Fractionalization of angular momentum at finite temperature around a magnetic vortex

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Abstract

Ambiguities in the definition of angular momentum of a quantum-mechanical particle in the presence of a magnetic vortex are reviewed. We show that the long-standing problem of the adequate definition is resolved in the framework of the second-quantized theory at nonzero temperature. Planar relativistic Fermi gas in the background of a point-like magnetic vortex with arbitrary flux is considered, and we find thermal averages, quadratic fluctuations, and correlations of all observables, including angular momentum, in this system. The kinetic definition of angular momentum is picked out unambiguously by the requirement of plausible behaviour for the angular momentum fluctuation and its correlation with fermion number.

Keywords: Bohm-Aharonov effect; field theory at finite temperature; fractional quantum numbers

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1 Introduction

Whereas in classical theory a comprehensive description of electromagnetism is given in terms of the electromagnetic field strength acting locally and directly on charged matter, this is not the case in quantum theory. Since the quantum-mechanical wave equation for a charged particle involves the electromagnetic vector potential, the state of charged matter can be influenced by electromagnetic effects even in the situation when the space-time region of nonvanishing field strength is not accessible to charged matter \[1, 2\]. The latter was demonstrated for the case of quantum-mechanical scattering on a tube of the magnetic force lines, which was impenetrable for scattered particles: the differential scattering cross section was found to be a periodic function of the total magnetic flux confined in the tube \[2\]. This phenomenon with all its various generalizations (see, e.g., Refs.\[3, 4, 5\]) has no classical analogues and is denoted usually as the Bohm-Aharonov effect.

Fractionalization of quantum numbers is a well-known feature of various field theory models that describe interaction between fermions and topological solitons \[6, 7, 8, 9, 10, 11, 12\]. It is usually convenient to interpret solitons as background fields and to quantize fermion fields in such backgrounds. In this respect it seems to be of interest to consider a case when the region where a background field is nonvanishing does not overlap with the region where a fermion field is quantized. Namely, the quantized fermion matter does not penetrate a tube of the magnetic force lines; for brevity, the impenetrable magnetic tube will be called the magnetic vortex in the following. One inquires about the dependence of the properties of this second-quantized system on the vortex flux and the boundary condition on the edge of the vortex; if a transverse size of the vortex is neglected, then a parameter of the boundary condition exhibits itself as a parameter of a self-adjoint extension of the one-particle hamiltonian operator.

A study of this second-quantized system started two decades ago \[13, 14\]. It was shown for a particular choice of the boundary condition that fermion number \[15\], magnetic flux \[16\], and angular momentum \[17\] are induced in the vacuum of quantized massive fermions. The induced vacuum quantum numbers under the most general set of boundary conditions were obtained in Refs.\[18, 19, 20, 21\]. All results, except angular momentum, are periodic in the value of the vortex flux, and such a type of behaviour is consistent with the above mentioned for the quantum-mechanical Bohm-Aharonov effect. As to angular momentum, the situation is not so far determinative, which is due to two alternatives for its definition. In quantum field theory, the angular momentum operator, as well as operators of all other observables, is obtained by sandwiching the appropriate quantum-mechanical (one-particle) operator between the fermion field operators (see Section 4 below). In quantum mechanics, there are two possibilities to define angular momentum of a fermion particle in the background of the Bohm–Aharonov magnetic vortex: the canonical one that is quantized with half-integer eigenvalues,

\[
\tilde{j} = n + \frac{1}{2},
\]
and the kinetic one that is quantized otherwise,

\[ j = n + \frac{1}{2} - e\Phi, \]

where \( e \) is the electric charge and \( \Phi \) is the vortex flux in units of \( 2\pi \); strictly speaking, \( \Phi \) is measured in units of \( 2\pi \hbar c \), but we use conventional units \( \hbar = c = 1 \). The motivation in favour of the canonical definition lies in that it yields angular momentum which is conserved when the vortex flux is varied in time \([3, 22, 23, 24]\). On the other hand, the kinetic definition proves to be rather rewarding, since it leads to such prolific concepts as anyons and fractional statistics \([25, 26]\) which are of fundamental value and possess fascinating phenomenological applications to some condensed matter systems \([27, 28, 29, 30]\). It should be noted that quantum-mechanical consequences of the kinetic definition are periodic in the value of \( e\Phi \) with the period equal to 1, and this can be regarded as one more of manifestations of the Bohm-Aharo nov effect. If we turn to quantum field theory and consider angular momentum which is induced in the vacuum around a magnetic vortex, then the kinetic definition yields the quantity which depends on the fractional part of \( e\Phi \), while the canonical definition yields the quantity which depends on both the fractional and integer parts of \( e\Phi \) \([17, 21]\).

In the present paper we shall study the effect of nonzero temperature on the induced angular momentum around a magnetic vortex, and consider its thermal average and quadratic fluctuation, as well as correlations with other observables. The previous analysis in Refs.\([31, 32]\) is augmented by taking account of the bulk volume contribution to thermal characteristics. In general, the expressions for thermal characteristics consist of two pieces: the one corresponding to the ideal gas contribution depends essentially on a size of the system, increasing by power law as the size increases, and the other one corresponding to the correction due to interaction with a magnetic vortex is finite in the limit of the infinite size; although, sometimes, as are the cases of the averages of fermion number and induced magnetic flux and their correlation, the ideal gas contribution is vanishing. We shall show that the canonical definition of angular momentum yields rather unconvincing results for the angular momentum fluctuation and its correlation with fermion number: these quantities contain pieces that are proportional to both powers of the system size and powers of the whole vortex flux, and such a mismatch of different contributions is hardly tolerable. On the contrary, the kinetic definition of angular momentum yields plausible results, and this allows one to conclude that quantum field theory at nonzero temperature favours for certain the latter definition.

In the next section, we review the definition of angular momentum for a quantum-mechanical charged particle placed into a classical static rotationally invariant magnetic field. In Section 3 we consider a situation when the particle does not penetrate the magnetic field region and discuss some manifestations of non-simply-connectedness of the accessible to the particle region. A general formalism for the treatment of thermal characteristics of observables in the framework of quantum field theory at finite temperature is introduced in Section 4. Section 5 deals with planar relativistic Fermi gas in the background of a magnetic vortex, and we determine relevant one-
particle spectral densities. Using formalism of Section 4 and results of Section 5, we obtain thermal characteristics of all observables in this system in Section 6. We discuss our results in Section 7. Some details in the derivation of the results are outlined in Appendices A-D.

2 Angular momentum of a particle in a magnetic field

Angular momentum of a free particle consists of the orbital and spin parts

\[ J_0 = -i \mathbf{x} \times \partial + \frac{1}{2} \mathbf{\sigma}. \]  \hspace{1cm} (3)

If a charged particle is posited into an external electromagnetic field, then a contribution of the latter has to be added to Eq.(3). It can be shown (see Appendix A, Eq.(A.18)) that this contribution in its turn consists of the purely field part

\[ M_{EM} = \int d^3x' (\mathbf{x}' \times (\mathbf{E} \times \mathbf{B})), \]  \hspace{1cm} (4)

and the particle-field interaction part

\[ M_I = - \int d^3x' x' \times A j^0 = - e \mathbf{x} \times \mathbf{A}(\mathbf{x}), \]  \hspace{1cm} (5)

where

\[ j^0 = e \delta^3(\mathbf{x}' - \mathbf{x}) \]  \hspace{1cm} (6)

is the charge density of a particle located at point \( \mathbf{x} \). Adding Eq.(5) to Eq.(3) results in the emergence of the covariant derivative, \( \partial \rightarrow \partial - ie \mathbf{A}(\mathbf{x}) \), in the orbital part.

Let us dwell now on the purely field contribution to the angular momentum, Eq.(4). Substituting \( \mathbf{B} = \partial \times \mathbf{A}_f \) (where \( \mathbf{A}_f \) is a certain, fixed, vector potential) into Eq.(4) and integrating it by parts, we get

\[ M_E^{i}_{EM} = \int d^3x' \partial'_l (\varepsilon^{ikl} x'^k \mathbf{A}_f^m \mathbf{E}^m - \varepsilon^{ikm} x'^k \mathbf{A}_f^l \mathbf{E}^m - \varepsilon^{ikm} x'^k \mathbf{A}_f^m \mathbf{E}^l) + \]
\[ + \int d^3x' \varepsilon^{ikm} x'^k [A^m_\nu \partial'_\nu \mathbf{E}^n + \mathbf{E}^m \partial'_\nu A^\nu_\nu + A^\nu_\nu (\partial'_\nu \mathbf{E}^m - \partial'_m \mathbf{E}^\nu)]]. \]  \hspace{1cm} (7)

Note that \( M_{EM} \) is the improved angular momentum which differs from the canonical angular momentum \( \tilde{M}_{EM} \) (see Eqs.(A.11) and (A.12)),

\[ M_E^{i}_{EM} - \tilde{M}_E^{i}_{EM} = \int d^3x' \partial'_l \varepsilon^{ikm} x'^k \chi^{m0l} = - \int d^3x' \partial'_l \varepsilon^{ikm} x'^k \mathbf{A}_f^m \mathbf{E}^l, \]  \hspace{1cm} (8)

where we have used Eq.(A.16) for \( \chi^{\mu\nu\lambda} \). We consider a static rotationally invariant magnetic field, \( (\mathbf{x} \times \partial) \cdot \mathbf{B} = 0 \), and choose \( \mathbf{A}_f \) satisfying conditions

\[ \mathbf{x} \cdot \mathbf{A}_f = 0, \quad \partial \cdot \mathbf{A}_f = 0. \]  \hspace{1cm} (9)
As to the electric field, it is generated by static charge density $j^0$ (6),

$$\partial' \cdot E = j^0, \quad E = \frac{e}{4\pi} \frac{x' - x}{|x' - x|^3},$$

(10)

and is curlless, $\partial' \times E = 0$. Then only the first term in square brackets in the second integral in the right hand side of Eq.(7) survives, yielding upon integration a contribution which is equal to the contribution of particle-field interaction, Eq.(5), taken with opposite sign and at $A = A_f$. As to the first integral in the right hand side of Eq.(7), it is transformed to an integral over a closed surface, as described in Appendix A, see Eqs.(A.4) and (A.13). Thus the behaviour of terms in brackets in this integral is crucial: the first two terms decrease as $|x'|^{-3}$ at large $|x'|$ yielding vanishing contribution upon integration, whereas the last one decreases as $|x'|^{-2}$ at large $|x'|$. This yields upon integration finite contribution which is equal to the total magnetic flux times $e(2\pi)^{-1}$ with the opposite sign [24]. Consequently, one gets

$$M_{EM} = e x \times A_f(x) - \frac{e}{2\pi} \int d^2 x' \mathbf{B},$$

(11)

where the integral is over a plane which is orthogonal to $\mathbf{B}$, and

$$\tilde{M}_{EM} = e x \times A_f(x).$$

(12)

Summing Eqs.(3), (5) and (12), one gets

$$\tilde{J} = -i x \times [\partial - ieA(x) + ieA_f(x)] + \frac{1}{2} \sigma.$$  

(13)

Taking into account relation

$$x \times A_f(x) = \frac{1}{2\pi} \int_D d^2 x' \mathbf{B},$$

(14)

where the integration is over region $D$ defined by $(x'^1)^2 + (x'^2)^2 < (x^1)^2 + (x^2)^2$, one can verify that $\tilde{J}$ (13) commutes with the one-particle hamiltonian operator; thus $\tilde{J}$ (13) can be regarded as the operator of total angular momentum of the system, which is conserved as a consequence of rotational invariance.

If the improved quantity, Eq.(11), is taken instead of the canonical one, Eq.(12), then, summing Eqs.(3), (5) and (11), one gets

$$J = -i x \times [\partial - ieA(x) + ieA_f(x)] - \frac{e}{2\pi} \int d^2 x' \mathbf{B} + \frac{1}{2} \sigma.$$  

(15)

Since the difference between operators $\tilde{J}$ (13) and $J$ (15) is $x$-independent and diagonal in spin indices, the latter operator commutes with the hamiltonian as well.

Although both the canonical and improved definitions of angular momentum of electromagnetic field are compatible with the conservation of total angular momentum,
the motivation in favour of the first one may be seen in the fact that we start initially from the canonical angular momentum. Then, at intermediate stage, the improved angular momentum is introduced as a convenient and well-defined quantity which is constructed directly from the gauge invariant and symmetric energy-momentum tensor, see Appendix A. This quantity plays a somewhat auxiliary role allowing one to express the canonical angular momentum through the explicitly gauge invariant entities. Note that angular momentum $\tilde{J} (13)$ in the gauge $A = A_f$ coincides with angular momentum of a free particle, Eq.(3). Thus, it is evident that spectra of operators $J_0$ and $\tilde{J}$ coincide, and this may serve as an additional argument in favour of the canonical definition of angular momentum.

Note also that the difference between $J (15)$ and $\tilde{J} (13)$ is equal to the return magnetic flux (times $e(2\pi)^{-1}$), see Eq.(8), which completes a flux loop at extremely large distances outside the position of the charged particle. If the particle is allowed to go beyond the region of the return flux, then, as it enters this region from outside and crosses it, one arrives at expression (13) for total angular momentum inside the region of the return flux.

In the case of a relativistic massive spinor particle, considered in the present paper, the spin matrix is $\sigma = (2i)^{-1}\alpha \times \alpha$ and the hamiltonian is

$$H = -i \alpha \cdot [\partial - ieA(x)] + \gamma^0 m, \quad (16)$$

where $\alpha = \gamma^0 \gamma$, and $\gamma^0, \gamma$ are the Dirac matrices.

### 3 Bohm–Aharonov configuration of the magnetic field

Of special physical interest, as it was noted in Introduction, is the case when the region of the magnetic flux lines is not accessible to a charged particle [2]. Namely, one considers a rotationally invariant magnetic field configuration in the form of a long solenoid with extremely distant return flux, and the charged particle is posited in the field-free region between the solenoid and the return flux. From now on we shall call the region of solenoid as the inner region and the field-free region as the outer region, implying that the return flux is all pushed out to infinity.

In the outer region, improved quantity (4) is obviously vanishing, $M_{EM} = 0$, whereas canonical quantity $\tilde{M}_{EM}$ is given by the same expression as Eq.(12) but is $x$-independent, since in this case it depends on the whole flux through the inner region, $x \times A_f(x) = \frac{1}{2\pi} \int d^2x' B$. Eq.(13) takes form

$$\tilde{J} = -i x \times [\partial - ieA(x)] + \frac{e}{2\pi} \int d^2x' B + \frac{1}{4i} \alpha \times \alpha, \quad (17)$$

whereas Eq.(15) takes form of the kinetic (or mechanical) angular momentum of the particle,

$$J = -i x \times [\partial - ieA(x)] + \frac{1}{4i} \alpha \times \alpha. \quad (18)$$
Note that the outer region is not simply connected, and one may feel free not to require the coincidence of the spectrum of total angular momentum with that of $J_0$ \((3)\). Also, vanishing of the improved angular momentum of electromagnetic field in this case may be regarded as an argument in favour of choosing $\tilde{J}$ \((17)\) or $J$ \((18)\) gives the physically meaningful angular momentum, was disputed in the literature, see, e.g., Refs.\([22, 23, 24, 25]\). In the present paper we shall show that namely the kinetic definition of angular momentum provides the physically reasonable behaviour of the thermal quadratic fluctuation and correlation in the second-quantized theory at nonzero temperature.

But before going to the second-quantized theory, let us discuss some manifestations of non-simply-connectedness of the outer region. Let magnetic field $B$ be directed along the $x^3$-axis in the inner region, and the outer region correspond to $(x^1)^2 + (x^2)^2 > r_c^2$ with $r_c$ being the transverse size of the inner region. Then vector potential $A_f$ takes form

$$
A_1^f = -\Phi \frac{x^2}{(x^1)^2 + (x^2)^2}, \quad A_2^f = \Phi \frac{x^1}{(x^1)^2 + (x^2)^2}, \quad A_3^f = 0,
$$

where

$$
\Phi = \frac{1}{2\pi} \int d^2 x' B^3
$$

is the total magnetic flux in units of $2\pi$, and Eq.(18) in gauge $A = A_f$ takes form

$$
J^3 = -i(x^1 \partial_2 - x^2 \partial_1) - e\Phi + \frac{1}{2i} \alpha_1 \alpha_2.
$$

Operator (21) acts on functions which are single-valued in the outer region:

$$
\langle r, \varphi, x^3 \rangle = \langle r, \varphi + 2\pi, x^3 \rangle,
$$

where $r$ and $\varphi$ are the polar coordinates in the $x^1 x^2$-plane. Since the outer region is not simply connected, one can consider functions satisfying much more general condition

$$
\langle r, \varphi, x^3 \rangle' = e^{i2\pi \Xi} \langle r, \varphi + 2\pi, x^3 \rangle',
$$

where $\Xi$ is a continuous real parameter. Functions satisfying conditions (22) and (23) are related by gauge transformation

$$
\langle r, \varphi, x^3 \rangle' = e^{-i\Xi \varphi} \langle r, \varphi, x^3 \rangle,
$$

$$
A' = A_f - e^{-1} \Xi \partial \varphi.
$$

Neglecting the transverse size of the inner region ($r_c \to 0$), and presenting the angular variable with range $0 < \varphi < 2\pi$ as

$$
\varphi = \arctan \left( \frac{x^2}{x^1} \right) + \pi \left[ \theta(-x^1) + 2\theta(x^1) \theta(-x^2) \right],
$$
where \( \theta(u) = \begin{cases} 1, & u > 0 \\ 0, & u < 0 \end{cases} \), one gets immediately

\[
\partial_1 \varphi = -\frac{x^2}{(x^1)^2 + (x^2)^2}, \quad \partial_2 \varphi = \frac{x^1}{(x^1)^2 + (x^2)^2} - 2\pi \theta(x^1) \delta(x^2),
\]

and, consequently,

\[
A'^1 = -(\Phi - e^{-1}\Xi)\frac{x^2}{(x^1)^2 + (x^2)^2},
\]

\[
A'^2 = (\Phi - e^{-1}\Xi)\frac{x^1}{(x^1)^2 + (x^2)^2} + 2\pi e^{-1} \theta(x^1) \delta(x^2),
\]

\[
A'^3 = 0.
\]

Although the curl of the vector potential remains invariant under gauge transformation (24),

\[
B^3 = \partial_1 A'^2 - \partial_2 A'^1 = \partial_1 A'^2 - \partial_2 A'^1 = 2\pi \Phi \delta(x^1) \delta(x^2),
\]

angular momentum (21) is changed to

\[
J'^3 = -i(x^1 \partial_2 - x^2 \partial_1) - e \Phi + \Xi + \frac{1}{2i} \alpha^1 \alpha^2,
\]

but its spectrum remains unchanged, because operator (29) acts on functions satisfying condition (23). In particular, choosing \( \Xi = e \Phi \) one gets the gauge with the vector potential eliminated everywhere excepting the semiaxis of positive \( x^1 \):

\[
A'^1 = 0, \quad A'^2 = 2\pi \Phi \theta(x^1) \delta(x^2), \quad A'^3 = 0.
\]

Kinetic angular momentum (18) in this gauge coincides with angular momentum of a free particle, Eq.(3), but their spectra differ: the operator of the latter acts on functions which are defined everywhere and, thus, single-valued (22), while the operator of the former acts on functions which are defined in space with a cut along the positive \( x^1 \) semiaxis and, thus, satisfying boundary condition (23) with \( \Xi = e \Phi \) on the sides of the cut. The cut can be continuously deformed, and, in general, its projection on the \( x^1 x^2 \)-plane can be a curved line starting from the origin and going to infinity. Eq.(30) in the case of such cut takes form

\[
A'^j = -2\pi \Phi \int_0^\infty e^{2ij'} \delta^2(x - x') dx'^j',
\]

where coordinate vectors \( x \) and \( x' \) lie in the \( x^1 x^2 \)-plane.

To conclude this section, we note that, obviously, canonical angular momentum (17) in gauge \( A = A_f \) takes form

\[
\bar{J}^3 = -i(x^1 \partial_2 - x^2 \partial_1) + \frac{1}{2i} \alpha^1 \alpha^2,
\]

(31)
and it is changed to
\[ \tilde{J}^3 = -i(x^1 \partial_2 - x^2 \partial_1) + \Xi + \frac{1}{2i} \alpha^1 \alpha^2 \] (32)
under gauge transformation (24). The spectrum of canonical angular momentum is given by Eq.(1), while the spectrum of kinetic angular momentum is given by Eq.(2).

4 Second-quantized theory at finite temperature

The operator of the second-quantized fermion field in a static background can be presented in the form
\[ \Psi(x, t) = \sum \int_{(E_\lambda > 0)} e^{-iE_\lambda t} \langle x| \lambda \rangle a_\lambda + \sum \int_{(E_\lambda < 0)} e^{-iE_\lambda t} \langle x| \lambda \rangle b_\lambda^+, \] (33)
where \( a_\lambda^+ \) and \( a_\lambda \) (\( b_\lambda^+ \) and \( b_\lambda \)) are the fermion (antifermion) creation and destruction operators satisfying anticommutation relations,
\[ [a_\lambda, a_{\lambda'}^+] = [b_\lambda, b_{\lambda'}^+] = \langle \lambda| \lambda' \rangle, \] (34)
and \( \langle x| \lambda \rangle \) is the solution to the stationary Dirac equation,
\[ H \langle x| \lambda \rangle = E_\lambda \langle x| \lambda \rangle, \] (35)
\( H \) is the Dirac (one-particle) hamiltonian, \( \lambda \) is the set of parameters (quantum numbers) specifying a one-particle state, and \( E_\lambda \) is the energy of the state; symbol \( \sum \int \) means the summation over discrete and the integration (with a certain measure) over continuous values of \( \lambda \). Ground state \( |\text{vac}\rangle \) of the second-quantized theory is defined as
\[ a_\lambda |\text{vac}\rangle = b_\lambda |\text{vac}\rangle = 0. \] (36)

Let \( J \) be an operator commuting with the hamiltonian in the first-quantized theory,
\[ [J, H]_- = 0. \] (37)
In the case of unbounded operators, commutation of their resolvents is implied, or, to be more specific, it is sufficient to require that operators \( H \) and \( J \) have a common set of eigenfunctions, i.e. relation
\[ J \langle x| \lambda \rangle = j_\lambda \langle x| \lambda \rangle \] (38)
holds as well as Eq.(35). Eigenfunctions \( \langle x| \lambda \rangle \) satisfy the conditions of completeness and orthonormality; in general, normalization to a delta function is implied. Thus,
in the second-quantized theory, the operators of the dynamical variables (physical observables) corresponding to \( H \) and \( J \) can be diagonalized:

\[
\hat{U} = \frac{1}{2} \int d^d x \left[ \Psi^+(x, t), H \Psi(x, t) \right] = \sum \int E_\lambda \left[ a_\lambda^+ a_\lambda - b_\lambda^+ b_\lambda - \frac{1}{2} \text{sgn}(E_\lambda) \right] \tag{39}
\]

and

\[
\hat{M} = \frac{1}{2} \int d^d x \left[ \Psi^+(x, t), J \Psi(x, t) \right] = \sum \int j_\lambda \left[ a_\lambda^+ a_\lambda - b_\lambda^+ b_\lambda - \frac{1}{2} \text{sgn}(E_\lambda) \right], \tag{40}
\]

\( d \) is the space dimension, and \( \text{sgn}(u) = \theta(u) - \theta(-u) \) is the sign function.

The thermal average of the observable corresponding to operator (40) is conventionally defined as (see, e.g., Ref.[33])

\[
M(T) = \langle \hat{M} \rangle_\beta \equiv \frac{S p \hat{M} e^{-\beta \hat{U}}}{S p e^{-\beta \hat{U}}}, \; \beta = (k_B T)^{-1}, \tag{41}
\]

where \( T \) is the equilibrium temperature, \( k_B \) is the Boltzmann constant, and \( S p \) is the trace or the sum over the expectation values in the Fock state basis created by operators in Eq.(34). Also one defines the thermal quadratic fluctuation

\[
\Delta(T; \hat{M}, \hat{M}) = \langle \hat{M}^2 \rangle_\beta - (\langle \hat{M} \rangle_\beta)^2. \tag{42}
\]

Quantities (41) and (42) can be expressed through the derivatives of the appropriate thermodynamic potential:

\[
M(T) = -\left. \frac{\partial \Omega(\beta, \mu)}{\partial \mu} \right|_{\mu=0}, \; \Delta(T; \hat{M}, \hat{M}) = -\left. \frac{1}{\beta} \frac{\partial^2 \Omega(\beta, \mu)}{\partial \mu^2} \right|_{\mu=0}, \tag{43}
\]

where \( \mu \) is the appropriate chemical potential, and

\[
\Omega(\beta, \mu) = \frac{1}{\beta} \ln S p \exp[-\beta(\hat{U} - \mu \hat{M})]. \tag{44}
\]

We show in Appendix B that thermodynamic potential (44) is presented as

\[
\Omega(\beta, \mu) = -\frac{1}{\beta} \text{Tr} \ln \cosh\left[\frac{1}{2} \beta (H - \mu J)\right], \tag{45}
\]

where \( \text{Tr} \) is the trace of an integro-differential operator in the functional space: \( \text{Tr} \ldots = \int d^d x \text{tr} \langle x | \ldots | x \rangle; \text{tr} \) denotes the trace over spinor indices only. Then, using Eq.(43), one can express average (41) and fluctuation (42) through functional traces of operators in the first-quantized theory:

\[
M(T) = -\frac{1}{2} \text{Tr} J \tanh\left(\frac{1}{2} \beta H\right). \tag{46}
\]
\[ \Delta(T; \hat{M}, \hat{M}) = \frac{1}{4} \text{Tr} J^2 \text{sech}^2 \left( \frac{1}{2} \beta H \right). \]  
(47)

Eqs. (46) and (47) are transformed into integrals over the energy spectrum,

\[ M(T) = -\frac{1}{2} \int_{-\infty}^{\infty} dE \tau_J(E) \tanh \left( \frac{1}{2} \beta E \right) \]  
(48)

and

\[ \Delta(T; \hat{M}, \hat{M}) = \frac{1}{4} \int_{-\infty}^{\infty} dE \tau_{J^2}(E) \text{sech}^2 \left( \frac{1}{2} \beta E \right), \]  
(49)

where

\[ \tau_J(E) = \text{Tr} J \delta(H - E) \]  
(50)

and

\[ \tau_{J^2}(E) = \text{Tr} J^2 \delta(H - E) \]  
(51)

are the appropriate spectral densities.

Note that in the case of \( J = I \), where \( I \) is the unit matrix in the space of Dirac matrices, the corresponding operator in the second-quantized theory is the operator of fermion number; \( \mu \) and \( \Omega \) are the conventional chemical and thermodynamic potentials in this case. In the \( d = 1 \) case fermion number is the only observable which is conserved in addition to energy. In more than one dimensions there are more conserved observables. In particular, in the \( d = 2 \) case, in addition to energy and fermion number, also total angular momentum is conserved when the system is rotationally invariant, as it was discussed in two previous sections.

If an observable is not conserved, then its operator in the first-quantized theory does not commute with hamiltonian, \([\Upsilon, H] \neq 0\), and its operator in the second-quantized theory,

\[ \hat{O} = \frac{1}{2} \int d^d x \left[ \Psi^\dagger(x, t), \Upsilon \Psi(x, t) \right]_-, \]  
(52)

is not diagonalizable. Similarly to Eq. (48), the thermal average of the nonconserved observable can be presented as

\[ O(T) = -\frac{1}{2} \int_{-\infty}^{\infty} dE \tau_{\Upsilon}(E) \tanh \left( \frac{1}{2} \beta E \right), \]  
(53)

where

\[ \tau_{\Upsilon}(E) = \text{Tr} \Upsilon \delta(H - E). \]  
(54)

Also one can consider the thermal correlation of the conserved and nonconserved observables,

\[ \Delta(T; \hat{O}, \hat{M}) = \langle \hat{O} \hat{M} \rangle_\beta - \langle \hat{O} \rangle_\beta \langle \hat{M} \rangle_\beta, \]  
(55)
which, similarly to Eq.(49), can be presented as

\[ \Delta(T; \hat{O}, \hat{M}) = \frac{1}{4} \int_{-\infty}^{\infty} dE \tau_{\hat{T}, \hat{J}}(E) \text{sech}^2\left(\frac{1}{2} \beta E\right), \]  

(56)

where

\[ \tau_{\hat{T}, \hat{J}}(E) = \text{Tr} \hat{T} \hat{J} \delta(H - E). \]

(57)

It should be noted that relations (45), (48), (49), (53), and (56) are somewhat formal, and the proper treatment of the normal ordering of operator product in the second-quantized theory is required. In the absence of interaction, the operators are normal ordered (see, e.g., Ref. [34]), i.e. the \(c\)-number pieces in Eqs.(39) and (40) are dropped. Consequently, in the presence of interaction with external fields, the \(c\)-number pieces in Eqs.(39) and (40) are renormalized by subtracting these dropped pieces. Correspondingly, thermodynamic potential (45) is decomposed as

\[ \Omega(\beta, \mu) = \Omega^{(0)}(\beta, \mu) + \Omega^{(1)}(\beta, \mu), \]

(58)

where

\[ \Omega^{(0)}(\beta, \mu) = -\frac{1}{\beta} \text{Tr} \ln \left\{ 1 + \exp \left[ -\beta (|H_0| - \mu J_0 \text{sgn}(H_0)) \right] \right\} \]

(59)

is the thermodynamic potential in the free field case, and

\[ \Omega^{(1)}(\beta, \mu) = -\frac{1}{\beta} \left\{ \text{Tr} \ln \cosh \left[ \frac{1}{2} \beta (H - \mu J) \right] - \text{Tr} \ln \cosh \left[ \frac{1}{2} \beta (H_0 - \mu J_0) \right] \right\} \]

(60)

is the addition which is due to interaction with external fields; here the operators in the first-quantized theory without interaction are denoted by \(H_0\) and \(J_0\). Note that term \(-\frac{1}{2} \text{Tr} [|H_0| - \mu J_0 \text{sgn}(H_0)]\) is dropped, which corresponds to the normal ordering at zero temperature in the free field case.

As a consequence of Eqs.(58)-(60) one gets

\[ M(T) = M^{(0)}(T) + M^{(1)}(T) \]

(61)

and

\[ \Delta(T; \hat{M}, \hat{\hat{M}}) = \Delta^{(0)}(T; \hat{M}, \hat{\hat{M}}) + \Delta^{(1)}(T; \hat{M}, \hat{\hat{M}}), \]

(62)

where

\[ M^{(0)}(T) = \int_{-\infty}^{\infty} dE \tau_{\hat{T}}^{(0)}(E) \frac{\text{sgn}(E)}{e^{\beta|E|} + 1}; \]

(63)

\[ M^{(1)}(T) = -\frac{1}{2} \int_{-\infty}^{\infty} dE \tau_{\hat{T}}^{(1)}(E) \tanh\left(\frac{1}{2} \beta E\right), \]

(64)

\[ \tau_{\hat{T}}^{(0)}(E) = \text{Tr} J_0 \delta(H_0 - E), \quad \tau_{\hat{T}}^{(1)}(E) = \text{Tr} J \delta(H - E) - \text{Tr} J_0 \delta(H_0 - E), \]

(65)
and
\[
\Delta^{(0)}(T; \hat{M}, \hat{M}) = \frac{1}{4} \int_{-\infty}^{\infty} dE \tau^{(0)}_{J^2}(E) \text{sech}^2\left(\frac{1}{2} \beta E\right), \tag{66}
\]
\[
\Delta^{(1)}(T; \hat{M}, \hat{M}) = \frac{1}{4} \int_{-\infty}^{\infty} dE \tau^{(1)}_{J^2}(E) \text{sech}^2\left(\frac{1}{2} \beta E\right), \tag{67}
\]
\[
\tau^{(0)}_{J^2}(E) = \text{Tr} J^2_0 \delta(H_0 - E), \quad \tau^{(1)}_{J^2}(E) = \text{Tr} J^2 \delta(H - E) - \text{Tr} J^2_0 \delta(H_0 - E). \tag{68}
\]

Using relation \(\delta(H - E) = \frac{1}{2\pi i} \left[(H - E - i0)^{-1} - (H - E + i0)^{-1}\right]\), one can transform integrals over the real energy spectrum in Eqs.(64) and (67) into integrals over a contour on the complex energy plane, thus yielding a representation of thermal characteristics through the renormalized resolvent traces,
\[
M^{(1)}(T) = -\frac{1}{2} \int_C \frac{d\omega}{2\pi i} \left[\text{Tr} J(H - \omega)^{-1}\right]_{\text{ren}} \tanh\left(\frac{1}{2} \beta \omega\right) \tag{69}
\]
and
\[
\Delta^{(1)}(T; \hat{M}, \hat{M}) = \frac{1}{4} \int_C \frac{d\omega}{2\pi i} \left[\text{Tr} J^2(H - \omega)^{-1}\right]_{\text{ren}} \text{sech}^2\left(\frac{1}{2} \beta \omega\right), \tag{70}
\]
where \(C\) is the contour consisting of two collinear straight lines, \((-\infty + i0, +\infty + i0)\) and \((+\infty - i0, -\infty - i0)\), in the complex \(\omega\)-plane, and the renormalized resolvent traces are
\[
\left[\text{Tr} J(H - \omega)^{-1}\right]_{\text{ren}} = \text{Tr} J(H - \omega)^{-1} - \text{Tr} J_0(H_0 - \omega)^{-1} \tag{71}
\]
and
\[
\left[\text{Tr} J^2(H - \omega)^{-1}\right]_{\text{ren}} = \text{Tr} J^2(H - \omega)^{-1} - \text{Tr} J^2_0(H_0 - \omega)^{-1}. \tag{72}
\]

To conclude this section, we note that thermal average (53) and correlation (56) can be treated in a similar way.

5 Traces of resolvents and spectral densities

Let us consider quantization of the spinor field on a plane \((d = 2)\) which is orthogonal to the Bohm–Aharonov magnetic field configuration. Dirac hamiltonian (16) in gauge \(A = A_f\), where \(A_f\) is given by Eq.(19), takes form
\[
H = -i\alpha^r \partial_r - i\alpha^\varphi (\partial \varphi - ie \Phi) + \gamma^0 m, \tag{73}
\]
where \(\Phi\) is the total magnetic flux of the solenoid (Bohm–Aharonov vortex), see Eq.(20), and
\[
\alpha^r = \alpha^1 \cos \varphi + \alpha^2 \sin \varphi, \quad \alpha^\varphi = -\alpha^1 \sin \varphi + \alpha^2 \cos \varphi.
\]
In 2 + 1-dimensional space-time the Clifford algebra has two inequivalent irreducible representations which can be differed in the following way:

\[ \alpha^1 \alpha^2 \gamma^0 = is, \quad s = \pm 1. \]  

Choosing the \( \gamma_0 \) matrix in the diagonal form

\[ \gamma^0 = \sigma_3, \]  

one gets

\[ \alpha^1 = -e^{i\frac{s}{2}\sigma_3 \chi_s} \sigma_2 e^{-i\frac{s}{2}\sigma_3 \chi_s}, \quad \alpha^2 = se^{i\frac{s}{2}\sigma_3 \chi_s} \sigma_1 e^{-i\frac{s}{2}\sigma_3 \chi_s}, \]  

where \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are the Pauli matrices, and \( \chi_1 \) and \( \chi_{-1} \) are the parameters varying in interval \( 0 < \chi_s < 2\pi \) to go over to equivalent representations. Note also that in odd-dimensional space-time the \( m \) parameter in Eq.(16) can take both positive and negative values; a change of sign of \( m \) corresponds to going over to the inequivalent representation.

In view of Eq.(74), the kinetic, Eq.(21), and canonical, Eq.(31), angular momenta of the planar system take form

\[ J = -i \partial_\varphi - e \Phi + \frac{1}{2} s \gamma^0 \]  

and

\[ \bar{J} = -i \partial_\varphi + \frac{1}{2} s \gamma^0, \]

respectively.

The kernel of the resolvent (the Green’s function) of the Dirac hamiltonian in the coordinate representation is defined as

\[ G^\omega(r, \varphi; r', \varphi') = \langle r, \varphi | (H - \omega)^{-1} | r', \varphi' \rangle, \]  

where \( \omega \) is a complex parameter with dimension of energy. Taking into account Eqs.(75) and (76), one expands Eq.(79) in modes and gets the following expression

\[ G^\omega(r, \varphi; r', \varphi') = \frac{1}{2\pi} \sum_{n = -\infty}^{\infty} e^{in(\varphi - \varphi')} \begin{pmatrix} a_n(r; r') & d_n(r; r') e^{-is(\varphi' - \chi_s)} \\ b_n(r; r') e^{is(\varphi - \chi_s)} & c_n(r; r') \end{pmatrix}. \]  

In the case of \( H \) given by Eq.(73) radial components of \( G^\omega(r, \varphi; r', \varphi') \) satisfy equations

\[
\begin{pmatrix}
-\omega + m \\
-\partial_r + s(n - e\Phi + s)r^{-1}
\end{pmatrix}
\begin{pmatrix}
\partial_r + s(n - e\Phi + s)r^{-1} \\
-\omega - m
\end{pmatrix}
\begin{pmatrix}
a_n(r; r') & d_n(r; r') \\
b_n(r; r') & c_n(r; r')
\end{pmatrix}
\] =

\[
\begin{pmatrix}
-\omega + m \\
-\partial_r + s(n - e\Phi + s)r'^{-1}
\end{pmatrix}
\begin{pmatrix}
\partial_r + s(n - e\Phi + s)r'^{-1} \\
-\omega - m
\end{pmatrix}
\begin{pmatrix}
a_n(r; r') & b_n(r; r') \\
d_n(r; r') & c_n(r; r')
\end{pmatrix}
\]
appropriately, radial components to the Weyl – von Neumann theory of self-adjoint operators (see, e.g., Ref.\[36\]). Aparthe vortex, free hamiltonian $H$ for all Refs.\[35, 18, 19\]. To be more precise, partial hamiltonians are essentially self-adjoint as a parameter of the self-adjoint extension of the hamiltonian operator, for details see of the boundary condition at the location of the vortex (at the origin) exhibits itself regularity at small distances ($r \to 0$ or $r' \to 0$), and this corresponds to the fact that free hamiltonian $H_0$ (i.e. $H (73)$ at $e\Phi = 0$) is essentially self-adjoint. In the presence of the vortex, $e\Phi \neq 0$, when a size of the inner region is neglected ($r_c \to 0$), the condition of regularity at small distances cannot be imposed on all radial components. This is due to the fact that hamiltonian $H (73)$ is not essentially self-adjoint, and a parameter of the boundary condition at the location of the vortex (at the origin) exhibits itself as a parameter of the self-adjoint extension of the hamiltonian operator, for details see Refs.\[35, 18, 19\]. To be more precise, partial hamiltonians are essentially self-adjoint for all $n$ with the exception of $n = n_c$, where

$$n_c = \lceil e\Phi \rceil + \frac{1}{2} - \frac{1}{2} s, \quad (82)$$

$[u]$ is the integer part of quantity $u$ (i.e., the largest integer which is less than or equal to $u$). The partial Hamiltonian for $n = n_c$ requires a self-adjoint extension according to the Weyl – von Neumann theory of self-adjoint operators (see, e.g., Ref.\[36\]). Appropriately, radial components $a_n$, $b_n$, $c_n$, and $d_n$ in Eq.\(80\) with $n \neq n_c$ are regular at $r \to 0$ and $r' \to 0$, whereas those with $n = n_c$ satisfy conditions (for details see Ref.\[31\]):

$$\cos \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^F a_{n_c}(r; r') = -\text{sgn}(m) \sin \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r' \to 0} (|m|r)^{1-F} b_{n_c}(r; r') \right),$$

$$\cos \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^F d_{n_c}(r; r') = -\text{sgn}(m) \sin \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r' \to 0} (|m|r)^{1-F} c_{n_c}(r; r') \right),$$

and

$$\cos \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r' \to 0} (|m|r')^F a_{n_c}(r; r') = -\text{sgn}(m) \sin \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r')^{1-F} b_{n_c}(r; r') \right),$$

$$\cos \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r' \to 0} (|m|r')^F b_{n_c}(r; r') = -\text{sgn}(m) \sin \left( s\frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r')^{1-F} c_{n_c}(r; r') \right),$$

where $\Theta$ is the self-adjoint extension parameter, and

$$F = s(e\Phi - [e\Phi]) + \frac{1}{2} - \frac{1}{2} s; \quad (85)$$

note here that Eqs.\(83\) and \(84\) imply that $0 < F < 1$, since in the case of $F = \frac{1}{2} - \frac{1}{2} s$ all radial components obey the condition of regularity at $r \to 0$ and $r' \to 0$. Note also that Eqs.\(83\) and \(84\) are periodic in $\Theta$ with period $2\pi$. 

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The radial components of the resolvent kernel in the presence and in the absence of the vortex are listed in Appendix C.

Let us consider quantities

\[ tr \ G^\omega(r, \varphi; r', \varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[ a_n(r; r') + c_n(r; r') \right], \quad (86) \]

\[ tr \ J \ G^\omega(r, \varphi; r', \varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left( n - e\Phi + \frac{1}{2} s \right) \left[ a_n(r; r') + c_n(r; r') \right], \quad (87) \]

\[ tr \ J^2 \ G^\omega(r, \varphi; r', \varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left( n - e\Phi + \frac{1}{2} s \right)^2 \left[ a_n(r; r') + c_n(r; r') \right]. \quad (88) \]

In the absence of the vortex, using Eqs.(C.14)-(C.17) and performing summation over \(n\), one gets in the case of \(Im\ k > |Re\ k|\):

\[ tr \ G_0^\omega(r, \varphi; r', \varphi) = \frac{\omega}{\pi} K_0(-ik|r - r'|), \quad (89) \]

\[ tr \ J_0 G_0^\omega(r, \varphi; r', \varphi) = \frac{sm}{2\pi} K_0(-ik|r - r'|), \quad (90) \]

\[ tr \ J_0^2 G_0^\omega(r, \varphi; r', \varphi) = \frac{\omega}{4\pi} \left[ K_0(-ik|r - r'|) - \frac{A_i k r'r'}{|r - r'|} K_1(-ik|r - r'|) \right], \quad (91) \]

where \(K_n(u)\) is the Macdonald function of order \(n\). In the presence of the vortex, quantities (86)-(88) consist of two pieces: one is finite in the limit \(r' \to r\), and another one, which is divergent in this limit, coincides with quantities (89)-(91), correspondingly. Moreover, the difference between appropriate quantities in the presence and in the absence of the vortex at \(r' = r\) is exponentially decreasing as \(r \to \infty\), and this allows us to perform integration over the two-dimensional infinite spatial volume, \(\int_0^{2\pi} d\varphi \int_0^{\infty} dr, r\), and obtain the renormalized traces (see Refs.31,32):

\[ [\text{Tr} \ (H - \omega)^{-1}]_{\text{ren}} = -\frac{1}{\omega^2 - m^2} \left[ F(\omega + m) \tan \nu_\omega + (1 - F)(\omega - m) e^{iF}\pi \right] - F(1 - F)\omega, \]

\[ [\text{Tr} \ J \ (H - \omega)^{-1}]_{\text{ren}} = -s(F - \frac{1}{2})[\text{Tr} \ (H - \omega)^{-1}]_{\text{ren}} + \frac{sF(1 - F)}{\omega^2 - m^2} \left[ \frac{1}{3}(F - \frac{1}{2})\omega + \frac{1}{2}m \right], \]

\[ [\text{Tr} \ J^2 \ (H - \omega)^{-1}]_{\text{ren}} = (F - \frac{1}{2})^2[\text{Tr} \ (H - \omega)^{-1}]_{\text{ren}} + \frac{F(1 - F)}{\omega^2 - m^2} \left[ \frac{1}{2}F(1 - F)\omega - \frac{1}{3}(F - \frac{1}{2})m \right] \]  

here \(\tan \nu_\omega\) is given by Eq.(C.13), and results (92)-(94) are analytically continued from region \(Im\ k > |Re\ k|\) to region \(Im\ k > 0\), i.e. to the whole complex \(\omega\)-plane.

To deal with the divergent at \(r' \to r\) pieces of the resolvent kernels, it is sufficient to regularize the resolvent kernel in the absence of the vortex and define

\[ G_0^{\omega, r}(r, \varphi; r', \varphi') = \langle r, \varphi | (H_0 - \omega)^{-1} \exp(-iH_0^2) | r', \varphi' \rangle, \quad (95) \]
where $t > 0$ is the regularization parameter. In Appendix D we obtain the following relations in the case of $\text{Im} k > |\text{Re} k|$:

\begin{align}
\text{tr} \, G^{\omega,t}_0 (r, \varphi; r, \varphi) &= \frac{\omega}{2\pi} e^{-t(m^2+k^2)} E_1(-tk^2), \\
\text{tr} \, J_0 G^{\omega,t}_0 (r, \varphi; r, \varphi) &= \frac{sm}{4\pi} e^{-t(m^2+k^2)} E_1(-tk^2), \\
\text{tr} \, J_0^2 G^{\omega,t}_0 (r, \varphi; r, \varphi) &= \frac{\omega}{4\pi} e^{-tm^2} \left[ r^2 t^{-1} + (r^2 k^2 + \frac{1}{2}) e^{-tk^2} E_1(-tk^2) \right],
\end{align}

where

$$E_1(u) = \int_u^\infty \frac{du}{u} e^{-u}$$

is the exponential integral (see, e.g., Ref. [37]); note that Eqs. (96)-(98) are analytically continued to region $\text{Im} k > 0$, i.e. to the whole complex $\omega$-plane. Integrating over the twodimensional spatial volume, \( \int \frac{d\varphi}{2\pi} \int \frac{dr}{R} \), where $R$ is the volume radius, we get

\begin{align}
\text{Tr} \, (H_0 - \omega)^{-1} e^{-tH^2_0} &= \frac{1}{2} R^2 \omega e^{-t\omega^2} E_1[t(m^2 - \omega^2)], \\
\text{Tr} \, J_0 (H_0 - \omega)^{-1} e^{-tH^2_0} &= \frac{1}{4} R^2 sm e^{-t\omega^2} E_1[t(m^2 - \omega^2)], \\
\text{Tr} \, J_0^2 (H_0 - \omega)^{-1} e^{-tH^2_0} &= \frac{1}{8} R^2 \omega \left\{ R^2 t^{-1} e^{-tm^2} + [R^2 (\omega^2 - m^2) + 1] e^{-t\omega^2} E_1[t(m^2 - \omega^2)] \right\}.
\end{align}

Although traces (99)-(101) are divergent in the limit $t \to 0_+$, the divergences do not contribute to the physical quantities, e.g., Eqs. (63) and (66). This is due to a specific form of a discontinuity of the exponential integral at negative real values of its argument, $\text{Im} \, E_1(-u \mp i0) = \pm i\pi$ ($u > 0$). Consequently, we get finite spectral densities

\begin{align}
\tau^{(0)}_I(E) &= \pm \lim_{t \to 0_+} \frac{1}{\pi} \text{Im} \text{Tr} (H_0 - E \mp i0)^{-1} e^{-tH^2_0} = \frac{1}{2} R^2 |E| \theta(E^2 - m^2), \\
\tau^{(0)}_J(E) &= \pm \lim_{t \to 0_+} \frac{1}{\pi} \text{Im} \text{Tr} J_0 (H_0 - E \mp i0)^{-1} e^{-tH^2_0} = \frac{1}{4} R^2 sm \text{sgn}(E) \theta(E^2 - m^2), \\
\tau^{(0)}_{J^2}(E) &= \pm \lim_{t \to 0_+} \frac{1}{\pi} \text{Im} \text{Tr} J_0^2 (H_0 - E \mp i0)^{-1} e^{-tH^2_0} = \frac{1}{8} R^2 |E|[R^2 (E^2 - m^2) + 1] \theta(E^2 - m^2).
\end{align}

## 6 Thermal averages, fluctuations, and correlations

As follows from two preceding sections, the expressions for thermal characteristics consist of two pieces, see Eqs. (61) and (62): the one, denoted by superscript \( ^{(0)} \) and
corresponding to the ideal gas contribution, depends essentially on the size of the system \( R \), see Eqs.(102)-(104), and the other one, denoted by superscript \(^{(1)}\) and corresponding to the correction due to interaction with a magnetic vortex, is independent of \( R \) in the limit \( R \to \infty \), see Eqs.(92)-(94). However, it appears that the ideal gas contribution to some characteristics is vanishing, and these characteristics are finite as the size of the system increases.

In particular, this is the case for the average fermion number of the system. Let us denote the operator of fermion number by \( \hat{N} \), then it is given by Eq.(40) with \( \hat{N}, \hat{I} \) and \( 1 \) substituted for \( \hat{M}, \hat{J} \) and \( j_\lambda \), respectively. Taking into account Eq.(102), one gets \( N^{(0)}(T) = 0 \), and using Eq.(92), one gets (see Ref.[31]):

\[
N(T) = -\frac{\sin(F\pi)}{\pi} \int_0^\infty \frac{du}{u\sqrt{u+1}} \tanh \left( \frac{1}{2} \beta m \sqrt{u+1} \right) \times \\
\frac{F u^F A - (1 - F)u^{1-F}A^{-1} + u \left[ (F - \frac{1}{2}) (u^F A + u^{1-F} A^{-1}) - \cos(F\pi) \right]}{[u^F A - u^{1-F} A^{-1} + 2 \cos(F\pi)]^2 + 4(u+1) \sin^2(F\pi)} \\
- \frac{1}{2} \theta(-\cos \Theta) \tanh \left( \frac{1}{2} \beta E_{BS} \right),
\]

(105)

where

\[
A = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan \left( \frac{s}{2} + \frac{\pi}{4} \right),
\]

(106)

\( \Gamma(u) \) is the Euler gamma function, \( E_{BS} \) is the energy of the bound state in the one-particle spectrum, which is determined as a real root of algebraic equation (for details see Ref.[19])

\[
\frac{(1 - m^{-1} E_{BS})^F}{(1 + m^{-1} E_{BS})^{1-F}} A = -1;
\]

(107)

note that the bound state exists at \( \cos \Theta < 0 \) \( (A < 0) \), and its energy is zero at \( A = -1 \), and, otherwise, one has \( 0 < |E_{BS}| < |m| \) and

\[
\text{sgn}(E_{BS}) = \frac{1}{2} \text{sgn}(m)[\text{sgn}(1 + A^{-1}) - \text{sgn}(1 + A)].
\]

(108)

Contrary to the average, the quadratic fluctuation of fermion number consists of two contributions. Using Eq.(102), one gets

\[
\Delta^{(0)}(T; \hat{N}, \hat{N}) = \frac{R^2}{\beta^2} \left[ \frac{\beta|m|}{e^{\beta|m|} + 1} + \ln(1 + e^{-\beta|m|}) \right],
\]

(109)

and, using Eq.(92), one gets [31]

\[
\Delta^{(1)}(T; \hat{N}, \hat{N}) = \frac{\sin(F\pi)}{2\pi} \int_0^\infty \frac{du}{u} \sech^2 \left( \frac{1}{2} \beta m \sqrt{u+1} \right) \times \\
\frac{F u^F A - (1 - F)u^{1-F}A^{-1} - u(2F - 1) \cos(F\pi)}{[u^F A - u^{1-F} A^{-1} + 2 \cos(F\pi)]^2 + 4(u+1) \sin^2(F\pi)} \\
+ \frac{1}{4} \theta(-\cos \Theta) \sech^2 \left( \frac{1}{2} \beta E_{BS} \right) - \frac{1}{4} F(1 - F) \sech^2 \left( \frac{1}{2} \beta m \right).
\]

(110)
Let us turn now to another conserved observable - angular momentum. The operator of this observable is given by Eq.(40) and, in this instance, we take kinetic angular momentum \( J \) as operator \( J \) in Eq.(40). Using Eq.(103), one gets

\[
M^{(0)}(T) = \frac{R^2 s m}{\beta} \ln(1 + e^{-\beta|m|}),
\]

and, using Eq.(93), one gets \[32\]

\[
M^{(1)}(T) = -s(F - \frac{1}{2}) N(T) + \frac{s}{4} F(1 - F) \tanh(\frac{1}{2} \beta m), \tag{112}
\]

where \( N(T) \) is the average fermion number, see Eq.(105). Contrary to the average angular momentum, the correlation of angular momentum with fermion number is finite in the infinite volume limit:

\[
\Delta(T; \hat{M}, \hat{N}) = -s(F - \frac{1}{2}) \Delta^{(1)}(T; \hat{N}, \hat{N}) - \frac{s}{12} (F - \frac{1}{2}) F(1 - F) \text{sech}^2(\frac{1}{2} \beta m), \tag{113}
\]

where \( \Delta^{(1)}(T; \hat{N}, \hat{N}) \) is the finite piece of the fermion number fluctuation, see Eq.(110); note that the latter relation follows from Eq.(93) and equality \( \Delta^{(0)}(T; \hat{M}, \hat{N}) = 0 \) following in its turn from Eq.(103). The quadratic fluctuation of angular momentum increases as the volume squared in the large volume limit:

\[
\Delta^{(0)}(T; \hat{M}, \hat{M}) = \frac{R^2}{2 \beta^2} \left[ \frac{3 R^2}{\beta^2} \int_0^{\infty} du \ln(1 + e^{-u}) + (R^2 m^2 + \frac{1}{2}) \ln(1 + e^{-|m|}) + \frac{1}{2} \frac{\beta |m|}{e^{\beta|m|} + 1} \right], \tag{114}
\]

where Eq.(104) is used. Using Eq.(94), one gets finite piece \[32\]

\[
\Delta^{(1)}(T; \hat{M}, \hat{M}) = (F - \frac{1}{2})^2 \Delta^{(1)}(T; \hat{N}, \hat{N}) - \frac{1}{8} F^2 (1 - F)^2 \text{sech}^2(\frac{1}{2} \beta m). \tag{115}
\]

Turning to nonconserved observables, see Eq.(52), one can take \( \Upsilon \) to be proportional either to \( x \cdot \alpha \) or to \( x \times \alpha \). One can verify that the spectral density in the first case is identically zero, whereas it is nonzero in the second case. To be more precise, let us take

\[
\Upsilon = \frac{e^2}{4\pi} x \times \alpha, \tag{116}
\]

then operator \( \hat{O}(52) \) corresponds to the observable with the physical meaning of the induced magnetic flux multiplied by \( e \) and divided by \( 2\pi \) (for details see, e.g., Refs.\[19, 32\]). The thermal average of this observable is finite in the infinite volume limit

\[
O(T) = -\frac{e^2 s F(1 - F)}{4\pi m} \left\{ \frac{\sin(F \pi)}{\pi} \int_0^{\infty} du \frac{\tan \left( \frac{1}{2} \beta m \sqrt{u + 1} \right)}{\sqrt{u + 1}} \right\} \times \]

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\[ u^F A - u^{1-F} A^{-1} - 2u \cos(F \pi) \]
\[ + \theta(- \cos \Theta) \frac{m}{(2F-1)E_{BS} + m} \tanh \left( \frac{1}{2} \beta E_{BS} \right) + \frac{1}{3} \left( F - \frac{1}{2} \right) \tanh \left( \frac{1}{2} \beta m \right) \} \], \quad (117)

as well as its correlation with fermion number

\[ \Delta(T; \hat{O}, \hat{N}) = \frac{e^2 s F(1 - F)}{8 \pi m} \left( \frac{\sin(F \pi)}{\pi} \right) \int_0^\infty \frac{du}{u} \text{sech}^2 \left( \frac{1}{2} \beta m \sqrt{u + 1} \right) \times \]
\[ \times \frac{u^F A - u^{1-F} A^{-1} + 2 \cos(F \pi)}{\left[ u^F A - u^{1-F} A^{-1} + 2 \cos(F \pi) \right]^2 + 4(u + 1) \sin^2(F \pi)} + \]
\[ + \theta(- \cos \Theta) \frac{m}{(2F-1)E_{BS} + m} \text{sech}^2 \left( \frac{1}{2} \beta E_{BS} \right) - \frac{1}{2} \text{sech}^2 \left( \frac{1}{2} \beta m \right) \} \] \quad (118)

and its correlation with angular momentum

\[ \Delta(T; \hat{O}, \hat{M}) = -s \left( F - \frac{1}{2} \right) \Delta(T; \hat{O}, \hat{N}), \] \quad (119)

see Ref. [32].

Other nonconserved observables are two pieces of angular momentum, i.e. spin and orbital angular momentum. The ideal gas contribution to average spin coincides with Eq.(111), thus the average orbital angular momentum is finite. Expressions for the averages of spin and orbital angular momentum and their correlations with conserved observables are given in Ref. [32].

It should be emphasized that the average of induced flux and its correlations with conserved observables are finite in the infinite volume limit. The average of fermion number and its correlation with angular momentum are also finite in this limit, whereas the fluctuation of fermion number, as well as the average of angular momentum, diverges as \( R^2 \), and the fluctuation of angular momentum diverges as \( R^4 \). Thus the ideal gas contribution to the thermal characteristics of angular momentum is predominant, unless temperature is zero.

In the limit \( T \to 0 \ (\beta \to \infty) \) the average of angular momentum tends to finite value:

\[
M(0) = \begin{cases} 
\frac{1}{4} s \text{sgn}(m)(1 - F)^2, & -1 < A < \infty \\
\frac{1}{4} s \text{sgn}(m)F^2, & A^{-1} = 0, -1 \\
\frac{1}{4} s \text{sgn}(m)[2F^2 - (1 - F)^2], & -\infty < A < -1 \\
\frac{1}{4} s \text{sgn}(m)F^2, & -1 < A^{-1} < \infty \\
\frac{1}{4} s \text{sgn}(m)(1 - F)^2, & A = 0, -1 \\
\frac{1}{4} s \text{sgn}(m)[2(1 - F)^2 - F^2], & -\infty < A^{-1} < -1 \\
\end{cases}, \quad 0 < F \leq \frac{1}{2}, \quad \frac{1}{2} \leq F < 1, \quad (120)
\]
whereas its fluctuation tends to zero for almost all values of $\Theta$ with exception of the one corresponding to the zero bound state energy, $E_{BS} = 0$ ($A = -1$):

$$\Delta(0; \hat{M}, \hat{M}) = \begin{cases} 0, & A \neq -1; \\ \frac{1}{4}(F - \frac{1}{2})^2, & A = -1; \end{cases}$$  \hspace{1cm} (121)$$

the behaviour of the fermion number fluctuation and the correlations in this limit is similar to Eq.(121), differing in the values at $A = -1$. It is instructive to present the average angular momentum in the form

$$M(T) = M^{(0)}(T) + M(0) + M^{(1)}(T),$$

where $M^{(0)}(T)$ and $M(0)$ are given by Eqs.(111) and (120), respectively, and

$$M^{(1)}(T) = -s(F - \frac{1}{2}) \theta(-\cos\Theta) - \frac{\text{sgn}(E_{BS})}{\exp(\beta|E_{BS}|) + 1} - \frac{1}{2} s \left[ \frac{1}{2} - F(1 - F) \right] \frac{\text{sgn}(m)}{\exp(\beta|m|) + 1} - \frac{s}{4\pi} \int_{1}^{\infty} dw \, \text{sech}^2\left( \frac{1}{2} \beta mw \right) \arctan \left[ \frac{A(w^2 - 1)^F - A^{-1}(w^2 - 1)^{1-F} + 2 \cos(F\pi)}{2w \sin(F\pi)} \right].$$

Thus, averages, as well as fluctuations and correlations, tend exponentially to their finite zero-temperature limiting values.

In the high-temperature limit the averages of fermion number and induced flux tend to zero by power law, see Refs.[31,32] respectively, whereas correlations tend to finite values:

$$\Delta(\infty; \hat{O}, \hat{N}) = \begin{cases} -\frac{s}{12} (F - \frac{1}{2})(1 - F)(3 - 2F), & A^{-1} \neq 0 \\ -\frac{s}{12} (F - \frac{1}{2}) F(1 + 2F), & A^{-1} = 0 \end{cases}, \hspace{1cm} 0 < F \leq \frac{1}{2};$$

$$\Delta(\infty; \hat{O}, \hat{N}) = \begin{cases} -\frac{s}{12} (F - \frac{1}{2}) F(1 + 2F), & A \neq 0 \\ -\frac{s}{12} (F - \frac{1}{2})(1 - F)(3 - 2F), & A = 0 \end{cases}, \hspace{1cm} \frac{1}{2} \leq F < 1,$$

$$\Delta(\infty; \hat{O}, \hat{N}) = \frac{e^2 s F(1 - F)}{8\pi m} \left\{ \frac{\theta(-\cos\Theta)}{(2F - 1)E_{BS} + m} - \frac{1}{2} + \frac{\sin(F\pi)}{\pi} \int_{0}^{\infty} du \left[ u^F A + u^{1-F} A^{-1} \right] \right\},$$

$$\Delta(\infty; \hat{O}, \hat{M})$$ is proportional to Eq.(125), see Eq.(119). Meanwhile, the average angular momentum, the fermion number fluctuation, and the angular momentum fluctuation increase as $T$, $T^2$, and $T^4$, in this limit; such a behaviour is obviously due to the ideal gas contribution, see Eqs.(111), (109), and (114).

Concluding this section, let us take canonical angular momentum (78) in the capacity of total angular momentum of the planar system, then the appropriate operator
in the second-quantized theory equals to $\hat{M} + e\Phi \hat{N}$. It is evident, how to obtain the thermal characteristics of this observable from our previous results. In particular, the increasing at large $R$ piece of the average remains unchanged, Eq.(111), whereas the finite piece of the average takes form

$$M^{(1)}(T) + e\Phi N(T) = ([e\Phi] + \frac{1}{2}) N(T) + \frac{s}{4} F(1 - F) \tanh(\frac{1}{2}\beta m);$$  

(126)

similarly, the correlation with the induced flux takes form

$$\Delta(T; \hat{O}, \hat{M} + e\Phi \hat{N}) = ([e\Phi] + \frac{1}{2}) \Delta(T; \hat{O}, \hat{N}).$$  

(127)

The correlation with fermion number consists of two pieces:

$$\Delta^{(0)}(T; \hat{M} + e\Phi \hat{N}, \hat{N}) = e\Phi \Delta^{(0)}(T; \hat{N}, \hat{N})$$  

(128)

and

$$\Delta^{(1)}(T; \hat{M} + e\Phi \hat{N}, \hat{M} + e\Phi \hat{N}) = ([e\Phi] + \frac{1}{2}) \Delta^{(1)}(T; \hat{N}, \hat{N}) - \frac{s}{12} (F - \frac{1}{2}) F(1 - F) \text{sech}^2(\frac{1}{2}\beta m),$$  

(129)

as well as the quadratic fluctuation does:

$$\Delta^{(0)}(T; \hat{M} + e\Phi \hat{N}, \hat{M} + e\Phi \hat{N}) = (e\Phi)^2 \Delta^{(0)}(T; \hat{N}, \hat{N}) + \Delta^{(0)}(T; \hat{M}, \hat{M})$$  

(130)

and

$$\Delta^{(1)}(T; \hat{M} + e\Phi \hat{N}, \hat{M} + e\Phi \hat{N}) = ([e\Phi] + \frac{1}{2}) \Delta^{(1)}(T; \hat{N}, \hat{N}) - \frac{1}{2} F(1 - F) \left[ \frac{s}{3} (F - \frac{1}{2}) e\Phi + \frac{1}{4} F(1 - F) \right] \text{sech}^2(\frac{1}{2}\beta m).$$  

(131)

It is straightforward to obtain zero-temperature and high-temperature limits of the above relations; in particular, the zero-temperature limit of Eq.(126) was first obtained in Ref. [21].

7 Discussion

In the present paper we consider fractionalization of angular momentum around a magnetic vortex in the framework of quantum field theory at finite temperature; for the sake of completeness, the results for fermion number and induced magnetic flux are also included. If the kinetic definition of angular momentum is chosen, then all thermal averages, fluctuations, and correlations are periodic in the value of the vortex flux, i.e. they depend on the fractional part of $e\Phi$. If the canonical definition of angular momentum is chosen, then those thermal characteristics which involve it depend on both the fractional and integer parts of $e\Phi$. The difference between two definitions
becomes especially striking in the case of the correlation of angular momentum with fermion number: the kinetic definition yields finite quantity, see Eq.(113), whereas the canonical definition yields a piece which increases in the large volume limit and is proportional to $e\Phi$, see Eq.(128); the latter looks rather unnatural, since such a piece should correspond to the ideal gas contribution. The same unnaturalness is relevant for the fluctuation of the canonically defined angular momentum: its increasing with the volume size piece which should correspond to the ideal gas contribution contains a term which is proportional to $(e\Phi)^2$, see Eq.(130).

According to the generally accepted paradigm, physical manifestations of the Bohm–Aharonov effect depend exclusively on the fractional part of $e\Phi$ (see, e.g., Refs.[3, 4, 5]), and this favours the kinetic definition of angular momentum. Moreover, the canonical definition of angular momentum yields rather embarrassing results for its fluctuation and correlation with fermion number.

Our analysis has been carried out for the whole variety of boundary conditions at the location of the vortex, and these conditions are specified by self-adjoint extension parameter $\Theta$. The nonvanishing of the angular momentum fluctuation signifies that angular momentum is not a sharp quantum observable and has to be understood as a thermal average only. In the high-temperature limit, angular momentum increases as $T$, see Eq.(111), and its fluctuation increases as $T^4$, see Eq.(114). In the zero temperature limit, angular momentum becomes a sharp quantum observable with finite value $M(0)$ (120). However, the last statement is true for all values of $\Theta$ in the case of $F = \frac{1}{2}$, whereas in the case of $F \neq \frac{1}{2}$ it is true for almost all values of $\Theta$ with the exception of one corresponding to the zero bound state energy, $E_{BS} = 0$ ($A = -1$), since in the latter case the zero-temperature fluctuation is nonzero, see Eq.(121).

Among the whole variety of boundary conditions, let us choose the condition of minimal irregularity, i.e. the condition corresponding to the radial components being divergent at $r \to 0$ at most as $r^{-p}$ with $p \leq \frac{1}{2}$ [15, 18, 19]:

$$\Theta = \begin{cases} s\frac{\pi}{2} (\text{mod } 2\pi), & 0 < F < \frac{1}{2} \\ 0 (\text{mod } 2\pi), & F = \frac{1}{2} \\ -s\frac{\pi}{2} (\text{mod } 2\pi), & \frac{1}{2} < F < 1 \end{cases}, \quad (132)$$

or $A^{-1} = 0$ at $0 < F < \frac{1}{2}$, $A = 1$ at $F = \frac{1}{2}$, $A = 0$ at $\frac{1}{2} < F < 1$. Under this condition thermal characteristics take rather simple form:

$$M^{(1)}(T) = \frac{1}{4}s \left( \frac{1}{2} - |F - \frac{1}{2}| \right)^2 \tanh(\frac{1}{2}\beta m), \quad (133)$$

$$\Delta^{(1)}(T; \hat{M}, \hat{M}) = -\frac{1}{8} \left( \frac{1}{2} - |F - \frac{1}{2}| \right)^2 \left[ |F - \frac{1}{2}| + F(1 - F) \right] \sech^2(\frac{1}{2}\beta m), \quad (134)$$

$$\Delta(T; \hat{M}, \hat{N}) = -\frac{1}{6}s \left( F - \frac{1}{2} \right) \left( \frac{1}{2} - |F - \frac{1}{2}| \right) \left( 1 - |F - \frac{1}{2}| \right) \sech^2(\frac{1}{2}\beta m), \quad (135)$$

$$\Delta(T; \hat{O}, \hat{M}) = 0. \quad (136)$$
We recall that the results of the present paper are relevant for planar fermions in an irreducible $2 \times 2$ representation of the Clifford algebra in $2 + 1$-dimensional space-time, see Eqs.(75) and (76); the mass of such fermions violates parity. As it should be expected, our results remain invariant under transitions to equivalent representations of the Clifford algebra, i.e. are independent of $\chi_s$. As to a transition to the inequivalent representation ($s \to -s$ or $m \to -m$), the averages of fermion number and angular momentum are odd and the average induced flux is even under this transition, see Eqs.(105), (111), (112), and (117); the quadratic fluctuations of fermion number and angular momentum, as well as their correlation, are even and the correlations of induced flux with fermion number and angular momentum are odd under this transition, see Eqs.(109), (110), (113)-(115), (118), and (119).

Finally, let us discuss consequences for planar fermions with parity-conserving mass; such fermions are assigned to a reducible $4 \times 4$ representation which is composed as a direct sum of two inequivalent irreducible representations. Thermal characteristics in the reducible representation are obtained by summing those in the irreducible one over $s = \pm 1$. In particular, the average induced flux takes form

$$O_{4\times4}(T) = 2 O(T) \bigg|_{F=e\Phi-[e\Phi]} \bigg|_{s=1 \atop m=|m|},$$

where $O(T)$ is given by Eq.(117). The averages of fermion number and angular momentum, as well as their correlations with induced flux, are vanishing,

$$N_{4\times4}(T) = M_{4\times4}(T) = 0, \quad \Delta_{4\times4}(T; \hat{O}, \hat{N}) = \Delta_{4\times4}(T; \hat{O}, \hat{M}) = 0,$$

while the fluctuations of fermion number and angular momentum, as well as their correlation, are nonvanishing,

$$\Delta_{4\times4}(T; \hat{N}, \hat{N}) = 2 \Delta(T; \hat{N}, \hat{N}) \bigg|_{F=e\Phi-[e\Phi]} \bigg|_{s=1 \atop m=|m|},$$

$$\Delta_{4\times4}(T; \hat{M}, \hat{M}) = 2 \Delta(T; \hat{M}, \hat{M}) \bigg|_{F=e\Phi-[e\Phi]} \bigg|_{s=1 \atop m=|m|},$$

$$\Delta_{4\times4}(T; \hat{M}, \hat{N}) = 2 \Delta(T; \hat{M}, \hat{N}) \bigg|_{F=e\Phi-[e\Phi]} \bigg|_{s=1 \atop m=|m|}.$$

Note that Eqs.(137) and (141) are finite in the large volume limit, thus depending essentially on the fractional part of $e\Phi$, whereas the dependence of Eqs.(139) and (140) on the vortex flux is not essential, since the ideal gas contribution to fluctuations is nonzero and prevailing in this limit. Obviously, our arguments in favour of the kinetic definition of angular momentum remain valid in this case also.
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Appendix A. Canonical and improved definitions of the moments of momentum tensor of classical fields

Let $\bar{T}^{\mu\nu}$ be the canonical tensor of energy and momentum of classical fields, and let us define tensor

$$T^{\mu\nu} = \bar{T}^{\mu\nu} + \partial_\lambda \chi^{\mu\nu\lambda}, \quad (A.1)$$

where $\chi^{\mu\nu\lambda}$ is an arbitrary third-rank tensor which is antisymmetric in two last indices, $\chi^{\mu\nu\lambda} = -\chi^{\nu\lambda\mu}$. Then

$$\partial_\nu T^{\mu\nu} = \partial_\nu \bar{T}^{\mu\nu}, \quad (A.2)$$

and conservation of $\bar{T}^{\mu\nu}$ leads to that of $T^{\mu\nu}$. Moreover, both tensors yield the same energy-momentum vector

$$P^\mu = \int d^3x \bar{T}^{\mu0} = \int d^3x T^{\mu0}, \quad (A.3)$$

since the contribution of $\chi^{\mu\nu\lambda}$ is transformed into an integral over a surface enclosing space,

$$\int d^3x \partial_\lambda \chi^{\mu0\lambda} = \int d^3x \partial_\lambda \chi^{\mu0\lambda} = \oint d\sigma l^{\mu0l}, \quad (A.4)$$

and the latter integral vanishes under mild assumptions on the decrease of $\chi^{\mu0l}$ at large distances.

Similarly, taking the canonical tensor of densities of moments of momentum

$$\bar{M}^{\mu\nu;\rho} = x^\mu \bar{T}^{\nu\rho} - x^{\nu} \bar{T}^{\mu\rho} + S^{\mu\nu;\rho}, \quad (A.5)$$

one defines tensor

$$M^{\mu\nu;\rho} = \bar{M}^{\mu\nu;\rho} + \partial_\lambda (x_\rho \chi^{\lambda\nu\mu} - x_\nu \chi^{\mu\rho\lambda}), \quad (A.6)$$

which satisfies divergence relation

$$\partial_\rho M^{\mu\nu;\rho} = \partial_\rho \bar{M}^{\mu\nu;\rho}, \quad (A.7)$$

and conservation of $\bar{M}^{\mu\nu;\rho}$ leads to that of $M^{\mu\nu;\rho}$.

In cases when tensors $\bar{T}^{\mu\nu}$ and $\bar{M}^{\mu\nu;\rho}$ fail to meet some plausible requirements, the transition to tensors $T^{\mu\nu}$ and $M^{\mu\nu;\rho}$ may allow one to eliminate such failings. In particular, the canonical energy-momentum tensor of the electromagnetic field is neither symmetric nor gauge invariant. By choosing a concrete form of $\chi^{\mu\nu\lambda}$ one can remove
both of these shortcomings. Incidentally, a convenient expression which does not involve spin part $S^{\mu\nu;\rho}$ is obtained for the moments of momentum tensor (Belinfante’s theorem \[38\]).

Namely, first, Eq.(A.6) with the use of Eqs.(A.1) and (A.5) is recast into the form

$$M^{\mu\nu;\rho} = x^{\mu}T^{\nu\rho} - x^{\nu}T^{\mu\rho} + \chi^{\mu\rho\nu} - \chi^{\mu\nu\rho} + S^{\mu\nu;\rho}. \quad (A.8)$$

Then defining $\chi$-tensor as

$$\chi^{\mu\nu\lambda} = \frac{1}{2}(-S^{\mu\nu;\lambda} + S^{\nu\lambda;\mu} - S^{\lambda\mu;\nu}) \quad (A.9)$$

(recall that $S$-tensor is antisymmetric in the first two indices and $\chi$-tensor is antisymmetric in the last two indices), one gets

$$M^{\mu\nu;\rho} = x^{\mu}T^{\nu\rho} - x^{\nu}T^{\mu\rho}. \quad (A.10)$$

Thus one gets following expressions for the canonical tensor of moments of momentum

$$\tilde{M}^{\mu\nu} = \int d^3x \left[ x^{\mu}T^{\nu0} - x^{\nu}T^{\mu0} - \partial_{\lambda}(x^{\mu}\chi^{\nu0\lambda} - x^{\nu}\chi^{\mu0\lambda}) \right] \quad (A.11)$$

and the improved one

$$M^{\mu\nu} = \int d^3x (x^{\mu}T^{\nu0} - x^{\nu}T^{\mu0}). \quad (A.12)$$

Although the difference between Eqs.(A.12) and (A.11) is transformed as previously, see Eq.(A.4), into an integral over a closed surface,

$$\int d^3x \partial_{\lambda}(x^{\mu}\chi^{\nu0\lambda} - x^{\nu}\chi^{\mu0\lambda}) = \oint d\sigma^l(x^{\mu}\chi^{\nu0l} - x^{\nu}\chi^{\mu0l}), \quad (A.13)$$

this integral may appear to be finite for certain, long-range, field configurations, since it contains an additional power of large distance, $|x|$. In this case one has to decide, whether Eq.(A.11) or Eq.(A.12) gives physically meaningful moments of momentum.

Let us consider a system characterized by lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A^{\mu}j_{\mu}, \quad (A.14)$$

where $F^{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the electromagnetic field strength, $A_{\mu}$ is the appropriate vector potential, $j_{\mu}$ is the charge current. The improved energy-momentum tensor of the system is proved to be \[38\] (see also, e.g., Ref.[34])

$$T^{\mu\nu} = \frac{1}{4}g^{\mu\rho}F^{\lambda\rho}F_{\lambda\nu} + F^{\mu\lambda}F_{\lambda\nu} + g^{\mu\nu}A^{\lambda}j_{\lambda} - A^{\mu}j^{\nu} \quad (A.15)$$

(note that purely electromagnetic part is symmetric and gauge invariant); incidentally, one gets

$$\chi^{\mu\nu\lambda} = A^{\mu}F^{\nu\lambda}. \quad (A.16)$$
Defining an axial vector from spatial tensor components
\[ M^i = \frac{1}{2} \epsilon^{ikl} M_{kl}, \]  
(A.17)
and using Eq.(A.12), one gets the following expression for the improved angular momentum
\[ M = \int d^3 x \left[ (\mathbf{x} \times (\mathbf{E} \times \mathbf{B})) - (\mathbf{x} \times \mathbf{A}) j^0 \right], \]  
(A.18)
where \( E^i = F^{i0} \) and \( B^i = -\frac{1}{2} \epsilon^{ikl} F_{kl} \) are the electric and magnetic field strengths.

Appendix B. Derivation of representation (45) for thermodynamic potential

The partition function of the fermionic system is presented as the functional integral over the Grassman fields
\[ \exp[-\beta \Omega(\beta, \mu)] = \int d\psi^+ d\psi e^{-S}, \]  
(B.1)
where
\[ S = \int_0^\beta d\tau \int d^4 x \psi^+ (\partial_\tau - \mu J + H) \psi \]  
(B.2)
is the Euclidean action, \( \tau \) is the imaginary time. The integral in Eq.(B.1) is of the Gauss type and can be immediately computed
\[ \exp[-\beta \Omega(\beta, \mu)] = \det(\partial_\tau - \mu J + H). \]  
(B.3)
Hence the thermodynamic potential is given by expression
\[ \Omega(\beta, \mu) = -\frac{1}{\beta} \ln \det(\partial_\tau - \mu J + H) = \]
\[ = -\frac{1}{\beta} \int_0^\beta d\tau \int d^4 x \text{tr} \langle \mathbf{x}, \tau \mid \ln(\partial_\tau - \mu J + H) \mathbf{x}, \tau \rangle. \]  
(B.4)
In the case of a static background, operators \( H \) and \( J \) are \( \tau \)-independent, and the integration over \( \tau \) is performed by using the antiperiodicity boundary condition at the ends of the imaginary time interval:
\[ \Omega(\beta, \mu) = -\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int d^4 x \text{tr} \langle \mathbf{x} \mid \ln(H - \mu J - i\omega_n) \mathbf{x} \rangle, \]  
(B.5)
where \( \omega_n = \frac{2\pi}{\beta} (n + \frac{1}{2}) \), and summation is over integer values of \( n \). Using the notation of functional trace, one gets further
\[ \Omega(\beta, \mu) = -\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \text{Tr} \ln(H - \mu J - i\omega_n) = -\frac{1}{\beta} \text{Tr} \ln \prod_{n=0}^{\infty} [(H - \mu J)^2 + \omega_n^2] = \]
= -\frac{1}{\beta} \left\{ \text{Tr} \ln \prod_{n=0}^{\infty} \left[ \left( \frac{H - \mu J}{\omega_n} \right)^2 + 1 \right] + \text{Tr} \ln \prod_{n=0}^{\infty} \omega_n^2 \right\}. \tag{B.6}

The second term in the figure brackets is dropped as an irrelevant infinite constant, and the infinite product in the first term is computed with the use of relation (C.7).

\cosh\left(\frac{\pi a}{2}\right) = \prod_{n=0}^{\infty} \left[ 1 + a^2(2n + 1)^{-2} \right].

As a result we get expression (45) for the thermodynamic potential.

**Appendix C. Radial components of \(G^\omega(r, \varphi; r', \varphi')\)**

The radial components of the resolvent kernel of \(H\) (73) take form (see Ref. [31]),

type 1 \((l = s(n - n_c) > 0)\):

\begin{align*}
a_n(r; r') &= \frac{i\pi}{2} (\omega + m) \left[ \theta(r - r') H_{l-F}^{(1)}(kr) J_{l-F}(kr') + \theta(r' - r) J_{l-F}(kr) H_{l-F}^{(1)}(kr') \right], \tag{C.1} \\
b_n(r; r') &= \frac{i\pi}{2} k \left[ \theta(r - r') H_{l+1-F}^{(1)}(kr) J_{l+1-F}(kr') + \theta(r' - r) J_{l+1-F}(kr) H_{l+1-F}^{(1)}(kr') \right], \tag{C.2} \\
c_n(r; r') &= \frac{i\pi}{2} (\omega - m) \left[ \theta(r - r') H_{l+1-F}^{(1)}(kr) J_{l+1-F}(kr') + \theta(r' - r) J_{l+1-F}(kr) H_{l+1-F}^{(1)}(kr') \right], \tag{C.3} \\
d_n(r; r') &= \frac{i\pi}{2} k \left[ \theta(r - r') H_{l+1+F}^{(1)}(kr) J_{l+1+F}(kr') + \theta(r' - r) J_{l+1+F}(kr) H_{l+1+F}^{(1)}(kr') \right]; \tag{C.4}
\end{align*}

type 2 \((l' = -s(n - n_c) > 0)\):

\begin{align*}
a_n(r; r') &= \frac{i\pi}{2} (\omega + m) \left[ \theta(r - r') H_{l+1+F}^{(1)}(kr) J_{l+1+F}(kr') + \theta(r' - r) J_{l+1+F}(kr) H_{l+1+F}^{(1)}(kr') \right], \tag{C.5} \\
b_n(r; r') &= -\frac{i\pi}{2} k \left[ \theta(r - r') H_{l-1+F}^{(1)}(kr) J_{l-1+F}(kr') + \theta(r' - r) J_{l-1+F}(kr) H_{l-1+F}^{(1)}(kr') \right], \tag{C.6} \\
c_n(r; r') &= \frac{i\pi}{2} (\omega - m) \left[ \theta(r - r') H_{l-1+F}^{(1)}(kr) J_{l-1+F}(kr') + \theta(r' - r) J_{l-1+F}(kr) H_{l-1+F}^{(1)}(kr') \right], \tag{C.7} \\
d_n(r; r') &= -\frac{i\pi}{2} k \left[ \theta(r - r') H_{l-1+F}^{(1)}(kr) J_{l-1+F}(kr') + \theta(r' - r) J_{l-1+F}(kr) H_{l-1+F}^{(1)}(kr') \right]; \tag{C.8}
\end{align*}

type 3 \((n = n_c)\):

\begin{align*}
a_{n_c}(r; r') &= \frac{i\pi}{\sin \nu_{\omega} + \cos \nu_{\omega} e^{iF \pi}} \left[ \theta(r - r') H_{-F}^{(1)}(kr) \sin \nu_{\omega} J_{-F}(kr') + \cos \nu_{\omega} J_{-F}(kr') \right] + \\
&\quad \frac{i\pi}{\sin \nu_{\omega} - \cos \nu_{\omega} e^{iF \pi}} \left[ \theta(r - r') H_{F}^{(1)}(kr) \sin \nu_{\omega} J_{F}(kr') - \cos \nu_{\omega} J_{F}(kr') \right] + \\
&\quad \frac{i\pi}{\sin \nu_{\omega} + \cos \nu_{\omega} e^{iF \pi}} \left[ \theta(r - r') H_{-F}^{(1)}(kr) \cos \nu_{\omega} J_{-F}(kr') + \sin \nu_{\omega} J_{-F}(kr') \right]
\end{align*}

28
Here we calculate its trace over spinor indices at $x$.

Appendix D. Derivation of Eqs.(96)-(98)

In the absence of the vortex the radial components take form:

$$
\tan \nu \omega = \frac{k^2 F}{\omega + m} \left[ 2 \left| m \right| \right]^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right). \tag{C.13}
$$

In the absence of the vortex the radial components take form:

$$
an(r; r')|_{\Phi=0} = \frac{i \pi}{2} (\omega + m) \left[ \theta(r - r') H_{sn}^{(1)}(kr) J_{sn}(kr') + \theta(r' - r) J_{sn}(kr) H_{sn}^{(1)}(kr') \right], \tag{C.14}
$$

$$
b_n(r; r')|_{\Phi=0} = \frac{i \pi}{2} k \left[ \theta(r - r') H_{sn+1}^{(1)}(kr) J_{sn}(kr') + \theta(r' - r) J_{sn+1}(kr) H_{sn+1}^{(1)}(kr') \right], \tag{C.15}
$$

$$
c_n(r; r')|_{\Phi=0} = \frac{i \pi}{2} (\omega - m) \left[ \theta(r - r') H_{sn+1}^{(1)}(kr) J_{sn+1}(kr') + \theta(r' - r) J_{sn+1}(kr) H_{sn+1}^{(1)}(kr') \right], \tag{C.16}
$$

$$
d_n(r; r')|_{\Phi=0} = \frac{i \pi}{2} k \left[ \theta(r - r') H_{sn}^{(1)}(kr) J_{sn+1}(kr') + \theta(r' - r) J_{sn}(kr) H_{sn+1}^{(1)}(kr') \right]. \tag{C.17}
$$

### Appendix D. Derivation of Eqs.(96)-(98)

Using integral representation of kernel (95)

$$
\langle x \left| (H_0 - \omega)^{-1} e^{-tH_0^2} \right| x' \rangle = \int \frac{d^2p}{(2\pi)^2} \exp \left[ i \mathbf{p} \cdot (x - x') - t(p^2 + m^2) \right] \frac{\alpha \cdot \mathbf{p} + \gamma^0 m + \omega}{p^2 - k^2}, \tag{D.1}
$$

we calculate its trace over spinor indices at $x' = x$

$$
tr \langle x \left| (H_0 - \omega)^{-1} e^{-tH_0^2} \right| x \rangle = 2\omega \int \frac{d^2p}{(2\pi)^2} \frac{e^{-t(p^2 + m^2)}}{p^2 - k^2}. \tag{D.2}
$$

29
Similarly, using relations

\[
J_0 \langle x \mid (H_0 - \omega)^{-1} e^{-tH_0^2} \mid x' \rangle = \int \frac{d^2 p}{(2\pi)^2} \exp \left[ ip \cdot (x - x') - t(p^2 + m^2) \right] \times \\
\times \left( x^1 p^2 - x^2 p^1 + \frac{1}{2} s\gamma^0 \right) \frac{\alpha \cdot p + \gamma^0 m + \omega}{p^2 - k^2},
\]

(D.3)

and

\[
J_0^2 \langle x \mid (H_0 - \omega)^{-1} e^{-tH_0^2} \mid x' \rangle = \int \frac{d^2 p}{(2\pi)^2} \exp \left[ ip \cdot (x - x') - t(p^2 + m^2) \right] \times \\
\times \left( x^1 p^2 - x^2 p^1 + \frac{1}{2} s\gamma^0 \right)^2 \frac{\alpha \cdot p + \gamma^0 m + \omega}{p^2 - k^2}.
\]

(D.4)

we find relations

\[
tr J_0 \langle x \mid (H_0 - \omega)^{-1} e^{-tH_0^2} \mid x \rangle = sm \int \frac{d^2 p}{(2\pi)^2} e^{-t(p^2 + m^2)} \times \\
\times \left( x^1 p^2 - x^2 p^1 + \frac{1}{2} s\gamma^0 \right) \frac{\alpha \cdot p + \gamma^0 m + \omega}{p^2 - k^2},
\]

(D.5)

and

\[
tr J_0^2 \langle x \mid (H_0 - \omega)^{-1} e^{-tH_0^2} \mid x \rangle = \omega \int \frac{d^2 p}{(2\pi)^2} \left( x^1 p^2 - x^2 p^1 + \frac{1}{2} s\gamma^0 \right) \frac{\alpha \cdot p + \gamma^0 m + \omega}{p^2 - k^2}.
\]

(D.6)

Eqs.(D.2), (D.5), and (D.6) in the case of \( Im k > |Re k| \) are reduced to Eqs.(96)-(98).

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