Lamb shift in muonic deuterium atom

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We present new investigation of the Lamb shift \((2P_{1/2} - 2S_{1/2})\) in muonic deuterium \((\mu d)\) atom using the three-dimensional quasipotential method in quantum electrodynamics. The vacuum polarization, nuclear structure and recoil effects are calculated with the account of contributions of orders \(\alpha^3, \alpha^4, \alpha^5\) and \(\alpha^6\). The results are compared with earlier performed calculations. The obtained numerical value of the Lamb shift 202.4139 meV can be considered as a reliable estimate for the comparison with forthcoming experimental data.

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I. INTRODUCTION

The muonic deuterium \((\mu d)\) is the bound state of negative muon and deuteron. The lifetime of this simple atom is determined by the muon decay in a time \(\tau_\mu = 2.19703(4) \times 10^{-6}\) s. When passing from electronic hydrogen to muonic hydrogen we observe the variation of the relative value of the nuclear structure and polarizability effects, the electron vacuum polarization corrections and recoil contributions to the fine and hyperfine structure of the energy spectrum \([1-6]\). Muonic atoms represent a unique laboratory for the determination of the nuclear properties. The experimental investigation of the \((2P - 2S)\) Lamb shift in light muonic atoms (muonic hydrogen, muonic deuterium, muonic helium ions) can give more precise values of the nuclear charge radii \([7-11]\). For more than forty years, a measurement of muonic hydrogen Lamb shift has been considered one of the fundamental experiments in atomic spectroscopy. Recently, the progress in muon beams and laser technology made such an experiment feasible. The first successful measurement of the \((\mu p)\) Lamb shift transition energy \(\left(2P_{3/2}^{F=2} - 2S_{1/2}^{F=1}\right)\) at PSI (Paul Scherrer Institute) produced the result 49881.88 (76) GHz \((206.2949(32)\) meV) \([12]\). It leads to new value of the proton charge radius \(r_p = 0.84184(36)(56)\) fm, where the first and second uncertainties originate respectively from the experimental uncertainty of 0.76 GHz and the uncertainty 0.0049 meV in the Lamb shift value which is dominated by the proton polarizability term. The new value of proton radius \(r_p\) improves the CODATA value \([13]\) by an order of magnitude. Another important project which exists now at PSI in the CREMA (Charge Radius Experiment with Muonic Atoms)
The program of the investigation of the energy levels in light muonic atoms suggests that the theoretical calculations of fine and hyperfine structure of states with \( n = 1, 2 \) will be performed with high accuracy. Note that a discrepancy in the new proton charge radius and CODATA value induced both a reanalysis of the earlier obtained contributions to the observed transition frequency and a study of the hypothetical muon-proton interaction \([14-17]\).

Theoretical investigations of the Lamb shift \((2P - 2S)\), fine and hyperfine structure of light muonic atoms was performed many years ago in Refs. \([1, 18-23]\) on the basis of the Dirac equation and nonrelativistic three-dimensional method (see other references in review articles \([1, 6]\)). Their calculation took into account different QED corrections with the accuracy \(0.0005 \text{ fm}\). Recently an approach of \([1]\) was extended to the case of muonic deuterium in \([2, 3]\) where fine and hyperfine structure was analyzed with high accuracy. Different corrections to fine and hyperfine structure of muonic hydrogen are calculated on the basis of three-dimensional method in quantum electrodynamics in \([4, 24–28]\). The vacuum polarization effects of order \(\alpha^5\) were considered in \([29–31]\). We consider such effects of the electron vacuum polarization, recoil and nuclear structure corrections which are crucial to attain high accuracy. With the exception of the nuclear structure and polarizability contribution, we calculate all corrections in the intervals \((2P_{1/2} - 2S_{1/2})\) and \((2P_{3/2} - 2P_{1/2})\) with a precision \(0.0001 \text{ meV}\) and \(0.00001 \text{ meV}\) correspondingly. Our purpose consists in a recalculation and improvement of the earlier obtained results \([1, 2]\) and derivation the reliable independent estimate for the \((2P_{1/2} - 2S_{1/2})\) and \((2P_{3/2} - 2S_{1/2})\) Lamb shift, which can be used for the comparison with forthcoming experimental data. Modern numerical values of fundamental physical constants are taken from Ref. \([13]\): the electron mass \(m_e = 0.510998910(13) \times 10^{-3} \text{ GeV}\), the muon mass \(m_\mu = 0.1056583668(38) \text{ GeV}\), the fine structure constant \(\alpha^{-1} = 137.035999084(51)\) \([33]\), the deuteron mass \(m_d = 1.875612793(47) \text{ GeV}\). Numerical values of the proton structure corrections are obtained with the 2010 year CODATA value for the deuteron charge radius \(r_d = 2.1424(21) \text{ fm}\) and \(r_d = 2.130 \pm 0.003 \pm 0.009 \text{ fm}\) from \([34]\).

II. EFFECTS OF VACUUM POLARIZATION IN THE ONE-PHOTON INTERACTION

Our approach to the investigation of the Lamb shift \((2P - 2S)\) in muonic deuterium is based on the use of quasipotential method in quantum electrodynamics \([27, 28, 35]\), where the two-particle bound state is described by the Schrödinger equation. In perturbation theory the basic contribution to the muon-deuteron interaction operator is determined by the Breit Hamiltonian \([1, 36]\):

\[
H_B = \frac{p^2}{2\mu} - \frac{Z\alpha}{r} - \frac{p^4}{8m_1^3} - \frac{p^4}{8m_2^3} + \frac{\pi Z\alpha}{2} \left(\frac{1}{m_1^2} + \frac{\delta l}{m_2^2}\right) \delta(r) - \frac{Z\alpha}{2m_1m_2r} \left(p^2 + \frac{r(rp)p}{r^2}\right) + \frac{Z\alpha}{r^3} \left(\frac{1}{4m_1^2} + \frac{1}{2m_1m_2}\right) (L\sigma_1) = H_0 + \Delta V^B,
\]

where \(L\) and \(\sigma_1\) are the orbital angular momentum and the Pauli matrices, respectively.
FIG. 1: Effects of one-loop and two-loop vacuum polarization in the one-photon interaction.

where \( H_0 = p^2/2\mu - Z\alpha/r \), \( m_1, m_2 \) are the muon and deuteron masses, \( \mu = m_1m_2/(m_1+m_2) \). The deuteron factor \( \delta_I = 0 \) because we used further the common definition of the deuteron charge radius

\[
\delta_2 = -6\frac{dF_C}{dQ^2}\bigg|_{Q^2=0} [37, 38].
\]

The wave functions of 2S- and 2P-states are equal to

\[
\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-\frac{Wr}{2}} \left( 1 - \frac{W_r}{2} \right), \quad \psi_{21m}(r) = \frac{W^{3/2}}{2\sqrt{6}} e^{-\frac{Wr}{2}} W^r Y_{1m}(\theta, \phi), \quad W = \mu Z\alpha. \quad (2)
\]

The ratio of the Bohr radius of muonic deuterium to the Compton wavelength of the electron \( m_e/W = 0.7 \), so, the basic contribution of the electron vacuum polarization (VP) to the Lamb shift is of order \( \alpha(Z\alpha)^2 \) (see Fig.1(a)). Accounting for the modification of the Coulomb potential due to the vacuum polarization in the coordinate representation

\[
V^C_{\text{VP}}(r) = \frac{\alpha}{3\pi} \int_1^\infty d\xi \rho(\xi) \left( -\frac{Z\alpha}{r} e^{-2m_e\xi r} \right), \quad \rho(\xi) = \frac{\sqrt{\xi^2 - 1}(2\xi^2 + 1)}{\xi^4},
\]

we present equations for the one-loop VP contributions to shifts of 2S-, 2P-states and the Lamb shift (2P − 2S):

\[
\Delta E_{\text{VP}}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi} \int_1^\infty \rho(\xi) d\xi \int_0^\infty x dx \left( 1 - \frac{x}{2} \right)^2 e^{-x(1+\frac{2m_e\xi}{W})} = -245.3194 \text{ meV}, \quad (4)
\]

\[
\Delta E_{\text{VP}}(2P) = -\frac{\mu(Z\alpha)^2\alpha}{72\pi} \int_1^\infty \rho(\xi) d\xi \int_0^\infty x^3 dx e^{-x(1+\frac{2m_e\xi}{W})} = -17.6847 \text{ meV}, \quad (5)
\]

\[
\Delta E_{\text{VP}}(2P-2S) = 227.6347 \text{ meV}, \quad (6)
\]

where we round for definiteness the number to four decimal digits. The subscript VP designates the contribution of electron vacuum polarization. Experimental error in a determination of the particle masses and fine structure constant does not influence on the digits given in (6). The muon one-loop vacuum polarization correction of order \( \alpha(Z\alpha)^4 \) is known in analytical form [8]. We included corresponding value \( \Delta E_{\text{MV,VP}}(2P-2S) = \alpha^5 \mu^3/30\pi m_1^3 = 0.01968 \text{ meV} \) to the total shift in section V (Eqs.(71)-(72)). This result agrees with that in [2]. Two-loop vacuum polarization effects in the one-photon interaction are shown in Fig.1(b,c,d). To obtain a contribution of the amplitude in Fig.1(b) to the interaction operator, it is necessary to use the following replacement in the photon propagator:

\[
\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \frac{1}{k^2 + 4m_e^2\xi^2}. \quad (7)
\]
FIG. 2: Effects of three-loop vacuum polarization in the one-photon interaction (a,b) and in third order perturbation theory (c). \(\tilde{G}\) is the reduced Coulomb Green function (33).

In the coordinate representation a diagram with two sequential loops gives the following particle interaction operator:

\[
V_{VP-VP}^C(r) = \frac{\alpha^2}{9\pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left( -\frac{Z\alpha}{r} \right) \frac{1}{(\xi^2 - \eta^2)} \left( \xi^2 e^{-2m_e\xi r} - \eta^2 e^{-2m_e\eta r} \right),
\]

where the subscript \((VP-VP)\) corresponds to two sequential loops in the Feynman amplitude (Fig.1b). Averaging (8) over the Coulomb wave functions (2), we find the contribution to the Lamb shift of order \(\alpha^2\):

\[
\Delta E_{VP-VP}(2P - 2S) = -\frac{\mu\alpha^2 (Z\alpha)^2}{18\pi^2} \int_1^\infty d\xi \int_1^\infty d\eta \frac{\rho(\xi)\rho(\eta)}{(\xi + \eta)} \times [4m_e^2 W^3 (4m_e\xi\eta + W(\xi + \eta)) \left( 8m_e^2\xi^2\eta^2 + 4m_eW \xi \eta (\xi + \eta) + W^2 (\xi^2 + \eta^2) \right)] = 0.2956 \text{ meV}.
\]

Higher order \(\alpha^2(Z\alpha)^4\) correction is determined by an amplitude with two sequential electron (VP) and muon (MVP) loops. Corresponding potential is given by

\[
\Delta V_{VP-MVP}(r) = -\frac{4(Z\alpha)^2}{45\pi^2 m_f^2} \int_1^\infty \rho(\xi) d\xi \left[ \pi\delta(r) - \frac{m_e^2\xi^2}{r} e^{-2m_e\xi r} \right].
\]

Its contribution to the shift \((2P - 2S)\) is equal to

\[
\Delta E_{VP-MVP}(2P - 2S) = 0.0001 \text{ meV}.
\]

The two-loop vacuum polarization graphs (Figs.1(b,c,d)) were first calculated in \([39, 40]\). The particle interaction potential, corresponding to two-loop amplitudes in Fig.1(c,d) with second order polarization operator, takes the form \([39]\):

\[
\Delta V_{2-loop VP}^C = -\frac{2}{3} \frac{Z\alpha}{r} \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 f(v) dv \frac{e^{-\frac{2m_e}{\sqrt{1-v^2}}}}{(1-v^2)^2},
\]

where the subscript \((2-loop VP)\) corresponds only to two-loop Feynman amplitudes shown in Fig.1(c,d), the spectral function

\[
f(v) = v \left( (3-v^2)(1+v^2) \left[ Li_2 \left( \frac{1-v}{1+v} \right) + 2Li_2 \left( \frac{1-v}{1+v} \right) + \frac{3}{2} \ln \frac{1+v}{1-v} \ln \frac{1+v}{1-v} - \ln \frac{1+v}{1-v} \ln v \right] \right)
\]

(13)
\[ + \left[ \frac{11}{16} (3-v^2)(1+v^2) + \frac{v^4}{4} \right] \ln \frac{1+v}{1-v} + \left[ \frac{3}{2} v(3-v^2) \ln \frac{1-v^2}{4} - 2v(3-v^2) \ln v \right] + \frac{3}{8} v(5-3v^2) \],

\( Li_2(z) \) is the Euler dilogarithm. The potential \( \Delta V_{2\text{loop}}^C V P(r) \) gives larger contribution as compared with (8) both to the hyperfine structure and Lamb shift \((2P-2S)\). In the case of the Lamb shift we find the following contribution:

\[ \Delta E_{2\text{loop}} V P(2P-2S) = 1.3704 \text{ meV}. \]  (14)

Changing in (12) the electron mass to the muon mass one can obtain two loop muon vacuum polarization correction. It is known in analytical form from the paper [41] (we present their result with five decimal digits):

\[ \Delta E_{2\text{loop} \ M V P}(2P-2S) = \frac{41}{324} \frac{\alpha^2 (Z\alpha)^4 \mu^3}{\pi^2 m_1^2} = 0.00017 \text{ meV} \]  (15)

The numerical values of corrections (9), (14) and the desired accuracy of the calculation show that it is important to consider three-loop contributions of the electron vacuum polarization (see Fig.2). One part of corrections to the potential from the diagrams of three-loop vacuum polarization in the one-photon interaction can be derived by means of equations (8)-(12) (sequential loops in Fig.2(a,b)) [28]. Corresponding contributions to the potential and the splitting \((2P-2S)\) are given by

\[ V_{V P-V P-V P}^C = -\frac{Z\alpha}{r (3\pi)^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta d\eta) \int_1^\infty \rho(\zeta) d\zeta \times \]

\[ \left[ e^{-2m_e \xi r} \frac{\zeta^4}{(\xi^2-\xi^2)(\eta^2-\xi^2)} + e^{-2m_e \eta r} \frac{\xi^4}{(\xi^2-\xi^2)(\eta^2-\xi^2)} + e^{-2m_e \eta r} \frac{\eta^4}{(\xi^2-\eta^2)(\xi^2-\eta^2)} \right], \]

\[ V_{V P}^{2\text{loop}} V P = -\frac{4\alpha^3 (Z\alpha)}{9\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \frac{f(\eta d\eta)}{\eta} \frac{1}{r(\eta^2-\xi^2)} \left( \eta^2 e^{-2m_e \eta r} - \xi^2 e^{-2m_e \xi r} \right), \]  (17)

\[ \Delta E_{V P-V P-V P}(2P-2S) = 0.0005 \text{ meV}, \]  (18)

\[ \Delta E_{V P-V P}(2P-2S) = 0.0034 \text{ meV}, \]  (19)

where subscripts \((VP-VP-VP)\) and \((VP-2-loop VP)\) designate only the Feynman amplitudes shown in Fig.2(a,b) respectively. Six order vacuum polarization contributions including Eqs.(18)-(19) were obtained in [2]. The contribution of other diagrams corresponding to the three loop contribution in \(1\gamma\) approximation were calculated in Refs. [29, 30] for muonic hydrogen. We estimated their contribution for the Lamb shift in \((\mu d)\) from the results given in Eqs.(18) and (23) of Ref. [29]; the result is 0.0021 meV. This gives a total three loop contribution 0.0060 meV in one-photon interaction, which is included in Table I. Two-loop and three-loop vacuum polarization corrections appearing in second order perturbation theory, are calculated in the next sections. Our sum of all three loop VP contributions 0.0086 meV is very close to the total three loop contribution 0.00842 meV given in Table I of [31] with the account of a rounding.

Additional one-loop vacuum polarization diagram is presented in Fig.3. In the energy spectrum it gives the correction of fifth order in \(\alpha\) (the Wichmann-Kroll correction) [42, 43].
The particle interaction potential can be written in this case in the integral form:

$$\Delta V_{WK}^{W K}(r) = \frac{\alpha (Z \alpha)^3}{\pi r} \int_0^\infty \frac{d\zeta}{\zeta^4} e^{-2m_e \zeta r} \left[ -\frac{\pi^2}{12} \sqrt{\zeta^2 - 1} \theta(\zeta - 1) + \int_0^\zeta dx \sqrt{\zeta^2 - x^2} f_{WK}(x) \right].$$

(20)

Exact form of the spectral function $f_{WK}$ is presented in Refs. [6, 42, 43]. Numerical integration in (20) with the wave functions (2) gives the following contribution to the Lamb shift:

$$\Delta E_{WK}^{WK}(2P - 2S) = -0.0011 \text{meV}.$$  

(21)

This agrees well with other calculations [2, 31]. The detailed calculation of all three light-by-light graphs is presented in [44]. We included in Table I their estimation using (21) and the results from [44].

III. RELATIVISTIC CORRECTIONS WITH THE VACUUM POLARIZATION EFFECTS

The electron vacuum polarization effects lead not only to corrections in the Coulomb potential (3), but also to the modification of other terms of the Breit Hamiltonian (1). The one-loop vacuum polarization corrections in the Breit interaction were obtained in [1, 25]:

$$\Delta V_{i,VP}^B(r) = \frac{\alpha (Z \alpha)^4}{3 \pi r} \int_1^\infty \rho(\xi) d\xi \sum_{i=1}^4 \Delta V_{i,VP}^B(r),$$

(22)

where the superscript $B$ designates the Breit interaction. In first order perturbation theory (PT) the potentials $\Delta V_{i,VP}^B(r)$ give necessary contributions of order $\alpha(Z \alpha)^4$ to the shift $(2P - 2S)$:
FIG. 4: Effects of one-loop and two-loop vacuum polarization in second order perturbation theory (SOPT). The dashed line shows the Coulomb photon. $\tilde{G}$ is the reduced Coulomb Green function (34). The potentials $\Delta V^B$, $\Delta V^C_{VP}$ and $\Delta V^B_{VP}$ are determined respectively by relations (1), (3) and (22).

$$\Delta E_{1,VP}^B(2P - 2S) = -0.0353 \text{ meV},$$
$$\Delta E_{2,VP}^B(2P - 2S) = 0.0011 \text{ meV},$$
$$\Delta E_{3,VP}^B(2P - 2S) = 0.0012 \text{ meV},$$
$$\Delta E_{4,VP}^B(2P - 2S) = -0.0023 \text{ meV}. \quad (27, 28, 29, 30)$$

The potentials $\Delta V^B_{2,VP}$, $\Delta V^B_{3,VP}$, $\Delta V^B_{4,VP}$ take into account the recoil effects over the ratio $m_1/m_2$. We have included in Table I the summary correction of order $\alpha(Z\alpha)^4$, which is determined by equations (27)-(30). The next to leading order correction of order $\alpha^2(Z\alpha)^4$ appears in the energy spectrum from two-loop modification of the Breit Hamiltonian. We consider in the potential the term of the leading order in $m_1/m_2$ (the function $f(v)$ is determined by expression (13)):

$$\Delta V^B_{2\text{-loop } VP}(r) = \frac{\alpha^2(Z\alpha)}{12\pi^2} \left( \frac{1}{m_1^2} + \frac{\delta t}{m_2^2} \right) \int_0^1 \frac{f(v)dv}{1 - v^2} \left[ 4\pi \delta(r) - \frac{4m_e^2}{(1 - v^2)r} e^{-\frac{2m_e}{\sqrt{1 - v^2}}} \right]. \quad (31)$$

Corresponding $(2P - 2S)$ shift is the following:

$$\Delta E_{2\text{-loop } VP}^B(2P - 2S) = -0.0002 \text{ meV}. \quad (32)$$

Other two-loop contributions to the Breit potential are omitted because they give the energy corrections which lie outside an accuracy of the calculation in this work.

In second order perturbation theory (SOPT) we have a number of the electron vacuum polarization contributions in orders $\alpha^2(Z\alpha)^2$ and $\alpha(Z\alpha)^4$, shown in Fig.4 (b,c):

$$\Delta E_{SOPT}^{VP} = \langle \psi | \Delta V^C_{VP} \tilde{G} \Delta V^C_{VP} | \psi \rangle + 2 \langle \psi | \Delta V^B \tilde{G} \Delta V^C_{VP} | \psi \rangle. \quad (33)$$
Laguerre polynomials. For a calculation of the Lamb shift (2P order PT. The second order perturbation theory corrections in the energy spectrum of hydrogen-like system are determined by the reduced Coulomb Green function $G$ (RCGF). It has a partial wave expansion [45]:

$$\tilde{G}_n(r, r') = \sum_{l,m} \tilde{g}_{nl}(r, r') Y_{lm}(n) Y_{lm}^*(n').$$

The radial function $\tilde{g}_{nl}(r, r')$ was presented in [45] in a form of the Sturm expansion in the Laguerre polynomials. For a calculation of the Lamb shift ($2P - 2S$) in muonic deuterium it is convenient to use the compact representation for the RCGF of $2S$- and $2P$-states, which was obtained in [4, 46]:

$$\tilde{G}(2S) = -\frac{Z\alpha\mu^2}{4x_1x_2} e^{-\frac{x_1+x_2}{2}} \frac{1}{4\pi} g_{2S}(x_1, x_2),$$

$$g_{2S}(x_1, x_2) = 8x_1 - 4x_2^2 + 8x_1 + 12x_1x_2 - 26x_1^2 + 2x_1^3 - 4x_1^2 - 26x_1x_2^2 + 23x_1^2x_2^2 - \frac{x_1^3x_2^2 + 2x_1x_2^3 - x_1^2x_2^3 + 4e^x(1-x)(x_2 - 2)x_2 + 4(x_1 - 2)x_1(x_2 - 2)x_2 \times}{-2C + Ei(x_1) - \ln(x_1) - \ln(x_2)},$$

$$\tilde{G}(2P) = -\frac{Z\alpha\mu^2}{36x_1x_2^2e^{-\frac{x_1+x_2}{2}}} \frac{3}{4\pi} \frac{x_1x_2}{g_{2P}(x_1, x_2)},$$

$$g_{2P}(x_1, x_2) = 24x_1^3 + 36x_1^3x_2 + 36x_1^3x_2^2 + 36x_1x_2^3 + 36x_1^2x_2^3 + 49x_1^3x_2^3 - 3x_1^4x_2^3 - 12e^x(2 + x_1 + x_2)x_1^3 - 3x_1^4x_2^3 - 12x_1^3x_2^3[ -2C + Ei(x_1) - \ln(x_1) - \ln(x_2)]$$

where $x_1 = min(x_1, x_2)$, $x_2 = max(x_1, x_2)$, $C = 0.57721566...$ is the Euler constant. As a result the two-loop vacuum polarization contribution to the first term of (33) can be presented originally in the integral form (Fig.4(c)). The subsequent numerical integration gives the following results:

$$\Delta E_{SOPT}^{VPVP}(2S) = -\frac{\mu^2(Z\alpha)^2}{72\pi^2} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times$$

$$\times \int_0^\infty \left(1 - \frac{x}{2}\right) e^{-x(1 + \frac{2m\xi}{w})} dx \int_0^\infty \left(1 - \frac{x'}{2}\right) e^{-x'(1 + \frac{2m\eta}{w})} dx' g_{2S}(x, x') = -0.1750 \text{ meV},$$
\begin{align}
\Delta E_{\text{SOPT}}^{VP,VP}(2P) &= -\frac{\mu \alpha^2 (Z\alpha)^2}{7776\pi^2} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta 
\times \int_0^\infty e^{-x(1+\frac{2m\xi}{w})}dx \int_0^\infty e^{-x'(1+\frac{2m\eta}{w})}dx' g_{2P}(x, x') = -0.0030 \text{ meV},
\end{align}

where the superscript \((VP,VP)\) designates the second order PT contribution when each of the perturbation potentials contains VP correction. The results (39), (40) agree with the calculation in [31]. Changing one electron VP potential by the muon VP potential we find that corresponding correction to the Lamb shift is very small:

\begin{equation}
\Delta E_{\text{SOPT}}^{VP,MVP}(2P - 2S) = 0.0001 \text{ meV}.
\end{equation}

The second term in (33) has the similar structure (see Fig.4(b)). A transformation of different matrix elements entering in it is carried out with the use of algebraic relations of the form:

\begin{equation}
<\psi|\frac{P^4}{(2\mu)^2} \sum' \sum_m |\psi_m><\psi_m| \Delta V_{VP}^C|\psi > = <\psi|(E_2 + \frac{Z\alpha}{r})\hat{H}_0 + \frac{Z\alpha}{r} \sum' \sum_m |\psi_m><\psi_m| \Delta V_{VP}^C|\psi > =
\end{equation}

\begin{equation}
<\psi| \left( E_2 + \frac{Z\alpha}{r} \right)^2 \tilde{G} \Delta V_{VP}^C|\psi > = - <\psi| \frac{Z\alpha}{r} \Delta V_{VP}^C|\psi > + <\psi| \frac{Z\alpha}{r} |\psi > <\psi| \Delta V_{VP}^C|\psi > .
\end{equation}

Omitting further details of the calculation of numerous matrix elements in (42), we present here the summary numerical contribution from second term in (33) to the shift \((2P - 2S)\):

\begin{equation}
\Delta E_{\text{SOPT}}^{B,VP}(2P - 2S) = 0.0530 \text{ meV}.
\end{equation}

Other contributions of second order PT (see Fig.4(d,e,f)) have the general structure similar to Eqs.(39), (40). They appear after the replacements \(\Delta V_{VP}^C \rightarrow \Delta V^B\) and \(\Delta V_{VP}^C \rightarrow \Delta V_{VP,VP}^C\) in the basic amplitude shown in Fig.4(c). The estimate of this contribution of order \(\alpha^2(Z\alpha)^4\) to the shift \((2P - 2S)\) can be derived if we take into account in the Breit potential the leading order term in the ratio \(m_1/m_2\). Its numerical value is

\begin{equation}
\Delta E_{\text{SOPT}}^{VP,VP,\Delta V^B}(2P - 2S) = 0.0004 \text{ meV}.
\end{equation}

The two-loop vacuum polarization contribution is determined also by the amplitude in Fig.4(a). To obtain its numerical value in the energy spectrum we have to use Eqs.(3) and (22). In the leading order in the ratio \(m_1/m_2\) we take again the potential (22), which leads to very small correction of order \(\alpha^2(Z\alpha)^4\):

\begin{equation}
\Delta E_{\text{SOPT}}^{VP,\Delta V^B}(2P - 2S) = -0.0001 \text{ meV}.
\end{equation}

Three-loop vacuum polarization contributions to the energy spectrum in second order perturbation theory are presented in Fig.5. Respective potentials required for their calculation are obtained earlier in relations (3), (8), (12). Considering an accuracy of the calculation we can restrict our analysis by a shift of 2S-level, which can be written in the form:

\begin{equation}
\Delta E_{\text{SOPT}}^{VP,VP,VP}(2S) = -\frac{\mu \alpha^3 (Z\alpha)^2}{108\pi^3} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \int_1^\infty \rho(\zeta)d\zeta \int_0^\infty dx(1 - \frac{x}{2}) \times
\end{equation}

\begin{equation}
\int_0^\infty dx'(1 - \frac{x'}{2})e^{-x'(1+\frac{2m\xi}{w})} \frac{1}{\xi^2 - \eta^2} \left[ \xi^2 e^{-x(1+\frac{2m\eta}{w})} - \eta^2 e^{-x(1+\frac{2m\eta}{w})} \right] g_{2S}(x, x') = -0.0007 \text{ meV}.
\end{equation}
FIG. 6: The leading order nuclear structure and vacuum polarization corrections. The thick point represents the nuclear vertex operator.

\[
\Delta E^{2-\text{loop}}_{V_{P,V_{P}}} (2S) = -\frac{\mu^3 (Z\alpha)^2}{18\pi^3} \int_0^1 \frac{f(v)dv}{1 - v^2} \int_1^\infty \rho(\xi) d\xi \times
\]
\[
\times \int_0^\infty dx \left(1 - \frac{x}{2}\right) e^{-x(1+\frac{2m_\xi}{\sqrt{1-v^2} W})} \int_0^\infty dx' \left(1 - \frac{x'}{2}\right) e^{-x'(1+\frac{2m_\xi}{\sqrt{1-v^2} W})} g_{2S}(x,x') = -0.0018 \text{ meV},
\]

In third order perturbation theory (TOPT) the three-loop VP contribution to the Lamb shift consists of two terms. One part of it is shown in Fig.2(c). This contribution can be calculated by means of (3), (35)-(38) [29, 31]. We carry out the coordinate integration analytically and the integration over three spectral parameters numerically. The result

\[
\Delta E_{V_{P,V_{P}}}^{V_{P},V_{P}}(2P - 2S) = 0.0001 \text{ meV}
\]
is in the agreement with [29, 31].

IV. NUCLEAR STRUCTURE AND VACUUM POLARIZATION EFFECTS

An influence of nuclear structure on the muon motion in muonic deuterium is determined in the leading order by the root mean square (rms) radius of the deuteron (charge radius). We present further all charge radius corrections at two values of \(r_d\): \(r_d = 2.1424(21) \text{ fm (CODATA 2010)}\) and \(r_d = 2.130(9) \text{ fm [34] (Fig.6(a))}\):

\[
\Delta E_{\text{str}}(2P - 2S) = -\frac{\mu^3 (Z\alpha)^4}{12} r_d^2 = -6.07313 \cdot r_d^2 = -27.8749(-27.5532) \text{ meV}.
\]

At this point and further the subscript \(\text{str}\) designates the structure correction. The precise value of the deuteron charge radius is needed for the interpretation of new data on transitions in muonic deuterium atom.

There are vacuum polarization corrections connected with the deuteron structure in first and second orders of perturbation theory (see diagrams in Fig.6(b,c)). The potential corresponding to the amplitude in Fig.6(b) can be written as follows:

\[
\Delta V_{\text{str}}^{VP}(r) = \frac{2Z\alpha^2}{9} r_d^2 \int_1^\infty \rho(\xi) d\xi \left[ \delta(r) - \frac{m_\xi^2\xi^2}{\pi r^2} e^{-2m_\xi \xi r} \right].
\]

Its contribution to the \(2P - 2S\) Lamb shift is determined by the formula:

\[
\Delta E_{\text{str}}^{VP}(2P - 2S) = -\frac{\mu^3 (Z\alpha)^4}{36\pi} r_d^2 \int_1^\infty \rho(\xi) d\xi \left[ 1 - \frac{16m_\xi^4\xi^4}{(2m_\xi + W)^2} \right] =
\]
FIG. 7: Nuclear structure corrections of order $(Z\alpha)^5$. The thick point is the deuteron vertex operator.

FIG. 8: Nuclear structure and two-loop vacuum polarization effects in one-photon interaction. The thick point is the nuclear vertex operator.

$$\Delta E_{str,SOPT}(2P - 2S) = -\frac{\mu^3\alpha(Z\alpha)^4}{36\pi} r_d^2 \int_1^\infty \rho(\xi) d\xi \times$$

$$\frac{-12 + 23b_1 - 8b_1^2 - 4b_1^3 + 4b_1^4 + 4b_1 (3 - 4b_1 + 2b_1^2) \ln b_1}{b_1^2} =$$

$$= -0.020487 \cdot r_d^2 \text{ meV} = -0.0940(-0.0929) \text{ meV}, \quad b_1 = 1 + \frac{2m_e}{W\xi}.$$

Factorizing $r_d^2$ in expressions (49), (51)-(52) we obtain the finite size correction in the form:

$$\Delta E_{str}(2P - 2S) + \Delta E_{str}^{VP}(2P - 2S) + \Delta E_{str,SOPT}^{VP}(2P - 2S) = -6.10712 \cdot r_d^2 = -28.0309(-27.7074) \text{ meV.}$$

The next important correction of order $(Z\alpha)^5$ is described by one-loop exchange diagrams (Fig.7). An investigation of elastic contribution to the Lamb shift and the deuteron polarizability contribution was performed in [22, 47–50]. Recently new detailed calculation of nuclear structure and polarizability corrections which improves previous theoretical results is presented in [51]. We have included in Table I the value of the $(2P - 2S)$ shift 1.680(16) meV from [51].
FIG. 9: Nuclear structure and two-loop vacuum polarization effects in second order perturbation theory. The thick point is the nuclear vertex operator. \( \tilde{G} \) is the reduced Coulomb Green function.

Two-loop vacuum polarization corrections with an account of nuclear structure are presented in Fig.8(a,b,c). The interaction operators constructed by means of Eq.(7) are determined by integral formulas:

\[
\Delta V_{str-VP}(r) = \frac{2Z\alpha^3}{27\pi^2 r_d^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \times \\
\times \left[\pi \delta(r) - \frac{m_e^2}{r(\xi^2 - \eta^2)} \left(\xi^4 e^{-2m_e \xi r} - \eta^4 e^{-2m_e \eta r}\right)\right], \tag{54}
\]

\[
\Delta V_{str-loopVP}(r) = \frac{4Z\alpha^3}{9\pi^2 r_d^2} \int_0^1 f(v) dv \left[\pi \delta(r) - \frac{m_e^2}{r(1-v^2)} e^{-2m_e \sqrt{1-v^2}}\right]. \tag{55}
\]

The sum of corrections from (54) and (55) to the Lamb shift \( (2P - 2S) \) is equal to

\[
\Delta E_{strVP}(2P - 2S) = -10.5 \cdot 10^{-5} \cdot r_d^2 = -0.0005(0.0005) \text{ meV}. \tag{56}
\]

Two-loop vacuum polarization and nuclear structure corrections of order \( \alpha^2 (Z\alpha)^4 \) in second order PT shown in Fig.9(a,b,c,d), also can be calculated by means of relations discussed in section III. The summary shift is equal to

\[
\Delta E_{str,SOPT}(2P - 2S) = -9.5 \cdot 10^{-5} \cdot r_d^2 = -0.0004(0.0004) \text{ meV}. \tag{57}
\]

There exists also the nuclear structure correction of order \( \alpha (Z\alpha)^5 \) coming from two-photon exchange diagrams with the electron vacuum polarization insertion (see Fig.10). It can be calculated as the elastic contribution of order \( (Z\alpha)^5 \) \[50\]. However, there is no need to calculate it because in this case we have the same cancelation between elastic two-photon correction and deuteron excited states correction as for the contribution of order \( (Z\alpha)^5 \) \[51\]. Indeed, using the notations of Ref.[51] we can present the muon matrix element \( P_{VP} \) for nonrelativistic two-photon exchange with an account of the vacuum polarization in the form:

\[
P_{VP} = \frac{2\alpha^3}{3\pi} \phi^2(0) \int_1^\infty \rho(\xi) d\xi \left[\frac{(4\pi)^2}{(2\pi)^3 q^2 (q^2 + 4m_e^2 \xi^2)} \frac{1}{E + \frac{q^2}{2m_1}} \right] e^{iq(R-R')} - 1 + \frac{q^2}{6}(R-R')^2, \tag{58}
\]

where \( R \) is the position of the proton with respect to nuclear mass center. Integrating (58) over \( q \) and expanding the resulting expression over small parameter \( \sqrt{2m_1 E |R - R'|} \) we obtain:

\[
P_{VP} = \frac{32\alpha^3}{3} m_1 \phi^2(0) |R - R'|^3 \int_1^\infty \rho(\xi) d\xi \left[a_\xi^2 - 3a_\xi^2 + 6a_\xi + 6e^{-a_\xi} - 6 \right] \frac{1}{12a_\xi^2} \tag{59}
\]

\[
2 \pi^2 m_e^2 \]
\[-2m_1 E |R - R'|^2 \frac{a_\xi^4 - 4a_\xi^3 + 12a_\xi^2 - 24a_\xi - 24e^{-a_\xi} + 24}{48a_\xi}, \quad a_\xi = 2m_e \xi |R - R'|.\]

It follows from (59) that in the leading order in $\sqrt{2m_1 E |R - R'|}$ elastic correction to atomic energy is canceled by the deuteron excited states correction (see more detailed discussion in [51]). An estimation of second term contribution in the square brackets of (59) to the energy spectrum can be derived if we take into account that the integral over $\xi$ is determined by the region near $\xi \approx 1$. Expanding second term in (59) at small $a_\xi$ we obtain $(-\pi/240a_\xi)$. Then performing analytical integration over $\xi$ and summing over excited states we obtain the contribution to the Lamb shift:

\[
\delta E_{\text{pol}}^{(2P - 2S)} = -\frac{m_1^2 a_\xi^3 \phi^2(0)}{1024m_e} \left[ \frac{1}{3} \langle \phi_D | R^2 H_D R^2 | \phi_D \rangle - \frac{4}{5} \langle \phi_D | R_i H_D R^2 R_i | \phi_D \rangle + \right.
\]

\[
\left. + \frac{2}{5} \langle \phi_D | (R_i R_j - \frac{1}{3} \delta_{ij} R^2) H_D (R_i R_j - \frac{1}{3} \delta_{ij} R^2) | \phi_D \rangle \right] = -0.0001 \text{ meV},
\]

where $\phi_D$ is the deuteron wave function. We make all integrations in (60) analytically using the deuteron wave function in the zero-range approximation [52]

\[
\phi_D(r) = \sqrt{\frac{\kappa}{2\pi r}} e^{-\kappa r},
\]

where $\kappa = 0.0457 \text{ GeV}$ is the inverse deuteron size.

Another term in the Lamb shift of order $\alpha(Z\alpha)^5$ is determined by muon-line radiative correction to the nuclear size effect. It was obtained in [53] in a suitable form for subsequent numerical estimate:

\[
\Delta E_{\text{str}}^{(Z\alpha)^5} (2P - 2S) = 1.985 \frac{\alpha(Z\alpha)^5 \mu^3}{8} r_d^2 = 9.62 \cdot 10^{-4} \cdot r_d^2 = 0.0044(0.0044) \text{ meV}. \tag{62}
\]

There exists also the correction of order $\alpha(Z\alpha)^5$ with muon vacuum polarization (see diagrams in Fig.(10)). Accounting for the result of its calculation from [6] the total coefficient in (62) should be changed: 1.985 $\rightarrow$ 1.485. However, we can consider together with the muon VP and nuclear structure amplitudes in Fig.(10) the contribution of the deuteron excited states. Calculating this summary contribution by means of equations similar to (58)-(60) (see also [51]) we observe the cancelation of elastic correction and excited states correction.

Nuclear structure corrections of order $(Z\alpha)^6$ can be derived with the use of relativistic corrections to nonrelativistic wave functions in matrix element (49) [6,22,54]. We present here total contribution to the Lamb shift $(2P - 2S)$ including additional state independent correction obtained in [22,54]:

\[
\Delta E_{\text{str}}^{(Z\alpha)^6} (2P - 2S) = \left( \frac{Z\alpha}{12} \mu^3 \right) \left\{ r_d^2 \left[ \ln \mu Z\alpha r + C - \frac{3}{2} \right] - \frac{1}{2} r_d^2 + \frac{1}{3} \left( r^3 \right) \right\} -
\]

\[
- I_2^\text{rel} - I_3^\text{rel} - \mu^2 F_{\text{NR}} + \frac{1}{40} \mu^2 \langle \alpha^4 \rangle \left\{ -21.28 \cdot 10^{-4} \cdot r_d^2 + 0.0029 = -0.0069(-0.0068) \text{ meV},
\]

where the quantities $I_{2,3}^\text{rel}, F_{\text{NR}}$ are written explicitly in [22,54]. We have extracted in the square brackets the frequently used quantity (main term) for an estimation of the contribution to the $(2P - 2S)$ Lamb shift in hydrogen atom because other corrections are very small (near 1%) and could be safely omitted. In the case of muonic deuterium they give the contribution near 25% of the main term and should be taken into account. Separate energy shifts for the $2S$ and $2P$ states are given in [22,54]. Numerical estimate is obtained on the basis of an exponential parametrization for the charge distribution from [22].
An investigation of different order corrections to the Lamb shift \((2P - 2S)\) of electronic hydrogen has been performed for many years. Modern analysis of the advances in the solution of this problem is presented in a review articles [6, 43, 55, 56]. The most part of the results was obtained in analytical form, so they can be used directly in muonic deuterium atom. In this section we analyze different contributions to the energy spectrum of \((\mu d)\) up to the sixth order in \(\alpha\) and derive their numerical estimations in the Lamb shift \((2P - 2S)\).

There are several recoil corrections of different order in \(\alpha\) which give important contributions in order to attain the necessary accuracy of the calculation. The recoil correction of order \((Z\alpha)^4\mu^3/m^2_1\) to the Lamb shift appears in the matrix element of the Breit potential with functions (2). It is calculated for muonic deuterium in [5, 37]:

\[
\Delta E_{rec}(2P - 2S) = \frac{\mu^3(Z\alpha)^4}{12m^2_2} = 0.0672 \text{ meV.}
\]  

(64)

The recoil correction of fifth order in \((Z\alpha)\) is determined by the expression [6, 55]:

\[
\Delta E_{rec}^{(Z\alpha)^5} = \frac{\mu^3(Z\alpha)^5}{m_1m_2\pi n^3} \left[ \frac{2}{3} \delta_{l0} \ln \frac{1}{3\alpha} + \frac{8}{3} \ln k_0(n, l) - \frac{1}{9} \delta_{r0} - \frac{7}{3} a_n \right] - \frac{2}{3} \delta_{l0} (m^2_2 \ln \frac{m^2_1}{\mu} - \frac{m^2_2}{\mu} \ln \frac{m^2_1}{\mu}) - \frac{1}{3} \ln \frac{1}{\alpha} - \frac{8}{3} \ln k_0(n, l) - \frac{1}{9} \delta_{r0} - \frac{7}{3} a_n \right] - m^2_2 \ln \frac{m^2_1}{\mu} - \frac{m^2_2}{\mu} \ln \frac{m^2_1}{\mu}
\]  

(65)
where $\ln k_0(n,l)$ is the Bethe logarithm:

\[
\ln k_0(2S) = 2.811769893120563, \tag{66}
\]

\[
\ln k_0(2P) = -0.030016708630213, \tag{67}
\]

\[
a_n = -2 \left[ \ln \frac{2}{n} + \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) + \frac{1}{2n} \right] \delta_{l0} + \frac{(1 - \delta_{l0})}{l(l+1)(2l+1)}. \tag{68}
\]

Eq. (65) gives the following numerical correction to the Lamb shift:

\[
\Delta E_{\text{rec}}^{(Z\alpha)^5}(2P - 2S) = -0.0266 \text{ meV}. \tag{69}
\]

The recoil correction of the sixth order in $(Z\alpha)$ was calculated analytically in [32, 57–60]:

\[
\Delta E_{\text{rec}}^{(Z\alpha)^6}(2P - 2S) = \frac{(Z\alpha)^6}{8m_e} \left( \frac{23}{6} - 4 \ln 2 \right) = 0.0001 \text{ meV}. \tag{70}
\]

Omitting explicit form of radiative-recoil corrections of orders $\alpha(Z\alpha)^5$ and $(Z^2\alpha)(Z\alpha)^4$ from Table 9 [6], we present their numerical value to the Lamb shift $(2P - 2S)$ of muonic deuterium atom:

\[
\Delta E_{\text{rad-rec}}(2P - 2S) = -0.0026 \text{ meV}. \tag{71}
\]

The energy contributions obtained in [6, 61, 62] from radiative corrections to the lepton line, the Dirac and Pauli form factors and muon vacuum polarization are given by

\[
\Delta E_{\text{MVP,MSE}}(2S) = \frac{\alpha(Z\alpha)^4}{8\pi} \frac{\mu^3}{m_1^2} \left[ \frac{4}{3} \ln \frac{m_1}{\mu(Z\alpha)^2} - \frac{4}{3} \ln k_0(2S) + \frac{38}{45} \right. \]

\[
+ \frac{\alpha}{\pi} \left( -\frac{9}{4} \zeta(3) + \frac{3}{2} \pi^2 \ln 2 - \frac{10}{27} \pi^2 - \frac{2179}{648} \right) + 4\pi Z\alpha \left( \frac{427}{384} - \frac{\ln 2}{2} \right) \right] = 0.7647 \text{ meV}, \tag{72}
\]

\[
\Delta E_{\text{MVP,MSE}}(2P) = \frac{\alpha(Z\alpha)^4}{8\pi} \frac{\mu^3}{m_1^2} \left[ -\frac{4}{3} \ln k_0(2P) - \frac{m_1}{6\mu} \right. - \frac{\alpha m_1}{3\pi \mu} \left( \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \ln 2 + \frac{\pi^2}{12} + \frac{197}{144} \right) \right] = -0.0100 \text{ meV}. \tag{73}
\]

The diagram in Fig.11(b) with electron loop polarization insertion in the radiative spectrum, which can be expressed in terms of the slope of the Dirac form factor $F'_1$ and the Pauli form factor $F_2$ [6]:

\[
\Delta E_{\text{rad+VP}}(nS) = \frac{\mu^3}{m_1^2} (Z\alpha)^4 \frac{4m_1^2 F'_1(0) \delta_{l0} + F_2(0) C_{jl}}{2l+1}, \tag{74}
\]

\[
C_{jl} = \delta_{l0} + (1 - \delta_{l0}) \frac{j(j + 1) - l(l + 1) - \frac{3}{7} \frac{m_1}{\mu}}{l(l + 1)}. \tag{75}
\]

Two-loop contribution to the form factors $F'_1(0)$ and $F_2(0)$ was calculated in [63] (see also [1, 6]):

\[
m_1^2 F'_1(0) = \left( \frac{\alpha}{\pi} \right)^2 \left[ \frac{1}{9} \ln^2 \frac{m_1}{m_e} - \frac{29}{108} \ln \frac{m_1}{m_e} + \frac{1}{9} \zeta(2) + \frac{395}{1296} \right], \tag{76}
\]

\[
m_1^2 F_2(0) = \left( \frac{\alpha}{\pi} \right)^2 \left[ \ln m_1^2 \right], \tag{77}
\]
TABLE I: Lamb shift \((2P_{1/2} - 2S_{1/2})\) in muonic deuterium atom.

| Contribution to the splitting | \(\Delta E(2P - 2S)\), meV | Equation, Reference |
|------------------------------|-----------------|---------------------|
| VP contribution of order \(\alpha(Z\alpha)^2\) in one-photon interaction | 227.6347 | (6), [2] |
| Two-loop VP contribution of order \(\alpha^2(Z\alpha)^2\) in one-photon interaction | 1.6660 | (9), (14), [2] |
| VP and MVP contribution in one-photon interaction | 0.0001 | (11), [2] |
| Three-loop VP contribution in one-photon interaction | 0.0060 | (17), (18), [29, 31] |
| The Wichmann-Kroll correction | -0.0011 | (21), [2, 31] |
| Light-by-light contribution | 0.0001 | [31] |
| Relativistic and VP corrections of order \(\alpha(Z\alpha)^4\) in first order PT | -0.0353 | (27)-(30), [5] |
| Relativistic and two-loop VP corrections of order \(\alpha^2(Z\alpha)^4\) in first order PT | -0.0002 | (32) |
| Two-loop VP contribution of order \(\alpha^2(Z\alpha)^2\) in second order PT | 0.1720 | (39)-(41), [31] |
| Relativistic and one-loop VP corrections of order \(\alpha(Z\alpha)^4\) in second order PT | 0.0530 | (43), [5] |
| Relativistic and two-loop VP corrections of order \(\alpha^2(Z\alpha)^4\) in second order PT | 0.0004 | (44)-(45) |
| Three-loop VP contribution in second order PT of order \(\alpha^3(Z\alpha)^2\) | 0.0025 | (46)-(47), [31] |
| Three-loop VP contribution in third order PT of order \(\alpha^3(Z\alpha)^2\) | 0.0001 | (48), [29, 31], |
| Nuclear structure contribution of order \((Z\alpha)^4\) | -27.8749 | (49), [2, 6] |
| Nuclear structure and polarizability contribution of order \((Z\alpha)^5\) | 1.6800 | [51] |
| Nuclear structure and VP contribution in \(1\gamma\) interaction of order \(\alpha(Z\alpha)^4\) | -0.0620 | (51) |
| Nuclear structure and VP contribution in second order PT of order \(\alpha(Z\alpha)^4\) | -0.0940 | (52) |
| Nuclear structure and two-loop VP contribution in \(1\gamma\) interaction of order \(\alpha^2(Z\alpha)^4\) | -0.0005 | (56) |
| Nuclear structure and two-loop VP contribution in second order PT of order \(\alpha^2(Z\alpha)^4\) | -0.0004 | (57) |
Table I (continued).

| 1 | 2 | 3 |
|---|---|---|
| Nuclear structure and polarizability contribution of order $\alpha(Z\alpha)^5$ with VP correction | -0.0001 | (60) |
| Nuclear structure contribution of order $\alpha(Z\alpha)^5$ with muon-line radiative correction | 0.0044 | (62), [61] |
| Nuclear structure contribution of order $\alpha(Z\alpha)^5$ with VP correction | 0.0069 | (63), [22, 54] |
| Recoil correction of order $(Z\alpha)^4$ | 0.0672 | (64), [5] |
| Recoil correction of order $(Z\alpha)^5$ | -0.0266 | (69), [2, 6, 55] |
| Recoil correction of order $(Z\alpha)^6$ | 0.0001 | (70), [6] |
| Recoil correction to VP of order $\alpha(Z\alpha)^5$ (seagull term) | 0.0002 | [5] |
| Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$ | -0.0026 | (71), Table 9 [6] |
| Muon self-energy and MVP contribution | -0.7747 | (72)-(73), [2, 6] |
| Muon form factor $F_2'(0)$, $F_2(0)$ contributions of order $\alpha^2(Z\alpha)^4$ | -0.0018 | (78), [4, 6, 63] |
| VP correction to muon self-energy | -0.0047 | (80), [4, 6] |
| HVP contribution | 0.0129 | [64, 65] |
| Total contribution | 202.4139 ± 0.0573 (rd = 2.1424(21) fm) | 202.7375 ± 0.2352 (rd = 2.130(9) fm) |

\[
F_2(0) = \left(\frac{\alpha}{\pi}\right)^2 \left[ \frac{1}{3} \ln \frac{m_1}{m_e} - \frac{25}{36} + \frac{\pi^2}{4} \ln \frac{m_1}{m_e} - 4 \frac{m_e^2}{m_1^2} \ln \frac{m_1}{m_e} - 3 \frac{m_e^2}{m_1^2} \right].
\]

Then the correction to the Lamb shift is equal to
\[
\Delta E_{\text{rad}+\text{VP}}(2P-2S) = -0.0018 \text{ meV.}
\]  

To estimate the muon self-energy and electron vacuum polarization contribution in Fig.11(a), we use the relation obtained in [4]:
\[
\Delta E_{\text{MSE}}^{\text{VP}}(2P-2S) = -0.0047 \text{ meV.}
\]  

The hadron vacuum polarization (HVP) contribution which can be taken into account on the basis of the numerical result obtained for muonic hydrogen in [64,65] is included in Table I. The error of the measurement of the cross section $\sigma(e^+e^- \to \pi^+\pi^-)$ was decreased to a few per cents. So, we estimate in 5% ($\pm0.0006$ meV) corresponding theoretical error of HVP correction.
VI. FINE STRUCTURE OF THE 2P-STATE

The leading order \((Z\alpha)^4\) contribution to fine structure is determined by the operator \(\Delta V^{fs}\):

\[
\Delta V^{fs}(r) = \frac{Z\alpha}{4m_1^2r^3} \left[ 1 + \frac{2m_1}{m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right] (L\sigma_1). \tag{81}
\]

\(\Delta V^{fs}\) includes the recoil correction and muon anomalous magnetic moment \(a_\mu\) correction. Fine structure interval \((2P_{3/2} - 2P_{1/2})\) for muonic deuterium can be written in the form \([66-68]\):

\[
\Delta E^{fs} = E(2P_{3/2}) - E(2P_{1/2}) = \frac{\mu^3(Z\alpha)^4}{32m_1^2} \left[ 1 + \frac{2m_1}{m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right] + \frac{5m_1(Z\alpha)^6}{256} - \frac{m_1^2(Z\alpha)^6}{64m_2} + \frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^2} \left[ \ln \left( \frac{\mu(Z\alpha)^2}{m_1} \right) + \frac{1}{5} \right] + \alpha(Z\alpha)^4 A_{VP} + \alpha^2(Z\alpha)^4 B_{VP} + A_{str}(Z\alpha)^6 \mu^2 \cdot r_d^2.
\]

This expression includes the relativistic correction of order \((Z\alpha)^6\), which can be calculated on the basis of the Dirac equation, relativistic recoil effects of order \(m_1(Z\alpha)^6/m_2\), correction of order \(\alpha(Z\alpha)^6\) enhanced by the factor \(\ln(Z\alpha)\) \([6]\), a number of terms of fifth and sixth order in \(\alpha\) which are determined by effects of the vacuum polarization and nuclear structure. Recoil correction \((-m_1^2(Z\alpha)^4/32m_2^2)\) (the Barker-Glover correction \([69]\)) is also taken into account in Eq.(82). This is evident from the expansion of first term in (82) over the mass ratio \(m_1/m_2\) up to second order terms: \(m_1(Z\alpha)^4(1 - m_1/m_2)/32\). The contributions to the coefficients \(A_{VP}\) and \(B_{VP}\) arise in first and second orders of perturbation theory. Numerical values of terms in the expression (82), which are presented in analytical form, are quoted in Table II for a definiteness with an accuracy 0.00001 meV. Fine structure interval (82) in the energy spectrum of electronic hydrogen is considered for a long time as a basic test of quantum electrodynamics \([6, 55]\).

Fine structure potential with the leading order vacuum polarization and its contribution to the coefficient \(A_{VP}\) are given by (84):

\[
\Delta V^{fs}_{VP}(r) = \frac{\alpha(Z\alpha)}{12\pi m_1^2r^3} \int_1^\infty \rho(s)ds \left[ 1 + \frac{2m_1}{m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right] e^{-2m_e sr}(1 + 2m_e sr)(L\sigma_1), \tag{83}
\]

\[
\Delta E^{fs}_1 = \frac{\mu^3 \alpha(Z\alpha)^4}{96\pi m_1^2} \left[ 1 + \frac{2m_1}{m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right] \int_1^\infty \rho(\xi)d\xi \frac{1 + \frac{6m_e}{W}\xi}{(1 + 2m_e/W\xi)^3} = 0.00346 \text{ meV}. \tag{84}
\]

Higher order corrections \(\alpha^2(Z\alpha)^4\) contributing to \(a_\mu\) are taken into account in this expression as well as recoil effects. The same order \(O(\alpha(Z\alpha)^4)\) contribution can be obtained in second order perturbation theory in the form:

\[
\Delta E^{fs}_{VP,SOPT} = \frac{\alpha(Z\alpha)^4 \mu^3}{1728\pi m_1^2} \left[ 1 + 2a_\mu + \left( 1 + a_\mu \right) \frac{2m_1}{m_2} \right] \int_1^\infty \frac{\rho(\xi)d\xi}{(1 + 2m_e/W\xi)^5} \times \tag{85}
\]

\[
\times \left[ 18 \frac{2m_e\xi}{W} \left( \frac{8m_e\xi}{W} + 11 \right) + 4 \left( 1 + \frac{2m_e\xi}{W} \right) \ln \left( 1 + \frac{2m_e\xi}{W} \right) + 3 \right] = 0.00229 \text{ meV}.
\]
Let us consider two-loop vacuum polarization contributions in the one-photon interaction shown in Fig.1. They give corrections to fine structure splitting of $P$-wave levels of order $\alpha^2(Z\alpha)^4$. In the coordinate representation, the interaction operator has the form [28, 35]:

$$\Delta V_{2-\text{loop}}^{fS, VP}(r) = \frac{Z\alpha}{r^3} \left[ \frac{1 + 2a_\mu}{4m_1^2} + \frac{1 + a_\mu}{2m_1m_2} \right] (L\sigma_1) \times$$

$$\times \left( \frac{\alpha}{3\pi} \right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \frac{\rho(\eta)d\eta}{(\xi^2 - \eta^2)^{3/2}} \left[ \xi^2 (1 + 2m_\xi \xi r)e^{-2m_\xi \xi r} - \eta^2 (1 + 2m_\eta \eta r)e^{-2m_\eta \eta r}. \right]$$

Averaging (86) over the wave functions (2), we obtain the following correction to the interval (82):

$$\Delta E_{2-\text{loop}}^{fS, VP} = \frac{\mu^3\alpha^2(Z\alpha)^4}{288\pi^2m_1^2} \left[ 1 + 2a_\mu + \frac{2m_1}{m_2}(1 + a_\mu) \right] \int_1^\infty \rho(\xi)d\xi \times$$

$$\times \int_1^\infty \rho(\eta)d\eta \left[ \xi^2 \left( \frac{6m_\xi}{W} + 1 \right)^{1/3} - \eta^2 \left( \frac{6m_\eta}{W} + 1 \right)^{1/3} \right] = 0.000003 \text{ meV}. \tag{87}$$

Two-loop vacuum polarization potential and the correction to fine structure $(2P_{3/2} - 2P_{1/2})$ are given by

$$\Delta V_{2-\text{loop}}^{fS, VP}(r) = \frac{2Z\alpha^3}{3\pi^2r^3} \left[ \frac{1 + 2a_\mu}{4m_1^2} + \frac{1 + a_\mu}{2m_1m_2} \right] \int_0^1 f(v)dv \frac{e^{-2m_e\xi r}}{1 - v^2} (1 + \frac{2m_e\xi r}{\sqrt{1 - v^2}})(L\sigma_1), \tag{88}$$

$$\Delta E_{2-\text{loop}}^{fS, VP} = \frac{\mu^3\alpha^2(Z\alpha)^4}{48\pi^2m_1^2} \left[ 1 + 2a_\mu + \frac{2m_1}{m_2}(1 + a_\mu) \right] \int_0^1 f(v)dv \left( \frac{6m_e}{W\sqrt{1 - v^2}} + 1 \right)^{1/3} = 0.00002 \text{ meV}. \tag{89}$$

Two-loop vacuum polarization contributions in second order perturbation theory shown in Fig.4(a,d-f) $(\Delta V^B \to \Delta V^{fS})$, have the same order $\alpha^2(Z\alpha)^4$. For their calculation it is necessary to employ the modified Coulomb potential by two-loop vacuum polarization [27, 28]. The amplitude in Fig.4(e-f) gives the following correction of order $\alpha^2(Z\alpha)^4$ to fine structure splitting:

$$\Delta E_{2-\text{loop}}^{fS, V, \text{SOPT}} = \frac{\mu^3\alpha^2(Z\alpha)^4}{3\pi^2m_1m_2} \left[ 1 + a_\mu + \frac{m_2}{2m_1}(1 + 2a_\mu) \right] \int_0^1 f(v)dv \frac{1}{1 - v^2} \times$$

$$\times \left[ \frac{1}{(1 + \frac{2m_e}{W\sqrt{1 - v^2}})^6} \left( 5 \frac{2m_e}{W\sqrt{1 - v^2}} + 4(1 + \frac{2m_e}{W\sqrt{1 - v^2}}) \ln(1 + \frac{2m_e}{W\sqrt{1 - v^2}}) \right) \right] = 0.000026 \text{ meV}. \tag{90}$$

Two other contributions from amplitudes in Fig.4(a,d) have the similar integral structure. Their numerical values are included in Table II.

There exists also the correction to fine structure splitting due to nuclear structure. In $1\gamma$-interaction it is related with the charge form factor of the deuteron. Fine structure potential (81) is obtained for the point deuteron. In the case of the deuteron of finite size we can express the contribution of nuclear structure to fine structure splitting in terms of the charge radius [68]:

$$\Delta E_{str}^{fS} = -\frac{\mu^5(Z\alpha)^6}{64m_1^2}d[1 + \frac{2m_1}{m_2} + 2a_\mu(1 + \frac{m_1}{m_2})] = -0.00028 \text{ meV}. \tag{91}$$

Earlier the calculation of the nuclear structure corrections to the energies of $P$-levels of order $(Z\alpha)^6$ was performed in [22]. Our numerical result (91) for the fine structure splitting agrees with the calculation in [22].
TABLE II: Fine structure of 2P-state in muonic deuterium atom.

| Contribution to fine splitting $\Delta E_{fs}$ | Numerical value in meV | Equation, Reference |
|-----------------------------------------------|------------------------|---------------------|
| Contribution of order $(Z\alpha)^4$ \[ \frac{\mu^3(Z\alpha)^4}{32m_2^2} (1 + \frac{2m_1}{m_2}) \] | 8.83848 | (82), [2, 6] |
| Muon AMM contribution \[ \frac{\mu^3(Z\alpha)^4}{16m_1^2} a_\mu (1 + \frac{m_1}{m_2}) \] | 0.01957 | (82), [2, 6] |
| Contribution of order $(Z\alpha)^6$ | 0.00031 | (82), [2, 6] |
| Contribution of order $(Z\alpha)^4 m_1/m_2$ | -0.00001 | (82), [2, 6] |
| Contribution of order $\alpha(Z\alpha)^4$ in first order PT $\langle \Delta V_{fs}^{VP} \rangle$ | 0.00346 | (84) |
| Contribution of order $\alpha(Z\alpha)^4$ in second order PT $\langle \Delta V_{fs}^{VP} \cdot \tilde{G} \cdot \Delta V_{fs}^{VP} \rangle$ | 0.00229 | (85) |
| Contribution of order $\alpha(Z\alpha)^6 \frac{\alpha(Z\alpha)^6}{32\pi m_1^4} \ln \frac{\mu(Z\alpha)^2}{m_1} + \frac{1}{5}$ | -0.00001 | (82), [6] |
| VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{fs}^{VP-VP} \rangle$ | 0.000003 | (87) |
| VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{fs}^{VP-VP} \rangle$ | 0.00002 | (89) |
| VP Contribution in second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{fs}^{VP-VP} \rangle$ | 0.000002 | Fig.4(a), $\Delta V^B \rightarrow \Delta V_{fs}^{VP}$ |
| VP Contribution in second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{fs}^{VP-VP} \rangle$ | -0.00001 | Fig.4(d), $\Delta V^B \rightarrow \Delta V_{fs}^{VP}$ |
| VP Contribution in second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{fs}^{VP-VP} \rangle$ | 0.000026 | (90), Fig.4(e-f), $\Delta V^B \rightarrow \Delta V_{fs}^{VP}$ |
| Nuclear structure correction in 1γ interaction | -0.00028 | (91), [22] |
| Summary contribution | 8.86386 | |

VII. SUMMARY AND CONCLUSION

In this work, various corrections of orders $\alpha^3$, $\alpha^4$, $\alpha^5$ and $\alpha^6$ are calculated to the Lamb shift $(2P_{1/2} - 2S_{1/2})$ and fine structure splitting $(2P_{3/2} - 2P_{1/2})$ in muonic deuterium atom. Contrary to earlier performed investigations of the energy spectra of light muonic atoms in [1, 2, 18], we have used the three-dimensional quasipotential approach for the description of two-particle bound state. Our analysis of different contributions to the Lamb shift ac-
counts for the terms of two groups. First group contains the specific corrections for muonic deuterium, connected with the electron vacuum polarization effects, nuclear structure and recoil effects in first and second order perturbation theory. As a rule the contributions of this group are obtained in integral form over auxiliary parameters and calculated numerically. The necessary order corrections of second group include analytical results known from the corresponding calculation in the electronic hydrogen Lamb shift. Recent advances in the physics of the energy spectra of simple atoms are presented in the review articles [6, 53, 56] which we use in this study. Numerical values of all corrections are written in Tables I, II, which contain also basic references on the earlier performed investigations (other references can be found in Ref. [1, 2, 6]). We compare our intermediate results for different corrections with calculations given in [2]. Most part of the results including the Uehling, Källen-Sabry, Wichmann-Kroll corrections, muon Lamb shift contribution, nuclear size and VP corrections and recoil terms agree well. Our results for relativistic contributions to the vacuum polarization are in agreement with those obtained in [5]. Second order VP correction (39) and (40) agrees with the result of [31] just as the three loop VP contribution which is determined in Table I by three lines corresponding to one-photon interaction (0.0060 meV), second order PT (0.0025 meV) and third order PT (0.0001 meV). Total numerical value 202.4139 meV of the Lamb shift (2P − 2S) in muonic deuterium atom from Table I is in good agreement with the theoretical result 202.263 meV obtained in [2]. The difference of our result from Ref. [2] is connected with the calculation of new contributions of higher order in α and m_1/m_2, the proton structure and polarizability correction [51] and slightly different numerical value of the charge radius of the deuteron r_d used in this work. Two-loop vacuum polarization contribution 0.1720 meV of order α^2(Zα)^2 in second order PT is absent in [2]. The value of the charge radius r_d = 2.139(3) fm is used in [2]. Fine structure splitting (2P_{3/2} − 2P_{1/2}) 8.86386 meV in Table II agrees also with the result 8.864 meV from [2]. Recently, improved analysis of different corrections to the Lamb shift in (µd) is performed in [3]. Total value of the Lamb shift (2P_{1/2} − 2S_{1/2}) for r_d = 2.130 fm according to Table 4 from [3] amounts 202.9440 meV. This value exceeds our result 202.7375 meV by 0.2065 meV. In our opinion the only two essential differences between our Table I and [3] are related with the Zemach correction 0.4329 meV and polarizability correction 1.5 meV [3]. It was shown in [51] that the Zemach correction is canceled by the deuteron excited states contribution. As a result the nuclear structure and polarizability contribution is equal to 1.680 meV [51] which we use in our work.

As has been mentioned above numerical values of corrections are obtained with an accuracy 0.0001 meV because certain contributions to the Lamb shift (2P − 2S) of order α^6 attain the value of tenth part of μeV. The theoretical error caused by uncertainties in fundamental parameters (fine structure constant, particle masses) entering the leading order contributions is around 10^{-5} meV. The other part of theoretical error is related to the QED corrections of higher order. This part can be estimated from the leading contribution of higher order in α: m_1α(Zα)^6ln(Zα)/πn^3 ≈ 0.0001 meV. Theoretical uncertainty connected with nuclear structure and polarizability contributions is equal to 0.0160 meV [51]. We have also small theoretical uncertainty determined by HVP contribution which we estimate in 5% (±0.0006 meV). This estimation is based on the experimental uncertainty in the cross section of e^+e^- annihilation into hadrons. The rounding errors can amount to 0.0001±0.0002 meV. Finally, the biggest theoretical error ±0.0550 meV (for r_d = 0.1424(21) fm) is related to the uncertainty of the deuteron charge radius. Thereby, the total theoretical error of the calculation is equal to ±0.0573 meV. To obtain this estimate we add the above
mentioned uncertainties in quadrature.

Let us summarize the basic particularities of the Lamb shift calculation performed above.

1. Numerical value of specific parameter $m_e/\mu Z\alpha = 0.7$ in muonic deuterium atom is sufficiently large, so the electron vacuum polarization effects play essential role in the interaction of the bound particles. We have considered the one-loop, two-loop and three-loop VP contributions to the Lamb shift ($2P_{1/2} - 2S_{1/2}$). A number of important vacuum polarization contributions from $1\gamma$-interaction agrees with the results obtained in $[2, 29-31]$.

2. Nuclear structure effects are expressed in the Lamb shift of muonic deuterium atom in terms of the deuteron charge radius $r_d$. We analyze complex effects due to nuclear structure and vacuum polarization in first and second orders of perturbation theory. The elastic nuclear structure contribution from two-photon exchange amplitudes is canceled by the deuteron polarizability correction $[51]$.

3. Nuclear structure and polarizability effects give the largest theoretical uncertainty in the total value of the Lamb shift ($2P - 2S$). It is useful to express the final theoretical value of the ($2P - 2S$) Lamb shift in the form $\Delta E^{\text{LS}}(2P - 2S) = (230.4511 - 6.108485 \cdot r_d^2) \text{ meV}$ with the value of the deuteron charge radius defined in fm. Then, comparing this expression with the experimental value of the Lamb shift measured with the precision 0.01 meV (50 ppm) we can obtain more accurate value of $r_d$ with an accuracy 0.0005 fm.

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