Control oriented modelling and modal analysis of the deformable mirror M4 of the extremely large telescope

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ABSTRACT
In this article, we derive a mechanical distributed parameter model for the annular sector plate segments of the Extremely Large Telescope’s deformable mirror M4. Additionally, we modally analyse the derived model via analytical and numerical approaches. The deformable mirror M4 is used to reject wavefront disturbances and enhance the optical imaging quality. We present a control oriented annular sector Kirchhoff–Love plate model featuring an elastic boundary condition and its modal analysis for one of the six identical M4 segments. Subsequently, we show that the well-known method of separation of variables is incompatible with the modal analysis of the presented distributed parameter model in cylindrical coordinates. Moreover, we successfully modally analyse the model using a finite difference approximation and a realistic construction of an M4 segment via a finite element approximation to compare the results. The modal analyses provide consistent results and therefore, both models underlying the analyses are consistent.

1. Introduction
Currently, the Extremely Large Telescope (ELT, see Figure 1a) is under construction in the Atacama desert, Chile. By featuring a primary mirror M1 of $\approx 39.3$ m in diameter (see Figure 2), it will be the largest ground-based telescope upon commissioning in 2025. As one of the first instruments of the ELT, the Mid-infrared ELT Imager and Spectrograph (METIS) is currently under development. METIS will search and characterize exoplanets, proto-planetary disks, and low-mass brown dwarfs, for example. To obtain measurements with maximum resolution despite atmospheric and wind-induced wavefront disturbances, METIS uses a single conjugate adaptive optics (SCAO) system.

The METIS-SCAO system (see Figure 3) comprises the deformable mirror M4 (see Figures 1b and Figures 2), the tip-tilt mirror M5, the infrared pyramid wavefront sensor, and a controller implemented on a real-time computer [1,2]. M4 is a thin annular mirror composed of six identical independent plate segments which are supported via flat springs at their outer curved edges. It can adopt arbitrary deformations with small amplitudes and high spatial frequencies. Due to the non-
negligible spatio-temporal dynamics of M4 and the second independent correction mirror M5, the design of SCAO controllers for METIS providing the best wavefront correction is complex. To control the METIS-SCAO system via a modal model-based concept as planned [3,4], a detailed model and modal analysis of the mirror M4 (or one of its segments) are necessary. This is the main focus of this paper.

Since the static and dynamic modelling of plates has been under intense investigation for several centuries, we focus on the most important results on thin annular sector Kirchhoff–Love plates such as the M4 mirror segments. The Kirchhoff–Love plate theory modelling thin plates of linear-elastic and isotropic material subject to small deformations is well established and described extensively [5–8]. Circular, annular, sector, and
annular sector plates are usually modelled in cylindrical coordinates to simplify the statement of their boundary conditions (BC) [5]. The most common BCs, in theory and practice, are clamped, simply supported, and free. Furthermore, the elastic BCs are known with transverse and bending elasticities counteracting the boundary forces and moments of the plate (see (9) and (10)) [7,9–12]. The common BCs represent special cases of elastic BCs, wherein the elasticities are zero or infinity [7].

Furthermore, the modal analysis of plates attracted intensive research interest for several decades. Typically, the modal analysis of a plate is performed using the following methods:

- analytical investigation of the fundamental plate partial differential equation (PDE) and BCs,
- method of weighted residuals (e.g. Galerkin method [6,13,14], Rayleigh–Ritz method [6,13], collocation method [6]),
- finite differences (FD) approximation of the plate and consecutive analysis of the resulting lumped parameter system [13,15],
- finite element (FE) approximation of the plate and consecutive analysis of the resulting lumped parameter system [13,15].

Moreover, many variations of the methods of weighted residuals (e.g. boundary collocation method [13]) exist and Fourier-series-based methods [10,11,16] were successfully applied in recent years.

The modal analyses for rectangular Kirchhoff–Love plates with common or elastic BCs were successfully conducted analytically [7,18,19] and via approximative methods [10,11,20], respectively. Additionally, the eigenmodes and -frequencies of circular and annular Kirchhoff–Love plates with common [6,7,18,19,21] and elastic [9] BCs are known. In particular, the damping in a commercial deformable mirror introduced by the air gap between the circular mirror plate and its suspension was modelled and modally analysed [22]. The literature on sector Kirchhoff–Love plates merely reports the eigenmodes and -frequencies with clamped or simply supported straight edges [7,23–25]. The analytical modal analysis of annular sector Kirchhoff–Love plates was successfully performed exclusively for simple supported straight edges until now [7,19,26]. By using approximative methods, the eigenmodes and -frequencies of annular sector Kirchhoff–Love plates were determined for the following BC configurations:

- straight edges clamped [27,28],
- inner curved edge clamped and other edges free [29,30],
- all edges clamped [31],
- straight edges simply supported [27,28,32],
- all edges elastically supported [12].

Using the modified Fourier-series-based method presented by Shi et al. [12] the modal analysis of Kirchhoff–Love plates with arbitrary BC-configurations should be possible, because elastic BCs can represent any common BCs. Unfortunately, we could not successfully transfer this method to the M4 mirror segment under investigation in this
paper. Moreover, Maruyama and Ichinomiya [33] presented experimental results for plates with all edges clamped.

The main contribution of this paper is the derivation of a mechanical distributed parameter model for one of the six identical M4 mirror segments and its modal analysis to design a controller for the METIS-SCAO system. The M4 segment under investigation is modelled as an annular sector Kirchhoff–Love plate with an elastically supported outer curved edge (i. e. elastic BCs) and free remaining edges. Subsequently, we show that the method of separation of variables, which is widely used for annular sector plates, is not compatible with the modal analysis of the presented model in cylindrical coordinates. Therefore, the presented distributed parameter M4 model is modally analysed for the first time via an FD approximation and a realistic construction of an M4 segment via an FE approximation to evaluate the FD results.

This paper is organized as follows: Section 2 introduces the METIS-SCAO system and the associated objectives. Subsequently, in Section 3, the distributed parameter model of the M4 mirror segment under investigation (i. e. annular sector Kirchhoff–Love plate with elastically supported outer curved edge) is deduced and presented. Section 4 describes the analytical and approximative modal analyses of the M4 mirror segment, demonstrating the incompatibility of model and method of separation of variables as well. Following, in Section 5, we show and examine the obtained analysis results. At the end, we summarize in Section 6 the presented results and provide directions for future work.

2. System description and objectives

In this section, we introduce the METIS-SCAO system (see Section 2.1) and in particular the deformable mirror M4 (see Section 2.2). Afterwards, we present our overall goals for the METIS-SCAO system (see Section 2.3) and the objectives of this paper (see Section 2.4).

2.1. METIS-SCAO system

The wavefront of light is the virtual surface of all wave loci featuring the same phase and is perpendicular to the local direction of propagation (cf. Huygens-Fresnel principle) [34,35]. Therefore, the wavefront is a spatial property and is given in microns or radians relative to a reference (typically a flat plane).

As the earth’s atmosphere is a turbulent fluid with cold and warm air cells and the refractive index of air is temperature-dependent, the incoming (star-, exoplanet-, etc.) light or wavefront is disturbed resulting in blurry images in METIS. Another major source of wavefront disturbances are wind-induced vibrations of the telescope structure, due to its enormous size.

In order to exploit the maximum resolution capability of the ELT, METIS is equipped with an SCAO system compensating for the disturbances and thus enabling very sharp images (close to the optical diffraction limit [35]) of the observation objects. Additionally, the SCAO system is essential to achieve the (challenging) scientific goals for METIS.

The METIS-SCAO system is a control loop rejecting arbitrary wavefront disturbances and depicted in Figure 3. Using the wavefront sensor of METIS, the incoming disturbed wavefront $y_{\text{wt}}$ is measured, which is the superposition of the disturbance $d$ (caused by e.g. atmosphere and wind) and the corrections $y_{\text{M4/M5}}$ of the active mirrors M4 and M5.
Subsequently, the SCAO controller computes the control inputs $u_{M4/M5}$ for the active mirrors M4 and M5 at a frequency of up to 1kHz to reduce the control error $e_{wf} = r_{wf} - y_{wf}$ between the reference and measured wavefront.

METIS’s wavefront sensor is an infrared pyramid sensor measuring the incoming (disturbed) wavefront at a frequency of 100 to 1000 Hz. The tip-tilt mirror M5 is a flat monolithic mirror sized $\approx 2.2 \, \text{m} \times 2.7\, \text{m}$ providing exclusively large tip and tilt corrections. M5 is actuated by three piezo-electric drives and its mirror tile is a lightweight structure made of silicon carbide [2,36].

### 2.2. Deformable mirror M4

The deformable mirror M4 is a flat annular plate mirror composed of six identical independent plate segments (see Figure 1b). The reflective annulus has an inner diameter of $\approx 0.5 \, \text{m}$, an outer diameter of $\approx 2.5 \, \text{m}$, and is realized via a thin ($\approx 10 \, \text{nm}$) aluminium layer on top of the segments. M4 can adopt arbitrary (out-of-plane) deformations with small amplitudes ($\pm 50 \, \mu\text{m}$) and high spatial frequencies utilizing $\approx 5350$ actuators [2,37].

Each M4 segment plate (see Figures 4a and Figures 8) is flat, constantly 1.95 mm thick, and made of the glass ceramic ‘Zerodur’, which is a linear isotropic material with nearly zero thermal expansion. The shapes of the M4 segments can be adjusted via non-contacting voice coil actuators and collocated capacitive sensors (892 per segment). The actuator-sensor pairs are arranged in a triangular grid across each segment and are spaced $\approx 31 \, \text{mm}$ apart. Moreover, the actuators’ permanent magnets are glued to the plate segments (opposing the aluminium layer) and their electric coils are mounted in an actively cooled support structure made of silicon carbide [2,37].

All plate segments are elastically supported (with respect to all translations and rotations) at their outer curved edges via flat springs directly glued to them (see Figures 4 and Figures 8). In contrast, all other edges of the segments are free, because the M4 segments hover $\approx 100 \, \mu\text{m}$ above the actuator mounting during operation. The flat springs themselves are clamped to the very stiff support structure [2,37]. Since there

\[ \text{Figure 3. Control loop of the METIS-SCAO system. The SCAO control commands the active mirrors M4 and M5 such that the error between the reference and measured wavefront is nearly zero.} \]
are no (mechanical, software, etc.) connections between adjacent segments, M4 consists of six independent annular sector mirrors.

2.3. Goals for the METIS-SCAO system

Our overarching goal for the METIS-SCAO system is the development of an SCAO controller providing the best wavefront correction to fully exploit the scientific capabilities of METIS. The following characteristics of the METIS-SCAO system significantly challenge the control development:

- non-negligible spatio-temporal dynamics of the active mirrors,
- very complex SCAO system (≈ 5350 actuator-sensor pairs, loop frequency up to 1 kHz, etc.),
- dual-stage system with the active mirrors M5 (large strokes, slow dynamics) and M4 (small strokes, fast dynamics; see Figure 3).

To develop a suitable SCAO controller for this demanding system setup, detailed models of all system components, especially the deformable mirror M4, are necessary.

We currently plan to realize the SCAO controller as a model-based modal controller utilizing the modes of the M4 segments. This concept reduces the complexity of the controller design and enables simple compensation of faulty M4 actuators [3,4].

2.4. Objectives

In order to control the METIS-SCAO system via a modal model-based concept and to establish a theoretical understanding of M4, we derive a distributed parameter model for one of its identical elastically supported segments (see Section 3). Next, we modally analyse...
this model (featuring a new configuration of BCs) represented in cylindrical coordinates using the well-known method of separation of variables. Unexpectedly, this method is incompatible with the analysis of the derived distributed parameter model (see Section 4.1) impeding the deduction of an analytical description of its eigenfrequencies and -modes.

Hence, the distributed parameter model is modally analysed via an FD approximation (see Section 4.2). Additionally, we conduct a modal analysis of a realistic construction of an M4 segment via an FE approximation, because no experiments can be conducted on M4 currently manufactured (see Section 4.3). Using these numerical results, we finally examine the match and consistency of the introduced FD and FE approximated models (see Section 5).

3. Distributed parameter model

We derive the Kirchhoff–Love plate model (see Section 3.1) and the BCs (see Section 3.2) for the M4 segment under investigation in this section. Finally, we summarize the obtained modelling results and present the overall model (see Section 3.3).

3.1. Plate

The M4 segment (see Figures 4a and Figures 8) under consideration is modelled in cylindrical coordinates \((r, \theta, z)\), because it is an annular sector and hence the associated sets are convex. Therefore, we define the following variables and sets for the M4 segment:

- inner segment radius: \(r_{\text{min}} = 0.267 \text{ m}\)
- outer segment radius: \(r_{\text{max}} = 1.27 \text{ m}\)
- extremal segment angles: \(\theta_{\text{min/max}} = \pm 30^\circ = \pm \frac{\pi}{6}\)
- open set of segment: \(\mathcal{S}^o = \{(r, \theta) \in \mathbb{R}^2 | r_{\text{min}} < r < r_{\text{max}}, \theta_{\text{min}} < \theta < \theta_{\text{max}}\}\)
- closed set of segment: \(\mathcal{S} = \{(r, \theta) \in \mathbb{R}^2 | r_{\text{min}} \leq r \leq r_{\text{max}}, \theta_{\text{min}} \leq \theta \leq \theta_{\text{max}}\}\)
- set of outer curved edge: \(B_1 = \{(r, \theta) \in \mathcal{S} | r = r_{\text{max}}\}\)
- set of inner curved edge: \(B_2 = \{(r, \theta) \in \mathcal{S} | r = r_{\text{min}}\}\)
- set of ‘left’ straight edge: \(B_3 = \{(r, \theta) \in \mathcal{S} | \theta = \theta_{\text{max}}\}\)
- set of ‘right’ straight edge: \(B_4 = \{(r, \theta) \in \mathcal{S} | \theta = \theta_{\text{min}}\}\)
- set of all edges: \(B = B_1 \cup B_2 \cup B_3 \cup B_4\)

As the M4 segment is made of a linear isotropic material (Young’s modulus \(E = 90.3 \text{ GPa}\), Poisson’s ratio \(\nu = 0.24\), specific mass \(\rho = 2530 \frac{\text{kg}}{\text{m}^3}\)), flat, subject to small deformations, and has a constant thickness \(h = 1.95 \text{ mm}\), it fulfils all assumptions of Kirchhoff–Love plate theory. The extremely thin reflective aluminium layer is neglected. Therefore, the segment is modelled as a Kirchhoff–Love plate

\[
D_E \Delta z(r, \theta, t) + Q_D(r, \theta, t) + \rho h \frac{\partial^2}{\partial t^2} z(r, \theta, t) = q(r, \theta, t) \quad \forall (r, \theta) \in \mathcal{S}^o \forall t > 0, \tag{1}
\]

with the time \(t\), the flexural rigidity \(D_E = \frac{E h^3}{12(1-\nu^2)}\), the Laplace operator for polar coordinates

\[
\n\]
a damping term \( Q_D(r, \theta, t) \) (described later in this section), and the external surface force \( q(r, \theta, t) \) normal to the plate (unit: N m\(^{-2}\)).

The external surface force of (1) comprises the forces of M4’s voice coil actuators and earth gravity, since M4 is mounted ‘upside down’ and slightly tilted in the ELT (see Figure 1b). As the actuators are very small compared to the M4 segment, we model them as point-like forces, i.e. their spatial influence characteristics are Dirac delta functions \( \delta(\cdot) \) in \( r \) and \( \theta \). Therefore, the external surface force of (1) is

\[
q(r, \theta, t) = \sum_{k=1}^{n_{act}} \frac{u_k(t)}{r_{a,k}} \delta(r - r_{a,k})\delta(\theta - \theta_{a,k}) + \rho h \cos(\alpha)g ,
\]

where \( u_k(t) \) is the force of the \( k \)-th actuator, \( r_{a,k} \) is the radial position of the \( k \)-th actuator, \( \theta_{a,k} \) is the angular position of the \( k \)-th actuator, \( \alpha = 7.75^\circ \approx 0.135 \text{ rad} \) is the mounting tilt of M4 (see Figure 1b, causing the ‘asymmetric’ light path in Figure 2), \( g \) is the earth’s gravitational acceleration, and \( n_{act} = 892 \) is the number of actuators of the M4 segment. The actuator dynamics are neglected here, because the dominant dynamics of the mechanical plate are significantly slower than the actuator dynamics due to underlying local controllers.

The capacitive sensors are collocated with the voice coil actuators and realized as annular sensors encircling the actuators. Therefore, the sensors return the ‘average’ displacements of the sensing annuli, i.e. the plate displacements at the actuator locations \( (r_{a,k}, \theta_{a,k}) \). Thus, their influence characteristics are also Dirac delta functions \( \delta(\cdot) \) in \( r \) and \( \theta \) and the output displacement of the \( k \)-th sensor (collocated with the \( k \)-th actuator) is

\[
y_k(t) = \int_{r_{a,k}}^{z(r, \theta, t)} \delta(r - r_{a,k})\delta(\theta - \theta_{a,k})dA = z(r_{a,k}, \theta_{a,k}, t),
\]

with the infinitely small area \( dA \) and the position \( (r_{a,k}, \theta_{a,k}) \) of the \( k \)-th actuator.

It is a well-known fact that deformable mirrors are usually subject to damping composed of structural and aerodynamic components. The aerodynamic damping results usually from an air gap of only a few 100 \( \mu \text{m} \) (air gap of M4 \( \approx 100 \mu \text{m} \)) between the mirror plate and the actuator mounting. Since M4 is currently being built, reliable information about the damping of the M4 segment under investigation is not available and thus we use the efficient and well-known Rayleigh damping model [6,38]. Therefore, the damping term \( Q_D(r, \theta, t) \) of (1) is

\[
Q_D(r, \theta, t) = (\lambda_D + \kappa_D \Delta \delta) \frac{\partial z}{\partial t}(r, \theta, t),
\]

where \( \lambda_D \) is the viscous and \( \kappa_D \) is the Kelvin–Voigt damping coefficient. Furthermore, this damping model does not influence the eigenmodes of the distributed parameter model (see Appendix A) and can be easily represented in a numerical model approximation via a superposition of the stiffness and mass matrices.
3.2. Boundary conditions

As described in Section 2.2 and shown in Figure 4, the outer curved edge $B_1$ of the M4 segment is supported via flat springs, whereas all other edges $B_2$ to $B_4$ are free. Therefore, the free BCs

$$M_r = M_r z(r, \theta, t) = -D_E \left( \frac{v}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{v}{r^2} \frac{\partial^2}{\partial \theta^2} \right) z(r, \theta, t) = 0, \quad (5)$$

$$V_r = Q_r - \frac{1}{r} \frac{\partial M_r}{\partial \theta} = V_r z(r, \theta, t)$$

$$= D_E \left( \frac{1 + \nu}{r^2} - \frac{\partial^2}{\partial r^2} + \frac{3 - \nu}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial r \partial \theta} - \frac{2}{r^2} \frac{\partial^2}{\partial r \partial \theta} \right) z(r, \theta, t) = 0 \quad (6)$$

apply to $B_2$ (inner curved edge), with the bending moment $M_r$ about $r$, the Kelvin–Kirchhoff $V_r$ and standard $Q_r$ shearing force perpendicular to $r$, the twisting moment $M_r \theta$ around $r$, and the moment $\mathcal{M}_r$ and shearing $V_r$ operators for $r$. Please note that forces and moments in these and the following BCs are length-normalized quantities in units N m$^{-1}$ and N m m$^{-1} = N$. Similarly, the free BCs

$$M_\theta = M_\theta z(r, \theta, t) = -D_E \left( \frac{1}{r} \frac{\partial}{\partial r} + \nu \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) z(r, \theta, t) = 0, \quad (7)$$

$$V_\theta = Q_\theta + \frac{\partial M_\theta}{\partial r} = V_\theta z(r, \theta, t)$$

$$= D_E \left( \frac{\nu - 2}{r^2} \frac{\partial^2}{\partial \theta \partial r} - \frac{1}{r^3} \frac{\partial^3}{\partial \theta^2} - \frac{\nu}{r} \frac{\partial^3}{\partial \theta^2} \right) z(r, \theta, t) = 0 \quad (8)$$

apply to $B_3$ and $B_4$ (straight edges), where $M_\theta$ is the bending moment about $\theta$, $V_\theta$ is the Kelvin–Kirchhoff and $Q_\theta$ the standard shearing force perpendicular to $\theta$, $M_\theta r$ is the twisting moment around $\theta$, and $M_r$ and $V_r$ are the moment and shearing operators for $\theta$.

Since the outer curved edge of the M4 segment is supported via flat springs, we model this support as elastic BCs (cf. [7, Section A.7]) for $B_1$ (outer curved edge)

$$M_r = M_r z(r, \theta, t) - M_E$$

$$= -D_E \left( \frac{v}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{v}{r^2} \frac{\partial^2}{\partial \theta^2} \right) z(r, \theta, t) - C_M(\theta) \frac{\partial z}{\partial r}(r, \theta, t) = 0, \quad (9)$$

$$V_r = V_r z(r, \theta, t) + Q_E$$

$$= D_E \left( \frac{1 + \nu}{r^2} - \frac{\partial^2}{\partial r^2} + \frac{3 - \nu}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial r \partial \theta} - \frac{2}{r^2} \frac{\partial^2}{\partial r \partial \theta} \right) z(r, \theta, t) + C_Q(\theta) z(r, \theta, t) = 0, \quad (10)$$

with the elastic bending moment $M_E$ and shearing force $Q_E$ generated by the flat springs and the corresponding $\theta$-dependent bending $C_M(\theta)$ and shearing $C_Q(\theta)$ stiffnesses (see Figure 4). The bending and shearing stiffnesses used in the following are depicted in Figure 5. Both stiffnesses are calculated via the FE approximation of the realistic construction of the M4 segment (cf. Section 4.3) and test-forces and -moments acting
on the outer curved edge to obtain the most realistic spatial stiffness profiles. Additionally, the stiffness peaks of $C_M$ and $C_Q$ match the attachment points of the flat springs (cf. Figure 8) and both stiffnesses do not go to zero 'between' the attachment points. Moreover, both stiffnesses feature in principle a hat- or bell-shaped profile. Possible reasons might be that the effects of a spring do not end abruptly at its physical edge but extend beyond it and the stiffnesses of neighbouring springs accumulate for $\theta \to 0^\circ$.

Instead of modelling the spring support of M4 as elastic BCs, the flat springs could be modelled individually as ordinary differential equations (ODEs, e.g. second order systems) or PDEs (e.g. beams) coupled with the mirror segment. We not opted for these potentially more precise modelling approaches, since analytical or numerical investigations for coupled ODE-PDE or PDE-PDE systems are much more complex than analysing the presented Kirchhoff–Love plate with elastic BCs. Furthermore, the elastic BCs are very versatile and at the same time easy to define. Moreover, the eigenmodes of the plate model with elastic BCs and the realistic construction of an M4 segment are consistent (see Section 5).

### 3.3. Overall system

Now we summarize the results of Sections 3.1 and 3.2 to establish the overall model of the M4 segment under examination. The complete distributed parameter plate model of the M4 segment is:

**Partial differential equation**

$$D_\varepsilon \Delta z(r, \theta, t) + Q_D(r, \theta, t) + \rho h \frac{\partial^2}{\partial t^2} z(r, \theta, t) = q(r, \theta, t) \quad \forall (r, \theta) \in S^\circ \forall t > 0 \quad (11a)$$

with
\[ Q_D(r, \theta, t) = (\lambda_D + \kappa_D \Delta \Delta) \frac{\partial z}{\partial t}(r, \theta, t) \] (11b)

\[ q(r, \theta, t) = \sum_{k=1}^{n_{act}} \frac{u_k(t)}{r_{a,k}} \delta(r - r_{a,k}) \delta(\theta - \theta_{a,k}) + \rho h \cos(\alpha) \] (11c)

\[
\begin{pmatrix}
 y_1(t) \\
 \vdots \\
 y_{n_{act}}(t)
\end{pmatrix} = 
\begin{pmatrix}
 z(r_{a,1}, \theta_{a,1}, t) \\
 \vdots \\
 z(r_{a,n_{act}}, \theta_{a,n_{act}}, t)
\end{pmatrix}
\] (11d)

**Boundary conditions**

\[ \mathcal{M}_r z(r, \theta, t) - C_M(\theta) \frac{\partial z}{\partial r}(r, \theta, t) = 0 \quad \forall (r, \theta) \in B_1 \ \forall t \geq 0 \] (11e)

\[ \mathcal{V}_r z(r, \theta, t) + C_Q(\theta) z(r, \theta, t) = 0 \quad \forall (r, \theta) \in B_1 \ \forall t \geq 0 \] (11f)

\[ \mathcal{M}_r z(r, \theta, t) = 0 \quad \forall (r, \theta) \in B_2 \ \forall t \geq 0 \] (11g)

\[ \mathcal{V}_r z(r, \theta, t) = 0 \quad \forall (r, \theta) \in B_2 \ \forall t \geq 0 \] (11h)

\[ \mathcal{M}_\theta z(r, \theta, t) = 0 \quad \forall (r, \theta) \in B_3 \cup B_4 \ \forall t \geq 0 \] (11i)

\[ \mathcal{V}_\theta z(r, \theta, t) = 0 \quad \forall (r, \theta) \in B_3 \cup B_4 \ \forall t \geq 0 \] (11j)

**Initial conditions**

\[ z(r, \theta, 0) = 0 \quad \forall (r, \theta) \in \overline{S} \] (11k)

\[ \frac{\partial z}{\partial t}(r, \theta, 0) = 0 \quad \forall (r, \theta) \in \overline{S} \] (11l)

The profiles of the bending \( C_M \) and shearing \( C_Q \) stiffnesses used in (11e) and (11f) are depicted in Figure 5. Additionally, trivial initial conditions are used, since these are a typical start-up configuration of the M4 segment.

**4. Modal analysis**

In the following section, we conduct the modal analysis of the M4 segment under investigation. For this purpose, we first use the analytical polar method of separation of variables, demonstrating the incompatibility of this approach and the distributed parameter model (11) (see Section 4.1). Subsequently, we modally analyse the distributed parameter model (11) via an FD approximation (see Section 4.2) and the realistic construction of the M4 segment via an FE approximation (see Section 4.3). Since no experiments can be conducted on M4 currently manufactured, the FE approximation is the most realistic representation of an M4 segment available. The (mechanical, geometric, etc.) parameters used for the model and the construction base on the publicly
available information regarding M4. Moreover, the damping (11b) is neglected in all modal analyses, since there is no reliable information available and Rayleigh damping is not influencing the eigenmodes (see Appendix A).

4.1. Analytical approach

To perform the analytical modal analysis of the distributed parameter model (11), we use its modal harmonic solution

\[ z(r, \theta, t) = m_k(r, \theta) \exp(-j\omega_k t) \]  \hspace{1cm} (12)

where \( m_k \) is the \( k \)-th eigenmode, \( j \) is the imaginary unit, and \( \omega_k = 2\pi f_k \) is the \( k \)-th angular eigenfrequency. Using (12) and the well-known and widely used ([6,7,9,25,26,39,40]) method of separation of variables in polar coordinates, the ansatz of the \( k \)-th eigenmode is

\[
m_k(r, \theta) = \underbrace{R_{k,1}(r)\Theta_{k,1}(\theta)}_{m_{k,1}(r, \theta)} + \underbrace{R_{k,2}(r)\Theta_{k,2}(\theta)}_{m_{k,2}(r, \theta)}
\]  \hspace{1cm} (13a)

subject to

\[
\left( r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \beta_k^2 r^2 - c_k^2 \right) R_{k,1}(r) = 0 \quad \forall (r, \theta) \in \mathcal{S},
\]  \hspace{1cm} (13b)

\[
\left( \frac{\partial^2}{\partial \theta^2} + c_k^2 \right) \Theta_{k,1}(\theta) = 0 \quad \forall (r, \theta) \in \mathcal{S},
\]  \hspace{1cm} (13c)

\[
\left( r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \beta_k^2 r^2 + c_k^2 \right) R_{k,2}(r) = 0 \quad \forall (r, \theta) \in \mathcal{S},
\]  \hspace{1cm} (13d)

\[
\left( \frac{\partial^2}{\partial \theta^2} + c_k^2 \right) \Theta_{k,2}(\theta) = 0 \quad \forall (r, \theta) \in \mathcal{S}.
\]  \hspace{1cm} (13e)

Here \( R_{k,1/2} : [r_{\text{min}}, r_{\text{max}}] \rightarrow \mathbb{R} \) and \( \Theta_{k,1/2} : [\theta_{\text{min}}, \theta_{\text{max}}] \rightarrow \mathbb{R} \) are the radial and angular components of the ansatz, \( \beta_k^2 = \frac{\omega^2 r^2}{k^2} \in (0, \infty) \) is the (constant) \( k \)-th eigenvalue, and \( c_k \in \mathbb{R} \) is a constant of the \( k \)-th eigenmode.

The modal harmonic solution (12) must fulfill the BC \( V_\theta z(r, \theta, t) = 0 \) for \( \forall (r, \theta) \in B_3 \) and \( \forall t \geq 0 \) (cf. (11)), hence the eigenmode \( m_k(r, \theta) \) has to satisfy

\[ V_\theta m_k(r, \theta) = D_E \left( \frac{\nu - 2}{r^2} \frac{\partial^2}{\partial \theta \partial r} - \frac{1}{r^3} \frac{\partial^3}{\partial \theta^3} - \frac{\nu}{r} \frac{\partial^2}{\partial \theta \partial r^2} \right) m_k(r, \theta) = 0 \quad \forall (r, \theta) \in B_3. \]  \hspace{1cm} (14)

Using the ansatz (13a), the associated conditions (13b) to (13e), and elementary mathematical conversions, (14) results in

\[
\beta_k^2 = (\nu - 1) \left( \frac{\partial^2}{\partial \theta^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( m_{k,1}(r, \theta) - m_{k,2}(r, \theta) \right) \right) \quad \forall (r, \theta) \in B_3
\]  \hspace{1cm} (15)
The right-hand side of (15) is a function of \( r \), because it cannot be simplified any further. Since the eigenvalue \( \beta_4^k \) is constant by assumption, (15) is contradictory and therefore the polar method of separation of variables cannot be used for modal analysis of the (undamped) distributed parameter model (11). This argumentation also holds for arbitrary annular sector Kirchhoff–Love plates featuring the (rather uncommon) BC (8), rendering the polar method of separation of variables incompatible with the modal analysis of these problem setups as well.

This incompatibility is also shown by the results of the modal analyses via FD and FE approximations presented in Sections 4.2 and 4.3. Figure 6 displays the corresponding eigenmodes of the M4 segments based on the FD and FE approximations in cylindrical coordinates. Both eigenmodes feature a curved 'horizontal' zero contour line (white line), rendering it impossible to represent them using the polar method of separation of variables. Furthermore, these results imply that the modal analysis (in cylindrical coordinates) of the M4 segment considered requires eigenmode ansatzes capable of generating curved zero contour lines.

### 4.2. Finite differences approximation

As the modal analysis of the distributed parameter model (11) via the method of separation of variables is not successful, we approximate the model via FD. Afterwards, we convert the obtained differential-algebraic equation (DAE) into an ODE system and modally analyse this approximation of the model (11).

The reasons for using the FD approximation instead of a method of weighted residuals to analyse the distributed parameter model (11) are: It is very challenging to choose suitable and ‘intuitive’ ansatz functions for the model’s eigenmodes. Typically, ansatzes such as \( m_k(r, \theta) \approx f(r) \cdot g(\theta) \) are widely used, where \( f \) and \( g : \mathcal{X} \subset \mathbb{R} \rightarrow \mathbb{R} \) are frequently used functions (e. g. polynomials, trigonometric functions), but these are ‘misleading’ based on the results of Section 4.1 (cf. Figure 6, Figures 9–18). Furthermore, the FD approximation of the model requires no prior knowledge of its eigenmodes.
The FD approximation of the distributed parameter model (11) is based on the polar uniform mesh shown in Figure 7. Table 1 lists the parameters of the FD mesh and their corresponding values. The link between the nodes of the FD mesh and the distributed parameter model (11) is:

Figure 7. Polar uniform mesh of the FD approximation of the distributed parameter model (11). The PDE nodes (■) correspond to the open segment set $S^0$, the BC nodes (●) to the edge set $B$, and the numeric nodes (○) are required to define the finite differences.

Figure 8. Model of the M4 segment under investigation in ANSYS. The mirror plate and springs are meshed via shell elements and connected by fixed contacts.
Table 1. Parameters of the FD mesh (see Figure 7) of the distributed parameter model (11) including their general and actual values used in this paper.

| Parameter                        | variable | general value | actual value |
|----------------------------------|----------|---------------|--------------|
| Number of nodes in θ-direction   | \( N_\theta \) | = 300         |              |
| Number of nodes in r-direction   | \( N_r \) | = 300         |              |
| Step size in θ-direction         | \( d_\theta \) | = \((\theta_{\text{max}} - \theta_{\text{min}})/(N_\theta + 2)\) | ≈ 3.47 mrad ≈ 0.2° |
| Step size in r-direction         | \( d_r \) | = \((r_{\text{max}} - r_{\text{min}})/(N_r + 2)\) | ≈ 3.31 mm |
| Number of PDE nodes              | \( N_{\text{PDE}} \) | = \( N_\theta N_r \) | 90000        |
| Number of BC nodes               | \( N_{\text{BC}} \) | = 2\( N_\theta + 2N_r + 4 \) | 1204         |
| Number of numeric nodes          | \( N_{\text{num}} \) | = 2\( N_\theta + 2N_r + 12 \) | 1212         |

- The innermost \( N_\theta \times N_r \) nodes of the mesh (■, referred to as PDE nodes) correspond to the open segment set \( \mathcal{S}^O \), which is the scope of the PDE ((11a) to (11d)).
- The penultimate layer of mesh nodes (●, referred to as BC nodes) is equivalent to the edge set \( \mathcal{B} \), which is the scope of the BCs ((11e) to (11j)).
- The outermost layer of mesh nodes (○, referred to as numeric nodes) are required for defining the difference quotients.

All difference quotients utilized for the FD approximation feature the numerical order \( \mathcal{O}(d^2) \) or \( \mathcal{O}(d^2_\theta) \) and wherever possible central quotients are used. Solely at the BCs or BC nodes, 3rd order decental difference quotients are applied.

Performing the FD approximation on the distributed parameter model (11) results in the DAE system

\[
\begin{pmatrix}
M & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\ddot{x}_1(t) \\
\ddot{x}_2(t)
\end{pmatrix}
+ \begin{pmatrix}
K_1 & K_2 \\
R_1 & R_2
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix}
= \begin{pmatrix}
B \\
0
\end{pmatrix}u(t),
\begin{pmatrix}
x_1(0) \\
x_2(0)
\end{pmatrix}
= 0,
\] (16a)

\[
y(t) = \begin{pmatrix}
C_1 & C_2
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix},
\] (16b)

where \( x_1(t) \in \mathbb{R}^{N_{\text{PDE}}} \) is the vector of displacements at the PDE nodes, \( x_2(t) \in \mathbb{R}^{N_{\text{BC}} + N_{\text{num}}} \) is the vector of displacements at the BC and numeric nodes, \( M \in \mathbb{R}^{N_{\text{PDE}} \times N_{\text{PDE}}} \) is the mass matrix, \( K_1 \in \mathbb{R}^{N_{\text{PDE}} \times N_{\text{PDE}}} \) and \( K_2 \in \mathbb{R}^{N_{\text{PDE}} \times (N_{\text{BC}} + N_{\text{num}})} \) are the components of the stiffness matrix associated with \( x_1(t) \) and \( x_2(t) \), \( R_1 \in \mathbb{R}^{(N_{\text{BC}} + N_{\text{num}}) \times N_{\text{PDE}}} \) and \( R_2 \in \mathbb{R}^{(N_{\text{BC}} + N_{\text{num}}) \times (N_{\text{BC}} + N_{\text{num}})} \) are the components of the BC matrix associated with \( x_1(t) \) and \( x_2(t) \), \( B \in \mathbb{R}^{N_{\text{PDE}} \times n_{\text{act}}} \) is the input matrix, \( C_1 \in \mathbb{R}^{n_{\text{act}} \times N_{\text{PDE}}} \) and \( C_2 \in \mathbb{R}^{n_{\text{act}} \times (N_{\text{BC}} + N_{\text{num}})} \) are the components of the output matrix associated with \( x_1(t) \) and \( x_2(t) \), \( u(t) = (u_1(t) \ldots u_{n_{\text{act}}}(t))^T \) is the vector of actuator forces \( u_k(t) \), and \( y(t) = (y_1(t) \ldots y_{n_{\text{act}}}(t))^T \) is the vector of sensor outputs \( y_k(t) \). The upper block line of (16a) represents the discretized PDE, the lower line corresponds to the BCs.

The DAE system (16) features the following characteristics: all matrices are sparse, \( M \) is diagonal, \( K_1 \) is diagonal dominant, and \( R_2 \) is regular and invertible. Therefore, the differential index of the DAE system is 1 and

\[
x_2(t) = -R_2^{-1}R_1x_1(t)
\] (17)

applies. By intuition, \( R_2 \) is regular because there are 2416 BC and numeric nodes and 2416 discretized, linear independent BCs (two BCs for each ‘ordinary’ BC node and four
BCs for each corner node). Using (17), the DAE system (16) is converted into the 2nd order ODE system approximating the distributed parameter model (11)

$$\begin{align*}
M \ddot{x}_1(t) + (K_1 - K_2 R_2^{-1} R_1) x_1(t) &= B u(t), \quad x_1(0) = 0, \\
y(t) &= \left( C_1 - C_2 R_2^{-1} R_1 \right) x_1(t),
\end{align*}$$

(18)

with the new stiffness matrix $K \in \mathbb{R}^{N_{\text{PDE}} \times N_{\text{PDE}}}$ and new output matrix $C \in \mathbb{R}^{n_{\text{out}} \times N_{\text{PDE}}}$. All matrices of (18) are sparse and $K$ is non-symmetric and not banded. $K$ features these properties, because it results from $K_1 - K_2 R_2^{-1} R_1$ and not the spatial differential operator, as usual when applying an FD approximation.

Using the harmonic solution $x_1(t) = \tilde{m}_k \exp(-j \omega_k t)$ of the ODE system (18), its eigenmodes and -frequencies fulfil the eigenvalue problem

$$K \tilde{m}_k = \omega_k^2 M \tilde{m}_k \Leftrightarrow M^{-1} K \tilde{m}_k = \omega_k^2 \tilde{m}_k$$

(19)

Note that $\tilde{m}_k$ represent the eigenmodes exclusively on the PDE nodes (corresponding to $\mathcal{S}$) and $\omega_k^2$ are the undamped eigenfrequencies. The eigenmodes of the distributed parameter model’s FD approximation on the PDE and BC nodes (corresponding to $\mathcal{S}$) are

$$m_k = \begin{pmatrix} \tilde{m}_k \\ -PR_1^{-1} R_1 \tilde{m}_k \end{pmatrix}$$

(20)

with the matrix $P$ extracting the modal displacements on the BC nodes and utilizing (17). Since $K$ is non-symmetric, the eigenmodes $\tilde{m}_k$ and $m_k$ are not necessarily orthogonal (see Section 5.2 and Figures 19 and Figures 20) [6, Ch. 4.5]. MATLAB 2017b is used to generate the matrices of the DAE (16) and ODE (18) systems and to numerically solve the associated eigenvalue problem (19).

### 4.3. Finite element approximation

In addition to the modal analysis of the distributed parameter model (11) approximated via FD, we modally analyse a realistic construction of an M4 segment (see Figure 8) using an FE approximation. This approximation is the most realistic representation of an M4 segment available, since M4 is currently manufactured. The additional FE-based modal analysis allows us to investigate the match of eigenmodes and -frequencies of the FD and FE approximated models. Moreover, we can draw conclusions based on this analysis regarding the agreement of both M4 segment models (see Section 5).

The realistic construction of an M4 segment depicted in Figure 8 is modally analysed via an FE approximation using ANSYS 19.0. This model comprises the annular sector mirror plate of Zerodur (neglecting the extremely thin reflective aluminium layer) and 25 independent flat steel springs. The plate and springs are connected via fixed contacts and the springs are clamped at their ‘outer’ ends reproducing their fixed connection to the very stiff support structure. Both plate and springs are meshed with two-dimensional triangular and quadrilateral shell elements with edge lengths $\leq 4$mm to obtain a
‘uniform’ spatial resolution. The resulting FE mesh is irregular and consists of 66,965 (rather) equal-sized elements and 69,146 nodes. Furthermore, no damping and no pre-loads are incorporated within the FE analysis.

5. Examination of analysis results

We present the eigenmodes and -frequencies of the FD- and FE-based modal analyses (see Section 5.1). Subsequently, we examine and compare these results to assess their match and the consistency of the M4 segment models studied (see Section 5.2).

5.1. Results

The modal analyses described in Sections 4.2 and 4.3 have been performed successfully. The first 10 eigenfrequencies and the corresponding eigenmodes (normalized with respect to (21)) of these analyses are shown in Table 2 and Figure 9 to Figures 18. Furthermore, the scalar products

\[ S_{k,n} = \int_{\mathcal{S}} \mathbf{m}_k(r, \theta) \mathbf{m}_n(r, \theta) r dr d\theta \]  

(21)

of the FD or FE eigenmodes with themselves are depicted in Figure 19, and the scalar products \( S \) of the FD and FE modes in Figure 20. The scalar products \( S \) are numerically calculated via a Gauß-Legendre quadrature (see [41, Ch. 7.2] and [42, Ch. 2.7]) based on the interpolated FE and FD modes.

| Mode no. | 1 | 2 | 3 | 4 | 5 |
|----------|---|---|---|---|---|
| FE results in Hz | 1.58 | 7.89 | 9.91 | 22.61 | 25.99 |
| FD results in Hz | 2.44 | 8.34 | 8.37 | 24.41 | 25.81 |
| Difference in % | -35.39 | -5.32 | 18.37 | -7.37 | 0.68 |

Table 2. Eigenfrequencies of the FD- and FE-based modal analyses.

| Mode no. | 6 | 7 | 8 | 9 | 10 |
|----------|---|---|---|---|----|
| FE results in Hz | 27.53 | 43.99 | 47.39 | 49.74 | 54.05 |
| FD results in Hz | 28.72 | 44.48 | 53.53 | 66.64 | 72.45 |
| Difference in % | -4.14 | -1.09 | -11.47 | -25.36 | -25.40 |

Figure 9. Deflections of normalized eigenmodes no. 1 of the FD- and FE-based modal analyses.
Table 2 shows a reasonable agreement of the corresponding eigenfrequencies and that the FD frequencies are usually higher than the FE frequencies. The FD and FE eigenmodes presented in Figure 9 to Figures 18 are usually consistent and sometimes sorted differently (e.g. FE mode no. 4 and FD mode no. 6; modes no. 7 and 8, cf. Figure 20). Figure 19 reveals that the FE eigenmodes are orthogonal and the FD eigenmodes are coupled and non-orthogonal. Nevertheless, the first 100 FE and FD eigenmodes each constitute a modal basis.

Figure 10. Deflections of normalized eigenmodes no. 2 of the FD- and FE-based modal analyses.

Figure 11. Deflections of normalized eigenmodes no. 3 of the FD- and FE-based modal analyses.

Figure 12. Deflections of normalized eigenmodes no. 4 of the FD- and FE-based modal analyses.
Figure 13. Deflections of normalized eigenmodes no. 5 of the FD- and FE-based modal analyses.

Figure 14. Deflections of normalized eigenmodes no. 6 of the FD- and FE-based modal analyses.

Figure 15. Deflections of normalized eigenmodes no. 7 of the FD- and FE-based modal analyses.

Figure 16. Deflections of normalized eigenmodes no. 8 of the FD- and FE-based modal analyses.
Comparison

As reported in Section 5.1, the FE eigenmodes create an orthogonal basis and are quite ‘intuitive’, because mode no. 1 corresponds to the plate’s first bending mode, mode no. 2

Figure 17. Deflections of normalized eigenmodes no. 9 of the FD- and FE-based modal analyses.

Figure 18. Deflections of normalized eigenmodes no. 10 of the FD- and FE-based modal analyses.

Figure 19. Scalar products $\phi$ (see (21)) of the normalized FD eigenmodes (left) or FE eigenmodes (right) with themselves. The FD eigenmodes provide a non-orthogonal base, whereas the FE eigenmodes constitute an orthogonal base.

5.2. Comparison

As reported in Section 5.1, the FE eigenmodes create an orthogonal basis and are quite ‘intuitive’, because mode no. 1 corresponds to the plate’s first bending mode, mode no. 2
to the first torsion mode, mode no. 3 to the second bending mode, and so on. In contrast, the FD eigenmodes are a non-orthogonal basis and partially intuitive, since, for example, mode no. 6 corresponds to the plate’s first bending-torsion mode, but modes no. 4 and 5 cannot be classified intuitively. Moreover, some FD and FE eigenmodes match very well, others match rather badly, and a couple of modes are sorted differently. Furthermore, the FD and FE eigenfrequencies agree reasonably but sometimes differ considerably.

The models used for the modal analyses differ in the following way: the FD approximation is based on the distributed parameter model (11). The FE approximation relies on the realistic construction of an M4 segment (see Figure 8), representing a coupling of several bodies and thus PDEs. Therefore, a discretized PDE is analysed in the modal analysis of the FD approximation and a coupled PDE-PDE system in the analysis of the FE approximation.

All these facts raise the question whether the differences in the analysis results are created by the different models or (inaccurate) algorithms underlying the analyses. The following facts strongly indicate that the differing results are caused by the different models: Both toolchains for FD- and FE-based modal analyses were tested with different annular sector Kirchhoff-Love plates (e.g. outer curved edge clamped, remaining edges free, all edges clamped) and provided closely matching results. Furthermore, the modal analyses were repeated with refined meshes (FD: N_r = N_θ = 350, FE: 129,453 nodes; cf. convergence analysis).

Figure 21 shows the deviations of eigenfrequency Δω and scalar product Δ_E = |1 − G_{n,k}| for each pair of normalized ‘standard’ and ‘refined’ modes. Since max(|Δω|) < 1% and max (Δ_E) < 4 · 10^{-4} (Δ_E = 0: identical modes, Δ_E ≫ 0.1: discrepant modes) for the first 15 pairs of modes, the combinations of meshes and algorithms used for both modal analyses are adequate. Moreover, several FD and FE modes match very well (e.g. modes no. 1, 2, 3) and therefore the presented results of the modal analyses can be considered reliable.

The reported results of the eigenfrequencies and -modes demonstrate that the analysis results obviously do not match exactly, but are consistent, as a significant amount of the results are similar. Therefore, the FE and FD approximations, respectively, the distributed parameter model (11) and construction of an M4 segment underlying the analyses are consistent in the same sense as the eigenmodes and -frequencies.

Table 2 shows the existence of groups (e.g. modes no. 2 and 3; modes no. 4 to 6) of closely packed eigenfrequencies. Within the first frequency group, the FE and FD eigenfrequencies match well, whereas within the second group, the modes are resorted (cf. FE mode no. 4 and FD mode no. 6). Furthermore, the eigenfrequencies of these groups match better than in general. The similarity of the frequency groups across the analyses supports the consistency of the analysis results and M4 segment models.

The matrix of scalar product depicted in Figure 20 is not symmetric (in contrast to the matrices in Figure 19), because two different sets of eigenmodes are considered. Moreover, Figure 20 provides valuable information, briefly presented in the following: The FD and FE eigenmodes no. 1 to 3 match. Furthermore, the FD eigenmodes no. 4 and 5 are a linear combination of the FE modes no. 1, 3, 5, and 6, with scaling factors featuring different absolute values and signs. More general, all ‘counter-intuitive’ modes can be represented as linear combination of a few FE modes. Moreover, the FD eigenmodes no. 9 and 10 feature their respective maximum scalar products with the FE modes no. 12 and 15 (outside the depicted ranges).
In this article, we presented a control oriented mechanical distributed parameter model and its modal analysis for one of the six identical segments of the deformable mirror M4 at the ELT. The M4 segment was modelled as an annular sector Kirchhoff–Love plate featuring an elastic BC at the outer curved edge and free BCs at the remaining edges. Following, we showed that the well-known method of separation of variables is not compatible with the modal analysis of the distributed parameter model in cylindrical coordinates. Subsequently, we successfully modally analysed the presented model using an FD approximation and a realistic construction of an M4 segment via an FE approximation to evaluate the FD results.

Figure 20. Scalar products $\bar{\varepsilon}$ (see (38)) of the normalized FD and FE eigenmodes. The maximum scalar products in each column are marked with a black dot (●) and are outside the depicted range for the 9th and 10th column.

Figure 21. Deviations of eigenfrequency $\Delta_\omega$ and scalar product $\bar{\varepsilon}$ for the first 15 pairs of normalized eigenmodes calculated via the presented and refined FD and FE approximations. □: comparison of FD results, ■ comparison of FE results.

6. Conclusion

In this article, we presented a control oriented mechanical distributed parameter model and its modal analysis for one of the six identical segments of the deformable mirror M4 at the ELT. The M4 segment was modelled as an annular sector Kirchhoff–Love plate featuring an elastic BC at the outer curved edge and free BCs at the remaining edges. Following, we showed that the well-known method of separation of variables is not compatible with the modal analysis of the distributed parameter model in cylindrical coordinates. Subsequently, we successfully modally analysed the presented model using an FD approximation and a realistic construction of an M4 segment via an FE approximation to evaluate the FD results.
The polar method of separation of variables is compatible neither with the modal analysis of the presented distributed parameter model nor with the analysis of an arbitrary annular sector Kirchhoff–Love plate featuring a free straight edge. Moreover, this result implies that the modal analysis of the M4 segment in cylindrical coordinates requires eigenmode ansatzes capable of generating curved zero contour lines. The modal analyses via the FD and FE approximations were successful and provided consistent results. Therefore, the distributed parameter model and construction of an M4 segment underlying the analysis are consistent as well.

Using the FD approximation of the distributed parameter model, we can now examine the influence of different stiffness profiles at the outer curved edge on the eigenmodes and -frequencies of the M4 segment. Furthermore, we will optimise these stiffness profiles to improve the match between the modal analysis results of the FD and FE approximations. Based on the determined eigenmodes of the M4 segments, we will continue the development and simulation of the model-based and modal METIS-SCAO controller. Moreover, the ODE and DAE systems resulting from the FD and FE approximations of the distributed parameter model can now undergo model reductions to enable dynamic simulations and system-theoretic analyses of the M4 segments.

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Abbreviations

| Acronym | Description |
|---------|-------------|
| BC      | Boundary condition |
| DAE     | Differential algebraic equation |
| ELT     | Extremely Large Telescope |
| FD      | Finite differences |
| FE      | Finite elements |
| METIS   | Mid-infrared ELT Imager and Spectrograph |
| ODE     | Ordinary differential equation |
| PDE     | Partial differential equation |
| SCAO    | Single conjugate adaptive optics |

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No potential conflict of interest was reported by the authors.

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Appendix A. Invariance of eigenmodes with respect to Rayleigh damping

First, we consider the undamped Kirchhoff–Love plate PDE

\[ D_E \Delta \Delta z(r, \theta, t) + \rho h \frac{\partial^2 z}{\partial t^2}(r, \theta, t) = 0 \quad \forall (r, \theta) \in \mathcal{A}^0 \quad \forall t > 0, \]  

(A1)

with the open set \( \mathcal{A}^0 \) defined by the plate geometry, (arbitrary) well-posed BCs and initial conditions, and a discrete eigenvalue spectrum. The modal harmonic solution of (A1) is

\[ z(r, \theta, t) = \sum_{k=0}^{\infty} a_k m_k(r, \theta) \exp(-j \omega_{k,0} t) = \sum_{k=0}^{\infty} a_k \mathfrak{z}_k(r, \theta, t), \]  

(A2)

where \( m_k \) is the \( k \)-th eigenmode of (A1), \( j \) is the imaginary unit, \( \omega_{k,0} \in (0, \infty) \) is the \( k \)-th angular eigenfrequency of (A1), \( a_k \in \mathbb{R} \) is the coefficient of the \( k \)-th eigenfrequency (dependent on the initial condition), and \( \mathfrak{z}_k \) is the \( k \)-th fundamental harmonic solution. Additionally, the eigenmodes \( m_k \) are subject to

\[ \beta_k^4 = \frac{\rho h \omega_k^2}{D_E} \quad \text{(eigen value of } m_k), \]  

(A3a)

\[ (\Delta \Delta - \beta_k^4) m_k(r, \theta) = 0 \quad \forall (r, \theta) \in \mathcal{A}^0, \]  

(A3b)

and the BCs of (A1).

Now, we move on to the damped Kirchhoff–Love plate PDE

\[ D_E \Delta \Delta z(r, \theta, t) + (\lambda_D + \kappa_D \Delta \Delta) \frac{\partial z}{\partial t}(r, \theta, t) + \rho h \frac{\partial^2 z}{\partial t^2}(r, \theta, t) = 0 \quad \forall (r, \theta) \in \mathcal{A}^0 \quad \forall t > 0 \]  

(A4)

with the same BCs and initial conditions as (A1), a discrete eigenvalue spectrum, and the viscous and Kelvin–Voigt damping coefficients \( \lambda_D \in (0, \infty) \) and \( \kappa_D \in (0, \infty) \), respectively. We assume that the modal solution of (A4) is

\[ z(r, \theta, t) = \sum_{k=0}^{\infty} b_k m_k(r, \theta) \exp\left(-\left(\sigma_k + j \omega_{k,d}\right)t\right) = \sum_{k=0}^{\infty} b_k \mathfrak{z}_k(r, \theta, t), \]  

(A5)

with the \( k \)-th eigenmode \( m_k \) of (A1) (sic!), the \( k \)-th angular eigenfrequency \( \omega_{k,d} \in (0, \infty) \) of (A4), the coefficient \( b_k \in \mathbb{R} \) and damping factor \( \sigma_k \in (0, \infty) \) of the \( k \)-th eigenfrequency, and the \( k \)-th fundamental solution \( \mathfrak{z}_k \). Moreover, we assume that each fundamental solution \( \mathfrak{z}_k \) individually fulfills the damped plate PDE (A4).

Inserting the fundamental solution \( \mathfrak{z}_k \) into (A4) and using both the conditions (A3a) to (A3b) and elementary mathematical conversions, results in

\[ m_k(r, \theta) \exp\left(-\left(\sigma_k + j \omega_{k,d}\right)t\right) \left(\rho h \omega_{k,0}^2 - (\lambda_D + \kappa_D \beta_k^4) (\sigma_k + j \omega_{k,d})
+ \rho h (\sigma_k + j \omega_{k,d})^2\right) = 0 \quad \forall (r, \theta) \in \mathcal{A}^0 \quad \forall t > 0 \]  

(A6)

Since neither the eigenmode \( m_k \) nor the exponential function \( \exp\left(-\left(\sigma_k + j \omega_{k,d}\right)t\right) \) are zero for \( \forall (r, \theta) \in \mathcal{A}^0 \) or \( \forall t > 0 \),

\[ \left(\rho h \omega_{k,0}^2 - (\lambda_D + \kappa_D \beta_k^4) (\sigma_k + j \omega_{k,d}) + \rho h (\sigma_k + j \omega_{k,d})^2\right) = 0 \]  

(A7)

must hold. By splitting (A7) into its real and imaginary part, we obtain the equation system
\[
\left(1 + \frac{\omega_{k,0}^2}{\sigma_k + \omega_{k,d}^2}\right)\sigma_k = \frac{\lambda_D + \kappa_D \rho_k^4}{\rho h}, \quad \left(1 - \frac{\omega_{k,0}^2}{\sigma_k + \omega_{k,d}^2}\right)\omega_{k,d} = 0, \quad (A8)
\]

respectively

\[
\sigma_k^2 = \omega_{k,0}^2 - \omega_{k,d}^2, \quad \sqrt{\omega_{k,0}^2 - \omega_{k,d}^2} = \frac{\lambda_D}{2 \rho h} + \frac{\kappa_D \omega_{k,0}^2}{2 D_E}. \quad (A9)
\]

On condition that all parameters fulfil

\[
\frac{\lambda_D}{2 \rho h} + \frac{\kappa_D \omega_{k,0}^2}{2 D_E} < \omega_{k,0} \quad \forall k \in \{1, 2, \ldots, \infty\}, \quad (A10)
\]

i.e. all fundamental solutions are weakly damped, \(\sigma_k \in (0, \infty)\) and \(\omega_{k,d} \in (0, \infty)\) exist \(\forall k \in \{1, 2, \ldots, \infty\}\) satisfying (A7) and (A9). Therefore, each individual fundamental solution \(\tilde{s}_k\) with \(\sigma_k\) and \(\omega_{k,d}\) compliant to (A9) fulfils the damped plate PDE (A4). Hence, (A5) is indeed the modal solution of the damped plate PDE (A4) and the eigenmodes \(m_k\) of (A1) and (A4) are identical. Consequently, the eigenmodes of a Kirchhoff–Love plate are invariant to Rayleigh damping under the condition (A10).