BEST POSSIBLE LOWER BOUNDS ON THE COEFFICIENTS OF EHRHART POLYNOMIALS

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ABSTRACT. For an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$, we recall $i(\mathcal{P}, n) = |n\mathcal{P}\cap\mathbb{Z}^d|$ the Ehrhart polynomial of $\mathcal{P}$. Let for $r = 0, \ldots, d$, $g_r(\mathcal{P})$ be the $r$-th coefficients of $i(\mathcal{P}, n)$. Martin Henk and Makoto Tagami gave the lower bounds on the coefficients $g_r(\mathcal{P})$ in terms of the volume of $\mathcal{P}$. In general, these bounds are not best possible. However, it is known that in the cases $r \in \{1, 2, d-2\}$, these bounds are best possible for any volume. In this paper, in the case $r = 3$ and in the cases $d-r$ is even, we show that their bounds are best possible, and we give a new and best possible lower bound on $g_{d-3}(\mathcal{P})$.

INTRODUCTION

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope, that is, a convex polytope whose vertices have integer coordinates, of dimension $d$. Given integers $n = 1, 2, \ldots$, we write $i(\mathcal{P}, n)$ for the number of integer points belonging to $n\mathcal{P}$, where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$. In other words,

$$i(\mathcal{P}, n) = |n\mathcal{P}\cap\mathbb{Z}^d|, \quad n = 1, 2, \ldots.$$

Late 1950’s Ehrhart did succeed in proving that $i(\mathcal{P}, n)$ is a polynomial in $n$ of degree $d$. We call $i(\mathcal{P}, n)$ the Ehrhart polynomial of $\mathcal{P}$. We refer the reader to [1, Chapter 3] of [3, Part II] for the introduction to the theory of Ehrhart polynomials. For $r = 0, \ldots, d$, let $g_r(\mathcal{P})$ be the $r$-th coefficients of $i(\mathcal{P}, n)$. The following properties are known:

- $g_0 = 1$;
- $g_d(\mathcal{P}) = \text{vol}(\mathcal{P})$;
- $g_{d-1}(\mathcal{P}) = \frac{1}{2} \sum_{\text{facet of } \mathcal{P}} \frac{\text{vol}_{d-1}(\mathcal{F})}{\text{det}(\text{aff } \mathcal{F}\cap \mathbb{Z}^d)}$ (II Theorem 5.6)],

where $\text{vol}(\cdot)$ denotes the usual volume and $\text{vol}_{d-1}(\cdot)$ denotes $(d-1)$-dimensional volume, and $\text{det}(\text{aff } \mathcal{F}\cap \mathbb{Z}^d)$ denotes the determinant of the $(d-1)$-dimensional sublattice contained in the affine hull of $\mathcal{F}$. All other coefficients $g_r(\mathcal{P})$, $1 \leq r \leq d-2$, have no such known explicit geometric meaning, except for special classes of polytopes.

For an integer $i$ and a variable $z$ we consider the polynomial

$$c_i(z) = (z+i)(z+i-1) \cdots (z+i-(d-1)) = d! \binom{z+i}{d},$$

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and we denote its \(r\)-th coefficient by \(C^d_{r,i}, 0 \leq r \leq d\). For instance, it is \(C^d_{d,i} = 1\), and for \(0 \leq i \leq d - 1\) we have \(C^d_{0,i} = 0\). For \(d \geq 3\) we set

\[ M_{r,d} = \min\{C^d_{r,i} : 1 \leq i \leq d - 2\}. \]

In [2, Theorem 1.1] Martin Henk and Makoto Tagami proved the following lower bounds on the coefficients \(g_r(P)\) in terms of the volume:

\[ g_r(P) \geq \frac{1}{d!} \left( (-1)^{d-r} \text{stirl}(d+1,r+1) + (d! \text{vol}(P) - 1) M_{r,d} \right), \quad 1 \leq r \leq d - 1, \tag{0.1} \]

where \(\text{stirl}(d, i)\) denote the Striling numbers of the first kind which can be defined via the identity \(\prod_{i=0}^{d-1}(z - i) = \sum_{i=1}^{d} \text{stirl}(d, i) z^i\). In general, these bounds are not best possible. However, it is known that in the cases \(r \in \{1, 2, d - 2\}\), they are best possible for any volume.

In section 1, we show that in the case \(r = 3\) and in the cases \(d - r\) is even, these bounds are best possible. In section 2, we give a new lower bound on \(g_{d-3}(P)\). In particular, we show the bound is best possible and Henk and Tagami’s bound on \(g_{d-3}(P)\) is not best possible.

1. THE CASES \(r = 3\) OR \(d - r\) IS EVEN

Let \(P \subset \mathbb{R}^d\) be an integral convex polytope of dimension \(d\), and let \(\partial P\) denote the boundary of \(P\). The generating function of the integral point enumerator, i.e., the formal power series

\[ \text{Ehr}_P(t) = 1 + \sum_{n=1}^{\infty} i(P, n)t^n \]

is called the Ehrhart series of \(P\). It is well known that it can be expressed as a rational function of the form

\[ \text{Ehr}_P(t) = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}}. \]

The sequence of the coefficients of the polynomial in the numerator

\[ \delta(P) = (\delta_0, \delta_1, \ldots, \delta_d) \]

is called the \(\delta\)-vector of \(P\).

The \(\delta\)-vector has the following properties:

- \(\delta_0 = 1, \delta_1 = |P \cap \mathbb{Z}^d| - (d + 1)\) and \(\delta_d = |(P \setminus \partial P) \cap \mathbb{Z}^d|\). Hence, \(\delta_1 \geq \delta_d\);
- Each \(\delta_i\) is nonnegative (\([1]\)));
- If \(\delta_d \neq 0\), then one has \(\delta_j \leq \delta_i\) for every \(1 \leq i \leq d - 1\) (\([1]\)));
- The leading coefficient \(\sum_{i=0}^{d} \delta_i / d!\) of \(i(P, n)\) is equal to the usual volume of \(P\), i.e., \(d! \text{vol}(P) = \sum_{i=0}^{d} \delta_i\) (\([1]\) Corollary 3.20, 3.21): \[ \text{vol}(P) = \sum_{i=0}^{d} \delta_i \]

There are two well-known inequalities on \(\delta\)-vector. Let \(s = \max\{i : \delta_i \neq 0\}\). One is

\[ \delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-1}, \quad 0 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor, \tag{1.1} \]
which is proved by Stanley [7], and another one is
\[
\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_1 + \delta_2 + \cdots + \delta_{i+1}, \quad 0 \leq i \leq \left\lfloor \frac{d-1}{2} \right\rfloor,
\]
which appears in the work of Hibi [4, Remark 1.4].

We can express the coefficients \( g_r(P) \) of the Ehrhart polynomial \( i(P, n) \) by using the \( \delta \)-vector \( \delta(P) \) ([2, Proof of Theorem 1.1]). In fact,
\[
g_r(P) = \frac{1}{d!} \sum_{i=0}^{d} \delta_i C_{r,d-i}.
\]

We will repeatedly use the following lemmas in this paper.

**Lemma 1.1** ([5, Theorem 1.1]). Let \( m, d, k \in \mathbb{Z}_{>0} \) be arbitrary positive integers satisfying \( m \geq 1, d \geq 2 \) and \( 1 \leq k \leq \lfloor (d+1)/2 \rfloor \). Then there exists an integral convex polytope \( P \) of dimension \( d \) such that \( \delta_0 = 1, \delta_k = m \) and for each \( i \notin \{0, k\} \), \( \delta_i = 0 \), where we let \( \delta(P) = (\delta_0, \ldots, \delta_d) \) be the \( \delta \)-vector of \( P \).

**Lemma 1.2** ([2, Lemma 2.2]). \( C_{r,i}^d = (-1)^{d-r} C_{r,d-1-i}^d \) for \( 0 \leq i \leq d-1 \).

First, we show that in the case \( r = 3 \) the bound (0.1) is best possible for any volume. In fact,

**Theorem 1.3.** Let \( d \) be an integer with \( d \geq 6 \) and \( P \) an integral convex polytope of dimension \( d \). Then
\[
g_3(P) \geq \begin{cases} 
\frac{1}{d!} \left( (-1)^{d-1} \text{stir}(d+1, 4) + (d! \text{vol}(P) - 1) C_{3,d-3}^d \right), & 6 \leq d \leq 9, \\
\frac{1}{d!} \left( (-1)^{d-1} \text{stir}(d+1, 4) + (d! \text{vol}(P) - 1) C_{3,d-2}^d \right), & d \geq 10.
\end{cases}
\]
And the bound is best possible for any volume.

In order to prove Theorem 1.3 we use the following lemma.

**Lemma 1.4.**
\[ M_{3,d} = \begin{cases} 
C_{3,d-3}^d, & 6 \leq d \leq 9, \\
C_{3,d-2}^d, & d \geq 10.
\end{cases} \]

**Proof.** For \( 1 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 1 \), we set
\[
f_k(z) = (z+k)(z+k-1) \cdots (z-1)z(z+1) \cdots (z-k+1)(z-k), \quad g_k(z) = (z-k-1)(z-k-2) \cdots (z+k-d+2)(z+k-d+1).
\]
Then \( c_k(z) = f_k(z)g_k(z) \). Also, for \( 1 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 2 \), we set
\[ I_k = \{ i \in \mathbb{Z} : k+1 \leq i \leq d-k-1 \} \]
We assume that \( d \geq 20 \). If \( 1 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 2 \), then
\[
\begin{align*}
f_k(z) &= (-1)^k (k!)^2 z + (-1)^{k-1} (k!)^2 \sum_{1 \leq i \leq k} \frac{1}{i^2} z^3 + \text{upper terms}, \\
g_k(z) &= (-1)^{d-2k-1} \frac{(d-k-1)!}{k!} + (-1)^{d-2k} \frac{(d-k-1)!}{k!} \sum_{i \in I_k} \frac{1}{i} z \\
&\quad + (-1)^{d-2k-1} \frac{(d-k-1)!}{k!} \sum_{\{i,j\} \subset I_k} \frac{1}{ij} z^2 + \text{upper terms}.
\end{align*}
\]
Hence for \( 1 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 2 \),
\[
C_{3,k}^d = (-1)^{d-k-1} k!(d-k-1)! \left( \sum_{\{i,j\} \subset I_1} \frac{1}{ij} - \sum_{1 \leq i \leq k} \frac{1}{i^2} \right).
\]
Also, if \( k = \left\lfloor \frac{d+1}{2} \right\rfloor - 1 \), then
\[
C_{3,k}^d = \begin{cases} 
(-1)^k (k!)^2 (k+1) \sum_{1 \leq i \leq k} \frac{1}{i^2} & (d: \text{even}) \\
(-1)^{k-1} (k!)^2 \sum_{1 \leq i \leq k} \frac{1}{i^2} & (d: \text{odd})
\end{cases}
\]
We show that for \( 2 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 1 \), \( |C_{3,1}^d| > |C_{3,k}^d| \). Since \( \sum_{\{i,j\} \subset I_1} \frac{1}{ij} - 1 > 0 \), we have \( |C_{3,1}^d| = (d-2)! \left( \sum_{\{i,j\} \subset I_1} \frac{1}{ij} - 1 \right) \). First, we show that the cases \( 2 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 2 \).
Since \( (d-2)! > k!(d-k-1)! \), we have
\[
\begin{align*}
(d-2)! \left( \sum_{\{i,j\} \subset I_1} \frac{1}{ij} - 1 \right) &+ k!(d-k-1)! \left( \sum_{\{i,j\} \subset I_k} \frac{1}{ij} - \sum_{1 \leq i \leq k} \frac{1}{i^2} \right) \\
>(d-2)! \left( \sum_{\{i,j\} \subset I_1} \frac{1}{ij} - 1 - \sum_{1 \leq i \leq \infty} \frac{1}{i^2} \right) \\
= (d-2)! \left( \sum_{\{i,j\} \subset I_1} \frac{1}{ij} - 1 - \frac{\pi^2}{6} \right) \\
>0,
\end{align*}
\]
and
\[
(d - 2)! \left( \sum_{\{i,j\} \subseteq I_1} \frac{1}{ij} - 1 \right) - k!(d - k - 1)! \left( \sum_{\{i,j\} \subseteq I_k} \frac{1}{ij} - \sum_{1 \leq i \leq k} \frac{1}{i^2} \right)
\]
\[
> (d - 2)! \left( \sum_{\{i,j\} \subseteq I_1} \frac{1}{ij} - 1 - \sum_{\{i,j\} \subseteq I_k} \frac{1}{ij} \right)
\]
\[
> 0.
\]
Hence since \(|C_{3,1}^d| + C_{3,k}^d > 0\) and \(|C_{3,1}^d| - C_{3,k}^d > 0\), we have \(|C_{3,1}^d| > |C_{3,k}^d|\). Next, we show that the case \(k = \left\lfloor \frac{d+1}{2} \right\rfloor - 1\). Since \((d - 2)! > (k!)^2(k + 1)\), we have
\[
|C_{3,1}^d| - |C_{3,k}^d| \geq (d - 2)! \left( \sum_{\{i,j\} \subseteq I_1} \frac{1}{ij} - 1 \right) - (k!)^2(k + 1) \sum_{1 \leq i \leq k} \frac{1}{i^2}
\]
\[
> (d - 2)! \left( \sum_{\{i,j\} \subseteq I_1} \frac{1}{ij} - 1 - \sum_{1 \leq i \leq \infty} \frac{1}{i^2} \right)
\]
\[
> 0.
\]
Therefore, for \(2 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 1\), \(|C_{3,1}^d| > |C_{3,k}^d|\). Since \(C_{3,1}^d > 0\), by Lemma 1.2 we have \(M_{3,d} = C_{3,d-2}^d\).

Finally, for \(6 \leq d \leq 19\) and for \(1 \leq k \leq d - 2\), by computing \(C_{3,k}^d\), we have
\[
M_{3,d} = \begin{cases} 
C_{3,d-3}^d, & 6 \leq d \leq 9, \\
C_{3,d-2}^d, & 10 \leq d \leq 19,
\end{cases}
\]
as desired. □

Now, we prove Theorem 1.3.

**Proof of Theorem 1.3**

We set
\[
jd = \begin{cases} 
2, & 6 \leq d \leq 9, \\
1, & d \geq 10.
\end{cases}
\]
Then by Lemma 1.4 we have \(M_{3,d} = C_{3,d-1-jd}^d\). Since \(jd + 1 \leq \left\lfloor \frac{d+1}{2} \right\rfloor\), by Lemma 1.1 for any volume, there exists an integral convex polytope \(P\) such that \(\delta_0 = 1\) and \(\delta_{jd+1} = d!\text{vol}(P) - 1\) and for each \(i \notin \{0, jd + 1\}\), \(\delta_i = 0\), where we let \(\delta(P) = (\delta_0, \ldots, \delta_d)\) be the \(\delta\)-vector of \(P\). Hence since \(C_{3,d}^d = (-1)^{d-1}\text{stirl}(d + 1, 4)\), we have
\[
d!g_3(P) = \sum_{i=0}^{d} \delta_i C_{3,d-i}^d
\]
\[
= (-1)^{d-1}\text{stirl}(d + 1, 4) + (d!\text{vol}(P) - 1)C_{3,d-1-jd}^d
\]
\[
= (-1)^{d-1}\text{stirl}(d + 1, 4) + (d!\text{vol}(P) - 1)M_{3,d},
\]
as desired. □
Next, we show that in the cases that $d - r$ is even, the bounds (0.1) are best possible for any volume. In fact,

**Theorem 1.5.** Let $d$ and $r$ be integers with $d \geq 6$ and $3 \leq r \leq d - 3$, and let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension $d$. Suppose that $d - r$ is even. Then

$$g_r(\mathcal{P}) \geq \frac{1}{d!} \left( (-1)^{d-r} \text{stir}(d+1,r+1) + (d! \text{vol}(\mathcal{P}) - 1)M_{r,d} \right).$$

And the bound is best possible for any volume.

**Proof.** We let $i$ be an integer with $1 \leq i \leq d - 2$ such that $C_{r,i} = M_{r,d}$. Since $d - r$ is even, by Lemma 1.2 we have $C_{r,i} = C_{r,d-i-1}$. We set $j := \max \{i, d - i - 1\}$. Then $1 \leq d - j \leq \lfloor \frac{d-1}{2} \rfloor$. Hence by Lemma 1.1 for any volume, there exists an integral convex polytope $\mathcal{P}$ such that $\delta_0 = 1$ and $\delta_{d-j} = d! \text{vol}(\mathcal{P}) - 1$ and for each $i \notin \{0,d-j\}$, $\delta_i = 0$, where we let $\delta(\mathcal{P}) = (\delta_0, \ldots, \delta_d)$ be the $\delta$-vector of $\mathcal{P}$. Hence since $C_{r,d} = (-1)^{d-r} \text{stir}(d+1,r+1)$, we have

$$d!g_r(\mathcal{P}) = (-1)^{d-r} \text{stir}(d+1,r+1) + (d! \text{vol}(\mathcal{P}) - 1)C_{r,j}$$

$$= (-1)^{d-r} \text{stir}(d+1,r+1) + (d! \text{vol}(\mathcal{P}) - 1)M_{r,d},$$

as desired. $\square$

2. A NEW LOWER BOUND ON $g_{d-3}(\mathcal{P})$

We assume that $d \geq 7$ and $r = d - 3$. Then since $d - r$ is odd and $r \geq 4$, it is not known whether the bound (0.1) on $g_{d-3}(\mathcal{P})$ is best possible for any volume. In this section, we give a new lower bound on $g_{d-3}(\mathcal{P})$. In particular, we show the bound is best possible, i.e., the bound (0.1) on $g_{d-3}(\mathcal{P})$ is not best possible.

We set

$$N_{d-3,d} = \min \{C_{d-3,i} : \left\lceil (d - 1)/2 \right\rceil \leq i \leq d - 2 \}.$$

Then $N_{d-3,d} \geq M_{d-3,d}$. In the following theorem, we give a new lower bound on $g_{d-3}(\mathcal{P})$.

**Theorem 2.1.** Let $d$ be an integer with $d \geq 7$ and $\mathcal{P}$ an integral convex polytope of dimension $d$. Then

$$g_{d-3}(\mathcal{P}) \geq \frac{1}{d!} \left( -\text{stir}(d+1,d-2) + (d! \text{vol}(\mathcal{P}) - 1)N_{d-3,d} \right).$$

And the bound is best possible for any volume. In particular, $M_{d-3,d} < N_{d-3,d}$.

In order to prove Theorem 2.1, we use the following lemma.

**Lemma 2.2.** For $0 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 2$, if $C_{d-3,k} \leq 0$, then $C_{d-3,k+1} \geq C_{d-3,k}$; and if $C_{d-3,k} \geq 0$, then $C_{d-3,k+1} \geq 0$.

Before proving lemma 2.2 we show the folloing lemma.
Lemma 2.3. Let $x$ be an integer and $y$ a positive integer. Then

$$
\sum_{-y \leq a < b \leq y} (x + a)(x + b) = \left(\frac{2y + 1}{2}\right) x^2 - \frac{1}{4} \left(\frac{2y + 2}{3}\right),
$$
and

$$
\sum_{-y \leq a < b < c \leq y} (x + a)(x + b)(x + c) = \left(\frac{2y + 1}{3}\right) x^3 - \left(\frac{2y + 2}{4}\right) x.
$$

Proof. By using induction on $y$, it immediately follows. \qed

Now, we prove Lemma 2.2.

Proof of Lemma 2.2. We let $f_k(z)$ and $g_k(z)$ be the polynomials in the proof of Lemma 1.4 and for $0 \leq k \leq \left\lfloor \frac{d+1}{2}\right\rfloor - 2$, we set $I_k = \{ i \in \mathbb{Z} : k + 1 \leq i \leq d - k - 1 \}.$

If $1 \leq k \leq \left\lfloor \frac{d+1}{2}\right\rfloor - 3$, then

$$
f_k(z) = z^{2k+1} - \sum_{1 \leq i \leq k} i^2 z^{2k-1} + \text{lower terms},
g_k(z) = z^{d-2k-1} - \sum_{i \in I_k} i z^{d-2k-2} + \sum_{\{i,j\} \subset I_k} ij z^{d-2k-3} - \sum_{\{i,j,l\} \subset I_k} i j l z^{d-2k-4} + \text{lower terms}.
$$

Hence for $1 \leq k \leq \left\lfloor \frac{d+1}{2}\right\rfloor - 3$,

$$
C_{d-3,k}^d = - \sum_{\{i,j\} \subset I_k} ij l + \left( \sum_{1 \leq i \leq k} i^2 \right) \cdot \left( \sum_{i \in I_k} i \right).
$$

Also, if $k = \left\lfloor \frac{d+1}{2}\right\rfloor - 2$, we have

$$
C_{d-3,k}^d = \begin{cases} 
- \sum_{\{i,j,l\} \subset I_k} ij l + \left( \sum_{1 \leq i \leq k} i^2 \right) \cdot \left( \sum_{i \in I_k} i \right) & (d : \text{even}), \\
\left( \sum_{1 \leq i \leq k} i^2 \right) \cdot \left( \sum_{i \in I_k} i \right) & (d : \text{odd}),
\end{cases}
$$

and if $k = \left\lfloor \frac{d+1}{2}\right\rfloor - 1$, we have

$$
C_{d-3,k}^d = \begin{cases} 
(k + 1) \sum_{1 \leq i \leq k} i^2 & (d : \text{even}), \\
0 & (d : \text{odd}).
\end{cases}
$$

Since $C_{d-3,0}^d = - \sum_{\{i,j,l\} \subset I_0} ij l < 0$ and since $I_1 \subset I_0$, we have $C_{d-3,1}^d - C_{d-3,0}^d > 0$,

Hence we should show the cases $1 \leq k \leq \left\lfloor \frac{d+1}{2}\right\rfloor - 2$.

First, we show the cases $d$ is even. If $k = \left\lfloor \frac{d+1}{2}\right\rfloor - 1$, then $C_{d-3,k}^d > 0$. Hence we should show the cases $1 \leq k \leq \left\lfloor \frac{d+1}{2}\right\rfloor - 3$. We set $x = \frac{d}{2}$ and for $1 \leq k \leq \left\lfloor \frac{d+1}{2}\right\rfloor - 2$, \[7\]
we set $J_k = \{ i \in \mathbb{Z} \mid -x + k + 1 \leq i \leq x - k - 1 \}$. For $1 \leq k \leq \lfloor \frac{d+1}{2} \rfloor - 2$, by Lemma 2.2 we have

$$C_{d-3,k}^d = - \sum_{\{i,j,l\} \subseteq J_k} (x + i)(x + j)(x + l) + \left( \sum_{1 \leq i \leq k} i^2 \right) \cdot \left( \sum_{i \in I_k} i \right) \leq \left( \frac{d - 2k - 1}{3} \right)x^3 - \left( \frac{d - 2k}{4} \right)x + \frac{k(k + 1)(2k + 1) d(d - 2k - 1)}{6} \frac{d}{2}$$

$$= - \frac{(d - 2)(d - 1)d(d - 2k - 1)(4k^2 - 4dk + 4k + d^2 - 3d)}{48}.$$

If $-\frac{\sqrt{d+1} - d+1}{2} \leq k \leq \frac{\sqrt{d+1} + d-1}{2}$, then $C_{d-3,k} \geq 0$, and if $1 \leq k \leq -\frac{\sqrt{d+1} - d+1}{2}$, then $C_{d-3,k} \leq 0$. Hence if $C_{d-3,k} \geq 0$, we have $C_{d-3,k+1} \geq 0$. If $1 \leq k \leq -\frac{\sqrt{d+1} - d+1}{2}$ and $1 \leq k \leq \lfloor \frac{d+1}{2} \rfloor - 3$, then $C_{d-3,k} \leq 0$ and

$$C_{d-3,k+1}^d - C_{d-3,k}^d = \frac{(d - 2)(d - 1)d(12k^2 - 12dk + 24k + 3d^2 - 13d + 12)}{24}.$$

Hence since $1 \leq k \leq \frac{3d - 6 - \sqrt{3d}}{6} \leq -\frac{\sqrt{d+1} - d+1}{2}$, we have $C_{d-3,k+1}^d - C_{d-3,k}^d \geq 0$.

Next, we show the cases $d$ is odd. If $k = \frac{d+1}{2} - 1$ or $k = \frac{d+1}{2} - 2$, then $C_{d-3,k}^d \leq 0$.

Hence we should show the cases $1 \leq k \leq \frac{d+1}{2} - 4$. We set $y = \frac{d+1}{2}$ and for $1 \leq k \leq \frac{d+1}{2} - 3$, we set $L_k = \{ i \in \mathbb{Z} \mid -y + k + 2 \leq i \leq y - k - 2 \}$. For $1 \leq k \leq \frac{d+1}{2} - 3$, by Lemma 2.2 we have

$$C_{d-3,k}^d = - \sum_{\{i,j\} \subseteq L_k \cup \{ -y+k+1 \}} (y + i)(y + j)(y + l) + \left( \sum_{1 \leq i \leq k} i^2 \right) \cdot \left( \sum_{i \in I_k} i \right)$$

$$= - (k + 1) \sum_{\{i,j\} \subseteq L_k} (y + i)(y + j) - \sum_{\{i,j\} \subseteq L_k} (y + i)(y + j)(y + l)$$

$$+ \left( \sum_{1 \leq i \leq k} i^2 \right) \cdot \left( \sum_{i \in I_k} i \right)$$

$$= - \frac{(d - 2)(d - 1)d(d - 2k - 1)(4k^2 - 4dk + 4k + d^2 - 3d)}{48}.$$

Hence it follows by the same argument, as desired.

Finally, we prove Theorem 2.1.
Proof of Theorem 2.1. Let \( \delta(\mathcal{P}) = (\delta_0, \ldots, \delta_d) \) be the \( \delta \)-vector of \( \mathcal{P} \). We assume that \( d \) is even. Then by Lemma 1.2, we have

\[
d!g_{d-3}(\mathcal{P}) = C_{d-3,d}^d + \sum_{i=1}^{\frac{d}{2}} (\delta_i - \delta_{d-i+1}) C_{d-3,i}^d
\]

\[
= C_{d-3,d}^d + \sum_{i=1}^{\frac{d}{2}-1} \left( \sum_{j=1}^{i} (\delta_j - \delta_{d-j+1})(C_{d-3,i}^d - C_{d-3,i-1}^d) \right)
\]

\[+ \sum_{j=1}^{\frac{d}{2}} (\delta_j - \delta_{d-j+1}) C_{d-3,i}^d.
\]

By Lemma 1.2 and the proof of Lemma 2.2, we have \( N_{d-3,d} < 0 \) and there exists an integer \( t \) such that for \( \frac{d}{2} \leq i \leq t \), \( C_{d-3,i}^d \leq 0 \), and for \( t+1 \leq i \leq d-1 \), \( C_{d-3,i}^d - C_{d-3,i-1}^d \geq 0 \). Hence by the inequality (1.2), we have

\[
d!g_{d-3}(\mathcal{P}) \geq C_{d-3,d}^d + \sum_{i=d-t}^{\frac{d}{2}-1} \left( \sum_{j=1}^{i} (\delta_j - \delta_{d-j+1})(C_{d-3,i}^d - C_{d-3,i-1}^d) \right)
\]

\[+ \sum_{j=1}^{\frac{d}{2}} (\delta_j - \delta_{d-j+1}) C_{d-3,i}^d
\]

\[= C_{d-3,d}^d + \sum_{i=d-t}^{i=d-t} (\delta_j - \delta_{d-j+1}) C_{d-3,i}^d + \sum_{i=d-t+1}^{\frac{d}{2}} (\delta_i - \delta_{d-i+1}) C_{d-3,i}^d.
\]

If for \( d - t + 1 \leq i \leq \frac{d}{2} \), \( \delta_i - \delta_{d-i+1} \leq 0 \), then since \( C_{d-3,i}^d \leq 0 \), we have \( (\delta_i - \delta_{d-i+1}) C_{d-3,i}^d \geq 0 \). Also, if for \( d - t + 1 \leq i \leq \frac{d}{2} \), \( \delta_i - \delta_{d-i+1} \geq 0 \), then \( (\delta_i - \delta_{d-i+1}) N_{d-3,d} \geq \delta_i - \delta_{d-i+1} N_{d-3,d} \). Hence since \( C_{d-3,d}^d = \text{stir}l(d+1, d-2) \) and \( N_{d-3,d} < 0 \), by the inequality (1.2), we have

\[
d!g_{d-3}(\mathcal{P}) \geq -\text{stir}l(d+1, d-2) + (d!\text{vol}(\mathcal{P}) - 1) N_{d-3,d}.
\]

Next, we assume that \( d \) is odd. Then by Lemma 1.2, we have

\[
d!g_{d-3}(\mathcal{P}) = \sum_{i=0}^{d} \delta_i C_{d-3,d-i}^d
\]

\[
= C_{d-3,d}^d + \sum_{i=1}^{\frac{d+1}{2}-1} \left( \sum_{j=1}^{i} (\delta_j - \delta_{d-j+1})(C_{d-3,d-i}^d - C_{d-3,d-i-1}^d) \right)
\]

\[+ \sum_{i=1}^{\frac{d+1}{2}} (\delta_j - \delta_{d-j+1}) C_{d-3,i}^d + \sum_{i=\frac{d+1}{2}+1}^{d} \delta_i C_{d-3,i}^d.
\]

Hence it follows by the same argument.

By Lemma 1.1, it follows that this bound is best possible.
Finally, we show that \( M_{d-3,d} < N_{d-3,d} \). In particular, we show that \( C_{d-3,1}^d < N_{d-3,d} \). For \( 2 \leq k \leq \lfloor \frac{d+1}{2} \rfloor - 3 \),
\[
C_{d-3,1} + C_{d-3,k}^d = -\frac{(d-2)(d-1)d(d-k-2)(4k^2 - 2dk - 2k + 2d - 5d + 6)}{24} < 0.
\]
Also, if \( d \) is even and \( k = \frac{d+1}{2} - 2 \), then \( C_{d-3,1} + C_{d-3,k}^d < 0 \), and if \( d \) is even and \( k = \frac{d+1}{2} - 1 \), then
\[
C_{d-3,1} + C_{d-3,k}^d = -\frac{(d-1)^2d(d^3 - 11d^2 + 37d - 47)}{48} < 0,
\]
and if \( d \) is odd and \( k = \frac{d}{2} - 2 \), then
\[
C_{d-3,1} + C_{d-3,k}^d = -\frac{(d-3)(d-2)d(2d^3 - 17d^2 + 32d - 8)}{96} < 0.
\]
Hence by Lemma 1.2 and Lemma 2.2, we have \( C_{d-3,1}^d < N_{d-3,d} \), as desired.

For \( d \geq 3 \) we set
\[
N_{r,d} = \min \{ C_{r,i}^d : \left\lceil \frac{(d-1)}{2} \right\rceil \leq i \leq d-2 \}.
\]
We recall the following lemma.

**Lemma 2.4 (Proposition 2.1)**. Let \( d \geq 3 \). Then
(i) \( M_{1,d} = C_{1,d-2}^d \),
(ii) \( M_{2,d} = C_{2,d-2}^d \),
(iii) \( M_{d-2,d} = C_{d-2,\lfloor \frac{d+1}{2} \rfloor}^d \).

If \( r \in \{1, 2, 3, d-3, d-2\} \) or \( d-r \) is even, then by this lemma and by the results so far, we have \( M_{r,d} = N_{r,d} \). Hence we immediately have the following corollary.

**Corollary 2.5.** Let \( P \subset \mathbb{R}^d, d \geq 3 \). Assume that \( r \in \{1, 2, 3, d-3, d-2\} \) or \( d-r \) is even. Then we have
\[
g_r(P) \geq \frac{1}{d!} \left( (-1)^{d-r} \text{stirl}(d+1, r+1) + (d! \text{vol}(P) - 1) N_{r,d} \right).
\]
And the bound is best possible for any volume.

Then the following conjecture naturally occurs:

**Conjecture 2.6.** Let \( P \subset \mathbb{R}^d, d \geq 3 \). Then for \( r = 1, \ldots, d-2 \)
\[
g_r(P) \geq \frac{1}{d!} \left( (-1)^{d-r} \text{stirl}(d+1, r+1) + (d! \text{vol}(P) - 1) N_{r,d} \right).
\]
And the bounds are best possible for any volume.

We set the sequence \( C_{r,d} = (C_{r,\lfloor \frac{d+1}{2} \rfloor}^d, \ldots, C_{r,d-1}^d) \). In order to solve this conjecture, we should study the properties of the sequence \( C_{r,d} \). The following proposition may be important to solve this conjecture.
Proposition 2.7. Let $\mathcal{P} \subset \mathbb{R}^d$, $d \geq 3$. Assume that $d-r$ is odd and the sequence $C_{r,d}$ satisfies the following condition: There exists an integer $t$ with $\lceil \frac{d-1}{2} \rceil + 1 \leq t \leq d-1$ such that for $t \leq i \leq d-1$, $C_{r,i}^d \geq C_{r,i-1}^d$ and $N_{r,d} = \min \{ C_{r,i}^d : \lceil \frac{d-1}{2} \rceil \leq i \leq t-1 \} = \min \{ -|C_{r,i}^d| : \lceil \frac{d-1}{2} \rceil \leq i \leq t-1 \}$. Then we have

$$g_r(\mathcal{P}) \geq \frac{1}{d!} \left( (-1)^{d-r} \text{stirl}(d+1, r+1) + (d! \text{vol}(\mathcal{P}) - 1)N_{r,d} \right).$$

And the bound is best possible for any volume.

Proof. We let $\delta(\mathcal{P}) = (\delta_0, \ldots, \delta_d)$ be the $\delta$-vector of $\mathcal{P}$. If for $\lceil \frac{d-1}{2} \rceil \leq i \leq t-1$, $\delta_{d-i} - \delta_{i+1} \leq 0$, then since $N_{r,d} \leq -C_{r,i}^d$, we have $(\delta_{d-i} - \delta_{i+1})C_{r,i}^d \geq (\delta_{d-i} - \delta_{i+1})N_{r,d}$. Also, if for $\lceil \frac{d-1}{2} \rceil \leq i \leq t-1$, $\delta_{d-i} - \delta_{i+1} \geq 0$, then since $N_{r,d} \leq C_{r,i}^d$, we have $(\delta_{d-i} - \delta_{i+1})C_{r,i}^d \geq (\delta_{d-i} - \delta_{i+1})N_{r,d}$. Hence since $N_{r,d} \leq 0$, by the same argument of the proof of Theorem 2.1, we have

$$g_r(\mathcal{P}) \geq \frac{1}{d!} \left( (-1)^{d-r} \text{stirl}(d+1, r+1) + (d! \text{vol}(\mathcal{P}) - 1)N_{r,d} \right).$$

In lower dimension, by computing $C_{r,d}$ we can know whether $C_{r,d}$ satisfies the condition of Proposition 2.7. In fact, for $d \leq 1000$, $C_{r,d}$ satisfies the condition. Hence by the results so far, this conjecture holds for $d \leq 1000$.

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