Galois characterization of Endoscopy for rational Siegel modular forms

Luis V. Dieulefait
Centre de Recerca Matemática
Bellaterra, Barcelona *

October 29, 2018

Abstract

We establish a relation between Galois reducibility and Endoscopy for genus 2 Siegel cusp forms which have rational eigenvalues and are unramified at 3.

1 The Theorem

Let $f$ be a genus 2 Siegel cuspidal Hecke eigenform of weight $k > 2$ and $\mathbb{Q}_f$ the number field generated by its eigenvalues. It is well known that if $f$ is not of Saito-Kurokawa type but it is “endoscopic” (i.e., it is in the image of the weak endoscopic lift) the compatible family of Galois representations $\{\rho_{f,\lambda}\}$ attached to $f$ (constructed by Taylor, Laumon and Weissauer for any Siegel cusp form) will be reducible over $\mathbb{Q}_f$, with two 2-dimensional irreducible components.

In this note we will prove that the converse statement is true, for the case $\mathbb{Q}_f = \mathbb{Q}$. We will have to impose a local condition at 3 and will assume that the determinants are minimally ramified. More precisely, the result is:

*e-mail: LDieulefait@crm.es. Supported by MECD postdoctoral fellowship at the Centre de Recerca Matemática from the Ministerio de Educación y Cultura
Theorem 1.1 Let $f$ be a genus 2 Siegel cusp form of weight $k > 2$ with $Q_f = \mathbb{Q}$, such that the corresponding automorphic representation $\pi_f$ has multiplicity one and $\pi_{f,3}$, its local component at 3, is unramified. Assume that the compatible family of Galois representations $\{\rho_{f,\ell}\}$ attached to $f$ reduces (over $\mathbb{Q}$) as follows:

$$\rho_{f,\ell} \simeq \sigma_{1,\ell} \oplus \sigma_{2,\ell}$$

(1.1)

for every prime $\ell$, where $\{\sigma_{1,\ell}\}$ and $\{\sigma_{2,\ell}\}$ are compatible families of 2-dimensional irreducible representations both with determinant $\chi^{2k-3}$.

Then, $f$ is endoscopic. More precisely, there exist two classical cuspidal modular forms $f_1, f_2$, of weights 2 and $2k - 2$ (respectively) such that the family of representations $\{\sigma_{1,\ell} \otimes \chi^{2-k}\}$ is attached to $f_1$ and the family $\{\sigma_{2,\ell}\}$ is attached to $f_2$.

Remarks: 1- The irreducibility assumption of the 2-dimensional components is equivalent to assume that $f$ is not of Saito-Kurokawa type. 2- Reducibility of the whole family $\{\rho_{f,\ell}\}$ as in the statement of the theorem is equivalent to a similar condition imposed only at a single prime $\ell$, provided that $\ell > 4k - 5$ and $\pi_{f,\ell}$ is unramified. This follows from a result of “existence of a family” proved in [D2].

2 Proof

Irreducibility of the 2-dimensional components implies that we are not in the Saito-Kurokawa case, and together with the multiplicity one assumption this implies (as proved by Weissauer) that the representations are pure, odd, and for every prime $\ell$ such that $\pi_{f,\ell}$ is unramified, the representations $\sigma_{1,\ell}$ and $\sigma_{2,\ell}$ are crystalline with Hodge-Tate weights $\{k - 2, k - 1\}$ and $\{0, 2k - 3\}$, respectively.

Let us first show modularity of the family $\{\sigma_{1,\ell} \otimes \chi^{2-k}\}$. By assumption, $\pi_{f,3}$ is unramified, thus $\sigma_{1,3} \otimes \chi^{2-k}$ is a Barsotti-Tate representation, irreducible, odd, with rational coefficients, and unramified outside a finite set of primes. Then, applying a combination of modularity results of Diamond- Taylor-Wiles and Skinner-Wiles (as done in [D1] and [D2]) we conclude that $\sigma_{1,3} \otimes \chi^{2-k}$ is modular, and this gives modularity of the family $\{\sigma_{1,\ell} \otimes \chi^{2-k}\}$. The corresponding modular form $f_1$ must have weight 2 because for almost
every $\ell$ the representations in this family are Barsotti-Tate.

This argument “à la Wiles” can not be applied to $\sigma_{2,3}$ because, even if we again have Wiles’ starting point (namely, we know that residually $\bar{\sigma}_{2,3}$ is either modular or reducible), the prime 3 is too small compared with the difference $2k - 3$ of the Hodge-Tate weights to make the strategy workable.

To show modularity of the family $\{\sigma_{2,\ell}\}$ we will explode the fact that (1.1) is telling us that the representations $\sigma_{2,\ell}$ can be obtained by “substracting” a modular representation from another modular representation.

A key ingredient is a result recently proved by Weselmann (yet unpublished, but see [BWW] and [W]), which states that $\pi_f$ has a weak lift to an automorphic representation $\pi'$ of $GL(4, \mathbb{A})$, where $\mathbb{A}$ are the rational adeles. Thus, by Cebotarev, the family $\{\rho_{f,\ell}\}$ is also attached to $\pi'$.

We want to apply a result of Jacquet and Shalika (which appears as theorem 3.3 in [T2]), in a similar way than what is done in [T2], section 533. We have from (1.1) the equality of $L$-functions:

$$L(\pi', s) = L(\sigma_{2,\ell}, s)L(f_1 \otimes \chi^{k-2}, s)$$

Observe that $\pi'$ is the weak lift of $\pi_f$, but it is not necessarily cuspidal.

To conclude that $\sigma_{2,\ell}$ is modular, as in section 533 of [T2], we must find a prime $\ell$ such that $L(\sigma_{2,\ell}^* \otimes \sigma_{2,\ell}, s)$ has a simple pole at $s = 1$, because in that case the result of Jacquet and Shalika implies $\sigma_{2,\ell} \simeq \sigma_{\pi_i,\ell}$, where $\pi_i$ is one of the cuspidal constituents of $\pi'$. Then, it only remains to find a prime satisfying this condition.

Take $\ell > 4k - 5$ such that the local components of $\pi'$ and $\pi_f$ at $\ell$ are unramified. For such a prime $\ell$ the representation $\sigma_{2,\ell}$ is crystalline with Hodge-Tate weights $\{0, 2k - 3\}$ and the main result of [T1] implies that there exists a totally real number field $F$ such that the restriction of $\sigma_{2,\ell}$ to the Galois group of $F$ is modular, i.e., it agrees with the Galois representation attached to a Hilbert modular form over $F$.

But, as explained in [T2], section 533, precisely from this potentially modular property (and the fact that it is preserved after solvable base change) one can deduce that $L(\sigma_{2,\ell}^* \otimes \sigma_{2,\ell}, s)$ does have a simple pole at $s = 1$, as we wanted. This shows modularity of the family $\{\sigma_{2,\ell}\}$ and it is clear from its Hodge-Tate decomposition that it corresponds to a modular form of weight $2k - 2$. We conclude that the Siegel cusp form $f$ is endoscopic.
3 Bibliography

[BWW] Ballmann, J., Weissauer, R., Weselmann, U., Remarks on the Fundamental Lemma for stable twisted Endoscopy of Classical Groups, preprint (2002)

[D1] Dieulefait, L., Modularity of Abelian Surfaces with Quaternionic Multiplication, Math. Res. Letters 10 no. 2-3 (2003)

[D2] Dieulefait, L., Existence of families of Galois representations and new cases of the Fontaine-Mazur conjecture, preprint (2003) available at http://www.arxiv.org/math.NT

[T1] Taylor, R., On the meromorphic continuation of degree two L-functions, preprint (2001) available at http://abel.math.harvard.edu/~rtaylor/

[T2] Taylor, R., Galois Representations, proceedings of ICM, Beijing, August 2002 longer version available at http://abel.math.harvard.edu/~rtaylor/

[W] Weissauer, R., A remark on the existence of Whittaker models for L-packets of automorphic representations of GSp(4), preprint