Decay of the pseudoscalar glueball into (axial-)vector and (pseudo)scalar mesons

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We resume the investigation of the ground-state pseudoscalar glueball, $J^{PC} = 0^{-+}$, by computing its two- and three-body decays into vector and axial-vector quark-antiquark meson fields additional to scalar and pseudoscalar mesons through the construction of an interaction chiral Lagrangian that produces these decays. We evaluate the branching ratio, with a parameter-free prediction, by setting the mass of the pseudoscalar glueball to 2.6 GeV as predicted by lattice QCD simulations. We duplicate the computation for the branching ratios for a pseudoscalar glueball mass 2.37 GeV which matches to the resonance $X(2370)$ mass in the BESIII experiment suppress measuring. We obtain that the decay mode $\rho \pi$ is the largest. Moreover, the decay into $\rho \pi$ and $K^+\eta K$ are sizable. The present channels and states are potentially reached and are interesting for the running BESIII experiment and the planned PANDA experiment at FAIR/GSI which will be able to detect the pseudoscalar glueball within the accessible energy range.

I. INTRODUCTION

Glueballs, the composite particles containing gluons without valence quarks, are predicted by the confinement properties of Quantum chromodynamics (QCD) $^1$ and the non-Abelian nature $^2$ of the $SU(3)_c$ colour symmetry by virtue of the gauge fields of QCD, the gluons, self-interaction and strong vacuum fluctuations. A scalar glueball, $J^{PC} = 0^{++}$, is the ground state and its mass range is estimated to be from 1000 to 1800 MeV. That followed by a pseudoscalar glueball, $J^{PC} = 0^{-+}$, at higher mass. Up to now, glueballs remain experimentally undiscovered $^3$ because of their mixing with ordinary meson states, and no meson is listed by the Particle Data Group (PDG) $^4$ to be unambiguously of predominant glueball nature. Therefore, the search for glueballs witnessed the extensive and intensive investigations by both theoretical and experimentally studies $^5$–$^{14}$. Actually, ten scalars including the isoscalars $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ are observed. Whereas theoretical studies come up with classification the resonance $f_0(1710)$ to be predominantly a scalar glueball that productivity to the numerical approach of the lattice QCD $^{15}$, $^{16}$ and effective approaches $^3$, $^7$–$^{12}$. Generally, theoretical studies of glueballs are performed by nonperturbative approaches, the flux tube model $^{19}$, constituent models $^{20}$, the holographic approach $^{21}$, effective chiral model $^7$–$^9$, $^{11}$, $^{12}$ and the Lattice QCD simulation $^{15}$, $^{16}$ which reaches to the spectrum of glueball states below 5 GeV. Whereas, the Lattice QCD plays an important role in the investigation of the low energy strong interaction phenomena. In the quenched approximation, Lattice QCD simulation computed the masses of glueballs $^3$, $^{15}$. For example, it predicts the mass of the ground-state pseudoscalar glueball around 2.6 GeV and the mass of the first excited state of the pseudoscalar glueball around 3.7 GeV. On the other hand, the production rates of glueballs in $J/\psi$ radiative decays an additional role in determining the glueballs owing to the gluon-rich environment. In the quenched approximation, the pure gauge glueballs are well-defined hadron states. Therefore, the electromagnetic form factors of $J/\psi$ radiatively decaying into glueballs are directly extracted from the calculation of the matrix elements of the electromagnetic current between glueballs and $J/\psi$. The BesIII collaboration studied the process $J/\psi \rightarrow \eta \gamma \pi^+\pi^-$ and observed in the complicated of the $\eta'\pi^+\pi^-$ decay the resonance $X(2370)$ with quantum number $J^{PC} = 0^{-+}$ $^{22}$. This resonance has the same quantum number of a pseudoscalar glueball and lies in the mass range of the pseudoscalar glueball in lattice QCD prediction. That leads us to suggest the correspondence of the pseudoscalar glueball also to the $X(2370)$ in our work previously published in Ref. $^7$ and therein.

The present study of the decay properties of the ground state pseudoscalar glueball (denoted as $\tilde{G}$) is based on the chiral symmetric model of low-energy QCD called the extended Linear Sigma Model (eLSM) $^{23}$. It contains all quark-antiquark mesons with (pseudo)scalar and (axial)vector as well as a scalar and a pseudoscalar glueball and implements the symmetries of the QCD and their breaking. The eLSM is interesting for the study the hadron phenomenology. It has been already succeeded to study the vacuum properties of glueballs $^7$, $^8$, $^{11}$, $^{12}$, $^{24}$, baryons $^{23}$, $^{26}$, excited mesons $^{23}$, hybrids $^{28}$, light mesons $^{23}$, $^{29}$ and surprised to has been studied successfully the phenomenology of open and hidden charmed mesons $^{30}$, $^{31}$.

We have studied the decay of the ground-state pseudoscalar glueball into scalar and pseudoscalar mesons $^7$, and into nucleons $^8$, as well as into the excited scalar and pseudoscalar meson fields, see Ref. $^{12}$. Moreover, the decay properties of the first excited pseudoscalar glueball have also been studied in Refs. $^{11}$, $^{12}$. These efforts on the
pseudoscalar glueball and its first excitation properties are important in the comprehension of the non-perturbative behaviour of QCD. Based on that, we are interested to continue in our investigations on the properties of glueballs.

In this paper, we used both masses \( M_G = 2.37 \) GeV relevant to BESIII experiment and \( M_G = 2.6 \) GeV predicted by the Lattice QCD prediction, to calculate the decay widths of the pseudoscalar glueball in the framework of the constructed effective model so to connect the pseudoscalar glueball, \( gg \), to \( gg \) vector and axial-vector mesons in addition to scalar and pseudoscalar mesons. This work is a further step in our investigations of the pseudoscalar glueball and predicts new decay channels including vector and axial-vector mesons. We can thus compute the widths for the decays \( G \to PS, G \to PV, G \to PPP, G \to PPA, G \to PPV, \) and \( G \to PSV, \) where \( P, S, V, \) and \( A \) stand for pseudoscalar, scalar, vector, and axial-vector quark-antiquark states, respectively. The pseudoscalar field \( P \) correspond to the well-known light mesons \( \{ \pi, K, \eta, \eta' \} \) and the scalar \( S \) correspond to the scalars above 1 GeV: \( \{ a_0(1450), K_0^*(1430), f_0(1370), f_0(1500) \} \), while the vector state \( V \) refers to \( \{ \rho(770), K^*(892), \omega, \phi \} \) and the axial-vector \( A \) refers to \( \{ a_1(1260), f_0(1285), f_1(1420), K_1(1200) \} \). The results presented as branching ratios to disregard the unknown coupling constant. We find that the largest decay mode is into \( \rho \tau \) following by \( K^*(892)K \). Both of them are sizable and could explain the \( \rho \tau \) puzzle of the charmonium state \( \psi(2S) \). The present results confirm some channels which are already predicted earlier in Ref. 7 where the decays of the pseudoscalar glueball into \( KK\eta \), and \( K \eta \) were calculated. That is particularly interesting. The present investigation for the two- and three-body decays of the ground-state pseudoscalar is a useful guideline for both the running BESIII/(Beijing, China) experiment and the planned PANDA experiment at the FAIR/(GSI, Germany) 32 which able to measure the proposed channels.

The present paper is organized as follows. In Sec. II we introduce the constructed effective Lagrangian which describes the two- and three-body decays of the ground-state pseudoscalar glueball into vector, axial-vector, scalar and pseudoscalar quark-antiquark degrees of freedom, allowing for the branching ratios prediction for the decays. Finally, in Sec. III we present the conclusions.

II. THE EFFECTIVE LAGRANGIAN

We consider a \( U(3)_R \times U(3)_L \) chiral Lagrangian, by following Ref. 33, 34, which couples the ground-state pseudoscalar glueball \( \tilde{G} \equiv \mid gg \rangle \) with quantum numbers \( J^{PC} = 0^{++} \) to the quark-antiquark vector, axial-vector, scalar and pseudoscalar field mesons

\[
\mathcal{L}^{int}_{G \Phi LR} = i c_{G \Phi LR} \tilde{G} \text{Tr} \left[ L_\mu \left( \partial^\mu \Phi \cdot \Phi \right) - \Phi \cdot \partial^\mu \Phi \right] + R_\mu \left( \partial^\mu \Phi \cdot \Phi \right),
\]

where \( c_{G \Phi LR} \) is a dimensionless and unknown coupling constant. The field \( \Phi \) is the \( 3 \times 3 \) scalar and pseudoscalar multiplet \( 25 \)

\[
\Phi = (S^a + iP^a)t^a = \frac{1}{\sqrt{2}} \begin{pmatrix} (s_N + a_0^T + i(s_N + a_0^T)) & a_0^+ + iN^+ & K_0^+ + iK^+ \\ a_0^+ + iN^- & (s_N - a_0^T + i(s_N - a_0^T)) & K_0^- + iK^- \\ K_S^+ + iK^- & K_S^- + iK^0 & \sigma_S \end{pmatrix},
\]

where \( t^a \) are the generators of the group \( U(N_f) \). The multiplet \( \Phi \) transforms under \( U_L(3) \times U_R(3) \) chiral transformations as \( \Phi \to U_L \Phi U_R \), whereas \( U_{L(R)} = e^{-i\theta_{L(R)}^T} \) are \( U(3)_{L(R)} \) matrices, and under the charge conjugation \( C \) as \( \Phi \to \Phi^T \) as well as under the parity \( P \) as \( \Phi(t, \vec{x}) \to \Phi^\dagger(t, \vec{x}) \). Moreover, the left- and right-handed \( L_\mu \) and \( R_\mu \) are multiplets containing the vector and axia-vector field mesons \( V^a \) and \( A^a \)

\[
L_\mu = (V^a + iA^a)_\mu t^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega^0_0 + \rho_0^0 & \frac{\omega^+ - \rho^+}{\sqrt{2}} & \frac{\omega^0_0 - \rho^0}{\sqrt{2}} \\ \frac{\omega^- + \rho^-}{\sqrt{2}} & \omega^- - \rho^- & \frac{\omega^- + \rho^-}{\sqrt{2}} \\ K^- + K^0 & K^- - K^0 & \omega + f_1 S \end{pmatrix}_\mu,
\]

and

\[
R_\mu = (V^a - iA^a)_\mu t^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega^0_0 - \rho^0}{\sqrt{2}} & \frac{\omega^+ + \rho^+}{\sqrt{2}} & \frac{\omega^- + \rho^-}{\sqrt{2}} \\ \frac{\omega^- + \rho^-}{\sqrt{2}} & \frac{\omega^0_0 + \rho^0}{\sqrt{2}} & \frac{\omega^- + \rho^-}{\sqrt{2}} \\ K^- - K^0 & K^- + K^0 & \omega - f_1 S \end{pmatrix}_\mu.
\]
Under $U_L(3) \times U_R(3)$ chiral transformations, the multiplets $L_\mu$ and $R_\mu$ transform as $L_\mu \rightarrow U_L L_\mu U L^*_R$ and $R_\mu \rightarrow U_R L_\mu U R^*_L$, respectively. Furthermore, the pseudoscalar glueball field $\tilde{G}$ is chirally invariant and transforms under charge conjugation as $\tilde{G} \rightarrow \tilde{G}$ and under the parity $P$ as $\tilde{G}(t, \vec{x}) \rightarrow -\tilde{G}(t, -\vec{x})$. Consequently, these transformation properties of the multiplets $\Phi$, $L_\mu, R_\mu$ and the pseudoscalar glueball $\tilde{G}$ have been used to construct the chirally invariant Lagrangian and the extended Linear Sigma Model (eLSM), see Appendix A and Ref. 23 as well.

The identification of the quark-antiquark fields in the model with the physical resonances presented in details in Ref. 23, is straightforward in the light (pseudo)scalar and (axial-)vector states with mass $\lesssim 2$. The Pseudoscalar sector $P^a$ includes the pion isorep $\pi^*$, the kaon isodoublet $K^0$, and the isoscalar fields $\eta_\pi \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$ and $\eta_S \equiv (\bar{s}s)$ which represent the non-strange and strangeness mixing components of the physical states $\eta$ and $\eta'$ [4] with mixing angle $\varphi \approx -44.6^\circ$.

\[
\eta = \eta_N \cos \varphi + \eta_S \sin \varphi, \quad \eta' = -\eta_N \sin \varphi + \eta_S \cos \varphi.
\]

The scalar sector $S^a$ contains the isorep $\bar{u}d$ which refers the physical resonance $a_0(1450)$ and the kaon field $K_S$ which is assigned to the physical isodoublet state $K^0(1430)$. In the scalar-isoscalar sector, the non-strange bare field $\sigma_N$, the bare strange field $\sigma_S$ and the scalar glueball $G$ mix and generate the three physical resonances $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ through the following mixing matrix as constructed in Ref. [9]:

\[
\begin{pmatrix}
  f_0(1370) \\
  f_0(1500) \\
  f_0(1710)
\end{pmatrix} = \begin{pmatrix}
  -0.91 & 0.24 & -0.33 \\
  0.30 & 0.94 & -0.17 \\
  -0.27 & 0.26 & 0.94
\end{pmatrix} \begin{pmatrix}
  \sigma_N \\
  \sigma_S \\
  G
\end{pmatrix}.
\]

However, the scalar-isoscalar fields, $f_0(1370)$, $f_0(1500)$, and $f_0(1700)$ are predominantly described by the bare configuration $\bar{u}u + \bar{d}d/\sqrt{2}$ state, $\bar{s}s$ states and a glueball $g$ state, respectively. We now turn to the assignment of the (axial-)vector states. The vector sector $V^a$ contains the iso-triplet $\vec{K}$, the kaon states $K^*$, and the isoscalar states $\omega_N$ and $\omega_S$ which are assigned to the $p(770), K^*(892), \omega$ and $\phi$ mesons, respectively 23. Note that the mixing between the strange and nonstrange isoscalars is vanishing in the extended linear sigma model eLSM 23, whereas this mixing is so small as obtained in Ref. 23. In the end, for the axial-vector sector $S^a$, the isorep $\vec{a}_1$, the isoscalar fields $f_{1N}$ and $f_{1S}$ correspond to the resonances $a_1(1260)$, $f_1(1285)$ and $f_1(1420)$ respectively. However, the four kaon states $K_1$ refer predominantly to the resonance $K_1(1200)$ and could also refer to $K_1(1400)$ because of the mixing between the pseudovector states and axial-vector states.

For computing the decays of the pseudoscalar glueball, we shift the scalar-isoscalar fields by their vacuum expectation values $\phi_N$ and $\phi_S$ to implement the effectiveness of the spontaneous symmetry breaking which takes place 23

\[
\sigma_N \rightarrow \sigma_N + \phi_N \quad \text{and} \quad \sigma_S \rightarrow \sigma_S + \phi_S.
\]

However, the (axial-)vector mesons are presented in the Lagrangian. Therefore, we shift the axial-vector fields that redefine the wave-function renormalization constants of the pseudoscalar fields

\[
\pi^* \rightarrow Z_{\pi} \pi^*, \quad K^i \rightarrow Z_K K^i, \quad \eta_j \rightarrow Z_{\eta_j} \eta_j,
\]

whereas $i = 1, 2, 3, 4$ indicates the four kaonic fields and $j$ refers to $N$ and $S$. The numerical values of the renormalization constants of the corresponding wave functions are listed in Ref. 23 as $Z_{\pi} = 1.709$, $Z_K = 1.604$, $Z_{KS} = 1.001$, $Z_{\eta_N} = Z_{\pi}$, $Z_{\eta_S} = 1.539$. The corresponding chiral condensates $\phi_N$ and $\phi_S$ read

\[
\phi_N = Z_{\pi} f_{\pi} = 0.158 \text{ GeV}, \quad \phi_S = \frac{2Z_K f_K - \phi_N}{\sqrt{2}} = 0.138 \text{ GeV},
\]

where the decay constant of the pion is $f_{\pi} = 0.0922$ GeV and the kaon is $f_K = 0.110$ GeV [4]. After performing the shift operations [1] and [3] in the Lagrangian, we obtain the relevant tree-level vertices for the decay processes of pseudoscalar glueball $\tilde{G}$.

The branching ratios of the pseudoscalar glueball $\tilde{G}$ for the decays into two-body $PS$ and $PV$ are reported in Table I for both choices of the pseudoscalar glueball masses, 2.6 GeV relevant for PANDA experiment at FAIR/GSI and 2.37 GeV related to BESIII experiment. The results are presented relative to the total decay width of the pseudoscalar glueball $\Gamma_{\tilde{G}}^{tot}$ to eliminate the unknown coupling constant. (For details of the two-body decay calculation, see Sec. A.3 of the Appendix.)
In Table II, the branching ratios of the pseudoscalar glueball $\tilde{G}$ for three-body decays are presented for both choices of $M_{\tilde{G}} = 2.6$ GeV and $M_{\tilde{G}} = 2.37$ GeV. That include the decay processes $\tilde{G} \rightarrow PPP$, $\tilde{G} \rightarrow PPA$, $\tilde{G} \rightarrow PPV$ and $\tilde{G} \rightarrow PSV$. The pseudoscalar glueball is able to decay into scalar-isoscalar states $f_0(1370)$ and $f_0(1500)$ within including the full mixing pattern above 1 GeV. (For details of the three-body decay calculation, see Sec. A 3 of the Appendix.)

| Quantity               | Case (i): $M_{\tilde{G}} = 2.6$ GeV | Case (ii): $M_{\tilde{G}} = 2.37$ GeV |
|------------------------|----------------------------------|--------------------------------------|
| $\Gamma_{\tilde{G} \rightarrow KK^+}/\Gamma_{\tilde{G}}^{tot}$ | 0.00014                          | 0.000063                              |
| $\Gamma_{\tilde{G} \rightarrow K_S^0}/\Gamma_{\tilde{G}}^{tot}$ | 0.008                            | 0.04                                   |
| $\Gamma_{\tilde{G} \rightarrow K_{S0}^*}/\Gamma_{\tilde{G}}^{tot}$ | 0.0019                           | 0.0009                                 |
| $\Gamma_{\tilde{G} \rightarrow K_0^+}/\Gamma_{\tilde{G}}^{tot}$ | 0.0149                           | 0.0059                                 |
| $\Gamma_{\tilde{G} \rightarrow K_{S0}^*}/\Gamma_{\tilde{G}}^{tot}$ | 0.0066                           | 0.0026                                 |
| $\Gamma_{\tilde{G} \rightarrow K_N}/\Gamma_{\tilde{G}}^{tot}$ | 0.0022                           | 0.0009                                 |
| $\Gamma_{\tilde{G} \rightarrow K_{S0}^*}/\Gamma_{\tilde{G}}^{tot}$ | 0.0014                           | 0.0056                                 |
| $\Gamma_{\tilde{G} \rightarrow K_0^+}/\Gamma_{\tilde{G}}^{tot}$ | 0.0025                           | 0.0002                                 |
| $\Gamma_{\tilde{G} \rightarrow K_{S0}^*}/\Gamma_{\tilde{G}}^{tot}$ | 0.0075                           | 0                                      |
| $\Gamma_{\tilde{G} \rightarrow f_0(1370)}/\Gamma_{\tilde{G}}^{tot}$ | 0.0093                           | 0.0038                                 |
| $\Gamma_{\tilde{G} \rightarrow f_0(1500)}/\Gamma_{\tilde{G}}^{tot}$ | 0.0011                           | 0                                      |
| $\Gamma_{\tilde{G} \rightarrow K_{S0}^*}/\Gamma_{\tilde{G}}^{tot}$ | 0.0112                           | 0                                      |

TABLE II: Branching ratios for the decay of the pseudoscalar glueball $\tilde{G}$ into a scalar, a pseudoscalar, a vector and an axial-vector meson.

Tables I and II present the results which depend only slightly on the glueball mass, which explains the similarity of there two columns. The channel $\pi \rho$ is the dominant one and the channel $KK^*$ is sizable. Generally, the presented three-body decays are subdominant. From the investigation of the decay of the pseudoscalar glueball in the present work and our previous papers [8], we found that the decay of the pseudoscalar glueball into three pions vanishes in our models

$$\Gamma_{\tilde{G} \rightarrow \pi \pi \pi} = 0,$$

which lead us to make addional testable predictions of our approach. Notice that in both our investigation of the decay of the pseudoscalar glueball were calculated, which presented in Ref. [8] and herein, together with the decay channels $\tilde{G} \rightarrow KK_S$, $\tilde{G} \rightarrow K_{S0}^*$, $\tilde{G} \rightarrow KK_{\eta'}$, and $\tilde{G} \rightarrow KK \pi$. This confirms the suggestion for experimentalists to search of glueball through these channels. Moreover, the two-body decay channel $\tilde{G} \rightarrow KK_S$ can proceed through a sequential instance, $K_0^*(1430) \rightarrow K \pi$, leading to the three-body decay $\tilde{G} \rightarrow KK \pi$. So to obtain the total three-body decay width for $\tilde{G} \rightarrow KK \pi$, the two- and three-body decay amplitudes of this channel have to be added coherently before taking the modulus square.

Fig. 1 shows the total decay action line of the pseudoscalar glueball, $\Gamma_{\tilde{G}}^{tot} = \Gamma_{\tilde{G} \rightarrow PS} + \Gamma_{\tilde{G} \rightarrow PV} + \Gamma_{\tilde{G} \rightarrow PPP} + \Gamma_{\tilde{G} \rightarrow PPA} + \Gamma_{\tilde{G} \rightarrow PPV} + \Gamma_{\tilde{G} \rightarrow PSV}$, as function of the coupling constant $c_{\tilde{G} \Phi LR}$ for both masses suggested in the present work, where the decay into baryons is negligible.
FIG. 1: Solid (orange) line: Total decay width of the pseudoscalar glueball with the bare mass $M_G = 2.6$ GeV as function of the coupling $c_{G_{\Phi LR}}$. Dashed (blue) line: the same curve for $M_G = 2.37$ GeV.

III. CONCLUSION

The two- and three-body decays of the ground state of the pseudoscalar glueball into a vector, an axial-vector, a scalar and a pseudoscalar quark-antiquark fields have been studied. We have started with the chiral invariant effective Lagrangian describing the interaction of the pseudoscalar glueball with (axial-)vector and (pseudo)scalar mesons for the three-flavour case $N_f = 3$. The size of the coupling constant intensity can not be determined. That leads to predict the branching ratios for the decay channels which are expected to dominate. According to the mass of the pseudoscalar glueball, we considered two options: (i) $M_G = 2.6$ GeV which is chosen to be in agreement with lattice QCD in the quenched approximation. In the planned PANDA experiment at FAIR/GSI [32], one can test the existence and decay properties of the pseudoscalar glueball resonances since glueballs could be directly formed in proton-antiproton fusion processes. (ii) $M_G = 2.37$ GeV which assumes that the measured resonance $X(2370)$ in the BESIII experiment is a pseudoscalar glueball (predominantly) [22]. The two-body decay of the pseudoscalar glueball produce $PS$ and $PV$ while the three-body decay produce $PPP, PPA, PPV$ and $PSV$ which includes the scalar-isoscalar states $f_0(1370)$ and $f_0(1500)$. The $M_G \rightarrow KK\pi$ channel is predicted in the dominant decay channel. The decay into three pions vanishes similarly to our previous study of the pseudoscalar glueball decays as seen in Ref. [7]. The decay channels $KK_S, KK_\eta, KK_\eta'$ and $KK\pi$ are predicted in this study, which confirm our previous study [7]. The present results of this work could be helpful and guideline for searching for the pseudoscalar glueball in the ongoing BESIII experiment and the future PANDA experiment at the FAIR/GSI.

In the framework of a chiral model, we progress the decay of the first excited pseudoscalar glueball into vector and axialvector mesons in addition to scalar and pseudoscalar mesons, and into nucleons [37] as well. Forcoming developments of the present work will be based on new results for the pseudoscalar glueball and its excitations when Lattice QCD will include the effect of dynamic fermions and works beyond the quenched approximation. That would be very beneficial for our model.
Appendix A: Details of the calculation

1. The full mesonic Lagrangian

The $U(N_f)_L \times U(N_f)_R$ linear sigma model with (axial-)vector and (pseudo)scalar quarkonia, a scalar glueball $G$ and a pseudoscalar glueball $\tilde{G}$ is given by

\[
\mathcal{L} = \mathcal{L}_{dil} + \text{Tr}[(D^\mu \Phi)\dagger(D^\mu \Phi)] - m_0^2 \left( \frac{G}{G_\theta} \right)^2 \text{Tr}(\Phi^\dagger \Phi) - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2
\]

\[
+ \text{Tr} \left\{ \left( \frac{G}{G_\theta} \right)^2 \frac{m_1^2}{2} + \Delta \right\} (L^\mu)^2 + (R^\mu)^2 \right] - \frac{1}{4} \text{Tr}[(L^\mu)^2 + (R^\mu)^2] - 2 \text{Tr}[\Phi^\dagger \Phi]
\]

\[
+ \text{Tr}[H(\Phi + \Phi^\dagger)] + c(\det \Phi - \det \Phi^\dagger)^2 + i \tilde{G} (\det \Phi - \det \Phi^\dagger) + \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi)\text{Tr}[(L^\mu)^2 + (R^\mu)^2]
\]

\[
+ h_2 \text{Tr}[(\Phi R^\mu)^2 + (L^\mu \Phi)^2] + 2h_3 \text{Tr}(\Phi R_\mu \Phi^\dagger L^\mu) + i \frac{g_2}{2} \{ \text{Tr}(L_{\mu
u}[L^\mu, L^\nu]) + \text{Tr}(R_{\mu
u}[R^\mu, R^\nu]) \} + \ldots,
\]

(A1)

where $D^\mu \Phi \equiv \partial^\mu \Phi - ig_0 (L^\mu \Phi - \Phi R^\mu)$ is the covariant derivative; and $L^{\mu\nu} = \partial^{\mu}L^{\nu} - \partial^{\nu}L^{\mu}$, and $R^{\mu\nu} = \partial^{\mu}R^{\nu} - \partial^{\nu}R^{\mu}$ refers to the left-handed and right-handed field strength tensors. The dilation Lagrangian

\[
\mathcal{L}_{dil} = \frac{1}{2} (\partial^\mu G)^2 - \frac{1}{4} \frac{m_0^2}{G_\theta^2} \left( G^4 \log \frac{G}{\Lambda} - \frac{G^4}{4} \right),
\]

(A2)

describes a scalar glueball $G \equiv |gg\rangle$ with quantum number $J^{PC} = 0^{++}$. For more details see Refs.\cite{22,33,38,39}.

The constants entering into the present paper decay expressions $Z_i$ and $w_i$ \cite{23}:

\[
Z_\pi = Z_{\eta N} = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2 \phi_N^2}}, \quad Z_K = \frac{2m_{K_1}}{\sqrt{4m_{K_1}^2 - g_1^2 (\phi_N + \sqrt{2} \phi_S)^2}}, \quad Z_{K_S} = \frac{2m_{K^*}}{\sqrt{4m_{K^*}^2 - g_1^2 (\phi_N - \sqrt{2} \phi_S)^2}}, \quad Z_{\eta_S} = \frac{m_{f_1}}{\sqrt{m_{f_1}^2 - 2g_1^2 \phi_S^2}},
\]

(A3)

and

\[
w_{f_1 N} = w_{a_1} = \frac{g_1 \phi_N}{m_{a_1}^2}, \quad w_{f_1 S} = \frac{\sqrt{2} g_1 \phi_S}{m_{f_1}^2},
\]

(A4)

\[
w_{K^*} = \frac{ig_1 (\phi_N - \sqrt{2} \phi_S)}{2m_{K^*}^2}, \quad w_{K_1} = \frac{g_1 (\phi_N + \sqrt{2} \phi_S)}{2m_{K_1}^2}.
\]

(A5)

2. Two-body decay

The single two-body decay channel which processes through a scalar and a pseudoscalar meson is $\tilde{G} \to KK_S$. The decay width follows from Eq. (11) as

\[
\Gamma_{\tilde{G} \to KK_S} = \frac{f_{\tilde{G} \to KK_S}}{8\pi M_{\tilde{G}}} | - i M_{\tilde{G} \to KK_S} |^2,
\]

(A7)

where $f_{\tilde{G} \to KK_S}$ is the isospin factor, $M_{\tilde{G}}$ is the pseudoscalar glueball mass, and

\[
| - i M_{\tilde{G} \to KK_S} |^2 = \frac{1}{4} c_{\tilde{G} \Phi LR}^2 \left[ \Phi_N (w_{K^*} Z_{K_S} + iw_{K_1} Z_K) + \sqrt{2} \Phi_S (w_{K^*} Z_{K_S} + iw_{K_1} Z_K) \right]^2 \times \left[ \frac{1}{2} (M_{\tilde{G}}^2 - m_{K^*}^2 - m_{K_S}^2) \right]^2
\]

(A8)
where $m_K$ and $m_{K_S}$ are the masses of the kaon and $K_S$ mesons, respectively, while $k_{GKK_S}$ is the center-of-mass momentum of kaon and $K_S$ and reads

$$k_{GKK_S} = \frac{1}{2M_G} \sqrt{M_G^4 + (m_K^2 - m_{K_S}^2)^2 - 2m_K^2 (m_K^2 + m_{K_S}^2)}.$$  

(A9)

Next we turn to the decay of $\tilde{G} \to \pi\rho$ is given from Eq. (1) as

$$\Gamma_{\tilde{G} \to \pi\rho} = \frac{f_{\tilde{G} \to \pi\rho} k_{\tilde{G}\pi\rho}}{8\pi M_G^2} \frac{1}{3} |c_{\tilde{G}4\Phi LR}^N|^2 \left[ -m_\pi^2 + \frac{M_G^2 - m_\pi^2 - m_\rho^2}{2m_\rho^2} \right]^2,$$  

(A10)

where $f_{\tilde{G} \to \pi\rho} = 3$ and $k_{\tilde{G}\pi\rho}$ is the central-of-mass momentum of pion and $\rho$. In an analogous way, all the decay processes $\tilde{G} \to PV$ in table I are calculated the corresponding change of the isospin factors, the decay products mass and the constants entering in the amplitudes.

3. Three-body decay

The general explicit expression for the three-body decay width for the process $\tilde{G} \to P_1 P_2 P_3$ is:

$$\Gamma_{\tilde{G} \to P_1 P_2 P_3} = \frac{f_{\tilde{G} \to P_1 P_2 P_3}}{32(2\pi)^3 M_G^4} \int_{(m_1 + m_2)^2}^{(M_{\tilde{G}} - m_3)^2} \int_{(m_{23})_{\text{min}}}^{(m_{23})_{\text{max}}} | -iM_{\tilde{G} \to P_1 P_2 P_3} |^2 \text{d}m_{23}^2$$

where

$$(m_{23})_{\text{min}} = (E_2^* + E_3^*)^2 - \left( \sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2} \right)^2,$$

(A11)

$$(m_{23})_{\text{max}} = (E_2^* + E_3^*)^2 - \left( \sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2} \right)^2,$$

(A12)

and

$$E_2^* = \frac{m_{12}^2 - m_1^2 + m_2^2}{2m_{12}}, \quad E_3^* = \frac{M_G^2 - m_{12}^2 - m_3^2}{2m_{12}}.$$  

(A13)

The quantities $m_1, m_2, m_3$ are the masses of the three decay products $P_1, P_2,$ and $P_3$. $M_{\tilde{G} \to P_1 P_2 P_3}$ is the corresponding tree-level decay amplitude, and $f_{\tilde{G} \to P_1 P_2 P_3}$ is a symmetrization factor which equals 1 if all decay products are different, equals 2 for two identical decay products in the final state, and it equals 6 if $P_1, P_2,$ and $P_3,$ are identical in the final state.

For example, the amplitude for the process $\tilde{G} \to K^{*0} K_S^{0} \pi^0$ is

$$| -iM_{\tilde{G} \to K^{*0} K_S^{0} \pi^0} |^2 = \frac{c_{\tilde{G}4\Phi LR}^2}{4} \left[ - (Z\pi^0 m_{K_S}^2 + 2Z\pi Z_{K_S} k_{\pi K_S} + m_\pi^2 Z_{K_S}^2) + \frac{1}{m_{K_S}^2} (Z\pi k_{\pi K_S} k_{K_S} + Z_{K_S} k_{K_S} k_{\pi K_S}) \right]^2,$$

(A14)

where

$$k_{\pi K_S} = \frac{m_{12}^2 - m_1^2 - m_{K_S}^2}{2},$$

(A15)

$$k_{\pi} = \frac{m_{23}^2 - m_\pi^2 - m_{K_S}^2}{2},$$

(A16)

$$k_{K_S} = \frac{m_{13}^2 - m_1^2 - m_{K_S}^2}{2}.$$  

(A17)

The other three-body decays are calculated in analogous way.
Notice that, there are several decay channels of the pseudoscalar glueball, \( \tilde{G} \) that appear in Eq. (1) but they are not kinematically allowed because the mass of the decaying particle is lower than the sum of the mass of the decay products \( M < \sum_i m_i \).

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