$ so that $C_{C_{H^*}(t)}(t_0)$ is connected. By Proposition 1.3.7, if $\xi = \Psi_{H(t),t_0}(\beta)$ as in (A) and $\xi$ covers $\lambda \in \cal E(G(s)^oF, s_0)$, then $\lambda$ has $|A_{C_{G^*}(s)}(ss_0)^F| H(t)^F$-conjugates, so that

$$(3.4.2.9) \quad \lambda(1)|A_{C_{G^*}(s)}(ss_0)^F| = \xi(1)$$

and $\lambda$ is covered by $|A_{C_{G^*}(s)}(ss_0)^F|/|A_{C_{G^*}(s)}(ss_0)^F|$ elements of $\cal E(H(t)^F, t_0)$. The multiplicative factor of Res$^{H(t)^F}_{G(s)^oF}$ above $\beta$ is

$$(3.4.2.10) \quad m_\beta(G(s)^oF \to H(t)^F) = |A_{C_{G^*}(s)}(ss_0)^F|^2/|A_{C_{G^*}(s)}(ss_0)^F|$$

(iii) From $G(s)^F$ to $G(s)^oF$ :

We need to compute the stabilizer in $G(s)^F$ of $\lambda \in \cal E(G(s)^oF, s_0)$. To study the action of $A_{C_{G^*}(s)}(ss_0)^F$ on $\cal E(G(s)^oF, s_0)$ we consider a regular embedding

$$\sigma_*: (G(s)^o, F) \to (\tilde{H}(\tilde{t}), F) = (T \times_{Z(G(s)^o)} G(s)^o, F)$$

where $T$ is the maximal $F$-stable torus of $G(s)^o$ giving rise to the dual root data $\cal D(G(s)^o, T), \cal D(C_{G^*}(s), T^*)$ that define $G(s)^o$ as a dual of $C_{G^*}(s)$. The group $A_{G^*}(s)$ acts on $T, G(s)^o, Z(G(s)^o)$. Hence $A_{G^*}(s)$ acts on $\cal D(\tilde{H}, \tilde{T})$, where $\tilde{T} = T \times_{Z(G(s)^o)} T$, by transport via $\cal D(\sigma_*)$ of its action on $\cal D(G(s)^o, T)$. Let $\sigma_*^*: \tilde{H}^* \to C_{G^*}(s)$ a morphism dual of $\sigma_*$. There exist $\tilde{t}_0 \in (\tilde{H}^*)_F^0$ such that $\sigma_*^*(\tilde{t}_0) = s_0$ and $A_{C_{G^*}(s)}(ss_0)^F$ fixes $\tilde{t}_0$ : if $\theta = \Psi_{T, t_0}(1)$, define $\tilde{t}_0$ by $\Psi_{\tilde{H}, \tilde{t}_0}(1) = \theta^{-1} \sigma_*^*(\tilde{t}_0) \theta$.

Acting on $\cal D(\tilde{H}, \tilde{T})$, and on $\cal D(\tilde{H}^*, \tilde{T}^*)$ by transposition, $A_{G^*}(s)$ has an image in the groups of outer automorphisms of $\tilde{H}$ and $\tilde{H}^*$. By construction for any $a \in A_{G^*}(s)^F$ there exist a pair $(\tau(a), \tau^*(a))$ of dual automorphisms of $\tilde{H}$ and $\tilde{H}^*$ with image $(a, \tau^*a)$ such that $\tau(a)$ and $\tau^*(a)$ commute with $F$, $\tau(a)$ (resp. $\tau^*(a)$) stabilizes $G(s)^o$ (resp $C_{G^*}(s)$). The stabilizer of $\cal E(\tilde{H}(t), \tilde{t}_0)$ in $A_{G^*}(s)^F$ is $A_{C_{G^*}(s)}(ss_0)^F$. The isotypic morphism $\sigma_*^*: C_{\tilde{H}, \tilde{t}_0} \to C_{G^*}(s)^F$ allows us to identify unipotent series, so that $\beta \in \cal E(C_{\tilde{H}, \tilde{t}_0}(1), 1)$. To $\beta$ there correspond $|A_{C_{G^*}(s)}(ss_0)^F|$ elements of $\cal E(G(s)^oF, s_0)$, the components of Res$^F_{G(s)^oF}(\Psi_{\tilde{H}, \tilde{t}_0}(\beta))$ (Proposition 1.3.6) , where $\lambda$.

By (iv) in Proposition 1.3.2, if $a \in A_{C^*_{G^*}(s)}(ss_0)^F$ and $^a\beta$ is the restriction of $\beta$ to $C_{\tilde{H}, \tilde{t}_0}(1)^F$ via $\tau^*(a)$ then $\Psi_{\tilde{H}, \tilde{t}_0}(\beta)$ is the restriction of $\Psi_{\tilde{H}, \tilde{t}_0}(^a\beta)$ via $\tau(a)$. As $(\tau(a), \tau^*(a))$ restricts to $(G(s)^oF, C_{G^*}(s)^F)$, Res$^F_{G(s)^oF}(\Psi_{\tilde{H}, \tilde{t}_0}(\beta)) = Res^F_{G(s)^oF}(\Psi_{\tilde{H}, \tilde{t}_0}(^a\beta)) \circ \tau(a)$. If $^a\beta \neq \beta$, then the orbits of $\beta$ and of $^a\beta$ under $A_{C_{G^*}(s)}(ss_0)^F$ are disjoint, thanks to the decomposition in direct product of $A_{G^*}(ss_0)$ (Proposition 1.2.4). That implies $A_{G^*}(ss_0)^F = A_{C^*_{G^*}(s)}(ss_0)^F \times A_{C^*_{G^*}(s)}(ss_0)^F$. Hence $G(s)^F/G(s)^oF \leq A_{C^*_{G^*}(s)}(ss_0)^F$ whereas the group $A_{C^*_{G^*}(s)}(ss_0)^F$ acts on the set of irreducible components of Res$^F_{G(s)^oF}(\Psi_{\tilde{H}, \tilde{t}_0}(\beta))$, that is a regular orbit under $A_{C^*_{G^*}(s)}(ss_0)^F$. As $[A_{C^*_{G^*}(s)}(ss_0), A_{C^*_{G^*}(s)}(ss_0)] = \{1\}$, any element of that orbit is fixed by $A_{C^*_{G^*}(s)}(ss_0)^F$. 73
We have find $G(s)^F / G(s)^F = A_{G^F, (s)}(s)^F$. If $\mu \in \text{Irr}(G(s)^F | \lambda)$,

$$(3.4.2.11) \quad \lambda(1) \cdot |A_{G^F, (s)}(s)^F| = \mu(1) \cdot |A_{G^F, (s)}(s)^F|$$

and $|A_{G^F, (s)}(s)^F|$ elements of $\mathcal{E}(G(s)^F, (s_0))$ are covering $|A_{G^F, (s)}(s)^F / A_{G^F, (s)}(s)^F|$ elements of $\text{Irr}(G(s)^F)$. If $\gamma$ is in the orbit of $\beta$ under $A_{G^F, (s)}(s_0)^F$ we have $|A_{G^F, (s)}(s)^F| = |A_{G^F, (s)}(s)^F|$. We have seen that $|A_{G^F, (s)}(s)^F / A_{G^F, (s)}(s)^F|$ series of $G(s)^F$ are fused under $G(s)^F$. Hence the multiplicative factor we are looking for is

$$m_{\beta}(G(s)^F \to G(s)^F) = |A_{G^F, (s)}(s)^F| / |A_{G^F, (s)}(s)^F|^2$$

Our claim on the existence of $\Psi_{G, b}$ follows from part (A) of the proof, (3.4.2.6), and the equality

$$m_{\beta}(G \to H^F) = m_{\beta}(G(s)^F \to H(t)^F) / m_{\beta}(G(s)^F \to G(s)^F)$$

easy to verify thanks to (3.4.2.8) and (3.4.2.10) and the preceding equality.

As for degrees, recall (3.4.2.3) for $\langle H, \xi \rangle$ proved in (A): $\eta(1) \cdot |C_{H^F, (t)}^F|_{\nu'} = \xi(1) \cdot |H^F|_{\nu'}$. In the morphism $\sigma^* ; H^* \to G^*$, $H^* F$ maps onto $G^* F$, $C_{H^F, (t)}^F$ onto $C_{G^F, (s)}^F$ and $C_{H^F, (t)}^F$ onto $C_{G^F, (s)}^F$, so that $|H^F|_{\nu'} = |G^F|_{\nu'}$. Thus

$$\eta(1) \cdot |G(s)^F|_{\nu'} = \xi(1) \cdot |G^F|_{\nu'} \cdot |A_{G^F, (s)}(s)^F|$$

With (3.4.2.7), (3.4.2.9), (3.4.2.11) and the isomorphism $C_{G^F, (s)}^F \to C_{G^F, (s)}^F \times C_{G^F, (s)}^F$ we obtain $\chi(1) \cdot |G(s)^F|_{\nu'} = \mu(1) \cdot |G^F|_{\nu'}$ that is (3.4.2.3) for $(G, \mu)$.
4. Brauer categories

4.1. On defect groups

The construction of a maximal subpair for a block $b_{G^F}(L, \lambda)$ (see 2.1.7) is given in [15] sections 4 and 5.1. First we describe defect groups.

4.1.1. Proposition. Assumption 2.2.3 on $(G,F,\ell,d)$. Let $s$ be a semi-simple element in $(G^*)^F$. Let $(L^*_s, \alpha)$ be a unipotent $d$-cuspidal datum of $(C^0_{G^*}(s), F)$. Let $(L, \lambda)$ be one of the $d$-cuspidal data in $(G,F)$ in series (s), associated to $(L^*_s, \alpha)$ by Proposition 2.1.4. Let $T^*$ be an $F$-stable maximal torus in $L^*_s$ such that $T^* \cap G^*_{b^*}$ is maximally split in $L^*_s \cap G^*_{b^*}$. Let $(T, \theta) (\theta \in \text{Irr}(T^*))$ be in duality with $(T^*, s)$ (where $s \in (T^*)^F$).

Let $D$ be an $\ell$-subgroup of $N_G(T)_F^\ell$, maximal for the property

“For any $r \in \Phi_{L^*_s}(T^*)$, the one parameter subgroup $Y_r$ is contained in $C_G(D)$”.

Then $D$ is a defect group of $b_{G^F}(L, \lambda)$.

Proof. If $s = 1$, then $\theta = 1$ and $L$ is in the dual class of $L^*_1 = L^*$ and the condition is a characterization of an $\ell$-Sylow subgroup of $C := C^0_{G^*}([L,L])^F$, that is an $\ell$-Sylow subgroup of $N_G(T \cap C)^F$ [12] Theorem 4.4.

Following 1.1.5 one has decompositions in central products $G = G_aG_b$, $G^* = G^*_aG^*_b$. The isogenies

\[ G^*_b \to (G_b)^*, G_b \to (G^*_b)^*(1.1.5.4) \]

restrict in isomorphisms between finite $\ell$-subgroups of $F$-fixed points. One has a central product

\[(4.1.1.1) \quad C^0_{G^*}(s) = C^0_{G^*_a}(s)C^0_{G^*_b}(s).\]

If $G = G_a$, then there is only one conjugacy class of unipotent $d$-cuspidal data, say $(T, 1_{T^F})$, in $(G,F)$, where $T$ is a diagonal torus. Hence $L^*_s \cap C^0_{G^*_a}(s)$ is a diagonal torus of $C^0_{G^*_a}(s)$ (see Proposition 2.1.5).

In [15] § 4 a defect-group $D$ of $b_{G^F}(L, \lambda)$ is described as follows:

Let $M$ and $M^*$ be dual $E$-split Levi subgroups of $G$, $G^*$ respectively such that

\[(4.1.1.2) \quad M \cap G_a = T \cap G_a, \quad M^* \cap G^*_b = C^0_{G^*_b}(Z(L^*(s) \cap G^*_b))^F_{\ell} \]

($T$ as above, duality around $(T,T^*)$). Then $D$ admits a unique maximal normal abelian subgroup $Z$ such that

\[(4.1.1.3) \quad M = C^0_{G^*}(Z), \quad Z \cap G_a = (T \cap G_a)^F_{\ell}, \quad Z \cap G_b = Z(M \cap G_b)^F_{\ell} \]

Let $Q$ be the subgroup of $W(C^0_{G^*}(s), T^*)$ generated by reflections relatives to the roots

\[ r \in \Phi_{C^0_{G^*}(s)}(T^*) \triangleleft \Phi_{L^*_s}(T^*) \]

let $V$ be an $\ell$-Sylow subgroup of $Q^F$. With these notations $N_{G^F}(Z) \cap N_G(T)$ contains a defect group $D$ of $b_{G^F}(L, \lambda)$ such that $D \cap T = Z$, $DT/T \subseteq W(G,T)$ is anti-isomorphic to $V$ and the extension $Z \to D \to D/Z$ is split [15] Lemma 4.16.

When $G = G_a$, $L^*_s$ is a diagonal torus of $C^0_{G^*}(s)$, the characterization of $Q$ reduces to “$D$ is an $\ell$-Sylow subgroup of $N_G(T)^F \cap C^0_{G^*}(\theta)$”, (where $C^0_{G^*}(\theta)$ is in the dual $G^F$-conjugacy class of the Levi subgroup $C^0_{G^*}(s)$) that is an extension of $Z = T^F_{\ell}$ by an $\ell$-Sylow subgroup of $W(C^0_{G^*}(\theta), T)^F \cong W(C^0_{G^*}(s), T^*)^F$.

Assume now $G = G_b$. We claim that

$M^*$ is the smallest of $E$-split Levi subgroups of $G^*$ that contain $L^*_s$. 75
By (4.1.1.2) one has $L_s^* = (T^* \cap G_{s, b}^*)(L_s^* \cap G_{b}^*) \subseteq M^*$, hence $Z(M^*)^F \subseteq Z(L_s^*)^F$. It follows, using (4.1.1.2) again, that $Z(L_s^*)^F = (T^* \cap G_{s, a}^*)^FZ(L_s^* \cap G_{b}^*)^F \subseteq Z(M^*)^F$. Thus $Z(M^*)^F = Z(L_s^*)^F$. But $Z(M)^F = (Z(M)^F \cap \Phi E)^F$, $Z(L_s^*)^F = (Z(L_s^* \cap G_{s, b}^*)^F \cap \Phi E)$ by Proposition 1.2.5, and $Z(M^*) \subseteq T^* \subseteq L_s^* \cap M^*$ so that $Z(M)^F \subseteq (Z(L_s^*)^F)_{|\Phi E}$. The definition of $E(1.1.5.1)$ implies $Z(M)^F \cap \Phi E = Z(L_s^*)_{|\Phi E}$.

Now let $Z_1 := T^F \cap (\cap_{\alpha \in \Phi L_s^*: (T^*)} \text{Ker} \alpha)$. We claim that $Z_1 = Z$. As $\Phi L_s^*: (T^*) \subseteq \Phi M^*: (T^*)$, using (4.1.1.3), one has $Z \subseteq Z(M)^F \subseteq Z_1$. Let $M_1 := \Phi G_{s}^*(Z_1)$. Thanks to our hypotheses "$t$ is good and $G = G_{b}^*$", $M_1$ is an $E$-split Levi subgroup of $G$. By definition of $Z_1$ and $M_1$, $Z_1 \subseteq Z(M_1)^F$ and, if $r^\vee \in \Phi L_s^*: (T^*)$, then $r \in \Phi M_1: (T)$. Let $M_1^*$ be the Levi subgroup of $G^*$ that contains $T^*$ and in duality with $M_1$, then $M_1^*$ is $E$-split and $\Phi L_s^*: (T^*) \subseteq \Phi M_1^*: (T^*)$. Thus $M_s^* \subseteq M_1^*$ so that $M^* \subseteq M_1^*$, $M \subseteq M_1$ and $Z \subseteq Z$.

Now let us consider Weyl’s groups : as $L_s^*$ is a Levi subgroup of $C_{G_{s}}^0(s)$, $Q$ is the subgroup of elements of $W(C_{G_{s}}^0(s), T^*)$ that fix any $r \in \Phi L_s^*: (T^*)$. Then Steinberg’s relations show that $Q$ is the image of $N \subseteq N_G(T)$ where $N$ centralizes any one parameter subgroup $Y_r$ of $G$ associated to some $r \in \Phi L_s^*: (T^*)$ and $N \cap T$ is a finite 2-group. As $|V|$ is odd there is a subgroup $\hat{V}$ in $N_G(T)^F$, with $\hat{V}.T/T = V$ and such that $[\hat{V}, Y_r] = \{1\}$ for any $\alpha \in \Phi L_s^*: (T^*)$. The semi-direct product $Z.\hat{V}$ is a defect group of $b_{GF}(L, \lambda)$.

Going from $G_a$ and $G_b$ to the central product $G = G_aG_b$ is quite easy. When $s$, $L_s^*$ et $T^*$ are fixed $(T, \theta)$ is defined mod $GF$-conjugacy, thus we have defined a $GF$-conjugacy class of $t$-subgroups of $GF$, it is the set of defect groups of $b_{GF}(L, \lambda)$.

As $G(s)^F/G(s)^oF$ is prime to $t$, if a block $b$ of $G(s)^F$ covers a block $b_0$ of $G(s)^oF$, a defect group of $b_0$ is a defect group of $b$.

4.1.2. Proposition. Assumption 2.2.3 on $(G, F, \ell, d)$. Let $s$ be a semi-simple element in $(G^*)^F$. Let $(L_s^*, \alpha)$ be a unipotent $d$-cuspidal datum in $(C_{G_{s}}^0(s), F)$. Let $(L, \lambda)$ (resp. $(L(s), \alpha)$) be one of the $d$-cuspidal data in $(G, F)$ in series $(s)$ (resp. in $(G(s)^o, F)$ in unipotent series) associated to $(L_s^*, \alpha)$ by Proposition 2.1.4. The defect groups of $b_{GF}(L, \lambda)$, $b_{G(s)^o}(L(s), \alpha)$ are isomorphic.

Proof. The central product in (4.1.1) gives

$$G(s)^o = G_a(s)^o.G_b(s)^o$$

(but $G_a(s)^o$ is not $(G(s)^o)^a$ and $G_b(s)^o$ is not exactly in duality with $C_{G_{s, b}^*}^0(s)$ ...)

Let $T$ et $T^*$ be dual tori as in 4.1.1 and let $T_1$ be a dual of $T^*$ in $G(s)^o$, so that the duality between $G(s)^o$ and $C_{G_{s}}^0(s)$ is defined around $(T_1, T^*)$. Then $L_s^*$ is in duality with a Levi subgroup $L(s)$ of $G(s)^o$ such that $T_1$ is maximally split in $L(s)$. There exists an $F$-compatible isomorphism between the couples $(T_1, T^*)$ and $(T, T^*)$, restricting as identity on $T^*$ and $X(T^*)$. Let

$$\rho: T_1 \rightarrow T$$

be its restriction to $T_1$.

We have $M^* \cap C_{G_{s, b}^*}^0(s) = L_s^* \cap G_{s, b}^*$ because $L_s^*$ is $E$-split in $C_{G_{s}}^0(s)$ as is $M^*$ in $G^*$.

We have seen in the proof of Proposition 4.1.1 how $Z$ is defined from $L_s^*$ : on $G_a$ side, $Z \cap G_a = (T \cap G_a)^F$. On $G_b$ side $Z^o(M \cap G_{s, b}^*)_{|\Phi E} = Z^o(L_s^* \cap G_{s, b}^*)_{|\Phi E}$ so that $Z^o(M \cap G_{s, b}^*)_{|\Phi E}$ is the biggest $\Phi E$-subgroup of $T \cap G_{s, b}^*$ in the kernel of any $r$ such that $r^\vee \in \Phi L_s^*: G_{s, b}^*(T^* \cap G_{s, b}^*)$.

Let $Z_1 \subseteq (T_1)^F$ be obtained by the same process from $G(s)^oF$ and the $d$-cuspidal unipotent datum $(L_s^*, \alpha)$, with $(G(s)^o, C_{G_{s}}^0(s), 1, T_1)$ instead of $(G, G^*, s, T)$. To compare $Z$ with $Z_1$, we have to consider the rational component $C_{G_{s, b}^*}^0(s)^a$ of $C_{G_{s, b}^*}^0(s)$. It is a central product $C_{G_{s, b}^*}^0(s)^a = C_{G_{s, b}^*}^0(s)[C_{G_{s, b}^*}^0(s)]^a$ and in
the dual side (dually up to some isogeny) is a similar decomposition \( G(s)^{\circ}_{\alpha} = G_a(s)^{\circ} \cdot [G_b(s)^{\circ}]_a \), as well as \( G(s)^{\circ}_{b} = [G_b(s)^{\circ}]_b \). Clearly \( T_1 \cap G(s)^{\circ}_{a} = (T_1 \cap G_a(s)^{\circ}) \cdot (T_1 \cap [G_b(s)^{\circ}]_a) \) and

\[
Z_{1} \cap G(s)^{\circ}_{a} = (T_1 \cap G_a(s)^{\circ})_F \cdot (T_1 \cap [G_b(s)^{\circ}]_a)_F
\]

(by Proposition 1.1.3 (b), \( Z([G_b(s)^{\circ}]_a)_F / Z([G_b(s)^{\circ}]_b)_F \) is prime to \( \ell \)). As \( (L^*_s \cap [C_{G^*_b(s)}(s)]_a)_F \) admits a cuspidal unipotent, \( L^*_s \cap [C_{G^*_b(s)}(s)]_a \) is a diagonal torus of \( [C_{G^*_b(s)}(s)]_a \).

Therefore \( L^*_s \cap [C_{G^*_b(s)}(s)]_a \) is \( T^* \cap [C_{G^*_b(s)}(s)]_a \) hence \( (T^* \cap [C_{G^*_b(s)}(s)]_a)_{\phi_E} = T^*_b \cap [C_{G^*_b(s)}(s)]_a \) and

\[
Z(L^*_s \cap G^*_b)_\phi_E = (T^* \cap [C_{G^*_b(s)}(s)]_a)_{\phi_E} \cdot Z(L^*_s \cap [C_{G^*_b(s)}(s)]_b)_{\phi_E}
\]

The central products

\[
T \cap G_b = S.R, \quad Z^0(M \cap G_b)_{\phi_E} = S_{\phi_E}H
\]

where \( H \) is defined from \( L^*_s \cap [C_{G^*_b(s)}(s)]_b \) as \( Z^0(M \cap G_b)_{\phi_E} \) is defined from \( L^*_s \cap G^*_b \). As \( \rho \) is the identity on \( X(T^*) \cong Y(T) \cong Y(T_1) \), one sees that \( \rho(T_1 \cap G(s)^{\circ}_{a}) = (T \cap G_a)S \) and \( \rho(T_1 \cap G(s)^{\circ}_{b}) = R \), so that \( \rho(Z_1 \cap G(s)^{\circ}_{a}) = (Z \cap G_a)(S_{\phi_E})_F \) and \( \rho(Z_1 \cap G(s)^{\circ}_{b}) = H^F \). We have obtained

\[
\rho(Z_1) = Z
\]

We note that \( Z_1 = Z(L(s))_F = Z^0(L(s))^F \) (see [13] § 4.3 and Proposition 3.3). Now we know by [13] again that \( N_{G(s)^{\circ}_F}(Z_1) \cap N_{G(s)^{\circ}}(T_1) \) contains a defect group \( D_1 \) of \( b_{G(s)^{\circ}_F}(L(s), \alpha) \) such that \( D_1 \cap T_1 = Z_1 \), \( D_1 T_1 / T_1 \) is an \( \ell \)-Sylow subgroup of \( Q^F_1 \), where \( Q_1 = W(T_1, C_{G(s)^{\circ}}([L(s), L(s)]), T_1) \) and the extension

\[
Z_1 \to D_1 \to D_1 / Z_1
\]

is split.

The anti-isomorphism between \( W(G(s)^{\circ}, T_1) \) and \( W(C_{G^*_s}(s), T^*) \) restricts to an anti-isomorphism between \( W(T_1, C_{G(s)^{\circ}}([L(s), L(s)]), T_1) \) and \( W(T^*, C_{G^*_s}(s) ([L^*_s, L^*_s]), T^*) \). By Proposition 4.1.1 and its proof our group \( Q_1 \) is anti-isomorphic to the group we denoted \( Q \). Thus the isomorphism from \( Z_1 \) onto \( Z \) is the restriction of an isomorphism from the defect group of \( b_{G(s)^{\circ}_F}(L(s), \alpha) \) onto a defect group of \( b_{G^F}(L, \lambda) \).

We see also that the defect groups of \( b_{G(s)^{\circ}_F}(L(s), \alpha) \) and \( b_{C_{G^*_s}(s)_F}(L^*_s, \alpha) \) are anti-isomorphic.

4.2. Subpairs and Brauer’s categories

Using notation 2.3.1, we enforce Proposition 2.3.2, where \( H \) was an \( E \)-split Levi subgroup of \( G \). The proofs are quite similar.

4.2.1. Proposition. Let \( Y \) be an \( \ell \)-subgroup of \( G^F \) and let \( H = C_{G^F}(Y) \). The relation \( (L_H, \alpha_H) \sim_{G^F}(L, \alpha) \) defines a bijection between the set \( \sim_{G^F} \)-classes of unipotent \( d \)-cuspidal data \( (L, \alpha) \) in \( (G, F) \) such that \([L, L] \subseteq H^g \) for some \( g \in G^F \) and \( \sim_{H^F} \)-classes of unipotent \( d \)-cuspidal data \( (L_H, \alpha_H) \) in \( (H, F) \).
Proof. Let \( G = G_a G_b \) be the decomposition defined in 1.1.5. A \( d \)-cuspidal unipotent datum in \( (G, F) \) is provided by the central product of unipotent \( d \)-cuspidal data \( (L_a, \alpha_a), (L_b, \alpha_b) \) in \( (G_a, F) \) and \( (G_b, F) \) respectively and \( L_a \) is a diagonal torus in \( G_a, \alpha_a = 1_{L_a} \) (Proposition 2.1.5). The set of \( \sim_{G,F} \)-classes of unipotent \( d \)-cuspidal data in \( (G, F) \) is in bijection with the set of \( \sim_{G_b}^{G,F} \)-classes of unipotent \( d \)-cuspidal data in \( (G_b, F) \).

If \( G_b \subseteq H \), then \( H_a = H \cap G_a \) and \( H_b = G_b \), hence our claim.

We assume \( G_b \not\subseteq H \) and use induction on the semi-simple rank of \( G \). There exists \( \ell \)-subgroups \( Y_a \subseteq G_a^{\ell} \) and \( Y_b \subseteq G_b^{\ell} \) such that \( C_G(Y) = C_{G_a}(Y_a) C_{G_b}(Y_b) \) (see (D) at the end of the proof of Proposition 1.2.6). By assumption on \( H, Y_b \not\neq 1 \), hence \( Z(Y_b) \not\neq 1 \). Let \( z \in Z(Y_b), z \not\neq 1 \), so that \( z \not\in Z(G_b) \), and \( C_G(z) \) is a proper \( E \)-split Levi subgroup of \( G \) by Proposition 1.2.5. Clearly we have \( Y.C_G(Y) \subseteq G_a.C_{G_b}(z) = C_G(z) \not\subseteq G \). There exists a proper \( d \)-cuspidal Levi subgroup \( M \) of \( G \) such that \( C_G(z) \subseteq M \) (1.1.5.3). By inductive hypothesis, given a unipotent \( d \)-cuspidal datum \( (L_H, \alpha_H) \) in \( (H, F) \) there exists \( (L_M, \alpha_M, \ell, d) \) such that \( C_{G_b}(z) \) is covered by exactly one unipotent block of \( G_b^{\ell} \). Thanks to Proposition 1.2.6, the \( L, \alpha \) of \( (L, \alpha) \sim_{G,F} (L_M, \alpha_M) \). 

In the following Proposition on unipotent blocks we describe all Brauer’s subpairs, a generalization of [16] Lemma 23.10, relying inclusion to the relation we have introduced in Proposition 4.2.1.

4.2.2. Proposition. Assumption 2.2.3 on \( (G, F, \ell, d) \). Let \( (L, \alpha) \) be a \( d \)-cuspidal unipotent datum in \( (G, F) \). Let \( (Y, b_Y) \) be a Brauer \( \ell \)-subpair of \( G^{\ell} \) and let \( (L_Y, \alpha_Y) \) be a unipotent \( d \)-cuspidal datum in \( (C_{G_a}(Y), E) \). Assume that \( b_Y \) covers \( b_{C_{G_a}^\ell(Y)^{\ell}}(L_Y, \alpha_Y) \).

One has \( \{(1), b_{G^\ell}(L, \alpha)\} \subseteq (Y, b_Y) \) if and only if \( (L_Y, \alpha_Y) \sim_{G^\ell} (L, \alpha) \).

Proof. Induction on the semi-simple rank of \( G \).

We know that \( b_{G^\ell}(L, \alpha) \) is defined by \( b_Y \) in the inclusion \( (1, b_{G^\ell}(L, \alpha)) \subseteq (Y, b_Y) \) and that \( b_Y \) is defined by \( b_{C_{G_a}^\ell(Y)^{\ell}}(L_Y, \alpha_Y) \). By Proposition 2.1.6, the \( G^\ell \)-conjugacy class of \( (L, \alpha) \) is then defined by the \( C_{G_a}^\ell(Y)^{\ell} \)-conjugacy class of \( (L_Y, \alpha_Y) \). Thanks to Proposition 4.2.1 we have only to show that the relation \( \sim_{G,F} \) between the two unipotent \( d \)-cuspidal data implies the inclusion of the so-defined \( \ell \)-subpairs.

Case (A): \( G_b \not\subseteq C_{G_a}(Y) \).

Let \( M \) be a proper \( d \)-split Levi subgroup of \( G \) such that \( G_a C_G(Y) \subseteq M \) (see the proof of Proposition 4.2.1). One has \( M = C_{G_a}^\ell(Z(M)^{\ell}) \) and \( M^F = C_{G^F}(Z(M)^{\ell}) \) by Proposition 1.2.3. If \( (L_1, \alpha_1) \) is a \( d \)-cuspidal unipotent datum in \( (M, F) \), induction applies: the relation \( (L_Y, \alpha_Y) \sim_{M^F} (L_1, \alpha_1) \) is equivalent to the inclusion \( (1, b_{M^F}(L_1, \alpha_1)) \subseteq (Y, b_Y) \) in \( M^F \).

That inclusion is equivalent to \( (Z(M)^{\ell}, b_{M^F}(L_1, \alpha_1)) \subseteq (Z(M)^{\ell}, Y, b_Y) \) in \( G^F \). As \( M \) est \( d \)-split, \( (L_1, \alpha_1) \) is a \( d \)-cuspidal datum in \( (G, F) \) and the \( M^F \)-conjugacy class of \( (L_1, \alpha_1) \) defines the \( G^F \)-conjugacy class of \( (L_1, \alpha_1) \). Clearly the \( G^F \)-conjugacy class of \( (L_1, \alpha_1) \) is equivalent to \( (L_Y, \alpha_Y) \sim_{G^F} (L_1, \alpha_1) \) (Proposition 4.2.1). By Propositions 2.1.6, 2.1.7 \( R_G^G(b_{M^F}(L_1, \alpha_1)) = b_{G^F}(L_1, \alpha_1) \) and \( (\{1\}, b_{G^F}(L_1, \alpha_1)) \subseteq (Z(M)^{\ell}, b_{M^F}(L_1, \alpha_1)) \) in \( G^F \). Transitivity of inclusion gives \( (1, b_{G^F}(L_1, \alpha_1)) \subseteq (Y, b_Y) \) in \( G^F \).

Case (B): \( G_b \subseteq C_{G_a}(Y) \).

The map \( \mathcal{E}(G^F, 1) \rightarrow \mathcal{E}(G_a^{\ell}, 1) \times \mathcal{E}(G_b^{\ell}, 1) \), defined by

\[
\text{Res}_{G_a^{\ell}, G_b^{\ell}}^{G^F} = \chi_a \otimes \chi_b, \quad \chi \mapsto (\chi_a, \chi_b)
\]

is one-to-one.

Thus any unipotent block of \( G_a^{\ell} G_b^{\ell} \) is covered by exactly one unipotent block of \( G^{\ell} \).
This applies to the decomposition $C_G^e(Y) = C_G^b(Y).G_b : L_Y = T_Y.(L_Y \cap G_b)$, where $T_Y$ is a diagonal torus of $C_L^a(Y)$, $L_Y \cap G_b$ is a $d$-split Levi subgroup of $G_b$ and $\alpha_Y \in E((T_Y.(L_Y \cap G_b))^F, 1)$ covers $1_{T_Y^F} \otimes \alpha_b$, $\alpha_b \in E(L_Y \cap G_b)^F, 1$, $\alpha_b$ is $d$-cuspidal (Proposition 2.1.5, (c)). Let $B_0(Y)$ be the principal block of $C_G^e(Y)^F$, $B_0(1)$ be the principal block of $G_a^F$ and $b_1 = b_{G_a^F}(L_Y \cap G_b, \alpha_b)$. There is exactly one unipotent block $b_Y^e$ of $C_G^e(Y)^F$ that covers $B_0 \otimes b_1$ and there is exactly one unipotent block $b_Y$ of $C_G^e(Y)$ which covers $b_Y^e$.

The inclusion in $G^F, G^F_b$ of subpairs

$$(1, B_0(1) \otimes b_{G_a^F}(L_Y \cap G_b, \alpha_b)) \subset (Y, B_0(1)) \otimes b_{G_a^F}(L_Y \cap G_b, \alpha_b))$$

is clear. By unicity of covering unipotent blocks we have an inclusion in $G^F$ of unipotent subpairs

$$(1, b_{G^F}(T_a, L \cap G_b, \alpha)) \subset (Y, b_Y)$$

where $T_a$ is a diagonal torus of $G_a$ and $\alpha \in E((T_a.(L \cap G_b))^F, 1)$ covers $1_{T_a^F} \otimes \alpha_b$.

Now we describe all Brauer subpairs and their inclusion in any series, under our general hypothesis.

**4.2.3. Proposition.** Assumption 2.2.3 on $(G, F, \ell, d)$. Let $Y$ be an $\ell$-subgroup of $G^F$ and $T, T_Y \subseteq T, Y', T_{Y'} \cong T_{Y'}^* \subseteq T^*$ as in Proposition 1.2.6, so that $C_G^e(Y)$ and $C_G^e(Y')$ are in duality. Let $(L_Y, \lambda_Y)$ be a $d$-cuspidal datum in $(C_G^e(Y), F)$ with $T_Y \subseteq L_Y.$ Assume $\lambda_Y \in E(L_Y^F, s)$ with $s \in (T_Y^*)^F_{G_a}$. Let $b_Y$ be the block of $C_{G^F}(Y)$ that covers $b_{C_G^e(Y)^F}(L_Y, \lambda_Y)$.

Let $(L_{Y,s}^*, \alpha_Y)$ be a unipotent $d$-cuspidal datum in $(C_{G_a^e(Y)}^*(s), F)$ associated to $(L_Y, \lambda_Y)$ by Proposition 2.1.4. Let $(L_{s}, \alpha)$ be a unipotent $d$-cuspidal datum in $(C_G^e(s), F)$ such that

$$(L_{Y,s}^*, \alpha_Y) \sim_{C_G^e(s)^F}(L_s^*, \alpha).$$

There exists a $d$-cuspidal datum $(L, \lambda)$ in $(G, F)$ such that $T \subset L, \lambda \in E(L^F, s)$, $(L, \lambda)$ is associated to $(L_s^*, \alpha)$ and

$$(4.2.3.1) \quad (\{1\}, b_{G^F}(L, \lambda)) \subset (Y, b_Y).$$

**Proof.** Clearly Proposition 4.2.3 goes through direct products.

(A) Assume the center of $G$ connected.

By Proposition 1.2.6, $C_G(s)$, $C_{G^*}(Y')$ and $Z(C_G^e(Y))$ are connected. Then the $C_{G_{G^*}(Y')^F}$-conjugacy class of $(L_{Y,s}^*, \alpha_Y)$ is well defined by $(L_Y, \lambda_Y)$ (Proposition 2.1.10). The $C_{G^*}(s)^F$-conjugacy class of $(L_{s}, \alpha)$ is well defined by $(L_{Y,s}^*, \alpha_Y)$ as in Proposition 4.2.2. We have to prove that the relation between the two unipotent $d$-cuspidal data inside $C_{G^*}(s)$ imply (4.2.3.1) in $G^F$. We use induction on the dimension of $G$.

(A.1) Assume $G = G_a$.

Let $G(s)$ be a Levi subgroup of $G$, in the dual $G^F$-conjugacy class of $C_{G^*}(s)$. There is only one block to consider in series $(s)$, denote it $b_{G^F}(s)$ and $b_Y$ covers $b_{C_G^e(Y)^F}(s)$. An $\ell$-Sylow subgroup of $G(s)^F$ is a defect group of $b_{G^F}(s)$. By our hypotheses, $s \in C_G^e(Y')$ and, up to $G^F$-conjugacy we may assume $Y \subseteq G(s)^F$. The condition $(L_{Y,s}^*, \alpha_Y) \sim_{C_{G^*}(s)^F}(L_s^*, \alpha)$ is always satisfied by $d$-cuspidal data. It is sufficient to prove the inclusion $(1, b_{G^F}(s)) \subset (Y, b_Y)$ and this is independent of choices of $Y'$, and of tori defining dualities.

The $\ell$-subpairs of $GL_n(q)$ are given in [7], where $G$ is identified with $G^*$, the $G^F$-conjugacy class of $(T, \theta)$ ($T$ a maximal $F$-stable torus, $\theta \in (T^F)^\wedge$) corresponds to the $G^*F$-conjugacy class of $(T^*, s)$ ($T^*$ a
maximal $F$-stable torus, $s \in T \cdot F)$. We know that $C_G(Y)$ is connected and a direct product of linear groups. From [7] we retain: if $B$ is a block of $GL_n(q)$ in series $(s)$ and $Y \subseteq G(s)^F$ there is an inclusion of subpairs $(1, B) \subset (Y, b_Y)$ in $GL_n(q)$ where $b_Y$ is in series $(s)$ in $C_G(Y)^F$. In our hypothesis $G = G_a$ that gives $(1, b_{G_a}(f) \in (Y, b_{C_G(Y)}^F(s))$ in $G^F$, (4.2.3.1) in that case.

Any group $G$ of type $A$ (resp. $(G, F)$ such that $G = G_a$) with connected center may be reached from general linear groups $GL_n$ (resp. $G = GL_n$ such that $G = G_a$) by a sequence of morphisms of three types:

1. direct product,
2. $(H, F) \to (G, F)$ is a regular covering between groups with connected centers (and $H = H_a$)
3. $(G, F) \to (H, F)$ is an isotypic embedding, where $Z(G)$ is connected, $[G, G]$ is simply connected and there is an isotypic embedding $G_0 \to H$, with $G_0 \cong GL_n$. We verify that our claim goes from $(H, F)$ to $(G, F)$ in 2. and 3.

In case 2, $G^F$ is a quotient of $H^F$, the dual map $G^* \to H^*$ is an embedding, so let $s \in H^*$. There is some $\ell$-group $Y_H$ in $H^F$ with image $Y$ in $G^F$ and $C_G^F(Y)$ is a quotient of $C_H^F(Y_H)$. There is a regular embedding $C_G(Y)^* \to C_H(Y_H)^*$, we assume $s \in C_H(Y_H)^* = Z^0(H^*).C_G(Y)^*) \subseteq H^*$. The two inclusions between $\ell$-subpairs $(1, b_{H^F}(s)) \subset (Y_H, b_{C_H(Y)}^F(s))$ in $H^F$ and $(1, b_{G^F}(s)) \subset (Y, b_{C_G(Y)}^F(s))$ in $G^F$ are true or not simultaneously.

In case 3 assuming $Y \subseteq G^F$, let $Y_0 = Y.Z^0(H)^F \cap G_0$. We have $H^F = Z^0(H)^F.G^F = Z^0(H)^F.G_0^F$ and $C_H(Y) = Z^0(H)^F.C_G(Y) = Z^0(H)^F.C_G^F(Y_0)$. Let $s_H \in (H^*)^F$ with images $s \in G^*, s_0 \in G_0^F$. Then $C_G^F(s)$ and $C_G^F(s_0)$ are quotients of $C_H^F(s_0)$ by central torii. Clearly $b_{H^F}(s_H)$ covers $b_{G^F}(s)$ and $b_{G_0^F}(s_0)$. The morphism $C_G(Y)^* \to G^*$ given by Proposition 1.2.6 may be deduced by quotient from a morphism $C_H(Y)^* \to H^*$ (with duality around $T_Y.Z^0(H)...$), giving $C_G^F(Y_0)^* \to G_0^F$ (duality around $T_Y.Z^0(H) \cap G_0$...).

Then $b_{C_H(Y)}^F(s_H)$ covers $b_{C_G^F(Y)}^F(s)$ and $b_{C_G^F(Y_0)}^F(s_0)$. The inclusion $(1, b_{G_0^F}(s_0)) \subset (Y_0, b_{C_G^F(Y_0)}^F(s_0))$ in $G_0^F$ implies $(1, b_{H^F}(s_H)) \subset (Y, b_{C_H(Y)}^F(s_H))$ which implies (4.2.3.1) in $G^F$.

(A.2) Assume $G_b \subseteq C_G(Y)$.

That case reduces to the preceding one by a standard description, with given dualities:

$s \mapsto (s_a, s_b)$ by $G^* \to (G_a)^*(G_b)^*$,

$C_G(Y) = C_G^F(Y_0)G_b$ where $Y_a \subseteq G_a^F$,

$C_G^F(Y')$ maps on $C_{G_a}(Y')_{G_a}(G_b)^*$ for some $Y' \subseteq (G_a)^*$ with $L_Y = L_Y.M, Y_a \subseteq C_G^F(Y_a), M \subseteq G_b, L_{Y_a}^F.L_{Y_a}^F \cdot \lambda_Y = \lambda_{Y_a} \otimes \mu$ where $\lambda_{Y_a}$ (resp. $\mu$) is $d$-cuspidal in series $(s_a)$ (resp. $(s_b)$) (see Proposition 2.1.5),

if $(L_{Y_a}, \lambda_{Y_a})$ (resp. $(M, \mu)$) is associated to the $d$-cuspidal unipotent datum $(L_{Y_a}^*, s_a, \alpha_{Y_a})$ (resp. $(M_{s_b}^*, \beta)$) then we may assume that $L_{Y_a}^*$ has image $(L_{Y_a,s_a}^* \times M_{s_b}^*)$ by the map $G^* \to (G_a)^* \times (G_b)^*$ and that $\text{Res}_{L_{Y_a}^*}^{L_{Y_a}^* \times M_{s_b}^*} = \alpha_{Y_a} \otimes \beta$.

the condition $(L_{Y_a}^*, \alpha_Y)_{C_{G_a}(Y_a)}^F(L_{Y_a}^*, \alpha)$ is then equivalent to :

$L_{s_a}^*$ has image $L_{s_a}^* \times M_{s_b}^*$ in $(G_a)^* \times (G_b)^*$, $\text{Res}_{L_{Y_a}^* \times M_{s_b}^*}^{L_{Y_a}^* \times M_{s_b}^*}(\alpha_{Y_a} \otimes \beta) = \alpha$ where $\alpha_a \in E(L_{s_a}^F, 1)$ and $(L_{Y_a,s_a}^*, \alpha_{Y_a})_{C_{G_a}(Y_a)}^F(L_{Y_a}^*, \alpha_{a})$

If $(L_a, \lambda_a)$ is a $d$-cuspidal datum in series $(s_a)$ in $(G_a, F)$ associated to $(L_{Y_a}^*, \alpha_a)$, then we may assume that $L = L_a.M$ and $\text{Res}_{L_{Y_a}^* M}^F \lambda = \lambda_a \otimes \mu$.

Then (4.2.3.1) is equivalent to $(1, b_{(G_a)^F}(L_a, \lambda_a)) \subset (Y_a, b_{Y_a})$. That last inclusion follows from (A.1).

(A.3) Assume now $G_b \subseteq C_G(Y)$. 80
As in the proof of Proposition 4.2.1, let \( z \neq 1 \) such that \( Y.C_G(Y) \subseteq C_G(z) \) where \( C_G(z) \) is a proper \( E \)-split Levi subgroup of \( G \). By Proposition 1.2.6 there exist an \( \ell \)-subgroup \( Y' \) in \( G^*F \) and inclusions of dual groups \( C_G(Y')^* = C_G^*(Y') \subseteq C_G(z)^* \subseteq G^* \) with \( s \in C_G^*(Y') \). By Proposition 4.2.2 we have unipotent \( d \)-cuspidal data \((L^*_\mu, \alpha_\mu) \) in \((C_G, (Y')^*(s), F), (L^*_s, \alpha_s) \) in \((C_G(z)^*, (s), F), (L^*_s, \alpha) \) in \((C_G, (s), F) \) such that 

\[
(L^*_\mu, \alpha_\mu)^*\sim C_{G^*(s)}^*(s)F(L^*_s, \alpha)^*\sim C_{G^*}(s)F(L^*_s, \alpha)
\]

These unipotent \( d \)-cuspidal data define \( d \)-cuspidal data in series \((s) : (L_Y, \lambda_Y) \) in \((C_G(Y), F), (L_z, \lambda_z) \) in \((C_G(z), F), (L, \lambda) \) in \((G, F) \). By Propositions 2.1.6, 2.1.7 we have an inclusion of subpairs:

\[
(1, b_{G^F}(L, \lambda)) \subseteq (1, b_{C_G(z)^F}(L_z, \lambda_z)) \in G^F
\]

By induction we have an inclusion of subpairs:

\[
(1, b_{C_G(z)^F}(L_z, \lambda_z)) \subseteq (Y, b_{C_G(Y)^F}(L_Y, \lambda_Y)) \in C_G(z)^F
\]

equivalent to

\[
(\langle z \rangle, b_{C_G(z)^F}(L_z, \lambda_z)) \subseteq (Y, b_{C_G(Y)^F}(L_Y, \lambda_Y)) \in G^F
\]

Transitivity of inclusion of subpairs in \( G^F \) gives (4.2.3.1).

(B) \( Z(G) \) connected or not.

The inclusion of subpairs in \( G^F \) may be described by inclusion of so-called “connected subpairs” in [15], 2.1, where \((1, b_{G^F}(L, \lambda)) \subseteq (Y, b_Y) \) is equivalent to its “restriction” \((1, b_{G^F}(L, \lambda)) \subseteq b_{C_G(Y)^F}(L_Y, \lambda_Y) \).

In a regular embedding \( G \to H \), let \( t \in H^* \) of image \( s \) by a dual map. Let \((M, \mu)\) be a \( d \)-cuspidal datum in series \((t) \) in \((H, F) \) such that \( L = M \cap G \) and \( \mu \) covers \( \lambda \) (see Proposition 2.1.5). The associated \( d \)-cuspidal data \((L^*_\mu, \alpha) \) and \((M^*_\mu, \beta)\) are related : \( \beta \) is the restriction of \( \alpha \) through the isotypic morphism \( M^*_\mu \to L^*_\mu \) induced by the restriction \( M^* \to L^* \) of \( H^* \to G^* \). By Proposition 2.4.2, Proposition 2.4.4 and its proof, a quotient \( H^F/K \) of \( H^F/G^F \cong M^F/L^F \) acts regularly on the set of blocks of \( G^F \) that are covered by \( b_{H^F}(M, \mu) \), as well as on the set of unipotent \( d \)-cuspidal elements of \( E((L^*_\mu, \lambda_\mu, \alpha_\mu, 1) \) that are covered by \( \mu \).

If \( D \) is a defect group of \( b_{G^F}(M, \mu) \), the defect groups of the various \( b_{G^F}(L, \lambda) \), \( \lambda \) covered by \( \mu \), form a \( H^F \)-conjugacy class and are the \( D \cap G^F \), where \( D \) is a defect group of \( b_{H^F}(M, \mu) \) (see Propositions 5.1.3, 5.1.4 in Appendix). The action of \( H^F \) on \( C_G(z)^F \) transforms inclusion of Brauer subpairs in \( G^F \) in inclusion of Brauer subpairs. Let \( Y \) be some subgroup of \( D \cap G^F \). We have a natural isomorphism \( H^F/G^F \cong C_H^*(Y)^F/C_G^*(F)^F \) giving an action of \( H^F/G^F \) on the blocks \( b_Y \) of \( C_G(Y) \) that are covered by a given block \( B_Y \) of \( C_H(Y) \). By this way \( H^F/K \) acts on the set of all Brauer subpairs in \( G^F \) containing \((1, b_{G^F}(L, \lambda)) \) such that \( \lambda \) is covered by \( \mu \). One sees that (4.2.3.1) follows from the inclusion \((1, b_{H^F}(M, \mu)) \subseteq (Y, b_Y) \).

The Proposition 4.2.4 is the decisive step to show isomorphism between Brauer categories of a unipotent block \( b_s \) of \( G(s)^F \) and its Jordan correspondant \( b_{G^F}(L, \lambda) \), a block of \( G^F \) in series \((s) \), as described in the proof of Proposition 3.4.1. The link between \( b_s \) and \( b_{G^F}(L, \lambda) \) is the \( d \)-cuspidal unipotent \( \alpha \in E(C_G^*(s)^F, 1) \), so we may assume, with notations of 3.4.1, that \( B_{G, s}(b_s) = b_{G^F}(L, \lambda) \). Given a subgroup \( Y \) of a common defect group, it appears a similar link between Brauer subpairs, a \( d \)-cuspidal \( \alpha_Y \in E(C_G^*(s)^F, 1) \). The inclusion of subpairs is a consequence of the relation we introduced in 2.3.1, by Proposition 4.2.2.

**4.2.4. Proposition.** Let \((G, F, \ell, d), (Y, Y', (L_Y, \lambda_Y)), b_Y, (L^*_s, \alpha), (L^*_Y, \alpha_Y) \) as in Proposition 4.2.3, hence (4.2.3.1) :

\[
(\{1\}, b_{G^F}(L, \lambda)) \subseteq (Y, b_Y).
\]
Let $G^o(s)$ and $G(s)$ be defined as in section 3.1. We may assume $Y \subset G(s)^o F$ by Proposition 4.1.2. The group $C^o_{C_{Gr}(Y^l)}(s) = C^o_{C_{Gr}(s)}(Y^l)$ is in duality with $C^o_{G(s^l)}(Y)$. Let $(L_s, \alpha)$ and $(L_{Y,s}, \alpha_Y)$ be unipotent $d$-cuspidal data of $(G(s)^o, F)$ and $(C^o_{G(s^l)}(Y), F)$ respectively, with dual data $(L^*_s, \alpha)$ and $(L^*_{Y,s}, \alpha_Y)$ in $(C^o_{C_{Gr}(s)}, F)$ and $(C^o_{C_{Gr}(s)}(Y^l), F)$ respectively (see (1.3.1, 1.3.5)).

Let $b_{Y,s}$ be a block of $C^o_{G(s^l)}(Y)$ that covers $b_{C^o_{G(s^l)}(Y^l)}(L_{Y,s}, \alpha_Y)$ and $b_{s}$ be the block of $G(s)^F$ such that $\{(1), b_{s}\} \subset (Y, b_{Y,s})$.

The groups $E_{G(s)^F}(Y, b_{Y,s})$ et $E_{GF}(Y, b_Y)$ are isomorphic.

Proof. $G(s)$ is defined around a couple of dual maximal tori $(T \subseteq G \cap G(s)^o, T^* \subseteq C_{Gr}(s))$. The $d$-cuspidal datum $(L, \lambda)$ in $(G, F)$ is associated to a set of unipotent $d$-cuspidal datum $(L_s, \alpha)$ in $(C^o_{C_{Gr}(s)}, F)$ (Propositions 2.1.4, 2.1.7). As a consequence of $(L^*_s, \alpha) = C^o_{C_{Gr}(s)}(L^*_s, \alpha)$, we assumed in Proposition 4.2.3, one has $(L^*_{Y,s}, \alpha_Y) = C^o_{C_{Gr}(s)}(L^*_{Y,s}, \alpha_Y)$. A subgroup of $A_{G(s)^F}$ acts transitively on the set of $d$-cuspidal $\alpha$ corresponding to $\lambda$ by Propositions 1.3.6 and 2.1.4. The groups $E_{G(s)^F}(Y, b_Y)$ defined by different $\alpha$ in the $A_L(\alpha^o)$-orbit are isomorphic. So we fix $\alpha$.

Transport the duality between $G^o(s)$ and $C^o_{C_{Gr}(s)}$ around dual $F$-stable maximal tori $S \subseteq G^o(s)$ and $S^* \subseteq L_s^*$, $L_s$ and $L_s^*$ being $d$-split Levi subgroups in dual conjugacy classes such that, with notations of Proposition 3.4.1, $B_{G(s)}(b_s) = b_{G(s)}(L, \lambda)$, where $b_s$ is one of the blocks of $G(s)^F$ that cover $b_{G(s)^F}(L_s, \alpha)$. Isomorphic or anti-isomorphic defect groups of $b_{C^o_{G(s)}(s)}(L^*_s, \alpha)$, $b_{G(s)^o}(L_s, \alpha)$, $b_{s}$ and $b_{G(s)}(L, \lambda)$ are known by Proposition 4.2.1. We may identify them and assume that, by a suitable choice of $S$, $Y \subseteq N_{G(s)}(S)$, $Y \subseteq N_{G(s)}(S)$ and $Y$ centralizes $[L_s, L_s]$. On dual side $Y^l \subseteq N_{(G^o)^o}(S^l) \cap C^o_{C_{Gr}(s)}$ and $Y^l$ centralizes $[L^*_s, L^*_s]$.

The blocks of $C^o_{G}(Y)^F$ that are covered by $b_Y$ are $C^o_{G}(Y)$-conjugate and $(L_Y, \lambda_Y)$ is defined up to $C^o_{G}(Y)^F$-conjugacy hence we have an isomorphism

$$E_{GF}(Y, b_Y) \cong N_{G(s)}(Y, L_Y, \lambda_Y)^F/N_{C^o_{G}}(Y)(L_Y, \lambda_Y)^F$$

(A) Assume $\mathbb{Z}(G)$ connected, hence $G(s)$ connected.

By Proposition 2.2.6, if $(L_{Y,s}, \alpha_Y)$ is a unipotent $d$-cuspidal datum in $(C^o_{C_{Gr}(s)}(Y), F)$ corresponding to $(L^*_{Y,s}, \alpha_Y)$ in $(C^o_{C_{Gr}(s)}, F)$, the groups $E_{C^o_{C_{Gr}(s)}(s)}(Y^l, b_{C^o_{C_{Gr}(s)}(s)}(L^*_s, \alpha_Y))$ and $E_{G(s)^o}(Y, b_{G(s)^o}(L_{Y,s}, \alpha_Y))$ are anti-isomorphic.

We may define duality between $C^o_{C_{Gr}}(Y)$ and $C^o_{C_{Gr}}(Y^l)$ around tori $(T_Y \subseteq C^o_{C_{Gr}}(Y))$, $(T^*_Y \subseteq C^o_{C_{Gr}}(Y^l))$ as in Proposition 1.2.6. We have $N_{G^o}(Y, b_Y) = N_{G^o}(Y, L_Y, \lambda_Y) \subseteq N_{G^o}(Y, L_Y, E(L_Y^F, s))$.

By Proposition 1.2.6, (c), the groups $N_{G^o}(Y, T_Y)/N_{C^o_{C_{Gr}}(Y)}(T_Y)$ and $N_{G^o}(Y^l, T^*_Y)/N_{C^o_{C_{Gr}}(Y^l)}(T^*_Y)$ are anti-isomorphic. So are $N_{G^o}(Y, T_Y, T^*_Y)/N_{C^o_{C_{Gr}}(Y)(L_Y, \lambda_Y)}(Y, Y^l, T_Y, T^*_Y)$ and $N_{G^o}(Y^l, T^*_Y)/N_{C^o_{C_{Gr}}(Y^l)}(Y^l, \lambda_Y^l)$. Here $L^*_Y$ is in the dual $C_{G^o}(Y^l)^F$-conjugacy class of $L_Y$. Then the groups $L^*_Y$ are $L^*_Y$ may be mutually defined, one by the other, as $L_Y$ and $L_{Y,s}$, by the relations $L^*_Y = L^*_Y \cap C_{G^o}(s)$ and $L^*_Y = C^o_{C_{Gr}(s)}(Y) \cap (Z(s))_{\phi, s}$ (see Propositions 2.1.4 and 3.1.1). We obtain an anti-isomorphism between $N_{G^o}(Y, L_Y, E(L_Y^F, s))F/L_Y^F$ and $N_{C^o_{C_{Gr}}(s)}(Y^l, L^*_Y)^F/(L^*_Y)^F$.

Finally, by Propositions 1.3.2, 2.1.4, 2.1.7 and 2.2.6 and isomorphism (4.2.4.1), $E_{G^o}(Y, b_Y)$ is anti-isomorphic to $N_{C^o_{C_{Gr}}(s)}(Y^l, L^*_Y)^F/N_{C^o_{C_{Gr}}(s)}(Y^l, L^*_Y)^F$, that is exactly $E_{C^o_{C_{Gr}}(s)}(Y^l, b_{C^o_{C_{Gr}}(s)}(L^*_Y, \alpha_Y))$ by (4.2.4.1).
As \( E_{G^s}(s)^F(Y', b_{G^s}(s)^F(L'_{Y,s}, \alpha_Y)) \) is anti-isomorphic to \( E_{G^s}(s)^F(Y, b_{G^s}(s)^F(L_{Y,s}, \alpha_Y)) \), that gives the claimed isomorphism.

(B) To obtain the claimed isomorphism in case \( Z(G) \) is not connected, consider as usual a regular embedding \( G \to H \), defining by restriction regular embeddings \( C^*_G(Y) \to C^*_H(Y) \), \( L_Y \to M_Y := Z(H).L_Y \) and dual maps \( H^* \to G^* \) (where \( t \in (H^*)^F \to s \), \( M_Y^* \to L_Y^* \).

The group \( N_{G^F}(Y, L_Y, \mathcal{E}(L^F, s))/L_Y^F \) is a split extension of \( N_{H^F}(Y, M_Y, \mathcal{E}(M^F, t))/M_Y^F \) by the stabilizer \( A_{G^s}(s)^F(Y^s')_{Y^s'}^s \) of the \( F \)-conjugacy class of \( (Y^s', L^s_{Y,s}) \) in \( A_{G^s}(s)^F \). But \( A_{G^s}(s)^F(Y^s')_{Y^s'}^s \) appears as a subgroup of \( N_{C^s_{G^s}(s)^F(Y')}}_{C^s_{G^s}(s)^F(Y')} \). By part (A) of the proof we have an exact sequence of anti-morphisms

\[
\text{(4.2.4.2)} \quad N_{C^s_G(s)}(Y^s', L^s_{Y,s})/L^s_Y \to N_{G^F}(Y, L_Y, \mathcal{E}(L^F, s))/L^F_Y \to A_{G^s}(s)^F(Y^s, L^s_{Y,s})
\]

By construction of \( G(s) \) and duality, \( A_{G^s}(s)^F(Y^s, L^s_{Y,s}) \) is the image in \( G(s)^F/G^oF \) of \( N_{G^s}(s)^F(Y, L_Y,s) \). The group \( A_{Y^s}^s(s)^F \) acts on \( \mathcal{E}(L^F, s) \) and on \( L^F \) through the Jordan decomposition, as said in section 1.3. The natural isomorphism between \( N_{G^F}(Y, L_Y)/L_Y^F \) and \( N_{H^F}(Y, M_Y)/M_Y^F \) exchange Jordan decomposition \( \Psi_{M_Y,t} \) and the action of \( A_{Y^s}^s(s)^F \) (Proposition 1.3.1, (iv)). Furthermore any element of \( N_{C^s_G(s)}(Y', L^s_{Y,s}) \) or of \( N_{G^F}(Y, L_Y, \mathcal{E}(L^F, s)) \) that fixes \( \lambda_Y \) fixes any of its \( N_{H^F}(Y, M_Y) \)-conjugate (Proposition 2.4.2). It follows that \( N_{G^F}(Y, L_Y, \lambda_Y)/L_Y^F \) has image in \( A_{G^s}(s)^F(Y^s, L^s_{Y,s}) \) the stabilizer of \( \alpha_Y \). The sequence (4.2.4.2) restricts in

\[
\text{(4.2.4.3)} \quad N_{C^s_{G^s}(s)}(Y^s', L^s_{Y,s}, \alpha_Y)^F/L^s_Y \to N_{G^F}(Y, L_Y, \lambda_Y)^F/L^F_Y \to A_{G^s}(s)^F(Y^s', L^s_{Y,s}, \alpha_Y) \to 1
\]

By (4.2.4.1), \( E_{G^F}(Y, b_Y) \) is a quotient of that extension by \( N_{C^s_{G^F}(Y)}(L_Y, \lambda_Y)^F/L_Y^F \).

Considering \( C^s_G(Y) \) instead of \( G \), we have a split exact subsequence of (4.2.4.3)

\[
\text{(4.2.4.4)} \quad N_{C^s_{C_{G^s}(s)}(Y')}(L^s_{Y,s}, \alpha_Y)^F/L^s_Y \to N_{C^s_{G^F}(Y)}(L_Y, \lambda_Y)^F/L^F_Y \to A_{C^s_{G^s}(s)}(Y)^F(L^s_{Y,s}, \alpha_Y) \to 1
\]

Here \( C^s_{C_{G^s}(s)}(Y') = C^s_{C_{G^s}(s)}(Y)^s \). The quotient group \( N_{C^s_{G^s}(s)}(Y^s', L^s_{Y,s}, \alpha_Y)^F/N_{C^s_{G^s}(s)}(Y)^s(L^s_{Y,s}, \alpha_Y)^F \) is anti-isomorphic to \( E_{G^s}^F(Y, b_{C^s_{G^s}(s)}(Y)^F(L_{Y,s}, \alpha_Y)) \). Furthermore \( A_{G^s}(s) \) is an \( \ell^s \)-group and \( C^s_{G^s} \) are \( \ell^s \)-groups.

From (4.2.4.3) we obtain as quotient a split exact sequence of morphisms

\[
1 \to E_{G^s}(s)^F(Y, b_{G^s}(s)^F(L_{Y,s}, \alpha_Y)) \to E_{G^s}(s)^F(Y, b_Y) \to A_{G^s}(s)^F(Y^s', L^s_{Y,s}, \alpha_Y)/A_{C^s_{G^s}(s)}(Y)^F(L^s_{Y,s}, \alpha_Y) \to 1
\]

A similar argument applies in \( G(s)^F \), where \( G(s)^F \cdot N_{G^s}(s)^F(Y, b_{G^s}(s)^F)/G(s)^F = A_{G^s}(s)^F(Y^s', L^s_{Y,s}, \alpha_Y) \). It gives easily he split extension

\[
1 \to E_{G^s}(s)^F(Y, b_{G^s}(s)^F(L_{Y,s}, \alpha_Y)) \to E_{G^s}(s)^F(Y, b_Y) \to A_{G^s}(s)^F(Y^s', L^s_{Y,s}, \alpha_Y)/A_{C^s_{G^s}(s)}(Y)^F(L^s_{Y,s}, \alpha_Y) \to 1
\]

hence the claim isomorphism. 

\[\Box\]
5. Appendix

5.1. Self-centralizing Brauer pairs, blocks and normal subgroups

In that section we collect folklore results on blocks, pairs and Clifford theory with abelian quotient.

Let $X$ be a finite group. The abelian group $(X/[X,X])^\wedge$ acts on the set $\text{Irr}(X)$ by tensor product, hence on the set of $\ell$-blocks of $X$ and on the set of Brauer $\ell$-subpairs in a coherent way as follows.

Let $B$ be a block of $X$, $Q$ be an $\ell$-subgroup of $X$, $\theta \in (X/[X,X])^\wedge$ and define $\theta_Q := \text{Res}^X_Q(\theta)$.

The blocks $B$ and $\theta \otimes B$ are isomorphic and one has $\text{Irr}(\theta \otimes B) = \{ \theta \otimes \chi \mid \chi \in \text{Irr}(B) \}$.

An inclusion of subpairs $(P,b_P) \subset (Q,b_Q)$ implies $(P,\theta_P \otimes b_P) \subset (Q,\theta_Q \otimes b_Q)$. The block $B$ being defined by a maximal subpair $(D,B_D)$, $\theta \otimes B = B$ is equivalent to $\theta_D \otimes b_D = b_D$ (note that $\theta_D$ is stable under $N_X(D)$).

The $\ell$-Sylow subgroup of $(X/[X,X])^\wedge$ fixes any $\ell$-block of $X$ ([28] Chapter 7, Corollary 5.6). Consider now a subgroup $Y$ of $X$ with $[X,X] \subseteq Y$. Define $I_Y^X(B)$ and $I_Y^X(\chi)$ for $\chi \in \text{Irr}(X)$ by

$$(X/I_Y^X(B))^\wedge = ((X/Y)^\wedge)_B, \quad (X/I_Y^X(\chi))^\wedge = ((X/Y)^\wedge)_\chi$$

Assume that $B$ covers a block $b$ of $Y$, let $X_b$ be the stabilizer of $b$ in $X$. If we consider blocks as primitive central idempotents in algebras we have $\sum_{\theta \in \text{Irr}(I_Y^X(B))} \theta \otimes B = \sum_{g \in X/X_b} b^g$. There exists a unique block $b'$ of $X_b$ such that $B = \text{Tr}^X_X(b')$. Then $\theta \otimes B = \text{Tr}^X_X((\text{Res}^X_{X_b}\theta) \otimes b')$. Thus $I_Y^X(B) = I_Y^X(b')$. By a theorem of Fong-Reynolds, ([28] Chapter 5, Theorem 5.10)

$$I_Y^X(B) = \{ \text{Ind}^X_{X_b}(\chi') \mid \chi' \in \text{Irr}(b') \}. \tag{5.1.1}$$

Hence for any $\theta \in (X/X_b)^\wedge$ and $\chi = \text{Ind}^X_{X_b}(\chi') \in \text{Irr}(B)$ we have $\theta \otimes \chi = \chi$. That proves $I_Y^X(\chi) \subseteq X_b$.

Now if $\zeta \in \text{Irr}(Y)$ one has clearly $X_{\zeta} \subseteq X_{B_Y(\zeta)}$. The formula (5.1.1) has a twin for irreducible representations that imply $I_Y^X(\chi) \subseteq X_\zeta$ when $\chi \in \text{Irr}(X \mid \zeta)$. Finally

$$I_Y^X(B_X(\chi)) \subseteq I_Y^X(\chi) \subseteq X_\zeta \subseteq X_{B_Y(\zeta)} \quad \text{when} \quad \chi \in \text{Irr}(X \mid \zeta). \tag{5.1.2}$$

If there is no multiplicity in restrictions from $X$ to $Y$, with notations used in (5.1.1), (5.1.2), $\zeta$ extends in some $\zeta'$ to $X_{\zeta}$, different extensions has the form $\theta \otimes \zeta'$ hence $I_Y^X(\chi) = X_{\zeta}$.

On defect groups recall a consequence of a theorem of Fong

5.1.3. ([28] Chapter 5, Theorem 5.16) Let $Y$ be an invariant subgroup of a finite group $X$ and $b$ be an $\ell$-block of $Y$. There exist an $\ell$-block $B$ of $X$ that cover $b$ and a defect group $D$ of $B$ contained in $X_b$ such that $D/D \cap Y$ is an $\ell$-Sylow subgroup of $X_b/Y$.

5.1.4. Proposition. Let $\varphi: X \to X/Y$ be a morphism of finite groups such that $X/Y$ is an abelian $\ell'$-group, let $b$ be an $\ell$-block of $Y$, $B$ an $\ell$-block of $X$.

(a) (i) Let $(U,b_U) \subset (V,b_V)$ be an inclusion of $\ell$-subpairs in $Y$. If $B_U$ is an $\ell$-block of $C_X(U)$ that covers $b_U$, there is an $\ell$-block of $C_X(V)$ that covers $b_V$ and such that $(U,b_U) \subset (V,b_V)$ is an inclusion in $X$. All covering blocks of $b_V$ are so obtained.

(ii) Let be $(D,b_D)$ a maximal subpair of $Y$, $(D,b_D)$ a maximal subpair of $X$ such that $(1,b) \subset (D,b_D)$ and $(1,1) \subset (D,b_D)$. Then $B$ covers $b$ if and only if some $N_X(D)$-conjugate of $B_D$ covers $b_D$.

(b) Let $(D,b_D)$ be a maximal subpair in $Y$ with canonical character $\lambda \in \text{Irr}(C_Y(D))$ and such that $(1,b) \subset (D,b_D)$. One has $X_b = Y : N_X(D) \lambda$. 

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(c) Assume \( B \) covers \( b \) and let \( \lambda \) as in (b). Assume non multiplicity in \( \text{Ind}_{C_X(D)}^{C_Y(D)} \lambda \) and let \( \mu \) a component of \( \text{Ind}_{C_X(D)}^{C_Y(D)} \lambda \).

There is a morphism \( f: N_X(D)_\lambda \to \text{Irr}(C_X(D)/C_Y(D)) \) whose kernel is \( N_X(D)_\lambda \cap N_X(D)_\mu \). Define \( J \) by the equality \( f(N_X(D)_\lambda) = \text{Irr}(C_X(D)/C_Y(D))/J \). One has \( \rho(J) = \rho(\overline{1}_Y(B)) \). The set of blocks of \( X \) that cover \( b \) is a regular orbit under \( \text{Irr}(\rho(J)) \).

Proof. As \( |X/Y| \) is prime to \( \ell \), if \( B \) covers \( b \), then \( B \) and \( \lambda \) have a common defect group by 5.1.3.

(a) (i) Consider a “normal inclusion of subpairs” in \( Y, (U, b_U) \prec (V, b_V) \) [16], Definition 5.2 : \( U \) and \( V \) are \( \ell \)-subgroups of \( Y \) such that \( U \) is normal in \( V \), \( b_U \), \( b_V \), are \( \ell \)-blocks of respectively \( C_Y(U) \), \( C_Y(V) \) and \( V \) fixes \( b_U \). The inclusion of subpairs in \( Y \) writes with Brauer’s morphism \( \text{Br}_Y \) on the images in characteristic \( \ell : \text{Br}_Y(b_U)b_Y = b_V \) and \( b_U \) is unique when \( b_Y \) is given. We have

\[
\sum_{x \in C_X(U)/C_X(U)_{b_U}} b_U^x = \sum_{j \in C_X(U)/C_X(U)_{b_U}} B_{U,j}, \quad \sum_{y \in C_X(V)/C_X(V)_{b_V}} b_V^y = \sum_{k \in C_X(V)/C_X(V)_{b_V}} B_{V,k}
\]

where \( \{B_{U,j}\} \) is the set of blocks of \( C_X(U) \) that cover \( b_U \), an orbit under the action of \( (C_X(U)/C_Y(U))^\wedge \), or of \( (X/Y)^\wedge \) via \( \theta \mapsto \theta b_U \), and \( \{B_{V,k}\} \) is the set of blocks of \( C_X(V) \) that cover \( b_V \), an orbit under the action of \( (C_X(V)/C_Y(V))^\wedge \), hence of \( (X/Y)^\wedge \).

One has \( \sum_{x \in C_X(U)/C_X(U)_{b_U}} \text{Br}_Y(b_U^x) = \sum_{j \in C_X(U)/C_X(U)_{b_U}} \text{Br}_V(B_{U,j}) \), an equality between sums of idempotents.

As \( b_Y \) appears on the left as primitive in the center of \( kC_Y(V) \) and the sum is stable under the actions of \( C_X(V)/C_Y(V) \) and of \( (X/Y)^\wedge \), any \( B_{V,k} \) appears on the right side. That proves a normal inclusion \( (U, B_{U,j}) \prec (V, B_{V,k}) \) in \( X \), in fact a map \( (k \mapsto j) \) that commutes with the action of \( (X/Y)^\wedge \).

As inclusion of subpairs is defined by transitivity from normal inclusions, we have (i) in (a).

(ii) Given \( B \), \( B_D \) is defined modulo \( N_X(D) \)-conjugacy. (ii) follows from (i).

(b) \( \lambda \) has central defect and defines \( b_D \). By Frattini argument, as all maximal pairs in \( Y \) containing \((1,b)\) are \( Y \)-conjugate we have \( X_\lambda = Y.N_X(D)_x \).

(c) We use (a) and consider the blocks of \( C_X(D) \) that cover \( b_D \). They are given by their canonical character \( \mu_j = \text{Ind}_{C_X(D)}^{C_Y(D)} \lambda_j \), \( \lambda_j \in \text{Irr}(C_X(D)_X|\lambda) \) (\( \lambda_j \) and \( \mu_j \) have central defect). As each \( \lambda_j \) extends \( \lambda \) (no-multiplicity hypothesis) we have \( m := |C_X(D)|/|C_Y(D)| \) distinct blocks \( B_j \) with central defect of \( C_X(D) \), a regular orbit under \( (C_X(D)/C_Y(D))^\wedge \), that cover \( |C_X(D)|/|C_X(D)_\lambda| \) blocks of \( C_Y(D) \). As \( X/Y \) is abelian we have \( [N_X(D), C_X(D)] \subseteq C_Y(D) \), hence \( N_X(D)_\mu = N_X(D)_B \) does not depend on \( j \), denote it \( N_X(D)_\mu \). If \( x \in N_X(D) \) and \( \mu_j^x = \mu_k \), then, by restriction to \( C_X(D) \), \( B_j^x = B_D \). By conjugacy of maximal subpairs containing a block, a block of \( X \) that covers \( b \) corresponds to an orbit under \( C_X(D).N_X(D)_\mu \) in the set of blocks of \( C_X(D) \) that cover \( b_D \). Recall that \( N_X(D)_b_D = N_X(D)_\lambda \). We have a regular orbit under \( C_X(D).N_X(D)_\lambda/C_X(D).N_X(D)_\lambda \cap N_X(D)_\mu \), a quotient isomorphic to \( N_X(D)/N_X(D)_\lambda \cap N_X(D)_\mu \).

We have obtained an exact sequence

\[
1 \to N_X(D)_\lambda \cap N_X(D)_\mu \to N_X(D)_\lambda \to \left( \text{Ind}_{C_Y(D)}^{C_X(D)}(B_D)/C_Y(D) \right)^\wedge \to \left( \text{Ind}_{C_Y(D)}^{C_X(D)}(B)/Y \right)^\wedge \to 1
\]

If, in Proposition 5.1.4, \( \lambda \) extends to \( \mu \in \text{Irr}(C_X(D)) \) and \( \mu \) is stable under \( N_X(D)_\lambda \) — and that is the case if \( B \) is the principal block — then \( \rho(C_Y(D)^\wedge) \) acts regularly on the set of blocks of \( X \) that cover \( b \).

5.1.5. Proposition. Let \( Y \subseteq X \) be finite groups, \( Y \) invariant in \( X \), and \( b \) a block of \( Y \), \( B \) a block of \( X \) such that \( B \) covers \( b \). Assume that \( B \) has central defect.
(a) If there is χ ∈ Irr(B) of height zero and whose restriction to Y has multiplicities prime to ℓ, then b has central defect group and Z(Y)ℓ = Y ∩ Z(X)ℓ.

(b) If X/Y is ℓ-solvable, then (a) applies and b has central defect group.

(c) If b has central defect, there is χ ∈ /IB as in (a).

Proof. There is some χ ∈ Irr(B) whose kernel contains the defect group, of height 0 : χ(1)ℓ = |X/Z(X)|ℓ.

Let ξ ∈ Irr(b) such that χ ∈ Irr(X | ξ) [28], Chapter 5, Lemma 5.7.

(a) If (ResV χ, ξ)Y is prime to ℓ, and that is clearly true if X/Y is an ℓ'-group or if X/Y is cyclic, then we have χ(1)ℓ = ξ(1)ℓ |X : Xξ|ℓ, hence ξ(1)ℓ = |Xξ/Z(X)|ℓ. We have Z(X) ⊆ Xξ ⊆ Xb. By 5.1.3 on defect groups of covering blocks we have |Xb/YZ(X)|ℓ = 1. Thus |Xb|ℓ = |Xξ|ℓ = |Y.Z(X)|ℓ, hence ξ(1)ℓ = |Y/Y ∩ Z(X)|ℓ. But Y ∩ Z(X) ⊆ Z(Y) and ξ(1) divides |Y/Z(Y)|. Hence Z(Y)ℓ = Y ∩ Z(X)ℓ and the defect group of b is Z(Y)ℓ.

(b) The condition Z(Y)ℓ = Z(X)ℓ ∩ Y is equivalent to Z(Y)ℓ ⊆ Z(X) and to Z(Y/Z(X))ℓ = Z(X)ℓ. If true that condition implies similar ones for any group R between Y and X: we have then Z(Y)ℓ ⊆ Z(R) and Z(R)ℓ ⊆ Z(X). If X/Y is ℓ-solvable there is a sequence 1 = K0 ⊆ K1 ⊆ . . . ⊆ Kk ⊆ Kk+1 = X such that each quotient Kk+1/Kk is a cyclic ℓ-group or an ℓ'-group, hence (a) applies.

(c) If b has central defect as B, then a := ⟨ResV χ, ξ⟩Y is prime to ℓ for some χ :

Let ξ be the canonical character of b and assume χ ∈ Irr(X | ξ). We have ξ(1)ℓ = |Y/Z(X)|ℓ and χ(1) = aξ(1)|X|/|Xξ|ℓ, |Xξ|ℓ = |Xξ/Z(X)|ℓ. Hence χ(1)ℓ, |Z(Y)/Z(X)|ℓ = aℓ|X/Z(X)|ℓ. But χ(1) divides |X/Z(X)| and Z(Y)ℓ = Y ∩ Z(X)ℓ, so that aℓ = 1.

An alternative proof when Y/X is an ℓ-group :

As Z(X)ℓ is in the kernel of the canonical character χ of B, Z(X)ℓ ∩ Y is contained in the kernel of any component ξ of Irr(b) ∩ ResV χ, we may assume Z(X)ℓ = 1. Then by Xb = Y, hence Xξ = Y. It follows that ξ is projective as is χ, so that b has null defect and Z(Y)ℓ = 1. Coming back to the general case Z(H)ℓ ⊆ Z(G)ℓ ∩ H.

5.2. On unipotent series

The following reminds us a result of Geck [25]. If d = 1, by the theory of Hecke algebras, in an Harish-Chandra series E(GF, (L, α)), to the sign representation of the associated Hecke algebra there corresponds a unique irreducible character whose degree has maximum p-value and with multiplicity 1 in RGF α. In Proposition 5.2, (a) we find in classical types an irreducible character whose degree has minimum p-value and with multiplicity 1 in RGF α. Assertion (b) of Proposition 5.2 is true for d = 1 by Geck’s argument. We use notations and definitions of section 2.2. Our reference is [26], Appendix.

5.2. Proposition. Let (G, F) be a connected reductive group defined on Fq. Let d be a positive integer and (L, α) a d-cuspidal unipotent datum of (G, F), defining the Generalized d-Harish-Chandra series E(GF, (L, α)) ⊂ E(GF, 1). Assume d > 1.

(a) Assume all components of G have classical type. Let χ0 ∈ E(GF, (L, α)) such that χ(1)p is minimal on χ0 for χ running in the G.d-HC series χ0 ∈ E(GF, (L, α)). Then χ0(1)p ≤ α(1)p, with equality if p > 2, and χ0(1) ≠ χ(1) for any other χ ∈ E(GF, (L, α)). Furthermore |⟨RGF α, χ0⟩GF | = 1.

(b) In any type there exist χ ∈ E(GF, (L, α)) such that ⟨RGF α, χ⟩GF ∈ {−1, 1} and ξ(1) ≠ χ(1) for any ξ ∈ E(GF, (L, α)) different from χ.

Proof. By inspection. As the unipotent Lusztig series is invariant in an isotypic morphism (see 1.3.1), we may assume that G is rationally irreducible, finally is irreducible.
(a.1) Type $A_n$.

Assume first that $G$ is split. One has a parametrization of $\mathcal{E}(G^F, 1)$ by partitions of $n + 1$. A partition $\Lambda = \{0 \leq \lambda_1 \leq \ldots \leq \lambda_m\}$ may be defined by a so-called “$\beta$-set” $B(\Lambda) = \{0 < b_1 < \ldots < b_m\}$ where $b_j = \lambda_j + j \ (1 \leq j \leq m)$. One may assume $m$ as large as we want, adding 0 at the beginning of $\{\lambda_j\}_j$. Assume $\Lambda$ define $\chi \in \mathcal{E}(G^F, 1)$. By the degree formula given in [26] Appendix, we have $\chi(1)_\rho = q^{v(B(\Lambda))}$, where

$$v(B(\Lambda)) = \sum_{1 \leq j < m} (m - j)b_j - \sum_{1 \leq j < m - 1} \binom{m - j}{2}$$


If $\Lambda_1$ is a partition of $(n + d + 1)$ with the same number of components than $\Lambda$, one says that $\Lambda$ may be deduced from $\Lambda_1$ by deleting a hook of length $d$ if and only if $B(\Lambda_1) = B(\Lambda) + \{b_j, b_j + d\}$ for some $j \in [1, m]$ (here $+$ is boolean addition and $b_j \in B(\Lambda)$, $b_j + d \in B(\Lambda_1)$). Clearly if $j = m$ then $v(B(\Lambda_1)) = v(B(\Lambda))$.

Let $k$ such that $b_k < b_j + d < b_{k+1}$ (with $k = m$ if $b_m < b_j + d$). By the above formula we have

$$v(B(\Lambda_1)) - v(B(\Lambda)) = \sum_{j<s \leq k} b_s + (m - k)d - (k - j)b_j = \sum_{j<s \leq k} (b_s - b_j) + (m - k)d$$

The two terms on the right are non negative and they are null only if $j = k$ and $m = k$, that is $j = m$.

A $d$-cuspidal element in $\mathcal{E}(G^F, 1)$ is defined by a $d$-core $\kappa$, or partition without any hook of length $d$, equivalently : if $b_j \in B(\kappa)$ and $b_j \geq d$, then $b_j + d \in B(\kappa)$. A $d$-cuspidal unipotent datum in $(G, F)$ is defined by a $d$-split Levi subgroup of type $A_{n-td}$ ($t \in \mathbb{N}$) and $\lambda \in \mathcal{E}(L^F, 1)$, $\lambda$ being defined by a $d$-core. The elements of $\mathcal{E}(G^F, (L, \lambda))$ corresponds to the set of partitions of $(n + 1)$ from which one can go down to $\kappa$ by deleting a sequence of hooks of length $d$. By what we saw in the case $t = 1$ there is a unique element $\Lambda_0$ in that set with minimal parameter $v$, precisely such that $v(B(\Lambda_0)) = v(B(\kappa))$. Furthermore there is only one way to go from $B(\Lambda_0) = \{0 < b_1 < \ldots < b_m < b_m + td\}$ to $B(\Lambda)$ by deleting a sequence of hooks of length $d$, and that implies $|\langle R_d^F \alpha, \chi \rangle_{G^F}| = 1$.

In connection with Ennola’s conjecture, there is a bijection from $\mathcal{E}(GL_{n+1}(q), 1)$ to $\mathcal{E}(GU_{n+1}(q), 1)$, $(\chi \mapsto \chi_1)$ that preserves the $p$-valuation of degrees. There is also a one-to-one map $\mathbb{N} \rightarrow \mathbb{N}$, $(d \mapsto d_1)$ and a correspondance $(L \mapsto L_1)$ between $GL_{n+1}(q)$-conjugacy classes of $d$-split Levi subgroups of $GL_{n+1}$ and $GU_{n+1}(q)$-conjugacy classes of $d_1$-split Levi subgroup of $GU_{n+1}$ that commutes with Lusztig induction so that $\chi$ is $d$-cuspidal if and only if $\chi_1$ is $d_1$-cuspidal, and $\mathcal{E}(GL_{n+1}(q), (L, \lambda))$ is sent onto $\mathcal{E}(GU_{n+1}(q), (L_1, \lambda_1))$.

Hence the property we claim goes from type $A$ to type $2A$.

(a.2) Types $B_n$, $C_n$.

We use the following elementary lemma : 

**Lemma.** Let $\{b_j\}_{1 \leq j \leq N}$ be a sequence of natural integers such that $b_j > b_1$ if $j \neq 1$ and $N$ is even. Let $\varepsilon : [1, N] \rightarrow \{-1, 1\}$, $q \in \mathbb{N}$, $q \geq 2$. Then

$$\prod_{1 \leq j \leq N} (q^{b_j} - \varepsilon(j)) \neq \prod_{1 \leq j \leq N} (q^{b_j} + \varepsilon(j))$$

**Proof of the lemma.** Assume equality. The general term in the developpement of each product is defined by a subset $Y$ of $[1, N]$ and writes on the right side $t(Y) = (\prod_{y \in Y} \varepsilon(y))q^{\sum_{y \in Y} b_y}$, on the left side $(-1)^{N-|Y|} t(Y)$.

If $|Y|$ is even these terms are equal, hence simplify. If $|Y|$ is odd, they differ by a minus sign.

If $q \neq 2$, reducing modulo $q^{b_1+1}$ we obtain $2q^{b_1} \equiv 0 \pmod{q^{b_1+1}}$, a contradiction.

If $q = 2$, reducing modulo $2^{b_1+2}$ we obtain $2^{b_1+1} \equiv 0 \pmod{2^{b_1+2}}$, a contradiction.

An element $\chi$ in $\mathcal{E}(G^F, 1)$ is defined by a symbol $(B, C)$ of rank $n$, where $B \subseteq \mathbb{N}$, $C \subseteq \mathbb{N}$,
$B = \{b_0 = 0 < b_1 < \ldots < b_j < \ldots b_s\}, \ C = \{c_0 = 0 < c_1 < \ldots < c_j < \ldots c_t\}$ and $(s-t) \in \mathbb{N}$ is odd. By [26] the $p$-valuation of $\chi(1)$ may be written $(2^{[B \cap C]}_p q_v^{(B, C)} + f(s+t))$ with

$$v(B, C) = \sum_{1 \leq j \leq s} (s-j + |C ∩ [b_j, \infty[|)b_j + \sum_{1 \leq j \leq t} (t-j + |B ∩ [c_j, \infty[|)c_j + \sum_{b \in B \cap C} b$$

We note that if $b_s > c_t$ (resp. $c_t > b_s$), then $v(B, C)$ is independant of $b_s$ (resp. $c_t$), but, as we’ll see, is growing up with the other values $b_j, c_j$.

(i) Assume $d$ odd. Then $\alpha$ is defined by a symbol $(B_0, C_0)$ of rank $k$, where $(n-k) ∈ d\mathbb{N}$, and such that the partitions associated to $B_0$ and $C_0$ respectively are $d$-cores. The symbol $(B, C)$ defines an element of $E(G^F, (L, \alpha))$ if and only if one can go from $B$ to $B_0$, and from $C$ to $C_0$ deleting a sequence of hooks of length $d$. Let $(B, C)$ as above and consider $(B_1, C_1)$ with $B_1 = B + \{b_j, b_j + d\}$ for some $j$ such that $b_j + d \notin B$. Let $r \in [j, s]$ maximum such that $b_r < b_j + d$. One has $v(B_1, C_1) - v(B, C) = \sum_{1 \leq j \leq r} (b_i - b_j) + (s-r)d + |C ∩ [b_j, \infty[|b_j + d, \infty[|d + \sum_{i(b_i < c_i \leq b_j + d)(c_i - b_j)$.

It is a sum of four non negative integers that are all null only under the conditions: $b_j$ is maximum in $B$ (that is $j = s$ and $C \subseteq [0, b_j]$), hence $b_s \geq c_t$. For $p \neq 2$ these are the conditions under which the $p$-valuation of the degree of corresponding irreducible unipotent character does not increase when going from $(B, C)$ to $(B_1, C_1)$.

For $p = 2$ we have to take account of the factor $(2^{[B \cap C]}_p)$: to obtain the variation of the $p$-contribution to the degree, multiply $q_v^{(B_1, C_1)} - v(B, C)$ by 2 if $b_j + d \in C$ but $b_j \notin C$, and by $1/2$ if $b_j \in C$ but $b_j + d \notin C$. The valuation in 2 may decrease strictly if $c_s = b_t$ and $j = t$.

Assume $\text{sup}(C_0) \neq \text{sup}(B_0)$. By our description in type $A$ and the value of $v(B, C)$ above, there exists one and only one symbol $(B, C)$ of rank $n$ in the $d$-series above $(B_0, C_0)$ such that $v(B, C) \leq v(B_0, C_0)$ and then $v(B, C) = v(B_0, C_0)$ if $b_s > c_t$, it is $(B_1, C_1) = (B_0 + \{b_s, b_s + n-k\}, C_0)$. If $\chi_1 \in E(G^F, 1)$ has parameter $(B_1, C_1)$, $\chi_1(1)_p$ is minimal in the $d$-series $E(G^F, (L, \alpha))$ (sketch specially the case $p = 2$, assuming $d > 1$). Then $|\text{Res}^F \chi_1|_{G^F} = 1$, because there is only one way to go from $(B_1, C_1)$ to $(B_0, C_0)$ by deleting successively $(n-k)/d$ hooks of length $d$.

Assume now $a := \text{sup}(C_0) = \text{sup}(B_0)$. There are exactly two different symbols

$$(B_1, C_1) = (B_0, C_0 + \{a, a+n-k\}), \quad (B_2, C_2) = (B_0 + \{a, a+n-k\}, C_0)$$

of rank $n$ in the $d$-series of $(B_0, C_0)$ such that $v(B_1, C_1) \leq v(B_0, C_0)$, $(i = 1, 2)$, and then $v(B_1, C_1) = v(B_0, C_0)$ (short of the case $p = 2$, see above). We have to compare the degrees of the $\chi_i \in E(G^F, 1)$ corresponding to $(B_i, C_i)$ $(i = 1, 2)$. The degree formula given in [26] shows that if $(B, C)$ is the parameter of $\chi \in E(G^F, 1)$, $\chi(1)_p$ is, up to a constant depending only of $(|G|, s, t)$ equal to

$$\prod_{\gamma} \frac{(2^{[B \cap C]}_p q_v^{(\gamma)} - 1)}{\prod_{\delta}(q_v^{(\gamma)} - 1)\prod_{\gamma} (q_v^{(\delta)} + 1)}$$

where $\gamma$ (resp. $\gamma_1, \delta$) runs on the set of hooks of $B$ (resp. hooks of $C$, cohooks of $(B, C)$) and the function $l$ design the length of the hook or cohook. We have

$$\left(2^{p'}(q^{a-k}_v - 1) \prod_{1 \leq a \leq n-k-1} (q^{2a}_v - 1) \frac{\chi_1(1)}{\alpha(1)}\right)^{p'} = \prod_{c < a, c \notin C_0} \frac{q^{a-c}_v - 1}{q^{a+n-k-c}_v - 1} \prod_{b < a, b \notin B_0} \frac{q^{a-b}_v + 1}{q^{a+n-k-b}_v + 1}$$
\[
\left(2p'.(q^{n-k} - 1)\right) \prod_{1 \leq u \leq n-k-1} (q^{2u} - 1) \left(\frac{\chi_2(1)}{\alpha(1)}\right)_{p'} = \prod_{c \leq a, c \notin C_0} \frac{q^{a-c} + 1}{q^{a+n-k-c} + 1} \prod_{b \leq a, b \notin B_0} \frac{q^{a-b} - 1}{q^{a+n-k-b} - 1}
\]
hence
\[
\left(\frac{\chi_1(1)}{\chi_2(1)}\right)_{p'} = \prod_{c \leq a, c \notin C_0} \frac{(q^{a-c} - 1)(q^{a+n-k-c} + 1)}{(q^{a+n-k-c} - 1)(q^{a-c} + 1)} \prod_{b \leq a, b \notin B_0} \frac{(q^{a-b} + 1)(q^{a+n-k-b} - 1)}{(q^{a+n-k-b} + 1)(q^{a-b} - 1)}
\]
Let \( \bar{B} := \{a - b \mid b < a, b \notin B_0\}, \bar{C} = \{a - c \mid c < a, c \notin C_0\}, D := n - k + \bar{B}, E = n - k + \bar{C} \). Assume \( \chi_1(1)_{p'} = \chi_2(1)_{p'} \). We have
\[
\prod_{x \in \bar{C}} (q^x - 1) \prod_{x \in \bar{B}} (q^x - 1) \prod_{x \in D} (q^x - 1) \prod_{x \in E} (q^x - 1) = \prod_{x \in \bar{C}} (q^x + 1) \prod_{x \in \bar{B}} (q^x + 1) \prod_{x \in D} (q^x + 1) \prod_{x \in E} (q^x - 1)
\]
in which we may exchange \((\bar{B}, E)\) and \((\bar{C}, D)\). If \( x \in \bar{B} \cap \bar{C} \), the factor \((q^{2x} - 1)\) appears in each side of that equality. So we may assume \( \bar{B} \cap \bar{C} = \emptyset \), as well as \( D \cap E = \emptyset \). Similarly if \( x \in \bar{C} \cap E \) or \( x \in \bar{B} \cap D \) there is a simplification by \((q^{2x} - 1)\). Finally we have an equality as above with \( \bar{C} \cap \bar{B} = \emptyset = D \cap E = \emptyset \), \( \bar{B} \neq \emptyset, |\bar{C}| = |E|, |\bar{B}| = |D| \). Then let \( y \) be the smallest element of \( \bar{B} \cup \bar{C} \cup D \cup E \). If \( y \in \bar{B} \) (resp. \( y \in \bar{C} \)), then \( y \notin D \), (resp. \( y \notin E \)) because \( \Inf(D) \geq n - k + \Inf(\bar{B}) \) (resp. \( \Inf(E) \geq n - k + \Inf(\bar{C}) \)). The Lemma applies : \( \chi_1(1)_{p'} \) and \( \chi_2(1)_{p'} \) are different.

The equalities \(|R^G_{L_0} \alpha, \chi_1\alpha|_{GF} = 1 = |R^G_{L_0} \alpha, \chi_2\alpha|_{GF} = 1\) are true as in the first case.

(ii) Assume \( d \) even.

In that case \( \alpha \) is defined by a symbol \((B_0, C_0)\) with no cohook of length \( d/2 \). If an element \( \chi \in \mathcal{E}(G^F, 1) \) defined by a symbol \((B, C)\), it belongs to the \( d \)-series \( \mathcal{E}(G^F, (L, \alpha)) \) if one can go down from \((B, C)\) to \((B_0, C_0)\) by deleting a sequence of cohooks of length \( d/2 \).

As in (i) the degree formula imply that if \( \Sup(B_0) > \Sup(C_0) \) there is only one symbol \((B_1, C_1)\) of rank \( n \) in the \( d \)-series of \((B_0, C_0)\) such that \( v(B_1, C_1) = v(B_0, C_0) \), that is \((B_0 + \{b_s\}, C_0 + \{b_s + n - k\})\) if \( 2(n-k)/d \) is odd, and \((B_0 + \{b_s, b_s + n - k\}, C_0)\) if \( 2(n-k)/d \) is even. The sequence of cohooks to go down from \((B_1, C_1)\) to \((B_0, C_0)\) is unique so that, if \( \chi_1 \in \mathcal{E}(G^F, 1) \) has parameter \((B_1, C_1)\) then \( (R^G_{L_0} \chi_1)_{GF} = 1 \).

Assume \( \Sup(B_0) = \Sup(C_0) = a \). We find two symbols of rank \( n \) in the \( d \)-series of \((B_0, C_0)\) : \((B_0 + \{a\}, C_0 + \{a + n-k\})\) and \((B_0 + \{a + n-k\}, C_0 + \{a\})\). Listing variations on the sets of hooks and cohooks between \((B_0, B_0)\) and \((B_i, C_i)\) \((i = 1, 2)\) one finds the same quotient \([\chi_1(1)/\chi_2(1)]_{p'}\) as above, hence \( \chi_1(1)_{p'} \neq \chi_2(1)_{p'} \).

(a) Types \( \text{D}_n, 2\text{D}_n \).

An element \( \chi \) in \( \mathcal{E}(G^F, 1) \) is defined by a symbol \((B, C)\) of rank \( n \), where \(|B| - |C|| \in 4\mathbb{N}\) if \((G, F)\) is split and \(|B| - |C| = 2 \pmod{4}\) if not. Furthermore, if \( B \neq C \), \((B, C)\) and \((C, B)\) define the same irreducible character, but \((B, B)\) defines two \( \chi \in \mathcal{E}(G^F, 1) \), said to be “twins”. A \( d \)-cuspidal \( \lambda \) is defined par a symbol \((B_0, C_0)\) without any hook of length \( d \) (in case \( d \) is odd) or without any cohook of length \( d/2 \) (in case \( d \) is even). The \( p \)-valuation and the \( p' \)-component of the degrees are given by the same formulas as in type \( \text{B} \), up to a constant factor. If \((B_0, C_0)\) is not symmetric the symbols \((B_i, C_i)\) given in (i) (if \( d \) is odd) or (ii) (if \( d \) is even) are not symmetric. The proofs given in (a.2) go on.

Thus we consider a symmetric symbol \((B_0, C_0)\). The twins element of \( \mathcal{E}(L^F, 1) \) defined by \((B_0, C_0)\) are \( N_{G^F}(L)-\text{conjugate} \) if \( L \neq G \). Then the symbols \((B_i, C_i)\), as defined in (a.2), are not symmetric and verify \((B_i, C_i) = (C_2, C_2)\), hence we obtain a unique \( \chi \in \mathcal{E}(G^F, 1) \) with claimed properties.

(b) Exceptional types.

One may conjecture that (a) generalizes to exceptional types as well as the following variation :
There exist $\chi \in \mathcal{E}(G^F,(L,\alpha))$ such that
for all $\xi \in \mathcal{E}(G^F,(L,\alpha))$, $\xi(1)_p$ divides $\chi(1)_p$; if $\xi(1)_p = \chi(1)_p$ and $\xi \neq \chi$ then $\xi(1) < \chi(1)$.
Furthermore $|\langle R^G_L \alpha, \chi \rangle|_{G^F} = 1$.

But the verification is quite boresome because in some G.d-HC unipotent series one may find different elements with equal generic polynomial degree. In all cases there is a unique element whose generic degree has minimal $q$-valuation, or a unique element whose generic degree has maximal $q$-valuation, and that gives (b).

If $L$ is a maximal torus $T$, so that $\alpha = 1_{TF}$, $1_{GF}$ has the property, and the Steinberg character the opposite one, thanks to [20], 12.7, 12.8.

The $d$-unipotent series $\mathcal{E}(G^F,(L,\alpha))$ when $L$ is not a torus are given in [10], Table 1, Table 2, with multiplicities $\langle \chi, R^G_L \alpha \rangle_{G^F}$. The generic degrees, polynomials in $q$, of element of $\mathcal{E}(G^F,1)$ are as in [26, Appendix]. The verification is immediate.

In series that are known only by the sum of two algebraic conjugate, as $15 + 16$, $40 + 41$, $42 + 43$ in notations of [10], two algebraic conjugate irreducible components (which have equal degree) are always in different series.

5.3. On a commutation formula in classical type

Here we prove the commutation formula (J3) we considered in section 1.3.3, for groups of classical type.

5.3. Proposition. Assume $G$ has classical type and a connected center. Let $s \in G^*^F$ be semi-simple, let $L^*$ and $M^*$ be $F$-stable Levi subgroups in $G^*$ such that $s \in L^* \subseteq M^*$, and $L \subseteq M$ be $F$-stable Levi subgroups of $G$ in dual $G^F$-conjugacy classes. Assuming Mackey decomposition formula between $F$-stable Levi subgroups of $G$, one has

$$R^M_L \circ \Psi_{L,s} = \Psi_{M,s} \circ R^{C_{M^*}(s)}_{C_{L^*}(s)}$$

Proof. By our general conventions we assume that the duality between $L$ and $L^*$, $M$ and $M^*$, $G$ and $G^*$ may be defined around the same pair of dual maximal tori $(T \subseteq L, T^* \subseteq L^*)$ and so on for any “dual” sets $\{L_j\}_j$, $\{L_j^*\}_j$ of Levi subgroups of $G$, $G^*$ with a common maximal torus $T \subseteq \cap_j L_j$, $T^* \subseteq \cap_j L_j^*$. Similar restrictions apply to dualities over $\mathbb{F}_q$ between $F$-stable groups. It follows that the choice of duality between $L$ and $L^*$, hence the eventual different choices of $\Psi_{L,s}$, if $s \in L^*$, do not affect $R^M_L \circ \Psi_{L,s}$.

As a consequence of Mackey formula $R^M_{L,P} \circ \Psi_{L,s}$ is independent of the choice of $P$. By duality (5.3.1) implies

$$\Psi_{L,s} \circ R^{C_{M^*}(s)}_{C_{L^*}(s)} = R^M_L \circ \Psi_{M,s}$$

We prove (5.3.1) by induction on the semi-simple rank of $M$. If $M$ is a torus, $L = M$, there is nothing to prove. If $L$ is a torus the formula follows from (ii) in Proposition 1.3.1.

(a) A “trivial case” is the following: it may happen that $C_{M^*}(s) = C_{L^*}(s)$ when $Z^s(L^*) \subseteq Z^s(C_{M^*}(s))$ (note that $Z^s(L^*) \subseteq C_{G^*}(s)$ and $C_{L^*}(s) = C_{G^*}(s)(Z^s(L^*))$). In that case $R^M_L$ restricts to and is defined by a bijection $\mathcal{E}(L^F,s) \to \mathcal{E}(M^F,s)$ that commutes with Jordan decomposition by (iii) of Proposition 1.3.2. So we have a special case of (5.3.1): $R^M_L \circ \Psi_{L,s} = \Psi_{M,s}$.
(b) By transitivity of Lusztig induction, if there exists some \( F \)-stable Levi subgroup \( K \) of \( G \) with \( L \subseteq K \subseteq M \), \( L^* \subseteq K^* \subseteq M^* \), such that \( \Psi_{M,s} \circ R^{C_{M^*}(s)}_{C_{K^*}(s)} = R^K_L \circ \Psi_{K,s} \) and \( \Psi_{K,s} \circ R^{C_{K^*}(s)}_{C_{L^*}(s)} = R^K_L \circ \Psi_{L,s} \), we are done. So we write \( R^M_L \) as a product of minimal "steps".

(c) To a Levi subgroup \( L \) with \( s \in L^* \) we associate an eventually smaller Levi subgroup \( L_s \) as follows: let \( L_s \subseteq L \) such that \( L_s \) is in the dual \( G^F \)-conjugacy class of

\[ L^*_s := C_{G^*}(Z^*(L^* \cap C_{G^*}(s))) \]

As \( Z^*(L^*) \subseteq Z^*(L^* \cap C_{G^*}(s)) \), one has \( L^*_s \subseteq L^* \), which allows to assume \( L_s \subseteq L \). By definition \( C_{L^*_s}(s) \) is \( C_{C_{G^*}(s)}(Z^*(L^* \cap C_{G^*}(s))) = C_{L^*}(s) \). The "trivial case" we described in (a) shows that \( R^M_L \circ \Psi_{L,s} = \Psi_{L,s} \).

Given \( L \) and \( M \) with \( L^* \subseteq M^* \), clearly \( L^*_s \subseteq M^*_s \) so we may assume that \( L_s \subseteq M_s \). Now it is sufficient to prove our formula between \( L_s \) and \( M_s \). Indeed, assuming that \( R^M_{L_s} \circ \Psi_{L,s} = \Psi_{L,s} \circ R^{C_{M^*}(s)}_{C_{L^*_s}(s)} \), we have by (a) and (b)

\[
R^M_L \circ \Psi_{L,s} = R^M_L \circ R^L_{L_s} \circ \Psi_{L,s} = R^M_{M_s} \circ R^M_L \circ \Psi_{L,s} = R^M_{M_s} \circ \Psi_{M,s} \circ R^{C_{M^*}(s)}_{C_{L^*_s}(s)}.
\]

(d) So now we assume that \( L = L_s \subseteq M = M_s \) with notations of (c) and \( L \neq M \). Thanks to (b) we assume that \( C_{L^*}(s) \) is a maximal \( F \)-stable Levi subgroup in \( C_{M^*}(s) \).

We claim that

\[ ||R^M_L(\Psi_{L,s}(\lambda))|| = ||R^{C_{M^*}(s)}_{C_{L^*_s}(s)}(\lambda)||. \tag{5.3.3} \]

Denote \( \zeta = \Psi_{L,s}(\lambda) \). The square norm of \( R^M_L \zeta \) is \( \langle R^M_L(R^L_L \zeta), \zeta \rangle_{L^F} \) and may be computed by Mackey formula. It is a sum, indexed on a set of double classes \( M^F \) in \( M^F \), of terms \( \langle R^L_{L^F \cap M^F}(R^L_{L^F \cap M^F} \zeta), \zeta \rangle_{L^F} \).

Here \( g \) is such that \( L^* \) and \( ^gL \) have a common maximal torus so that \( L \cap ^gL \) is a Levi subgroup of \( ^gL \) and \( L \). When \( g_1 \) runs in \( L^F g L^F \), \( L \cap ^gL \) runs in a complete \( L^F \)-conjugacy class of Levi subgroups of \( L \). The scalar product above is

\[ \langle R^L_{K(g)}(R^L_{K(g)}(\zeta)), \zeta \rangle_{L^F} \tag{5.3.4} \]

As \( \zeta \in E(L^F, s) \), \( n(g) \) is non zero only if for any couple \( (T, \theta) \) with \( T \subseteq K(g), \theta \in \text{Irr}(T^F) \) and \( \langle R^L_{T^F}(\theta) \rangle \neq 0 \) the \( L^F \)-conjugacy class of \( (T, \theta) \) is associated to the \( (L^*)^F \)-conjugacy class of some \( (S^*, s) \). That condition on \( \zeta \) is equivalent to \( \langle R^L_{T^F}(\theta) \rangle \neq 0 \). As \( g \in M^F \) we have \( \langle R^L_{T^F}(\theta) \rangle \neq 0 \) and \( \langle T^g, \theta \rangle \) are \( L^F \)-conjugate. But \( g \) is defined modulo \( L^F \), so we may assume

(A) \( g \in N_{M^F}(T, \theta) \) for some \( (T, \theta) \) in the \( L^F \)-conjugacy class corresponding to the \( (L^*)^F \)-conjugacy class of \( (T^*, s) \).

and then

\[ ||R^M_L(\Psi_{L,s}(\lambda))||^2 = \sum_{L^F g L^F \cap \{\lambda\}} n(g) \tag{5.3.5} \]

Let \( w = gT \in W(M, T)^F \) and \( w^* \) the image of \( w \) in the anti-isomorphism \( W(M, T) \to W(M^*, T^*) \), \( w^* \in W(C_{M^*}(s), T^*)^F \) and \( w^* = g^*T^* \) for some \( g^* \in N_{C_{M^*}(s)^F}(T^*) \). Let \( \Phi(L, T) \subseteq X(T) \) be the set of roots
of $L$ with respect to $T$. With our choice of $g$ we have $\Phi(K(g),T) = \Phi(L,T) \cap w\Phi(L,T)$. We define a dual Levi subgroup $K^*(g)$ in $L^*$ by $\Phi(K^*(g),T^*) = \Phi(L^*,T^*) \cap w^*\Phi(L^*,T^*)$. As $w^* \in W(C_{M^*}(s),T^*)$, we have $\Phi(C_{K^*(-)}(s),T^*) = \Phi(C_{L^*}(s),T^*) \cap w^*\Phi(C_{L^*}(s),T^*)$, that is

$$(5.3.6) \quad K^*(g) = L^* \cap g^* L^*, \quad C_{K^*(-)}(s) = C_{L^*}(s) \cap g^* C_{L^*}(s)$$

To $K(g)$, $g$ as in (A), we have associate a Levi subgroup $C_{L^*}(s) \cap g^* C_{L^*}(s)$ of $C_{M^*}(s)$, hence one of the terms of the Mackey formula for $R_{C_{L^*}(s)}^{C_{M^*}(s)} \circ R_{C_{L^*}(s)}^{C_{M^*}(s)}$. The contribution of that term to the square product $||R_{C_{L^*}(s)}^{C_{M^*}(s)} \lambda||^2$ is

$$(5.3.7) \quad n(g^*) := (\mu(g^*),\lambda)_{C_{L^*}(s)F} \quad \text{for} \quad \mu(g^*) := R_{C_{K^*(-)}(s)}^{C_{M^*}(s)}(R_{C_{K^*(-)}(s)}^{g^* \lambda}(g^*))$$

with $K^*(g)$ given by (5.3.6). We have

$$(5.3.8) \quad ||R_{C_{L^*}(s)}^{C_{M^*}(s)} \lambda||^2 = \sum_{C_{L^*}(s)^F g^* C_{M^*}(s)^F \subseteq M^*F} n(g^*)$$

Assume $h$ satisfy condition (A) with a torus $S$ as $g$ with $T$ and define $(K^*(h),h^*)$ from $(h,S)$ as $(K^*(g),g^*)$ from $(g,T)$. If $K^*(h)(s)$ is $C_{M^*}(s)^F$-conjugate to $K^*(g)(s)$, then $C_{L^*}(s)^F h^* C_{L^*}(s)^F = C_{L^*}(s)^F g^* C_{L^*}(s)^F$ therefore $(L^*)^F h^*(L^*)^F = (L^*)^F g^*(L^*)^F$ so that $K^*(g)$ and $K^*(h)$ are $L^*F^*$-conjugate. Then $K(h)$ and $K(g)$ are $L^*F^*$-conjugate and $L^*F h L^*F = L^*F g L^*F$.

To any of the double class $L^*F g L^*F$ ($g \in M^F$) we have to consider in (5.3.5) we associate a double class $C_{L^*}(s)^g C_{L^*}(s)$ ($g^* \in C_{M^*}(s)^F$) and that application is injective. Given a double class $C_{L^*}(s)^g C_{L^*}(s)$ where $g^* \in C_{M^*}(s)^F$ and $C_{L^*}(s) \cap g^* C_{L^*}(s)$ contains a maximal torus $T^*$ we may choose $g^* \in N_{M^*}(T^*)^F$, so that $g^* T^* = w^* \in W(C_{L^*}(s),T^*)^F \subseteq W(L^*,T^*)^F$. For $w^* \mapsto w$ and $gT = w$ ($g \in M^F$), $w$ in the isomorphic group $W(L,T)$, $C_{L^*}(s)^g C_{L^*}(s)$ is associated to $L^*F g L^*F$.

So we have defined a bijection $L^*F g L^*F \rightarrow C_{L^*}(s)^g C_{L^*}(s)$ between terms of the two decomposition formulas (5.3.5) and (5.3.8). Now (5.3.3) will follow from the equality $n(g) = n(g^*)$ for any such couple $(g,g^*)$.

On $M$-side one has $n(g) = (R_{K(g)}^L g \zeta, R_{K(g)}^L g \zeta)(K^*)^F$. As $g$ induces an isomorphism $(L,F) \rightarrow (gL,F)$, $g^*$ induces a dual isomorphism $(L^*,F) \rightarrow g^* L^*$ that fixes $s$ so that we may assume $\zeta = \Psi_{g^* s}(g^* \lambda)$. If $g \in N_M(L)^F$, then $K(g) = L$ and $K^*(g) = L^*$. In that case $n_g \neq 0$ if and only if $\zeta = \zeta$, if ond only if $g^* \lambda = \lambda$ and then $n(g) = 1 = (g^* \lambda,\lambda)_{C_{L^*}(s)^F} = n(g^*)$.

If $L \cap g^* L \neq L$, induction hypothesis applies : we have, by (5.3.2),

$$R_{K(g) \cap L^*F}^L g \zeta = \Psi_{K(g),s}(R_{C_{K^*(-)}(s)}^{C_{L^*}(s)}(g^* \lambda)), \quad R_{K(g)}^L \zeta = \Psi_{K(g),s}(R_{C_{K^*(-)}(s)}^{C_{L^*}(s)}(\lambda))$$

hence $n(g) = (R_{C_{K^*(-)}(s)}^{C_{L^*}(s)}(g^* \lambda), R_{C_{K^*(-)}(s)}^{C_{L^*}(s)}(\lambda))_{C_{K^*(-)}(s)^F} = n(g^*)$.

(5.3.3) is proved.

(e) To conclude we need a

5.3.9. Lemma. Let $(G,F)$ be any algebraic reductive group defined on $F_q$, all components of which have classical type. Let $L$ be a maximal proper $F$-stable Levi subgroup of $G$ and $\lambda \in E(L^*,1)$. Then
$R_{L,P}^G \lambda$ is uniquely defined as the element of minimal norm among the set of $\mu \in ZE(G^F,1)$ such that $\pi^G_{un}(\mu) = R^G_L (\pi^L_{un}(\lambda))$, so is independant of $P$.

(Proof of the Lemma after the end of the proof of the formula (5.3.1))

We have commutation formulas (1.3.1.4) and (1.3.2.4):

$$R^M_L \circ \pi^L_{un} = \pi^M_{un} \circ R^M_L, \quad \Psi_{M,s} \circ \pi^M_{un}(s) \circ R^C_{L,s} = \pi^M_{un} \circ R^M_L \circ \Psi_{L,s}$$

Apply the Lemma between $C_{L,s}(s)$ and $C_{M,s}(s)$, and isometry $\Psi_{M,s}$, knowing that Jordan decomposition commute with projection on the space of uniform function ((J.1) in Proposition 1.3.2) : $\Psi_{M,s}(R_{C_{L,s}}^M(s) \lambda)$ is uniquely defined as the element of minimal norm among the set of $\Lambda \in ZE(M^F,1)$ such that $\pi^M_{un}(\Lambda) = \pi^M_{un}(\Psi_{M,s}(R_{C_{L,s}}^M(s) \lambda))$, its norm being precisely $\|R_{C_{L,s}}^M(s) \lambda\|$.

We have $\pi^M_{un}(R^M_L (\Psi_{L,s} \lambda)) = \Psi_{M,s}(\pi^M_{un}(R_{C_{L,s}}^M(s) \lambda)) = \pi^M_{un}(\Psi_{M,s}(R_{C_{L,s}}^M(s) \lambda)).$ By (5.3.3), (5.3.1) is proved.

Proof of the Lemma 5.3.9. We use the fact that in classical type the application $(\lambda \mapsto \pi^G_{un}(\lambda))$ from $ZE(G^F,1)$ to $Klir(G^F)$, is one-to-one.

Let $[G,G] = G_1 G_2 \ldots G_k$ be a decomposition in a central product of rationally integrable components. The Levi subgroup $L$ writes $L = Z(G) . (L \cap G_1) \ldots (L \cap G_k)$ where $L \cap G_i$ is a Levi subgroup of $G_i$. As $L$ is maximal there is only one $i$, say $i = 1$, such that $L \cap G_i \neq G_i.$ As the set of unipotent irreducible characters is indifferent to central quotients, we may write $\lambda = \lambda_1 \mathbin{\scriptstyle \otimes} \lambda_2 \mathbin{\scriptstyle \otimes} \ldots \mathbin{\scriptstyle \otimes} \lambda_k$ where $\lambda_1 \in E((L \cap G_1)^F,1)$, $\lambda_i \in E(G_i^F,1)$ for $i \neq 1$ and $\mu \in ZE(G_1^F,1) \otimes \ldots \otimes E(G_k^F,1)$, so that $R^L_L \lambda = R^G_{L \cap G_1} \lambda_1 \mathbin{\scriptstyle \otimes} \lambda_2 \mathbin{\scriptstyle \otimes} \ldots \mathbin{\scriptstyle \otimes} \lambda_k$. Hence $\|R^L_L \lambda\|^2 = \|R^G_{L \cap G_1} \lambda_1\|^2$ and $\pi^G_{un}(R^L_L \lambda) = \pi^G_{un}(R^G_{L \cap G_1} \lambda_1 \mathbin{\scriptstyle \otimes} \pi^G_{un}(\lambda_2) \mathbin{\scriptstyle \otimes} \ldots \mathbin{\scriptstyle \otimes} \pi^G_{un}(\lambda_k))$.

(a) If $G_1$ has type $A$, every central unipotent function on $G_1^F$ is a uniform function. The condition $\pi^G_{un}(\mu) = R^L_L (\pi^L_{un}(\lambda))$ gives $\pi^G_{un}(\mu) = R^G_{L \cap G_1} \lambda_1 \mathbin{\scriptstyle \otimes} \ldots \mathbin{\scriptstyle \otimes} \pi^G_{un}(\lambda_k)$ ($L \cap G_1$ maximal in $G_1$ or not). From the condition $||\mu||^2 \leq ||R^L_L \lambda||^2$, it follows that $\mu = R^G_{L \cap G_1} \lambda_1 \mathbin{\scriptstyle \otimes} \ldots \mathbin{\scriptstyle \otimes} \mu_k$ with $||\mu_i||^2 = 1$ for $i > 1$ and $\pi^G_{un}(\mu_i) = \pi^G_{un}(\lambda_i)$, hence $\mu_i = \lambda_i$ for $i > 1$.

(b) If $G_1$ has classical type $X \in \{B,C,D\}$, then there is some integer $d$ such that $L \cap G_1$ has type $X_{k-d}$ (the rational types of $G_1$ and $(L \cap G_1)$ may differ in types $D$, $2D$, $Z^2(L \cap G_1)$ has polynomial degree $(x^d-1)$ if $d$ is odd) or $(x^{d/2}+1)$ if $d$ is even). Then $R^G_{L \cap G_1} \lambda_1$ is given by one of Asai’s formulas, a $d$-hook or $(d/2)$-cohook formula, see [9], (3.5), (3.9). These formulas show that $R^G_{L \cap G_1} \lambda_1$ is a sum $\sum_{1 \leq i \leq r} \pm \chi_j$, all $\chi_j$ in distinct families, families defined from $(W(G_1),F)$. But $\chi_j$ is uniquely defined in $E(G_1^F,1)$ by $\pi^G_{un}(\chi_j)$, and $\pi^G_{un}(R^G_{L \cap G_1} \lambda_1)$ is unique in $ZE(G_1^F,1)$ with uniform projection $\sum \pm \pi^G_{un}(\chi_j)$ and minimal square norm $r = ||R^G_{L \cap G_1} \lambda||^2 = ||R^G_L \lambda||^2$. We conclude as in (a).

5.4. More on Generalized $d$-Harish-Chandra theory

We use here results of Bonnafé [3] on type $A$ with a Frobenius endomorphism, completed by Cabanes [12] in twisted type.

5.4.1. Facts on wreath products

We need properties of the Weyl groups of centralizers of semi-simple elements in groups of type $A$.

In that section $\Sigma$ is a finite set, a finite group $B$ acts by permutations on $\Sigma$, $W^0$ is a direct product indexed on $\Sigma : W^0 = \times_{\beta \in \Sigma} W_\beta$. If $\beta$ belongs to the orbit $B\alpha \in \Sigma/B$, then $W_\beta$ is isomorphic to $W_\alpha$. 93
We identify the groups $W_\beta$ ($\beta \in B$) to $W_\alpha$ so that $B$ acts on $\times_{\beta \in B} W_\beta$ as on the set of functions from $B_\alpha$ to $W_\alpha$, with its natural group structure. When $\Omega$ is contained in an orbit under $B$ on $\Sigma$ we denote $W^\Omega = \times_{\beta \in \Omega} W(\beta)$, a subgroup of $W^0$, with a projection morphism $W^0 \rightarrow W^\Omega$, $(w \mapsto w^\Omega)$.

The fundamental property of the action of $B$ on $W^0$ is that $b \in B$ and $b\beta = \beta$ imply $b\omega = \omega$ for any $\omega \in W_\beta$. We say that $W := W^0 \times B$ is a wreath product. Note that in that sense a direct product $W^0 \times B$ is a wreath product! With standard notations, if $B$ acts faithfully and is transitive on $\Sigma$, then $W^0 \times B \cong W_\times B$.

The proofs of all properties we recall here may be reduced to the transitive case. To ease references we use notations similar to notations in [12] or [3].

As $B$ acts on $\text{Irr}(W^0) \cong \times_{\beta \in \Sigma} \text{Irr}(W_\beta)$, we identify $\text{Irr}(W^0)$ with $\times_{\beta \in \Sigma} \text{Irr}(W_\beta)$ and we use notation $\chi^\Omega \in \text{Irr}(W^\Omega)$ if $\Omega$ is as above. For any subgroup $C$ of $B$ we have a natural one-to-one map

\[(\text{Irr}(W^0)^C \rightarrow \text{Irr}(W^0)_C), \quad \chi = \chi_1 \mapsto \chi_C\]

and simplify $\chi(b)$ in $\chi_b$. Let $\chi \in \text{Irr}(W^0)$. Once the complement $B$ is fixed, there is a special canonical extension of $\chi$ to $W^0 \times B$, as well as to any subgroup $W^0 \times C$ if $C \subseteq B_\chi$, design it $\chi \times C$, as in [12]. It is characterized as the unique irreducible extension of $\chi$ to $W^0 \times C$ with natural integer value on $c$ for any $c \in C$.

The restriction of $\chi \times C$ to $W^0 c$, we denoted $\chi \times c$, may be defined as follows: for any $\omega = (c)\alpha \in \Sigma / \langle c \rangle$, an orbit under $\langle c \rangle$ in $\Sigma$, any $w^\omega = (w(\beta))_{\beta \in \omega} \in W^\omega$, put

\[(\chi^\omega \times c)(w^\omega c) = \chi^\omega_c(w(e^{[\omega \cdot -1}]^{\alpha}) \ldots w(e^{[\omega \cdot -j]}^{\alpha}) \ldots w(\alpha))\]

If $(\beta \mapsto w(\beta))$ is constant on $\omega$, then $(\chi^\omega \times c)(w^\omega c) = \chi^\omega_c(w^{[\omega \cdot]})$. Then $\chi \times C$ is defined by

\[(\chi \times C)(wc) = (\chi \times c)(wc) = (\chi \times c)(wc) = [\otimes_{\omega \in \Sigma / \langle c \rangle} (\chi^\omega \times c)](wc)\]

Thus by (4.4.1.2) $(\chi \times C)(c) = (\chi \times c)(c) = \chi_c(1)$.

Let $\hat{\chi}$ be in $\text{Irr}(W \mid \chi)$, $\chi$ be defined by $\hat{\chi}$ mod $B$-action, $\hat{\chi}$ is one of the distinct

\[\Gamma^W(\chi * \theta) := \text{Ind}_{W^0 \times B_\chi}^W(\theta \otimes (\chi \times B_\chi)) \quad \chi \in \text{Irr}(W^0)_B, \theta \in \pi B_\chi\]

5.4.1.5. Let be $W := W^0 \times B$, $\chi \in \text{Irr}(W^0)$ as above.

(i) If $C \subseteq B_\chi$, then $\text{Res}_{W^0 \times B}^\chi(\chi \times B_\chi) = \chi \times C$. For any $b \in B$, $(\chi \times B_\chi)^b = \chi \times B_\chi^b$.

(ii) If $B_\chi \subseteq C \subseteq D \subseteq B$ and $\theta \in \text{Irr}(B_\chi)$, then

\[\text{Ind}_{W^0 \times D}^{W \times C}(\Gamma^W(\chi * \theta)) = \Gamma^W(\chi * \theta)\]

(iii) Assume $B$ abelian. Let $C$ be a subgroup of $B$, $\psi \in B^\wedge$, $\theta \in (B_\chi)^\wedge$, $\rho = \text{Res}_{B_\chi} B_\chi \theta$. One has

$\psi \otimes \Gamma^W(\chi * \theta) = \Gamma^W(\chi * (\text{Res}_{B_\chi}^B \psi \otimes \theta))$ and $\text{Res}_{W^0 \times C}^\wedge(\psi \otimes \theta)) = \sum_{b \in B / C} \text{Res}_{W^0 \times C}^\wedge(\chi^b * \theta)$.

(iv) Assume $B$ abelian. Let $V \times B \times B$ be a wreath product of $W^0 \times B$, hence $V \cap W^0 = V^0$, $B_V \subseteq B$.

One has, for any $\theta \in (B_\chi)^\wedge$, $\zeta \in \text{Irr}(V_0)$, $\psi \in (B_{V,\zeta})^\wedge$,

\[\text{Res}_{W_{V,\zeta}^0 \times B}(\Gamma^W(\chi * \theta), \psi \zeta \otimes \theta)) \equiv \sum_{b \in B / B_\chi} \text{Res}_{W_{V,\zeta}^0 \times B}^\wedge(\zeta b \psi \otimes \theta)) \times \text{Res}_{B_\chi} B_\chi \theta \times \text{Res}_{B_\chi} B_\chi \zeta \psi \otimes \theta) \times \text{Res}_{B_\chi} B_\chi \zeta \psi \otimes \theta)\]

On proof. (i) and (ii) are clear from definition and characterization of canonical extensions, (iii) is a special case of (iv) with $V^0 = W^0$ and is easily deduced from definition (5.4.1.4) and Mackey's formula. Using (iii), (ii) and transitivity of restriction, one sees that (iv) is true if it is true when $\theta = 1 = \psi$. We may
extend \( \chi \mapsto \chi \times b \) by \( \mathbb{Z} \)-linearity on \( \mathbb{Z}[\text{Irr}(W^0)b] \) so that (5.4.1.2) gives \( \text{Res}_{W^0b}^W(\chi \times b) = (\text{Res}_{W^0}^W \chi) \times b \). The equality in (iv) follows by Mackey’s formula, \( B/B_\chi B_{\nu,\zeta} \) represents the set of doubles classes for \( W^0 \times B_\chi \) and \( V^0 \times (B_{\nu,\zeta}) \) in \( W^0 \times B \).

5.4.1.6. Assume \( B \) abelian. By Mellin transform one defines for any \( b \in B \) and \( \chi \in \text{Irr}(W^0)b \)

\[
\hat{\Gamma}^W(\chi \ast b) = \sum_{\theta \in (B_\chi)\wedge} \theta(b^{-1})\Gamma^W(\chi \ast \theta)
\]

Then \( \hat{\Gamma}(\chi \ast b) \) extends \( |B_\chi| \text{Res}_{W^0b}^W(\Gamma(\chi \ast 1)) \) by 0 outside \( W^0b \) and \( \langle \hat{\Gamma}(\chi \ast b), \hat{\Gamma}(\chi \ast b) \rangle_{W^0b} = |B_\chi| |B| \). The family \( \{\chi \ast b\}_{\chi \in \text{Irr}(W^0)b} \) is an orthonormal basis of \( \text{Cent}(W^0b) \).

5.4.2. G.d-HC in type A

Bonnafé introduced the following conjecture, assuming \( p \) is good for \( G \), and \( \Gamma^M_\nu, \Gamma^L_\nu \), are Gelfand-Graev characters of \( M \) and \( L \) respectively associated to regular unipotent elements \( u_M \) (of \( M \)) and \( u_L \) (of \( L \)) [3] 14.E.

5.4.2.0. Conjecture \( \mathcal{G} \). If \( L \) is an \( F \)-stable Levi complement of a parabolic subgroup \( P \) of an \( F \)-stable Levi subgroup \( M \) of \( G \), with a coherent choice of regular unipotent elements \( u_M, u_L \) one has \( \text{Res}_{L \subset P}^M \Gamma^M_{u_M} = \Gamma^L_{u_L} \). (recall that we have included the sign \( \epsilon_L \epsilon_M \) in our definition of \( \text{Res}_{L \subset P}^M \)).

Conjecture \( \mathcal{G} \) is verified if \( p \) is good and \( [\mathbb{Z}(G) \) is connected or \( F \) is a Frobenius \( \mathbb{F}_q \)-endomorphism and \( q \) large enough \].

5.4.2. Proposition. Assume all components of \( G \) has type \( A \) and Conjecture \( \mathcal{G} \) holds in \((G, F)\). Then Generalized \( d \)-Harish-Chandra theory holds in any series \( \mathcal{E}(G^F, s) \), any \( d \).

Proof. In part (A) we describe G.d-HC series in type \( A \) when the center of \( G \) is connected. In part (B), thanks to the description of \( \mathcal{E}(G^F, s) \) in \( \text{SL}_n(q) \) and \( \text{SU}_n(q) \) due to Bonnafé and Cabanes, we describe G.d-HC series in the general case by their image in the set of irreducible characters of the Weyl group of \( \mathcal{C}_{G^F}(s) \), see (B.3.2).

(A) Connected center.

(A.1) The origin of G.d-HC for \( G := \text{GL}_n \), \( G^F = \text{GL}_n(\mathbb{F}_q) \) is Murnaghan-Nakayam formula in symmetric groups. We recall briefly these facts. The Weyl group \( W \) with respect to a maximal diagonal torus in \( G \) is the symmetric group \( \mathfrak{S}_n \) on which \( F \) acts trivially. One has a one-to-one map, defined using Deligne-Lusztig induction, [20] 15.8

\[
\text{Irr}(W) \to \mathcal{E}(G^F, 1), \quad \mu \mapsto R_\mu
\]

and the map \( \mu \mapsto R_\mu \) transform induction (on \( W \)-side) in Lusztig induction (on \( (G, F) \)-side).

Let \( \delta \) be the order of \( \mathbb{F}_q \) modulo \( \ell \), so that \( \delta \) differs from \( d \) in twisted type : if \( \epsilon = -1 \) and \( \ell \neq 2 \), then \( d = \delta, \delta/2, 2\delta \) resp. when \( \delta \equiv 0, 2, (1 \text{ or } 3) \pmod{4} \) resp.). If \( \delta = 1 \), then there is only one G.d-HC series in \( \mathcal{E}(G^F, 1) \). Assume \( \delta \neq 1 \).

Let \( S \) be a subgroup of \( \mathfrak{S}_n \) generated by \( m \) cycles of length \( \delta \) with disjoints supports. The product \( v \) of these \( m \) cycles is a parameter with respect to a diagonal maximal torus of \( G \) for a maximal torus \( T_v \) in \( G \), the centralizer of \( v \) in \( \mathfrak{S}_n \) has the form \( \mathfrak{S}_{n-m\delta} \times (S \times \mathfrak{S}_n) \) and \( \mathfrak{S}_{n-m\delta} \) is the Weyl group of the \( d \)-split Levi subgroup \( L_v := C_G((T_v)_{\Phi_q}) \) with respect to \( T_v \). Any Levi subgroup of \( G \) occurring in a unipotent \( d \)-cuspidal datum in \((G, F)\) is so obtained. A \( d \)-split Levi subgroup of \( G \) that contains \( L_v \) up to \( G^F \)-conjugacy is defined
by a partition of $m$, say $m = \sum_j m_j$, and has Weyl group $\mathfrak{S}_{n-m\delta} \times (\times_j [\mathfrak{S}_{m_j}]^\delta)$ where $F$ acts regularly on each direct product $[\mathfrak{S}_{m_j}]^\delta$. Any $d$-split Levi subgroup of $G$ contains some $L_v$ (with $m$ large enough).

The Murnaghan-Nakayama formula allows to compute $\text{Res}_W^W(\mu)$. When we consider that restriction to the class $W(L_v)v$ as a central function on $W(L_v)$, we write it $\text{Res}_W^W(\mu)(L_v)v$. With these notations all unipotent central functions being uniform functions, $\text{Res}_W^W(\mu)(L_v)v$ gives $R^W_{L_v} \mu$ (see [20], 15.7, 15.8). As well we may define $\text{Ind}_W^W(\mu)$ so that $\text{Ind}_W^W(\mu)(L_v)v$ gives $R^W_{L_v} \mu$ when $v \in \text{Irr}(W(L_v))$.

If $\mu = [\lambda] \in \text{Irr}(\mathfrak{S}_n)$ is defined by a partition $\lambda$ of $n$ with no hook of length $\delta$ — we say that $\lambda$ is a $\delta$-core — then, by Murnaghan-Nakayama formula, $\mu$ has null value on any element whose cycle decomposition has a cycle whose length belongs to $\delta N$. The type of any maximal torus in a proper $d$-split Levi subgroup of $G$ contains such a cycle, that’s why the corresponding $R_{\mu}$ is $d$-cuspidal in our first sense (2.4). The set of partitions of $n$ is a disjoint union of “$\delta$-HC series” defined for each $\delta$-core $\kappa$ of size $(n - m\delta)$ for some $m$, hence $\text{Irr}(\mathfrak{S}_n)$ is a disjoint union of “$\delta$-HC series” defined for each “$\delta$-cuspidal” $\mu = [\kappa] \in \text{Irr}(\mathfrak{S}_{n-m\delta})$ ($\kappa$ a $\delta$-core) for some $m$. With $v$ of cycle type $\delta^m$ as above is defined a series in $\text{Irr}(\mathfrak{S}_n)$:

$$\text{Sp}^\delta_n(m, \delta, \mu) = \{ \lambda \in \text{Irr}(\mathfrak{S}_n) \mid \text{Res}_{\mathfrak{S}_{n-m\delta} \times (\times_j [\mathfrak{S}_{m_j}]^\delta \mu)}^\lambda = c\mu, c \neq 0 \}$$

these series form a partition of $\text{Irr}(\mathfrak{S}_n)$ when $\mu = [\kappa]$ and $\kappa$ runs on the set of $\delta$-cores such that $n - ||\kappa|| \in \delta N$. In (5.4.2.2) $c \in \mathbb{Z}$.

On $(G, F)$-side, $(L_v, R_v)$ is a unipotent $d$-cuspidal datum. The relation $\langle \lambda, \text{Ind}_{\mathfrak{S}_{n-m\delta} \times (\times_j [\mathfrak{S}_{m_j}]^\delta \mu)}^\delta \rangle$ is not 0 is equivalent to $\langle R_{\lambda}, R^W_{L_v} \mu \rangle_{G F} \neq 0$. Furthermore, the two scalar products are equal up to a sign $\epsilon_\mu \epsilon_\lambda$ and this sign is 1 in non twisted type. Thus one has a partition of $E(G^F, 1)$ in $G.d$-HC series $E(G^F, (L_v, R_v))$,

$$\text{(5.4.2.3)}$$

$$E(G^F, (L_v, R_v)) = \{ R_\lambda \mid \lambda \in \text{Sp}^\delta_n(m, \delta, \mu) \}$$

(A.2) Unipotent $G.d$-HC series.

The generalization of $G.d/(\epsilon G)$-HC to any direct product of symmetric groups is immediate.

From the “invariance” of the set of unipotent irreducible characters in isotypic morphisms and properties of direct products, we may assume that $G$ is rationally irreducible. It may happen that the $\mathfrak{F}_e$-endomorphism acts non trivially on the Dynkin diagram of $G$ or/and of $L_v$. If it is the case we have to consider $F$-conjugacy classes in $W$, central functions on $WF$, to recover Lusztig induction.

So we consider the scalar descent : the Weyl group $W$ with respect to a maximal diagonal torus is a wreath product $W = \mathfrak{S}_n \times (\times_j [\mathfrak{S}_{m_j}]^\delta)$, so $E$ acts on $\Sigma$ as a transitive permutation, and $\phi$ is defined by $F$ (see 5.4.1 with $B = (\phi)$).

One has a one-to-one map $\text{Irr}(W^0)^\phi \to E(G^F, 1)$, $(\chi \mapsto \chi \times \phi \mapsto R_\lambda)$ that generalizes (5.4.2.1).

Let $n$ such that $W_\beta \cong \mathfrak{S}_n \cong W_0^0$, let $r$ be the order of $\phi$, $r = ||\Sigma||$. Let $m$ be a positive integer such that $m\delta < n$, the set of elements $(v_\beta)_{\beta \in \Sigma}$ in $W_0^0$ such that $v_\beta$ has cycle type $\delta^m$ for any $\beta \in \Sigma$ is an $F$-stable $W_0^0$-conjugacy class whose intersection with $W_0^0 F$ is a conjugacy class of $W_0^0$. Take $v$ inside with $F(v) = v$.

We have $C_{W_0^0}(v) = V_0^0.\mathfrak{S}_0$ where $\mathfrak{S}_0$ is generated by the cycles of $v$, $V_0^0 \cong \times_{\beta \in \Sigma} [\mathfrak{S}_{m-\delta\beta}]^\delta$, $S^0$ and $V_0^0$ are $F$-stable subgroups of $W_0^0$, $S^0 \times (\phi)$ and $V_0^0 \times (\phi)$ are sub-wreath products of $W$ and $V_0^0$ is the Weyl group of the $d$-split Levi subgroup $L_v$ of $G$, $C_{G}((T_v)_\phi, d)$ with respect to $T_v$. The action of $F$ on $W(L_v, T_v)$ is given by $v\phi$, acting as $\phi$.
We want to compute $\text{Res}_{V^\phi}^W (\tau)$ (the transpose map $\text{Ind}_{V^\phi}^W$ is called induction tordue in [1], 3.1). With notations introduced in 5.4.1, (5.4.1.4), if $\chi \in \text{Irr}(W^\phi)$, $\theta \in \langle \phi \rangle^\wedge$, $h \in V^0$, then

$$\Gamma^W(\chi \star \theta)(hv\phi) = \theta(\phi)(\chi \times \phi)(hv\phi)$$

But, as $\phi v = v \phi$ and $[v, V^0] = 1$, $(\chi \times \phi)(hv\phi) = \chi(h'v'^r)$ where $h' \in V^0 \phi$ is given by (5.4.1.2), with $(\phi, h)$ instead of $(c, w^o)$. Let $\delta = \delta'(\delta, r)$, where $(\delta, r)$ denote the GCD of $\delta$ and $r$, so that $v'^r$ has cycle type $\delta \cdot (\delta, r)$. Assume that $\chi_\phi$ belongs to $S^{\text{G}_n}(m_1, \delta_1, \mu)$, with notations of (5.4.2.2), $\mu$ being defined by a $\delta_1$-core.

Let be $\text{Res}_{V^\phi}^W(\phi) = \sum_{\zeta \in \text{Irr}(V^\phi)} t(\zeta) \zeta_\phi$. Then $t(\zeta) \neq 0$ implies $\zeta \in \text{Irr}(V^\phi)$ and

$$\text{Res}_{V^\phi}^W(\phi) \chi = \sum_{\zeta \in \text{Irr}(V^\phi)} t(\zeta) \zeta_\phi$$

By (5.4.1.2) $(\chi \times \phi)(hv\phi) = \sum_{\zeta \in \text{Irr}(V^\phi)} c(\zeta) \zeta_\phi(h')$ and $\zeta_\phi(h') = (\zeta \times \phi)(hv\phi)$. Extending linearly the isometry $(\chi \mapsto \chi \times \phi)$ to spaces of central functions $\text{Cent}(V^\phi) \to \text{Cent}(V^\phi)$, we obtain

$$(\chi \times \phi)(hv\phi) = ((\text{Res}_{V^\phi}^W(\phi)) \times v\phi)(hv\phi), \quad \text{Res}_{V^\phi}^W(\phi) = \text{Res}_{V^\phi}^W(\phi) \times v\phi)$$

(5.4.2.4)

The partition in series like (5.4.2.2) applies to the basis $\{\chi \times \phi\}_{\zeta \in \text{Irr}(V^\phi)}$ of the space of central functions on $W \phi$ (or $F$-central functions on $W$) (see 5.4.1.6). Assume $\zeta \in \text{Irr}(V^\phi)$ is defined by a $\delta_1$-core $\kappa$, partition of $n - m_1 \delta_1$, so that $\zeta_\phi = [\kappa]$. Then $\text{Res}_{V^\phi}^W(\phi) = \text{Res}_{V^\phi}^W(\phi) \times \phi = c(\phi)\zeta_\phi$ with $c \in \mathbb{Z}$ and $c \neq 0$ if and only if $\chi \in \text{Irr}(W^\phi)$ and $\chi_\phi \in S^{\text{G}_n}(m_1, \delta_1, [\kappa])$ by (5.4.2.4). So are defined $G.d$-HC unipotent series in $G^F$ from $G.\delta$-HC series in $W^F$:

$$(5.4.2.5) \quad \mathcal{E}(G^F, (L_v, R_\zeta)) = \{R_\lambda | \chi_\phi \in S^{\text{G}_n}(m_1, \delta_1, [\kappa])\}, \quad \zeta_\phi = [\kappa]$$

where $\zeta \in \text{Irr}(V^\phi)$ and $\zeta_\phi$ is $\delta_1$-cuspidal in $S_{n-m_1 \delta_1}$.

(A.3) In any Lusztig series.

Assume $s$ is a semi-simple element of $G^F$ and $C_{G^*}(s)$ is connected. As an example, think to a central product $G$ of $\text{GL}_{n_1}$ for some $(n_1)_j$. Then $C_{G^*}(s)$ is a central product of connected groups $\prod_{\beta \in \Sigma} C_{\beta}$ of type $A$, the product is indexed on the set $\Sigma = \cup_j \Sigma_j$ of eigenvalues of $s$ in the standard representation. One has $W(s) = \times_{\beta \in \Sigma} W_{\beta}$. The preceding description applies to $C_{G^*}(s)$ by direct product on $j \in J$ and on each $\langle F \rangle$-orbit on $\Sigma$. Going down from $\chi_{\omega \in \Sigma}(F)(\chi_{\beta \in \omega}(W_\beta)) \times \langle \phi \rangle$ to $W \times \langle \phi \rangle$ is easy. That’s why $G.d$-HC theory holds in $\mathcal{E}(C_{G^*}(s)^F, 1)$.

Assuming $Z(G)$ connected we have commutation formula (J3) : $R^F_L \circ \Psi_{L, s} = \Psi_{G, s} \circ R^G_{C_{L^*}(s)}$ (when $s \in L^*$). It gives $G.d$-HC theory in $\mathcal{E}(G^F, s)$.

If $L$ is $d$-split in $(G, F)$ then $L^*$ is $d$-split in $(G^*, F)$ hence $C_{L^*}(s)$ is $d$-split in $(C_{G^*}(s), F)$, $\Psi_{L, s}(\lambda)$ is $d$-cuspidal in $E(L^*, s)$ if and only if $\lambda$ is $d$-cuspidal in $E(C_{L^*}(s)^F, 1)$ and $G^*$ is the only $d$-split Levi subgroup of $G^*$ that contains $C_{G^*}(s)$, Any $d$-cuspidal unipotent datum $(L^*_s, \alpha)$ in $(C_{G^*}(s), F)$ defines a $G^F$-conjugacy class of $d$-cuspidal data $(L, \lambda)$ in series $s$ of $(G, F)$ as follows : $L$ is in the dual class of the
$d$-split Levi subgroup $C_{G^*}(Z(L^*_{\omega}))$ and $\lambda = \Psi_{L,s}(\alpha)$. By (J3) again one obtains a partition of $E(G^F, s)$ in $G.d$-HC series

\[ E(G^F, (L, \Psi_{L,s}(\lambda))) = \Psi_{G,s}[E(C_{G*}(s)^F, (C_{L^*}(s), \lambda))], \quad L^* = C_{G^*}(Z(L^*_{\omega})) \]

(B) General case, $Z(G)$ connected or not.

From section 2 we may consider a regular embedding $G, F \subseteq (H, F)$. Let $t$ be a semi-simple element of $H^*$ that maps on $s$ by a dual map. We need the following elementary fact :

(B.1) Let $M_t^* = \times_{\omega \in \Sigma/(\phi)} M_{\omega}^*$ be a $d$-split Levi subgroup of $C_{H^*}(t) = \times_{\omega} C_{\omega} (\Sigma$ as in (A.3)). Define $M^*$ by $M^* = C_{H^*}(Z(M^*_{\omega}))$ and let $L^*$ be the image of $M^*$ in $G^*$. Let $\omega \in \Sigma/(\phi) and assume $M^*_{\omega} \not= C_{\omega}$. Then $\omega$ is stable under $A_{L^*}(s)^F$.

Proof of (B.1).

(a) Reduction.

$(G, F)$ is a central product of rationally irreducible components $G_i$. Let $\nu: (G^*, F) \rightarrow (x_iG_i^*, F)$ be a dual map of $(G_0, F) = (x_iG_i, F) \rightarrow (G, F)$. From Proposition 1.1.4, (d) there exist regular embeddings $G_i \rightarrow H_i$, $G \rightarrow H$, an isotypic morphism $x_iH_i \rightarrow H$ and by duality a commutative diagram of isotypic morphisms

\[
\begin{array}{ccc}
H^* & \xrightarrow{\nu} & G^* \\
\downarrow{\nu_1} & & \downarrow{\nu} \\
H_0^* := x_iH_i^* & \xrightarrow{\nu_0} & G_0^* := x_iG_i^*
\end{array}
\]

So are defined $(t_i)_i = \nu_1(t), (s_i)_i = \nu_0((t_i)_i) = \nu(s)$. Using the four maps above we may identify the Weyl groups $W(t), W((t_i)_i), W^\circ(s), W^\circ((s_i)_i)$, as well as the sets of eigenvalues $\Sigma(t), \cup_i\Sigma(t_i)$ with $\phi$-action (see (A.3)). Furthermore $\nu$ induces morphisms $W(s) \rightarrow W((s_i)_i), A_{G^*}(s) \rightarrow A_{G^*_i}(s_i))$ commuting with actions on $\Sigma(t)$ (see Proposition 1.2.3). Given $M_t^*$, we may consider $M_{(t_i)_i}^* = \nu_1(M_t^*)$, and then $M_0^* = C_{H^*_0}(Z^\circ(M_{(t_i)_i}^*)) = \nu_1(M^*) = x_iM_{0,i}^*$, so that $\rho_0(M_{0,i}^*) = \nu(L^*) = x_iL_{0,i}^*$, where $\rho(M_{0,i}^*) = L_{0,i}^*$. The restriction of $\nu$ induces a $F$-morphism $A_{L^*}(s) \rightarrow A_{\nu(L^*)}(s_i)$, the last one is a subgroup of $x_iA_{L^*_0}(s_i)$. Now it is clear that if (B.1) is verified for $(G_i \rightarrow H_i, t_i, s_i, M_i^*)$, any $i$, it is true for $(G \rightarrow H, t, s, M_t^*)$. So we may assume $G$ rationally irreducible.

In a scalar descent (see (A.2)) we may replace $G$ by one of its component and $F$ by some convenient power. Then $H = GL_n$ for some $n$, so that $G$ contains $SL_n$. The above argument shows that if (B.1) is true for $SL_n$ it is true for $G$. So we assume $G = SL_n$.

(b) $G = SL_n \subset GL_n$.

By definition $M^*$ and $L^*$ are $d$-split. Thanks to (b) in Proposition 1.2.4, we may assume that $M^*_\omega$ is a maximal proper $d$-split Levi subgroup of $C_{\omega}$, and that $M^*_\omega' = C_{\omega'}$ if $\omega' \not= \omega$.

Then we have $M^*_\omega = S_{\omega} \times K_{\omega}$, a direct product defined on $F_q$, where $S_{\omega} \cong [GL_m]^d_{\omega}$ for some $m$, any $\beta \in \omega$, and $(S_{\omega})^F \cong GL_m((eq)^{|\omega|/|\delta|}), K_{\omega} = GL_h, (K_{\omega})^F \cong GL_h((eq)^{|\omega|})$. We have $M^* = M_1^* \times M_2^*$, where $M_1^* \cong [GL_m]^d_{\omega}$ acts on the space $V_1 \oplus V_2 \ldots \oplus V_\delta$ of fixed points of $K_{\omega} \times (\times_{\omega \neq \omega'} C_{\omega'})$ in $F^n$ (so that $m_1 = m|\omega|/|\delta|$) and $(M_1^*)^F \cong (GL_m((eq)^{\delta})), M_2^* \cong GL_m(m_2 = n-m_1 \delta)$ and $(M_2^*)^F \cong GL_m((eq))$.

A dual Levi subgroup $M$ of $M^*$ in $(H, F)$ is isomorphic to $M^*$. Note that the center of $L = M \cap G$ may be disconnected : the center of $M$ has rank $(\delta + 1)$ and the function determinant on it is $\bar{F}^\times \rightarrow \bar{F}$, $(\lambda_1, \ldots, \lambda_\delta, \lambda_0) \mapsto (\lambda_1 \ldots \lambda_\delta)^{m_1} \lambda_0^{m_2}$. 

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An element $a$ in $A$ acts on $\Sigma$ as an element of $\bar{F}$ by multiplication and so semi-regularly on the eigenspaces $V_\beta$ of $t$, or analogous in $H$. One sees that $a$ may be found in $M^*$ only if $a \omega = \omega$. Any $\beta \in \omega$, $C_\beta = (C_\beta \cap S_\omega) \times (C_\beta \cap K_\omega)$ and $\phi$ acts regularly on these products, hence such a $a = (a_1, a_2) \in M^*_1 \times M^*_2$ may exists, if the spectrum of $s$ in each $V_\beta$ is $a_1$-stable. The order of $a_1$ has to divide $|\omega|/|\omega|, \delta)$.

(B.2) Bonnafé-Cabanes parametrization.

Recall the decomposition $C_{H^*}(t) = \times_{\beta \in \Sigma} C_{C_\beta}$, $\Sigma$ the set of eigenvalues of $t$ in the standard representation of $H^*$ on $\bar{F}^n$. We have seen that $C_{G^*}(s)$ is a semi-direct product of $C_{G^*}(s)$ by the abelian group $A_{G}(s)$.

Let $A := A_{G^*}(s)F$, $W^\phi(s), W(s)$ be the Weyl groups of $C_{G^*}(s), C_{C^*}(s)$ with respect to a diagonal torus $T_s^*$ of $C_{C^*}(s)$, that is $W(s) = N_{C_{G^*}(s)}(T_s^*)/T_s^*$. The map $H^* \rightarrow G^*$ restricts to $C_{H^*}(t) \rightarrow C_{C^*}(s)$, $T_t^* \rightarrow T_s^*$ and an isomorphism $W(t) \rightarrow W^\phi(s)$. Design by $\phi$ the action of $F$ on $W^\phi(s)$ and $W(s)$, hence on $\Sigma$, so that $B := (A, \phi)$ is considered as a subgroup of $\mathfrak{S}(\Sigma)$. Then $W(s), W^\phi(s) \ltimes (A, F)$ and $W^\phi(s)^\phi \ltimes A$ appear as wreath products with base group $W^\phi(s) := W(C_{C^*}(s)) = \times_{\beta \in \Sigma} W_\beta$ or $W^\phi(s)^\phi = \times_{\phi \in \Sigma/\langle \phi \rangle} W_\omega^\phi$. We may have $W_\beta = 1$ for some $\beta \in \Sigma$, then the following combinatorial description of $E(G^*, s)$ keep sense.

For each orbit $\omega \in \Sigma/\langle \phi \rangle$ is given an integer $\delta_\omega$ so that elements of cycle type $(\delta_\omega)^m$ in $W_\beta (\beta \in \omega)$ provide parameters with respect to the diagonal torus of $C_\omega$ of $\phi_\delta$-split Levi subgroups of $C_\omega$.

Bonnafé [3] Théorème 23.9 (non twisted type) and Cabanes [12] Theorem 3.6 (twisted type) have shown that, under conjecture $\mathfrak{S}$ (5.4.2.0), there is a one-to-one map

\[(5.4.2.7)\]  
\[R^G[s] = R[s]: \pm \text{Irr}(W(s)^\phi)/\{1, -1\} \rightarrow \pm E(G^*, s)/\{1, -1\}\]

In other words there exists a sign function $v: \text{Irr}(W(s)^\phi) \rightarrow \{-1, 1\}$ such that $vR[s]$ is one-to-one from $\text{Irr}(W(s)^\phi)$ onto $E(G^*, s)$.

Simplify $W^\phi(s) \ltimes B$ (resp. $W(s)$) in $W$ (resp. $W^\phi$). If $\chi = (\chi_\beta)_{\beta \in \Omega} \in \times_{\beta \in \Omega} \text{Irr}(W_\beta)$, we denote $(\chi_\beta)_{\beta \in \Omega} \in \text{Irr}(W_\Omega)$, by $\chi^\Omega$ so that $\chi^\Sigma = \chi \in \text{Irr}(W^\phi)$.

We first describe the map $R[s]$, following [3], [12]. That map is defined by restriction of a linear map between spaces of central functions.

Let Cent($G^*$, $s$) be the space of central functions on $G^*$ with basis $E(G^*, s)$. Using notations of 1.3.5 we knows that in the regular embedding $G \subseteq H$, $H^F/\tau_H, s(A)$ acts on $E(G^*, s)$. By (1.3.5.1) $(H^F/\sigma_H, s(A))^{\wedge}$ is isomorphic to $A$. So Cent($G^F, s$) decomposes as a direct sum of isotypic ($H^F/\tau_H, s(A)$)-spaces Cent($G^F, s, a$).

One obtains an orthogonal decomosition which is independant of the regular embedding [3] (11.15) corresponding on $W$-side to the decomposition $W^\phi = \cup_{a \in A} W^\phi \circ a$:

\[(5.4.2.8)\]  
\[\text{Cent}(G^F, s) = \perp_{a \in A} \text{Cent}(G^F, s, a), \quad \text{Cent}(W^\phi) = \perp_{a \in A} \text{Cent}(W^\phi \circ a)^A\]

Isometries are defined in [3] 23.C

\[(5.4.2.9)\]  
\[R^G[s, a] = R[s, a]: \text{Cent}(W^\phi \circ a)^A \rightarrow \text{Cent}(G^F, s, a), \quad R[s] = \oplus_{a \in A} R[s, a]\]

The scalar product on $\text{Cent}(W^\phi \circ a)^A$ has to be defined as $1/|A|$ times the usual scalar product on $a$-central functions on $W^\phi$.

When $C_{G^*}(s)$ is connected $R^G[s] = R^G[s, 1]$ may be obtained as in part (A) thanks to Jordan decomposition and the unipotent case $\text{Irr}(W(C_{G^*}(s)^F)) \rightarrow E(C_{G^*}(s)^F, 1)$, or simply because $C_{G^*}(s)$ is a Levi subgroup of $G^*$ (see 1.3.2 and 1.3.3 (b)). Up to a sign the map $(\mu \mapsto R_\mu)$ in (5.4.2.1) is just $\mu \mapsto R^G[1](\mu)$.

By properties of wreath products (5.4.1.6), Cent($W^\phi \circ a$) has an orthonormal basis $\{\chi \circ a \}_\chi$ where $\chi$ runs in $\text{Irr}(W^\phi)^a$. Using the natural one-to-one maps $\text{Irr}(W^\phi)^a \rightarrow \text{Irr}(W^\phi)^{a, \phi} \cong \text{Irr}(W^0)^{a, \phi} \rightarrow \text{Irr}(W^0)^\phi$
is defined an isometry $\text{Cent}(W^0 \phi_a) \to \text{Cent}(W^0 a \phi)$, sending $\chi \times a$ on $\chi_a \times \phi$ for any $\chi \in \text{Irr}(W^0)^{(a, \phi)}$, and that isometry commutes with $A$-action by (5.4.1.5) (i). So is defined an isometry

\[(5.4.2.10) \quad \sigma_{s,a}^W = \sigma_{s,a}: \text{Cent}(W^0 \phi_a)^A \to \text{Cent}(W^0 a \phi)^A \]

Let ([3] 23.C)

\[R^G[s, a] := R^G[s, a] \circ (\sigma_{s,a}^W)^{-1} \]

A fact is that $R^G[s, a]$ may be defined for any type from $\text{Cent}(W^0 a \phi)$ to $\text{Cent}(G^F, s, a)$, and then commute with $A$-action [3] (17.17), Proposition 17.18. So we compose it with the orthogonal projection

\[\pi'_A: \text{Cent}(W^0 a \phi) \to \text{Cent}(W^0 a \phi)^A \]

Furthermore $R^G[s, a]$ commute in some sense with Lusztig induction. Let $L$ be an $F$-stable Levi subgroup of $G$, with dual $L^*$ in $G^*$ such that $s \in L^*$, assume that some diagonal maximal torus of $C_L^\circ(s)$ has parameter $w_L \in W^0$ with respect to the maximal diagonal torus of $C_L^\circ(s)$, let $A(L) = A_L \circ (s)^F$, a subgroup of $A$. One has a commutation formula for any $a \in A(L)$ [3] Proposition 17.24:

\[(5.4.2.11) \quad |A(L)| R^G_L \circ R^L[s, a] = |A| |A| \text{Cent}(W^0 a \phi)^A \]

Therefore the inverse of Lusztig induction $R^G_L$ by $R[s]$ in (5.4.2.7) is given by

\[(5.4.2.12) \quad T^W_{W_L} = |A| A(L) [\oplus_{\alpha \in A(L)} ([\sigma_{s,a}^{-1} \circ \pi'_A \circ \text{Ind}^{W^0 a \phi}_{W^0 a \phi} \circ \sigma_{s,a}^W] \]

(we don’t write the orthogonal projections given by (5.4.2.8)). We would like to compute its value on $\eta_{w_L \phi}$ when $(L, R^L[s](\eta_{w_L \phi}))$ would define a $d$-cuspidal datum in series $[s]$ in $(G, F)$ for some $\eta \in \text{Irr}(W^0)^{w_L \phi}$. Our claim is that $R^L[s](\eta_{w_L \phi})$ is a component of $\text{Res}^{G^F}_L[\chi]$ where $M = L.Z(H)$ and $\chi$ is $d$-cuspidal.

By [3], (23.15) for any $\zeta \in \text{Irr}(W^0)^{\phi}$ and with notations of (5.4.1.4) one has

\[(5.4.2.13) \quad \text{Res}^{G^F}_L(R^H[t](\zeta)) = \sum_{\theta \in (A, \zeta)^\wedge} R^G[s](\Gamma W^\sigma(\zeta \phi \ast \theta)) \]

(B.3) $d$-cuspidal data and $G.d$-HC series in type A.

From our description in part (A.3) of that proof and (5.4.2.6), a $d$-cuspidal datum in series $(t)$ in $(H, F)$ is defined on $W^0$-side as follows: for each orbit $\omega \in \Sigma/\langle \phi \rangle$ are given $W_\omega \cong \times_{\beta \in \omega} \mathcal{S}_{n_\omega}$, an integer $\delta_\omega$, and a “$\delta_\omega$-cuspidal” element $\zeta_\omega$ in $\text{Irr}(\mathcal{S}_{n_\omega})$ where $\delta_\omega$ divides $(n_\omega - m_\omega)$, $n_\omega \in W^0_\omega$, $V_\omega \subseteq W^0_\omega$. We obtain a “$(\delta_\omega)_{\omega \in \Sigma/\langle \phi \rangle}$-cuspidal datum in $(W^0, \phi)$” : $(\times_{\omega \in \Sigma/\langle \phi \rangle} V_\omega, (\zeta_\omega)_{\omega \in \Sigma/\langle \phi \rangle})$. For each $\omega \in \Sigma/\langle \phi \rangle$ is defined a $d$-split Levi subgroup $M_{\omega \in \Sigma/\langle \phi \rangle}$ of $C_\omega := \Pi_{\beta \in \omega} C_{\beta} \subseteq C_{H^*}(t)$. Then (5.4.2.4) apply in each $W_\omega$, giving a similar formula in $W^0$. So is defined a $d$-cuspidal datum $(M, \lambda)$ where $M = \times_{\omega \in \Sigma/\langle \phi \rangle} M_{\omega \in \Sigma/\langle \phi \rangle}$, $w_M = (v_\omega)_{\omega \in \Sigma/\langle \phi \rangle}$, $W(M) = \times_{\omega \in \Sigma/\langle \phi \rangle} V_\omega$, $\lambda = \otimes_{\omega \in \Sigma/\langle \phi \rangle} R^M[t](\zeta_\omega)$. The $G.d$-HC series in series $(t)$ is a set product on $\Sigma/\langle \phi \rangle$ of $G.d$-HC series, each one defined from a series in $W^\circ$, as (5.4.2.2) gives (5.4.2.5).

Furthermore, if $\zeta = (\zeta_\omega)_{\omega \in \Sigma/\langle \phi \rangle}$, $R^H[t](\zeta)$ is $d$-cuspidal if and only if $Z(C_{H^*}(t)) \phi_d \subseteq Z(H^*)$ and $\zeta_{\phi} = (\zeta_\omega)_{\omega \in \Sigma/\langle \phi \rangle}$ is $(\delta_\omega)_{\omega \in \Sigma/\langle \phi \rangle}$-cuspidal. A first step is to show that $R^G[s](\Gamma W^\sigma(\zeta \phi \ast \theta))$ is $d$-cuspidal if and only if $R^H[t](\zeta)$ is $d$-cuspidal, to obtain a generalization of 2.1.5, where the order of $s$ is prime to $\ell$ :
(B.3.1) Let \((G, F) \subseteq (H, F)\) be a regular embedding, \(\sigma: H^* \to G^*\) a dual morphism, \(t\) a semi-simple element of \((H^*)^F, s = \sigma^*(t), \mu \in \mathcal{E}(H^F, t), \lambda\) an irreducible component of \(\text{Res}^H_{G\mu} \mu\). Then \(\lambda\) is \(d\)-cuspidal if and only if \(\mu\) is \(d\)-cuspidal.

**Proof of (B.3.1).**

(a) If \(\mu\) is \(d\)-cuspidal, \(\lambda\) is \(d\)-cuspidal:

Let \(\mu = R^H[t](\zeta_d)\), assumed to be \(d\)-cuspidal. Let \(L\) be a maximal proper \(d\)-split Levi subgroup of \((G, F)\), a dual \(L^*\) in \((G^*, F)\) such that \(s \in L^*, M^* = L^*Z(H^*), t \in M^*\). As \(Z(C_{H^*}(t))_{d}\) \(\subseteq Z(H^*)\), \(C_{M^*}(t) \neq C_{H^*}(t)\) and \(C_{M^*}(t)\) (resp. \(C_{L^*}(s)\)) is a proper maximal \(d\)-split Levi subgroup of \((C_{M^*}(t), F)\) (resp. \((C_{G^*}(s), F)\)).

(B.1.1) and the description we made in its proof apply. We have \(A(L)(\omega) = \omega\).

Let \(\xi \in \text{Irr}(W_M)^{w_L^0}\) and \(\psi \in (A(L)_{\xi})^\wedge\) defining \(\eta = \Gamma_{w_L^0}^W(\xi_{w_L^0} \ast \psi)\) in \(\text{Irr}(W_M)^{w_L^0}\). To compute the scalar product of \(\mathcal{I}_{W_L}^W(\eta)\) with \(\Gamma_{\xi}^{w_L}(\zeta \ast \theta)\) using the orthogonal decompositions in (5.4.2.8) we may restrict \(a\) in \(A(L)_{\xi} \cap A_{\xi}\). The projection of \(\eta\) on \(\text{Cent}(W_M)^{w_L^0}\) is \(\psi(a) \sum_{b \in A(L)/A(L)_{\xi}} (\xi_b \ast w_L \phi)\)

To compute \(\text{Ind}_{W_M^0}^{W_M}(\alpha_{W_L}(\eta))\) in (5.4.2.12) we use (5.4.2.4) applied in \(W_0^\circ\). Then \(w_L^\circ = (v(\beta))_{\beta \in \omega}\) and \(\zeta^\circ := (\zeta(\beta))_{\beta \in \omega}\) are constant on \(\omega\), so are \(v_{\alpha}\) and \(c_{\alpha}\) on \(\omega_{\alpha} := \omega/\langle\alpha\rangle\). The cardinal \(r_{\alpha}\) of \(\omega_{\alpha}\) is a divisor of \(r = |\omega|\). In the computation we made to obtain (5.4.2.4), if we consider \((W_0^\circ, W_0^0, \zeta_{\circ})\) instead of \((W_0^0, \chi)\), we have to replace \(r\) by \(r_{\alpha}\). Then \(\delta_{\omega} = \delta/\langle\delta, r_{\alpha}\rangle\) divides \(\delta_{\alpha} = \delta/\langle\delta, r_{\alpha}\rangle\). The description of partitions and hooks by so-called \(\beta\)-sets shows that any \(\delta_{\alpha}\)-core is a \(\delta_{\alpha}\)-core, because deleting a hook of length \(\delta_{\alpha}\) may always be obtained by deleting successively \(\delta_{\alpha}/\delta_{\omega}\) hooks of length \(\delta_{\omega}\). As \(\zeta_{\circ}\) is defined by \(\delta_{\omega}\)-cuspidal element, then \(\text{Res}_{W_M^0}^{W_M}(\zeta \ast \phi) = 0\). That implies \(\text{Ind}_{W_M^0}^{W_M}(\alpha_{W_L}(\eta)) = 0\).

(b) If \(\mu\) is not cuspidal, \(\lambda\) is not \(d\)-cuspidal:

By (A.3), if \(Z(H^*)_{d} \neq Z(C_{H^*}(t))_{d}\), then \(R^H[t](\zeta_{d})\) is not \(d\)-cuspidal whenever \(\xi \in \text{Irr}(W_0^\circ)\).

In that case let \(M^* = C_{H^*}(Z(C_{H^*}(t))_{d})\), \(L^*\) the image of \(M^*\) in \(G^*\), \(M\) a \(d\)-split Levi subgroup of \((H, F)\) in duality with \(M^*\), \(L = M \cap G\) : we have \(W(C_M(t)) = W(C_{H^*}(t)), W(C_{G^*}(s)) = W(C_{L^*}(s))\) so that \(R^H[t](\zeta) = R^H_M(R^M[t](\zeta))\) and \(A_{M^*}(s) \simeq A_{G^*}(s)\). By (1.3.1.1) and (5.4.2.13)

\[
\text{Res}^H_{G^*}(R^H[t](\zeta)) = R^G_L(R^M(t)(\zeta)) = R^G_G(\sum_{\theta \in A^\wedge} R^L[s](\Gamma_{\Gamma^*(\zeta)} \otimes \theta)).
\]

By Clifford theory the isomorphic quotient groups \(H^F/G^F\) and \(M^F/L^F\) act regularly and the same way on the sets \(\{R^G_G(s)(\Gamma_{\Gamma^*(\zeta)} \otimes \theta)\}_\theta\) and \(\{R^L[s](\Gamma_{\Gamma^*(\zeta)} \otimes \theta)\}_\theta\). Thus \(R^G_G\) induces a one-to-one map between these two sets (one may show that \(R^G_G(R^L[s](\Gamma_{\Gamma^*(\zeta)} \otimes \theta)) = R^G_G(s)(\Gamma_{\Gamma^*(\zeta)} \otimes \theta)\) hence \(R^G_G(s)(\Gamma_{\Gamma^*(\zeta)} \otimes \theta)\) is a \(G\)-\(d\)-HC component in series \(\{t\}\) in \((H, F)\).

More generally (b) is implied by the following description of \(G\)-\(d\)-HC series in series \(s\) in \(G\):

(B.3.2) Let \((M, \mu)\) be a \(d\)-cuspidal datum in \((H, F)\), where \(\mu = R^M[t](\xi_{w_M^0})\), \(\xi \in \text{Irr}(W_M)^{w_M^0}\), and let \(L = M \cap G, \lambda = R^L[s](\Gamma_{w_M^0}(\xi \ast \psi))\) (\(W_M = W_L = W(C_{H^*}(t)), \psi \in (A(L)_{\xi})^\wedge\) ). Then \(R^G_G[s](\Gamma_{\Gamma^*(\zeta \ast \theta)})\) is a component of \(R^G_G \lambda\) if and only if \(R^H[t](\zeta) \in \mathcal{E}(H^F, (M, \mu))\) and \(\text{Res}_{A(L)_{\xi}}^A \theta = \psi\).
Proof of (B.3.2). Let \( \chi := R^H[t](\zeta_\phi) \in \mathcal{E}(H^F, (M, R^M[t](\xi_{w\phi})) (\zeta \in \text{Irr}(W^0)^\phi) \). We may assume \( M \neq H \), thanks to (a) and (5.4.2.13).

The last condition in (B.3.2) assume \( A(L) \subseteq A_\zeta \), a consequence of (B.1):

We have \( C_G(t) = \times_{\omega}C_{\omega}, C_M(t) = \times_{\beta}M_\omega \). For any \( \omega \in \Sigma/(\phi) \), if \( M_\omega = C_\omega, \zeta^\omega = \zeta^\omega \) and \( A(L) \subseteq \zeta \), then \( M_\omega \) is a maximal proper \( d \)-split Levi subgroup of \( C_\omega \) because \( M_\omega = S_\omega \times K_\omega \), where \( S_\omega \) is a torus (as in the proof of (B.1.1) with \( m = 1 \)). By (B.1), \( A(L) \omega = \omega \). As \( \zeta \in \text{Irr}(W^0)^\phi \), \( \zeta(\beta) \) is constant on \( \omega \) hence \( \zeta^\omega \) is fixed by \( A(L) \). Thus we have \( A(L) \subseteq A_\zeta \).

We have by (5.4.2.13)

\[
\text{Res}_{L_F}^{M_F} (R^M[t](\xi_{w\phi})) = \sum_{\psi \in (A(L)\subseteq)^\land} R^L[s](\Gamma^{W^0}_{L}(\xi_{w\phi} \ast \psi))
\]

and know by (a) that \( R^L[s](\Gamma^{W^0}_{L}(\xi_{w\phi} \ast \psi)) \) is \( d \)-cuspidal for any \( \psi \in (A(L)\subseteq)^\land \). Note that for any \( nM^F \) in \( N_H(F)(M^F, [t])/M^F \), the isomorphisms \( N_H(F)(M^F, [t])/M^F \cong N_{W^0}(W_M^M)/W_M^{nM^F} \cong N_{G[F]}(F^F, [s])/F^F \) imply \( A(L) \subseteq A_\zeta \) and allow to write

\[
R^M[t](\xi_{w\phi})^n = R^M[t](\xi_{w\phi}^n), \text{Res}_{L_F}^{M_F} (R^M[t](\xi_{w\phi}^n)) = \sum_{\phi \in (A(L))\subseteq} R^L[s](\Gamma^{W^0}_{L}(\xi_{w\phi}^n \ast \psi)).
\]

Thanks to the functorial behaviour of formula (5.4.2.4) with respect to the direct product on \( \Sigma/(\phi, A) \), \( \zeta_\phi \times \phi \) is a component of \( \text{Ind}_{W^0}^{W^0}(\xi_{w\phi}) \) if, for any \( \Omega \in \Sigma/(a, \phi) \), \( \zeta_\phi(\Omega) = (\Omega_\lambda)_\lambda \) appears as a component of some \( \text{Ind}_{W^0}^{W^0}(\xi_{w\phi}) \), where \( b \in A \) and \( \omega_\lambda \in (\Sigma/(a, \phi))/\phi \cong \Sigma/(a, \phi) \).

Coming back to the \( d \)-cuspidal datum \( (L, R^L[s](\eta_{w\phi})) \) we know that if \( \zeta_\phi \notin S^{W^0}(\xi_{w\phi} \ast \psi) \) for some \( b \in A \), then \( R^G[s](\Gamma^{W^0}(\chi \ast \theta)) \) is not a component of \( R^G[s](\eta_{w\phi}) \).

So assume \( \zeta_\phi \in S^{W^0}(\xi_{w\phi} \ast \psi), a \in (A(L)\subseteq) \), so that \( R^H[t](\zeta_\phi) \in \mathcal{E}(H^F, (M, \mu)) \). On \( W^0 \phi \) we have a decomposition \( \text{Ind}_{W_L^{0\phi} \ast w_L \phi}^{W_L^{0\phi}}(\xi_{w\phi}) = \sum \lambda d(\xi, \lambda) \lambda \phi \) where \( d(\xi, \lambda) = (\text{Res}_{W_L^{0\phi} \ast w_L \phi}(\lambda \phi))^{W_L^{0\phi}} \).

As \( (W_L, w_L, \xi) = (W_L^{0\phi}, \xi_{w\phi}), d(\xi, \lambda) = d(\xi, \lambda_\phi) \). On \( W^0 \phi \), we have \( \text{Ind}_{W_L^{0\phi} \ast w_L \phi}(\xi_{w\phi}) = \sum \lambda_\phi d(\xi, \lambda_\phi) \lambda_\phi \).

Furthermore \( d(\xi, \lambda) = d(\xi, \lambda_\phi) \lambda_\phi \) for any \( b \in A \) and \( \pi^a_\lambda(\chi \ast \phi) = 1\lvert A \rvert \sum_{b \in A}(\lambda_\phi) \phi_\phi \). Finally we obtain

\[
|A/A(L)|(\oplus_{a \in A(L)} \sigma_{s,a}^{W}(z_{w\phi}))(\eta_{w\phi}) = |A/A(L)| \psi(a) \sum_{\lambda_\phi} \sum_{b \in A/A(L)} d(\xi, \lambda_\phi)(\lambda_\phi) \phi_\phi
\]

Let \( \theta \in (A_\zeta)^\land \). If \( a \notin A_\zeta \), the projection of \( \Gamma^{W^0}(\chi \ast \theta) \) on \( \text{Cent}(W^0)^a \) is null. If \( a \in A_\zeta \), the projection of \( \Gamma^{W^0}(\chi \ast \theta) \) on \( \text{Cent}(W^0)^a \) is \( \theta(a) \sum_{b \in A/A_\zeta}(\lambda_\phi \lambda_\phi) \lambda_\phi \). Its image by \( \sigma_{s,a}^{W}(z_{w\phi})(\lambda_\phi) \phi_\phi \).

As \( \sigma_{s,a}^{W}(z_{w\phi}) \) is an isometry, the scalar product of \( \Gamma^{W^0}(\chi \ast \theta) \) with \( \gamma_{W_L}^{W}(\eta_{w\phi}) \) is equal to

\[
|A/A_V| \cdot |A_\zeta/A(L)\subseteq| \sum_{a \in A_\zeta} \theta(a) \psi(a)^{-1} d(\xi, \zeta_\phi). \quad \text{One sees that it is non zero and only if } \text{Res}_{A_\zeta}(\psi_\phi) = \theta. \]
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