Blade Products and Angles Between Subspaces

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Communicated by Leo Dorst

Abstract. Principal angles are used to define an angle bivector of subspaces, which fully describes their relative inclination. Its exponential is related to the Clifford geometric product of blades, gives rotors connecting subspaces via minimal geodesics in Grassmannians, and decomposes giving Plücker coordinates, projection factors and angles with various subspaces. This leads to new geometric interpretations for this product and its properties, and to formulas relating other blade products (scalar, inner, outer, etc., including those of Grassmann algebra) to angles between subspaces. Contractions are linked to an asymmetric angle, while commutators and anticommutators involve hyperbolic functions of the angle bivector, shedding new light on their properties.

Mathematics Subject Classification. 15A66, 15A75.

Keywords. Clifford algebra, Geometric algebra, Grassmann algebra, Blade product, Angle between subspaces, Angle bivector, Asymmetric angle.

1. Introduction

Much of the usefulness of dot and cross products of vectors comes from the formulas relating them to angles. Clifford geometric algebra has similar results for the scalar product and contraction of blades [6], but the relation between other products and angles has been mostly ignored. An exception is Hitzer’s formula [19] relating the Clifford geometric product of blades to principal angles [23,35], but it is too complex to really help us understand the product, and its main purpose was to compute the angles.

A difficulty in relating blade products to angles is that blades can represent high dimensional subspaces, for which there are various angle concepts.
Measuring the separation of subspaces is useful in geometry, linear algebra, functional analysis, statistics, computer vision, data mining, etc., but a full description of their relative inclination requires a list of principal angles, which can be cumbersome. So, depending on the purpose, one usually takes the smallest, largest, or some function of principal angles describing whatever relation between the subspaces is most relevant [3, 8, 36]. This can lead to misunderstandings, as distinct concepts are often called the angle between subspaces, despite each having its own properties and limitations.

The geometric algebra literature has some (not fully equivalent) definitions for the angle between blades or subspaces [6, 18, 19], but it does not seem to be well understood, being usually described only in simple cases, by comparison with usual angles in $\mathbb{R}^3$ (sometimes erroneously, as we show). This angle is similar to the angles introduced in [10, 13, 22, 35] and which measure separation of subspaces in terms of how volumes contract when orthogonally projected between them (so, projection factors [29]).

These angles have been unified and generalized into an asymmetric angle of subspaces [28], which has better properties. Its unusual asymmetry for subspaces of distinct dimensions turns out to be an advantage, making its use more efficient and leading to more general results. This angle has been linked to the Grassmann algebra and the geometry of Grassmannians, and here we show it is also deeply connected with the Clifford algebra.

Instead of a scalar angle, Hawidi [14] proposed an angle operator carrying all data about the relative inclination of subspaces, but it was never widely adopted. Fortunately, geometric algebra has a better way to store such information: describing relative inclination is akin to telling how to rotate one subspace into another, which calls for the use of rotors and bivectors.

Using principal angles and vectors, we define angle bivectors whose exponentials give rotors connecting subspaces through minimal geodesics in Grassmannians. When properly decomposed, such exponentials give angles and projection factors for various subspaces, and also Plücker coordinates. Hitzer’s formula is turned into a simple relation between the Clifford product of blades and the exponential of the angle bivector.

Clifford algebra gives strong results with such ease that the geometry behind them can be missed just as easily. Our results provide new geometric interpretations for the Clifford product and some of its well known algebraic properties. For example, Plücker coordinates stored in the product allow the invertibility of non-null blades, the relation $\|AB\| = \|A\|\|B\|$ for blades is linked to a Pythagorean theorem for volumes [29], and the dualities [6] $(AB)^* = AB^*$, $(A \wedge B)^* = A|B|^*$ and $(A \| B)^* = A \wedge B^*$ reflect a symmetry in the exponentials of angle bivectors.

The formula relating the Clifford product to the angle bivector yields others for the various geometric algebra products of blades: Lounesto’s asymmetric contractions [25] are linked to the asymmetric angle, the outer product to a complementary angle, and the scalar product, Hestenes inner product and Dorst’s dot product [5] to symmetrized angles. Blade commutators and anticommutators are related to hyperbolic functions of the angle bivector. We also give formulas for Grassmann algebra products, which are used in
[28] to obtain formulas for computing asymmetric angles and identities that lead to properties of projection factors [29].

The symmetrized angle, related to Hestenes inner product, has worse properties than the asymmetric one, and this supports Dorst’s case for the use of contractions instead of that product. Their asymmetry is linked to that of asymmetric angles, and in both cases it leads to better results with simpler proofs.

Section 2 reviews some concepts and results. Section 3 introduces the Grassmann algebra inner product and studies its exponential. Section 4 relates the Clifford product to the angle bivector, and interprets geometrically some of its properties. Section 5 relates other geometric algebra products to angles. Appendix A does the same for Grassmann algebra products, and requires only Sect. 2. develops Appendix B some properties of hyperbolic functions of multivectors.

2. Preliminaries

In this article1, X is a n-dimensional Euclidean space. A p-subspace is a subspace of dimension p. For a multivector \( M \in \wedge X \), \( \langle M \rangle_p \in \wedge^p X \) denotes its component of grade p. We use \( \wedge^p X = \{0\} \) for \( p < 0 \).

A p-blade \((p = 1, 2, \ldots)\) is a simple multivector \( B = v_1 \wedge \cdots \wedge v_p \in \wedge^p X \), with \( v_1, \ldots, v_p \in X \). Its reversion is \( \tilde{B} = v_p \wedge \cdots \wedge v_1 = (-1)^{\frac{p(p-1)}{2}} B \), which extends linearly to an involution of \( \wedge X \). If \( B \neq 0 \), its p-subspace is \( [B] = \text{span}(v_1, \ldots, v_p) \), and \( \wedge^p [B] = \text{span}(B) \). A scalar \( B \in \wedge^0 X \) is a 0-blade, with \( [B] = \{0\} \). Since \( 0 \in \wedge^p X \) for all p, it is considered a blade of all grades [6, p. 45], and \( [0] = \{0\} \).

For \( M, N \in \wedge X \), the scalar product \([18] M \ast N = \langle MN \rangle_0 \) is related to the Grassmann algebra inner product by\(^2\) \( \tilde{M} \ast N = \langle MN \rangle \), and the norm is \( \|M\| = \sqrt{\tilde{M} \ast M} \). For \( B = v_1 \wedge \cdots \wedge v_p \), \( \|B\| = \sqrt{BB} = \sqrt{\det(v_i \cdot v_j)} \) gives\(^3\) the p-dimensional volume of the parallelotope spanned by \( v_1, \ldots, v_p \).

For subspaces \( V, W \subset X \), \( P_W: X \to W \) and \( P_V^V: V \to W \) are orthogonal projections. We also write \( P_B \) for \( P_{[B]} \), and extend \( P = P_W \) to an orthogonal projection \( P: \wedge X \to \wedge W \), with \( P(v_1 \wedge \cdots \wedge v_p) = Pv_1 \wedge \cdots \wedge Pv_p \).

A little more geometric algebra notation, for those who are mainly interested in Appendix A: for vectors \( v, w \in X \), the (dot) product \( v \cdot w \) equals the Grassmann algebra inner product \( \langle v, w \rangle \), and the Clifford geometric product (indicated by juxtaposed elements, with no product symbol between them) is \( vw = v \cdot w + v \wedge w \). When \( v_1, \ldots, v_k \in X \) are orthogonal, the geometric product \( v_1 v_2 \ldots v_k \) equals the exterior product \( v_1 \wedge v_2 \wedge \cdots \wedge v_k \).

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1Except for Appendix A which includes complex spaces. Any changes needed for the complex case are indicated in footnotes (only in Sect. 2).
2In the complex case, the \( M \) on the left is also conjugated, but as the same happens wherever we use this product, one can simply always replace \( \tilde{M} \ast N \) with \( \langle MN \rangle \).
3For complex blades, \( \|B\|^2 \) gives the 2p-dimensional volume of the parallelotope spanned by \( v_1, iv_1, \ldots, v_p, iv_p \).
2.1. Principal Angles and Vectors

In high dimensions, no single scalar angle can fully describe the relative inclination of subspaces. This requires a list of principal angles \([1,9–11,35]\), also called canonical or Jordan angles. These angles were introduced in 1875 by Jordan \([23]\), and have important applications in statistics (in Hotelling’s theory of canonical correlations \([21]\)), numerical analysis and other areas.

Definition 2.1. Let \(V, W \subset X\) be nonzero subspaces, \(p = \text{dim} V\), \(q = \text{dim} W\) and \(m = \min\{p, q\}\). Their principal angles are \(0 \leq \theta_1 \leq \cdots \leq \theta_m \leq \frac{\pi}{2}\), and orthonormal bases \(\beta_V = (e_1, \ldots, e_p)\) and \(\beta_W = (f_1, \ldots, f_q)\) are associated principal bases, formed by principal vectors, if, for all \(1 \leq i \leq p\) and \(1 \leq j \leq q\),

\[
e_i \cdot f_j = \delta_{ij} \cos \theta_i.
\]

(1)

We also say that \(\beta_W\) is a principal basis of \(W\) w.r.t. \(V\). Note that

\[
P_We_i = \begin{cases} 
  f_i \cos \theta_i & \text{if } i \leq m, \\
  0 & \text{if } i > m.
\end{cases}
\]

(2)

Principal bases and angles can be obtained via a singular value decomposition, in a method introduced by Björck and Golub \([2]\) (see also \([7,9,11]\)). The \(e_i\)’s and \(f_i\)’s are orthonormal eigenvectors of \(P_W^V P_W^V\) and \(P_W^V P_W^V\), respectively, and the \(\cos \theta_i\)’s are the square roots of the eigenvalues of \(P_W^V P_W^V\), if \(p \leq q\), or \(P_W^V P_W^V\) otherwise (Fig. 1).

A recursive characterization of principal angles is that, for \(1 \leq i \leq m\),

\[
\theta_i = \min\{\theta_{v,w} : v \in V\{0\}, w \in W\{0\}, v \cdot e_j = w \cdot f_j = 0 \forall j < i\},
\]

where \(\theta_{v,w}\) is the angle\(^4\) between \(v\) and \(w\), and unit vectors \(e_i \in V\) and \(f_i \in W\) are chosen so that (1) holds for all \(j \leq i\). A geometric interpretation is that the unit sphere of \(V\) projects orthogonally to an ellipsoid in \(W\) and, for \(i \leq m\), \(e_i\) projects onto a semi-axis of length \(\cos \theta_i\) along \(f_i\) (Fig. 1).

\(^4\)In complex spaces there are different concepts of angle between vectors \([33]\). For the present characterization, Euclidean or Hermitian angles give the same result \([9]\).
Note that the number of null principal angles, for which \( e_i = f_i \), equals \( \dim(V \cap W) \). Also, \( V \perp W \) if, and only if, all principal angles are \( \frac{\pi}{2} \).

Principal angles are uniquely determined, but principal bases are not. If \( \theta_i \) is not repeated then \( e_i \) and \( f_i \) are determined up to a sign, but if some \( \theta_i \)'s are equal then orthogonal transformations of the corresponding eigenspaces in \( V \) and \( W \) yield new principal bases. For \( i \leq m \) with \( \theta_i \neq \frac{\pi}{2} \), a choice of \( e_i \)'s or \( f_i \)'s with \( i > m \) can be chosen freely to complete an orthonormal basis.

2.2. Partial Orthogonality

Let \( V, W \subset X \) be subspaces. As usual, we write \( V \perp W \) if \( v \cdot w = 0 \) for all \( v \in V \) and \( w \in W \), and \( W^\perp \) is the orthogonal complement of \( W \). We will also need a weaker concept of orthogonality [28].

**Definition 2.2.** \( V \) is partially orthogonal to \( W \) (we write \( V \prec W \)) if there is a nonzero \( v \in V \) such that \( v \cdot w = 0 \) for all \( w \in W \).

**Proposition 2.3.** For any subspaces \( V, W \subset X \):

(i) \( V \prec W \) if \( \dim V > \dim W \) or a principal angle of \( V \) and \( W \) is \( \frac{\pi}{2} \).

(ii) \( V \prec W \) if \( \dim V > \dim P_W(V) \).

(iii) If \( \dim V = \dim W \) then \( V \perp W \) if \( V \perp W \).

*Proof.* (i) When \( \dim V \leq \dim W \), the largest principal angle equals the largest angle between a nonzero \( v \in V \) and \( W \). (ii) Immediate. (iii) Follows from item i. \( \square \)

Note that \( \prec \) is not a symmetric relation when \( \dim V \neq \dim W \): any plane is partially orthogonal to a line, but the converse is not true.

**Definition 2.4.** Let \( A, B \in \wedge X \) be blades.

(i) If \( [A] \prec [B] \) we say \( A \) is partially orthogonal to \( B \), and write \( A \perp B \).

(ii) If \( [A] \perp [B] \) we say \( A \) and \( B \) are completely orthogonal.

Orthogonality in the sense of \( A \ast B = 0 \) is weaker than (ii), and for nonzero blades of same grade it equals (i) (see Proposition 2.7 below).

2.3. Principal Decomposition and Relative Orientation

Let \( A \in \wedge^p X \) and \( B \in \wedge^q X \) be nonzero blades, and \( \beta_A = (e_1, \ldots, e_p) \) and \( \beta_B = (f_1, \ldots, f_q) \) be associated principal bases of \( [A] \) and \( [B] \).

**Definition 2.5.** A principal decomposition of \( A \) and \( B \) is

\[
A = \epsilon_A \|A\| e_1 e_2 \cdots e_p, \quad B = \epsilon_B \|B\| f_1 f_2 \cdots f_q,
\]

where \( \epsilon_A, \epsilon_B = \pm 1 \). We also define\(^5\) \( \epsilon_{A,B} = \epsilon_A \epsilon_B \), which we call relative orientation of \( A \) and \( B \) (w.r.t. \( \beta_A \) and \( \beta_B \)).

**Lemma 2.6.** If \( p = q \) then \( \hat{A} \ast B = \epsilon_{A,B} \|A\| \|B\| \prod_{i=1}^p \cos \theta_i \).

*Proof.* Follows from (1) and (3). \( \square \)

\(^5\)In complex spaces \( \epsilon_A \) and \( \epsilon_B \) are phase factors \( e^{i\varphi} \), and we define \( \epsilon_{A,B} = \hat{\epsilon}_A \epsilon_B \).
Proposition 2.7. For nonzero blades of same grade \( A, B \in \bigwedge^p X \) we have 
\( A \perp B \Leftrightarrow A \ast B = 0 \).

Proof. Follows from Propositions 2.6 and 2.3i. \( \square \)

If \( A \ast B \neq 0 \) then \( \epsilon_{A,B} = \frac{\tilde{A} \ast B}{|A \ast B|} \) is uniquely determined by the orientations of \( A \) and \( B \), being such that \( \epsilon_{A,B} P_B A \) has the same orientation as \( B \) (i.e., \( \epsilon_{A,B} P_B A = c B \) for some \( c > 0 \)). If \( A \ast B = 0 \) (distinct grades, or partially orthogonal blades), the orientations of \( A \) and \( B \) can not really be compared, and \( \epsilon_{A,B} \) becomes less meaningful, depending on the choice of principal bases, but in a way that makes it useful to track orientation changes in them.

2.4. Asymmetric Angle of Subspaces

The asymmetric angle\(^6\) \([28]\) of subspaces \( V, W \subset X \) measures their separation in terms of a projection factor \( \pi_{V,W} \) \([29]\), describing how volumes in \( V \) contract when orthogonally projected on \( W \).

Definition 2.8. Let\(^7\) \( p = \dim V, S \subset V \) be a \( p \)-dimensional parallelotope, and \( \text{vol}_p \) be the \( p \)-dimensional volume. The projection factor of \( V \) on \( W \) is
\[ \pi_{V,W} = \frac{\text{vol}_p P_W(S)}{\text{vol}_p S}. \]

Definition 2.9. The asymmetric angle \( \Theta_{V,W} \in [0, \frac{\pi}{2}] \) of \( V \) with \( W \) is given\(^8\) by
\[ \cos \Theta_{V,W} = \pi_{V,W}. \]

In simple cases having a clear and unique concept of the angle between the subspaces, \( \Theta_{V,W} \) coincides with it (e.g., when \( V \) is a line, or \( V \) and \( W \) are hyperplanes). This angle has many useful properties \([28]\), of which we mention just a few.

Proposition 2.10. Given subspaces \( V, W \subset X \) with principal angles \( \theta_1, \ldots, \theta_m \), where \( m = \min\{p, q\} \) for \( p = \dim V \) and \( q = \dim W \), let \( A, B \in \bigwedge X \) be nonzero blades such that \( [A] = V \) and \( [B] = W \). Then:

(i) \( \cos \Theta_{V,W} = \frac{|P_B A|}{|A|} \).
(ii) \( \Theta_{V,W} = \Theta_{V,P_{W}(V)} \).
(iii) \( \Theta_{V,W} \) is the angle\(^9\) in \( \bigwedge^p X \) between \( \bigwedge^p V \) and \( \bigwedge^p W \).
(iv) If \( \dim V = \dim W \) then \( \cos \Theta_{V,W} = \frac{|A \ast B|}{|A||B|} \), and \( \Theta_{V,W} = \Theta_{W,V} \).
(v) If \( p > q \) then \( \Theta_{V,W} = \frac{\pi}{2} \), otherwise
\[ \cos \Theta_{V,W} = \prod_{i=1}^{m} \cos \theta_i. \]
(vi) $\Theta_{V,W} = 0 \iff V \subset W$.
(vii) $\Theta_{V,W} = \frac{\pi}{2} \iff V \perp W$.

Proof. (i) Follows from the relation between blade norm and volume. (ii) Follows from i. (iii) $\bigwedge^p V = \text{span}(A)$ and, by i, $\Theta_{V,W}$ is the angle between $A$ and its projection on $\bigwedge^p W$. (iv) In this case $\bigwedge^p W = \text{span}(B)$ and so, by iii, $\Theta_{V,W}$ is the smallest angle between $A$ and $\pm B$. (v) Follows from i, since $P_B A = 0$ if $p > q$, otherwise $\|P_B A\| = \|A\| \prod_{i=1}^{m} \cos \theta_i$, by (2) and (3). (vi) Follows from v. (vii) Follows from v and Proposition 2.3i. \[\Box\]

As its name indicates, $\Theta_{V,W}$ is asymmetric: in general, $\Theta_{V,W} \neq \Theta_{W,V}$ if $\dim V \neq \dim W$. This feature sets it apart from similar angles which also measure volume contraction, but in projections from the smaller subspace to the larger one (e.g., those in [10,13,22,35] and in Sect. 2.4.3). This asymmetry may seem odd, but it reflects the lack of symmetry between subspaces of distinct dimensions. A way to understand it is to note that, as Proposition 2.10vi indicates, $\Theta_{V,W}$ measures, in a sense, how far $V$ is from being contained in $W$. If $\dim V > \dim W$, no rotation of $V$ will bring this any closer to happening, and as $V$ will always have a nonzero vector orthogonal to $W$, $\Theta_{V,W}$ will never be less than $\frac{\pi}{2}$, by Proposition 2.10vii. On the other hand, $W$ can be rotated into $V$, and $\Theta_{W,V}$ can assume any value in $[0, \frac{\pi}{2}]$.

The asymmetry turns out to be useful, leading to more general results with simpler proofs, as the angle ‘keeps track’ of special cases depending on which subspace is larger. For example, if we had defined the angle in a symmetric way (projecting from the smaller subspace to the larger one, as in [6], or using (4) without the exception for $p > q$, as in [19]), the results in Proposition 2.10 would require extra conditions.

Like other angles between high-dimensional subspaces, $\Theta_{V,W}$ captures some properties of their relative inclination, but misses other information, and it is important to note its peculiarities. Given two pairs of subspaces $(V,W)$ and $(V',W')$, even if all dimensions are the same and $\Theta_{V,W} = \Theta_{V',W'}$ there may be no orthogonal transformation of $X$ matching the pairs (this requires both to have the same list of principal angles [10]). And if $\dim V > 1$ then $\Theta_{V,W}$ tends to be larger than any (usual) angle between a line of $V$ and $W$.

2.4.1. Related Angles. Other angles related to $\Theta_{V,W}$ are useful at times.

Definition 2.11. The max- and min-symmetrized angles are, respectively, $\hat{\Theta}_{V,W} = \max\{\Theta_{V,W}, \Theta_{W,V}\}$ and $\tilde{\Theta}_{V,W} = \min\{\Theta_{V,W}, \Theta_{W,V}\}$.

Symmetrizing with min corresponds to projecting from the smaller subspace to the larger one, and leads to worse properties: $\hat{\Theta}_{V,W}$ does not satisfy a triangle inequality, while $\tilde{\Theta}_{V,W}$ gives a metric on the full Grassmannian of all subspaces of $X$ [28]. On the other hand, we always have $\tilde{\Theta}_{V,W} = \frac{\pi}{2}$ for different dimensions, which is less helpful. The asymmetric $\Theta_{V,W}$ strikes a good balance between nice properties and useful information, giving the Fubini-Study metric on the Grassmannian of subspaces of a given dimension, and an asymmetric metric on the full Grassmannian [28].

Definition 2.12. The complementary angle is $\Theta_{V,W}^\perp = \Theta_{V,W \perp}$.
The term ‘complementary’ refers to the orthogonal complement $W^\perp$, and this should not be confused with the usual complement of an angle. When $V$ is a line we do have $\Theta_{V,W} = \frac{\pi}{2} - \Theta_{V,W}$, but, in general, the relation between these two angles is complicated [28]. Assuming for simplicity that $V$, $W$ and $W^\perp$ have the same dimension $p > 1$, this can be understood by noting that, in the Plücker embedding of the Grassmannian of $p$-subspaces, these angles measure geodesic distances in the ambient space [28], so if $V$ is not in the geodesic from $W$ to $W^\perp$ then $\Theta_{W,V} + \Theta_{V,W^\perp} > \Theta_{W,W^\perp} = \frac{\pi}{2}$.

The following result was proven in [28] by showing (among other things) that each $\theta_i \neq 0$ gives a principal angle $\frac{\pi}{2} - \theta_i$ of $V$ and $W^\perp$, while a $\theta_i = 0$ implies $V \perp W^\perp$. We now give a simpler proof using Theorem A.2 (whose proof, in Appendix A, uses only Proposition 2.10i, so there is no circularity).

**Proposition 2.13.** Given subspaces $V, W \subset X$ with principal angles $\theta_1, \ldots, \theta_m$, where $m = \min\{\dim V, \dim W\}$, we have

$$\cos \Theta_{V,W}^\perp = \prod_{i=1}^m \sin \theta_i. \quad (5)$$

**Proof.** Let $A = e_1 \wedge \cdots \wedge e_p$ and $B = f_1 \wedge \cdots \wedge f_q$ for principal bases $(e_1, \ldots, e_p)$ and $(f_1, \ldots, f_q)$ of $V$ and $W$, and assume, without loss of generality, $p \leq q$. Theorem A.2 gives $\cos \Theta_{V,W}^\perp = \|A \wedge B\|$, and with (1) we obtain $\|A \wedge B\| = \|e_1 \wedge f_1\| \cdots \|e_p \wedge f_p\| \|f_{p+1}\| \cdots \|f_q\| = \sin \theta_1 \cdots \sin \theta_p$. \qed

**Proposition 2.14.** Let $V, W \subset X$ be subspaces.

(i) $\Theta_{V,W}^\perp = 0$ $\iff$ $V \perp W$.

(ii) $\Theta_{V,W}^\perp = \frac{\pi}{2}$ $\iff$ $V \cap W \neq \{0\}$.

(iii) $\Theta_{V,W}^\perp = \Theta_{W,V}^\perp$.

**Proof.** (i) By (5), $\Theta_{V,W}^\perp = 0$ $\iff$ $\theta_i = \frac{\pi}{2}$ for all $i$. (ii) By (5), $\Theta_{V,W}^\perp = \frac{\pi}{2}$ $\iff$ $\theta_i = 0$ for some $i$. (iii) Follows from (5), as principal angles do not depend on the order of $V$ and $W$. We can also obtain it directly from Theorem A.2. \qed

Note that (5) holds regardless of the dimensions of $V$ and $W$, unlike Proposition 2.10v, and $\Theta_{V,W}^\perp$ is always symmetric. A way to understand this is to note that $\dim V > \dim W^\perp \iff \dim W > \dim V^\perp$. What is surprising is that Proposition 2.14i, ii and (5) depend on the asymmetry of $\Theta_{V,W}$, without which these results would not even hold for two planes in $\mathbb{R}^3$.

2.4.2. Oriented Angles. It is also convenient to define an angle that takes the orientation of subspaces into account. Let $V$ and $W$ be oriented by blades $A, B \in \wedge X$, respectively, with relative orientation $\epsilon_{A,B}$ (w.r.t. given principal bases of $V$ and $W$).

**Definition 2.15.** The oriented asymmetric angle $^{10} \Theta_{A,B} \in [0, \pi]$ of $V$ with $W$ is given by $\cos \Theta_{A,B} = \epsilon_{A,B} \cos \Theta_{V,W}$.

$^{10}$In complex spaces, it is a complex-valued angle $\Theta_{A,B} \in \mathbb{C}$. Complex-valued angles between complex vectors have been considered, for example, in [33].
We have $\Theta_{A,B} = \Theta_{V,W}$ if $\tilde{A} \ast B > 0$, $\Theta_{A,B} = \pi - \Theta_{V,W}$ if $\tilde{A} \ast B < 0$, and if $\tilde{A} \ast B = 0$ it depends on the principal bases (which will be useful).

We call $\Theta_{A,B}$ an ‘oriented angle’ for short, as the ‘oriented’ refers to the subspaces. To be clear about our notation: any $\Theta_{V,W}$, with subspaces as subscripts, is a non-oriented angle, while any $\Theta_{A,B}$, with blades as subscripts, is the oriented angle of the subspaces $[A]$ and $[B]$, oriented by these blades. Such $\Theta_{A,B}$ should not be confused with the usual angle between $A$ and $B$ in $\bigwedge X$ (though they do coincide when grades are equal). For the non-oriented angle of $[A]$ and $[B]$ we write $\Theta_{[A],[B]}$. The same convention will apply to non-oriented and oriented versions of other concepts.

Definition 2.16. $\pi_{A,B} = \epsilon_{A,B} \pi_{V,W}$ is an oriented projection factor, and oriented max-symmetrized and complementary angles $\hat{\Theta}_{A,B}, \Theta_{\perp A,B} \in [0, \pi]$ are given by $\cos \hat{\Theta}_{A,B} = \epsilon_{A,B} \cos \hat{\Theta}_{V,W}$ and $\cos \Theta_{\perp A,B} = \epsilon_{A,B} \cos \Theta_{\perp V,W}$.

Note that in general $\hat{\Theta}_{A,B} \neq \max\{\Theta_{A,B}, \Theta_{B,A}\}$, and for distinct grades $\hat{\Theta}_{A,B} = \frac{\pi}{2}$. Also, $\Theta_{\perp A,B}$ encodes information about the relative orientation of $V$ and $W$, not $V$ and $W^\perp$ (this will be important in Proposition 5.4).

2.4.3. Geometric Algebra Angles. The geometric algebra literature has different definitions for the angle between blades or subspaces, all closely related to our angles.

Hestenes [18] defines an angle between multivectors by $\cos \phi = \frac{\tilde{A} \ast B}{\|A\| \|B\|}$. He says it has a simple geometric interpretation for same grade blades, but only describes it for intersecting planes, as a dihedral angle. For same grade blades it equals $\Theta_{A,B}$, and for distinct grades it is always $\frac{\pi}{2}$ (even for a line contained in a plane), so that it corresponds to $\hat{\Theta}_{A,B}$.

Dorst et al. [6] use Hestenes definition for same grade blades, and if $A$ has a lower grade than $B$ they take the angle with its projection on $B$, which is the same as $\Theta_{[A],[B]}$. They erroneously see it as a dihedral angle: if, after taking out common factors, there is at most one vector left in each blade, it is the angle between these vectors, otherwise “no single scalar angle can be defined geometrically, and this geometric nonexistence is reflected in the algebraic answer of 0 for the scalar product” [6, p. 70]. This is incorrect: if $(e_1, e_2, e_3, e_4)$ is the canonical basis of $\mathbb{R}^4$, $A = (e_1 + e_2) \land (e_3 + e_4)$ and $B = e_1 \land e_3$ have no common factors (as $A \land B \neq 0$) but $A \ast B \neq 0$.

Hitzer [19] defines the angle for subspaces of same dimension as in (4), but uses it for different dimensions as well, without the exception for $p > q$, so it corresponds to the min-symmetrized angle $\Theta_{V,W}$. He is silent on its geometric interpretation, and recovers Hestenes formula for equal dimensions.

In a survey of the theory [27], the angle is defined in terms of a ratio of volumes, by $\cos \phi = \frac{\|P_B A\|}{\|A\|}$, which would make it equal to $\Theta_{[A],[B]}$. But it is not clear if the case of $A$ having larger grade is admitted.
3. Angle Bivector of Subspaces

As seen, the asymmetric angle has many useful properties, but it does not fully describe the relative inclination of subspaces (no single scalar angle can do this). Alternatively, an angle bivector can conveniently store all data about the relative inclination of two subspaces of same dimension. In Sect. 3.5 we consider the case of distinct dimensions.

Let $V, W \subset X$ be $p$-subspaces, with associated principal bases $\beta_V = (e_1, \ldots, e_p)$ and $\beta_W = (f_1, \ldots, f_p)$, and principal angles $\theta_1 \leq \cdots \leq \theta_p$. Also, let $d = \dim(V \cap W)$, $E = e_1e_2 \cdots e_p$ and $F = f_1f_2 \cdots f_p$.

**Definition 3.1.** For $1 \leq i \leq p$, span$(e_i, f_i)$ is a principal plane, with principal bivector $I_i$ (oriented from $V$ to $W$) given by

$$I_i = \begin{cases} 0 & \text{if } i \leq d, \\ \frac{e_i \wedge f_i}{\|e_i \wedge f_i\|} & \text{if } i > d. \end{cases}$$

For $d < i \leq p$, $e_i^\perp = I_i f_i$ and $f_i^\perp = e_i I_i$ are orthoprincipal vectors.

The term ‘plane’ is used broadly, as it degenerates to a line if $i \leq d$. For $d < i \leq p$, $e_i^\perp$ is the normalized component of $e_i$ orthogonal to $W$, and likewise for $f_i^\perp$ (see Fig. 2). Note that $I_i = e_i^\perp f_i = e_i f_i^\perp$ anti-commutes with $e_i$ and $f_i$, and for $j \neq i$ it commutes with $e_j$, $f_j$ and $I_j$, since principal planes are mutually orthogonal.

**Definition 3.2.** $\Phi_{V,W} = \sum_{i=1}^p \theta_i I_i$ is an angle bivector from $V$ to $W$.

Note that $\tilde{\Phi}_{V,W} = -\Phi_{V,W}$ is an angle bivector from $W$ to $V$. Also, $\Phi_{V,W}$ may depend on the choice of principal bases. In Sect. 3.2 we interpret this non-uniqueness in terms of the geometry of the Grassmannians.

**Proposition 3.3.** $\Phi_{V,W}$ is uniquely defined $\iff \theta_p \neq \frac{\pi}{2}$. 

---

**Figure 2.** Principal vectors, bivectors and planes, and orthoprincipal vectors, for planes $V, W \subset \mathbb{R}^3$.
Proof. Switching signs of both $e_i$ and $f_i$ does not affect $I_i$. If $\theta_p = \frac{\pi}{2}$ we can keep $e_p$ and replace $f_p$ with $-f_p$, so now $I_p$ switches sign. If $\theta_p \neq \frac{\pi}{2}$ but some $\theta_i$’s are repeated, the choice of principal bases can affect their $I_i$’s, but does not change $\Phi_{V,W}$. To prove it, let $\theta_i = \theta \neq \frac{\pi}{2}$ for all $i$, for simplicity. An orthogonal change of principal basis $e' = \sum_j c_{ij} e_j$ in $V$ must be accompanied by a corresponding change $f' = \sum_j c_{ij} f_j$ in $W$, so the bases remain associated $(e'_i \cdot f'_j = \delta_{ij} \cos \theta)$. A calculation shows $\sum_i \theta I'_i = \sum_i \theta I_i$ for the new $I'_i = \sum_{j,k} c_{ij} c_{ik} e'_i k \sin \theta_i$.

With more than one $\theta_i = \frac{\pi}{2}$, we can obtain a continuous family of $\Phi_{V,W}$’s via independent orthogonal transformations of $V \cap W \perp$ and $W \cap V \perp$.

In Sect. 3.3 we analyze the exponentials of angle bivectors in detail. For now, we show that they give rotors connecting the subspaces.

**Proposition 3.4.** $F = e^{-\frac{\Phi}{2}} E e^{\frac{\Phi}{2}} = E e^\Phi = e^{-\Phi} E$, where $\Phi = \Phi_{V,W}$.

**Proof.** Commutativity and $f_i = e^{-I_i \theta_i / 2} e_i e^{I_i \theta_i / 2}$ give the first equality. And since $e_i$ anti-commutes with $I_i$ we have $e^{-I_i \theta_i / 2} e_i = e_i e^{I_i \theta_i / 2}$. □

The operation $E \mapsto e^{-\frac{1}{2} \Phi_{V,W}} E e^{\frac{1}{2} \Phi_{V,W}}$ takes $V = [E]$ onto $W = [F]$ via independent rotations, by principal angles along principal planes, each produced by a plane rotor $e^{I_i \theta_i / 2}$. The simpler process $E \mapsto E e^{\Phi_{V,W}}$ also takes $V$ to $W$, but while $e^{-\frac{1}{2} \Phi_{V,W}} e_i e^{\frac{1}{2} \Phi_{V,W}} = f_i$, in general $e_i e^{\Phi_{V,W}} \neq f_i$. So, while the first operation takes each vector of $V$ to another in $W$, the second one relates the whole subspaces, without mapping individual vectors.

Since $E$ and $F$ are unit $p$-blades in $V$ and $W$, Proposition 3.4 implies $e^{\Phi_{V,W}}$ (but not $e^{\frac{1}{2} \Phi_{V,W}}$) is uniquely defined up to a sign, even when $\Phi_{V,W}$ is not unique.

**Example 3.5.** Let $\{e_1, e_2, e_3, f_2, f_3\}$ be an orthonormal set, $E = e_1 e_2 e_3$, $F = e_1 f_2 f_3$, $V = [E]$ and $W = [F]$. With the associated principal bases $\beta_V = (e_1, e_2, e_3)$ and $\beta_W = (e_1, f_2, f_3)$ we obtain $\Phi_{V,W} = \frac{\pi}{2} (e_2 f_2 + e_3 f_3)$, $e_{\Phi_{V,W}} = e_2 f_2 e_3 f_3$ and $e^{\frac{1}{2} \Phi_{V,W}} = (1 + e_2 f_2 + e_3 f_3 + e_2 f_2 e_3 f_3) / 2$. Calculations confirm Proposition 3.4 and show that $-\frac{1}{2} \Phi_{V,W} e_2 e^{\frac{1}{2} \Phi_{V,W}} = f_2$ but $e_2 e^{\Phi_{V,W}} = f_2 e_3 f_3 \neq f_2$. Note that the angle bivector ignores $e_1 \in V \cap W$.

If $f'_2 = (f_2 + f_3) / \sqrt{2}$ and $f'_3 = (f_2 - f_3) / \sqrt{2}$ then $\beta_W = (e_1, f'_2, f'_3)$ is another principal basis of $W$ associated with $\beta_V$. With it, we now find $\Phi_{V,W} = \frac{\pi}{2} (e_2 f'_2 + e_3 f'_3) \neq \Phi_{V,W}, e^{\frac{1}{2} \Phi_{V,W}} = e_2 f'_2 e_3 f'_3 = -e_2 f_2 e_3 f_3 = e^{\Phi_{V,W}}$ and $e^{\frac{1}{2} \Phi_{V,W}} = (1 + e_2 f'_2 + e_3 f'_3 + e_2 f'_2 e_3 f'_3) / 2 \neq \pm e^{\frac{1}{2} \Phi_{V,W}}$. Also, $-\frac{1}{2} \Phi_{V,W} e_2 e^{\frac{1}{2} \Phi_{V,W}} = f'_2$ and likewise for $e_3$, so $E$ is rotated to $E' = e_1 f'_2 f'_3 = -F$.

### 3.1. Oriented Angle Bivector

Let $V$ and $W$ be oriented by nonzero blades $A, B \in \wedge^P X$, respectively, with relative orientation $\epsilon_{A,B}$ (w.r.t. $\beta_V$ and $\beta_W$).

**Definition 3.6.** The oriented principal angles and bivectors of $V$ and $W$ are $\theta_i^+ = \theta_i$ and $I_i^+ = I_i$ for $i < p$ and also for $i = p$ if $\epsilon_{A,B} = 1$, otherwise
\[ \theta^+_p = \pi - \theta_p \] and \[ I^+_p = -I_p. \] The oriented angle bivector from \( V \) to \( W \) is
\[
\Phi_{A,B} = \sum_{i=1}^{p} \theta^+_i I^+_i = \begin{cases} 
\Phi_{V,W} & \text{if } \epsilon_{A,B} = 1, \\
\Phi_{V,W} - \pi I_p & \text{if } \epsilon_{A,B} = -1.
\end{cases}
\] (6)

Again, we use \( \Phi_{V,W} \) for the non-oriented angle, \( \Phi_{A,B} \) for the oriented one. Note that \( 0 \leq \theta^+_1 \leq \cdots \leq \theta^+_{p-1} \leq \frac{\pi}{2} \) and \( \theta^+_{p-1} \leq \pi - \theta^+_p \leq \pi - \theta^+_{p-1}. \)

**Proposition 3.7.** \( \Phi_{A,B} \) is uniquely defined \( \Leftrightarrow \theta^+_{p-1} + \theta^+_p \neq \pi. \)

**Proof.** \( \theta^+_{p-1} + \theta^+_p = \pi \Leftrightarrow \theta_{p-1} = \theta_p = \frac{\pi}{2}, \) or \( \theta_{p-1} = \theta_p \neq \frac{\pi}{2} \) and \( \epsilon_{A,B} = -1. \) Swapping \( f_{p-1} \) and \( f_p \) in the first case, or \( (e_{p-1}, f_{p-1}) \) and \( (e_p, f_p) \) in the second one, changes \( \Phi_{A,B}. \)

If \( \theta_{p-1} \neq \theta_p = \frac{\pi}{2}, \) replacing \( f_p \) with \( -f_p \) as in Proposition 3.3 does not change \( I^+_p = \epsilon_{A,B} I_p, \) as both \( I_p \) and \( \epsilon_{A,B} \) switch signs. If \( \theta_p \neq \frac{\pi}{2} \) then \( \Phi_{V,W} \) is uniquely defined, and so is \( I_p \) if \( \theta_{p-1} \neq \theta_p \) as well. \( \square \)

**Proposition 3.8.** \( e^{\Phi_{A,B}} = \epsilon_{A,B} e^{\Phi_{V,W}}. \)

**Proof.** Follows from (6). \( \square \)

**Proposition 3.9.** If \( \|A\| = \|B\| = 1 \) then \( B = e^{-\frac{\pi}{2}} A e^{\frac{\pi}{2}} = A e^{\Phi} = e^{-\Phi} A, \)

where \( \Phi = \Phi_{A,B}. \)

**Proof.** By (3), \( A = \epsilon_A E \) and \( B = \epsilon_B F, \) so the result follows from Propositions 3.4 and 3.8, and also \( e^{\frac{1}{2} \Phi_{A,B}} = -I_p e^{\frac{1}{2} \Phi_{V,W}} \) if \( \epsilon_{A,B} = -1. \) \( \square \)

Now \( V \) is rotated onto \( W \) matching orientations, and \( e^{\Phi_{A,B}} \) (but not \( e^{\frac{1}{2} \Phi_{A,B}} \)) is uniquely determined by \( A \) and \( B. \) In Sect. 3.5 we show this is not valid in case of different dimensions.

**Example 3.10.** In Example 3.5, let \( V \) and \( W \) be oriented by \( E \) and \( F. \) Using \( \beta_V \) and \( \beta_W \) we find \( \epsilon_{E,F} = 1 \) and \( \Phi_{E,F} = \Phi_{V,W}. \) As \( F' = -F, \) with \( \beta'_W \) we have \( \epsilon'_{E,F} = -1 \) and \( \Phi'_{E,F} = \frac{\pi}{2}(e_2f'_2 - e_3f'_3), \) which does not equal \( \Phi_{V,W} \) nor \( \Phi'_{V,W}. \) Still, \( e^{\Phi'_{E,F}} = -e_2f'_2e_3f'_3 = e^{\Phi_{E,F}}, \) so that, either way, \( V \) is taken to \( W \) with the correct orientation. However, \( e^{\frac{1}{2} \Phi'_{E,F}} \neq e^{\frac{1}{2} \Phi_{E,F}}, \) since vector-wise we have distinct rotations: \( (e_1, e_2, e_3) \) is taken by the first rotor to \( (e_1, f'_2, -f'_3) \), and by the second one to \( (e_1, f_2, f_3). \)

### 3.2. Minimal Geodesics in the Grassmannians

Let \( G_p(X) \) (resp. \( G^+_p(X) \)) be the Grassmannian of non-oriented (resp. oriented) \( p \)-subspaces of \( X, \) identified with its Plücker embedding in the projective space \( \mathbb{P}(\bigwedge^p X) \) (resp. unit sphere \( S(\bigwedge^p X) \)). We relate \( \Phi_{V,W} \) and \( \Phi_{A,B} \) to the geometry of these manifolds, using results from [24].

The curve given by \( F(t) = e^{-\frac{1}{2} \Phi_{V,W}} E e^{\frac{1}{2} \Phi_{V,W}} = f_1(t) \wedge \cdots \wedge f_p(t), \) where \( t \in [0,1] \) and \( f_i(t) = \cos(t \theta_i) e_i + \sin(t \theta_i) f_i^L, \) is a minimal geodesic in \( G_p(X) \) connecting \( V \) to \( W, \) and \( \|\Phi_{V,W}\| = (\sum_{i=1}^{p} \theta_i^2)^{\frac{1}{2}} \) gives the arc-length distance between them. Note that this is the distance along geodesics inside \( G_p(X), \) while the Fubini-Study distance given by \( \Theta_{V,W} \) measures geodesics in the ambient space \( \mathbb{P}(\bigwedge^p X). \)
A minimal geodesic in $G_p^+(X)$ connecting unit $p$-blades $A$ and $B$ is given by $B(t) = e^{-\frac{t}{2} \Phi_{A,B}} A e^{\frac{t}{2} \Phi_{A,B}} = f_1(t) \wedge \cdots \wedge f_p(t)$, with $t \in [0,1]$, $f_i(t)$ as before for $i < p$ and $f_p(t) = \cos(t\theta_p) e_p + \epsilon_{A,B} \sin(t\theta_p) f_p^\perp$. Its length $\|\Phi_{A,B}\|$ equals $\|\Phi_{V,W}\|$ if $\epsilon_{A,B} = 1$, otherwise $\|\Phi_{A,B}\|^2 = \|\Phi_{V,W}\|^2 + \pi(\pi - 2\theta_p)$.

The minimal geodesic is unique unless $\theta_p = \frac{\pi}{2}$ in the non-oriented case, or $\theta_{p-1} + \theta_p = \frac{\pi}{2}$ in the oriented one, precisely the cases in which $\Phi_{V,W}$ or $\Phi_{A,B}$ depend on the choice of principal bases. In fact, we have:

**Proposition 3.11.** There is a one-to-one correspondence between angle bivectors $\Phi_{V,W}$ (resp. $\Phi_{A,B}$) and minimal geodesics connecting the subspaces in $G_p(X)$ (resp. $G_p^+(X)$).

**Proof.** Any $\Phi_{V,W}$ determines a minimal geodesic in $G_p(X)$ given by $F(t)$ as above. An angle bivector of $V$ with its middle point $U = F(\frac{1}{2})$ is given by $\Phi_{V,U} = \Phi_{V,W}/2$, and the principal angles of $V$ and $U$ are at most $\frac{\pi}{4}$. By Proposition 3.3, $\Phi_{V,U}$ is uniquely defined, so if $\Phi_{V,W}'$ gives the same geodesic then $\Phi_{V,W}' = 2\Phi_{V,U} = \Phi_{V,W}$.

And given a minimal geodesic $\gamma$ from $V$ to $W$, the minimal geodesic from $V$ to its middle point $U$ is unique. So, given an angle bivector $\Phi_{V,U}$, the geodesic it determines must coincide with the first half of $\gamma$. Thus $\Phi_{V,W} = 2\Phi_{V,U}$ is an angle bivector determining $\gamma$.

The proof for the oriented case is similar, using Proposition 3.7. □

### 3.3. Exponentials of Angle Bivectors

Exponentials of angle bivectors decompose into rotors of principal planes, or in terms of principal angles, asymmetric angles, or projection factors, as follows. In Sect. 4.3 we show how to obtain these decompositions explicitly.

**Definition 3.12.** For $1 \leq i \leq p$, $R_i = e_i f_i = e^{I_i \theta_i} = \cos \theta_i + I_i \sin \theta_i$ is a principal rotor.

**Proposition 3.13.** $e^{\Phi_{V,W}} = R_1 R_2 \cdots R_p$.

**Proof.** As the $I_i$’s commute, $e^{\Phi_{V,W}} = \prod_{i=1}^p e^{I_i \theta_i}$. □

**Proposition 3.14.** Let $c_i = \cos \theta_i$ and $s_i = \sin \theta_i$. Then

$$e^{\Phi_{V,W}} = c_1 c_2 \cdots c_p + s_1 c_2 \cdots c_p I_1 + c_1 s_2 c_3 \cdots c_p I_2 + \cdots + c_1 \cdots c_{p-1} s_p I_p$$

$$+ s_1 s_2 c_3 \cdots c_p I_1 I_2 + s_1 c_2 s_3 \cdots c_p I_1 I_3 + \cdots + c_1 \cdots s_{p-1} s_p I_{p-1} I_p$$

$$+ c_1 s_2 \cdots s_p I_2 \cdots I_p + \cdots + s_1 \cdots s_{p-1} c_p I_1 \cdots I_{p-1} I_p$$

$$+ s_1 s_2 \cdots s_p I_1 I_2 \cdots I_p.$$

**Proof.** $e^{\Phi_{V,W}} = \prod_{i=1}^p R_i = \prod_{i=1}^p (c_i + s_1 I_i)$. □

This expression appears in Hitzer’s geometric product formula [19], and this is understandable since Proposition 3.4 gives $e^{\Phi_{V,W}} = \tilde{E} F$. We can simplify it using the asymmetric angles and some multi-index notation.
Definition 3.15. Given $a,b,k \in \mathbb{N}$ with $a \leq b$ and $1 \leq k \leq b - a + 1$, let $\mathcal{T}_{0}^{a,b} = \{0\}$ and $\mathcal{T}_{0}^{a,b} = \{(i_{1},\ldots,i_{k}) \in \mathbb{N}_{k} : a \leq i_{1} < \cdots < i_{k} \leq b\}$. Also, let $\mathcal{T}^{a,b} = \bigcup_{k=0}^{b-a+1} \mathcal{T}_{0}^{a,b}$. When $a = 1$ we omit it and write $\mathcal{T}^{b}$.

Definition 3.16. Let $I_{0} = 1$ and $I_{1} = I_{i_{1}} \cdots I_{i_{k}}$ for $i = (i_{1},\ldots,i_{k}) \in \mathcal{T}^{d+1,p}$, where $d = \dim(V \cap W)$. For any $i \in \mathcal{T}^{d+1,p}$, let $F_{i} = I_{1}F$ and $W_{i} = [F_{i}]$, where $F = f_{1}f_{2} \cdots f_{p}$ as before.

Lemma 3.17. For any $i \in \mathcal{T}^{d+1,p}$:

(i) $F_{i}$ is the unit $p$-blade obtained from $F$ by replacing $f_{i}$ with $e_{i}^{\perp}$ for each $i \in i$.

(ii) $\beta_{i} = \{f_{i} : i \not\in i\} \cup \{e_{i}^{\perp} : i \in i\}$ is a principal basis of $W_{i}$ associated to $\beta_{V}$.

(iii) The (unordered) principal angles of $V$ and $W_{i}$ are $\theta_{i}$ for $i \not\in i$ and $\frac{\pi}{2} - \theta_{i}$ for $i \in i$.

Proof. (i) As $I_{i}$ and $f_{j}$ commute if $i \neq j$, and $I_{i}f_{i} = e_{i}^{\perp}$, we have $I_{1}F = I_{i_{1}} \cdots I_{i_{k}}f_{1} \cdots f_{p} = f_{1} \cdots I_{i_{1}}f_{i_{1}} \cdots I_{i_{k}}f_{i_{k}} \cdots f_{p} = f_{1} \cdots e_{i_{1}}^{\perp} \cdots e_{i_{k}}^{\perp} \cdots f_{p}$. (ii, iii) By the previous item, $\beta_{i}$ is a basis of $[F_{i}]$, obtained from $\beta_{W}$ by replacing, for $i \in i$, $f_{i}$ with $e_{i}^{\perp}$, which is in the same principal plane and makes with $e_{i}$ an angle $\frac{\pi}{2} - \theta_{i}$. Thus condition (1), with the appropriate substitutions, is satisfied for $\beta_{V}$ and $\beta_{i}$.

Theorem 3.18. $e^{\Phi_{V,W}} = \sum_{i \in \mathcal{T}^{d+1,p}} \cos \Theta_{V,W_{i}}I_{i} = \sum_{i \in \mathcal{T}^{d+1,p}} \pi_{V,W_{i}}I_{i}$.

Proof. Follows from Proposition 3.14, since $\cos \Theta_{V,W_{i}} = \prod_{i \not\in i} \cos \theta_{i} \prod_{i \in i} \sin \theta_{i}$ by Proposition 2.10v and Lemma 3.17iii.

With $e^{\Phi_{V,W}}$ decomposed like this, each component shows how volumes in $V$ contract when orthogonally projected on a $W_{i}$. In particular:

Proposition 3.19. $\langle e^{\Phi_{V,W}} \rangle_{0} = \cos \Theta_{V,W} = \pi_{V,W}$, and also $\|\langle e^{\Phi_{V,W}} \rangle_{2p} \| = \cos \Theta_{V,W}^{\perp} = \pi_{V,W^{\perp}}$.

Proof. In Theorem 3.18, $\langle e^{\Phi_{V,W}} \rangle_{0}$ is given by $i = 0$, and $W_{0} = W$. If $d \neq 0$ then $\langle e^{\Phi_{V,W}} \rangle_{2p} = 0$ and $\cos \Theta_{V,W} = \frac{\pi}{2}$ by Proposition 2.14ii. If $d = 0$, $P_{W^{\perp}}(V) = W_{1}$ for $i = (1,\ldots,p)$, and $\Theta_{V,W_{i}} = \Theta_{V,W^{\perp}}$ by Proposition 2.10ii.

The theorem can be adapted for oriented angles, once we orient $W_{i}$.

Definition 3.20. For $i \in \mathcal{T}^{d+1,p}$, we give $W_{i}$ the orientation of $B_{i} = I_{1}B = \epsilon_{B}\|B\|F_{i}$, and set $\epsilon_{A,B_{i}}$ in terms of $\beta_{V}$ and $\beta_{i}$.

Proposition 3.21. $e^{\Phi_{A,B}} = \sum_{i \in \mathcal{T}^{d+1,p}} \cos \Theta_{A,B_{i}}I_{i} = \sum_{i \in \mathcal{T}^{d+1,p}} \pi_{A,B_{i}}I_{i}$.

Proof. Follows from Proposition 3.8 and Theorem 3.18, as $\epsilon_{A,B_{i}} = \epsilon_{A,B}$.
3.4. Plücker Coordinates

It will be interesting to rewrite Theorem 3.18 in a different form. This will require some more notation.

**Definition 3.22.** Extend $\beta_W$ to the orthonormal basis of $Y = V + W$ given by $\beta_Y = (f_1, \ldots, f_d, e^1_{d+1}, f_{d+1}, \ldots, e^1_p, f_p)$ = $(y_1, \ldots, y_{2p-d})$. Its coordinate blades are $C_0 = 1$ and $C_j = y_{j_1}y_{j_2} \cdots y_{j_k}$ for $j = (j_1, \ldots, j_k) \in \mathcal{I}^{2p-d}$, forming orthonormal bases $\beta_{\wedge^k Y} = (C_1)_j \in \mathcal{I}^{2p-d}$ and $\beta_{\wedge^j Y} = (C_j)_j \in \mathcal{I}^{2p-d}$ of $\wedge^k Y$ and $\wedge Y$. Each $Y_j = [C_j]$ with $j \in \mathcal{I}_k^{2p-d}$ is a coordinate $k$-subspace.

Coordinates of a blade $A \in \wedge^k Y$ in $\beta_{\wedge^k Y}$ give homogeneous Plücker coordinates of $[A]$ w.r.t. $\beta_Y$.

**Definition 3.23.** Let $\sigma : \wedge Y \to \wedge Y$ be given by $\sigma(C) = CF^{-1} = Cf_p \cdots f_1$ for any $C \in \wedge Y$.

This map produces a permutation of $\beta_{\wedge Y}$. Also, $I_i = e^1_{i_1}f_{i_1} \cdots e^1_{i_k}f_{i_k}$ and $F_i$ are elements of $\beta_{\wedge Y}$, with $I_i = \sigma(F_i)$.

Among all coordinate $p$-subspaces of $\beta_Y$, the $W_i$’s are those having either $f_i$ or $e^1_i$ for each $i$, and the projection factor of $V$ on any other vanishes. So we can extend the sum in Theorem 3.18 as follows.

**Proposition 3.24.** $e^{\Phi_{V, W}} = \sum_{j \in \mathcal{I}^{2p-d}} \cos \Theta_{V, Y_j} \sigma(C_j) = \sum_{j \in \mathcal{I}^{2p-d}} \pi_{V, Y_j} \sigma(C_j)$.

**Proof.** Given $i \in \mathcal{I}^{d+1,p}$ we have $F_i = C_j$ for some $j \in \mathcal{I}^{2p-d}$, and so $W_i = Y_j$ and $I_i = \sigma(C_j)$. And given $j \in \mathcal{I}^{2p-d}$, if $C_j$ has either $f_i$ or $e^1_i$ for each $1 \leq i \leq p$ then forming $i \in \mathcal{I}^{d+1,p}$ with the indices for which it has $e^1_i$ we obtain $C_j = F_i$. Otherwise $e_i \perp Y_j$ for some $i$ and $\pi_{V, Y_j} = 0$. So the nonzero terms in the sums above are the same as in Theorem 3.18. □

By Proposition 3.4, $E = \sigma^{-1}(e^{\Phi_{V, W}}) = \sum_{j \in \mathcal{I}^{2p-d}} \cos \Theta_{V, Y_j} C_j$, so the coefficients in Proposition 3.24 are Plücker coordinates of $V$ w.r.t. $\beta_Y$. They are normalized, with
\[
\sum_{j \in \mathcal{I}^{2p-d}} \cos^2 \Theta_{V, Y_j} = 1, \tag{7}
\]
but also scrambled by $\sigma$. To relate each coefficient in $e^{\Phi_{V, W}}$ to the correct coordinate blade or subspace, note that the only ones that do not vanish are those from Theorem 3.18, and $I_i$ corresponds to $F_i$ via $\sigma^{-1}$.

Though useful for theoretical purposes, Plücker coordinates are computationally expensive if $n = \dim X$ is large [34, p. 197]. It takes $\binom{n}{p}$ coordinates to represent $p$-subspaces, but Plücker relations reduce the dimension of their Grassmannian to $p(n-p)$. When $p$ is not close to 1 or $n$, blades become sparse among multivectors, and this representation becomes quite inefficient. There are more economical, even if less elegant, ways to locate a subspace (e.g., reduced simplex representations can use as low as $p(n-p)+1$ coordinates [34, p. 204]). The angle bivector strikes a nice balance between economy and theoretical convenience.
3.5. Distinct Dimensions and Projective-Orthogonal Decomposition

Now we treat the case of different dimensions. Let \( A \in \bigwedge^p X \) and \( B \in \bigwedge^q X \) be nonzero blades, \( m = \min\{p, q\} \), \( \beta_V \) and \( \beta_W = (f_1, \ldots, f_q) \) be associated principal bases of \( V = [A] \) and \( W = [B] \), and \( \epsilon_B \) be as in (3).

Definition 3.25. A projective-orthogonal (PO) decomposition of \( B \) w.r.t. \( A \) is \( B = B_P B_{\perp} \), where \( B_P = \epsilon_B \|B\| f_1 f_2 \cdots f_m \) and \( B_{\perp} = f_{m+1} f_{m+2} \cdots f_q \) \((= 1 \text{ if } p \geq q)\) are, respectively, projective and orthogonal subblades. We also decompose \( W = W_P \oplus W_{\perp} \), where \( W_P = [B_P] \) and \( W_{\perp} = [B_{\perp}] \) are, respectively, projective and orthogonal subspaces of \( W \) w.r.t. \( V \).

Note that \( B_{\perp} \) is completely orthogonal to \( A \) and \( B_P \), so \( B = B_P \wedge B_{\perp} \). Also, \( \beta_{W_P} = (f_1, \ldots, f_m) \) is a principal basis of \( W_P \) associated to \( \beta_V \), for which \( \epsilon_{A, B_P} = \epsilon_{A, B_1} \) and the principal angles of \( V \) and \( W_P \) are the same as those of \( V \) and \( W \). If \( p \leq q \) then \( \text{grade}(B_P) = \text{grade}(A) \).

If \( p \geq q \) then \( B_P = B \), \( B_{\perp} = 1 \), \( W_P = W \) and \( W_{\perp} = \{0\} \). If \( p < q \) and \( V \not\perp W \), (2) gives \( W_P = P_W(V) \) and \( W_{\perp} = W \cap W_{\perp} \), and the blades are unique up to signs, with \( B_P \) having the orientation of \( \epsilon_{A, B} P_B A \). If \( p < q \) and \( V \perp W \), both decompositions depend on the choice of \( \beta_W \), with \( W_P \supseteq P_W(V) \) and \( W_{\perp} \subseteq W \cap V_{\perp} \).

Proposition 3.26. \( \Theta_{V, W} = \Theta_{V, W_P} \), \( \Theta_{V, W}^\perp = \Theta_{V, W_P}^\perp \), \( \Theta_{A, B} = \Theta_{A, B_P} \) and \( \Theta_{A, B}^\perp = \Theta_{A, B_P}^\perp \) (for oriented angles w.r.t. \( \beta_V \), \( \beta_W \) and \( \beta_{W_P} \)).

Proof. Follows from (4) and (5). \( \square \)

We extend Definitions 3.2 and 3.6 to the case of distinct dimensions.

Definition 3.27. \( \Phi_{V, W} = \Phi_{V, W_P} \) and \( \Phi_{A, B} = \Phi_{A, B_P} \) (w.r.t. \( \beta_V \), \( \beta_W \) and \( \beta_{W_P} \)).

When \( p < q \), the non-uniqueness of the decompositions can increase the ambiguity of the angle bivectors. Not even \( e^{\Phi_{A, B}} \) is uniquely defined anymore, as we can switch the orientation of \( B_P \), and if \( V \perp W \) we can swap \( f_p \) with any unit vector in \( W_{\perp} \).

Still, our results readily adapt. For example, if \( p \leq q \) and \( \|A\| = \|B\| = 1 \) then \( e^{-\frac{1}{2} \Phi_{A, B}} A e^{\frac{1}{2} \Phi_{A, B}} = B_P \), and so \( V \) rotates onto \( W_P \). And as \( B_{\perp} \) is completely orthogonal to all principal bivectors in \( \Phi_{B, A} = -\Phi_{A, B} \), we have \( e^{-\frac{1}{2} \Phi_{B, A}} B_P B_{\perp} e^{\frac{1}{2} \Phi_{B, A}} = AB_{\perp} \), so that \( W \) rotates to \( V \oplus W_{\perp} \).

4. Clifford Geometric Product

We relate the geometric product of blades to the angle bivector, and interpret geometrically some of its well known algebraic properties. We consider first equal grades, leaving the general case for Sect. 4.2, and for completeness we define \( \Phi_{A, B} = 0 \) if \( A \) or \( B \) is 0.

Theorem 4.1. \( \tilde{A} \tilde{B} = \|A\| \|B\| e^{\Phi_{A, B}} \) for same grade blades \( A, B \in \bigwedge^p X \).

Proof. For \( A, B \neq 0 \), Proposition 3.9 gives \( \frac{B}{\|B\|} = \frac{A}{\|A\|} e^{\Phi_{A, B}} \). \( \square \)
Note that $\tilde{AB} = \epsilon_{A,B} \|A\| \|B\| e^{\Phi_{[A],[B]}}$ carries the relative orientation of $A$ and $B$, which makes sense as $\epsilon_{A,B}$ is the sign of $\tilde{A}*B = \langle \tilde{A}B \rangle_0$ (if $A*B \neq 0$). Still, this makes relating their orientations in $AB = \epsilon_{A,B} \|A\| \|B\| e^{\Phi_{[A],[B]}}$ less immediate. In Sect. 5.1.1 we discuss how this affects other products.

With Proposition 3.21 we obtain
\[ \tilde{AB} = \|A\| \|B\| \sum_{i \in I^{d+1,p}} \pi_{A,B_i} I_i = \epsilon_{A,B} \sum_{i \in I^{d+1,p}} \|P_{B_i}A\| \|B\| I_i. \]  
(8)

The projections from $A$ to the $B_i$’s make $A$ and $B$ seem to play very different roles in the product. But $A_i = AI_i$ satisfies $\pi_{A,B_i} = \pi_{B,A_i}$, as $e_i \cdot e_i^\perp = f_i \cdot f_i^\perp$, and so $\tilde{AB} = \epsilon_{A,B} \sum_i \|A\| \|P_{A_i}B\| I_i$ as well.

With $\tilde{AB}$ decomposed in terms of products of principal bivectors oriented from $[A]$ to $[B]$, as above, all coefficients have the same sign $\epsilon_{A,B}$. The only effect of swapping $A$ and $B$ (of same grade) is to reorient principal planes, from $[B]$ to $[A]$. Applying a reversion to (8) we obtain $\tilde{BA} = \epsilon_{A,B} \sum_i \|P_{B_i}A\| \|B\| \tilde{I}_i$, so that components change sign depending on whether $\tilde{I}_i = \pm I_i$ has an even or odd number of $I_i$’s. In Sect. 5.2 we relate this to the commutator of blades.

Example 4.2. Let $\{f_1, f_2, g_1, g_2\}$ be orthonormal, $e_1 = \frac{f_1 + 3g_1}{\sqrt{10}}$, $e_2 = \frac{2f_2 + g_2}{\sqrt{5}}$, $A = e_1 e_2$ and $B = f_1 f_2$. Then $\beta_A = (e_1, e_2)$ and $\beta_B = (f_1, f_2)$ are associated principal bases of $[A]$ and $[B]$, with $I_i = g_i f_i$, and $e_i^\perp = g_i$. As $\epsilon_{A,B} = 1$, all coefficients in $\tilde{AB} = (2 + 6I_1 + I_2 + 3I_1 I_2)/5\sqrt{2}$ are positive. And since $\|A\| = \|B\| = 1$, the coefficients are, in order, projection factors of $[A]$ on $[B] = [f_1 f_2, [e_1 f_2], [f_1 e_2]]$ and $[e_1 e_2] = ([A]^\perp)_P$ (the projective subspace of $[B]^\perp$ w.r.t. $[A]$).

In $\tilde{BA} = (2 + 6I_1 + I_2 + 3I_1 I_2)/5\sqrt{2}$, $\|A\| = \|B\| = e_1 e_2, f_1 f_2, [e_1 f_2]^\perp$ and $[f_1 e_2]^\perp = ([A]^\perp)_P$, where $f_1^\perp = \frac{3f_1 - g_1}{\sqrt{10}}, f_2^\perp = \frac{f_2 - 2g_2}{\sqrt{5}}$ and $([A]^\perp)_P$ is the projective subspace of $[A]^\perp$ w.r.t. $[B]$.

4.1. Plücker Coordinates in the Geometric Product

Consider Proposition 3.24 with $V = [A]$ and $W = [B]$. As the coefficients in that decomposition give Plücker coordinates of $V$ w.r.t. $\beta_Y$, and these are homogeneous, the same holds for the coefficients in
\[ \tilde{AB} = \epsilon_{A,B} \|A\| \|B\| \sum_{j \in I^{2p-d}} \cos \Theta_{V,Y_j} \sigma(C_j). \]
(9)

Example 4.3. In Example 4.2, $[A]$ has Plücker coordinates $\langle 2:6:1:3:0:0 \rangle$ in the basis $(f_1 f_2, e_1^\perp f_2, f_1 e_2^\perp, e_1^\perp e_2^\perp, f_1 e_1^\perp, f_2 e_2^\perp)$ of $\wedge^2 Y$ obtained from $\beta$. The last coordinates vanish as $A$ is partially orthogonal to any coordinate blade having neither $e_i^\perp$ nor $f_i$ for some $i$.

With $\tilde{BA}$ in terms of $\tilde{I}_i$’s, we find the same coordinates for $[B]$ in the basis $(e_1 e_2, f_1^\perp e_2, e_1 f_2^\perp, f_1 f_2^\perp, e_1 f_1^\perp, e_2 f_2^\perp)$ obtained from $\{e_1, e_2, f_1^\perp, f_2^\perp\}$. 

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Example 4.4. In Fig. 3, $[A]$ has normalized Plücker coordinates $(\frac{3}{5} : \frac{4}{5} : 0)$ w.r.t. $(f_1 f_2, f_1 g_2, f_2 g_2)$, and $B = f_1 f_2$. From this, the norms and orientations, we obtain $\tilde{A} B = -3 - 4I_2$, where $I_2 = g_2 f_2$ is one of the principal bivectors, oriented from $A$ to $B$. As the first principal plane is degenerate, its principal bivector is $I_1 = 0$, so $\tilde{A} B$ has no 4-vector (this is another way to look at the usual result that $[A] \cap [B] \neq \{0\} \Rightarrow A \land B = 0$).

The lack of economy of Plücker coordinates is reflected in the geometric product of high grade blades in even larger spaces, which tends to produce a large number of components, linked by many relations. It is an interesting question whether this is a price to be paid for the algebraic simplicity of this product: if a more economical operation could accomplish the same tasks, would its algebra necessarily be more complicated?

4.2. Distinct Grades

We can adapt Theorem 4.1 for blades of different grades using PO decompositions. In the following result, the same principal bases must be used for the decomposition and to form $\Phi_{A,B}$.

Proposition 4.5. Let $A \in \bigwedge^p X$ and $B \in \bigwedge^q X$ be blades.

(i) If $p \leq q$ then $\tilde{A} B = \|A\|\|B\|e^{\Phi_{A,B}} B_\perp$, where $B_\perp$ is an orthogonal subblade of $B$ w.r.t. $A$.

(ii) If $p \geq q$ then $\tilde{A} B = \|A\|\|B\|e^{\Phi_{A,B}} \tilde{A}_\perp$, where $A_\perp$ is an orthogonal subblade of $A$ w.r.t. $B$.

Proof. (i) Follows from Theorem 4.1, since $\tilde{A} B = (\tilde{A} B_P) B_\perp$, $B_P$ has the same grade as $A$, $\|B_P\| = \|B\|$ and $\Phi_{A,B_P} = \Phi_{A,B}$. (ii) Similar, using $\tilde{A} B = \tilde{A}_\perp(\tilde{A}_P B)$ and also that $\tilde{A}_\perp$ commutes with $e^{\Phi_{A,B}}$, as it is completely orthogonal to all principal bivectors. \hfill \Box
As the orthogonal subblade is completely orthogonal to all components of $e_{\Phi A,B}$, we actually have $e_{\Phi A,B} \wedge B_\perp$ and $e_{\Phi A,B} \wedge A_\perp$ in these formulas. So, in the geometric product of blades, the smaller one operates only with the projective subblade of the larger blade, preserving the orthogonal subblade.

This result also shows that, while $e_{\Phi A,B}$ and $B_\perp$ (or $A_\perp$) can depend on the choice of principal bases, their product cannot.

Example 4.6. Let \{f_1, \ldots, f_4, g_1, g_2\} be orthonormal, $e_1 = \sqrt{3}/2 f_1 + 1/2 g_1$, $e_2 = g_2$, $A = e_1 e_2$ and $B = f_1 f_2 f_3 f_4$. Then $I_1 = g_1 f_1$, $I_2 = g_2 f_2$, $\Phi_{A,B} = \Phi_{A,f_1 f_2} = \pi_6 I_1 + \pi_2 I_2$, $e_{\Phi A,B} = \sqrt{3}/2 I_2 + 1/2 I_1 I_2$ and $\tilde{AB} = (\sqrt{3}/2 + 1/2 I_1) I_2 f_3 f_4$. The common factor $I_2$ is due to $\theta_2 = \pi/2$ (as $e_2 \perp f_2$, projections of $A$ on coordinate subspaces having $f_2$ instead of $e_2$ vanish), and the rest is $B_\perp = f_3 f_4$. The coefficients are projection factors of $[A]$ on $[f_1 g_2 f_3 f_4]$ and $[g_1 g_2 f_3 f_4]$, and $[A]$ has Plücker coordinates $(0 : \sqrt{3} : 0 : 1 : 0 : 0)$ in the basis $(f_1 f_2, f_1 g_2, f_1 g_2, f_1 g_2, f_1 g_2, f_2 g_2)$ of $\wedge^2 [f_1 f_2 g_1 g_2]$. Signs in $\tilde{BA} = (\sqrt{3}/2 - 1/2 I_1) I_2 f_3 f_4$ are the result of reverting the $I_i$’s and $B_\perp$.

4.3. Principal Angles via Geometric Algebra

Hitzer’s method \cite{19} to find principal angles via geometric product can be used to decompose $e_{\Phi v,w}$ or $\tilde{AB}$ in terms of $I_i$’s, as follows.

Given subspaces $V, W \subset X$ with $\dim V = p \leq q = \dim W$, compute $\tilde{AB}$ for unit blades $A \in \wedge^p V$ and $B \in \wedge^q W$ with $e_{A,B} = 1$. The result is a $(p + q)$-blade if, and only if, all principal angles are $\pi/2$, in which case any orthonormal bases are associated principal bases.

Assume otherwise, and let $d = \dim(V \cap W)$ and $D = \max\{i : \theta_i \neq \pi/2\}$, whose values we might not know yet. Using Proposition 3.13, with $R_i = 1$ for $i \leq d$, $R_i = I_i$ for $i > D$, and expanding the other $R_i$’s as in Proposition 3.14, we find that the nonzero components of $\tilde{AB} = e_{\Phi A,B} B_\perp$ are:

\[
\tilde{AB} = R_{d+1} \cdots R_{D} I_{D+1} \cdots I_{p} B_\perp = \left( c_{d+1} \cdots c_{D} \right) + s_{d+1} c_{d+2} \cdots s_{D} I_{d+1} + \cdots + c_{d+1} \cdots c_{D-1} s_{D} I_{D} + \cdots + s_{d+1} c_{d+2} \cdots s_{D} I_{d+1} \cdots I_{D-1} \right) I_{D+1} \cdots I_{p} B_\perp.
\]

So the lowest and highest non-vanishing grades are $p + q - 2D$ and $p + q - 2d$, and this tells us which $\theta_i$’s are 0 or $\pi/2$. From orthonormal bases of $V \cap W \perp$ and $W \cap V \perp$ we get principal vectors $e_{D+1}, \ldots, e_p$ and $f_{D+1}, \ldots, f_q$ to form $I_{D+1}, \ldots, I_p$ and $B_\perp$.

Also, the product of the non-zero components of second lowest grade (10c) by the inverse of the lowest grade blade (10b) is a bivector. Decomposing it into commuting blades we obtain $\tan \theta_{d+1} I_{d+1} + \cdots + \tan \theta_D I_D$ and find $\theta_i$ and $I_i$ for $d < i \leq D$. 

With this we can write the decompositions. If desired, we can also obtain principal vectors \( e_i = f_i \) for \( i \leq d \) from an orthonormal basis of \( V \cap W \), and for \( d < i \leq D \) take unit vectors \( e_i \in V \cap [I_i] \) and \( f_i \in W \cap [I_i] \) with \( e_i \cdot f_i > 0 \).

### 4.4. The Hidden Geometry of Algebraic Properties

In this section, we use our results to reveal the rich geometry that lies behind some simple and well known algebraic properties of the Clifford product.

#### 4.4.1. Invertibility.

The invertibility of non-null blades \( A \in \bigwedge^p X \) and \( B \in \bigwedge^q X \) can be seen as the result of \( AB \) carrying all geometric data needed to, given one blade, recover the other one.

If \( p = q \) we can see in \( A = AB \frac{B}{\|B\|^2} = \frac{A}{\|B\|^2}B = (\epsilon_{A,B} \frac{\|A\|}{\|B\|} e^{-\Phi_{\{B\},\{A\}}}B) \) how \( AB \) has all we need to change the orientation, norm and subspace (via Proposition 3.4) of \( B \) into those of \( A \). More precisely, the Plücker coordinates stored in \( AB = \sum_i \epsilon_{\tilde{A},B_i} \|A\|\|B\|\pi_{A,B_i} \) allow us to locate \([A] \) relative to \([B] \). Each component has all information needed to, using \( B \), find one of \( A = \sum_{j \in T^2_{d-p}} P_{C_j}A = \sum_{i \in T_{d+1,p}} P_{B_i}A: \)

\[
\begin{align*}
\left( \epsilon_{\tilde{A},B} \|A\|\|B\|\pi_{A,B_i} I_i \right) \left( \tilde{B}/\|B\|^2 \right) &= \epsilon_{A,B} \|P_{B_i} A\| I_i B/\|B\| \\
&= \epsilon_{A,B} \|P_{B_i} A\| I_i \epsilon_B F \\
&= \epsilon_A \|P_{B_i} A\| F_i \\
&= P_{B_i} A.
\end{align*}
\]

If \( p < q \), we obtain the same result with components of \( AB = (AB_P)B_\perp \). If \( p > q \), each component of \( AB = (-1)^{p(q-q)}A_\perp (APB) \) multiplied by \( B^{-1} \) gives \( (P_{B_i} A_F)A_\perp \).

In particular, inverting \( B \) means finding \( A \) such that \( AB = 1 \). This 1 may seem to carry too little information, but it has all we need. The fact that it is a scalar means \( AB \) has no orthogonal subblade, so \( p = q \), and no \( I_i \)'s, so \([A] = [B] \) as all other Plücker coordinates vanish. Its norm implies \( \|A\| = 1/\|B\| \), and its sign shows \( \epsilon_{\tilde{A},B} = 1 \), so \( A \) has the orientation of \( \tilde{B} \). Putting it all together we find \( A = \tilde{B}/\|B\|^2 \), as expected.

#### 4.4.2. The Geometry of \( \|AB\| = \|A\|\|B\| \).

The relation (9) between the geometric product and the asymmetric angles is more complicated than those we give in Sect. 5.1 for other products. This is understandable, since this product includes the others as its components, and carries information about projections on various subspaces.

Surprisingly, this complexity is behind one of its simplest properties: for blades, \( \|AB\| = \|A\|\|B\| \). The algebraic proof is deceivingly easy, but not very illuminating, and does not explain what makes this product special in this respect, while others are submultiplicative.

A geometric proof can use (9) and (7) to obtain, for same grade blades, \( \|AB\|^2 = \|A\|^2\|B\|^2 \sum_j \cos^2 \Theta_{V,Y_j} = \|A\|^2\|B\|^2 \). And if \( B \), for example, has larger grade, using a PO decomposition, and since \( B_\perp \) is completely orthogonal to \( A \) and \( B_P \), we also find \( \|AB\| = \|AB_P\|\|B_\perp\| = \|A\|\|B_P\| = \|A\|\|B\| \).
To show what is behind this, we note that (7) is in fact a general identity for asymmetric angles with coordinate $p$-subspaces of orthogonal bases [28], that leads to a generalized Pythagorean theorem [29] stating that projections on all such subspaces preserve the total squared volume\(^{11}\), i.e., \( \|A\|^2 = \sum_j \|P_j A\|^2 \). Writing (9) as $AB = \epsilon_{A,B} \|B\| \sum_j \|P_j A\| \sigma(C_j)$, we again find $\|AB\|^2 = \|B\|^2 \sum_j \|P_j A\|^2 = \|A\|^2 \|B\|^2$.

So, while the products in (13) are submultiplicative for blades because they involve projections on single subspaces, which shrink volumes and thus reduce norms, the geometric product preserves norms precisely because it involves projections on all coordinate $p$-subspaces $Y_j$.

4.4.3. Duality. The relation\(^{12}\) $(AB)^* = AB^*$, where $*$ denotes the dual obtained via product with $J^{-1}$ for a given unit pseudoscalar $J$, is algebraically trivial, a mere consequence of the associativity of the geometric product. But it expresses a duality between $AB$ and $AB^*$ which, as we show, is linked to another one between $e^{\Phi_{V,W}}$ and $e^{\Phi_{V,W^\perp}}$, reflecting a symmetry that swaps sines and cosines in the components of Proposition 3.14.

Let $A \in \bigwedge^p X$ and $B \in \bigwedge^q X$ be unit blades, $V = [A]$ and $W = [B]$, with associated principal bases $\beta_V = (e_1, \ldots, e_p)$ and $\beta_W = (f_1, \ldots, f_q)$, and principal angles $\theta_1, \ldots, \theta_m$ for $m = \min\{p, q\}$, and let $e_i^\perp, I_i, R_i$ be as before.

We consider first a case with $p = q$ and $V \cap W = \{0\}$, and take duals w.r.t. $J = I_1 I_2 \cdots I_p$ (for which $[J] = V \oplus W$). Completing $(e_1^\perp, \ldots, e_p^\perp)$ to a principal basis of $W^\perp$ w.r.t. $V$, we obtain $\Phi_{V,W^\perp} = \sum_i^p (\frac{1}{2} - \theta_i) I_i$ and principal rotors $R_i^\perp = e_i e_i^\perp = R_i I_i = \sin \theta_i + I_i \cos \theta_i$.

**Proposition 4.7.** Under the above conditions we have $(e^{\Phi_{V,W}})^* = e^{\Phi_{V,W^\perp}}$ and $(e^{\Phi_{A,B}})^* = e^{\Phi_{A,B^*}}$.

**Proof.** Proposition 3.13 gives $(e^{\Phi_{V,W}})^* = R_1 \cdots R_p \tilde{I}_p \cdots \tilde{I}_1 = R_1^\perp \cdots R_p^\perp = e^{\Phi_{V,W^\perp}}$. And using (3) we find $B^* = \epsilon_B f_1 f_2 \cdots f_p \tilde{I}_p \cdots \tilde{I}_1 = \epsilon_B e_1^\perp \cdots e_p^\perp$, so $\epsilon_{A,B^*} = \epsilon_{A,B}$ and $[B^*]$ is the projective subspace of $W^\perp$ w.r.t. $V$. Thus the second identity follows from the first one and Proposition 3.8.\(\square\)

Theorem 4.1 shows, at least under the above conditions, that this duality of exponentials lies behind $(AB)^* = AB^*$. Even better, expanding principal rotors as in Proposition 3.14, we observe a component-wise duality,

$$(e^{\Phi_{V,W}})^* = (c_1 \cdots c_p)^* + (s_1 c_2 \cdots c_p I_1)^* + \cdots + (s_1 \cdots s_p I_1 \cdots I_p)^* = c_1 \cdots c_p \tilde{I}_1 \cdots \tilde{I}_p + s_1 c_2 \cdots c_p \tilde{I}_2 \cdots \tilde{I}_p + \cdots + s_1 \cdots s_p = e^{\Phi_{V,W^\perp}},$$

so each component of grade $k$ in $AB$ is dual to one of grade $2p - k$ in $AB^*$, generalizing the dualities [6, p.82] $(A \land B)^* = A \lvert B^*$ and $(A \lvert B)^* = A \land B^*$ between outer product and contraction\(^{13}\) (or Hestenes inner product [18, p.23]).

In the general case ($p \neq q$, $V \cap W \neq \{0\}$, duals w.r.t. the whole space), the duality between exponentials becomes more complicated, but it

\(^{11}\)In complex spaces, the (non-squared) volume is the sum of volumes of projections [29].

\(^{12}\)We adopt the convention that $*$ takes precedence over the product, so $AB^*$ means $A(B^*)$.

\(^{13}\)See Definition 5.1.
still has the same kind of symmetry, even if the corresponding grades are different. Taking duals in (10a) w.r.t. \( J = f_1 \cdots f_d I_{d+1} \cdots I_p f_{p+1} \cdots f_q \) (for which \(|J| = V + W|\)) we find \((AB)^* = (R_{d+1} \cdots R_D I_{D+1} \cdots I_p f_{p+1} \cdots f_q)^* = R_D^* \cdots R_{d+1}^* f_d \cdots f_1\), and likewise for \(\tilde{AB}^*\). So, except for the extra blades shifting the grade correspondence, the duality between \(AB\) and \(AB^*\) is again due to \(R_i\)'s turning into \(R_i^*\)’s. Taking duals w.r.t. the whole space, another grade shift makes each component of grade \(k\) in \(AB\) dual to one of grade \(n - k\) in \(AB^*\), where \(n = \dim X\).

5. Other Geometric Algebra Products

Let \(A \in \bigwedge^p X\) and \(B \in \bigwedge^q X\) be blades. As is known, \(AB\) can have nonzero components of grades \(|q - p|, |q - p| + 2, \ldots, p + q\), with the first and last ones (and also some of the vanishing ones) giving useful component subproducts.\(^{14}\) Most of these products are well known, and we have already been using some, but we provide here a definition for easy reference (for general multivectors, they are extended linearly).

Definition 5.1. The scalar product \(A \star B\), left and right contractions\(^{15}\) \(A|B\) and \(A[B\), (fat) dot product \(A \cdot B\), and outer product \(A \wedge B\) are, respectively, the components of grades 0, \(q - p\), \(p - q\), \(|q - p|\) and \(p + q\) of \(AB\) (a negative grade means the component is 0). Hestenes inner product \(A \cdot B\) is the \(|q - p|\) component if \(p, q \neq 0\), vanishing otherwise.\(^{16}\)

Contractions and fat dot product, introduced by Lounesto [25] and Dorst [5], are less known alternatives to Hestenes inner product. These products are related by

\[
A \cdot B = \begin{cases} 
A|B & \text{if } p \leq q, \\
A\overline{B} & \text{if } p \geq q,
\end{cases}
\quad \text{and} \quad
A \cdot B = \begin{cases} 
A \cdot B & \text{if } p, q \neq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Left and right contractions are related by \(A|B = (\overline{B}|A)^\sim\), and using a PO decomposition we obtain \(A|B = (A \star B P)B_\perp\). When grades are distinct, they are asymmetric (in general, \(A|B \neq B|A\) and \(A[B \neq B[A\), with \(A|B = 0\) if \(p > q\), and \(A[B = 0\) if \(p < q\).

As noted by Lounesto [26, p. 291], contractions have better properties than \(A \cdot B\) (or \(A \ast B\)). Dorst [4, 5] also advocates for their use, arguing that identities involving \(A \cdot B\) often depend on grade conditionals, which tend to accumulate as they are combined, while with contractions “known results are simultaneously generalized and more simply expressible, without conditional exceptions” [4, p. 136], since their asymmetry allows them to ‘switch off’ automatically when conditions fail to hold.

\(^{14}\)The geometric product can be defined axiomatically and the other products obtained as its components [5, 18, 20], or it can be defined using Grassmann algebra products, taken as more fundamental ones [6, 12]. The different approaches are discussed in [26].

\(^{15}\)These are Dorst’s symbols [5]. Lounesto [25] uses \(\wedge\) and \(\vee\), which we will reserve for a slightly different contraction in Sects. 5.1.1 and 5.2.

\(^{16}\)This is the definition from [18]. In later works [17], Hestenes removes the exceptionality of the scalar case, so that \(A \cdot B = A \ast B\) for all \(p, q \geq 0\).
In Sect. 5.1 we relate these subproducts to our angles. Section 5.2 discusses other products: the usual commutator and an anticommutator.

5.1. Component Subproducts

There is a reason why only certain components of the geometric product give interesting subproducts. If \( p = q \), rewriting (8) as

\[
AB = \epsilon_{A,B} \|A\| \|B\| \sum_{i \in \mathcal{I}^{d+1,p}} \cos \Theta_{[A],[B]} I_i, \tag{12}
\]

we see how the components of grades 0, 2, 4, ..., \( 2p \) (in fact, at most \( 2(p - d) \), where \( d = \dim([A] \cap [B]) \)) are formed. We can also understand why \( \langle AB \rangle_0 \) and \( \langle AB \rangle_{2p} \) are the most relevant ones, giving the products of Definition 5.1: they describe projections on \([B]\) and \([B]^\perp\), while other components involve less important subspaces. If \( p \neq q \), the orthogonal subblade increases grades by \(|q - p|\), so now \( \langle AB \rangle_{|q-p|} \) and \( \langle AB \rangle_{p+q} \) are most relevant.

Identifying the appropriate components in (12), we obtain formulas relating the subproducts to our various angles.

**Theorem 5.2.** For any blades \( A, B \in \bigwedge X \),

\[
\begin{align*}
|A \ast B| &= \|A\| \|B\| \cos \hat{\Theta}_{[A],[B]}, \tag{13a} \\
\|A \rfloor B\| &= \|A\| \|B\| \cos \Theta_{[A],[B]}, \tag{13b} \\
\|A \lfloor B\| &= \|A\| \|B\| \cos \Theta_{[B],[A]}, \tag{13c} \\
\|A \cdot B\| &= \|A\| \|B\| \cos \Theta_{[A],[B]} (A, B \text{ non-scalars}), \tag{13d} \\
\|A \wedge B\| &= \|A\| \|B\| \cos \Theta^\perp_{[A],[B]}, \tag{13f}
\end{align*}
\]

**Proof.** Let \( p = \text{grade}(A) \) and \( q = \text{grade}(B) \). If \( p = q \), the first 5 products are the \( i = 0 \) component in (12), with angle \( \Theta_{[A],[B]} = \Theta_{[B],[A]} = \hat{\Theta}_{[A],[B]} = \Theta_{[A],[B]^\perp} \), by Proposition 2.10iv. If, moreover, \( d = 0 \), then \( A \wedge B \) is the \( i = (1, \ldots, p) \) component, for which \( \Theta_{[A],[B]} = \hat{\Theta}_{[A],[B]^\perp} \), otherwise \( A \wedge B = 0 \) and \( \Theta_{[A],[B]^\perp} = \frac{\pi}{2} \), by Proposition 2.14ii.

If \( p \neq q \) then \( A \ast B = 0 \) and \( \hat{\Theta}_{[A],[B]} = \frac{\pi}{2} \), so (13a) holds. Using a PO decomposition of the larger blade, the unit orthogonal subblade increases grades but preserves norms. So if \( p < q \) we have \( \|\langle AB \rangle_{r+q-p}\| = \|\langle AB \rangle_{r}\| \), and taking \( r = 0 \) and \( r = 2p \) we find, using Proposition 3.26, that (13b) and (13f) remain valid. Also, in this case both sides of (13c) vanish. The case \( p > q \) is similar, and (13d) and (13e) follow from (11). \( \square \)

The geometric algebra literature has analogous results for the angles of Sect. 2.4.3. Hestenes angle definition is similar to (13a). Hitzer also gives it, and a result like (13f) but with the product of sines of principal angles, which is not interpreted in terms of a single angle. Dorst’s description of contraction norm for \( p \leq q \) corresponds to (13b).

Dorst’s contention against \( A \cdot B \) and \( A \ast B \) is supported by (13d) and (13e), as the min-symmetrized angle has worse properties. Contractions, on the other hand, are related to the asymmetric angle, and their asymmetries
Proposition 5.4. Let \( \Theta \) which can be obtained from the contractions via (11). Then \( e \) gives more detailed formulas, with oriented angles. We omit \( \tilde{\theta} \) then \( \Theta = 0 \) or \( \dim(X) = n \), requiring 0 grade \( A \leq \) grade \( B \).

Corollary 5.3. For any blades \( A, B \in \bigwedge X \), \( \|A \wedge B\| = \|B \wedge A\| \). Also, \( A \wedge B = 0 \Leftrightarrow A \perp B \).

The symmetry of \( \Theta \) is reflected in (13f), which, ironically, depends on the asymmetry of \( \Theta \). For example, \( A \wedge B = 0 \) for any \( A, B \in \bigwedge \mathbb{R}^3 \), but \( \Theta_{[A], [B]} \) could assume any value if it were the usual (symmetric) angle between a plane \([A]\) and a line \([B] \). Without asymmetry, (13f) would need the hypothesis \([A] \cap [B] = \{0\}\), as in analogous results relating volumes of parallelotopes [1] and matrix volumes [31] to products of sines of principal angles.

For \( v, w \in X \), (13f) gives the usual \( \|v \wedge w\| = \|v\| \|w\| \sin \theta_{v,w} \), as (5) reduces to a single sine. We could put the formula for \( \|A \wedge B\| \) in this familiar form using the sine of an angle \( \Theta_{V,W} = \frac{\pi}{2} - \Theta_{V,W} \), which however does not have a nice interpretation in \( \bigwedge X \) as \( \Theta_{V,W} \) does [28].

For simplicity, Theorem 5.2 presented just the product norms. Now we give more detailed formulas, with oriented angles. We omit \( \tilde{\Lambda} \cdot B \) and \( \tilde{\Lambda} \cdot B \), which can be obtained from the contractions via (11).

Proposition 5.4. Let \( A \in \bigwedge^p X \) and \( B \in \bigwedge^q X \) be nonzero blades, \((e_1, \ldots, e_p)\) and \((f_1, \ldots, f_q)\) be associated principal bases of \([A]\) and \([B]\), with orthoprincipal vectors \( e_{d+1}^\perp, \ldots, e_m^\perp \) and \( f_{d+1}^\perp, \ldots, f_m^\perp \), where \( d = \dim([A] \cap [B]) \) and \( m = \min\{p, q\} \), and \( A_L \) and \( B_L \) be the corresponding orthogonal subblades. If \( d \neq 0 \) let \( J = 0 \), otherwise let \( J = e_1 e_2^\perp \cdots e_p^\perp f_1 f_2 \cdots f_q \) if \( p \leq q \), and \( J = e_1 e_2 \cdots e_p f_1^\perp f_2^\perp \cdots f_q^\perp \) if \( p \geq q \). Then

\[
\begin{align*}
\tilde{\Lambda} \ast B &= \|A\| \|B\| \cos \tilde{\Theta}_{A,B}, \\
\tilde{\Lambda} \bigwedge B &= \|A\| \|B\| \cos \theta_{A,B} B_L, \\
\tilde{\Lambda} \bigwedge B &= \|A\| \|B\| \cos \tilde{\Theta}_{B,A} \tilde{A}_L, \\
A \wedge B &= \|A\| \|B\| \cos \Theta_{A,B} J.
\end{align*}
\]

Proof. (14a) Follows from (13a), Definition 2.16 and the relation between \( e_{A,B} \) and \( \tilde{\Lambda} \ast B \). (14b) \( \tilde{\Lambda} \bigwedge B = (\tilde{\Lambda} \ast B) B_L \), and \( \tilde{\Theta}_{A,B} = \Theta_{A,B} \), by Proposition 3.26. (14c) Similar. (14d) Assume \( d = 0 \), otherwise both sides vanish. For \( p \leq q \), (12) gives \( A \wedge B = e_{A,B} \|A\| \|B\| \cos \Theta_{[A], [B]}^\perp I_1 \cdots I_p B_L \). The reordering of \( e_i^\perp \)'s from \( I_i = e_i^\perp f_i \) cancels the reversion in \( e_{A,B} \), and we use Definition 2.16. The case \( p > q \) is similar.

Note that both definitions of \( J \) give the same result when \( p = q \). If \( d = 0 \) then \( [J] = [A] \oplus [B] \). For \( d \neq 0 \) we defined \( J = 0 \) for simplicity, but, since \( \Theta_{A,B}^\perp = \frac{\pi}{2} \) anyway (by Proposition 2.14 ii), one might as well use some \( J \) for
which \([J] = [A] + [B]\). If \(p \neq q\) then \(\Theta_{A,B}\) and \(\Theta_{B,A}\) depend on the choice of principal bases, but in a way that offsets changes in \(B_{\perp}\) and \(A_{\perp}\).

### 5.1.1. Products and Reversions

As noted after Theorem 4.1, the geometric product \(AB\) involves the relative orientation of \(\tilde{A}\) and \(B\). The same holds for its component subproducts, and this makes interpreting the resulting orientations less immediate, even unfeasible if grade\((A)\) is unknown. To compensate and obtain more geometrically meaningful orientations, these products are often combined with a reversion. For example, the Grassmann algebra inner product is \(<A, B> = \tilde{A}^*B\), the norm is \(\|A\| = (\tilde{A}^*A)^{1/2}\), and a slightly different contraction given by \(A_B = \tilde{A}|B\) (used in [32] and in Appendix A) has its orientation directly related to those of \(A\) and \(B\) (Corollary A.7). This is another geometric price to be paid for algebraic simplicity. A product given by \(A \odot B = \tilde{A}B\) would involve \(\epsilon_{A,B}\) directly, but would not be associative, making inverses less useful.

The outer product is the only subproduct inheriting the associativity of \(AB\), and the only one whose orientation with a reversion \((\tilde{A} \wedge B)\) seems less natural. The proof of (14d) shows why \(A \wedge B\) reflects the orientations of \(A\) and \(B\) directly: the reversion in \(\epsilon_{A,B}\) disappears as we reorder the \(e_{i}^\perp\)'s to match the usual convention of having first the vectors of \(A\) (or their components orthogonal to \(B\)) then those of \(B\), ordered according to the orientation of each blade.

But this begs the question of the reason for such convention. Of course, it is appropriate that the orientations of the geometric object formed by joining \(A\) and \(B\) and of the algebraic element \(A \wedge B\) used to represent it match each other. But, geometrically, orienting by \(A \triangle B = \tilde{A} \wedge B\) (first the vectors of \(A\) in reverse order, then those of \(B\)) would have been fine. Again, the answer lies in algebraic simplicity: \(A \triangle B\) would have been a worse product, having neither associativity nor alternativity.

### 5.2. Commutator and Anticommutator

Let \(M, N \in \bigwedge X\). The commutator \(M \times N = (MN - NM)/2\) is mainly used with \(M\) or \(N\) being a bivector, as in this case it preserves grades, and \((\bigwedge^2 X, \times)\) is a Lie algebra [18]. But the whole \((\bigwedge X, \times)\) is also a Lie algebra, and, as we show, the commutator of blades has nice properties for all grades, suggesting that this product may be more interesting than usually recognized.

The anticommutator \(M \triangleright N = (MN + NM)/2\) turns the Clifford algebra into a special Jordan algebra [30]. It does not seem to have attracted the attention of geometric algebraists, which is strange, given that Clifford algebras are often built from generators satisfying some anticommutation relation (e.g., for the Dirac algebra [15] we have \(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_\mu\nu\), which reduces to \(\eta_\mu\nu\).

These products are linked by the following identities.

**Proposition 5.5.** Let \(M, N \in \bigwedge X\).

(i) \(M \triangleright N + M \times N = MN\).

(ii) \(M \triangleright N - M \times N = NM\).
(iii) \( \|M \otimes N\|^2 + \|M \times N\|^2 = \frac{\|MN\|^2 + \|NM\|^2}{2} \).

**Proof.** (iii) \( (\widetilde{MN} + \widetilde{NM}) \ast (MN + NM) + (\widetilde{MN} - \widetilde{NM}) \ast (MN - NM) = 2(\widetilde{MN}) \ast (MN) + 2(\widetilde{NM}) \ast (NM) \). \( \square \)

The trivial identity (i) becomes more interesting for blades \( A \) and \( B \), in which case the components of \( AB \) are distributed between \( A \otimes B \) and \( A \times B \) according to their grades. This can be proven algebraically, analyzing the signs of \( (A \otimes B)^\sim = \tilde{A} \otimes \tilde{B} = \pm A \otimes B \) and \( (A \times B)^\sim = -\tilde{A} \times \tilde{B} = \mp A \times B \). We give a more geometric argument, which will be useful later.

**Proposition 5.6.** Given blades \( A \in \bigwedge^p X \) and \( B \in \bigwedge^q X \), with \( p \leq q \), and a PO decomposition of \( B \) w.r.t. \( A \), we have

\[
A \times B = \begin{cases} 
(A \times Bp) \wedge B_\perp = \sum_{k=0}^\infty \langle AB \rangle_{q-p+4k+2} \ if \ p(q-1) is even, \\
(A \otimes Bp) \wedge B_\perp = \sum_{k=0}^\infty \langle AB \rangle_{q-p+4k} \ if \ p(q-1) is odd,
\end{cases}
\]

and likewise for the anticommutator, with the conditions swapped.

**Proof.** As seen in Sect. 4, if \( p = q \) the only difference between \( \tilde{AB} \) and \( \bar{BA} \) is that components with an odd number of \( I_i \)'s switch signs. Therefore \( A \times B = (-1)^{\frac{p(p-1)}{2}} (\tilde{AB} - \bar{BA})/2 \) has the components of grades \( 4k + 2 \) \((k \in \mathbb{N})\) of \( AB \), while \( A \otimes B \) has those of grades \( 4k \). And when \( p < q \) we have \( 2A \times B = AB_p B_\perp - B_p B_\perp A = (AB_P - (-1)^{p(q-p)} B_P A) B_\perp \), and likewise for the anticommutator. \( \square \)

**Example 5.7.** In \( \bigwedge \mathbb{R}^2 \), for \( M = i \) and \( N = 1 + ij \) we have \( M \otimes N = i \) and \( M \times N = j \), which decompose \( MN = i + j \), but not by grades. This shows it is not enough that one of the multivectors be a blade.

**Proposition 5.8.** Let \( A, B \in \bigwedge X \) be blades.

(i) \( (A \otimes B) \times (A \times B) = 0 \).

(ii) \( (A \otimes B) \ast (A \times B) = 0 \).

(iii) \( (A \otimes B)^2 - (A \times B)^2 = A^2 B^2 \).

(iv) For unit blades of same grade, \( (A \otimes B)^2 - (A \times B)^2 = 1 \).

(v) \( \|A \otimes B\|^2 + \|A \times B\|^2 = \|A\|^2 \|B\|^2 \).

**Proof.** (i) \( (AB + BA)(AB - BA) - (AB - BA)(AB + BA) = 2BAAB - 2ABBA = 0 \), since for blades \( A^2 \) and \( B^2 \) are scalars. (ii) By Proposition 5.6, \( A \otimes B \) and \( A \times B \) have no common grades. (iii) \( (AB + BA)^2 - (AB - BA)^2 = 2ABBA + 2BAAB = 4A^2 B^2 \). (iv) For unit \( p \)-blades, \( A^2 = B^2 = (-1)^{\frac{p(p-1)}{2}} \).

(v) Follows from Proposition 5.5iii. \( \square \)

**Example 5.9.** Let \( M = N = 1 + i \in \bigwedge \mathbb{R}^2 \). Then \( MN = NM = M \otimes N = 2 + 2i \) and \( M \times N = 0 \). One can check that, while Proposition 5.5iii holds, Proposition 5.8v does not (as these are not blades).

These products are related to hyperbolic functions (see Appendix B) of the angle bivector.
**Theorem 5.10.** For blades \( A, B \in \bigwedge^p X \) we have
\[ \tilde{A} \otimes B = \|A\|\|B\| \cosh \Phi_{A,B} \]
and
\[ \tilde{A} \times B = \|A\|\|B\| \sinh \Phi_{A,B}. \]

**Proof.** As grades are equal, Theorem 4.1 gives
\[ \tilde{A} \otimes B = (\tilde{A}B + \tilde{B}A)/2 = \|A\|\|B\|(e^{\Phi_{A,B}} + e^{-\Phi_{A,B}})/2, \]
and likewise for \( \tilde{A} \times B \).

These formulas may give the impression that these products are not submultiplicative for blades, which is false, by Proposition 5.5v. What happens is that, as the \( I_i \)'s in \( \Phi_{A,B} \) square to \(-1\), these hyperbolic functions expand in terms of trigonometric functions of the \( \theta_i \)'s. Indeed, for \( V = [A] \) and \( W = [B] \) we have
\[ \cosh \Phi_{A,B} = \epsilon_{A,B} \cosh \Phi_{V,W} \]
(likewise for sinh), and \( \cosh \Phi_{V,W} \) (resp. sinh) is given by the terms in Proposition 3.14 with an even (resp. odd) number of \( I_i \)'s, so that \( \|\cosh \Phi_{A,B}\|^2 + \|\sinh \Phi_{A,B}\|^2 = \|e^{\Phi_{A,B}}\|^2 = 1 \) (as in Proposition B.3vi).

**Example 5.11.** Let \( c_i = \cos \theta_i, \ s_i = \sin \theta_i \), and \( p = \dim V = \dim W \). If \( p = 2 \) then \( \cosh \Phi_{V,W} = c_1c_2 + s_1s_2I_1I_2 \) and \( \sinh \Phi_{V,W} = s_1c_2I_1 + c_1s_2I_2 \). If \( p = 3 \) then \( \cosh \Phi_{V,W} = c_1c_2c_3 + s_1s_2c_3I_1I_2 + s_1c_2s_3I_1I_3 + c_1s_2s_3I_2I_3 \) and \( \sinh \Phi_{V,W} = s_1c_2c_3I_1 + c_1s_2c_3I_2 + c_1c_2s_3I_3 + s_1s_2s_3I_1I_2I_3 \).

With Theorem 5.10, we can see that many properties presented here reflect others of hyperbolic functions given in Appendix B. For example, Proposition 5.8iv corresponds directly to B.2vi. If grades are distinct, the orthogonal subblade can demand some extra effort from us to see the correspondence, but it exists. For example, consider Proposition 5.8iii with \(m=n\).

**Proposition 5.12.** Given blades \( A, B \in \bigwedge^p X \), let \( r \) be the number of principal angles that are \( \frac{\pi}{2} \). Then \( \cosh \Phi_{A,B} = 0 \) (resp. sinh) if, and only if, \( r \) is odd (resp. even) and any other principal angle is 0.

**Proof.** If \( \theta_1 \leq \cdots \leq \theta_p \) are the principal angles then \( c_i = \cos \theta_i \neq 0 \) for \( i \leq p - r \), and \( s_i = \sin \theta_i \neq 0 \) for \( i > p - r \). As \( \Phi_{A,B} \) has components with all possible products of an even number of \( s_i \)'s, multiplied by \( c_i \)'s of the other indices, if \( r \) is even it has a component with \( c_1 \cdots c_{p-r} s_{p-r+1} \cdots s_p \neq 0 \). So \( \cosh \Phi_{A,B} = 0 \) implies \( r \) is odd. But then it will have a component with \( c_1 \cdots c_{p-r-1} s_{p-r} \cdots s_p \), which must vanish, so \( \theta_{p-r} = 0 \). Conversely, if \( r \) is odd and \( \theta_{p-r} = 0 \), all components will have some \( c_i = 0 \) or some \( s_i = 0 \). The proof for sinh \( \Phi_{A,B} \) is similar.

**Proposition 5.13.** Blades \( A \in \bigwedge^p X \) and \( B \in \bigwedge^q X \), with \( p \leq q \), commute (resp. anticommute) if, and only if, \( A = MA' \) and \( B = MB'B'' \) for completely orthogonal blades \( M, A', B', B'' \), with grade(\( A' \)) = grade(\( B' \)) having the same (resp. opposite) parity of \( p(q - 1) \).
Proof. Follows from Propositions 5.10 and 5.12 if \( p = q \) (with \( B'' = 1 \)), and also Proposition 5.6 if \( p < q \) (with \( B'' = B \perp \)).

Example 5.14. In Examples 3.5 and 3.10, the principal angles are \( (0, \frac{\pi}{2}, \frac{\pi}{2}) \), \( E \boxtimes F = \cosh \Phi_{E,F} = e_2 f_2 e_3 f_3 \) and \( E \times F = \sinh \Phi_{E,F} = 0 \). We also have \((E \boxtimes F)^2 = 1\) and \(E^2 = F^2 = -1\).

Example 5.15. In Example 4.2, \( A \boxtimes B = -(2 + 3I_1 I_2)/5\sqrt{2} \) has the terms of \( AB = -\bar{A}B \) that do not depend on the direction of the projections (from \( [A] \) to \([B]\) or vice-versa), and \( A \times B = -(6I_1 + I_2)/5\sqrt{2} \) has those do. We have \((A \boxtimes B)^2 = (13 + 12I_1 I_2)/50\), \((A \times B)^2 = (-37 + 12I_1 I_2)/50\) and \(A^2 = B^2 = -1\). Also, \(\|A \boxtimes B\|^2 = \frac{13}{50}, \|A \times B\|^2 = \frac{37}{50}\), and \(\|A\| = \|B\| = 1\). Finally, \((A \boxtimes B) \times (A \times B) = 0\), since the \( I_k \)'s commute, and \((A \boxtimes B) \ast (A \times B) = 0\), as their components have no common grades.

Example 5.16. In Example 4.6, \( \cosh \Phi_{A,B} = \frac{1}{3} I_1 I_2 \) and \( \sinh \Phi_{A,B} = \frac{\sqrt{3}}{2} I_2, A \boxtimes B = (-\bar{A}B + \bar{B}A)/2 = -\frac{1}{2} I_1 I_2 f_3 f_4 \) and \( A \times B = (-\bar{A}B - \bar{B}A)/2 = -\frac{\sqrt{3}}{2} I_2 f_3 f_4 \). We have \( p = 2 \) and \( q = 4 \), so \( p(q - 1) \) is even, and \( B \perp = f_3 f_4 \). Note that \( A \boxtimes B = (e_1 e_2 \boxtimes f_1 f_2)B \perp = -\cosh \Phi_{A,B}B \perp \) has grade \( 6 = q - p + 4 \), while \( A \times B = (e_1 e_2 \times f_1 f_2)B \perp = -\sinh \Phi_{A,B}B \perp \) has grade \( 4 = q - p + 2 \). Also, they commute, \((A \boxtimes B)^2 = -\frac{1}{4}, (A \times B)^2 = \frac{3}{4}, A^2 = -1\) and \(B^2 = 1\).

With \( e_1 \) instead of \( A \), we have \( \Phi_{e_1,B} = \Phi_{e_1,f_1} = \frac{\pi}{3} I_1 \), \( \cosh \Phi_{e_1,B} = \frac{\sqrt{3}}{2} \) and \( \sinh \Phi_{e_1,B} = \frac{1}{2} I_1 \). Now \( p = 1 \) and \( q = 4 \), so \( p(q - 1) \) is odd, and \( B \perp = f_2 f_3 f_4 \). We find that \( e_1 \boxtimes B = \frac{1}{2} I_1 f_2 f_3 f_4 = (e_1 \times f_1)B \perp = \sinh \Phi_{e_1,B}B \perp \) has grade \( 5 = q - p + 2 \), while \( e_1 \times B = \frac{\sqrt{3}}{2} f_2 f_3 f_4 = (e_1 \boxtimes f_1)B \perp = \cosh \Phi_{e_1,B}B \perp \) has grade \( 3 = q - p \). They commute, \((e_1 \boxtimes B)^2 = \frac{1}{4}\) and \((e_1 \times B)^2 = -\frac{3}{4}\).

Acknowledgements
The author would like to thank Dr. K. Scharnhorst for his comments and for suggesting references, and the anonymous referees who encouraged the conversion of previous versions of the manuscript (and the author himself) to the formalism of geometric algebra. Note. This article has been posted to the arXiv e-print repository, with the identifier arXiv:1910.07327

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A. Appendix: Blade products in Grassmann algebra
We present some results of Sect. 5.1 in terms of Grassmann algebra, to make them more accessible to researchers who might not be familiarized with Clifford algebra. We provide direct proofs requiring (mostly) only Sect. 2.

Here \( X \) can be a real or complex vector space, with inner product \((\cdot, \cdot)\) (Hermitian product in the complex case, with conjugate-linearity in the first argument). The complex case is important for applications, but requires some adjustments in Sect. 2, which are indicated in footnotes.
Formulas relating the inner product of same grade blades to some angle are well known in the real case [10,22]. The following one also works in complex spaces and for distinct grades.

**Theorem A.1.** \( \langle A, B \rangle = \|A\|\|B\| \cos \hat{\Theta}_{A,B} \) for blades \( A, B \in \wedge X \).

**Proof.** If grades are equal, follows from Lemma 2.6 (as \( \langle A, B \rangle = \tilde{A} \ast B \)), (4) and Definition 2.16. Otherwise both sides vanish. \( \square \)

The exterior product satisfies a similar formula.

**Theorem A.2.** \( \|A \wedge B\| = \|A\|\|B\| \cos \Theta_{\perp[A], [B]} \) for blades \( A, B \in \wedge X \).

**Proof.** Let \( P_{\perp} = P_{B \perp} \). Then \( \|A \wedge B\| = \|(P_{\perp}A) \wedge B\| = \|P_{\perp}A\|\|B\| \), and the result follows from Proposition 2.10i. \( \square \)

This result corresponds to (13f), and our comments about that formula (after Corollary 5.3) also apply here.

Contraction or interior product by a vector is widely used in Geometry and Physics, but as its generalization for multivectors is less known, we give a brief description here. The following contraction is related to that of Sect. 5.1 by \( A \ll B = \tilde{A} | B \). For more details, see [6,32].

**Definition A.3.** The (left) contraction \( A \ll B \) of \( A \in \wedge^p X \) on \( B \in \wedge^q X \) is the unique element of \( \wedge^{q-p} X \) such that \( \langle C, A \ll B \rangle = \langle A \wedge C, B \rangle \) for all \( C \in \wedge^{q-p} X \).

If \( p = q \) then \( A \ll B = \langle A, B \rangle \), so the contraction generalizes the inner product for distinct grades, but giving a \((q-p)\)-vector instead of a scalar. For \( p \neq q \) this product is asymmetric (in general, \( A \ll B \neq B \ll A \)), with \( A \ll B = 0 \) if \( p > q \). In the complex case it is conjugate-linear in \( A \) and linear in \( B \).

Let \( A \in \wedge^p X \) and \( B \in \wedge^q X \) be nonzero blades, and \( B = P_{B \perp} \wedge B_{\perp} \) be a PO decomposition\(^{17} \) of \( B \) w.r.t. \( A \).

**Proposition A.4.** \( A \ll B = \langle A, B \rangle \ll B_{\perp} \).

**Proof.** As \( B_{\perp} \) is completely orthogonal to \( A \), for any \( C \in \wedge^{q-p} X \) we have \( \langle A \wedge C, B_{\perp} \rangle = \langle A, B \rangle \langle C, B_{\perp} \rangle = \langle C, \langle A, B \rangle \ll B_{\perp} \rangle \). \( \square \)

So \( A \ll B \) performs an inner product of \( A \) with a subblade of \( B \) where it projects, leaving another subblade of \( B \) completely orthogonal to \( A \).

**Corollary A.5.** \( B_{\perp} = B_{\perp} / \|B\|^2 \).

**Theorem A.6.** \( A \ll B = \|A\|\|B\| \cos \Theta_{A,B} \ll B_{\perp} \).

**Proof.** Follows from Propositions A.1, A.4 and 3.26 if \( p \leq q \), otherwise \( \Theta_{A,B} = \frac{\pi}{2} \) and both sides vanish. \( \square \)

**Corollary A.7.** \( A \ll B = \epsilon_{A,B} \|P_{B\perp}A\| B_{\perp} \).

**Corollary A.8.** \( A \ll B = 0 \iff A \perp B \).

\(^{17}\) See Sect. 3.5.
B. Appendix: Hyperbolic Functions of Multivectors

Hyperbolic functions of multivectors are defined as usual, in terms of exponentials or power series [16].

**Definition B.1.** For any $M \in \bigwedge X$, $\cosh M = \frac{e^M + e^{-M}}{2} = \sum_{k=0}^{\infty} \frac{M^{2k}}{(2k)!}$ and $\sinh M = \frac{e^M - e^{-M}}{2} = \sum_{k=0}^{\infty} \frac{M^{2k+1}}{(2k+1)!}$.

Though we only need the bivector case for Sect. 5.2, we present properties of these functions in more generality, as the literature on them is rather scarce (a few other properties, mostly for blades, can be found in [16, p. 77]).

**Proposition B.2.** Let $M, N \in \bigwedge X$.

(i) $\cosh M + \sinh M = e^M$.

(ii) $\cosh M - \sinh M = e^{-M}$.

(iii) $\cosh(-M) = \cosh M$ and $\sinh(-M) = -\sinh M$.

(iv) $(\cosh(M))^\sim = \cosh \tilde{M}$ and $(\sinh(M))^\sim = \sinh \tilde{M}$.

(v) If $M \times N = 0$ then $\cosh(M) \times \sinh(N) = 0$.

(vi) $\cosh^2 M - \sinh^2 M = 1$.

**Proof.** (i–iv) Immediate. (v) If $M \times N = 0$ then $e^M e^N = e^{M+N}$, and so $\cosh M \sinh N = \left(e^{M+N} - e^{M-N} + e^{N-M} - e^{M-N}\right)/4 = \sinh N \cosh M$.

(vi) $\cosh^2 M = (e^{2M} + 2 + e^{-2M})/4$ and $\sinh^2 M = (e^{2M} - 2 + e^{-2M})/4$.

**Proposition B.3.** Let $H \in \bigwedge^p X$ be homogeneous of grade $p$, and $r = p \mod 4$.

(i) $(\cosh H)^\sim = \cosh H$ and $(\sinh H)^\sim = (-1)^{\frac{p(p-1)}{2}} \sinh H$.

(ii) $\cosh H \in \bigoplus_{k \in \mathbb{N}} \bigwedge^{4k} X$ and $\sinh H \in \bigoplus_{k \in \mathbb{N}} \bigwedge^{4k+r} X$.

(iii) If $r \neq 0$ then $\cosh H \ast \sinh H = 0$.

(iv) If $r = 0$ then

\[
\frac{\|\cosh H\|^2 + \|\sinh H\|^2}{\|e^H\|^2 + \|e^{-H}\|^2} = 1,
\]

(v) If $r = 1$ then

\[
\frac{\|\cosh H\|^2 + \|\sinh H\|^2}{\|e^H\|^2} = 1,
\]

(vi) If $r = 2$ or $3$ then

\[
\frac{\|\cosh H\|^2 + \|\sinh H\|^2}{\|e^H\|^2} = 1,
\]

\[
\frac{\|\cosh H\|^2 - \|\sinh H\|^2}{\|e^{2H}\|^2} = 1,
\]

\[
\|\cosh H\| \leq 1, \|\sinh H\| \leq 1 \text{ and } \|e^H\| = \|e^{-H}\| = 1.
\]

**Proof.** (i) Follows from Proposition B.2 iii and iv, as $\tilde{H} = (-1)^{\frac{p(p-1)}{2}} H$. (ii) Follows from i, as components of $\cosh H$ are even, and those of $\sinh H$ have the parity of $p$. (iii) By ii, these functions have no components of same grade when $r \neq 0$. (iv) $\tilde{H} = H$, so $\|\cosh H\|^2 = (e^H + e^{-H}) \ast (e^H + e^{-H}) = \|e^H\|^2 + 2 + \|e^{-H}\|^2$ and $\|\sinh H\|^2 = \|e^H\|^2 - 2 + \|e^{-H}\|^2$. (v) Likewise,
but by ii and Proposition B.2i we also have \( \| \cosh H \|^2 + \| \sinh H \|^2 = \| e^H \|^2 \), which implies \( \| e^H \| = \| e^{-H} \| \). (vi) Now \( \bar{H} = -H \), so that \( 4\| \cosh H \|^2 = 2 + \langle e^{2H} + e^{-2H} \rangle_0 = 2 + 2\langle \cosh(2H) \rangle_0 \) and \( 4\| \sinh H \|^2 = 2 - 2\langle \cosh(2H) \rangle_0 \), while ii and Proposition B.2i imply \( \langle \cosh(2H) \rangle_0 = \langle e^{2H} \rangle_0 \). Finally, \( \| e^H \|^2 = e^{\bar{H}} * e^H = \langle e^{-H} e^H \rangle_0 = 1 \). □

Note that (ii) and Proposition B.2i restrict which grades \( e^H \) can include, and when \( r \neq 0 \) its components are divided between \( \cosh H \) and \( \sinh H \) according to their grades.

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Received: September 27, 2020.
Accepted: August 9, 2021.