Entropy Production in Continuously Measured Quantum Systems

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Introduction.— Entropy production, a fundamental concept in non-equilibrium thermodynamics, provides a measure of the degree of irreversibility of a physical process. It is thus of paramount importance for the characterization of an ample range of systems across all scales, from macroscopic to microscopic [1–11].

The lack of a continuity equation for entropy prevents entropy production from being a physical observable, in general. Its quantification must thus pass through the construction of inference strategies that connect the values taken by entropy production to physically accessible observables, such as energy [12–14]. This approach has recently led to the possibility to experimentally measure entropy production in microscopic [15] and mesoscopic quantum systems [16]. It has also opened up intriguing opportunities for its experimental control [17]. Alternative approaches to the quantification of entropy production are based on the ratio between forward and time-reversed path probabilities of stochastic trajectories followed by systems undergoing non-equilibrium processes [18–20]. Quite an extensive body of work has been produced in an attempt to bypass the bottleneck embodied by such a lack of generally applicable theories of entropy production [11, 21]. Remarkably, this has allowed the identification of important contributions to the degree of irreversibility emerging from a given process stemming from system-environment correlations [17, 22, 23], quantum coherence [24–26], and potential finiteness of the environment with which the system interacts [23, 27].

Among the directions along which the ongoing developments of an entropy production framework should be extended, one which is particularly relevant is the inclusion of the back-action resulting from measuring a quantum system. Measurements can have a dramatic effect on both the state and the ensuing dynamics of a quantum system: while the randomness brought about by a quantum measurement adds stochasticity to the evolution of a system, the information gained through a measurement process unlocks effects akin to those of a Maxwell daemon [28, 29]. Both such features are intuitively expected to affect the entropy production and its rate [cf. Fig. 1]

The ways such modifications occur have been the focus of some attention recently. Ref. [29] tackled the problem by focusing on the stochastic energy fluctuations that occur during measurements, while Refs. [30–36] addressed the case of weak quantum measurements of a system, which allowed for the introduction of trajectory-dependent work and heat quantifiers. (See also Refs. [37, 38]). In line with a Landauer-like framework, Sagawa and Ueda focused on the minimum thermodynamic cost implied by a measurement, highlighting the information theoretical implications of the latter on the stochastic thermodynamics of a two-level system [39, 40], an approach that can both be generalized to general quantum measurements [41] and assessed experimentally [42].

In this Letter we contribute to the quest for a general framework for entropy production and its rate in a general system brought out of equilibrium and being continuously monitored.

In particular, in an attempt at addressing a situation of vast theoretical applicability and experimental interest, we sift through open Gaussian systems subject to continuous Gaussian measurements. Such a class of systems and processes plays a substantive role in the broad panorama of quantum optics, condensed matter physics, and quantum information science in general. Moreover, Gaussian measurements are some of the most widely used techniques in quantum labs, and their role in stochastic non-classical thermodynamics is thus both

![FIG. 1. (a) A quantum system – prepared in an arbitrary state – interacts with an environment, being possibly externally driven. The dynamics is associated with a (unconditional) rate of entropy production $\Pi_{uc}$. (b) The introduction of a continuous-measurement mechanism alters the dynamics of the system, resulting in a new entropy production rate $\Pi$ that will include a – in general non-zero – information-theoretical contribution $I$ determined by the amount of information extracted through the measurement process.](http://example.com/fig1.png)
interesting and physically very well motivated.

We lay down a widely applicable formalism able to identify the measurement-affected rate of entropy production and the associated flux of entropy to or from the environment that is connected to the monitored system. We nail down the thermodynamic consequences of measuring by way of both general arguments and specific case studies, showing that the entropy production rate can be split in a term that is intrinsically dynamical and one that is informational in nature. When pushed further, such splitting leads to a refined, and tightened, 2nd law for continuously measured Gaussian systems, and the demonstration of the possibility to control the non-equilibrium thermodynamics of a quantum system by means of suitable measurement strategies, as we illustrate by addressing the case of a thermal quench of an harmonic oscillator and a driven-dissipative optical parametric oscillator. Our approach paves the way to the assessment of the non-equilibrium thermodynamics of continuously monitored mesoscopic Gaussian systems — such as (ensembles of) trapped ions and quadratically-confined levitated optomechanical systems — whose energetics will require tools that are exquisitely designed to tackle the intricacies of quantum dynamics and the stochasticity of quantum measurements, which are currently lacking [43–51].

Continuously measured Gaussian systems.— The dynamics of a continuously measured open quantum system subjected to a Markovian evolution can be described by a Stochastic Master Equation (SME) that describes the evolution of the quantum state of the system conditioned on the outcomes of the continuous measurement. [52–54]. Upon averaging over all stochastic trajectories, weighted by the outcomes probabilities, the stochastic part vanishes leaving a deterministic Lindblad master equation for the open system dynamics, whose dynamics will be called unconditional throughout the paper.

While elegantly formulated, solving the SME is in general a tall order. Luckily, the intricacy of such an approach is greatly simplified when dealing with Gaussian systems, as their description can be reduced to the knowledge of the first two statistical moments of the quadratures of the system. The problem of solving the SME can thus be superseded in favor of a much simpler system of stochastic equations.

Let us consider a system of \( n \) modes, each described by a pair of quadrature operators \( \{\hat{q}_i, \hat{p}_i\} \) with \( [\hat{q}_i, \hat{p}_j] = i \), and define a vector of operators \( \mathbf{x} = \{\hat{q}_1, \hat{p}_1, \hat{q}_2, \ldots, \hat{q}_n, \hat{p}_n\} \). When restricting to Gaussian systems, the Hamiltonian appears at most quadratic in the quadrature operators and can be written in the general form \( \hat{H} = \frac{1}{2} \hat{\mathbf{x}}^T \hat{H}_c \hat{\mathbf{x}} + \mathbf{b}^T \Omega \mathbf{x} \), where \( \hat{H}_c \) is a \( 2n \times 2n \) matrix, \( \mathbf{b} \) is a \( 2n \)-dimensional vector accounting for a (possibly time-dependent) linear driving, and \( \Omega = \sum_{j=1}^{n} \sigma_{y,j} \) is the \( n \)-mode symplectic matrix (\( \sigma_{y,j} \) is the \( y \)-Pauli matrix of subsystem \( j \)).

When the environment is modelled by Lindblad generators that are linear in the quadratures of the system and the latter is monitored through Gaussian measurements, the SME preserves the Gaussianity of any initial state. In this case, the vector of average momenta \( \mathbf{\bar{x}} = \langle \mathbf{x} \rangle \) and the Covariance Matrix (CM) \( \sigma_{ij} = \langle (\mathbf{x}_i - \mathbf{\bar{x}}_i)(\mathbf{x}_j - \mathbf{\bar{x}}_j) \rangle / 2 - \langle \mathbf{x}_i \rangle \langle \mathbf{x}_j \rangle \) of the modes are sufficient to completely describe the dynamics, which is accounted for by the equations [53–55]

\[
\begin{align*}
\dot{\mathbf{x}} &= \mathbf{A} \mathbf{x} + \sigma \mathbf{x}^T \mathbf{D} - \chi(\mathbf{x}), \\
\mathbf{d}\mathbf{x} &= (\mathbf{A} \mathbf{x} + \mathbf{b}) \, dt + (\sigma \mathbf{x}^T + \mathbf{T}) \, d\mathbf{w},
\end{align*}
\]

where \( d\mathbf{w} \) is a \( 2\ell \)-dimensional vector of Wiener increments (\( \ell \) is the number of output degrees of freedom being monitored), \( \mathbf{A} \) (\( \mathbf{D} \)) is the drift (diffusion) matrix characterizing the unconditional open dynamics of the system, and \( \chi(\mathbf{x}) = (\sigma \mathbf{x}^T + \mathbf{T}) (\mathbf{x} + \mathbf{G}) \) \( \theta \) is defined in terms of the \( 2\ell \times 2n \) matrices \( \mathbf{C} \) and \( \mathbf{G} \) that describe the measurement process. While the explicit form of such matrices is inessential for our scopes [cf. Ref. [53–55] for details], it is useful to highlight the fact that the drift matrix \( A \) can be decomposed as \( A = \Omega H_c + A_{\text{int}} \), where the first term accounts for the unitary evolution and the second one for the diffusive one. Notwithstanding the stochasticity of the overall dynamics, the equation for the covariance matrix is deterministic. Thus, \( \sigma(t) \) does not depend on the explicit outcomes of the measurement (i.e., the trajectory followed by the system), while it depends on the specific measurement carried out [54]. The dynamics of the corresponding unconditional quantities \( \sigma_{uc} \) and \( \mathbf{x}_{uc} \) is achieved from Eqs. (1) by taking \( C = \Gamma = 0 \) with \( \Omega \) the null \( 2\ell \times 2n \) matrix. This then implies \( \chi(\mathbf{x}) = 0 \), so that one recovers the dynamical equation for the evolution of \( \sigma_{uc} \).

Entropy production rate and flux.— The conditional dynamics in Eqs. (1) can be conveniently cast in the phase space as the continuity equation [56]

\[
dW = -\text{div}[\mathbf{J}_{\text{rev}} - \mathbf{J}_{\text{sto}}],
\]

where \( W = e^{-\frac{1}{2}(\mathbf{x} - \mathbf{\bar{x}})^T \sigma^{-1}(\mathbf{x} - \mathbf{\bar{x}})}/(2\pi)^n \sqrt{\det \sigma} \) is the Wigner function associated with the state of the \( n \)-mode system and we have introduced the deterministic phase-space current \( J \), which can be divided as \( J = J_{\text{rev}} + J_{\text{int}} \), and its stochastic counterpart \( J_{\text{sto}} \). Here, \( J_{\text{rev}} = \Omega H_c \mathbf{x} + \mathbf{b} \), \( J_{\text{int}} = \mathbf{A}_{\text{int}} \mathbf{x} \), and \( \mathbf{dW} = (\sigma \mathbf{x}^T + \mathbf{T}) \, d\mathbf{w} \) accounts for the irreversible dissipative evolution. It should be noted that these currents are equal to the ones of the unconditional dynamics with the replacement \( W \to W_{\text{uc}} \), where \( W_{\text{uc}} \) is the Wigner function of the unconditional state obtained through the replacements \( \sigma \to \sigma_{uc} \) and \( \mathbf{x} \to \mathbf{x}_{uc} \). The stochastic term \( J_{\text{sto}} = W(\sigma \mathbf{x}^T + \mathbf{T}) d\mathbf{w} \) depends entirely on the conditional dynamics, through \( \sigma \) and \( W \), and the measurement strategy being chosen.

In order to characterize the entropy of the \( n \)-mode system we adopt the Wigner entropy \( S = -\int W \ln W \, d^n\mathbf{x} \) as our entropic measure. For Gaussian systems [57]

\[
S = -\text{ln}(\mathcal{P}) + k_n = \frac{1}{2} \text{ln}(\text{det } \sigma) + k_n,
\]

with \( k_n \) an inessential constant that depends only on the number \( n \) of modes involved and \( \mathcal{P} \) the purity of the state of the system, which for a Gaussian state reads \( \mathcal{P} = (\text{det } \sigma)^{-1/2} \). Therefore, \( S \) coincides (modulo the constant \( k_n \)) with the
Rényi-2 entropy and tends to the von-Neumann entropy in the classical limit. This quantity was recently used to build a quantifier for the entropy production rate of an unconditional Gaussian system that bypasses some of the controversies linked to more standard von Neumann entropy-based formulations [58]. As the Wigner entropy only depends on the CM of the system, its evolution is deterministic even for continuously measured system. The same then holds true for the *entropy rate* which is given by

\[
\frac{dS}{dt} = \frac{1}{2} \frac{d}{dt} (\text{Tr}[\log \sigma]) = \frac{1}{2} \text{Tr}[2A + \sigma^{-1}(D - \chi(\sigma))].
\]  

Let us focus for a moment on the unconditional case: the entrap part of the phase-space current as rate [11]. Both quantities crucially depends on the irreversible with expressions for these quantities written in terms of \(A\) in the supplementary information [56] we report explicit ex-

\[
\Phi_{\text{uc}} = -2 \int d^2x J_{\text{irr}}^T D^{-1} A_{\text{irr}} x, \quad \Pi_{\text{uc}} = 2 \int \frac{d^2x}{W_{\text{uc}}} J_{\text{irr}}^T D^{-1} J_{\text{irr}}.
\]  

In the supplementary information [56] we report explicit expressions for these quantities written in terms of \(A_{\text{irr}}, \sigma_{\text{uc}}\), and \(D\). For a system interacting with an environment in thermal equilibrium and at high temperature, \(\Phi_{\text{uc}}\) coincides with the energy flux from the system to the environment. Thus, this formalism generalizes the usual thermodynamic description in a meaningful manner.

For the continuously measured case, while Eq. (4) presents a deterministic quantity, both the entropy flux and production rate are inherently stochastic, as they depend on the first moments of the quadrature operators. It is however possible to show that the entropy rate can indeed be written as \(dS = d\Phi_{\text{uc}} + d\Pi_{\text{uc}}\) with \(d\Phi_{\text{uc}}, d\Pi_{\text{uc}}\) the conditioned trajectory-dependent entropy flux and production rates. Upon taking the average \(\mathbb{E}\) over the outcomes of the measurement, we can rewrite the entropy rate in a way akin to the unconditional counterpart, i.e.

\[
\mathbb{E}[d\Phi_{\text{uc}}/dt] + \mathbb{E}[d\Pi_{\text{uc}}/dt] = \Phi + \Pi,
\]  

which is suggestive of a direct link with the formal result valid in the case of no measurement. Indeed, as demonstrated in Ref. [56], Eq. (6) can be recast into the very elegant form

\[
\dot{S} = \dot{S}_{\text{uc}} + \dot{I},
\]  

where the term \(\dot{I}\) accounts for the excess entropy production resulting from the measurement process, and it is thus information theoretical in nature.

Indeed, by integrating Eq. (7), one immediately recognizes that \(\dot{I} = \ln(\mathcal{P}_{\text{uc}}/\mathcal{P})\), showing that \(\dot{I}\) quantifies the noise that one needs to add to the conditional state in order to bring it back to its unconditional form. As the purity of the conditional state is in general larger than that of the un-monitored one \(\mathcal{P}_{\text{uc}}\), we gather that \(\dot{I} \leq 0\). In fact, it can be further shown that \(\dot{I} = -I(\mathbf{X} : \mathbf{X}) \leq 0\), with \(I\) the classical mutual information between the phase-space position \(\mathbf{X}\) in the unconditional case and the stochastic first momenta \(\mathbf{X}\), which evolve according to the Itô equation in Eq. (1). The inequality is obviously saturated if and only if \(\sigma = \sigma_{\text{uc}}\) [56].

A straightforward calculation based on Eq. (4) leads to

\[
\dot{J} = \frac{1}{2} \text{Tr}[\sigma^{-1} D - \sigma_{\text{uc}}^{-1} D],
\]  

where, as it can be appreciated from Eq. (1), \(\dot{D} = D - \chi(\sigma)\) accounts for a modification to the diffusion matrix of the dynamics which occurs due to the measurement strategy being chosen: the measurement combines with the environment and acts on the system so as to modify its diffusive dynamics. Eq. (8) embodies clearly the effects of the continuous detection, which modifies the CM of the system and its open dynamics.

By comparing Eq. (6) and (7), and under the assumption that the stochastic entropy flux \(d\Phi_{\text{uc}}\) is linear in the conditioned state of the system (see below), so that \(\mathbb{E}[d\Phi_{\text{uc}}/dt] = \Phi_{\text{uc}}\), we also obtain a similar splitting for the entropy production rate, i.e.

\[
\dot{\Pi} = \dot{\Pi}_{\text{uc}} + \dot{I}.
\]  

Eq. (9) is the main result of this work. It connects the entropy production rate of the un-monitored open Gaussian system to the homonymous quantity for the monitored one via the informational term \(\dot{I}\). The second law for the un-monitored system, which reads \(\dot{\Pi}_{\text{uc}} \geq 0\), can now be used to obtain the refined 2nd law for continuously measured Gaussian systems \(\dot{\Pi} \geq \dot{I}\), which epitomizes the connection between non-equilibrium thermodynamics and information theory, as pioneered by Landauer’s principle. Indeed, a Landauer-like statement can be formulated by considering the integrated version of such refined 2nd law \(\Sigma \geq I\) – with \(\Sigma\) the total entropy production – which states that the degree of irreversibility of the dynamics being considered, which is associated with a change in entropy of the state of the system, is lower-bounded by an information theoretical cost that crucially depends on the measurement strategy that has been chosen and that is in general more stringent than that associated with un-measured dynamics. Needless to say, this echoes and extends significantly the earlier results reported in Ref. [40], which was valid for discrete measurements and equilibrium environments.

Let us now go back to the assumed linearity of the entropy flux \(\mathbb{E}[d\Phi_{\text{uc}}/dt] = \Phi_{\text{uc}}\): this request can be justified, in general, by referring to the standard case where the entropy flux is identified with the heat flux from the system to an (equilibrium) environment at a reference temperature. In that case, the heat flux rate is by definition linear in the conditioned state. Another argument in support of such an assumption can be found in [8], where the entropy flux is associated to the occurrence of a quantum jump of the system. For the case of Gaussian dynamics, this assumption can be verified adopting the phase-space formalism discussed above, thus making it into a property of the system more than an assumption. Indeed, the quantity \(d\Phi_{\text{uc}}\) can be derived explicitly and shown to
FIG. 2. (Color online). (a)-(c): Thermal quench of a harmonic oscillator interacting with an external single mode in thermal equilibrium subject to homodyne and heterodyne measurements. The single mode environment has a mean occupation number $\bar{n}_\text{th}$ such that $2\bar{n}_\text{th} + 1 = 100$. The initial state of the harmonic oscillator, which is externally driven by a pump of amplitude $E$, is thermal with an energy ten times larger than that of the environmental mode. Panels (a) and (b) show the unconditioned entropy production rate $\Pi_{\text{uc}}$ (dashed blue curve), the averaged conditioned entropy production rate $\Pi$ (solid red curve), the averaged conditioned entropy flux $\Phi$ (black dotted curve), and the informational term $\dot{I}$ (dot-dashed green curve) for homodyne (monitoring of the system’s momentum quadrature) and heterodyne detection. Panel (c) shows $I$ for (homodyne) monitoring of the position and momentum quadratures of the system (dot-dashed green curve and dotted red curve, respectively), and heterodyne measurements (dashed blue curve). Panels (d)-(f): Optical parametric oscillator in contact with an external single mode in thermal equilibrium – subject to homodyne and heterodyne measurements with additive Gaussian noise – with $\bar{n}_\text{th} = 0$. Panels (d) and (e) are analogous to (a) and (b). Panel (f) shows $I$ for different measurement detection schemes. See Ref. [56] for additional details.

yield the unconditional entropy flux rate on average, see [56] for details. Nevertheless, we deem such an assumption relevant beyond the case of Gaussian systems as the basis for obtaining the entropy production rate unambiguously once an entropic measure has been chosen.

Case study.— In order to corroborate and illustrate our framework, we consider two simple yet physically relevant examples: a thermal quench and the driven-dissipative optical parametric oscillator. In both cases, the system (either a driven simple harmonic oscillator or an optical parametric oscillator) is monitored by a single mode in thermal equilibrium subjected to general-dyne measurements [53], with which it is coupled via an excitation-exchange interaction [56]. In Fig. 2, the evolution of the entropy production rate, entropy flux rate, and $\dot{I}$ is shown for both systems under either homodyne or heterodyne measurements. The inspection of the results for a thermal quench shows that $\dot{I} \to 0$ at the steady state [cf. Fig. 2 (c)] as the steady state of conditional and unconditional evolution is the same irrespective of the general-dyne measurement being performed. In contrast, Fig. 2 (f), which refers to the parametric oscillator, shows quite different long-time behavior of $I$ for different measurement schemes due to the additive Gaussian noise that is introduced by the choice of detection strategy. This has the effect of making the steady state of the conditioned dynamics a mixed state with different degrees of purity depending on the specific measurement schemes. While in both examples $\dot{I}$ takes both positive and negative values, the regions where $\dot{I} > 0$ showcase how our refined 2nd law offers a more stringent constraint than the standard one.

Conclusions.— We have characterized the entropy production rate of continuously monitored non-equilibrium Gaussian quantum processes in terms of the effects induced on the dynamics of a system by the measurement strategy. This led us to the formulation of a refined 2nd law reminiscent of Landauer’s principle and in line with previous results valid for discrete measurements and systems.

On the one hand, our results shine new light on the emerging field of information thermodynamics, highlighting the tight fundamental link between non-equilibrium thermodynamics and information gains. On the other hand, they offer a general, unambiguous way to define the entropy production and flux rates for Gaussian systems, thus overcoming the limitations of previous approaches.

Such framework will be invaluable to analyze and characterize the non-equilibrium dynamics of experimental systems of strong current interest. In particular, levitated quantum optomechanics offers fertile ground for the application of our
formalism [44, 45, 47–51].

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Supplemental Materials: Entropy Production in Continuously Measured Systems

In this supplementary material we show how to connect $I$ to the mutual information and how to derive the stochastic entropy production and flux rates from the phase-space dynamics. We furthermore discuss how to obtain the averaged version of these quantities encountered in the main text. For completeness, we also report the specifics of the two physical examples considered in the main text in order to make all the results reproducible.

**MUTUAL INFORMATION AND NEGATIVITY OF THE INFORMATIONAL TERM**

Here we show that, the integrated informational quantity $I$ is equal to minus the mutual information between the random variable $X$, representing the position in phase-space, and $\bar{X}$ described by the stochastic process in Eq. (1). In order to do so, we note that the conditional and unconditional CMs are related by $\sigma_{\text{uc}} = \sigma + V$, where $V(t)$ is the positive defined CM of the stochastic process $\bar{X}$. In order to properly characterize $V$, note that for the conditional dynamics the first momenta $\bar{x}_{\text{uc}} = \mathbb{E}[\bar{x}]$ (corresponding to the first momenta of the unconditional evolution and evolving according to $d\bar{x}_{\text{uc}}/dt = A\bar{x}_{\text{uc}} + b$) and the corresponding noise covariance matrix $V$. The evolution of $V$ can be found directly from the Langevin equation in (1) and reads

$$\frac{dV}{dt} = AV + VA^T + \chi(\sigma).$$

Interestingly, we see that the quantum covariance matrix $\sigma(t)$ affects the diffusion coefficient of the noise CM.

Consider the unconditional and conditional Wigner functions $W_{\text{uc}} = N(\bar{x}_{\text{uc}}, \sigma_{\text{uc}})$ and $W = N(\bar{x}, \sigma)$ (here $N$ stands for the normal distribution). We can identify the conditional Wigner function with the conditional probability (density) of getting $X = x$ given $\bar{X} = \bar{x}$ and write, with a small abuse of notation, $W = p(x | \bar{x})$. In the same way we identify $W_{\text{uc}} = p(x)$ in such a way that, given the normal distribution for the stochastic variable $\bar{X}$, we have

$$p(x) = \sum_{\bar{x}} p(\bar{x}) p(x | \bar{x}),$$

which, for Gaussian states, can be shown to be in accordance with the relation $\sigma_{\text{uc}} = \sigma + V$. Here $p(\bar{x}) = N(\bar{x}_{\text{uc}}, V)$ is the Gaussian pdf describing the statistics of the Kalman filtered outcomes $\bar{x}$ [55]. Note that, we use sums instead of integrals for the sake of clarity, however the process that we are considering are always continuous and described properly by probabilities densities.

The quantity $I$, whose rate appears in Eq. (7) of the main text, being the difference between the Wigner entropies of the conditional and unconditional dynamics, can be written as

$$I = \sum_x p(x | \bar{x}) \log p(x | \bar{x}) - H(X) = H(X | \bar{X}) - H(X),$$

where we have used the fact that the Wigner entropy of the conditional dynamics is actually independent of the outcome $\bar{x}$ of the stochastic process, thus $\sum_x p(x | \bar{x}) \log p(x | \bar{x}) = \sum_{\bar{x}} p(\bar{x}) \sum_x p(x | \bar{x}) \log p(x | \bar{x})$. Finally, we can use the properties of the conditional entropy

$$H(A | B) - H(B) = H(B | A) - H(A)$$

$$2I(A : B) = H(A) + H(B) - H(A | B) - H(B | A),$$

to conclude that

$$I = -I(X : \bar{X}) \leq 0.$$  \hspace{1cm} (S6)

While we were able to connect $I$ with a mutual information, the result about its negativity can be readily obtained from the convexity of Wigner entropy. In particular we have

$$I = S(W) - S(W_{\text{uc}}) \leq S(W) - \int d\bar{x} S(W)p(\bar{x}) = 0,$$  \hspace{1cm} (S7)

[57] G. Adesso, D. Girolami, and A. Serafini, Phys. Rev. Lett. 109, 190502 (2012).
[58] J. P. Santos, G. T. Landi, and M. Paternostro, Phys. Rev. Lett. 118, 220601 (2017).
where in the last equality we have used again the fact that the Wigner entropy of the conditioned dynamics is a deterministic quantity, i.e., it does not depend on the realization of the stochastic process. We stress once more that this is a peculiarity of the Gaussian formalism.

Finally, we prove that the inequality is saturated if and only if \( \sigma(t) = \sigma_{uc}(t) \), i.e., only when the CM of the conditioned dynamics is equal to the one of the unconditional dynamics. To show this, let us consider the definition of \( I \) in terms of the purity, we have

\[
I = \frac{1}{2} \log \left( \frac{|\sigma|}{|\sigma_{uc}|} \right) = -\frac{1}{2} \log \left( \frac{|\sigma + V|}{|\sigma|} \right),
\]

Using the properties of the determinant, and the fact that the CM \( \sigma \) is positive definite, we can write the argument of the logarithm as

\[
|\sigma + V| \cdot |\sigma|^{-1} = \| I + \sqrt{\sigma^{-1}} V \sqrt{\sigma^{-1}} \|.
\]

From the positivity of \( V \), which guarantees that it can be written in the form \( LL^T \), we deduce that \( \sqrt{\sigma^{-1}} V \sqrt{\sigma^{-1}} \) is also positive. Thus, we can write

\[
I = -\frac{1}{2} \log \left( \| I + \sqrt{\sigma^{-1}} V \sqrt{\sigma^{-1}} \| \right) = -\frac{1}{2} \log (\Pi_{i=1,2}(1 + \lambda_i)),
\]

where \( \lambda_i \geq 0 \) are the eigenvalues of the positive definite matrix \( \sqrt{\sigma^{-1}} V \sqrt{\sigma^{-1}} \). From this expression it is immediate to conclude that \( I = 0 \) if and only if \( V = 0 \), i.e., \( \sigma = \sigma_{uc} \).

**DERIVATION OF THE STOCHASTIC FOKKER-PLANCK EQUATION USING ITO-CALCULUS**

Given that, for the conditioned dynamics, the first momenta \( \bar{x} \) follow a stochastic process, the Wigner function is a function of stochastic variables. Thus, in order to obtain the corresponding Kushner-like dynamical equation in phase-space, we apply Ito-lemma to the Wigner function. The Ito-formula for diffusive processes reads (index summation convention understood)

\[
dW = \left( \partial_{\sigma} W \partial_{\sigma} \sigma + \frac{\partial W}{\partial \bar{x}} (A\bar{x} + b) + \frac{1}{2} \frac{\partial^2 W}{\partial \bar{x} \partial \bar{x}} \chi(\sigma) \right) dt + \frac{\partial W}{\partial \sigma} (\sigma C^T + \Gamma^T) d\sigma
\]

which gives

\[
\partial_{\sigma} W \partial_{\sigma} \sigma = -\text{div}[J_a]
\]

\[
J_a = A(\bar{x} - \bar{x}) W - \frac{1}{2} \left( D - \chi(\sigma) \right) V W,
\]

while the other terms give

\[
\frac{1}{2} W(\bar{x} - \bar{x})^T \sigma^{-1} \chi(\sigma) \sigma^{-1} (\bar{x} - \bar{x}) - \frac{1}{2} W \text{Tr}[\sigma^{-1} \chi] = -\frac{1}{2} \text{div}(-\chi V W)
\]

\[
W(\bar{x} - \bar{x})^T \sigma^{-1} (A\bar{x} + b) = -\text{div}(W(A\bar{x} + b))
\]

\[
W(\bar{x} - \bar{x})^T \sigma^{-1} (\sigma C^T + \Gamma^T) d\sigma = -\text{div}(W(\sigma C^T + \Gamma^T) d\sigma)
\]

Putting all the pieces together we finally obtain the stochastic Fokker-Planck equation

\[
dW = -\text{div}[J_a + W(A\bar{x} + b)] - \frac{1}{2} \chi V W] dt - \text{div}[W(\sigma C^T + \Gamma^T) d\sigma]
\]

\[
= -\text{div}[(A\bar{x} + b) W - \frac{1}{2} D V W] dt - \text{div}[W(\sigma C^T + \Gamma^T) d\sigma]
\]

\[
= -\text{div}[J dt + J_{sto}],
\]

where \( J = (A\bar{x} + b) W - \frac{1}{2} D V W \) and \( J_{sto} = W(\sigma C^T + \Gamma^T) d\sigma \). It is interesting to note that, the deterministic current coincides (in form) with the one we would have for the unconditional dynamics.
STOCHASTIC ENTROPY FLUX AND PRODUCTION RATES

Let us consider the case of conditional dynamics, in which the evolution of the system in phase-space is described by the stochastic Kushner-like equation (S19). Note that, while the Wigner entropy rate is a deterministic quantity (since it depends on the deterministic dynamics of the CM), the Wigner function is not. Thus, in order to work out the Wigner entropy rate we need to resort to Ito-lemma. It is then easy to see that the entropy increment

\[ dS = -\int dx d(W \log W) \]

results in

\[ dS = \int dx \left[ \left( A_x W + b W - \frac{1}{2} (D - \chi)^T \chi W \right) dt + (\sigma C^T + \Gamma^T) dW \right] \log W \]

where we have introduced \( J_2 = \frac{1}{2} \chi^T \nabla W \), while \( J \) and \( J_{sto} \) are the ones defined in the main text. It is interesting to note that the term containing the second derivatives of \( W \) with respect to the first momenta vanishes identically.

Note that, in analogy with the unconditional dynamics case [58], the Hamiltonian part of the current, i.e., \( J_{rev} \), does not contribute to the entropy rate. Indeed, such a term is divergence-less and can be dropped out. The same is true also for the \( J_{sto} \) part of the current, but not for \( J_2 \) which encodes the measurement matrices. Thus, integrating by parts, we remain with

\[ dS = -\int dx \frac{d}{W} (J_{intr} + J_2)^T \nabla W \ dt \]

Expressing the gradient of the Wigner function in terms of \( J_{intr} \) and \( J_2 \), the entropy increment can be cast in the form

\[ dS = 2 \int dx J_{intr}^T D^{-1} J_{intr} dt - 2 \int dx J_{intr}^T D^{-1} \chi J_2 dt - 2 \int dx J_{intr}^T D^{-1} A_{intr} \mathbf{x} dt. \]

We identify the last terms as the entropy flux increment and the first two terms as the entropy production one. In the unconditional dynamics case, only the first term would be present for the entropy production and, in the literature, such term is identified with the entropy production (since it is a positive semi-definite quadratic form of the currents). In the conditional case, we see that on top of the positive semi-definite quadratic form, the second term quadratic in the currents is negative semi-definite due to the fact that \( \chi \) is a positive semi-definite matrix by construction. Nonetheless, we incorporate this term in the entropy production and recognize in the last term the entropy flux with the same form as in the unconditional case. Thus, we end up with the following expressions for the entropy flux and production in terms of the parameters of the model

\[ \frac{dS}{dt} = \frac{1}{2} \text{Tr} [2A + \sigma^{-1} (D - \chi)], \]

\[ d\phi_x = -\text{Tr} [A_{intr} dt] - 2 \text{Tr} [A_{intr}^T D^{-1} A_{intr} \sigma] dt - 2 \mathbf{x}^T A_{intr}^T D^{-1} A_{intr} \mathbf{x} dt \]

\[ d\pi_x = 2 \text{Tr} [A_{intr} dt] + 2 \text{Tr} [A_{intr}^T D^{-1} A_{intr} \sigma] dt + 2 \mathbf{x}^T A_{intr}^T D^{-1} A_{intr} \mathbf{x} dt + \frac{1}{2} \text{Tr} [\sigma^{-1} (D - \chi)]. \]

Note that these quantities still depends on the stochastic variable \( \mathbf{x} \). In order to obtain the entropy flux and production rates, in which we are interested, we need to average these expressions over all possible trajectories of the system. Given the Gaussian nature of the problem, the average is trivial to perform using the Gaussian pdf

\[ p(\mathbf{x}, t; \mathbf{x}_{inc}, V) = \frac{1}{(2\pi)^n \sqrt{\det V}} e^{-\frac{1}{2}(\mathbf{x} - \mathbf{x}_{inc})^T V^{-1} (\mathbf{x} - \mathbf{x}_{inc})}, \]
where $V$ is the noise matrix we introduced above. The result of the average is

$$\mathbb{E} \left[ \frac{d\mathbf{\phi}_B}{dt} \right] = -2\text{Tr}[A_{in}] - 2\text{Tr}[A_{in}^T D^{-1} A_{in} \sigma_{uc}] - 2\tilde{\mathbf{\sigma}}_{in}^T D^{-1} A_{in} \tilde{\mathbf{\phi}}_{uc} = \Phi_{uc}$$

(S31)

$$\mathbb{E} \left[ \frac{d\mathbf{\sigma}_{uc}}{dt} \right] = 2\text{Tr}[A_{in}] + 2\text{Tr}[A_{in}^T D^{-1} A_{in} \sigma_{uc}] + 2\tilde{\mathbf{\sigma}}_{in}^T A_{in}^T D^{-1} A_{in} \mathbf{\sigma}_{uc} + \frac{1}{2} \text{Tr}[\sigma_{uc}^T D]$$

(S32)

$$+ \frac{1}{2} \text{Tr}[\sigma_{uc}^{-1} D] - \sigma_{uc}^{-1} \mathbf{\chi} = \Pi_{uc} + \mathbf{I},$$

where we have used the explicit expressions for the entropy flux ($\Phi_{uc}$) and production ($\Pi_{uc}$) rates for the unconditional dynamics given by

$$\Phi_{uc} = -2\text{Tr}[A_{in}] - 2\text{Tr}[A_{in}^T D^{-1} A_{in} \sigma_{uc}] - 2\tilde{\mathbf{\sigma}}_{in}^T A_{in}^T D^{-1} A_{in} \mathbf{\sigma}_{uc}$$

$$\Pi_{uc} = 2\text{Tr}[A_{in}] + 2\text{Tr}[A_{in}^T D^{-1} A_{in} \sigma_{uc}] + \frac{1}{2} \text{Tr}[\sigma_{uc}^{-1} D] + 2\tilde{\mathbf{\sigma}}_{in}^T A_{in}^T D^{-1} A_{in} \mathbf{\sigma}_{uc},$$

We see that the stochastic quantity $d\Phi_B$ averages exactly to the unconditional entropy flux. This result strengthen our assumption in the main text about the linearity of the stochastic flux with respect to the state of the system. Indeed, we have shown here how to arrive at an expression for the stochastic flux which is linear in the state, in terms of the dynamics of the Wigner function in phase-space. The crucial step is to identify the entropy production rate as the term quadratic in the currents. While this is less direct in this case compared to the unconditional dynamics, in which the quadratic term is positive definite, it squares well with the presence of the current $J_2$ which is related to the measurement and can, in principle, make the entropy production rate negative.

**DETAILS OF THE CASE STUDIES REPORTED IN THE MAIN PAPER**

Here we give some specifics of the two examples considered in the main text, i.e., the thermal quench of a simple harmonic oscillator and the optical parametric oscillator coupled to a thermal bath. In doing so we follow closely [54] where the example of the optical parametric oscillator is described in details. The bath is described, in both our examples, by the initial single mode CM $\sigma_B = (n_0 + 1/2)\mathbf{I}$. The excitation-exchange interaction Hamiltonian, also common to both the examples, is given by

$$\hat{H}_{int} = \sqrt{s} \mathbf{\hat{X}} \mathbf{\hat{X}}_B = \frac{1}{2} \left( s \mathbf{\hat{X}} \mathbf{\hat{X}}_B \right) \left( \begin{array}{c} 0 \\ C \\ 0 \end{array} \right) \left( \begin{array}{c} \mathbf{\hat{X}} \\ 0 \\ \mathbf{\hat{X}}_B \end{array} \right), \quad C = \left( \begin{array}{cc} \sqrt{s} & 0 \\ 0 & \sqrt{t} \end{array} \right).$$

(S33)

where we have introduced the quadratures of the bath $\mathbf{\hat{X}}_B$. A general-dyne, noisy measurement is described, in the Gaussian formalism, by giving the CM corresponding to the state over which one project. For an ideal general-dyne measurement on the single output mode, the state is pure and the CM is given by the $2 \times 2$ matrix

$$\sigma_m = R[\varphi]^T \left( \begin{array}{cc} s & 0 \\ 0 & 1/s \end{array} \right) R[\varphi],$$

(S34)

where $R[\varphi]$ is a rotation matrix and $s > 0$. Note that, given the form of the excitation-exchange interaction Hamiltonian, $s = 0$ corresponds to homodyne detection of the $x$-quadrature of the output mode and thus the indirect monitoring of the $p$-quadrature of the system, $s = 1$ corresponds to heterodyne detection on the output mode, and $s = \infty$ to indirect monitoring of the $x$-quadrature of the system (homodyne detection of the $p$-quadrature of the output mode). In order to account for noisy measurements, the CM $\sigma_m$ needs to be modified by acting on it with the dual of a CP Gaussian map. The reason for this stems from the fact that a noisy measurement can be seen as the action of a CP Gaussian map on the state of the system previous to an ideal general-dyne measurement [54]. For our simple case, the CM $\sigma_m$ for a general-dyne detection with efficiency $\eta \in [0, 1]$ and additive Gaussian noise $\Delta$ is given by

$$\sigma_m = R[\varphi]^T \left( \begin{array}{cc} s/\eta & 0 \\ 0 & 1/(s\eta) \end{array} \right) R[\varphi] + \left( \begin{array}{cc} (1 - \eta)/\eta + \Delta & 0 \\ 0 & (1 - \eta)/\eta + \Delta \end{array} \right).$$

(S35)

Given the measurement CM and the interaction Hamiltonian, we can obtain the measurements matrices $\Gamma, C$. For both our examples, these are given by

$$\Gamma^T = \frac{1}{\sqrt{2}} \sqrt{2} \Omega C \sigma_B + \sigma_m \right)^{-1/2}$$

(S36)

$$C^T = -\sqrt{\Delta} \Omega C \sigma_B + \sigma_m \right)^{-1/2}$$

(S37)
It should be noted that, in the limit of $\eta \to 0$ and/or $\Delta \to \infty$, i.e., for zero efficiency of the detectors and/or infinite additive Gaussian noise, the conditional dynamics converges to the unconditional one. This is intuitive, given that in both these situations no information about the system is acquired by the inefficient/very noisy detection scheme. However, this highlight also another interesting aspect of this type of noisy general-dyne detection: no matter how large the additive noise or how inefficient the detectors, the conditional dynamics will never increase the uncertainty on the state of the system more than the unconditional dynamics.

Finally, we specify the Hamiltonians of the two systems that we consider and we explicit the parameters chosen in Fig. 2. For the thermal quench of the harmonic oscillator, we chose a thermal occupation number of the bath such that $2n_{\text{th}} + 1 = 100$. The Hamiltonian of the system, comprising a quadratic term and a linear drive, is given by

$$\hat{H}_s = \frac{1}{2} \hat{x}^T \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \hat{x} + d^T \Omega \hat{x},$$

(S38)

where $d = -\sqrt{2} \hat{E} \cos \theta, \sqrt{2} \hat{E} \sin \theta)$ and the driving term correspond, in terms of annihilation and creation operators of the system oscillator, to $\hat{H}_{\text{drive}} = i \hat{E} (\hat{a} e^{i\theta} - \hat{a}^\dagger e^{-i\theta})$. Here as in the second example, we work in natural units. Furthermore, we set $\omega = 1$ so that all other quantities appearing are in units of $\omega$. In Fig. 2 we have chosen $\theta = 0, \hat{E} = 2$, and $\gamma = 1/10$. For the measurement, we chose the rotation matrix to be the identity, i.e., $\varphi = 0$, and we consider an ideal measurement $\Delta = 0, \eta = 1$, with $s = \{0, 1, \infty\}$. Furthermore, the initial mean values of the oscillator quadratures is $\langle \hat{x}(t = 0) \rangle = (1, 1)$. The drift and diffusion matrices are readily obtained as

$$A = \begin{pmatrix} -\gamma/2 & \omega \\ -\omega & -\gamma/2 \end{pmatrix}$$

(S39)

$$D = \gamma (n_{\text{th}} + 1/2) \mathbb{I}.$$ 

(S40)

For the optical parametric oscillator, we chose the bath in the vacuum state $n_{\text{th}} = 0$, which is a reasonable assumption for optical modes. The (effective) Hamiltonian of the system is given by

$$\hat{H}_s = \frac{1}{2} \hat{x}^T \begin{pmatrix} 0 & -\chi \\ -\chi & 0 \end{pmatrix} \hat{x}.$$ 

(S41)

The unconditional dynamics of the oscillator is stable only if $\gamma > 2\chi$. The drift and diffusion matrices are given by

$$A = \begin{pmatrix} -\chi - \gamma/2 & 0 \\ 0 & \chi - \gamma/2 \end{pmatrix}$$

(S42)

$$D = \gamma (n_{\text{th}} + 1/2) \mathbb{I}.$$ 

(S43)

Note that, as expected, only the reversible part of the drift matrix ($A$) changes with respect to the previous example. In Fig. 2, we set $\chi = 1$ so that all other quantities appearing are in units of $\chi$. We chose the coupling constant to be $\gamma = 2.001$, i.e., the parametric oscillator is close to the instability point $\gamma = 2\chi$. The initial expectation value of the quadratures is chosen in the origin of phase-space. For the measurement, we chose the rotation matrix to be the identity, i.e., $\varphi = 0$, and we consider an efficient measurement ($\eta = 1$) with a small additive noise $\Delta = 0.1$. As before $s = \{0, 1, \infty\}$. We add Gaussian noise in such a way to have an unconditional dynamics which, for different detection schemes, leads the system to steady states with different purities. It should be noted that, for a thermal bath at zero temperature and ideal general-dyne measurements, the steady state of the conditional dynamics would always be a pure state (in contrast to the unconditional steady state which is always a mixed state).