PERFECT CRYSTALS FOR $U_q(D_4^{(1)})$

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1. Introduction

In [KMN1] the notion of perfect crystal was introduced. In the subsequent paper [KMN2] a perfect crystal of an arbitrary level was given for the quantum affine algebra corresponding to every non-exceptional affine algebra. Later studies revealed that there are more perfect crystals than listed in [KMN2]. See [BFKL, Ko, JMO, SS, Y, K3, NS] for example. Actually, there is a conjecture originating from fermionic formulas [HKOTY], saying that a certain finite-dimensional module, called Kirillov-Reshetikhin module, KR module for short, of a quantum affine algebra has a crystal base. KR modules are parametrized by two integers $r, s$. $r$ corresponds to a vertex of the Dynkin diagram of the affine algebra except a distinguished vertex 0 as in [Kac], and $s$ is a positive integer. The conjecture also states that if $s$ is a multiple of $t_r := \max(1, 2/(\alpha_r, \alpha_r))$, then the conjectural crystal base of the corresponding KR module is perfect of level $s/t_r$. Here $(\cdot, \cdot)$ stands for the standard bilinear form on the weight lattice as in [Kac]. The latter conjecture was derived by taking a suitable limit of the corresponding fermionic formula. Up to now there is no counterexample to this conjecture. We remark that apart from the existence of crystal base, KR modules have marvelous properties, such as algebraic relations among characters [KR, KNS, N, H], relations with Demazure modules [KMOTU, K4, FL, CM, FSS], that with fusion products [CL, AK] and so on.

In this paper, we add another evidence for the conjecture to be valid. To explain what we have done more precisely let us recall basic notations and the definition of perfect crystal. Let $\mathfrak{g}$ be an affine algebra, $U_q(\mathfrak{g})$ the associated quantum affine algebra and $U'_q(\mathfrak{g})$ its subalgebra without the degree operator. Let $P$ be the weight lattice, i.e., $P = \sum_i Z \Lambda_i \oplus Z \delta$ where $\Lambda_i$ is a fundamental weight and $\delta$ is the generator of the null roots, and set $P_{cl} = P/Z \delta$, $P_{cl}^+ = \{ \lambda \in P_{cl} \mid \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \} = \sum_i Z_{\geq 0} \Lambda_i$ and $(P_{cl}^+)_l = \{ \lambda \in P_{cl}^+ | \langle c, \lambda \rangle = l \}$ for $l \in Z_{\geq 0}$. Here $c$ is the canonical central element. Let $\text{Mod}^l(\mathfrak{g}, P_{cl})$ be the category of finite-dimensional $U'_q(\mathfrak{g})$-modules. The modules in this category have the weight decomposition with respect to $P_{cl}$. Let $B$ be a $P_{cl}$-weighted crystal. For $b \in B$, we set $\varepsilon_i(b) = \sum_i \varepsilon_i(b) \Lambda_i$ and $\varphi_i = \sum_i \varphi_i(b) \Lambda_i$, where $\varepsilon_i(b) = \max \{ k \mid e_i^k b \neq 0 \}$ and $\varphi_i(b) = \max \{ k \mid f_i^k b \neq 0 \}$.

Definition 1.1. ([KMN1]) For $l \in Z_{>0}$ we say $B$ is a perfect crystal of level $l$ if $B$ satisfies the following conditions.

(P1) $B \otimes B$ is connected.

(P2) There exists $\lambda_0 \in P_{cl}$ such that $\text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} Z_{\leq 0} \alpha_i$ and that $\varphi_i(B_{\lambda_0}) = 1$.

(P3) There is a $U'_q(\mathfrak{g})$-module in $\text{Mod}^l(\mathfrak{g}, P_{cl})$ with a crystal pseudobase $(L, B')$ such that $B$ is isomorphic to $B'/\{ \pm 1 \}$.
(P4) For any \( b \in B \), we have \( \langle c, \varepsilon(b) \rangle \geq l \).

(P5) The maps \( \varepsilon \) and \( \varphi \) from \( B_{\min} := \{ b \in B \mid \langle c, \varepsilon(b) \rangle = l \} \) to \( (P_3^+) \) are bijective.

We consider the quantum affine algebra \( U_q'(g) \) corresponding to the exceptional affine algebra \( g = D_4^{(3)} \). The KR modules we treat in this paper correspond to the vertex 1 in the Dynkin diagram in section 2.1. According to the above conjecture the KR module for the pair \((r, s) = (1, l)\) is to be perfect of level \(l\). Let us look through the main text in order. In section 2 we construct a fundamental representation \( V^1 \) of \( U_q(D_4^{(3)}) \) and calculate the quantum \( R \)-matrix that will be used in the next section. In section 3 we construct a family of \( U_q'(D_4^{(3)}) \)-modules \( \{ V^l \}_{l \geq 1} \) by so-called fusion construction. This \( V^l \) is nothing but the KR module for \((1, l)\). Using a technique developed in [KMN2] one sees that \( V^l \) has a crystal pseudobase (Theorem 3.2), which confirms (P2) and (P3) of Definition 1.1. In section 4 we introduce a \( U_q(G_2) \)-crystal \( B_l \) that has the same decomposition as \( V^l \) and define the 0-action by hand. It was obtained by the investigation of the result of [Y] and computer experiments. We check in section 5 that with respect to the \((0, 1)\)-actions \( B_l \) turns into a disjoint union of \( U_q(A_2) \)-crystals (Theorem 5.3). Hence by Theorem 6.1 that states the uniqueness of such crystal, one concludes that \( B_l \) is the crystal base of \( V^l \). We stress here that this theorem can be applied to affine algebras of other types. (P1), (P4) and (P5) are checked in the last section. Thus we have obtained

**Theorem 1.2.** For \( l \in \mathbb{Z}_{>0} \) \( B_l \) is a perfect crystal of level \( l \).

The value of this theorem lies in the following fact. Let \( B(\lambda) \) denote the crystal of the irreducible highest weight \( U_q(g) \)-module of a dominant integral weight \( \lambda \). If \( B_l \) is a perfect crystal of level \( l \), then for any dominant integral weight \( \lambda \) of level \( l \) there exists a unique dominant integral weight \( \mu \) and an isomorphism of crystals

\[
B(\lambda) \simeq B(\mu) \otimes B_l.
\]

By iterating it, one can obtain the so-called Kyoto path model of \( B(\lambda) \). By analogy one may consider a path model for the crystal \( B(\infty) \) of the negative part \( U_q^-(g) \) of \( U_q(g) \). For this purpose one needs a coherent family of perfect crystals (Definition 7.2). At the final stage we show that our family of perfect crystals \( \{ B_l \}_{l \geq 1} \) are coherent. It implies that there is a limit \( B(\infty) \) of \( \{ B_l \}_{l \geq 1} \) and an isomorphism of crystals

\[
B(\infty) \simeq B(\infty) \otimes B(\infty).
\]

2. \( U_q'(D_4^{(3)}) \) and Its Fundamental Representation

2.1. Quantum affine algebra \( U_q'(D_4^{(3)}) \). We collect necessary data for the affine Lie algebra \( D_4^{(3)} \). Let \( \{ \alpha_0, \alpha_1, \alpha_2 \} \), \( \{ h_0, h_1, h_2 \} \) and \( \{ \Lambda_0, \Lambda_1, \Lambda_2 \} \) be the set of simple roots, simple coroots and fundamental weights, respectively. The generalized Cartan matrix \( (\langle h_i, \alpha_j \rangle)_{i,j=0,1,2} \) is given by

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -3 \\
0 & -1 & 2
\end{pmatrix},
\]

and its Dynkin diagram is depicted as follows.

0 → 1 → 2
The standard null root $\delta$ and the canonical central element $c$ are given by 
$$\delta = \alpha_0 + 2\alpha_1 + \alpha_2 \quad \text{and} \quad c = h_0 + 2h_1 + 3h_2.$$ 

The affine Lie algebra $D_4^{(3)}$ contains the finite-dimensional simple Lie algebra $G_2$. Its fundamental weights are given by $\Lambda_1 = \Lambda_1 - 2\Lambda_0$, $\Lambda_2 = \Lambda_2 - 3\Lambda_0$.

The quantum affine algebra $U'_q(D_4^{(3)})$ is the associative algebra over $\mathbb{Q}(q)$ generated by $\{e_i, f_i, t_i^{\pm 1} \mid i = 0, 1, 2\}$ satisfying the relations:
$$t_i t_j = t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1,$$
$$t_i e_j t_i^{-1} = q_i^{\langle h_i, \alpha_j \rangle} e_j, \quad t_i f_j t_i^{-1} = q_i^{-\langle h_i, \alpha_j \rangle} f_j, \quad [e_i, f_j] = \delta_{ij} t_i^{-1} - t_i^{-1},$$
$$\sum_{n=0}^l (-1)^n e_i^{(n)} e_j^{(l-n)} f_j f_i^{(l-n)} = 0 \text{ where } i \neq j, l = 1 - \langle h_i, \alpha_j \rangle.$$ 

Here we use the following notations: $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{m=1}^n [m]_i$, $e_i^{(n)} = e_i^n/[n]_i!$, $f_i^{(n)} = f_i^n/[n]_i!$ with $q_0 = q_1 = q$, $q_2 = q^3$.

The algebra $U_q(D_4^{(3)})$ is defined by introducing another generator $q^d$, so $U'_q(D_4^{(3)})$ is its subalgebra. The algebra $U'_q(D_4^{(3)})$ has subalgebras $U_q(G_2)$ and $U_q(A_2)$ generated by $\{e_i, f_i, t_i^{\pm 1} \mid i = 1, 2\}$ and $\{e_i, f_i, t_i^{\pm 1} \mid i = 0, 1\}$, respectively.

2.2. Fundamental representation. Let $V^1 = V^{G_2}(\Lambda_1) \oplus V^{G_2}(0)$ denote the direct sum of the irreducible highest weight $U_q(G_2)$-modules $V^{G_2}(\Lambda_1)$ and $V^{G_2}(0)$ with highest weights $\Lambda_1$ and 0. Let $\{v_1, v_2, v_3, v_0, v_2, v_1, v_0\}$ be a basis of $V^1$, where $v_0$ belongs to $V^{G_2}(0)$. $V^1$ is endowed with the $U'_q(D_4^{(3)})$-module structure.

First, if $v$ is a weight vector of weight $\lambda$, we have $t_i v = q_i^{\langle h_i, \lambda \rangle} v$. The weight of each basis vector is given respectively by $\Lambda_1 - 2\Lambda_0, \Lambda_2 - \Lambda_1 - \Lambda_0, \Lambda_1 - \Lambda_2 - \Lambda_0, 0, \Lambda_0 + \Lambda_2 - 2\Lambda_1, \Lambda_0 + \Lambda_1 - \Lambda_2, 2\Lambda_0 - \Lambda_1, 0$. The actions of $e_i, f_i$ are given as follows.

$$e_0 v_1 = v_0 + \frac{1}{2} v_0, \quad e_0 v_2 = v_2, \quad e_0 v_3 = v_3, \quad e_0 v_0 = v_1, \quad e_0 v_0 = \frac{[3]}{[2]} v_1,$$
$$f_0 v_1 = v_0 + \frac{1}{2} v_0, \quad f_0 v_2 = v_2, \quad f_0 v_3 = v_3, \quad f_0 v_0 = v_1, \quad f_0 v_0 = \frac{[3]}{[2]} v_1,$$
$$e_1 v_2 = v_1, \quad e_1 v_0 = [2] v_3, \quad e_1 v_3 = v_0, \quad e_1 v_1 = v_2,$$
$$f_1 v_2 = v_1, \quad f_1 v_0 = [2] v_3, \quad f_1 v_3 = v_0, \quad f_1 v_1 = v_2,$$
$$e_2 v_3 = v_2, \quad e_2 v_2 = v_3, \quad e_2 v_1 = v_0,$$
$$f_2 v_3 = v_2, \quad f_2 v_2 = v_3.$$

If the action of some basis vector is not written, then we should understand that it is 0. Hereafter $[m]$ always means $[m]_0$.

Set $A_2 = \{f(q)/g(q) \mid f(q), g(q) \in \mathbb{Z}[q], g(0) = 1\}$ and $K_2 = A_2[q^{-1}]$. We define $U'_q(\mathfrak{g})_{K_2}$ as the $K_2$-subalgebra of $U'_q(\mathfrak{g})$ generated by $\{e_i, f_i, t_i^{\pm 1}\}$. It is easy to see that $V^1$ admits an $U'_q(\mathfrak{g})_{K_2}$-submodule $V^1_{K_2}$. In the subsequent section we need a polarization ( . ) on $V^1$ such that $(V^1_{K_2}, V^1_{K_2}) \subset K_2$. See section 2 of [KMN2] for the polarization. It is constructed as follows. It is known ([K1]) that for any dominant integral weight $\lambda$ the irreducible highest weight $U_q(G_2)$-module $V^{G_2}(\lambda)$ of highest
weight \lambda has a polarization. Let \((\ , \ )_1\) be such polarization on \(V^{G_2}(A_1)\) normalized as \((v_1, v_1)_1 = 1\). We define a symmetric bilinear form \((\ , \ )\) on \(V^1\) by requiring
\[
(u, v) = (u, v)_1 \quad \text{for} \quad u, v \in V^{G_2}(A_1),
\]
\[
(u, v_\phi) = 0 \quad \text{for} \quad u \in V^{G_2}(A_1),
\]
\[
(v_\phi, v_\phi) = q^{|2|/2}.
\]
Then \((\ , \ )\) satisfies
\[
(t_i u, v) = (u, t_i v), \quad (e_i u, v) = (u, q^{-1}_{i}t^{-1}_i f_i v), \quad (f_i u, v) = (u, q^{-1}_i t_i e_i v)
\]
for all \(u, v \in V^1\) and it becomes a polarization. \((V^1_{K_2}, V^1_{K_2}) \subset K_2\) can also be checked.

As a \(U_q(G_2)\)-module, the tensor product \(V^1 \otimes V^1\) decomposes into
\[
(2.1) \quad V^1 \otimes V^1 \cong V^{G_2}(2A_1) \oplus V^{G_2}(2A_2) \oplus V^{G_2}(A_1)^{\otimes 3} \oplus V^{G_2}(0)^{\otimes 2}.
\]

A highest weight vector in each irreducible component is listed below.
\[
\begin{align*}
\quad u_{2A_1} &= v_1 \otimes v_1, \\
\quad u_{A_2} &= v_1 \otimes v_2 - qv_2 \otimes v_1, \\
\quad u_{A_1}^{(1)} &= v_1 \otimes v_0, \\
\quad u_{A_1}^{(2)} &= v_\phi \otimes v_1, \\
\quad u_{A_1}^{(3)} &= v_1 \otimes v_0 - qv_0 \otimes v_1 - q^2|2|v_2 \otimes v_3 + q^3|2|v_3 \otimes v_2, \\
\quad u_0^{(1)} &= v_\phi \otimes v_0, \\
\quad u_0^{(2)} &= v_1 \otimes v_1 + q^{10}v_1 \otimes v_1 - qv_2 \otimes v_2 - q^9v_2 \otimes v_2 + q^4v_3 \otimes v_3 \\
&\quad \quad + q^6v_3 \otimes v_3 - q^4|2|v_0 \otimes v_0.
\end{align*}
\]

Here lower indices signify the highest weights. For the action on the tensor product, we use the lower coproduct. Namely,
\[
\Delta(e_i) = e_i \otimes t_{i}^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + t_{i} \otimes f_i.
\]

2.3. Calculation of the \(R\)-matrix. Let \(V^1_x = Q[x, x^{-1}] \otimes V^1\) be the \(U'_q(D_4^{(3)})\)-module with the actions of \(e_i, f_i, t_i\) replaced with \(x^{h_{i0}}e_i, x^{-h_{i0}}f_i, \ t_i\), respectively. The \(R\)-matrix \(R(x, y)\) for \(V^1 \otimes V^1\) is an operator
\[
R(x, y) : V^1_x \otimes V^1_y \rightarrow V^1_y \otimes V^1_x
\]
commuting with the actions of \(U'_q(D_4^{(3)})\). It is unique up to a scalar multiple and satisfies the following properties.
\[
\begin{align*}
(1) \quad & R(x, y) \in Q(q)[x/y, y/x] \otimes \text{End}(V^1 \otimes V^1). \\
(2) \quad & \text{The Yang-Baxter equation holds:}
\quad (R(y, z) \otimes 1)(1 \otimes R(x, z))(R(x, y) \otimes 1)
\quad = (1 \otimes R(x, y))(R(x, z) \otimes 1)(1 \otimes R(y, z)).
\end{align*}
\]
\[
(3) \quad R(x, y)R(y, x) \in Q(q)[x/y, y/x].
\]
By $U_\ell(G_2)$-linearity and (2.1), we have

$$R(u_{2\Lambda_1}) = a^{2\Lambda_1}u_{2\Lambda_1}, \quad R(u_{\Lambda_2}) = a^{\Lambda_2}u_{\Lambda_2},$$

$$R(u^{(i)}_{\Lambda_1}) = \sum_{j=1}^{3} a^{\Lambda_1}_{ij}u^{(j)}_{\Lambda_1} \ (i = 1, 2, 3), \quad R(u^{(i)}_{0}) = \sum_{j=1}^{2} a^{0}_{ij}u^{(j)}_{0} \ (i = 1, 2).$$

To calculate these coefficients we prepare

**Lemma 2.1.** We have the following relations.

1. $f_0f_1f_2u_{2\Lambda_2} = (qxy)^{-1}(x-q^2y)u_{2\Lambda_1}$
2. $f_0u^{(1)}_{\Lambda_1} = (q^2y)^{-1}[3]/[2]u_{2\Lambda_1}$
3. $f_0u^{(2)}_{\Lambda_1} = x^{-1}[3]/[2]u_{2\Lambda_1}$
4. $f_0u^{(3)}_{\Lambda_1} = (q^2xy)^{-1}(x-q^8y)u_{2\Lambda_1}$
5. $f_0^2f_1f_2f_1u^{(1)}_{\Lambda_1} = (qxy)^{-1}[3]u_{2\Lambda_1}$
6. $f_0^2f_1f_2f_1u^{(2)}_{\Lambda_1} = (qxy)^{-1}[3]u_{2\Lambda_1}$
7. $f_0^2f_1f_2f_1u^{(3)}_{\Lambda_1} = (qxy)^{-2}[2](2)(x^2-q^8y^2) - q^3(1-q^2)xyu_{2\Lambda_1}$
8. $f_0^3(f_1f_2f_1)^2u^{(1)}_{\Lambda_1} = (x^2y)^{-1}[2][3]^2u_{2\Lambda_1}$
9. $f_0^3(f_1f_2f_1)^2u^{(2)}_{\Lambda_1} = (q^2xy)^{-1}(2)[3]^2u_{2\Lambda_1}$
10. $f_0^3(f_1f_2f_1)^2u^{(3)}_{\Lambda_1} = (qxy)^{-2}[2][3](x-q^8y)u_{2\Lambda_1}$
11. $f_0^3u^{(1)}_{0} = (qxy)^{-1}[3]^2/[2]u_{2\Lambda_1}$
12. $f_0^3u^{(2)}_{0} = (q^2xy)^{-2}[2](x^2+q^{14}y^2) - q^7xyu_{2\Lambda_1}$
13. $f_0^3(f_1f_2f_1)^2f_0u^{(1)}_{0} = (q^2xy)^{-1}[3]^3(x^2+q^2y^2)u_{2\Lambda_1}$
14. $f_0^3(f_1f_2f_1)^2f_0u^{(2)}_{0} = (q^2xy)^{-1}[2][3](x-q^8y)(x-q^8y) + q^3[3](1+q^{10})xyu_{2\Lambda_1}$

From this one can calculate the coefficients $a^{2\Lambda_1}, a^{\Lambda_2}, a^{\Lambda_1}_{ij} (i, j = 1, 2, 3), a^{0}_{ij} (i, j = 1, 2)$. Let $P_{2\Lambda_1}, P_{\Lambda_2}, P^{(i)}_{\Lambda_1} (i = 1, 2, 3), P^{(i)}_{0} (i = 1, 2)$ be the projections from $V^1 \otimes V^1$ onto $U_q(G_2)$-submodule $V^{G_2}(\mathcal{T}_1), V^{G_2}(\mathcal{T}_2), U_q(G_2)u^{(i)}_{\Lambda_1}, U_q(G_2)u^{(i)}_{0}$, respectively. Let $\iota^{(i,j)}_{\Lambda_1} (i, j = 1, 2, 3)$ (resp. $\iota^{(i,j)}_{0} (i, j = 1, 2)$) be the $U_q(G_2)$-isomorphism sending $u^{(j)}_{\Lambda_1}$ to $u^{(i)}_{\Lambda_1}$ (resp. $u^{(j)}_{0}$ to $u^{(i)}_{0}$). Then we have the spectral decomposition of the $R$-matrix.

**Proposition 2.2.** Let $z = x/y$. Up to a multiple of an element of $\mathbb{Q}(q)(z)$, the $R$-matrix is of the following form

$$R(x, y) = (1-q^2z)(1-q^6z)(1+q^4z+q^8z^2)P_{2\Lambda_1}$$
$$+ (z-q^2)(1-q^6z)(1+q^4z+q^8z^2)P_{\Lambda_2}$$
$$+ \sum_{i,j=1}^{3} a^{\Lambda_1}_{ij} \iota^{(j,i)}_{\Lambda_1} P^{(i)}_{\Lambda_1} + \sum_{i,j=1}^{2} a^{0}_{ij} \iota^{(j,i)}_{0} P^{(i)}_{0},$$
Moreover, $a_{ij}^1, a_{ij}^0$ are given by

\begin{align*}
  a_{11}^1 &= a_{22}^1 = (1 - q^6)z(1 - q^{12}z)/(1 + q^2), \\
  a_{12}^1 &= a_{21}^3 = q^2(1 - z)(1 - q^6z)((1 + q^2)(1 + q^6z^2) + (q^2 + q^6)z)/(1 + q^2), \\
  a_{13}^1 &= a_{23}^3 = q(1 - q^6)z(1 - z^3)/(1 + q^2), \\
  a_{31}^3 &= a_{32}^3 = (1 + q^2)^2(1 + q^8)a_{13}^3, \\
  a_{33}^3 &= (1 - q^6z)(q^2(1 + q^2)(z^3 - q^6) + (1 - q^2)(1 - q^6)z(z - q^4))/(1 + q^2), \\
  a_{11}^0 &= (1 - q^2)(q^2 + q^{14}z^2) - (1 + q^8)z(q^2 + q^8z^2) \\
  &+ (1 - q^4)(1 - q^6)(1 + q^8)z^2/(1 + q^2), \\
  a_{12}^0 &= q(1 - q^6)z(1 - z^2)/(1 + q^2), \\
  a_{21}^0 &= q(1 - q^{14})(1 - q^4 + q^8)z(1 - z^2), \\
  a_{22}^0 &= z^2(a_{11}^0|_{z \to 1/z}).
\end{align*}

Moreover,

\begin{align*}
  \det(a_{ij}^1) &= \frac{(z - q^2)^2(z^2 + q^4z + q^8)}{(1 - q^2z)^2(1 + q^4z + q^8z^2)}(a_{22}^1)^3, \\
  \det(a_{ij}^0) &= \frac{(z - q^2)(z - q^6)(z^2 + q^4z + q^8)}{(1 - q^2z)(1 - q^6z)(1 + q^4z + q^8z^2)}(a_{22}^0)^2.
\end{align*}

### 3. Fusion Construction

In this section we construct a $U'_q(D_4^{(3)})$-module $V^l$ from $V^1$ by so-called fusion construction. It is then shown that $V^l$ admits a crystal pseudobase.

#### 3.1. Review

Following section 3 of [KMN2], we review the fusion construction and rewrite a necessary proposition.

Let $l$ be a positive integer and $\mathcal{S}_l$ the $l$-th symmetric group. Let $s_i$ be the simple reflection which interchanges $i$ and $i + 1$, and let $l(w)$ be the length of $w \in \mathcal{S}_l$. Let $\mathfrak{g}$ be an affine Lie algebra and $V$ a finite-dimensional $U'_q(\mathfrak{g})$-module which has a $U'_q(\mathfrak{g})_{K_2}$-submodule $V_{K_2}$. Assume that $V$ has a polarization $(\ , \ )$ such that $(V_{K_2}, V_{K_2}) \subset K_2$. Assume also that $V$ admits a crystal base which is perfect of level 1. Let $R(x,y)$ denote the $R$-matrix for $V \otimes V$. For any $w \in \mathcal{S}_l$ we construct a $U'_q(\mathfrak{g})$-linear map $R_w(x_1, \ldots, x_l) : V_{x_1} \otimes \cdots \otimes V_{x_l} \to V_{x_{w(l)}} \otimes \cdots \otimes V_{x_{w(l)}}$ by

\begin{align*}
  R_1(x_1, \ldots, x_l) &= 1, \\
  R_w(x_1, \ldots, x_{l}) &= R(x_{i_1}, x_{i_2}) \otimes \left( \bigotimes_{j<i} \text{id}_{V_{x_j}} \right), \\
  R_{ww'}(x_1, \ldots, x_{l}) &= R_w(x_{w(1)}, \ldots, x_{w(l)}) \circ R_{w'}(x_1, \ldots, x_{l}) \text{ for } w, w' \text{ such that } l(ww') = l(w) + l(w').
\end{align*}

Fix $r \in \mathbb{Z}_{>0}$. For each $l \in \mathbb{Z}_{>0}$, we put

\begin{align*}
  R_l &= R_{qw}(q^{r(l-1)}, q^{r(l-3)}, \ldots, q^{r(l-1)}), \\
  V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}} &\to V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}},
\end{align*}
where $w_0$ is the longest element of $G$. Then $R_l$ is a $U'_q(g)$-linear homomorphism. Define

$$V^l = \text{Im } R_l.$$ 

Let us denote by $W$ the image of

$$R(q^r, q^{-r}) : V_q^r \otimes V_{q^{-r}} \longrightarrow V_{q^{-r}} \otimes V_q^r$$ 

and by $N$ its kernel. Then we have

$V^l$ considered as a submodule of $V^l = V_{q^{-r}(l-1)} \otimes \cdots \otimes V_q^{r(l-1)}$

is contained in $\bigcap_{i=0}^{l-2} V^i \otimes W \otimes V^{l-2-i}$. 

Similarly, we have

$V^l$ is a quotient of $V^l / \sum_{i=0}^{l-2} V^i \otimes N \otimes V^{l-2-i}$. 

Let $P$ be the weight lattice of $g$ and set $P_{cl} = P / \mathbb{Z} \delta$. Let $\lambda_0$ be an element of $P_{cl}$ such that

$$\langle h_0, \lambda_0 + j\alpha_0 \rangle \geq 0$$ 

for $i \neq 0$ and $0 \leq j \leq m$. 

Let $I$ be the index set of the simple roots of $g$ and $g_{I\setminus\{0\}}$ the finite-dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-vertex of that of $g$. Let $V(\lambda)$ be the irreducible $U'_q(g_{I\setminus\{0\}})$-module with highest weight $\lambda$.

**Proposition 3.1.** (Proposition 3.4.5 of [KMN2]) Let $m$ be a positive integer and assume the following conditions:

(i) $\langle h_i, 1_{\lambda_0 + j\alpha_0} \rangle \geq 0$ for $i \neq 0$ and $0 \leq j \leq m$.

(ii) $\dim(V^l)_{\lambda_0 + k\alpha_0} \leq \sum_{j=0}^{m} \dim V(l_{\lambda_0 + j\alpha_0})_{\lambda_0 + k\alpha_0}$ for $0 \leq k \leq m$.

(iii) There exists $i_1 \in I$ such that $\{ i \in I \mid \langle h_0, \alpha_i \rangle < 0 \} = \{ i_1 \}$.

(iv) $-\langle h_0, 1_{\lambda_0 - \alpha_{i_1}} \rangle \geq 0$.

Then we have

$V^l \simeq \bigoplus_{j=0}^{m} V(l_{\lambda_0 + j\alpha_0})$ as a $U'_q(g_{I\setminus\{0\}})$-module 

and $V^l$ admits a crystal pseudobase as a $U'_q(g)$-module.
3.2. Our case.  Set \( G = D_4^{(3)} \) and let \( V \) be the representation \( V^1 \) constructed in section 2.2. We have checked that \( V \) has a polarization such that \( (V^1_{K_1}, V^1_{K_2}) \subset K_1 \). We have also calculated the \( R \)-matrix for \( V^1 \otimes V^1 \). Set \( r = 1 \) and \( \lambda_0 = \overline{\lambda}_1 \). Then \( \overline{\lambda}_1 \) is satisfied. From Proposition 3.1 we have

\[
\varphi(z) = (1 - q^2 z)(1 - q^6 z)(1 + q^4 z + q^8 z^2).
\]

Hence \( \overline{\lambda}_1 \) is also satisfied.

**Theorem 3.2.** The \( U_q'(D_4^{(3)}) \)-module \( V^l \) constructed by the fusion construction admits a crystal pseudobase. Moreover, we have

\[
V^l \simeq \bigoplus_{j=0}^l V(j\overline{\lambda}_1) \quad \text{as a } U_q(G_2)\text{-module}.
\]

**Proof.** We use Proposition 3.1. It suffices to check the conditions (i)-(iv). Set \( \lambda_0 = \overline{\lambda}_1, m = l \). Note that \( \overline{\lambda}_1 = -\lambda_0 \). (i), (iii) and (iv) are easily checked as

(i) \( \langle h_i, l\lambda_0 + j\alpha_0 \rangle = (l - j)\delta_{ij} \geq 0 \) for \( i \neq 0 \) and \( 0 \leq j \leq l \).

(ii) \( \{ i \in I \mid \langle h_0, \alpha_i \rangle < 0 \} = \{ 1 \} \).

(iv) \( -\langle h_0, l\lambda_0 - \alpha_1 \rangle = 2l - 1 \).

We are to show (ii). By the direct calculation using Proposition 2.2, we see \( N = \ker R(q, q^{-1}) \ contains \ u_{\overline{\lambda}_1}, u^{(1)}_{\overline{\lambda}_1} - u^{(2)}_{\overline{\lambda}_1}, q(1 - q^4)u^{(1)}_{\overline{\lambda}_1} - u^{(3)}_{\overline{\lambda}_1}, 2(1 - q^4 + q^8)u^{(1)}_{\overline{\lambda}_1} - [3]u^{(2)}_{\overline{\lambda}_1} \). Hence by the explicit form of the highest weight vectors, at \( q = 1 \), \( N \) contains \( \Lambda^2 V(\overline{\lambda}_1), V(0) \wedge V(\overline{\lambda}_1) \) and \( v_\phi \otimes v_\phi + u \), where \( u \) is an element of \( V(\overline{\lambda}_1)^{\otimes 2} \). Hence, at \( q = 1 \),

\[
V^{\otimes l} / \sum V^{\otimes j} \otimes N \otimes V^{\otimes (l-2-j)}
\]

is generated by \( S^l(V(\overline{\lambda}_1)) \) and \( V(0) \otimes S^{l-1}(V(\overline{\lambda}_1)) \). Hence so is for a generic \( q \). Thus we obtain

\[
\sum_{l \geq 0} \chi (V^l)t^l \leq \sum_{l \geq 0} \chi S^l(V(\overline{\lambda}_1))t^l + \sum_{l \geq 1} \chi S^{l-1}(V(\overline{\lambda}_1))t^l
\]

\[
= (1 + t) \sum_{l \geq 0} \chi S^l(V(\overline{\lambda}_1))t^l
\]

\[
= \frac{1 + t}{(1 - t)^{\prod_{\beta \in S}(1 - e^{\beta t})(1 - e^{-\beta t})}}.
\]

where \( S = \{ \alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \} \). On the other hand, from the description of the crystal base of \( U_q(G_2)\)-module \( V(j\overline{\lambda}_1) \) in section 4.1

\[
\sum_{l \geq 0} \left( \sum_{0 \leq j \leq l} \chi V(j\overline{\lambda}_1) \right) t^j
\]

\[
= \frac{1}{1 - t} \sum_{j \geq 0} \chi V(j\overline{\lambda}_1)t^j
\]

\[
= \frac{1 + t}{(1 - t)^{\prod_{\beta \in S}(1 - e^{\beta t})(1 - e^{-\beta t})}}.
\]

Thus we have

\[
\dim(V^l) \lambda \leq \sum_{j=0}^l \dim V(j\overline{\lambda}_1) \lambda
\]

for any \( \lambda \). The proof is completed.
4. $U'_q(D_4^{(3)})$-CRYSTAL

In this section we define a $U'_q(D_4^{(1)})$-crystal $B_l$. As a $U_q(G_2)$-crystal, $B_l$ is isomorphic to the crystal $\bigoplus_{j=0}^{1} B^{G_2}(j\Lambda_1)$ for the $U_q(G_2)$-module $\bigoplus_{j=0}^{1} V^{G_2}(j\Lambda_1)$.

4.1. $U_q(G_2)$-crystal. In [KM] the crystal graph for any finite-dimensional irreducible $U_q(G_2)$-module was given. For our purpose the description of the crystal $B^{G_2}(j\Lambda_1)$ for the highest weight module with highest weight $j\Lambda_1$ is necessary. Any element of $B^{G_2}(j\Lambda_1)$ is represented as a one-row semistandard tableau whose entries are $1,2,3,0,\bar{3},\bar{2},\bar{1}$ with the total order $1 \prec 2 \prec 3 \prec 0 \prec \bar{3} \prec \bar{2} \prec \bar{1}$ as

$$
\begin{array}{cccccccc}
1 & \cdots & 1 & 2 & \cdots & 2 & 3 & \cdots & 3 \\
\bar{w}_1 & \bar{w}_2 & \bar{w}_0 & \bar{w}_3 & \bar{w}_4 & \bar{w}_5 & \bar{w}_2 & \bar{w}_1 \\
\end{array}
$$

In the tableau, $0$ occurs at most once and the length is $j$, i.e., $w_0 = 0$ or $1$, $\sum_{i=1}^{3}(w_i+\bar{w}_i)+w_0 = j$. For instance $$\begin{array}{cccccccc}
1 & 2 & 2 & 3 & 0 & 1 & 1 & \\
\bar{w}_1 & \bar{w}_2 & \bar{w}_3 & \bar{w}_0 & \bar{w}_4 & \bar{w}_5 & \bar{w}_2 & \bar{w}_1 \\
\end{array}$$
is an element of $B^{G_2}(7\Lambda_1)$. It is also useful to introduce a coordinate representation for an element of $B^{G_2}(j\Lambda_1)$ by

$$
x_i = w_i, \quad \bar{x}_i = \bar{w}_i \quad (i = 1, 2),
\bar{x}_3 = 2\bar{w}_3 + w_0.
$$

Then we have

$$
B^{G_2}(j\Lambda_1) = \left\{ b = (x_1, x_2, x_3, \bar{x}_2, \bar{x}_3, \bar{x}_1) \in (\mathbb{Z}_{\geq 0})^6 \mid \begin{array}{c}
x_3 \equiv \bar{x}_3 \pmod{2}, \\
\sum_{i=1,2}(x_i + \bar{x}_i) + (x_3 + \bar{x}_3)/2 = j
\end{array} \right\}.
$$

Below we give the explicit crystal structure of $B^{G_2}(j\Lambda_1)$ with this parametrization. Set $(x)_+ = \max(x,0)$, then we have

$$
\begin{align*}
\tilde{e}_1 b & = \left\{ \begin{array}{ll}
\{ \ldots, \bar{x}_2 + 1, \bar{x}_1 - 1 \} & \text{if } \bar{x}_2 - \bar{x}_3 \geq (x_2 - x_3)_+ , \\
\{ \ldots, x_3 + 1, \bar{x}_3 - 1, \ldots \} & \text{if } \bar{x}_2 - \bar{x}_3 < 0 \leq x_3 - x_2 , \\
\{ x_1 + 1, x_2 - 1, \ldots \} & \text{if } (\bar{x}_2 - \bar{x}_3)_+ < x_2 - x_3 ,
\end{array} \right. \\
\tilde{f}_1 b & = \left\{ \begin{array}{ll}
\{ x_1 - 1, x_2 + 1, \ldots \} & \text{if } (\bar{x}_2 - \bar{x}_3)_+ \leq x_2 - x_3 , \\
\{ \ldots, x_3 - 1, \bar{x}_3 + 1, \ldots \} & \text{if } \bar{x}_2 - \bar{x}_3 \leq 0 < x_3 - x_2 ,
\end{array} \right. \\
\tilde{e}_2 b & = \left\{ \begin{array}{ll}
\{ \ldots, \bar{x}_3 + 2, \bar{x}_2 - 1, \ldots \} & \text{if } \bar{x}_3 \geq x_3 ,
\{ \ldots, x_2 + 1, x_3 - 2, \ldots \} & \text{if } \bar{x}_3 < x_3 ,
\end{array} \right. \\
\tilde{f}_2 b & = \left\{ \begin{array}{ll}
\{ \ldots, x_2 - 1, x_3 + 2, \ldots \} & \text{if } \bar{x}_3 \leq x_3 ,
\{ \ldots, \bar{x}_3 - 2, \bar{x}_2 + 1, \ldots \} & \text{if } \bar{x}_3 > x_3 ,
\end{array} \right.
\end{align*}
$$

$$
\begin{align*}
\varepsilon_1(b) & = \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+) , \\
\varepsilon_2(b) & = \bar{x}_2 + \frac{1}{2}(x_3 - \bar{x}_3)_+ , \\
\varphi_1(b) & = x_1 + (x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_+) , \\
\varphi_2(b) & = x_2 + \frac{1}{2}(\bar{x}_3 - x_3)_+ .
\end{align*}
$$

If $\tilde{e}_i b$ or $\tilde{f}_i b$ does not belong to $B^{G_2}(j\Lambda_1)$, namely, if $x_j$ or $\bar{x}_j$ for some $j$ becomes negative, we should understand it to be $0$. 


4.2. **Action of** \( \tilde{e}_0, \tilde{f}_0 \). For a positive integer \( l \) we introduce a \( U'_q(\mathfrak{g}) \)-crystal \( B_l \). As a \( U_q(\mathfrak{g}_2) \)-crystal,

\[
B_l = \bigoplus_{j=0}^{l} B^{G_2}(j\Lambda_1),
\]

where \( B^{G_2}(j\Lambda_1) \) is the \( U_q(\mathfrak{g}_2) \)-crystal explained in the previous subsection. To define the actions of \( \tilde{e}_0 \) and \( \tilde{f}_0 \), we introduce conditions \((E_1)-(E_6)\) and \((F_1)-(F_6)\). Set

\[
z_1 = \bar{x}_1 - x_1, \quad z_2 = \bar{x}_2 - \bar{x}_3, \quad z_3 = x_3 - x_2, \quad z_4 = (\bar{x}_3 - x_3)/2,
\]

and

\[
(F_1) \quad z_1 + z_2 + z_3 + 3z_4 \leq 0, z_1 + z_2 + 3z_4 \leq 0, z_1 + z_2 \leq 0, z_1 \leq 0,
\]

\[
(F_2) \quad z_1 + z_2 + z_3 + 3z_4 \leq 0, z_2 + 3z_4 \leq 0, z_2 \leq 0, z_1 > 0,
\]

\[
(F_3) \quad z_1 + z_3 + 3z_4 \leq 0, z_3 + 3z_4 \leq 0, z_4 \leq 0, z_2 > 0, z_1 + z_2 > 0,
\]

\[
(F_4) \quad z_1 + 2z_3 + 3z_4 > 0, z_1 + z_3 > 0, z_4 > 0, z_3 \leq 0, z_1 + z_3 \leq 0,
\]

\[
(F_5) \quad z_1 + z_2 + z_3 + 3z_4 > 0, z_3 + 3z_4 > 0, z_3 > 0, z_1 \leq 0,
\]

\[
(F_6) \quad z_1 + z_2 + z_3 + 3z_4 > 0, z_1 + z_3 + 3z_4 > 0, z_1 + z_3 > 0, z_1 > 0.
\]

\((E_i)\) \((1 \leq i \leq 6)\) is defined from \((F_i)\) by replacing \(>\) (resp. \(\leq\)) with \(\geq\) (resp. \(<\)).

We define

\[
\tilde{e}_0 b = \begin{cases} 
E_1 b := (x_1 - 1, \ldots) & \text{if } (E_1), \\
E_2 b := (\ldots, x_3 - 1, \bar{x}_3 - 1, \ldots, \bar{x}_1 + 1) & \text{if } (E_2), \\
E_3 b := (\ldots, x_3 - 2, \ldots, \bar{x}_2 + 1, \ldots) & \text{if } (E_3), \\
E_4 b := (\ldots, x_2 - 1, \ldots, \bar{x}_3 + 2, \ldots) & \text{if } (E_4), \\
E_5 b := (x_1 - 1, \ldots, x_3 + 1, \bar{x}_3 + 1, \ldots) & \text{if } (E_5), \\
E_6 b := (\ldots, \bar{x}_1 + 1) & \text{if } (E_6).
\end{cases}
\]

\[
\tilde{f}_0 b = \begin{cases} 
F_1 b := (x_1 + 1, \ldots) & \text{if } (F_1), \\
F_2 b := (\ldots, x_3 + 1, \bar{x}_3 + 1, \ldots, \bar{x}_1 - 1) & \text{if } (F_2), \\
F_3 b := (\ldots, x_3 + 2, \ldots, \bar{x}_2 - 1, \ldots) & \text{if } (F_3), \\
F_4 b := (\ldots, x_2 + 1, \ldots, \bar{x}_3 - 2, \ldots) & \text{if } (F_4), \\
F_5 b := (x_1 + 1, \ldots, x_3 - 1, \bar{x}_3 - 1, \ldots) & \text{if } (F_5), \\
F_6 b := (\ldots, \bar{x}_1 - 1) & \text{if } (F_6).
\end{cases}
\]

**Remark 4.1.**

(i) Set

\[
A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4)
\]

and \( z_1, z_2, z_3, z_4 \) are given in \((\overline{1})\). Denote the \( i \)-th component of \( A \) by \( A_i \).

Then, for \( 1 \leq i \leq 6 \), \((F_i)\) holds if and only if \( \max A = A_i \) and \( A_j < A_i \) for any \( j \) such that \( 1 \leq j < i \). Similarly, \((E_i)\) holds if and only if \( \max A = A_i \) and \( A_j < A_i \) for any \( j \) such that \( j > i \).

(ii) By (i), we have

\[
B_l = \bigcup_{1 \leq i \leq 6} \{ b \in B_l \mid b \text{ satisfies } (E_i) \} = \bigcup_{1 \leq i \leq 6} \{ b \in B_l \mid b \text{ satisfies } (F_i) \}.
\]
For $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1)$ we set

\[(4.3) \quad s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1.\]

Suppose $b \in B_t$. Looking at the rule of the action of $\tilde{\phi}_0$, carefully, we see that the coordinates of $\tilde{\phi}_0 b$ never get negative. It means that $\tilde{\phi}_0 b = 0$ occurs only when $s(b) = l$ and $b$ satisfies $(F_1)$. The case of $\tilde{e}_0$ is similar. Checking directly one can show that $B_t$ satisfies the condition: for $b, b' \in B_t$,

\[b' = \tilde{\phi}_0 b \iff b = \tilde{e}_0 b'.\]

Hence one can draw the crystal graph of $B_t$ with arrows of color $0, 1, 2$.

**Example 4.2.** Let us denote the elements of $B_1$ by

\[
\begin{align*}
1 &= (1, 0, 0, 0, 0), & 2 &= (0, 1, 0, 0, 0), & 3 &= (0, 0, 2, 0, 0), \\
0 &= (0, 0, 1, 0, 0), & 3' &= (0, 0, 0, 2, 0), & 2' &= (0, 0, 0, 1, 0), \\
1' &= (0, 0, 0, 0, 1), & \phi &= (0, 0, 0, 0, 0).
\end{align*}
\]

then, the crystal graph of $B_1$ is given as follows:

![Crystal Graph](image_url)

The arrows without number are 0-arrows.

The next two propositions are related to the action of $\tilde{e}_0, \tilde{\phi}_0$. The first one is easily proved.

**Lemma 4.3.**

1. Suppose that $b \in B_1$ satisfies $(F_1)$ and $\tilde{\phi}_0 b \in B_1$. Then $\tilde{\phi}_0 b$ also satisfies $(F_1)$.

2. Suppose that $b \in B_1$ satisfies $(E_6)$ and $\tilde{e}_0 b \in B_1$. Then $\tilde{e}_0 b$ also satisfies $(E_6)$.

**Proposition 4.4.** The values of $\varepsilon_0$ and $\varphi_0$ of an element $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1)$ of $B_t$ is given by

\[
\begin{align*}
\varphi_0(b) &= l - s(b) + \max A, \\
\varepsilon_0(b) &= l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4),
\end{align*}
\]

where $A$ is as in (4.2).

**Proof.** Notice that if $\tilde{\phi}_0 b = 0$ occurs for $b \in B_t$, then $b$ satisfies $(F_1)$. From Lemma 4.3(1), one verifies that the formula of $\varphi_0$ is correct when $b$ satisfies $(F_1)$. Thus we are left to show that $\varphi_0(\tilde{\phi}_0 b) = \varphi_0(b) - 1$ if $b, \tilde{\phi}_0 b \in B_t$. It can be checked case by case. Let $A'$ be the list $A$ for $\tilde{\phi}_0 b$ and $A_i'$ be its $i$-th component. Notice that if $b$ satisfies $(F_1)$, then $\max A' = A_i'$. □
5. Decomposition of $B_l$ as a $U_q(A_2)$-crystal

5.1. Review on $U_q(A_2)$-crystal. We review on the $U_q(A_2)$-crystal $B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1)$ of the highest weight module of highest weight $j_0\Lambda_0 + j_1\Lambda_1$. We use $\{0,1\}$ as the index set of simple roots of $A_2$. It is known that any element of $B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1)$ is uniquely expressed as $\hat{f}_0^r \hat{f}_q^p u$ for some $p,q,r$ such that $0 \leq p \leq j_0$, $0 \leq q \leq j_1$, and $0 \leq r \leq j_0 - 2p + q$. Here $u$ stands for the highest weight vector. By [KN] an element of $B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1)$ is also represented by a two-row tableau. $\hat{f}_0^r \hat{f}_q^pu$ corresponds to

$$t(p,q,r) = \frac{j_0 + j_1 - p - r}{2} \frac{2q'}{3q' - p}. $$

Below we give the crystal structure.

$$\tilde{e}_0 t(p,q,r) = \begin{cases} t(p,q,r - 1) & \text{if } r > 0, \\ 0 & \text{if } r = 0, \end{cases}$$

$$\tilde{e}_1 t(p,q,r) = \begin{cases} t(p-1,q-1,r+1) & \text{if } p > 0, p - q + r \geq 0, \\ t(p,q,r) & \text{if } p - q + r < 0, \\ 0 & \text{if } p = 0, p - q + r \geq 0, \end{cases}$$

$$\tilde{f}_0 t(p,q,r) = \begin{cases} t(p,q,r+1) & \text{if } 0 \leq r < j_0 + q - 2p, \\ 0 & \text{if } r = j_0 + q - 2p, \end{cases}$$

$$\tilde{f}_1 t(p,q,r) = \begin{cases} t(p+1,q+1,r-1) & \text{if } p \leq q < p + r, \\ t(p,q+1,r) & \text{if } p + r \leq q < j_1 + p, \\ 0 & \text{if } p + r \leq q = j_1 + p. \end{cases}$$

The remaining data $\varepsilon_i, \varphi_i$ of $t(p,q,r)$ are given by

$$\varepsilon_0 = r, \quad \varphi_0 = j_0 - 2p + q - r,$$

$$\varepsilon_1 = p + (q - p - r)_+, \quad \varphi_1 = (p - q + r)_+ + j_1 + p - q.$$

The following proposition is immediate.

Proposition 5.1. The lowest weight vector of $B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1)$ is given by $t(j_0,j_0 + j_1,j_1)$. Moreover, we have

$$t(p,q,r) = \tilde{e}_0^p \tilde{e}_1^q \tilde{f}_0^r t(j_0,j_0 + j_1,j_1),$$

where $p' = j_1 - q + p$, $q' = j_0 + j_1 - q$, $r' = j_0 + q - 2p - r$.

5.2. $U_q(A_2)$-crystal structure. In what follows in this section we investigate the structure of the crystal subgraph of $B_l$ obtained by forgetting 2-arrows.

Definition 5.2. For $l \in \mathbb{Z}_{>0}$ take integers $i, j_0, j_1$ such that

$$0 \leq i \leq l/2, \quad i \leq j_0, j_1 \leq l - i \quad \text{and} \quad j_0, j_1 \equiv l - i \pmod{3},$$

and set $y_a = (l - i - j_a)/3$ for $a = 0, 1$. We define the element $\tilde{b}^{l,i}_{j_0,j_1}$ of $B_l$ by

$$\tilde{b}^{l,i}_{j_0,j_1} = \begin{cases} (0, y_1, -2y_1 + 3y_0 + i, y_0 + i, y_0 + j_0, 0) & \text{if } j_0 \leq j_1, \\ (0, y_0, y_0 + i, 2y_1 - y_0 + i, -y_1 + 2y_0 + j_0, 0) & \text{if } j_0 > j_1. \end{cases}$$

We also define the subset $B^{l,i}_{j_0,j_1}$ of $B_l$ to be the connected component of $B_l$ generated by $\tilde{e}_a, \tilde{f}_a$ ($a = 0, 1$) that contains $\tilde{b}^{l,i}_{j_0,j_1}$. 
Our main theorem of this section is given as follows.

**Theorem 5.3.** Forgetting 2-arrows, the crystal graph $B_1$ decomposes into connected components in the following manner.

$$B_l = \bigsqcup_{i=0}^{1|\frac{1}{2}|} \iota_{0, j_0, j_1} \leq l \equiv l-1 \ (\text{mod} \ 3) B_{j_0, j_1}^{i, i}.$$  

Moreover, $B_{j_0, j_1}^{i, i}$ is isomorphic to the $U_q(A_2)$-crystal $B^{A_2}(\Lambda_0 + j_1 \Lambda_1)$.

For the proof we introduce some notations. Set

$$B_{\geq 0} = \{ (x_1, x_2, x_3, \tilde{x}_3, \tilde{x}_2, \tilde{x}_1) \in \mathbb{Z}_{\geq 0}^6 \ | \ (x_3 + \tilde{x}_3) / 2 \in \mathbb{Z}_{\geq 0} \}.$$  

Note that $B_l = \{ b \in B_{\geq 0} \ | \ s(b) \leq l \}$ where $s(b)$ was defined in (4.3). One can endow $B_{\geq 0}$ with the crystal structure by applying the same rule for $\tilde{e}_i, \tilde{f}_i$ as section 4.1 and 4.2 with $l = \infty$. Namely, $\tilde{e}_i, \tilde{f}_i$ vanish only when some coordinate becomes negative. Note that $b_{j_0, j_1}^{i, i}$ is $U_q(A_2)$-highest, i.e., $\tilde{e}_0 b_{j_0, j_1}^{i, i} = 0$ for $a = 0, 1$, as an element of $B_l$, but $\tilde{e}_0 b_{j_0, j_1}^{i, i} \neq 0$ as an element of $B_{\geq 0}$.

### 5.3. Some relations on $B_{\geq 0}$

We prepare two relations that hold on $B_{\geq 0}$.

**Lemma 5.4.** Suppose that $j_0 \leq j_1$. On $B_{\geq 0}$ we have

$$\tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1^{i, i} = \tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1^{i-1, i} \text{ if } i < j_0, p < j_0, p < q \leq j_1 + p.$$  

**Proof.** We use the table in Appendix A. Under the assumption one can show that all the cases satisfy $(F_0)$ of the rule of 0-action. Hence, if we write $x = \tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1^{i, i} = (x_1, x_2, x_3, \tilde{x}_3, \tilde{x}_2, \tilde{x}_1)$, then $\tilde{f}_0 x = (x_1, x_2, x_3, \tilde{x}_3, \tilde{x}_2, \tilde{x}_1 - 1)$. On the other hand, in each case we also have

$$x |_{(j_0, j_1, q) \rightarrow (j_0 - 1, j_1 - 1, q - 1)} = (x_1, x_2, x_3, \tilde{x}_3, \tilde{x}_2, \tilde{x}_1 - 1).$$

Hence we have the desired relation. Note that $l$ should be replaced by $l - 1$ so that $y_0, y_1$ remain the same.

**Lemma 5.5.** Suppose that $j_0 \leq j_1$. On $B_{\geq 0}$ we have

$$\tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1^{i, i} = \tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1^{i+1, i} \text{ if } q \leq p < j_0.$$  

**Proof.** We again use the table in Appendix A. Under the assumption the cases that occur are

1. $0 \leq p \leq i$, (i) $0 \leq q \leq j_0 - i + p$,
2. $i \leq p$, (i) $0 \leq q \leq j_0 - p + i$,
3. $i \leq p$, (ii) $j_0 - p + i \leq q \leq j_1 + p - i$.

Each case satisfies $(F_0), (F_3), (F_2)$, respectively. In each case the action of $\tilde{f}_0$ is realized by replacing $p$ with $p + 1$.

### 5.4. Proof of Theorem 5.3

We frequently use the following condition for $(p, q, r)$.

(C) $0 \leq p \leq j_0, \ p \leq q \leq j_1 + p, \ 0 \leq r \leq j_0 + q - 2p$.

[Step 1] We show $B_{j_0, j_1}^{i, i} \simeq B^{A_2}(\Lambda_0 + j_1 \Lambda_1)$ for $j_0 \leq j_1$.

**Proof.** Due to the fact that for $b, b' \in B_l$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$ $(i = 0, 1)$, it suffices to show for (C)
(i) $\tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{b}_{j_0,j_1}^{r,1,i} \in B_l$.
(ii) $b = \tilde{f}_0^{q-2p} \tilde{f}_1^{q} \tilde{b}_{j_0,j_1}^{r,1,i}$ satisfies $(F_1)$ and $s(b) = l$,
and as an element of $B_l$

(iii) $\tilde{f}_1 \tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{b}_{j_0,j_1}^{r,1,i} = \begin{cases} 
\tilde{f}_0^{-1} \tilde{f}_1 \tilde{f}_0^{q+1} \tilde{b}_{j_0,j_1}^{r,1,i} & \text{if } p < q < p + r, \\
\tilde{f}_0 \tilde{f}_1 \tilde{f}_0^{q+1} \tilde{b}_{j_0,j_1}^{r,1,i} & \text{if } p + r < q < j_1 + p, \\
0 & \text{if } p + r < j_1 + p,
\end{cases}$

(iv) $\tilde{e}_0 \tilde{f}_0 \tilde{f}_1 \tilde{b}_{j_0,j_1}^{r,1,i} = 0$.
(v) $\tilde{e}_1 \tilde{f}_1 \tilde{f}_0 \tilde{b}_{j_0,j_1}^{r,1,i} = 0$ if $r \geq q$.

We prove (i)-(v) by using induction on $l$.

(i) Suppose that $r > 0, i < j_0, p < j_0, p < q$. By Lemma 5.4 we have

$$\tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{b}_{j_0,j_1}^{r,1,i} = \tilde{f}_0^{-1} \tilde{f}_1 \tilde{f}_0^{q-1} \tilde{b}_{j_0,j_1}^{r,1,i}.$$ 

If $r < j_0 + q - 2p$, we get $\tilde{f}_0^{-1} \tilde{f}_1 \tilde{f}_0^{q-1} \tilde{b}_{j_0,j_1}^{r,1,i}$ satisfies $(F_1)$ and $s(b) = l$.

(ii) The claim can be checked directly from Appendix D.

(iii) $\tilde{e}_0 \tilde{f}_0 \tilde{f}_1 \tilde{b}_{j_0,j_1}^{r,1,i} = 0$.

(iv) $\tilde{e}_1 \tilde{f}_1 \tilde{f}_0 \tilde{b}_{j_0,j_1}^{r,1,i} = 0$ if $r \geq q$.

We prove (i)-(v) by using induction on $l$.

(i) Suppose that $r > 0, i < j_0, p < j_0, p < q$. By Lemma 5.4 we have

$$\tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{b}_{j_0,j_1}^{r,1,i} = \tilde{f}_0^{-1} \tilde{f}_1 \tilde{f}_0^{q-1} \tilde{b}_{j_0,j_1}^{r,1,i}.$$ 

If $r < j_0 + q - 2p$, we get $\tilde{f}_0^{-1} \tilde{f}_1 \tilde{f}_0^{q-1} \tilde{b}_{j_0,j_1}^{r,1,i}$ satisfies $(F_1)$ and $s(b) = l$.

(ii) The claim can be checked directly from Appendix D.

(iii) $\tilde{e}_0 \tilde{f}_0 \tilde{f}_1 \tilde{b}_{j_0,j_1}^{r,1,i} = 0$.

(iv) $\tilde{e}_1 \tilde{f}_1 \tilde{f}_0 \tilde{b}_{j_0,j_1}^{r,1,i} = 0$ if $r \geq q$.
If \( r < j_0 + q \), the RHS is 0 by induction hypothesis. If \( r = j_0 + q, \) \( \tilde{p} f_{j_0} p f_{j_0+j_1} = 0 \)
can be checked directly from Appendix A.

Note that \( r = 0 \) implies \( q = 0 \) and \( j_0 = 0 \) implies \( i = j_0 \). If \( i = j_0 \), the claim is
checked from Appendix A. If \( q = 0 \), it is checked from Appendix A. \( \square \)

[Step 2] Next we show

\[
B_{j_0,j_1}^{l,i} \simeq B^{A_2}(j_0 A_0 + j_1 A_1) \text{ for } j_0 > j_1.
\]

Define an involution on \( B_{\geq 0} \) by

\[
b = (x_1, x_2, x_3, x_4, x_5) \mapsto (\bar{x}_1, \bar{x}_2, \bar{x}_3, x_3, x_1) = b^\vee.
\]

We prove two lemmas related to this involution. The next one follows immediately
from the definitions.

**Lemma 5.6.** Let \( b \in B_1. \) For \( i = 0, 1, 2, \)

\( 1) \) if \( \tilde{e}_i b \neq 0, \) then \( (\tilde{e}_i b)^\vee = \tilde{f}_i (b^\vee). \)

\( 2) \) if \( \tilde{f}_i b \neq 0, \) then \( (\tilde{f}_i b)^\vee = \tilde{e}_i (b^\vee). \)

**Lemma 5.7.** Suppose that \( j_0 \leq j_1. \) As an element of \( B_1, \) we have for (C)

\[
(f_0 \tilde{f}_1 \tilde{p} f_{j_0+j_1}^{l,i})^\vee = \tilde{f}_1 \tilde{f}_0 \tilde{p} f_{j_0}^{l,i},
\]

where

\[
p' = j_1 - q + p, \quad q' = j_0 + j_1 - q, \quad r' = j_0 + q - 2p - r.
\]

**Proof.** By the result of Step 1 and Proposition 5.1, we have

\[
\tilde{f}_1 \tilde{f}_0 \tilde{p} f_{j_0+j_1}^{l,i} = \tilde{e}_0' \tilde{e}_1' \tilde{e}_0 \tilde{p} f_{j_0+j_1}^{l,i},
\]

where \( \tilde{f}_{j_0,j_1}^{l,i} = \tilde{f}_{j_1} \tilde{f}_{j_0} \tilde{f}_{j_0+j_1}^{l,i}. \) Apply \( \vee \) on both sides and use
the previous lemma. We obtain \( (f_0 \tilde{f}_1 \tilde{p} f_{j_0+j_1}^{l,i})^\vee = \tilde{f}_1 \tilde{f}_0 \tilde{p} f_{j_0}^{l,i}, \)
which can be shown from the table in Appendix A and the definition (5.6).

**Proof of Step 2.** Apply \( \vee \) on both sides of (i)-(v) in the proof of Step 1. Use Lemma 5.7
and interchange \( (p, q, r) \) and \( (p', q', r'). \) Substituting \( (p', q', r') \) with \( j_0 \) and \( j_1 \)
interchanged we obtain for \( 0 \leq p \leq j_1, p \leq q \leq j_0 + p, 0 \leq r \leq j_1 - 2p + q, \)

\( i') \quad f_0 \tilde{f}_1 \tilde{p} f_{j_0}^{l,i}, j_0 \in B_1,
\)

\( ii') \quad \tilde{e}_0 f_0 \tilde{f}_1 \tilde{p} f_{j_0+j_1}^{l,i} = \begin{cases} \tilde{f}_{j_0} \tilde{f}_1 \tilde{p} f_{j_0}^{l,i} & \text{if } r > 0, \\ 0 & \text{if } r = 0, \end{cases} \)

\( iii') \quad \tilde{e}_1 f_0 \tilde{f}_1 \tilde{p} f_{j_0+j_1}^{l,i} = \begin{cases} \tilde{f}_0 \tilde{f}_1 \tilde{p} f_{j_1}^{l,i} & \text{if } p - q + r < 0, \\ 0 & \text{if } p > 0, p - q + r \geq 0, \end{cases} \)

\( iv') \quad f_0 \tilde{f}_1 + q - 2p \tilde{f} f_{j_0+j_1}^{l,i} = 0, 
\)

\( v') \quad \tilde{f}_1 f_0 \tilde{f}_1 \tilde{p} f_{j_0+j_1}^{l,i} = 0 \text{ if } p + r \leq q = j_0 + p. \)

These relations are enough to check our claim. \( \square \)

[Step 3] We are left to show

\[
B_1 = \bigcup_{j_0,j_1 \equiv \ell \text{ (mod 3)}} B_{j_0,j_1}^{l,i}.
\]
Proof. It suffices to show
\[ \sharp B_l = \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \sum_{j_0, j_1 \leq l-i \, (\text{mod} \, 3)} \sharp B_{j_0, j_1}^{l,i}. \]
By Step 1 and 2 we have
\[ \sharp B_{j_0, j_1}^{l,i} = \sharp B^{A_2}(j_0 \Lambda_0 + j_1 \Lambda_1) = \frac{(1 + j_0)(1 + j_1)(2 + j_0 + j_1)}{2}. \]
By direct calculation we have
\[ \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \sum_{j_0, j_1 \leq l-i \, (\text{mod} \, 3)} \sharp B_{j_0, j_1}^{l,i} = \frac{(l+1)(l+2)(l+3)^2(l+4)(l+5)}{360}. \]
On the other hand, computing \( \sharp B_l \) from the definition of \( B_l \) reads
\[ \sharp B_l = \frac{(l+6)!}{l!6!} + \frac{((l-1)+6)!}{(l-1)!6!} = \frac{(l+1)(l+2)(l+3)^2(l+4)(l+5)}{360}. \]
\[ \square \]
Thus the proof of Theorem 6.3 is completed.

6. Uniqueness Problem

In this section we deal with a certain uniqueness problem of crystals in a more general situation. In order to state our theorem precisely, we prepare some notations. Let \( \mathfrak{g} \) be an affine Lie algebra and \( I \) the index set of vertices of the corresponding Dynkin diagram. Let \( \{a_i\}_{i \in I}, \{h_i\}_{i \in I}, \{\Lambda_i\}_{i \in I} \) be the set of simple roots, simple coroots, fundamental weights. Let \( \delta \) and \( c \) be the generator of null roots and the canonical central element. Let 0 be the vertex of the Dynkin diagram as above. Let \( (\alpha_{ij})_{i,j \in I} \) be the generalized Cartan matrix of \( \mathfrak{g} \). We assume the following conditions for \( \mathfrak{g} \):
\begin{align*}
\{i \in I \mid a_{0i} < 0\} &= \{1\},
\quad (6.1) \quad a_{01} = a_{10} = -1. \quad (6.2)
\end{align*}

Namely, in the Dynkin diagram of \( \mathfrak{g}, 0 \) is connected only with 1 by a single bond. This implies \( a_0 = 2\Lambda_0 - \Lambda_1 \). We note that \( D_4^{(3)} \) we treat in this paper satisfies these conditions. We also remark that the labeling of \( I \) does not always agree with that of \( Kac \). Let \( \mathfrak{g}_{01} \) (resp. \( \mathfrak{g}_{\neq 0}, \mathfrak{g}_{\neq 0,1} \)) denote the Levi subalgebra of \( \mathfrak{g} \) corresponding to the index set \( \{0, 1\} \) (resp. \( I \setminus \{0\}, I \setminus \{0, 1\} \)). For an integral weight \( \lambda \) such that \( \langle h_i, \lambda \rangle \geq 0 \) for \( i \neq 0 \), let us denote by \( B_{\neq 0}(\lambda) \) the \( U_q(\mathfrak{g}_{\neq 0}) \)-crystal with highest weight \( \lambda \).

**Theorem 6.1.** Let \( \mathfrak{g} \) be an affine Lie algebra satisfying the conditions (6.1), (6.2). Let \( B, B' \) be \( U_q(\mathfrak{g}) \)-crystals which decompose into \( \bigoplus_{0 \leq k \leq l} B_{\neq 0}(-k\alpha_0) \) as \( U_q(\mathfrak{g}_{\neq 0}) \)-crystals. Then they are isomorphic to each other as \( U_q(\mathfrak{g}) \)-crystals.

The theorem says that under the assumptions (6.1), (6.2), there is a unique way to draw 0-arrows in the crystal graph of the \( U_q(\mathfrak{g}_{\neq 0}) \)-crystal \( \bigoplus_{0 \leq k \leq l} B_{\neq 0}(-k\alpha_0) \).

For an element \( x \) of the Weyl group \( W \) of \( \mathfrak{g} \), let \( x = s_{i_1} s_{i_2} \cdots s_{i_l} \) be a reduced expression of \( x \) by simple reflections. We define \( e_x^{\max} = e_{i_1}^{\max} e_{i_2}^{\max} \cdots e_{i_l}^{\max} \) and \( S_x = S_{i_1} S_{i_2} \cdots S_{i_l} \). Here \( e_{i}^{\max}a = e_{i}^{\varepsilon_{i}(b)}a \) and \( S_i \) is the Weyl group action on crystals. Note that \( e_{i}^{\max} \) or \( S_i \) do not depend on the choice of a reduced expression. For these matters along with basic notations on crystals, see \( K2 \).
The rest of this section is devoted to the proof of Theorem 6.3. Let $B$ and $B'$ be as in Theorem 6.3. Let $\psi : B \to B'$ be a unique $U_q(\mathfrak{g}_{\neq 0})$-crystal isomorphism. It is enough to show that $\psi$ commutes with $\tilde{e}_0$ and $\tilde{f}_0$.

Set $\lambda = -\alpha_0$. For a weight $\mu$ of the form $w(k\lambda)$ ($w \in W$, $0 \leq k \leq l$), we denote by $u_\mu$ a unique element of $B_{\neq 0}(k\lambda)$ with weight $\mu$. We denote by the same letter the corresponding element of $B$.

For $b \in B$ or $b \in B'$, let us denote by $B_{01}(b)$ the connected $U_q(\mathfrak{g}_{01})$-subcrystal containing $b$. Similarly, we denote by $B_{0}(b)$ the connected $U_q(\mathfrak{g}_0)$-subcrystal containing $b$.

We prepare several lemmas. The next lemma is the same as Sublemma 6.2 of [KS]. Let us denote by $w$ the longest element of the Weyl group of $\mathfrak{g}_{\neq 0,1}$.

**Lemma 6.2.** If $\omega_1 = \delta - \alpha_0 - \alpha_1 = s_1(\delta - \alpha_0)$. Moreover, the length $l(s_1 w s_1 w s_1)$ of $s_1 w s_1 w s_1$ is equal to $2l(w) + 3$.

Hence, $u\omega_1 = -s_1\alpha_0$ as elements of $P_{\geq 1}$, and $s_0 = s_1 w s_1 w s_1$ as automorphisms of $P$. Moreover $\tilde{e}_{s_2 w s_1 w s_1}^{\max} = \tilde{e}_{w}^{\max} \tilde{e}_{w}^{\max} \tilde{e}_{w}^{\max} \tilde{e}_{w}^{\max}$.

**Lemma 6.3.** $\tilde{e}_0^u u_{l\lambda} = u_{(l(k)-k)\lambda}$ for $0 \leq k \leq 2l$.

Proof. One knows that $S_1 u_{l\lambda}$ is the lowest weight vector of the $U_q(\mathfrak{g}_{01})$-crystal $B^{A_2}(l(\alpha_0 + \alpha_1))$. With the notations in section 5.1, $\tilde{e}_0^u u_{l\lambda}$ is identified with $l(0,l,2l-k)$. If $0 \leq k \leq l$, then $\varepsilon_i(\tilde{e}_0^u u_{l\lambda}) = 0$ from (5.5). We also have $\varepsilon_i(\tilde{e}_0^u u_{l\lambda}) = \varepsilon_i(u_{l\lambda}) = 0$ for $i \in \Gamma \setminus \{0,1\}$. Hence $\tilde{e}_0^u u_{l\lambda}$ is a $\mathfrak{g}_{\neq 0}$-highest vector of weight $(l(k)-k)\lambda$ and coincides with $u_{(l(k)-k)\lambda}$.

Similarly, for $0 \leq k \leq l$, one can show $\tilde{f}_0^u u_{-l\lambda} = u_{-(l(k)-k)\lambda}$, which completes the proof.

By this lemma, $B_{01}(S_1 u_{l\lambda})$ contains all the $\mathfrak{g}_{\neq 0}$-highest weight vectors.

**Lemma 6.4.** The restriction of $\psi$ gives a $U_q(\mathfrak{g}_{01})$-crystal isomorphism

$$B_{01}(S_1 u_{l\lambda}) \cong B_{01}(\psi(S_1 u_{l\lambda})).$$

Proof. Any element of $B_{01}(S_1 u_{l\lambda})$ can be written as $f_1^d f_0^c f_1^a f_1^b S_1 u_{l\lambda}$ with $a \leq l$, $a \leq c \leq a + l$ and $d \leq l - 2a + c$. Set $b = f_0^c f_1^b S_1 u_{l\lambda}$. We can see easily by section 5.1

(i) If $c \geq l$, we have $b = S_u f_1^b f_1^{a-c} u(a-c)_{a+1}$. Indeed, we have $S_u b = f_0^c S_u f_1^b f_1^a S_1 u_{l\lambda}$ and $S_u f_1^b S_1 u_{l\lambda}$ has weight $a(\alpha_0 + \alpha_1) - l\alpha_1$. Since the multiplicity of $B$ at this weight is one, we have $S_u b = f_0^c S_1 u_{l\lambda} = \tilde{e}_0^l + c \tilde{e}_1^{l-a} u_{-l(\alpha_0 + \alpha_1)}$. Hence $S_u b = \tilde{e}_0^l + c \tilde{e}_1^{l-a} u_{-l(\alpha_0 + \alpha_1)} = \tilde{e}_0^l + c \tilde{e}_1^{l-a} u_{-l(\alpha_0 + \alpha_1)} = f_1^b f_0^c u(a-c)_{a+1}$.

(ii) If $a \leq c \leq l$, we have $b = S_1 S_u \tilde{e}_0^c u(l-a)_{a+1}$. In this case $b$ is 1-highest, and $S_1 b = \tilde{e}_0^{2l-a} \tilde{e}_1^{l-a} u_{l(\alpha_0 + \alpha_1)}$. Hence by applying (i) by reversing the arrows, we have $S_1 b = S_u \tilde{e}_0^{a-c} u(l-a)_{a+1}$.

**Lemma 6.5.** (i) For any $b \in B$, $\tilde{e}_{s_1 w s_1 w s_1}^{\max} b$ is $\mathfrak{g}_{\neq 0,1}$-highest and 1-lowest, and $S_u b$ is 1-highest. Then $\tilde{e}_{s_1 w s_1 w s_1}^{\max} b$ is $\mathfrak{g}_{\neq 0,1}$-highest.
Proof: (i) The claim follows from the fact that \( s_1 w s_1 \lambda = -\lambda \), which can be checked by Lemma 6.2.

(ii) \( \tilde{e}_{s_1 w}^{\text{max}} b = \tilde{e}_{s_1 w s_1 u s_1}^{\text{max}} S_i S_u b \) is \( g_{\neq 0} \)-highest by (i). \( \square \)

Lemma 6.6. For \( k \) such that \( 0 \leq k \leq l \) and an element \( b \) of \( B_{\neq 0}(k\lambda) \), suppose that \( b \) is \( 1 \)-highest, \( S_i b \) is \( g_{\neq 0, 1} \)-highest, and \( S_u S_i b \) is \( 1 \)-highest. Then we have \( \langle h_0, \text{wt } b \rangle \leq -k \).

Proof. By applying the previous lemma for \( S_i w s_1 \), one knows that \( \tilde{e}_{s_1 w}^{\text{max}} b = \tilde{e}_{s_1 w s_1 u s_1}^{\text{max}} S_i S_u b \) is \( g_{\neq 0} \)-highest. Hence we have

\[
\text{wt}(\tilde{e}_w^{\text{max}} b) \in s_1(k\lambda) + Z_{\geq 0} \alpha_1 \quad \text{and} \quad \text{wt } b \in s_1(k\lambda) + Z_{\geq 0} \alpha_1 + \sum_{i \neq 0, 1} Z_{\leq 0} \alpha_i.
\]

Hence we have

\[
\langle h_0, \text{wt } b \rangle \leq \langle h_0, s_1(k\lambda) \rangle = \langle h_0 + h_1, k\lambda \rangle = -k.
\]

\( \square \)

Let \( A(r) \) (\( i = 1, 2, 3 \)) be the following statements:

\( A(r)_1 \): for \( b \in B \) such that \( \| \text{wt } b \|^2, \| \text{wt } b + \alpha_0 \|^2 \geq r \),

we have \( \varepsilon_0(b) = \varepsilon_0(\psi(b)) \) and \( \psi(\tilde{e}_0 b) = \tilde{e}_0 \psi(b) \),

\( A(r)_2 \): for \( b \in B \) such that \( \| \text{wt } b \|^2, \| \text{wt } b - \alpha_0 \|^2 \geq r \),

we have \( \varphi_0(b) = \varphi_0(\psi(b)) \) and \( \psi(\tilde{f}_0 b) = \tilde{f}_0 \psi(b) \),

\( A(r)_3 \): for \( b \in B \) such that \( \| \text{wt } b \|^2 \geq r \), we have \( \psi(S_0 b) = S_0 \psi(b) \).

We prove these statements for any \( r \geq 0 \) by the descending induction on \( r \). Assume \( A(r') \) for \( r' > r \).

Then from \( A(r') \) we have

Lemma 6.7. For \( b \in B \) such that \( \| \text{wt } b \|^2 > r \), there exists a \( U_q(\mathfrak{g}_{01}) \)-crystal isomorphism \( \xi: B_{01}(b) \xrightarrow{\sim} B_{01}(\psi(b)) \). Moreover, we have \( \xi(b') = \psi(b') \) for any \( b' \in B_{01}(b) \) such that \( \| \text{wt } b' \|^2 > r \).

The following lemma implies \( A(r)_1 \) and \( A(r)_2 \).

Lemma 6.8. Assume that \( b \in B \) satisfies \( \| \text{wt } b \|^2 \geq r \). Then there exists a \( U_q(\mathfrak{g}_{0}) \)-crystal isomorphism \( \eta: B_0(b) \xrightarrow{\sim} B_0(\psi(b)) \) such that \( \eta(\tilde{e}_0 b) = \tilde{e}_0 \psi(b) \) for any \( b' \in B_0(b) \) such that \( \| \text{wt } b' \|^2 > r \). In particular, \( \varepsilon_0(b) = \varepsilon_0(\psi(b)) \).

Proof. Assume first that \( b \) is not \( 1 \)-extremal. Then \( \| \text{wt } \tilde{e}_1^{\text{max}} b \| > \| \text{wt } b \| \), and hence by the preceding lemma, there exists a \( U_q(\mathfrak{g}_{01}) \)-isomorphism \( \xi: B_{01}(b) \xrightarrow{\sim} B_{01}(\psi(b)) \) such that \( \xi(b') = \psi(b') \) for \( b' \in B_{01}(b) \) such that \( \| \text{wt } b' \|^2 > r \). Since \( \xi(\tilde{e}_1^{\text{max}} b) = \psi(\tilde{e}_1^{\text{max}} b) \), we have \( \xi(b) = \psi(b) \).

Similarly, if \( S_u b \) is not \( 1 \)-extremal, the assertion holds.

Now we may assume further that \( b \) is \( g_{\neq 0} \)-highest. Moreover, we may assume that \( b \) and \( S_u b \) are \( 1 \)-extremal. If \( b \) is \( 1 \)-highest, then \( b \) is \( g_{\neq 0} \)-highest, and the assertion follows from Lemma 6.4. Hence one can assume that \( b \) is \( 1 \)-lowest. If \( S_u b \) is \( 1 \)-lowest, then \( S_u b \) is \( g_{\neq 0} \)-lowest and the assertion holds. Hence one can assume that \( S_u b \) is \( 1 \)-highest.
By Lemma 6.11, \( \tilde{e}^\text{max}_w S_1 b \) is \( g_{\geq 0} \)-highest. Hence, \( \tilde{e}^\text{max}_w S_1 b \in B_0(S_1 u_{1\lambda}) \), and \( \tilde{e}^\text{max}_w S_1 b \in B_0(S_1 u_{1\lambda}) \). Then Lemma 6.11 implies that \( \zeta = \psi|_{B_0(S_1 u_{1\lambda})} \) gives a \( U_q(g_0) \)-crystal isomorphism

\[
\zeta : B_0(\tilde{e}^\text{max}_w S_1 b) \xrightarrow{\sim} B_0(\psi(\tilde{e}^\text{max}_w S_1 b)),
\]

which implies that

\[
(6.3) \quad \varepsilon_0(\tilde{e}^\text{max}_w S_1 b) = \varepsilon_0(\psi(\tilde{e}^\text{max}_w S_1 b)) \quad \text{and} \quad \tilde{e}^\text{max}_w \psi(\tilde{e}^\text{max}_w S_1 b) = \psi(\tilde{e}^\text{max}_w \tilde{e}^\text{max}_w S_1 b).
\]

Hence, \( \psi \) induces a \( U_q(g_0) \)-crystal isomorphism \( B_0(\tilde{e}^\text{max}_w S_1 b) \xrightarrow{\sim} B_0(\psi(\tilde{e}^\text{max}_w S_1 b)) \).

Since \( \tilde{e}^\text{max}_w \) commutes with \( \tilde{e}_0, f_0 \) and \( \psi \), \( \psi \) gives a \( U_q(g_0) \)-crystal isomorphism \( B_0(S_1 b) \xrightarrow{\sim} B_0(\psi(S_1 b)) \) which sends \( S_1 b \) to \( \psi(S_1 b) \) and \( \tilde{e}^\text{max}_w S_1 b \) to \( \psi(\tilde{e}^\text{max}_w S_1 b) \).

In particular, \( \varepsilon_0(S_1 b) = \varepsilon_0(\psi(S_1 b)) \) and \( \psi(\tilde{e}^\text{max}_w S_1 b) = \tilde{e}^\text{max}_w \psi(S_1 b) \).

Since \( S_1 b \) is 1-highest, \( \tilde{e}^\text{max}_1 \tilde{e}^\text{max}_0 S_1 b \) is the highest weight element of \( B_0(1) \). Similarly, \( \psi(\tilde{e}^\text{max}_1 \tilde{e}^\text{max}_0 S_1 b) = \tilde{e}^\text{max}_1 \tilde{e}^\text{max}_0 \psi(S_1 b) \) is \( (0,1) \)-highest. Hence we have a \( U_q(g_0) \)-crystal isomorphism \( \eta : B_0(0) \xrightarrow{\sim} B_0(\psi(0)) \) which sends \( \tilde{e}^\text{max}_0 \tilde{e}^\text{max}_0 S_1 b \) to \( \psi(\tilde{e}^\text{max}_0 \tilde{e}^\text{max}_0 S_1 b) \), and therefore \( \tilde{e}^\text{max}_0 S_1 b \) to \( \psi(\tilde{e}^\text{max}_0 S_1 b) \) and \( S_1 b \) to \( \psi(S_1 b) \). Hence \( \eta \) sends \( b \) to \( \psi(b) \).

If \( b' \in B_0(1) \subseteq B_0(1) \) satisfies \( \|\psi(b')^2 > r \), then we have \( \|\psi(b')^2 > r \) and Lemma 6.7 implies that \( \eta(b') = \psi(b') \).

**Lemma 6.9.** Assume that \( b \in B \) satisfies \( \|\psi(b)^2 > r \). Then, there exists a \( U_q(g_0) \)-crystal isomorphism \( \eta : B_0(0) \xrightarrow{\sim} B_0(\psi(0)) \) such that \( \eta(b') = \psi(b') \) for any \( b' \in B_0(1) \) with \( \|\psi(b')^2 > r \).

**Proof.** By Lemma 6.8, there exists a \( U_q(g_0) \)-crystal isomorphism \( B_0(1) \xrightarrow{\sim} B_0(\psi(1)) \) which sends \( b' \) to \( \psi(b') \) and \( b \) to \( \psi(b) \).

**Lemma 6.10.** If \( \|\psi(b)^2 > r \) and \( b \) is not 0-extremal, then \( \psi(S_0 b) = S_0 \psi(b) \).

**Proof.** By Lemma 6.8, there exist \( U_q(g_0) \)-crystal isomorphisms \( B_0(0) \xrightarrow{\sim} B_0(\psi(0)) \) and \( \eta' : B_0(S_0 b) \xrightarrow{\sim} B_0(\psi(S_0 b)) \) such that \( \eta(b) = \psi(b) \) and \( \eta'(S_0 b) = \psi(S_0 b) \). By the assumption, \( \|\psi(b)^2 > r \). Hence \( \eta(\tilde{e}^\text{max}_0 b) = \psi(\tilde{e}^\text{max}_0 b) = \tilde{e}^\text{max}_0 \psi(b) = \eta'(\tilde{e}^\text{max}_0 S_0 b) \), which implies that \( \eta = \eta' \). Hence \( \psi(S_0 b) = \eta(S_0 b) = S_0 \eta(b) = S_0 \psi(b) \).

**Lemma 6.11.** Assume that \( \|\psi(b)^2 > r \) and the \( (0,1) \)-highest weight \( \mu \) of \( B_0(1) \) satisfies \( \|\mu\|^2 > r \). Then there exists a \( U_q(g_0) \)-crystal isomorphism \( \eta : B_0(1) \xrightarrow{\sim} B_0(\psi(1)) \) such that \( \eta(b') = \psi(b') \) for any \( b' \in B_0(1) \) such that \( \|\psi(b')^2 > r \).

**Proof.** Let \( b_0 := \tilde{e}^\text{max}_0 \tilde{e}^\text{max}_1 \tilde{e}^\text{max}_0 \) be the \( (0,1) \)-highest weight vector of \( B_0(1) \). Since \( \|\psi(b_0)^2 > r \), Lemma 6.8 implies that there exists a \( U_q(g_0) \)-crystal isomorphism \( \eta : B_0(b_0) \xrightarrow{\sim} B_0(\psi(b_0)) \) such that \( \eta(b') = \psi(b') \) for any \( b' \in B_0(b_0) \) such that \( \|\psi(b')^2 > r \). In particular \( \eta(b_0) = \psi(b_0) \). It is enough to show that \( \eta(b) = \psi(b) \).

Assume first that \( b \) is not 0-extremal. Then \( \tilde{e}^\text{max}_0 b \) has square length greater than \( r \), and \( \tilde{e}^\text{max}_0 \tilde{e}^\text{max}_0 b = \psi(\tilde{e}^\text{max}_0 b) \). Then Lemma 6.8 implies that \( \eta(b) = \tilde{e}^\text{max}_0 b \).

Hence we may assume that \( b \) is 0-extremal. Since the case where \( b \) is 0-lowest is similarly proved by reversing the arrows, we assume that \( b \) is 0-highest. Then \( b_0 = \tilde{e}_0^\text{max} \tilde{e}_1^\text{max} b_0 \). By Lemma 6.8, \( \eta(\tilde{e}_0^\text{max} b_0) = \psi(\tilde{e}_0^\text{max} b_0) \). Hence, we have \( \eta(b) = \psi(b) \).

Now we are ready to complete the proof of \( A(r)_3 \).

**Lemma 6.12.** If \( \|\psi(b)^2 > r \), then \( \psi(S_0 b) = S_0 \psi(b) \).
Proof: We shall argue by the descending induction on the length of the 0-string containing $b$. If $b$ is not $(0,1)$-extremal, then the preceding lemma implies the desired result. Hence we may assume that $b$ is $(0,1)$-extremal.

Since the case when $b$ is 0-lowest is similar, we assume that $b$ is 0-highest. One can assume that $b$ is $g_{\lambda, 0}$-highest. If $b$ is 1-highest, it is $g_{\lambda, 0}$-highest and the assertion follows from Lemma 6.3. Hence one can assume that $b$ is 1-lowest. Similarly, one can assume that $S_0b$ is 1-highest.

We divide the proof into two cases: (1) $S_1b$ is 0-highest, (2) $S_1b$ is not 0-highest but is 0-lowest.

First we consider the case (1). Since $S_1b$ is $(0,1)$-highest in this case, we have $\langle h_0, \text{wt } S_1b \rangle \geq 0$. On the other hand, Lemma 6.6 implies that $\langle h_0, \text{wt } b \rangle \leq -k$ with $k \geq 0$ defined by $S_1b \in B_{\neq 0}(k\lambda)$. Hence we obtain $k = 0$, and $\text{wt } b = 0$, which implies that $S_0b = b$.\[\square\]

Next, we consider the case (2). In this case, $S_0b$ is $(0,1)$-lowest and $S_0S_1b$ is $(0,1)$-highest. Since $\tilde{e}_1^\max e_1^\max S_1b$ is $g_{\neq 0}$-highest by Lemma 6.3, Lemma 6.3 implies that $\psi(S_0e_1^\max S_1b) = S_0\psi(e_1^\max S_1b)$. Write $e_1^\max S_1b = \tilde{e}_{i_1} \cdots \tilde{e}_{i_m} S_1b$ with $i_1, \ldots, i_m \in I \setminus \{0,1\}$. Since $\tilde{e}_i$ commutes with $S_0$ and $\psi$, we have $\tilde{e}_i \cdots \tilde{e}_{i_m} \psi(S_0S_1b) = \psi(S_0e_1^\max S_1b) = S_0\psi(e_1^\max S_1b) = \tilde{e}_{i_1} \cdots \tilde{e}_{i_m} S_0\psi(S_1b)$. Hence we have

$$\psi(S_0S_1b) = S_0\psi(S_1b) = S_0S_1\psi(b).$$

Let $\mu$ be the weight of $S_0S_1b$, and set $\mu_i = \langle h_i, \mu \rangle$. We have $\mu_0 > 0$, for, otherwise, $S_0b$ is 0-highest.

Note that $\varphi_0(S_0S_1b) = \mu_0 + \mu_1 > \varphi_0(b) = \mu_1$. By the descending induction on the length of the 0-string containing $b$, we see that $\psi(S_0S_1b) = S_0\psi(S_1b)$. Hence $S_0S_1\psi(S_0b) = \psi(S_0S_1b) = S_1\psi(S_0S_1b) = S_1\psi(S_0S_1b) = S_0S_1\psi(b)$. Here, we used (6.3). Hence we obtain $\psi(S_0b) = S_0\psi(b).\square$

Thus, the descending induction on $r$ proceeds, and the proof of Theorem 6.1 is complete.

7. Perfectness of the Crystal $B_l$

7.1. Connectedness of $B_l \otimes B_l$ (Proof of (P1)). We show that any element $b \otimes b'$ of $B_l \otimes B_l$ can be connected with $\phi \otimes \phi$. Here $\phi$ stands for $(0,0,0,0,0) \in B_l$. Like this we use in this section the tableau representation for an element of $B_l$. By applying $f_i$ and $f_2$ sufficiently many times, one can assume that $f_i(b \otimes b') = 0$ ($i = 1, 2$). This implies $\tilde{f}_i b' = 0$ ($i = 1, 2$), namely,

$$b' = \tilde{1}^m$$

for some $0 \leq m \leq l$.

Note that the 0-string containing $\tilde{1}^m$ is given by

$$\tilde{1}^l \rightarrow \cdots \rightarrow \tilde{1}^m \rightarrow \cdots \rightarrow \tilde{1} \rightarrow \phi \rightarrow 1 \rightarrow \cdots \rightarrow 1^l.$$

Set $\gamma(b) = m + (\varphi_0(b) - l + m)_+$, then from the tensor product rule of crystals we have

$$\tilde{f}_0^\gamma(b \otimes \tilde{1}^m) = \tilde{f}_0^\gamma(b) \otimes \tilde{f}_0^m(\tilde{1}^m) = \tilde{b} \otimes \phi,$$

where $\tilde{b} = \tilde{f}_0^\gamma(b) - m b$.

Since $\tilde{f}_i \phi = 0$ for $i = 1, 2$ there exists a sequence $\{i_1, \ldots, i_k\} \subset \{1, 2\}$ and a non-negative integer $m'$ such that $\tilde{f}_{i_k} \cdots \tilde{f}_{i_1} (\tilde{b} \otimes \phi) = \tilde{1}^m' \otimes \phi$. Thus we have

$$\tilde{f}_0^m' \tilde{f}_{i_k} \cdots \tilde{f}_{i_1} \tilde{f}_0^\gamma(b \otimes \tilde{1}^m) = \phi \otimes \phi.$$
7.2. Minimal elements in $B_l$ (Proof of (P4) and (P5)). First we are to show $\langle c, \varphi(b) \rangle \geq l$ for $b \in B_l$. From Proposition 4.3 and formulas of $\varepsilon_i, \varphi_i$ ($i = 1, 2$) in section 4.1, we have

$$\langle c, \varphi(b) \rangle = \varphi_0(b) + 2\varphi_1(b) + 3\varphi_2(b) = l + \max A + 2(z_3 + (z_2)_+) + (3z_4)_+ - (z_1 + 2z_2 + 2z_3 + 3z_4),$$

where $z_j$ ($1 \leq j \leq 4$) are given in (4.1) and $A$ is given in (4.2).

Lemma 7.1. For $(z_1, z_2, z_3, z_4) \in \mathbb{Z}^4$ set

$$\psi(z_1, z_2, z_3, z_4) = \max A + 2(z_3 + (z_2)_+) + (3z_4)_+ - (z_1 + 2z_2 + 2z_3 + 3z_4).$$

Then we have $\psi(z_1, z_2, z_3, z_4) \geq 0$ and $\psi(z_1, z_2, z_3, z_4) = 0$ if and only if $(z_1, z_2, z_3, z_4) = (0, 0, 0, 0)$.

Proof. Note that for $z \in \mathbb{Z}$, $2(z)_+ \geq z$ and $2(z)_+ = z$ implies $z = 0$. We prove by dividing the cases of the values that attain the maximum in $\max A$. Suppose $\max A = 0$. Using the above inequality, we have

$$\psi \geq (z_2)_+ + (3z_4)_+ - (z_1 + 2z_2 + 2z_3 + 3z_4).$$

Since $z_1 + 2z_2 + 2z_3 + 3z_4 \leq 0$, we have $\psi \geq 0$. $\psi = 0$ holds if and only if $z_1 + 2z_2 + 2z_3 + 3z_4 = 0, z_3 = 0, z_2, z_1 \leq 0$.

Since we have $z_1 \leq 0$ in this case, one can conclude that $\psi = 0$ implies $(z_1, z_2, z_3, z_4) = (0, 0, 0, 0)$.

The other cases are similar. In particular, if $\max A > 0$, $\psi > 0$. \hfill $\square$

Thanks to the lemma, we have

$$\langle c, \varphi(b) \rangle - l = \psi(z_1, z_2, z_3, z_4) \geq 0.$$

Since $\langle c, \varphi(b) - \varepsilon(b) \rangle = 0$, we also obtain $\langle c, \varepsilon(b) \rangle \geq l$, which proves (P4).

Suppose $\langle c, \varepsilon(b) \rangle = l$. It implies $\psi = 0$. Hence from the lemma one can conclude that such element $b = (x_1, x_2, x_3, x_2, x_1)$ should satisfy $x_1 = \bar{x}_1, x_2 = x_3 = \bar{x}_3 = \bar{x}_2$. Therefore we have

$$(B_l)_{\min} = \{ (\alpha, \beta, \beta, \beta, \alpha) | \alpha, \beta \in \mathbb{Z}_{\geq 0}, 2\alpha + 3\beta \leq l \}.$$

For $b = (\alpha, \beta, \beta, \beta, \alpha) \in B_l$ one calculates

$$\varepsilon(b) = \varphi(b) = (l - 2\alpha - 3\beta)\Lambda_0 + \alpha\Lambda_1 + \beta\Lambda_2.$$

Thus we have also shown (P5).

7.3. Coherent family of perfect crystals. We review the notion of a coherent family of perfect crystals introduced in [KKM]. Let $\{B_l\}_{l \geq 1}$ be a family of perfect crystals $B_l$ of level $l$ and $(B_l)_{\min}$ be the subset of minimal elements of $B_l$. Set $J = \{ (l, b) | l \in \mathbb{Z}_{\geq 0}, b \in (B_l)_{\min} \}$. Let $\sigma$ denote the isomorphism of $(P^+_l)_l$ defined by $\sigma = \varepsilon \circ \varphi^{-1}$.

Definition 7.2. A crystal $B_{\infty}$ with an element $b_{\infty}$ is called a limit of $\{B_l\}_{l \geq 1}$ if it satisfies the following conditions:

- $\text{wt} b_{\infty} = 0, \varepsilon(b_{\infty}) = \varphi(b_{\infty}) = 0$,
- for any $(l, b) \in J$, there exists an embedding of crystals

$$I_{(l, b)} : T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \rightarrow B_{\infty}$$

sending $t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}$ to $b_{\infty}$,
• $B_{\infty} = \bigcup_{(l,b) \in J} \text{Im} f_{(l,b)}$.

If a limit exists for the family $\{B_i\}$, we say that $\{B_i\}$ is a coherent family of perfect crystals.

For $\lambda \in P_{cl}$, $T_{\lambda}$ denotes a crystal with a unique element $t_{\lambda}$. See [K2] for the details. For our purpose the following facts are sufficient. For any $P_{cl}$-weighted crystal $B$ and $\lambda, \mu \in P_{cl}$ consider the crystal

$$T_{\lambda} \otimes B \otimes T_{\mu} = \{t_{\lambda} \otimes b \otimes t_{\mu} \mid b \in B\}.$$ 

The crystal structure is given by

$$\hat{e}_i(t_{\lambda} \otimes b \otimes t_{\mu}) = t_{\lambda} \otimes \hat{e}_i b \otimes t_{\mu}, \quad \hat{f}_i(t_{\lambda} \otimes b \otimes t_{\mu}) = t_{\lambda} \otimes \hat{f}_i b \otimes t_{\mu},$$

$$\varepsilon_i(t_{\lambda} \otimes b \otimes t_{\mu}) = \varepsilon_i(b) - \langle h_i, \lambda \rangle, \quad \varphi_i(t_{\lambda} \otimes b \otimes t_{\mu}) = \varphi_i(b) + \langle h_i, \mu \rangle,$$

$$\text{wt}(t_{\lambda} \otimes b \otimes t_{\mu}) = \lambda + \mu + \text{wt} b.$$ 

Let us now consider the following set

$$B_{\infty} = \{b = (\nu_1, \nu_2, \nu_3, \nu_1, \nu_2, \nu_3) \mid \nu_1, \nu_2, \nu_3 \in \mathbb{Z}, \nu_3 \equiv \nu_3 \pmod{2}\},$$

and set $b_{\infty} = (0, 0, 0, 0, 0, 0)$. We introduce the crystal structure on $B_{\infty}$ as follows. The actions of $\hat{e}_i$ and $\hat{f}_i$ are given in (4.1) with $x_i$ replaced with $\nu_i$ and $\bar{\nu}_i$. The only difference lies in the fact that $\hat{e}_i$ and $\hat{f}_i$ never become 0, since we allow a coordinate to be negative and there is no restriction for the sum $s(b) = \sum_{i=1}^{2} (\nu_i + \bar{\nu}_i)$. For $\varepsilon_i, \varphi_i$, with $i = 1, 2$ we adopt the formulas in section 4.1. For $\varepsilon_0, \varphi_0$ we define

$$\varepsilon_0(b) = -s(b) + \text{max} A - (2z_1 + z_2 + z_3 + 3z_4),$$

$$\varphi_0(b) = -s(b) + \text{max} A,$$

where

$$A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4)$$

and $z_1, z_2, z_3, z_4$ are given in (4.1) with $x_i$, $\bar{x}_i$ replaced with $\nu_i$, $\bar{\nu}_i$. Note that $\text{wt} b_{\infty} = 0$ and $\varepsilon_i(b_{\infty}) = 0$ for $i = 0, 1, 2$.

Let $b_0 = (\alpha, \beta, \beta, \beta, \beta, \alpha)$ be an element of $(B_l)_{\text{min}}$. Since $\varepsilon(b_0) = \varphi(b_0)$, one can set $\sigma = \text{id}$. Let $\lambda = \varepsilon(b_0)$. For $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in B_l$ we define a map

$$f_{(l,b_0)} : T_{\lambda} \otimes B_l \otimes B_{-\lambda} \to B_{\infty}$$

by

$$f_{(l,b_0)}(t_{\lambda} \otimes b \otimes t_{-\lambda}) = b' = (\nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_2, \bar{\nu}_1)$$

where

$$\nu_1 = x_1 - \alpha, \quad \bar{\nu}_1 = \bar{x}_1 - \alpha,$$

$$\nu_j = x_j - \beta, \quad \bar{\nu}_j = \bar{x}_j - \beta \quad (j = 2, 3).$$

Then we have

$$\text{wt}(t_{\lambda} \otimes b \otimes t_{-\lambda}) = \text{wt} b = \text{wt} b',$$

$$\varphi_0(t_{\lambda} \otimes b \otimes t_{-\lambda}) = \varphi_0(b) + \langle h_0, -\lambda \rangle$$

$$\varphi_0'(b') + (l - s(b')) + s(b') - (l - 2\alpha - 3\beta) = \varphi_0(b'),$$

$$\varphi_1(t_{\lambda} \otimes b \otimes t_{-\lambda}) = \varphi_1(b) + \langle h_1, -\lambda \rangle = \varphi_1'(b') + \alpha - \alpha = \varphi_1(b'),$$

$$\varphi_2(t_{\lambda} \otimes b \otimes t_{-\lambda}) = \varphi_2(b) + \langle h_2, -\lambda \rangle = \varphi_2(b') + \beta - \beta = \varphi_2(b').$$
it is straightforward to check that if \( b, \tilde{e}_i b \in B_l \) (resp. \( b, \tilde{f}_i b \in B_l \)), then

\[
\tilde{e}_i f_{(l,b_0)}(t_{\lambda} \otimes b \otimes t_{-\lambda}) = \tilde{e}_i f_{(l,b_0)}(t_{\lambda} \otimes b \otimes t_{-\lambda}) \quad (\text{resp.} \quad f_{(l,b_0)}(\tilde{f}_i t_{\lambda} \otimes b \otimes t_{-\lambda}) = f_{(l,b_0)}(\tilde{f}_i t_{\lambda} \otimes b \otimes t_{-\lambda})).
\]

Hence \( f_{(l,b_0)} \) is a crystal embedding. It is easy to see that

\[
f_{(l,b_0)}(t_{\lambda} \otimes b_0 \otimes t_{-\lambda}) = b_\infty.\]

We can also check \( B_\infty = \bigcup_{(l,b) \in J} \text{Im} f_{(l,b)}. \) Therefore we have shown that the family of perfect crystals \( \{B_l\}_{l \geq 1} \) forms a coherent family.

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APPENDIX A. TABLE OF \( f_{1j,0}^{\tilde{p}j,0,q,i} \)

Assume \( j_0 \leq j_1 \). We give the table of \( x = \tilde{f}_{1j,0}^{\tilde{p}j,0,q,i} \) on \( B_{\geq 0} \) for \( 0 \leq p \leq j_0, 0 \leq q \leq j_1 + p \).

1. \( 0 \leq p \leq i \) case:
   
   (i) \( 0 \leq q \leq j_0 - i + p \) case:
   
   \[ x = (p, y_1, 3 y_0 - 2 y_1 + i - p, y_0 + i - p, y_0 + j_0 - q, q). \]
   
   (ii) \( j_0 - i + p \leq q \leq j_1 \) case:
   
   \[ x = (p, y_1, 3 y_0 - 2 y_1 - q + j_0, y_0 + 2 i - 2 p + q - j_0, y_0 + i - p, j_0 - i + p). \]
   
   (iii) \( j_1 \leq q \leq j_1 + p \) case:
   
   \[ x = (p - q + j_1, y_1 + q - j_1, y_1 + y_0 + 2 i - 2 p + j_1 - j_0, y_0 + i - p, j_0 - i + p). \]

2. \( i \leq p \leq j_0 \) case:
   
   (i) \( 0 \leq q \leq j_0 - p + i \) case:
   
   \[ x = (i, y_1, 2 p - 2 i + 3 y_0 - 2 y_1, y_0 + j_0 - p + i - q, q). \]
   
   (ii) \( j_0 - p + i \leq q \leq j_0 - i + j_1 \) case:
   
   \[ x = (i, y_1, 2 y_1 - i + j_0 - p + q, y_0 + q - j_0 + p - i, y_0 + j_0 - p + i). \]
   
   (iii) \( p - i + j_1 \leq q \leq j_1 + p \) case:
   
   \[ x = (p + j_1 - q, y_1 + q - p + i - j_1, y_1 + 2 p - 2 i + j_1 - j_0 + y_0, y_0, y_0, j_0 - p + i). \]

APPENDIX B. TABLE OF \( f_{0j,0}^{\tilde{r}i,0,q,i} \)

We give the table of \( x = \tilde{f}_{0j,0}^{\tilde{r}i,0,q,i} \) on \( B_{\geq 0} \) for \( 0 \leq p \leq q \leq j_1 + p, 0 \leq r \leq i + q - 2 p \).

1. \( p \leq i, p \leq q, r \leq i + q - 2 p \) case:
   
   (i) \( q \leq p + \frac{1}{2}(j_1 - i), 0 \leq r \leq i + q - 2 p \) case:
   
   \[ x = (p + r, y_1, j_1 + y_1 - q - r, y_0 + i - 2 p + q + r, y_0 + i - p, p). \]
   
   (ii) \( p + \frac{1}{2}(j_1 - i) \leq q \leq j_1, 0 \leq r \leq j_1 - q \) case: same as (i).
Assume $j_0 \leq j_1$. We give the table of $x = \hat{f}^r_{p,j_1} \hat{f}^{b,0}_{j_0,j_1}$ on $B_{\geq 0}$ for $0 \leq q \leq j_0 + j_1$, $0 \leq r$.

I. $0 \leq q \leq i$ case:

$$x = (i + r, y_1, y_1 + j_0 + j_1 - i - q, y_0, y_0 + i - q, q).$$

II. $i \leq q \leq j_0$ case:

$$x = (i + r, y_1, y_1 + j_0 + j_1 - i - q, y_0 - i + q, y_0, i).$$

III. $j_0 \leq q \leq j_0 + j_1 - i$ case:

(i) $j_0 \leq q \leq j_0 + \frac{2i}{1+q}$, $0 \leq r \leq q - j_0$ or $j_0 + \frac{2i}{1+q} \leq q \leq j_0 + j_1 - i$, $0 \leq r \leq j_0 + j_1 - i - q$ case:

$$x = (i + r, y_1, y_1 + j_0 + j_1 - i - q - r, y_0 - i + q - r - y_0, i).$$

(ii) $j_0 \leq q \leq j_0 + \frac{2i}{1+q}, r \geq q - j_0$ case:

$$x = (i + r, y_1, y_1 + 2j_0 + j_1 - i - 2q, y_0 + j_0 - i, y_0, i).$$

(iii) $j_0 + \frac{2i}{1+q} \leq q \leq \frac{4j_0+2j_1}{3} - i, j_0 + j_1 - i - q \leq r \leq q - j_0$ or $\frac{4j_0+2j_1}{3} - i \leq q \leq j_0 + j_1 - i, j_0 + j_1 - i - q \leq r \leq \frac{j_0+2j_1}{3} - i$ case:

$$x = (j_0 + j_1 - q, y_1 - j_0 - j_1 + i + q + r, y_1, y_0 + j_0 + j_1 - 2i - 2r, y_0, i).$$

(iv) $j_0 + \frac{2i}{1+q} \leq q \leq \frac{4j_0+2j_1}{3} - i, i \geq q - j_0$ case:

$$x = (2j_0 + j_1 - 2q + r, y_1 - 2j_0 - j_1 + i + 2q, y_1, y_0 + 3j_0 + j_1 - 2i - 2q, y_0, i).$$
\( j_0 + j_1 - q, 2y_1 - y_0 - j_0 + q, 2y_0 - y_1 - 2j_0 - 2j_1 + 2i + 2q, y_1, y_1 + j_0 + j_1 - i - q, i \).

**IV.** \( j_0 + j_1 - i \leq q \leq j_0 + j_1 \) case:

(i) \( 0 \leq r \leq \frac{2q + 2j_1}{3} \) case: same as III (iii).

(ii) \( \frac{2q + 2j_1}{3} \leq r \leq j_1 - i \) case: same as III (v).

(iii) \( j_1 - i \leq r \leq q - j_0 \) case:

\[
x = (j_0 + j_1 - q, 2y_1 - y_0 - j_0 + q, 2y_0 - y_1 - j_1 + i + r, y_1 - j_1 + i + r, y_1, j_1 - r).
\]

(iv) \( r \geq q - j_0 \) case:

\[
x = (2j_0 + j_1 - 2q + r, 2y_1 - y_0 - j_0 + q, 2y_0 - j_0 - j_1 + i + q, y_1 - j_0 - j_1 + i + q, y_1, j_0 + j_1 - q).
\]

**APPENDIX D. TABLE OF** \( \tilde{f}_0^{j_0 + q - 2p} \tilde{f}_1^{j_0} \tilde{f}_0^{p} \tilde{f}_1^{i} \)

Assume \( j_0 \leq j_1 \). We give the table of \( x = \tilde{f}_0^{j_0 + q - 2p} \tilde{f}_1^{j_0} \tilde{f}_0^{p} \tilde{f}_1^{i} \) on \( B_{\geq 0} \) for \( 0 \leq p \leq j_0, p \leq q \leq j_1 + p \).

**I.** \( 0 \leq p \leq i \) case:

(i) \( i \leq q - p + i \leq \frac{j_0 + j_1}{2} \) case:

\[
x = (i - p + q, y_1, y_1 + j_0 + j_1 - 2i + 2p - 2q, y_0, y_0 + i - p, p).
\]

(ii) \( \frac{j_0 + j_1}{2} \leq q - p + i \leq \frac{2q + 2j_1}{3} \) case:

\[
x = (j_0 + j_1 - i + p - q, y_1 - j_0 - j_1 + 2i - 2p + 2q, y_1, y_0 + j_0 + j_1 - 2i + 2p - 2q, y_0 + i - p, p).
\]

(iii) \( q - p + i \geq \frac{2q + 2j_1}{3}, q \leq j_1 \) case:

\[
x = (j_0 + j_1 - i + p - q, 2y_1 - y_0 - j_0 + i + p + q, 2y_0 - y_1 - 2j_1 + 2i - 2p + 2q, y_1, y_1 + j_1 - q, p).
\]

(iv) \( j_1 \leq q \leq j_1 + p \) case:

\[
x = (j_0 + j_1 - i + p - q, 2y_1 - y_0 - j_0 + i + p + q, 2y_1 - y_0 - j_0 + 2i - 2p + q, y_1 - j_1 + q, y_1, j_1 + p - q).
\]

**II.** \( i \leq p \leq j_0 \) case:

(i) \( 2p \leq 2q \leq j_0 + j_1 + p - i \) case:

\[
x = (i - p + q, y_1, y_1 + j_0 + j_1 - i + p - 2q, y_0 - i + p, y_0, i).
\]

(ii) \( 2q \geq j_0 + j_1 + p - i, q - p + i \leq \frac{j_0 + j_1}{2} \) case:

\[
x = (j_0 + j_1 - q, y_1 - j_0 - j_1 + i + p + 2q, y_1, y_0 + j_0 + j_1 - 2i + 2p - 2q, y_0, i).
\]

(iii) \( \frac{j_0 + j_1}{2} \leq q - p + i \leq j_1 \) case:

\[
x = (j_0 + j_1 - q, 2y_1 - y_0 - j_0 + q, 2y_0 - y_1 - 2j_1 + 2i - 2p + 2q, y_1, y_1 + j_1 - i + p - q, i).
\]

(iv) \( j_1 \leq q - p + i \leq j_1 + i \) case:

\[
x = (j_0 + j_1 - q, 2y_1 - y_0 - j_0 + q, 2y_0 - y_1 - j_1 + i - p + q, y_1 - j_1 + i - p + q, y_1, j_1 + p - q).
\]
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