\( \mathcal{N} = 1 \) supersymmetric solutions of IIB supergravity from Killing spinors

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We present a new class of “dielectric” \( \mathcal{N} = 1 \) supersymmetric solutions of IIB supergravity. This class contains not just the ten-dimensional lift of the Leigh-Strassler renormalization group flow, but also the Coulomb branch deformation of this flow in which the branes are allowed to spread in a radially symmetric manner, preserving the \( SU(2) \) global symmetry. We use the “algebraic Killing spinor” technique, illustrating how it can be adapted to \( \mathcal{N} = 1 \) supersymmetric flows.
1. Introduction

Understanding and characterizing supersymmetric backgrounds with fluxes is very important in string theory, particularly those backgrounds that lead to $\mathcal{N}=1$ supersymmetry in four dimensions. In this paper we construct explicitly a new family of such backgrounds in type IIB supergravity thereby extending the approach in [1,2] to backgrounds with less supersymmetry.

As in [1,2] the starting point in our construction is a specific Ansatz for the form of the Killing spinors, $\epsilon^{(i)}$, and the metric. This Ansatz is motivated by the symmetry and general structure of the holographically dual supersymmetric field theory in four dimensions. We then show that the Killing spinor equations for unbroken supersymmetries together with the Bianchi identities for the fluxes completely determine the background in terms of a single “master function,” $\Psi(u,v)$, satisfying a non-linear second order partial differential equation:

$$u^3 \frac{\partial}{\partial u} \left( \frac{1}{u^3} \frac{\partial}{\partial u} \Psi \right) + \frac{1}{2v} \frac{\partial}{\partial v} \left( \frac{1}{v^3} \frac{\partial}{\partial v} e^{2\Psi} \right) = 0. \quad (1.1)$$

The fact that we find that the complete solution is characterized by a single PDE should not be surprising: We are seeking a solution that is dual to a field theory with a Coulomb branch and so one should expect to find some generalization of the “harmonic rule” that characterizes pure Coulomb branch flows. Our flow is massive and required to have an $SU(2)$ global symmetry: Hence the non-linearity and the reduction in the number of variables.

Our approach is closely related to the recent work on the classification of supersymmetric backgrounds in terms of $G$-structures (for a review see [3]). Since the analysis of those structures is still quite complicated, most of the work thus far has focused on M-theory [4,5] and on theories in lower dimensions and/or with fewer supersymmetries (see, for example, [6,7]). In particular, we are not aware of any systematic study of $G$-structures in the IIB theory.

The basic objects in the $G$-structure approach are the differential $p$-forms

$$\Omega^{ij}_{M_1 M_2 \ldots M_p} \equiv \epsilon^{(i)} \gamma_{M_1 M_2 \ldots M_p} \epsilon^{(j)} \quad \text{and} \quad C^{ij}_{M_1 M_2 \ldots M_p} \equiv \epsilon^{(i)} T \gamma_{M_1 M_2 \ldots M_p} \epsilon^{(j)}, \quad (1.2)$$

constructed as bilinears in the Killing spinors. Note that $\Omega$ involves Dirac conjugation whereas $C$ involves the Majorana conjugate, which means that the former has $R$-charge 0 and the latter has $R$-charge +1. In particular, the one forms, $\Omega^{ij}_M$, give rise to Killing
vectors of the background \( \mathbb{R}^4 \). This observation will prove crucial for the integration of the Killing spinor equations. The forms (1.2) satisfy a system of first order differential equations together with non-linear algebraic relations that follow from Fierz identities. Geometrically they encode the reduction of the holonomy of the supercovariant connection on the spinor bundle. It has been shown recently [8] that this holonomy group for a type IIB background with Killing spinors \( \epsilon^{(i)}, \ i = 1, \ldots, \nu \), where \( \nu \) is the number of unbroken supersymmetries between 0 and 32, is a subgroup of the semi-direct product\( SL(32 - \nu, \mathbb{R}) \times ( \oplus^\nu R^{32-\nu} ) \), in particular, \( SL(32, \mathbb{R}) \) is the largest holonomy of a generic background.

Our emphasis here is on explicit construction of supersymmetric backgrounds with fluxes and we find it simpler to start directly with the invariant spinors of the \( G \)-structure. More precisely, we make a very general Ansatz for the metric and to some extent for the fluxes, and then write some projection conditions that define the supersymmetries. The supersymmetry variations then become (over-determined) algebraic equations for the background tensor-gauge fields, and the equations of motion then emerge from both the over-determined algebraic equations and the Bianchi identities for the field strengths.

In this paper we will find a family of solutions of IIB supergravity with four supersymmetries that are also holographic duals of \( \mathcal{N} = 1 \) supersymmetric field theories in four dimensions. In particular, we find a family of solutions that generalizes the flows of [1,10,11], which contain the holographic dual of a “Leigh-Strassler” (LS) renormalization group flow [12,13]. We obtain a family of flows because the solution is completely determined by a single function of two variables that is required to satisfy (1.1). Our restriction of the number of variables is needed to make the problem manageable, but we expect that there should be generalizations to more variables. Our approach closely parallels that of [14] in which families of solutions with four supersymmetries are found in \( M \)-theory. One of our purposes here is to show that the ideas of [1,2] can be adapted, in a very simple manner, to address problems with less supersymmetry.

Our approach is similar to others involving prescriptions for the spinors that make up the supersymmetry. The natural first step is to use the Poincaré invariance on the brane to break the supersymmetry into a “4 + 6” split [15]:

\[
\epsilon = \zeta \otimes \chi^{(1)} + \zeta^* \otimes \chi^{(2)*}, \tag{1.3}
\]

where \( \Gamma^{(4)} \zeta = +\zeta \) and \( \Gamma^{(6)} \chi^{(i)} = -\chi^{(i)} \) denote the helicity components in 4 and 6 dimensions respectively. The issue of supersymmetry then hinges upon how \( \chi^{(1)} \) is related to
Interesting classes of solutions arise from relatively simple relationships, such as \( \chi^{(2)} = 0 \) (type B) or \( \chi^{(2)} = e^{i\psi} \chi^{(1)} \) (types A and C). However, the type of solution that we wish to obtain, and that arises naturally in physically important massive flows of holographic gauge theories, do not fit into such simple schemes: The relationship between \( \chi^{(1)} \) and \( \chi^{(2)} \) is significantly more complicated.

Fortunately, the underlying physics provides us with an invaluable guide to solving the problem. The massive flows we seek involve fluxes for the 3-form field strengths, and so one should expect some dielectric polarization of the D3-branes into D5-branes through the Myers effect \[20\]. One of the surprising results of [1] was that there was a concomitant “dielectric” deformation of the canonical supersymmetry projector. That is, the presence of tensor gauge fields caused the supersymmetry projector transverse to the original branes to be rotated so as to receive a component in the internal directions: The product \( \gamma^{1234} \) was rotated into a term of the form \( \gamma^{1234AB} \), for some choice of \( A \) and \( B \). We believe that this should be interpreted as polarizing some of the D3-branes into a mixture of D5 and NS5 branes. We will find a very similar structure here for the flows with four supersymmetries, showing that such dielectric deformations are an essential part of holographic RG flows.

More generally, we believe that it will be important to consider the dielectric deformation of the canonical supersymmetry projector in broader classes of string compactification, including backgrounds based on compact manifolds. In the context of holographic RG flows we expect that our methods can be used to study other families of supergravity solutions, in particular (i) to find the explicit solution, and generalizations, of the Klebanov-Witten flow \[21\], (ii) to test the conjecture duality of \[22,23\] that relates the flows that we obtained in \[9,10\] to those of Klebanov and Witten \[21\], (iii) to generalize the duality cascade of Klebanov and Strassler \[24\] to other UV fixed point theories.

In section 2 we describe our general Ansatz for the metric, tensor gauge fields and supersymmetries. While we can motivate our choice of Ansatz rather generally, we actually arrived at it by a detailed study of the supersymmetry in the solutions of \[9,10\]. To simplify the presentation, we describe the particular solutions in section 4. Section 3 contains the solution to the general Ansatz, and section 5 contains some final remarks. Throughout this paper we use the conventions of \[25\].
2. The supersymmetry Ansatz

2.1. The underlying holographic field theory

To understand and motivate the supergravity calculation it is useful to recall some of the details of the holographic flow in field theory. We consider a relevant deformation of \( \mathcal{N} = 4 \) super-Yang-Mills theory by a mass term for one of the three \( \mathcal{N} = 1 \) adjoint chiral superfields. That is, we consider a superpotential of the form

\[
W = \text{Tr} \left( \Phi_3 [\Phi_1, \Phi_2] \right) + \frac{1}{2} m \text{Tr} (\Phi_3^2)
\]

The first term is the superpotential inherited from the original \( \mathcal{N} = 4 \) supersymmetric gauge theory, while the second term breaks conformal invariance, reduces the supersymmetry from \( \mathcal{N} = 4 \) to \( \mathcal{N} = 1 \), and drives the theory to a new, non-trivial \( \mathcal{N} = 1 \) superconformal fixed point in the infra-red \([12]\).

This infra-red fixed fixed point has a four-dimensional Coulomb branch that may be described in terms of the vevs of the operators \( \Phi_1 \) and \( \Phi_2 \). A two parameter family of flows on this Coulomb branch were studied in \([11,26]\), and a brane-probe study can be found in \([27,28]\).

Our purpose here is to find a family of flows that correspond to branes with an arbitrary, rotationally symmetric distribution of branes on this Coulomb branch. As usual, the vevs of the scalar fields correspond to directions perpendicular to the branes in the supergravity solution, and we will use polar coordinates \((u, \phi)\) to describe the \(\Phi_3\)-direction, and \((v, \varphi_1, \varphi_2, \varphi_3)\) to describe the \((\Phi_1, \Phi_2)\)-direction. The \(\varphi_j\) may be thought of as Euler angles on the \(S^3\)'s at constant \(v\) in the \((\Phi_1, \Phi_2)\)-direction. The mass deformation in (2.1) preserves an \(SU(2) \times U(1) \times U(1)\) subgroup of the original \(SU(4)\) \(R\)-symmetry. The \(SU(2) \times U(1) = U(2)\) acts on \((\Phi_1, \Phi_2)\) as a doublet, while the last \(U(1)\) is a \(\phi\)-rotation. In the finite \(N\) field theory this \(U(1)\) is anomalous, but in large \(N\) this is restored and it is thus a symmetry of the supergravity solution. The brane moduli space is at \(u = 0\) and is spanned by \((v, \varphi_j)\), and the solution we seek has the branes spread out with a density that is an arbitrary function, \(\rho(v)\), of the radial variable on the moduli space. This choice keeps the problem relatively simple in that it preserves all the symmetries.
2.2. The supergravity background

The supersymmetry variations for the gravitino, $\psi_M$, and the spin-$\frac{1}{2}$ field, $\lambda$, in IIB supergravity read [25]:

$$\delta \psi_M = D_M \epsilon + \frac{i}{480} F_{PQRST} \gamma^{PQRST} \gamma_M \epsilon + \frac{1}{96} \left( \gamma_M^{PQR} - 9 \delta^M_P \gamma^{QR} \right) G_{PQR} \epsilon^*, \quad (2.2)$$

and

$$\delta \lambda = i P_M \gamma^M \epsilon^* - \frac{i}{24} G_{MNP} \gamma^{MNP} \epsilon, \quad (2.3)$$

where $\epsilon$ is a complex chiral spinor satisfying

$$\gamma^{11} \epsilon = -\epsilon. \quad (2.4)$$

We now take the metric to have the form:

$$ds^2 = H_1^2 (dx_a)^2 - H_5^2 dv^2 - H_6^2 (du^2 + u^2 d\phi^2) - H_7^2 (\sigma_1^2 + \sigma_2^2) - (H_0 \sigma_3 + H_0 d\phi)^2, \quad (2.5)$$

and we use the frames:

$$e^a = H_1 dx^a, \quad a = 1, \ldots, 4, \quad e^5 = H_5 dv, \quad e^6 = H_6 du,$$

$$e^7 = H_7 \sigma_1, \quad e^8 = H_7 \sigma_3, \quad e^9 = -u H_6 d\phi, \quad e^{10} = H_0 \sigma_3 + H_0 d\phi, \quad (2.6)$$

where the $H_I = H_I(u, v)$ are functions of both $u$ and $v$, and the $\sigma_i$ are the left-invariant one-forms parametrized by the Euler angles $\varphi_i$, $i = 1, 2, 3$, and normalized so that $d\sigma_1 = \sigma_2 \wedge \sigma_3$.

This Ansatz is based upon the form of the holographic field theory outlined above and the ten-dimensional lift of the LS-flow in [10]. The primary difference between the result in [10] and the Ansatz is that we have introduced the more natural coordinates, $(u, v)$. Indeed, the solution of [10] exactly fits our Ansatz if one changes variable according to

$$u = e^{\frac{A}{2}} \sqrt{\sinh \chi} \sin \theta, \quad v = e^A \rho \cos \theta. \quad (2.7)$$

---

1 We use the same notation and $\gamma$-matrix conventions as in [25], except that we label the indices from 1 to 10. Also, see appendix A of [1].

2 As with the $\mathcal{N} = 2$ solutions of [1], one can show that this change of coordinates can be associated with a Lorentz rotation which brings the Killing spinors of unbroken supersymmetries into a canonical form.
One of the important features of the new variables is the simple form of the metric in the \((u, \phi)\) direction. Indeed, the metric has a natural almost-complex structure on the internal space:

\[
J \equiv -e^6 \wedge e^9 + e^7 \wedge e^8 + e^5 \wedge e^{10}.
\] (2.8)

Note that there are some arbitrary choices of sign that can be made in each term. The choices that we have made here will correlate with helicity projections that define the supersymmetry.

Since the dilaton and axion backgrounds were trivial in [10], we will will also seek such backgrounds here.

Following the observations of [29], we make an Ansatz for the two-form potential in which all the indices are holomorphic with respect to (2.8):

\[
A_{(2)} = i e^{-i\phi} \left[ a_1 (e^6 + ie^9) + a_2 (e^5 - ie^{10}) \right] \wedge (e^7 - ie^8)
\] (2.9)

for some functions, \(a_1(u,v)\) and \(a_2(u,v)\). The factor of \((e^7 - ie^8)\) in (2.9) is required by the action of the \(U(1)\) symmetries, but beyond this one could make a more general Ansatz for \(A_{(2)}\). In principle one should be able to fix this using the form of the Killing spinors defined below, combined with the variations (2.2) and (2.3). In practice, solving such a system is hard, and so we have made the holomorphic Ansatz above based upon the observations of [29].

Because the dilaton and axion are trivial, the three-form field strength, \(G_{(3)}\), is simply

\[
G_{(3)} = dA_{(2)}.
\] (2.10)

The foregoing Ansatz for the three-index tensor in (2.9) and (2.10) is more restrictive than the one for the \(\mathcal{N} = 2\) supersymmetric solutions in [1], where the only requirement was that certain components, \(G_{MNP}\), of the three-index field strength vanished. Here we start with an Ansatz for the potential, and the basic reason is the smaller amount of supersymmetry (\(\mathcal{N} = 1\)) in the present problem. If one makes a very general Ansatz, then the system of Killing equations and Bianchi identities that would result here would considerably more difficult to analyze than that encountered in [1].

Finally, we define:

\[
C_{(4)} = w dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,
\] (2.11)
for some function, \( w(u, v) \), and then take the five-form field strength to be:

\[
F^{(5)} = dC^{(4)} + \ast dC^{(4)}.
\]  

\[ (2.12) \]

2.3. The supersymmetries

Having made the Ansatz for the metric and the tensor fields, we now restrict the form of the Killing spinors through sets of projectors. First, there is the helicity condition \((2.4)\) on the spinors of the IIB theory:

\[
\Pi_{11} \epsilon = \epsilon, \quad \text{where} \quad \Pi_{11} = \frac{1}{2} (1 - \gamma^{11}).
\]  

\[ (2.13) \]

Next, we follow the philosophy outlined in [2], and see how the supersymmetry must be defined on the moduli space of the brane probes and then assume that whatever projection conditions are needed on that space will lift, without modification, to the full space. To reduce to one-quarter supersymmetry on the four-dimensional moduli space one must reduce the four-component spinors to a single helicity component. That is, one needs to fix the helicity of \( \gamma^{78} \) and \( \gamma^{510} \). These pairs of \( \gamma \)-matrices are the natural ones given by the almost complex structure of \((2.8)\). We therefore introduce the projectors:

\[
\Pi_{78} = \frac{1}{2} \left( 1 - i \gamma^7 \gamma^8 \right), \quad \Pi_{510} = \frac{1}{2} \left( 1 - i \gamma^5 \gamma^{10} \right),
\]  

\[ (2.14) \]

and impose the further conditions:

\[
\Pi_{78} \epsilon = \epsilon, \quad \Pi_{510} \epsilon = \epsilon.
\]  

\[ (2.15) \]

There are choices of sign to be made in the definitions of \((2.14)\). As we will see, the choices here are fixed by the choices of signs in \((2.8)\), \((2.9)\) and \((2.6)\).

The final projector is a dielectric deformation of the standard projector for the \(D3\)-branes:

\[
\Pi_{1234} = \frac{1}{2} \left[ 1 + i \gamma^1 \gamma^2 \gamma^3 \gamma^4 (\cos \beta - e^{-i \phi} \sin \beta \gamma^7 \gamma^{10} \ast) \right],
\]  

\[ (2.16) \]

where \( \beta = \beta(u, v) \) is a function to be determined. We then impose the condition

\[
\Pi_{1234} \epsilon = \epsilon.
\]  

\[ (2.17) \]

We will describe below how we arrived at this projector, however its form is essentially fixed by the physics and mathematics of the problem. First, because we are dielectrically
polarizing the $D3$-branes into $D5$-branes and $NS5$-branes, the deformation term must be a product of six $\gamma$-matrices (for the five-branes) containing $\gamma^1\gamma^2\gamma^3\gamma^4$ (for the $D3$-branes). Thus we need to find the two extra $\gamma$-matrices. The result must commute with $\Pi_{78}$ and $\Pi_{5\,10}$, and be a “true deformation.” For example, $\gamma^1\gamma^2\gamma^3\gamma^4\gamma^7\gamma^8$ will not suffice because (2.15) means that it is the same as $\gamma^1\gamma^2\gamma^3\gamma^4$. This leads one to choose one of $\gamma^7, \gamma^8$ and one of $\gamma^5, \gamma^{10}$, and (2.15) means that it does not matter which ones we choose. Having made the choice, the complex conjugation, $\ast$, operation is essential for $\Pi_{1234}$ to commute with $\Pi_{78}$ and $\Pi_{5\,10}$. Finally, the factor of $e^{-i\phi}$ is required to correct the $\phi$-dependence after complex conjugation. Put more physically, $\phi$-rotations generate the $U(1)$ $\mathcal{R}$-symmetry, and the $e^{-i\phi}$ term is essential for the projector to preserve $\mathcal{R}$-symmetry.

It is worth noting that the projection conditions (2.15) and (2.16) are natural generalizations of the corresponding conditions used to define the Killing spinors on a Calabi-Yau manifold.

Having defined the space of supersymmetries, we need to fix their dependence on the various coordinates. The angular dependence can be fixed using the Lie derivative on spinors [30,31]:

$$L_K \epsilon \equiv K^M \nabla_M \epsilon + \frac{1}{4} \nabla_{[M} \gamma_{N]} \epsilon.$$  \hfill (2.18)

The fact that the spinors are singlets of the $SU(2)$ symmetry means that $L_K \epsilon = 0$ for Killing vectors, $K^M_{(j)}$, in these directions. If one chooses the $\sigma_j$ with the appropriate handedness, the connection term and the $\nabla_{[M} K_{N]}$ term cancel, and one is left with $\partial_{\varphi_j} \epsilon = 0, j = 1, 2, 3$. A similar cancellation takes place for the Killing vector in the $\phi$-direction, except that this is the residual $\mathcal{R}$-symmetry that acts on the supersymmetry, and $\epsilon$ has charge $+\frac{1}{2}$. The tensor gauge field (2.9) has $\mathcal{R}$-charge $+1$, and therefore one has:

$$\partial_{\phi} \epsilon = \gamma^6 \gamma^9 \epsilon - \frac{i}{2} \epsilon.$$  \hfill (2.19)

The $\phi$-phase dependence in (2.16) can also be fixed by requiring that the projector commute with this Lie derivative operator.

Finally, the dependence on $(u, v)$ can be fixed by using the fact that $K^M \equiv \bar{\epsilon} \gamma^M \epsilon$ is always a Killing vector. Using spinors that satisfy the projection conditions, one finds that $K^M$ is either zero, or it is parallel to the brane, in which case it must be a constant vector. This fixes the normalization of $\epsilon$ in terms of $H_1$. The end-result is that we have completely determined the form of supersymmetries up to an arbitrary function, $\beta$. 

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We conclude by observing that the foregoing can be re-written rather more directly by introducing the “rotation” matrix:

$$\mathcal{O}^*(\beta) = \cos(\frac{\beta}{2}) + \sin(\frac{\beta}{2}) \gamma^7 \gamma^{10} * .$$

(2.20)

The Killing spinor is then given explicitly by:

$$\epsilon = H^{1/2}_1 e^{-i\phi/2} \mathcal{O}^*(\beta) e^{i\phi} \epsilon_0,$$

(2.21)

where $\epsilon_0$ is a constant spinor satisfying:

$$\Pi_{78} \epsilon_0 = \epsilon_0, \quad \Pi_{510} \epsilon_0 = \epsilon_0, \quad \Pi_{1234}^{(0)} \epsilon_0 = \epsilon_0,$$

(2.22)

and where

$$\Pi_{1234}^{(0)} = \frac{1}{2} [1 + i \gamma^1 \gamma^2 \gamma^3 \gamma^4].$$

(2.23)

Conjugating $\Pi_{1234}^{(0)}$ by $e^{-i\phi/2} \mathcal{O}^*(\beta)$ results in the deformed projector (2.16),

$$\Pi_{1234} = e^{-i\phi/2} \mathcal{O}^*(\beta) \Pi_{1234}^{(0)} \mathcal{O}^*(\beta)^{-1} e^{i\phi/2}.$$

(2.24)

Thus the whole family of solutions considered here may be thought of as a duality rotation of a standard, $\mathcal{N} = 1$ supersymmetric brane compactification whose supersymmetries are defined by (2.22).

3. Solving the Ansatz

We now use our Ansatz in (2.2) and (2.3) and solve for all the undetermined functions.

3.1. The step-by-step process

The easiest one to solve is (2.3) which, because of the trivial dilaton/axion background, collapses to $G_{MNP} \gamma^{MNP} \epsilon = 0$. This leads immediately to the projection condition $\Pi_{78} \epsilon = \epsilon$. Next, we observe that in the combination

$$2\gamma^1 \delta \psi_1 + \gamma^7 \delta \psi_7 + \gamma^8 \delta \psi_8 = 0$$

(3.1)

all terms with the antisymmetric tensors cancel and the entire contribution on the left-hand-side comes from the spin connection. It is this that leads us to the second projection...
condition, \( \Pi_{10} \epsilon = \epsilon \). We thus see how the choices of signs in the Ansatz result in the sign choices for the projectors (2.13). From this we also obtain the conditions:

\[
H_{03} = \frac{v}{2 H_1^2 H_5}, \quad \partial_u (H_1 H_7) = 0.
\]  

(3.2)

Motivated by this we set:

\[
H_7 = \frac{1}{2} v H_1^{-1}.
\]  

(3.3)

We are free to choose this solution because we have not yet fixed the coordinate freedom in \( u \) and \( v \). Specifically, we can redefine \( v \to 2H_1 H_7 \), and then redefine \( u \) so as to remove any \( du \, dv \) cross-terms in the metric. The metric Ansatz is now reduced to:

\[
ds^2 = H_1^2 (dx^\mu)^2 - H_5^2 \, dv^2 - H_6^2 (du^2 + u^2 d\phi^2) - \frac{v^2}{4 H_1^2} (\sigma_1^2 + \sigma_2^2) - \frac{v^2}{4 H_1^4 H_5^2} (\sigma_3 + 2 H_0 d\phi)^2,
\]  

(3.4)

where \( H_0 \equiv v^{-1} H_0 \, H_1^2 \, H_5 \).

The remaining supersymmetry variations yield an entangled system of first order differential equations. However, by taking suitable linear combinations of those equations it is quite straightforward to determine \( H_1 \) and \( w \) in terms of the other functions in the Ansatz:

\[
H_1^2 = \frac{1}{2 v H_5 H_6} \frac{\partial}{\partial v} \left( \frac{v^2 H_6}{H_5} \right), \quad w = -\frac{1}{4} H_1^4 \cos \beta.
\]  

(3.5)

At this point the algebra gets tougher. In addition, one needs to consider separately the two cases depending on whether \( \beta \) is zero or not. In the appendix we have summarized the remaining equations that one must still solve at this point.

If \( \beta = 0 \) then the solution is degenerate in the sense that the supersymmetry variations alone do not determine all the functions in the Ansatz. That is, the \( \mathcal{N} = 1 \) supersymmetry does not determine a solution without a complete analysis of the field equations. This also happens in the more familiar, standard harmonic solutions with unbroken supersymmetry. We refer the reader to the appendix for some additional discussion of the solution with \( \beta = 0 \) and in the following assume that \( \beta \neq 0 \).

For \( \beta \neq 0 \), the main conclusion of a detailed analysis of the equations in the appendix is that using the supersymmetry variations one can determine explicitly all functions in the Ansatz in terms of just two of them, \( H_5 \) and \( H_6 \). In particular, in addition to (3.3), we have

\[
\sin \beta = \frac{2 u}{H_1^3 H_6},
\]  

(3.6)
Then, using (3.5), we can integrate (A.9) to obtain
\[ H_0 = -\frac{1}{2} u \frac{\partial}{\partial u} \log \left( \frac{H_6}{H_5} \right). \] (3.7)

Finally, the two-index potential is given by
\[ a_1 = \frac{v H_1 H_0}{u^2 H_5}, \quad a_2 = -\tan(\frac{1}{2} \beta). \] (3.8)

### 3.2. Building the solution

To solve for \( H_5 \) and \( H_6 \) one must disentangle the remaining first order system of differential equations. It is convenient to define:
\[ \Psi = \log \left( \frac{v^2 H_6}{H_5} \right). \] (3.9)

This function must satisfy the “master equation:"
\[ u^3 \frac{\partial}{\partial u} \left( \frac{1}{u^3} \frac{\partial}{\partial u} \Psi \right) + \frac{1}{2} v \frac{\partial}{\partial v} \left( \frac{1}{v^3} \frac{\partial}{\partial v} e^{2\Psi} \right) = 0. \] (3.10)

Associated with \( \Psi \) there is a conjugate function, \( S \), defined by:
\[ \frac{\partial S}{\partial u} = -\frac{1}{2} v^3 \frac{\partial}{\partial u} (e^{2\Psi}), \quad \frac{\partial S}{\partial v} = \frac{v}{u^3} \frac{\partial \Psi}{\partial u}. \] (3.11)

The “master equation” is the integrability condition for \( S \). If one solves (3.10) and integrates the solution to obtain \( S \), one can then determine all other functions as follows. First one determines \( \beta \) from:
\[ \tan^2 \left( \frac{1}{2} \beta \right) = -\left( 1 + \frac{1}{2} u \frac{\partial}{\partial u} \log S \right) = \frac{e^{2\Psi} \frac{\partial}{\partial v} \Psi}{2 u^2 v^3 S} = \frac{\partial_v e^{2\Psi}}{4 u^2 v^3 S}. \] (3.12)

Then one has:
\[ H_6 = \frac{1}{v^2} e^{\Psi}, \quad H_5^2 H_6 = \frac{2 u}{\sin \beta}, \quad H_1^2 H_5^2 = \frac{1}{2} v \frac{\partial \Psi}{\partial v}. \] (3.13)

from which one can algebraically determine \( H_1, H_5 \) and \( H_6 \). In particular, one can see that
\[ H_1^4 = \frac{u^2}{v^3 \sin^2 \beta} \frac{\partial_v (e^{2\Psi})}{}, \] (3.14)

and then one can easily read off \( H_5 \) and \( H_6 \). The remaining functions are then given by:
\[ H_0 = -\frac{1}{2} u \frac{\partial \Psi}{\partial u}, \] (3.15)
\[ a_1 = \frac{v H_1 H_0}{u^2 H_5}, \quad a_2 = -\tan(\frac{1}{2} \beta), \] (3.16)
\[ w = -\frac{1}{4} H_1^4 \cos \beta. \] (3.17)

Thus, once one solves (3.10), one has solved the entire Ansatz.

We have also verified that given these equations, the Ansatz satisfies all the Bianchi identities and the equations of motion of the IIB theory.
3.3. A comment on the perturbation expansion

Finally, we note that for solutions that are asymptotic to $AdS_5 \times S^5$, or asymptotic to any Coulombic brane distribution, one wants $H_5 \rightarrow H_6$, or $\Psi \rightarrow \log(v^2)$. Define $\tilde{\Psi} \equiv \Psi - \log(v^2)$ and observe that $\tilde{\Psi} \rightarrow 0$ at infinity and that $\tilde{\Psi}$ satisfies the master equation if and only if $\Psi$ satisfies the same equation. This means that we can make a simple asymptotic perturbation expansion for $\tilde{\Psi}$:

$$
\tilde{\Psi}(u,v) = \sum_{n=1}^{\infty} \xi^n \chi_n(u,v),
$$

where $\xi$ is a small parameter and the $\chi_n$ are to be determined. The function $\chi_1$ satisfies the linearized master equation:

$$
\mathcal{L} \chi_1 \equiv u^2 \frac{\partial}{\partial u} \left( \frac{1}{u^2} \frac{\partial}{\partial u} \chi_1 \right) + \frac{1}{v} \frac{\partial}{\partial v} \left( \frac{1}{v^3} \frac{\partial}{\partial v} \chi_1 \right) = 0,
$$

while the $\chi_n$ must satisfy a linear equation of the form:

$$
\mathcal{L} \chi_n = \mathcal{P}(\chi_1, \ldots, \chi_{n-1}),
$$

for some polynomial, $\mathcal{P}$. There is, of course, the ambiguity of adding in more of the homogeneous solution at each step, but such ambiguities are easily resolved in terms of a re-definition of $\chi_1$. The function $\chi_1$ is thus the “seed function” for the complete solution. Since it satisfies the linear PDE (3.19), we may choose it to have a source defined by a function, $\rho(v)$, in the plane $u = 0$. While this function may not ultimately be true brane distribution of the non-linear problem, our argument shows that while we have a non-linear problem, there is a good perturbation theory, and that there is indeed a family of solutions determined by the choice of an arbitrary function $\rho(v)$. Moreover, this function will determine the multipole expansion of the brane distribution as seen from infinity.

4. Some special solutions

4.1. The FGPW-flow

The flow solution obtained in [13] was based upon five-dimensional supergravity and involved the metric:

$$
ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu - dr^2.
$$

(4.1)
The equations of motion were:

$$\frac{d\varphi_j}{dr} = \frac{1}{L} \frac{\partial W}{\partial \varphi_j}, \quad \frac{dA}{dr} = -\frac{2}{3L} W,$$  \hfill (4.2)

with

$$W \equiv \frac{1}{4\rho^2} \left[ \cosh(2\chi) \left( \rho^6 - 2 \right) - (3\rho^6 + 2) \right], \hfill (4.3)$$

where $\chi = \varphi_1$, $\rho = \exp(\frac{1}{\sqrt{6}} \varphi_2)$.

The lift of this solution to ten dimensions was presented in [10] and further simplified in [29]. As remarked above, the ten-dimensional solution is obtained from our Ansatz by taking:

$$u(r,\theta) = e^{\frac{2}{3}A} \sqrt{\sinh \chi} \sin \theta, \quad v(r,\theta) = e^A \rho \cos \theta. \hfill (4.4)$$

For completeness we also note that one has:

$$\Psi(u,v) = \log \left( \frac{e^{\frac{2}{3}A} \cosh \chi \cos^2 \theta}{\sqrt{\sinh \chi}} \right), \quad S(u,v) = \frac{e^{-4A}}{\rho^4 \sinh^2 \chi \sin^2 \theta}. \hfill (4.5)$$

One can then solve (4.4) and (4.5) to obtain

$$\cosh \chi = \frac{e^\Psi}{uv^2 \sqrt{S}}, \quad \frac{\cos^4 \theta}{\sin^2 \theta} = \frac{e^{2\Psi}}{u^2} - v^4 S, \hfill (4.6)$$

but there is no similarly simple formula to express $\rho$ as a function of $u$ and $v$.

4.2. The KPW fixed point

The general flows considered here should correspond to the Coulomb branch of the $\mathcal{N} = 1$ supersymmetric fixed point theory described in [12]. This means that the supergravity solution will generically be singular in its core, and the singularity will merely reflect the appropriate continuum distribution of branes. There is, however, one non-singular background, discovered in [32,9] that represents the conformal IR fixed point field theory [33,13], and in which the space-time has an $\text{AdS}_5$ factor. In terms of the parameterization above, this point has:

$$\cosh \chi(r) = \frac{2}{\sqrt{3}}, \quad \rho(r) = 2^{1/6}, \quad A(r) = \frac{2}{3} \frac{2^{2/3}}{r}, \hfill (4.7)$$

\[3\] In the following we set $L = 1$. 

13
which gives:

\[ u = \frac{1}{3^{1/4}} e^{2^{2/3} r} \sin \theta, \quad v = 2^{1/6} e^{(22^{2/3}/3) r} \cos \theta. \]  

(4.8)

The master function is

\[ \Psi(u, v) = \frac{1}{2} \log \left( v^3 F \left( \frac{9}{\sqrt{2}} \frac{u^2}{v^3} \right) \right), \]  

(4.9)

where

\[ F(x) = \frac{\sqrt{2}}{x} \left( \left( x + \sqrt{x^2 - 1} \right)^{2/3} + \left( x - \sqrt{x^2 - 1} \right)^{2/3} - 1 \right). \]  

(4.10)

One then finds that:

\[ S(u, v) = \frac{27}{2 v^4} H \left( \frac{9}{\sqrt{2}} \frac{u^2}{v^3} \right), \]  

(4.11)

where

\[ H(x) \equiv \frac{(x + \sqrt{x^2 - 1})^{1/3}}{x \left( 1 + (x + \sqrt{x^2 - 1})^{1/3} \right)}. \]  

(4.12)

4.3. Other solutions

Motivated by the form of (4.9), one can seek solutions of the form:

\[ e^{2\Psi(u,v)} = v^a u^b F(v^c u^d). \]  

(4.13)

Then the function \( F(x) \) satisfies an ordinary differential equation provided

\[ a = 6 + \frac{c(2 + b)}{d}, \]  

(4.14)

and the coefficients in the eqs have only integer powers of \( x \) when

\[ \frac{b + 2}{d} = n, \quad n \in \mathbb{Z}. \]  

(4.15)

The critical point solution has

\[ a = 3, \quad b = 0, \quad c = -3, \quad d = 2, \quad n = 1. \]  

(4.16)

Another solution in this spirit is obtained by setting

\[ a = 2, \quad b = c = -2, \quad d \to 0, \quad n = 2, \]  

(4.17)

and is simply

\[ e^{2\Psi(u,v)} = -\frac{2}{3} \frac{v^6}{u^2}. \]  

(4.18)
Here the differential equation is linear:

\[ x^4 F'' - 3x^3 F' + 8 = 0 \]  

(4.19)

For \( d = 0 \) we have to divide out some singular terms to arrive at the differential equation (4.19), and it turns out that only the solution (4.18) gives rise to a solution of the master equation.

Finally, there are obvious solutions where we set each term in the master equation to zero. This yields solutions in which:

\[ \Psi(u, v) = \frac{1}{2} \log(c_1 + v^4) + c_2 u^4 + c_3 \]  

(4.20)

The corresponding solution for \( S \) is then:

\[ S(u, v) = \frac{1}{2} c_2 v^2 - e^{2c_3} \sqrt{\frac{\pi c_2}{2}} \text{Erfi} \left( \sqrt{\frac{c_2}{2}} u^2 \right) + \frac{e^{2c_3 + c_2 u^4/2}}{u^2} + c_4 \]  

(4.21)

where \( c_1, c_2, c_3 \) and \( c_4 \) are integration constants.

5. Final comments

We have shown that the algebraic Killing spinor techniques proposed in [1,2] can be successfully applied to problems with fewer supersymmetries. The ideas adapt very directly to the new class of problems considered here, and yield infinite families of solutions that once again generalize the harmonic Ansatz. We therefore believe that these techniques will find broader applications within string theory, and as we indicated in the introduction, research on these issues is continuing.

On a more physical level, we have once again seen that interesting supersymmetric flow solutions involve a dielectric polarization of \( D3 \)-branes into five-branes. This polarization does not break the supersymmetry by itself: The supersymmetry projector undergoes a duality rotation in which the original \( D3 \)-branes directions are all parallel to the emergent distribution of five-branes. Indeed, this represents a unifying thread between this paper and our earlier work: In more standard compactifications the supersymmetries are defined by simple geometric projection conditions, and we are essentially replacing the canonical projector associated with the branes by some dielectric deformation, while leaving all the other projectors untouched. Since the solutions we are studying are holographic
duals of interesting flows in supersymmetric gauge theories, we believe that these dielectric deformations of branes will play a significant role in understanding supersymmetric compactifications and supersymmetry breaking within string theory.

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Appendix A. Supersymmetry constraints

In this appendix we summarize the supersymmetry equations which remain when the partial solutions (3.2), (3.3) and (3.5) are used in the Killing spinor equations (2.2) for the $\mathcal{N}=1$ supersymmetry.

We observe that Ansatz for the two-form potential (2.9) implies, in particular, that the corresponding field strength is of the form

$$G^{(3)} = (i \, g_{56} \, e^5 \wedge e^6 + i \, g_{90} \, e^9 \wedge e^{10} + g_{59} \, e^5 \wedge e^9 + g_{50} \, e^5 \wedge e^{10} + g_{69} \, e^6 \wedge e^9 + g_{60} \, e^6 \wedge e^{10}) \wedge (e^7 - i \, e^8),$$

where all functions $g_\omega(u,v)$ are real. Since the supersymmetry variations constitute a system of linear equations for the components of the three-index field strength, it is convenient to first determine $g_\omega$’s in terms of the other functions in the Ansatz. It is at this point that the $\beta \neq 0$ case differs from that of $\beta = 0$.

A.1. Solving with $\beta \neq 0$

One finds that all the functions $g_\omega$ are determined by the supersymmetry variations through the following system of equations:

$$g_{56} - g_{60} = \frac{1}{H_6} \frac{\partial \beta}{\partial u}, \quad g_{59} + g_{90} = -\frac{1}{H_1^4 H_6} \frac{\partial}{\partial u}(H_1^4 \sin \beta),$$

$$g_{50} = -\frac{1}{H_5} \frac{\partial \beta}{\partial v}, \quad g_{69} = \frac{1}{H_1^4 H_5} \frac{\partial}{\partial v}(H_1^4 \sin \beta),$$

and

$$3g_{56} + g_{60} = \frac{H_1^2}{2 \, u^2 \, v \, H_5^2} \left( \frac{2v^2}{H_1} \frac{\partial}{\partial v}(H_0) - \frac{uv}{H_1^3 H_5^2} \frac{\partial}{\partial u} \left( H_1^4 H_5^4 \cos \beta \right) \right),$$

$$g_{90} - 3g_{59} = \frac{H_1^2}{2 \, u^2 \, v \, H_5^2} \left( \frac{2v^2 \cos \beta}{H_1} \frac{\partial}{\partial v}(H_0) - \frac{uv \cos \beta}{H_1^3 H_5^2} \frac{\partial}{\partial u} \left( H_1^4 H_5^4 \cos \beta \right) - \frac{8u^3 v}{H_1^5 H_6^2} \frac{\partial}{\partial u}(H_5^2) \right).$$

Here $H_1$ is given by (3.3) and, in addition, one finds that

$$\sin \beta = \frac{2u}{H_1^4 H_6}.$$ 

Then the following two equations

$$\frac{12v}{H_1^4 H_6} \frac{\partial}{\partial u}(H_0) + \frac{u^8 H_1}{2H_6^2} \frac{\partial}{\partial v}\left( \frac{H_6^8 \sin(2\beta)}{u^8} \right) - 5 H_1 H_6^2 \frac{\partial \beta}{\partial v} = 0,$$
and
\[ \frac{v}{2H_1^2H_5^2} \frac{\partial}{\partial v}(H_0) - \frac{u^8 H_1^3}{48 H_6^5} \frac{\partial}{\partial u} \left( \frac{H_5^3 \sin(2\beta)}{u^8} \right) + 2H_0 + \frac{5}{24} H_1^3 H_6 \frac{\partial \beta}{\partial u} = 1, \] (A.7)

exhaust the remaining supersymmetry constraints.

In deriving (A.2)-(A.7) we have only used the general form of \( G_3 \) in (A.1). An additional restriction on the functions \( g_\omega \), that follows from the holomorphic Ansatz for the two-form potential (2.9), is
\[ g_{60} + g_{59} + g_{56} - g_{90} = 0, \] (A.8)

and it brings a dramatic simplification of the problem. Indeed, if we substitute (A.2)-(A.4) in (A.8) and then use (A.6) and (A.7), we obtain a remarkably simple result
\[ \frac{\partial}{\partial u} H_0 = -\frac{u}{v} \frac{\partial}{\partial u} (H_1 H_5)^2, \] (A.9)

which is crucial for solving the Ansatz in section 3.

A.2. Solving with \( \beta = 0 \)

For \( \beta = 0 \), several equations become dependent and the supersymmetry variations alone do not determine all fields in the Ansatz. In particular, the functions \( g_\omega \) satisfy
\[ g_{50} = g_{69} = 0, \quad g_{56} = -g_{59} = g_{60} = g_{90}, \] (A.10)

and thus (A.8) does not provide any additional constraint. The remaining supersymmetry variations consist of three equations which can be written as
\[ \frac{\partial}{\partial u} H_0 = -\frac{u}{v} \frac{\partial}{\partial v} (H_1 H_6)^2, \quad \frac{\partial}{\partial v} H_0 = \frac{u}{v} \frac{\partial}{\partial v} (H_1 H_5)^2, \] (A.11)

together with
\[ H_0 = \frac{1}{2} u \frac{\partial \Phi}{\partial u}, \] (A.12)

where \( \Phi \) is the “master function”
\[ \Phi = \log \left( \frac{v^2 H_6}{H_5} \right). \] (A.13)

\footnote{The reader might be puzzled about the sign difference between (3.15) and (A.12). However, there is no contradiction here, as the initial set of equations that lead to (3.15) and (A.12) are different for \( \beta \neq 0 \) and \( \beta = 0 \), respectively.}
A consistency between (A.11) and (A.12) using (3.5) requires that $\Phi$ satisfies the master equation

$$\frac{1}{u} \frac{\partial}{\partial u} \left( u \frac{\partial}{\partial u} \Phi \right) + \frac{1}{2v} \frac{\partial}{\partial v} \left( \frac{1}{v^3} \frac{\partial}{\partial v} e^{2\Phi} \right) = 0,$$

which is different from (3.10), which applies for $\beta \neq 0$!

To summarize, we find that, unlike in the $\beta \neq 0$ case, the supersymmetry variations leave one function in the metric, e.g., $H \equiv H_5^4$, and the two-form potential independent of the master function $\Phi$.

**Example: AdS$_5 \times S^5$**

An obviously important example of the $\beta = 0$ case is the AdS$_5 \times S^5$ compactification. It is easy to check that this solution corresponds to

$$\Phi(u, v) = \log(v^2), \quad H(u, v) = \frac{1}{(u^2 + v^2)^2}, \quad g_\omega(u, v) = 0,$$

where $H$ satisfies the harmonic equation

$$\frac{1}{v^3} \frac{\partial}{\partial v} \left( v^3 \frac{\partial}{\partial v} H \right) + \frac{1}{u} \frac{\partial}{\partial u} \left( u \frac{\partial}{\partial u} H \right) = 0,$$

which follows from the Bianchi identity for $F(5)$ or, equivalently, from the Einstein and/or the Maxwell field equations.
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