QUANTUM ARITHMETIC DYNAMICS

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Abstract. We study dynamics of the Lattès maps in the complex plane in terms of the Cuntz-Krieger algebras associated to the endomorphisms of the non-commutative tori. In particular, it is shown that iterations of the Lattès maps can be reduced to the dynamics of the subshifts of finite type. Using such a reduction, we calculate the zeta function of the Lattès maps.

1. Introduction

The aim of our note is an interplay between the arithmetic dynamics [Silverman 2007] [7] and the quantum dynamics, i.e. the crossed product C*-algebras [Williams 2007] [9]. Namely, we construct a functor from the category of Lattès dynamical systems [Silverman 2007] [7], Section 6.4 to such of the Cuntz-Krieger algebras [Cuntz & Krieger 1980] [2], see theorem 1.3. Such a functor reduces complex dynamics of the Lattès maps to the symbolic dynamics on the alphabet consisting of only two symbols [Lind & Marcus 1995] [5], see corollary 1.4. Our construction is based on the fundamental correspondence between elliptic curves and non-commutative tori [6, Section 1.3].

Let $E$ be an elliptic curve, i.e. the subset of the complex projective plane of the form $\{(x, y, z) \in \mathbb{CP}^2 \mid y^2 z = x^3 + axz^2 + bx^2 + cz^3\}$, where $a, b$ and $c$ are some constant complex numbers. Recall that a rational map $\phi: \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree $d \geq 2$ is called the Lattès map if there exist an elliptic curve $E$, a morphism $\psi: E \to E$, and a finite covering $\pi: E \to \mathbb{CP}^1$, such that $\pi \circ \phi = \psi \circ \pi$.

Example 1.1. Consider a double covering $\pi$ of the $\mathbb{CP}^1$ defined by the involution $(x, y, z) \mapsto (x, -y, z)$ of $E$. Let $\psi: E \to E$ be the duplication map $(x, y, z) \mapsto (2x, 2y, 2z)$ commuting with the involution $\pi$. Taking projection of $E$ to the first coordinate and using the Bachet duplication formulas, one gets the Lattès map:

$$\phi(x) = \frac{x^4 - 2bx^2 - 8cx + b^2 - 4ac}{x^3 + 4ax^2 + bx + c}. \quad (1.1)$$

The non-commutative torus $A_\theta$ is a universal C*-algebra $\langle u, v \mid vu = e^{2\pi i \theta} uv \rangle$, where $u, v$ are unitary operators and $\theta \in \mathbb{R}$ is a constant. The $A_\theta$ is said to have real multiplication if $\theta \in \mathbb{Q}(\sqrt{D})$ is a quadratic irrationality. The non-commutative tori and elliptic curves are related by a fundamental correspondence saying that there exists a covariant functor $\mathcal{F}: \{E\} \to \{A_\theta\}$ which maps the morphisms between elliptic curves to the endomorphisms of non-commutative tori [6, Section 1.3].

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Example 1.2. Let $D > 1$ be an integer. If $E_{CM}$ is an elliptic curve with complex multiplication by $\sqrt{-D}$, then $\mathcal{F}(E_{CM}) = \mathcal{A}_{RM}$, where $\mathcal{A}_{RM}$ is the non-commutative torus with real multiplication by $\sqrt{-D}$ [6, Theorem 6.1.2].

Recall that the two-dimensional Cuntz-Krieger algebra $O_A$ is a universal $C^*$-algebra $(s_1, s_2 | s_1^* s_1 = a_{11} s_1^* + a_{12} s_2^* s_2, s_2^* s_2 = a_{21} s_1^* + a_{22} s_2^* s_2, s_1 s_1^* + s_2 s_2^* = 1)$, where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a matrix with the non-negative integer entries, while $s_1, s_2$ are partial isometries and $s_1^*, s_2^*$ are their conjugates. The $O_A$ is a quantum dynamical system, i.e. it can be represented as a crossed product $C^*$-algebra [Williams 2007] [9]. Namely,

$$O_A \otimes \mathcal{K} \cong \mathbb{A}_\theta \rtimes \mathbb{Z}, \quad (1.2)$$

where $\mathcal{K}$ is the $C^*$-algebra of all compact operators on a Hilbert space and the crossed product at the RHS of (1.2) is taken by a shift endomorphism $\sigma$ defined by a transposed matrix $A'$ corresponding to the stationary AF-algebra $\mathbb{A}_\theta$ containing a dense copy of $\mathbb{A}_\theta$, see e.g. [Blackadar 1986] [1, Exercise 10.11.9 (b)], [6, Sections 3.5 and 3.7] or Section 2.3 of this paper.

Let $K \subset \mathbb{C}$ be a number field, such that $K \cong \mathbb{Q}$ or a finite extension of $\mathbb{Q}$. Let $PGL_2(K) = GL_2(K)/K^*$ be the matrix group with entries in $K$, see Section 2.1. Denote by $\mathcal{L}$ a category of all dynamical systems generated by iterations of the Lattès maps $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$ with the coefficients in $K$ and such that the projection maps $\pi$ and $\pi'$ associated to $\phi$ and $\phi'$ in Figure 1 both have degree 2. The arrows (morphisms) of $\mathcal{L}$ are the linear conjugacies $\{f^{-1} \circ \phi \circ f | f \in PGL_2(K)\}$ between the Lattès maps $\phi$. Likewise, denote by $\mathcal{Q}$ a category of all two-dimensional Cuntz-Krieger algebras $O_A$. The arrows (morphisms) of $\mathcal{Q}$ are the Morita equivalencies of the Cuntz-Krieger algebras $O_A \otimes \mathcal{K} \cong O_A \otimes \mathcal{K}$, i.e. the similarity classes $\{A' = T^{-1} A T | T \in GL_2(\mathbb{Z})\}$ of the matrices $A$. We define the map $F : \mathcal{L} \to \mathcal{Q}$ as a composition of the above maps:

$$\begin{align*}
(\mathbb{CP}^1, \phi) &\xrightarrow{\pi^{-1}} (E_{CM}, \psi) \xrightarrow{\mathcal{F}} (\mathbb{A}_{RM}, \sigma) \xrightarrow{(1.2)} O_A,
\end{align*} \quad (1.3)
$$

Our main results can be formulated as follows.

Theorem 1.3. The map $F : \mathcal{L} \to \mathcal{Q}$ is a functor transforming the conjugate Lattès dynamical systems $(\mathbb{CP}^1, \phi)$ to the Morita equivalent Cuntz-Krieger algebras $O_A$.

Denote by $(X_A, \sigma_A)$ a subshift of finite type corresponding to the matrix $A$ [Lind & Marcus 1995] [5]. Recall that the subshifts $(X_A, \sigma_A)$ and $(X_{A'}, \sigma_{A'})$ are shift equivalent (over $\mathbb{Z}^+$) if there exist non-negative matrices $R$ and $S$ and a positive integer $k$ (a lag), satisfying the equations $A R = R A', A' S = S A, A^k = R S$ and $S R = (A')^k$. Theorem 1.3 implies a reduction of the complex dynamics of the Lattès maps to the symbolic dynamics of the subshifts of finite type. (To the best of our knowledge, such a reduction is known only for the Hénon and real rational maps so far.)

Corollary 1.4. The Lattès dynamical systems $(\mathbb{CP}^1, \phi)$ and $(\mathbb{CP}^1, \phi')$ are $PGL_2(K)$-conjugate if and only if $(X_A, \sigma_A)$ and $(X_{A'}, \sigma_{A'})$ are shift equivalent.
Let $\phi : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a Lattès map defined over the number field $K$. Denote by $\operatorname{Per}_n(\phi) = \{ x \in \mathbb{C}P^1 \mid \phi^n(x) = x \}$ the set of $n$-periodic points of the map $\phi$. Recall that the zeta function of $\phi$ is defined by the formula:

$$
\zeta_\phi(t) := \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} |\operatorname{Per}_n(\phi)| t^n \right).
$$

(1.4)

**Corollary 1.5.** Let $F(\phi) = O_A$. Denote by $\operatorname{tr}(A)$ and $\det(A)$ the trace and the determinant of matrix $A$, respectively. Then

$$
\zeta_\phi(t) = \frac{1}{\det(A) t^2 - \operatorname{tr}(A) t + 1}.
$$

(1.5)

**Remark 1.6.** A general formula of the zeta function for a rational map of the $\mathbb{C}P^1$ was obtained by [Hinkkanen 1994] [4]. Formula (1.5) can be viewed a refinement of Hinkkanen’s formula to the Lattès maps.

The article is organized as follows. The preliminary facts can be found in Section 2. Theorem 1.3 and corollaries 1.4-1.5 are proved in Section 3. An example is considered in Section 4.

2. Preliminaries

We briefly review the Lattès maps, symbolic dynamics, non-commutative tori and Cuntz-Krieger algebras. The reader is referred to [Silverman 2007] [7, Section 6.4], [Lind & Marcus 1995] [5], [Cuntz & Krieger 1980] [2] and [6, Section 1] for a detailed exposition.

2.1. Lattès maps. Denote by $\mathcal{E}$ an elliptic curve, i.e. the subset of the complex projective plane of the form \{(x, y, z) \in \mathbb{C}P^2 \mid y^2z = x^3 + ax^2z + bxz^2 + cz^3\}, where $a$, $b$ and $c$ are some constant complex numbers.

**Definition 2.1.** A rational map $\phi : \mathbb{C}P^1 \to \mathbb{C}P^1$ of degree $d \geq 2$ is called a Lattès map if there are an elliptic curve $\mathcal{E}$, a morphism $\psi : \mathcal{E} \to \mathcal{E}$, and a finite covering $\pi : \mathcal{E} \to \mathbb{C}P^1$ such that the diagram in Figure 1 is commutative.

Recall that the Mōbius transformation is a map $\mathbb{C}P^1 \to \mathbb{C}P^1$ of the form:

$$
z \mapsto \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad a, b, c, d \in \mathbb{C}.
$$

(2.1)

Such a transformation defines an automorphism of $\mathbb{C}P^1$ and each automorphism of $\mathbb{C}P^1$ has the form (2.1). The Mōbius transformations can be represented by
matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and they make the matrix group \( GL_2(\mathbb{C}) \) under the composition operation. Two matrices define the same Möbius transformation if and only if they are scalar multiples of one another. Thus the automorphism group of \( \mathbb{C}P^1 \) is given by the formula:

\[
\text{Aut } \mathbb{C}P^1 \cong GL_2(\mathbb{C})/\mathbb{C}^* \cong PGL_2(\mathbb{C}).
\] (2.2)

**Definition 2.2.** Two rational maps \( \phi \) and \( \phi' \) of \( \mathbb{C}P^1 \) are said to be \( PGL_2(\mathbb{C}) \)-conjugate if for some \( f \in PGL_2(\mathbb{C}) \) holds

\[
\phi' = f^{-1} \circ \phi \circ f.
\] (2.3)

**Remark 2.3.** In view of (2.2), the \( PGL_2(\mathbb{C}) \)-conjugate rational maps correspond to the same map of \( \mathbb{C}P^1 \) up to a coordinate change. Clearly, the iterations of such maps generate the same dynamical systems.

Let \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C} \) be an algebraic number field, i.e. a finite degree extension of the field \( \mathbb{Q} \). The following result is proved in [Silverman 2007] \[7, \text{Theorem 6.46}\].

**Theorem 2.4.** Let \( \phi \) and \( \phi' \) be Lattès maps defined over \( \mathbb{K} \) that are associated, respectively, to elliptic curves \( \mathcal{E} \) and \( \mathcal{E}' \). Assume that the projection maps \( \pi \) and \( \pi' \) associated to \( \phi \) and \( \phi' \) both have degree 2. If \( \phi \) and \( \phi' \) are \( PGL_2(\mathbb{K}) \)-conjugate to one another, then \( \mathcal{E} \) and \( \mathcal{E}' \) are isomorphic.

### 2.2. Subshifts of finite type

A full Bernoulli \( n \)-shift is the set \( X_n \) of bi-infinite sequences \( x = \{x_k\} \), where \( x_k \) is a symbol taken from a set \( S \) of cardinality \( n \). The set \( X_n \) is endowed with the product topology, making \( X_n \) a Cantor set. The shift homeomorphism \( \sigma_n : X_n \to X_n \) is given by the formula \( \sigma_n(x_{-\infty}x_{-1}x_0x_1x_{+1} \ldots) = (x_{+1}x_0x_{-1}x_{-\infty} \ldots) \). The homeomorphism defines a dynamical system \( \{X_n, \sigma_n\} \) given by the iterations of \( \sigma_n \).

Let \( A \) be an \( n \times n \) matrix, whose entries \( a_{ij} := a(i, j) \) are 0 or 1. Consider a subset \( X_A \subseteq X_n \) consisting of the bi-infinite sequences, which satisfy the restriction \( a(x_k, x_{k+1}) = 1 \) for all \(-\infty < k < \infty\). (It takes a moment to verify that \( X_A \) is indeed a subset of \( X_n \) and \( X_A = X_n \), if and only if, all the entries of \( A \) are 1’s.) By definition, \( \sigma_A = \sigma_n \mid X_A \) and the pair \( \{X_A, \sigma_A\} \) is called a subshift of finite type (SFT). A standard edge shift construction described in [Lind & Marcus 1995] \[5\] allows to extend the notion of SFT to any matrix \( A \) with the non-negative entries.

It is well known that the SFT’s \( \{X_A, \sigma_A\} \) and \( \{X_B, \sigma_B\} \) are topologically conjugate (as the dynamical systems), if and only if, the matrices \( A \) and \( B \) are strong shift equivalent (SSE), see [Lind & Marcus 1995] \[5\] for the corresponding definition. The SSE of two matrices is a difficult algorithmic problem, which motivates the consideration of a weaker equivalence between the matrices called a shift equivalence (SE). Namely, the matrices \( A \) and \( B \) are said to be shift equivalent (over \( \mathbb{Z}^+ \)), when there exist non-negative matrices \( R \) and \( S \) and a positive integer \( k \) (a lag), satisfying the equations \( AR = RB, BS = SA, A^k = RS \) and \( SR = B^k \). Finally, the SFT’s \( \{X_A, \sigma_A\} \) and \( \{X_B, \sigma_B\} \) (and the matrices \( A \) and \( B \)) are said to be flow equivalent (FE), if the suspension flows of the SFT’s act on the topological spaces, which are homeomorphic under a homeomorphism that respects the orientation of the orbits of the suspension flow. We shall use the following implications

\[
\text{SSE } \Rightarrow \text{SE } \Rightarrow \text{FE}. \quad (2.4)
\]
2.3. $C^*$-algebras. The $C^*$-algebra is an algebra $\mathcal{A}$ over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $\{a \mapsto a^* \mid a \in \mathcal{A}\}$ such that $\mathcal{A}$ is complete with respect to the norm, and such that $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a||^2$ for every $a, b \in \mathcal{A}$. Each commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$. Any other algebra $\mathcal{A}$ can be thought of as a noncommutative topological space.

2.3.1. AF-algebras. An AF-algebra (approximately finite $C^*$-algebra) is defined to be the norm closure of an ascending sequence of the finite dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with the entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents a semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \cdots$, where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. (It is easy to see, that each $\varphi_i$ is given by an integer matrix $A_i$ with non-negative entries.) The set-theoretic limit $\mathcal{A} = \lim_{\to} M_n$ has a natural algebraic structure given by the formula $a_m + b_k \to a + b$; here $a_m \to a, b_k \to b$ for the sequences $a_m \in M_{n_m}, b_k \in M_k$.

If the homomorphisms $\varphi_1 = \varphi_2 = \cdots = Const$ then the AF-algebra $\mathcal{A}$ is called stationary. In this case, the AF-algebra is given by a single matrix $A$ and by taking a power of $A$ one gets a strictly positive integer matrix, which we always assume to be the case. The stationary AF-algebra $\mathcal{A}$ is therefore the limit $M_1 \xrightarrow{\Delta} M_2 \xrightarrow{\Delta} \cdots$, and shifting the sequence by one generates a map $\sigma : \mathcal{A} \to \mathcal{A}$ called a shift endomorphism of $\mathcal{A}$.

2.3.2. Dimension groups. By $M_\infty(\mathcal{A})$ one understands the algebraic direct limit of the $C^*$-algebras $M_n(\mathcal{A})$ under the embeddings $a \mapsto \text{diag}(a,0)$. The direct limit $M_\infty(\mathcal{A})$ can be thought of as the $C^*$-algebra of infinite-dimensional matrices whose entries are all zero except for a finite number of the non-zero entries taken from the $C^*$-algebra $\mathcal{A}$. Two projections $p, q \in M_\infty(\mathcal{A})$ are equivalent, if there exists an element $v \in M_\infty(\mathcal{A})$, such that $p = vv^*v$ and $q = vv^*$. The equivalence class of projection $p$ is denoted by $[p]$. We write $V(\mathcal{A})$ to denote all equivalence classes of projections in the $C^*$-algebra $M_\infty(\mathcal{A})$, i.e. $V(\mathcal{A}) := \{[p] : p = p^* = p^2 \in M_\infty(\mathcal{A})\}$. The set $V(\mathcal{A})$ has the natural structure of an abelian semi-group with the addition operation defined by the formula $[p] + [q] := \text{diag}(p, q) = [p' \oplus q']$, where $p' \sim p, q' \sim q$ and $p' \perp q'$. The identity of the semi-group $V(\mathcal{A})$ is given by $[0]$, where $0$ is the zero projection. By the $K_0$-group $K_0(\mathcal{A})$ of the unital $C^*$-algebra $\mathcal{A}$ one understands the Grothendieck group of the abelian semi-group $V(\mathcal{A})$, i.e. a completion of $V(\mathcal{A})$ by the formal elements $[p] - [q]$. The image of $V(\mathcal{A})$ in $K_0(\mathcal{A})$ is a positive cone $K_0^+(\mathcal{A})$ defining the order structure $\leq$ on the abelian group $K_0(\mathcal{A})$.

The pair $(K_0(\mathcal{A}), K_0^+(\mathcal{A}))$ is known as a dimension group of the $C^*$-algebra $\mathcal{A}$. The scale $\Sigma(\mathcal{A})$ is the image in $K_0^+(\mathcal{A})$ of the equivalence classes of projections in the $C^*$-algebra $\mathcal{A}$. Each scale can always be written as $\Sigma(\mathcal{A}) = \{a \in K_0^+(\mathcal{A}) \mid 0 \leq a \leq u\}$, where $u$ is an order unit of $K_0^+(\mathcal{A})$. The pair $(K_0(\mathcal{A}), K_0^+(\mathcal{A}), \Sigma(\mathcal{A}))$ are invariants of the Morita equivalence and isomorphism class of the $C^*$-algebra $\mathcal{A}$, respectively.

If $\mathcal{A}$ is an AF-algebra, then its scaled dimension group (dimension group, resp.) is a complete invariant of the isomorphism (Morita equivalence, resp.) class of $\mathcal{A}$, see e.g. [6, Theorem 3.5.2].
2.3.3. Non-commutative tori. By a non-commutative torus one understands the universal $C^*$-algebra $A_\Theta$ generated by unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \Theta} uv$ for a real constant $\Theta$. The $A_\Theta$ is said to have real multiplication if $\Theta \in \mathbb{Q}(\sqrt{D})$ is a quadratic irrationality.

The non-commutative tori and elliptic curves are related by a fundamental correspondence saying that there exists a covariant functor $F : \{\mathcal{E}\} \to \{A_\Theta\}$ which maps the morphisms between elliptic curves to the endomorphisms of non-commutative tori [6, Section 1.3].

2.3.4. Effros-Shen algebras. The Effros-Shen algebra $A_\Theta$ is an AF-algebra given by the matrices

$$A_i = \prod_{j=0}^i \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix},$$

where $\Theta = [a_0, a_1, \ldots]$ is the continued fraction of a real number $\Theta$. The Pimsner-Voiculescu Theorem says that $A_\Theta$ is a dense subalgebra of the Effros-Shen algebra $A_\Theta$ under an embedding $A_\Theta \hookrightarrow A_\Theta$.

2.3.5. Cuntz-Krieger algebras. A Cuntz-Krieger algebra, $O_A$, is the $C^*$-algebra generated by partial isometries $s_1, \ldots, s_n$ that act on a Hilbert space in such a way that their support projections $Q_i = s_i^* s_i$ and their range projections $P_i = s_i s_i^*$ are orthogonal and satisfy the relations $Q_i = \sum_{j=1}^n b_{ij} P_j$, for an $n \times n$ matrix $A = (a_{ij})$ consisting of 0’s and 1’s [Cuntz & Krieger 1980] [2]. The notion is extendable to the matrices $A$ with the non-negative integer entries [Cuntz & Krieger 1980] [2, Remark 2.16]. It is known, that the $C^*$-algebra $O_A$ is simple, whenever matrix $A$ is irreducible (i.e. a certain power of $A$ is a strictly positive integer matrix). It was established in [Cuntz & Krieger 1980] [2], that $K_0(O_A) \cong \mathbb{Z}^n/(I - A^t)\mathbb{Z}^n$ and $K_1(O_A) = \text{Ker} \left( I - A^t \right)$, where $A^t$ is a transpose of the matrix $A$. It is not difficult to see, that whenever $\text{det} \left( I - A^t \right) \neq 0$, the $K_0(O_A)$ is a finite abelian group and $K_1(O_A) = 0$. The both groups are invariants of the stable isomorphism class of the Cuntz-Krieger algebra.

Let $\mathcal{A}_\Theta$ be the stationary Effros-Shen algebra given by a matrix $A$ and let $\sigma$ be the a shift endomorphism of $\mathcal{A}_\Theta$. Denote by $\mathcal{K}$ the $C^*$-algebra of compact operators. The Cuntz-Krieger algebra can be written as the crossed product $C^*$-algebra:

$$O_A \otimes \mathcal{K} \cong \mathcal{A}_\Theta \rtimes_{\sigma} \mathbb{Z}.$$

3. Proof

3.1. Proof of theorem 1.3. We shall split the proof in two lemmas.

Lemma 3.1. Let $\mathcal{A}_{RM}$ be the Effros-Shen algebra containing a non-commutative torus $\mathcal{A}_{RM}$ given by the embedding (2.6). For an endomorphism $\epsilon \in \text{End} \mathcal{A}_{RM}$ there exists a stationary AF-algebra $\mathcal{A} \subseteq \mathcal{A}_{RM}$ given by a matrix $T \in GL_2(\mathbb{Z})$, such that:

$$\mathcal{A}_{RM} \rtimes_{\epsilon} \mathbb{Z} \cong \mathcal{A} \rtimes_{\sigma} \mathbb{Z}.$$  

where $\sigma$ is the shift automorphism of $\mathcal{A}$.

Proof. Roughly speaking, the isomorphism (3.1) follows from the Unimodular Conjecture for the dimension groups [Effros 1981] [3, p.34]. Such a conjecture is known to be true for the dimension groups associated to the Effros-Shen algebras. We
refer the reader to [Effros 1981] [3] for the notations and details. Let us proceed step by step.

(i) Let $\Lambda = \mathbb{Z} + \mathbb{Z} \theta \subset \mathbb{R}$ and $\Lambda^+ = \{ \lambda \mid \mathbb{Z} + \mathbb{Z} \theta > 0 \}$. Denote by $(\Lambda, \Lambda^+)$ a dimension group associated to the Effros-Shen algebra $\mathbb{A}_{RM}$. Since our Effros-Shen algebra has real multiplication, we conclude that $\Lambda \subset k$, where $k = \mathbb{Q} (\sqrt{D})$ is a real quadratic number field.

(ii) Let $O_k$ be the ring of integers of the field $k$. The multiplication of $\Lambda$ by an element of $O_k$ generates an endomorphism of $\Lambda$. It is not hard to see, that $\text{End} \Lambda \cong O_k$ or to an order in the ring $O_k$.

(iii) Since each $\epsilon : \mathbb{A}_{RM} \to \mathbb{A}_{RM}$ induces an endomorphism of $\Lambda$, we have $\text{End} \mathbb{A}_{RM} \cong \text{End} \Lambda$.

(iv) Let us determine matrix $T \in \text{GL}_2(\mathbb{Z})$ in lemma 3.1. Denote by $A \in M_2(\mathbb{Z})$ the matrix form of the algebraic integer $\epsilon \in O_k$ under the isomorphism $\text{End} \mathbb{A}_{RM} \cong O_k$. Consider a dimension group given by the inductive limit:

$$\mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \ldots$$

(v) The dimension group (3.2) defines a stationary AF-algebra $\mathbb{A} \subseteq \mathbb{A}_{RM}$ corresponding to the dimension group $(\Lambda_A, \Lambda_A^+) \subseteq (\Lambda, \Lambda^+)$ invariant under iterations of the endomorphism $A$. The index of $\Lambda_A$ in $\Lambda$ is equal to the degree $|\text{det} A|$ of the endomorphism $\epsilon$.

(vi) Since the Unimodular Conjecture is true for the Effros-Shen algebras [Effros 1981] [3, p.34], we conclude that inductive limit (3.2) can be written as:

$$\mathbb{Z}^2 \xrightarrow{T} \mathbb{Z}^2 \xrightarrow{T} \ldots,$$

where $T \in \text{GL}_2(\mathbb{Z})$.

(vii) Our matrix $T$ can be obtained from the continued fraction of $\epsilon \in O_k$ as follows. Since $\epsilon$ is a quadratic irrationality, the corresponding continued fraction must be eventually $k$-periodic for some $k \geq 1$, i.e. $\epsilon = [b_1, \ldots, b_N; a_1, \ldots, a_k]$. Consider the following inductive limit:

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix}} \ldots \xrightarrow{\begin{pmatrix} b_N & 1 \\ 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{A} \ldots,$$

where

$$T = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$ (3.5)

(viii) Since (3.3) and (3.4) differ only in a finite number of terms, we conclude that the corresponding inductive limits define the same dimension group $(\Lambda_A, \Lambda_A^+) \subseteq (\Lambda, \Lambda^+)$. Thus $T$ in (3.3) is given by the formula (3.5).

(ix) The isomorphism (3.1) follows from such between the dimension groups (3.2) and (3.3).

This argument finishes the proof of lemma 3.1. □
Corollary 3.2. $\mathbb{A}_{RM} \rtimes_{\epsilon} \mathbb{Z} \cong \mathcal{O}_A \otimes \mathcal{K}$.

Proof. The corollary follows from formulas (2.7) and (3.1). □

Lemma 3.3. If $\phi$ and $\phi'$ are the $\text{PGL}_2(K)$-conjugate Lattès maps, then the corresponding Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ are Morita equivalent.

Proof. In outline, the proof of lemma 3.3 follows from Theorem 2.4 and corollary 3.2. Let us pass to a detailed argument.

(i) Let $\phi$ and $\phi'$ be the $\text{PGL}_2(K)$-conjugate Lattès maps lying in the category $\mathcal{L}$. Since the covering maps $\pi$ and $\pi'$ both have degree 2, one can apply Theorem 2.4.

(ii) It follows from 2.4 that there exists an isomorphism $\psi : \mathcal{E} \to \mathcal{E}'$ between the corresponding elliptic curves.

(iii) The isomorphism $\psi$ induces an isomorphism of the non-commutative tori $\mathcal{A}_{RM} \to \mathcal{A}'_{RM}$. The same is true for the Effros-Shen algebras $\mathbb{A}_{RM} \to \mathbb{A}'_{RM}$ under the embedding (2.6).

(iv) We conclude from (iii) that the crossed product $C^*$-algebras $\mathbb{A}_{RM} \rtimes_{\epsilon} \mathbb{Z}$ and $\mathbb{A}'_{RM} \rtimes_{\epsilon} \mathbb{Z}$ must be isomorphic.

(v) From (iv) and corollary 3.2 one gets an isomorphism of the $C^*$-algebras $\mathcal{O}_A \otimes \mathcal{K} \cong \mathcal{O}_{A'} \otimes \mathcal{K}$. In other words, the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ are Morita equivalent.

This argument finishes the proof of lemma 3.3. □

Theorem 1.3 follows from lemma 3.3.

3.2. Proof of corollary 1.4. We split the proof in the following steps.

(i) Let $\phi, \phi' \in \mathcal{L}$ be conjugate Lattès map. Denote by $\mathcal{O}_A, \mathcal{O}_{A'} \in \mathcal{O}$ the corresponding Cuntz-Krieger algebras (theorem 1.3).

(ii) Using lemma 3.1 when $\epsilon$ is an automorphism of $\mathbb{A}_{RM}$, we conclude that dimension groups $(\Lambda_A, \Lambda_A^+) \text{ and } (\Lambda_{A'}, \Lambda_{A'}^+)$ are isomorphic.

(iii) In view of (ii), one can apply Krieger’s Theorem (\cite{Wagoner 1999}, Theorem 2.11) saying that $(X_A, \sigma_A)$ and $(X_{A'}, \sigma_{A'})$ are shift equivalent over $\mathbb{Z}^+$ if and only if $(\Lambda_A, \Lambda_A^+)$ and $(\Lambda_{A'}, \Lambda_{A'}^+)$ are isomorphic dimension groups.

(iv) The converse statement is proved similarly and is left to the reader.

Corollary 1.4 follows from items (i)-(iv).
3.3. **Proof of corollary 1.5.** Roughly speaking, corollary 1.5 follows from the Lefschetz fixed-point theorem applied to the compact topological space $X_A$ and corollary 1.4. Let us pass to a step by step argument.

(i) In view of corollary 1.4, the dynamics of the Lattès map $\phi$ is encoded by the subshift $(X_A, \sigma_A)$, where $O_A = F(\phi)$. In particular, the set $Per_n(\phi)$ has the same cardinality as the set $Per_n(\sigma_A)$ of all $n$-periodic points $x \in X_A$.

(ii) It is well known, that $|Per_n(\sigma_A)| = \text{tr}(A^n)$, see e.g. [Lind & Marcus 1995][5, p.195] or [Wagoner 1999] [8, p. 273]. This formula follows from the Lefschetz fixed-point theorem applied to the compact topological space $X_A$.

(iii) Let $\lambda$ and $\bar{\lambda}$ be the complex conjugate eigenvalues of the matrix $A$. Bringing $A$ to the diagonal form $A = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$, one gets $A^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \bar{\lambda}^n \end{pmatrix}$ and therefore $\text{tr}(A^n) = \lambda^n + \bar{\lambda}^n$.

(iv) One can write (1.4) in the form:

$$
\zeta_\phi(t) = \exp \left( \sum_{n=1}^{\infty} \frac{(\lambda t)^n + (\bar{\lambda} t)^n}{n} \right) = \exp \left( \ln \frac{1}{(1-\lambda t)(1-\bar{\lambda} t)} \right), \quad (3.6)
$$

where the second equality follows from the formula for the sum of geometric progression and formal integration of the obtained series. On gets from (3.6):

$$
\zeta_\phi(t) = \frac{1}{(1-\lambda t)(1-\bar{\lambda} t)} = \frac{1}{\det(I-tA)} = \frac{1}{\det(A) t^2 - \text{tr}(A) t + 1}. \quad (3.7)
$$

Corollary 1.5 follows from formula (3.7).

4. **Example**

We conclude by an example illustrating theorem 1.3 and corollaries 1.4-1.5. For simplicity, we consider the Lattès map coming from an elliptic curve with complex multiplication. In this case, an explicit formula for the functor $F : \mathcal{C} \to \mathcal{O}$ can be derived from example 1.2.

**Example 4.1.** Let $\mathcal{E}_{\text{CM}}$ be an elliptic curve with complex multiplication by $\sqrt{-2}$ given by the equation:

$$
y^2 = x^3 + 4x^2 + 2x. \quad (4.1)
$$

(i) One can write $\mathcal{E}_{\text{CM}} \cong \mathbb{C}/L_{\text{CM}}$, where $L_{\text{CM}} = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$ is a lattice in the complex plane $\mathbb{C}$. Let $\psi : \mathcal{E}_{\text{CM}} \to \mathcal{E}_{\text{CM}}$ be an endomorphism defined as multiplication of the $L_{\text{CM}}$ by $\sqrt{-2}$. The norm $N(\sqrt{-2}) = 2$ and therefore the degree of $\psi$ is equal to 2. The duplication formulas (1.1) with $a = 4$, $b = 2$ and $c = 0$ imply that the corresponding Lattès map has the form:

$$
\phi(x) = \frac{(x^2 - 2)^2}{x(x^2 + 4x + 2)}. \quad (4.2)
$$
(ii) Recall that $\mathcal{F}(\mathcal{E}_{CM}) = \mathcal{M}_\mathcal{F}$ (example 1.2). The endomorphism $\psi$ transforms into an endomorphism of the pseudo-lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\sqrt{2} \subset \mathbb{R}$ given as multiplication by $\sqrt{2}$. Denoting by $N$ and $Tr$ the norm and the trace of an algebraic number, one gets the matrix form of the algebraic number:

$$A = \begin{pmatrix} 0 & 1 \\ -N(\sqrt{2}) & Tr(\sqrt{2}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}. \quad (4.3)$$

(iii) It is easy to see, that $\sqrt{2} \Lambda = 2\mathbb{Z} + \mathbb{Z}\sqrt{2}$ and therefore $\Lambda_A \cong \mathbb{Z} + \mathbb{Z}\frac{\sqrt{2}}{2}$. Since the continued fraction $\frac{\sqrt{2}}{2} = [0; 1, 2]$ has period $2$, we conclude that

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.4)$$

(iv) The zeta function of the Lattès map (4.2) can be calculated by a substitution of the matrix (4.3) in the formula (1.5):

$$\zeta_\phi(t) = \frac{1}{1 - 2t^2}. \quad (4.5)$$

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