On the Algebraic Structure of Linear Trellises

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Abstract

Trellises are crucial graphical representations of codes. While conventional trellises are well understood, the general theory of (tail-biting) trellises is still under development. Iterative decoding concretely motivates such theory. In this paper we first develop a new algebraic framework for a systematic analysis of linear trellises which enables us to address open foundational questions. In particular, we present a useful and powerful characterization of linear trellis isomorphy. We also obtain a new proof of the Factorization Theorem of Koetter/Vardy and point out unnoticed problems for the group case.

Next, we apply our work to: describe all the elementary trellis factorizations of linear trellises and consequently to determine all the minimal linear trellises for a given code; prove that nonmergeable one-to-one linear trellises are strikingly determined by the edge-label sequences of certain closed paths; prove self-duality theorems for minimal linear trellises; analyze quasi-cyclic linear trellises and consequently extend results on reduced linear trellises to nonreduced ones. To achieve this, we also provide new insight into mergeability and path connectivity properties of linear trellises.

Our classification results are important for iterative decoding as we show that minimal linear trellises can yield different pseudocodewords even if they have the same graph structure.

Index Terms – Linear tail-biting trellises, linear block codes, nonmergeable trellises, minimal trellises.

1 Introduction

Trellis representations of block codes play a prominent role in coding theory and practice as they provide combinatorial insight into algebraic codes and enable the design of efficient decoding.

Traditionally trellises were separated in two classes, conventional trellises (introduced in [1]) and tail-biting trellises (introduced in [35]), however it is no harm to see conventional ones as a subclass of tail-biting ones, and we will do so. In fact for theoretical purposes it is convenient to do so (this point of view was already adopted for example in [18,19]).

Also, the actual objects of study in trellis theory are linear trellises, i.e. trellises with a linear structure (trellises without any algebraic structure are infeasible to control). The notion of linear trellis was formalized though only at a late stage, by Koetter/Vardy [23] (while McEliece [28] acknowledged a bit earlier the linear structure of minimal conventional trellises).

The study of trellises was confined to conventional ones until late in the 90s. The interest for a general theory of (linear tail-biting) trellises surged along with the interest for suboptimal iterative
decoding (sparked by Wiberg’s thesis [37] and the invention of Turbo Codes) as its complexity benefits from the long known fact proven in [35] that nonconventional representations can achieve smaller size (while optimal decoding does not, see for example [31]).

The first works [5, 22, 23, 24] towards such general theory provided a rigorous basis to the subject and had a strong influence on what came next. In particular, Koetter/Vardy [23] considered the trellis product operation (introduced first for conventional trellises in [26] and then extended to all trellises in [5]) and proved that all linear trellises factor as products of elementary trellises (Factorization Theorem), which can be more easily handled. This groundbreaking result enabled them to achieve in [24] breakthrough on the minimality problem (which is far more complicated in the general case than in the conventional case) by narrowing down the search for minimal linear trellises to computable characteristic sets of elementary trellises, which inspired much of the subsequent research.

Steady subsequent research (e.g. [3, 14, 18, 19, 30, 31, 32, 33, 34]) on the top of the seminal works has led to a fairly rich development of linear trellis theory.

However, some important foundational questions have not been addressed, and as a consequence the problem of classifying minimal linear trellis representations has been addressed only partially. This problem is important not only for theoretical purposes but also for iterative/LP decoding (as we point out in Subsection 5.5).

In this paper we build an algebraic framework that gives extra insight into linear trellises, answers such fundamental questions, and provides new algebraic tools that we apply to address the classification problem and more. Mathematically, we provide a thorough analysis of the monoid of linear trellises with trellis product.

1.1 Contents and contributions of the paper

Remark 1. All trellises in Sections 3, 4 and Appendix B will be reduced.

Section II (Preliminaries and basics on trellises): In this section we fix the notation/terminology and go over the necessary background with the intent to make our treatment as self-contained as possible. The reader versed in trellis theory may skim through this part, except for paying some attention to our notation for spans (given in Subsection 2.4).

Section III (Algebraic framework for linear trellises and new foundational insights): In this section we introduce the core machinery and results on the top of which the rest of the paper is built on. The structure of linear trellises is analyzed from an algebraic perspective and related to the trellis product operation. Our methodology consists in studying the label code $S(T)$ and showing how its properties correspond to properties of $T$. The key tools that we introduce are span subcodes of $S(T)$ (Subsection 3.1) and product bases (Subsection 3.3), which describe the structure of $S(T)$.

The foundational paper of Koetter/Vardy [23] on linear trellises lacks a framework that goes beyond proving the existence of elementary trellis factorizations. Our algebraic framework enables us to answer the following fundamental questions:

1. How is the trellis product operation encoded by the label code $S(T)$?

2. When are two linear trellises linearly isomorphic (i.e. equivalent)?

3. Is the linear structure of a linear trellis essentially unique? Equivalently, if two linear trellises are isomorphic are they linearly isomorphic too?
In answering 1) we provide a new proof of the Factorization Theorem, which is simpler and more explanatory. In answering 2) we provide a fundamental characterization of linear trellis isomorphy (Theorem 7) that tells us how span subcodes discriminate linear trellises. This is the first criterion of such type in the literature and it is crucial in proving the main following results of the paper, in particular for trellis classification purposes (see Section V). It also allows us to answer 3) positively. So (for possible implementation purposes) there is no need to search for the best linear structure of a linear trellis, and to look for linear isomorphisms of linear trellises is equivalent to look for isomorphisms (which also resolves the confusion in the literature where some authors use the first notion while others use the second one).

At the end of this section we point out some important overlooked aspects of group trellises by showing that the notion of elementary group trellis given in [23] is too strict for the Factorization Theorem to hold also in that case, and discuss a possible remedy.

Section IV (Elementary trellis factorizations of linear trellises): In this section we apply the work of the previous one to answer the following fundamental questions:

1. How does a linear trellis \( T \) determine its elementary trellis factorizations?

2. How can we compute (all the possible) elementary trellis factorization of \( T \)?

In particular, we show that the span distribution of any such factorization of \( T \) is unique (a striking fact which passed unnoticed in [23]) and is only determined by the underlying graph structure of \( T \). We also determine precise conditions under which \( T \) has a unique elementary trellis factorization, and give a formula for the number of such factorizations for linear trellises with no repeated spans.

Besides the theoretical value of the above questions, to be able to compute elementary trellis factorizations of \( T \) is important since such a factorization yields important data that we can use to check more easily whether certain properties hold or not for \( T \). Note that there are important classes of linear trellises that are not presented as elementary trellis products, e.g. BCJR trellises (see [18, 19, 30]). Also, our results yield a method to find out (efficiently) whether two products of elementary trellises are equal or not.

The work of this section is crucial for being able to classify and determine all the minimal linear trellises for a fixe code as these are precisely constructed as elementary trellis products.

Section V (Insights into the nonmergeable property, classification of nonmergeable linear trellises through multicyles, and complete classification/computation of minimal linear trellises): It is known that biproper nonconventional linear trellises may be mergeable, even in the one-to-one case (see the trellis depicted after Observation 9), while for conventional trellises this never happens ([36]). No explanation of this phenomenon has been given so far though. In this section we present first a new characterization of the nonmergeable property for one-to-one linear trellises which explains when and how such trellises fail to satisfy that property, and reconciles the conventional and nonconventional cases. This characterization amounts to a one-to-one correspondence between all long enough paths and their edge-label sequences.

We then use this result along with Theorem 7 to prove the striking fact that such trellises (in particular minimal linear trellises) are completely determined (and so classifiable) by their codes of edge-label sequences of closed paths of length greater than the trellis length (multicyles). In other words, a trellis \( T \) of length \( n \) does not only represent a single code \( C(T) \), but a sequence of codes \( C(T), C^2(T), C^3(T), \ldots \), of respective length \( n, 2n, 3n, \ldots \), and for nonmergeable linear one-to-one trellises \( C^2(T) \) completely determines \( T \).
Next, we show how to determine/compute all the minimal linear trellises with same graph structure for a given code \( C \) from the knowledge of its so-called characteristic matrix from [24]. In particular we give a formula for their number. Combining this with the results of [24] yields then a method to compute and so efficiently classify all the minimal linear trellises for a fixed code. Besides its theoretical value, we then discuss how to be able to do is very important for iterative (or LP) decoding applications as we show that two minimal linear trellises with the same graph structure for the same code can still yield drastically different pseudocodewords, a striking phenomenon which was never observed before.

We also apply the above results to deduce some very interesting results on self-duality of linear trellises, e.g. we show that a KV-trellis (and so in particular a minimal linear trellis) \( T \) is self-dual if and only if \( C^2(T) \) is.

**Section VI (Factorizations and isomorphisms of quasi-cyclic linear trellises, and extension of results on reduced linear trellises to nonreduced ones):** In this section we further demonstrate the power of the framework developed in Section 3 by giving some other interesting applications. We first define quasi-cyclic trellises and prove that in the linear case their elementary trellis factorizations and isomorphisms allow a quasi-cyclic version. We use then this result to show that our Theorems 8, 9 and 20 extend to nonreduced linear trellises as well. In order to achieve that we also prove some independent interesting results on the structure of trellises, in particular, we show how the nonreduced case and the reduced case can be linked to each other by means of trellis covers (which we define in the previous section in order to deal with duality questions), for which we provide some basic results. This link is very useful for extending results on reduced linear trellises to nonreduced ones. Note that nonreduced trellises naturally arise by taking duals of reduced trellises or wrapped fragments of quasi-cyclic trellises and it is thus worth to have results concerning them.

**Appendix A (Connectivity and graphical properties of linear trellises):** In this appendix we prove some important facts on connectivity and path properties of linear trellises that have been overlooked and have not appeared in the literature, and which we make use of in the paper. In particular, we prove that a linear trellis is connected as a directed graph if and only if it is as an undirected graph, and a characterization of the “reduced” property for connected linear trellises is given.

**Appendix B (Graphical characterization of span distributions of linear trellises):** In this appendix we present an alternative proof of the uniqueness of the span distribution of elementary factorizations of linear trellises and an alternative method to compute such distribution that are based on a graphical approach given by considering intersections of paths.

# 2 Preliminaries

## 2.1 Some general notation

The ring of integers modulo \( n \) is denoted by \( \mathbb{Z}_n \). Finite fields are denoted by \( \mathbb{F} \). The cardinality of a set \( S \) is denoted by \( |S| \). The support of \( v = (v_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} V_i \) (where the \( V_i \)'s are vector spaces) is \( \text{supp}(v) := \{ i \in \mathcal{I} | v_i \neq 0 \} \). The coordinate indices of products \( V_0 \times \ldots \times V_{n-1} \) (e.g. \( \mathbb{F}^n \)) will be seen as lying inside \( \mathbb{Z}_n \) in order to perform modular operations on them. For \( v \in V_0 \times \ldots \times V_{n-1} \) we denote by \( \sigma \) the left cyclic shift given by

\[
\sigma(v_0v_1\ldots v_{n-1}) := v_1\ldots v_{n-1}v_0
\]

We use angle brackets \( () \) to indicate the subspace generated by the elements of a vector space \( V \) given within the same brackets. A sum \( \sum_{i \in \mathcal{I}} V_i \) of a family of subspaces \( \{V_i\}_{i \in \mathcal{I}} \) of a vector space
V is a direct sum if given \( v^i \in V_i \) such that \( \sum_{i \in I} v^i = 0 \) then \( v^i = 0 \) for all \( i \). In that case we write it also as \( \oplus_{i \in I} V_i \).

### 2.2 Basics on trellises

A (tail-biting) trellis of length \( n \) over \( F \) is a directed graph \( T = (V, E) \) with \( F \)-labeled edges and a partition into vertex sets \( V = \sqcup_{i \in \mathbb{Z}_n} V_i(T) \) such that any edge starting in \( V_i(T) \) must end in \( V_{i+1}(T) \).

We assume that parallel edges (i.e. edges starting and ending at same vertices) must have different labels. The set of edges of \( T \) that start in \( V_i(T) \) and end in \( V_{i+1}(T) \) is thus a subset 

\[
E_i(T) \subseteq V_i(T) \times F \times V_{i+1}(T)
\]

We call it an edge set of \( T \). A linear trellis over \( F \) is a trellis \( T \) over \( F \) with an \( F \)-vector space structure on each \( V_i(T) \) such that each \( E_i(T) \) is a vector subspace of \( V_i(T) \times F \times V_{i+1}(T) \). We assume that trellises are trim, i.e. each vertex has an outgoing and an incoming edge. If the edge-labels of \( T \) are all equal then they are essentially irrelevant. In that case we call \( T \) also an unlabeled trellis and simply assume that all edge-labels are equal to 0. If \( |V_0(T)| = 1 \) then \( T \) is said to be conventional.

A subtrellis of \( T \) is a trim subgraph \( T' \subseteq T \). If in addition \( T \) is linear and \( V_i(T'), E_i(T') \) are vector subspaces respectively of \( V_i(T), E_i(T) \) for all \( i \), then \( T' \) is a linear subtrellis, and we write \( T' \leq T \). The cyclic shift of \( T \) by \( j \in \mathbb{Z} \) positions is the trellis \( \sigma^j(T) \) defined by

\[
V_i(\sigma^j(T)) := V_{i+j}(T) \\
E_i(\sigma^j(T)) := E_{i+j}(T)
\]

We write \( \sigma(T) \) for \( \sigma^1(T) \).

Trellises are usually visualized by diagrams like the one below for a linear trellis over the binary field \( F_2 \). Vertex sets are plotted vertically. The leftmost vertices are identified with the rightmost ones, and make up \( V_0(T) \). The edge-labels are represented by full lines for 1 and dashed lines for 0. The vector space structure of the \( V_i(T) \)'s is given by vertex-labels. All the examples in this paper will be for trellises over \( F_2 \), except in Subsection 3.8.

![Diagram of a trellis](image)

#### 2.2.1 Morphisms

A morphism of trellises \( f : T \to T' \) is a collection of maps \( f_i : V_i(T) \to V_i(T') \), \( i \in \mathbb{Z}_n \), such that

\[
v\alpha w \in E_i(T) \implies f_i(v) \alpha f_{i+1}(w) \in E_i(T')
\]

for all \( i \). If the \( f_i \)'s are bijective and

\[
v\alpha w \in E_i(T) \iff f_i(v) \alpha f_{i+1}(w) \in E_i(T')
\]
for all $i$, we say that $f$ is an isomorphism and that $T$ and $T'$ are isomorphic. In that case we write $T \sim T'$. Isomorphic trellises must be regarded as equal (since by renaming their vertices they are exactly the same trellis). If $T$, $T'$, and the $f_i$’s are all linear we say that $f$ is linear. If $f$ is a linear isomorphism then we say that $T$ and $T'$ are linearly isomorphic, and write $T \simeq T'$. Two trellises $T, T'$ are said to be structurally isomorphic if there exist bijective maps $f_i : V_i(T) \to V_i(T')$, $i \in \mathbb{Z}_n$, such that for all $v \in V_i(T)$, $w \in V_i+1(T)$, there are as many edges from $v$ to $w$ as from $f_i(v)$ to $f_{i+1}(w)$, that is, by forgetting all edge-labels $T$ and $T'$ have the same (ordered) graph structure.

### 2.2.2 Paths

A (directed) path $p$ of length $m$ of a trellis $T$ is an ordered sequence

$$v_0\alpha_0v_1\alpha_1 \ldots v_{m-1}\alpha_{m-1}v_m$$

such that $v_j\alpha_jv_{j+1}$ is an edge of $T$ for all $j = 0, \ldots, m - 1$. The path is closed if $v_0 = v_m$. We put

$$L(p) := \alpha_0 \ldots \alpha_{m-1}$$

$$v_j(p) := v_j$$

By deinterleaving vertices and edge-labels, paths of length $m$ starting in $V_i(T)$ can be seen as vectors of $\prod_{j=i}^{i+m} V_j(T) \times \mathbb{F}^n$. If $v, w \in V_i(T)$ we denote by $P(v, w)$ the set of paths in $T$ from $v$ to $w$ of same length as $T$. If $T$ is linear and all the $v_j$’s and $\alpha_j$’s are zero then $p$ is a zero path. The zero element of $V_i(T)$ will be denoted also by $0_i$.

### 2.2.3 Cycles and associated codes

Let $n$ be the length of $T$. A cycle $\lambda$ of $T$ is a closed path of $T$ of length $n$ starting in $V_0(T)$. Cycles can be written as sequences $v_0\alpha_0 \ldots v_{n-1}\alpha_{n-1}$ or (by deinterleaving) as pairs $(v, \alpha) \in \prod_{i \in \mathbb{Z}_n} V_i(T) \times \mathbb{F}^n$. The label code of $T$ is

$$S(T) := \{ \lambda | \lambda \text{ is a cycle of } T \}$$

The code represented by $T$ is

$$C(T) := L(S(T)) \subseteq \mathbb{F}^n$$

We say that $T$ is a trellis for $C(T)$. If $T$ is linear so are $S(T)$ and $C(T)$. If each vertex of $T$ belongs to some cycle we say that $T$ is almost reduced. If also each edge belongs to some cycle of $T$ we say that $T$ is reduced. A reduced trellis is identified by its label code. The trellis $T(S)$ spanned by a subset $S \subseteq S(T)$ is the (largest) subtrellis of $T$ covered by the cycles in $S$. The shift map

$$\sigma^j : S(T) \to S(\sigma^j(T))$$

is defined by $\sigma^j((v, \alpha)) := (\sigma^j(v), \sigma^j(\alpha))$.

**Remark 2.** While much of our terminology goes back to [23, 24], within behavioral system theory trellises can be seen also as dynamical systems (cf. [13, 14]). So it makes sense to think of: the indexing set $\mathbb{Z}_n$ as a (circular) time axis; indices as time indices; vertices as states. Also, in such terminology $S(T)$ is the “behavior” of $T$, $E_i(T)$ is a “constraint code”, and “almost reduced” becomes “state-trim”.
2.3 Classes of trellises

Let $T$ be a trellis. Then $T$ is:

- **connected** if for any vertices $v \neq w$ there exists a path from $v$ to $w$ (in Appendix A we show that a linear trellis is connected if and only if it is connected as an undirected graph)

- **one-to-one** if $L : \mathcal{S}(T) \to C(T)$ is injective

- **biproper** if different edges with same label never start or end in the same vertex

- **mergeable** if there exist vertices $v \neq w$ in $V_i(T)$ for some $i$ such that the trellis resulting from merging $v$ with $w$ represents the same code as $T$, otherwise nonmergeable

Clearly, a nonmergeable trellis is connected, and a biproper conventional trellis is one-to-one (the “conventional” hypothesis is necessary here).

2.3.1 Minimal trellises

Given trellises $T, T'$ of same length, we say that $T'$ is smaller than $T$ if $|V_i(T')| \leq |V_i(T)|$ for all $i$, with at least one strict inequality. If there exists no $T'$ smaller than $T$ such that $C(T') = C(T)$ then we say that $T$ is a minimal trellis (for $C(T)$). While this is the principal notion of trellis size/minimality, other notions can be given, especially by refining this one (see [24]). However all notions of minimality coincide for conventional linear trellises, as the following holds:

**Theorem 1** (Minimal Conventional Trellis [29, 27, 30]). A linear code has a unique minimal conventional trellis representation (up to isomorphism). Such trellis is linear and minimizes all $|V_i(T)|$ and $|E_i(T)|$ simultaneously.

We denote the minimal conventional trellis for a linear code $C$ by $T^*(C)$. For nonconventional trellises the situation is not as easy, and we will go back to that in Section 5. For much on minimal conventional trellises see the comprehensive survey [30], while for the nonconventional case see [2, 5, 24, 31, 32, 34].

**Remark 3.** If a linear trellis $T$ is minimal amongst all linear trellises is it so also amongst all trellises? By Theorem 1 if $T$ is conventional then the answer is yes. But in general the answer is not clear (a nonconventional minimal trellis for a linear code may be not linear, see [27]). This question is overlooked in the literature as only “linear minimality” is really dealt with. Indeed there is no systematic way to construct/control nonlinear trellis representations, while linearity makes things feasible. We will stay in this better world, so by a minimal linear trellis for $C$ we will always mean a linear trellis which is minimal amongst all linear trellises for $C$. Note also that the usual assumption in the literature that minimal linear trellises be reduced is redundant (see Theorem 15).

2.3.2 Self-dual trellises

We say that a linear trellis $T$ is self-dual if $T \sim T^\perp$. Here $T^\perp$ is the dual trellis of $T$ as defined in [12], which satisfies $C(T^\perp) = (C(T))^\perp$. For $\mathbb{F} = \mathbb{F}_2$, $T^\perp$ is given by

$$V_i(T^\perp) := V_i(T) \equiv \mathbb{F}_2^n,$$

$$E_i(T^\perp) := (E_i(T))^\perp.$$
where the dual of $E_i(T) \leq \mathbb{F}_2^{r_i+1+r_i+1}$ is with respect to the standard scalar product. The definitions of dual trellis given in [30] and [21] coincide with the above one when $T$ is minimal. Note that $T^\perp$ may be not trim, even if $T$ is reduced. However, $T^\perp$ is minimal (and so reduced) if and only if $T$ is. See also [19] for more on trellis dualization.

2.4 Spans

Let $n \geq 0$ be fixed. For $a, b \in \mathbb{Z}_n$ we put

$$[a, b] := \{a, a + 1, \ldots, a + l\} \subseteq \mathbb{Z}_n$$

where $l = \min\{l \geq 0 \mid a + l \equiv b \mod n\}$. This is a (circular) interval of $\mathbb{Z}_n$. We also put $(a, b) := [a, b] \setminus \{a\}$. Let now $\prod_{i\in \mathbb{Z}_n} V_i$ be a product of vector spaces which we think of as alphabets. Given $a \in \mathbb{Z}_n$, $0 \leq l \leq n - 1$, and $v \in \prod_{i \in \mathbb{Z}_n} V_i$, we say that the pair $(a, l)$ is a span of $v$ of length $l$ if $\text{supp}(v) \subseteq [a, a + l]$. The starting and ending point of $(a, l)$ are respectively $a$ and $a + l$. We also say that $\emptyset$ is a span of length $-1$ of $0 \in V$ and dually that $\mathbb{Z}_n$ is a span of length $n$ of all $v \in V$. No starting and ending point are associated to $\emptyset$ and $\mathbb{Z}_n$, so we also say that these spans are degenerate. Nevertheless we will use respectively the notation $(a, -1)$, $(a, n)$ for the spans $\emptyset$, $\mathbb{Z}_n$ too. A generic span will be also simply denoted by the letter $s$ when we do not need to specify its starting point and length. If no alphabets are specified when talking of spans of vectors in $\mathbb{F}^n$ we tacitly assume that they are all one-dimensional.

Remark 4. In the classical terminology one calls $[a, b]$ a span of $v$ if $\text{supp}(v) \subseteq [a, b]$. However in tail-biting trellis theory we want to distinguish between spans with different starting and ending points, and so for $c \neq d$ we want $[c, c - 1]$ and $[d, d - 1]$ to be different spans, which clashes with $[c, c - 1] = [d, d - 1]$. Similarly, in [24] and subsequent works $(a, b)$ is called a span of $v$ if $\text{supp}(v) \subseteq [a, b]$, leading for example to $(0, 0)$ being a span of 100 and $(1, 1)$ being not, while $(1, 1) = (0, 0)$. Our terminology avoids such formal abuses while remaining compatible with the literature. In fact a span should still be thought of as an interval. The notation $(a, l)$ rather serves to parametrize spans and to discriminate between them. It is also convenient for proofs by induction on span length.

Spans have a natural partial order: we put

$$(a_1, l_1) \leq (a_2, l_2)$$

if and only if $l_1 \leq l_2 < n - 1$ and $[a_1, a_1 + l_1] \subseteq [a_2, a_2 + l_2]$, or $l_2 = n - 1$ and $(a_1, a_1 + l_1) \subseteq (a_2, a_2 + l_2)$, or $l_1 = -1$, or $l_2 = n$. This yields the following Hasse diagram.

\[ \text{Diagram here} \]
If \((a_1,l_1) \leq (a_2,l_2)\) we also say that \((a_1,l_1)\) is contained in \((a_2,l_2)\). A span \((a,l)\) is said to be conventional if \((a,l) \leq (0,n-1)\).

2.5 Elementary trellises

Let \((a,l)\) be a span of \(\alpha \in \mathbb{F}^n\), with \(0 \leq l \leq n\). We denote by \(\alpha|(a,l)\) the elementary trellis for \(\alpha\) of span \((a,l)\). This is defined as follows:

- \(V_i(\alpha|(a,l)) := 0\) for all \(i \in \mathbb{Z}_n \setminus (a,a+l]\)
- \(V_i(\alpha|(a,l)) := \mathbb{F}\) for all \(i \in (a,a+l]\)
- \(E_i(\alpha|(a,l)) := \langle (v_i, \alpha_i, v_{i+1}) \rangle\) for all \(i\), where \(v_i := 0\) if \(i \in \mathbb{Z}_n \setminus (a,a+l]\) and \(v_i := 1\) otherwise

Clearly \(\alpha|(a,l)\) is: reduced and linear; minimal if and only if \((a,l)\) is a minimal span of \(\alpha\); one-to-one if and only if \(\alpha \neq 0\). Also, a linear trellis \(T\) is (isomorphic to) an elementary trellis if and only if \(\dim \mathbb{S}(T) = 1\). This justifies the adjective “elementary”.

Note that elementary trellis graph structures correspond to spans, thanks to our terminology. In fact the graph structure of \(\alpha|(a,l)\) does not depend on \(\alpha\).

**Example 1.** All the possible elementary trellises for 10 are given by

\[
\begin{align*}
10|(0,1) & = \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ
\end{array}
\end{array}  \\
10|(1,1) & = \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ
\end{array}
\end{array} \\
10|(0,0) & = \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \\
\circ
\end{array}
\end{array}  \\
10|\{2\} & = \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ
\end{array}
\end{array}
\end{align*}
\]

Two more elementary trellis graph structures of length 2 are missing, namely those corresponding to the spans \((1,0)\) and \(\emptyset\) (which are not spans of 10). These can be given by

\[
\begin{align*}
01|(1,0) & = \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ
\end{array}
\end{array}  \\
00|\emptyset & = \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \quad \circ
\end{array}
\end{array}
\end{align*}
\]

2.6 Trellis product

Let \(T\) and \(T'\) be trellises of same length. The trellis product \(T \otimes T'\) is given by

\[
\begin{align*}
V_i(T \otimes T') & := V_i(T) \times V_i(T') \\
E_i(T \otimes T') & := \{vv'(\alpha + \alpha')ww'|(vw, v'\alpha'w') \in E_i(T) \times E_i(T')\} 
\end{align*}
\]

for all \(i\). Below is an easy example:
The trellis product has the following basic properties:

1. \(T \otimes T' \sim T' \otimes T\) (commutativity)

2. \(T \otimes (T' \otimes T'') \sim (T \otimes T') \otimes T''\) (associativity)

3. \(0 \otimes T \sim T\), where 0 is the zero trellis given by \(V_i(T) := 0\) and \(E_i(T) := 0\) for all \(i\) (identity element)

4. \(T' \sim T'' \implies T \otimes T' \sim T \otimes T''\)

5. if \(T\) and \(T'\) are linear/reduced/one-to-one so is \(T \otimes T'\)

6. we can replace \(\sim\) with \(\simeq\) in 1), 2), 3), and 4) when the trellises involved are linear

It also satisfies the crucial identity

\[C(T \otimes T') = C(T) + C(T')\]

So one can construct trellises for linear codes by taking products of elementary trellises for their generators. Remarkably, the Factorization Theorem from \([23]\) (which we reprove as Corollary \([1]\)) says that this is exhaustive, i.e. any reduced linear trellis factors into elementary trellises. Finally, we call

\[\otimes_{i=1}^r \alpha_i|(a_i, l_i)\]

an elementary trellis factorization of \(T\) if \(T \sim \otimes_{i=1}^r \alpha_i|(a_i, l_i)\).

**Remark 5.** We will always assume that in a product of elementary trellises at most one elementary trellis with span \((a, 0)\) appears for each \(a \in \mathbb{Z}_n\), in order to avoid the degeneracy \(\alpha|(a, 0) = \alpha|(a, 0) \otimes \alpha|(a, 0)\) which would require cumbersome distinctions in the statements of our theorems.

## 3 An algebraic framework for linear trellises

### 3.1 Spans of cycles and span subcodes

We introduce here our main tool: span subcodes of \(S(T)\). Let \(T\) be a linear trellis of length \(n\). Given \(a \in \mathbb{Z}_n\), \(0 \leq l \leq n - 1\), and \(\lambda = (v, \alpha) \in S(T)\), we say that \((a, l)\) is a span starting at \(a\) of length \(l\) of \(\lambda\) if \((a + 1, l - 1)\) is a span of \(v \in \prod_{i \in \mathbb{Z}_n} V_i(T)\) (with respect to the alphabets \(V_i(T)\)) and \((a, l)\) is a span of \(\alpha \in F^n\), i.e. if \(\text{supp}(v) \subseteq (a, a + l)\) and \(\text{supp}(\alpha) \subseteq [a, a + l]\). Like in \([2, 4]\) we say as well that \(\emptyset\) is a span of length \(-1\) of \(0 \in S(T)\) and \(\mathbb{Z}_n\) is a span of length \(n\) of all \(\lambda \in S(T)\), and write also \((a, -1)\) for \(\emptyset\) and \((a, n)\) for \(\mathbb{Z}_n\). The linear subcode of \(S(T)\) defined as

\[S_{(a, l)}(T) := \{\lambda \in S(T) | (a, l)\ is \ a \ span \ of \ \lambda\}\]

is then called the \((a, l)\)-span subcode of \(S(T)\) (or simply of \(T\)).

Considering the partial order for spans defined in Subsection \([2, 4]\) we have that

\[(a_1, l_1) \leq (a_2, l_2) \implies S_{(a_1, l_1)}(T) \subseteq S_{(a_2, l_2)}(T)\]
and so the Hasse diagram of spans yields a diagram giving containments amongst the span subcodes $S_{(a,l)}(T)$. In particular $S_0(T) = 0 \leq S_{(a,l)}(T) \leq S(T) = S_{\mathbb{Z}_n}(T)$ for all $(a,l)$. For elementary trellises the picture of span subcodes is simple:

$$S_{(a',t')}(\alpha|(a,l)) = \begin{cases} S(\alpha|(a,l)) & \text{if } (a,l) \leq (a',t') \\ 0 & \text{otherwise} \end{cases}$$

Here is a less trivial example:

**Example 2.** Consider the below linear trellis with label code $S(T) = \langle (000, 100), (010, 100), (001, 011) \rangle$.

![Diagram of Example 2](image)

This yields the diagram:

![Diagram of Example 2](image)

Note that $S_{(a,0)}(T)$ has either dimension 0 or 1, since between adjacent vertices there exists either 1 edge (labelled with 0) or $|F|$ parallel edges (whose labels span $F$).

We call the minimum length of spans of $\lambda$ the **span length** of $\lambda$ and denote it by $\ell(\lambda)$. The subcode generated by all cycles of span length less or equal than a given $l$ is denoted by

$$S_l(T) := \sum_{a \in \mathbb{Z}_n, l' \leq l} S_{(a,l')}(T)$$

We also put

$$S_{<}(a,l) := \sum_{(a',l') \leq (a,l)} S_{(a',l')}(T)$$

If $\lambda$ has a minimum span we will denote it by $[\lambda]$ and refer to it simply as the **span** of $\lambda$. If $\lambda$ has not a minimum span then it is easy to see that $\lambda \in S_{\ell(\lambda) - 1}(T)$.

**Example 3.** Let $T$ be the unlabeled linear trellis

![Diagram of Example 3](image)
Then \([0\,1\,1\,0,\,0]\) = (0, 3). On the other hand, \([1\,0\,1\,0,\,0]\) has two minimal incomparable spans, (1, 4) and (4, 3). We see that \([1\,0\,1\,0,\,0] = (0\,0\,1\,0,\,0) + (1\,0\,0\,0,\,0) \in S_{(1,\,1)}(T) + S_{(4,\,1)}(T),\) while \(\ell((1\,0\,1\,0,\,0)) = 3.\)

Finally, for each \((a,l)\) we put

\[
C_{(a,l)}(T) := L(S_{(a,l)}(T))
\]

Obviously \(C_{(a,l)}(T) \subseteq \{\alpha \in C(T)\mid (a,l)\) is a span of \(\alpha\}\), but in general the containment is strict.

### 3.2 Label code maps as trellis maps

Let \(T,\, T'\) be linear trellises. Any linear morphism \(f : T \to T'\) induces a linear map \(S(f) : S(T) \to S(T')\) given by

\[
S(f)(v_0a_0 \ldots v_{n-1}a_{n-1}) = f_0(v_0)a_0 \ldots f_{n-1}(v_{n-1})a_{n-1}
\]

We say that a linear map \(F : S(T) \to S(T')\) is a trellis map if \(F = S(f)\) for some linear morphism \(f : T \to T'\). It is easily seen that \(F\) is a trellis map if and only if it preserves edge-labels and \(\nu_i(\lambda) = \nu_i(\lambda') \Rightarrow \nu_i(F(\lambda)) = \nu_i(F(\lambda'))\) for all \(i \in \mathbb{Z}_n\) and all \(\lambda,\lambda' \in S(T)\). In fact if those conditions are satisfied then \(F = S(f)\) with \(f_i(v) := \nu_i(F(\lambda))\) for all \(v \in V_i(T)\) and any \(\lambda\) such that \(\nu_i(\lambda) = v\). By linearity we thus have also that \(F\) is a trellis map if and only if it preserves edge-labels and

\[
F(S_{(a,n-1)}(T)) \subseteq S_{(a,n-1)}(T')
\]

for all \(a \in \mathbb{Z}_n\) (which implies that \(F(S_{(a,l)}(T)) \subseteq S_{(a,l)}(T')\) for all \((a,l)\)).

Note that \(S(f \circ g) = S(f) \circ S(g)\) (the composition of trellis morphisms being the obvious one) and \(f = g\) if \(S(f) = S(g)\). So if \(F = S(f)\) and \(F\) has an inverse which is also a trellis map then \(f\) is a linear isomorphism (and vice versa). We thus conclude that:

**Observation 1.** \(T \simeq T'\) if and only if there exists a linear isomorphism \(F : S(T) \to S(T')\) that preserves edge-labels and such that

\[
F(S_{(a,n-1)}(T)) = S_{(a,n-1)}(T')
\]

for all \(a \in \mathbb{Z}_n\) (in which case \(F(S_{(a,l)}(T)) = S_{(a,l)}(T')\) for all \((a,l)\)).

This observation allows us to prove that two linear trellises are isomorphic by focusing locally on the span subcodes. See our core Theorem 7 for this crucial approach in its proof.

Note that the existence of a linear isomorphism between \(S(T)\) and \(S(T')\) which is a trellis map only in one direction is not sufficient for \(T\) and \(T'\) to be isomorphic. For example, if \(T\) and \(T'\) are the two nonisomorphic trellises below and \(f : T \to T'\) is given by \(f_0 = \text{Id},\, f_1 = \text{Id},\, f_2(01) = f_2(11) = 1\), then \(S(f)\) is a linear isomorphism.

\[
T = \begin{array}{c}
0 & 11 & 11 \\
10 & 00 & 00 \\
11 & 01 & 10
\end{array}
\]

\[
T' = \begin{array}{c}
0 & 11 & 11 \\
10 & 00 & 00 \\
11 & 01 & 10
\end{array}
\]
3.3 Algebraic structure of $S(T)$: product bases

We now isolate the key properties of the label code $S(T)$ (and its family of subcodes $\{S_{(a,l)}(T)\}$) related to how trellis factorizations are encoded in its structure.

Consider two linear trellises $T$ and $T'$. Naturally, $T$ is identified with the linear subtrellis $T \otimes T' \subseteq T \times T'$ via the (injective) linear morphism $T \to T \otimes T'$ given by

$$V_i(T) \ni v \mapsto (v,0) \in V_i(T) \times V_i(T').$$

Consequently each span subcode $S_{(a,l)}(T)$ is identified with a subcode of $S_{(a,l)}(T \otimes T')$ by the map $(v,\alpha) \mapsto ((v,0),\alpha)$. With those identifications in mind we have:

**Observation 2.** $S_{(a,l)}(T \otimes T') = S_{(a,l)}(T) + S_{(a,l)}(T')$ for each span $(a,l)$. Moreover, if for all $b \in [a,a+l]$ there are no parallel edges at time index $b$ in $T$ and $T'$ simultaneously then

$$S_{(a,l)}(T \otimes T') = S_{(a,l)}(T) \oplus S_{(a,l)}(T').$$

**Proof.** The equality $S_{(a,l)}(T \otimes T') = S_{(a,l)}(T) + S_{(a,l)}(T')$ is immediate. It remains to prove the second statement, i.e. that $S_{(a,l)}(T) \cap S_{(a,l)}(T') = 0$ under the given hypothesis. Assume that we have

$$((v,0),\alpha) = ((0,v'),\alpha') \in S_{(a,l)}(T) \cap S_{(a,l)}(T')$$

Then $v = 0, v' = 0$, and $\alpha = \alpha'$. This implies that for all $b \in \text{supp}(\alpha) \subseteq [a,a+l]$ there are parallel edges at time index $b$ both in $T$ and $T'$. So $\alpha = 0$, and we are done.

**Remark 6.** By obvious inductive arguments the above extends to any finite linear trellis product $\otimes_{i=1}^r T_i$. That is, identifying naturally $S_{(a,l)}(T_i)$ as a subspace of $S_{(a,l)}(\otimes_{i=1}^r T_i)$ we can write $S_{(a,l)}(\otimes_{i=1}^r T_i) = \sum_{i=1}^r S_{(a,l)}(T_i)$, where the sum is direct if $\sum_{i=1}^r \dim S_{(b,0)}(T_i) \leq 1$ (i.e. at most one $T_i$ has parallel edges at $b$) for all $(b,0) \leq (a,l)$.

We can use Observation 2 to analyze the structure of the label code of a product of elementary trellises. Recall that if $T = \alpha|(a,l)$ then $S_{(a',l')}(T) = S(T)$ if $(a',l') \geq (a,l)$ and $S_{(a',l')}(T) = 0$ otherwise. It then follows that:

**Observation 3.** Let $T = \otimes_{i=1}^r \alpha^i|(a_i,l_i)$. For each $i$ let $\lambda^i$ be a generator of $S(\alpha^i|(a_i,l_i))$, which we identify with a subspace of $S(T)$ as usual. Then $\{\lambda^i\}_{i \leq (a,l), i \leq (a,l)}$ is a basis of $S_{(a,l)}(T)$.

Thus the label code $S(\otimes_{i=1}^r T_i)$ of an elementary trellis product has a basis $B$ such that $B \cap S_{(a,l)}(\otimes_{i=1}^r T_i)$ generates $S_{(a,l)}(\otimes_{i=1}^r T_i)$, for each span $(a,l)$. We are now going to show that any linear trellis label code possesses such a basis. In order to do so we first prove a more fundamental property of label codes.

**Theorem 2.** Let $T$ be a linear trellis, and let $\{(a_i,l_i)\}_{i \in I}$ be a family of spans. If $(a,l)$ is a span such that $(a,l) \notin (a_i,l_i)$ for all $i \in I$ then

$$S_{(a,l)}(T) \cap \sum_{i \in I} S_{(a,l_i)}(T) \leq S_{(a,l)}(T).$$

**Proof.** Let $(v,\alpha) \in S_{(a,l)}(T) \cap \sum_{i \in I} S_{(a,l_i)}(T)$. If $l = -1$ or $l = n$, then the statement is trivially true. If $l = 0$, then it is clear that $(v,\alpha) = (0,0)$, since any cycle in $S_{(a,l)}(T)$ yields a codeword whose support does not contain $a$. We, assume now $n > l \geq 1$. If $v_{a+1} = 0$ then the statement is
always true as in that case \((v, \alpha)\) is clearly a sum of a cycle in \(S_{(a,0)}(T)\) and a cycle in \(S_{(a+1,l-1)}(T)\), namely, \((0, \alpha') \in S_{(a,0)}(T)\) where \(\alpha_j' = \delta_{ja} \alpha_a\), and

\[(v, \alpha) - (0, \alpha') \in S_{(a+1,l-1)}(T)\]

So we can also assume that \(v_{a+1} \neq 0\). Now, write \((v, \alpha) = \sum_{i \in I} (v^i, \alpha^i)\) for some \((v^i, \alpha^i) \in S_{(a,l)}(T)\). Let

\[I' := \{i \in I | v^i_{a+1} \neq 0\}\]

Note that \(I' \neq \emptyset\), as \(v_{a+1} \neq 0\). If \(i \in I'\) then obviously \(a + 1 \in (a_i, a_i + l_i]\), and so we must have \(\langle a, a_i + l_i \rangle \subseteq \langle a + l, a_i + l_i \rangle\), because otherwise \(\langle a, a + l \rangle \subseteq \langle a_i, a_i + l_i \rangle\), which would contradict the hypothesis. We deduce that \(l \geq 2\) and so that for each \(i \in I'\) we can construct a path \(p^i\) in \(T\) of length \(l - 1\) from \(v^i_{a+1}\) to \(0_{a+l} \in V_{a+l}\) given by concatenating the path

\[v^i_{a+1} \alpha^i_{a+1} \ldots v^i_{a+l} \alpha^i_{a+l} 0_{a+l+1}\]

with the zero path from \(0_{a+l+1}\) to \(0_{a+l}\). By linearity of \(T\) we can add all those paths to get a path \(p = \sum_{i \in I'} p^i\) from \(v_{a+1}\) to \(0_{a+l}\). Concatenating the edge \(0_{a+l} a v_{a+1}\) with \(p\), and then extending on right and left by the zero path, we get a cycle \((v', \alpha') \in S_{(a+1,l-1)}(T)\) such that \((v, \alpha) - (v', \alpha') \in S_{(a+1,l-1)}(T)\). Hence we are done.

**Theorem 3.** Let \(T\) be a linear trellis. For each span \((a,l)\) lift an arbitrary basis of

\[S_{(a,l)}(T)/S_{<a,l)}(T)\]

to a subset \(B_{(a,l)} \subseteq S_{(a,l)}(T)\). Then \(B := \bigcup_{(a,l)} B_{(a,l)}\) is a basis of \(S(T)\) such that \(B \cap S_{(a,l)}(T)\) generates \(S_{(a,l)}(T)\) for all \((a,l)\).

**Proof.** We will prove by induction on \(l = -1, \ldots, n-1\), that \(B_l := \bigcup_{(a,l')} B_{(a,l')}\) is a basis of \(S_l(T)\) such that \(B_l \cap S_{(a,l)}(T)\) generates \(S_{(a,l)}(T)\) for all \((a,l')\) with \(l' \leq l\). Then clearly \(\bigcup_{(a,l)} B_{(a,l)}\) will satisfy our statement. If \(l = -1\) there is nothing to prove. Assume now \(B_l\) satisfies the above property for some \(-1 \leq l < n - 1\). We claim that \(B_{l+1}\) is a basis of \(S_{l+1}(T)\). So, suppose

\[\sum_{\lambda \in B_{l+1}} x_{\lambda} \lambda = 0\]

for some coefficients \(x_{\lambda}\) in \(F\). Then for \(a \in \mathbb{Z}_n\) we have that

\[\sum_{\lambda \in B_{l+1}} x_{\lambda} \lambda \in \left(S_{(a,l+1)}(T) \cap \left(\sum_{\alpha' \neq a} S_{(a',l+1)}(T) + S_l(T)\right)\right)\]

\[\subseteq S_{<a,l+1)}(T)\]

where the containment follows from Theorem 2. Hence \(x_{\lambda} = 0\) for all \(\lambda \in B_{a,l+1}\) and all \(a \in \mathbb{Z}_n\). So \(x_{\lambda} = 0\) for all \(\lambda \in B_l\) too, since \(B_l\) is a basis of \(S_l(T)\), and our claim is proven. Now, clearly \(B_{l+1} \cap S_{(a,l')}(T)\) generates \(S_{(a,l')}(T)\) for all \((a,l')\) with \(l' \leq l\), as \(B_l \cap S_{(a,l')}(T)\) does. For the same reason \(B_l \cap S_{<a,l+1)}(T)\) generates \(S_{<a,l+1)}(T)\). So by definition of \(B_{a,l+1}\) we get that

\[(B_l \cup B_{a,l+1}) \cap S_{(a,l+1)}(T)\]

generates \(S_{(a,l+1)}(T)\). This concludes our proof.

**Remark 7.** Given a pair \((V, \{V_i\}_{i \in I})\) consisting of a vector space \(V\) and a family of subspaces of \(V\), in general is not true that there exists a basis \(B\) of \(V\) such that \(B \cap V_i\) generates \(V_i\) for all \(i \in I\). A counterexample is given by \(F^5\) with the family of subspaces \(\{\langle e^0 \rangle, \langle e^1 \rangle, \langle e^2 \rangle, \langle e^3 \rangle, \langle e^0, e^1, e^4 \rangle, \langle e^1, e^2, e^0 + e^4 \rangle, \langle e^2, e^3, e^1 + e^5 \rangle, \langle e^3, e^0, e^2 + e^4 \rangle\}\) (where \(e^i\) is the \(i\)-th canonical basis element). Only special pairs \((V, \{V_i\}_{i \in I})\) have such bases.
A basis of \( S(T) \) satisfying the property of Theorem 3 will be called a product basis of \( S(T) \) (or simply of \( T \)). This denomination is justified by the fact that product bases correspond to factorizations of \( T \) in terms of elementary trellises, as we will see in the next subsection.

The following observation gives some useful properties of product bases.

**Observation 4.** Let \( B \) be a product basis of \( T \). Then:
1. Every \( \lambda \in B \) has a minimum span.
2. \( \sum_{i \in I} S(a_i, l_i)(T) = \langle \lambda \in B \mid |\lambda| \leq (a_i, l_i) \text{ for some } i \in I \rangle \)
3. \( S_{< (a, l)}(T) = \langle \lambda \in B \mid |\lambda| < (a, l) \rangle \)
4. \( \{ \lambda + S_{< (a, l)}(T) \mid \lambda \in B, |\lambda| = (a, l) \} \) is a basis of \( S_{(a, l)}(T)/S_{< (a, l)}(T) \)
5. A cycle \( \lambda \in S(T) \) belongs to some product basis of \( T \) if and only if \( \lambda \in S_{(a, l)}(T) \setminus S_{< (a, l)}(T) \) for some \( (a, l) \)

Proof. Let \( \lambda \in B \) be a product basis element, and let \( (a, l) \) be a span of minimum length \( l = \ell(\lambda) \) of \( \lambda \). Assume that \( \lambda \) does not have a minimum span. Then \( \lambda \in S_{< (a, l)}(T) \). Since \( B \) is a product basis it follows that

\[
B \cap \left( \bigcup_{(a', l') < (a, l)} S_{(a', l')}(T) \right)
\]

is a basis of \( S_{< (a, l)}(T) \), so that \( \lambda \in S_{(a', l')}(T) \) for some \( (a', l') < (a, l) \), which contradicts our assumption on \( l \). As for the identities: identity 2) follows easily from the definition of product basis; identity 3) is an instance of 2); identity 4) is a plain consequence of 2) and 3). As for 5) the “if” part follows from Theorem 3 while the “only if” part follows from 4).

### 3.4 Product bases and elementary trellis factorizations

In Observation 3 we have observed that an elementary trellis factorization translates into a product basis. We now show that the converse is true too.

**Theorem 4.** \( T \) has a product basis \( \{(v^i, \alpha^i)\}_{i=1,...,r} \) with \( [v^i, \alpha^i] = (a_i, l_i) \) if and only if \( T \simeq \otimes_{i=1}^r \alpha^i|(a_i, l_i) \)

Proof. Let \( T' = \otimes_{i=1}^r \alpha^i|(a_i, l_i) \). For each \( i = 1, \ldots, r \), take a generator \( \lambda^i \) of \( S(\alpha^i|(a_i, l_i)) \) such that \( L(\lambda^i) = \alpha^i \). Now, assume \( \{(v^i, \alpha^i)\}_{i=1,...,r} \) is a product basis of \( T \) with \( [v^i, \alpha^i] = (a_i, l_i) \). In particular, \( \{(v^i, \alpha^i)\}_{i=1,...,r} \) is a basis of \( S_{(a_i, l_i)}(T) \) for all spans \( (a, l) \). Similarly, by Observation 3 \( \{\lambda^i\}_{i=1,...,r} \) is a basis of \( S_{(a, l)}(T') \), for all \( (a, l) \). So, by Observation 1 the linear isomorphism \( F : S(T') \rightarrow S(T) \) that sends \( \lambda^i \) to \( (v^i, \alpha^i) \) yields a linear isomorphism of \( T' \) and \( T \). The “if” part is clear, since a linear isomorphism of trellises sends a product basis to a product basis, and \( \{\lambda^i\}_{i=1,...,r} \) is a product basis of \( T' \) by Observation 3

**Example 4.** Let \( T \) be the linear trellis with \( S(T) = \{(010, 000), (001, 011)\} \). The only product basis of \( T \) is \( \{(010, 000), (001, 011)\} \). This corresponds to the only elementary trellis factorization, depicted just below. The basis \( \{(010, 000), (011, 011)\} \) is not a product basis as the second product below is not \( T \).
With such dictionary Theorems 2 and 3 turn out to yield a new and compact proof of the acclaimed Factorization Theorem [23] of Koetter/Vardy.

Corollary 1 (Factorization Theorem). Any linear trellis is linearly isomorphic to a product of elementary trellises.

Remark 8. Note that the Factorization Theorem is actually stated in a weaker form in [23] since it “only” says that any linear trellis $T$ is isomorphic to a product of elementary trellises. No linear isomorphism is mentioned in [23], as no importance is given to the particular linear structure of the spaces $V_i(T)$ (see the preamble of Section 6 therein). However, a careful read of [23] reveals that linear isomorphy is a plain consequence of what is proven therein. In fact, it can be easily seen that a representation matrix $G$ in product form of $T$ (Definition 6.1 in [23]) yields a linear isomorphism between $T$ and the product of elementary trellises corresponding to the rows of $G$, and existence of such matrices is precisely what is proven in [23] (Theorem 6.2). Actually we will see that isomorphic linear trellises must be also linearly isomorphic (Theorem 8), but this can be proven only by using our approach.

Despite the rather intricate proof of the Factorization Theorem given in [23], no other proof has appeared in the literature since then (note though that recently, following our new proof, which we first had announced and illustrated in [9], Gluesing-Luerrsen came up with one further proof — private communication). Our new proof is simpler (for example, we do not need elaborate transformations of vertex-labels and investigations on the connection properties of vertices as in [23]) and shorter (the core part of it being contained in Theorems 2 and 3). It is also more explanatory. In fact, it came out as a byproduct of our studies on the structure of linear trellises via label codes.

Span subcodes and product bases yield an algebraic point of view and methodology which is technically powerful and gives more comprehensive understanding. Our framework allows to go beyond “just” proving that $T$ admits one factorization into elementary trellises. Indeed by our approach in the next sections we will be able to give a complete description of trellis factorizations (while the proof of [23] does not shed light on what are the possible factorizations of $T$ and how these are encoded in $S(T)$) and consequently to obtain also results concerning the classifications of trellises.

3.5 Structure of $S(T)$: correspondence with trellis factorizations and atomic cycles

The following theorem shows that there is a correspondence between decompositions $S(T) = \bigoplus_{i=1}^{r} S^i$ satisfying $S_{(a,l)}(T) = \bigoplus_{i=1}^{r} (S^i \cap S_{(a,l)}(T))$ for all $(a,l)$ and factorizations $T \simeq \otimes_{i=1}^{r} T_i$, and so it completes the picture on how the label code encodes the trellis product. To simplify notation, for $S \leq S(T)$ we will put

$$S_{(a,l)} := S \cap S_{(a,l)}(T)$$

Theorem 5. If $S^1,\ldots,S^r \leq S(T)$ satisfy

$$S_{(a,l)}(T) = \bigoplus_{i=1}^{r} S^i_{(a,l)}$$
for all \((a,l)\) then \(S^i = S(T(S^i))\) for all \(i\) and there exists a linear isomorphism

\[
f : T \to \bigotimes_{i=1}^r T(S^i)
\]

such that \(f(T(S^i)) = 0 \otimes \ldots \otimes 0 \otimes T(S^i) \otimes 0 \otimes \ldots \otimes 0\) for all \(i\).

Vice versa, given \(T_1, \ldots, T_r\) with no simultaneous parallel edges (i.e., \(\sum_{i=1}^r \dim S_{(a,0)}(T_i) \leq 1\) for all \(a \in \mathbb{Z}_n\)), if there exists a linear isomorphism \(f : T \to \bigotimes_{i=1}^r T_i\) then for all \((a,l)\) we have

\[
S_{(a,l)}(T) = \bigoplus_{i=1}^r S_{(a,l)}(f^{-1}(0 \otimes \ldots \otimes 0 \otimes T_i \otimes 0 \otimes \ldots \otimes 0))
\]

**Proof.** The “vice versa” part is an immediate consequence of Observation 2. So let us go the first part. By induction it suffices to consider the case of two subcodes \(S', S'' \leq S(T)\). Let \(\lambda \in S(T(S'))\). By hypothesis \(\lambda \in S' \oplus S''\). Let \(-1 \leq l \leq n\) be the least \(l\) such that \(\lambda \in S' \oplus S''(a,l)\) for some \((a,l)\). If \(l = -1\) then \(S''(a,l) = 0\), and we are done. So, assume \(l \geq 0\). Write \(\lambda = \lambda' + \lambda''\) for some \(\lambda' \in S'\), \(\lambda'' \in S''(a,l)\). By assumption \((a,l)\) is a minimal span of \(\lambda''\). Since \(\lambda \in S(T(S'))\), there exists a cycle \(\tilde{\lambda} \in S'\) which agrees with \(\lambda\) in the \(a\)-th edge. Then

\[
\lambda - \tilde{\lambda} \in S_{(a+1,n-2)}(T) = S'_{(a+1,n-2)} \oplus S''_{(a+1,n-2)}
\]

So \(\lambda' - \tilde{\lambda} + \lambda'' = \lambda'' + \lambda'''\) for some \(\lambda''' \in S'_{(a+1,n-2)}\), \(\lambda''' \in S''_{(a+1,n-2)}\). But \(S' \cap S'' = 0\) by hypothesis, hence \(\lambda'' = \lambda''' \in S''_{(a+1,n-2)}\), contradicting the minimality of \((a,l)\) for \(\lambda''\). Hence \(l = -1\) in the first place, and so \(\lambda \in S'\). Since \(S' \subseteq S(T(S'))\) is always true we have proved that \(S' = S(T(S'))\). Symmetrically, \(S'' = S(T(S''))\).

Now, let \(T' := T(S'), T'' := T(S'')\). The map

\[
(\lambda', \lambda'') \mapsto \lambda' + \lambda''
\]

is an isomorphism and satisfies \(G(S_{(a,l)}(T') \times S_{(a,l)}(T'')) = S_{(a,l)}(T)\) since \(S'_{(a,l)} \oplus S''_{(a,l)} = S_{(a,l)}(T)\). On the other hand, by Observation 2 we have a natural isomorphism

\[
F : S(T') \times S(T'') \to S(T' \times T'')
\]

satisfying \(F(S_{(a,l)}(T') \times S_{(a,l)}(T'')) = S_{(a,l)}(T' \times T'')\). So \(F \circ G\) and its inverse are trellis maps. This yields a linear isomorphism from \(T' \times T''\) which by construction satisfies our statement.

By the defining property of product bases and the above theorem, if \(B = B^1 \sqcup \ldots \sqcup B^r\) is a partition of a product basis of \(T\) then

\[
T \simeq \bigotimes_{i=1}^r T(\langle B^i \rangle)
\]

This is a more general formulation of the dictionary given by Theorem 4. The following interesting corollary also follows.

**Corollary 2.** A product basis of \(T' \leq T\) extends to a product basis of \(T\) if and only if there exists \(T''\) and a linear isomorphism \(f : T \to T' \times T''\) such that \(f(T') = T' \times 0\) (or equivalently, if and only if there exists \(S \leq S(T)\) such that \(S_{(a,l)}(T) = S_{(a,l)}(T') \oplus S_{(a,l)}\) for all \((a,l)\)).

The condition \(T \simeq T' \oplus T''\) alone is not sufficient for a product basis of \(T' \leq T\) to extend to one of \(T\), even though that condition is equivalent to \(T' \simeq T''\) for some \(T'' \leq T\) whose product basis extends. In other words, isomorphic subtrellises can contribute differently to the structure of \(T\).
Example 5. Let $T$ be as below. Then

$$T(((00101, 0))) \simeq T(((01010, 0)))$$

but $\{(00101, 0)\}$ cannot be extended to a product basis of $T$, while $\{(01010, 0)\}$ can.

We can get more insight into this phenomenon by adopting the following point of view. The additive structure of a linear trellis $T$ allows for the decomposition of cycles into sums of smaller cycles, i.e. cycles with shorter span length. However, some cycles in $T$ cannot be written as sums of other smaller cycles. In other words, some cycles are *atomic*. More precisely, we say that a nonzero cycle $\lambda \in \mathcal{S}(T)$ is *atomic* if $\lambda \notin \mathcal{S}_{\ell(\lambda)-1}(T)$. We now have:

**Observation 5.** A cycle $\lambda \in \mathcal{S}(T)$ is atomic if and only if it belongs to some product basis of $T$.

**Proof.** Assume $\lambda \in \mathcal{S}_{(a, \ell(\lambda))}(T)$ is atomic. Then $\lambda \notin \mathcal{S}_{\ell(\lambda)-1}(T)$. In particular $\lambda \notin \mathcal{S}_{< (a, \ell(\lambda))}(T)$. By 5) of Observation 4, then $\lambda$ belongs to some product basis. Vice versa, if $\lambda$ belongs to a product basis of $T$ then by Observation 4, again we conclude that $\lambda \notin \mathcal{S}_{\ell(\lambda)-1}(T)$, i.e. $\lambda$ is atomic. \qed

In Example 5, we see that $(01010, 0)$ is atomic while $(00101, 0)$ is not: the different contributions of the associated (elementary) trellises to the structure of $T$ can now be intrinsically explained by the atomic property. Note though that while the atomic property characterizes cycles belonging to product bases, in general it is not true that a linearly independent set of atomic cycles is part of a product basis. Nevertheless, the following theorem holds:

**Theorem 6.** A basis $\mathcal{B}$ of $\mathcal{S}(T)$ is a product basis if and only if it minimizes total span length.

It is easily checked in Examples 4 and 5 that the product bases are precisely those bases that minimize the total span length. This theorem enables to find product bases (and so elementary trellis factorizations) of $T$ via bases of minimum total span length. We omit its proof here since it is rather technical and it requires extra machinery which goes beyond the scope of this paper. The reader is referred to [6] for the proof and more on this.

### 3.6 Characterization of linear trellis isomorphy

In the previous subsections we have seen how structural properties of the ordered family of span subcodes of a linear trellis $T$ correspond to factorizations of $T$. We now prove a central theorem of this paper which describes how isomorphy is encoded by the same family. Our theorem, which yields an effective method for checking whether or not two linear trellises are isomorphic, will be crucial in determining all factorizations of linear trellises and in our later results on the classification of nonmergeable/minimal linear trellises (Sections 4 and 5). We actually prove our theorem for linear isomorphy, but we shall see a posteriori that the same result is then true for nonlinear isomorphy too (see Theorem 8). First, an easy “mathematical folklore” lemma:
Lemma 1. Assume we have linear maps $V \xrightarrow{f} U$ and $W \xrightarrow{g} U$ of vector spaces $V$, $W$, $U$, such that $f(V) = g(W)$ and $\dim V = \dim W$. Then there exists an isomorphism $V \xrightarrow{h} W$ such that $g \circ h = f$, i.e. the below diagram commutes.

$$
\begin{array}{ccc}
V & \xrightarrow{f} & U \\
\downarrow{h} & & \downarrow{g} \\
W & \xrightarrow{g} & U
\end{array}
$$

Proof. Take a basis $u_1, \ldots, u_m$ of $f(V) = g(W)$. Take linearly independent elements $v_1, \ldots, v_m \in V$ and linearly independent elements $w_1, \ldots, w_m \in W$ such that $f(v_i) = u_i = g(w_i)$ for $i = 1, \ldots, m$. Obviously $\langle v_1, \ldots, v_m \rangle \cap \ker f = 0$, so that we can complete $v_1, \ldots, v_m$ to a basis $v_1, \ldots, v_n$ of $V$ such that $v_{m+1}, \ldots, v_n \in \ker f$. Similarly, we obtain a basis $w_1, \ldots, w_n$ of $W$ such that $w_{m+1}, \ldots, w_n \in \ker g$. Then the linear map $h$ defined by $h(v_i) := w_i$ obviously satisfies our statement.

Theorem 7. Two linear trellises $T$ and $T'$ (of same length) are linearly isomorphic if and only if for all $(a, l)$ the equalities

$$
\dim S_{(a,l)}(T) = \dim S_{(a,l)}(T')
$$

$$
C_{(a,l)}(T) = C_{(a,l)}(T')
$$

hold true.

Proof. The “only if” part is trivial, so we need only to prove the “if” part. We need to show that there exists a linear isomorphism $F : S(T) \rightarrow S(T')$ such that $F(S_{(a,l)}(T)) = S_{(a,l)}(T')$ for all $(a, l)$, and $L(F(\lambda)) = L(\lambda)$ for all cycles $\lambda$. For increasing $l = -1, 0, \ldots, n$ we will construct for all $a \in \mathbb{Z}_n$ linear isomorphisms

$$
F_{(a,l)} : S_{(a,l)}(T) \rightarrow S_{(a,l)}(T')
$$

such that

$$
L(F_{(a,l)}(\lambda)) = L(\lambda)
$$

$$
F_{(a,l)}|_{S_{(a',l')}(<(a,l))} = F_{(a',l')}
$$

for all $\lambda \in S_{(a,l)}(T)$ and all $(a', l') \leq (a, l)$. The map $F_{\mathbb{Z}_n}$ will then clearly be our sought $F$. For the first step $l = -1$, $F_{-1}$ can only be the zero map, and there is nothing to prove.

Now, for $l \geq 1$ assume we have constructed for each $a \in \mathbb{Z}_n$ and $l' \leq l - 1$ linear isomorphisms $F_{(a,l')}$ satisfying the above properties, and let $B$ be a product basis of $T$. Fix $a \in \mathbb{Z}_n$. The first step towards the construction of our sought map $F_{(a,l)}$ is to extend all the isomorphisms $F_{(a',l')}$, for $(a', l') < (a, l)$, to an isomorphism $F_{<(a,l)} : S_{<(a,l)}(T) \rightarrow S_{<(a,l)}(T')$ which also preserves edge-labels.

So, define $F_{<(a,l)}$ by

$$
F_{<(a,l)}(\lambda) := F_{|\lambda}(\lambda)
$$

for all $\lambda \in B \cap S_{<(a,l)}(T)$. Note that if an element $\lambda \in B$ satisfies $|\lambda| \leq (a', l') < (a, l)$ then, by our assumption, $F_{|\lambda}(\lambda) = F_{(a',l')}(\lambda)$, so that $F_{<(a,l)}$ extends $F_{(a',l')}$ for all $(a', l') < (a, l)$. It follows that $F_{<(a,l)}$ is surjective, and hence that

$$
\dim S_{<(a,l)}(T') \leq \dim S_{<(a,l)}(T)
$$

Symmetrically, $\dim S_{<(a,l)}(T) \leq \dim S_{<(a,l)}(T')$. Therefore $F_{<(a,l)}$ is an isomorphism. Obviously by construction we also have that $L(F_{<(a,l)}(\lambda)) = L(\lambda)$ for all $\lambda \in S_{<(a,l)}$. 

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We will now construct \( F_{(a,l)} \) by extending \( F_{< (a,l)} \). Let
\[
C := L(S_{< (a,l)}(T)) = L(S_{< (a,l)}(T'))
\]
where the equality follows from the hypothesis. We then have two naturally induced maps
\[
\tilde{L} : S_{(a,l)}(T)/S_{< (a,l)}(T) \to \mathbb{F}^n/C \\
\tilde{L} : S_{(a,l)}(T')/S_{< (a,l)}(T') \to \mathbb{F}^n/C
\]
We have seen that \( \dim S_{< (a,l)}(T) = \dim S_{< (a,l)}(T') \). By hypothesis \( \dim S_{(a,l)}(T) = \dim S_{(a,l)}(T') \), and so also \( \dim S_{(a,l)}(T)/S_{< (a,l)}(T) = \dim S_{(a,l)}(T')/S_{< (a,l)}(T') \). Moreover, since by hypothesis \( C_{(a,l)}(T) = C_{(a,l)}(T') \), we have that
\[
\tilde{L}(S_{(a,l)}(T)/S_{< (a,l)}(T)) = \tilde{L}(S_{(a,l)}(T')/S_{< (a,l)}(T'))
\]
Hence by Lemma 1 we can find an isomorphism
\[
G : S_{(a,l)}(T)/S_{< (a,l)}(T) \to S_{(a,l)}(T')/S_{< (a,l)}(T')
\]
such that \( \tilde{L} \circ G = \overline{L} \), where \( G = \overline{g} \) for some linear map \( g : S_{(a,l)}(T) \to S_{(a,l)}(T') \) (it is easily seen that any linear map from a quotient of vector spaces \( V_1/V_2 \) to another quotient \( W_1/W_2 \) is the reduction of a linear map from \( V_1 \) to \( W_1 \)). The situation is depicted in the following commutative diagram.

Now, for \( \lambda \in S_{(a,l)}(T) \) we have \( \overline{L}(\overline{g}(\overline{\lambda})) = \overline{L}(\overline{\lambda}) \) (where \( \overline{\lambda} \) is the reduction of \( \lambda \) modulo \( S_{< (a,l)}(T) \)) and therefore
\[
L(g(\lambda)) - L(\lambda) \in C = L(S_{< (a,l)}(T'))
\]
i.e. \( L(g(\lambda)) = L(\lambda) + L(\overline{\lambda}) \) for some \( \overline{\lambda} \in S_{< (a,l)}(T') \). So, putting
\[
\begin{align*}
F_{(a,l)}(\lambda) & := F_{< (a,l)}(\lambda) \\
F_{(a,l)}(\lambda) & := g(\lambda) - \overline{\lambda}
\end{align*}
\]
for each basis element \( \lambda \) with span \( [\lambda] < (a,l) \), and
\[
\begin{align*}
F_{(a,l)}(\lambda) & := \overline{g}(\overline{\lambda}) - \overline{\lambda}
\end{align*}
\]
for each basis element \( \lambda \) with \( [\lambda] = (a,l) \), we get that \( F_{(a,l)} \) extends \( F_{(a',l')} \) whenever \( (a',l') \leq (a,l) \), and that \( L(F_{(a,l)}(\lambda)) = L(\lambda) \) for all \( \lambda \in S_{(a,l)}(T) \).

It remains to prove that the map \( F_{(a,l)} : S_{(a,l)}(T) \to S_{(a,l)}(T') \) so constructed is an isomorphism. By the hypothesis it is sufficient to show that \( F_{(a,l)} \) is injective. So assume that \( F_{(a,l)}(\lambda') = 0 \) for \( \lambda' \in S_{(a,l)}(T) \). We can write
\[
\lambda' = \sum_{\lambda \in S[\lambda] = (a,l)} x_\lambda \lambda + \mu
\]
for some $x_\lambda \in \mathbb{F}$ and $\mu \in S_{<}(a,l)(T)$. Applying $F_{(a,l)}$ then

$$\sum_{\lambda \in \mathcal{B} [\lambda] = (a,l)} x_\lambda g(\lambda) - \sum_{\lambda \in \mathcal{B} [\lambda] = (a,l)} x_\lambda \lambda + F_{<}(a,l)(\mu) = 0$$

Reducing modulo $S_{<}(a,l)(T')$ and $S_{<}(a,l)(T)$ we thus get that

$$\overline{g}(\sum_{\lambda \in \mathcal{B} [\lambda] = (a,l)} x_\lambda \lambda) = 0$$

and so since $\overline{g} = G$ is an isomorphism

$$\sum_{\lambda \in \mathcal{B} [\lambda] = (a,l)} x_\lambda \lambda = 0$$

But by 4) of Observation 4 we know that $\{\lambda | \lambda \in \mathcal{B}, [\lambda] = (a,l)\}$ is a basis of $S_{<}(a,l)(T)/S_{<}(a,l)(T)$. So we conclude that $x_\lambda = 0$ for all $\lambda$. Then $\lambda' = \mu$ and $F_{<}(a,l)(\mu) = F_{<}(a,l)(\lambda') = 0$, hence $\mu = 0$ by injectivity of $F_{<}(a,l)$. Hence $F_{(a,l)}$ satisfies all the wanted properties, and therefore the proof is finished.

For one-to-one trellises linear isomorphism reduces to equality of the represented span subcodes. In fact, $T$ is one-to-one if and only if $L : S(T) \to C(T)$ is one-to-one, in which case $\dim S_{<}(a,l)(T) = \dim C_{(a,l)}(T)$ for all $(a,l)$, so that we get the following corollary.

**Corollary 3.** Two one-to-one linear trellises $T, T'$ are linearly isomorphic if and only if $C_{(a,l)}(T) = C_{(a,l)}(T')$ for all $(a,l)$.

### 3.7 Uniqueness of linear structure of linear trellises

A trellis $T$ is said to be linearizable if all its vertex sets $V_i(T)$ admit a vector space structure that make $T$ a linear trellis (i.e., more concretely, if we can label all vertices with words over $\mathbb{F}$ so that $S(T)$ is a linear code). While in [23] it was shown how to find out if a trellis is linearizable, the fundamental question whether a linearizable trellis admits an essentially unique linear structure has not been addressed so far. This question can be rephrased as: are isomorphic linear trellises essentially equal? Since for linear trellises to be “essentially equal” means to be linearly isomorphic, the question is then whether isomorphy of linear trellises implies also linear isomorphy.

Thanks to Theorem 7 we can now establish that for linear trellises there is no difference between being linearly isomorphic and being isomorphic. In particular, in Theorem 7 we can drop the “linearly” adverb with no need to change anything.

**Theorem 8.** Let $T, T'$ be linear trellises. If $T$ and $T'$ are isomorphic then they are also linearly isomorphic.

**Proof.** Assume we have a nonlinear trellis isomorphism $f : T \to T'$ given by $f_i : V_i(T) \to V_i(T')$, $i \in \mathbb{Z}_n$. Let $F = S(f)$. Define a map from $S(T)$ to $S(T')$ by

$$\lambda \mapsto F(\lambda + F^{-1}(0))$$

for all cycles $\lambda \in S(T)$. Clearly this map preserves edge labels and it is injective, as $F$ and so $F^{-1}$ do. Now, let $\lambda \in S_{<}(a,l)(T)$, for some $0 \leq i \leq n - 1$. Then $\nu_i(\lambda) = 0$ for all $i \notin (a, a + \ell)$. So, for $i \notin (a, a + \ell]$ we have that

$$\nu_i(\lambda + F^{-1}(0)) = \nu_i(F^{-1}(0)) = f_i^{-1}(0_i)$$
and therefore \( \nu_i(F(\lambda + F^{-1}(0))) = f_i(\nu_i(\lambda + F^{-1}(0))) = 0_i \). So our map is an injective and edge-label preserving map from \( S_{(a,t)}(T) \) to \( S_{(a,t)}(T') \). Symmetrically, there is also such a map from \( S_{(a,t)}(T') \) to \( S_{(a,t)}(T) \). From this follows that the hypothesis of Theorem \( \square \) are satisfied, hence the two trellises are linearly isomorphic.

The essential uniqueness of the linear structure of linearizable trellises now follows:

**Theorem 9.** Let \( T = (\sqcup V_i(T), \sqcup E_i(T)) \) be a trellis. Assume that each vertex set \( V_i(T) \) has two addition operations \( +_1, +_2 \), and two scalar multiplication operations \( \cdot_1, \cdot_2 \), such that both \( (+_1, \cdot_1) \) and \( (+_2, \cdot_2) \) make \( T \) linear. Then the two resulting linear trellises \( T_1 \) and \( T_2 \) are linearly isomorphic.

The reader must be aware that Theorem \( S \) is not saying that an isomorphism of linear trellises is also a linear isomorphism. Equivalently, Theorem \( J \) is not saying that the identity map of \( T \) necessarily results in a linear isomorphism between \( T_1 \) and \( T_2 \). Counterexamples can be easily constructed indeed. Still, the proofs of the above results explicitly tell us how to construct a linear isomorphism from the given isomorphism.

**Remark 9.** There is a suggestive parallel between vector spaces and linear trellises. Any \( K \)-vector space \( V \) has a basis and thus decomposes into parts of dimension one (\( V \cong K^n \)), and if two \( K \)-vector spaces are equivalent as sets (i.e., have the same cardinality) then they must be linearly isomorphic. Similarly, any linear trellis \( T \) has a product basis, so that the Factorization Theorem holds (\( T \cong \bigotimes_{i=1}^r T_i \), with \( \dim S(T_i) = 1 \) for all \( i \)), and we also have Theorem \( S \). However, while for vector spaces such fundamental properties are easy to prove, for linear trellises this is not at all the case.

When the trellises under consideration are one-to-one we get a much stronger result, which turns out to be even much simpler to prove.

**Theorem 10.** Let \( f : T \to T' \) be an isomorphism of one-to-one linear trellises. Then \( f \) is linear.

**Proof.** We need to show that \( S(f) \) is linear. Since \( f \) is a trellis morphism we have that \( L \circ S(f) = L' \), where \( L \) and \( L' \) are the edge-label sequence maps of cycles respectively in \( T \) and \( T' \). But since \( T \) and \( T' \) are one-to-one and linear, \( L \) and \( L' \) must be injective linear maps, and so \( S(f) = L^{-1} \circ L' \) is linear (where the domain of \( L^{-1} \) is \( C(T) = L'(S(T)) \)).

**Corollary 4.** Assume we are in the same situation as in Theorem \( \square \). Assume also that \( T \) is one-to-one. Then the identity map of \( T \) is a linear isomorphism between \( T_1 \) and \( T_2 \).

The above corollary tells us that there is literally only one vector space structure on each space \( V_i(T) \) that can possibly make a one-to-one trellis \( T \) linear, so that the situation is very rigid in that case. In other words, any two different labelings of vertices of a one-to-one trellis \( T \) that make \( T \) linear are one the linear transformation of the other.

The function-theoretical argument used in the above proof leads us also to a necessary and sufficient condition for a one-to-one trellis to admit a linear structure.

**Observation 6.** Let \( T \) be a one-to-one trellis. Put \( G_i := \nu_i \circ L^{-1} \). Then \( T \) is linearizable if and only if for all \( \mathbf{v}, \mathbf{v}', \mathbf{w}, \mathbf{w}' \in C(T), \alpha \in F, i \in \mathbb{Z}_n \) such that \( G_i(\mathbf{v}) = G_i(\mathbf{v}') \) and \( G_i(\mathbf{w}) = G_i(\mathbf{w}') \) the following equalities hold

\[
G_i(\mathbf{v} + \mathbf{w}) = G_i(\mathbf{v}' + \mathbf{w}')
\]
\[
G_i(\alpha \mathbf{v}) = G_i(\alpha \mathbf{v}')
\]
Proof. If $T$ is linear then $G_i$ is linear, and so it obviously satisfies the stated conditions. Vice versa, if the stated conditions are satisfied then they induce a unique vector space structure on each $V_i(T)$ such that $G_i$ is a linear map. It is then easy to check that the induced structures makes $T$ linear.

Note that in [23] no theoretical characterization is given of the linearizable property for (one-to-one) trellises. Instead, Koetter/Vardy present therein an algorithm that halts if $T$ is non-linearizable or otherwise outputs a complete set of vertex-labels which make $T$ linear.

3.8 Group trellises: remarks on their structure and on extending results from the linear case

Replacing $\mathbb{F}$ with a group $G$ in the definition of trellis yields trellises over groups. Definitions and statements for trellises over fields can be thus translated (right away or after the appropriate adaptation to group theoretical language) into definitions/statements for trellises over groups. In particular, we can talk of group trellises, i.e. trellises with a group structure over $G$ and which represent group codes (i.e. subgroups of $G^n$).

Now, while in [23] (first Remark of Section 3 therein) and [24] (first Remark of Section IV therein) it is stated that all the results therein “hold essentially without change for group trellises over an abelian group”, the Factorization Theorem does not hold for (abelian) group trellises if we do not revise the definition of elementary group trellis described therein (which naturally generalizes the one for the linear case and on the base of which such a trellis must represent a group code). For example, consider the below group trellis over the cyclic group $\mathbb{Z}_4$, where edge-labels are identified by the arrow filling pattern (dashed $\equiv 0$; dash-dotted $\equiv 1$; dotted $\equiv 2$; full $\equiv 3$).

If $T \sim T_1 \otimes \ldots \otimes T_r$ for some group trellises $T_1, \ldots, T_r$, then $C(T_i) = \langle 112 \rangle$ for some $i$, since $\langle 112 \rangle = \sum_{i=1}^r C(T_i)$. Then it’s easy to see that $T_i \sim T$, and so also $T_j = 0$ for all $j \neq i$. On the other hand $T$ is not elementary (according to the mentioned definition an elementary group trellis consists of cycles that run disjointly inside its defining span while coinciding with the zero cycle outside it, like in the linear case). In fact the following is the only conventional elementary group trellis which represents $\langle 112 \rangle$:

Nevertheless, we have that $T \sim T_1 \otimes T_2$, where
So, we can still say that a factorization theorem holds for $T$. The only catch is that $T_1$ is not elementary according to the previous definition, as it represents $\{000, 112\} \subseteq \mathbb{Z}_4^3$, which is not a group code. Still, it makes sense to assign the “elementary” adjective to it too, since it is not (isomorphic to) a product of smaller trellises. Moreover we can think of $T_1$ as representing a group in some sense, since $\{000, 112\}$ corresponds (bijectively) to the quotient group $\langle 112 \rangle / \langle 220 \rangle$ (it must be kept in mind though that this correspondence/information is not given by $T_1$ alone, i.e. it is not encoded in it, but it is only provided by the factorization $T \sim T_1 \otimes T_2$).

After having checked more examples, it seems likely that the above happens in general, i.e. any (abelian) group trellis is a product of trellises which cannot be further factored and which represent sets corresponding to quotient groups of prime order. However this still requires a proof. While the approach given in [23] does not seem to be adaptable for yielding such a proof, our framework can be extended to group trellises and we believe that with some modifications it can yield the sought proof (we leave this for future research).

Note though that in [17] related problems have been treated.

**Remark 10.** Some substantial modifications may be necessary when extending our framework to the group case: for example, the below group trellises over $\mathbb{Z}_4$ are isomorphic but their group structures are not, contrarily to what the straight translation of Theorem 8 to the group case would tell us (note though that Theorem 10 and Corollary 4 extend without saying to group trellises).

The Factorization Theorem can make proving certain results for linear trellises easier, however, because of the above situation, when proving a result for such trellises it is preferable to avoid that theorem if possible and give instead proofs that exploit only the additive structure and can be thus immediately extended to the (abelian) group case too.

4 Applications: factoring linear trellises

4.1 Uniqueness of span distribution of linear trellises

Knowing from the Factorization Theorem that every linear trellis factors into elementary trellises, a natural following question is whether such a factorization is unique. In general the answer is negative. For example, $111|(0, 2) \otimes 010|(1, 0)$ and $101|(0, 2) \otimes 010|(1, 0)$ yield the same trellis, but the two factorizations are different since $111|(0, 2) \neq 101|(0, 2)$. One sees though that the list of
spans of the two factorizations in this case are the same. This is no coincidence, since here we are dealing with a minimal conventional trellis and it is well known that all the elementary trellis factorizations of such a trellis give rise to the same spans, more precisely, the atomic spans of the represented code (see Subsection 5.1.3). However, this uniqueness of span distribution had been proven only as a byproduct of the minimality assumption. More recently in [18] (Proposition III.14 therein), with arguments based again on atomic spans, this result was extended to the class of so-called KV-trellises (see Subsection 5.1.3), which is a subclass of the class of nonmergeable, one-to-one, linear trellises containing the class of minimal linear trellises.

We show here that this actually holds true for any linear trellis, and furthermore, that the edge-labels really play no role in determining the spans. The underlying graph structure alone determines them. We will prove these two claims as easy consequences of the algebraic framework developed in the previous section.

First, let us make the terminology precise: by the span distribution of the elementary trellis factorization $\otimes_{i=1}^{r} \alpha^i(a_i, l_i)$ of a linear trellis we mean the multiset

$$\{ (a_i, l_i) | i = 1, \ldots, r \}$$

Now, given a product basis $B$ of a linear trellis $T$, recall that by 4) of Observation 4 the number of cycles $\lambda \in B$ with span $[\lambda] = (a, l)$ is given precisely by $\dim S_{(a,l)}(T)/S_{<(a,l)}(T)$, and thus it depends only on $T$, not on the particular choice of the product basis. In particular, all product bases yield the same span distribution. But then by Theorem 8 and the correspondence between product bases and elementary trellis factorizations given by Theorem 4 we conclude that the following holds:

**Theorem 11.** Two elementary trellis factorizations of a linear trellis $T$ have the same span distribution. More precisely, the number of times a span $(a, l)$ appears in an elementary trellis factorization of $T$ is equal to

$$\dim S_{(a,l)}(T)/S_{<(a,l)}(T) = \dim S_{(a,l)}(T) - \dim S_{<(a,l)}(T)$$

This shows that our first claim holds, and also justifies talking about the span distribution of a linear trellis $T$, which we will henceforth denote by $S(T)$ (recall that this is a multiset). We also put

$$S_+(T) := \{ (a, l) \in S(T) | l > 0 \}$$

and

$$S_0(T) := S(T) \setminus S_+(T)$$

(note that $S_0(T)$ is a set, because of our assumption in Remark 5). We now prove our second claim.

**Theorem 12.** Two linear trellises are structurally isomorphic if and only if they have the same span distribution.

**Proof.** Given a trellis $T$ we denote by $\overline{T}$ its underlying unlabeled trellis, i.e. $\overline{T}$ is defined by putting all the edge-labels of $T$ equal to 0, so that $V_i(\overline{T}) = V_i(T)$ and $E_i(\overline{T})$ is the image of $E_i(T)$ under the map $vaw \mapsto v0w$, for all $i$. In particular

$$\overline{\alpha^i(a, l)} = \begin{cases} 0 & \text{if } l > 0 \\ 0 & \text{if } l = 0 \end{cases}$$

Also, clearly $\overline{T}$ commutes with trellis products, i.e.

$$\overline{T_1 \otimes T_2} = \overline{T_1} \otimes \overline{T_2}$$
So it follows immediately that $S(T) = S_b(T)$. On the other hand, two linear trellises $T_1$, $T_2$ are structurally isomorphic if and only if $S_0(T_1) = S_0(T_2)$ and $T_1 \sim T_2$. We can thus conclude by the above theorem that structurally isomorphic linear trellises must have the same span distribution. The “if” part is trivial, and we have included it for the sake of completeness. 

Thus the span distribution of a linear trellis does not depend on its edge-labels, but only on its underlying graph. Note that by the arguments in the above proof it also follows that the multiplicity of a span $(a, l) \in S(T)$ with positive length $l > 0$ is equal to $\dim S_{(a,l)}(T)/S_{< (a,l)}(T)$ (where $T$ is defined in the same proof).

Remark 11. The mathematically keen reader may have recognized at this point that the above theorem along with the Factorization Theorem imply that the class of unlabeled linear trellises (up to isomorphism) equipped with the trellis product operation is a unique factorization monoid.

We conclude this subsection by showing that we can compute all the span multiplicities if we know the dimension of each span subcode $S_{(a,l)}(T)$. The following identity makes that possible.

Observation 7. Let $T$ be a linear trellis. Then for each $(a, l)$ we have

$$\dim S_{(a,l)}(T)/S_{< (a,l)}(T) = \dim S_{(a,l)}(T) - \sum_{(a', l')<(a,l)} \dim S_{(a',l')}(T)/S_{< (a',l')}(T)$$

Equality (1) can be used recursively for increasing span lengths to compute all the multiplicities $\dim S_{(a,l)}(T)/S_{< (a,l)}(T)$ from the span subcode dimensions $\dim S_{(a,l)}(T)$. Indeed, starting with length equal to $-1$, i.e. $S_{(a,-1)}(T) = 0$, we get

$$\dim S_{(a,-1)}(T) \equiv \dim S_{(a,0)}(T) = 0$$
$$\dim S_{(a,0)}(T)/S_{< (a,0)}(T) = \dim S_{(a,0)}(T)$$
$$\dim S_{(a,1)}(T)/S_{< (a,1)}(T) = \dim S_{(a,1)}(T) - \sum_{(a',0)<(a,1)} \dim S_{(a',0)}(T)$$
$$\dim S_{(a,2)}(T)/S_{< (a,2)}(T) = \dim S_{(a,2)}(T) - \sum_{(a',1)<(a,2)} \dim S_{(a',1)}/S_{(a',1)}(T) = \dim S_{(a,2)}(T) - \sum_{(a',1)<(a,2)} \left( \dim S_{(a',1)}(T) - \sum_{(a'',0)<(a',1)} \dim S_{(a'',0)}(T) \right)$$

and so on.

This can be practically worked out by replacing each span $(a, l)$ by $\dim S_{(a,l)}(T)$ in the Hasse diagram for spans, and then processing the entries in the diagram from bottom to top according to the above equations, where a single step amounts to subtracting from an entry in the diagram the sum of all the other entries below it in the diagram and update the entry. The final entries will then give the multiplicity of each span. For example, if we take the trellis $T$ from Example 2 we get the following sequence, where we underline the entries which have been processed so far at each stage:
The final diagram tells us that the span distribution of \( T \) is \( \{((0,0),(0,1),(1,1))\} \), which is indeed the case.

**Remark 12.** We had announced Theorem 12 first in [8]. Therein indeed we had illustrated how the span distribution \( S(T) \) of a linear trellis \( T \) can be recovered from simple graphical characteristics of \( T \). In Appendix B we provide the complete details of that worthy alternative perspective.

### 4.2 Edge-labels of elementary trellis factorizations

After showing that a linear trellis determines uniquely the span distribution of its elementary trellis factorizations and how this can be determined, the naturally following problem is to find out what are the possible edge-labels of the elementary factors. We have already pointed out before that unique factorization does not hold, which means that edge-labels are not necessarily unique. Nevertheless we give here a method for finding all the legitimate edge-labelings. We will also deduce necessary and sufficient conditions for a linear trellis \( T \) to have a unique elementary trellis factorization. Again, our results will be easy consequences of what proven in Section 3.

**Observation 8.** Let \( T, T' \) be linear trellises such that \( S(T) = S(T') \). Then \( T \) and \( T' \) are isomorphic if and only if

\[
C_s(T) = C_s(T')
\]

for all spans \( s \).

**Proof.** By our Isomorphy Theorem 7 we need to prove that \( \dim S_s(T) = \dim S_s(T') \) for all spans \( s \). As the two span distributions are equal, we have that \( \dim S_s(T)/S_{<s}(T) = \dim S_s(T')/S_{<s}(T') \) for all \( s \). The sought equalities follow then from equation 1. \( \square \)

Now, assume that we know the span distribution \( S(T) \) of a linear trellis \( T \). For a generic span \( s \) let

\[
m(s, T)
\]

denote the multiplicity of \( s \) in \( S(T) \). In particular, \( m(s, T) > 0 \) if \( s \in S(T) \), and \( m(s, T) = 0 \) otherwise. Any elementary trellis factorization of \( T \) can be then written as

\[
\otimes_{s \in S(T)} \otimes_{i=1}^{m(s,T)} \alpha_{s,i} | s
\]
for some $\alpha^{s,i} \in \mathbb{F}^n$. The question then is: what are the possible $\alpha^{s,i}$? Here is the complete answer:

**Theorem 13.** Let $T$ be a linear trellis. Then

$$\otimes_{s \in S(T)} \otimes_{i=1}^{m(s,T)} \alpha^{s,i} | s$$

(2)

is an elementary trellis factorization of $T$ if and only if for each span $s \in S(T)$ we have

$$C_s(T) = \langle \alpha^{s,1}, \ldots, \alpha^{s,m(s,T)} \rangle + \sum_{s' < s} C_{s'}(T)$$

(3)

**Proof.** We know by Observation 3 that

$$C_s(\otimes_{s \in S(T)} \otimes_{i=1}^{m(s,T)} \alpha^{s,i} | s) =
\langle \alpha^{s',i} | s' \in S(T), s' \leq s, i = 1, \ldots, m(s',T) \rangle$$

for all $s$, and in particular for all $s \in S(T)$. Thus, if (2) is an elementary trellis factorization of $T$ equation (3) follows. Vice versa, assume (3) holds for all $s \in S(T)$. First, note that if $s \notin S(T)$ then $C_s(T) = \sum_{s' < s} C_{s'}(T)$. Now, by iterative substitutions in these equations and in (3) for decreasing span lengths we deduce that

$$C_s(T) =
\langle \alpha^{s',i} | s' \in S(T), s' \leq s, i = 1, \ldots, m(s',T) \rangle =
C_s(T')$$

for all $s$. The conclusion follows now from Observation 8.

The above theorem can be used to compute all the possible labelings of the elementary factors of $T$, and so, preceded by the computation of $S(T)$ via Theorem 11 (or via the graphical intersection data as described in Appendix B), all the possible elementary factors of $T$.

Note that for a given span $s \in S(T)$ only the labels for elementary factors with that same span appear in equation (3). Therefore, when computing all the possible labelings of the factors of $T$ this can be worked out independently for each span $s \in S(T)$.

**Example 6.** Let $T$ be

One can check that $S(T) = \{\{(1,2), (3,3), (3,4)\}\}$ (see also Example 13). Then one gets:

- $C_{(1,2)}(T) = \langle 01010 \rangle$
- $C_{(3,3)}(T) = \langle 01011 \rangle$
• \( C_{(3,4)}(T) = \langle 01011 \rangle \)

Therefore by the above theorem there are two and only two distinct factorizations of \( T \), namely

\[
01010| (1, 2) \otimes 01011| (3, 3) \otimes 01011| (3, 4) \\
01010| (1, 2) \otimes 01011| (3, 3) \otimes 00000| (3, 4)
\]

According to the same theorem

\[
01010| (1, 2) \otimes 00000| (3, 3) \otimes 01011| (3, 4)
\]

is not an elementary trellis factorization of \( T \) (because \( C_{(3,3)}(T) \neq 0 \)), i.e. it yields a different (i.e. nonisomorphic) trellis \( T' \), which we have depicted below. From the diagram one can indeed check that \( T' \neq T \) as it is possible to go out from \( 0 \in V_3(T') \) along two paths with all edge-labels equal to \( 0 \) that meet again at \( V_2(T') \), while this is not possible in \( T \).

Since by the Factorization Theorem all linear trellises can be written as elementary trellis products the above theorem is crucial for their classification. We will use it indeed in the next section for classifying minimal linear trellises.

We want now to count the number of distinct elementary trellis factorizations of a linear trellis \( T \) such that \( m(s, T) = 1 \) for all \( s \in S(T) \). First, let \( \otimes_{s \in S(T)} \alpha_s|s \) be an elementary trellis factorization of \( T \) (which we know to exist by the Factorization Theorem). Now, by \( \exists \alpha' \) is a valid labeling of the elementary factor with span \( s \) if and only if

\[
\langle \alpha_s \rangle + \sum_{s' < s} C_{s'}(T) = \langle \alpha' \rangle + \sum_{s' < s} C_{s'}(T)
\]

As \( \alpha'|s = \alpha''|s \) if and only if \( \alpha' = y\alpha'' \) for some \( y \in \mathbb{F}_q^* \), the elementary factors with span \( s \) are precisely given by \( (\alpha^s + \mathbf{w})|s \) for arbitrary \( \mathbf{w} \in \sum_{s' < s} C_{s'}(T) \). Thus, putting

\[
k_s = \dim(\sum_{s' < s} C_{s'}(T))
\]

we have that:

• if \( \alpha^s \in \sum_{s' < s} C_{s'}(T) \) then there are precisely

\[
1 + \frac{q^{k_s} - 1}{q - 1}
\]

valid elementary factors with span \( s \)
• if \( \alpha^s \not\in \sum_{s'<s} C_{s'}(T) \) then \((\alpha^s + w)|s \neq (\alpha^s + w')|s \) whenever \( w \neq w' \), and so there are precisely
\[
q^{k_s}
\]
valid elementary factors with span \( s \)

Note that if \( T \) is also one-to-one then \( \alpha^s \not\in \sum_{s'<s} C_{s'}(T) \) and
\[
\dim(\sum_{s'<s} C_{s'}(T)) = |\{s' \in S(T)|s' < s\}|
\]

Resuming:

**Corollary 5.** Assume that \( m(s,T) = 1 \) for all \( s \in S(T) \). Put
\[
S' := \{s \in S(T)|C_s(T) = \sum_{s'<s} C_{s'}(T)\}
\]
\[
S'' := S(T) \setminus S'
\]

Then \( T \) has precisely
\[
\prod_{s \in S'} q^{k_s} + q - 2 \prod_{s \in S''} q^{k_s}
\]
distinct elementary trellis factorizations. If \( T \) is also one-to-one then it has precisely
\[
\exp_q(\sum_{s \in S(T)} |\{s' \in S(T)|s' < s\}|)
\]
distinct elementary trellis factorizations.

For example, the trellis \( T \) of the above example is one-to-one with span distribution given by
\( S(T) = \{(1,2), (3,3), (3,4)\} \), and so it has precisely two distinct factorizations. We also get the following corollary.

**Corollary 6.** Let \( T \) be a one-to-one linear trellis. Then \( T \) has a unique elementary trellis factorization if and only if all spans in \( S(T) \) have multiplicity equal to 1 and are incomparable.

This extends the well known result that if there are no containments between different atomic spans of a linear code then the associated minimal conventional trellis factors uniquely, since (up to scalar multiplication) all the atomic generators are uniquely determined.

## 5 Applications: classifying nonmergeable and minimal linear trellises

### 5.1 Preliminaries

We review here some known fundamental results on the top of which we will build our new results. Some new notation will be also introduce in Subsection 5.1.3. We actually start by discussing and proving some important facts which have been overlooked in the literature.
5.1.1 Overlooked facts on nonmergeability and minimality

While it is obvious that a minimal trellis is nonmergeable the same property for minimal linear trellises (i.e. linear trellises that are minimal only amongst linear trellises — see Remark 3) requires a proof. The proof essentially amounts to showing that a mergeable linear trellis can be merged to a smaller linear trellis for the same code. Such fact was actually observed in [21] (see Lemma 4 therein), but the proof there is cloudy. In fact it turns out that the “almost reduced” hypothesis (i.e. each vertex belongs to some cycle) is necessary, which was overlooked in [21]. Below we prove the correct result and then give an example of the necessity of the extra hypothesis. We will also prove that a minimal linear trellis is reduced, which is tied up with the same result.

**Theorem 14.** Let \( T \) be an almost reduced linear trellis for \( C \). If \( T \) is mergeable then it can be merged to a linear trellis for \( C \) smaller than \( T \).

**Proof.** Suppose \( T \) is mergeable at \( v \neq w \in V_i(T) \). By shifting \( T \) we can assume \( i = 0 \). Let \( u \in V_0(T) \) and \( x \in \mathbb{F}^* \). By hypothesis there exist paths \( p \in P(v, v), p' \in P(u, u) \). By linearity

\[
P(u, u + x(v - w)) = p' + xp - xP(v, w)
\]

so that \( L(P(u, u + x(v - w))) \subseteq C \). Similarly \( L(P(u + x(v - w), u)) \subseteq C \). Thus the vertices in each coset \( u + \mathbb{F}(v - w) \) can be merged together without affecting the code. The resulting trellis is clearly linear and smaller than \( T \), proving the statement. \( \square \)

**Example 7.** The below linear trellis for \( \langle 111 \rangle \) is mergeable, and it is so only at the pair of vertices \( 00, 11 \in V_0(T) \). Once those two vertices are merged no other merging is possible, so that there is no way we can obtain a smaller linear trellis by merging vertices. The trellis is clearly not almost reduced.

![Linear Trellis](image)

**Theorem 15.** Any minimal linear trellis is reduced.

**Proof.** Let \( T \) be a minimal linear trellis for \( C \). By deleting all the edges of \( T \) that do not belong to cycles we get a linear subtrellis \( T' \leq T \) which is reduced and has the same vertices as \( T \). Thus \( T' \) must be minimal for \( C \) too, and so by Theorem 14 it must be nonmergeable, and therefore also connected. Now the statement follows from Theorem 31. \( \square \)

**Corollary 7.** Any minimal linear trellis is nonmergeable.

**Proof.** Put together Theorem 14 and Theorem 15. \( \square \)

**Remark 13.** The definition of minimal linear trellises commonly given in the literature requires the reduced property in the hypothesis. Theorem 15 shows that by minimizing the state-complexity profile \( (|V_0(T)|, \ldots, |V_{n-1}(T)|) \) we get automatically rid of the unnecessary edges and so there is no such thing as a nonreduced minimal linear trellis, a fact which was not recognized before.
5.1.2 Well known facts ([24, 36])

For a linear trellis $T$:

- $T$ is minimal $\implies$ $T$ is one-to-one.
- If $T$ is reduced then: $T$ is nonmergeable $\implies$ $T$ is biproper, but the converse is not true.
- If $T$ is nonmergeable it is not necessarily minimal.
- If $T$ is conventional then:
  $T$ is minimal $\iff$ $T$ is nonmergeable $\iff$ $T$ is biproper

5.1.3 Atomic bases, atomic spans, and minimal trellises

First, some notation. Henceforth by $C$ we will denote an $[n, k]$ linear code of full support (i.e. $\text{supp}(C) = \mathbb{Z}_n$). We denote by $v^*$ the minimum conventional span of $v \in \mathbb{F}^n$ (which clearly exists), and we call it the (conventional) span of $v$.

For any span $(a, l)$ we put $\sigma((a, l)) := (a - 1, l)$

Now, we have:

**Theorem 16 ([26, 28])**. There exists a basis $\{v^1, \ldots, v^k\}$ of $C$ such that the spans $[v^i]^*$, $i = 1, \ldots, k$, all start and end at different positions, and if $\{w^1, \ldots, w^k\}$ is another basis with the same property then $\{[v^i]^*\}_{i=1,\ldots,k} = \{[w^i]^*\}_{i=1,\ldots,k}$.

Such bases of $C$ are called atomic since their elements are atomic codewords, i.e. they cannot be written as sums of codewords with shorter conventional span. The uniquely determined set $\{[v^i]^*\}_{i=1,\ldots,k}$ is called the atomic span set of $C$. We denote it by $S^*(C)$

Atomic bases are closely tied up with minimal conventional trellises:

**Theorem 17 ([26, 27, 36])**. Let $\{v^i\}_{i=1,\ldots,k}$ be a basis of $C$. Then $\otimes^k_{i=1}v^i|[v^i]^*$ is the minimal conventional trellis $T^*(C)$ if and only if $\{v^i\}_{i=1,\ldots,k}$ is an atomic basis of $C$.

This gives a method to construct $n$ (some of them possibly equal) minimal linear trellises for $C$: for all $i = 0, \ldots, n - 1$, construct the minimal conventional trellis for $\sigma^i(C)$ from an atomic basis $\{v^1, \ldots, v^k\}$ of $\sigma^i(C)$ with span set $S^*(\sigma^i(C))$, and then shift it backwards by applying $\sigma^{-i}$. As a consequence, the set $S(C) := \bigcup_{i=0}^{n-1} \sigma^{-i}(S^*(\sigma^i(C)))$

contains precisely all the spans appearing in all the minimal trellises so constructed. Koetter/Vardy [24] proved the striking result that no other spans are needed to describe the graph structure of any other possible minimal linear trellis for $C$:

**Theorem 18 ([24])**. The following holds:

1. If $\otimes^k_{i=1}v^i|s^i$ is a minimal linear trellis for $C$ then all the spans $s^i$ are different and $\{s^i\}_{i=1,\ldots,k} \subseteq S(C)$.

2. $S(C)$ contains precisely $n$ spans, and they all start and end at different positions.
3. For all \( i \geq 0 \) there are precisely \( k \) conventional spans in \( \sigma^i(S(C)) \), and these are the atomic spans of \( \sigma^i(C) \).

4. If \( S(C) = \{(a_1, l_1), \ldots, (a_n, l_n)\} \) then \( S(C^\perp) = \{(a_1 + l_1, n - l_1), \ldots, (a_n + l_n, n - l_n)\} \).

The set \( S(C) \) is thus called the characteristic span set of \( C \). Note that all the spans in \( S(C) \) are nondegenerate, corresponding to the fact that any minimal linear trellis must be connected. A set of \( n \) elementary trellises \( v^1|s^1, \ldots, v^n|s^n \), such that \( v^i \in C \), \( s^i \) is a minimal span of \( v^i \) for all \( i \), and \( \{s^i\}_{i=1,\ldots,n} = S(C) \), will be called a characteristic set of \( C \) (or characteristic matrix of \( C \) if the trellises are listed in array form). If \( \{s^i\}_{i=1,\ldots,n} \) is a characteristic set of \( C \), by the above theorems it follows that

\[ \{\sigma^j(v^i) | \sigma^j(s^i) \text{ is conventional}, i = 1, \ldots, n\} \]

is an atomic basis of \( \sigma^j(C) \) for all \( j = 0, \ldots, n - 1 \).

**Example 8.** Assume that \( C \) is cyclic. It is clear that if \( s \) is the span of a generating codeword of \( C \) then \( S^*(C) = \{\sigma^i(s)\}_{i=0,\ldots,k-1} \) is the atomic span set of \( C \). But then, as \( C \) is cyclic, it also follows that

\[ S(C) = \{\sigma^i(s)\}_{i=0,\ldots,n-1} \]

See [31, 33] for more on minimal linear trellises of cyclic codes.

Any one-to-one trellis for \( C \) which is isomorphic to a product of elementary trellises from a characteristic set of \( C \) is called a KV-trellis (notice that Nori [30] calls any product of elementary trellises a KV-trellis, while we are following the terminology used in [18]). Theorem 18 tells us that any minimal linear trellis is a KV-trellis. The converse is known to be false. Also, although all minimal linear trellises for \( C \) are structurally isomorphic to KV-trellises produced from a single fixed characteristic set of \( C \) (see Corollary 12), in general it is not true that they are also all isomorphic to the KV-trellises produced from a single fixed characteristic set, as wrongly stated in Thm 5.5 of [21]. The simple code of Example 9 yields an easy counterexample. This subtlety was also independently recognized before in [18]. In general one needs several characteristic sets of \( C \) to describe all minimal linear trellises as KV-trellises. Nevertheless, we will show in Subsection 5.4 how to determine and count all minimal linear trellises from a single characteristic set.

### 5.2 New insight into the nonmergeable property

We give here a new characterization of the nonmergeable property for one-to-one linear trellises. Let us start with a couple of definitions. We say that a trellis \( T \) of length \( n \) is pathwise one-to-one if different paths of length \( n \) starting at \( V_0(T) \) never yield the same edge-label sequence. We then say that a trellis \( T \) of length \( n \) is fragment one-to-one if all its cyclic shifts \( \sigma^i(T), i = 0, \ldots, n - 1 \), are pathwise one-to-one, that is, if any two distinct paths in \( T \) of length \( n \) starting at the same time index never yield the same edge-label sequence.

When \( T \) is biproper any two paths of length \( n \) that start at a same time index \( i \) and yield the same edge-label sequence cannot intersect. Hence a biproper conventional trellis \( T \) must be fragment one-to-one. Vice versa, if a conventional trellis is fragment one-to-one then it is easy to see that it is biproper. So, for conventional trellises the two concepts are equivalent. On the other hand, for conventional linear trellises the biproper property is also equivalent to the nonmergeable property. Thus the following holds:

**Observation 9.** A conventional linear trellis is nonmergeable if and only if it is fragment one-to-one.
In the nonconventional case it is known that biproper linear trellises may be mergeable. Koetter/Vardy \[24\] give the following example of such a situation:

\[
T =
\begin{array}{cccc}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

Note that this trellis is one-to-one but is not fragment one-to-one: the paths 101101110, 111000101 of length 3 starting at \( V_2(T) \) both yield the word 101. This is precisely the reason why \( T \) is mergeable, as we are now going to show that the “fragment one-to-one” property is strongly related to the nonmergeable property. Our result can be seen as a general extension of Observation \[9\].

**Theorem 19.** Let \( T \) be a linear trellis. If \( T \) is nonmergeable and one-to-one then \( T \) is connected and fragment one-to-one. The converse also holds if \( T \) is almost reduced.

**Proof.** Assume \( T \) is nonmergeable and one-to-one. First, since \( T \) is nonmergeable it must be also connected. We prove now that \( T \) is fragment one-to-one. By using cyclic shifts it is sufficient to prove that \( T \) is pathwise one-to-one. So assume there is a path \( p \in \mathbb{P}(v,w), v,w \in V_0(T) \), with edge-label sequence \( L(p) = 0 \). Then \( L(p') \in C(T) \) for all \( p' \in \mathbb{P}(v,w) \), since \( p' - p \) is a cycle. Similarly, \( L(p') \in C(T) \) for all \( p' \in \mathbb{P}(w,v) \), since \( p' + p \) is a cycle. Thus we can merge \( v \) and \( w \) without affecting the represented code. So it must be \( v = w \). But since \( T \) is one-to-one, we conclude that \( p \) is the zero cycle. Hence \( T \) is pathwise one-to-one.

As for the converse, assume \( T \) is almost reduced, connected and fragment one-to-one. Obviously then \( T \) is one-to-one. Assume that \( T \) is mergeable at \( v \neq w \in V_i(T) \). By a cyclic shift we can assume \( i = 0 \). By Corollary \[22\] we know that there exists \( p \in \mathbb{P}(v,w) \). So \( L(p) \in C(T) \). But since \( T \) is fragment one-to-one \( p \) must then be a cycle, i.e. \( v = w \), a contradiction. So \( T \) is nonmergeable. \( \square \)

Since minimal linear trellises are nonmergeable (Corollary \[7\] and one-to-one, we conclude that:

**Corollary 8.** Any minimal linear trellis is fragment one-to-one.

In \[18\] it was proven that any KV-trellis is nonmergeable, thus we can extend the above corollary (recall that any minimal linear trellises is a KV-trellises).

**Corollary 9.** Any KV-trellis is fragment one-to-one.

However, Example IV.12 of \[18\] shows that a nonmergeable one-to-one linear trellis does not have to be a KV-trellis. Thus not all fragment one-to-one linear trellises are KV-trellises.

Note that the proof given in \[18\] of KV-trellises being nonmergeable requires quite a long detour into BCJR-trellises (see the same reference for the definition). In fact it is proven there that special types of BCJR-trellises are nonmergeable and that KV-trellises are isomorphic to such special trellises. In sight of our new characterization of nonmergeable (one-to-one) trellises, it would be interesting to see if it is possible to give a direct proof that KV-trellises are fragment one-to-one. This would also make more clear in what circumstances linear trellises fail to be KV-trellises. We leave this problem for future investigations.
5.3 Classifying nonmergeable trellises via multicycle codes

In general a linear code has several minimal linear trellis representations, and even more non-
mergeable trellis representations. The problem of how can we distinguish and thus classify these
representations has never been posed before. Thanks to the characterization from the previous
subsection we can now address this problem for the class of nonmergeable, one-to-one, reduced,
linear trellises (and so in particular for minimal linear trellises). Our result will tell us that it is
possible to classify these trellises by means of edge-label sequences of closed paths of length longer
than the length of $T$.

To that purpose we first introduce some terminology. Let $T$ be a trellis of length $n$. A closed
path in $T$ starting at $V_0(T)$ of length greater than $n$ will be called a multicycle of $T$. Clearly a
multicycle must have length in for some $i > 1$. A multicycle of length in will be also called an
$i$-cycle of $T$. The code of edge-label sequences of $i$-cycles will be denote by $C^i(T)$ and it will be referred to also as the $i$-th code represented by $T$. So, a trellis does not represent
merely a single code, but rather the sequence of codes \{ $C^i(T)$ $\} \infty_{i=1}$.

We can now state and prove the announced result:

**Theorem 20.** Let $T$, $T'$ be fragment one-to-one, connected, and reduced linear trellises. Suppose
that

$$C^i(T) = C^i(T')$$

for some $i > 1$. Then $T$ and $T'$ are isomorphic.

**Proof.** We use Corollary 3 of the Isomorphy Theorem \cite{7}. We need to show that $C_{(a,l)}(T) = C_{(a,l)}(T')$ for all nondegenerate spans $(a, l)$. By a cyclic shift, it is sufficient to show it for conventional spans. So, let $(a, l)$ be a conventional span, and let $\alpha \in C_{(a,l)}(T)$. Then clearly

$$\beta := \alpha 0 \ldots 0 \in C^i(T)$$

where we are appending $(i - 1)n$ zeros to the right of $\alpha$. Since $C^i(T) = C^i(T')$, there must exist $v_j \in V_j(T')$, $j = 0, \ldots, in - 1$, such that

$$\lambda := v_0\beta_0v_1\beta_1 \ldots v_{in-1}\beta_{in-1}$$

is an $i$-cycle of $T'$. Now, since $\alpha \in C_{(a,l)}(T)$, the edge-label sequence of the path

$$v_{a+l+1}\beta_{a+l+1}v_{a+l+2}\beta_{a+l+2} \ldots v_{in-1}\beta_{in-1}v_0\beta_0 \ldots v_a$$

is the zero word. But since $i > 1$, this path has length at least $n$, where $n$ is the length of $T'$, so all its vertices must be zero, because $T'$ is fragment one-to-one. So we conclude that

$$v_0\beta_0v_1\beta_1 \ldots v_{n-1}\beta_{n-1} \in S_{(a,l)}(T')$$

and thus $\alpha \in C_{(a,l)}(T')$. So $C_{(a,l)}(T) \subseteq C_{(a,l)}(T')$. Symmetrically $C_{(a,l)}(T') \subseteq C_{(a,l)}(T)$, and we are done.

**Corollary 10.** Let $T$, $T'$ be nonmergeable, one-to-one, and reduced linear trellises (e.g. minimal linear trellises, or KV-trellises). If $C^i(T) = C^i(T')$ for some $i > 1$ then $T$ and $T'$ must be isomorphic.
This is a striking result since it tells us that the code of 2-cycles $C^2(T)$ (or $C^i(T)$ for any $i > 1$) completely determines a trellis $T$ in the mentioned class (which is a large enough class for potential applications, especially since it contains the whole class of KV-trellises). In particular, while in general a linear code $C$ has several different minimal linear trellis representations $T_1, \ldots, T_r$ which thus satisfy $C = C(T_1) = \ldots = C(T_r)$, they must also satisfy $C^2(T_i) \neq C^2(T_j)$ for $i \neq j$. See Example 9 for instance.

Note also that this result extends what is known for conventional trellises. In fact, it is known that a nonmergeable (equivalently, minimal) conventional trellis for a linear code $C$ is completely determined by $C$. But if $T$ and $T'$ are conventional trellises then $C(T) = C(T')$ if and only if $C^i(T) = C^i(T')$, since $C^i(T) = C(T) \times \ldots \times C(T)$. Thus the above theorem includes as a special case the uniqueness of minimal conventional trellises. In addition, it proves that a minimal linear trellis $T$ is conventional (and so it is the unique minimal conventional trellis) if and only if

$$C^2(T) = C(T) \times C(T)$$

Moreover this result gives a new method to determine whether or not two minimal linear trellises are equal (i.e. isomorphic). Indeed this method is alternative to the methods given in Sections 3 and 4 for general linear trellises, which require the knowledge respectively of all span subcodes and of an elementary trellis factorization, while this information is not required in the above result. In sight of potential applications it would be interesting to determine which method is quicker, i.e. which one yields the algorithm with lowest complexity. We leave this question for future investigations.

We can also apply the above result to deduce an interesting characterization of self-duality for KV-trellises. In order to do so, we need first a definition and a lemma. Given a trellis $T$ of length $n$ and $i \geq 1$ we define the $i$-cover of $T$ as the trellis $T^i$ of length $in$ given by

$$V_j(T^i) := V_j(T)$$
$$E_j(T^i) := E_j(T)$$

for all $j \in \mathbb{Z}_{in}$. By definition we have $V_j(T^i) = V_{j+n}(T^i)$ and $E_j(T^i) = E_{j+n}(T^i)$ for all $j \in \mathbb{Z}_{in}$. Thus the trellis diagram of $T^i$ is obtained by continuing $i$ times the trellis diagram of $T$. In graph-theoretical language $T^i$ is the unique (directed) graph $i$-cover of $T$ whose girth is $i$ times the girth of $T$, which justifies the terminology. Clearly

$$C^i(T) = C(T^i)$$

i.e. multicycles of $T$ are cycles in trellis covers of $T$.

Now, we have the following:

**Lemma 2.** Let $i \geq 1$. If $T$ is linear then so is $T^i$, and

$$(T^i)^\perp = (T^\perp)^i$$

As a consequence

$$(C^i(T))^\perp = C^i(T^\perp)$$

In particular if $T$ is self-dual then $C^i(T)$ is self-dual for all $i \geq 1$.

**Proof.** By definition of $T^i$ it is clear that if $T$ is linear so is $T^i$. On the other hand $T^\perp$ is defined locally (i.e. $V_i(T^\perp)$, $V_{i+1}(T^\perp)$, and $E_i(T^\perp)$ depend only on $V_i(T)$, $V_{i+1}(T)$, and $E_i(T)$), and so,
Corollary 11. Let \( T \) and \( T' \) be KV-trellises. Then \( T^\perp \sim T' \) if and only if
\[
(C^i(T))^\perp = C^i(T')
\]
for some \( i > 1 \). In particular, if \( C^i(T) \) is self-dual for some \( i > 1 \) then \( T \) is self-dual (and so \( C^i(T) \) is self-dual for all \( i \geq 1 \)).

Proof. By Theorem IV.3 of [19] we know that \( T^\perp \) is also a KV-trellis, so the statement follows from Corollary 10 and the above lemma.

5.4 Determining and counting minimal linear trellises

Let \( C \) be an \([n,k]\) linear code (with full support). We want to determine and consequently count all the minimal linear trellises for \( C \) with the same underlying graph structure. As we pointed out before taking products of elementary trellises from a single characteristic set of \( C \) is not sufficient for that task. We will make use of what we have proved in Section 4.

We start with proving the following important theorem.

Theorem 21. Let \( T = \alpha^1|s_1 \otimes \ldots \otimes \alpha^k|s_k \) be a minimal linear trellis for \( C \). Let \( \beta^1, \ldots, \beta^k \in C \) such that \( s_1 \) is a minimal span of \( \beta^i \) for all \( i \). Then \( \beta^1, \ldots, \beta^k \) is a basis of \( C \). As a consequence, \( T' = \beta^1|s_1 \otimes \ldots \otimes \beta^k|s_k \) is also a minimal linear trellis for \( C \).

Proof. By changing generators one at a time it is clearly sufficient to prove the case where we change only one generator, say \( \alpha^i \neq \beta^i \) and \( \alpha^i = \beta^i \) for \( i = 2, \ldots, k \). Now, assume that \( \beta^1, \alpha^2, \ldots, \alpha^k \) are linearly dependent. Then
\[
\beta^1 = x_2\alpha^2 + \ldots + x_k\alpha^k
\]
for \( x_i \in \mathbb{F} \). Since \( s_1 \) is a minimal span for both \( \alpha^1 \) and \( \beta^1 \), there exists \( v \in \mathbb{F} \) such that \( v := \alpha^1 - y\beta^1 \) has a span \( s \leq s_1 \). It follows that \( v, \alpha^2, \ldots, \alpha^k \) is a basis: if not then \( v \in \mathbb{F}\alpha^2 + \ldots + \mathbb{F}\alpha^k \), and so also
\[
\alpha^1 \not\in \mathbb{F}\beta^1 + \mathbb{F}\alpha^2 + \ldots + \mathbb{F}\alpha^k \subseteq \mathbb{F}\alpha^2 + \ldots + \mathbb{F}\alpha^k
\]
which is impossible by our assumption. But then we conclude that the trellis \( T' := v|s \otimes \alpha^2|s_2 \ldots \otimes \alpha^k|s_k \) represents \( C \) and is smaller than \( T \), which is a contradiction.

Note that the above theorem was given as an open problem in [18]. We have been later informed by Gluesing-Luerssen that Elizabeth Weaver has independently proved it in the related context of counting characteristic matrices.

Combined with Theorem 18 the above theorem yields the following crucial corollary:
Corollary 12. Let $T$ be a minimal linear trellis for an $[n,k]$ linear code and let $\chi$ be a characteristic set of $C$. Then $T$ is structurally isomorphic to $\bigotimes_{i=1}^{k} \alpha^i | s_i$ for some $\alpha^1 | s_1, \ldots, \alpha^k | s_k \in \chi$.

Now, let
\[ \chi = \{ \alpha^s | s \in S(C) \} \]
be a fixed characteristic set of $C$, where $S(C)$ is the characteristic span set of $C$. Let $S \subseteq S(C)$ be a subset of $k$ spans for which there exists a minimal linear trellis for $C$ whose span set is precisely $S$. By the above theorem $\bigotimes_{s \in S} \alpha^s | s$ is also a minimal linear trellis for $C$. Then we have the following main result:

Theorem 22. Let $T$ be a minimal linear trellis. Then $T$ is a minimal linear trellis for $C$ which is structurally isomorphic to the minimal linear trellis $\bigotimes_{s \in S} \alpha^s | s$ for $C$ if and only if
\[ T \sim \bigotimes_{s \in S} (\alpha^s + w^s) | s \]
for some $w^s \in \langle \alpha^s | s' < s, s' \in S(C) \setminus S \rangle$, $s \in S$. Moreover, if
\[ \bigotimes_{s \in S} (\alpha^s + w^s) | s \sim \bigotimes_{s \in S} (\alpha^s + w'^s) | s \]
for some $w^s, w'^s \in \langle \alpha^s | s' < s, s' \in S(C) \setminus S \rangle$, $s \in S$, then $w^s = w'^s$ for all $s \in S$.

Proof. Throughout we will reserve the notation $w^s$ for elements of $\langle \alpha^s | s' < s, s' \in S(C) \setminus S \rangle$. Now, assume that $T$ is a minimal linear trellis for $C$ which is structurally isomorphic to $\bigotimes_{s \in S} \alpha^s | s$. Then by the Factorization Theorem and Theorem 12 we have that
\[ T \sim \bigotimes_{s \in S} (\alpha^s + w^s) | s \]
for some $\beta^s$, $s \in S$. Since $T$ is minimal, $s$ must be a minimal span of $\beta^s$. By Theorem 18 for each $s \in S$ there exists $r \geq 0$ such that $\sigma^r(s)$ is an atomic span of $\sigma^r(C)$. Since $\chi = \{ \alpha^s | s \in S(C) \}$ is a characteristic set, by properties of atomic bases it follows that $\beta^s \in \langle \alpha^s | s' < s, s' \in S(C) \rangle$ for each $s \in S$. Thus for each $s \in S$ we have
\[ \beta^s = x^s \alpha^s + v^s + w^s \]
for some $x^s \in \mathbb{F}$, $v^s \in \langle \alpha^s | s' < s \rangle$, and $w^s$. By rescaling we can assume that $\beta^s = \alpha^s + v^s + w^s$. In particular $\beta^s = \alpha^s + w^s$ for all the minimal spans $s$ in $S$. Now, by Theorem 13 we know that if $\bigotimes_{s \in S} c^s | s$ is a factorization of $T$ and $b^s \in \langle \alpha^s | s' < s \rangle$ then
\[ \bigotimes_{s \in S} c^s + b^s | s \]
is also a factorization of $T$. Thus starting with the spans $s \in S$ directly above the minimal spans of $S$ and going up, we can transform each codeword $\beta^s$ into the form $\alpha^s + w^s$ by adding an appropriate multiple of the ones with smaller span, and so get the sought factorization of $T$.

Vice versa, assume $T \sim \bigotimes_{s \in S} (\alpha^s + w^s) | s$. Clearly $T$ is structurally isomorphic to $\bigotimes_{s \in S} \alpha^s | s$, since the spans of the factors are the same. Also, it is clear that $s \in S$ is a minimal span of $\alpha^s + w^s$ since the characteristic spans all start and end at different positions. But then by Theorem 21 we deduce that $T$ must be minimal for $C$.

Finally, assume that
\[ T \sim \bigotimes_{s \in S} (\alpha^s + w^s) | s \sim \bigotimes_{s \in S} (\alpha^s + w'^s) | s \]
By Theorem 13 we have that

\[ \langle \alpha^s + w^s \rangle + \sum_{s' < s} C_{s'}(T) = \langle \alpha^s + w'^s \rangle + \sum_{s' < s} C_{s'}(T) \]

for every \( s \in S \). Since \( T \) is one-to-one we must have

\[ \alpha^s + w^s \notin \sum_{s' < s} C_{s'}(T) \]

So from the above equality we deduce that

\[ w^s - w'^s \in \sum_{s' < s} C_{s'}(T) = \langle \alpha^{s'} + w'^s \mid s' < s, s' \in S \rangle \]

On the other hand \( w^s - w'^s \in \langle \alpha^{s'} \mid s' < s, s' \in S(C) \rangle \). But if \( A \) and \( B \) are disjoint sets of characteristic spans all sitting inside the same span then \( \langle \alpha^s \mid s \in A \rangle \cap \langle \alpha^s \mid s \in B \rangle = 0 \), since different characteristic spans start and end at different positions. So \( w^s = w'^s \) for all \( s \in S \). \( \square \)

A straight application of the above theorem gives us the possibility to count minimal linear trellises with the same underlying graph structure.

**Corollary 13.** Let \( T = \otimes_{s \in S} \alpha^s|s \) be a minimal linear trellis for \( C \). Then the number of distinct minimal linear trellises for \( C \) that are structurally isomorphic to \( T \) is

\[ \exp_q \left( \sum_{s \in S} \# \{ s' < s, s' \in S(C) \setminus S \} \right) \]  

(4)

From the above we deduce immediately the following two corollaries for trellises and codes with special characteristic span distributions.

**Corollary 14.** Let \( T = \otimes_{s \in S} \alpha^s|s \) be a minimal linear trellis for \( C \) such that no span in \( S \) contains a characteristic span of \( C \) not in \( S \). Then there are no other minimal trellises for \( C \) with same graph structure as that of \( T \). In particular, if \( C \) is self-dual and \( T \) is structurally isomorphic to \( T^{-1} \) then \( T \) is self-dual.

**Corollary 15.** If there are no containments between the characteristic spans of \( C \) then two distinct minimal linear trellises for \( C \) are never structurally isomorphic. In particular, this holds true for cyclic codes.

**Remark 14.** By similar arguments to the ones used in the proof of Theorem 22 one can show that Corollary 14 (and hence also Corollary 15) holds for KV-trellises too.

By Corollary 12 and Theorem 22 to determine and count all the minimal linear trellises for a given \([n, k]\) code \( C \) we can proceed as follows:

1. Compute a characteristic set \( \chi \) of \( C \) (using for example “Algorithm A” from [24]).
2. Find all the possible KV-trellises yielded by \( \chi \) (i.e. find all the possible subsets of \( k \) elements of \( \chi \) whose codewords are linearly independent).
3. Apply a sorting algorithm to the so found list of KV-trellises to find out which of those are minimal.
4. Apply Theorem 22 (and Corollary 13) to each minimal linear trellis so found.

Example 9. Let $C = \langle 01010, 11111 \rangle$. Then a characteristic matrix for $C$ is given by

$$
\begin{pmatrix}
01010 & (1, 2) \\
01010 & (3, 3) \\
10101 & (0, 4) \\
10101 & (2, 3) \\
10101 & (4, 3)
\end{pmatrix}
$$

A trellis for $C$ from this characteristic matrix can be built only as a product of one of the first rows with one of the last three rows, which gives 6 possibilities. All these possibilities turn out to be minimal linear trellises. Five of them are shifted conventional trellises. The remaining one is the product of the second and third row, and has span set $S = \{(3, 3), (0, 4)\}$. Note that $(3, 3)$ contains no other characteristic span, while $(0, 4)$ contains $(1, 2)$. Thus there are precisely two nonisomorphic minimal linear trellises for $C$ with span distribution equal to $S$. These are depicted just below:

01010$(3, 3) \otimes 10101$(0, 4) =

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,0) node[midway,above] {11} node[below] {00};
\draw[->] (1,0) -- (2,0) node[midway,above] {11} node[below] {01};
\draw[->] (2,0) -- (3,0) node[midway,above] {11} node[below] {01};
\draw[->] (3,0) -- (4,0) node[midway,above] {11} node[below] {00};
\end{tikzpicture}
\end{center}

01010$(3, 3) \times 11111$(0, 4) =

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,0) node[midway,above] {11} node[below] {00};
\draw[->] (1,0) -- (2,0) node[midway,above] {11} node[below] {01};
\draw[->] (2,0) -- (3,0) node[midway,above] {11} node[below] {01};
\draw[->] (3,0) -- (4,0) node[midway,above] {11} node[below] {00};
\end{tikzpicture}
\end{center}

In particular, the minimal trellis 01010$(3, 3) \otimes 11111$(0, 4) is not equal to any KV-trellis coming from the above characteristic set, while 01010$(3, 3) \otimes 10101$(0, 4) is not equal to any KV-trellis coming from the only other possible characteristic set of $C$. We conclude also that in total $C$ has 7 minimal linear trellises.

Note that the 5 minimal linear trellises which are shifted conventional trellises give rise to the following codes of 2-cycles:

$$C \times C, \sigma(C \times C), \sigma^2(C \times C), \sigma^3(C \times C), \sigma^4(C \times C)$$

The other two trellises instead give rise to:

$$C^2(01010)(3, 3) \otimes 10101$(0, 4) =
= \langle 0101010101, 0101001010, 0110100100, 011111101 \rangle$$

$$C^2(01010)(3, 3) \otimes 11111$(0, 4) =
= \langle 1010110101, 0101001010, 1110101000, 1101101111 \rangle$$

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As predicted by Corollary 10 one can check that these 7 codes of length 10 are all different.

Below we give another interesting example of the potential application of the above results, where we deduce that the Golay trellis \([5]\) is self-dual.

**Example 10.** Consider the Golay trellis \(T_G\) presented in \([2]\) (see also \([4]\)) as

\[
T_G = \bigotimes_{i=0}^{2} \sigma^8_i (T_1 \otimes T_2 \otimes T_3 \otimes T_4)
\]

where

- \(T_1 = 110111011100000000000000\) \((0, 9)\)
- \(T_2 = 001111010111000000000000\) \((2, 9)\)
- \(T_3 = 000110111101110000000000\) \((4, 9)\)
- \(T_4 = 000001110111011100000000\) \((6, 9)\)

\(T_G\) is a minimal trellis for the binary \([24, 12, 8]\) Golay code \(G\) (with the given special coordinate ordering). It is actually a very special trellis, since it achieves simultaneously the minimum value for \(\prod_i |V_i(T)|\) and \(\max_i \{|V_i(T)|\}\), for any possible \(T\) for \(G\) under any possible coordinate ordering. Now, the spans of its factors have all length 9. Thus by Theorem 18, since \(G\) is self-dual, the other 12 characteristic spans have all length 15, and so cannot be contained in any span of \(T_G\). Hence by Corollary 15 there is no other minimal trellis for \(G\) with the same graph structure as that of \(T_G\). Moreover, since the dual trellis \(T_G^\perp\) has the same state-complexity profile as that of \(T_G\), we have that

\[
\prod_i |V_i(T_G^\perp)| = \prod_i |V_i(T_G)|
\]

from which follows that \(S(T_G^\perp)\) must be also made up of the 12 spans of length 9 in \(S(G)\) (as \(T_G^\perp\) also represents and hence is minimal for \(G = G^\perp\)), and so \(T_G^\perp\) is structurally isomorphic to \(T_G\). But then \(T_G \sim T_G^\perp\), i.e. \(T_G\) is self-dual.

### 5.5 Improving iterative/LP trellis decoding through the complete classification of minimal linear trellises

The complexity of trellis decoding is directly proportional to trellis size, so, to achieve low complexity it is necessary to search for the smallest trellis representations of codes. If trellis size is all what one is interested in then obviously the way labels are arranged on a trellis does not matter (as long as the trellis represents the prescribed code), and it is thus sufficient to classify all trellis representations up to structural isomorphism. The work done by Koetter/Vardy \([24]\) goes precisely in that direction as it actually focuses on the structural classification of minimal linear trellises for linear codes.

On the other hand, the performance and behavior of iterative and LP trellis decoding is affected by so-called pseudocodewords (see \([10, 15, 16, 20, 25]\)), which gives importance to sorting trellises also with respect to their pseudocodewords. In fact, one wants to find trellises that yield few bad pseudocodewords when performing iterative/LP decoding.

Now, consider the two minimal linear trellises \(T = 01010|\langle 3, 3\rangle|01010|0, 4\rangle\) and \(T' = 01010|\langle 3, 3\rangle|11111|0, 4\rangle\) for the code \(C = \langle 01010, 10101 \rangle\) depicted in Example 9. \(T'\) yields the (unscaled) pseudocodeword 12101 \(\in \mathbb{R}^5\) (arising from the unique 2-cycle with edge-label sequence 1110101000), while as one can easily check all the (unscaled) pseudocodewords of \(T\) are sums of the codewords of \(C\) seen as a subset of \(\mathbb{R}^5\) (via the map \(\mathbb{F}_2 \ni 0 \mapsto 0 \in \mathbb{R}, \mathbb{F}_2 \ni 1 \mapsto 1 \in \mathbb{R}\)). In particular the convex cones generated by the pseudocodewords of \(T\) and \(T'\) are different (the shape of such cones strongly influences the behavior of iterative/LP decoding).
This shows that it is possible to have two different (i.e. nonisomorphic) but structurally isomorphic (minimal) linear trellises for the same code yielding different pseudocodewords. In other words, a rearrangement of edge-labels that preserves the represented code may still change the yielded pseudocodewords. Such phenomenon (whose discovery we actually had announced first in \cite{7}) was never observed before in the literature, and it implies that in order to sort all the (minimal) linear trellises for a given linear code with respect to pseudocodewords it is not sufficient to do a classification of such trellises up to structural isomorphism. We need instead a complete classification. For minimal linear trellises this completed classification can be feasibly carried out as described in the paragraph before Example \cite{9} which is the result of joining our work with the one of Koetter/Vardy \cite{24}. By carrying out this classification we can thus find the trellises which at the same time achieve the lowest decoding complexity and have the best behavior and performance for iterative/LP decoding.

**Remark 15.** One can easily check that if a linear trellis is optimized with respect to iterative/LP decoding (in the sense that it yields the fewest possible pseudocodewords) then it must be one-to-one. So, for such decoding purposes, even if we want to explore nonminimal linear trellis representations we can still restrict ourselves to the classification of those that are one-to-one.

6 Further applications to quasi-cyclic and nonreduced linear trellises

6.1 Quasi-cyclic factorizations and isomorphisms of quasi-cyclic linear trellises

Let $m \geq 1$ divide the length of $T$. We say that $T$ is $m$-quasi-cyclic if it is isomorphic to a trellis $T'$ satisfying

$$V_i(T') = V_{i+m}(T')$$
$$E_i(T') = E_{i+m}(T')$$

for all $i \in \mathbb{Z}_n$. Note that if $T$ is linear then the isomorphic trellis $T'$ is also linear, and they are linearly isomorphic.

Given a trellis $T$ obviously the $i$-cover $T^i$ is an $i$-quasi-cyclic trellis, since by its very definition $T^i$ satisfies the above equalities. Another natural way to construct quasi-cyclic trellises is given as follows: given a trellis $T$ of length $n$ and an $m$ dividing $n$ then it is easily proven that

$$T \otimes \sigma^m(T) \otimes \sigma^{2m}(T) \otimes \ldots \otimes \sigma^{(\frac{n}{m}-1)m}(T)$$

is $m$-quasi-cyclic. For example, if $T = 101|0, 2)$ and $m = 1$, by definition of trellis product we get that $T \otimes \sigma(T) \otimes \sigma^2(T)$ is precisely equal to

![Diagram of trellis product](attachment:image.png)
which is isomorphic to (just swap the vertices labeled 100 and 001 at time index 2, and then relabel all the vertices)

Similarly, the Golay trellis of Example 10 (which is depicted in [5]) is 3-quasi-cyclic.

Now, we have the following theorem:

**Theorem 23.** Let \( T \) be a connected, reduced, linear trellis of length \( n \) such that \( V_i(T) = V_{i+m}(T) \) and \( E_i(T) = E_{i+m}(T) \) for all \( i \in \mathbb{Z}_n \), and some \( m \) dividing \( n \). Consider the shift map of cycles

\[
\beta := \sigma^m : S(T) \to S(\sigma^m(T)) = S(T)
\]

Let \( B \) be a product basis of \( T \) and let \( B_{[0,m)} \) be the subset of those \( \lambda \in B \) whose span starting point \( a \) satisfies \( 0 \leq a < m \). Then

\[
\bigcup_{i=0}^{(n/m) - 1} \beta^i(B_{[0,m)})
\]

is a product basis of \( T \).

**Proof.** Note that \( \beta \) is a linear isomorphism of \( S(T) \) with itself. We clearly have that \( \beta(S_{(a,m,l)}(T)) = S_{(a,l)}(T) \) for all spans \( (a,l) \). So, a subset \( S \subset S_{(a,m,l)}(T) \) is a lifting of a basis of \( S_{(a,m,l)}(T) \) if and only if \( \beta(S) \subset S_{(a,l)}(T) \) is a lifting of a basis of \( S_{(a,l)}(T) \). Thus, by Observation 4 and Theorem 3 it follows that the union \( \bigcup_{i=0}^{(n/m) - 1} \beta^i(B_{[0,m)}) \) is a product basis of

\[
S(T) = S_{n-1}(T) = \sum_{a \in \mathbb{Z}_n, l \leq n-1} S_{(a,l)}(T)
\]

where the first equality follows from \( T \) being connected, and so we are done.

By combining the above theorem with the Factorization Theorem and Theorem 4 we get also the following corollaries:

**Corollary 16.** The span distribution \( S(T) \) of any connected, \( m \)-quasi-cyclic, reduced, linear trellis \( T \) of length \( n \) is decomposed into disjoint orbits of order \( n/m \) under the action of \( \sigma^m \).

**Corollary 17.** Let \( \otimes_{i=1}^r T_i \) be a product of connected elementary trellises of length \( n \). If \( \otimes_{i=1}^r T_i \) is \( m \)-quasi-cyclic then

\[
\otimes_{i=1}^r T_i \sim \otimes_{j=0}^{(n/m)-1}(\beta^j(T_{i_1}) \otimes \ldots \otimes \beta^j(T_{i_s}))
\]

where \( T_{i_1}, \ldots, T_{i_s} \) are those elementary trellises whose span starting point \( a \) satisfies \( 0 \leq a < m \) and \( \beta = \sigma^m \).

The above tells us that the above type of trellises admit a product basis/elementary trellis factorization with a quasi-cyclic structure.

We can also deduce the following theorem which we will use in the next subsection to extend results for reduced trellises to nonreduced ones. This theorem tells us that if two aforementioned trellises are isomorphic then they admit an isomorphism with a quasi-cyclic structure too.
Theorem 24. Let $T$ and $T'$ be two connected, reduced, linear trellises of same length $n$ and satisfying $V_i(T) = V_{i+m}(T)$, $V_i(T') = V_{i+m}(T')$, $E_i(T) = E_{i+m}(T)$, and $E_i(T') = E_{i+m}(T')$ for all $i \in \mathbb{Z}_n$, and some $m$ dividing $n$. Assume that $T$ and $T'$ are (linearly) isomorphic. Then there exists a linear isomorphism $f : T \to T'$ such that
\[ \beta \circ S(f) = S(f) \circ \beta \]
for the shift operator $\beta = \sigma^m$, i.e. $f_i = f_{i+m}$ for all $i$.

Proof. Let $\mathcal{B}$ be a product basis of $T$ and $g : T \to T'$ a linear isomorphism. The induced isomorphism $S(g) : S(T) \to S(T')$ sends $\mathcal{B}$ to a product basis $\mathcal{B}'$ of $T'$ and satisfies $S(g)(\mathcal{B}_{[0,m]}) = \mathcal{B}'_{[0,m]}$ (we are using the notation from Theorem 23). By Theorem 23
\[ \hat{\mathcal{B}} := \bigcup_{i=0}^{n/m-1} \beta^i(\mathcal{B}_{[0,m]}), \quad \hat{\mathcal{B}}' := \bigcup_{i=0}^{n/m-1} \beta^i(\mathcal{B}'_{[0,m]}) \]
are product bases respectively of $T, T'$. So we can define a linear isomorphism $F : S(T) \to S(T')$ by the formula
\[ F(\beta^i(\lambda)) := \beta^i(S(g)(\lambda)) \]
for all $i$ and all $\lambda \in \mathcal{B}_{[0,m]}$. By construction $F(\hat{\mathcal{B}}) = \hat{\mathcal{B}}'$ and $\beta \circ F = F \circ \beta$.

Now, since $L \circ S(g) = L$, i.e. $S(g)$ preserves edge-labels, it follows that
\[ L(F(\beta^i(\lambda))) = L(\beta^i(S(g)(\lambda))) = \beta^i(L(S(g)(\lambda))) = \beta^i(L(\lambda)) = L(\beta^i(\lambda)) \]
for all $i$ and all $\lambda \in \mathcal{B}_{[0,m]}$, and so $F$ preserves edge-labels too. Also, being $g$ a linear isomorphism we have $[S(g)(\lambda)] = [\lambda]$ for all $\lambda \in \mathcal{B}$, and so
\[ [F(\beta^i(\lambda))] = [\beta^i(S(g)(\lambda))] = \beta^i([S(g)(\lambda)]) = \beta^i([\lambda]) = [\beta^i(\lambda)] \]
for all $i$ and all $\lambda \in \mathcal{B}_{[0,m]}$, i.e. $[F(\lambda)] = [\lambda]$ for all $\lambda \in \hat{\mathcal{B}}$, from which also follows that
\[ [F^{-1}(\lambda)] = [F(F^{-1}(\lambda))] = [\lambda] \]
for all $\lambda \in \hat{\mathcal{B}}$. Thus $F(S_{(a,l)}(T)) = S_{(a,l)}(T')$ for all $(a,l)$. We conclude that $F = S(f)$ for some linear isomorphism $f : T \to T'$ (see Subsection 3.2). Finally, $\beta \circ F = F \circ \beta$ means that $\beta \circ S(f) = S(f) \circ \beta$.

6.2 Extending results to nonreduced trellises

We start with proving an important property which makes it possible to extend results for reduced linear trellises to nonreduced linear trellises, as we shall see.

Theorem 25. Let $T$ be a linear trellis. Then there exists $i \geq 1$ such that $T^i$ is reduced.

Proof. By Corollary 21 we know that for each edge $e$ of $T$ there exists $i \geq 1$ such that $e$ belongs to an $i$-cycle of $T$. Also, clearly, if $e$ belongs to an $i$-cycle, then it belongs to an $ir$-cycle for all $r \geq 1$. Thus, by taking a common multiple, there exists $m \geq 1$ such that each edge of $T$ belongs to some $m$-cycle. But then it is clear that $T^m$ is reduced (since its diagram is just $m$ concatenated copies of the diagram of $T$).
We can apply the above result to prove that connected linear trellises are determined by their covers. First an important lemma:

**Lemma 3.** If $T$ is a connected linear trellis and $i \geq 1$ then $T^i$ is also connected.

**Proof.** Let $v \in V_j(T^i) = V_j(T)$. Then there exists a path $p$ in $T$ from $v$ to $0 \in V_0(T)$. Clearly $p$ yields a path in $T^i$ from $v \in V_j(T^i)$ to $0 \in V_{rn}(T^i)$ for some $r \geq 0$. But all the zero vertices are connected, thus $T^i$ is connected. \qed

**Theorem 26.** Let $T$ and $T'$ be connected linear trellises, and let $i > 1$. Then

$$T \simeq T' \iff T^i \simeq (T')^i$$

**Proof.** Clearly $T \simeq T' \Rightarrow T^i \simeq (T')^i$. By the previous lemma and theorem there exist $s, s' \geq 1$ such that $T^s$ and $(T')^{s'}$ are both reduced and connected. So, if $T^i \simeq (T')^i$ then $T^{iss'}$ and $(T')^{iss'}$ are linearly isomorphic, connected and reduced (if a trellis $T$ is reduced then $T^j$ is clearly reduced too for any $j \geq 1$). Then by Theorem 24 we are done. \qed

Note that linearity is necessary both for Theorem 25 and Lemma 3 as one can check by easy examples. Also, the connectedness hypothesis is necessary for Theorem 26. We depict below nonisomorphic trellises $T$ and $T'$ ($T$ has 3 connected components while $T'$ has 4) such that $T^2 \simeq (T')^2$:

$$T = \begin{array}{cccc}
1 & 1 & 1 \\
10 & 10 & 10 \\
01 & 01 & 01 \\
00 & 00 & 00
\end{array} \quad \neq \quad \begin{array}{cccc}
1 & 1 & 1 \\
10 & 10 & 10 \\
01 & 01 & 01 \\
00 & 00 & 00
\end{array} = T'$$

$$T^2 = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
10 & 10 & 10 & 10 \\
01 & 01 & 01 & 01 \\
00 & 00 & 00 & 00
\end{array} \simeq (T')^2$$

An immediate consequence of the above two theorems is that we can extend the important Theorems 8 and 9 to the nonreduced case too:

**Corollary 18.** Two connected, nonreduced, linear trellises are isomorphic if and only they are linearly isomorphic.

**Corollary 19.** The linear structure of a connected, nonreduced, linear trellis is essentially unique (as in theorem 9).

We can also extend Theorem 20 to nonreduced trellises, i.e. prove that the sequence $\{C^i(T)\}_{i=1}^\infty$ determines trellises (in the mentioned class) even in the nonreduced case. We will need the following theorem.
Theorem 27. Let $T$ be a connected linear trellis. If $T_i$ is reduced then $T_{i+1}$ is reduced.

Proof. By Lemma 3 $T_i$ and $T_{i+1}$ are connected. By Theorem 31 we know that a connected linear trellis of length $n$ is reduced if and only if each vertex can be connected in both directions to a zero vertex by paths of length $n-1$. Thus for every vertex $v \in T_i$ there exist paths $v \to 0$ and $0 \to v$ of length $n-1$, where $n$ is the length of $T$. But then it is clear that the same holds for $T_{i+1}$. So $T_{i+1}$ is reduced too.

The connectedness hypothesis is necessary. For example, here we have a linear disconnected trellis $T$ such that $T^2$ is reduced while $T^3$ is not:

$$T = \begin{array}{c}
11 \\
10 \\
01 \\
00
\end{array} \quad T^2 = \begin{array}{c}
11 \\
10 \\
01 \\
00
\end{array} \quad T^3 = \begin{array}{c}
11 \\
10 \\
01 \\
00
\end{array}$$

Now we can prove our extension of Theorem 20

Theorem 28. Let $T$ and $T'$ be fragment, one-to-one, connected linear trellises. Let $i, j \geq 1$ such that $T_i$ and $(T')^j$ are reduced (by Theorem 25 such $i, j$ exist). Assume that $C^s(T) = C^s(T')$ for some $s > \max\{i, j\}$. Then $T \simeq T'$.

Proof. Let $h := \max\{i, j\}$ and $s > h$ such that $C^s(T) = C^s(T')$. By Theorem 27 we have that $T^h$ and $(T')^h$ are both reduced. But then an immediate adaption of the arguments used in the proof of Theorem 20 yields that $T^h \simeq (T')^h$, which by Theorem 26 implies that $T \simeq T'$.

We conclude this subsection by observing that a motivation for studying nonreduced linear trellises comes from the fact that these naturally arise by taking duals of reduced linear trellises or wrapped fragments of quasi-cyclic trellises (i.e. cutting $T$ at time indices $i, j$ such that $V_i(T) = V_j(T)$ and wrapping). For example the nonreduced linear trellis of length 8 depicted in [5] which represents the $[8, 4, 4]$ Hamming code is a fragment of the Golay trellis $T_G$ (see Example 10) from the same paper.

A Connectivity of linear trellises

We prove in this appendix some fundamental results on connectivity of linear trellises which have not appeared before in the literature. We will also make use of them in the paper. We use the notation $v \to w$ to mean a path from $v$ to $w$. 

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Remark 16. Connectedness is closely related to the notion of controllability in systems theory. See [13, 14] for relations between connectedness and other trellis properties from the “controllability” point of view.

We start with proving that for linear trellises there is no distinction between being connected by directed paths and being connected by undirected paths.

Theorem 29. Let $T$ be a linear trellis. Let $e$ be an edge in $T$ from $v$ to $w$. Then there exists a path in $T$ from $w$ to $v$.

Proof. Put $v_0 := v$, $v_1 := w$. Since all our trellises are trim the outdegree and indegree of each vertex of $T$ are positive, and so we can construct a doubly infinite sequence of vertices

$$\ldots v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$$

such that for each $i \in \mathbb{Z}$ there exists an edge in $T$ from $v_i$ to $v_{i+1}$. In particular, for all $i < j$ there exists a path from $v_i$ to $v_j$ of length $j - i$. Now, $T$ has finitely many vertices so we can find $i, j > 0$ such that $v_{-i} = v_{-i-n}$ and $v_j = v_{j+n}$, where $n$ is the length of $T$. Thus we get closed paths $v_{-i} \rightarrow p_{-i}, v_j \rightarrow q_{j}$ of length $n$. Take $i \leq i' < i + n$ such that $-i' \equiv j \mod n$. Changing the starting point, we can assume that $p$ starts (and so ends) at $v_{-i'}$. We also know that there exists a path $v_{-i'} \rightarrow s_{-i'}$ of length $j + i'$. Note that $v_{-i'}$ and $v_j$ belong to the same vertex set $V_h(T)$ for some $h \in \mathbb{Z}_n$, so $i' + j = rn$, for some $r \geq 1$. Let $v_{-i'} \rightarrow p_{-i'}$ be the closed path of length $rn$ given by cycling $r$ times around $p$. Define $v_j \rightarrow q_{j}$ similarly. By linearity we get a path

$$s' = p^r + q^r - s$$

of length $rn$ from $v_j = v_{-i'} + v_j - v_{-i'}$ to $v_{-i'} = v_{-i'} + v_j - v_j$. Since the vertices $v = v_0$ and $w = v_1$ belong by construction to the path $v_{-i'} \rightarrow s_{-i'}$, we can use it in conjunction with $v_j \rightarrow q_{j}$ to reach $v$ from $w$, and so our proof is concluded.

Corollary 20. A linear trellis is connected if and only if it connected as an undirected graph.

Corollary 21. Each vertex and edge of a linear trellis belongs to some closed path.

Example 11. The following is a linear, connected, and nonreduced trellis.

```
00 01 10 11
01 10 11 00
11 10 00 01
```

Note that some edges of $T$ belong to cycles, i.e. closed paths of length 2, while other edges belong only to closed paths of length 4 or even length 6. For example, this happens respectively for the edges $000000 \in E_0(T)$, $10001 \in E_0(T)$, $00001 \in E_1(T)$.

The above results do not hold for nonlinear trellises as one can easily check.

The following theorem is another important consequence of linearity. It was also observed by Heide Gluesing-Luerssen (private communication). Note also that Lemma 6.8 of [23] can be obtained as a special case of it.
Theorem 30. Let $T$ be a linear trellis of length $n$. Suppose $T$ is almost reduced. Let $v, w \in V_i(T)$ for some $i \in \mathbb{Z}_n$, and suppose there exists a path in $T$ from $v$ to $w$. Then there exists a path from $v$ to $w$ of length $n$.

Proof. Assume we have a path $v \overset{p}{\rightarrow} w$ of length $rn$, $r > 1$. Then there are vertices $v_j \in V_i(T)$ for $j = 1, \ldots, r$, with $v_0 = v$, $v_r = w$, and paths $v_j \overset{p_j}{\rightarrow} v_{j+1}$ of length $n$ for $j = 0, \ldots, r - 1$. By hypothesis we also have closed paths $v_j \overset{q_j}{\rightarrow} v_j$ of length $n$ for $j = 1, \ldots, r$. By linearity we then get a path $\sum_{i=0}^{r-1} p_i - \sum_{j=1}^{r-1} q_i$ of length $n$ from

$v = v_0 = \sum_{j=0}^{r-1} v_j - \sum_{j=1}^{r-1} v_j$

to

$w = v_r = \sum_{j=0}^{r-1} v_{j+1} - \sum_{j=1}^{r-1} v_j$.

\(\square\)

Corollary 22. If $T$ is a connected, almost reduced, linear trellis of length $n$ then for each pair of vertices $v, w \in V_0(T)$ there exists paths $v \rightarrow w$ and $w \rightarrow v$ of length $n$.

Corollary 23. Let $T$ be a reduced linear trellis of length $n$. Assume $v$ is connected to some (and thus each) zero vertex. Then there exist paths $v \rightarrow 0$ and $0 \rightarrow v$ of length $n - 1$.

Proof. By inverting the direction of all edges it is sufficient to prove that there exists a path $v \rightarrow 0$ of length $n - 1$. Now, since all zero vertices are connected, from the assumption it follows that there exists a path $v \rightarrow 0$ of length $rn$ for some $r \geq 1$. Thus, from Theorem 30 there exists a path $v \overset{p}{\rightarrow} 0$ of length $n$, i.e. a path $v \overset{p'}{\rightarrow} w$ of length $n - 1$ and a path $w \overset{q'}{\rightarrow} 0$ of length 1, for some vertex $w$. Since $T$ is reduced there must exist a path $0 \overset{q}{\rightarrow} w$ of length $n - 1$. So $p' - q$ is a path of length $n - 1$ from $v = v - 0$ to $0 = w - w$. \(\square\)

As a consequence of the last corollary we can give an alternative characterization of reduced trellises in the connected case.

Theorem 31. Let $T$ be a connected linear trellis of length $n$. Then $T$ is reduced if and only if for each vertex $v$ of $T$ there exist paths $v \rightarrow 0$ and $0 \rightarrow v$ of length $n - 1$.

Proof. The “only if” part is due to Corollary 23. Vice versa, assume $e = vaw$ is an edge of $T$. We want to show that $e$ belongs to a cycle of $T$, i.e. that there exists a path $w \rightarrow v$ of length $n - 1$. By hypothesis we have paths $0 \overset{p}{\rightarrow} v$ and $w \overset{q}{\rightarrow} 0$ of length $n - 1$. The starting time indices of $p$ and $q$ are equal. Thus we can add them and get the path $p + q$ of length $n - 1$ from $w$ to $v$. \(\square\)

B Graphical characterization of span distributions

In this appendix we show how Theorem 12 can be proven by means of a direct graphical approach. This approach involves looking at the earliest intersections of paths starting along different edges from a fixed vertex. It turns out also that from this intersection data one can completely determine $S(T)$. As argued in the proof of Theorem 12 we need to consider only unlabeled trellises, so all trellises in this appendix will be unlabeled.
Now, let us first give some notation. Given a multiset $S$, we write $m(x,S)$ for the multiplicity of $x$ in $S$. If $e = vv'$ is an edge of a trellis $T$ then

$$
h(e) := v' \quad t(e) := v$$

are respectively the head and tail of $e$. Given two different edges $e \neq e'$ of $T$ such that $t(e) = t(e')$, we define

$$l(e,e')$$

to be the smallest $r \geq 0$ such there exist two (directed) paths $p = v_0 \ldots v_{r+1}$, $p' = v'_0 \ldots v'_{r+1}$ in $T$ satisfying $e = v_0v_1$, $e' = v'_0v'_1$ (so that $v_0 = v'_0$), and $v_{r+1} = v'_{r+1}$. If there is no path satisfying those conditions then we put $l(e,e') := \infty$. We define then the multiset

$$I(e) := \{l(e,e') | e' \neq e, t(e') = t(e)\}$$

**Example 12.** Consider the nonlinear trellis

Then $I(e) = \{\{1\}\}$, $I(e') = \{\{2\}\}$, and $I(e'') = \{\{1,1\}\}$.

For linear trellises to compute $l(e,e')$ one can fix $p$ and let only $p'$ vary. In fact the following holds.

**Observation 10.** Let $T$ be a linear trellis. Fix a path $p = v_0 \ldots v_{r+1}$ such that $e = v_0v_1$. Then $l(e,e') = \min\{r \geq 0 | \exists p' = v'_0 \ldots v'_{r+1} \text{ such that } e' = v'_0v'_1, v'_{r+1} = v_{r+1}\}$. 

**Proof.** This is an immediate consequence of linearity.

The highly symmetrical graph structure of linear trellises is further reflected in the following fundamental lemma.

**Lemma 4.** Let $T$ be a linear trellis of length $n$. Then:

- $l(e,e') \leq n - 1$
- $I(e) = I(e')$ if $t(e), t(e') \in V_a(T)$ for some $a \in \mathbb{Z}_n$

**Proof.** The inequality is clearly true for elementary trellises, and so it is also true for product of elementary trellises, i.e. linear trellises. Finally, the equality is an immediate consequence of the linearity of $T$.

In sight of the above, for a linear trellis $T$ and $a \in \mathbb{Z}_n$ it is legitimate to define

$$I_a(T) := I(e)$$

where $e$ is any edge of $T$ such that $t(e) \in V_a$. The next lemma tells us that $I_a(T)$ is determined only by those elementary factors of $T$ whose span starts at $a$. 

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Lemma 5. Let $T$ be a linear trellis, and let $T'$ be an elementary trellis with span not starting at $a \in \mathbb{Z}_n$. Then $I_a(T \otimes T') = I_a(T)$.

Proof. The equality follows immediately from the fact that any path $p$ in $T \otimes T'$ is given by a sequence of vertices $(v_0, v'_0), \ldots, (v_r, v'_r)$ where $(v_i, v'_i) \in V_i(T \otimes T') = V_i(T) \times V_i(T')$ for all $i$.

The above lemma is crucial for the following theorem, which is the key result:

Theorem 32. Let $T$ be an unlabeled linear trellis. Fix $a \in \mathbb{Z}_n$. Then for $l = 0, \ldots, n - 1$

$$1 + \sum_{i=0}^{l} m(i, I_a(T)) = \exp_q \left( \sum_{i=0}^{l} m((a, i), T) \right)$$

(5)

Proof. Since $T$ is an unlabeled linear trellis we have that $T = 0|(a_1, l_1) \otimes \cdots \otimes 0|(a_s, l_s)$ for some $a_j \in \mathbb{Z}_n$ and $l_j > 0$. By the above lemma we can assume without loss of generality that $a_j = a$ and $l_j < n$ for all $j = 1, \ldots, s$. By a shift we can also clearly assume that $a = 0$, so that $T$ is conventional. Since all the spans start at $a$, the outgoing degree of any $v \in V_a(T)$ with $a > 0$ is equal to one. But then both sides of (5) are equal to the number of cycles in $T$ that pass through $0 \in V_l(T)$.

For any fixed $a \in \mathbb{Z}_n$, equation (5) can be recursively solved for $m((a, i), T)$, $i = 0, \ldots, n - 1$, and the solution is unique. Thus the graph-theoretical data given by the multisets $I_a(T)$ gives us the possibility to find all nondegenerate spans along with their multiplicities. The multiplicity of the degenerate span $\mathbb{Z}_n$ is instead given by the logarithm of the number of connected components of $T$. Also, as the left-hand side of (5) depends only on $T$ and not on its factorization, this gives another proof of Theorem 12.

Example 13. Let $T$ be the linear unlabeled trellis

Then one easily computes:

- $I_0(T) = I(e^0) = \emptyset$
- $I_1(T) = I(e^1) = \{2\}$
- $I_2(T) = I(e^2) = \emptyset$
- $I_3(T) = I(e^3) = \{3, 4, 4\}$
- $I_4(T) = I(e^4) = \emptyset$

So, by equation (5) we get that $S_0(T) = S_2(T) = S_4(T) = \emptyset$, $S_1(T) = \{(1, 2)\}$, and $S_3(T) = \{(3, 3), (3, 4)\}$, where $S_i(T) := \{(a, l) \in S(T) | l = i\}$. Therefore $T = 0|(1, 2) \otimes 0|(3, 3) \otimes 0|(3, 4)$.
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