Optimal quadrature formulas of closed type in the space $L^2_2(0,1)$
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Abstract

It is discussed the problem on construction of optimal quadrature formulas in the sense of Sard in the space $L^2_2(0,1)$, when the nodes of quadrature formulas are equally spaced. Here the representations of optimal coefficients for any natural numbers $m$ and $N$ are found.

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Consider quadrature formulas of the form

$$\int_0^1 \varphi(x) dx \approx \sum_{\beta=0}^N C[\beta] \varphi[\beta]$$  (1)

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N C[\beta]\delta(x - h\beta),$$  (2)

where $C[\beta]$ are the coefficients of formula (1), $[\beta] = h\beta$, $h = 1/N$, $N = 1, 2, 3, ...$, the function $\varphi(x)$ belongs to the space $L^2_2(0,1)$. Norm of functions in the space $L^2_2(0,1)$ is determined by formula

$$\|\varphi(x)|L^2_2(0,1)\| = \left( \int_0^1 (\varphi^{(m)}(x))^2 dx \right)^{\frac{1}{2}}.$$

Below $[\beta] = h\beta$.

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Quadrature formulas of the form (1) is closed type, since the points $x = 0$ and $x = 1$ are the nodes of the formula.

The difference

$$((\ell(x), \varphi(x)) = \int_0^1 \varphi(x)dx - \sum_{\beta=0}^N C[\beta] \varphi[\beta]$$

is called the error of formula (1).

The problem of construction of optimal quadrature formulas (1) in the sense of Sard in the space $L_2^{(m)}(0, 1)$ consists of computation the quantity

$$\|\ell(x)\|_{L_2^{(m)}(0, 1)} = \inf_{C[\beta]} \sup_{\|\varphi(x)\| \neq 0} \frac{|(\ell(x), \varphi(x))|}{\|\varphi(x)\|_{L_2^{(m)}(0, 1)}}.$$  \hfill (3)

where $L_2^{(m)}(0, 1)$ is the conjugate space to the space $L_2^{(m)}(0, 1)$.

The coefficients $C[\beta]$ satisfying equality (3) (if there exist) are called optimal and are denoted as $\hat{C}[\beta]$. Formulas of the form (1) with coefficients $\hat{C}[\beta]$ are called optimal quadrature formulas in the sense of Sard.

The problem of construction of optimal quadrature formulas in the sense of Sard in different spaces are investigated by many mathematicians. In the space $L_2^{(m)}$ this problem was investigated by S.L.Sobolev in [1] and for optimal coefficients $\hat{C}[\beta]$ the system of linear equations was obtained, which for $x_{\beta} = [\beta] = h \beta$ ($\beta = 0, 1$) has the form

$$\sum_{\gamma=0}^N \hat{C}[\gamma][\beta - \gamma]^{2m-1} + \sum_{\alpha=0}^{m-1} \lambda_\alpha[\beta]^{\alpha} = \frac{[\beta]^{2m} + (1 - [\beta])^{2m}}{2(2m)!}, \quad [\beta] \in [0, 1],$$ \hfill (4)

$$\sum_{\gamma=0}^N \hat{C}[\gamma][\gamma]^{\alpha} = \frac{1}{\alpha + 1}, \quad \alpha = 0, m - 1.$$ \hfill (5)

In this system $\hat{C}[\beta]$ and $\lambda_\alpha$, $\alpha = 0, 1, \ldots, m - 1$ are unknowns.

Note that the existence and uniqueness of the solution of the system (4)-(5) was proved by S.L.Sobolev in [1].

The aim of this work is finding the explicit forms of the optimal coefficients $\hat{C}[\beta]$.

Later for convenience optimal coefficients $\hat{C}[\beta]$ we remain as $C[\beta]$. 

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Here mainly is used the concept of discrete argument functions and operations on them. For completeness we give some definitions about functions of discrete argument.

Assume that $\varphi(x)$ and $\psi(x)$ are real-valued functions of real variable and are defined in real line $\mathbb{R}$.

**Definition 1.** Function $\varphi(h\beta)$ is called the function of discrete argument, if it is given on some set of integer values of $\beta$.

**Definition 2.** By inner product of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the number
\[
[\varphi, \psi] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),
\]
if the series on right hand side of the last equality converges absolutely.

**Definition 3.** By convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the inner product
\[
\varphi(h\beta) \ast \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).
\]

Suppose $C[\beta] = 0$ for $\beta < 0$ and $\beta > N$. Then, using definition 3, system (4)-(5) can be rewritten in the following form
\[
G_{m,1}[\beta] \ast C[\beta] + P_{m-1}[\beta] = f_m[\beta] \text{ when } [\beta] \in [0,1],
\]
\[
C[\beta] = 0 \text{ when } [\beta] \not\in [0,1],
\]
\[
\sum_{\beta=0}^{N} C[\beta] \left[\beta\right]^\alpha = \frac{1}{\alpha + 1}, \quad \alpha = 0, 1, 2, ..., m - 1,
\]
where $P_{m-1}[\beta]$ is polynomial of degree $m - 1$ with respect to $[\beta]$ and
\[
G_{m,1}[\beta] = \frac{[\beta]^{2m-1}\text{sign}[\beta]}{2(2m - 1)!},
\]
\[
f_m[\beta] = \frac{[\beta]^{2m} + (1 - [\beta])^{2m}}{2(2m)!}.
\]

Consider the following problem.

**Problem A.** Find the discrete function $C[\beta]$ and unknown polynomial $P_{m-1}[\beta]$.

Denote
\[
v[\beta] = G_{m,1}[\beta] \ast C[\beta]
\]
and
\[ u[\beta] = v[\beta] + P_{m-1}[\beta]. \] (12)

Then we have to express \( C[\beta] \) with the help of \( u[\beta] \). For this we must construct discrete operator \( D_m[\beta] \), which satisfies the equation
\[ hD_m[\beta] * G_{m,1}[\beta] = \delta[\beta], \] (13)
where \( \delta[\beta] \) equals 0 when \( \beta \neq 0 \) and equals 1 when \( \beta = 0 \), i.e. \( \delta[\beta] \) is the discrete delta-function, \( G_{m,1}[\beta] \) is defined by formula (9).

In connection with this in the work [2] the discrete analogue \( D_m[\beta] \) of the differential operator \( d^{2m}/dx^{2m} \) was constructed.

In [2] the following was proved.

**Theorem 1.** The discrete analogue of the differential operator \( d^{2m}/dx^{2m} \) have the form
\[
D_m[\beta] = \frac{(2m-1)!}{h^{2m}} \begin{cases}
    \sum_{k=1}^{m-1} \frac{(1 - q_k)^{2m+1} q_k^{[\beta]}}{q_k E_{2m-1}(q_k)} & \text{for } |\beta| \geq 2, \\
    1 + \sum_{k=1}^{m-1} \frac{(1 - q_k)^{2m+1}}{E_{2m-1}(q_k)} & \text{for } |\beta| = 1, \\
    -2^{2m-1} + \sum_{k=1}^{m-1} \frac{(1 - q_k)^{2m+1}}{q_k E_{2m-1}(q_k)} & \text{for } \beta = 0,
\end{cases}
\] (14)
where \( E_{2m-1}(x) \) is the Euler-Frobenius polynomial of degree \( 2m-1 \), \( q_k \) are roots of the polynomial \( E_{2m-2}(x) \), \( |q_k| < 1 \), \( h \) is small parameter.

**Theorem 2.** For the operator \( D_m[\beta] \) and monomials \( [\beta]^k = (h\beta)^k \) the following are true
\[
\sum_{\beta} D_m[\beta] [\beta]^k = \begin{cases}
    0 & \text{when } 0 \leq k \leq 2m - 1, \\
    (2m)! & \text{when } k = 2m,
\end{cases}
\] (15)
\[
\sum_{\beta} D_m[\beta] [\beta]^k = \begin{cases}
    0 & \text{when } 2m + 1 \leq k \leq 4m - 1, \\
    \frac{h^{2m} (4m)! B_{2m}}{(2m)!} & \text{when } k = 4m.
\end{cases}
\] (16)

Taking into account (13) and theorems 1, 2 for optimal coefficients we have
\[ C[\beta] = hD_m[\beta] * u[\beta]. \] (17)
Thus, if we find the function $u[\beta]$, then optimal coefficients will be found from (17).

In order to calculate convolution (17) we need representation of the function $u[\beta]$ in all integer values of $\beta$. From (6) we have $u[\beta] = f_m[\beta]$ when $[\beta] \in [0, 1]$. Now we must find representation of $u[\beta]$ for $\beta < 0$ and $\beta > N$.

Since $C[\beta] = 0$ when $[\beta] \notin [0, 1]$, then

$$C[\beta] = hD_m[\beta] * u[\beta] = 0, \quad [\beta] \notin [0, 1].$$

We calculate the convolution $v[\beta] = G_{m,1}[\beta] * C[\beta]$ when $[\beta] \notin [0, 1]$.

Suppose $\beta < 0$, then taking into account (4), (5) we have

$$v[\beta] = G_{m,1}[\beta] * C[\beta] = -\sum_{\gamma=-\infty}^{\infty} C[\gamma] (\beta - [\gamma])^{2m-1} \frac{1}{2(2m-1)!} = -\frac{1}{2} \sum_{j=0}^{m-1} \frac{[\beta]^{2m-1-j}(-1)^j}{(j+1)!(2m-1-j)!} - \frac{1}{2} \sum_{j=m}^{2m-1} \frac{[\beta]^{2m-1-j}(-1)^j}{j!(2m-1-j)!} \sum_{\gamma=0}^{N} C[\gamma][\gamma]^j. \quad (18)$$

Denote

$$Q_{2m-1}[\beta] = \frac{1}{2} \sum_{j=0}^{m-1} \frac{[\beta]^{2m-1-j}(-1)^j}{(j+1)!(2m-1-j)!}, \quad R_{m-1}[\beta] = \frac{1}{2} \sum_{j=m}^{2m-1} \frac{[\beta]^{2m-1-j}(-1)^j}{j!(2m-1-j)!} \sum_{\gamma=0}^{N} C[\gamma][\gamma]^j. \quad (21)$$

Then from (18) for $v[\beta]$ when $\beta < 0$ we obtain

$$v[\beta] = -Q_{2m-1}[\beta] - R_{m-1}[\beta]. \quad (19)$$

Similarly for the case $\beta > N$ we have

$$v[\beta] = Q_{2m-1}[\beta] + R_{m-1}[\beta]. \quad (20)$$

Denoting

$$R_{m-1}^-[\beta] = P_{m-1}[\beta] - R_{m-1}[\beta], \quad (21)$$

$$R_{m-1}^+[\beta] = P_{m-1}[\beta] + R_{m-1}[\beta], \quad (22)$$

and taking into account (19), (20), (12) we get following problem.

**Problem B.** Find solution of the equation

$$hD_m[\beta] * u[\beta] = 0, \quad [\beta] \notin [0, 1]. \quad (23)$$
having the form

\[ u[\beta] = \begin{cases} 
- Q_{2m-1}[\beta] + R_{m-1}^-[\beta], & \beta < 0, \\
 f_m[\beta], & 0 \leq \beta \leq N - 1, \\
 Q_{2m-1}[\beta] + R_{m-1}^+[\beta], & \beta > N, 
\end{cases} \tag{24} \]

where \( R_{m-1}^-[\beta] \) and \( R_{m-1}^+[\beta] \) unknown polynomials of degree \( m - 1 \).

If we find \( R_{m-1}^-[\beta] \) and \( R_{m-1}^+[\beta] \), then from (21), (22) we get

\[
P_{m-1}[\beta] = \frac{1}{2} \left( R_{m-1}^+[\beta] + R_{m-1}^-[\beta] \right),
\]

\[
R_{m-1}[\beta] = \frac{1}{2} \left( R_{m-1}^+[\beta] - R_{m-1}^-[\beta] \right).
\]

Unknowns \( R_{m-1}^-[\beta] \) and \( R_{m-1}^+[\beta] \) can be found from (23), using the function \( D_m[\beta] \). Then we obtain explicit form of \( u[\beta] \) and from (17) will be found the optimal coefficients \( C[\beta] \). Thus, the problem B and respectively the problem A will be solved.

But here we will not find \( R_{m-1}^-[\beta] \) and \( R_{m-1}^+[\beta] \), instead of them, using \( D_m[\beta] \) and taking into account (17), we will find expressions for the optimal coefficients \( C[\beta] \).

The main result of the present work is the following.

**Theorem 3.** Optimal coefficients of quadrature formulas of the form (1) on the space \( L_2^{(m)}(R) \) have the form

\[
C[\beta] = h \begin{cases} 
\frac{1}{2} - \sum_{k=1}^{m-1} d_k \frac{q_k - q_k^N}{1 - q_k} & \text{for } \beta = 0, N, \\
1 + \sum_{k=1}^{m-1} d_k \left( q_k^\beta + q_k^{N-\beta} \right) & \beta = 1, 2, \ldots, N - 1, \\
0 & \beta < 0, \beta > N,
\end{cases}
\]

where

\[
\sum_{k=1}^{m-1} d_k \sum_{i=1}^j \frac{q_k + (-1)^{i+1} q_k^{N+i}}{(q_k - 1)^i} \Delta^i q^j = \frac{B_{j+1}}{j+1}, \quad j = 1, 2, \ldots, m - 1.
\]

\( q_k \) are roots of the Euler-Frobenius polynomial \( E_{2m-2}(x) \) of degree \( 2m - 2 \).

In the proof of theorem 3 the following preliminary results are used.

The following formula is true \[3\]

\[
\sum_{\gamma=0}^{n-1} q^{i\gamma} \gamma^k = \frac{1}{1 - q} \sum_{i=0}^{k} \left( \frac{q}{1 - q} \right)^i \Delta^i q^{\gamma} - \frac{q^n}{1 - q} \sum_{\gamma=0}^{k} \left( \frac{q}{1 - q} \right)^i \Delta^i q^{\gamma} |_{\gamma = n}, \tag{25}\]

\]

\]
where $\Delta^i \gamma^n$ is finite difference of order $i$ from $\gamma^n$, $\Delta^0 \gamma^n = \Delta^i \gamma^n |_{\gamma=0}$.

And also we use the following well-known formula (see, for example, [4])

$$
\sum_{i=0}^{\beta-1} \gamma^k = \sum_{j=1}^{k+1} \frac{k!B_{k+1-j}}{j!(k+1-j)!} \beta^j.
$$

It is known in [5], that the Euler-Frobenius polynomials $E_k(x)$ have the form

$$
x E_k(x) = (1-x)^{k+2} D^k x \frac{x}{(1-x)^2},
$$

where

$$
D = x \frac{d}{dx}, \quad D^k = x \frac{d}{dx} D^{k-1}.
$$

In [5] it was shown that all roots $q_j^{(k)}$ of the Euler-Frobenius polynomials $E_k(x)$ are real, negative and distinct:

$$
q_1^{(k)} < q_2^{(k)} < \ldots < q_k^{(k)} < 0.
$$

Furthermore for the roots (28) the following is true:

$$
q_j^{(k)} \cdot q_{k+1-j}^{(k)} = 1.
$$

If we denote $E_k(x) = \sum_{s=0}^{k} a_s^{(k)} x^s$, then coefficients $a_s^{(k)}$ of the Euler-Frobenius polynomial $E_k(x)$ are expressed by formula

$$
a_s^{(k)} = \sum_{j=0}^{s} (-1)^j \binom{k+2}{j} (s+1-j)^{k+1}.
$$

This formula was obtained by Euler.

From definition of $E_k(x)$ we get following statements.

**Lemma 1.** For the polynomials $E_k(x)$ the following recurrence relation holds

$$
E_k(x) = (kx + 1)E_{k-1}(x) + x(1-x)E'_{k-1}(x),
$$

where $E_0(x) = 1$, $k = 1, 2, \ldots$.

**Lemma 2.** The polynomial $E_k(x)$ satisfies the identity

$$
E_k(x) = x^k E_k \left( \frac{1}{x} \right)
$$
or otherwise \(a_s^{(k)} = a_k^{(k-s)}\), \(s = 0, 1, 2, \ldots, k\).

**Lemma 3.** Polynomials

\[
P_k(x) = (x - 1)^{k+1} \sum_{i=1}^{k+1} \frac{\Delta^i 0^{k+1}}{(x - 1)^i}
\]

and

\[
P_k\left(\frac{1}{x}\right) = \left(\frac{1}{x} - 1\right)^{k+1} \sum_{i=1}^{k+1} \left(\frac{x}{1-x}\right)^i \Delta^i 0^{k+1}
\]

are the Euler-Frobenius polynomials \(E_k(x)\) and \(E_k\left(\frac{1}{x}\right)\) respectively.

**Lemma 4.** The operator \(D_m[\beta]\) satisfies the relation

\[
[D_m[\beta], [\beta]^{2m}] = \sum_{\beta=-\infty}^{\infty} D_m[\beta][\beta]^{2m} = (2m)!.
\]

**Proofs of Lemmas.**

**Proof of lemma 1.** From (27) obviously that

\[
E_{k-1}(x) = x^{-1}(1-x)^{k+1}D_{k-1}\frac{x}{(1-x)^2}.
\]

Differentiating by \(x\) the polynomial \(E_{k-1}(x)\), we obtain

\[
E'_{k-1}(x) = -(1-x)^kx^{-2}(kx+1)D_{k-1}\frac{x}{(1-x)^2} + \frac{E_k(x)}{x(1-x)}.
\]

Hence and from (34) implies that

\[
(kx+1)E_{k-1}(x) + x(1-x)E'_{k-1}(x) = (kx+1)x^{-1}(1-x)^{k+1}D_{k-1}\frac{x}{(1-x)^2} -
\]

\[-(1-x)^{k+1}x^{-1}(kx+1)D_{k-1}\frac{x}{(1-x)^2} + E_k(x) = E_k(x).
\]

Thus lemma 1 is proved.

**Proof of lemma 2.** Lemma 2 we will proof by induction. For \(k = 1\) from (27) we find

\[
E_1(x) = x + 1.
\]

Assume that for \(k \geq 1\) the equality \(a_n^{(k-1)} = a_{k-1-n}^{(k-1)}\), \(n = 0, 1, \ldots, k-1\) is fulfilled. Suppose that \(a_n^{(k-1)} = 0\) for \(n < 0\) and \(n > k-1\).
From (29) we get
\[ a_s^{(k)} = (s + 1)a_s^{(k-1)} + (k - s + 1)a_{s-1}^{(k-1)}; \]
then, using assumptions of the induction, we obtain
\[ a_{k-s}^{(k)} = (k - s + 1)a_{k-s}^{(k-1)} + (s + 1)a_{k-s-1}^{(k-1)} = (k - s + 1)a_{s-1}^{(k-1)} + (s + 1)a_s^{(k-1)} = a_s^{(k)}; \]
and Lemma 2 is proved.

**Proof of lemma 3.** Consider relations (29)
\[ E_k(x) = (kx + 1)E_{k-1}(x) + x(1 - x)E'_{k-1}(x), \]
\[ E_0(x) = 1. \]
If \( P_k(x) \) also satisfies this relations then lemma 3 will be proved. From (31) evidently that \( P_0(x) = 1 \). We denote
\[ V_k(x) = (kx + 1)P_{k-1}(x) + x(1 - x)P'_{k-1}(x). \]
Using the equalities
\[ P_{k-1}(x) = \sum_{i=1}^{k} \frac{(x - 1)^k}{(x - 1)^i} \Delta^i 0^k, \]
\[ P'_{k-1}(x) = k(x - 1)^{k-1} \sum_{i=1}^{k} \frac{\Delta^i 0^k}{(x - 1)^i} - (x - 1)^k \sum_{i=1}^{k} \frac{i\Delta^i 0^k}{(x - 1)^{i+1}}, \]
after some simplifications we have
\[ V_k(x) = (kx + 1)P_{k-1}(x) + x(1 - x)P'_{k-1}(x) = \]
\[ = P_{k-1}(x) + x(x - 1)^k \sum_{i=1}^{k} \frac{\Delta^i 0^k}{(x - 1)^i} = \sum_{i=1}^{k} \Delta^i 0^k (1 + xi)(x - 1)^{k-i}. \]
Doing change of variables \( x - 1 = y \), we obtain
\[ V_k(y + 1) = \sum_{i=0}^{k} y^{k-i}(\Delta^i 0^k + \Delta^{i+1} 0^k)(i + 1), \]
\[ P_k(y + 1) = \sum_{i=0}^{k} y^{k-i} \Delta^{i+1} 0^{k+1}. \]
Consider coefficients of the polynomial $V_k(y + 1)$

$$(\Delta^i 0^k + \Delta^{i+1} 0^k)(i + 1) = \left[ \sum_{\alpha=0}^{i} (-1)^{i-\alpha} \binom{i}{\alpha} \alpha^k + \sum_{\alpha=0}^{i+1} (-1)^{i+1-\alpha} \binom{i+1}{\alpha} \alpha^k \right] (i + 1) =$$

$$= \left[ \sum_{\alpha=0}^{i} (-1)^{i-\alpha} \binom{i}{\alpha} \alpha^k + \sum_{\alpha=0}^{i} (-1)^{i+1-\alpha} \binom{i+1}{\alpha} \alpha^k + (i + 1)^k \right] (i + 1) =$$

$$= \left[ \sum_{\alpha=0}^{i} (-1)^{i-\alpha} \binom{i}{\alpha} \alpha^k + \sum_{\alpha=0}^{i} (-1)^{i+1-\alpha} \left( \binom{i}{\alpha} + \binom{i}{\alpha - 1} \right) \alpha^k + (i + 1)^k \right] (i + 1) =$$

$$= \left[ \sum_{\alpha=0}^{i} (-1)^{i+1-\alpha} \binom{i}{\alpha - 1} \alpha^k + (i + 1)^k \right] (i + 1) =$$

$$= \sum_{\alpha=0}^{i+1} (-1)^{i+1-\alpha} \binom{i+1}{\alpha} \alpha^{k+1} = \Delta^{i+1} 0^{k+1}.$$  

Hence we get $V_k(y + 1) = P_k(y + 1)$, i.e.

$$P_k(x) = E_k(x).$$

Lemma 3 is proved.

**Proof of lemma 4.** Using (14) we have

$$[D_m[\beta], [\beta]^{2m}] = (2m - 1)! h^{-2m} \left( 2 \sum_{\beta=1}^{\infty} \sum_{k=1}^{m-1} \frac{(1 - q_k)^{2m+1}}{q_k E_{2m-1}(q_k) q_k^\beta q_k^{2m+1}} + 2[1]^{2m} \right) = 2 \cdot (2m - 1)! \left( \sum_{\beta=1}^{\infty} \frac{(1 - q_k)^{2m+1}}{q_k E_{2m-1}(q_k) q_k^{2m+1}} \right).$$

Hence, by virtue of well-known formula

$$\sum_{\gamma=0}^{\infty} q^\gamma \gamma^k = \frac{1-q}{1-q^k} \sum_{i=0}^{k} \left( \frac{q}{1-q} \right)^i \Delta^i 0^k,$$

we obtain

$$[D_m[\beta], [\beta]^{2m}] = 2(2m - 1)! \left( \sum_{k=1}^{m-1} \frac{(1 - q_k)^{2m+1}}{q_k E_{2m-1}(q_k) q_k} \cdot \frac{1}{1-q_k} \sum_{i=0}^{2m} \left( \frac{q_k}{1-q_k} \right)^i \Delta^i 0^{2m+1} \right).$$

According to lemma 3

$$[D_m[\beta], [\beta]^{2m}] = 2(2m - 1)! \left( \sum_{k=1}^{m-1} \frac{(1 - q_k)^{2m+1}}{q_k E_{2m-1}(q_k) q_k} \cdot \frac{E_{2m-1}(1/q_k)}{(1-q_k)(1/q_k - 1)^{2m+1}} \right).$$
Hence, using (14), (15) and (33), we get
\[ [D_m[\beta], [\beta]^{2m}] = 2(2m - 1)! \left( \sum_{k=1}^{m-1} \frac{(1 - q_k)^{2m+1}}{q_k E_{2m-1}(q_k)} \cdot \frac{E_{2m-1}(q_k)}{(1 - q_k)^{2m} (1/q_k - 1)^{2m} + 1} \right) = 2 \cdot (2m - 1)! \cdot m = (2m)! \]

Lemma 4 is proved.

**Proof of theorem 3.**

Suppose \( \beta = \frac{1}{N-1} \). Then from (17), using (24) and definition of convolution of discrete functions, we have
\[
C[\beta] = h D_m[\beta] * u[\beta] = h \sum_{\gamma=-\infty}^{\infty} D_m[\beta - \gamma] u[\gamma] = h \left( \sum_{\gamma=-\infty}^{-1} D_m[\beta - \gamma] (-Q_{2m-1}[\gamma] + R_m^{-}[\gamma]) + \sum_{\gamma=0}^{N} D_m[\beta - \gamma] f_m[\gamma] + \sum_{\gamma=N+1}^{\infty} D_m[\beta - \gamma] (Q_{2m-1}[\gamma] + R_m^{+}[\gamma]) \right) = h \left( \sum_{\gamma=-\infty}^{\infty} D_m[\beta - \gamma] f_m[\gamma] + \sum_{\gamma=1}^{\infty} D_m[\beta + \gamma] (-Q_{2m-1}[\gamma] + R_m^{-}[\gamma] - f_m[-\gamma]) + \sum_{\gamma=1}^{\infty} D_m[N + \gamma - \beta](Q_{2m-1}[N + \gamma] + R_m^{+}[\gamma] - f_m[N + \gamma]) \right) = h \left( D_m[\beta] * f_m[\beta] + \sum_{\gamma=1}^{\infty} D_m[\beta + \gamma] (-Q_{2m-1}[\gamma] + R_m^{-}[\gamma] - f_m[-\gamma]) + \sum_{\gamma=1}^{\infty} D_m[N + \gamma - \beta](Q_{2m-1}[N + \gamma] + R_m^{+}[\gamma] - f_m[N + \gamma]) \right).
\]

Hence, using (14), (15) and (33), we get
\[
C[\beta] = h \left( 1 + \sum_{k=1}^{m-1} q_k \gamma^{(2m - 1)} \frac{(1 - q_k)^{2m+1}}{h^{2m} q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} (-Q_{2m-1}[\gamma] + R_m^{-}[\gamma] - f_m[-\gamma]) + \sum_{k=1}^{m-1} q_k N - \beta \gamma^{(2m - 1)} \frac{(1 - q_k)^{2m+1}}{h^{2m} q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} (Q_{2m-1}[N + \gamma] + R_m^{+}[\gamma] - f_m[N + \gamma]) \right).
\]
We denote
\[ d_k = \frac{(2m-1)!}{h^{2m}} \frac{(1-q_k)^{2m+1}}{q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^\gamma (-Q_{2m-1}[\gamma] + R_{m-1}[-\gamma] - f_m[-\gamma]), \]
\[ p_k = \frac{(2m-1)!}{h^{2m}} \frac{(1-q_k)^{2m+1}}{q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^\gamma (Q_{2m-1}[N+\gamma] + R_{m-1}[N+\gamma] - f_m[N+\gamma]), \] (36)
where \( k = 1, m - 1 \). Then when \( \beta = 1, N - 1 \) for the optimal coefficients
\[ C[\beta] = h \left( 1 + \sum_{k=1}^{m-1} \left( d_k q_k^\beta + p_k q_k^{N-\beta} \right) \right). \] (37)

Now from (8) when \( \alpha = 0, 1 \) keeping in mind (37) for \( C[0] \) and \( C[N] \) we obtain the expressions
\[ C[0] = h \left( \frac{1}{2} + \sum_{k=1}^{m-1} \left( \frac{d_k q_k - p_k q_k^N}{q_k - 1} - \frac{h(q_k^{N+1} - q_k)(d_k - p_k)}{(q_k - 1)^2} \right) \right), \] (38)
\[ C[N] = h \left( \frac{1}{2} + \sum_{k=1}^{m-1} \left( \frac{p_k q_k - d_k q_k^N}{q_k - 1} + \frac{h(q_k^{N+1} - q_k)(d_k - p_k)}{(q_k - 1)^2} \right) \right), \] (39)
where \( d_k \) and \( p_k \) are determined from (36).

Now in order to prove theorem 3 sufficiently to show the equalities \( d_k = p_k, k = 1, m - 1 \). Now we will prove, that \( d_k = p_k, k = 1, m - 1 \).

Consider the convolution in equality (6) and we rewrite it in the form
\[ g[\beta] = G_{m,1}[\beta] * C[\beta] = \sum_{\gamma=0}^{N} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{2(2m-1)!} = \]
\[ = \sum_{\gamma=0}^{\beta} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{(2m-1)!} - \sum_{\gamma=0}^{N} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{2(2m-1)!} = T_1 - T_2, \] (40)
where
\[ T_1 = \sum_{\gamma=0}^{\beta} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{(2m-1)!}, \quad T_2 = \sum_{\gamma=0}^{N} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{2(2m-1)!}. \] (41)

First we consider \( T_1 \). For \( T_1 \) using (34) we have
\[ T_1 = C[0] \frac{(h\beta)^{2m-1}}{(2m-1)!} + \sum_{\gamma=1}^{\beta} h \left( 1 + \sum_{k=1}^{m-1} \left( d_k q_k^\gamma + p_k q_k^{N-\gamma} \right) \right) \frac{(h\beta - h\gamma)^{2m-1}}{(2m-1)!} = \]
\[ = C[0] \frac{(h\beta)^{2m-1}}{(2m-1)!} + \frac{h^{2m}}{(2m-1)!} \left( \sum_{\gamma=0}^{\beta-1} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{(2m-1)!} \right). \]
Using by formulas (25), (26) the expression for $T_1$ we reduce to the form

$$T_1 = C[0] \frac{(h \beta)^{2m-1}}{(2m-1)!} + \frac{h^{2m}}{(2m-1)!} \left[ \sum_{j=1}^{2m} \frac{(2m-1)! B_{2m-j}}{j!(2m-j)!} \left( \frac{\beta}{2m} + \frac{2m-1}{2m} h(q_k^{N+1} - q_k)(p_k - d_k) \right) + \sum_{i=0}^{m-2} \frac{(2m-1)! B_{2m-j}}{j!(2m-j)!} \beta^j \right] + \sum_{k=1}^{m-1} \left( \frac{d_k q_k^2 \sum_{\gamma=0}^{\beta-1} q_\gamma \gamma^{2m-1} + p_k q_k^{N-\beta} \sum_{\gamma=0}^{\beta-1} q_\gamma \gamma^{2m-1}}{2} \right).$$

Keeping in mind that $q_k$ is the root of the Euler-Frobenius polynomial $E_{2m-2}(x)$ of degree $2m - 2$, using lemma 3 and equality (38), after simplifications, from (42) we get

$$T_1 = \frac{h^{2m}}{(2m-1)!} \left[ \beta^{2m-1} - \frac{h(q_k^{N+1} - q_k)(p_k - d_k)}{(q_k - 1)^2} \right] + \sum_{j=0}^{m-2} \frac{(2m-1)! B_{2m-j}}{j!(2m-j)!} \beta^j + \sum_{k=1}^{m-1} \frac{-d_k q_k + (-1)^i q_k^{N+i} p_k \Delta i \beta^{2m-1}}{(q_k - 1)^{i+1}}.$$

Hence using the formula

$$\Delta i \beta^{2m-1} = \sum_{j=0}^{2m-2} \binom{2m-1}{j} \Delta i \beta^{2m-1-j}$$

and grouping in powers of $\beta$ we have

$$T_1 = \frac{(h \beta)^{2m}}{(2m)!} + \frac{h^{2m+1}}{(2m-1)!} \beta^{2m-1} \left( \frac{q_k^{N+1} - q_k}{(q_k - 1)^2} \right) + \frac{h^{2m}}{(2m-1)!} \sum_{j=1}^{m-2} \frac{(2m-1)! B_{2m-j}}{j!(2m-1-j)!} \left[ \frac{B_{j+1}}{j+1} + \sum_{k=1}^{m-1} \frac{-d_k q_k + (-1)^i q_k^{N+i} p_k \Delta i \beta^{2m-1-j}}{(q_k - 1)^{i+1}} \right] + \frac{h^{2m}}{(2m-1)!} \sum_{k=1}^{m-1} \frac{-d_k q_k + (-1)^i q_k^{N+i} p_k \Delta i \beta^{2m-1}}{(q_k - 1)^{i+1}}.$$

Now consider $T_2$. Using binomial formula and equalities (5) the expression for $T_2$ we reduce to the form

$$T_2 = \sum_{\gamma=0}^{N} C[\gamma] \frac{(h \beta - h \gamma)^{2m-1}}{2(2m-1)!} =$$
\[
= \frac{1}{2} \sum_{j=0}^{m-1} \frac{(h\beta)^{2m-1-j}(-1)^j}{(j+1)!(2m-1-j)!} \sum_{j=1}^{m} \frac{(h\beta)^{m-j}(-1)^{m+j-1}}{(m+j-1)!(m-j)!} C[\gamma][\gamma]^{m+j-1}.
\]

And for the right hand side \( f_m[\beta] \) of equation (4) (see (10)) using binomial formula we get

\[
f_m[\beta] = \frac{(h\beta)^{2m}}{(2m)!} + \frac{1}{2} \sum_{j=0}^{m-1} \frac{(h\beta)^{2m-1-j}(-1)^j}{(j+1)!(2m-1-j)!} + \frac{1}{2} \sum_{j=1}^{m} \frac{(h\beta)^{m-j}(-1)^{m+j}}{(m+j)!(m-j)!}.
\]

Then substituting (43), (44) into (40) taking into account (45) after some simplifications for the difference \( f_m[\beta] - g[\beta] \) we get

\[
f_m[\beta] - g[\beta] = \frac{h^{2m+1}}{(2m-1)!} h^{2m-1} \sum_{k=1}^{m-1} \frac{(q_k^{N+1} - q_k)(d_k - p_k)}{(q_k - 1)^2} \]

\[- \frac{h^{2m}}{(2m-1)!} \sum_{j=1}^{2m-2} \frac{(2m-1)!}{j!(2m-1-j)!} \sum_{i=1}^{m} \frac{-d_k q_k + (-1)^i q_k^{N+i} p_k}{(q_k - 1)^{i+1}} \Delta^i 0^{2m-1}.
\]

Hence note that \( f_m[\beta] - g[\beta] \) is polynomial of degree \( (2m-1) \) of \( [\beta] = h\beta \), i.e.

\[
f_m[\beta] - g[\beta] = \sum_{j=0}^{2m-1} a_j [\beta]^j.
\]

Here

\[
a_j = \begin{cases} 
\frac{h^2}{(2m-1)!} \sum_{k=1}^{m} \frac{(q_k^{N+1} - q_k)(d_k - p_k)}{(q_k - 1)^2} & \text{for } j = 2m - 1, \\
b_j & \text{for } m \leq j \leq 2m - 2, \\
\frac{b_j}{2((2m-1)!)^{j}} \left( \frac{1}{2m-j} - \sum_{\gamma=0}^{N} C[\gamma][\gamma]^{2m-j-1} \right) & \text{for } 1 \leq j \leq m - 1, \\
- \frac{h^{2m}}{(2m-1)!} \sum_{k=1}^{m-1} \frac{2m-1}{(q_k - 1)^{i+1}} \Delta^i 0^{2m-1} + \\
+ \frac{1}{2(2m-1)!} \left( \frac{1}{2m-j} - \sum_{\gamma=0}^{N} C[\gamma][\gamma]^{2m-1} \right) & \text{for } j = 0,
\end{cases}
\]
where
\[ b_j = -\frac{h^{2m-j}}{j!(2m-j-1)!} \left( \frac{B_{2m-j}}{2m-j} + \sum_{i=1}^{2m-j-1} \sum_{k=1}^{m-1} \frac{-d_k q_k + (-1)^i q_k^{N+i} p_k \Delta^i q^{2m-j-1}}{(q_k - 1)^{i+1}} \right). \]

On the other hand from (6) for difference \( f_m[\beta] - g[\beta] \) the following is true
\[ f_m[\beta] - g[\beta] = P_{m-1}[\beta]. \] (47)

If
\[ \sum_{k=1}^{m-1} \sum_{i=1}^{j} d_k q_k + (-1)^{i+1} q_k^{N+i} p_k \Delta^i 0^j = \frac{B_{j+1}}{j+1}, \quad j = 1, m-1, \] (48)
\[ \sum_{k=1}^{m-1} \frac{(q_k^{N+1} - q_k)(d_k - p_k)}{(q_k - 1)^2} = 0, \] (49)
then equality (47) is take placed. From (46) and (47) we find unknown polynomial \( P_{m-1}[\beta] \) in the system (6)-(8)
\[ P_{m-1}[\beta] = \sum_{j=1}^{m-1} a_j[\beta]^j. \] (50)

Thus, for unknowns \( d_k \) and \( p_k \) \((k = 1, m-1)\) we have got system of linear equations (48)-(49).

Later, from (8) for \( \alpha = \frac{2}{m-1} \), using equalities (37), (39), (25), (26) and (49) after some simplifications we obtain
\[ h^{\alpha+1} \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} \left( \frac{d_k q_k^i + (-1)^{i+1} q_k^{N+i} p_k \Delta^i 0^\alpha}{(1 - q_k)^{i+1}} - \frac{d_k q_k^{N+i} + (-1)^{i+1} q_k p_k \Delta^i 0^\alpha}{(1 - q_k)^{i+1}} \right) + \]
\[ + \sum_{j=2}^{\alpha} \frac{\alpha! h^j}{(j-1)!(\alpha + 1 - j)!} \left[ \frac{B_j}{j} - \sum_{k=1}^{m-1} \sum_{i=0}^{j-1} \frac{d_k q_k^{N+i} + (-1)^{i+1} q_k p_k \Delta^i 0^{j-1}}{(1 - q_k)^{i+1}} \right] = 0. \] (51)
The left hand side of the equation (51) is the polynomial of degree \((\alpha + 1)\) with respect to \( h \). Then from (51) obviously that all coefficients of this polynomial are zero, i.e.
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{j} \frac{d_k q_k^{N+i} + (-1)^{i+1} q_k p_k \Delta^i 0^j}{(1 - q_k)^{i+1}} = \frac{B_{j+1}}{j+1}, \quad j = 1, \alpha - 1, \] (52)
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} \frac{d_k q_k^i + (-1)^{i+1} q_k^{N+i} p_k \Delta^i 0^\alpha}{(1 - q_k)^{i+1}} = \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} \frac{d_k q_k^{N+i} + (-1)^{i+1} q_k p_k \Delta^i 0^\alpha}{(1 - q_k)^{i+1}}, \quad \alpha = \frac{2}{m-1}. \] (53)
Thus for unknowns $d_k$ and $p_k$ we have got the system (52), (53).

Using lemmas 2, 3 for the left hand side of the system (53) we obtain
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} d_k q_k^i + (-1)^{i+1} q_k^{N+i} p_k \frac{\Delta^i 0^\alpha}{(q_k - 1)^{i+1}} = (-1)^{\alpha+1} \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} d_k q_k + (-1)^{i+1} q_k^{N+i} p_k \Delta^i 0^\alpha, \]
where $\alpha = 2, 3, \ldots, m - 1$.

Hence taking into account (48) we get
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} d_k q_k^i + (-1)^{i+1} q_k^{N+i} p_k \frac{\Delta^i 0^\alpha}{(q_k - 1)^{i+1}} = (-1)^{\alpha+1} \frac{B_{\alpha+1}}{\alpha + 1}, \quad \alpha = 2, 3, \ldots, m - 1. \] (54)

Since for $\alpha = 2\ell, \ell = 1, 2, \ldots$ Bernoulli numbers $B_{\alpha+1} = B_{2\ell+1}$ are zero, then combining (54) and (53) we have
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} d_k q_k^i + (-1)^{i+1} q_k^{N+i} p_k \frac{\Delta^i 0^\alpha}{(q_k - 1)^{i+1}} = \frac{B_{\alpha+1}}{\alpha + 1}, \quad \alpha = 2, 3, \ldots, m - 1. \] (55)

Comparing systems (52) and (55) it is easy to see, that from (52) for $j = 1$ we get only one equation with respect to unknowns $d_k$ and $p_k$. And for $j = 2, \ldots, \alpha - 1$ obtained equations from system (52) are part of the system (55).

Thus, from (48), (52), (55) we get the following system
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{j} d_k q_k + (-1)^{i+1} q_k^{N+i} p_k \frac{\Delta^i 0^j}{(q_k - 1)^{i+1}} = \frac{B_{j+1}}{j + 1}, \quad j = 1, 2, \ldots, m - 1, \] (56)
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{j} d_k q_k^{N+i} + (-1)^{i+1} q_k p_k \frac{\Delta^i 0^j}{(q_k - 1)^{i+1}} = \frac{B_{j+1}}{j + 1}, \quad j = 1, 2, \ldots, m - 1. \] (57)

Subtracting equality (57) from (56) gives
\[ \sum_{k=1}^{m-1} \sum_{i=0}^{j} (q_k - q_k^{N+i} (-1)^{i+1}) (d_k - p_k) \frac{\Delta^i 0^j}{(q_k - 1)^{i+1}} = 0, \quad j = 1, 2, \ldots, m - 1. \] (58)

From system (58) for $j = 1$ we get equation (49). Therefore for finding unknowns $d_k$ and $p_k$ it is sufficiently to solve the system (56), (57).

Further we investigate the system (58). After some transformations homogenous system (58) we reduce to the form
\[ \sum_{k=1}^{m-1} \frac{(q_k - q_k^{N+j} (-1)^{j+1}) (d_k - p_k)}{(q_k - 1)^{j+1}} = 0, \quad j = 1, 2, \ldots, m - 1, \] (59)
where $d_k - p_k$, $k = 1, 2, ..., m - 1$ are unknowns. Obviously, that the main matrix of the system (59) is the Vandermonde type matrix, i.e. system (59) has a unique solution, which identically zero. This means

$$d_k = p_k, \quad k = 1, 2, ..., m - 1. \quad (60)$$

Then keeping in mind (60) from (37), (38), (39) and (56) we respectively get the following system

$$C[\beta] = h \left( 1 + \sum_{k=1}^{m-1} d_k \left( q_k^\beta + q_k^{N-\beta} \right) \right), \quad \beta = 1, N - 1,$$

$$C[0] = h \left( \frac{1}{2} - \sum_{k=1}^{m-1} d_k \frac{q_k - q_k^N}{1 - q_k} \right),$$

$$C[N] = h \left( \frac{1}{2} - \sum_{k=1}^{m-1} d_k \frac{q_k - q_k^N}{1 - q_k} \right),$$

$$\sum_{k=1}^{m-1} d_k \sum_{i=1}^{j} \frac{q_k + (-1)^{i+1} q_k^{N+i}}{(q_k - 1)^{i+1}} \Delta^i q^j = \frac{B_{j+1}}{j+1}, \quad j = 1, 2, ..., m - 1.$$

Theorem 3 is proved.

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