W-translated Schubert divisors and transversal intersections

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Received June 8, 2021; accepted December 22, 2021; published online April 15, 2022

Abstract We study the toric degeneration of Weyl group translated Schubert divisors of a partial flag variety \(F_{\ell_1, \ldots, \ell_k}^{n_1, \ldots, n_k}\) via Gelfand-Cetlin polytopes. We propose a conjecture that Schubert varieties of appropriate dimensions intersect transversally up to translation by Weyl group elements, and verify it in various cases, including the complex Grassmannian \(Gr(2, n)\) and the complete flag variety \(F_{1, 2, 3; 4}\).

Keywords W-translated Schubert varieties, transversal intersection, Gelfand-Cetlin polytope, toric degeneration, flag varieties

MSC(2020) 14N15, 14M15, 14M25

Citation: Hwang D S, Lee H, Lee J-H, et al. W-translated Schubert divisors and transversal intersections. Sci China Math, 2022, 65: 1997–2018, https://doi.org/10.1007/s11425-021-1940-6

1 Introduction

The partial flag variety \(X = F_{\ell_1, \ldots, \ell_k}^{n_1, \ldots, n_k}\), parameterizing the space of partial flags \(\{V_{n_1} \subseteq \cdots \subseteq V_{n_k} \subseteq \mathbb{C}^n\}\), is a smooth projective \(SL(n, \mathbb{C})\)-homogeneous variety. The cohomology ring \(H^*(X, \mathbb{Z})\) is torsion-free, and has a \(\mathbb{Z}\)-additive basis of Schubert classes \(\sigma^w\) labeled by a subset of the permutation group \(S_n\) of \(n\) objects. The structure constants \(N_{u_1, \ldots, u_m, u_{m+1}}^w\) in the cup product

\[\sigma^{u_1} \cup \cdots \cup \sigma^{u_{m+1}} = \sum_{\ell(w) = \ell(u_1) + \cdots + \ell(u_{m+1})} N_{u_1, \ldots, u_m, u_{m+1}}^w \sigma^w\]

are nonnegative integers, where \(\ell : S_n \to \mathbb{Z}_{\geq 0}\) denotes the standard length function. One of the most central problems in Schubert calculus is finding a manifestly positive combinatorial formula of \(N_{u_1, u_2}^w\), which determines a manifestly positive formula for general \(N_{u_1, \ldots, u_{m+1}}^w\). However, this problem is widely open except for two-step flag varieties \(F_{\ell_1, \ell_2}^{n_1, n_2}\) [5, 6, 9], beyond the famous Littlewood-Richardson rules for complex Grassmannians \(Gr(m, n) = F_{m, n}\). In addition, there is a recent work [25] on three-step flag varieties, and there has been a nice algorithm for the Schubert structure constants [11] (with sign

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are Schubert varieties $\Delta$. leads to the strategy that $W$-translated Schubert varieties $X^{u,v}$’s of complex codimension $\ell(w)$ defined by imposing dimension conditions with respect to the complete flag opposite to the fixed one. Kleiman’s transversality theorem [21] says that for generic $g,g' \in SL(n,\mathbb{C})$, the varieties $g \cdot X^u$, $g' \cdot X^v$ and $X_w$ intersect transversally along a Zariski dense open subset of every component of their intersection. Moreover, the transversal intersection $g \cdot X^u \cap g' \cdot X^v \cap X_w$ is a finite set of cardinality $\mathcal{N}_{u,v}^{g,g'}$ provided that $\ell(w) = \ell(u) + \ell(v)$. Along this direction, one may ask the following question.

**Question 1.1.** How do you specify a pair $(g,g')$ of elements in $SL(n,\mathbb{C})$ that satisfies the Kleiman’s transversality for given $u,v$ and $w$?

It would be great if one could characterize the pair $(g,g')$ of generic elements and give a combinatorial description of the transversal intersection, which would result in a manifestly positive formula of $\mathcal{N}_{u,v}^{g,g'}$. The desired pair $(g,g')$ exists almost everywhere. However, it is still like looking for a needle in a haystack if one really wants to specify one such $(g,g')$. In this paper, we consider the finitely many subvarieties $uX^v$’s where $u \in S_n$, and call them $W$-translated Schubert varieties. We will study the transversality of intersections among them. We remark that varieties of the form $g_1X^{u_1} \cap \cdots \cap g_mX^{u_m}$ for $g_1,\ldots,g_m \in SL(n,\mathbb{C})$ are called intersection varieties in [2], whose singular loci were studied therein.

Our study of transversal intersections of $W$-translated Schubert varieties uses toric degenerations and Gelfand-Cetlin polytopes. This was inspired by the works [19,20] of Kiritchenko et al., which related Schubert calculus with Gelfand-Cetlin polytopes closely in an algebraic/representation-theoretic way, in addition to the early study in [26]. Algebraically, the polytope ring [32,36] associated with the Gelfand-Cetlin polytopes $\mathcal{N}$ for a complete flag variety $F_{\ell_1,\ldots,\ell_n}$ is isomorphic to $H^*(F_{\ell_1,\ldots,\ell_n},\mathbb{Z})$ [18]; descriptions of the Schubert classes modeled by such Gelfand-Cetlin polytope through the associated polytope ring are given in [19,20]. Geometrically, there are toric degenerations of partial flag varieties to the toric varieties $X_\Delta$ of the Gelfand-Cetlin polytopes $\Delta$ [1,13,24,27,31]. The standard (resp. opposite) Schubert varieties of the partial flag variety $X$ behave well with respect to the toric degeneration in the sense that they degenerate to reduced unions of toric subvarieties of $X_\Delta$ [27, Theorem 8 and Remark 10]. The $W$-translated Schubert varieties are also expected to enjoy similar nice degeneration properties. This leads to the strategy that intersection problems in the generic fiber are likely transferred to intersection problems of toric subvarieties of $X_\Delta$, and hence to that of faces of the Gelfand-Cetlin polytope $\Delta$. Indeed, as we will show in Theorem 3.5, every $W$-translated Schubert divisor, which is defined by a positive path, degenerates to the union of toric divisors, defined by effective edges that lie on the positive path. Each $W$-translated Schubert variety $X^{u,v}$ is a scheme-theoretical intersection of $X$ with appropriate $W$-translated Schubert divisors. Therefore, it corresponds to a set-theoretic intersection $\Delta(u,v)$ of unions of faces $F \subset \Delta$ with codimension one. The above strategy for $X = F_{\ell_n}$ can be achieved in the following way, which builds a bridge connecting geometry and combinatorics with the help of toric degeneration.

**Theorem 1.2.** Let $v_1,\ldots,v_{m+1}, w \in S_n$ with $\ell(w) = \sum_{i=1}^{m+1} \ell(v_i)$, where $m \geq 1$. Suppose that there exist $u_1,\ldots,u_{m+1} \in S_n$ with $S := \bigcap_{i=1}^{m+1} \Delta(u_i,v_i) \cap \Delta(w_0,w,w)$ consisting of part of the regular vertices of $\Delta$ for $F_{\ell_n}$. Then $u_1X^{v_1},\ldots,u_{m+1}X^{v_{m+1}}, X_w$ intersect transversely, and $\mathcal{N}_{w_0}^{u_1,\ldots,u_{m+1}} = zS$.

Here, $w_0$ denotes the longest element of $S_n$, and we refer to Theorem 4.5 for more precise statements for general $F_{\ell_1,\ldots,\ell_n}$. We notice that all the $W$-translated Schubert varieties $(wX^u)_{w \in W}$ represent the same Schubert class $\sigma^{u}$. In [19], Kiritchenko assigned to $\sigma^{u}$ different choices of faces of the Gelfand-Cetlin polytopes $\Delta$ for complete flag varieties in a representation-theoretic way. Instead of the one-to-many correspondence between Schubert classes and faces of the polytope $\Delta$, we believe that it is more natural to make a one-to-one correspondence between $W$-translated Schubert varieties and combinations of faces of the polytope, arising from the toric degeneration. This will probably fill the gap between geometry and the algebraic model in [20]. For example, Remark 2.5 therein can be explained in this way.
(namely, the toric divisor of the combination of the two faces in the remark is the degeneration of the corresponding $W$-Schubert divisor, which should be treated as one single object since the very beginning from the viewpoint of geometry). For the case of Schubert divisor classes, we will describe the expected relationship precisely in Remark 3.6.

To explore an answer to Question 1.1, we will construct a partition $\mathcal{U}$ of the set $\{(u,v,w) \mid u,v,w \in S_n, \ell(w) = \ell(u) + \ell(v)\}$ in Subsection 4.2. Each triple $(u,v,w)$ represents a Schubert structure constant $N^w_{u,v}$. Therein we will also modify the partition a little bit to obtain $\hat{\mathcal{U}}$, by adding elements of the form $(u_1',\ldots,u_{m+1}',w')$ to each class $[(u,v,w)] \in \mathcal{U}$. As we will see, the way of constructing $\hat{\mathcal{U}}$ ensures that Schubert structure constants represented by elements in a same class are equal to each other, namely, we have $N^w_{u,v} = N^w_{u_1',\ldots,u_{m+1}'}$. We make the following conjecture.

**Conjecture 1.3.** For any $[(u,v,w)] \in \hat{\mathcal{U}}$, there exists $(u_1',\ldots,u_{m+1}',w') \in [(u,v,w)]$ such that $\hat{u}_1 \cdot X^{w_1} \cap \cdots \cap \hat{u}_{m+1} \cdot X^{w_{m+1}} \cap X^{w'}$ is a transversal intersection for some $\hat{u}_1,\ldots,\hat{u}_{m+1} \in S_n$.

We remark that $m = 1$ is usually taken, while $m > 1$ is sometimes necessary as we will see in the example for $Gr(3,6)$ in Subsection 4.4. If Conjecture 1.3 is true, then we will obtain a Littlewood-Richardson rule by Theorem 4.5. On one hand, we could reduce the searching of $(\lambda,\mu)$ by studying the transversality of intersections of Schubert varieties in different ways [33,34] or by different degeneration methods [37].

As one byproduct of the study of toric degeneration, we give explicit defining equations for certain toric subvarieties of $X_\Delta$, including all the toric divisors in Proposition 3.4. As another byproduct, we give a combinatorial description of an anti-canonical divisor $-K_X$ of $X$ in Proposition 5.4, using the notions of special paths. Such anti-canonical divisor degenerates to that of the Gelfand-Cetlin toric variety $X_0$ and is potentially useful in the study of Strominger-Yau-Zaslow (SYZ) mirror symmetry [35] for a partial flag variety $X$. For example, Chan et al. [7] constructed a special Lagrangian fibration for the open Calabi-Yau manifold $X \setminus -K_X$ for the two-step flag variety $F\ell_{1,n-1,n}$, which is one key step in the study of SYZ mirror symmetry.

The rest of this paper is organized as follows. In Section 2, we review basic facts on Gelfand-Cetlin toric varieties. In Section 3, we introduce the notion of $W$-translated Schubert varieties, and study the toric degeneration of certain $W$-translated Schubert varieties. In Section 4, we study transversal intersections of $W$-translated Schubert varieties by using toric degenerations in various examples. Finally in Section 5, we specify an anti-canonical divisor of $X$.

## 2 Gelfand-Cetlin polytopes and toric varieties

Throughout this paper, we fix a sequence $0 = n_0 < n_1 < n_2 < \cdots < n_k < n_{k+1} = n$ of integers, together with a decreasing sequence of nonnegative integers

$$\lambda = \{\lambda_1^{(n)} = \lambda_2^{(n)} = \cdots = \lambda_{n_1}^{(n)} > \lambda_{n_1+1}^{(n)} = \cdots = \lambda_{n_2}^{(n)} > \cdots > \lambda_{n_k+1}^{(n)} = \cdots = \lambda_{n_k}^{(n)}\}.$$

In this section, we review notions and facts on Gelfand-Cetlin polytopes and toric varieties. We mainly follow [1,27] while give the descriptions in a uniform language by ladder diagrams and positive paths,
which will take advantages in the later study of transversal intersections. In what follows, a unit square is called a box.

**Definition 2.1.** Let $Q$ be an $n \times n$ square, i.e., a square consisting of $n^2$ boxes, and $Q_l$ ($l = 1, 2, \ldots, k+1$) be squares of size $(n_l - n_{l-1}) \times (n_l - n_{l-1})$ placed on the diagonal of $Q$ in the lower right direction. The ladder diagram

$$\Lambda := \Lambda(n_1, \ldots, n_k; n)$$

is the set of boxes below the diagonal squares $Q_l$'s.

For each index $l$, we denote the lower right (resp. lower left) vertex of $Q_l$ by $O_l$ (resp. $L_l$) as in Figure 1. We denote by $O_0$ the lower left vertex of $\Lambda$ (or of $Q$).

**Definition 2.2.** We refer to the upper right-hand side of the boundary of $\Lambda$ as the roof of $\Lambda$. An edge of a box is said to be effective, if it is either in the interior of $\Lambda$ or on the roof meeting some vertex $L_l$.

**Remark 2.3.** One can associate, to a ladder diagram $\Lambda$, a graph $\Gamma$ by using the so-called extended plane duality introduced in [1, Definition 2.1.3].

Each effective edge of $\Lambda$ defines a facet of the associated Gelfand-Cetlin polytope to be described below. Recall that $\{\lambda_j^{(n)}\}_{j}$ are fixed already.

**Definition 2.4.** An array $\{\lambda_j^{(i)}\}_{1 \leq j \leq i \leq n}$ of real numbers is called a Gelfand-Cetlin pattern for $\lambda$ if all the inequalities $\lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_j^{(i+1)+1}$ hold for $1 \leq j \leq i \leq n - 1$. The associated Gelfand-Cetlin polytope, denoted by $\Delta_\lambda$ or simply $\Delta$, is the convex hull of all the integer Gelfand-Cetlin patterns for $\lambda$ in $\mathbb{R}^{n(n+1)/2}$.

Every Gelfand-Cetlin pattern can be put in the boxes below the diagonal boxes of $Q$. Precisely, we assign $\lambda_j^{(i)}$ in a box along the direction as illustrated in Figure 2. In particular, all the boxes inside the square $Q_l$ are assigned to the constant quantity $\lambda_{n_l}^{(n)}$ for each $1 \leq l \leq k+1$, due to the inequalities. We have

$$\dim \mathbb{R} \Delta = \sum_{i=1}^{k} (n_i - n_{i-1})(n - n_i),$$

which equals the number of boxes in the ladder diagram $\Lambda(n_1, \ldots, n_k; n)$.

Clearly, the following are equivalent:

1. an edge $e$ in $\Lambda$ is effective;
2. each edge $e$ of $\Lambda$ presents an equation $a_e = b_e$, where $a_e$ and $b_e$ are the assigned quantities of the two boxes or squares containing $e$. Here, the equation $a_e = b_e$ defines a facet $F_e$ of $\Delta$ (i.e., $F_e := \Delta \cap \{a_e = b_e\}$ is a face of codimension one).

![Figure 1](image-url)  
*Figure 1* The ladder diagram $\Lambda$ and its associated dual graph $\Gamma$
The polytope $\Delta$ defines a normal toric variety $X_\Delta$ together with its projective embedding, the so-called Gelfand-Cetlin toric variety. Moreover, it is well known that faces $F$ of $\Delta$ are in one-to-one correspondence with toric subvarieties $X_{\Delta(F)}$ of $X_\Delta$. Here by a toric subvariety, we always assume it to be reduced and irreducible, and refer to [10] for the background on toric varieties.

**Defining equations of $X_\Delta$.** The Gelfand-Cetlin toric variety $X_\Delta$ can be described in terms of explicit defining equations, as we will review below.

**Definition 2.5.** A positive path $\pi$ on the ladder diagram $\Lambda(n_1,\ldots,n_k;n)$ is a path starting at the lower left vertex $O_0$ and moving either upward or to the right along edges, towards one of $O_j$ ($1 \leq j \leq k$). We define a partial order on the set of positive paths as follows: $\pi' \preceq \pi$ if $\pi$ runs above $\pi'$.

A pair $(\pi, \pi')$ is called incomparable, if neither $\pi \preceq \pi'$ nor $\pi' \preceq \pi$ holds.

We denote by $\pi_{i_1,\ldots,i_\ell}$ the positive path $\pi$ being horizontal exactly at the $i_1$-th, ..., $i_\ell$-th steps. Given $I = [i_1,\ldots,i_\ell]$ and $J = [j_1,\ldots,j_m]$, it follows from the definition that $\pi_I \preceq \pi_J$ if and only if $\ell \geq m$ and $i_r \leq j_r$ for $r = 1,\ldots,m$.

**Example 2.6.** For $\Lambda(4,7,9,12;14)$ as in Figure 3, we have $\pi = \pi_{4,5,8,9,10,13,14}$, $\pi' = \pi_{1,2,5,6,7,8,11,12,14}$ and $\pi'' = \pi_{1,2,3,4,5,6,7,8,14}$. Moreover, $\pi \preceq \pi' \preceq \pi''$.

**Definition 2.7.** Given any pair $(\pi_I, \pi_J)$ (not necessarily incomparable), where $I = [i_1,\ldots,i_\ell]$ and $J = [j_1,\ldots,j_m]$ with $\ell \geq m$, the meet $\pi_I \wedge \pi_J$ and the join $\pi_I \vee \pi_J$ are defined, respectively, by

$$I \wedge J := \{\min\{i_1,j_1\},\ldots,\min\{i_\ell,j_\ell\},i_{\ell+1},\ldots,i_\ell\},$$

$$I \vee J := \{\max\{i_1,j_1\},\ldots,\max\{i_\ell,j_\ell\}\}.$$
Let $X_0$ be the subvariety of $\prod_{i=1}^k \mathbb{P}(\Lambda^{n_i} \mathbb{C}^n)$ defined by the quadratic equations

$$p_{I}p_{J} - p_{I\cup J}p_{I\setminus J} = 0,$$

one for each incomparable pair $(\pi_I, \pi_J)$; here, $\{p_I := p_{\pi_I}\}_I$ are naturally regarded as the multi-homogeneous coordinates of the product of projective spaces. Let $\lambda^{(n)}_{n+1} = 0$ and set $b_j = \lambda^{(n)}_j - \lambda^{(n)}_{j+1}$ for each $1 \leq j \leq n$. In particular, $b_j = 0$ if $j \neq n_i$ for any $i$. We have the following proposition.

**Proposition 2.8** (See [27, Proposition 7]). The projective embedding

$$X_0 \hookrightarrow \mathbb{P}\left(\bigotimes_{i=1}^{k+1} \text{Sym}^{b_{ij}}\left(\Lambda^{n_i} \mathbb{C}^n\right)\right)$$

is the projective embedding of the Gelfand-Cetlin toric variety $X_\Delta$ associated with $\Delta$.

We remark that the above proposition was stated for $(n_1, \ldots, n_k; n) = (1, 2, \ldots, n-1; n)$ in [27], while it holds in general. The coincidence was obtained in [27] by making a bijection between the set $\Pi_\lambda$ of lattice points of $\Delta$ and the set of global sections of $\mathcal{O}_{X_0}(b_{n_1}, \ldots, b_{n_k+1})$ (by which we mean the result of tensoring together with the pullbacks of the bundles $\mathcal{O}_{\mathbb{P}(\Lambda^{n_i} \mathbb{C}^n)}(b_{n_i})$ to $X_0$). For the later study of toric subvarieties, we reinterpret the identification in the current framework here.

Let us define an endomorphism of $\mathbb{Z}^{n(n+1)}$ by

$$\phi : \mathbb{Z}^{n(n+1)} \rightarrow \mathbb{Z}^{n(n+1)}_{\pi}, \quad \{b_{ij}\}_{1 \leq i \leq j \leq n} \mapsto \{\lambda^{(i)}_{j} := b_{i,j} + b_{i-1,j} + \cdots + b_{j,j}\}.$$

In other words, if we label every $\lambda^{(i)}_{j}$-box as a $b_{ij}$-box, then the value $\lambda^{(i)}_{j}$ equals the sum of values on the $b_{ij}$-box and all the boxes below $b_{ij}$-box in the column $j$. Clearly, $\phi$ is a bijection with its inverse map $\psi$ defined by $b_{ij} = \lambda^{(i)}_{j} - \lambda^{(i-1)}_{j}$ (where $\lambda^{(i-1)}_{j} := 0$ for convention). For each positive path $\pi_I$, we define a vector $\beta_I = (\beta_{ij}) \in \mathbb{Z}^{n(n+1)}_{\pi}$ by setting $\beta_{ij} = 1$ if the $b_{ij}$-box is right above the path $\pi_I$, or $\beta_{ij} = 0$ otherwise (see Figure 3 for an illustration of the positive path $\pi = \pi_{1,2,5,7,8}$ for $F(3,5,8)$, where all the boxes assigned with 1 are shown)\(^{1}\)). Here for convention, we have treated the path $\pi_{12 \ldots n}$ (which starts at $O_0$, and moves along the bottom boundary of the square $Q$, towards the lower right corner of $Q$) as a positive path. As in [27]\(^{2}\), the vector space of global sections $\mathcal{O}_{X_0}(b_{n_1}, \ldots, b_{n_k+1})$ decomposes into a multiplicity-one direct sum of weight spaces. Denote by $\left(\binom{n}{j}\right)$ the set of subsequences of $[1, 2, \ldots, n]$ of cardinality $j$. The set of weights occurring in this decomposition is given by

$$\Upsilon_\lambda = \left\{ \sum_I \beta_I \left| \text{there are exactly } b_j \text{ sequences } I \in \left(\binom{n}{j}\right) \text{ for } 1 \leq j \leq n \right. \right\}.$$

**Proposition 2.9.** The bijection $\phi$ sends $\Upsilon_\lambda$ onto $\Pi_\lambda$.

The statement follows from exactly the same arguments as on [27, p.10] with the replacement of notation by $\lambda^{(i)}_{j} = \lambda_{n+1-i,j}$ and $b_{ij} = a_{n+1-i,j}$, and hence yields Proposition 2.8.

### 3 Toric degenerations of $W$-translated Schubert varieties

In this section, we introduce the notion of $W$-translated Schubert varieties, and study their toric degenerations.

#### 3.1 $W$-translated Schubert varieties

Let $G = SL(n, \mathbb{C})$ and $B \subset G$ be the Borel subgroup of upper triangular matrices. Let $P \supset B$ be the parabolic subgroup of $G$ consisting of block-upper triangular matrices with diagonal blocks of the form

\(^{1}\) An exponent vector $a_I \in \mathbb{Z}^{n^2}$ is equipped with $p_I$ in [27].

\(^{2}\) The case $b_{n_1} = b_{n_2} = \cdots = b_{n_k+1} = 1$ was proved in [1].
\[ \text{diag}(M_1, \ldots, M_{k+1}), \text{ where } M_i \text{ is an } (n_i - n_{i-1}) \times (n_i - n_{i-1}) \text{ matrix. The (partial) flag variety } X = G/P \text{ is smooth projective, parameterizing partial flags in } \mathbb{C}^n: \]

\[ X = \{ V_{n_1} \subset \cdots \subset V_{n_k} \subset \mathbb{C}^n \mid \dim V_{n_i} = n_i, i = 1, \ldots, k \} =: \mathcal{F}_{n_1, \ldots, n_k}/G. \]

The Weyl group \( W = S_n \) of \( G \) is generated by the transpositions \( s_i = (i, i+1), i = 1, \ldots, n-1 \). Let \( T \subset G \) consist of diagonal matrices, and \( N(T) \) denote the normalizer of \( T \) in \( G \). There is a standard isomorphism of groups

\[ W \xrightarrow{\sim} N(T)/T, \quad w = s_{i_1} \cdots s_{i_m} \mapsto \hat{s}_{i_1} \cdots \hat{s}_{i_m}. \]

Here,

\[ \hat{s}_i = \left( \sum_{j=1}^{i-1} E_{jj} \right) + E_{ii+1} - E_{i+1,i} + \sum_{j=i+2}^{n} E_{jj} \in G, \]

where \( E_{jj} \) denotes the \( n \times n \) matrix with the \((i, j)\)-entry being 1, and 0 otherwise. Although \( \hat{w} := \hat{s}_{i_1} \cdots \hat{s}_{i_m} \) depends on the expression of \( w = s_{i_1} \cdots s_{i_m} \), the coset \( \hat{w}T \) does not.

Let \( \ell : W \to \mathbb{Z}_{\geq 0} \) be the standard length function. The flag variety \( X = G/P \) admits a well-known Bruhat decomposition

\[ X = \bigcup_{w \in W^\ell} B\hat{w}P/P, \]

where \( B\hat{w}P/P \) is isomorphic to the affine space \( \mathbb{C}^\ell(w) \), and \( W^\ell \) is the subset of \( S_n \) given by

\[ W^\ell = \{ w \in S_n \mid w(n_{i-1} + 1) < w(n_{i-1} + 2) < \cdots < w(n_i), i = 1, \ldots, k + 1 \}. \]

The (opposite) Schubert varieties \( X_w \) (resp. \( X^w \)) of complex (co)dimension \( \ell(w) \) are defined by

\[ X_w := B\hat{w}P/P, \quad X^w := \hat{w}_0 B\hat{w}_0\hat{w}P/P, \]

where \( \hat{w}_0 \) is the longest element in \( W \). The Schubert cohomology classes \( \sigma^w := P.D.[X^w] \in H^2(\mathbb{C}\ell(w))(X, \mathbb{Z}) \) form an additive \( \mathbb{Z} \)-basis of \( H^\ast(X, \mathbb{Z}) \).

**Definition 3.1.** A subvariety \( Y \subset SL(n, \mathbb{C})/P \) is called a \emph{W-translated Schubert variety} if \( Y = \hat{u}X_v \) for some \( u \in W \) and \( v \in W^\ell \).

Clearly, the (opposite) Schubert varieties are special cases of the \( W \)-translated Schubert varieties. Moreover, \( W \)-translated Schubert varieties \( \hat{u}X_v \)'s represent the same cohomology class as \( X_v \)'s, and are all \( T \)-invariant. Although the Weyl group \( W \) does not act on \( SL(n, \mathbb{C})/P \), it does act transitively on the set of \( W \)-translated Schubert varieties \( \{ \hat{u}X_v \mid u \in W, v \in W^\ell \} \) in the obvious way. We will simply define \( \hat{u}X_v \) as \( uX_v \) by abuse of notation.

The \( W \)-translated Schubert divisors are bijective to the Plücker coordinates, or equivalently, to the positive paths of the ladder diagram \( \Lambda(n_1, \ldots, n_k; n) \). More precisely, there is a well-known Plücker embedding

\[ P_l : \mathcal{F}_{n_1, \ldots, n_k}/n \to \mathbb{P}^{n_1} \left( \wedge \mathbb{C}^{n_1} \right) \times \cdots \times \mathbb{P}^{n_k} \left( \wedge \mathbb{C}^{n_k} \right). \]

It can be defined by the straightening relations (see [13, Section 9]) \( p_I p_J = p_{I \lor J} p_{I \land J} \pm \sum_{(I', J')} p_{I'} p_{J'} \), one for each incomparable pair \( (\pi_I, \pi_J) \) (where each \( p_I p_J \) occurring in the sum satisfies \( I < I \land J < I \lor J \)). Given a positive path \( \pi_I \), we define

\[ D_{\pi_I} := P_l^{-1}(\{ p_I = 0 \}) \cap P_l(X), \quad \text{or simply } D_{\pi_I} := \{ p_I = 0 \} \cap X, \]

whenever there is no confusion. As a standard fact, the opposite Schubert divisor \( X^{\pi_0} \) is defined by a single coordinate hyperplane for each \( i \), precisely given by

\[ X^{\pi_0_i} = D_{\pi_I} \quad \text{for } I = [1, 2, 3, \ldots, n_i]. \]
We observe that $W = N(T)/T$ acts transitively on the set $\{D_{p_{ij}}\}_J$ via

$$w \cdot D_{p_{ij}} = D_{p_{i\alpha(j)}},$$

where the usual notation convention $p_{ij;1j_2\ldots j_n} = -p_{i,j_2\ldots j_n}$ for Plücker coordinates is adopted. This yields the following proposition.

**Proposition 3.2.** The set \{uX^{s_{i\alpha}} \mid u \in W, 1 \leq i \leq k\} of $W$-translated Schubert divisors coincides with the set $\{D_{p_I}\}_J$, precisely given by $uX^{s_{i\alpha}} = D_{p_{i(1\ldots n)}}$.

Every Schubert variety $X_w$ is scheme-theoretically the intersection of $X$ with all the Plücker coordinate hyperplanes containing $X_w$ (see, e.g., [3, Subsection 2.10]). Consequently, every $W$-translated Schubert variety is scheme-theoretically the intersection of $X$ with all the $W$-translated Schubert divisors containing it. Moreover, the $W$-translated Schubert divisors behave very well with respect to the toric degeneration of $X$, as we will see in the next subsection.

### 3.2 Toric degenerations of $W$-translated Schubert divisors for $F_{n,\ldots,n;k,n}$

There have been lots of studies of the toric degenerations of the partial flag variety $X$ to the Gelfand-Cetlin toric variety $X_{\Delta}$ [1, 13, 24, 27, 31]. Although the degenerations are realized in different ways, they satisfy the following common property: the degeneration is a flat family

$$X \xrightarrow{\varphi} \text{Proj}(\Lambda^{n_1} \mathbb{C}^n) \times \cdots \times \text{Proj}(\Lambda^{n_k} \mathbb{C}^n) \times \mathbb{C}$$

such that $X|_{\mathbb{C}^*} \cong X \times \mathbb{C}^*$ and $X|_{t=0} = X_{\Delta} \times \{0\}$. It is then natural to ask whether a flat subfamily exists and how it looks like for a reasonably nice subvariety of $X$. In the case of the complete flag variety $F_{n,\ldots,n-1,n}$, this has been well studied by Kogan and Miller [27] for (opposite) Schubert varieties, which are special $W$-translated Schubert varieties. They degenerate to the reduced union of the toric subvarieties of the so-called (dual) Kogan faces of the Gelfand-Cetlin polytope $\Delta$. Here, we restrict us to divisors, but study the general $W$-Schubert divisors of partial flag varieties $F_{n,\ldots,n,k,n}$.

Let us use the degeneration given by Gonciulea and Lakshmibai [13], which is defined by equations of the form

$$p_1p_{ij} = p_{i\vee j}p_{i\wedge j} \pm \sum_{(I',J')} t^{N_{I'} + N_{J'} - N_{I} - N_{J}} p_{I'}p_{J'},$$

one for each incomparable pair $(\pi_I, \pi_J)$. The isomorphism $X|_{t=1} \xrightarrow{\cong} X|_{t}$ (where $t \neq 0$) is simply given by $p_I \mapsto t^{N_I} p_I$, where the exponents $N_I = N_{|1\ldots|i}$ can be defined for example by the number $\sum_{r=1}^t (2n)^{n-r}i_r$. It follows immediately that any closed subvariety $Z \subset X$ given by the intersection of Plücker coordinate hyperplanes is flat over $\mathbb{C}^*$. Such $Z|_{\mathbb{C}^*}$ admits a flat extension to the central fiber due to the next proposition.

**Proposition 3.3** (See [14, Proposition III. 9.8]). Let $Y$ be a regular, integral scheme of dimension 1, $p \in Y$ be a closed point, and $Z^* \subset \mathbb{P}^N_{Y-p}$ be a closed subscheme which is flat over $Y - p$. Then there exists a unique closed subscheme $\mathcal{Z}^* \subset \mathbb{P}^N_{Y}$, flat over $Y$, whose restriction to $\mathbb{P}^N_{Y-p}$ is $Z^*$.

Now we consider the case where the subvariety of $X$ is a $W$-translated Schubert divisor. Recall that every effective edge $e$ in the ladder diagram $\Lambda = \Lambda(n_1,\ldots,n_k;n)$ defines a facet $F_e$ of the Gelfand-Cetlin polytope $\Delta$. We have denoted by $X_{F_e}$ the toric divisor of the Gelfand-Cetlin toric variety $X_{\Delta}$ associated with the facet $F_e$.

**Proposition 3.4.** For any effective edge $e$ of $\Lambda$, the corresponding toric divisor $X_{F_e}$, as a subvariety of $X_0$ in $\prod_{i=1}^k \text{Proj}(\Lambda^{n_i} \mathbb{C}^n)$, is given by

$$X_{F_e} = X_0 \cap \{p_J = 0 \mid \pi_J \text{ is a positive path containing } e\}.$$
It is sufficient to show the coincidence of both sides under the embedding in Proposition 2.8. The set of weights of global sections of $\mathcal{O}_{\mathcal{F}_{\lambda}}(b_1,\ldots,b_{n+1})$ is the subset $\mathcal{T}_{\lambda}^\ast \subset \mathcal{T}_{\lambda}$ that satisfies $\phi(\mathcal{T}^\ast_{\lambda}) \subset \mathcal{F}_{\lambda}$.

We claim

$$\mathcal{T}_{\lambda} = \left\{ \sum_I \beta_I \bigg| \text{there are } b_j \text{ sequences } I \in \begin{pmatrix} [n]_j \\ j \end{pmatrix} \text{ for all } j; \text{ none of the positive paths } \pi_I \text{ contain } e \right\}.$$ 

Assuming this claim first, we denote by $D(e)$ the subvariety of $X_0$ defined by the equations on the right-hand side in the statement. Clearly, $D(e)$ is a torus-invariant closed subvariety of $X_0$ with respect to the maximal torus action on $X_0$, and $\mathcal{T}_{\lambda}$ is a subset of the set of weights of global sections of $\mathcal{O}_{D(e)}(b_1,\ldots,b_{n+1})$. Hence, $\mathcal{F}_{\lambda} \subset D(e)$ (though set-theoretically, a priori). On the other hand, given an incomparable pair $(\pi_I, \pi_J)$, we observe that either of the positive paths $\pi_I$ and $\pi_J$ contains the edge $e$ if and only if both of the positive paths $\pi_{I \lor J}$ and $\pi_{I \land J}$. It follows that $\{ \pi_I \mid \pi_I \text{ does not contain } e \}$ is a finite distributive sublattice of $\{ \pi_I \}_{I}$, and that $D(e)$ is the (reduced, irreducible) toric variety associated with such a distributive sublattice (see, for example, [13, Theorem 4.3] for the standard fact about toric varieties associated with distributive lattices). Clearly, $D(e) \subseteq X_0$. Hence, $\mathcal{F}_{\lambda} = D(e)$ as varieties.

It remains to prove the claim. Indeed, if $e$ is a horizontal (resp. vertical) effective edge of $\Lambda$, then it is the common edge of $\lambda^{(i)}_j$-box and $\lambda^{(i-1)}_j$-box (resp. $\lambda^{(i)}_{j+1}$-box) for some $i$ and $j$. We notice that a positive path $\pi_I$ does not contain $e$ if and only if $\pi_I$ is either above $\lambda^{(i)}_j$-box or below $\lambda^{(i-1)}_j$-box (resp. either above or below both of the two boxes). Take $\sum \beta_I \in \mathcal{T}_{\lambda}$. If $e$ is horizontal (resp. vertical), then for every such $\beta_I$, the $b_{j+i}$-box always has the value 0 (resp. the sum of the values of the boxes in the column between the $b_{j+i}$-box and the bottom is equal to that between the $b_{j+i+1}$-box and the bottom). It follows from the definition of $\phi$ that $\lambda^{(i)}_j = \lambda^{(i-1)}_j$ (resp. $\lambda^{(i)}_j = \lambda^{(i+1)}_j$). Hence, $\phi(\mathcal{T}_{\lambda}) \subset \mathcal{F}_{\lambda}$. Conversely, given $\chi \in \mathcal{F}_{\lambda}$, we write $\psi(\chi) = \sum \beta_I$. With the same arguments above, we conclude that none of the positive paths $\pi_I$ contain $e$ so that $\psi(\mathcal{F}_{\lambda}) \subset \mathcal{T}_{\lambda}$.

Theorem 3.5. For any positive path $\pi_I$, the subvariety $\mathcal{D}_{\pi_I} := \{ \pi_I = 0 \} \cap \mathcal{X}$ of $\mathcal{X}$ is a flat subfamily, whose generic fiber is isomorphic to the $W$-translated Schubert divisor $D_{\pi_I}$ of the partial flag variety $X$, and whose fiber at $t = 0$ is a (reduced) union of toric divisors of the Gelfand-Cetlin toric variety $X_0 = X_{\Delta}$:

$$\bigcup_{e \in \pi_I \text{ is effective}} X_{\mathcal{F}_{\lambda}}.$$

Proof. It follows immediately from the aforementioned realization of $\mathcal{X}$ that $\mathcal{D}_{\pi_I} |_{t \neq 0}$ is flat over $\mathbb{C}^*$. (In fact, more is true: $p_I \mapsto \Gamma_{W}p_I$ induces an isomorphism $\mathcal{D}_{\pi_I} |_{t \neq 0} \cong D_{\pi_I} \times \mathbb{C}^*$.) We notice that $\mathcal{D}_{\pi_I} |_{t = 0} = X_{\Delta} \cap \{ \pi_I = 0 \}$ is a reduced torus-invariant divisor of $X_0$. In particular, it is of pure dimension $\dim X_0 - 1 = \dim D_{\pi_I}$. Hence, $\mathcal{D}_{\pi_I}$ is irreducible. (Otherwise, $\mathcal{D}_{\pi_I}$ would have a component, which is a hypersurface in $\mathcal{X}$ and whose image under $\phi$ is a point. Such a component of dimension $(1 + \dim D_{\pi_I})$ can only occur in $t = 0$, resulting in a contradiction.) Since $\mathcal{D}_{\pi_I}$ is closed, it follows that the unique flat extension $\mathcal{D}_{\pi_I} |_{t \neq 0}$ by Proposition 3.3 coincides with $\mathcal{D}_{\pi_I}$. The description of the fiber of $\mathcal{D}_{\pi_I}$ at $t = 0$ follows from the explicit defining equations of toric divisors as in Proposition 3.4.

Remark 3.6. Apply Theorem 3.5 to the case $X = \mathcal{F} \times \mathcal{N}$ with $\pi_I = \pi_{n+1-i_0,n+2-i_0,\ldots}$, where $1 \leq i_0 \leq n - 1$. Then the facets of $\Delta$ occurring in the toric degeneration of the $W$-translated Schubert divisor $D_{\pi_I} = uX^{s_{\pi_I}}$ are precisely given by $\Gamma_{W} \in \pi_I$ in terms of the notation in $[20]$. In other words, it is consistent with the representation of the divisor class $[uX^{s_{\pi_I}}] = \sigma^{s_{\pi_I}}$ by elements of a polytope ring in $[20$, Example 4.4]. Moreover, by induction and $[20$, Proposition 3.2], one can show that the following two collections of sets of facets coincide with each other:

$$\left\{ \{ F_e \}_e \bigg| \text{there exists } uX^{s_{\pi}} \text{ that degenerates to } \bigcup_e X_{\mathcal{F}_{\lambda}} \right\},$$

$$\left\{ \{ F_e \}_e \bigg| \text{there exists } X^{s_{\pi}} \text{ such that } \sum_e \pi([F_e]) = \sigma^{s_{\pi}} \text{ in the sense of } [20, \text{Subsection 4.3}] \right\}.$$
3.3 Toric degeneration of \( W \)-translated Schubert varieties for \( Gr(m, n) \)

Toric degeneration of Schubert varieties of complex Grassmannians \( Gr(m, n) = F\ell_{m,n} \) have been studied well. Although it is known to the experts, a precise description of such degenerations in terms of faces of the Gelfand-Cetlin polytopes seems to be missing in the literature. In this subsection, we give precise descriptions for the (opposite) Schubert varieties of \( Gr(m, n) \) as well as for certain \( W \)-translated Schubert varieties of \( Gr(2, n) \).

3.3.1 The indexing set by partitions

Schubert varieties of \( Gr(m, n) \) are traditionally indexed by the set of partitions inside an \( m \times (n - m) \) rectangle, i.e.,

\[
\mathcal{P}_{m,n} := \{ \mu = (\mu_1, \ldots, \mu_m) \in \mathbb{Z}^m \mid n - m \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0 \},
\]

which is equipped with a natural partial order \( \leq \) as for vectors in \( \mathbb{Z}^m \). There is an isomorphism of partially ordered sets \((\{\pi_I \}, \leq) \sim \rightarrow (\mathcal{P}_{m,n}, \leq)\) defined by

\[
p_{i_1 i_2 \cdots i_m} \mapsto \mu = (i_m - m, 2, i_2 - 2, i_1 - 1).
\]

The partition \( \mu \) can be directly read off from the ladder diagram by counting the number of boxes below the positive path column-to-column from right to left. The complement \( \mu^\vee := (n - m - \mu_m, \ldots, n - m - \mu_1) \) can also be directly read off from the ladder diagram by counting the number of boxes above the positive path column-to-column from left to right.

3.3.2 Defining equations of toric subvarieties for \( \Lambda(m; n) \)

Given a partition \( \mu = \mu_\pi \) corresponding to a positive path \( \pi \), we define a face \( F_{\mu_\pi} \) (resp. \( F_{\mu_\pi}^\vee \)) of \( \Delta \) simply by letting all the boxes below (resp. above) the path \( \pi \) take the same value \( b := \lambda_\pi(n) \) (resp. \( a := \lambda_\pi(m) \)). (See Figure 4 for an illustration.) By the similar arguments to the proof of Proposition 3.4, we have the following proposition.

**Proposition 3.7.** The toric subvarieties \( X_{F_{\mu_\pi}} \) and \( X_{F_{\mu_\pi}^\vee} \), as subvarieties of \( X_0 \) in \( \mathbb{P}(\wedge^m \mathbb{C}^n) \), are respectively given by

\[
X_{F_{\mu_\pi}} = X_0 \cap \{ p_I = 0 \mid \pi_I \not\geq \pi \}, \quad X_{F_{\mu_\pi}^\vee} = X_0 \cap \{ p_I = 0 \mid \pi_I \not\leq \pi \}.
\]

**Proof.** We notice that \( \pi_I \not\geq \pi \) if and only if either \( \pi_I \not\leq \pi \) or the pair \((\pi_I, \pi_\pi)\) is incomparable. Under the embedding

\[
X_0 \hookrightarrow \mathbb{P} \left( \text{Sym}^a \wedge \mathbb{C}^n \otimes \text{Sym}^b \wedge \mathbb{C}^n \right)
\]

as in Proposition 2.8, the set \( \Upsilon_\lambda \) of weights of global sections is given by

\[
\Upsilon_\lambda = \left\{ b\beta_{i_2 \cdots i_m} + \sum_{j=1}^{a-b} \beta_{i_j} \mid \pi_I \text{ is a positive path with } m \text{ horizontal steps for each } j \right\}.
\]

**Figure 4** (Color online) Partitions and faces associated with a positive path in \( \Lambda(5;8) \)
Moreover, $\Upsilon_\lambda$ is sent onto the lattice points of the Gelfand-Cetlin polytope $\Delta$ via the bijection $\phi$. Consider the subsets

$$\Upsilon^{\mu^+}_\lambda := \left\{ b\beta_{12\cdots n} + \sum_{j=1}^{a-b} \beta_{I_j} \mid \pi_{I_j} \geq \pi \text{ for each } j \right\} \cap \Upsilon_\lambda \subset \Upsilon_\lambda,$$

$$\Upsilon^{\mu^0}_\lambda := \left\{ b\beta_{12\cdots n} + \sum_{j=1}^{a-b} \beta_{I_j} \mid \pi_{I_j} \leq \pi \text{ for each } j \right\} \cap \Upsilon_\lambda \subset \Upsilon_\lambda.$$

The vector $\beta_{12\cdots n}$ takes the value 1 on the entries labeled by the boxes on the bottom row of the square $Q$, or 0 otherwise. For any $\beta_{I_j}$ in an element $x$ of $\Upsilon^{\mu^0}_\lambda$ (resp. $\Upsilon^{\mu^+}_\lambda$), the nonzero entries of the vector $\beta_{I_j}$ occur only on some boxes right above the positive path $\pi_{I_j}$, and hence above $\pi$ (resp. and hence make contributions 1 for any box above $\pi$ in the image $\phi(x)$). It follows immediately that $\phi(\Upsilon^{\mu^0}_\lambda) \subset F_{\mu^0}$ and $\phi(\Upsilon^{\mu^+}_\lambda) \subset F_{\mu^+}$. Now take any element $y$ of $F_{\mu^0}$ (resp. $F_{\mu^+}$). If $\pi_{I_j} \geq \pi$ (resp. $\pi_{I_j} \leq \pi$) did not hold for some $\beta_{I_j}$ in $\psi(y)$, then there must exist a box having a top (resp. bottom) edge on the path $\pi$ and being above (resp. below) $\pi_{I_j}$. It would make contribution 1 (resp. 0) to such box in the image $\phi(\psi(y))$. Consequently, this box would take the value larger than $b$ (resp. smaller than $a$) in $y = \phi(\psi(y))$, resulting a contradiction. Hence, there is a bijection between $\Upsilon^{\mu^0}_\lambda$ (resp. $\Upsilon^{\mu^+}_\lambda$) and the lattice points in $F_{\mu^0}$ (resp. $F_{\mu^+}$) via the map $\phi$.

On the other hand, we define $X_0(F_{\mu^0}) := X_0 \cap \{ p_I = 0 \mid \pi_I \notin \pi \}$ and $X_0(F_{\mu^+}) := X_{F_{\mu^+}} = X_0 \cap \{ p_I = 0 \mid \pi_I \notin \pi \}$. Clearly, they are both torus-invariant, and the sets of weights of the global sections of the restriction of $\mathcal{O}_{X_0(\mathbb{A}^n \cap \mathbb{C})}$ to them contain $\Upsilon^{\mu^0}_\lambda$ and $\Upsilon^{\mu^+}_\lambda$, respectively. Furthermore, they are the toric varieties, respectively, associated with the finite distributive sublattices $\{ \pi_{I} \mid \pi_I \geq \pi \}$ and $\{ \pi_{I} \mid \pi_I \leq \pi \}$. It follows that $X_{F_{\mu^0}} \subset X_0(F_{\mu^0})$ and $X_{F_{\mu^+}} \subset X_0(F_{\mu^+})$ as reduced, irreducible varieties. The ranks of these lattices are, respectively, given by

$$\max\{ r - 1 \mid \pi_{I_1} > \pi_{I_2} > \cdots > \pi_{I_r} \geq \pi \} = \sum_{i=1}^{m} (n - m - \mu_i) = \dim \mathbb{R} F_{\mu^0},$$

$$\max\{ r - 1 \mid \pi_{I_1} < \pi_{I_2} < \cdots < \pi_{I_r} \leq \pi \} = \sum_{i=1}^{m} \mu_i = |\mu| = \dim \mathbb{R} F_{\mu^+}.$$

It is well known that the rank of a finite distributive lattice is equal to the complex dimension of the associated projective toric variety (see, e.g., [15, Section 1]). Therefore, $X_0(F_{\mu^0})$ (resp. $X_0(F_{\mu^+})$) is of (co)dimension $|\mu|$, the same as that of $X_{F_{\mu^0}}$ (resp. $X_{F_{\mu^+}}$). Hence, they must coincide with each other.  

3.3.3 Toric degeneration of Schubert varieties

The (opposite) Schubert varieties of $X = Gr(m,n) = SL(n,\mathbb{C})/P$ are indexed by the subset $W^P = \{ w \in S_n \mid w(1) < \cdots < w(m); w(m+1) < \cdots < w(n) \}$ of permutations. There are bijections of sets

$$W^P \xrightarrow{\cong} \{ p_\mu \} \xrightarrow{\cong} \mathcal{P}_{m,n}, \quad w \mapsto p_{w(1),\ldots,w(m)} \mapsto w = (w(m) - m, \ldots, w(1) - 1).$$

Traditionally, the (opposite) Schubert varieties $X_{\mu} := X_w$ and $X^{\mu^0} := X^w$ are indexed by the partitions $\mu \in \mathcal{P}_{m,n}$. We simply define $\mu = \mu_w$. The defining equations of (opposite) Schubert varieties, as subvarieties of $\mathbb{P}(\wedge^n \mathbb{C}^n)$ via the Plücker embedding, are just given by (see, e.g., [28, Subsection 4.3.4])

$$X_{\mu} = X \cap \{ p_I = 0 \mid \pi_I \notin \pi_{\mu} \}, \quad X^{\mu} = X \cap \{ p_I = 0 \mid \pi_I \notin \pi_{\mu} \}.$$

**Proposition 3.8.** The subfamilies $\mathcal{X}_{\mu} := X \cap \{ p_I = 0 \mid \pi_I \notin \pi_{\mu} \}$ and $X^{\mu} := X \cap \{ p_I = 0 \mid \pi_I \notin \pi_{\mu} \}$ are flat over $\mathbb{C}$. Moreover,

$$\mathcal{X}_{\mu} |_{t=1} = X_{\mu}, \quad \mathcal{X}_{\mu} |_{t=0} = X_{F_{\mu^0}}, \quad X^{\mu} |_{t=1} = X^{\mu}, \quad X^{\mu} |_{t=0} = X_{F_{\mu^+}}.$$

Furthermore, for any partition $\eta \in \mathcal{P}_{m,n}$ with $\eta \geq \mu$, the subfamily $X^{\mu} \cap \mathcal{X}_\eta$ is flat.
Proof. We notice that \( X_\mu |_{C} \) is flat, isomorphic to \( X_\mu \times \mathbb{C}^* \). Hence, it extends to the flat subfamily \( \mathcal{X}_\mu |_{C} \subset \mathcal{X}_\mu \) with \( \mathcal{X}_\mu |_{C} \big|_{t=0} \subset \mathcal{X}_\mu |_{t=0} \). By Proposition 3.7, \( \mathcal{X}_\mu |_{t=0} = X_{F_\mu} \). In particular, it is reduced, irreducible, and of dimension \(|\mu| = \dim X_\mu \). Since \( \dim \mathcal{X}_\mu |_{C} |_{t=0} = \dim \mathcal{X}_\mu |_{C} |_{t=1} = |\mu| \), it follows that \( \mathcal{X}_\mu |_{C} |_{t=0} = X_{F_\mu} \) and \( \mathcal{X}_\mu |_{C} = \mathcal{X}_\mu \). The arguments for \( \mathcal{X}^\mu \) are the same.

For \( \eta \geq \mu \), \( X_{F_\eta} \cap X_{F_\mu} \) is a (reduced, irreducible) toric subvariety of dimension \(|\eta| - |\mu| = \dim X_\nu \cap X_\eta \). Thus the last statement follows as well. \( \Box \)

3.3.4 Toric degeneration of special \( W \)-translated Schubert varieties in \( Gr(2, n) \)

Let \( C \) denote the cyclic permutation \((2, 3, \ldots, n, 1)\). Here, we study the toric degeneration of the \( W \)-translated Schubert varieties \( C^k \cdot X^{(1,1)} \) of \( Gr(2, n) \). We remark that the permutation \( C \) plays an interesting role in the study of mirror symmetry for complex Grassmannians \([17]\). Note that \( C \) is of order \( n \), and that \( C^0 \cdot X^{(1,1)} = X^{(1,1)} \) and \( C^{n-1} \cdot X^{(1,1)} = X_{(n-3,n-3)} \) have been studied above. The Gelfand-Cetlin polytope of \( Gr(2, n) \) are parameterized by variables \( \{\lambda_1^{(i)}, \lambda_2^{(i+1)} | 1 \leq i \leq n-2\} \). Let \( \Delta_{(k)} \) denote the face of \( \Delta \) of codimension two defined by \( \Delta_{(k)} = \Delta \cap \{\lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_2^{(n-k)}\} \), where we define \( \lambda_2^{(1)} = b \) and \( \lambda_2^{(n-1)} = a \) for conventions.

Proposition 3.9. For \( 1 \leq k \leq n-2 \), the \( W \)-translated Schubert variety \( C^k \cdot X^{(1,1)} \) of \( X = Gr(2, n) \) degenerates to the toric subvariety of \( X_0 \) defined by the face \( \Delta_{(k)} \).

Proof. The opposite Schubert variety \( X^{(1,1)} \) is given by the scheme-theoretical intersection

\[
X \cap \{p_{ij} = 0 \mid 2 \leq j \leq n\}.
\]

Therefore, \( C^k \cdot X^{(1,1)} \) is given by the scheme-theoretical intersection \( X \cap \{p_{k+1,j} = 0 \mid 1 \leq j \leq n, j \neq k+1\} \), and it degenerates to a (codimension two) closed subvariety of \( X_0 \cap \{p_{k+1,j} = 0 \mid 1 \leq j \leq n, j \neq k+1\} \). By direct verifications, \( \Delta_{(k)} \) is the set-theoretically given by \( \bigcap_{1 \leq j \leq n, j \neq k+1} \bigcup_{k \in C^{1}} F_i \). Hence, \( C^k \cdot X^{(1,1)} \) degenerates to \( X_0 \cap \{p_{k+1,j} = 0 \mid 1 \leq j \leq n, j \neq k+1\} \). It follows that the multiplicity is equal to one, i.e., the statement follows. \( \Box \)

We remark that \( C \cdot X^{(r,1)} \) degenerates to the union of more than one toric subvarieties of \( X_0 \) for general \( n-2 > r > t > 0 \). For example, the subvariety \( C \cdot X^{(2,0)} \) of \( Gr(2, 5) \) degenerates to three distinct toric subvarieties of \( X_0 \subset \mathbb{P}^9 \), each of which is isomorphic to \( \mathbb{P}^4 \).

4 Applications: Transversal intersections of Schubert varieties

In this section, we study the transversality of intersections of \( W \)-translated Schubert varieties by using toric degenerations. We restate Conjecture 1.3 in Conjecture 4.9, which leads to a Littlewood-Richardson rule for complete flag varieties. We verify our conjecture in some cases.

4.1 Transversal intersections by toric degenerations

The natural projection from \( F_{\ell_n} := F_{\ell_1, \ldots, \ell_{n-1}, n} \) to \( F_{\ell_1, \ldots, \ell_k, n} \) induces an injective homomorphism of algebras \( H^*(F_{\ell_1, \ldots, \ell_k, n}, \mathbb{Z}) \rightarrow H^*(F_{\ell_n}, \mathbb{Z}) \), which sends every Schubert class \( \sigma_p^\mu \) of the former one to the Schubert class \( \sigma_p^\nu \) of the latter one. Therefore, we skip the subscripts \( P \) and \( B \) whenever it is well understood, and use the same notation for the corresponding Schubert structure constants. The Schubert classes \( \{\sigma^\mu\} \) form an additive \( \mathbb{Z} \)-basis of \( H^*(F_{\ell_n}, \mathbb{Z}) \) so that for any \( u_1, \ldots, u_m, v \in S_n \), we can write

\[
\sigma^u_1 \cup \cdots \cup \sigma^u_m \cup \sigma^v = \sum_{w \in S_n} N_{u_1, \ldots, u_m, v}^w \sigma^w.
\]

We usually consider the case \( m = 1 \) only, since it determines the cases \( m \geq 2 \). The structure constant \( N_{u_1, \ldots, u_m, v}^w \) is nonnegative, counting the number of intersection points of \( g_1 \cdot X^{u_1} \cap \cdots \cap g_m \cdot X^{u_m} \cap \cdots \cap g_v \cdot X^{v} \).
\( g_{m+1} X^n \cap X_w \) for generic \((g_1, \ldots, g_{m+1}) \in \prod_{i=1}^{n+1} SL(n, \mathbb{C}) \) if \( \ell(w) = \ell(u_1) + \cdots + \ell(u_m) + \ell(v) \), or equal to zero otherwise. One of the central problems in Schubert calculus is finding a Littlewood-Richardson rule, namely, a manifestly combinatorial formula/algorihm for the Schubert structure constants. However, this problem is widely open except for complex Grassmannians and for two-step flag varieties [5, 6, 9].

We make the statement valid for any partial flag variety \( X = SL(n, \mathbb{C})/P = F\ell_{n_1, \ldots, n_r} \). Denote by \( V \) the set of vertices of \( \Delta \) for \( X \), and let \( V^X := \{ z \in V \mid X_0(z) \in X \} \), where \( X_0(z) \) denotes the 0-dimensional toric subvariety of \( X \) corresponding to \( z \); here, we consider \( X_0(z) \) as an element of the product of projective spaces that contains both \( X_0 \) and \( X \). We call \([z_0, z_1, \ldots, z_N] \) in \( \mathbb{P}^N \) a coordinate point if all \( z_j \)'s but one are equal to 0.

**Lemma 4.1.** Let \( X = Gr(m, n) \). Then \( \{X_0(z) \mid z \in V \} \) is the set of the coordinate points of \( \mathbb{P}(\omega)^{-1} \).

**Proof.** Notice that a vertex in the polytope \( \Delta \) is an assignment of the boxes by either \( a = \lambda^{(i)}_n \) or \( b = \lambda^{(i)}_n \). Since \( \lambda^{(i+1)}_j = \lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+1} \), for any \( 2 \leq j \leq m \), we have the inequality \( 0 \leq 2(i + 1) \leq j \leq n - m \). In other words, it determines a unique partition and hence the corresponding positive path \( \pi_1 \). By Proposition 3.4, the corresponding point in \( X_0 \) does not lie in the coordinate plane \( \{p_1 = 0\} \) but lies in any other coordinate plane, and hence is a coordinate point. The number of positive paths is equal to \( \binom{m}{2} \), which also equals the number of coordinate points of \( \mathbb{P}(\omega)^{-1} \). Therefore, the correspondence is one-to-one, and hence the first statement follows. The second statement also follows immediately, by noting that every coordinate point satisfies all the defining equations of \( X \), which are all the sum of the form \( p_{j} \neq K \) up to a sign.

**Corollary 4.2.** Let \( X = F\ell_{n_1, \ldots, n_r} \). For any \( z \in V \), \( X_0(z) \) is a coordinate point of \( \mathbb{P}(\omega)^{-1} \times \cdots \times \mathbb{P}(\omega)^{-1} \).

**Proof.** The restriction of the natural projection

\[
\mathbb{P}(\omega)^{-1} \times \cdots \times \mathbb{P}(\omega)^{-1} \rightarrow \mathbb{P}(\omega)^{-1}
\]

to the Gelfand-Cetlin toric variety \( X_0 \) has image in the Gelfand-Cetlin toric variety with respect to \( \Lambda(n_j, n) \). Moreover, it sends \( X_0(z) \) to a point of the latter toric variety that corresponds to a vertex of the latter polytope. Therefore, the statement follows from Lemma 4.1.}

**Proposition 4.3.** If \( X = F\ell_{n_0} \), then \( V^X \) consists of vertices of \( \Delta \) such that for any \( 1 \leq j \leq i < i + 2 \leq n \), not all the equalities \( \lambda^{(i)}_j = \lambda^{(i+1)}_j = \lambda^{(i+1)}_{j+1} = \lambda^{(i+2)}_{j+1} \) hold.

**Proof.** Let \( z = \{\lambda^{(i)}_c\} \) be a vertex in \( \Delta \). For \( 1 \leq r \leq n \), let us simply call \( \{\lambda^{(i)}_c \mid \lambda^{(i)}_c = \lambda^{(i)}_c \} \) the \( \lambda^{(i)}_c \)-block. The inequalities for the Gelfand-Cetlin patterns ensure that the \( \lambda^{(i)}_c \)-block and the \( \lambda^{(i+1)}_{r+1} \)-block are adjacent by a unique positive path \( \pi_{I_r} \) with \(|I_r| = r \) for any \( 1 \leq r \leq n - 1 \). These positive paths define a coordinate point of the product of projective spaces.

For the given vertex \( z \), we notice that the equalities of the form \( \lambda^{(i)}_r = \lambda^{(i+1)}_r = \lambda^{(i+1)}_{r+1} = \lambda^{(i+2)}_{r+1} \) can at most occur in the \( \lambda^{(i)}_c \)-block for some \( 2 \leq r \leq n - 1 \). If they do occur in some \( \lambda^{(i)}_c \)-block, we can assume \( r = 2 \) without loss of generality. Let \( k_3 = \min\{i \mid \lambda^{(i)}_j = \lambda^{(i)}_j \} \) for \( j \in \{1, 2\} \) and \( k_3 = \max\{i + 1 \mid \lambda^{(i)}_j = \lambda^{(i)}_j \} \). It follows that \( k_1 < k_2 < k_3 \). In this case, the coordinates of \( X_0(z) \) satisfy \( p_{\lambda(k_2)} p_{\lambda(k_1)} = 0 \) and \( p_{\lambda(k_3)} p_{\lambda(k_2)} = 0 \). Since one defining equation of \( F\ell_{n_0} \) is given by \( p_{\lambda(k_2)} p_{\lambda(k_1)} = 0 \), \( X_0(z) \) cannot belong to \( F\ell_{n_0} \).

Now we consider the case where for any \( i \), not all the equalities \( \lambda^{(i)}_1 = \lambda^{(i+1)}_2 = \lambda^{(i+1)}_1 = \lambda^{(i+2)}_1 \) hold for the given vertex \( z \). The nonvanishing coordinates \( p_{I_1} \) and \( p_{I_2} \) of \( X_0(z) \), where \(|I_1| = i \) for \( i \in \{1, 2\} \), satisfy either (a) \( I_1 = \{k\} \) and \( I_2 = \{k, k\} \) or (b) \( I_1 = \{k\} \) and \( I_2 = \{k, k\} \) for some \( k < k \). In particular, \( p_{I_1} p_{I_2} \) never occurs in the defining equations with respect to incomparable pairs \( \pi_{I_1}, \pi_{I_2} \) with \(|J| = |I_1| \) and \(|K| = |I_2| \). In other words, \( X_0(z) \) satisfy all the corresponding defining equations. The arguments for the nonvanishing coordinates \( p_{I_1} \) and \( p_{I_2} \) for general \( 1 \leq a < b \leq n - 1 \) are similar. It follows that \( X_0(z) \) satisfies all the defining equations of \( F\ell_{n_0} \), and hence belongs to \( F\ell_{n_0} \).
Remark 4.4. The vertices in the above statement are precisely the regular vertices of \( \Delta \), namely, the vertices that lie in exactly \( \frac{n(n-1)}{2} \) facets of \( \Delta \). There are exactly \( n! = |S_n| \) regular vertices [19, Subsection 5.1]. The fibers of the Gelfand-Cetlin system \( F\ell_n \to \Delta \) were well studied in [4, 8], which are diffeomorphic to the product of a real torus and a precisely described smooth manifold. The regular vertices are exactly those points in \( \Delta \) whose fibers are points.

For any \( v \in W^P \subset W = S_n \), the opposite Schubert variety \( X^v \) of \( X = F\ell_{n_1,\ldots,n_k} \) is given by the intersection of finitely many \( W \)-translated Schubert divisors \( \{D_{p_{I_j}}\} \) with \( X \). For any \( u \in W \), by \( \pi_u(I_j) \) we mean the positive path in the ladder diagram of \( X \) whose horizontal steps are precisely elements of the set \( u(I_j) \). We consider the following subset of \( \Delta \):

\[
\Delta(u,v) := \bigcap_j \bigcup_{a \in \text{effective}} F_{e_a}.
\]

Notice that \( X_v = w_0X^\pi(w_0v) \) where \( w_0 \) denotes the longest element of \( W \), and \( \pi(u_0v) \) denotes the unique element in \( W^P \) such that \( \pi(u_0v)^{-1}u_0v \) belongs to the subgroup \( W^P \) generated by \( \{s_j \mid j \in \{1,\ldots,n-1\} \} \setminus \{n_1,\ldots,n_k\} \).

Theorem 4.5. Let \( v_1,\ldots,v_{m+2},w \in W^P \) with \( m \geq 1 \) and \( \ell(w) = \sum_{i=1}^{m+1} \ell(v_i) \). Suppose that there exist \( u_1,\ldots,u_{m+1} \in W \) such that \( S := \bigcap_{i=1}^{m+1} (\Delta(u_i,v_i) \cap \Delta(w_0,\pi(w_0v))) \) is a subset of \( V^X \). Then \( u_1X^{v_1},\ldots,u_{m+1}X^{v_{m+1}},X_w \) intersect transversely, and \( N_w^{v_1,\ldots,v_{m+1}} = \mathbb{Z}S \).

Proof. Define \( u_{m+2} := w_0 \) and \( v_{m+2} := \pi(w_0v) \). For \( 1 \leq i \leq m+2 \), we let \( u_iX^{v_i} := \bigcap_{l=1}^i \{p_{u_l(I_j)} = 0\} \cap X \) be the closed subvariety of \( X \) in which \( \bigcap_{l=1}^i \{p_{u_l(I_j)} = 0\} \) defines the opposite Schubert variety \( X^{v_i} \) of \( X \). Clearly, \( Y := \bigcap_{i=1}^{m+2} u_iX^{v_i} \) is a closed subfamily of \( X \); \( Y|_C \) is flat over \( \mathbb{C}^* \), isomorphic to the trivial family \( Y \times \mathbb{C}^* \) with \( Y := \bigcap_{i=1}^{m+2} u_iX^{v_i} \). Hence, it admits a flat extension \( Y|_{C^*} \), sitting inside \( Y \), by Proposition 3.3. It follows from the flatness and Theorem 3.5 that

\[
\dim Y = \dim Y|_{C^*}|_{t=0} = \dim Y|_{C^*}|_{t \geq 0} \leq \dim Y|_{t=0}.
\]

Notice that \( Y|_{t=0} \) is a coordinate point of \( X_0 \) corresponding to \( S \). In particular, if \( S \) is empty, then \( Y|_{t=0} = \emptyset \). Hence, we have \( Y = \emptyset \) and consequently, \( N_w^{u_1,\ldots,u_{m+1}} = 0 \) due to the geometrical meaning of the structure constants.

By Corollary 4.2, \( X_0(z) \) is a coordinate point of the product of projective spaces for any \( z \in V \). By Theorem 3.5, the subspace of \( X_0 \) corresponding to \( S \) is given by the set-theoretical intersection of coordinate planes \( \{p_{I_1} = 0\} \) with \( X_0 \). It follows that this subspace is given by intersections of the form \( \{p_{I_1} = 0\}_{I_1} \) and \( \{p_{I_1} = 0\}_{I_2} \) with \( J_i \neq K_i \) for all \( i \). Hence, each point in the subvariety of \( X_0 \) corresponding to \( S \) is reduced in the scheme-theoretical intersection of these coordinate planes.

Therefore, the cardinality of \( Y \) with multiplicities counted is less than or equal to \( \mathbb{Z}S \) due to the flatness of the family \( Y|_{C^*} \). Moreover, for any \( z \in S \subset V^X \), we have \( X_0(z) \in X \) by definition. Hence, \( X_0(z) \) is also in the intersection of the corresponding coordinate planes \( \{p_{I_1} = 0\} \) with \( X \) so that \( \mathbb{Z}Y \geq \mathbb{Z}S \). It follows that \( Y = \bigcap_{i=1}^{m+2} u_iX^{v_i} \) is a transversal intersection and \( N_w^{u_1,\ldots,u_{m+1}} = \mathbb{Z}Y = \mathbb{Z}S \). Moreover, \( Y \) coincides with the subspace of \( X_0 \) corresponding to \( S \).

\[\square\]

Remark 4.6. We expect that \( S = \bigcap_{i=1}^{m+1} \Delta(u_i,v_i) \cap \Delta(w_0,\pi(w_0v)) \) is always a subset of \( V^X \), if \( S \) is a subset of \( V \).

4.2 Towards a Littlewood-Richardson rule

In this subsection, we construct a modified partition of the following set:

\[
A := \{(u,v,w) \in S_n \times S_n \times S_n \mid \ell(w) = \ell(u) + \ell(v)\}
\]

so that Conjecture 1.3 makes sense.

Firstly, we divide the above set into the set

\[
A_1 := \{(u,v,w) \in S_n \times S_n \times S_n \mid \ell(w) = \ell(u) + \ell(v), u \leq w, v \leq w\}
\]
and its complement $A_0$. Here, $u \leq w$ is with respect to the Bruhat order. It is a well-known fact that $N_{w,w}^w = 0$ whenever $(u,v,w) \in A_0$.

Secondly, for every $(u,v,w) \in A_1$, we start with $B_{u,v,w}^{(1)} := \{(u,v,w), (v,u,w)\}$, $B_{u,v,w}^{(2)} := \{(u,v,w), (w_0u,w_0v,w_0w)\}$ and $B_{u,v,w}^{(3)} := \{(u,v,w), (u,w_0v,w_0v)\}$. We notice the identities

$$N_{w,w}^w = N_{v,v}^w = N_{u,v,w}^{w_0v} \quad \text{and} \quad N_{u,v}^w = N_{w_0u,w_0v,w_0w}^{w_0w}.$$  \hspace{1cm} (4.1)

There is an automorphism of the Dynkin diagram of $SL(n,\mathbb{C})$ given by $\alpha_i \mapsto -w_0(\alpha_i) = \alpha_{n-i}$. It induces an automorphism of $X = F\ell_n = SL(n,\mathbb{C})/B$, which sends $X^w$ to $X^{w_0w}$. Then the induced automorphism of the cohomology $H^*(F\ell_n, \mathbb{Z})$ results in the last identity in (4.1).

There is another type of identities among (equivariant) genus zero, three-point Gromov-Witten invariants for the complete flag variety $G/B$ of general Lie types in [16,29]. The special case of degree zero Gromov-Witten invariants recovers the recurrence formula in [22]. For the classical cohomology of $F\ell_n$, we have the following proposition.

**Proposition 4.7.** For any $u,v,w \in S_n$ and any $1 \leq i \leq n-1$, we have

$$N_{u,v}^w = \begin{cases} N_{us_i,v}^{w_i} & \text{if } \ell(us_i) > \ell(u), \ell(vs_i) > \ell(v), \ell(ws_i) > \ell(w), \\ 0, & \text{if } \ell(us_i) > \ell(u), \ell(vs_i) > \ell(v), \ell(ws_i) < \ell(w). \end{cases}$$

Consequently, we consider

$$B_{u,v,w}^{(4)} := \{(u,v,w)\} \cup \{(us_i,v,ws_i) \mid \ell(us_i) > \ell(u), \ell(vs_i) > \ell(v), \ell(ws_i) > \ell(w) \text{ for some } i\}.$$  \hspace{1cm} (4.2)

In this way, we obtain a covering of $A$, given by

$$U_0 := \{A_0\} \cup \{B_{u,v,w}^{(j)} \mid (u,v,w) \in A_1, 1 \leq j \leq 4\}.$$  \hspace{1cm} (4.3)

It defines a consistent relation $R_0 \subset A \times A$, and hence gives rise to an equivalence relation $R_1 \subset A \times A$ by letting $R_1$ be the transitive closure of $R_0$. The quotient space $U_1 := A/R_1$ is a partition of $A$ with $A_0$ being an equivalence class in $U_1$. Now we set

$$C := \{[(\hat{u}, \hat{v}, \hat{w})] \in U_1 \setminus \{A_0\} \mid \text{there is } (u,v,w) \in [(\hat{u}, \hat{v}, \hat{w})] \text{ such that all the inequalities} \hspace{1cm} \ell(us_i) > \ell(u), \ell(vs_i) > \ell(v), \ell(ws_i) < \ell(w) \text{ hold for some } i\}.$$  \hspace{1cm} (4.4)

It follows that $\mathcal{U} := (U_1 \setminus (C \cup \{A_0\})) \cup \{A_0 \cup \bigcup_{\alpha \in C} \alpha\}$ is still a partition of $A$. Moreover, $N_{u_1,v_1}^{w_1} = N_{u_2,v_2}^{w_2}$ whenever $[(u_1,v_1,w_1)] = [(u_2,v_2,w_2)]$; $N_{u_1,v_1}^{w_1} = 0$ whenever $(u_1,v_1,w_1) \in A_0 \cup \bigcup_{\alpha \in C} \alpha$.

Finally we construct $\mathcal{U}$ as follows. We say that $u \in S_n$ satisfies the property $(\ast)$ at level $m$ if $u$ can be written as $u = u_1u_2 \cdots u_m$ with $\ell(u_i) \geq 1$ and the sets $\Xi_i := \{s_k \mid s_k \text{ occurs in a reduced expression of } u_i\}$ satisfy $\Xi_i \cap \Xi_j = \emptyset$ and $ss' = s's$ for any $s \in \Xi_i$ and $s' \in \Xi_j$, whenever $i \neq j$. We have the following proposition.

**Proposition 4.8.** Let $u \in S_n$. If $u = u_1u_2$ satisfies the property $(\ast)$ at level 2, then $\sigma^{u_1 \cup u_2} = \sigma^u$.

**Proof.** For $w \in S_n$, $N_{u_1,u_2}^{w_1,w_2} \neq 0$ only if $u_1 \leq w$ with respect to the Bruhat order for $i = 1, 2$, i.e., $u_i$ can be obtained from a subexpression of a reduced expression of $w$. Then it follows from the definition of the property $(\ast)$ that $w = u_1u_2$ and $\ell(w) = \ell(u_1) + \ell(u_2)$. By Proposition 4.7, we conclude $N_{u_1,u_2}^{w_1,w_2} = N_{u_1,u_2}^{w_1,w_2} = 1$. Therefore, the statement follows.

We remark that many Schubert classes cannot be represented as a positive multiple of Schubert classes in $H^{\ast,0}(F\ell_n)$. For example, $\sigma^{u_1 \cup u_2 \cup u_3}$ does not satisfy the property $(\ast)$ at level 2 nor at level 3.

Now for any $[(\hat{u}, \hat{v}, \hat{w})] \in \mathcal{U} \setminus \{A_0 \cup \bigcup_{\alpha \in C} \alpha\}$, we check every $(u,v,w) \in [(\hat{u}, \hat{v}, \hat{w})]$. If $u$ satisfies the property $(\ast)$ at some level larger than 1, then we take the maximal level $m$ at which $u = u_1 \cdots u_m$ satisfies the property $(\ast)$. So does for $v = v_1 \cdots v_m$. Then we add $(u_1, \ldots, u_m, v_1, \ldots, v_m, w)$ to the set $[(\hat{u}, \hat{v}, \hat{w})]$. We denote by $\mathcal{U}$ the union of $\{A_0 \cup \bigcup_{\alpha \in C} \alpha\}$ and all the modified classes $[(\hat{u}, \hat{v}, \hat{w})]$, and call it a modified partition of $A$. In a summary, the elements $\{(u,v,w)\}$ in $\mathcal{U}$ satisfy either of the following conditions.
(a) \([u,v,w] = A_0 \cup \bigcup_{s \in C} \alpha; \) every element in this class is a triple \((u',v',w')\) and \(N^w_{u',v'} = 0.\)

(b) Elements in \([u,v,w]\) are \((m + m' + 1)\)-tuples \((u_1, \ldots, u_m, v_1, \ldots, v_{m'}, w')\), for most of which \(m = m' = 1; N^w_{u_1, \ldots, u_m, v_1, \ldots, v_{m'}} = N^w_{u', v'}\).

Now we rewrite both \((u',v',w')\) and \((u_1, \ldots, u_m, v_1, \ldots, v_{m'})\) uniformly in the form \((u'_1, \ldots, u'_{m+1})\), and restate Conjecture 1.3 as follows.

**Conjecture 4.9.** For any \([u,v,w] \in \tilde{U}\), there exists \((u'_1, \ldots, u'_{m+1}, w') \in [u,v,w]\) such that \(\tilde{u}_1 \cdot X^{w_1} \cap \cdots \cap \tilde{u}_{m+1} \cdot X^{w_{m+1}} \cap X^{w'} = 1\) is a transversal intersection for some \(\tilde{u}_1, \ldots, \tilde{u}_{m+1} \in S_n\).

We notice that the above conjecture holds for the case (a). In this case, \([u,v,w]\) contains an element \((u',v',w') \in A_0\), for which \(u' \not\subset w'\) or \(v' \not\subset w'\) holds; consequently, either of the intersections \(X^{u'} \cap X^{w'}\) and \(X^{v'} \cap X^{w'}\) is an emptyset already.

### 4.3 Special Schubert structure constants for \(Gr(k,n)\)

The partitions \(1^r := (1, \ldots, 1, 0, \ldots, 0)\) and \(r := (r, 0, \ldots, 0)\) for \(Gr(m,n)\) are special. Geometrically, the Schubert class \(\sigma^r = P.D. [X^r]\) (resp. \(\sigma^r = P.D. [X^r]\)) is the r-th Chern class of the tautological quotient bundle (resp. dual subbundle) over the \(Gr(m,n)\). Recall that a partition \(\mu\) corresponds to a positive path \(\pi_{\mu}\) and hence a Grassmannian permutation \(w_{\mu} \in W_P\).

**Proposition 4.10.** For any partitions \(\mu, \eta \in P_{m,n}\) satisfying \(\mu \leq \eta\) and \(|\eta| = |\mu| + 1\), the intersection \(w_{\mu} X^\eta \cap X^{\mu} \cap X^{\eta}_{\eta} = 0\). 

**Proof.** The argument can be read off immediately from Figure 5. Indeed, the face \(\Delta(\text{id}, w_{\mu})\) (resp. \(\Delta(w_0, \pi(w_0 w_{\mu}))\)) of \(\Delta\) corresponds to the toric subvariety to which \(X^{w_{\mu}}\) (resp. \(w_0 X^{\pi(w_0 w_{\mu})} = X_{\eta}\)) degenerates. Thus it is given by the face \(F_\mu\) (resp. \(F_\eta\)) by Proposition 3.8. The intersection \(F_\eta \cap F_\mu\) is a one-dimensional face with a parameter in the un-valued box bounded by the positive paths \(\pi_{\mu}\) and \(\pi_{\eta}\) (see the third diagram in Figure 5 for an illustration). All the boxes above \(\pi_{\eta}\) (resp. below \(\pi_{\mu}\)) take the value \(a = \lambda_{0}^{(n)}\) (resp. \(b = \lambda_{n}^{(n)}\)). Notice that \(w_{\mu} X^{s_{m}} = w_{\mu} X^{1}\) and \(\Delta(w_{\mu}, s_{m}) = \bigcup_{\epsilon \subset \pi_{\mu}} F_{\epsilon}\). Thus any effective edge \(e\) on \(\pi_{\mu}\) is the common edge of a \(b\)-valued box with either an \(a\)-valued box or the un-valued box. In the former case, the intersection \(F_\eta \cap F_\mu \cap F_\epsilon\) is empty; in the latter case, the intersection results in a same vertex of the latter diagram by evaluating \(\epsilon\) on the un-valued box. Hence, the set \(\Delta(w_{\mu}, s_{m}) \cap \Delta(\text{id}, w_{\mu}) \cap \Delta(w_0, \pi(w_0 w_{\mu}))\) consists of a point in \(V\), and therefore the statement follows by Theorem 4.5.

**Corollary 4.11 (Pieri-Chevalley formula for \(Gr(m,n)\)).** For any partition \(\mu \in P_{m,n},\n
\sigma^1 \cup \sigma^\mu = \sum_{|\eta| = |\mu| + 1} \sigma^\eta.

**Proof.** Notice that \(\sigma^\eta\) occurs in \(\sigma^1 \cup \sigma^\mu\) only if \(\eta \geq \mu\) and \(|\eta| = |\mu| + 1\), for which the Richardson variety \(X^\mu \cap X^{\eta}\) is nonempty. We are done by Proposition 4.10.

**Proposition 4.12.** Let \(r, q \in P_{m,n}\) be special partitions with \(r + q \leq n - m\). There exists \(u \in S_n\) such that the intersection \(u X^r \cap X^q \cap X_{r+q}\) is transversal and consists of a point. In particular, \(N^r_{n+1} = 1.\)

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**Figure 5** (Color online) Triple intersections for \(Gr(5,8)\)
Proposition 4.14. Conjecture 1.3 holds for $C$.

Recall that $\pi_i$ denotes the cyclic permutation $(2\ldots i)$.

Proof. Take $u \in S_n$ such that

$$u(j) = \begin{cases} j, & \text{if } 1 \leq j \leq m-1, \\ j+q, & \text{if } m \leq j \leq m+r-1. \end{cases}$$

Denote by $w_r$ the Grassmannian permutation of the special partition $r = (r,0,\ldots,0)$ in $P_{m,n}$. Notice that $\Delta(id,w_0) \cap \Delta(w_0,\pi(w_0w_{r+q})) = F_q \cap F_{r+q}$ is the face of $\Delta$ that takes the value $a$ on every box above the positive path $\pi_1,\ldots,\pi_{m-1,m+r+q}$ and takes the value $b$ on every box below the positive path $\pi_1,\ldots,\pi_{m-1,m+q}$. Clearly, the intersection of $F_q \cap F_{r+q}$ with

$$\bigcup_{c \subseteq \pi_1,\ldots,\pi_{m-1,m+q}} F_c = \{\lambda^{(m+q)} = \lambda^{(m+q+1)}\} \cup \bigcup_{i=1}^q \{\lambda^{(m-1+i)} = \lambda^{(m+i)}\}$$

gives the face $F_{q+1} \cap F_{r+q}$, the union in which consists of facets given by $\pi_1,\ldots,\pi_{m-1,j+q}$. By induction, the intersection of $F_q \cap F_{r+q}$ with the intersection of the union of facets given by $\{\pi_1,\ldots,\pi_{m-1,j+q} \mid m \leq j \leq m+r-1\}$ gives the vertex $F_{r+q} \cap F_{r+q}$. It is easy to see that this point belongs to the union of facets given by any one of the remaining positive paths $\pi_{m(l)}$ satisfying $\pi_j \neq \pi_1,\ldots,\pi_{m-1,m+q}$. Therefore, $\Delta(u,w_r) \cap \Delta(id,w_0) \cap \Delta(w_0,\pi(w_0w_{r+q})) = F_{r+q} \cap F_{r+q}$ consists of a vertex of $\Delta$. Hence, the statement follows from Lemma 4.1 and Theorem 4.5.

Remark 4.13. The case $m = 1$ tells that Conjecture 1.3 holds for $Gr(1,n) = \mathbb{P}^{n-1}$.

4.4 Transversal intersections in $Gr(2,n)$

Recall that $C$ denotes the cyclic permutation $(2,3,\ldots,n,1)$. We will see in Theorem 4.16 that Conjecture 1.3 holds for $X = Gr(2,n)$.

Proposition 4.14. Let $\mu,\eta \in P_{2,n}$. If $\eta = \mu + (1,1)$, then $X^{(1,1)} \cap C \cdot X^\mu \cap X_\eta$ is a transversal intersection and consists of a point. If $\eta - \mu$ equals $(2,0)$ or $(0,2)$, then $C^\mu \cdot X^{(1,1)} \cap X^\mu \cap X_\eta = \emptyset$ for some $k \in \mathbb{Z}$.

Proof. If $(\eta_1,\eta_2) = (\mu_1,\mu_2) + (1,1)$, then we conclude that $X^{(1,1)} \cap C \cdot X^\mu \cap X_\eta = \{p_1 = 0 \mid \pi_1 \neq \pi_\eta\}$, which is a reduced coordinate point of $\mathbb{P}^{(2)}$ so that the first statement follows. Indeed, we notice that $X_\eta = X \cap \{p_1 = 0 \mid \pi_1 \neq \pi_\eta\}$, $C^X = X \cap \{p_{C(l)} = 0 \mid \pi_{1,l} \neq \pi_\mu\}$ and $X^{(1,1)} = X \cap \{p_1 = 0 \mid 2 \leq j \leq n\}$. Thus $p_1 = 0$ with $I = (i_1,i_2)$ does not occur in the triple intersection if and only if $\pi_1 \subseteq \pi_\eta$, $i_1 > 1$ and $\pi_{(i_1-1,i_2-1)} \supseteq \pi_\mu$ all hold, i.e., $(i_1,i_2) \subseteq (\eta_1,\eta_2)$, $i_1 > 1$ and $(i_1-1,i_2-1) \supseteq (\mu_1,\mu_2) = (\eta_1,\eta_2) - (1,1)$. Hence, $\eta = \pi_1$ and we are done.

Now we assume $(\eta_1,\eta_2) = (\mu_1,\mu_2) + (2,0)$ and let $v = C^{\mu_2}$. Then $\mu_1 + 2 \leq n - 2$. Notice that

$$eX^{(1,1)} = X \cap \{p_{1+\mu_2,j} = 0 \mid 1 \leq j \leq n, j \neq 1 or \mu_2\},$$

and that $F_\mu \cap F_\eta$ takes the value $a$ (resp. $b$) on each box above (resp. below) the positive path $\pi_\eta$ (resp. $\pi_\mu$), and the only un-valued boxes are labeled by $\lambda_2^{(\mu_1+3)}$ and $\lambda_2^{(\mu_1+4)}$. Thus for any effective edge $e$ on the positive path $\pi_\mu = \pi_{(1+\mu_2,2+\mu_1)}$, the intersection of the facet $F_e$ with $F_\mu \cap F_\eta$ is empty unless $e$ is the common edge of the boxes labeled by $\lambda_2^{(\mu_2+2)}$ and $\lambda_2^{(\mu_1+3)}$, which forces the $\lambda_2^{(\mu_2+3)}$-box to take the value $b$. The further intersection with the facets determined by $\pi_{(1+\mu_2,3+\mu_1)}$ will make the $\lambda_2^{(\mu_2+4)}$-box to take the value $b$. This will result in an empty set after one more intersection with the facets determined by $\pi_{(1+\mu_2,\mu_2+4)}$. Therefore, we have

$$\Delta(v,w_{(1,1)}) \cap \Delta(id,w_\mu) \cap \Delta(w_0,\pi(w_0w_\eta)) = \emptyset \subset \Delta.$$
Proposition 4.7 repeatedly, we conclude that \( w = \cdot \) the statement follows from Proposition 4.12.

The elements \( s_{a+1} \cdots s_3, s_{b+1} \cdots s_3, s_d \cdots s_{c+1} \cdots s_3 \) are Grassmannian permutations associated with the partitions \((a,0),(b,0)\) and \((c,d)\), respectively. By Proposition 4.7, we have

\[
N^w_{u,v} = N^{w_1}_{u_1,v_1} = N^{w_2}_{u_2,v_2} = N^{w_3}_{u_3,v_2} = N^{w_4}_{u_2,v_2} = N^{w_5}_{u_3,v_3}.
\]

Theorem 4.16. For any \( \lambda, \mu, \eta \in \mathcal{P}_{2,n} \), there exist \( \lambda', \mu', \eta' \in \mathcal{P}_{m,n} \) for some \( m \) such that \( N^\eta_{\lambda,\mu} = N^{\eta'}_{\lambda',\mu'} \) and \( uX^{\lambda'} \cap X^{\mu'} \cap X^{\eta'} \) is a transversal intersection in \( Gr(m,n) \) for some \( u \in S_n \).

Proof. By Corollary 4.15, we have \( N^\eta_{\lambda,\mu} = N^{(c,d)}_{(a,0),(b,0)} \), where \( a = \lambda_1 - \lambda_2, b = \mu_1 - \mu_2, c = \eta_1 - \lambda_2 - \mu_2 \) and \( d = \eta_2 - \lambda_2 - \mu_2 \). Clearly, \( N^{(c,d)}_{(a,0),(b,0)} \) is nonzero only if \( n - 2 \geq c \geq d \geq 0, c + d = a + b \) and \( c \geq \max\{a,b\} \), which implies \( d \leq \min\{a,b\} \). Notice that \( u = s_{a+1}s_3 \cdots s_2, v = s_{b+1}s_3 \cdots s_2 \) and \( w = s_d \cdots s_{c+1}s_c \cdots s_2 \) are the Grassmannian permutations associated with the partitions \((a,0),(b,0)\) and \((c,d)\), respectively. By Proposition 4.7, we have

\[
N^w_{u,v} = N^{w_1}_{u_1,v_1} = N^{w_2}_{u_2,v_2} = N^{w_3}_{u_3,v_2} = N^{w_4}_{u_2,v_2} = N^{w_5}_{u_3,v_3}.
\]

The elements \( s_{a+1} \cdots s_3, s_{b+1} \cdots s_3, s_d \cdots s_{c+1} \cdots s_3 \) are Grassmannian permutations associated with the partitions \((a-1,0,0),(b-1,0,0)\) and \((c-1,d-1,0)\) for \( Gr(3,n) \). By induction and using Proposition 4.7 repeatedly, we conclude that \( N^\eta_{\lambda,\mu} = N^w_{u,v} \) is equal to the Schubert structure constant \( N^{\eta'}_{\lambda',\mu'} \) for \( Gr(n+2,n) \) with \( \lambda' = (a-d,0,\ldots,0), \mu' = (b-d,0,\ldots,0)\) and \( \eta' = (c-d,0,\ldots,0) \). Hence, the statement follows from Proposition 4.12.

4.5 Discussions for complete flag varieties

A face of the Gelfand-Cetlin polytope \( \Delta \) for the complete flag variety \( F_{\ell,n} \) is called a (dual) Kogan face, if all the equalities are of the form \( \lambda_j^{(i)} = \lambda_j^{(i-1)} \) (resp. \( \lambda_j^{(i)} = \lambda_j^{(i+1)} \)). In other words, a (dual) Kogan face \( F \) (resp. \( F^* \)) is indexed by a set of horizontal (resp. vertical) effective edges of the ladder diagram \( \Lambda \) (which were defined in terms of the dual graph of \( \Lambda \) in [20]). For each edge on the bottom (resp. right) of a box on the \( i \)-th row (resp. column) where \( 1 \leq i \leq n-1 \), we assign the simple reflection \( s_i \). Then every Kogan face \( F \) defines a permutation \( w(F) \) by going from the bottom to the top and then by going from the left to the right among those edges of \( \Delta \) that define \( F \). Similarly, every dual Kogan face \( F^* \) defines a permutation \( w(F^*) \) by going from the left to the right and then by going from the bottom to the top among those edges defining \( F^* \). A (dual) Kogan face is called reduced, if the resulting expression is a reduced decomposition of the associated permutation. For example, the first and third faces in Figure 6 are reduced, while the second Kogan face is not. Recall that we denote by \( X^\Phi \) the toric subvariety of \( X_{\Delta} \) defined by the face \( F \) of \( \Delta \), and notice that the number of edges defining a (dual) Kogan face equals the codimension of the face in the Gelfand-Cetlin polytope. Thanks to this identification, we have the following proposition.

Proposition 4.17 (See [26, Theorem 8 and Remark 10]). The flat family \( \mathcal{X} \) contains two kinds of flat subfamilies, giving the flat degeneration of (opposite) Schubert varieties \( X_u \) and \( X^v \), respectively, to the reduced union of toric subvarieties of reduced (dual) Kogan faces \( \sum_{w(F)=w_0} X_F \) and \( \sum_{w(F^*)=v} X_{F^*} \).

![Figure 6](image_url) (Dual) Kogan faces and the associated permutations.
Example 4.18. Let us consider the Schubert structure constant $N^0$ of $Gr(3, 6)$ where $\lambda = \mu = (2, 1, 0)$ and $\eta = (3, 2, 1)$. As $X_\eta = w_0 X^\mu$, this is the most complicated case in the sense that it counts the cardinality of the triple self-intersection $g_1 X^\mu \cap g_2 X^\mu \cap w_0 X^\mu$ up to a generic translation. For any $u, v \in S_0$, it follows from direct calculations that $uX^\mu \cap vX^\mu \cap X_\eta$ is of positive dimension, and hence it cannot be a transversal intersection. Nevertheless, we have the following equalities. The first equality is simply the identification of permutations with Grassmannian permutations and the natural treatment of Structure constants for $Gr(3, 6)$ as that for $F\ell_6$. The seventh equality applies the identity (2) in Subsection 4.2, where we notice $w_0 s_1 w_0 = s_{0-1}$ in this example. The eighth equality uses the fact $\sigma^{s_2} \cup \sigma^{s_4} = \sigma^{s_2 s_4}$. The remaining equalities follow from Proposition 4.7. It holds that

$$
N^0_{\lambda, \mu} = N_{s_2 s_4 s_3 s_2 s_1} = N_{s_2 s_4 s_3 s_2 s_1} = N_{s_2 s_4 s_3 s_2 s_1} = N_{s_2 s_4 s_3 s_2 s_1} = N_{s_2 s_4 s_3 s_2 s_1} = N_{s_2 s_4 s_3 s_2 s_1} = N_{s_2 s_4 s_3 s_2 s_1}.
$$

Define

$$
v = s_4 s_2 s_3 s_2 s_4 \quad \text{and} \quad w = s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1.
$$

There is a unique subexpression $s_4 s_2 s_3 s_2 s_4 s_3 s_4 s_3$ of $w_0 = s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1 s_2 s_1 s_2$, of length 7 whose product equals $v$, given by the $(2, 3, 4, 5, 8, 9, 12)$-th positions. This gives the unique reduced dual Kogan face $F^*$ together with the toric subvariety $X_{F^*}$, to which $X^v$ degenerates. There is a unique subexpression of $w_0 = s_4 s_2 s_3 s_2 s_4 s_3 s_4 s_3 s_2 s_1 s_2 s_1$, of length $(15 - 9)$ whose product equals $w_0 w$, given by the $(2, 3, 4, 5, 8, 9)$-th positions. This gives the unique reduced dual Kogan face $F$ together with the toric subvariety $X_F$, to which $X_w$ degenerates. The faces $F^*$ and $F$ are precisely given by $\Delta(id, v)$ and $\Delta(w_0, w_0 w)$, respectively, in Figure 7. Let $\tilde{u}$ (resp. $\tilde{v}$) denote the Grassmannian permutation associated with the positive path $\pi_{(2, 3)}$ (resp. $\pi_{(1, 4, 5, 6)}$). The intersection of $F^* \cap F$ with $\Delta(\tilde{u}, s_2)$ (i.e., with the union of faces corresponding to the $W$-translated Schubert divisor $\tilde{u} X^{s_2}$) is the union of two one-dimensional faces of $\Delta$. The intersection $F^* \cap F \cap \Delta(\tilde{u}, s_2) \cap \Delta(\tilde{v}, s_1)$ consists of two regular vertices of $\Delta$ for $F\ell_6$. Therefore, it follows from Theorem 4.5 that $\tilde{a} X^{s_2} \cap \tilde{a} X^{s_4} \cap X^v \cap X_w$ is a transversal intersection and consists of two regular vertices of $\Delta$, and consequently $N_{w, s_2, v}^w = 2$.

Example 4.19. Let $X = F\ell_4$. It suffices to consider the Schubert structure constants $N_{w, u, v}^w$, where $u, v, w \in S_4$ satisfying $\ell(w) = \ell(u) + \ell(v)$ and $u \leq w$, $v \leq w$ with respect to the Bruhat order.

![Figure 7](Color online) Intersections corresponding to $\tilde{a} X^{s_2} \cap \tilde{a} X^{s_4} \cap X^v \cap X_w$ in $F\ell_6$
The case for $u = \text{id}$ or $v = \text{id}$ is trivial. Noting that $X^u = u_0 X_{u_0 u}$ is of dimension $(6 - \ell(u))$, we have $N^w_{u, v} = N^w_{u_0 u, u_0 v}$. Thus we can also assume $\ell(u) \leq 4$ and use the identity (2) (i.e., $N^w_{u, v} = N^w_{u_0 u, u_0 v}$) in Subsection 4.2, where $u_0 u v_0$ is obtained from a reduced expression of $u$ by interchanging $s_1$ and $s_3$.

Then we do a further reduction by using the Proposition 4.7. It turns out that all the remaining $N^w_{u, v}$'s are either equal to 0 (for example, we have $N^w_{s_1 s_1 s_2 s_1} = N^w_{s_1 s_1 s_2 s_1} = 0$) or reduced to one of the following cases:

(1) It holds that

\[
N^w_{s_1 s_1 s_2 s_1} = N^w_{s_1 s_1 s_2 s_1} = N^w_{s_1 s_1 s_2 s_1} = N^w_{s_1 s_1 s_2 s_1} = N^w_{s_1 s_1 s_2 s_1} = N^w_{s_1 s_1 s_2 s_1},
\]

(2) It holds that

\[
N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2},
\]

(3) It holds that

\[
N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2} = N^w_{s_2 s_2 s_2 s_2}.
\]

In the case (1), it is reduced to the trivial case $N^w_{u, v} = 1$. In the case (2), it is reduced to the Schubert structure constants $N^w_{s_1 s_1 s_2 s_1}$ and $N^w_{s_1 s_1 s_2 s_1}$ for $Gr(1, 4)$, and hence is done by Remark 4.13. In the case (3), it is reduced to the Schubert structure constants $N^w_{s_2 s_2 s_2 s_2}$ and $N^w_{s_2 s_2 s_2 s_2}$ for $Gr(2, 4)$, and hence is done by Propositions 4.10 and 4.14. The above identities result in a consistent relation on the set $A$. Without going further to the modified partition $U$, we have already verified the transversality of the corresponding triple intersections. Therefore, Conjecture 1.3 holds for $F\ell_4$.

5 An anti-canonical divisor $-K_X$

As an application of $W$-translated Schubert divisors, we specify an anti-canonical divisor $-K_X$ of $X = F_{\ell_1, \ldots, \ell_n} = SL(n, \mathbb{C})/P$, which may have potential applications in the study of Strominger-Yau-Zaslow mirror symmetry for $X$. For example, for $X = F\ell_1, n - 1, n$, a special Lagrangian fibration for the open Calabi-Yau manifold $X \setminus -K_X$ was constructed in [7] with respect to such $-K_X$.

The anti-canonical divisor class is given by (see [12, Lemma 3.5] for the formula of $c_1(T_{G/P})$ for general Lie types) $[\pi X] = c_1(T_X) = \sum_{i=1}^{k} (\pi_{i+1} - \pi_{i-1}) \sigma^{\pi_i}$. By Proposition 3.2, all the positive paths towards $O_i$ represent the same Schubert divisor class $\sigma^{\pi_i}$. Now we introduce the following notation, in order to specify a representative of $[\pi X]$ with a property compatible with the toric degeneration of $X$.

Definition 5.1. Let $e$ be an edge on the roof of $\Lambda$. We define the special path associated with $e$ to be the (unique) positive path $p$ with the fewest corners among those positive paths having a corner containing $e$.

Example 5.2. In Figure 3, $\pi''$ is the special path associated with the edge $e'$, while $\pi'$ is not.

Remark 5.3. The special paths defined above correspond to the partitions of the zero rectangle or extremal rectangles in [30], i.e., rectangles whose length equal to $m$ or width equal to $n - m$. See Figure 8 for the case of $Gr(4, 7)$.

We observe that there are exactly $\sum_{i=1}^{k} (n_{i+1} - n_{i-1})$ edges on the roof of $\Lambda$. Furthermore, for each $1 \leq i \leq k$, there are precisely $(n_{i+1} - n_{i-1})$ special paths towards $O_i$. It follows immediately as follows.
Proposition 5.4. An anti-canonical divisor of $X$ is given by

$$-K_X := \sum_{\pi_I \text{ is a special path}} D_{p_I}. \quad (5.1)$$

Moreover, $-K_X$ degenerates to the anti-canonical divisor $-K_{X_\Delta}$ of the Gelfand-Cetlin toric variety $X_\Delta$ along the flat subfamily $\{\prod_{\pi_I \text{ is a special path}} p_I = 0\} \cap X$.

Here, $-K_{X_\Delta}$ denotes the canonical representative of the anti-canonical divisor class of the toric variety $X_\Delta$, given by the sum $\sum_{e}$ is an effective edge of $\Lambda X_F$ of toric divisors. The second statement is a direct consequence of Theorem 3.5 together with the definition of $-K_X$.

Remark 5.5. The above $-K_X$ coincides with the anti-canonical divisor $-K'$ given by the sum of projected Richardson hypersurfaces by Knutson et al. [23] in the cases of complex flag varieties and complex Grassmannians, while they are different in general. Indeed, the number of irreducible divisors in $-K_X$ is $n + n_k - n_1 = \sum_{i=1}^k (n_{i+1} - n_{i-1})$, while that for $-K'$ is equal to $n + k - 1$.

Acknowledgements The first author was supported by the Samsung Science and Technology Foundation (Grant No. SSTF-BA1602-03). The third author was supported by the National Research Foundation of Korea (Grant No. NRF-2019R1F1A1058962). The fourth author was supported by National Natural Science Foundation of China (Grant Nos. 11771455, 11822113 and 11831017) and Guangdong Introducing Innovative and Enterpreneurial Teams (Grant No. 2017ZT07X355). The authors thank Cheol-Hyun Cho, Leonardo C. Mihalcea, Ezra Miller, Ziv Ran, Vijay Ravikumar and Chi Zhang for useful discussions and helpful comments. The authors also thank the anonymous referees for their very helpful comments.

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