Some considerations in relation to the matrix equation $AXB = C$

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Abstract
In this paper we represent a new form of condition for the consistency of the matrix equation $AXB = C$. If the matrix equation $AXB = C$ is consistent, we determine a form of general solution which contains both reproductive and non-reproductive solutions. Also, we consider applications of the concept of reproductivity for obtaining general solutions of some matrix systems which are in relation to the matrix equation $AXB = C$.

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1. The reproductive equations

The general concept of the reproductive equations was introduced by S.B. Prešić in 1968. In this part of the paper we give the definition of reproductive equations and the most important statements related to the reproductive equations. Using the concept of reproductivity, in the next section, we obtain the general solutions of some matrix systems which are in relation to the matrix equation $AXB = C$.

Let $S$ be a given non-empty set and $J$ be a given unary relation of $S$. Then an equation $J(x)$ is consistent if there is at least one element $x_0 \in S$, so-called the solution, such that $J(x_0)$ is true. A formula $x = \phi(t)$, where $\phi : S \rightarrow S$ is a given function, represents the general solution of the equation $J(x)$ if and only if

$$(\forall t)J(\phi(t)) \land (\forall x)(J(x) \Rightarrow (\exists t)x = \phi(t)).$$

In this part of the paper we give the definition of reproductive equations and the fundamental statements related to the reproductive equations.

Definition 1.1. The reproductive equations are the equations of the following form:

$$x = \varphi(x),$$

where $x$ is a unknown, $S$ is a given set and $\varphi : S \rightarrow S$ is a given function which satisfies the following condition:

$$\varphi \circ \varphi = \varphi. \quad (1)$$

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The condition (1) is called the condition of reproductivity. The fundamental properties of the reproductive equations are given by the following two statements S.B. Prešić [30] (see also [8], [32]-[34] and [42]).

**Theorem 1.2.** For any consistent equation $J(x)$ there is an equation of the form $x = \varphi(x)$, which is equivalent to $J(x)$ being in the same time reproductive as well.

**Theorem 1.3.** If a certain equation $J(x)$ is equivalent to the reproductive one $x = \varphi(x)$, the general solution is given by the formula $x = \varphi(y)$, for any value $y \in S$.

Let us remark that a formula $x = \varphi(t)$, where $\varphi : S \rightarrow S$ is a given function, represents the reproductive general solution [6] of the equation $J(x)$ if and only if

$$\forall t)J(\varphi(t)) \land (\forall t)(J(t) \implies t = \varphi(t)).$$

S.B. Prešić was the first one who considered implementations of reproductivity on some matrix equations [30] (see also [17], [18] and [29]). The concept of reproductivity allows us to analyse various forms of the solution. General applications of the concept of reproductivity were also considered by J.D. Kečkić in [19], [20], J.D. Kečkić and S.B. Prešić in [24], S. Rudeanu in [36]-[38] and D. Banković in [2]-[6].

2. The matrix equation $AXB = C$

Let $m, n \in \mathbb{N}$ be natural numbers and $\mathbb{C}$ is the field of complex numbers. The set of all matrices of order $m \times n$ over $\mathbb{C}$ is denoted by $\mathbb{C}^{m\times n}$. By $\mathbb{C}^{m\times n}_a$ we denote the set of all $m \times n$ complex matrices of rank $a$. For $A \in \mathbb{C}^{m\times n}$, the rank of $A$ is denoted by rank$(A)$. The unit matrix of order $m$ is denoted by $I_m$ (if the dimension of unit matrix is known from the context, we shall omit the index which indicates the dimension and use the symbol $I$). Let $A = [a_{i,j}] \in \mathbb{C}^{m\times n}$. By $A_{i\rightarrow}$ and $A_{\downarrow j}$ we denote the $i$-th row of $A$ and the $j$-th column of $A$, respectively. Therefore,

$$A_{i\rightarrow} = (a_{i,1}, a_{i,2}, ..., a_{i,n}), \quad i = 1, ..., m$$

and

$$A_{\downarrow j} = (a_{1,j}, a_{2,j}, ..., a_{m,j})^T, \quad j = 1, ..., n.$$}

The matrix equation

$$AXB = C \tag{2}$$

was considered by many authors ([11]-[13], [17], [18], [21]-[26], [29], [34], [39] and [40]). In the papers [14]-[18], [21] and [23] the matrix equation (2) was studied as a part of different matrix systems or as a special case of corresponding matrix equations. Special case of the matrix equation (2) is the following matrix equation:

$$AXA = A \tag{3}.$$

Any solution of this equation is called $\{1\}$-inverse of $A$ and is denoted by $A^{(1)}$. The set of all $\{1\}$-inverses of $A$ is denoted by $A\{1\}$.
For the matrix \( A \), let regular matrices \( Q \in \mathbb{C}^{m \times m} \) and \( P \in \mathbb{C}^{n \times n} \) be determined such that the following equality is true:
\[
QAP = E_a = \begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix},
\]
(4)
where \( a = \text{rank}(A) \). In [35] C. Rohde showed that the general form of \( \{1\}\)-inverse \( A^{(1)} \) can be represented as:
\[
A^{(1)} = P \begin{bmatrix} I_a & X_1 \\ X_2 & X_3 \end{bmatrix} Q,
\]
(5)
where \( X_1, X_2 \) and \( X_3 \) are arbitrary matrices of suitable sizes (see also [7] and [9]). Considerations which follow are described in terms of \( \{1\}\)-inverse of matrices.

This section of the paper is organized as follows: In subsection 2.1. we represent a new form of condition for the consistency of the matrix equation (2). An extension of Penrose’s theorem related to the general solution of the matrix equation (2) is given in subsection 2.2. Namely, we represent the formula of general solution of the matrix equation (2) if any particular solution \( X_0 \) is known. In subsection 2.3. we give a form of particular solution \( X_0 \) such that the formula of general solution of the matrix equation (2), which is given in subsection 2.2., is reproductive. The main results of this paper are obtained in subsections 2.1.–2.3. and additionally in subsection 2.4. we give two applications of the concept of reproducibility to some matrix systems which are in relation to the matrix equation (2).

2.1. Let \( A \in \mathbb{C}_a^{m \times n}, B \in \mathbb{C}_b^{p \times q} \) and \( C \in \mathbb{C}^{m \times q} \). The matrix \( A \in \mathbb{C}_a^{m \times n} \) has \( a \) linearly independent rows and \( a \) linearly independent columns. Let \( T_{A_r} \) be a \( m \times m \) permutation matrix such that multiplying the matrix \( A \) by the matrix \( T_{A_r} \) on the left, we can permute the rows of the matrix \( A \) and let \( T_{A_c} \) be a \( n \times n \) permutation matrix such that multiplying the matrix \( A \) by the matrix \( T_{A_c} \) on the right, we can permute the columns of the matrix \( A \). Then, for the matrix \( A \) there are permutation matrices \( T_{A_r} \) and \( T_{A_c} \) such that the matrix
\[
\hat{A} = T_{A_r} A T_{A_c}
\]
(6)
has linearly independent rows and linearly independent columns at the first \( a \) positions. Analogously, for the matrix \( B \) there are permutation matrices \( T_{B_r} \) and \( T_{B_c} \) such that the matrix
\[
\hat{B} = T_{B_r} B T_{B_c}
\]
(7)
has linearly independent rows and linearly independent columns at the first \( b \) positions.

The considerations which follow are valid for any choice of matrices \( T_{A_r}, T_{A_c}, T_{B_r} \) and \( T_{B_c} \) such that \( \hat{A} \) has linearly independent rows and linearly independent columns at the first \( a \) positions and \( \hat{B} \) has linearly independent rows and linearly independent columns at the first \( b \) positions. Let
\[
\hat{C} = T_{A_c} C T_{B_c}
\]
(8)
Next, let for the matrices \( A \) and \( B \) regular matrices \( Q_1, P_1 \) and \( Q_2, P_2 \) be determined such that the following equalities are true:
\[
Q_1 A P_1 = E_a = \begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_2 B P_2 = E_b = \begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix}
\]
(9)
i.e.
\[
A = Q_1^{-1} E_a P_1^{-1} \quad \text{and} \quad B = Q_2^{-1} E_b P_2^{-1}.
\]
(10)
Then, from (6), (7) and (10) we get that
\[ \hat{A} = T_A Q_1^{-1} E_a P_1^{-1} T_A c \quad \text{and} \quad \hat{B} = T_B Q_2^{-1} E_b P_2^{-1} T_B c \]
i.e.
\[ \hat{A} = (Q_1 T_A^{-1})^{-1} E_a (T_A c P_1)^{-1} \quad \text{and} \quad \hat{B} = (Q_2 T_B^{-1})^{-1} E_b (T_B c P_2)^{-1}. \]
If we introduce the following notations:
\[ \hat{Q}_1 = Q_1 T_A^{-1}, \quad \hat{P}_1 = T_A c P_1 \quad \text{and} \quad \hat{Q}_2 = Q_2 T_B^{-1}, \quad \hat{P}_2 = T_B c P_2 \]
we get that
\[ \hat{A} = \hat{Q}_1^{-1} E_a \hat{P}_1^{-1} \quad \text{and} \quad \hat{B} = \hat{Q}_2^{-1} E_b \hat{P}_2^{-1}. \]
(11)
Considering Rohde’s general form of \( \{1\}\)-inverses \( \hat{A}^{(1)} \) and \( \hat{B}^{(1)} \):
\[ A^{(1)} = P_1 \begin{bmatrix} I_a \\ X_2 \\ X_3 \end{bmatrix} Q_1 \quad \text{and} \quad B^{(1)} = P_2 \begin{bmatrix} I_b \\ Y_2 \\ Y_3 \end{bmatrix} Q_2, \]
(13)
where \( X_1, X_2, X_3 \) and \( Y_1, Y_2, Y_3 \) are arbitrary matrices of suitable sizes, we obtain that:
\[ \hat{A}^{(1)} = \hat{P}_1 \begin{bmatrix} I_a \\ X_2 \\ X_3 \end{bmatrix} \hat{Q}_1 \quad \text{and} \quad \hat{B}^{(1)} = \hat{P}_2 \begin{bmatrix} I_b \\ Y_2 \\ Y_3 \end{bmatrix} \hat{Q}_2 \]
i.e.
\[ \hat{A} \hat{A}^{(1)} = \hat{Q}_1^{-1} E_a \hat{P}_1^{-1} \hat{P}_1 \begin{bmatrix} I_a \\ X_2 \\ X_3 \end{bmatrix} \hat{Q}_1 = \hat{Q}_1^{-1} \begin{bmatrix} I_a \\ 0 \\ 0 \end{bmatrix} \hat{Q}_1 \]
and
\[ \hat{B} \hat{B}^{(1)} = \hat{P}_2 \begin{bmatrix} I_b \\ Y_2 \\ Y_3 \end{bmatrix} \hat{Q}_2 \hat{Q}_2^{-1} E_b \hat{P}_2^{-1} = \hat{P}_2 \begin{bmatrix} I_b \\ 0 \\ 0 \end{bmatrix} \hat{P}_2^{-1}. \]
(14)
(15)
As we mentioned, the matrix \( \hat{A} \) has linearly independent rows and linearly independent columns at the first \( a \) positions and the matrix \( \hat{B} \) has linearly independent rows and linearly independent columns at the first \( b \) positions.
Let
\[ \hat{A}_{i \rightarrow} = \sum_{l=1}^{a} \alpha_{i,l} \hat{A}_{l \rightarrow}, \quad i = a + 1, \ldots, m, \]
(16)
\[ \hat{A}_{i j} = \sum_{k=1}^{a} \alpha'_{k,j} \hat{A}_{i k}, \quad j = a + 1, \ldots, n, \]
(17)
and
\[ \hat{B}_{i \rightarrow} = \sum_{l=1}^{b} \beta_{i,l} \hat{B}_{l \rightarrow}, \quad i = b + 1, \ldots, p, \]
(18)
\[ \hat{B}_{i j} = \sum_{k=1}^{b} \beta'_{k,j} \hat{B}_{i k}, \quad j = b + 1, \ldots, q, \]
(19)
for some scalars $\alpha_{i,t}$, $\alpha'_{k,j}$ and $\beta_{i,t}$, $\beta_{k,j}$. As we know, the matrices $\widehat{Q}_1$, $\widehat{P}_1$ and $\widehat{Q}_2$, $\widehat{P}_2$ are not uniquely determined, but we shall use, without loss of generality, their following forms:

$$\widehat{Q}_1 = \begin{bmatrix} I_a & 0 \\ L_1 & I_{m-a} \end{bmatrix}, \quad \widehat{P}_1 = \begin{bmatrix} W^{-1} & L'_1 \\ 0 & I_{n-a} \end{bmatrix}$$ (20)

and

$$\widehat{Q}_2 = \begin{bmatrix} U^{-1} & 0 \\ L'_2 & I_{p-b} \end{bmatrix}, \quad \widehat{P}_2 = \begin{bmatrix} I_b & L_2 \\ 0 & I_{q-b} \end{bmatrix}$$ (21)

for

$$L_1 = \begin{bmatrix} -\alpha_{a+1,1} & \cdots & -\alpha_{a+1,a} \\ \vdots & \ddots & \vdots \\ -\alpha_{m,1} & \cdots & -\alpha_{m,a} \end{bmatrix}, \quad L'_1 = \begin{bmatrix} -\alpha'_{1,a+1} & \cdots & -\alpha'_{1,n} \\ \vdots & \ddots & \vdots \\ -\alpha'_{a,a+1} & \cdots & -\alpha'_{a,n} \end{bmatrix},$$

$$L'_2 = \begin{bmatrix} -\beta'_{b+1,1} & \cdots & -\beta'_{b+1,b} \\ \vdots & \ddots & \vdots \\ -\beta'_{p,1} & \cdots & -\beta'_{p,b} \end{bmatrix}, \quad L_2 = \begin{bmatrix} -\beta_{1,b+1} & \cdots & -\beta_{1,q} \\ \vdots & \ddots & \vdots \\ -\beta_{b,b+1} & \cdots & -\beta_{b,q} \end{bmatrix}$$

and where $W$ is a $a \times a$ submatrix of $\widehat{A}$ such that $\widehat{A} = \begin{bmatrix} W & \tilde{A}_2 \\ A_3 & A_4 \end{bmatrix}$ and $U$ is a $b \times b$ submatrix of $\widehat{B}$ such that $\widehat{B} = \begin{bmatrix} U & \tilde{B}_2 \\ B_3 & B_4 \end{bmatrix}$.

Let us emphasize that the following statement is true.

**Lemma 2.1.** Let $A \in \mathbb{C}_a^{m \times n}$, $B \in \mathbb{C}_b^{p \times q}$, $C \in \mathbb{C}^{m \times q}$. Suppose that $\widehat{A}$ and $\widehat{B}$ are determined by (6) and (7). Then, the conditions

$$AA^{(1)}CB^{(1)} = C \quad (22)$$

and

$$\widehat{A}\widehat{A}^{(1)}\widehat{C}\widehat{B}^{(1)}\widehat{B} = \widehat{C} \quad (23)$$

are equivalent.

**Proof.** The following equivalences are true $AA^{(1)}CB^{(1)}B = C \iff T_A^{-1}\widehat{A}\widehat{A}^{(1)}T_ACT_B\widehat{B}^{(1)}\widehat{B}T_B^{-1} = C \iff \widehat{A}\widehat{A}^{(1)}T_ACT_B\widehat{B}^{(1)}\widehat{B} = T_ACT_B \iff \widehat{A}\widehat{A}^{(1)}\widehat{C}\widehat{B}^{(1)}\widehat{B} = \widehat{C}$. \hfill $\Diamond$

Let us remark that (22) is Penrose’s condition of consistency for the matrix equation (2), and if the matrix equation (2) is consistent, then the formulas of general solution are given in Theorem 2.7 and 2.9. In the following statement we give a condition which is equivalent to Penrose’s condition of consistency for the matrix equation (2). So, we can use this new condition to test the consistency of the matrix equation (2).
Theorem 2.2. Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$, $C \in \mathbb{C}^{m \times q}$. Suppose that $\hat{A}$ and $\hat{B}$ are determined by (6) and (7) and that (16)–(19) are satisfied. Then, the condition (22) is true for any choice of $\{I\}$-inverses $A^{(1)}$ and $B^{(1)}$ iff

$$
\hat{C} = \begin{bmatrix}
c_{1,1} & \ldots & c_{1,b} & \sum_{k=1}^{b} \beta_{k,b+1}c_{1,k} & \ldots & \sum_{k=1}^{b} \beta_{k,q}c_{1,k} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c_{a,1} & \ldots & c_{a,b} & \sum_{k=1}^{b} \beta_{k,b+1}c_{a,k} & \ldots & \sum_{k=1}^{b} \beta_{k,q}c_{a,k} \\
\sum_{l=1}^{a} \alpha_{a+1,l}c_{l,1} & \ldots & \sum_{l=1}^{a} \alpha_{a+1,l}c_{l,b} & \sum_{k=1}^{b} \alpha_{a+1,l} \beta_{k,b+1}c_{l,k} & \ldots & \sum_{k=1}^{b} \alpha_{a+1,l} \beta_{k,q}c_{l,k} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\sum_{l=1}^{a} \alpha_{m,l}c_{l,1} & \ldots & \sum_{l=1}^{a} \alpha_{m,l}c_{l,b} & \sum_{k=1}^{b} \alpha_{m,l} \beta_{k,b+1}c_{l,k} & \ldots & \sum_{k=1}^{b} \alpha_{m,l} \beta_{k,q}c_{l,k}
\end{bmatrix}, \quad (24)
$$

where $c_{i,j}$ are arbitrary elements of $\mathbb{C}$.

Proof. $(\Rightarrow)$: Suppose that the condition (22) is valid for any choice of $\{I\}$-inverses $A^{(1)}$ and $B^{(1)}$. Based on Lemma 2.1, the condition (23) is also valid. Then, considering the equalities (14) and (15), we get the following equality

$$
\hat{Q}_{1}^{-1} \begin{bmatrix}
I_{a} & X_{1} \\
0 & 0
\end{bmatrix} \hat{Q}_{1} \hat{C} \hat{P}_{2} \begin{bmatrix}
I_{b} & 0 \\
Y_{2} & 0
\end{bmatrix} \hat{P}_{2}^{-1} = \hat{C}.
$$

By multiplying the previous equality by $\hat{Q}_{1}$ on the left and by $\hat{P}_{2}$ on the right we get

$$
\begin{bmatrix}
I_{a} & X_{1} \\
0 & 0
\end{bmatrix} \hat{Q}_{1} \hat{C} \hat{P}_{2} \begin{bmatrix}
I_{b} & 0 \\
Y_{2} & 0
\end{bmatrix} = \hat{Q}_{1} \hat{C} \hat{P}_{2}.
$$

Suppose that

$$
\hat{C} = \begin{bmatrix}
c_{1,1} & \ldots & c_{1,q} \\
\vdots & \ddots & \vdots \\
c_{m,1} & \ldots & c_{m,q}
\end{bmatrix}.
$$

We are going to show that $\hat{C}$ has the form (24). Let

$$
E = \hat{Q}_{1} \hat{C} \hat{P}_{2} \quad \text{and} \quad F = \begin{bmatrix}
I_{a} & X_{1} \\
0 & 0
\end{bmatrix} E \begin{bmatrix}
I_{b} & 0 \\
Y_{2} & 0
\end{bmatrix}.
$$

From (20) and (21) we obtain that
for $i = 1, \ldots, a, j = 1, \ldots, b$

$$E_{i,j} = c_{i,j},$$

for $i = 1, \ldots, a, j = b + 1, \ldots, q$

$$E_{i,j} = c_{i,j} - \sum_{k=1}^{b} \beta_{k,j} c_{i,k},$$

for $i = a + 1, \ldots, m, j = 1, \ldots, b$

$$E_{i,j} = c_{i,j} - \sum_{l=1}^{a} \alpha_{i,l} c_{i,j},$$

for $i = a + 1, \ldots, m, j = b + 1, \ldots, q$

$$E_{i,j} = c_{i,j} - \sum_{l=1}^{a} \alpha_{i,l} c_{i,j} - \sum_{k=1}^{b} \beta_{k,j} (c_{i,k} - \sum_{l=1}^{a} \alpha_{i,l} c_{i,k})$$

and

for $i = 1, \ldots, a, j = 1, \ldots, b$

$$F_{i,j} = c_{i,j} + \sum_{l=a+1}^{m} x_{i,l} (c_{l,j} - \sum_{l=1}^{a} \alpha_{i,l} c_{l,j})$$

$$+ \sum_{k=b+1}^{m} \gamma_{k,j} c_{i,k} - \sum_{k=1}^{b} \beta_{k,j} c_{i,k}$$

$$+ \sum_{l=a+1}^{m} x_{i,l} (c_{l,k} - \sum_{l=1}^{a} \alpha_{i,l} c_{l,k})$$

$$- \sum_{k=1}^{b} \beta_{k,j} (c_{l,k} - \sum_{l=1}^{a} \alpha_{i,l} c_{l,k})$$

for $i = 1, \ldots, a, j = b + 1, \ldots, q$

$$F_{i,j} = 0,$$

for $i = a + 1, \ldots, m, j = 1, \ldots, b$

$$F_{i,j} = 0,$$

for $i = a + 1, \ldots, m, j = b + 1, \ldots, q$

$$F_{i,j} = 0.$$

Finally, from (25) and (26) i.e. $E = F$ we get that

for $i = 1, \ldots, a, j = 1, \ldots, b$

$c_{i,j}$ are arbitrary elements of $\mathbb{C},$

for $i = a + 1, \ldots, m, j = 1, \ldots, b$

$$c_{i,j} = \sum_{l=1}^{a} \alpha_{i,l} c_{i,j},$$

for $i = 1, \ldots, a, j = b + 1, \ldots, q$

$$c_{i,j} = \sum_{k=1}^{b} \beta_{k,j} c_{i,k},$$

for $i = a + 1, \ldots, m, j = b + 1, \ldots, q$

$$c_{i,j} = \sum_{l=1}^{a} \sum_{k=1}^{b} \alpha_{i,l} \beta_{k,j} c_{l,k}.$$  

($\iff$): Suppose that the matrix $\hat{C}$ has the form (24). Then,

$$\hat{Q}_1 \hat{C} \hat{P}_2 = \ldots = \begin{bmatrix}
  c_{1,1} & \ldots & c_{1,b} & 0 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \ldots & 0 & \ldots & \ldots & 0 \\
\end{bmatrix}$$

(27)
and

\[
\begin{bmatrix}
I_a & X_1 \\
0 & 0
\end{bmatrix} \hat{Q}_1 \hat{C} \hat{P}_2 \begin{bmatrix}
I_b & 0 \\
Y_2 & 0
\end{bmatrix} = \ldots = \begin{bmatrix}
c_{1,1} & \ldots & c_{1,b} & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
c_{a,1} & \ldots & c_{a,b} & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & \ldots & 0
\end{bmatrix}.
\quad (28)
\]

From (27) and (28) we conclude that

\[
\begin{bmatrix}
I_a & X_1 \\
0 & 0
\end{bmatrix} \hat{Q}_1 \hat{C} \hat{P}_2 \begin{bmatrix}
I_b & 0 \\
Y_2 & 0
\end{bmatrix} = \hat{Q}_1 \hat{C} \hat{P}_2.
\]

By multiplying the previous equality by \(\hat{Q}_1^{-1}\) on the left and by \(\hat{P}_2^{-1}\) on the right we obtain the following equality:

\[
\hat{Q}_1^{-1} \begin{bmatrix}
I_a & X_1 \\
0 & 0
\end{bmatrix} \hat{Q}_1 \hat{C} \hat{P}_2 \begin{bmatrix}
I_b & 0 \\
Y_2 & 0
\end{bmatrix} \hat{P}_2^{-1} = \hat{C}.
\quad (29)
\]

From (29), considering the equalities (14) and (15), we see that the condition (23) is true. Based on Lemma 2.1 we conclude that the condition (22) is true. ◊

**Remark 2.3.** Let us remark that the general form of matrix \(C\), such that the matrix equation (2) is consistent, always exists. The matrix equation (2) is consistent for an arbitrary matrix \(C\) iff a matrix \(A\) has full row rank and a matrix \(B\) has full column rank (see also Exercises 10.50 from [1]).

**Remark 2.4.** In the paper [41] author considered some forms which are equivalent to Penrose’s condition of consistency for matrix equation (2).

The application of Theorem 2.2 will be illustrated by the following examples.

**Example 2.5.** Let be given the following matrices:

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
1 & -3 & 2 \\
2 & 1 & -1 \\
-1 & -4 & 3 \\
3 & -2 & 1
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
0 & 1 & 2 & 3 & -1 \\
0 & 3 & 1 & 4 & 2 \\
0 & 4 & 1 & 5 & 3 \\
0 & 2 & 3 & 5 & -1
\end{bmatrix}.
\]

Then, \(\text{rank}(A) = 2\), \(\text{rank}(B) = 2\) and for
satisfies the condition (22).

Based on Theorem 2.2, each matrix $\hat{C}$ which has the following form

$$\hat{C} = \begin{bmatrix}
    c_{1,1} & c_{1,2} & 0 & c_{1,1} + c_{1,2} & c_{1,1} - c_{1,2} \\
    c_{2,1} & c_{2,2} & 0 & c_{2,1} + c_{2,2} & c_{2,1} - c_{2,2} \\
    c_{1,1} - c_{2,1} & c_{1,2} - c_{2,2} & 0 & c_{1,1} - c_{2,1} + c_{1,2} - c_{2,2} & c_{1,1} - c_{2,1} - c_{1,2} + c_{2,2} \\
    c_{1,1} + c_{2,1} & c_{1,2} + c_{2,2} & 0 & c_{1,1} + c_{2,1} + c_{1,2} + c_{2,2} & c_{1,1} + c_{2,1} - c_{1,2} - c_{2,2} \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

satisfies the condition (22). From that we conclude that each matrix $C = T_{A^{r}}^{-1}\hat{C}T_{B^{c}}^{-1}$ which has the following form

$$C = \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 \\
    0 & c_{1,1} & c_{1,2} & c_{1,1} + c_{1,2} & c_{1,1} - c_{1,2} \\
    0 & c_{2,1} & c_{2,2} & c_{2,1} + c_{2,2} & c_{2,1} - c_{2,2} \\
    0 & c_{1,1} - c_{2,1} & c_{1,2} - c_{2,2} & c_{1,1} - c_{2,1} + c_{1,2} - c_{2,2} & c_{1,1} - c_{2,1} - c_{1,2} + c_{2,2} \\
    0 & c_{1,1} + c_{2,1} & c_{1,2} + c_{2,2} & c_{1,1} + c_{2,1} + c_{1,2} + c_{2,2} & c_{1,1} + c_{2,1} - c_{1,2} - c_{2,2}
\end{bmatrix}$$

satisfies the condition (22).
Example 2.6. Let $A$ and $B$ be the matrices as in Example 2.5. and

\[
\begin{align*}
a) \quad C &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & -4 \\ 0 & 3 & -2 & 1 & 5 \\ 0 & -1 & 2 & 1 & -3 \end{bmatrix}, \\
b) \quad C &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & 4 \\ 0 & 3 & -2 & 1 & 5 \\ 0 & -1 & 2 & 1 & -3 \end{bmatrix}.
\end{align*}
\]

If we compare the matrix $C$ from a) and from b) with the general form of matrix $C$ which satisfies the condition (22) (see Example 2.5) we see that the matrix $C$ from a) satisfies the condition (22) and the matrix $C$ from b) does not satisfy the condition (22). Therefore, the matrix equation (2) is consistent for the matrix $C$ from a), but it is not consistent for the matrix $C$ from b). ♦

2.2. Recall that the matrix equation $AXB = C$ is marked with (2) for $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$, $C \in \mathbb{C}^{m \times q}$. Methods for solving the consistent matrix equation (2) are considered in the book [10] (Chapter X). In the paper [27] R. Penrose proved the following theorem related to the matrix equation (2).

Theorem 2.7. The matrix equation (2) is consistent iff for some choice of $\{I\}$-inverses $A^{(1)}$ and $B^{(1)}$ of the matrices $A$ and $B$ the condition (22) is true. The general solution of the matrix equation (2) is given by the formula

\[
X = f(Y) = A^{(1)}CB^{(1)} + Y - A^{(1)}AYB^{(1)},
\]

where $Y \in \mathbb{C}^{n \times p}$ is an arbitrary matrix.

Remark 2.8. If the matrix equation (2) is consistent, the equivalence

\[
AXB = C \iff X = f(X) = X - A^{(1)}(AXB - C)B^{(1)}
\]

is true. Therefore, the starting equation is equivalent to some reproductive equation. Based on Theorem 1.3 we can also conclude that (30) is the general solution of the matrix equation (2).

In this paper we give a simple extension of Theorem 2.7.

Theorem 2.9. If $X_0$ is any particular solution of the matrix equation (2), the general solution of the matrix equation (2) is given by the formula

\[
X = g(Y) = X_0 + Y - A^{(1)}AYB^{(1)},
\]

where $Y \in \mathbb{C}^{n \times p}$ is an arbitrary matrix. The function $g$ satisfies the condition of reproductivity (7) iff $X_0 = A^{(1)}CB^{(1)}$. 

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Proof. It is easily to see that the solution of the matrix equation (2) is given by (32). On the contrary, let $X$ is any solution of the matrix equation (2), then

$$
X = X - A^{(1)}CB^{(1)} + A^{(1)}CB^{(1)}
= X - A^{(1)}AXBB^{(1)} + A^{(1)}AX_0BB^{(1)}
= X - A^{(1)}A(X - X_0)BB^{(1)}
= X_0 + (X - X_0) - A^{(1)}A(X - X_0)BB^{(1)}
= X_0 + Y - A^{(1)}AYBB^{(1)} = g(Y),
$$

where $Y = X - X_0$. From this we see that every solution $X$ of the matrix equation (2) can be represented in the form (32). Based on the following matrix equality:

$$
g^2(Y) = g(Y) + (X_0 - A^{(1)}CB^{(1)})
$$

we see that the function $g$ satisfies the condition (1) iff $X_0 = A^{(1)}CB^{(1)}$. ♦

Remark 2.10. Using the previous theorem and the appropriate choice of particular solution $X_0$ we can obtain the general solutions for different cases of the matrix equation (2). It was considered in the papers [17] and [29].

The general solution (32) of the matrix equation (2) is reproductive iff $X_0 = A^{(1)}CB^{(1)}$. Therefore, Penrose’s general solution (30) of the matrix equation (2) is the reproductive solution. If the condition (22) is not true, the matrix equation (2) is solved approximately as described in the paper [27] and books [1], [7] and [9].

2.3. Using the obtained form of matrix $\hat{C}$ we obtain the form of particular solution $X_0$ of the matrix equation (2) such that the general solution (32) of the matrix equation (2) is reproductive.

Theorem 2.11. Let $X_0$ any particular solution of the matrix equation (2). The general solution (32) of the matrix equation (2) is reproductive iff

$$
X_0 = P_1 \begin{bmatrix} C_1 & C_1Y_1 \\ X_2C_1 & X_2C_1Y_1 \end{bmatrix} Q_2
$$

(33)

where $P_1$, $Q_2$, $X_2$, $Y_1$ are the matrices from (13) and $C_1$ is the submatrix of the matrix $\hat{C}$ and it has the following form:

$$
C_1 = \begin{bmatrix}
    c_{1,1} & \ldots & c_{1,b} \\
    \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots \\
    c_{a,1} & \ldots & c_{a,b}
\end{bmatrix},
$$

where $c_{i,j}$ are some elements of $\mathbb{C}$.
Proof. For general \( \{1\}\)-inverses \( A^{(1)} \) and \( B^{(1)} \) the statement follows from Theorem 2.9, because

\[
X_0 = A^{(1)}CB^{(1)} = P_1 \begin{bmatrix} I_a & X_1 \\ X_2 & X_3 \end{bmatrix} Q_1CP_2 \begin{bmatrix} I_b & Y_1 \\ Y_2 & Y_3 \end{bmatrix} Q_2
\]

\[
= P_1 \begin{bmatrix} I_a & X_1 \\ X_2 & X_3 \end{bmatrix} Q_1T_A^{-1}CT_B^{-1}P_2 \begin{bmatrix} I_b & Y_1 \\ Y_2 & Y_3 \end{bmatrix} Q_2
\]

\[
= P_1 \begin{bmatrix} I_a & X_1 \\ X_2 & X_3 \end{bmatrix} Q_1\hat{C}P_2 \begin{bmatrix} I_b & Y_1 \\ Y_2 & Y_3 \end{bmatrix} Q_2
\]

\[
= P_1 \begin{bmatrix} I_a & X_1 \\ X_2 & X_3 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_b & Y_1 \\ Y_2 & Y_3 \end{bmatrix} Q_2
\]

\[
= P_1 \begin{bmatrix} C_1 \\ X_2C_1 \end{bmatrix} \begin{bmatrix} C_1Y_1 \\ X_2C_1Y_1 \end{bmatrix} Q_2.\qquad \Box
\]

Corollary 2.12. (i) If a matrix \( A \) has full row rank and a matrix \( B \) has full column rank, then parameters from submatrices \( X_2 \) and \( Y_1 \) don’t exist and don’t appear in the matrix \( X_0 \) of form (33).

(ii) If either a matrix \( A \) has full row rank or a matrix \( B \) has full column rank, then the matrix \( X_0 \) of form (33) has the structure of an affine linear space with parameters from either submatrix \( X_2 \) or submatrix \( Y_1 \), respectively.

(iii) If a matrix \( A \) doesn’t have full row rank and a matrix \( B \) doesn’t have full column rank, then the matrix \( X_0 \) of form (33) doesn’t have the structure of an affine linear space relative to parameters from submatrices \( X_2 \) and \( Y_1 \).

Remark 2.13. In the paper [25] authors proved that there is a matrix equation (2) and its particular solution \( X_1 \) such that \( X_1 \neq A^{(1)}CB^{(1)} \) for any choice of \( \{1\}\)-inverses \( A^{(1)} \) and \( B^{(1)} \).

Remark 2.14. According to Theorem VI, pp. 345-346, from [10], it is possible to extract \( a \cdot b \) parameters in a matrix \( Y \) such that, these parameters are expressed, in the solution (32), as a non-homogeneous linear functions of the other \( n \cdot p - a \cdot b \) independent parameters.

2.4. In this part of the paper we analysed two applications the concept of reproductivity on some matrix systems which are in relation to the matrix equation (2).

Application 2.15. In [27] R. Penrose studied a matrix system

\[
\begin{align*}
(34a) \quad AX &= B \\
(34b) \quad XD &= E,
\end{align*}
\]

where \( A, B, D \) and \( E \) are given complex matrices corresponding dimensions. He proved that

\[
X_1 = A^{(1)}B + ED^{(1)} - A^{(1)}AED^{(1)}
\]

is one common solution of the matrix equations (34a) and (34b) if \( AE = BD \) and the matrix equations (34a) and (34b) are consistent.

In [7] A. Ben-Israel and T.N.E. Greville proved that the matrix equations (34a) and (34b) have a common solution iff each equation separately has a solution and \( AE = BD \). Also, they proved that if \( X_0 \) is any common solution of the matrix equations (34a) and (34b), the general solution of the matrix system (34) is given by the formula

\[
X = g(Y) = X_0 + (I - A^{(1)}A)Y(I - DD^{(1)}),
\]

where \( Y \) is an arbitrary matrix corresponding dimensions.
We will prove that if the matrix system (34) is consistent, the general reproductive solution is given by the formula

\[ X = f(Y) = A^{(1)}B + ED^{(1)} - A^{(1)}AED^{(1)} + (I - A^{(1)}A)Y(I - DD^{(1)}), \quad (37) \]

where \( Y \) is an arbitrary matrix corresponding dimensions.

If the matrix system (34) is consistent, the following equivalence is true

\[ (AX = B \land XD = E) \iff X = f(X). \quad (38) \]

The direct implication of (38) follows by implications (see Remark 2.8 in the subsection 2.1):

\[
AX = B \implies X = f_1(X) = A^{(1)}B + X - A^{(1)}AX, \\
XD = E \implies X = f_2(X) = ED^{(1)} + X - XDD^{(1)}, \\
AXD = BD = AE \implies X = f_3(X) = A^{(1)}AED^{(1)} + X - A^{(1)}AXDD^{(1)}.
\]

From the previous implications we can conclude

\[ (AX = B \land XD = E) \implies X = f(X) = f_1(X) + f_2(X) - f_3(X). \]

The reverse implication of (38) is trivial. Notice that the function \( f \) is reproductive. Therefore, if the matrix system (34) is consistent, it is equivalent to the reproductive matrix equation

\[ X = f(Y). \]

Based on Theorem 1.3, we conclude that \( X = f(Y) \) is the general reproductive solution of the matrix system (34). If there is a particular solution \( X_0 \) of the matrix system (34) so that \( X_0 \neq X_1 \), then \( X = g(Y) \) is the general non-reproductive solution. At the end of these application let us remark that equality \( X = g(Y - X_0) = f(Y) \) also represents one simple proof of the Statement 1 from [18].

**Application 2.16.** Let \( A \in \mathbb{C}^{n \times n} \) be a singular matrix. In this section we consider a matrix system

\[ AXA = A \land AX =XA. \quad (39) \]

The consistency of the matrix system (39) is determined by Theorem 1 in [21] (see also [15] and [23]). Let \( \bar{A} \) is commutative \( \{1\} \)-inverse, [21]. Based on the reproducibility, we give a new proof that the formula from [21]:

\[ X = f(Y) = \bar{A}AA + Y - \bar{A}AY - YA\bar{A} + \bar{A}AY\bar{A}, \quad (40) \]

where \( Y \) is an arbitrary matrix corresponding dimensions, represents the general solution of the consistent matrix system (39).

Namely, if the matrix system (39) is consistent, the equivalence

\[ (AXA = A \land AX =XA) \iff X = f(X) \quad (41) \]

is true. The direct implication of (41) is based on the following simple matrix equalities:

\[
\bar{A}AXA = \bar{A}AXA = \bar{A}AXA = \bar{A}AXA = \bar{A}AX = \bar{A}AX.
\]

and

\[
\bar{A}AXA = \bar{A}AXA = \bar{A}AXA = \bar{A}AXA = \bar{A}AXA. \]

13
From this we get that $X = X + \bar{A}\bar{A} - \bar{A}AX + \bar{A}AX\bar{A} - X\bar{A}\bar{A} = f(X)$. The reverse implication of (41) is trivial. Notice that the function $f$ is reproductive. Therefore, if the matrix system (39) is consistent, it is equivalent to the reproductive matrix equation $X = f(X)$. Based on Theorem 1.3, we conclude that $X = f(Y)$ is the general reproductive solution of the matrix system (39). If $X_0$ is any solution of the matrix system (39), the formula

$$X = g(Y) = X_0 + Y - \bar{A}AY - Y\bar{A}\bar{A} + \bar{A}AY\bar{A},$$

also determines a form of the general solution of the matrix system (39) because the equality $g(Y) = f(X_0 + Y)$ is true. If there is a particular solution $X_0$ of the matrix system (39) such that $X_0 \neq \bar{A}\bar{A}$, then $X = g(Y)$ is the general non-reproductive solution. Additional applications of the concept of reproductivity for some matrix equations and systems were considered in the paper [26].

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References

[1] K.M. Abadir and J.R. Magnus, Matrix Algebra, Econometric exercises, Volume 1, Cambridge, 2005.

[2] D. Banković, On general and reproductive solutions of arbitrary equations, Publications de l’institut mathématique, Nouvelle serie, tome 26 (40), Beograd 1979, 31 - 33.

[3] D. Banković, All solutions of finite equations, Discrete Mathematics Vol. 137 (1-3), 1995, 1 - 6.

[4] D. Banković, General reproductive solutions of Postian equations, Discrete Mathematics Vol. 169 (1-3), 1997, 163 - 168.

[5] D. Banković, All general solutions of Prešić’s equation, Facta universitatis, Ser. Math. Inform. Vol. 17, Niš 2002, 1 - 4.

[6] D. Banković, General Solutions of System of Finite Equations, Scientific Publications of the State University of Novi Pazar Ser. A: Appl. Math. Inform. and Mech. vol. 3, 2 (2011), 117 - 121.

[7] A. Ben-Israel and T.N.E. Greville, Generalized Inverses: Theory and Applications, Springer, 2003.

[8] M. Božić, A Note On Reproductive Solutions, Publications de l’institut mathématique, Nouvelle serie, tome 19 (33), Beograd 1975, 33 - 35. (http://publications.mi.sanu.ac.rs/)

[9] S.L. Campbell and C.D. Meyer, Generalized Inverses of Linear Transformations, Society for Industrial and Applied Mathematics, 2009.

[10] C.E. Cullis, Matrices and determinoids - Volume I, Cambridge, University Press Publ. 1913. (http://archive.org/details/matricesdetermin01cull1)
[11] D.S. Cvetković-Ilić, The reflexive solutions of the matrix equation $AXB = C$, Comp. Math. Appl., 51 (2006), 897 - 902.

[12] D.S. Cvetković-Ilić, A. Dajić, and J.J. Koliha, Positive and real-positive solutions to the equation $axa^* = c$ in $C^* -$ algebra, Linear & Multilinear algebra, 55, (6) (2007), 535 - 543.

[13] D.S. Cvetković-Ilić, Re-nnd solutions of the matrix equation $AXB = C$, Journal of the Australian Mathematical Society, 84 (2008), 63 - 72.

[14] A. Dajić and J.J. Koliha, Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators, Lin. Alg. and its Appl. 429 (2008) 1779 - 1809.

[15] L.D. Dobryakov, Commuting generalized inverse matrices, Mathematical Notes, Volume 36, Number 1, 500 - 504, 1985 (Translated from Matematicheskie Zametki, Vol. 36, No. 1, 17 - 23, 1984.).

[16] V. Harizanov, On the functional equation $f \circ f = f$, Publications de l’institut mathématique, Nouvelle serie, tome 29 (43), Beograd 1981, 61 - 64.

[17] M. Haverić, Formulae for general reproductive solutions of certain matrix equations, Publications de l’institut mathématique, Nouvelle serie, tome 34 (48), Beograd 1983, 81 - 84.

[18] M. Haverić, On solutions of a matrix equations system $AX = B$ and $XD = E$, Matematički Vesnik 36 (1), Beograd 1984, 11 - 16.

[19] J.D. Kečkić, Reproductivity of some equations of analysis I, Publications de l’institut mathématique, Nouvelle serie, tome 31(45), Beograd 1982, 73 - 81.

[20] J.D. Kečkić, Reproductivity of some equations of analysis II, Publications de l’institut mathématique, Nouvelle serie, tome 33(47), Beograd 1983, 109 - 118.

[21] J.D. Kečkić, Commutative weak generalized inverses of a square matrix and some related matrix equations, Publications de l’institut mathématique, Nouvelle serie, tome 38 (52), Beograd 1985, 39 - 44.

[22] J.D. Kečkić, On some generalized inverses of matrices and some linear matrix equations, Publications de l’institut mathématique, Nouvelle serie, tome 45 (59), Beograd 1989, 57 - 63.

[23] J.D. Kečkić, Some remarks on possible generalized inverses in semigroups, Publications de l’institut mathématique, Nouvelle serie, tome 61 (75), Beograd 1997, 33 - 40.

[24] J.D. Kečkić and S.B. Prešić, Reproductivity - A general approach to equations, Facta universitatis, Ser. Math. Inform. Vol. 12, Niš 1997, 157 - 184.

[25] B. Malešević and B. Radičić, Non-reproductive and reproductive solutions of some matrix equations, Proceedings of the International conference Mathematical and Informational Technologies, MIT - 2011, Vrnjačka Banja, Serbia, 2011, 246 - 251. (http://mit.rs/)

[26] B. Malešević and B. Radičić, Some considerations of matrix equations using the concept of reproductive, Kragujevac Journal of Mathematics, 36(1) (2012), 151 - 161.
[27] R. Penrose, *A generalized inverses for matrices*, Math. Proc. Cambridge Philos. Soc. **51** (1955), 406 - 413.

[28] S.B. Prešić, *Methode de resolution d’une classe d’équations fonctionnelles lineaires*, Comptes rendus de l’Académie des Sciences Paris, **257** (1963), 2224 - 2226.

[29] S.B. Prešić, *Certaines équations matricielles*, Publ. Elektrotehn. Fak. Ser. Mat.-Fiz., N° 121, Beograd 1963. (http://pefmath.etf.rs/)

[30] S.B. Prešić, *Une classe d’équations matricielles et l’équation fonctionnelle $f^2 = f$*, Publications de l’institut mathématique, Nouvelle serie, tome **8** (22), Beograd 1968, 143 - 148.

[31] S.B. Prešić, *Une methode de resolution des equations dont toutes les solutions appartiennent a un ensemble fini donne*, Comptes rendus de l’Académie des Sciences Paris, **272** (1971), 654 - 657.

[32] S.B. Prešić, *Ein Satz Über Reproduktive Lösungen*, Publications de l’institut mathématique, Nouvelle serie, tome **14** (28), Beograd 1972, 133 -136.

[33] S.B. Prešić, *All reproductive solutions of finite equations*, Publications de l’institut mathématique, Nouvelle serie, tome **44** (58), Beograd 1988, 3 -7.

[34] S.B. Prešić, *A generalization of the notion of reproductivity*, Publications de l’institut mathématique, Nouvelle serie, tome **67** (81), Beograd 2000, 76 - 84.

[35] C.A. Rohde, *Contribumtion to the theory, computation and application of generalized inverses*, Doctoral dissertation, University of North Carolina at Releigh, May 1964.

[36] S. Rudeanu, *On general solutions of arbitrary equations*, Publications de l’institut mathématique, Nouvelle serie, tome **24** (38), Beograd 1978, 143 - 145.

[37] S. Rudeanu, *On general and reproductive solutions of finite equations*, Publications de l’institut mathématique, Nouvelle serie, tome **63** (77), Beograd 1998, 26 - 30.

[38] S. Rudeanu, *Lattice Functions and Equations*, Springer, 2001.

[39] Y. Tian, *On additive decompositions of solutions of the matrix equation $AXB = C$*, Calcolo, Vol. **47** (4), 2010, 193 - 209.

[40] Y. Tian, *On Additive Decomposition of the Hermitian Solution of the Matrix Equation $AXA^* = B$*, Mediterranean Journal of Mathematics, **9** (2012), 47 - 60.

[41] Q.-W. Wang, *A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity*, Lin. Alg. and its Appl. **384**, (2004), 43 - 54.

[42] A. Krapež (editor), *A tribute to S.B. Prešić: Papers Celebrating his 65-th Birthday*, Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, publ. 2001. (http://elibrary.matf.bg.ac.rs/handle/123456789/448)