The dual model for an Ising model with nearest and next-nearest neighbors

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Abstract
We construct and analyze a dual model to the next-nearest-neighbor Ising (NNNI) model on the rectangular lattice. The Hamiltonian of the dual model turns out to contain two- and four-spin interactions. The free fermion approximation suggests that an increase in the critical temperature of the dual model caused by the four-spin interactions is limited to a finite range.

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1. Introduction

Duality [1–4] is an important symmetry of Ising models, as it implies a relationship between the partition functions of two Ising models with Hamiltonians $H_1(J)$ and $H_2(K)$

$$\frac{Z_1(H_1(J))}{f_1(J)} = \frac{Z_2(H_2(K))}{f_2(K)},$$

(1.1)

where $f_1$ and $f_2$ are some nonsingular functions of interaction constants $J, K$ appearing in Hamiltonians $H_1$ and $H_2$, respectively. Similar functional relations between the partition functions of two different Ising models can also be obtained with the help of the decoration, star–triangle or star–square transformations [4]. Duality combined with these transformations enables one to derive the expressions for the critical temperature for a wide range of Ising models. It should be stressed, however, that in the case of duality the total number of spin interactions must be the same for both Hamiltonians, $H_1$ and $H_2$, which allows one to introduce the concept of a dual lattice. For example, the triangular lattice turns out to be dual to the honeycomb lattice and the rectangular lattice is dual to itself. Using the concept of the dual lattice, the duality relation can be interpreted as a relationship between high-temperature (low-temperature) expansion for the model with Hamiltonian $H_1$ and the low-temperature (high-temperature) expansion for the model described by $H_2$.

The concept of duality has been extended by Wegner [5] for a wider class of Ising models $M_{dn}$ on d-dimensional lattices characterized by a number $n = 1, 2, \ldots, d$, where $n = 1$ corresponds to the Ising model with a two-spin interaction. In this paper we construct and
investigate analytically the dual model to the Ising model on a rectangular lattice with the nearest- and next-nearest-neighbor interactions (NNNI model) [6–9]. Although this simple extension of the Onsager model has been widely investigated for decades now, no analytical expression for the partition function of the model has been found, nor has its dual model been presented. To this end we use the usual algebraic approach and express the partition function as the trace of a power of the transfer matrix. In section 2 we employ the classic 2D Ising model on a rectangular lattice to introduce the formalism and some basic formulas necessary to investigate the duality of a more complex model in section 3. In particular, we show how the self-dual symmetry of this model can be related to the similarity transformation (U) of the transfer matrix. We show that the difference between two forms of the transfer matrix can be related to a change of the direction in which the matrix was constructed and present the explicit form of U. In section 3 we apply the similarity transformation to the transfer matrix of the NNNI model. We show that there exists a simple Ising model for which the transformed matrix is the transfer matrix. The Hamiltonian of this model contains a two-spin nearest-neighbor interaction between nodes of the brick-wall lattice (which is topologically equivalent to a honeycomb lattice) and additional four-spin interactions (henceforth this model will be called the (2 + 4) BWI model, where BWI stands for ‘brick-wall Ising’). The number of the four-spin interactions is equal to the number of bricks in the lattice. Since the two Hamiltonians have the same number of all interactions, the model can be regarded as the dual model to the NNNI model. As a result we obtain the duality relation between the partition functions of these two models. Moreover, we show that the dual model becomes self-dual in the presence of an external magnetic field. In section 4 the critical temperature of the (2 + 4) BWI model is investigated. We show that an increase in the critical temperature of the dual model caused by the four-spin interactions is limited to a finite range. We derive the equations for both the lower and upper limits of this temperature range and find that they depend on the magnitude of the two-spin interactions only.

2. Self-duality of the Ising model on the rectangular lattice

Consider the classical Ising model with the nearest-neighbor interaction constants $J_1$ and $J_2$ (for simplicity, we assume that $J = J_1 = J_2$) on a rectangular lattice with $N$ columns and $2M$ rows. We assume the cyclic boundary conditions in the direction of the transfer matrix action and we adjust the boundary conditions in the perpendicular direction to the transformations performed on the transfer matrix to get the result in a closed form (we adopt this convention throughout this paper). The partition function for this model reads [3, 10, 11]

$$Z_{N,2M}(\kappa) = \text{Tr} V^N = (2 \sinh \kappa)^{MN} \text{Tr}(e^{iA} e^{iC})^N,$$

(2.1)

where $\kappa = \beta J = J/k_B T$, $\tilde{\kappa} = -\frac{1}{2} \ln \tanh \kappa$, and $A$ and $C$ are $2^{2M} \times 2^{2M}$ matrices defined by

$$A = \sigma^x_1 + \sum_{j=1}^{2M-1} \sigma^x_j \sigma^x_{j+1},$$

(2.2)

$$C = \sum_{j=1}^{2M} \sigma^z_j,$$

(2.3)

with

$$\sigma^\alpha_{j} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma^\alpha \otimes 1 \otimes \cdots \otimes 1.$$  (2.4)
where the symbol \( \otimes \) denotes the tensor product, \( \mathbb{1} \) is the \( 2 \times 2 \) unit matrix, and \( \sigma^\alpha (\alpha = x, y, z) \) represent the Pauli matrices

\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.5}
\]

and \( j \) runs through the \( 2M \) nodes of a lattice column.

It turns out that \( A \) and \( C \) are related to each other by a similarity transformation \( U \),

\[
UAU^{-1} = C, \quad UCU^{-1} = A. \tag{2.6}
\]

Relations of this type were found to be essential in proving the duality or self-duality of many Ising models, including the self-duality of the 1D Ising model with an external transverse magnetic field \([12–14]\).

Applying (2.6) under the trace operator in (2.1), one obtains

\[
Z_{N;2M}(\kappa) = (2 \sinh 2\kappa)^{MN} \text{Tr}(U(e^{\kappa A}e^{\kappa C})^N U^{-1})
\]

\[
= (2 \sinh 2\kappa)^{MN} \text{Tr}(e^{\kappa (C+1) - UC(U^{-1})})^N
\]

\[
= (2 \sinh 2\kappa)^{MN} \text{Tr}(e^{\kappa C+1})^N. \tag{2.7}
\]

Using the cyclicity of the trace operator, (2.7) can be rewritten in a well-known form

\[
Z_{N;2M}(\kappa) = (2 \sinh 2\kappa)^{MN} \text{Tr}(e^{\kappa A}e^{\kappa C})^N
\]

\[
= \left( \frac{2 \sinh 2\kappa}{\sinh 2\kappa} \right)^{MN} \text{Tr}(e^{\kappa A}e^{\kappa C})^N
\]

\[
= \left( \frac{2 \sinh 2\kappa}{\sinh 2\kappa} \right)^{MN} Z_{N;2M}(\tilde{\kappa})
\]

\[
= (\sinh 2\kappa)^{2MN} Z_{N;2M}(\tilde{\kappa}), \tag{2.8}
\]

which proves the self-duality of the discussed model.

Equation (2.8) was usually derived by constructing the transfer matrix along different directions of the lattice, without deriving the explicit form of \( U \), which however will be necessary in our study further below. To obtain it, one can assume that it can be expressed as a product of two \( 2^{2M} \times 2^{2M} \) matrices

\[
U = SR, \tag{2.9}
\]

where \( R \) is a permutation matrix, whereas \( S \) exchanges Pauli matrices \( \sigma_i^x \) with \( \sigma_j^y \) and changes the signs of the interaction coefficients appropriately. A particularly simple form of \( R \) reads

\[
R = \prod_{j=1}^{2M-1} R_{j,j+1}, \tag{2.10}
\]

where

\[
R_{j,j+1} = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes r \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \tag{2.11}
\]

and

\[
r = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.12}
\]
It follows that $r^{-1} = r$, $[R_{j,j+1}, R_{j+1,j+2}] \neq 0$, and

$$R^{-1} = \prod_{l=1}^{2M-1} R_{2M-l,2M-(l-1)}. \quad (2.13)$$

The following relations

$$R_{l,l+1}\sigma_{l}^{x} R_{l,l+1}^{-1} = \sigma_{l+1}^{x}$$

$$R_{l,l+1}\sigma_{l}^{y} R_{l,l+1}^{-1} = \sigma_{l+1}^{y}$$

$$R_{l,l+1}\sigma_{l}^{z} R_{l,l+1}^{-1} = -\sigma_{l+1}^{z}$$

$$R_{l,l+1}\sigma_{l}^{z} R_{l,l+1}^{-1} = \sigma_{l+1}^{z}$$

$$R_{l,l+1}\sigma_{l}^{z} R_{l,l+1}^{-1} = -\sigma_{l+1}^{z}$$

$$R_{l,l+1}\sigma_{l}^{z} R_{l,l+1}^{-1} = \sigma_{l+1}^{z}$$

(2.14)

can be used to show how $A$ and $C$ transform under $R$.

$$R A R^{-1} = \sum_{j=2}^{2M} \sigma_{j}^{x},$$

$$R C R^{-1} = \sum_{j=1}^{2M-1} \sigma_{j}^{x} \sigma_{j+1} + \sigma_{2M}^{x}. \quad (2.15)$$

The cyclic boundary conditions in $A$ are mathematically troublesome. However, since the thermodynamics of the system does not depend on the boundary conditions in the thermodynamic limit, they can be neglected [12, 13]. As for $S$, it can be expressed as

$$S = P \prod_{j=1}^{2M} s_{j}, \quad (2.16)$$

where

$$s_{j} = \underbrace{1 \otimes 1 \otimes \cdots 1 \otimes s \otimes 1 \otimes \cdots \otimes 1}_{j-1} \otimes \underbrace{1 \otimes 1 \otimes \cdots 1 \otimes 1}_{2M-j},$$

(2.17)

with

$$s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (2.18)$$

whereas $\mathcal{P}$ is a transformation exchanging matrices $\sigma_{i}^{\alpha}$ with $\sigma_{2M-i+1}^{\alpha}$.

$$\mathcal{P} = \prod_{i=1}^{M} P_{i,2M-i+1}, \quad (2.19)$$

where

$$P_{i,j} = \frac{1}{2} \left( 1 + \sigma_{i}^{x} \sigma_{j}^{x} + \sigma_{i}^{y} \sigma_{j}^{y} + \sigma_{i}^{z} \sigma_{j}^{z} \right) \quad (2.20)$$

and

$$P_{i,j} \sigma_{i}^{\alpha} P_{i,j}^{-1} = \sigma_{j}^{\alpha}, \quad \alpha = x, y, z \text{ and } i, j = 1, 2, \ldots, 2M. \quad (2.21)$$

On applying the full transformation $U = SR$ to $A$ and $C$, one arrives at (2.6).
Figure 1. The Ising model with the nearest- and next-nearest-neighbor interactions on a rhomboidal lattice (the NNNI model). \( \bar{J}_1, \bar{J}_2, \) and \( \bar{J}_3 \) are the interaction constants for the diagonal, horizontal and vertical directions, respectively.

3. The dual model to the NNNI model

The transfer matrix for the Ising model with the nearest- and next-nearest-neighbor interactions was found in 1956 by Temperley [15], who constructed it along the diagonal of the basic rectangular lattice defined by the nearest-neighbor interactions. Although this matrix is not of the lowest possible rank, its form is particularly simple. We will use this matrix for the rhomboidal lattice presented in figure 1. While in most papers devoted to the NNNI model only two interaction constants are used [7, 8], \( \bar{J}_1, \bar{J}_2 = \bar{J}_3 \), here we consider the general case with three arbitrary constants controlling the ferromagnetic interactions \( \bar{J}_1, \bar{J}_2, \bar{J}_3 \) (\( \bar{J}_i < 0 \)).

The transfer matrix and the partition function for this model on the rhomboidal lattice consisting of \( 2M \times N \) nodes satisfy

\[
\frac{Z_{N,2M}}{(2 \sinh 2L_2)^{NM}} = \text{Tr}(e^{L_1A} e^{L_2B_{2Z}^{-1}} e^{L_3C_{2Z}^{-1}} e^{L_1A} e^{L_2B_{2Z}} e^{L_3C_{2Z}})^N,
\]

(3.1)

where \( L_i = \beta \bar{J}_i \) (\( i = 1, 2, 3 \)), \( L_2 = \frac{1}{2} \ln \tanh L_2 \), \( C_{2Z} \) and \( C_{2Z}^{-1} \) are matrices of size \( 2^{2M} \times 2^{2M} \), defined as follows,

\[
A = \sigma_1^x + \sum_{k=1}^{2M-1} \sigma_k^x \sigma_{k+1}^x,
\]

(3.2)

\[
B_{2Z} = \sigma_2^x + \sum_{k=1}^{M-1} \sigma_{2k+1}^x \sigma_{2k+2}^x,
\]

(3.3)

\[
B_{2Z}^{-1} = \sum_{k=1}^{M-1} \sigma_{2k-1}^x \sigma_{2k+1}^x + \sigma_1^x \sigma_{2M-1}^x \sigma_M^x,
\]

(3.4)

\[
C_{2Z} = \sum_{k=1}^{M} \sigma_{2k}^z,
\]

(3.5)

\[
C_{2Z}^{-1} = \sum_{k=1}^{M} \sigma_{2k-1}^z,
\]

(3.6)
with \( \mathbb{Z} \) denoting the set of even numbers. The correspondence between these matrices and the interaction constants are shown in figure 2. Similar transfer matrices for the NNNI model were already used in [16], without giving their explicit forms, however.

On applying the similarity transformation \( U \) to the matrices of the NNNI model we arrive at

\[
UAU^{-1} = \sum_{k=1}^{2M} \sigma_k^z = C,
\]

\[
UB_{2Z-1}U^{-1} = \sum_{k=2}^{M} \sigma_{2k-2}^z\sigma_{2k-1}^z + \sigma_1^z\sigma_M^z = D_{2Z},
\]

\[
UC_{2Z}U^{-1} = \sigma_1^x + \sum_{k=1}^{M-1} \sigma_{2k-1}^z\sigma_{2k+1}^x = A_{2Z},
\]

\[
UB_{2Z}U^{-1} = \sum_{k=1}^{M} \sigma_{2k-1}^z\sigma_{2k}^z = D_{2Z-1},
\]

\[
UC_{2Z-1}U^{-1} = \sum_{k=1}^{M} \sigma_{2k-1}^z\sigma_{2k}^z = A_{2Z-1},
\]

which also defines \( A_{2Z}, A_{2Z-1}, D_{2Z}, \) and \( D_{2Z-1} \). These relations lead to

\[
\frac{Z_{N,2M}}{(2 \sinh 2L)^{NM}} = \text{Tr} \tilde{V}^N,
\]

where

\[
\tilde{V} = e^{ijC_{2Z}} e^{iL_{2Z}} e^{iJ_{2Z}} e^{iL_{2Z-1}} e^{iJ_{2Z-1}}.
\]

Equation (3.12) ensures that if there exists an Ising model whose transfer matrix is equal to \( \tilde{V} \), this model will be dual to the NNNI model. This is a nontrivial requirement, as while the transfer matrix of any Ising model can be expressed as a product of exponential matrices, the product of exponential matrices is hardly ever the transfer matrix of an Ising model. In the

\[
\text{Figure 2. Correspondence between the two-spin interactions and the matrix operators.}
\]
case considered here, however, such a model exists and is described by the Hamiltonian
\[ H = - \sum_{\langle i, j \rangle} J_{ij} \sigma_i \sigma_j - J_4 \sum_{\langle i, j, k, l \rangle} \sigma_i \sigma_j \sigma_k \sigma_l \] (3.14)
defined on the brick-wall lattice [17], which is topologically equivalent to the honeycomb lattice. As explained in figure 3, $J_4$ is the four-spin interaction constant for the four spins occupying the right-hand side of a brick, and the values of the nearest-neighbor interaction constants $J_{ij}$ are non-zero only if $i$ and $j$ are on the same side of a brick, in which case they are equal $J_1$ and $J_2$ for interactions along the vertical and horizontal direction, respectively. All interaction constants are ferromagnetic in nature ($J_1, J_2, J_4 < 0$).

The dual lattice is composed of $2M$ horizontal chains, each containing $2N$ nodes. The partition function of the model satisfies
\[ Z_{2N,2M}(K_2, K_4, K_1) \]
\[
\frac{Z_{2M,2N}(K_2, K_4, K_1)}{2^{2MN}[\sinh^2 2K_2(e^{2K_4} \cosh^2 2K_2 - 1)]^{2MN/2}} = \text{Tr}(e^{\tilde{K}_2 C} e^{\tilde{K}_4 D^2} e^{K_1 A_2} e^{\tilde{K}_1 A_2 - 1} e^{K_1 A_{2-1}})^N, \] (3.15)
where $K_i = \beta J_i$, ($i = 1, 2, 4$) and
\[ \tilde{K}_2 = -\frac{1}{4} \ln \frac{\cosh 2K_2 - e^{-2K_4}}{\cosh 2K_2 + e^{-2K_4}}, \] (3.16)
\[ \tilde{K}_4 = -\frac{1}{4} \ln \frac{\cosh^2 2K_2 - 1}{\cosh^2 2K_2 - e^{-4K_4}}. \] (3.17)

The transfer matrix in (3.15) acts on the vertical groups of nodes and advances the partition function in the horizontal direction [18]. Its explicit form can be obtained from Onsager’s classical transfer matrix for the rectangular lattice [3], see (2.1). First, we add the four-spin interactions that modify only two horizontal two-spin bonds lying one above the other. Next, half of the vertical two-spin interactions connecting the horizontal chains has to be removed. While adding the four-spin interactions to the transfer matrix, we can use
a relationship
\[ \Delta^{1/4} e^{i2(\sigma_i + \sigma_{i+1})} e^{i2\sigma_i \sigma_{i+1}} = \cosh K_{ik} \sigma_i \sigma_{i+1} + \sinh K_{ik} \sigma_i \sigma_{i+1}, \]  
where \( \Delta = \frac{1}{4} \sinh^2 2K_2 (e^{4K_2} \cosh^2 2K_2 - 1) \) and
\[ c_k = \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ \c_k & \c_k & \cdots & \c_k & \cdots & \c_k \\ & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \]  
(3.19)
and \( k = 1, \ldots, 2M \). Finally, we find the duality relationship between the NNNI model, figure 1, and the (2 + 4) BWI model, figure 3,
\[ \frac{Z_{N,2M}(L_1, L_2, L_3)}{(2 \sinh 2L_2)^{MN}} = \frac{Z_{2N,2M}(K_2, K_4, K_1)}{2^{2MN}[\sinh^2 2K_2 (e^{4K_2} \cosh^2 2K_2 - 1)]^{MN/2}}, \]  
(3.20)
where the relations between the interaction constants for both models are given by
\[ L_1 = K_2, \quad L_3 = \bar{K}_4, \quad \bar{L}_2 = K_1. \]  
(3.21)

4. The critical temperature for the (2 + 4) BWI model

While the Ising model on the brick-wall lattice can be solved exactly [19], the addition of the four-spin interaction makes it intractable analytically. However, it turns out that it is still possible to draw some important conclusions about the way in which the critical temperature of the Ising model defined by Hamiltonian (3.14) depends on the strength of the four-spin interactions.

In the Ising model with nearest- and next-nearest-neighbor interactions (NNNI model, figure 1) there are two important limiting cases: \( \bar{J}_1 = 0 \) and \( \bar{J}_1 = 0 \). In the former case the NNNI model reduces to the Ising model with nearest-neighbor interactions on the triangular lattice with the interaction constants \( \bar{J}_1, \bar{J}_2 \). Since in this case \( L_3 = \beta \bar{J}_1 = 0 \), matrices \( e^{i\beta \bar{J}_3} \) and \( e^{i\beta \bar{J}_2} \) appearing in (3.13) become the identity matrices and the critical point of this model can be determined exactly. It is worth noting that the duality transformation converts the transfer matrix of this model into the transfer matrix of the Ising model with two nearest-neighbor interactions on a honeycomb lattice equivalent to the brick-wall lattice. This explains why the dual model to NNNI model is defined on the brick-wall lattice. In the second case \( \bar{J}_1 = 0 \), the NNNI model splits into two identical independent self-dual Ising models with nearest-neighbor interactions on the rectangular lattice. Similarly to the previous case, matrices \( e^{iA} \) in (3.13) become the identity matrices and the critical point for this model is exactly determinable.

Equation (3.22), which describes relationships between the interactions of the NNNI and (2 + 4) BWI models, reduces to
\[ 0 = L_3 = \bar{K}_4 = -\frac{1}{4} \ln \frac{\cosh^2 2K_2 - 1}{\cosh^2 2K_2 - e^{-4K_1}}, \]  
(4.1)
for \( \bar{J}_1 = 0 \) and to
\[ 0 = L_1 = \bar{K}_2 = -\frac{1}{4} \ln \frac{\cosh 2K_2 - e^{-2K_1}}{\cosh 2K_2 + e^{-2K_1}}, \]  
(4.2)
Figure 4. Relation between the critical temperature ($kT_C$) and the four-spin interaction constant ($J_4$) in the dual model with $J_1 = J_2 = 1$. The solid and dotted lines are the solution of equations (4.3) and (4.4), respectively, and the crosses indicate the possible location of the phase transition. $kT_{CBW}$ is the critical temperature for the Ising model on a brick-wall lattice and $kT_{C_{max}}$ is the maximum critical temperature for the ferromagnetic case of the $(2 + 4)$ BWI model.

The critical temperatures satisfy

$$e^{-4 \frac{J_4}{kT_C}} = \cosh^2 \left( \frac{2J_2}{kT_C} \right) \frac{\tan^2 \left( \frac{J_1}{kT_C} \right)}{\tan^2 \left( \frac{J_2}{kT_C} \right) + 1},$$  \hspace{1cm} (4.3)

$$e^{-4 \frac{J_4}{kT_C}} = \cosh^2 \left( \frac{2J_2}{kT_C} \right) - e^{4 \frac{J_1}{kT_C} \sinh^2 \left( \frac{2J_2}{kT_C} \right)},$$  \hspace{1cm} (4.4)

for $J_1 = 0$ and $J_1 = 0$, respectively.

Equation (4.1) is equivalent to $K_4 = 0$, whereas (4.2) can be satisfied only in the limit of $K_4 \to \infty$. The critical temperatures satisfy

$$e^{-4 \frac{J_4}{kT_C}} = \cosh^2 \left( \frac{2J_2}{kT_C} \right) \frac{\tan^2 \left( \frac{J_1}{kT_C} \right)}{\tan^2 \left( \frac{J_2}{kT_C} \right) + 1},$$

which yields $kT_{C_{max}} \approx 3.28204$. The fact that (4.4) has only the left-hand side asymptote at the critical point defined by $K_4 \to \infty$ is in contradiction to the Griffiths inequalities [20], which imply that if an additional ferromagnetic interaction is introduced into a ferromagnetic Ising model, the critical temperature $T_C$ will increase with the magnitude of this additional interaction. Therefore, this solution has to be rejected as nonphysical. Our preliminary computer simulations (which will be discussed elsewhere) suggest that the relation between $J_4$ and $T_C$ is located along the line marked with crosses in figure 4, with the right vertical asymptote at the point given by (4.5).

Our considerations show that the additional four-spin interaction in the Ising model on a brick-wall lattice does not yield an infinite increase of the critical temperature. Instead, the critical temperature is always limited by a condition involving two-spin interactions only.

Figure 4 presents a sketch of the relation between $J_4$ and the critical temperature $kT_C$ in a particular case of $J_1 = J_2 = 1$. The solid line represents the asymptotic solution for the case $K_4 \to 0$ obtained from (4.3), and the dotted line depicts the asymptotic solution of (4.4) obtained for $K_4 \to \infty$. The solution of (4.4) has only the left-hand vertical asymptote at the temperature $kT_{C_{max}}$ given as the solution to

$$\cosh \left( \frac{2}{kT_{C_{max}}} \right) = \exp \left( \frac{2}{kT_{C_{max}}} \right) \sinh \left( \frac{2}{kT_{C_{max}}} \right),$$

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order to confirm this qualitative result obtained for a particular choice of \( J_1 \) and \( J_2 \), we plan to perform detailed computer simulations of the model.

5. Self-duality of the \((2 + 4)\) BWI model in the presence of an external magnetic field

As in other pairs of Ising models, the inclusion of an external magnetic field makes the duality relationship between the NNNI and the \((2 + 4)\) BWI models cease to exist: duality relations (3.21)–(3.22) were obtained only in the absence of an external magnetic field. However, the Ising model described by the Hamiltonian (3.14) has an additional internal symmetry. If one adds to the Hamiltonian an interaction with an external magnetic field \(-H \sum_i \sigma_i\), then the partition function will take the form

\[
Z_{2N,2M}(K_2, K_4, h, K_1) = \frac{\text{Tr}(V)^N}{2^{2MN} \sinh^2(2K_2(\cosh^2(2K_2 - 1))^{MN/2}}
\]

where

\[
V = e^{\tilde{K}_2 C} e^{\tilde{K}_4 D} e^{hE} e^{K_1 A \tilde{E}} e^{\tilde{K}_2 C} e^{\tilde{K}_4 D} e^{hE} e^{K_1 A \tilde{E}}
\]

is the transfer matrix, \( h = \beta H \), and \( E \) is a matrix defined as

\[
E = \sum_{k=1}^{2M} \sigma_k^x
\]

Let

\[
\mathcal{S} = \prod_{i=1}^{2M} s_i
\]

where \( s_i \) are given by (2.17). This operator exchanges \( \sigma_i^x \) with \( \sigma_i^z \). Applying it as a similarity transformation for \( V \) leads to

\[
V' = \mathcal{S} V \mathcal{S}^{-1}
\]

where

\[
V' = e^{\tilde{K}_2 C} e^{\tilde{K}_4 D} e^{hE} e^{K_1 A \tilde{E}} e^{\tilde{K}_2 C} e^{\tilde{K}_4 D} e^{hE} e^{K_1 A \tilde{E}}
\]

Thus, \( V' \) can be interpreted as the transfer matrix of the \((2 + 4)\) BWI model in which the interactions have been changed as follows,

\[
\begin{align*}
 h &\rightarrow \tilde{K}_2, \\
 K_1 &\rightarrow \tilde{K}_4.
\end{align*}
\]

The second difference is that while in the initial Ising model the four-spin interactions act on the right-hand side of the bricks, the four-spin interactions appear on their left-hand sides of the transformed model. The self-duality relation for this model is described by the equation

\[
Z_{2N,2M}(K_2, K_4, h, K_1) = \alpha Z_{2N,2M}(\tilde{K}_2, \tilde{K}_4, \tilde{h}, \tilde{K}_1)
\]

where

\[
\alpha = \left[ \frac{\sinh^2(2K_2(\cosh^2(2K_2 - 1))^{MN/2}}{\sinh^2(2h(\cosh^2(2h - 1))^{1/2}} \right]^{\frac{MN}{2}}
\]

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and

\[ \tilde{K}_2 = -\frac{1}{4} \ln \frac{\cosh 2K_2 - e^{-2K_1}}{\cosh 2K_2 + e^{-2K_1}}, \]

\[ \tilde{K}_4 = -\frac{1}{4} \ln \frac{\cosh^2 2K_2 - 1}{\cosh^2 2K_2 - e^{-4K_1}}, \]

\[ \tilde{K}_1 = -\frac{1}{4} \ln \frac{\cosh^2 2h - 1}{\cosh^2 2h - e^{-4K_1}}, \]

\[ \tilde{h} = -\frac{1}{4} \ln \frac{\cosh 2h - e^{-2K_1}}{\cosh 2h + e^{-2K_1}}. \]

(5.10)

Unfortunately the model under consideration is a ferromagnetic one, so the phase transition in an external magnetic field does not exist. Self-duality only enables one to find a number of interesting relations between correlation function for the (2 + 4) BWI model.

6. Summary

The main goal of this paper was to find and investigate the dual model for the 2D Ising model with the isotropic nearest- and anisotropic next-nearest-neighbor interactions on a rectangular lattice. The dual model turned out to be a 2D Ising model with two-spin nearest-neighbor anisotropic interactions and additional four-spin interactions on the brick-wall lattice. The appearance of the four-spin interactions is associated with the presence of non-planar next-nearest-neighbor interactions in the original model. The way we constructed the dual model is quite general, does not require the introduction of the dual lattice concept and can be applied to other Ising models.

Investigation of the impact of the four-spin interactions on the critical temperature of the dual model revealed that while the four-spin interactions can increase the critical temperature of the model, this increase is restricted to a finite temperature range. Moreover, the limits of this range are bounded by the two-spin interaction constants of the dual model. This rather unexpected result needs to be analyzed further, e.g. by computer simulation, for example, using Landau’s approach [21]. Computer simulations are also necessary to investigate the exact location of the phase transition as well as its critical properties, including the universality class. They will also be useful in relating the results for the dual model to the properties of the original Ising model.

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