Determinants of matrices related to the Pascal triangle

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Abstract

In this note we prove an assertion made by M. Levin in 1999: the Pascal matrix modulo 2 has the property that each of the square sub-matrices laying on the upper border or on the left border has determinants, computed in $\mathbb{Z}$, equal to 1 or $-1$.

1 Introduction

In this note we prove an assertion made by M. Levin in 1999: the Pascal matrix modulo 2 has the property that each of the square sub-matrices laying on the upper border or on the left border has determinants, computed in $\mathbb{Z}$, equal to 1 or $-1$. This extends some of the results in [1, 2] on determinants related to the Pascal triangle.

The Pascal triangle matrix has been used in the theory of uniform distribution modulo one to construct sequences of real numbers in the unit interval with smallest possible discrepancy: the first $N$ terms have discrepancy at most $(\log N)/N$ times a constant (see [4] and the references therein). When we restrict to sequences of the form $\{b^n x \pmod{1}\}_{n=1,2,3,...}$ for any integer $b$ greater than or equal to 2 and for real numbers $x$, the smallest exact discrepancy that can be achieved by some $x$ is not known. The question dates back to Korobov in 1956 (cfr. [6]).

Using the Pascal triangle matrix modulo 2, in [5] M. Levin constructs numbers $x$ such that the sequence $\{b^n x \pmod{1}\}_{n=0,1,2,...}$ has discrepancy of the first $N$ terms bounded from above by $(\log N)^{2/3}/N$ times a constant. Becher and Carton in [3] defined variants of the Pascal triangle matrix modulo 2 that have the same property of the invertibility of the square sub-matrices laying on the upper or left border. They obtain a family of numbers with the same property as Levin’s. Larcher and Hofer recently showed that for Levin’s number constructed for $b = 2$ the discrepancy estimate $(\log N)^{2/3}/N$ is the best possible (cfr. [5]).

The property that all square matrices in the upper and left border of the Pascal matrix modulo 2 have determinants, computed in $\mathbb{Z}$ equal to 1 or $-1$ ensures that if these determinants are computed in $\mathbb{Z}/b\mathbb{Z}$, for any $b$, they are also equal to 1 or $-1$. Thus, indeed, Levin’s method yields numbers $x$ such that the $\{b^n x \pmod{1}\}_{n=1,2,...}$ has the small discrepancy property.

The article is organized as follows. In section 2 we introduce some notation, define the infinite matrix $U$ and state the main result (Theorem 1). Section 3 is devoted to its proof. In section 4 we define a whole family of matrices sharing the property of having all its sub-matrices laying on the upper or left border invertible, compute its number and give some examples.
2 Pascal matrices

We want to study determinants of sub-matrices of certain infinite matrix. It will be convenient to index the rows and columns with non-negative numbers.

Let $U$ be the infinite matrix whose entry in the $i, j$ position is the remainder when the binomial coefficient $\binom{j}{i}$ is divided by 2. Namely

$$U_{i,j} = \begin{cases} 1 & \text{if } \binom{j}{i} \text{ is odd}, \\ 0 & \text{if } \binom{j}{i} \text{ is even}. \end{cases}$$

**Remark 1.** By the well known result of Kummer [7] we know that the $(i, j)$-entry of $U$ is 1 if $j \geq i$ and the binary representations of both $i$ and $j - i$ don’t share a 1 in the same position, and 0 otherwise.

Writing $U_{I,J}$ for the sub-matrix of $U$ corresponding to the rows and columns indexed respectively by the sets $I, J \subseteq \mathbb{Z}_{\geq 0}$ introduce the following notation. The subset of integers $k$ greater than or equal to $m$ but smaller than $n$ is denoted by $[m:n) = \{k \in \mathbb{Z} : m \leq k < n \}$. The principal minors $U_{[0:n),[0:n)}$ are denoted $U(n)$. For the top-most minors $U_{[0:n),[m:m+n)}$ we write $U_{0,m}(n)$. Finally, $U_{i,j}(n, m)$ stands for $U_{[i:i+n)[j:j+m)}$.

The main result of this article is the following:

**Theorem 1.** The sub-matrix $U_{0,m}(n)$ has determinant $\pm 1$, for all choices of $n, m \geq 0$.

**Remark 2.** From the celebrated formula by Lucas [9] it follows that $U$ is the infinite tensor of the matrix $U(2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Here we think of this infinite tensor as the stable top-left square matrix in the sequence $U(2^k) = U(2)^\otimes k$ for $k \geq 1$. For example:

$$U(2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U(4) = \begin{pmatrix} U(2) & U(2) \\ 0 & U(2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U(8) = \begin{pmatrix} U(2) & U(2) & U(2) & U(2) \\ 0 & U(2) & 0 & U(2) \\ 0 & 0 & U(2) & U(2) \\ 0 & 0 & 0 & U(2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Remark 3.** The first $2^k$ rows of $U$ are $2^k$-periodic .

**Remark 4.** All the sub-matrices $U(n)$ are upper triangular with only 1’s on the diagonal.

**Remark 5.** $U(2^k)$ are symmetric with respect to the anti-diagonal.
3 Proof of Theorem 1

Proof of Theorem 1. We proceed by induction on the size $n$ of the sub-matrix. Since the first row of $U$ is made entirely of 1's, the statement is true for $n = 0$. By Remark 4 the statement is also true for $m = 0$. Consider $k$ such that $2^{k-1} < n \leq 2^k$. By periodicity (cfr. Remark 3) we may assume $m < 2^k$. We separate in two cases according to whether $m$ is less than $2^{k-1}$ or not.

Case $m < 2^{k-1}$:

Let us compare the matrices $U_{0,m}(n)$ and $U_{0,m+2^{k-1}}(n)$. The first $2^{k-1}$ rows are identical by $2^{k-1}$-periodicity (cfr. Remark 3).

For the remaining $n - 2^{k-1}$ rows we apply elementary row operations to the first matrix and obtain the second one, up to sign.

Subtracting $U_{0,m}(n - 2^{k-1}, n)$ from $U_{2^{k-1},m}(n - 2^{k-1}, n)$ we get exactly (by Remark 1) the sub-matrix $U_{2^{k-1},m+2^{k-1}}(n - 2^{k-1}, n)$ multiplied by $-1$ (Fig. 1).

Therefore, the determinant of $U_{0,m}(n)$ is that of $U_{0,m+2^{k-1}}(n)$ multiplied by $(-1)^{(n-2^{k-1})}$ and this case reduces to the next one.

Case $m \geq 2^{k-1}$:

Subdivide $U_{0,m}(n)$ into 4 blocks as follows (Fig. 2): taking $J = [m : 2^k)$, $J' = [2^k : m + n)$, $I = [0 : m + n - 2^k)$ and $I' = [m + n - 2^k : n)$, we get the partition

$$U_{0,m}(n) = \begin{pmatrix} U_{I,J} & U_{I,J'} \\ U_{I',J} & U_{I',J'} \end{pmatrix}$$

as $[0 : n) = I \cup I'$ and $[m : m + n) = J \cup J'$.
Note that the bottom-right block $U_{I',J'}$ is full of zeros (by $2^k$-periodicity and Remark 4) and the top-right one $U_{I,J}$ agrees with $U_{0,2^k}(m+n-2^k)$ (again by Remark 3) and therefore has determinant 1.

To find the determinant of $U_{0,m}(n)$ we consider the block matrix obtained by swapping the blocks

$$
\begin{pmatrix}
U_{I,J} & U_{I,J'} \\
U_{I',J'} & U_{I',J''}
\end{pmatrix}
$$

which is upper triangular by blocks with first block $U_{I,J}$ having determinant 1.

\[ Figure 2: \text{Position of } U_{0,m}(n) \text{ when } m \geq 2^k-1. \]

The determinant of $U_{0,m}(n)$ ends up being $(-1)^{(n-1)(m+n-2^k)} = (-1)^{(n-1)m}$ times that of $U_{I,J}$.

Note that by Remark 5 the anti-transpose $U^*_{I,J}$ of this last sub-matrix $U_{I,J}$ is $U_{0,2^k-n}(2^k - m)$ (Fig. 3).

Since $2^k - m \leq 2^k - 2^{k-1} = 2^{k-1} < n$

its determinant is $\pm 1$ by inductive hypothesis.

\[ \square \]

4 A family of Pascal-like matrices

Our matrix $U$ appeared in [2] to prove that all symmetric Pascal matrices (mod 2) have determinant $\pm 1$.

Consider $P$ the infinite matrix\[\text{i.e.: the reflection with respect to the anti-diagonal.}\]

\[ P_{i,j} = \begin{cases} 
1 & \text{if } \binom{j+i}{i} \text{ is odd,} \\
0 & \text{if } \binom{j+i}{i} \text{ is even.}
\end{cases} \]

\[ \text{it is noted } P(\infty)_2 \text{ in [2].} \]

4
For its \textit{LU}-decomposition we define \( L \) as the transpose of \( U \) and \( D \) as the infinite diagonal with the Thue–Morse sequence. Namely

\[
D_{i,j} = \begin{cases} 
1 & \text{if } i = j \text{ has an even number of 1's in base 2,} \\
-1 & \text{if } i = j \text{ has an odd number of 1's in base 2,} \\
0 & \text{if } i \neq j.
\end{cases}
\]

Adopting the same notation for sub-matrices as with \( U \) we have \( P, L \) and \( D \) are the infinite tensor product of

\[
\begin{align*}
P(2) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & L(2) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{and} & D(2) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

respectively.

Since

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

we get that \( P(n) = L(n)D(n)U(n) \) for all \( n \geq 1 \).

As \( L \) is lower triangular and \( D \) is diagonal with only ±1’s, Theorem \( \text{[1]} \) implies

\textbf{Corollary 1}. \textit{Given } \( n, m \geq 0 \), \textit{the sub-matrix } \( P_m(n) \) \textit{has determinant } ±1.

The same result applies for infinite matrices having a similar \textit{LU}-decomposition.

By the symmetry of \( P \) we deduce that every square sub-matrix laying on the upper or left border of \( P \) has determinant ±1.

\[\text{https://oeis.org/A106400}\]
We say that a matrix is *Pascal-like* if every such sub-matrix is invertible. When working over the integers this means having determinant $\pm 1$ so our matrix $P$ is Pascal-like by Corollary 1.

Another example over the integers is provided by the honest Pascal matrix $M_{i,j} = \binom{i+j}{j}$ as can be seen by a routine application of Vandermonde determinant and elementary row operations.

More is true according to the following

**Proposition 1.** Let $R$ be a commutative ring with finite group of units $R^\times$. The number of Pascal-like matrices $M \in R^{n \times m}$ is exactly $\#(R^\times)^{nm}$.

**Proof.** Each entry $M_{i,j}$ is the bottom-right entry of exactly one square sub-matrix laying on the top or left side whose determinant is to be a unit. There are precisely $\#(R^\times)^{nm}$ ways to prescribe those determinants. For each such prescription there is a unique way of solving for each entry $M_{i,j}$ recursively in $i + j$ by row expansion.

**Corollary 2.** There are exactly $2^{nm}$ Pascal-like matrices in $\mathbb{Z}^{n \times m}$. All of them are congruent (mod 2) but at the same time they cover all the possible Pascal-like matrices when reduced modulo 3, 4 or 6.

**Proof.** This follows from Proposition 1 and the facts that

$$\#(\mathbb{Z}^\times) = \varphi(3) = \varphi(4) = \varphi(6) = 2$$

and $\varphi(2) = 1$.

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