GRADED LIMITS OF MINIMAL AFFINIZATIONS OVER THE QUANTUM LOOP ALGEBRA OF TYPE $G_2$

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Abstract. The aim of this paper is to study the graded limits of minimal affinizations over the quantum loop algebra of type $G_2$. We show that the graded limits are isomorphic to multiple generalizations of Demazure modules, and obtain defining relations of them. As an application, we obtain a polyhedral multiplicity formula for the decomposition of minimal affinizations of type $G_2$ as a $U_q(\mathfrak{g})$-module, by showing the corresponding formula for the graded limits. As another application, we prove a character formula of the least affinizations of generic parabolic Verma modules of type $G_2$ conjectured by Mukhin and Young.

Key words: minimal affinizations; quantum loop algebras; current algebras

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1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra, $L_\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the corresponding loop algebra, and $U_q(L_\mathfrak{g})$ the corresponding quantum loop algebra. Minimal affinizations of representations of quantum groups are an important family of simple $U_q(L_\mathfrak{g})$-modules which was introduced in [Cha95]. The celebrated Kirillov-Reshetikhin modules are examples of minimal affinizations.

Graded limits of minimal affinizations, which are graded analogs of the classical limits defined over the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$, were studied in [Cha01], [CM06], [Mou10], [MP11], [Nao13], [Nao14]. Minimal affinizations over the quantum loop algebra of type $G_2$ were studied in [Cha95], [CM07], [MP07], [LM13], [QL14]. The aim of this paper is to study the graded limits of minimal affinizations over the quantum loop algebra of type $G_2$.

Assume that $\mathfrak{g}$ is of type $G_2$. Let $L(m)$ be the graded limit of a minimal affinization with highest weight $\lambda$, and let $M(\lambda)$ be the $\mathfrak{g}[t]$-module generated by a nonzero vector $v_\lambda$ with certain relations. Our first main result (Theorem 3.2) is that $M(\lambda) \cong L(m) \cong T(\lambda)$, where $T(\lambda)$ is some generalized Demazure module. These isomorphisms were previously conjectured by Moura in [Mou10].

Let $\omega_1$ (resp. $\omega_2$) be the fundamental weight with respect to the long (resp. short) simple root. Assume that $\lambda = k\omega_1 + l\omega_2$. Using the above isomorphisms, we obtain the following polyhedral multiplicity formula as a $\mathfrak{g}$-module (Theorem 3.3)

\[
L(m) \cong \bigoplus_{(a_1, \ldots, a_5) \in S_\lambda} V((k - a_1 + a_3 + a_4 - a_5)\omega_1 + (l - a_2 - 3a_3 - 3a_4)\omega_2)
\]

where

\[
S_\lambda = \{(a_1, \ldots, a_5) \in \mathbb{Z}_+^5 \mid a_1 \leq k, a_1 - a_3 + a_5 \leq k, 2a_2 + 3a_3 + 3a_4 \leq l, 2a_2 + 3a_4 + 3a_5 \leq l\}.
\]

Here $V(\mu)$ denotes the simple $\mathfrak{g}$-module with highest weight $\mu$. As an immediate corollary, we obtain a similar formula for the multiplicity of minimal affinizations as a $U_q(\mathfrak{g})$-module.
This formula is a generalization of the one given in [CM07], in which the formula for Kirillov-Reshetikhin modules (i.e. the case \( k = 0 \) or \( l = 0 \)) is given.

We also give a formula for the limit of normalized characters (Corollary 3.3), which yields the character formula of least affinizations of generic parabolic Verma modules of type \( G_2 \) conjectured by Mukhin and Young [MY14, Conjecture 6.3].

The paper is organized as follows. In Section 2 we give some background information about the quantum loop algebra of type \( G_2 \). In Section 3 we describe our main results in this paper. In Section 4 we prove Theorem 3.2. In Section 5 we prove Theorem 3.3.

2. Background

Let \( \mathbb{Z} \) be the set of integers, and \( \mathbb{Z}_+ \) the set of nonnegative integers. In this paper, we take \( \mathfrak{g} \) to be the complex simple Lie algebra of type \( G_2 \). Let \( \mathfrak{h} \) be a Borel subalgebra containing \( \mathfrak{h} \). Let \( I = \{1, 2\} \). We choose simple roots \( \alpha_1, \alpha_2 \) and scalar product \( (\cdot, \cdot) \) such that

\[
(\alpha_1, \alpha_1) = 6, \quad (\alpha_1, \alpha_2) = -3, \quad (\alpha_2, \alpha_2) = 2.
\]

Therefore \( \alpha_1 \) is the long simple root and \( \alpha_2 \) is the short simple root. The set of long positive roots is

\[
\{\alpha_1, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}.
\]

The set of short positive roots is

\[
\{\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}.
\]

Let \( \Delta_+ \) denote the set of positive roots. Denote by \( \Delta \) the root system of \( \mathfrak{g} \), and by \( \Delta_+ \) the set of positive roots. Let \( W \) denote the Weyl group with simple reflections \( s_i \) (\( i \in I \)). Denote by \( \mathfrak{g}_\alpha \) (\( \alpha \in \Delta \)) the corresponding root space, and for each \( \alpha \in \Delta_+ \) fix nonzero elements \( e_\alpha \in \mathfrak{g}_\alpha \), \( f_\alpha \in \mathfrak{g}_{-\alpha} \) and \( \alpha^\vee \in \mathfrak{h} \) such that

\[
[e_\alpha, f_\alpha] = \alpha^\vee, \quad [\alpha^\vee, e_\alpha] = 2e_\alpha, \quad [\alpha^\vee, f_\alpha] = -2f_\alpha.
\]

We also use the notation \( e_i = e_{\alpha_i} \), \( f_i = f_{\alpha_i} \), for \( i \in I \), and \( e_{-\alpha} = f_{\alpha} \) for \( \alpha \in \Delta_+ \). Set \( \mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm \alpha} \).

Let \( \omega_i \) (\( i \in I \)) be the fundamental weight. We have \( \omega_1 = 2\alpha_1 + 3\alpha_2 \), \( \omega_2 = \alpha_1 + 2\alpha_2 \). Let \( P \) be the weight lattice, and

\[
P_+ = \sum_{i \in I} \mathbb{Z}_+ \omega_i \subseteq P, \quad Q_+ = \sum_{i \in I} \mathbb{Z}_+ \alpha_i \subseteq P.
\]

Note that \( P \) coincides with the root lattice \( \sum_{i \in I} \mathbb{Z} \alpha_i \), but \( P_+ \neq Q_+ \). We write \( \lambda \leq \mu \) for \( \lambda, \mu \in P \) if \( \mu - \lambda \in Q_+ \). For \( \lambda \in P_+ \), denote by \( V(\lambda) \) the simple \( \mathfrak{g} \)-module with highest weight \( \lambda \).

Let \( \widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d \) be the affine Kac-Moody Lie algebra associated with \( \mathfrak{g} \), where \( K \) is the canonical central element and \( d \) is the degree operator. Let \( \hat{I} = \{0, 1, 2\} \), and

\[
e_0 = f_{2\alpha_1 + 3\alpha_2} \otimes t, \quad f_0 = e_{2\alpha_1 + 3\alpha_2} \otimes t^{-1}.
\]

In this paper, we put \( \hat{\cdot} \) to denote the objects associated with \( \widehat{\mathfrak{g}} \). For example, \( \hat{P} \) and \( \hat{Q} \) denote the weight and root lattices of \( \widehat{\mathfrak{g}} \) respectively, and so on. Let \( \delta \in \hat{P} \) be the null root, and denote by \( \Lambda_0 \in \hat{P}_+ \) the unique dominant integral weight of \( \widehat{\mathfrak{g}} \) satisfying

\[
\langle \alpha_i^\vee, \Lambda_0 \rangle = 0 \text{ for } i \in I, \quad \langle K, \Lambda_0 \rangle = 1, \quad \langle d, \Lambda_0 \rangle = 0.
\]
Let $L_q = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $g[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the loop algebra and the current algebra associated with $\mathfrak{g}$ respectively, whose Lie algebra structures are given by

$$[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t)g(t).$$

Note that $g[t]$ is naturally considered as a Lie subalgebra of $\hat{\mathfrak{g}}$.

Quantum groups are introduced independently by Jimbo [Jim85] and Drinfeld [Dri87]. Quantum loop algebra are infinite-dimensional quantum groups. The quantum loop algebra $U_q(L_q)$ in Drinfeld’s new realization is a $\mathbb{C}(q)$-algebra generated by $x_{i,n}^\pm (i \in I, n \in \mathbb{Z})$, $k_i^\pm (i \in I)$, $h_{i,n} (i \in I, n \in \mathbb{Z}\setminus \{0\})$, subject to certain relations, see [Dri87]. Denote by $U_q(g)$ the subalgebra of $U_q(L_q)$ generated by $x_{i,0}^\pm (i \in I), k_i^\pm (i \in I)$, which is isomorphic to the quantized enveloping algebra associated with $\mathfrak{g}$. For $\lambda \in P_+$, let $V_q(\lambda)$ denote the finite-dimensional simple $U_q(g)$-module of type 1 with highest weight $\lambda$.

Simple $U_q(L_q)$-modules are parametrized by dominant monomials in $Z[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}(q)^\times}$, where $Y_{i,a}^{\pm 1}$s are formal variables, and a monomial $m = \prod_{i \in I, a \in \mathbb{C}(q)^\times} Y_{i,a}^{u_{i,a}}$ is dominant if $u_{i,a} \geq 0$ for all $i$ and $a$ (see [CP95a], or [FR99] for the present formulation). For a dominant monomial, denote by $L_q(m)$ the corresponding simple $U_q(L_q)$-module. Let $P_+$ be the monoid generated by $\{Y_{i,a} | i \in I, a \in \mathbb{C}^\times q^\mathbb{Z}\}$.

Let $\lambda = k\omega_1 + l\omega_2$, $k, l \in \mathbb{Z}_+$. A simple $U_q(L_q)$-module $L_q(m)$ is a minimal affinization of $V_q(\lambda)$ if and only if $m$ is one of the following monomials

$$\left( \prod_{i=0}^{l-1} Y_{1, a q^i} \right) \left( \prod_{i=0}^{k-1} Y_{2, a q^{2i+1}} \right) \left( \prod_{i=0}^{l-1} Y_{2, a q^{2i}} \right) \left( \prod_{i=0}^{k-1} Y_{1, a q^{2i+6i+5}} \right),$$

for some $a \in \mathbb{C}(q)^\times$, see [CP95a].

3. Main results

The aim of this paper is to study the graded limits of minimal affinizations in type $G_2$. So let us recall the definition of the graded limits.

Let $\lambda = k\omega_1 + l\omega_2$, and $m$ be one of the monomials in (2.1). Without loss of generality, we may assume that $a \in \mathbb{C}^\times$. Let $A = \mathbb{C}[q, q^{-1}], U_A(L_q)$ be the $A$-lattice of $U_q(L_q)$ (see [CP94]), and $L_A(m) = U_A(L_q)v_m$ where $v_m$ is a highest $\ell$-weight vector of $L_q(m)$. Then

$$\overline{L_q(m)} = L_A(m) \otimes_A \mathbb{C}$$

becomes a finite-dimensional $L_q$-module called the classical limit of $L_q(m)$, where we identify $\mathbb{C}$ with $A/(q - 1)$. Define a Lie algebra automorphism $\varphi_a : g[t] \to g[t]$ by

$$\varphi_a(x \otimes f(t)) = x \otimes f(t-a) \quad \text{for} \ x \in \mathfrak{g}, f \in \mathbb{C}[t].$$

Now we consider $\overline{L_q(m)}$ as a $g[t]$-module by restriction, and define a $g[t]$-module $L(m)$ by the pull-back $\varphi_a^* (\overline{L_q(m)})$. We call $L(m)$ the graded limit of $L_q(m)$. By the construction we have for every $\mu \in P_+$ that

$$\left[ L_q(m) : V_q(\mu) \right] = \left[ L(m) : V(\mu) \right],$$

where the left- and right-hand sides are the multiplicities as a $U_q(g)$-module and $g$-module, respectively.

Now we shall state our first main theorem, which gives isomorphisms between $L(m)$ and other two $g[t]$-modules. Let $M(\lambda)$ be the $g[t]$-module generated by a nonzero vector $v_M$.
with relations
\[
\begin{align*}
    n_+(t^i) v_M &= 0, \\
    (h \otimes t^k) v_M &= \delta_{k0}(h, \lambda) v_M &\text{for} & h \in \mathfrak{h}, \\
    f_{\alpha_i} (\alpha_i^\vee, \lambda)^{+1} v_M &= 0 &\text{for} & i \in I, \\
    (f_{\alpha_1} \otimes t) v_M &= 0, \\
    (f_{\alpha_2} \otimes t) v_M &= 0, \\
    (f_{\alpha_1+\alpha_2} \otimes t) v_M &= 0.
\end{align*}
\] (3.2)

The other \( g[t] \)-module is a multiple generalization of a Demazure module defined as follows. Let \( \xi_1, \ldots, \xi_p \) be a sequence of elements of \( \hat{P} \), and assume for each \( 1 \leq i \leq p \) that there exists \( \Lambda^i \in \hat{P}_+ \) such that \( \xi_i \) belongs to the affine Weyl group orbit \( \hat{W}\Lambda^i \) of \( \Lambda^i \). Let \( \hat{V}(\Lambda^i) \) denote the simple highest weight \( g \)-module with highest weight \( \Lambda^i \), and \( v_{\xi_i} \in \hat{V}(\Lambda^i)_{\xi_i} \) be an extremal weight vector with weight \( \xi_i \). We define a \( \hat{b} \)-module \( D(\xi_1, \ldots, \xi_p) \) by
\[
D(\xi_1, \ldots, \xi_p) = U(\hat{b})(v_{\xi_1} \otimes \cdots \otimes v_{\xi_p}) \subseteq \hat{V}(\Lambda^1) \otimes \cdots \otimes \hat{V}(\Lambda^p).
\] (3.3)

Here \( \hat{b} = b \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus t\mathfrak{g}[t] \) is the standard Borel subalgebra of \( \hat{g} \).

**Remark 3.1.** For any \( c_1, \ldots, c_p \in \mathbb{Z} \), it obviously holds that
\[
D(\xi_1 + c_1\delta, \ldots, \xi_p + c_p\delta) \cong D(\xi_1, \ldots, \xi_p)
\]
as \( (b \oplus t\mathfrak{g}[t]) \)-modules.

Now write \( l = 3r + s \) with \( r \in \mathbb{Z}_+, s \in \{0, 1, 2\} \), and set
\[
T(\lambda) = \begin{cases} 
D(k(-\omega_1 + \Lambda_0), r(-3\omega_2 + \Lambda_0)) & \text{if } s = 0, \\
D(k(-\omega_1 + \Lambda_0), r(-3\omega_2 + \Lambda_0), -s\omega_2 + \Lambda_0) & \text{otherwise}.
\end{cases}
\]

Note that \( T(\lambda) \) is extended to a module over \( g[t] \oplus \mathbb{C}K \oplus \mathbb{C}d \), and as a \( g[t] \)-module \( T(\lambda) \) is generated by the one-dimensional weight space \( T(\lambda)_\lambda \).

Our first main theorem is the following theorem.

**Theorem 3.2.** As a \( g[t] \)-module, we have
\[
M(\lambda) \cong L(m) \cong T(\lambda).
\]

The second main theorem gives a multiplicity formula for \( L(m) \) as a \( g \)-module. For \( \lambda = k\omega_1 + l\omega_2 \), define a subset \( S_\lambda \subseteq \mathbb{Z}_+^5 \) by
\[
S_\lambda = \{(a_1, \ldots, a_5) \mid a_1 \leq k, a_1 - a_3 + a_5 \leq k, 2a_2 + 3a_3 + 3a_4 \leq l, 2a_2 + 3a_4 + 3a_5 \leq l\}.
\]

**Theorem 3.3.** As a \( g \)-module,
\[
L(m) \cong \bigoplus_{(a_1, \ldots, a_5) \in S_\lambda} V((k - a_1 + a_3 + a_4 - a_5)\omega_1 + (l - a_2 - 3a_3 - 3a_4)\omega_2).
\]

By \( \mathfrak{S}_1 \), we immediately obtain the following corollary.

**Corollary 3.4.** As a \( \mathcal{U}_q(\mathfrak{g}) \)-module,
\[
L_q(m) \cong \bigoplus_{(a_1, \ldots, a_5) \in S_\lambda} V_q((k - a_1 + a_3 + a_4 - a_5)\omega_1 + (l - a_2 - 3a_3 - 3a_4)\omega_2).
\]

From Theorem 3.2, we also obtain the following formula for the limit of the (normalized) characters of minimal affinizations.

**Corollary 3.5.** Let \( J \) be a subset of \( I \), and suppose that \( \lambda_1, \lambda_2, \ldots \) is an infinite sequence of elements of \( P_+ \) such that
\[
\lim_{n \to \infty} \langle \alpha_i^\vee, \lambda_n \rangle = \infty \text{ for all } i \in J \text{ and } \langle \alpha_i^\vee, \lambda_n \rangle = 0 \text{ for all } i \notin J, n \in \mathbb{Z}_{>0}.
\]
Let \( m_1, m_2, \ldots \) be an infinite sequence of elements of \( P_+ \) such that \( L_q(m_n) \) is a minimal affinization of \( V_q(\lambda_n) \). Then \( \lim_{n \to \infty} e^{-\lambda_n} \text{ch} L_q(m_n) \) exists, and
\[
\lim_{n \to \infty} e^{-\lambda_n} \text{ch} L_q(m_n) = \prod_{\alpha \in \Delta_+} \left( \frac{1}{1 - e^{-\alpha}} \right)^{\max_{\beta \in \Delta} \langle \beta, \alpha \rangle}.
\]

**Proof.** This result follows from Theorem 5.2 and the proof is the same as the proof of [Nao13 Corollary 4.13]. \( \square \)

This corollary, together with [MY14 Corollary 5.6], yields the character formula of the least affinizations of generic parabolic Verma modules of type \( G_2 \) conjectured by Mukhin and Young [MY14 Conjecture 6.3].

4. PROOF OF THEOREM 3.2

Throughout the rest of this paper, we fix \( \lambda = k \omega_1 + l \omega_2 \in P_+ \) and set \( r = \mathbb{Z}_+ \) and \( s \in \{0, 1, 2\} \) to be such that \( l = 3r + s \). Let \( m \) be one of the monomials in \((2.1)\). In this section, we shall prove one by one the existence of three surjective homomorphisms
\[
M(\lambda) \twoheadrightarrow L(m), \quad L(m) \twoheadrightarrow T(\lambda), \quad T(\lambda) \twoheadrightarrow M(\lambda),
\]
which completes the proof of Theorem 3.2.

4.1. **Proof of** \( M(\lambda) \twoheadrightarrow L(m) \). Let \( v_m \) be a highest \( \ell \)-weight vector of \( L_q(m) \), and \( W = U_q(\mathfrak{g}) v_m \subseteq L_q(m) \) the simple \( U_q(\mathfrak{g}) \)-submodule generated by \( v_m \). It follows from [Cha02 Proposition 5.5] that \( \bigoplus_{\mu \geq \lambda - \alpha_1 - \alpha_2} L_q(m)_{\mu} \subseteq W \), where \( L_q(m)_{\mu} \) denotes the weight space with weight \( \mu \). Hence we have
\[
x_{1,1}^- v_m \in W, \quad x_{\alpha_1,1}^- v_m \in W, \quad [x_{\alpha_1,1}^-, x_{\alpha_2}^-] v_m \in W.
\]
Then it is proved from the definition of the graded limit that the vector \( \bar{v}_m = v_m \otimes_A 1 \in L(m) \) satisfies
\[
(f_{\alpha_1} \otimes t) \bar{v}_m = (f_{\alpha_2} \otimes t) \bar{v}_m = (f_{\alpha_1 + \alpha_2} \otimes t) \bar{v}_m = 0
\]
(see [Nao13 Subsection 4.1]). The other relations in \( (3.2) \) are easily checked from the construction. Hence \( M(\lambda) \twoheadrightarrow L(m) \) follows.

4.2. **Proof of** \( L(m) \twoheadrightarrow T(\lambda) \). Here we only consider the case where the monomial \( m \) is of the form \( \prod_{i=0}^{k-1} Y_{1, aq^{\delta_{2i}}} \cdot \prod_{i=0}^{l-1} Y_{2, aq^{\delta_{2i+1}}} \). The proof of the other case is similar. Set
\[
m_1 = \prod_{i=0}^{k-1} Y_{1, aq^{\delta_{2i}}}, \quad m_2 = \prod_{i=0}^{l-1} Y_{2, aq^{\delta_{2i+1}}}.
\]
By [Cha02 Theorem 5.1] (or more precisely, the dualized statement of it), there exists an injective homomorphism
\[
L_q(m) \hookrightarrow L_q(m_1) \otimes L_q(m_2)
\]
mapping a highest \( \ell \)-weight vector to the tensor product of highest \( \ell \)-weight vectors. Then by the definition of graded limits, we obtain a \( \mathfrak{g}[[t]] \)-module homomorphism
\[
L(m) \to L(m_1) \otimes L(m_2)
\]
mapping a highest weight vector to the tensor product of highest weight vectors. Now the existence of a surjection \( L(m) \twoheadrightarrow T(\lambda) \) is proved from the following lemma.
Lemma 4.1. (i) $L(m_1)$ is isomorphic to $D(k(-\omega_1 + \Lambda_0))$ as a $\mathfrak{g}[t]$-module.
(ii) $L(m_2)$ is isomorphic to $D(r(-3\omega_2 + \Lambda_0))$ (resp. $D(r(-3\omega_2 + \Lambda_0), -s\omega_2 + \Lambda_0)$) if $s = 0$ (resp. $s = 1, 2$) as a $\mathfrak{g}[t]$-module.

Proof. The graded limit $L(m_1)$ is isomorphic to the Kirillov-Reshetikhin module $KR(k\omega_1)$ for $\mathfrak{g}[t]$ defined in [CM06, CM07], which is proved from the facts that there exists a surjection $KR(k\omega_1) \twoheadrightarrow L(m_1)$ (see Subsection 4.1) and the characters of two modules are the same (see [HKO+99, Her06, CM07]). Hence the assertion (i) follows from [FL07, Theorem 4]. Similarly $L(m_2)$ is isomorphic to $KR(k\omega_2)$, and hence by [CM07, Corollary 2.3] it is isomorphic to the $\mathfrak{g}[t]$-submodule of $KR(3r\omega_2) \otimes KR(s\omega_2)$ generated by the tensor product of highest weight vectors. Now $KR(3r\omega_2) \cong D(r(-3\omega_2 + \Lambda_0))$ follows from [FL07, Theorem 4], and $KR(s\omega_2) \cong D(-s\omega_2 + \Lambda_0)$ is verified by the Demazure character formula (see [FL07]). Hence the assertion (ii) is proved.

4.3. Proof of $T(\lambda) \rightarrow M(\lambda)$. First we introduce the following notation, as in [Nao13, Nao14]. Assume that $V$ is a $\hat{\mathfrak{g}}$-module and $D$ is a $\mathfrak{b}$-submodule of $V$. For $i \in \hat{I}$ let $\hat{\mathfrak{b}}_i$ denote the parabolic subalgebra $\mathfrak{b} \oplus \mathfrak{d}_j \subseteq \hat{\mathfrak{g}}$, and set $F_i D = U(\hat{\mathfrak{b}}_i)D \subseteq V$ to be the $\hat{\mathfrak{b}}_i$-submodule generated by $D$. It is easily seen that, if $\xi_1, \ldots, \xi_p \in \hat{W}(\hat{\mathfrak{h}}_+)$ satisfy $\langle \alpha^\vee, \xi_j \rangle \geq 0$ for all $1 \leq j \leq p$, then

$$F_i D(\xi_1, \ldots, \xi_p) = D(s_i \xi_1, \ldots, s_i \xi_p) \quad (4.1)$$

(see [Nao13, Lemma 2.4]).

Let $\hat{\Delta}^\mathfrak{re} = \Delta + \mathbb{Z}\delta$ be the set of real roots of $\hat{\mathfrak{g}}$, and $\hat{\Delta}_+^\mathfrak{re} = \Delta_+ \cup (\Delta + \mathbb{Z}_{>0}\delta)$ the set of positive real roots. For $\gamma = \alpha + p\delta \in \hat{\Delta}^\mathfrak{re}$, set

$$\gamma^\vee = \alpha^\vee + \frac{6p}{(\alpha, \alpha)}K,$$

and define a number $\rho(\gamma)$ by

$$\rho(\gamma) = \max\{0, -\langle \gamma^\vee, k(\omega_1 + \Lambda_0) \rangle\} + \max\{0, -\langle \gamma^\vee, r(3\omega_2 + \Lambda_0) \rangle\} + \max\{0, -\langle \gamma^\vee, s\omega_2 + \Lambda_0 \rangle\}.$$

The explicit values of $\rho(\gamma)$ for $\gamma \in \hat{\Delta}_+^\mathfrak{re}$ are given as follows:

$$\begin{align*}
\rho(- (\alpha_1 + 2\alpha_2) + \delta) &= 3r + \delta, \\
\rho(- (\alpha_1 + 3\alpha_2) + \delta) &= 2r + \delta, \\
\rho(- (2\alpha_1 + 3\alpha_2) + \delta) &= k + 2r + \delta, \\
\rho(- (\alpha_1 + 3\alpha_2) + 2\delta) &= \rho(- (2\alpha_1 + 3\alpha_2) + 2\delta) = r,
\end{align*}$$

and $\rho(\gamma) = 0$ for all the other $\gamma \in \hat{\Delta}_+^\mathfrak{re}$. Here $\delta_{\alpha, \beta}$ denotes the Kronecker's delta. For $\alpha + p\delta \in \hat{\Delta}^\mathfrak{re}$ set $x_{\alpha + p\delta} = e_\alpha \otimes t^p$.

Recall that $v_\xi$ denotes an extremal weight vector in $\hat{V}(\Lambda)$ with weight $\xi$, where $\Lambda \in \hat{\mathfrak{h}}_+$ is the element satisfying $\xi \in \hat{W}\Lambda$. Let $v_T \in T(\lambda)$ be the tensor product of the extremal weight vectors:

$$v_T = \begin{cases} v_{k(\omega_1 + \Lambda_0)} \otimes v_{r(3\omega_2 + \Lambda_0)} & s = 0, \\
\frac{v_{k(\omega_1 + \Lambda_0)} \otimes v_{r(3\omega_2 + \Lambda_0)} \otimes v_{s\omega_2 + \Lambda_0}}{2} & s = 1, 2.
\end{cases}$$

Note that $T(\lambda)$ is generated by $v_T$ as a $\mathfrak{g}[t]$-module. Throughout the rest of this paper, we will abbreviate $X \otimes t^p$ as $Xt^p$ to shorten the notation.
Lemma 4.2. We have

$$\text{Ann}_{U(\mathfrak{h}_+)}(v_T) = U(\mathfrak{h}_+)(\bigoplus_{\gamma \in \Delta^+_t} \mathbb{C} \nu^\gamma + 1 + \mathbb{C} f_{a_1 + a_2} t f_{a_1 + 2a_2} t^{3r-2} + th[t]),$$

where $\mathbb{C} f_{a_1 + a_2} t f_{a_1 + 2a_2} t^{3r-2}$ is omitted if $r = 0$.

Proof. First assume that $s = 0$, and set $\Lambda = r(-2 \omega_1 + 3 \omega_2 + \Lambda_0)$. Note that

$$F_0 D(k \Lambda_0, \Lambda) \cong D(k(\omega_1 + \Lambda_0), r(3 \omega_2 + \Lambda_0)) ( = U(\mathfrak{h}) v_T)$$

holds by (4.1), and we have

$$\text{Ann}_{U(\mathfrak{h}_+)}(v_{k \Lambda_0} \otimes v_\Lambda) = \text{Ann}_{U(\mathfrak{h}_+)}(v_\Lambda)$$

since $\mathfrak{h}_+$ acts trivially on $v_{k \Lambda_0}$. We shall check that $D(k \Lambda_0, \Lambda)$ satisfies the conditions (i) – (iii) (for $T$) in [Nao13, Lemma 5.3]. Note that the condition (iii) holds by [LLM02, Theorem 5]. By [Mat88, Lemma 26], we have

$$\text{Ann}_{U(\mathfrak{h}_+)}(v_\Lambda) = U(\mathfrak{h}_+) \left( \bigoplus_{\gamma \in \Delta^+_t \setminus \{\alpha_0\}} \mathbb{C} \nu^\gamma \right)$$

It follows that

$$\max\{0, -\Lambda(\nu^\gamma)\} = \begin{cases} 3r & \gamma = \alpha_1 + \alpha_2, \\ 2r & \gamma = \alpha_1, \\ r & \gamma = \alpha_1 + \delta \text{ or } 2\alpha_1 + 3\alpha_2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{h}_0$ be the Lie subalgebra $\bigoplus_{\gamma \in \Delta^+_t \setminus \{\alpha_0\}} \mathbb{C} \nu^\gamma$ of $\mathfrak{h}_+$, and define a left $U(\mathfrak{h}_0)$-ideal $I$ by

$$I = U(\mathfrak{h}_0) \left( \bigoplus_{\gamma \in \Delta^+_t \setminus \{\alpha_0\}} \mathbb{C} \nu^\gamma \otimes th[t] \right).$$

It is directly checked for every $p \in Z_+$ that

$$\text{ad}(e_0)(e_{\alpha_1 + \alpha_2}^p) \in \mathbb{C} e_{\alpha_1 + \alpha_2} f_{a_1 + a_2} t + \mathbb{C} e_{\alpha_1 + \alpha_2} f_{a_1 + 2a_2} t + \mathbb{C} e_{\alpha_1 + \alpha_2} f_{\alpha_2} t + \mathbb{C} e_{\alpha_1 + \alpha_2} e_{\alpha_1 + \alpha_2} t,$$

where we set $e_{\alpha_1 + \alpha_2}^p = 0$ if $q < 0$. Using this we see that $I$ is $\text{ad}(e_0)$-invariant, and

$$\text{Ann}_{U(\mathfrak{h}_+)}(v_\Lambda) = U(\mathfrak{h}_+) e_0 + U(\mathfrak{h}_+) I.$$

Now the assertion (for $s = 0$) follows by [Nao13, Lemma 5.3].

The case $s = 1$ is easily proved from the case $s = 0$ since $\mathfrak{h}_+$ acts trivially on $v_{\omega_2 + \Lambda_0}$ and hence

$$\text{Ann}_{U(\mathfrak{h}_+)}(v_{k(\omega_1 + \Lambda_0)} \otimes v_{r(3 \omega_2 + \Lambda_0)} \otimes v_{\omega_2 + \Lambda_0}) = \text{Ann}_{U(\mathfrak{h}_+)}(v_{k(\omega_1 + \Lambda_0)} \otimes v_{(3 \omega_2 + \Lambda_0)}).$$

For the case $s = 2$, notice by (4.1) that

$$D(r(3 \omega_2 + \Lambda_0), 2 \omega_2 + \Lambda_0) \cong F_0 F_1 F_2 F_1 F_0 D(r \Lambda_0, 2 \omega_2 + \Lambda_0).$$

Then this is isomorphic to

$$F_0 F_1 F_2 F_1 F_0 D(\omega_2 + (r + 1) \Lambda_0) \cong D((3r + 2) \omega_2 + (r + 1) \Lambda_0)$$
Proposition 4.3. The vector \( v_M \in M(\lambda) \) satisfies the relations
\[
x_\gamma^{(\gamma)+1}v_M = 0 \quad \text{for} \quad \gamma \in \Delta^+ \quad \text{and} \quad \theta h[i]v_M = 0, \quad f_{\alpha_1+3\alpha_2}t^2(f_{\alpha_1+2\alpha_2}t)^{3r-2}v_M = 0,
\]
where the last one is omitted when \( r = 0 \).

Now Lemma 4.2 and the following proposition yield a \((\mathfrak{h} \oplus \hat{\mathfrak{n}}_+)\)-module homomorphism from \( U(\mathfrak{h} \oplus \hat{\mathfrak{n}}_+)v_T \) to \( M(\lambda) \) sending \( v_T \) to \( v_M \) since their weights are both \( \lambda \), and then the existence of a surjection \( T(\lambda) \to M(\lambda) \) is proved by the same argument with [Nao13] two paragraphs below Lemma 5.2.

**Proposition 4.3.** The vector \( v_M \in M(\lambda) \) satisfies the relations
\[
x_\gamma^{(\gamma)+1}v_M = 0 \quad \text{for} \quad \gamma \in \Delta^+ \quad \text{and} \quad \theta h[i]v_M = 0, \quad f_{\alpha_1+3\alpha_2}t^2(f_{\alpha_1+2\alpha_2}t)^{3r-2}v_M = 0,
\]
where the last one is omitted when \( r = 0 \).

The rest of this subsection is devoted to prove Proposition 4.3. For simplicity we assume that \( s = 0 \) in the rest of this subsection, and prove the proposition only in this case. The proof of the other cases are almost the same. Note that the relations \( x_\gamma v_M = 0 \) for \( \gamma \not\in \{-(\alpha_1+2\alpha_2)+\delta, -(\alpha_1+3\alpha_2)+\delta, -(2\alpha_1+3\alpha_2)+\delta, -(\alpha_1+3\alpha_2)+2\delta, -(2\alpha_1+3\alpha_2)+2\delta\} \) and \( \theta h[i]v_M = 0 \) are easily proved from the definition. For example when \( \gamma = -(\alpha_1+2\alpha_2)+2\delta \), \( x_\gamma v_M = 0 \) follows since \( [x_{-(\alpha_1+2\alpha_2)+\delta}, x_{-\alpha_2+\delta}]v_M = 0 \).

For computational convenience, we assume from now on that the root vectors are normalized so that
\[
[e_{\alpha_2}, f_{\alpha_1+3\alpha_2}] = f_{\alpha_1+2\alpha_2}, \quad [e_{\alpha_2}, f_{\alpha_1+2\alpha_2}] = f_{\alpha_1+\alpha_2}, \quad [e_{\alpha_2}, f_{\alpha_1+\alpha_2}] = f_{\alpha_1}, \quad [f_{\alpha_1+\alpha_2}, f_{\alpha_1+2\alpha_2}] = 0 f_{2\alpha_1+3\alpha_2}.
\]

For an element \( X \) in an algebra and \( p \in \mathbb{Z}_+ \) denote by \( X^{(p)} \) the divided power \( X^p/p! \), and set \( X^{(p)} = 0 \) if \( p < 0 \).

**Lemma 4.4.** (i) For \( q \in \mathbb{Z}_+ \), we have
\[
e_{\alpha_2}f_{\alpha_1+2\alpha_2}^{(q)} = 3f_{2\alpha_1+3\alpha_2}f_{\alpha_1+2\alpha_2}^{(q-2)} \mod U(\mathfrak{g})(C e_{\alpha_2} \oplus C f_{\alpha_1} \oplus C f_{\alpha_1+\alpha_2}).
\]
(ii) For \( p, q \in \mathbb{Z}_+ \), we have
\[
e_{\alpha_2}f_{\alpha_1+3\alpha_2}^{(q)} = \sum_i f_{2\alpha_1+3\alpha_2}f_{\alpha_1+3\alpha_2}^{(q-p+i)} f_{\alpha_1+2\alpha_2}^{(p-3i)} \mod U(\mathfrak{g})(C e_{\alpha_2} \oplus C f_{\alpha_1} \oplus C f_{\alpha_1+\alpha_2}),
\]
where \( i \) runs over the set of integers such that \( \max\{0, p - q\} \leq i \leq p/3 \).

**Proof.** We have
\[
e_{\alpha_2}f_{\alpha_1+2\alpha_2}^{(q)} = \frac{1}{q!} \sum_{i=1}^{q} f_{\alpha_1+2\alpha_2}^{(i-1)} f_{\alpha_1+2\alpha_2}^{(q-i)} = \frac{1}{q!} \sum_{i=1}^{q} 6(q-i)f_{2\alpha_1+3\alpha_2}f_{\alpha_1+2\alpha_2}^{(q-2)}
\]
\[
= \frac{1}{q!} \cdot 3q(q-1)f_{2\alpha_1+3\alpha_2}f_{\alpha_1+2\alpha_2}^{(q-2)} = 3f_{2\alpha_1+3\alpha_2}f_{\alpha_1+2\alpha_2}^{(q-2)}.
\]
and the assertion (i) holds. The assertion (ii) with $p = 1$ is immediate. Then we have by induction and (i) that

$$(p + 1) e_{α_2}^{(p+1)} f_{α_1+3α_2}^{(q)} = e_{α_2} \sum_i f_{2α_1+3α_2}^{(q)} f_{α_1+3α_2}^{(p+1)} f_{α_1+2α_2}^{(p-3i)}$$

$$= \sum_i f_{2α_1+3α_2}^{(i)} (f_{α_1+3α_2}^{(q−p+1−1)} f_{α_1+2α_2}^{(p−3i)} + 3 f_{2α_1+3α_2}^{(p−3i−2)} f_{α_1+2α_2}^{(p−3i−2)})$$

$$= \sum_i (p − 3i + 1) f_{2α_1+3α_2}^{(i)} f_{α_1+2α_2}^{(p−3i+1)} + \sum_i 3(i + 1) f_{2α_1+3α_2}^{(i+1)} f_{α_1+2α_2}^{(p−3i−2)}$$

$$= (p + 1) \sum_i f_{2α_1+3α_2}^{(i)} f_{α_1+2α_2}^{(p−3i+1)}.$$

Hence the assertion (ii) holds. □

By Lemma 4.4 (ii), we also see that

$$e_{α_2}^{(p)} (f_{α_1+3α_2} t)^{(q)} = \sum_{i=\max\{0, p-q\}}^{\lfloor p/3 \rfloor} (f_{2α_1+3α_2} t^2)^{(i)} (f_{α_1+3α_2} t)^{(q−p+i)} (f_{α_1+2α_2} t)^{(p−3i)} \mod U(\mathfrak{g}) (C_{α_2} \oplus C_{α_1} t \oplus C f_{α_1+α_2} t).$$

**Lemma 4.5.** The relations $(f_{α_1+3α_2} t)^{(2r+1)} v_M = 0$ and $(f_{2α_1+3α_2} t)^{(k+2r+1)} v_M = 0$ hold.

**Proof.** We have

$$\langle α_2^\vee, \text{wt}((f_{α_1+3α_2} t)^{(2r+1)} v_M) \rangle = \langle α_2^\vee, λ − (2r+1)(α_1 + 3α_2) \rangle = −(3r + 3).$$

On the other hand, it follows from (12) that

$$e_{α_2}^{3r+3} (f_{α_1+3α_2} t)^{(2r+1)} v_M = 0,$$

and hence we have $(f_{α_1+3α_2} t)^{(2r+1)} v_M = 0$ since $M(λ)$ is an integrable $\mathfrak{g}$-module. Now it is an elementary fact that this relation and $f_{α_1}^{k+1} v_M = 0$ imply $(f_{2α_1+3α_2} t)^{(k+2r+1)} v_M = 0$ (for example, see [Nao12, Lemma 4.5]). □

**Lemma 4.6.** The relations $(f_{2α_1+3α_2} t)^{(r+1)} v_M = 0$ and $(f_{α_1+3α_2} t)^{(r+1)} v_M = 0$ hold.

**Proof.** By Lemma 15 and (4.2), we have

$$0 = f_{α_2}^{(3r+3)} (f_{α_1+3α_2} t)^{(2r+2)} v_M = (f_{2α_1+3α_2} t^2)^{(r+1)} v_M,$$

and hence $(f_{2α_1+3α_2} t^2)^{(r+1)} v_M = 0$ follows. From this we see that

$$0 = e_{α_1}^{(r+1)} (f_{2α_1+3α_2} t^2)^{(r+1)} v_M = c(f_{α_1+3α_2} t^2)^{(r+1)} v_M$$

with some nonzero $c$. Hence $(f_{α_1+3α_2} t^2)^{(r+1)} v_M = 0$ also holds. □

**Lemma 4.7.** The relation $(f_{α_1+2α_2} t)^{(3r+1)} v_M = 0$ holds.

**Proof.** By Lemma 15 and (4.2), we have for $p \geq 2r + 1$

$$0 = e_{α_2}^{(p)} (f_{α_1+3α_2} t)^{(p)} v_M = \sum_{i=0}^{\lfloor p/3 \rfloor} \frac{1}{(p−3i)!} (f_{2α_1+3α_2} t^2)^{(i)} (f_{α_1+3α_2} t)^{(i)} (f_{α_1+2α_2} t)^{(p−3i)} v_M.$$
Hence in order to prove \((f_{a_1+2a_2}t)^{3r+1}v_M = 0\), it is enough to show that the matrix \(A = (a_{ij})_{0 \leq i, j \leq r}\) with
\[
a_{ij} = \begin{cases} 
\frac{1}{(3r+1-3i-j)!} & \text{if } 3r + 1 - 3i - j \geq 0, \\
0 & \text{otherwise}
\end{cases}
\]
is invertible. Assume that \(v_0, v_1, \ldots, v_r\) satisfy \(\sum a_{ij}v_j = 0\) for all \(i\), and consider the polynomial
\[
f(x) = \frac{v_0}{(3r+1)!}x^{3r+1} + \frac{v_1}{(3r-2)!}x^{3r-2} + \cdots + \frac{v_r}{(3r+1)!}x^1 + v_r.
\]
Then \(\frac{df}{dx}(1) = 0\) holds for all \(0 \leq j \leq r\), which implies that \(f(x)\) is divisible by \((x - 1)^{r+1}\).
Since \(f(\zeta x) = \zeta f(x)\) holds where \(\zeta\) is a third primitive root of unity, we see that \(f(x)\) is divisible by \((x^3 - 1)^{r+1}\). By the degree consideration we have \(f(x) = 0\), and the proof is complete.

Now the following lemma completes the proof of Proposition 4.3.

**Lemma 4.8.** The relation \(f_{a_1+3a_2}t^2(f_{a_1+2a_2}t)^{3r-2}v_M = 0\) holds when \(r \neq 0\).

**Proof.** Let \(p \geq 2r - 1\). By Lemma 4.5 we have
\[
0 = e_{a_1+3a_2}(f_{a_1+3a_2}t)^{p+1}v_M = \frac{1}{(p+2)!} \sum_{i=0}^{p+1} (f_{a_1+3a_2}t)^p (a_1+3a_2)^i t^i (f_{a_1+3a_2}t)^iv_M
\]
\[
= \frac{1}{(p+2)!} \sum_{i=0}^{p+1} -2i(f_{a_1+3a_2}t)^p f_{a_1+3a_2}t^2v_M = -(f_{a_1+3a_2}t)^p f_{a_1+3a_2}t^2v_M. \tag{4.3}
\]
We easily see that all the elements \(e_{a_2}, f_{a_1}, f_{a+1}t\) annihilate the vector \(f_{a_1+3a_2}t^2v_M\), and hence we have from (4.2) and (4.3) that
\[
0 = e_{a_2}^{[p/3]} f_{a_1+3a_2}t^2(v_M
\]
\[
= \sum_{i=0}^{[p/3]} \frac{1}{(p-3i)!} f_{a_1+3a_2}t^2(f_{a_1+3a_2}t^2)^{(i)}(f_{a_1+3a_2}t)(f_{a_1+2a_2}t)^{p-3i}.
\]
Now the lemma is proved by a similar argument as in the proof of Lemma 4.7. \(\square\)

5. PROOF OF THEOREM 4.3

5.1. A basis of the space of highest weight vectors. For \(a = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{Z}_+^5\), set
\[
f_a = (f_{2a_1+3a_2}t^2)^{(a_5)}(f_{a_1+3a_2}t^2)^{(a_1)}(f_{a_1+3a_2}t)^{a_3}(f_{a_1+2a_2}t)^{(a_2)}(f_{2a_1+3a_2}t)^{(a_1)},
\]
and
\[
\text{wt}(a) = (2a_1 + a_2 + a_3 + a_4 + 2a_5)\omega_1 + (3a_1 + 2a_2 + 3a_3 + 3a_4 + 3a_5)\omega_2
\]
\[
= (a_1 - a_3 - a_4 + a_5)\omega_1 + (a_2 + 3a_3 + 3a_4)\omega_2 \in Q_+.
\]
Note that \(\text{wt}(f_a) = -\text{wt}(a)\). In this section, we denote by \(v\) a highest weight vector of \(L(m)\). Since \(L(m) \cong M(\lambda)\), we easily see from Proposition 4.3 and the PBW theorem that
\[
L(m) = \sum_{a \in \mathbb{Z}_+^5} U(g)f_a v.
\]
Let \(\alpha \in Q_+\), and set \(L(m)_{> \lambda - \alpha} = \bigoplus_{\mu > \lambda - \alpha} L(m)_\mu\). The \(g\)-submodule \(U(g)L(m)_{> \lambda - \alpha}\) of \(L(m)\) coincides with the sum of simple \(g\)-submodules whose highest weights are larger than
λ − α. Hence we see that the multiplicity of \( V(λ − α) \) in \( L(m) \) is equal to the dimension of the weight space of the quotient \( \mathfrak{g}\)-module \( L(m)/U(\mathfrak{g})L(m)_{>λ − α} \) with weight \( λ − α \), that is

\[
\left[ L(m) : V(λ − α) \right] = \dim \left( L(m)/U(\mathfrak{g})L(m)_{>λ − α} \right)_{λ − α}.
\]

Therefore, in order to prove Theorem [3.3] it suffices to show the following proposition, which is proved in the next subsections.

**Proposition 5.1.** For every \( α \in Q_+ \), the projection images of \( \{ f_a v \mid a \in S_λ, \ wt(a) = α \} \) form a basis of \( \left( L(m)/U(\mathfrak{g})L(m)_{>λ − α} \right)_{λ − α} \).

5.2. The space is spanned by the vectors. For \( α \in Q_+ \), set

\[
Z_5^5 [α] = \{ a \in Z_5^5 | wt(a) = α \}, \quad S_λ [α] = S_λ \cap Z_5^5 [α].
\]

In this subsection, we shall show the following.

**Lemma 5.2.** For every \( α \in Q_+ \), the projection images of \( \{ f_a v \mid a \in S_λ [α] \} \) span the space \( \left( L(m)/U(\mathfrak{g})L(m)_{>λ − α} \right)_{λ − α} \).

We denote by \( \leq \) the lexicographic order on \( Z_5^5 \), that is, \( (a_1, \ldots, a_5) < (b_1, \ldots, b_5) \) if and only if there exists \( i \) such that \( a_j = b_j \) for \( j < i \) and \( a_i < b_i \). Fix \( α \in Q_+ \). Following [CM07 Subsection 3.5], we define a finite sequence \( r_1, \ldots, r_p \) of elements of \( Z_5^5 [α] \) inductively as follows. Set \( r_1 \) to be the least element (with respect to the lexicographic order) of \( Z_5^5 [α] \) such that \( f_{r_1} v \notin U(\mathfrak{g})L(m)_{>λ − α} \). Assume that \( r_1, \ldots, r_p \) are defined. We set \( r_{p+1} \) to be the least element of \( Z_5^5 [α] \) such that

\[
f_{r_{p+1}} v \notin \bigoplus_{i=1}^p \mathbb{C} f_{r_i} v + U(\mathfrak{g})L(m)_{>λ − α}
\]

if such an element exists, and otherwise we set \( t = p \).

Set \( K[α] = \{ r_1, \ldots, r_t \} \). By the definition the projection images of \( \{ f_a v \mid a \in K[α] \} \) span \( \left( L(m)/U(\mathfrak{g})L(m)_{>λ − α} \right)_{λ − α} \), and every \( r \in K[α] \) satisfies that

\[
f_r v \notin \bigoplus_{a \in Z_5^5 [α], a < r} \mathbb{C} f_a v + U(\mathfrak{g})L(m)_{>λ − α}.
\]

(5.1)

It is enough to show that every \( r = (r_1, \ldots, r_5) \in K[α] \) satisfies

\[
r_1 \leq k, \quad r_1 − r_3 + r_5 \leq k, \quad 2r_2 + 3r_3 + 3r_4 \leq l, \quad 2r_2 + 3r_4 + 3r_5 \leq l,
\]

since this implies \( K[α] \subseteq S_λ [α] \).

Fix \( r = (r_1, \ldots, r_5) \in K[α] \), and first assume that \( r_1 > k \). The Lie subalgebra of \( \mathfrak{g}[t] \) spanned by \( f_{α_1}, f_{α_1+3α_2} t, \) and \( f_{2α_1+3α_2} t \) is isomorphic to the 3-dimensional Heisenberg algebra. Then [CM07 Lemma 1.5] and \( f_{α_1+1} t \) imply that

\[
(f_{α_1+3α_2} t)^s (f_{2α_1+3α_2} t)^r v \in \sum_{0<p,q<0,0\leq s\leq k} \mathbb{C} f_{α_1} (f_{α_1+3α_2} t)^p (f_{2α_1+3α_2} t)^q v.
\]

From this we easily see that

\[
f_r v = \sum_{a \in Z_5^5 [α], a < r} \mathbb{C} f_a v + U(\mathfrak{g})L(m)_{>λ − α},
\]
which contradicts (5.1).

Next assume that \( r_1 - r_3 + r_5 > k \). Let \( e_i \) (\( 1 \leq i \leq 5 \)) denote the standard basis of \( \mathbb{Z}^5 \), and set \( s = r - r_4 e_4 + r_4 e_5 \). We easily see that

\[
e_{\alpha_1}^r f_s v \in \mathbb{C}^5 f_v + \sum_{a \in \mathbb{Z}_5^+ [\alpha]} \mathbb{C} f_a v.
\]

(5.2)

Note that

\[\text{wt}(f_s v) = \lambda - \alpha - r_4 \alpha_1 = (k - r_1 + r_3 - r_4 - r_5) \omega_1 + (l - r_2 - 3r_3) \omega_2,\]

and hence we have

\[s_1 \text{wt}(f_s v) = \lambda - \alpha + (r_1 - r_3 + r_5 - k) \alpha_1 > \lambda - \alpha,\]

which implies \( f_s v \in U(g)L(m)_{\lambda - \alpha} \). Then this and (5.2) contradict (5.1).

The inequality \( 2r_2 + 3r_3 + 3r_4 \leq l \) is proved in the same way as in [CM07, Subsection 3.5].

Finally assume that \( 2r_2 + 3r_4 + 3r_5 > l \). Then \( r_5 > r_3 \) follows, since otherwise we have \( 2r_2 + 3r_4 + 3r_5 \leq 2r_2 + 3r_3 + 3r_3 \leq l \). Set

\[s_j = (r_1, 0, r_2 + r_3 + 2r_5 - 2j, r_4 - j) \quad \text{ for } 0 \leq j \leq r_3.\]

We have

\[\text{wt}(f_{s_j} v) = \lambda - \alpha - (r_2 + 3r_5 - 3j) \alpha_2, \quad \langle \text{wt}(f_{s_j} v), \alpha_2^\vee \rangle = l - 3r_2 - 3r_3 - 3r_4 - 6r_5 + 6j.\]

Then by a similar argument as in the proof of \( r_1 - r_3 + r_5 \leq k \), we can show that

\[f_{s_j} v \in U(g)L(m)_{\lambda - \alpha} \quad \text{ for all } 0 \leq j \leq r_3.\]

(5.3)

It follows from (4.2) that

\[
e_{\alpha_2}^{(r_2 + 3r_5 - 3j)} f_{s_j} v = \sum_{i = \max(0, r_5 - r_3 - j)}^{r_5 - j + |r_2/3|} \binom{i + j}{j} f_{(r_1, r_2 + 3r_5 - 3i - j, r_3 - r_5 + i + j, r_4, i, j)} v
\]

\[
= \sum_{i = -[r_2/3]}^{\min\{r_5 - j, r_3\}} \binom{r_5 - i}{j} f_{(r_1, r_2 + 3i, r_3 - i, r_4, r_5 - i)} v
\]

\[
\in \sum_{i = 0}^{\min\{r_5 - j, r_3\}} \binom{r_5 - i}{j} f_{r + i(3e_2 - e_3 - e_5)} v + \sum_{a \in \mathbb{Z}_5^+ [\alpha], a < r} \mathbb{C} f_a v,
\]

and then by (5.3) we have for every \( 0 \leq j \leq r_3 \) that

\[
\sum_{i = 0}^{\min\{r_5 - j, r_3\}} \binom{r_5 - i}{j} f_{r + i(3e_2 - e_3 - e_5)} v \in \sum_{a \in \mathbb{Z}_5^+ [\alpha], a < r} \mathbb{C} f_a v + U(g)L(m)_{\lambda - \alpha}.
\]

From this we can show that

\[f_v v \in \sum_{a \in \mathbb{Z}_5^+ [\alpha], a < r} \mathbb{C} f_a v + U(g)L(m)_{\lambda - \alpha},\]

by a similar argument as in Lemma 4.7, in which we use a polynomial

\[f(x) = v_0 x^{r_5} + v_1 x^{r_5 - 1} + \cdots + v_{r_3} x^{r_5 - r_3}.\]
instead. Now this contradicts \([5.1]\).

5.3. Linearly independence. Proposition \([5.1]\) is proved from the following lemma, together with Lemma 5.2.

**Lemma 5.3.** For every \(\alpha \in Q_+\), the images of \(\{ f_\alpha v \mid a \in S_\lambda[\alpha] \}\) under the canonical projection \(L(m) \to L(m)/U(g)L(m)_{>\lambda-\alpha}\) are linearly independent.

Fix \(\alpha \in Q_+\). Let \(\overline{L(m)} = L(m)/U(g)L(m)_{>\lambda-\alpha}\), and \(pr\) denote the canonical projection \(L(m) \to \overline{L(m)}\). We shall show the lemma by the induction on \(k\). The case \(k = 0\) is proved in [CM07].

Assume that \(k > 0\), and a sequence \(\{ c_a \}_{a \in S_\lambda[\alpha]}\) of complex numbers satisfies
\[
\sum_{a \in S_\lambda[\alpha]} c_a pr(f_\alpha v) = 0. \tag{5.4}
\]

First we shall show that
\[
c_a = 0 \quad \text{for all} \quad a \in S_\lambda[\alpha] \quad \text{such that} \quad a_1 > 0. \tag{5.5}
\]
Let \(L_1\) and \(L_2\) be the graded limits of minimal affinizations of \(V_q(\omega_1)\) and \(V_q(\lambda - \omega_1)\) respectively, and \(v_1, v_2\) be respective highest weight vectors. Set \(\lambda_2 = \lambda - \omega_1\). It follows that
\[
L(m) \cong T(\lambda) \hookrightarrow T(k\omega_1) \otimes T(l\omega_2) \hookrightarrow T(\omega_1) \otimes T((k-1)\omega_1) \otimes T(l\omega_2),
\]
and from this we see that \(L(m) \cong U(g)[t](v_1 \otimes v_2) \subseteq L_1 \otimes L_2\). It is known that
\[
L_1 = U(g)(v_1) \oplus U(g)f_1 v_1 \cong V(\omega_1) \oplus V(0)
\]
as a \(g\)-module, and \(f_\alpha v_1 = 0\) if \(a \not\in \{0, e_1\}\).

Let \(pr^1: L_1 \to V(0)\) be the projection with respect to the \(g\)-module decomposition, and \(pr^2_{\lambda-\alpha}: L_2 \to L_2/U(g)(L_2)_{>\lambda-\alpha}\) the canonical projection. Since
\[
(L_1 \otimes L_2)_{>\lambda-\alpha} = \bigoplus_{\mu \in P} (L_1)_\mu \otimes (L_2)_{>\lambda-\alpha-\mu} \subseteq V(0) \otimes (L_2)_{>\lambda-\alpha} \oplus V(\omega_1) \otimes L_2,
\]
we have
\[
U(g)(L_1 \otimes L_2)_{>\lambda-\alpha} \subseteq V(0) \otimes U(g)(L_2)_{>\lambda-\alpha} \oplus V(\omega_1) \otimes L_2.
\]
Hence the composition
\[
\kappa: L(m) \to L_1 \otimes L_2 \xrightarrow{pr^1 \otimes pr^2_{\lambda-\alpha}} V(0) \otimes \left( L_2/U(g)(L_2)_{>\lambda-\alpha} \right) \cong L_2/U(g)(L_2)_{>\lambda-\alpha}
\]
induces a \(g\)-module homomorphism \(\pi: \overline{L(m)} \to L_2/U(g)(L_2)_{>\lambda-\alpha}\). It is easily seen for \(a = (a_1, \ldots, a_5)\) that
\[
f_\alpha(v_1 \otimes v_2) = \begin{cases} v_1 \otimes f_\alpha v_2 + f_\alpha v_1 \otimes f_{a-e_1} v_2 & \text{if } a_1 > 0, \\ v_1 \otimes f_\alpha v_2 & \text{otherwise.} \end{cases} \tag{5.6}
\]

Hence we see from the definition of \(\kappa\) that \([5.4]\) yields
\[
0 = \pi \left( \sum_{a \in S_\lambda[\alpha]} c_a pr(f_\alpha v) \right) = \sum_{a \in S_\lambda[\alpha]} c_a \kappa(f_\alpha v) = \sum_{a \in S_\lambda[\alpha] : a_1 > 0} c_a pr^2_{\lambda-\alpha}(f_{a-e_1} v_2).
\]
Since \(\lambda - \alpha = \lambda_2 - (\alpha - \omega_1)\) and \(\{ a - e_1 \mid a \in S_\lambda[\alpha], a_1 > 0 \} \subseteq S_{\lambda_2}[\alpha - \omega_1]\), \([5.5]\) follows from the induction hypothesis, as required.

Set
\[
S_0^0[\alpha] = \{ a \in S_\lambda[\alpha] : a_1 = 0 \} \quad \text{and} \quad S_0^{k,0}[\alpha] = \{ a \in S_\lambda[\alpha] : a_1 = 0, -a_3 + a_5 = k \} \subseteq S_\lambda[\alpha].
\]
It is easily checked that
\[ S^0_\lambda [\alpha] = S^0_{\lambda_2}[\alpha] \sqcup S^{0,k}_\lambda [\alpha]. \] (5.7)

Next we would like to prove that
\[ e_a = 0 \quad \text{for all } a \in S^0_{\lambda_2}[\alpha], \] (5.8)
and in order to do that we will first prove that
\[ f_a v_2 \in \mathbb{C}^x f_a f_{a+(e_4-e_5)} v_2 + U(\mathfrak{g})(L_2)_{>\lambda_2-(\alpha-a)} \] if \( a \in S^{0,k}_\lambda [\alpha]. \) (5.9)

Assume that \( r = (0, r_2, r_3, r_4, r_5 + k) \in S^{0,k}_\lambda [\alpha]. \) We see by a direct calculation that
\[ e_{a_1}^{r_4} f_{r+r_4(e_4-e_5)} v_2 \in \mathbb{C}^x f_r v_2 \quad \text{and} \quad e_{a_1}^{r_4} f_{r+r_4(e_4-e_5)} v_2 \in \mathbb{C}^x f_{r+(e_4-e_5)} v_2. \] (5.10)

Since
\[ \text{wt}(f_{a_1} f_{r+r_4(e_4-e_5)} v_2) = -(r_4 + 3)\omega_1 + (l - r_2 - 3r_3 + 3)\omega_2, \]
\[ s_1 \text{wt}(f_{a_1} f_{r+r_4(e_4-e_5)} v_2) = \text{wt}(f_r v_2) + 2\alpha_1 > \lambda_2 - (\alpha - \alpha_1), \]
which implies \( f_{a_1} f_{r+r_4(e_4-e_5)} v_2 \in U(\mathfrak{g})(L_2)_{>\lambda_2-(\alpha - \alpha_1)}. \) Hence it follows that
\[ f_{a_1} e_{a_1}^{r_4} f_{r+r_4(e_4-e_5)} v_2 = (e_{a_1}^{r_4} f_{a_1} + [f_{a_1}, e_{a_1}^{r_4+1}]) f_{r+r_4(e_4-e_5)} v_2 \]
\[ \in \mathbb{C}^x e_{a_1} f_{r+r_4(e_4-e_5)} v_2 + U(\mathfrak{g})(L_2)_{>\lambda_2-(\alpha - \alpha_1)}, \]
which together with (5.10) imply (5.9). Let \( \text{pr}^2_{\lambda_2-\alpha} : L_2 \to L_2 \otimes U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha} \) be the canonical projection. Since \( U(\mathfrak{g})(L_1 \otimes L_2)_{>\lambda_2-\alpha} \subseteq L_1 \otimes U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha}, \) the composition
\[ L(m) \to L_1 \otimes L_2 \to L_1 \otimes (L_2 \otimes U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha}) \]
induces a \( \mathfrak{g} \)-module homomorphism \( \overline{L(m)} \to L_1 \otimes (L_2 \otimes U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha}). \) We see from (5.4) that \( \text{pr}^2_{\lambda_2-\alpha}(f_a v_2) = 0 \) if \( a \in S^{0,k}_\lambda [\alpha], \) and then (5.5), (5.6), (5.7) and the induced homomorphism yield
\[ v_1 \otimes \left( \sum_{a \in S^{0,k}_\lambda [\alpha]} c_a \text{pr}^2_{\lambda_2-\alpha}(f_a v_2) \right) = 0. \]

By the induction hypothesis this implies (5.8), as required.

We have
\[ \sum_{a \in S^{0,k}_\lambda [\alpha]} c_a \text{pr}(f_a v) = 0 \] (5.11)
by (5.4), (5.5) and (5.8), and it remains to show that \( c_a = 0 \) for \( a \in S^{0,k}_\lambda [\alpha]. \) Fix \( r = (r_1, \ldots, r_5) \in S^{0,k}_\lambda [\alpha], \) and set \( s = r + e_4 - e_5. \) We define a \( \mathfrak{g} \)-submodule \( L'_2 \) of \( L_2 \) by
\[ L'_2 = \sum_{a \in S_{\lambda_2} \atop \text{wt}(a) < \alpha, a \neq s} U(\mathfrak{g}) f_a v_2. \]
We have \((L_2)_{>\lambda_2-\alpha} \subseteq \mathbb{C} f_s v_2 + L'_2 \) by Lemma 5.2 and from this we see that
\[ (L_1 \otimes L_2)_{>\lambda_2-\alpha} = \mathbb{C} v_1 \otimes (L_2)_{>\lambda_2-\alpha} \bigoplus_{\beta > 0} (L_1)_{>\lambda_2-\alpha} \otimes (L_2)_{>\lambda_2-\alpha+\beta} \]
\[ \subseteq \mathbb{C} v_1 \otimes f_s v_2 + L_1 \otimes L'_2, \]
which implies \( U(\mathfrak{g})(L_1 \otimes L_2)_{>\lambda_2-\alpha} \subseteq U(\mathfrak{g})(v_1 \otimes f_s v_2) + L_1 \otimes L'_2. \) Hence the composition
\[ \rho : L(m) \to L_1 \otimes L_2 \to (L_1 \otimes L_2) \bigg/ (U(\mathfrak{g})(v_1 \otimes f_s v_2) + L_1 \otimes L'_2) \]
induces a $g$-module homomorphism

$$
\overline{\rho}: L(m) \to (L_1 \otimes L_2)/(U(g)(v_1 \otimes f_s v_2) + L_1 \otimes L'_2).
$$

If $a \in S^{0,k}([\alpha] \setminus \{r\})$, then we have $a + e_4 - e_5 \in S_{\lambda_2}([\alpha - \alpha_1] \setminus \{s\}$ and hence it follows by (5.9) that

$$
f_a(v_1 \otimes v_2) = v_1 \otimes f_a v_2 \in L_1 \otimes L'_2.
$$

Hence we have from (5.11) that

$$
0 = \overline{\rho} \left( \sum_{a \in S^{0,k}_1([\alpha])} c_a \rho(f_a v) \right) = \sum_{a \in S^{0,k}_1([\alpha])} c_a \rho(f_a v) = c_r \rho(f_r v).
$$

Assume that $c_r \neq 0$, which implies $\rho(f_r v) = 0$. Let $\text{pr}'_2$ denote the canonical projection $L_2 \to L_2/L'_2$. We easily see that $\rho(f_r v) = 0$ is equivalent to

$$
v_1 \otimes \text{pr}'_2(f_r v_2) \in U(g)(v_1 \otimes \text{pr}'_2(f_s v_2)).
$$

Note that $\text{pr}'_2(f_s v_2) \neq 0$ by the induction hypothesis, and this also implies $\text{pr}'_2(f_r v_2) \neq 0$ since $e_{a_1} \text{pr}'_2(f_s v_2) \in C^\times \text{pr}'_2(f_s v_2)$ by (5.10). Since

$$
\text{wt}(v_1 \otimes \text{pr}'_2(f_s v_2)) = \text{wt}(v_1 \otimes \text{pr}'_2(f_s v_2)) - \alpha_1,
$$

(5.12) implies

$$
v_1 \otimes \text{pr}'_2(f_r v_2) \in C f_{a_1}(v_1 \otimes \text{pr}'_2(f_s v_2)).
$$

However this contradicts $f_{a_1} v_1 \neq 0$. Hence $c_r = 0$ holds, as required.

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**References**

[Cha95] V. Chari. Minimal affinizations of representations of quantum groups: the rank 2 case. *Publ. Res. Inst. Math. Sci.*, 31(5):873–911, 1995.

[Cha01] V. Chari. On the fermionic formula and the Kirillov-Reshetikhin conjecture. *Int. Math. Res. Not. IMRN*, (12):629–654, 2001.

[Cha02] V. Chari. Braid group actions and tensor products. *Int. Math. Res. Not.*, (7):357–382, 2002.

[CM06] V. Chari and A. Moura. The restricted Kirillov-Reshetikhin modules for the current and twisted current algebras. *Comm. Math. Phys.*, 266(2):431–454, 2006.

[CM07] V. Chari and A. Moura. Kirillov-Reshetikhin modules associated to $G_2$. In *Lie algebras, vertex operator algebras and their applications*, volume 442 of *Contemp. Math.*, pages 41–59. Amer. Math. Soc., Providence, RI, 2007.

[CP94] V. Chari and A. Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.

[CP95a] V. Chari and A. Pressley. Minimal affinizations of representations of quantum groups: the nonsimply-laced case. *Lett. Math. Phys.*, 35(2):99–114, 1995.

[CP95b] V. Chari and A. Pressley. Quantum affine algebras and their representations. In *Representations of groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 59–78. Amer. Math. Soc., Providence, RI, 1995.

[Dri87] V. G. Drinfel’d. A new realization of Yangians and of quantum affine algebras. *Dokl. Akad. Nauk SSSR*, 296(1):13–17, 1987.

[FL07] G. Fourier and P. Littelmann. Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. *Adv. Math.*, 211(2):566–593, 2007.
E. Frenkel and N. Reshetikhin. The $q$-characters of representations of quantum affine algebras and deformations of $W$-algebras. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 163–205. Amer. Math. Soc., Providence, RI, 1999.

D. Hernandez. The Kirillov-Reshetikhin conjecture and solutions of $T$-systems. *J. Reine Angew. Math.*, 596:63–87, 2006.

G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

M. Jimbo. A $q$-difference analogue of $U(g)$ and the Yang-Baxter equation. *Lett. Math. Phys.*, 10(1):63–69,1985.

D. Hernandez. The Kirillov-Reshetikhin conjecture and solutions of $T$-systems. *J. Reine Angew. Math.*, 596:63–87, 2006.

G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

M. Jimbo. A $q$-difference analogue of $U(g)$ and the Yang-Baxter equation. *Lett. Math. Phys.*, 10(1):63–69,1985.

V. Lakshmibai, P. Littelmann, and P. Magyar. Standard monomial theory for Bott-Samelson varieties. *Compositio Math.*, 130(3):293–318, 2002.

J. R. Li, E. Mukhin. Extended $T$-system of type $G_2$. *SIGMA Symmetry, Integrability Geom. Methods Appl.*, 9:Paper 054, 28, 2013. 20, 2014.

O. Mathieu. Construction du groupe de Kac-Moody et applications. *C. R. Acad. Sci. Paris Sér. I Math.*, 306(5):227–230, 1988.

A. Moura. Restricted limits of minimal affinizations. *Pacific J. Math.*, 244(2):359–397, 2010.

A. Moura and F. Pereira. Graded limits of minimal affinizations and beyond: the multiplicity free case for type $E_6$. *Algebra Discrete Math.*, 12(1):69–115, 2011.

M. G. Moakes, A. N. Pressley. $q$-characters and minimal affinizations. *Int. Electron. J. Algebra*, 1:55–97, 2007.

E. Mukhin, C. A. S. Young, Affinization of category $O$ for quantum groups. *Trans. Amer. Math. Soc.*, 366(9):48154847, 2014.

K. Naoi. Weyl modules, Demazure modules and finite crystals for non-simply laced type. *Adv. Math.*, 229(2):875–934, 2012.

K. Naoi. Demazure modules and graded limits of minimal affinizations. *Represent. Theory*, 17:524–556, 2013.

K. Naoi. Graded limits of minimal affinizations in type $D$. *SIGMA Symmetry, Integrability Geom. Methods Appl.*, 10:Paper 047, 20, 2014.

L. Qiao, J. R. Li, Cluster algebras and minimal affinizations of representations of the quantum group of type $G_2$. [arXiv:1412.3884](http://arxiv.org/abs/1412.3884) 1–17, 2014.

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