A FINITELY PRESENTED $E_\infty$-PROP I: DIFFERENTIAL GRADED CONTEXT

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Abstract. We introduce a finitely presented prop $S = \{S(n, m)\}$ in the category of differential graded modules whose associated operad $U(S) = \{S(1, m)\}$ is a model for the $E_\infty$-operad. This finite presentation allows us to describe a natural $E_\infty$-coalgebra structure on the chains of any simplicial sets in terms of only three maps: The Alexander-Whitney diagonal, the augmentation map and a differential graded version of the join of simplices. One of the appendices connects our construction and the Surjection operad of McClure-Smith and Berger-Fresse.

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1. INTRODUCTION

A careful and purposeful construction of a model for the $E_\infty$-operad is central in most contexts where commutativity up to coherent homotopies plays a role. The usual pattern is as follows. First establish some homotopical consequence taking advantage of the special features of a specific model and then, using the homotopical nature of the $E_\infty$-operad, argue the independence and generality of the result. This claim is supported by the large number of available models having deep levels of specialization. In this paper, we work in the differential graded context and the property that we are interested in is the size of a presentation in terms of generators and relations. No finitely presented model for the $E_\infty$-operad can exists, but as we show in this note, passing to the more general setting of props allows for a finite presentation, described in section Section 3, of a prop whose associated operad is a model for the $E_\infty$-operad. We use this finitely presented prop to describe, in Section 4, an $E_\infty$-structure on the chains of any simplicial set induced by only
three operations: The Alexander-Whitney chain approximation to the diagonal map, the augmentation map and a differential graded version of the join of simplices.

The outline of this paper is as follows. In the second section, we review the material on operads and props needed for the rest of the paper. In the third section, we give the finite presentation of the prop $S$ and compute its homology. In the fourth section, we construct an $S$-coalgebra structure on the chains of each standard simplex, natural with respect to simplicial maps, and based solely on the Alexander-Whitney map, the augmentation map and a differential graded join map. These $S$-coalgebra structures induce a natural $U(S)$-coalgebra structure on the chains of any simplicial set, where $U(S)$ is the canonical $E_\infty$-operad with $U(S)(m) = S(1, m)$ for all $m \geq 0$. In the first appendix, we describe a relationship between $S$ and the operad of McClure-Smith [MS03] and Berger-Fresse [BF04]. Explicitly, we construct a sequence of prop morphisms $S \hookrightarrow Sh \rightarrow MS$ such that the associated operad $U(MS)$ is isomorphic to this previously defined and widely used algebraic model. We also describe how the $E_\infty$-structure defined by these authors on chains of simplicial sets is generated by the Alexander-Whitney, augmentation and join maps. In the second appendix, we construct natural $S$-algebra structures on the chains of each standard augmented simplex. This construction exhibits an interesting duality between the diagonal and join maps in the usual and augmented simplicial contexts.

Acknowledgments. I would like to thank Bruno Vallette, Stephan Stolz, Ralph Cohen and Dennis Sullivan for their insights, questions and comments about this project.

2. Preliminaries about operads and props

Conventions. We will work in the symmetric monoidal category $(\text{Mod}_k, \otimes, k)$ of differential homologically graded $k$-modules, with $k$ a commutative ring. We will use the notation $\mathfrak{m}$ for $\{1, \ldots, n\}$ with $\mathfrak{m}$ standing for the empty set.

2.1. $E_\infty$-Operads. We say that an operad $O$ is $\Sigma$-free if $O(m)$ is a $k[\Sigma_m]$-free module for every $m$. A $\Sigma$-free resolution of an operad $O$ is a morphism from a $\Sigma$-free operad to $O$ inducing a homology isomorphism in each arity $m$.

For any $C \in \text{Mod}_k$, we have two naturally associated operads given for any $m \geq 0$ by

$$\operatorname{End}(C)(m) = \operatorname{Hom}(C, C^{\otimes m}) \quad \text{and} \quad \operatorname{End}^\text{op}(C)(m) = \operatorname{Hom}(C^{\otimes m}, C).$$

(Notice that this is not the usual convention.) For any $C \in \text{Mod}_k$ there are two types of representations of an operad $O$ on $C$, they are referred to as $O$-coalgebra and $O$-algebra structures and are respectively given by operad morphisms

$$O \to \operatorname{End}(C) \quad \text{and} \quad O \to \operatorname{End}^\text{op}(C).$$

The operad $\operatorname{End}(k)$ is of particular importance, with $\operatorname{End}(k)$-(co)algebras defining usual (co)commutative, (co)associative and (co)unital (co)algebras.

Following May, for example [KM95], an operad $O$ is called an $E_\infty$-operad if it is a $\Sigma$-free resolution of $\operatorname{End}(k)$ and $O(0) = k$.

2.2. Props. A prop is a strict symmetric monoidal category $\mathcal{P} = (\mathcal{P}, \circ, 0)$ enriched in $\text{Mod}_k$ generated by a single object. For a prop $\mathcal{P}$ with generator $p$ denote $\operatorname{Mor}_\mathcal{P}(p^{\otimes n}, p^{\otimes m})$ by $\mathcal{P}(n, m)$ and notice, induced from the symmetry of the monoidal structure, the commuting right and left actions of $\Sigma_n$ and $\Sigma_m$ respectively. Therefore, we think of the data of a prop as a $\Sigma$-bimodule, i.e., a collection $\mathcal{P} = \{\mathcal{P}(n, m)\}_{n,m \geq 0}$ of differential graded $k$-modules with commuting actions of $\Sigma_n$ and $\Sigma_m$ together with three types of maps

$$o_h : \mathcal{P}(n_1, m_1) \otimes \cdots \otimes \mathcal{P}(n_s, m_s) \to \mathcal{P}(n_1 + \cdots + n_s, m_1 + \cdots + m_s),$$

$$o_a : \mathcal{P}(k, m) \otimes \mathcal{P}(k, n) \to \mathcal{P}(n, m),$$

$$\eta : k \to \mathcal{P}(n, n).$$
These types of maps are referred to respectively as horizontal compositions, vertical compositions and units. They come from the monoidal product, the categorical composition of \( \mathcal{P} \) and its identity morphisms.

For any \( C \in \mathcal{Mod}_k \) we have two naturally associated props given for any \( n, m \geq 0 \) by

\[
\text{End}(C)(n, m) = \text{Hom}(C^\otimes n, C^\otimes m) \quad \text{and} \quad \text{End}^{\mathcal{P}}(C)(n, m) = \text{Hom}(C^\otimes m, C^\otimes n).
\]

There are two types of representations of a prop \( \mathcal{P} \) on \( C \), they are referred to as \( \mathcal{P} \)-coalgebra and \( \mathcal{P} \)-algebra structures and are respectively given by prop morphisms

\[
\mathcal{P} \to \text{End}(C) \quad \text{and} \quad \mathcal{P} \to \text{End}^{\mathcal{P}}(C).
\]

Let \( U \) be the functor from the category of props to that of operads given by restricting the vertical composition structure of a prop \( \mathcal{P} \) to the \( \Sigma \)-module \( U(\mathcal{P}) = \{ \mathcal{P}(1, m) \}_{m \geq 0} \) together with \( \eta : k \to \mathcal{P}(1, 1) \). Notice that a \( \mathcal{P} \)-coalgebra (resp. \( \mathcal{P} \)-algebra) structure on \( C \) induces a \( U(\mathcal{P}) \)-coalgebra (resp. \( U(\mathcal{P}) \)-algebra) structure on \( C \).

Following Boardman and Vogt [BV06], a prop \( \mathcal{P} \) is called an \( E_\infty \)-prop if \( U(\mathcal{P}) \) is an \( E_\infty \)-operad.

### 2.3. Free props

As described for example in [Mar08] or [Val07], the free prop generated by a \( \Sigma \)-bimodule is constructed using open directed graphs with no directed loops that are enriched with the following labeling. We think of each directed edge as built from two compatibly directed half-edges. For each vertex \( v \) of a directed graph \( G \), we have the sets \( \text{in}(v) \) and \( \text{out}(v) \) of half-edges that are respectively incoming to and outgoing from \( v \). Half-edges that do not belong to \( \text{in}(v) \) or \( \text{out}(v) \) for any \( v \) are divided into the disjoint sets \( \text{in}(G) \) and \( \text{out}(G) \) of incoming and outgoing external half-edges. The labeling is given by bijections

\[
\text{in}(G) \to \text{in}(G) \quad \text{and} \quad \text{out}(G) \to \text{out}(G)
\]

and

\[
\text{in}(v) \to \text{in}(v) \quad \text{and} \quad \text{out}(v) \to \text{out}(v)
\]

for every vertex \( v \). We refer to the isomorphism classes of such labeled directed graphs with no directed loops as \((n,m)\)-graphs. We consider the right action of \( \Sigma_n \) and the left action of \( \Sigma_m \) on a \((n,m)\)-graph given respectively by permuting the labels of \( \text{in}(G) \) and \( \text{out}(G) \).

A \( k \)-module basis for the \((n,m)\)-part of the \( \Sigma \)-bimodule underlying the free prop generated by a \( \Sigma \)-bimodule \( \mathcal{P} \) is given by all isomorphism classes of \((n,m)\)-graphs which are \( \mathcal{P}(n,m) \)-decorated in the following way. To every vertex \( v \) of one such \( G \) one assigns an element \( p \in \mathcal{P}(\text{in}(v), \text{out}(v)) \) and introduces the equivalence relations:

![Diagram](image)

where \( p, q \in \mathcal{P}(n,m), \sigma \in \Sigma_n, \tau \in \Sigma_m \) and \( a \in k \). Therefore, we have a description of the form

\[
\mathcal{P}(n,m) = \left( \bigoplus_{G} \mathcal{P}(\text{in}(v), \text{out}(v)) \right) / \sim
\]
that gives the $\mathcal{P}(n, m)$ the structure of a differential grade $k$-module. The vertical compositions are given by grafting, whereas the horizontal composition is given by unions of graphs with the corresponding relabeling of external half-edges.

2.4. **Presentations.** We will describe what we mean by a presentation $(G, \partial, R)$ of a prop. The first piece of data is a collection $G = \{G(n, m)_d\}$ of sets thought of as generators of biarity $(n, m)$ and homological degree $d$. Consider the free $\Sigma$-bimodule over the category of graded $k$-modules having basis $G$, i.e., in biarity $(n, m)$ the $k[\Sigma_n^0 \times \Sigma_m]$-module in homological degree $d$ is free with basis $G(n, m)_d$. The free prop over the category of graded $k$-modules generated by this $\Sigma$-bimodule is denoted $F(G)$. The second piece of data is a map $\partial : G \to F(G)$ of degree $-1$, thought of as the boundary of the generators. Extending this map as a derivation we promote $F(G)$ to a prop over $\text{Mod}_k$. The third piece of data is a collection $R = \{R(n, m)_d\}$ of subsets of $F(G)(n, m)$ thought of as relations. Denote by $\langle R \rangle$ the (differential graded) prop ideal generated by $R$ in $F(G)$. We say that the triple $(G, \partial, R)$ is a presentation of the prop $F(G)/\langle R \rangle$.

2.5. **Immersion convention.** If a graph immersed in the plane appears representing a labeled directed graphs with no directed loops, the convention we will follow is that the direction is given from top to bottom and the labeling from left to right. For example:

![Diagram](image)

3. **The prop $\mathcal{S}$**

In this section we define the central object of this note: the prop $\mathcal{S}$. We give a finite presentation of $\mathcal{S}$ and show it is an $E_\infty$-prop.

3.1. **The definition of $\mathcal{S}$.**

**Definition 1.** Let $\mathcal{S}$ be the prop generated by

- $1 \in \mathcal{S}(1, 0)_0$
- $\loom \in \mathcal{S}(1, 2)_0$
- $\rhood \in \mathcal{S}(2, 1)_1$

with differential

- $\partial 1 = 0$
- $\partial \loom = 0$
- $\partial \rhood = \rhood\loom$

and restricted by the relations

- $\rhood \loom - \loom \rhood$
- $\loom - \loom$

3.2. **The homology of $\mathcal{S}$.**

**Lemma 2.** Let $\text{End}(k)(n, m) = \begin{cases} \text{End}(k)(n, m) & \text{if } n > 0 \\ 0 & \text{if } n = 0. \end{cases}$

The map $\mathcal{S} \to \text{End}(k)$ defined on generators by

- $1 \mapsto (1 \mapsto 1) \in \text{Hom}(k, k^{\otimes 0})$
- $\loom \mapsto (1 \mapsto 1 \otimes 1) \in \text{Hom}(k, k^{\otimes 2})$
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\[ \gamma \mapsto 0 \in \text{End}(k^{\otimes 2}, k) \]

induces a homology isomorphism in each biarity.

Proof. For $n = 0$, $S(0, m) = 0 = \text{End}(k)(0, m)$. For $n > 0$ and $m \geq 0$ we start by showing the complexes $S(n, m)$ and $S(n, m + 1)$ are chain homotopy equivalent. Consider the collection of maps \( \{i : S(n, m) \to S(n, m + 1)\} \) described by the following diagram

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
1 \ldots m \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
12 \ldots m + 1
\end{array}
\end{array}
\]

Consider also the collection of maps \( \{r : S(n, m + 1) \to S(n, m)\} \) described by

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
12 \ldots m + 1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
1 \ldots m
\end{array}
\end{array}
\]

The diagram below shows that $r \circ i$ is the identity

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
1 \ldots m \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
12 \ldots m + 1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
1 \ldots m
\end{array}
\end{array}
\]

Let us compute diagramatically the composition \( i \circ r \)

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
12 \ldots m \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
1 \ldots m \end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
1 \ldots m \\
\end{array}
\end{array}
\]

Define the collection of homotopies \( \{H : S(n, m)_\bullet \to S(n, m)_{\bullet + 1}\} \) by

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
G \\
1 \ldots m
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \ldots n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \ldots m
\end{array}
\end{array}
\]
We now compute the value of $H \circ \partial$

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
\hline
G
\end{array} & \begin{array}{c}
1 \ldots n \\
\hline
\partial G
\end{array} & \begin{array}{c}
1 \ldots n \\
\hline
\partial G
\end{array}
\end{array}
\end{array}
\end{array}
$$

And the value of $\partial \circ H$

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
\hline
H 1 \ldots n
\end{array} & \begin{array}{c}
1 \ldots n \\
\hline
1 \ldots n
\end{array} & \begin{array}{c}
1 \ldots n \\
\hline
1 \ldots n
\end{array}
\end{array}
\end{array}
\end{array}
$$

These computations show that $i$ and $r$ are chain homotopy inverses.

We verify that $S(n,0)$ has the homology of a point. In fact, the whole complex $S(n,0)$ is generated by the following element

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ldots n \\
\hline
\end{array}
\end{array}
\end{array}
\end{array}
$$

To see this, consider any basis element in $S(n,0)$ and follow up any terminal strand. If you encounter $\Upsilon$, then that linear generator is 0. If you encounter $\Lambda$, then the strand can be removed. We can perform this simplification until the chosen basis element equals the one above.

Finally, if $\psi_{(n,m)} : S(n,m) \to \text{End}(n,m)$ is the biarity $(n,m)$ part of the map, then we can show it is a homology isomorphism with an induction argument using the commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
S(n,m) & \text{End}(k)(n,m) \\
\hline
\psi_{(n,m)} & \psi_{(n,m+1)}
\end{array}
\end{array}
\end{array}
\end{array}
$$

where the map in the bottom is induced from the isomorphism $k \cong k \otimes k$.

**Theorem 3.** The prop $S$ is an $E_\infty$-prop.

**Proof.** Since by construction the action of $\Sigma_m$ on $U(S)(m) = S(1,m)$ is free, the theorem follows from the previous lemma.

**Remark 4.** Notice that the operad obtained by restricting to $\{S(n,1)\}$ is not $\Sigma$-free. For example,

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ldots 3 \\
\hline
1 \ldots 3
\end{array}
\end{array}
\end{array}
\end{array}
$$

in $S(3,1)$ is fixed by the transposition $(1,2)$.
4. The simplicial category and the prop $\mathcal{S}$

In this section we describe a family, natural with respect to simplicial maps, of $\mathcal{S}$-coalgebra structures on the chains of the standard simplices, i.e., natural prop morphisms

$$\{ \mathcal{S} \to \text{End}(C_{\bullet}(\Delta^d)) \}_{d \in \Delta}.$$  

The image of the three generators of $\mathcal{S}$ are the Alexander-Whitney map, the augmentation map and a differential graded version of the join of simplices. For any simplicial set $X$ this family induces natural $E_\infty$-structures

$$U(\mathcal{S}) \to \text{End}(C_{\bullet}(X)) \quad \text{and} \quad U(\mathcal{S}) \to \text{End}^{op}(C^{\bullet}(X)).$$

4.1. The natural $\mathcal{S}$-coalgebra on the chains of standard simplices.

**Notation 5.** Consider the simplicial category $\Delta$ and the functors of normalized chains $C_{\bullet}(-)$ and cochains $C^{\bullet}(-)$ with coefficients in $k$. For every object $d$ in the simplicial category we denote its image with respect to these functors respectively by $C_{\bullet}(\Delta^d)$ and $C^{\bullet}(\Delta^d)$.

**Theorem 6.** For every object $d$ of the simplicial category, the following assignment of generators defines a prop morphism $\mathcal{S} \to \text{End}(C_{\bullet}(\Delta^d))$ which is natural with respect to simplicial maps:

- $1 \in \text{Hom}(C_{\bullet}(\Delta^d), k)_0$ is the augmentation map, i.e.
  $$[v_0, \ldots, v_d] = \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d > 0. \end{cases}$$

- $\delta \in \text{Hom}(C_{\bullet}(\Delta^d), C^{\bullet}(\Delta^d) \otimes 2)_0$ is the Alexander-Whitney map, i.e.
  $$[v_0, \ldots, v_q] = \sum_{i=0}^{d} [v_0, \ldots, v_i] \otimes [v_i, \ldots, v_q].$$

- $\gamma \in \text{Hom}(C_{\bullet}(\Delta^d) \otimes 2, C^{\bullet}(\Delta^d))_1$ is the join map, i.e.
  $$\gamma ([v_0, \ldots, v_p] \otimes [v_{p+1}, \ldots, v_q]) = \begin{cases} (-1)^p \text{sign}(\pi) \cdot [v_{\pi(0)}, \ldots, v_{\pi(q)}] & \text{if } i \neq j \text{ implies } v_i \neq v_j \\ 0 & \text{if not} \end{cases}$$
  where $\pi$ is the permutation that orders the totally ordered set of vertices.

**Proof.** Throughout this proof we identify $\partial$, $\delta$ and $\gamma$ with their images in $\text{End}(C_{\bullet}(\Delta^d))$. We need to verify that the relations of $\mathcal{S}$ are satisfied by these images.

The fact that the four maps $\delta \gamma$, $-\delta \gamma$, $\partial \delta$, $\partial \delta$ are equal to the corresponding zero map is classical and can be easily verified.

The join map $\gamma$ is of degree 1 and the augmentation map $1$ vanishes on chains of degree greater than 0, so $\gamma$ is the zero map.

In order to establish $\partial \gamma = 1$, we must consider six cases for the basis elements which the map is applied to:

1. Both of degree 0 and not sharing a vertex
(2) Both of degree 0 and sharing a vertex

(3) Only one of degree 0 and sharing a vertex

(4) Only one of degree 0 and not sharing a vertex

(5) None of degree 0 and sharing a vertex

(6) None of degree 0 and not sharing a vertex
4.2. The associated $E_\infty$-coalgebra on the chains of simplicial sets.

**Remark 7.** The natural family of prop morphisms \( \{ S \to \text{End}(C_\bullet(\Delta^d)) \}_{d \in \Delta} \) of Theorem 6 restricts to a natural family of operad morphisms \( \{ U(S) \to \text{End}(C_\bullet(\Delta^d)) \}_{d \in \Delta} \).

**Corollary 8.** The families of prop morphisms described in Remark 7 induce, for every simplicial set \( X \), natural operad morphisms
\[
U(S) \to \text{End}(C_\bullet(X)) \quad \text{and} \quad U(S) \to \text{End}^{\text{op}}(C_\bullet(X)).
\]

**Proof.** This is a straightforward application of the Kan extension construction. The details can be found for example in [May03], where such a family is thought of as an operad morphism from \( U(S) \) to the canonical Eilenberg-Zilber operad. \( \square \)

**Remark 9.** It is not true that the family of representations described in Theorem 6 induces for every simplicial set \( X \) a prop morphism \( S \to \text{End}(C_\bullet(X)) \). For example, consider the simplicial set containing only two 0-simplices.

**Example 10.** For any simplicial set \( X \), the following elements in \( S(1, 2) \) are send to representatives of the first three Steenrod cup-i coproducts in \( C_\bullet(X) \).

As an example we compute \( \Delta_1[0, 1, 2] \in C_\bullet(\Delta^2) \otimes C_\bullet(\Delta^2) \).

\[
\Delta_1[0, 1, 2] = \bigtriangleup_1 [0, 1, 2] = \bigtriangleup_1 \otimes \bigtriangleup_1 [0, 1, 2].
\]

We compute the iterated Alexander-Whitney map \( \bigtriangleup_1 \otimes [0, 1, 2] \) to be equal to
\[
[0] \otimes [0] \otimes [0, 1, 2] + [0] \otimes [0, 1] \otimes [1, 2] + [0] \otimes [0, 1, 2] \otimes [2] + [0, 1] \otimes [1] \otimes [1, 2] + [0, 1] \otimes [1, 2] \otimes [2] + [0, 1, 2] \otimes [2] \otimes [2].
\]

Applying \( \bigtriangleup_1 \) to it gives
\[
-[0, 1, 2] \otimes [0, 1] + [0, 2] \otimes [0, 1, 2] + [0, 1, 2] \otimes [1, 2]
\]
and therefore,
\[
\Delta_1[0, 1, 2] = -(0, 1, 2) \otimes (0, 1) + (0, 2) \otimes (0, 1, 2) + (0, 1, 2) \otimes (1, 2).
\]
Appendix A. The surjection operad and the prop $S$

In this section we construct a sequence of prop morphisms

$$S \twoheadrightarrow Sh \rightarrow MS$$

and an isomorphism with $\mathbb{F}_2$-coefficients

$$U(MS) \cong Sur,$$

where $Sur$ is the Surjection operad constructed by McClure-Smith [MS03] and Berger-Fresse [BF04].

Furthermore, we compare the natural map $\phi: Sur \rightarrow \text{End}(C_\bullet(X))$ defined by McClure-Smith and Berger-Fresse for any simplicial set $X$ and the natural map $U(MS) \rightarrow \text{End}(C_\bullet(X))$ induced from Corollary 8 and the above sequence of prop morphisms. The result is that with $\mathbb{F}_2$-coefficients the induced dashed arrow makes the following diagram commutative

$$\begin{array}{ccc}
U(S) & \xrightarrow{U(\phi)} & \text{End}(C_\bullet(X)) \\
\uparrow & & \uparrow \\
U(Sh) & \xrightarrow{\cong} & U(MS) \\
\downarrow & & \\
U(MS) & \xrightarrow{\cong} & Sur
\end{array}$$

A.1. The sequence of props $S \twoheadrightarrow Sh \rightarrow MS$.

**Notation 11.** We will utilize graphs with higher valence vertices to represent labeled directed graphs resulting from iterated grafting of $\bigtriangledown$ and of $\bigtriangleup$ in an unspecified order. We use the following diagrams for them

\[
\begin{array}{c}
1 \ldots n \\
\bigtriangleup
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\backslash \ldots \bigtriangledown \\
\text{or}
\end{array}
\]

with the case $n = 1$ representing the identity element.

**Definition 12.** A $(1,m)$-graph, with $m \geq 1$, is called surjection-like if it is of the form

\[
\begin{array}{c}
1 \quad 2 \quad 3 \quad \cdots \quad n \\
\bigtriangleup
\end{array}
\]

where there are no internal vertices (in particular $\sum_{i=1}^m k_i = n$) and for every $1 \leq r \leq m$ the associated map $\overline{k_r} \rightarrow \pi$ is order preserving.

The surjection $\pi \rightarrow \overline{m}$ is called the associated surjection to the surjection-like graph.

Define the prop $Sh$ to be the smallest subprop of $S$ containing all elements supported on surjection-like graphs.
Definition 13. Let $\mathcal{I}$ be the ideal of $\mathcal{S}$ generated by the associative, coassociative, Leibniz and involutive elements:

Definition 14. The prop $\mathcal{MS}$ is defined as the quotient of $\mathcal{Sh}$ by $\mathcal{I} \cap \mathcal{Sh}$.

Definition 15. The inclusion and quotient maps define a sequence of prop morphism

$$S \hookrightarrow \mathcal{Sh} \rightarrow \mathcal{MS}.$$ 

A.2. Comparing the surjection operad and $U(\mathcal{MS})$. The following definition is due to McClure-Smith and Berger-Fresse. We refer to the resulting operad as the surjection operad. We present it over the field with two elements $\mathbb{F}_2$ and remit the interested reader to [MS03] and [BF04] for their respective sign conventions.

Definition 16. For a fixed $m \geq 1$, consider the free differential graded $\mathbb{F}_2$-module with a basis given by all functions $s : \pi \rightarrow M$ with any $n \geq 1$. The degree of a basis element $s : \pi \rightarrow M$ is $(n - m)$ and its differential is defined to be

$$\partial s = \sum_{k=1}^{n} s \circ \iota_k$$

where $\iota_k : \overline{n - 1} \rightarrow \overline{m}$ is the order preserving injection that misses $k$. Consider the free differential graded $\mathbb{F}_2$-submodule generated by functions $s : \pi \rightarrow M$ which are either non-surjective or for which $s(i)$ equals $s(i + 1)$ for some $i$. Define $\text{Sur}(m)$ to be the associated quotient. The collection $\text{Sur} = \{\text{Sur}(m)\}_{m \geq 1}$ is a $\Sigma$-module with the action of $\Sigma_m$ on $\text{Sur}(m)$ given by postcomposition. The $\Sigma$-module $\text{Sur}$ is an operad with partial composition $\circ_r : \text{Sur}(m') \otimes \text{Sur}(m) \rightarrow \text{Sur}(m + m' - 1)$ defined on two generators $s : \pi \rightarrow M$ and $s' : \pi' \rightarrow M'$ as follows. Represent the surjections $s$ and $s'$ by sequences $(s(1), \ldots, s(n))$ and $(s'(1), \ldots, s'(n'))$ and suppose that $r$ appears $k$ times in the sequence representing $s'$ as $s'(i_1), \ldots, s'(i_k)$. Denote the set of all tuples

$$1 = j_0 \leq j_1 \leq \cdots \leq j_k = n$$

by $J(k,n)$ and for each such tuple consider the subsequences

$$(s(j_0), \ldots, s(j_1)) \quad (s(j_1), \ldots, s(j_2)) \quad \cdots \quad (s(j_{k-1}), \ldots, s(j_k)).$$

Then, in $(s'(1), \ldots, s'(n))$, replace the term $s'(i_t)$ by the sequence $(s(j_{i-1}), \ldots, s(j_i))$. In addition, increase the terms $s(j)$ by $r - 1$ and the terms $s'(i)$ such that $s'(i) > r$ by $m - 1$. The surjection $s' \circ_r s$ is represented by the sum, parametrized by $J(k,n)$, of these resulting sequences.

terminology. An element in $\mathcal{MS}$ supported on a surjection-like graph is called a surjection-like element and we refer to the surjection associated to the supporting graph as the surjection associated to the element.

Theorem 17. The assignment mapping a surjection-like element to its associated surjection extends, in the category of operad over the category of differential graded $\mathbb{F}_2$-module, to an isomorphism from $U(\mathcal{MS})$ to $\text{Sur}$.

Lemma 18. Consider in $\mathcal{MS}$ a linear combination $\sum \alpha_i S_i$ of surjection-like elements. Then, $\sum \alpha_i S_i = 0$ if and only if $\alpha_i \neq 0$ implies $S_i$ contains a copy of $\hat{\mathcal{S}}$.

Proof. The “only if” direction follows directly from the involutive identity. The “if” direction amounts to show that $\mathcal{I} \cap \mathcal{Sh}$ has no relations coming from combinations of relations. This follows from showing, in the terminology of Gröbner bases, that “all critical monomials are confluent”. We verify this next.

Studying the interaction of involution and the Leibniz rule we have
Studying the interaction of counitality and the Leibniz rule we have

Studying the interaction of coassociativity and the Leibniz rule we have

Studying the interaction of associativity and the Leibniz rule we have
The remaining pairs of relations are straightforward and left to the interested reader to check. □

Continuing the study of Theorem 17, notice that a surjection-like element contains a copy of $\hat{\ast}$ if and only if its associated surjection has two equal consecutive values, i.e., if and only if it is 0 in $\text{Sur}$. By definition, elements supported on surjection-like graphs form a set of operadic generators for $U(\mathcal{MS})$. We claim that those elements which contain no copy of $\hat{\ast}$ form a linear basis for each $U(\text{Sh})(m)$. From the claim we would then obtain for each $m$ a bijection between the linear bases of $U(\text{Sh})(m)$ and $\text{Sur}(m)$ that is easily seen to respect degree, differential and symmetric action. The fact that this elements form linear bases for each $U(\mathcal{MS})(m)$ and that the corresponding bijections respect operadic composition follow from the next

**Lemma 19.** For any pair $r,n \geq 1$ denote by $A(r,n)$ the set of all sequences $a = (a_1, \ldots, a_r)$ of non-negative integers satisfying $1 + a_1 + \cdots + a_r = n$ and for any such sequence $a = (a_1, \ldots, a_r)$ consider the element in $(S/I)(r,n)$ defined by

$$
\langle (a) \rangle \overset{\text{def}}{=} a_1 a_2 \cdots a_r
$$

Then, for any pair $k, n \geq 1$ we have the following identity in $S/I$

$$
\binom{k}{n} = \sum_{a \in A(k,n)} \langle (a) \rangle
$$

**Corollary 20.** For every $m \geq 1$, the set of surjection-like elements in $U(\mathcal{MS})(m)$ containing no copy of $\hat{\ast}$ is a linear basis for it. Explicitly, the $r$-th partial composition of two surjection-like elements satisfies
Proof. The above expression shows that the surjection-like elements form a set of linear generators and Lemma 18 shows that surjection-like elements containing no copy of $\phi$ are linearly independent.

From this, the proof of Theorem 17 now follows using the fact that for $k, m \geq 1$ the set $J(k, m)$ (used for operadic composition of surjections) can be identify with $A(k, m)$ through the map sending

$$1 = j_0 \leq \cdots \leq j_k = m$$

to

$$\langle j_1 - j_0, \ldots, j_m - j_{m-1} \rangle.$$ 

Keeping this in mind, one can unwind the definitions and verify the required composition compatibility of the map $U(MS) \to Sur$.

**Proof of Lemma 19.** For $n = k = 2$ the statement is precisely the Leibniz relation. For $n = 2$ and $k > 2$ we use the following inductive argument.

which by induction equals

which in turns equals

as desired. Now, we proceed by induction on $n$.  

\begin{align*}
\sum_{i=0}^{k-1} k_{-i} \cdot x_i &= \sum_{a \in A(k+1,2)} (e) \\
\end{align*}
where we used the case \( n = 2 \) proven before. Now, by induction, the sum above equals

\[
\sum_{i=0}^{k-1} \sum_{a \in A(k-i-1,n)} (a) = \sum_{b \in A(k,n+1)} (b)
\]

where the assignment takes \( a \in A(k-i-1,n) \) to \( b \in A(k,n+1) \) with

\[
b_j = \begin{cases} 
    a_j & \text{if } j < k - i - 1 \\
    a_{k-i} + 1 & \text{if } j = k - i - 1 \\
    0 & \text{if } j > k - i - 1.
\end{cases}
\]

This concludes the proof of the lemma and of Theorem 17. \( \square \)

A.3. Comparing \( E_\infty \)-coalgebra structures. Next we describe the natural \( \text{Sur} \)-coalgebra structures on the normalized chains with \( \mathbb{F}_2 \)-coefficients of the standard simplices. These operad morphisms

\[
\phi[d] : \text{Sur} \rightarrow \text{End}(C_*(\Delta^d))
\]

are due to McClure-Smith and Berger-Fresse, see [MS03] and [BF04] for their respective sign conventions.

Definition 21. Fix an object \( d \) in the simplicial category. The maps

\[
\phi[d](m) : \text{Sur}(m) \otimes C_*(\Delta^d) \rightarrow C_*(\Delta^d)^{\otimes m}
\]

are defined on bases elements as follows. For any \( \sigma = [v_0, \ldots, v_d] \in C_*(\Delta^d) \) and \( s = (s(1), \ldots, s(n)) \in \text{Sur}(m) \) the element \( \phi[d](m)(s \otimes [v_0, \ldots, v_d]) \) is of the form

\[
\sum_I \sigma_I^1 \otimes \cdots \otimes \sigma_I^m
\]

where the sum ranges over all sequences \( I = \{i_0, \ldots, i_n\} \) such that \( 0 = i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i_n = d \), and each \( \sigma_I^r \) is a face of \( \sigma \) that we now define. For a fix sequence \( I \) and \( 1 \leq r \leq m \), let \( \{j_1^r, \ldots, j_k^r\} = s^{-1}(r) \); define \( \sigma_I^r \) to be equal to either

\[
[v_{i_1^1}, \ldots, v_{i_k^1}, v_{i_1^2}, \ldots, v_{i_k^2}, \ldots, v_{i_1^r}, \ldots, v_{i_k^r}]
\]

if all vertices are distinct and to be 0 if they are not.

Lemma 22. Setting the underlying ring \( k = \mathbb{F}_2 \). The natural family of prop morphisms

\[
\{\varphi[d] : S \rightarrow \text{End}(C_*(\Delta^d))\}_{d \in \Delta}
\]

described in Theorem \( \square \) factors through \( \text{MS} \) when restricted to \( \text{Sh} \). That is to say, the dashed arrow making the diagram
commutative exists for every object \( d \) in the simplicial category.

**Proof.** The fact that the join map is associative with \( \mathbb{F}_2 \)-coefficients follows from the associativity of union of sets. The Alexander-Whitney map being coassociative is a classical result that can be easily verified. To see that the involution element is sent to the zero map, recall that the Alexander-Whitney map of any simplex is a sum of tensor pairs of simplices that share a vertex, and that the join of two simplices that contain a common vertex equals 0. To verify that the Leibniz element is sent to the map that takes any \([v_0, \ldots, v_p] \otimes [w_0, \ldots, w_q]\) to 0, we notice that the order preserving condition imposed on the generators of \( Sh \) allows us to assume \( v_i \leq w_j \) for all \( i \) and \( j \). Therefore, the images of both

\[
\sum_{i=0}^{p} [v_0, \ldots, v_i] \otimes [v_i, \ldots, v_p, w_0, \ldots, w_q] + \sum_{j=0}^{q} [v_0, \ldots, v_p, w_0, \ldots, w_j] \otimes [w_j, \ldots, w_q],
\]

which concludes the verification. \( \square \)

**Remark 23.** One could also consider the quotient of the full \( S \) by the ideal \( I \) but, since the natural family \( \{ S \to \text{End}(C_\bullet(\Delta^d)) \}_{d \in \Delta} \) does not factor through it, we do not. For example, the image in \( \text{End}(C_\bullet(\Delta^2)) \) of the Leibniz element does not map \([0, 2] \otimes [1]\) to 0.

**Remark 24.** We thank Bruno Vallette for pointing out that the join map, as defined in Theorem 6, is not associative with \( \mathbb{Z} \)-coefficients.

**Theorem 25.** For every object \( d \) in the simplicial category we have a natural commutative diagram in the category of operads over the category of differential graded \( \mathbb{F}_2 \)-modules:

\[
\begin{array}{ccc}
U(S) & \xrightarrow{U(\varphi[d])} & \text{End}(C_\bullet(\Delta^d)) \\
\uparrow & & \uparrow \\
U(Sh) & & \\
\downarrow & & \\
U(MS) & \xrightarrow{\varphi[d]} & \text{Sur}
\end{array}
\]

where the horizontal map is the isomorphism of Theorem 17 and the diagonal arrow is the operad morphism induced from the dashed arrow of Lemma 22.

**Proof.** Let \( s \in \text{Sur}(m) \) be a surjection of degree \((n - m)\) and for every \( r = 1, \ldots, m \) let

\[
s^{-1}(r) = \{ i_1^r, \ldots, i_{k_r}^r \}.
\]

Then, the surjection-like element whose associated surjection is \( s \) factors as follows:
Recall from Theorem 6 that
\[
\sum_{I} [v_{i_0}, \ldots, v_{i_n}] = \sum_{I} [v_{i_0} \otimes [v_{i_1}, \ldots, v_{i_2}] \otimes \cdots \otimes [v_{i_{n-1}}, \ldots, v_{i_n}]
\]
where the sum ranges over all sequences \(0 = i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i_n = d\).

If we then apply
\[
\sum_{I} \sigma_I^1 \otimes \cdots \otimes \sigma_I^m
\]
with \(\sigma_I^r\) agreeing with (2) in Definition 21. This concludes the proof. \(\square\)

**Corollary 26.** For every simplicial set \(X\) the following diagram is commutative
\[
\begin{array}{ccc}
U(S) & \xrightarrow{U(\varphi)} & \text{End}(C_\bullet(X)) \\
\uparrow & & \downarrow \\
U(Sh) & \xrightarrow{\phi} & \text{Sur} \\
\downarrow & & \\
U(MS) & \xrightarrow{\simeq} & \text{Sur}
\end{array}
\]
with \(\mathbb{F}_2\)-coefficients.

**Proof.** Holds since the previous theorem shows the result for representable simplicial sets. See Corollary 8 and the reference there for further details. \(\square\)

**Appendix B. The augmented simplicial category and the prop \(S\)**

In this appendix we describe a family, natural with respect to augmented simplicial maps, of \(S\)-algebra structures on the chains of the standard augmented simplices, i.e., natural prop prop morphisms
\[
\{S \to \text{End}^{op}(C_\bullet(\Delta^d_+))\}_{d \in \Delta^d_+}.
\]
The image of the three generators of $\mathcal{S}$ are the join map, the empty simplex and the Alexander-Whitney map. Comparing with the family of $\mathcal{S}$-coalgebra structures on the chains of the standard simplices described in Section 4, we see a duality relationship between diagonal approximations and join operations in the usual and augmented simplicial contexts.

**Notation 27.** Consider the augmented simplicial category whose objects are the sets

$$\bar{0} = \emptyset, \quad \bar{1} = \{1\}, \quad \bar{2} = \{1, 2\}, \ldots$$

and whose morphisms are order preserving maps. The normalized chain complex functor is defined analogously to the non-augmented case. It satisfies $\partial[v] = [\emptyset]$ for all vertices and it is graded by cardinality instead of dimension. The functor of normalized cochains is defined by postcomposing with the functor of linear duality. We denote the respective images of an object $\bar{d}$ by $\mathcal{C}^a\bullet(\Delta^d_\pm)$ and $\mathcal{C}^a_{\bullet}(\Delta^d_\pm)$.

**Theorem 28.** For every object $\bar{d}$ of the augmented simplicial category, the following assignment of generators defines an $\mathcal{S}$-algebra $\mathcal{S} \to \text{End}(\mathcal{C}^a(\Delta^d_+))$ natural with respect to augmented simplicial maps:

\[ I \in \text{Hom}(\mathcal{C}^a(\Delta^d_+))_0 \text{ is defined by } I(1) = [\emptyset]. \]

\[ \prec \in \text{Hom}(\mathcal{C}^a(\Delta^d_+) \otimes^2, \mathcal{C}^a(\Delta^d_+))_0 \text{ is defined by } \]

\[ \prec \left( [v_0, \ldots, v_p] \otimes [v_{p+1}, \ldots, v_q] \right) = \begin{cases} \text{sign}(\pi) \cdot [v_{\pi(0)}, \ldots, v_{\pi(p)}] & \text{if } i \neq j \text{ implies } v_i \neq v_j \\ 0 & \text{if not.} \end{cases} \]

where $\pi$ is the permutation that orders the totally ordered set of vertices.

\[ \uparrow \in \text{Hom}(\mathcal{C}^a(\Delta^d_+), \mathcal{C}^a(\Delta^d_+) \otimes^2)_1 \text{ is defined by } \]

\[ \uparrow [v_0, \ldots, v_q] = \sum_{i=0}^d (-1)^i [v_0, \ldots, v_i] \otimes [v_i, \ldots, v_q]. \]

**Proof.** Throughout this proof we identify $I$, $\prec$, and $\uparrow$ with their images in $\text{End}(\mathcal{C}^a(\Delta^d_+))$. We need to verify that the relations of $\mathcal{S}$ are satisfied by these images.

The fact that the maps $\prec - I$ and $I - \prec$ are equal to the zero map follows from the fact that the empty set is the unit for the union of sets. The formula defining $\uparrow$ as well as the fact that it has degree 1 makes the definition of $\uparrow$ as the zero map the only sensible one. In order to establish $\partial \uparrow = I - ||\cdot||$, we must consider two cases for the basis element to which the map is applied:

1. Degree 0

\[ \begin{array}{ccc} \uparrow & \downarrow & \partial \\ ||\cdot|| & \uparrow & ||\cdot|| \\ 0 & 0 & 0 \end{array} \]

2. Degree greater than 0
In order to establish $\partial \bigwedge = 0$, we must consider four cases for the basis elements to which the map is applied:

1. Both of degree 0

2. Only one of degree 0

3. None of degree 0 and sharing a vertex

4. None of degree 0 and not sharing a vertex
Remark 29. Adamaszek and Jones have also explored a relationship between the higher joins and the Steenrod diagonal in [AJ13].

REFERENCES

[AJ13] Michal Adamaszek and John DS Jones. The symmetric join operad. Homology, Homotopy and Applications, 15(2):245–265, 2013.

[BF04] Clemens Berger and Benoit Fresse. Combinatorial operad actions on cochains. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 137, pages 135–174. Cambridge University Press, 2004.

[BV06] John Michael Boardman and Rainer M Vogt. Homotopy invariant algebraic structures on topological spaces, volume 347. Springer, 2006.

[KM95] Igor Kriz and J. P. May. Operads, algebras, modules and motives. Astérisque, (233):iv+145pp, 1995.

[Mar08] Martin Markl. Operads and props. Handbook of algebra, 5:87–140, 2008.

[May03] JP May. Operads and sheaf cohomology. Preprint, December, 2003.

[MS03] James McClure and Jeffrey Smith. Multivariable cochain operations and little n-cubes. Journal of the American Mathematical Society, 16(3):681–704, 2003.

[Val07] Bruno Vallette. A Koszul duality for props. Transactions of the American Mathematical Society, 359(10):4865–4943, 2007.