Diffraction on random fractal structures

O A Mossoulina¹, S G Volotovsky²

¹Samara National Research University, 34, Moskovskoye Shosse, Samara, Russia, 443086
²Image Processing Systems Institute – Branch of the Federal Scientific Research Centre “Crystallography and Photonics” of Russian Academy of Sciences, 151, Molodogvardeyskaya, Samara, Russia, 443001

Abstract. In this paper, the formation of diffraction patterns for regular and random fractals of Cantor is numerically studied. The calculation of diffraction on one-dimensional, two-dimensional and radial fractal structures is performed using a fractional Fourier transform. This propagation operator makes it possible to obtain a field distribution (with the accuracy of scale) in any paraxial region, both in the Fresnel diffraction zone and in the far field.

1. Introduction
Laser beams with increased resistance to the effects of random fluctuations of the optical medium, including atmospheric turbulence, are an important factor in improving the quality of optical communication in free space, as well as remote sensing systems and laser radar systems [1, 2]. Among such beams, the laser modes of high order [3, 4], including those with a vortex phase singularity [5-9]. Coherence and polarization are also important characteristics of beam stability [10, 11]. When using the Bessel type communication beams, the possibility of coding the polarization state [12] provides an additional degree of information protection, which previously assumed only amplitude and phase coding.

There are various ways of modeling random environments [2, 13-16], including the turbulence of the atmosphere and reservoirs. One way is using of the random fractal structures [17, 18]. The regular fractals are characterized by self-similarity and fractional dimension [19]. The spatial spectrum of such structures also has self-similarity properties, and therefore the diffraction pattern of optical radiation on the fractal structure [20] can be used to determine the characteristics of the structure itself [21, 22]. Moreover, the property of self-similarity makes it possible to use even a small part of the spatial spectrum to obtain an image of the original object [23].

The investigation of diffraction by fractal gratings [20, 24, 25] is promising for solving other important problems – the formation of periodically self-reproducing fields [26-29], the creation of multifocus [30-33] or specified longitudinal distributions [34, 35], and in achromatic depicting systems [36-39].

One of the most important characteristics of fractals is the spatial spectrum [21, 22, 40, 41]. For optical fractal structures, it is also important to have a longitudinal distribution, namely, the formation of a number of foci, which is characteristic of binary zone plates [42].

Random fractal structures preserve some properties of regular fractals, in particular, dimension. Moreover, in the diffraction pattern it is possible to observe the speckle structures, that differ from the
speckle patterns of the ordinary random fields [41, 43]. Thus, the spatial spectrum contains important
information even about random structures. Even more information can be obtained about the object of
research, if to consider not only the focal plane, but some region of diffraction, although not
exclusively in the far zone (the Fraunhofer zone).
In this paper, the simulation of formation of the diffraction patterns for regular and random fractals of
Cantor on the basis of the fractional Fourier transform is performed [44]. This operator makes it
possible to obtain the field distribution (with the accuracy of scale) in any paraxial region, both in the
Fresnel diffraction zone and in the far zone [45].

2. Theoretical part and simulation results
The spatial spectrum for the one-dimensional triadic Cantor fractal [46] can be estimated from the
formula:

$$ F_S(u) = 2^{-S} \prod_{i=0}^{S-1} \cos \left(2\pi \cdot 3^i \cdot u \right) \frac{\sin \left(\pi u \right)}{\pi u}, \quad (1) $$

where $S$ – level of the fractal.

As follows from formula (1) in the Fraunhofer diffraction zone (or in the focal plane of the lens), a
superposition of the set of cosines with increasing frequency will increase as the level of the fractal
increases. The envelope of this superposition depends on the size of the fractal structure.

Figure 1 shows the distributions of the spatial spectrum for fractals of various levels, from which a
self-similar structure is clearly visible.

![Figure 1](image1.png)

**Figure 1.** Spatial spectrum (intensity) for $S = 2$ (black colour), $S = 3$
(blue colour), $S = 4$ (red colour).

The spatial spectrum shown in figure 2 corresponds to the Fraunhofer diffraction zone or the focal
plane of the lens. To obtain the field distribution in any paraxial region, including in the Fresnel
diffraction zone, as well as during the passage of the lens system, a fractional Fourier transform (FFT)
can be used [44, 45, 47, 48]:

$$ E(u, z) = \left( -\frac{ik}{2\pi f \sin (\alpha z)} \right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \exp \left( \frac{ikx^2 \cos (\alpha z)}{2f \sin (\alpha z)} - \frac{ikxz}{f \sin (\alpha z)} \right) dx \quad (2) $$

where $k = \frac{2\pi}{\lambda}$, $\lambda$ – the wavelength of the radiation, $f$ – the focal length of the lens, $\alpha = \frac{\pi}{2f}$, $z$ – the
distance from the input plane, $2R$ – the size of the field within input plane.

Figure 2 shows the longitudinal diffraction patterns, formed at the distance of two foci ($f = 100$ mm) from the input plane, when the fractal structure of 1 mm in size is illuminated by
radiation with the wavelength of 633 nm.
The fields in figure 2 in the central part with the accuracy of scale, correspond to the Fraunhofer diffraction zone (focal region), the vertical cross section at the center gives a picture of the spatial spectrum (as in figure 2). The rest part allows to see the distribution in the Fresnel diffraction zone and get more information about the fractal structure. In particular, at the edges of the pictures, traces of diffraction on fine details, characteristic of high-level fractals are clearly visible. Note, that the diffraction properties of real fractals are different from theoretical consideration [49], especially in the near-field diffraction.

Let’s consider further examples of formation of the random one-dimensional fractals. Fractal modeling takes place by generating the random Cantor set. Each random Cantor set corresponds to the same random process, that is, the removal of the part of the segment, and the length of the segment itself and the parts separated by it are randomly assigned within the allowed value. At the same time, it retains a certain degree of regularity, since at each step of the process of constructing the fractal, the deletion is repeated at random on both sides of the remote segment. Figure 3 shows examples of the random fractals, derived from the Cantor set and the Cantor ladder.

It is possible to construct the two-dimensional fractal, simply as the product of the one-dimensional fractures, and the scale transformation can be introduced along different axes. Moreover, in this case, it is possible to use fractals of different levels. Then the spatial spectrum will have the following form:

$$F_{S,p}(u,v) = 2^{-S-P} \prod_{s=0}^{S} \cos\left(2\pi \cdot 3^s \alpha u\right) \prod_{p=0}^{P} \cos\left(2\pi \cdot 3^p \beta v\right) \text{sinc}(\alpha u) \text{sinc}(\beta v),$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$, $\alpha, \beta$ – the scale factors.
\[
E(u, v, z) = -\frac{ik}{2\pi f \sin(\alpha z)} \exp \left\{ \frac{ik(u^2 + v^2)\cos(\alpha z)}{2f \sin(\alpha z)} \right\} \times
\]
\[
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \exp \left\{ \frac{ik(x^2 + y^2)\cos(\alpha z)}{2f \sin(\alpha z)} - \frac{ik(xu + yv)}{f \sin(\alpha z)} \right\} \, dx \, dy \tag{4}
\]

To generate the random two-dimensional fractal, a unit square on the first step are used \(E_0 = [0,1] \times [0,1]\). At the next step (level), the fractal is specified as \(E_1 = [(0,a_1] \cup [b_1,1]] \times [(0,a_2] \cup [b_2,1])\), where \(a_1, a_2, b_1\) and \(b_2\) – the fractal parameters, specified in the range \((0,1)\), with \(a_1 < b_1\) and \(a_2 < b_2\). Generation of the two-dimensional fractal, as for the one-dimensional case, occurs according to the principle, shown in figure 1, with some corrections, so that the length at each step is given randomly. If the parameters will be set as equal, then the fractals, which are shown in figure 4, will be received.

The spatial spectrum was calculated using the direct Fourier transform:
\[
F(X) = \mathfrak{F}\left[ f(X) \right](U) = \int_{-\infty}^{\infty} f(X) \exp(-2\pi i X U) \, d^2X, \tag{5}
\]

where \(f(X)\) – the input function, given in the form of the vector, representing the given structure; \(X = (x_1, x_2, ..., x_n)\) – the input values of the function; \(F(U)\) – the output function received; \(U = (u_1, u_2, ..., u_n)\) – the output values of the function; \(\mathfrak{F}\left[ \cdot \right]\) – Fourier transform operator.

![Figure 4. The form of the random fractal structure (on the left) and its spatial spectrum (on the right) for different fractal levels: a) \(S=3\); b) \(S=4\); c) \(S=5\).](image)

An interesting type of fractals are radial fractals [50]. On their basis, the fractal lenses and axicons were considered [30, 51-52].

The propagation of the radially symmetric field through the paraxial lens system can be described with using of the FFT in polar coordinates [49, 50]:
\[
E(\rho, \theta, z) = -\frac{ik}{2\pi f \sin(\alpha z)} \exp\left\{ ikz \right\} \exp\left\{ \frac{ik\rho^2}{2f \tan(\alpha z)} \right\} \times
\]
\[
\times \int_{0}^{2\pi} \int_{0}^{\infty} E_0(r, \phi) \exp\left\{ \frac{ikr^2}{2f \tan(\alpha z)} \right\} \exp\left\{ -\frac{ik}{f \sin(\alpha z)} \rho r \cos(\theta - \phi) \right\} \, r \, dr \, d\phi \tag{6}
\]

Figure 5 shows the longitudinal diffraction patterns formed at the distance of two focal points (\(f = 300\) mm) from the input plane, when the fractal structure with the radius of 1 mm is illuminated by radiation with the wavelength of 633 nm.
Figure 5. The form of the fractal structures (on the left) and the longitudinal diffraction patterns (amplitude, \( z \in [0.1f, 1.9f] \)) for different fractal levels: a) \( S=1 \); b) \( S=2 \); c) \( S=3 \).

In Figure 5, the appearance and amplification of the local foci in the near-field diffraction zone is clearly visible, due to the increase in the level of the fractal and the narrowing of the width of the ring lines.

3. Conclusions

In this paper, the one-dimensional, two-dimensional, and radial case of obtaining the fractal structure of the regular and random type are presented. The spatial spectra of the formed fractals are obtained, as well as diffraction patterns in the Fresnel zone. It is shown, that the diffraction properties of the regular fractals in the near diffraction zone differ substantially from the diffraction patterns in the Fraunhofer zone. The difference is particularly noticeable for fractals with high levels, which is due to appearance of many local foci in the near zone due to diffraction on small details.

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