Low-energy QCD: Chiral coefficients
and the quark-quark interaction

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Abstract

A detailed investigation of the low-energy chiral expansion is presented within a model truncation of QCD. The truncation allows for a phenomenological description of the quark-quark interaction in a framework which maintains the global symmetries of QCD and permits a $1/N_c$ expansion. The model dependence of the chiral coefficients is tested for several forms of the quark-quark interaction by varying the form of the running coupling, $\alpha(q^2)$, in the infrared region. The pattern in the coefficients that arises at tree level is consistent with large $N_c$ QCD, and is related to the model truncation.

I. INTRODUCTION AND SUMMARY

Phenomenological approaches to quantum chromodynamics (QCD) continue to provide useful intuition into the nature of the strong interaction, and compliment the more direct evaluation via lattice techniques. The utility of these treatments is perhaps most apparent in the study of chiral observables where lattice calculations are subject to large uncertainties due to the extrapolation to light quarks. At low energies, this aspect of QCD is characterized
by chiral perturbation theory (χPT) [1–3]. The coefficients of the chiral expansion are input parameters to χPT, and their values are determined from experimental observables. These coefficients therefore provide a convenient representation of a large body of data relevant to low-energy QCD.

Quantum chromodynamics is formulated in terms of unobserved degrees of freedom – quarks and gluons. The presence of these fundamental constituents of hadrons is inferred through the analysis of deep inelastic lepton scattering. Nevertheless, the successful application of effective theories such as χPT to a broad range of low-energy strong-interaction phenomena suggests that quarks and gluons may be replaced by local effective hadronic degrees of freedom in the low-energy domain. This success is largely due to the separation in the hadron spectrum between the Goldstone modes and higher mass states. At intermediate energies it is not clear that such a description remains effective [4], nor is it clear that explicit quark and gluon degrees of freedom are essential. An ideal perspective on this problem would be provided if composite hadron fields and their interactions could be modeled in a manageable form in terms of the point fields of QCD. Functional integral calculus formulates this problem as an exercise in changing the variables of integration from quark and gluon fields to hadron fields [5]. An obvious advantage of this approach is that the effective hadron-field interactions retain knowledge of their subhadronic origin.

The notion that such a change of variables exists for QCD in the low energy domain is implicit in the success of the above mentioned hadronic formulations. The explicit operation of changing variables allows the underlying dynamics of the microscopic description to influence interactions at the macroscopic level. The goal of this “matching” program is then to perform the appropriate change of integration variables in the functional integral formulation of QCD;

\[
\int D\bar{q}DqDA \exp \left( -S[\bar{q}, q, A] \right)
= \int D\pi...D\bar{N}DN... \exp \left( -S'[\pi, ..., \bar{N}, N, ...] \right).
\]

(1.1)

Significant progress toward this goal has recently been achieved [5].
As the local integration variables in (1.1) are identified with the bare hadron fields, their effective interactions are simultaneously defined. This process is the result of an expansion about the chiral symmetry breaking ground state [6], and an allocation of internal and center-of-mass dynamics. The latter is prescribed by the normal-mode expansion of the free kinetic operator of the composite particle in a manner analogous to the interaction picture of standard quantum field theory [7,8]. The tree-level effective interactions thereby obtained occur through a dynamically regulated “constituent-quark” loop and thus reflect the underlying description. The low-momentum (gradient) expansion of these tree-level nonlocalities produce finite coefficients, and for the Nambu-Goldstone modes is structurally consistent with $\chi$PT.

The direct derivation of the chiral coefficients from QCD is presently inaccessible. However, they can be derived from a class of chiral invariant quark-based field-theory models of QCD which are distinguished by the form of the quark-quark interaction. In this investigation the sensitivity of the chiral coefficients to the underlying quark-quark interaction is tested for a variety of forms to determine their utility in constraining these models. Previous work [9,10] has demonstrated how these techniques can be used to extract the second-order coefficients and those at fourth order associated with $\pi - \pi$ scattering. In the following we extend the previous work by calculating the chiral coefficients $L_1$–$L_8$, and further by investigating the sensitivity of these coefficients to the infrared form of the quark-quark interaction. The coefficients $L_9$ and $L_{10}$ are left for a future investigation, however work in that direction has been initiated [11].

We find that in general, for momentum-space interactions of the form $\alpha_s(q^2)/q^2$, the reproduction of the accepted values of the chiral coefficients requires the running coupling, $\alpha_s(q^2)$, to have a sufficiently large integrated strength to produce dynamical chiral symmetry breaking ($D\chi$SB), but is otherwise not acutely dependent on the detailed form. This constraint is implemented here by fixing the value of the pion decay constant, $f_\pi$, which determines the overall scale. The low-momentum strength of $\alpha_s(q^2)$ is then implied by the scale at which the infrared phenomenology is matched to the known ultraviolet form. This scale
is allowed to vary to investigate the sensitivity of the coefficients, and several two-parameter models for $\alpha_s(q^2)$ are employed.

We also find in particular that the coefficients $L_5$ and $L_8$ are most sensitive to the form of the interaction in the infrared. These coefficients are primarily responsible for distinguishing the pion, kaon, and eta decay constants and providing higher order corrections to their masses [11,12]. The sensitivity of these mass dependent coefficients is an indication that the hadron spectrum is playing a role in the determination of the quark-quark interaction. This result is consistent with a previous investigation which shows that the convergence radius of the chiral expansion in the current quark mass alone strongly depends on the form of the quark-quark interaction in the infrared region [13,14]. In particular it was found there that the running coupling has to be strong in the infrared region in order to obtain convergence of the chiral series in the strange quark sector.

Finally, we find that a pattern in the coefficients emerges at tree level which is consistent with the large $N_c$ expansion in QCD and can be traced to approximations that are made to QCD here in deriving the low-energy expansion[1]. In this way the consequences of the model assumptions can be directly observed. The present investigation further provides a significant reduction in the number of parameters needed to represent the low-energy QCD data mentioned above, and thereby deepens our understanding of low-energy QCD.

The paper is organized as follows. In section II the path from QCD to $\chi$PT is explored. In Section III the consequences of the approximations made in Section II, along with the model dependence of the results, are investigated. Finally conclusions are offered in Section IV.

II. QCD, THE EFFECTIVE QUARK-QUARK INTERACTION, AND $\chi$PT

1The role of the $\eta_0$ and the associated anomaly are neglected here in considering the $N_c$ dependence of the chiral coefficients. This question was first addressed in Ref. [15] and is reviewed in Ref. [12].
A. From QCD to the effective quark-quark interaction

A global color symmetry model (GCM) \[6,9\] that is based upon an effective quark-quark interaction can be defined through a truncation of QCD as follows. The generating functional for QCD in the Euclidean metric is given by

\[
Z[\psi, \bar{\eta}, \eta] = \int D\bar{q}DqDA \exp \left( -S[\bar{q}, q, A_\mu^a] - \bar{q}\psi q + \bar{\eta}q + \bar{q}\eta \right) \tag{2.1}
\]

and can be rewritten as

\[
Z[\psi, \bar{\eta}, \eta] = \int D\bar{q}Dq \exp \left[ -\int \bar{q}(\partial + \psi)q + \bar{\eta}q + \bar{q}\eta \right] \exp \left( W \left[ ig \frac{\lambda^a}{2} \gamma_\mu q \right] \right) \tag{2.2}
\]

with \( W[J] \) given by \( \exp (W[J]) = \int DA \exp \left( -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + J^a_\mu A_\mu^a \right) \). Here \( \bar{\eta}, \eta \) and \( \psi \) are external source fields.

The quantity \( W[J] \) has an expansion in gluon \( n \)-point functions starting at second order;

\[
W[J] = \frac{1}{2} \int D_{\mu\nu}^{ab}(x, y) J^a_\mu(x) J^b_\nu(y) + W_R[J], \tag{2.3}
\]

where \( W_R[J] \) involves gluon \( n(\geq 3) \)-point functions. It is worth noting that the \( n \)-point functions have mass dimension \([mass]^n\). One might therefore hope that for low-energy hadron physics the low-dimension functions would provide a good description.

By replacing the quark field variables in \( W_R[J] \) by their source derivatives, the generating functional of QCD can be written as

\[
Z[\psi, \bar{\eta}, \eta] = \exp \left( W_R \left[ ig \frac{\delta}{\delta \bar{q}} \lambda^a \frac{\delta}{\delta \eta} \right] \right) Z_{GCM}[\psi, \bar{\eta}, \eta] \tag{2.4}
\]

where \( Z_{GCM}[\psi, \bar{\eta}, \eta] \equiv \int D\bar{q}Dq \exp \left( -S_{GCM}[\psi, \bar{q}, q] + \bar{\eta}q + \bar{q}\eta \right) \) with

\[
S_{GCM}[\psi, \bar{q}, q] \equiv \int d^4xd^4y \left\{ \bar{q}(x) \left[ \left( \partial_x + \psi(x) \right) \delta(x-y) \right] q(y) \\
+ \frac{g^2}{2} j^a_\nu(x) D(x-y) j^a_\nu(y) \right\}. \tag{2.5}
\]

Here \( j^a_\nu(x) \equiv \bar{q}(x) \frac{\lambda^a}{2} \gamma_\nu q(x) \) is the quark color current, and for convenience a gauge for the gluon propagator \( D_{\mu\nu}^{ab}(x-y) = \delta_{ab} \delta_{\mu\nu} D(x-y) \) is employed. From here forward we work within the model truncation defined in (2.5).
The primary benefit of this truncation is that a reasonably solvable model is obtained, which is nevertheless sufficiently general to address a variety of phenomenological issues such as the role of quark-quark interactions in effective hadronic field theories. This model as well maintains the global symmetries of QCD and permits a $1/N_c$ expansion.

The primary loss of working at this level is that of the local color gauge invariance of QCD. The consequences of this loss are unclear, but are determined by the operation of $W_R$ in (2.4). The approximation of a local symmetry by a global symmetry is similar to the approximation of general relativity by special relativity. If the relevant field is sufficiently weak in the region of interest, then such an approximation is reasonable. In the case of localized color-singlet states one might hope that color neutrality could provide such a scenario [16].

It should be noted that the Nambu–Jona-Lasinio (NJL) model [17] is obtained from (2.5) in the limit $D(x - y) = \delta(x - y)/M^2$, with $M$ the appropriate mass scale. The chiral coefficients in the NJL model have been investigated [18–20], and are a limiting case of the present investigation. Our interest here is with the more general question of the model dependence of these coefficients. The present description also allows the discussion of higher mass excitations due to the nonlocal interaction.

B. From the quark-quark interaction to $\chi$PT

1. Bosonization and saddle-point expansion

The meson sector of the variable change implied in (1.1) is revealed by first identifying field combinations (currents) with the transformation properties of mesons. This is achieved through a Fierz reordering of the current-current term of the action (2.5) to obtain

$$\frac{g^2}{2} j_{\mu}^{a}(x) D(x - y) j_{\mu}^{a}(y) = -\frac{g^2}{2} J^{a}(x, y) D(x - y) J^{a}(y, x),$$

(2.6)

where $J^{a}(x, y) \equiv \bar{q}(x) \Lambda^{a} q(y)$ and the minus sign in (2.6) arises from the Grassmann nature of the quark field variables. Here the quantity $\Lambda^{a}$ is the direct product of Dirac, flavor $SU(3)$
and color matrices;

\[ \Lambda^{\theta} = \frac{1}{2} \left( 1_D, i\gamma_5, i\sqrt{2} \gamma_\nu, i\sqrt{2} \gamma_\nu \gamma_5 \right) \otimes \left( \frac{1}{\sqrt{3}} 1_F, \frac{1}{\sqrt{2}} \lambda^a_F \right) \otimes \left( \frac{4}{3} 1_C, \frac{i}{\sqrt{3}} \lambda^a_C \right), \]  

(2.7)

which contains, in particular, color singlet \( q\bar{q} \) combinations. It should be noted, however, that there are also color octet \( q\bar{q} \) combinations present in (2.7). An alternate color Fierz reordering,

\[ \sum_{a=1}^{8} (\lambda_a)_{ij} (\lambda_a)_{kl} = \frac{4}{3} \delta_{il} \delta_{kj} + \frac{2}{3} \sum_{m=1}^{3} \epsilon_{mik} \epsilon_{mlj}, \]  

(2.8)

eliminates the color octet \( q\bar{q} \) sector in favor of color triplet-antitriplet \( qq \) combinations and leads naturally to baryons [5]. This alternate approach, although natural for the investigation of baryons, is unnecessary for the investigation of meson interactions of interest here. The interested reader is encouraged to consult Ref. [5] and references therein for details of the baryon sector.

Having identified field combinations with the transformation properties of mesons, the current-current term of the action (2.5) is eliminated by multiplying the partition function by unity in the Gaussian form

\[ 1 = \mathcal{N} \int DB \exp \left[ - \int d^4 x d^4 y \frac{B^\theta(x,y)B^\theta(y,x)}{2g^2 D(x-y)} \right] \]  

(2.9)

and shifting the bilocal-field integration variables as \( B^\theta(x,y) \to B^\theta(x,y) + g^2 D(x-y) J^\theta(y,x) \) [21]. This requires in particular that the bilocal fields \( B^\theta(x,y) \) display the same symmetry transformations as the bilocal currents \( J^\theta(y,x) \) [11].

The partition function now has the form \( Z[\psi] = \mathcal{N} \int DB D\bar{q} Dq \ e^{-S[\psi,\bar{q},q,B]} \) where

\[ S[\psi, \bar{q}, q, B] = \int d^4 x d^4 y \bar{q}(x) \left[ (\partial_x + \psi(x)) \delta(x-y) + \Lambda^\theta B^\theta(x,y) \right] q(y) + \frac{B^\theta(x,y)B^\theta(y,x)}{2g^2 D(x-y)}. \]  

(2.10)

The action (2.10) is quadratic in the quark fields which allows the Grassmann integration to be performed by standard methods. The resulting expression for the partition function in terms of the bilocal-field integration is \( Z[\psi] = \mathcal{N} \int DB \ e^{-S[\psi,B]} \) where the action is given by.
\[ S[\psi, B] = -\text{Tr}\ln \left[ G^{-1} \right] + \int d^4x d^4y \frac{B^\theta(x, y)B^\theta(y, x)}{2g^2 D(x - y)}, \] (2.11)

and the quark inverse Green’s function, \( G^{-1} \), is defined as

\[ G^{-1}(x, y) \equiv (\bar{\psi}_x + \psi(x))\delta(x-y) + \Lambda^\theta B^\theta(x, y). \] (2.12)

This replacement of the quark-field integration with the bilocal-field integration represents an exact functional change of variables. Observable quantities extracted from the partition function are unaffected by the variable change, but are now expressed in terms of effective (meson) degrees of freedom. A benefit of this is that the effective mesonic interactions, which are generated from the quark-field determinant in (2.11), represent a summation of quark processes, and are easily exposed by expanding in powers of the bilocal fields. The structure of these interactions is illustrated in Fig.1. At this level the bilocal fields interact through a bare quark loop as in Fig.1a, and do not readily display the dynamics expected of quark bound states of QCD. However, as the notion of bare mesons is developed, this picture of their interactions is simultaneously refined.

In anticipation of dynamical chiral symmetry breaking, bare mesons are defined in terms of the fluctuations about the saddle point of the action (which is equivalent to the classical vacuum). This choice of an expansion point harbors profound dynamical consequences in that it largely determines both the structure and interactions of the bare mesons. In particular, this choice leads to the rainbow Dyson-Schwinger equation of the quark self energy, and the ladder Bethe-Salpeter equation for the internal structure of the bare mesons. More importantly, as a result of grouping this particular class of diagrams into bare mesons, the expansion about the classical vacuum leads to results for the chiral coefficients which are consistent with large \( N_c \) QCD, as is discussed in Section III.

The saddle-point of the action is defined as \( \frac{\delta S}{\delta B} |_{B_{\text{0}}, \psi = 0} = 0 \) and is given by

\[ B^\theta_0(x - y) = g^2 D(x - y) \text{tr} \left[ \Lambda^\theta G_0(x - y) \right]. \] (2.13)

These configurations are related to nonlocal vacuum condensates \cite{22} and provide self-energy dressing of the quarks through the definition \( \Sigma(p) \equiv \Lambda^\theta B^\theta_0(p) = i \not{p} [A(p^2) - 1] + B(p^2) \), where
\[ [A(p^2) - 1] p^2 = g^2 \frac{8}{3} \int \frac{d^4 q}{(2\pi)^4} D(p-q) \frac{A(q^2)q \cdot p}{q^2 A^2(q^2) + B^2(q^2)}, \]  

(2.14)

and

\[ B(p^2) = g^2 \frac{16}{3} \int \frac{d^4 q}{(2\pi)^4} D(p-q) \frac{B(q^2)}{q^2 A^2(q^2) + B^2(q^2)}. \]  

(2.15)

This dressing comprises the notion of “constituent” quarks by providing a mass

\[ M(p^2) = \frac{B(p^2)}{A(p^2)}. \]  

(2.16)

Their role as constituents is further displayed by expanding the bilocal fields about the saddle point,

\[ \mathcal{B}^\theta(x, y) = \mathcal{B}^\theta_0(x - y) + \hat{\mathcal{B}}^\theta(x, y), \]  

(2.17)

then examining the effective interactions of the fluctuations, \( \hat{\mathcal{B}} \). These interactions are produced by the quark-field determinant \( \text{TrLn} \left( \vartheta + \Sigma + \Lambda^\theta \hat{\mathcal{B}}^\theta \right) \), as is illustrated in Fig.1b. There it is seen that the fluctuation-field interactions now occur through the constituent-quark loops.

The connection between the bilocal fluctuation fields and the local fields of standard hadronic field-theory phenomenology remains to be shown. The bilocal fields contain information about internal excitations of the \( q\bar{q} \) pair in addition to their net collective or center-of-mass motion which is to be associated with the usual local field variables. A separation of the internal and center-of-mass dynamics is achieved by considering the normal modes of the free kinetic operator of the bilocal fields in a manner which is analogous to the interaction representation of standard quantum field theory. Details of the localization procedure can be found in Refs. [8] and [23]. The process amounts to a projection of the bilocal field \( \hat{\mathcal{B}}^\theta \) onto a complete set of internal excitations \( \Gamma^\theta_n \) with the remaining center-of-mass degree of freedom represented by the coefficients \( \phi^\theta_n(P) \equiv \int d^4 q \hat{\mathcal{B}}^\theta(P, q)\Gamma^\theta_n(P, q) \). The bilocal fluctuations can thus be written as

\[ \hat{\mathcal{B}}^\theta(P, q) = \sum_n \phi^\theta_n(P)\Gamma^\theta_n(P, q). \]  

(2.18)
The functions $\Gamma_{\theta}^n$ are in general eigenfunctions of the the free kinetic operator of the bilocal fields. At the mass shell point, $P^2 = -M_n^2$, they satisfy the homogeneous Bethe-Salpeter equation in the ladder approximation for the given quantum numbers $\theta$ and mode $n$. This modal expansion is then used to localize the action.

At tree level the local fields $\phi_{\theta}^n$ interact through a dynamically regulated constituent-quark loop, as is illustrated, for example, in Fig.2. These “effective interactions” thus reflect the underlying QCD structure. The intrinsic nonlocality plays a dual role in the subsequent description of physical phenomena. First, when sufficiently short length scales are probed as in the large momentum behavior of hadronic form factors, the nonlocal structure is directly observed \[11,24\]. Second, independent of external probes, the nonlocality provides a regulation of internal loop integrations, and serves to suppress hadron-loop effects \[25\]. The present approach can also accommodate extensions of low-energy effective theories through the consideration of the higher mass states, and therefore provides a consistent framework in which many of the issues facing hadronic field theories might be addressed.

2. Dynamical chiral symmetry breaking

In the following discussion, $D_{\chi\text{SB}}$ is associated with the occurrence of a massless Goldstone mode that is related to the dynamical generation of a scalar amplitude in the quark self energy in the limit of vanishing quark mass. We begin by considering the axial-vector Ward identity in the chiral limit given by \[26,27\]

$$P_\mu \Gamma_5^\mu(P,q)\bigg|_{m=0} = G^{-1}(q + P/2)\gamma_5 + \gamma_5 G^{-1}(q - P/2).$$

(2.19)

It is well known \[26,27\] that in the chiral limit the axial-vector vertex contains a zero-momentum pole of the form

$$\Gamma_5^\mu(P,q) \rightarrow \frac{P_\mu}{P^2} \Gamma_5(0,q)f_\pi,$$

(2.20)

associated with the massless Goldstone mode. It should be noted that in (2.20) the quark-pseudoscalar vertex $\Gamma_5$ is also evaluated in the vicinity of $P = 0$, and is therefore a solution
of the homogeneous Bethe-Salpeter equation for the pseudoscalar bound state.

Operating on \((2.20)\) with \(P_\mu\) and comparing with \((2.19)\) obtains

\[
\Gamma_5(0, q) = 2\gamma_5 \frac{B(q^2)}{f_\pi}.
\]

(2.21)

This is the Goldberger-Treiman relation for the quark-pseudoscalar vertex. The fact that the quark self-energy function \(B\) occurs as the residue of the zero momentum pole in the quark-axial-vector vertex is equivalent to a statement of Goldstone’s theorem in this context. It is also readily verified that the ladder Bethe-Salpeter equation for the pseudo-scalar Goldstone mode reduces to the self-energy equation \((2.15)\) [26,27].

Since our interest here is the effective action for the Goldstone modes, we neglect all of the higher mass fluctuations present in the bilocal fields. This implies that the full bilocal field of Eq. \((2.17)\) can be written, using the expressions \((2.13)\) and \((2.21)\) for the saddle point and the Bethe-Salpeter amplitude of the Goldstone modes respectively, as

\[
\Lambda^\theta B^\theta(x, y) = \Sigma(x - y) + B(x - y) \left[ U_5 \left( \frac{x + y}{2} \right) - 1 \right],
\]

(2.22)

where \(U_5(x) = P_R U(x) + P_L U^\dagger(x)\) with \(P_{R,L}\) the standard right-left projection operators. For the \(SU(3)\) flavor case under consideration here the chiral field \(U\) is defined as \(U \equiv e^{i\lambda^a \psi^a/f_\pi}\).

It should be stressed at this point that we have not integrated over the higher mass states, but have simply neglected them. The effect of including and integrating over the higher mass states is addressed in Section III.

3. The low-energy expansion

For the application to low-energy observables, an expansion of the action to fourth order is now considered. The usual chiral power counting is observed [12]. In order to preserve the chiral invariance of the full action \((2.3)\), the quarks have to be coupled to the external source field \(\psi(x)\), which transforms in a certain way under chiral rotations [12]. In performing the gradient expansion it is important to keep the \(x\) dependence of this external field. After
carrying out the gradient expansion to fourth order, we will employ the equation of motion which is obtained at second order and depends on the external field $\psi(x)$, and then finally we will identify $\psi(x)$ with the current quark mass matrix. Failure to keep the $x$ dependence of $\psi$ to the very end violates chiral invariance and will render unphysical results for some of the low-energy coefficients. This approach differs somewhat from the previous work of Refs. [9] and [10] where the equation of motion is not employed. However, there the mass-dependent fourth-order coefficients are not considered.

We consider here only the real contribution to the effective action. The imaginary contribution, which contains the Wess-Zumino term, has also been investigated in Refs. [9] and [10], and the interested reader is encouraged to consult these references for more details.

The restriction of the fluctuations to Goldstone modes with $UU^\dagger = 1$, as in (2.22), entails that the second term of the action in Eq.(2.11) is independent of the fields $U$ and can therefore be neglected. The real contribution to the action is then given by

$$S \equiv \text{Re}[S] = -\frac{1}{2} \text{Tr} \ln \left( G^{-1} \left[ G^{-1} \right]^\dagger \right),$$

(2.23)

where $G^{-1}$ is, from (2.12) and (2.22), given by

$$G^{-1}(x,y) = \gamma \cdot \partial_x A(x-y) + \psi \left( \frac{x+y}{2} \right) \delta(x-y) + B(x-y)U_5 \left( \frac{x+y}{2} \right).$$

(2.24)

By expanding the logarithm and dropping irrelevant constant terms, Eq.(2.23) can further be written as

$$S = \frac{1}{2} \sum_{n=1}^{\infty} \left( -1 \right)^n \frac{1}{n} \text{Tr} \left( a + b + c + d \right)^n,$$

(2.25)

where $a$, $b$, $c$, and $d$ are non-commuting operators formed from $A$, $B$, $U_5$, and $\psi$, and are at least of order one, one, two, and three in chiral counting respectively. The explicit form of these operators is given in the appendix.

The effective chiral action to the desired order is now obtained by expanding the sum in Eq.(2.25) and expanding the operators $a$–$d$ in gradients. The result to fourth order is (in Euclidean space)
\[ S = \int d^4x \left\{ \frac{f^2}{4} \operatorname{tr} \left[ (\partial_\mu U)(\partial_\mu U^\dagger) \right] - \frac{f^2}{4} \left[ U\chi^\dagger + \chi U^\dagger \right] - L_1 \left( \operatorname{tr} \left[ (\partial_\mu U)(\partial_\mu U^\dagger) \right] \right)^2 - L_2 \operatorname{tr} \left[ (\partial_\mu U)(\partial_\mu U^\dagger) \right] \cdot \operatorname{tr} \left[ (\partial_\nu U)(\partial_\nu U^\dagger) \right] \\
\quad - L_3 \operatorname{tr} \left[ (\partial_\mu U)(\partial_\nu U)(\partial_\mu U^\dagger) \right] + L_5 \operatorname{tr} \left[ (\partial_\mu U)(\partial_\mu U^\dagger)(U\chi^\dagger + \chi U^\dagger) \right] \\
\quad - L_8 \operatorname{tr} \left[ \chi U^\dagger \chi U^\dagger + U\chi U^\dagger \chi U^\dagger \right] \right\}, \tag{2.26} \]

where \( \chi(x) = -2\langle \bar{q}q(x) \rangle / f^2 \) and the remaining trace is over flavor. In obtaining this result the equation of motion

\[ (\partial^2 U)U^\dagger + (\partial_\mu U)(\partial_\mu U^\dagger) + \frac{1}{2} (\chi U^\dagger - U\chi^\dagger) = 0 \tag{2.27} \]

and the \( SU(3) \) relation \[12\]

\[ \operatorname{tr} \left[ (\partial_\mu U)(\partial_\nu U)(\partial_\mu U^\dagger) \right] = \frac{1}{2} \left( \operatorname{tr} \left[ (\partial_\mu U)(\partial_\mu U^\dagger) \right] \right)^2 \]

\[ + \operatorname{tr} \left[ (\partial_\mu U)(\partial_\nu U^\dagger) \right] \cdot \operatorname{tr} \left[ (\partial_\mu U)(\partial_\nu U^\dagger) \right] - 2\operatorname{tr} \left[ (\partial_\mu U)(\partial_\mu U^\dagger)(\partial_\nu U)(\partial_\nu U^\dagger) \right] \tag{2.28} \]

have been used. Explicit forms of the coefficients are given in the appendix.

III. RESULTS AND DISCUSSION

Several conclusions can be drawn directly from the low-energy expansion (2.26). It is immediately apparent that the coefficients \( L_4, L_6, \) and \( L_7 \) vanish. It is also evident, by application of the \( SU(3) \) relation (2.28) (see appendix), that \( L_2 = 2L_1 \). These relationships are expected in the large \( N_c \) limit\[2\] of QCD \[12\]. The fact that they are produced here is perhaps not too surprising and can be linked to our truncation of the QCD action to include only the gluon two-point function.

With only a two-point quark-quark interaction, the large \( N_c \) limit leads to a description of mesons as a sum of ladder exchanges. Our description of mesons as fluctuations about

\[^2\text{In the presence of the } U_A(1) \text{ anomaly the coefficient } L_7 \text{ is of order } N_c^2 \text{ \[13,12\]. Our neglect of the } \eta_0 \text{ here leads to the vanishing of } L_7.\]
the saddle point of the action is equivalent to the $1/N_c$ expansion and, in this model truncation, leads directly to the ladder approximation. Our further neglect of higher mass states, explicitly excludes intermediate states of pure glue which are “$N_c$ suppressed”. A departure from this tree-level pattern in the coefficients would therefore have to arise in the present formalism by including and integrating over the higher mass mesons, which we have explicitly excluded in Eq.(2.22). The role of the underlying description is thus clearly displayed in the pattern of the chiral coefficients.

Examples of the diagrams that are generated by integrating over higher mass mesons are illustrated in Fig.3. The diagram of Fig.3a is of order one in $N_c$ counting and produces departures from the large $N_c$ relations, while the diagram of Fig.3b is of order $N_c$ and produces, for example, the $\rho$-pole in $\pi-\pi$ scattering. All of the contributions that we are presently considering are of order $N_c$ and arise from a single quark loop.\(^3\)

The remaining nonzero coefficients must be evaluated numerically. These depend explicitly of the values of the self-energy functions $A$ and $B$ in Eqs.(2.14) and (2.15) respectively, and are therefore implicitly dependent on the quark-quark interaction $D$. The procedure is then to select a form for the function $D$, solve the coupled nonlinear equations (2.14) and (2.15) for $A$ and $B$ respectively, and then evaluate the pion decay constant $f_\pi$, the condensate $\langle \bar{q}q \rangle$, and the coefficients $L_1, L_3, L_5, \text{and } L_8$.

The quark-quark interaction $D$ has the form

$$g^2 D(s) = \frac{4\pi\alpha(s)}{s},$$

where $s = q^2$, and we investigate three different two-parameter models for $\alpha(s)$;

$$\alpha_1(s) = 3\pi s \chi^2 e^{-s/\Delta} + \frac{\pi d}{4\Delta^2 + \ln(s/\Lambda^2 + e)}$$

$$\alpha_2(s) = \pi d \left[ \frac{s\chi^2}{s^2 + \Delta} + \frac{1}{\ln(s/\Lambda^2 + e)} \right].$$

\(^3\)The double trace terms proportional to $L_1$ and $L_2$ in (2.26) arise here from the $SU(3)$ relation (2.28) and originate from a single quark loop.
\[ \alpha_3(s) = \pi d \left[ \frac{1 + \chi e^{-s/\Delta}}{\ln(s/\Lambda^2 + e)} \right]. \]

Each of these forms incorporates the one-loop perturbative result for large \( s \) (here \( \Lambda = 0.2\text{GeV} \) and \( d = 12/(33 - 2N_F) = 12/27 \)), and extrapolates differently into the low-momentum region. The two low-momentum parameters, \( \chi \) and \( \Delta \), are varied with the pion decay constant held fixed at \( f_\pi = 86\text{MeV} \). This value is appropriate at zero-momentum rather than the pion-mass-shell value of 93MeV, however the results are not very sensitive to this small difference. By fixing \( f_\pi \) the overall scale of \( D\chi\text{SB} \) is fixed. The remaining independent parameter is associated with the matching scale to the perturbative form.

The running coupling for the three cases listed in Eq.\( (3.2) \) are plotted in Figs. 4, 5, and 6 along with the corresponding solutions of Eqs.\( (2.14) \) and \( (2.15) \) for the self-energy functions. In all three cases as the matching point to the perturbative form is decreased to lower momentum, the infrared strength must be increased to maintain the fixed value of \( f_\pi \). Thus the integrated strength of \( \alpha \) is largely constant. This trend is also present to a lesser extent in the self-energy functions.

The first model, \( \alpha_1 \) in Eq.\( (3.2) \), has been used in previous investigations of the present type [28]. There the parameters were fixed at \( \Delta = 0.002\text{GeV}^2 \) and \( \chi = 1.14\text{GeV} \), which leads to a slightly lower value of \( f_\pi \). The small \( \Delta \) limit of this model obtains a matching point near zero momentum and a delta-function behavior in the quark-quark interaction \( D \). This limit has been used previously to model confinement [27]. The infrared contribution to the second model, \( \alpha_2 \), generates a \( 1/q^4 \) singularity in the quark-quark interaction \( D \) in the limit as \( \Delta \to 0 \). Such a singularity has also been considered previously as a model of confinement [30]. The \( 1/q^4 \) form falls much slower than the Gaussian in \( \alpha_1 \), and hence leads to much higher matching scales. Finally the third model \( \alpha_3 \) has been chosen here to be structurally different from \( \alpha_1 \) and \( \alpha_2 \) in order to further illustrate the independence of the results to the details of the low-momentum parameterization. The corresponding results for the low energy coefficients \( L_1, L_3, L_5, \) and \( L_8 \) are displayed in Tables I-III, respectively.

In all of the three cases the same pattern is observed: The coefficients \( L_1 \) and \( L_3 \), which
are responsible for $\pi-\pi$ and $K-K$ scattering, are nearly independent of the form of $\alpha(s)$ and therefore on the form of the quark-quark interaction, provided that the integrated strength of $\alpha(s)$ is fixed by $f_\pi$. On the other hand, the mass dependent coefficients, $L_5$ and $L_8$, are more strongly dependent on the actual form of the interaction. For example with $L_5$, in order to reproduce the experimental value, forms of $\alpha(s)$ with a small matching scale, i.e. which are relatively strong in the infrared region, are required. This observation is in coincidence with the result of Ref. [14], where it is shown that quark-quark interactions with a low matching scale are also required to achieve convergence of the chiral series in the strange quark sector. Furthermore an explanation for the success of the “delta-function-plus-tail” type models (obtained for example from $\alpha_1$ in the limit $\Delta \to 0$) in describing chiral observables [29] is offered by this fact.

We also find that the results for the fourth-order coefficients are rather insensitive to the asymptotic UV tale of $\alpha(s)$; even omitting this tail completely gives no significant changes, again provided that $f_\pi$ is fixed.

An increased accuracy in the experimental determination of the coefficients would make tighter restrictions on the quark-quark interaction, however the additional investigation of higher mass excitations is clearly required to gain detailed information on its infrared form [31].

It has frequently been stated with regard to the fourth order coefficients that QCD “seems to predict that deviations from the lowest order chiral relation must be in such a form as to reproduce the low energy tails of the light resonances, in particular the $\rho$.” [12]. Here we have explicitly neglected in Eq.(2.22) the $q\bar{q}$ fluctuation associated with the $\rho$, and illustrated in Fig.3 how the $\rho$-pole contribution would arise. One might then ask: What is the mechanism that produces the $\rho$-tail-like contribution to the coefficients here?

This question is easily answered by again considering the diagram of Fig.2. There is a $q\bar{q}$ pair in the intermediate state which arises from the quark loop structure of the interactions. The integrands of these quark loops are peaked at a momentum $q_{\text{peak}}$ such that the constituent mass of Eq.(2.16) gives
\[ M(q^2) \rightarrow M(q_{\text{peak}}^2) \approx 300 - 400 \text{MeV}. \] (3.3)

This \( q\bar{q} \) pair can have the quantum numbers of the \( \rho \), and carries sufficient mass to contribute the \( \rho \)-tail effect away from the pole.

IV. CONCLUSIONS

We have made a detailed examination of the low-energy chiral expansion from the standpoint of the model truncation of QCD given in Eq.(2.5). The structure of the model maintains the global symmetries of QCD (including global color symmetry), and permits a \( 1/N_c \) expansion. The infrared momentum dependence of the quark-quark interaction is phenomenological input to the model; here three different two-parameter forms are investigated.

We find that by truncating QCD to include only a two-point quark-quark interaction and describing mesons as fluctuations about the saddle point of the effective action, one obtains a pattern in the chiral coefficients which is consistent with large \( N_c \) results in QCD. This conclusion can be understood by considering the \( 1/N_c \) expansion within the model truncation, and provides a direct link between the model assumptions and consequences for physical observables in QCD, independent of the phenomenological treatment of the quark-quark interaction. The structure of the underlying theory is in this way displayed by the pattern in the chiral coefficients. The departure from the large \( N_c \) result is provided here by the integration over higher mass states.

We find that by fixing the pion decay constant, an integrated strength of the running coupling is prescribed. This sets the scale for \( D\chi_{\text{SB}} \). The remaining independent parameter is associated with the matching scale to the perturbative form of the running coupling. The chiral coefficients \( L_1 \) and \( L_3 \), which are related to \( \pi-\pi \) and \( K-K \) scattering data are nearly insensitive to this scale. It appears, therefore, that any chiral quark-quark interaction which is capable of \( D\chi_{\text{SB}} \) can be expected to reproduce these coefficients and the corresponding low energy meson scattering data. However, some constraint on the matching scale is provided by the sensitivity of the mass-dependent coefficients \( L_5 \) and \( L_8 \), which favor interaction forms
that are strong in the infrared domain.

Finally we conclude that the model truncation that is employed here reproduces low-energy QCD, as represented by \( \chi_{PT} \), quite well. More importantly, this model is not limited to low energy and might therefore be used to extend low-energy effective theories.

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APPENDIX A:

The operators in Eq(2.25) are defined as

\[
\begin{align*}
  a & \equiv \frac{1}{x}[\gamma_\mu \bar{A}_\mu U_5^\dagger B + BU_5 \gamma_\mu \bar{A}_\mu^\dagger] \\
  b & \equiv \frac{1}{x}B[U_5 U_5^\dagger - 1]B \\
  c & \equiv \frac{1}{x}[BU_5 \bar{\psi}^\dagger + \psi U_5^\dagger B + \bar{\psi}^\dagger \psi] \\
  d & \equiv \frac{1}{x}[\gamma_\mu \bar{A}_\mu \bar{\psi}^\dagger + \psi \gamma_\mu \bar{A}_\mu^\dagger] \\
  x & \equiv \gamma_\mu \bar{A}_\mu \gamma_\nu \bar{A}_\nu^\dagger + B^2,
\end{align*}
\]

where for example

\[
\begin{align*}
  <x_1|\bar{A}_\mu|x_2> &= \partial_{\mu x_1} A(x_1 - x_2), <p_1|\bar{A}_\mu|p_2> = i p_{1 \mu} A(p_1^2) \delta(p_1 - p_2) \\
  <x_1|B|x_2> &= B(x_1 - x_2), <p_1|B|p_2> = B(p_1^2) \delta(p_1 - p_2) \\
  <x_1|BU_5|x_2> &= B(x_1 - x_2) U_5 \left( \frac{x_1 + x_2}{2} \right), <p_1|BU_5|p_2> = \frac{1}{(2\pi)^2} B \left( \frac{p_1 + p_2}{2} \right) U_5(p_1 - p_2)
\end{align*}
\]
$< x_1 | U_5^I B | x_2 > = B(x_1 - x_2)U_5^I \left( \frac{x_1 + x_2}{2} \right)$, $< p_1 | U_5^I B | p_2 > = \frac{1}{(2\pi)^2} B \left( \frac{p_1 + p_2}{2} \right) U_5^I (p_2 - p_1)$.

From the discussion presented in the text, one can then obtain the coefficients

$$f^2_\pi = F \int dss \left( \frac{B}{x} \right)^2 \left[ A^2 + sAA' + s^2(A')^2 + s(B')^2 \right]$$

$$- \frac{BB' + \frac{s}{2}[(B')^2 + BB'']}{x}$$

(A3)

$$\langle \bar{q}q \rangle_{1\text{GeV}} = -F \int^{1\text{GeV}^2} dss \frac{B}{x}$$

(A4)

$L_1 = -\frac{1}{2} \lambda_3$

$L_2 = -\lambda_3$

$L_3 = -(\lambda_2 - 2\lambda_3 + \lambda_1)$

$L_5 = \lambda_4 - \lambda_6$

$L_8 = -(\lambda_5 - \frac{1}{4} \lambda_1 + \frac{1}{2} \lambda_6)$

(A5)

where

$$\lambda_1 = \int ds(\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15})$$

(A6)

with

$$\lambda_{11} = \frac{F s^2 BB'}{32 x^2} \left( B'^2 + BB'' + \frac{1}{3} sBB''' + sB'B'' \right)$$

$$\lambda_{12} = \frac{F}{64} s^2 \left( -8s( BB')^2 \frac{x''x - x'^2}{x^4} - 8(BB')^2 \frac{x'}{x^3} + 2 \frac{BB'}{x^2} [BB''' - B'B''] 
+ 2 \frac{BB'}{x^2} [3BB'' - (B')^2] \right)$$

$$\lambda_{13} = -\frac{Fs}{96x} \left( 3[(B')^2 + BB''] + 3s[BB''' + 3B'B''] + \frac{s^2}{2} [BB'''' + 4B'B''' + 3(B'')^2] \right)$$

$$\lambda_{14} = \frac{F}{8} s \left( \frac{B}{x} \right)^2 \left( \frac{3}{2} s^2 A'A' + sAA' + \frac{1}{3} s^3 A'A''' + \frac{1}{6} s^2 AA'' + \frac{1}{2} s(A')^2 + AA' \right)$$

$$\lambda_{15} = \frac{F}{8} sB^2 \left( \frac{s^3 (A')^2 + s^2 AA' + \frac{1}{2} sA^2}{x^4} \right) \frac{(x')^2 - xx''}{x^3} - \left[ s^2 (A')^2 + sAA' + A^2 \right] \frac{x'}{x^3}$$
\[
\lambda_2 = \int ds(\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} + \lambda_{25} + \lambda_{26} + \lambda_{27} + \lambda_{28} + \lambda_{29} + \lambda_{210}) \quad (A7)
\]

with

\[
\lambda_{21} = \frac{F}{16} \frac{s}{x^2} \left( (BB')^2 + sBB'[BB'' + \frac{s^2}{3}(BB')^2 + BB''] + sBB' \left[ (BB')^2 + BB'' + \frac{s}{3}BB'' + sBB' \right] \right)
\]

\[
\lambda_{22} = \frac{F}{32} s^2 \left( -\frac{8}{3} s(BB')^2 \frac{x''}{x} - (x')^2 + 2[BB'' - (BB')^2] \frac{BB'}{x^2} + \frac{2}{3} s(BB') \frac{BB'' - BB'}{x^2} \right)
\]

\[
\lambda_{23} = -\frac{2F}{3} s^2 \left( \frac{1}{8} (BB')^3 + \frac{s}{4} (BB')^2 (BB' + BB'') \right)
\]

\[
\lambda_{24} = \frac{Fs^3}{6} \left( \frac{BB'}{x} \right)^4
\]

\[
\lambda_{25} = 2Fs \left( \frac{B^2 B'}{x^2} \right)^2 \left( \frac{s^2}{3} AA' + \frac{s^3}{3} (A')^2 \right)
\]

\[
\lambda_{26} = -Fs \left( \frac{B^2 B'}{x^2} \right)^2 \left( \frac{s^2}{3} AA' + \frac{s^3}{3} (A')^2 \right)
\]

\[
\lambda_{27} = -\frac{sF}{4} \frac{B^2}{x^3} \left( A^2 BB' + \frac{s^2}{2} A^2 (BB')^2 + sBB'[AA' + s(A')^2] + \frac{2}{3} s^2 [AA' + s(A')^2] (BB')^2 + BB' \right)
\]

\[
\lambda_{28} = \frac{Fs^3}{4} \frac{B^3 B'}{x^3} (A')^2
\]

\[
\lambda_{29} = \frac{sF}{4} \frac{B}{x^2} \left( -\frac{4}{3} s^2 B^2 B' \frac{x}{x} - [s(A')^2 + AA'] - \frac{s}{2} x' B^2 A^2 [A] - \frac{4}{3} s^2 B ((B')^2 + BB') \frac{AA'}{x} \right) + \frac{2}{3} s^2 AA' + s(A')^2
\]

\[
\lambda_{210} = \frac{sF}{4} \frac{(B)}{x^4} \left( A^4 + 2 s A^3 A' + \frac{8}{3} s^2 (AA')^2 + \frac{4}{3} s^3 A(A')^3 + \frac{2}{3} s^4 (A')^4 \right)
\]

\[
\lambda_3 = \int ds(\lambda_{31} + \lambda_{32} + \lambda_{33} + \lambda_{34} + \lambda_{35} + \lambda_{36} + \lambda_{37} + \lambda_{38} + \lambda_{39} + \lambda_{310}) \quad (A8)
\]

with

\[
\lambda_{31} = \frac{sF}{16 x^2} \left( \frac{s^2}{6} (BB')^2 + BB' (BB'' + s BB'' + sBB') \right)
\]

\[
\lambda_{32} = \frac{s^2 F}{64} \left( -\frac{8}{3} s(BB')^2 \frac{x''}{x} - (x')^2 - 8(BB')^2 \frac{x'}{x^3} + 2[BB'' + (BB')^2] \frac{BB'}{x^2} \right)
\]
\[
\lambda_{33} = -\frac{2}{3}s^2F \left( \frac{1}{4} \left( \frac{BB'}{x} \right)^3 + \frac{s}{8} \left( BB'' \right)^2 \left( \frac{B'}{x} \right)^2 + BB'' \right)
\]
\[
\lambda_{34} = \frac{s^3F}{12} \left( \frac{BB'}{x} \right)^4
\]
\[
\lambda_{36} = -Fs \left( \frac{B^2B'}{x^2} \right)^2 \left( \frac{s}{4}A^2 + \frac{s^2}{6}AA' + \frac{s^3}{6} (A')^2 \right)
\]
\[
\lambda_{35} = Fs \left( \frac{B^2B'}{x^2} \right)^2 \left( \frac{s^2}{3} AA' + \frac{s^3}{3} (A')^2 \right)
\]
\[
\lambda_{37} = -\frac{sF B^2}{4 x^3} \left( \frac{B^2}{3} \right) \left( AA' + s(A')^2 \right) \left( -s B'' + (B')^2 + BB'' \right)
\]
\[
\lambda_{38} = -\frac{s^3F B^3 B'}{4 x^3} (A')^2
\]
\[
\lambda_{39} = \frac{sF B}{4 x^2} \left( -\frac{2}{3} s^2 B^2 B' x' \right) \left( s(A')^2 + A' A \right) - s B^2 B' A' x' x^2 + \frac{2}{3} s^2 B [(B')^2 + BB'' \frac{AA'}{x} + s(A')^2]
\]
\[
\lambda_{310} = \frac{s F}{8} \left( \frac{B}{x} \right)^4 \left( -A^4 - 2s A^3 A' - \frac{4}{3} s^2 (AA')^2 + \frac{4}{3} s^3 A (A')^3 + \frac{2}{3} s^4 (A')^4 \right)
\]
\[
\lambda_4 = \frac{F}{2 B_0} \int dss \left( \frac{B B'}{4 x^2} \left( BB' + \frac{s}{2} (B')^2 + BB'' \right) - \frac{1}{2} \left( \frac{B}{x} \right)^3 \left[ s (B')^2 + A^2 + s AA' + s^2 (A')^2 \right] \right)
\]
\[
\lambda_5 = \frac{F}{16 B_0} \int dss \left( \frac{B}{x} \right)^2
\]
\[
\lambda_6 = \frac{1}{2 B_0} \int ds (\lambda_{61} + \lambda_{62} + \lambda_{63})
\]

with
\[
\lambda_{61} = -\frac{sF B}{4 x^2} \left[ A^2 + s AA' + s^2 (A')^2 \right]
\]
\[
\lambda_{62} = \frac{s F B' + \frac{s}{2} B''}{8 x}
\]
\[
\lambda_{63} = -\frac{F}{4 B} \left( \frac{s B'}{x} \right)^2
\]
and finally

\[ F \equiv \frac{4N_c}{16\pi^2} \quad \text{and} \quad B_0 \equiv -\frac{\langle \bar{q}q \rangle}{f^2}. \]  

(A12)

In the above expressions, the arguments of the functions are \( s = q^2 \) and the prime indicates differentiation with respect to \( s \).
REFERENCES

[1] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) 158, 142 (1983); Nucl.Phys. B 250, 465 (1985).

[2] G. Ecker, Prog. Part. Nucl. Phys. 35, 1 (1995), and references therein.

[3] U.G. Meissner, Rep. Prog. Phys. 56, 903 (1993); V. Bernard, N. Kaiser and U.G. Meissner, Int. J. Mod. Phys. E 4, 193 (1995), and references therein.

[4] N. Isgur, in From Fundamental Fields to Nuclear Phenomena, Eds., J.A. McNeil and C.E. Price (World Scientific, Singapore 1991) pp. 46-54; R.L. Jaffe and P.F. Mende, Nucl. Phys. B 369, 189 (1992).

[5] R.T. Cahill, Nucl. Phys. A 543, 63 (1992).

[6] R.T. Cahill and C.D. Roberts, Phys. Rev. D 32, 2419 (1985).

[7] H. Kleinert, Phys. Lett. B 62, 429 (1976); in Proc. of the 1976 School of Subnuclear Physics, Erice (Plenum, 1978); Fortschr. Phys. 30, 351 (1982).

[8] R.T. Cahill, Aust. J. Phys. 42, 171 (1989).

[9] C.D. Roberts, R.T. Cahill, and J. Praschifka, Ann. Phys. (N.Y.) 188, 20 (1988).

[10] C.D. Roberts, R.T. Cahill, M.E. Sevior, and N. Iannella, Phys. Rev. D 49, 125 (1994).

[11] M.R. Frank and P.C. Tandy, Phys. Rev. C 49, 478 (1994).

[12] John F. Donoghue, Eugene Golowich, and Barry R. Holstein, Dynamics of the Standard Model (Cambridge University Press, 1992), and references therein.

[13] T. Hatsuda, Phys. Rev. Lett. 65, 543 (1990).

[14] T. Meissner, Phys. Lett. B 340, 226 (1994).

[15] E. Witten, Nucl. Phys. B156, 269 (1979).
[16] M.R. Frank, B.K. Jennings, and G.A. Miller, “The role of color neutrality in nuclear physics: Modifications of the nucleonic wave functions”, submitted to Physical Review C (1995), and references therein.

[17] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); 124, 246 (1961).

[18] E. Ruiz Arriola, Phys. Lett. B 253, 430 (1991); Phys. Lett. B 264, 178 (1991); C. Schueren, E. Ruiz Arriola and K. Goeke, Nucl. Phys. A 547, 612 (1993).

[19] J. Bijnens, C. Bruno and E. de Rafael, Nucl. Phys. B 390, 501 (1993); J. Bijnens, “Chiral Lagrangians and Nambu-Jona- Lassino-like models”, NORDITA-95-10-N-P, Feb 1995.

[20] J. Mueller and S. Klevansky, Phys. Rev. C 50, 410 (1994).

[21] E. Shrauner, Phys. Rev. D16,1887 (1977).

[22] M.R. Frank, P.C. Tandy, and G. Fai, Phys. Rev. C 43, 2808 (1991); P.C. Tandy and M.R. Frank, Aust. J. Phys. 44, 181 (1991); M.R. Frank and P.C. Tandy, Phys. Rev. C 46, 338 (1992).

[23] M.R. Frank in Few-Body Problems in Physics, Ed.,Franz Gross ( AIP Conference Proceedings 334) pp. 15-30.

[24] C.D. Roberts, it Electromagnetic Pion Form Factor and Neutral Pion Decay Width, ANL preprint no. ANL-PHY-7842-TH-94 (1994); M.R, Frank, K.L. Mitchell, C.D. Roberts, and P.C. Tandy, Phys. Lett. B 359, 17 (1995); C. J. Burden, C. D. Roberts and M. J. Thomson, Electromagnetic Form Factors of Charged and Neutral Kaons, ANL preprint no. ANL-PHY-8240-TH-95 (1995).

[25] L.C.L. Hollenberg, C.D. Roberts, and B.N.J. McKellar, Phys. Rev. C 46, 2057 (1992); R. Alkofer, A. Bender, and C.D. Roberts, “Pion loop contribution to the electromagnetic pion charge radius”, ANL-PHY-7663-TH-93, UNITUE-THEP-13/1993 (1993).
[26] R. Delbourgo and M.D. Scadron, J. Phys. G 5, 1621 (1979).

[27] C.D. Roberts and A.G. Williams, Dyson-Schwinger equations and their application to hadronic physics, in Progress in Particle and Nuclear Physics, Ed., A. Faessler (Pergamon Press, Oxford 1994); and references therein.

[28] J. Praschifka, R.T. Cahill, and C.D. Roberts, Int. J. Mod. Phys. A 4, 4929 (1989).

[29] M.R. Frank and C.D. Roberts, “Model gluon propagator and pion and rho-meson observables”, to appear in Physical Review C, January (1996).

[30] W. Marciano and H. Pagles, Phys. Rep. C 36, 137 (1978).

[31] M.R. Frank, Phys. Rev. C 51, 987 (1995).
FIGURES

FIG. 1. The effective interactions obtained from the quark determinant are illustrated both before and after the saddle-point expansion.

FIG. 2. An example of the effective interactions between the localized mesons is shown. The quark lines and vertices are dressed in the rainbow and ladder approximations respectively.

FIG. 3. Examples of the diagrams generated by integrating over higher mass mesons are shown. The diagram in (a) is of order one in $N_c$ counting while that of (b) is of order $N_c$.

FIG. 4. The running coupling $\alpha_1$, and quark self-energy functions $A$ and $B$ as shown versus $s = q^2$. The parameter choices maintain $f_\pi = 86$MeV.

FIG. 5. Same as Fig. 4 using $\alpha_2$.

FIG. 6. Same as Fig. 4 using $\alpha_3$. 
$$\alpha_1(s) = 3\pi s\chi^2 e^{-s/\Delta}/(4\Delta^2) + \pi d/\ln(s/\Lambda^2 + e)$$

| $\Delta$ (GeV²) | $\chi$ (GeV) | $-\langle\bar{q}q\rangle^{1/3}$ (MeV) | $L_1(0.7\pm0.5)$ | $L_3(-3.6\pm1.3)$ | $L_5(1.4\pm0.5)$ | $L_8(0.9\pm0.3)$ |
|-----------------|--------------|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| 0.002           | 1.4          | 150                             | 0.84            | -4.4            | 1.0             | 0.88            |
| 0.02            | 1.5          | 160                             | 0.82            | -4.4            | 1.14            | 0.84            |
| 0.2             | 1.65         | 173                             | 0.81            | -4.0            | 1.66            | 0.83            |
| 0.4             | 1.84         | 177                             | 0.80            | -3.8            | 2.0             | 0.87            |

**TABLE I.** The chiral coefficients, calculated using $\alpha_1$, are displayed. The parameter choices listed maintain $f_\pi = 86$MeV. The “experimental” values in parenthesis are taken from Ref.[3]. The quark condensate $\langle \bar{q}q \rangle$ is evaluated at 1 GeV.

$$\alpha_2(s) = \pi d(s\chi^2/(s^2 + \Delta) + 1/\ln(s/\Lambda^2 + e))$$

| $\Delta$ (GeV⁴) | $\chi$ (GeV) | $-\langle\bar{q}q\rangle^{1/3}$ (MeV) | $L_1(0.7\pm0.5)$ | $L_3(-3.6\pm1.3)$ | $L_5(1.4\pm0.5)$ | $L_8(0.9\pm0.3)$ |
|-----------------|--------------|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| $10^{-7}$        | 0.83         | 162                             | 0.82            | -4.4            | 1.28            | 0.87            |
| $10^{-4}$        | 1.02         | 167                             | 0.81            | -4.2            | 1.60            | 0.91            |
| $10^{-1}$        | 1.83         | 173                             | 0.79            | -3.8            | 2.36            | 1.00            |
| 1               | 2.73         | 173                             | 0.79            | -3.5            | 3.0             | 1.17            |

**TABLE II.** Same as Table I using $\alpha_2$.  

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$$\alpha_3(s) = \pi d(1 + \chi e^{-s/\Delta})/\ln(s/\Lambda^2 + \epsilon)$$

| $\Delta$ (GeV$^2$) | $\chi$ | $-\langle \bar{q}q \rangle^{1/3}$ (MeV) | $L_1(0.7\pm0.5)$ | $L_3(-3.6\pm1.3)$ | $L_5(1.4\pm0.5)$ | $L_8(0.9\pm0.3)$ |
|-------------------|--------|----------------------------------|----------------|----------------|----------------|----------------|
| 0.1               | 61.0   | 163                              | 0.82           | -4.3           | 1.22           | 0.83           |
| 0.4               | 24.0   | 169                              | 0.81           | -4.2           | 1.48           | 0.84           |
| 1.0               | 15.3   | 171                              | 0.80           | -4.1           | 1.73           | 0.88           |
| 2.0               | 12.2   | 172                              | 0.80           | -4.0           | 1.97           | 0.95           |

**TABLE III.** Same as Table I using $\alpha_3$. 

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(a) \[ \mathcal{B} \]

(b) \[ \hat{\mathcal{B}} \]

\[ [\mathcal{B}]^{-1} \]

\[ [\mathcal{B} + \Sigma]^{-1} \]
(a) Running Coupling Constant \( \alpha(s) = 3\pi s \frac{\chi}{s^2} e^{-\Delta/(4s^2)} + \pi d \ln(s/\Lambda^2 + e) \)

\[ \alpha(s) = \pi d \ln(s/\Lambda^2 + e) \]

\[ \Delta = 0.4 \text{(GeV)}^2, \chi = 1.84 \text{(GeV)} \]

\[ \Delta = 0.2 \text{(GeV)}^2, \chi = 1.65 \text{(GeV)} \]

\[ \Delta = 0.02 \text{(GeV)}^2, \chi = 1.53 \text{(GeV)} \]

\[ \Delta = 0.002 \text{(GeV)}^2, \chi = 1.41 \text{(GeV)} \]

(b) Quark Self-Energy Function \( B(s) \)

(c) Quark Self-Energy Function \( A(s) \)
(a) Running Coupling Constant $\alpha(s) = \pi d(s^2/(s^2 + \Delta) + 1/\ln(s/\Lambda^2 + e))$

\[ \alpha(s) = \pi d/\ln(s/\Lambda^2 + e) \]

\[ \Delta = 1.0 \times 10^4 \text{(GeV)}^4, \chi = 2.73 \text{(GeV)} \]

\[ \Delta = 1.0 \times 10^4 \text{(GeV)}^4, \chi = 1.83 \text{(GeV)} \]

\[ \Delta = 1.0 \times 10^4 \text{(GeV)}^4, \chi = 1.02 \text{(GeV)} \]

\[ \Delta = 1.0 \times 10^4 \text{(GeV)}^4, \chi = 0.83 \text{(GeV)} \]

(b) Quark Self-Energy Function $B(s)$

(c) Quark Self-Energy Function $A(s)$
(a) Running Coupling Constant $\alpha(s) = \pi d \left(1 + \chi e^{-\frac{s}{\Delta}}\right) \ln\left(s/\Lambda^2 + e\right)$

- $\alpha(s) = \pi d / \ln(s/\Lambda^2 + e)$
- $\Delta = 2.0$ (GeV$^2$), $\chi = 12.2$
- $\Delta = 1.0$ (GeV$^2$), $\chi = 15.3$
- $\Delta = 0.4$ (GeV$^2$), $\chi = 24.0$
- $\Delta = 0.1$ (GeV$^2$), $\chi = 61.0$

(b) Quark Self-Energy Function $B(s)$

(c) Quark Self-Energy Function $A(s)$