In this paper, we present an overview of different types of random walk strategies with local and non-local transitions on undirected connected networks. We present a general approach to analyzing these strategies by defining the dynamics as a discrete time Markovian process with probabilities of transition expressed in terms of a symmetric matrix of weights. In the first part, we describe the matrices of weights that define local random walk strategies like the normal random walk, biased random walks, random walks in the context of digital image processing and maximum entropy random walks. In addition, we explore non-local random walks like Lévy flights on networks, fractional transport and applications in the context of human mobility. Explicit relations for the stationary probability distribution, the mean first passage time and global times to characterize the random walk strategies are obtained in terms of the elements of the matrix of weights and its respective eigenvalues and eigenvectors. Finally, we apply the results to the analysis of particular local and non-local random walk strategies; we discuss their efficiency and capacity to explore different types of structures. Our results allow to study and compare on the same basis the global dynamics of different types of random walk strategies.

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I. INTRODUCTION

Since their introduction as an informal question posted in the journal Nature in 1905 by Rayleigh, random walks have had an important impact in science with applications in a broad range of fields like biology, physics, chemistry, economy, computation, among many others [1, 2]. The success of random walk models in different applications lies in their simplicity, typically defined as a walker in a particular space that moves randomly without memory of its previous path, making this characteristic a good candidate in the description of processes like the diffusive transport, chemical reactions, fluctuations in the economy and even the foraging of some animal species [1, 2]. Despite the mentioned simplicity in the definition, the consequences of the dynamics of a random walker are non-trivial and continue to surprise us with new results and with all the complexity that emerges from its simple rules.

In recent years, much of the interest in random walks have migrated to the study of complex systems described through networks [7–9]. In this context, the interplay between the topology of the network and the dynamical processes taking place on this structure are of utmost importance [7, 10, 11]. In particular, random walk strategies that allow transitions from one node to one of its nearest neighbors on the network constitutes a paradigmatic case and are the natural framework to study diffusive transport [10, 12–14], navigation and search processes in networks [15–18], multiplex networks [19], with applications in a variety of systems like the propagation of epidemics and spreading phenomena [20–24], the dynamics on social networks [22], the analysis of information [23], human mobility [24], among others [2, 10].

On the other hand, in different cases full or partial knowledge of the network structure is available to define a random walk capable to use this information to increase the capacity to visit nodes with hops to the nearest neighbors but also long-range transitions beyond this local neighborhood. In Fig. 1 we illustrate local and non-local transitions in a network. In this case, the walker can visit one of the nearest-neighbors with a local transition...
but also there is the option of a non-local transition. By using this long-range dynamics the random walker can contact directly long-distance nodes without the intervention of intermediate nodes and without altering the topology of the network. As we will see, some non-local random walk strategies consider the shortest path connecting two nodes whereas others include quantities that contain all the possible paths between nodes.

In addition, it is important to mention that random walks with a non-local character have been explored in the literature. This is the case of Lévy flights on networks where random transitions occur to non-nearest neighbors with a probability that decays as a power law of the distance separating two nodes [25]; the capacity of this strategy to explore networks has been studied in [25] [31]. Lévy flights on networks were generalized by Estrada et. al. by using a random multi-hopper model defined in terms of decaying functions of the shortest-path distances; this approach is explored in detail in [32]. Furthermore, we also found non-local dynamics in the fractional transport on networks defined in terms of the fractional Laplacian of a graph [33]. In this case, long-range displacements on the network emerge from a formalism that is introduced as the equivalent of the fractional diffusion equation on networks [33] [34]. This strategy is studied in the context of transport in networks and lattices [33] [35], in connection with information analysis [30] and quantum transport on networks [41]. The fractional transport is a particular case of a series of strategies that can be defined in terms of functions of the Laplacian of a network [42]. In general, the study of random walks with long-range displacements on networks opens several questions regarding the way in which these large displacements can appear or be induced in different applications. Moreover, it is necessary the introduction of new methods and quantities that allow us to compare in the same background the efficiency to visit the nodes of a network through random walk strategies.

In this paper, we explore different types of local and non-local random walks on networks. We present a general approach to study these processes on the same basis by using the information contained in a symmetric matrix of weights used to define the probability of transition between nodes. We model the dynamics as a discrete time Markovian process. In the first part, we describe the matrices of weights that define local random walk strategies: traditional random walks, biased random walks, random walks in the context of digital image processing and, maximum entropy random walks. In the same way, examples of non-local random walks are described: Lévy flights on networks, fractional dynamics and applications in the context of human mobility. In all these cases, explicit relations for the stationary probability distribution of the random walker are obtained in terms of the elements of the matrix of weights that defines each strategy. After analyzing the transition matrix for these different processes, in a second part of the paper, a general formalism to calculate the mean first passage time and global times to characterize the dynamics is presented. Analytical expressions in terms of eigenvalues and eigenvectors of the transition matrix are obtained for all these quantities. Finally, we apply the results to the analysis of local and non-local random walk strategies to discuss and compare their efficiency and capacity to explore networks.

II. RANDOM WALKS ON WEIGHTED NETWORKS

In this section, we introduce different concepts about random walks on weighted networks and the notation implemented to describe this process. We introduce a general random walker with probabilities of transition defined in terms of a network and a matrix of weights, the respective temporal evolution is modeled as a discrete time Markovian process for which we find an analytical result for its stationary probability distribution.

We consider undirected weighted networks with \( N \) nodes \( i = 1, \ldots, N \). The topology of the network is described by an adjacency matrix \( \mathbf{A} \) with elements \( A_{ij} = A_{ji} = 1 \) if there is an edge between the nodes \( i \) and \( j \) and \( A_{ij} = 0 \) otherwise; in particular, \( A_{ii} = 0 \) to avoid lines connecting a node with itself. The degree of the node \( i \) is the number of neighbors that this node has and is given by \( k_i = \sum_{i=1}^{N} A_{il} \). Additionally to the network structure, we have a \( N \times N \) symmetric matrix of weights \( \mathbf{\Omega} \) with elements \( \Omega_{ij} = \Omega_{ji} \geq 0 \) and \( \Omega_{ii} = 0 \). The matrix \( \mathbf{\Omega} \) can include information of the structure of the network or incorporate additional data describing characteristics of links and nodes. By definition, the strength of the node \( i \) is given by \( S_i = \sum_{l=1}^{N} \Omega_{il} \) and represents the total weight of the node \( i \).

In the following, we study discrete time random walks on connected weighted networks with transition probabilities between nodes determined by the elements of the matrix of weights \( \mathbf{\Omega} \). The occupation probability to find the random walker in the node \( j \) at time \( t \) starting from \( i \) at \( t = 0 \) is given by \( P_{ij}(t) \) and obeys the master equation [12] [43]:

\[
P_{ij}(t + 1) = \sum_{m=1}^{N} P_{im}(t)\pi_{m \rightarrow j}, \quad (1)
\]

where the transition probability \( \pi_{i \rightarrow j} \) between the nodes \( i \) and \( j \) is given by

\[
\pi_{i \rightarrow j} = \frac{\Omega_{ij}}{\sum_{l=1}^{N} \Omega_{il}} = \frac{\Omega_{ij}}{S_i}. \quad (2)
\]

The transition matrix \( \mathbf{\Pi} \), with elements \( \pi_{i \rightarrow j} \), in the general case is not symmetric; however, as a consequence of Eq. (2) and the symmetry of the matrix \( \mathbf{\Omega} \), we obtain \( S_i \pi_{i \rightarrow j} = S_j \pi_{j \rightarrow i} \), a result that establishes a connection between the transition probabilities \( \pi_{i \rightarrow j} \) and \( \pi_{j \rightarrow i} \). On
the other hand, iterating the master equation \( \Pi \), the probability \( P_{ij}(t) \) takes the form

\[
P_{ij}(t) = \sum_{j_1,\ldots,j_{t-1}} \pi_{i\rightarrow j_1} \cdot \pi_{j_1\rightarrow j_2} \cdots \pi_{j_{t-1}\rightarrow j}
\]  

(3)

and, using Eq. (3), we obtain

\[
P_{ij}(t) = \sum_{j_1,\ldots,j_{t-1}} \frac{S_{j_1}}{S} \cdots \frac{S_{j_{t-1}}}{S} \pi_{j_{t-1}\rightarrow j_1} \cdots \pi_{j\rightarrow i}
\]  

\[
= \frac{S_i}{S} P_{ij}(t).
\]

(4)

In this way, the detailed balance condition

\[
S_i P_{ij}(t) = S_j P_{ji}(t)
\]

(5)

is deduced as a direct consequence of the symmetry of \( \Omega \). The relation in Eq. (5) allows to obtain the stationary probability distribution \( P_{ij}^{\infty} = \lim_{t \to \infty} P_{ij}(t) \), that gives the probability to find the random walker in the node \( j \) when \( t \to \infty \). We have

\[
P_{i\rightarrow j}^{\infty} = \frac{S_i}{\sum_{l=1}^{N} S_l},
\]

(6)

showing that the stationary distribution \( P_{i\rightarrow j}^{\infty} \) of the node \( i \) is directly proportional to its strength \( S_i \). The stationary distribution \( P_{i\rightarrow j}^{\infty} \) in Eq. (6) is a general result that characterizes the global behavior of the random walker. As we will see in the next section, this quantity allows to rank and classify the nodes of the network with a measure that combines the topological characteristics of the network structure with their capacity of transport modeled by the master equation \( \Pi \) and the transition matrix \( \Pi \). Furthermore, it is well known in the context of Markovian processes that the value \( 1/P_{i\rightarrow j}^{\infty} \) is the average number of steps required for the random walker to return for the first time to the node \( i \). \[23\ 45\].

III. RANDOM WALK STRATEGIES

Diverse types of random walk strategies can be explored in terms of the matrix of weights formalism described before. The only restrictions to this approach are the symmetry of the elements of the matrix of weights \( \Omega_{ij} = \Omega_{ji} \), the condition \( \Omega_{ij} \geq 0 \) and \( \Omega_{ii} = 0 \). In this section, we present particular cases of navigation strategies that can be described by using this method. We divide our discussion into local strategies, for which the transitions of the random walker are restricted to adjacent sites on the network, and long-range strategies, for which the walker can hop with displacements beyond its nearest neighbors.

A. Local random walks

In local random walk navigation strategies, the random walker always hops from a node to one of its nearest neighbors on the network. As a consequence, the elements of the matrix of weights take the form \( \Omega_{ij} = g_{ij} A_{ij} \), where, as we explain in the following part, the value \( g_{ij} \) is related to quantities assigned to each node or to the weight of the link that connects the nodes \( i \) and \( j \).

1. Normal random walk

In this case, the weights coincide with the elements of the adjacency matrix; therefore \( \Omega_{ij} = A_{ij} \). As a consequence, from Eq. (2), the transition matrix is given by

\[
\pi_{i\rightarrow j} = \frac{A_{ij}}{k_i}.
\]

(7)

By definition, the normal random walker hops with equal probability from a node to one of its nearest neighbors in the network. In addition, from Eq. (3), the stationary distribution is \( P_{i\rightarrow j}^{\infty} = \frac{k_i}{\sum_{i=1}^{N} k_i} \). Normal random walks have been extensively studied in different contexts with applications in diverse types of networks; in particular, lattices \[12\ 43\], general graphs \[13\ 46\], complex networks \[17\ 17\ 49\], fractal and recursive structures \[50\], among others \[23\].

2. Preferential navigation

In the preferential navigation strategy, a random walker hops with transition probabilities \( \pi_{i\rightarrow j} \) that depend of the quantity \( q_i > 0 \) assigned to each node \( i \) of the network. The value \( q_i \) can represent a topological feature of the respective node (e.g., the degree, the betweenness centrality, the eigenvector centrality, the clustering coefficient, among other measures \[7\]) or a value, independent of the network structure, that quantifies an existing resource at each node. We define preferential random walks with local information by means of the weights \( \Omega_{ij} = (q_i q_j)^\beta A_{ij} \), where the exponent \( \beta \) is a real parameter. Then, from Eq. (2), we have

\[
\pi_{i\rightarrow j} = \frac{A_{ij} q_j^\beta}{\sum_{l=1}^{N} A_{il} q_i^\beta}.
\]

(8)

In Eq. (8), \( \beta > 0 \) describes the tendency to hop to neighbor nodes with large values of \( q \), whereas for \( \beta < 0 \) this behavior is inverted and the walker tends to hop to nodes with lower values of \( q \). On the other hand, for \( \beta = 0 \) the normal random walk is recovered. By means of Eq. (8), the stationary distribution for the preferential random walk is

\[
P_{i\rightarrow j}^{\infty} = \frac{\sum_{l=1}^{N} (q_i q_l)^\beta A_{il}}{\sum_{l,m=1}^{N} (q_i q_m)^\beta A_{lm}}.
\]

(9)

As we will see in the next part, the general preferential strategy defined by Eq. (8) determines different types of local random walks depending of the election of the quantities \( q_i \).
3. Degree biased random walks

This type of random walk is a particular case of the preferential navigation with $q_i = k_i$ in Eq. [5]. The resulting strategy is known as degree biased random walks [10, 51]. For this particular case, the stationary distribution $P_i^\infty$ takes the form

$$P_i^\infty = \frac{\sum_{l=1}^{N}(k_l k_l)^\beta A_{il}}{\sum_{l,m=1}^{N}(k_l k_m)^\beta A_{lm}}. \quad (10)$$

Degree biased random walks have been studied extensively in the literature in different contexts as varied as routing processes [51], chemical reactions [52], extreme events [53, 54], among others [16, 55, 56]. Additionally, mean field approximations have been explored for diverse cases [16, 52, 57]. For example, in networks with no degree correlations is valid the approximation $P_i^\infty \approx \frac{1}{\sum_{l} k_l^\beta}$. In Fig. 1 we present the values of the stationary distribution $P_i^\infty$ for degree biased random walks on an Erdős-Rényi network (ER) and, a scale-free network (SF) of the Barabási-Albert type, in which each node has a degree that follows asymptotically a power-law distribution $p(k) \approx k^{-\gamma}$. We calculate the stationary distribution by direct evaluation of the Eq. (10) and we depict $P_i^\infty$ as a function of the degree $k_i$. The results reveal that in the ER network is valid the mean-field approximation whereas in a SF network, this is only valid for nodes with high degrees [16].

4. Maximal entropy random walks

Maximum entropy random walks (MERW) are a particular strategy derived from Eq. [8] for which the random walker uses information of the neighbor nodes. In this case, the transition probability is defined in terms of a matrix of weights $\Omega_{ij}$ that satisfies $A \vec{\xi} = \chi \vec{\xi}$, where $\chi$ is the maximum eigenvalue of $A$. In the study of topological features of networks, the components $\xi_i$ of the eigenvector centrality quantify the global influence of the node $i$ in the whole structure [7].

In this way, MERW are defined in the formalism of weighted networks with the election of weights $\Omega_{ij} = \xi_i \xi_j A_{ij}$. Then, the value of the strength $S_i$ is

$$S_i = \sum_{l=1}^{N} \Omega_{il} = \sum_{l=1}^{N} \xi_l \xi_i A_{il} = \xi_i \sum_{l=1}^{N} A_{il} \xi_l = \chi \xi_i^2, \quad (11)$$

where the last result is a consequence of the relation $\sum_{l=1}^{N} A_{il} \xi_l = \chi \xi_i$ that satisfy the components of the eigenvector centrality. In this way, by using Eq. [2], the transition rule $\pi_{i\to j}$ is given by

$$\pi_{i\to j} = A_{ij} \frac{\xi_i \xi_j}{\chi \xi_i^2} = \frac{\xi_j}{\chi}, \quad (12)$$

relation that defines a maximal entropy random walk [58]. Additionally, by using the Eq. [6], the stationary distribution of the maximal entropy random walk is

$$P_i^\infty = \frac{\chi \xi_i^2}{\sum_{l=1}^{N} \chi \xi_l^2} = \xi_i^2, \quad (13)$$

It is worth to mention that, the MERW defined by the transition probabilities in Eq. (12) maximizes the entropy rate production $h$ of the process given by [58]

$$h = -\sum_{i=1}^{N} P_i^\infty \sum_{j=1}^{N} \pi_{i\to j} \log \pi_{i\to j}. \quad (14)$$

Combining this expression with Eqs. (12) and (13), $h = \log \chi$ [58]. In this case, the trajectories that follow the random walker are maximally random [58, 59]. Diverse variations of the MERW and applications of this process have been explored in [59, 62].

5. Random walks for image segmentation

An important application of random walks on networks emerges in the context of the processing and segmentation of digital images [64]. In this case, the statistical description of the diffusive transport from seed regions to specific pixels allows to detect and differentiate objects and structures in a digital image [61]. The network is a square lattice where each node represents a pixel and the normalized intensity $I_i$ of $i$ is a quantity associated to the norm of the vector $\vec{p}_i$ that contains the values RGB (red, green, blue) of the respective pixel, $0 \leq I_i \leq 1$. In terms of a matrix of weights $\Omega$, a local random walker is defined by [61]

$$\Omega_{ij} = \exp \left[-(I_i - I_j)^2/\sigma^2\right] A_{ij}. \quad (15)$$

Here, the real parameter $\sigma$ satisfies $\sigma > 0$ and the values $A_{ij}$ give the elements of the adjacency matrix of a square lattice associated to the pixels positions and interactions between nearest neighbors. The resulting random walker follows a strategy given by Eq. (2) to visit the pixels; this transition probability gives high probability to the pass to pixels with the same intensity and $\sigma$ determines the interaction between the pixels establishing a characteristic scale for the differences of intensity in the model controlling the capacity to hop to sites with a different color. In Fig. 3 we plot the strength $S_i$ for each pixel, this quantity is proportional to the stationary probability distribution for a random walker in a digital image that follows a strategy determined by the weights given by Eq. (15). It is observed how with this strategy, $S_i$
FIG. 2. (Color online) Stationary distribution $P_i^\infty$ as a function of $k_i$ for degree biased random walks. The values are obtained by direct evaluation of Eq. (10). We use three values of the parameter $\beta$ and we study two types of networks with $N = 5000$ nodes. (a) An Erdős-Rényi network (ER) with an average degree $\langle k \rangle = 50$; the dashed lines represent the results obtained by the mean field approximation. (b) A scale-free network (SF) with $\langle k \rangle = 6$.

In addition to the local strategy mentioned before, it is worth mentioning that exists different variations of these models; this is the case of the topological biased random walks for which $\Omega_{ij} = e^{\beta y_{ij}} A_{ij}$, where the quantity $y_{ij}$ describes the properties of the edge that connects $i$ with $j$. A similar idea is explored for image segmentation in [64], showing the vast applicability of random walks in different scenarios.

B. Non-local random walks

Non-local random walks on networks are motivated by the possibility of hopping from one node to sites on the network beyond the neighbor nodes in cases where the total structure of the network is available. Random walk strategies with long-range displacements have shown an unprecedented applicability in the context of web searching. The PageRank introduced to classify pages on the Web [67] and variations of this non-local strategy have been explored to rank the importance of nodes in a broad range of systems. In the following part, we present diverse non-local strategies on undirected networks that can be expressed in terms of a matrix of weights that includes information about the whole structure of the network to define the dynamical process. As particular examples of this case we have the Lévy flights on networks [25], the fractional diffusion on networks [33, 34], the dynamics of agents moving visiting specific locations in a city [24] and different strategies in the context of the random multi-hopper model [32]. The study and possible applications of non-local dynamical processes on networks are relatively new and open questions related with the exploration of the effects that non-locality introduces as well as the search of global quantities that allow us to compare the performance of non-local against local dynamics.

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1. Lévy flights on networks

The term Lévy flights makes reference to a random walk with displacements of length \( l \) that appear with a probability distribution \( \mathcal{K}(l) \) that asymptotically is described by an inverse power law relation \( \mathcal{K}(l) \sim l^{-\gamma} \). For Lévy flights in the \( n \)-dimensional space \( \mathbb{R}^n \), \( \mathcal{K}(0) = 0 \) and \( \mathcal{K}(l) \sim \frac{1}{l^{\gamma}} \) if \( l \neq 0 \) for \( 0 < \gamma < 1 \). With this definition, the variance of the displacements diverges; this characteristic differentiates Lévy flights from the Brownian motion for which the variance is finite \( [13] \). In Fig. 4 we present Monte Carlo simulations for Brownian motions and Lévy flights in a plane. Lévy flights have a fractal behavior consisting of trajectories that alternate between groups described by local movements (similar to the observed in the Brownian motion) interrupted by long-range jumps; this structure is repeated at all levels. In this way, Lévy flights combine local movements, that appear with high probability, with long-range displacements that emerge with low but non-null probability. These characteristics are illustrated in Fig. 4(b). Lévy flights constitute an active area of research in different complex systems. For example, Lévy flights are encountered in the modelling of animal dynamics and foraging \( [5, 6, 70, 72] \), human mobility \( [73, 74] \), among many others \( [68, 69, 76] \).

In the context of networks, Lévy flights are introduced in reference \( [25] \). In this case, the transitions are defined in terms of the distance \( d_{ij} \) that gives the number of lines in the shortest path connecting the nodes \( i \) and \( j \). All the information about the distances between nodes is contained in the distance matrix \( D \) with elements \( d_{ij} \) for \( i, j = 1, 2, \ldots, N \). The distance matrix \( D \) contains more information about the structure of the network than the adjacency matrix \( A \), but \( D \) can be calculated efficiently from \( A \) using different algorithms \( [7] \). In Fig. 5 we depict the relative frequency of distances in the entries of the matrix \( D \) for large-world networks (square lattice and tree) and small-world networks (Erdős–Rényi network and scale-free network of the Barabási–Albert type). The histograms reveal the marked difference between the distances in these two types of structures.

Lévy flights on networks can be described in terms of the weights \( \Omega_{ij} = d_{ij}^{-\alpha} \) for \( i \neq j \) and \( \Omega_{ii} = 0 \). Here \( \alpha \) is a real parameter in the interval \( 0 \leq \alpha < \infty \). For the elements of the transition matrix, we have \( \pi_{i \rightarrow i} = 0 \) and by using Eq. \( [3] \) for \( i \neq j \) is obtained \( [25] \)

\[
\pi_{i \rightarrow j} = \frac{d_{ij}^{-\alpha}}{\sum_{l \neq i} d_{il}^{-\alpha}}. \tag{16}
\]

The dynamics inspired in Lévy flights allows long-range transitions on the network. For a finite non-null value of \( \alpha \), the transitions to the nearest neighbors appear with high probability, but hops beyond these nodes are allowed generalizing the dynamics observed in the normal random walker in Eq. \( [7] \). In the limit \( \alpha \to \infty \) we have \( \lim_{\alpha \to \infty} d_{ij}^{-\alpha} = A_{ij} \), then \( \pi_{i \rightarrow j} = \frac{A_{ij}}{\sum_{l \neq i} d_{il}^{-\alpha}} \) and the Lévy strategy recovers the normal random walk. Another interesting limit case is obtained when \( \alpha \to 0 \), in this case \( \lim_{\alpha \to 0} d_{ij}^{-\alpha} = 1 \) if \( i \neq j \) and the dynamics induces the possibility to reach with equal probability any node of the network \( [23] \).

Once defined Lévy flights in terms of the elements \( \Omega_{ij} = d_{ij}^{-\alpha} \) for \( i \neq j \); for this particular model we denote the strength \( S_i = \sum_{l=1}^{N} \Omega_{il} \) as \( D_i^{(\alpha)} = \sum_{l \neq i} d_{il}^{-\alpha} \) and by us-

![FIG. 4. (Color online) Monte Carlo simulations of two different types of random walks on a plane. (a) Brownian motion. (b) Lévy flights. We depict \( 10^3 \) steps for each realization of the dynamics. The colorbar codifies the number of each step that in this case is a measure of a discrete time.](image)

![FIG. 5. (Color online) Frequencies \( \nu(d_{ij}) \) of the non-null distances \( d_{ij} \) in the entries of the distance matrix \( D \). We analyze networks with \( N = 5000 \). (a) Square lattice with dimensions \( 50 \times 100 \). (b) Tree. (c) Erdős–Rényi network with probability of connection \( p = \frac{\log N}{N} \). (d) Scale-free network of the Barabási–Albert type. The results are expressed as a fraction of the value \( N(N-1)/2 \) that gives the total number of different non-null entries in the distance matrix \( D \).](image)
Different aspects of Lévy flights and their capacity to explore networks have been studied in [26–29], as well as in the context of multiplex networks [30]. A general approach to study the random walker in Eq. (16) and other strategies defined in terms of a function of the distances in a network are analyzed in detail by Estrada et al. in [32]. In this context is introduced the exponential strategy that in terms of our matrix of weights formalism is defined by $\Omega_{ij} = e^{-sd_{ij}}$ for $i \neq j$ and using $s > 0$. By following a similar approach to the presented in Eqs. (17)-(18), can be deduced analytical expressions for the stationary probability distribution of the exponential strategy.

2. Gravity law, spatial networks, and human mobility

In diverse situations networks are embedded in a metric space, then, spatial locations are assigned to each node. This is the case of spatial networks that describe several real systems like social networks, airports and transportation networks, among others [27, 77, 78]. On the other hand, it has been suggested that migration and human movements are well described in terms of a “Gravity Law” that models the number of trips from a location $i$ to the location $j$ as $g_{ij} = C p_i p_j / l_{ij}^\alpha$. Here $p_i$ and $p_j$ denote the population of the respective locations, $l_{ij}$ is the geometric distance between the nodes $i, j$, $C$ is a constant and $\alpha > 0$ is a free parameter. This type of model suggests a similar algorithm for a random walker on networks described by the weights

$$\Omega_{ij} = \frac{q_i q_j}{d_{ij}^\alpha}$$

for $i \neq j$. Here the value $q_i$ is a quantity associated to the node $i$ in the network and $d_{ij}$ is the topological distance in the network. The general formalism in terms of weighted networks also applies to the model presented in Eq. (19), but with geometric distances $l_{ij}$. In this model, the structure of the network is absent and it is assumed...
as a complete graph.

In the gravity law model, the resulting random walk contains characteristics of the biased random walks determined by Eq. (3) and the Lévy flights on networks with transition probabilities given by Eq. (10). The random walk defined in Eq. (19) has been explored in order to characterize co-occurrences of words on web pages [79]. In addition, there are different variations of the gravity law in spatial networks (see [77] and references therein). Some of these models are described in the weighted network approach by weights \( \Omega_{ij} \) proportional to a positive function of the distance \( f(d_{ij}) \) [77].

As an example of random walks that take place in a continuous space but can be modeled with the formalism of random walks defined in terms of a matrix of weights, in reference [24] is introduced a strategy to visit randomly specific locations in a spatial region modeling characteristics of human mobility in urban settlements. In this case, \( N \) points are located in a 2D plane and integer numbers \( i = 1, 2, \ldots, N \) label each location. In addition, the coordinates of the locations are known and we denote as \( l_{ij} \) the distance between the places \( i \) and \( j \). The distance \( l_{ij} = l_{ji} \geq 0 \) can be calculated by different metrics; for example, in some applications could be appropriated the use a Euclidean metric, whereas, in other contexts, a Manhattan distance could be more useful. In order to define a discrete time random walker that at each step visits one of the locations, the transition probability \( \pi_{i \rightarrow j}^{(\alpha)}(R) \) to hop from site \( i \) to site \( j \) is given by [24]

\[
\pi_{i \rightarrow j}^{(\alpha)}(R) = \frac{\Omega_{ij}^{(\alpha)}(R)}{\sum_{m=1}^{N} \Omega_{im}^{(\alpha)}(R)},
\]

where the weights \( \Omega_{ij}^{(\alpha)}(R) \) are defined by the relation

\[
\Omega_{ij}^{(\alpha)}(R) = \begin{cases} 
1 & \text{for } 0 \leq l_{ij} \leq R, \\
(R/l_{ij})^\alpha & \text{for } R < l_{ij}.
\end{cases}
\]

Here \( \alpha \) and \( R \) are positive real parameters. The radius \( R \) determines a neighborhood around which the random walker can go from the initial site to any of the locations in this region with equal probability; this transition is independent of the distance between the respective sites. That is, if there are \( S \) sites inside a circle of radius \( R \), the probability of going to any of these sites is a constant. Additionally, for places beyond the local neighborhood, for distances greater than \( R \), the transition probability decays as an inverse power law of the distance and is proportional to \( l_{ij}^{-\alpha} \) [24]. In this way, the parameter \( R \) defines a characteristic length of the local neighborhood and \( \alpha \) controls the capacity of the walker to hop with long-range displacements. In particular, in the limit \( \alpha \rightarrow \infty \) the dynamics becomes local, whereas the case \( \alpha \rightarrow 0 \) gives the possibility to go from one location to any different one with the same probability. In this limit, we have \( \pi_{i \rightarrow j}^{(0)}(R) = N^{-1} \). This model is then a combination of a rank model [80,82] for shorter distances and a gravity-like model for larger ones [78,83]. It is important to mention that in the strategy defined by the weights in Eq. (21), we choose \( \Omega_{ij}^{(\alpha)}(R) \neq 0 \), in this way the walker also can stay in the node \( i \) with non-null probability. All the results presented are also valid for this case whenever the value of \( R \) is such that the random walker can reach any of the \( N \) sites used in the definition of the transition matrix.

In Fig. 7(a) we illustrate the model for the random strategy introduced in Eq (20). In Fig. 7(b), we present Monte Carlo simulations of the random walker described by Eqs (20)-(21). We generate \( N \) random locations (points) on a 2D plane on the region \([0,1] \times [0,1]\) in \( \mathbb{R}^2 \).
and, for different values of the exponent α, we depict the trajectories described by the walkers. In the case of $\alpha \to \infty$, it is observed how the dynamics is local and only allows transitions to sites in a neighborhood determined by a radius $R = 0.17$ around each location. In this case, all the possible trajectories in the limit $t \to \infty$ form a random geometric graph \cite{84,85}; we can identify features of this structure in our simulation. On the other hand, finite values of $\alpha$ model spatial long-range displacements of the fractional diffusion on networks \cite{33,34,35–37,94,95}. In the limit of infinite-dimensional lattices with periodic boundary conditions has been addressed in different contexts \cite{33,34,35–37,94,95}. For this type of periodic structures, it is obtained the analytical relation \cite{36}. On the other hand, the elements of the Laplacian matrix satisfy $L_{ij} = \delta_{ij}k_i - A_{ij}$, where $\delta_{ij}$ denotes the Kronecker delta; in particular, $L_{ii} = k_i$. The Laplacian matrix is introduced in graph theory and in the modeling of dynamical processes on networks \cite{10,14,86–92,24}. In the context of the fractional diffusion on networks is introduced the fractional Laplacian matrix $\mathbf{L}^\gamma$, where $\gamma$ is a real number ($0 < \gamma < 1$). The resulting process models the fractional dynamics on general networks \cite{33,34,88,89}.

Since the Laplacian matrix $\mathbf{L}$ is a symmetric matrix, by using the Gram-Schmidt orthonormalization of the eigenvectors of $\mathbf{L}$, we obtain a set of eigenvectors $\{|\Psi_j\rangle\}_{j=1}^N$ that satisfy the eigenvalue equation $\mathbf{L} |\Psi_j\rangle = \mu_j |\Psi_j\rangle$ for $j = 1, \ldots, N$ and $\langle \Psi_j | \Psi_j \rangle = \delta_{jj}$, where $\mu_j$ are the eigenvalues, which are real and nonnegative. For connected networks, the smallest eigenvalue is $\mu_1 = 0$ and $\mu_m > 0$ for $m = 2, \ldots, N$. We define the matrix $\mathbf{Q}$ with elements $Q_{ij} = \langle i | \Psi_j \rangle$ and the diagonal matrix $\mathbf{\Delta} = \text{diag}(0, \mu_2, \ldots, \mu_N)$. These matrices satisfy $\mathbf{L} \mathbf{Q} = \mathbf{Q} \mathbf{\Delta}$, therefore $\mathbf{L} = \mathbf{Q} \mathbf{\Delta} \mathbf{Q}^\dagger$, where $\mathbf{Q}^\dagger$ denotes the conjugate transpose of $\mathbf{Q}$. Therefore \cite{33,88,89,92}

$$L^\gamma = \mathbf{Q} \mathbf{\Delta}^\gamma \mathbf{Q}^\dagger = \sum_{m=2}^{N} \mu_m^\gamma |\Psi_m\rangle \langle \Psi_m|,$$  

(22)

where $\mathbf{\Delta}^\gamma = \text{diag}(\mu_2^\gamma, \ldots, \mu_N^\gamma)$. It is worth noticing that the diagonal elements of the fractional Laplacian matrix defined in Eq. \cite{24,29} introduce a generalization of the degree $k_i = (\mathbf{L})_{ii}$ to the fractional case. In this way, the fractional degree $k_i^{(\gamma)}$ of the node $i$ is given by \cite{33,88,89,92}

$$k_i^{(\gamma)} \equiv (\mathbf{L}^\gamma)_{ii} = \sum_{m=2}^{N} \mu_m^\gamma |i \rangle \langle \Psi_m| |\Psi_m\rangle \langle i|.$$  

(23)

The fractional random walk is the random walk associated to the fractional diffusion in networks \cite{33,88,89,92}. In the formalism of weighted networks is defined by the elements $\Omega_{ii} = 0$ and, for $i \neq j$

$$\Omega_{ij} = -\langle \mathbf{L}^\gamma \rangle_{ij}$$  

(24)

with $0 < \gamma < 1$. On the other hand, the elements of the Laplacian matrix satisfy $k_i = -\sum_{j \neq i} L_{ij}$ and, in the fractional case we have $k_i^{(\gamma)} = -\sum_{j \neq i} (\mathbf{L}^\gamma)_{ij}$. As result the strength of the node $i$ is given by

$$S_i = \sum_{l=1}^{N} \Omega_{il} = -\sum_{l \neq i} (\mathbf{L}^\gamma)_{il} = k_i^{(\gamma)},$$  

(25)

then, by using Eq. \cite{2}, the transition probability $\pi_{i \to j}$ is given by

$$\pi_{i \to j} = \delta_{ij} - \frac{\langle \mathbf{L}^\gamma \rangle_{ij}}{k_i^{(\gamma)}}.$$  

(26)

In the limit $\gamma \to 1$, the normal random walk strategy is recovered. In addition, by using the Eq. \cite{2}, the stationary distribution is

$$P_i^\infty = \frac{k_i^{(\gamma)}}{\sum_{l=1}^{N} k_l^{(\gamma)}}.$$  

(27)

This is a generalization of the result $P_i^\infty \propto k_i$ for normal random walks discussed before and recovered from Eq. \cite{27} when $\gamma = 1$.

The fractional random walk is the process associated to the fractional diffusion on networks and the transition probabilities in Eq. \cite{20} define a navigation strategy with long-range displacements on the network \cite{33}. The case of infinite $n$-dimensional lattices with periodic boundary conditions has been addressed in different contexts \cite{33,37,94,95}. For this type of periodic structures, it is obtained the analytical relation \cite{36}

$$\pi_{i \to j} \sim d_{ij}^{-n-2\gamma} \quad \text{for} \quad d_{ij} \gg 1.$$  

(28)

The result in Eq. \cite{28} establishes a connection between Lévy flights on networks \cite{24} and the fractional strategy defined by Eq. \cite{20}. On the other hand, in networks with constant degree $k$, the fractional Laplacian can be expressed as \cite{33,88,89,92,94,95}

$$(\mathbf{L}^\gamma)_{ij} = \sum_{m=0}^{\infty} \frac{\Gamma(x)}{\Gamma(x+1)} \left(\frac{\gamma}{m+\gamma}\right) (-1)^m k_i^{\gamma-m}(\mathbf{A}^m)_{ij}$$  

(29)

where $\gamma_{ij} \equiv \Gamma(x+1)/(\Gamma(x+1)\Gamma(x+1))$ and $\Gamma(x)$ denotes the Gamma function \cite{96}. The result in Eq. \cite{29} relates the fractional Laplacian matrix with the integer powers of the adjacency matrix $\mathbf{A}^m$ for $m = 1, 2, \ldots$ for which the element $(\mathbf{A}^m)_{ij}$ is the number of all the possible trajectories connecting the nodes $i$, $j$ with $m$ links \cite{97}. In this way, the fractional strategy defined by the transition matrix with elements $\pi_{i \to j}$ in Eq. \cite{20} incorporates
global information about all the possible trajectories connecting the nodes $i$ and $j$ \[34\].

In order to illustrate the effect of the fractional dynamics of a random walker on a network, in Fig. 8 we present Monte Carlo simulations of discrete-time random walks on a tree. The discrete time $t$ denotes the number of steps of the random walker as it moves from one node to the next node on the network. Given the topology of the network, we calculate the adjacency matrix and the corresponding Laplacian matrix $L$ of the network. Then we obtain its eigenvalues and eigenvectors that allow us in turn to get the fractional Laplacian matrix $L_\gamma$. Finally, using Eq. (26), we determine the transition probabilities for different values of the parameter $\gamma$. The dynamics starts at $t = 0$ from an arbitrary node. We show three discrete times $t = 25, 50, 100$ for three values of the parameter $\gamma = 1, 0.75, 0.5$. The case $\gamma = 1$ corresponds to a normal random walker whereas the cases with $\gamma = 0.75, 0.5$ correspond to a fractional random walk leading to anomalous transport. We represent with different colors the unvisited nodes, visited nodes and the position of the random walker at time $t$.

The introduction of the fractional random walks is motivated by the search of an equivalent on networks of the fractional diffusion and its relation with Lévy flights. Recently, other types of functions of matrices with local information have shown interesting properties associated with long-range dynamics and the global structure of networks; this is the case of the concept communicability \[91\] and the accessibility random walk introduced in \[98\]. Particular functions of matrices can be used to define different types of long-range strategies and characterized with the formalism reviewed in this work.

As a generalization of Eq. (26), other functions of the Laplacian $g(L)$ can be applied to define random walk strategies on networks. The functions $g(x)$ to define random walk strategies should satisfy the following conditions \[42\].

- **Condition I:** The matrix $g(L)$ must be positive semidefinite, i.e., the eigenvalues of $g(L)$ are restricted to be positive or zero. In this way, the property of the Laplacian eigenvalues $\mu_i \geq 0$ for $i = 2, \ldots, N$ is preserved by the function $g(x)$. 

FIG. 8. (Color online) Monte Carlo simulation of a discrete-time fractional random walker on a tree with transition probabilities given by Eq. (26). The dynamics starts at $t = 0$ from an arbitrary node. We show three discrete times $t = 25, 50, 100$ for three values of the parameter $\gamma = 1, 0.75, 0.5$. The case $\gamma = 1$ corresponds to a normal random walker whereas the cases with $\gamma = 0.75, 0.5$ correspond to a fractional random walk leading to anomalous transport. We represent with different colors the unvisited nodes, visited nodes and the position of the random walker at time $t$. 

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As a generalization of Eq. (26), other functions of the Laplacian $g(L)$ can be applied to define random walk strategies on networks. The functions $g(x)$ to define random walk strategies should satisfy the following conditions \[42\].
• **Condition II:** The elements of the matrix $g(L)$ denoted as $g_{ij}(L)$, for $i, j = 1, 2, . . . , N$, should satisfy

$$
\sum_{j=1}^{N} g_{ij}(L) = 0.
$$

(30)

Therefore, the function $g(x)$ maintains the property $\sum_{j=1}^{N} L_{ij} = 0$ associated to the elements of the Laplacian matrix.

• **Condition III:** All the non-diagonal elements of $g(L)$ must satisfy

$$
g_{ij}(L) \leq 0.
$$

(31)

For this type of functions, transition probabilities are defined by the relation

$$
\pi_{i \to j} = \delta_{ij} - \frac{g_{ij}(L)}{K_{i}},
$$

(32)

where we use the generalized degrees $K_{i}$ defined by the diagonal elements of $g(L)$ that satisfy \cite{42}

$$
K_{i} = g_{ii}(L) = -\sum_{j \neq i} g_{ij}(L).
$$

(33)

Examples of functions that satisfy the conditions I-III are the fractional Laplacian of a graph $g(L) = L^{\gamma}$ with $0 < \gamma < 1$, the logarithmic function $g(L) = \log(1 + \alpha L)$ for $\alpha > 0$ and the function $g(L) = 1 - e^{-aL}$ with $a > 0$. In all these cases is observed that the random walker hops with long-range displacements on the network \cite{42}.

In terms of the formalism of the matrix of weights, the generalized random walk strategy in Eq. (32) can be analyzed by using the weights $\Omega_{ij} = -g_{ij}(L)$ for $i \neq j$ and $\Omega_{ii} = 0$. In this way, as a consequence of the condition in Eq. (31), the weights satisfy $\Omega_{ij} \geq 0$; also, the strength of each node is given by the generalized degree $K_{i}$ allowing us to write the stationary probability distribution of the process as

$$
P_{i}^{\infty} = \frac{K_{i}}{\sum_{l=1}^{N} K_{l}}.
$$

(34)

In the general case described in Eq. (32), the values of $g_{ij}(L)$ can be obtained by using the spectral methods described before for the fractional Laplacian (see \cite{42} for details).

We conclude this section with a compilation of the types of random walk strategies represented by specific types of weighted networks. In Table I we summarize the matrices of weights that define the local and non-local strategies analyzed in this section. Each model is presented with the respective parameters that define the random walker and key references to works analyzing these strategies.

### IV. MEAN FIRST PASSAGE TIME AND GLOBAL CHARACTERIZATION

Once described a general formalism that allows us to define different types of local and non-local random walks strategies on networks and analytical results for their respective stationary distributions; in this section, we explore the mean first passage time (MFPT) \cite{3}, that gives the average number of steps needed by the random walker to reach a specific node for the first time. We also study global times to quantify and compare the capacity of local and non-local random walks to explore different types of networks.

#### A. MFPT

In order to calculate the MFPT for strategies defined in terms of weighted networks, we use a similar approach to the formalism presented in \cite{12,15} where normal random walks are studied. We start representing the probability $P_{ij}(t)$ in the master equation in Eq. (1) as

$$
P_{ij}(t) = \delta_{0} \delta_{ij} + \sum_{t'=0}^{t} P_{jj}(t-t') F_{ij}(t') .
$$

(35)

The first term in Eq. (35) represents the initial condition and $F_{ij}(t)$ is the probability to start in the node $i$ and reach the node $j$ for the first time after $t$ steps, by definition $F_{ij}(0) = 0$. Now, by using the discrete Laplace transform $\tilde{f}(s) \equiv \sum_{t=0}^{\infty} e^{-st} f(t)$, the relation in Eq. (35) takes the form

$$
\tilde{F}_{ij}(s) = (\tilde{P}_{ij}(s) - \delta_{ij})/\tilde{P}_{jj}(s) .
$$

(36)

By definition, using the quantity $F_{ij}(t)$, the MFPT $\langle T_{ij} \rangle$ for a random walker that starts in the node $i$ and reach for the first time the node $j$ is given by \cite{12}

$$
\langle T_{ij} \rangle \equiv \sum_{t=0}^{\infty} t F_{ij}(t) = -\tilde{F}_{ij}'(0).
$$

(37)

Now, by means of the moments $R^{(n)}_{ij}$ of the probability $P_{ij}(t)$ defined as

$$
R^{(n)}_{ij} \equiv \sum_{t=0}^{\infty} t^{n} \{ P_{ij}(t) - P_{jj}^{\infty} \} ,
$$

(38)

the expansion in series of $\tilde{P}_{ij}(s)$ is

$$
\tilde{P}_{ij}(s) = P_{jj}^{\infty} \frac{1}{(1 - e^{-s})} + \sum_{n=0}^{\infty} (-1)^{n} R^{(n)}_{ij} s^{n} n^{n} .
$$

(39)

Introducing this result in Eq. (36), the MFPT is obtained

$$
\langle T_{ij} \rangle = \frac{1}{P_{jj}^{\infty}} \left[ R^{(0)}_{ij} - R^{(0)}_{jj} + \delta_{ij} \right] .
$$

(40)
As result of the relation in Eq. (43)

tic processes is denominated Kemeny’s constant [102, 103].

The quantity

\[ \langle T_{ij} \rangle = \frac{1}{P_i} \]

which is a time independent of the initial node and the first return time \( T_{ij} \), as consequence

\[ \langle T_{ij} \rangle - \langle T_{ji} \rangle = \tau_j - \tau_i, \]

relation that describes the asymmetry of navigation [15].

The time \( \tau_i \) is interpreted as the average time needed to reach the node \( i \) from a randomly chosen initial node of the network; on the other hand, the quantity \( C_i \equiv \tau_i^{-1} \) is the random walk centrality introduced for the analysis of random walks with local information [15]. The centrality \( C_i \) combines information of the network and the random walk strategy implemented to visit nodes and gives a high value to nodes easy to reach and small values to nodes for which the random walker takes, in average, many steps to hit the node for the first time starting from any node of the network [13, 25].

Additional to the times \( \langle T_{ij} \rangle \) and \( \tau_i \), from Eq. (40) we have

\[ \sum_{j=1}^{N} \langle T_{ij} \rangle P_j^\infty = \sum_{j=1}^{N} R_{ij}^{(0)} - \sum_{j=1}^{N} R_{ji}^{(0)} + 1 = \sum_{j=1}^{N} R_{ij}^{(0)} + 1. \]

The quantity \( K \equiv \sum_{m=1}^{N} R_{mm}^{(0)} \) in the context of stochastic processes is denominated Kemeny’s constant [102, 103]. As result of the relation in Eq. (43)

\[ K = \sum_{m=1}^{N} R_{mm}^{(0)} = \sum_{j \neq i} \langle T_{ij} \rangle P_j^\infty, \]

Locally, \( \sum_{j=1}^{N} R_{ij}^{(0)} P_j^\infty = P_i \) the time obtained by averaging the mean first passage times \( \langle T_{ij} \rangle \) weighted with the stationary distribution \( P_j^\infty \).

### B. Linear algebra approach

Once defined general quantities that characterize the performance of a random walk strategy to explore a network, it is important to have an algorithm that allows us to calculate these values by using the information consigned in the transition probability matrix \( \Pi \) in Eq. (2), defined in terms of the matrix of weights \( \Omega \) and that essentially contains all the information about the random walker. Therefore, in the following part we deduce expressions for the MFPT \( \langle T_{ij} \rangle \), the time \( \tau_i \) and the Kemeny’s constant \( K \) in terms of the eigenvalues and eigenvectors of the transition matrix \( \Pi \).

In order to calculate \( \tau_i \) and \( \langle T_{ij} \rangle \) is necessary to find \( P_{ij}(t) \). We start with the matricial form of Eq. (1)

\[ \vec{P}(t) = \vec{P}(0) \Pi^t. \]

Here \( \vec{P}(t) \) is the probability vector at time \( t \). Using Dirac’s notation

\[ P_{ij}(t) = \langle i | \Pi^t | j \rangle, \]

where \( \{|m\}\}_{m=1}^{N} \) represents the canonical base of \( \mathbb{R}^N \). Due to the existence of a detailed balance condition, the matrix \( \Pi \) can be diagonalized and its spectrum has real values [4]. For right eigenvectors of \( \Pi \) we have \( \Pi \phi_i = \lambda_i \phi_i \) for \( i = 1, \ldots, N \), where the set of eigenvalues is ordered in the form \( \lambda_1 = 1 \) and \( \lambda_2 \geq \ldots \geq \lambda_N \geq -1 \). On the other hand, from right eigenvectors we define a

| Strategy | Weights \( \Omega_{ij}, i \neq j \) | Parameters | References |
|----------|-------------------------------|------------|----------|
| 1. Normal random walk | \( A_{ij} \) | | [12, 13, 15] |
| 2. Biased random walk | \( (q_i q_j)^{\beta} A_{ij} \) | \( \beta \in \mathbb{R}, q_i > 0 \) | [102] |
| 3. Degree biased random walk | \( (k_i k_j)^{\beta} A_{ij} \) | \( \beta \in \mathbb{R} \) | [16] |
| 4. Maximal entropy random walk | \( \xi_i A_{ij} \) | | [58, 60, 62] |
| 5. Random walks for image segmentation | \( e^{-\frac{1}{2} (x_i - x_j)^2} A_{ij} \) | \( \sigma > 0 \) | [44] |
| 6. Topologically biased random walk | \( e^{\beta w_{ij}} A_{ij} \) | \( \beta \in \mathbb{R}, y_{ij} = y_{ji} \) | [65, 66] |

**TABLE I.** Diverse types of random walks described in terms of a symmetric matrix of weights \( \Omega \) with transition probabilities \( \pi_{i \rightarrow j} \) defined by Eq. (2). For all the random walk strategies \( \Omega_{ii} = 0 \) and, the non-diagonal elements are presented in the table with a short description of the quantities and parameters involved in the definition. The detailed description of each random walk strategy is presented in Section III.
matrix $Z$ with elements $Z_{ij} = \langle i | \phi_j \rangle$. The matrix $Z$ is invertible, and a new set of vectors $\langle \tilde{\phi}_i |$ is obtained by means of $(Z^{-1})_{ij} = \langle \tilde{\phi}_i | j \rangle$, then

$$\delta_{ij} = (Z^{-1}Z)_{ij} = \sum_{l=1}^{N} \langle \tilde{\phi}_i | l \rangle \langle l | \phi_j \rangle = \langle \tilde{\phi}_i | \phi_j \rangle$$

and

$$\mathbb{I} = ZZ^{-1} = \sum_{i=1}^{N} |\phi_i \rangle \langle \tilde{\phi}_i|,$$}

where $\mathbb{I}$ is the $N \times N$ identity matrix.

In different cases, especially when it is necessary to calculate numerically the eigenvalues and eigenvectors of the transition matrix, it is convenient to use the symmetry of the matrix of weights $\Omega$. In this way, the eigenvectors $|\phi_i \rangle$ and $\langle \tilde{\phi}_i| \rangle$ can alternatively be deduced from the analysis the symmetric matrix $M$ with elements

$$M_{ij} = \Omega_{ij}/\sqrt{S_i S_j}. \quad (49)$$

From an orthonormal set of eigenvectors $|\varphi_l \rangle$ that satisfy $M|\varphi_l \rangle = \lambda_l |\varphi_l \rangle$ for $l = 1, \ldots, N$, it is obtained $|\phi_i \rangle = S^{-1/2} |\varphi_i \rangle$ and $\langle \tilde{\phi}_i| = \langle \varphi_i| S^{-1/2}$ where $S$ is the $N \times N$ diagonal matrix $S = \text{diag}(S_1, \ldots, S_N)$.

Once obtained the spectrum and the left and right eigenvectors of the transition matrix, we can deduce different analytical expressions for quantities that characterize the random walker. By using the diagonal matrix $\Delta \equiv \text{diag}(\lambda_1, \ldots, \lambda_N)$ is obtained $\Pi = ZZ^{-1}$, therefore Eq. (46) takes the form

$$P_{ij}(t) = \langle i | Z \Delta^t Z^{-1} | j \rangle = \sum_{l=1}^{N} \lambda_l^t \langle i | \phi_l \rangle \langle \tilde{\phi}_l| j \rangle. \quad (50)$$

From Eq. (50), the stationary probability distribution $P_{ij}^\infty = \langle i | \phi_1 \rangle \langle \tilde{\phi}_1| j \rangle$, where the result $\langle i | \phi_1 \rangle = \text{constant}$ makes $P_{ij}^\infty$ independent of the initial condition. Now, by means of the definition of $R_{ij}^{(0)}$, we have

$$R_{ij}^{(0)} = \sum_{l=2}^{N} \frac{1}{1 - \lambda_l} \langle i | \phi_l \rangle \langle \tilde{\phi}_l| j \rangle. \quad (51)$$

Therefore, the time $\tau_i$ is given by

$$\tau_i = \sum_{l=2}^{N} \frac{1}{1 - \lambda_l} \langle i | \phi_l \rangle \langle \tilde{\phi}_l| i \rangle, \quad (52)$$

and, for $i \neq j$ in Eq. (40), the MFPT $\langle T_{ij} \rangle$ is

$$\langle T_{ij} \rangle = \sum_{l=2}^{N} \frac{1}{1 - \lambda_l} \frac{\langle j | \phi_l \rangle \langle \tilde{\phi}_l| j \rangle - \langle i | \phi_l \rangle \langle \tilde{\phi}_l| i \rangle}{\langle j | \phi_1 \rangle \langle \tilde{\phi}_1| j \rangle}, \quad (53)$$

whereas $T_{ii} = (\langle i | \phi_1 \rangle \langle \tilde{\phi}_1| i \rangle)^{-1}$. Finally, from Eqs. (44) and (51) is obtained the Kemeny’s constant

$$K = \sum_{m=1}^{N} \sum_{l=2}^{N} \frac{1}{1 - \lambda_l} \langle \bar{\phi}_l|m \rangle \langle m|\phi_l \rangle = \sum_{l=2}^{N} \frac{1}{1 - \lambda_l} \quad (54)$$

result that only depends on the eigenvalues of the transition matrix $\Pi$.

### C. Global characterization

In this part we define global times that quantify the capacity of a random walk to reach any site of the network; by using these global times is possible to compare the efficiency of the different strategies defined through Eq. (2). Global quantities like entropy rates [58, 104], the global mean first passage time [17], and the cover time [10, 12] have been used to study random walks on networks. We use the global quantity [25]

$$\tau \equiv \frac{1}{N} \sum_{i=1}^{N} \tau_i, \quad (55)$$

that gives an estimate of the average time to reach any site of the network. The values $\tau_i$ can present a huge dispersion due to the fact that in some irregular networks there are nodes easily accessible to the random walker and other sites that are hardly reached; despite this fact, the mean value of the times $\tau_i$ is an important quantity that characterize the capacity of a random walker to visit the nodes of a network. In the following section we explore the time $\tau$ for different random walk strategies.

On the other hand, in the particular case of random walks on weighted networks for which the value $S_i = \sum_{j=1}^{N} \Omega_{ij}$ is constant, the stationary distribution given by Eq. (4) is $P_{ij}^\infty = 1/N$. In this type of regular cases, using Eqs. (52) and (55) we have for the global time $\tau$

$$\tau_{\text{reg}} = \frac{1}{N} \sum_{i=1}^{N} \frac{R_{ii}^{(0)}}{P_{ii}^\infty} = \sum_{i=1}^{N} R_{ii}^{(0)} = \sum_{i=2}^{N} \frac{1}{1 - \lambda_l}. \quad (56)$$

Then, in regular cases $\tau_{\text{reg}}$ is equal to the Kemeny’s constant. Examples of this simplification are the normal random walks on a complete graph. This case illustrates the best scenario for the exploration of a network by means of normal random walks since all the nodes are connected. For a complete graph $A_{ij} = 1 - \delta_{ij}$ and $\pi_{i \rightarrow j} = \frac{1 - \delta_{ij}}{N-1}$ [11]. The eigenvalues of the matrix $\Pi$ are $\lambda_1 = 1$ and $\lambda_2 = \ldots = \lambda_N = -(N-1)^{-1}$, then the Kemeny’s constant given by Eq. (56) for unbiased random walks on a complete network is

$$\tau_0 = \frac{(N - 1)^2}{N}, \quad (57)$$

this is the lowest value that the time $\tau$ can take.
FIG. 9. (Color online) Time $\tau_i$ vs. $k_i$ for degree biased random walks on networks obtained using Eq. (52). (a) Erdös-Rényi (ER) network with $N = 5000$ and $\langle k \rangle = 50$. (b) Scale-free (SF) network with $N = 5000$ $\langle k \rangle = 6$. We use three values of the parameter $\beta$.

V. EFFICIENCIES OF PARTICULAR STRATEGIES

In this section, we apply the results in Eqs. (52)-(54) that allow to calculate exact values of $\langle T_{ij} \rangle$, $\tau_j$, and $K$ for random walks on weighted networks. In particular, we analyze the global time $\tau$ and the Kemeny’s constant for the preferential navigation, the Lévy flights on networks, the fractional transport and the model in Eq. (20). All these are defined in terms of a matrix of weights through the approach presented in Section III. Similar methods can be implemented to study different types of random walks described in Table I.

A. Preferential navigation

The preferential navigation defined in Eq. (8) can represent different types of local random walkers; in particular, degree biased random walkers with transitions described in Section III A 3. In order to characterize this process, in Fig. 9 we depict the values of the time $\tau_i$ for two different networks, the respective stationary distribution was presented in Fig. 2. The obtained values of $\tau_i$ give the average number of steps needed by a degree biased random walker to reach the node $i$ from a random site in the network for different values of the parameter $\beta$.

In the ER network is observed the validity of the result $\tau_i \approx 1/P_{i}^{\infty}$ obtained by a mean field approximation [16]. On the other hand, in the SF network is not valid this approximation and it is observed that, compared with the case $\beta = 0$ that recovers the normal random walk, any degree biased random walk is a bad strategy to reach efficiently nodes with a lower degree in the SF network. Our findings also reveal that, in comparison with the result for $\beta = 0$, in the ER network the value $\beta = -2$ reduces the number of steps needed to reach nodes with few connections, whereas the parameter $\beta = 2$ reduces the value of $\tau_i$ in nodes with large degree $k_i$.

Now, we calculate the global time $\tau$ for different cases of the preferential navigation defined by Eq. (8). We start generating the network and obtaining the transition matrix $\Pi$ for specific values of the parameter $\beta$ using the definition in Eq. (8). Once obtained $\Pi$ we calculate the respective left and right eigenvectors $\langle \bar{\varphi} \rangle$ and $|\varphi_l\rangle$ by the method described in Section IV B. Then, we use the Eq. (52) to calculate the values of $\tau_i$ and finally, the mean value of the times $\tau_i$ gives $\tau$. In a similar way, the Kemeny’s constant is obtained from the spectrum of $\Pi$ by means of Eq. (54). This process is repeated for different types of local strategies. We study cases for which the value $q_i$ in Eq. (8) is determined by common quantities used to describe the role of the node $i$ in each network, we explore the effect of the following $q_i$ choices given by:

- The node degree $k_i$, therefore, the dynamics is the degree biased random walk explored in Figs. 2 and 9.

- The average degree of the neighbors of $i$ given by $k_i^{(nn)} = \sum_{l=1}^{N} A_{il}k_l/k_i$. In this case, the transition probabilities $\pi_{i \rightarrow j}$ depend on the average degree of the first neighbors of the node $j$.

- The closeness centrality $g_i$ of the node $i$ given by $g_i = (\sum_{j \neq i} d_{ij})^{-1}$ where $d_{ij}$ is the distance between $i$ and $j$ [7].

- The eigenvector centrality $\xi_i$ of the node $i$. For the particular case $\beta = 1$, the probabilities $\pi_{i \rightarrow j}$ are determined by Eq. (12) that defines a maximal entropy random walk.

From the quantities $k_i$, $k_i^{(nn)}$, $g_i$, $\xi_i$, we analyze the global time $\tau$ for the preferential strategy with different
Furthermore, we are interested in the values of the parameter \( \beta \) that modules the biased random walk. From the numerical value of \( \tau \) we can compare the efficiency of the strategies to visit the nodes on the network. In Fig. 10 we analyze each of these strategies in a scale-free network. In Fig. 10(a) we depict the time \( \tau \) as a function of \( \beta \). In this case, we observe that unbiased random walks (\( \beta = 0 \)) have the lowest values of \( \tau \). In addition, for the random walk with \( q_i = g_i \) the values of \( \tau \) do not change significantly with variations of the parameter \( \beta \). In addition, it is observed that, for a given value of \( \beta \), the strategy defined by \( q_i = \xi_i \) has the largest values of \( \tau \) making this method to visit the nodes of the network inefficient to reach easily any node of the structure.

In addition, the results reveal that, in comparison to the unbiased case, some node biased random walks need much more time to explore the network.

Furthermore, we are interested in the values of \( K \) and \( \tau \) as a measure of the efficiency of the random walker to explore sites on the network. In this way, we use Monte Carlo simulations of each preferential random walk with the values \( \beta = -2 \) and \( \beta = 2 \). We depict the obtained results in Figs. 10(b)-(c) for the average number of visited nodes \( \langle N_v \rangle \) as a function of time. Based on our findings for \( \tau \), we know that in some cases, for example when \( q_i = k_i \) or \( q_i = \xi_i \), the average time \( \tau \) is much bigger for \( \beta = 2 \) than the result obtained for \( \beta = -2 \). These results and the behavior observed in the Monte Carlo simulations are in agreement with the predictions that \( \tau \) gives for the different orders of magnitude of the time needed to visit any node on the network. In contrast, the Kemeny’s constant (presented in the inset in Fig. 10(a)) does not describe the results obtained with Monte Carlo simulations. In this way, we can infer that for the preferential navigation, the eigenvectors of the transition matrix \( \Pi \) contain relevant information about the efficiency of the processes, by considering only the spectrum of \( \Pi \) this information is lost. This does not apply in regular cases for which \( P_i^\infty = 1/N \), where \( \tau_{\text{reg}} = K \), and as a consequence important information to describe the efficiency of the process is contained in the spectrum of \( \Pi \).

All the analysis discussed for the preferential random walk in Fig. 10 reveals that the time \( \tau \) is a measure that describes appropriately the global efficiency of random walk strategies. On the other hand, the Kemeny’s constant is a useful simplification to analyze only regular cases when the strength \( S_i = \sum_{i=1}^N \Omega_{il} \) is constant.

**B. Lévy flights**

In this part, we analyze the efficiency of Lévy flights to explore a network. We use the formalism introduced in Section IV B valid for all the strategies represented in terms of the matrix of weights \( \Omega \). In this way we have exact analytical values that allow us to evaluate the mean first passage time \( \langle T_{ij} \rangle \), the time \( \tau_i \) and the global quantity \( \tau \) by using Eqs. (52)-(55). In Fig. 11 we present the results obtained for \( \tau_i \) as a function of \( k_i \) for an Erdős–Rényi network and a scale-free network. In a similar way to the results observed for the preferential random walk in Fig. 9 we obtain that for Lévy flights on these small-world networks \( R_{ii}^{(0)} \approx 1 \) and, as a consequence \( \tau_i \approx 1/P_i^\infty \).

On the other hand, the efficiency of Lévy flights can be explored with the quantity \( \tau \). In networks for which the

![FIG. 10. (Color online) Preferential random walks on a scale-free network (SF) of the Barabási-Albert type with \( N = 5000 \) nodes. (a) Global time \( \tau \) vs. \( \beta \) for different types of random walks defined by Eq. (8) with \( q_i \) given by the node degree \( k_i \), the average degree of neighbors \( k^{(nn)}_i \), the closeness centrality \( g_i \), and the eigenvector centrality \( \xi_i \); in the inset we plot the values of the Kemeny’s constant \( K \) vs. \( \beta \). The continuous lines are used as a guide. In (b) and (c) we present Monte Carlo simulations of the number of visited sites \( N_v \) as a function of time for node biased random walks with \( \beta = -2 \) and \( \beta = 2 \). The average of the number of visited nodes \( \langle N_v \rangle \) is obtained from 1000 different realizations of the random walker, the results are expressed as a fraction of the total number of nodes multiplied by 100.](image-url)
long-range degree $D_i^{(\alpha)}$ is a constant for all the nodes on
the network, the stationary distribution is $P_i^\infty = \frac{1}{N}$, i.e.
each node has the same probability to be visited in the
limit $t \to \infty$. This is the case of some regular networks
like rings, square lattices with periodic boundary conditions,
complete graphs, among others. For this type of
networks $\tau$ is the Kemeny’s constant given by Eq. (54),
result that only depends on the spectrum of the transi-
tion matrix $\Pi$ with elements given by Eq. (2). In the
following part we calculate analytically the value of
$\tau$ for Lévy flights on rings and we explore numerically the ef-
ficiency in other structures.

First, we study Lévy flights on a ring. In this particular
case, the matrix of weights $\Omega$ and the transition matrix
$\Pi$ are circulant matrices [11]. The eigenvalues and eigenvectors of circulant matrices are well known [11] and,
in this way, we can obtain exact analytical expressions for
the different quantities presented in Section IV. For ex-
ample, for a ring with an even number of nodes $N$, the
spectrum of the transition matrix $\Pi$ is

$$
\lambda_l = \frac{2 \sum_{n=1}^{N/2-1} n^{-\alpha} \cos [n\theta_l] + \left(\frac{N}{2}\right)^{-\alpha} \cos \left[\frac{N}{2}\theta_l\right]}{2 \sum_{n=1}^{N/2-1} n^{-\alpha} + \left(\frac{N}{2}\right)^{-\alpha}},
$$

(58)

with $\theta_l = 2\pi(l-1)/N$. In a similar way, for an odd value
of $N$, we have

$$
\lambda_l = \frac{\sum_{n=1}^{(N-1)/2} n^{-\alpha} \cos [n\theta_l]}{\sum_{n=1}^{(N-1)/2} n^{-\alpha}}.
$$

(59)

Introducing the eigenvalues in Eqs. (58) and (59) in the
relation in Eq. (56), we obtain analytically the value of
$\tau$ for Lévy flights on a ring. In particular, when $\alpha \to \infty$
we deduce $\tau$ for normal random walks on a ring

$$
\tau = \sum_{l=2}^{N} \frac{1}{1 - \cos \left[\frac{\pi}{N}(l-1)\right]}.
$$

(60)

In Fig. 12 we analyze the values of $\tau$ for Lévy flights on
finite rings. In Fig. 12(a) we present the result for
$\tau/\tau_0$ as a function of the ring size $N$ for different values
of $\alpha$. From the obtained results, we observe that in the
limit $\alpha \to \infty$, the time $\tau$ behaves as $\tau/\tau_0 \sim N$, therefore
$\tau \sim N^2$. In a similar way, the reduction of the parameter
$\alpha$ gradually changes $\tau$ to $\tau \sim N^{1+\delta}$, where $\delta$ takes values
in the interval $0 < \delta < 1$ for $0 < \alpha < \infty$. Finally, in
the limit $\alpha \to 0$, we have $\tau \sim N$. In addition, in Fig.
12(b) we depict the results for $\tau/\tau_0$ as a function of $\alpha$
for different values of the ring size $N$. In this case, it is
observed how the relation $\tau/\tau_0$ maintains a similar be-
havior for different values of the number of nodes $N$. In
the interval $0 < \alpha < 2$ is observed that $\tau/\tau_0$ are close to
the value $\tau_0$ obtained in the limit $\alpha \to 0$ or in a complete
graph. Moreover, in the range $2 < \alpha \leq 5$, $\tau/\tau_0$ presents
an increase that can be of several orders. For $\alpha > 6$, the
results are again constant and close to the limit $\alpha \to \infty$
given by Eq. (60).

Once analyzed the Lévy flights on rings, for which the
eigenvalues of the transition have the analytic form pre-
seated in Eqs. (58) and (59), it is important to explore the
value of $\tau$ for other structures. In Fig. 13 we present
the results obtained for the average number of steps $\tau$ as
a function of the parameter $\alpha$ for Lévy flights on dif-
ferent types of networks. In regular networks (ring and
square lattice), the value of $\tau$ is obtained from Eq. (56).
For the tree, the ER and the SF networks, the results
are calculated using Eqs. (52) and (55). The obtained
values for $\tau$ suggest that, compared with the normal ran-
don walker, in large-world networks the average number

![FIG. 11. (Color online) Time $\tau_i$ vs. $k_i$ for Lévy flights on networks; the quantities are obtained evaluating numerically Eq. (52). We use three values of the parameter $\alpha$ and we study two types of networks with $N = 5000$. (a) Erdős-Rényi network (ER) at the percolation limit $p = (\log N)/N$. (b) Scale-free (SF) network with $\langle k \rangle = 6$. Dashed lines denote the value $\tau_0$ obtained for $\alpha = 0$.](image-url)

![FIG. 12. (Color online) Values of $\tau$ for Lévy flights on finite rings. (a) Erdős-Rényi network (ER) and (b) Scale-free (SF) network with $\langle k \rangle = 6$. Dashed lines denote the value $\tau_0$ obtained for $\alpha = 0$.](image-url)

![FIG. 13. (Color online) Values of $\tau$ for Lévy flights on different types of networks. (a) Tree, (b) Erdős-Rényi network (ER) and (c) Scale-free (SF) network with $\langle k \rangle = 6$. Dashed lines denote the value $\tau_0$ obtained for $\alpha = 0$.](image-url)
of steps required to reach any node in the network is lower for the Lévy flight strategy. In small-world networks, the differences are smaller, but even in this case, the Lévy strategy improves the results obtained for the normal random walker. This result is reasonable due to the fact that in large-world networks, Lévy flights define a dynamics that induces small-world property. In the case of small-world networks, the nodes in the network are separated by short distances and, in this way, Lévy flights and the normal random walk explore the network with efficiencies of the same order of magnitude.

C. Fractional transport

In this part we apply the approach described before for preferential random walks and Lévy flights to the case of the fractional transport on networks with transition probabilities between nodes given by the Eq. (26). In the case of rings with \( N \) nodes, the eigenvalues of the transition matrices can be obtained analytically [34, 42]. For this regular structure, the fractional degree in Eq. (25) is a constant \( k^{(\gamma)} \) given by [42]

\[
k^{(\gamma)} = (L^{\gamma})_{ii} = \frac{1}{N} \sum_{l=1}^{N} \left( 2 - 2 \cos \left( \frac{2\pi}{N} (l - 1) \right) \right)^{\gamma}
\]

and the eigenvalues \( \{\lambda_i\}_{i=1}^{N} \) for the transition matrix \( \Pi \), with elements in Eq. (26), are

\[
\lambda_i = 1 - \frac{1}{k^{(\gamma)}} \left( 2 - 2 \cos \left( \frac{2\pi}{N} (i - 1) \right) \right)^{\gamma}.
\]

Now, as a consequence of the results in Eqs. (61) and (62), the time \( \tau \) that characterizes the global performance of the random strategy in Eq. (26) to explore a ring coincides with the Kemeny’s constant and is given by [42]

\[
\tau = k^{(\gamma)} \sum_{m=2}^{N} \frac{1}{(2 - 2 \cos \phi_m)^{\gamma}},
\]

where \( \phi_i \equiv \frac{2\pi}{N} (i - 1) \).

In Fig. 12 we represent the numerical values of the global time \( \tau/\tau_0 \) obtained for the fractional transport on rings. The results are calculated by direct evaluation of the result in Eq. (63). We explore the parameter \( \gamma \) that defines the fractional strategy for different values of the size of the ring \( N \). In Fig. 12 we observe that the dynamics with \( 0 < \gamma < 1 \) always improves the capacity to explore the ring in comparison with a normal random walk recovered in the case \( \gamma = 1 \). This effect is observed in the reduction of the quantity \( \tau/\tau_0 \) for \( \gamma = 0.25, 0.5 \) and \( \gamma = 0.75 \). On the other hand, in the limit \( \gamma \to 0 \) the dynamics is equivalent to a normal random walker on a

FIG. 12. (Color online) Global time \( \tau \) for Lévy flights to reach any site of a finite ring. The results are expressed in terms of the value \( \tau_0 = (N - 1)^2/N \) obtained for a complete graph. The values are calculated using the analytical expressions for the spectra in Eqs. (58)–(60) and the Eq. (61). In (a) the time \( \tau \) is presented as a function of \( N \) for \( \alpha = 1, 2, 3 \) and the limit case \( \alpha \to \infty \); these results are complemented in (b) where the time \( \tau \) is plotted as a function of \( \alpha \) for Lévy flights with different values of \( N \).

FIG. 13. (Color online) Global time \( \tau \) that gives the average number of steps needed for Lévy flights to reach any site of the network in different structures; the results are expressed in terms of the value \( \tau_0 = (N - 1)^2/N \). In figures (a) and (b) we depict the time \( \tau \) for different values of \( \alpha \) in networks with \( N = 5000 \). We calculated the values numerically using the Eqs. (52) and (55) for large-world networks (ring, tree, and a square lattice of size 50 \( \times \) 100 and periodic boundary conditions) and small-world networks (Erdős–Rényi network (ER) in the percolation limit and a scale-free network (SF) of the Barabási–Albert type).

In Fig. 13 we represent the numerical values of the global time \( \tau/\tau_0 \) for different structures.

\[
N \in \{10^{10}, 10^{10}, (N = 10^3), (N = 5 \times 10^3), (N = 10^4)\}
\]

\[
\alpha \in \{1, 2, 3, \alpha \to \infty\}
\]

\[
\tau / \tau_0 (\text{Ring}), \tau / \tau_0 (\text{Tree}), \tau / \tau_0 (\text{Lattice})
\]

\[
\tau / \tau_0 (\text{SF}), \tau / \tau_0 (\text{ER})
\]

In Fig. 12 we observe that the dynamics with \( 0 < \gamma < 1 \) always improves the capacity to explore the ring in comparison with a normal random walk recovered in the case \( \gamma = 1 \). This effect is observed in the reduction of the quantity \( \tau/\tau_0 \) for \( \gamma = 0.25, 0.5 \) and \( \gamma = 0.75 \). On the other hand, in the limit \( \gamma \to 0 \) the dynamics is equivalent to a normal random walker on a
works generated from the Watts-Strogatz (WS) model with connected networks with \( N \). In this way, the efficiency or global performance of the random walker is quantified by the time \( \tau \) given by the average of the times Eq. (52) that depends on the eigenvectors and eigenvalues of the transition matrix \( \Pi \) with elements given by Eq. (26). In Fig. 15 we show the global time \( \tau \) for networks with \( N = 5000 \) nodes. We analyze a deterministic tree created by an iterative method for which an initial node ramifies with two leaves that also repeat this process until the size \( N \), the final structure is a large-world network with average distances \( \langle d \rangle \) between nodes that scale as the size of the network. On the other hand, we analyze random networks generated with the Watts-Strogatz model for which an initially regular network is generated and then rewired uniformly randomly with probability \( p \); for values of \( p \to 0 \) this random network has the large world property of the original lattice; however, the rewiring introduces shortcuts that reduce the average path lengths with the increasing of \( p \). In addition, small-world networks generated with the Erdös-Rényi model and a scale-free (SF) network of the Barabási-Albert type are explored \cite{107,108}. We observe that the generalized dynamics defined in terms of the fractional Laplacian \( L^\gamma \) with \( 0 < \gamma < 1 \) always improves the efficiency to explore the networks, the effects are marked in large-world networks with a significant change in the value \( \tau / \tau_0 \), but the dynamics also improves the results for small-world networks.

As we discussed in Section III B 3, the random walk defined in terms of the fractional Laplacian \( L^\gamma \) is a particular case of non-local strategies expressed using functions of the Laplacian \( g(L) \). The same approach presented here applies to other types of strategies \cite{12}; for example, when we use weights defined in terms of the logarithmic function \( \log(1 + aL) \) for \( \alpha > 0 \) and the function \( \Pi - e^{-aL} \) with \( a > 0 \).

**D. Random walks to visit specific locations**

We end this section with an application of the formalism developed in terms of a matrix of weights but now for a model in the context of human mobility that not necessarily requires the definition of a network. We analyze the random walk strategy with transition probabilities defined by Eqs. (20) - (21) to visit specific locations in a region. In this case, the matrix of weights with elements \( \Omega^{(\gamma)}(R) \) depends on the characteristic length \( R \), that defines a local neighborhood, and the parameter \( \alpha \) that controls a dynamics with similar characteristics to the Lévy flights on networks but now to visit the locations. In order to quantify the capacity of the random walker to visit the \( N \) locations in space, we use the time \( \tau^{(\alpha)}(R) \), that gives the average number of steps needed.
The value $\tau^{(\alpha)}(R)$ gives the average number of steps needed to reach any of the $N$ sites, independently of the initial condition; we use $N = 100$ random sites in the region $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. The results are obtained from the analytical expressions in Eqs. (52)-(55) and the numerical evaluation of the eigenvectors and eigenvalues of the transition matrix with elements $w_{ij}(R)$. In Fig. 16 we present the time $\tau^{(\alpha)}(R)$ for different values of the parameters $\alpha$ and $R$ to visit $N = 100$ random locations on the plane. The values are obtained using the exact analytical results in terms of the eigenvectors and eigenvalues of the transition matrix defined by Eq. (20). It is observed how, for $\alpha \gg 1$, different values of $R$ define diverse ways to visit the $N$ sites in the plane; in particular, $R \ll 1$ characterizes a local strategy that requires many steps to reach the locations. Strategies with $\alpha \leq 1$ are optimal and in this interval the results are independent of the parameter $R$. The results observed with the help of the global time $\tau^{\alpha}(R)$ suggest that long-range strategies always improve the capacity of the random walker to reach any of the $N$ locations [24].

VI. CONCLUSIONS

In this work, we presented a general approach to examine different random walk strategies on undirected networks described as a discrete time Markovian process with transition probabilities defined in terms of a symmetric matrix of weights. This formalism is explored for different types of random walks with transitions restricted to nearest neighbors in the case of local strategies and hops from one node to any site of the network in the case non-local strategies. We obtained the stationary probability distribution and other quantities that characterize the capacity of each random walk strategy to reach the nodes of the network like the mean first passage time and average times to reach a specific node. All these quantities are obtained in terms of the eigenvectors and eigenvalues of the transition matrix $\Pi$; in a similar way, we calculate global quantities like the Kemeny’s constant and the average time to reach any site of the network $\tau$. We explore in detail these quantities for the preferential random walk, the Lévy flights and the fractional transport on networks. For these cases, we conclude that the value of $\tau$ is a good measure that allows comparing the performance of random walk strategies since it quantifies the efficiency of the random walker to explore new sites on the network. On the other hand, the Kemeny’s constant is a useful simplification that depends only on the spectrum of the transition matrix and describes correctly the efficiency of random walk strategies with a constant stationary probability distribution.

Finally, it is worth mentioning that the formalism introduced here can be implemented to the study of other processes described in terms of a matrix of weights. An application in a context different from the field of network science is presented for the analysis of a random walk introduced in the context of human mobility and implemented to visit locations in the plane. In this case, the global time $\tau$ help us to understand how strategies that combine local and non-local displacements improve the capacity of the exploration of specific locations in a region in the plane.

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