On a Notion of Ring Groupoid

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Abstract. By a ring groupoid we mean an animated ring whose $i$-th homotopy groups are zero for all $i > 1$.

In this expository note we give an elementary treatment of the $(2,1)$-category of ring groupoids (i.e., without referring to general animated rings and without using $n$-categories for $n > 2$). The note is motivated by the fact that ring stacks play a central role in the Bhatt-Lurie approach to prismatic cohomology.

1. Introduction

1.1. Subject of this note. This note is expository. Following an idea of Lawvere [Law1, Law2], we introduce a notion of ring groupoid and a slightly more general notion of ring object in a $(2,1)$-category. Then we recall an elementary description of the $(2,1)$-category of ring groupoid; the idea (which goes back to B. Noohi [N1]) is to use the magic word “extension”.

The $(2,1)$-category of ring groupoids is a full subcategory of the $\infty$-category of animated rings, see §2.4.2. But we do not emphasize this point of view. On the contrary, our exposition of the notion of ring groupoid goal is elementary (we do not use $n$-categories for $n > 2$).

1.2. Motivation. The notion of ring stack plays a central role in the Bhatt-Lurie approach to prismatic cohomology, see [Dr] §1.3-1.4.

1.3. Organization. In §2, we define the $(2,1)$-category of ring groupoids. In §3, we define and describe the naive 1-category of ring groupoids. In §4, we use the description of the 1-category to describe the $(2,1)$-category; the main result (Theorem 4.5.5) goes back to [N1, AN1]. In §5, we recall the notion of anafunctor from M. Makkai’s work [Mak] (this notion is closely related to the material from §4).

Let us note that §3 and the related §4.6 can be read independently of the rest of the article.

1.4. Acknowledgements. I thank A. Mathew and N. Rozenblyum for useful discussions. In particular, they recommended me to define the $(2,1)$-category of ring groupoids using Lawveres’s approach. Moreover, §3.3.5 and §3.7 are due to A. Mathew.

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2. Definition of the $(2,1)$-category of ring groupoids

2.1. Lawvere’s observation.

\footnote{We do not claim that it is the only reasonable notion, see §2.4.4.}

\footnote{By a ring stack on a site $S$ we mean a ring object in the $(2,1)$-category of stacks on $S$. Equivalently, it is a prestack of ring groupoids which happens to be a stack.}
2.1.1. **Convention.** All rings are assumed to be commutative, associative, and unital (unless said otherwise).

2.1.2. **Notation.** Let Rings be the category of all rings. Let Pol ⊂ Rings be the full subcategory of free rings. Let Pol_fin ⊂ Pol be the full subcategory of finitely generated free rings (i.e., rings isomorphic to \( \mathbb{Z}[x_1, \ldots, x_n] \) for some \( n \geq 0 \)).

2.1.3. **Lawvere’s observation.** Consider the functors

\[
\text{(2.1) } \text{Rings} \to \text{Funct}_\Pi(\text{Rings}^{\text{op}}, \text{Sets}) \to \text{Funct}_\Pi(\text{Pol}^{\text{op}}, \text{Sets})
\]

where Funct_\Pi stands for the category of those functors that commute with products and the first arrow in (2.1) is the Yoneda embedding. We also have a canonical functor

\[
\text{(2.2) } \text{Funct}_\Pi(\text{Pol}^{\text{op}}, \text{Sets}) \to \text{Funct}_\Pi(\text{Pol}_{\text{fin}}^{\text{op}}, \text{Sets}),
\]

where Funct_\Pi(\text{Pol}_{\text{fin}}^{\text{op}}, \text{Sets}) is the category of those functors \( \text{Pol}_{\text{fin}}^{\text{op}} \to \text{Sets} \) that commute with finite products.

In [Law1, Law2] Lawvere observed that the functor (2.2) and the composite functor (2.1) are equivalences. He also observed that the inverse functor Funct_\Pi(\text{Pol}_{\text{fin}}^{\text{op}}, \text{Sets}) \to \text{Rings} takes a functor \( F \in \text{Funct}_\Pi(\text{Pol}^{\text{op}}, \text{Sets}) \) to the following ring \( R_F \): as a set, \( R_F = F(\mathbb{Z}[x]) \), and the addition (resp. multiplication) map \( R_F \times R_F \to R_F \) comes from the homomorphism \( \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes \mathbb{Z}[x] \) that takes \( x \) to \( x \otimes 1 + 1 \otimes x \) (resp. to \( x \otimes x \)). Moreover, Lawvere observed that the word “ring” can be replaced by any type of algebraic structure.

2.2. **Definition of the (2, 1)-category of ring groupoids.**

2.2.1. **Notation.** We keep the notation of §2.1.2. Let Grpds be the (2, 1)-category of essentially small groupoids. It contains Sets as a full subcategory.

2.2.2. **Definition.** Let Funct_\Pi(\text{Pol}^{\text{op}}, \text{Grpds}) be the (2, 1)-category of those functors

\[
\text{Pol}^{\text{op}} \to \text{Grpds}
\]

that commute with products. This (2, 1)-category is called the (2, 1)-category of ring groupoids and denoted by RGrpds.

RGrpds identifies with Funct_\Pi(\text{Pol}_{\text{fin}}^{\text{op}}, \text{Grpds}) , where Funct_\Pi(\text{Pol}_{\text{fin}}^{\text{op}}, \text{Grpds}) is the (2, 1)-category of those functors \( \text{Pol}_{\text{fin}}^{\text{op}} \to \text{Grpds} \) that commute with finite products.

2.2.3. **The fully faithful functor** Rings \( \to \) RGrpds. By §2.1.3, the fully faithful embedding Sets \( \to \) Grpds induces a fully faithful embedding

\[
\text{(2.3) } \text{Rings} = \text{Funct}_\Pi(\text{Pol}^{\text{op}}, \text{Sets}) \hookrightarrow \text{Funct}_\Pi(\text{Pol}^{\text{op}}, \text{Grpds}) =: \text{RGrpds}.
\]

\(^3\)Informally, Lawvere’s idea was to consider a ring \( R \) as a set with \textit{infinitely many} operations: any \( f \in \mathbb{Z}[X_1, \ldots, X_n] \) defines an operation \( (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n), \ x_i \in R \).

\(^4\)A category is said to be essentially small if it is equivalent to a small one.
2.2.4. The functor $\pi_0: RGrpds \to Rings$. The set of isomorphism classes of objects of a groupoid $\Gamma$ is denoted by $\pi_0(\Gamma)$. The functor $\pi_0: Grpds \to Sets$ induces a functor $RGrpds := \text{Funct}_\Pi(Pol^{op}, Grpds) \to \text{Funct}_\Pi(Pol^{op}, Sets) = Rings,$

which will be denoted by $\pi_0: RGrpds \to Rings$. This functor is left adjoint to $\mathcal{2.3}$. The unit of the adjunction provides a canonical 1-morphism $R \to \pi_0(R)$ for any $R \in RGrpds$.

**Lemma 2.2.5.** Let $\mathcal{R} \in RGrpds$ and $R \in Pol$. Then any homomorphism $R \to \pi_0(\mathcal{R})$ lifts to a 1-morphism $R \to \mathcal{R}$; moreover, this 1-morphism is unique up to 2-isomorphism.

**Proof.** We have to show that the natural map $\pi_0(\text{Mor}(R, \mathcal{R})) \to \text{Mor}(R, \pi_0(\mathcal{R}))$ is bijective, where $\text{Mor}(R, \mathcal{R})$ is the groupoid of 1-morphisms. By definition, $\mathcal{R}$ is a product-preserving functor $F: Pol^{op} \to Grpds$. By Yoneda’s lemma, $\text{Mor}(R, \mathcal{R}) = F(R)$ and $\text{Mor}(R, \pi_0(\mathcal{R})) = \pi_0(F(R))$. □

2.2.6. The functor $RGrpds \to Grpds$. The “forgetful” functor $RGrpds \to Grpds$ is defined as follows: $F \in RGrpds := \text{Funct}_\Pi(Pol^{op}, Grpds)$ goes to $F(\mathbb{Z}[x]) \in Grpds$. One has commutative diagrams

\[
\begin{array}{ccc}
\text{Rings} & \longrightarrow & \text{Sets} \\
\downarrow & & \downarrow \\
\text{RGrpds} & \longrightarrow & \text{Grpds}
\end{array}
\quad
\begin{array}{ccc}
\text{Rings} & \longrightarrow & \text{Sets} \\
\pi_0 & & \pi_0 \\
\text{RGrpds} & \longrightarrow & \text{Grpds}
\end{array}
\]

whose horizontal arrows are the forgetful functors.

2.2.7. Fiber products in $RGrpds$. In the $(2, 1)$-category $Grpds$ fiber products always exist. The same is true for $RGrpds$; moreover, the functor $RGrpds \to Grpds$ commutes with fiber products. Let us note that if $\mathcal{R} \in RGrpds$ and $R_1, R_2$ are usual rings equipped with morphisms to $\mathcal{R}$ then $R_1 \times_{\mathcal{R}} R_2$ is a usual ring.

2.3. Variants of the definition of the $(2, 1)$-category of ring groupoids.

2.3.1. Reformulation in terms of fibered categories. As suggested to me by J. Lurie, one could equivalently define a ring groupoid as a pair $(C, U)$, where $C$ is a 1-category with finite coproducts and $U: C \to Pol_{\text{fin}}$ is a functor which preserves finite coproducts and is a fibration in groupoids. To a ring groupoid $\mathcal{R}$ in the sense of $\mathcal{2.2}$ one associates the pair $(C, U)$ defined as follows: $C$ is the category of pairs $(P, f)$, where $P \in Pol_{\text{fin}}$ and $f$ is a 1-morphism $P \to \mathcal{R}$; the functor $U: C \to Pol_{\text{fin}}$ forgets $f$. (This procedure is called Grothendieck construction.)

Lurie’s definition is “elementary”: it does not involve the notion of functor from an ordinary category to a 2-category.

Recall that $Pol_{\text{fin}}$ was defined to be the category of those rings that are isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]$ for some $n \geq 0$. One could replace $Pol_{\text{fin}}$ by its full subcategory formed by the rings $\mathbb{Z}[x_1, \ldots, x_n]$ themselves. Then Lurie’s definition become even more elementary.
2.3.2. Ring groupoids as Picard groupoids with additional structure. Let $\mathcal{Pic}$ denote the symmetric monoidal $(2,1)$-category of strictly commutative Picard groupoids. It is known that a ring groupoid in the sense of §2.2 is the same as a “strictly” commutative monoid in the symmetric monoidal $(2,1)$-category $\mathcal{P}$. We will not use this fact.

Let us note that §2 of [JP] contains a reformulation in “concrete” terms of the notion of (noncommutative) monoid in the symmetric monoidal $(2,1)$-category of (nonstrictly) commutative Picard groupoids.

2.4. Some generalizations.

2.4.1. Ring objects of an $n$-category. If $\mathcal{C}$ is an $n$-category with products we define a ring object in $\mathcal{C}$ to be a product-preserving functor $\text{Pol}^{\text{op}} \to \mathcal{C}$. We will use this definition only for $n \in \{1, 2\}$, except a brief digression in §2.4.2.

Without assuming the existence of products in $\mathcal{C}$, one can define a ring object in $\mathcal{C}$ to be a product-preserving functor $F : \text{Pol}^{\text{op}} \to \text{Funct}(\mathcal{C}^{\text{op}}, \text{Sets})$ such that $F(\mathbb{Z}[x])$ is a representable functor $\mathcal{C}^{\text{op}} \to \text{Sets}$.

2.4.2. Animated rings. Let $\mathcal{C}$ be the $\infty$-category of $\infty$-groupoids, which are also known as animated sets, see [CS]. One can also describe $\mathcal{C}$ as the $\infty$-category of spaces or simplicial sets.

Ring objects in $\mathcal{C}$ are called animated rings, see [CS]. They form an $\infty$-category, which can also be described as the $\infty$-category of simplicial rings.

The 2-category of groupoids is a full subcategory of the $\infty$-category $\mathcal{C}$. So the 2-category of ring groupoids is a full subcategory of the $\infty$-category of animated rings (namely, the full subcategory of 1-truncated animated rings).

Let us note that the exposition of “animation” in [CS] relies on [Lu, §5.5.8].

2.4.3. Replacing rings by other types of algebraic structure. One can replace rings by any type of algebraic structure (groups, Lie algebras, etc.). Let $\text{Ab-Grpds}$ denote the analog of $\text{RGrpds}$ obtained by replacing rings with abelian groups and replacing $\text{Pol}$ with the category of free abelian groups.

2.4.4. Example: Picard groupoids. It is known that the above $(2,1)$-category $\text{Ab-Grpds}$ is canonically equivalent to the $(2,1)$-category of strictly commutative Picard groupoids in the sense of [SGA4] Exposé XVIII, §1.4. So the $(2,1)$-category $\text{Ab-Grpds}$ is “reasonable”. On the other hand, the bigger $(2,1)$-category of all commutative Picard groupoids in the sense of [SGA4] is no less reasonable.

3. The naive 1-category of ring groupoids

3.0.1. Notation. Let $\text{SSets}$ (resp. $\text{SRings}$) be the 1-category of simplicial sets (resp. simplicial rings).

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5The $(2,1)$-category of strictly commutative Picard groupoids is defined in §1.4.1-1.4.6 of Exposé XVIII of [SGA4] (strictness means that for every object $X$, the commutativity isomorphism $X + X \xrightarrow{\sim} X + X$ equals the identity). The symmetric monoidal structure on this $(2,1)$-category is defined in §1.4.8 of the same Exposé XVIII.

6By “strict” commutativity of the monoid $\mathcal{P}$ we mean the following. First, it is commutative, so for any objects $X, Y$ of the Picard groupoid $\mathcal{P}$ we have the commutativity isomorphism $X \cdot Y \xrightarrow{\sim} Y \cdot X$. Second, if $X = Y$ this isomorphism is required to be the identity.
3.1. Three incarnations of the 1-category of groupoids.

3.1.1. Let Grpds′_1 be the most naive 1-category of small groupoids (its morphisms are functors on the nose). It contains Sets as a full subcategory. In §3.1.1-3.1.2 below we define categories Grpds′_2 and Grpds′_3 canonically equivalent to Grpds′_1.

3.1.2. Associating to a groupoid Γ its nerve $N\Gamma$, one gets a fully faithful embedding

$$\text{Grpds′}_1 \hookrightarrow \text{SSets}.$$ 

Let Grpds′_2 ⊂ SSets be its essential image. A simplicial set $X$ belongs to Grpds′_2 if and only if it has the following property: for any $n \geq 2$ and any horn $\Lambda$ in the simplex $\Delta^n$, every map $\Lambda \rightarrow X$ has one and only one extension to a map $\Delta^n \rightarrow X$. Passing from Grpds′_1 to Grpds′_2 is a convenient “book-keeping device”.

3.1.3. Here is a way to relate the 1-category Grpds′_1 to the $(2,1)$-category Grpds. Let $[1]$ denote the ordered set ${0, 1}$ viewed as a category. Let Funct([1], Grpds) be the $(2,1)$-category of functors $[1] \rightarrow \text{Grpds}$. Now define Grpds′_3 ⊂ Funct([1], Grpds) to be the full subcategory of functors $\Phi : [1] \rightarrow \text{Grpds}$ such that $\Phi(0) \in \text{Sets}$ and the functor $\Phi(0) \rightarrow \Phi(1)$ is essentially surjective. Given $\Gamma \in \text{Grpds}_1$, define $\Phi_\Gamma \in \text{Grpds}_3$ as follows: $\Phi_\Gamma(1) = \Gamma$, $\Phi_\Gamma(0)$ is the set $\text{Ob}\Gamma$ (viewed as a discrete groupoid), and the map $\Phi_\Gamma(0) \rightarrow \Phi_\Gamma(1)$ is the obvious one. Thus we get a functor

$$\text{Grpds}_1 \rightarrow \text{Grpds}_3, \quad \Gamma \mapsto \Phi_\Gamma.$$ 

This functor is an equivalence (so Grpds′_3 is a 1-category rather than merely a $(2,1)$-category). The inverse functor takes $\Phi \in \text{Grpds}_3$ to the following groupoid $\Gamma$: the set of objects of $\Gamma$ is $\Phi(0)$, and for every $x, y \in \Phi(0)$ one has $\text{Mor}_\Gamma(x, y) := \text{Mor}_{\Phi(1)}(x_1, y_1)$, where $x_1, y_1 \in \Phi(1)$ are the images of $x$ and $y$ (composition of $\Gamma$-morphisms comes from composition of $\Phi(1)$-morphisms).

3.2. Three incarnations of the 1-category of ring groupoids. For $n = 1, 2, 3$ define RGrpds′_n to be the category of ring objects in the category Grpds′_n. Similarly to §2.2.6 the “forgetful” functor RGrpds′_n → Grpds′_n is defined as follows: $F \in \text{RGrpds}_n := \text{Funct}_n(\text{Pol}^{\text{op}}, \text{Grpds}_n')$ goes to $F(\mathbb{Z}[x]) \in \text{Grpds}_n'$.

3.2.1. On RGrpds′_2. By 2.1.3 RGrpds′_2 identifies with the category of simplicial rings such that the underlying simplicial set is a nerve of a groupoid.

3.2.2. A convenient way to think of RGrpds′_2. For any category $\mathcal{C}$, there is a notion of groupoid internal to $\mathcal{C}$. The category RGrpds′_2 identifies with the category of groupoids internal to Rings. We will mostly think of RGrpds′_2 in this way.

The forgetful functor Rings → Sets commutes with projective limits, so a groupoid internal to Rings is just a usual groupoid $\Gamma$ plus a ring structure on $\text{Ob}\Gamma$ and on $\text{Mor}\Gamma$ (where $\text{Mor}\Gamma$ is the set of all morphisms in $\Gamma$) such that the following maps are ring homomorphisms:

(i) the map $\text{Mor}\Gamma \rightarrow \text{Ob}\Gamma$ that takes a morphism to its source (resp. target);
(ii) the map $\text{Ob}\Gamma \rightarrow \text{Mor}\Gamma$ that a takes $a \in \text{Ob}\Gamma$ to $\text{id}_a$;
(iii) the map $(f, g) \mapsto g \circ f$, which is defined on the ring of all composable pairs of $\Gamma$-morphisms.

5
3.2.3. **On RGrpds′.** The category RGrpds′\(3\) identifies with the category formed by functors \(\Phi : [1] \to \text{RGrpds}\) such that \(\Phi(0) \in \text{Rings}\) and the functor \(\Phi(0) \to \Phi(1)\) is essentially surjective (by this we mean that it is essentially surjective as a functor between “plain” groupoids).

**Proposition 3.2.4.** The functor RGrpds′\(1\) \(\to\) RGrpds is essentially surjective.

**Proof.** It suffices to show that the functor RGrpds′\(3\) \(\to\) RGrpds is essentially surjective. By §3.2.3, the problem is to show that for any ring groupoid \(R\) there exists an essentially surjective 1-morphism \(R \to \mathcal{R}\), where \(R\) is a ring.

Choose a polynomial ring \(R\) and an epimorphism \(R \twoheadrightarrow \pi_0(\mathcal{R})\). By Lemma 2.2.5, it lifts to a 1-morphism \(R \to \mathcal{R}\). The latter is essentially surjective. □

3.3. **Quasi-ideals, DG rings, and 1-truncated simplicial rings.** In this subsection we define a category \(\mathcal{Q}\) and some categories which are obviously equivalent to it. In §3.4 we will show that these categories are equivalent to the categories RGrpds′\(1\), RGrpds′\(2\), RGrpds′\(3\) from §3.2 in other words, they can be considered as incarnations of the 1-category of ring groupoids. These incarnations are more manageable than those from §3.2.

3.3.1. **Quasi-ideals and the category \(\mathcal{Q}\).** By a quasi-ideal in a ring \(C\) we mean a pair \((I,d)\), where \(I\) is a \(C\)-module and \(d : I \to C\) is a \(C\)-linear map such that

\[
d(x) \cdot y = d(y) \cdot x
\]

for all \(x, y \in I\).

Let \(\mathcal{Q}\) be the category of all triples \((C,I,d)\), where \(C\) is a ring and \((I,d)\) is a quasi-ideal in \(C\).

3.3.2. **Remarks.** (i) A quasi-ideal \((I,d : I \to C)\) with \(\text{Ker} \ d = 0\) is essentially the same as an ideal in \(C\).

(ii) If \((I,d)\) is a quasi-ideal in \(C\) then \(I\) is a (non-unital) ring with respect to the multiplication operation \((x,y) \mapsto d(x) \cdot y\).

3.3.3. **Quasi-ideals and DG rings.** If \((I,d)\) is a quasi-ideal in a ring \(C\) then one can define a DG ring \(R\) as follows: \(R^0 = C\), \(R^{-1} = I\), \(R^i = 0\) for \(i \neq 0,1\), the differential in \(R\) is given by \(d : I \to C\), and the multiplication maps

\[
R^0 \times R^0 \to R^0, \quad R^0 \times R^{-1} \to R^{-1}
\]

come from the ring structure on \(C\) and the \(C\)-module structure on \(I\); note that the Leibnitz rule in \(R\) is equivalent to (3.1). Thus one gets an equivalence between the category \(\mathcal{Q}\) from §3.3.1 and the category of DG rings \(R\) such that \(R^i = 0\) for \(i \neq 0, -1\).

3.3.4. **1-truncated simplicial sets and rings.** Recall that a simplicial set is a functor

\[
\Delta^{op} \to \text{Sets},
\]

where \(\Delta\) is the category of finite linearly ordered sets. By a 1-truncated simplicial set we mean a functor \(\Delta^{op}_{\leq 1} \to \text{Sets}\), where \(\Delta_{\leq 1} \subset \Delta\) is the full subcategory formed by linearly ordered sets of order \(\leq 2\). The category of 1-truncated simplicial sets will be denoted by \(\text{SSets}_{\leq 1}\). We have the restriction (a.k.a. truncation) functor \(\text{SSets} \to \text{SSets}_{\leq 1}\).
Similarly, we have the category of $1$-truncated simplicial rings, denoted by $\text{SRings}_{\leq 1}$, and the truncation functor $\text{SRings} \to \text{SRings}_{\leq 1}$. Explicitly, an object of $\text{SRings}_{\leq 1}$ is a collection 

$$(A_0, A_1, \partial_0 : A_1 \to A_0, \partial_1 : A_1 \to A_0, s : A_0 \to A_1),$$

where $A_0, A_1$ are rings and $\partial_0, \partial_1, s$ are ring homomorphisms such that 

$$\partial_0 \circ s = \partial_1 \circ s = \text{id}_{A_0}.$$ 

3.3.5. The subcategory $\text{SRings}_{\leq 1}^{\text{good}}$. Let $\text{SRings}_{\leq 1}^{\text{good}} \subset \text{SRings}_{\leq 1}$ be the full subcategory of collections $(A_0, A_1, \partial_0, \partial_1, s) \in \text{SRings}_{\leq 1}$ such that 

$$(3.2) \quad (\ker \partial_0) \cdot (\ker \partial_1) = 0.$$ 

Lemma 3.3.6. The categories $Q$ and $\text{SRings}_{\leq 1}^{\text{good}}$ are equivalent.

Proof. Given $(A_0, A_1, \partial_0, \partial_1, s) \in \text{SRings}_{\leq 1}^{\text{good}}$, set $C := A_0$, $I := \ker \partial_0$, define $d : I \to C$ by $d := \partial_1|_I$, and define the $C$-module structure on $I$ using the homomorphism $C = A_0 \to s A_1$. Let us prove (3.1). If $x, y \in I$ then $x(y - s(\partial_1(y)) = 0$ by (3.2), which means that the r.h.s. of (3.1) equals $xy$. By symmetry, this is also true for the l.h.s. of (3.1).

We have constructed a functor $\text{SRings}_{\leq 1}^{\text{good}} \to Q$. It is easy to see that it is an equivalence and the inverse functor $Q \to \text{SRings}_{\leq 1}^{\text{good}}$ is as follows: $A_0 := C, A_1 := C \oplus I, s : A_0 \to A_1$ is the inclusion $C \hookrightarrow C \oplus I$, the maps $\partial_0, \partial_1 : C \oplus I \to C$ are given by 

$$\partial_0(c, x) := c, \quad \partial_1(c, x) := c + dx, \quad c \in C, x \in I,$$

and the ring structure on $A_1$ comes from the $C$-module structure on $I$ and the operation on $I$ defined in (3.3.2)(ii); one checks that $\partial_1$ is a ring homomorphism and (3.2) holds. \hfill \Box

3.3.7. Remark. We will always use the equivalence $Q \xrightarrow{\sim} \text{SRings}_{\leq 1}^{\text{good}}$ constructed in the above proof. One can check that it is isomorphic to the equivalence $Q \xrightarrow{\sim} \text{SRings}_{\leq 1}^{\text{good}}$ that one gets by reversing the roles of $\partial_0$ and $\partial_1$.

3.4. The equivalence $Q \xrightarrow{\sim} \text{RGrpds}_{2}$. We will first construct a canonical equivalence $Q \xrightarrow{\sim} \text{RGrpds}_{2}$ using the category $\text{SRings}_{\leq 1}^{\text{good}}$ as an intermediate step. Then we describe this equivalence directly, see (3.4.1). The reader may prefer to look at (3.4.7) before reading (3.4.1).

3.4.1. The 1-truncated nerve of a category. The nerve of a category $C$ is denoted by $N_C$. The image of $NC$ under the functor $\text{SSets} \to \text{SSets}_{\leq 1}$ will be denoted by $N_{\leq 1} C$ and called the 1-truncated nerve of $C$.

In other words, $N_{\leq 1} C$ remembers the set of objects of $C$, the set of morphisms in $C$, the source and target of each morphism, and the morphisms id$_c$ for all $c \in \text{Ob} C$; however, it forgets the composition of morphisms. Thus $N_{\leq 1} C$ is not really interesting.

Similarly, if $C$ is a category internal to Rings one has the nerve $NC \in \text{SRings}$ and the 1-truncated nerve $N_{\leq 1} C \in \text{SRings}_{\leq 1}$. The next proposition shows that in this setting $N_{\leq 1} C$ is quite interesting.

Proposition 3.4.2. (i) The above functor

$$N_{\leq 1} : \{\text{Categories internal to Rings}\} \to \text{SRings}_{\leq 1}$$

is fully faithful.
(ii) Its essential image equals \( \text{SRings}^{\text{good}}_{\leq 1} \).

(iii) Every category internal to \( \text{Rings} \) is a groupoid.

The proposition will be deduced from Lemmas 3.4.3-3.4.5. The first two of them describe categories internal to \( \text{Ab} \), where \( \text{Ab} \) is the category of abelian groups.

**Lemma 3.4.3.** Let \( \mathcal{C} \) be a category internal to \( \text{Ab} \) and let \((A_0, A_1, \partial_0, \partial_1, s)\) be its 1-truncated nerve.

(i) Suppose that \( f, g \in A_1 \) and \( \partial_1(f) = \partial_0(g) \) (in other words, \( f \) and \( g \) form a composable pair of morphisms). Then
\[
(3.3) \quad g \circ f = f + g - s(a), \quad \text{where } a = \partial_1(f) = \partial_0(g).
\]

(ii) \( \mathcal{C} \) is a groupoid. The inverse of \( f \in A_1 \) equals \( s(\partial_0(f)) + s(\partial_1(f)) - f \).

**Proof.** Let \( B \) be the group of all pairs \((f, g)\) as in (i). The map
\[
B \to A_1, \quad (f, g) \mapsto g \circ f
\]
is a group homomorphism by the definition of “category internal to \( \text{Ab} \)”. The map \( B \to A_1 \) given by (3.3) is also a group homomorphism. The group \( B \) is generated by \( B_1 \) and \( B_2 \), where \( B_1 \) (resp. \( B_2 \)) is the group of all \((f, g) \in B\) such that \( f \) (resp. \( g \)) is an identity morphism. So it suffices to check (3.3) if either \( f = \text{id}_a \) or \( g = \text{id}_a \). This is clear because \( s(a) \) is just another name for \( \text{id}_a \).

We have proved (i). Statement (ii) follows. \( \square \)

Similarly to Lemma 3.4.3(ii), one proves the following converse statement.

**Lemma 3.4.4.** Let \((A_0, A_1, \partial_0, \partial_1, s)\) be a 1-truncated simplicial abelian group. Then the operation (3.3) makes it into a category internal to \( \text{Ab} \). \( \square \)

Proposition 3.4.2 follows from Lemmas 3.4.3-3.4.4 and the following one.

**Lemma 3.4.5.** Let \((A_0, A_1, \partial_0, \partial_1, s)\) be a 1-truncated simplicial ring. Let
\[
B := \{(f, g) \mid f, g \in A_1, \partial_1(f) = \partial_0(g)\}.
\]
In this situation, the map \( B \to A_1 \) defined by (3.3) is a ring homomorphism if and only if \((\text{Ker} \partial_0) \cdot (\text{Ker} \partial_1) = 0\).

**Proof.** Let \( \varphi : B \to A_1 \) be the map (3.3). Let \( B_1, B_2 \) be as in the proof of Lemma 3.3. Then \( \varphi|_{B_1} \) and \( \varphi|_{B_2} \) are automatically ring homomorphisms. One has
\[
B_1 = (B_1 \cap B_2) + J_1, \quad B_2 = (B_1 \cap B_2) + J_2,
\]
where \( J_1 := \{(0, g) \mid g \in \text{Ker} \partial_0\}, \quad J_2 := \{(f, 0) \mid f \in \text{Ker} \partial_1\} \). Moreover, \( J_1 \cdot J_2 = 0 \). So \( \varphi : B \to A_1 \) is a ring homomorphism if and only if \( \varphi(J_1) \cdot \varphi(J_2) = 0 \), which means that \((\text{Ker} \partial_0) \cdot (\text{Ker} \partial_1) = 0\). \( \square \)

3.4.6. The equivalence \( \mathcal{Q} \xrightarrow{\sim} \text{RGrpds}^{\text{good}}_2 \). Recall that \( \text{RGrpds}^{\text{good}}_2 \) is the category of groupoids internal to \( \text{Rings} \), and \( \mathcal{Q} \) is the category of all triples \((C, I, d)\), where \( C \) is a ring and \((I, d : I \to C)\) is a quasi-ideal in \( C \). In the proof of Lemma 3.3.6 we constructed an equivalence \( \mathcal{Q} \xrightarrow{\sim} \text{SRings}^{\text{good}}_{\leq 1} \). Composing it with the equivalence \( \text{SRings}^{\text{good}}_{\leq 1} \xrightarrow{\sim} \text{RGrpds}^{\text{good}}_2 \) from Proposition 3.4.2 we get an equivalence
\[
\mathcal{Q} \xrightarrow{\sim} \text{RGrpds}^{\text{good}}_2.
\]
We denote it as follows:

\[(C, I, d) \mapsto \text{Cone}(d) = \text{Cone}(I \xrightarrow{d} C).\]

Let us now give a description of \(\text{Cone}(d)\) (the reader may prefer to use it as a definition).

### 3.4.7. Explicit description of \(\text{Cone}(d)\)

Let \(C\) be a ring and \((I, d : I \to C)\) a quasi-ideal in \(C\). Then \(\text{Cone}(I \xrightarrow{d} C)\) is a groupoid internal to Rings, whose set of objects is the ring \(C\) and whose morphisms are labeled by \(C \times I\). The morphism corresponding to \(c \in C\) and \(x \in I\) is a morphism \(c \to c + dx\), denoted by \(f_{c,x}\). Morphisms are composed as follows:

\[f_{c+dx,y} \circ f_{c,x} = f_{c,x+y} .\]

Finally, the ring structure on the set of morphisms is given by

\[f_{c,x} + f_{c',x'} = f_{c+c',x+x'},\]

(3.4) \[f_{c,x} \cdot f_{c',x'} = f_{cc',y}, \text{ where } y = cx' + c'x + x \cdot dx' = cx' + c'x + x' \cdot dx.\]

Note that as a usual groupoid (rather than a groupoid internal to Rings), \(\text{Cone}(d)\) is just the quotient groupoid of \(C\) by the following action of \(I\): an element \(x \in I\) takes \(c \in C\) to \(c + dx\).

### 3.4.8. Quasi-isomorphisms in \(Q\)

Let \(f : (C, I, d) \to (C', I', d')\) be a morphism in \(Q\). It induces a functor \(\text{Cone}(d) \to \text{Cone}(d')\) between the corresponding groupoids. Using §3.4.7, one checks that this functor is an equivalence if and only if \(f\) is a quasi-isomorphism (which means that \(f\) induces isomorphisms \(\text{Ker} d \xrightarrow{\sim} \text{Ker} d'\) and \(\text{Coker} d \xrightarrow{\sim} \text{Coker} d'\)).

### 3.4.9. Motivation of the \(\text{Cone}\) notation

If \(C\) is any ring then \(\text{Cone}(0 \to C)\) identifies with \(C\) (viewed as discrete groupoid). In general, the morphism \((C, I, d) \to (C/d(I), 0, 0)\) induces a functor

\[\text{Cone}(I \xrightarrow{d} C) \to \text{Cone}(0 \to C/d(I)) = C/d(I),\]

and if \(\text{Ker} d = 0\) this functor is an equivalence (but not an isomorphism, unless \(I = 0\)).

Let us note that in the context of abelian groups (instead of rings and quasi-ideals) the groupoid \(\text{Cone}(d)\) is considered in [SGA4, Exposé XVIII, §1.4], where it is denoted by \(\text{ch}(d)\). It is proved there that the \((2, 1)\)-category of strictly commutative Picard groupoids is canonically equivalent to the full subcategory of the DG category of complexes of abelian groups formed by complexes with cohomology concentrated in degrees \(-1\) and \(0\); moreover, this equivalence takes \(\text{ch}(d)\) to the usual cone of \(d\) (i.e., to the complex \(0 \to I \xrightarrow{d} C \to 0\) placed in degrees \(-1\) and \(0\)). This is our main motivation for writing \(\text{Cone}(d)\) instead of \(\text{ch}(d)\).

### 3.5. The parallel story for groups

This subsection and §3.6 can be skipped by the reader.

#### 3.5.1. Abelian groups

By Lemmas 3.4.3-3.4.4, the \(1\)-category of categories internal to \(\text{Ab}\) (or equivalently, groupoids internal to \(\text{Ab}\)) identifies with the category of all \(1\)-truncated simplicial abelian groups. Similarly to Lemma 3.3.6, the latter identifies with the category of triples \((C, I, d)\), where \(C, I \in \text{Ab}\) and \(d : I \to C\) is a homomorphism (this is a “baby version” of the Dold-Kan equivalence).

---

In our language, this is the \((2, 1)\)-category \(\text{Ab-Grpds}\), see §2.4.3-2.4.4.
3.5.2. Arbitrary groups. Let Groups denote the category of all groups. Let SGroups\textsubscript{good} \leq 1 be the category of 1-truncated simplicial groups \((G_0, G_1, \partial_0, \partial_1, s)\) such that \(\text{Ker} \partial_0\) centralizes \(\text{Ker} \partial_1\) (this condition is somewhat similar to \((3.2)\)). One can check that Proposition 3.4.2 and Lemma 3.3.6 remain valid if one replaces Rings, SGroups\textsubscript{good} \leq 1 and also replaces the category \(Q\) from \((3.3.1)\) by the category of crossed modules.

The notion of crossed module is due to J. H. C. Whitehead \cite{Wh}. For an overview of it, see \cite{Wei}, \S6.6.12, \cite{N2}, and references therein.

3.6. Remarks on DG rings. This subsection can be skipped by the reader.

3.6.1. Notation. Let DGRings\textsubscript{\leq 0} (resp. DGRings\textsubscript{0,−1}) be the category of DG rings \(R\) such that \(R^i = 0\) for \(i > 0\) (resp. for \(i \neq 0, -1\)).

3.6.2. DG rings via Eilenberg-Zilber. The normalized chain complex of a simplicial ring has a DG ring structure, which is defined via the Eilenberg-Zilber map. One can check that the functor \(Q \to \text{DGRings}^{\leq 0}\) from \((3.3.3)\) is isomorphic to the composite functor

\[Q \to \text{RGrpds}_2' \hookrightarrow \text{SRings} \xrightarrow{\mathcal{N}} \text{DGRings}^{\leq 0},\]

where the first arrow is as in \((3.4.6)\) the second one takes a groupoid to its nerve, and \(\mathcal{N}\) is the functor of normalized chains. We will not use this fact.

3.6.3. Simplicial rings via Alexander-Whitney. Recall that the functor \(Q \to \text{DGRings}^{\leq 0}\) induces an equivalence \(Q \xrightarrow{\sim} \text{DGRings}^{0,−1} \subset \text{DGRings}^{\leq 0}\). Let us discuss the composite functor

\[(3.5) \quad \xymatrix{
\text{DGRings}^{0,−1} \ar[r] & Q \ar[r]^-{\mathcal{N}} & \text{DGRings}^{\leq 0},
}

Let \text{Rings} be the category of unital associative but not necessarily commutative rings. The functor \(\mathcal{N} : \text{SRings} \to \text{DGRings}^{\leq 0}\) extends to a functor

\[\mathcal{N} : \text{SRings} \to \text{DGRings}^{\leq 0}.\]

The latter has a canonical right inverse \(\Gamma : \text{DGRings}^{\leq 0} \to \text{SRings}\), which is defined using the cup product on the cochain complexes of certain simplicial sets (i.e., using the Alexander-Whitney map). The functor \(\Gamma\) does not preserve commutativity (because the \(\cup\)-product is not commutative at the level of cochains); in other words,

\[(3.6) \quad \Gamma(\text{DGRings}^{\leq 0}) \not\subset \text{SRings}.
\]

However, one can check that

\[(3.7) \quad \Gamma(\text{DGRings}^{0,−1}) \subset \text{SRings},
\]

and the functor \((3.5)\) is isomorphic to \(\Gamma : \text{DGRings}^{0,−1} \to \text{SRings}\). We will not use this fact.

Here is a way to believe in formulas \((3.6)-(3.7)\) (or even to prove them): in formula \((3.4)\) we have \(x \cdot dx' = x' \cdot dx\) because \(x' \cdot dx - x \cdot dx' = d(xx')\) and \(xx' = 0\) (thus the condition \(xx' = 0\) is very essential).
4. The Aldrovandi-Noohi model of the (2,1)-category RGrpds

In this section we define a (2,1)-category RGrpds\textsubscript{AN} and prove Theorem 4.5.5 which says that RGrpds\textsubscript{AN} is canonically equivalent to RGrpds, i.e., to the (2,1)-category of ring groupoids defined in §2. This theorem is a variant of the main result of [Al12] (the main difference is that in [A12] rings are not assumed to be commutative).

The definition of RGrpds\textsubscript{AN} follows the ideas\textsuperscript{8} of B. Noohi [N1], which were further developed in the works by E. Aldrovandi and B. Noohi [AN1, AN2, Al1, Al2, N1, N3]. The symbol AN stands for Aldrovandi-Noohi and also for “anamorphism” (see §4.2.2) and “anafunctor” (the latter notion will be recalled in §5).

4.1. The 2-category of correspondences. Let \( C \) be a category in which finite fiber products always exist. Then one defines the 2-category of correspondences Corr(\( C \)) as follows.

(i) The objects of Corr(\( C \)) are those of \( C \).

(ii) For \( c_1, c_2 \in C \), the category of Corr(\( C \))-morphisms is defined to be the category of diagrams\textsuperscript{9} \( c_1 \leftarrow c_1 \rightarrow c_2 \) in \( C \). This category is denoted by Corr(\( c_1, c_2 \)), and its objects are called correspondences from \( c_1 \) to \( c_2 \).

(iii) The composition of correspondences \( c_1 \leftarrow c_1 \rightarrow c_2 \) and \( c_2 \leftarrow c_2 \rightarrow c_3 \) is defined to be the correspondence \( c_1 \leftarrow c_1 \times_{c_2} c_2 \rightarrow c_3 \).

Let us note that correspondences are also called spans (e.g., in [N1] §9.1).

4.2. Correspondences in DGRings\textsuperscript{0,−1}.

4.2.1. Recollections on DGRings\textsuperscript{0,−1}. Recall that DGRings\textsuperscript{0,−1} stands for the category of DG rings \( R \) such that \( R^i = 0 \) for \( i \neq 0,−1 \). This category is one of the incarnations of the 1-category of ring groupoids (see §3.3.3 and §3.4).

4.2.2. Three classes of correspondences. Let \( R_1, R_2 \in \text{DGRings}^{0,−1} \). According to §4.1\textsuperscript{10}, a correspondence from \( R_1 \rightarrow R_2 \) is just a diagram

\[
\begin{array}{ccc}
R_1 & \xleftarrow{f} & R_{12} \\
& \xrightarrow{g} & R_2
\end{array}
\]

in DGRings\textsuperscript{0,−1}.

We say that a correspondence (4.1) is admissible (resp. weakly admissible) if \( f \) is a quasi-isomorphism and the map \( R_{12} \rightarrow R_1 \times R_2 \) is an isomorphism (resp. epimorphism). We say that a (4.1) is an anamorphism\textsuperscript{11} from \( R_1 \) to \( R_2 \) if \( f \) is a surjective quasi-isomorphism.

Let Corr\textsubscript{adm}(\( R_1, R_2 \)) (resp. Corr\textsubscript{wadm}(\( R_1, R_2 \))) be the category of admissible (resp. weakly admissible) correspondences from \( R_1 \) to \( R_2 \). Let Corr\textsubscript{ana}(\( R_1, R_2 \)) be the category of anamorphisms from \( R_1 \) to \( R_2 \). Then

\[
\text{Corr\textsubscript{adm}(}R_1, R_2\text{)} \subset \text{Corr\textsubscript{wadm}(}R_1, R_2\text{)} \subset \text{Corr\textsubscript{ana}(}R_1, R_2\text{)}.
\]

\textsuperscript{8}The only difference is that Noohi [N1] considers (noncommutative) groups rather than rings and crossed modules rather than objects of DGRings\textsuperscript{0,−1}.

\textsuperscript{9}A morphism from a diagram \( c_1 \leftarrow c_{12} \rightarrow c_2 \) to a diagram \( c_1 \leftarrow c_{12} \rightarrow c_2 \) is a morphism \( h : c_{12} \rightarrow c_{12} \) such that \( f' h = f \) and \( g' h = g \).

\textsuperscript{10}Anamorphisms are analogous to anafunctors, see §5.1.1 below.

\textsuperscript{11}The class of surjective quasi-isomorphisms is good for us because it is closed under pullbacks (i.e., if \( f : R' \rightarrow R \) is a surjective quasi-isomorphism then so is \( \tilde{f} : R' \times_R \tilde{R} \rightarrow \tilde{R} \)). Without surjectivity this would be false. See also (14.5.01) below.
4.2.3. Admissible correspondences via “butterflies”. Admissible correspondences have the following description, which I learned from the works by Aldrovandi and Noohi (e.g., see [AN1, Def. 8.1] or [AN1, §4.1.3]). Note that given an admissible correspondence (4.1), one can use the isomorphism $R_{12}^{-1} \sim R_{1}^{-1} \times R_{2}^{-1}$ to write $d : R_{12}^{-1} \to R_{12}^{0}$ as a pair of maps $h_{i} : R_{i}^{-1} \to R_{i}^{0}$ for $i = 1, 2$. Thus we see that an admissible correspondence (4.1) is the same as a commutative diagram

![Diagram](image)

with the following properties:

(i) $R_{12}^{0}$ is a ring, and $f^{0}, g^{0}$ are ring homomorphisms;
(ii) $h_{1}$ and $h_{2}$ are $R_{12}^{0}$-module homomorphisms assuming that the $R_{12}^{0}$-module structure on $R_{1}^{-1}$ (resp. $R_{2}^{-1}$) is defined via $f^{0}$ (resp. $g^{0}$);
(iii) the NW-SE sequence in (4.2) is a complex, and the NE-SW sequence is exact, i.e.,

$$\text{Ker } h_{2} = 0, \quad \text{Im } h_{2} = \text{Ker } f^{0}, \quad \text{Coker } f^{0} = 0.$$ 

A commutative diagram (4.2) with properties (i)-(iii) is called a butterfly.

Lemma 4.2.4. The category $\text{Corr}_{adm}(R_{1}, R_{2})$ is a groupoid.

Proof. We have to show that any morphism in $\text{Corr}_{adm}(R_{1}, R_{2})$ is an isomorphism. By §4.2.3 it suffices to prove a similar statement for morphisms of butterflies. This is clear because the NE-SW sequence in (4.2) is exact. □

Lemma 4.2.5. The composition of weakly admissible correspondences is weakly admissible. The composition of anamorphisms is an anamorphism. □

4.2.6. Remarks. (i) The class of correspondences $R_{1} \xleftarrow{f} R_{12} \to R_{2}$ such that $f$ is a quasi-isomorphism is not closed under composition of correspondences.

(ii) The composition of admissible correspondences is not admissible, in general. To cure this, one uses the admisibilization functor constructed in the next subsection.

4.3. Admissibilization.

Proposition 4.3.1. The inclusion functor $\text{Corr}_{adm}(R_{1}, R_{2}) \hookrightarrow \text{Corr}_{ana}(R_{1}, R_{2})$ has a left adjoint functor $\text{Adm} : \text{Corr}_{ana}(R_{1}, R_{2}) \to \text{Corr}_{adm}(R_{1}, R_{2})$.

The functor $\text{Adm} : \text{Corr}_{ana}(R_{1}, R_{2}) \to \text{Corr}_{adm}(R_{1}, R_{2})$ is called admisibilization. Let us note that the restriction of $\text{Adm}$ to $\text{Corr}_{wadm}(R_{1}, R_{2}) \subset \text{Corr}_{ana}(R_{1}, R_{2})$ has a very simple description, see Lemma 4.3.3(iv) below.

Proof. Let $R_{1} \xleftarrow{f} R_{12} \xrightarrow{g} R_{2}$ be an anamorphism. Let $L := R_{1} \times R_{2}.$ Let $\varphi : R_{12} \to L$ be given by $(f, g)$. Note that since $f$ is a quasi-isomorphism, we have

$$\text{Ker}(H^{-1}(R_{12}) \to H^{-1}(L)) = 0.$$ 

This is the only place in the proof where we use that $f$ is an anamorphism. In particular, we do not use surjectivity of $f$. □
Let \( C \) be the category of factorizations of \( \varphi \) as
\[
R_{12} \xrightarrow{\psi} \tilde{R}_{12} \xrightarrow{\chi} L
\]
such that the correspondence
\[
R_1 \leftarrow \tilde{R}_{12} \rightarrow R_2
\]
given by \( \chi \) is admissible. By Lemma 4.2.4, \( C \) is a groupoid. We have to show that \( C \) is a point.

Admissibility of (4.5) is equivalent to the following properties of (4.4): \( \psi \) is a quasi-isomorphism, and \( \chi^{-1} : \tilde{R}_{12}^{-1} \rightarrow L^{-1} \) is an isomorphism. So as a complex of abelian groups, \( \tilde{R}_{12} \) has to be as follows: \( \tilde{R}_{12}^{-1} = L^{-1} \), and the pair \( (\tilde{R}_{12}^{0}, \tilde{d} : \tilde{R}_{12}^{-1} \rightarrow \tilde{R}_{12}^{0}) \) is determined by the push-out diagram
\[
\begin{array}{ccc}
R_{12}^{-1} & \xrightarrow{\varphi^{-1}} & L^{-1} = \tilde{R}_{12}^{-1} \\
\downarrow{d} & & \downarrow{\tilde{d}} \\
R_{12}^{0} & \xrightarrow{\psi^{0}} & \tilde{R}_{12}^{0}
\end{array}
\]

The morphism \( R_{12} \rightarrow \tilde{R}_{12} \) defined by this diagram is a quasi-isomorphism by (4.3).

Now we have a commutative diagram of complexes of abelian groups
\[
\begin{array}{ccc}
R_{12}^{-1} & \xrightarrow{\varphi^{-1}} & \tilde{R}_{12}^{-1} \xrightarrow{\text{id}} L^{-1} \\
\downarrow{d} & & \downarrow{\tilde{d}} \\
R_{12}^{0} & \xrightarrow{\psi^{0}} & \tilde{R}_{12}^{0} \xrightarrow{\varphi^{0}} L^{0}
\end{array}
\]

and the problem is to define a ring structure on \( \tilde{R}_{12} \) which makes (4.6) into a diagram of DG rings. This problem has at most one solution because \( \tilde{R}_{12}^{0} = R_{12}^{0} + \tilde{d}(\tilde{R}_{12}^{-1}) \) and from the right square of (4.6) we see that for \( x, y \in \tilde{R}_{12}^{-1} \) we must have
\[
(\tilde{d}x) \cdot y = (d'x) \cdot y, \quad (\tilde{d}x) \cdot (\tilde{d}y) = \tilde{d}((d'x) \cdot y) = \tilde{d}((d'x) \cdot y).
\]

It is straightforward to check that these formulas indeed define a solution to our problem. \( \square \)

4.3.2. Admissibilization as localization. The functor
\[
\text{Adm} : \text{Corr}_{\text{ana}}(R_1, R_2) \rightarrow \text{Corr}_{\text{adm}}(R_1, R_2)
\]
identifies \( \text{Corr}_{\text{adm}}(R_1, R_2) \) with the groupoid obtained from \( \text{Corr}_{\text{ana}}(R_1, R_2) \) by inverting all morphisms. This follows from Lemma 4.2.4 and the adjunction from Proposition 4.3.1

The admissibilization of a weakly admissible correspondence is described in part (iv) of the next lemma.

Lemma 4.3.3. Suppose that a correspondence
\[
R_1 \xleftarrow{f} R_{12} \xrightarrow{g} R_2
\]
is weakly admissible. Then
(i) \( f \) is surjective;
(ii) Ker $f$ is acyclic;
(iii) the DG ideal $I \subset R_{12}$ generated by Ker($R_{12}^{-1} \to R_1^{-1} \times R_2^{-1}$) is acyclic;
(iv) the correspondence

$$R_1 \leftarrow R_{12}/I \to R_2$$

is the admissibilization of $I$. \hfill \Box

4.3.4. Admissibilization of the composition. Let $\alpha \in \text{Corr}_{\text{ana}}(R_1, R_2)$, $\beta \in \text{Corr}_{\text{ana}}(R_2, R_3)$, where $R_1, R_2, R_3 \in \text{DGRings}^{0,-1}$. The canonical morphisms $\alpha \to \text{Adm}(\alpha)$ and $\beta \to \text{Adm}(\alpha)$ induce a morphism $\beta \circ \alpha \to \text{Adm}(\beta) \circ \text{Adm}(\alpha)$. The corresponding morphism

$$\text{Adm}(\beta \circ \alpha) \to \text{Adm}(\text{Adm}(\beta) \circ \text{Adm}(\alpha))$$

is an isomorphism by Lemma 4.2.4. Thus $\text{Adm}(\beta \circ \alpha) = \text{Adm}(\text{Adm}(\beta) \circ \text{Adm}(\alpha))$.

4.3.5. The map $\text{adm} : \text{Hom}(R_1, R_2) \to \text{Corr}_{\text{adm}}(R_1, R_2)$. Define a map

$$\text{adm} : \text{Hom}(R_1, R_2) \to \text{Corr}_{\text{adm}}(R_1, R_2)$$

to be the composition $\text{Hom}(R_1, R_2) \to \text{Corr}_{\text{ana}}(R_1, R_2) \xrightarrow{\text{Adm}} \text{Corr}_{\text{adm}}(R_1, R_2)$, where the first map takes $\varphi \in \text{Hom}(R_1, R_2)$ to the correspondence

$$R_1 \xleftarrow{\text{id}} R_1 \xrightarrow{\varphi} R_2.$$

**Lemma 4.3.6.** (i) Let $\alpha \in \text{Corr}_{\text{adm}}(R_1, R_2)$ be an admissible correspondence

$$R_1 \xleftarrow{f} R_{12} \xrightarrow{g} R_2$$

Let $\text{adm}^{-1}(\alpha)$ be the fiber of (4.9) over $\alpha$ (i.e., the set of isomorphism classes of pairs consisting of an element $\varphi \in \text{Hom}(R_1, R_2)$ and an isomorphism $\text{adm}(\varphi) \sim \alpha$). Then $\text{adm}^{-1}(\alpha)$ canonically identifies with the set of splittings

$$\{s : R_1 \to R_{12} \mid f \circ s = \text{id}_{R_1}\},$$

and after this identification the map $\text{adm}^{-1}(\alpha) \to \text{Hom}(R_1, R_2)$ is given by $s \mapsto g \circ s$.

(ii) The map from the set (4.12) to the set

$$\{\sigma : R_1^0 \to R_{12}^0 \mid f^0 \circ \sigma = \text{id}_{R_1^0}\}$$

given by $\sigma = s^0$ is bijective.

The lemma implies that an admissible correspondence (4.11) belongs to the essential image of (4.9) if and only if the homomorphism $f^0 : R_{12}^0 \to R_1^0$ admits a splitting. In terms of butterflies, this means that the NE-SW exact sequence in (4.2) admits a splitting; in §4.5 of [AN1] such butterflies are called splittable.

**Proof.** Combining the definitions of $\text{adm}$ and $\text{Adm}$ with Lemma 4.2.4 we see that an element of $\text{adm}^{-1}(\alpha)$ is the same as an element $\varphi \in \text{Hom}(R_1, R_2)$ plus a morphism from the correspondence (4.10) to the correspondence (4.11). Such a morphism is the same as a homomorphism $s : R_1 \to R_{12}$ such that $g \circ s = \varphi$. This proves (i).

Statement (ii) easily follows from $f$ being a surjective quasi-isomorphism. \hfill \Box
4.3.7. Explicit description of $\text{adm}(\varphi)$. By Lemma 4.3.6, the groupoid of admissible correspondences (4.11) equipped with a splitting $\sigma : R_1^0 \to R_{12}^0$ is a set, which identifies with $\text{Hom}(R_1, R_2)$ as follows: given an admissible correspondences (4.11) and a splitting $\sigma : R_1^0 \to R_{12}^0$ one defines $\varphi \in \text{Hom}(R_1, R_2)$ by $\varphi = g \circ s$, where $s : R_1 \to R_{12}$ is the unique splitting of $f : R_{12} \to R_1$ with $s^0 = \sigma$.

Let us describe the construction in the opposite direction in terms of butterflies. Given $\varphi \in \text{Hom}(R_1, R_2)$, we have to construct a butterfly

\[
\begin{array}{c}
\begin{tikzcd}
R_{12}^0 \arrow{d}{\rho^0} \\
R_1^0 \arrow{r}{g^0} \arrow{u}{h_2} & R_2^0 \arrow{u}{h_1}
\end{tikzcd}
\end{array}
\]

equipped with a splitting $\sigma : R_1^0 \to R_{12}^0$. One checks that the answer is as follows: as an additive group,

\[R_{12}^0 = \{(x, y) \mid x \in R_1^0, y \in R_2^{-1}\},\]

the multiplication operation in $R_{12}^0$ is given by

\[(x, y) \cdot (x', y') = (xx', \varphi^0(x)y' + \varphi^0(x')y + dy \cdot y'),\]

and the maps $h_1, h_2, f^0, g^0$ and $\sigma : R^0 \to R_{12}^0$ are given by

\[
\begin{align*}
&h_2(y) = (0, y), \quad f^0(x, y) = x, \quad \sigma(x) = (x, 0), \\
g^0(x, y) = \varphi^0(x) + dy, \quad h_1(z) = (dz, -\varphi^{-1}(z)).
\end{align*}
\]

4.4. The $(2,1)$-category $\text{RGrpds}_{AN}$.

4.4.1. Definition. Following [N1, AN1] and other works by Aldrovandi and Noohi, we define a $(2,1)$-category $\text{RGrpds}_{AN}$ as follows:

(a) its objects are those of $\text{DGRings}^{0,-1}$;

(b) for $R_1, R_2 \in \text{DGRings}^{0,-1}$, the groupoid of 1-morphisms from $R_1$ to $R_2$ is $\text{Corr}_{\text{adm}}(R_1, R_2)$;

(c) the composition of 1-morphisms is the admissibilization of their composition as correspondences.

Good news: the composition of admissible correspondences is weakly admissible, so its admissibilization is as described in Lemma 4.3.3(iii). So the reader can easily describe composition of 1-morphisms in $\text{RGrpds}_{AN}$ using the language of butterflies from §4.2.3 (on the other hand, the answer can be found in [N1, §10.1] or [AN1, §5.1.1]).

To check that $\text{RGrpds}_{AN}$ is indeed a 2-category, one has to prove the existence of the identity 1-morphisms. In fact, the identity endomorphism of the object of $\text{RGrpds}_{AN}$ corresponding to $R \in \text{DGRings}^{0,-1}$ is $\text{adm}(\text{id}_R)$, where $\text{adm} : \text{Hom}(R, R) \to \text{Corr}_{\text{adm}}(R, R)$ is as in §4.3.5 this follows from §4.3.4. An explicit description of $\text{adm}(\text{id}_R)$ was given in §4.3.7.

4.4.2. Exercise. The 1-morphism given by an admissible correspondence $R_1 \xleftarrow{f} R_{12} \xrightarrow{g} R_2$ is invertible if and only if $f$ is a quasi-isomorphism; in this case the inverse 1-morphism is the correspondence $R_2 \xleftarrow{g} R_{12} \xrightarrow{f} R_1$. 

15
4.4.3. The functor $\text{DGRings}^{0,-1} \to \text{RGrpds}_{\text{AN}}$. Recall that $\text{DGRings}^{0,-1}$ and $\text{RGrpds}_{\text{AN}}$ have the same objects. We define a functor

$$\text{DGRings}^{0,-1} \to \text{RGrpds}_{\text{AN}}$$

as follows: at the level of objects, it is the identity, and at the level of morphisms it is given by the map $\text{adm} : \text{Hom}(R_1, R_2) \to \text{Corr}_{\text{adm}}(R_1, R_2)$ from §4.3.5. Compatibility with composition of morphisms follows from §4.3.4.

4.5. The equivalence $\text{RGrpds}_{\text{AN}} \sim \to \text{RGrpds}$.

4.5.1. The 2-category $\text{RGrpds}_{\text{ana}}^{\text{AN}}$. Define a 2-category $\text{RGrpds}_{\text{ana}}^{\text{AN}}$ as follows:

(a) its objects are those of $\text{DGRings}^{0,-1}$;
(b) for $R_1, R_2 \in \text{DGRings}^{0,-1}$, the category of 1-morphisms from $R_1$ to $R_2$ is $\text{Corr}_{\text{ana}}(R_1, R_2)$;
(c) the composition of 1-morphisms is the usual composition of correspondences.

4.5.2. The functor $\text{DGRings}^{0,-1} \to \text{RGrpds}$. Recall that $\text{RGrpds}$ is the $(2, 1)$-category of ring objects in the $(2, 1)$-category of groupoids. So we have a tautological functor $\text{RGrpds}'_1 \to \text{RGrpds}$, where $\text{RGrpds}'_1$ is the category of ring objects in the naive 1-category of groupoids. In §3.3-3.4 we constructed equivalences

$$\text{RGrpds}'_1 \sim \to Q \sim \to \text{DGRings}^{0,-1}.$$ 

So we get a functor

$$\text{(4.14)} \quad \text{DGRings}^{0,-1} \to \text{RGrpds}.$$ 

4.5.3. The functor $\text{RGrpds}_{\text{ANA}}^{\text{ana}} \to \text{RGrpds}$. By §3.4.8 the functor (1.14) takes quasi-isomorphisms in $\text{DGRings}^{0,-1}$ to equivalences between ring groupoids. So the functor (1.14) canonically extends to a functor

$$\text{(4.15)} \quad \text{RGrpds}_{\text{ANA}}^{\text{ana}} \to \text{RGrpds}.$$ 

4.5.4. The functor $\text{RGrpds}_{\text{AN}} \to \text{RGrpds}$. We have a canonical functor

$$\text{(4.16)} \quad \text{RGrpds}_{\text{ANA}} \to \text{RGrpds}_{\text{AN}},$$

which acts as identity at the level of objects and as $\text{Adm} : \text{Corr}_{\text{ana}}(R_1, R_2) \to \text{Corr}_{\text{adm}}(R_1, R_2)$ at the level of morphisms. By §4.3.2 the functor (1.16) has the following universal property: for any $(2, 1)$-category $C$, any functor $\text{RGrpds}_{\text{ANA}}^{\text{ana}} \to C$ uniquely factors through $\text{RGrpds}_{\text{AN}}$. In particular, the functor (1.15) induces a functor

$$\text{(4.17)} \quad \text{RGrpds}_{\text{AN}} \to \text{RGrpds}.$$ 

Theorem 4.5.5. The functor (4.17) is an equivalence.
4.5.6. Reducing Theorem 4.5.5 to Proposition 4.5.7. Let $R_i \in \text{DGRings}^{0,-1}$ and let $R_i \in \text{RGrpds}$ be the image of $R_i$ under the functor (4.14). The functor (4.15) induces functors

\begin{align*}
(4.18) \quad \text{Corr}_{\text{ana}}(R_1, R_2) &\rightarrow \text{Mor}_{\text{RGrpds}}(R_1, R_2), \\
(4.19) \quad \text{Corr}_{\text{adm}}(R_1, R_2) &\rightarrow \text{Mor}_{\text{RGrpds}}(R_1, R_2),
\end{align*}

where $\text{Mor}_{\text{RGrpds}}(R_1, R_2)$ is the groupoid of 1-morphisms $R_1 \rightarrow R_2$.

By Proposition 3.2.4, the functor (4.17) is essentially surjective, so it remains to show that the functor (4.19) is an equivalence. Since $\text{Mor}_{\text{RGrpds}}(R_1, R_2)$ is a groupoid, this follows from the next proposition.

**Proposition 4.5.7.** Let $\Phi \in \text{Mor}_{\text{RGrpds}}(R_1, R_2)$. Let $\text{Corr}^\Phi_{\text{ana}}(R_1, R_2)$ and $\text{Corr}^\Phi_{\text{adm}}(R_1, R_2)$ be the fibers of (4.18) and (4.19) over $\Phi$. Then

(i) the category $\text{Corr}^\Phi_{\text{ana}}(R_1, R_2)$ has a final object;
(ii) the final object of $\text{Corr}^\Phi_{\text{ana}}(R_1, R_2)$ belongs to $\text{Corr}^\Phi_{\text{adm}}(R_1, R_2)$.

The proof given below uses the fiber product in the $(2, 1)$-category of ring groupoids, see §2.2.7.

**Proof.** Statement (ii) follows from (i): if $X \in \text{Corr}_{\text{ana}}(R_1, R_2)$ is the image of a final object of $\text{Corr}^\Phi_{\text{ana}}(R_1, R_2)$ then the morphism $X \rightarrow \text{Adm}(X)$ is an isomorphism, so $X$ is admissible.

To prove (i), we will use the equivalence

\begin{equation}
(4.20) \quad \text{DGRings}^{0,-1} \sim \rightarrow \text{RGrpds}'_3,
\end{equation}

from §3 where $\text{RGrpds}'_3$ is as in §3.2.3. In particular, we think of $R_n \in \text{DGRings}^{0,-1}$ as a pair $(R_n, \pi_n : R^0_n \rightarrow R_n) \in \text{RGrpds}'_3$. The equivalence (4.20) identifies $\text{Corr}^\Phi_{\text{ana}}(R_1, R_2)$ with the category of 2-commutative diagrams

\[
\begin{array}{cccc}
R^0_1 & \xleftarrow{\Phi \pi_1} & R^0_{12} & \xrightarrow{\pi_2} & R^0_2 \\
\downarrow \Phi \pi_1 & & \downarrow \pi_2 & & \\
R & & R_2 & & \\
\end{array}
\]

So $\text{Corr}^\Phi_{\text{ana}}(R_1, R_2)$ has a final object: it corresponds to the diagram

\begin{equation}
(4.21) \quad \begin{array}{cccc}
R^0_1 & \xleftarrow{\Phi \pi_1} & R^0_{12} & \xrightarrow{\pi_2} & R^0_2 \\
\downarrow & & \downarrow & & \\
R & & R_2 & & \\
\end{array}
\end{equation}

(the map $R^0_1 \times_{R_2} R^0_2 \rightarrow R^0_1$ is surjective because $\pi_2 : R^0_2 \rightarrow R_2$ is essentially surjective). □

The above proof of Proposition 4.5.7(ii) was somewhat indirect (we used the functor $\text{Adm}$ and the morphism $\text{Id} \rightarrow \text{Adm}$). In §4.7.2 we will give a direct proof of this fact. But first we have to discuss the notion of admissible correspondence in a more abstract context.

4.6. Admissible correspondences between categories.
4.6.1. Notation. Let $\text{Cats}$ be the 2-category of essentially small categories. Let $\text{Cats}'$ be the most naive 1-category of small categories (its morphisms are functors on the nose). Recall that if $C_1, C_2$ are categories we write $\text{Funct}(C_1, C_2)$ for the category of functors $C_1 \to C_2$.

Given $C_1, C_2 \in \text{Cats}'$, let $\text{Corr}(C_1, C_2)$ be the category of correspondences from $C_1$ to $C_2$ in $\text{Cats}'$ (see §4.1(ii)). Thus objects of $\text{Corr}(C_1, C_2)$ are diagrams (4.22)

$$C_1 \xleftarrow{F} C_{12} \xrightarrow{G} C_2$$

in $\text{Cats}'$.

4.6.2. The graph of a functor. By the graph of a functor $\Phi : C_1 \to C_2$ we mean the category of triples $(c_1, c_2, \psi)$, where $c_i \in C_i$ and $\psi$ is an isomorphism $\Phi(c_1) \sim c_2$. We denote this category by $\text{Graph}_\Phi$. The correspondence $C_1 \leftarrow \text{Graph}_\Phi \rightarrow C_2$ will be denoted by $\text{Graph}(\Phi)$.

Thus we get a functor (4.23) $\text{Graph} : \text{Funct}(C_1, C_2) \to \text{Corr}(C_1, C_2), \quad \Phi \mapsto \text{Graph}(\Phi)$.

\[ \text{Lemma 4.6.3.} \quad \text{The functor (4.23) is fully faithful.} \quad \square \]

4.6.4. Admissible correspondences. Let $\text{Corr}_{\text{adm}}(C_1, C_2)$ denote the essential image of (4.23). We say that a correspondence (4.22) is admissible if it belongs to $\text{Corr}_{\text{adm}}(C_1, C_2)$. In this case $F : C_{12} \to C_1$ is a strictly surjective equivalence.

\[ \text{Lemma 4.6.5.} \quad \text{Let (4.24) be a correspondence in } \text{Cats}' \text{ such that } F \text{ is an equivalence. Then the following are equivalent:} \]

(a) the correspondence (4.24) is admissible;

(b) the canonical map (4.25)

$$\text{Ob}_{C_{12}} \to \text{Ob}_{C_1 \times C_2} \text{ Ob}_{C_2}$$

is bijective; here

$$\text{Ob}_{C_1 \times C_2} \text{ Ob}_{C_2} := \{(c_1, c_2, \psi) \mid c_i \in \text{Ob} C_i, \quad \psi : G F^{-1}(c_1) \sim c_2\}$$

is the fiber product in the 2-category $\text{Cats}$, and the map (4.25) takes $c \in C_{12}$ to the triple $(c_1, c_2, \psi)$, where $c_1 = F(c)$, $c_2 = G(c)$, and $\psi : G F^{-1}(c) \sim G(c)$ comes from the isomorphism $F^{-1}F(c) \sim c$;

c) the functor

$$H : C_{12} \to C_1 \times C_2, \quad H := (F,G)$$

has the following property: for every $c \in C_{12}$ every $(C_1 \times C_2)$-isomorphism with source $H(c)$ has one and only lift to a $C_{12}$-isomorphism with source $c$. \quad \square

\[ ^{13} \text{Our terminology is not standard. In [Mak] admissible correspondences are called saturated anafunctors.} \]
4.7. Comparing the two notions of admissibility. Let
\[ R^1 \leftrightarrow R_{12} \rightarrow R_2 \]
be a correspondence in RGrpds' (i.e., in the 1-category of ring groupoids). It is said to be admissible if its image under the equivalence RGrpds' \( \sim \) DGRings\(^{0,-1}\) is an admissible correspondence in DGRings\(^{0,-1}\) (as defined in \( \S 4.2.2\)).

On the other hand, applying to (4.26) the functor RGrpds' \( \rightarrow \) Grpds' \( \subset \) Cats' from \( \S 3.2\) we get a correspondence in Cats'. For such correspondences we have the notion of admissibility from \( \S 4.6.4\).

**Lemma 4.7.1.** A correspondence \( R^1 \leftrightarrow R_{12} \rightarrow R_2 \) in RGrpds' is admissible if and only if its image in Cats' is an admissible correspondence in the sense of \( \S 4.6.4\).

**Proof.** Use the equivalence (a)\( \iff \) (c) from Lemma 4.6.5 (combined with \( \S 3.4.7\); \( \S 3.4.8\)). \( \square \)

4.7.2. A direct proof of Proposition 4.5.7(ii). We have to show that the correspondence in DGRings\(^{0,-1}\) (or equivalently, in RGrpds' corresponding to diagram (4.21)) is admissible. This follows from Lemma 4.7.1 and the equivalence (a)\( \iff \) (c) from Lemma 4.6.5. \( \square \)

4.7.3. On the functor \( \text{Mor}_{\text{RGrpds}}(R^1, R_2) \sim \text{Corr}_{\text{adm}}(R_1, R_2) \). Let \( R^1, R_1, R_2 \in \text{RGrpds} \) correspond to \( R_1, R_2 \in \text{DGRings}^{0,-1}\). In the proof of Proposition 4.5.7 we constructed the functor \( \text{Mor}_{\text{RGrpds}}(R^1, R_2) \rightarrow \text{Corr}_{\text{adm}}(R_1, R_2) \) inverse to (4.19). The same functor can be described as follows.

First of all, \( \text{Corr}_{\text{adm}}(R_1, R_2) = \text{Corr}_{\text{adm}}(R^1, R^2) \), where \( R^i \in \text{RGrpds}' \) is the image of \( R_i \) under the equivalence DGRings\(^{0,-1}\) \( \sim \) RGrpds'. To construct the functor
\[ \text{Mor}_{\text{RGrpds}}(R^1, R_2) \rightarrow \text{Corr}_{\text{adm}}(R^1, R^2), \]
recall that by definition, \( R^i \) is a functor \( F_i : \text{Pol}^{\text{op}} \rightarrow \text{Grpds}' \). A 1-morphism \( R^1 \rightarrow R_2 \) defines for each \( A \in \text{Pol}^{\text{op}} \) a functor from the groupoid \( F_1(A) \) to the groupoid \( F_2(A) \). Its graph (in the sense of \( \S 4.6.2\)) is a correspondence \( F_1(A) \leftrightarrow F_{12}(A) \rightarrow \) F\(_2\)(A). Note that \( F_{12}(A) \) is defined up to unique isomorphism, i.e., as an object of Grpds'. The assignment \( A \mapsto F_{12}(A) \) is a functor
\[ \text{Pol}^{\text{op}} \rightarrow \text{Grpds}', \]
commuting with products, i.e., an object of RGrpds'. Thus we get an object of Corr\((R^1, R^2)\). By Lemma 4.7.1 it is in Corr\(_{\text{adm}}\)(\( R^1, R^2 \)).

5. Anafunctors

We keep the notation of \( \S 4.6.1, \S 4.6.2\) in particular, Cats' stands for the most naive 1-category of categories (its morphisms are functors on the nose). In this section we continue the discussion of correspondences in Cats', which began in \( \S 4.6\). We mostly follow M. Makkai [Mak].

5.1. **Four classes of correspondences in** Cats'. Given \( C_1, C_2 \in \text{Cats}' \), we defined in \( \S 4.6.4\) a strictly full subcategory \( \text{Corr}_{\text{adm}}(C_1, C_2) \subset \text{Corr}(C_1, C_2) \). Now we are going to define strictly full subcategories
\[ \text{Corr}_{\text{adm}}(C_1, C_2) \subset \text{Corr}_{\text{ana}}(C_1, C_2) \subset \text{Corr}_{\text{eq}}(C_1, C_2) \subset \text{Corr}(C_1, C_2) \]
containing \( \text{Corr}_{\text{adm}}(C_1, C_2) \).
5.1.1. Corr\(_{\text{eq}}(C_1, C_2)\) and Corr\(_{\text{ana}}(C_1, C_2)\). Define strictly full subcategories
\[
    \text{Corr}_{\text{ana}}(C_1, C_2) \subset \text{Corr}_{\text{eq}}(C_1, C_2) \subset \text{Corr}(C_1, C_2)
\]
as follows: a correspondence
\[
    C_1 \xleftarrow{F} C_{12} \xrightarrow{G} C_2
\]
is in Corr\(_{\text{eq}}(C_1, C_2)\) (resp. in Corr\(_{\text{ana}}(C_1, C_2)\)) if and only if \(F : C_{12} \to C_1\) is an equivalence (resp. a strictly surjective equivalence).

Objects of Corr\(_{\text{ana}}(C_1, C_2)\) are called anafunctors; this terminology goes back to M. Makkai’s work [Mak]. Warning: Makkai’s notion of morphism of anafunctors is different from ours (his category of anafunctors is equivalent to Funct\((C_1, C_2)\)).

5.1.2. The diagram Corr\(_{\text{eq}}(C_1, C_2) \rightrightarrows \text{Funct}(C_1, C_2)\). (i) In §4.6.2 we defined a fully faithful functor
\[
    \text{Graph} : \text{Funct}(C_1, C_2) \to \text{Corr}_{\text{ana}}(C_1, C_2) \subset \text{Corr}_{\text{eq}}(C_1, C_2).
\]
(ii) On the other hand, we have the following functor Corr\(_{\text{eq}}(C_1, C_2) \to \text{Funct}(C_1, C_2)\): to a diagram (5.1) such that \(F\) is an equivalence we associate \(G \circ F^{-1} \in \text{Funct}(C_1, C_2)\). Note that \(F^{-1}\) and \(G \circ F^{-1}\) are defined up to unique isomorphism.

**Lemma 5.1.3.** The functor Corr\(_{\text{eq}}(C_1, C_2) \to \text{Funct}(C_1, C_2)\) from (5.1.2(ii) is left adjoint to the functor (5.2). The unit of the adjunction is given by the functor
\[
    C_{12} \to \text{Graph}_{GF^{-1}}
\]
that takes \(c \in C_{12}\) to \((c_1, c_2, \psi)\), where \(c_1 = F(c),\) \(c_2 = G(c)\), and \(\psi : GF^{-1}F(c) \xrightarrow{\sim} G(c)\) comes from the isomorphism \(F^{-1}F(c) \xrightarrow{\sim} c\).

5.1.4. Admissible and weakly admissible correspondences. In §5.2 we defined the category Corr\(_{\text{adm}}(C_1, C_2)\) of admissible correspondences to be the essential image of the functor (5.2). By Lemma 5.1.3 a correspondence (5.1) is admissible if and only if the functor (5.3) is an isomorphism.

We say that a correspondence (5.1) is weakly admissible if the functor (5.3) is strictly surjective. Note that in our situation essential surjectivity of (5.3) is automatic; in fact, the functor (5.3) is automatically an equivalence.

Let Corr\(_{\text{wadm}}(C_1, C_2) \subset \text{Corr}_{\text{eq}}(C_1, C_2)\) be the full subcategory of weakly admissible correspondences. One has
\[
    \text{Corr}_{\text{wadm}}(C_1, C_2) \subset \text{Corr}_{\text{ana}}(C_1, C_2)
\]
because strict surjectivity of (5.3) implies strict surjectivity of \(F : C_{12} \to C_1\).

In Lemmas 4.6.5 we gave two criteria for admissibility. This lemma remains valid if one replaces “admissible” by “weakly admissible”, replaces “bijective” by “surjective” in (b) and removes the words “only one” from (c).

5.1.5. Admissibilization. The functor Corr\(_{\text{eq}}(C_1, C_2) \to \text{Corr}_{\text{wadm}}(C_1, C_2)\) obtained by composing the two functors from §5.1.2 is called admissibilization\(^{14}\).

If a correspondence (5.1) is weakly admissible then its admissibilization is the correspondence \(C_1 \leftarrow \tilde{C}_{12} \to C_2\) obtained by setting \(\text{Ob} \tilde{C}_{12} := \text{Ob}(C_{12})/R\), where \(R\) is the following

\(^{14}\)The name used by Makkai [Mak] is saturation. His name for “admissible correspondence” is “saturated anafunctor”.

20
equivalence relation: \( c \sim c' \) if \( F(c) = F(c'), G(c) = G(c') \), and the unique isomorphism \( \alpha : c \sim \sim c' \) with \( F(\alpha) = \text{id} \) satisfies \( G(\alpha) = \text{id} \). (We do not have to worry about the morphisms of \( \tilde{C}_{12} \) and their composition: they come from \( C_1 \)).

5.2. Composition of functors and correspondences. As before, we follow [Mak].

5.2.1. Composition of correspondences. According to the general definition from §4.1(iii), correspondences in \( \text{Cats}' \) are composed as follows: the composition of correspondences

\[
C_1 \leftarrow C_{12} \rightarrow C_2 \quad \text{and} \quad C_2 \leftarrow C_{23} \rightarrow C_3
\]

is the correspondence \( C_1 \leftarrow C_{12} \times_{C_2} C_{23} \rightarrow C_3 \), where \( C_{12} \times_{C_2} C_{23} \) is the fiber product in \( \text{Cats}' \) (rather than in \( \text{Cats} \)). In general, the fiber product in \( \text{Cats}' \) is “not really good”. But it is good enough if the correspondences in question are anafunctors.

One checks that the composition of anafunctors is an anafunctor, and the composition of weakly admissible correspondences is weakly admissible. Moreover, the construction of §5.1.2(ii) takes composition of anafunctors to composition of functors.

Thus we get the 2-category whose objects are categories, whose 1-morphisms are anafunctors, and whose 2-morphisms are 2-morphisms between correspondences; we also get a functor from this 2-category to \( \text{Cats} \).

Let us note that the composition of admissible correspondences is usually not admissible. In other words, given functors \( C_1 \xrightarrow{\Phi} C_2 \xrightarrow{\Psi} C_3 \), we have a canonical surjective equivalence \( \text{Graph}_\Phi \times_{C_2} \text{Graph}_\Psi \rightarrow \text{Graph}_{\Psi \circ \Phi} \), but it is not an isomorphism. Indeed, an object of \( \text{Graph}_{\Psi \circ \Phi} \) is given by objects \( c_1 \in C_1, c_3 \in C_3 \), and an isomorphism \( \Psi(\Phi(c_1)) \sim c_3 \); on the other hand, to specify an object of \( \text{Graph}_\Phi \times_{C_2} \text{Graph}_\Psi \), one needs, in addition, an object \( c_2 \in C_2 \) and an isomorphism \( \Phi(c_1) \sim c_2 \).

5.2.2. \( \text{Cats}_{\text{AN}} \) as a “model” for \( \text{Cats} \). Thus we see that the 2-category \( \text{Cats} \) is equivalent to the 2-category \( \text{Cats}_{\text{AN}} \) defined as follows:

(a) the objects of \( \text{Cats}_{\text{AN}} \) are categories;
(b) the category of 1-morphisms from an object \( C_1 \) to and object \( C_2 \) is \( \text{Corr}_{\text{adm}}(C_1, C_2) \);
(c) the composition of 1-morphisms is the admissibilization of their composition as correspondences in \( \text{Cats}' \).

The above definition of \( \text{Cats}_{\text{AN}} \) is parallel to the definition of \( \text{RGrpds}_{\text{AN}} \) from §4.4.1.

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