FIRST EXTENSIONS AND FILTRATIONS OF STANDARD MODULES FOR GRADED HECKE ALGEBRAS

KEI YUEN CHAN

ABSTRACT. In this paper, we establish connections between the first extensions of simple modules and certain filtrations of standard modules in the setting of graded Hecke algebras. The filtrations involved are radical filtrations and Jantzen filtrations. Our approach involves the use of information from the Langlands classification as well as some deeper understanding on some structure of some modules. Such module arises from the image of a Knapp-Stein type intertwining operator and is a quotient of a generalized standard module.

As an application, we compute the Ext-groups for irreducible modules in a block for the graded Hecke algebra of type $C_3$, assuming a version of Jantzen conjecture.

1. Introduction

1.1. Let $R$ be a reduced root system and let $\Pi$ be a fixed set of simple roots in $R$. Let $H$ be the Lusztig’s graded (affine) Hecke algebra associated to a root datum $(R, V, R^\vee, V^\vee, \Pi)$ and a parameter function $k$ (see Definition 2.1). This paper continues our study of the Ext-groups in the category of $H$-modules. The representation theory of graded Hecke algebras can be transferred to the counterpart of affine Hecke algebras by Lusztig’s reduction theorems and hence is also useful in understanding the representation theory of $p$-adic groups.

The graded Hecke algebra is a deformation of a skew group ring. The category of representations of a graded Hecke algebra however does not possess a (natural) tensor product, which differs from the one of a skew group ring. An $H$-module still admits a Koszul type resolution analogous to the one for skew group ring, which allows one to establish a Poincaré duality on Ext-groups and establish a formula for the Euler-Poincaré pairing depending only on the reflection group structure of $H$-modules.

The extensions between a discrete series and a tempered module have been determined independently in [Me], [OS] and [Ch2] (for different but related settings). Then it is natural to consider modules outside those pairs. One possible direction is to study extensions between tempered modules in the setting of graded Hecke algebras analogous to the results of Opdam-Solleveld [OS2]. Another possible direction is to consider extensions for some non-tempered modules. Our study mainly arises when we consider extensions of the latter pairs.

The main result (Theorem 7.1) in this paper is to establish connections between first extensions of simple $H$-modules and certain filtrations of standard modules. Such result is then applied to compute some examples at the end. It is not surprising that filtrations
contain information of extensions between simple modules and one may see that our work is to explain in a more concrete terms how to obtain such information.

Jantzen filtrations and radical filtrations are involved in our study. It is interesting to see if various filtrations are compatible. A number of such study for Verma modules as well as Weyl modules can be found in the literature, and are closely related to the Kazhdan-Lusztig theory and an interpretation of Kazhdan-Lusztig polynomials by Vogan [Vo]. In our case of graded Hecke algebras, radical filtrations and Jantzen filtrations in general do not coincide (see Example 5.3). In that example, the socle filtration is also not the same as the Jantzen filtration. It would be an interesting question to see if there are some deeper relations between those filtrations and with related geometry.

Our approach of the study makes use of an indecomposable module, which arises from the image of an intertwining operator and is a quotient of an indecomposable generalized standard module. The structure of such module in terms of composition factors can be described in some details (Theorem 6.4) and hence provides a platform for transferring information between Jantzen filtrations and Ext-groups.

1.2. In order to describe our results in more detail, we need more notations. We first recall the Langlands classification for $\mathbb{H}$. For each $J \subseteq \Pi$, one can associate a parabolic subalgebra $\mathbb{H}_J$ of $\mathbb{H}$ (see Notation 2.2). The Langlands classification states that simple $\mathbb{H}$-modules can be parametrized by the set $\Xi_L$ of pairs $(J, U)$, where $J \subseteq \Pi$ and $U$ is certain irreducible $\mathbb{H}_J$-module (analogous to the tempered representation twisted by a character for reductive groups).

The more explicit construction of simple modules from those Langlands classification data $(J, U)$ is as follows. Given a pair $(J, U) \in \Xi_L$, one constructs a parabolically induced modules $I(J, U) := \mathbb{H} \otimes_{\mathbb{H}_J} U$, which is usually referred to standard modules in the literature. The Langlands classification asserts that $I(J, U)$ has a unique simple quotient, denoted by $L(J, U)$, and all simple modules arises from this way.

We first state a useful vanishing result on Ext-groups between some standard modules and some simple modules. For each $(J, U) \in \Xi_L$, we associate an element $\nu(J, U) \in V^\vee$ (see Definition 3.1). Recall that there is a partial ordering $\leq$ on the set $\Xi_L$ determined by the dominance ordering on the elements $\nu(J, U) \in V^\vee$.

**Proposition 1.1.** (Proposition 3.6) Let $(J_1, U_1), (J_2, U_2) \in \Xi_L$. If $\nu(J_1, U_1) \nleq \nu(J_2, U_2)$, then

$$\text{Ext}^1_{\mathbb{H}}(I(J_1, U_1), L(J_2, U_2)) = \text{Ext}^1_{\mathbb{H}}(I(J_1, U_1), I(J_2, U_2)) = 0.$$
understanding the first self-extension of a simple module is much harder and has less clue from the standard information of Langlands classification.

For determining self-extensions of simple modules, our approach of using standard modules logically leads to study a structure of generalized standard modules. Here a generalized standard module is an indecomposable module which admits a certain filtration by standard modules (see Definition 3.2 and Proposition 3.8 for the details).

An approach to understand such structure is to use an intertwining operator between a generalized standard module and a certain dual of a generalized standard module. The intertwining operator is normalized in a way to have properties analogous to a Knapp-Stein intertwining operator for real reductive groups [KS] (see Proposition 4.13). One may also compare with other study of intertwining operators by Rogawski [R1], by Reeder [R2], by Kriloff-Ram [KR], by Delorme-Opdam [DO] and by Solleveld [So]. We also remark that the analogous intertwining operator for standard modules in the affine Hecke algebra setting is also studied in an unpublished work of Delorme-Opdam.

Such intertwining operator defines a Jantzen type filtration and in particular defines an interesting quotient from the first layer of the Jantzen filtration. In the case of standard modules, the first layer of Jantzen filtration is simply the simple quotient and so we may regard such quotient to be a generalization of the simple quotient (see Definition 4.14).

A main technical result (Theorem 6.4) in this paper is to describe the structure of such quotient for special cases in terms of the Jantzen filtration of the (ordinary) standard module, and characterize such module by certain property with respect to the composition factors. This is also the pathway to connect some first self-extensions of simple modules and the Jantzen filtration.

The special cases of generalized standard modules considered in Theorem 6.4 are called of $S(V)$-type, meaning that extensions are arising from extensions of representations of a polynomial algebra. Then one can exploit the well-known extensions for polynomials algebras.

1.3. To describe another result, we need more notations. Let $(J, U) \in \Xi_L$. Let $V_J$ be the space spanned by simple roots in $J$. Let $V_J^{\perp, \perp}$ be the subspace of $V_J^\vee$ containing functionals vanishing on $V_J$ (Notation 2.2). For each $\eta^\vee \in V_J^{\perp, \perp}$ (possibly zero for our definition), we can deform the induced module $H \otimes_{H_J} U$ along the direction $\eta^\vee$ to obtain a family of $H$-modules $H \otimes_{H_J} U_{t\eta^\vee}$, where $t$ is an indeterminate. One can associate an intertwining operator $\Delta_{t\eta^\vee}$ for $H \otimes_{H_J} U_{t\eta^\vee}$ and the vanishing order of elements under $\Delta_{t\eta^\vee}$ defines a Jantzen-type filtration, denoted $JF^t_{\eta^\vee}(J, U)$ on the standard module $H \otimes_{H_J} U$. For the details of the notations, see Section 5. As discussed before, the Jantzen type filtration is also defined for generalized standard modules in Section 5 as the intertwining operator $\Delta_{t\eta^\vee}$ is defined for all generalized standard modules.

We define

$$V_{bad}^\perp(J, U) = \left\{ \eta^\vee \in V_J^{\perp, \perp} : JF^1_{\eta^\vee}(J, U) = JF^2_{\eta^\vee}(J, U) \right\},$$

which forms a vector space. We now give a version of Theorem 6.11.
Theorem 1.2. (part of Theorem 6.11) Let \((J, U) \in \Xi_L\). Let \(H_J \cong H_J^{ss} \otimes S(V_J)\) be a decomposition as in Notation 2.2. Suppose \(\text{Res}_{H_J^{ss}} U\) is a discrete series, or more generally \(\text{Ext}^1_{H_J^{ss}}(\text{Res}_{H_J^{ss}} U, \text{Res}_{H_J^{ss}} U) = 0\). Here \(\text{Res}_{H_J^{ss}}\) is the restriction functor to \(H_J^{ss}\)-module. Then

\[
\text{Ext}^1_{H_J^{ss}}(L(J, U), L(J, U)) \cong V_{\text{bad}}^{ss}(J, U).
\]

Using Theorem 1.1 and Theorem 1.2, the first extensions of simple modules can be much determined by information from filtrations of standard modules. We state a version of Theorem 7.1:

Theorem 1.3. (Theorem 7.1) Let \((J_1, U_1), (J_2, U_2) \in \Xi_L\).

1. Suppose \(\nu(J_1, U_1) \neq \nu(J_2, U_2)\). If \(\text{Ext}^1_{H_J^{ss}}(L(J_1, U_1), L(J_2, U_2)) \neq 0\), then either \(L(J_1, U_1)\) is isomorphic to a (simple) subquotient of \(I(L_2, U_2)\) or \(L(J_2, U_2)\) is isomorphic to a (simple) subquotient of \(I(J_1, U_1)\).

2. Suppose \(\nu(J_1, U_1) = \nu(J_2, U_2)\) (and in particular \(J_1 = J_2\)). Suppose

\[
\text{Ext}^1_{H_J^{ss}}(\text{Res}_{H_J^{ss}} U_1, \text{Res}_{H_J^{ss}} U_2) = 0.
\]

Then \(\text{Ext}^1_{H_J^{ss}}(L(J_1, U_1), L(J_2, U_2))\) is determined by the second layer of Jantzen filtrations in the sense of Theorem 1.2 when \(U_1 \cong U_2\) and is zero when \(U_1 \not\cong U_2\).

As an application of our results, we compute Ext-groups between some simple modules based on some computations of composition factors for standard modules in [Ci] at the end. We remark that results in this paper are independent of [Ch2], but we shall need some results in [Ch2] for computing some examples.

1.4. It leaves some questions from our study. First, we do not consider non-\(S(V)\)-type extensions, which roughly means dropping the hypothesis in Theorem 1.2. Second, finding an effective way to compute radical filtration and Jantzen filtration in general is still an open problem. For the second question, there is a conjectural way to compute Jantzen filtration by geometric means (see e.g. [BC]) using Kazhdan-Lusztig polynomials [La3] [Lu4] (also see [CC], [Lu]) and moreover the Arakawa-Suzuki functor [AS] [Su] as well as results of Rogawski [Ro] determine some Jantzen filtrations in type \(A_n\) case. This sheds some light on understanding first self-extensions in such directions.

1.5. We give an organization of this paper. Section 1 is the introduction. Section 2 recalls basic definitions and states some basic results. Section 3 defines the generalized standard modules and deduces several results about indecomposability and extensions from the Langlands classification. Section 4 studies a certain normalized intertwining operator for generalized standard modules and show it has properties analogous to a Knapp-Stein intertwining operator. Section 5 studies the Jantzen filtration of a generalized standard modules, which is defined from the intertwining operation in Section 4. Such intertwining operator defines a certain quotient on the generalized standard modules. Section 6 specifies on certain generalized standard modules and we describe the structure of the quotient in...
terms of the Jantzen filtration of (ordinary) standard modules. Section 7 summarizes our study by giving a connection between some first extensions of simple modules and filtrations on standard modules. Theorem 7.1 is our main result.

1.6. Notation. For an algebra \( A \) and an \( A \)-module \( X \), we write \( \pi_X(a)x \) or \( a.x \) or \( ax \) for the action of \( a \) on \( x \in X \).

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2. Preliminaries

2.1. Root systems. Let \( R \) be a reduced root system. Let \( \Pi \) be a fixed choice of simple roots in \( R \). Then \( \Pi \) determines the set \( R^+ \) of positive roots. For \( \alpha \in R^+ \), sometimes write \( \alpha > 0 \). For \( \alpha \in R \setminus R^+ \), write \( \alpha < 0 \). Let \( W \) be the finite reflection group of \( R \). Let \( V_0 \) be a real vector space spanned by \( R \). For any \( \alpha \in \Pi \), let \( s_\alpha \) be the simple reflection in \( W \) associated to \( \alpha \) (i.e. \( \alpha \in V_0 \) is in the \((-1)\)-eigenspace of \( s_\alpha \)). For \( \alpha \in R \), let \( \alpha^\vee \in \text{Hom}_R(V_0, \mathbb{R}) \) such that

\[
    s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha,
\]

where \( \langle v, \alpha^\vee \rangle = \alpha^\vee(v) \). Let \( R^\vee \subset \text{Hom}_R(V_0, \mathbb{R}) \) be the collection of all \( \alpha^\vee \ (\alpha \in R) \). Let \( V_0^\vee = \text{Hom}_R(V_0, \mathbb{R}) \). For each \( \alpha \in \Pi \), let \( \omega_\alpha \in V \) (resp. \( \omega_\alpha^\vee \in V^\vee \)) be the fundamental weight (resp. coweight) associated to \( \alpha \) i.e. \( \beta^\vee(\omega_\alpha) = \delta_{\alpha, \beta} \) for any \( \beta \in \Pi \).

By extending the scalars, let \( V = \mathbb{C} \otimes_R V_0 \) and let \( V^\vee = \mathbb{C} \otimes_R V_0^\vee \). We call \( (R, V, R^\vee, V^\vee, \Pi) \) to be a root datum. Let \( n = \dim_{\mathbb{C}} V = |\Pi| \). We remark that in [Ch2] we do not assume \( \dim_{\mathbb{C}} V = |\Pi| \) in general.

2.2. Graded Hecke algebras. Let \( k : \Pi \to \mathbb{C} \) be a parameter function such that \( k(\alpha) = k(\alpha') \) if \( \alpha \) and \( \alpha' \) are in the same \( W \)-orbit. We shall simply write \( k_\alpha \) for \( k(\alpha) \).

Definition 2.1. [Lu2] Section 4] The graded (affine) Hecke algebra \( \mathbb{H} = \mathbb{H}(\Pi, k) \) associated to a root datum \( (R, V, R^\vee, V^\vee, \Pi) \) and a parameter function \( k \) is an associative algebra with an unit over \( \mathbb{C} \) generated by the symbols \( \{t_w : w \in W\} \) and \( \{f_v : v \in V\} \) satisfying the following relations:

(1) The map \( w \mapsto t_w \) from \( \mathbb{C}[W] = \bigoplus_{w \in W} \mathbb{C}w \to \mathbb{H} \) is an algebra injection,

(2) The map \( v \mapsto f_v \) from \( S(V) \to \mathbb{H} \) is an algebra injection,

For simplicity, we shall simply write \( v \) for \( f_v \) from now on.

(3) the generators satisfy the following relation:

\[
    t_{s_\alpha} v - s_\alpha(v)t_{s_\alpha} = k_\alpha \langle v, \alpha^\vee \rangle \quad (\alpha \in \Pi, \ v \in V).
\]

In the remainder of this paper, we shall assume that \( \mathbb{H} \) is a graded Hecke algebra obtained from an extended affine Hecke algebra in the sense of [Lu2] Section 9 (also see BM, Section 3]). The only place we need this assumption is Lemma 2.4 and the results depending on
Lemma 2.4. Other parts are also valid for more general graded Hecke algebras, including non-crystallographic types.

Notation 2.2. For any subset $J$ of $\Pi$, define $V_J$ to be the complex subspace of $V$ spanned by elements in $J$ and define $V_J^\vee$ be the dual space of $V_J$ lying in $V^\vee$. Let $R_J = V_J \cap R$ and let $R_J^\vee = V_J^\vee \cap R^\vee$. Let $R^*_J = R_J \cap R^*$. Let $W_J$ be the subgroup of $W$ generated by the elements $s_\alpha$ for $\alpha \in J$. For $J \subseteq \Pi$, let $W_J$ be the subgroup of $W$ generated by all $s_\alpha$ with $\alpha \in J$. Let $w_{0,J}$ be the longest element in $W_J$. Let $W^J$ be the set of minimal representatives in the cosets in $W/W_J$.

Let
\[ V_J^{\vee,\perp} = \{ \gamma^\vee \in V^\vee : \gamma^\vee(v) = 0 \text{ for all } v \in V_J \} \]
and
\[ V_J^\perp = \{ v \in V : \gamma^\vee(v) = 0 \text{ for all } \gamma^\vee \in V_J^\vee \}. \]

For $J \subseteq \Pi$, let $H_J$ be the subalgebra of $H$ generated by all $v \in V$ and $t_w$ ($w \in W_J$) and let $H_J^\ast$ be the subalgebra of $H$ generated by all $v \in V_J$ and $t_w$ ($w \in W_J$). We have
\[ H_J \cong H_J^\ast \otimes S(V_J^\perp). \]

For a complex associative algebra $A$ with an unit and for $A$-modules $X$ and $Y$, denote by $\text{Ext}_A^i(X,Y)$ the $i$-th derived functor of $\text{Hom}_A(\cdot,Y)$ on $X$ in the category of $A$-modules.

Definition 2.3. For a finite-dimensional $H_J$-module $U$ and for $\gamma^\vee \in V^\vee$, an element $0 \neq u \in U$ is called a generalized $\gamma^\vee$-weight vector (or simply generalized weight vector) if $(v - \gamma^\vee(v))^ru = 0$ for some positive integer $r$. Such $\gamma^\vee$ is called a weight of $U$.

Let $U$ be an $H_J$-module. Let $\text{Wgt}(U)$ be the set of weights of $U$. Let $M(U,\gamma^\vee)$ be the space spanned by generalized weight vectors of weight $\gamma^\vee$.

2.3. Central characters. Let $J \subseteq \Pi$. The center $Z(H_J)$ of $H_J$ is naturally isomorphic to $S(V)^{W_J}$ [Lu2 Theorem 6.5], the $W_J$-invariant polynomials in $S(V)$. Each element in $Z(H_J)$ acts on a simple $H_J$-module $X$ by a scalar, and determines a map from $S(V)^{W_J}$ to $\mathbb{C}$. Those maps can be associated to a unique $W_J$-orbits on $V^\vee$. We shall call such orbit to be the central character of such module $X$.

We shall use the following result freely in the remainder of this paper. See for example [BW] Theorem I. 4.1] for similar arguments, also see [Ch Proposition 4.2.32].

Lemma 2.4. Let $J \subseteq \Pi$. Let $X$ and $Y$ be simple $H_J$-modules. If $X$ and $Y$ have different central characters, then $\text{Ext}_{H_J}^i(X,Y) = 0$ for all $i$.

2.4. $\bullet$-dual. We recall an anti-involution $\bullet$ studied by Opdam [Op] and Barbasch-Ciubotaru [BC, BC2] (with a slight variation). Define $\bullet : H \to H$ to be a linear anti-involution determined by
\[ v^\bullet = v \quad \text{for } v \in V, \quad t_w^\bullet = t_w^{-1} \quad \text{for } w \in W. \]

For an $H$-module $X$, the $\bullet$-operation defines a dual module denoted $X^\bullet$.

Lemma 2.5. All simple $H$-modules are self $\bullet$-dual.
Proof. Let $X$ be a simple $\mathbb{H}$-module. Note that $X^\ast$ has the same weight space as $X$ from the definition of $\ast$. To prove the lemma, it suffices to show that if two simple modules have the same set of weights, then they are isomorphic, that is [EM Theorem 5.5] and the Lusztig’s reduction theorem [Lu2]. □

2.5. Künneth formula. We have the following form of Künneth formula. For a proof, see for example [Be, Theorem 3.5.6] for similar arguments.

Lemma 2.6. Let $J \subset \Pi$. Let $U_1, U_2$ be finite-dimensional $\mathbb{H}_J^\ast$-modules and let $L_1, L_2$ be finite-dimensional $S(V^\dagger_J)$-representations. Then

$$\text{Ext}^i_{\mathbb{H}_J^\ast \otimes S(V^\dagger_J)}(U_1 \otimes L_1, U_2 \otimes L_2) \cong \bigoplus_{j+k=i} \text{Ext}^j_{\mathbb{H}_J^\ast}(U_1, U_2) \otimes \text{Ext}^k_{S(V^\dagger_J)}(L_1, L_2).$$

3. Generalized standard modules

The main goal of this section is to see some extensions between standard modules can be constructed from extensions of corresponding tempered modules for a parabolic subalgebra. Then we shall make some constructions and further study in Section 6.

3.1. Tempered modules. Since $V^\dagger$ admits a natural real form, we can talk about the real part of elements in $V^\dagger$.

Definition 3.1. Let $J \subset \Pi$. A (not necessarily irreducible) $\mathbb{H}_J$-module $U$ is said to be an $\mathbb{H}_J$-tempered module if the real part of any weight $\gamma^\dagger \in V^\dagger$ of $X$ has the form:

$$\text{Re} \gamma^\dagger = \sum_{\alpha \in J} a_\alpha \alpha^\dagger + \sum_{\alpha \in \Pi \backslash J} b_\alpha \omega_\alpha^\dagger, \quad \text{where } a_\alpha \leq 0, \ b_\alpha > 0.$$ 

If $U$ is an irreducible $\mathbb{H}_J$-tempered module, denote

$$\nu(J, U) = \sum_{\alpha \in \Pi \backslash J} b_\alpha \omega_\alpha^\dagger,$$

where $\sum_{\alpha \in \Pi \backslash J} b_\alpha \omega_\alpha^\dagger$ is the second term of the left hand side of (3.1). Note that our assumption of the irreducibility on $U$ assures that the term $\sum_{\alpha \in \Pi \backslash J} b_\alpha \omega_\alpha^\dagger$ is independent of the choice of a weight for $U$.

An $\mathbb{H}$-module $X$ is said to be a discrete series if $X$ is an $\mathbb{H}$-tempered module with all the inequalities for $a_\alpha$ in (3.1) being strict (i.e. $a_\alpha < 0$ for all $\alpha \in \Pi$).

Here our terminology of $\mathbb{H}_J$-tempered modules follows [KR]. It sometimes does not coincide with other definitions in the literature (e.g. [Ex Definition 1.4]). Our terminology is more convenient for discussions, although it is not quite a direct analog of tempered representations of $p$-adic groups.

3.2. Generalized standard modules and the Langlands classification. For any $J \subset \Pi$ and any $\mathbb{H}_J$-module $U$, we sometimes write $I(J, U)$ for $\mathbb{H} \otimes_{\mathbb{H}_J} U$. For any $\mathbb{H}$-module $X$, write $\text{Res}_{\mathbb{H}_J} X$ to be the restriction of $X$ to an $\mathbb{H}_J$-module. (The notation Res will also be similarly used for other algebras such as $\mathbb{H}_J^\ast$.)

**Definition 3.2.** Let $\Xi_J^U$ be the collection of all pairs $(J, U)$ with $J \subseteq \Pi$ and $U$ being a finite-dimensional indecomposable $\mathbb{H}_J$-tempered module. Let $\Xi_L$ be the subset of $\Xi_J^U$ containing all pairs $(J, U)$ with $U$ being an irreducible $\mathbb{H}_J$-tempered module. For $(J, U) \in \Xi_L$, $I(J, U)$ is usually called a standard module in the literature. For $(J, U) \in \Xi_J^U$, we shall call the module $I(J, U)$ to be a generalized standard module.

For $(J, U) \in \Xi_L$, let $L(J, U)$ be the unique simple quotient of $I(J, U)$ and let $N(J, U)$ be the unique maximal proper submodule of $I(J, U)$. For any irreducible $\mathbb{H}$-module $X$, $X$ is isomorphic to $L(J, U)$ for some $(J, U) \in \Xi_L$. (See Langlands classification [EV, KR] for the details.) For such $X$, denote $\nu(X) = \nu(J, U)$.

For $(J, U) \in \Xi_J^U$, we can also define $\nu(J, U') = \nu(J, U')$, where $U'$ is a composition factor of $U$. It is well-defined by Lemma 3.3 below. We shall extend the notion of $L(J, U)$ to all $(L, U) \in \Xi_J^U$ in Definition 4.14 later.

**Lemma 3.3.** Let $(J, U) \in \Xi_J^U$. Let $U_1, U_2$ be (simple) composition factors of $U$. Then $\nu(J, U_1) = \nu(J, U_2)$.

**Proof.** Recall that $\mathbb{H}_J \cong \mathbb{H}_J \otimes S(V_J^+)$. Note that any irreducible $\mathbb{H}_J$-module is isomorphic to $U \otimes L$ for some irreducible $\mathbb{H}_J$-module $U$ and 1-dimensional $S(V_J^+)$-module $L$. For two simple $\mathbb{H}_J$-modules $U_1 \otimes L_1$ and $U_2 \otimes L_2$, if $L_1 \not\cong L_2$ (equivalently $\nu(J, U_1 \otimes L_1) \not\cong \nu(J, U_2 \otimes L_2)$), then

$$\text{Ext}_{\mathbb{H}_J}^1(U_1 \otimes L_1, U_2 \otimes L_2) = 0$$

by Lemma 2.6 and $\text{Ext}_{S(V_J^+)}^1(L_1, L_2) = 0$.

Now we consider $(J, U) \in \Xi_J^U$ and prove by an induction on the length of $U$. If the length of $U$ is 1, then the statement is clear. Let $U'$ be a maximal submodule of $U$ such that any composition factors $U_1, U_2$ of $U'$ satisfy $\nu(J, U_1) = \nu(J, U_2)$. Note that $U' \not= 0$ since simple submodules of $U$ always satisfy that property. Now we have a short exact sequence of the following form:

$$0 \to U' \to U \to U/U' \to 0.$$ 

Since $U/U'$ is of finite length, we can write $U/U'$ as $U/U' = N_1 \oplus \ldots \oplus N_i$, where all $N_j$ are indecomposable modules. By the inductive hypothesis, all $N_j$ have the property that all the composition factors $N_j'$ of $N_j$ satisfy $\nu(J, N_j') = \nu(J, N_j')$. Now by using the maximality of our choice of $U'$ and using [3, 4], we have

$$\text{Ext}_{S(V_J^+)}^1(U', N_j) = 0$$

for all $j$. Now we have that $U \cong U' \oplus N_1 \oplus \ldots \oplus N_i$. By the indecomposability of $U$, we have $U = U'$ as desired.

We recall the dominance ordering on $V_0^\vee$. For any two elements $\gamma_1^\vee, \gamma_2^\vee \in V_0^\vee$, write $\gamma_1^\vee \leq \gamma_2^\vee$ if

$$\gamma_2^\vee - \gamma_1^\vee = \sum_{\alpha \in \Pi} a_\alpha \alpha^\vee$$

for some $a_\alpha \geq 0$. We write $\gamma_1^\vee < \gamma_2^\vee$ if $\gamma_1^\vee \leq \gamma_2^\vee$ and $\gamma_1^\vee \not= \gamma_2^\vee$. 

We recall some results related to the Langlands classification. We need more notations. 

For any $\gamma^\vee \in V^\vee$, the real part of $\gamma^\vee$ can be uniquely written as
\begin{equation}
\operatorname{Re} \gamma^\vee = \sum_{\alpha \in J} a_\alpha \alpha^\vee + \sum_{\alpha \in \Pi \setminus J} b_\alpha \omega_\alpha^\vee,
\end{equation}
for some $J \subset \Pi$, $a_\alpha < 0$ and $b_\alpha \geq 0$ ([Kn] Ch VIII Lemma 8.59], c.f. $\nu(J,U)$ in Definition 3.1]. For such $\gamma^\vee$, define
\begin{equation*}
\nu(\gamma^\vee) = \sum_{\alpha \in \Pi \setminus J} b_\alpha \omega_\alpha^\vee.
\end{equation*}

We shall freely use the fact that $\langle \omega_\alpha, \omega_\beta^\vee \rangle \geq 0$ for $\alpha, \beta \in \Pi$ ([Kn] Ch VIII Lemma 8.57] below.

**Lemma 3.4.** Let $(J,U) \in \Xi^0_L$ and let $\gamma^\vee \in V^\vee$. If $\nu(\gamma^\vee) < \nu(J,U)$, then $\gamma^\vee$ is not $W_J$-conjugate to any weight of $U$.

**Proof.** By definitions, there exists $J' \subset \Pi$ such that
\begin{equation*}
\operatorname{Re} \gamma^\vee = \sum_{\alpha \in J'} a_\alpha \alpha^\vee + \nu(\gamma^\vee)
\end{equation*}
for $a_\alpha \leq 0$ and $b_\alpha > 0$. By simple linear algebra we can also write
\begin{equation*}
\operatorname{Re} \gamma^\vee = \sum_{\alpha \in J} a'_\alpha \alpha^\vee + \sum_{\alpha \in \Pi \setminus J} b'_\alpha \omega_\alpha^\vee,
\end{equation*}
where $a'_\alpha, b'_\alpha \in \mathbb{R}$. Set $\nu' = \sum_{\alpha \in \Pi \setminus J} b'_\alpha \omega_\alpha^\vee$. To prove the lemma, it suffices to show that $\nu'$ is not equal to $\nu(J,U)$. For $\alpha_0 \in J$,
\begin{equation*}
a'_\alpha + \langle \omega_\alpha, \nu' \rangle = \langle \omega_\alpha, \operatorname{Re} \gamma^\vee \rangle \leq \langle \omega_\alpha, \nu(\gamma^\vee) \rangle \leq \langle \omega_\alpha, \nu(J,U) \rangle.
\end{equation*}
Hence if $\nu' = \nu(J,U)$, then all $a'_{\alpha_0} \leq 0$. By the uniqueness property in the expression 3.3, we have $J = J'$ and so $\nu(\gamma^\vee) = \nu' = \nu(J,U)$. This gives a contradiction. Hence $\nu' \neq \nu(J,U)$ as desired. \hfill \Box

**Lemma 3.5.** Let $(J,U) \in \Xi^0_L$. Then
\begin{equation*}
\operatorname{Res}_H, I(J,U) \cong U \oplus Y,
\end{equation*}
where $Y$ is an $H_J$-module such that for any weight $\gamma^\vee$ of $Y$, $\nu(\gamma^\vee) < \nu(J,U)$. Moreover,

1. any composition factors of $Y$ have $H_J$-central characters different from any composition factors of $U$,
2. Suppose $(J,U) \in \Xi_L$. Then for any composition factors $Z$ of $I(J,U)$, if $Z$ is not isomorphic to $L(J,U)$, then $\nu(Z) < \nu(J,U)$.

**Proof.** This is the consequence of the proof for the Langlands classification [Ev]. We provide some explanations.

By Frobenius reciprocity, we have the following short exact sequence for $H_J$-modules:
\begin{equation*}
0 \to U \to \operatorname{Res}_H, I(J,U) \to Y \to 0,
\end{equation*}
where $Y = \operatorname{Res}_H, I(J,U) / U$. Then using the arguments as in [Ev] which use Geometric Lemmas of Langlands (also see proof of [KR] Theorem 2.4] and proof of [Ch] Proposition
from what we have just proved, we have $\text{Ext}^1_{\mathbb{H}_J}(Y,U) = 0$. Hence

$$\text{Ext}^1_{\mathbb{H}_J}(Y,U) = 0$$

and so $\text{Res}_{\mathbb{H}_J} I(J,U) \cong U \oplus N$ as desired. Other assertions and details can be referred to [Ev] [KR].

3.3. Extensions. The following vanishing result is useful in computing some extensions.

**Proposition 3.6.** Let $(J_1, U_1), (J_2, U_2) \in \Xi_L$. If either $\nu(J_1, U_1)$ and $\nu(J_2, U_2)$ are incomparable or $\nu(J_2, U_2) < \nu(J_1, U_1)$, then

$$\text{Ext}^1_{\mathbb{H}_J}(I(J_1, U_1), L(J_2, U_2)) = \text{Ext}^1_{\mathbb{H}_J}(I(J_1, U_1), I(J_2, U_2)) = 0.$$ 

**Proof.** Suppose $\text{Ext}^1_{\mathbb{H}_J}(I(J_1, U_1), L(J_2, U_2)) \neq 0$, equivalently $\text{Ext}^1_{\mathbb{H}_J}(U_1, \text{Res}_{\mathbb{H}_J} L(J_2, U_2)) \neq 0$ by Shapiro’s Lemma. Then by considering the $\mathbb{H}_J$-central character and using Lemma 2.4 there exists an $\mathbb{H}_{J_1}$-composition factor $U'$ of $\text{Res}_{\mathbb{H}_{J_1}} L(J_2, U_2)$ such that $U'$ has a weight whose real part has the form

$$\sum_{\alpha \in J_1} a_\alpha \alpha^\vee + \nu(J_1, U_1).$$

Using the Langlands classification on the irreducible $\mathbb{H}_{J_1}$-module $\text{Res}_{\mathbb{H}_{J_1}} U'$, we can further get a weight, denoted $\gamma^\vee$, of $U'$ of the following form:

$$\text{Re}(\gamma^\vee) = \left(\sum_{\alpha \in J_1} a_\alpha \alpha^\vee + \sum_{\alpha \in J_1 \setminus J'} b_\alpha \overline{\omega}_\alpha^J\right) + \nu(J_1, U_1),$$

with $J' \subset J_1$, $a_\alpha \leq 0$ and $b_\alpha > 0$, where $\overline{\omega}_\alpha^J$ are fundamental coweights in $V_{J_1}^\vee$ corresponding to $\alpha$ (i.e. $\overline{\omega}_\alpha^J \in V_{J_1}^\vee$, $\overline{\omega}_\alpha^J(\beta) = \delta_{\alpha,\beta}$ for $\alpha, \beta \in R_{J_1}$). By [Kn] Ch VIII Lemma 8.57, $\overline{\omega}_\alpha^J$ is non-negative sum of simple coroots. Hence

$$\sum_{\alpha \in J'} a_\alpha \alpha^\vee + \nu(J_1, U_1) \leq \text{Re}(\gamma^\vee).$$

Then by geometric lemmas of Langlands ([La], see [Kn] Ch VIII Lemmas 8.56 and 8.59),

$$\nu(J_1, U_1) = \nu\left(\sum_{\alpha \in J'} a_\alpha \alpha^\vee + \nu(J_1, U_1)\right) \leq \nu(\gamma^\vee).$$

Now by Lemma 3.3, we have

$$\nu(J_1, U_1) \leq \nu(\gamma^\vee) \leq \nu(J_2, U_2).$$

This proves $\text{Ext}^1_{\mathbb{H}_J}(I(J_1, U_1), L(J_2, U_2)) = 0$ if $\nu(J_1, U_1) \leq \nu(J_2, U_2)$.

We now consider $\text{Ext}^1_{\mathbb{H}_J}(I(J_1, U_1), I(J_2, U_2))$ and suppose $\nu(J_1, U_1) \leq \nu(J_2, U_2)$. Then for any composition factor $X$ of $I(J_2, U_2)$, $\nu(J_1, U_1) \leq \nu(X)$ by Lemma 3.3. Hence from what we have just proved, we have $\text{Ext}^1_{\mathbb{H}_J}(I(J_1, U_1), X) = 0$. This implies that $\text{Ext}^1_{\mathbb{H}_J}(I(J_1, U_1), I(J_2, U_2)) = 0$. \qed
Remark 3.7. One may compare with a vanishing result of Kato \[Ka\] (see the paragraph below \[Ka\] Theorem C], also see \[Lm\] Section 8).

3.4. Indecomposability. We show below that a generalized standard module is indecomposable.

**Proposition 3.8.** Let \((J, U) \in \Xi_L^J\). Then \(I(J, U)\) is also an indecomposable \(\mathbb{H}\)-module.

**Proof.** Let \((J, U) \in \Xi_L^J\) and let \(X = I(J, U)\). Let

\[
0 \to Y \xrightarrow{f} X \xrightarrow{g} Z \to 0
\]

be a short exact sequence. Suppose the short exact sequence splits and we shall show that either \(Y\) or \(Z\) is zero. Since the short exact sequence splits, there exists an \(\mathbb{H}\)-map \(t : Z \to X\) such that \(g \circ t = \text{Id}\). Since \(\text{Res}_{\mathbb{H}, J}\) is an exact functor, we have a short exact sequence

\[
0 \to \text{Res}_{\mathbb{H}, J} Y \to \text{Res}_{\mathbb{H}, J} X \to \text{Res}_{\mathbb{H}, J} Z \to 0.
\]

We also have an exact sequence of the form

\[
0 \to f^{-1}(U) \xrightarrow{\text{Res}_{\mathbb{H}, J} f^{-1}(U)} U \xrightarrow{g|U} g(U) \to 0,
\]

where we identify the \(\mathbb{H}_J\)-subspace \(1 \otimes U\) of \(I(J, U)\) with \(U\). Since \(t(g(U))\) has an \(\mathbb{H}_J\)-central character as \(U\), \(t(g(U)) \subset U\) by Lemma \[3.3\]. Thus it makes sense to write \(g|U \circ t|_{g(U)} = \text{Id}\).

Since \(U\) is an indecomposable \(\mathbb{H}_J\)-module, we have either \(f^{-1}(U) = 0\) or \(g(U) = 0\). Suppose \(g(U) = 0\). Then by Frobenius reciprocity, \(g = 0\) and hence \(Z = 0\). Suppose \(f^{-1}(U) = 0\). Then \(t|_{g(U)}\) is surjective onto \(U\). This implies that \(1 \otimes U \subset t(g(U)) \subset t(Z)\). By applying the \(\mathbb{H}\)-action on both sides of the inclusion, we have \(X \subset t(Z)\) and so \(t(Z) = X\) (i.e. \(t\) is surjective). Hence \(t\) is an isomorphism and so is \(g\). Thus \(Y = 0\) as desired. \(\square\)

**Remark 3.9.** Proposition \[3.3\] is not true in general if \(U\) is not \(\mathbb{H}_J\)-tempered. For example, let \(R\) be the root system of type \(A_1\). Assume \(k_\alpha \neq 0\) for all \(\alpha \in \Pi\). Let \(U'\) be the unique 2-dimensional indecomposable \(S(V)\)-module with the weight equal to 0. In this case, \(U'\) is not \(\mathbb{H}_0\)-tempered and one can verify that \(I(0, U')\) is semisimple and of length 2. (In our terminology, \(I(0, U')\) is \(\mathbb{H}\)-tempered.)

The following result and its proof mainly guides for constructions in Section \[3\].

**Lemma 3.10.** Let \(X\) be a finite-dimensional \(\mathbb{H}\)-module. Suppose there exists a fixed \(J \subset \Pi\) and a fixed \(\nu^\vee \in V_J^{\vee, -1} \cap V_0^\vee\) such that \(X\) admits a filtration \(0 = F_0 \subset \ldots \subset F_{r-1} \subset F_r = X\) with the subquotients \(F_i/F_{i+1} \cong I(J, U_i)\) for some \((J, U_i) \in \Xi_L^J\) satisfying \(\nu(J, U_i) = \nu\). Then there exists an \(\mathbb{H}_J\)-tempered module \(U\) such that \(X\) is isomorphic to \(I(J, U)\). Moreover, the \(\mathbb{H}_J\)-tempered module \(U\) is unique, up to isomorphism.

**Proof.** By using an inductive argument, it suffices to prove the case when \(X\) admits such filtration of length 2 (i.e. \(r = 2\)). For \(i = 1, 2\), let \(P^?\) be a projective resolution of \(U_i\). We write as

\[
P^2_{U_i} \xrightarrow{d_2} P^1_{U_i} \xrightarrow{d_1} P^0_{U_i} \to U_i \to 0.
\]
Then $I(J, P_{U_1}^k)$ is still a projective resolution for $I(J, U_1)$ with the differential maps denoted by $d'_{k^*}$. The map is determined by

$$
\cdots \longrightarrow I(J, P_{U_1}^1) \overset{d'_2}{\longrightarrow} I(J, P_{U_1}^0) \longrightarrow I(J, U_1) \longrightarrow 0
$$

where $M$ is the pushout of $d'_2 : I(J, P_{U_1}^1) \to I(J, P_{U_1}^0)$ and $f' : I(J, P_{U_1}^1) \to I(J, U_2)$. Explicitly,

$$
X \cong (I(J, U_2) \oplus I(J, P_{U_1}^0))/ \{(f(p), -d'_2(p)) : p \in I(J, P_{U_1}^1)\}.
$$

Recall that $I(J, P_{U_1}^k) = \mathbb{H} \otimes_{\mathbb{H}_J} P_{U_1}^k$. Now we consider the identification

$$
\text{Hom}_{\mathbb{H}}(I(J, P_{U_1}^1), I(J, P_{U_2}^0)) \cong \text{Hom}_{\mathbb{H}_J}(P_{U_1}^1, \text{Res}_{\mathbb{H}_J}(I(J, P_{U_2}^0))) \cong \text{Hom}_{\mathbb{H}_J}(P_{U_1}^1, U_2) \oplus \text{Hom}_{\mathbb{H}_J}(P_{U_1}^1, N),
$$

where $N$ is an $\mathbb{H}_J$-module whose composition factors have weights $\nu'_v$ satisfying $\nu'_v < \nu(J, U_2) = \nu_v$ as in Lemma 3.5. Under such identification, we write $f$ as $(f_1, f_2)$. Now since $\text{Ext}_{\mathbb{H}_J}^1(U_1, N) = 0$ by using Proposition 3.6, $(f_1, 0)$ and $(f_1, f_2)$ determine the same cohomology class of $\text{Ext}_{\mathbb{H}_J}^1(I(J, U_1), I(J, U_2)) \cong \text{Ext}_{\mathbb{H}_J}^1(U_1, U_2) \oplus \text{Ext}_{\mathbb{H}_J}^1(U_1, N)$. Let $f' = (f_1, 0)$. By the identification, we now have $f'(1 \otimes P_{U_1}^1) \subset 1 \otimes U_2$. By definition, we have $d'_2(1 \otimes P_{U_1}^1) \subset 1 \otimes P_{U_1}^0$. Hence, we now have maps $\tilde{f}' : P_{U_1}^1 \to U_2$ and $\tilde{d}_2' : P_{U_1}^1 \to P_{U_1}^0$ naturally arisen from $f'$ and $d'_{2}$ respectively. Let

$$
M = (I(J, U_2) \oplus I(J, P_{U_1}^0))/ \{(f'(p), d'_2(p)) : p \in I(J, P_{U_1}^1)\}.
$$

By the construction of the Yoneda extension, $X \cong M$. We now also define

$$
M_U = (U_2 \oplus P_{U_1}^0))/ \{(\tilde{f}'(1 \otimes p), \tilde{d}_2' (1 \otimes p)) : p \in P_{U_1}^1\}.
$$

It is straightforward to verify $M \cong I(J, M_U)$. Hence $X \cong I(J, M_U)$ as desired.

It remains to prove the uniqueness. Suppose there exists $\mathbb{H}_J$-tempered modules $U_1$ and $U_2$ such that $X \cong I(J, U_1) \cong I(J, U_2)$. Let $f : I(J, U_1) \to I(J, U_2)$ be an isomorphism. By using Proposition 3.5, we may first reduce $U_1$ and $U_2$ to be indecomposable and then show that $f(1 \otimes U_1) = 1 \otimes U_2$ by Lemma 3.5. Then $f$ defines an $\mathbb{H}_J$-isomorphism between $U_1$ and $U_2$. \hfill $\square$

3.5. $S(V)$-type extensions. We shall refine our study to a certain class of extensions which arises from extensions of representations for polynomial rings. Such extensions shall be called $S(V)$-type extensions and have close connections to Jantzen filtrations discussed later. Constructions and more study on $S(V)$-type extensions will be carried out in Section 6 after the necessary tools have been explained.

**Lemma 3.11.** Let $(J, U_1), (J, U_2) \in \mathcal{E}_L$ with $\nu(J, U_1) = \nu(J, U_2)$. Via the natural identification of $\mathbb{H}_J \cong \mathbb{H}_J^{ss} \otimes S(V_J^\perp)$, we have

$$
U_1 \cong \overline{U}_1 \otimes L_1, \ U_2 \cong \overline{U}_2 \otimes L_2.
$$
such that \( \overline{U}_1 \) is an \( \mathbb{H}_j^+ \)-module and \( L_i \) is a 1-dimensional \( S(V_j^+) \)-representation. Then

\[
\text{(3.5)} \quad \text{Ext}^1_{\mathbb{H}_j^+}(I(J, U_1), I(J, U_2)) \cong \text{Hom}_{\mathbb{H}_j^+}(\overline{U}_1, \overline{U}_2) \otimes \text{Ext}^1_{S(V_j^+)}(L_1, L_2)
\]

\[
\text{(3.6)} \quad \oplus \text{Hom}_{S(V_j^+)}(L_1, L_2) \otimes \text{Ext}^1_{\mathbb{H}_j^+}(\overline{U}_1, \overline{U}_2).
\]

**Proof.** By Shapiro’s Lemma, Lemma 3.4 and Lemma 3.5

\[
\text{Ext}^1_{\mathbb{H}_j^+}(I(J, U_1), I(J, U_2)) \cong \text{Ext}^1_{\mathbb{H}_j^+}(U_1, U_2) \oplus \text{Ext}^1_{\mathbb{H}_j^+}(U_1, Y).
\]

By Lemma 3.5, the \( \mathbb{H}_j \)-central characters of \( U_1 \) and \( Y \) are different and hence \( \text{Ext}^1_{\mathbb{H}_j}(U_1, Y) = 0 \). The \( \text{Ext}^1_{\mathbb{H}_j}(U_1, U_2) \) is naturally isomorphic to the term in the left hand side of (3.5) by Lemma 2.6. \( \square \)

**Definition 3.12.** We use the notation in Lemma 3.11. By composing the isomorphism in Lemma 3.11 and a projection map to one of factors, we obtain a map

\[
\text{pr}_{\mathbb{H}_j^+} : \text{Ext}^1_{\mathbb{H}_j^+}(I(J, U_1), I(J, U_1)) \to \text{Hom}_{\mathbb{H}_j^+}(\overline{U}_1, \overline{U}_1) \otimes \text{Ext}^1_{\mathbb{H}_j^+}(\overline{U}_1, \overline{U}_1),
\]

\[
\text{pr}_{S(V_j^+)} : \text{Ext}^1_{\mathbb{H}_j^+}(I(J, U_1), I(J, U_1)) \to \text{Hom}_{S(V_j^+)}(\overline{U}_1, \overline{U}_1) \otimes \text{Ext}^1_{S(V_j^+)}(L_1, L_1) \cong \text{Ext}^1_{S(V_j^+)}(L_1, L_1).
\]

Let \( \zeta \in \text{Ext}^1_{\mathbb{H}_j^+}(I(J, U_1), I(J, U_2)) \). We say the element \( \zeta \) is a \( S(V) \)-extension if \( \text{pr}_{\mathbb{H}_j^+}(\zeta) = 0 \) and \( \text{pr}_{S(V_j^+)}(\zeta) \neq 0 \).

Let \( (J, U) \in \Xi_L \). Let \( \eta' \in \text{Ext}^1_{\mathbb{H}_j^+}(I(J, U), I(J, U)) \) be a \( S(V) \)-extension. We call \( X \) is \( (J, U, \eta') \)-\( S(V) \)-type (or simply strict \( S(V) \)-type) if there exists a filtration on \( X \) of the form:

\[
0 \subset X_1 \subset X_2 \subset \ldots \subset X_l = X
\]

such that for all \( i \), \( X_i/X_{i-1} \cong \text{Ext}(J, U) \) and the short exact sequence

\[
\text{(3.7)} \quad 0 \to X_i/X_{i-1} \to X_{i+1}/X_{i-1} \to X_{i+1}/X_i \to 0
\]

corresponds to \( \eta' \) under the Yoneda correspondence.

It is indeed not necessary to fix one \( \eta' \) for all short exact sequences in the above definition and allow a larger class of modules. Our approach later can also deal with some of those modules, but it requires more set-up. For our purpose of studying first extension, those strict \( S(V) \)-types will suffice.

4. **Intertwining operators**

Intertwining operators for parabolically induced modules are the major tools for our computation in this paper. Some treatments in this section and the Appendix B are similar to [KR] and [IC]. The intertwining operator defines the Jantzen filtration of a generalized standard module which will be discussed in Section 5.

The main result in this section is Proposition 4.13 which gives a description of an intertwining operator for a generalized standard module. The image of the intertwining operator also defines a quotient \( L(J, U) \) for \( (J, U) \in \Xi_L^p \) in Definition 4.14.
4.1. **Intertwining elements.** In this section, we fix $J \subseteq \Pi$ and fix a finite-dimensional $\mathbb{H}_J$-module $U$. We shall define some intertwining element, which involves inverting some elements in $S(V)$. To deal with such matter in a proper way, we define some notations below.

**Definition 4.1.** Suppose $J \neq \Pi$. Let $A$ be the multiplicative closed set in $S(V)$ which contains 1 and all the elements of the form $(v_1 - c_1) \cdots (v_k - c_k)$, where $v_i \in V \setminus V_J$ and $c_k \in \mathbb{C}$. Let $\mathcal{O}(J) = A^{-1} S(V)$ be the localization of the ring $S(V)$ by $A$. If $J = \Pi$, simply set $\mathcal{O}(J) = S(V)$.

**Lemma 4.2.** $\mathbb{H}_J \otimes_{S(V)} \mathcal{O}(J)$ has a natural algebra structure such that $\mathbb{H}_J$ embeds naturally into $\mathbb{H}_J \otimes_{S(V)} \mathcal{O}(J)$ as a subalgebra.

**Proof.** For $v \in V \setminus V_J$, the relation between $t_{s_n} \in \mathbb{H}_J$ ($\alpha \in J$) and $\frac{1}{t} \in \mathcal{O}(J)$ is given by

$$(t_{s_n} \otimes 1)(1 \otimes \frac{1}{v-c}) - (1 \otimes \frac{1}{s_n(v) - c})(t_{s_n} \otimes 1) = \frac{k_n \alpha^\vee(v)}{(v-c)(s_n(v)-c)}.$$

It is straightforward to check that $s_n(v) \notin V_J$ and so the relation is well-defined. The map $h \mapsto h \otimes 1$ from $\mathbb{H}_J$ to $\mathbb{H}_J \otimes_{S(V)} \mathcal{O}(J)$ defines the natural embedding. 

Define $\mathcal{H}_J = \mathbb{H}_J \otimes_{S(V)} \mathcal{O}(J)$ (which is an algebra by Lemma 4.2). Define $\mathcal{H}^J = \mathbb{H} \otimes_{\mathbb{H}_J} (\mathbb{H}_J \otimes_{S(V)} \mathcal{O}(J)) = \mathbb{H} \otimes_{\mathbb{H}_J} \mathcal{H}_J$ (which does not have a natural algebraic structure) and we shall regard $\mathcal{H}_J$ as an $(\mathbb{H}, \mathcal{H}_J)$-bimodule (by the left and right multiplications respectively). For $w \in W$, we shall simply write $t_w$ for $t_w \otimes (1 \otimes 1)$ as an element in $\mathcal{H}_J$. For $q \in \mathcal{O}(J)$, we shall simply write $q$ for $(1 \otimes 1 \otimes q)$ as an element in $\mathcal{H}^J$. We also have other similar notations such as $t_w q$ for $t_w \otimes 1 \otimes q$.

**Definition 4.3.** Let $p = \dim V_J$ and let $\{\omega^\vee_1, \ldots, \omega^\vee_{n-p}\}$ be a basis for $V_J^\vee$. Denote $\mathbb{C}(a_1, \ldots, a_{n-p})$ be the algebra of rational functions with indeterminantes $a_1, \ldots, a_{n-p}$ over $\mathbb{C}$.

Let $U$ be an $\mathbb{H}_J$-module. Let $\eta_n^\vee = a_1 \omega^\vee_1 + \ldots + a_{n-p} \omega^\vee_{n-p}$, which will be regarded as a (natural) function from $V$ to $\mathbb{C}(a_1, \ldots, a_{n-p})$. Define $U_n$ to be an $\mathcal{H}_J$-module such that $U_n$ is isomorphic to $\mathbb{C}(a_1, \ldots, a_{n-p}) \otimes \mathbb{C}U$ as vector spaces and the action of $\mathcal{H}_J$ is determined by

$$\pi_{U_n}(t_w)(b \otimes u) = b \otimes \pi_{U_n}(t_w)u \quad \text{for } w \in W$$

$$\pi_{U_n}(v)(b \otimes u) = b \otimes \pi_{U_n}(v)u + \eta_n^\vee(v)b \otimes u \quad \text{for } v \in V$$

For $v \in V \setminus V_J$, $\pi_{U_n}(v)$ is invertible for generic values of $(a_1, \ldots, a_{n-p})$. Hence the $\mathcal{H}_J$-action on $U_n$ is well-defined. For an element $b \otimes u \in U_n$, we shall simply write $bu$ for $b \otimes u$. There is a natural multiplication of $\mathbb{C}(a_1, \ldots, a_{n-p})$ on $U_n$ and we shall consider $U_n$ to be an $\mathcal{H}_J$-module over $\mathbb{C}(a_1, \ldots, a_{n-p})$.

Fix a $\mathbb{C}$-basis $\{u_1, \ldots, u_k\}$ for $U$. We consider the tensor product $\mathcal{H}^J \otimes_{\mathcal{H}_J} U_n$, which will be regarded as an $\mathbb{H}$-module via the left multiplication of $\mathbb{H}$ on $\mathcal{H}^J$. For any element
Proposition 4.6. One advantage to have such expression as in Proposition 4.6 is the nice Remark 4.7. Proposition 4.4.

Proof. It suffices to verify that for \( \sum a \in H \otimes H, \) \( a \) can be written into the form

\[
(4.8) \quad x_a = \sum_{w \in W^J} t_w \otimes \left( \sum_{i=1}^{k} b_{w,i} u_i \right),
\]

where \( b_{w,i} \in \mathbb{C}(a_1, \ldots, a_{n-p}) \). We say that \( x_a \) is holomorphic at 0 if each \( b_{w,i} \) is holomorphic at \((0, \ldots, 0)\). It is easy to see that the definition of holomorphicity is independent of the choice of a basis for \( U \). Then for an holomorphic element \( x_a \in H^J \otimes H J U_a \) with the form \( 4.8 \), define the specialization \( |a=0 \) as follows:

\[
(4.9) \quad x_a|_{a=0} = \sum_{w \in W^J} t_w \otimes \left( \sum_{i=1}^{k} b_{w,i}(0, \ldots, 0) u_i \right) \in H \otimes H J U.
\]

Let \( w \in W^J \). Let \( w = s_{\alpha_1} \ldots s_{\alpha_1} \) be a reduced expression of \( w \). Let \( R(w) = \{ \alpha \in R^+ : w(\alpha) < 0 \} \). Define the intertwining element:

\[
(4.10) \quad \tau_w = (t_{s_{\alpha_1} \alpha_1 - k_{\alpha_1}} \ldots (t_{s_{\alpha_1} \alpha_1 - k_{\alpha_1}} \left( \prod_{\alpha \in R(w)} \alpha^{-1} \right) \in H J.
\]

The way of normalization for \( \tau_w \) will become clear in Proposition 4.13. The well-definedness of \( \tau_w \) follows from the following result:

Proposition 4.4. [KR Proposition 2.5(e)] For \( w \in W^J \), \( \tau_w \) is independent of the choice of a reduced expression.

Proof. Note that there is an assumption in [KR Proposition 2.5(e)] but the proof still applies.

Lemma 4.5. Let \( w \in W^J \) and let \( v \in V \subset O(J) \). Then \( v \tau_w = \tau_w w^{-1}(v) \).

Proof. It suffices to verify that for \( v \in V \), \( v(t_{s_{\alpha} \alpha - k_{\alpha}}) = (t_{s_{\alpha} \alpha - k_{\alpha}}) s_{\alpha}(v) \), which follows from \( k_{\alpha}(v - s_{\alpha}(v)) = k_{\alpha} v - s_{\alpha}(v) \) and a commutation relation of the graded Hecke algebra.

Proposition 4.6. Let \( x \in H \otimes H J U \). Then there exists a holomorphic element \( x_a \) in \( H^J \otimes H J U_a \) of the form

\[
\sum_{w \in W^J} \tau_w \left( \sum_{i=1}^{N_w} q_{w,i} \otimes u_{w,i} \right) \quad (q_{w,i} \in O(J), \ u_{w,i} \in U, N_w \in \mathbb{Z})
\]

such that \( x_a|_{a=0} = x \).

Proof. Any element in \( H \otimes H J U \) can be written into the form of \( \sum_{w \in W^J} t_w \otimes u_w \). The statement then follows from the fact that \( \tau_w \) forms an \( O(J) \)-basis for \( H J \).

Remark 4.7. One advantage to have such expression as in Proposition 4.6 is the nice commutation relation with the subalgebra \( S(V) \). A drawback is hard to obtain certain uniqueness statement for such expression. For example, for \( R = \{ \alpha \} \) of type \( A_1 \), let \( U = Cu \) be the 1-dimensional \( S(V) \)-representation with the weight 0. We have \( (t_{s_{\alpha} \alpha + k_{\alpha}}) u|_{a=0} = 0 \).
Let $U$ be a finite-dimensional $\mathbb{H}_J$-module. For $\gamma^\vee \in V^\vee$, define

$$W(J, U, \gamma^\vee) = \{ (w, \lambda^\vee) \in W^J \times \text{Wgt}(U) : w(\lambda^\vee) = \gamma^\vee \}.$$ 

**Proposition 4.8.** Let $U$ be a finite-dimensional $\mathbb{H}_J$-module. Let $\gamma^\vee \in V^\vee$. Let $x \in \mathbb{H} \otimes \mathbb{H}_J U$ be a generalized weight vector with weight $\gamma^\vee$. Then there exists a holomorphic element $x_a \in H^J \otimes H_J U_a$ of the form

$$\sum_{(w, \lambda^\vee) \in W(J, U, \gamma^\vee)} \tau_w \left( \sum_{i=1}^{N_w, \lambda^\vee} q_{w, \lambda^\vee, i} \otimes u_{w, \lambda^\vee, i} \right) (q_{w, \lambda^\vee, i} \in \mathcal{O}(J), u_{w, \lambda^\vee, i} \in 1 \otimes U \subset U_a, N_{w, \lambda^\vee} \in \mathbb{Z})$$

such that $x_a|_{a=0} = x$ and each $u_{w, \lambda^\vee, i}$ is a generalized weight vectors with the weight $\lambda^\vee$.

Since a complete proof for Proposition 4.8 is lengthy, we separate out into an appendix. The idea of our proof is to construct weight vectors via an induction of the length of Weyl groups elements.

**4.2. Generalized standard modules.** In this section, we shall refine our intertwining operator to the case of $(J, U) \in \mathbb{E}_L$. Recall that $W^J$ is the set consisting of all the minimal representatives for $W/W_J$. Let $w_{0, J}$ be the longest element in $W_J$ and let $w^J$ be the longest element in $W^J$.

We now consider two involutions. The first one is the map $\theta$ given by $\theta(\alpha) = -w_0(\alpha)$ ($\alpha \in J$). The second one is the map $\theta_J$ given by $\theta_J(\alpha) = -w_{0, J}(\alpha)$ ($\alpha \in J$). We also define $\phi_J = \theta \circ \theta_J$, which is not an involution in general. Since we only work for one fixed $J$, we shall simply write $\phi$ for $\phi_J$ most of time.

**Lemma 4.9.** Let $J \subset \Pi$. Then

1. the map $w \mapsto w^J$ from $W^{\theta(J)}$ to $W^J$ is a bijective well-defined function.
2. For $w \in W^{\theta(J)}$, $l(w) + l(w^J) = l(w^J)$.

**Proof.** For (1), we first show that the map is well-defined i.e. $ww^J \in W^J$ for $w \in W^{\theta(J)}$. It is equivalent to show that $ww^J(\alpha) > 0$ for any $\alpha \in J$. Since $w^J = w_0w_{0, J}$, $w^J$ sends simple roots in $J$ to simple roots in $\theta(J)$, which implies $ww^J(\alpha) > 0$ for $\alpha \in J$. To show the map is bijective, we shall show that the map $w \mapsto ww^J$ from $W^J$ to $W^{\theta(J)}$ gives the inverse map. This indeed follows from the following equations:

$$w^J w^{\theta(J)} = w^J w^{\theta(J)} w_{0, \theta(J)} w_{0, \theta(J)} = w^J w_0 w_{0, \theta(J)} = w^J w_{0, J} w_0 = \text{Id}.$$ 

This proves (1).

For (2), first we have $l(ww^J w_{0, J}) = l(ww_0) = l(w_0) - l(w)$. On the other hand, we have

$$l(ww^J w_{0, J}) = l(ww^J) + l(w_{0, J}) \quad \text{by (1)}$$

Now (2) is obtained by combining two equations. \hfill $\square$

**Definition 4.10.** Let $\delta : V \to V$ be a linear isomorphism such that $\delta(J) \subset \Pi$. The map $\delta$ induces a linear isomorphism $\delta : V^\vee \to V^\vee$ such that $\delta^{-1}(\gamma^\vee)(v) = \gamma^\vee(\delta(v))$. Then $\delta$ induces a map $\delta : \mathbb{H}_J \to \mathbb{H}_{\delta(J)}$ given by $\delta(t_{s, \alpha}) = t_{s, \delta(\alpha)}$, and $\delta(v) = \delta(v)$ for $v \in V$. It is
straightforward to verify that \( \delta \) is an algebra map. We shall simply write \( \delta \) for \( \tilde{\delta} \) later. The map \( \delta \) can also be similarly extended to \( \mathcal{H}_J \).

Let \( U \) be an \( \mathbb{H}_J \)-module. Define \( \delta(U) \) to be the \( \mathbb{H}_{\tilde{\delta}(J)} \)-module such that \( U \) is isomorphic to \( \delta(U) \) as vector spaces via a map still denoted \( \delta_U : U \rightarrow \delta(U) \) and the \( \mathbb{H}_{\tilde{\delta}(J)} \)-module is determined by

\[
\pi_{\delta(U)}(h)\delta_U(u) = \delta_U(\pi_U(\delta^{-1}(h))u).
\]

For lightening the notation, we simply write \( \delta \) if the meaning is clear from the context.

**Lemma 4.11.** Let \( (J, U) \in \mathcal{E}_L^0 \). Let \( \gamma^\vee \) be a weight of \( U \). Then

1. \( W(J, U, \gamma^\vee) = \{(1, \gamma^\vee)\} \) and
2. \( W(\theta(J), \phi(U), \gamma^\vee) = \{\{w^\theta(J), w^J(\gamma^\vee)\} \} \).

**Proof.** Note that \( W(J, U, \gamma^\vee) \) is the set of weights of \( I(J, U) \). Then (1) follows from Lemma 3.5 (also see the proof of [Ev] and [KR, Theorem 2.4]).

For (2), by definitions, \( w^J(\gamma^\vee) = \phi(\gamma^\vee) \) is a weight of \( \phi(U) \). Since \( w^\theta(J)(w^J(\gamma^\vee)) = \gamma^\vee \) by Lemma 4.9 \( \{u^\theta(J), w^J(\gamma^\vee)\} \in W(\theta(J), \phi(U), -w_0(\gamma^\vee)) \). To prove another inclusion, by Lemma 4.9(1), it is equivalent to show that if there exists \( w \in W^\theta(J) \) such that \( w(-w_0(\gamma^\vee)) = -w_0(\gamma^\vee) \) for some weight \( \gamma^\vee \) of \( U \), then \( w = 1 \). This is essentially the same as the proof of (1).

We also define an analog for an \( \mathbb{H}_{\tilde{\delta}(J)} \)-module \( \delta(U_a) \).

**Definition 4.12.** Define \( \delta(U_a) \) to be an \( \mathcal{H}_J \)-module such that \( \delta(U_a) \) is isomorphic to \( \mathbb{C}(a_1, \ldots, a_r) \otimes_{\overline{\mathbb{C}}} \delta(U) \) as vector spaces and the action of \( \mathcal{H}_J \) is determined by

\[
\pi_{\delta(U_a)}(h)\delta_{U_a}(u_a) = \delta_{U_a}(\pi_{U_a}(\delta^{-1}(h))u_a)
\]

Again, we shall simply write \( \delta \) for \( \delta_{U_a} \) if there is no confusion.

Recall \( \theta \) is defined in the beginning of this section. We define the evaluation \( |_{a=0} \) for holomorphic elements of \( \mathcal{H}^J \otimes_{\mathbb{H}_J} \phi(U_a) \) as \([10]\) for \( \mathcal{H}^J \otimes_{\mathcal{H}_J} U_a \).

We also remark that it does not really make sense to write \( \phi(U_a) \) since the definition of \( U_a \) depends on a choice of basis.

We now use the intertwining element \( \tau_{\omega(J)} \) to define an intertwining operator from \( \mathcal{H}^J \otimes_{\mathcal{H}_J} U_a \) to \( \mathcal{H}^J \otimes_{\mathcal{H}_J} \phi(U_a) \). One may consider as an analogue of a Knapp-Stein intertwining operator ([KS], also see [ALTMOV, Section 14]). There are similar results for standard modules in the affine Hecke algebra setting in an unpublished work of Delorme-Opdam (also see relevant work in [DO]).

We also remark that \( (\theta(J), \phi(U)) \) is not in \( \mathcal{E}_L^0 \) in general.

**Proposition 4.13.** Let \( (J, U) \in \mathcal{E}_L^0 \). Then:

1. Any element \( \tau_{\omega(J)} \otimes \phi(u) \in \mathcal{H}^J \otimes_{\mathcal{H}_J} \phi(U_a) \) with \( u \in 1 \otimes U \subset U_a \) is holomorphic.
2. The subspace

\[
\{\tau_{\omega(J)} \otimes \phi(u_a) : u_a \in U_a\}
\]
of $\mathcal{H}^{\theta(J)} \otimes_{H^{\theta(J)}} U_a$ is invariant under the $H^\gamma$-action and is isomorphic to $U_a$ as an $H^\gamma$-module. The isomorphism is characterized by the map:

$$1 \otimes u_a \mapsto \tau_{w^{\theta(J)}} \otimes \phi(u_a)$$

for $u \in 1 \otimes U \subset U_a$.

(3) The isomorphism in (2) induces an $H$-module isomorphism from $\mathcal{H}^J \otimes_{H,J} U_a$ to $\mathcal{H}^{\theta(J)} \otimes_{H^{\theta(J)}} \phi(U_a)$.

(4) The subspace

$$\{ (\tau_{w^{\theta(J)}} \otimes \phi(u))|_{a=0} : u \in 1 \otimes U \subset U_a \}$$

of $H \otimes H^{\theta(J)} \phi(U)$ is invariant under the $H^\gamma$-action and is isomorphic to $U$ as $H^\gamma$-modules via the following map:

$$1 \otimes u \mapsto (\tau_{w^{\theta(J)}} \otimes \phi(u))|_{a=0}.$$

(5) The map in (4) induces an $H$-map $I(J,U) \to I(\theta(J), \phi(U))$. Moreover, if $(J,U) \in \Xi_L$, then the image of the map is isomorphic to the unique simple quotient $L(J,U)$ of $I(J,U)$.

**Proof.** Let $\gamma^\vee$ be a weight of $U$. Fix a set of generalized $\gamma^\vee$-weight vectors $\{u_1, \ldots, u_k\}$ in $U$. It is not hard to show from linear algebra that there exists a generalized weight vector $x$ with the weight $w^{\theta(J)}(\phi(\gamma^\vee)) = \gamma^\vee$ of the form

$$x = t_{w^{\theta(J)}} \otimes \phi(u_i) + \sum_{w \in W^{\theta(J)} \setminus \{w^{\theta(J)}\}} t_w \otimes \phi(u_w),$$

where $u_w \in U$. By Proposition 4.8 and Lemma 2.11 there exists a holomorphic element $x_a \in \mathcal{H}^J \otimes_{H,J} U_a$ of the form

$$x_a = \sum_{i=1}^k \tau_{w^{\theta(J)}} \otimes b_i \phi(u_i)$$

such that $x_a|_{a=0} = x$. By considering the term $t_{w^{\theta(J)}} \otimes \phi(u_i)$, we have $q_i$ is holomorphic for each $i$. Furthermore $b_i(0, \ldots, 0) \neq 0$. Hence we also have $b_i^{-1}$ is holomorphic at $a = 0$. Thus $\tau_w \otimes u_i = \tau_w \otimes b_i^{-1}(b_iu_i)$ is holomorphic for each $l$. Then by linearity, we obtain (1).

We now consider (2). For notation simplicity, set $w = w^{\theta(J)}$. For any $v \in V$, we have $v^\tau_w = \tau_w v^{-1}(v)$ (Lemma 4.5). For any $a \in J$, $s_\alpha w \notin W^J$ and $l(s_\alpha w) = l(w) + 1$ (see the proof of Lemma 4.9) and hence Lemma 9.11 in Appendix B implies $t_{s_\alpha \tau_w} = \tau_w t_{s_{l(w) - 1}(a)}$.

By $\phi^{-1}(w^{-1}(v)) = v$, we can verify (2).

For (3), we see the induced map sends $\tau_{\gamma^\vee} \otimes u$ to $q \otimes u$ for some invertible $q \in \mathcal{O}(J)$ and $u \in U$. Explicitly,

$$q = \prod_{\alpha \in R^+ \setminus R^+_J} \frac{\alpha^2 - k_\alpha}{\alpha^2}.$$  

Hence the induced map is invertible.

For (4), the map is well-defined by (1). It follows from (2) that the map is an $H$-map.

The first assertion for (5) follows directly from (4) and Frobenius reciprocity. For the second assertion, in a paper of Barbasch-Moy [BM2], there is a notion of anti-involution.
Remark 4.16. For where to be the element position factor intertwining operator along arbitrary directions. We keep using notations in the earlier notation of \( L \) with the earlier notation of \( L \).

Remark 4.15. In general, for \((J, U) \in \Xi^J_L \setminus \Xi_L\), the space of intertwining operators from \( \mathbb{H} \otimes_{\mathbb{H}_J} U \) to \( \mathbb{H} \otimes_{\mathbb{H}_{\theta(J)}} \phi(U) \) has dimension greater than 1.

4.3. Intertwining operator along arbitrary directions. We keep using notations in previous subsections. Let \( t \) be an indeterminate and let \( \mathbb{C}(t) \) be the rational function ring over \( t \). Let \((J, U) \in \Xi^J_L\). Let \( \eta^\vee \in V^J_J \) (possibly zero). Let \( t \) be an indeterminate. Let \( U_{t\eta^\vee} = \mathbb{C}(t) \otimes_{\mathbb{C}} U \). We shall consider \( U_{t\eta^\vee} \) as an \( \mathbb{H}_J \)-module over \( \mathbb{C}(t) \) such that the action is given by

\[
\pi_{U_{t\eta^\vee}}(t_u)(b \otimes u) = b \otimes \pi_U(t_u)u \quad \text{for } w \in W,
\]

\[
\pi_{U_{t\eta^\vee}}(v)(b \otimes u) = b \otimes \pi_U(v)u + t\eta^\vee(v)b \otimes u \quad \text{for } v \in V,
\]

where \( b \in \mathbb{C}(t) \). We define \( \phi(U_{t\eta^\vee}) \) analogous to the notion of \( \phi(U_a) \) in Definition 4.12.

We consider an element \( 1 \otimes u \in \mathbb{H} \otimes_{\mathbb{H}_J} U_{t\eta^\vee} \), where \( u \in 1 \otimes U \subset U_{t\eta^\vee} \). The element \( 1 \otimes u \) is also naturally inside \( 1 \otimes U \subset U_a \) and by Proposition 4.13(1), \( \tau_{w_{\theta(J)}} \otimes \phi(u) \in \phi(U_a) \) is an holomorphic element and so we can specialize \( \tau_{w_{\theta(J)}} \otimes \phi(u) \in \mathcal{H}^{\theta(J)} \otimes_{\mathbb{H}_{\theta(J)}} \phi(U_a) \) at \( a = t\eta^\vee \) for small \( t \). (Here the precise meaning of \( a = t\eta^\vee \) is to specialize \( \eta^\vee = t\eta^\vee \).) Now the element \( \tau_{w_{\theta(J)}} \otimes u|_{a=t\eta^\vee} \) is in \( \mathbb{H} \otimes_{\mathbb{H}_{\theta(J)}} \phi(U_{t\eta^\vee}) \). We now define

\[
\Delta_{t\eta^\vee}^U : \mathbb{H} \otimes_{\mathbb{H}_J} U_{t\eta^\vee} \to \mathbb{H} \otimes_{\mathbb{H}_{\theta(J)}} \phi(U_{t\eta^\vee})
\]

to be the \( \mathbb{H} \)-map extending

\[
1 \otimes u \mapsto (\tau_{w_{\theta(J)}} \otimes \phi(u))|_{a=t\eta^\vee},
\]

where \( u \in 1 \otimes U \subset U_{t\eta^\vee} \). Again, we may simply write \( \Delta_{t\eta^\vee} \) or \( \Delta_{t} \) if there is no confusion.

Remark 4.16. For \( u \in 1 \otimes U \subset U_{t\eta^\vee} \), the notion \( \tau_{w_{\theta(J)}} \otimes u \) is well-defined for generic \( \eta^\vee \) but not for all \( \eta^\vee \in V^J_J \). Hence we have to pass to \( U_a \) to construct the intertwining operator.
5. Jantzen filtrations

One may also compare the way to define a Jantzen filtration of this section with the paper \[BC\]. Those two Jantzen filtrations (for the case of standard modules) coincide by the uniqueness of the intertwining operator. We shall not need this fact except in the computations of Example 7.3 which assume the truth of a version of Jantzen conjecture.

5.1. Jantzen filtrations. The Jantzen filtration is defined through $\Delta_{t\eta^\vee}$ in Section 143. One may define a multivariable version of Jantzen filtration through $\Delta_{a}$. However, if we hope to compute the Jantzen filtration from geometric way (e.g. a conjecture in [BC] using [Lu3, Lu4]) or from Arakawa-Suzuki functor [AS, Su] or results in [Ro], then perhaps a single variable is a better notion.

Definition 5.1. (Jantzen filtration [Ja]) Let $(J, U) \in \Xi^g_L$. Let $\eta^\vee \in V^\vee_{J,1}$, which is possibly zero. Set $a = t\eta^\vee$. Let $\Omega(t) \subset \mathbb{C}(t)$ be the set of all functions in $\mathbb{C}(t)$ holomorphic at $t = 0$. We can define a Jantzen filtration as follows. For each integer $i$, let

$$U^i_{t\eta^\vee} = t^i \Omega(t) \otimes U \subset U_{t\eta^\vee}.$$ 

We shall regard $U^i_{t\eta^\vee}$ as an $\mathbb{H}_J$-module. Recall that $\phi = \theta \circ \theta_J$. Set $T^i(U_{t\eta^\vee}) = \mathbb{H} \otimes_{\mathbb{H}_{\eta^\vee}} \phi(U^i_{t\eta^\vee})$ and $T^i(\phi(U^i_{t\eta^\vee})) = \mathbb{H} \otimes_{\mathbb{H}_{\eta^\vee}} \phi(U^i_{t\eta^\vee})$. From this point, we shall consider $T^i(U_{t\eta^\vee})$ and $T^i(\phi(U^i_{t\eta^\vee}))$ to be $\mathbb{H}$-modules over $\mathbb{C}$, but the $\mathbb{H}$-action is not significant sometimes and we may simply write $T^i(U)$ and $T^i(\phi(U))$ respectively to lighten notations.

Via the specialization at $t = 0$, there is a natural identification

$$T^i(U_{t\eta^\vee})/T^{i+1}(U_{t\eta^\vee}) \sim \mathbb{H} \otimes_{\mathbb{H}_J} U.$$

The Jantzen filtration is, roughly speaking, to get the information of the vanishing degree of elements in $\mathbb{H} \otimes_{\mathbb{H}_J} U$ under the intertwining operator.

By Proposition 4.11 (1), we have the map

$$\Delta_{t\eta^\vee}^U = pr_i \circ \Delta_{t\eta^\vee}^U : T^0(U) \to T^0(\phi(U))/T^i(\phi(U)),$$

where $\Delta_{t\eta^\vee}^U$ is the map in 4.11 restricted to the space $T^0(U_{t\eta^\vee})$ and $pr_i$ is the natural projection map from $T^0(\phi(U))$ to $T^0(\phi(U))/T^i(\phi(U))$. We may drop the superscript $U$ and write $\Delta_{t\eta^\vee}^i$ sometimes.

We then have the filtration of the form:

$$F^i_{\eta^\vee}(J, U) = (\ker \Delta_{t\eta^\vee}^i + T^1(U))/T^1(U).$$

By (5.12), this gives a filtration, denoted $JF^i_{\eta^\vee}(J, U)$ or simply $JF^i_{\eta^\vee}$, for the generalized standard module $\mathbb{H} \otimes_{\mathbb{H}_J} U$:

$$\mathbb{H} \otimes_{\mathbb{H}_J} U = JF^0_{\eta^\vee}(J, U) \supset JF^1_{\eta^\vee}(J, U) \supset JF^2_{\eta^\vee}(J, U) \supset \ldots$$

We shall call the filtration to be the Jantzen filtration of $I(J, U)$ associated to $\eta^\vee$. A more intuitive way to describe $JF^i_{\eta^\vee}(J, U)$ via the identification (5.12) is as follows. The space $JF^i_{\eta^\vee}(J, U)$ contains elements $x \in \mathbb{H} \otimes_{\mathbb{H}_J} U$ such that there exists an element $x_t \in \mathbb{H} \otimes_{\mathbb{H}_J} U_{t\eta^\vee}$ satisfying the conditions that $x_t|_{t=0} = x$ and $\Delta_{t\eta^\vee}(x_t)$ has zero of order at least $i$. 

From the definitions, we have
\[ \text{JF}_t (J,U)/\text{JF}_t (J,U) \cong L(J,U) \]
for any \( \eta^\vee \in V_{\eta^\vee}^1 \).

Note that our definition allows \( \eta^\vee \) to be in arbitrary direction, and possibly zero. In general, we do not have \( \bigcap_i \text{JF}_t \neq 0 \). When \( \eta^\vee = 0 \), we have \( \text{JF}_t^0 (J,U)/\text{JF}_t (J,U) \cong L(J,U) \) for all \( i \geq 1 \).

5.2. Linearly independence. We keep using notations in the previous subsection. For each integer \( i \) and each \( x \in F^i (J,U_{t^\eta^\vee}) \setminus F^{i+1} (J,U_{t^\eta^\vee}) \), let \( x_t \in \ker \Delta^i_{t^\eta^\vee} \) be a representative of \( x \). In particular, \( \Delta^i_{t^\eta^\vee} (x_t) \) has zeros of order \( i \), i.e.
\[ \frac{1}{t^i} \Delta^i_{t^\eta^\vee} (x_t) \in T^0 (\phi(U)) \quad \text{and} \quad \frac{1}{t^{i+1}} \Delta^i_{t^\eta^\vee} (x_t) \notin T^0 (\phi(U)) \]

For each \( i \), fix the representatives \( x_t^{i,1}, \ldots, x_t^{i,d_i} \in T^0 (U) \) for \( F^i (J,U_{t^\eta^\vee}) \setminus F^{i+1} (J,U_{t^\eta^\vee}) \) such that the projections of \( x_t^{i,1}, \ldots, x_t^{i,d_i} \) form a basis for \( F^i (J,U_{t^\eta^\vee})/F^{i+1} (J,U_{t^\eta^\vee}) \). Let \( \{ x_t^{i,k} \}_{k=1}^{d_i} \subset T^0 (U) \) whose projection in \( (\ker \Delta^i_{t^\eta^\vee} + T^1 (U))/T^1 (U) \) forms a basis. By definition \( \Delta^i_{t^\eta^\vee} (x_t^{i,k}) = 0 \) for all \( i \).

Let \( X_t^i (J,U) \) (or simply \( X_t^i \)) be the space spanned by \( x_t^{i,1}, \ldots, x_t^{i,d_i} \). Let \( X_t \) be the space spanned by all \( X_t^i \) (which is not an \( \mathbb{H} \)-module in general). From definitions, we have
\[ X_t \cap T^1 (U) = 0. \]

We now prove a statement of linearly independence.

**Proposition 5.2.** The set
\[ \left\{ \frac{1}{t^i} \Delta^i_{t^\eta^\vee} (x_t^{i,1}) \bigg|_{t=0}, \ldots, \frac{1}{t^i} \Delta^i_{t^\eta^\vee} (x_t^{i,d_i}) \bigg|_{t=0} \right\}_{i \in \mathbb{Z}} \]
is linearly independent. Equivalently,
\[ \text{span} \left\{ \frac{1}{t^i} \Delta^i_{t^\eta^\vee} (x_t^{i,1}), \ldots, \frac{1}{t^i} \Delta^i_{t^\eta^\vee} (x_t^{i,d_i}) \right\}_{i \in \mathbb{Z}} \cap T^1 (\phi(U)) = 0 \]

**Proof.** We consider a linear equation of the following form
\[ \sum_{i \in \mathbb{Z}} \frac{1}{t^i} \sum_{k=1}^{d_i} a_{i,k} \Delta^i_{t^\eta^\vee} (x_t^{i,k}) = y, \]
where \( y \in T^1 (\phi(U_{t^\eta^\vee})) \). Let \( i' \) be the greatest integer such that \( a_{i',k} \neq 0 \) for some \( k = 1, \ldots, d_{i'} \). Suppose such \( i' \) exists.
\[ \sum_{k=1}^{d_{i'}} a_{i',k} \Delta^i_{t^\eta^\vee} (x_t^{i',k}) \in T^{i'+1} (\phi(U)) \]
Therefore \( \sum_{k=1}^{d_{i'}} a_{i',k} \Delta^{i'+1}_{t^\eta^\vee} (x_t^{i',k}) = 0 \). Hence \( \sum_{k=1}^{d_{i'}} a_{i',k} x_t^{i',k} \in F^{i'+1} (J,U) \) and this gives a contradiction to our choice of \( x_t^{i',k} \). \( \square \)
5.3. **Comparing filtrations.** In general, the Jantzen filtration for a standard module does not coincide with the radical filtration or socle filtration. This gives a discrepancy from the picture of real groups \[BB\ Corollary 5.3.2\] (also see \[Ba, Ir\] for Verma module cases). We shall give an example for type $B_2$.

We recall some definitions. The radical of an $\mathbb{H}$-module $X$, denoted $\text{rad}(X)$, is the minimal submodule of $X$ such that the quotient is semisimple. This defines the radical filtration for $X$:

$$X \supset \text{rad}^1(X) \supset \text{rad}^2(X) \supset \ldots \supset 0$$

The socle, denoted $\text{soc}(X)$, of an $\mathbb{H}$-module $X$ is the maximal semisimple module of $X$. Define inductively $\text{soc}^{i+1}(X)$ by $\text{soc}^{i+1}(X)/\text{soc}^i(X) \cong \text{soc}(X/\text{soc}^i(X))$. This gives the socle filtration for $X$:

$$0 \subset \text{soc}^1(X) \subset \text{soc}^2(X) \subset \ldots \subset X$$

**Example 5.3.** We consider the case of type $B_2$ with $k_\alpha = k_\beta = 2$. Denote the simple roots $\alpha$ and $\beta$ such that

$$\alpha^\vee(\alpha) = \beta^\vee(\beta) = -\alpha^\vee(\beta) = 2, \quad \beta^\vee(\alpha) = -1.$$  

We consider the central character $\gamma^\vee = \alpha^\vee$. There exists two non-isomorphic irreducible tempered module of the central character $\gamma^\vee$. Denote by $T_0$ the tempered module which contains a sign representation as $W$-representation and denote by $T_1$ the tempered module which is 1-dimensional. There is another 1-dimensional irreducible $\mathbb{H}$-module with the central character $\gamma^\vee$. Denote the module by $Z$. The weight space of $Z$ is $\alpha^\vee$.

Using the projective resolution in \[Ch2\ Section 3\], simple computations for Hom-space of $W$-representations give

$$\dim \text{Ext}^i_{\mathbb{H}}(Z, Z) = \dim \text{Ext}^i_{\mathbb{H}}(T_1, T_1) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

(5.13)

$$\dim \text{Ext}^i_{\mathbb{H}}(A, B) = \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases},$$

where $(A, B)$ can be one of the following pairs: $(T_1, Z), (Z, T_1), (T_1, T_0), (T_0, T_1), (Y, Z), (Z, Y)$.

To compute Ext-groups for the pairs $(T_0, T_0)$ and $(Y, Y)$, using the projective resolution of \[Ch2\ Section 3\] requires some further structural information for $T_0$ and $Y$. Instead of computing such information, we apply the duality result \[Ch2\ Theorem 4.15\] and then obtain

$$\dim \text{Ext}^2_{\mathbb{H}}(Y, Y) = \dim \text{Hom}_{\mathbb{H}}(T_0, Y) = 0.$$  

Then using the $W$-structure, we obtain the Euler-Poincare pairing $\text{EP}(T_0, T_0) = \text{EP}(Y, Y) = 1$. Combining with the fact that the global dimension of $\mathbb{H}$ is 2, we can deduce that

$$\text{Ext}^i_{\mathbb{H}}(Y, Y) = \text{Ext}^i_{\mathbb{H}}(T_0, T_0) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

(5.14)

Let $J = \{\alpha\}$ and let $U$ be the irreducible tempered $\mathbb{H}_J$-module with the weight 0. Let $\nu^\vee = \alpha^\vee + 2\beta^\vee \in V_J^\vee$. We consider the standard module $X = \mathbb{H} \otimes_{\mathbb{H}_J} (U \otimes \mathbb{C}_{\nu^\vee})$, where $\mathbb{C}_{\nu^\vee}$ is 1-dimensional $S(V_J^\vee)$-module with the weight $\nu^\vee$. By a slight consideration on the
weight of $X$, one can deduce that $T_0$, $T_1$, $Y$ and $Z$ have multiplicity one in the composition series of $X$.

The simple quotient of $X$ is $Y$ and hence we have the following short exact sequence:

$$0 \to N \to X \to Y \to 0,$$

where $N$ is the maximal proper submodule of $X$.

In order to determine the radical $\text{rad}(N)$ of $N$, we have to compute $\text{Hom}_H(N,C)$, where $C = Z, T_0, T_1$. By using Proposition 3.8 and the associated long exact sequence of the functor $\text{Hom}_{H}(.,C)$, we have

$$\text{Hom}_H(N,C) \cong \text{Ext}^1_H(Y,C).$$

and so the Hom-space can be computed from (5.13) and (5.14). This implies there exists a surjective map from $N$ to $Z \oplus T_1$ and $\text{rad}(N) = T_0$. In summary, the radical filtration of $X$ is given by: $\text{rad}^0(X)/\text{rad}^1(X) = Y$, $\text{rad}^1(X)/\text{rad}^2(X) \cong T_1 \oplus Z$, $\text{rad}^2(X)/\text{rad}^3(X) \cong T_0$.

On the other hand, the Jantzen filtration of $X$ can be computed directly (see Appendix A for the details, also c.f. [3] Section 4.3, [BC] Section 6.7):

$$\text{JF}^0_{\nu \nu}/\text{JF}^1_{\nu \nu} \cong Y, \quad \text{JF}^1_{\nu \nu}/\text{JF}^1_{\nu \nu} \cong T_1$$

$$\text{JF}^2_{\nu \nu}/\text{JF}^3_{\nu \nu} \cong Z, \quad \text{JF}^3_{\nu \nu}/\text{JF}^3_{\nu \nu} \cong T_0$$

For the socle filtration, we apply the • anti-involution. By a weight consideration, we have $X^* \cong IM(X)$, where $IM$ is the Iwahori-Matsumoto involution (see [EX] for the definition of $IM$). In particular, $IM(T_0) = Y$ and $IM(T_1) = Z$. The radical filtration of $X^* \cong IM(X)$ and Lemma 3.5 now determine the socle filtration of $X$.

6. The quotient $L(J,U)$ of generalized standard modules

Recall that $S(V)$-extensions are defined in Definition 3.12. In this section, we deal with generalized standard modules of strict $S(V)$-type. Since those extensions come from extensions of representations of a polynomial ring, we can write down the structure of corresponding modules in a fairly explicit way.

However, in order to describe the structure in terms of composition factors, we need some more work and make use of Jantzen filtrations.

We shall make use of notations in Sections 2 and 5 (e.g. $U_a$, $\mathcal{H}, \Delta$, etc).

6.1. Realization of $S(V)$-extensions. Let $(J, U) \in \Xi$. (Some constructions are also valid for $(J, U) \in \Xi'$, but those are not our main concern.) Then $U = \overline{U} \otimes L$ for some $\mathbb{H}_{\nu \nu}$-module $\overline{U}$ and some one dimensional $S(V_{\nu \nu})$-module $L$. From the discussions on Section 3 we can naturally construct Yoneda first extensions between two $I(J,U)$ through the Yoneda first extensions between two $U$. Using discussions in Section 3 we can and shall identify the following:

$$\text{Ext}^1_{H}(I(J,U), I(J,U)) \cong \text{Ext}^1_{H}(U, U) \cong \text{Ext}^1_{H_{\nu \nu}}(\overline{U}, \overline{U}) \oplus \text{Ext}^1_{S(V_{\nu \nu})}(L, L).$$
where $L$ appears $r$ times in the right hand side. The $H$-action is given by:

\begin{align}
\pi_{U^r,\eta^r}(t_w)(u_1, \ldots, u_r) &= (\pi_U(t_w)u_1, \ldots, \pi_U(t_w)u_r) \quad (w \in W), \\
\pi_{U^r,\eta^r}(v)(u_1, \ldots, u_r) &= (\pi_U(v)u_1, \ldots, \pi_U(v)u_r + \eta^r(v)(0, u_1, \ldots, u_{r-1}) \quad (v \in V).
\end{align}

By definitions, $(J, U^{r,\eta^r}) \in \Xi_L^n$.

**Lemma 6.1.** Let $(J, U) \in \Xi_L$. Then $I(J, U^{r,\eta^r})^* \cong I(\theta(J), \phi(U^{r,\eta^r}))$.

**Proof.** Note that $I(J, U^{r,\eta^r})^* \cong \theta(I(J, U^{r,\eta^r})^*)$ (see e.g. [Ch2, Lemma 4.5]). Here $\ast$ is a linear anti-involution defined similarly as in [BM2, Section 1]. Then by [BM2, Corollary 1.3], $I(J, U^{r,\eta^r})^* \cong I(\theta(J), (U^{r,\eta^r})^*)$. Now by considering weights of $U^{r,\eta^r}$ and using [EM, Theorem 5.5], we have $U^{r,\eta^r} \cong \theta_I(U)$. Then $(U^{r,\eta^r})^* \cong \theta_J(U^{r,\eta^r})$ (which we also need to use the strict $S(V)$-extensions for $U^{r,\eta^r}$). Hence $I(J, U^{r,\eta^r})^* \cong I(\theta(J), \phi(U^{r,\eta^r}))$. \hfill \Box

Note that $I(J, U^{r,\eta^r})$ can be naturally identified with $T^0(U)/T^r(U)$ via the map

\[ t_w \otimes (u_1, \ldots, u_r) \mapsto \sum_{i=1}^r t_w \otimes t^{i-1} u_i, \]

where $u_i$ on the left-hand side is also regarded as an element in $1 \otimes U \subset U_{\eta^r}$. Similarly, we can identify $I(\theta(J), \phi(U^{r,\eta^r}))$ with $T^0(\phi(U))/T^r(\phi(U))$.

Since $\Delta_{t_{\eta^r}}(T^r(U)) \subset T^r(\phi(U))$, the map $\Delta_{t_{\eta^r}}$ induces a map, denoted

\[ \Delta_{U^{r,\eta^r}} : T^0(U)/T^r(U) \to T^0(\phi(U))/T^r(\phi(U)). \]

**Lemma 6.2.** $\text{im} \Delta_{U^{r,\eta^r}} \cong \text{im} \Delta_{U^{r,\eta^r}}^r$.

**Proof.** By using Frobenius reciprocity, Lemma 2.5, Lemma 6.1 and Lemma 3.5, we have

\[ \text{Hom}_H(I(J, U), I(\theta(J), \phi(U))) \cong \text{Hom}_H(U^{r,\eta^r}, U^{r,\eta^r}) \cong \mathbb{C}^r. \]

Furthermore, for an element $\psi \in \text{Hom}_H(I(J, U^{r,\eta^r}), I(\theta(J), \phi(U^{r,\eta^r})))$, $\text{im} \psi$ is determined by $\text{im} \psi|_{U^{r,\eta^r}}$. On the other hand by counting dimensions, one can conclude that

\[ \Delta_{U^{r,\eta^r}}(1 \otimes U^{r,\eta^r}) \cong \Delta_{U^{r,\eta^r}}(1 \otimes U^{r,\eta^r}) \cong 1 \otimes U^{r,\eta^r}. \]

Combining all these, we prove the lemma. \hfill \Box

### 6.2. A quotient of generalized standard modules

**Lemma 6.3.** Let $\Delta$ be a complex associative algebra with an unit. Let $X$ be a finite-dimensional $\Delta$-module. Fix a finite collection $\{L_1, \ldots, L_r\}$ of simple $\Delta$-modules. There exists a unique submodule $N$ such that

1. the composition factors of the socle of $X/N$ are isomorphic to some $L_i$,
2. no composition factors of $N$ are isomorphic to some $L_i$. 


Proof. Let $\mathcal{M}$ be the set of all proper submodules of $X$ whose composition factors are not isomorphic to any of $L_i$. $\mathcal{M}$ is nonempty since the zero module is in $\mathcal{M}$. Suppose $N_1$ and $N_2$ are two maximal element (with respect to the inclusion) in $\mathcal{M}$. By considering $N_1 + N_2$ which has no composition factor of $L_i$ and using the maximality of $N_1$ and $N_2$, we have $N_1 = N_2$. (To see $N_1 + N_2$ has no composition factor of $L_i$, we can use $(N_1 + N_2)/N_2 \cong N_1/(N_1 \cap N_2)$ and apply the Jordan-Hölder Theorem for composition factors by finite dimensionality.) Hence $N$ has a unique maximal element. Let $N$ be the maximal element in $\mathcal{N}$. Then $N$ automatically satisfies (2). To show $N$ also satisfies the property (1), suppose there exists a simple module $L$ and $L$ be as in Section 6.1. Let $\ell : I(J, U^{-1, r, \eta}) \hookrightarrow I(J, U^{r, \eta}) \cong \text{Hom}_{\mathfrak{g}}(L, X/N)$ be the inclusion map (unique up to a scalar). Then

(1) The map $\iota$ induces an injective map from $L(J, U^{-1, r, \eta})$ to $L(J, U^{r, \eta})$.
(2) $\text{JF}_{\eta^r}(J, U^{-1, r})/\text{JF}_{\eta^r}(J, U)$ is isomorphic to $L(J, U^{-1, r, \eta})/L(J, U^{-1, r, \eta})$.
(3) $\text{JF}_{\eta^r}(J, U^{r, \eta})/\text{JF}_{\eta^r}(J, U^{-1, \eta}) \cong \text{L}(J, U^{r, \eta})$ is the unique indecomposable quotient of $\text{I}(J, U^{r, \eta})$ with the properties that: (a) the module has unique simple quotient and unique simple submodule, both of which are isomorphic to $L(J, U)$ and (b) the multiplicity of $L(J, U)$ in the composition series of the module is the same as that in the composition series of $\text{I}(J, U)^{r, \eta}$.
(4) $L(J, U^{r, \eta})$ is self-dual.

Proof. For simplicity, we shall write $\Delta_i$ for $\Delta_i^{U^{r, \eta}}$.

We first prove (1). By Lemma 6.2, we have isomorphisms $\text{coker} \Delta U^{r, \eta} \cong \text{coker} \Delta U^{r, \eta} \cong L(J, U^{r, \eta})$ (i.e., $r = 1, r$). The map $\iota$ induces a natural map from $\iota' : I(J, U^{-1, r, \eta}) \cong T(U)/T^{-1}(U) \hookrightarrow T(U)/T^{r'}(U) \cong I(J, U^{r, \eta})$ given by a multiplication of $\iota$. Hence we have a map $\tilde{\iota} : T(U)/T^{r}(U) \rightarrow \text{coker} \Delta_r$. By comparing the maps $\Delta_{r-1}, \Delta_r$, it is straightforward to verify that $\ker \tilde{\iota} = \ker \Delta_{r-1}$. Hence we obtain an induced injective map from $\text{coker} \Delta U^{r-1, \eta}$ to $\text{coker} \Delta U^{r, \eta}$. 

We now describe quotients of some generalized standard modules of strict $S(V)$-type in terms of the Jantzen filtration of (ordinary) standard modules. In the case that the generalized standard module is the standard module, the quotient simply gives the unique simple quotient in the Langlands classification.

Theorem 6.4. Let $(J, U) \in \Xi_L$ (Definition 3.2) and let $0 \neq \eta^r \in V^{-1, r, \eta}$. Let $U^{r-1, \eta}$ and $U^{r, \eta}$ be as in Section 6.1. Let $\iota : I(J, U^{r-1, \eta}) \hookrightarrow I(J, U^{r, \eta}) \cong \text{Hom}_{\mathfrak{g}}(L, X/N)$ be the inclusion map (unique up to a scalar). Then

(1) The map $\iota$ induces an injective map from $L(J, U^{r-1, \eta})$ to $L(J, U^{r, \eta})$.
(2) $\text{JF}_{\eta^r}(J, U^{-1, r})/\text{JF}_{\eta^r}(J, U)$ is isomorphic to $L(J, U^{-1, r, \eta})/L(J, U^{-1, r, \eta})$.
(3) $\text{JF}_{\eta^r}(J, U^{r, \eta})/\text{JF}_{\eta^r}(J, U^{-1, \eta}) \cong L(J, U^{r, \eta})$ is the unique indecomposable quotient of $\text{I}(J, U^{r, \eta})$ with the properties that: (a) the module has unique simple quotient and unique simple submodule, both of which are isomorphic to $L(J, U)$ and (b) the multiplicity of $L(J, U)$ in the composition series of the module is the same as that in the composition series of $\text{I}(J, U)^{r, \eta}$.
(4) $L(J, U^{r, \eta})$ is self-dual.
We now prove (2).

\[
\begin{array}{ccccccccc}
0 & \rightarrow & T^0(U)/T^{r-1}(U) & \stackrel{\iota'}{\rightarrow} & T^0(U)/T^r(U) & \stackrel{F}{\rightarrow} & T^0(U)/T^1(U) & \rightarrow & 0 \\
& & \downarrow\Delta_r & & \downarrow\Sigma & & \\
& & T^0(\phi(U))/T^r(\phi(U)) & & T^0(\phi(U))/(T^r(\phi(U)) + \Delta_{t\eta'}(T^1(U))) & & \\
\end{array}
\]

where \(F\) is the natural surjective map so that the top sequence is exact. Define \(F' = \Delta^3 \circ F\).

We claim that \(\ker F' = \text{im}\iota' + \ker\Delta_r\). From this, we have

\[
\frac{(T^0(U)/T^r(U))}{\ker F'} \cong \frac{(T^0(U)/T^r(U))}{(\text{im}\iota' + \ker\Delta_r)/\ker\Delta_r} \cong \frac{L(J,F_{\eta'}(J,U))}{L(J,F_{\eta'}(J,U))}.
\]

The last isomorphism follows from (1). On the other hand, we have

\[
\frac{(T^0(U)/T^r(U))}{\ker\Delta} \cong \frac{T^0(U)}{\ker\Delta_{t\eta'} + T^1(U)} \cong \frac{JF_{\eta'}^0(J,U)}{JF_{\eta'}^0(J,U)}.
\]

This shall prove (2). The first isomorphism above follows from the fact that \(F\) is surjective. For the details of the second isomorphism, one can deduce from the discussions in Section 5.2. (More precisely, using Proposition 5.2, one can show that \(\ker\Delta \cong (\ker\Delta_{t\eta'} + T^1(U))/T^1(U)\).)

We now turn to prove the claim \(\ker F' = \text{im}\iota' + \ker\Delta_r\). The inclusions \(\text{im}\iota' \subset \ker F'\) and \(\ker\Delta_r \subset \ker F'\) follow from definitions. We now pick a representative \(\bar{x}_t \in T^0(U)\) of an element in \(\ker F'\). Write \(\bar{x}_t\) as the form:

\[
\sum_{i=0}^{r-1} t^i \sum_{p \in \mathbb{Z} \geq 0} x^{p,i}_t + t^r z_t,
\]

where \(x^{p,i}_t \in X^p_t\) and \(z_t \in T^0(U)\). Here \(X^p_t\) is defined as in Section 5.2. For simplicity, set \(y^{p,i}_t = \frac{1}{t^i}\Delta_{t\eta'}(x^{p,i}_t)\). Let \(N\) be the least integer such that there exists a pair \((p,i)\) with \(p + i = N\) and \(x^{p,i} \neq 0\). If such integer exists and \(N \leq r - 1\), then the element \(\Delta_{t\eta'}(\bar{x}_t)\) can be written as

\[
t^N(y^{0,N}_t + y^{1,N-1}_t + \ldots + y^{N,0}_t) + tz'_t,
\]

where \(z'_t \in T^0(\phi(U))\). Recall that \(\bar{x}_t\) is a representative of an element in \(\ker\Delta_r\). We must then have \(y^{0,N}_t = 0\) by Proposition 5.2 and so \(x^{0,N}_t = 0\). Let \(x^0_t = tz^0_t + \ldots + t^N y^{N,0}_t \in T^1(U)\). Now we consider the element \(x_t - x^0_t\). Repeating the above process we shall eventually obtain an element \(x''_t \in T^1(U)\) such that \(\Delta_{t\eta'}(x_t - x''_t) \in T^r(\phi(U))\) and \(x''_t \in T^1(U)\). By definition \(x''_t + T^r(U) \in \text{im}\iota'\) and hence \(x_t + T^r(U) \in \text{im}\iota' + \ker F'\) as desired.

We now prove (3). Let \(Z = JF_{\eta'}^0(J,U + \eta')(J,U + \eta')\). Since \(JF_{\eta'}^0(J,U + \eta')(J,U + \eta') \) is a quotient of \(I(J,U + \eta')\), we have a surjective map from \(\mathbb{H} \otimes_{\mathbb{H}_J} U + \eta'\) to \(JF_{\eta'}^0(J,U + \eta')\). By the left-exactness of \(\text{Hom}_\mathbb{H}(\_, Y)\), we have an injection from \(\text{Hom}_\mathbb{H}(Z, Y)\) to \(\text{Hom}_\mathbb{H}(I(J,U + \eta'), Y)\). Then \(Z\) having a unique simple quotient isomorphic to \(L(J,U\eta')\) follows from the fact that
I(J, U) has a unique simple quotient isomorphic to L(J, U). This also implies Z is indecomposable. By definitions,
\[ T^0(U^{r, n^\vee})/\ker \Delta_{U^{r, n^\vee}} \cong \text{im} \Delta_{U^{r, n^\vee}} / T^1(\phi(U^{r, n^\vee})). \]
Hence Z is also isomorphic to a submodule of \( T^0(\phi(U^{r, n^\vee}))/T^1(\phi(U^{r, n^\vee})) \cong I(\theta(J), \phi(U)) \).
Now by Lemma 3.3 and Frobenius reciprocity, I(J, U^{r, n^\vee}) has a unique simple quotient and so does L(J, U^{r, n^\vee}). By considering I(\theta(J), \phi(U))^\ast as in the proof of Proposition 4.4, we have \( I(\theta(J), \phi(U^{r, n^\vee}))^\ast \) has a unique simple quotient and so I(\theta(J), \phi(U^{r, n^\vee})) has a unique simple submodule. Hence L(J, U^{r, n^\vee}) also has a unique simple submodule. This proves the property (a). Property (b) follows from an induction argument using (2), which we have proved.

The uniqueness for (3) follows from Lemma 6.3.

We now consider (4). For notation simplicity, set \( G = \Delta_{U^{r, n^\vee}} \). Then we have a dual map \( G^\ast : I(\theta(J), \phi(U^{r, n^\vee}))^\ast \rightarrow I(J, U^{r, n^\vee}) \). Then the image and the cokernel of \( G^\ast \) is isomorphic to L(J, U^{r, n^\vee})^\ast. On the other hand, by Lemma 6.1 we have
\[ I(\theta(J), \phi(U^{r, n^\vee}))^\ast \cong I(J, U^{r, n^\vee}), \quad I(J, U^{r, n^\vee}) \cong I(\theta(J), \phi(U^{r, n^\vee})). \]
By using (2) and Lemma 2.5 we have \( \text{im} G^\ast \cong \Delta_{U^{r, n^\vee}} \). Hence
\[ L(J, U^{r, n^\vee}) \cong L(J, U^{r, n^\vee})^\ast. \]

**Remark 6.5.** Generalizing Theorem 6.3 to other layers of Jantzen filtrations of I(J, U^{r, n^\vee}) seems to be harder or less direct from our approach (because the formula involves derivatives and it is hard to apply). We give an example to illustrate the Jantzen filtration of a higher layer may be more complicated. Let \( H \) be of type A_1 and let \( k = 1 \). Let \( \alpha \) be the unique simple root of type A_1. Let \( U \) be the unique irreducible \( S(V) \)-module with the weight \( \frac{1}{2} \alpha^\vee \).

We consider the Jantzen filtration of \( H \otimes_{S(V)} U^{2, n^\vee} \), where \( n^\vee = \frac{1}{2} \alpha^\vee \). Take \( v_0 = \alpha \). Let
\[ \bar{x}_t = \left( t_{s_{\alpha}} - \frac{1}{\alpha} \right) \otimes (0, u). \]
We have
\[ \Delta_{U^{n^\vee}}(\bar{x}_t) = 1 \otimes \left( 0, \frac{t^2 + 2t}{(t + 1)^2} u \right) \in T^1(\phi(U^{n^\vee})) \setminus T^2(\phi(U^{n^\vee})). \]

It might be tempted to think in the beginning that \( x_t \mid t=0 \in \text{JF}^1(J, U^{2, n^\vee}) \setminus \text{JF}^2(J, U^{2, n^\vee}) \), which however is false. Let
\[ y_t = - \left( t_{s_{\alpha}} - \frac{1}{\alpha} \right) \otimes (u, 0). \]
Then \( \Delta_{U^{n^\vee}}(t y_t + x_t) \in T^2(\phi(U)) \). Indeed we have \( \text{JF}^1(J, U^{2, n^\vee}) = \text{JF}^2(J, U^{2, n^\vee}) \) and \( \text{JF}^3(J, U^{2, n^\vee}) = 0 \).

**6.3. Good and bad directions.** We now define a set, which will be used to parametrize certain self-extensions of simple modules.
Definition 6.6. For $\eta^\vee \in V^\perp^\vee_J$, we say $\eta^\vee$ is in a bad direction (with respect to $J$ and $U$) if

$$JF^1\eta^\vee(J, U) = JF^2\eta^\vee(J, U).$$

Otherwise we say $\eta^\vee$ is a good direction. Denote by $V^\perp_{\text{bad}}(J, U)$, or simply $V^\perp_{\text{bad}}$, the set of vectors in a bad direction.

Example 6.7. We consider $\mathbb{H}$ to be of type $A_2$ with $k \equiv 1$. Let $\alpha, \beta$ be the simple roots. Consider the central character $\gamma^\vee = \frac{1}{2} \alpha^\vee + C(\beta^\vee + 2\alpha^\vee)$, where $C$ is taken to be a sufficiently large positive number. In particular, $\gamma^\vee$ is in the dominant chamber. We consider the standard module $X = \mathbb{H} \otimes_{S(V)} C_{\gamma^\vee}$. By a simple computation, we have that $X$ contains two composition factors.

The element $\frac{1}{2}(\beta^\vee + 2\alpha^\vee)$ is in a bad direction. We have

$$JF^1\frac{1}{2}(\beta^\vee + 2\alpha^\vee)(\emptyset, C_{\gamma^\vee}) \cong L(\emptyset, C_{\gamma^\vee})$$

for all $i \geq 1$. In contrast, the element $\frac{1}{2}\alpha^\vee$ is a good direction since $JF^1\frac{1}{2}\alpha^\vee(\emptyset, C_{\gamma^\vee})$ is the unique simple submodule of $X$ and $JF^2\frac{2}{2}\alpha^\vee(\emptyset, C_{\gamma^\vee}) = 0$.

Remark 6.8. When the central character supports a tempered module, we expect that those corresponding standard modules have "less" bad directions (or even no bad directions). In contrast, for the principal series of a generic central character, the whole space $V^\vee$ is the set of bad directions. Example 6.7 illustrates an example in between of the previous two cases. We expect that the occurrence of bad directions depends on how "generic" the central character of the standard module is.

Proposition 6.9. The set $V^\perp_{\text{bad}}(J, U)$ forms a vector space i.e. closed under addition and scalar multiplication.

Proof. We can naturally identify $T^i(\phi(U_{t\eta^\vee}))$, $T^1(\phi(U_{t\eta^\vee}))$ and $T^1(\phi(U_{t(\eta^\vee + \eta^\vee)}))$ as vector spaces (via the grading by $t$) and simply write $T^i(\phi(U))$. We also similarly do for $T^0(U)$. Let $x_t \in T^0(U)$. If $\Delta_{t(\eta^\vee)}(x_t), \Delta_{t(\eta^\vee)}(x_t) \in T^1(\phi(U))$, then direct calculation from definitions gives that

$$\Delta_{t(\eta^\vee)}(x_t) + \Delta_{t(\eta^\vee)}(x_t) - \Delta_{t(\eta^\vee + \eta^\vee)}(x_t) \in T^2(\phi(U)).$$

Note that we also have $0 \in V^\perp_{\text{bad}}(J, U)$. \hfill \Box

6.4 First self-extension. Theorem 6.4 provides structural information for a quotient of generalized standard modules. We shall show in Theorem 6.11 how to obtain some information for Ext-group from those information. One may expect to obtain some other information from those quotients (see Example 7.5).

Consider the short exact sequence

$$0 \to N(J, U) \to I(J, U) \overset{pr}{\to} L(J, U) \to 0.$$

Lemma 6.10. Let $(J, U) \in \Xi_L$. Then $pr$ induces an isomorphism $\text{Ext}^1_{\text{pr}}(I(J, U), I(J, U)) \cong \text{Ext}^1_{\text{pr}}(I(J, U), L(J, U))$. 

Proof. We apply the functor $\text{Hom}_{\mathbb{H}}(I(J, U), .)$ to the short exact sequence before the lemma to obtain a long exact sequence. Then by Proposition 5.5 and Lemma 5.3, we have $\text{Ext}^1_{\mathbb{H}}(I(J, U), N(J, U)) = 0$. Thus $\text{Ext}^1_{\mathbb{H}}(I(J, U), I(J, U)) \cong \text{Ext}^1_{\mathbb{H}}(I(J, U), L(J, U))$ via the induced map.

\[\square\]

**Theorem 6.11.** Let $(J, U) \in \Xi_L$ (Definition 6.2). Recall that $V^+_{\text{bad}}(J, U)$ is defined in Definition 6.7. Let

$$\text{pr}^{*, i} : \text{Ext}^1_{\mathbb{H}}(L(J, U), L(J, U)) \to \text{Ext}^1_{\mathbb{H}}(I(J, U), L(J, U))$$

be the natural map induced from the surjective map $I(J, U) \to L(J, U)$. Then

1. Identify $\text{Ext}^1_{\mathbb{H}}(I(J, U), I(J, U)) \cong \text{Ext}^1_{\mathbb{H}}(I(J, U), L(J, U))$ via Lemma 6.16. Identify

$$\text{Ext}^1_{\mathbb{H}}(I(J, U), I(J, U)) \cong \text{Ext}^1_{S(V^{J, +})}(L, L) \oplus \text{Ext}^1_{S(V^{J, +})}(T, T)$$

as in Lemma 5.11 and identify $\text{Ext}^1_{S(V^{J, +})}(L, L) \cong V^{J, +}$. Then

$$\text{im} \text{pr}^{*, 1} \cap V^{J, +} \cong V^+_{\text{bad}}(J, U).$$

2. Suppose $T$ is a discrete series or more generally $\text{Ext}^1_{S(V^{J, +})}(T, T) = 0$. Then

$$\text{Ext}^1_{\mathbb{H}}(L(J, U), L(J, U)) \cong V^+_{\text{bad}}(J, U).$$

**Proof.** We first consider (1). Consider the following short exact sequence:

$$0 \to N(J, U) \to I(J, U) \to L(J, U) \to 0.$$ 

This induces a long exact sequence of the following form:

$$\ldots \to \text{Hom}_{\mathbb{H}}(N(J, U), L(J, U)) \to \text{Ext}^1_{\mathbb{H}}(L(J, U), L(J, U)) \text{pr}^{*, 1} \to \text{Ext}^1_{\mathbb{H}}(I(J, U), L(J, U)) \to \ldots$$

Suppose $\text{im} \text{pr}^{*, 1} \cap V^{J, +} \setminus V^+_{\text{bad}} \neq \emptyset$. Let $\eta^\vee \in \text{im} \text{pr}^{*, 1} \cap V^{J, +} \setminus V^+_{\text{bad}}(J, U)$ and let $\eta' \in \text{Ext}^1_{\mathbb{H}}(L(J, U), L(J, U))$ such that $\text{pr}^{*, 1}(\eta') = \eta^\vee$. Let $E(\eta')$ be the $\mathbb{H}$-modules constructed from the Yoneda first extension (see e.g. [WG, Theorem 3.4.3]) for $\eta'$ respectively. Recall that we are working for several identification. We now regard $\eta^\vee$ as an element in $\text{Ext}^1_{\mathbb{H}}(I(J, U), I(J, U))$ and let $E(\eta^\vee)$ be the $\mathbb{H}$-module constructed from the Yoneda first extension for $\eta^\vee$.

From the construction of the Yoneda extension and tracing identifications, the map $I(J, U) \to L(J, U)$ being surjective implies that the induced map $E(\eta^\vee) \to E(\eta')$ is also surjective. This implies that there exists a subquotient of $E(\eta^\vee)$, in which all the composition factors are isomorphic to $L(J, U)$. By the uniqueness statement in Theorem 6.12, $\text{JF}^0_{\eta^\vee}(J, U^{2, \eta^\vee})/\text{JF}^1_{\eta^\vee}(J, U^{2, \eta^\vee}) \cong L(J, U^{2, \eta^\vee})$ is isomorphic to $E(\eta')$.

On the other hand, by Theorem 6.11, the composition factors of $\text{JF}^0_{\eta^\vee}(J, U^{2, \eta^\vee})/\text{JF}^1_{\eta^\vee}(J, U^{2, \eta^\vee})$ contains composition factors of $\text{JF}^0_{\eta^\vee}(J, U)/\text{JF}^1_{\eta^\vee}(J, U)$. However, since $\text{JF}^1_{\eta^\vee}(J, U) \neq \text{JF}^2_{\eta^\vee}(J, U), \text{JF}^0_{\eta^\vee}(J, U)/\text{JF}^2_{\eta^\vee}(J, U)$ contains a composition factor other than $L(J, U)$ and does $L(J, U^{2, \eta^\vee})$. This gives a contradiction to the above conclusion that $L(J, U^{2, \eta^\vee}) \cong E(\eta')$. This proves $\text{im} \text{pr}^{*, 1} \cap V^{J, +} \subset V^+_{\text{bad}}(J, U)$.

For the converse inclusion, let $0 \neq \eta^\vee \in V^+_{\text{bad}}$. Then

$$\text{JF}^0_{\eta^\vee}(J, U)/\text{JF}^2_{\eta^\vee}(J, U) = \text{JF}^0_{\eta^\vee}(J, U)/\text{JF}^1_{\eta^\vee}(J, U) \cong L(J, U)$$
by the definition of bad directions. Hence, by Theorem 6.3(1),
$$JF^\nu_v(J, U^{2, \eta^\nu})/(JF^1_v(J, U^{2, \eta^\nu}) + \text{im} \iota) \cong L(J, U),$$
where $\iota$ is the natural embedding from $I(J, U)$ to $I(J, U^{2, \eta^\nu})$. We also have $\text{im} \iota/(\text{im} \iota \cap JF^1_v(J, U^{2, \eta^\nu})) \cong L(J, U)$ by the definition of intertwining operators and the definition of $L(J, U)$ (see Theorem 6.4(1)). For simplicity, let $E = JF^0_v(J, U^{2, \eta^\nu})/JF^1_v(J, U^{2, \eta^\nu})$. The above facts imply that $E$ is an indecomposable module of length 2 and all the composition factors isomorphic to $L(J, U)$. Since $I(J, U^{2, \eta^\nu}) \cong JF^0_v(J, U^{2, \eta^\nu})$, we have a surjection from $I(J, U^{2, \eta^\nu})$ to $E$. Since $E$ has a unique submodule isomorphic to $L(J, U)$, the surjection map induces a commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & I(J, U) & \rightarrow & I(J, U^{2, \eta^\nu})/N(J, U) & \rightarrow & I(J, U) & \rightarrow & 0 \\
& & pr' & \downarrow & pr & & \\
0 & \rightarrow & L(J, U) & \rightarrow & E & \rightarrow & L(J, U) & \rightarrow & 0
\end{array}
$$

Here $pr'$ is a non-zero scalar multiple of $pr$. Then we have a natural commutative diagram of the following form:

$$
\begin{array}{cccccc}
0 & \rightarrow & I(J, U)/N(J, U) & \rightarrow & I(J, U^{2, \eta^\nu})/N(J, U) & \rightarrow & I(J, U) & \rightarrow & 0 \\
& & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L(J, U) & \rightarrow & E & \rightarrow & L(J, U) & \rightarrow & 0
\end{array}
$$

Recall that $I(J, U)/N(J, U) \cong L(J, U)$. Denote by $\eta' \in \text{Ext}^1_H(L(J, U), L(J, U))$ for the corresponding element of the bottom short exact sequence (under the Yoneda correspondence). It follows from definitions that $pr^{*^{-1}}(\eta') = \eta^\nu$. This completes the proof for (1).

For (2), we have the long exact sequence:

$$\ldots \rightarrow \text{Hom}_H(N(J, U), L(J, U)) \rightarrow \text{Ext}^1_H(L(J, U), L(J, U)) \rightarrow \text{Ext}^1_H(I(J, U), L(J, U)) \rightarrow \ldots$$

Since $\text{Hom}_H(N(J, U), L(J, U)) = 0$, $pr^{*^{-1}}$ is injective and hence $\text{Ext}^1_H(L(J, U), L(J, U)) \cong \text{im} pr^{*^{-1}}$. Combining with the result of (1) and assumptions, we obtain (2).

7. First extensions and filtrations

7.1. First extensions. We now summarize our study and state our main result concerning $\text{Ext}^1_H$ for some simple modules.

**Theorem 7.1.** Let $\mathcal{H}$ be the graded Hecke algebra as in Definition 2.1. Let $(J_1, U_1), (J_2, U_2) \in \Xi_L$ (Definition 6.2). Set $X = L(J_1, U_1)$ and let $Y = L(J_2, U_2)$ (see Definition 7.2 for notations). Then

1. If $\nu(Y) < \nu(X)$, then $\text{Ext}^1_H(X, Y) \cong \text{Hom}_H(N(J_1, U_1), Y)$.
2. If $\nu(X) < \nu(Y)$, then $\text{Ext}^1_H(X, Y) \cong \text{Ext}^1_H(Y, X) \cong \text{Hom}_H(N(J_2, U_2), X)$.
3. If $\nu(Y)$ and $\nu(X)$ are incomparable, then $\text{Ext}^1_H(X, Y) = 0$.
4. Suppose $\nu(X) = \nu(Y)$ (and in particular $J_1 = J_2$). Further suppose that $\text{Ext}^1_H(X, Y) \cong \text{Res}^{s_0}_{\mathcal{H}}(U_1, s_0_{\mathcal{H}} U_2) = 0$. 

(a) If $U_1 \not\cong U_2$, then $\text{Ext}_{\mathbb{H}}^1(X,Y) = 0$.
(b) If $U_1 \cong U_2$, then $\text{Ext}_{\mathbb{H}}^1(X,Y) \cong V_{\text{bad}}^1(J_1,U_1)$.

\textbf{Proof.} We have the following short exact sequence:
\[0 \to N(J_1,U_1) \to I(J_1,U_1) \to L(J_1,U_1) \to 0.\]

By applying the $\text{Hom}_{\mathbb{H}}(Y)$ functor, we have
\[\ldots \to \text{Hom}_{\mathbb{H}}(I(J_1,U_1),Y) \to \text{Hom}_{\mathbb{H}}(N(J_1,U_1),Y) \to \text{Ext}_{\mathbb{H}}^1(L(J_1,U_1),Y) \to \text{Ext}_{\mathbb{H}}^1(I(J_1,U_1),Y) \to \ldots\]

We first consider (1) and (3) and so suppose $\nu(X) \not\leq \nu(Y)$. Then using Proposition 3.6, we have $\text{Ext}_{\mathbb{H}}^1(L(J_1,U_1),Y) \cong \text{Hom}_{\mathbb{H}}(N(J_1,U_1),Y)$. This proves (1). For $\nu(Y)$ and $\nu(X)$ being incomparable, we also have $\text{Hom}_{\mathbb{H}}(N(J_1,U_1),Y) = 0$ by Lemma 3.5. This proves (3).

For (2), there is a natural isomorphism between $\text{Ext}_{\mathbb{H}}^1(X,Y) \cong \text{Ext}_{\mathbb{H}}^1(Y^\bullet,X^\bullet)$. (2) then follows from (1) and Lemma 3.5.

We now consider (4). Write $U_i = \overline{U}_i \otimes L_i$ as $\mathbb{H}_J \cong \mathbb{H}_J^{\text{ss}} \otimes S(V_J^+)\text{-algebras}$, where $\overline{U}_i$ is $\mathbb{H}_J^{\text{ss}}$-tempered module and $L_i$ is a one-dimensional $S(V_J^+)$-module. By Lemma 3.11, we have
\[\text{Ext}_{\mathbb{H}}^1(I(J_1,U_1),L(J_2,U_2)) \cong \text{Ext}_{\mathbb{H}_J}^1(\overline{U}_1,\overline{U}_2) \otimes \text{Hom}_{S(V_J^+)}(L_1,L_2) \otimes \text{Hom}_{S(V_J^+)}(\overline{U}_1,\overline{U}_2) \otimes \text{Ext}_{S(V_J^+)}^1(L_1,L_2).\]

For (4)(a), by using $\text{Hom}_{\mathbb{H}_J}^1(\overline{U}_1,\overline{U}_2) = 0$ and $\text{Ext}_{\mathbb{H}_J}^1(\overline{U}_1,\overline{U}_2) = 0$, we have
\[\text{Ext}_{\mathbb{H}}^1(I(J_1,U_1),L(J_2,U_2)) \cong 0.
\]

This implies $\text{Ext}_{\mathbb{H}}^1(L(J_1,U_1),L(J_2,U_2)) = 0$ by using a long exact sequence from the short exact sequence
\[0 \to N(J_1,U_1) \to I(J_1,U_1) \to L(J_1,U_1) \to 0.\]

(4)(b) follows from Theorem 6.11(2). \hfill \square

\textbf{Remark 7.2.} (1) For Theorem 7.1(1) and (2), that is related to the second layer of the radical filtration of the corresponding standard module. For Theorem 7.1(4), that is related to the second layer of the Jantzen filtration.

(2) For Theorem 7.1(a), our approach only deals with the assumption that $\text{Ext}_{\mathbb{H}_J}^1(\text{Res}_{\mathbb{H}_J} U_1,\text{Res}_{\mathbb{H}_J} U_2) = 0$. Nevertheless, the assumption in Theorem 7.1(a) is satisfied independently in [Mc, OS] and [Ch2] that $\text{Ext}_{\mathbb{H}_J}^1(\text{Res}_{\mathbb{H}_J} U_1,\text{Res}_{\mathbb{H}_J} U_2) = 0$ ($U_2$ can be any arbitrary tempered modules). If $\text{Res}_{\mathbb{H}_J} U_1$ is elliptic and not a discrete series, then it can be checked from the result of [OS2] Theorem 5.2 that $\text{Ext}_{\mathbb{H}_J}^1(\text{Res}_{\mathbb{H}_J} U_1,\text{Res}_{\mathbb{H}_J} U_1) = 0$ for a number of cases.

7.2. \textbf{Some computations on Ext-groups.} In this section, we discuss some computations on Ext-groups from results in this paper and [Ch2] and some computations from [C].

\textbf{Example 7.3.} Here we give an example on computing Ext-groups from our results. We shall assume a version of the Jantzen conjecture (see e.g. [HC] Conjecture 6.2.2) to compute Jantzen filtrations from some Kazhdan-Lusztig polynomials [C] (which uses [Lu3, Lu4]). We mainly use to get information for the second layer of the Jantzen filtration.
Consider $\mathfrak{H}$ of type $C_3$ as in [Ch3 Section 4.4] and use the notation in that section. Denote by $I(4_b), I(3_b, s)$, etc (resp. $L(4_a), L(3_b, s)$, etc) the standard modules (resp. simple modules) associated to $4_a, 3_b, s$, etc respectively. Since $L(5_s)$ and $L(5_t)$ are discrete series, the Ext-groups follow from [Ch2 Theorem 7.2].

The standard module $I(4_b)$ satisfies the hypothesis in Theorem [7.14] and the Jantzen filtration (with assuming the truth of the conjecture), we have $\text{Ext}^1_{\mathfrak{H}}(L(4_a), L(4_a)) = 0$ and thus we have

$$\text{Ext}^2_{\mathfrak{H}}(L(4_a), L(4_a)) \cong \mathbb{C}, \quad \text{Ext}^3_{\mathfrak{H}}(L(4_a), L(4_a)) = 0 \text{ for } i \neq 0, 2.$$

We also have

$$\text{Ext}^5_{\mathfrak{H}}(L(4_a), L(5_s)) = \text{Ext}^1_{\mathfrak{H}}(L(4_a), L(5_s)) = 0 \text{ for } i \neq 0,$$

and

$$\text{Ext}^1_{\mathfrak{H}}(L(4_a), L(5_s)) \cong \text{Ext}^1_{\mathfrak{H}}(L(4_a), L(5_t)) \cong \mathbb{C}.$$

We now turn to $3_b, s$. There are two possible radical filtration based on the Jantzen filtration. Suppose $\text{rad}^1(I(3_b, s)) \cong L(4_a) \oplus L(5_s)$. For such case, applying the $\text{Hom}_{\mathfrak{H}}(\cdot, L(5_s))$-functor and using Proposition [3.6] we have $\text{Ext}^2_{\mathfrak{H}}(L(3_b, s), L(4_a)) \cong \text{Ext}^2_{\mathfrak{H}}(L(4_a) \oplus L(5_s), L(5_s)) \cong \mathbb{C}$. Now by applying the duality [Ch2 Theorem 4.15], we have $\text{Ext}^3_{\mathfrak{H}}(L(4_a), L(4_b)) \cong \mathbb{C}$ which contradicts to Theorem [7.11] and the data in [Ch3 Section 4.4]. Thus we can only have

$$\text{rad}^1(I(3_b, s))/\text{rad}^2(I(3_b, s)) \cong L(4_a), \quad \text{rad}^2(I(3_b, s))/\text{rad}^3(I(3_b, s)) \cong L(5_s).$$

Then by standard homological algebra, we have

$$\text{Ext}^1_{\mathfrak{H}}(L(3_b, s), L(4_a)) \cong \text{Ext}^2_{\mathfrak{H}}(L(3_b, s), L(5_s)) \cong \mathbb{C}$$

and

$$\text{Ext}^1_{\mathfrak{H}}(L(3_b, s), L(4_a)) = \text{Ext}^1_{\mathfrak{H}}(L(3_b, s), L(5_s)) = \text{Ext}^2_{\mathfrak{H}}(L(3_b, s), L(5_s)) = 0$$

for all $i, j$ not as in (7.19). By Theorem [7.11] and Jantzen filtrations, we have

$$\text{Ext}^1_{\mathfrak{H}}(L(3_b, s), L(3_b, s)) = 0.$$

By using Theorem [7.1] we have

$$\text{Ext}^1_{\mathfrak{H}}(L(3_b, s), L(3_b, s)) \cong \text{Ext}^1_{\mathfrak{H}}(L(3_b, s), L(4_b)) = 0.$$

We now compute $\text{Ext}^1_{\mathfrak{H}}(L(3_b, t), L(3_b, t))$. By Theorem [7.1]

$$\text{Ext}^1_{\mathfrak{H}}(L(3_b, t), L(3_b, t)) = 0,$$

$$\text{Ext}^2_{\mathfrak{H}}(L(3_b, t), L(3_b, t)) \cong \text{Ext}^1_{\mathfrak{H}}(L(3_b, t), L(2_b)) \cong 0,$$

$$\text{Ext}^3_{\mathfrak{H}}(L(3_b, t), L(3_b, t)) \cong \text{Ext}^1_{\mathfrak{H}}(L(3_b, t), L(2_b)) \cong 0.$$

By using the duality [Ch2 Theorem 4.15], we also have

$$\text{Ext}^1_{\mathfrak{H}}(L(2_b), L(2_b)) = \text{Ext}^2_{\mathfrak{H}}(L(2_b), L(2_b)) = 0.$$

This is also compatible with the result of Theorem [7.14] and the Jantzen filtration of the first layer for $I(2_b)$.

Other pairs can be computed by similar manner, or using suitable duality to reduce to known Ext-groups. We have also checked many cases that the resulting Ext-groups
and the structure of standard modules in terms of the (conjectured) Jantzen filtration are compatible.

**Remark 7.4.** We remark that applying the Kazhdan-Lusztig polynomials and the Jantzen conjecture, one expect to obtain information for a generic vector in $V^+_\text{bad}$ (which is also assumed in Example 7.3). Thus it is easier to determine $\text{Ext}^1_{\mathcal{H}}(L(J, U), L(J, U))$ for $|J| = |\Pi| - 1$. For cases of larger $J$, one sometimes needs some more information.

**Example 7.5.** Here we show that one can obtain some other structure for the extension algebra from Theorem 6.4. We use the notation in Example 7.3. We shall show that the Yoneda product
\begin{equation}
\text{Ext}^1_{\mathcal{H}}(L(5_s), L(4_a)) \otimes \text{Ext}^1_{\mathcal{H}}(L(4_a), L(5_s)) \to \text{Ext}^2_{\mathcal{H}}(L(4_a), L(4_a))
\end{equation}
is a non-zero map.

Let $X = I(4_a)/L(5_t)$. We consider the short exact sequence:
\[0 \to L(5_s) \to X \to L(4_a) \to 0.\]
By applying the $\text{Hom}_{\mathcal{H}}(\_, L(4_a))$ functor, we have the long exact sequence
\[
\ldots \to \text{Ext}^1_{\mathcal{H}}(L(4_a), L(4_a)) \to \text{Ext}^1_{\mathcal{H}}(X, L(4_a)) \to \text{Ext}^1_{\mathcal{H}}(L(5_s), L(4_a)) \overset{\partial}{\to} \text{Ext}^2_{\mathcal{H}}(L(4_a), L(4_a)) \to \ldots
\]
The map $\partial$ coincides with the Yoneda product in (7.20). Hence if $\partial$ is zero, we have $\text{Ext}^1_{\mathcal{H}}(X, L(4_a)) \cong \mathbb{C}$. Then by comparing dimensions, the natural surjective map $I(4_a) \to X$ induces an isomorphism $\text{Ext}^1_{\mathcal{H}}(X, L(4_a)) \cong \text{Ext}^1_{\mathcal{H}}(I(4_a), L(4_a))$. Now considering the Yoneda construction of the modules for $\text{Ext}^1_{\mathcal{H}}(X, L(4_a))$ and $\text{Ext}^1_{\mathcal{H}}(I(4_a), L(4_a))$, we obtain a module $L(J, U^{2, \eta'})$ which does not have a unique simple module. This gives a contradiction and hence we have the Yoneda product to be non-zero. Here $(J, U) \in \Xi_L$ such that $L(J, U) = L(4_a)$.

8. **Appendix A: Jantzen filtration for an example of type $B_2$**

We keep using the notation in Example 5.3. Fix a basis $\{u_1, u_2\}$ of $U$ such that the action of $S(V)$ on $\phi(U_{t, \nu})$ in matrix form with respect to the basis $\{\phi(u_1), \phi(u_2)\}$ is as follows:
\[
\alpha = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]
and
\[
\beta = \begin{bmatrix} -(2 + 2t) & 0 \\ -1 & -(2 + 2t) \end{bmatrix}.
\]
We also have
\[
(2\alpha + \beta)^2 - 4 = 4 \begin{bmatrix} t(2 + t) & 0 \\ -(1 + t) & t(2 + t) \end{bmatrix}
\]
and
\[
(\alpha + \beta)^2 - 4 = 4 \begin{bmatrix} t(2 + t) & 0 \\ 0 & t(2 + t) \end{bmatrix}
\]
and
\[
\beta^2 - 4 = 4 \begin{bmatrix} t(2 + t) & 0 \\ (1 + t) & t(2 + t) \end{bmatrix}.
\]
We now regard $\phi(u_1)$ and $\phi(u_2)$ as elements in $\phi(U_{\nu'})$. Set $u_1^0 = \beta^{-1}(\alpha + \beta)^{-1}(2\alpha + \beta)^{-1}\phi(u_1) \subseteq \phi(U_{\nu'})$ and $u_2^0 = \beta^{-1}(\alpha + \beta)^{-1}(2\alpha + \beta)^{-1}\phi(u_2) \subseteq \phi(U_{\nu'})$. Note that $1 \otimes u_1^0$ and $1 \otimes u_2^0$ are holomorphic and both of them specialized at $t = 0$ are nonzero.

Now we compute the image of $\Delta_{\nu'}$ of the following elements.

$$\begin{align*}
\Delta_{\nu'}(1 \otimes u_1) &= \tau\mathbb{S}_{s,\alpha,\beta} \otimes u_1^0 \\
\Delta_{\nu'}(1 \otimes u_2) &= \tau\mathbb{S}_{s,\alpha,\beta} \otimes u_2^0 \\
\Delta_{\nu'}(\tau s) \otimes u_1 &= 4t(2 + t)\tau s, s \otimes u_1^0 - 4(1 + t)\tau s, s \otimes u_2^0 \\
\Delta_{\nu'}(\tau s) \otimes u_2 &= 4t(1 + t)\tau s, s \otimes u_2^0 \\
\Delta_{\nu'}(\tau s, s \otimes u_1) &= 16t^2(2 + t)^2\tau s, s \otimes u_1 - 16t(1 + t)(2 + t)\tau s, s \otimes u_2^0 \\
\Delta_{\nu'}(\tau s, s \otimes u_2) &= 16t^2(2 + t)^2\tau s, s \otimes u_1^0 \\
\Delta_{\nu'}(\tau s, s \otimes u_1) &= 64t^3(2 + t)^3 \otimes u_1^0 \\
\Delta_{\nu'}(\tau s, s \otimes u_2) &= 64t^3(2 + t)^3 \otimes u_2^0
\end{align*}$$

Note that the image of $1 \otimes u_1$, $1 \otimes u_2$, $\tau s \otimes u_1$ span $JF^0_{\nu'}/JF^1_{\nu'}$. Similarly the image of $\tau s, s \otimes u_1$ spans $JF^1_{\nu'}/JF^2_{\nu'}$. The image of $\tau s, s \otimes u_2 \otimes u_1$, $\tau s, s \otimes u_2 \otimes u_1$, $\tau s, s \otimes u_2$ spans $JF^3_{\nu'}/JF^4_{\nu'}$.

9. Appendix B: Proof of Proposition 4.8

In the following proofs, we assume the reader is familiar with the standard properties of Bruhat-Chevalley ordering (see e.g. [HR]). Some facts are used without mentioning explicitly. Define $l : W \rightarrow \mathbb{Z}_{\geq 0}$ be the length function on $W$.

Lemma 9.1. Let $w \in W'. Let s_\alpha$ be a simple reflection. Then we have one of the following cases:

1. $l(s_\alpha w) = l(w) + 1$ and $s_\alpha w \in W'$. In this case, $w^{-1}(\alpha) \in R \setminus R^\circ$. Furthermore, $(t s_\alpha - k_\alpha) w = t s_\alpha w - w^{-1}(\alpha)$ and $t s_\alpha w = t s_\alpha w - k_\alpha w^{-1}(\alpha)^{-1}$. 

2. $l(s_\alpha w) = l(w) + 1$ and $s_\alpha w \notin W'$. In this case, $s_\alpha w = w s_\alpha'$ for some $\alpha' \in J$. Furthermore $t s_\alpha w - w^{-1}(\alpha)$ and $(t s_\alpha - k_\alpha) w = k_\alpha w^{-1}(\alpha)^{-1}$.

3. $l(s_\alpha w) = l(w) - 1$. In this case, $s_\alpha w \in W^J$ and $w^{-1}(\alpha) \in R \setminus R^\circ$. Furthermore, $(t s_\alpha - k_\alpha) w = (t s_\alpha, s_\alpha - k_\alpha) w = t s_\alpha w - k_\alpha w^{-1}(\alpha)^{-2} + k_\alpha w^{-1}(\alpha)^{-1}$.

Proof. (1) follows from definitions (and some details are similar to (2) below).

For (2), suppose $s_\alpha w \notin W'$. Let $R(w) = \{\beta \in R^+: w(\beta) < 0\}$. Since $R(w) \subset R(s_\alpha w)$ and $R(w) \cap R_J = \emptyset$, $|R(s_\alpha) \cap R_J| \leq 1$. The condition that $s_\alpha w \notin W'$ implies that $|R(s_\alpha) \cap R_J| = 1$. By a unique factorization of an element into the product of an element in $W_J$ and an element in $W_J$, we have $s_\alpha w = w s_\alpha'$ for some $\alpha' \in \Pi$. The assertion that $(t s_\alpha - k_\alpha) w = (t s_\alpha, s_\alpha - k_\alpha) w$ follows again from the proof of [KR] Proposition 2.5] (c.f. Proposition 3.4). Since $w^{-1}(\alpha) = \alpha'$, we also have $\alpha w = t w s_\alpha'$ (see Lemma 3.3). Then one can verify that the action of $t s_\alpha w$ and $\tau w t s_\alpha'$ is the same on a faithful $\mathbb{H}$-module described in the proof of [KR] Proposition 2.8(e)]. Hence $t s_\alpha w = t w s_\alpha'$. 


For (3), since \( l(s_n w) = l(w) - 1, \) \( R(s_n w) \subset R(w) \) and so \( s_n w \in W^J. \)

**Proof of Proposition 4.8**

Let \( w \in W^J. \) Let

\[
\Lambda(w, \gamma^\vee) = \{ \lambda^\vee \in \text{Wgt}(U) : w(\lambda^\vee) = \gamma^\vee \text{ for some } \lambda^\vee \in \text{Wgt}(U) \}.
\]

For each \( \lambda^\vee \in \text{Wgt}(U) \), let \( u_1, \ldots, u_{r_{\lambda^\vee}} \) form a basis of the generalized \( \lambda^\vee \)-weight space and regard those elements in \( 1 \otimes U \subset U_a. \)

For each \( w_1 \in W^J \) and \( \lambda_1^\vee \in \text{Wgt}(U) \) such that \( \gamma^\vee = w_1(\lambda_1^\vee) \), and for each \( k = 1, \ldots, r_{\lambda_1^\vee} \), we shall construct vectors \( x_a \in \tilde{U} \) of the form

\[
(9.21) \quad x_a = \tau_{w_1} p_{w_1, \lambda_1^\vee, k} \otimes u_{\lambda_1^\vee, k} + \sum_{w \in W(J, U, \gamma^\vee)} \sum_{w_1 > w} \sum_{i=1}^{r_{\lambda_1^\vee}} \tau_{w, w_1, \lambda_1^\vee, i} \otimes u_{\lambda_1^\vee, i}
\]

with

(i) \( x_a \) is holomorphic;
(ii) \( p_{w_1, \lambda_1^\vee, k} \in S(V) \subset O(J); \)
(iii) \( x_a|_{a=0} \) has weight \( \gamma^\vee; \)
(iv) \( \lambda_1^\vee(p_{w_1, \lambda_1^\vee, k}) \neq 0. \)

By the definition of \( |a=0 \) and property (iv) above, we see that \( x \) is a non-zero scalar multiple of an element of the form

\[
t_{w_1} \otimes u_{\lambda_1^\vee, k} + \sum_{w \in W^J, w_1 > w} t_w \otimes u_w
\]

for some \( u_w \in U. \)

Fix \( \lambda^\vee \in \text{Wgt}(U). \) For \( w = 1 \in W^J, \) there is nothing to prove. Let \( w_1 \in W^J \) and with \( w' \neq 1 \) and let \( w_1 = s_{\alpha_1} \ldots s_{\alpha_r} \) be a reduced expression of \( w_1. \) Then \( w_2 = s_{\alpha_2} \ldots s_{\alpha_r}, \) which is also in \( W^J \) by definitions. By our inductive construction, we can assume there exists an element \( x_a \) the form \( (9.21) \) starting with the term \( \tau_{w_2} p_{w_2, \lambda_1^\vee, k} \otimes u_{\lambda_1^\vee, k} \) satisfying properties (i) to (iii).

Here we divide into few cases. Before that, set \( \alpha = \alpha_1 \) for simplicity of notations. For the first case, suppose \( s_\alpha w_2(\lambda^\vee) = w_2(\lambda^\vee), \) equivalently \( w_2(\lambda^\vee)(\alpha) = \alpha. \) In this case, set \( \overline{x}_a = t_{s_\alpha} x_a. \) Note that \( t_{s_\alpha} \overline{x}_a \) is also holomorphic and \( (t_{s_\alpha} \overline{x}_a)|_{a=0} = t_{s_\alpha} (\overline{x}|_{a=0}). \) Set \( \gamma^\vee = w_2(\lambda^\vee). \) We also have

\[
(9.22) \quad (v - \gamma^\vee(v)) t_{s_\alpha} (\overline{x}_a)|_{a=0} = t_{s_\alpha} (s_\alpha(v) - \gamma^\vee(v))(\overline{x}_a)|_{a=0} + \alpha^\vee(v)(\overline{x}_a)|_{a=0}
\]

\[
(9.23) \quad = t_{s_\alpha} (s_\alpha(v) - \gamma^\vee(s_\alpha(v)))(\overline{x}_a)|_{a=0} + \alpha^\vee(v)(\overline{x}_a)|_{a=0} \quad (\text{by } s_\alpha(\gamma^\vee) = \gamma^\vee)
\]

and so \( (v - \gamma^\vee(v)) t_{s_\alpha} (\overline{x}_a)|_{a=0} = 0 \) for sufficiently large \( l. \) Thus \( t_{s_\alpha}(\overline{x}_a)|_{a=0} \) is a generalized weight vector with the weight \( w_1(\lambda^\vee) = w_2(\lambda^\vee). \) This shows that \( \overline{x}_a \) satisfies property (iii).
We now rewrite \( \tilde{x}_a \) to the form as in (9.21). We consider the leading term \( t_{s_a} \tau_{w_2} p_{w_2, \lambda^\vee, k} \otimes u_{\lambda^\vee, k} \) and write as

\[
t_{s_a} \tau_{w_2} p_{w_2, \lambda^\vee, k} \otimes u_{\lambda^\vee, k}
= t_{s_a} \alpha - k_\alpha \tau_{w_2} (w_2^{-1}(\alpha)^{-1} p_{w_2, \lambda^\vee, k}) \otimes u_{\lambda^\vee, k} + k_\alpha \tau_{w_2} (w_2^{-1}(\alpha)^{-1} p_{w_2, \lambda^\vee, k}) \otimes u_{\lambda^\vee, k}
\]

Other terms can be rewritten in a similar fashion with the use of Lemma 9.1. We remark that if the term falls in the case of Lemma 9.12, the algebra structure from Lemma 9.2 is also needed.

Recall that from our inductive construction, \( p_{w_2, \lambda^\vee, k} \in S(V) \) and \( \lambda^\vee(\lambda^\vee, k) \neq 0 \). By looking at the leading term of \( t_{s_a} \tilde{x}_a \) (in the form as in (9.21)), we see that \( t_{s_a} \tilde{x}_a \) satisfies properties (ii) and (iv). Hence \( t_{s_a} \tilde{x}_a \) gives the desired element in the case of \( s_{\alpha_1}(w_2(\lambda^\vee)) = w_2(\lambda^\vee) \). This completes the verification for the case \( s_{\alpha}(w_2(\lambda^\vee)) = w_2(\lambda^\vee) \).

We now consider the case \( s_{\alpha}(w_2(\lambda^\vee)) \neq w_2(\lambda^\vee) \). In this case, let \( \tilde{x}_a = (t_{s_a} \alpha - k_\alpha)x \).

We have to rewrite \( \tilde{x}_a \) to the form (9.21). Again we only do it for the leading terms and other terms can be rewritten similarly with the use of Lemma 9.1

(9.24) \[
(t_{s_a} \alpha - k_\alpha) \tau_{w_2} p_{w_2, \lambda^\vee, k} \otimes u_{\lambda^\vee, k}
\]

(9.25) \[
= t_{s_a} \tau_{w_2} (w_2^{-1}(\alpha)p_{w_2, \lambda^\vee, k}) \otimes u_{\lambda^\vee, k} \quad \text{(by Lemma 9.1 (1))}
\]

To check property (i), one can use Lemma 4.5. Properties (ii) and (iv) follow from the facts that \( (w_2^{-1}(\alpha)p_{w_2, \lambda^\vee, k}) \in S(V) \) and \( \lambda^\vee(w_2^{-1}(\alpha)p_{w_2, \lambda^\vee, k}) = (w_2(\lambda^\vee)(\alpha))\lambda^\vee(p_{w_2, \lambda^\vee, k}) \neq 0 \).

Here \( w_2(\lambda^\vee)(\alpha) \neq 0 \) because of our assumption that \( s_{\alpha}(w_2(\lambda^\vee)) \neq w_2(\lambda^\vee) \).

From (9.21), we see that the weight vectors we constructed are linearly independent. By counting the dimension, those weight vectors form a basis for \( \mathbb{H} \otimes_{\mathbb{H}_J} U \). Then using the property (ii), we obtain the statement.

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**Korteweg-de Vries Institute for Mathematics, Universiteit van Amsterdam**

*E-mail address: K.Y.Chan@uva.nl*