DEFORMATIONS AND HOMOTOPY THEORY OF ROTA-BAXTER ALGEBRAS OF ANY WEIGHT

KAI WANG AND GUODONG ZHOU

Abstract. This paper studies the formal deformations and homotopy of Rota-Baxter algebras of any given weight. We define an $L_\infty$-algebra that controls simultaneous the deformations of the associative product and the Rota-Baxter operator of a Rota-Baxter algebra. As a consequence, we develop a cohomology theory of Rota-Baxter algebras of any given weight and justify it by interpreting the lower degree cohomology groups as formal deformations and abelian extensions. The notion of homotopy Rota-Baxter algebras is introduced and it is shown that the operad governing homotopy Rota-Baxter algebras is a minimal model of the operad of Rota-Baxter algebras.

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Date: September 9, 2021.
2010 Mathematics Subject Classification. 16E40 16S80 16S70.
Key words and phrases. cohomology, abelian extension, formal deformation, $L_\infty$-algebra, minimal model, operad, Rota-Baxter algebra, homotopy Rota-Baxter algebra.
INTRODUCTION

A general philosophy of deformation theory of mathematical structures, as evolved from ideas of Gerstenhaber, Nijenhuis, Richardson, Deligne, Schlessinger, Stasheff, Goldman, Millson, is that the deformation theory of any given mathematical object can be described by a certain differential graded (=dg) Lie algebra or more generally a $\mathcal{L}_\infty$-algebra associated to the mathematical object (whose underlying complex is called the deformation complex). This philosophy has been made into a theorem in characteristic zero by J. Lurie [52] and J. Pridham [57], expressed in terms of infinity categories. It is an important question to construct explicitly this dg Lie algebra or $\mathcal{L}_\infty$-algebra governing deformation theory of this mathematical object.

Another important question about algebraic structures is to study their homotopy versions, just like $A_\infty$-algebras for usual associative algebras. The nicest result would be providing a minimal model of the operad governing an algebraic structure. When this operad is Koszul, there exists a general theory, the so-called Koszul duality for operads [31][30][51], which defines a homotopy version of this algebraic structure via the cobar construction of the Koszul dual cooperad, which, in this case, is a minimal model. However, when the operad is NOT Koszul, essential difficulties arise and few examples of minimal models have been worked out. For instance, Gálvez-Carrillo, Tonks and Vallette [24] gave a cofibrant resolution of the Batatlin-Vilkovisky operad using inhomogeneous Koszul duality theory. However, their cofibrant resolution is not minimal and in another paper of Drummond-Cole and Vallette [21], the authors succeeded in finding a minimal model which is a deformation retract of the cofibrant resolution found in the previous paper. Dotsenko and Khoroshkin [20] constructed resolutions for shuffle monomial operads by the inclusion-exclusion principle and for operads presented by a Gröbner basis [19] by deformation of the monomial case.

These two questions, say, describing controlling $\mathcal{L}_\infty$-algebras and constructing homotopy versions, are closed related. In fact, given a cofibrant resolution, in particular, a minimal model, of the operad in question, one can form the deformation complex of the algebraic structure and construct its $\mathcal{L}_\infty$-structure as explained by Kontsevich and Soibelman [45] and van der Laan [70, 71]. This method has been generalised to properads by Markl [53], Merkulov and Vallette [54, 55], and to colored props by Frégier, Markl and Yau [23].

In this paper, we follow a somehow inverse direction and make use of an ad hoc method. Given an algebraic structure on a space $V$ realised as an algebra over an operad, by considering the formal deformations of this algebraic structure, we first construct the deformation complex and find an $\mathcal{L}_\infty$-structure on the underlying graded space of this complex such that the Maurer-Cartan elements are in bijection with the algebraic structures on $V$. When $V$ is graded, we define a homotopy version of this algebraic structure as Maurer-Cartan elements in the $\mathcal{L}_\infty$-algebra constructed above. Finally under suitable conditions, we could show that the operad governing the homotopy version is a minimal model of the original operad.

The algebraic structure investigated in this paper is Rota-Baxter algebras of any weight.
Rota-Baxter algebras (previously known as Baxter algebras) originated with the work of Baxter [4] in his study on probability theory. Baxter’s work was further investigated by, among others, Rota [60] (hence the name “Rota-Baxter algebras”), Cartier [8] and Atkinson [2] etc. The subject was revived beginning with the work of Guo et al. [37, 38, 33, 1]. Nowadays, Rota-Baxter algebras have numerous applications and connections to many mathematical branches, to name a few, such as combinatorics [32, 61], renormalization in quantum field theory [9], multiple zeta values in number theory [41], operad theory [1, 5], Hopf algebras [9], Yang-Baxter equation [3]. For basic theory about Rota-Baxter algebras, we refer the reader to the short introduction [35] and to the comprehensive monograph [36].

The deformation theory and cohomology theory of Rota-Baxter algebras had been absent for a long time despite the importance of Rota-Baxter algebras. Recently there are some breakthroughs in this direction. Tang, Bai, Guo and Sheng [66] developed the deformation theory and cohomology theory of $O$-operators (also called relative Rota-Baxter operators) on Lie algebras, with applications to Rota-Baxter Lie algebras in mind. Das [10] developed a similar theory for Rota-Baxter associative algebras of weight zero. Lazarev, Sheng and Tang [49] succeeded in establishing deformation theory and cohomology theory of relative Rota-Baxter Lie algebras of weight zero and found applications to triangular Lie bialgebras. They determined the $L_\infty$-algebra that controls deformations of a relative Rota-Baxter Lie algebra and introduced the notion of a homotopy relative Rota-Baxter Lie algebra. The same group of authors also related homotopy relative Rota-Baxter Lie algebras and triangular $L_\infty$-bialgebras via a functorial approach to Voronov’s higher derived brackets construction [50]. Later Das and Misha also determined the $L_\infty$-structures underlying the cohomology theory for Rota-Baxter associative algebras of weight zero [18]. There are some other related work [67, 68, 11, 12, 13, 16, 17, 43, 59]. These work all concern Rota-Baxter operators of weight zero.

A recent paper by Pei, Sheng, Tang and Zhao [56] considered cohomologies of crossed homomorphisms for Lie algebras and they found a DGLA controlling deformations of crossed homomorphisms. Another exciting progress in this subject is the introduction of the notion of Rota-Baxter Lie groups by Guo, Lang and Sheng [39]; as a successor to this work, Jiang, Sheng and Zhu considered cohomology of Rota-Baxter operators of weight 1 on Lie groups and Lie algebras and relationship between them [44]. While this paper is ready to submit, another paper appeared [14] in which Das investigated cohomology of Rota-Baxter operators of arbitrary weights on associative algebras and which has some overlap with Sections 5.1 and 6.2 of this paper; in another paper [15], he studied twisted Rota-Baxter operators on Leibniz algebras. It seems that these are the only papers which investigates Rota-Baxter operators of nonzero weight (for a related work on differential algebras of nonzero weight, see [40]). In these papers, the authors dealt with the deformations of only the Rota-Baxter operators with the Lie algebra or associative algebra structure unchanged. The goal of the present paper is to study simultaneous deformations of Rota-Baxter operators of nonzero weight and of associative algebra structures. One of the reasons is that when one structure remains undeformed, the homotopy version obtained could not be a minimal model of the operad of Rota-Baxter Lie algebras or Rota-Baxter associative algebras.

In this paper, we follow a somehow inverse direction to the classical approach from cofibrant resolutions to $L_\infty$-algebras. Given an algebraic structure on a space $V$ realised as an algebra over an operad, by considering the formal deformations of this algebraic structure, we first construct the deformation complex and find an $L_\infty$-structure on the underlying graded space of this complex such that the Maurer-Cartan elements are in bijection with the algebraic structures on $V$. When
V is graded, we define a homotopy version of this algebraic structure as Maurer-Cartan elements in the $L_\infty$-algebra constructed above. Finally under suitable conditions, we could show that the operad governing the homotopy version is a minimal model of the original operad.

While the above mentioned papers of Sheng et al. use derived brackets [46, 72, 73] as a main tool, our method is somehow ad hoc. We give a direct proof of the constructed $L_\infty$-structure. Finally, we could show that the resulting homotopy version is indeed the minimal model of the operad of Rota-Baxter associative algebras. It might be appropriate to explain here the relationship of our result with the paper of Dotsenko and Khoroshkin [20]. In that paper, the authors tried to deform the minimal model of the corresponding monomial operads obtained by Gröbner basis of the Rota-Baxter operad and they got the generators of the operad of homotopy Rota-Baxter algebras. It seems that it is not easy to obtain all the relations. While our generators of homotopy Rota-Baxter algebras are the same, we could determine all the relations in an indirect way with the aid of $L_\infty$-structure on the deformation complex found using our ad hoc method. However, our method to verify the minimal model was inspired by Dotsenko and Khoroshkin [20]. Dotsenko kindly pointed out another proof based the paper [20]; see Remark 10.12.

Another remark is in order. Once we show that the obtained operad of homotopy Rota-Baxter algebras is the minimal model of that of Rota-Baxter algebras, one could use the method of Kontsevich and Soibelman [45] and van der Laan [70, 71] to produce the $L_\infty$-structure on the deformation complex instead of our direct method. We offer two reasons. One reason is that the direct method is not much more complicated than the latter method and another reason is that we want to exhibit the effectiveness of our ad hoc method to deal with deformation theory and homotopy theory of algebraic structures.

It would be an interesting problem to give a general approach for operated algebras in the sense of Guo [34] and other algebraic structures. For instance, it is desirable to deal with differential algebras (continuing [40]) and averaging algebras (continuing and completing [69]) etc. We are working on these projects using our method.

This paper is organised as follows. The first section contains some preliminaries. Section 2 recalls the language of differential graded Lie algebras and $L_\infty$-algebras. Associative algebras are taken as baby model of our method in the third section. Basic definitions and facts about Rota-Baxter algebras which are mostly well known are recalled in Section 4. After defining a cohomology complex of Rota-Baxter operators, with the help of the usual Hochschild cocochain complex, a cochain complex, whose cohomology groups should control deformation theory of Rota-Baxter algebras, is exhibited in Section 5. We justify this cohomology theory by interpreting lower degree cohomology groups as formal deformations (Section 6) and abelian extensions of Rota-Baxter algebras (Section 7). Rota-Baxter algebra structures over the underlying space of this cochain complex is then realized as the Maurer-Cartan elements of an $L$-infinity algebra structure over the cochain complex, as is done in the eighth section. With the help of this $L$-infinity algebra, one introduces the notion of homotopy Rota-Baxter algebras of any weight in the ninth section. Finally it is shown that the operad governing homotopy Rota-Baxter algebras is a minimal model of the operad of Rota-Baxter algebras in the tenth section. We postpone the lengthy proof of the central result Theorem 8.1 to Appendix A and Appendix B contains a proof of another technical result Proposition 9.2.
1. Preliminaries

Throughout this paper, let \( k \) be a field of characteristic 0. All vector spaces are defined over \( k \), all tensor products and Hom-spaces are taken over \( k \).

A (homologically) graded vector space is a family of vector spaces \( V = \{V_n\}_{n \in \mathbb{Z}} \) indexed by integers. Elements of \( \bigcup_{n \in \mathbb{Z}} V_n \) are called homogeneous and the degree of \( v \in V_n \) is written as \( |v| := n \).

We use both homological and cohomological gradings. For a homologically graded space \( V = \bigoplus_{n \in \mathbb{Z}} V_n \), write \( V^n = V_{-n} \) will transform homological grading to cohomological grading and vice versa.

Let \( V \) and \( W \) be graded vector spaces. A graded map \( f : V \to W \) of degree \( r \) is by definition a linear map \( f : V \to W \) such that \( f(V_n) \subseteq W_{n+r} \) for all \( n \). In this case, denote \( |f| = r \). Write

\[
\text{Hom}(V, W)_r = \prod_{p \in \mathbb{Z}} \text{Hom}(V_p, W_{p+r})
\]

the space of graded maps of degree \( r \) and denote

\[
\text{Hom}(V, W) = \{\text{Hom}(V, W)_r\}_{r \in \mathbb{Z}}
\]
to be the graded space of graded linear maps from \( V \) to \( W \).

Let \( V \) and \( W \) be graded vector spaces. The tensor product \( V \otimes W \) of \( V \) and \( W \) is graded whose grading is given by

\[
(V \otimes W)_n := \bigoplus_{p+q=n} V_p \otimes W_q.
\]

Denote by \( \mathbb{Z} \) the 1-dimensional graded vector space spanned by \( s \) with \( |s| = 1 \). The suspension of \( V \) is \( sV := \mathbb{Z} \otimes V \), so \( (sV)_i \) can be identified with \( V_{i−1} \) for any \( i \in \mathbb{Z} \). Note that for \( v \in V_n \), \( sv \in sV \) is of degree \( n + 1 \) and the map \( s : V \to sV, v \mapsto sv \) is a graded map of degree 1. One can also define another 1-dimensional graded vector space \( \mathbb{Z}^{-1} \) with \( |\mathbb{Z}^{-1}| = −1 \). The desuspension of \( V \) is \( s^{-1}V := \mathbb{Z}^{-1} \otimes V \) and the desuspension map \( s^{-1} : V \to s^{-1}V, v \mapsto s^{-1}v \) is a graded map of degree −1.

We will encounter many signs in the graded world. The basic principle to determine signs is the so-called Koszul rule, that is, when we exchange the positions of two graded objects in an expression, we need to multiply the expression by a power of \(-1\) whose exponent is the product of their degrees. For instance, given two graded maps \( f : V \to V', g : W \to W' \), define \( f \otimes g : V \otimes W \to V' \otimes W' \) via

\[
(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).
\]

Another example is given as follows: for four graded maps \( f, f' : V \to V', g, g' : W \to W' \), the composition of \( f \otimes g \) and \( f' \otimes g' \) is defined to be

\[
(f \otimes g) \circ (f' \otimes g') = (-1)^{|g||f'|} (f \circ f') \otimes (g \circ g').
\]

For \( v_1, \ldots, v_n \in V \), write \( v_1, \ldots, v_n := v_1 \otimes \cdots \otimes v_n \in V^{\otimes n} \) and also \( sv_1, \ldots, v_n = sv_1 \otimes \cdots \otimes sv_n \in (sV)^{\otimes n} \).

Let \( n \geq 1 \). Recall \( S_n \) denotes the symmetric group in \( n \) variables. For \( 0 \leq i_1, \ldots, i_r \leq n \) with \( i_1 + \cdots + i_r = n \), \( \text{Sh}(i_1, i_2, \ldots, i_r) \) is the set of \((i_1, \ldots, i_r)\)-shuffles, i.e., those permutation \( \sigma \in S_n \) such that

\[
\sigma(1) < \sigma(2) < \cdots < \sigma(i_1), \ \sigma(i_1 + 1) < \cdots < \sigma(i_1 + i_2), \ldots, \ \sigma(i_r - 1 + 1) < \cdots < \sigma(n).
\]

The following fact is well known:
Lemma 1.1. Let \( n \geq 1 \), \( 1 \leq i \leq n-1 \). Then for any \( \delta \in S_n \), there exists a unique triple \((\tau, \sigma, \pi)\) with \( \sigma \in \text{Sh}(i, n-i) \), \( \tau \in S_i \), \( \pi \in S_{n-i} \) such that \( \delta(l) = \sigma \tau(l) \) for \( 1 \leq l \leq i \), and \( \delta(i+m) = \sigma(i+\pi(m)) \) for \( 1 \leq m \leq n-i \).

Let \( V \) be a graded vector space. Define the graded symmetric algebra \( S(V) \) of \( V \) to be \( T(V)/I \) where the two-sided ideal \( I \) is generated by \( x \otimes y - (-1)^{|x||y|} y \otimes x \) for all homogeneous elements \( x, y \in V \). For \( x_1 \otimes \cdots \otimes x_n \in T(V) \), write \( x_1 \otimes \cdots \otimes x_n \) to be the corresponding element in \( S(V) \). Define the weight of \( x_1 \otimes \cdots \otimes x_n \) to be \( n \), so \( S(V) \) is weight graded whose weight \( n \)-th component is written as \( S(V)^{(n)} \), \( n \geq 0 \).

For homogeneous elements \( x_1, \ldots, x_n \in V \) and \( \sigma \in S_n \), the Koszul sign \( \varepsilon(\sigma; x_1, \ldots, x_n) \) is defined by

\[
\varepsilon(\sigma; x_1, \ldots, x_n) = \varepsilon(\sigma_1, x_1, \ldots, x_n) \varepsilon(\sigma_2, x_{1+1}, \ldots, x_{1+n}) \cdots \varepsilon(\sigma_n, x_{n-1}, \ldots, x_n) \in S(V).
\]

Define the graded exterior algebra \( \Lambda(V) \) of \( V \) to be \( T(V)/J \) where the two-sided ideal \( J \) is generated by \( x \otimes y - (-1)^{|x||y|} y \otimes x \) for all homogeneous elements \( x, y \in V \). For \( x_1 \otimes \cdots \otimes x_n \in T(V) \), write the corresponding element in \( \Lambda(V) \) as \( x_1 \wedge \cdots \wedge x_n \). Define the weight of \( x_1 \wedge x_2 \wedge \cdots \wedge x_n \) to be \( n \), so \( \Lambda(V) \) is weight graded whose weight \( n \)-th component is written as \( \Lambda(V)^{(n)} \), \( n \geq 0 \). For homogeneous elements \( x_1, \ldots, x_n \in V \) and \( \sigma \in S_n \), the Koszul sign \( \chi(\sigma; x_1, \ldots, x_n) \) is defined by

\[
\chi(\sigma; x_1, \ldots, x_n) = \chi(\sigma(1), x_{1+1}, \ldots, x_{1+n}) \chi(\sigma(2), x_{1+n+1}, \ldots, x_{1+2n}) \cdots \chi(\sigma(n), x_{1+(n-1)n}, \ldots, x_n) \in S(V).
\]

Obviously

\[
\chi(\sigma; x_1, \ldots, x_n) = \text{sgn}(\sigma) \varepsilon(\sigma; x_1, \ldots, x_n),
\]

where \( \text{sgn}(\sigma) \) is the sign of \( \sigma \).

Fix an isomorphism \( S_s(V)^{(n)} \cong s^n \Lambda(V)^{(n)} \) by sending \( sx_1 \otimes \cdots \otimes sx_n \) to

\[
(-1)^{\sum_{j=1}^{n-1} \sum_{j=1}^{k} |x_j|} s^n(x_1 \wedge x_2 \wedge \cdots \wedge x_n).
\]

Under this isomorphism, we have the equality:

\[
\chi(\sigma; x_1, \ldots, x_n)(-1)^{\sum_{j=1}^{n-1} |x_{\sigma(j)}|} = \varepsilon(\sigma; sx_1, \ldots, sx_n)(-1)^{\sum_{j=1}^{n-1} |x_j|},
\]

for any \( \sigma \in S_n \) and homogeneous elements \( x_1, \ldots, x_n \in V \).

For permutations \( \delta, \sigma, \pi, \tau \) appearing in Lemma 1.1, we have the following equality:

\[
\chi(\delta; x_1, \ldots, x_n) = \chi(\sigma; x_1, \ldots, x_n) \chi(\pi; x_{\sigma(1)+1}, \ldots, x_{\sigma(i+n-1)}) \chi(\tau; x_{\sigma(1)+1}, \ldots, x_{\sigma(i)}),
\]

whose proof is left to the reader.

2. Differential graded Lie algebras and \( L_{\infty} \)-algebras

In this section, we will recall some preliminaries on differential graded Lie algebras and \( L_{\infty} \)-algebras. For more background on differential graded Lie algebras and \( L_{\infty} \)-algebras, we refer the reader to [64, 47, 48, 28].
2.1. Differential graded Lie algebras and Maurer-Cartan elements.

**Definition 2.1.** A differential graded (=dg) Lie algebra is a triple \((L, l_1, l_2)\) where \(L = \bigoplus_{i \in \mathbb{Z}} L_i\) is a graded \(k\)-space, \(l_1 : L \to L\) and \(l_2 : L^\otimes 2 \to L\) are two graded linear maps with \(|l_1| = -1\) and \(|l_2| = 0\), subject to the following conditions:

(i) \(l_1 \circ l_1 = 0\);
(ii) \(l_1 \circ l_2 = l_2 \circ (l_1 \otimes \text{Id} + \text{Id} \otimes l_1)\);
(iii) (anti-symmetry) \(l_2(x \otimes y) + (-1)^{|x||y|} l_2(y \otimes x) = 0, \forall x, y \in L\);
(iv) (Jacobi identity)
\[
l_2(l_2(x \otimes y) \otimes z) + (-1)^{|x||y|} l_2(l_2(y \otimes z) \otimes x) + (-1)^{|x||y|+|z|} l_2(l_2(z \otimes x) \otimes y) = 0, \forall x, y, z \in L.
\]

When \(l_1 = 0\), the pair \((L, l_2)\) is called a graded Lie algebra.

**Definition 2.2.** Let \((L, l_1, l_2)\) be a dg Lie algebra. An element \(\alpha \in L_{-1}\) is called a Maurer-Cartan element if it satisfies the Maurer-Cartan equation
\[
l_1(\alpha) - \frac{1}{2} l_2(\alpha \otimes \alpha) = 0.
\]

Given an arbitrary Maurer-Cartan element in a dg Lie algebra, one can get a new dg Lie algebra by twisting the original dg Lie algebra structure using this element.

**Lemma 2.3.** Let \((L, l_1, l_2)\) be a dg Lie algebra and \(\alpha \in L_{-1}\) be a Maurer-Cartan element. Define new operations \(l_1^\alpha\) and \(l_2^\alpha\) on \(L\) as
\[
l_1^\alpha(x) = l_1(x) - l_2(\alpha \otimes x), \forall x \in L\]
and \(l_2^\alpha = l_2\).

Then \((L, l_1^\alpha, l_2^\alpha)\) is a dg Lie algebra as well. This new dg Lie algebra is called the twisted dg Lie algebra (by the Maurer-Cartan element \(\alpha\)).

2.2. \(L_\infty\)-algebras and Maurer-Cartan elements.

**Definition 2.4.** Let \(L = \bigoplus_{i \in \mathbb{Z}} L_i\) be a graded space over \(k\). Assume that \(L\) is endowed with a family of graded linear operators \(l_n : L^\otimes n \to L, n \geq 1\) with \(|l_n| = n - 2\) subject to the following conditions: for any \(n \geq 1\), \(\sigma \in S_n\) and \(x_1, \ldots, x_n \in L\),

(i) (generalised anti-symmetry)
\[
l_n(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) = \chi(\sigma; x_1, \ldots, x_n) l_n(x_1 \otimes \cdots \otimes x_n);
\]
(ii) (generalised Jacobi identity)
\[
\sum_{i=1}^{n} \sum_{\sigma \in \text{Sh}(n-i)} \chi(\sigma; x_1, \ldots, x_n)(-1)^{(n-i)} l_{n-i+1}(l_i(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n)}) = 0,
\]

where recall that \(\text{Sh}(i, n-i)\) is the set of \((i, n-i)\) shuffles.

Then \((L, \{l_n\}_{n \geq 1})\) is called an \(L_\infty\)-algebra.

**Remark 2.5.** Let us consider the generalised Jacobi identity for \(n \leq 3\) with the assumption of generalised anti-symmetry.

(i) \(n = 1\), \(l_1 \circ l_1 = 0\), that is, \(l_1\) is a differential,
(ii) \(n = 2\), \(l_1 \circ l_2 = l_2 \circ (l_1 \otimes \text{Id} + \text{Id} \otimes l_1)\), that is \(l_1\) is a derivation for \(l_2\),
(iii) $n = 3$, for homogeneous elements $x_1, x_2, x_3 \in L$

\[
l_2(l_2(x_1 \otimes x_2) \otimes x_3) + (-1)^{|x_1||x_2|+|x_3|}l_2(l_2(x_2 \otimes x_3) \otimes x_1) + (-1)^{|x_2||x_3|+|x_1|}l_2(l_2(x_3 \otimes x_1) \otimes x_2)
= -\left(l_1(l_3(x_1 \otimes x_2 \otimes x_3)) + l_3(l_1(x_1) \otimes x_2 \otimes x_3) + (-1)^{|x_1|}l_3(x_1 \otimes l_1(x_2) \otimes x_3) +
\right. \\
\left. (-1)^{|x_1|+|x_2|}l_3(x_1 \otimes x_2 \otimes l_1(x_3)) \right),
\]

does not necessarily hold. That is, $l_2$ does not satisfy the Jacobi identity up to homotopy.

In particular, if all $l_n = 0$ with $n \geq 3$, then $(L, l_1, l_2)$ is just a dg Lie algebra.

One can also define Maurer-Cartan elements in $L_\infty$-algebras.

**Definition 2.6.** Let $(L, \{l_n\}_{n \geq 1})$ be an $L_\infty$-algebra. An element $\alpha \in L_{-1}$ is called a Maurer-Cartan element if it satisfies the Maurer-Cartan equation:

\[
\sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} l_n(\alpha \otimes \cdots \otimes \alpha) = 0,
\]

whenever this infinite sum exists.

Lemma 2.3 can be generalised to $L_\infty$-algebras.

**Proposition 2.7** (Twisting procedure). Given a Maurer-Cartan element $\alpha$ in $L_\infty$-algebra $L$, one can introduce a new $L_\infty$-structure $\{l^\alpha_n\}_{n \geq 1}$ on graded space $L$, where $l^\alpha_n : L^{\otimes n} \to L$ is defined as:

\[
l^\alpha_n(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{\infty} \frac{1}{i!} (-1)^{\frac{i(i+1)}{2}} l_{n+i}(\alpha \otimes x_1 \otimes \cdots \otimes x_n), \ \forall x_1, \ldots, x_n \in L,
\]

whenever these infinite sums exist. The new $L_\infty$-algebra $(L, \{l^\alpha_n\}_{n \geq 1})$ is called the twisted $L_\infty$-algebra (by the Maurer-Cartan element $\alpha$).

**Remark 2.8.**

(i) The signs in Definition 2.6 and Proposition 2.7 are different from those appearing in [49], as the conventions in [49] are essentially about $L_\infty[1]$-algebras. We refer the reader to [74] for the translation between $L_\infty$-structures and $L_\infty[1]$-structures.

(ii) Proposition 2.7 is essentially contained in [28, Section 4]. Notice that here we only ask the existence of the infinite sums, although in many references, nilpotent or weakly filtered $L_\infty$ algebras [28, 49] are used to guarantee the convergence of these sums.

3. **Formal deformations, Hochschild cohomology and homotopy theory of associative algebras**

In this section, we will recall the formal deformations and Hochschild cohomology of associative algebras. We will see how the dg Lie algebra structure on the underlying graded space of the Hochschild cochain complex introduced by Gerstenhaber will enable defining $A_\infty$-algebras which is the homotopy version of associative algebras.

This example is our baby model for deformation theory and homotopy theory of Rota-Baxter algebras.
3.1. Hochschild cohomology of associative algebras.

Let \((A, \mu)\) be an associative \(k\)-algebra. We often write \(\mu(a \otimes b) = a \cdot b = ab\) for any \(a, b \in A\). Let \(M\) be a bimodule over \(A\). The Hochschild cochain complex of \(A\) with coefficients in \(M\) is

\[
C^\bullet_{\text{Alg}}(A, M) := \bigoplus_{n=0}^{\infty} C^n_{\text{Alg}}(A, M),
\]

where \(C^n_{\text{Alg}}(A, M) = \text{Hom}(A^\otimes n, M)\) and the differential \(\delta^n : C^n_{\text{Alg}}(A, M) \to C^{n+1}_{\text{Alg}}(A, M)\) is defined as:

\[
\delta^n(f)(a_1, \ldots, a_{n+1}) = (-1)^{n+1}a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n}(-1)^{n-i+1} f(a_1, a_i, a_{i+1}, \ldots, a_{n+1}) + f(a_1, \ldots, a_n) a_{n+1}
\]

for all \(f \in C^n_{\text{Alg}}(A, M), a_1, \ldots, a_{n+1} \in A\).

The cohomology of the Hochschild cochain complex \(C^\bullet_{\text{Alg}}(A, M)\) is called the Hochschild cohomology of \(A\) with coefficients in \(M\), denoted by \(\text{HH}^\bullet(A, M)\). When the bimodule \(M\) is the regular bimodule \(A\) itself, we just denote \(C^\bullet_{\text{Alg}}(A, A)\) by \(C^\bullet_{\text{Alg}}(A)\) and call it the Hochschild cochain complex of associative algebra \((A, \mu)\). Denote the cohomology \(\text{HH}^\bullet(A, A)\) by \(\text{HH}^\bullet(A)\), called the Hochschild cohomology of associative algebra \((A, \mu)\).

3.2. Formal deformations of associative algebras.

Given an associative \(k\)-algebra \((A, \mu)\), consider \(k[[t]]\)-bilinear associative products on

\[
A[[t]] = \{ \sum_{i=0}^{\infty} a_i t^i \mid a_i \in A, \forall i \geq 0 \}.
\]

Such a product is determined by

\[
\mu_t = \sum_{i=0}^{\infty} \mu_i t^i : A \otimes A \to A[[t]],
\]

where for all \(i \geq 0, \mu_i : A \otimes A \to A\) are linear maps. When \(\mu_0 = \mu\), we say that \(\mu_t\) is a formal deformation of \(\mu\) and \(\mu_1\) is called the infinitesimal of formal deformation \(\mu_t\).

The only constraint is the associativity of \(\mu_t\):

\[
\mu_t(\mu_t(a \otimes b) \otimes c) = \mu_t(a \otimes \mu_t(b \otimes c)), \forall a, b, c \in A
\]

where is equivalent to the following family of equations:

\[
\sum_{i,j \geq 0} \left( \mu_i(\mu_j(a \otimes b) \otimes c) - \mu_i(a \otimes \mu_j(b \otimes c)) \right) = 0, \forall a, b, c \in A, n \geq 0.
\]

Looking closely at the cases \(n = 0\) and \(n = 1\), one obtains:

(i) when \(n = 0\), \((a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in A\), which is exactly the associativity of \(\mu\);

(ii) when \(n = 1\),

\[
a \mu_1(b \otimes c) - \mu_1(ab \otimes c) + \mu_1(a \otimes bc) - \mu_1(a \otimes b)c = 0, \forall a, b, c \in A,
\]

which says that the infinitesimal \(\mu_1\) is a 2-cocycle in the Hochschild cochain complex \(C^\bullet_{\text{Alg}}(A)\).
In general, we can rewrite Equation (10) as

$$\delta^2(\mu_i) = \frac{1}{2} \sum_{i=1}^{n-1} [\mu_i, \mu_{n-i}]_G$$

where \([-,-]_G\) is the Gerstenhaber bracket; see the next subsection.

3.3. Gerstenhaber brackets.
Let \(V = \bigoplus_{i \in \mathbb{Z}} V_i\) be a graded space and recall that \(sV\) denotes the suspension of \(V\), i.e., \((sV)_n = V_{n-1}, \forall n \in \mathbb{Z}\). The free conilpotent tensor coalgebra \(T^c(sV)\) is defined to

$$T^c(sV) = \mathbb{k} \oplus sV \oplus (sV)^{\otimes 2} \oplus \cdots \oplus (sV)^{\otimes n} \oplus \cdots$$

with the usual deconcatenation coproduct. Let \(\mathcal{C}_{\text{Alg}}(V) = \text{Hom}(T^c(sV), sV)\).

Let \(m \geq 1, 1 \leq n \leq m\). Given homogeneous elements \(f \in \text{Hom}((sV)^{\otimes m}, V)\) and \(g_i \in \text{Hom}((sV)^{\otimes i}, V), i = 1, \ldots, n\) with all \(l_j \geq 0\), then \(sf, sg_1, sg_2, \ldots, sg_n \in \mathcal{C}_{\text{Alg}}(V) = \text{Hom}(T^c(sV), sV)\), define the brace operation

$$sf\{sg_1, \ldots, sg_n\} \in \text{Hom}((sV)^{\otimes u}, sV)$$

to be

$$sf\{sg_1, \ldots, sg_n\}(sa_{1,u}) = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} (-1)^{\xi} sf(sa_{i_1,i_2} \otimes sg_1(sa_{i_1+i_2,i_3}) \otimes sa_{i_1+i_2+i_4} \otimes sg_2(sa_{i_2+i_3+i_4}) \otimes \cdots \otimes sg_n(sa_{i_{n-1}+i_n} \otimes sa_{i_n+i_n+1,u}),$$

where \(a_1, \ldots, a_u \in V, u = m + l_1 + \cdots + l_n - n\) and \(\xi = \sum_{k=1}^{n} (l_k + 1) \sum_{j=1}^{i_k} (a_j + 1)\).

In particular, for homogeneous elements \(f \in \text{Hom}((sV)^{\otimes m}, V)\) with \(m \geq 1\) and \(g \in \text{Hom}((sV)^{\otimes n}, V)\) with \(n \geq 0\), for each \(1 \leq i \leq m\), write

$$sf \circ_i sg = sf \circ (\text{Id}^{\otimes (i-1)} \otimes sg \otimes \text{Id}^{\otimes (m-i)}).$$

These notations will be very useful while dealing with operads.

For two homogeneous elements \(sf \in \text{Hom}((sV)^{\otimes m}, sV), sg \in \text{Hom}((sV)^{\otimes n}, sV)\), define

$$[sf, sg]_G = sf\{sg\} - (-1)^{(|f|+1)(|g|+1)} sg\{sf\},$$

called the Gerstenhaber bracket of \(sf\) and \(sg\).

**Theorem 3.1** ([25]). For any given graded space \(V\), the Gerstenhaber bracket makes the graded space \(\mathcal{C}_{\text{Alg}}(V)\) into a graded Lie algebra.

Moreover, the brace operation on \(\mathcal{C}_{\text{Alg}}(V)\) satisfies the following pre-Jacobi identities:

**Proposition 3.2** ([25, 27, 26]). For any homogeneous elements \(sf, sg_1, \ldots, sg_m, sh_1, \ldots, sh_n\) in \(\mathcal{C}_{\text{Alg}}(V)\), the following identity holds:

$$[sf\{sg_1, \ldots, sg_m\}\{sh_1, \ldots, sh_n\} = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq j_n \leq n} (-1)^{\sum_{k=1}^{m} (|g_{i_k}|+1) \sum_{j=1}^{i_k} (|h_{j_k}|+1)} sf\{sh_{1,i_1}, sg_1\{sh_{i_1+i_1,j_1}\}, \ldots, sg_m\{sh_{i_m+i_m,j_n}\}, sh_{j_n+i_n+1,u}\}. $$
The following two results are well known and we include sketch of proofs to fix the notations. Fix two isomorphisms

\[ \text{Hom}((sV)^{\otimes n}, sV) \cong \text{Hom}(V^{\otimes n}, V), f \mapsto \tilde{f} := s^{-1} \circ f \circ s^{\otimes n} \]

for \( f \in \text{Hom}((sV)^{\otimes n}, sV) \) and

\[ \text{Hom}((sV)^{\otimes n}, V) \cong \text{Hom}(V^{\otimes n}, V), g \mapsto \hat{g} := g \circ s^{\otimes n} \]

for \( g \in \text{Hom}((sV)^{\otimes n}, V) \)

**Proposition 3.3.** Let \( V \) be an ungraded space considered as a graded space concentrated in degree 0. Then there is a bijection between the set of Maurer-Cartan elements in the graded Lie algebra \( \mathcal{C}_{\text{Alg}}(V) \) and the set of associative algebra structure on space \( V \).

**Sketch of Proof:** Since \( V \) is concentrated in degree 0, the degree \(-1\) part of \( \mathcal{C}_{\text{Alg}}(V) \) is \( \text{Hom}((sV)^{\otimes 2}, sV) \). Given an element \( \alpha \in \mathcal{C}_{\text{Alg}}(V) \) of degree \(-1\), we define an operation \( \mu : V^{\otimes 2} \to V \) as

\[ \mu = \tilde{\alpha} = s^{-1} \circ \alpha \circ (s \otimes s). \]

Then it can be checked that the fact that \( \alpha \) satisfying the Maurer-Cartan equation in graded Lie algebra \( \mathcal{C}_{\text{Alg}}(V) \) is equivalent the associativity of the operation \( \mu \).

**Proposition 3.4.** Let \( (A, \mu) \) be an associative algebra and \( \alpha \) be the corresponding Maurer-Cartan element in \( \mathcal{C}_{\text{Alg}}(A) \). Then the underlying complex of the twisted dg Lie algebra \( (\mathcal{C}_{\text{Alg}}(A), l_1^n, l_2^n) \) is exactly \( sC^*_{\text{Alg}}(A) \), the shift of the Hochschild cochain complex of associative algebra \( A \).

**Sketch of Proof:** Recall that \( \alpha = -s \circ \mu \circ (s^{-1} \otimes s^{-1}) : sV \otimes sV \to sV \). Then one checks that

\[ l_1^n(f) = -[\tilde{\alpha}, f]_G = -\delta^n(\tilde{f}) \]

for any \( f \in \text{Hom}((sA)^{\otimes n}, sA) \). This shows that the complex \( (\mathcal{C}_{\text{Alg}}(A), l_1^n) \) is isomorphic to \( sC^*_{\text{Alg}}(A) \), the shift of the Hochschild cochain complex of associative algebra \( A \).

### 3.4. Defining \( A_{\infty} \)-algebras via Maurer-Cartan elements.

Let \( V \) be a graded vector space. Recall that the reduced cofree conilpotent coalgebra is

\[ \overline{T}(sV) = sV \oplus (sV)^{\otimes 2} \oplus \cdots \]

with the usual deconcaternation coproduct. Define \( \overline{\mathcal{C}}_{\text{Alg}}(V) = \text{Hom}(\overline{T}(sV), sV) \). It is easy to verify that \( \overline{\mathcal{C}}_{\text{Alg}}(V) \) is a graded Lie subalgebra of \( \mathcal{C}_{\text{Alg}}(V) \).

**Definition 3.5.** An \( A_{\infty} \)-algebra structure on graded space \( V \) is defined to be a Maurer-Cartan element in graded Lie algebra \( \overline{\mathcal{C}}_{\text{Alg}}(V) \).

By definition, an \( A_{\infty} \)-algebra structure on \( V \) consists of a family of operators

\[ b_n : (sV)^{\otimes n} \to sV, \forall n \geq 1 \]

with \( |b_n| = -1 \) satisfying

\[ \sum_{j=1}^{n} b_{n-j+1}b_j = 0, \forall n \geq 1. \]

Define operators \( m_n = \tilde{b}_n = s^{-1} \circ b_n \circ s^{\otimes n} : V^{\otimes n} \to V \). Then one can get the following equivalent definition of \( A_{\infty} \)-algebras, which is the original definition due to Stasheff.
Definition 3.6 ([63]). An $A_\infty$-algebra structure on $V$ consists of a family of operators
\[ m_n : V^\otimes n \to V, n \geq 1 \]
with $|m_n| = n - 2$, which fulfill the Stasheff identities:
\[ \sum (-1)^{i+j+k} m_{i+1+k} \circ (\text{Id}^i \otimes m_j \otimes \text{Id}^k) = 0, \forall n \geq 1. \]

For later use, we record here the definition of $A_\infty$-morphisms.

Definition 3.7. Let $V, W$ be two $A_\infty$-algebras and $b = \sum_{i \geq 1} b_i \in \overline{\mathfrak{C}}_{\text{Alg}}(V), b' = \sum_{i \geq 1} b'_i \in \overline{\mathfrak{C}}_{\text{Alg}}(W)$ be the corresponding Maurer-Cartan elements respectively. An $A_\infty$-morphism $\phi$ from $V$ to $W$ consists of a family of operators $\phi_i : (sV)^{\otimes i} \to sW, i \geq 1$ of degree 0 satisfying the following equations:
\[ \sum_{i \geq 1} \phi_{i+1+k}(\text{Id}^i \otimes b_j \otimes \text{Id}^k) = \sum_{m \geq 1} \sum_{i_1, \ldots, i_m \geq 1} b'_m \circ (\phi_{i_1} \otimes \ldots \otimes \phi_{i_m}), \forall n \geq 1. \]

Remark 3.8. (i) In Definition 3.5, we use the reduced version $\overline{\mathfrak{C}}_{\text{Alg}}(V)$ to define $A_\infty$-algebras, while Maurer-Cartan elements in the full version $\mathfrak{C}_{\text{Alg}}(V)$ would give curved $A_\infty$-algebras [29].

(ii) We introduce $A_\infty$-algebras using the naive approach in this section. However, one can use Koszul duality theory for operads [31, 51] instead, as the operad governing $A_\infty$-algebras is a minimal cofibrant resolution of the operad of associative algebras in the model category of operads [42, 6].

Nevertheless, as we will see soon in the next section, the operad of Rota-Baxter algebras is NOT quadratic, so it seems that the Koszul duality theory for operads [31][51] can not apply directly. This is why we adopt the naive approach in this paper while developing homotopy theory of Rota-Baxter algebras.

4. ROTA-BAXTER ALGEBRAS AND ROTA-BAXTER BIMODULES

In this section, we recall some basic definitions and facts about Rota-Baxter algebras.

Definition 4.1. Let $(A, \mu = \cdot)$ be an associative algebra over field $k$ and $\lambda \in k$. A linear operator $T : A \to A$ is said to be a Rota-Baxter operator of weight $\lambda$ if it satisfies
\[ T(a) \cdot T(b) = T(a \cdot T(b)) + T(a) \cdot b + \lambda a \cdot b \]
for any $a, b \in A$, or in terms of maps
\[ \mu \circ (T \otimes T) = T \circ (\text{Id} \otimes T + T \otimes \text{Id}) + \lambda T \circ \mu. \]

In this case, $(A, \mu, T)$ is called a Rota-Baxter algebra of weight $\lambda$. Denote by $RBA_\lambda$ the category of Rota-Baxter algebras of weight $\lambda$ with obvious morphisms.

Remark 4.2. As mentioned by Remark 3.8 (ii), although the associativity of $\mu$ is quadratic, the defining relation Equation (18) of Rota-Baxter operator $T$ is not quadratic and not even homogeneous when $\lambda \neq 0$, so the Koszul duality theory for operads [31][51] could not be applied directly to develop a cohomology theory of Rota-Baxter algebras.
**Definition 4.3.** Let \((A, \mu, T)\) be a Rota-Baxter algebra and \(M\) be a bimodule over associative algebra \((A, \mu)\). We say that \(M\) is a bimodule over Rota-Baxter algebra \((A, \mu, T)\) or a Rota-Baxter bimodule if \(M\) is endowed with a linear operator \(T_M : M \to M\) such that the following equations
\[
T(a)T_M(m) = T_M(aT_M(m) + T(a)m + \lambda am),
\]
(19)
\[
T_M(m)T(a) = T_M(mT(a) + T_M(m)a + \lambda ma).
\]
(20)
hold for any \(a \in A\) and \(m \in M\).

Of course, \((A, \mu, T)\) itself is a bimodule over the Rota-Baxter algebra \((A, \mu, T)\), called the regular Rota-Baxter bimodule.

The following result is easy whose proof is left to the reader:

**Proposition 4.4.** Let \((A, \mu, T)\) be a Rota-Baxter algebra and \(M\) be a bimodule over associative algebra \((A, \mu)\). It is well known that \(A \oplus M\) becomes an associative algebra whose multiplication is
\[
(a, m)(b, n) = (a \cdot b, an + mb).
\]
(21)
Write \(\iota : A \to A \oplus M, a \mapsto (a, 0)\) and \(\pi : A \oplus M \to A, (a, m) \mapsto a\). Then \(A \oplus M\) is a Rota-Baxter algebra such that \(\iota\) and \(\pi\) are both morphisms of Rota-Baxter algebras if and only if \(M\) is a Rota-Baxter bimodule over \(A\).

This new Rota-Baxter algebra will be denoted by \(A \ltimes M\), called the semi-direct product (or trivial extension) of \(A\) by \(M\).

In fact, we will see that the above result is a special case of Propositions 7.3 and 7.7.

There is a definition of bimodules over two Rota-Baxter algebras in [58].

**Remark 4.5.** One can use monoid objects in certain slice categories to justify Definition 4.3 following [22]. In fact, one can show an equivalence between the category of monoids in the slice category \(\text{RBA}_A/A\) and that of Rota-Baxter bimodules over a Rota-Baxter algebra \(A\).

Recall first the following interesting observation:

**Proposition 4.6 ([36, Theorem 1.1.17]).** Let \((A, \mu, T)\) be a Rota-Baxter algebra. Define a new binary operation as:
\[
a \star b := a \cdot T(b) + T(a) \cdot b + \lambda a \cdot b
\]
(22)
for any \(a, b \in A\). Then
(i) the operation \(\star\) is associative and \((A, \star)\) is a new associative algebra;
(ii) the triple \((A, \star, T)\) also forms a Rota-Baxter algebra of weight \(\lambda\) and denote it by \(A\star\);
(iii) the map \(T : (A, \star, T) \to (A, \mu, T)\) is a morphism of Rota-Baxter algebras.

One can also construct new Rota-Baxter bimodules from old ones.

**Proposition 4.7.** Let \((A, \mu, T)\) be a Rota-Baxter algebra of weight \(\lambda\) and \((M, T_M)\) be a Rota-Baxter bimodule over it. We define a left action \(\triangleright\) and a right action \(\triangleleft\) of \(A\) on \(M\) as follows: for any \(a \in A, m \in M\),
\[
a \triangleright m := T(a)m - T_M(am),
\]
(23)
\[
m \triangleleft a := mT(a) - T_M(ma).
\]
(24)
Then these actions make \(M\) into a Rota-Baxter bimodule over \(A\star\) and denote this new bimodule by \(_\lambda M\).
Proof. Firstly, we show that \((M, \triangleright)\) is a left module over \((A, \star)\).

\[
\begin{align*}
a \triangleright (b \triangleright m) &= a \triangleright (T(b)m - T_M(bm)) \\
&= T(a)(T(b)m - T_M(bm)) - T_M(aT(b)m - aT_M(bm)) \\
&= T(a)T(b)m - T_M(aT_M(bm)) + T(a)bm + \lambda abm \\
&- T_M(aT(b)m) + T_M(aT_M(bm)) \\
&= T(a)T(b)m - T_M(T(a)bm) - \lambda T_M(abm) - T_M(aT(b)m),
\end{align*}
\]

\[
(a \star b) \triangleright m = T(a \star b)m - T_M((a \star b)m) = T(a)T(b)m - T_M(aT(b)m + T(a)bm + \lambda abm).
\]

So we have

\[
a \triangleright (b \triangleright m) = (a \star b) \triangleright m.
\]

Thus the operation \(\triangleright\) makes \(M\) into a left module over \((A, \star)\). Similarly, one can check that operation \(\triangleleft\) defines a right module structure on \(M\) over \((A, \star)\).

Now, we are going to check the compatibility of operations \(\triangleright\) and \(\triangleleft\). We have the following equations:

\[
(a \triangleright m) \triangleleft b = (T(a)m - T_M(am)) \triangleleft b = (T(a)m - T_M(am))T(b) - T_M(T(a)mb - T_M(am)b) = T(a)mT(b) - T_M(T_M(am)b + amT(b) + \lambda amb) - T_M(T(a)mb) + T_M(T_M(am)b) = T(a)mT(b) - T_M(T_M(amT(b)) - \lambda T_M(abm) - T_M(T(a)mb),
\]

\[
a \triangleright (m \triangleleft b) = a \triangleright (mT(b) - T_M(mb)) = T(a)(mT(b) - T_M(mb)) - T_M(amT(b) - aT_M(mb)) = T(a)mT(b) - T_M(aT_M(mb)) + T(a)mb + \lambda amb - T_M(amT(b)) + T_M(aT_M(mb)) = T(a)mT(b) - T_M(T(a)mb) - \lambda T_M(abm) - T_M(amT(b)).
\]

Thus we have

\[
(a \triangleright m) \triangleleft b = a \triangleright (m \triangleleft b),
\]

that is, operations \(\triangleright\) and \(\triangleleft\) make \(M\) into a bimodule over associative algebra \((A, \star)\).

Finally, we show that \(\mathcal{R}_M\) is a Rota-Baxter bimodule over \(A_\star\). that is, for any \(a \in A\) and \(m \in M\),

\[
T(a) \triangleright T_M(m) = T_M(a \triangleright T_M(m)) + T(a) \triangleright m + \lambda a \triangleright m, \\
T_M(m) \triangleleft T(a) = T_M(m \triangleleft T(a)) + T_M(m \triangleleft a) + \lambda m \triangleleft a.
\]

We only prove the first equality, the second being similar.

In fact,

\[
T(a) \triangleright T_M(m) = T^2(a)T_M(m) - T_M(T(a)T_M(m)) = T_M(T(a)T_M(m)) + T^2(a)m + \lambda T(a)m - T_M(aT_M(m)) + T(a)m + \lambda am = T_M(T^2(a)m + \lambda T(a)m)
\]
and
\[
T_M(a \triangleright T_M(m) + T(a) \triangleright m + \lambda a \triangleright m),
\]
\[
= T_M(T(a)T_M(m) - T_M(aT_M(m)) + T^2(a)m - T_M(T(a)m) + \lambda T(a)m - \lambda T_M(am))
\]
\[
= T_M(T_M(aT_M(m) + T(a)m + \lambda am) - T_M(aT_M(m)) + T^2(a)m - T_M(T(a)m) + \lambda T(a)m - \lambda T_M(am))
\]
\[
= T_M(T^2(a)m + \lambda T(a)m)
\]
\[
= T(a) \triangleright T_M(m).
\]

5. Cohomology theory of Rota-Baxter algebras

In this section, we will define a cohomology theory for Rota-Baxter algebras of any weight.

5.1. Cohomology of Rota-Baxter Operators.

Firstly, let’s study the cohomology of Rota-Baxter operators.

Let \((A, \mu, T)\) be a Rota-Baxter algebra and \((M, T_M)\) be a Rota-Baxter bimodule over it. Recall that Proposition 4.6 and Proposition 4.7 give a new associative algebra \(A_\bullet\) and a new Rota-Baxter bimodule \(\triangleright M_\circ\) over \(A_\bullet\). Consider the Hochschild cochain complex of \(A_\bullet\) with coefficients in \(\triangleright M_\circ\):

\[
C^\bullet_{\text{Alg}}(A_\bullet, \triangleright M_\circ) = \bigoplus_{n=0}^{\infty} C^n_{\text{Alg}}(A_\bullet, \triangleright M_\circ).
\]

More precisely, for \(n \geq 0\),

\[
C^n_{\text{Alg}}(A_\bullet, \triangleright M_\circ) = \text{Hom}(A^\otimes n, M)
\]

and its differential

\[
\partial^n : C^n_{\text{Alg}}(A_\bullet, \triangleright M_\circ) \to C^{n+1}_{\text{Alg}}(A_\bullet, \triangleright M_\circ)
\]

is defined as:

\[
\partial^n(f)(a_{1,n+1}) = (-1)^{n+1} a_1 \triangleright f(a_{2,n+1}) + \sum_{i=1}^{n} (-1)^{n-i+1} f(a_{1,i-1} \otimes a_i \star a_{i+1} \otimes a_{i+2,n+1}) + f(a_{1,n}) \triangleright a_{n+1}
\]

\[
= (-1)^{n+1} \left( T(a_1) f(a_{2,n+1}) - T_M(a_1 f(a_{2,n+1})) \right)
\]

\[
+ \sum_{i=1}^{n} (-1)^{n-i+1} \left( f(a_{1,i-1} \otimes a_i T(a_{i+1}) \otimes a_{i+2,n+1}) + f(a_{1,i-1} \otimes T(a_i) a_{i+1} \otimes a_{i+2,n+1}) + \lambda f(a_{1,i-1} \otimes a_{i+1} \otimes a_{i+2,n+1}) \right)
\]

\[
+ \left( f(a_{1,n}) T(a_{n+1}) - T_M(f(a_{1,n}) a_{n+1}) \right)
\]

for any \(f \in C^n_{\text{Alg}}(A_\bullet, \triangleright M_\circ)\) and \(a_1, \ldots, a_{n+1} \in A\).

**Definition 5.1.** Let \(A = (A, \mu, T)\) be a Rota-Baxter algebra of weight \(\lambda\) and \(M = (M, T_M)\) be a Rota-Baxter bimodule over it. Then the cochain complex \((C^\bullet_{\text{Alg}}(A_\bullet, \triangleright M_\circ), \partial)\) is called the cochain complex of Rota-Baxter operator \(T\) with coefficients in \((M, T_M)\), denoted by \(C^\bullet_{\text{RBO}_\lambda}(A, M)\). The cohomology of \(C^\bullet_{\text{RBO}_\lambda}(A, M)\), denoted by \(H^\bullet_{\text{RBO}_\lambda}(A, M)\), are called the cohomology of Rota-Baxter operator \(T\) with coefficients in \((M, T_M)\).
When \((M, T_M)\) is the regular Rota-Baxter bimodule \((A, T)\), we denote \(C^\bullet_{\text{RBA}, A}(A, A)\) by \(C^\bullet_{\text{RBO}, A}(A)\) and call it the cochain complex of Rota-Baxter operator \(T\), and denote \(H^\bullet_{\text{RBO}, A}(A, A)\) by \(H^\bullet_{\text{RBA}, A}(A)\) and call it the cohomology of Rota-Baxter operator \(T\).

### 5.2. Cohomology of Rota-Baxter algebras.

In this subsection, we will combine the Hochschild cohomology of associative algebras and the cohomology of Rota-Baxter operators to define a cohomology theory for Rota-Baxter algebras.

Let \(M = (M, T_M)\) be a Rota-Baxter bimodule over a Rota-Baxter algebra of weight \(\lambda\) \(A = (A, \mu, T)\). Now, let’s construct a chain map

\[
\Phi^\bullet : C^\bullet_{\text{Alg}, M}(A, M) \to C^\bullet_{\text{RBO}, A}(A, M),
\]

i.e., the following commutative diagram:

\[
\begin{array}{cccccc}
C^0_{\text{Alg}, M}(A, M) & \xrightarrow{\delta^0} & C^1_{\text{Alg}, M}(A, M) & \xrightarrow{\delta^1} & \cdots & \xrightarrow{\delta^n} C^{n+1}_{\text{Alg}, M}(A, M) \\
\downarrow{\phi^0} & & \downarrow{\phi^1} & & \cdots & \downarrow{\phi^n} \\
C^0_{\text{RBO}, A}(M) & \xrightarrow{\delta^0} & C^1_{\text{RBO}, A}(M) & \xrightarrow{\delta^1} & \cdots & \xrightarrow{\delta^n} C^{n+1}_{\text{RBO}, A}(M)
\end{array}
\]

Define \(\Phi^0 = \text{Id}_{\text{Hom}(k, M)} = \text{Id}_M\), and for \(n \geq 1\) and \(f \in C^n_{\text{Alg}, M}(A, M)\), define \(\Phi^n(f) \in C^n_{\text{RBO}, A}(A, M)\) as:

\[
\Phi^n(f)(a_1 \otimes \cdots \otimes a_n) = f(T(a_1) \otimes \cdots \otimes T(a_n))
\]

\[- \sum_{k=0}^{n-1} \lambda^{n-k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} T_M \circ f(a_1, i_{i-1} \otimes T(a_{i_1}) \otimes a_{i_1+1}, i_{i-1} \otimes T(a_{i_2}) \otimes \cdots \otimes T(a_{i_k}) \otimes a_{i_k+1}i_n).
\]

### Proposition 5.2. The map \(\Phi^\bullet : C^\bullet_{\text{Alg}, M}(A, M) \to C^\bullet_{\text{RBO}, A}(A, M)\) is a chain map.

This result follows from the \(L_\infty\)-structure over the cochain complex of Rota-Baxter algebras, so we omit it; see Proposition 8.3.

### Definition 5.3. Let \(M = (M, T_M)\) be a Rota-Baxter bimodule over a Rota-Baxter algebra of weight \(\lambda\) \(A = (A, \mu, T)\). We define the cochain complex \((C^\bullet_{\text{RBA}, A}(A, M), d^\bullet)\) of Rota-Baxter algebra \((A, \mu, T)\) with coefficients in \((M, T_M)\) to the negative shift of the mapping cone of \(\Phi^\bullet\), that is, let

\[
C^0_{\text{RBA}, A}(A, M) = C^0_{\text{Alg}, M}(A, M) \quad \text{and} \quad C^n_{\text{RBA}, A}(A, M) = C^n_{\text{Alg}, M}(A, M) \oplus C^{n-1}_{\text{RBO}, A}(A, M), \quad \forall n \geq 1,
\]

and the differential \(d^n : C^n_{\text{RBA}, A}(A, M) \to C^{n+1}_{\text{RBA}, A}(A, M)\) is given by

\[
d^n(f, g) = (\delta^n(f), -\delta^{n-1}(g) - \Phi^n(f))
\]

for any \(f \in C^n_{\text{Alg}, M}(A, M)\) and \(g \in C^{n-1}_{\text{RBO}, A}(A, M)\). The cohomology of \((C^\bullet_{\text{RBA}, A}(A, M), d^\bullet)\), denoted by \(H^\bullet_{\text{RBA}, A}(A, M)\), is called the cohomology of the Rota-Baxter algebra \((A, \mu, T)\) with coefficients in \((M, T_M)\). When \((M, T_M) = (A, T)\), we just denote \(C^\bullet_{\text{RBA}, A}(A, A)\) by \(C^\bullet_{\text{RBA}, A}(A)\), \(H^\bullet_{\text{RBA}, A}(A, A)\) by \(H^\bullet_{\text{RBA}, A}(A)\) respectively, and call them the cochain complex, the cohomology of Rota-Baxter algebra \((A, \mu, T)\) respectively.

There is an obvious short exact sequence of complexes:

\[
0 \to sC^\bullet_{\text{RBO}, A}(A, M) \to C^\bullet_{\text{RBA}, A}(A, M) \to C^\bullet_{\text{Alg}, A}(A, M) \to 0
\]
which induces a long exact sequence of cohomology groups
\[ 0 \to H^0_{RBA_i}(A, M) \to HH^0(A, M) \to H^0_{RBO_i}(A, M) \to H^1_{RBA_i}(A, M) \to HH^1(A, M) \to \cdots \]
\[ \cdots \to HH^0(A, M) \to H^0_{RBO_i}(A, M) \to H^{p+1}_{RBA_i}(A, M) \to HH^{p+1}(A, M) \to \cdots \]

6. Formal deformations of Rota-Baxter algebras and cohomological interpretation

In this section, we will study formal deformations of Rota-Baxter algebras and interpret them via lower degree cohomology groups of Rota-Baxter algebras defined in last section.

6.1. Formal deformations of Rota-Baxter algebras.

Let \((A, \mu, T)\) be a Rota-Baxter algebra of weight \(\lambda\). Consider a 1-parameterized family:
\[ \mu_t = \sum_{i=0}^{\infty} \mu_i t^i, \quad \mu_i \in C^i_{\text{Alg}}(A), \quad T_t = \sum_{i=0}^{\infty} T_i t^i, \quad T_i \in C^i_{\text{RBO}}(A). \]

**Definition 6.1.** A 1-parameter formal deformation of Rota-Baxter algebra \((A, \mu, T)\) is a pair \((\mu_t, T_t)\) which endows the flat \(k[[t]]\)-module \(A[[t]]\) with a Rota-Baxter algebra structure over \(k[[t]]\) such that \((\mu_0, T_0) = (\mu, T)\).

Power series \(\mu_t\) and \(T_t\) determine a 1-parameter formal deformation of Rota-Baxter algebra \((A, \mu, T)\) if and only if for any \(a, b, c \in A\), the following equations hold:
\[ \mu_i(a \otimes \mu(b \otimes c)) = \mu_i(\mu_i(a \otimes b) \otimes c), \]
\[ \mu_i(T_i(a) \otimes T_i(b)) = T_i(\mu_i(a \otimes T_i(b)) + \mu_i(T_i(a) \otimes b) + \lambda \mu_i(a \otimes b)). \]

By expanding these equations and comparing the coefficient of \(t^n\), we obtain that \(\mu_i\) and \(T_i\) have to satisfy: for any \(n \geq 0\),
\[ \sum_{i=0}^{n} \mu_i \circ (\mu_{n-i} \otimes \text{Id}) = \sum_{i=0}^{n} \mu_i \circ (\text{Id} \otimes \mu_{n-i}), \]
\[ \sum_{i+j+k=n, i,j,k \geq 0} \mu_i \circ (T_j \otimes T_k) = \sum_{i+j+k=n, i,j,k \geq 0} T_i \circ \mu_j \circ (\text{Id} \otimes T_k) \]
\[ + \sum_{i+j+k=n, i,j,k \geq 0} T_i \circ \mu_j \circ (T_k \otimes \text{Id}) + \lambda \sum_{i+j+k=n, i,j,k \geq 0} T_i \circ \mu_j. \]

Obviously, when \(n = 0\), the above conditions are exactly the associativity of \(\mu = \mu_0\) and Equation (17) which is the defining relation of Rota-Baxter operator \(T = T_0\).

**Proposition 6.2.** Let \((A[[t]], \mu_t, T_t)\) be a 1-parameter formal deformation of Rota-Baxter algebra \((A, \mu, T)\) of weight \(\lambda\). Then \((\mu_1, T_1)\) is a 2-cocycle in the cochain complex \(C^*_{\text{RBA}}(A)\).

**Proof.** When \(n = 1\), Equations (26) and (27) become
\[ \mu_1 \circ (\mu \otimes \text{Id}) + \mu \circ (\mu \otimes \text{Id}) = \mu_1 \circ (\text{Id} \otimes \mu) + \mu \circ (\text{Id} \otimes \mu_1), \]
and
\[ \mu_1(T \otimes T) - (T \circ \mu_1 \circ (\text{Id} \otimes T) + T \circ \mu_1 \circ (T \otimes \text{Id}) + \lambda T \circ \mu_1) \]
\[ = -(\mu \circ (T \otimes T) - T \circ \mu \circ (\text{Id} \otimes T_1)) + (T_1 \circ \mu \circ (\text{Id} \otimes T) + T_1 \circ \mu \circ (T \otimes \text{Id}) + \lambda T_1 \circ \mu) \]
\[ - (\mu \circ (T_1 \otimes T) - T \circ \mu \circ (T_1 \otimes \text{Id})). \]
Note that the first equation is exactly \( \delta^2(\mu_1) = 0 \in C^*_{\text{Alg}}(A) \) and that second equation is exactly to
\[
\Phi^2(\mu_1) = -\delta^1(T_1) \in C^*_{\text{RBA}_1}(A).
\]
So \((\mu_1, T_1)\) is a 2-cocycle in \(C^*_{\text{RBA}_1}(A)\).

\[\square\]

**Definition 6.3.** The 2-cocycle \((\mu_1, T_1)\) is called the infinitesimal of the 1-parameter formal deformation \((A[[t]], \mu_1, T_1)\) of Rota-Baxter algebra \((A, \mu, T)\).

In general, we can rewrite Equation (26) and (27) as
\[
\delta^2(\mu_n) = \frac{1}{2} \sum_{i=1}^{n-1} [\mu_i, \mu_{n-i}] \delta
\]
\[
\partial^1(T_n) + \Phi^2(\mu_n) = \sum_{i \neq j \neq k \neq \ldots \neq n} \mu_i \circ (T_j \otimes T_k) - \sum_{i \neq j \neq k \neq \ldots \neq n} T_i \circ \mu_j \circ (\text{Id} \otimes T_k) - \sum_{i \neq j \neq k \neq \ldots \neq n} T_i \circ \mu_j - \sum_{i \neq j \neq k \neq \ldots \neq n} T_i \circ \mu_j.
\]

**Definition 6.4.** Let \((A[[t]], \mu_1, T_1)\) and \((A[[t]], \mu'_1, T'_1)\) be two 1-parameter formal deformations of Rota-Baxter algebra \((A, \mu, T)\). A formal isomorphism from \((A[[t]], \mu_1, T_1)\) to \((A[[t]], \mu'_1, T'_1)\) is a power series \(\psi_t = \sum_{i=0}^{\infty} \psi_i t^i : A[[t]] \rightarrow A[[t]]\), where \(\psi_i : A \rightarrow A\) are linear maps with \(\psi_0 = \text{Id}_A\), such that:
\[
\psi_t \circ \mu'_i = \mu_i \circ (\psi_t \otimes \psi_t),
\]
\[
\psi_t \circ T'_i = T_i \circ \psi_t.
\]
In this case, we say that the two 1-parameter formal deformations \((A[[t]], \mu_1, T_1)\) and \((A[[t]], \mu'_1, T'_1)\) are equivalent.

Given a Rota-Baxter algebra \((A, \mu, T)\), the power series \(\mu_t, T_t\) with \(\mu_t = \delta_{t,0} \mu, T_t = \delta_{t,0} T\) make \((A[[t]], \mu, T_t)\) into a 1-parameter formal deformation of \((A, \mu, T)\). Formal deformations equivalent to this one are called trivial.

**Theorem 6.5.** The infinitesimals of two equivalent 1-parameter formal deformations of \((A, \mu, T)\) are in the same cohomology class in \(H^*_{\text{RBA}_1}(A)\).

**Proof.** Let \(\psi_t : (A[[t]], \mu'_1, T'_1) \rightarrow (A[[t]], \mu_1, T_1)\) be a formal isomorphism. Expanding the identities and collecting coefficients of \(t\), we get from Equations (30) and (31):
\[
\mu'_1 = \mu_1 + \mu \circ (\text{Id} \otimes \psi_1) - \psi_1 \circ \mu + \mu \circ (\psi_1 \otimes \text{Id}),
\]
\[
T'_1 = T_1 + T \circ \psi_1 - \psi_1 \circ T,
\]
that is, we have
\[
(\mu'_1, T'_1) - (\mu_1, T_1) = (\delta^1(\psi_1), -\Phi^1(\psi_1)) = d^1(\psi_1, 0) \in C^*_{\text{RBA}_1}(A).
\]

\[\square\]

**Definition 6.6.** A Rota-Baxter algebra \((A, \mu, T)\) is said to be rigid if every 1-parameter formal deformation is trivial.

**Theorem 6.7.** Let \((A, \mu, T)\) be a Rota-Baxter algebra of weight \(\lambda\). If \(H^2_{\text{RBA}_1}(A) = 0\), then \((A, \mu, T)\) is rigid.
Proof. Let \((A[[t]], \mu_t, T_t)\) be a 1-parameter formal deformation of \((A, \mu, T)\). By Proposition 6.2, \((\mu_1, T_1)\) is a 2-cocycle. By \(H^2_{\text{RBA}}(A) = 0\), there exists a 1-cochain

\[
(\psi'_1, x) \in C^1_{\text{RBA}}(A) = C^1_{\text{Alg}}(A) \oplus \text{Hom}(k, A)
\]
such that \((\mu_1, T_1) = d^1(\psi'_1, x)\), that is, \(\mu_1 = \delta^1(\psi'_1)\) and \(T_1 = -\delta^0(x) - \Phi^1(\psi'_1)\). Let \(\psi_1 = \psi'_1 + \delta^0(x)\). Then \(\mu_1 = \delta^1(\psi_1)\) and \(T_1 = -\Phi^1(\psi_1)\), as it can be readily seen that \(\Phi^1(\delta^0(x)) = \delta^0(x)\).

Setting \(\psi_t = \text{Id}_A - \psi_1 t\), we have a deformation \((A[[t]], \overline{\mu}_t, \overline{T}_t)\), where

\[
\overline{\mu}_t = \psi_t^{-1} \circ \mu_t \circ (\psi_t \times \psi_t)
\]
and

\[
\overline{T}_t = \psi_t^{-1} \circ T_t \circ \psi_t.
\]

It can be easily verify that \(\overline{\mu}_1 = 0, \overline{T}_1 = 0\). Then

\[
\overline{\mu}_t = \mu + \overline{\mu}_t t^2 + \cdots,
\]

\[
T_t = T + \overline{T}_t t^2 + \cdots.
\]

By Equations (28) and (29), we see that \((\overline{\mu}_t, \overline{T}_t)\) is still a 2-cocycle, so by induction, we can show that \((A[[t]], \mu_t, T_t)\) is equivalent to the trivial extension \((A[[t]], \mu, T)\). Thus, \((A, \mu, T)\) is rigid. \(\Box\)

6.2. Formal deformations of Rota-Baxter operator with product fixed.

Let \((A, \mu = \cdot, T)\) be a Rota-Baxter algebra of weight \(\lambda\). Let us consider the case where we only deform the Rota-Baxter operator with the product fixed. So \(A[[t]] = \{\sum_{i=0}^{\infty} a_i t^i \mid a_i \in A, \forall i \geq 0\}\) is endowed with the product induced from that of \(A\), say,

\[
(\sum_{i=0}^{\infty} a_i t^i)(\sum_{j=0}^{\infty} b_j t^j) = \sum_{n=0}^{\infty} (\sum_{i+j=n, i,j \geq 0} a_i b_j) t^n.
\]

Then \(A[[t]]\) becomes a flat \(k[[t]]\)-algebra, whose product is still denoted by \(\mu\).

In this case, a 1-parameter formal deformation \((\mu_t, T_t)\) of Rota-Baxter algebra \((A, \mu, T)\) satisfies \(\mu_t = 0, \forall i \geq 1\). So Equation (26) degenerates and Equation (27) becomes

\[
\mu \circ (T_i \otimes T_j) = T_i \circ (\mu \circ (\text{Id} \otimes T_j) + \mu \circ (T_i \otimes \text{Id}) + \lambda \mu).
\]

Expanding these equations and comparing the coefficient of \(t^n\), we obtain that \(\{T_i\}_{i \geq 0}\) have to satisfy: for any \(n \geq 0\),

\[
\sum_{i+j=n, i,j \geq 0} \mu \circ (T_i \otimes T_j) = \sum_{i+j=n} T_i \circ \mu \circ (\text{Id} \otimes T_j) + \sum_{i+j=n} T_i \circ \mu \circ (T_j \otimes \text{Id}) + \lambda T_n \circ \mu.
\]

Obviously, when \(n = 0\), Equation (32) becomes exactly Equation (17) defining Rota-Baxter operator \(T = T_0\).

When \(n = 1\), Equation (32) has the form

\[
\mu \circ (T_1 \otimes T_1 + T_1 \otimes T) = T \circ \mu \circ (\text{Id} \otimes T_1) + T_1 \circ \mu \circ (\text{Id} \otimes T) + T \circ \mu \circ (T_1 \otimes \text{Id}) + T_1 \circ \mu \circ (T \otimes \text{Id}) + \lambda T_1 \circ \mu
\]

which says exactly that \(\delta^1(T_1) = 0 \in C^*_\text{RBO}(A)\). This proves the following result:

**Proposition 6.8.** Let \(T_t\) be a 1-parameter formal deformation of Rota-Baxter operator \(T\) of weight \(\lambda\). Then \(T_t\) is a 1-cocycle in the cochain complex \(C^*_\text{RBO}(A)\).

This means that the cochain complex \(C^*_\text{RBO}(A)\) controls formal deformations of Rota-Baxter operators.
6.3. Formal deformations of associative product with Rota-Baxter operator fixed.

Let \((A, \mu, T)\) be a Rota-Baxter algebra of weight \(\lambda\). Let us consider the case where we only deform the associative product with Rota-Baxter operator fixed. So the induced Rota-Baxter operator on \(A[[r]]\) is given by \(\sum_{i=0}^{\infty} a_i r^i \mapsto \sum_{i=0}^{\infty} T(a_i r^i)\), still denoted by \(T\).

In this case, a 1-parameter formal deformation \((\mu_i, T_i)\) of Rota-Baxter algebra \((A, \mu, T)\) satisfies \(T_i = 0, \forall i \geq 1\). So Equation (26) remains unchanged and Equation (27) becomes for any \(n \geq 0\),

\[
\mu_n \circ (T \otimes T) = T \circ \mu_n \circ (\text{Id} \otimes T + T \otimes \text{Id}) + \lambda T \circ \mu_n.
\]

As usual, Equation (26) for \(n = 1\) says that \(\delta^2(\mu_1) = 0 \in C^2_{\text{Alg}}(A)\), but Equation (33) implies that \(\mu_n\) lies in \(\text{Ker}(\Phi^2 : C^2_{\text{Alg}}(A) \rightarrow C^2_{\text{RBO}_\lambda}(A))\).

This proves the following result:

**Proposition 6.9.** Let \(\mu_i\) be a 1-parameter formal deformation of associative product \(\mu\) with Rota-Baxter operator \(T\) fixed. Then \(\mu_1\) is a 2-cocycle in the cochain complex \(\text{Ker}(\Phi^2 : C^*_{\text{Alg}}(A) \rightarrow C^*_{\text{RBO}_\lambda}(A))\).

This means that the cochain complex \(\text{Ker}(\Phi^* : C^*_{\text{Alg}}(A) \rightarrow C^*_{\text{RBO}_\lambda}(A))\) controls formal deformations of associative product with Rota-Baxter operator fixed.

7. Abelian extensions of Rota-Baxter algebras

In this section, we study abelian extensions of Rota-Baxter algebras and show that they are classified by the second cohomology, as one would expect of a good cohomology theory.

Notice that a vector space \(M\) together with a linear transformation \(T_M : M \rightarrow M\) is naturally a Rota-Baxter algebra where the multiplication on \(M\) is defined to be \(uv = 0\) for all \(u, v \in M\).

**Definition 7.1.** An abelian extension of Rota-Baxter algebras is a short exact sequence of morphisms of Rota-Baxter algebras

\[
0 \rightarrow (M, T_M) \xrightarrow{i} (\hat{A}, \hat{T}) \xrightarrow{\hat{p}} (A, T) \rightarrow 0,
\]

that is, there exists a commutative diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M & \xrightarrow{i} & \hat{A} & \xrightarrow{\hat{p}} & A & \longrightarrow & 0 \\
\text{\(T_M\)} & \downarrow & \text{\(\hat{T}\)} & \downarrow & \text{\(T\)} & & \\
0 & \longrightarrow & M & \xrightarrow{i} & \hat{A} & \xrightarrow{\hat{p}} & A & \longrightarrow & 0,
\end{array}
\]

where the Rota-Baxter algebra \((M, T_M)\) satisfies \(uv = 0\) for all \(u, v \in M\).

We will call \((\hat{A}, \hat{T})\) an abelian extension of \((A, T)\) by \((M, T_M)\).

**Definition 7.2.** Let \((\hat{A}_1, \hat{T}_1)\) and \((\hat{A}_2, \hat{T}_2)\) be two abelian extensions of \((A, T)\) by \((M, T_M)\). They are said to be isomorphic if there exists an isomorphism of Rota-Baxter algebras \(\zeta : (\hat{A}_1, \hat{T}_1) \rightarrow (\hat{A}_2, \hat{T}_2)\) such that the following commutative diagram holds:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & (M, T_M) & \xrightarrow{i} & (\hat{A}_1, \hat{T}_1) & \xrightarrow{p} & (A, T) & \longrightarrow & 0 \\
\text{\(\|\)} & & \downarrow & \text{\(\zeta\)} & \downarrow & \text{\(\|\)} & & \\
0 & \longrightarrow & (M, T_M) & \xrightarrow{i} & (\hat{A}_2, \hat{T}_2) & \xrightarrow{p} & (A, T) & \longrightarrow & 0.
\end{array}
\]
A section of an abelian extension \((\hat{A}, \hat{T})\) of \((A, T)\) by \((M, T_M)\) is a linear map \(s: A \to \hat{A}\) such that \(p \circ s = \text{Id}_A\).

We will show that isomorphism classes of abelian extensions of \((A, T)\) by \((M, T_M)\) are in bijection with the second cohomology group \(H^2_{\text{RBA}}(A, M)\).

Let \((\hat{A}, \hat{T})\) be an abelian extension of \((A, T)\) by \((M, T_M)\) having the form Equation (34). Choose a section \(s: A \to \hat{A}\). We define

\[
am := s(a)m, \quad ma := ms(a), \quad \forall a \in A, m \in M.
\]

**Proposition 7.3.** With the above notations, \((M, T_M)\) is a Rota-Baxter bimodule over \((A, T)\).

**Proof.** For any \(a, b \in A, m \in M\), since \(s(ab) - s(a)s(b) \in M\) implies \(s(ab)m = s(a)s(b)m\), we have

\[
(ab)m = s(ab)m = s(a)s(b)m = a(bm).
\]

Hence, this gives a left \(A\)-module structure and the case of right module structure is similar.

Moreover, \(\hat{T}(s(a)) - s(T(a)) \in M\) means that \(\hat{T}(s(a))m = s(T(a))m\). Thus we have

\[
T(a)T_M(m) = s(T(a))T_M(m) = \hat{T}(s(a))T_M(m) = \hat{T}(\hat{T}(s(a))m + s(a)T_M(m) + \lambda s(a)m) = T_M(T(a)m + aT_M(m) + \lambda am)
\]

It is similar to see \(T_M(m)T(a) = T_M(T_M(m)a + mT(a) + \lambda ma)\).

Hence, \((M, T_M)\) is a Rota-Baxter bimodule over \((A, T)\). \(\square\)

We further define linear maps \(\psi: A \otimes A \to M\) and \(\chi: A \to M\) respectively by

\[
\psi(a \otimes b) = s(a)s(b) - s(ab), \quad \forall a, b \in A,
\]

\[
\chi(a) = \hat{T}(s(a)) - s(T(a)), \quad \forall a \in A.
\]

**Proposition 7.4.** The pair \((\psi, \chi)\) is a 2-cocycle of Rota-Baxter algebra \((A, T)\) with coefficients in the Rota-Baxter bimodule \((M, T_M)\) introduced in Proposition 7.3.

The proof is by direct computations, so it is left to the reader.

The choice of the section \(s\) in fact determines a splitting

\[
0 \xrightarrow{i} M \xrightarrow{t} \hat{A} \xrightarrow{p} A \xrightarrow{s} 0
\]

subject to \(t \circ i = \text{Id}_M, t \circ s = 0\) and \(it + sp = \text{Id}_{\hat{A}}\). Then there is an induced isomorphism of vector spaces

\[
\begin{pmatrix} p & t \end{pmatrix}: \hat{A} \cong A \oplus M : \begin{pmatrix} s \\ i \end{pmatrix}.
\]

We can transfer the Rota-Baxter algebra structure on \(\hat{A}\) to \(A \oplus M\) via this isomorphism. It is direct to verify that this endows \(A \oplus M\) with a multiplication \(*_\psi\) and an Rota-Baxter operator \(T_\chi\) defined by

\[
(a, m) *_\psi (b, n) = (ab, an + mb + \psi(a, b)), \quad \forall a, b \in A, m, n \in M,
\]

\[
T_\chi(a, m) = (T(a), \chi(a) + T_M(m)), \quad \forall a \in A, m \in M.
\]
Moreover, we get an abelian extension

\[
0 \to (M, T_M) \xrightarrow{(s \ i)} (A \oplus M, T_\chi) \xrightarrow{(p \ t)} (A, T) \to 0
\]

which is easily seen to be isomorphic to the original one (34).

Now we investigate the influence of different choices of sections.

**Proposition 7.5.**
(i) Different choices of the section \(s\) give the same Rota-Baxter bimodule structures on \((M, T_M)\);
(ii) the cohomological class of \((\psi, \chi)\) does not depend on the choice of sections.

**Proof.** Let \(s_1\) and \(s_2\) be two distinct sections of \(p\). We define \(\gamma : A \to M\) by \(\gamma(a) = s_1(a) - s_2(a)\).

Since the Rota-Baxter algebra \((M, T_M)\) satisfies \(uv = 0\) for all \(u, v \in M\),

\[
s_1(a)m = s_2(a)m + \gamma(a)m = s_2(a)m.
\]

So different choices of the section \(s\) give the same Rota-Baxter bimodule structures on \((M, T_M)\);

We show that the cohomological class of \((\psi, \chi)\) does not depend on the choice of sections. Then

\[
\psi_1(a, b) = s_1(a)s_1(b) - s_1(ab)
\]

\[
= (s_2(a) + \gamma(a))(s_2(b) + \gamma(b)) - (s_2(ab) + \gamma(ab))
\]

\[
= (s_2(a)s_2(b) - s_2(ab)) + s_2(a)\gamma(b) + \gamma(a)s_2(b) - \gamma(ab)
\]

\[
= (s_2(a)s_2(b) - s_2(ab)) + a\gamma(b) + \gamma(a)b - \gamma(ab)
\]

\[
= \psi_2(a, b) + \delta(\gamma)(a, b)
\]

and

\[
\chi_1(a) = \hat{T}(s_1(a)) - s_1(T(a))
\]

\[
= \hat{T}(s_2(a) + \gamma(a)) - (s_2(T(a)) + \gamma(T(a)))
\]

\[
= (\hat{T}(s_2(a)) - s_2(T(a))) + \hat{T}(\gamma(a)) - \gamma(T(a))
\]

\[
= \chi_2(a) + T_m(\gamma(a)) - \gamma(T_A(a))
\]

\[
= \chi_2(a) - \Phi^1(\gamma)(a).
\]

That is, \((\psi_1, \chi_1) = (\psi_2, \chi_2) + d^1(\gamma)\). Thus \((\psi_1, \chi_1)\) and \((\psi_2, \chi_2)\) form the same cohomological class in \(H_{RBA}^2(A, M)\).

We show now the isomorphic abelian extensions give rise to the same cohomology classes.

**Proposition 7.6.** Let \(M\) be a vector space and \(T_M \in \text{End}_k(M)\). Then \((M, T_M)\) is a Rota-Baxter algebra with trivial multiplication. Let \((A, T)\) be a Rota-Baxter algebra. Two isomorphic abelian extensions of Rota-Baxter algebra \((A, T)\) by \((M, T_M)\) give rise to the same cohomology class in \(H_{RBA}^2(A, M)\).

**Proof.** Assume that \((\hat{A}_1, \hat{T}_1)\) and \((\hat{A}_2, \hat{T}_2)\) are two isomorphic abelian extensions of \((A, T)\) by \((M, T_M)\) as is given in (35). Let \(s_1\) be a section of \((\hat{A}_1, \hat{T}_1)\). As \(p_2 \circ \zeta = p_1\), we have

\[
p_2 \circ (\zeta \circ s_1) = p_1 \circ s_1 = \text{Id}_A.
\]
Hence, \( \zeta \circ s_1 \) is a section of \((\hat{A}_2, \hat{T}_2)\). Denote \( s_2 := \zeta \circ s_1 \). Since \( \zeta \) is a homomorphism of Rota-Baxter algebras such that \( \zeta|_M = \Id_M, \zeta(am) = \zeta(s_1(a)m) = s_2(a)m = am \), so \( \zeta|_M : M \to M \) is compatible with the induced Rota-Baxter bimodule structures. We have

\[
\psi_2(a \otimes b) = s_2(a)s_2(b) - s_2(ab) = \zeta(s_1(a))\zeta(s_1(b)) - \zeta(s_1(ab))
\]

\[
= \zeta(s_1(a)s_1(b) - s_1(ab)) = \zeta(\psi_1(a,b))
\]

and

\[
\chi_2(a) = \hat{T}_2(s_2(a)) - s_2(T(a)) = \hat{T}_2(\zeta(s_1(a))) - \zeta(s_1(A(a)))
\]

\[
= \zeta(\hat{T}_1(s_1(a)) - s_1(T(a))) = \zeta(\chi_1(a))
\]

\[
= \chi_1(a).
\]

Consequently, two isomorphic abelian extensions give rise to the same element in \( H^2_{RBA}(A, M) \).

\[\square\]

Now we consider the reverse direction.

Let \((M, T_M)\) be a Rota-Baxter bimodule over Rota-Baxter algebra \((A, T)\), given two linear maps \( \psi : A \otimes A \to M \) and \( \chi : A \to M \), one can define a multiplication \( \cdot_\psi \) and an operator \( T_\chi \) on \( A \otimes M \) by Equations (36)(37). The following fact is important:

**Proposition 7.7.** The triple \((A \otimes M, \cdot_\psi, T_\chi)\) is a Rota-Baxter algebra if and only if \((\psi, \chi)\) is a 2-cocycle of the Rota-Baxter algebra \((A, T)\) with coefficients in \((M, T_M)\). In this case, we obtain an abelian extension

\[
0 \to (M, T_M) \xrightarrow{\left(\begin{array}{cc} 0 & \Id \\ \Id & 0 \end{array}\right)} (A \oplus M, T_\chi) \xrightarrow{(A, T)} (A, T) \to 0,
\]

and the canonical section \( s = \left(\begin{array}{cc} \Id & 0 \end{array}\right) : (A, T) \to (A \oplus M, T_\chi) \) endows \( M \) with the original Rota-Baxter bimodule structure.

**Proof.** If \((A \oplus M, \cdot_\psi, T_\chi)\) is a Rota-Baxter algebra, then the associativity of \( \cdot_\psi \) implies

\[
(38) \quad a\psi(b \otimes c) - \psi(ab \otimes c) + \psi(a \otimes bc) - \psi(a \otimes b)c = 0,
\]

which means \( \delta^2(\phi) = 0 \) in \( C^*(A, M) \). Since \( T_\chi \) is an Rota-Baxter operator, for any \( a, b \in A, m, n \in M \), we have

\[
T_\chi((a,m)) \cdot_\psi T_\chi((b,n)) = T_\chi(T_\chi(a,m) \cdot_\psi (b,n) + (a,m) \cdot_\psi T_\chi(b,n) + \lambda(a,m) \cdot_\psi (b,n))
\]

Then \( \chi, \psi \) satisfy the following equations:

\[
T(a)\chi(b) + \chi(a)T(b) + \psi(T(a) \otimes T(b))
\]

\[
= T_M(\chi(a)b) + T_M(\psi(T(a) \otimes b)) + \chi(T(a)b)
\]

\[
+ T_M(a\chi(b)) + T_M(\psi(a \otimes T(b))) + \chi(aT(b))
\]

\[
+ \lambda T_M(\psi(a \otimes b)) + \lambda \chi(ab)
\]

That is,

\[
\partial^1(\chi) + \Phi^2(\psi) = 0.
\]

Hence, \((\psi, \chi)\) is a 2-cocycle.
Conversely, if \((\psi, \chi)\) is a 2-cocycle, one can easily check that \((A \oplus M, \cdot \psi, T_\chi)\) is a Rota-Baxter algebra. The last statement is clear.

Finally, we show the following result:

**Proposition 7.8.** Two cohomologous 2-cocycles give rise to isomorphic abelian extensions.

**Proof.** Given two 2-cocycles \((\psi_1, \chi_1)\) and \((\psi_2, \chi_2)\), we can construct two abelian extensions \((A \oplus M, \cdot \psi_1, T_{\chi_1})\) and \((A \oplus M, \cdot \psi_2, T_{\chi_2})\) via Equations (36) and (37). If they represent the same cohomology class in \(H^2_{\text{RBA}}(A, M)\), then there exists two linear maps \(\gamma_0 : k \to M, \gamma_1 : A \to M\) such that

\[
(\psi_1, \chi_1) = (\psi_2, \chi_2) + (\delta^1(\gamma_1), -\Phi^1(\gamma_1) - \delta^0(\gamma_0)).
\]

Notice that \(\delta^0 = \Phi^1 \circ \delta^0\). Define \(\gamma : A \to M\) to be \(\gamma_1 + \delta^0(\gamma_0)\). Then \(\gamma\) satisfies

\[
(\psi_1, \chi_1) = (\psi_2, \chi_2) + (\delta^1(\gamma), -\Phi^1(\gamma)).
\]

Define \(\zeta : A \oplus M \to A \oplus M\) by

\[
\zeta(a, m) := (a, -\gamma(a) + m).
\]

Then \(\zeta\) is an isomorphism of these two abelian extensions \((A \oplus M, \cdot \psi_1, T_{\chi_1})\) and \((A \oplus M, \cdot \psi_2, T_{\chi_2})\).

\(\square\)

8. \(L_{\infty}\)-algebra structure on the cochain complex of a Rota-Baxter algebra

In this section, we will consider \(L_{\infty}\)-algebra structures controlling deformations of Rota-Baxter algebras. Rota-Baxter algebra structures on a vector space will be realised as Maurer-Cartan elements in an explicitly constructed \(L_{\infty}\)-algebra and it will be seen that the shift of the cochain complex of a Rota-Baxter algebra is exactly the underlying complex of the twisted \(L_{\infty}\)-algebra by the corresponding Maurer-Cartan element corresponding to the Rota-Baxter algebra structure.

8.1. \(L_{\infty}\)-algebra structure on \(\mathcal{C}_{\text{RBA},i}(V)\).

Let \(V\) be a graded vector space. We define a graded space \(\mathcal{C}_{\text{RBA},i}(V)\) as :

\[
\mathcal{C}_{\text{RBA},i}(V) = \mathcal{C}_{\text{Alg}}(V) \oplus \mathcal{C}_{\text{RBO},i}(V),
\]

where

\[
\mathcal{C}_{\text{Alg}}(V) = \text{Hom}(T^c(sV), sV) \quad \text{and} \quad \mathcal{C}_{\text{RBO},i}(V) = \text{Hom}(T^c(sV), V).
\]

Notice that for a Rota-Baxter algebra \(A\), \(\mathcal{C}_{\text{RBA},i}(A)\) is just the underlying space of the cochain complex of Rota-Baxter algebra \(A\) up to shift.

Now, we are going to build an \(L_{\infty}\)-algebra structure on \(\mathcal{C}_{\text{RBA},i}(V)\). The operators \(\{l_n\}_{n \geq 1}\) on \(\mathcal{C}_{\text{RBA},i}(V)\) are defined as follows:

(I) For \(sh \in sV = \text{Hom}(k, sV) \subset \mathcal{C}_{\text{Alg}}(V)\) with \(h \in V\), define

\[
l_1(sh) = h \in V = \text{Hom}(k, V) \subset \mathcal{C}_{\text{RBO},i}(V).
\]

(II) For homogeneous elements \(sf, sh \in \mathcal{C}_{\text{Alg}}(V)\), define

\[
l_2(sf \otimes sh) := [sf, sh]_G \in \mathcal{C}_{\text{Alg}}(V),
\]

where \([-,-]_G\) is the Gerstenhaber bracket defined in Equation (11).
Theorem 8.1. Given a graded space $V$ and a scalar $\lambda \in k$, the graded space $\mathfrak{c}_{RBA,\lambda}(V)$ endowed with operations $\{l_n\}_{n \geq 1}$ defined above forms an $L_\infty$-algebra.

This theorem is one of the main results in this paper, whose proof requires quite a lot of technical details, so we postpone it to Appendix A.

8.2. Realising Rota-Baxter algebra structures as Maurer-Cartan elements.

Theorem 8.2. Let $V$ be an ungraded space considered as a graded space concentrated in degree 0. Then a Rota-Baxter algebra structure of weight $\lambda$ on $V$ is equivalent to a Maurer-Cartan element in the $L_\infty$-algebra $\mathfrak{c}_{RBA,\lambda}(V)$ introduced above.

Proof. Since $V$ is concentrated in degree 0, the degree $-1$ part of $\mathfrak{c}_{RBA,\lambda}(V)$ is $\text{Hom}(\langle sV \rangle, sV) \oplus \text{Hom}(sV, V)$. Let $\alpha = (m, \tau) \in \mathfrak{c}_{RBA,\lambda}(V)_{-1}$. Then

$$l_2(\alpha \otimes \alpha) = (l_2(m \otimes m), l_2(m \otimes \tau) + l_2(\tau \otimes m))$$

$$= (l_2(m \otimes m), 2l_2(m \otimes \tau))$$

$$= ([m, m]_G, 2\lambda \tau \circ m),$$
\[ l_3(\alpha^{\otimes 3}) = (0, l_3(m \otimes \tau \otimes \tau) + l_3(\tau \otimes m \otimes \tau) + l_3(\tau \otimes \tau \otimes m)) = (0, 3l_3(m \otimes \tau \otimes \tau)) = (0, -6 \left((s^{-1}m) \circ (\tau \otimes \tau) - s^{-1}(\tau)m\right)) \]

and \( l_n(\alpha^{\otimes n}) = 0 \) for \( n \neq 2, 3 \). By expanding the Maurer-Cartan equation (8)

\[
\sum_{i=1}^{\infty} \frac{1}{i!}(-1)^{\frac{i(i-1)}{2}} l_i(\alpha^{\otimes n}) = 0,
\]

we get:

\[
\begin{align*}
[m, m]_G &= 0, \\
-\lambda &\circ m + (s^{-1}m) \circ (\tau \otimes \tau) - \tau \circ m \circ (\tau \otimes \text{Id} + \text{Id} \otimes \tau) = 0.
\end{align*}
\]

Set  = ̃m = s^{-1} \circ m \circ s^{\otimes 2} : V^{\otimes 2} \to V and T = ̃T = \tau \circ s : V \to V via the fixed isomorphisms (13) and (14). Equation (39) is equivalent to saying that \( \mu \) is associative, see also Proposition 3.3; Equation (40) is equivalent to

\[
\lambda T \circ \mu - \mu \circ (T \otimes T) + T \circ \mu \circ (T \otimes \text{Id} + \text{Id} \otimes T) = 0,
\]

which says exactly that \( T \) is a Rota-Baxter operator of weight \( \lambda \) on associative algebra \((V, \mu)\).

Conversely, Given a Rota-Baxter algebra structure \((\mu, T)\) of weight \( \lambda \) on vector space \( V \), define

\[
m = -s \circ \mu \circ (s^{-1})^{\otimes 2} : (sV)^{\otimes 2} \to sV and \tau = T \circ s^{-1} : sV \to V.
\]

Then \((m, \tau)\) is a Maurer-Cartan element in \( C_{\text{RBA}}(V) \). \(\square\)

**Proposition 8.3.** Let \((A, \mu, T)\) be a Rota-Baxter algebra of weight \( \lambda \). Twist the \( L_\infty \)-algebra \( C_{\text{RBA}}(A) \) by the Maurer-Cartan element corresponding to the Rota-Baxter algebra structure \((A, \mu, T)\), then its underlying complex is exactly \( sC_{\text{RBA}}^*(A) \), the shift of the cochain complex of Rota-Baxter algebra \((A, \mu, T)\) defined in Section 5.2.

**Proof.** By Theorem 8.2, the Rota-Baxter algebra structure \((A, \mu, T)\) is equivalent to a Maurer-Cartan element \( \alpha = (m, \tau) \) in the \( L_\infty \)-algebra \( C_{\text{RBA}}(A) \) with

\[
m = -s \circ \mu \circ (s^{-1})^{\otimes 2} : (sV)^{\otimes 2} \to sV and \tau = T \circ s^{-1} : sV \to V.
\]

By Proposition 2.7, the Maurer-Cartan element induces a new \( L_\infty \)-algebra structure \( \{l_n\}_{n \geq 1} \) on the graded space \( C_{\text{RBA}}(A) \). By definition, for any \( sf \in \text{Hom}((sA)^{\otimes n}, sA) \subset C_{\text{Alg}}(A) \),

\[
l_1(sf) = \sum_{i=0}^{\infty} \frac{1}{i!}(-1)^{\frac{i(i-1)}{2}} l_i(\alpha^{\otimes i} \otimes sf) = \left(-l_2(m \otimes sf), \sum_{i=1}^{n} \frac{1}{i!}(-1)^{\frac{i(i-1)}{2}} l_{i+1}(\tau^{\otimes i} \otimes sf)\right).
\]

By definition of \( \{l_n\}_{n \geq 1} \) on \( C_{\text{RBA}}(A) \), \( -l_2(m \otimes sf) = -[m, sf]_G \), which corresponds to \( -\delta^\alpha(\tilde{f}) \) in \( sC_{\text{Alg}}^*(A) \) under the fixed isomorphism (13); for details, see Proposition 3.4.

On the other hand, we have

\[
\sum_{i=0}^{\infty} \frac{1}{i!}(-1)^{\frac{i(i-1)}{2}} l_{i+1}(\tau^{\otimes i} \otimes sf)
\]
Proof. Consider \( \{ \)

\( \text{Cendowed with a dg Lie algebra structure, and a Rota-Baxter operator } T \) of weight \( \Phi \) which corresponds to \( \alpha \) element \( L \). By the construction of \( \tau \) over \( \lambda \) is fixed, the graded space \( \lambda^{-k} \otimes \alpha \) is an \( \mathbf{L} \mathbf{A} \) on \( \lambda \) is \( \lambda^{-k} \mathbf{m} \otimes (s \mathbf{t} \otimes s \mathbf{g}) \), which can be seen to be correspondent to \( \Phi^k(f) \) via the fixed isomorphism (14).

For any \( g \in \text{Hom}(s\mathbf{A})^{\otimes(n-1)}, A) \subset \mathcal{C}_{\mathbf{RBA},}(A) \), we have

\[
\ell^n_1(g) = \sum_{k=0}^n \frac{1}{k!}(-1)^{\frac{n(n+1)}{2}} l_{n+1}(\alpha \otimes g)
\]

\[
= - l_2(m \otimes \tau \otimes s \mathbf{g}) + \frac{1}{2!} \left( l_3(m \otimes \tau \otimes s \mathbf{g}) + l_3(\tau \otimes m \otimes g) \right)
\]

\[
= (-1)^n \lambda g \circ m + s^{-1} m \circ (s \mathbf{t} \otimes s \mathbf{g}) - \tau \circ (m[s \mathbf{g}]) + s^{-1} m \circ (s \mathbf{g} \otimes s \mathbf{t}) - (-1)^n g[m[s \mathbf{t}]].
\]

which corresponds to \( \partial^{n-1}(\hat{g}) \) via the fixed isomorphism (14).

We have shown that the underlying complex of twisted \( \mathbf{L}_\infty \)-algebra \( \mathcal{C}_{\mathbf{RBA},}(A) \) by Maurer-Cartan element \( \alpha \) is exactly the complex \( s\mathbf{C}_{\mathbf{RBA},}(A) \), the shift of the complex \( \mathcal{C}_{\mathbf{RBA},}(A) \) defined in Section 5.2.

Although \( \mathcal{C}_{\mathbf{RBA},}(A) \) is an \( \mathbf{L}_\infty \)-algebra, the next result shows that once the associative algebra structure \( \mu \) over \( A \) is fixed, the graded space \( \mathcal{C}_{\mathbf{RBO},}(A) \), which, after twisting procedure, controls deformations of Rota-Baxter operators, is a genuine differential graded Lie algebra.

**Proposition 8.4.** Let \( (A, \mu) \) be an associative algebra. Then the graded space \( \mathcal{C}_{\mathbf{RBO},}(A) \) can be endowed with a dg Lie algebra structure, and a Rota-Baxter operator \( T \) of weight \( \lambda \) on \( (A, \mu) \) is equivalent to a Maurer-Cartan element in this dg Lie algebra. Given a Rota-Baxter operator \( T \) on associative algebra \( (A, \mu) \), the underlying complex of the twisted dg Lie algebra \( \mathcal{C}_{\mathbf{RBO},}(A) \) by the corresponding Maurer-Cartan element is exactly the cochain complex of Rota-Baxter operator \( \mathbf{C}_{\mathbf{RBO},}(A) \).

**Proof.** Consider \( A \) as graded space concentrated in degree 0. Define \( m = - s \circ \mu \circ (s^{-1} \otimes s^{-1}) : (s\mathbf{A})^{\otimes 2} \to s\mathbf{A} \). Then by Equations (39)(40), \( \alpha = (m, 0) \) is naturally a Maurer-Cartan element in \( \mathbf{L}_\infty \)-algebra \( \mathcal{C}_{\mathbf{RBA},}(A) \). By the construction of \( \ell_n \) on \( \mathcal{C}_{\mathbf{RBA},}(A) \), the graded subspace \( \mathcal{C}_{\mathbf{RBO},}(A) \) is closed under the action of operators \( \ell^n_1 \). Since the arity of \( m \) is 2, the restriction of \( \ell^n_1 \) on \( \mathcal{C}_{\mathbf{RBO},}(A) \) is 0 for \( n \geq 3 \). Thus \( (\mathcal{C}_{\mathbf{RBO},}, (\ell^n_1)_{n=1,2}) \) forms a dg Lie algebra. More explicitly, for \( f \in \text{Hom}(s\mathbf{A})^{\otimes n}, A) \), \( g \in \text{Hom}(s\mathbf{A})^{\otimes k}, A) \),

\[
\ell^n_1(f) = - l_2(m \otimes f) = (-1)^{l+1} \lambda f[m] = (-1)^n \lambda f[m]
\]

\[
\ell^n_1(f \otimes g) = l_3(m \otimes f \otimes g)
\]


(−1)^{\lceil l \rceil}(s^{-1}m \circ (sf \otimes sg) − (−1)^{\lceil l \rceil+1}f[m[sg]])
+ (−1)^{\lceil l \rceil+1+|g|}(s^{-1}m \circ (sg \otimes sf) − (−1)^{\lceil l \rceil+1}g[m[sg]])
− s^{-1}m \circ (sf \otimes sg) + f[m[sg]]
+ (−1)^{nk+1+k} s^{-1}m \circ (sg \otimes sf) − (−1)^{nk}g[m[sg]].

Since \( A \) is concentrated in degree 0, we have \( \mathcal{C}_{RBO,}(A)_{−1} = \text{Hom}(sA, A) \). Take an element \( \tau \in \text{Hom}(sA, A) \). Then \( \tau \) satisfies the Maurer-Cartan equation:

\[ l^\omega_1(\tau) − \frac{1}{2}l^\omega_2(\tau \otimes \tau) = 0, \]

if and only if

\[ −λτ \circ m + s^{-1}m \circ (st \otimes st) − τ \circ (m[st]) = 0. \]

Define \( T = \tau \circ s : A \to A \). The above equation is exactly the statement that \( T \) is a Rota-Baxter operator of weight \( λ \) on associative algebra \((A, μ)\).

Now let \( T \) be a Rota-Baxter operator on associative algebra \((A, μ)\). By the first statement, it corresponds to a Maurer-Cartan element \( β \) in the dg Lie algebra \((\mathcal{C}^{RBA,}_{\omega,}, l^\omega_1, l^\omega_2)\). More precisely, \( β \in \mathcal{C}_{RBO,}(A)_{−1} = \text{Hom}(sA, A) \) is defined to be \( β = T \circ s^{-1} \). For \( f \in \text{Hom}((sA)^{\otimes n}, A) \), we compute \((l^\omega_1)^\beta(f)\). In fact,

\[ (l^\omega_1)^\beta(f) = l^\omega_1(f) − l^\omega_1(β \circ f) \]
\[ = (−1)^{\lceil l \rceil}f[m] + s^{-1}m \circ (sf \otimes sβ) − β[m[sf]] + s^{-1}m \circ (sf \otimes sβ) + (−1)^{\lceil l \rceil}f[m[sβ]], \]

which corresponds to \( \partial^n(\hat{f}) \) via the fixed isomorphism (14). So the underlying complex of the twisted dg Lie algebra \( \mathcal{C}_{RBO,}(A) \) by the corresponding Maurer-Cartan element \( β \) is exactly the cochain complex of Rota-Baxter operator \( C^*_{RBO,}(A) \).

\[ \Box \]

9. Homotopy Rota-Baxter Algebras

In this subsection, we will introduce the notion of homotopy Rota-Baxter algebras of any weight.

Recall \( \overline{\mathcal{T}^c}(sV) = \bigoplus_{n=1}^{∞}(sV)^{\otimes n} \) and \( \overline{\mathcal{C}_{\text{Alg}}}(V) = \text{Hom}(\overline{\mathcal{T}^c}(sV), sV) \subset \mathcal{C}_{\text{Alg}}(V) \). Denote \( \overline{\mathcal{C}_{RBO,}}(V) = \text{Hom}(\overline{\mathcal{T}^c}(sV), V) \subset \mathcal{C}_{RBO,}(V) \), and set \( \overline{\mathcal{C}_{RBA,}}(V) = \overline{\mathcal{C}_{\text{Alg}}}(V) \oplus \overline{\mathcal{C}_{RBO,}}(V) \subset \mathcal{C}_{RBA,}(V) \).

It is not difficult to see that \( \overline{\mathcal{C}_{RBA,}}(V) \) is an \( L_∞ \)-subalgebra of \( \mathcal{C}_{RBA,}(V) \).

**Definition 9.1.** Let \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) be a graded space. Then a homotopy Rota-Baxter algebra structure of weight \( λ \) on \( V \) is defined to be a Maurer-Cartan element in the \( L_∞ \)-algebra \( \overline{\mathcal{C}_{RBA,}}(V) \).
Let’s make the definition explicit. Given an element
\[ \alpha = (b_1)_{i \geq 1}, (R_i)_{i \geq 1} \in \operatorname{RBA}_n(V)_{-1} = \operatorname{Hom}(\mathcal{T}(sV), sV)_{-1} \oplus \operatorname{Hom}(\mathcal{T}(sV), V)_{-1} \]
with \( b_1 : (sV)^{\otimes i} \to sV \) and \( R_i : (sV)^{\otimes i} \to V, \alpha \) satisfies the Maurer-Cartan equation if and only if for each \( n \geq 1 \), the following equalities hold:

\[ \sum_{i+j+1+k, i \geq 0, j \geq 1} b_{i+1+k} \circ (\operatorname{Id}^{\otimes i} \otimes b_j \otimes \operatorname{Id}^{\otimes k}) = 0, \]

\[ \sum_{l \in l_{i+j}^k, i \geq j \geq 1} b_k \circ (sR_{l_1} \otimes \cdots \otimes sR_{l_k}) = \sum_{l \in l_{i+j}^k, i \geq j \geq 1} \lambda^{p-q} (sR_{l_1})\{b_{p}\{sR_{r_2}, \ldots, sR_{r_q}\}\}. \]

Define two family of operators \( \{m_n\}_{n \geq 1} \) and \( \{T_n\}_{n \geq 1} \) as:
\[ m_n = s^{-1} \circ b_n \circ s^{\otimes n} : V^{\otimes n} \to V \quad \text{and} \quad T_n = R_n \circ s^{\otimes n} : V^{\otimes n} \to V. \]
For each \( n \geq 1 \), Equations (41) and (42) are equivalent, respectively, to:

\[ \sum_{i+j+1+k, i \geq j \geq 1} (-1)^{i+j+k} m_{i+1+k} \circ (\operatorname{Id}^{\otimes i} \otimes m_j \otimes \operatorname{Id}^{\otimes k}) = 0 \]

and

\[ \sum_{l \in l_{i+j}^k, i \geq j \geq 1} (-1)^{p-q} m_k \circ (T_{l_1} \otimes \cdots \otimes T_{l_k}) = \sum_{l \in l_{i+j}^k, i \geq j \geq 1} \sum_{l \in l_{i+j}^k, i \geq j \geq 1} \lambda^{p-q} (T_{l_1})\{m_{p}\{T_{r_2} \otimes \cdots \otimes T_{r_q} \otimes \operatorname{Id}^{\otimes j_q}\} \otimes \operatorname{Id}^{\otimes k}) \]

where
\[ \alpha = \frac{k(k-1)}{2} + \frac{n(n-1)}{2} + \sum_{j=1}^{k} (k-j)l_j, \]
\[ \beta = \frac{p(p-1)}{2} + \sum_{j=1}^{q} \frac{r_j(r_j-1)}{2} + k + \sum_{i=2}^{q} \frac{\sum_{r_i=1}^{(i-1)} j_r + \sum_{i=2}^{r_i} r_i + p_i}{2} \]
\[ = \frac{n(n-1)}{2} + i + \sum_{j=2}^{q} (r_j-1)k + \sum_{i=2}^{q} \sum_{r_i=1}^{q} j_r + q - i \]

As introduced in Subsection 3.4, Equation (43) is exactly the Stasheff identity (15) in the definition of \( A_{\infty} \)-algebras. In particular, the operator \( m_1 \) is a differential on \( V \), and the operator \( m_2 \) induces an associative algebra structure on the homology space \( H_{\bullet}(V, m_1) \).

Equation (44) for \( n = 1, 2 \) gives
\[ m_1 \circ T_1 = T_1 \circ m_1, \]
and
\[ m_2 \circ (T_1 \otimes T_1) - T_1 \circ m_2 \circ (\operatorname{Id} \otimes T_1) - T_1 \circ m_2 \circ (T_1 \otimes \operatorname{Id}) - \lambda T_1 \circ m_2 \]
\[ = -(m_1 \circ T_2 + T_2 \circ (\operatorname{Id} \otimes m_1) + T_2 \circ (m_1 \otimes \operatorname{Id})). \]

Equation (45) implies that \( T_1 : (V, m_1) \to (V, m_1) \) is a chain map, thus \( T_1 \) is well-defined on the \( H_{\bullet}(V, m_1) \); Equation (46) indicates that \( T_1 \) is a Rota-Baxter operator of weight \( \lambda \) with respect to
m_2 up to homotopy, whose obstruction is just operator T_2. As a consequence, \((H_*(V,m_1),m_2,T_1)\) is a Rota-Baxter algebra.

Now, we will give a homotopy version of Proposition 4.6.

Let V be a graded vector space. Let \(\{b_n\}_{n \geq 1}, \{R_n\}_{n \geq 1}\) be a Maurer-Cartan element in \(C_{RBA}(V)\). So we have the corresponding operators \(\{m_n\}_{n \geq 1}\) and \(\{T_n\}_{n \geq 1}\) which define a homotopy Rota-Baxter algebra structure on V.

Define a new family of operators \(\{\overline{b}_n\}_{n \geq 1}\)

\[
\overline{b}_n = \sum_{p=1}^{n} \sum_{q=0}^{p-1} \sum_{l_1+q+p-q=n \atop l_1 \geq 1} \lambda^{p-q-1} b_p(sR_{l_1}, \ldots, sR_{l_q})
\]

and set \(\overline{m}_n := s^{-1} \overline{b}_n \circ s^{\otimes n} : V^{\otimes n} \to V\). Introduce another family of operators \(\{\overline{R}_n : (sV)^{\otimes n} \to V\}_{n \geq 1}\) as follows:

(i) put \(R_n^1 := \lambda^{n-1} R_n : (sV)^{\otimes n} \to V\) for any \(n \geq 1\);

(ii) define \(R_n^2 := \sum_{1 \leq q < p \leq n} \sum_{l_1+q+p-q=n} \lambda^{p-q-1} s^1(sR_p)(sR_{l_1}, \ldots, sR_{l_q})\);

(iii) taking induction, define \(R_n^k = \sum_{1 \leq q < p \leq n} \sum_{l_1+q+p-q=n} \lambda^{p-q-1} s^1(sR_p)(sR_{l_1}, \ldots, sR_{l_q})\);

(iv) define \(\overline{R}_n = \sum_{k=1}^{\infty} R_n^k\).

Note that for any given \(n \geq 1\), this is always a finite sum, thus \(\overline{R}_n\) is a well-defined map of degree \(-1\) in Hom((sV)^{\otimes n}, V). Impose \(\overline{T}_n = \overline{R}_n \circ s^{\otimes n} : V^{\otimes n} \to V\).

**Proposition 9.2.**  
(i) The pair \((V,\{\overline{m}_n\}_{n \geq 1})\) forms an \(A_{\infty}\)-algebra. And the family of operators \(\{T_n\}_{n \geq 1}\) defines an \(A_{\infty}\)-morphism from \((V,\{\overline{m}_n\}_{n \geq 1})\) to \((V,\{m_n\}_{n \geq 1})\).

(ii) These two family of operators \(\{\overline{b}_n\}_{n \geq 1} \cup \{\overline{R}_n\}_{n \geq 1}\) is also a Maurer-Cartan element in \(C_{RBA}(V)\), thus a homotopy Rota-Baxter algebra structure of weight \(\lambda\) on V.

For a proof, see Appendix B.

10. The Minimal model for the Operad of Rota-Baxter Algebras

In the last section, we have defined the notion of homotopy Rota-Baxter algebras of any weight. In this section, we will prove that the dg operad governing homotopy Rota-Baxter algebras of weight \(\lambda\) is a minimal model of the operad for Rota-Baxter algebras of weight \(\lambda\). Therefore, the cohomology theory for Rota-Baxter algebras defined before is the right cohomology theory for Rota-Baxter algebras in the sense of operad theory.

For basic theory of operads, we refer the reader to the textbooks [51, 7]. As we will only care about nonsymmetric operads in this paper, we will delete the adjective “nonsymmetric” everywhere. For a collection \(M = \{M(n)\}_{n \geq 1}\) of (graded) vector spaces, denote by \(\mathcal{F}(M)\) the free (graded) operad generated by \(M\). Recall that a dg operad is called quasi-free if its underlying graded operad is free.

**Definition 10.1** ([21]). A minimal model for an operad \(P\) is a quasi-free dg operad \((\mathcal{F}(M), d)\) together with a surjective quasi-isomorphism of operads \((\mathcal{F}(M), \partial) \twoheadrightarrow P\), where the dg operad \((\mathcal{F}(M), \partial)\) satisfies the following conditions:
(i) the differential $d$ is decomposable, i.e. $\partial$ takes $M$ to $\mathcal{F}(M)^{\geq 2}$, the subspace of $\mathcal{F}(M)$ consisting of elements with weight $\geq 2$;
(ii) the generating collection $M$ admits a decomposition $M = \bigoplus_{i \geq 1} M(i)$ such that $\partial(M(k+1)) \subset \mathcal{F}\left( \bigoplus_{i=1}^{k} M(i) \right)$ for any $k \geq 1$ (usually $M(i)$ is the degree $i$ part).

**Theorem 10.2** ([21]). When an operad $P$ admits a minimal model, it is unique up to isomorphisms.

The operad for Rota-Baxter algebras of weight $\lambda$, denoted by $\mathcal{RB}^\lambda$, is generated by a unary operator $T$ and a binary operator $\mu$ with the operadic relation generated by

$$
\mu \circ_1 \mu - \mu \circ_2 \mu \quad \text{and} \quad (\mu \circ_1 T) \circ_2 T - (T \circ_1 \mu) \circ_1 T - (T \circ_1 \mu) \circ_2 T - \lambda T \circ_1 \mu.
$$

Recall that a homotopy Rota-Baxter algebra structure on a graded space $V$ consists of two families of operators $\{m_n\}_{n \geq 1}$ and $\{T_n\}_{n \geq 1}$ satisfying Equations (43)-(44). As operator $-m_1$ makes $V$ into a complex, it induces a differential operator $\partial$ on graded space $\text{Hom}(V^{\otimes n}, V)$ containing $m_n, T_n$. Rewriting Equations (43)-(44) gives:

$$
\partial m_n = (-m_1) \circ m_n - (-1)^{n-2} \sum_{i=1}^{n} m_n \circ_i (-m_1) = \sum_{j=2}^{n} \sum_{i=1}^{n-j+1} (-1)^{j+1} m_{n-j+1} \circ_i m_j
$$

$$
\partial T_n = (-m_1) \circ T_n - (-1)^{n-1} \sum_{i=1}^{n} T_n \circ_i (-m_1)
$$

$$
= \sum_{k=2}^{n} \sum_{\substack{l_1+\cdots+l_k=n \\ l_1, \ldots, l_k \geq 1}} (-1)^{\gamma} \left( \cdots (m_k \circ_1 T_{l_1} \circ_1 T_{l_2}) \cdots \circ_1 T_{l+\cdots+l_k-1+1} T_{l_k} + \sum_{\substack{2 \leq p \leq n \\ 1 \leq q \leq \gamma \\ r_1, \ldots, r_q \geq 1 \\ 1 \leq r_k < \cdots < r_{k-1} < p}} (-1)^{\beta} A^{p-q} \left( T_{r_1} \circ_1 \left( \cdots (m_p \circ_{k_1} T_{r_2}) \circ_{k_2+r_2-1} T_{r_3} \cdots \circ_{k_{p-1}+r_{p-1}-1+1} T_{r_p} \right) \right) \right),
$$

where

$$
\alpha' = \frac{k(k-1)}{2} + \sum_{j=1}^{k} (k-j)l_j = \sum_{j=1}^{k} (k-j)(l_j-1),
$$

$$
\beta' = i + (p + \sum_{j=2}^{q} (r_j-1))(r_1-1) + \sum_{j=2}^{q} (r_j-1)(p-k_{j-1})
$$

Now we introduce the dg operad of homotopy Rota-Baxter algebras of weight $\lambda$.

**Definition 10.3.** Let $M = (M(0), M(1), \ldots, M(n), \ldots)$ be the graded collection where $M(0) = 0$, arity 1 part $M(1)$ is the one-dimensional graded space spanned by $T_1$ with $|T_1| = 0$, and for $n \geq 2$, arity $n$ part $M(n)$ is the two-dimensional graded space spanned by $T_n, m_n$ with $|T_n| = n-1$, $|m_n| = n-2$. The dg operad for homotopy Rota-Baxter algebras of weight $\lambda$, denoted by $\mathcal{RB}^\lambda$, is
the free graded operad generated by $M$ endowed with differential $\partial$ subject to

\begin{equation}
\partial m_n = \sum_{j=2}^{n-1} \sum_{i=1}^{n-j+1} (-1)^{i+1+j(n-i)} m_{n-j+1} \circ_i m_j
\end{equation}

and

\begin{equation}
\partial T_n = \sum_{k=2}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{\alpha'} \left( \cdots (m_{i_k} \circ_{i_{k-1}} \cdots (m_{i_2} \circ_{i_1} T_{i_1}) \cdots) \circ_{i_{k-1}+\cdots+i_2} T_{i_k} + \right.

\sum_{2 \leq p \leq n} \sum_{1 \leq q \leq p} (-1)^{\beta'} \lambda^{p-q} \left( T_{i_1} \circ_{i_2} \left( \cdots (m_{i_p} \circ_{i_{p-1}} \cdots (m_{i_2} \circ_{i_1} T_{i_1}) \cdots) \circ_{i_{p-1}+\cdots+i_2} T_{i_p} \right) \right).
\end{equation}

where the signs $(-1)^{\alpha'}$ and $(-1)^{\beta'}$ are determined by Equations (47)-(48) respectively.

We will use planar rooted trees to display elements in the dg operad $\mathcal{R} \mathcal{B}_{\lambda, \infty}^1$. We use the corolla with $n$ leaves and a black vertex to represent generators $m_n, n \geq 2$ and the corolla with $n$ leaves and a white vertex to represent generators $T_n, n \geq 1$:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree.png}
\caption{Planar rooted trees representing generators $m_n$ and $T_n$.}
\end{figure}

By [7, Chapter 3], a planar rooted tree with all internal vertices dyed white or black (such a tree will be called a tree monomial) gives an element in $\mathcal{R} \mathcal{B}_{\lambda, \infty}^1$ by composing its vertices clockwise. Conversely, any element in $\mathcal{R} \mathcal{B}_{\lambda, \infty}^1$ can be represented by such a unique planar rooted tree in this way. For example, the element

\[ (((m_3 \circ_1 T_4) \circ_3 m_4) \circ_9 m_2) \circ_{10} T_2 \]

can be represented by the following tree:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree_monomial.png}
\caption{Tree representation of an element in $\mathcal{R} \mathcal{B}_{\lambda, \infty}^1$.}
\end{figure}

In this means, the action of differential operator $\partial$ on generators can be expressed by trees as follows:
The following result is the main result of this section, whose proof occupies the rest of this section.

**Theorem 10.4.** The dg operad $\mathcal{RB}^\lambda_\infty$ is the minimal model of the operad $\mathcal{RB}^\lambda$.

Now, we are going to prove that there exists a quasi-isomorphism of dg operads $\mathcal{RB}^\lambda_\infty \to \mathcal{RB}^\lambda$, where $\mathcal{RB}^\lambda$ is considered as a dg operad concentrated in degree 0.

The degree zero part of $\mathcal{RB}^\lambda_\infty$ is the free graded operad generated by $\{m_2\} \cup \{T_1\}$. The image of $\partial$ in this degree zero part is the operadic ideal generated by $\partial T_2$, $\partial m_3$. By definition, we have:

\[
\partial(m_3) = m_2 \circ_1 m_2 - m_2 \circ_2 m_2,
\]

\[
\partial(T_2) = -T_1 \circ_1 (m_2 \circ_1 T_1) - T_1 \circ_1 (m_2 \circ_2 T_1) - \lambda T_1 \circ m_2 + (m_2 \circ_1 T_1) \circ_2 T_1.
\]

Thus $H_0(\mathcal{RB}^\lambda_\infty) \cong \mathcal{RB}^\lambda$.

To prove the natural map $\phi : \mathcal{RB}^\lambda_\infty \to \mathcal{RB}^\lambda$ is a quasi-isomorphism, we just need to prove that $H_i(\mathcal{RB}^\lambda_\infty) = 0$ for all $i \geqslant 1$.

We need the following notion of graded path-lexicographic ordering on $\mathcal{RB}^\lambda_\infty$.

Each tree monomial gives rise to a path sequence; for details, see [7, Chapter 3]. More precisely, to any tree monomial $T$ with $n$ leaves (written as $\text{arity}(T) = n$), we can associate with a sequence $(x_1, \ldots, x_n)$ where $x_i$ is the word formed by generators of $\mathcal{RB}^\lambda_\infty$ corresponding to the vertices along the unique path from the root of $T$ to its $i$-th leaf.

For two graded tree monomials $T, T'$, we compare $T, T'$ in the following way:

(i) If $\text{arity}(T) > \text{arity}(T')$, then $T > T'$;
(ii) if $\text{arity}(T) = \text{arity}(T')$, and $\deg(T) > \deg(T')$, then $T > T'$, where $\deg(T)$ is the sum of the degrees of all generators of $RB^\lambda_{\infty}$ appearing in tree monomial $T$;
(iii) if $\text{arity}(T) = \text{arity}(T')(= n)$, $\deg(T) = \deg(T')$, then $T > T'$ if the path sequences $(x_1, \ldots, x_n), (x'_1, \ldots, x'_n)$ associated to $T, T'$ satisfies $(x_1, \ldots, x_n) > (x'_1, \ldots, x'_n)$ with respect to the length-lexicographic order of words induced by $T_1 < m_2 < T_2 < m_3 < \cdots < T_n < m_{n+1} < T_{n+1} < \cdots$.

It is ready to see that this is a well order. Under this order, the leading term in the expansion of $\partial(m_n), \partial(T_n)$ are the following trees respectively:

\[
\begin{aligned}
\text{Typical Divisor} & \quad \text{Leaf}
\end{aligned}
\]

**Definition 10.5.** Let $S$ be a generator of degree $\geq 1$ in $RB^\lambda_{\infty}$. Denote the leading monomial of $\partial S$ by $\hat{S}$ and the coefficient of $\hat{S}$ in $\partial$ is written as $l_S$. A tree monomial of the form $\hat{S}$ is called typical.

It can be easily seen that the coefficient $l_S$ is always $\pm 1$.

To prove that $H_i(\mathbb{RB}^\lambda_{\infty}) = 0, i \geq 1$, we are going to construct a homotopy map $\delta$, i.e., a map of degree 1, $\delta : \mathbb{RB}^\lambda_{\infty} \to \mathbb{RB}^\lambda_{\infty}$ satisfying $\partial \delta + \delta \partial = \text{Id}$ in positive degrees.

**Definition 10.6.** A tree monomial $T$ in $\mathbb{RB}^\lambda_{\infty}$ is called effective if $T$ satisfies the following conditions:

(i) There exists a typical divisor $T' = \hat{S}$ in $T$ such that: on the path from the root of $T'$ to the leftmost leaf $l$ of $T$ above the root of $T'$, there are no other typical divisors, and there are no vertex of positive degree on this path except the root of $T'$ possibly.
(ii) For any leaf $l'$ of $T$ which lies on the left of $l$, there are no vertices of positive degree and no typical divisors on the path from the root of $T$ to $l'$.

The typical divisor $T'$ is called the effective divisor of $T$ and the leaf $l$ is called the typical leaf of $T$.

Morally, the effective divisor of a tree monomial $T$ is the left-upper-most typical divisor of $T$. It can be easily see that for the effective divisor $T'$ in $T$ with effective leaf $l$, any vertex in $T'$ doesn’t belong to the path from root of $T$ to any leaf $l'$ located on the left of $l$. 
Example 10.7. Consider three tree monomials with the same underlying tree:

For the three trees displayed above, each has two typical divisors.

- \( T_1 \) is effective and the divisor in the blue dashed circle is its effective divisor and \( l \) is its effective leaf.
- \( T_2 \) is not effective, since the first leaf belongs to a vertex of degree 1, say the root of \( T_2 \), which violates Condition (ii) in Definition 10.6.
- \( T_3 \) is not effective since there is a vertex of degree 1 on the path from the root of the typical divisor in the blue dashed circle to the leftmost leaf above it, which violates Condition (i) in Definition 10.6.

Definition 10.8. Let \( T \) be an effective tree monomial in \( \mathbb{RB}_\infty^l \) and \( T' \) be its effective divisor. Assume that \( T' = \hat{S} \), where \( S \) is a generator of positive degree. Then define

\[
\overline{\delta}(T) = (-1)^\omega \frac{1}{lS} m_{T,S}(T),
\]

where \( m_{T,S}(T) \) is the tree monomial obtained from \( T \) by replacing the effective divisor \( T' \) by \( S \), \( \omega \) is the sum of degrees of all the vertices on the path from root of \( T' \) to the root of \( T \) (except the root vertex of \( T' \)) and on the left of this path.

Then we define a map \( \delta \) of degree 1 on \( \mathbb{RB}_\infty^l \) as

(i) If \( T \) is not an effective tree monomial, then define \( \delta(T) = 0 \);
(ii) If \( T \) is effective, denote by \( \overline{T} \) is obtained from \( T \) by replacing \( T' \) by \( T' - \frac{1}{lS} \partial S \) with \( T' \) being the leading term of \( \partial S \). Define \( \delta(T) = \overline{\delta}(T) + \delta(\overline{T}) \), where, since each tree monomial in \( \overline{T} \) is strictly smaller than \( T \), define \( \delta(\overline{T}) \) by taking induction on leading terms (this can be done by Lemma 10.9).

Let’s explain more on the definition of \( \delta \). Denote \( T \) by \( T_1 \). By definition above, \( \delta(T) = \overline{\delta}(T_1) + \delta(\overline{T_1}) \). Since \( \delta \) vanishes on non-effective tree monomials, we have \( \delta(T_1) = \delta(\sum_{i_1 \in I_1} T_{i_1}) \) where \( \{T_{i_1}\}_{i_1 \in I_1} \) is the set of effective tree monomials together with their coefficients appearing in the expansion of \( T_1 \). Then by definition of \( \overline{\delta} \), \( \delta(\sum_{i_1 \in I_1} T_{i_1}) = \overline{\delta}(\sum_{i_1 \in I_1} T_{i_1}) \) \( + \delta(\sum_{i_1 \in I_1} \overline{T}_{i_1}) \), then we have

\[
\delta(T) = \overline{\delta}(T_1) + \overline{\delta}(\sum_{i_1 \in I_1} T_{i_1}) + \overline{\delta}(\sum_{i_2 \in I_2} \overline{T}_{i_2}).
\]

Take induction on leading terms, \( \delta(T) \) is the following series:

\[
\delta(T) = \overline{\delta}(T_1) + \overline{\delta}(\sum_{i_1 \in I_1} T_{i_1}) + \overline{\delta}(\sum_{i_2 \in I_2} T_{i_2}) + \cdots + \overline{\delta}(\sum_{i_n \in I_n} T_{i_n}) + \ldots,
\]
Lemma 10.10. Let \( \mathcal{T}_{i_n} \) be in the set of the effective tree monomials with their nonzero coefficients appearing in the expansion of \( \sum_{i_n \in I_n} \mathcal{T}_{i_n} \).

Lemma 10.9. For any effective tree monomial \( \mathcal{T} \), the expansion of \( \mathcal{S}(\mathcal{T}) \) in Equation (51) is always a finite sum, i.e., there exists some large integer \( n \) such that all tree monomials in \( \sum_{i_n \in I_n} \mathcal{T}_{i_n} \) are not effective.

Proof. It is easy to see that \( \max\{\mathcal{T}_{i_n}|i_k \in I_k\} > \max\{\mathcal{T}_{i_k+1}|i_k+1 \in I_{k+1}\} \) for all \( k \geq 1 \) (by convention, \( i_1 \in I_1 = \{1\} \)), so Equation (51) cannot be an infinite sum, as \( > \) is a well order.

Lemma 10.10. Let \( \mathcal{T} \) be an effective tree monomial. Then \( \partial \mathcal{S}(\mathcal{T}) + \mathcal{S}(\partial(\mathcal{T} - \mathcal{T})) = \mathcal{T} - \mathcal{T} \).

Proof. We can write \( \mathcal{T} \) as a compositions in the following way:

\[
(\cdots(((X_1 \circ_{i_1} X_2) \circ \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{i_j} Y_1) \circ_{i_j} Y_2) \cdots) \circ_{i_q} Y_q,
\]

where \( \mathcal{S} \) is the effective divisor of \( \mathcal{T} \) and \( X_1, \ldots, X_p \) are generators of \( \mathcal{R}^1_{\geq 0} \) corresponding to the vertices which live on the path from root of \( \mathcal{T} \) and root of \( \mathcal{S} \) (except the root of \( \mathcal{S} \)) and on the left of this path in the underlying tree of \( \mathcal{T} \).

By definition,

\[
\partial \mathcal{S}(\mathcal{T}) = \frac{1}{l_S} (-1)^{\frac{p}{2} \sum_{j=1}^{p} |X_j|} \partial((\cdots(((X_1 \circ_{i_1} X_2) \circ \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{i_j} Y_1) \circ_{i_j} Y_2) \cdots) \circ_{i_q} Y_q)
\]

and

\[
\mathcal{S}(\partial(\mathcal{T} - \mathcal{T})) = \frac{1}{l_S} \mathcal{S}(\partial((\cdots(((X_1 \circ_{i_1} X_2) \circ \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} \partial S) \circ_{i_j} Y_1) \circ_{i_j} Y_2) \cdots) \circ_{i_q} Y_q)
\]

where \( \partial \mathcal{S}(\mathcal{T}) \) and \( \mathcal{S}(\partial(\mathcal{T} - \mathcal{T})) \) are generators of \( \mathcal{R}^1_{\geq 0} \) corresponding to the vertices which live on the path from root of \( \mathcal{T} \) and root of \( \mathcal{S} \) (except the root of \( \mathcal{S} \)) and on the left of this path in the underlying tree of \( \mathcal{T} \).
By the definition of the effective divisor in an effective tree monomial, it can be easily seen that each tree monomial in the expansion of

$$(\cdots(((\cdots((X_1 \circ i_1 \cdots) \circ i_{k-1} \circ X_k) \circ i_k \cdots) \circ i_{p-1} X_p) \circ i_p S) \circ j_1 Y_1) \circ j_2 \cdots \circ j_q Y_q)$$

and of

$$(\cdots(((\cdots((X_1 \circ i_1 X_2) \circ i_2 \cdots) \circ i_{p-1} X_p) \circ i_p S) \circ j_1 Y_1) \circ j_2 \cdots \circ j_q Y_q)$$

is still effective tree monomial whose effective divisor is still $S$. Thus we have

$$\partial S(\mathcal{T} - \mathcal{T}) = \frac{1}{l_S} \sum_{k=1}^{p} (-1)^{\sum X_j + \sum Y_j - 1} (\cdots(((\cdots((X_1 \circ i_1 \cdots) \circ i_{k-1} \circ X_k) \circ i_k \cdots) \circ i_{p-1} X_p) \circ i_p S) \circ j_1 Y_1) \circ j_2 \cdots \circ j_q Y_q \right)$$

$$+ \frac{1}{l_S} \sum_{k=1}^{q} (-1)^{\sum X_j + \sum Y_j - 1} (\cdots(((\cdots((X_1 \circ i_1 X_2) \circ i_2 \cdots) \circ i_{p-1} X_p) \circ i_p S) \circ j_1 Y_1) \circ j_2 \cdots \circ j_q Y_q \right).$$

Take sum of the above expansion, then we get $\partial S(\mathcal{T}) + \partial S(\mathcal{T} - \mathcal{T}) = \mathcal{T} - \mathcal{T}$. 

**Proposition 10.11.** The degree 1 map $\partial S$ defined above satisfies $\partial S + \partial S = \text{Id}$ in all positive degrees of $\mathfrak{RB}^A$.

**Proof.** Let $\mathcal{T}$ be an effective tree monomial. Since the leading term of $\mathcal{T}$ is strictly smaller than $\mathcal{T}$, by induction, we have

$$\partial S(\mathcal{T}) + \partial S(\mathcal{T} - \mathcal{T}) = \mathcal{T}.$$

By the definition of $\partial S$, $\partial S(\mathcal{T}) = \mathcal{T}(\mathcal{T}) + \partial S(\mathcal{T})$ and we have $\partial S(\mathcal{T}) = \partial S(\mathcal{T}) + \partial S(\mathcal{T})$. Thus,

$$\partial S(\mathcal{T}) + \partial S(\mathcal{T}) = \partial S(\mathcal{T}) + \partial S(\mathcal{T}) + \partial S(\mathcal{T} - \mathcal{T}) + \partial S(\mathcal{T})$$

$$= \partial S(\mathcal{T}) + \partial S(\mathcal{T} - \mathcal{T})$$

$$= \mathcal{T} - \mathcal{T} + \mathcal{T}$$

$$= \mathcal{T},$$

where in the third equality we have used the induction hypothesis and

$$\partial S(\mathcal{T}) + \partial S(\mathcal{T} - \mathcal{T}) = \mathcal{T} - \mathcal{T}$$

by Lemma 10.10.

Next let’s prove that for a non-effective tree monomial $\mathcal{T}$, the equation $\partial S(\mathcal{T}) + \partial S(\mathcal{T}) = \mathcal{T}$ holds.

By the definition of $\partial S$, since $\mathcal{T}$ is not effective, $\partial S(\mathcal{T}) = 0$, thus we just need to check that $\partial S(\mathcal{T}) = \mathcal{T}$. Since $\mathcal{T}$ has positive degree, there must exists at least one vertex of positive degree.

Let’s pick a special vertex $S$ satisfying the following conditions:

(i) on the path from $S$ to the leftmost leaf $l$ of $\mathcal{T}$ above $S$, there are no other vertices of positive degree;

(ii) for any leaf $l'$ of $\mathcal{T}$ located on the left of $l$, the vertices on the path from the root of $\mathcal{T}$ to $l'$ are all of degree 0.
It is easy to see such a vertex always exists in $T$. Morally, this vertex is the “left-upper-most” vertex of positive degree. Then the tree monomial $T$ can be written as

$$(\cdots(((\cdots(X_1\circ_{i_1} X_2)\circ_{i_2} \cdots)\circ_{i_{p-1}} X_p)\circ_{i_p} S)\circ_{j_1} Y_1)\circ_{j_2} \cdots)\circ_{j_q} Y_q,$$

where $X_1, \ldots, X_p$ corresponds to the vertices located on the path from the root of $T$ to $S$ and on the left of this path in the plane.

By definition,

$$\partial_{\delta T} = \{ \sum_{k=1}^{p} (-1)^{\sum |X_k|} \cdots (((\cdots((\cdots(X_1\circ_{i_1} X_2)\circ_{i_2} \cdots)\circ_{i_{k+1}} X_k)\circ_{i_k} S)\circ_{j_1} Y_1)\circ_{j_2} \cdots)\circ_{j_q} Y_q + \sum_{k=1}^{q} (-1)^{\sum |X_k|+|X_{k+1}|} \cdots (((\cdots((\cdots(X_1\circ_{j_1} X_2)\circ_{j_2} \cdots)\circ_{i_{p-1}} X_p)\circ_{i_p} S)\circ_{j_1} Y_1)\circ_{j_2} \cdots)\circ_{j_k} Y_k) \cdots \circ_{j_q} Y_q \}
$$

By the assumption, the divisor consisting of the path from $S$ to $l$ must be of the following forms

(A) $m_p(n \geq 3)$
(B) $m_q(n \geq 3)$
(C) $T_p(n \geq 2)$
(D) $T_q(n \geq 2)$

By the assumption that $T$ is not effective and the speciality of the position of $S$, one can see that the effective tree monomials in $\partial_{\delta T}$ will only appear in the expansion of

$$(-1)^{\sum |X_k|} \cdots (((\cdots((\cdots(X_1\circ_{i_1} X_2)\circ_{i_2} \cdots)\circ_{i_{p-1}} X_p)\circ_{i_p} S)\circ_{j_1} Y_1)\circ_{j_2} \cdots)\circ_{j_q} Y_q.$$

Consider the tree monomial

$$((\cdots((\cdots(X_1\circ_{i_1} X_2)\circ_{i_2} \cdots)\circ_{i_{p-1}} X_p)\circ_{i_p} S)\circ_{j_1} Y_1)\circ_{j_2} \cdots)\circ_{j_q} Y_q$$

in $\partial T$. The path connecting root of $\hat{S}$ and $l$ must be of the following form:
So the tree monomial
\[(\cdots (((X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{j_1} Y_1) \circ_{j_2} Y_2 \cdots) \circ_{j_q} Y_q\]
is effective and its effective divisor is exactly \(S\) itself. Then we have
\[
\mathcal{S} \partial \mathcal{T} = S((-1)\Sigma_{i=1}^p [X_i] (((X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{j_1} Y_1) \circ_{j_2} Y_2 \cdots) \circ_{j_q} Y_q
\]
\[
= l_S S((-1)\Sigma_{i=1}^p [X_i] (((X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} (S - l_S S)) \circ_{j_1} Y_1) \circ_{j_2} Y_2 \cdots) \circ_{j_q} Y_q
\]
\[
= l_S S((-1)\Sigma_{i=1}^p [X_i] (((X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{j_1} Y_1) \circ_{j_2} Y_2 \cdots) \circ_{j_q} Y_q
\]
\[
= (\cdots (((X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{j_1} Y_1) \circ_{j_2} Y_2 \cdots) \circ_{j_q} Y_q
\]
\[
= \mathcal{T}.
\]
This completes the proof. \(\square\)

**Proof of Theorem 10.4:**
We have proved that the natural morphism \(\phi : \mathcal{RB}_\infty \rightarrow \mathcal{RB}_1\) is a surjective quasi-isomorphism, and it can be easily seen that the differential \(\partial\) on \(\mathcal{RB}_\infty\) satisfies the conditions (1)(2) in Definition 10.1.

**Remark 10.12.** We are grateful to Dotsenko who kindly pointed out an alternative proof of Theorem 10.4.
Let \( \mathfrak{T} \) be the free operad generated by a unary operation \( T \) and a binary operation \( \mu \). Then \( \mathcal{RB} \cong \mathfrak{T} / \triangleleft G \) where \( G \) is the defining relations for Rota-Baxter algebras. In [20], the authors prove that \( G \) is a Gröbner-Shirshov basis and they constructed the minimal model for the operad \( \mathfrak{T} / \triangleleft G \) where \( G \) is the set of leading monomials in \( \tilde{G} \). Denote this minimal model by \( m \mathcal{RB}_\infty^d \).

Now, let’s introducing a new ordering \( \prec \) on the the set \( \mathfrak{T}(n) \) as follows: For two tree monomials \( T, T' \) in \( \mathfrak{T}(n) \),

1. If weight\((T) < \) weight\((T')\), then \( T < T' \).
2. If weight\((T) = \) weight\((T')\), compare \( T \) with \( T' \) via the natural path-lexicographic induced by setting \( T < \mu \).

With respect to the ordering \( \prec \), the set \( \mathfrak{T}(n) \) becomes a totally ordered set: \( \mathfrak{T}(n) = \{ x_1 < x_2 < x_3 < \ldots \} \cong \mathbb{N}^+ \). Now, we define a map \( \omega : \mathcal{RB}_\infty^d \to \mathfrak{T} \) by replacing vertices \( m_n(n \geq 2), T_n(n \geq 1) \) by \( \mu_1 \circ \mu_1 \circ \cdots \circ \mu_1 \) and \( T_1 \circ \mu_1 \circ T_1 \circ \cdots \circ T_1 \circ \mu_1 \circ T_1 \) in \( \mathfrak{T} \) respectively. Notice that different tree monomials in \( \mathcal{RB}_\infty^d \) may have same image in \( \mathfrak{T} \) under the action \( \omega \). Define \( F_i^n \) to be the subspace of \( \mathcal{RB}_\infty^d \) spanned by the tree monomials \( R \) with \( \omega(R) \) smaller than or equal to \( x_i \) with respect to the ordering \( \prec \) on \( \mathfrak{T}(n) \). Then we get a bounded below and exhaustive filtration for \( \mathcal{RB}_\infty^d \)

\[
0 = F_0^n \subset F_1^n \subset F_2^n \subset \ldots.
\]

It can be easily seen that the filtration in compatible with the differential \( \partial \) on \( \mathcal{RB}_\infty^d \). Moreover, one can prove that there is an isomorphism of complexes

\[
\bigoplus_{i \geq 0} F_{i+1}^n / F_i^n \cong m \mathcal{RB}_\infty^d(n).
\]

Since all positive homologies of \( m \mathcal{RB}_\infty^d(n) \) vanishes, by classical spectral sequence argument, we have that all positive homologies of \( \mathcal{RB}_\infty^d(n) \) are trivial.

This provides another proof for Theorem 10.4.

**Appendix A: Proof of Theorem 8.1**

In this appendix, we will prove Theorem 8.1.

**Theorem 8.1.** Given a graded space \( V \) and an element \( \lambda \in k \), the graded space \( \mathfrak{c}_{\mathcal{RB}_\lambda}(V) \) endowed with operations \( \{ l_n \}_{n \geq 1} \) defined above forms an \( L_\infty \)-algebra.

By the definition of \( L_\infty \)-algebras, we need to check that the operators \( \{ l_n \}_{n \geq 1} \) on \( \mathfrak{c}_{\mathcal{RB}_\lambda}(V) \) satisfy the generalised anti-symmetry and the generalised Jacobi identity in Definition 2.4.

The operators \( \{ l_n \}_{n \geq 1} \) is automatically anti-symmetric by construction.

Now, we are going to check that \( \{ l_n \}_{n \geq 1} \) satisfies the generalised Jacobi identity, i.e., the following equation:

\[
\sum_{i=1}^{m} \sum_{\sigma \in \mathfrak{Sh}(i, m-i)} \chi(\sigma; x_1, \ldots, x_m)(-1)^{(m-i)} l_{m-i+1}(l_i(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(m)}) = 0
\]

for any \( x_1, \ldots, x_m \in \mathfrak{c}_{\mathcal{RB}_\lambda}(V) \), \( m \geq 1 \).

By Remark 2.5 (i), when \( m = 1 \), Equation (52) holds by definition of \( l_1 \); when \( m = 3 \) and all \( x_1, x_2, x_3 \in \mathfrak{c}_{\mathfrak{Alg}}(V) \), the LHS of Remark 2.5 (iii), is just the usual Jabobi identity of the graded Lie
algebra $\mathcal{C}_{\text{Alg}}(V)$ endowed with Gerstenhaber Lie bracket and the RHS of Remark 2.5(iii) vanishes because $l_1$ sends an element of $\mathcal{C}_{\text{Alg}}(V)$ to zero or an element of $\mathcal{C}_{\text{RBO}_1}(V)$ in which case we have two elements of $\mathcal{C}_{\text{Alg}}(V)$ and one in $\mathcal{C}_{\text{RBO}_1}(V)$, then $l_3$ applied to these three elements would give zero.

We have seen that one only needs to check Equation (52) with some $x_j \in \mathcal{C}_{\text{RBO}_1}(V)$.

By definition, $l_m(x_1, \ldots, x_m) = 0$ when $x_1, \ldots, x_m$ are all contained in $\mathcal{C}_{\text{RBO}_1}(V)$. So we have

$$l_m-x_{i+1}(l_i(x_{x(1)} \otimes \cdots \otimes x_{x(i)}) \otimes x_{x(i)+1} \otimes \cdots \otimes x_{x(m)}) = 0$$

unless there are exactly two elements in $\{x_1, \ldots, x_m\}$ belonging to $\mathcal{C}_{\text{Alg}}(V)$. Write $m = n + 2$ and assume that

$$x_1 = sh_1 \in \text{Hom}((sV)^{\otimes n}, sV) \subset \mathcal{C}_{\text{Alg}}(V), x_2 = sh_2 \in \text{Hom}((sV)^{\otimes 2}, sV) \subset \mathcal{C}_{\text{Alg}}(V)$$

and

$$x_3 = g_1, \ldots, x_{n+2} = g_n \in \mathcal{C}_{\text{RBO}_1}(V).$$

Since the expansion of the Equation (52) depends on the integers $n_1, n_2, n$, we classify the following cases:

(I) $n < \min(n_1, n_2)$,

(II) $\min(n_1, n_2) < n < \max(n_1, n_2)$,

(III) $\max(n_1, n_2) < n < n_1 + n_2 - 1$,

(IV) $n = n_1 + n_2 - 1$.

Now, we check Equation (52) for Case (I).

Assume first that $n_1, n_2 \geq 1$. Given $\sigma \in S_m$ and graded elements $g_1, \ldots, g_m$, denote $\varepsilon_i = \sum_{j=1}^{i} |g_{i(\sigma)}|$ and $\eta_i = \sum_{j=1}^{i} (|g_{i(\sigma)}| + 1)$. The expansion of Equation (52) contains the following three parts:

(1) The terms that $sh_1, sh_2$ are both contained in $l_i(\ldots)$:

$$A = l_{n+1}(l_2(sh_1 \otimes sh_2) \otimes g_1 \otimes \cdots \otimes g_n)$$

$$= l_{n+1}([sh_1, sh_2]_{G} \otimes g_1 \otimes \cdots \otimes g_n)$$

$$= l_{n+1}(sh_1 \otimes g_1 \otimes \cdots \otimes g_n) + (-1)^{1+|g_{i(1)}|+1}l_{n+1}(sh_2 \otimes g_1 \otimes \cdots \otimes g_n)$$

$$= \sum_{\sigma \in S_1} (-1)^{\delta_1} \lambda_1^{n_1, n_2-1-n} g_{\delta(1)}(sh_1) \{sg_{\delta(2)}, \ldots, sg_{\delta(n)} \}$$

$$+ \sum_{\sigma \in S_1} (-1)^{\delta_1} (-1)^{1+|g_{i(1)}|+1} \lambda_1^{n_1, n_2-1-n} g_{\delta(1)}(sh_2) \{sg_{\delta(2)}, \ldots, sg_{\delta(n)} \}$$

$$= \sum_{k=0}^{n-i} \sum_{\sigma \in S_1} \sum_{\lambda_1^{n_1, n_2-1-n}} g_{\delta(1)}(sh_1) \{sg_{\delta(2)}, \ldots, sg_{\delta(i)}, sh_2 \{sg_{\delta(i+1)}, \ldots, sg_{\delta(i+k)} \}, \}$$

$$+ \sum_{k=0}^{n-i} \sum_{\sigma \in S_1} \sum_{\lambda_1^{n_1, n_2-1-n}} g_{\delta(1)}(sh_2) \{sg_{\delta(2)}, \ldots, sg_{\delta(i)}, sh_1 \{sg_{\delta(i+1)}, \ldots, sg_{\delta(i+k)} \}, \}$$
Then we have

\[
(-1)^{\alpha_1} = \chi(\delta; g_1, \ldots, g_n)(-1)^{\sum_{i=1}^{n-1} e_i^{i+1+n(h_1+n)+e_i^{i+1+n(h_1+n)}}},
\]

\[
(-1)^{\alpha_2} = \chi(\delta; g_1, \ldots, g_n)(-1)^{\sum_{i=1}^{n} e_i^{i+1+n(h_1+|h_2|)+e_i^{i+1+n(h_1+|h_2|)}}},
\]

\[
(-1)^{\alpha_3} = \chi(\delta; g_1, \ldots, g_n)(-1)^{\sum_{i=1}^{n} e_i^{i+1+n(h_1+|h_2|)+e_i^{i+1+n(h_1+|h_2|)}}}.
\]

(2) The terms with \(sh_1\) contained in \(l_1(\ldots)\):

\[
B = \sum_{i=1}^{n} \sum_{\sigma \in S_{n-i}} (-1)^{\beta_i} l_{n-i+2}(l_{i+1}(sh_1, g_{\sigma(1)}, \ldots, g_{\sigma(i)}), sh_2, g_{\sigma(i+1)}, \ldots, g_{\sigma(n)})
\]

\[
= \sum_{i=1}^{n} \sum_{\sigma \in S_{n-i}} (-1)^{\beta_i} l_{n-i+2}(\underbrace{sh_2, l_{i+1}(sh_1, g_{\sigma(1)}, \ldots, g_{\sigma(i)}), g_{\sigma(i+1)}, \ldots, g_{\sigma(n)})}_{\mathcal{h}_i}
\]

\[
= \sum_{i=1}^{n} \sum_{\sigma \in S_{n-i}} \sum_{\pi \in S_{n-i}} (-1)^{\beta_i} \lambda_{\pi_2}^{n-i+1} \mathcal{h}_i \{sh_2\{g_{\sigma(i+\pi(1))}, \ldots, g_{\sigma(i+\pi(n-i))}\}
\]

\[
+ \sum_{i=1}^{n} \sum_{k=1}^{n-i} \sum_{\sigma \in S_{n-i}} \sum_{\pi \in S_{n-i}} (-1)^{\beta_k} \lambda_{\pi_2}^{n-i+1} \mathcal{h}_i \{g_{\sigma(i+\pi(1))}, \ldots, g_{\sigma(i+\pi(n-i))}\}
\]

By definition,

\[
\mathcal{h}_i = l_{i+1}(sh_1, g_{\sigma(1)}, \ldots, g_{\sigma(i)})
\]

\[
= \sum_{\tau \in S_{i}} \chi(\tau; g_{\sigma(1)}, \ldots, g_{\sigma(i)})(-1)^{1+e_i^{i+1+n(h_1+n)+e_i^{i+1+n(h_1+n)}}} \lambda_{\pi_1}^{n-i} \mathcal{h}_i \{sh_1\{g_{\sigma(2)}, \ldots, g_{\sigma(i)}\}\}
\]

Then we have

\[
B = \sum_{i=1}^{n} \sum_{\sigma \in S_{n-i}} \sum_{\pi \in S_{n-i}} \sum_{\tau \in S_{i}} (-1)^{\beta_i} \lambda_{\pi_1}^{n-i+1} \{sh_2\{g_{\sigma(i+\pi(1))}, \ldots, g_{\sigma(i+\pi(n-i))}\}\}
\]

\[
+ \sum_{k=1}^{n} \sum_{i=1}^{n-i} \sum_{\sigma \in S_{n-i}} \sum_{\pi \in S_{n-i}} (-1)^{\beta_k} \lambda_{\pi_1}^{n-i+1} \{g_{\sigma(i+\pi(1))}, \ldots, g_{\sigma(i+\pi(n-i))}\}
\]

(12)
\[
+ \sum_{k=1}^{i} \sum_{i=1}^{n} \sum_{s \in S_{i-1}} \sum_{t \in S_{t}} \sum_{r \in S_{r}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}}.
\]

\[
g_{\sigma^{(1)}} \{ s_{1} g_{\sigma^{(2)}}, \ldots, s_{g_{\sigma^{(k)}}}, s_{h_{2}}[s_{g_{\sigma^{(i+\pi(1))}}}, \ldots, s_{g_{\sigma^{(i+\pi(n-1))}}}], s_{g_{\sigma^{(k+1)}}, \ldots, s_{g_{\sigma^{(i)}}}}] \}
\]

\[
+ \sum_{k=2}^{i} \sum_{i=1}^{n} \sum_{s \in S_{i-1}} \sum_{t \in S_{t}} \sum_{r \in S_{r}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}}.
\]

\[
g_{\sigma^{(1)}} \{ s_{h_{1}}[s_{g_{\sigma^{(i+\pi(1))}}}, \ldots, s_{g_{\sigma^{(i+\pi(n-1))}}}], s_{g_{\sigma^{(k+1)}}, \ldots, s_{g_{\sigma^{(i)}}}}] \}
\]

\[
+ \sum_{k=1}^{n-i} \sum_{i=1}^{n} \sum_{s \in S_{i-1}} \sum_{t \in S_{t}} \sum_{r \in S_{r}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}}.
\]

\[
g_{\sigma^{(i+\pi(1))}} \{ s_{h_{2}}[s_{g_{\sigma^{(i+\pi(2))}}}, \ldots, s_{g_{\sigma^{(i+\pi(k))}}}], s_{h_{1}}[s_{g_{\sigma^{(i+\pi(2))}}}, \ldots, s_{g_{\sigma^{(i)}}}] \}
\]

\[
= \sum_{i=1}^{n} \sum_{s \in S_{n}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}} g_{\delta^{(1)}} \{ s_{1} g_{\delta^{(2)}}, \ldots, s_{g_{\delta^{(i)}}}, s_{h_{2}}[s_{g_{\delta^{(i+1)}}}, \ldots, s_{g_{\delta^{(n)}}}] \}
\]

\[
+ \sum_{i=1}^{n} \sum_{s \in S_{n}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}} g_{\delta^{(1)}} \{ s_{h_{1}}[s_{g_{\delta^{(i)}}}, \ldots, s_{g_{\delta^{(i+1)}}}], s_{1} g_{\delta^{(2)}}, \ldots, s_{g_{\delta^{(n)}}}] \}
\]

\[
+ \sum_{k=0}^{n-i} \sum_{s \in S_{n}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}} g_{\delta^{(1)}} \{ s_{h_{1}}[s_{g_{\delta^{(i)}}}, \ldots, s_{g_{\delta^{(i+1)}}}], s_{h_{2}}[s_{g_{\delta^{(i+1)}}}, \ldots, s_{g_{\delta^{(i+k)}}}], s_{g_{\delta^{(i+k+1)}}, \ldots, s_{g_{\delta^{(n)}}}] \}
\]

\[
+ \sum_{k=0}^{n-i} \sum_{s \in S_{n}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}} g_{\delta^{(1)}} \{ s_{h_{1}}[s_{g_{\delta^{(i)}}}, \ldots, s_{g_{\delta^{(i+1)}}}], s_{h_{2}}[s_{g_{\delta^{(i+1)}}}, \ldots, s_{g_{\delta^{(i+k)}}}], s_{g_{\delta^{(i+k+1)}}, \ldots, s_{g_{\delta^{(n)}}}] \}
\]

\[
+ \sum_{k=0}^{n-i} \sum_{s \in S_{n}} (-1)^{\tilde{\delta} \lambda^{n_{1}+n_{2}-n-1}} g_{\delta^{(1)}} \{ s_{h_{2}}[s_{g_{\delta^{(i-1)}}, \ldots, s_{g_{\delta^{(i)}}}], s_{h_{1}}[s_{g_{\delta^{(i+1)}}}, \ldots, s_{g_{\delta^{(i+k)}}}], s_{g_{\delta^{(i+k+1)}}, \ldots, s_{g_{\delta^{(n)}}}] \}.
\]
where
\[( -1)^{\beta_3} = \chi(\sigma; g_1, \ldots, g_n)(-1)^{(i+1)(n-i+1)+i+(h_2+1)\eta'}, \]
\[( -1)^{\beta_4} = \chi(\sigma; g_1, \ldots, g_n)(-1)^{(i+1)(n-i+1)+i+(h_2+1)\eta'+(h_1+1)\eta''}, \]
\[( -1)^{\beta_5} = ( -1)^{\beta_2} \chi(\pi; g_{\sigma(i+1)}, \ldots, g_{\sigma(i+n-i)})(-1)^{1+(n-i+1)(h_2+1)+i+(h_1+1)(\sum_{k=1}^{i}(\sum_{j=1}^{k} \sigma_{\sigma(i+j)})+i(\eta')+(\eta''))}, \]
\[( -1)^{\beta_6} = ( -1)^{\beta_2} \chi(\pi; g_{\sigma(i+1)}, \ldots, g_{\sigma(i+n-i)})(-1)^{1+(i(h_2+1)+\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]
\[( -1)^{\beta_7} = ( -1)^{\beta_3} ( -1)^{(h_1+1)+\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(1)}(\eta''), \]
\[( -1)^{\beta_8} = ( -1)^{\beta_3} ( -1)^{1+(i(h_2+1)+\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]
\[( -1)^{\beta_9} = \chi(\delta; g_1, \ldots, g_n)(-1)^{n(h_1+1)+\sum_{j=1}^{i} \sigma_{\sigma(i+j)}(\eta'')}}, \]
\[( -1)^{\beta_10} = ( -1)^{\beta_3} ( -1)^{1+(i(h_2+1)+\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]
\[( -1)^{\beta_11} = \chi(\delta; g_1, \ldots, g_n)(-1)^{1+(\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]
\[( -1)^{\beta_12} = \chi(\delta; g_1, \ldots, g_n)(-1)^{1+(\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]
\[( -1)^{\gamma_1} = ( -1)^{\beta_3}, \]
\[( -1)^{\gamma_2} = \chi(\delta; g_1, \ldots, g_n)(-1)^{1+(\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]
\[( -1)^{\gamma_3} = \chi(\delta; g_1, \ldots, g_n)(-1)^{1+(\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]
\[( -1)^{\gamma_4} = \chi(\delta; g_1, \ldots, g_n)(-1)^{1+(\sum_{j=1}^{i} \sigma_{\sigma(i+j)})(\eta'')}}, \]

Notice that in the step $\ast$ above, we replace the triple $(\tau, \sigma, \pi)$ by its corresponding permutation $\delta \in S_n$ as in Lemma 1.1, and we use Equation (5).

(3) The computation of the terms with $sh_2$ contained in $l_i(\ldots)$ is almost the same as (II)

\[
C = \sum_{i=1}^{n} \sum_{\sigma \in S_{i}(n-i)} (-1)^{\gamma_1} l_{n-i+2}(l_{i+1}(sh_2, g_{\sigma(1)}, \ldots, g_{\sigma(i)}), sh_1, g_{\sigma(i+1)}, \ldots, g_{\sigma(n)}) = \sum_{i=1}^{n} \sum_{\sigma \in S_{i}(n-i)} (-1)^{\gamma_2} l_{n-i+2}(sh_1, l_{i+1}(sh_2, g_{\sigma(1)}, \ldots, g_{\sigma(i)}), g_{\sigma(i+1)}, \ldots, g_{\sigma(n)}) = \sum_{i=1}^{n} \sum_{\delta \in S_{n}} (-1)^{\gamma_3} l_{n+2-n-1}(g_{\delta} \{sh_2, sg_{\delta(2)}, \ldots, sg_{\delta(i)}\}, sh_1 \{sg_{\delta(i+1)}, \ldots, sg_{\delta(n)}\})
\]
\[ + \sum_{i=1}^{n} \sum_{\delta \in S_n} (-1)^{s_i} A^{n_1+n_2-n-1} g_{\delta(1)} \left\{ s h_1 \left[ s g_{\delta(2)}, \ldots, s g_{\delta(i)} \right], s h_2 \left[ s g_{\delta(i+1)}, \ldots, s g_{\delta(n)} \right] \right\} \]
\[ + \sum_{k=0}^{n} \sum_{i=1}^{n} \sum_{\delta \in S_n} (-1)^{s_i} A^{n_1+n_2-n-1} \cdot \]
\[ g_{\delta(1)} \left\{ s h_2 \left[ s g_{\delta(2)}, \ldots, s g_{\delta(i)}, s h_1 \left[ s g_{\delta(i+1)}, \ldots, s g_{\delta(i+k)} \right], s g_{\delta(i+k+1)}, \ldots, s g_{\delta(n)} \right] \right\} \]
\[ + \sum_{k=0}^{n} \sum_{i=2}^{n} \sum_{\delta \in S_n} (-1)^{s_i} A^{n_1+n_2-n-1} \cdot \]
\[ g_{\delta(1)} \left\{ s h_2 \left[ s g_{\delta(2)}, \ldots, s g_{\delta(i-1)}, s g_{\delta(i)} \left[ s h_2 \left[ s g_{\delta(i+1)}, \ldots, s g_{\delta(i+k)} \right], s g_{\delta(i+k+1)}, \ldots, s g_{\delta(n)} \right] \right] \right\} \]

where

\[ (-1)^{s_1} = \chi(\sigma, g_1, \ldots, g_n)(-1)^{i_1+1+(i_1+1)(i_2+1)+e^\alpha}, \]
\[ (-1)^{s_2} = (-1)^{s_1}(1)(i_2+1)(i_1+1), \]
\[ (-1)^{s_3} = \chi(\delta; g_1, \ldots, g_n)(-1)^{i_3+1+(i_1+1)(i_2+1)}, \]
\[ (-1)^{s_4} = \chi(\delta; g_1, \ldots, g_n)(-1)^{i_4+1+n(i_1+1)}, \]
\[ (-1)^{s_5} = \chi(\delta; g_1, \ldots, g_n)(-1)^{i_5+1+1} \cdot \]
\[ (-1)^{s_6} = \chi(\delta; g_1, \ldots, g_n)(-1)^{i_6+1+(i_1+1)(i_2+1)} \cdot \]

Then the expansion of Equation (52) is just \( A + B + C \). And one can see that the same term appears exactly twice in \( A + B + C \) with opposite signs. Thus we have

\[ A + B + C = 0. \]

For the situation that \( n_1 = 0 \) or \( n_2 = 0 \), i.e., \( s h_1 \) or \( s h_2 \) belongs to \( \text{Hom}(k, sV) \), the computation for Equation (52) is similar, but notice that \( l_1(s h_1) \) or \( l_1(s h_2) \) may be nonzero in this situation.

For the cases (II) (III) (IV), the computation is also similar, but there may be more terms in the expansion of Equation (52). For example, in case (II), assuming \( n_1 \leq n < n_2 \), there will be terms of the following forms:

\[ h_1 \circ \left( s g_{\delta(1)}, \ldots, s g_{\delta(i)} \right) \left[ s h_2 \left[ s g_{\delta(i+1)}, \ldots, s g_{\delta(i+n-n_1)} \right] \right] \]
\[ g_{\delta(1)} \left[ s h_2 \left[ s g_{\delta(2)}, \ldots, s g_{\delta(i)}, s h_1 \circ \left( s g_{\delta(i+1)}, \ldots, s g_{\delta(i+n)} \right) \right] \right] \]

in both \( B \) and \( C \). Tracking their signs, one can find that these terms will be eliminated. We are done!
APPENDIX B: PROOF OF PROPOSITION 9.2

In this appendix, we will prove Proposition 9.2.

Proposition 9.2

(i) The pair \((V, \{\mu_n\}_{n \geq 1})\) forms an \(A_\infty\)-algebra. And the family of operators \(\{T_n\}_{n \geq 1}\) defines an \(A_\infty\)-morphism from \((V, \{\mu_n\}_{n \geq 1})\) to \((V, \{m_n\}_{n \geq 1})\).

(ii) These two family of operators \(\{\bar{b}_n\}_{n \geq 1} \cup \{\bar{R}_n\}_{n \geq 1}\) is also a Maurer-Cartan element in \(\mathfrak{c}_{\text{RBA}}(V)\), thus a homotopy Rota-Baxter algebra structure of weight \(\lambda\) on \(V\).

For (i), we show that operators \(\{\bar{b}_n\}_{n \geq 1}\) satisfy the following equation:

\[
\sum_{j=1}^{n} \bar{b}_{n-j+1}(\bar{b}_j) = \sum_{p+k \leq n, j \geq 1} (\text{Id} \otimes \bar{b}_j \otimes \text{Id}^\otimes k) = 0,
\]

which says that \((V, \{\bar{m}_n\}_{n \geq 1})\) is an \(A_\infty\)-algebra.

In fact,

\[
\sum_{j=1}^{n} \bar{b}_{n-j+1}(\bar{b}_j) = \sum_{j=1}^{n} \sum_{p=1}^{n-j+1} \sum_{k=0}^{p-1} \lambda^{p-k-1} (b_p(sR_{l_1}, \ldots, sR_{l_p}))(\bar{b}_j)
\]

\[
= \sum_{j=1}^{n} \sum_{p=1}^{n-j+1} \sum_{k=0}^{p-1} \sum_{l_1, \ldots, l_p, l_{j-1} \geq 1} \lambda^{p-k-1} b_p(sR_{l_1}, \ldots, sR_{l_p}, \bar{b}_j, sR_{l_{j-1}}, \ldots, sR_{l_k})
\]

\[
+ \sum_{j=1}^{n} \sum_{p=1}^{n-j+1} \sum_{k=0}^{p-1} \sum_{l_1, \ldots, l_p, l_{j-1} \geq 1} \lambda^{p-k-1} b_p(sR_{l_1}, \ldots, sR_{l_p}, \bar{b}_j, sR_{l_{j-1}}, \ldots, sR_{l_k})
\]

\[
= \sum_{j=1}^{n} \sum_{p=1}^{n-j+1} \sum_{k=0}^{p-1} \sum_{l_1, \ldots, l_p, l_{j-1} \geq 1} \lambda^{p+q-k-s-2} b_p(sR_{l_1}, \ldots, sR_{l_k}, b_q(sR_{l_1}, \ldots, sR_{l_k}), sR_{l_{j-1}}, \ldots, sR_{l_k})
\]

\[
+ \sum_{j=1}^{n} \sum_{p=1}^{n-j+1} \sum_{k=0}^{p-1} \sum_{l_1, \ldots, l_p, l_{j-1} \geq 1} \lambda^{p+q-k-s-2} b_p(sR_{l_1}, \ldots, sR_{l_k}, b_q(sR_{l_1}, \ldots, sR_{l_k}), sR_{l_{j-1}}, \ldots, sR_{l_k})
\]

\[
= \sum_{j=1}^{n} \sum_{p+q-2}^{n-j+1} \sum_{k=0}^{p-1} \sum_{m=0}^{s=0} \Lambda^{p+q-k-s-2} b_p(sR_{l_1}, \ldots, sR_{l_k}, b_q(sR_{l_1}, \ldots, sR_{l_k}), sR_{l_{j-1}}, \ldots, sR_{l_k})
\]

\[
+ \sum_{j=1}^{n} \sum_{p=1}^{n-j+1} \sum_{k=0}^{p-1} \sum_{l_1, \ldots, l_{j-1} \geq 1} \Lambda^{p+q-k-s-2} b_p(sR_{l_1}, \ldots, sR_{l_k}, b_q(sR_{l_1}, \ldots, sR_{l_k}), sR_{l_{j-1}}, \ldots, sR_{l_k})
\]
\[
\lambda^{p-k-1} b_p \{ sR_{j_1}, \ldots, sR_{j_k}, \sum_{1 \leq i < q < j} \sum_{1 \leq l_i \leq q, l_j \leq i, \ell_{i-j} \geq 1} \lambda^{q-i} sR_{i} \{ b_q [sR_{i_2}, \ldots, sR_{i_1}], sR_{j_1}, \ldots, sR_{j_k} \} \}
\]

\[
= \sum_{1 \leq p \leq q+1} \sum_{k=0}^{q-1} \sum_{r=0}^{p-1} \sum_{m=0}^{p+q-2m} \lambda^{p+q-m-2} b_p \{ sR_{i_1}, \ldots, sR_{i_r}, b_q \circ (sR_{i_1}, \ldots, sR_{i_1}), \ldots, sR_{i_m} \}
\]

\[
+ \sum_{1 \leq p \leq q+1} \sum_{k=1}^{m-q} \lambda^{p+q-m} b_p \{ sR_{i_1}, \ldots, sR_{i_1} \}
\]

\[
= 0,
\]

where we get the fourth equality by reindexing, the fifth equality is obtained from Equation (42) and the last equality uses Equation (41).

By the definition of homotopy Rota-Baxter algebra of weight \( \lambda \), the three family of operators \( \overline{b}_n \), \( \overline{\{ b_n \}} \), \( \{ R_n \} \) fulfill the equation:

\[
\sum_{k=1}^{n} \sum_{1 \leq l_1 \leq l_2 \leq \ldots \leq l_n \geq 1} b_k (sR_{l_1} \otimes \ldots \otimes sR_{l_k}) = \sum_{p=1}^{n} sR_p [\overline{b}_{p+1}].
\]

Thus \( \{ T_n \} \) is an \( A_{\infty} \)-morphism from \( (V, [m_n]) \) to \( (V, [m_n]) \).

For (ii), we just need to check that \( \overline{b}_n \cup \overline{R}_n \) fulfill Equation (44). By the definition of \( \overline{R}_n \), one can check that the following equation holds:

\[
\overline{R}_n = \sum_{k=1}^{\infty} \sum_{l_1 \leq \ldots \leq l_k \geq 1} \sum_{0 \leq q \leq p-1} \sum_{t_1 \leq \ldots \leq t_q \geq 1} \lambda^{p-q-1} s^{-1}(sR_p) [sR_{l_1}^q, \ldots, sR_{l_k}^q]
\]

Now, let's prove Equation (42) holds for \( \overline{b}_n \cup \overline{R}_n \), i.e., the following equation holds for any \( n \geq 1 \):

\[
\sum_{k=1}^{n} \sum_{t_1 \leq \ldots \leq t_k \geq 1} s^{-1} \overline{b}_k \circ (sR_{t_1} \otimes \ldots \otimes sR_{t_k}) = \sum_{p=1}^{n} \sum_{0 \leq q \leq p-1} \sum_{t_1 \leq \ldots \leq t_q \geq 1} \lambda^{p-q-1}s^{-1}(sR_{t_1}^q) [b_p [sR_{t_2}, \ldots, sR_{t_q}]].
\]
We prove this by taking induction on \( n \). When \( n = 1 \), it is easy to see that \( \bar{R}_1 = R_1 \) and \( \bar{b}_1 = b_1 \). The Equation (54) holds naturally for \( n = 1 \). Now, assume that Equation (54) holds for all integers \( \leq n - 1 \). Firstly, we have the following equation holds:

\[
\sum_{m=1}^{n} \sum_{l_1 + \cdots + l_m = n} s^{-1} \bar{b}_m \circ \left( s\bar{R}_{l_1} \otimes \cdots \otimes s\bar{R}_{l_m} \right) = \sum_{m=1}^{n} \sum_{l_1 + \cdots + l_m = n} \sum_{0 \leq p < m} \lambda^{p-k-1} \left( b_p \{ sR_{i_1}, \ldots, sR_{i_k} \} \right) \circ \left( s\bar{R}_{l_1} \otimes \cdots \otimes s\bar{R}_{l_m} \right) = \sum_{m=1}^{n} \sum_{l_1 + \cdots + l_m = n} \sum_{0 \leq p < m} \sum_{0 \leq i_1 \leq \cdots \leq i_l \leq \lambda m} \lambda^{p-k-1} \cdot
\]

\[
\sum_{m=1}^{n} \sum_{l_1 + \cdots + l_m = n} b_p \circ \left( \sum_{0 \leq i_1 \leq \cdots \leq i_l \leq \lambda m} \lambda^{i_1-\cdots-i_l} sR_{i_1} \{ s\bar{R}_{l_1} \} \right) \otimes \cdots 
\]

\[
\cdots \otimes \left( \sum_{0 \leq i_1 \leq \cdots \leq i_l \leq \lambda m} \lambda^{i_1-\cdots-i_l} sR_{i_1} \{ s\bar{R}_{l_1} \} \right) \otimes \cdots \otimes \left( \sum_{0 \leq j_1 \leq \cdots \leq j_l \leq \lambda m} \lambda^{j_1-\cdots-j_l} sR_{j_1} \{ s\bar{R}_{j_1} \} \right) 
\]

\[
= \sum_{p+q \leq m+n} \lambda^{p-q-1} \sum_{j_1 + \cdots + j_p = p} s^{-1} \left( b_h \circ \left( sR_{i_1} \otimes \cdots \otimes sR_{i_k} \right) \right) \left[ s\bar{R}_{i_1} \otimes \cdots \otimes s\bar{R}_{i_q} \right].
\]

In the Equality \( \ast \) above, we replace all \( \bar{R}_{l_j}, j \notin \bigcup \{ t_r, 1, \ldots, t_r + i_r \} \) by their expansions in the last line of Equation (53).

Now, let’s compute the RHS of Equation (54). We have:

\[
\sum_{p=1}^{n} \sum_{1 \leq q \leq p} \sum_{r_1 + \cdots + r_q + p \leq m} \lambda^{p-q-1} s^{-1} (s\bar{R}_{r_1}) [b_p \{ s\bar{R}_{r_1}, \ldots, s\bar{R}_{r_q} \}]
\]

\[
= \sum_{p=1}^{n} \sum_{0 \leq q \leq p} \sum_{m+r_1+\cdots+r_q+p \leq n} \lambda^{p-q-1} s^{-1} (s\bar{R}_{m}) [b_p \{ s\bar{R}_{r_1}, \ldots, s\bar{R}_{r_q} \}]
\]

\[
= \sum_{p=1}^{n} \sum_{0 \leq q \leq p} \sum_{m+r_1+\cdots+r_q+p \leq n} \lambda^{p-q-1} \sum_{0 \leq j_1 \leq \cdots \leq j_q \leq m} \lambda^{j_1-\cdots-j_q} s^{-1} (sR_{j_1} \otimes \cdots \otimes sR_{j_q}) [b_p \{ s\bar{R}_{r_1}, \ldots, s\bar{R}_{r_q} \}]
\]
\[ \sum_{i_1 + \cdots + i_j + p \cdot k - r - 1 = n} \sum_{0 \leq q \leq p - 1, 0 \leq j \leq r < q < k - 1} \alpha^{k-(r-q)+1-p-q-1} s^{-1}(s R_k) \left\{ \bar{s} R_{i_1}, \ldots, \bar{s} R_{i_j}, \bar{b}_p [s R_{j_{1+1}}, \ldots, s R_{j_{1+q}}], s R_{j_{1+q+1}}, \ldots, s R_{i_1} \right\} \]

\[ = \sum_{i_1 + \cdots + i_j + p \cdot k - r - 1 = n} \sum_{0 \leq q \leq p - 1, 0 \leq j \leq r < q < k - 1} \alpha^{k-(r-q)+1-p-q-1} s^{-1}(s R_k) \left\{ \bar{s} R_{i_1}, \ldots, \bar{s} R_{i_j}, \bar{b}_p [s R_{j_{1+1}}, \ldots, s R_{j_{1+q}}], s R_{j_{1+q+1}}, \ldots, s R_{i_1} \right\} \]

Notice that in the last step of the above expansion, \( m = i_j + i_{j+1} + \cdots + i_{j+q} + p - q - 1 \leq n - 1 \).

By assumption, Equation (54) holds for all integers \( \leq n - 1 \), so we have:

\[ \sum_{i_j + i_{j+1} + \cdots + i_{j+q} + p \cdot q - 1 = n} \alpha^{p-q-1} s^{-1} R_{i_j} \left[ \bar{b}_p [s R_{j_{1+1}}, \ldots, s R_{j_{1+q}}] \right] = \sum_{p=1}^{m} \sum_{i_1 + \cdots + i_p = m} \bar{b}_p \circ (s R_{i_1} \otimes \cdots \otimes s R_{i_p}) \]

Replacing the underlined part in the expansion by the RHS above and reindexing, we have

\[ \sum_{p=1}^{n} \sum_{1 \leq q \leq p} \sum_{r_1 + \cdots + r_q + p - q = n} \alpha^{p-q} s^{-1}(s R_{r_1}) [\bar{b}_p [s R_{r_2}, \ldots, s R_{r_q}]] \]

\[ = \sum_{i_1 + \cdots + i_j + p \cdot k - r - 1 = n} \sum_{0 \leq j \leq r < q < k - 1} \alpha^{k-(r-q)+1-p-q-1} s^{-1}(s R_k) \left\{ \bar{s} R_{i_1}, \ldots, \bar{s} R_{i_j}, \bar{b}_p [s R_{j_{1+1}}, \ldots, s R_{j_{1+q}}], s R_{j_{1+q+1}}, \ldots, s R_{i_1} \right\} \]

\[ + \sum_{i_1 + \cdots + i_j + p \cdot k - r - 1 = n} \sum_{0 \leq j \leq r < q < k - 1} \alpha^{k-(r-p+1)+1-p-q-1} s^{-1}(s R_k) \left\{ \bar{s} R_{i_1}, \ldots, \bar{s} R_{i_j}, \bar{b}_p \circ (s R_{j_{1+1}} \otimes \cdots \otimes s R_{j_{1+p}}), \ldots, s R_{i_1} \right\} \]

\[ = \sum_{r_1 + \cdots + r_q + p - q = n} \sum_{0 \leq q \leq p - 1} \alpha^{p-q-1} s^{-1} \left\{ s R_k \left[ \bar{b}_{p-k+1} \right] \right\} \left\{ \bar{s} R_{r_1}, \ldots, \bar{s} R_{r_q} \right\} \]

Since \( \{b_k\}_{k \geq 1} \cup \{R_k\}_{k \geq 1} \) fulfill Equation (42), we have the equation

\[ \sum_{j_1 + \cdots + j_k = p} s^{-1} \left( s b_k \circ (s R_{j_1} \otimes \cdots \otimes s R_{j_k}) \right) = \sum_{k=1}^{p} s^{-1} \left( s R_k \left[ \bar{b}_{p-k+1} \right] \right) \]

holds for all positive integer \( p \). Then Equation (54) holds for integer \( n \). Thus \( \{\bar{b}_k\}_{k \geq 1} \cup \{\bar{R}_k\}_{k \geq 1} \) gives a homotopy Rota-Baxter algebra structure of weight \( \lambda \) on \( V \).

\textbf{Acknowledgements:}\ The authors were supported by NSFC (No. 11971460, 12071137) and by STCSM (No. 18dz2271000).

The authors are grateful to Jun Chen, Xiaojun Chen, Li Guo, Yunnan Li, Zihao Qi, Yunhe Sheng, Rong Tang etc for many useful comments. Li Guo read carefully part of this paper and gave very detailed suggestions and kind encouragements. Xiaojun Chen imposed a question which led to Subsection 6.3 and Zihao Qi added a remark which simplified the proof of Lemma 10.9. Dotensko kindly gave an alternative proof of Theorem 10.4 which is reproduced.
in Remark 10.12. Das draw our attention to his papers [14, 15]. We are very grateful to these researchers for their interests and comments.

The authors lectured about this paper in various occasions, in particular, at ICCM meeting in December 2020, at Capital Normal University in January 2021, at Southeast University in May 2021, at Beijing Normal University and at Northeast Normal University in June 2021 etc. We would like to express our sincere gratitude to the organisers for the invitations and their useful remarks.

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