A DECOMPOSITION FOR ADDITIVE FUNCTIONALS OF LÉVY PROCESSES

LUIS ACUÑA VALVERDE*

ABSTRACT. Motivated by the recent results of Nualart and Xu [13] concerning limits laws for occupation times of one dimensional symmetric stable processes, we prove by means of analytic and probabilistic tools the existence of a decomposition for functionals of one dimensional symmetric Lévy processes under certain conditions on their characteristic exponent and transition densities.

Keywords: Lévy processes, characteristic exponent, Fourier Transform, weak convergence, relativistic stable processes.

1. INTRODUCTION

Let $X = \{X_t \}_{t \geq 0}$ be a one-dimensional symmetric Lévy process started at zero on the probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ with characteristic function given by

$$\mathbb{E} [e^{-i x X_s}] = e^{-s \Psi(x)}. \quad (1.1)$$

In this paper we are interested in finding suitable positive and increasing sequences $\{a(n), n \in \mathbb{N}\}$ and $\{b(n,t), n \in \mathbb{N}, t > 0\}$, both tending to $\infty$ as $n \to \infty$ such that under appropriate conditions on the characteristic exponent $\Psi(x)$ and the function $f$, the additive functional

$$\frac{1}{a(n)} \int_0^{b(n,t)} f(X_s) ds \quad (1.2)$$

can be decomposed as a sum of two processes

$$I^{(1)}_n(t) + I^{(2)}_n(t), \quad (1.3)$$

where $I^{(1)}_n(t)$ converges to zero in $L^p(\mathbb{P})$, for some in $p \geq 1$, and $\{I^{(2)}_n(t), n \in \mathbb{N}\}$ is a uniformly integrable sequence with finite moments and with further probabilistic properties.

The foregoing decomposition is of great interest since employing certain techniques it is possible to prove weak convergence of (1.2) to a non–degenerate random variable. One of these techniques, that may immediately provide weak convergence, consists of proving the existence of local times $\{L_t(x), t \geq 0, x \in \mathbb{R}\}$ for the process $X$. This is equivalent to showing [12] that

$$\int_{\mathbb{R}} \Re \left( \frac{1}{1 + \Psi(x)} \right) dx < \infty.$$

* Supported in part by NSF Grant # 0603701-DMS under PI Rodrigo Bañuelos.
The local time describes the amount of time spent by the process at $x$ in the interval $[0,t]$ and it is defined as [2]

$$L_t(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_s-x|<\epsilon\}} ds.$$ 

We also have the occupation density formula

$$\int_0^t f(X_s) ds = \int_\mathbb{R} f(x) L_t(x) dx.$$ 

Particular examples of processes having Local times are the symmetric $\alpha$-stable processes with $\Psi(x) = |x|^\alpha$, $1 < \alpha \leq 2$, for which can be shown by appealing to the well-known scaling property $X_t \overset{\mathcal{L}}{=} t^{1/\alpha} X_1$ that

$$n^{\frac{1}{2\alpha}} \int_0^{n^2} f(X_s) ds \overset{\mathcal{L}}{=} L_t(0) \int_\mathbb{R} dx f(x),$$

for $f \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ [13]. Here, $\overset{\mathcal{L}}{=}$ means equality in law. We also refer the reader to [11] for a probabilistic approach to Local times.

A further approach used to prove weak convergence is the method of moments ([4], pg. 307). This technique is very restrictive since it requires the existence and finiteness of the quantities $m_k(t) = \lim_{n \to \infty} \mathbb{E} \left[ \left( I_n^{(2)}(t) \right)^k \right]$. Moreover, $\{m_k, k \geq 1\}$ must uniquely determine the distribution of a random variable. In this direction, Carleman’s condition [4]

$$\sum_{k=1}^{\infty} m_k^{-1/2k} = \infty$$

is sufficient to guarantee uniqueness. The moment techniques have been used in the recent paper by Nualart and Xu [13] where a decomposition similar to (1.3) is proved for the symmetric Cauchy process $X$, $\Psi(x) = |x|$ (the case $\alpha = 1$). There, it is also proved that

$$\frac{1}{n} \int_0^{n^2} f(X_s) ds \overset{\mathcal{L}}{=} Z(t) \int_\mathbb{R} dx f(x),$$

for bounded functions $f$ with $\int_\mathbb{R} |x||f(x)| dx < \infty$, where $Z(t)$ is an exponential random variable with parameter $t^{-1}$.

Our results in this paper are motivated by the Nualart–Xu result [13] and the Fourier transform techniques used by Bañuelos and Sá Barreto [6] and in the author’s paper [1] to compute heat invariants for Schrödinger operators.

In order to state our main theorem, we will impose some conditions not only for the characteristic exponent $\Psi(x)$ but also for the transition densities for the process $X$, denoted here by $p_t(x)$, and on the function $f$ to be considered in (1.2). To begin with, we will assume that the characteristic exponent $\Psi(x) = \Psi(|x|) \geq 0$ is a non-decreasing function on $[0, \infty)$. Furthermore, we assume the existence of a constant $\delta_0 > 0$ and a non-decreasing function $\ell(x)$ defined on $[0, \delta_0)$, right continuous at zero with $\ell(0) > 0$ such that the following inequality

$$\frac{x^2}{\ell(b)} \leq \Psi(x) \leq \frac{x^2}{\ell(0)}$$

(1.4)

holds for every $0 \leq x \leq b \leq \delta_0$. At this point, it is worth mentioning that the function $\ell(x)$ stated above may not be unique, however the main property to take into consideration
and which is a consequence of the previous inequality is that
\[
\lim_{x \to 0^+} \frac{\Psi(x)}{x^2} = \frac{1}{\ell(0)}.
\]

We now proceed to exhibit, in a general context, examples of symmetric Lévy processes satisfying condition (1.4) which are known in the literature as Subordinated Brownian Motions. Henceforth, \(a \land b\) will stand for \(\min\{a, b\}\).

**Example 1.1.** Subordinated Brownian Motions are defined, according to [7], as Lévy processes whose characteristic exponent \(\Psi(x)\) can be expressed as
\[
\Psi(x) = \phi(x^2),
\]
for some Bernstein function \(\phi(x)\). That is, \(\phi : (0, \infty) \to [0, \infty)\) is a \(C^\infty\) function that admits the following representation
\[
\phi(x) = \int_0^\infty \left(1 - e^{-tx}\right) \mu(ds),
\]
where \(\mu\) is a \(\sigma\)-finite measure on \((0, \infty)\) satisfying \(\int_0^\infty (s \land 1) \mu(ds) < \infty\). Thus, we notice that (1.5) transforms into
\[
\lim_{\lambda \to 0^+} \frac{\phi(\lambda)}{\lambda} = \frac{1}{\ell(0)}.
\]

In addition, in [7] is also shown that
\[
\lim_{\lambda \to 0^+} \frac{\phi(\lambda)}{\lambda} = \int_0^\infty s \mu(ds).
\]

Here, \(\mu\) is the aforementioned measure in (1.6). Then, we conclude that \(\ell(0) = \left(\int_0^\infty s \mu(ds)\right)^{-1}\), whenever the integral term in the last expression is positive and finite. Regarding the function \(\ell(x)\), due to the fact that for Subordinated Brownian Motions \(\Psi \in C^\infty\), we have, by the Taylor expansion of order 2 at zero that
\[
\Psi(x) = 2^{-1} \Psi''(\zeta(x)) x^2.
\]
for \(0 < \zeta(x) < x < \delta_0\) and some \(\delta_0\). Therefore, the condition (1.4) will hold with
\[
\ell(x) = 2 / \Psi''(x),
\]
provided that \(\Psi''\) is a non-increasing and positive function on \([0, \delta_0]\).

To state our assumptions on the transition density \(p_t(x)\), we remark that by (1.1) and the Fourier inversion formula we have
\[
p_t(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi} e^{-t\Psi(\xi)} d\xi,
\]
which implies that \(p_t(x)\) is a non-negative radial function because of our assumptions on \(\Psi(x)\). The proof of our main result requires an integrability conditions for \(p_t(0)\) and the existence of a limit as follows. For every \(a > 0\), we require that
\[
\int_0^a p_s(0) ds < \infty
\]
and that
\[
\lim_{a \to \infty} \frac{1}{a} \int_0^a p_s(0) ds = 0.
\]
The most relevant Lévy processes that share the properties previously mentioned are the relativistic \(\alpha\)-stable processes with \(1 < \alpha < 2\) to be introduced in the example below. These processes belong to the class of Subordinated Brownian Motions and play an important role in physics and Schrödinger operator theory (see [10, 14, 5, 6]).

**Example 1.2.** Let \(m > 0\) and \(1 < \alpha < 2\). The one-dimensional symmetric Lévy process with characteristic exponent given by

\[
\Psi_{m,\alpha}(x) = (x^2 + m^{2/\alpha})^{\alpha/2} - m
\]

will be denoted by \(X_{m,\alpha} = \{X_{t}^{m,\alpha}\}_{t \geq 0}\). This process is called the relativistic \(\alpha\)-stable process of index \(m\). It is a well known fact that \(\phi(x) = (x + m^{2/\alpha})^{\alpha/2} - m\) is a Bernstein function [7] with measure \(\mu\) as defined in example 1.1 given by

\[
\mu(ds) = \frac{\alpha/2}{\Gamma(1 - \alpha/2)} e^{m^{2/\alpha}s} e^{-s} ds.
\]

In [8] and [9] is proved that the transition densities \(p_t^{m,\alpha}(x)\) of \(X_{m,\alpha}\) satisfy the following two sided estimates:

(i) For \(0 < t \leq 1\), we have

\[
C_{m,\alpha}^{-1} \left( t^{-1/\alpha} \wedge \frac{t \Lambda(m^{1/\alpha}|x|)}{|x|^{1+\alpha}} \right) \leq p_t^{m,\alpha}(x) \leq C_{m,\alpha} \left( t^{-1/\alpha} \wedge \frac{t \Lambda(m^{1/\alpha}|x|)}{|x|^{1+\alpha}} \right),
\]

where

\[
\Lambda(r) = 2^{-1+1+\alpha} \int_0^\infty \frac{s^{\frac{1+\alpha}{2}}}{s^{\frac{1+\alpha}{2}}} e^{-s - \frac{r^2}{4}} ds = \text{is a bounded non-increasing function on } [0, \infty) \text{ with } \Lambda(0) = 1.
\]

(ii) For \(t > 1\), we have

\[
c_{\alpha, m}^{-1} t^{-1/2} e^{-\beta_1 \left( |x|^{1+\alpha} \right)/t} \leq p_t^{m,\alpha}(x, y) \leq c_{\alpha, m} t^{-1/2} e^{-\beta_2 \left( |x|^{1+\alpha} \right)/t}.
\]

In particular, we observe that

\[
p_t^{m,\alpha}(0) \leq C_{m,\alpha} t^{-1/\alpha} \mathbb{1}_{(0,1]}(t) + c_{m,\alpha} t^{-1/2} \mathbb{1}_{(1,\infty]}(t),
\]

which in turn implies

\[
\int_0^a p_t^{m,\alpha}(0) ds_1 \leq C \left( a^{(\alpha-1)/\alpha} \mathbb{1}_{[0,1]}(a) + \sqrt{a} \mathbb{1}_{(1,\infty]}(a) \right),
\]

for some constant \(C = C(m, \alpha) > 0\). Hence, it follows that these processes satisfy the conditions (1.10) and (1.11).

On the other hand, \(\Psi_{m,\alpha}(x)\) is infinitely differentiable for \(x \geq 0\) with

\[
\Psi_{m,\alpha}'(x) = \alpha x (x^2 + m^{2/\alpha})^{\alpha/2 - 1},
\]

\[
\Psi_{m,\alpha}''(x) = \alpha (x^2 + m^{2/\alpha})^{\alpha/2 - 2} \left[ x^2 (\alpha - 1) + m^{2/\alpha} \right],
\]

\[
\Psi_{m,\alpha}'''(x) = -\alpha (2 - \alpha) x (x^2 + m^{2/\alpha})^{\alpha/2 - 3} \left[ x^2 (\alpha - 1) + 3m^{2/\alpha} \right].
\]

Thus, we conclude that \(\Psi_{m,\alpha}''(x) > 0\) for all \(x \in \mathbb{R}\) and non–increasing on \([0, \infty)\) (because the third derivative is non-positive). Thus, by example 1.1, we set

\[
\ell(x) = 2/\Psi_{m,\alpha}'(x),
\]
so that condition (1.4) holds and in this particular case we obtain
\[ \ell(0) = 2 \alpha^{-1} m(2-\alpha)/\alpha. \]

As for the function \( f \), we impose the following conditions.
(C0) \( f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \)
(C1) \( \hat{f} \in L^1(\mathbb{R}). \)
(C2) There exist \( \delta_f > 0 \) and a non-negative function \( G(x) \) so that
\[ |\hat{f}(x) - \hat{f}(0)| \leq G(x), \quad |x| \leq \delta_f, \]
(1.12)
\[ \int_{|x| \leq \delta_f} \frac{G(x)}{x^2} \, dx < \infty. \]

We denote by \( \mathcal{D} \) the set of functions satisfying (C0)-(C2). Next, we exhibit some examples of functions that belong to \( \mathcal{D} \).

**Example 1.3.** A typical element of \( \mathcal{D} \) is the Gaussian kernel \( \gamma_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \) since clearly \( \hat{\gamma}_1(\xi) = e^{-|\xi|^2/2} \) satisfies (C1). Regarding (C2), we observe that
\[ |e^{-|\xi|^2/2} - 1| = \left| \int_0^{|\xi|^2/2} e^{-u} \, du \right| \leq |\xi|^2/2. \]

Another element of \( \mathcal{D} \) is the Jackson–Vallée–Poussin Kernel
\[ \gamma_2(x) = \frac{12}{\pi} \left( \sin(\frac{x}{2}) \right)^4, \]
for which is known ([3], pg. 23) that
\[ \hat{\gamma}_2(\xi) = \begin{cases} 
1 - \frac{3\xi^2}{4} + \frac{4|\xi|^3}{3} & \text{for } |\xi| \leq 1, \\
\frac{1}{4}(2 - |\xi|^2) & \text{for } 1 \leq |\xi| \leq 2, \\
0 & \text{for } |\xi| \geq 2.
\end{cases} \]

Clearly, \( \gamma_i \ast \gamma_j \) with \( i, j \in \{1, 2\} \) are also members of \( \mathcal{D} \) as well.

The main result of the paper is the following:

**Theorem 1.1.** Let \( a : [0, \infty) \to (0, \infty) \) be a differentiable and a strictly increasing function such that \( a(T) \to \infty \) as \( T \to \infty \). Fix \( f \in \mathcal{D} \) and consider \( \delta_f \) and \( \delta_0 \) as given in (1.12) and (1.4), respectively. For \( t > 0 \) and \( n \in \mathbb{N} \), set
\[ I_n(t) = \frac{1}{a(n)} \int_0^t a^2(n) f(X_s) \, ds. \]
Then, the process \( I_n(t) \) can be written as
\[ I^{(1)}_{n, \delta}(t) + \hat{f}(0) I^{(2)}_{n, \delta}(t), \]
where
(i) for any \( 0 < \delta \leq \delta_f \wedge \delta_0 \),
\[ I^{(1)}_{n, \delta}(t) \to 0 \text{ in } L^2(\mathbb{P}) \text{ as } n \to \infty, \]
and
ii) \[ I^{(2)}_{n, \delta}(t) \to 0 \text{ as } n \to \infty. \]
Corollary 1.1. For any \(0 < \delta < \delta_0/2\) the sequence \(\{I_{n\delta}(t), n \in \mathbb{N}\}\) is uniformly integrable. Furthermore, for \(k \in \mathbb{N}\), we have

\[
(1.13) \quad \lim_{\delta \to 0^+} \lim_{n \to \infty} \mathbb{E} \left[ I_{n\delta}(t) \right]^k = \lim_{\delta \to 0^+} \lim_{n \to \infty} \mathbb{E} \left[ I_{n\delta}(t) \right] = 2^{1-k} \ell(0)^{k/2} (k-1)! E \Gamma(k/2).
\]

As an application of this theorem, we obtain the following limits.

Corollary 1.1. Under the assumptions of Theorem 1.1, we have

\[
(1.14) \quad \lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{a(n)} \int_0^{X^2} f(X_s) ds \right] = t \left( \int \mathbb{R} dx f(x) \right) \sqrt{\ell(0)/\pi},
\]

\[
(1.15) \quad \lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{a^2(n)} \left( \int_0^{X^2} f(X_s) ds \right)^2 \right] = t^2 \left( \int \mathbb{R} dx f(x) \right)^2 \frac{\ell(0)}{2}.
\]

Remark 1.1. For relativistic \(\alpha\)-stable processes with index \(m\) and \(1 < \alpha < 2\), (1.14) and (1.15) hold with

\[
\ell(0) = 2\alpha^{-1} m^{(2-\alpha)/\alpha}.
\]

The paper is organized as follows. In §2, we give some notation and state a basic identity (Lemma 2.1) which will be employed later. In §3, we provide the proof of Theorem 1.1 as a consequence of a series of propositions. Finally, the proof of corollary 1.1 is given in §4.

2. Notation and preliminaries

For any elements \(x_1, \ldots, x_n \in \mathbb{R}^d, d \in \mathbb{N}\), we will denote

\[
x^{(n)} = (x_1, \ldots, x_n) \in \mathbb{R}^{dn} \quad \text{and} \quad dx^{(n)} = dx_1 dx_2 \cdots dx_n.
\]

Also for any \(L > 0\), we define

\[
D_k(L) = \{(s_1, s_2, \ldots, s_k) \in [0, L]^k : s_0 = 0 < s_1 < s_2 < \cdots < s_k < L\}.
\]

Henceforth, we will use the following basic identity [15] which holds for every integrable function \(V\) over \([0, L]\) and \(k \in \mathbb{N}\).

\[
(2.1) \quad \left( \int_0^L V(s) ds \right)^k = k! \int_{D_k(L)} \prod_{i=1}^k V(s_i) ds^{(k)}.
\]

We also observe that for \(0 = s_0 < s_1 < \cdots < s_k < \infty\) and \(x_1, \ldots, x_k \in \mathbb{R}\), we have

\[
(2.2) \quad \mathbb{E} \left[ \exp \left( \sum_{j=1}^k x_j X_{s_j} \right) \right] = \exp \left( - \sum_{i=1}^k \Psi \left( \sum_{j=i}^k x_j \right) (s_i - s_{i-1}) \right),
\]

where the last equality is a consequence of the independence of increments of the process \(X\).

Let \(k > 1\) be an integer. For \(r > 0\), let \(x_i = r \Theta_i(\theta^{(k-1)})\), \(i \in \{1, \ldots, k\}\) be the polar coordinate representation in the Euclidean space \(\mathbb{R}^k\), where we will denote by \(S^{k-1}\) the boundary of the open unit ball with surface area \(\mathbb{S}^{k-1}\). It is an elementary fact that the Jacobian of the polar coordinates transformation can be written as \(r^{k-1} J_k(\theta^{(k-1)})\) for some function \(J_k\).
Set
\[ S_+^k := S_+^{k-1} \cap \{ (x_1, \ldots, x_k) : x_i > 0 \text{ for all } i \in \{ 1, \ldots, k \} \} = \left\{ \theta^{(k-1)} \in [0, 2\pi)^{k-1} : \Theta_i(\theta^{(k-1)}) > 0 \text{ for all } i \in \{ 0, \ldots, k \} \right\}. \]

We define for a bounded and non-decreasing function \( F : [0, \infty) \to [0, \infty) \),
\begin{equation}
H_F^{(k)}(r, \xi) := \begin{cases} 
F(r\xi) & \text{if } k = 1, \\
\int_{S_+^{k-1}} \prod_{i=1}^k F(\xi \cdot \Theta_i(\theta^{(k-1)}))J_i(\theta^{(k-1)})d\theta^{(k-1)} & \text{if } k \geq 2,
\end{cases}
\end{equation}
for any \( r, \xi \geq 0 \). This function has the following properties:

(i) Fix \( \xi \geq 0 \). Then, \( H_F^{(k)}(r, \xi) \) is a non-decreasing function with respect to the variable \( r \) since \( F \) is non-decreasing by assumption.

(ii) If \( F(\lambda) = 1 \) for all \( \lambda \geq 0 \), then
\[ H_F^{(k)}(r, \xi) = \begin{cases} 1 & \text{if } k = 1, \\
2^{-k}|S^{k-1}_+| & \text{if } k \geq 2.
\end{cases} \]

(iii) An application of either Monotone Convergence Theorem (\( F \) non-decreasing) or
Lebesgue Dominated convergence Theorem (\( F \) bounded and (ii) above) shows
that for any sequence \( \{ r_n : n \in \mathbb{N} \} \) with \( r_n \to \infty \) as \( n \to \infty \),
\begin{equation}
\lim_{n \to \infty} H_F^{(k)}(r_n, \xi) = \begin{cases} ||F||_\infty & \text{if } k = 1, \\
2^{-k}||F||_\infty |S^{k-1}_+| & \text{if } k \geq 2,
\end{cases}
\end{equation}
for \( \xi \geq 0 \) fixed (notice when \( \xi = 0 \), we have to replace \( ||F||_\infty \) with \( F(0) \)). Thus, we have shown
that \( \lim_{r \to \infty} H_F^{(k)}(r, \xi) \) exists and is given by the right hand side of (2.4).

**Lemma 2.1.** For any \( L > 0 \) and \( k \in \mathbb{N} \), we have
\[ \int_{D_k(L)} ds^{(k)} \prod_{i=1}^k \frac{F(\xi \sqrt{s_i-s_{i-1}})}{\sqrt{s_i-s_{i-1}}} = 2^k \int_0^\sqrt{\pi} dr \ r^{k-1} H_F^{(k)}(r, \xi). \]

**Proof.** The change of variables \( u_i = \sqrt{s_i-s_{i-1}} \) transform \( D_k(L) \) into
\[ \left\{ (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k : u_i > 0, \sum_{i=1}^k u_i^2 < L \right\} \]
with Jacobian satisfying
\[ \frac{\partial(s_1, \ldots, s_k)}{\partial(u_1, \ldots, u_k)} = 2^k \prod_{i=1}^k u_i. \]
Consequently, the change to polar coordinates yields the desired result. \( \square \)

### 3. Proof of Theorem 1.1

The proof consists of several steps. To begin with, the Fourier inversion formula can be applied to any function \( f \in \mathcal{D} \) to obtain
\begin{equation}
(3.1) \quad \int_0^{2\pi a^2(n)} f(X_s)ds = (2\pi)^{-1} \int_0^{2\pi a^2(n)} ds \int_\mathbb{R} dx \hat{f}(x)e^{ixX_s}. \end{equation}
Proposition 3.1. Let \( \delta > 0 \) and set
\[
F_{n,1}(\delta, t) = \frac{1}{a(n)} \int_{0}^{t} a^2(n) \int_{|x| > \delta} ds \int_{|x| > \delta} dx \hat{f}(x) e^{i\pi X_s}.
\]
Then \( F_{n,1}(\delta, t) \to 0 \) in \( L^2(\mathbb{P}) \), as \( n \to \infty \).

Proof. Observe that by (2.1) and (2.2) with \( k = 2 \), we have after a suitable change of variables that
\[
\mathbb{E} \left[ F_{n,1}^2(\delta, t) \right] = \frac{2}{a^2(n)} \int_{0}^{t} a^2(n) \int_{s_1}^{s_2} ds_1 \int_{|x_1| > \delta} ds_2 \int_{|x_2| > \delta} dx \hat{f}(x_2) \hat{f}(x_1) e^{-\Psi(x_2)(y_2 - s_1) - \Psi(x_2 + x_1)s_1} e^{-\Psi(y_2)(y_2 - s_1) - \Psi(y_1)s_1} dy_2.
\]
Since \( \Psi(x) \) is non-decreasing on \([0, \infty)\), we have for \( |y_2| > \delta \) that
\[
\int_{s_1}^{s_2} ds_2 e^{-\Psi(y_2)s_2} \leq e^{-\Psi(y_2)s_1} \Psi(y_2)^{-1} \leq e^{-\Psi(y_2)s_1} \Psi(\delta)^{-1}.
\]
The latter inequality, \( \hat{f} \in L^1(\mathbb{R}) \) and (1.9) show that
\[
\mathbb{E} \left[ F_{n,1}^2(\delta, t) \right] \leq \frac{2}{a^2(n) \Psi(\delta)} \int_{0}^{t} a^2(n) \int_{|y_2| > \delta} ds_1 \int_{|y_1 - y_2| > \delta} dy_2 \hat{f}(y_2) \hat{f}(y_1 - y_2) e^{-\Psi(y_1)s_1} dy_2 dy_1 e^{-\Psi(y_1)s_1} \leq \frac{4\pi \| \hat{f} \|_{\infty} \| \hat{f} \|_1}{\Psi(\delta)} \left( \frac{1}{a^2(n)} \int_{0}^{t} a^2(n) \int_{|x| > \delta} dx \hat{f}(x) \hat{f}(0) e^{i\pi X_s} \right).
\]
Next, due to the conditions (1.10) and (1.11) the last term in the above inequality vanishes as \( n \to \infty \) and this completes the proof. \( \square \)

Proposition 3.2. Consider \( 0 < \delta \leq \delta_f \wedge \delta_0 \) and set
\[
F_{n,2}(\delta, t) := \frac{1}{a(n)} \int_{0}^{t} a^2(n) \int_{|x| \leq \delta} dx \int_{|x| \leq \delta} dx \left( \hat{f}(x) - \hat{f}(0) \right) e^{i\pi X_s}.
\]
Then, \( \mathbb{E} \left[ F_{n,2}^2(\delta, t) \right] \to 0 \), as \( n \to \infty \).

Proof. First, we observe by (1.4) that \( \Psi(y_2) \geq y_2^2 (\ell(\delta))^{-1} \) for any \( |y_2| \leq \delta \) since by assumption \( \delta < \delta_0 \). Next, by appealing to (1.12) and the fact that \( \delta \leq \delta_f \), we have by
mimicking the proof of the latter proposition that
\[
|\mathbb{E} \left[ F_{n,2}^2(\delta, t) \right]| \\
\leq \frac{2}{\alpha^2(n)} \int_0^{\pi^2 a^2(n)} ds_1 \int_{|y_2| \leq \delta} \int_{|y_1 - y_2| \leq \delta} \frac{|\hat{f}(y_2) - \hat{f}(0)|}{\psi(y_2)} \left| \hat{f}(y_1 - y_2) - \hat{f}(0) \right| e^{-\psi(y_1)s_1} dy_2
\]
\[
\leq \frac{2f(\delta)}{\alpha^2(n)} \int_0^{\pi^2 a^2(n)} ds_1 \int_{|y_2| \leq \delta} \int_{|y_1 - y_2| \leq \delta} \frac{|\hat{f}(y_2) - \hat{f}(0)|}{y_2^2} \left| \hat{f}(y_1 - y_2) - \hat{f}(0) \right| e^{-\psi(y_1)s_1} dy_2
\]
\[
\leq \frac{8\pi \ell(\delta \wedge \delta)}{2^2} \left( \int_{|y_2| \leq \delta \wedge \delta} \frac{G(y_2)}{y_2^2} dy_2 \right) \left( \frac{1}{\alpha^2(n)} \int_0^{\pi^2 a^2(n)} p_{s_1}(0) ds_1 \right),
\]
where the last term converges to 0 by assumption (1.11).

Proposition 3.3. Let \( \delta > 0 \). Set
\[
F_n(\delta, t) := \frac{1}{a(n)} \int_0^{\pi^2 a^2(n)} ds \int_{|x| \leq \delta} dx e^{ixX_s} = \frac{2}{a(n)} \int_0^{\pi^2 a^2(n)} \frac{\sin(\delta X_s)}{X_s} ds.
\]

Then, the sequence \( \{ F_n(\delta, t), n \in \mathbb{N} \} \) has the following properties.
(i) The sequence is uniformly integrable for any \( \delta < \delta_0/2 \).
(ii) For any \( \delta < \delta_0/k, k \in \mathbb{N} \), the sequence has positive and finite k-th moments. Moreover,
\[
\lim_{\delta \to 0^+} \lim_{n \to \infty} \mathbb{E} \left[ F_n^k(\delta, t) \right] = \lim_{\delta \to 0^+} \lim_{n \to \infty} \mathbb{E} \left[ F_n^k(\delta, t) \right] = 2 \pi^k (k - 1)! t^k \ell(0)^{k/2} \Gamma^{-1}(k/2).
\]

Proof. Under the notation given in the previous section and due to (2.2), we have that
\[
\mathbb{E} \left[ F_n^k(\delta, t) \right] \text{ is equal to}
\]
\[
\frac{k!}{\alpha^k(n)} \int_{D_k(t^2 a^2(n))} ds(k) \int_{\{|x_i| \leq \delta, i = 1, \ldots, k\}} dx^{(k)} e^{-\sum_{i=1}^k \frac{\Psi(\sum_{j=1}^k x_j)}{s_i - s_{i-1}}}
\]
\[
= \frac{k!}{\alpha^k(n)} \int_{D_k(t^2 a^2(n))} ds(k) \int_{\{|y_i| \leq \delta, |y_i - y_{i+1}| \leq \delta, i = 1, \ldots, k - 1\}} dy^{(k)} e^{-\sum_{i=1}^k \frac{\Psi(y_i)}{s_i - s_{i-1}}},
\]
where in the second equality we have applied the change of variables \( y_i = \sum_{j=i}^k x_j \).

We observe that
\[
\{ |y_i| \leq \delta/2, i = 1, \ldots, k \} \subset \{ |y_k| \leq \delta, |y_i - y_{i+1}| \leq \delta, i = 1, \ldots, k - 1 \}
\]
\[
\subset \{ |y_i| \leq k\delta, i = 1, \ldots, k \}.
\]

Before proceeding, let us define
\[
F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{|y| \leq \lambda} dy e^{-y^2/2},
\]
so that we have for every \( A, M > 0 \) that
\[
\int_{\{|y| \leq M\delta, i = 1, \ldots, k \}} dy^{(k)} e^{-\sum_{i=1}^k \frac{y_i^2(s_i - s_{i-1})A}{s_i - s_{i-1}}} = \left( \frac{\pi}{A} \right)^{k/2} \prod_{i=1}^k \frac{F(M\delta \sqrt{2A(s_i - s_{i-1})})}{\sqrt{s_i - s_{i-1}}}.
\]

Let \( k \in \mathbb{N} \). Notice that for any \( \delta < \delta_0/k, \) we are allowed to use inequality (1.4) with any real number \( x \) satisfying either \( |x| \leq k\delta \) or \( |x| \leq \delta/2 \). Therefore, by combining (3.3)
and Lemma 2.1, we obtain that $\mathbb{E} \left[ F_n^k(\delta, t) \right]$ is bounded above and below by terms of the form

$$\left( \frac{\pi}{A} \right)^{k/2} \frac{2^k k!}{a^k(n)} \int_0^{ta(n)} dr \ r^{k-1} H_F^{(k)}(r, M \delta \sqrt{2A})$$

where

(a) for the upper-bound, $A = 1/\ell(k\delta)$ and $M = k$,

(b) for the lower-bound, $A = 1/\ell(0)$ and $M = 1/2$.

Hence, by appealing to the fact that $a(T)t > a(0)t > 0$ for every $T > 0$ and $H_F^{(k)}$ is non-decreasing in the variable $r$ for every $\xi \geq 0$ fixed, we have

$$\int_0^{ta(T)} dr \ r^{k-1} H_F^{(k)}(r, M \delta \sqrt{2A}) \geq \left( \frac{a_k(T) - a_k(0)}{k} \right) t^k H_F^{(k)}(a(0)t, M \delta \sqrt{2A}),$$

which allows us to obtain together with (3.5) the following lower bound for the $k$-th moment when we replace $T$ with $n$:

$$\left( \frac{\pi}{A} \right)^{k/2} 2^k (k-1)! H_F^{(k)} \left( a(0)t, \frac{\delta}{\sqrt{2\ell(0)}} \right) \left( 1 - \frac{a_k(0)}{a_k(n)} \right) t^k \leq \mathbb{E} \left[ F_n^k(\delta, t) \right].$$

We notice that the term $H_F^{(k)}$ which was defined in (2.3) appearing on the left hand side of the latter inequality has to be positive because of the definition of $F$ in (3.4).

Next, it is a basic fact that the function $F$ satisfies $\|F\|_{\infty} = 1$ which leads according to (2.4) to the following upper bound

$$\mathbb{E} \left[ F_n^k(\delta, t) \right] \leq \left( \pi \ell(k\delta) \right)^{k/2} (k-1)! t^k \left( \|S^{k-1}\| \cdot 2 \cdot \mathbb{I}_{\{k \geq 2\}} \right).$$

Thus, we have shown that all the $k$-th moments are positive and finite. On the other hand, uniformly integrability easily follows from the last inequality since for $k = 2$ and any $\delta < \delta_0/2$, we obtain that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[ F_n^2(\delta, t) \right] \leq 2\pi^2 \ell(2\delta)t^2,$$

and with this we have proved (i).

We now proceed to observe that inequality (3.6) shows that the term on the left hand side of the same inequality tends to $\infty$ as $T \to \infty$. By applying L’Hôpital rule and using (2.4) with $\|S^{k-1}\| = 2\pi^{k/2} \Gamma^{-1}(k/2)$ for $k \geq 2$, we conclude that

$$\lim_{T \to \infty} \frac{1}{a^k(T)} \int_0^{ta(T)} dr \ r^{k-1} H_F^{(k)}(r, M \delta \sqrt{2A}) = \frac{(ta(T))^{k-1} H_F^{(k)}(ta(T), M \delta \sqrt{2A}) t \frac{d}{dT} a(T)}{ka^{k-1}(T) \frac{d}{dT} a(T)} = \frac{2\pi^{k/2} t^k}{k^{2^k} \Gamma(k/2)},$$

for any $\delta, M, A > 0$ fixed and for all $k \geq 1$. Therefore, we also have that

$$\lim_{n \to \infty} \frac{1}{a^k(n)} \int_0^{ta(n)} dr \ r^{k-1} H_F^{(k)}(r, M \delta \sqrt{2A}) = \frac{2\pi^{k/2} t^k}{k^{2^k} \Gamma(k/2)}.$$

Hence, using one more time that $\mathbb{E} \left[ F_n^k(\delta, t) \right]$ is bounded by terms of the form (3.5) we arrive at

$$2 \pi^k (k-1)! t^k \ell(0)^{k/2} \Gamma^{-1}(k/2) \leq \lim_{n \to \infty} \mathbb{E} \left[ F_n^k(\delta, t) \right].$$
and
\[ \lim_{n \to \infty} \mathbb{E} \left[ k_n^k(\delta, t) \right] \leq 2 \pi^k (k - 1)! \ell_k \ell(k\delta)^{k/2} \Gamma^{-1}(k/2). \]
Thus, the proof of (ii) is completed by letting \( \delta \to 0^+ \) and using the right continuity of \( \ell \) at zero. \( \square \)

We now observe that the proof of the Theorem 1.1 follows by setting
\[ I_{n,\delta}^{(1)}(t) = (2\pi)^{-1} (F_{n,1}(\delta, t) + F_{n,2}(\delta, t)) \quad \text{and} \quad I_{n,\delta}^{(2)}(t) = (2\pi)^{-1} F_n(\delta, t). \]

**Remark 3.1.** We point out that Proposition 3.3 remains true for any Lévy process with characteristic exponent of the form \( \Psi(x) = x^2/\ell(x) \), with \( \ell(x) \) radial, \( \ell(0) > 0 \), right continuous at 0, and either non-decreasing or non-increasing in \([0, \infty)\). This is so because the key ingredient in the proof is to have either (in the non-increasing case)
\[ \frac{x^2}{\ell(0)} \leq \Psi(x) \leq \frac{x^2}{\ell(b)} \]
or (in the non-decreasing case)
\[ \frac{x^2}{\ell(b)} \leq \Psi(x) \leq \frac{x^2}{\ell(0)} \]
for \(|x| < b\). For instance, we can apply Proposition 3.3 to relativistic processes with \( \alpha = 2/N \), \( N \in \mathbb{N} \setminus \{0, 1\} \) and \( m > 0 \) since the characteristic exponent \( \Psi_{2/N,m}(x) \) of these processes can be written as \( x^2/\ell(x) \), with
\[ \ell(x) = \sum_{j=0}^{N-1} (x^2 + mN)^{(N-1-j)/N} m^j. \]

4. PROOF OF COROLLARY 1.1

**Proof.** For \( f \in \mathcal{D}, \) consider the decomposition provided by Theorem 1.1. That is,
\[ \frac{1}{a(n)} \int_0^{t^2 a(n)^2} f(X_s) ds = I_{n,\delta}^{(1)}(t) + \hat{f}(0) I_{n,\delta}^{(2)}(t). \]

Henceforth, we only consider \( \delta < (\delta_f \wedge \delta_0)/2 \).

Let us denote
\[ A_n(t) = \mathbb{E} \left[ \frac{1}{a^2(n)} \left( \int_0^{t^2 a^2(n)} f(X_s) ds \right)^2 \right]. \]

Recall that \( I_{n,\delta}^{(2)}(t) = (2\pi)^{-1} F_n(t, \delta) \). Then, by inequality (3.7), we obtain that
\[ \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left| I_{n,\delta}^{(2)}(t) \right|^2 \right] \leq 2^{-1} t^2 \ell(2\delta). \]

Thus, we derive by the Cauchy–Schwarz inequality that
\[ \left| \mathbb{E} \left[ I_{n,\delta}^{(1)}(t) I_{n,\delta}^{(2)}(t) \right] \right|^2 \leq \mathbb{E} \left[ \left| I_{n,\delta}^{(1)}(t) \right|^2 \right] 2^{-1} t^2 \ell(2\delta). \]
By part (i) of Theorem 1.1, we know that \( I_{n,\delta}(t) \) converges to zero in \( L^2(\mathbb{P}) \) as \( n \to \infty \) (which also holds in \( L^1(\mathbb{P}) \) again because of the Cauchy-Schwarz inequality) so that we conclude
\[ \lim_{n \to \infty} \mathbb{E} \left[ I_{n,\delta}^{(1)}(t) I_{n,\delta}^{(2)}(t) \right] = 0, \]
for any $\delta < (\delta_f \wedge \delta_0)/2$.

Combining the facts given above and decomposition (4.1), we arrive at

$$\lim_{n \to \infty} A_n(t) = (\hat{f}(0))^2 \lim_{n \to \infty} \mathbb{E} \left[ \left| f_{n,t}^{(2)}(t) \right|^2 \right],$$

$$\lim_{n \to \infty} A_n(t) = (\hat{f}(0))^2 \lim_{n \to \infty} \mathbb{E} \left[ \left| f_{n,t}^{(2)}(t) \right|^2 \right],$$

for any $\delta < (\delta_f \wedge \delta_0)/2$. Since the left hand side of these equalities does not depend on $\delta$, by letting $\delta \to 0^+$ and appealing to (1.13), we have

$$\lim_{n \to \infty} A_n(t) = \lim_{n \to \infty} A_n(t) = t^2 (\hat{f}(0))^2 2^{-1} \ell(t),$$

which shows (1.15).

Next, (1.14) is easy to compute by using part of the arguments given above and is left to the reader. This completes the proof. \qed

Finally, as we mentioned in the introduction, the results in this paper were motivated by the Nualart–Xu results [13]. It is interesting to note that many of the computations for the Fourier transform in [13] are similar to those used by Bañuelos and Sá Barreto [6] and in the author’s paper [1] to compute the heat invariants for Schrödinger operators for the Laplacian and the fractional Laplacian. In these papers one uses Fourier transform methods to obtain estimates on Feynman–Kac expressions of the form

$$E^{t}_{x,x} \left[ e^{-\int_{0}^{t} V(X_s) ds} \right].$$

(4.2)

Here, $X$ is the symmetric $\alpha$-stable process and $E^{t}_{x,x}$ stands for the expectation with respect to the process (stable bridge) starting at $x$ and conditioned to be at $x$ at time $t$. The function (potential $V$) is infinitely differentiable of compact support. One interesting problem is to obtain estimates and properties of (4.2) with less regularity on the functions $V$. In this direction, in [5], Bañuelos and Selma employed the Taylor expansion of the exponential function and probabilistic techniques to investigate the $k$-th moment of $\int_{0}^{t} V(X_s) ds$ with respect to the stable bridge, for $V$'s which are Hölder continuous. Expressions similar to those in Nualar–Xu [13] for computation of moments are derived.

Acknowledgements: I am grateful to my supervisor, Professor Rodrigo Bañuelos, for his valuable suggestions and time while preparing this paper.

References

[1] L. Acuña Valverde, Trace asymptotics for Fractional Schrödinger operators. J. Funct. Anal., 266, 514-559, (2014).
[2] Applebaum. D, Lévy processes and Stochastic Calculus. Second Edition, (2009).
[3] N. I. Akhiezer, Lectures on Integral Transforms. Translation of Mathematical Monograph, AMS, 70.
[4] K. Athreya and S. Lahiri, Measure theory and Probability Theory. Springer, (2006).
[5] R. Bañuelos and S. Yildirim, Heat trace of non-local operators. J. London Math. Society, 87, 304-318, (2013).
[6] R. Bañuelos and A. Sá Barreto, On the heat trace of Schrödinger operators. Comm in Partial Differential equations, 20, 2153-2164, (1995).
[7] K. Bogdan, Potential Analysis of Stable Processes and its extensions. Lecture notes in mathematics 1980. Springer-Verlag, (2009).
[8] Z. Chen, P. Kim, R. Song, Global Heat Kernel Estimates for Relativistic Stable Processes in Half-space-like Open Sets. Potential Analysis, 36, 235–261, (2012).
[9] Z. Chen, P. Kim, T. Kumagai, Global heat kernel estimates for symmetric jump processes. Trans. Amer. Math. Soc, 363, 5021–5055, (2011).
[10] T. Byczkowski, M. Ryznar and J. Malecki, Bessel Potentials, Hitting Distributions and Green Functions. *American Mathematical Society*, 361, 4871-4900, (2009).

[11] J. Rosen, Second order limits laws for the local times of stable processes. *Séminaire de probabilités (Strasbourg)*, 25, 407-424, (1991).

[12] M. Marcus and J. Rosen, *Markov Processes, Gaussian Processes, and the local times of symmetric Levy Processes*. Cambridge University Press, First edition, (2006).

[13] D. Nualart and Fangjun Xu, Limits laws for occupation times of stables processes, http://arxiv.org/abs/1305.0241, (2013).

[14] M. Ryznar, Estimates of Green Function for Relativistic \( \alpha \)-stable process. *Potential Analysis*, 17, 1-23, (2002).

[15] T. Trif, *Multiple Integrals of Symmetric Functions*. *The American Mathematical Monthly*, 104, 605-608, (1997).

**DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907, USA**

*E-mail address: lacunava@math.purdue.edu*