Riemannian Metrics and Harmonic Sections of Spinor Bundles

Simone Farinelli
Aumühlstrasse 20
CH-8906 Bonstetten
Email: simone@coredynamics.ch

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Abstract

We study the clustering of the lower eigenvalues of the Dirac operator on a spinor bundle when the metric structure is varied. We then apply the general results to show that any closed spin manifold of dimension $m \geq 4$ has a Riemannian metric admitting non-trivial harmonic spinors.

1 Introduction

On a closed connected $m$-dimensional Riemannian spin manifold $(M, g, P)$ with metric $g$ and spin structure $P = P_{\text{Spin}}(M)$ we consider the Dirac operator $D$, a first order differential operator which acts on sections of the spinor bundle $\Sigma$. It is known that the dimension $h(M, g, P) = \dim \ker(D)$ of the space $\ker(D)$ of harmonic spinors is not a topological invariant. For instance, the one parameter family of Berger metrics on $S^3$ admits non-trivial harmonic spinors only for non-generic values of the parameter, cf. [Hi74].

There are, however, topological bounds for the metric invariant $h(M, g, P)$. In even dimension $\Sigma$ splits into the direct sum $\Sigma = \Sigma^+ \oplus \Sigma^-$ of the bundles of positive and negative Weyl spinors and $D$ is of the form

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}. \quad (1)$$

We write $h(M, g, P) = h^+(M, g, P) + h^-(M, g, P)$ where $h^\pm(M, g, P) := \ker(D^\pm)$. If $m = 4k$, by the Atiyah-Singer index Theorem

$$h^+(M, g, P) - h^-(M, g, P) = \hat{A}(M), \quad (2)$$

where $\hat{A}(M)$ is the $\hat{A}$-genus of $M$, cf. [AS68], therefore $h(M, g, P) \geq |\hat{A}(M)|$ for any Riemannian metric and spin structure. There is a similar index
if \( m \equiv 1,2 \mod 8 \), cf. [Mi65, AS71]:

\[
\begin{align*}
m = 8k + 1 &\Rightarrow h(M,g,P) \equiv \alpha(M) \mod 2, \\
m = 8k + 2 &\Rightarrow \frac{h(M,g,P)}{2} \equiv \alpha(M) \mod 2,
\end{align*}
\]

(3)

where \( \alpha(M) \in \mathbb{Z}_2 \) is Milnor’s \( \alpha \)-genus, a topological invariant which depends on the choice of the spin structure on \( M \). In particular \( h(M,g,P) \neq 0 \) if \( \alpha(M) \neq 0 \).

In more recent years, there have been extensive studies on \( D \)-minimal Riemannian metrics, that is metrics for which the lower bound given by the index theorem is attained:

| \( \dim(M) \) | Genus | \( h(M,g,P) \) for \( D \)-minimal metrics |
|----------------|-------|---------------------------------------|
| \( m = 4k \)   | \( \tilde{A}(M) \geq 0 \) | \( h^+(M,g,P) = \tilde{A}(M) \) \( h^-(M,g,P) = 0 \) |
|                | \( \tilde{A}(M) < 0 \) | \( h^+(M,g,P) = 0 \) \( h^-(M,g,P) = -\tilde{A}(M) \) |
| \( m = 8k + 1 \) | \( \alpha(M) = 0 \) | \( h(M,g,P) = 0 \) |
|                | \( \alpha(M) = 1 \) | \( h(M,g,P) = 1 \) |
| \( m = 8k + 2 \) | \( \alpha(M) = 0 \) | \( h^+(M,g,P) = 0 \) \( h^-(M,g,P) = 1 \) |
|                | \( \alpha(M) = 1 \) | \( h^+(M,g,P) = h^-(M,g,P) = 1 \) |
| \( m \equiv 3,5,6,7 \ mod 8 \) | \( \tilde{A}(M) = 0 \) | \( h(M,g,P) = 0 \) |

In [Ma97] it is proved using a variational approach that any generic metric on a closed connected spin manifold of dimension \( \leq 4 \) is \( D \)-minimal. In [BD02] the same result is proved for simply connected manifolds of dimension at least 5, using surgery and spin bordism arguments. These results were later established in [ADH09] for any dimension and without the simply connectedness hypothesis; the essential step in the proof is the construction of a single \( D \)-minimal metric on any given closed connected spin manifold.

On the other hand, there is the following long-standing conjecture, cf. f.i. [BD02].

**Conjecture.** Let \((M,P)\) be a closed connected spin manifold of dimension \( \geq 3 \). Then, there exists a Riemannian metric \( g \) on \((M,P)\) with non-trivial harmonic spinors, i.e. \( h(M,g,P) > 0 \).

The conjecture does not hold in dimension 2, because for any eigenvalue \( \lambda \) of the Dirac operator on a closed surface \( \Sigma \) of genus zero, one has

\[
\lambda^2 \geq \frac{4\pi}{\text{vol}(\Sigma)}.
\]

(4)
in particular, there are no harmonic spinors (for any metric and spin structure), cf. [Bär92]. If the genus is \( \leq 2 \) then all metrics are \( D \)-minimal and the spectrum of \( D \) can be explicitly computed when the genus is 1, cf. [BS92]. For results on spin structures for all hyper-elliptic surfaces \( \Sigma \) of genus \( \geq 2 \) we refer the reader to [BS92].

The conjecture has been shown to be true for \( m \equiv 0,1,7 \mod 8 \) by Hitchin, using the Atiyah-Singer index theorem and the theory of exotic spheres in [Hi74], and for \( m \equiv 3,7 \mod 8 \) by Bär using an analytic approach in [Bär96]. Both these results follow also from Dahl’s theorem on invertible Dirac operators, cf. [Da08].

For the special case of \( m \)-spheres, explicit metrics with non-trivial harmonic spinors have been constructed when \( m \equiv 0 \mod 4 \) in [Se00] and in the case of Berger spheres when \( m \equiv 3 \mod 4 \) in [Bär96].

In particular Dahl proves in [Da06] that for any closed connected spin manifold of dimension \( m \geq 3 \) and any \textit{non-zero} \( \lambda \in \mathbb{R} \) it is possible to construct a Riemannian metric for which the Dirac operator has \( \lambda \) as eigenvalue. The main contribution of this paper is to extend Dahl’s result to \( \lambda = 0 \) and establish that the conjecture holds in all dimensions \( m \geq 4 \):

**Theorem 1.1.** Let \( (M,P) \) be a closed connected spin manifold of dimension \( m \geq 4 \). Then, there exists a Riemannian metric \( g \) on \( (M,P) \) admitting non-trivial harmonic spinors.

This paper is structured as follows. Section 2 recalls the basics of the theory of Dirac bundles. In Section 3 we establish several results on the spectrum of Dirac operators, preliminary to the proof of the main theorem in Section 4. Section 5 concludes with some observations.

# 2 Preliminaries

## 2.1 Dirac bundles

The purpose of this section is to recall the basic definitions and set the notations concerning the theory of Dirac operators. General references are [LM89], [BW93] and [BGV96].

**Definition 1.** Let \( \Sigma \to M \) be a complex vector bundle on a Riemannian manifold endowed with a Hermitian structure \( \langle \cdot, \cdot \rangle \) and a Hermitian connection \( \nabla \). Then \( \Sigma \) is a **Dirac bundle** if it is a bundle of Clifford modules such that:

(i) Clifford multiplication \( X \cdot \) by vectors is skew-adjoint w.r.t. \( \langle \cdot, \cdot \rangle \);

(ii) Clifford multiplication is compatible with \( \nabla \), i.e.

\[
\nabla(X \cdot \varphi) = \nabla X \cdot \varphi + X \cdot \nabla \varphi
\]

for all \( \varphi \in \Gamma(\Sigma) \).
Given a Dirac bundle $\Sigma$, the associated **Dirac operator** $D : \Gamma(\Sigma) \to \Gamma(\Sigma)$ is the first-order differential operator defined by the composition

$$\Gamma(\Sigma) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma) \xrightarrow{} \Gamma(\Sigma),$$

of $\nabla$ with Clifford multiplication. The square $D^2$ of the Dirac operator is the **Dirac Laplacian**.

A particularly important example of Dirac bundle is given by the spinor bundle. We recall that an oriented Riemannian manifold $(M, g)$ of dimension $m \geq 3$ is a **spin manifold** if it admits a spin bundle, i.e. a double cover $\pi : P \to SO(M)$ of the oriented orthonormal frame bundle $SO(M)$ of $M$ with structure group $Spin(m)$ inducing the canonical covering $\Theta : \Spin(m) \to SO(m)$ on each fiber. The vector bundle

$$\Sigma = P \times_{\Spin(m)} Cl^l, \quad l = 2^{[{m\over 2}]}$$

associated with the spin representation $Cl^l$ of $\Spin(m)$ is the **spinor bundle**. It is a Dirac bundle with respect to the Levi-Civita connection $\nabla$ on $M$.

The restriction of $\Sigma$ to an oriented hypersurface $N \subset M$ is a Dirac bundle, as we now briefly recall. Let $\nu$ be the unit normal vector field and $B : TN \to TN$ the form operator with respect to $\nu$, $B(X) = -\nabla_X \nu$. We have the classical Gauss formula

$$\nabla_X Y = \nabla_X^N Y + g(B(X), Y)\nu$$

for all $X, Y \in \mathfrak{X}(N)$, where $\nabla^N$ is the Levi-Civita connection on $(N, g|_N)$.

**Proposition 2.1** (see e.g. [Bär96]). Let $\Sigma$ be the spinor bundle on a Riemannian manifold $M$ and $N \subset M$ an oriented hypersurface. Then $\Sigma|_N$ has a natural structure of Dirac bundle on $N$ where

$$X \cdot \sigma = X \cdot \nu \cdot \sigma,$$

$$\nabla_X \sigma = \nabla_X^N \sigma + \frac{1}{2} B(X) \cdot \nu \cdot \sigma$$

for all $X \in TN$ and $\sigma \in \Gamma(\Sigma|_N)$.

### 2.2 Spectral Resolutions

We review here the spectral properties of the Dirac operator and Laplacian over a compact Riemannian manifold $M$. Consider first the case $\partial M = \emptyset$, see e.g. [BW93, Gi95]. The operators $D, D^2 : \Gamma(\Sigma) \to \Gamma(\Sigma)$ are elliptic and formally self-adjoint; taking as domains the closure of $\Gamma(\Sigma)$ in the Sobolev $H^1$- and $H^2$- topology leads to two self-adjoint operators:
Proposition 2.2. For compact $M$ the operator $D$ admits a discrete spectral resolution, i.e., there exists a sequence $(\varphi_j, \lambda_j)_{j \geq 0}$ such that $(\varphi_j)_{j \in \mathbb{N}}$ is an orthonormal basis of $L^2(M, \Sigma)$ consisting of smooth sections and, for all $j \geq 0$,

$$D\varphi_j = \lambda_j \varphi_j,$$

with $\lambda_j \in \mathbb{R}$. The absolute values of the sequence $(\lambda_j)_{j \geq 0}$ can be ordered in an increasing diverging sequence. An analogous result holds for $D^2$, in this case $\lambda_j \geq 0$ for all $j \geq 0$.

The existence of a discrete spectral resolution for Dirac and Laplacian operators on manifolds with boundary is more involved. In particular, while for the Dirac Laplacian is always possible to find local elliptic boundary conditions allowing for a discrete spectral resolution, this is not always the case for the Dirac operator [Gi95, §1.11.6], see also [Gr96, Hö85].

The Dirac Laplacian on a Dirac bundle $\Sigma \to M$ over a compact Riemannian manifold $M$ with boundary is a formally self-adjoint operator on the smooth sections satisfying the Dirichlet boundary condition $B_D \varphi := \varphi|_{\partial M} = 0$ or the Neumann boundary condition $B_N \varphi = \nabla_\nu \varphi = 0$, where $\nu$ is the inward normal vector field on $\partial M$. Taking the closure in the Sobolev $H^2$-topology, leads to a self-adjoint operator.

The following analogue of Proposition 2.2 is a special case of classical elliptic boundary theory developed by Seeley in [Sl66, Sl69] and Greiner in [Gre70, Gre71]. Therein one fixes $B = B_D$ or $B = B_N$.

Proposition 2.3. There exists an orthonormal basis $(\varphi_j)_{j \geq 0}$ of $L^2(M, \Sigma)$ consisting of smooth sections and such that, for all $j \geq 0$,

(i) $D^2 \varphi_j = \lambda_j \varphi_j$,

(ii) $B \varphi_j = 0$,

with $\lambda_j \in \mathbb{R}$. The absolute values of the eigenvalues $(\lambda_j)_{j \geq 0}$ can be ordered in an increasing diverging sequence, with only a finite number of the $\lambda_j$ negative. The Dirichlet eigenvalues are all strictly positive. The Neumann eigenvalues are all but for a finite number strictly positive.

For an example of negative Neumann eigenvalues see [Fa98] Chapter 5. For an extensive overview of Dirac and Dirac Laplacian spectra on manifolds without and with boundary and their elliptic boundary conditions see [Fa23] subsections 3.1 and 3.2.

2.3 Spectral Estimates

We review the Courant-Hilbert Theorem, originally formulated for the scalar Laplacian on $\mathbb{R}^m$ [CH93, Cha84] and extended to the Dirac Laplacian in [Bär91, Fa98].
Proposition 2.4 (Spectral Upper Bounds). Let $\Sigma \to M$ be a Dirac bundle over a compact, oriented, Riemannian manifold $M$ with (smooth) boundary $\partial M$. For some integer $N \geq 1$, consider the decomposition

$$M = \bigcup_{k=1}^{N} M_k$$

(9)

of $M$ into 0-codimensional submanifolds $M_k$ with (smooth) boundaries $\partial M_k$ and pairwise disjoint interiors. We require that if a boundary $\partial M_k$ intersects $\partial M$ then it agrees with the corresponding connected component of $\partial M$. Let $(\lambda_j)$ be the Dirichlet spectrum of $D^2$ on $M$, likewise $(\mu^k_j)$ for $M_k$.

Choose any $N' \leq N$ and set

$$(\mu_j) = \bigcup_{k=1}^{N'} (\mu^k_j),$$

(10)

where the sequences are in non-decreasing order and the eigenvalues are counted according to their multiplicities. Then

$$\lambda_j \leq \mu_j,$$

(11)

for all $j \geq 0$.

There exists an analogous result for the Neumann boundary condition [Bär91, Fa98], but we will not need it in the following.

3 Warped products and Variations of metrics

3.1 Warped Products

In this section we compute the eigenvalues of the Dirac Laplacian under the Dirichlet boundary condition on a cylindrical manifold $M := [0, 1] \times N$, where $(N, g_N)$ is a closed and oriented Riemannian manifold of dimension $\dim N = m - 1$. We equip $M$ with the warped product metric

$$g := dt^2 + \rho^2(t)g_N,$$

(12)

where $\rho$ is a smooth function, and assume that $M$ is a spin manifold with spinor bundle $\Sigma \to M$.

By Proposition 2.1 we have an induced Dirac bundle structure on each hypersurface $(N_t := \{t\} \times N, \rho^2(t)g_N)$. We note that $\nu = \frac{\partial}{\partial t}$ is the unit normal vector to any $N_t$ and denote by $\nabla$ and $\nabla^{N_t}$ the Levi-Civita connections on $M$ and $N_t$. We now compare the Dirac and Laplacian operators $D$, $D^2$ on $M$ with the analogous operators $D_{N_t}$, $(D_{N_t})^2$ on $N_t$.

By [Bär96] one has

$$D \varphi = \nu \cdot D_{N_t} \varphi - \frac{m - 1}{2} H \nu \cdot \varphi + \nu \cdot \nabla_\nu \varphi$$

(13)
for \( \varphi \in \Gamma(\Sigma) \). Here and in the following \( H = -\rho'/\rho \) is the mean curvature of \( N_t \). The following lemma deals with the Laplacians.

**Lemma 3.1.** Let \( \varphi \in \Gamma(\Sigma) \) then

\[
D^2 \varphi = (D^{N_t})^2 \varphi + [D^{N_t}, \nabla_\nu] \varphi + \left( \frac{m-1}{2} H' - \left( \frac{m-1}{2} \right)^2 H^2 \right) \varphi \quad (14)
\]

\[
+ (m-1) H \nabla_\nu \varphi - \nabla^2_\nu \varphi.
\]

**Proof.** It follows from a straightforward computation making use of (13), the fact that \( \nabla_\nu \nabla_\nu = 0 \) and the relations

\[
D^{N_t}(\nu \cdot \varphi) = -\nu \cdot D^{N_t} \varphi, \quad -\nu \cdot \nabla_\nu (H \nu \cdot \varphi) = H' \varphi + H \nabla_\nu \varphi,
\]

which hold for all \( \varphi \in \Gamma(\Sigma) \).

\[\square\]

For any \( n \in N \) let \( \Pi_{t_0}^t \) be parallel transport in \( \Sigma \) with respect to \( \nabla \) along the curve \( u \mapsto (u,n) \) from \( (t_0,n) \) to \( (t,n) \), where \( t_0, t \in [0,1] \). For the sake of brevity, we omit the proof of the following useful lemma.

**Lemma 3.2.** We have

\[
D^{N_t} \Pi_{t_0}^t \sigma = \frac{\rho(t_0)}{\rho(t)} \Pi_{t_0}^t D^{N_{t_0}} \sigma, \quad (16)
\]

for all \( \sigma \in \Gamma(\Sigma|N_{t_0}) \).

Let \( \{ \sigma_j \}_{j \geq 0} \) be an \( L^2 \)-orthonormal eigenbasis of the Dirac operator \( D^{N_{t_0}} \) at a fixed slice \( N_{t_0} \) with eigenvalues \( \{ \mu_j \} \). We note that \( \nu \cdot \sigma_j \) is an eigenvector with eigenvalue \( -\mu_j \), as Clifford multiplication by \( \nu \) anticommutes with \( D^{N_{t_0}} \). We may hence assume \( \sigma_{j+1} = \nu \cdot \sigma_j \), \( \mu_{j+1} = -\mu_j \) for, say, all \( j \) even.

Parallel transport along \( t \)-lines yields sections

\[
\varphi_j = \Pi_{t_0}^t \sigma_j
\]

of the spinor bundle \( \Sigma \) on \( M \). By Proposition 3.2, the collection \( \{ \varphi^t_j \}_{j \geq 0} \) of their restrictions \( \varphi^t_j = \varphi_j|_{N_t} \) to \( N_t \) is an \( L^2 \)-orthonormal eigenbasis of \( D^{N_t} \) with the eigenvalues

\[
\mu_j(t) = \frac{\rho(t_0)}{\rho(t)} \mu_j.
\]

Clearly \( \varphi^0_j = \sigma_j \) and \( \varphi^t_{j+1} = \nu \cdot \varphi^t_j \) at all \( t \), since \( \nu \cdot \) is parallel along \( t \)-lines.

Let now \( \varphi \) be a smooth section over \( M \). Restriction to \( N_t \) yields a smooth section over \( N_t \) which can be expressed in the basis \( \{ \varphi^t_j \}_{j \geq 0} \), therefore

\[
\varphi = \sum_{j \geq 0} a_j \varphi_j, \quad (17)
\]

for some functions \( a_j = a_j(t) \). We use (17) to rewrite the eigenvalue equation and boundary condition for the Dirac Laplacian on \( M \) as follows.
Proposition 3.3. The equation $D^2\varphi = \lambda \varphi$ subject to the Dirichlet boundary condition $\varphi|_{\partial M} = 0$ is equivalent to the system

$$-a''_j + (m-1)Ha'_j + \left(\mu_j^2 - \mu'_j + \frac{m-1}{2}H' - \frac{(m-1)^2}{4}H^2 - \lambda\right)a_j = 0,$$

$$a_j(0) = a_j(1) = 0$$

(18)

for all $j \geq 0$. If $N$ admits a non-trivial harmonic spinor, then the equation has a non-trivial harmonic spinor for any eigenvalue of the form $\lambda = \pi^2n^2$, where $n$ is some positive integer.

Proof. Inserting the decomposition (17) into the equation $(D^2 - \lambda)\varphi = 0$ and using Lemma 3.1 and $\nabla_\nu \varphi_j = 0$, we obtain

$$\sum_{j \geq 0} \left[-a''_j + (m-1)Ha'_j + \left(\mu_j^2 - \mu'_j + \frac{m-1}{2}H' - \frac{(m-1)^2}{4}H^2 - \lambda\right)a_j\right] \varphi_j = 0$$

(19)

and the first equation of (18) follows immediately. The second equation is just the Dirichlet boundary condition.

Now, under the substitution

$$\bar{a}_j := \exp\left[-\frac{m-1}{2} \int_0^t H \right] a_j,$$

(20)

the system (18) becomes

$$-\bar{a}''_j + (\mu_j^2 - \mu'_j - \lambda) \bar{a}_j = 0,$$

$$\bar{a}_j(0) = \bar{a}_j(1) = 0$$

(21)

for all $j \geq 0$. If $N$ admits a non-trivial harmonic spinor then we have $\mu_k = 0$ for some $k$ and hence $\mu_k(t) = 0$ identically. We set $\bar{a}_j = 0$ for all $j \neq k$ and reduce system (21) to

$$-\bar{a}''_j - \lambda\bar{a}_j = 0,$$

$$\bar{a}_j(0) = \bar{a}_j(1) = 0,$$

(22)

which can be easily seen to have non-trivial solutions with $\lambda$ as required. 

3.2 Continuity of the Eigenvalues

We recall that the uniform $C^k$-topology for sections of a vector bundle $E \rightarrow M$ over a compact manifold $M$ is usually introduced by means of a good
presentation of $E$ and a partition of unity, which allow to globalize the corresponding topology for (locally defined) functions, see e.g. [LM89].

For our purposes, however, it is more convenient to consider the following equivalent definition. If $(M, g)$ is a compact Riemannian manifold and $E$ an Hermitian vector bundle with a compatible connection, we set

$$\|\sigma\|^2_{C^k} = \sup_{x \in M} \left( \sum_{j=0}^{k} |\nabla \cdots \nabla \sigma|^2 \right)$$

for all $\sigma \in \Gamma(E)$, where $\nabla$ is the combination of the connection on $E$ and the Levi-Civita connection of $(M, g)$ and $|\cdot|$ is the pointwise norm on the fibres of $TM$ and $E$.

The norm (23) does depend on the choice of metrics and connections but different norms are all equivalent and induce the uniform $C^k$-topology. For $k = 0$, this is the $L^\infty$-topology and we write $\|\sigma\|_{L^\infty} = \|\sigma\|_{C^0}$ for the corresponding norm. Crucially for our aims, this norm still depends on the choice of metrics.

Nowaczyk proves in [No13] the following result.

**Theorem 3.4.** Let $M$ be a compact spin manifold and $\mathcal{R}(M)$ be the normed space of the Riemannian metrics over $M$ with the norm inducing the uniform $C^1$-convergence over $M$. For any choice of the Riemannian metric $g \in \mathcal{R}(M)$ let $D^g$ be the Dirac operator over $M$. The spectrum of $D^g$ is a pure point spectrum which can be ordered as

$$\text{spec}(D^g) = (\lambda_j(g))_{j \in \mathbb{Z}},$$

where the eigenvalues are repeated according to their multiplicities, and for all $j \in \mathbb{Z}$ and all $g \in \mathcal{R}(M)$

$$\lambda_j(g) \leq \lambda_{j+1}(g).$$

All the eigenvalues $\lambda_j : \mathcal{R}(M) \to \mathbb{R}$ are continuous with respect to the uniform $C^1$-topology.

**Remark 3.1.** There are different possibilities of ordering the spectrum:

- If we fix the order in such a way that $\lambda_0(g)$ is the smallest non-negative eigenvalue near to 0, then in general the $\lambda_j$’s will not be continuous functions of the Riemannian metric with respect to uniform $C^1$-convergence. Nowaczyk constructs an explicit counter-example in the proof of his Main Theorem 3 in [No13].

- Nowaczyk introduces an enumeration for the eigenvalues such that the $\lambda_j$’s are continuous functions of the Riemannian metric with respect to uniform $C^1$-convergence, see the proof of his Main Theorem 2 in [No13].
If we fix the ordering such that the eigenvalues are repeated according to their multiplicity, and for all \( j \in \mathbb{Z} \) and all \( g \in \mathcal{R}(M) \)

\[
0 \leq \lambda_j^2(g) \leq \lambda_{j+1}^2(g),
\]

the \( \lambda_j^2 \)'s and the \( \lambda_j \)'s are continuous functions of the Riemannian metric with respect to uniform \( C^\infty \)-convergence, as proved by Canzani in the proof Theorem 2.7 in [Ca14].

### 3.3 Differentiability of the Eigenvalues and First Variation

We recall that there exists a geometric process to compare spinor fields for two different Riemannian metrics [BG92]. By a result of Rellich, for an analytic variation of metrics with associated one-parameter family \((\Sigma^t, \langle \cdot, \cdot \rangle^t, \nabla^t)\) of spinor bundles, there is an analytic discrete spectral resolution \((\varphi^t_j, \lambda^t_j)_{j \geq 0}\) of the corresponding family of Dirac operators. As usual, the absolute values of the sequence \((\lambda^t_j)_{j \geq 0}\) can be ordered in an increasing diverging sequence.

Explicit formulas for the first derivative of the analytic branches of the eigenvalues are provided by the following.

**Theorem 3.5.** If \( g^t = g + tk \) is a linear variation of the Riemannian metric \( g \) on a compact \( m \)-dimensional spin manifold \( M \), where \( k \) is some symmetric 2-tensor, then any eigenvalue \( \lambda_j(g^t) \) of the Dirac operator \( D g^t \) of multiplicity \( m \) can be written as

\[
\lambda_j(g^t) = \begin{cases} 
\mu^p_j(t), & (t \geq 0) \\
\mu^q_j(t), & (t \leq 0)
\end{cases}
\]

for appropriate \( p, q \in \{1, 2, \ldots, m\} \), where \( \{\mu^1_j, \ldots, \mu^m_j\} \) are real analytic real function of \( t \) in an open real neighbourhood of \( 0 \in \mathbb{R} \). Moreover, the directional derivatives of the \( \mu^p_j \)'s at \( g \) in direction \( k \) read

\[
\frac{d}{dt} \bigg|_{t=0} \mu^p_j(g^t) = -\frac{1}{2} \int_M g \left( Q_{\varphi_j}^p, k \right) \text{vol}_g \\
= -\frac{1}{2} \int_M \left\langle \sum_{i=1}^m e_j \cdot \nabla K_g(e_j) \varphi_j^0, \varphi_j^0 \right\rangle \text{vol}_g,
\]

where \((e_j)\) is a local \( g \)-orthonormal frame, \( Q_{\varphi} \) is the symmetric 2-tensor

\[
Q_{\varphi}(X, Y) = \frac{1}{2} \text{Re} \left\langle X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \varphi \right\rangle
\]

defined for any \( \varphi \in \Gamma(\Sigma^0) \), and \( K_g \) is the endomorphism of \( TM \) defined by \( k(X, Y) = g(K_g(X), Y) \).
Proof. By Theorem A.1 (Theorem VII.3.9 (Rellich) in [Ka80, pp. 392–393]), for any symmetric 2-tensor $k$, the eigenvalue $\lambda_j^t = \lambda_j(g + tk)$ has real analytic branches $\mu_j^1(t), \ldots, \mu_j^m(t)$, and by adapting Theorem 2.1 in [EI08] from the Laplace-Beltrami operator to the Dirac operator we obtain equation (30). Thereby, we have utilized the continuity of $\lambda_j^t$ at $t = 0$. See also Subsection 2.1 in [KMP23]. Formula (28) is proved in [BG92, pp. 593–595].

Corollary 3.6. If $g' = g + tk$ is a linear variation of the Riemannian metric $g$ on a compact $m$-dimensional spin manifold $M$, where $k$ is some symmetric 2-tensor, then any eigenvalue $\lambda_j(g')$ of the Dirac operator $D^g$ of multiplicity $m$ can be written as

$$\lambda_j(g') = \begin{cases} \tilde{\mu}_j^p(g'), & (t \in [0, +\varepsilon[) \\ \tilde{\mu}_j^q(g'), & (t \in ]-\varepsilon, 0]) \end{cases}$$

for appropriate $p, q \in \{1, 2, \ldots, m\}$, where $\{\tilde{\mu}_j^1, \ldots, \tilde{\mu}_j^m\} : \mathcal{R}(M) \to \mathbb{R}$ are infinitely Fréchet differentiable functions for any $\varepsilon > 0$. In particular, if $\lambda_j(g)$ is a simple eigenvalue of $D^g$, then it is infinitely Fréchet differentiable.

Proof of Corollary 3.6.

Let us consider the proof of Theorem 3.5.

- By Theorem A.1 (Theorem VII.3.9 (Rellich) in [Ka80, pp. 392–393]), for any symmetric 2-tensor $k$ the eigenvalue $\lambda_j^t = \lambda_j(g + tk)$ has real analytic branches $\{\mu_j^1, \ldots, \mu_j^m\}$, which are functions of $t$ on an open neighbourhood $]-\varepsilon, +\varepsilon[$ of $0 \in \mathbb{R}$. In particular, for any $g \in \mathcal{R}(M)$ and any direction $k \in T\mathcal{R}(M)$ all the directional derivatives of any order of the branches $\mu_j^p$, for any $p \in \{1, 2, \ldots, m\}$, at $g$ in direction $k$ are well defined for all $j \geq 0$.

- Hence, by multiple application of Proposition A.2 (Proposition E.5.2 in [Ta18]), the branch $\tilde{\mu}_j^p(g + tk)$ for any $p \in \{1, 2, \ldots, m\}$ is infinitely many times Gateaux differentiable with respect to $g$ and $t$ and the Gateaux derivatives are continuous. The continuity with respect to $g$ is meant in terms of the $C^1$-uniform topology.

- Hence, by multiple application of Proposition A.3 (Proposition E.5.3 in [Ta18]), $\tilde{\mu}_j^p(g + tk)$ is infinitely many times Fréchet differentiable with respect to $g$ and $t$ and the Fréchet derivatives are continuous. The continuity with respect to $g$ is meant in terms of the $C^1$-uniform topology.

\[\square\]
**Theorem 3.7.** Let M be a compact m-dimensional spin manifold which supports a sequence of Riemannian metrics \((h_n)_{n \geq 0}\) such that

(i) the sequence is contained in a compact set of \(\Gamma(S^2 T^* M)\) with respect to the uniform \(C^1\)-topology;

(ii) there exists an \(j \geq 0\) such that the eigenvalue \(\lambda_j(h_n) \to 0\) for \(n \to +\infty\).

Then there exists a Riemannian metric \(h\) on \(M\) with non-trivial harmonic spinors.

**Remark 3.2.** The metric \(h\) is not the limit of the sequence \((h_n)\), rather it is obtained as an appropriate linear variation of one of its elements.

**Proof. Step I.**

We first need some preliminary observations. Let \(g\) be a Riemannian metric on \(M\) and \(g^t = g + tk\) be the linear variation determined by a tensor of the form

\[
k = \|Q\|_{L^\infty}^{-1} Q, \tag{31}
\]

where \(Q\) is a non-trivial, i.e., not identically vanishing, symmetric 2-tensor (possibly depending on \(g\)) and the \(L^2\)-norm is computed using \(g\).

Using a local \(g\)-orthonormal frame \((e_i)\) of \(TM\) which diagonalizes \(Q\), it is easy to see that \(g^t\) is positive-definite, hence a metric, for all \(|t| < 1\).

We will be interested in the case where

\[Q = Q_\varphi\]

is as in (29) for some eigenspinor. First note that if \(Q_\varphi \equiv 0\) for some metric \(g\) and a eigenspinor \(\varphi \in \Gamma(\Sigma)\) for the corresponding Dirac operator then

\[
(D \varphi, \varphi) = \sum_{i=1}^{m} (e_i \cdot \nabla e_i \varphi, \varphi) = \sum_{i=1}^{m} Q_\varphi(e_i, e_i) = 0,
\]

so that \(\varphi\) would be an harmonic spinor and our result would immediately follow. We will therefore assume from now on that \(Q_\varphi\) is non-trivial, for any metric \(g\) and eigenspinor \(\varphi\). In particular, the associated tensors (31) are always well-defined and not identically vanishing. This concludes our preliminary facts.

**Step II.**

Let \(g\) be a metric on \(M\) and \(g^t = g + tk\) be the linear variation associated to \(Q = Q_\varphi\) for some \(j \geq 0\), with the discrete spectral resolution \((\varphi^t_j, \lambda^t_j)_{j \geq 0}\).
As Theorem 3.5 let $\tilde{\mu}_j^p(g')$ be one branch of the eigenvalue $\lambda_j^p$, and the real valued function

$$F(g, t) = -2d\tilde{\mu}_j^p(g')/dt$$

$$= \int_M g' \left( Q_{\varphi_j^p}, k \right) \text{vol}_{g'}$$

$$= \int_M \left( \sum_{i=1}^m e_i \cdot \nabla_{(I+tK_g)^{-2}K_g(e_i)}\varphi_j^p, \varphi_j^p \right)^t \text{vol}_{g'} \quad (32)$$

is the derivative of an analytic function, hence analytic in $t$. For the ease of notation we suppress the $j$ and $p$ parameter dependence of the function $F$. Furthermore, for our special choices of $k$ and $Q$, the function $F$ is locally Lipschitz continuous, hence locally uniformly continuous, in $(g, t)$. The proof of this technical fact is a consequence of following considerations:

- By Theorem 3.5, the branches of $\lambda_j(g + tk)$ are infinitely many times Fréchet derivable with respect to $g$ and $t$ and the Fréchet derivatives are continuous. The continuity with respect to $g$ is meant in terms of the $C^1$ uniform topology.

- We choose now $k = k_g := \|Q_{\varphi_j^0}\|_{L^\infty}^{-1} Q_{\varphi_j^0}$ and compute the first Gateaux derivative of $F$ with respect to $g$:

$$F(g, t) = -2d\tilde{\mu}_j^p(g + tk_g), k_g$$

$$dF(g, t).w = -2d[d\tilde{\mu}_j^p(g + tk_g), k_g].w =$$

$$= -2d^2 \tilde{\mu}_j^p(g + tk_g).d(g + tk_g).w, k_g - 2d\tilde{\mu}_j^p(g + tk_g).dk_g.w, \quad (33)$$

where $df.w$ denotes the first Gateaux derivative of $f$ (linear in $w$) and $d^2 f.(w, v)$ the second Gateaux derivative of $f$ (bilinear, i.e. linear in both $w$ and $v$). The first Gateaux derivative of $k_g$ is

$$dk_g = \frac{dQ_{\varphi_j^0}}{\|Q_{\varphi_j^0}\|_{L^2}^2} - \frac{d||Q_{\varphi_j^0}||_{L^\infty}}{||Q_{\varphi_j^0}||_{L^\infty}^2} Q_{\varphi_j^0} \quad (34)$$

Inserting (34) into (33) we conclude that the first Gateaux derivative of $F$ with respect to $g$ exists and is continuous in $g$.

- Hence, the first Gateaux derivative of $F$ with respect to $(g, t)$ exists and is continuous in $(g, t)$. The continuity with respect to $g$ is meant in terms of the $C^1$ uniform topology.

- Hence, by Proposition A.2 (Proposition E.5.2 in [Ta18]), the first Fréchet derivative of $F$ with respect to $(g, t)$ exists, and it is continuous in $(g, t)$. 13
Hence, by proposition A.4 (Proposition E.4.6 in [Ta18]), the function $F = F(g, t)$ is locally Lipschitz continuous in $(g, t)$.

**Step III.**

By (i), up to taking a subsequence, we may assume that $(h_n)$ is convergent in the uniform $C^1$-topology; we stress that the limit $h_\infty$ is not a metric in general but some possibly degenerate symmetric 2-tensor. Let now $G = (h_n) = (h_n) \cup h_\infty$ be the closure of $(h_n)$ and define

$$D = \left\{ (h_n, t) \mid |t| \leq \frac{1}{2} \right\}$$

and remark that the function (32) is well-defined on this set by construction. By (i), the set $G$ is compact, hence the closure

$$\overline{D} = D \cup \left\{ (h_\infty, t) \mid |t| \leq \frac{1}{2} \right\}$$

of $D$ is compact too. We note that identity (36) is a simple consequence of the rectangular form of $D$. The function (32) is (locally and hence globally on $D$) uniformly continuous by Step II, hence it admits a (unique) continuous extension to $\overline{D}$, which we still denote by the same symbol $F$. Clearly $F$ is uniformly continuous also on $\overline{D}$.

**Step IV.**

From now on, for simplicity of exposition, we will refer to any element of $G$ different from $h_\infty$ directly as a metric $g \in G$. Now

$$F(g, 0) = \frac{\|Q_{\varphi_j}\|_2^2}{\|Q_{\varphi_j}\|_\infty} \geq \inf_{g \in G} \frac{\|Q_{\varphi_j}\|_2^2}{\|Q_{\varphi_j}\|_\infty} =: C \geq 0$$

for all metrics $g \in G$.

If $C = 0$, since $G$ is compact and $F(\cdot, 0)$ is continuous by Step II, there exists a $g \in G$ such that $Q_{\varphi_j} = 0$, the spinor $\varphi_j$ is harmonic and we are finished. From now on, we assume that $C > 0$.

By uniform continuity on $\overline{D}$

$$F(g, t) \geq \frac{C}{2},$$

for all $|t| < \frac{1}{2}$ for some $\delta$ independent of $g \in G$. 

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For this range of $t$, we shall now consider the Taylor expansion at $t = 0$ of the $p$-th branch of $j$-th eigenvalue. For any metric $g \in \mathcal{G}$, we have

$$\tilde{\mu}_j^p(g^t) = \tilde{\mu}_j^p(g) + \alpha(g, \theta) t$$

with

$$\alpha(g, t) = \frac{d\tilde{\mu}_j^p(g^t)}{dt} = -\frac{1}{2} F(g, t),$$

where $\theta : (-\delta, \delta) \to (-\delta, \delta)$ is a continuous function such that $\theta(0) = 0$.

We then have

$$\alpha(g, t) \leq -\frac{C}{4},$$

for any metric $g \in \mathcal{G}$ and $|t| < \delta$.

Now, observe that $\lambda_j(g) = \tilde{\mu}_j^p(g) = 0$ for some metric $g \in \mathcal{G}$ means that $g$ has non-trivial harmonic spinors. From now on, we therefore assume $\tilde{\mu}_j^p(g) \neq 0$ for all metrics $g \in \mathcal{G}$.

Given a metric $g \in \mathcal{G}$ and the associated linear variation $g^t$ for $|t| < \delta$, we need to consider two separate cases:

- **Case $\tilde{\mu}_j^p(g) \geq 0$.**
  
  We have
  $$0 \leq \tilde{\mu}_j^p(g^t) \leq \tilde{\mu}_j^p(g) - \frac{C}{4} t,$$
  
  for all $0 \leq t < \delta$. So, the right hand side vanishes if
  $$t = \frac{4\tilde{\mu}_j^p(g)}{C}.$$ 
  
  By (ii) we can choose $g \in \mathcal{G}$ such that $t < \delta$ and, therefore, $\tilde{\mu}_j^p(g^t) = 0$.

- **Case $\tilde{\mu}_j^p(g) \leq 0$.**
  
  We have
  $$\tilde{\mu}_j^p(g) - \frac{C}{4} t \leq \tilde{\mu}_j^p(g^t) \leq 0,$$
  
  for all $-\delta < t \leq 0$. So, the left hand side vanishes if
  $$t = \frac{4\tilde{\mu}_j^p(g)}{C}.$$ 
  
  By (ii) we can choose $g \in \mathcal{G}$ such that $t < \delta$ and, therefore, $\tilde{\mu}_j^p(g^t) = 0$.

Hence, by Corollary 3.6, the Dirac operator $D^{g^t}$ has an harmonic spinor and the proof is completed.

**Remark 3.3.** It is straightforward to see that Theorem 3.5, Corollary 3.6 and Theorem 3.7 hold true for a generic Dirac bundle.
During the proof of the main Theorem 1.1 we will need the following auxiliary result, whose proof is straightforward and therefore omitted.

**Lemma 3.8.** Let \((M, g)\) be a Riemannian spin manifold and \(g^t = t g\) be the linear variation of Riemannian metrics, for all \(t > 0\). Then
\[ \lambda_j^t = t^{-1/2} \lambda_j, \]
for all \(j \geq 0\).

### 3.4 Berger Metrics

The one-parameter family of metrics \((g_s)_{s>0}\) on \(S^{2k+1}\) known as the Berger metrics can be conveniently described in terms of the Hopf fibration \(S^{2k+1} \to \mathbb{C}P^k, k \geq 1\), where \(S^{2k+1}\) is equipped with the round metric of constant curvature 1 and \(\mathbb{C}P^k\) with the Fubini-Study metric. Recall that the Hopf fibration is a Riemannian submersion with typical fiber \(S^1\). The Berger metric \(g_s\) is obtained rescaling the length of the \(S^1\) fibers by a positive constant \(s\) while keeping the metric on the orthogonal complement to the fibers unchanged.

**Proposition 3.9.** *(see [Bär96, pp. 8–9])* The Berger sphere \((S^{2k+1}, g_s), k\) odd, admits non-trivial harmonic spinors for \(s = 2(k + 1)\).

### 4 Proof of the Main Theorem

We split the proof into two steps. First we give conditions under which a sequence of Riemannian metrics on a compact manifold has a subsequence convergent in the uniform \(C^1\)-topology. Next, we prove the main theorem about the existence of harmonic spinors.

Let \((M, g)\) be a compact Riemannian manifold and \(\nabla\) its Levi-Civita connection. For any non-negative integer \(k\), we consider the Sobolev space \(H^k(M, S^2T^*M)\) of symmetric 2-tensors on \(M\), with norm given by
\[ ||h||_{H^k}^2 = \sum_{j=0}^k \int_M |\nabla \cdots \nabla h|^2 dvol_g \]
for all \(h \in \Gamma(S^2T^*M)\). Sobolev norms associated to different Riemannian metrics are all equivalent and induce the same topology of a Hilbert space.

**Lemma 4.1.** Let \((M, g)\) be a compact Riemannian manifold and \((h_n)\) a sequence of Riemannian metrics bounded in Sobolev norm for a sufficiently large \(k \geq 0\). Then, up to taking a subsequence, \((h_n)\) is contained in a compact set with respect to the uniform \(C^1\)-topology.
Proof. By Rellich Lemma, up to taking a subsequence, we may assume that \((h_n)\) is convergent in the uniform \(C^1\)-topology to some \(h_\infty\). The closure 
\[ \mathcal{G} = \overline{(h_n)} = (h_n) \cup \{h_\infty\} \]
is sequentially compact. \(\square\)

We recall that an oriented hypersurface of a spin manifold inherits a natural Dirac bundle structure (see Proposition 2.1).

**Theorem 4.2.** Let \((M, g)\) be a closed connected Riemannian \(m\)-dimensional spin manifold with spinor bundle \(\Sigma \to M\) and \(N\) a 0-codimensional submanifold with an oriented boundary. Assume there exists a Riemannian metric on \(\partial N\) with non-trivial harmonic sections of the Dirac bundle \(\Sigma|_{\partial N} \to \partial N\). Then there is a Riemannian metric on \(M\) with harmonic spinors.

**Proof.** For any \(t > 0\) the manifold \(M\) is diffeomorphic to the iterated connected sum along a hypersurface \(\partial N\):
\[ M^t = (M \setminus \text{Int}(N)) \# U \# V^t \# W^t \# N, \]
where
\begin{align*}
U &:= [-1, 0] \times \partial N, \\
V^t &:= [0, t] \times \partial N, \\
W^t &:= [t, t+1] \times \partial N.
\end{align*}
(44)

In other words, \(M\) admits a decomposition as in Proposition 2.4. We equip \(M^t\) with the Riemannian metric \(g^t\), defined by
\[ g^t = \begin{cases} 
g & \text{on } M \setminus \text{Int}(N), \\
\, du^2 + \left(1 - \psi(u) + \psi(u)\rho^2(u)\right)g|_{\partial N} & \text{on } U, \\
\, du^2 + \rho^2(u)g|_{\partial N} & \text{on } V^t, \\
\, du^2 + \left(\rho^2(u) - \chi(u)\rho^2(u) + \chi(u)K\right)g|_{\partial N} & \text{on } W^t, \\
Kg & \text{on } N, \end{cases} \]
(45)

where \(K\) is the constant \(\rho^2(t+1)\) for some smooth function \(\rho\) to be specified later, and \(\psi, \chi : \mathbb{R} \rightarrow [0,1]\) smooth functions such that
\[ \psi(u) = \begin{cases} 0 & \text{for } u \leq -\frac{3}{4}, \\
1 & \text{for } u \geq -\frac{1}{4} \end{cases}, \quad \chi(u) = \begin{cases} 0 & \text{for } u \leq t + \frac{1}{4}, \\
1 & \text{for } u \geq t + \frac{3}{4}. \end{cases} \]
(46)

In order for the metrics on \(M \setminus \text{Int}(N)\) and \(U\) to join smoothly, it is sufficient to identify a tubular neighbourhood of \(\partial(M \setminus \text{Int}(N)) = \partial N\) with an open subset of \(U\), following the flow of the outer normal to \(\partial N\) as in, e.g., Theorem
9.20 in [Le18]. We note that $\partial N$ is a closed embedded submanifold of $M$. A similar remark applies to $W^t$ and $N$.

The spin bundle over $(M,g)$ can be stretched over $(M^t,g^t)$. By Propositions 2.4 and 3.3, any eigenvalue $\lambda_j^2(t)$ of the Dirac Laplacian on $(M^t,g^t)$ is dominated by the corresponding Dirichlet eigenvalue $\mu_j(t)$ on $(V^t,g^t)$ and there is $j$ so that

$$\lambda_j^2(t) \leq \mu_j(t) = \frac{\pi^2}{t^2}. \quad (47)$$

We are going to consider the one-parameter family of metrics

$$\tilde{g}^t = \frac{1}{t^\alpha} g^t,$$

where $0 < \alpha < 2$. For them the inequality becomes $\tilde{\lambda}_j^2(t) \leq \mu_j(t) = \pi^2 t^{\alpha-2}$ hence

$$0 \leq \tilde{\lambda}_j^2(t) \xrightarrow{t \to \infty} 0.$$

This implies condition (ii) of Theorem 3.7.

We now specify

$$\rho(u) = \exp\left(-\frac{u}{2(m-1)}\right)$$

and compute the Sobolev norm of the metric $\tilde{g}^t$ over $V^{t^\alpha}$. Taking the metric $g^1$ as reference, this Sobolev norm satisfies for all $k \geq 0$ the growth condition

$$\| \tilde{g}^{t^\alpha} \|_{H^k(V^{t^\alpha})}^2 = \sum_{|\alpha| \leq k} \int_{V^{t^\alpha}} dvol_{g^1} \langle \nabla^\alpha \tilde{g}^{t^\alpha}, \nabla^\alpha \tilde{g}^{t^\alpha} \rangle \leq \text{const}, \quad (48)$$

for some positive constant independent of the parameters $t$ and $\alpha$, as long as $\alpha > 0$. The diffeomorphism

$$\Phi^{t^\alpha} : V^1 \to V^{t^\alpha}, \quad (u, y) \mapsto (t^{\alpha} u, y) \quad (49)$$

induces the metric $(\Phi^{t^\alpha})^* \tilde{g}^{t^\alpha}$ on $V^1$ for which, taking $(\Phi^1)^* g^1$ as reference metric in equation (43) for the computation of the Sobolev norm,

$$\| (\Phi^{t^\alpha})^* \tilde{g}^{t^\alpha} \|_{H^k(V^1)}^2 \leq \text{const} \quad (50)$$

holds true for all $k \geq 0$.

If $\Psi$ denotes a fixed diffeomorphism mapping $M$ to $M^1$, then, by Lemma 3.8 the metrics $h_j := (\Psi)^* (\Phi^{t^\alpha})^* \tilde{g}^{t^\alpha}$ for a fixed $\alpha \in [0, 2]$ satisfy the assumptions of Lemma 4.1 and Theorem 3.7 for a sequence $\{t_j\}_{j \geq 0}$ such that $t_j \to +\infty$ as $j \to \infty$. Hence, there exists a Riemannian metric on $M$ such that the corresponding Dirac operator has zero as eigenvalue and the proof is completed. \qed
We can proceed now with the proof of the main theorem.

Proof of Theorem 1.1. Let $p \in M$ be fixed. If $\delta(p) > 0$ is the injectivity radius of $p$, then

$$ N := \exp_p \left( \frac{1}{\delta(p)} B^{R^m}(0, 1) \right) \subset M $$

(51)

is a geodesic ball centered at $p$, whose boundary $\partial N$ is diffeomorphic to $S^{m-1}$. We analyze several cases:

- $m = 2$: the circle $S^1$ has two spin structures and only one admits harmonic spinors. Therefore Theorem 4.2 cannot be applied. This is in accordance with the fact that $S^2$ has no harmonic spinors for all Riemannian metrics.

- $m = 3$: as we have just seen the classic Dirac operator on $S^2$ has no vanishing eigenvalues and, again, Theorem 4.2 cannot be applied.

- $m \geq 4$: recalling that there is a unique spin structure on $S^{m-1}$, we have to distinguish several subcases:
  - $m = 4k + 4$, for $k \in \mathbb{N}_0$. By Proposition 3.9, the Berger metric on $S^{m-1}$ admits a non trivial harmonic spinor, so that Theorem 4.2 applies and the statement of Theorem 1.1 holds.
  - $m = 4k + 5$, for $k \in \mathbb{N}_0$. We can now apply Theorem 1.1 to $S^{4k+4}$, as we have just seen, to conclude that there is a metric on $S^{4k+4}$ admitting non trivial harmonic spinors. We continue with Theorem 4.2 and the statement of the Theorem is proved.
  - $m = 4k + 6$, for $k \in \mathbb{N}_0$. We apply Theorem 1.1 to $S^{4k+5}$ and the rest follows analogously.
  - $m = 4k + 7$, for $k \in \mathbb{N}_0$. We apply Theorem 1.1 to $S^{4k+6}$ and the rest follows analogously.

The proof is completed.

Remark 4.1. Why does the proof of Theorem 1.1 not work for the Laplace-Beltrami operator or for the complex Laplacian as well? Of course Theorem 4.2 holds for general Dirac bundles and since the constant functions on $S^{m-1}$ are harmonic differential forms of degree 0, we can trivially conclude that there are always non trivial harmonic forms on any Riemannian manifolds, namely the constant functions. The question now is if it is possible to say something for differential forms of fixed degree $k > 0$. Unfortunately, both Euler and Clifford operators do not preserve the graduation of the differential form bundles and there is no version of Theorem 3.5 for the Laplace-Beltrami
operator or for the complex Laplacian acting on forms of fixed degree. So, the constructions in the proof of Theorem 4.2 cannot be mimicked for pure forms. If this were the case, we could conclude for instance that the De Rham cohomology $H_k(S^m, \mathbb{R})$ does not vanish for $k \neq 0, m$, which is of course a contradiction.

5 Conclusion

We have studied the spectrum of a general Dirac Operator under variation of the Riemannian metric and extended a result of Bourguignon and Gauduchon on the first derivative of the eigenvalues from the special case of the spinor bundle to the general Dirac bundle case. In conjunction with the extension of the Courant-Hilbert Theorem for upper bounds of the Dirac Laplacian, we proved a result on the existence of Riemannian metrics allowing for harmonic Dirac bundle sections. In particular, we could show that any closed spin manifold of dimension $m \geq 4$ can be always be provided with a Riemannian metric admitting harmonic spinors. Since in dimension $m = 1, 2$, there are counterexamples, the conjecture remains open for $m = 3$.

A Some Results from Functional Analysis

Theorem A.1 (Rellich). Let $T(x)$ be a selfadjoint holomorphic family of type (A) defined for $x$ in a neighbourhood of an interval $I_0$ of the real axis. Furthermore, let $T(x)$ have compact resolvent. Then, all eigenvalues of $T(x)$ can be represented by functions which are holomorphic on $I_0$. More precisely, there is a sequence of scalar-valued functions $\mu_n(x)$ and a sequence of vector-valued functions $\varphi_n(x)$, all holomorphic on $I_0$, such that if $x \in I_0$, the $\mu_n(x)$ represent all the repeated eigenvalues of $T(x)$ and the $\varphi_n(x)$ form a complete orthonormal family of the associated eigenvectors of $T(x)$.

Proposition A.2. Let $X, Y$ be normed linear spaces. Assume that $f: X \rightarrow Y$ has a directional variation in $X$, i.e. there exists a directional variation $f'(x)(\eta)$ for all $x, \eta \in X$. Assume further that:

(i) For fixed $x$, $f'(x)(\eta)$ is continuous in $\eta$ at $\eta = 0$.

(ii) For fixed $\eta$, $f'(x)(\eta)$ is continuous in $x$ for all $x \in X$.

Then, $f'(x) \in \mathcal{L}(X, Y)$, i.e. $f'(x)$ is a bounded linear operator.

Proposition A.3. Consider $f: X \rightarrow Y$, where $X$ and $Y$ are normed linear spaces. Suppose that $f$ is Gateaux differentiable in an open set $U \subset X$. If $f'(x)$ is continuous at $x \in U$, then $f'(x)$ is a Fréchet derivative.
Proposition A.4. Consider $f : X \rightarrow Y$, where $X$ and $Y$ are normed linear spaces. If $f$ is Fréchet differentiable at $x \in X$, then $f$ is continuous at $x$. If the Fréchet derivative $f'(x)$ is continuous for all $x \in X$, then $f$ is locally Lipschitz continuous.

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References

[ADH09] B. AMMAN, M. DAHL and E. HUMBERT, *Surgery and Harmonic Spinors*, Adv. Math. 220 no. 2, (523-539), 2009.

[AS68] M. F. ATIYAH and I. M. SINGER, *The Index of Elliptic Operators III*, Ann. Math. 87, (546-604), 1968.

[AS71] M. F. ATIYAH and I. M. SINGER, *The Index of Elliptic Operators V*, Ann. Math. 93, (139-149), 1971.

[Bä91] C. BÄR, *Das Spektrum von Dirac-Operatoren*, Bonner mathematische Zeitschriften, 1991.

[Bä92] C. BÄR, *Lower Eigenvalue Estimates for Dirac Operators*, Math. Ann. 293 (39-46), 1992.

[Bä96] C. BÄR, *Metrics with Harmonic Spinors*, Geometric and Functional Analysis 6, (899-942), 1996.

[BS92] C. BÄR and P. SCHMUTZ, *Harmonic Spinors on Riemann Surfaces*, Ann. Glob. Anal. Geom. 10 (263-273), 1992.

[BD02] C. BÄR and M. DAHL, *Surgery and the Spectrum of the Dirac Operator*, J. Reine Angew. Math. 552, (53-76), 2002.

[BGV96] N. BERLINE, E. GETZLER and M. VERGNE, *Heat Kernels and Dirac Operators*, Corrected Second Printing, Grundlehren der mathematischen Wissenschaften, Springer Verlag, 1996.

[BW93] B. BOOSS/BAVNBEK, K.P. WOJCIECHOWSKI, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, 1993.
[BG92] J.-P. BOURGUIGNON and P. GAUDUCHON, *Spineurs, Opérateurs de Dirac et Variations de Métriques*, Commun. Math. Phys. 144, (581-599), 1992.

[Ca14] Y. CANZANI, *On the Multiplicity of Eigenvalues of Conformally Covariant Operators*, Annales de l’Institut Fourier, 64, (947-970), 2014.

[Cha84] I. CHAVEL, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

[CH93] R. COURANT and D. HILBERT, *Methoden der mathematischen Physik*, Springer, 1993.

[Da05] M. DAHL, *Prescribing Eigenvalues of the Dirac Operator*, Manuscripta Mathematica 118, (191-199), 2005.

[Da08] M. DAHL, *On the Space of Metrics with Invertible Dirac Operators*, Comment. Math. Helv. 83, (451-469), 2008.

[EL08] A. EL SOUFI and S. ILIAS, *Laplacian Eigenvalue Functionals and Metric Deformations on Compact Manifolds*, Journal of Geometry and Physics, Vol. 58, Issue 1, (89-104), 2008.

[Fa98] S. FARINELLI, *Spectra of Dirac Operators over a Family of Degenerating Hyperbolic Three Manifolds*, ETH Diss. 12690, 1998.

[Fa23] S. FARINELLI, *Dirac Cohomology on Manifolds with Boundary and Spectral Lower Bounds*, Partial Differ. Equ. Appl. 4, 46, 2023.

[Gi95] P. B. GILKEY, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, Second Edition, Studies in Advanced Mathematics, CRC Press, 1995.

[Gr70] P. GREINER, *An Asymptotic Expansion for the Heat Equation*, Global Analysis, Berkeley 1968, Proc. Symp. Pure Math. 16, (133-137), Amer. Math. Soc., Providence, 1970.

[Gr71] P. GREINER, *An Asymptotic Expansion for the Heat Equation*, Arch. Rat. Mech. Anal. 41, 1971, (163-218).

[Gr96] G. GRUBB, *Functional Calculus of Pseudodifferential Boundary Problems*, Second Edition, Birkhäuser, 1996.

[Hi74] N. HITCHIN, *Harmonic Spinors*, Adv. Math. 14, (1-55), 1974.

[Hö85] L. HÖRMANDE, *The Analysis of Linear Partial Differential Operators III*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, 1985.
[KMP23] M. KARPUKHIN, A. MÉTRAS and I. POLTEROVICH, Dirac Eigenvalue Optimisation and Harmonic Maps to Complex Projective Spaces, arXiv:2308.07875, 2023.

[Ka80] T. KATO, Perturbation Theory for Linear Operators, Springer, 1980.

[LM89] H. B. LAWSON and M.-L. MICHELSOHN, Spin Geometry, Princeton University Press, 1989.

[Le18] J. M. LEE, Introduction to smooth manifolds, Second edition. Graduate Texts in Mathematics, 218, Springer, New York, 2013.

[Ma97] S. MAIER, Generic Metrics and Connections on Spin- and Spin^c Manifolds, Commun. Math. Phys., 188, (407-437), 1997.

[Mi65] J. W. MILNOR, Remarks Concerning Spin Manifolds, S. Cairns (Ed.) Differential and Combinatorial Topology, Princeton, (55-62), 1965.

[No13] N. NOWACZYK, Continuity of Dirac Spectra, Ann. Glob. Anal. Geom. 44, (541-563), 2013.

[Se00] L. SEEGER, Metriken mit harmonischen Spinoren auf geradedimensionalen Sphären, Dissertation, Universität Hamburg, 2000.

[Sl66] R. T. SEELEY, Singular integrals and Boundary Problems, Amer. J. Math. 88, 1966, (781-809).

[Sl69] R. T. SEELEY, The Resolvent of an Elliptic Boundary Value Problem, Amer. J. Math. 91, 1969, (889-920).

[Ta18] R. A. TAPIA, A Unified Approach to Mathematical Optimization Theory for Scientists and Engineers, Course notes for CAAM 560: Optimization Theory, Rice University, Houston, TX, 2018.