Strong formulations of the generalised Navier-Stokes momentum equation

Josip Bašića,∗, Martina Bašića, Branko Blagojevića

aFaculty of Electrical Engineering, Mechanical Engineering and Naval Architecture, University of Split, R. Boskovica 32, 21000 Split, Croatia

Abstract

In this paper, the strong formulation of the generalised Navier-Stokes momentum equation is investigated. Specifically, the formulation of shear–stress divergence is investigated, due to its effect on the performance and accuracy of computational methods. It is found that the term may be expressed in two different ways. While the first formulation is commonly used, the alternative derivation is found to be potentially more convenient for direct numerical manipulation. The alternative formulation relocates a part of strain information under the variable-coefficient Laplacian operator, thus making future computational schemes potentially simpler with larger time-step sizes.

Keywords: incompressible flow; non-Newtonian flow; generalised Navier-Stokes; variable viscosity; strong formulation; Laplacian

1. Introduction

The motion of incompressible viscous fluids with variable viscosity is governed by generalised Navier–Stokes (GNS) equations. Existence of weak solution for the GNS equations is well researched [1, 2, 3], and variational formulations are attractive since they contain only first-order derivatives [4] if the viscosity is solved by projection [5]. On the other hand, the research on solution methods for strong formulations of GNS equations for variable–viscosity flows is sparse. Peng et al. [6] presented how the strong form of generalised Navier-Stokes equations in Lagrangian context may be used to simulate granular materials, and Bašić et al. [7] extended the proposed method to simulate any viscoplastic material. During the research, it was found that the crucial point that defines the performance and accuracy of a strong-form GNS computational method is the discretisation of the shear–stress divergence. This paper introduces two perspectives on preparing the GNS momentum equation for its direct discretisation.

The mass conservation (continuity or incompressibility) equation for an incompressible fluid defines that the divergence of velocity is always zero:

$$\nabla \cdot \mathbf{u} = 0,$$

(1)

where $\mathbf{u} = f(t, \mathbf{x})$ is the velocity vector field, dependent on time $t$ and location $\mathbf{x}$. The GNS momentum equation is written as:

$$\frac{D(\rho \mathbf{u})}{Dt} = -\nabla p + \nabla \cdot \mathbf{\tau} + \mathbf{F}_{\text{ext}}.$$

(2)
where $\text{D}/\text{D}t$ is the Lagrangian derivative, $\text{D}/\text{D}t \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$, $p$ is the pressure scalar field, $\mathbf{\tau}$ is the shear–stress tensor, and $\mathbf{F}_{\text{ext}}$ is the vector field of external forces that act on the fluid. Spatial and temporal dependency for all terms in Eq. (2) is implied. The shear–stress tensor $\mathbf{\tau}$ is defined as:

$$\mathbf{\tau} = 2\mu \mathbf{E},$$  \hspace{1cm} (3)

where $\mu$ is the dynamic–viscosity scalar field, and the strain–rate tensor $\mathbf{E}$ is defined as:

$$\mathbf{E} = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right],$$  \hspace{1cm} (4)

where $^\top$ denotes the transpose operation. Within numerical schemes, $\mathbf{u}$ for the next time step is obtained by solving Eq. (2), and directly discretising $\nabla \cdot \mathbf{\tau}$ yields inaccurate explicit schemes with rigorous time-step restrictions. Therefore, the motivation for his work is establishing a strong form of the momentum equation, that is convenient for direct numerical manipulation and implicit solving.

2. Divergence of shear stress

The “divergence of the scalar–tensor product” identity for a tensor $\mathbf{T}$ and scalar $\phi$ is defined as:

$$\nabla \cdot (\phi \mathbf{T}) = \phi \nabla \cdot \mathbf{T} + \mathbf{T} \nabla \phi.$$  \hspace{1cm} (5)

The divergence of shear–stress may be expanded by applying the identity (5) to Eq. (3):

$$\nabla \cdot \mathbf{\tau} = \nabla \cdot (2\mu \mathbf{E})$$

$$= 2\mu \nabla \cdot \mathbf{E} + 2\mathbf{E} \nabla \mu.$$  \hspace{1cm} (6)

Eq. (4) is substituted into Eq. (6) to obtain the following:

$$\nabla \cdot \mathbf{\tau} = \mu \nabla \cdot \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right] + 2\mathbf{E} \nabla \mu$$

$$= \mu \nabla^2 \mathbf{u} + \mu \nabla \cdot (\nabla \mathbf{u})^\top + 2\mathbf{E} \nabla \mu$$

$$= \mu \nabla^2 \mathbf{u} + 2\mathbf{E} \nabla \mu.$$  \hspace{1cm} (7)

**Remark 2.1.** The term $\nabla \cdot (\nabla \mathbf{u})^\top$ is null vector, since the “divergence of the transpose of a vector gradient” is equivalent to the “gradient of the divergence of a vector”, which is evident using the index notation:

$$\nabla \cdot (\nabla \mathbf{u})^\top = \frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} = \nabla (\nabla \cdot \mathbf{u}) = \mathbf{0}.$$  

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3. Alternative expression for divergence of shear stress

Alternatively, Eq. (4) may be substituted into Eq. (3) and \( \nabla \cdot \tau \) may be expressed as:

\[
\nabla \cdot \tau = \nabla \cdot (2\mu E) = \nabla \cdot \left\{ \mu \left[ \nabla u + (\nabla u)^\top \right] \right\} = \nabla \cdot (\mu \nabla u) + \nabla \cdot \left[ \mu (\nabla u)^\top \right]. 
\]

The identity (5) may be applied to the second term of the right-hand-side in Eq. (8), therefore obtaining:

\[
\nabla \cdot \tau = \nabla \cdot (\mu \nabla u) + \mu \nabla \cdot (\nabla u)^\top + (\nabla u)^\top \nabla \mu. 
\]

Example 3.1. The variable–coefficient Laplacian discretised using second–order finite differences on a uniform \( n \)-dimensional grid is defined using the summation form as:

\[
\begin{align*}
\nabla \cdot (c_i \nabla \phi_i) & := \sum_j c_j + c_i \phi_j - \phi_i h^2, \\
\text{where} \quad j & \text{ are stencil nodes–neighbours to the central node } i, \quad \text{and} \quad h \text{ is the grid spacing.}
\end{align*}
\]

4. Momentum equation formulations

Due to two different ways of expressing \( \nabla \cdot \tau \) using Eqs. (7) and (9), the momentum Eq. (2) can be written using the standard Laplacian (with the strain-rate) notation:

\[
\frac{D (\rho u)}{Dt} - \mu \nabla^2 u = \left[ \nabla u + (\nabla u)^\top \right] \nabla \mu - \nabla p + F_{\text{ext}}, 
\]

or using the novel variable–coefficient Laplacian (with a part of the strain-rate) notation:

\[
\frac{D (\rho u)}{Dt} - \nabla \cdot (\mu \nabla u) = (\nabla u)^\top \nabla \mu - \nabla p + F_{\text{ext}}. 
\]

Completeness of including velocity terms on the left-hand-side defines the accuracy and performance of a computational method. In order to have flexible simulation schemes, \( \mu \) and \( \nabla \mu \) are computed before solving Eq. (7) for \( u \). Due to this aspect and complexity of discretising \( E \nabla \mu \) product implicitly, it is commonly imposed on the right-hand-side. This limits the time-step. However, Eq. (9) allows to transfer a part of the strain rate information on the left-hand-side without adding numerical complexity compared to Eq. (7). Eq. (11) is a convenient representation for computational methods that discretise the strong formulation (e.g. finite difference methods), because the variable–coefficient Laplacian and standard Laplacian usually depend on the same discretisation procedure. In the discrete context, the continuity Eq. (1) cannot be truly satisfied; in the best case it is satisfied to the machine precision. The commonly used momentum Eq. (10) that is derived based on Eq. (7) neglects the velocity divergence due to Eq. (1), i.e. retains the complete strain rate information on the right-hand-side. On the contrary, the term \( \nabla \cdot (\mu \nabla u) \) in Eq. (9), which is used to derive the new momentum Eq. (11), implicitly includes part of the strain-rate information on the left-hand-side, therefore having better numerical balance between left-hand-side and right-hand-side.
5. Concluding remarks

This paper presents a derivation of two expressions for the momentum equation of the generalised Navier-Stokes equation in strong form. Due to the computational complexity of $E \nabla \mu$ in the so-called Laplacian formulation, it is often translated to the right-hand-side when solving the momentum equation for the velocity, therefore sacrificing accuracy and limiting time step sizes. The newly derived formulation of the momentum equation allows to use part of the strain information using the same Laplacian numerics on the left-hand-side, therefore potentially making future computational schemes simpler with larger time-step sizes, while retaining the solving performance.

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