GRAVOTHERMAL EXPANSION IN AN N-BODY SYSTEM

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Abstract. This paper describes the numerical evolution of an $N$-body system with a slight “temperature inversion”; i.e. the maximum velocity dispersion occurs not at the centre but further out. Fluid models predict that the core of such a system expands on a time-scale of thousands of central relaxation times, and here this behaviour is qualitatively confirmed for an $N$-body system of over 3000 bodies. With certain qualifications, this demonstrates the existence in $N$-body systems of one of the fundamental mechanisms which, in fluid models, drive the gravothermal oscillations discovered by Bettwieser & Sugimoto.

1. Introduction

The dynamical evolution of globular clusters is an old problem with a rich history. One of the most unexpected developments of the 1980s was the discovery by Sugimoto & Bettwieser (1983, Bettwieser & Sugimoto 1984) of gravothermal oscillations. These affect the core of the system when the initial collapse of the core has finished. The collapse of the core is a consequence of two-body relaxation alone, but it can be brought to an end only by the intervention of some other mechanism, and in the context of the $N$-body problem this mechanism almost certainly involves the formation and evolution of binary stars (see Spitzer 1987 for a review). It is the subsequent interplay between these two mechanisms which appears to cause the core to oscillate between phases of high and low density.

Why do these mechanisms give rise to oscillations? Bettwieser & Sugimoto themselves gave an explanation which subsequent work (mainly by Goodman 1987) has confirmed and extended. Essentially, what happens is that the binaries modify the temperature of the core by supplying energy when its density is high; the gravothermal instability (Lynden-Bell & Wood 1968) determines how the system responds to the distribution of temperature. Core collapse happens because the core is slightly warmer than its surroundings. When the density in the core is high enough, binaries become active and cause the core to expand, which cools it to a temperature slightly lower than its surroundings. At this stage the warmest part of the system is outside the core, and this loses heat to both the core and the
outer part of the entire system. The flux of energy into the core drives the gravothermal instability in reverse, and the core expands. Ultimately, however, the heat flux to the outside cools the intermediate warm zone sufficiently to arrest the flux into the core. In due course a normal distribution of temperature (decreasing monotonically from the centre to the outside) reestablishes itself, and collapse of the core sets in once again. Allen & Heggie (1992) constructed a simple model illustrating that these mechanisms are sufficient to cause oscillations like those observed by Bettwieser and Sugimoto and later authors.

Core collapse is relatively well understood, but in this paper we concentrate on the expansion phase. Now the systems in which gravothermal oscillations and expansions have been observed are almost all continuum models, using either equations of gas dynamics (as in the original research of Bettwieser & Sugimoto) or the Fokker-Planck equation (Cohn et al. 1989). Certainly a better model would be an \( N \)-body model, and so the question naturally arises whether such a model also would exhibit gravothermal oscillations. Unfortunately our physical understanding of this phenomenon places the answer in some doubt. The foregoing explanation of core expansion clearly hinges on the response of the system to the temperature inversion, i.e. the fact that the maximum temperature, \( T_{\text{max}} \), occurs not at the centre but outside the core. The magnitude of this inversion is quite modest; if \( T_c \) is the temperature in the core, typical values for \( (T_{\text{max}} - T_c)/T_c \) are 0.12 (Bettwieser & Sugimoto 1984, Fig. 3) and 0.07 (Cohn et al. 1989, Fig. 6). Could such a modest temperature inversion be masked by statistical fluctuations in an \( N \)-body system containing \( \lesssim 10^5-10^6 \) particles, especially as only the innermost few thousand (at most) would participate in the oscillations?

This question is not easy to answer from the point of view of theory. It is not simply a statistical question of the standard deviation of the instantaneous mean square speed of a sample of stars, because such fluctuations will be greatly reduced if one averages over the long timescales characteristic of gravothermal expansions. Rather it is a question of fluctuations in the average change of energy in stars of one population as a result of two-
body interactions with stars from a different population with a slightly different kinetic
temperature.

While it might be possible to give a theoretical discussion along these lines, this paper
is devoted to a rather more direct method of establishing whether gravothermal expansions
can take place in the presence of statistical fluctuations—direct simulation by $N$-body
 techniques. This is not an easy method of solution either. Gas-dynamic models of post-
collapse evolution (Goodman 1987, Heggie & Ramamani 1989) indicate that gravothermal
oscillations occur only in systems with $N \gtrsim 7000$, and this is twice as large as any $N$-body
system for which useful results, extending sufficiently far into the post-collapse regime,
have been obtained by direct integration (Inagaki 1986).

The first step in making the $N$-body method practicable is to observe that the gravo-
thermal effects are confined mainly to the inner parts of the system. The outer parts
mainly have the role of providing enough pressure to maintain near-hydrostatic equilib-
rium, and this may be provided equally effectively by a rigid enclosure. (This assertion is
illustrated in more detail below.) Even so, the evolution of the core should still proceed
at the same very slow rate that it would in a large system, and it would take a very long
computational effort to pass through core collapse and beyond, to the point where the
predicted temperature inversion is established. For this reason, a second simplifying as-
sumption in the calculations reported below was adopted: it was decided to construct the
initial conditions with a distribution of kinetic temperature very close to that predicted
by fluid models, in particular with a temperature inversion of the same size. Such an $N$-
body experiment should therefore establish whether gravothermal expansion should take
place in an $N$-body system with a suitable initial distribution of temperatures. It does
not establish whether such a distribution would actually be set up (by binary activity, for
example) in an $N$-body system.

This limited goal is nevertheless a significant one, because evidence for the occurrence
of gravothermal expansions in $N$-body systems is still rather meagre. Makino & Sugimoto
(1987) studied a 1000-body system, and observed a few core oscillations whose time-scale is not very different from that expected on the basis of fluid calculations (Heggie & Ramamani 1989). On the other hand the interpretation has to be somewhat different, since the 1000-body system is much smaller (in number of particles) than the smallest fluid systems which exhibit gravothermal oscillations. Indeed Makino & Sugimoto proposed that it is the stochastic nature of binary activity in small $N$-body systems which can give rise to the required temperature inversions, and the work of Takahashi & Inagaki (1991), who incorporated stochastic heating effects into a gas model, shows that these tend to complicate the nature of the oscillations even in large systems. At any rate, so far the system of Makino & Sugimoto is the only $N$-body calculation in which gravothermal oscillations might have been observed. (Their occurrence in a much smaller 100-body system [Makino et al 1986] is debatable [Heggie & Ramamani 1989].)

The present paper begins (§2) with a fluid-dynamical discussion of the occurrence of gravothermal expansions in systems enclosed by a reflecting wall, and then describes the setting up of initial conditions for an $N$-body calculation. Section 3 describes and interprets the results of the $N$-body computations. Brief conclusions and further comments comprise the last section.

2. Choice of initial conditions

2.1 Gravothermal expansion in fluid systems

The purpose of this paper is to investigate whether gravothermal expansions can occur in $N$-body systems in a way which resembles their occurrence in fluid systems. Therefore we begin by summarising some known results about the evolution of gaseous models of star clusters.

As discussed in §1, one of the characteristic signs of gravothermal expansion is the occurrence of a temperature inversion, i.e. a zone in which $\partial \sigma / \partial r > 0$, where $\sigma(r)$ is the one-dimensional velocity dispersion at a distance $r$ from the centre of the system. In a
A long-term numerical integration of such a large $N$-body system is not yet feasible (Hut et al 1988). But the gravothermal expansions which occur in such systems are largely confined to a modest fraction of the mass near the centre of the cluster (see e.g. Cohn, Hut & Wise 1989, Fig. 5, for a system with $N = 5 \times 10^4$, evolved using a Fokker–Planck program). Since the outer parts of the system evolve only slightly during the time-scale of a typical gravothermal expansion, it is plausible to conjecture that they may be replaced by a rigid spherical enclosure. Fig. 1 shows that this conjecture is correct provided that the radius of the enclosure is not too small. It shows what happens to a 20000-body gas model, which is undergoing a gravothermal expansion, if the inner parts are suddenly enclosed in a reflecting sphere. Very small systems expand to a certain extent, and thermalise to a stable isothermal configuration (cf. the case in Fig. 1 of an enclosure containing the innermost 2000 stars; the fact that the equilibrium reached is isothermal is not evident from this figure, but is confirmed by the computational results). Systems of an intermediate size expand to a comparable extent but eventually recollapse (cf. the case with 3090 stars). Incidentally, in the results shown with solid curves in Fig. 1 there is no energy generation.
The expansion is almost entirely gravothermal. This is confirmed by the dashed curve, which shows what happens if energy generation is maintained: the expansion and recollapse are affected to a rather slight extent, though the recollapse is eventually arrested when the rate of emission of energy becomes large enough.

Let us recall that the aim of the $N$-body experiment which we shall describe is to test the occurrence of gravothermal expansion in $N$-body systems, in the presence of fluctuations. It might have been possible to demonstrate this for a system with 2000 particles, or even much less. However, Fig. 1 suggests that the isothermal equilibrium to which these small systems tend has a sufficiently large domain of attraction that even statistical fluctuations will not significantly affect the endpoint of the evolution. It would be preferable to test a system which can undergo both expansion and contraction, and Fig. 1 implies that the radius of the enclosing sphere must be large enough that the sphere contains at least about 3000 stars. In fact the sphere was chosen to enclose about 3120 stars. The radius of the sphere was exactly 0.34, in standard units (Heggie & Mathieu 1986) appropriate to the initial Plummer model of the fluid calculations. Incidentally, these are the units used in all subsequent references to our $N$-body calculations.

2.2 Structure and evolution of the fluid model

Fig. 2 shows the initial profiles of density, $\rho$, and one-dimensional velocity dispersion, $\sigma$, of the chosen model. Note the “temperature inversion,” i.e. a zone in which $\sigma$ increases with increasing $r$. The size of the inversion is modest: the maximum value of $\sigma$ exceeds the central value by only about 1.7%.

In order to construct an $N$-body realisation of this, the information in Fig. 2 is not enough. One requires the number-density in phase space, i.e. the distribution function $f(\varepsilon)$, where $\varepsilon$ is the energy per unit mass. A standard result (e.g. Binney & Tremaine 1987) gives the formula

$$f(\varepsilon) = \frac{1}{2\sqrt{2}\pi m} \frac{d}{d\varepsilon} \int_{\varepsilon}^{\infty} \frac{\partial \rho / \partial \phi}{\sqrt{\phi - \varepsilon}} d\phi,$$
where \( m \) is the stellar mass and \( \phi \) is the gravitational potential per unit mass. This equation was used to determine \( f(\varepsilon) \) from the functions \( \rho(r) \), which is known numerically, and \( \phi(r) \), which is determined from the solution of Poisson’s equation in spherical symmetry. Note that we require values of \( \rho \) and \( \phi \) outside the largest radius plotted in Fig. 2; these were taken from the entire fluid model, i.e. before truncation at the radius of the enclosure. In order to test the resulting distribution function, it was used to recalculate all the data depicted in Fig. 2. Values of \( \rho \) were reproduced to better than 0.5\%, and values of \( \sigma \) to better than 0.3\%. (The latter is an entirely independent test, since data for \( \sigma \) were not used for deriving \( f(\varepsilon) \).) Because the temperature inversion is so small, the magnitude of the inversion in the computed data has a much larger relative error, of about 25\% in fact. Further tests with a gas model suggest that such a relative change in the temperature inversion would have a comparable relative effect on the expansion rate.

The only other information needed about the fluid model is its rate of evolution. The rate of evolution of the central density is given by \( \dot{\rho}_c/\rho_c \simeq -0.037 \), and the rate of evolution of Lagrangian shells (containing a fixed fraction of the mass within the enclosing radius) is given in Fig. 3. Surprisingly, perhaps, most of the matter is flowing in! This behaviour has nothing to do with the boundary (at which \( d\ln r/dt = 0 \), necessarily), but it is explained in the context of self-similar evolution by Inagaki & Lynden-Bell (1983). They show, in effect, that the sign of \( d\ln r/dt \) is determined by a mass-weighted average of the quantity \( (d\ln \rho(r)/d\ln r - d\ln \rho_c(t)/d\ln r_c(t)) \), where \( \rho_c \) is the central density and \( r_c \) is the core radius. The subtle variations in the slope of the logarithmic density profile in Fig. 2 are enough to explain the inward motion of the outer matter.
2.3 Initial conditions of the $N$-body model

The fluid model gives information on the mass distribution in the form of $M(r)$, the mass contained within a sphere of radius $r$. We use units in which the entire mass of the fluid system (including matter outside the enclosure) is unity. Let the stellar mass be $m$ ($= 1/N$, since all masses will be chosen equal, where $N = 20000$). Also, let $M_i$ be the value of $M(r)$ at the distance of the $i$th star, in order of increasing distance from the centre of the system. Then $M_i$ is distributed like the time of arrival of the $i$th event in a Poisson process with rate $1/m$, and so $M_{i+1} - M_i$ has an exponential distribution with mean $m$. This formulation was used to generate the values of $M$ for the stars, and hence their radii, starting at the centre. In this way, not only is the distribution of radii statistically correct, but the number of stars inside the enclosure is subject to statistical fluctuation also. Indeed this process resulted in 3151 stars inside $r = 0.34$, compared with a value of 3118.7 for the fluid model. Fig. 4 compares the distribution from the fluid model with that of the $N$-body model. Since the fluctuation at $r = 0.34$ is less than one standard deviation, it is evident that the agreement is satisfactory. (Fig. 4 in effect provides a pictorial Kolmogorov-Smirnov test.)

The radii of the particles having been specified, their directions relative to the centre were chosen at random. The potential at each star was known from the gas model, and so the speed of each star could be chosen using the distribution $f(\varepsilon)$ described in §2.2. This was performed using a standard rejection procedure. The directions of the velocities were chosen isotropically, in keeping with the underlying assumptions of the gas model.

Verifying that the $N$-body model correctly reproduces the thermal structure of the fluid model is harder than checking the distribution of mass. It is readily seen that there is little prospect, even with over 3000 stars, of observing directly the very modest temperature inversion visible (for the fluid model) in Fig. 2, even if the data are binned as coarsely as possible in radius. On the other hand, $\sigma^2$ is an average over the distribution function, and averaging diminishes the details in a function. Therefore it is better to check $f$ as
directly as possible. This is difficult to do with discrete data, and so what is plotted in Fig. 5 is $F$, where $F(\varepsilon) \equiv \int_{\varepsilon}^{\infty} f(\varepsilon')d\varepsilon'$. To see how this may be obtained for the $N$-body model, observe that $f(\varepsilon)$, the number density in phase space, is related to $n(\varepsilon)$, the number density in energy space, by the relation

$$n(\varepsilon) = \frac{ds}{d\varepsilon} f(\varepsilon),$$

where $s(\varepsilon)$ is the volume of phase-space contained within an energy hypersurface; i.e.

$$s(\varepsilon) = \frac{16\pi^2}{3} \int_{0}^{r_{\text{max}}} [2(\varepsilon - \phi)]^{3/2} r^2 dr,$$

where $\phi(r_{\text{max}}) = \varepsilon$ if $\varepsilon < \phi(r_e)$, otherwise $r_{\text{max}} = r_e$, the radius of the enclosure. For an $N$-body system $n(\varepsilon) = \sum_{i=1}^{N} \delta(\varepsilon - \varepsilon_i)$, where $\varepsilon_i$ is the energy (per unit mass) of the $i$th star, whence

$$F(\varepsilon) = \sum_{\varepsilon_i > \varepsilon} \frac{1}{\frac{ds}{d\varepsilon}(\varepsilon_i)}.$$

To bring out a subtle feature in $F$, in Fig. 5 it is normalised by a function proportional to the Boltzmann distribution which best fits the central velocity dispersion of the fluid model. With a little thought it can be seen that the slight rise in the solid curve (which represents the fluid result) is the cause of the temperature inversion visible in Fig. 2. It is quite faithfully reflected in the $N$-body data. Limited experience suggests that the amplitude of the deviations for $E \gtrsim -1$ is quite typical of the variations exhibited by different realisations of the model.

2.4 Hardware and software aspects

The first $N$-body run using the initial conditions just described extended to time $t \approx 11$ (see Table 1). It was carried out on an ICL Distributed Array Processor with a parallelised version of the code NBODY1 (Aarseth 1985). Slight softening of the potential was introduced to avoid difficulties with close encounters. Results were described in Heggie (1988), and it can now be regarded as a pilot run.
The other runs listed in Table 1 extended for much longer and used better software, so that no softening was required. Despite their similar initial conditions, they may be regarded as independent of one another because they were performed on different hardware, and it is well known (Goodman et al. 1993) that minute differences in the positions and velocities of stars in N-body systems grow on a time scale much shorter than a crossing time. Therefore, differences in rounding carried out by different computers will quickly lead to large deviations in the motions of the particles. (The crossing time for the enclosed N-body system is less than one time unit.) Runs 2 and 3 are independent because, for Run 2, the initial velocities were reversed, leading to an equally valid, but dynamically distinct, realization of the gas-sphere initial conditions. The independence of the runs is also evident empirically (e.g. Fig 6 below).

3. The N-body calculations

3.1 Error control

The purpose of the N-body calculation was to detect evolution of the system on the time-scale indicated in Fig. 3. One possible mechanism, however, for expansion (or contraction) of the system is numerical error, and so the constancy of the total energy was frequently monitored. The maximum deviation observed in the sets of calculations can be measured by the quantity $|\delta E|/T$, where $\delta E$ is the change in energy, and $T$ is the total kinetic energy of the stars inside the enclosure. Values are listed in Table 1. Now it is probable that the numerical errors arose chiefly in the innermost parts of the system. The kinetic energy of the innermost fraction $\mu$ (by mass) of the system is roughly $\mu T$. If we suppose that the fractional change in the corresponding Lagrangian radius should exceed the fractional change in energy over the duration of the calculation we find that numerical errors should not invalidate the calculation provided that $|\delta E|/T \lesssim \mu \delta r/r$. Since we shall see that the values of $\delta r/r$ are of order unity for the innermost radii, and of order 0.02 for the outermost radii, we can conclude that the energy associated with
numerical errors should not have a significant on our results. These considerations are somewhat inappropriate at intermediate radii, which are almost stationary. Note also that the widely differing duration of the runs should be taken into account in any comparison of the maximum relative errors.

Another potential source of energy which could complicate the interpretation of our results is the formation of hard binaries. In fact, this occurred in Run 4 at $t \sim 20$. The maximum energy $\varepsilon$ reached by this binary was approximately $0.02T$, and its last energy-releasing interaction (in which its binding energy doubled) occurred at $t \sim 68$. This energy is about the same as the initial core kinetic energy, and therefore the binary may possibly account for some of the rise in the inner Lagrangian radii in this run during this time interval (Fig 9b below). Certainly the inner parts of this model expand further than those of the other models (Figs 10, 11 below). Nevertheless, the binary should have a negligible effect on the long-term evolution of the outer Lagrangian radii.

3.2 Motion of the core

This paper is largely concerned with the evolution of Lagrangian radii, i.e. the radii of spheres containing a fixed fraction of the mass of the system, but these must be measured with respect to some suitable centre. The geometric centre of the enclosure is not appropriate, because it is known from previous studies (Makino & Sugimoto 1987, Heggie 1988) that the densest part of the system soon moves to a considerable distance from the geometric centre.

The problem of determining the position of the densest part of an $N$-body system is not as simple as it seems. The “density centre” (Casertano & Hut 1985) is most commonly used, but in the pilot study (Heggie 1988) the concept of a “potential centre” was introduced, defined as the location of the particle with the minimum smoothed potential (i.e. calculated with an interparticle potential of the form $(r^2 + a^2)^{-1/2}$). However, it has been found in our longer runs that the motion of the potential centre is qualitatively similar to
that of the density centre, except that it is considerably noisier (cf. also Sweatman 1993), and it is much more expensive to calculate. Accordingly, in what follows we restrict attention to the density centre. Operationally, this is determined by a variant on the procedure used by Casertano & Hut, as described in more detail by McMillan et al. (1990).

Fig. 6 shows the changes with time in one coordinate (the $x$-component) of the density centre in Runs 1-4. Clearly there is motion of the centre on a variety of time scales. Some of this is presumably due to the motions of stars in the core, and would happen even if the core were quite stationary. But the amplitude of the motion exceeds the core radius ($\simeq 0.0101$ initially), and so it is clear that some of the movements must be attributed to genuine motion of the potential well of the cluster, and especially the core.

In order to assist in the interpretation of these results, we show in Fig 7 a power-spectrum analysis of the data in Fig. 6. Here the power at frequency $f$ is defined to be $P(f) = |\hat{x}(f)|^2$, where $\hat{x}$ is the Fourier transform of the coordinate $x$. At high frequencies, all runs show a noisy power-law spectrum with mean logarithmic slope near $-2$. The character of the noise is similar to that reported by McMillan et al. (1988) in a study of relaxation in $N$-body systems, and most likely has the same explanation, namely effectively random motion on suborbital time scales. At lower frequencies ($f \lesssim 0.5$), the power spectrum turns down, though the effect is not clearly visible in the shorter runs. At still lower frequencies, the power spectrum rises again, reaching a maximum at a frequency corresponding to a period just above 5 time units. (This is best determined in the longest run, Run 4.) Below this frequency, little can be said from the shorter runs (Runs 1 and 2), but the two longest runs show a fairly flat spectrum.

Though no definitive interpretation of these motions will be offered, it is helpful to have for comparison some fundamental time scales for this system. The period of small-amplitude stellar orbits at the centre of the system (i.e. $\sqrt{6\pi/G\rho_c}$) is $t_{\text{orb}} \simeq 0.11$, which is comparable to the core crossing time. The central relaxation time is 0.058 if $\Lambda$, the argument of the Coulomb logarithm, is set equal to the total number of particles in the
simulation; if $\Lambda$ is set equal to $1.9N_c$ (Spitzer 1987, p.149), where $N_c \simeq 72$ is the number of stars in the core, the value is increased to 0.095.

The time scales just mentioned all fall within the range of frequencies where the power spectrum nearly follows a power law. The density at a point will also fluctuate on the time scale on which stars traverse a distance a distance of order the interparticle separation, which may be expected to be of order $t_{\text{orb}}/N_c^{1/3} \simeq 0.03$. In a general way, therefore, we may suggest that this part of the power spectrum has a purely kinematic origin; it would be observed also in a system with a fixed, core-like potential. There is also a flattening at high frequencies in Runs 1–3, which were sampled more often than Run 4. At these frequencies we presumably resolve the smooth motions of the particles.

A power spectrum with slope $-2$ corresponds to a random walk, or Brownian motion (McMillan et al. 1988). (The derivative of Brownian motion is white noise, which has a flat spectrum, and so $P(f) \propto f^{-2}$ for Brownian motion.) We have no detailed model which proves that this would follow from the qualitative kinematic explanation offered above, although it is a plausible consequence. We offer no explanation for the downturn in the power spectrum below frequencies of order 0.5, though the motion of the density centre is bounded, unlike Brownian motion over long time scales, and this suggests that there should be less power at low frequencies.

The maximum at a frequency of about 0.5 corresponds to a period which is quite noticeable in some of the plots in Fig. 6, especially Run 4, where the sampling interval was longer, and the plot is less confused by high-frequency noise. Also noticeable in Fig. 6, especially in Run 1, is a period of order 5. This was even detected in the much shorter Run 0, cf. Heggie 1988, and corresponds to the low-frequency maxima ($f \sim 0.2$) in Fig. 7. We believe that this motion is associated with the boundary, as the following calculation confirms.

If we suppose that the entire system (within the enclosure) moves rigidly like an isothermal model centred at the moving density centre, but truncated by the fixed enclo-
sure, it is easy to estimate the acceleration of a star at the centre of the core. If the core has a small vector displacement $\epsilon$, the acceleration is $-(4/3)\pi G \rho_e \epsilon$, where $\rho_e$ is the density at the radius of the enclosure. Treating this as the acceleration of a simple harmonic oscillator with displacement $\epsilon$, the corresponding period is 5.6. This is surprisingly close to the time scale of the largest oscillation noticeable in Fig. 6, and the frequency at which the power is greatest. Clearly, however, if this interpretation is correct, different behaviour would be expected from a system with a different-sized enclosure, or one with no enclosure at all.

Satisfactory as the above remarks may be, a closer examination of Fig. 6 reveals other phenomena. In particular there are numerous instances of very rapid motion, where the speed of the density centre is comparable to that of individual stars (and actually may be substantially larger, since our estimates of the “speed” are limited by the sampling interval). These motions are hard to understand if the core moves en masse as we have supposed. As an alternative, it is possible that the core expands and dissolves, and that a new core forms around another condensation some distance away. Some support for this view comes from Fig. 8, derived from data of Run 0. In this run a similar sudden movement of the core was observed near $t = 4.4$. Fig. 8 shows all stars within a distance of 0.1 of the geometrical centre at times before and immediately after this instant. In the earlier figure the potential centre matches an obvious condensation, which is separated by an apparent low-density region from another condensation. This latter condensation is not far from the site of the potential centre at the later time. In fact, we have also noted this phenomenon in other, unrelated, $N$-body simulations. It appears that situations where multiple density centers exist, and where the first and second density maxima do not even lie within the computed core, are fairly common.

Study of the autocorrelation of the data in Fig. 6 suggests that the interpretation of the core motion may be further complicated on long time scales, up to the order of the relaxation time at the outer boundary. The statistical significance of this finding is in
doubt, however, as the results are not exhibited consistently by all three coordinates of the density centre, even in the same run.

3.3 Evolution of the mass distribution

Whatever the cause of motions of the core, they are real and significant. Therefore fluid models, which assume spherical symmetry, fail to predict the evolution of an $N$-body system in this respect. Since most of the motions take place on a time scale short compared to the time scale for evolution of the core according to the fluid models (of order $10^2$, cf. Fig. 3), it would also be surprising if the evolution of the core should follow the predictions of these models.

The purpose of the present section is to test this by examining the radii containing fixed fractions of the mass, measured from the density centre. (It is clear, and confirmed by the data, that the evolution of these radii, if measured from the geometric centre, will be dominated by the motion of the core, at least for small mass-fractions comparable with the fraction of mass in the core.)

Fig. 9a and b show the variation with time of several Lagrangian radii in Runs 3 and 4. The points mark the actual numerically obtained data. The initial core radius was $r_c \simeq 0.0101$, and so the innermost radius shown in these figures falls within the core (and hence is determined by only a few stars, leading to substantial fluctuations), while the outermost lies well outside the half-mass radius (of the material within the enclosure). Evidently the inner radii show a tendency to increase with time, whereas for outer radii the trend is inward. Inner radii appear to show a generally random scatter about the trend, but also fluctuations on a variety of time scales. The outer radii, when examined closely, show marked, persistent oscillations. We now discuss these several aspects of the results quantitatively.

First we consider the underlying trend. The solid lines in Figs. 9 show the same Lagrangian radii computed from the gas model. It is evident that the trend exhibited by
the $N$-body results is very similar to the outcome of the gas-sphere simulation. For the inner radii, Fig. 9 already gives an adequate quantitative representation of the extent of the agreement, but the results are less clear at larger radii. Therefore in Fig. 10 we show the changes in the Lagrangian radii over the first 120 time units, for Runs 3 and 4 and the gas model. (This interval was chosen so that comparable results for both runs could be exhibited together; besides, the gas results show that the expansion is almost complete by this time.) For the $N$-body runs the changes in the radii have been obtained by averaging results over the first and last twenty time units of the interval $0 < t < 120$, while for the gas model results at $t = 10$ and 110 were used. Fig. 11 shows a similar comparison between the gas model and Runs 1–4 over just the first 50 time units.

These comparisons indicate that the spatial evolution of the $N$-body and gas models are in agreement qualitatively. Both models expand at small radii and contract at large radii. The transition between the two kinds of behaviour occurs at a very similar value of the mass fraction $\mu$. Quantitatively, it is tempting to deduce that the gas model evolves too fast (by about 40%), but a glance at Figs. 9 suggests rather that it evolves too far: at late times the inner parts of the $N$-body model have not expanded as much as in the gas model, and similarly the outer parts have not contracted as much. Incidentally, Fig. 10 is similar in many ways to the results presented in Heggie (1988) for Run 0, even though its duration was far shorter. In that run, however, the $N$-body model evolved farther than the gas model, and there was poorer agreement in the radius separating the expanding from the contracting regions.

One additional point should be borne in mind when comparing gas and $N$-body models in this situation: the expression for the thermal conductivity in the fluid model includes an arbitrary constant, the value of which can only be obtained by comparison with some less idealised model. (See, for example, Heggie & Stevenson 1988). The data in Fig. 10 were based on a calibration against the isotropised Fokker-Planck solution for core collapse (Cohn 1980), and there is no reason to suppose that the same calibration is valid for a
gravothermal expansion, especially when account is taken of the anisotropy which presumably develops in the \(N\)-body model. Also, there are significant differences in detail between the different \(N\)-body realizations of the system. The apparent good agreement between Run 4 and the gas model may be due in part to the initial dip in the inner Lagrangian radii in the \(N\)-body run (see Fig. 9b), which is most likely the result of a stochastic fluctuation, or to the enhanced expansion caused by the binary (§3.1). Another possible complicating factor is the error in the initial magnitude of the temperature inversion (§2.2).

So much, then, for the extent of the agreement between the fluid and \(N\)-body models. We now mention a number of caveats concerning the reliability of the \(N\)-body data plotted in Fig. 10. We have not plotted formal error estimates here, because the separate data points are not statistically independent. Careful examination of Figs. 9 shows that the larger radii exhibit persistent oscillations, and similar oscillations of shorter period are evident at all radii in the data from Run 0 which were sampled more frequently. In addition, the inner radii in Figs. 9 exhibit apparently irregular intermediate- and long-period fluctuations. Nevertheless the consistency of the results from the \(N\)-body models shown in Fig. 10 suggests that these phenomena do not significantly degrade the results.

Let us turn first to the sustained oscillations, whose presence is most noticeable in Fig. 9a at large radii. At first sight it is surprising to observe oscillations which look so persistent, because one would expect such behaviour to be strongly damped by phase mixing. Similar oscillations in Plummer models have been studied by Sweatman (1993), and he shows convincingly that such motions are almost purely kinematic, that is, this behaviour occurs even in a system of \(N\) stars moving in a fixed potential. The only effect of returning to a self-consistent \(N\)-body model is that the period and amplitude of the oscillations are slightly altered. As might be expected for a kinematic phenomenon, the period of the oscillations at a given radius is closely related to the basic dynamical time scales there, e.g. the period of a circular orbit, or the epicyclic period.

Finally we briefly examine the irregular fluctuations, on a variety of time scales, which
appear in the inner Lagrangian radii. Fig. 12 shows the power spectra of the innermost two Lagrangian shells displayed in Figs. 9a and b. The only obvious features of these graphs are the rather broad peaks seen near \( f = 8 \) and \( f = 6 \), respectively, in the upper-left and upper-right frames (corresponding to Run 3). These peaks are not resolvable in Run 4 (lower frames), because of the lower sampling frequency, though the upturn of the lower plots at high frequencies suggests that they would be found to behave similarly if the data extended farther to the right. Aside from the peaks, all four power spectra are quite well fitted by straight lines of slope \(-2\) (parallel to the lines marked on each plot). We interpret the peak frequency as corresponding to the orbit period of the “average” star crossing each shell, since similar data at larger radii show that the frequency at which the peak occurs varies nearly as the frequency of a circular orbit at the corresponding radius. Motions near this frequency are, therefore, presumably to be interpreted in the same way as those discussed by Sweatman (1993). The roughly constant slope of the lower-frequency portions of the power spectra indicates that the microscopic processes driving the expansion have no preferred time scale, just as would be expected from the theory of two-body relaxation.

4. Discussion and Conclusions

The \( N \)-body experiments described in this paper were formulated to determine whether gravothermal expansion can occur in an \( N \)-body system. At lowest order Fig. 10 strongly suggests that the answer is affirmative, and it is the aim of the following remarks to consider some possible objections.

One concerns boundary conditions, i.e. the replacement of an entire self-gravitating isolated cluster by a system restrained by a surrounding reflecting enclosure. The evidence from fluid models is that the cores of the two systems (isolated and enclosed) should expand in very similar ways. Furthermore the enclosed system, just like the isolated one, should eventually recollapse, though the time taken for this to occur is longer for the enclosed system. At any rate, the initial expansion of the enclosed system does not arise simply
because the only thermodynamic equilibrium available to it is one with an expanded core. The effects of an enclosure can, however, be more subtle. Rapidly moving stars, caused by interactions involving binaries, cannot escape if there is an enclosure. Though such stars relax slowly against the remaining stars, their effects in the long term could be significant, and may lead to a further difference in the behaviour of isolated and enclosed systems. On the other hand the absence of any observed really hard binaries in our $N$-body runs suggests that this is unimportant.

A second qualification concerns the initial conditions. The $N$-body system was initially endowed with a temperature inversion comparable to that which develops (as a result of energy input from binary activity) in fluid models. The $N$-body system described in this paper therefore provides no evidence on whether such a temperature profile could ever arise in an $N$-body system as a result of binary activity. The only direct evidence on this is provided by the calculation of Makino & Sugimoto (1987), which suggests that temperature inversions are indeed caused by binary interactions in $N$-body systems with 1000 particles. Those authors argue that the stochastic nature of binary activity in such a small system is an important factor favouring the creation of a temperature inversion. If so, and if energy generation by binaries is in some sense smoother in larger systems, then the creation of a suitable temperature profile may be less assured. On the other hand, the evidence from fluid models, using both stochastic and smooth energy-generation rates, leaves little doubt that temperature inversions do occur, at least in systems with $N > 7000$.

The final qualification concerns numerical errors. We have tried to ensure that these are relatively small by insisting that the numerically generated change in energy, if concentrated in the innermost few percent of the system, should cause an evolution only on a time-scale much longer than the time-scale for gravothermal expansions.

The foregoing remarks summarise the main qualifications and the main result of the $N$-body experiments described in this paper. But the experiment also revealed evidence of two further interesting phenomena: core motion, and radial oscillations. The nature
of the latter has been elucidated in recent work by Sweatman (1993), and here we have considered only the movement of the core, which is still poorly understood.

Similar motions in an $N$-body system of comparable size were reported some time ago by Makino & Sugimoto (1987), who found that the most rapid motions could be associated with the ejection of fast particles from triple interactions. We have also observed fast core motions, at a speed reaching about half that of the single particles, on occasions. Because little activity of hard binaries was noticed in our computations, it is less likely that these motions are associated with binary ejection. Furthermore our core, with about 70 members, is relatively more massive than that in the study of Makino & Sugimoto, and so recoil effects should be weaker. On longer time-scales, it is possible that the motion of the core is influenced by the enclosure. On all time-scales, down to those of the fast motions referred to above, the amplitude of the motions is comparable with the core radius.

The interest in core motions is not simply a matter of curiosity. Our standard picture of the evolution of a star cluster is based on spherical symmetry, and this assumption may be invalidated by the motion of the core. Whether or not this plays any significant role in modifying the late-time evolution of the cluster remains to be seen.

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Table 1

Summary of the Computations

| Run | $t_{\text{max}}$ | Hardware  | Author | Code   | $\max(|\delta E|/T)$ |
|-----|-----------------|-----------|--------|--------|---------------------|
| 0   | 10.8            | ICL DAP   | DCH    | NBODY1 | $2.3 \times 10^{-4}$ |
| 1   | 50.1            | FACOM VP-400 | SI/SLWM | NBODY5 | $5.6 \times 10^{-4}$ |
| 2   | 64.0            | Sun 4/370 | SI     | NBODY5 | $1.5 \times 10^{-3}$ |
| 3   | 123.3           | Sun 4/370 | SI     | NBODY5 | $2.8 \times 10^{-3}$ |
| 4   | 234.8           | Cray Y/MP | SLWM   | NBODY5 | $9.6 \times 10^{-4}$ |

Figure Captions

**Fig. 1.** Dependence of central density, $\rho_c$, on time, for fluid systems truncated by reflecting spheres of different radii. Solid curves show three systems without energy generation, the number giving the number of stars enclosed. For the largest ($N = 20000$) the radius of the enclosure is effectively infinite. The dashed line shows how this system evolves if energy generation is maintained. The initial conditions are drawn from a fluid model of 20000 stars with energy generation, which was evolved from a Plummer model, through core collapse and on into gravothermal expansion. With these initial conditions, all truncated models continue to expand, but only those with more than about 3000 stars eventually recollapse. Time here is measured from the time at which the inner parts of the system are enclosed, and not the time of the initial Plummer model. It is given in units of the initial half-mass relaxation time $t_{rh}(0)$ of the Plummer model, which is approximately 207 time units in the units used in the N-body models in this paper.

**Fig. 2.** (a) Density and (b) velocity dispersion profiles, $\rho(r)$ and $\sigma(r)$, of the fluid model used to generate the initial conditions. Only the part inside the reflecting enclosure at radius $r = 0.34$ is shown.
Fig. 3. Initial rate of evolution of the fluid model inside the reflecting enclosure. If $r(t)$ is the radius which always contains a given fraction $\mu$ of the mass inside the enclosure, the ordinate gives $\dot{r}/r$.

Fig. 4. Mass distribution in the fluid and $N$-body models. $N(r)$ is the number of stars within radius $r$, up to the radius of the enclosure ($r = 0.34$). The smooth curve gives the result for the fluid model.

Fig. 5. Normalised integral $F(\varepsilon)$ of the distribution function $f(\varepsilon)$. The central velocity dispersion of the fluid model is $\sigma_c$. The smooth curve gives the fluid result, the points give the initial values for the $N$-body system.

Fig. 6. The $x$-coordinate of the “density centre” plotted against time, for Runs 1-4. The different runs have been displaced vertically to avoid overlap.

Fig. 7. Power spectrum of the data in Fig. 6. The abscissa is the reciprocal of the period, and the ordinate is defined in the text. The solid black line in each case indicates a slope of -2.

Fig. 8. Stars within 0.1 of the geometric centre, at times (a) 4.117 and (b) 4.467, in Run 0. The cross marks the ‘potential centre’ defined in Sweatman (1993).

Fig. 9. Radii containing constant mass fractions ($\mu$) of the mass within the reflecting enclosure, as functions of time in (a) Run 3 and (b) Run 4. The radii are measured to the density centre, and the plots show results for $\mu = 0.010, 0.025, 0.063, 0.100, 0.158, 0.251, 0.398, \text{ and } 0.631$ (Run 3) and $\mu = 0.013, 0.032, 0.050, 0.079, 0.126, 0.200, 0.316, 0.501$ (Run 4). (In each case, the masses are equally spaced in log $\mu$, except that some radii near the centre have been omitted to avoid confusion by overlap.) The solid lines are the results of the gas model.

Fig. 10. Fractional expansion or contraction (defined more carefully in the text) over the first 120 time units, as a function of mass fraction. The continuous curve gives results for the fluid model (cf. Fig. 3); the points give the results of Runs 3 and 4. The horizontal
dashed line at an expansion factor of unity is included only as an aid to the eye.

**Fig. 11** As for Fig. 10, except that now Runs 1–4 are compared with the gas model over the first 50 time units. The results of Runs 1–4 are represented by triangles, filled circles, crosses, and squares, respectively.

**Fig. 12.** Power spectra of the innermost Lagrangian radii shown in Fig. 9. The solid black line in each case indicates a slope of -2.