Optimal control of a tetrahedral configured reaction wheels for Quaternion rotation based on rigid satellite model

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Abstract. Reaction wheel is the most popular actuator for the attitude manoeuvring of a satellite system due to its compact size, reliability and able to produce precise torque. However, for redundancy purposes, the reaction wheels are assembled in a tetrahedral configuration that is able to generate torque in any direction even when one of the reaction wheel fails. The tetrahedral configured reaction wheel causes the maximum torque generation to be uneven in any direction. Hence, the quaternion rotation is completed at different rate in different direction. In order to optimize the rotation rate, an optimal control using iterative method via GPOPS toolbox. It is then compared with the traditional Eigen-axis Quaternion Feedback control. The improvement shown by the optimal control to be between 3.49% to 25.11% improvement in manoeuvre time depending on the direction of manoeuvre. The optimal control is able to outperform the traditional control method.

1. Introduction
Since 1957, artificial satellites have been placed into the Earth’s orbit. These satellites play important in various purposes, namely, military intelligence and defence, communications, space explorations and the scientific study [1].

The orientation of satellites need to be changed on every mission to achieve desired outcome. Hence, attitude control is needed to alter the satellite attitude in the desired direction. Generally, the control methods are classified as active or passive. Passive method has been utilized in [2]. Some of the active methods include generalized predictive control method [3], control method based on Lyapunov [4] and nonlinear control based on linear matrix inequality [5].

The main purpose of this paper is to introduce a control function on each of the three satellite’s kinetic equations. These control functions are within the framework of an optimal control problem via PMP. Also, we show that these controls are able to drive the satellite’s attitude to desired direction at minimum time compared to the traditional control method.

2. The reaction wheel
The attitude control of a satellite system involves the cooperation between a controller and actuator where the controller provides the desired input while the actuator generates the input to the overall system based on the controller. A typical satellite system is equipped with four or more wheels of Reaction Wheel (RW) for the purpose of redundancy in case when one or more RW fails [6]. A attitude
manoeuvring actuator system with a configuration of four RW is shown in Figure 1 in reference to RazakSAT® satellite [7]. The tetrahedral configuration is crucial for redundancy purpose because if one of the RW fails, the other three RW is still able to generate torque in any of the three axis.

**Figure 1.** Configuration of reactions wheels.

### 3. The equation of motion

The momentum of the RW is defined as the product of RW’s inertia and angular velocity as shown in Equation 1.

\[
h_{r,j} = I_{r,j} \Omega_j
\]

where \( I_{r,j} \) and \( \Omega_j \) are the moment of inertia and the angular velocity of the RW respectively. \( h_{r,j} \) is the angular momentum with reference to the satellite body frame. The torque with respect to the wheel frame, \( \tau_{r,i} \) is the time derivative of Equation 2.

\[
\tau_{r,j} = \frac{dh_{r,j}}{dt} = I_{r,j} \dot{\Omega}_j
\]

Hence, the torque generated by the RW, \( \tau_{out,i} \) is

\[
\tau_{out,i} = \frac{dh_{out,j}}{dt} = c R^w I_{r,j} \dot{\Omega}_j
\]

\( h_{out} \) is the angular momentum with reference to the satellite body frame and \( c R^w \) is the transformation matrix for the RW orientation with respect to the body frame. By Euler’s transport theorem adopted from [8], the inertial Equation 4 is obtained.

\[
\tau_{out,j} = c R^w \tau_{r,j} + \dot{s} \times I_{r,j} \Omega_j
\]

Hence, the RW array generated a torque that is equivalent but in opposite direction in the body frame due to the moment exchange between the RW and the satellite body, as shown in Equation 5.

\[
\begin{bmatrix}
\tau_x \\
\tau_y \\
\tau_z
\end{bmatrix}
= \tau_{out,j}^N = c R^w \tau_{r,j} + \dot{s} \times I_{r,j} \Omega_j
\]

### 4. The Eigen-axis Quaternion Feedback
The common controller applied in satellite system is the Eigen-axis Quaternion Feedback (EQF). The controller uses an Eigen-axis manoeuvre so that the attitude manoeuvring problem forms a simple single angle rotation. The model obtained from the EQF control logic often used in satellite attitude control system is shown in Equation 6 [9].

\[
\tau_{eq} = -k_c I_s q_{123} - c_c I_s \dot{\theta}_s + \dot{\theta}_s \times I_s \dot{\theta}_s \tag{6}
\]

The torque \(\tau_{eq}\) is what the EQF controller is trying to control. Equations 6 contain the gyroscopic coupling term which is not desired in solving a linear motion along an arc. \(I_s\) is the total moment of inertia. The nonlinearity can be removed by simply adding the gyroscopic term into the EQF equation to remove the nonlinearity that exists in the control Equation 6. Hence, Equation 7 is obtained.

\[
\tau_{req} = \tau_{eq} - \dot{\theta}_s \times I_s \dot{\theta}_s = -k_c I_s q_{123} - c_c I_s \dot{\theta}_s = I_s \dot{\theta}_s \tag{7}
\]

The control torque \(\tau_{req}\) is effective control of the satellite body frame. Equation 8 can be rearranged as Equation 8.

\[
\tau_{req} + k_c I_s q_{123} + c_c I_s \dot{\theta}_s = 0 \tag{8}
\]

The inertia is a constant, hence it is taken out to obtain Equation 9.

\[
\dot{\theta}_s = -k_c q_{123} - c_c \dot{\theta}_s \tag{9}
\]

The satellite attitude is measured in three-dimensional space. However, by applying the Eigen Axis which simplifies the rotation into a single axis, the trajectory of the rotation can also be reduced to single angle, while the body rate of the satellite can also be measured as a single angular rate, as shown in Equation 10.

\[
\theta_E = \theta_s - \theta_d \quad \ddot{\theta}_E = -k_c (\theta_s - \theta_d) - c_c (\dot{\theta}_s - \dot{\theta}_d) \tag{10}
\]

Assume that a desired final attitude and rate are known. The controller will manoeuvre the error from initial state to zero. For a satellite to change its attitude, it is considered a rest to rest motion. Hence, the desired rate is zero, and the Equation 11 is obtained.

\[
\dot{\theta}_d = 0 \quad \frac{u}{I_{tot}} = \ddot{\theta}_E = -k_c (\theta_s - \theta_d) - c_c \dot{\theta}_s \tag{11}
\]

As shown in Equation 11, the EQF controller equation is similar to the PD control system.

5. The Optimal Control

The minimum time optimal control is done using GPOPS for developing the ideal control function with the shortest time to complete desired attitude rotation. The optimal control problem is obtained by compiling the dynamic models and states of the system from Equations 1 to 11 shown in Equation 12.

Given \(x^T = [q_1, q_2, q_3, q_4, \dot{\theta}_s, \dot{\theta}_r, \dot{\theta}_d, \Omega_1, ..., \Omega_4]\)
\begin{equation}
\mathbf{u}^T = [\tau_{r,1}, \ldots, \tau_{r,4}] \nonumber
\end{equation}

Minimize

\begin{equation}
J = t_f 
\end{equation}

Subject to

\begin{equation}
\dot{\theta}_s = \frac{r - \dot{\theta}_s \times I_s \dot{\theta}_s}{I_s} \nonumber
\end{equation}

\begin{equation}
\mathbf{x}_o^T = [q_{0,1}, q_{0,2}, q_{0,3}, q_{0,4}, 0, 0, 0, 0, 0, 0] \nonumber
\end{equation}

\begin{equation}
\mathbf{x}_f^T = [q_{f,1}, q_{f,2}, q_{f,3}, q_{f,4}, 0, 0, 0, 0, 0, 0] \nonumber
\end{equation}

\begin{equation}
-\tau_{max} < \tau_{r,j} < \tau_{max} - \Omega_{max} < \Omega < \Omega_{max} \nonumber
\end{equation}

Where, \( x^T \) refers to the states, \( x_o^T \) is the initial state and \( x_f^T \) is the final state. \( u^T \) is control variable, while \( J \) is the cost function.

6. The Simulation

The parameters applied for the attitude manoeuvring simulation is shown in Table 1.

| Variable | Value |
|----------|-------|
| \( I_s \) | \[
\begin{bmatrix}
200 & 0 & 0 \\
0 & 200 & 0 \\
0 & 0 & 400
\end{bmatrix}
k\cdot m^3
\] |
| \( \tau_{max} \) | 0.403 N\cdot m |
| \( \Omega_{max} \) | 314 rad/s |

The results from the attitude maneuvering parameters in quaternion are shown in Figure 2.

(a)

(b)

(c)

(d)

Figure 2. Attitude rotation of satellite in Quaternion (a) \( q_1 \) (b) \( q_2 \) (c) \( q_3 \) (d) \( q_4 \)

The results from Figure 2 are translated into Euler angles as shown in Figure 3. The figure shows that the GPOPS solution takes 55.45 seconds to complete the manoeuvring which is 25.11% faster than the EQF manoeuvre at 74.04 second. Regardless of the axis of rotation, the GPOPS which generates time optimal manoeuvre will certainly be as fast as or faster than the Eigen axis manoeuvre.
The optimal solution manoeuvre is able to achieve the final desired attitude much faster for two main reasons. Firstly, the optimized solution is able to take advantage of Off-Eigen axis motion. Hence, this enables the satellite to manoeuvre at a greater magnitude in both torque and momentum. Secondly, the manoeuvre rate of the EQF is confined by the pseudo-inverse restriction. On the other hand, the optimal solution is able to achieve momentum and torque at the full physical capacity of the RW. Table 2 presents a comparison of a manoeuvre-time between the EQF and GPOPS about a 120° manoeuvre on different Eigen axis. Each case moves from the initial origin [0, 0, 0, 1] quaternion to the designated attitude. Interestingly, the time-optimal maximum body rate consistently outperforms the EQF, this accounts for much off-Eigen Axis movement.

Table 2. Manoeuvring performance at difference Quaternion.

| q | Eigen Axis | GPOPS (s) | EQF (s) | Improvement (%) |
|---|---|---|---|---|
| [0, 0, 1, 0.5] | [0, 0, 1.155] | 63.6371 | 65.94 | 3.49 |
| [0.5, 0.5, -0.5, 0.5] | [0.577, 0.577, -0.577] | 55.5748 | 61.18 | 9.16 |
| [-0.612, -0.433, -0.433, 0.5] | [0.7071, 0.5, 0.5] | 53.3832 | 60.83 | 12.24 |
| [0.707, 0, 0.5, 0.5] | [0.8165, 0, 0.577] | 56.7654 | 65.11 | 12.82 |

7. Conclusion

Time optimal control is performed to obtain the control which will generate desired attitude change at the minimum. This is done via the application of GPOPS toolbox while verification of the optimality is also done. The works presented shows a 3.49% to 25.11% improvement in manoeuvre time. The performance increase is due to two major factors. Firstly, the optimal control is able to fully utilize the all the capacity of the RW while the EQF is plagued by the pseudo-inverse limitation. Secondly, the application of optimal control allows the trajectory to deviate from the effective Eigen axis to achieve faster manoeuvre by utilizing the torque that is unavailable to the effective Eigen axis manoeuvre.

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