CONFORMAL DEFORMATIONS OF A SPECIFIC CLASS OF LORENTZIAN MANIFOLDS WITH NON-IRREDUCIBLE HOLonomy REPRESENTATION

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Abstract. Concerning holonomy theory or in the context of the existence of parallel spinors, Lorentzian manifolds with indecomposable, but non-irreducible holonomy representation have considerable significance. In this paper, we have comprehensively concentrated on conformal deformations of a particular class of four dimensional Lorentzian manifolds with indecomposable, non-irreducible holonomy representation which admit a recurrent light-like vector field. This type of Lorentzian manifolds are denoted by pr-waves and their holonomy algebra is contained in the parabolic algebra \( (\mathbb{R} \oplus \text{so}(2)) \times \mathbb{R}^2 \). Moreover, it is mainly illustrated that for an arbitrary conformal diffeomorphism by inducing some specific structural conditions a pr-wave manifold behaves totally analogous to Einstein manifolds. Particularly, it is demonstrated that in some special circumstances the structure of a pr-wave manifold is precisely the same as a manifold equipped with a warped product metric.

1. Introduction. In the context of holonomy theory or the existence of parallel spinors, Lorentzian manifolds with indecomposable, but non-irreducible holonomy representation are of undeniable significance. This considerable importance is mainly due to the well-known de Rham /Wu decomposition theorem according which any arbitrary complete semi-Riemannian manifold is isometric to a product of complete semi-Riemannian manifolds whose holonomy representation is indecomposable [6, 14]. Consequently, any Lorentzian manifold, based on this theorem, is isometric to a product of flat or irreducible Riemannian manifolds and a Lorentzian manifold which is equipped with either indecomposable, irreducible or trivial holonomy representation (refer to [10] for more complete details).

Let \((M, h)\) be an arbitrary semi-Riemannian manifold and \(\nabla\) be the Levi-Civita connection. A vector field \(X\) is called recurrent if \(\nabla X = \Theta \otimes X\) where \(\Theta\) is a one-form on \(M\). The noticeable fact is that a Lorentzian manifold whose holonomy representation is non-irreducible and indecomposable, admits a recurrent light-like vector field. In addition, according to [10] provided that the dimension of the manifold is \(n + 2\), the corresponding holonomy algebra is contained in the parabolic algebra \( (\mathbb{R} \oplus \text{so}(n)) \times \mathbb{R}^n \). The holonomy group of an \((n+2)\)-dimensional Lorentzian

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manifold \((M, h)\) in \(p \in M\) admits the one-dimensional light-like invariant subspace \(\mathbb{R} \cdot X_p\), whenever \((M, h)\) equipped with a recurrent light-like vector field \(X\). Accordingly, in \(p \in M\), the holonomy group does not act irreducibly and two parallel distributions are resulted; A one-dimensional distribution \(\Omega\) with \(X \in \Gamma(\Omega)\) which is totally isotropic and its \((n + 1)\)-dimensional orthogonal complement defined by:

\[
\Omega^\perp = \left\{ V \in TM \mid h(V, X) = 0 \right\}.
\]

Consequently, the manifold \((M, h)\) is foliated into light-like lines which are the flow of \(X\) and light-like hypersurfaces [10]. This foliation leads to creation of a particular coordinate system. In other words, an \((n + 2)\)-dimensional Lorentzian manifold \((M, h)\) where \(n + 2 > 2\) carries a recurrent light-like vector field \(X\) if and only if there exists the coordinates \((U, \Phi = (x, (y_i)^n_{i=1}, z))\) in which the metric \(h\) for \(f \in C^\infty(M)\) has locally the following shape:

\[
h = 2dxdz + \sum_{i=1}^n u_i dy_i dz + fdz^2 + \sum_{i,j=1}^n g_{ij} dy_idy_j, \quad \frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0. \tag{1}
\]

These coordinates are denoted by Walker coordinates. Furthermore, the vector field \(X\) is parallel if and only if the function \(f\) does not depend on \(x\). These coordinates are referred to as Brinkmann coordinates. A Lorentzian manifold which carries a light-like parallel vector field is denoted by a Brinkmann wave manifold [5, 11]. In addition, a pp-wave manifold is a Brinkmann wave manifold whose curvature tensor \(\mathcal{R}\) satisfies the trace condition \(\text{tr}_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0\).

In [13] Schimming proved that an \((n+2)\)-dimensional Lorentzian manifold \((M, h)\) is a pp-wave if and only if there exists local coordinates \((U, \Phi = (x, (y_i)^n_{i=1}, z))\) in which the metric \(h\) has the following structure:

\[
h = 2dxdz + fdz^2 + \sum_{i=1}^n dy_i^2, \quad \frac{\partial f}{\partial x} = 0. \tag{2}
\]

A pp-wave is Ricci-isotropic and has vanishing scalar curvature. This fact provides a fruitful setting in order to generalize pp-waves by inducing the condition of existence of a recurrent light-like vector field. Similar to pp-waves, in terms of local coordinates an \((n + 2)\)-dimensional Lorentzian manifold \((M, h)\) is a pr-wave if and only if around any point \(p \in M\) there exists coordinates \((U, \Phi = (x, (y_i)^n_{i=1}, z))\) in which the metric \(h\) has the following local representation:

\[
h = 2dxdz + f dz^2 + \sum_{i=1}^n dy_i^2, \quad f \in C^\infty(M). \tag{3}
\]

Meanwhile, it is worth mentioning that in pp-waves the abbreviation “pp” stands for “plane fronted with parallel rays” and in pr-waves, the abbreviation “pr” denotes “plane fronted with recurrent rays” (refer to [10] for more details).

Conformal transformations on a semi-Riemannian manifold can be considered as one of the most significant diffeomorphisms of a semi-Riemannian manifold. In this paper, we have thoroughly analyzed the four-dimensional pr-wave manifolds under the conformal deformations. In addition, we have indicated that considering specific circumstances the behavior of Lorentzian pr-wave manifolds is completely similar to Einstein manifolds under the influence of conformal deformations. For this purpose, in section 2, Christoffel Symbols, Levi-Civita connection and the curvature
tensor of four-dimensional pr-Wave manifolds are precisely computed. Section 3 is
dedicated to the comprehensive investigation of the four-dimensional Lorentzian pr-
wave manifolds under the conformal deformations. Some concluding remarks are
mentioned at the end of the paper.

2. Curvature Tensor of Four-Dimensional Pr-Wave Manifolds. In this section,
the Christoffel symbols, Levi-Civita connection and the curvature tensor of
four-dimensional Lorentzian pr-wave manifolds will be determined in terms of co-
ordinate vector fields \{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \}. Let \((M, h)\) be a four-dimensional Lorentzian
pr-wave manifold and \(\nabla\) be the Levi-Civita connection of \((M, h)\) and \(\mathcal{R}\) denotes its
curvature tensor. Then with respect to the basis coordinate vector fields for which
(3) holds, the non-vanishing components of the metric are given by:

\[ g_{y_1 y_1} = 1, \quad g_{y_2 y_2} = 1, \quad g_{zz} = 1, \quad g_{zz} = f. \tag{4} \]

Taking into account (4), the non-vanishing Christoffel symbols are as follows:

\[
\Gamma^x_{zz} = \frac{1}{2} \frac{\partial f}{\partial x}, \quad \Gamma^x_{zz} = \frac{1}{2} \left( f \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right), \quad \Gamma^y_{y_1 z} = \frac{1}{2} \frac{\partial f}{\partial y_1}, \quad \Gamma^z_{zz} = -\frac{1}{2} \frac{\partial f}{\partial x},
\Gamma^z_{y_1 z} = \frac{1}{2} \frac{\partial f}{\partial y_2}, \quad \Gamma^y_{y_2 z} = -\frac{1}{2} \frac{\partial f}{\partial y_2}. \tag{5} \]

Thus the non-vanishing covariant derivatives of the basis coordinate vector fields
are as follows:

\[
\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial f}{\partial x} \frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial y_1}} \frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial f}{\partial y_1} \frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial y_2}} \frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial f}{\partial y_2} \frac{\partial}{\partial x},
\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial z} = \frac{1}{2} \left( f \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right) \frac{\partial}{\partial x} - \frac{1}{2} \left( \frac{\partial f}{\partial y_1} \frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2} \frac{\partial}{\partial y_2} \right) - \frac{1}{2} \frac{\partial f}{\partial x} \frac{\partial}{\partial x}. \tag{6} \]

According to [2] by applying (6) in \(\mathcal{R}(X, Y) = \nabla_{[X, Y]} \nabla_X - \nabla_X \nabla_Y\) the non-vanishing
curvature components are computed as:

\[
\mathcal{R}(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) \frac{\partial}{\partial x} = -\frac{1}{2} \frac{\partial^2 f}{\partial z^2} \frac{\partial}{\partial x},
\mathcal{R}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}) \frac{\partial}{\partial x} = \mathcal{R}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z}) \frac{\partial}{\partial x} = \frac{1}{2} \frac{\partial^2 f}{\partial y_1^2} \frac{\partial}{\partial x},
\mathcal{R}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y_2}) \frac{\partial}{\partial x} = \mathcal{R}(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial z}) \frac{\partial}{\partial x} = \frac{1}{2} \frac{\partial^2 f}{\partial y_2^2} \frac{\partial}{\partial x},
\mathcal{R}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z}) \frac{\partial}{\partial y_1} = -\frac{1}{2} \frac{\partial^2 f}{\partial z^2} \frac{\partial}{\partial y_1}, \quad \mathcal{R}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}) \frac{\partial}{\partial y_1} = \frac{1}{2} \frac{\partial^2 f}{\partial y_1^2} \frac{\partial}{\partial y_1},\mathcal{R}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z}) \frac{\partial}{\partial y_1} = -\frac{1}{2} \frac{\partial^2 f}{\partial y_1^2} \frac{\partial}{\partial z},
\mathcal{R}(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial z}) \frac{\partial}{\partial y_2} = \frac{1}{2} \frac{\partial^2 f}{\partial y_2^2} \frac{\partial}{\partial y_2}, \quad \mathcal{R}(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial z}) \frac{\partial}{\partial y_2} = -\frac{1}{2} \frac{\partial^2 f}{\partial y_2^2} \frac{\partial}{\partial z},
\mathcal{R}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z}) \frac{\partial}{\partial y_2} = \frac{1}{2} \left( f \frac{\partial f}{\partial y_1} \frac{\partial}{\partial y_2} + \frac{\partial^2 f}{\partial x \partial y_1} \frac{\partial}{\partial y_1} + \frac{\partial^2 f}{\partial y_1 \partial y_2} \frac{\partial}{\partial y_2} + \frac{\partial^2 f}{\partial z^2} \frac{\partial}{\partial z} \right),
\mathcal{R}(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial z}) \frac{\partial}{\partial y_1} = \frac{1}{2} \left( f \frac{\partial f}{\partial y_2} \frac{\partial}{\partial y_1} + \frac{\partial^2 f}{\partial x \partial y_2} \frac{\partial}{\partial y_2} + \frac{\partial^2 f}{\partial y_1 \partial y_2} \frac{\partial}{\partial y_2} + \frac{\partial^2 f}{\partial z^2} \frac{\partial}{\partial z} \right),
\mathcal{R}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z}) \frac{\partial}{\partial y_2} = \frac{1}{2} \left( f \frac{\partial^2 f}{\partial x \partial y_1} \frac{\partial}{\partial y_2} + \frac{\partial^2 f}{\partial y_1 \partial y_2} \frac{\partial}{\partial y_1} + \frac{\partial^2 f}{\partial y_2^2} \frac{\partial}{\partial y_2} + \frac{\partial^2 f}{\partial x \partial y_2} \frac{\partial}{\partial z} \right). \tag{7} \]
3. Conformal Deformations of Four-Dimensional Pr-Wave Manifolds. Conformal transformations on a semi-Riemannian manifold can be considered as one of the most noteworthy diffeomorphisms of a semi-Riemannian manifold. Let \((M, g)\) and \((\tilde{M}, \tilde{g})\) be two arbitrary semi-Riemannian manifolds, a diffeomorphism \(\Psi : M \to \tilde{M}\) is called a conformal transformation if: \(\Psi^* (\tilde{g}) = e^{\Phi} g\), where \(\Phi \in C^\infty (M)\). Thus two semi-Riemannian metrics \(g\) and \(\tilde{g}\) are called conformally related whenever there exist such a diffeomorphism between them. In addition if \(\Phi\) is a constant, then \(\Psi\) is called a homothety and if \(\Phi = 0\), then \(\Psi\) is said to be an isometry.

In this section, we will comprehensively focus on conformal deformations of four-dimensional pr-wave manifolds which can be regarded as a particular class of four dimensional Lorentzian manifolds with indecomposable, non-irreducible holonomy representation which admit a recurrent light-like vector field and their holonomy algebra is contained in the parabolic algebra. Moreover, we will mainly illustrate that for an arbitrary conformal diffeomorphism by inducing some specific structural conditions a pr-wave manifold behaves totally analogous to Einstein manifolds. Particularly, it is demonstrated that in some special circumstances the structure of a pr-wave manifold is precisely the same as a manifold equipped with a warped product metric.

Let \((M, h)\) be an arbitrary four-dimensional pr-wave manifold. We want to obtain the conditions under which there exist a smooth function \(\Psi \in C^\infty (M)\) such that \((M, \tilde{h})\) where \(\tilde{h} = \Psi^{-2} g\) is an Einstein manifold. For this purpose, first of all we need the following proposition [1, 8, 9]:

**Proposition 1.** Let \((M, g)\) be an arbitrary \(m\)-dimensional semi-Riemannian manifold and \(\tilde{g} = \Psi^{-2} g, \Psi = e^\Omega\). Suppose \(\text{Ric}\) and \(\tilde{\text{Ric}}\) denote the Ricci tensor associated to \(g\) and \(\tilde{g}\), respectively. Then the following significant identity holds:

\[
\tilde{\text{Ric}}(X, Y) = \text{Ric}(X, Y) + \frac{m-2}{\Psi} \nabla^2 \Psi (X, Y) + (\Delta \Omega - (m-2)\|\nabla \Omega\|^2)g(X, Y). \tag{8}
\]

**Proof.** Firstly, consider \(\{E_1, E_2, \cdots, E_m\}\) as a local orthogonal frame on \((M, g)\), then we have \(\tilde{g}(e^{\Omega} E_1, e^{\Omega} E_i) = e^{2\Omega} \tilde{g}(E_1, E_i) = e^{2\Omega} e^{-2\Omega} g(E_1, E_i) = 1\). Consequently, \(\{e^\Omega E_1, e^\Omega E_2, \cdots, e^\Omega E_m\}\) is a local orthogonal frame on \((M, \tilde{g})\). Hence, we have:

\[
\tilde{\text{Ric}}(Y, Z) = \sum_{i=1}^{m} \tilde{g} (\tilde{\nabla} (e^\Omega E_i, Y) Z, e^\Omega E_i) = e^{2\Omega} \sum_{i=1}^{m} e^{-2\Omega} \tilde{g} (\tilde{\nabla} (E_i, Y) Z, E_i) \tag{9}
\]

Taking into account [1], the following identity holds:

\[
\tilde{\text{Ric}}(X, Y) Z = \text{Ric}(X, Y) Z - \left\{ g(X, Z) H_{\Omega}(Y) - g(Y, Z) H_{\Omega}(X) \right\}
+ \left\{ \nabla^2 \Omega (Y, Z) + Y(\Omega) Z(\Omega) - g(Y, Z) \|\nabla \Omega\|^2 \right\} X
+ \left\{ \nabla^2 \Omega (X, Z) + Z(\Omega) X(\Omega) - g(X, Z) \|\nabla \Omega\|^2 \right\} Y
+ \left\{ X(\Omega) g(Y, Z) - Y(\Omega) g(X, Z) \right\} \nabla \Omega. \tag{10}
\]
By substituting (10) in (9) and after a series of straightforward calculations we have:

\[
\tilde{\text{Ric}}(Y,Z) = \text{Ric}(Y,Z)Z + (m - 2)\nabla^2\Omega(Y,Z) + (m - 2)Y(\Omega)Z(\Omega) - (m - 2)g(Y,Z)\|\nabla\Omega\|^2 + g(Y,Z)\Delta\Omega.
\]  

(11)

For more details refer to [1]. Now considering \(\Psi = e^\Omega\), \(Y(\Psi) = e^\Omega Y(\Omega)\), it is deduced that:

\[
\nabla Y Z(\Psi) = e^\Omega \nabla Y Z(\Omega) \quad \text{and} \quad Y(Z(\Psi)) = Y(e^\Omega Z(\Omega)) = e^\Omega Y Z(\Omega) + e^\Omega Y(\Omega)Z(\Omega).
\]

Overall, according to above identities it is inferred that:

\[
\nabla^2 \Psi(Y,Z) = \nabla^2\Omega(Y,Z) + Y(\Omega)Z(\Omega).
\]  

(12)

Finally, by inserting relation (12) in (11) the identity (8) is obtained.

In the context of Riemannian geometry, Riemannian manifolds with constant scalar curvature as well as Einstein manifolds can be considered as two noteworthy families of Riemannian manifolds which are of specific significance in differential geometry. We denote these classes by \(\mathcal{C}\) and \(\mathcal{E}\), respectively; In addition assume that \(\mathcal{P}\) is the class of manifolds with parallel Ricci tensor. Every Einstein manifold has parallel Ricci tensor. On the other hand, not every manifold with constant scalar curvature has parallel Ricci tensor. Consequently, we have: \(\mathcal{E} \subset \mathcal{P} \subset \mathcal{C}\). In [7], A. Gray introduces a family of Riemannian manifolds denoted by Einstein-like spaces which can be regarded as a natural extension of the class \(\mathcal{E}\) of Einstein manifolds. It is worth mentioning that apart from Einstein manifolds whose Ricci tensor satisfies \(\varrho = \lambda g\) for a constant \(\lambda\) and the class \(\mathcal{P}\) of Ricci-parallel manifolds identified by the condition \(\nabla \varrho = 0\), the family of Einstein-like manifolds contain two extensive families of Riemannian manifolds. These two notable classes are denoted by \(\mathcal{A}\) and \(\mathcal{B}\) and lie between \(\mathcal{P}\) and \(\mathcal{C}\) and are characterized via the following identities:

\[
\mathcal{A} : \nabla_i \varrho_{jk} + \nabla_k \varrho_{ij} + \nabla_j \varrho_{ki} = 0,
\]

\[
\mathcal{B} : \nabla_i \varrho_{jk} - \nabla_j \varrho_{ik} = 0,
\]

where \(\varrho_{ij}\) is the Ricci tensor. Taking into account the fact that these significant class of manifolds are defined by imposing some restrictions on the Ricci tensor, above definitions can be naturally extended at one to the semi-Riemannian case. Now, according to [2, 7], we can state the following theorem:

**Theorem 3.1.** Let \((M,h)\) be a compact four-dimensional pr-wave manifold of negative sectional curvature. Suppose that \((M,\tilde{h} = \Psi^{-2}h, \Psi = e^\Omega)\) is a conformal deformation of \((M,h)\) and in (3) the function \(f\) satisfies the following relation:

\[
f \frac{\partial^3 f}{\partial x^2 \partial y} + \left( \frac{\partial f}{\partial y_1} \frac{\partial^2 f}{\partial x \partial y_1} - \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial y_1^2} \right) - \left( \frac{\partial^2 f}{\partial y_1 \partial y_2} - \frac{\partial^2 f}{\partial x \partial y_2} \right) = 0.
\]

(13)

then \((M,\tilde{h} = \Psi^{-2} h, \Psi = e^\Omega)\) is an Einstein manifold if and only if \(\nabla^2 \Psi = \frac{\Delta \Psi}{4} h\).

**Proof.** First of all, as it is well-known for an arbitrary semi-Riemannian manifold \((M,h)\), its Ricci tensor \(\text{Ric}\) is defined as the contraction of the curvature tensor as follows:

\[
\text{Ric}(X,Y) = \text{tr}\{Z \rightarrow \mathcal{R}(X,Z)Y\}.
\]

(14)
Hence, the components of the Ricci tensor with respect to the basis of coordinate vector fields \( \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z} \right\} \) are computed as:

\[
\begin{align*}
\text{Ric}_{xz} &= \text{Ric}(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}, \\
\text{Ric}_{yz} &= \text{Ric}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z}) = \frac{1}{2} \frac{\partial^2 f}{\partial x \partial y_1}, \\
\text{Ric}_{zz} &= \text{Ric}(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial z}) = \frac{1}{2} \left( \frac{\partial^2 f}{\partial y_1^2} - \frac{\partial^2 f}{\partial y_2^2} \right), \\
\text{Ric}_{yy} &= \text{Ric}(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_1}) = \frac{1}{2} \frac{\partial^2 f}{\partial x \partial y_2}.
\end{align*}
\]

In the next step, we compute the covariant derivative \( \nabla_{\text{Ric}} \) of the metric (3) in the four-dimensional case. Taking into account (7) and (15) it can be demonstrated that the non-vanishing components of the covariant derivative of Ric are given by:

\[
\begin{align*}
\left( \nabla_{\frac{\partial}{\partial x}} \text{Ric} \right)(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) &= \frac{1}{2} \frac{\partial^3 f}{\partial x^3}, \\
\left( \nabla_{\frac{\partial}{\partial y_1}} \text{Ric} \right)(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z}) &= \frac{1}{2} \frac{\partial^3 f}{\partial x \partial y_1^2}, \\
\left( \nabla_{\frac{\partial}{\partial y_2}} \text{Ric} \right)(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial z}) &= \frac{1}{2} \frac{\partial^3 f}{\partial x \partial y_2^2}, \\
\left( \nabla_{\frac{\partial}{\partial z}} \text{Ric} \right)(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) &= 1.
\end{align*}
\]

Let \( h \) be a Lorentzian metric described by (3). A semi-Riemannian manifold \((M, h)\) belongs to class \( \mathcal{A} \) if and only if its Ricci tensor \( \text{Ric} \) is cyclic-parallel i.e. for all tangent vector fields \( X, Y, Z \) the following relation holds:

\[
\left( \nabla_X \text{Ric} \right)(Y, Z) + \left( \nabla_Y \text{Ric} \right)(Z, X) + \left( \nabla_Z \text{Ric} \right)(X, Y) = 0,
\]

(17)
This equation is equivalent to imposing the condition that $\text{Ric}$ is a Killing tensor, in other words:

$$\left(\nabla_X \text{Ric}\right)(X, X) = 0.$$  \hspace{1cm} (18)

Now by applying (18) to (16) the relation (13) is deduced. Consequently, the four-dimensional pr-wave manifold $(M, h)$ is an Einstein-like manifold of class $A$ if and only if the function $f$ satisfies (13). If $(M, \tilde{h})$ is Einstein then there exist a constant $\kappa_1$ such that $\tilde{\text{Ric}} = \kappa_1 \tilde{h}$. Since $(M, h)$ is compact and has negative sectional curvature, according to [7], $(M, h)$ is an Einstein manifold. Hence, there exists a constant $\kappa_2$ such that $\text{Ric} = \kappa_2 g$. Now by applying (8) and considering $\tilde{h} = \Psi^{-2} h$, the following relation is deduced:

$$\kappa_1 \Psi^{-2} h = \kappa_2 h + \frac{2}{\Psi} \nabla^2 \Psi + (\Delta \Psi - 2) \|\nabla \Omega\|^2 h,$$  \hspace{1cm} (19)

or equivalently:

$$\nabla^2 \Psi = \frac{\Psi}{2} \left\{ \kappa_1 \Psi^{-2} - \kappa_2 - \Delta \Psi + 2 \|\nabla \Omega\|^2 \right\} h.$$  \hspace{1cm} (20)

Consequently, we have $\nabla^2 \Psi = \zeta h$ where

$$\zeta = \frac{\Psi}{2} \left\{ \kappa_1 \Psi^{-2} - \kappa_2 - \Delta \Psi + 2 \|\nabla \Omega\|^2 \right\}.$$  \hspace{1cm} (21)

At the final stage, by taking trace in (21) we obtain $\Delta \Psi = 4 \zeta$, hence $\zeta = \frac{\Delta \Psi}{4}$ or $\nabla^2 \Psi = \frac{\Delta \Psi}{4} h$. Now conversely if $\nabla^2 \Psi = \frac{\Delta \Psi}{4} h$ then by applying (8) and $\text{Ric} = \kappa_2 h$ it is inferred that $\text{Ric} = \Phi h$ where $\Phi : M \rightarrow \mathbb{R}$ and subsequently according to Schur's theorem $\Phi$ is constant and $(M, \tilde{h})$ is an Einstein manifold.

**Theorem 3.2.** Let $(M, h)$ be a compact four-dimensional pr-wave manifold of negative sectional curvature. Suppose that $(M, \tilde{h} = \Psi^{-2} h, \Psi = e^{\Omega})$ is a conformal deformation of $(M, h)$ and in (3) the function $f$ satisfies the relation (13). Then the metric $h$ is locally expressed by the warped product metric $ds^2 = d\xi^2 + (\Psi(\xi))^2 ds^*_\xi$ and $ds^*_\xi$ is the line element of a metric on an appropriate regular level hypersurface of $\Psi$.

**Proof.** First of all, it is worth noticing that due to assumptions the manifold $(M, h)$ can be regarded of class $E$ [7]. Consequently, according to theorem (3.1) we have to prove that on the domain of the local solutions of the differential equation $\nabla^2 \Psi = \frac{\Delta \Psi}{4} h$ the metric $h$ becomes as a warped product metric. In other words, it should be illustrated that there exist local coordinates $(\xi, \xi_1, \xi_2, \xi_3)$ in a neighborhood of $p$ and a function $\Psi = \Psi(\xi)$ with $\Psi(p) \neq 0$ and a three-dimensional Riemannian metric $h_\ast = h_\ast(\xi_1, \xi_2, \xi_3)$ such that

$$h(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}) = 1, \quad h(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi_i}) = 0 \quad \text{and} \quad h(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_i}) = (\Psi(\xi))^2 h_\ast(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_i}), \quad i, j = 1, 2, 3.$$  \hspace{1cm} (22)

Now let $c = \Psi(p)$ and $\mathcal{L}_c = \{ q : \Psi(q) = c \}$. Then $\mathcal{L}_c$ can be considered as our desired level hypersurface of $\Psi$ [1, 8]. Hence we can select a coordinate system...
\((\xi, \xi_1, \xi_2, \xi_3)\) on \(\mathcal{L}_c\) and expand this to geodesic parallel coordinates \((\xi, \xi_1, \xi_2, \xi_3)\) in an arbitrary neighborhood of \(p \in M\). Subsequently, it is demonstrated that:

\[
h(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}) = 1, \quad h(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}) = 0, \quad i = 1, 2, 3.
\]  

(23)

Furthermore, the following significant relation can be displayed:

\[
\frac{\partial}{\partial \xi} h(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}) = h(\nabla_{\xi_1} \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}) + h(\nabla_{\xi_2} \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_1}) \\
\quad = \frac{1}{\Psi} h(\nabla_{\xi_1} \nabla_\Psi, \frac{\partial}{\partial \xi_2}) + h(\nabla_{\xi_2} \nabla_\Psi, \frac{\partial}{\partial \xi_1}) \\
\quad = \frac{1}{\Psi} (2\nabla^2 \Psi h(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})) = \frac{2}{\Psi^4} \frac{\Delta \Psi}{4} h_{ij} = \frac{2\Psi''}{\Psi^4} h_{ij}.
\]

Therefore for fixed \(\xi_1, \xi_2, \xi_3, h_{ij} = (h_{ij}(\xi))\) satisfies \(h'_{ij} = \frac{2\Psi''}{\Psi^4} h_{ij}\) or \(\left(\frac{h_{ij}}{(\Psi')^2}\right)^2 = 0\).

Consequently, it is illustrated that \((\Psi')^2 h_{ij}(\xi, \xi_1, \xi_2, \xi_3)\) has no dependence on \(\xi\) and this expression is reckoned as \(h_{ij}(\xi, \xi_1, \xi_2, \xi_3)\) which completes the proof of the theorem. \(\Box\)

Now, taking into account theorem (3.2), we can prove the existence of two complementary orthogonal totally geodesic and holonomy invariant foliations on \(TM\).

**Theorem 3.3.** Let \((M, h)\) be a compact four-dimensional pr-wave manifold of negative sectional curvature. Suppose that \((M, h = \Psi^{-2} h, \Psi = e^{H})\) is a conformal deformation of \((M, h)\) and in (3) the function \(f\) satisfies the relation (13). Then for any \((x, y) \in TM\) there exists a neighborhood \(N_T^2 \subset TM\) and two submanifolds \(N_T^2\) and \(N_T^2\) of dimensions two and six, admitting the metric structures \(g\) and \(g^\perp\) such that \((N_T^2, h)\) is the product of \((N_T^2, g)\) and \((N_T^2, g^\perp)\). Furthermore, there exists two complementary orthogonal totally geodesic and holonomy invariant foliations on \(TM\).

**Proof.** As demonstrated in theorem (3.2), the metric \(h\) is locally expressed by a warped product metric. Now assume that \(\mathcal{F}\) be a parallel non-degenerate 1-foliation on an \((1 + 3)\)-dimensional manifold \((M, h)\). Hence, the corresponding tangent distribution \(D\) to \(\mathcal{F}\) is non-degenerate and parallel with respect to the Levi-Civita connection \(\nabla\) on \((M, h)\). Consequently, the distribution \(D^\perp\) is parallel, non-degenerate and complementary orthogonal to \(D\). This yields the second parallel 3-foliation \(\mathcal{F}'\).

Therefore, \((M, D, D^\perp)\) admits the geometric structure of an almost product manifold and the pair \((\mathcal{F}, \mathcal{F}'\perp)\) is a \(\nabla\)-grid with respect to the Levi-Civita connection. Now consider \(L\) and \(L\) to be the leaves through \(x^*\) of \(\mathcal{F}\) and \(\mathcal{F}'\perp\), respectively. Then there exists a foliated chart \((\mathcal{V}, \gamma)\) about \(x^*\) with local coordinates \((x^1, x^2, x^3, x^4)\) such that each plaque of \(\mathcal{F}\) is given by the equations:

\[
x^2 = c^2, \quad x^3 = c^3, \quad x^4 = c^4.
\]

(25)
noticeable point is that we can assign \((x^1, x^2, x^3, x^4)\) as a coordinate system on \(N^*\) compatible with both foliations \(\mathcal{F}\) and \(\mathcal{F}^\perp\). That is to say:
\[
D = \text{span}\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\},
\]
\(D^\perp = \text{span}\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}, \frac{\partial}{\partial y^4}\}\),

(26)
on \(N^*\). The coordinate system \(\{(V \times \mathcal{W}, \gamma \times \theta) : (x^1, x^\alpha), \alpha = 2, 3, 4\}\) on \(N^* = N \times N^\perp\) defines a coordinate system \(\{(V^* \times \mathcal{W}^*, \Gamma \times \Theta) : (x^1, x^\alpha, y^1, y^\alpha), \alpha = 2, 3, 4\}\) on \(TN^* \cong TN \oplus TN^\perp\), where for any \(x \in V\) and \(y_x \in T_xN\), we have: \(V^* = \pi^{-1}_V(V)\) and \(\Gamma : V^* \to R^2\) is a diffeomorphism of \(V^*\) on \(\gamma(V) \times R\), and \((x^1, x^2, x^3, x^4; y^1, y^2, y^3, y^4) = \Gamma(y_x)\). Correspondingly for any \(v \in \mathcal{W}\) and \(v_u \in T_uM_2\), we have: \(W^* = \pi^{-1}_V(W)\) and \(\Theta : W^* \to R^6\) is a diffeomorphism of \(W^*\) on \(\theta(W) \times R^3\) and \((x^1, x^2, x^3; y^1, y^2, y^3, y^4) = \Theta(v_u)\). For simplicity, we denote by \((x, y) = (x^1, y^1)\) the coordinate of \(y_x\) (likewise \((u, v) = (x^1, x^2, x^3, y^2, y^3, y^4)\) the coordinate of \(v_u\)). Hence, we have \((x, u, y, v) \in T_xN \oplus T_uN^\perp\). Now we consider another coordinate system \(\{(\tilde{V}, \tilde{W}, \tilde{\gamma} \times \tilde{\theta}) : (\tilde{x}^1, \tilde{x}^\alpha), \alpha = 2, 3, 4\}\) on \(N^* = N \times N^\perp\) such that \(V \cap \tilde{V} \neq \emptyset\) and \(W \cap \tilde{W} \neq \emptyset\). Then the local coordinates \((x, u, y, v)\) and \((\tilde{x}, \tilde{u}, \tilde{y}, \tilde{v})\) on \(TN^* = TN \oplus TN^\perp\) are related by [4]:
\[
\begin{align*}
\tilde{x}^1 &= \tilde{x}^1(x^1), & x^\alpha &= \tilde{x}^\alpha(x^1, x^2, x^3, x^4), \\
\tilde{y}^1 &= y^1, & \tilde{y}^\alpha &= \tilde{\partial}_{\alpha} \tilde{x}^\beta \tilde{y}^\gamma, \quad \alpha, \beta = 2, 3, 4.
\end{align*}
\]
(27)

In the following we will denote by \(N^*_T = TN^*, N_T = TN\) and \(N^*_T = TN^\perp\). If \(\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}\}\) and \(\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}\}\) are the basis on \(N_T\) and \(N^*_T\) respectively, then we shall assume that \(\{\frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial y^\alpha}\}\) are their lifts to \(N_T \times N^*_T\). As a consequence of (27) the local frame fields \(\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^\alpha}\}\) and \(\{\frac{\partial}{\partial \tilde{x}^1}, \frac{\partial}{\partial \tilde{x}^\beta}, \frac{\partial}{\partial \tilde{y}^1}, \frac{\partial}{\partial \tilde{y}^\beta}\}\) satisfy the following identities with respect to the change of coordinates as described above [12]:
\[
\begin{align*}
\frac{\partial}{\partial x^1} &= \frac{\partial}{\partial \tilde{x}^1} + \frac{\partial}{\partial \tilde{x}^\beta} \frac{\partial}{\partial \tilde{y}^\gamma}, & \frac{\partial}{\partial x^\alpha} &= \frac{\partial}{\partial \tilde{x}^\beta} \frac{\partial}{\partial \tilde{x}^\alpha} + \frac{\partial}{\partial \tilde{y}^\beta} \frac{\partial}{\partial \tilde{y}^\gamma}, \\
\frac{\partial}{\partial y^1} &= \frac{\partial}{\partial \tilde{y}^1}, & \frac{\partial}{\partial y^\alpha} &= \frac{\partial}{\partial \tilde{x}^\beta} \frac{\partial}{\partial \tilde{x}^\alpha} \frac{\partial}{\partial \tilde{y}^\gamma},
\end{align*}
\]
(28)
\[
\text{rank } \begin{bmatrix}
\frac{\partial}{\partial \tilde{x}^\beta} & 0 \\
0 & \frac{\partial}{\partial \tilde{x}^\alpha} \\
\end{bmatrix} = 1 + 3, \quad i, j = 1, 2, 3, 4, \quad \alpha, \beta = 2, 3, 4.
\]

It is worth noticing that the \((1 + 3)\)-dimensional product manifold \(N^* = N \times N^\perp\) can be structurally reckoned as the configuration space of a dynamical system which is governed by the following system of second order ordinary differential equations:
\[
\begin{align*}
\frac{d^2x^1}{dt^2} + 2G^1(x, u, \frac{dx}{dt}, \frac{du}{dt}) &= 0, \\
\frac{d^2x^\alpha}{dt^2} + 2G^\alpha(x, u, \frac{dx}{dt}, \frac{du}{dt}) &= 0, \quad \alpha = 2, 3, 4.
\end{align*}
\]
(29)

where system (29) is defined over a local chart on \(N^*_T \simeq N_T \oplus N^*_T\). The functions \(G^1(x, u, y, v)\) and \(G^\alpha(x, u, y, v)\) are of class \(C^\infty\) on \(N^*_T \setminus \{0\}\) and only continuous
on the null section. Hence, we have a collection of systems (29) on every induced local chart on $N^\alpha_T$ that are thoroughly compatible on the intersection of induced local chart. This is equivalent to this fact that under the change (27) of local induced coordinates on $N^\alpha_T$, the functions $G^1(x, u, y, v)$ and $G^\alpha(x, u, y, v)$ transform as follows:

$$ 2\hat{G}^1 = 2G^1 - \frac{\partial y^1}{\partial x^1}y^1, \quad 2\hat{G}^\alpha = \frac{\partial x^\alpha}{\partial x^1}2G^1 - \frac{\partial y^\alpha}{\partial x^1}y^\alpha. $$

(30)

Now taking into account the change of local coordinates (27) on $TM$ and by considering (28) it is inferred that:

$$ y^1\frac{\partial}{\partial x^1} - 2G^1(x, u, y, v)\frac{\partial}{\partial y^1} + y^\alpha\frac{\partial}{\partial x^\alpha} - 2G^\alpha(x, u, y, v)\frac{\partial}{\partial y^\alpha} $$

$$ = \hat{y}^1\frac{\partial}{\partial x^1} - 2\hat{G}^1(\hat{x}, \hat{u}, \hat{y}, \hat{v})\hat{\partial}y^1 + \hat{y}^\alpha\frac{\partial}{\partial x^\alpha} - 2\hat{G}^\alpha(\hat{x}, \hat{u}, \hat{y}, \hat{v})\hat{\partial}y^\alpha. $$

if and only if the functions $G^1, \hat{G}^1, G^\alpha$ and $\hat{G}^\alpha$ are related by (30). Thus a vector field $S \in \chi(TM)$ is called a semispray or a second order vector field if on every domain of local charts of $N^\alpha_T$ we have a collection of the functions $\{G^1, G^\alpha\}$ such that:

$$ S = y^1\frac{\partial}{\partial x^1} - 2G^1(x, u, y, v)\frac{\partial}{\partial y^1} + y^\alpha\frac{\partial}{\partial x^\alpha} - 2G^\alpha(x, u, y, v)\frac{\partial}{\partial y^\alpha}. $$

(31)

The functions $\{G^1, G^\alpha\}$ are called the local coefficients of the semispray. Assume that $S$ is a semispray to form (31) with local coefficients $\{G^1, G^\alpha\}$. We define:

$$ N = (G^i_j) = \begin{pmatrix}
G^1_0 := \frac{\partial G^1}{\partial y^1} & G^1_1 := \frac{\partial G^\alpha}{\partial y^1} \\
G^1_1 := \frac{\partial G^1}{\partial y^\alpha} & G^1_2 := \frac{\partial G^\alpha}{\partial y^\alpha}
\end{pmatrix}, \quad i, j = 1, 2, 3, 4. $$

(32)

Consequently, $N$ is a non-linear connection on $N^\alpha_T = N^\alpha_T \oplus N^\alpha_T$. In local coordinates the semispray induced by the nonlinear connection $N = (G^i_j)$ with coefficients (32), is expressed by:

$$ S = y^1\frac{\partial}{\partial x^1} - (G^1_1y^1 + G^1_2y^\alpha)\frac{\partial}{\partial y^1} + y^\alpha\frac{\partial}{\partial x^\alpha} - (G^1_1y^1 + G^1_2y^\alpha)\frac{\partial}{\partial y^\alpha}. $$

(33)

In other words, the coefficients of the induced semispray are identified as follows:

$$ 2G^1(x, u, y, v) = G^1_1y^1 + G^1_2y^\alpha, \quad 2G^\alpha(x, u, y, v) = G^1_1y^1 + G^1_2y^\alpha. $$

By applying (32), we define:

$$ \begin{aligned}
\frac{\delta^*}{\delta^* x^1} &= \frac{\partial}{\partial x^1} - G^1_1\frac{\partial}{\partial y^1} - G^1_2\frac{\partial}{\partial y^\alpha}, \\
\frac{\delta^*}{\delta^* x^\alpha} &= \frac{\partial}{\partial x^\alpha} - G^1_1\frac{\partial}{\partial y^1} - G^1_2\frac{\partial}{\partial y^\alpha}, \quad \alpha, \beta = 2, 3, 4.
\end{aligned} $$

(34)

Thus if we put: $\mathcal{V}(TM) := \text{span}\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^\alpha}\}$, $\mathcal{H}(TM) := \text{span}\{\frac{\delta^*}{\delta^* x^1}, \frac{\delta^*}{\delta^* x^\alpha}\}$ then we can write, $TTM = \mathcal{V}(TM) \oplus \mathcal{H}(TM)$. It is worth noticing that we can designate
the coordinate system \( \{(V^* \times W^*, \Gamma \times \Theta) : (x^1, x^\alpha; y^1, y^\beta)\} \) as a coordinate system on \( \mathcal{N}^T \) compatible with both foliations \( \mathcal{T}^\perp \) and \( \mathcal{F}^\perp \). This means that:

\[
T(\mathcal{T}) = \text{span}\{ \frac{\partial}{\partial y^1}, \frac{\delta^*}{\delta^{*}x^1} \}, \quad T(\mathcal{F}^\perp) = \text{span}\{ \frac{\partial}{\partial y^\alpha}, \frac{\delta^*}{\delta^{*}x^\alpha} \}, \quad \alpha = 2, 3, 4. \tag{35}
\]

It is clear that with respect to above coordinate system the matrix of the local components of the metric \( h \) has the following form:

\[
[h_{ij}(x, y)] = \begin{bmatrix}
g_{11}(x, y) & 0 & g_{12}(x, y) \\
0 & 0 & g_{23}(x, y)
g_{12}(x, y) & g_{23}(x, y)
\end{bmatrix}, \quad i, j \in \{1, 2, 3, 4\}, \quad \alpha, \beta \in \{2, 3, 4\}. \tag{36}
\]

In other words, taking into account (36), with respect to the frame field \( \{\frac{\delta^*}{\delta^{*}x^i}, \frac{\partial}{\partial y^i}\} \) which is locally defined on \( TM \), the following can be stated:

\[
\begin{align*}
g = g_{11}(x, y) &= h\left( \frac{\delta^*}{\delta^{*}x^1}, \frac{\delta^*}{\delta^{*}x^1} \right) = h\left( \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1} \right), \\
h\left( \frac{\delta^*}{\delta^{*}x^1}, \frac{\partial}{\partial y^1} \right) &= 0, \quad \text{and} \\
g^\perp = g_{\alpha\beta}(x, y) &= h\left( \frac{\delta^*}{\delta^{*}x^\alpha}, \frac{\delta^*}{\delta^{*}x^\beta} \right) = h\left( \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right), \\
h\left( \frac{\delta^*}{\delta^{*}x^\alpha}, \frac{\partial}{\partial y^\beta} \right) &= 0, \quad \alpha, \beta \in \{2, 3, 4\}.
\end{align*} \tag{37}
\]

Now by applying the metric \( h \), the following decomposition can be reached:

\[
TT(M) = T(\mathcal{T}) \oplus T(\mathcal{F}^\perp). \tag{38}
\]

By \( \pi \) and \( \bar{\pi} \) the projection morphism can be denoted on \( T(\mathcal{T}) \) and \( T(\mathcal{F}^\perp) \) respectively. Additionally, according to [3] with respect to above decomposition there exists unique linear connections \( \nabla^T \) and \( \nabla^\perp \) on \( T(\mathcal{T}) \) and \( T(\mathcal{F}^\perp) \), respectively and are denoted by intrinsic connections. Now by considering the Levi-Civita connection \( \nabla \) on \( (TM, h) \) the following can be stated:

\[
\begin{cases}
(a) : \nabla^T_X \pi Y = \pi \nabla_X \pi Y + \pi[\bar{\pi}X, \pi Y], \\
(b) : \nabla^\perp_X \bar{\pi} Y = \bar{\pi} \nabla_X \pi Y + \pi[\pi X, \bar{\pi} Y].
\end{cases} \tag{39}
\]

We call \( \mathbb{H} : \Gamma(T(\mathcal{T})) \times \Gamma(T(\mathcal{T})) \to \Gamma(T(\mathcal{T}^\perp)) \) and \( \bar{\mathbb{H}} : \Gamma(T(\mathcal{F}^\perp)) \times \Gamma(T(\mathcal{F}^\perp)) \to \Gamma(T(\mathcal{F}^\perp)) \), given by:

\[
\mathbb{H}(\pi X, \pi Y) = \bar{\pi} \nabla_{\pi X} \pi Y, \quad \bar{\mathbb{H}}(\pi X, \bar{\pi} Y) = \pi \nabla_{\bar{\pi} X} \bar{\pi} Y. \tag{40}
\]

the second fundamental forms of \( T(\mathcal{T}) \) and \( T(\mathcal{F}^\perp) \) respectively. Since \( \mathcal{T} \) is a parallel non-degenerate foliation on \( TM \), then for any \( X, Y \in \Gamma(T(\mathcal{T})) \) we have \( \nabla_X Y \in \Gamma(T(\mathcal{T^\perp})) \). Thus by (40) it follows that the second fundamental form \( \mathbb{H} \) of \( \mathcal{T} \) vanishes identically on the tangent bundle of the four-dimensional pr-wave manifold \( (M, h) \). Hence, \( \mathcal{T} \) defines a totally geodesic foliated cocycle on \( TM \). Furthermore, considering the fact that \( h \) is parallel with respect to \( \nabla \), it is inferred that:

\[
\forall X \in \Gamma(TT(M)), \ Y \in \Gamma(T(\mathcal{T})), \ Z \in \Gamma(T(\mathcal{F}^\perp)), \\
h(\nabla_X Y, Z) + h(Y, \nabla_X Z) = 0. \tag{41}
\]

As \( \nabla_X Y \in \Gamma(T(\mathcal{T})), \) we have \( h(\nabla_X Y, Z) = 0 \). Hence \( h(Y, \nabla_X Z) = 0 \), which implies that \( \nabla_X Z \in \Gamma(T(\mathcal{F}^\perp)) \). Thus \( T(\mathcal{F}^\perp) \) is also parallel with respect to
∇ and therefore integrable. In an analogous manner as declared above, for any $X, Y \in \Gamma(T(F^\perp_T))$ we have $\nabla_X Y \in \Gamma(T(F^\perp_T))$. As a result, according to (40) it follows that the second fundamental form $\tilde{H}$ of $F^\perp_T$ vanishes identically on $TM$. Hereupon, $F^\perp_T$ is likewise totally geodesic. Ultimately, we obtain that $[g_{11}]$ and $[g_{\alpha\beta}]$ represent the matrices of two adapted transverse metrics on $N^T$ and $N^\perp_T$, respectively. Consequently, it is illustrated that both of the foliations constructed above are holonomy invariant and the proof completes.

4. Conclusion. This paper is mainly dedicated to thorough investigation of conformal deformations of a particular class of four dimensional Lorentzian manifolds with indecomposable, non-irreducible holonomy representation which admit a recurrent light-like vector field. The holonomy algebra of this type of Lorentzian manifolds is contained in the parabolic algebra $(\mathbb{R} \oplus \text{so}(2)) \ltimes \mathbb{R}^2$ and they are denoted by pr-waves. Furthermore, it is demonstrated that by inducing some particular geometric conditions on an arbitrary four-dimensional pr-wave manifold, it behaves totally analogous to Einstein manifolds under the influence of a conformal diffeomorphism. Particularly, it is illustrated that in some special circumstances the structure of a pr-wave manifold is precisely the same as a manifold equipped with a warped product metric. Moreover, we have proved the existence of two complementary orthogonal totally geodesic and holonomy invariant foliations on the tangent bundle of a compact four-dimensional pr-wave manifold of negative sectional curvature.

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