Real Forms of Non-abelian Toda Theories
and their \(\mathcal{W}\)-algebras

J.M. Evans\(^1\), DAMTP, University of Cambridge\(^1\)
and
J.O. Madsen\(^2\), Laboratoire de Physique Théorique ENSLAPP\(^2\), Groupe d’Annecy\(^3\)

Abstract
We consider real forms of Lie algebras and embeddings of \(sl(2)\) which are consistent with the construction of integrable models via Hamiltonian reduction. In other words: we examine possible non-standard reality conditions for non-abelian Toda theories. We point out in particular that the usual restriction to the maximally non-compact form of the algebra is unnecessary, and we show how relaxing this condition can lead to new real forms of the resulting \(\mathcal{W}\)-algebras. Previous results for abelian Toda theories are recovered as special cases. The construction can be extended straightforwardly to deal with \(osp(1|2)\) embeddings in Lie superalgebras. Two examples are worked out in detail, one based on a bosonic Lie algebra, the other based on a Lie superalgebra leading to an action which realizes the \(N = 4\) superconformal algebra.

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\(^1\)Silver Street, Cambridge CB3 9EW, UK.
\(^2\)e-mail: madsen@lapphp.in2p3.fr
\(^3\)URA 14-36 du CNRS, associée à l’Ecole Normale Supérieure de Lyon et à l’Université de Savoie
\(^4\)Chemin de Bellevue BP 110, F-74941 Annecy-le-Vieux Cedex, France.
1 Introduction

Over the last twelve years there has been intensive effort devoted to the study of two-dimensional, conformally-invariant quantum field theories. The tractability of these models relies on the fact that they contain infinite-dimensional chiral symmetry algebras which organize the states or operators into manageable numbers of representations. Such chiral algebras always contain the Virasoro algebra, but they may also incorporate supersymmetries, Kac-Moody currents, or other, higher-spin, quantities. The generators of these algebras typically do not close onto linear combinations of themselves, but rather onto non-linear expressions, and symmetries of this general type have come to be known as $\mathcal{W}$-algebras [1].

There are field theory models which realize $\mathcal{W}$-symmetries: these are the so-called reduced WZW models or non-abelian Toda theories. They are constructed using a version of Hamiltonian reduction [2, 3] which can be implemented at the Lagrangian level by gauging a WZW model [4] (whereas the original construction of Drinfeld and Sokolov can be applied directly to a Kac-Moody algebra, without ever mentioning a Lagrangian). The construction results in actions of the general form

$$S(g) = \frac{1}{\kappa} \left[ S_{\text{WZW}}(g) - \int d^2 z \text{Tr} \{ M_+ g M_- g^{-1} \} \right]. \quad (1.1)$$

$G$ is some real Lie group with corresponding real Lie algebra $\mathbf{g}$; the field $g$ takes values in some subgroup $G_0 \subset G$ with corresponding Lie algebra $\mathbf{g}_0$; $M_\pm$ are specially chosen elements of $\mathbf{g}$; and $\kappa$ is a coupling constant. The first term is the usual WZW action [5]

$$S_{\text{WZW}}(g) = \frac{1}{2} \int d^2 z \text{Tr} (\partial g \bar{\partial} g^{-1}) + \frac{1}{3} \int_D \text{Tr} (g^{-1}dg)^3 \quad (1.2)$$

in which two-dimensional space-time with (complexified) coordinates $(z, \bar{z})$ is regarded as the boundary of a three-dimensional disc $D$. This is invariant under a pair of commuting $\hat{G}_0$ Kac-Moody algebras, but the potential term involving $M_\pm$ may break some or all of this symmetry, so that the theory as a whole has only some smaller Kac-Moody invariance (at least as far as the linearly realized symmetries are concerned). In the case when the subalgebra $\mathbf{g}_0$ is abelian one recovers, for certain choices of $M_\pm$, the ‘standard’ or ‘abelian’ Toda theories based on the algebra $\mathbf{g}$. The ways in which $\mathbf{g}_0$ and $M_\pm$ can be chosen for a given $\mathbf{g}$ will be discussed in more detail below.

The specific issues we shall consider in this paper are: the freedom that is available in choosing different real forms for the groups $G$ and $G_0$, or algebras $\mathbf{g}$ and $\mathbf{g}_0$, appearing in this construction; and the effect that these choices have on the $\mathcal{W}$-algebra arising in the model. In the standard presentations of reduced WZW theories (see [4] and references given there) attention has been confined almost exclusively to the case in which $\mathbf{g}$ is chosen to be a maximally non-compact or split Lie algebra. The issues we shall address arise from the possibility of taking various other real forms for $\mathbf{g}$. It is perhaps not widely appreciated that these additional possibilities exist, but in fact it is not difficult to see that the standard arguments for integrability [4, 6] can be extended to cover these circumstances, and they are certainly encompassed by the most general reduction schemes set out in [4]. The new integrable theories that emerge are interesting from a number of points of view.

First, it is natural to anticipate that choosing different real forms for $\mathbf{g}$ will lead to different real forms of the resulting $\mathcal{W}$-algebra, and we shall see below that this is indeed...
the case. By comparison with finite-dimensional Lie algebras, one expects these real forms to differ in some profound ways—as regards their representation theory, for example. Such issues have recently begun to be investigated in the literature for the simpler case of finite $\mathcal{W}$-algebras [7].

Second, the construction we shall describe gives a systematic way of finding consistent ‘non-standard reality conditions’ for Toda theories. For the special case of abelian Toda theories it has already been pointed out some time ago [3] that alternative reality conditions are permissible consistent with integrability. As well as extending such results to the case of non-abelian Toda, the work here explains the connection between these non-standard reality conditions and Hamiltonian reduction, an issue which was never addressed in [3].

Third, the ideas can be immediately extended to the reduction of Lie superalgebras, and they then play an important part in constructing Lagrangian models with extended superconformal symmetry. It was shown in [3] how non-standard reality conditions are needed to obtain Toda theories with the correct $N = 2$ superconformal invariance. We shall see that similar considerations for non-abelian Toda models allow the construction of a Lagrangian theory which is invariant under the ‘large’ $N = 4$ superconformal algebra [9].

In the next section we discuss in more detail the class of models in which we are interested, and the general approach to constructing new variations of them. The subsequent section applies these ideas to two examples and we conclude with some suggestions for future work.

2 Integrable models and real forms

There is a wide class of integrable systems which can be defined by a Lagrangian of the form (1.1) and the all-important property of integrability can be established in a variety of different ways. Thus in [4] such models are obtained by gauging, or constraining certain currents in the WZW theory associated with $G$, while the original approach [6] makes use of Lax pairs for the resulting partial differential equations. The class of models we shall consider here is by no means the most general, but it will be sufficiently broad to explain our ideas and derive some new results.

Following closely the approach set out in [3] and [13, 14] we can define a conformally-invariant field theory of the form (1.1) by specifying two things:

(i) An embedded $sl(2)$ subalgebra of $G$. It is crucial to understand that we mean here an embedding of the real Lie algebra $sl(2) \equiv sl(2, \mathbb{R})$ into the real Lie algebra $G$ (the relationship with complex embeddings will be discussed shortly). Let the generators of this $sl(2)$ subalgebra be $M_0, M_\pm$ obeying $[M_0, M_\pm] = \pm M_\pm, [M_+, M_-] = M_0$. The eigenvalues of $M_0$ must be integers or half-integers, in which cases we say that the embedding is integral or half-integral respectively. The algebra $G$ can be decomposed into finite-dimensional irreducible representations of this embedded $sl(2)$, each labeled by its spin, $j$, in the usual way.

(ii) A compatible, integral and non-degenerate grading of $G$. By a compatible grading we mean a choice of some generator $H = M_0 + Y$ in $G$ such that $Y$ commutes with the entire embedded $sl(2)$: $[Y, M_0] = [Y, M_\pm] = 0$. Under the adjoint action of $H$, the algebra $G$ decomposes into eigenspaces $G = \oplus_n G_n$ labeled by their eigenvalues $n$. The condition
for an integral grading is that these eigenvalues are integers, irrespective of whether the $sl(2)$ subalgebra is defined via an integral or half-integral embedding. We can now distinguish the zero-grade part $G_0$ which has the special property of being a subalgebra of $G$ and to which we can therefore associate a corresponding subgroup $G_0$ of $G$. When $G$ is decomposed into irreducible representations of the embedded $sl(2)$, each of which has a certain spin $j$, the generator $Y$ must be constant on each such irreducible representation, with eigenvalue $y$, say. The condition for a non-degenerate grading is that this eigenvalue should not exceed the spin: $|y| \leq j$ on each irreducible representation [13]. The two pieces of data (i) and (ii) provide all the ingredients necessary to define an integrable, conformally-invariant model of type [14]. Our task now is to characterize such embeddings and gradings in a way that is concrete enough to allow us to calculate the resulting Lagrangians explicitly.

We have already emphasized that the construction rests on finding suitable embeddings of real Lie algebras. Nevertheless, it is very convenient to approach this via the corresponding problem for complex Lie algebras. To avoid any possible confusion, we shall always use a superscript ‘c’ to denote a complex Lie algebra. Thus, we shall write $sl(2)$ to mean the real Lie algebra $sl(2,\mathbb{R})$ (as introduced above) and $sl(2)^c$ to mean the complex Lie algebra $sl(2,\mathbb{C})$. More generally, given a real Lie algebra $G$, its complexification will be denoted $G^c$.

The theory of embeddings of $sl(2)^c$ into a complex Lie algebra $G^c$ is very well-established and dates back to Dynkin [10]. Let us introduce a Cartan-Weyl basis $t_a$ for $G^c$ which consists of Cartan generators $H_i$ together with step operators $E_{\alpha i}$ for each root $\alpha$. These obey $[H_i, E_{\alpha j}] = (\alpha_i \cdot \alpha) E_{\alpha j}$ and $[E_{\alpha i}, E_{-\alpha j}] = \delta_{ij} H_i$ where $\alpha_i$ are the simple roots. There is an essentially unique principal embedding in any $G^c$, in which $M_\pm$ are taken to be linear combinations of step operators for all the positive/negative simple roots:

$$M_0 = \sum_i \kappa_i H_i, \quad M_\pm = \sum_i \sqrt{\kappa_i} E_{\pm \alpha_i}$$  \hspace{1cm} (2.3)

where $\kappa_i = \sum_j K_{ij}^{-1}$ and $K_{ij} = \alpha_i \cdot \alpha_j$. The added significance of this type of embedding is that all other possible $sl(2)^c$ embeddings in $G^c$ are given (up to a number of exceptions for the D-type and E-type algebras) by principal embeddings in some regular subalgebra $H^c \subset G^c$, ie. a subalgebra whose roots are a subset of the roots of $G^c$.

To pass from complex to real Lie algebras, we use the idea of an automorphism $\tau$ of $G^c$ which is anti-linear, meaning $\tau(a_1 \Phi_1 + a_2 \Phi_2) = a_1^* \tau(\Phi_1) + a_2^* \tau(\Phi_2)$ for any $\Phi_1, \Phi_2 \in G^c$ and $a_1, a_2 \in \mathbb{C}$, and involutive, meaning $\tau^2 = 1$ (also called an involutive semimorphism). Such an automorphism essentially corresponds to a notion of complex conjugation on $G^c$, and so it can be used to define a real Lie algebra consisting of those elements which are fixed by the automorphism: $\tau(\Phi) = \Phi$. Any real form $G$ can be obtained from $G^c$ in this way. (For more details, see eg. [14].)

To elaborate on this, consider an element in $G^c$ written in terms of the Cartan-Weyl basis introduced above:

$$\Phi = \sum_a \phi_a t_a$$  \hspace{1cm} (2.4)

for arbitrary complex parameters $\phi_a$. We can define an automorphism $\tau$ by its action on the basis $t_a$, checking that $\tau^2 = 1$, and then extending to the whole complex algebra using
the property of anti-linearity. This means that we can specify \( \tau \) by writing
\[
\tau(t_a) = \sum_b t_b \tau_{ba}
\]
for some matrix \( \tau_{ab} \). By definition, the real form \( \mathcal{G} \) corresponding to \( \tau \) consists precisely of the elements \( \Phi \) in (2.4) for which
\[
(\phi_a)^* = \sum_b \tau_{ab} \phi_b.
\]
There are two simple possibilities which illustrate this. One obvious choice is to take \( \tau(t_a) = t_a \). The associated real Lie algebra is then, by definition, made up of all combinations (2.4) in which the parameters \( \phi_a \) are real. This gives the \textit{maximally non-compact}, or \textit{split}, real form. A second possibility is to take \( \tau(H_i) = -H_i \), \( \tau(E_{\pm \alpha}) = -E_{\mp \alpha} \).

This results in the \textit{compact} real form. These two real forms exist for any complex Lie algebra \( \mathcal{G}^c \) but in general there will be many others. For examples, see [12].

Using this point of view, we have a natural way to construct \( sl(2) \) embeddings (and gradings) of a real Lie algebra \( \mathcal{G} \). We first consider an embedding of \( sl(2)^c \) (and a grading) corresponding to a regular sub-algebra \( \mathcal{H}^c \subset \mathcal{G}^c \). Then, using an automorphism \( \tau \) to define the real forms \( \mathcal{H} \subset \mathcal{G} \), we need only check that the embedding and grading are consistent with the automorphism, in the sense that
\[
\tau(M_{\pm}) = M_{\pm}, \quad \tau(M_0) = M_0, \quad \tau(Y) = Y.
\]
Notice that \( \tau(t_a) = t_a \), giving the maximally non-compact form, always satisfies these consistency conditions, and it is this case that is the focus of most of the literature. By contrast, the compact real form, given by (2.7), is never consistent with (2.8). Our main interest here is to explore what happens for real forms which lie somewhere between these two extremes.

The consistency conditions (2.8) imply that \( \tau \) restricts to an automorphism of \( \mathcal{G}_0^c \subset \mathcal{G}^c \), and so the equations (2.4)–(2.6) apply in just the same fashion to this subalgebra, defining a real form \( \mathcal{G}_0 \) by specifying the allowed elements \( \Phi \). The corresponding group elements (in some neighbourhood of the identity) can be written \( g = \exp \Phi \) and in specific cases the Lagrangian (1.1) can be calculated explicitly. Despite the modified reality conditions defined by (2.6), the entire Lagrangian is real, by construction.

### 2.1 Abelian Toda theories

The simplest possible examples arise when we consider a principal embedding of \( sl(2)^c \) in \( \mathcal{G}^c \). Whatever real form \( \mathcal{G} \) we choose, \( \mathcal{G}_0 \) will be abelian, corresponding to a Cartan subalgebra of \( \mathcal{G} \), which can be parameterized by a set of Toda fields \( \phi_i \), each associated with a simple root. If \( \mathcal{G} \) is maximally non-compact, then all the fields \( \phi_i \) are real. For some algebras, however, there may be a non-trivial outer automorphism \( \tau \) which we can define by a permutation \( \sigma \) of the simple roots:
\[
\tau(H_i) = H_{\sigma(i)}, \quad \tau(E_{\pm \alpha}) = E_{\mp \sigma(i)}.
\]
The condition (2.6) then implies \( \phi_i^* = \phi_{\sigma(i)} \), and we recover exactly the construction introduced previously in [3].
2.2 \( \mathcal{W} \)-algebras

We recall (without justification) how the generators of \( \mathcal{W} \)-symmetry appear in the framework of Hamiltonian reduction. Concentrating on the holomorphic sector of the original WZW model, there is a conserved Kac-Moody current \( J \) with values in \( \mathcal{G} \). In passing to the reduced theory, the structure of this current can be fixed by choosing the highest weight gauge. In general we have \( J_{hw} = M_+ + \sum_{a \in \Lambda} W^a t_a \), where the set \( \Lambda \) specifies those generators \( t_a \) which are highest-weight vectors of the embedded \( sl(2) \), and \( W^a \) are the desired conserved quantities. By analogy with (2.6), one can show that the induced reality conditions for the \( \mathcal{W} \)-algebra generators are

\[
(W^a)^* = \sum_b \tau_{ab} W^b
\]  

(2.9)

where \( \tau_{ab} \) is defined in equation (2.5). Notice that the energy momentum tensor always remains real and that the conformal weights of any \( \mathcal{W} \)-generator are unaffected by the choice of real form.

This makes explicit how different real forms of the \( \mathcal{W} \)-algebra arise but we should also point out that some care may be needed in interpreting these conditions. If we regard the theory as defined on Minkowski space (with the spatial direction compactified to form a cylinder) then the interpretation is clear. But if we pass via a conformal transformation to the complex plane, then the notions of reality, complex conjugation etc. are replaced by various hermiticity conditions on fields (depending on their conformal weights) or on their modes (see eg. [21]). It is this latter interpretation which will be relevant when we write down operator product expansions in the examples below.

2.3 Supersymmetric models

We have described how bosonic integrable models arise from embeddings of \( sl(2) \) into Lie algebras; to construct supersymmetric integrable models we can consider instead embeddings of \( osp(1|2) \) into real Lie superalgebras \( \mathcal{G} \) (see [14] for a classification with comprehensive references; related work can be found in [16, 17, 18, 19]). The resulting non-abelian Toda theories are defined in \( N = 1 \) superspace with coordinates \( Z = (z, \theta) \) and derivatives \( D = \theta \partial_z + \partial_\theta \) (and similarly in the anti-holomorphic sector). The general superspace action is a natural extension of (1.1) in which \( S_{WZW}(g) \) appears as an \( N = 1 \) version of the WZW model [20, 15] for a super-group-valued superfield, and in which the superpotential takes the form \( \text{Str}\{M_+gM_-g^{-1}\} \) where \( M_\pm \) are the fermionic generators of \( osp(1|2) \) defining the embedding. One simplifying feature of the supersymmetric case is that we can, without loss of generality, set \( Y = 0 \) so that the embedding fixes the grading uniquely. Embeddings of \( osp(1|2) \) are also classified (up to a few exceptions) as super-principal embeddings into regular sub-superalgebras. The existence of different integrable models corresponding to different real forms of a complex superalgebra \( \mathcal{G}^c \) can therefore be investigated just as in the bosonic case, by searching for automorphisms which respect the \( osp(1|2) \) embedding.

3 Non-abelian examples

Having discussed the general ideas and how these apply in the abelian case, we now wish to consider a couple of non-abelian examples. For simplicity we fix \( Y = 0 \) in the bosonic
case. We can then define related non-abelian Toda theories by a pair \((G^c, H^c)\) which specifies a \(sl(2)^c\) (or \(osp(1|2)^c\)) embedding in \(G^c\) as a (super-)principal embedding in a regular subalgebra \(H^c\). We denote the corresponding complex \(W\)-algebra by \(W(G^c, H^c)\) and we shall see below how the real form of this algebra changes for suitable choices of automorphisms \(\tau\). We shall be dealing exclusively with classical \(W\)-algebras but it is convenient, nevertheless, to present them using the language of operator product expansions.

### 3.1 A bosonic example: \((so(5)^c, so(3)^c)\)

This is the simplest example of a non-principal embedding into a classical Lie algebra which meets all our requirements for the existence of a new real form (including the simplifying assumption \(Y = 0\)). Let the long and short simple roots of \(so(5)^c\) be \(\alpha_1\) and \(\alpha_2\) respectively. We take the \(sl(2)^c\) embedding in \(so(5)^c\) defined by the short root \(\rho\), i.e. we choose \(M_\pm = E_{\pm \alpha_2}\) and \(M_0 = H_2\), which indeed defines an integral grading. The zero-grade subalgebra is \(G_0^c = so(3)^c \oplus gl(1)^c\) with elements \(\Phi + uM_0\), where

\[
\Phi = (\phi_-/\sqrt{2})E_{-(\alpha_1+\alpha_2)} + \phi_0(H_1 + H_2) + (\phi_+/\sqrt{2})E_{\alpha_1+\alpha_2}
\]

(3.10)

parametrizes the non-abelian factor. The group-elements which appear in the action can be written \(g = \exp(\Phi)\exp(um_0)\).

The maximally non-compact real form is \(G = so(3, 2)\) with \(G_0 = so(2, 1) \oplus gl(1)\). This is realized by taking \(\phi_0, \phi_\pm\) and \(u\) to be real and the WZW part of the resulting action is

\[
S_{WZW}(g) = S^{so(3, 1)}_{WZW} + S^{gl(1)}_{WZW} = S^{so(3)}_{WZW} - S^{gl(1)}_{WZW}
\]

But an alternative real form can be defined by using the automorphism

\[
\tau(H_1) = -(H_1 + 2H_2), \quad \tau(H_2) = H_2, \\
\tau(E_{\pm \alpha_1}) = -E_{\mp \alpha_1 \mp 2\alpha_2}, \quad \tau(E_{\pm \alpha_2}) = E_{\pm \alpha_2}.
\]

(3.11)

It is simple to check that this is involutive and compatible with the \(sl(2)^c\) embedding. The real Lie algebra which it defines is \(G = so(1, 4)\) and the zero-grade subalgebra is \(G_0 = so(3) \oplus gl(1)\). This follows from (2.3) applied to (3.10) and (3.11):

\[
(\phi_+)^* = -\phi_- \Rightarrow \phi_\pm = \pm \phi_1 + i\phi_2, \\
(\phi_0)^* = -\phi_0 \Rightarrow \phi_0 = i\phi_3,
\]

(3.12)

with \(\phi_i (i = 1, 2, 3)\) real. With this new choice of real form the WZW action is

\[
S_{WZW}(g) = S^{so(3)}_{WZW} + S^{gl(1)}_{WZW} = S^{so(3)}_{WZW} - S^{gl(1)}_{WZW}
\]

The first term is the WZW action for the compact real algebra \(so(3)\) and as such it has positive-definite kinetic part, in contrast to the previous case. However, we must emphasize that for \textit{either} real form the second term is the action for a single free field with an overall \textit{negative} sign. This relative minus sign between the two terms cannot be altered. The potential term can also be calculated, and we find

\[
\text{Tr}\{M_+ gM_- g^{-1}\} = \frac{2e^{-u}}{\rho^2} \left[ \phi_0^2 + \phi_+ \phi_- \cosh \rho \right], \quad \rho^2 = \phi_0^2 + \phi_+ \phi_- ,
\]

\[
= \frac{2e^{-u}}{\phi^2} \left[ \phi_3^2 + (\phi_1^2 + \phi_2^2) \cos \phi \right], \quad \phi^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 .
\]

\[1\text{This reduction has also been considered in }[22]\text{ and, from a rather different point of view, in }[23].\]
It is clearly real, as we know it must be, for either choice of real form.

The classical algebra $\mathcal{W}(so(5)^c, so(3)^c)$ contains 4 generators: the energy-momentum tensor $T$, a $U(1)$ generator $U$, and two spin-two primary fields $V^\pm$, with $U(1)$-charge $\pm 1$. They appear in the expression for the current in the highest weight gauge as follows:

$$J_{hw} = M_\tau + T E_{\alpha_2} + U(H_1 + H_2) + V^+ E_{\alpha_1 + 2\alpha_2} + V^- E_{-\alpha_1}.$$ 

In addition to the standard operator products involving the energy-momentum tensor, we find

$$U(z)U(w) \sim \frac{k}{(z-w)^2}, \quad U(z)V^\pm (w) \sim \frac{\pm V^\pm}{z-w},$$

$$V^+(z)V^-(w) \sim \frac{-6k^3}{(z-w)^4} + \frac{-6k^2U}{(z-w)^3} + \frac{-4kU^2 + 2k^2T - 3k^2 \partial U}{(z-w)^2}$$

$$+ \frac{2kUT - 2U^3 + k^2 \partial T - k^2 \partial^2 U - 4kU \partial U}{z-w},$$

where $k$ is related to the coupling $\kappa$, and on the right-hand side the argument $w$ is implied. For the maximally non-compact form, these generators are all real. But for the new real form defined by $\tau$ in (3.11), we follow the prescription (2.9) and find that (in addition to $T^* = T$) we have

$$(V^+)^* = -V^- \quad \Rightarrow \quad V^\pm = \pm V^1 + iV^2$$

$$U^* = -U \quad \Rightarrow \quad U = iJ$$

where the quantities $V^i$ $(i = 1, 2)$ and $J$ are now real. This results in an algebra

$$J(z)J(w) \sim \frac{-k}{(z-w)^2}, \quad J(z)V^i(w) \sim \frac{\epsilon^{ij}V^j}{z-w},$$

$$V^i(z)V^j(w) \sim \frac{3k^3 \delta^{ij}}{(z-w)^4} - \frac{3k^2 \epsilon^{ij} J}{(z-w)^3} - \frac{\delta^{ij}(2kJ^2 + k^2 T) + \frac{3}{2} k^2 \epsilon^{ij} \partial J}{(z-w)^2}$$

$$- \frac{\delta^{ij} \left( \frac{1}{2} k^2 \partial T + 2kJ \partial J \right) + \epsilon^{ij} (-kJ - J^3 + \frac{1}{2} k^2 \partial^2 J)\}}{z-w}.$$ 

Notice that for the compact real form, the potential and the operator-product expansions are manifestly invariant under an $so(2)$ symmetry (generated by $J$) rather than under the larger $so(3)$ symmetry of the WZW action. This is in keeping with a remark made in the introduction.

### 3.2 A supersymmetric example: $(osp(4|2)^c, osp(2|2)^c)$

This is not the simplest example, but it has a number of especially interesting features. We choose as simple roots of $osp(4|2)^c$ bosonic roots $\alpha_1$ and $\alpha_2$ corresponding to an $so(4)^c$ subalgebra, together with a single fermionic root $\alpha_3$. The embedded $osp(1|2)^c$ is defined by its fermionic generators $M_\pm = E_{\pm (\alpha_1 + \alpha_2 + \alpha_3)} + E_{\pm \alpha_3}$, and the zero-grade subalgebra is
then $G_0^c = so(4)^c \oplus gl(1)^c = so(3)^c \oplus so(3)^c \oplus gl(1)^c$. We write a general element of this algebra in the form $\Phi + \Phi' + uM_0$ where

$$
\Phi = \phi^- E_{-a_1} + \phi_0 H_1 + \phi_+ E_{a_1} , \quad \Phi' = \phi'_- E_{a_2} - \phi'_0 H_2 + \phi'_+ E_{-a_2} , \quad (3.13)
$$

parametrizes the non-abelian factors. The super-group element appearing in the action is therefore $g = \exp(\Phi) \exp(\Phi') \exp(uM_0)$.

There is, as always, the option of choosing the split real form, but the main interest for us lies in the existence of another real form defined by the automorphism

$$
\tau(H_1) = -H_1 \quad \tau(E_{\pm a_1}) = -E_{\mp a_1} \\
\tau(H_2) = -H_2 \quad \tau(E_{\pm a_2}) = -E_{\mp a_2} \\
\tau(H_3) = H_3 \quad \tau(E_{\pm a_3}) = E_{\pm(a_1+a_2+a_3)} \quad (3.14)
$$

It is easy to verify that this is involutive, and we see immediately that it is compatible with the embedded $osp(1|2)^c$. The corresponding real Lie superalgebra has a zero-grade subalgebra $so(4) \oplus gl(1)$, with compact non-abelian part and the appropriate reality conditions for the fields appearing in $\Phi$ and $\Phi'$ above are just those written in $\{3.12\}$ in the last section.

These conditions lead to a WZW action

$$
S_{WZW}(g) = S_{WZW}^{so(4)} - S_{WZW}^{gl(1)} = S_{WZW}^{so(4)} + S_{WZW}^{gl(1)}
$$

and it is very striking that this is positive-definite. This was not the case in the previous bosonic example, despite the emergence there of a compact non-abelian factor, because the single free scalar field still appeared in the action with the wrong sign. In this example, however, there is a super-trace involved in defining the WZW action on a Lie super-group, and this contributes exactly the additional minus sign required to make the scalar field term positive. The potential term can also be computed:

$$
\text{Str}\{M_+ g M_- g^{-1}\} = 4 e^{-u} \left[ \cos \phi \cos \phi' + \frac{1}{\phi_0 \phi'_0} (\phi_1 \phi'_1 + \phi_2 \phi'_2 + \phi_3 \phi'_3) \sin \phi \sin \phi' \right]
$$

where we use the same notation as in $\{3.12\}$ for the parameters in $\Phi$ and $\Phi'$, with $\phi^2 = \sum_i \phi_i^2$ and $(\phi')^2 = \sum_i (\phi'_i)^2$. Once again we confirm that the result is real.

Let us now consider the implications for the $\mathcal{W}$-symmetries of this model. The complex algebra $\mathcal{W}(osp(4|2)^c, osp(2|2)^c)$ is equivalent to the complex version of the so-called ‘large’ $N = 4$ algebra, which contains an $so(4)$ Kac-Moody symmetry [14, 15]. But, since we are working in an $N = 1$ superspace formalism, this algebra will arise in an unusual basis consisting of the super-stress tensor $T$, three spin-one superfields $G^0, G^\pm$ and three spin-half superfields $J^0, J^\pm$. In detail, these appear in the expression for the current in the highest weight gauge as follows:

$$
J_{hw} = M_- + J^-(E_{a_1} + E_{-a_2}) + J^+(E_{a_2} + E_{-a_1}) + J^0 \frac{1}{2}(H_2 - H_1) + G^- E_{a_1+a_3} + G^+ E_{a_2+a_3} + \frac{G^0}{2} (E_{a_1+a_2+a_3} - E_{a_3}) + TE_{a_1+a_2+2a_3}.
$$

To recover the usual generators of the $N = 4$ superconformal algebra, it is possible to expand these superfields in ordinary fields and to factorize the three spin-half fermions...
but we shall not pursue those details here. Omitting, for brevity, the standard
operator products involving the super-stress tensor, we find

\[ J^0(Z)J^\pm(W) \sim \frac{\theta - \eta}{Z - W} \left\{ \pm J^\pm \right\} \]
\[ J^0(Z)J^0(W) \sim \frac{2k}{Z - W} \]
\[ J^+(Z)J^-(W) \sim \frac{k}{Z - W} + \frac{\theta - \eta}{Z - W} \left\{ \frac{1}{2} J^0 \right\} \]
\[ J^0(Z)G^\pm(W) \sim \frac{\theta - \eta}{Z - W} \left\{ \pm G^\pm \right\} \]
\[ J^\pm(Z)G^0(W) \sim \frac{\theta - \eta}{Z - W} \left\{ G^\pm \right\} \]
\[ G^\pm(Z)G^\pm(W) \sim -\frac{\theta - \eta}{Z - W} \left\{ \frac{1}{2k} DJ^\pm J^\pm \right\} \]
\[ G^0(Z)G^0(W) \sim \frac{2k}{(Z - W)^2} + \frac{\theta - \eta}{Z - W} \left\{ 2T - \frac{1}{2k} J^0 D J^0 \right\} \]
\[ G^0(Z)G^\pm(W) \sim -\frac{\theta - \eta}{(Z - W)^2} \left\{ J^\pm \right\} - \frac{D J^\pm + \frac{1}{2k} J^0 J^\pm}{Z - W} - \frac{\theta - \eta}{Z - W} \left\{ \partial J^\pm + \frac{1}{2k} J^0 D J^0 \right\} \]
\[ G^+(Z)G^-(W) \sim -\frac{k}{(Z - W)^2} - \frac{\theta - \eta}{(Z - W)^2} \left\{ \frac{1}{2} J^0 \right\} - \frac{\frac{1}{2} DJ^0 + \frac{1}{2k} J^- J^+}{Z - W} \]
\[ -\frac{\theta - \eta}{Z - W} \left\{ T + \frac{1}{2} \partial J^0 - \frac{1}{2k} J^- J^+ \right\} \]

where \( k \) is related to the coupling \( \kappa \), the ‘covariant difference’ in
superspace is \( Z - W = z - w - \theta \eta \), and the argument \( W \) of the operators is implied on the right hand side.

For the reduction of the maximally non-compact real superalgebra, all the generators
introduced above would be real. When we change the real form using \( \tau \) on the other
hand, we find, following the prescription \( \text{(2.9)} \), that the reality conditions of the generators
become:

\( (J^-)^* = -J^+ \Rightarrow J^\pm = \frac{1}{2} (\pm J^1 + \imath J^2) \)
\( (J^0)^* = -J^0 \Rightarrow J^0 = \imath J^3 \)
\( (G^-)^* = -G^+ \Rightarrow G^\pm = \frac{1}{2} (G^1 \pm \imath G^2) \)
\( (G^0)^* = -G^0 \Rightarrow G^0 = -\imath G^3 \)

where \( J^i \) and \( G^i \) \((i = 1, 2, 3)\) are real. With these definitions the operator products
simplify considerably:

\[ J^i(Z)J^j(W) \sim -\frac{2k \delta^{ij}}{Z - W} + \frac{\theta - \eta}{Z - W} \left\{ \epsilon^{ijk} J^k \right\} \]
\[ J^i(Z)G^j(W) \sim \frac{\theta - \eta}{Z - W} \left\{ \epsilon^{ijk} G^k \right\} \]
\[ G^i(Z)G^j(W) \sim -\frac{2k \delta^{ij}}{(Z - W)^2} + \frac{\theta - \eta}{(Z - W)^2} \left\{ \epsilon^{ijk} J^k \right\} + \frac{\epsilon^{ijk} D J^k - \frac{1}{2k} J^j J^i}{Z - W} \]
\[ + \frac{\theta - \eta}{Z - W} \left\{ -2 \delta^{ij} T + \epsilon^{ijk} \partial J^k - \frac{1}{2k} D J^i J^j \right\} . \]
It is this real form which is related to the standard $N = 4$ superconformal algebra. It is interesting that both the potential and the operator product expansions written above are manifestly invariant under an $so(3)$ symmetry corresponding to the currents $J^i$, whereas the WZW action is based on the larger algebra $so(4)$. In this case we know that the symmetry of the whole model is actually $so(4)$, but in the $N = 1$ superspace formulation used here only the $so(3)$ subalgebra is linearly realized. In passing to component fields, a second copy of $so(3)$ emerges from the currents $G^i$.

4 Discussion

As we mentioned in the introduction, the study of Hamiltonian reduction has, up until now, been confined largely to the case of maximally non-compact or split real Lie algebras. This may be due partly to the fact that $sl(2)$ embeddings into maximally non-compact real Lie algebras have a particularly direct relationship to embeddings into complex Lie algebras, a subject which has been extensively investigated. But it seems also that it may not be universally appreciated that Hamiltonian reduction is possible at all for other real forms. In this paper we have introduced a method which allows us to study systematically the Hamiltonian reduction of arbitrary real Lie algebras (except for the compact Lie algebras, since these have no $sl(2, \mathbb{R})$ subalgebras) and we have made a number of points, illustrated by two examples, to explain why the resulting theories can be physically and mathematically interesting.

On the one hand, our method can be viewed as a way of finding various consistent reality conditions for the fields in a non-abelian Toda theory. On the other hand, we have explained how this corresponds to carrying out Hamiltonian reduction of a WZW model based on different real forms of the same complex Lie algebra, and, as a result, we have seen that it provides a technique for studying the different real forms of extended conformal algebras. In a following article [25], we shall consider in more detail the technical and mathematical aspects of these ideas. We will show in particular that all real forms of extended conformal algebras can be found using this method; we will also show that we can obtain explicitly all real forms of a complex Lie algebra compatible with a given Hamiltonian reduction and we shall develop a method to classify these.

There are a number of other interesting questions which we hope to pursue in the future; we mention just a few. First, there is the detailed study of the new forms of the $\mathcal{W}$-algebras we have been discussing, including their representation theory (see [1, 2] for some related work). Second, it would be worthwhile studying more systematically the reduction of real Lie superalgebras, which we have just touched on in this paper. The example we gave showed that there is the possibility to construct actions with positive-definite kinetic parts. Third, we have been considering exclusively the case of conformal reductions. It would be very interesting to extend our methods to massive Toda models, and to make contact with work such as [27], which implicitly contains a number of similar ideas. Finally, all the considerations here have been classical, and it is clearly important to confirm that these results can be transferred to the quantum case.

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