Fock Space Distributions, Structure Functions, Higher Twists and Small x

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Abstract

We compute quark structure functions and the intrinsic Fock space distribution of sea quarks in a hadron wavefunction at small x. The computation is performed in an effective theory at small x where the gluon field is treated classically. At $Q^2$ large compared to an intrinsic scale associated with the density of gluons $\mu^2$, large compared to the QCD scale $\Lambda_{QCD}^2$, and large compared to the quark mass squared $M^2$, the Fock space distribution of quarks is identical to the distribution function measured in deep inelastic scattering. For $Q^2 \leq M^2$ but $Q^2 \gg \mu^2$, the quark distribution is computed in terms of the gluon distribution function and explicit expressions are obtained. For $Q^2 \leq \mu^2$ but $Q^2 \gg \Lambda_{QCD}^2$ we obtain formal expressions for the quark distribution functions in terms of the glue. An evaluation
of these requires a renormalization group analysis of the gluon distribution function in the regime of high parton density. For light quarks at high $Q^2$, the DGLAP flavor singlet evolution equations for the parton distributions are recovered. Explicit expressions are given for heavy quark structure functions at small $x$. 
1 Introduction

One of more interesting problems in perturbative QCD is the behaviour of structure functions at small values of Bjorken $x$. In deep inelastic scattering (DIS) for instance, for a fixed $Q^2 \gg \Lambda_{QCD}^2$, the operator product expansion (OPE) eventually breaks down at sufficiently small $x$ \cite{1}. Therefore at asymptotic energies, the conventional approaches towards computing observables based on the linear DGLAP \cite{2} equations are no longer applicable and novel techniques are required. Even at current collider energies such as those of HERA where the conventional wisdom is that the DGLAP equations successfully describe the data, there is reason to believe that effects due to large logarithms in $\alpha_S \log(1/x)$ (or large parton densities) are not small and we may be at the threshold of a novel region where non–linear corrections to the evolution equations are large \cite{3, 4}. One reason violations of DGLAP evolution have not been seen clearly thus far at HERA is the small phase volume for gluon emission \cite{5}. A straightforward way to further probe this region at current collider energies would be by using nuclear beams at HERA \cite{6} or in electron–proton collisions at the LHC collider where the phase volume for gluon emission is significantly larger \cite{7}.

In recent years, a non–OPE based effective field theory approach to small $x$ physics has been developed by Lipatov and collaborators \cite{8}. Their initial efforts resulted in an equation known popularly as the BFKL equation \cite{9}, which sums the leading logarithms of $\alpha_S \log(1/x)$ in QCD. In marked contrast to the leading twist Altarelli–Parisi equations for instance, it sums all twist operators that contain the leading logarithms in $x$. The solutions to the BFKL equation predict a rapidly rising gluon density and there was much initial euphoria when the H1 and ZEUS data at HERA showed rapidly rising parton densities \cite{10}. However, it was shown since:

- The rapid rise of the structure functions can arguably be accounted for by the next to leading order (NLO) DGLAP equations by appropriate choices of
initial parton densities [11, 12].

- The next to leading logarithmic corrections to the BFKL equation computed in the above mentioned effective field theory approach are very large [13, 14, 15].

As a consequence, the theoretical situation is wide open and novel approaches need to be explored.

An alternative effective field theory approach to QCD at small $x$ was put forward in a series of papers [16]-[20]. In the approach of Lipatov and collaborators, the fields of the effective theory are composite reggeons and pomeron. This approach is motivated by the reggeization of the gluon that occurs in the leading log result. Our approach based on Refs.[16]-[20] is instead a Wilson renormalization group approach where the fields are those of the fundamental theory but the form of the action at small $x$ is obtained by integrating out modes at higher values of $x$. Integrating out the higher $x$ modes results in a set of non–linear renormalization group equations [20]. If the parton densities are not too high, the renormalization group equations can be linearized and have been shown to agree identically with the leading log BFKL and small $x$ DGLAP equations [19]. There is much effort underway to explore and make quantitative predictions for the non–linear regime beyond [21, 23].

In this paper, we apply the above effective action approach to study the fermionic degrees of freedom at small $x$. At small values of $x$, gluon degrees of freedom dominate and the fermionic degrees of freedom present are essentially the sea quarks that are radiatively generated from the glue [24] (and are therefore $O(\alpha_S)$ corrections). Nevertheless, the sea quark distributions are extremely important since they are directly measured in deep inelastic scattering experiments. In this paper, we will develop a formalism, in the context of the renormalization group approach, which relates structure functions at small $x$ to the sea quark distributions, and therefore to the gluon distribution. We derive analytic expressions summing a particular class
of all twist operators which we argue give the dominant contribution at small $x$. At leading twist, in light cone gauge, these reduce to the well known simple relation between the structure function $F_2$ and the sea quark Fock space distribution function:

$$F_2(x, Q^2) = \int_0^{Q^2} dk^2 \frac{dN_{sea}}{dk^2/\Delta x}.$$  

Above, $x$ and $Q^2$ are the usual invariants in deep inelastic scattering. We show explicitly that for light quarks and at high $Q^2$, we reproduce the Altarelli–Parisi evolution equations for the quark distributions at small $x$.

There has been much interest in heavy quark distributions motivated partly by the significant contribution of heavy quarks to the structure functions at HERA. Until recently, heavy quarks were treated as infinitely massive for $Q^2$ equal to or less than the quark mass squared and massless below. There now exist approaches which study quark distributions in a unified manner for a range of $Q^2$ and quark masses (for a discussion and further references see Refs. [31, 32]). A nice feature of our formalism is that heavy quark evolution is treated on the same footing as light quarks and specific predictions can be made for $F_2^{charm/bottom}/F_2$ within our formalism. These can also be related to the diffractive cross section at small $x$ but that issue will not be addressed in this paper. This issue and the relation of our work to the above cited work and other recent works on heavy quark production at small $x$ [33, 34] will be addressed at a later date.

The results of our analysis are the following. In our theory of the gluon distribution functions, a dimensionful scale appears which measures the density of gluons per unit area,

$$\mu^2 = \frac{1}{\sigma} \frac{dN}{dy},$$  

where $\sigma$ is the hadronic cross section of interest. Here $y = y_0 - \ln(1/x), y_0$ is an arbitrarily chosen constant and $x$ is Bjorken $x$. When this parameter satisfies
\( \mu >> \Lambda_{QCD} \) the gluon dynamics, while nonperturbative, is both weakly coupled and semiclassical. We shall always assume that at sufficiently small \( x \) this is satisfied.

At small \( x \), the gluon field, being bosonic, has to be treated non-perturbatively. This is analogous to the strong field limit used in Coulomb problems. Fermions, on the other hand, do not develop a large expectation value and may be treated perturbatively. To lowest order in \( \alpha_{QCD} \), the gluon distribution function is determined by knowing the fermionic propagator in the classical gluon background field. In general, this propagator must be determined to all orders in the classical gluon field as the field is strong. This can be done due to the simple structure of the background field.

At this point it is useful to distinguish between two different quantities which are often used interchangeably. One is the quark structure functions as measured in deep inelastic scattering and the second is the Fock space distribution of quarks. At high \( Q^2 \) which is usually the case considered in perturbative QCD, these two quantities are essentially identical. However, for massive quarks when \( Q^2 \leq M^2 \) even if \( Q^2 >> \Lambda_{QCD}^2 \), the two quantities differ. For \( Q^2 >> \mu^2 \), regardless of mass, the quark Fock space distribution function and the quark structure functions may both be simply expressed as linear functions of the gluon distribution functions. Therefore, with knowledge of the gluon distribution function, one can compute the quark distribution function.

For the case of massless quarks, when \( \mu^2 \geq Q^2 >> \Lambda_{QCD}^2 \), we are still in the weak coupling limit. However, we must keep all orders in the gluon field. In this region, the integration over the gluon fields in our effective field theory cannot be directly performed as yet since it requires a full renormalization group analysis of the theory. In other words, the measure of integration for the high density regime in the effective theory has not yet been computed. Nevertheless, we obtain an explicit functional dependence on the gluon fields which must be integrated over with the right measure.
The power of the technique which we use to analyze this problem is that it does not rely on a high twist expansion. It uses only the weak coupling nature of the theory which must be true at small \( x \) if the gluon density is very high. We are therefore in a position to find non-trivial relations between these various parameters in a region where the weak coupling analysis is valid but where perturbation theory and leading order operator product expansion methods are not valid.

This paper is organized as follows. In section 2 we write down and review an effective action for the small \( x \) modes in QCD. The action is imaginary and the modes are averaged over with a statistical weight

\[
\exp \left[ -F[\rho] \right],
\]

where \( F[\rho] \) is a functional over the color charge density \( \rho \) of the higher \( x \) modes. The functional \( F[\rho] \) obeys a non-linear renormalization group equation. Of particular interest in this paper is the saddle point solution of the effective action since the sea quark distributions are computed in the classical background field of this action. This section is a quick review of known results which are necessary to understand the remainder of the paper.

An expression relating the electromagnetic current–current correlator to the fermion propagator in the classical background field is derived in section 3. We also discuss the light cone Fock space distributions and their relation to the structure functions in this section.

In section 4, we solve the Dirac equation in the classical background field and obtain an explicit expression for the fermion Green’s function in the classical background field.

The Green’s function is used in section 5 to compute the sea quark distribution function and the leading twist contribution to the structure function \( F_2(x, Q^2) \) at small \( x \). The color averaging over the functional \( F[\rho] \) in the distribution function is compactly represented by a function \( \tilde{\gamma}(p_t) \), where \( p_t \) can be interpreted as the...
intrinsic transverse momentum of the glue at high parton densities. It is shown explicitly that for light quark masses the flavor singlet Altarelli–Parisi equations at small x are recovered.

The current–current correlator and the structure functions are computed explicitly in section 6. As a check of our computation, it is shown that the leading twist results are recovered in the appropriate limit. The heavy quark structure functions are computed explicitly. The phenomenological implications and the connections to the recent literature on heavy quark production will not be addressed in this work but will be considered at a later date.

Section 7 contains a summary of our results and a discussion of future work.

The first of two appendices contains a discussion of our notation and conventions. In the second appendix we present an explicit form for $\tilde{\gamma}(p_t)$ for the particular case of Gaussian color fluctuations.

2 Effective Field Theory for Small x Partons in QCD: Review of Results

We will discuss below an effective action for the wee parton modes in QCD. The action contains an imaginary piece which involves functional $F[\rho]$ which satisfies a non-linear Wilson renormalization group equation. In the weak field limit of this renormalization group equation, the BFKL equation is recovered. In the double logarithmic region, the evolution equation is also equivalent to DGLAP $^1$. We next discuss the classical background field which is the saddle point solution of this action. It is this background field that the sea quarks couple to at small x and the properties of the background field will be relevant for the discussion in later sections.

$^1$This was first noticed by Yuri Dokshitzer in his paper in Ref. $^3$. 

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2.1 The Effective Action and the Wilson Renormalization Group at Small x

In the infinite momentum frame $P^+ \to \infty$, the effective action for the soft modes of the gluon field with longitudinal momenta $k^+ << P^+$ (or equivalently $x \equiv k^+/P^+ << 1$) can be written in light cone gauge $A^+ = 0$ as

$$S_{\text{eff}} = -\int d^4x \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} + \frac{i}{N_c} \int d^2x_t dx^- \rho^a(x_t, x^-) \text{Tr} (\tau^a W_{-\infty,\infty}[A^-](x^-, x_t)) + i \int d^2x_t dx^- F[\rho^a(x_t, x^-)].$$

(2)

Above, $G_{\mu\nu}^a$ is the gluon field strength tensor, $\tau^a$ are the $SU(N_c)$ matrices in the adjoint representation and $W$ is the path ordered exponential in the $x^+$ direction in the adjoint representation of $SU(N_c)$,

$$W_{-\infty,\infty}[A^-](x^-, x_t) = P \exp \left[ -ig \int dx^+ A_0^- (x^-, x_t) \tau^a \right].$$

(3)

The above is the most general gauge invariant form [19] of the action that was proposed in Ref. [16].

This is an effective action valid in a limited range of $P^+ << \Lambda^+$ where $\Lambda^+$ is an ultraviolet cutoff in the plus component of the momentum. The degrees of freedom at higher values of $P^+$ have been integrated out and their effect is to generate the second and third terms in the action.

The first term in the above is the usual field strength piece of the QCD action and describes the dynamics of the wee partons at the small x values of interest. The second term in the above is the coupling of the wee partons to the hard color charges at higher rapidities, with x values corresponding to values of $P^+ \geq \Lambda^+$. When expanded to first order in $A^-$ this term gives the ordinary $J \cdot A$ coupling for classical fields. The higher order terms are needed to ensure a gauge invariant coupling of the fields to current.
In the infinite momentum frame, only the $J^+$ component of the current is large (the other components being suppressed by $1/P^+$). The longer wavelength wee partons do not resolve the higher rapidity parton sources to within $1/P^+$ and for all practical purposes, one may write

$$\rho^a(x_t, x^-) \longrightarrow \rho^a(x_t)\delta(x^-).$$ (4)

The last term in the effective action is imaginary. It can be thought of as a statistical weight resulting from integrating out the higher rapidity modes in the original QCD action. Expectation values of gluonic operators $O(A)$ are then defined as

$$<O(A)> = \frac{\int [d\rho] \exp (-F[\rho]) \int [dA] O(A) \exp (iS[\rho, A])}{\int [d\rho] \exp (-F[\rho]) \int [dA] \exp (iS[\rho, A])},$$ (5)

where $S[\rho, A]$ corresponds to the first two terms in Eq. 2. The color averaging procedure for fermionic observables is discussed further in section 5.1.

In Ref. [16] a Gaussian form for the action

$$\int d^2x_t \frac{1}{2\mu^2} \rho^a \rho^a,$$ (6)

was proposed, where $\mu^2$ was the average color charge squared per unit area of the sources at higher rapidities than is appropriate for our effective action, that is. For large nuclei $A >> 1$ it was shown that

$$\mu^2 = \frac{1}{\pi R^2} \frac{N_q}{2N_c} \sim A^{1/3}/6 \text{ fm}^{-2}.$$ (7)

This result was independently confirmed in a model constructed in the nuclear rest frame [22]. If we include the contribution of gluons which have been integrated out by the renormalization group technique, one finds that [44]

$$\mu^2 = \frac{1}{\pi R^2} \left( \frac{N_q}{2N_c} + \frac{N_cN_g}{N_c^2} \right)$$ (8)

Here $N_q$ is the total number of quarks with $x$ above the cutoff

$$N_q = \sum_i \int_{x}^{1} dx' q_i(x')$$ (9)
where the sum is over different flavors, spins, quarks and antiquarks. For gluons, we also have

\[ N_g = \int_1^x dx' g(x') \]   \hspace{1cm} (10)

The value of \( \pi R^2 \) is well defined for a large nucleus. For a smaller hadron, we must take it to be \( \sigma \), the total cross section for hadronic interactions at an energy corresponding to the cutoff. This quantity will become better defined for a hadron in the renormalization group analysis.

The above equation for \( \mu^2 \) is subtle because, implicitly, on the right hand side, there is a dependence on \( \mu \) through the structure functions themselves. This is the scale at which they must be evaluated. Calculating \( \mu \) therefore involves solving an implicit equation. Note that because the gluon distribution function rises rapidly at small \( x \), the value of \( \mu \) grows as \( x \) decreases.

The Gaussian form of the functional \( F[\rho] \) is reasonable when the color charges at higher rapidity are uncorrelated and are random sources of color charge. This is true for instance in a very large nucleus. It is also true if we study the Fock space distribution functions or deep inelastic structure functions at a transverse momentum scale which is larger than an intrinsic scale set by \( \alpha_s \mu \). In this equation \( \alpha_s \) is evaluated at the scale \( \mu \). At smaller transverse momenta scales, one must do a complete renormalization group analysis to determine \( F[\rho] \). This analysis is not yet complete, but should be feasible in the context of the weakly coupled field theory as long as the transverse momentum scale remains larger than that \( \Lambda_{QCD} \). The color averaging procedure for the case of Gaussian fluctuations is discussed in appendix B.

A final comment about the limit of applicability of the classical action above concerns limitations in the transverse momentum range. The action above is valid only when probing transverse momenta scales \( p_T \leq \mu \). This includes the Gaussian region since \( \alpha_s \ll 1 \). At higher transverse momenta, one must use DGLAP evolu-
tion, with the structure functions as determined at lower values of $Q^2$ as boundary conditions [17]. This will be important for the case of heavy quarks as the transverse momentum scale there is very large.

The above comments on the renormalization group analysis show the limitations of our analysis with respect to quarks. For transverse momentum scales $p_t \gg \alpha_s \mu$, one can use a Gaussian source and all relevant quantities can be computed explicitly. At smaller scales, one can derive a formal expression, which, hopefully, will be directly computed in the near future. For heavy quarks, the Gaussian analysis should be adequate.

To complete a review of the renormalization group, we briefly review the procedure used to determine $F[\rho]$. It was shown that a Wilson renormalization group procedure [18] could be applied to derive a non-linear renormalization group equation for $F[\rho]$. The procedure, briefly, is as follows. The gauge field is split as

$$A^a_\mu(x) = b^a_\mu(x) + \delta A^a_\mu(x) + a^a_\mu(x), \quad (11)$$

where $b^a_\mu(x)$ is the saddle point solution of Eq. 2 and corresponds to the hard modes above the longitudinal momentum scale $\Lambda$. The fluctuation field $\delta A^a_\mu(x)$ contains the soft modes $\Lambda^{t+} < k^+ < \Lambda^+$ and $a_\mu(x)$ are soft fields with longitudinal momenta $k^+ < \Lambda^{t+}$. The cutoffs are chosen such that $\alpha_s \log(\Lambda^{t+}/\Lambda^+) << 1$. Small fluctuations are performed about the saddle point solution $b^a_\mu(x)$ to the effective action at the scale $\Lambda^+$, to obtain the effective action for the fields $a^a_\mu(x)$ at the scale $\Lambda^{t+}$. The new charge density at this scale $\rho'$ is given by $\rho'' = \rho^a + \delta \rho^a$, where $\delta \rho^a$ can be expressed as the sum of linear and bi-linear terms in the fluctuation field $\delta A^a_\mu(x)$.

To leading order in $\alpha_s$, the effective action at the scale $\Lambda^{t+}$ can be expressed in the same form as Eq. 2 with the functional $F[\rho']$ satisfying a non–linear renormalization group equation [19]-[20]. In terms of the statistical weight $Z = \exp(-F[\rho])$,
it can be expressed as
\[ \frac{dZ}{d \log(1/x)} = \alpha S \left[ \frac{1}{2} \frac{\delta^2}{\delta \rho \delta \rho} (Z \chi_{\mu \nu}) - \delta \frac{\delta}{\delta \rho} (Z \sigma_{\mu}) \right], \tag{12} \]
where \( \sigma[\rho] \) and \( \chi[\rho] \) are respectively one and two point functions obtained by integrating over \( \delta A \) for fixed \( \rho \). The one point function \( \sigma \) includes the virtual corrections to \( F[\rho] \) while the two point function \( \chi \) includes the real contributions to \( F[\rho] \). Both of these can be computed explicitly from the small fluctuations propagator in the classical background field. The propagator was first computed by fixing the residual gauge freedom to be \( \delta A^- (x^- = 0) \) \[17\] but a less restrictive gauge choice was later found which may be useful for computing \( \sigma \) and \( \chi \) \[20\].

For weak fields, the free gluon propagator can be used to obtain the well known

BFKL equation for the unintegrated gluon density, which is defined as
\[ \frac{dN}{dk_t^2} = \int d^2 p_t e^{-i k_t^2 x_t} <\rho(x_t)\rho(0)>_\rho. \tag{13} \]
Performing the renormalization group procedure defined above to obtain the charge density \( \rho' = \rho + \delta \rho \), one obtains
\[ <\rho' \rho'>_\rho - <\rho \rho>_\rho = \alpha S \log(1/x) [2 <\rho \sigma>_\rho + <\chi>_\rho]. \tag{14} \]

The one and two point functions \( \sigma \) and \( \chi \) respectively can be computed to linear order in the classical background field and the results are \[19\]
\[ \sigma^a(k_t) = -\frac{g^2 N_c}{2(2\pi)^3} \rho^a(k_t) \int d^2 p_t \frac{k_t^2}{p_t^2 (p_t - k_t)^2}, \tag{15} \]
and
\[ \chi = \frac{2g^2 N_c}{(2\pi)^3} \int d^2 p_t \rho^a(p_t) \rho^a(-p_t) \frac{k_t^2}{p_t^2 (k_t - p_t)^2}. \tag{16} \]
The above are respectively the virtual and real contributions to the change in the color change density after integrating out the modes \( \Lambda^+ < k^+ < \Lambda^+ \). Substituting these into Eq. \[14\], one obtains the well known BFKL equation
\[ \frac{dN}{dk_t^2} \frac{dN}{d k_t^2} dx = \alpha S N_c \frac{k_t^2}{2 \pi^2} \int d^2 p_t \frac{k_t^2}{p_t^2 (p_t - k_t)^2} \left[ \frac{dN}{dk_t^2} - 2 \frac{dN}{dp_t^2} \right]. \tag{17} \]
Strenuous efforts are currently underway to compute $\sigma$ and $\chi$ to all orders in the background field and thereby solve the full non–linear Wilson renormalization group equation for $F[\rho]$ \cite{21, 23}.

2.2 The Classical Background Field at Small $x$

The effective action in Eq. 2 has a remarkable saddle point solution \cite{16, 18, 22}. It is equivalent to solving the Yang–Mills equations

$$D_\mu G^{\mu\nu} = J^\nu \delta^{\nu+}, \quad (18)$$

in the presence of the source $J^{+;a} = \rho^a(x_t, x^-)$. Here we will allow the source to be smeared out in $x^-$ as this is useful in the renormalization group analysis. It is also useful for intuitively understanding the nature of the field. One finds a solution where $A^\pm = 0$ and

$$A^i = -\frac{1}{ig} V \partial^i V^\dagger, \quad (19)$$

for $i = 1, 2$ is a pure gauge field which satisfies the equation

$$D_i \frac{dA^i}{dy} = g \rho(y, x_\perp). \quad (20)$$

Here $D_i$ is the covariant derivative $\partial_i + V \partial_i V^\dagger$ and $y = y_0 + \log(x^-/x_0^-)$ is the space–time rapidity and $y_0$ is the space-time rapidity of the hard partons in the fragmentation region. At small x we will use the space–time and momentum space notions of rapidity interchangeably \cite{15}. The momentum space rapidity is defined to be $y = y_0 - \ln(1/x)$ where $x$ is Bjorken $x$. The solution of the above equation is

$$A^i_\rho(x_t) = \frac{1}{ig} \left(P e^{ig \int_{y_0}^{y} dy' \frac{1}{2\tau^\perp} \rho(y', x_t)} \right) \nabla_i \left(P e^{ig \int_{y_0}^{y} dy' \frac{1}{2\tau^\perp} \rho(y', x_t)} \right)^\dagger. \quad (21)$$

To compute the classical nuclear gluon distribution function, for instance,

$$\frac{dN}{d^3k} = \frac{1}{(2\pi)^3} 2|k^+| \int d^3x d^3x' e^{ik\cdot(x-x')} < A^a_t(x^-, x_t) A^a_t(x'^-, x'_t) > \rho, \quad (22)$$

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one needs in general to average over the product of the classical fields at two space–
time points with the weight \( F[\rho] \) as shown in Eq. \( \text{3} \) or for the Gaussian measure
with the weight in Eq. \( \text{3} \). In the latter case, exact analytical solutions are available
for correlators in the classical background field. For the case of interest here, the
fermion Green’s function in the classical background field will depend on correlators
of the form \( \langle V(x_t)V^\dagger(y_t) \rangle_\rho \) where the \( V \)’s are \( SU(N_c) \) gauge transformation
matrices defined above. In appendix B, we discuss in detail the computation of this
correlator for the case of Gaussian fluctuations.

3 The Current–Current Correlator at Small \( x \)

In this section, we will derive a formal expression for the hadron tensor \( W_{\mu\nu} \) at
small \( x \) relating it to the fermion Green’s function in the classical background field.
We also derive a relation between the light cone quark distribution function and
the fermion Green’s function. To leading twist, the structure functions are simply
related to the light cone quark distribution function. In general (for example, for
heavy quark distributions) this is not true. Nevertheless, a simple relationship may
be found between the gluon distribution functions and that of the quarks. This is
because the quarks distribution functions is given by an integral over a propagator
in the classical gluon background field described above.

3.1 Derivation

In deep inelastic electroproduction, the hadron tensor can be expressed in terms of
the forward Compton scattering amplitude \( T_{\mu\nu} \) by the relation \( \text{[38]} \)
\[
W^{\mu\nu}(q^2, P \cdot q) = 2 \text{Disc} \ T^{\mu\nu}(q^2, P \cdot q) \equiv \frac{1}{2\pi} \text{Im} \int d^4x \exp(iq \cdot x) \times \langle P|T(J^\mu(x)J^\nu(0))|P > ,
\]

\( (23) \)
where “T” denotes time ordered product, $J^\mu = \bar{\psi} \gamma^\mu \psi$ is the hadron electromagnetic current and “Disc” denotes the discontinuity of $T_{\mu\nu}$ along its branch cuts in the variable $P \cdot q$. Also, $q^2 \to \infty$ is the momentum transfer squared of the virtual photon and $P$ is the momentum of the target. In the infinite momentum frame, $P^+ \to \infty$ is the only large component of the momentum. The fermion state above is used in the expectation value for the current operators is normalized as $< P | P' > = \frac{(2\pi)^3}{E/m} \delta^{(3)}(P - P')$ where $m$ is the mass of the target hadron. This definition of $W^{\mu\nu}$ and normalization of the state is traditional, and we will abide by these conventions in spite of the awkward factors of $m$. We will see in the end that all factors of $m$ cancel from the definition of quantities of physical interest. (The normalization we will use in this paper for quark and lepton states will have $E/m$ replaced by $2P^+$.)

Let us first describe the computation of $< P | T(J^\mu(x)J^\nu(0)) | P >$. In our computation, we have an external source corresponding to the particle whose state vector is denoted by $| P >$. Our source is located at some fixed position. We must therefore consider the generalization of $W^{\mu\nu}$ for such a source which has a position dependence. Note that for a given source, we also have a lack of translational invariance in the transverse direction. Transverse translational invariance is restored after integration over the source. There is no dependence of our source on $x^+$. Therefore the relevant variable is $x^-$. We now argue that the relevant definition of $W^{\mu\nu}$ is

$$W^{\mu\nu}(q^2, P \cdot q) = \frac{1}{2\pi} \frac{P^+}{m} \text{Im} \int d^4x dX^- e^{i q \cdot x} < T(J^\mu(X^- + x/2)J^\nu(X^- - x/2)) > . \tag{24}$$

To see this let us first verify that we can write $W^{\mu\nu}$ in this form for the conventional definition valid for plane wave states $| P >$. Notice that we can define $< O > = < P | O | P > / < P | P >$ where $O$ is any operator. As mentioned above, the

\footnote{Note that in our metric convention, a space–like photon has $q^2 = Q^2 > 0$.}
expectation value \(< P \mid P > = (2\pi)^3 E/m \delta^{(3)}(0) = (2\pi)^3 E/m \ V. Here we shall take the spatial volume \(V\) to be \(\sigma\) times an integral over the longitudinal extent of the state. Using these conventions, we see that we reproduce the above definition of \(W^{\mu\nu}\).

This definition corresponds to treating the variable \(X^-\) as a center of mass coordinate and \(x^-\) as a relative longitudinal position. For a translationally invariant state, this would give the longitudinal dimension of the system. The definition is Lorentz covariant as will be shown explicitly in section 6. The integration over \(X^-\) is required since we must include all of the contributions from quarks at all \(X^-\) to the distribution function. In our external source language, the variable \(P^+\) can be taken to be the longitudinal momentum corresponding to the fragmentation region. In the end all of the \(P^+\) (and \(m\)) dependence will disappear upon taking the infinite momentum limit. To check this definition later, we shall show that it reproduces the conventional results in the high \(q^2\) region.

The expectation value is straightforward to compute in the limit where the gluon field is treated as a classical background field. If we write

\[
< T(\mathcal{J}^\mu(x)\mathcal{J}^\nu(y)) > < T(\bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(y)\gamma^\nu \psi(y)) > ,
\]

then when the background field is classical, and one ignores quantum corrections arising from either loops of fermions or loops of gluons, (a good approximation in the weak coupling limit of high parton densities), we obtain

\[
< T(\mathcal{J}^\mu(x)\mathcal{J}^\nu(y)) > = \text{Tr}(\gamma^\mu S_A(x))\text{Tr}(\gamma^\nu S_A(y)) + \text{Tr}(\gamma^\mu S_A(x,y)\gamma^\nu S_A(y,x)) .
\]

In this expression, \(S_A(x,y)\) is the Green’s function for the fermion field in the external field \(A\)

\[
S_A(x,y) = -i < \psi(x)\bar{\psi}(y) >_A
\]

for fixed \(A\) (before averaging over \(A\)).
The first term on the right hand side of Eqn. 28 is a tadpole contribution which does not involve a non–zero imaginary part. It therefore does not contribute to $W^\mu\nu$. We find then that

$$W^\mu\nu(q^2, p \cdot q) = \frac{1}{2\pi} \frac{P^+}{m} \text{Im} \int dX^- d^4x e^{iq \cdot x} \langle \text{Tr} \left( \gamma^\mu S_A(X^- + x/2, X^- - x/2) \times \gamma^\nu S_A(X^- - x/2, X^- + x/2) \right) \rangle. \quad (28)$$

### 3.2 The Light Cone Fock Distribution Function and Structure Functions

The expression we derived above for $W^\mu\nu$ is entirely general and makes no reference to the operator product expansion. In particular, it is relevant at the small $x$ values and moderate $q^2$ where the operator product expansion is not reliable [1]. At sufficiently high $q^2$ though (and for massless quarks) it should agree with the usual leading twist computation of the structure functions. The fact that we do not have a valid operator product expansion forces us to distinguish between two quantities which are identical in leading twist. The first is the Fock space distribution of partons within a hadron. The second are the parton structure functions which are measured in deep inelastic scattering. In our analysis, at high values of $q^2$, these expressions are identical. At smaller values, say those values typical of the intrinsic transverse momentum scale $\alpha_S \mu$, they are no longer the same and must be differentiated between.

We will derive below an expression for the light cone quark Fock distribution in terms of the propagator in light cone quantization [40, 41]. The quark Fock space distribution is then simply related to the structure function $F_2$ for $q^2 \gg \alpha_S^2 \mu^2$. In light cone quantization, only the two component spinor projection $\psi_+$ is dynamical. (Note: notation and conventions are discussed in appendix A.) The other two spinor components $\psi_-$ (recall that $\psi = \psi_- + \psi_+$) are defined via the light cone constraint relation defined below in Eq. [41]. The dynamical fermions can then be written in
terms of creation and annihilation operators as

\[
\psi_+ = \int_{k^+>0} \frac{d^3k}{2^{1/4}(2\pi)^3} \sum_{s=\pm 1/2} \left[ e^{ik\cdot x} w(s) b_s(k) + e^{-ik\cdot x} w(-s) d_s^\dagger(k) \right].
\]  \hspace{1cm} (29)

Above \( b_s(k) \) is a quark destruction operator and destroys a quark with momentum \( k \) while \( d_s^\dagger(k) \) is an anti–quark creation operator and creates an anti–quark with momentum \( k \). Also above the unit spinors \( w(s) \) are defined as

\[
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}; \hspace{1cm}
\begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix},
\]  \hspace{1cm} (30)

Note that since

\[
d^\dagger(\vec{k}_t, k^+; x^+; +\frac{1}{2}) = b(-\vec{k}_t, -k^+; x^+; -\frac{1}{2}) \\
d^\dagger(\vec{k}_t, k^+; x^+; -\frac{1}{2}) = b(-\vec{k}_t, -k^+; x^+; +\frac{1}{2}),
\]

one can show that

\[
w(s)b_s(k) = 2^{1/4} \int d^3x \ e^{-ik\cdot x} \psi_+^\dagger(x). \hspace{1cm} (31)
\]

The light cone Fock distribution function is defined in terms of the creation and annihilation operators as

\[
\frac{dN}{d^3k} = \frac{1}{(2\pi)^3} \sum_s \left[ b_s^\dagger b_s + d_s^\dagger d_s \right] = \frac{2}{(2\pi)^3} \sum_s b_s^\dagger b_s. \hspace{1cm} (32)
\]

We have assumed above that the sea is symmetric between quarks and anti–quarks. Combining the two equations above, we get

\[
\frac{dN}{d^3k} = \frac{2\sqrt{2}}{(2\pi)^3} \int d^3x \ d^3y e^{ik\cdot(x-y)} \psi_+^\dagger(x)\psi_+^\dagger(y). \hspace{1cm} (33)
\]

Using the light cone identity

\[
\text{Tr} \left[ \gamma^+ \psi(x) \bar{\psi}(y) \right] = \sqrt{2} \psi_+(x) \psi_+^\dagger(y), \hspace{1cm} (34)
\]
we obtain the following expression for the sea quark distribution function

\[
\frac{dN}{d^3k} = \frac{2i}{(2\pi)^3} \int d^3x \, d^3y \, e^{-i k \cdot (x-y)} \text{Tr} \left[ \gamma^+ S_A(x,y) \right]
\] (35)

where the fermion propagator \( S_A(x,y) \) is the light cone time ordered product \( S_A(x,y) = -i < T(\psi(x)\bar{\psi}(y)) > \) in the background field \( A^\mu \). In our effective action approach, as discussed in sections 2 and 3.1, we can replace \( S_A(x,y) \rightarrow S_{A,cl}(x,y) \) to obtain the sea quark distribution in the classical background field.

In a nice pedagogical paper, (see Ref. [25] and references within), Jaffe has shown that the Fock space distribution function can be simply related to the leading \textit{twist} structure function \( F_2 \) by the relation

\[
F_2(x,Q^2) = \int_0^{Q^2} dk_1^2 \int_0^{dN/dk_1^2} dx.
\] (36)

Actually, Jaffe’s expression is defined as the sum of the quark and anti–quark distributions. At small \( x \), these are identical and the resulting factor of 2 is already included in our definition of the light cone quark distribution function. In the following section, we will compute the fermion Green’s function in the classical background field. We will then use it in Eq. 35 and the above equation to show that we do indeed recover the standard perturbative result for \( F_2 \). For heavy quarks and/or moderate \( Q^2 \), structure functions should be computed by inserting the fermion Green’s function in the definition for \( W_{\mu\nu} \). We will see that in general the structure functions and the Fock space distribution functions are not the same.

\section{The Fermion Green’s Function in the Classical Background Field}

In this section we shall derive an expression for the fermion propagator in the classical gluon background field described in section 2.2. The field strength carried
by these classical gluons is highly singular, being peaked about the source (corresponding to the parton current at $x$ values larger than those in the field) localized at $x^- = 0$. Away from the source, the field strengths are zero and the gluon fields are pure gauges on both sides of $x^- = 0$. The fermion wavefunction is obtained by solving the Dirac equation in the background field on either side of the source and matching the solutions across the discontinuity at $x^- = 0$. Once the eigenfunctions are known, the fermion propagator can be constructed in the standard fashion. We begin this section with a discussion of the notation and conventions, proceed to write down the solution of the Dirac equation and finally, construct explicitly the fermion propagator in the classical gluon background field. This expression is formally exact and is valid to all orders in the source color charge density.

4.1 The Dirac Equation in the Classical Background Field

In order to compute the propagator for a spinor field in the fundamental representation of the gauge group propagating in the background gauge field

\[
\begin{align*}
A^a_+ &= 0 \\
A^a_- &= 0 \\
\tau \cdot A_t &= \theta(x^-)\kappa_t(x_t),
\end{align*}
\]

(37)

where $\kappa_t(x_t), t = 1, 2$ is a two dimensional pure gauge,

\[
\kappa_t(x_t) = -\frac{1}{ig}V(x_t)\nabla_t V^\dagger(x_t),
\]

(38)

we first need to solve the Dirac eigenvalue equation which can be written as

\[
\begin{align*}
\{ \bar{\alpha}_t \cdot (\bar{p}_t - g\bar{A}_t) - \sqrt{2}p^+\alpha^- - \sqrt{2}p^-\alpha^+ + \beta M \} \psi_\lambda(x) &= \lambda \psi_\lambda(x) 
\end{align*}
\]

(39)

for the spinor field $\psi$ and a corresponding equation for $\bar{\psi}(x)$. The $\alpha$’s and $\beta$ above are defined in appendix A and and $p^\mu = -i\partial^\mu$. For $x^- < 0$, the solution is trivial.
and is just the free spinor plane wave solution. For \( x^- > 0 \) the solution is less trivial and is given by the non–Abelian analogue of the ‘Baltz’ ansatz \([39]\). The full solution of the Dirac equation in the classical background field is

\[
\psi^{\alpha,s}_{\lambda q}(x) = \theta(-x^-)e^{iq\cdot x}u^{\alpha}_{s,\lambda}(q) + \theta(x^-) \frac{1}{\sqrt{2}} \int \frac{d^2 p_t}{(2\pi)^2} dz_t(V(x_t)V^+(z_t))^{\alpha\beta}e^{ip\cdot x - iq^- x^+} \\
\times e^{z \cdot (q - p_t)} \exp \left( -i\frac{(p_t^2 + M^2 - \lambda)}{2p^-} x^- \right) \left\{ 1 + \frac{(\alpha_t \cdot p_t + \beta M)}{\sqrt{2q^-}} \right\} u^{\beta,s}_{\lambda}(q),
\]

(40)

Above, the superscripts \( \alpha, \beta \) denote the color index in the fundamental representation and \( s \) the spinor index. The elementary spinors are normalized as \( \bar{u}^{s,\lambda}(q) u^{s',\lambda'}(q) = 2M\delta_{ss'}\delta_{\lambda\lambda'} \) and summed over spins \( (u\bar{u})_{\mu\nu} = (M - q')_{\mu\nu} \).

The interested reader will notice that the above equation is not continuous across \( x^- = 0 \). This is because though \( \psi_-(x) \) is continuous across \( x^- = 0 \), \( \psi_- \) is related to \( \psi_+ \) via the light cone constraint equation

\[
\psi_+ = \frac{1}{\sqrt{2q^-}} \left[ \alpha_t \cdot \left( \frac{1}{i} \partial_t - gA_t \right) + \beta M \right] \psi_- ,
\]

(41)

which is discontinuous across \( x^- = 0 \) by the same amount as in the previous equation.

### 4.2 Computation of the Fermion Propagator

Having obtained the eigenfunctions for the Dirac equation in the classical background field we are now in a position to compute the fermion propagator in the classical background field. This is given by the relation

\[
S(x, y) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2 - i\varepsilon} \sum_{pol} \psi_q(x)\bar{\psi}_q(y) ,
\]

(42)

after identifying \( q^+ = (q^2 + M^2 - \lambda)/2q^- \). It is straightforward to check from the above expression that \( (q + M)S(x, y) = (2\pi)^4\delta^{(4)}(x - y) \). Substituting the
eigenfunctions from Eq. 40 in the above, we have
\begin{align*}
S(x, y) &= \theta(-x^-)\theta(-y^-)S_0(x - y) + \theta(x^-)\theta(y^-) \left( V(x_t)S_0(x - y)V^\dagger(y_t) \right) \\
&+ \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2 - i\varepsilon} e^{iq \cdot (x - y)} \int \frac{d^2p_t}{(2\pi)^2} d^2z_t \left\{ \theta(x^-)\theta(-y^-) \\
&\times e^{ip_t \cdot (x_t - z_t)} \left[ -i \frac{(pt + qt)^2 - q^2_t}{2q^-} \right] \left( V(x_t)V^\dagger(z_t) \right) \frac{1}{2q^-} (M - q^- - p^\mu) \gamma^- (M - q^-) \\
&+ \theta(-x^-)\theta(y^-) e^{-ip_t \cdot (y_t - z_t)} \exp \left[ i \frac{(pt + qt)^2 - q^2_t}{2q^-} y^- \right] \left( V(z_t)V^\dagger(y_t) \right) \\
&\times \frac{1}{2q^-} (M - q^-) \gamma^- (M - q^- - p^\mu) \right\}, \tag{43}
\end{align*}

where the free fermion Green’s function is
\begin{equation}
S_0(x - y) = \int \frac{d^4q}{(2\pi)^4} e^{iq \cdot (x - y)} \frac{M - q^-}{q^2 + M^2 - i\varepsilon}. \tag{44}
\end{equation}

The translational symmetry of the Green’s function in the \( x^- \) direction is of course broken by the presence of the source at \( x^- = 0 \). In the absence of the source of color charge, it may be confirmed that the free fermion propagator is recovered by putting \( V = I \), where \( I \) denotes the unit matrix in the fundamental representation.

The reader will note that the propagator between two points on the same side of the source, for either \( x^-, y^- < 0 \) or \( x^-, y^- > 0 \) is the free propagator or a gauge transform of it. The only non–trivial contribution comes from the pieces connecting points on the opposite sides of the source. The \( \theta(x^-)\theta(-y^-) \) piece can be written more simply as
\begin{equation}
S(x, y) = -i \int d^4z V(x_t) S_0(x, z) \gamma^- \delta(z^-) S_0(z, y) V^\dagger(z_t). \tag{45}
\end{equation}

An analogous expression holds for the \( \theta(-x^-)\theta(y^-) \) piece of the propagator. A similar simple expression for the scalar quark propagator \[46\] was found recently by Hebecker and Weigert \[47\] (see also the recent work of Balitskii \[48\]). The only difference between the form of the above result and that for scalar quarks is the \( \gamma^- \) matrix present here due to the different spinor structure and a partial derivative \( \partial_{z^+} \) absent here due to the \( 1/2q^- \) factor in Eq. 43.
If we define

$$G(x_t, x^-) = \theta(-x^-) + \theta(x^-)V(x_t) \quad (46)$$

which is the gauge transformation matrix which transforms the gluon field at hand to a singular field which has only a plus component, $A^\mu = \delta^{\mu+}\alpha(x_t)$, we then see that our propagator has the form

$$S_A(x, y) = G(x)S_0(x - y)G^\dagger(y) - i \int d^4z G(x) \left( \theta(x^-)\theta(-y^-)(V^\dagger(z_t) - 1) - \theta(-x^-)\theta(y^-)(V(z_t) - 1) \right) G^\dagger(y)S_0(x - z)\gamma^-\delta(z^-)S_0(z - y) \quad (47)$$

This very simple form of the propagator is useful in the manipulations below.

In fact the current-current correlation function is explicitly gauge invariant. We may therefore use the singular gauge form of the propagator for computing the current-current correlation function

$$S_A^{\text{sing}}(x, y) = S_0(x - y) - i \int d^4z \left( \theta(x^-)\theta(-y^-)(V^\dagger(z_t) - 1) - \theta(-x^-)\theta(y^-)(V(z_t) - 1) \right) S_0(x - z)\gamma^-\delta(z^-)S_0(z - y) \quad (48)$$

A diagrammatic representation of the form of the propagator above is shown in Fig. 1 In the expressions below for $W^{\mu\nu}$ we will drop the superscript sing and simply use the singular gauge expression for the propagator.

The Fock space distribution function however is gauge dependent. In computing it we must therefore either use the form of the propagator with the explicit gauge matrices above, or go back to our original form. In what follows, we shall use the original form of Eq. 43 for computing the Fock space distribution function.

Our result for the fermion propagator in the classical background field was obtained for a $\delta$–function source in the $x^-$ direction. This assumption was motivated by the observation that small $x$ modes with wavelengths greater than $1/P^+$ perceive a source which is a $\delta$-function in $x^-$. The propagator above can also be derived for
the general case where the source has a dependence on $x^-$. The gauge transforms above are transformed from $V(x_t) \to V(x_t, x^-)$, to path ordered exponentials, where $V(x_t, x^-)$ is given by Eq. 21. Our result for the propagator is obtained as a smooth limit of $\Delta x^- = 1/xP^+ >> x^- (= 1/P^+)$. In other words, our form for the propagator is the correct one as long as we interpret the $\theta$-functions and $\delta$-functions in $x^-$ to be so only for distances of interest greater than $1/P^+$, the scale of the classical source.

5 The Leading Twist Computation of $F_2$

Now that we have computed the fermion propagator in the classical background field, we are in a position to calculate the sea quark distributions in this background field, and in turn the structure function $F_2$ and evolution equations using Eq. 36. This calculation is accurate to lowest order in $\alpha_s$ but to all orders in $\alpha_s \mu$. Due to the singular nature of the propagators in the background field, the actual computation of the distribution function is a little subtle and will be outlined below.
Before we go ahead to the computation, we will begin with a discussion of the averaging procedure over the labels of color charges at rapidities higher than those of interest. We will obtain a compact expression for it below.

5.1 Color Averaging Over the Sources of Color Charge at Higher Rapidities

Our expression for the propagator in the previous section makes no particular assumption about the color averaging over the color labels of the external sources corresponding to the valence quarks and/or gluons at higher rapidities than the rapidity of interest. The expression we quote before averaging is the quantity which will be useful in loop graph computations. To relate our expression to physical observables, as discussed in section 2 (see Eq. 5) we need to average over all the color labels of the external color charge density corresponding to the color charge density $\rho^a(x_t, y)$ at rapidities greater than the rapidity of interest.

Here we have smeared out the source in $x^-$ and are no longer treating it as a delta function. This means that the sources acquire a rapidity dependence and that the weight for the Gaussian fluctuations over sources is replaced by

$$\int d^2x_t dy \frac{1}{2d\mu^2/dy} \rho^2(x_t, y)$$ (49)

which leads us to define

$$\mu^2 = \int_y^\infty dy' \frac{d\mu^2}{dy'}$$ (50)

Here the lower limit would be the rapidity of interest for evaluating the structure functions.

If we average the Green’s function in Eq. 43 over all possible values of the color labels corresponding to the partons at higher rapidities, we can employ the following
definitions for future reference. Defining

\[ \frac{1}{N_c} \langle \text{Tr} \left( V(x_t)V^\dagger(y_t) \right) \rangle = \gamma(x_t - y_t), \] (51)

we see that

\[ \gamma(0) = 1, \] (52)

which follows from the unitarity of the matrices \( V \). Now defining the Fourier transform

\[ \tilde{\gamma}(p_t) = \int d^2 x_t e^{-ip_t x_t} [\gamma(x_t) - 1], \] (53)

we have the sum rule

\[ \int \frac{d^2 p_t}{(2\pi)^2} \tilde{\gamma}(p_t) = 0. \] (54)

The function \( \tilde{\gamma}(p_t) \) will appear frequently in our future discussions and as we shall see, can be related to the gluon density at small \( x \).

For the particular case of a large nucleus \[16\] the averaging procedure has the form

\[ \langle O \rangle = \int [d\rho] O(\rho) \exp \left( - \int_0^\infty dy \int d^2 x_t \text{Tr} \rho^2(x_t, y) \frac{d\mu^2(y)}{dy} \right), \] (55)

where \( \mu^2 \) is the average color charge density per unit transverse area per unit rapidity. For an extensive discussion of the above averaging procedure we refer the reader to Refs. \[16, 22, 19\]. In appendix B we will explicitly derive an expression for \( \tilde{\gamma}(p_t) \) for Gaussian fluctuations. A Gaussian form for the averaging over color charges at higher rapidities is likely valid for very large nuclei or for realistic nuclei and hadrons for \( x << 1 \) but not at too small \( x \)'s (or large enough parton densities) where non-linear corrections to the renormalization group equations are important.

\(^3\)We define the Fourier transform in this way because it corresponds to only the connected pieces in the correlator.
5.2 Sea Quark Fock Space Distribution Function

The relation Eq. 35 can now be combined with our expression in Eq. 43 to compute the sea quark distribution function. We shall use below the following identities:

\[
\frac{1}{2q^-} \text{Tr} [(M - q' - p t)\gamma^-(M - q)\gamma^+] = \frac{2}{q^-}(M^2 + q_t^2 + p_t \cdot q_t).
\] (56)

We then obtain the following expression for the distribution function

\[
\frac{dN_{\text{ferm}}}{d^3k} = \frac{2iN_c}{(2\pi)^3} \int d^3x \, d^3y \, \int \frac{d^4q}{(2\pi)^4} \, \frac{e^{i(q-k) \cdot (x-y)}}{q^2 + M^2 - i\epsilon} \\
\times \left\{ 4q^+ \left[ \theta(x^-)\theta(y^-) + \theta(x^-)\theta(y^-)\gamma(x_t - y_t) \right] \\
+ \int \frac{d^2p_t}{(2\pi)^2} \, d^2z_t \, \frac{2}{q^-} \left( M^2 + p_t \cdot q_t + q_t^2 \right) \left[ \theta(x^-)\theta(y^-)e^{i(p_t \cdot (x_t - z_t))} \right. \\
\times \exp \left( -\frac{(p_t + q_t)^2 - q_t^2}{2q^-}x^- \right) \gamma(x_t - z_t) + \theta(x^-)\theta(y^-)e^{-i(p_t \cdot (y_t - z_t))} \right] \\
\times \exp \left( i\frac{(p_t + q_t)^2 - q_t^2}{2q^-}y^- \right) \gamma(z_t - y_t) \right\}.
\] (57)

We will now sketch below the procedure used to simplify the above equation.

a) First perform the integrals over the transverse co–ordinates. This introduces a common factor \(\pi R^2\) and for the \(\theta(x^-)\theta(y^-)\) term a factor \(\delta^{(2)}(q_t - k_t)\) (a factor \(\tilde{\gamma}\) which was defined in Eq. 53 pops up in the other three pieces).

b) Perform the integrals over \(x^-\) and \(y^-\). For the \(\theta(\pm x^-)\theta(\pm y^-)\) pieces, we obtain the factors

\[
\frac{1}{q^+ - k^+ \pm i\epsilon} \quad \frac{1}{q^+ - k'^+ \mp i\epsilon},
\] (58)

respectively. We have introduced above (to ensure smooth convergence) a slight difference in the momenta \((k^+ \text{ and } k'^+)\) respectively multiplying \(x^-\) and \(y^-\) in the phases. In the final step we take the limit \(k'^+ - k^+ \to 0\).
c) The simple contour integral over $q^+$ is done next. This introduces the factors $\theta(\pm q^-)$.

d) The final step is to perform the (logarithmic) integral over $q^-$. The ultraviolet cutoff $\Lambda \to \infty$ cancels among the different terms and we obtain a finite result. Putting all the terms together and using the identity in Eq. 54, we obtain the general result for the sea quark distribution function

$$\frac{1}{\pi R^2} \frac{dN^{\text{ferm}}}{dk^+ d^2 k_t} = \frac{N_c}{2 \pi^4} \frac{1}{k^+} \int \frac{d^2 p_t}{(2\pi)^2} \tilde{\gamma}(p_t) \left[ 1 - \frac{(k_t^2 + k_t \cdot p_t + M^2)}{p_t^2 + 2 k_t \cdot p_t} \log \left( \frac{(k_t + p_t)^2 + M^2}{k_t^2 + M^2} \right) \right].$$

(59)

In the region where the logarithm can be expanded, it can be checked analytically that the argument of the above expression is positive definite. We have checked numerically that the argument remains positive definite in the entire $(k_t, p_t)$ phase space (as it should be).

In the next section we will study the above result in different limits and relate it to the well known evolution equations for sea quark distributions.

5.3 Evolution Equations for Sea Quark Distributions at Small $x$

In this section, we will show that for large $q^2$, the Fock space seaquark distribution we derived above in Eq. 59 gives us the Altarelli–Parisi evolution equation for seaquark evolution at small $x$.

Towards that end, consider the Fock space distribution in the limit of large $k_t$. In the integral in Eq. 59 we approximate $k_t \gg p_t$. Then expanding the logarithm and defining

$$\frac{(k_t + p_t)^2 - k_t^2}{k_t^2 + m^2} = 1 + \kappa,$$

(60)
we find that the terms in the square brackets \([\cdots]\) in Eq. (59) can be approximated by

\[
[\cdots] \approx \left[ \frac{\kappa_2}{2} - \frac{\kappa_2^2}{3} - \frac{k_t \cdot p_t}{k_t^2 + M^2} + \frac{(k_t \cdot p_t) \kappa}{k_t^2 + M^2} - \frac{\kappa_2^2 (k_t \cdot p_t)}{2 k_t^2 + M^2} \right].
\] (61)

Now \(\tilde{\gamma}(p_t)\) has rotational symmetry in the transverse plane. This helps simplify our expression above since only even terms in \(k_t \cdot p_t\) survive. To leading order in \(p_t^2 / k_t^2\), then, Eq. (59) reduces to

\[
\frac{1}{\pi R^2} k^+ \frac{d N^\text{ferm}}{dk^+d^2k_t} = \frac{N_c}{2\pi^4} \int \frac{d^2p_t}{(2\pi)^2} \frac{1}{2} \left[ \frac{p_t^2}{k_t^2 + M^2} - \frac{4}{3} \frac{k_t^2 p_t^2}{(k_t^2 + M^2)^2} + \frac{k_t^2 p_t^2}{(k_t^2 + M^2)^2} \right] \tilde{\gamma}(p_t).
\] (62)

Since we are interested in the limit \(k_t^2 >> M^2\), the above expression can be further simplified to read

\[
\frac{1}{\pi R^2} k^+ \frac{d N^\text{ferm}}{dk^+d^2k_t} = \frac{N_c}{2\pi^4} \frac{1}{3} \frac{1}{k_t^2} \int \frac{d^2p_t}{(2\pi)^2} p_t^2 \tilde{\gamma}(p_t).
\] (63)

To make contact with the evolution equations we will now obtain a relation between \(\tilde{\gamma}(p_t)\) and the gluon distribution function at small \(x\). We begin with the relation we defined in the last section–Eq. (58):

\[
\tilde{\gamma}(p_t) = \int d^2 x_t e^{-ip_t x_t} [\gamma(x_t) - 1].
\]

Then

\[
p_t^2 \tilde{\gamma}(p_t) = -\int d^2 x_t e^{ip_t x_t} \partial_{x_t}^2 \gamma(x_t).
\] (64)

Recall that \(\gamma(x_t) = \frac{1}{N_c} < \text{Tr}(V(x_t) V(0)^\dagger) >_\rho\). Expanding out the matrix \(V = 1 + i\Lambda(x_t) - \Lambda^2(x_t)/2 + \cdots\), and doing the same for the correlator of gauge fields \(A^i = \frac{1}{ig} V \partial^i V^\dagger\), we obtain the relation

\[
p_t^2 \tilde{\gamma}(p_t) = \frac{g^2}{2N_c} \int d^2 x_t e^{ip_t x_t} <A_t^a(x_t) A_t^a(0) >_\rho.
\] (65)
The correlator \(< A_\eta^a A_\eta^a >\) can be related to the gluon distribution function by the formula
\[
\frac{dN}{d^3l} = \frac{2 |l^+|}{(2\pi)^3} \int d^3x d^3x' e^{-i l^+ x^-} e^{i l^+ x'^-} \theta(x^-) \theta(x'^-) e^{i l^+ x_t} < A_\eta^a(x_t) A_\eta^a(0) >_\rho . (66)
\]
Integrating both sides over \(l^+\), we obtain
\[
\int d^2 x_t e^{i l^+ x_t} \langle A_\eta^a(x_t) A_\eta^a(0) \rangle = \frac{k^+ (2\pi)^3}{2} \int_{k^+}^{P^+} \frac{dl^+}{|l^+|} dN^{\text{glue}} . (67)
\]
Substituting the RHS of the above equation in Eq. 65 and the resulting expression for \(p_t^2 \gamma(p_t)\) into Eq. 63, we obtain
\[
\frac{d}{dx} \frac{d^2 N}{dx dk_t^2} = \frac{\alpha_S}{2\pi} \frac{1}{3} \int_x^1 \frac{dy}{y} \int_0^{k_t^2} dp_t^2 \frac{dN^{\text{glue}}}{dp_t^2} , (68)
\]
where we defined \(x = k^+ / P^+\) and \(y = l^+ / P^+\).

Recall that the structure function to leading twist and lowest order in \(\alpha_S\) is
\[
F_2(x, Q^2) = \int_0^{Q^2} dk_t^2 x \frac{dN^{\text{perm}}}{dx dk_t^2} = x[q(x, Q^2) + \bar{q}(x, Q^2)] \equiv 2xq(x, Q^2) , (69)
\]
and similarly,
\[
xG(x, Q^2) = \int_0^{Q^2} x \frac{dN^{\text{glue}}}{dx dk_t^2} , (70)
\]
where \(xq(x, Q^2)\) and \(xG(x, Q^2)\) are the quark and gluon momentum distributions respectively. At small \(x\), the sea is symmetric and we take \(q(x, Q^2) = \bar{q}(x, Q^2)\). In the limit of light quark masses \(Q^2 >> M^2\) we find then that
\[
\frac{d(xq(x, Q^2))}{d\log(Q^2)} = \alpha_S \frac{2}{4\pi} \frac{1}{3} x \int_x^1 \frac{dy}{y^2} yG(y, Q^2) . (71)
\]
Now at small \(x\), \(yG(y, Q^2)\) is slowly varying. One can for instance parametrize it by a power law \(x^{\alpha_S C}\), where \(C\) is some constant. The scale of variation of the structure function then corresponds to higher orders in \(\alpha_S\). We can therefore take \(yG(y, Q^2)\) out of the integrand. (The same is true for any other slowly varying function.)

We then get finally for the seaquark evolution equation at small \(x\) the result
\[
\frac{d(xq(x, Q^2))}{d\log(Q^2)} = \alpha_S \frac{2}{4\pi} \frac{1}{3} xG(x, Q^2) + O(\alpha_S) . (72)
\]
Thus to lowest order in \( \alpha_s \), the seaquark evolution at small \( x \) is local and simply proportional to the gluon density at that \( x \).

Now consider the Altarelli–Parisi evolution equation

\[
Q^2 \frac{d\Sigma}{dQ^2} = \frac{\alpha_s(Q^2)}{2\pi} \left[ \Sigma \otimes P_{qq} + G \otimes 2fP_{qG} \right].
\]  \( (73) \)

Above the operation \( \otimes \) denotes

\[
A \otimes B \equiv \int_x^1 \frac{dy}{y} A(y)B\left(\frac{x}{y}\right),
\]

and \( \Sigma = \sum_f (q + \bar{q}) \), where \( f \) is the number of flavors. Also, \( P_{qq} \) and \( P_{qG} \) are the well known Altarelli–Parisi splitting functions. Since at small \( x \) the quark distribution is \( \alpha_s \) suppressed relative to the glue, taking \( q = \bar{q} \) and \( f = 1 \) the leading contribution to the seaquark evolution is

\[
\frac{d(xq(x,Q^2))}{d\log(Q^2)} = \frac{1}{2} \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} G(y,Q^2) \left[ \frac{x^2}{y^2} + (1 - \frac{x}{y})^2 \right].
\]  \( (74) \)

Above we have made use of the relation \( 2fP_{qG}(z) = f[z^2 + (1 - z)^2] \). Let \( z = y/x \).

Then the above relation can be re-written as

\[
\frac{d(xq(x,Q^2))}{d\log(Q^2)} = \frac{\alpha_s}{4\pi} \int_1^\frac{1}{z} \frac{dz}{z^2} zG(zx,Q^2) \left[ \frac{2}{z^2} + 1 - \frac{2}{z} \right].
\]  \( (75) \)

Again, as previously, we can argue that since at small \( x \) \( zG(zx,Q^2) \) is slowly varying, we can take it out of the integral. Doing that and performing the integral, we obtain finally

\[
\frac{d(xq(x,Q^2))}{d\log(Q^2)} = \frac{\alpha_s}{4\pi} \frac{2}{3} xG(x,Q^2) + O(\alpha_s).
\]  \( (76) \)

which is the same as Eq. \( 74 \). This form of the sea quark evolution equation at small \( x \) was first obtained by Ellis, Kunzt and Levin \( 35 \).
6 Computation of Current-Current Correlator to All Twists in the Classical Background Field at small $x$.

In the previous section we used the fermion Green’s function derived in section 4.2 to compute the lightcone seaquark distributions at small $x$ and subsequently the leading twist expression for the structure functions. It was shown that these structure functions obeyed evolution equations which were precisely the small $x$ Altarelli–Parisi evolution equations.

In this section we will again use the fermion Green’s function in Eq. 43 to derive an explicit result for the hadronic tensor $W^{\mu\nu}$. This result will be valid to all twists at small $x$ and for arbitrary quark masses. For light quarks, we will compute the structure functions $F_1$ and $F_2$ and show that the leading twist result in the previous section is recovered as a limit of our general result. We will also use our general result to obtain expressions for heavy quark structure functions at small $x$. We should note here that Levin and collaborators have studied screening corrections to the structure functions for light and heavy quarks in the Glauber–Gribov framework \[37\].

6.1 Analytic Result for $W^{\mu\nu}$ at Small $x$

As in the previous sections, we define

$$ W^{\mu\nu}(q, P, X^-) = \text{Im} \int d^4z \, e^{iq \cdot z} < T(J^\mu(X^- + \frac{z}{2})J^\nu(X^- - \frac{z}{2})>, \quad (77) $$

where “Imaginary” stands for the discontinuity in $q^-$. Then

$$ W^{\mu\nu}(q, P) = \frac{1}{2\pi} \sigma \frac{P^+}{m} \int dX^- W^{\mu\nu}(q, P, X^-) \equiv \frac{1}{2\pi} \sigma P^+ \text{Im} \int dX^- \int d^4z \, e^{iq \cdot z} $$

$$ \times \text{Tr} \left( S_{Acl} \left( X^- + \frac{z}{2}, X^- - \frac{z}{2} \right) \gamma^\nu S_{Acl} \left( X^- - \frac{z}{2}, X^- + \frac{z}{2} \right) \gamma^\mu \right). \quad (78) $$
The only terms in the propagator that contribute to the above are the \( \theta(\pm x^-)\theta(\mp y^-) \) pieces. Using our representation for the propagator in Eq. [45],

\[
W^{\mu
u}(q, P, X) = \text{Im} N_c \int d^4 z e^{i q \cdot z} \int \frac{d^4 p d^4 l'}{(2\pi)^8} \frac{d^4 p' d^4 l'}{(2\pi)^8} d^2 u_t d^2 u'_t (2\pi) \delta(p^--l^-)
\]

\[
\times (2\pi) \delta(p'^--l'^-)e^{i p'(X+\frac{z}{2}-u_t)} e^{-i l'(X-\frac{z}{2}-u_t)} e^{i l'(X-\frac{z}{2}-u'_t)} e^{-i p'^-(X+\frac{z}{2}-u'_t)}
\]

\[
\times \text{Tr} \left\{ \frac{(M-p')\gamma^- (M-l')\gamma^\mu (M-l')\gamma^- (M-p')\gamma^\nu}{(p^2+M^2-i\varepsilon)(l'^2+M^2-i\varepsilon)(l'^2+M^2-i\varepsilon)(p'^2+M^2-i\varepsilon)} \right\}
\]

\[
\times \left( \theta(X^-+\frac{z}{2})\theta(\frac{z}{2}-X^-) + \theta(X^-+\frac{z}{2})\theta(-X^-+\frac{z}{2}) \right) \gamma(u_t-u'_t) \left( \begin{array}{c}
\frac{z^-}{2} \\
(p^--l^+-p'^++l'^+) \end{array} \right) \right).
\] (79)

We have used above the condition \( \gamma(u_t, u'_t) = \gamma(u'_t-u_t) = \gamma(u_t-u'_t) \), since the correlation functions are translationally, rotationally and parity invariant in the transverse plane. We now perform the integral over \( X^- \) and use the identity

\[
\int dX^- e^{i(-p^++l^++l'^++p'^-)} X^- \left( \theta(X^-+\frac{z}{2})\theta(\frac{z}{2}-X^-) + \theta(X^-+\frac{z}{2})\theta(-X^-+\frac{z}{2}) \right) = \frac{\epsilon(z^-)}{(p^--l^+-p'^++l'^+)} 2 \sin \left( \frac{z^-}{2} (p^--l^+-p'^++l'^+) \right).
\] (80)

We then obtain

\[
W^{\mu\nu}(q, P) = \frac{\sigma P^+ N_c^{\pi m}}{2\pi m} \text{Im} \int d^4 z e^{i q \cdot z} \epsilon(z^-) \int \frac{d^4 p d^4 l}{(2\pi)^8} \frac{d^4 p' d^4 l'}{(2\pi)^8} (2\pi)^2 \delta(p^--l^-)\delta(p'^--l'^-)
\]

\[
\times (2\pi)^2 \delta^{(2)}(l_t-p_t+p'_t-l'_t) \frac{1}{2} e^{iz \cdot (p_t-l_t+p'_t-l'_t)/2} e^{iz^+ (p^--p'^-)}
\]

\[
\times \text{Tr} \left\{ \frac{(M-p')\gamma^- (M-l')\gamma^\mu (M-l')\gamma^- (M-p')\gamma^\nu}{(p^2+M^2-i\varepsilon)(l'^2+M^2-i\varepsilon)(l'^2+M^2-i\varepsilon)(p'^2+M^2-i\varepsilon)} \right\}
\]

\[
\times \frac{1}{(p^--l^+-p'^++l'^+)} 2 \sin \left( \frac{z^-}{2} (p^--l^+-p'^++l'^+) \right).
\] (81)

The subsequent procedure of solving the integrals is as follows:

a) Perform first the integral over \( l'_t \). Then perform the integral over \( z_t \) and \( z^+ \).

Defining \( k_t = (p_t-l_t+p'_t-l'_t)/2 \), this sets \( l'_t = p_t-k_t-q_t \), \( l_t = p_t-k_t \) and \( q_t = p_t-p'_t \). Also, we get \( q^- = p^- - p'^- \).

b) Next perform the integral over \( z^- \). This gives us

\[
\int dz^- \epsilon(z^-) e^{iq^+z^-} \left( e^{-iz^- (p^++l'^+)} - e^{-iz^- (l^+-l'^+)} \right)
\]
\[
= i \left[ \frac{\ell'^+ - \ell^- - p'^+ + p^+}{(q^+ - p^+ + p'^+ + i\varepsilon)(q^+ - l^+ + \ell'^+ + i\varepsilon)} + (i\varepsilon \rightarrow -i\varepsilon) \right]. \tag{82}
\]

The numerator above cancels the term \(1/(p^+ - l^+ - p'^+ + \ell'^+)\) in \(W^{\mu\nu}\).

c) Lastly, we do the integrals over \(p'^+\) and \(l'^+\). This sets \(p'^+ = p^+ - q^+\) and \(l'^+ = l^+ - q^+\). Then we can define in \(W^{\mu\nu}\), \(p' = p - q\) and \(l' = l - q\) with \(l = p - k\) and \(k^- = 0\).

After these considerations, we can write \(W^{\mu\nu}\) as

\[
W^{\mu\nu}(q, P) = \frac{\sigma P^+ N_c}{2\pi m} \text{Im} \int \frac{d^4p}{(2\pi)^4} \frac{d^2k_t}{(2\pi)^2} \frac{dk^+}{(2\pi)} \bar{\gamma}(k_t) \\
\times \text{Tr} \left\{ (p^2 + M^2 - i\varepsilon)(l'^2 + M^2 - i\varepsilon)(l'^2 + M^2 - i\varepsilon)(p'^2 + M^2 - i\varepsilon) \right\}, \tag{83}
\]

where \(l', p'\) and \(l\) are defined as in step ‘c’ above. Correspondingly, we can write \(W^{\mu\nu}\) as the imaginary part of the diagram shown in Fig. 2.

For the case of deep inelastic scattering, \(q^2 > 0\) (see footnote 1), and we can cut the above diagram only in the two ways shown in Fig. 3 (the diagram where both insertions from the external field are on the same side of the cut is forbidden by the kinematics).

Also interestingly, the contribution to \(W^{\mu\nu}\) can be represented solely by the diagram in Fig. 4 and not, as is usually the case, from the sum of this diagram and the standard box diagram. This is because in our representation of the propagator multiple insertions from the external field on a quark line can be summarized into a single insertion. See for instance Eq. 48 which makes this point clear.

Applying the Landau-Cutkosky rule, and making the shift \(p \rightarrow p + k\), Eq. \ref{eq:83} can be written as

\[
W^{\mu\nu}(q, P) = \frac{\sigma P^+ N_c}{2\pi m} \int \frac{d^4p}{(2\pi)^4} \frac{d^2k_t}{(2\pi)^2} \frac{dk^+}{(2\pi)} \bar{\gamma}(k_t) \\
\times \text{Tr} \left\{ (M - p' - k')\gamma^-(M - p')\gamma^\mu(M - p' + q')\gamma^- (M - p' - k' + q')\gamma^\nu \right\}
\]
Figure 2: Polarization tensor with arbitrary number of insertions from the classical background field. The wavy lines are photon lines, the solid circle denotes the fermion look and the dashed lines are the insertions from the background field (see Fig. 1). The imaginary part of this diagram gives $W^{\mu\nu}$.

\[
\times \left[ \theta(p^+ + k^+)\theta(q^+ - p^+)(2\pi)^2\delta((p + k)^2 + M^2)\delta((p - q)^2 + M^2)) \frac{1}{p^2 + M^2} \right. \\
\times \left. \frac{1}{(p + k - q)^2 + M^2} + \theta(p^+)\theta(q^+ - p^+)(2\pi)^2\delta(p^2 + M^2) \delta((p + k - q)^2 + M^2)) \right]
\]

(84)

With an appropriate change of variables, the second term is the same as the first except that now $\mu \leftrightarrow \nu$. We then get

\[
W^{\mu\nu}(q, P) = \frac{\sigma P^+ N_c}{2\pi m} \int \frac{d^4p}{(2\pi)^4} \frac{d^2k_t}{(2\pi)^2} \frac{dk^+}{2\pi} \gamma(k_t)M^{\mu\nu} \theta(p^+ + k^+)\theta(-p^+) \\
\times (2\pi)^2\delta((p + k)^2 + M^2) \delta((p - q)^2 + M^2)) \frac{1}{p^2 + M^2} \frac{1}{(p + k - q)^2 + M^2},
\]

(85)
Figure 3: Cut diagrams corresponding to the imaginary part of $W^{\mu\nu}$.

where above the trace is represented by $\mathcal{I}$

$$M^{\mu\nu} = \text{Tr} \left\{ (M - q^i - k^j)\gamma^- (M - q^i)\gamma^\mu (M - q^i + q^j)\gamma^- (M - q^i - k^j + q^j)\gamma^\nu + \mu \leftrightarrow \nu \right\}.$$ (86)

Performing the $\delta$–function integrations above, our expression for $W^{\mu\nu}$ can be simplified to

$$W^{\mu\nu} = -\frac{\sigma P^+ N_c}{16\pi m} \frac{1}{(q^-)^2} \int \frac{d^2 k_t}{(2\pi)^2} \frac{d^2 p_t}{(2\pi)^2} \gamma^\mu(k_t) \int_{-\infty}^{-\frac{M_{p+k-q}^2}{2\pi}} \frac{dp^+}{2\pi} \frac{M^{\mu\nu}}{M_{p+k-q}^2} \cdot I(k_t, p_t, q, p^+),$$ (87)

with the definitions $M_{p-k-q}^2 = (p_t - q_t)^2 + M^2$ and $M_{p+k-q}^2 = (p_t + k_t - q_t)^2 + M^2$.

4Kinematic note: the observant reader will notice we have put $q^+ = 0$ here. Since we are working in the infinite momentum frame, the hadron has only one large momentum component, $P^+$. The rest are put to zero. For the photon, we choose a left moving frame such that $q^0 = |q^0|$ and $q^+ = 0$. Then, $q^2 = q_t^2 > 0$, $P \cdot q = -P^+ q^-$ and $x_{Bj} = -q^2/(2P \cdot q) \equiv q_t^2/(2P^+ q^-)$. Since in the infinite momentum frame $0 < x_{Bj} < 1$, this gives $q^- > 0$. We are at liberty to choose the above frame since the hadron tensor is clearly Lorentz invariant and hence can be expressed purely as a function of $q^2$ and $P \cdot q$. The explicit presence of $P^+$ in Eq. 74 may give the reader cause for concern. It arises from a relativistic normalization of the vacuum states. One can show that, despite appearances, Eq. 73 is Lorentz invariant. Of course our later results will confirm this fact.
Figure 4: In the singular gauge representation for the propagator (see Eq. 48 and Fig. 1), multiple, higher twist contributions from the classical gluon background field to the current–current correlator (imaginary part of LHS) is equivalent to the imaginary part of RHS.

and

\[
I(p_t, k_t, q, p^+) = \frac{1}{p^+ - \frac{(M_p^2 - M_{p-k}^2)}{2q^2}} - \frac{1}{p^+ - \frac{M_{p-q}^2 (M_{p+k}^2 - M_{p+k-q}^2)}{(2q^-)M_{p+k-q}^2}}.
\]

Eq. 87 is our general result for the hadronic tensor. We shall now study the different components of the above tensor and extract from these, different limits of interest to us.

6.2 Structure Functions at Small x

The hadronic tensor \( W^{\mu\nu} \) can be decomposed in terms of the structure functions \( F_1 \) and \( F_2 \) as \( 88 \)

\[
mW^{\mu\nu} = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)F_1
\]
\[ (P^\mu - q^\mu(P \cdot q)/q^2) (P^\nu - q^\nu P \cdot q)/q^2 (P \cdot q) F_2, \]  
(89)

where \( P^\mu \) is the four–momentum of the hadron or nucleus and \( P^2 = m^2 \approx 0 \) \((<< q^2)\). In the infinite momentum frame, we have \( P^+ \to \infty \) and \( P^-, P_t \approx 0 \).

The above equation can be inverted to obtain expressions for \( F_1 \) and \( F_2 \) in terms of components of \( W^{\mu \nu} \). Since in our kinematics \( q^+ = 0 \) (see footnote 3 for a kinematic note)

\[ F_1 = \frac{F_2}{2x} + \left( \frac{q^2}{(q^-)^2} \right) W^{--}, \]  
(90)

with

\[ \frac{1}{2x} F_2 = - \left( \frac{(q^-)^2}{q^2} \right) W^{++}. \]  
(91)

It is useful to verify explicitly that our expression for \( W^{\mu \nu} \) derived in an external field can be written in the form of Eqn. 89. Recall that \( W^{\mu \nu} \) can be written in Lorentz covariant form by using the vector \( n^\mu = \delta^\mu_+ \). Using \( n \cdot \gamma = -\gamma^- \) in Eqn. 83, we see that \( W^{\mu \nu} \) is a Lorentz covariant function of the only vectors in the problem–\( q^\mu \) and \( n^\mu \). Identifying \( n^\mu = P^\mu/P^+ \) in Eqn. 89, we see that these forms are in complete agreement. We also see that all factors of \( m \) disappear from \( F_1 \) and \( F_2 \) by the explicit forms of Eqns. 89 and 83. Henceforth we will take \( m = 1 \) since it disappears from the quantities of interest, and was in fact only introduced due to historical normalization conventions.

We also see that the structure functions can only be functions of \( q^2 \) and \( n \cdot q \) by Lorentz invariance. We can therefore take \( q^+ = 0 \) for the purpose of computing \( F_1 \) and \( F_2 \).

To compute \( W^{++} \) and \( W^{--} \), we need to know the the traces \( M^{++} \) and \( M^{--} \), respectively in Eq. 87. We can compute them explicitly and the results can be represented compactly as

\[ \frac{1}{16} M^{++} = \frac{1}{2} \left( M_{p^2}^2 M_{p-k-q}^2 + M_{p+k}^2 M_{p-q}^2 - q^2 k_t^2 \right), \]  
(92)
and

\[ M^{-} = 32(p^{-})^2(p^{-} - q^{-})^2. \]  \hfill (93)

From the relations above of \( F_1 \) and \( F_2 \) to \( W^{++} \) and \( W^{--} \), we obtain from Eq. 87 the following general results for the structure functions for arbitrary values of \( Q^2 \), \( M^2 \) and the intrinsic scale \( \mu^2 \):

\[
F_2 = \frac{\sigma N_c}{16\pi^2} \int \frac{d^2 p_t}{(2\pi)^2} \frac{d^2 k_t}{(2\pi)^2} \frac{\tilde{\gamma}(k_t)}{M^{++}} \frac{M^{++}}{M_{p-q}^2 M_{p+k-q}^2 M_P^2} \log \left( \frac{M_{p+k}^2 M_{p-q}^2}{M_{p+k-q}^2 M_P^2} \right),
\]

and

\[
F_1 = \frac{F_2}{2x} \frac{q^4 \sigma N_c}{2x 2\pi^4} \int d^2 p_t \int \frac{d^2 k_t}{(2\pi)^2} \frac{\tilde{\gamma}(k_t)}{M_{p-q}^2 M_{p+k-q}^2} \frac{\mathcal{F}(\alpha, \beta)}{M_{p-q}^2 M_{p+k-q}^2}.
\]

Above, we defined the function \( \mathcal{F} \) in terms of \( \alpha \) and \( \beta \) (in turn functions of \( p_t, k_t, q_t \) and \( M \)) which are defined as

\[
\alpha = \left( 1 - \frac{M_{p+k}^2}{M_{p+k-q}^2} \right) \quad \beta = \left( 1 - \frac{M_P^2}{M_{p-q}^2} \right).
\]

The general expression for \( \mathcal{F} \) is

\[
\mathcal{F}(\alpha, \beta) = \frac{1}{\alpha \beta} \left[ \frac{1}{3} - \frac{3}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) + \left( \frac{1}{\alpha^2} + \frac{1}{\alpha \beta} + \frac{1}{\beta^2} \right) \right] + \frac{\alpha \beta}{2(\beta - \alpha)} \left( \frac{1}{\alpha^2}(1 - \frac{1}{\alpha})^2 \log(1 - \alpha)^2 - \frac{1}{\beta^2}(1 - \frac{1}{\beta})^2 \log(1 - \beta)^2 \right).
\]

Eqn. 94 and Eqn. 95 are the central results of this work. Since they are the most general possible expressions for arbitrary values of \( Q^2 \), \( M^2 \), and \( \mu^2 \), it is inevitable that they look complicated. We shall show in the following sub-section that they simplify considerably in the high \( q^2 \) limit.

\footnote{which is implicitly contained in the function \( \tilde{\gamma}(k_t) \) in Eq. 87}

\footnote{Our expression for \( F_2 \) in Eq. (94) can be further simplified—a result which will be discussed in a forthcoming paper.}

40
6.3 Structure functions in the limit $q^2 \to \infty$.

We shall now obtain the leading twist limits of Eq. 95 and Eq. 94. In particular we will show that our structure function for $M^2 \to 0$ and $q^2 \to \infty$ is identical to the structure function obtained by integrating the light cone Fock distribution in Eq. 59. That this should be the case is a well known property of the leading twist structure functions [25, 26]. Further, we will recover the Callan–Gross result [27] $F_1 = F_2/2x$ in this limit.

Consider first our general formula for $F_2$ in Eq. 94. The trace simplifies considerably when we put $M = 0$. For $q_t \gg p_t, k_t$, we obtain

$$M^{++} \to 16q_t^2 \left( p_t^2 + p_t \cdot k_t \right).$$

In the logarithm, the ratio $M_{p-q}^2/M_{p+k-q}^2 \to 1$. Finally, in the denominator of the integrals,

$$\left( M_{p-q}^2 M_{p+k}^2 - M_{p+k-q}^2 M_p^2 \right) \to M_q^2 \left( M_{p+k}^2 - M_p^2 \right).$$

Then putting these back into our general expression we obtain

$$F_2 = \frac{\sigma N_c}{2\pi^4} \int d^2 p_t \int \frac{d^2 k_t}{(2\pi)^2} \gamma(k_t) \left[ 1 - \frac{(p_t^2 + k_t \cdot p_t)}{(k_t^2 + 2k_t \cdot p_t)} \log \left( \frac{(k_t + p_t)^2}{p_t^2} \right) \right]. \quad (98)$$

A comparison with Eq. 59 immediately reveals that, setting $k_t \leftrightarrow p_t$, and integrating the latter over $p_t$ upto $q^2$ gives an identical result to the one above. Thus we have recovered a well known, non–trivial leading twist result as the limit of our general expression for $q^2 \to \infty$ and $M \to 0$.

Even though our general expression for the longitudinal structure function $F_1$ looked terribly complicated, in the limit considered here it is remarkably simple. In this limit $\alpha, \beta \to 1$ and hence Eq. 77 for $F \to \frac{1}{3}$ a constant! The product in

\footnote{The contribution in the $p_t$ integral in the region $p_t \sim q_t$ is identical to that from $p_t \ll q_t$. This provides a factor of 2 that must be taken into account.}
the denominator of Eq. 95, \( M_{p-q}^2 M_{p+k-q}^2 \rightarrow q^4 \) and cancels the \( q^4 \) factor outside the integral. From the sum rule Eq. 54, we find remarkably that the complicated integral vanishes and Eq. 95 reduces to

\[
F_1 = \frac{F_2}{2x}.
\]

(99)

The above is the well known Callan–Gross relation.

We should clarify the result obtained above to avoid confusion. The reader may note above that the deviation from the Callan–Gross relation vanishes as a power law as \( q^2 \rightarrow \infty \). On the other hand, it is well known in QCD [28, 29] that the violations of the Callan–Gross relation only disappear logarithmically as \( q^2 \rightarrow \infty \). The apparent contradiction is resolved by one realizing that the logarithmic violations at large \( q^2 \) in QCD come from diagrams where the sea quark emits a gluon (thereby violating Feynman’s parton model helicity argument). These diagrams are of higher order in our picture and are therefore not included. In fact, the deviations from the Callan–Gross relation of the sort discussed above (at small \( x \)) should die off faster than logarithmically at very large \( q^2 \) because for sufficiently large \( q^2 \), the violations of the Callan–Gross relation should come from precisely the diagrams not included here. At moderate \( q^2 \) however, the contributions we have discussed above should be important.

7 Summary

In this paper we have used a classical theory of the gluon field to derive expressions for Fock space distribution functions of quarks and structure functions for deep inelastic scattering. This theory is valid at small \( x \) when the gluon density is large. In this region, the coupling constant evaluated at this density scale is small. With this density scale denoted by \( \mu^2 \), we have seen that when \( q^2 >> \alpha^2 \mu^2 \) and \( q^2 >> M^2 \), where \( M \) is the mass of the heavy quark being probed, all leading twist results are
reproduced. For \( q^2 \leq \alpha^2 \mu^2 \) or \( q^2 \leq M^2 \), we derived an expression valid to all orders in twist but only to leading order in \( \alpha_S \). In this kinematic region, the Callan-Gross relation is not valid, and there is no simple relation between the Fock space distribution function and the structure functions for quarks.

The structure function of heavy quarks deserves more study. If \( M^2 \gg \alpha^2 \mu^2 \), then for \( q^2 \gg \alpha^2 \mu^2 \), the heavy quark distribution is a linear function of the gluon distribution function. The gluon distribution function may nevertheless be computed to all orders in twist. The leading twist relation between the structure function and the quark distribution function is however not valid for large \( M^2 \). As we go to smaller values of \( x \) corresponding to larger values of \( \mu^2 \), the non-linearities corresponding to higher powers of the gluon density turn on and can be studied systematically in our weak coupling formalism.

The situation for light quarks is also amusing and needs more study as well. If \( q^2 \leq \alpha^2 \mu^2 \), then the non-linearities in the gluon distribution function become important. In this kinematic region, we expect saturation of the gluon distribution function, and our function \( \gamma(k_t) \sim 1/k_t^2 \) up to logarithms of \( k_t \). If this is the case, a look at the definition of \( F_2 \) and \( F_1 \) shows that these distributions should be dimensionally of order \( q^2 \). In this region a precise analytic estimate is difficult since in the integral representations for \( F_1 \) and \( F_2 \) (Eqs. 95 and 94 respectively), there is no hierarchy of momentum scales. All momenta are of order \( q \), and the integrand does not simplify much. Nevertheless, we see that the saturation of the gluon density is sufficient to imply saturation of the quark density.

Both the study of the heavy quark and light quark distributions merit more theoretical and phenomenological work within the framework described in this paper.
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Appendix A: Notation and Conventions

We start by defining our convention and notations. Our metric is the $+2$ metric $\hat{g}^{\mu\nu} = (-, +, +, +)$. The gamma matrices in space–time co–ordinates are denoted by carets. In the chiral representation,

\[
\hat{\gamma}^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \hat{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \hat{\gamma}^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\]

and \{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = -2\hat{g}^{\mu\nu}$. Above, $\sigma^i, i = 1, 2, 3$ are the usual $2 \times 2$ Pauli matrices and $I$ is the $2 \times 2$ identity matrix. In light cone co–ordinates, $\gamma^\pm = (\hat{\gamma}^0 \pm \hat{\gamma}^3)/\sqrt{2}$ and $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, where $g^{++} = g^{--} = 0, g^{+-} = g^{-+} = -1$ and $g_{t_1,t_2} = 1$ where $t_1, t_2 = 1, 2$ here stand for the two transverse co–ordinates. Note for instance that in this convention $A_+ = -A^-$ and $A_t = +A^t$. Also, $q^2 = -2q^- q^+ + q_t^2$ hence a “space–like” $q^2$ implying large space–like components would correspond to $q^2 > 0$.

We now define the projection operators

\[
\alpha^\pm = \frac{\hat{\gamma}^0 \gamma^\pm}{\sqrt{2}} \equiv \frac{\gamma^\mp \gamma^\pm}{2},
\]

which project out the two component spinors $\psi_\pm = \alpha^\pm \psi$. Some relevant
properties of the projection operators $\alpha^\pm$ are
\[
(\alpha^\pm)^2 = \alpha^\pm; \quad \alpha^+\alpha^- = 0; \quad \alpha^+ + \alpha^- = 1; \quad (\alpha^\pm)^\dagger = \alpha^\pm.
\] }

(101)

It follows from the above that $\psi_+ + \psi_- = \psi$. In the following, we will also use the familiar Dirac conventions $\beta = \hat{\gamma}^0$ and $\alpha_\perp = \hat{\gamma}^0\gamma_\perp$.

The two component spinor is the dynamical spinor in the light cone QCD Hamiltonian $P_{QCD}^-$ and it is defined in terms of creation and annihilation operators as
\[
\psi_+ = \int_{k^+ > 0} \frac{d^3k}{(2\pi)^3} \sum_{s=\pm\frac{1}{2}} \left[ e^{i\vec{k}\cdot\vec{x}}w(s)b_s(k) + e^{-i\vec{k}\cdot\vec{x}}w(-s)d_s^\dagger(k) \right].
\] }

(102)

Above $b_s(k)$ is a quark destruction operator and destroys a quark with momentum $k$ while $d_s^\dagger(k)$ is an anti–quark creation operator and creates an anti–quark with momentum $k$. Also above the unit spinors $w(s)$ are defined as
\[
w(\frac{1}{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad w(-\frac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
\] }

(103)

The creation and annihilation operators obey the equal light cone time ($x^+$) commutation relations
\[
\{b_s(\vec{k}, x^+), b_{s'}^\dagger(\vec{k}', x^+)\} = \{d_s(\vec{k}, x^+), d_{s'}^\dagger(\vec{k}', x^+)\} = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}')\delta_{ss'}.
\] }

(104)

The above definitions ensure that the light cone QCD Hamiltonian can be defined as $P_{QCD}^- = P_0^- + V_{QCD}$, where the non–interacting piece of the Hamiltonian is defined as
\[
P_0^- = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm\frac{1}{2}} \frac{(k^2 + M^2)}{2k^+} \left( b_s^\dagger(k)b_s(k) + d_s^\dagger(k)d_s(k) \right).
\] }

(105)

The definition of quark distribution functions is further discussed in the text of section 3.2
The dynamical components of the gauge fields $A^a_i(x)$ with $i = 1, 2$ in light cone gauge $A^+ = 0$ are defined as

$$A^a_i(x) = \int_{k^+ > 0} \frac{d^3k}{\sqrt{2|k^+|(2\pi)^3}} \sum_{\lambda = 1,2} \delta_{\lambda i} \left[e^{ik\cdot x}a^a_\lambda(k) + e^{-ik\cdot x}a^a_\lambda(k)^\dagger\right], \quad (106)$$

where the $\lambda$'s here correspond to the two independent polarizations and $a^a_\lambda (a^a_\lambda)^\dagger$ creates (destroys) a gluon with momentum $k$. They obey the commutation relations

$$[a^a_\lambda(\vec{k}), a^b_\lambda(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')\delta_{ab}\delta_{\lambda\lambda'}. \quad (107)$$

The gluon distribution function is then defined as

$$\frac{dN}{d^3k} = \frac{a_i^a a_i^a}{(2\pi)^3}. \quad (108)$$

Performing the Fourier transform of the gauge field above, we obtain Eq. 66 in section 5.3 For a more extensive discussion of the above formalism see Ref. [42].

We should mention here that there are several conventions in use. For a discussion of some of these, see the review article by Brodsky, Pauli and Pinsky [43]. Our convention is alike that of Kogut and Soper [40] but differs from theirs for quark spinor and gauge field normalizations by a factor $\sqrt{|k^+|}/\sqrt{(2\pi)^3}$.

**Appendix B: Derivation of the Function $\tilde{\gamma}(p_t)$ for Gaussian Fluctuations**

Since $\tilde{\gamma}(p_t)$ is defined as the Fourier transform of $\frac{1}{N_c} < \text{Tr}(U(x_t)U^\dagger(y_t)) >_\rho$, we need to compute this correlator in co–ordinate space first. Note that the symbol $< \cdots >_\rho$ denotes the averaging over with a Gaussian weight. Now, as discussed by Jalilian–Marian et al., if

$$U(y, x_t) = U_{\infty, y}(x_t) = \text{P exp} \left[i \int_y^\infty dy' A(y', x_t)\right], \quad (109)$$

46
where $U_{\infty,y}(x_t)$ is the path ordered exponential (in rapidity) which corresponds to the pure gauge potential $A^i = -U_{\infty,y}(x_t)\nabla^i U_{\infty,y}^\dagger(x_t)/ig$, then $A^i$ satisfies the Yang–Mills equation

$$D_i \frac{dA}{dy} = g\rho^a(x_t, y),$$

if $\Lambda$, the argument of the path ordered exponential, satisfies the Laplace equation

$$\nabla^2 \Lambda^a(x_t, y) = \rho^a(x_t, y).$$

The measure for the functional integral is then

$$\int [dp] \exp \left( -\int_0^\infty dy \int d^2 x_t \frac{\text{Tr} \rho^2(y)}{\mu^2(y)} \right) \rightarrow \int [d\Lambda] \exp \left( -\int_0^\infty dy \int d^2 x_t \frac{\text{Tr}(\Lambda \nabla^4 \Lambda)}{g^4\mu^2(y)} \right).$$

As argued by Jalilian–Marian et al., we can write

$$U_{\infty,y}(x_t) =: U_{\infty,y}(x_t) : \exp \left( -\frac{g^4 N_c \Gamma(0)}{2} \int_y^{y'} dy' \mu^2(y') \right),$$

where $\cdots$ denotes normal ordering and

$$\Gamma(x_t) = \frac{1}{\sqrt{4}} \equiv \Gamma(0) + \frac{x_t^2}{16\pi} \log(x_t^2 L^2) + \text{finite pieces}….$$  

Above, $\Gamma(0) \propto 1/L^2$ where $L$ corresponds to an infrared cut-off. Writing $U_{\infty,y}(x_t)$ in the above normal ordered form enables us to isolate and exponentiate the infrared singular terms coming from disconnected graphs.

Our correlator has then the form

$$\gamma(x_t, \bar{x}_t; y, \bar{y}) = N_c \gamma'(x_t, \bar{x}_t; y, \bar{y}) \exp \left( -\frac{g^4 N_c \Gamma(0)}{2} \left( \int_y^\infty \int_{y'}^\infty dy' \mu^2(y') \right) \right),$$

where

$$\gamma'(x_t, \bar{x}_t; y, \bar{y}) = \int [d\Lambda] \exp \left( -\int_0^\infty dy \int d^2 x_t \frac{\text{Tr}(\Lambda \nabla^4 \Lambda)}{g^4\mu^2(y)} \right) \left( : U_{\infty,y}(x_t) ; \right) \left( : U_{\infty,y}(\bar{x}_t) \right)^\dagger.$$
Expanding out first few terms in the path ordered exponentials above, we have

\[
\begin{align*}
\gamma'(0) &= 1, \\
\gamma'(1) &= \frac{N^2_c - 1}{2N_c} \Gamma(x_t - \bar{x}_t) g^4 (y - \bar{y}) \int_y^{\infty} dy' \mu^2(y'), \\
\gamma'(2) &= \frac{1}{2!} \left[ \frac{N^2_c - 1}{2N_c} \Gamma(x_t - \bar{x}_t) g^4 \int_y^{\infty} dy' \mu^2(y') \right]^2.
\end{align*}
\] (117)

In the expression for \( \gamma'(2) \) above only one of the two possible terms survive on account of the path ordering. From similar considerations it can be argued that in general

\[
\gamma'(n) = \frac{1}{n!} \left[ \frac{N^2_c - 1}{2N_c} \Gamma(x_t - \bar{x}_t) g^4 \int_y^{\infty} dy' \mu^2(y') \right]^2,
\] (118)

where \( \xi(y) = \int_y^{\infty} dy' \mu^2(y') \). Resumming the terms above and including the disconnected pieces, we have

\[
\gamma(x_t - \bar{x}_t; y, \bar{y}) = \exp \left( \frac{g^4(N^2_c - 1)}{2N_c} \xi(y) \right). \] (119)

The above expression is the complete non–perturbative result for \( \gamma(x_t - \bar{x}_t; y, y') \). Note that trivially \( \gamma(0; y, \bar{y}) = 1 \) as we would expect from the definition of \( \gamma \).

Taking the Fourier transform of \( \gamma \),

\[
\tilde{\gamma}(p_t) = \int d^2 x_t e^{i p_t \cdot x_t} \gamma(x_t) = \int d^2 x_t e^{i p_t \cdot x_t} \exp \left[ \kappa(\Gamma(x_t) - \Gamma(0)) \right],
\] (120)

where \( \kappa = g^4(N^2_c - 1)\xi(y)/2N_c \), and expanding out \( \gamma(x_t) \), we obtain

\[
\tilde{\gamma}(p_t) = \left[ \delta^{(2)}(p_t) + \kappa \int d^2 x_t e^{i p_t \cdot x_t} \Gamma(x_t) - \kappa \Gamma(0) \delta^{(2)}(p_t) + \cdots \right].
\] (121)

If we now recall the definition of \( \Gamma(x_t) \) from Eq. [114],

\[
\Gamma(x_t) = \int \frac{d^2 k_t}{(2\pi)^2} \frac{e^{i k_t \cdot x_t}}{k_t^4},
\] (122)
and substitute for $\Gamma(x_t)$ in the above, we find

$$\tilde{\gamma}(p_t) = (1 - \kappa \Gamma(0)) \delta^{(2)}(p_t) + \frac{N_c^2 - 1}{2N_c} \frac{(4\pi)^2 \alpha_s^2 \xi}{p_t^4} + \cdots. \quad (123)$$

The first term in $\tilde{\gamma}(p_t)$ is not relevant for our computation of the sea quark distributions since the relevant momenta are $p_t >> \Lambda_{QCD}$. The second term is the perturbative expression computed by us previously [46]—upto a factor of two which was missing in that paper. Note also that the perturbative $\tilde{\gamma}(p_t)$ above explicitly satisfies the sum rule condition of Eq. 54.

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