SINGULAR SHOCK WAVES IN INTERACTIONS

MARKO NEDELJKOV

Abstract. In a number of papers it was shown that there are one-dimensional systems such that they contain solutions with, so called, overcompressive singular shock waves besides the usual elementary waves (shock and rarefaction ones as well as contact discontinuities).

One can see their definition for a general $2 \times 2$ system with fluxes linear in one of dependent variables in [8]. This paper is devoted to examining their interactions with themselves and elementary waves. After a discussion of systems given in a general form, a complete analysis will be given for the ion-acoustic system given in [6].

Keywords: conservation law systems, singular shock wave, interaction of singularities, generalized functions

1. Introduction

Consider the system

$$
(f_2(u))_t + (f_3(u)v + f_4(u))_x = 0
$$

$$
(g_1(u)v + g_2(u))_t + (g_3(u)v + g_4(u))_x = 0.
$$

(1)

where $f_i, g_j, i = 2, \ldots, 4, j = 1, \ldots, 4$ are polynomials with the maximal degree $m$, $(u, v) = (u(x, t), v(x, t))$ are unknown functions with a physical range $\Omega$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. We shall fix the following notation for the rest of the paper:

$$
f_i(y) = \sum_{k=0}^{m} a_{i,k} y^k, \quad g_j(y) = \sum_{k=0}^{m} b_{j,k} y^k, \quad i = 2, 3, 4, \quad j = 1, 2, 3, 4.
$$

There are cases when there is no classical solution to Riemann problem for the above system. Sometimes, there is a solution in the form of delta or singular shock wave. In [8] one can see when a system in evolution form (i.e. when $f_2 = u$, $g_1 = 1$ and $g_2 = 0$) permits a solution in the shape of singular shock wave. With the same type of reasoning and a more effort, one can give the answer to the same question in the case system [6].

The aim of this paper is to investigate what happens during and after an interaction of a singular shock wave with another wave. After a general statement about new initial data taken at interaction point (of course, true for delta shock waves, too) in Section 3, we shall present a detailed investigation in the case of the system (so called ion-acoustic system)

$$
u_t + (u^2 - v)_x = 0
$$

$$
u_t + (u^3/3 - u)_x = 0
$$

(2)

given in [6].

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Definitions and concepts used here are from [8], based on the use of Colombeau generalized functions defined in [11]. They will be briefly described in the Section 2. If one is not familiar with these concepts, he/she can assume that a solution to the above system is given by nets of smooth functions with equality substituted by a distributional limit. The reason why the generalized functions are used is to give opportunity for extending the procedure in this paper for arbitrary initial data when a system posses singular or delta shock wave as a solution.

Few interesting facts observed during the investigations of system (2) are arising a question about possibilities in a general case. Observed facts are:

1. The singular shock wave solution to a Riemann problem for (2) always has an increasing strength of the rate \( O(t), t \to \infty \). (The strength of the shock is a function which multiplies the delta function contained in a solution, \( s(t) \) in (7)). After the interaction, the resulting singular shock wave is supported by a curve, not necessary straight line as before, and its strength can be an increasing, but also a constant or a decreasing function with the respect to the time variable.

2. When the resulting singular shock wave has a decreasing strength (this can occurs during an interaction of a admissible singular shock wave with a rarefaction wave), after some time it can decompose into two shock waves. This is a quite new phenomenon.

The structure of this paper can be described in the following way.

In the second section we will introduce necessary notation and give basic notions based on the papers [11] and [8].

In the third section, one can find a way how to continue a solution to the general case of system (1) after an interaction point (Theorem 1). The basic assumption is that a left-hand side of the first, and the right-hand side of the second wave can be connected by a new singular shock wave. The conditions for such a possibility are formulated through a notion of second delta singular locus, see Definition 7. Explicit calculations for a geometric description of the locus are possible to perform for system (1), but we shall omit it, to preserve readers attention on the further topics.

The results given in these sections are used in the next one devoted to special case (2).

The first part of the fourth section is devoted to description of a situation which can occur after a singular shock and a shock wave interact. In the same way one can do the same for two singular shock waves, as one can at the end of this section.

The final, 5th section, contains the most interesting and important results about singular shock and rarefaction wave interaction. In that case the decoupling of a singular shock into a pair of shock waves, already mentioned before, can occur. The analysis is done when a singular shock wave is on the left-hand side of a rarefaction wave. But one can easily see that these results can be obtained using the same procedure when a singular shock is on the other side of a rarefaction wave.

2. Notation

We shall briefly repeat some definitions of Colombeau algebra given in [11] and [8]. Denote \( \mathbb{R}^2_+ := \mathbb{R} \times (0, \infty) \), \( \mathbb{R}^2_- := \mathbb{R} \times [0, \infty) \) and let \( C_c^\infty(\Omega) \) be the algebra of smooth functions on \( \Omega \) bounded together with all their derivatives. Let \( C_c^\infty_b(\mathbb{R}^2) \) be a set of all functions \( u \in C^\infty(\mathbb{R}^2_+) \) satisfying \( u|_{\mathbb{R} \times (0, T)} \in C_c^\infty_b(\mathbb{R} \times (0, T)) \) for every
$T > 0$. Let us remark that every element of $C^\infty_b(\mathbb{R}_+^2)$ has a smooth extension up to the line $\{t = 0\}$, i.e. $C^\infty_b(\mathbb{R}_+^2) = C^\infty(\mathbb{R}_+^2)$. This is also true for $C^\infty_b(\mathbb{R}_+^2)$.

**Definition 1.** $\mathcal{E}_{M,g}(\mathbb{R}_+^2)$ is the set of all maps $G : ((0,1) \times \mathbb{R}_+^2) \to \mathbb{R}$, $(\varepsilon, x, t) \mapsto G_\varepsilon(x, t)$, where for every $\varepsilon \in (0,1)$, $G_\varepsilon \in C^\infty_b(\mathbb{R}_+^2)$ satisfies:

For every $(\alpha, \beta) \in \mathbb{N}_0^2$ and $T > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{(x,t) \in \mathbb{R} \times (0,T)} |\partial_\varepsilon^\alpha \partial_t^\beta G_\varepsilon(x, t)| = \mathcal{O}(\varepsilon^{-N}), \text{ as } \varepsilon \to 0.$$ 

$\mathcal{E}_{M,g}(\mathbb{R}_+^2)$ is an multiplicative differential algebra, i.e. a ring of functions with the usual operations of addition and multiplication, and differentiation which satisfies Leibniz rule.

$\mathcal{N}_g(\mathbb{R}_+^2)$ is the set of all $G \in \mathcal{E}_{M,g}(\mathbb{R}_+^2)$, satisfying:

For every $(\alpha, \beta) \in \mathbb{N}_0^2$, $a \in \mathbb{R}$ and $T > 0$

$$\sup_{(x,t) \in \mathbb{R} \times (0,T)} |\partial_\varepsilon^\alpha \partial_t^\beta G_\varepsilon(x, t)| = \mathcal{O}(\varepsilon^a), \text{ as } \varepsilon \to 0.$$ 

$\square$

Clearly, $\mathcal{N}_g(\mathbb{R}_+^2)$ is an ideal of the multiplicative differential algebra $\mathcal{E}_{M,g}(\mathbb{R}_+^2)$, i.e. if $G_\varepsilon \in \mathcal{N}_g(\mathbb{R}_+^2)$ and $H_\varepsilon \in \mathcal{E}_{M,g}(\mathbb{R}_+^2)$, then $G_\varepsilon H_\varepsilon \in \mathcal{N}_g(\mathbb{R}_+^2)$.

**Definition 2.** The multiplicative differential algebra $\mathcal{G}_g(\mathbb{R}_+^2)$ of generalized functions is defined by $\mathcal{G}_g(\mathbb{R}_+^2) = \mathcal{E}_{M,g}(\mathbb{R}_+^2) / \mathcal{N}_g(\mathbb{R}_+^2)$. All operations in $\mathcal{G}_g(\mathbb{R}_+^2)$ are defined by the corresponding ones in $\mathcal{E}_{M,g}(\mathbb{R}_+^2)$. $\square$

If $C^\infty_b(\mathbb{R})$ is used instead of $C^\infty_b(\mathbb{R}_+^2)$ (i.e. drop the dependence on the $t$ variable), then one obtains $\mathcal{E}_{M,g}(\mathbb{R})$, $\mathcal{N}_g(\mathbb{R})$, and consequently, the space of generalized functions on a real line, $\mathcal{G}_g(\mathbb{R})$.

In the sequel, $G$ denotes an element (equivalence class) in $\mathcal{G}_g(\Omega)$ defined by its representative $G_\varepsilon \in \mathcal{E}_{M,g}(\Omega)$.

Since $C^\infty_b(\mathbb{R}_+^2) = C^\infty_b(\mathbb{R}_+^2)$, one can define a restriction of a generalized function to $\{t = 0\}$ in the following way.

For given $G \in \mathcal{G}_g(\mathbb{R}_+^2)$, its restriction $G|_{t=0} \in \mathcal{G}_g(\mathbb{R})$ is the class determined by a function $G_\varepsilon(x, 0) \in \mathcal{E}_{M,g}(\mathbb{R})$. In the same way as above, $G(x - ct) \in \mathcal{G}_g(\mathbb{R})$ is defined by $G_\varepsilon(x - ct) \in \mathcal{E}_{M,g}(\mathbb{R})$.

If $G \in \mathcal{G}_g$ and $f \in C^\infty(\mathbb{R})$ is polynomially bounded together with all its derivatives, then one can easily show that the composition $f(G)$, defined by a representative $f(G_\varepsilon)$, $G \in \mathcal{G}_g$ makes sense. It means that $f(G_\varepsilon) \in \mathcal{E}_{M,g}$ if $G_\varepsilon \in \mathcal{E}_{M,g}$, and $f(G_\varepsilon) - f(H_\varepsilon) \in \mathcal{N}_g$ if $G_\varepsilon - H_\varepsilon \in \mathcal{N}_g$.

The equality in the space of the generalized functions $\mathcal{G}_g$ is strong for our purpose, so we need to define a weaker relation called association.

**Definition 3.** A generalized function $G \in \mathcal{G}_g(\Omega)$ is said to be **associated with** $u \in \mathcal{D}'(\Omega)$, $G \approx u$, if for some (and hence every) representative $G_\varepsilon$ of $G$, $G_\varepsilon \to u$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$. Two generalized functions $G$ and $H$ are said to be associated, $G \approx H$, if $G - H \approx 0$. The rate of convergence in $\mathcal{D}'$ with respect to $\varepsilon$ is called the **rate of association**. $\square$

A generalized function $G$ is said to be of a bounded type if

$$\sup_{(x,t) \in \mathbb{R} \times (0,T)} |G_\varepsilon(x, t)| = \mathcal{O}(1) \text{ as } \varepsilon \to 0,$$
for every $T > 0$.

Let $u \in \mathcal{D}''_L(\mathbb{R})$. Let $\mathcal{A}_0$ be the set of all functions $\phi \in C_0^\infty(\mathbb{R})$ satisfying $\phi(x) \geq 0$, $x \in \mathbb{R}$, $\int \phi(x)dx = 1$ and $\text{supp} \phi \subset [-1, 1]$, i.e.

$$\mathcal{A}_0 = \{ \phi \in C_0^\infty : (\forall x \in \mathbb{R})\phi(x) \geq 0, \int \phi(x)dx = 1, \text{supp} \phi \subset [-1, 1] \}.$$

Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, $x \in \mathbb{R}$. Then

$$\iota_\phi : u \mapsto u * \phi_\varepsilon/\mathcal{N}_g,$$

where $u * \phi_\varepsilon/\mathcal{N}_g$ denotes the equivalence class with respect to the ideal $\mathcal{N}_g$, defines a mapping of $\mathcal{D}'_L(\mathbb{R})$ into $\mathcal{G}_g(\mathbb{R})$, where $*$ denotes the usual convolution in $\mathcal{D}'$. It is clear that $\iota_\phi$ commutes with the derivation, i.e.

$$\partial_x \iota_\phi(u) = \iota_\phi(\partial_x u).$$

**Definition 4.**

(a) $G \in \mathcal{G}_g(\mathbb{R})$ is said to be a generalized step function with value $(y_0, y_1)$ if it is of bounded type and

$$G_\varepsilon(y) = \begin{cases} y_0, & y < -\varepsilon \\ y_1, & y > \varepsilon \end{cases}$$

Denote $[G] := y_1 - y_0$.

(b) $D \in \mathcal{G}_g(\mathbb{R})$ is said to be generalized split delta function (S$\delta$-function, for short) with value $(\alpha_0, \alpha_1)$ if $D = \alpha_0 D^- + \alpha_1 D^+$, where $\alpha_0 + \alpha_1 = 1$ and

$$DG \approx (y_0\alpha_0 + y_1\alpha_1)\delta,$$

for every generalized step function $G$ with value $(y_0, y_1)$.

(c) Let $m$ be an odd positive integer. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be $m'$-singular delta function ($m'$-SD-function, for short) with value $(\beta_0, \beta_1)$ if $d = \beta_0 d^- + \beta_1 d^+$, $\beta_0^{m-1} + \beta_1^{m-1} = 1$, $d^\pm \in \mathcal{G}_g(\mathbb{R})$, $(d^\pm)^i \approx 0$, $i \in \{1, \ldots, m-2, m\}$, $(d^\pm)^{m-1} \approx \delta$, and

$$d^{m-1}G \approx (y_0\beta_0^{m-1} + y_1\beta_1^{m-1})\delta,$$

for every generalized step function $G$ with value $(y_0, y_1)$.

(d) Let $m$ be an odd positive integer. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be $m$-singular delta function ($m$-SD-function, for short) with value $(\beta_0, \beta_1)$ if $d = \beta_0 d^- + \beta_1 d^+$, $\beta_0^m + \beta_1^m = 1$, $d^\pm \in \mathcal{G}_g(\mathbb{R})$, $(d^\pm)^i \approx 0$, $i \in \{1, \ldots, m-1\}$, $(d^\pm)^m \approx \delta$, and

$$d^mG \approx (y_0\beta_0^m + y_1\beta_1^m)\delta,$$

for every generalized step function $G$ with value $(y_0, y_1)$.

\(\square\)

In this paper we shall assume the compatibility condition $Dd \approx 0$, where $D$ is S$\delta$- and $d$ is S$\delta$- or $m'$Sd-function.

Suppose that the initial data are given by

$$u|_{t=T} = \begin{cases} u_0, & x < X \\
 u_1, & x > X \end{cases} \quad v|_{t=T} = \begin{cases} v_0, & x < X \\
 v_1, & x > X \end{cases}$$
Definition 5. Singular shock wave (DSSW for short) is an associated solution to \( u((x - X), (t - T)) = G((x - X) - c(t - T)) + s(t)G_{\partial}^-(c(t - T)) + \alpha_1G^+(c(t - T)) \) with the initial data \( (u_0, v_0) \) of the form
\[
\begin{align*}
  u((x - X), (t - T)) &= G((x - X) - c(t - T)) \\
  v((x - X), (t - T)) &= H((x - X) - c(t - T)) \\
  s(t) &= (\beta_0G^- + \beta_1G^+) \\
  d_t &= (\gamma_0G^- + \gamma_1G^+) \\
  m &= \frac{1}{v - c} \\
\end{align*}
\]
where
\( (i) \) \( c \in \mathbb{R} \) is the speed of the wave,
\( (ii) \) \( s(t) \), \( \tilde{s}(t) \) and \( \tilde{s} \) are smooth functions for \( t \geq 0 \), and equal zero at \( t = T \).
\( (iii) \) \( G \) and \( H \) are generalized step functions with values \( (u_0, u_1) \) and \( (v_0, v_1) \) respectively,
\( (iv) \) \( d_1 = \alpha_0G^- + \beta_1G^+ \) and \( d_2 = \gamma_0G^- + \gamma_1G^+ \) are \( msD- \) or \( ms'D- \) functions,
\( (v) \) \( D = \alpha_0G^- + \alpha_1G^+ \) is an \( S_\delta \)-function compatible with \( d \).
The singular part of the wave is
\[
\tilde{s}(t)(\alpha_0G^- + \alpha_1G^+) + \beta_0G^- + \beta_1G^+ + \tilde{s}(t)(\gamma_0G^- + \gamma_1G^+) \]
The wave is overcompressive if its speed is less or equal to the left- and greater or equal to the right-hand side characteristics i.e.
\[
\lambda_2(u_0, v_0) > \lambda_1(u_0, v_0) \geq c \geq \lambda_1(u_1, v_1) > \lambda_1(u_1, v_1).\]

Remark 1. (a) In \( \mathbb{F} \) one can find special choice for \( S_\delta \)- and and \( d \) is \( msD- \) or \( ms'D- \) functions. For example \( D^\pm \in \mathcal{G}_\partial(\mathbb{R}) \) are given by the representatives
\[
D^\pm_\varepsilon(y) := \frac{1}{\varepsilon} \phi\left(\frac{y - (\pm 2\varepsilon)}{\varepsilon}\right), \phi \in \mathcal{A}_0.
\]
m\( \delta \)- and \( m'\delta \)-functions can be chosen in the same manner.
(b) Compatibility condition for an \( S_\delta \)-function \( D \) and an \( msD- \) or \( ms'D- \) function \( d \) is automatically fulfilled if
\[
\supp d^+_\varepsilon \cap \supp D^+_\varepsilon = \supp d^-_\varepsilon \cap \supp D^-_\varepsilon = \emptyset
\]
(c) Idea behind the above definition of products \( \mathbb{G} \), \( \mathbb{G} \) and \( \mathbb{G} \) is the following. Starting point is that we know nothing about infinitesimal values of the initial data (carried on by step functions \( G \) and \( H \) above) around zero, but only that any such unmeasurable influence stops at the points \( \pm \varepsilon \). The above mentioned definitions are made in order to get uniqueness of all products where step functions, \( S_\delta \), \( msD- \) and \( ms'D- \) functions appear. With an additional information for \( G_\varepsilon \) and \( H_\varepsilon \) around zero, one can choose \( D \) and \( d \) much more freely. For example, in \( G_\varepsilon \) and \( H_\varepsilon \) are monotone functions (which is quite natural assumption), relation \( \mathbb{G} \) can be substituted by
\[
DG \approx \gamma \delta, \gamma \text{ can be any real between } \min\{y_0, y_1\} \text{ and } \max\{y_0, y_1\}.
\]
The possibilities in Colombeau algebra are even wider for specific systems instead of general case \( \mathbb{G} \). One can look in \( \mathbb{G} \) for a good review of such possibilities. We
dealing with a system in a general form and it is the reason for using the above definition.

(d) Due to absence of known additional facts for the general case (hyperbolicity, additional conservation laws,...), one can use the overcompressibility as an admissibility condition.

Definition 6. The set of all points \((u_1, v_1) \in \Omega\) such that there exists an singular shock wave solution (called corresponding DSSW) to Cauchy problem \((1)\) is called delta singular locus. We shall write \((u_1, v_1) \in DSL(u_0, v_0)\). If the corresponding DSSW is overcompressive, then it is called overcompressive delta singular locus. We shall write \((u_1, v_1) \in DSL^*(u_0, v_0)\). □

In the sequel, the term “solution” will denote generalized function which solves a system in the association sense.

3. The new initial data

Suppose that system \((1)\) posses a DSSW solution for some initial data. Assume one of the following.

(i) If an \(m\) SD-function is contained in the above DSSW, then assume
\[
\deg(g_1) < m - 1, \quad \deg(g_2) < m, \quad \deg(f_2) < m. \tag{8}
\]

(ii) If an \(m'\) SD-function is contained in the above DSSW, then assume
\[
\deg(g_1) < m - 2, \quad \deg(g_2) < m - 1, \quad \deg(f_2) < m - 1. \tag{9}
\]

Take the new initial data
\[
\begin{align*}
  u|_{t=T} &= \begin{cases} 
    u_0, & x < X \\
    u_1, & x > X
  \end{cases}, \\
  v|_{t=T} &= \begin{cases} 
    v_0, & x < X \\
    v_1, & x > X + \zeta \delta(x,X)
  \end{cases}, \tag{10}
\end{align*}
\]

for system \((1)\), where \(\zeta\) is a non-zero real.

Definition 7. The set of all points \((u_1, v_1) \in \Omega\) such that there exists an DSSW solution (called corresponding DSSW) to Cauchy problem \((1)\) for some \(\zeta\) is called second delta singular locus of initial strength \(\zeta\) for \((u_0, v_0)\). We shall write \((u_1, v_1) \in SDSL_\zeta(u_0, v_0)\) if the the corresponding DSSW is overcompressive, then it is called overcompressive second delta singular locus, and write \((u_1, v_1) \in SDSL_\zeta^*(u_0, v_0)\). □

Before the main theorem, let us give a useful lemma.

Lemma 1. Suppose that \((u_1, v_1) \in DSL(u_0, v_0)\). Then \((u_1, v_1) \in SDSL_\zeta(u_0, v_0)\), if \(\zeta > 0\).

If the corresponding DSSW contains \(m\) SD-function, and \(m\) is an odd number, then the statement holds true for every real \(\zeta\).

Additionally, \(\beta_i, i = 1, 2\), from Definition 4 for the corresponding DSSW do not depend on \(\zeta\).

Proof. We shall give the proof for a DSSW containing \(m\) SD-function \((7)\). The other case can be proved in the same way.
Inserting functions $u$ and $v$ from 7 into system 11 with initial data 10 and taking account relations 5 or 2, one gets

\begin{align*}
  f_2(u) & \approx f_2(G) \\
  g_1(u) & \approx g_1(G) \\
  g_2(u) & \approx g_2(G) \\
  f_3(u) & \approx f_3(G) + \tilde{s}(t)^{m-1} (u_1 \alpha_0^{m-1} d^- + u_0 \alpha_1^{m-1} d^+) ma_{3,m-1} \\
  & \quad + \tilde{s}(t)^m (\alpha_0^m d^- + \alpha_1^m d^+) a_{3,m} \tilde{s}(t)^m a_{3,m} \delta \\
  f_4(u) & \approx f_4(G) + \tilde{s}(t)^m (\alpha_0^{m-1} d^- + \alpha_1^{m-1} d^+) a_{4,m} \tilde{s}(t)^m a_{4,m} \delta \\
  g_3(u) & \approx g_3(G) + s(t)^{m-1} (u_1 \beta_0^{m-1} d^- + u_0 \beta_1^{m-1} d^+) mb_{3,m-1} \\
  & \quad + \tilde{s}(t)^m (\beta_0^m d^- + \beta_1^m d^+) b_{3,m} \tilde{s}(t)^m b_{3,m} \delta \\
  g_4(u) & \approx f_4(G) + \tilde{s}(t)^m (\beta_0^{m-1} d^- + \beta_1^{m-1} d^+) b_{4,m} \tilde{s}(t)^m b_{4,m} \delta 
\end{align*}

There are two possible cases. Either $\tilde{s} \neq 0$ and $a_{3,m} = b_{3,m} = 0$ (i.e. deg($f_3$) $\leq m-1$ and deg($g_3$) $\leq m-1$), or $\tilde{s} \equiv 0$. In both the cases, the procedure which follows is the same, so take $\tilde{s} \neq 0$ for definiteness. From the first equation of 11 one gets

\[
\left( f_2(u) \right)_t + \left( f_3(u) v + f_4(u) \right)_x 
\approx -c \left[ f_2(G) \right] + f_3(G) H + f_4(G) \delta 
\approx s(t)^{m-1} \tilde{s}(t) (u_1 \alpha_0^{m-1} \gamma_0 + u_0 \alpha_1^{m-1} \gamma_1) ma_{3,m-1} \delta' 
+ (f_3(u_0) \beta_0 + f_3(u_1) \beta_1) \delta' + \tilde{s}(t)^m \delta' \approx 0.
\]

One immediately gets the speed of DSSW,
\[
c = \frac{[f_3(G) H + f_4(G)]}{[f_2(G)]},
\]
and the relations
\[
\kappa_1 s(t) = \tilde{s}(t)^{m-1} \tilde{s}(t) \text{ and } \kappa_2 s(t) = \tilde{s}(t)^m,
\]
for some reals $\kappa_1$ and $\kappa_2$. Finally, one gets
\[
\kappa_1 (u_1 \alpha_0^{m-1} \gamma_0 + u_0 \alpha_1^{m-1} \gamma_1) ma_{3,m-1} + f_3(u_0) \beta_0 + f_3(u_1) \beta_1 + \kappa_2 b_{4,m} = 0. \quad (11)
\]

Inserting all these relations into the second equation, one gets
\[
\left( g_1(u) v + g_2(u) \right)_t + \left( g_3(u) v + g_4(u) \right)_x 
\approx -c [g_1(G) H + g_2(G)] + [g_3(G) H + g_4(G)] + s'(t) (g_1(u_0) \beta_0 + g_1(u_1) \beta_1) \delta 
+ s(t) (g_1(u_0) \beta_0 + g_1(u_1) \beta_1 + g_3(u_0) \beta_0 + g_3(u_1) \beta_1 
+ \kappa_1 (u_1 \alpha_0^{m-1} \gamma_0 + u_0 \alpha_1^{m-1} \gamma_1) mb_{3,m-1} + \kappa_2 b_{4,m}) \delta' \approx 0.
\]

The function $s$ must be a linear one, say $s'(t) = \sigma$, and the above functional equation gives the last two equations in \( \mathbb{R} \),
\[
-c [g_1(G) H + g_2(G)] + [g_3(G) H + g_4(G)] + \sigma (g_1(u_0) \beta_0 + g_1(u_1) \beta_1) = 0 \quad (12)
\]
and
\[
-c ((g_1(u_0) + g_3(u_0) \beta_0 + (g_1(u_1) + g_3(u_1) \beta_1) 
+ \kappa_1 (u_1 \alpha_0^{m-1} \gamma_0 + u_0 \alpha_1^{m-1} \gamma_1) mb_{3,m-1} + \kappa_2 b_{4,m} = 0. \quad (13)
\]

In the above equations, only important fact about $s$ is its derivative. Thus one can safely put $s(t) = \sigma t + \zeta$ and if the above system 11 13 has a solution, then
\((u_1, v_1) \in \text{SDSL}_\zeta(u_0, v_0)\) provided that \(\tilde{s}\) and \(\tilde{s}'\) can be recovered. This is certainly the case when \(\zeta > 0\). If \(m\) is an odd number, then \(\tilde{s} = s(t)^{1/m}\) and \(\tilde{s}' = \tilde{s}\) are always determined.

The second part of the assertion, that \(\beta_i, i = 1, 2\) are independent of \(\zeta\) is obvious from the above. \(\square\)

Remark 2. From the proof of the lemma one can see that it is actually possible for \(\zeta\) to take negative values, i.e. it is enough that \(\zeta \geq -s(T)\), where \(T\) is a time of interaction when new initial data are given.

The following assertion is crucial for the construction of weak solution (a solution in an associated sense) to \(\text{II}\) after an interaction: At an interaction point of a DSSW and some other wave one can consider the new initial value problem which contains delta function.

Suppose that the initial data are given by

\[
u(x, 0) = \begin{cases} u_0, & x < a \\ u_1, & a < x < b \\ u_2, & x > b \end{cases}
\]

and \(v(x, 0) = \begin{cases} v_0, & x < a \\ v_1, & a < x < b \\ v_2, & x > b \end{cases}\) (14) such that there exist a singular shock wave starting from the point \(x = a\) and a shock wave (or another singular shock wave) starting from the point \(x = b\), \(a < b\). They can interact if \(c_1 > c_2\), where \(c_i\) is the speed of the \(i\)-th wave, \(i = 1, 2\). For the simplicity we shall assume that \(b = 0\).

Let \((X, T)\) be the interaction point of the overcompressive singular shock wave starting at the point \(x = a\)

\[
u^1(x, t) = G^1(x - c_1 t - a) + \tilde{s}^1(t) \left( \alpha_0 d^- (x - c_1 t - a) + \alpha_1 d^+ (x - c_1 t - a) \right)
\]

\[
v^1(x, t) = H^1(x - c_1 t - a) + \tilde{s}^1(t) \left( \beta_0 D^- (x - c_1 t - a) + \beta_1 D^+ (x - c_1 t - a) \right)
\]

and the admissible (singular) shock wave

\[
u^2(x, t) = G^2(x - c_2 t) + \tilde{s}^2(t) \left( \alpha_0' d^- (x - c_2 t) + \alpha_1 d^+ (x - c_2 t) \right)
\]

\[
v^2(x, t) = H^2(x - c_2 t) + \tilde{s}^2(t) \left( \beta_0' D^- (x - c_2 t) + \beta_1' D^+ (x - c_2 t) \right)
\]

where \(G^1, G^2, H^1\) and \(H^2\) are the generalized step functions with values \((u_0, u_1), (u_1, u_2), (v_0, v_1)\) and \((v_1, v_2)\), respectively. Also, \(\alpha_i^{m_1} + (\alpha_i')^{m_1} = (\gamma_i')^{m_1} + (\gamma_i')^{m_1} = \beta_0^i + \beta_1^i = 1\), \(i = 1, 2\). Here, \(m_1 = m\) if singular part of singular shock wave is \(m\text{SD}-\text{function}\) and \(m_1 = m - 1\) in the case of \(m\text{SD}-\text{function}\). If the second wave is a shock one, then one can put \(s^2 = \tilde{s}^2 = \tilde{s}'^2 = 0\).

The speed of a singular shock wave (as well as for a shock wave) can be found using the first equation in \(\text{II}\) because of assumptions \(\text{II}\) or \(\text{I}\). For the first singular shock wave \(\text{II}\) we have

\[
(f_2(u))_t + (f_3(u)) v + f_4(u)_x \approx (f_2(G))_t + (f_3(G)) H + f_4(G)_x + (\text{const } s^1(t) \delta)_x \\
\approx (-c_1 [f_2(G)] + [f_3(G) H + f_4(G)]) \delta + \text{const } s^1(t) \delta' \approx 0,
\]
where the term $\text{const} s^1(t)$ is determined, but we shall not write the exact value since it is not needed for the assertion. Missing argument in the above expression is $x - c_1 t - a$.

Let $\Gamma_1 = \{ x = c_1 t + a \}$ and $\Gamma_2 = \{ x = c_2 t \}$. Then $[\cdot]_{\Gamma_i}$ denotes the jump at the curve $\Gamma_i$, $i = 1, 2$. Thus, one can see that the speed of that singular shock wave has the same value as in the case of shock wave,

$$c_1 = \frac{[f_3(G)H + f_4(G)]_{\Gamma_1}}{[f_2(G)]_{\Gamma_1}}.$$

Also,

$$c_2 = \frac{[f_3(G)H + f_4(G)]_{\Gamma_2}}{[f_2(G)]_{\Gamma_2}}.$$

Finally, one can see that the waves given by (15) and (16) will interact at the point $(X, T)$ if $a < 0$ and $c_1 > c_2$, where

$$T = \frac{a[f_2(G)]_{\Gamma_1}[f_4(G)]_{\Gamma_1} - [f_3(G)H + f_4(G)]_{\Gamma_1}[f_2(G)]_{\Gamma_2}}{a[f_3(G)H + f_4(G)]_{\Gamma_1}[f_2(G)]_{\Gamma_1} - [f_3(G)H + f_4(G)]_{\Gamma_2}[f_2(G)]_{\Gamma_2}}.$$

$$X = \frac{a[f_2(G)]_{\Gamma_1}[f_4(G)]_{\Gamma_1} - [f_3(G)H + f_4(G)]_{\Gamma_1}[f_2(G)]_{\Gamma_2}}{a[f_3(G)H + f_4(G)]_{\Gamma_1}[f_2(G)]_{\Gamma_1} - [f_3(G)H + f_4(G)]_{\Gamma_2}[f_2(G)]_{\Gamma_2}}.$$

Denote by $(\hat{u}(x, t), \hat{v}(x, t))$ a solution before interaction time $t = T$ consisting of waves (15) and (16).

**Remark 3.** In the case of system (2) one can easily calculate speeds of the above shocks and coordinates of the interaction point. The speeds of singular shock and entropy shock wave are

$$c_1 = \frac{u_1^2 - v_1 - u_0^2 + v_0}{u_1 - u_0} \quad \text{and} \quad c_2 = \frac{u_2^2 - v_2 - u_1^2 + v_1}{u_2 - u_1}.$$

If $c_1 > c_2$, then one gets

$$X = \frac{-ac_2}{c_2 - c_1} \quad \text{and} \quad T = \frac{a}{c_2 - c_1},$$

for the interaction point $(X, T)$.

**Theorem 1.** Let system (14) be given. Suppose that $(u_2, v_2) \in \text{SDSL}_c(u_0, v_0)$, $\zeta = (\zeta_1 + \zeta_2)/(g_1(u_0)\beta_0 + g_1(u_1)\beta_1)$, where the constants $\zeta_i$, $i = 1, 2$, are defined by

$$g_1(u^1)v^1 + g_2(u^1)|_{(t=T)} \approx \zeta_1\delta(X, T),$$

$$g_1(u^2)v^2 + g_2(u^2)|_{(t=T)} \approx \zeta_2\delta(X, T).$$

The corresponding DSSW, $(\hat{u}, \hat{v})(x, t)$ is given by

$$\hat{u}(x, t) = \begin{cases} G(x - X - c(t - T)) \\ + \hat{s}(t)\left(\alpha_0 d^-(x - X - c(t - T)) + \alpha_1 d^+(x - X - c(t - T))\right) \end{cases}$$

$$\hat{v}(x, t) = \begin{cases} H(x - X - c(t - T)) \\ + \hat{s}(t)\left(\beta_0 D^-(x - X - c(t - T)) + \beta_1 D^+(x - X - c(t - T))\right) \end{cases}$$

$$\begin{cases} \gamma_0 d^-(x - X - c(t - T)) + \gamma_1 d^+(x - X - c(t - T)) \end{cases} \quad (17)$$
for \( t > T \). By Lemma 1, \( \beta_0 \) and \( \beta_1 \) are determined independently on \( \zeta \), so the definition of DSSW makes sense.

Then there exist a solution to \( \text{(1)} \) in the association sense such that it equals \((\hat{u}, \hat{v})(x, t)\) for \( t < T - \varepsilon \), and it equals \((\check{u}, \check{v})(x, t)\) for \( t > T + \varepsilon \).

Proof. Take a constant \( t_0 \) such that singular parts of the waves \((u_2^1(x, t), v_2^1(x, t))\) and \((u_2^2(x, t), v_2^2(x, t))\) have disjoint supports (i.e. \( c_1 t - a - c_2 t > 4 \varepsilon \), for \( t < T - t_0 \varepsilon \), if one uses the construction of the S\( \delta \), mSD and m' SD-functions defined above).

Let us denote
\[
\Delta_\varepsilon = \{(x, t) : |x - X| \leq t_0 \varepsilon + \varepsilon, |t - T| \leq t_0 \varepsilon + \varepsilon\},
\]
\[
\check{\Delta}_\varepsilon = \{(x, t) : |x - X| \leq t_0 \varepsilon, |t - T| \leq t_0 \varepsilon\},
\]
\[
A_\varepsilon = \{(x, t) : |x - X| \leq t_0 \varepsilon + \varepsilon, t = T - t_0 \varepsilon - \varepsilon\},
\]
\[
B_\varepsilon = \{(x, t) : x = X + t_0 \varepsilon + \varepsilon, |t - T| \leq t_0 \varepsilon + \varepsilon\},
\]
\[
C_\varepsilon = \{(x, t) : |x - X| \leq t_0 \varepsilon + \varepsilon, t = T + t_0 \varepsilon + \varepsilon\},
\]
\[
D_\varepsilon = \{(x, t) : x = X - t_0 \varepsilon - \varepsilon, |t - T| \leq t_0 \varepsilon + \varepsilon\}.
\]

Define a cut-off function \( \xi_\varepsilon(x, t) \) which equals zero for \((x, t) \in \Delta_\varepsilon \) and 1 for \((x, t) \in \check{\Delta}_\varepsilon \). Let
\[
(u_{\text{temp}}, v_{\text{temp}})(x, t) = \begin{cases} 
(\hat{u}(x, t), \check{v}(x, t)), & t < T \\
(\check{u}(x, t), \check{v}(x, t)), & t > T.
\end{cases}
\]

We shall prove that the generalized functions \( u \) and \( v \) represented by
\[
u_\varepsilon(x, t) = u_{\text{temp}}(x, t) \xi_\varepsilon(x, t), \quad \text{and} \quad v_\varepsilon(x, t) = v_{\text{temp}}(x, t) \xi_\varepsilon(x, t), \quad x \in \mathbb{R}, \ t \geq 0 \]
solve \( \text{(1)} \) in the association sense.

Denote
\[
\begin{align*}
F(u, v) &= \begin{bmatrix}
  f_2(u) \\
g_1(u)v + g_2(u)
\end{bmatrix} \quad \text{and} \quad G(u, v) = \begin{bmatrix}
  f_3(u)v + f_4(u) \\
g_3(u)v + g_4(u)
\end{bmatrix}.
\end{align*}
\]

We have
\[
\begin{align*}
\int_{\mathbb{R}_+^2} F(u, v) \Psi_t + G(u, v) \Psi_x dx dt & = \int_{\mathbb{R}_+^2} F(u, v) \Psi_t + G(u, v) \Psi_x dx dt \\
& = \int_{\mathbb{R}_+^2} F(u, v) \Psi_t + G(u, v) \Psi_x dx dt \\
& = \int_{\mathbb{R}_+^2 \setminus \Delta_\varepsilon} F(u, v) \Psi_t + G(u, v) \Psi_x dx dt,
\end{align*}
\]
for every test function \( \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in C_0^\infty(\mathbb{R}_+^2) \).

The measure of the set \( \Delta_\varepsilon \) is \( O(\varepsilon^2) \), as \( \varepsilon \to 0 \), while
\[
\| F(u, v) \Psi_t + G(u, v) \Psi_x \|_{L^\infty(\mathbb{R}_+^2)} \leq \text{const} \varepsilon^{-1+1/m}
\]
due to the assumptions in Definition 1. Thus,
\[
\int_{\Delta_\varepsilon} F(u, v) \Psi_t + G(u, v) \Psi_x dx dt \sim \varepsilon^{1/m} \to 0, \quad \text{as} \ \varepsilon \to 0.
\]
Using the divergence theorem for the second integral one gets
\[
\int \int_{\mathbb{R}^2 \setminus \tilde{\Delta}_\varepsilon} F(u,v) \Psi_t + G(u,v) \Psi_x \, dx \, dt \\
= \int_{\partial \tilde{\Delta}_\varepsilon} F(u,v) \Psi_t + \int G(u,v) \Psi_x \, ds \\
- \int \int_{\mathbb{R}^2 \setminus \tilde{\Delta}_\varepsilon} F(u,v) \Psi + G(u,v) \Psi_x \, dx \, dt.
\]
The last integral in the above expression tend to zero as \( \varepsilon \to 0 \) since \((u,v)\) solves (1) in \( \mathbb{R}^2 \setminus \tilde{\Delta}_\varepsilon \) due to the construction. For the other integral one gets
\[
\int_{\partial \tilde{\Delta}_\varepsilon} F(u,v) \Psi_t + \int G(u,v) \Psi_x \, ds \\
= \int_{A_\varepsilon} F(u,v) \Psi \, dx - \int_{C_\varepsilon} F(u,v) \Psi \, dx + \int_{D_\varepsilon} G(u,v) \Psi \, dt - \int_{B_\varepsilon} G(u,v) \Psi \, dt.
\]
Functions \( u_\varepsilon \) and \( v_\varepsilon \) are \( L^\infty \)-bounded uniformly in \( \varepsilon \) on the sides \( B_\varepsilon \) and \( D_\varepsilon \). Since their lengths are \( O(\varepsilon) \), integrals over them tends to zero as \( \varepsilon \to 0 \).

Using the fact that \( f_2(\hat{u}) \approx 0 \) one gets
\[
\lim_{\varepsilon \to 0} F(\tilde{\hat{u}}, \tilde{\hat{v}})|_{t=T} = \begin{bmatrix} 0 \\ (\zeta_1 + \zeta_2) \delta(X,T) \end{bmatrix},
\]
as well as the construction of \( S\delta \)- and \( m'\)SD (or \( m\)SD)-functions, one gets
\[
\lim_{\varepsilon \to 0} \int_{A_\varepsilon} F(u_\varepsilon, v_\varepsilon) \, dx = \begin{bmatrix} 0 \\ \zeta_1 + \zeta_2 \end{bmatrix} \cdot \Psi(X,T).
\]
Thus, there has to be true that
\[
\lim_{\varepsilon \to 0} \int_{C_\varepsilon} F(u_\varepsilon, v_\varepsilon) \, dx = - \begin{bmatrix} 0 \\ \zeta_1 + \zeta_2 \end{bmatrix} \cdot \Psi(X,T).
\]
This implies \( f_2(\hat{u})|(X,T) \approx 0 \) and
\[
g_1(\hat{u}) \hat{v} + g_2(\hat{u})|(X,T) \approx (\zeta_1 + \zeta_2) \delta(X,T).
\]
Due to conditions (8) or (9) one immediately gets \( f_2(\hat{u})|(X,T) \approx 0 \). Put \( \zeta = (\zeta_1 + \zeta_2)/(g_1(u_0)\beta_0 + g_1(u_1)\beta_1) \). Then
\[
g_1(\hat{u}) \hat{v} + g_2(\hat{u})|_{t=T} \approx \hat{G} \hat{H} + \hat{G} + s(T)(g_1(u_0)\beta_0 + g_1(u_1)\beta_1) \delta(X)
\]
and after another restriction on \( x = X \),
\[
g_1(\hat{u}) \hat{v} + g_2(\hat{u})|(X,T) \approx (\zeta_1 + \zeta_2) \delta(X,T).
\]
This concludes the proof. \( \square \)
Remark 4. The distributional limit of the result of the interaction is given by

\[ u(x, t) = \begin{cases} 
  u_0, & x < c_1 t - a, \ t < t \\
  u_1, & c_1 t - a < x < c_2 t, \ t < T \\
  u_2, & x > c_2 t, \ t < T \\
  u_0, & x < ct + X, \ t > T \\
  u_2, & x > ct + X, \ t > T 
\end{cases} \]

\[ v(x, t) = \begin{cases} 
  v_0, & x < c_1 t - a, \ t < t \\
  v_1, & c_1 t - a < x < c_2 t, \ t < T \\
  v_2, & x > c_2 t, \ t < T \\
  v_0, & x < ct + X, \ t > T \\
  v_2, & x > ct + X, \ t > T 
\end{cases} + s_1(t)\delta_{S_1} + s_2(t)\delta_{S_2} + s(t)\delta_S, \]

where \( S_1 = \{(x, t) : x = c_1 t + a, t \in [0, T]\} \), \( S_2 = \{(x, t) : x = c_2 t, t \in [0, T]\} \) and \( S = \{(x, t) : x - X = c(t - T), t \in [T, \infty)\} \). If the second wave \((16)\) is a shock one, then \( s_2 \equiv 0 \).

The above solution is continuous in \( t \) with values in \( \mathcal{D}'(\mathbb{R}) \). This fact can be used in the approach similar to \((5)\), where the variable \( t \) is treated separately, i.e. when system \((1)\) is considered to be in evolution form.

The theorem shows that after an interaction of a singular shock with some shock or another singular shock the problem reduces to solving system \((1)\) with the new initial data \((10)\).

Remark 5. (i) The solution to the interaction problem from Theorem \((1)\) is always associated with a lower association rate than the solution of the original Riemann problem. For specific system it seems possible to make more sophisticated construction in order to improve the rate. (ii) It appears that \( d_1^\pm \) are unavoidable correction factors even their distributional limit equals zero.

The conditions \((8)\) and \((9)\) ensures that the new initial data at intersection point do not depend on \( m_{SD} \) or \( m'_{SD} \)-functions in the solution. We have used them because the real nature of \( m_{SD} \) and \( m'_{SD} \)-functions is not so clear yet.

The above theorem will be used in the rest of the paper for investigation of interactions between singular shock waves and other types of waves in the special case of system \((2)\).

4. Applications

Consider now system \((2)\) which a special case to \((1)\). The authors of \((6)\) defined and proved existence of singular shock wave solutions for some Riemann problems of this system.

In the present paper, we will investigate interactions of such solutions with the other solutions to Riemann problem for \((2)\). In order to familiarize a reader with the presented results, let us give some basic remarks about such solutions.

For a given Riemann data \((u_0, v_0), (v_0, v_1)\), there are three basic solution types:

(a) Shock waves

\[
 u(x, y) = \begin{cases} 
  u_0, & x < ct \\
  u_1, & x > ct 
\end{cases} \quad v(x, y) = \begin{cases} 
  v_0, & x < ct \\
  v_1, & x > ct 
\end{cases}
\]
where \( c = [u^2 - v]/u \) and \((u_1, v_1)\) lies in an admissible part of Hugoniot locus of the point \((u_0, v_0)\).

(b) **Centered rarefaction waves**

\[
\begin{align*}
u(x, t) &= \begin{cases} 
 u_0, x < (u_0 - 1)t \\
 x/t + 1, (u_0 - 1)t \leq x \leq (u_1 - 1)t \\
 u_1, x > (u_1 - 1)t 
\end{cases} \\
v(x, t) &= \begin{cases} 
 v_0, x < (u_0 - 1)t \\
 (x/t)^2/2 + 2x/t + C_1, (u_0 - 1)t \leq x \leq (u_1 - 1)t \\
 v_1, x > (u_1 - 1)t 
\end{cases}
\end{align*}
\]

(21)

(1-rarefaction wave), where \( C_1 = v_0 - u_0^2/2 - u_0 - 1/2 \), when \((u_1, v_1)\) lies in an 1-rarefaction curve starting at the point \((u_0, v_0)\).

Or

\[
\begin{align*}
u(x, t) &= \begin{cases} 
 u_0, x < (u_0 + 1)t \\
 x/t - 1, (u_0 + 1)t \leq x \leq (u_1 + 1)t \\
 u_1, x > (u_1 + 1)t 
\end{cases} \\
v(x, t) &= \begin{cases} 
 v_0, x < (u_0 + 1)t \\
 (x/t)^2/2 - 2x/t + C_2, (u_0 + 1)t \leq x \leq (u_1 + 1)t \\
 v_1, x > (u_1 + 1)t 
\end{cases}
\end{align*}
\]

(22)

(2-rarefaction wave), where \( C_2 = v_0 - u_0^2/2 + u_0 - 1/2 \), when \((u_1, v_1)\) lies in an 2-rarefaction curve starting at the point \((u_0, v_0)\).

(c) **Singular shock waves** (see Definition 7) of 3SD-type,

\[
\begin{align*}
u(x, y) &= \begin{cases} 
 u_0, x < ct \\
 u_1, x > ct 
\end{cases} + s(t)(\alpha_0 d_x^- (x - ct) + \alpha_1 d_x^+ (x - ct)) \\
v(x, y) &= \begin{cases} 
 v_0, x < ct \\
 v_1, x > ct 
\end{cases} + s(t)(\beta_0 D_x^- (x - ct) + \beta_1 D_x^+ (x - ct)),
\end{align*}
\]

(23)

where \( c = [u^2 - v]/u \), and all other terms are as in that definition. That means

\[ D_x \approx \delta, \ (d_x^\pm)^i \approx 0, \ i = 1, 3, \ (d_x^\pm)^2 \approx \delta, \]

(24)

while \((u_1, v_1)\) lies in a region denoted by \( Q_7 \) in (7) of the point \((u_0, v_0)\) (see Figure 1).

For an arbitrary Riemann problem to (2) one can construct a solutions by the means of these waves or their combinations (3).

While interactions of the first two types can be handled in a usual way, interactions involving singular shock waves are quite different and far more interesting, so they become a topic of this paper.

The procedure for the singular shock wave interactions can be also used for systems (11). But a complete after-interaction solution highly depends on a particular system. That is the reason why we treat system (2) only.

In order to simplify notation, we shall substitute the point \((X, T)\) in (10) by \((0, 0)\) and then solve the Cauchy problem (2110). There are no multiplication of \( v \) with \( u \) in system (2), so in the sequel it will be enough to take \( D^- = D^+, \alpha_0(t) := \alpha_0 \hat{s}(t), \alpha_1(t) := \alpha_1 \hat{s}(t) \) and \( \beta(t) := s(t) \), i.e. to
look for a solution of the form
\[ u = G(x - ct) + (\alpha_0(t)d^- (x - ct) + \alpha_1(t)d^+ (x - ct)) \]
\[ v = H(x - ct) + \beta(t)D(x - ct), \]
where \( G \) and \( H \) are generalized step functions, while \( d \) is 3’SD- and \( D \) is Sδ-function and \( c \in \mathbb{R} \).

Let us determine SDSL of (2) for some \((u_0, v_0) \in \mathbb{R}^2\).

Substitution of (25) into the first equation of the system gives
\[ c = \frac{u_1^2 - v_1 - u_0^2 + v_0}{u_1 - u_0} \]
\[ \alpha_0^2(t) + \alpha_1^2(t) = \beta(t), \]
where \( c \) is the speed of the wave. After neglecting all terms converging to zero as \( \varepsilon \to 0 \), the second equation becomes
\[ \partial_t H_c(x - ct) + \beta'(t)\delta(x - ct) - c\beta(t)\delta'(x - ct) + \partial_x \left( \frac{1}{3} G^3_c - G \right) \]
\[ + (u_1\alpha_0^2(t) + u_0\alpha_1^2(t))\delta'(x - ct) = 0. \]
Thus, the following relations has to hold.
\[ \beta'(t) = c(v_1 - v_0) - \left( \frac{1}{3} u_0^3 - u_0 - \frac{1}{3} u_1^3 + u_1 \right) =: k, \]
i.e.
\[ \beta(t) = kt + \zeta, \text{ since } \beta(0) = \zeta \]
and
\[ u_1\alpha_0^2(t) + u_0\alpha_1^2(t) = c\beta(t). \]

Like in [6] one can see that the overcompressibility means
\[ u_0 - 1 \geq c \geq u_1 + 1, \]
i.e., \( v_1 \) lies between the curves
\[ D = \{ (u, v) : v = v_0 + u^2 + u - u_0 u - u_0 \} \]
\[ E = \{ (u, v) : v = v_0 - u + u_0 u - u_0^2 + u_0 \}, \]
and \( u_0 - u_1 \geq 2. \)

Denote by \( J_1 \) the union of the parts of admissible Hugoniot locus
\[ S_1 = \left\{ (u, v) : v - v_0 = (u - u_0) \left( \frac{u_0 + u}{2} + \sqrt{1 - \frac{(u_0 - u)^2}{12}} \right) \right\}, \]
and
\[ S_2 = \left\{ (u_1, v_1) : v - v_0 = (u - u_0) \left( \frac{u_0 + u}{2} - \sqrt{1 - \frac{(u_0 - u)^2}{12}} \right) \right\}, \]
for \( u \in [u_0 - \sqrt{12}, u_0 - 3] \). Note that \( S_1 \) is not an \( i \)th shock curve but only a label.

The points between the curves \( D \) and \( E \), and on the left-hand side of \( J_1 \) defines the area denoted by \( Q_7 \) in [6]. Here, this area is called delta singular locus.

One can easily check that system (26,28) has a solution if and only if \( \beta(t) > 0 \). Depending on \( k \), defined in (27), there are three possibilities for a resulting wave:

(i) If \( k > 0 \), then \( \beta'(t) > 0 \) and \((u_1, v_1) \in Q_7\). The resulting singular shock has the same properties as before, i.e. its strength increases with the time.
(ii) If \( k = 0 \), then \( \tilde{\beta} \equiv \text{const} = \zeta > 0 \) and the corresponding part of a singular overcompressive locus is \( J_1 \). The result of the interaction is a new kind of singular shock wave, its strength is a constant with respect to the time.

(iii) If \( k < 0 \) (this means that the point \( (u_1, v_1) \) is on the left-hand side of \( J_1 \)), then the resulting singular shock wave has much more differences from the usual one (with an increasing strength). Its initial strength equals \( \zeta, \beta(0) = \zeta > 0 \), but linearly decreases in time. At some point \( T_0 \) the strength of the singular shock equals zero and the singular shock wave does not exist after that. In the rest of the paper we shall see some cases when this happens. The new initial data for time \( t = T_0 \) are the Riemann ones, and the solution after that time can be find in the usual way, by using the results in \([6] \).

All the above facts are collected in the following theorem.

**Theorem 2.** The SDSL \( \zeta, \zeta > 0 \), for \((2,10)\) is the area bounded by the curves \( D, E, S_2 \setminus J_1 \) and \( S_1 \setminus J_1 \). (The area \( Q_7 \) is a subset of this one, as known from Lemma \([4] \).) The overcompressive SDSL \( \zeta, \zeta > 0 \), is a part of the SDSL bounded by the curves \( D \) and \( E \) such that \( u_1 \leq u_0 - 2 \).

### 4.1. Interaction of a singular shock and an admissible shock wave.

Suppose that a singular shock wave with a speed \( c_1 \) and a left- and right-hand values \( U_0 = (u_0, v_0) \) and \( U_1 = (u_1, v_1) \), respectively, interact with an admissible shock wave with a speed \( c_2 < c_1 \) having left-hand and right-hand values \( U_1 = (u_1, v_1) \) and \( U_2 = (u_2, v_2) \), respectively, at a point \((X,T)\).

**Lemma 2.** If the above singular shock and shock wave are admissible, \((u_2, v_2)\) lies between the lines \( D \) and \( E \). Thus, the solution after the interaction is a single overcompressive singular shock wave.
Proof. Since $u_0 \geq u_1 + 3$ and $u_1 > u_2$ (because of the admissibility conditions for singular and shock wave), we have $u_0 > u_2 + 3$. The point $(u_2, v_2)$ lies on the curve $S_1$ or $S_2$ with the origin at the point $(u_1, v_1)$. Thus

$$v_2 = v_1 + (u_2 - u_1)\left(\frac{u_1 + u_2}{2} \pm \sqrt{1 - \frac{(u_1 - u_2)^2}{12}}\right).$$

The point $(u_1, v_1)$ lies in the area denoted by $Q_7$, thus below or at the curve $D$ with the origin at $(u_0, v_0)$. Therefore

$$v_1 \leq v_0 + u_1^2 + u_1 - u_0 u_1 - u_0.$$

Let the point $(u_0, v_0)$ be the origin. The point $(u_2, v_2)$ will be below the curve $D$ if

$$v_0 + u_1^2 + u_1 - u_0 u_1 - u_0 + (u_2 - u_1)\left(\frac{u_1 + u_2}{2} \pm \sqrt{1 - \frac{(u_1 - u_2)^2}{12}}\right)$$

$$\leq v_0 + u_1^2 + u_2 - u_0 u_2 - u_0.$$

Non-positivity of $u_1 - u_2$ gives

$$\pm \sqrt{1 - \frac{(u_1 - u_2)^2}{12}} \leq \frac{1}{2}(u_0 - u_1) + \frac{1}{2}(u_0 - u_2) - 1.$$

The left-hand side of the above inequality is less than 2, while the right-hand side is greater than 2. Thus, the point $(u_2, v_2)$ really lies below the curve $D$.

In the same way one can prove that the point $(u_2, v_2)$ lies above the curve $E$. \qed

Remark 6. In the same manner as above, one can prove that the situation is the same when singular shock and shock wave change sides. That is, when an admissible singular shock wave interacts with an admissible shock wave from the right-hand side, then the solution is again a single admissible singular shock wave.

4.2. Double singular shock wave interaction. Suppose that an admissible singular shock wave with a speed $c_1$ and left- and right-hand side values $U_0 = (u_0, v_0)$ and $U_1 = (u_1, v_1)$, respectively, interacts with an another singular shock wave with a speed $c_2 < c_1$ and left-hand (right-hand) side values $U_1 = (u_1, v_1)$ ($U_2 = (u_2, v_2)$) at the point $(X, T)$. Since the conditions for the existence of singular shock waves include $u_0 - u_1 \geq 3$ and $u_1 - u_2 \geq 3$, then $u_0 - u_2 \geq 6$, i.e. the point $(u_2, v_2)$ is on the left-hand side of the line $u = u_0 - \sqrt{12}$. Concerning the position of the point $(u_2, v_2)$ in the plane of wave regions with the origin at $(u_0, v_0)$ there are three possibilities:

(i) The point $(u_2, v_2)$ is between or at the curves $D$ and $E$. The result of the interaction is a single singular shock wave (with increasing strength).

(ii) The point $(u_2, v_2)$ is above the curve $D$. The result of the interaction is an 1-rarefaction wave followed with a singular shock wave.

(iii) The point $(u_2, v_2)$ is below the curve $E$. The result of the interaction is a singular shock wave followed by a 2-rarefaction wave.

SDSL’s always have increasing strength in these three cases.

5. Intersection of a singular shock wave and a rarefaction wave

The last possibility of singular shock wave interactions is between a singular shock wave and a rarefaction wave. That possibility is omitted from a considerations of the general case due to a richness of possible behaviors. Nevertheless, the most
of specific Riemann problems can be treated similarly as system (2) was here, at least up to some point.

For a given point \((u_0, v_0)\), the rarefaction curves are given by (see [6])

\[
R_1 = \{(u, v) : v = v_0 - \frac{1}{2}u_0^2 + \frac{1}{2}u^2 + u - u_0\},
\]

\[
R_2 = \{(u, v) : v = v_0 - \frac{1}{2}u_0^2 + \frac{1}{2}u^2 - u + u_0\}.
\]

Suppose that a singular shock wave with left- and right-hand side values \(U_0 = (u_0, v_0)\) and \(U_1 = (u_1, v_1)\), from the left-hand side interacts with a rarefaction wave at some point \((X,T)\). If the rarefaction wave is approximated with a number of small amplitude (non-admissible) shock waves like in wave front tracking algorithm (see [1] for example), intuition given in Theorem 1, such that the first task should be to look at the singular shock wave and the interaction of singular shock and non-admissible shock wave. It is possible to extend Theorem 1 for such a case, providing that a non-admissible shock wave has amplitude small enough (of the rate \(\varepsilon^2\), say). Denote by \((u_r, v_r)\) the end-point in a rarefaction curve. Let us note that the starting point of the curve \((u_1, v_1)\) is in \(Q_7\).

In what follows, we shall abuse the notation and denote by \((u_1, v_1)\) the left-hand side of an approximated non-admissible shock wave. Denote by \((u_1, v_1) \in Q_7\) the left-hand side and by \((u_2, v_2)\) the right-hand side value of a part from the rarefaction curve. If \((u_2, v_2) \in Q_7\), then the result of the interaction is a single singular shock wave, with the left-hand side value equals \((u_0, v_0)\). The speed depends on initial values as in [20]. So, one can continue the procedure taking approximate points from the rarefaction curve as the right-hand values of the non-admissible shock wave until it reaches the border of \(Q_7\).

After looking at the above discrete model we are back in a real situation.

Let us denote by \((c(t), t)\), \(t\) belonging to some interval, a path of the resulting singular shock wave through \(Q_7\). It is possible to explicitly calculate the above path. For example if a singular shock wave interacts with a centered 1-rarefaction waves, substituting

\[
\begin{align*}
    u(x, t) &= \begin{cases} 
        u_1, & x < c(t) \\
        \phi_1(x/t), & x > c(t)
    \end{cases} + \alpha_0(t)\phi_{2}^-(x-c(t)) + \alpha_1(t)\phi_{2}^+(x-c(t)) \\
    v(x, t) &= \begin{cases} 
        v_1, & x < c(t) \\
        \phi_1(x/t), & x > c(t)
    \end{cases} + \beta(t)\phi_{2}^-(x-c(t))
\end{align*}
\]

in system (2), one obtains

\[
\begin{align*}
    \dot{\alpha}_0^2(t) + \dot{\alpha}_1^2(t) &= \tilde{\beta}(t) \\
    c(t) &= \left(t(1 - 2(u_1 - v_0 + v_1 + u_0^2 - u_1^2))ight) + T(1 - 2(u_0 - v_1 - u_0u_1 + u_0^2 - u_1^2))/(2(u_0 - 1)) \\
    \tilde{\beta}'(t) &= c'(t)\left(\frac{1}{2} \left(\frac{c(t)}{t} + 1\right) + \frac{c(t)}{t} + 1\right) + v_1 - \frac{1}{2}u_1^2 - u_1 - v_0 \right) \\
    &- \left(\frac{1}{3} \frac{c(t)}{t} + 1\right)^3 - \left(\frac{c(t)}{t} + 1\right) - \frac{1}{3}u_0^3 + u_0)
\end{align*}
\]
where the initial data for $\beta$ at the point $t = T$ is the initial strength of the singular shock wave $\beta(T)$. The above calculations means that a form of the resulting singular shock curve and its strength are uniquely determined through the area $Q_7$. If $(u_\tau, v_\tau) \in Q_7$, then the analysis is finished. Suppose that this is not true. The main problem is to analyze situations when rarefaction curve intersects the boundary of $Q_7$. Let us try to find out what is happening by using a discrete model.

Thus, the first real problem is to find a form of solution when the points from the rarefaction curve satisfy: $(u_1, v_1) \in Q_7$ and $(u_2, v_2) \notin Q_7$.

Denote by $D$ and $G$ the intersection points of the curve $J_1$ (or the line $u = u_0 - 3$) with the curves $E$ and $D$, respectively (see Figure 2).

5.1. **The first critical case.** Denote by $J$ the 1-rarefaction curve starting from the point $\tilde{G}$ and by $J_2$ the 2-rarefaction curve starting from the point $\tilde{D}$. The region where $(u_2, v_2)$ can lie consist of five subregions:

(i) **The rarefaction curve which starts at $(u_1, v_1)$ intersects the curve $D$ out of point $\tilde{G}$.** The point $(u_2, v_2)$ lies in the region above the curve $D$ and left of the line $u = u_0 - 3$. The final result of the interaction is a 1-rarefaction wave $(R_1)$ followed by a singular shock wave with increasing strength.

(ii) **The rarefaction curve which starts at $(u_1, v_1)$ intersects the curve $E$ out of point $\tilde{D}$.** The point $(u_2, v_2)$ lies in the region below the curve $E$ and on the left-hand side of the line $u = u_0 - 3$. The result of the interaction is a singular shock wave with increasing strength followed by a 2-rarefaction wave $(R_2)$.

(iii) **The rarefaction curve which starts at $(u_1, v_1)$ intersects the curve $J_1$ out of points $\tilde{D}$ and $\tilde{G}$.** Since an amplitude of a non-admissible shock wave can be as small as necessary, one can assume that the point $(u_2, v_2)$ lies in the second delta singular locus and the resulting singular shock wave has a negative strength. The strength-function $\beta(t) = \zeta + k(t - T_0)$ of the resulting singular shock is decreasing, so, there could exists a point $T_1 = T_0 - \frac{\zeta}{k}$ such that $\beta(T_1) = \alpha_0 = \alpha_1 = 0$. Let $X_1 = cT_1 + (X - T)$, where $c$ is the speed of the resulting singular shock wave (space coordinate of the point where strength reaches zero). Therefore, in the time $t = T_1$, we have to solve new Riemann problem

$$u|_{t=T_1} = \begin{cases} u_0, & x < X_1 \\ u_2, & x > X_1 \end{cases}, \quad v|_{t=T_1} = \begin{cases} v_0, & x < X_1 \\ v_2, & x > X_1 \end{cases}.$$

This problem has a unique entropy solution consists from two shock waves, since the point $(u_2, v_2)$ is between the curves $S_1$ and $S_2$, with respect to the origin at the point $(u_0, v_0)$. This means that the singular shock wave decouples into a pair of admissible shock waves. If $u_\tau \leq u_0 - 2$, this pair of the shock waves are the final solution. The case when $u_\tau < u_0 - 2$ belongs to the following subsection, i.e. the second critical case.

(iv) **The rarefaction curve $R_j$, $j = 1$ or $2$, which starts at $(u_1, v_1)$ intersects the curve $J_1$ in the point $\tilde{G}$.** We can take $\tilde{G} = (u_2, v_2)$ for convenience. The set of such points $(u_1, v_1)$ lies on the inverse rarefaction curve, which starts from the right-hand side values, i.e.

$$\tilde{R}_1 = \{(u, v) : v = v_0 + (u_0^2 - u^2)/2 + u_0 - u\}$$

and

$$\tilde{R}_2 = \{(u, v) : v = v_0 + (u_0^2 - u^2)/2 - u_0 + u\}$$
(the same explanation will be used in Remark 7 below). Straightforward calculation shows that this curve lies in the region $Q_7$, thus this situation is possible, as one can see using the inverse rarefaction curves $\hat{R}_1$ and $\hat{R}_2$ given above.

If $j = 1$, then the point $(u_2, v_2)$ belongs to $J$ and the solution after the interaction is an $R_1$-wave followed by a singular shock wave with a constant strength.

If $j = 2$, then the point $(u_2, v_2)$ lies in the area bellow the curve $J$. This can be verified by direct calculation, taking into account that the amplitude of a non-admissible shock is small enough, $u_2 < u_0 - 2$. The solution after the interaction is an admissible singular shock wave with a decreasing strength. Further explanations of a such singular shock wave is given in the following subsection.

(v) The rarefaction curve $R_j$ which starts at $(u_1, v_1)$ intersects the curve $J_1$ in the point $\hat{D}$. Again, let $\hat{D} = (u_2, v_2)$. Simple calculation, as in the case (iv), shows that this situation is also possible since the inverse rarefaction curves starting from $\hat{D}$ stay in $Q_7$. If $j = 2$, the point $(u_2, v_2)$ belongs to $J_2$, and then the solution after the interaction is singular shock wave with a constant strength followed by an $R_2$-wave.

If $j = 1$, then use of the same arguments as above gives that the point $(u_2, v_2)$ lies in the area above the curve $J_2$ and the result of the interaction is an admissible singular shock wave with a decreasing strength. Again, one can see the following subsection for the further analysis.

### 5.2. The second critical case

Now we are dealing with the problem when the rarefaction wave after passing through $J_1$, after passes trough the curves $D$ or $E$.

One can see that this is the continuation of the cases (iii)-(v) from the previous part.

(a) Denote by $\hat{D}$ the area above the curve $D$, bellow $S_1$ and on the left-hand side of the line $u = u_0 - 2$. Also denote by $\hat{D}$ the area above the curve $E$, bellow $S_1$ and on the right-hand side of the line $u = u_0 - 2$. If $(u_2, v_2)$ lies in one of these regions, the solution is combination of a rarefaction wave $R_1$ and an overcompressive singular shock wave with a decreasing or constant strength.

(b) Denote by $\hat{E}$ the area bellow the curve $E$, above $S_2$ and on the left-hand side of the line $u = u_0 - 2$. Also denote by $\hat{E}$ the area bellow the curve $D$, above $S_2$ and on the right-hand side of the line $u = u_0 - 2$. If $(u_2, v_2)$ lies in one of these regions, the solution is then a combination of an overcompressive singular shock wave with a decreasing or constant strength and a rarefaction wave $R_2$. Denote by $D_0$ the area bounded by the curves $D$, $E$, $S_1$ and $S_2$ such that $u < u_0 - 2$ in $D_0$.

One can see that a rarefaction curve cannot enter into $D_0$ since it has to pass trough the intersection point $(u_0 - 2, -2u_0 + v_0 + 2)$ of $D$ and $E$, but

\[2u - 2u_0 - u_0 + u^2/2 + u_0^2/2 + 2 > 0 \text{ (ie. } R_1 \text{ is above the curve } E)\]

\[2u_0 - 2u + u_0 - u^2/2 - u_0^2/2 - 2 < 0 \text{ (ie. } R_2 \text{ is bellow the curve } D)\]

Therefore, a rarefaction curve which passes trough the point $D \cap E$ goes either into $\hat{D}$ or $\hat{E}$, and these cases are analysed above.

Thus, we have described all important points of the interactions between singular shock and rarefaction waves. When a result of a single interaction is known, the question about further singular shock path could be answered by a successive use of the above procedures.
Remark 7. One can use the similar analysis of all possible cases when a rarefaction wave which interacts with a singular shock wave is on the left-hand side of it. Instead of direct rarefaction and singular shock curves, the inverse ones should be used, i.e. \((u_2, v_2)\) is a starting point and one is able to calculate \(v_0\) from formulas of \(E, D, S_1, S_2, R_1\) and \(R_2\).

Remark 8. In the contrast with the case in [9], where interaction can generate some “strange” solution containing unbounded \(L^1_{loc}\) function, in the presented system one can find only bounded functions and singular shock waves as a result on an interaction.

For a system (1) with \(g_1 \not\equiv \text{const}\), or \(g_2 \not\equiv 0\), interaction of singular shock and rarefaction waves cannot be treated as easy as here.

Thus, we have proved the following assertion for the interaction in the case of system (2).

**Theorem 3.** Suppose that a singular shock wave interacts with a rarefaction wave at the time \(T\). For some time period \(T < t < T_1\) the solution is represented by a singular shock wave supported by a uniquely defined curve (not a line) followed by a new rarefaction wave. Depending on the right-hand value of the primary rarefaction wave, one has the following possible cases for a solution after \(t > T_1\).

(a) Single singular shock wave (supported by a line) with an increasing strength.
(b) 1-rarefaction wave followed by singular shock wave with an increasing strength.
(c) Singular shock wave with an increasing strength followed by 2-rarefaction wave.
(d) Singular shock wave with a decreasing strength prolonged by either a single singular shock wave with an increasing strength, or a pair of admissible shock waves.

(e) 1-rarefaction wave followed by singular shock wave with a constant strength.

(f) Singular shock wave with a constant strength followed by 2-rarefaction wave.

(g) Singular shock wave with a decreasing strength prolonged by either 1-rarefaction wave followed by singular shock wave with decreasing or constant strength, or singular shock wave with decreasing or constant strength followed by 2-rarefaction wave.

“Prolonged” is the state after strength of singular shock wave becomes zero. Such wave can also stop with non-zero strength, and then there is obviously no prolongation described above.

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Department of Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia

E-mail address: markonne@uns.ns.ac.yu