Soliton Solutions of Generalized Third-Order Nonlinear Schrödinger Equation by Using GKM

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ABSTRACT: In this study, we have worked on GKM in order to obtain the soliton solutions of the generalized third-order nonlinear Schrödinger equation. Thus, we have acquired some new soliton solutions of the generalized third-order nonlinear Schrödinger equation which has an important usage area in optical fiber. Also, we have drawn some 2D and 3D surfaces of these obtained results by using Wolfram Mathematica 12. Then, we have shown the validity of the obtained solutions.

Keywords: Generalized Kudryashov method, generalized third-order nonlinear Schrödinger equation, soliton solutions
INTRODUCTION

The nonlinear Schrödinger equations (NLSEs), a category of nonlinear evolution equations (NLEEs), are used in much areas of engineering and applied sciences kind of fluid mechanics, hydrodynamics, applied mathematics, biophysics, optical fibers, mathematical physics, plasma physics, fluid dynamics and so on (Ma, 2019; Liu et al., 2015; Chettouh et al., 2017; Chowdury et al., 2014; Azzouzi et al., 2009; Triki and Taha, 2012; Xu and Zhang, 2007; Triki et al., 2018; Arshad et al., 2017a; Arshad et al., 2017b; Biswas et al., 2017; Seadawy et al., 2018).

Generalized third-order (NLSE), which is a class of the NLSEs, has been the subject of some research recently. Generalized third-order (NLSE) is given as:

$$i \left( \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} \right) + |u|^2 \left( \beta_1 u + i \beta_2 \frac{\partial u}{\partial x} \right) + i \beta_3 \frac{\partial (|u|^2)}{\partial x} u = 0.$$  \hspace{1cm} (1)

Where value of function is a complex and value of coefficients and are real. Generalized third-order (NLSE) given by Eq. (1) is an important model which is used to model ultra-short pulses in optical fibers. Solitons and solitary wave solutions of this model have recently been tried to be found by many researchers. Various methods have been studied such as the extended simple equation method and the exp (−ϕ(ξ))-expansion method (Lu et al., 2019), the generalized Riccati mapping method (Nasreen et al., 2019), the exp-a function and unified methods (Hosseini et al., 2020), F-expansion method (Seadawy et al., 2020b) and modified extended direct algebraic method (Seadawy et al., 2020a).

Our aim in this article is ascertain the soliton solutions of generalized third-order (NLSE) through GKM (Tuluce Demiray and Bulut, 2015; Pandir et al., 2016; Tuluce Demiray and Bulut, 2016; Tuluce Demiray and Bulut, 2017; Tuluce Demiray and Bulut, 2019). In Section 2, GKM’s basic structure is given. In Section 3, some soliton solutions of generalized third-order (NLSE) have been obtained by applying GKM.

MATERIALS AND METHODS

We take into account a general nonlinear partial differential equation (NLPDE) in the form:

$$P(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0.$$  \hspace{1cm} (2)

Step1: Firstly, we consider the travelling wave solution as following form;

$$u(x, y, t) = u(\xi) e^{i P(x, t)}, \xi = kx + wt, P(x, t) = \delta x + \lambda t.$$  \hspace{1cm} (3)

Where k, w, δ and λ arbitrary constants. Equation (2) is turned into ordinary differential equation by Eq. (3):

$$P(u, u', u'', u''', ...) = 0.$$  \hspace{1cm} (4)

Where superscripts denote ordinary derivatives with respect to ξ.

Step2: Suppose that we consider the solutions of Eq. (4) as:

$$u(\xi) = \sum_{i=0}^{N} a_i Z_i(\xi) = \frac{A[Z(\xi)]}{B[Z(\xi)]}.$$  \hspace{1cm} (5)
Where $Z$ is $\frac{1}{1 \pm e^{\xi}}$ dir. We should point out that $Z$ is the solution to the following equation.

$$Z_{\xi} = Z^2 - Z. \quad (6)$$

**Step3:** We can ascertain the values of $M$ and $N$ in Eq. (5) through the homogeneous balance principle. Therefore we balance the highest order nonlinear terms in Eq. (4).

**Step4:** We substitute Eq. (5) into Eq. (4). Thus we obtain a polynomial of $R(Z)$ of $Z$. Then equating the all coefficients of $R(Z)$ to zero, we find an algebraic equation system. By solving this system, we determine $c$ and the variable coefficients of $a_0, a_1, a_2, ..., a_N, b_0, b_1, b_2, ..., b_M$. Finally we can obtain the exact solutions of Eq. (4).

**Application of GKM to the equation**

To find the traveling wave solutions of Eq. (1) we consider the following transformation:

$$u(x, y, t) = u(\xi)e^{i\eta(x, t)}, \xi = kx + wt, P(x, t) = \delta x + \lambda t. \quad (7)$$

Replace Eq. (2) into Eq. (1) and we get the following

$$3k^2\delta u'' + (\lambda - \delta^3)u + (\delta \beta_2 + \beta_1)u^3 = 0. \quad (8)$$

And

$$k^3u'' + (w - 3\delta^2k)u + \frac{k(2\beta_3 + \beta_2)}{3}u^3 = 0. \quad (9)$$

By using balance principle in Eq. (9), we obtain

$$N - M + 2 = 3N - 3M \Rightarrow N = M + 1. \quad (10)$$

If we select $M = 1$ and $N = 2$ we find the following solution

$$u(\xi) = \frac{a_0 + a_1Z + a_2Z^2}{b_0 + b_1Z}, \quad (11)$$

$$u'(\xi) = (Z^2 - Z)\left[\frac{(a_1 + 2a_2Z)(b_0 + b_1Z) - b_1(a_0 + a_1Z + a_2Z^2)}{(b_0 + b_1Z)^2}\right], \quad (12)$$

$$u''(\xi) = \frac{Z^2 - Z}{(b_0 + b_1Z)^2}(2Z - 1)[(a_1 + 2a_2Z)(b_0 + b_1Z) - b_1(a_0 + a_1Z + a_2Z^2)] + \frac{(Z^2 - Z)^2}{(b_0 + b_1Z)^3}$$

$$+ [2a_2(b_0 + b_1Z)^2 - 2b_1(a_1 + 2a_2Z)(b_0 + b_1Z)$$

$$+ 2b_1^2(a_0 + a_1Z + a_2Z^2)]. \quad (13)$$

We obtain the soliton solutions of Eq. (1) in the following different cases;

**Case1:**

$$a_0 = -\frac{i\sqrt{3}k}{\sqrt{\beta_2 + 2\beta_3}}, a_1 = -\frac{a_2}{2} + \frac{i\sqrt{6}k b_0}{\sqrt{\beta_2 + 2\beta_3}}, b_1 = -\frac{ia_2\sqrt{\beta_2 + 2\beta_3}}{\sqrt{6}k},$$

$$w = \frac{1}{2}k(k^2 + 6\delta^2). \quad (14)$$
Substituting the above values in Eq. (11), we acquire the soliton solution of Eq. (1)

\[ u_1(x,t) = -\frac{i\sqrt{\frac{3}{2}} k \tanh \left[ \frac{1}{2} \left( kx + \frac{1}{2} kt (k^2 + 6\delta^2) \right) \right]}{\sqrt{\beta_2 + 2\beta_3}}. \]  

(15)

\[ u_2(x,t) = -\frac{i\sqrt{\frac{3}{2}} k \coth \left[ \frac{1}{2} \left( kx + \frac{1}{2} kt (k^2 + 6\delta^2) \right) \right]}{\sqrt{\beta_2 + 2\beta_3}}. \]  

(16)

**Figure 1:** The 3D graph of the solution (15) for \( k = 1, \delta = 2, \beta_2 = 2, \beta_3 = 2, -20 < x < 20, -20 < t < 20 \) and 2D graph for this values and \( t = 0.1 \).

**Case 2:**

\[ a_0 = \frac{\sqrt{3}}{2} \frac{-3kb_1\sqrt{-\beta_2 - 2\beta_3} + \sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{4(\beta_2 + 2\beta_3)}, a_1 = -\frac{\sqrt{6}kb_1}{\sqrt{-\beta_2 - 2\beta_3}}, a_2 = \frac{\sqrt{6}kb_1}{\sqrt{-\beta_2 - 2\beta_3}}, \]

\[ b_0 = -\frac{b_1}{2}, w = \frac{1}{4} k \left( 5k^2 + 12\delta^2 - \frac{3k\sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{b_1\sqrt{-\beta_2 - 2\beta_3}} \right). \]  

(17)

Substituting the above values in Eq. (11), we acquire the soliton solution of Eq.
\[ u_3(x, t) = \frac{-\sqrt{3}}{4\sqrt{2}b_1(-\beta_2 - 2\beta_3)^{3/2}} \left( -2k \left( -1 + 3\cosh \left[ kx + \frac{1}{4}kt \left( 5k^2 + 12\delta^2 - \frac{3k\sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{b_1\sqrt{-\beta_2 - 2\beta_3}} \right) \right] \right) \times \text{csch} \left[ kx + \frac{1}{4}kt \left( 5k^2 + 12\delta^2 - \frac{3k\sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{b_1\sqrt{-\beta_2 - 2\beta_3}} \right) \right] b_1(\beta_2 + 2\beta_3) \\
- 2\coth \left( \frac{1}{2} \left( kx + \frac{1}{4}kt \left( 5k^2 + 12\delta^2 - \frac{3k\sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{b_1\sqrt{-\beta_2 - 2\beta_3}} \right) \right) \right) \times \sqrt{-\beta_2 - 2\beta_3} \left( -k^2b_1^2(\beta_2 + 2\beta_3) \right). \] 

(18)

\[ u_4(x, t) = \frac{-\sqrt{3}}{4\sqrt{2}b_1(-\beta_2 - 2\beta_3)^{3/2}} \left( -2k \left( -1 + 3\cosh \left[ kx + \frac{1}{4}kt \left( 5k^2 + 12\delta^2 - \frac{3k\sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{b_1\sqrt{-\beta_2 - 2\beta_3}} \right) \right] \right) \times \text{csch} \left[ kx + \frac{1}{4}kt \left( 5k^2 + 12\delta^2 - \frac{3k\sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{b_1\sqrt{-\beta_2 - 2\beta_3}} \right) \right] b_1(\beta_2 + 2\beta_3) \\
- 2\tanh \left( \frac{1}{2} \left( kx + \frac{1}{4}kt \left( 5k^2 + 12\delta^2 - \frac{3k\sqrt{-k^2b_1^2(\beta_2 + 2\beta_3)}}{b_1\sqrt{-\beta_2 - 2\beta_3}} \right) \right) \right) \times \sqrt{-\beta_2 - 2\beta_3} \left( -k^2b_1^2(\beta_2 + 2\beta_3) \right). \] 

(19)

\textbf{Figure 2:} The 3D graph of the solution (18) for \( k = 0.5, \delta = 2, b_1 = 1, \beta_2 = 0.2, \beta_3 = 4, -20 < x < 20, -2 < t < 2 \) and 2D graph for this values and \( t = 0.5 \).

\textbf{Case3:}

\[ a_0 = -\frac{\sqrt{6}kb_0}{\sqrt{-\beta_2 - 2\beta_3}}, a_1 = \frac{2\sqrt{6}kb_0}{\sqrt{-\beta_2 - 2\beta_3}}, a_2 = -\frac{2\sqrt{6}kb_0}{\sqrt{-\beta_2 - 2\beta_3}} \\
b_1 = -2b_0, w = (2k^2 + 3k\delta^2). \] 

(20)
Substituting the above values in Eq. (11), we acquire the soliton solution of Eq. (1)

\[ u_5(x, t) = -\frac{\sqrt{6}k\coth(kx + t(2k^3 + 3k\delta^2))}{\sqrt{-\beta_2 - 2\beta_3}}. \]  

\[ u_6(x, t) = -\frac{\sqrt{6}k\tanh(kx + t(2k^3 + 3k\delta^2))}{\sqrt{-\beta_2 - 2\beta_3}}. \]  

Figure 3: The 3D graph of the solution (21) for \( k = 0.2, \delta = 2, \beta_2 = 0.2, \beta_3 = 3, -25 < x < 25, -25 < t < 25 \) and 2D graph for this values and \( t = 0.5 \).

RESULTS AND DISCUSSION

We obtained some soliton solutions of the generalized third-order (NLSE) equation by applying GKM. We proved their accuracy by graphically representing these obtained results by aid of Wolfram Mathematica 12. Several methods were previously applied by some authors to obtain the solutions of the generalized third-order (NLSE) equation. When we check the solutions we found with those of other authors, our (15) and (22) solutions are similar to the (19) and (37) solutions given by Lu et al., the (12) solution given by Nasreen et al. and the (19) solution given by Seadawy et al. In addition to our (16) and (21) solutions are similar to the (17) solution given by Lu et al., the (13) solution given by Nasreen et al. and the (42) solution given by Seadawy et al. According to our research our (18) and (19) solutions are not given before and are new.

CONCLUSION

In this made study, We obtained the soliton solutions of generalized third-order (NLSE) describing ultra-short pulses in optical fiber. Thus, GKM, which is easier to apply than other methods, is a very effective and reliable method for finding solutions to NLEEs. In addition, the accuracy of the obtained solutions has been shown with graphical representations.

Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author’s Contributions

The authors declare that they have contributed equally to the article.
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