GEOMETRIC SYZYGY CONJECTURE IN POSITIVE CHARACTERISTIC

YI WEI

Abstract. We study the syzygies of canonical curves of even genus \( g = 2k \) over an algebraically closed field \( \mathbb{F} \) with \( \text{char}(\mathbb{F}) = p > 0 \). For \( p \geq \frac{g+4}{2} = k+2 \), we prove a structure theorem for the last syzygy space. As a corollary, this gives a new proof of Green’s Conjecture for general canonical curves of even genus in positive characteristic. Assuming \( p > 2k \), we prove Geometric Syzygy Conjecture for a general smooth curve of even genus \( g = 2k \).

1. Introduction

Throughout this paper, \( \mathbb{F} \) is any algebraically closed field with characteristic \( p > 0 \). A pair \((X, L)\) of a K3 surface \( X \) and a line bundle \( L \) is called polarized K3 surface of degree \( 2d \) if \( L \) is ample and \( L^2 = 2d \). The pair \((X, L)\) is primitively polarized if, further, \( L \) is primitive. Over any algebraically closed field, there exist primitively polarized K3 surfaces \( (X, L) \) of degree \( 2g - 2 \) for any \( g \geq 3 \). [Huy16 §II. Thm 4.2]

Assume \( \text{char}(\mathbb{F}) \neq 2 \). Let \((X, L)\) be any primitively polarized K3 surface over \( \mathbb{F} \) of even genus \( g = 2k \geq 4 \), i.e. degree \( 2g - 2 \). A general member \( C \in |L| \) is irreducible and smooth of genus \( g \). [Huy16 §II. Cor 3.6]. Moreover, the hyperplane restriction theorem [Gre84] implies the equality of Koszul cohomologies

\[
K_{k,1}(C, \omega_C) = K_{k,1}(X, L)
\]

whenever \( C \) is a smooth and connected hyperplane section of a K3 surface \( X \).

To any pair \((C, A)\) consisting of a smooth curve \( C \in |L| \) and a base-point free line bundle \( A \in W^{1}_{k+1}(C) \) with \( h^0(A) = 2 \), one associates a rank 2 Lazarsfeld-Mukai bundle \( E = E_{C, A} \) on \( X \). [Laz86]. The dual bundle \( E^\vee \) fits into the exact sequence

\[
0 \rightarrow E^\vee \rightarrow H^0(C, A) \otimes O_X \rightarrow i_*A \rightarrow 0,
\]

where \( i : C \hookrightarrow X \) is the inclusion. The vector bundle \( E \) has numerical invariants \( c_1(E) = \det(E) = L, c_2(E) = k + 1; h^0(E) = k + 2, h^1(E) = h^2(E) = 0, h^0(E^\vee) = h^1(E^\vee) = 0 \). A general section \( s \in H^0(E) \) gives a general zero-dimensional \( Z(s) \) on a general curve \( C \in |L| \) with \( \deg Z(s) = k + 1 \). Moreover, all \( g^1_{k+1} \)'s on curves arise as sections of \( E \). Our first result is the following.

Theorem 1.1. Fix an algebraically closed field \( \mathbb{F} \) with \( \text{char}(\mathbb{F}) = p > 0 \). Let \((X, L)\) be a general member in any component of the moduli space of primitively polarized K3 surfaces of even genus \( g = 2k \geq 4 \) over \( \mathbb{F} \). Assume \( p \geq k + 2 \). Let \( E \) be the rank two Lazarsfeld-Mukai bundle defined above. For any nonzero \( s \in H^0(E) \), the space \( K_{k-1,1}(X, L; H^0(L \otimes I_{Z(s)})) \) is a one-dimensional subspace of \( K_{k-1,1}(X, L) \). The morphism

\[
\psi : \mathbb{P}(H^0(E)) \rightarrow \mathbb{P}(K_{k-1,1}(X, L))
\]

\[
[s] \mapsto [K_{k-1,1}(X, L; H^0(L \otimes I_{Z(s)}))]
\]

is the Veronese embedding of degree \( k - 2 \).

In particular, \( \psi \) induces a natural isomorphism \( \text{Sym}^{k-2}H^0(X, E) \approx K_{k-1,1}(X, L) \).
Note that the above theorem implies $K_{k,1}(X, L) = 0$ by standard dimension computations. \cite[IV.1]{Far17}. Hence $K_{k,1}(C, \omega_C) = 0$ by \cite{F}. It is well-known that this single vanishing suffices to prove Green’s conjecture for a general canonical curve.

**Corollary 1.1** (Green’s Conjecture in positive characteristic). Let $\mathbb{F}$ be any algebraically closed field with $\text{char}(\mathbb{F}) = p > 0$. Let $C$ be a general curve over $\mathbb{F}$ of even genus $g = 2k$ for $k \geq 2$. Assume $p \geq \frac{g+4}{2} = k + 2$. Then $K_{k,1}(C, \omega_C) = 0$.

Corollary 1.1 is first proved in \cite[Thm 1.2]{AFP} with full generality, i.e. including the case of odd genus, assuming a little different bound on the characteristic. See also \cite{RS21} for an improved bound. The authors used totally different method which turned out to be quite a technical proof. Our approach is considerably simple, based on \cite[Thm 0.2]{Kem20}, which used only homological algebra. But note that the effective lower bound on the characteristic we give is almost identical as that given in \cite[Thm 1.2]{AFP}, which is $\frac{g+2}{2}$.

**Remark 1.1.** (i) Theorem 1.1 in characteristic zero is proved in \cite[Thm 0.2]{Kem20}, where K3 surface $(X, L)$ is assumed to have $\text{Pic}(X) = \mathbb{Z}[L]$, indeed a general case in characteristic zero. Kemeny’s proof makes essential use of the property $\text{Pic}(X) = \mathbb{Z}[L]$, which could be a huge issue in positive characteristic. In particular, the Tate conjecture implies that any K3 surface over $\mathbb{F}_p$ has even Picard number \cite[§XVII. Cor 2.9]{Huy16}.

(ii) The existence of such a base-point free line bundle $A \in W^1_{k,1}(C)$ with $h^0(A) = 2$ on a general smooth curve $C \in |L|$ for a general primitively polarized K3 surface $(X, L)$ is guaranteed by the following proposition, for which the proof is saved until Section \ref{sec3} (One simply sets $r = 1, d = k + 1, g = 2k$)

**Proposition 1.1.** Fix an algebraically closed field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$. Let $(X, L)$ be a general member of any component of the moduli space of primitively polarized K3 surfaces over $\mathbb{F}$ of genus $g$ for $g \geq 3$. Let $r, d, g$ be such integers that the Brill-Noether number $\rho(r, d, g) = 0$. Consider the Brill-Noether loci $W^r_g(C)$ for a general smooth curve $C \in |L|$, then each $A \in W^r_g(C)$ has $h^0(A) = r + 1$ and $A$ base-point free.

For any polarized variety $(X, L)$, a natural question is whether one can find a spanning set of the spaces $K_{q,1}(X, L)$ consisting of elements of low rank. Here one defines the rank of a syzygy $\alpha \in K_{q,1}(X, L)$ as the minimum dimension of the subspace $V \subseteq H^0(X, L)$ such that $\alpha \in K_{q,1}(X, L; V)$, i.e. $\alpha$ is represented by an element of $\wedge^q V \otimes H^0(X, L)$ \cite{Bot07}.

For a general primitively polarized K3 surface $(X, L)$ of even genus $g = 2k$, we will show that, any non-zero section $s \in H^0(E)$ is regular i.e. the zero locus $Z(s) \subseteq X$ is zero-dimensional, and $h^0(X, L \otimes I_{Z(s)}) = k + 1$. If we restrict the subspaces $K_{k,1}(X, L; H^0(L \otimes I_{Z(s)})) \subseteq K_{k,1}(X, L)$ to a curve $C \in |L|$ containing the locus $Z(s)$, then $K_{k,1}(X, L; H^0(L \otimes I_{Z(s)}))$ restricts to

$$K_{k,1}(C, \omega_C; H^0(\omega_C \otimes A^{-1})) \subseteq K_{k,1}(C, \omega_C),$$

where $A \in W^1_{k+1}(C)$ is the line bundle associated to the divisor $Z(s) \subseteq C$. In particular, the rank $k + 1$ syzygies $\alpha \in K_{k,1}(X, L; H^0(L \otimes I_{Z(s)}))$ drop rank by one when restricted to $C$.

Following a similar strategy to the proof of \cite[Cor 0.4]{Kem20} and with a slightly different assumption on the lower bound on the characteristic, Theorem 1.1 combined with an observation of Voisin \cite[Prop 7]{Voi02}, implies the Geometric Syzygy Conjectures for even genus in positive characteristic.

**Theorem 1.2** (Geometric Syzygy Conjecture for Even Genus). Fix an algebraically closed field $\mathbb{F}$ with $\text{char}(\mathbb{F}) = p > 0$. Let $C$ be a general curve over $\mathbb{F}$ of even genus $g = 2k$ for $k \geq 2$. Assume $p > 2k$. Then $K_{k,1}(C, \omega_C)$ is generated by syzygies of lowest possible rank $k$. More precisely, $K_{k,1}(C, \omega_C)$ is generated by the rank $k$ syzygies

$$\alpha \in K_{k,1}(C, \omega_C; H^0(\omega_C \otimes A^{-1})), \quad A \in W^1_{k+1}(C).$$
Remark 1.3. The inclusions of $H^0(\omega_C \otimes A^{-1}) \hookrightarrow H^0(\omega_C)$ are parameterized by sections $H^0(A)$. And so are $K_{k-1,1}(C, \omega_C; H^0(\omega_C \otimes A^{-1})) \hookrightarrow K_{k-1,1}(C, \omega_C)$. Different sections in $H^0(A)$ give different subspaces $K_{k-1,1}(C, \omega_C; H^0(\omega_C \otimes A^{-1}))$ of $K_{k-1,1}(C, \omega_C)$. One may need multiple sections to have enough such subspaces of $K_{k-1,1}(C, \omega_C)$. The above theorem asserts that those subspaces generate $K_{k-1,1}(C, \omega_C)$. Since $K_{k-1,1}(C, \omega_C)$ is finite dimensional, one needs only finitely many rank $k$ syzygies to generate $K_{k-1,1}(C, \omega_C)$.

Theorem 1.1 is proved by two steps. We first adapt [Kem20, Thm 0.2] under generalized assumptions as follows.

Proposition 1.2. Let $(X, L)$ be a primitively polarized K3 surface of even genus $g = 2k$ for $k \geq 2$ over an algebraically closed field $\mathbb{F}$. Assume $\text{char}(\mathbb{F}) = p \geq \frac{9+4}{2} = k+2$. Assume on a smooth curve $C \subseteq |L|$, there exists a line bundle $A \in W^{1}_{k+1}(C)$ with $h^0(A) = 2$ and $A$ base-point free. Let $E$ be a rank 2 Lazarsfeld-Mukai bundle associated to such a pair $(C, A)$. If any non-zero $s \in H^0(E)$ is regular, i.e. the vanishing locus $Z(s)$ is zero-dimensional, then the space $K_{k-1,1}(X, L; H^0(L \otimes I_{Z(s)}))$ is a one-dimensional subspace of $K_{k-1,1}(X, L)$. The morphism

$$\psi : \mathbb{P}(H^0(E)) \to \mathbb{P}(K_{k-1,1}(X, L))$$

$$[s] \mapsto [K_{k-1,1}(X, L; H^0(L \otimes I_{Z(s)}))]$$

is the Veronese embedding of degree $k-2$.

In particular, $\psi$ induces a natural isomorphism $\text{Sym}^{k-2}H^0(X, E) \cong K_{k-1,1}(X, L)$.

Remark 1.3. (i) If $\text{Pic}(X) = \mathbb{Z}[L]$, then every non-zero section $s \in H^0(E)$ is regular. Indeed, if $Z(s)$ is one-dimensional for some non-zero $s$, then $H^0(E(-D)) \neq 0$ for some effective divisor $D$, which has to be a multiple of $L$, i.e. $E(-D) = E(-mL)$ for some positive $m$. However, $H^0(E(-mL)) \subseteq H^0(E(-L))$ and $E(-L) \cong E^\vee$ with $h^0(E^\vee) = 0$. A contradiction.

(ii) Let $Z \subseteq X \times \mathbb{P}(H^0(E))$ be the reduced closed subvariety defined by $\{(x, s) \mid s(x) = 0\}$. If any non-zero section $s \in H^0(E)$ is regular, then $Z$ is locally complete intersection, hence Cohen-Macaulay. By miracle flatness theorem, the second projection $Z \to \mathbb{P}(H^0(E))$ is finite and flat.

The converse is clear.

Secondly, we conclude by a deformation type argument. The main tool we use is Ogus’s Theorem as follows.

Theorem 1.3 (Ogu99). Let $\mathbb{F}$ be an algebraically closed field with $\text{char}(\mathbb{F}) = p > 0$. Let $(\mathcal{X}/T, \mathcal{L})$ be a versal $\mathbb{F}$-deformation of a primitively polarized K3 surface $(X, L)$ of degree $2d$, with $(L) \subseteq \text{Pic}(X)$ a direct summand. Then the geometric generic fiber $(X_{\tau}, \mathcal{L}_\tau)$ is ordinary with degree $2d$, and $\text{Pic}(X_{\tau})$ is generated by $\mathcal{L}_\tau$.

We consider a versal $\mathbb{F}$-deformation of an arbitrary primitively polarized K3 surface $(X, L)$ over $\mathbb{F}$ with degree $2g-2$ and construct a universal Lazarsfeld-Mukai bundle associated to it in Section 3. We will show that a general primitively polarized K3 surface $(X, L)$ satisfies the assumption of Proposition 1.2, i.e. any non-zero section $s \in H^0(E)$ is regular. By generic flatness theorem and flat base change, it suffices to prove the case for the geometric generic fiber $(X_{\tau}, \mathcal{L}_\tau)$.

Note that the geometric generic fiber $(X_{\tau}, \mathcal{L}_\tau)$ has Picard number one, which easily satisfies the assumption.

(Remark 1.3 (i))

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2. Preliminaries

2.1. Koszul Cohomology. Let $V$ be a vector space and denote $S_V := \bigoplus_{d \geq 0} \text{Sym}^d(V)$ symmetric tensor algebra associated to $V$. Let $M$ be a graded $S_V$-module, one defines Koszul cohomology $K_{p,q}(M, V)$ as the middle cohomology of

$$\bigwedge^{p+1} V \otimes M_{q-1} \to \bigwedge^p V \otimes M_q \to \bigwedge^{p-1} V \otimes M_{q+1}.$$ 

As computed in [Laz86],

$$E$$

we simplify the notions $K_{2}$ variety , $\deg$,

Dualizing it gives

may identify $s$ section $A$ on $X$. Hence the Lazarsfeld-Mukai bundle is of rank 2. By the relation between Chern classes and $K$ Koszul Cohomology.

2.2. Lazarsfeld-Mukai Bundle. In [Laz86], Lazarsfeld first introduced such vector bundle $E = E_{C,A}$ on a K3 surface $X$ associated to a pair $(C, A)$, where $C$ is a smooth curve lying on $X$, and $A$ a base-point free line bundle on $C$ with $h^0(A) = r + 1$ and $\deg A = d$. As mentioned in the Introduction, $E^\vee$ fits into an exact sequence

$$0 \to E^\vee \to H^0(C, A) \otimes O_X \to i_* A \to 0,$$

where $i : C \hookrightarrow X$ is the inclusion. Dualizing it gives

(2)

$$0 \to H^0(C, A)^* \otimes O_X \to E \to i_* (\omega_C \otimes A^{-1}) \to 0.$$ 

As computed in [Laz86], $E$ is a vector bundle of rank $r + 1$, and $c_1(E) = O_X(C)$, $c_2(E) = d$; $h^0(E) = h^0(A) + h^1(A)$, $h^1(E) = h^2(E) = 0$, $h^0(E^\vee) = h^1(E^\vee) = 0$. Moreover $\chi(E \otimes E^*) = 2h^0(E \otimes E^*) - h^1(E \otimes E^*) = 2 - 2\rho(A)$ where $\rho(A) = g(C) - h^0(A) \cdot h^1(A)$.

This paper focuses on the case when $d = k + 1$ and $r = 1$, and $C \in |L|$ for an ample line bundle on $X$. Hence the Lazarsfeld-Mukai bundle is of rank 2. By the relation between Chern classes and degeneracy locus of sections of a vector bundle. A general section $s \in H^0(E)$ gives a general zero-dimensional $Z(s)$ on a general curve $C \in |L|$ with $\deg Z(s) = k + 1$. Thus, we have the following exact sequence

$$0 \to O_X \xrightarrow{s} E \xrightarrow{\Lambda s} I_{Z(s)} \otimes L \to 0.$$ 

Take another $s' \in H^0(E)$ independent of $s$ such that their degeneracy locus is $C$, then the above exact sequence adapts to

(3)

$$0 \to \langle s, s' \rangle \otimes O_X \to E \to i_* (\omega_C \otimes O_C(-Z(s))) \to 0.$$ 

Denote $O_C(Z(s))$ as $A$, then the above exact sequence implies $h^0(\omega_C \otimes A^{-1}) = h^1(A) = k$. Dualizing it gives

$$0 \to E^\vee \to \langle s, s' \rangle^* \otimes O_X \to i_* A \to 0.$$ 

Hence $\deg(A) = c_2(E) = k + 1$. By Riemann-Roch theorem, $h^0(A) = 2$. Compare (2) and (3), we may identify $H^0(C, A) \simeq \langle s, s' \rangle^*$. Therefore $A$ is base-point free. In a word, for a non-zero regular section $s \in H^0(E)$, i.e. $Z(s)$ is zero-dimensional, and it is associated to a base-point free line bundle $A \in W^1_{k+1}(C)$ for a curve $C \in |L|$ containing $Z(s)$.
2.3. Brill-Noether Theory. Classical Brill-Noether theory concerns special divisors on a smooth curve $C$ of genus $g$ over an algebraically closed field. For given integers $r, d$, one defines the Brill-Noether number $\rho(r, d, g) = g - (r + 1)(r - d + g)$. Define Brill-Noether loci

$$W^r_d(C) := \{ \text{line bundle } A \text{ on } C \mid \deg(A) = d, h^0(A) \geq r + 1 \}.$$ 

Let $G_d^r(C) := \{(V, A) \mid r + 1 \text{ dimensional subspace } V \subseteq H^0(A), A \in W^r_d(C) \}$ i.e. parametrize all linear systems of degree $d$ and dimension $r$ on $C$. Brill-Noether asserts that for a general curve $C$, (i) if $\rho < 0$, $W^r_d(C)$, hence $G_d^r(C)$, is empty; (ii) if $\rho \geq 0$, $G_d^r(C)$, hence $W^r_d(C)$, is not empty.

When working over $\mathbb{C}$, Lazarsfeld [Laz86] showed that, if $\text{Pic}(X) = \mathbb{Z}[L]$, then any smooth member $C$ in $|L|$ is general in the Brill-Noether sense. Moreover, a curve $C$ is called to satisfy Petri's condition, if the map $H^0(A) \otimes H^0(\omega_C \otimes A^*) \to H^0(\omega_C)$ is injective for any line bundle $A$ on $C$. Gieseker proved that a general curve satisfies Petri’s condition, which refines the Brill-Noether theory of the loci $W^r_d(C)$. [ACGH85, §IV] In fact, Lazarsfeld [Laz86] even proves that a general smooth member of $C \in |L|$ is Petri general. In his elegant proof, he essentially used the generic smoothness theorem in characteristic 0, which can be a huge problem in positive characteristic.

By Theorem [L3] we may assume $(X, L)$ is a polarized K3 surface over some algebraically closed field $\mathbb{F}$ such that $\text{Pic}(X) = \mathbb{Z}[L]$. By a similar argument of [Laz86] Cor 1.3, any Lazarsfeld-Mukai bundle associated to a prescribed pair $(C, A)$ is simple, hence $\chi(E \otimes E^*) = 2 - 2\rho(A) = 2h^0(E \otimes E^*) - h^1(E \otimes E^*) \leq 2$, i.e. $\rho(A) \geq 0$. One can further prove that, for any smooth member $C \in |L|$ and every line bundle $A$ on $C$, $\rho(A) \geq 0$. [Laz86] Cor 1.4 i.e. any smooth curve $C \in |L|$ is Brill-Noether general.

2.4. Green’s Conjecture. Green’s Conjecture on syzygies of canonical curves asserts that one can read off existence of special linear series on an algebraic curve, by looking at the syzygies of its canonical embedding. Precisely, if $C$ is a smooth projective curve of genus $g$, let $K_{p,q}(C, \omega_C)$ denote the $(p, q)$-th Koszul cohomology group of the canonical bundle $\omega_C$ and $\text{Cliff}(C)$ be the Clifford index of $C$, then Green [Gre84] conjectured the following:

$$K_{p,2}(C, \omega_C) = 0, \quad \forall p < p' \iff \text{Cliff}(C) > p'.$$

In the case of even genus $g = 2k$, a single vanishing $K_{k,1}(C, \omega_C) = 0$ suffices to prove Green’s Conjecture for a general canonical curve. In [Vois01], Voisin studied curves lying on a polarized K3 surface $(X, L)$ of even genus $g = 2k$ with $\text{Pic}(X) = \mathbb{Z}[L]$. As mentioned in the Introduction, for $\text{char}(\mathbb{F}) \neq 2$, the hyperplane restriction theorem [Gre84] implies

$$K_{k,1}(C, \omega_C) = K_{k,1}(X, L)$$

whenever $C$ is a smooth and connected (i.e. general) hyperplane section of $L$ on $X$ (note that $L|_C = \omega_C$ in this case). Indeed, the above identity holds true under the hypothesis that $H^1(X, qL) = 0$ for all $q \geq 0$. Although Kodaira-Ramanujam vanishing theorem fails in positive characteristic in general, it does hold for an ample line bundle $L$ on K3 surfaces, [Huy16] §II. Prop 3.1.

Theorem [L1] implies in particular that, for a general primitively polarized K3 surface $(X, L)$, we have $\text{Sym}^{k-2}H^0(X, E) \cong K_{k-1,1}(X, L)$ and $\text{dim} K_{k-1,1}(X, L) = \binom{2k-1}{k-2}$. This implies $K_{k,1}(X, L) = 0$ by standard dimension computations [Par17] §IV.1. Therefore, one deduces Corollary [L1] i.e. Green’s Conjecture for general canonical curves of even genus in positive characteristic.

Now we proceed to prove Proposition [L2]. Let $F$ be an algebraically closed field with $\text{char}(\mathbb{F}) = p$, we have a natural isomorphism

$$(\text{Sym}^j F)^\vee \cong \text{Sym}^j F^\vee, \quad \text{for } 1 \leq j \leq p - 1$$

for any vector bundle $F$ on an algebraic $\mathbb{F}$-variety $X$. Indeed, $(\text{Sym}^j F^\vee)^\vee$ is naturally isomorphic to the divided algebra

$$D^j F := \{ x \in F \otimes S_j \mid \sigma(x) = x, \text{ for all } \sigma \in S_j \}$$
and we define an isomorphism

$$D^j F \to \text{Sym}^j F$$

$$x \mapsto \frac{1}{j!} \sum_{\sigma \in S_j} \sigma(x)$$

Note that this isomorphism fails to be defined for $j \geq p$.

Moreover, let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence of vector bundles on $X$. Provided, $1 \leq j \leq p - 1$, there are two long exact sequences. [ABW82 §V]:

$$\cdots \to \bigwedge^{j-2} F_2 \otimes \text{Sym}^2 F_1 \to \bigwedge^{j-1} F_2 \otimes F_1 \to \bigwedge^j F_2 \to \bigwedge^j F_3 \to 0,$$

$$\cdots \to \text{Sym}^{j-2} F_2 \otimes \bigwedge^2 F_1 \to \text{Sym}^{j-1} F_2 \otimes F_1 \to \text{Sym}^j F_2 \to \text{Sym}^j F_3 \to 0.$$
such that for every $t \in \mathcal{T}$, $C_t \subseteq X_t$ is a smooth curve of genus $g$ with $O_{X_t}(C_t) \cong \mathcal{L}_t$. We may interchangeably use the notion $(X, \mathcal{C}) \to \mathcal{T}$ as a smooth family of primitively polarized K3 surfaces with degree $2g - 2$.

**Proof.** By reducing to an irreducible component, we may assume $\mathcal{T}$ to be irreducible. Since for every $t \in \mathcal{T}$, $\dim H^0(X_t, \mathcal{L}_t) = g + 1$, then $\pi_* \mathcal{L}$ is a locally free sheaf on $\mathcal{T}$ by Grauert’s theorem. Consider the projective bundle $\mathbb{P}(\pi_* \mathcal{L}) \to \mathcal{T}$, there is a universal family

$$\xymatrix{ \mathcal{C} \ar[r] & \mathcal{X} \times \mathcal{T} \ar[d]^p \ar[r]_{\pi|_\mathcal{C}} & \mathbb{P}(\pi_* \mathcal{L}) }$$

where $\mathcal{C} := \{(x, [s]_t) \in \mathcal{X} \times \mathcal{T} \mathbb{P}(\pi_* \mathcal{L}) \mid x \in V([s]_t) \subseteq X_t$ for $[s]_t \in H^0(X_t, \mathcal{L}_t) \text{ and } t \in \mathcal{T}\}$, i.e. the fiber of $[s]_t \in \mathbb{P}(\pi_* \mathcal{L})$ along $\pi|_\mathcal{C}$ is just the vanishing set of $[s]_t \in H^0(X_t, \mathcal{L}_t)$ in $X_t$. Denote it as $C_{[s]_t} \subseteq X_t$, it is a curve of arithmetic genus $g$ (possibly singular).

A general member $C_t \in |\mathcal{L}|$ is smooth. After restricting to a non-empty Zariski open of $\mathbb{P}(\pi_* \mathcal{L})$, one may assume each fiber $C_{[s]_t}$ is smooth. Moreover, choose a local section of $\mathbb{P}(\pi_* \mathcal{L}) \to \mathcal{T}$ over some non-empty open $\mathcal{T}'$ of $\mathcal{T}$, i.e. $s : \mathcal{T}' \to \mathbb{P}(\pi_* \mathcal{L})$, the pullback of the above diagram by the base change to $\mathcal{T}' \to \mathbb{P}(\pi_* \mathcal{L})$ gives us the following diagram, satisfying all the assumptions.

$$\xymatrix{ \mathcal{C}' \ar@{^{(}->}[r] \ar[d]^{\pi'|_{\mathcal{C}'}} & \mathcal{X}' \ar[d]^{\pi'} \ar[r] \ar[d]_{\pi} & \mathcal{X} \ar[d]_{\pi} \ar[l] }$$

\[\square\]

**Lemma 3.2.** Let $\pi : \mathcal{C} \to \mathcal{T}$ be a smooth family of genus $g$ curves. Assume the geometric generic fiber $C_\pi$ is Brill-Noether general. Let $r, d$ be such integers such that the Brill-Noether number $\rho(r, d, g) = 0$. Then up to a base change to an étale open of $\mathcal{T}$, there exists a line bundle $\mathcal{A}$ on $\mathcal{C}$, such that for every $t \in \mathcal{T}$, $\mathcal{A}_t \in W^r_d(C_t)$ with $h^0(C_t, \mathcal{A}_t) = r + 1$ and $\mathcal{A}_t$ base-point free.

**Proof.** Without loss of generality, we may assume $\pi : \mathcal{C} \to \mathcal{T}$ admits a section, since any smooth family admits an étale local section. Let $Pic^d_{\mathcal{C}/\mathcal{T}}$ be the relative Picard scheme with degree $d$ associated to $\pi : \mathcal{C} \to \mathcal{T}$. The (relative) Brill-Noether variety $\mathcal{W}^r_d(\pi) \subseteq Pic^d_{\mathcal{C}/\mathcal{T}}$ associated to $\pi : \mathcal{C} \to \mathcal{T}$ is a $\mathcal{T}$-scheme representing the functor (see [ACG1] §XXI.3)

$$\xymatrix{ \text{Sch}/\mathcal{T} \ar[r] & \text{Sets} \\ S \ar@{|->}[r] & \{ [\mathcal{L}] \in Pic^d_{\mathcal{C}/\mathcal{T}}(S) \mid \text{the Fitting-rank}(R^1_{pS_*}, \mathcal{L}) \geq g - d + r \} }$$

where $pS : \mathcal{C} \times \mathcal{T} \to S$ is the projection. More precisely, set-theoretically,

$$\text{supp}(\mathcal{W}^r_d(\pi)) = \{ (t, L) \mid t \in \mathcal{T}, L \in Pic^d(C_t) \text{ such that } L \in W^r_d(C_t) \}.$$ 

Since $\rho \geq 0$ implies $W^r_d(C) \neq \emptyset$ for every curve $C$, the $\mathcal{T}$-points $W^r_d(\pi)(\mathcal{T}) \neq \emptyset$. Choose any $\mathcal{A} \in W^r_d(\pi)(\mathcal{T})$, i.e. for every $t \in \mathcal{T}$, $\mathcal{A}_t \in W^r_d(C_t)$ i.e. $h^0(C_t, \mathcal{A}_t) \geq r + 1$. By assumption, the geometric generic fiber $C_\pi$ is Brill-Noether general with $\rho(r, d, g) = 0$.

**Claim.** For any $\mathcal{A} \in W^r_d(C_t)$ over the original fiber $C_\pi$, we have

$$\begin{cases} h^0(\mathcal{A}) = r + 1 , \\ \mathcal{A} \text{ is base-point free} \end{cases} \quad \text{in particular,} \quad \begin{cases} h^0(\mathcal{A}_t) = r + 1 , \\ \mathcal{A}_t \text{ is base-point free} \end{cases}$$

Indeed, denote $\overline{\mathcal{A}}$ as the pullback of $\mathcal{A}$ by $C_\pi \to C_\pi$. By flat base change, we have $H^0(C_\pi, \overline{\mathcal{A}}) = H^0(C_\pi, \mathcal{A}) \otimes_{\kappa(\pi)} \kappa(\pi)$. If $h^0(\mathcal{A}) = h^0(\overline{\mathcal{A}}) \geq r + 2$, then $W^{r+1}_d(C_\pi) \neq \emptyset$, but it contradicts with
\( \rho(r+1,d,g) < 0 \) as \( \mathcal{C}_\tau \) is Brill-Noether general. Moreover, as the field extension is faithfully flat, if \( A \) is not base-point free, equivalently \( A \) globally generated, so is \( \overline{A} \). Take the base-point free part \( \overline{A}_b \) (i.e. \( \text{im}(H^0(\overline{A}) \otimes \mathcal{O}_{\mathcal{C}_\tau} \rightarrow \overline{A}) \)), whose degree is strictly less than that of \( \overline{A} \) but with the same global sections. Hence \( W_{d-p}^r(C_{\tau}) \neq \emptyset \) for some \( p > 0 \), but it contradicts with \( \rho(r,d-p,g) < 0 \).

Finally, by the upper semi-continuity of the function \( t \mapsto h^0(C_t, \mathcal{A}_t) \) and base-point freeness being an open condition, we may shrink the base again to get the required property: there exists a line bundle \( A \) on \( C \) such that for every \( t \in \mathcal{T} \), \( A_t \in W_{d}^r(C_t) \) with \( h^0(C_t, \mathcal{A}_t) = r+1 \) and \( A_t \) base-point free. \( \square \)

With the argument in the above lemma, we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** For any primitively polarized K3 surface \((X, L)\) of degree \(2g-2\) over an algebraically closed field \( \mathbb{F} \). Let \( \pi : (\mathcal{X}, \mathcal{L}) \rightarrow \mathcal{T} \) be a smooth versal \( \mathbb{F}\)-deformation of \((X, L)\), i.e. smooth family of primitively polarized K3 surfaces with line bundle \( \mathcal{L} \) of (relative) degree \(2g-2\).

The geometric generic fiber \((X_{\tau}, \mathcal{L}_{\tau})\) has \( \text{Pic}(X_{\tau}) = \mathbb{Z}[L_{\tau}] \). In particular, any smooth \( C_{\tau} \in |\mathcal{L}_{\tau}| \) is Brill-Noether general.

Similar in the proof of Lemma 3.1 with a focus on an open dense "smooth" part \( \mathbb{P}(H^0(\pi_{*}\mathcal{L})) \) whose points are those \((t, [s]_t)\) such that \([s]_t\) defines a smooth curve \( C \in |\mathcal{L}_t| \). There is a universal family

\[
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & X_{\mathbb{P}} := \mathcal{X} \times_{\mathcal{T}} \mathbb{P}(\pi_{*}\mathcal{L})_s \\
p|\mathcal{C} & \downarrow & p \\
\mathbb{P}(\pi_{*}\mathcal{L})_s & \longrightarrow & \mathbb{P}(\pi_{*}\mathcal{L})_s
\end{array}
\]

such that the fiber over the point \((t, [s]_t) \in \mathbb{P}(\pi_{*}\mathcal{L})_s \) along \( p|\mathcal{C} \) is the smooth curve \( C_{[s]_t} \subseteq X_t \) defined by section \([s]_t \in H^0(L_t).

**Claim.** For a general fiber \((X_t, L_t)\) over \( \mathcal{T} \) and a general smooth curve \( C \in |L_t| \), i.e. a general fiber over \( \mathbb{P}(\pi_{*}\mathcal{L})_s \), each \( A_t \in W_{d}^r(C_t) \) has (i) \( h^0(A) = r+1 \) and (ii) \( A \) base-point free.

If (i) fails for a general fiber, consider the relative Brill-Noether variety \( W_{d+1}^r(p|\mathcal{C}) \) associated to the smooth family of curves \( p|\mathcal{C} : \mathcal{C} \rightarrow \mathbb{P}(\pi_{*}\mathcal{L})_s \), then \( \text{supp}(W_{d+1}^r(p|\mathcal{C})) \) is not supported on any divisor of \( \mathbb{P}(\pi_{*}\mathcal{L})_s \), hence also contains the generic points of \( \mathbb{P}(\pi_{*}\mathcal{L})_s \) (maybe reducible). But as the Brill-Noether number \( \rho(r,d,g) = 0 \), by the claim in the proof of Lemma 3.2, those generic fibers, which are all over the generic point \( \tau \in \mathcal{T} \), are geometrically Brill-Noether general. Hence they do not admit any line bundle with \( h^0(A) \geq r+2 \). A contradiction.

Similarly, if (ii) fails for a general fiber, consider the relative Brill-Noether variety \( W_{d-i}^r(p|\mathcal{C}) \) for \( 0 < i < d \), then at least for some \( i \), \( \text{supp}(W_{d-i}^r(p|\mathcal{C})) \) is not supported on any divisor of \( \mathbb{P}(\pi_{*}\mathcal{L})_s \), hence also contains the generic points of \( \mathbb{P}(\pi_{*}\mathcal{L})_s \). A contradiction by the same reason as above. \( \square \)

Let \( \pi : (\mathcal{X}, C) \rightarrow \mathcal{T} \) be a smooth family of primitively polarized K3 surfaces of degree \(2g-2\). Let \( i : C \hookrightarrow \mathcal{X} \) denote the closed embedding. Assume \( \mathcal{A} \) is a line bundle \( \mathcal{A} \) over \( C \) such that \( \mathcal{A}_t \in W_{d}^r(C_t) \) and for every \( t \in \mathcal{T} \), \( h^0(C_t, \mathcal{A}_t) = r+1 \) and \( \mathcal{A}_t \) base-point free. By Grauert’s theorem, one easily see that \( \pi|_{C_*} \mathcal{A} = \pi_* i_* \mathcal{A} \) is a locally free sheaf on \( \mathcal{T} \) of rank \( r+1 \). We construct a universal Lazarfield-Mukai bundle \( \mathcal{E} \) on \( \mathcal{X} \) associated to the line bundle \( \mathcal{A} \) on \( C \) by the following steps. Take the kernel \( \mathcal{F} \) of the natural surjective evaluation map \( \pi^* \pi_* i_* \mathcal{A} \rightarrow i_* \mathcal{A} \) to fit into an exact sequence

\[
0 \rightarrow \mathcal{F} \rightarrow \pi^* \pi_* i_* \mathcal{A} \rightarrow i_* \mathcal{A} \rightarrow 0.
\]

**Remark 3.1.** The natural map \( \pi^* \pi_* i_* \mathcal{A} \rightarrow i_* \mathcal{A} \) being surjective means \( i_* \mathcal{A} \) is globally generated along fibers, i.e. for any \( t \in \mathcal{T} \), \( H^0(C_t, \mathcal{A}_t) \otimes_{\mathcal{O}_{\mathcal{T}}} \mathcal{O}_{\mathcal{X}_t} \rightarrow i_{t,*} \mathcal{A}_t \), which is true under our assumption.
Proposition 3.1. Let $\pi : (X, \mathcal{C}) \to \mathcal{T}$ and $A$ defined as above. Denote the associated line bundle $L = \mathcal{O}_X(\mathcal{C})$. Then

(i) The kernel $F$ is a locally free sheaf of rank $r+1$ on $X$. Its dual $F^\vee$ is denoted as the universal Lazarsfeld-Mukai bundle $E$ for $\pi : (X, \mathcal{C}) \to \mathcal{T}$ associated to the line bundle $A$ on $\mathcal{C}$. We have $\det(E) = L$.

(ii) For every $t \in \mathcal{T}$, the following pullback of the exact sequence (4) along the fiber $X_t$ is exact.

$0 \to F_t \to H^0(C_t, A_t) \otimes \kappa(t) \mathcal{O}_{X_t} \to i_{t*} A_t \to 0$

where $i_t : C_t \hookrightarrow X_t$ denotes the closed embedding. In particular, $E_t := F_t^\vee$ is the Lazarsfeld-Mukai bundle associated to a pair $(C_t, A_t)$ on $X_t$. Moreover, it satisfies the following numerical invariants: $\det(F_t) = L_t = \mathcal{O}_{X_t}(C_t), h^0(E_t) = h^0(A_t) + h^1(A_t), h^1(E_t) = h^2(E_t) = 0$.

Proof. For (i), note that $\mathcal{C} \subseteq X$ is a divisor, $i_* A$ being locally isomorphic to $i_* \mathcal{O}_C$, has homological dimension 1 over $\mathcal{O}_X$. Hence $F$ is locally free. It is clear that $E = F^\vee$ has rank $r+1$. As the map $F \to \pi^* i_* A$ drops rank along $C$, $\det(F) = \mathcal{O}_X(-\mathcal{C})$, i.e. $\det(E) = L$.

For (ii), by applying $j^*$ to the sequence (4), where $j : X_t \hookrightarrow X$ denotes the embedding above any point $t \in \mathcal{T}$, we have

$\text{Tor}_r^j(\mathcal{O}_X, j^! i_* A, \mathcal{O}_{X_t}) \to F_t \to \pi_t|_{C_t} A_t \otimes \kappa(t) \otimes \kappa(t) \mathcal{O}_{X_t} \to i_{t*} A_t \to 0$.

In fact, $i_* A$ is a flat sheaf of $\mathcal{O}_X$-module since flatness is a local property and $i_* \mathcal{O}_C$ is a flat sheaf of $\mathcal{O}_X$-module. Also, $j^{-1}$ is exact. This implies $\text{Tor}_r^j(\mathcal{O}_X, j^! i_* A, \mathcal{O}_{X_t}) = 0$. In addition, by proper base change theorem, we have the natural isomorphism $\pi|_{C_t} A \otimes \kappa(t) \simeq H^0(C_t, A_t)$. Hence, we obtain the exact sequence (5). By Serre duality, it gives

$0 \to H^0(C_t, A_t) \otimes \kappa(t) \mathcal{O}_{X_t} \to E_t \to i_{t*} (\omega_{C_t} \otimes A_t^\vee) \to 0$

Then $h^0(E_t) = h^0(A_t) + h^1(A_t)$. Other numerical invariants are similarly computed as in [Laz86].

Now we are going to use the construction of universal Lazarsfeld-Mukai bundle for a smooth versal deformation of K3 surfaces to prove Theorem 1.1.

Proof of Theorem 1.1. For any primitively polarized K3 surface $(X, L)$ of genus $g = 2k$ over $\mathbb{F}$. Let $\pi : (X, \mathcal{C}) \to \mathcal{T}$ be a smooth versal $\mathbb{F}$-deformation of $(X, L)$, of which the geometric generic fiber $(X, \mathcal{C}, L)$ has $\text{Pic}(X) = \mathbb{Z}[\mathcal{C}]$. Since the Brill-Noether number $\rho(1, k + 1, 2k) = 0$, according to Lemma 3.2, by passing to an étale open of $\mathcal{T}$, one may assume $A$ to be a line bundle on $\mathcal{C}$, flat over $\mathcal{T}$, such that for every $t \in \mathcal{T}$, $A_t \in W_{k-1}^1(C_t)$ with $h^0(C_t, A_t) = 2$ and $A_t$ base-point free. Let $E$ be the rank 2 universal Lazarsfeld-Mukai bundle over $X$ associated to the line bundle $A$ on $\mathcal{C}$. In particular, for every $t \in \mathcal{T}$, $h^1(A_t) = k$ by Riemann-Roch theorem, hence $h^0(E_t) = h^0(A_t) + h^1(A_t) = k + 2$. By Grauert’s theorem, $\pi_* E$ is a locally free sheaf of rank $k + 2$ on $\mathcal{T}$.

Consider the projective bundle $\varphi : \mathbb{P}(\pi_* E) \to \mathcal{T}$. Any point $p \in \mathbb{P}(\pi_* E)$ can be identified with $(t, [s])$ for some $t \in \mathcal{T}$ and $[s] \in \mathbb{P}(H^0(E_t))$. Take the fiber product with $\pi : X \to \mathcal{T}$ to form a smooth family of K3 surfaces $X_p := X \times_{\mathcal{T}} \mathbb{P}(\pi_* E) \to \mathcal{T}$. Without too much abuse of notions, we still denote the pullback of $\mathcal{L}$ (resp. $\mathcal{E}$) to $X_p$ as $\mathcal{L}_p$ (resp. $\mathcal{E}_p$). Above any point $p = (t, [s]) \in \mathbb{P}(\pi_* E)$, denote the fiber $X_p \subseteq X_p$. Note that we have natural isomorphism of fibers $(X_p, \mathcal{L}_p, \mathcal{E}_p) \simeq (X_t, \mathcal{L}_t, \mathcal{E}_t)$. Define the reduced closed subvariety $\mathcal{Z} \subseteq X_p$ on the closed set

$\mathcal{Z} = \{(x, [s]) \in X \times_{\mathcal{T}} \mathbb{P}(\pi_* E) | x \in V([s]) \subseteq X_t \text{ for } [s] \in \mathbb{P}(\pi_* E)\}$.

For any point $p = (t, [s]) \in \mathbb{P}(\pi_* E)$, we have $\mathcal{Z}_p = \{x \in X_p \simeq X_t | s(x) = 0\}$. On the other hand, for any point $t \in \mathcal{T}$, denote the projection $\mathcal{Z} \to \mathbb{P}(\pi_* E)$ under the base change $\mathbb{P}(H^0(E_t)) \hookrightarrow \mathbb{P}(\pi_* E)$ as $\mathcal{Z}_t \subseteq X_t \times \mathbb{P}(H^0(E_t)) \to \mathbb{P}(H^0(E_t))$. Since the geometric generic fiber $(X, L)$ has Picard number one, $\mathcal{Z}_t \rightarrow \mathbb{P}(H^0(E_t))$ is finite and flat. (Remark 1.3.) As the field extension is faithfully flat, $\mathcal{Z}_\tau \rightarrow \mathbb{P}(H^0(E_\tau))$ is also flat.
By generic flatness theorem, we may choose a non-empty Zariski open set \( U \hookrightarrow \mathbb{P}(\pi^*E) \) such that \( Z|_U \) is flat over \( U \). May assume \( U \) contains the whole generic fiber \( \mathbb{P}(H^0(E_{\tau})) \subset \mathbb{P}(\pi^*E) \) as \( \mathbb{P} \rightarrow \mathbb{P}(H^0(E_{\tau})) \) is flat. Since \( \varphi : \mathbb{P}(\pi^*E) \rightarrow T \) is proper and surjective, the image of the complement of \( U \) under \( \varphi \) is a proper closed subset of \( T \) avoiding the generic point \( \tau \in T \). By possibly shrinking \( U \supseteq \mathbb{P}(\pi^*E) \), we may take open \( \mathcal{V} \subset T \) such that \( \varphi^{-1}|_U : \mathcal{V} \rightarrow \mathbb{P}(\pi^*E) \) is flat. Moreover, since over the generic point \( \tau \), \( Z_{\tau} \rightarrow \mathbb{P}(H^0(E_{\tau})) \) is a surjection between two projective varieties of the same dimension \( k + 1 \), the map is quasi-finite. It follows that \( Z_t \rightarrow \mathbb{P}(H^0(E_t)) \) is also quasi-finite, hence finite, for any \( t \in \mathcal{V} \). Now it is clear that, for any closed point \( t \in \mathcal{V} \), \((X_t, L_t)\) satisfies the assumption of Proposition 1.2. Therefore, we prove that a general primitively polarized K3 surface \((X, L)\) satisfies the expected properties. \( \square \)

**Proposition 3.2.** Let \((X, L)\) be a general primitively polarized K3 surface of even genus \( g = 2k \). For a general smooth curve \( C \in |L| \), let \( A \in W_{k+1}^1(C) \) with \( h^0(A) = 2 \) and base-point free. Let \( E \) be the rank 2 Lazarsfeld-Mukai bundle associated to such a pair \((C, A)\). We have a well-defined finite surjective map \( d : \text{Gr}_2(H^0(E)) \rightarrow |L| \) with degree \( \frac{1}{k+1} \binom{2k}{k} \). Moreover, the fiber \( d^{-1}(C) \) over a general smooth curve \( C \in |L| \) may be identified naturally with the Brill-Noether loci \( W_{k+1}^1(C) \).

**Proof.** The existence of such line bundle \( A \) over a general smooth curve \( C \in |L| \) is guaranteed by Proposition 1.1. Taking degree 2 wedge power of \( H^0(E) \otimes \mathcal{O}_X \rightarrow E \) gives \( \bigwedge^2 H^0(E) \otimes \mathcal{O}_X \rightarrow \bigwedge^2 E = L \). Taking the global sections gives

\[
\text{det} : \bigwedge^2 H^0(E) \rightarrow H^0(L).
\]

A rational map \( d : \text{Gr}_2(H^0(E)) \dasharrow |L| \) is given by taking a general two dimensional subspace \( W \subset H^0(E) \) to the degeneracy locus of the evaluation map \( W \otimes \mathcal{O}_X \xrightarrow{\text{ev}} E \), which is given by a section of \( L \) via taking degree 2 wedge power of \( \text{ev} \)

\[
s : \mathcal{O}_X \simeq \bigwedge^2 W \otimes \mathcal{O}_X \xrightarrow{\bigwedge^2 \text{ev}} \bigwedge^2 E = L.
\]

To sum up, we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_2(H^0(E)) & \xrightarrow{\text{Plücker embedding}} & \mathbb{P}(\bigwedge^2 H^0(E)) \\
\downarrow d & & \downarrow \text{det} \\
\mathcal{L} & & \mathbb{P}(\mathcal{L})
\end{array}
\]

Let \( \mathcal{L} \) denote the open dense subset of \( |L| \) contained by the locus of irreducible smooth curves. Consider a relative Brill-Noether variety \( p : \mathcal{W}_{k+1}^1(|\mathcal{L}|_{gs}) \rightarrow |\mathcal{L}|_{gs} \). By Proposition 1.1, we may further shrink \( |\mathcal{L}|_{gs} \) if necessary, to assume that, for any curve \( C \in |\mathcal{L}|_{gs} \), each \( A \in W_{k+1}^1(C) \) has \( h^0(C, A) = 2 \) and \( A \) base point free. Define a morphism \( \psi : \mathcal{W}_{k+1}^1(|\mathcal{L}|_{gs}) \rightarrow \text{Gr}_2(H^0(E)) \) as follows. Given \( A \in W_{k+1}^1(C) \), we associate it to \( [H^0(C, A)^*] \in \text{Gr}_2(H^0(E)) \) via the following exact sequence

\[
0 \rightarrow H^0(C, A)^* \otimes \mathcal{O}_X \xrightarrow{\text{ev}} E \rightarrow i_* (\omega_C \otimes A^{-1}) \rightarrow 0.
\]

The map \( \text{ev} \) drops rank along \( C \), hence the vanishing locus of the corresponding section \( s : \mathcal{O}_X \rightarrow \bigwedge^2 E = L \) is exactly \( C \). That is to say, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{W}_{k+1}^1(|\mathcal{L}|_{gs}) & \xrightarrow{\psi} & d^{-1}(|\mathcal{L}|_{gs}) \subseteq \text{Gr}_2(H^0(E)) \\
p & \downarrow & d \\
|\mathcal{L}|_{gs} & \xrightarrow{|\mathcal{L}|_{gs}} & |\mathcal{L}|_{gs}
\end{array}
\]
On the other hand, for any \([W] \in \text{Gr}_2(H^0(E))\) such that \(d(W) = C \in |L|_{gs}\), the cokernel is a rank one torsion free sheaf supported on \(C\), i.e. a line bundle on \(C\). We may write as the following where \(A\) is some line bundle on \(C\).

\[ 0 \to W \otimes \mathcal{O}_X \to E \to i_*(\omega_C \otimes A^{-1}) \to 0. \]

This gives \(h^0(\omega_C \otimes A^{-1}) = k\). Dualizing it gives

\[ 0 \to E^\vee \to W^* \otimes \mathcal{O}_X \to i_*A \to 0. \]

Hence \(\text{deg}(A) = c_2(E) = k + 1\). By Riemann-Roch, \(h^0(A) = 2\). This identifies \(W = H^0(C, A)^*\) and \(A\) is base-point free, i.e. \(A \in W^1_{k+1}(C)\). This process gives an inverse of \(\psi\), which identifies \(d^{-1}(C) \cong W^1_{k+1}(C)\) for \(C \in |L|_{gs}\).

Let \(\pi : (X, C) \to \mathcal{T}\) be a versal \(\mathbb{F}\)-deformation of arbitrary primitive polarized K3 surface. Over the geometric generic fiber, \(\text{Pic}(X_{\mathbb{F}}) = \mathbb{Z}[^{\mathfrak{L}}_\mathfrak{C}]\). By Lemma 3.2 and Proposition 3.1, we may further assume that there is a line bundle \(A \) on \(C\) such that for every \(t \in \mathcal{T}\), \(A_t \in W^1_{k+1}(C_t)\) and \(h^0(C_t, A_t) = 2\) with \(A_t\) base-point free, and \(E\) is the rank 2 universal Lazarsfeld-Mukai bundle associated to the pair \((C, A)\). Let \(\pi^* \pi_* E \to E\) be the natural evaluation map, taking the degree 2 wedge power gives

\[ \bigwedge^2 \pi^* \pi_* E = \pi^* \bigwedge^2 \pi_* E \to \det E = L. \]

Hence after pushing forward to \(\mathcal{T}\) and by projection formula, we have the determinant map of vector bundles over \(\mathcal{T}\)

\[ \det : \pi_* \pi^* \bigwedge^2 \pi_* E = \pi_* \mathcal{O}_X \otimes \bigwedge^2 \pi_* E \to \pi_* L, \]

where \(\pi_* \mathcal{O}_X\) is a line bundle over \(\mathcal{T}\) by Grauert’s theorem. Take the stalk at \(t \in \mathcal{T}\), the determinant map reduces to

\[ \det|_t : \bigwedge^2 H^0(E_t) \to H^0(L_t). \]

Consider the induced map between the associated projective bundles and a related Grassmannian bundle over \(\mathcal{T}\). Note that the projective bundle is insensitive to a twist by a line bundle, i.e. \(\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{E} \otimes \mathcal{L})\) for any line bundle \(\mathcal{L}\) and vector bundle \(\mathcal{E}\). We have the following commutative diagram of \(\mathcal{T}\)-schemes

\[ \text{Gr}_2(\pi_* \mathcal{E}) \xrightarrow{\text{Pl"ucker embedding}} \mathbb{P}(\bigwedge^2 \pi_* \mathcal{E}) \]

\[ \downarrow d \quad \downarrow \text{F(det)} \]

\[ \mathbb{P}(\pi_* \mathcal{L}) \]

Over each point \(t \in \mathcal{T}\), the above diagram reduces to the following

\[ \text{Gr}_2(H^0(E_t)) \xrightarrow{\text{Pl"ucker embedding}} \mathbb{P}(\bigwedge^2 H^0(E_t)) \]

\[ \downarrow d|_{X_t} \quad \downarrow \text{F(det)|}_{X_t} \]

\[ |L_t| \]

Over the geometric generic fiber, \(\text{Pic}(X_{\mathbb{F}}) = \mathbb{Z}[L_{\mathbb{F}}]\). It implies the determinant map \(\det|_{X_{\mathbb{F}}} : \bigwedge^2 H^0(E_{\mathbb{F}}) \to H^0(L_{\mathbb{F}})\) does not vanish on any element of rank 2. [Vol02 proof of equ (3.18)]. Hence it defines a regular map \(d|_{X_{\mathbb{F}}} : \text{Gr}_2(H^0(E_{\mathbb{F}})) \to |L_{\mathbb{F}}|\). It is clearly dominant. Moreover, it is finite since both are projective varieties with the same dimension 2\(k\).

It follows from flat base change that over the generic fiber, we have a well-defined surjective regular map \(d|_{X_t} : \text{Gr}_2(H^0(E_t)) \to |L_t|\), which is finite as well by the same reason as above. Note that there is a largest dense open \(U\) of \(\text{Gr}_2(H^0(\pi_* \mathcal{E}))\) where the rational map \(d : \text{Gr}_2(H^0(\pi_* \mathcal{E})) \dashrightarrow \mathbb{P}(\pi_* \mathcal{L})\).
is defined. Apparently, \( \mathcal{U} \) contains the whole generic fiber \( \text{Gr}_2(\mathcal{H}^0(\mathcal{E}_s)) \). Since the Grassmannian bundle \( \varphi : \text{Gr}_2(\pi_*\mathcal{E}) \to \mathcal{T} \) is proper and surjective, the image of the complement of \( \mathcal{U} \) under \( \varphi \) is a proper closed subset \( \mathcal{S} \) of \( \mathcal{T} \) avoiding the generic point \( \tau \). Let \( \mathcal{V} = \mathcal{T} \setminus \mathcal{S} \) be a non-empty open set, then \( d \) is a regular map on \( \varphi^{-1}(\mathcal{V}) \subseteq \mathcal{U} \subseteq \text{Gr}_2(\pi_*\mathcal{E}) \), i.e. over each fiber \( t \in \mathcal{V} \subseteq \mathcal{T} \), \( d|_{X_t} : \text{Gr}_2(\mathcal{H}^0(\mathcal{E}_t)) \to |L_t| \) is regular. As the map \( d|_{X_t} : \text{Gr}_2(\mathcal{H}^0(\mathcal{E}_t)) \to |L_t| \) over the generic fiber is finite, finiteness being an open condition, implies over a general fiber \( X_t, \ d|_{X_t} : \text{Gr}_2(\mathcal{H}^0(\mathcal{E}_t)) \to |L_t| \) is finite. In particular, it implies \( d|_{X_t} \) is surjective as well. Therefore, one may shrink \( \mathcal{V} \) if necessary to show that a general \( (X, \mathcal{L}) \) satisfies the prescribed property.  

Now we proceed to compute the degree of the finite surjective map \( d : \text{Gr}_2(\mathcal{H}^0(\mathcal{E})) \to |\mathcal{L}| \) for a general \((X, \mathcal{L})\). To ease the notation, let \( Z = \text{Gr}_2(\mathcal{H}^0(\mathcal{E})) \) and \( Y = |\mathcal{L}| \). Note that the finite map \( d : Z \to Y \) between two smooth schemes is automatically flat. Assume its degree is \( n \). Now for any \( y \in Y \), the fiber \( d^{-1}(y) = Z_y \) is a finite subscheme of \( Z \), and \( \mathcal{O}(Z_y) \) is a finite dimensional \( \mathbb{P} \) vector space with \( \text{dim}_{\mathbb{C}} \mathcal{O}(Z_y) = n \). [Liu02] p.176 In fact, \( Z_y \) is a complete intersection of hyperplane sections of \( \text{Gr}_2(\mathcal{H}^0(\mathcal{E})) \) in the space \( \mathbb{P}(\wedge^2 \mathcal{H}^0(\mathcal{E})) \). [Voi02] p.29 Moreover, the number of hyperplane sections \( \text{dim} \wedge^2 \mathcal{H}^0(\mathcal{E}) - \text{dim} \mathcal{H}^0(\mathcal{L}) = \left( \frac{k+2}{2} \right) - (2k+1) = \text{codim}(X, \mathbb{P}(\wedge^2 \mathcal{H}^0(\mathcal{E}))) \) has the expected intersection dimension. Hence, the degree of the finite map \( n = \text{dim}_{\mathbb{C}} \mathcal{O}(Z_y) \) coincides with the degree of the Grassmannian \( \text{Gr}_2(\mathcal{H}^0(\mathcal{E})) \) via its Plücker embedding. A formula for its degree can be found in [Har92] p.247  

\[
\deg \text{Gr}(2, k+2) = (2k)! \prod_{i=0}^{k} \frac{i!}{(k+i)!} = \frac{(2k)!}{k!(k+1)!} = \frac{1}{k+1} \left( \begin{array}{c} 2k \\ k \end{array} \right) .
\]

\[\Box\]

It is well-known that Theorem 1.1 combined with an observation of Voisin, implies Theorem 1.2, see the unpublished [Bot01] §XI.  

**Proof of Theorem 1.2** Let \((X, \mathcal{L})\) be a general primitively polarized K3 surface with even genus, such that the regular map \(d : \text{Gr}_2(\mathcal{H}^0(\mathcal{E})) \to |\mathcal{L}|\) is surjective and finite with degree \( \frac{1}{k+1} \left( \begin{array}{c} 2k \\ k \end{array} \right) \). By assumption \( p > 2k \), hence coprime to the degree \( \frac{1}{k+1} \left( \begin{array}{c} 2k \\ k \end{array} \right) \). It follows that the corresponding extension of function fields is separable. By generic smoothness for a generically separable map, for a general smooth \( C \in |\mathcal{L}| \), the fiber \( d^{-1}(C) \cong W_{k+1}^1(C) \) is reduced. Note that \( A \in W_{k+1}^1(C) \) is mapped to \( |\mathcal{H}^0(\mathcal{A})^*| \subseteq \text{Gr}_2(\mathcal{H}^0(\mathcal{E})) \).  

By [Voi02] Prop 7, the spaces \( \text{Sym}^{k-2} \mathcal{H}^0(\mathcal{A})^* \), for \( A \in W_{k+1}^1(C) \) generate \( \text{Sym}^{k-2} \mathcal{H}^0(\mathcal{E}) \). Each \( \mathcal{H}^0(\mathcal{A})^* \) corresponds to a line \( T_A \) in \( \mathbb{P}(\mathcal{H}^0(\mathcal{E})) \). A section \( [\ell]_A \in \mathcal{H}^0(\mathcal{A})^* \subseteq \mathcal{H}^0(\mathcal{E}) \) corresponds to a point \( [\ell]_A \) sitting on the line \( T_A \), whose image under \( \psi : \mathbb{P}(\mathcal{H}^0(\mathcal{E})) \to \mathbb{P}(K_{k-1,1}(X, \mathcal{L})) \) is \( [K_{k-1,1}(X, \mathcal{L}; \mathcal{H}^0(\mathcal{L} \otimes I_{Z([\ell]_A)}))] \). (see Proposition 1.2) Let the set \( T := \bigcup_{A \in W_{k+1}^1(C)} T_A \) be the union of lines. The image of \( T \) under \( \psi \) is non-degenerate. [Bot01] Thm 11.2 This implies \( K_{k-1,1}(X, \mathcal{L}) \) is generated by the subspaces \( K_{k-1,1}(X, \mathcal{L}; \mathcal{H}^0(\mathcal{L} \otimes I_{Z(s)})) \), where \( Z(s) \) given by a non-zero regular \( s \in \mathcal{H}^0(\mathcal{E}) \) corresponds to a line bundle \( A \in W_{k+1}^1(C) \) on a fixed curve \( C \). (see Section 2.2) After restricting to \( C \), such spaces are identified with subspaces of the form \( K_{k-1,1}(C, \omega_C; \mathcal{H}^0(\omega_C \otimes A^{-1})) \cong K_{k-1,1}(C, \omega_C), \ A \in W_{k+1}^1(C) \) under the isomorphism \( K_{k-1,1}(X, \mathcal{L}) \cong K_{k-1,1}(C, \omega_C) \). This completes the proof. \[\Box\]
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