Markov Chain Analysis of Musical Dice Games

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Abstract

We have studied entropy, redundancy, complexity, and first passage times to notes for 804 pieces of 29 composers. The successful understanding of tonal music calls for an experienced listener, as entropy dominates over redundancy in musical messages. First passage times to notes resolve tonality and feature a composer. We also discuss the possible distances in space of musical dice games and introduced the geodesic distance based on the Riemann structure associated to the probability vectors (rows of the transition matrices).

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1 Musical dice game as a Markov chain

A system for using dice to compose music randomly, without having to know neither the techniques of composition, nor the rules of harmony, named Musikalisches Würfelspiel (Musical dice game) had become quite popular throughout Western Europe in the 18th century [1]. Depending upon the results of dice throws, the certain pre-composed bars of music were patched together resulting in different, but similar, musical pieces. "The Ever Ready Composer of Polonaises and Minuets" was devised by Ph. Kirnberger, as early as in 1757. The famous chance music machine attributed to W.A. Mozart (K 516f) consisted of numerous two-bar fragments of music named after the different letters of the Latin alphabet and destined to be combined together either at random, or following an anagram of your beloved had been known since 1787.

We can consider a note as an elementary event in the musical dice game, as notes provide a natural discretization of musical phenomena that facilitate their performance and analysis. Given the entire keyboard $K$ of 128 notes corresponding to a pitch range of 10.5 octaves, each divided into 12 semitones, we regard a note as a discrete random variable $X$ that maps the musical event to a value of a $n$-set of pitches $P = \{x_1, \ldots, x_n\} \subseteq K$. In the musical dice game, a piece is generated by patching notes $X_t$ taking values from the set of pitches $P$ that sound good together into a temporal sequence $\{X_t\}_{t \geq 1}$. Herewith, two consecutive notes, in which the second pitch is a harmonic of the first one are considered to be pleasing to the ear, and therefore can be patched to the sequence. Thus tonal harmony sets up the Markov property for the sequence $\{X_t\}_{t \geq 1}$ that can be assessed in terms of the transition probabilities between consecutive notes, in the framework of a simple time – homogeneous model called Markov chain [2],

$$\Pr \left[ X_{t+1} \mid X_t = y, X_{t-1} = z, \ldots \right] = \Pr \left[ X_{t+1} \mid X_t = y \right] = T_{yx},$$

$$\sum_{x \in P} T_{yx} = 1,$$

(1)

where the stochastic transition matrix $T_{yx}$ weights the chance of a pitch $x$ going directly to another pitch $y$ independently of time. The model (1) obviously does not impose a severe limitation on melodic variability, since there are many possible combinations of notes considered consonant, as sharing some harmonics and making a pleasant sound together. The relations between notes in (1) are rather described in terms of probabilities and expected numbers of random
steps than by physical time. Under such circumstances, the actual length \( N \) of a composition is formally put \( N \to \infty \), or as long as you keep rolling the dice. Markov chains are widely used in algorithmic music composition, as being a standard tool, in music mix and production software.

Interactions between humans via speech and music constitute the unifying theme of research in modern communication technologies. As with music, speech and written language also have the sets of rules (crucial for establishing effective communication) that govern which particular combinations of sounds and letters may or may not be produced. However, while communications by the spoken and written forms of human languages have been paid much attention from the very onset of information theory \([3, 4]\), not very much is known about the relevant information aspects of music \([5]\). Although we use the acoustic channel in both music and speech, the acoustical and structural features we implement to encode and perceive the signals in music and speech are dramatically different, as "speech is communication of world view as the intellection of reality while music is communication of world view as the feeling of reality" \([6]\). With the Markov chain model \([11]\), we could precisely quantify this difference, since it allows to appraise tonal music as a generalized communication process, in which a composer sends a message transmitted by a performer to a listener.

In our work, we report some results on the Markov chain analysis of the musical dice games encoded by the transition matrices between pitches in the MIDI representations of the 804 musical compositions attributed to 29 composers: J.S. Bach (371), L.V. Beethoven (58), A.Berg (7), J. Brahms (8), D. Buxtehude (3), F. Chopin (26), C. Debussy (26), G. Fauré (5), C. Franck (7), G.F. Händel (45), F. Liszt (4), F. Mendelssohn Bartholdi (19), C. Monteverdi (13), W.A. Mozart (51), J. Pachelbel (2), S. Rachmaninoff (4), C. Saint-Saëns (2), E. Satie (3), A. Schönberg (2), F. Schubert (55), R. Schumann (30), A. Scriabin (7), D. Shostakovitch (12), J. Strauss (2), I. Stravinsky (5), P. Tchaikovsky (5), J. Titelouze (20), A. Vivaldi (4), R. Wagner (8). The MIDI representations of many musical pieces are freely available on the Web \([7]\).

The paper is organized as follows. In Sec. \([2]\) we discuss the MIDI representations of music and the different methods to encode them into a Markov chain transition matrix. The encoding problem is not trivial, as ambiguities would arise provided a piece has more than one voice. We then consider a music as a
generalized communication process in Sec. While the elements of the transition matrix indicate the possibility to consequently find the two notes in the musical score, an infinite number of powers of the transition matrix must be considered to estimate the eventual distance between them with respect to the entire structure of the musical dice game. First passage times to notes and the classification of composers with respect to their tone scale preferences are discussed in Sec. The possible distances between the different musical dice games are discussed in Sec. We conclude in the last section.

2 Encoding of a discrete model of music (MIDI) into a transition matrix

While analyzing the statistical structure of musical pieces, we used the MIDI representations providing a computer readable discrete time model of music by a sequence of the 'note' events, note_on and note_off: In the MIDI representation, each event (like that one shown in Tab. 1) is characterized by the four variables: 'time', 'channel', 'note', and 'velocity'. A MIDI file has a specific value of discreteness 'ticks/quarter' indicating the number of 'ticks' that make up a quarter note. The value of 'time' then gives the number of 'ticks' between two consequent note events. In the example given in Tab. 1 the event of C4 starts after 192 'ticks' have passed. The 'channel' indicate one of 16 channels (0 to 15) this event may belong to. Notes are not encoded by their names like C or A. Instead, the harmonic scale is mapped onto numbers from 0 to 127 with chromatic steps. For instance, the identification number 60 corresponds to the C4, in musical notation. Then, note number 61 is C4#, 62 is D4 etc. (see Tab. 2 for some octaves and their MIDI note ID numbers)

Finally, the 'velocity' (0 – 127) describes the strength with which the note is played. As MIDI files contain all musically relevant data, it is possible to determine the probabilities of getting from one note to another for all notes in
Octave | C  | C# | D  | D# | E  | F  | F# | G  | G# | A  | A# | B  |
---|---|---|---|---|---|---|---|---|---|---|---|---|
3  | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
4  | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 |
5  | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 |

Table 2: MIDI note ID numbers corresponding to musical notation.

a musical composition by analyzing its MIDI file with a computer program. To get transition matrices for tonal sequences, we need only 'time', 'channel', and 'note' to be considered.

The MIDI files of 804 musical compositions were processed by a program written in Perl; the MIDI parsing was done using the module Perl::MIDI [8], which allowed the conversion of the MIDI data into a more convenient form called MIDI::Score where each two consequent note_on and note_off events are combined to a single note event. Each note event contains an absolute time, the starting time of the event, and a duration which gives the duration of the event in ticks.

Figure 1: The first three bars from the fugue of BWV846. Also shown are MIDI note numbers.

To give an example of the process of getting to a transition matrix from a musical score, we consider the first bars of the Fugue BWV846 of J.S. Bach shown in Fig. 1. The numbers below the first notes in Fig. 1 indicate the corresponding MIDI ID note numbers. In Tab. 3 we show the representation of these notes in MIDI and in MIDI::Score format. Here, the value of velocity is omitted.

For the first notes shown in Fig. 1 the definition of a transition is easy as there is only one voice. In particular, from Tab. 3 we can conclude that there would be the consequent transitions 60 → 62 and 62 → 64. However, like most
musical pieces, this Fugue then contains several voices that play simultaneously, so that an additional convention is required to define a transition from note to note.

Table 4: MIDI::Score data from the middle of the second bar of Fugue I BWV846 where the second voice starts playing. The note names and the voices of the events are also shown.

In the middle of the second bar shown in Fig. 1, a second voice is starting. Some note events starting from there are given in Tab. in MIDI::Score form. From Tab. it is clear that in principle it is not necessary to put the upper voice into a different channel than that of the lower voice. In the example shown in Tab. the notes 67 and 64 both start at time 2496. As note 64 has a duration of 96 ticks, it is obvious that note 62 at time 2592 belongs to the same voice as note 64. However, for the notes 69 and 60 starting at 2688, it is unclear to which voice each note belongs to, and how they might be encoded into a transition matrix. It is important to note that such an ambiguity is not a
The problem of MIDI representation itself, but rather of music. It depends upon the experience of a listener how she distinguishes voices while listening to a musical composition that contains several simultaneous voices. Even if the musical score explicitly separates those voices by placing them atop of each other, our personal impression of them might not coincide with that one notated, rather arising from live audio mixing of all simultaneous voices during the performance. Thus, to get transition matrices from MIDI files, we have to answer the following important question: "Which transitions between which note events have to be accounted?"

In our approach, we sort note events ascending by time and channel. By surfing over the list of events, a transition between two subsequent occurrences is accounted when the moment of time of the second event is greater than that of the first one. When several events occur simultaneously, we give a priority to the event belonging to a small number channel. Let us emphasize that under the used method not all possible transitions between note events contribute into the transition matrix. For example, let us consider the musical bar shown in

| time (ms) | dur (ms) | ch | note | name |
|----------|---------|----|------|------|
| 13056    | 288     | 0  | 76   | (E5) * |
| 13056    | 288     | 0  | 72   | (C5)  |
| 13056    | 192     | 1  | 60   | (C4)  |
| 13152    | 96      | 1  | 59   | (B3) * |
| 13248    | 96      | 1  | 60   | (C4) * |
| 13344    | 96      | 0  | 78   | (F#5) * |
| 13344    | 48      | 0  | 74   | (D5)  |
| 13344    | 96      | 1  | 57   | (A3)  |
| 13392    | 48      | 0  | 72   | (C5) * |

Figure 2: Example from our fugue. [7]

Fig. 2 and its list of events given in the adjacent table. The resulting transitions accounted in the matrix would be those between events marked with '*': 76 → 59, 59 → 60, 60 → 78, 78 → 72. Note events with small channel values are favored over those with higher values. For simultaneous note events occurring in the same channel, only the first one is considered that mostly means the topmost voice, in musical notation. We believe that the encoding method we use is quite efficient for unveiling the individual melodic lines and identifying a creative character of a composer from musical compositions because of the appearance
Figure 3: Transition matrices for the F. Chopin “Revolutionary Etude” (Op.10, No.12) (left) and the I. Stravinsky “The Fire-bird” suite (right) generated accordingly to the encoding method we used.

of the resulting transition matrices. Those matrices generated with respect to the chosen encoding method look differently, from piece to piece and from composer to composer (see the examples shown in Fig. 3). However, if we were treated each voice in a musical composition separately (the transitions of the upper voice and those of the lower voice might be accounted independently while computing the probabilistic vector forming a row of the transition matrix), the transition matrices were clearly dominated by a region along the main diagonal, similarly for all compositions.

It is important to mention that no matter which encoding method is used the resulting transition matrices appear to be essentially not symmetric: if $T_{xy} > 0$, for some $x, y$, it might be that $T_{yx} = 0$. A musical composition can be represented by a weighted directed graph, in which vertices are associated with pitches and directed edges connecting them are weighted accordingly to the probabilities of the immediate transitions between those pitches. Markov’s chains determining random walks on such the graphs are not ergodic: it may be impossible to go from every note to every other note following the score of the musical piece.
3 Musical dice game as a communication process

Contrary to the alphabets used in human languages, the sets of pitches underlying the different musical compositions can be very distinct and may not overlap (even under chromatic transposition). The cardinality of the set of pitches $\mathcal{P}$ also changes from piece to piece demonstrating a tendency of slow growth, with the length of composition $N$. In Fig. 4, we have sketched how the number of different pitches $n$ used to compose a piece depends upon the size of composition $N$. The data collected over 610 pieces created by the six classical music composers show that the growth can be well approximated by a logarithmic curve indicating that $n \sim \log N$ can be used as the simplest parameter assessing complexity of a classical musical composition. Let us suppose that a musical piece generated as an output of the musical dice game (1) can be encoded by a sequence of independent and identically-distributed random variables representing notes which can take values of different pitches. To measure the uncertainty...

Figure 4: The number of different pitches used in a composition grows approximately logarithmically with the size of compositions. The data have been collected over 371 chorales of J.S. Bach, 58 various compositions of L.V. Beethoven, 55 compositions of F. Schubert, 51 compositions of W.A. Mozart, 45 compositions of G.F. Händel, and 30 compositions of R. Schumann.
associated with a pitch, we can then use the Shannon entropy [9],

\[ H = - \sum_{x \in \mathcal{P}} \pi_x \log_n \pi_x \]  

(2)

where \( \pi_x \) is the probability to find the note \( x \in \mathcal{P} \) in the musical score, and the base of the logarithm is \( n = |\mathcal{P}| \). Since the entropy of a musical piece defined by (2) is affected by the number of used pitches \( n \), the parameter of information redundancy,

\[ R = 1 - \frac{H}{\max H}, \quad \max H = \log n, \]  

(3)

where \( \max H \) is the theoretical maximum entropy, might be used for comparing different musical compositions. Accordingly to information theory [10], redundancy quantifies predictability of a pitch in the piece, as being a natural counterpart of entropy.

As we have mentioned above, a Markov chain encoding the musical dice game is not ergodic, and therefore the probability to find a pitch in the musical score cannot be found simply as the entry in the left eigenvector of the transition matrix \( T \) belonging to the maximal eigenvalue \( \mu = 1 \). In order to find the probability of observing the note in the musical score, we can use the method of group generalized inverse [11, 12] that might be applied for analyzing every Markov chain regardless of its structure. As the Laplace operator corresponding to the Markov chain (1),

\[ L = 1 - T, \]  

(4)

where \( 1 \) is a unit matrix, is always a member of a multiplicative matrix group, it always possesses a group inverse \( L^\dagger \), a special case of the Drazin generalized inverse [13, 14, 11] satisfying the Erdélyi conditions [15]:

\[ LL^\dagger L = L, \quad L^\dagger LL^\dagger = L^\dagger, \quad [L, L^\dagger] = 0 \]  

(5)

where \([A, B] = AB - BA\) denotes the commutator of the two matrices. The role of group inverses (5) in the analysis of Markov chains have been discussed in details in [11, 12, 16]. Here, we only mention that the stationary vector of a Markov chain can be calculated as

\[ \pi_{x_i} = \left(1 - LL^\dagger\right)_{x_i, x_j}; \]  

(6)

the rows of (6) are all equal to the corresponding components of the stationary vector \( \pi \).
Determining the entropy of texts written in a natural language is an important problem of language processing. The entropy of current written and spoken languages (English, Spanish) has been estimated experimentally as ranged from 0.5 to 1.3 bit per character [4, 17]. An approximately even balance (50:50) of entropy and redundancy is supposed as necessary to achieve effective communication in transmitting a message, as it makes easier for humans to perceive information [17].

![Bach chorales: entropy(gray) & redundancy (black)](image)

Figure 5: The box plots show the statistic of the magnitudes of entropy and redundancy vs. the number of pitches used in a composition, for 371 chorales of J.S. Bach. In a box plot, a central line of each box shows the median; a lower line shows the first quartile; an upper line shows the third quartile; two lines extending from the central box of maximal length 3/2 the interquartile range but not extending past the range of the data; eventually, the outliers are those points lying outside the extent of the previous elements.

For all musical compositions we studied, the magnitudes of entropy fluctuate in a range between 0.7 and 1.1 bit per note well fitting with the entropy range of usual languages. In classical music where the tonal method of composition is widely used, pieces involving more pitches appear to have lower magnitudes of entropy but higher values of redundancy (predictability). In Fig. 5 we have presented the statistics of entropy and redundancy vs. the number of pitches.
through their five-number summaries, for 371 chorales of J.S. Bach. A central line of each box in the box plot (Fig. 5) shows the median (not the mean), the value separating the higher half of the data sample from the lower half, that is found by arranging all the observations from lowest value to highest value and picking the middle one. Other lines of the box plot indicate the quartile values which divide the sorted data set into four equal parts, so that each part represents one fourth of the sample. A lower line in each box shows the first quartile, and an upper line shows the third quartile. Two lines extending from the central box of maximal length 3/2 the interquartile range but not extending past the range of the data. The outliers are those points lying outside the extent of the previous elements.

The entropy and redundancy statistics suggests that compositions in classical music might contain some repeated patterns, or motives in which certain combinations of notes are more likely to occur than others. In particular, the dramatic increase of redundancy as the range of pitches expands up to 7.5 octaves implies that musical compositions involving many pitches might convey mostly conventional, predictable blocks of information to a listener. However, in contrast to human languages where entropy and redundancy are approximately equally balanced [17], in classical music entropy clearly dominates over redundancy. While decoding a musical message requires the listener to invest nearly as much efforts as in everyday decoding of speech, the successful understanding of the composition would call for an experienced listener ready to invest his or her full attention to a communication process that would span across cultures and epochs.

Another possible information measure that can be applied to the analysis of musical dice games is the past-future mutual information (complexity) introduced in studies of the symbolic sequences generated by dynamical systems [18] (see also [10]). It estimates the information content of the blocks of notes and can be formally derived as the limiting excess of the block entropy

\[ C = \lim_{N \to \infty} \left( H(S^N) - H \cdot N \right). \] (7)
Figure 6: The box plot (left) represents complexity (measured by the past-future mutual information) vs. the number of pitches used in a composition, for 371 chorales of J.S. Bach. The trend (shown by the solid lines) is the cubic splines interpolating between the mean values of complexity over the data ranges. The scatter plot of complexity vs. the magnitude of entropy in 480 pieces written by classical composers (given in the log-linear scale) suggests that a strong positive correlation exists between the value of entropy and the logarithm of complexity. The reference line indicates an exponential growth, in the log-linear scale.

Following [19], we use the fact that the transition probability between states in a Markov chain determined by the matrix (1) is independent of $N$, so that complexity (7) can be computed simply as

$$C = -\sum_{x \in P} \pi_x \log n \frac{\pi_x}{\prod_{y \in P} T_{xy}^{\pi_y}}. \quad (8)$$

In Fig. 6 (left), we have presented the statistics of complexity values for the Bach’s chorales. The main trend (shown in Fig. 6 (left) by a solid line) is given by a cubic spline interpolating between the mean (not the median) complexity values. Complexity decreases rapidly with the number of pitches used in a composition suggesting that the musical pieces might contain a few types of melodic lines translated over the entire diapason of pitches by chromatic transposition. Finally, in Fig. 6 (right), we have sketched a scatter plot showing the pace of complexity with entropy in 480 compositions of classical music that implies that a strong positive correlation exists between the value of entropy and the logarithm of complexity, in compositions of classical music.
4 First passage times to notes resolve tonality and feature a composer

Statistics of entropy, redundancy, and complexity in the discrete time models of classical musical compositions suggests that tonal music generated by the musical dice game (1) constitutes the well structured data that contain conventional patterns of information. Obviously, some notes might be more “important” than others, with respect to such a structure.

In music theory [20], the hierarchical pitch relationships are introduced based on a tonic key, a pitch which is the lowest degree of a scale and that all other notes in a musical composition gravitated toward. A successful tonal piece of music gives a listener a feeling that a particular (tonic) chord is the most stable and final. The regular method to establish a tonic through a cadence, a succession of several chords which ends a musical section giving a feeling of closure, may be difficult to apply without listening to the piece.

Figure 7: The histograms show the first passage times to the notes for the Duet I, BWV 802 (E minor) of J.S. Bach (left) and for the Cello Sonata No.3, Op.69 of L.V. Beethoven (E major, A major) (right) mapped into a single octave. Bars are shaded with the intensity of gray scale 0-100%, in proportion to the magnitude of the first passage time. Therefore, the basic pitches of a tonal scale are rendered with light gray color, as being characterized by short first passage times, and the tonic key by the smallest magnitude of all.

While in a musical dice game, the intuitive vision of musicians describing
the tonic triad as the "center of gravity" to which other chords are to lead acquires a quantitative expression. Namely, every pitch in a musical piece is characterized with respect to the entire structure of the Markov chain by its level of accessibility estimated by the first passage time to it \[21, 22\], that is the average length of the shortest random path toward the pitch from any other one randomly chosen in the musical score. Analyzing the first passage times in scores of tonal musical compositions, we have found that they can help in resolving tonality of a piece, as they precisely render the hierarchical relationships between pitches.

The majority of tonal music assumes that notes spaced over several octaves are perceived the same way as if they were played in one octave \[23\]. Using this assumption of octave equivalency, we can chromatically transpose each musical piece into a single octave getting the $12 \times 12$ transition matrices, uniformly for all musical pieces, independently of the actual number of pitches used in composition. Given a stochastic matrix $T$ describing transitions between notes within a single octave $O$, the first passage time to the note $i \in O$ is computed \[22\] as the ratio of the diagonal elements,

$$F_i = \frac{(L^\#)_{ii}}{1 - (LL^\#)_{ii}},$$

where $L$ is the Laplace operator corresponding to the transition matrix $T$, and $L^\#$ is its group generalized inverse. Let us note that in the case of ergodic Markov chains the result \[9\] coincides with the classical one on the first passage times of random walks defined on undirected graphs \[24\].

In Fig. 7, we have shown the two examples of the arrangements of first passage times to notes in one octave, for the E minor scale (left) and E major, A major scales (right). The basic pitches for the E minor scale are E, F#, G, A, B, C, and D. The E major scale is based on E, F#, G#, A, B, C#, and D#. Finally, the A major scale consists of A, B, C#, D, E, F#, and G#. The values of first passage times are strictly ordered in accordance to their role in the tone scale of the musical composition. Herewith, the tonic key is characterized by the shortest first passage time (usually ranged from 5 to 7 random steps), and the values of first passage times to other notes collected in ascending order reveal the entire hierarchy of their relationships in the musical scale.

By analyzing the typical magnitudes of first passage times to notes in one octave, we can discover an individual music language of a composer and track
Figure 8: Statistics of first passage times of in the musical pieces of J.S. Bach, A. Berg, F. Chopin, and C. Franck are represented through their five-number summaries in the box plots.

...out the stylistic influences between different composers. The box plots shown in Fig. 8 depict the data on first passage times to notes in a number of compositions written by J.S. Bach, A. Berg, F. Chopin, and C. Franck through their five-number summaries: 3/2 the interquartile ranges, the lower quartile, the third quartile, and the median. In tonal music, the magnitudes of first passage times to the notes are completely determined by their roles in the hierarchy of tone scales. Therefore, a low median in the box plot (Fig. 8) indicates that the note was often chosen as a tonic key in many compositions. Correlation and
Figure 9: The correlogram displays the correlation matrix for the medians of the first passage times to notes of one octave, for 23 composers. In the shaded rows, each cell is shaded from violet to red depending on the sign of the correlation, and with the intensity of color scaled 0-100%, in proportion to the magnitude of the correlation.

covariance matrices calculated for the medians of the first passage times in a single octave provide the basis for the classification of composers, with respect to their tonality preferences. For our analysis, we have selected only those musical compositions, in which all 12 pitches of the octave were used. The tone scale symmetrical correlation matrix has been calculated for 23 composers, with the elements equal to the Pearson correlation coefficients between the medians of the first passage times. For exploratory visualization of the tone scale correlation matrix, we arranged the "similar" composers contiguously. Following [25], while ordering the composers, we considered the eigenvectors (principal components) of the correlation matrix associated with its three largest eigenvalues. Since the cosines of angles between the principal components approximate...
the correlations between the tonal preferences, we used an ordering based on
the angular positions of the three major eigenvectors to place the most similar
composers contiguously, as it is shown in Fig. 9.

The correlogram presented on Fig. 9 allows for identifying the three groups of
composers exhibiting similar preferences in the use of tone scales, as correlations
are positive and strong within each tone group while being weak or even negative
between the different groups. The smaller subgroups might be seen within the
first largest group (from J. Strauss to G. Fauré), in the left upper corner of
the matrix on Fig. 9. Most of composers that appeared in the largest group
are traditionally attributed to the Classical Period of music. The strongest
positive correlations we observed in the choice of a tonic key (about 97%) is
between the compositions of J. Strauss and A. Vivaldi who led the way to a
more individualistic assertion of imaginative music. The tonality statistics in
the masterpieces of R. Wagner appears also quite similar to them. Other subgroups
are formed by G.F. Händel and D. Shostakovitch, J.S. Bach and R. Schumann.
The Classical Period boasted by L.V. Beethoven and W.A. Mozart who led
the way further to the Romantic period in classical music. F. Mendelssohn
Bartholdi was deeply influenced by the music of J.S. Bach, L.V. Beethoven, and
W.A. Mozart, as often reflected by his biographers [26] – not surprisingly, he
found his place next to them. Furthermore, the piano concerts of C. Saint-Saëns
were known to be strongly influenced by those of W.A. Mozart, and, in turn,
appear to have influenced those of S. Rachmaninoff that receives full exposure in
the correlogram (Fig. 9). Moreover, we also get the evidence of affinity between
I. Stravinsky and A. Berg, F. Schubert, F. Chopin, and G. Faure, as well as of the
strong correlation between the tonality styles of A. Scriabin and F. Liszt. The
last group, in the lower right corner of the matrix are occupied by the Middle
and Late Romantic era composers: P. Tchaikovsky, J. Brahms, C. Debussy,
and C. Franck. Interestingly, the names of composers that are contiguous in the
correlogram (Fig. 9) are often found together in musical concerts and on records
performed by commercial musicians.
5 On possible distances in space of musical dice games

Most music is written for playing on standard keyboards and involves mostly overlapping sets of pitches. Given two pieces modeled by the different musical dice games but defined on the same set of pitches, a natural idea arises to compare their Markov chains in order to estimate their similitude.

Let us note that the Kullback-Leibler divergence \[ D_{KL}(\pi^{(1)} | \pi^{(2)}) = \sum_{i=1}^{n} \pi^{(1)}_i \log \left( \frac{\pi^{(1)}_i}{\pi^{(2)}_i} \right) \] (10), a measure of the difference between two probability distributions playing the important role in information theory, cannot help us much with the Markov transition matrices since the transition probability vectors (rows of the transition matrices) in general are not the probability distributions, as many of their components might be equal to zero (even for quite a long composition mapped into one octave) thus prohibiting transitions between some states of the Markov chain. Nevertheless, the Kullback-Leibler divergence can be used in order to compare two different musical dice games defined on the same set of pitches by means of their stationary vectors \([6]\).

The Kullback-Leibler divergence (10) is neither symmetric, nor satisfies the triangle inequality.

The Euclidean distance between the two transition matrices, \(T_A\) and \(T_B\), is defined by

\[ D^{(E)}_{AB} = \|T_A - T_B\|_F \] (11)

where \(\| \ldots \|\) is the Frobenius norm induced by the Euclidean inner product for matrices, \((T_A, T_B) = \text{Tr}(T_A^\top T_B)\), in which \(T^\top\) denotes a transposed matrix. However, there is no any indication of that probabilistic space of musical dice games possesses the structure of Euclidean space.

Another possibility to compare the musical dice games by their transition matrices is to use the Riemann structure associated to the probability vectors (rows of the transition matrices) instead of (11). Let us discuss such a distance in more details, for the case of 12 × 12 transition matrices \(T\).

First, let us introduce the new matrix \(Q_{ij} = \sqrt{T_{ij}}\) and note that the 12 rows of \(Q\) define the 12 points on the surface of a unit sphere \(S_1^{11}\). It is obvious that
under any change to the musical dice game the rows of the matrix $Q$ remain on the surface of $S_1^{11}$, and therefore the difference between a pair of musical compositions chromatically transposed into one octave is always described by a set of 12 rotations $\{\omega_1, \ldots, \omega_{12}\} \in SO(12)$ relating the two sets of 12 points on $S_1^{11}$.

Second, given $Q_A$ and $Q_B$, representing the two different musical dice games, $A$ and $B$, on $S_1^{11}$, we can approximate the set of rotations $\{\omega_1, \ldots, \omega_{12}\}$ by a single one $\Omega_{AB} \in SO(12)$ that minimizes the Frobenius norm of a possible discrepancy,

$$\min_{\Omega \in SO(12)} \|Q_A \Omega_{AB} - Q_B\|_F.$$ 

Indeed, such a minimization is nothing else but the orthogonal Procrustes problem \cite{27}, which is equivalent to the singular value decomposition of the matrix $Q_A^T Q_B$,

$$Q_A^T Q_B = U \Sigma V^T, \quad \Omega_{AB} = UV^T. \quad (12)$$

The matrix $\Omega_{AB} \in SO(12)$ defined in (12) describes the optimal rotation (with respect to the Frobenius norm) translating $Q_A$ to $Q_B$, while the transposed matrix, $\Omega_{AB}^T$, makes the backward translation, $Q_B$ to $Q_A$. Obviously, $\Omega_{AB} = 1$ if and only if $Q_A = Q_B$.

We define the Riemann distance between the two musical dice games, $A$ and $B$, as the length of a geodesic curve connecting $Q_A$ and $Q_B$ on the surface of $S_1^{11}$

$$D_{AB}^{(R)} = \|\log \Omega_{AB}\|_F. \quad (13)$$

It follows from the definition that the metric (13) satisfies the conditions of non-negativity, identity of indiscernibles, symmetry, and subadditivity. The triangle inequality is satisfied, as the length of the geodesic curve on the unit sphere, $\exp(t \log \Omega_{AB})$, $0 \leq t \leq 1$, is a strictly positive function.

6 Conclusions

We have studied the musical dice games encoded by the transition matrices between pitches of the 804 musical compositions. Contrary to the language where the alphabet is independent of a message, musical compositions might involve different sets of pitches; the number of pitches used to compose a piece grows approximately logarithmically with its size.
Entropy dominates over redundancy in the musical dice games derived from the compositions of classical music. Thus the successful understanding of a musical composition requires much attention and experience from a listener. Statistics of complexity measured by the past-future mutual information suggests that pieces in classical music might contain a few melodic lines translated over the diapason of pitches by chromatic transposition. The hierarchical relations between pitches in tonal music can be rendered by means of first passage time to them, in musical dice games. Correlations between the medians of the first passage times to the notes of one octave provide the basis for the classification of composers, with respect to their tonality preferences. Finally, we have discussed the possible distances in space of musical dice games and introduced the geodesic distance based on the Riemann structure associated to the probability vectors (rows of the transition matrices).

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