Significance of Deformity on the Mass Quadrupole Moment of Non-Rotating Neutron Stars

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Abstract. Generally, when modeling non-rotating neutron stars, assumptions of perfect spherical symmetry are made; however, this assumption is not correct if the interior composition of the star is described by an anisotropic equation of state. Due to extremely high magnetic fields, magnetars and/or neutron stars that contain color-superconducting quark matter cores can exhibit such anisotropies. Recent publications on models of the global structure of highly magnetized neutron stars indicate that these objects can be deformed making them either oblate or prolate spheroids. Due to these deformations, the gravitational quadrupole moment is expected to be non-zero resulting in a non-homogeneous mass distribution in either the equatorial or polar directions, thus resulting in different masses from the spherical case. In this work, we examine this inhomogeneity by calculating the gravitational mass quadrupole moment of non-rotating deformed neutron stars in the framework of general relativity and investigate any changes from traditional spherical models.

1. Introduction
Non-rotating neutron stars are compact stellar objects whose masses lie between $\sim 1-2$ solar masses with a radii of $\sim 10-15$ kilometers. Such a large mass enclosed in a small volume results in densities on the nuclear scales. Traditionally, non-rotating neutron stars are modeled with the assumption that they are perfect spheres whose said stellar properties are described by the well known Tolman-Oppenheimer-Volkoff (TOV) equation [1,2],

$$\frac{dP}{dr} = -\frac{\varepsilon \left(1 + \frac{P}{\varepsilon}\right) m \left(1 + \frac{4\pi P r^3}{m}\right)}{r^2 \left(1 - \frac{2m}{r}\right)}, \quad (1)$$

where the normalized units of $G = c = 1$. In Eq. (1) $P$ is the pressure, $\varepsilon$ is the energy density, and $m$ is the mass of a spherical shell of radius $r$ described by

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon. \quad (2)$$

Equations (1) and (2) can be easily solved numerically with a use of a given model for the equation of state (EoS), which describes the interior composition of a neutron star.
This assumption of perfect spherical symmetry may not always be correct. If high magnetic fields (up to approximately $10^{18–20}$ Gauss) are present such as for Magnetars [3–5], and/or the pressure of the matter in the cores of neutron stars is anisotropic, as predicted by some models of color superconducting quark matter [6–11], then deformation of neutron stars can occur [6,12–17], thus making them oblong spheroids as shown in Fig. 1.

Figure 1. (Color on-line) Graphical representation of an oblate (a) and prolate (b) spheroid. The polar direction ($z$) is orthogonal to the equatorial ($x–y$) plane.

where these oblate and prolate stars have distinct polar ($z$) and equatorial ($x–y$) directions. The geometry of these axial symmetric objects are described mathematically by the Weyl metric [18,19]

$$ds^2 = e^{2\lambda}dt^2 - e^{-2\lambda}\left[e^{2\nu}(dr^2 + dz^2) + r^2d\phi^2\right] = g_{\mu\nu}dx^\mu dx^\nu,$$

(3)

where $t$ is the time component and $r$, $z$, and $\phi$ are the spatial components in cylindrical coordinates. The terms $\lambda$ and $\nu$ are the unknown metric functions that depend on both the radial $r$ and polar $z$ directions such that $\lambda = \lambda(r,z)$ and $\nu = \nu(r,z)$. Due to the fact that there are distinct polar and radial directions which are both coupled together, as described in Eq. (3), the hydrostatic equilibrium of deformed neutron stars must take these distinct directions into consideration.

Recent work on deformation models of compact stars suggest that the masses of these deformed objects can change appreciably from the spherical case [20–23]. In particular, oblate stars can have an increase in mass and prolate stars can have a decrease in mass [21–23]. In this work, we further examine the deformation of highly magnetized neutron stars by calculating the mass quadropole moment which is expected to be non-zero.

2. Non-Spherical Geometry for Oblong Spheroids

The stellar structure of oblong spheroids with distinct polar and radial directions can be described by applying a parametrization on the polar radius in terms of the equatorial radius along with deformation constant $\gamma$ on the metric described by Eq. (3), the metric will then read as

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-\gamma}dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2,$$

(4)

described in [21], which is analogous to the metric described in [19]. In Eq. (4), $\gamma$ determines the degree of deformation either in the oblate ($\gamma < 1$) or prolate case ($\gamma > 1$) (see Fig. 1), where
the parametrization is described by \( z = \gamma r \). Using this parametrization, the mass quadropole moment is then

\[
Q = \frac{2}{3} M^3 \left( 1 - \gamma^2 \right),
\]

as generalized by [19] where \( M \) is the total gravitational mass of an oblate or prolate star. In order to calculate the gravitational mass quadropole moment, we first need to determine the total gravitational mass \( M \). Second, we need values for the deformation constant \( \gamma \). To determine the total gravitational mass of a deformed neutron star \( M \), we utilize a parameterized TOV equation as described in [21],

\[
\frac{dP}{dr} = - \left( \epsilon + P \right) \left( \frac{1}{2} r + 4\pi r^3 P - \frac{1}{2} r \left( 1 - \frac{2m}{r} \right) \gamma \right)
\]

where in the case when \( \gamma = 1 \), Eq. (6) simplifies to the well-known Tolman-Oppenheimer-Volkoff equation [1, 2] which describes the stellar structure of perfectly spherically symmetric object. The gravitational mass of a deformed neutron star parameterized with the deformation constant \( \gamma \) is then described by

\[
\frac{dm}{dr} = 4\pi r^2 \epsilon \gamma,
\]

so that the total gravitational mass, \( M \), of a deformed neutron star with an equatorial radius \( R \) follows as [19, 21] \( M = \gamma m(R) \). For a given deformation, \( \gamma \), this is a very straightforward and useful model for the description of deformed compact stars. Using any EoS model in the limiting case of isotropy, the parameterized structure equations can be easily solved numerically and produce stellar properties such as masses and radii.

Solving Eqs. (6) and (7), stellar properties such as masses and radii are produced as generalized by [20, 21] shown in Fig. 2.

![Figure 2](image-url)  
**Figure 2.** (Color-online) Mass-Radius relations for deformed neutron stars using a Quark-Hadron equation of state model [24, 25]. The dots on each curve represent the maximum mass for each stellar sequence.
The black solid line in Fig. 2, represents the spherical case which is in good agreement from results in the literature [25]. However, if we change the deformation constant \( \gamma \) by various amounts, our results show an appreciable change in the maximum mass from the spherical case.

The mass increases as the oblateness increases and the mass decreases as the prolateness increases. The maximum mass can increase up to about approximately 13.5\% for a 10\% change in deformity in oblateness. For increasing prolateness, the mass decreases on the order of approximately 12\% for the same change in deformity. These interesting results about the maximum masses leads us to investigate the deformity further by examining the gravitational quadrupole moment \( Q \).

3. Deformity

The shapes of oblate and prolate neutron stars are distinct. One must be able to distinguish the differences from oblate and prolate stars. At a first glance at Fig. 1, it may appear that the oblate star and a prolate star is the same object just rotated 90 degrees. This is not the case in these models. The magnetic field is pointing from the polar (z) direction, therefore we must take that direction as our reference axis as shown in Figs. 3 and 4.

![Figure 3](image1.png)  
**Figure 3.** (Color on-line) Illustration of oblateness for deformed neutron stars. The magnetic field \( \vec{B} \) is oriented along the polar (z) – axis for both cases. If the star is being observed head-on (a), it is clear the star is oblate. If we view the star at 90 degrees (b), the star is still an oblate star since the magnetic field is still oriented along the polar axis.

![Figure 4](image2.png)  
**Figure 4.** (Color on-line) Same as Fig. 3 but for prolate stars. The magnetic field is oriented along the polar (z) – axis for both cases.

Both objects in Figs. 3 and 4 are oblate and prolate stars respectively, regardless of the angle from which they are viewed. The important distinction is the z-axis and where the magnetic field is oriented from.
4. Results and Conclusions

Using the same values of \( \gamma \) in addition to mass results from Fig. 2, we compute the gravitational mass quadropole moment as described by Eq. (5). We calculate this mass quadropole moment as we numerically construct the star, thus producing the graphically results shown in Fig. 5.

\[
\gamma = \begin{cases} 
0.80, & M = 3.02 \text{ M}_\odot \\
0.90, & M = 2.62 \text{ M}_\odot \\
1.00, & M = 2.29 \text{ M}_\odot \\
1.10, & M = 2.03 \text{ M}_\odot \\
1.20, & M = 1.81 \text{ M}_\odot 
\end{cases}
\]

**Figure 5.** (Color-online) Quadropole moment as a function of mass for various values of \( \gamma \) using the Quark-Hadron EoS model. The black solid line represents the quadropole moment for a perfect spherically symmetric object. Oblate stars are for values of \( \gamma < 1 \) while prolate stars are for values of \( \gamma > 1 \). The results here show how asymmetric the mass distributions are for oblate versus prolate stars. The results shown here are in agreement with the change in masses from Fig. 2.

In Fig. 5, the gravitational mass quadropole moment is non-zero as expected for deformed neutron stars. There is also a clear distinction of oblate and prolate stars due to the fact that the quadropole moment is not symmetric indicating the mass distribution is different for oblate and prolate stars. If this was not the case then all the curves corresponding to values of \( \gamma \neq 1 \) would be symmetric about the solid black line. This is obviously not the case. We also see that there is more uneven mass distribution as the star gets more oblate. For example, a 20% deformation in the oblate case results in a higher quadropole moment. This is in agreement with the results in Fig. 2 which indicate higher masses for oblate stars and lower masses for prolate stars. This can be more easily seen in the heatmaps (i.e. colormaps to show how the mass distribution is inhomogeneous throughout the star) in Figs. 6 and 7.

The heatmaps in Figs. 6 and 7 show how different the mass distribution is for oblate and prolate stars. Examining the right vertical axis of the mass quadrapole moment \( Q \), it is apparent that the mass is inhomogeneously distributed throughout the star thus resulting in either an oblate or prolate star. We can also graphically illustrate this inhomogeneity another way by looking how the quadropole moment \( Q \) changes as function of polar and equatorial radii as shown in Fig. 8.
Figure 6. (Color on-line) Heatmap for quadropole moment $Q$ as a function of the polar $z$ and equatorial $r$ directions for $\gamma = 0.90$ (a) and $\gamma = 0.80$ (b) for oblate stars. In these figures, blue represents zero quadropole moments and thus even homogeneous mass distribution. The heatmap shows how the mass distribution is unevenly distributed along the equatorial and polar directions. The more we deviate from the spherical case, the more uneven the mass distribution gets for these oblate stars. However, we see the increase in deviation as we move vertically away from the equatorial plane.

Figure 7. (Color on-line) Heatmap for quadropole moment $Q$ as a function of the polar $z$ and equatorial $r$ directions for $\gamma = 1.10$ (a) and $\gamma = 1.20$ (b) for prolate stars. In these figures, yellow represents zero quadropole moments and thus even homogeneous mass distribution. The heatmap shows how the mass distribution is unevenly distributed along the equatorial and polar directions. The more we deviate from the spherical case, the more uneven the mass distribution gets for these prolate stars. However, we see the increase in deviation as we move horizontally away from the polar direction.

In Fig. 8, the black solid line represents even mass distribution as expected for spherically symmetric stars. However, for oblate stars the mass distribution is more unevenly distributed in the polar direction than it is in the equatorial direction. That is, there is more deviation in the polar region than in the equatorial plane. For prolate stars, we see the opposite effect. The mass distribution is more unevenly distributed in the equatorial direction and we see more deviation in the equatorial plane than in the polar direction. From these results, we also see how asymmetric the deviations are allowing us to conclude that the mass distribution is not symmetric for oblate and prolate stars, thus these objects are distinct oblong spheroids.
Figure 8. (Color-online) Quadrupole-Radii relations for various values of $\gamma$ using the Quark-Hadron EoS model. The black solid line represents the quadrupole moment for a perfect spherically symmetric object. Oblate stars are for values of $(\gamma<1)$ while prolate stars are for values of $(\gamma>1)$.

4.1. Observational Implications

One should also investigate the observational implications of these deformed objects. The bulk stellar properties such as masses and radii obtained from solving Eqs. (6) and (7) allow us to compute the total gravitational redshift ($Z$) which is described by

$$Z = \frac{1}{(1 - \frac{2M}{R})^{\gamma/2}} - 1.$$  

(8)

For the full derivation of Eq. (8), see [21]. In Eq. (8) $M$ and $R$ are the total gravitational mass and equatorial radii respectively which are obtained by solving Eqs. (6) and (7) for a given equation of state model. Using the same values of $\gamma$ as in Fig. 2, we compute the gravitational redshift numerically thus producing the results shown in Fig. 9. From these results it is clear that the gravitational redshift changes significantly from the spherical case due to these deformations. Since the redshift depends on the total mass, and that total mass is different for oblate and prolate stars, one can then see that these oblong spheriods are distinct objects.

5. Conclusions

Using the results from a 1-D parameterized model for deformed neutron stars, we were able to calculate the gravitational mass quadrupole moment of non-rotating neutron stars. From our results shown in Figs. 2 - 9, the stellar properties such as masses, radii, gravitational redshift, and quadrupole moment are greatly affected by deformation. Thus deformation plays a pivotal role in the stellar stellar structure of these compact objects. Hence, the deformation does not need to be high to see significant changes in said stellar properties.
Figure 9. (Color-online) Gravitational Redshift vs. Mass for the Quark-Hadron equation of state model. The dots on each curve represent the maximum mass of each stellar sequence corresponding to the values in Fig. 2. The redshift values here are calculated from mass and radii values from Fig. 2 for the same values of $\gamma$. As it shows, the redshift values highly depend on the deformation of the object.

We investigated the inhomogeneity of the mass distribution in oblate and prolate stars and we conclude that the mass distribution is not symmetric among oblate and prolate stars, thus indicating that these oblong spheroids are distinct objects. Due to a non-zero quadropole moment, this work can lead to a more detailed study on these deformed objects and their possible connection to gravitational waves.

Deformation plays an important role on the stellar structure on highly magnetized compact stars. This work on deformed neutron stars can also be applied to highly magnetized White Dwarf stars. Recent publications on highly magnetized white dwarfs indicate that these objects can also be either oblate or prolate spheroids [11,26–28]. The stellar properties such as masses, radii, and quadropole moment can be easily calculated by using the models presented here and in previous work performed on deformed neutron stars.

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