1-quasi-hereditary algebras: Examples and invariant submodules of projectives

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Abstract

In [8] it was shown that every 1-quasi-hereditary algebra $A$ affords a particular basis which is related to the partial order $\leq$ on the set of simple $A$-modules. In this paper we show that the modules generated by these basis-elements are also modules over the endomorphism algebra of some projective indecomposable modules. In case the Ringel-dual of 1-quasi-hereditary algebra is also 1-quasi-hereditary, all local $\Delta$-good submodules of projective indecomposable modules are also $\text{End}_A(P)^{op}$-modules for the projective-injective indecomposable module $P$.

Introduction

The class of (basic) 1-quasi-hereditary algebras, introduced in [8] is characterized by the fact that all possible non-zero Jordan-Hölder multiplicities of standard modules as well as $\Delta$-good multiplicities of indecomposable projectives are equal to 1. Many factor algebras (related to a saturated subsets) of an algebra $A$ associated to a block of category $\mathcal{O}(g)$ of a semisimple $\mathbb{C}$-Lie algebra $g$ are 1-quasi-hereditary.

The class of 1-quasi-hereditary algebras has a non-empty intersection with some other subclasses of quasi-hereditary algebras: BGG-algebras [12], quasi-hereditary algebras having Borel subalgebras [6], Ringel self-dual algebras etc.. However, 1-quasi-hereditary algebras are in general not BGG-algebras and the class of 1-quasi-hereditary algebras is not closed under Ringel duality. The selected examples in this paper serve to illustrate this.

Several properties of 1-quasi-hereditary algebras only depend on the related partial order (Ext-quiver, good-filtrations, etc. see [8]). In particular, a 1-quasi-hereditary algebra $(A, \leq)$ has a $K$-basis $\mathfrak{B}(A) = \bigcup_{j \in Q_0(A)} \mathfrak{B}_j(A)$ containing distinguished paths, which are linked only to $\leq$. The paths in $\mathfrak{B}_j(A)$ form a $K$-basis of the projective indecomposable $A$-module $P_A(j)$ for any $i \in Q_0$ (see [8, Theorem 3.2]). This basis plays an important role for the structure of the endomorphism algebras of the projective $A$-modules.

**Theorem A.** Let $(A, \leq)$ be a 1-quasi-hereditary algebra and $j \in Q_0(A)$. The submodules of $P_A(j)$ generated by the paths in $\mathfrak{B}_j(A)$ are $\text{End}_A(P_A(j))^{op}$-modules.

Any 1-quasi-hereditary algebra $A$ has (up to isomorphism) a unique projective-injective indecomposable module $P(A)$ (see [8] Lemma 2.1). In case the Ringel-dual $R(A)$ of $A$ is also 1-quasi-hereditary, there exist elements in $P(A)$ which generate modules satisfying remarkable properties.

**Theorem B.** Let $(A, \leq)$ and $(R(A), \geq)$ be 1-quasi-hereditary algebras. There exists a $K$-basis $\mathfrak{B}$ of the projective-injective indecomposable $A$-module $P(A)$, such that

(1) The set $\{A \cdot b \mid b \in \mathfrak{B}\}$ is the set of local, $\Delta$-good submodules of all projective indecomposable $A$-modules.

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(2) For every \( b \in \mathcal{B} \) the space \( A \cdot b \) is an \( \text{End}_A(P(A))^{op} \)-submodule of \( P(A) \).

The paper is organised in the following way: In Section 1, the definition of 1-quasi-hereditary algebra is recalled, and we present some examples of 1-quasi-hereditary algebras. In particular, we define a 1-quasi-hereditary algebra \( A_n(C) \) for some \( C \in \text{GL}_{n-2}(K) \) for \( n \geq 3 \). We show that these algebras are not BGG-algebras in general and that the Ringel-dual of \( A_n(C) \) is 1-quasi-hereditary, namely \( R(A_n(C)) \cong A_n(C^{-1}) \). Moreover, we give an example of a 1-quasi-hereditary algebra, whose Ringel-dual is not 1-quasi-hereditary. Using this algebra we illustrate the connection between the Jordan-Hölder-filtrations of standard resp. costandard and good-filtrations of injectives resp. projectives indecomposable modules described in [8, Sec. 4].

Section 2 and 3 is devoted to the proof of Theorems A and B respectively. These are based on order reversing and preserving bijections between certain subsets of \( Q_0(A) \) with respect to \( \leq \) and the lattice of modules generated by \( p \in \mathcal{B}(A) \). The connections between the modules and the partial order \( \leq \) given in Theorem A and B will be illustrated using the algebra associated to the regular block of category \( \mathcal{O}(\text{sl}_3(\mathbb{C})) \).

1. Preliminaries and examples of 1-quasi-hereditary algebras

Throughout the paper, any algebra \( \mathcal{A} \) is a finite dimensional, basic \( K \)-algebra over an algebraically closed field \( K \) given by a quiver \( Q(\mathcal{A}) \) and relations \( \mathcal{I}(\mathcal{A}) \). The vertices of \( Q(\mathcal{A}) \) are parameterized by the natural numbers. Any \( \mathcal{A} \)-module is a finite dimensional left module and for any \( i \in Q_0(\mathcal{A}) \) we denote by \( P(i) \), \( I(i) \) and \( S(i) \) corresponding projective, injective and simple \( \mathcal{A} \)-module, respectively. The product of arrows \( (i \to j) \) and \( (k \to i) \) is given by \( (k \to i \to j) = (i \to j) \cdot (k \to i) \).

If \( \{ i \to j \mid \alpha \in Q_1(\mathcal{A}) \} \leq 1 \) for all \( i, j \in Q_0(\mathcal{A}) \), then a path \( p = (i_1 \to i_2 \to \cdots \to i_m) \) in \( Q(\mathcal{A}) \) (and also in \( \mathcal{A} \)) we denote by \( (i_1 i_2 \ldots i_m) \). For the corresponding \( \mathcal{A} \)-map \( f_p : P(i_m) \to P(i_1) \) given by \( f(e_{i_m}) = p \) holds \( f_p = f_{(i_1 i_2)} \circ f_{(i_2 i_3)} \circ \cdots \circ f_{(i_m \cdots i_1)} \). The \( \mathcal{A} \)-module generated by \( p \) we denote by \( \langle p \rangle \), i.e. \( \langle p \rangle = \mathcal{A} \cdot p = \text{im}(f_p) \). By \( [M : S(i)] = \dim_K M_i \) we denote the Jordan-Hölder multiplicity of \( S(i) \) in an \( \mathcal{A} \)-module \( M \).

The equivalent definition of quasi-hereditary algebras introduced by Cline-Parshall-Scott [3] is given by Dlab and Ringel in [5]. The following is a brief review of relevant terminology and notations: Let \( (Q_0(\mathcal{A}), \leq) \) be a partially ordered set. For every \( i \in Q_0 \) the standard module \( \Delta(i) \) is the largest factor module of \( P(i) \) such that \( [\Delta(i) : S(k)] = 0 \) for all \( k \in Q_0 \) with \( k \not< i \), resp. the costandard module \( \nabla(i) \) is the largest submodule of \( I(i) \) such that \( [\nabla(i) : S(k)] = 0 \) for all \( k \in Q_0 \) with \( k \not> i \). We denote by \( \mathcal{F}(\Delta) \) the full subcategory of mod \( \mathcal{A} \) consisting of the modules having a filtration such that each subquotient is isomorphic to a standard module. The modules in \( \mathcal{F}(\Delta) \) are called \( \Delta \)-good and these filtrations are \( \Delta \)-good filtrations (resp. \( \nabla \)-good modules have \( \nabla \)-good filtrations and belongs to \( \mathcal{F}(\nabla) \)). For \( M \in \mathcal{F}(\Delta) \) we denote by \( (M : \Delta(i)) \) the (well-defined) number of subquotients isomorphic to \( \Delta(i) \) in some \( \Delta \)-good filtration of \( M \) (resp. \( \nabla(i) \) appears \( (M : \nabla(i)) \) times in some \( \nabla \)-good filtration of \( M \in \mathcal{F}(\nabla) \)).

The algebra \( \mathcal{A} = (KQ/\mathcal{T}, \leq) \) is quasi-hereditary if for all \( i, k \in Q_0 \) the following holds:

- \( [\Delta(i) : S(i)] = 1 \),
- \( P(i) \) is a \( \Delta \)-good module with \( (P(i) : \Delta(k)) = 0 \) for all \( k \not< i \) and \( (P(i) : \Delta(i)) = 1 \).
1.1 Definition. A quasi-hereditary algebra $A = (KQ/I, \leq)$ is called 1-quasi-hereditary if for all $i, j \in Q_0 = \{1, \ldots, n\}$ the following conditions are satisfied:

1. There is a smallest and a largest element with respect to $\leq$, without loss of generality we will assume them to be 1 resp. $n$,
2. $[\Delta(i) : S(j)] = (P(j) : \Delta(i)) = 1$ for $j \leq i$,
3. $\text{soc } P(i) \cong \text{top } I(i) \cong S(1)$,
4. $\Delta(i) \hookrightarrow \Delta(n)$ and $\nabla(n) \twoheadrightarrow \nabla(i)$.

The projective indecomposable module of a 1-quasi-hereditary algebra $A$ which corresponds to the minimal vertex 1 is also injective with $P(1) \cong I(1)$ (see [3, Lemma 2.1]). Axiom (3) shows that any projective indecomposable $A$-module can be considered as a submodule of $P(1)$. Axiom (2) implies that for any $i \in Q_0(A)$ there exist a uniquely determined submodule $M(i)$ of $P(1)$ with $M(i) \cong P(i)$. Consequently, for all $i, j \in Q_0$ with $j \leq i$ there exists a uniquely determined submodule of $P(j)$ isomorphic to $P(i)$ (see [3, Lemma 2.2]). We will often make use of this fact in the following.

Example 1. Let $\mathcal{A}$ be an algebra associated to a block of the category $\mathcal{O}(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ defined in [2]. Then $A$ is 1-quasi-hereditary if $\text{rank } (\mathfrak{g}) \leq 2$. The quivers and relations of these algebras are to be found in [11]. To illustrate some statements we use the algebra corresponding to a regular block of category $\mathcal{O} (\mathfrak{sl}_3(\mathbb{C}))$ (see also [7]):

$$
\begin{array}{cccccc}
6 & 1 & 2 & 3 & 4 & 5 \\
646 & = & 0 & 656 & = & 0 \\
643 & = & 653 & 652 & = & 642 \\
346 & = & 356 & 256 & = & 246 \\
421 & = & 431 & 521 & = & 531 \\
124 & = & 134 & 125 & = & 135
\end{array}
$$

The quiver and relations of the algebra $\mathcal{A}$ corresponding to a regular block of $\mathcal{O}(\mathfrak{sl}_4(\mathbb{C}))$ are calculated in [11] (in this notations we have $24 \leq i \leq 1$ for all $i \in Q_0(\mathcal{A})$). The algebra $\mathcal{A}$ is not 1-quasi-hereditary, since $[\Delta(3) : S(16)] = 2$. However, the factor algebra $\mathcal{A}/(\mathcal{A}e(j)\mathcal{A})$ with $\epsilon(j) = \sum_{i \not\in j} e_i$ is 1-quasi-hereditary for every $j \in \{i \in Q_0 \mid i \leq x\} \text{ for some } x = 5, 6, 7, 9$.

Example 2. Dlab, Heath and Marko described in [1] quasi-hereditary algebras which are obtained in the following way: Let $B$ be a commutative local self-injective $K$-algebra, $\text{dim}_K B = n$. Let $\mathfrak{X} = \{\mathcal{X}(\lambda) \mid \lambda \in \Lambda\}$ be a set of local ideals of $B$ with $B = \mathcal{X}(\lambda_1) \in \mathfrak{X}$ indexed by a finite partially ordered set $\Lambda$ reflecting inclusions: $\mathcal{X}(\lambda') \subseteq \mathcal{X}(\lambda)$ if and only if $\lambda' > \lambda$. Then $A = \text{End}_B (\bigoplus_{\lambda \in \Lambda} \mathcal{X}(\lambda))$ is a quasi-hereditary algebra with respect to $(\Lambda, \leq)$ if and only if $|\Lambda| = n$ and $\text{rad } \mathcal{X}(\lambda) = \sum_{\lambda' \leq \lambda} \mathcal{X}(\mu)$ for every $\mathcal{X}(\lambda) \in \mathfrak{X}$.

Quasi-hereditary algebras obtained in this way satisfy all axioms for 1-quasi-hereditary algebra (see [1] Section 4)).

The algebras in the examples 1 and 2 are BGG-algebras: A quasi-hereditary algebra $\mathcal{A} = (KQ/I, \leq)$ with a duality functor $\delta$ on $\text{mod } \mathcal{A}$ [$\delta$ is a contravariant, exact, additive functor such that $\delta \cdot \delta$ is the identity on $\text{mod } A$ and $\delta$ induces a $K$-map on the $K$-spaces

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Hom_A(M, N) for all M, N ∈ mod A] is called a BGG-algebra if δ(P(i)) ≅ I(i) for all i ∈ Q_0(A) (see [12, Remark 1.4]).

The functor δ for an algebra A in the previous examples is induced by the anti-automorphism defined by ε : A → A via ε(e_i) = e_i and ε(i → j) = (j → i) for all (i → j) ∈ Q_1. The next example shows that 1-quasi-hereditary algebras are in general not BGG-algebras.

**Example 3.** Let A_n(C) for n ≥ 3 be the algebra given by the following quiver and relations: For all i, j ∈ {2, . . . , n − 1} the following holds:

```
  1 → 2 → 3 → . . . → n
  i
```

The order is given by 1 < i < n for all i ∈ {2, . . . , n − 1}, thus axiom (1) of 1.1 holds.

It’s easy to verify that the set ℂ(i) forms a K-basis of P(i) for any k ∈ Q_0, where

\[ ℂ(1) := \{e_i, (1 2 n 1) \cup \{1 1 i | 2 ≤ i ≤ n − 1\} \cup \{(1 i), (1 2 n i) | 2 ≤ i ≤ n − 1\} \cup \{(1 2 n)\} \]

\[ ℂ(2) := \{(i 1), (i n 1) \cup e_i, (i n i) \cup \{(i n j) | 2 ≤ j ≤ n − 1, j ≠ i\} \cup \{(i n)\}, 2 ≤ i ≤ n − 1, \in P(i) \} \]

\[ ℂ(3) := \{n 1 2 n 1 \cup \{(n i) | 2 ≤ i ≤ n − 1\} \cup \{e_n\} \} \]

Axiom (2): The A-map f_{jk} : P(k) → P(j) with f(e_k) = (j → k) is injective for (j, k) ∈ \{(i, n), (1, i) | 1 < i < n\}. Moreover we have Hom_A(P(n), P(j)) = span_K \{f_{jn}\} and Hom_A(P(i), P(j)) = span_K \{f_{jn} \circ f_{ni}\} for all 1 < i < n. The definition of Δ(j) and im(f_{jn} \circ f_{ni}) ⊂ im(f_{jn}) implies Δ(j) = P(j) / \left( \sum_{j \neq i} \sum_{f \in \text{Hom}_A(P(i), P(j))} \text{im}(f) \right) = P(j) / \left( \sum_{j \neq i} \text{im}(f_{jn}) \right) = \text{span}_K \{ζ_j(j)\} for 1 < j < n as well as Δ(1) ≅ S(1), Δ(n) = P(n).

For all i, j ∈ Q_0 with i ≤ j we have \{Δ(j) : S(i)\} = 1.

The filtration 0 ⊆ M_1 ⊆ M_2 ⊆ . . . ⊆ M_{n-1} ⊆ P(1) with M_1 = im(f_{12} \circ f_{2n}) ≅ Δ(n) and M_k = \sum_{i=2}^k \text{im}(f_{ii}) is Δ-good, since P(1)/M_{n-1} ≅ S(1) ≅ Δ(1) and M_k/M_{k-1} ≅ \text{im}(f_{kk}) / \left( \text{im}(f_{1k}) \cap \sum_{m=2}^{k-1} \text{im}(f_{im}) \right) ≅ P(k) / \text{im}(f_{kn}) ≅ Δ(k) for any 2 ≤ k ≤ n − 1. The filtration 0 ⊆ im(f_{jn}) ⊂ P(j) is Δ-good for any 1 < j < n. Thus (P(j) : Δ(i)) = 1 for all i, j ∈ Q_0 with j ≤ i.

Axiom (3): Since soc Δ(i) ≅ S(1) for all i ∈ Q_0 and P(1) ∈ ℂ(Δ), we have soc P(1) ≅ S(1)^m for m ∈ N. A simple submodule of P(1) is generated by some non zero element q ∈ P(1) with (1 → j) · q = 0 for all 2 ≤ j ≤ n − 1. The basis ℂ(1) of P(1) shows q = λ_1 e_1 + \sum_{i=2}^{n-1} λ_i(i 1) + λ_n(1 2 n 1) with λ_i ∈ K. If λ_1 ≠ 0, then \langle q \rangle = A · q = P(1)(q \neq soc P(1)). Let λ_1 = 0, for every j ∈ {2, . . . , n − 1} it is

\[ 0 = (1 → j) · q = \sum_{i=2}^{n-1} λ_i(i 1 1 j) + λ_n(1 2 n 2 1 j) \]

\[ = \sum_{i=2}^{n-1} λ_i c_{ij}(1 i n j) + λ_n c_{2j}(1 2 n 2 n \ j) = \sum_{i=2}^{n-1} λ_i c_{ij}(1 j n j) \]
if and only if \( \sum_{i=2}^{n-1} \lambda_i c_{ij} = 0 \). Since \( \det C \neq 0 \), we obtain \( \lambda_i = 0 \) for all \( 2 \leq i \leq n - 1 \). Hence, \( q \in P(1)_1 \) generates a simple module if \( q = \lambda(12n21) \) for some \( \lambda \in K \setminus \{0\} \). For all \( \lambda, \mu \in K \setminus \{0\} \) we have \( \langle \lambda(12n21) \rangle = \langle \mu(12n21) \rangle \). Thus \( 12n21 = \operatorname{soc} P(1) \cong S(1) \). Since \( f_{(ii)} : P(i) \hookrightarrow P(1) \), we obtain \( \operatorname{soc} P(i) \cong S(1) \) for all \( i \in Q_0 \). The algebra \( (A_n(C))^\text{op} \) is defined by the quiver and relations of \( A_n(C^{tr}) \). Using the same procedure we obtain \( \operatorname{soc} P_{A_n(C)^\text{op}}(i) \cong S(1) \) and the standard duality implies top \( I(i) \cong S(1) \) for all \( i \in Q_0 \).

Axiom (4): For any \( 1 < i < n \) we have \( \ker(f_{(mi)} : P(i) \hookrightarrow P(n)) = \im(f_{(mi)} : P(n) \hookrightarrow P(i)) \), therefore \( \Delta(i) \cong \Delta(P(i)/\im(f_{(mi)})) \cong \im(f_{(mi)}) \hookrightarrow P(n) = \Delta(n) \). Since \( \Delta(1) \cong \Delta(0) \), we obtain \( \Delta(i) \hookrightarrow \Delta(n) \), for all \( i \in Q_0 \). Using the same procedure we obtain also \( \Delta_{A^\text{op}}(i) \hookrightarrow \Delta_{A^\text{op}}(n) \). The standard duality provides \( \nabla(n) \hookrightarrow \nabla(i) \) for every \( i \in Q_0 \). The algebra \( A_n(C) \) is 1-quasi-hereditary.

For the algebra \( A := A_4(C) \) with \( C = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \) and \( q \neq 0 \) the submodules of \( P(3) \) and of \( P_{A^\text{op}}(3) \) are represented in the submodule diagrams (here \( \langle p \rangle = A \cdot p \) resp. \( \langle p \rangle = A^\text{op} \cdot p \)):

By duality, the number of factor modules of \( P_{A^\text{op}}(3) \) is the number of submodules of \( I_A(3) \):

- \( 10 = |\{\text{submodules of } P(3)\}| \)
- \( 9 = |\{\text{submodules of } I(3)\}| \)

Therefore on \( \text{mod } A \) there can not exist a duality functor \( \delta \) with \( \delta(P(3)) \cong I(3) \).

The 1-quasi-hereditary algebra \( A_4 \left( \begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix} \right) \) with \( q \neq 0 \) is not a BGG-algebra.

If \( q = 0 \) then \( A_4(C) \) is a BGG-algebra with the duality induced by an anti-automorphism.

Specific for the class of quasi-hereditary algebras is the concept of Ringel-duality: Let \( \mathcal{A} \) be a quasi-hereditary algebra. For any \( i \in Q_0(\mathcal{A}) \) there exists up to isomorphism an unique indecomposable \( \mathcal{A} \)-module \( T(i) \) having a \( \Delta \)-good and \( \nabla \)-good filtration with \( (T(i) : \Delta(i)) = (T(i) : \nabla(i)) = [T(i) : S(i)] = 1 \) and \( (T(i) : \Delta(j)) = (T(i) : \nabla(j)) = [T(i) : S(j)] = 0 \) for all \( j \neq i \), moreover there exists a submodule \( Y(i) \in \mathfrak{F}(\nabla) \) of \( T(i) \) with \( T(i)/Y(i) \cong \nabla(i) \) and a factor module \( X(i) \in \mathfrak{F}(\Delta) \) with \( \ker(T(i) \to X(i)) \cong \Delta(i) \). The algebra \( R(\mathcal{A}) := \operatorname{End}_{\mathcal{A}}(\bigoplus_{i \in Q_0} T(i))^{\text{op}} \) with the opposite order \( \succ \) is also quasi-hereditary, where \( T = \bigoplus_{i \in Q_0} T(i) \) is the characteristic tilting module of \( \mathcal{A} \). In particular, \( \mathfrak{F}(\Delta) \cap \mathfrak{F}(\nabla) = \operatorname{add}(T) \) and \( \mathcal{A} \) is isomorphic to \( R(R(\mathcal{A})) \) as a quasi-hereditary algebra (see [7]). The algebra \( R(\mathcal{A}) \) is called Ringel-dual of \( \mathcal{A} \).

Some properties of the characteristic tilting module and the Ringel-dual of a 1-quasi-hereditary algebra are considered in section 5 and 6 in [8]. According to [8, Remark 5.3] for the direct summands of the characteristic tilting \( A_n(C) \)-module we obtain \( T(1) \cong S(1) \), \( T(n) \cong P(1) \) and \( T(i) \cong P(1)/\left( \sum_{j=2}^{n-1} P(j) \right) \cong \bigcap_{j=2}^{n-1} \ker(P(1) \to I(j)) \) for all \( 2 \leq i \leq n - 1 \), since any vertex \( i \in Q_0 \setminus \{1, n\} \) is a neighbor of 1. Consequently, \( T(i) \) is a submodule (and a factor module) of \( P(1) \), thus there exists some element in \( P(1)_1 \), which generates \( T(i) \). In particular, the Ringel-dual of \( A_n(C) \) is also 1-quasi-hereditary (see [8, Theorem 6.1]).

1.2 Lemma. Let \( C = (c_{ij})_{2 \leq i, j \leq n-1} \in \text{GL}_{n-2}(K) \), the algebra \( A_n(C) \) given in Example 3 and \( C^{-1} = (d_{ij})_{2 \leq i, j \leq n-1} \). Then the following hold:
(1) \( t(i) := \sum_{j=2}^{n-1} d_{ij} \cdot (j \ 1) \) generates \( T(j) \) for \( 2 \leq i \leq n - 1 \).

Moreover, \( t(1) := (12n21) \) generates \( T(1) \) and \( t(n) := e_1 \) generates \( T(n) \).

(2) \( R(A_n(C)) \cong A_n(C^{-1}) \).

**Proof.** (1) The \( A_n(C) \)-module \( \{t(i)\} \) is local with top \( \{t(i)\} \cong S(1) \cong \Delta(1) \) since \( t(i) \in P(1) \) for all \( i \in Q_0 \). Using the calculations in Example 3, we obtain \( T(1) \cong (12n21) \) and \( T(n) \cong \langle e_1 \rangle \), since \( T(1) \cong \text{soc} \, P(1) \) and \( T(n) \cong P(1) \). For all \( i, k \in \{2, \ldots, n - 1\} \) we have

\[
(1 \to k) \cdot t(i) = \sum_{j=2}^{n-1} d_{ij} \cdot (j \ 1 \ k) = \sum_{j=2}^{n-1} d_{ij} \cdot c_{jk} \cdot (1 \ n \ k) = \begin{cases} \{1 \ i \ n \ i\} & \text{if } k = i, \\ 0 & \text{else}. \end{cases}
\]

Consequently, \( \text{rad} \{t(i)\} = \langle 1 \ i \ n \ i \rangle \cong \Delta(i) \) and \( 0 \subset \text{soc} \{t(i)\} = \langle t(1) \rangle \subset \text{rad} \{t(i)\} \subset \langle t(i)\rangle \) is the unique Jordan-Hölder-filtration of \( \{t(i)\} \). The filtration \( 0 \subset \text{rad} \{t(i)\} \subset \langle t(i)\rangle \) resp. \( 0 \subset \text{soc} \{t(i)\} \subset \langle t(i)\rangle \) is \( \Delta \)-good resp. \( \nabla \)-good with the properties of \( T(i) \), thus \( \{t(i)\} \cong T(i) \) for every \( 2 \leq i \leq n - 1 \).

(2) We consider \( T(i) \) as a submodule of \( P(1) \). Since \( R(A_n(C)) \cong \text{End}_{A_n(C)}(T)^{op} \) is 1-

\( \text{quasi-hereditary}, \) the quivers of \( (A_n(C), \leq) \) and \( (R(A_n(C)), \geq) \) have the same shape (see [8, Theorem 2.7]). The vertex \( i \) in \( Q_0(R(A_n(C))) \) corresponds to the direct summand \( T(i) \) of \( T = \bigoplus_{i \in Q_0} T(i) \). For \( (l, m) \in \{(1, j), (j, 1), (n, j), (j, n) \mid 2 \leq j \leq n - 1\} \) we denote by \( \tau_{l,m}(i) \) the following maps:

\[
\tau_{(1,j)} : T(1) \to T(j) \text{ with } \tau_{(1,j)}(t(1)) = t(1) \quad \text{and} \quad \tau_{(j,1)} : T(j) \to T(1) \text{ with } \tau_{(j,1)}(t(j)) = t(1)
\]

\[
\tau_{(j,n)} : T(j) \to T(n) \text{ with } \tau_{(j,n)}(t(j)) = t(j) \quad \text{and} \quad \tau_{(n,j)} : T(n) \to T(j) \text{ with } \tau_{(n,j)}(t(n)) = t(j).
\]

It is easy to compute that the space of maps in \( \text{Hom}_{A_n(C)}(T(l), T(m)) \) which do not factors through \( T \) is spanned by \( \tau_{l,m}(i) \), thus \( \tau_{l,m}(i) \) corresponds to the arrow \( (l \to m) \) for any \( (l, m) \). We can compute the relations: For all \( 2 \leq i, j \leq n - 1 \) we have \( \tau_{(i,n)} \circ \tau_{(1,i)} = \tau_{(j,n)} \circ \tau_{(1,j)} \) and \( \tau_{(i,1)} \circ \tau_{(n,i)} = \tau_{(j,1)} \circ \tau_{(n,j)} \) thus \( (1 \ j) = (1 \ i \ n) \) and \( (n \ j) = (n \ i) \). Moreover, \( \langle \tau_{(1,j)} \circ \tau_{(i,1)} \rangle \{t(i)\} = \{t(1)\} \) and \( \langle \tau_{(n,j)} \circ \tau_{(i,n)} \rangle \{t(i)\} = \{t(i) \cdot t(j) \}

We obtain \( \tau_{(n,j)} \circ \tau_{(i,n)} = d_{ij} \cdot (\tau_{(1,j)} \circ \tau_{(i,1)}) \), thus \( (i \ n \ j) = d_{ij} \cdot (1 \ 1 \ j) \). The map \( \tau_{(j,1)} \circ \tau_{(1,j)} \) is zero-map, thus \( (1 \ j \ n) = 0 \) for any \( j \in Q_0 \setminus \{1, n\} \). There are no relations between paths starting and ending in \( n \), since the set \( \{\tau_{(j,n)} \circ \tau_{(n,j)} \} \{t(n)\} = \{t(j) \mid 2 \leq j \leq n - 1\} \cup \{t(1)\} \) is linearly independent.

By interchanging the notations \( 1 \mapsto n \) and \( n \mapsto 1 \) we obtain that the quiver and relations of \( R(A_n(C)) \) are those of the algebra \( A_n(C^{-1}) \).

The class of 1-

\( \text{quasi-hereditary} \) algebras is not closed under Ringel-duality. According to [8, Theorem 6.1] the algebra \( R(A) \) is 1-

\( \text{quasi-hereditary} \) if and only if the factor algebra
A(i) = A/ \left( A \left( \sum_{j \neq i} e_j \right) A \right) is 1-quasi-hereditary for every i \in Q_0 (see [8, Section 5]). The next example presents a 1-quasi-hereditary algebra \( A \) such that there exists \( i \in Q_0(A) \) with \( A(i) \) not being 1-quasi-hereditary.

**Example 4.** The algebra \( A \) given by the following quiver and relations is 1-quasi-hereditary with the partial order \( 1 \prec 2 \prec 3 \prec 5 \prec 6 \) and \( 1 \prec 4 \prec 5 \).

![Diagram of quiver](image)

The Auslander-algebra \( A_m \) of \( K[x]/(x^m) \) is obtained by the construction in Example 2 with \( B = K[x]/(x^m) \) and the set of ideals \( \mathcal{X} = \{ X(i) := \langle x^i \rangle \ | \ 1 \leq i \leq m \} \).

For \( i \in Q_0 \) with \( i = 1, 2, 3 \), the algebra \( A(i) \) is isomorphic to the Auslander-algebra \( A_i \) and \( A(4) \cong A_2 \).

Thus \( A(i) \) is 1-quasi-hereditary for \( i = 1, 2, 3, 4, 6 \) and we can compute an element \( t(i) \in P(1) \) which generates \( T(i) \):

\[
\begin{align*}
t(1) &= (1456541), \\
t(2) &= (14541), \\
t(3) &= q \cdot (12321) - (141), \\
t(6) &= e_1. \\
t(4) &= (141) - (12321).
\end{align*}
\]

The quiver of a 1-quasi-hereditary algebra \( A \) depends only on \( (Q_0(A), \leq) \): If \( i \xrightarrow{a} j \in Q_1(A) \), then \( i \prec j \) resp. \( i \nVDash j \) and \( | \{ \alpha \in Q_1(A) \ | \ i \xrightarrow{a} j \} | = | \{ \alpha \in Q_1(A) \ | \ j \xrightarrow{a} i \} | = 1 \) (see [8, Theorem 2.7]). All Jordan-Hölder-filtrations of \( \Delta(i) \) and \( \nabla(i) \) as well as good filtrations of \( P(i) \) and \( I(i) \) are connected to the sequences from \( T(i) \) resp. \( \mathcal{L}(i) \) which also depend only on the structure of \( \leq \) for every \( i \in Q_0 \) (see [8, Propositions 4.1 and 4.2]). Each of these sequences can be completed to some sequence from \( T(n) = \mathcal{L}(1) = \{ i = (i_1, i_2, \ldots, i_n) \in Q_0(A^n) \ | \ i_k \not\geq i_t, \ 1 \leq k < t \leq n \} \), thus we have

\[
\mathcal{L}(1) \leftrightarrow \begin{cases} \text{Jordan-Hölder} & \text{of } \Delta(n) \\ \text{f. of } S(i) \end{cases} \leftrightarrow \begin{cases} \text{Jordan-Hölder} & \text{of } \nabla(n) \\ \text{f. of } S(i) \end{cases} \leftrightarrow \begin{cases} \Delta\text{-good f.} & \text{of } P(1) \\ \mathcal{D}(i) \end{cases} \leftrightarrow \begin{cases} \nabla\text{-good f.} & \text{of } I(1) \\ \mathcal{N}(i) \end{cases}
\]

The sequence of indices of the simple factors in \( \mathcal{S}(i) \) and \( \Delta\text{-good factors in } \mathcal{D}(i) \) are the same (and in the same order). The same holds for the indices of factors in \( S(i) \) and \( \mathcal{N}(i) \). Furthermore the indices of simple factors of \( \mathcal{S}(i) \) and \( S(i) \) are the same, but in reversed order. Since \( \Delta(n) \) (resp. \( \nabla(n) \)) has finitely many submodules, all filtrations of \( \Delta(n) \) can be represented in the submodule diagram of \( \Delta(n) \), i.e. the Hasse diagram of \( \{ \text{submodules of } \Delta(n) \} \subseteq \).

The last observation shows that \( \nabla\text{-good filtrations of } I(1) \) can be represented in a diagram (we will call it \( \nabla\text{-good diagram of } I(1) \)) whose shape coincides with the submodule diagram of \( \Delta(n) \). By standard duality, the submodule diagram of \( \nabla(n) \) and the \( \Delta\text{-good diagram of } P(1) \) have the same shape. Moreover, the shape of the diagram of \( \Delta(n) \) is obtained from the diagram of \( \nabla(n) \) by turning it upside down. In particular, all \( \nabla\text{-good filtrations}
of $I(i)$ are parts of $\nabla$-diagram of $I(1)$ resp. the $\Delta$-good diagram of $P(i)$ is a $\Delta$-subdiagram of $P(1)$ for any $i \in Q_0$. We illustrate this using Example 4.

The set $\mathcal{L}(1) = \{1, 2, 3, 4, 5, 6\}$ shows that there exist three Jordan-Hölder filtrations of $\Delta(6)$ resp. $\nabla(6)$ and three good filtrations of $P(1)$ resp. $I(1)$. All Jordan-Hölder filtrations of $\Delta(j)$ resp. $\nabla(j)$ are subfiltrations of $\Delta(6)$ resp. factor filtrations of $\nabla(6)$ (in the picture $\mathfrak{H}(j) = \ker(\nabla(n) \to \nabla(j))$). Similar $\Delta$-good filtrations of $P(j)$ are subfiltrations of $P(1)$ and $\nabla$-good filtrations of $I(1)$ are factor filtrations of those of $I(1)$ (here $\mathcal{K}(j) = \ker(I(1) \to I(j))$) for all $j \in Q_0$.

The subquotients are illustrated as follows: The module $N$ pointing to the line which connects the modules $M$ and $M'$ such that $M \subset M'$ has the meaning $M'/M \cong N$.

### 2. Modules generated by the paths $p(j, i, k)$

In this section $(A, \leq)$ is a $1$-quasi-hereditary algebra. In the following we use some notations introduced in [S]: For any $i \in Q_0$ we have the sets

$$\Lambda(i) := \{j \in Q_0 \mid j \leq i\} \quad \text{and} \quad \Lambda^{(i)} := \{j \in Q_0 \mid j \geq i\}.$$

If $i$ is a small resp. large neighbor of $j$ w.r.t. $\leq$, we write $i \triangleleft j$ resp. $i \triangleright j$. Recall that $(p) = A \cdot p$.

We recall the definition of $p(j, i, k)$ from [S Sec. 3]: Let $j, i, k$ be in $Q_0$ with $i \in \Lambda^{(j)} \cap \Lambda^{(k)}$. There exist $\lambda_0, \ldots, \lambda_r \in \Lambda^{(j)}$ and $\mu_0, \ldots, \mu_m \in \Lambda^{(k)}$ with $j = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_r = i$ and $i = \mu_0 \geq \mu_1 \geq \cdots \geq \mu_m = k$ giving a path

$$p(j, i, k) := (j \to \lambda_1 \to \cdots \to \lambda_{r-1} \to i \to \mu_1 \to \cdots \to \mu_{m-1} \to k)$$

(a path $p(j, i, k)$ [black] resp. $p(k, i, j)$ [gray] is visualised in the picture to the right). We fix a path of the form $p(j, i, i)$ and $p(i, i, k)$, we obviously have $p(j, i, k) = p(i, i, k) \cdot p(j, i, i)$. We denoted by $f_{(j,i,k)}$ the $A$-map corresponding to $p(j, i, k)$, i.e. $f_{(j,i,k)}: P(k) \to P(j)$ with $f_{(j,i,k)}(a \cdot e_k) = a \cdot p(j, i, k)$ for all $a \in A$. We obtain $f_{(j,i,k)} = f_{(j,i,i)} \circ f_{(i,i,k)}$. A path $p(j, i, i)$ resp. $p(i, i, k)$ is increasing resp. declining and $e_i = p(i, i, i)$ is the trivial path $e_j$. In the following, we point out of some facts which are presented in [S]:

![Diagram showing submodule and Δ-good diagrams](attachment:image.png)

The submodule diagrams are illustrated as follows: The module $N$ pointing to the line which connects the modules $M$ and $M'$ such that $M \subset M'$ has the meaning $M'/M \cong N$.
(1) The set \( \{ p(j, i, k) \mid i \in \Lambda^{(j)} \cap \Lambda^{(k)} \} \) is a \( K \)-basis of \( P(j)_k \) for all \( j, k \in Q_0 \) (see \[\text{Remark 3.2}\]).

(2) The module generated by \( p(j, i, k) \) is the (unique) submodule of \( P(j) \) isomorphic to \( P(i) \) for any \( i \in \Lambda^{(j)} \). The map \( f_{(j,i,k)} : P(i) \to P(j) \) is an inclusion, thus \( \langle p(j, i, k) \rangle = \text{im} \left( P(k) \xrightarrow{f_{(i,j,k)}} P(i) \xrightarrow{f_{(j,i,k)}} P(j) \right) \) is a submodule of \( P(i) \subseteq P(j) \) (see \[\text{Remark 3.1}\]).

(3) We have \( \text{im} \left( P(k) \xrightarrow{f_{(i',j',k)}} P(i') \xrightarrow{f_{(j',i',k)}} P(j) \right) \subseteq P(i') \subseteq P(i) \subseteq P(j) \) for \( i', i \in \Lambda^{(j)} \cap \Lambda^{(k)} \) with \( i \leq i' \), thus \( p(j, i', k) \in P(i)_k \subseteq P(j)_k \). We obtain that the set \( \{ p(j, i', k) \mid i' \in \Lambda^{(i)} \} \) is a \( K \)-basis of the space \( P(i)_k \) of the submodule \( P(i) \) of \( P(j) \), since \( \dim_K P(i)_k = [P(i) : S(k)] = |\Lambda^{(i)} \cap \Lambda^{(k)}| = |\Lambda^{(i)}| \) (see \[\text{Lemma 2.1}\]).

We now consider the \( A \)-modules generated by \( p(j, i, k) \) for all \( i, j, k \in Q_0 \) with \( i \in \Lambda^{(j)} \cap \Lambda^{(k)} \). We show that the submodule \( \langle p(j, i, k) \rangle \) of \( P(j) \) is \( F \)-invariant for every \( F \in \text{End}_{A}(P(j)) \), i.e. \( F(\langle p(j, i, k) \rangle) \subseteq \langle p(j, i, k) \rangle \). Thus \( \langle p(j, i, k) \rangle \) is an \( \text{End}_{A}(P(j))^{op} \)-module.

2.1 Theorem. Let \( A = (KQ/\mathcal{I}, \preceq) \) be a 1-quasi-hereditary algebra and \( j, k \in Q_0 \). For any \( i \in \Lambda^{(j)} \cap \Lambda^{(k)} \) the module generated by \( p(j, i, k) \) is \( F \)-invariant for every \( F \in \text{End}_{A}(P(j)) \).

The proof of this Theorem follows from some properties of the lattice of submodules of \( P(j) \) generated by \( p(j, i, k) \). They provide a relationship between the Hasse diagrams for the posets \( \{ \{ p(j, i, k) \mid i \in \Lambda^{(j)} \cap \Lambda^{(k)} \} , \subseteq \} \) and \( \{ \{ p(j, i, k) \mid k \in \Lambda^{(i)} \} , \subseteq \} \) as well as \( \{ \{ p(j, i, k) \mid k \in \Lambda^{(i)} \} , \subseteq \} \) and \( \{ \{ p(j, i, k) \mid k \in \Lambda^{(i)} \} , \subseteq \} \).

2.2 Lemma. Let \( A = (KQ/\mathcal{I}, \preceq) \) be a 1-quasi-hereditary algebra and \( j \in Q_0 \). For all \( i \in \Lambda^{(j)} \) and \( k \in \Lambda^{(i)} \) the following holds:

(a) \( \langle p(j, i', k) \rangle \subseteq \langle p(j, i, k) \rangle \) if and only if \( i' > i \).

(b) \( \langle p(j, i, k') \rangle \subseteq \langle p(j, i, k) \rangle \) if and only if \( k' < k \).

The picture to the right visualizes the presentation of \( P(1) \) over the 1-quasi-hereditary algebra \( A \) given by the quiver and relations in Example 1 (this algebra corresponds to a regular block of \( \mathcal{O}(\text{sl}_3(\mathbb{C})) \)). The circles represents the spaces \( P(1)_k \) for any \( k \in Q_0 \) which is spanned by \( \{ p(i, k) \mid i \in \Lambda^{(k)} \} \) (see (1)), where

\[
p(i, k) := p(1, i, k).
\]

The meaning of the arrows \( p \to q \) and \( p \longrightarrow q \), is \( \langle p \rangle \subseteq \langle q \rangle \) (they illustrate Lemma 2.2 (a) and (b) respectively). The set \( \{ p(i, k) \mid i \in \Lambda^{(j)}, k \in \Lambda^{(i)} \} \) is a \( K \)-basis of the submodule \( P(j) \) of \( P(1) \) which is generated by \( p(j, j) \). The set \( \{ p(6, k) \mid k \in \Lambda^{(j)} \} \) is a \( K \)-basis of the submodule \( \Delta(j) \) of \( P(1) \) which is generated by \( p(6, j) \) (see \[\text{Remark 3.1}\]).
2.3 Remark. General theory of modules over basic algebras says that an $A$-module $M$ has finitely many local submodule with top isomorphic to $S(k)$ if and only if there exist a $K$-basis of $\{x_1, x_2, \ldots, x_m\}$ of $M_k$ with $\langle x_1 \rangle \subset \langle x_2 \rangle \subset \cdots \subset \langle x_m \rangle$. If $M_k$ has such a $K$-basis for any $k \in Q_0(A)$ then the number of submodules of $M$ is finite.

Lemma 2.2 implies that a projective indecomposable $A$-module $P(j)$ over 1-quasi-hereditary algebra $A$ has finitely many submodules if and only if the set $\{A^0, \leq\}$ totally ordered. Let $A$ be an algebra given in Example 4, then the $A$-module $P(i)$ has finitely many submodules for every $i \in Q_0\{1\}$.

The proof of Lemma 2.2 is based on some properties of the factor algebras $A(l) \coloneqq A/A\left(\sum_{j \in Q_0\Lambda(l)} e_j\right)$ of $A$ considered in [8 Sect. 5]: Any $A(l)$-module is an $A$-module $M$ with $[M : S(t)] = 0$ for all $t \in Q_0\Lambda(l)$. The algebra $A(l)$ is quasi-hereditary with $\Delta_{A(l)} = \Delta(l)$ for all $l \in \Lambda(l)$ (the quiver of $A(l)$ is the full subquiver of $Q$ containing the vertices of $\Lambda(l)$). Let $p$ be a path of $A$ passing through the vertices in $\Lambda(l)$, then this path is also a path of $A(l)$ which we denote by $p(l)$. The $A(l)$-module generated by $p(l)$ [also denoted by $\langle p(l) \rangle$] is the largest factor module of the $A$-module $(p)$ with the property $[M : S(t)] = 0$ for all $t \in Q_0\Lambda(l)$. In particular, $\langle p(l) \rangle \subset \langle q(l) \rangle$ implies $(p) \subset (q)$. For any $i \in \Lambda(l)$ a path of the form $p(j, i, k)$ runs over the vertices of $\Lambda(l)$ and the set $\{p(l)(j, i, k) \mid j, i, k \in \Lambda(l), j, k \in \Lambda(l)\}$ is a $K$-basis of $A(l)$ (see [8 Lemma 5.2(a)]).

Proof of Lemma 2.2. (a) $\Rightarrow$ The property (2) shows that $p(j, i', k) \in P(i)k$ can be considered as an element of a $K$-basis $\{p(j, l, k) \mid l \in \Lambda(l)\}$ of $P(i)k$ (we consider $P(i)$ as a submodule of $P(j)$). Thus $i' \in \Lambda(l)$ and $i' \neq i$, since $p(j, i', k) \neq p(j, i, k)$.

"$\Leftarrow$" Let $i < l < \cdots < i'$. We consider the algebra $A(l)$ and show $\langle p(l)(j, l, k) \rangle \subset \langle p(l)(j, i, k) \rangle$. This implies $\langle p(j, l, k) \rangle \subset \langle p(j, i, k) \rangle$ and iteratively we have $\langle p(j, l, k) \rangle \subset \cdots \subset \langle p(j, i, k) \rangle \subset \langle p(j, i, k) \rangle$.

Since $l$ is maximal in $\Lambda(l)$, we obtain $P(l)(l) = \Delta_{A(l)}(l) \cong \Delta(l)$. The equality $[P(l)(v) : S(l)] = (P(l)(v) : \Delta_{A(l)}(l)) = 1$ implies that $\Delta_{A(l)}(l)$ is an unique submodule of $P(l)(v)$ with top isomorphic to $S(l)$ for any $v \in \Lambda(l)$ (see [8 Lemma 5.2]). Moreover, it is a fact that a local submodule $M$ of $\Delta_{A(l)}(l)$ with top$(M) \cong S(l)$ is unique and isomorphic to $\Delta_{A(l)}(l)$ (see [8 Lemma 2.5]). For the $(l)$-map $f_{(j,l,k)}$ corresponding to $p(l)(j, l, k)$ we have

$$f_{(j,l,k)}: P(l)(k) \xrightarrow{f_{(j,l,k)}} \Delta_{A(l)}(l) \xrightarrow{f_{(j,l,k)}} P(l)(j)$$

and $f_{(j,l,k)} \neq 0$, since $p(l)(j, l, k) \neq 0$. Thus $f_{(j,l,k)}$ is injective and $\text{im}(f_{(j,l,k)}) = \langle p(l)(j, l, k) \rangle$ is a submodule of $\Delta_{A(l)}(l) \subset P(l)(j)$ with top isomorphic to $S(k)$. We obtain $\langle p(l)(j, l, k) \rangle \cong \Delta_{A(l)}(k)$ and $\langle p(l)(j, i, k) \rangle \cong P(l)(i) \Delta_{A(l)}(l)$ because $p(l)(j, i, k)$ and $p(l)(j, l, k)$ are linearly independent.

Let $W := \Delta_{A(l)}(l) \cap \langle p(l)(j, i, k) \rangle$. We show now existence of $t \in \Lambda(l)$ with $t \triangleright k$ such that $S(t)$ is a direct summand of top$(W)$. This implies an existence of a local submodule $L$ of $\Delta_{A(l)}(l)$ with top$L \cong S(t)$ such that $L \subset \langle p(l)(j, i, k) \rangle$. The fact $(\ast)$ provides $L = \Delta_{A(l)}(l)$ and consequently $\langle p(l)(j, i, k) \rangle = \Delta_{A(l)}(k) \subset \Delta_{A(l)}(t) \subset \langle p(l)(j, i, k) \rangle$ (see [8 Remark 2.6]): The filtration $0 \subset \Delta_{A(l)}(l) \subset P(l)(i)$ is $\Sigma$-good, since $P(l)(i) : \Delta_{A(l)}(j) = 1$ for $j = i, l$ (see [8 Lemma 5.2]). Since $\langle p(l)(j, i, k) \rangle / W \twoheadrightarrow P(l)(i) / \Delta_{A(l)}(l) \cong \Delta_{A(l)}(i)$, we have $\langle p(l)(j, i, k) \rangle / W \cong \Delta_{A(l)}(k)$ (see $(\ast)$). For the $A$-module $\langle p(l)(j, i, k) \rangle$ there exists a surjective map $F : P(k) \twoheadrightarrow \langle p(l)(j, i, k) \rangle$. According to [8 Lemma 2.4] we have $P(k) / (\sum_{k \in l} P(t)) \cong \Delta(k)$, thus $F(\sum_{k \in l} P(t)) = W$. Since $W$ is an $A(l)$-module we obtain $\text{top}(W) \in \text{add}(\text{top}(\sum_{k \in l} P(t))) \cap \text{add}(\bigoplus_{t \in \Lambda(l)} S(t)) = \text{add}(\bigoplus_{t \in \Lambda(l)} S(t))$ (see [11].
Corollary 3.9].

(b) ’’ ⇔ ’’ It is \( \langle p(j, l, k') \rangle \subset \langle p(j, l, k) \rangle \) if and only if \( \langle p(j, l, k') \rangle \subset \langle p(j, l, k) \rangle \). We have \( \langle p(j, l, k') \rangle \cong \Delta(l)\langle k' \rangle \cong \Delta(k') \) and \( \langle p(j, l, k) \rangle \cong \Delta(l)\langle k \rangle \cong \Delta(k) \). According to \[8\] Lemma 2.5 we obtain \( \Delta(k') \subset \Delta(k) \) if and only if \( k' < k \).

Proof of Theorem. Let \( F \in \text{End}_A(P(j)) \). The submodule \( P(i) \) of \( P(j) \) is \( F \)-invariant according to \[8\] Remark 2.3. Since \( (p(j, i, k) \subset P(i) \subset P(j) \), we have \( F(p(j, i, k)) \in P(i) \) and we obtain \( \Delta(k') \subset \Delta(k) \) if and only if \( k' < k \).

3. Generators of the characteristic tilting module

In this section we consider a \( 1 \)-quasi-hereditary algebra \( A = KQ/\mathcal{I} \) for which the corresponding Ringel-dual \( R(A) \) is also \( 1 \)-quasi-hereditary. According to \[8\] Theorem 6.1] this is exactly the case if the factor algebra \( A(i) := A/\left(A \left( \sum_{t \in Q_0 \setminus \Lambda(i)} e_t \right) A \right) \) is \( 1 \)-quasi-hereditary for every \( i \in Q_0 \) (for \( A(i) \)-modules we use the index \( i \)).

The properties of a \( 1 \)-quasi-hereditary algebra \((A, \leq)\) (with \( \{1\} = \min\{Q_0(A), \leq\} \) yields the following: If \( x \in A \) generates \( P_A(1) \), then \( p(1,k,k) \cdot x \) generates the submodule \( P_A(k) \) of \( P_A(1) \) for any increasing path \( p(1,k,k) \) from 1 to \( k \) (see \[8\] Remark 3.1]).

Let \( A(i) \) be \( 1 \)-quasi-hereditary for all \( i \in Q_0 \). The direct summand \( T(i) \) of the characteristic tilting \( A \)-module \( T \) is a local submodule of \( P(1) \) with \( \text{top} \{ T(i) \cong S(1) \) (see \[8\] Theorem 5.1]). There exists \( (i, 1) \in P(1) \) which generates \( T(i) \) for all \( i \in Q_0 \). Because \( i \in Q_0(A(i)) = \Lambda(i) \) is maximal, for \( A(i) \)-module \( T(i) \) we have \( P(i)(1) \cong I(i)(1) \cong T(i) \cong T(i)(i) \) (see \[8\] Lemma 5.2]). Since \( P(i)(1) \cong (t(i, 1)) \), for any \( k \in \Lambda(i) \) we obtain that

\[
t(i, k) := p(1,k,k) \cdot t(i, 1) \ \text{generates the submodule} \ P(i)(k) \ \text{of} \ P(i)(1).
\]

Since \( t(i, k) \in P(1,k) \), we have \( t(i, k) = \sum_{t \in \Lambda(i)} c_t \cdot p(1,l,k) \) for some \( c_t \in K \) (see Section 2 (1)). In particular, for any \( j \in \Lambda(i) \) the algebra \( A(i) \) is a factor algebra of \( A(j) \), namely \( A(i) = (A(j))(i) \), thus \( T(i) \) is a direct summand of the characteristic tilting \( A(j) \)-module \( T(j) = \bigoplus_{k \in \Lambda(i)} T(j)(k) \) with \( T(i) \cong T(j)(i) \). The next theorem shows particular features of \( t(i, k) \) and of \( A \)-modules generated by \( t(i, k) \).

3.1 Theorem. Let \( A \) and \( R(A) \) be \( 1 \)-quasi-hereditary and \( t(i, k) \) be an element defined as above. Then the following hold:

(1) The set \( \{ t(i, k) \mid i \in Q_0, k \in \Lambda(i) \} \) is a \( K \)-basis of \( P(1) \).

(2) \( \{ t(i,k) \mid i \in Q_0, k \in \Lambda(i) \} = \{ \text{local submodules of } P(1) \text{ in } \mathfrak{F}(\Delta) \} \).

Let \( j \in Q_0 \), then a local submodule of \( P(j) \) in \( \mathfrak{F}(\Delta) \) is isomorphic to some \( \langle t(i, k) \rangle \).

(3) Let \( F \in \text{End}_A(P(1)) \), then \( \langle t(i, k) \rangle \) is \( F \)-invariant for all \( i, k \in Q_0 \) with \( k \in \Lambda(i) \).

Proof. (1) Obviously, \( t(i, k) \) belongs to \( P(1)_k \) for all \( i \in \Lambda(k) \). Let \( r = \left| \Lambda(k) \right| \) and \( \mathcal{L}(k) := \{ (i_1, i_2, . . . , i_r) \mid i_t \in \Lambda(k), i_v \neq i_t, 1 \leq v < t \leq r \} \) and \( (i_1, . . . , i_t, . . . , i_r) \in \mathcal{L}(k) \).

By induction on \( t \) we show that \( t(i_1, k), . . . , t(i_r, k) \) are linearly independent. This implies that the set \( \{ t(i, k) \mid i \in \Lambda(k) \} \) is a \( K \)-basis of \( P(1)_k \), since \( \dim_K P(i)_k = r \) (see \[8\] Lemma
2.1): For \( t = 1 \) we have \( i_1 = k \), thus \( t(i_1, i_1) \neq 0 \) is linearly independent. By definition of \( \mathcal{L}(k) \) we have \( i_t \not\in i_{t+1} \), thus \( [t(i_t, k)) : S(i_{t+1})] = [P(i_t)(k) : S(i_{t+1})] = 0 \) for all \( 1 \leq l \leq t \), however \([t(i_{t+1}, k) : S(i_{t+1})] = [P(i_{t+1})(k) : S(i_{t+1})] = 1 \), since \( A(i_{t+1}) \) is 1-quasi-hereditary. Hence \( t(i_{t+1}, k) \) is a submodule of \( P(i_{t+1}) \) and consequently \( t(i_{t+1}, k) \not\subseteq \text{span}_K \{ t(i_1, k), \ldots, t(i_t, k) \} \).

(2) Hence \( \mathcal{L}(k) \) implies \( i_1 \leq l \leq i_t \) for all \( l \in \{ i_1, \ldots, i_t \} \) and \( \Lambda(i_t) \cap \{ i_{t+1}, \ldots, i_r \} = \emptyset \). Thus \( \Lambda(i_1), \ldots, i_t = \Lambda^{(k)}(\Lambda(i_1)) = \Lambda^{(k)}(\Lambda(i_t)) = \emptyset \). According to \( \mathcal{L}(k) \) we have \( \Lambda(i_t) \) is a submodule of \( P(i_t) / \left( \sum_{t \in \Lambda(i_t) \cap \{ i_{t+1}, \ldots, i_r \}} P(l) \right) \approx P(i_t)(k), \) thus \( M = \langle t(i_t, k) \rangle . \) Since \( \Psi(\Lambda(i_t)) \subseteq \Psi(\Delta) \), we have \( " \subseteq " \).

Any submodule of \( P(j) \) is isomorphic to a submodule of \( P(1) \), since \( P(j) \approx P(1) \) for all \( j \in Q_0 \). Thus any local submodule of \( P(j) \) in \( \Psi(\Delta) \) is a submodule of \( P(1) \) in \( \Psi(\Delta) \).

(3) First, we show that any \( \Lambda(i) \)-submodule \( M \) of \( P(1) \) is contained in \( \langle t(i, 1) \rangle \approx P(1)(1) \approx T(i) \). Let \( i \neq n \). We have \( P(1) / T(i) \rangle \in \Psi(\Delta) \), since the category \( \Psi(\Delta) \) is closed under cokernels of injective maps (see \( [9] \)). If \( P(1) / T(i) \rangle \neq 0 \), then \( t \in Q_0 \setminus \Lambda(i) \) because \( P(1) : \Delta(t) \rangle = 1 \) for all \( t \in Q_0 \) and \( (T(i) : \Delta(t) \rangle = (I(i) : \Delta(t) \rangle = 1 \) for all \( t \in \Lambda(i) \). Thus \( \text{soc}\{P(1) / T(i) \rangle \} = \sum_{t \in Q_0 \setminus \Lambda(i)} S(t) \). Consequently, for any submodule \( N \) of \( P(1) \) with \( \langle t(i, 1) \rangle \approx T(i) \rangle \subseteq N \) we have \( \text{soc}\{N / S(t) \rangle \neq 0 \) for some \( t \in Q_0 \setminus \Lambda(i) \), i.e. \( N \) is not a \( \Lambda(i) \)-module. Thus \( \langle t(i, 1) \rangle \) is the largest \( \Lambda(i) \)-submodule of \( P(1) \).

Let \( M \) be a submodule of \( P(1) \) with \( M : S(t) = 0 \) for all \( t \in Q_0 \setminus \Lambda(i) \), then for any \( F \in \text{End}_A(P(1)) \) it is \( F(M) : S(t) = 0 \) for all \( t \in Q_0 \setminus \Lambda(i) \). We have \( F(M) \subseteq \langle t(i, 1) \rangle \). Since \( A(i) \) is 1-quasi-hereditary, any submodule \( L \) of \( P(i)(1) \) with \( \text{top} L \approx S(k) \) is contained in \( P(i)(k) \approx \langle t(i, k) \rangle \) (see \( [8] \) Lemma 2.2). Thus \( F(\langle t(i, k) \rangle) \subseteq \langle t(i, k) \rangle \) for every \( F \in \text{End}_A(P(1)) \), since \( F(\langle t(i, k) \rangle) \) is a submodule of \( P(i)(1) \) with \( \text{top} F(\langle t(i, k) \rangle) \approx S(k) \).

A similar relationship to that between the modules generated by \( p(1, i, k) \) and the posets \( (\Lambda(i), \leq) \) as well as \( (\Lambda^{(k)}, \geq) \) in Lemma 2.2 exists also between the submodules generated by \( t(i, k) \) and \( (\Lambda^{(k)}, \leq) \) as well as \( (\Lambda(i), \geq) \): The Hasse diagrams of the sets \( \{ (t(i, k)) : i \in \Lambda^{(k)} \}, \subseteq \) and \( \{ (t(i, k)) : k \in \Lambda(i) \}, \leq \) as well as the diagrams of \( \{ (t(i, k)) : k \in \Lambda(i) \}, \subseteq \) and \( \{ (t(i, k)) : k \in \Lambda(i) \}, \geq \) coincide.

3.2 Lemma. Let \( A \) and \( R(A) \) be 1-quasi-hereditary and \( t(i, k) \) be an element defined as above. Then for all \( i, i', k, k' \in Q_0 \) it is

(a) \( \langle t(i', k) \rangle \subseteq \langle t(i, k) \rangle \) if and only if \( i' < i \).

(b) \( \langle t(i, k') \rangle \subseteq \langle t(i, k) \rangle \) if and only if \( k' > k \).

Proof. (a) If \( \langle t(i', k) \rangle \subseteq \langle t(i, k) \rangle \), then \( [P(i')(k) : S(i')] \neq 0 \). Thus \( [P(i)(k) : S(i')] \neq 0 \) so we get \( i' \in \Lambda(i) \) (see \( [8] \) Lemma 5.2 (c)). Since \( \langle t(i', k) \rangle \neq \langle t(i, k) \rangle \), we obtain \( i \neq i' \).
On the other hand, if \( i' < i \), then \( T(i') \) is the direct summand \( T_{(i)}(i') \) of the characteristic \( \Lambda(i) \)-module and therefore a submodule of \( P_{(i)}(1) \), since \( \Lambda(i) = \Lambda(i) \) \( (i') \) is 1-quasi-hereditary (see [8, Theorem 5.1]). We have \( P_{(i)}(1) \to P_{(i')}(1) \cong T(i') \cong T_{(i)}(i') \to P_{(i)}(1) \), thus \( P_{(i)}(k) \) is a local submodule of \( P_{(i)}(1) \) with top isomorphic to \( S(k) \). According to [8, Lemma 2.2], we obtain \( P_{(i)}(k) \subseteq P_{(i)}(k) \). Since \( i' < i \) we have \( [P_{(i')}(k) : S(i)] = 0 \) and \( [P_{(i)}(k) : S(i)] = 1 \). Therefore \( P_{(i')}(k) \neq P_{(i)}(k) \) and consequently \( t(i', k) \subseteq t(i, k) \).

(b) According to [8, Lemma 2.2] we have \( P_{(i)}(k') \subseteq P_{(i)}(k) \) if and only if \( k' > k \). □

Algebras associated to the blocks of category \( \mathcal{O}(\mathfrak{g}) \) are Ringel self-dual (see [10]). The 1-quasi-hereditary algebra \( A \) given by quiver and relations in the Example 1 corresponds to the regular block of \( \mathcal{O}(\mathfrak{sl}_2(\mathbb{C})) \), thus the Ringel-dual of \( A \) is also 1-quasi-hereditary. Analogously to the previous picture we can give the inclusion diagram of the submodules \( P_{(i)}(k) \cong \langle t(i, k) \rangle \) of \( P(1) \), in view of Lemma 3.2. The meaning of circles and arrows is the same as in the last picture (the arrows \( p \to q \) and \( p \to q \) illustrates Lemma 3.2(a) and (b) respectively). We additionally pointed out the elements which generates the direct summands of the characteristic tilting module, projective and standard submodules of \( P(1) \). The notation \( M \leftrightarrow p \) resp. \( p \leftrightarrow M \) means \( M = (p) \). These \( \dim_K P(1) = 19 \) submodules of \( P(1) \) are the local submodules of \( P(1) \) from \( \mathfrak{f}(\Delta) \).

The element \( t(i, k) \) is the following linear combination of \( \{ p(l, k) := p(1, l, k) \mid l \in \Lambda^{(k)} \} \):

\[
\begin{align*}
t(2, 1) &= p(5, 1), & t(4, 3) &= p(5, 3), \\
t(3, 1) &= p(4, 1), & t(5, 3) &= p(4, 3) + p(5, 3), \\
t(4, 1) &= p(3, 1), & t(4, 2) &= p(4, 2) + p(5, 2), \\
t(5, 1) &= p(2, 1), & t(5, 2) &= p(4, 2), \\
t(i, i) &= p(6, i), & t(i, k) &= p(i, i) & \text{for every } i \in Q_0.
\end{align*}
\]

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