Quantum Mechanics on a Real Hilbert Space

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Abstract

The complex Hilbert space of standard quantum mechanics may be treated as a real Hilbert space. The pure states of the complex theory become mixed states in the real formulation. It is then possible to generalize standard quantum mechanics, keeping the same set of physical states, but admitting more general observables. The standard time reversal operator involves complex conjugation, in this sense it goes beyond the complex theory and may serve as an example to motivate the generalization. Another example is unconventional canonical quantization such that the harmonic oscillator of angular frequency $\omega$ has any given finite or infinite set of discrete energy eigenvalues, limited below by $\hbar \omega/2$.

1 Introduction

There are well known mathematical arguments saying that the Hilbert space of quantum mechanics could be real or quaternionic, as alternatives to the standard complex theory [1–3]. The real case was dealt with mainly by Stueckelberg and collaborators, who concluded that it is essentially equivalent to the complex case [4–10] (see also [11, 16]). The interest in the quaternionic case is more alive [11–16]. A more exotic subject is octonionic quantum theory [17, 18].

The compromise proposed here is a genuine extension of the complex theory, but is not quite the full quantum theory on a real Hilbert space. The set of physical states is taken to be exactly the same as in the complex theory, but the complex Hilbert space is reinterpreted as a real space, and the set of observables is enlarged from the set of all complex Hermitean matrices to the set of all real symmetric matrices.

There exists some physical motivation for such a generalization in the fact that the time reversal operator $T$ is antilinear in the complex theory. All transformations involving time reversal are antilinear, among them the fundamental $CPT$ symmetry of quantum field theory. It is true that the effect of time reversal can be described easily enough in standard quantum theory, but strictly
speaking, as soon as time reversal is introduced, the step from the complex to the real Hilbert space has already been taken

As another example, it is shown in Section 4 below how to represent the canonical Poisson bracket relation \( \{x, p\} = 1 \) in terms of operators on a finite dimensional real Hilbert space. As is well known, the canonical commutation relation \( [x, p] = -i\hbar I \) can not be represented on a finite dimensional complex Hilbert space, simply because the commutator on the left hand side must then have zero trace, while the identity operator \( I \) on the right hand side has nonzero trace. The argument does not apply in the real Hilbert space, because there the operator \( J = iI \) has zero trace.

The main argument of Stueckelberg for the equivalence of real and complex quantum mechanics is the need for an uncertainty principle. On the real Hilbert space it is very useful, if not strictly necessary, to have an operator \( J \) commuting with all observables and having the property that \( J^2 = -1 \), if one wants to derive a general inequality for the product of the variances of any pair of observables. However, the argument is not compelling, partly because, as Stueckelberg points out, there might in principle be one separate operator \( J \) for every pair of observables, and partly because quantum mechanics makes sense even without a general uncertainty principle. In the example of Section 4 below, the uncertainty principle for position and momentum holds indeed in all physical states, even though the position and momentum operators both anticommute with the operator \( J \) defining the complex structure.

Stueckelberg and collaborators also discussed field quantization with fields that are either linear or antilinear, in the sense of commuting or anticommuting with \( J \). Again their main conclusion is that quantization with antilinear fields is impossible for bosons, and possible for fermions but then essentially equivalent to quantization with linear fields, so that the real case reduces to the complex case. If the conclusion is valid in the present case, it means that the unconventional quantization of the harmonic oscillator does not lead to any interesting new quantum field theory. However, one should perhaps reexamine the arguments, keeping in mind in particular that there might be several different square roots of \( -1 \), as discussed briefly in Section 3.

If antilinear field operators are ever going to be useful, it would most likely be in the quantization of the Dirac field. If the Dirac matrices are chosen real, then the Dirac equation is seen to be a real equation, and it is not unnatural to go one step further and formulate the quantum field theory in terms of a real Hilbert space, with linear or (possibly?) antilinear field operators, symmetric with respect to the real scalar product. The standard complex notation, both for the field equation and for the Hilbert space, hides the fact that the theory contains several \( i = \sqrt{-1} \) that are logically different, although they are identical by the notation.

One \( i \) is the generator of electromagnetic gauge transformations in the Dirac equation for a charged particle, it commutes with the mass term in the equation. A second \( i \) appears in the massless Weyl equation, it generates chiral
gauge transformations, and does not commute with the Majorana mass term. For this reason, the Majorana and Dirac mass terms are claimed to be different, although one might just as naturally have concluded that there are two different $i$'s and only one kind of mass term. A third $i$ turns up in the Fourier transformation connecting the position and momentum representations of the fields, by definition it commutes with the field operators. The fourth $i$, acting on the complex Hilbert space where all the fields act as operators, need not in principle be identified with any of the three.

2 Complex and real Hilbert spaces

For simplicity, we will consider here mostly finite dimensional Hilbert spaces. The complex Hilbert space $\mathbb{C}^D$ of complex dimension $D$ corresponds to the real Hilbert space $\mathbb{R}^{2D}$ of real dimension $2D$. The imaginary unit $i$ on $\mathbb{C}^D$ is then a linear operator $J$ on $\mathbb{R}^{2D}$ with $J^2 = -I$. Here $I$ is the identity operator on $\mathbb{R}^{2D}$. We define the correspondence so that we have for $D = 2$, as an example,

$$
\psi = \begin{pmatrix} \psi_{1r} & \psi_{1i} \\ \psi_{2r} & \psi_{2i} \end{pmatrix} \in \mathbb{C}^2 \leftrightarrow \psi = \begin{pmatrix} \psi_{1r} \\ \psi_{1i} \\ \psi_{2r} \\ \psi_{2i} \end{pmatrix} \in \mathbb{R}^4.
$$

Then $J$ is an antisymmetric matrix,

$$
J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

The complex scalar product on $\mathbb{C}^D$,

$$
\phi^\dagger \psi = \sum_{j=1}^{D} (\phi_{jr} \psi_{jr} + \phi_{ji} \psi_{ji}) + i \sum_{j=1}^{D} (\phi_{jr} \psi_{ji} - \phi_{ji} \psi_{jr}),
$$

has a real part which is the symmetric scalar product $\phi^T \psi = \psi^T \phi$ on $\mathbb{R}^{2D}$, and an imaginary part which is the antisymmetric symplectic scalar product $-\phi^T J \psi = \psi^T J \phi$ on $\mathbb{R}^{2D}$.

To any complex $D \times D$ matrix $A$ corresponds a real $2D \times 2D$ matrix, which we choose to call by the same name $A$. The correspondence is such that, for example,

$$
\begin{pmatrix} A_{11r} + iA_{11i} & A_{12r} + iA_{12i} \\ A_{21r} + iA_{21i} & A_{22r} + iA_{22i} \end{pmatrix} \leftrightarrow \begin{pmatrix} A_{11r} & -A_{11i} & A_{12r} & -A_{12i} \\ A_{11i} & A_{11r} & A_{12i} & A_{12r} \\ A_{21r} & -A_{21i} & A_{22r} & -A_{22i} \\ A_{21i} & A_{21r} & A_{22i} & A_{22r} \end{pmatrix}.
$$
In tensor product notation we may write
\[
A = \left( \begin{array}{rr}
1 & 0 \\
0 & 1 \\
\end{array} \right) \otimes \left( \begin{array}{rr}
A_{11r} & A_{12r} \\
A_{21r} & A_{22r} \\
\end{array} \right) + \left( \begin{array}{rr}
0 & -1 \\
1 & 0 \\
\end{array} \right) \otimes \left( \begin{array}{rr}
A_{11i} & A_{12i} \\
A_{21i} & A_{22i} \\
\end{array} \right).
\] (5)

The Hermitian conjugate \(A^\dagger\) of the complex matrix \(A\) corresponds to the transposed \(A^T\) of the real matrix \(A\). The distinguishing property of those real matrices that correspond to complex matrices, is that they commute with \(J\). Any real \(2D \times 2D\) matrix \(A\) can be written in a unique way as \(A = A_+ + A_-\), where \(A_+ J = JA_+\) and \(A_- J = -JA_-\), in fact the explicit solution is
\[
A_{\pm} = \frac{1}{2} (A \mp JA J).
\] (6)

\(A_+\) is complex linear, \(A_-\) is complex antilinear, thus it is a product of complex conjugation and a complex linear operator. In the \(4D^2\) dimensional space of all real matrices, the complex linear and the complex antilinear matrices form two complementary subspaces of complex dimension \(D^2\) and real dimension \(2D^2\).

The complex \(D \times D\) matrix \(A\) is Hermitian if \(A^\dagger = A\) and unitary if \(A^\dagger = A^{-1}\). The real \(2D \times 2D\) matrix \(A\) is symmetric if \(A^T = A\), antisymmetric if \(A^T = -A\), orthogonal if \(A^T = A^{-1}\), and symplectic if \(A^T J = JA^{-1}\). An orthogonal matrix is symplectic if and only if it commutes with \(J\). Thus, the complex Hermitian matrices correspond to those real matrices that are symmetric and commute with \(J\), whereas the complex unitary matrices correspond to precisely those real matrices that are orthogonal and symplectic.

In other words, an orthogonal matrix is a real linear operator on \(\mathbb{R}^{2D}\) which is invertible and preserves the ordinary real scalar product \(\phi^T \psi\). A symplectic matrix is invertible and preserves the symplectic scalar product \(-\phi^T J \psi\). And a unitary matrix is a complex linear operator on \(\mathbb{C}^D\) which is invertible and preserves the complex scalar product \(\phi^\dagger \psi\). (In the finite dimensional case, but not in the infinite dimensional case, the invertibility is a consequence of the preservation of scalar products.)

An infinitesimal linear transformation \(U = I + \epsilon A\) on \(\mathbb{R}^{2D}\), with \(\epsilon\) infinitesimal, is orthogonal if and only if the generator \(A\) is antisymmetric, \(A^T = -A\). It is symplectic if and only if \(A^T J = -JA\), which means that the matrix \(B = JA\) is symmetric. Equivalently, \(A = -JB\) with \(B\) symmetric. \(U\) is both orthogonal and symplectic if and only if \(A = -JB = -BJ\) with \(B\) symmetric.

The dimension of the orthogonal group \(O(2D)\) is \(2D^2 - D\), the dimension of the symplectic group \(Sp(2D)\) is \(2D^2 + D\), and the dimension of the unitary group \(U(D)\) is \(D^2\).
3 Quantum Mechanics

3.1 Observables and probabilities

In standard quantum mechanics a pure state of a given physical system is represented by a unit vector $\psi \in \mathbb{C}^D$, or equivalently by the Hermitean density matrix $\rho = \psi \psi^\dagger$, which is a projection operator, since $\rho^2 = \rho$. An observable of the system is represented by a Hermitean $D \times D$ matrix $A$, and the theory predicts the expectation value in the pure state $\psi$ as

$$\langle A \rangle = \psi^\dagger A \psi = \text{Tr}(\rho A).$$  \hspace{1cm} (7)

More generally, any pure or mixed state is represented by a Hermitean density matrix $\rho$ which is positive definite, i.e. has nonnegative eigenvalues, and has unit trace, $\text{Tr} \rho = 1$. The expectation value of the observable $A$ in this state is $\langle A \rangle = \text{Tr}(\rho A)$. The theory also predicts the variance of $A$, $\text{var}(A) = (\Delta A)^2$, as

$$\text{var}(A) = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2. \hspace{1cm} (8)$$

$A$ has a sharp value in the state $\rho$ if and only if $\text{var}(A) = 0$.

This probability interpretation makes sense because of the spectral theorem for Hermitean matrices, which guarantees the existence of the spectral representation

$$A = \sum_{n=1}^{N} a_n P_n. \hspace{1cm} (9)$$

Here $a_1, \ldots, a_N$ are distinct real eigenvalues, $N \leq D$, and $P_1, \ldots, P_N$ are Hermitean projection operators, with the properties that $P_m P_n = 0$ for $m \neq n$, and

$$I = \sum_{n=1}^{N} P_n. \hspace{1cm} (10)$$

This implies that

$$\langle A \rangle = \sum_{n=1}^{N} p_n a_n, \quad \text{var}(A) = \sum_{n=1}^{N} p_n (a_n - \langle A \rangle)^2, \hspace{1cm} (11)$$

where $p_n = \langle P_n \rangle = \text{Tr}(\rho P_n)$. According to the probability interpretation, the possible results of a measurement of $A$ are the eigenvalues $a_1, \ldots, a_N$, and $p_n$ is the probability of the result $a_n$ in the state $\rho$.

The fact that the spectral theorem for complex Hermitean matrices is valid for all real symmetric matrices, with no more than the obvious changes in wording, allows us to generalize standard quantum mechanics by admitting as observables all the real symmetric matrices.
3.2 States

In this generalization we have two options for choosing the set of states. The straightforward choice is to admit all real unit vectors as possible pure states of the system, and all real symmetric and positive definite matrices of unit trace as possible mixed states. This enlarges the set of possible states as compared to standard quantum mechanics, since not all such real density matrices are complex linear. In particular, it doubles the total number of states in the system, and it doubles the degeneracy of the spectrum of all standard observables, i.e. those that are complex linear. It seems that the degeneracy doubling is unphysical, at least if we want to describe systems that are well described by standard quantum theory.

The second option, more interesting from the physical point of view, is to admit exactly the same states as in the complex theory. It means that we enlarge the class of observables, including all real symmetric matrices, but we admit only those density matrices that commute with $J$. For example, with $J$ as in equation (2) the most general physical density matrix has the form

$$
\rho = \begin{pmatrix}
\alpha & 0 & \gamma & \delta \\
0 & \alpha & -\delta & \gamma \\
\gamma & -\delta & \beta & 0 \\
\delta & \gamma & 0 & \beta
\end{pmatrix},
$$

(12)

with $2(\alpha + \beta) = 1$, $\alpha \geq 0$, $\beta \geq 0$, and $\alpha \beta - \gamma^2 - \delta^2 \geq 0$.

This has the advantage that the physical degeneracies are unchanged, but it also has the somewhat strange consequence that there are no pure states in the theory. In fact, the pure states in the complex theory correspond to mixed states in the real theory. One way to see this is to observe that the meaning of the trace is different in the real theory as compared to the complex theory, because the number of basis vectors is doubled. Thus, if $\text{Tr} \rho = 1$ when the density matrix $\rho$ is regarded as a complex $D \times D$ matrix, we have $\text{Tr} \rho = 2$ when the same $\rho$ is regarded as a real $2D \times 2D$ matrix. Therefore the proper correspondence is that the complex density matrix $\rho$ must correspond to the real density matrix $\rho/2$, which can never represent a pure state since it has no eigenvalues larger than $1/2$.

In this generalization of quantum mechanics as a theory defined on $\mathbb{R}^{2D}$, every observable will have a complete set of $2D$ real eigenvalues and eigenvectors. However, one eigenvector alone does not represent a physical state. If we say that two physical states $\rho$ and $\rho'$ are orthogonal when $\rho \rho' = \rho' \rho = 0$, the maximum number of orthogonal physical states is $D$. Just by counting we see that if an observable $A$ has more than $D$ different eigenvalues, not every one of these can possess its own “physical eigenstate”, in which a measurement of $A$ gives this particular value with probability one.

More explicitly stated, if an observable $A$ does not commute with $J$, then it will have at least one eigenvalue which is not sharply realized in any physical
state, and conversely, there will exist no complete set of physical states such that \( \text{var}(A) = 0 \) in every state belonging to the complete set.

### 3.3 Poisson brackets

To the classical Poisson bracket \( \{ A, B \} \) of two classical observables \( A \) and \( B \) corresponds the “quantum Poisson bracket”

\[
\{ A, B \} = -\frac{i}{\hbar} [A, B] = -\frac{i}{\hbar} (AB - BA) .
\]

(13)

The proper way to write the same quantity in the real formulation is

\[
\{ A, B \} = A\Omega B - B\Omega A ,
\]

(14)

with \( \Omega = -J/\hbar \). The antisymmetry, \( \{ A, B \} = -\{ B, A \} \), and the Jacobi identity,

\[
\{ A, \{ B, C \} \} + \{ B, \{ C, A \} \} + \{ C, \{ A, B \} \} = 0 ,
\]

(15)

are easily verified. The most important property of the matrix \( \Omega \) is that it is antisymmetric, \( \Omega^T = -\Omega \), because that ensures that \( \{ A, B \} \) is symmetric whenever \( A \) and \( B \) are both symmetric. Another way to write the relation \( \{ A, B \} = C \) for symmetric matrices \( A, B \) and \( C \) is as

\[
[-JA, -JB] = -\hbar JC .
\]

(16)

This is then a commutation relation in the Lie algebra of the symplectic group.

The real density matrix \( \rho \) must in general be explicitly time dependent and satisfy the Liouville equation

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{ \rho, H \} = 0 .
\]

(17)

Here \( d\rho/dt \) is the absolute time derivative, \( \partial \rho/\partial t \) is the explicit time derivative, and \( H \) is the Hamiltonian. Thus the explicit time dependence of \( \rho \) is given by the equation of motion

\[
\frac{\partial \rho}{\partial t} = \{ H, \rho \} = H\Omega \rho - \rho \Omega H .
\]

(18)

The equation of motion must preserve \( \text{Tr} \, \rho \). A sufficient condition is that either \( H \) or \( \rho \) commute with \( \Omega \), because then we have either \( \rho \Omega H = \rho H \Omega \) or \( \rho \Omega H = \Omega \rho H \), and in both cases

\[
\frac{\partial (\text{Tr} \, \rho)}{\partial t} = \text{Tr} (H\Omega \rho - \rho \Omega H) = 0 .
\]

(19)

Thus, if we accept all symmetric and positive definite matrices of unit trace as density matrices, we should impose the condition on the Hamiltonian \( H \) that it commute with \( J = -\hbar \Omega \).
We should impose the same condition on $H$ even in the case where we accept only density matrices that commute with $J$. The point is that the equation of motion must preserve the condition of commutation with $J$, that is, the Poisson bracket $\{H, \rho\} = H\Omega \rho - \rho \Omega H$ must commute with $J$. A sufficient condition, when $\rho$ commutes with $J$, is that $H$ also commutes with $J$.

When $H$ commutes with $J$, and is not explicitly time dependent, the equation of motion can be integrated explicitly to give

$$\rho(t) = U(t) \rho(0) U(-t),$$  \hspace{1cm} (20)

where $U(t) = e^{-\frac{i}{\hbar} JH}$ is the unitary time development operator.

In conclusion, not every real symmetric matrix is an acceptable Hamiltonian in the generalized quantum mechanics as formulated here. It is necessary, or at least natural, to require the Hamiltonian to be a complex linear matrix. If we also require the density matrices to be complex linear matrices, it would seem that we are back to the point of departure, which was the standard complex quantum mechanics. However, we have enlarged the class of observables, even though we do not accept the new observables to be Hamiltonians governing the time development.

4 The harmonic oscillator

Time reversal was mentioned in the introduction as a motivation for the proposed generalization of the complex formalism. A more unconventional example of the generalizations that become possible, is the representation of the canonical Poisson bracket $\{x, p\} = I$ in any finite and even dimension. For example, in the four dimensional case considered above, any two positive lengths $\xi_1, \xi_2$ define a representation of the form

$$x = \begin{pmatrix} \xi_1 & 0 & 0 & 0 \\ 0 & -\xi_1 & 0 & 0 \\ 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 0 & -\xi_2 \end{pmatrix}, \quad p = \frac{\hbar}{2} \begin{pmatrix} 0 & 1/\xi_1 & 0 & 0 \\ 1/\xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\xi_2 \\ 0 & 0 & 1/\xi_2 & 0 \end{pmatrix}. \hspace{1cm} (21)$$

The generalization to any even dimension $2D$, or to infinite dimension, is obvious. Then both $x$ and $p$ anticommute with the imaginary unit $J$, as opposed to standard quantum mechanics where they commute with $J$. With the most general physical density matrix $\rho$, equation (12), this gives $\langle x \rangle = 0$, $\langle \Delta x \rangle^2 = \langle x^2 \rangle = 2(\alpha \xi_1^2 + \beta \xi_2^2)$, and similarly for $p$. Thus the Heisenberg uncertainty relation holds in every physical state,

$$\Delta x \Delta p = \hbar \sqrt{(\alpha \xi_1^2 + \beta \xi_2^2) \left( \frac{\alpha}{\xi_1^2} + \frac{\beta}{\xi_2^2} \right)}$$

$$= \hbar \sqrt{(\alpha + \beta)^2 + \alpha \beta \left( \frac{\xi_1}{\xi_2} - \frac{\xi_2}{\xi_1} \right)^2} \geq \frac{\hbar}{2}. \hspace{1cm} (22)$$
With these definitions the Hamiltonian of the harmonic oscillator of angular frequency $\omega$,

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2,$$  \hspace{1cm} (23)

is diagonal and has the energy eigenvalues

$$E_i = \frac{\hbar^2}{8m\xi_i^2} + \frac{1}{2} m\omega^2 \xi_i^2.$$  \hspace{1cm} (24)

For any given value $E_i \geq \hbar\omega/2$ this equation has the positive solutions

$$\xi_i = \sqrt{\frac{2E_i \pm \sqrt{4E_i^2 - \hbar^2 \omega^2}}{2m\omega^2}}.$$  \hspace{1cm} (25)

By the obvious generalization, we may assign to the harmonic oscillator any finite number, or an infinite number, of arbitrary energy levels above the lower bound $\hbar\omega/2$.

Note that the Hamiltonian $H$ does commute with $J$, even though $x$ and $p$ here do not, implying that the time development operator $e^{-\frac{\bar{\hbar}}{\hbar} JH}$ is unitary. On the other hand, the operator $e^{-\frac{\bar{\hbar}}{\hbar} Jp}$, representing translation in space by a distance $d$, is not unitary when $d \neq 0$, but only symplectic. The problem is that it is not orthogonal, because the matrix $Jp$ in the exponent is symmetric rather than antisymmetric. Thus it does not conserve probabilities, and is not a symmetry transformation, as is evident from the fact that the position operator $x$ has a discrete spectrum.

The case $\xi_1 = \xi_2 = \xi$ in equation (21) is interesting because it is a realization of the canonical relation $\{x, p\} = I$, but is also, in a certain sense, a fermionic quantization of the harmonic oscillator, with

$$x^2 = \xi^2, \quad p^2 = \frac{\hbar^2}{4\xi^2}, \quad xp + px = 0.$$  \hspace{1cm} (26)

In fact, the last relations are just one particular form of the canonical anticommutation relations

$$aa^T + a^Ta = I, \quad a^2 = (a^T)^2 = 0.$$  \hspace{1cm} (27)

To see this in more detail, let us introduce another antisymmetric matrix $K$ so that it is an imaginary unit, $K^2 = -I$, and commutes with both $x$ and $p$. Then we take

$$a = \frac{1}{2\xi} x + \frac{\xi}{\hbar} Kp, \quad a^T = \frac{1}{2\xi} x - \frac{\xi}{\hbar} Kp.$$  \hspace{1cm} (28)

9
The Hamiltonian $H$ as defined in equation (23) is just a constant, and a more interesting quantity is the usual Hamiltonian of the fermionic oscillator, which is

$$H' = \hbar \omega \left( a^T a - \frac{1}{2} \right) = -\frac{\omega}{2} K (xp - px) = \frac{\hbar \omega}{2} J K .$$

(29)

The time development operator with this alternative Hamiltonian is

$$U'(t) = e^{-\frac{\hbar}{J} H'} = e^{\frac{\hbar}{J} K} .$$

(30)

In some respects this theory, where $x$ and $p$ anticommute with the imaginary unit $J$, is equivalent to another theory where the imaginary unit is $K$, which commutes with $x$ and $p$. An explicit representation for $K$ might be

$$K = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = SJS^{-1} ,$$

(31)

with, for example,

$$S = S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

(32)

If we define

$$\tilde{\rho} = S \rho S^{-1} = \begin{pmatrix} \alpha & \gamma & 0 & \delta \\ \gamma & \beta & -\delta & 0 \\ 0 & -\delta & \alpha & \gamma \\ \delta & 0 & \gamma & \beta \end{pmatrix} ,$$

(33)

then this is a density matrix which commutes with $K$ instead of with $J$. If now $\rho(t) = U'(t) \rho(0) U'(-t)$, then we have $\tilde{\rho}(t) = \tilde{U}(t) \tilde{\rho}(0) \tilde{U}(-t)$, where

$$\tilde{U}(t) = S U'(t) S^{-1} = e^{-\frac{\hbar}{K} H'} = e^{\frac{\hbar}{J} K} ,$$

(34)

since $SH' = H'S$. The energy spectrum is the same in the two theories, for example we have

$$\text{Tr}(\rho H') = \text{Tr}(\tilde{\rho} H') = 2\delta \hbar \omega .$$

(35)

However, the expectation values for $x$ and $p$ are not the same. For example,

$$\text{Tr}(\rho x) = 0 , \quad \text{Tr}(\tilde{\rho} x) = 2(\alpha - \beta) \xi .$$

(36)
5 More than one degree of freedom

With two degrees of freedom, referred to here by indices $a$ and $b$, and belonging for example to two fields commuting with each other, the Hilbert space is a tensor product,

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b.$$  \hspace{1cm} (37)

However, the complex and real tensor products are mathematically different constructions. The complex tensor product of spaces of complex dimensions $D_a$ and $D_b$ has complex dimension $D_aD_b$ and real dimension $2D_aD_b$, whereas the real tensor product of spaces of real dimensions $2D_a$ and $2D_b$ has dimension $4D_aD_b$.

The relation between the two types of tensor product can be understood as follows. In the complex case the following relations hold for tensor products of vectors,

$$\phi \otimes \psi = -i(\phi \otimes i\psi) = -(i\phi \otimes i\psi).$$  \hspace{1cm} (38)

In the real case the four tensor products $\phi \otimes \psi$, $(J\phi) \otimes \psi$, $\phi \otimes (J\psi)$ and $(J\phi) \otimes (J\psi)$ are linearly independent vectors, thus there exist two imaginary units $J_a = J \otimes I$ and $J_b = I \otimes J$ defined as operators on the real tensor product space $\mathcal{H}$. It follows that $\mathcal{H}$ is a direct sum, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm = P_\pm \mathcal{H}$, and the two operators

$$P_\pm = \frac{1}{2} (I \mp J_aJ_b)$$  \hspace{1cm} (39)

are complementary orthogonal projection operators. Each of the two subspaces $\mathcal{H}_\pm$ has dimension $2D_aD_b$. Both $J_a$ and $J_b$ commute with $P_\pm$, and hence act within the subspaces $\mathcal{H}_\pm$ separately. On $\mathcal{H}_+$ the relation $J_a = J_b$ holds, whereas $J_a = -J_b$ holds on $\mathcal{H}_-$.

The complex tensor product space can be identified in a natural way with $\mathcal{H}_+$. Hence a physical density matrix $\rho$ on the product space $\mathcal{H}$ must be a real $(4D_aD_b) \times (4D_aD_b)$ matrix that commutes with both $J_a$ and $J_b$, and in addition it must have the property that $\rho = P_+ \rho = \rho P_+ = P_+ \rho P_+$.

If an operator $x$ acts on $\mathcal{H}_a$ and anticommutes with $J$, then the corresponding operator $x_a = x \otimes I$ on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ anticommutes with $J_a$, but commutes with $J_b$. This means that $x_a$ maps $\mathcal{H}_+$ into $\mathcal{H}_-$, and vice versa. Such an operator could not be constructed at all within the complex quantum theory, because the subspace $\mathcal{H}_-$ simply would not exist. It is essential for the construction that there exist two mutually commuting imaginary units $J_a$ and $J_b$ on the real tensor product Hilbert space. The possibility of having operators that are antilinear with respect to $J_a$ and at the same time linear with respect to $J_b$, or vice versa, is at least a partial answer to one of the problems with antilinear field operators recognized by Stueckelberg and collaborators.
The operator $x_a$ above is “unphysical” in the sense that it maps the physical space $H_+$ out of itself. But similar unphysical operators are well known in physics. For example, a general isospin rotation does not respect the superselection rule for electric charge, it transforms a physical state into an unphysical superposition of states with different values of the charge. Another example is the relative position of two identical particles, which is an operator changing the symmetry properties of the two-particle wave functions.

The generalization to more than two factors in the tensor product is easy, just introduce one new pair of projection operators for each new factor. If for example $H = H_a \otimes H_b \otimes H_c$, then define $P_\pm$ as above, and in addition

$$Q_\pm = \frac{1}{2} (I \mp J_a J_c).$$

(40)

Since $P_\epsilon$ and $Q_\eta$ commute, for $\epsilon = \pm$ and $\eta = \pm$, the products $P_\epsilon Q_\eta$ are also projection operators. Decompose $H$ as a direct sum of subspaces $H_{\epsilon\eta} = P_\epsilon Q_\eta H$, then in each such subspace we have $J_a = \epsilon J_b = \eta J_c$. The physical subspace is $H_{++}$, it corresponds to the complex tensor product, since the relations $J_a = J_b = J_c$ hold there.

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