An Approach To Potential Scattering

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Abstract

Recently developed time-independent bound-state perturbation theory is extended to treat the scattering domain. The changes in the partial wave phase shifts are derived explicitly and the results are compared with those of other methods.

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1 Introduction

In our previous papers [1-6] published recently, a time-independent novel perturbation theory has been developed in the bound state domain, which is non-perturbative, self-consistent and systematically improvable, and used to treat successfully significant problems in different fields of physics. Gaining confidence from these applications, we aim through the present work to show that similar techniques can also be used in the continuum.

In the next section we summarize the main ideas of our approach. The extension of the model for scattering states and the relationship to some other perturbation approaches are discussed in Section 3. The paper ends with a brief summary and concluding remarks.

2 The Model

Let us start with a brief introduction of the formalism to remind the compact form of the method, which would provide an easy access of the scheme to understanding of the treatments in the continuum. For the consideration of spherically symmetric potentials, the corresponding Schrödinger equation in the bound state domain for the radial wave function has the form ($\hbar = 2m = 1$)
\[ \frac{\Psi''(r)}{\Psi_n(r)} = [V(r) - E_n], \quad V(r) = \left[ V_0(r) + \frac{\ell(\ell + 1)}{r^2} \right] + \Delta V(r), \quad n = 0, 1, 2, \ldots, (1) \]

where \( V_0 \) is an exactly solvable unperturbed potential together with the angular momentum barrier while \( \Delta V \) is a perturbing potential. Expressing the wave function \( \Psi_n \) as a product

\[ \Psi_n(r) = \chi_n(r) \phi_n(r), \quad (2) \]

in which \( \chi_n \) is the known normalized eigenfunction of the unperturbed Schrödinger equation whereas \( \phi_n(r) \) is a moderating function corresponding to the perturbing potential. Substituting (2) into (1) yields

\[ \left( \frac{\chi''_n}{\chi_n} + \frac{\phi''_n}{\phi_n} + 2 \frac{\chi'_n}{\chi_n} \frac{\phi'_n}{\phi_n} \right) = V - E_n. \quad (3) \]

Instead of setting the functions \( \chi_n \) and \( \phi_n(r) \), we will set their logarithmic derivatives

\[ W_n = -\frac{\chi'_n}{\chi_n}, \quad \Delta W_n = -\frac{\phi'_n}{\phi_n} \quad (4) \]

which leads to

\[ \frac{\chi''_n}{\chi_n} = W_n^2 - W'_n = \left[ V_0(r) + \frac{\ell(\ell + 1)}{r^2} \right] - \varepsilon_n, \quad (5) \]

where \( \varepsilon_n \) is the eigenvalue of the exactly solvable unperturbed potential, and

\[ \left( \frac{\phi''_n}{\phi_n} + 2 \frac{\chi'_n}{\chi_n} \frac{\phi'_n}{\phi_n} \right) = \Delta W_n^2 - \Delta W'_n + 2W_n \Delta W_n = \Delta V(r) - \Delta \varepsilon_n \quad (6) \]

in which \( \Delta \varepsilon_n \) is the energy value for the perturbed potential leading to \( E_n = \varepsilon_n + \Delta \varepsilon_n \). If the whole potential, involving the perturbing piece \( \Delta V \), can be analytically solvable, then Eq.(1) through (5) and (6) reduces to

\[ (W_n + \Delta W_n)^2 - (W_n + \Delta W_n) = V - E_n, \quad (7) \]

which is known as the usual supersymmetric quantum mechanical treatment [7] in the literature.

However, if the whole potential has no analytical solution as the case considered in this Letter, which means Eq.(6) cannot be exactly solvable for \( \Delta W \), then one can expand the functions in terms of the perturbation parameter \( \lambda \),

\[ \Delta V(r; \lambda) = \sum_{N=1}^{\infty} \lambda^N \Delta V_N(r), \quad \Delta W_n(r; \lambda) = \sum_{N=1}^{\infty} \lambda^N \Delta W_{nN}(r), \quad \Delta \varepsilon_n(\lambda) = \sum_{N=1}^{\infty} \lambda^N \Delta \varepsilon_{nN} \quad (8) \]
where $N$ denotes the perturbation order. Substitution of the above expansion into Eq.(6) and equating terms with the same power of $\lambda$ on both sides yields up to for instance $O(\lambda^3)$

$$2W_n \Delta W_n - \Delta W'_n = \Delta V_1 - \Delta \varepsilon_{n1},$$
$$\Delta W_{n1}^2 + 2W_n \Delta W_{n2} - \Delta W'_{n2} = \Delta V_2 - \Delta \varepsilon_{n2}$$
$$2(W_n \Delta W_{n3} + \Delta W_{n1} \Delta W_{n2}) - \Delta W'_{n3} = \Delta V_3 - \Delta \varepsilon_{n3}$$  (9-11)

Eq.(6) and its expansion through Eqs.(9-11) give a flexibility for the easy calculations of the perturbative corrections to energy and wave functions for the $n$th state of interest through an appropriately chosen perturbed superpotential. It has been shown [1-6] that this feature of the present model leads to a simple framework in obtaining the corrections to all states without using complicated mathematical procedures.

### 3 Application to the scattering domain

It is well known that there are many scattering problems in which the interaction between the projectile and the target decomposes naturally into two parts ($V = V_0 + \Delta V$). This division is especially useful if the scattering wave function under the action one part can be obtained exactly ($V_0$), while the effect of the other ($\Delta V$) can be treated in some approximation as in the present formalism.

For simplicity, we here confine ourselves to $s-$wave scattering from a potential which is assumed that vanishes beyond a finite radius $R$. The associated total wavefunction behaves at large distances

$$\Psi(r) = \frac{1}{k} \sin(kr + \delta), \quad r \geq R,$$  (12)

where $\delta$ is the $s-$wave phase shift.

Our present treatment of scattering has concerned itself primarily with determining how the solutions of the free Schrödinger equation are affected by the presence of the interaction. Within the framework of the present formalism we suppose that the solutions of Eq.(5) are known, or are easily found, to give the corresponding phase shift $\delta_0$. Considering the expansion $\delta = \delta_0 + \lambda \delta_1 + \lambda^2 \delta_2 + ..., \,$ as in Eq.(8), we aim here to derive explicitly solvable and easily accessible expressions for the phase shift contributions at successive perturbation orders.

#### 3.1 First-order phase shift correction

Keeping in mind Eq.(12) and considering the discussion in Section 2, at the first perturbation order one has

$$(W + \lambda \Delta W_1) = -k \cot(kr + \delta_0 + \lambda \delta_1), \quad W_n = -\frac{\chi'}{\chi} = -k \cot(kr + \delta_0),$$  (13)
from where the superpotential relating to the perturbing interaction

$$\Delta W_1(r) = \frac{k \delta_1}{\sin^2(kr + \delta_0)}, \quad (14)$$

is obtained assuming that $\sin \lambda \delta_1 \approx \lambda \delta_1$ and $\cos \lambda \delta_1 \approx 1$.

In the second step, one needs to employ Eq. (9) to arrive at another expression for $\Delta W_1$. Rearranging the terms, $\Delta W_1' - 2W_1 \Delta W_1 = (\Delta \varepsilon_1 - \Delta V_1)$ and multiply both sides by the integrating factor $\exp(-2 \int_0^r W(z) dz)$, which is the square of the unperturbed wave function $\chi^2(r)$ through Eq.(4), one obtain

$$\frac{d}{dr} \left[ \chi^2(r) \Delta W_1(r) \right] = \chi^2(r)(\Delta \varepsilon_1 - \Delta V_1). \quad (15)$$

The integration, and the remove of $\Delta \varepsilon_1$ term due to the consideration of elastic scattering process here, yields

$$\Delta W_1(r) = -\frac{1}{\chi^2(r)} \int_0^r \chi^2(z) \Delta V_1(z) dz. \quad (16)$$

As $\chi = \frac{1}{k} \sin(kr + \delta_0)$ in the asymptotic region, comparison of Eqs.(14) and (16) reproduces the first-order change in the phase shift

$$\delta_1 = -k \int_0^\infty \chi^2(r) \Delta V_1(r) dr. \quad (17)$$

If necessary, the corresponding change in the wavefunction can easily be obtained by the substitution of Eq.(16) into (4), $\phi_1 = \exp(-\int \Delta W_1)$. For the reliability of the present expression obtained, Eq (17), one may compare it with that reproduced by other methods. For example, in the limiting case where the unperturbed potential vanishes, the unperturbed $s$–wave function is reduced to a plane wave $\chi(r) = \sin(kr)/k$, and the first-order change in the phase shift becomes

$$\delta_1 = -\frac{1}{k} \int_0^\infty \sin^2(kr) \Delta V_1(r) dr \quad (18)$$

which is just the first Born approximation for the phase shift [8]. In addition, the well known expression for $s$–wave scattering amplitude by the two-potential formula in scattering theory [8],

$$f_1 = -e^{2i\delta_0} \int_0^\infty \chi^2(r) \Delta V_1(r) dr \quad (19)$$

where the phase factor in front of the integration arises because of the standing wave boundary conditions, justifies once more our result since $f_1 = -e^{2i\delta_0} \delta_1/k$ and, equating this to the above equation leads immediately to Eq.(17).

The present result has a widespread applicability, which may also be used in the treatment of scattering length problems. At low-energy limit, the phase shift
is related to the scattering length $\delta_k \to 0 \to -ka$ where $a = a_0 + \lambda a_1 + \lambda^2 a_2 + ...$ may be expanded in a perturbation series similar to the phase shift. Outside the range of the potential, the unperturbed wave function behaves as $\chi \to (r - a_0)$. Thus, the first correction to the scattering length is

$$a_1 = \lim_{r \to \infty} \left[ \int_0^r (z - a_0)^2 \Delta V_1(z) dz \right]$$

which can be calculated for a given $\Delta V_1$. The scattering length has an important physical significance. In the low-energy limit only the $s$-wave makes a nonzero contribution to the cross section, so that the angular distribution of the scattering is spherically symmetric and the total cross section is $4\pi(a_0 + \lambda a_1 + ...)^2$. This is also exactly the result obtained in most textbooks for the low-energy scattering of a hard sphere of radius. Thus the scattering length is the effective radius of the target at zero energy.

As a last example, consider the case of the angular momentum barrier as the unperturbed potential $V_0 = \ell(\ell + 1)/r^2$ that produces $[r j_{\ell}(kr)]$ with a phase shift $\delta_0 = -\ell\pi/2$. For a trivial perturbation let us choose $\Delta V_1 = \lambda/r^2$, due to which the angular momentum is slightly perturbed $\ell \approx \ell + \lambda/(2\ell + 1) + O(\lambda^2)$. Therefore the phase shift correction at first-order is $\delta_1 = -\pi/2(2\ell + 1)$. Again, this exact result confirms the reliability of Eq.(17).

3.2 Second-order phase shift correction

To solve Eq.(10) for $\Delta W_2$ we mimic the preceding calculation. The integration factor is the same. In fact, examining Eqs.(9) and (10), the only difference is that the quantity $\Delta V_1 - \Delta \varepsilon_1$ is replaced by $\Delta V_2 - \Delta W_1^2 - \Delta \varepsilon_2$. As $\Delta \varepsilon_2$ term is zero due to the process of interest, $\Delta W_2$ is thus

$$\Delta W_2(r) = -\frac{1}{\chi^2(r)} \int_0^r \chi^2(z) \left[ \Delta W_1^2(z) - \Delta V_2(z) \right] dz.$$  

Bearing in mind that $\chi = \frac{1}{k} \sin(kr + \delta_0)$ for the region $r \geq R$, the second-order expansion in the superpotential similar to Eq.(13) provides another expression for $\Delta W_2$ which is

$$\Delta W_2(r) = k\delta_1^2 \frac{\cot(kr + \delta_0)}{\sin^2(kr + \delta_0)} + \frac{k\delta_2}{\sin^2(kr + \delta_0)}$$

Comparison of Eqs.(21) and (22), together with the substitution of (14) in (21), leads to an auxiliary function for the second order phase shift correction,

$$\delta_2(r) = -\frac{1}{k} \int_0^r \Delta V_2(z) \sin^2(kz + \delta_0) dz + k\delta_1^2 \int_0^r \frac{dz}{\sin^2(kz + \delta_0)} - \delta_1^2 \cot(kr + \delta_0).$$
where a singularity appears in the second integral at \( z = 0 \). This problem can be circumvented by replacing the lower limit of the integral with \( R \). Assuming \( \Delta V = \Delta V_1 \) as in realistic problems of nuclear physics, which means that \( \Delta V_2 = 0 \), the \( r \)-dependent phase shift correction in the second-order is given in the form of

\[
\delta_2(r) = \delta_1^2 \cot(kR + \delta_0) - 2\delta_1^2 \cot(kr + \delta_0). \tag{24}
\]

As an alternative treatment, which leads to a concrete comparison, one can go back to Eq.(21) and split \( \chi^2 \Delta W_1^2 \) term in two parts as \( (\chi^2 \Delta W_1)(\Delta W_1) \) allowing to invoke Eq.(16). In this case the comparison of the result with the expansion in (22) gives

\[
\delta_2 = -k \int_0^\infty \chi^2(r) \Delta V_1(r) dr \int_{R}^{r} \frac{dz}{\chi^2(z)} \left[ \int_{z}^{R} \chi^2(y) \Delta V_1(y) dy - \frac{\delta_1}{k} \right] + \delta_1^2 \cot(kR + \delta_0) \tag{25}
\]

which is in agreement with the work in \[10\]. In addition, the use of (17) in (24) transforms it into Eq. (25). Furthermore, the reader is reminded that the second Born approximation for the phase shift can be most easily derived using the variable phase equation approach \[11\],

\[
\delta_2 = 2k^2 \int_0^\infty \chi^2(r) \Delta V_1(r) \cot(kr) dr \int_0^r \chi^2(y) \Delta V_1(y) dy \tag{26}
\]

which, in the light of Eq. (15), is the same result as we find from Eq (25), by putting \( \delta_0 = 0 \). Higher order terms can also be evaluated in the same manner.

4 Concluding Remarks

The recently introduced time-independent perturbation theory has been successfully extended from the bound state region to the scattering domain. For the clarification, the work has been carried out with the consideration of \( s \)-wave scattering only. However, generalization of the formalism to higher partial waves in the scattering domain does not cause any problem. The inclusion of the centrifugal barrier contribution in the effective potential for instance leads to the replacement of the \( s \)-wave phase shift with \( \delta_\ell - \ell \pi/2 \) due to the related wave function \( \chi(r) = \sin(kr + \delta_\ell - \ell \pi/2)/k \) in the asymptotic region, supposing both the unperturbed and perturbed potentials vanish at a large \( r > R_1 \) which means that in the region \( R_1 < r \leq R \) there is then only the centrifugal barrier contribution. This inclusion requires simply to repeat the present calculations for the replacement in the phase shift.

It should be stress that, anything that can be achieved from the present formalism must also be obtainable from the works \[9,10\] in the literature. For instance, considering the bound state region, Bender’s formalism \[9\] can be simplified by introducing the auxiliary function \( F_N(r) \) such that the whole wave function
$\Psi_N(r) = \chi(r)F_N(r)$ where denotes the perturbation order. The first-order correction can then be written as $\frac{d}{dr}\left[\chi^2 \frac{dF}{dr}\right] = (\Delta V_1 - \Delta \varepsilon_1)\chi^2$ which corresponds exactly to the present treatment by Eq. (15) when we identify $\Delta W_1 = \frac{dF}{dr}$. The higher order calculations can be linked to ours in the similar manner. Whereas, the works of Milward and Wilkin [10] may be related to the present formalism in both domain, the bound and scattering region by making a relation between their probability density distributions/derivatives and our $\Delta W$ functions, such as $\Delta W_0 = -P_0'/2P_0$ at the zeroth order, $\Delta W_1 = (-P_1/2P_0)'$ at the first order and $\Delta W_2 = (-P_2/2P_0)'$ at the second order etc. Nevertheless, the present technique provides a clean and explicit route for the calculations without tedious and cumbersome integrals.

The energy variation of the scattering wave function and phase shift can also be studied by perturbing in the energy. We wish to stress that all these effects depend purely upon the perturbation and the unperturbed wave function; explicit knowledge of the unperturbed potential is not necessary. This exposition will be deferred to a later publication.

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