Almost periodicity in complex analysis *

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Abstract

This is a brief survey of up-to-date results on holomorphic almost periodic functions and mappings in one and several complex variables, mainly due to the Kharkov mathematical school.

Keywords: Almost periodic divisor, almost periodic function, Bohr compactification.

While the notion of almost periodic function on $\mathbb{R}$ (or $\mathbb{R}^m$) seems to be quite understood, it is not the case for holomorphic almost periodic functions on a strip in $\mathbb{C}$ (or, more generally, on a tube domain in $\mathbb{C}^m$). Investigation of the zero sets of holomorphic almost periodic functions leads to such objects as almost periodic divisors and holomorphic chains. The central problem is to determine if a divisor (holomorphic chain) is generated by a holomorphic almost periodic function (mapping). Partial results in the one-dimensional situation ([14], [15], [30], [31]) indicate that the problem is highly non-trivial. We present here a brief survey of recent results in this direction, together with some related topics, mainly due to the Kharkov mathematical school.

Definitions and basic properties. We start with standard notions of the theory of almost periodic functions.

Definition 1 A continuous mapping $f$ from $\mathbb{R}^m$ to a metric space $X$ is called almost periodic if its orbit $\{T_t f\}_{t \in \mathbb{R}^m}$ is a relatively compact set in $C(\mathbb{R}^m, X)$ with respect to the topology of uniform convergence on $\mathbb{R}^m$, where $T_t$ is the translation by $t \in \mathbb{R}^m$: $(T_t f)(x) = f(x + t)$.

Definition 2 A continuous mapping $f$ from a tube domain $T_\Omega := T^m + i\Omega = \{z = x + iy : x \in \mathbb{R}^m, y \in \Omega\}$ to a metric space $X$ is called almost periodic on $T_\Omega$ if its orbit $\{T_t f\}_{t \in \mathbb{R}^m}$ is a relatively compact subset of $C(T_\Omega, X)$ with respect to the topology of uniform convergence on each tube subdomain $T_{\Omega'}$, $\Omega' \subset \subset \Omega$.

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We reserve the word "function" for mappings to \(C\). Any almost periodic function \(f\) on \(T_\Omega\) has its mean value
\[
\mathcal{M}[f](y) = \lim_{s \to \infty} (2s)^{-m} \int_{[-s,s]^m} f(x + iy) dx, \quad y \in \Omega.
\]
The spectrum of \(f\) is the set
\[
\text{sp}f = \{ \lambda \in \mathbb{R}^m : a_\lambda(y) := \mathcal{M}[f(x + iy)e^{-i\langle x, \lambda \rangle}] \neq 0 \},
\]
\(\langle x, \lambda \rangle\) being the scalar product in \(\mathbb{R}^m\); this set is at most countable. Then a Fourier series is assigned to each function \(f\),
\[
f \sim \sum_n a_\lambda_n(y)e^{i\langle x, \lambda_n \rangle};
\]
in addition, if \(f\) is holomorphic on a tube domain, then
\[
f \sim \sum_n a_\lambda_n e^{i\langle z, \lambda_n \rangle}, \quad a_\lambda_n \in \mathbb{C}.
\]
Any almost periodic function on a tube domain is uniformly approximated by Bochner-Fejer’s sums on every subtube \(T_\Omega', \Omega' \subset \subset \Omega\):
\[
\sum_n \beta_j(\lambda_n)a_\lambda_n(y)e^{i\langle x, \lambda_n \rangle}
\]
with \(0 \leq \beta_j < 1, \beta_j \to 1\) as \(j \to \infty\).

The converse statement is also true: if a function \(f(z)\) is uniformly approximated by finite sums \(\sum a_n(y)e^{i\langle x, \lambda_n \rangle}\) (or \(\sum a_n e^{i\langle z, \lambda_n \rangle}\)) on every tube subdomain \(T_\Omega', \Omega' \subset \subset \Omega\), then \(f(z)\) is almost periodic (holomorphic almost periodic) on \(T_\Omega\).

All this can be extended to more general objects (namely, to distributions), see [28], [25]. Here we will use the following notion.

**Definition 3** A complex-valued measure \(g\) on \(T_\Omega\) is called almost periodic if for every continuous function \(\varphi\) compactly supported in \(T_\Omega\), the action of \(g\) on \(T_\Omega\varphi\),
\[
g_\varphi(t) := \int T_\Omega \varphi dg,
\]
is an almost periodic function in \(t \in \mathbb{R}^m\).

Let us remark that the spectrum of \(g\) is the union of the spectra of \(g_\varphi\) over all functions \(\varphi\).

Since every holomorphic almost periodic function on a tube domain \(T_\Omega\) has a unique extension to the convex hull of the tube, the base \(\Omega\) will be always assumed to be convex.
Zeros of almost periodic functions. It was proved in [21] that the function \( \log |f(z)| \) is almost periodic, in the sense of distributions, for any holomorphic almost periodic function \( f \) on a tube domain. Therefore, the divisor \( df \) of such a function \( f \) is almost periodic in the sense of distributions. This means that the current

\[
\partial \bar{\partial} \log |f(z)| = \sum_{j,k} \frac{\partial^2 \log |f(z)|}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k
\]

is almost periodic, i.e., all the terms in this sum (that are always complex-valued measures) are almost periodic. Note that, on the other hand, almost periodicity of this current does not imply that the function \( f \) itself is almost periodic.

For \( m = 1 \), the divisor \( df \) of a holomorphic function \( f \) on an open strip \( S = T(\alpha, \beta) \) is a positive discrete measure on \( S \). Its mass at each point \( a \) equals the multiplicity \( k(a) \) of the zero of the function \( f \) at this point. The divisor is almost periodic whenever the convolution

\[
\sum_{a_n \in \text{supp} df} k(a_n) \chi(a_n + t)
\]

is an almost periodic function in the variable \( t \in \mathbb{R} \) for every function \( \chi \in C(S) \) with \( \text{supp} \chi \subset \subset S \). When \( m > 1 \), almost periodicity of the divisor \( df \) of a function \( f \) holomorphic in \( T_\Omega \) means that the function

\[
\int_{Z_f} k(a) \chi(a + t)
\]

is almost periodic in \( t \in \mathbb{R}^m \) for any form \( \chi \) of bidegree \( (m - 1, m - 1) \) with continuous coefficients compactly supported in \( T_\Omega \); here \( Z_f \) is the zero set of \( f \).

**Theorem 1** ([11], [12]). If the trace measure

\[
\sum_{k=1}^m \frac{\partial^2 \log |f(z)|}{\partial z_k \partial \bar{z}_k}
\]

is almost periodic, then so is the divisor \( df \).

**Jessen’s function.** Since \( \log |f| \) is almost periodic (in the sense of distributions), there exists the mean value \( \mathcal{M}[\log |f|] \). The function

\[
J_f(y) := \mathcal{M}[\log |f|](y)
\]

is called Jessen’s function of \( f \). It is convex in \( y \in \Omega \) and generates the Riesz measure

\[
\mu_J = \theta_m \sum_j \partial^2 J_f / \partial y_j^2.
\]
**Theorem 2** ([21]). Let \( V_f(\Omega_0, s), \Omega_0 \subset \subset \Omega, \) be the volume of the zero set of \( f(z) \) inside the set \( \{ z : x \in [-s, s]^m, y \in \Omega_0 \} \). Then
\[
\lim_{s \to \infty} (2s)^{-m} V_f(\Omega_0, s) = \frac{\theta_{2m} \mu_J(\Omega_0)}{\theta_m}
\]
for every \( \Omega_0 \subset \subset \Omega \) with \( \mu_J(\partial \Omega_0) = 0 \). Furthermore, \( \mu_J(\Omega_0) = 0 \) (i.e., \( J_f(y) \) is linear on \( \Omega_0 \)) if and only if \( f(z) \neq 0 \) on \( T_{\Omega_0} \).

For \( m = 1 \), the result was proved in [14]. Moreover, that seminal work contains a necessary and sufficient (albeit quite sophisticated) condition for a function to be Jessen’s function of an almost periodic holomorphic function on a strip. As a consequence, every convex and piece-wise linear function is Jessen’s; when the spectrum of \( f \) is a subset of some free countable additive subgroup of \( \mathbb{R} \), the number of linearity components for \( J_f \) is locally finite.

The first statement is no longer true for \( m > 1 \):

**Theorem 3** ([23]). A convex piece-wise linear function \( J(y) \) on \( \Omega \) is Jessen’s function of an almost periodic holomorphic function \( f \) on \( T_{\Omega} \) if and only if it has the form
\[
J(y) = \sum_{j=1}^\omega \gamma_j \max \{ \langle y, \nu^{(j)} \rangle - h_j, 0 \} + \gamma_0 \langle y, \nu^{(0)} \rangle,
\]
where \( \omega \leq \infty, \gamma_j > 0, \lambda_j \in \mathbb{R}^m, h_j \in \mathbb{R} \); in this case,
\[
f(z) = \prod_{j=1}^\omega f_j \left( \langle z, \nu^{(j)} \rangle - ih_j \right),
\]
\( f_j \) being entire almost periodic functions on \( \mathbb{C} \) with real zeros.

No complete description of Jessen’s functions is known in general situation. Some necessary condition is given in [5], which implies

**Theorem 4** ([5]; for the case of sections of periodic divisors, see [27]). If the spectrum of \( f \) is a subset of some free countable additive subgroup of \( \mathbb{R}^m \), then the number of connected components of the set \( \Omega' \setminus \text{supp} \mu_{J_f} \) (linearity components) is finite for each \( \Omega' \subset \subset \Omega \).

The assertion is false in the class of holomorphic almost periodic functions with arbitrary spectrum.

**Almost periodic divisors.** There exist almost periodic divisors which cannot be generated by holomorphic almost periodic functions (for \( m = 1 \), see [31]; for \( m > 1 \), see [26]). This invokes the problem of characterization for divisors of almost periodic holomorphic functions. The first steps in this direction were the following results.
Theorem 5 ([30]). A skew-symmetric $N \times N$-matrix $M_d$ with integer entries can be assigned to each almost periodic divisor $d$ on a strip with the spectrum in $\text{Lin}\{\lambda_1, \ldots, \lambda_N\}$, $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$, such that $d$ is the divisor of a holomorphic almost periodic function on the strip if and only if $M_d = 0$.

Theorem 6 ([24]). A skew-symmetric $N \times N$-matrix $M_D$ with integer entries can be assigned to each $N$-periodic (with real periods) divisor $D$ on a tube such that $D$ is the divisor of a holomorphic $N$-periodic function on the tube if and only if $M_D = 0$.

The both matrices $M_d$ and $M_D$ are generated by means of the increments of log $\Phi$ for certain functions $\Phi$ constructed from $d$ and $D$, respectively. A relation between these two theorems is given by

Theorem 7 ([26]). Let $d$ be a section of an $N$-periodic divisor $D$. Then $d$ is almost periodic; moreover, $d$ is the divisor of some holomorphic almost periodic function if and only if $D$ is the divisor of some holomorphic $N$-periodic function.

In order to give a complete description of divisors for almost periodic holomorphic functions, we need the following notion.

Definition 4 Bohr’s compact set $K$ is the compactification of $\mathbb{R}^m$ with respect to the topology generated by the sets

$$\{x \in \mathbb{R}^m : |e^{i(x, \mu_k)} - e^{i(x_0, \mu_k)}| < \delta, 1 \leq k \leq N\}$$

over all $x_0 \in \mathbb{R}^m$, $\delta > 0$, $N < \infty$, $\mu_1, \ldots, \mu_N \in \mathbb{R}^m$.

We will consider functions on the “Bohr’s tubes” $K_{i\Omega} := K + i\Omega$, so we need a definition of a function holomorphic on $K_{i\Omega}$:

Definition 5 A continuous function $\hat{f}(\tau + iy)$ is holomorphic on an open set $\omega \subset K_{i\Omega}$ if the function $T_x \hat{f}(\tau + iy)$ is a smooth function of $x, y \in \mathbb{R}^m$ for small $x$ and

$$\frac{\partial}{\partial z_k} T_x \hat{f}(\tau + iy) |_{x=0} = 0, \quad k = 1, \ldots, m,$$

for all $\tau + iy \in \omega$.

It is well known that almost periodic functions on $\mathbb{R}^m$ are just the restrictions of continuous functions on $K$. The corresponding results for holomorphic almost periodic functions and almost periodic divisors in tube domain follow.

Proposition 1 ([6]). Holomorphic almost periodic functions on $T_{i\Omega}$ are just the restrictions to $T_{i\Omega}$ of holomorphic functions on $K_{i\Omega}$.
Proposition 2 ([6]). For any almost periodic divisor \( d \) on \( T_\Omega \) and any point \( \tau_0 + iy_0 \in K_\Omega \), there exists a function \( r(\tau + iy) \), holomorphic on a neighborhood \( \omega \) of \( \tau_0 + iy_0 \), with the following property: for each \( x_1 + iy_1 \in \omega \cap T_\Omega \), the restriction of \( d \) to some ball in \( T_\Omega \) with center in \( x_1 + iy_1 \) coincides with the divisor of the restriction of \( r(x+iy) \) to this ball; the function \( r(\tau + iy) \) is unique up to a holomorphic factor without zeros on \( \omega \).

Note that for the divisor of a holomorphic almost periodic function \( f(z) \) we can take \( r(\tau + iy) = \hat{f}(\tau + iy) \); on the other hand, if the function \( r(\tau + iy) \) from the proposition is well-defined on the whole set \( K \times \Omega \), then \( d \) is the divisor of the holomorphic almost periodic function \( r(x+iy) \). Using the notion of bundle, we obtain the following result.

Theorem 8 ([6]). A line bundle \( \mathcal{F}_d \) over \( K_\Omega \) corresponds to each almost periodic divisor \( d \) on the tube \( T_\Omega \) such that trivial bundles correspond just to the divisors of almost periodic holomorphic functions.

It can be proved that the bundle \( \mathcal{F}_d \) is trivial if and only if the first Chern class of the restriction of the bundle to the base \( K + iy_0 \) is trivial. Thus

Theorem 9 ([4] for \( m = 1 \), [6] for \( m > 1 \)). A cohomology class \( c(d) \in H^2(K, \mathbb{Z}) \) is assigned to each almost periodic divisor \( d \) on \( T_\Omega \) such that the trivial class corresponds just to all the divisors of holomorphic almost periodic functions on \( T_\Omega \). In addition, \( c(d) \) is a homomorphism of the semigroup of all almost periodic divisors to \( H^2(K, \mathbb{Z}) \).

When considering only divisors with spectrum in \( \text{Lin}_\mathbb{Z}\{\lambda_1, \ldots, \lambda_N\} \) and the vectors \( \lambda_1, \ldots, \lambda_N \in \mathbb{R}^m \) linearly independent over \( \mathbb{Z} \), Bohr’s compact set \( K \) in Theorems 8 and 9 can be replaced by the \( N \)-dimensional torus \( T^N \). The group \( H^2(T^N, \mathbb{Z}) \) coincides with the group of all skew-symmetric \( N \times N \)-matrix with integer entries, so this particular case of Theorem 9 is a multidimensional version of Theorem 5.

Theorem 9 allows to obtain the following sufficient conditions.

Theorem 10 ([4] for \( m = 1 \), [6] for \( m > 1 \)). If the restriction of an almost periodic divisor \( d \) on a tube \( T_\Omega \) to some tube \( T_{\Omega'} \), \( \Omega' \subset \Omega \), is the divisor of a holomorphic almost periodic function \( f \) on \( T_{\Omega'} \), then \( d \) is the divisor of a holomorphic almost periodic function \( f_1 \) on \( T_\Omega \).

Corollary 1 ([10] for \( m = 1 \), [6] for \( m > 1 \)). If the projection of the support \( d \) to \( \Omega \) is not dense in \( \Omega \), then \( d \) is the divisor of a holomorphic almost periodic function \( f \) on \( T_\Omega \).

Theorem 11 ([10] for \( m = 1 \), [6] for \( m > 1 \)). If an almost periodic divisor \( d \) is invariant with respect to the map \( y \mapsto -y \), then \( d \) is the divisor of some holomorphic almost periodic function.
It is not hard to prove that the spectrum \( df \) for an almost periodic holomorphic \( f \) is contained in the minimal additive group \( G(sp f) \) containing \( sp f \). The converse statement is also true:

**Theorem 12** ([4] for \( m = 1 \), [6] for \( m > 1 \)). If \( d \) is the divisor of a holomorphic almost periodic function \( f \), then \( d \) is the divisor of some holomorphic almost periodic function \( f_1 \) with spectrum in \( G(sp d) \).

There exists a convenient representation for the classes \( c(d) \). Let \( g(\zeta) \) be an entire function on \( \mathbb{C} \) with simple zeros at all points with integer coordinates. Given \( \lambda, \mu \in \mathbb{R}^m \), denote by \( \lambda \wedge Q \mu \) the Chern class of the divisor \( d^{\lambda, \mu} \) of the function

\[
g^{\lambda, \mu}(z) := g((z, \lambda) + i(z, \mu)), \quad z \in \mathbb{C}^m;
\]

it can be proved that \( \wedge Q \) is an inner product over \( Q \). Furthermore, if \( \lambda, \mu \) are linearly independent over \( \mathbb{R} \), then \( d^{\lambda, \mu} \) is periodic, and if \( \lambda, \mu \) are linearly dependent over \( \mathbb{R} \) and independent over \( Q \), then \( d^{\lambda, \mu} \) is almost periodic.

**Theorem 13** ([6]). For any almost periodic divisor \( d \) on \( T_\Omega \) there exists a finite number of the standard divisors \( d^{\lambda_k, \mu_k} \), \( k = 1, \ldots, n \), such that

\[
c(d) = \lambda_1 \wedge Q \mu_1 + \ldots + \lambda_n \wedge Q \mu_n.
\]

In addition, the divisor

\[
d + \sum_k d^{\mu_k, \lambda_k}
\]

is the divisor of a holomorphic almost periodic function on \( T_\Omega \); when \( m > 1 \), all the divisors \( d^{\mu_k, \lambda_k} \) can be taken periodic.

**Almost periodic holomorphic mappings into affine space.** A holomorphic mapping \( f \) from a tube domain into \( \mathbb{C}^q \), \( q > 1 \), is almost periodic if all its components \( f_j \), \( 1 \leq j \leq q \), are holomorphic almost periodic functions. It turns out that the zero set of such a mapping (more precisely, the corresponding holomorphic chain) need not be almost periodic [12]. The nature of this phenomenon is that the codimension of the zero set can drop at infinity. This leads to consideration of the so-called regular almost periodic holomorphic mappings introduced in [22]:

**Definition 6** A holomorphic almost periodic mapping \( f : T_\Omega \rightarrow \mathbb{C}^q \), \( q \leq m \), is said to be regular if the dimension of the zero set of each mapping from the closure of the orbit \( \{ T_t f \}_{t \in \mathbb{R}} \) of \( f \) is at most \( m - q \).

A sufficient condition, in terms of spectrum, for a mapping to be regular [22] shows that such mappings are, in a sense, 'generic' almost periodic mappings.

The zero set of \( f \) can be represented by means of the Monge-Ampère current \((i\partial\bar{\partial}\log|f|)^q\), and an application of the machinery of Monge-Ampère operators gives
Theorem 14 ([11], [12]). The zero set of any regular almost periodic holomorphic mapping is almost periodic.

As a result, the zero set has a density $\mu_f$ which is a non-negative measure on $\Omega$. On the other hand, since the function $\log |f|$ is almost periodic (in the sense of distributions) for every almost periodic holomorphic mapping $f$, one can also define Jessen’s function $J_f = M[\log |f|]$, which is convex on $\Omega$ (see [17], [18]). Nevertheless, there is no direct analog to Theorem 2: there exists a regular mapping with no zeros on $T\Omega$ such that the real Monge-Ampère measure of $J_f$ is strictly positive [19]. Besides, Jessen’s functions of regular mappings have the following rigidity: if $J_f$ is linear on some open set $\omega \subset \Omega$, then $J_f$ is linear everywhere (see [18], [19]).

A more satisfactory result can be obtained in terms of Jessen’s functions of the components of the mappings. Namely, for a wide subclass of regular mappings to $\mathbb{C}^q$, $q \leq m$ (namely, for mappings with independent components), the support of the real mixed Monge-Ampère measure of $J_{f_1}, \ldots, J_{f_q}$ coincides with the union of the supports of $\mu_g$ over all $g(z) = (f_1(z + s^1), \ldots, f_q(z + s^q))$, $s^k \in \mathbb{C}^m$ (see [20]). A sufficient condition for a mapping to have independent components (in terms of the spectra of the components) is given in [20]: the condition means that the spectra are in general position, so such mappings are still ‘generic’.

Note that, similar to Theorem 1, almost periodicity of any holomorphic chain $Z$ of pure dimension $q$ is equivalent to almost periodicity of its trace measure $Z \wedge (i\partial \bar{\partial} |z|^2)^{m-q}$ ([11], [12]).

Almost periodic meromorphic functions and mappings into projective space. Let $F = (f_0(z) : \ldots : f_q(z))$ be a holomorphic almost periodic mapping of a tube $T\Omega \subset \mathbb{C}^m$ to the projective space $\mathbb{CP}^q$ equipped with the Fubini-Study metric. Note that the functions $f_0(z), \ldots, f_q(z)$ are not uniquely defined, but their zeros ("coordinate divisors") are well-defined; we will suppose that each function is not identically zero.

In the particular case $q = 1$ this gives the class of meromorphic almost periodic functions; for $m = q = 1$ the class was introduced in [29]. Note that this class is not closed with respect to either multiplication or addition.

Theorem 15 ([7] for $m = 1$, [6] for $m > 1$). The product of two meromorphic almost periodic functions is almost periodic if and only if the zeros and poles of the product are uniformly separated in $T\Omega$ for any $\Omega' \subset \subset \Omega$.

Theorem 16 ([7] for $m = q = 1$, [16] for $m = 1$, $q > 1$, [6] for $m > 1$, $q > 1$). There exists a holomorphic almost periodic mapping from a tube $T\Omega \subset \mathbb{C}^m$ to $\mathbb{CP}^q$ with given coordinate divisors $d_0, \ldots, d_q$ if and only if

A) all the divisors $d_0, \ldots, d_q$ are almost periodic,

B) all the divisors $d_0, \ldots, d_q$ have the same Chern class,
C) for any \( \Omega' \subset \subset \Omega \), every ball with center in \( T_{\Omega'} \) and radius \( r(\Omega') \) intersects at most \( q \) supports of the divisors \( d_l, l = 0, \ldots, q \).

The mapping can be represented by holomorphic almost periodic functions \( f_0(z), \ldots, f_q(z) \) without common zeros if and only if all the divisors \( d_0, \ldots, d_q \) have the trivial Chern class.

**Holomorphic functions with almost periodic modulus.** A complete description for almost periodic divisors on a strip, in terms of holomorphic functions, is given in the following theorem.

**Theorem 17** ([10]) A divisor \( d \) on a strip \( S \) is almost periodic if and only if there exists a holomorphic function \( f \) such that \( df = d \) and \(|f(z)|\) is an almost periodic function on \( S \).

It follows from [22] that if a function \( f \), holomorphic on a tube \( T_{\Omega} \subset \mathbb{C}^m \), has almost periodic modulus, then its divisor is almost periodic and the matrix

\[
M_f := \left( \right. \text{Im} \mathcal{M}[\partial^2 \log |f|/\partial \bar{z}_k \partial z_l] \left. \right]_{l,k=1}^m
\]

is zero (since the measures \( \partial^2 \log |f|/\partial \bar{z}_k \partial z_l \) are almost periodic, all the mean values exist).

The matrix \( M_f \) depends only on the Chern class \( c(d) \) of the divisor \( d \). Moreover, if the divisors \( d \) and \( d_0 \) have the same Chern class and \( d \) is the divisor of a holomorphic function \( f \) with almost periodic modulus, then \( d_0 \) is the divisor of some holomorphic function \( f_0 \) with almost periodic modulus, too. Using Theorem 13, we get the following result.

**Theorem 18** ([6]) If \( d \) is an almost periodic divisor in \( T_{\Omega} \) and the matrix \( M_f \) is zero for some \( f \) with \( df = d \), then \( d \) is the divisor of a holomorphic function \( f_1 \) whose modulus is almost periodic in \( T_{\Omega} \).

**Almost periodicity of slices.** It is well-known that a function, holomorphic and bounded on a strip \( S \), is almost periodic in \( S \) if it is almost periodic on at least one straight line in \( S \). In [32], [33] this result was extended to holomorphic almost periodic functions in tube domains; besides the usual uniform metric, the integral Stepanoff’s, Weyl’s, and Besicovich’s metrics were examined as well.

Such a criterion is not true for meromorphic functions in a strip (which are bounded as mappings to \( \mathbb{CP} \)). The following theorem is valid instead.

**Theorem 19** ([8]) Let \( F(z) \) be a holomorphic mapping from a tube domain \( T_{\Omega} \) to a compact complex manifold, uniformly continuous on every domain \( T'_{\Omega} \) with \( \Omega' \subset \subset \Omega \). If \( F(z) \) is almost periodic on some hyperplane \( \mathbb{R}^m + iy_0 \subset T_{\Omega} \), then \( F(z) \) is almost periodic on \( T_{\Omega} \).
Almost periodic solutions of functional equations. It is easy to see that a continuous solution of the equation \( w^2 = f(t) \) with an almost periodic (nonnegative) function \( f(t) \) need not be almost periodic. Nevertheless, the following theorem is true.

**Theorem 20** ([3] for \( m = 1 \), [2] for \( m > 1 \)). Let \( w(z) \) be a continuous solution of the equation

\[
a_n(z)w^n + a_{n-1}(z)w^{n-1} + \ldots + a_1(z)w + a_0(z) = 0
\]

on a tube domain \( T_\Omega \subset \mathbb{C}^m \) with holomorphic almost periodic functions \( a_0(z), \ldots, a_n(z) \). Then \( w(z) \) is holomorphic almost periodic, too.

In [1] this theorem was extended to solutions of \( F(z, w) = 0 \) with \( F \) holomorphic, almost periodic in \( z \) and such that the mapping \((F(z, w), F'_w(z, w))\) is regular in the sense of Definition 6.

Almost periodic functions with spectrum in a cone. H. Bohr proved that an almost periodic function \( f \) on the real axis has nonnegative spectrum if and only if the function \( f \) extends to the upper half-plane as a bounded holomorphic function; next, the spectrum of an almost periodic function is bounded if and only if this function extends to the complex plane as an entire function of exponential growth.

The following theorems generalize Bohr’s results to functions of several variables.

**Theorem 21** ([13]). In order that an almost periodic function \( f \) on \( \mathbb{R}^m \) have spectrum in a convex cone \( \Gamma \subset \mathbb{R}^m \), it is necessary and sufficient that \( f \) extends to the tube domain \( T_\Gamma \) as a holomorphic function, bounded in every tube domain \( T_{\Gamma'} \).

Here

\[
\hat{\Gamma} = \{ y \in \mathbb{R}^m : \langle y, \mu \rangle \geq 0 \quad \text{for all } \mu \in \Gamma \}
\]

is the conjugate cone to \( \Gamma \), and \( \Gamma' \) is an internal subcone of \( \hat{\Gamma} \).

**Theorem 22** ([13]). In order that an almost periodic function \( f \) on \( \mathbb{R}^m \) have bounded spectrum, it is necessary and sufficient that \( f \) extends to \( \mathbb{C}^m \) as an entire function of the growth at most \( C \exp|z| \). Moreover,

\[
\sup_{x \in \mathbb{R}^m} \limsup_{r \to \infty} \frac{1}{r} \log |f(x + iry)| = \sup_{\lambda \in \text{sp}f} \langle -y, \lambda \rangle.
\]

Theorems 21 and 22 are also valid for integral Stepanoff’s metric instead of the usual uniform one.
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