CANONICAL MODELS OF ARITHMETIC
(1; ∞)-CURVES

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Abstract. In 1983 Takeuchi showed that up to conjugation there are exactly 4 arithmetic subgroups of $\text{PSL}_2(\mathbb{R})$ with signature $(1; \infty)$. Shinichi Mochizuki gave a purely geometric characterization of the corresponding arithmetic $(1; \infty)$-curves, which also arise naturally in the context of his recent work on inter-universal Teichmüller theory.

Using Belyi maps, we explicitly determine the canonical models of these curves. We also study their arithmetic properties and modular interpretations.

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Let $(E, O)$ be a pointed curve of genus 1 over $\mathbb{C}$, and let $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. We can then construct the universal ramified cover $U \to E$ that ramifies over $O$ with index $e$. As a complex analytic space $U$ is isomorphic to the upper half plane $\mathcal{H}$, and the covering map $\mathcal{H} \to E$ can be described as a quotient by a subgroup $\Gamma$ of $\text{PSL}_2(\mathbb{R})$. As a subgroup of $\text{PSL}_2(\mathbb{R})$, the group $\Gamma$ is well-defined up to conjugacy. We can ask when the pair $(E, O)$ and the ramification index $e$ determine an arithmetic group $\Gamma$. This is the same as asking which arithmetic subgroups $\Gamma$ of $\text{PSL}_2(\mathbb{R})$ have signature $(1; e)$, or alternatively a presentation of the form

\begin{equation}
(0.1.1) \quad \Gamma = \langle \alpha, \beta \mid [\alpha, \beta]^e = 1 \rangle.
\end{equation}
This question was answered by Takeuchi [19], who provided the finite list of arithmetic groups of signature \((1; e)\). When \(e\) is finite, then the group \(\Gamma\) has no cusps and comes from a non-trivial quaternion algebra. The resulting quotient curves

\[(0.1.2) \quad X(\Gamma) = \Gamma \backslash \mathcal{H}\]

were studied in [15], where their canonical models in the sense of Shimura [12] were determined.

The present article considers the curves in Takeuchi’s list that were not treated in [15], namely those for which \(e = \infty\). This is the case for 4 conjugacy classes of groups \(\Gamma\). These groups have a single conjugacy class of cusps, and are commensurable with the modular group \(\Gamma(1) = \text{PSL}_2(\mathbb{Z})\). This means that classical methods are available to compute canonical models of the corresponding quotient curves \(X(\Gamma)\).

It should be mentioned that determining these 4 curves did not require any fundamentally new theory or concepts to be developed. Still, our computations were rather involved in practice, and there seems to be no way to determine equations for these curves by a short theoretical detour. We have used the computer algebra system MAGMA [1] to facilitate these calculations. All the code needed to obtain the result in this article can be found online at [16]. Along the way, we find some interesting phenomena: an explicit Shimura curve whose Atkin-Lehner quotients has smaller canonical field of moduli than the original curve (case III below), and a modular description of the cover corresponding to the commutator subgroup of \(\text{SL}_2(\mathbb{Z})\) (case IV below).

However, the main motivation for this computation is not simply that these 4 curves “are there”, but that they play a role in the work of Shinichi Mochizuki. In the articles [9, 10] he showed that the property of these curves being arithmetic can be rephrased purely geometrically, in terms of a certain category \(\text{Loc}(X)\) introduced in [10, Definition 2.1] not having a terminal object (or core). In Mochizuki’s later inter-universal Teichmüller theory these curves are exceptional, in the sense that the theory developed does not apply to these curves and the corresponding categories, requiring them to be excluded from consideration. For instances of this happening in the literature on this theory, we refer to Fesenko’s expository work [5, Footnote 27] as well as the original work of Mochizuki [11, Definition 3.1(d)].

Because of this, the question of determining these curves was raised by Fesenko at the workshop on inter-universal Teichmüller theory at the University of Oxford in December 2015. This question then went unanswered until the models in this article were found.
We will use classical methods, as well as the more recent methods by Shimura \cite{Shimura}, to determine the canonical models of the 4 arithmetic \((1; \infty)\)-curves in Takeuchi’s list. Moreover, we describe the correspondence of these curves with the classical modular curve \(X(1)\), and we give the arithmetic and modular properties of these canonical models.

A main tool that we will use is that of Belyĭ maps. For a survey that shows the ubiquity of these maps, we refer to \cite{Zagier}; they also play a significant role in \cite{Voight}. Expressed briefly, given an arithmetic group \(P\Gamma\) commensurable with a triangle group \(P\Delta\), the inclusion of groups \(P\Gamma \cap P\Delta \subset P\Delta\) gives rise to a Belyĭ map \(X(P\Gamma \cap P\Delta) \to X(P\Delta)\), which is a cover of a projective line ramified above at most three points. The combinatorics of this inclusion yield equations for \(X(P\Gamma)\) via algorithms such as those in \cite{Oda}, and the correspondence with the curve \(X(P\Delta)\) can be used to obtain significant arithmetic information regarding the curve \(X(P\Gamma)\), for instance properties of its canonical model. These considerations apply in particular to the classical case \(P\Delta = P\Gamma(1) = PSL_2(\mathbb{Z})\) in which we find ourselves.

Section 1 introduces the relevant groups and their associated orders. Section 2 gives the resulting Belyĭ maps, which are used in Section 3 to determine the canonical models and their arithmetic properties. Finally, the modular interpretation of these models is described in Section 4.

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1. Uniformizations and orders

In this section, and in fact throughout the article, our exposition will be rather condensed; a more comprehensive exposition of these topics is in \cite{Tak, Tak2, Tak3}.

In \cite[Proposition 3.1, Theorem 3.4]{Tak}, Takeuchi describes his \((1; \infty)\)-groups as follows.

**Lemma 1.1.1.** Let \(P\Gamma \subset PSL_2(\mathbb{R})\) be an arithmetic \((1; \infty)\)-group. Then the inverse image \(\Gamma\) of \(P\Gamma\) in \(SL_2(\mathbb{R})\) is generated by two elements \(\alpha\) and \(\beta\) whose commutator is parabolic and for which the traces of \(\alpha\), \(\beta\) and \(\alpha\beta\) are given in Table 1. Conversely, these traces determine \(\Gamma\) up to \(SL_2(\mathbb{R})\)-conjugacy.
As in [13, §1.2], we construct $\alpha$ and $\beta$ by taking
\begin{equation}
\alpha = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \beta = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.
\end{equation}
Here $\lambda$ is the largest real solution of $\lambda^{-1} + \lambda = \text{Tr}(\alpha)$. Having determined $\lambda$, we recover $a$ and $d$ by from $\text{Tr}(\beta)$ and $\text{Tr}(\alpha\beta)$ by solving a system of linear equations. Finally, we choose $b$ to be the positive real solution of $ad - b^2 = 1$.

The matrices $\alpha$, $\beta$ thus obtained are far from having entries in $\mathbb{Z}$. However, in the proof of [19, Theorem 3.4] (and see also his work [18]) Takeuchi shows the following more precise statement.

**Lemma 1.1.3.** Let $S$ be the set $\{1, \alpha^2, \beta^2, \alpha^2\beta^2\}$, and let $\Gamma^{(2)} = \langle \gamma^2 : \gamma \in \Gamma \rangle$.

1. The group $\Gamma^{(2)}$ has signature $(1; \infty^4)$, and the image $\text{P}\Gamma^{(2)}$ of $\Gamma^{(2)}$ in $\text{PSL}_2(\mathbb{R})$ is normal in the image $\text{P}\Gamma \subset \text{PSL}_2(\mathbb{R})$. The quotient group $\text{P}\Gamma / \text{P}\Gamma^{(2)}$ is a Klein Vierergruppe generated by $\alpha$ and $\beta$.
2. The $\mathbb{Q}$-vector space generated by $S$ is a quaternion algebra over the rationals.
3. The $\mathbb{Z}$-module generated by $\Gamma^{(2)}$ is an order $\mathcal{O}$ of $B$, and the group $\mathcal{O}^1$ of elements of $\mathcal{O}$ with reduced norm 1 contains $\Gamma^{(2)}$.

**Remark 1.1.4.** In [19, Proposition 3.1] Takeuchi also gives a list of generators of $\Gamma^{(2)}$, so that we can explicitly construct the order $\mathcal{O}$ in part (ii) of the Lemma.

In our cases $\Gamma$ has a cusp, hence so does $\Gamma^{(2)}$. Therefore $\mathcal{O}$ is in fact conjugate to an order in the trivial quaternion algebra $M_2(\mathbb{Q})$ over $\mathbb{Q}$. This simplifies our considerations, since the original matrices $\alpha$ and $\beta$ will have entries in an extension of $\mathbb{Q}$; in case II this extension is even a biquadratic number field. To find an order in $M_2(\mathbb{Q})$ to which $\mathcal{O}$ is conjugate, we can let $\Gamma$ act on $S$ by left multiplication and identify the resulting associative algebra with $M_2(\mathbb{Q})$ by using the algorithms in [20].
We have slightly tweaked this approach; instead of taking $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$, we take $\mathcal{O}' = \mathbb{Z}[\Gamma']$, where $\Gamma'$ is the largest subgroup between $\Gamma^{(2)}$ and $\Gamma$ that generates some order in the algebra $B$.

**Proposition 1.1.5.** In the cases I-IV, the largest groups $\Gamma'$ inbetween $\Gamma^{(2)}$ and $\Gamma$ that generate an order in the matrix algebra $B$ in Lemma 1.1.3 are given in Table 2.

| Case | Largest $\Gamma'$ generating an order |
|------|---------------------------------------|
| I    | $\langle \Gamma^{(2)}, \alpha\beta \rangle$ |
| II   | $\Gamma^{(2)}$ |
| III  | $\langle \Gamma^{(2)}, \alpha\beta \rangle$ |
| IV   | $\Gamma$ |

Table 2. Groups used to generate orders.

**Proof.** The quotient $\mathbb{P}\Gamma / \mathbb{P}\Gamma^{(2)}$ is a Klein Vierergruppe generated by $\alpha$ and $\beta$, so it suffices to see which of $\alpha, \beta, \alpha\beta$ can be written as a $\mathbb{Q}$-linear combination of the basis $S$ of $B$, which is a straightforward computation as $S$ is also an $\mathbb{R}$-basis of $M_2(\mathbb{R})$. □

The groups $\Gamma'$ are indicated in Table 2. After trivializing the quaternion algebras $B$ involved, we have conjugated the corresponding orders $\mathcal{O}$ of $M_2(\mathbb{Q})$ into $M_2(\mathbb{Z})$ by an ad hoc calculation. In general, one could determine a maximal order of $M_2(\mathbb{Q})$ containing $\mathcal{O}$. Such a maximal order is conjugate to $M_2(\mathbb{Z})$, so that an explicit conjugating element can be used to map $\mathcal{O}$ into $M_2(\mathbb{Z})$.

Identifying the groups $\Gamma'$ and $\Gamma$ with their images under the preceding conjugations, we now have reduced to a simpler situation, where we have an inclusion $\Gamma^{(2)} \subset \Gamma'$ as well as inclusions

\[(1.1.6) \quad \Gamma \supset \Gamma' \subset \mathbb{Z}[\Gamma'] = \mathcal{O}^1 \subset \text{SL}_2(\mathbb{Z}).\]

Proposition 2.1.3 will show that in fact $\Gamma' = \Gamma \cap \text{SL}_2(\mathbb{Z})$. Note that it is not clear that $\Gamma'$ equals $\mathbb{Z}[\Gamma']^1$; in fact in case IV the order $\mathcal{O}'$ is a non-Eichler order of index 4 whose group of units is of index 2 in $\text{SL}_2(\mathbb{Z})$ and hence properly contains in $\Gamma$. In all the other cases, Proposition 2.1.2 will show that indeed $\mathcal{O}^1 = \Gamma'$. In fact, in case IV we still have that $\mathbb{Z}[\Gamma^{(2)}]^1 = \Gamma^{(2)}$, so that we could have worked with the original order $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$ defined by $\Gamma^{(2)}$ instead. Regardless, we will see below that case IV is less complicated than the others.
2. Belyi maps

2.1. Monodromy triples. The reason for our using the unit group \( \mathcal{O}' \) is that it is less complicated to determine whether a given element \( \gamma \in \text{SL}_2(\mathbb{Z}) \) belongs to this group; it suffices to see whether \( \gamma \in \mathcal{O}' \), which gives a straightforward additive criterion. In this way, we can quickly determine the monodromy of the cover (and in fact Belyi map) corresponding to the inclusion \( \mathcal{O}' \subset \text{SL}_2(\mathbb{Z}) \) by using the standard generators \( S, T \) and \((ST)^{-1}\) of \( \text{SL}_2(\mathbb{Z}) \) of order \( 2, 3, \infty \) and using the methods of [7, §3].

Remark 2.1.1. It is also possible to work directly with one of the groups \( \Gamma^{(2)}, \Gamma' \) or \( \Gamma \) by using Dirichlet domains. For details, we again refer to [7, §3], especially Algorithms 3.8 and 3.14.

In case IV the group \( \Gamma \) is itself a subgroup of \( \text{SL}_2(\mathbb{Z}) \). Considering the possible ramification then shows that it is of index 6, and that the cover \( X(\Gamma) \rightarrow X(1) \) has ramification type \((3^2), (3^3), (6^1)\) above \( j = 0, 1728, \infty \) respectively. Geometrically, this is nothing but the quotient of an elliptic curve with \( j \)-invariant 0 by its automorphism group; in the next section, we determine the correct model of this cover over its canonical field of definition \( \mathbb{Q} \).

In the other cases, we have to put in more effort. The permutation triples \((\sigma_0, \sigma_1, \sigma_\infty)\) that describe the monodromy of the covers defined by the inclusion \( \mathcal{O}' \subset \text{SL}_2(\mathbb{Z}) \) above \( j = 0, 1728, \infty \) are given in Table 3. We have followed the convention \( \sigma_\infty \sigma_1 \sigma_0 = 1 \). Note that as mentioned above we get a subgroup of index 2, not 6, in case IV. However, we have the following.

Proposition 2.1.2. In the cases I-III, we have \( \mathbb{Z}[\Gamma']^1 = \Gamma' \) in Table 2.

Proof. This follows from Table 3 since clearly \( \Gamma' \subset \mathbb{Z}[\Gamma']^1 \) and the index of \( \mathbb{Z}[\Gamma']^1 \), as read off from that table, coincides with the index of \( \Gamma' \), as determined by the covolume of this group. \( \square \)

Corollary 2.1.3. In all cases, we have \( \Gamma' = \Gamma \cap \text{SL}_2(\mathbb{Z}) \).

Proof. The inclusion \( \Gamma' \subset \Gamma \cap \text{SL}_2(\mathbb{Z}) \) clearly holds, and if the intersection were larger, then the corresponding group would generate an order in \( M_2(\mathbb{Z}) \) larger than that generated by \( \Gamma' \), a contradiction with the definition of the latter group. \( \square \)

We have to determine the Belyi maps corresponding to the monodromy triples in Table 3. These maps are of sufficiently low degree to be accessible directly by general methods such as those developed in [7], and the functionality under development in [6] quickly returns
explicit formulas for them. Still, we indicate a useful technique in the calculation of covers that facilitates these computations.

2.2. Decomposing covers. Let $\overline{f} : \overline{Z} \to \overline{X}$ be a ramified cover of degree $d$ of a known base space $\overline{X}$ that is the projective completion of an unramified cover $f : Z \to X$. We wish to determine whether $f$ can be written as a non-trivial composition $Z \to Y \to X$ (which will imply a similar statement for $\overline{f}$), and if so to describe the maps $Z \to Y$ and $Y \to X$ combinatorially. When studying Belyi maps, we are of course considering $\overline{X} = \mathbb{P}^1$, $X = \mathbb{P}^1 - \{0, 1, \infty\}$.

The cover $f$ itself corresponds to a homomorphism $\varphi : \pi_1(X, x) \to S_d$, where $x$ is a chosen base point of $X$. Let $G$ be the image of $\varphi$, and let $H$ be the stabilizer of $1 \in \{1, \ldots, d\}$. Then $G$ is canonically isomorphic to a quotient of $\pi_1(X, x)$, and in this way the set of cosets $G/H$ becomes a $\pi_1(X, x)$-set. After having chosen a presentation of $\pi_1(X, x)$, its generators are mapped by $\varphi$ to elements of $S_d$, which group we can identify with $\text{Sym}(G/H)$. In the case where such a generator represents a simple loop around a branch point in the completion $\overline{X}$ of $X$, its image under $\varphi$ describes the local monodromy.

Via the dictionary of covering theory [7, 14], a factorization of $f$ as $Z \to Y \to X$ corresponds to a subgroup $K$ inbetween $H$ and $K$. In our cases $X = \mathbb{P}^1 - \{0, 1, \infty\}$, and we have determined the representation $\pi_1(X, x) \to S_d$ up to conjugacy in Table 3. Since the groups involved

| Case | Monodromy |
|------|-----------|
| I    | $\sigma_0 = (1, 3, 8)(2, 7, 4)(5, 11, 9)(6, 12, 10)$ |
|      | $\sigma_1 = (1, 2)(3, 5)(4, 6)(7, 10)(8, 9)(11, 12)$ |
|      | $\sigma_\infty = (1, 4, 10, 2, 8, 11, 6, 7, 12, 5)(3, 9)$ |
| II   | $\sigma_0 = (1, 3, 8)(2, 7, 4)(5, 11, 9)(6, 12, 10)$ |
|      | $\sigma_1 = (1, 2)(3, 5)(4, 6)(7, 11)(8, 12)(9, 13)$ |
|      | $\sigma_\infty = (1, 4, 10, 5)(2, 8, 16, 20, 21, 13, 14, 15, 19, 24, 18, 11)(3, 9, 17, 6, 7, 12)(22, 23)$ |
| III  | $\sigma_0 = (1, 3, 8)(2, 7, 4)(5, 11, 9)(6, 12, 10)$ |
|      | $\sigma_1 = (1, 2)(3, 5)(4, 6)(7, 11)(8, 10)(9, 12)$ |
|      | $\sigma_\infty = (1, 4, 10, 3, 9, 6, 7, 5)(2, 8, 12, 11)$ |
| IV   | $\sigma_0 = (1)$ |
|      | $\sigma_1 = (1, 2)$ |
|      | $\sigma_\infty = (1, 2)$ |

Table 3. Monodromy generators for $\mathcal{O}^1 \subset \text{SL}_2(\mathbb{Z})$
are very small, we can calculate the lattice of subgroups of $G$ that contain $H$. In practice we do not merely find a factorization in one step, but a succession of inclusions $H \subset K \subset K' \subset \cdots \subset G$. We wish to describe the resulting covers, for which we may without loss of generality consider the case $H \subset K \subset G$ of a single intermediary subgroup.

For a start, the inclusion $K \subset G$ corresponds to the $\pi_1(X, x)$-set $G/K$. This means that we can reuse the generators of $\pi_1(X, x)$ on this smaller set of cosets to describe the monodromy, which furnishes a combinatorial description of the cover $Y \to X$.

The cover corresponding to the inclusion $H \subset K$ can be more difficult to describe, since in this case we have to describe a cover of $Y$; in particular, the fundamental group used changes from $\pi_1(X, x)$ to $\pi_1(Y, y)$, where $y$ is some element of $Y$ that is in the fiber of $Y$ over $x$. Using the methods of [7], it is possible to calculate a presentation of $\pi_1(Y, y)$ in terms of the elements of $\pi_1(X, x)$, and with it the representation of this group on the set of cosets $K/H$. In the coming lemma, we determine some weaker invariants, namely the local monodromy around the points of $Y$. In all of the cases under consideration in this article, this suffices for the calculation of the covers involved. In stating it, we done the projective completion of a curve $X$ by $\overline{X}$.

**Lemma 2.2.1.** Let $f : Z \to X$ be a cover described by the monodromy morphism $\varphi : \pi_1(X, x) \to G \subset S_d$, and let $K$ be a group in between $H = \text{Stab}(1)$ and $G$. Let $Z \to Y \to X$ be the resulting factorization of $f$. Let $\gamma$ be an element of $\pi_1(X, x)$ that represents a simple loop around an element $\overline{\gamma}$ of $\overline{X} - X$, and let $\sigma = \varphi(\gamma)$ be the corresponding local monodromy. Then the monodromy of the cover $Z \to Y$ around the points $\overline{y}$ of $\overline{Y}$ over $\overline{\gamma}$ can be described as follows.

(i) The points $\overline{y}$ correspond to the equivalence classes in $G/K$ under left multiplication by $\sigma$.

(ii) Let $c \in G/K$ be a coset representing a point as in (i). Define $\tau \in K$ by

$$(2.2.2) \quad \tau = (c^{-1} \sigma c)^e = c^{-1} \sigma^e c$$

where $e$ is the length of the cycle obtain by multiplying with $\sigma$. Then the monodromy around the point $\overline{y}$ is described by $\tau$.

**Proof.** This follows by considering the loop around $\overline{\gamma}$ in $\pi_1(X, x)$. It lifts to various segments of loops on $Y$, which need not again be loops. However, the smallest powers described $\tau$ described in the Lemma correspond to simple loops on $Y$ with various centers in the fiber of $\overline{Y}$.
over \(\pi\). Since all loops on \(Y\) around a point in such a fiber come from exactly one such orbit under powering, our statement is proved. \(\Box\)

We can now determine the covers that we need by the decomposition procedure described above.

**Lemma 2.2.3.** In case II, the Belyi map \(X(\Gamma) \rightarrow X(1)\) is described by the map

\[
(x, y) \mapsto 6912 \frac{(2x^3 - 6x^2 - 1)^3}{(x - 2)^6(x + 1)^5x^2(x - 3)}
\]

from the curve \(y^2 = x(x - 1)(3x - 2)(3x + 1)\).

**Proof.** In this case the monodromy group has cardinality 288 and trivial center. We use Lemma 2.2.1. There multiple ways in which we can use the subgroup lattice between \(H\) and \(G\). We have chosen a particularly simple one; between \(H\) and \(G\) there is a group that contains \(H\) of index 2 and that gives rise to a cover of \(\mathbb{P}^1 - \{0, 1, \infty\}\) with permutation triple

\[
\sigma_1 = (1, 2)(3, 4)(5, 10)(6, 9)(7, 8)(11, 12)
\]

(2.2.5) \[
\sigma_0 = (1, 3, 6)(2, 5, 4)(7, 9, 10)(8, 11, 12)
\]

\[
\sigma_\infty = (2, 8, 7)(3, 5, 9)(6, 10, 12)
\]

Riemann-Hurwitz shows that the resulting cover has genus 0. Moreover, since all ramification above the elliptic points of index 2 and 3 has been absorbed, we see that the corresponding curve has signature \((0; 4^4)\), where the cusps correspond to the 4 elements above \(\infty\). We can then recover \(X(\Gamma(2))\) by taking a degree 2 cover that ramifies above these 4 points. So we instead consider the cover \(Z \rightarrow X\) associated to (2.2.5) and the associated groups \(G\) and \(H\).

Using the subgroup lattice shows that once more there is an intermediate subgroup \(K\) between \(G\) and \(H\), generated by the elements

\[
(1, 4)(2, 6, 7, 12, 8, 10)(3, 5, 9),
\]

(2.2.6) \[
(1, 11, 4)(2, 3, 12)(5, 6, 8)(7, 9, 10),
\]

\[
(2, 8, 7)(3, 5, 9)(6, 10, 12).
\]

It contains \(H\) of index 3. We get a factorization \(Z \rightarrow Y \rightarrow X\). Considering cosets of \(K\) shows that \(Y \rightarrow X\) is described by the triple

\[
\sigma_1 = (1, 2)(3, 4)
\]

(2.2.7) \[
\sigma_0 = (1, 2, 3)
\]

\[
\sigma_\infty = (2, 3, 4)
\]
This gives the cover
\[(2.2.8) \quad x \mapsto 6912 \frac{x^3(x + 2)}{4x - 1}\]
of the \(j\)-line.

The cover \(Z \to Y\) ramifies over the points \(x = -2\) in the fiber over \(j = 0\) and the points \(x = 1/4, \infty\) in the fiber over \(j = \infty\). Applying Lemma 2.2.1 shows that the ramification over these points is given by \(3^1, 2^11^1, 2^11^1\). The Belyi map \(b : x \mapsto (-27/4)x^2(x - 1)\) ramifies in this way over 0, 1 and \(\infty\), so by moving these branch points we get our cover \(Z \to Y\), which we can take to be \((-8b - 1)/(4b - 4)\). We obtain \(Z \to X\) by substituting this latter map in the former map (2.2.8) and then polishing by substituting \(x/3\) for \(x\).

\[\square\]

Similarly, we obtain:

**Lemma 2.2.9.** In case I, the Belyi map \(X(\Gamma) \to X(1)\) is described by the map
\[(2.2.10) \quad (x, y) \mapsto \frac{-(x^2 - 10x + 5)^3}{x}\]
from the curve \(y^2 = x(x^2 - 22x + 125)\).

**Lemma 2.2.11.** In case III, the Belyi map \(X(\Gamma) \to X(1)\) is described by the map
\[(2.2.12) \quad (x, y) \mapsto 256 \frac{(x^2 + 1)^3}{x^4}\]
from the curve \(y^2 = x(4x^2 + 1)\).

### 3. Canonical models

We now determine canonical models for our \((1; \infty)\)-curves, using two different methods.

**3.1. First method: \(q\)-expansions.** The first method is elementary and arguably still the most insightful; we choose the defining equations that we encounter in such a way that the resulting \(q\)-expansions of the coordinates are rational.

To illustrate this method, we consider case II, where we first try the model furnished in Lemma 2.2.3, namely \(y^2 = x(x + 1)(x - 2)(x - 3)\). In this case the fiber of the Belyi map above \(\infty\) has 4 elements with distinct ramification indices. All of these will therefore give rise to a branch that gives a rational \(q\)-expansion of \(x\). However, we also need to consider the square root that we have to draw when determining \(y\), and this only gives a rational \(q\)-expansion for that coordinate if we switch to
the quadratic twist by $-1$. This fully determines the canonical model over $\mathbb{Q}$ of the curve $X(\Gamma(2))$. It is independent of the particular point over $\infty$ that is chosen.

Since $\Gamma(2) \neq \Gamma$, it still remains to determine canonical model of $X(\Gamma)$ in this case. This is obtained by dividing out the 2-torsion of the Jacobian, which in fact leads to an isomorphic curve. Case IV corresponds to a subgroup of $\text{SL}_2(\mathbb{Z})$, so we do not have to take any further isogeny, while in case I we can recover a model of $X(\Gamma)$ from the given one for $X(\Gamma')$ be taking an isogeny with kernel $(0,0)$.

The case III leads to a subtlety, due to an automorphism of the cover obtained from the group $\Gamma'$ (which is the map $x \mapsto -x$ in Lemma 2.2.11). In this case there is no common quadratic twist that makes both branches over $\infty$ have rational $q$-expansions. Either choice of branch leads to a model of $X(\Gamma')$ over $\mathbb{Q}$. Neither of them can be called canonical over $\mathbb{Q}$, however, but only over $\mathbb{Q}(i)$ where either chosen model over $\mathbb{Q}$ leads to all branches being rational. On the other hand, the model of the codomain $X(\Gamma)$ of the resulting 2-isogeny (which again has kernel $(0,0)$) does not depend on the choice of branch, and we get rational $q$-expansions at the single cusp of this curve. Therefore also in this case we obtain a canonical model for $X(\Gamma)$ over $\mathbb{Q}$, even though $X(\Gamma')$ admits no such model.

Table 4 summarizes the canonical models found; the parabolic point corresponds to the point at infinity in all cases. Besides a reference to the LMFDB [8], where more detailed information on these curves can be found, we have also included their $j$-invariant and Faltings height $h_{\text{Falt}}$. Table 5 gives some of the resulting $q$-expansions of the coordinates $x$ and $y$.

| Case | Curve | LMFDB label | $j$-invariant | $h_{\text{Falt}}$ |
|------|-------|-------------|---------------|----------------|
| I    | $y^2 = x^3 - 44x^2 - 16x$ | 20.a1 | $2^{14}31^4/5^4$ | 3.814 |
| II   | $y^2 = x^3 - 4x^2 - 384x - 2304$ | 24.a3 | $2^{7}3^3/3^4$ | 3.152 |
| III  | $y^2 = x(x^2 - 256)$ | 32.a4 | 1728 | -1.311 |
| IV   | $y^2 = x^3 - 1728$ | 36.a3 | 0 | -1.321 |

Table 4. Canonical models

The curves that we found have minimal conductor among all their possible quadratic twists over $\mathbb{Q}$. Note that while case III gives a curve that is isomorphic to $X_0(32)$ over $\mathbb{Q}$, the latter curve is in fact a $(1;\infty^8)$-curve. The corresponding groups are commensurable, with neither being a subgroup of the other up to conjugacy.
3.2. Second approach: Shimura’s viewpoint. A second approach to obtain canonical models is to use work by Shimura [3, 12] that gives a more general description of canonical models and their properties. In [15], this served as an essential tool to determine canonical models in the presence of a non-split quaternion algebra.

In this approach we further enlarge the orders $O'$ from Section 1 to orders $O'' \supset O'$ with $O''^1 = O'^1$ but with the property that the norm surjects to as large a subset of $\mathbb{Z}$ as possible. Shimura then describes the canonical field of definition of $X(O'^1)$ and indicates how the traces of Frobenius of the canonical model over this field can be determined. We briefly sketch the results here.

In cases I and II the overorder $O''$ can be chosen in such a way that the norm surjects to $\mathbb{Z}$. In this case the canonical field of definition equals $\mathbb{Q}$. The corresponding curve has good reduction outside the primes that divide the index of $O''$ in $M_2(\mathbb{Z})$, and its traces of Frobenius can be determined using Shimura congruence, as in [13, Chapter 4]. In combination with the known $j$-invariant, these data suffice to determine the canonical model completely. We checked that the results obtained agree with the first approach.

In case III we can at most obtain an order $O''$ whose norm map attains all primes that are congruent to 1 modulo 4. Shimura’s results then show that the canonical field of definition of the curve $X(O'^1) = X(\langle \Gamma(2), \alpha \beta \rangle)$ is $\mathbb{Q}(i)$, which is in line with the subtlety encountered when using the first method above. The methods from [13] can again be used to compute the traces of Frobenius of Deligne’s non-connected model at all primes, and once more the results agree with the more elementary method above.

Case IV is quite surprising. In this case $\Gamma$ is the image of the commutator subgroup of $\text{SL}_2(\mathbb{Z})$ in $\text{PSL}_2(\mathbb{R})$. Indeed, like the former

| Case | Expansions of $x$ and $y$ |
|------|--------------------------|
| I    | $x = q^{-1/5} + 16 + 134q^{1/5} + 760q^{3/5} + 3345q^{4/5} + 12256q^{7/5} + \ldots$  
   | $y = q^{-3/10} + 2q^{-1/10} - 129q^{9/10} - 1778q^{3/10} - 13725q^{1/2} + \ldots$ |
| II   | $x = q^{-1/6} + 2 + 79q^{1/6} + 352q^{1/3} + 1431q^{1/2} + 4160q^{2/3} + \ldots$  
   | $y = q^{-1/4} + q^{-1/12} - 76q^{1/12} - 778q^{1/4} - 5224q^{5/12} + \ldots$ |
| III  | $x = q^{-1/4} + 52q^{1/4} + 834q^{3/4} + 4760q^{5/4} + 24703q^{7/4} + \ldots$  
   | $y = q^{-3/8} - 50q^{1/8} - 2599q^{5/8} - 29154q^{9/8} - 238728q^{13/8} + \ldots$ |
| IV   | $x = q^{-1/3} + 824q^{2/3} - 613348q^{5/3} + 831470016q^{8/3} + \ldots$  
   | $y = q^{-1/2} + 372q^{1/2} + 29250q^{3/2} - 134120q^{5/2} + \ldots$ |

Table 5. $q$-expansions
group, the latter is abelian of index 6 in $\text{PSL}_2(\mathbb{R})$. By the universal property of the commutator subgroup, these groups therefore have to coincide.

The commutator subgroup of $\text{SL}_2(\mathbb{Z})$ is described in [2, Remark 3.9]. Let
\begin{equation}
(3.2.1) \quad \gamma_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}
\end{equation}
be generators of the commutator subgroup of $\text{SL}_2(\mathbb{Z})$. Recall that the order $\mathcal{O}' = \mathbb{Z}[\Gamma]$ had the property $\Gamma \not\subseteq \mathcal{O}'$ and is therefore too large to study $\Gamma$. We can instead consider the order $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$. Then
\begin{equation}
(3.2.2) \quad \Gamma = \langle \mathcal{O}^1, \gamma_1, \gamma_2 \rangle.
\end{equation}
It turns out that $\mathcal{O}$ can be enlarged to an order of index 36 in $M_2(\mathbb{Z})$ to which none of $\gamma_1, \gamma_2, \gamma_1\gamma_2$ belongs. Moreover, the groups $\langle \Gamma^{(2)}, \alpha \rangle, \langle \Gamma^{(2)}, \beta \rangle, \langle \Gamma^{(2)}, \alpha\beta \rangle$ all generate additional orders of index 36 to which exactly one of these commutators belongs. These orders are distinct, but have the same index in $M_2(\mathbb{Z})$ and give rise to curves with canonical field of definition $\mathbb{Q}$ whose Jacobians are all isogenous.

### 3.3. Orders obtained

Bases of the orders $\mathcal{O}''$ can be found by running the code at [16]. None of these orders is Eichler, and in all cases the index in $M_2(\mathbb{Z})$ equals the corresponding conductor in Table 4. In the next section we will construct these orders $\mathcal{O}''$ as intersections.

### 4. Modular interpretations

The curves $X(\Gamma)$ also parametrize the points of certain moduli functors, which we describe in this section. A more advanced general exposition is given in [4].

#### 4.1. Over $\mathbb{C}$

Consider pairs
\begin{equation}
(4.1.1) \quad (E, (\lambda_1, \lambda_2))
\end{equation}
where $E$ is an elliptic curve over $\mathbb{C}$ and where $(\lambda_1, \lambda_2)$ is an ordered basis of the homology group $H_1(E, \mathbb{Z})$, which we assume to be positively oriented in the sense that $\text{im}(\lambda_1/\lambda_2) > 0$ in the one-dimensional $\mathbb{C}$-vector space $V = H_1(E, \mathbb{Z}) \otimes \mathbb{R} = H^0(E, \omega_E)^\vee$ that is the universal cover of $E$.

The group $\text{SL}_2(\mathbb{Z})$ acts on the left on the set of objects (4.1.1) as follows: if $\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, then
\begin{equation}
(4.1.2) \quad \gamma'(E, (\lambda_1, \lambda_2)) = (E, (a\lambda_1 + b\lambda_2, c\lambda_1 + d\lambda_2)).
\end{equation}
Let \( \Gamma' \) be a subgroup of \( \text{SL}_2(\mathbb{Z}) \) (not necessarily congruence). We say that two pairs \((E, (\lambda_1, \lambda_2)), (E', (\lambda'_1, \lambda'_2))\) are \( \Gamma' \)-equivalent if and only if there exists an isomorphism \( \varphi : E \to E' \) such that
\[
(4.1.3) \quad (E', (\varphi_* (\lambda_1), \varphi_* (\lambda_2))) = \gamma'(E', (\lambda'_1, \lambda'_2))
\]
for some \( \gamma' \) in \( \Gamma' \).

**Proposition 4.1.4.** The points of the quotient space \( X(\Gamma') \) bijectively correspond to the \( \Gamma' \)-equivalence classes of pairs (4.1.1).

**Proof.** Given a pair as in (4.1.1), there exists a unique \( \tau \in \mathcal{H} \) such that
\[
(4.1.5) \quad (E, (\lambda_1, \lambda_2)) \sim (E_\tau, (\tau, 1)).
\]
Here \( E_\tau = V/\Lambda_\tau \). The action of \( \Gamma' \) is then given via the usual action of \( \text{PSL}_2(\mathbb{R}) \) on the upper half plane \( \mathcal{H} \). Moreover, two pairs \((E_\tau, (\tau, 1)), (E_{\tau'}, (\tau', 1))\) are equivalent under if and only if \( \tau \) and \( \tau' \) are related by this action. \( \square \)

In the cases I-III, the group \( \Gamma \) is not itself a subgroup of \( \text{SL}_2(\mathbb{Z}) \), and we first restrict to \( \Gamma' = \Gamma \cap \text{SL}_2(\mathbb{Z}) \). In all of these cases, the group \( \Gamma \) can be obtained from \( \Gamma' \) by adjoining elements \( w \in M_2(\mathbb{Z}) \) that are not in \( \text{SL}_2(\mathbb{Z}) \) but whose square is a still scalar. These elements \( w \) normalize the order \( O' \) defined by \( \Gamma' \).

The modular interpretation of these involutions in terms of the pairs (4.1.1) is as follows. Let \( s \) be the square of \( w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), considered as a scalar, let \( w' = w^{-1} \), and let \( \Lambda^w = w'(\Lambda) \) be the overlattice of \( \Lambda \) defined by \( w' \). Then \( \Lambda^w \) contains \( \Lambda \) of index \( s \). We define
\[
(4.1.6) \quad E^w = V/\Lambda^w
\]
and define
\[
(4.1.7) \quad (\lambda_1^w, \lambda_2^w) = w'(\lambda_1, \lambda_2) = (a\lambda_1 + b\lambda_2, c\lambda_1 + d\lambda_2)
\]
as in (4.1.2). Then the association
\[
(4.1.8) \quad (E, (\lambda_1, \lambda_2)) \mapsto (E^w, (\lambda_1^w, \lambda_2^w))
\]
gives rise to a well-defined map on \( \Gamma' \)-orbits because \( w^{-1}\Gamma'w = \Gamma' \).

Restricting to pairs of the form \((E_\tau, (\tau, 1))\) as above, we see the following.

**Proposition 4.1.9.** Suppose that \( \Gamma = \langle \Gamma', S \rangle \), where \( S \) is a set of involutions \( w \) whose scalar is a square and that satisfy \( w\Gamma'w^{-1} = \Gamma' \). Then the points of the quotient space \( X(\Gamma) \) bijectively correspond to the equivalence classes of pairs (4.1.1), up to the additional identifications generated by
\[
(4.1.10) \quad (E, (\lambda_1, \lambda_2)) \sim (E^w, (\lambda_1^w, \lambda_2^w))
\]
for $w \in S$.

4.2. Over general base fields. Now suppose that $\Gamma' \subset \text{SL}_2(\mathbb{Z})$ is a congruence subgroup, a situation that applies to our examples. Then another description is possible, which lends itself to generalization to arbitrary fields. Suppose that the full-level subgroup $\Gamma(N)$ is a subgroup of $\Gamma'$, and let $\Gamma'$ be the image of $\Gamma'$ in $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Consider pairs

\[(4.2.1) \quad (E, (\lambda_1, \lambda_2))\]

where $E$ is an elliptic curve and where $(\lambda_1, \lambda_2)$ is an ordered basis of the $N$-torsion group $E[N]$.

The group $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on the left on the set of objects $(4.2.1)$ as follows: if $\gamma' = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$, then

\[(4.2.2) \quad \gamma'(E, (\lambda_1, \lambda_2)) = (E, (a\lambda_1 + b\lambda_2, c\lambda_1 + d\lambda_2)).\]

Via this action, we also get an action of $\Gamma'$, which projects to $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

We say that two pairs $(E, (\lambda_1, \lambda_2)), (E', (\lambda_1', \lambda_2'))$ are $\Gamma'$-equivalent if and only if there exists an isomorphism $\varphi : E \to E'$ such that

\[(4.2.3) \quad (E', (\varphi(\lambda_1), \varphi(\lambda_2))) = \gamma'(E', (\lambda_1', \lambda_2'))\]

for some $\gamma'$ in $\Gamma'$. Then we again have the following.

**Proposition 4.2.4.** The points of the algebraic curve $X(\Gamma')$ over an algebraically closed field $k$ bijectively correspond to the $\Gamma'$-equivalence classes of pairs $(4.2.1)$ over $k$.

Note that for a non-algebraically closed field $k$, such an equivalence class can define a $k$-rational point of $X(\Gamma')$ without its constituents being $k$-rational; if $k$ is perfect, then it suffices that the class is closed under conjugation by the absolute Galois group of $k$.

We can also give a description of the involution defined by an element $w \in \Gamma$ with integral scalar square $s$ as above. We define $w'' = Nw^{-1} \in M_2(\mathbb{Z}/N\mathbb{Z})$. Say $w'' = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then defining

\[(4.2.5) \quad w''(\lambda_1, \lambda_2) = (a\lambda_1 + b\lambda_2, c\lambda_1 + d\lambda_2),\]

we let

\[(4.2.6) \quad E^w = E/\langle w''(\lambda_1, \lambda_2) \rangle.\]

Let $s$ be the denominator of $w^{-1}$. We choose a basis $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ of $E[Ns]$ in such a way that $s\tilde{\lambda}_i = \lambda_i$. Let $w''' = sw^{-1}$ and define

\[(4.2.7) \quad (\lambda_1^w, \lambda_2^w) = w'''(\tilde{\lambda}_1, \tilde{\lambda}_2).\]
which is an element of $E^w[N]$. Then the association
\begin{equation}
(E, (\lambda_1, \lambda_2)) \mapsto (E^w, (\lambda_1^w, \lambda_2^w))
\end{equation}
gives rise to a well-defined map on $\Gamma'$-orbits, and gives the requested modular description. The reader will readily formulate the analogue of Proposition 4.1.9. Again, over perfect fields $k$ the rational points of $X(\Gamma)$ parametrize equivalence classes that are closed under Galois conjugation.

In our concrete examples, we can give slightly more elegant prime-by-prime descriptions of the moduli problems, essentially because when $\Gamma' = \Gamma_0(N)$ we obtain the classical description in terms of torsion subgroups (instead of bases). We now proceed to give these descriptions, up to conjugacy.

4.3. Case I. In this case the order $O''$ from the end of the previous section is the intersection of the index 4 order
\begin{equation}
O_4 = \langle (1 0, 0 1), (0 1, 1 0), (0 1, -1 0) \rangle
\end{equation}
and the index 5 order
\begin{equation}
O_5 = \langle (0 1, 1 0), (0 1, 0 0), (0 0, 0 1) \rangle.
\end{equation}
We obtain a congruence subgroup of level 10.

The corresponding modular description admits the following simplification: the curve $X(\Gamma')$ parametrizes equivalence classes of triples
\begin{equation}
(E, H_5, (\lambda_1, \lambda_2))
\end{equation}
where $E$ is an elliptic curve, $H_5$ is a subgroup of $E[5]$ of order 5, and $(\lambda_1, \lambda_2)$ is a basis of $E[2]$. Two such triples $(E, H_5, (\lambda_1, \lambda_2))$ and $(E', H'_5, (\lambda'_1, \lambda'_2))$ are equivalent if and only there exists an isomorphism $\varphi : E \to E'$ that maps $H_5$ into $H'_5$ and such that $\varphi(\lambda_1, \lambda_2) = \alpha(\lambda'_1, \lambda'_2)$, where $\alpha'$ is a power of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{F}_2)$.

There is a single involution defined by the element
\begin{equation}
w_5 = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}
\end{equation}
of determinant 5. In terms of the triples (4.3.3), its action can be described by
\begin{equation}
w_5(E, H_5, (\lambda_1, \lambda_2)) = (E', H'_5, (\lambda'_1, \lambda'_2)),
\end{equation}
where $E'$ is the quotient of $E$ by $H_5$, $H'_5$ is the image of $E[5]$ on $E'$, and $(\lambda'_1, \lambda'_2)$ is the image of $(\lambda_2, \lambda_1)$ on $E'$. 

4.4. **Case II.** In this case the order $\mathcal{O}''$ from the end of the previous section is the intersection of the index 8 order

\[ \mathcal{O}_8 = \langle (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}), (\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}) \rangle \]

and the index 3 order

\[ \mathcal{O}_3 = \langle (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \rangle \].

We obtain a congruence subgroup of level 12. Its elements parametrize triples

\[ (E, H_3, (\overline{\lambda}_1, \overline{\lambda}_2)) \]

where this time $H_3$ is a subgroup of order 3 of $H_3$, and where $(\overline{\lambda}_1, \overline{\lambda}_2)$ is a basis of $E[4]$ up to the equivalence defined by the elements of the order in $\text{SL}_2(\mathbb{Z}/4\mathbb{Z})$ determined by the elements in the basis of $\mathcal{O}_8$.

There are three involutions, defined by the elements

\[ w_2 = (\begin{smallmatrix} -2 & -2 \\ 3 & 2 \end{smallmatrix}), \]

\[ w_3 = (\begin{smallmatrix} -3 & -4 \\ 3 & 3 \end{smallmatrix}), \]

\[ w_6 = (\begin{smallmatrix} 0 & 2 \\ 3 & 0 \end{smallmatrix}), \]

of determinant 2, 3, 6 respectively. Note that these elements only commute modulo the group $\mathcal{O}'''$.

The action of the involution $w_2$ is given by

\[ w_2(E, H_3, (\overline{\lambda}_1, \overline{\lambda}_2)) = (E', H'_3, (\overline{\lambda}'_1, \overline{\lambda}'_2)), \]

where $E'$ is the quotient of $E$ by $2\overline{\lambda}_1$, where $H'_3$ is the image of $H_3$ on $E'$, and where $(\overline{\lambda}'_1, \overline{\lambda}'_2)$ is some point on $E'$ such that $(\overline{\lambda}'_1, 2\overline{\lambda}'_2)$ equals the image of $(\overline{\lambda}_2, -\overline{\lambda}_1)$ on $E'$.

The action of the involution $w_3$ is given by

\[ w_3(E, H_3, (\overline{\lambda}_1, \overline{\lambda}_2)) = (E', H'_3, (\overline{\lambda}'_1, \overline{\lambda}'_2)), \]

where $E'$ is the quotient of $E$ by $H_3$, where $H'_3$ is the image of $E[3]$ on $E'$, and where $(\overline{\lambda}'_1, \overline{\lambda}'_2)$ is the image of $(\overline{\lambda}_1, \overline{\lambda}_1 - \overline{\lambda}_2)$ on $E'$.

The action of the involution $w_6$ is the composition of that of $w_2$ and $w_3$. It is given by

\[ w_6(E, H_3, (\overline{\lambda}_1, \overline{\lambda}_2)) = (E', H'_3, (\overline{\lambda}'_1, \overline{\lambda}'_2)), \]

where $E'$ is the quotient of $E$ by the group generated by $H_3$ and $2\overline{\lambda}_1$, where $H'_3$ is the image of $E[3]$ on $E'$, and where $(\overline{\lambda}'_1, \overline{\lambda}'_2)$ is some point on $E'$ such that $(\overline{\lambda}'_1, 2\overline{\lambda}'_2)$ equals the image of $(\overline{\lambda}_2, \overline{\lambda}_1)$ on $E'$. 
4.5. Case III. In this case the order $O''$ is of index 32, given by
\[(4.5.1)\quad O_{32} = \langle (1\ 0\ 0\ 1), (0\ 2\ -2\ 0), (2\ -1\ 1\ -2), (0\ 1\ 3\ 0) \rangle\]
We obtain a congruence subgroup of level 8. There is a single involution defined by the element
\[(4.5.2)\quad w_2 = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\]
of determinant 2. In this case we get a modular description in terms of the 8-torsion for which a simpler description than the generic one in Section 4.2 does not seem readily available.

4.6. Case IV. We consider the order $O''$ obtained by enlarging $\Gamma^{(2)}$. This is the intersection of the index 4 order
\[(4.6.1)\quad O_4 = \langle (1\ 0\ 0\ 1), (0\ 0\ 1\ 1), (0\ 1\ 1\ 1) \rangle\]
and the index 9 order
\[(4.6.2)\quad O_9 = \langle (1\ 0\ 0\ 1), (0\ 1\ 1\ 1), (0\ 0\ 3\ 0) \rangle\]
We obtain a congruence subgroup of level 6, from which we can obtain $\Gamma$ by adding the commutators $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$. This group corresponds to the unique normal subgroup of $SL_2(\mathbb{Z}/6\mathbb{Z})$ that gives rise to a quotient that is cyclic of order 6. Using the corresponding character makes it easier to check the equivalence (4.2.3).

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