Approximation of Random Functions by Random Polynomials in the Framework of Choquet’s Theory of Integration

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Abstract. Given a submodular capacity space, we prove the uniform convergence in capacity and also the uniform convergence in the Choquet-mean of order \( p \geq 1 \) with a quantitative estimate, of the multivariate Bernstein polynomials associated to a random function.

1. Introduction

In this paper we extend some old results on the approximation of random functions by Bernstein random polynomials to the framework of Choquet’s theory of integrability. As is well known, these polynomials are among the most studied and the most interesting polynomials used in the probabilistic framework of approximation theory. We mention here the classical book of G. G. Lorentz [16] and the papers of O. Onicescu and V. I. Istratescu [17, 18], Gh. Cenusa and I. Sacuin [2], S. G. Gal [6, 7], and S. G. Gal and A. R. Villena [11]. In [15], E. Kowalski connects the random Bernstein polynomials with the almost surely time-continuous Brownian motion \( B(t, \omega) \), \( t \in [0, +\infty) \). He starts by proving the existence of the Brownian motion as the limit in distribution of the random Bernstein polynomials attached to some suitable Gaussian random variables. See Corollary 4.2 and Theorems 4.3 and 4.4 in [15]. Then, knowing that the Brownian motion exists and using the properties of the random Bernstein polynomials attached to the Brownian motion, the almost everywhere nondifferentiability property of the Brownian motion is proved and then the zeroes of the attached random Bernstein polynomials are studied.

The papers cited above have motivated us to study the extension of the approximation properties of random Bernstein polynomials in the much more general framework provided by capacity spaces and the Choquet integral. Unlike the case of probability measures, the capacities are nonadditive set functions, and precisely the lack of additivity makes them useful in risk theory (especially in decision making under risk and uncertainty). See H. Föllmer, A. Schied [5] and M. Grabisch [12].

In Section 2 we present preliminaries on capacities and Choquet integral. Section 3 is devoted to a description of various concepts of continuity of random functions.

Date: February 16, 2020.

2000 Mathematics Subject Classification. Primary: 60G99, 41A10, 41A36, Secondary: 28A25.

Key words and phrases. Choquet integral, submodular capacity, random Bernstein polynomials, approximation in Choquet-mean, approximation in capacity, Choquet \( L^p \)-modulus of continuity.

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and of the convergence of sequences of random functions in the setting of Choquet integral. Section 4 deals with approximation results by random Bernstein polynomials of several variables in the framework of capacities and Choquet integral. Our main results are Theorem 3 devoted to the approximation in the Choquet-mean of order $p \in [1, \infty)$, and Theorem 4 devoted to the uniform approximation in capacity by sequences of multivariate random Bernstein polynomials. In the probabilistic case (and for functions of one real variable and $p = 1$ for Theorem 3), these results were previously proved respectively in [2] and [17].

2. Preliminaries on capacities and Choquet integral

For the convenience of the reader we will briefly recall some basic facts concerning Choquet’s theory of integrability with respect to a nondecreasing set function (not necessarily additive). Full details are to be found in the books of D. Denneberg [4] and Grabisch [12].

Let $(\Omega, \mathcal{A})$ be an arbitrarily fixed measurable space, that is, a nonempty abstract set $\Omega$ endowed with a $\sigma$-algebra $\mathcal{A}$ of subsets of $\Omega$.

Definition 1. A set function $\mu : \mathcal{A} \to \mathbb{R}_+$ is called a capacity if it verifies the following two conditions:

(a) $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$;
(b) $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{A}$, with $A \subset B$.

A capacity $\mu$ is called subadditive if

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

and submodular (or strongly subadditive) if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$$

for all $A, B \in \mathcal{A}$.

A simple way to construct nontrivial examples of submodular capacities is to start with a probability measure $P : \mathcal{A} \to [0, 1]$ and to consider any nondecreasing concave function $u : [0, 1] \to [0, 1]$ such that $u(0) = 0$ and $u(1) = 1$; for example, one may chose $u(t) = t^\alpha$ with $0 < \alpha < 1$. Then $\mu = u(P)$ is a submodular capacity on the $\sigma$-algebra $\mathcal{A}$, called a distorted probability.

The capacity spaces (that is, the triplets $(\Omega, \mathcal{A}, \mu)$, where $\Omega$ is a nonempty abstract set endowed with a $\sigma$-algebra $\mathcal{A}$ of subsets of $\Omega$ and $\mu : \mathcal{A} \to \mathbb{R}_+$ is a capacity) represent a generalization of the classical concept of probability space.

To a capacity space $(\Omega, \mathcal{A}, \mu)$ one can attach several spaces of functions, starting with the space $L^0(\Omega, \mathcal{A}, \mu)$ of all random variables $f : \Omega \to \mathbb{R}$ (that is, of all functions $f$ verifying the condition of $\mathcal{A}$-measurability, $f^{-1}(A) \in \mathcal{A}$ for every Borel subset $A \subset \mathbb{R}$). At the end of this section, the analogs of the classical Lebesgue spaces $L^p(\Omega, \mathcal{A}, \mu)$ (for $p \geq 1$) will be presented (under the requirement that the capacity $\mu$ is submodular).

The key ingredient is the integrability of random variables $f \in L^0(\Omega, \mathcal{A}, \mu)$ with respect to the capacity $\mu$. 
Definition 2. The Choquet integral of a random variable $f : \Omega \to \mathbb{R}$ on a set $A \in \mathcal{A}$ is defined by the formula
\begin{equation}
(C) \int_A f \, d\mu = \int_0^{+\infty} \mu(\{\omega \in \Omega : f(\omega) > t\} \cap A) \, dt
+ \int_{-\infty}^0 \left[\mu(\{\omega \in \Omega : f(\omega) > t\} \cap A) - \mu(A)\right] \, dt,
\end{equation}
where the integrals in the right hand side are generalized Riemann integrals.

If $(C) \int_A f \, d\mu$ exists in $\mathbb{R}$, then $f$ is called Choquet integrable on $A$.

Notice that if $f \geq 0$, then the last integral in the formula (2.1) is 0.

The Choquet integral agrees with the Lebesgue integral in the case of probabilistic measures. See Denneberg [4], p. 62.

The next remark summarizes the basic properties of the Choquet integral:

Remark 1. (a) If $f, g \in L^0(\Omega, \mathcal{A}, \mu)$ are Choquet integrable on $A$, then

$f \geq 0$ implies $(C) \int_A f \, d\mu \geq 0$ (positivity)

$f \leq g$ implies $(C) \int_A f \, d\mu \leq (C) \int_A g \, d\mu$ (monotonicity)

$(C) \int_A a f \, d\mu = a \cdot (C) \int_A f \, d\mu$ for all $a \geq 0$ (positive homogeneity)

$(C) \int_A 1 \cdot d\mu(t) = \mu(A)$ (calibration).

(b) In general, the Choquet integral is not additive but, if $f$ and $g$ are comonotonic (that is, $(f(\omega) - f(\omega')) \cdot (g(\omega) - g(\omega')) \geq 0$, for all $\omega, \omega' \in A$), then

$(C) \int_A (f + g) \, d\mu = (C) \int_A f \, d\mu + (C) \int_A g \, d\mu.$

An immediate consequence is the property of translation invariance,

$(C) \int_A (f + c) \, d\mu = (C) \int_A f \, d\mu + c \cdot \mu(A)$

for all $c \in \mathbb{R}$ and $f$ integrable on $A$.

(c) If $\mu$ is a subadditive capacity and $f$ is nonnegative and Choquet integrable on the sets $A$ and $B$, then

$(C) \int_{A \cup B} f \, d\mu \leq (C) \int_A f \, d\mu + (C) \int_B f \, d\mu.$

For (a) and (b) see Denneberg [4], Proposition 5.1, p. 64; (c) follows in a straightforward way from the definition of the Choquet integral.

Remark 2. (The Subadditivity Theorem) If $\mu$ is a submodular capacity, then the associated Choquet integral is subadditive, that is,

$(C) \int_A (f + g) \, d\mu \leq (C) \int_A f \, d\mu + (C) \int_A g \, d\mu$

for all $f$ and $g$ integrable on $A$. See [4], Theorem 6.3, p. 75. In addition, the following two integral analogs of the modulus inequality hold true,

\[\|(C) \int_A f \, d\mu\| \leq \int_A |f| \, d\mu\]
and
\[ |(C) \int_A f \, d\mu - (C) \int_A g \, d\mu| \leq (C) \int_A |f - g| \, d\mu; \]
the last assertion is covered by Corollary 6.6, p. 82, in [4].

The analogs of the Lebesgue spaces in the context of capacities can be introduced for \(1 \leq p < +\infty\) via the formulas
\[ L^p(\Omega, \mathcal{A}, \mu) = \{ f : f \in L^0(\Omega, \mathcal{A}, \mu) \text{ and } (C) \int_\Omega |f(\omega)|^p \, d\mu < +\infty \}. \]
When \(\mu\) is a subadditive capacity (in particular, when \(\mu\) is submodular), the functionals \(\| \cdot \|_{L^p(\Omega, \mathcal{A}, \mu)}\) defined by the formula
\[ \| f \|_{L^p(\Omega, \mathcal{A}, \mu)} = \left( (C) \int_\Omega |f(\omega)|^p \, d\mu \right)^{1/p} \]
satisfy the triangle inequality too (see, e.g. Theorem 2, p. 5 in [3], or Proposition 9.4, p. 109-110 in [4]).

Under the stronger hypothesis that \(\mu\) is a submodular capacity, the quotient space
\[ L^p(\Omega, \mathcal{A}, \mu) = L^p(\Omega, \mathcal{A}, \mu) / N_p, \]
where
\[ N_p = \{ f \in L^p(\Omega, \mathcal{A}, \mu); \left( (C) \int_\Omega |f(\omega)|^p \, d\mu \right)^{1/p} = 0 \}, \]
becomes a normed vector space relative to the norm
\[ \| f \|_{L^p(\Omega, \mathcal{A}, \mu)} = \left( (C) \int_\Omega |f(\omega)|^p \, d\mu \right)^{1/p} . \]
See [4], Proposition 9.4, p. 109, for \(p = 1\) and p. 115 for arbitrary \(p \geq 1\).

If \(\mu\) is not only submodular, but also lower continuous in the sense that
\[ \lim_{n \to \infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n) \]
for every nondecreasing sequence \((A_n)_n\) of sets in \(\mathcal{A}\), then \(L^p(\Omega, \mathcal{A}, \mu)\) is a Banach space. See [4], Proposition 9.5, p. 111 and the comment at page 115. Under the same hypotheses, \(h \in N_p\) if and only if
\[ h = 0 \quad \mu\text{-a.e.} \]
meaning that the existence of a set \(N \subset \Omega\) such that
\[ \mu^*(N) = \inf \{ \mu(A) : A \in \mathcal{A}, \ A \supset N \} = 0 \]
and \(h(\omega) = 0\) for all \(\omega \in \Omega \setminus N\). See [4], p. 107, Corollary 9.2 and pp. 107-108.

3. Continuity of Random functions associated to a capacity space

Given a capacity space \((\Omega, \mathcal{A}, \mu)\) and a subset \(D\) of the Euclidean space \(\mathbb{R}^N\), we will refer to the functions \(F : D \to L^0(\Omega, \mathcal{A}, \mu)\) as random functions. It is usual to interpret \(F\) as a stochastic process \(F : D \times \Omega \to \mathbb{R}, F(x, \omega) = F(x)(\omega)\). For fixed \(\omega, F(x, \omega)\) is a deterministic function of \(x\), called a sample function.

Following the case of probabilistic spaces one can consider several kinds of continuity, of interest for us being the following ones.
A random function $F$ is continuous in capacity at the point $x_0 \in D$, if $x \to x_0$ implies $F(x) \to F(x_0)$ in capacity, that is, for every $\varepsilon > 0$ and $\eta > 0$ there exists $\delta = \delta(\varepsilon, \eta, x_0) > 0$ such that

$$\mu(\{\omega \in \Omega : |F(x, \omega) - F(x_0, \omega)| \geq \varepsilon\}) < \eta$$

whenever $x \in E$ and $\|x - x_0\| < \delta$.

- A random function $F$ is called uniformly continuous in capacity, if for every $\varepsilon > 0$ and $\eta > 0$ there exists $\delta = \delta(\varepsilon, \eta) > 0$, such that

$$\mu(\{\omega \in \Omega : |F(x', \omega) - F(x'', \omega)| \geq \varepsilon\}) < \eta$$

whenever $x', x'' \in D$ and $\|x' - x''\| < \delta$.

When $F$ takes values in a space $L^p(\Omega, \mathcal{A}, \mu)$ (for some $p \in [1, \infty]$), then one can speak of its continuity in the Choquet-mean of order $p$.

- A random function $F$ is called continuous in the Choquet-mean of order $p$ at the point $x_0 \in D$, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x_0) > 0$, such that for all $x \in E$ with $\|x - x_0\| < \delta$, we have

$$(C) \int_{\Omega} |F(x, \omega) - F(x_0, \omega)|^p d\mu < \varepsilon.$$ 

- A random function $F$ is called uniformly continuous in Choquet-mean of order $p$ if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that for all $x', x'' \in E$ with $\|x' - x''\| < \delta(\varepsilon)$ we have

$$(C) \int_{\Omega} |F(x', \omega) - F(x'', \omega)|^p d\mu < \varepsilon.$$ 

In the next section we will be interested in the approximation of random functions by random Bernstein polynomials. The notions of approximation in capacity and approximation in Choquet-mean are defined as follows:

- A sequence $(F_n)_n$ of random functions converges in capacity to the random function $F$ if for every $\varepsilon, \eta > 0$ and $x \in D$ there exists $N(\varepsilon, \eta, x) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon, \eta, x)$ we have

$$\mu(\{\omega \in \Omega : |F_n(x, \omega) - F(x, \omega)| \geq \varepsilon\}) < \eta.$$ 

If $N(\varepsilon, \eta, x)$ does not depend on $x$, then we say that $(F_n)_n$ converges uniformly in capacity to $F$.

- A sequence $(F_n)_n$ of random functions converges in Choquet-mean of order $p$ to the random function $F$ if for every $\varepsilon > 0$ and $x \in D$, there exists $N(\varepsilon, x) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon, x)$ we have

$$(C) \int_{\Omega} |F_n(x, \omega) - F(x, \omega)|^p d\mu < \varepsilon.$$ 

If $N(\varepsilon, x)$ does not depend on $x$, then we will say that $(F_n)_n$ converges uniformly to $F$ in the Choquet-mean of order $p$.

- For $1 \leq p < +\infty$ and $\delta_i \geq 0$, $i = 1, \ldots, N$, the multivariate Choquet $L^p$-modulus of continuity of $F$ will be defined by

$$\omega(f; \delta_1, \ldots, \delta_N)_p = \left( \sup_{|t_i - s_i| \leq \delta_i, i = 1, \ldots, N} (C) \int_{\Omega} |F(t_1, \ldots, t_N, \omega) - F(s_1, \ldots, s_N, \omega)|^p d\mu(\omega) \right)^{1/p}.$$
A big source of convergence in capacity is provided by convergence in distribution.

If \( f : \Omega \to \mathbb{R} \) is a random variable, then its distribution function with respect to the capacity \( \mu \) is defined by the formula

\[
F_f(x) = \mu(\{\omega \in \Omega : f(\omega) \leq x\}), \quad x \in \mathbb{R}.
\]

A sequence of random variables \((f_n)_n\) is called convergent in distribution to the random variable \( f : \Omega \to \mathbb{R} \) if

\[
\lim_{n \to \infty} F_{f_n}(x) = F_f(x)
\]
at each point \( x \in \mathbb{R} \) where \( F_f \) is continuous.

**Theorem 1.** Let \( \mu \) be a subadditive capacity. If \((f_n)_n\) converges in capacity to \( f \), then \((f_n)_n\) converges in distribution to \( f \).

**Proof.** For any fixed \( x \in \mathbb{R} \) and \( f, g \) random variables, we have

\[
\{g \leq x\} \subset \{f \leq x + \varepsilon\} \cup \{|f - g| > \varepsilon\}
\]
for if \( g \leq x \) and \( |f - g| \leq \varepsilon \), then \( f \leq x + \varepsilon \). Since \( \mu \) is monotone and subadditive, it follows

\[
\mu(\{g \leq x\}) \leq \mu(\{f \leq x + \varepsilon\}) + \mu(\{|f - g| > \varepsilon\}).
\]

We show that the sequence of cumulative distribution functions \((F_{f_n})_n\) converges to the \( F_f \) at every point \( x \) where \( F_f \) is continuous. Indeed, for every \( \varepsilon > 0 \), due to the previous inequality, we get

\[
\mu(\{f_n \leq x\}) \leq \mu(\{f \leq x + \varepsilon\}) + \mu(\{|f_n - f| > \varepsilon\})
\]
and

\[
\mu(\{f \leq x - \varepsilon\}) \leq \mu(\{f_n \leq x\}) + \mu(\{|f_n - f| > \varepsilon\}).
\]

So, we have

\[
\mu(\{f \leq x - \varepsilon\}) - \mu(\{|f_n - f| > \varepsilon\}) \leq \mu(\{f_n \leq x\}) \leq \mu(\{f \leq x + \varepsilon\}) + \mu(\{|f_n - f| > \varepsilon\}).
\]

Taking the limit as \( n \to \infty \), we obtain

\[
F_f(x - \varepsilon) \leq \lim_{n \to \infty} \mu(\{f_n \leq x\}) \leq F_f(x + \varepsilon).
\]

By the continuity of \( F_f \) at \( x \), it follows that both \( F_f(x - \varepsilon) \) and \( F_f(x + \varepsilon) \) converge to \( F_f(x) \) as \( \varepsilon \searrow 0 \). Therefore, taking these limits, we obtain

\[
\lim_{n \to \infty} \mu(\{f_n \leq x\}) = \mu(\{f \leq x\}),
\]
which means that \( f_n \) converges to \( f \) in distribution. \( \Box \)

An important property of the Choquet \( L^p \)-modulus of continuity used in approximation is the following ones, stated and proved here only for simplicity for two variables.

**Theorem 2.** Let \( 1 \leq p < +\infty \). If \( \mu \) is a submodular capacity, then

\[
\omega(F; \alpha \delta, \beta \eta)_p \leq (1 + \alpha + \beta)\omega(F; \delta, \eta)_p,
\]
for all \( \alpha, \beta > 0 \).
Indeed, let $t_1, r_1, s_1$ with $|t_1 - s_1| \leq \delta_1 + \delta_2, |t_1 - r_1| \leq \delta_1, |r_1 - s_1| \leq \delta_2$ and $t_2, r_2, s_2$ with $|t_2 - s_2| \leq \eta_1 + \eta_2, |r_2 - r_2| \leq \eta_1, |r_2 - s_2| \leq \eta_2$.

Since $\mu$ is submodular, by e.g. Theorem 2, p. 5 in [3] or Proposition 9.4, p. 109-110 in [4], the Minkowski’s inequality holds in the space $L_p(\Omega, \mathcal{A}, \mu)$. This implies

$$\left(\int_{\Omega} |F(t_1, t_2, \omega) - F(s_1, s_2, \omega)|^p d\mu(\omega)\right)^{1/p} \leq \left(\int_{\Omega} |F(t_1, t_2, \omega) - F(r_1, r_2, \omega)|^p d\mu(\omega)\right)^{1/p} + \left(\int_{\Omega} |F(r_1, r_2, \omega) - F(s_1, s_2, \omega)|^p d\mu(\omega)\right)^{1/p}.$$ 

Passing now to the corresponding supremums, firstly in the right-hand side and then in the left-hand side, immediately lead us to (3.1).

Now, by (3.1) we easily obtain

$$\omega(F; n\delta, m\eta)_p \leq \max\{n, m\} \omega(F; \delta, \eta)_p,$$

which by $\alpha < [\alpha] + 1, \beta < [\beta] + 1$ and $\max\{[\alpha] + 1, [\beta] + 1\} = \max\{[\alpha], [\beta]\} + 1 < \alpha + \beta + 1$, immediately implies the required inequality in the statement. \hfill \Box

4. Approximation via random Bernstein polynomials

The approximation of random functions defined on a compact $N$-dimensional interval in $\mathbb{R}$ (that is, on a product of $N$ compact intervals of $\mathbb{R}$) can be easily reduced (via an affine transformation) to the particular case where the domain is the $N$-dimensional unit cube $[0, 1]^N$. In this context it is important to study the approximation of random functions $F : [0, 1]^N \to L^p(\Omega, \mathcal{A}, \mu)$ via the associated Bernstein polynomials,

$$B_{n_1, \ldots, n_N}(F)(x_1, \ldots, x_N, \omega) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_N=0}^{n_N} p_{k_1, n_1}(x_1) \cdot \cdots \cdot p_{k_N, n_N}(x_N) \cdot F\left(\frac{k_1}{n_1}, \ldots, \frac{k_N}{n_N}, \omega\right),$$

where $p_{k_j, n_j}(x_j) = \binom{n_j}{k_j} x_j^{k_j} (1 - x_j)^{n_j - k_j}$, $k_j \in \{0, \ldots, n_j\}$, $n_j \in \mathbb{N}$ and $x_j \in [0, 1]$ for $j = 1, \ldots, N$.

These polynomials have a number of nice property related to shape preservation, that make them useful to computer aided geometric design. Details are available in the book of Gal [4] (and the references therein).

The approximation of random functions by Bernstein polynomials will be discussed in the context of submodular capacity spaces $(\Omega, \mathcal{A}, \mu)$, that is, when the capacity $\mu$ under attention is submodular. We start with the case of approximation in the Choquet-mean of order $p \in [1, \infty)$. 

Proof. Firstly we prove

$$\omega(F; \delta_1 + \delta_2, \eta_1 + \eta_2)_p \leq \omega(F; \delta_1, \eta_1)_p + \omega(F; \delta_2, \eta_2)_p.$$ 

Indeed, let $t_1, r_1, s_1$ with $|t_1 - s_1| \leq \delta_1 + \delta_2, |t_1 - r_1| \leq \delta_1, |r_1 - s_1| \leq \delta_2$ and $t_2, r_2, s_2$ with $|t_2 - s_2| \leq \eta_1 + \eta_2, |r_2 - r_2| \leq \eta_1, |r_2 - s_2| \leq \eta_2$.

Since $\mu$ is submodular, by e.g. Theorem 2, p. 5 in [3] or Proposition 9.4, p. 109-110 in [4], the Minkowski’s inequality holds in the space $L_p(\Omega, \mathcal{A}, \mu)$. This implies

$$\left(\int_{\Omega} |F(t_1, t_2, \omega) - F(s_1, s_2, \omega)|^p d\mu(\omega)\right)^{1/p} \leq \left(\int_{\Omega} |F(t_1, t_2, \omega) - F(r_1, r_2, \omega)|^p d\mu(\omega)\right)^{1/p} + \left(\int_{\Omega} |F(r_1, r_2, \omega) - F(s_1, s_2, \omega)|^p d\mu(\omega)\right)^{1/p}.$$ 

Passing now to the corresponding supremums, firstly in the right-hand side and then in the left-hand side, immediately lead us to (3.1).

Now, by (3.1) we easily obtain

$$\omega(F; n\delta, m\eta)_p \leq \max\{n, m\} \omega(F; \delta, \eta)_p,$$

which by $\alpha < [\alpha] + 1, \beta < [\beta] + 1$ and $\max\{[\alpha] + 1, [\beta] + 1\} = \max\{[\alpha], [\beta]\} + 1 < \alpha + \beta + 1$, immediately implies the required inequality in the statement. \hfill \Box
Theorem 3. Suppose that $(\Omega, \mathcal{A}, \mu)$ is a submodular capacity space and $F : [0, 1]^N \rightarrow L^p(\Omega, \mathcal{A}, \mu)$ is a random function which is continuous in the Choquet-mean of order $p$ at each $x \in [0, 1]^N$ and verifies the boundedness condition

\begin{equation}
M = \sup_{x \in [0, 1]^N} (C) \int_\Omega |F(x, \omega)|^p d\mu < +\infty.
\end{equation}

Then the sequence of random Bernstein polynomials $(B_{n_1, ..., n_N}(F))_{n_1, ..., n_N}$ converges uniformly to $F$ in the Choquet-mean of order $p$ as $\min\{n_1, ..., n_N\} \to \infty$.

Proof. For simplicity, we will detail the proof in the case $N = 2$ (the general case being similar).

Our first remark is the uniform continuity of $F$ in the Choquet-mean of order $p$ (motivated by the compactness of its domain $[0, 1]^2$). Indeed, supposing by reductio ad absurdum that the contrary is true, then, would exist $\varepsilon > 0$ and two sequences of elements $x_n', x_n'' \in [0, 1]^2$ such that $\|x_n' - x_n''\| < \frac{1}{n}$ for every $n$ and

\[(C) \int_\Omega |F(x_n', \omega) - F(x_n'', \omega)|^p d\mu \geq \varepsilon.
\]

Replacing the two sequences above by appropriate subsequences we may assume that they converge to a same limit, say $x_0 \in [0, 1]^2$. Taking into account the numerical inequality $|F + G|^p \leq 2^p(|F|^p + |G|^p)$ and the subadditivity of Choquet integral (since $\mu$ is submodular), we have

\[0 < \varepsilon \leq (C) \int_\Omega |F(x_n', \omega) - F(x_n'', \omega)|^p d\mu \leq 2^p \cdot (C) \int_\Omega |F(x_n', \omega) - F(x_0, \omega)|^p d\mu + 2^p \cdot (C) \int_\Omega |F(x_0, \omega) - F(x_n'', \omega)|^p d\mu.
\]

However, since $F$ is continuous at $x_0$ in Choquet-mean of order $p$, the right hand side of the above inequality tends to zero as $n \to \infty$. This contradicts the fact that $\varepsilon > 0$ and the proof of the uniform continuity of $F$ is done.

Taking into account the identity

\[\sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) = 1\]

and the convexity of the function $x^p$ for $p \geq 1$, we infer from Jensen’s inequality that

\[|F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|^p \leq \left[ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)| \right]^p \leq \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)|^p.
\]

Integrating side by side and using Remark 3 (a) and (c) we arrive at the estimate

\[(C) \int_\Omega |F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|^p d\mu
\]
In particular

Let \( \varepsilon \in (0, 1] \) arbitrarily fixed. Since \( F \) is uniformly continuous, there exists \( \delta(\varepsilon) > 0 \), such that for all \( x_1', x_2', x_1'', x_2'' \in [0, 1] \) with \( \max \{|x_1' - x_1''|, |x_2' - x_2''|\} < \delta(\varepsilon) \) we have

\[
(4.2) \quad \int_{\Omega} |F(x_1', x_2', \omega) - F(x_1'', x_2'', \omega)|^p \, d\mu < \varepsilon.
\]

There also exists a number \( T(\varepsilon, p) \in \mathbb{N} \) such that

\[
\frac{2^{p-1}M}{n\delta^2} < \frac{\varepsilon}{2} \quad \text{for all } n \geq T(\varepsilon, p).
\]

In particular \( \frac{2^{p-1}M}{n\delta^2} < \frac{\varepsilon}{2} \) if \( n_1, n_2 \geq T(\varepsilon, p) \). Now fix arbitrarily a pair of integers \( n_1, n_2 \geq T(\varepsilon, p) \) and put

\[
I'_1 = \{ 0 \leq k_1 \leq n_1 : |k_1/n_1 - x_1| < \delta(\varepsilon) \}
\]

\[
I''_1 = \{ 0 \leq k_1 \leq n_1 : |k_1/n_1 - x_1| \geq \delta(\varepsilon) \},
\]

and

\[
I'_2 = \{ 0 \leq k_2 \leq n_2 : |k_2/n_2 - x_2| < \delta(\varepsilon) \}
\]

\[
I''_2 = \{ 0 \leq k_2 \leq n_2 : |k_2/n_2 - x_2| \geq \delta(\varepsilon) \}.
\]

Then

\[
(4.2) \quad \int_{\Omega} |F(x_1, x_2, \omega) - B_n(F)(x_1, x_2, \omega)|^p \, d\mu(\omega)
\]

\[
\leq \sum_{k_1 \in I'_1} \sum_{k_2 \in I'_2} p_{k_1, n_1}(x_1)p_{k_2, n_2}(x_2)(C) \int_{\Omega} |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)|^p \, d\mu
\]

\[
+ \sum_{k_1 \in I'_1} \sum_{k_2 \in I''_2} p_{k_1, n_1}(x_1)p_{k_2, n_2}(x_2)(C) \int_{\Omega} |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)|^p \, d\mu
\]

\[
+ \sum_{k_1 \in I''_1} \sum_{k_2 \in I'_2} p_{k_1, n_1}(x_1)p_{k_2, n_2}(x_2)(C) \int_{\Omega} |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)|^p \, d\mu
\]

\[
+ \sum_{k_1 \in I''_1} \sum_{k_2 \in I''_2} p_{k_1, n_1}(x_1)p_{k_2, n_2}(x_2)(C) \int_{\Omega} |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)|^p \, d\mu
\]

\[
\leq \frac{\varepsilon}{2} + 2^p \left[ 2M \sum_{k_2 \in I''_2} p_{k_2, n_2}(x_2) \right] + 2^p \left[ 2M \sum_{k_1 \in I''_1} p_{k_1, n_1}(x_1) \right]
\]

\[
+ 2^p \left[ 2M \sum_{k_1 \in I''_1} p_{k_1, n_1}(x_1) \cdot \sum_{k_2 \in I''_2} p_{k_2, n_2}(x_2) \right]
\]

\[
\leq \frac{\varepsilon}{2} + 2^p \frac{2M}{4n_2\delta^2} + 2^p \frac{2M}{4n_1\delta^2} + 2^p \frac{2M}{4n_1\delta^2} \cdot 2^p \frac{2M}{4n_2\delta^2}
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \frac{1}{2^p} < 2\varepsilon.
\]
We used the estimate
\[
\sum_{k_j \in I''} p_{k_j,n_j}(x_j) \leq \frac{1}{4n_0^2} \quad \text{for } j = 1, 2,
\]
which is inequality (7) in G. Lorentz [10], p. 6. The proof of Theorem 3 is now complete. 

\[\square\]

**Remark 3.** For the approximation result in Theorem 3 we can deduce a quantitative estimate too in terms of the Choquet $L^p$-modulus of continuity, as follows. Using the inequality (4.2) in the proof of Theorem 3 and then Theorem 2, we get
\[
(C) \int_{\Omega} |F(x_1, x_2, \omega) - B_{n_1,n_2}(F)(x_1, x_2, \omega)|^p \, d\mu \leq \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{k_1,n_1}(x_1)p_{k_2,n_2}(x_2)
\]
\[
\cdot \left[ \omega(F; \frac{1}{\sqrt{n_1}} \cdot (\sqrt{n_1}|x_1 - k_1/n_1|), \frac{1}{\sqrt{n_2}} \cdot (\sqrt{n_2}|x_2 - k_2/n_2|) \right]^p
\]
\[
\leq \left[ \omega \left( F; \frac{1}{\sqrt{n_1}}, \frac{1}{\sqrt{n_2}} \right) \right]^p
\]
\[
\cdot \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{k_1,n_1}(x_1)p_{k_2,n_2}(x_2)(1 + \sqrt{n_1}|x_1 - k_1/n_1| + \sqrt{n_2}|x_2 - k_2/n_2|)^p.
\]

But by the general estimate of the moments of Bernstein polynomials
\[
\sum_{k=0}^{n} p_{k,n}(x)|\sqrt{n}|x - k/n| |^j \leq 2\Gamma(1+j/2), \quad j = 0, 1, \ldots, p,
\]
where $\Gamma$ denotes the Gamma function (see Theorem 1 in [1]), it is immediate that
\[
\sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{k_1,n_1}(x_1)p_{k_2,n_2}(x_2)(1 + \sqrt{n_1}|x_1 - k_1/n_1| + \sqrt{n_2}|x_2 - k_2/n_2|)^p \leq C_p,
\]
where $C_p$ is independent of $n_1$, $n_2$ and $x_1, x_2 \in [0,1]$. Concluding, we obtain
\[
\left[ (C) \int_{\Omega} |F(x_1, x_2, \omega) - B_{n_1,n_2}(F)(x_1, x_2, \omega)|^p \, d\mu \right]^{1/p} \leq [C_p]^{1/p} \omega \left( F; \frac{1}{\sqrt{n_1}}, \frac{1}{\sqrt{n_2}} \right)_p.
\]

The next result concerns the approximation in capacity.

**Theorem 4.** Suppose that $(\Omega, A, \mu)$ is a submodular capacity space and $F : [0,1]^N \to L^0(\Omega, A, \mu)$ is a random function which is continuous in capacity at each point $x \in [0,1]^N$ and verifies the boundedness condition
\[
M = \sup_{x \in [0,1]^N} |F(x, \omega)| < +\infty.
\]
for all $\omega \in \Omega$, excepting possibly a set $E$ of capacity zero.

Then the sequence of random Bernstein polynomials $(B_{n_1,\ldots,n_N}(F))_{n_1,\ldots,n_N}$ converges uniformly to $F$ in capacity as $\min \{n_1,\ldots,n_N\} \to \infty$. 

\[\square\]
Proof: For simplicity, we will detail the proof in the case $N = 2$ (the general case being similar).

We start by noticing that the uniform convergence in capacity is implied by the convergence in the semi-metric

\[ d(F, G) = \sup_{x \in [0,1]^2} (C) \int_{\Omega} \frac{|F(x, \omega) - G(x, \omega)|}{1 + |F(x, \omega) - G(x, \omega)|} d\mu. \]

The fact that $d$ satisfies the triangle inequality is immediate from the fact that the function $\varphi(t) = \frac{1}{1+t}$ is increasing and subadditive on $[0, +\infty)$ and from the properties of the Choquet integral as mentioned in Remark II (a) and (c).

The fact that the convergence with respect to $d$ implies the uniform convergence in capacity, is a direct consequence of Markov’s inequality (for the Choquet integral). Keeping fixed $x \in [0,1]^2$ and assuming that $H(x, \omega)$ is a nonnegative random variable, then for each $a > 0$ we have

\[ (C) \int_{\Omega} H(x, \omega) d\mu \geq (C) \int_{\Omega \setminus \{\omega \in \Omega : H(x, \omega) \geq a\}} H(x, \omega) d\mu \]

\[ \geq (C) \int_{\Omega \cap \{\omega \in \Omega : H(x, \omega) \geq a\}} a d\mu = a \cdot \mu(\{\omega \in \Omega : H(x, \omega) \geq a\}), \]

which is Markov’s inequality. It can be generalized by considering a positive and strictly increasing function $\varphi$ on $[0, +\infty)$. Indeed,

\[ \mu(\{\omega \in \Omega : H(x, \omega) \geq a\}) = \mu(\{\omega \in \Omega : \varphi(H(x, \omega)) \geq \varphi(a)\}) \leq \frac{(C) \int_{\Omega} \varphi(H(x, \omega)) d\mu}{\varphi(a)}. \]

Choosing $\varphi(t) = \frac{1}{1+t}$ and $H(x, \omega) = |F(x, \omega) - G(x, \omega)|$ in (4.4), one can easily see that the convergence in the metric $d$ implies the uniform convergence in capacity.

Concerning the set $E$ in the hypothesis, let us notice that any random variable $G(x, \omega) \geq 0$ verifies

\[ (C) \int_{E} G d\mu = \int_{0}^{\infty} \mu(\{x : f(x) \geq t\} \cap E) dt = 0. \]

According to assertions (a) and (c) of Remark II

\[ (C) \int_{\Omega} G d\mu \leq (C) \int_{E} G d\mu + (C) \int_{\Omega \setminus E} G d\mu = (C) \int_{\Omega \setminus E} G d\mu \]

and thus

\[ (C) \int_{\Omega} G d\mu = (C) \int_{\Omega \setminus E} G d\mu. \]

Next, notice that due to the compactness of the $[0,1]^2$, the function $F$ is uniformly continuous in capacity. This can be done by reductio ad absurdum (as in the proof of Theorem III).

As a consequence, for $\varepsilon > 0$ arbitrary fixed there exists $\delta(\varepsilon)$ such that

\[ (C) \mu(\{\omega \in \Omega : |F(x_1', x_2', \omega) - F(x_1'', x_2'', \omega)| \geq \varepsilon/2\}) < \frac{\varepsilon}{2M} \]

for all $x_1', x_2', x_1'', x_2'' \in [0,1]$ with $|x_1' - x_1''|, |x_2' - x_2''| < \delta(\varepsilon)$. 

\[ (C) \int_{\Omega} G d\mu = (C) \int_{\Omega \setminus E} G d\mu. \]
One can also choose an integer $N(\varepsilon)$ such that

$$\frac{M}{2n\delta^2} < \varepsilon/2$$

for all $n \geq N(\varepsilon)$.

Fix arbitrarily a pair of integers $n_1, n_2 \geq N(\varepsilon)$ and define the sets $I_1', I_2'$ and $I_2''$ as in the proof of Theorem 3. Put

$$\Omega_1 = \{\omega \in \Omega : |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)| < \varepsilon/2\},$$

$$\Omega_2 = \{\omega \in \Omega : |F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega)| \geq \varepsilon/2\}.$$

Then the sets

(4.7) $E, \Omega_1' = \Omega_1 \setminus E$ and $\Omega_2' = \Omega_2 \setminus E$

constitutes a partition of $\Omega$.

Taking into account Remark 1 for all $x_1, x_2 \in [0, 1]$ and $n_1, n_2 \geq N(\varepsilon)$ we have

$$(C) \int_{\Omega} \frac{|F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|}{1 + |F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|} d\mu$$

$$= (C) \int_{\Omega} \left| \sum_{k_1 = 0}^{n_1} \sum_{k_2 = 0}^{n_2} p_{k_1, n_1}(x_1)p_{k_2, n_2}(x_2) \Delta F(x_1, x_2; k_1/n_1, k_2/n_2) \right| d\mu$$

$$\leq (C) \int_{\Omega} \left| \sum_{k_1 = 0}^{n_1} \sum_{k_2 = 0}^{n_2} p_{k_1, n_1}(x_1)p_{k_2, n_2}(x_2) \Delta F(x_1, x_2; k_1/n_1, k_2/n_2) \right| d\mu$$

$$\leq \Delta F(x_1, x_2; k_1/n_1, k_2/n_2) = F(x_1, x_2, \omega) - F(k_1/n_1, k_2/n_2, \omega).$$

The last double sum can be decomposed (according to the partition (4.7) and the equation (4.5)) into a sum of four terms as follows

$$\sum_{k_1 \in I_1'} \sum_{k_2 \in I_2'} \Delta F(x_1, x_2; k_1/n_1, k_2/n_2) \int_{\Omega_1' \cup \Omega_2' \cup E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| d\mu$$

$$+ \sum_{k_1 \in I_1'} \sum_{k_2 \in I_2'} \Delta F(x_1, x_2; k_1/n_1, k_2/n_2) \int_{\Omega_1' \cup \Omega_2' \cup E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| d\mu$$

$$+ \sum_{k_1 \in I_1'} \sum_{k_2 \in I_2''} \Delta F(x_1, x_2; k_1/n_1, k_2/n_2) \int_{\Omega_1' \cup \Omega_2' \cup E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| d\mu$$

$$+ \sum_{k_1 \in I_1'} \sum_{k_2 \in I_2''} \Delta F(x_1, x_2; k_1/n_1, k_2/n_2) \int_{\Omega_1' \cup \Omega_2' \cup E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| d\mu.$$
Denote the last four double sums above respectively $A_1, A_2, A_3$ and $A_4$. Then, based on Remark 1 (c) and equation (4.5), we have

$$A_1 \leq \sum_{k_1 \in k'_1} \sum_{k_2 \in k'_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_1} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k'_1} \sum_{k_2 \in k'_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_2} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k'_1} \sum_{k_2 \in k'_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$\leq \frac{\varepsilon}{2} + 2M \cdot \mu(\Omega'_2) + 0 = \frac{\varepsilon}{2} + \varepsilon = \frac{3\varepsilon}{2}.$$ 

Next, taking into account the estimate (4.3),

$$A_2 \leq \sum_{k_1 \in k'_1} \sum_{k_2 \in k''_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_1} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k'_1} \sum_{k_2 \in k''_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_2} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k'_1} \sum_{k_2 \in k''_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$\leq \frac{\varepsilon}{2} + 2M \cdot \sum_{k_2 \in k''_2} p_{k_2, n_2}(x_2) + 0 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

We continue with

$$A_3 \leq \sum_{k_1 \in k''_1} \sum_{k_2 \in k'_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_1} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k''_1} \sum_{k_2 \in k'_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_2} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k''_1} \sum_{k_2 \in k'_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$\leq \frac{\varepsilon}{2} + 2M \cdot \sum_{k_1 \in k''_1} p_{k_1, n_1}(x_1) + 0 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and

$$A_4 \leq \sum_{k_1 \in k''_1} \sum_{k_2 \in k''_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_1} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k''_1} \sum_{k_2 \in k''_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{\Omega_2} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$+ \sum_{k_1 \in k''_1} \sum_{k_2 \in k''_2} p_{k_1, n_1}(x_1) \cdot p_{k_2, n_2}(x_2) \left( \int_{E} |\Delta F(x_1, x_2; k_1/n_1, k_2/n_2)| \, \mathrm{d}\mu \right)$$

$$\leq \frac{\varepsilon}{2} + 2M \cdot \sum_{k_2 \in k''_2} p_{k_2, n_1}(x_2) + 0 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Therefore
\[ (C) \int_{\Omega} \frac{|F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|}{1 + |F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|} d\mu < \frac{3\varepsilon}{2} + \varepsilon + \varepsilon < 5\varepsilon \]
for all \( x_1, x_2 \in [0, 1] \), and \( n_1, n_2 \geq N(\varepsilon) \), which shows that
\[ d(F, B_{n_1, n_2}(F)) = \sup_{x \in [0,1]^2} (C) \int_{\Omega} \frac{|F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|}{1 + |F(x_1, x_2, \omega) - B_{n_1, n_2}(F)(x_1, x_2, \omega)|} d\mu \]
does not exceed \( 5\varepsilon \), whenever \( n_1, n_2 \geq N(\varepsilon) \). According to a remark above this implies the uniform convergence in capacity of \( B_{n_1, n_2}(F)(x_1, x_2, \omega) \) to \( F(x_1, x_2, \omega) \) as \( \min \{ n_1, n_2 \} \to \infty \).

In the case when \( \mu \) is a \( \sigma \)-additive measure and \( F \) is a real-valued random function defined on \([0, 1] \), Theorem 3 case \( p = 1 \), was proved in [2] and Theorem 4 was proved in [17]. Also note that the case of one variable, when \( \mu \) is a \( \sigma \)-additive measure and \( p = 2 \), the quantitative estimate in Remark 3 was obtained in [13] and [14]. Finally, combining Theorem 4 with Theorem 4 it follows that the random Bernstein polynomials also converge in distribution with respect to the submodular capacity \( \mu \).

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