Dirichlet problem for the constant mean curvature equation and CMC foliation in the extended Schwarzschild spacetime

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Abstract
We prove the existence and uniqueness of the Dirichlet problem for the spacelike, spherically symmetric, constant mean curvature equation with symmetric boundary data in the extended Schwarzschild spacetime. As an application, we completely solve the CMC foliation conjecture which is proposed by Malec and Murchadha (2003 Phys. Rev. D 68 124019).

Keywords: constant mean curvature hypersurfaces, Schwarzschild spacetime, Dirichlet problem, CMC foliaton

(Some figures may appear in colour only in the online journal)

1. Introduction

Spacelike constant mean curvature (CMC) hypersurfaces in spacetimes have been considered important objects in general relativity. This is because CMC hypersurfaces are used in the analysis on Einstein constraint equations [6, 10] and in the gauge condition in the Cauchy problem of the Einstein equations [2, 5]. In addition, York suggested the concept of the CMC foliation and the CMC time function on relativistic cosmology [13], so it is possible to characterize the global structure of cosmological spacetimes by CMC hypersurfaces.

Some results of spacelike, spherically symmetric constant mean curvature hypersurfaces (SS-CMC) in the Schwarzschild spacetime and in the Kruskal extension can be found in a paper by Brill et al [3], and in Malec and Murchadha’s papers [11, 12]. In papers [8, 9], the authors considered the initial value problem of the SS-CMC equation in the Schwarzschild spacetime. These SS-CMC solutions are completely solved and characterized by two constants of integration. Furthermore, the authors discussed behaviors of SS-CMC hypersurfaces near the coordinate singularities such that correspondences between SS-CMC hypersurfaces in the Schwarzschild spacetime and SS-CMC hypersurfaces in the Kruskal extension are
established. Thus, the initial value problem of the SS-CMC equation in the Kruskal extension is solved as well.

In this paper, we consider the Dirichlet problem for the SS-CMC equation in the Kruskal extension with $T$-axisymmetric boundary data. The result is comprehensively stated below:

**Theorem.** The Dirichlet problem for the SS-CMC equation with symmetric boundary data in the Kruskal extension is solvable and the solution is unique.

The precise setting of the SS-CMC Dirichlet problem is in section 3.1. For the existence part, we use the shooting method since the boundary value problem can be reduced to the initial value problem, which is solved in papers [8, 9]. For the uniqueness part, we introduce the Lorentzian distance function with respect to some spacelike hypersurface. Laplacian of the Lorentzian distance function restricted on another spacelike hypersurface at the maximum point has a good estimate related to the mean curvatures of both spacelike hypersurfaces. Thus, if the solution of the SS-CMC Dirichlet problem is not unique, we apply the Lorentzian distance function estimate to two of SS-CMC solutions, and it implies a contradiction.

As an application, we prove that the $T$-axisymmetric SS-CMC foliation (we use TSS-CMC foliation for short) property conjectured by Malec and Murchadha in [11] is true. In [11], they constructed a family of TSS-CMC hypersurfaces with the same mean curvature for each slice and conjectured this family foliates the Kruskal extension. In [9], the authors reformulated the TSS-CMC foliation conjecture and proved some partial results. In this paper, the author uses a totally different approach from [9] and completely answers the conjecture. The key observation is that the existence of the Dirichlet problem is equivalent that the TSS-CMC hypersurfaces family covers the Kruskal extension, and the uniqueness of the Dirichlet problem is equivalent that any two TSS-CMC hypersurfaces are disjoint. Thus the TSS-CMC foliation conjecture is proved by our main theorem. This Lorentzian geometric analysis approach avoids the subtle calculations appearing in [9], but some inevitable case-by-case arguments are still needed because the structure of the Kruskal extension is built by two Schwarzschild spacetimes.

The organization of this paper is as follows. We give a brief introduction to the Schwarzschild spacetime and Kruskal extension in section 2.1, and then summarize the main results of the initial value problem of the SS-CMC equation in the Kruskal extension in section 2.2. Properties of the Lorentzian distance function are discussed in section 2.3. All results in section 2 are used to solve the SS-CMC Dirichlet problem in section 3. In section 4, we prove the TSS-CMC foliation conjecture.

2. Preliminary

2.1. The Schwarzschild spacetime and the Kruskal extension

The Schwarzschild spacetime is a four-dimensional time-oriented Lorentzian manifold with metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2,$$

where $M > 0$ is a constant. After coordinates change, the Schwarzschild metric can be written as
\[ ds^2 = \frac{16M^2 e^{-\frac{r}{2M}}}{r} (-dT^2 + dX^2) + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 = \frac{16M^2 e^{-\frac{r}{2M}}}{r} dUdV + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2, \]

where

\[
\begin{align*}
(r - 2M) e^{\frac{r}{2M}} &= X^2 - T^2 = VU \\
\frac{1}{2M} &= \ln \left| \frac{X + T}{X - T} \right| = \ln \left| \frac{V}{V'} \right|.
\end{align*}
\]

From (1), we know that \( r = 2M \) is only a coordinate singularity. The Schwarzschild spacetime has a maximal analytic extension, called the Kruskal extension. It is the union of regions \( I, II, I', \) and \( II' \), where regions \( I \) and \( II \) correspond to the exterior and the interior of one Schwarzschild spacetime, respectively, and regions \( I' \) and \( II' \) correspond to the exterior and interior of another Schwarzschild spacetime. Figure 1 indicates their correspondences and coordinate systems \( TX \) or \( UV \).

Sometimes we will use other null coordinates \((u, v)\) by

\[
u = t - (r + 2M \ln|r - 2M|) \quad \text{and} \quad v = t + (r + 2M \ln|r - 2M|),\]

and relations between \((U, V)\) and \((u, v)\) are given by

\[
\begin{array}{c|c|c|c|c}
\text{region} & I & II & I' & II' \\
\hline
U & e^{\frac{r}{2M}} & -e^{\frac{r}{2M}} & -e^{\frac{r}{2M}} & e^{\frac{r}{2M}} \\
V & e^{\frac{r}{2M}} & e^{\frac{r}{2M}} & -e^{\frac{r}{2M}} & -e^{\frac{r}{2M}}.
\end{array}
\]

In this article, we will take \( \partial_T \) as a future-directed timelike vector field in the Kruskal extension. Note that \( \partial_T \) in two Schwarzschild spacetimes has different directions and it is shown in figure 1. Once \( \partial_T \) is chosen, for a spacelike hypersurface \( \Sigma \), we will choose the normal vector \( \vec{n} \) of \( \Sigma \) as future-directed in the Kruskal extension, and the mean curvature \( H \) of \( \Sigma \) is defined by \( H = \frac{1}{2} \delta^{ij} \langle \nabla_i \vec{n}, e_j \rangle \), where \( \{e_i\}_{i=1}^{3} \) is a basis on \( \Sigma \).

### 2.2. SS-CMC hypersurfaces in the Kruskal extension

In this subsection, we will summarize the results in papers [8, 9] about spherically symmetric, spacelike constant mean curvature (SS-CMC) hypersurfaces in the Schwarzschild spacetime.
and Kruskal extension. These formulae and arguments will be used to prove the SS-CMC Dirichlet problem in section 3.

In papers [8, 9], we considered the initial value problem of the SS-CMC equation in the Schwarzschild spacetime and Kruskal extension. We first studied SS-CMC hypersurfaces in the standard Schwarzschild coordinates. In most of cases, an SS-CMC hypersurface $\Sigma$ can be locally written as a graph $(t = f(r), r, \theta, \phi)$ in either Schwarzschild exterior or interior. Direct computation gives the SS-CMC equation, which is a second-order ordinary differential equation:

$$f'' + \left(1 - (f')^2 h\right) \left(\frac{2h'}{r} + \frac{h'}{h} + \frac{h'}{h^{1/2}}\right) f' \pm 3H \left(1 - (f')^2 h\right) = 0,$$  \hspace{1cm} (4)

where $h(r) = 1 - \frac{2m}{r}$, $H \in \mathbb{R}$ is the CMC, and the spacelike condition is equivalent to $\frac{1}{h} - (f')^2 h > 0$. The choice of $\pm$ signs in the equation (4) depends on different regions that cause different directions of the normal vector (as figure 1 shows) and different pieces of an SS-CMC hypersurface. In each region and where $f(r)$ is defined, the solution $f(r)$ is uniquely determined by two constants of integration, denoted by $c$ and $\tilde{c}$, where $c$ controls the slope of the function of $f(r)$ and $\tilde{c}$ represents the $t$-direction translation. Thus, if we require an SS-CMC hypersurface $\Sigma_{d,e,c}$ to pass through a point $(t_0, r_0)$, then the constant $\tilde{c}$ is determined.

All solutions of the SS-CMC equation can be expressed in the integration form, which are completely discussed in [8, 9]. Because of symmetry, only parts of expressions of $\Sigma_{d,e,c}$ are needed in this paper, and we summarize these formulae below.

(A) For $\Sigma_{d,e,c}$ in the Schwarzschild exterior (maps to region $I$)

$$f(r; H, c, \tilde{c}) = \int_{r_1}^{r} \frac{l(x; H, c)}{h(x) \sqrt{1 + f'(x; H, c)}} \, dx + \tilde{c},$$

where $l(r; H, c) = \frac{1}{\sqrt{h(r)}} \left(Hr + \frac{c}{r}\right)$, and $r_1 \in (2M, \infty)$ is a fixed number.

If $c < -8M^2H$, then $\lim_{r \to 2M} f(r) = \infty$ because $f'(r) = -\frac{1}{r(h(r))} + \tilde{f}'(r)$, where

$$\tilde{f}'(r) = \frac{\left(Hr^3 + c\right)^2 + r^3(r - 2M) - \left(Hr^3 + c\right)\sqrt{(Hr^3 + c)^2 + r^3(r - 2M)}}{r^4}.$$  \hspace{1cm} (5)

If $c > -8M^2H$, then $\lim_{r \to 2M} f(r) = -\infty$ because $f'(r) = -\frac{1}{h(r)} + \tilde{f}'(r)$, where

$$\tilde{f}'(r) = -\frac{\left(Hr^3 + c\right)^2 + r^3(r - 2M) + \left(Hr^3 + c\right)\sqrt{(Hr^3 + c)^2 + r^3(r - 2M)}}{r^4}.$$  \hspace{1cm} (6)

If $c = -8M^2H$, then $\lim_{r \to 2M} f(r)$ is finite with order $f'(r) = O((r - 2M)^{-\frac{1}{2}})$.

(B) For $\Sigma_{d,e,c}$ in the Schwarzschild interior (maps to region $I'$)

$$f(r; H, c, \tilde{c}) = \int_{r_2}^{r} \frac{l(x; H, c)}{\pm h(x) \sqrt{f'(x; H, c) - 1}} \, dx + \tilde{c},$$

where $l(r; H, c) = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c}{r}\right) > 1$, and $r_2$ is in the domain of $f(r)$.

If $f'(r) > 0$, then $\lim_{r \to 2M} f(r) = \infty$ since $f'(r) = -\frac{1}{h(r)} + \tilde{f}'(r)$, where $\tilde{f}'(r)$ is (5).

(C) For $\Sigma_{d,e,c}$ in the Schwarzschild interior (maps to region $I''$), then

$$f(r; H, c, \tilde{c}) = \int_{r_1}^{r} \frac{l(x; H, c)}{\pm h(x) \sqrt{f'(x; H, c) - 1}} \, dx + \tilde{c},$$
where $l(r; H, c) = \frac{1}{\sqrt{-H(r)}} \left( H(r) + \frac{c}{r} \right) > 1$, and $r_4$ is in the domain of $f(r)$.

If $f'(r) < 0$, then $\lim_{r \to 2M} f(r) = -\infty$ since $f'(r) = \frac{1}{H(r)} + \hat{f}'(r)$, where $\hat{f}'(r)$ is (6). (D) If $c < -8M^2H$, we can find $\hat{c}$ such that hypersurfaces in region $I$ and $\bar{I}$ can be smoothly glued together at $U = 0$ ($r = 2M$ and $t = \infty$).

If $c > -8M^2H$, we can find $\hat{c}$ such that hypersurfaces in region $I$ and $\bar{I}'$ can be smoothly glued together at $V = 0$ ($r = 2M$ and $t = -\infty$).

If $c = -8M^2H$, we can find $\hat{c}$ such that hypersurfaces in region $I$ and $\bar{I}'$ can be smoothly glued together at $(U, V) = (0, 0)$ ($r = 2M$ and $t$ is finite).

Only constant slices $r = r_0$, $r_0 \in (0, 2M)$ are SS-CMC hypersurfaces which can not be written as a graph of the form $t = f(r)$. These hypersurfaces are called cylindrical hypersurfaces and each hypersurface has CMC

$$H(r_0) = \frac{2r_0 - 3M}{3\sqrt{r_0^3(2M - r_0)}}$$ in region $\bar{I}$, or $H(r_0) = \frac{3M - 2r_0}{3\sqrt{r_0^3(2M - r_0)}}$ in region $\bar{I}'$.

The behavior of an SS-CMC hypersurface $\Sigma_{H, c, c}$ in the Kruskal extension highly depends on the constant of integration $c$. The followings we will describe the effect of $c$. When we study the domain of $f(r)$ in Schwarzschild interior $I$ or $I'$, the condition will be $l = \frac{1}{\sqrt{-H(r)}} (-H(r) - \frac{c}{r^2}) > 1$ or $l = \frac{1}{\sqrt{-H(r)}} (H(r) + \frac{c}{r^2}) > 1$, so it is natural to define two functions $k_H(r)$ and $\hat{k}_H(r)$ on $(0, 2M)$ by

$$k_H(r) = -H r^3 - r^2 (2M - r)^{\frac{3}{2}}$$ and $\hat{k}_H(r) = -H r^3 + r^2 (2M - r)^{\frac{3}{2}}$.

Thus for each $c$, the domain of $f(r)$ contains the preimage of $c < k_H(r)$ or $c > \hat{k}_H(r)$.

Figure 2(a) shows graphs of $k_H(r)$ and $\hat{k}_H(r)$ in $H > 0$ case. Cases $H = 0$ or $H < 0$ are similarly discussed. Denote $r_H$, $c_H$ by the minimum point of $k_H(r)$ and $(R_H, C_H)$ by the maximum point of $\hat{k}_H(r)$. We use $k_H(r)$, $\hat{k}_H(r)$, $\hat{k}_H(r)$, and $\hat{k}_H(r)$ to represent their increasing or decreasing part. Notice that both $r = r_H$ and $r = R_H$ are cylindrical SS-CMC hypersurfaces with CMC $H$ in region $I$ and $\bar{I}'$, respectively. They correspond to hyperbolas in the Kruskal extension, and thus they divide the Kruskal extension into three regions, called the top region, the middle region, and the bottom region, as figure 2(b) shows.
Now we summarize results of the initial value problem of the SS-CMC equation in the Kruskal extension. Given \( H \in \mathbb{R} \), for \( (T_0, X_0) \) in the middle region shown in figure 3, and for \( c \in \mathbb{R} \), there exists a unique SS-CMC hypersurface \( \Sigma_{H,c} \) passing through \( (T_0, X_0) \). There are three values \( c = c_H, c = -8M^3H, \) and \( c = C_H \) such that the value \( c \) in different interval determines different behavior of the SS-CMC hypersurface \( \Sigma_{H,c} \), which is also illustrated in figure 3. Notice that for \( c \in (c_H, C_H) \), when we solve \( c = k_H^+(r) \) or \( c = \tilde{k}_H^-(r) \), the solution is denoted by \( r = r_{H,c} \), and \( (t_{H,c} = f(t_{H,c}), r_{H,c}) \) is called the ‘throat’ of the SS-CMC hypersurface \( \Sigma_{H,c} \). Remark that for \( c \in (c_H, C_H) \), \( \Sigma_{H,c} \) is defined for \( r \geq r_{H,c} \). Since each \( \Sigma_{H,c} \) is diffeomorphic to \( I \times S^2 \), where \( I \subset \mathbb{R} \), it implies the smallest radius of the Schwarzschild coordinates sphere in \( \Sigma_{H,c} \). Furthermore, every SS-CMC hypersurface \( \Sigma_{H,c} \), \( c \in (c_H, C_H) \) is symmetric with respect to \( t = t_{H,c} \) in the Schwarzschild coordinates. See the discussion in [8, proposition 2.6].

For \( (T_0, X_0) \) in the top region (or the bottom region), as figure 4, denote \( (t_0, r_0) \) by its Schwarzschild coordinates, then for \( c \leq k_H^+(r_0) \) (or \( c \geq \tilde{k}_H^-(r_0) \)), we can find an SS-CMC hypersurface \( \Sigma_{H,c} \) with \( f'(r_0) < 0 \) (or \( f'(r_0) > 0 \)) in the Schwarzschild interior passing through \( (T_0, X_0) \). There is also an SS-CMC hypersurface \( \Sigma_{H,c} \) with \( f'(r_0) > 0 \) (or \( f'(r_0) < 0 \)) in the Schwarzschild interior passing through \( (T_0, X_0) \), but it is not illustrated in figure 4.

Figure 3. SS-CMC initial value problem when \( (T_0, X_0) \) is in the middle region.

Figure 4. SS-CMC initial value problem when \( (T_0, X_0) \) is in the top or bottom region.
2.3. The Lorentzian distance function

In this subsection, we will summarize some properties of the Lorentzian distance function from a fixed achronal spacelike hypersurface. Our goal is to quote the Laplacian of the distance function (7), and the estimate (8) is crucial to prove the uniqueness of the Dirichlet problem of the SS-CMC equation in section 3.

Here we focus on the spacetime case. Remark that these results hold in general n-dimensional Lorentzian manifold. We recommend [1, 7] for more details.

Let $N^0$ be an achronal spacelike hypersurface in a spacetime $(L^4, \langle \cdot , \cdot \rangle)$. Given $p, q \in L$, define the Lorentzian distance function $d(p, q) : L \times L \to [0, \infty]$ as follows:

(a) If $q \in J^+(p)$, then $d(p, q)$ is the supremum of the Lorentzian lengths of all the future-directed causal curves from $p$ to $q$.

(b) If $q \not\in J^+(p)$, then $d(p, q) = 0$.

Notice that the Lorentzian distance function is not symmetric, that is, $d(p, q) \neq d(q, p)$. For $q \in L$, we can further define the Lorentzian distance function with respect to $N$ $d_N : L \to [0, \infty]$ as $d_N(q) = \sup_{p \in N} d(p, q)$.

Let $\nu$ be the future-directed unit timelike normal vector field of $N$. Define a function $s_N : N \to [0, \infty]$ by $s_N(p) = \max \{ t \geq 0 : d_N(\gamma(t)) = t \}$, where $\gamma(t)$ is the future inextendible geodesic such that $\gamma(0) = p$ and $\gamma'(0) = \nu$. In general, the Lorentzian distance function $d_N$ is not smooth for every point in $L$, but it is smooth in a sufficiently near chronological future of $N$, called $\tilde{I}^+(N) = \exp_N(\text{int} \tilde{I}^+(N))$, where $\tilde{I}^+(N) = \{ \nu \in N^0 : \text{for all } p \in N \text{ and } 0 < t < s_N(p) \}$, and where $\exp_N$ is the exponential map with respect to $N$. See [1, lemma 5.1], or [7, proposition 3.6].

If we restrict the Lorentzian distance function $d_N$ on a spacelike hypersurface $\Sigma$, which denotes $d_{N|\Sigma} : \Sigma \to [0, \infty]$, by the decomposition of ambient spacetime connection into the submanifold connection and the second fundamental form (or shape operator), relations between $\text{Hess}(d_{N|\Sigma})$ and $\overline{\text{Hess}}(d_N)$ will be

$$\text{Hess}(d_{N|\Sigma})(X, X) = \overline{\text{Hess}}(d_N)(X, X) - \sqrt{1 + |\nabla(d_{N|\Sigma})|^2} \langle A_{N|\Sigma}(X), X \rangle,$$

where $A_{N|\Sigma}$ is the shape operator of $\Sigma$ with respect to $L$. After taking trace on the basis of $\Sigma$, we get for all $q \in \Sigma$,

$$\Delta_{\Sigma}(d_{N|\Sigma})(q) = \overline{\Delta} d_N(q) + \overline{\text{Hess}}(d_N)(q)(\nu, \nu) + 3H_{\Sigma}(q) \sqrt{1 + |\nabla(d_{N|\Sigma})(q)|^2}.$$  \hspace{1cm} (7)

Detailed discussions of (7) can be found in [1, p 5091].

Next, if a spacetime $L$ satisfies the timelike convergence condition, which means $\text{Ric}(Z, Z) \geq 0$ for every unit timelike vector $Z$, then we have $\overline{\Delta} d_N(q) \geq -3H_N(p)$, where $p$ is the orthogonal projection of $q$ on $N$. The complete argument of getting this inequality is in [1, lemma 5.7] and here we briefly state the main idea.

By definition, $\overline{\Delta} d_N$ is the trace of the Hessian of the distance function, and it can be expressed in terms of the normal $N$-Jacobi field $J$, curvature $R$ of the spacetime $L$, and the shape operator $A_N$.

This expression will induce the concept of the index form of the geodesic with respect to $N$, and the normal $N$-Jacobi field will maximize the index form in Lorentzian case. Thus the inequality holds because the timelike convergence condition implies the curvature term is non-negative, and the shape operator term is $-3H_N(p)$.

To sum up, we have the following proposition:

**Proposition 1** [1, proposition 5.9]. Let $L$ be a spacetime such that $\text{Ric}(Z, Z) \geq 0$ for every unit timelike vector $Z$, and let $N$ be an achronal spacelike hypersurface with $\tilde{I}^+(N) \neq \emptyset$. 

7
Suppose that a spacelike hypersurface $\Sigma$ satisfies $\Sigma \subset \mathcal{I}'(N)$. Denote $d_{N}|_{\Sigma}$ by the Lorentzian distance function with respect to $N$ on $\Sigma$. Then
\begin{equation}
\Delta_{\Sigma}(d_{N}|_{\Sigma})(q) \geq \text{Hess}(d_{N}(q))(\nu, \nu) + 3H_{\Sigma}(q)\sqrt{1 + |\nabla (d_{N}|_{\Sigma})(q)|^2} - 3H_{\Sigma}(p),
\end{equation}
where $\nu$ and $H_{\Sigma}$ are the future-directed unit timelike vector field and the mean curvature of $\Sigma$, respectively, $H_{N}$ is the mean curvature of $N$ along the orthogonal projection of $\Sigma$ on $N$, and $p$ is the orthogonal projection of $q$ on $N$.

Finally, the following proposition states that the longest timelike geodesic between two spacelike hypersurfaces will perpendicular to both hypersurfaces.

**Proposition 2.** Let $N$ and $\Sigma$ be two submanifolds of $L$ such that $\Sigma \subset \mathcal{I}'(N)$, and let $\gamma : [0, 1] \to L$ be a future-directed timelike geodesic such that $\gamma(0) \in N$, $\gamma(1) \in \Sigma$ and $\gamma$ is the longest curve from $N$ to $\Sigma$. Then $\gamma'(0)$ is perpendicular to $N$ and $\gamma'(1)$ is perpendicular to $\Sigma$.

Remark that this perpendicular property is also true for shortest geodesics in Riemannian case. See [4, proposition 1.5, pp 6–7] for the discussion.

3. Dirichlet problem for SS-CMC equation

3.1. Setting the SS-CMC Dirichlet problem

Let $\Sigma : (T = F(X), X, \theta, \phi)$ be an SS-CMC hypersurface in the Kruskal extension. In [8], we have computed the SS-CMC equation:
\begin{equation}
F''(X) + e^{-\frac{TM}{2}}\left(\frac{6M}{r^2} - \frac{1}{r}\right)(-F(X) + F'(X)X)(1 - (F'(X))^2) + \frac{12HM}{r}e^{-\frac{TM}{2}} (1 - (F'(X))^2)^{\frac{3}{2}} = 0,
\end{equation}
where the spacelike condition is equivalent to $1 - (F'(X))^2 > 0$, and where $H$ is the CMC. Remark that $r = r(T, X) = r(F(X), X)$ satisfies the equation (2), namely, $(r - 2M)e^{\frac{TM}{2}} = X^2 - T^2 = X^2 - (F(X))^2$; spherically symmetric condition means that the function $T = F(X)$ is independent of $\theta$ and $\phi$.

We can formulate the SS-CMC Dirichlet problem as follows:

(*) Dirichlet problem for the SS-CMC equation with symmetric boundary data. Given $H \in \mathbb{R}$ and boundary data $(T_0, X_0, \theta, \phi), (T_0, -X_0, \theta, \phi)$ in the Kruskal extension, does there exist a unique hypersurface $\Sigma : (T = F(X), X, \theta, \phi)$ satisfying the SS-CMC equation (9), the spacelike condition $1 - (F'(X))^2 > 0$, and the boundary value condition $F(X_0) = F(-X_0) = T_0$?

Since only SS-CMC hypersurfaces are considered in this paper, we write the $T$-$X$ coordinates $(T, X)$ instead of $(T, X, \theta, \phi)$ for convenience in the following paragraphs.
3.2. Existence of the SS-CMC equation

**Theorem 3.** Dirichlet problem for the SS-CMC equation with symmetric boundary data (*) is solvable.

The idea to prove theorem 3 is the shooting method. Take boundary data \((T_0, X_0)\) and \((-T_0, -X_0)\) in the middle region for example, and see figure 5. Consider the family of SS-CMC hypersurfaces \(\Sigma_{t, \epsilon, c}\) passing through \((T_0, X_0)\), where \(c \in (c_H, C_H)\), then \(\{\Sigma_{t, \epsilon, c}\}\) is continuously varied with respect to \(c\). We will detect the position of the ‘throat’ \((t_{H, \epsilon, c}, r_{H, \epsilon, c})\) for each \(\Sigma_{t, \epsilon, c}\). Since \(\Sigma_{t, \epsilon, c}\) is symmetric about \(t = t_{H, \epsilon, c}\) in the standard Schwarzschild coordinates, by studying the limit behavior as \(c \rightarrow c_H\) and \(c \rightarrow C_H\) and by the intermediate value theorem, we can find some \(\epsilon'\) such that \(t_{H, \epsilon', c} = 0\), which implies the symmetric axis is the \(T\)-axis in the Kruskal extension. Thus \(\Sigma_{t, \epsilon', c}\) will pass through the other boundary data \((-T_0, -X_0)\).

**Proof.** Case 1: For \((T_0, X_0)\) and \((-T_0, -X_0)\) in the region between \(r = r_H\) and \(r = R_H\), denote \((t_0, r_0)\) by the standard Schwarzschild coordinates of \((T_0, X_0)\). For \(c \in (c_H, C_H)\), consider the following curve in the Kruskal extension (in null coordinates):

\[
\alpha(c) = (U(c), V(c)) = (U(t_{H, \epsilon, c}, r_{H, \epsilon, c}), V(t_{H, \epsilon, c}, r_{H, \epsilon, c})),
\]

where

\[
U(c) = \begin{cases} 
-e^{-\frac{1}{M}(t_0-2r_{H, \epsilon, c}-4M \ln |r_{H, \epsilon, c}+2M|+r_0+2M \ln |r_0-2M|+\int_{r_0}^{r_{H, \epsilon, c}} f'(x, c) \, dx) } & \text{in region } \mathbb{I} \\
e^{-\frac{1}{M}(t_0-2r_{H, \epsilon, c}-2M \ln |r_{H, \epsilon, c}+2M|+t_0+2M \ln |r_0-2M|+\int_{r_0}^{t_{H, \epsilon, c}} f'(x, c) \, dx) } & \text{in region } \mathbb{I}', \\
e^{-\frac{1}{M}(t_0+2r_{H, \epsilon, c}+4M \ln |r_{H, \epsilon, c}-2M|-r_0+2M \ln |r_0-2M|+\int_{r_0}^{t_{H, \epsilon, c}} f'(x, c) \, dx) } & \text{in region } \mathbb{I} \\
e^{-\frac{1}{M}(t_0+2r_{H, \epsilon, c}+4M \ln |r_{H, \epsilon, c}-2M|-r_0+2M \ln |r_0-2M|+\int_{r_0}^{t_{H, \epsilon, c}} f'(x, c) \, dx) } & \text{in region } \mathbb{I}', 
\end{cases}
\]

and where \(f'\) and \(f^{'}\) are (5) and (6), respectively. The curve \(\alpha(c)\) will record the position of the throat \((t_{H, \epsilon, c}, r_{H, \epsilon, c})\) for each SS-CMC hypersurface \(\Sigma_{t, \epsilon, c}\) which passes through \((T_0, X_0)\) in the
Kruskal extension. Remark that an SS-CMC hypersurface is symmetric to \( t = t_{H,c} \) in the standard Schwarzschild coordinates.

Because of the relation \( \frac{V(c')}{U(c')} \bigg|_{x(c')} = -e^{\frac{\pi}{2}H_c} \) from (2), we will claim that \( \frac{V(c')}{U(c')} = -1 \) for some \( c' \in (c_H, c_H) \), or \( \lim_{c \to c_H} V(c') \frac{V(c)}{U(c)} = -1 \), which means the symmetric axis of \( \Sigma_{H,c} \) is the \( T \)-axis in the Kruskal extension. By symmetry, the hypersurface \( \Sigma_{H,c} \) must pass through \( (T_0, -X_0) \), and thus we prove the existence of the Dirichlet problem (\( \ast \)).

To prove the claim, since \( f'(x; c) \) and \( \tilde{f}'(x; c) \) have different order behaviors when \( c \to -8M^3H \), we need to take care of them by the following arguments. From \( (2) \), we know that the curve \( \alpha(c) \) lies in region \( \mathbb{I} \mathbb{I} \). Denote

\[
\alpha(c) = \int_{r_{H,c}} f'(x; c) \, dx.
\]

Using relation \( (2) \) will imply

\[
\frac{V(c)}{U(c)} \bigg|_{x(c)} = -e^{\frac{\pi}{2}H_c} = -\left( e^{\frac{\pi}{2}H_c} \left( e^{\frac{\pi}{2}H_c} - \tilde{t}_{H,c} \right) \right).
\]

When \( c \to -8M^3H \), we have

\[
\lim_{c \to -8M^3H} e^{\frac{\pi}{2}H_c} = \lim_{c \to -8M^3H} e^{\frac{\pi}{2}H_c} \left( f(\alpha +) + \int_{\alpha} f'(x; c) \, dx \right) = e^{\frac{\pi}{2}H_c} \left( f(\alpha +) + \int_{\alpha} f'(x; -8M^3H) \, dx \right).
\]

The limit exists because the function \( f'(x; -8M^3H) \) is of order \( O((x - 2M)^{-1}) \) (See (A) in section 2.2). Next, we investigate the other limit:

\[
\lim_{c \to -8M^3H} e^{\frac{\pi}{2}H_c} = \lim_{c \to -8M^3H} e^{\frac{\pi}{2}H_c} \left( f(\alpha +) + \int_{\alpha} f'(x; c) \, dx \right)
\]

Since \( f'(x; c) \) can be rewritten as \( f'(x; c) = \frac{x^2}{(x - r_{H,c})^2P(x)} \), where \( P(x; c) > 0 \) is a smooth function, let \( y = x - r_{H,c} \), we get

\[
0 \leq \int_{r_{H,c}} f'(x, c) \, dx = \int_0^{2(2M - r_{H,c})} \frac{y^2P(y + r_{H,c}; c)}{(y + r_{H,c})^2} \, dy \leq \int_0^{2(2M - r_{H,c})} \frac{M}{y^2} \, dy = 4M(2M - r_{H,c}) \to 0 \quad \text{as} \quad c \to -8M^3H.
\]

Hence the limit

\[
\lim_{c \to -8M^3H} \frac{V(c)}{U(c)} \bigg|_{x(c)} = -e^{\frac{\pi}{2}H_c} \left( f(\alpha +) + \int_{\alpha} f'(x; -8M^3H) \, dx \right)
\]

exists. We denote the limit by \( L_0 \). In addition, from the limit behavior of SS-CMC hypersurfaces in paper [8], we know

\[
\lim_{c \to -2} \frac{V(c)}{U(c)} \bigg|_{x(c)} = 0.
\]

If \( c > -8M^3H \), we know that the curve lies in region \( \mathbb{I} \mathbb{I} \), and we get

\[
\frac{V(c)}{U(c)} = -e^{\frac{\pi}{2}H_c} = -e^{\frac{\pi}{2}H_c} \left( \tilde{t}_{H,c} + 2M \ln |\tilde{t}_{H,c} - 2M| - \tilde{t}_{H,c} - 2M |\tilde{t}_{H,c} - 2M| \right) + \int_{\alpha} f'(x; c) \, dx.
\]
Similar discussion gives the result that

\[
\lim_{c \to -8M^3 H} \frac{V(c)}{U(c)} \bigg|_{\alpha(c)} = -e^{\frac{3}{2M}} \left( f'(r_0) + \int_{r_0}^{2M} f'(c; -8M^3 H) \, dc \right) = L_0.
\]

Furthermore, from the limit behavior of SS-CMC hypersurfaces in paper [8], we know

\[
\lim_{c \to C_H} \frac{V(c)}{U(c)} \bigg|_{\alpha(c)} = -\infty.
\]

In conclusion, by the intermediate value theorem, we get

1. If \( L_0 < -1 \), there exists \( c' \in (c_H, -8M^3 H) \) such that \( \frac{V(c')}{U(c')} = -1 \).
2. If \( L_0 = -1 \), then \( c' = -8M^3 H \) satisfies \( \lim_{c \to c'} \frac{V(c)}{U(c)} = -1 \).
3. If \( L_0 > -1 \), there exists \( c' \in (-8M^3 H, C_H) \) such that \( \frac{V(c')}{U(c')} = -1 \).

Case 2: For \((T_0, X_0)\) and \((T_0, -X_0)\) in the region between \( r = r_H \) and \( r = 0 \), denote \((T_0, X_0)\) by the standard Schwarzschild coordinates of \((T_0, X_0)\). Let \( c_0 = -Hr_0^3 - r_0^2 (2M - r_0)^{\frac{3}{2}} \). For \( c \in (c_H, c_0) \), the curve

\[
\alpha(c) = (U(c), V(c)) = \left( -e^{\frac{3}{2M}} \left( \int_{r_0}^{r_0} f'(c; c) \, dc - r_0 - 2M \ln|n_0 - 2M| \right), e^{\frac{3}{2M}} \left( \int_{r_0}^{r_0} f'(c; c) \, dc + r_0 + 2M \ln|n_0 - 2M| \right) \right)
\]

will record the position of the throat \((T_H, r_H, c)\) of the SS-CMC hypersurface \( \Sigma_{H, c} \) which passes through \((T_0, X_0)\) in null coordinates. Since \( \lim_{c \to c_0} \frac{V(c)}{U(c)} = -e^{\frac{3}{2M}} \ln|n_0 - 2M| < -1 \) and \( \lim_{c \to c_0} \frac{V(c)}{U(c)} = 0 \), by the intermediate value theorem, there exists \( c' \in (c_0, c_H) \) such that \( \frac{V(c')}{U(c')} = -1 \).

Case 3: For \((T_0, X_0)\) and \((T_0, -X_0)\) in the region between \( r = R_H \) and \( r = 0 \), denote \((T_0, X_0)\) by the standard Schwarzschild coordinates of \((T_0, X_0)\). Let \( c_0 = -Hr_0^3 + r_0^2 (2M - r_0)^{\frac{3}{2}} \). For \( c \in (c_0, C_H) \), the curve

\[
\alpha(c) = (U(c), V(c)) = \left( e^{-\frac{3}{2M}} \left( \int_{r_0}^{r_0} f'(c; c) \, dc - r_0 - 2M \ln|n_0 - 2M| \right), e^{-\frac{3}{2M}} \left( \int_{r_0}^{r_0} f'(c; c) \, dc + r_0 + 2M \ln|n_0 - 2M| \right) \right)
\]

will record the position of the throat \((T_H, r_H, c)\) of the SS-CMC hypersurface \( \Sigma_{H, c} \) which passes through \((T_0, X_0)\) in null coordinates. Since \( \lim_{c \to c_0} \frac{V(c)}{U(c)} = -e^{\frac{3}{2M}} \ln|n_0 - 2M| > -1 \) and \( \lim_{c \to c_0} \frac{V(c)}{U(c)} = -\infty \), by the intermediate value theorem, there exists \( c' \in (c_0, C_H) \) such that \( \frac{V(c')}{U(c')} = -1 \).

It is easy to find that solutions of the SS-CMC equation with symmetric boundary data are symmetric about the \( T \)-axis in the Kruskal extension.
Definition 4. We say an SS-CMC hypersurface $\Sigma : (T = F(X), X, \theta, \phi)$ in the Kruskal extension is $T$-axisymmetric if $F(-X) = F(X)$ for all $X$.

Theorem 5. All SS-CMC hypersurfaces satisfying the Dirichlet problem (*) are $T$-axisymmetric. That is, $F(-X) = F(X)$ for all $X$.

Proof. Suppose that an SS-CMC hypersurface $\Sigma$ satisfying (*) is not $T$-axisymmetric. We consider its $T$-axisymmetric reflection hypersurface, called $\Sigma$, then $\Sigma$ and $\Sigma$ have different parameter $c$ because these two hypersurfaces have different slopes at the boundary point. On the other hand, the $T$-axisymmetric reflection does not change the radius of the throat $r = r_{H, c}$, so $\Sigma$ and $\Sigma$ must have the same value $c$, and it leads to the contradiction.

3.3. Uniqueness of the SS-CMC equation

We will prove the uniqueness of the SS-CMC Dirichlet problem in this subsection. Before that, we need to find more properties about the SS-CMC hypersurfaces. Theorems 3 and 5 imply that for boundary data $(T_0, X_0)$ and $(T_0, -X_0)$, and for every $H \in \mathbb{R}$, there exists a $T$-axisymmetric SS-CMC (TSS-CMC) hypersurface $\Sigma_{H, c}$ satisfying the Dirichlet problem (*). Next theorem shows that these hypersurfaces $\Sigma_{H, c}$ are continuously varied with respect to the mean curvature $H$.

Theorem 6. For every $H \in \mathbb{R}$, denote $\Sigma_{H, c, (H)} : (T = F_{H}(X), X, \theta, \phi)$ an SS-CMC hypersurface satisfying the SS-CMC equation (9), the spacelike condition $1 - (F_{H}(X))^2 > 0$, and the boundary value condition $F_{H}(X_0) = F_{H}(-X_0) = T_0$. Then the family $\{\Sigma_{H, c, (H)}\}$ (the function $F_{H}(X)$) is continuously varied with respect to the mean curvature $H$.

Proof. Case 1: If $(T_0, X_0)$ and $(T_0, -X_0)$ lie between $r = r_H$ and $r = R_H$, denote $(t_0, r_0)$ by the standard Schwarzschild coordinates of $(T_0, X_0)$. We will define two functions $G(R, H)$ and $\tilde{G}(R, H)$ as follows: If $c \in (t_H, -8M^3H)$, let $R$ be the solution of $c = k_H(r)$; that is, $R$ satisfies $c = -HR^3 - R^2(2M - R)^2$. Define

$$G(R, H) = t_0 - R - 2M \ln |R - 2M| + r_0 + 2M \ln |r_0 - 2M| - \int_{R}^{t_0} f'(r; H, R) \, dr,$$

where $f'(r; H, R)$ is the equation (5). Similarly, if $c \in [-8M^3H, R_H)$, let $R$ be the solution of $c = \tilde{k}_H(r)$; that is, $R$ satisfies $c = -HR^3 + R^2(2M - R)^2$. Define

$$\tilde{G}(R, H) = t_0 + R + 2M \ln |R - 2M| - r_0 - 2M \ln |r_0 - 2M| - \int_{R}^{t_0} \tilde{f}'(r; H, R) \, dr,$$

where $\tilde{f}'(r; H, R)$ is the equation (6).

Both domains of $G(R, H)$ and $\tilde{G}(R, H)$ are a subset of $(0, 2M] \times \mathbb{R}$. The reason we consider these two functions is that we have the relation
Since theorem 3 shows that for every \( H \in \mathbb{R} \), there exists \( R = R(H) \) such that \( \bar{G}(R, H) = 0 \) or \( \tilde{G}(R, H) = 0 \), we get that the set \( (R, H) \) satisfying \( \bar{G}(R, H) = 0 \) or \( \tilde{G}(R, H) = 0 \) will correspond to TSS-CMC hypersurfaces \( \Sigma_{H_c(r H)} \).

We compute the partial derivative of the function \( G(R, H) \) with respect to \( H \):

\[
\frac{\partial G}{\partial H} = - \int_{r_0}^{r_0} \frac{\partial f'}{\partial H}(r; H, R) \, dr = - \int_{r_0}^{r_0} \frac{\partial f'}{\partial H}(r; H, R) \, dr
\]

\[
= - \int_{r_0}^{r_0} \frac{1}{h(r)} \left(1 + l^2(r; H, R)\right)^2 \, dr < 0.
\]

The last inequality holds because \( R < r_0 \) and \( \frac{\partial l}{\partial r} > 0 \) from relations (A) and (B) in section 2.2. By the implicit function theorem, for every \( (R, H) \) satisfying \( G(R, H) = 0 \), there exists an interval \( (R - \delta, R + \delta) \) such that the solution of \( G(R, H) = 0 \) in the interval can be written as a graph of a function \( H = H(R) \).

Similarly, we compute

\[
\frac{\partial G}{\partial H} = - \int_{r_0}^{r_0} \frac{\partial f'}{\partial H}(r; H, R) \, dr = - \int_{r_0}^{r_0} \frac{\partial f'}{\partial H}(r; H, R) \, dr
\]

\[
= - \int_{r_0}^{2M} \frac{1}{h(r)} \left(1 + l^2(r; H, R)\right)^2 \, dr - \int_{2M}^{r_0} \frac{1}{h(r)} \left(1 + l^2(r; H, R)\right)^2 \, dr < 0.
\]

The last inequality holds because of the signs of \( \frac{\partial l}{\partial r} \) from relations (A) and (C) in section 2.2. By the implicit function theorem, for every \( (R, H) \) with \( \tilde{G}(R, H) = 0 \), there exists an interval \( (R - \delta, R + \delta) \) such that the solution of \( \tilde{G}(R, H) = 0 \) in the interval can be written as a graph of a function \( H = H(R) \).

Case 2: If \( (T_0, X_0) \) and \( (T_0, -X_0) \) lie between \( r = r_H \) and \( r = 0 \) (between \( r = 0 \) and \( r = R_H \)), consider the function

\[\text{Figure 6. (a) and (b): Curves } \beta_1(R), \beta_2(R), \text{ and } \gamma_1 \cup \gamma_2(R) \text{ are solutions of } \bar{G}(R, H) = 0, \tilde{G}(R, H) = 0, \text{ and } G(R, H) = 0, \text{ respectively. Each point on the curves corresponds to a TSS-CMC hypersurface } \Sigma_{H_c(r H)}. \text{ (c) We map all curves in figures 6(a) and (b) to the } T-H \text{ plane.} \]
in region II (in region II'). Since \( \frac{\partial G}{\partial M} > 0 \) in region II \((\frac{\partial G}{\partial M} > 0 \) in region II'), by the implicit function theorem, for every \( RH \),

\[
(0,1)
\]

such that the solution of \( G(R, H) = 0 \) in the interval can be written as a graph of a function \( H = H(R) \).

Now we rephrase the result of theorem 6. We will study the set \( G(R, H) = 0, G(R, H) = 0, \) and \( G(R, H) = 0 \) by plotting their curves in the \( R – H \) plane. They are \( bb \), and \( g_2 \) in figures 6(a) and (b).

In figure 6(a), the dotted curve \( \alpha_1(R) \) is the graph of \( H(R) = \frac{2R - 3M}{3\sqrt{R^2(2M - R)}} \). Each point on \( \alpha_1(R) \) represents a cylindrical hypersurface with CMC \( H \) in region II. Curves \( \beta_1(R) \) and \( \gamma_1(R) \) are solutions of \( \tilde{G}(R, H) = 0 \) and \( G(R, H) = 0 \) respectively. Similarly, in figure 6(b), the dotted curve \( \alpha_2(R) \) is the graph of \( H(R) = \frac{3M - 2R}{\sqrt{R^2(2M - R)}} \). Curves \( \beta_2(R) \) and \( \gamma_2(R) \) are solutions of \( \tilde{G}(R, H) = 0 \) and \( G(R, H) = 0 \) respectively. Theorem 6 implies that each of curves \( \beta_1, \beta_2, \gamma_1, \) and \( \gamma_2 \) is the graph of some continuous function \( H = H(R) \). Furthermore, theorem 3 implies that \( bb \).

Consider a map \( P \) from these two \( R – H \) planes in figures 6(a) and (b) to the \( T – H \) plane in figure 6(c) by

\[
H \mapsto H \quad \text{and} \quad R \mapsto T = \begin{cases} 
- \sqrt{2M - R} e^{\frac{R}{2M}} & \text{if } R \in (0, 2M] \text{ in region II} \\
- \sqrt{2M - R} e^{\frac{R}{2M}} & \text{if } R \in (0, 2M] \text{ in region II'} 
\end{cases}
\]

Theorem 6 and previous discussion shows that each curve \( \beta_1 \cup \beta_2(T), \gamma_1(T), \) and \( \gamma_2(T) \) is the graph of some continuous function in the \( T – H \) plane. Through the mapping \( P, \) we are ready to prove the uniqueness of the SS-CMC Dirichlet problem (+), it is equivalent to prove that each curve \( \beta_1 \cup \beta_2(T), \gamma_1(T), \) and \( \gamma_2(T) \) is the graph of a ‘strictly decreasing’ function in the \( T – H \) plane.

Figure 7. Choose two hypersurfaces \( N \) and \( \Sigma \) in the increasing part of \( \beta_1 \cup \beta_2(T) \), and then find a contradiction to the nonuniqueness of the SS-CMC Dirichlet problem.

\[
G(R, H) = t_0 + \int_0^R f' r(e; c) \, dr \quad \text{with} \quad \frac{V(R, H)}{U(R, H)} = -e^{-\frac{t_0}{G(R, H)}}
\]
Theorem 7. The solution of the Dirichlet problem for SS-CMC equation with symmetric boundary data \( (*) \) is unique.

Proof. Here we prove the case of \( (T_0, X_0) \) and \( (T_0, -X_0) \) in the middle region, and other cases are proved similarly. Given \( H_0 \in \mathbb{R} \), suppose that there are two SS-CMC hypersurfaces \( \Sigma_1 : (T = F_1(X), X) \) and \( \Sigma_2 : (T = F_2(X), X) \) with CMC \( H_0 \) satisfying the Dirichlet problem \( (*) \), and suppose that \( F_1(X) < F_2(X) \) in the interval \( (-X_0, X_0) \). See figures 7(a) and (b), then the curve \( \beta_1 \cup \beta_2(T) \) in the \( T-H \) plane passes through \( (T_0 = F_1(0), H_0) \) and \( (T_2 = F_2(0), H_0) \), and \( \beta_1 \cup \beta_2(T) \) is no longer the graph of a strictly decreasing function in the \( T-H \) plane, which implies that there must be some increasing part on some interval \( I \subset [T_1, T_2] \). Then we can take two SS-CMC hypersurfaces \( N \) and \( \Sigma \) such that \( \Sigma \subset \mathcal{T}(N) \) in the region \( T \times [-X_0, X_0] \) in the Kruskal extension and \( H_N < H_C \), as figure 7(c) showed.

Consider \( d_{N|\Sigma} \) the Lorentzian distance function restricted on the spacelike hypersurface \( \Sigma \). Since \( N \neq \Sigma \), we know that the maximum value of \( d_{N|\Sigma} \) is positive and will achieve at some point \( q \), which is in the interior of \( \Sigma \cap T \times (-X_0, X_0) \) in the Kruskal extension. We apply the estimate \( (8) \) to \( N \) and \( \Sigma \) and get

\[
0 \geq \Delta_S(d_{N|\Sigma})(q) \geq \text{Hess}(d_{N}(q))(\nu, \nu) + 3H_C(q)\sqrt{1 + |\nabla(d_{N|\Sigma})(q)|^2} - 3H_N(p) = 3H_C(q) - 3H_N(p) > 0,
\]

which is a contradiction. Remark that \( \text{Hess}(d_{N}(q))(\nu, \nu) = 0 \) because of the perpendicular property in proposition 2. \( \square \)

4. Applications to CMC foliation conjecture

Malec and Murchadha in [11] constructed a family of \( T \)-axisymmetric SS-CMC (TSS-CMC) hypersurfaces in the Kruskal extension. Each TSS-CMC hypersurface in this family shares the same mean curvature \( H \). They conjectured this TSS-CMC family will foliate the Kruskal extension in [11] without rigorous mathematical proof.

From the viewpoint of the SS-CMC Dirichlet problem, we can prove the existence of the TSS-CMC foliation in the Kruskal extension by the following argument: Given \( H \in \mathbb{R} \), for all symmetric pairs \( (T_0, X_0) \) and \( (T_0, -X_0) \) in the Kruskal extension, we collect all TSS-CMC hypersurfaces with mean curvature \( H \) and pass through \( (T_0, X_0) \) and \( (T_0, -X_0) \). We denote this TSS-CMC family by \( \{ \Sigma_{\ell} \} \). Since \( (T_0, X_0) \) and \( (T_0, -X_0) \) are taken in the Kruskal extension, \( \{ \Sigma_{\ell} \} \) must cover the whole Kruskal extension. If any two hypersurfaces \( \Sigma_1, \Sigma_2 \in \{ \Sigma_{\ell} \} \) are not disjoint, then they must intersect at some symmetric pairs. However, the uniqueness of the SS-CMC Dirichlet problem implies \( \Sigma_1 = \Sigma_2 \). Therefore, we conclude the following theorem:

Theorem 8. The existence and uniqueness of the Dirichlet problem \( (*) \) is equivalent to the existence of TSS-CMC foliation in the Kruskal extension.

From theorem 8, we can conclude Malec and Murchadha’s CMC foliation conjecture in [11] is true. The statement of the theorem is the following.
Theorem 9. For any $H \in \mathbb{R}$, there exists a $T$-axisymmetric, spacelike, spherically symmetric, CMC hypersurfaces foliation in the Kruskal extension.

Finally, we remark that every TSS-CMC foliation can be changed as a SS-CMC foliation without $T$-axisymmetric property by the Lorentzian isometry.

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