Entanglement Sudden Death in a Quantum Memory

Yankov S. Weinstein

1Quantum Information Science Group, MITRE, 260 Industrial Way West, Eatontown, NJ 07724, USA

I explore entanglement dynamics in examples of quantum memories, decoherence free subspaces (DFS) and noiseless subsystems (NS), to determine how a complete loss of entanglement affects the ability of these techniques to protect quantum information. Using negativity and concurrence as entanglement measures, I find that in general there is no correlation between the complete loss of entanglement in the system and the fidelity of the stored quantum information. These results complement previous results in which quantum protocols not explicitly based on entanglement exhibit little correlation between ESD and the accuracy of the given protocol.

PACS numbers: 03.67.Mn, 03.67.Bg, 03.67.Pp

I. INTRODUCTION

Protecting quantum information from the effects of decoherence, unwanted interactions between the system and its environment, is a vital requirement of any hoped for quantum computer implementation [1]. Such protection can be achieved via active or passive techniques. Active techniques, quantum error correction, identify and correct errors that may have affected the quantum information. Alternatively, passive techniques store the quantum information in such a way that it is a priori immune to error. Two general schemes which allow for the passive avoidance of quantum errors are decoherence free subspaces (DFS) and noiseless subsystems (NS). DFSs store quantum information in specific states with an inherent symmetry such that the information is then immune to decoherence generators that respect those symmetries [2, 3]. NSs store quantum information in certain symmetries of other degrees of freedom of the system [4] which are not the states themselves. The error avoidance properties of DFSs and NSs makes them especially well suited for the construction of quantum memories since the system storing the information can, ideally, never be addressed until the information is needed. However, manifestations of both schemes may utilize states that are highly entangled and thus may be subject to entanglement sudden death (ESD).

Entanglement is a uniquely quantum mechanical phenomenon in which quantum systems exhibit correlations not possible for classical systems [5]. Decoherence may be especially detrimental to highly entangled states [6] such as those used for protecting quantum information. An extreme negative manifestation of this is ESD in which entanglement is completely lost in finite time [7, 8] despite the fact that the coherence loss of the system is asymptotic. Recently [9], there has been a call to develop techniques to counteract ESD so as to protect quantum memory from its harmful consequences.

In this paper I explore the affect of ESD on entangled states that are specifically examples of DFSs and NSs. My goal is to explore whether ESD is really a threat to quantum memories built from error aviodance schemes above and beyond that of typical decoherence. The direct study of the affect of ESD on quantum protocols has only been undertaken recently. In [10] it was shown that a three (physical) qubit error correction code capable of protecting a qubit of quantum information from phase flips is indifferent to the phenomenon of ESD. In [10] it was shown that in cluster states capable of primitive quantum gates via cluster state (or one-way) quantum computing protocols, correlations exist between ESD and the point at which the fidelity of the decohered state equals .5. In this paper such explorations are extended to the affect of ESD on error avoidance protocols. Other related work addressing ESD of multi-particle systems can be found in [10, 11, 12, 13] and there have been several initial experimental studies of this phenomenon [14].

As mentioned above when constructing a quantum memory it would be most practical to store the information and not have to address the memory again until the information is needed. This can be achieved using error avoidance techniques. In addition, it is reasonable to assume that in a quantum memory the dominant decoherence generators would be far field, such that they affect all of the qubits collectively. Thus, we begin by studying a four-qubit DFS and a three-qubit NS that protect quantum information from collective decoherence. However, it is most likely that some additional decoherence that is not collective, but rather qubit independent, will affect the system. This decoherence will degrade the stored quantum information and, if strong enough may cause ESD. We will explore the relationship between the decoherence strength at which ESD is exhibited due to this decoherence and the fidelity of the stored qubit of quantum information.

The decoherence models we explore are the independent qubit dephasing and depolarizing environments. The dephasing environment is fully described by the Kraus operators

$$K_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & \sqrt{1-p} \end{array} \right); \quad K_2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & \sqrt{p} \end{array} \right)$$

(1)

where $p$ is the dephasing strength. When all $n$ qubits undergo dephasing we have $2^n$ Kraus operators each of the form $A_l = (K_i \otimes K_j \otimes K_k)$ where $l = 1, 2, \ldots, 2^n$ and
\( i, j, k = 1, 2 \). The depolarizing environment is described by Kraus operators

\[
K_1 = \sqrt{1 - \frac{3p}{4}}, \quad K_c = \frac{\sqrt{p}}{2} \sigma_c, \quad c = 2, 3, 4, \tag{2}
\]

where \( \sigma_c \) are the Pauli spin matrices and now \( p \) is the depolarizing strength. For the depolarizing environment there are \( 4^n \) Kraus operators. Though all of the below calculations are done with respect to \( p \), I implicitly assume that \( p \) increases with time, \( \tau \), at a rate \( \kappa \), such that \( p = 1 - e^{-\kappa \tau} \) and \( p \rightarrow 1 \) only at infinite times. I also assume equal decoherence strength on all qubits (this can be viewed as the worst-case scenario).

To monitor the occurrence of ESD I will utilize a number of entanglement metrics. The first is the negativity, \( N^{(i)} \), for which I will simply use the (absolute value of the) most negative eigenvalue of the partial transpose of the system density matrix \( \text{Tr}_b \rho \). When there are more than two qubits in the system the partial transpose can be taken with respect to different sets of qubits \( i \) giving, in general, inequivalent negativities. I will also make use of the two qubit concurrence \( C_{jk} \). The concurrence between two qubits \( j \) and \( k \) with density matrix \( \rho_{jk} \) is usually defined as the maximum of zero and \( \Lambda \), where \( \Lambda = \sqrt{\lambda_1 - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}} \) and the \( \lambda_i \) are the eigenvalues of \( \rho_{jk} (\sigma_y^k \otimes \sigma_y^k) \rho_{jk}^* (\sigma_y^k \otimes \sigma_y^k) \) in decreasing order and \( \sigma_y \) is the \( y \) Pauli matrix of qubit \( i \). For the purposes of clearly seeing at what point ESD occurs we will use \( \Lambda \) as the concurrence noting that ESD occurs when \( \Lambda = 0 \) in finite time (i.e., before \( p \rightarrow 1 \)). To measure entanglement between general states of three qubits I will use the tri-partite negativity \( N_3 \), which is simply the third root of the product of the negativities with respect to each of the three qubits \( N_3 = \sqrt[3]{N^{(1)} N^{(2)} N^{(3)}} \).

II. FOUR QUBIT DFS

The smallest possible DFS that can protect one qubit of quantum information from the affects of collective decoherence is comprised of four physical qubits. The two (orthogonal) basis states for this DFS are \( 1 \):

\[
\begin{align*}
|0\rangle_L &= \frac{1}{2} (|01\rangle - |10\rangle)_{1,2} \otimes (|01\rangle - |10\rangle)_{3,4} \\
|1\rangle_L &= \frac{1}{\sqrt{12}} (2|0011\rangle + 2|1100\rangle - |0101\rangle \\
&\quad - |1010\rangle - |0110\rangle - |1001\rangle).
\end{align*}
\tag{3}
\]

I assume an initial state of \( |\psi\rangle_{DFS} = \cos a |0\rangle_L + e^{ib} \sin a |1\rangle_L \). Any such state will not evolve under collective decoherence but is degraded by independent (physical) qubit decoherence. To determine the affect of ESD on the storage of quantum information in this DFS we compare the decoherence strength at which ESD is exhibited for different entanglement measures to the fidelity of the degraded state.

In an independent dephasing environment the fidelity of the four qubit system in any state \( |\psi\rangle_{DFS} \) is given by:

\[
F(a, b, p) = \frac{1}{16} (4 + p(12p - 48) + p^2 (\cos 4a + 2\cos 2b)(\sin 2a)^2). \tag{4}
\]

Yet, despite the fact that the fidelity of the state of the system can be degraded below .5, the system negativity does not exhibit ESD. Thus, unlike \( 16 \), ESD is not an indicator that the fidelity of the stored information falls below .5. The independent qubit dephasing environment does cause ESD for the entanglement between any two of the qubits in the system as measured by the concurrence. For the evolution of concurrence between the first qubit and each of the other qubits, \( C_{ij}, j = 2, 3, 4 \), ESD is exhibited at different points based on the initial state as shown in Fig. 1.

If concurrence in the dephasing environment does exhibit ESD while the negativity does not we may ask what type of entanglement is present after the concurrence goes to zero. To answer this we can look at the tri-partite negativity, \( N_3 \), after tracing over one of the four qubits. In an independent dephasing environment none of the tri-partite negativities exhibit ESD. Thus, the remaining entanglement after the sudden disappearance of the concurrence between two qubits is, at least, the tri-partite entanglement measured by the tri-partite negativity.

Fig. 1 displays the evolution of the various entanglement metrics as a function of the initial state and decoherence strength and allows us to compare the onset (or not) of ESD to the evolution of the fidelity. There is no discontinuity or change of behavior that occurs in the fidelity evolution at the point where ESD sets in. In fact, there is not even a clear correlation between the entanglement evolution and that of the fidelity. This implies that the affect of ESD is no more or less than that of typical decoherence.

In an independent qubit depolarizing environment the fidelity of the four qubit system in any state \( |\psi\rangle_{DFS} \) is given by:

\[
F(a, b, p) = \frac{1}{16} (4 + p^2 (p - 1)^2 \cos 4a + \cos 2b (1 - \cos 4a) + 8p^4 - 34p^3 + 59p^2 - 48p + 16). \tag{5}
\]

However, in this case the negativities with the partial transpose taken with respect to one, \( N^{(i)} \), or two qubits, \( N^{(jk)} \), do exhibit ESD. The two qubit concurrence (taken between the same qubit combinations as studied in the dephasing environment) also exhibits ESD, though at lower decoherence strengths than in a dephasing environment, as does the the tri-partite negativity. The evolution of these entanglement metrics is shown in Fig. 2.

In asking what type of entanglement lasts the longest we note that in the plots below, the negativity metrics disappear at \( p \) slightly greater than 0.4. The concurrence between any of the two-qubit combinations usually disappears at much weaker decoherence strengths while the
FIG. 1: (Color online) Negativity, $N^{(1)}$, with respect to the first qubit, (top left), negativity, $N^{(12)}$, with respect to the first two qubits (top right), curves where the concurrence is equal to zero (center left), tri-partite negativity (center right) and fidelity (bottom) of the state $|\psi\rangle_{DFS}$, as a function of the initial state, parameterized by $a$, and dephasing strength $p$. For the negativity and fidelity plots $b = 0$. The concurrence plot shows $C_{12}$ (solid line), $C_{13}$ (large dashed line), and $C_{14}$ (small dashed line). ESD for $C_{12}$ is independent of $b$ and ESD ($C_{jk} = 0$) for the other concurrences are shown for (bottom to top) $b = \frac{\pi}{2}$, $\frac{\pi}{6}$, and 0. The two states in the tri-partite negativity plot are $b = 0$ (light) and $b = \frac{\pi}{4}$ (dark). Note that the negativities and tri-partite negativity do not exhibit ESD despite the fidelity going below .5. None of the entanglement measures seem to be at all correlated with the fidelity of the state of the DFS.

Tri-partite negativity disappears at $p < 0.4$. Thus, the type of entanglement that drives the negativity to not exhibit ESD until $p > 0.4$ is some sort of three qubit entanglement not measured by the tri-partite entanglement or genuine four-partite entanglement.

The behavior of the negativity $N^{(1)}$ is similar to that of the fidelity in that there is little dependence on $a$ or $b$. $N^{(1)}$ for initial state $a = b = 0$ exhibits ESD at $p \simeq 0.227$ and at that value the fidelity is about .6220. However, this should be compared to the dephasing environment where the fidelity can fall below .5 while no ESD is exhibited. This comparison highlights the lack of correlation between ESD and the proper functioning of a quantum memory. In addition, both $N^{(1)}$ and the fidelity differ from the behavior of the negativity where the partial transpose is taken with respect to two qubits. Finally, neither the behavior of the concurrence nor of the tri-partite negativity are at all correlated with the behavior of the fidelity. The most we can say is that as depolarization strength decreases so does the fidelity and the amount of entanglement but there is no correlation between these parameters.

III. THREE QUBIT NS

While a DFS requires four physical qubits to protect against collective decoherence, a noiseless subsystem (NS) can provide the same protection using only three physical qubits. This is done by storing the quantum information in the total angular momentum $S = 1/2$ subspace and storing the quantum information in the two pathways leading to the total angular momentum.

Storing the information in this way, and not in the state of the system, means that the quantum information is protected and can be efficiently extracted despite the fact that the actual state of the system is affected by the decoherence. For the three qubit NS the states which span each of the two logical qubit basis states are:

$$
|0\rangle_L \otimes |\frac{1}{2}\rangle_Z = \frac{1}{\sqrt{3}}(|001\rangle + \omega|010\rangle + \omega^2|100\rangle)
$$

$$
|0\rangle_L \otimes |\frac{-1}{2}\rangle_Z = \frac{1}{\sqrt{3}}(|110\rangle + \omega|101\rangle + \omega^2|011\rangle)
$$

$$
|1\rangle_L \otimes |\frac{1}{2}\rangle_Z = \frac{1}{\sqrt{3}}(|001\rangle + \omega^2|010\rangle + \omega|100\rangle)
$$

$$
|1\rangle_L \otimes |\frac{-1}{2}\rangle_Z = \frac{1}{\sqrt{3}}(|110\rangle + \omega^2|101\rangle + \omega|011\rangle)
$$

where $L$ refers to the logical qubit which is protected against collective errors, $Z$ is the subsystem that experiences the errors, and $\omega = e^{\pm \pi i/3}$. Following the protocols of [21, 22], we encode the initial single qubit state $\cos a|0\rangle + e^{ib} \sin a|1\rangle$ into the state $(\cos a|0\rangle_L + e^{ib} \sin a|1\rangle_L) \otimes |\frac{-1}{2}\rangle_Z$. To calculate the fidelity of the stored information after decoherent evolution we apply the decoding circuit of [21, 22] and compare the single qubit output to the initial state (which is equivalent to encoding and decoding with no applied decoherence).

All of the above states spanning the $S = 1/2$ subspace contain some entanglement. Does the finite time loss of this entanglement affect the system’s ability to protect the stored quantum information? As mentioned above, a qubit of information stored in this NS is perfectly protected against collective decoherence despite the fact that the state of the system after application of the decoherence is not equivalent to the initial encoded state. In
and fidelity (bottom) of the state equal to zero (center left), tri-partite negativity (center right) (small dashed line). ESD of $C_F$ or the negativity and fidelity plots $b$ initial state, parameterized by $a$, and depolarizing strength $p$. The concurrence between any two of the three qubits. The fidelity of the stored quantum information, given by

$$F(a, p) = \frac{1}{12}(12 - 5p - p(2\cos 2a + \cos 4a)),$$

is not dependent on $b$, and can go as low as $\frac{5}{4}$.

In a qubit-independent depolarizing environment, with depolarization strength $p$ on each qubit, the system does exhibit ESD for a host of entanglement measures. We compare the decoherence value at the onset of ESD to the fidelity of the stored quantum information which is given by

$$F(a, p) = \frac{1}{4}(4 - p(5 + p(p - 4)) - p(p - 1)^2 \cos 4a).$$

Note that the fidelity is again not dependent on $b$, and at $p = 1$ the fidelity goes to $1/2$. ESD of the negativity occurs at similar decoherence strengths irrespective of the choice of partial transpose. The fidelity of the stored quantum information and the negativity with partial transpose taken with respect to the first qubit are shown in Fig. 3. Note that for $p \approx .42486$ ESD is observed for states where $a = 0, \frac{\pi}{2}$. The fidelity where ESD is exhibited for those states is $\approx .595$. Compare this to the case of dephasing where no ESD is exhibited and the fidelity can be as low as $\frac{1}{3}$.

Under independent qubit depolarization the concurrence between any of the two qubits of the system also exhibits ESD as shown in Fig. 3. In general ESD of concurrence occurs at lower decoherence strengths than ESD of the negativity implying remaining tri-partite entanglement in the system after the disappearance of the bi-partite entanglement. Again no correlation is seen between the decoherence strength where ESD is exhibited and the fidelity.

**IV. OTHER PROTECTED SUBSYSTEMS**

There are some DFS and NS variants which do not exhibit ESD, or even utilize any entanglement at all. For example, the states $|0\rangle_L = |01\rangle$, $|1\rangle_L = |10\rangle$, form a two qubit DFS to protect against collective dephasing [21]. General states within this logical basis have EPR-type entanglement which do not exhibit ESD in a depolarizing environment (the two qubit density matrix does not have the form [8]).

The parity of two qubits forms an NS that can protect against collective bit flip errors [22]. Initial states within this space are not entangled at all. While collective $\sigma_x$ rotations not of $\pi/2$ can cause entanglement, such entanglement is again not subject to ESD.

For completeness, I have looked at a couple of additional examples of DFSs where the system does exhibit ESD to see if any correlation can be found between
the finite-time loss of entanglement and the fidelity of stored quantum information. Specifically, I looked at DFSs which consist of a doubly degenerate ground state of a three-qubit and four-qubit system with always-on Heisenberg couplings. The energy gap between the logical qubit states and other states of the system forces decoherence generators to add energy in order to affect the system state.

In a series of calculations comparing ESD of different types of entanglement with the fidelity of the state of the DFS, no correlation is found. This again shows that ESD does not affect the workings of a quantum memory.

V. CONCLUSION

In conclusion, I have explored several qubit systems that one would typically utilize as a quantum memory. I have shown that the disappearance of entanglement from these systems does not correlate with the loss of fidelity of the stored quantum information. Certainly there is no change in the behavior of the fidelity when the system undergoes ESD, and there is not even a correlation between the decoherence strengths where ESD is exhibited and a given fidelity measure. The systems I explored were a four-qubit decoherence-free subspace and a three-qubit noiseless subsystem. Both of these systems can protect a qubit of quantum information from collective decoherence but cannot protect quantum information from independent qubit decoherence which can thus cause ESD.

It is always necessary to protect quantum information from the possibly debilitating affects of decoherence. ESD may be caused by certain decoherence generators but protection of quantum information from the ESD phenomenon does not require any special attention.

It is a pleasure to thank L. Viola and G. Gilbert for helpful feedback and acknowledge support from the MITRE Technology Program under MTP grant #07MSR205.

[1] M. Nielsen, I. Chuang, Quantum information and Computation (Cambridge University Press, Cambridge, 2000).
[2] P. Zanardi and R. Rosetti, Phys. Rev. Lett. 79, 3306 (1997).
[3] L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. 79, 1953 (1997).
[4] E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. 84, 2525 (2000).
[5] For a recent review see R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, [arXiv:quant-ph/0702225].
[6] C. Simon and J. Kempe, Phys. Rev. A 65, 052327 (2002);
W. Dur and H.-J. Briegel, Phys. Rev. Lett. 92, 180403 (2004);
M. Hein, W. Dur, and H.-J. Briegel, Phys. Rev. A 71, 032350 (2005);
S. Bandyopadhyay and D.A. Lidar, Phys. Rev. A 72, 042339 (2005);
O. Guhne, F. Boudakly, and M. Blaauboer, Phys. Rev. A 78, 060301 (2008).
[7] L. Diosi, in Irreversible Quantum Dynamics, edited by F. Benatti and R. Floreanini, Lect. Notes Phys. 622, (Springer-Verlag, Berlin) 157 (2003); P.J. Dodd and J.J. Halliwell, Phys. Rev. A 69, 052105 (2004).
[8] T. Yu and J.H. Eberly, Phys. Rev. Lett. 93, 140404 (2004); ibid. 97, 140403 (2006).
[9] T. Yu and J.H. Eberly, Science 323, 598 (2009).
[10] I. Sainz and G. Bjork, Phys. Rev. A 76, 042313 (2007).
[11] L. Aolita, R. Chaves, D. Cavalcanti, A. Acin, and L. Davidovich, Phys. Rev. Lett. 100, 080501 (2008).
[12] C.E. Lopez, G. Romero, F. Lastra, E. Solano, and J.C. Retamal, Phys. Rev. Lett. 110, 080503 (2008).
[13] M. Yonac, T. Yu, J.H. Eberly, J. Phys. B 39, 5621 (2006); ibid. 40, 545 (2007).
[14] M.P. Almeida, et al., Science 316, 579 (2007); J. Laurat, K.S. Choi, H. Deng, C.W. Chou, and H.J. Kimble, Phys. Rev. Lett. 99, 180504 (2007); A. Salle, F. de Melo, M.P. Almeida, M. Hor-Meyll, S.P. Walborn, P.H. Souto
Ribeiro, and L. Davidovich, Phys. Rev. A 78, 022322 (2008).
[15] Y.S. Weinstein, Phys. Rev. A 79, 023318 (2009).
[16] Y.S. Weinstein, arXiv:0902.2997.
[17] J. Kempe, D. Bacon, D.A. Lidar, and K.B. Whaley, Phys. Rev. A 63, 042307, (2001).
[18] G. Vidal and R.F. Werner, Phys. Rev. A 65 032314 (2002).
[19] S. Hill and W.K. Wootters, Phys. Rev. Lett 78, 5022 (1997).
[20] C. Sabin and G. Garcia-Alcaine, Eur. Phys. J. D 48, 435 (2008).
[21] L. Viola, E.M. Fortunato, M.A. Pravia, E. Knill, R. Laflamme, and D.G. Cory, Science 293, 2059 (2001).
[22] E.M. Fortunato, L. Viola, M.A. Pravia, E. Knill, R. Laflamme, T.F. Havel, and D.G. Cory, Phys. Rev. A 67, 062303 (2003).
[23] It is interesting that the lowest possible fidelity for the independent depolarizing environment is higher than the lowest possible fidelity for the independent dephasing environment.
[24] E.M. Fortunato, L. Viola, J. Hodges, G. Teklemariam, and D.G. Cory, New J. Phys. 4, 5 (2002).
[25] Y.S. Weinstein and C.S. Hellberg, Phys. Rev. A 72, 022319 (2005).
[26] D. Bacon, K.R. Brown, and K.B. Whaley, Phys. Rev. Lett. 87, 247902 (2001); Y.S. Weinstein, C.S. Hellberg and J. Levy, Phys. Rev. A 72, 020304(R) (2005); Y.S. Weinstein and C.S. Hellberg, Phys. Rev. Lett. 98, 110501 (2007).