SELECTION GAMES ON HYPERSPACES

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Abstract. In this paper we connect selection principles on a topological space to corresponding selection principles on one of its hyperspaces. We unify techniques and generalize theorems from the known results about selection principles for common hyperspace constructions. This includes results of Lj. D. R. Kočinac, Z. Li, and others. We use selection games to generalize selection principles and we work with strategies of various strengths for these games. The selection games we work with are primarily abstract versions of the selection principles of Rothberger, Menger, and Hurewicz type, as well as games of countable fan tightness and selective separability. The hyperspace constructions that we work with are the Vietoris and Fell topologies, both upper and full, generated by ideals of closed sets. Using a new technique we are able to extend straightforward connections between topological constructs to connections between selection games related to those constructs. This extension process works regardless of the length of the game, the kind of selection being performed, or the strength of the strategy being considered.

1. Introduction

In [16, pp. 10] Scheepers proved what he referred to as monotonicity laws for selection principles. The monotonicity laws allow one to draw implications between selection principles from the subset relations between the collections defining the selection principle (for example: if $\mathcal{A} \subseteq \mathcal{B}$, and $\mathcal{C}$ is some other collection, then $S_1(\mathcal{B}, \mathcal{C}) \implies S_1(\mathcal{A}, \mathcal{C})$). In a prior paper [3], the current authors created a new kind of monotonicity law. Instead of requiring direct inclusion, this new law uses translation functions which can be lined up to play games. Throughout that paper the translation functions are used to concisely prove some general theorems regarding the connections between selection games on a space and one of its corresponding spaces of continuous functions. Theorem 40 of [2] identifies a relationship between a class of selection games on a space and the space of its compact subsets endowed with the Vietoris topology. In this paper we seek to elaborate on that relationship using the translation theorems from [3].

The three main results from this paper are vaguely described below. We direct the reader to Section 2 for notation and definitions.

**Theorem 1.** Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are collections so that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Additionally, suppose that there exists a bijection $\beta : \bigcup \mathcal{A} \to \bigcup \mathcal{B}$ with the following features:

- $A \in \mathcal{A} \iff \beta[A] \in \mathcal{B}$, and
- $C \in \mathcal{C} \iff \beta[C] \in \mathcal{D}$.

Then the selection game defined by $\mathcal{A}$ and $\mathcal{C}$, and the game defined by $\mathcal{B}$ and $\mathcal{D}$ are equivalent. This equivalence holds whether the games use single selections or finite selections.

**Theorem 2.** Fix a topological space $X$. Then

(i) The generalized Rothberger/Menger game on $X$ is equivalent to the Selective Separability game played on the generalized upper Fell topology,
(ii) The generalized local Rothberger/Menger game on $X$ is equivalent to the Countable Fan Tightness game played on the generalized upper Fell topology, and

(iii) The generalized Hurewicz game on $X$ is equivalent to a modified Selective Separability game played on the generalized upper Fell topology.

**Theorem 3.** The previous theorem holds between $X$ and the generalized full Fell topology if the covers of $X$ are replaced by $\mathcal{A}_f$-covers.

Significant prior work on selection principles for the upper Fell topology and upper Vietoris topology was done by Di Maio, Kočinac, and Meccariello in 2015 [6]. In 2016, Li developed the right definitions to work on selection principles for the full Fell topology and full Vietoris topology [12]. Aside from generalizing the results of [6] and [12] for selection principles to selection games, we also provide a unifying infrastructure which highlights the similarities, not only in the results, but in the arguments employed.

In Section 2 we provide definitions for most of the topological concepts we will be discussing. We also summarize some of the relationships between those concepts. In Section 3, we elaborate on the strategy translation results of [3]. The section begins with the most general and technical version of the translation theorem. We then use that to generate less general but more easily applied versions of the translation theorem.

In Section 4 we establish various equivalences involving different kinds of open sets and other classes of sets. Finally, in Section 5, we tie the equivalences of Section 4 to selection principles and selection games using the results of Section 3. As an ending note, in 5.3, we look at some forms of tightness and determine that the results of Section 3 also link to certain selection principles.

Throughout, all ground spaces $X$ are assumed to be Hausdorff (and non-compact).

## 2. Definitions

**Definition 1.** For a topological space $X$, we let

- $\mathcal{T}_X$ denote the collection of all proper, non-empty open subsets of $X$,
- $[X]^{<\omega}$ denote the collection of all non-empty finite subsets of $X$,
- $\mathcal{K}(X)$ denote the collection of all proper, non-empty compact subsets of $X$, and
- $\mathcal{F}(X)$ denote the collection of all proper, non-empty closed subsets of $X$.

Notice that the map $U \mapsto X \setminus U$, $\mathcal{T}_X \to \mathcal{F}(X)$, is a bijection. We will be using this often.

**Definition 2.** Given a topological space $X$, by an *ideal of closed sets* we mean a collection $\mathcal{A}$ consisting of proper closed sets such that

- for all $A, B \in \mathcal{A}$, if $A \cup B \neq X$, then $A \cup B \in \mathcal{A}$ and
- if $A \in \mathcal{A}$ and $B \subseteq A$ is closed, then $B \in \mathcal{A}$.

Examples of ideals of closed sets include

- the collection of all finite subsets of a space,
- the collection of all compact subsets of a non-compact space,
- the collection of closed and real-bounded sets of a space,
- the collection of all closed subsets with empty interior inside a Baire space,
- the collection of all closed subsets with measure zero in a measure space, and
- the collection of all proper closed subsets of a space.

### 2.1. Types of Covers

Generally, for an open cover $\mathcal{U}$ of a topological space $X$, we say that $\mathcal{U}$ is *non-trivial* provided that $X \notin \mathcal{U}$. We let $\mathcal{O}_X$ denote the collection of all non-trivial open covers of $X$. If $Y \subseteq X$, we say that a collection of open sets $\mathcal{U}$ is a *non-trivial open cover of $Y$* provided that $Y \subseteq \bigcup \mathcal{U}$ and, for each $U \in \mathcal{U}$, $Y \not\subseteq U$.

The first general class of covers we define here generalized the idea of an $\omega$-cover or a $k$-cover.
Definition 3. For a space $X$ and an ideal of closed sets $\mathcal{A}$, a non-trivial open cover $\mathcal{U}$ is an $\mathcal{A}$-cover if, for every $A \in \mathcal{A}$, there exists $U \in \mathcal{U}$ so that $A \subseteq U$. We let $O(X, \mathcal{A})$ denote the collection of all $\mathcal{A}$-covers of $X$. We also define $\Gamma(X, \mathcal{A})$ to be all infinite non-trivial open covers $\mathcal{U}$ so that for every $A \in \mathcal{A}$, $\{U \in \mathcal{U} : A \not\subseteq U\}$ is finite.

Remark 1. Note that
- $O(X, [X]^{<\omega})$ is the collection of all $\omega$-covers of $X$.
- $O(X, \mathcal{K}(X))$ is the collection of all $k$-covers of $X$.
- $\Gamma(X, \mathcal{K}(X))$ is the collection of all $\gamma_k$-covers of $X$.

A kind of localized cover construction occurs naturally when studying the upper Vietoris and Fell topologies. We generalize that in the following definition.

Definition 4. For a space $X$, $Y \subseteq X$, and an ideal of closed sets $\mathcal{A}$, a non-trivial open cover $\mathcal{U}$ of $Y$ is an $\mathcal{A}$-cover of $Y$ if, for every $A \in \mathcal{A}$ with $A \subseteq Y$, there exists $U \in \mathcal{U}$ so that $A \subseteq U$. We let $O(X, Y, \mathcal{A})$ denote the collection of all $\mathcal{A}$-covers of $Y$.

Remark 2. Note that $O(\mathbb{R}, (-1, 1), \mathcal{F}(\mathbb{R}))$ is not the same as $O((-1, 1), \mathcal{F}((-1, 1)))$. So in general, the localized covers in Definition 4 are not the same thing as the relatively open covers.

Notation. Suppose $\mathcal{A}$ is a collection of subsets of $X$. We define
- $c.A$ to be the set $\{X \setminus A : A \in \mathcal{A}\}$.
- $\neg A$ to be the complement of $A$ relative to the power set of $X$.

The notion of a groupable cover is used to provide alternate characterizations of the Gerlits-Nagy $(*)$ property, introduced in [8], and Hurewicz property, introduced in [9], as seen in [11] and [10]. The idea of groupable and weakly groupable are added to the concept of an $\mathcal{A}$-cover in the following definitions. Normally, groupable covers are assumed to be countable. Here we allow the covers to be uncountable.

Definition 5. For a space $X$ and an ideal of closed sets $\mathcal{A}$, a non-trivial open cover $\mathcal{U}$ is an $\omega$-groupable $\mathcal{A}$-cover if there is a partition $\varphi : \mathcal{U} \to |\mathcal{U}|$ so that
- $\varphi^{-1}(\xi)$ is finite for all $\xi < |\mathcal{U}|$, and
- for every $A \in \mathcal{A}$, there is a $\xi_0 < |\mathcal{U}|$ so that for all $\xi \geq \xi_0$, there is a $U_\xi \in \varphi^{-1}(\xi)$ so that $A \subseteq U_\xi$.

Note that if $\mathcal{U}$ is countable, then $\omega$-groupable corresponds to the notion of groupability as discussed in [6]. We include the $\omega$ modifier to indicate that the partition consists of finite sets. We will observe this convention in forthcoming definitions as well.

We let $O^{gp}(X, \mathcal{A})$ denote the collection of all $\omega$-groupable $\mathcal{A}$-covers of $X$.

Definition 6. For a space $X$ and an ideal of closed sets $\mathcal{A}$, a non-trivial open cover $\mathcal{U}$ is a weakly $\omega$-groupable $\mathcal{A}$-cover if there is a partition $\varphi : \mathcal{U} \to |\mathcal{U}|$ so that
- $\varphi^{-1}(\xi)$ is finite for all $\xi < |\mathcal{U}|$, and
- for every $A \in \mathcal{A}$, there is a $\xi < |\mathcal{U}|$ so that $A \subseteq \bigcup_{\xi} \varphi^{-1}(\xi)$.

Note that if $\mathcal{U}$ is countable, then being weakly $\omega$-groupable corresponds to the notion of being weakly groupable as discussed in [1].

We let $O^{wgp}(X, \mathcal{A})$ denote the collection of all weakly $\omega$-groupable $\mathcal{A}$-covers of $X$.

In [12], Li defined cover types $c_\mathcal{V}$ and $k_\mathcal{F}$ which captured features of the full Vietoris and Fell topologies, respectively. Let $\mathcal{K}_\mathcal{F}(X)$ denote the $k_\mathcal{F}$-covers and $\mathcal{C}_\mathcal{V}(X)$ denote the $c_\mathcal{V}$-covers of $X$. In the following definition we establish a cover type which generalizes both $\mathcal{K}_\mathcal{F}(X)$ and $\mathcal{C}_\mathcal{V}(X)$.

Definition 7. For a space $X$ and an ideal of closed sets $\mathcal{A}$, a non-trivial open cover $\mathcal{U}$ is an $\mathcal{A}_\mathcal{F}$-cover iff for all $A \in \mathcal{A}$ and all finite sequences of open sets $V_1, \ldots, V_n \subseteq X$ with the property that $(X \setminus A) \cap V_j \neq \emptyset$ for $1 \leq j \leq n$, there are an open set $U \in \mathcal{U}$ and a finite set $F \subseteq X$ so that
• $A \subseteq U$,
• $F \cap V_j \neq \emptyset$ for $1 \leq j \leq n$, and
• $F \cap U = \emptyset$.

Let $O_F(X,A)$ denote the $A_F$-covers of $X$.

Note that, for any collection $A$, $O_F(X,A) \subseteq O(X,A)$. We have chosen the subscript $F$ here to reflect the added condition regarding finite sets and to remind the reader that these covers are used to study the Fell topology.

**Proposition 4.** Fix a space $X$. Then $K_F(X) = O_F(X,K(X))$ and $C_F(X) = O_F(X,F(X))$.

**Proof.** The fact that $K_F(X) = O_F(X,K(X))$ is immediate from Definition 2.1 of [12]. We show that $C_F(X) = O_F(X,F(X))$.

Suppose $\mathcal{U} \in C_F(X)$. Let $A \subseteq X$ be a closed set and $V_1, \ldots, V_n \subseteq X$ be open sets with the property that $(X \setminus A) \cap V_j \neq \emptyset$ for $1 \leq j \leq n$. Set $W_j = (X \setminus A) \cap V_j$. Then $W_1, \ldots, W_n$ are non-empty open sets, so we can find a finite set $F \subseteq X$ and $U \in \mathcal{U}$ so that $F \cap W_j \neq \emptyset$ for $1 \leq j \leq n$, $\bigcap_{j=1}^n (X \setminus W_j) \subseteq U$, and $F \cap U = \emptyset$.

Since $\bigcup_{j=1}^n W_j \subseteq (X \setminus A)$,

$$A \subseteq X \setminus \left( \bigcup_{j=1}^n W_j \right) = \bigcap_{j=1}^n (X \setminus W_j) \subseteq U.$$  

This shows that $\mathcal{U} \in O_F(X,F(X))$.

Now suppose that $\mathcal{U} \in O_F(X,F(X))$. Let $V_1, \ldots, V_n \subseteq X$ be open sets. Set $A = \bigcap_{j=1}^n (X \setminus V_j)$. Then we can find $U \in \mathcal{U}$ and $F \subseteq X$ so that $A \subseteq U$, $F \cap V_j \neq \emptyset$ for $1 \leq j \leq n$, and $F \cap U = \emptyset$. Thus $\bigcap_{j=1}^n (X \setminus V_j) \subseteq U$ and $\mathcal{U}$ is a $C_F$-cover.

Recall that all finite sets are compact and that therefore all $k$-covers are $\omega$-covers. This is part of a more general trend that we formalize below.

**Definition 8.** Recall that a collection $B$ refines $A$, denoted $B \prec A$, if, for every $B \in B$, there exists $A \in A$ so that $B \subseteq A$.

**Proposition 5.** For ideals of closed sets $A$ and $B$, the following are equivalent.

(i) $B \prec A$
(ii) $B \subseteq A$
(iii) $O(X,A) \subseteq O(X,B)$
(iv) $O_F(X,A) \subseteq O_F(X,B)$.

**Proof.** The equivalence of (i) and (iii) was demonstrated in Lemma 6 of [3].

We prove the equivalence of (i) and (ii). It is immediate that that $B \subseteq A$ implies $B \prec A$. Now, if $B \subseteq A$ and $B \in B$, let $A \in A$ be so that $B \subseteq A$. Then, since $B$ is closed and $A$ is an ideal of closed sets, $B \in A$.

We now prove the equivalence of (ii) and (iv). Suppose $B \subseteq A$ and let $\mathcal{U} \in O_F(X,A)$. Let $B \in B$ and $V_1, V_2, \ldots, V_n$ be open so that $V_j \setminus B \neq \emptyset$ for each $j$. Notice that $B \in A$. Since $\mathcal{U} \in O_F(X,A)$ and $B \in A$, there exists a $U \in \mathcal{U}$ with the desired properties.

By way of the contrapositive, suppose $B \nsubseteq A$. Let $B \in B$ be so that $B \notin A$. Notice that since $A$ is an ideal, this means that $B \nsubseteq A$ for all $A \in A$. Choose $x_A \in B \setminus A$ for all $A \in A$ and set $U_A = X \setminus \{x_A\}$. Consider $\mathcal{U} = \{U \in F_X : (\exists A \in A)[U \subseteq U_A]\}$.

To see that $\mathcal{U} \notin O_F(X,B)$, we note that $\mathcal{U} \notin O(X,B)$. To see that $\mathcal{U} \in O_F(X,A)$, let $A \in A$ and $V_1, V_2, \ldots, V_n$ be open sets with $V_j \setminus A \neq \emptyset$ for each $j$. Let $x_j \in V_j \setminus A$ for each $j = 1, 2, \ldots, n$. Then consider $F = \{x_A, x_1, x_2, \ldots, x_n\}$ and $U = X \setminus F$. Observe that $A \subseteq U$, $F \cap V_j \neq \emptyset$ for each $j$, and that $F \cap U = \emptyset$. Since $U \in \mathcal{U}$, we see that $\mathcal{U} \in O_F(X,A)$. □
Just as localized $\mathcal{A}$-covers show up when studying the upper hyperspace topologies, localized versions of the $\mathcal{A}F$-covers naturally occur when studying the full hyperspace topologies. Li defined localized versions of $K_F$ and $C_V$ in Definitions 2.2 and 2.4 of [12]. Let $K_F(X,Y)$ denote the $k_F$-covers of $Y$ and $C_V(X,Y)$ denote the $c_V$-covers of $Y$. The following definition generalizes these two notions.

**Definition 9.** For a space $X$, non-empty $Y \subseteq X$ and an ideal of closed sets $\mathcal{A}$, a non-trivial open cover $\mathcal{U}$ of $Y$ is an $\mathcal{A}F$-cover of $Y$ if for all $A \in \mathcal{A}$ with $A \subseteq Y$ and all finite sequences of open sets $V_1, \ldots, V_n \subseteq X$ with the property that $(X \setminus Y) \cap V_j \neq \emptyset$ for $1 \leq j \leq n$, there are open sets $U \in \mathcal{U}$ and a finite set $F \subseteq X$ so that

- $A \subseteq U$,
- $F \cap V_j \neq \emptyset$ for $1 \leq j \leq n$, and
- $F \cap U = \emptyset$.

Let $\mathcal{O}_F(X,Y,\mathcal{A})$ denote the $\mathcal{A}F$-covers of $Y$.

**Proposition 6.** Fix a space $X$, non-empty $Y \subseteq X$, and a non-trivial open cover $\mathcal{U}$ of $X$. Then $K_F(X,Y) = \mathcal{O}_F(X,Y,\mathcal{K}(X))$ and $C_V(X,Y) = \mathcal{O}_F(X,Y,\mathcal{F}(X))$.

**Proof.** The fact that $K_F(X,Y) = \mathcal{O}_F(X,Y,\mathcal{K}(X))$ follows immediately from Definition 2.2 of [12]. So we address $C_V(X,Y) = \mathcal{O}_F(X,Y,\mathcal{F}(X))$.

Suppose $\mathcal{U} \in C_V(X,Y)$. Let $A \subseteq Y$ be closed and $V_1, V_2, \ldots, V_n$ be open with the property that $V_j \setminus Y \neq \emptyset$ for each $j$. Set $W_0 = X \setminus A$ and, for each $j = 1, 2, \ldots, n$, $W_j = V_j \setminus A$. Notice that each $W_j$ is a non-empty open set. Moreover, $W_j \setminus Y \neq \emptyset$ for each $j$. Notice that

$$\bigcap_{j=0}^n (X \setminus W_j) \subseteq X \setminus W_0 = A \subseteq Y$$

so we can find $U \in \mathcal{U}$ and a finite set $F \subseteq X$ so that $\bigcap_{j=0}^n (X \setminus W_j) \subseteq U$, $F \cap W_j \neq \emptyset$ for each $j$, and $F \cap U = \emptyset$. As $W_j \subseteq X \setminus A$, we see that $A \subseteq X \setminus W_j$ for each $j$. Hence, $A \subseteq \bigcap_{j=0}^n (X \setminus W_j) \subseteq U$. That is, $\mathcal{U} \in \mathcal{O}_F(X,Y,\mathcal{F}(X))$.

Now, suppose $\mathcal{U} \in \mathcal{O}_F(X,Y,\mathcal{F}(X))$. Let $V_1, V_2, \ldots, V_n$ be open sets with $\bigcap_{j=1}^n (X \setminus V_j) \subseteq Y$ and $V_j \setminus Y \neq \emptyset$ for each $j$. Let $A = \bigcap_{j=1}^n (X \setminus V_j)$ and notice that $A$ is closed. Then we can find $U \in \mathcal{U}$ and finite $F \subseteq X$ so that $A \subseteq U$, $F \cap V_j \neq \emptyset$ for each $j$, and $F \cap U$. This establishes that $\mathcal{U} \in C_V(X,Y)$. \qed

Following Definitions 5.1 and 5.3 of [12], where the classes $K_F^{gp}$ of $k_F$-groupable covers and $C_V^{gp}$ of $c_V$-groupable covers are introduced, we define the general notion of an $F$-groupable $\mathcal{A}F$-cover.

**Definition 10.** For a space $X$, an ideal $\mathcal{A}$ of closed sets, and a non-trivial open cover $\mathcal{U}$ of $X$, we say $\mathcal{U}$ is $\omega$-$F$-groupable if there is a function $\varphi : \mathcal{U} \rightarrow |\mathcal{U}|$ so that

- $\varphi^{-1}(\xi)$ is finite for all $\xi < |\mathcal{U}|$ and
- for all $A \in \mathcal{A}$ and all finite sequences of open sets $V_1, \ldots, V_m \subseteq X$ with the property that $(X \setminus A) \cap V_j \neq \emptyset$ for $1 \leq j \leq m$, there is a $\xi_0 < |\mathcal{U}|$ so that for all $\xi \geq \xi_0$, there is an open set $U_\xi \in \varphi^{-1}(\xi)$ and a finite set $F_\xi \subseteq X$ so that
  - $A \subseteq U_\xi$,
  - $F_\xi \cap V_j \neq \emptyset$ for $1 \leq j \leq m$, and
  - $F_\xi \cap U_\xi = \emptyset$.

Let $\mathcal{O}_F^{gp}(X,\mathcal{A})$ denote the $\omega$-$F$-groupable $\mathcal{A}F$-covers of $X$.

**Proposition 7.** Fix a space $X$. Generally, $K_F^{gp}(X) \subseteq \mathcal{O}_F^{gp}(X,\mathcal{K}(X))$ and $C_V^{gp}(X) \subseteq \mathcal{O}_F^{gp}(X,\mathcal{F}(X))$.

In the case that we have a countable family $\mathcal{U}$ of open sets, $\mathcal{U} \in \mathcal{O}_F^{gp}(X,\mathcal{K}(X)) \implies \mathcal{U} \in K_F^{gp}(X)$ and $\mathcal{U} \in \mathcal{O}_F^{gp}(X,\mathcal{F}(X)) \implies \mathcal{U} \in C_V^{gp}(X)$. 

Proof. The fact that $K_F^{gp}(X) \subseteq O_F^{gp}(X, \mathbb{K}(X))$ follows from the observation that Definition 5.1 of [12] necessitates that $\mathcal{U} \in K_F^{gp}(X)$ be countable. To see that $C^{gp}_F(X) \subseteq O_F^{gp}(X, \mathbb{K}(X))$, let $\mathcal{U} \in C^{gp}_F(X)$. By Definition 5.3 of [12], we have $\varphi : \mathcal{U} \to \omega$ so that $\varphi^{-1}(n)$ is finite for each $n$ and some other properties hold. Let $A \subseteq X$ be closed and $V_1, V_2, \ldots, V_m$ be open so that $V_j \setminus A \neq \emptyset$ for all $j$. Set $W_j = V_j \setminus A$ and notice that each $W_j$ is a non-empty open set. Then we can find $n_0 \in \omega$ so that, for each $n \geq n_0$, there exist $U_n \subseteq \varphi^{-1}(n)$ and $F_n \subseteq [X]^{<\omega}$ so that $\bigcap_{j=1}^m (X \setminus W_j) \subseteq U_n$, $F_j \cap W_j \neq \emptyset$ for each $j$, and $F_n \cap U_n = \emptyset$. Since $\bigcup_{j=1}^m W_j \subseteq X \setminus A$, we see that $A \subseteq \bigcap_{j=1}^m (X \setminus W_j) \subseteq U_n$. This demonstrates that $\mathcal{U} \in O_F^{gp}(X, \mathbb{F}(X))$.

If $\mathcal{U} \in O_F^{gp}(X, \mathbb{K}(X))$ is countable, then we see that $\mathcal{U} \in K_F^{gp}(X)$ by [12, Definition 5.1]. So suppose $\mathcal{U} \in O_F^{gp}(X, \mathbb{K}(X))$ is a countable. Let $\varphi : \mathcal{U} \to \omega$ satisfy the membership criteria for $O_F^{gp}(X, \mathbb{F}(X))$. Notice that $\{\varphi^{-1}(n) : n \in \omega\}$ represents $\mathcal{U}$ as a union of infinitely many finite sets. Let $V_1, V_2, \ldots, V_m$ be non-empty open sets and notice that $A = \bigcap_{j=1}^m (X \setminus V_j)$ is a proper closed subset of $X$. Moreover, $(X \setminus A) \cap V_j \neq \emptyset$ for each $j$. Then we obtain $n_0 \in \omega$ so that, for every $n \geq n_0$, there exists $U \subset \varphi^{-1}(n)$ and $F \subseteq [X]^{<\omega}$ so that $F \cap V_j \neq \emptyset$ for each $j$, and $F \cap U = \emptyset$. That is, $\mathcal{U} \in C_F^{gp}(X)$. \hfill \Box

**Definition 11.** For a space $X$, an ideal $\mathcal{A}$ of closed sets, and a non-trivial open cover $\mathcal{U}$ of $X$, we say $\mathcal{U}$ is weakly $\omega$-groupable if there is a function $\varphi : \mathcal{U} \to |\mathcal{U}|$ so that

- $\varphi^{-1}(\xi)$ is finite for all $\xi < |\mathcal{U}|$ and
- for all $A \in \mathcal{A}$ and all finite sequences of open sets $V_1, V_2, \ldots, V_m \subseteq X$ with the property that $(X \setminus A) \cap V_j \neq \emptyset$ for $1 \leq j \leq m$, there is a $\xi < |\mathcal{U}|$ and a finite set $F \subseteq X$ so that
  - $A \subseteq \bigcup \varphi^{-1}(\xi)$,
  - $F \cap V_j \neq \emptyset$ for $1 \leq j \leq m$, and
  - $F \cap \bigcup \varphi^{-1}(\xi) = \emptyset$.

Let $O_F^{wg}(X, \mathcal{A})$ denote the weakly $\omega$-groupable $\mathcal{A}_F$-covers of $X$.

**Proposition 8.** Fix a space $X$. Then $K_F^{wg}(X) \subseteq O_F^{wg}(X, \mathbb{K}(X))$ and $C_F^{wg}(X) \subseteq O_F^{wg}(X, \mathbb{F}(X))$. In the case that $\mathcal{U}$ is a countable family of open sets, $\mathcal{U} \in O_F^{wg}(X, \mathbb{K}(X)) \implies \mathcal{U} \in K_F^{wg}(X)$ and $\mathcal{U} \in O_F^{wg}(X, \mathbb{F}(X)) \implies \mathcal{U} \in C_F^{wg}(X)$.

Proof. The fact that $K_F^{wg}(X) \subseteq O_F^{wg}(X, \mathbb{K}(X))$ follows immediately from Definition 5.5 of [12] and the observation that every $\mathcal{U} \in K_F^{wg}(X)$ must be countable.

To see that $C_F^{wg}(X) \subseteq O_F^{wg}(X, \mathbb{F}(X))$, let $\mathcal{U} \in C_F^{wg}(X)$ and let $\varphi : \mathcal{U} \to \omega$ be the partition as in Definition 5.7 of [12]. Note that $|\mathcal{U}| = \omega$. Let $A \subseteq X$ be a proper closed set and $V_1, V_2, \ldots, V_m$ be open so that $V_j \setminus A \neq \emptyset$ for each $j$. Set $W_j = V_j \setminus A$ for each $j$. Then, we can find $n \in \omega$ and a finite set $F \subseteq X$ so that $\bigcap_{j=1}^m (X \setminus W_j) \subseteq \varphi^{-1}(n)$, $F \cap V_j \neq \emptyset$ for each $j$, and $F \cap \bigcup \varphi^{-1}(n) = \emptyset$. Since $A \subseteq \bigcap_{j=1}^m (X \setminus W_j)$, we see that $\mathcal{U} \in O_F^{wg}(X, \mathbb{F}(X))$.

Let $\mathcal{U} \in O_F^{wg}(X, \mathbb{K}(X))$ be countable and notice that Definition 5.5 of [12] is clearly satisfied. So let $\mathcal{U} \in O_F^{wg}(X, \mathbb{K}(X))$ be countable. Let $\varphi : \mathcal{U} \to \omega$ be as specified in the definition and $V_1, V_2, \ldots, V_m$ be arbitrary proper non-empty open sets. Notice that $A = \bigcap_{j=1}^m (X \setminus V_j)$ is a closed set. Since $(X \setminus A) \cap V_j \neq \emptyset$ for each $j$, we can find $n_0 \in \omega$ and a finite set $F \subseteq X$ that satisfy the criteria listed above. This satisfies Definition 5.7 of [12]. \hfill \Box

The following proposition and examples show how the different cover types defined in this section relate to each other.
Proposition 9. For a space $X$ and an ideal $\mathcal{A}$ consisting of closed sets,

$$
\Gamma(X, \mathcal{A}) \subseteq \mathcal{O}^{gp}(X, \mathcal{A}) \subseteq \mathcal{O}(X, \mathcal{A}) \subseteq \mathcal{O}^{wgp}(X, \mathcal{A}) \subseteq \mathcal{O}_X
$$


Example 1. In general, $\Gamma(X, \mathcal{A})$ is a proper subset of $\mathcal{O}^{gp}(X, \mathcal{A})$.

Consider $X = \mathbb{R}$ and $\mathcal{A} = [\mathbb{R}]^{<\omega}$. Let $U_n = (-n-2, n+2)$ and $V_n = (-2^{-n}, 2^{-n})$ for each $n \in \omega$. Notice that

$$
\mathcal{U} = \{U_n : n \in \omega\} \cup \{V_n : n \in \omega\} \in \mathcal{O}^{gp}(X, \mathcal{A}) \setminus \Gamma(X, \mathcal{A}).
$$

Example 2. In general, $\mathcal{O}^{gp}(X, \mathcal{A})$ is a proper subset of $\mathcal{O}(X, \mathcal{A})$.

Consider $\omega_1$ with the discrete topology, $\mathcal{A}$ the ideal of finite subsets, and $\mathcal{U}$ consisting of precisely the set of finite subsets of $\omega_1$. Note that $\mathcal{U} \in \mathcal{O}(\omega_1, \mathcal{A})$. By way of contradiction, let $\phi: \mathcal{U} \to \omega_1$ be so that $\phi^{-1}(\alpha)$ is a finite collection of finite subsets for each $\alpha < \omega_1$ that satisfies the groupability criterion. Since $\phi^{-1}(\alpha)$ is a finite family of finite sets, we assume that $\phi^{-1}(\alpha)$ is a finite subset of $\omega_1$. For $n \in \omega$, let $\alpha_n < \omega_1$ be so that, for all $\beta \geq \alpha_n$, $\{k : k < n\} \subseteq \phi^{-1}(\beta)$. Now, let $\alpha = \sup\{\alpha_n : n \in \omega\}$ and notice that $\alpha < \omega_1$. Since $\alpha_n \leq \alpha$, we know that $\{k : k < n\} \subseteq \phi^{-1}(\alpha)$ for all $n \in \omega$. However, $\phi^{-1}(\alpha)$ was assumed to be finite. Therefore, $\mathcal{U} \notin \mathcal{O}^{gp}(\omega_1, \mathcal{A})$.

Example 3. In general, $\mathcal{O}(X, \mathcal{A})$ is a proper subset of $\mathcal{O}^{wgp}(X, \mathcal{A})$.

Consider $\mathbb{R}$ and $\mathcal{A} = [\mathbb{R}]^{<\omega}$. Let

$$
\mathcal{U}_n = \left\{ \left( \frac{j-1}{2^n}, \frac{j+1}{2^n} \right) : -3^n < j < 3^n \right\},
$$

and set $\mathcal{U} = \bigcup_n \mathcal{U}_n$. Then $\{-1, 1\} \nsubseteq U$ for any $U \in \mathcal{U}$, so $\mathcal{U}$ is not an $\omega$-cover. However, $(-\frac{3}{2}, \frac{3}{2}) \subseteq \bigcup \mathcal{U}_n$ for each $n$. So $\mathcal{U}$ is $\omega$-weakly groupable.

Example 4. In general, $\mathcal{O}^{wgp}(X, \mathcal{A})$ is a proper subset of $\mathcal{O}_X$.

Consider $\omega$ and let $\mathcal{U} = \{\{n\} : n \in \omega\}$. Let $\varphi: \mathcal{U} \to \omega$ be arbitrary and consider $\varphi^{-1}(0)$. Then $F = \bigcup \varphi^{-1}(0) \in [\omega]^{<\omega}$. Let $m = \max F$ and consider $E = F \cup \{m+1\} \in [\omega]^{<\omega}$. Clearly, $E \nsubseteq \bigcup \varphi^{-1}(0)$. Let $n \in \omega$ be so that $n > 0$ and notice that $F \cap \bigcup \varphi^{-1}(n) = \emptyset$ so $E \nsubseteq \bigcup \varphi^{-1}(n)$. So $\mathcal{U} \in \mathcal{O}_X \setminus \mathcal{O}^{wgp}(X, \mathcal{A})$.

2.2. Dense Sets and Blades.

Notation. We will be using the following classes.

- $\Omega_{X,x}$ to be the set of all $A \subseteq X$ with $x \in \text{cl}_X(A)$. We also call $A \in \Omega_{X,x}$ a blade of $x$.
- $D_X$ to be the collection of all dense subsets of $X$.

Definition 12. For a space $X$ and a dense $D \subseteq X$, we say $D$ is $\omega$-groupable if there is a function $\varphi: D \to |D|$ so that

- $\varphi^{-1}(\xi)$ is finite for all $\xi < |D|$ and
- for all open $U \subseteq X$, there is a $\xi_0 < |D|$ so that for all $\xi \geq \xi_0$, $U \cap \varphi^{-1}(\xi) \neq \emptyset$.

Let $D_X^{gp}$ denote the $\omega$-groupable dense subsets of $X$.

Observe that if $D$ is countable, then the notion of $\omega$-groupable is equivalent to the standard notion of being groupable as discussed in [6].

The following modification of denseness has shown up repeatedly in the literature without name: see [6, Theorems 23 and 24] and [12, Theorems 5.6 and 5.8]. We give it a symbol here for notational convenience. This definition does not seem to be a general topological notion, but rather is specific to the context of hyperspaces (or perhaps topologies on lattices).
Definition 13. Fix a space $X$. Consider the space of closed sets $\mathcal{F}(X)$ with topology $\mathcal{T}$. Then $D \subseteq \mathcal{F}(X)$ is weakly $\omega$-groupable if there is a function $\varphi : D \to |D|$ so that

- $\varphi^{-1}(\xi)$ is finite for all $\xi < |D|$ and
- for all open $V \subseteq \mathcal{F}(X)$, there is an $\xi < |D|$ so that $\bigcap \varphi^{-1}(\xi) \in V$.

Let $\mathcal{D}_{\omega g}^{\mathcal{F}(X)}$ denote the weakly $\omega$-groupable dense subsets of $\mathcal{F}(X)$.

2.3. The Hyperspace Topologies. The Vietoris and Fell hyperspace topologies are probably the most commonly studied hyperspace topologies. Originally, the Fell topology, introduced in [7], was studied in the context of functional analysis and $C^*$-algebras. For a detailed study of the Vietoris topology, see [13]. We present a general version of these topologies in this section.

Definition 14. Suppose $X$ is a topological space and $\mathcal{A}$ consists of closed sets. The upper $\mathcal{A}$-topology or co-$\mathcal{A}$ topology on $\mathcal{F}(X)$, denoted $\mathcal{T}_{\mathcal{A}^+}$, is generated by subbasic open sets of the form

$$(X \setminus A)^+ = \{ F \in \mathcal{F}(X) : F \subseteq X \setminus A \} = \{ F \in \mathcal{F}(X) : F \cap A = \emptyset \}$$

along with $\mathcal{F}(X)$. If $\mathcal{A}$ is an ideal, then these are the basic open sets. Set $\mathcal{F}(X, \mathcal{A}^+) = (\mathcal{F}(X), \mathcal{T}_{\mathcal{A}^+})$.

- The upper Fell topology (or co-compact topology), $\mathcal{T}_F$ is the case where $\mathcal{A}$ is the collection of compact subsets of $X$. Set $\mathcal{F}(X, F^+) = (\mathcal{F}(X), \mathcal{T}_{F^+})$.
- The upper Vietoris topology, $\mathcal{T}_V$ is the case where $\mathcal{A}$ is the collection of closed subsets of $X$. Set $\mathcal{F}(X, V^+) = (\mathcal{F}(X), \mathcal{T}_{V^+})$.
- The co-finite topology, $\mathcal{T}_{Z^+}$ is the case where $\mathcal{A}$ is the collection of finite subsets of $X$. Set $\mathcal{F}(X, Z^+) = (\mathcal{F}(X), \mathcal{T}_{Z^+})$.

Definition 15. Suppose $X$ is a topological space. Let $\mathcal{A}$ be an ideal consisting of closed subsets of $X$. The topology on $\mathcal{F}(X)$ generated by $\mathcal{A}$, denoted $\mathcal{T}_{\mathcal{A}}$, has basic open sets of the form

$$(A; V_1, \ldots, V_n) = \{ F \in \mathcal{F}(X) : F \cap A = \emptyset \text{ and } F \cap V_j \neq \emptyset \text{ for } 1 \leq j \leq n \}$$

where $A \in \mathcal{A}$ and $V_1, \ldots, V_n \subseteq X$ are open. Set $\mathcal{F}(X, \mathcal{A}) = (\mathcal{F}(X), \mathcal{T}_{\mathcal{A}})$.

- The Fell topology, $\mathcal{T}_F$ is the case where $\mathcal{A}$ is the collection of compact subsets of $X$. Set $\mathcal{F}(X, F) = (\mathcal{F}(X), \mathcal{T}_F)$.
- The Vietoris topology, $\mathcal{T}_V$ is the case where $\mathcal{A}$ is the collection of closed subsets of $X$. Set $\mathcal{F}(X, V) = (\mathcal{F}(X), \mathcal{T}_V)$.

One ought to notice that the Vietoris topology in this context is defined on the space of proper, non-empty closed sets and not just on the space of non-empty compact subsets of $X$.

Proposition 10. Suppose $X$ is a topological space. Let $\mathcal{A}$ be an ideal consisting of closed subsets of $X$. If the basic open set $[A; V_1, \ldots, V_n] \in \mathcal{F}(X, \mathcal{A})$ is non-empty, then $V_j \cap (X \setminus A) \neq \emptyset$ for all $j$.

Proof. Let $G \in [A; V_1, \ldots, V_n]$. Then $G \subseteq X \setminus A$ and $G \cap V_j \neq \emptyset$ for all $j$. Thus $(X \setminus A) \cap V_j \neq \emptyset$ for all $j$. 

Lemma 11. If $\mathcal{B} \subseteq \mathcal{A}$ are ideals of closed sets in a space $X$, then

- $\mathcal{T}_{\mathcal{B}^+} \subseteq \mathcal{T}_{\mathcal{A}^+}$ and
- $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}_{\mathcal{A}}$.

Proof. Consider an open set in $\mathcal{F}(X, \mathcal{B}^+)$ which is necessarily of the form $(X \setminus B)^+$ for $B \in \mathcal{B}$. Since $B \in \mathcal{A}$, we see that $(X \setminus B)^+$ is open in $\mathcal{F}(X, \mathcal{A}^+)$.

Now suppose we have an open set $U$ in $\mathcal{F}(X, \mathcal{B})$. Let $F \in U$ be arbitrary and let $B \in \mathcal{B}$ and $V_1, \ldots, V_n$ open in $X$ be so that

$F \in [B; V_1, \ldots, V_n] \subseteq U$.

Since $B \in \mathcal{A}$, notice that $[B; V_1, \ldots, V_n]$ is a $\mathcal{T}_{\mathcal{A}}$ neighborhood of $F$ and, as $F$ was chosen to be arbitrary, $U$ is open in $\mathcal{F}(X, \mathcal{A})$.  

□
2.4. Networks. Recall the notion of a $\pi$-network.

**Definition 16.** For a space $X$ and $\mathcal{V} \subseteq \wp(X)$, $\mathcal{V}$ is said to be a $\pi$-network if, for every non-empty open subset $U$ of $X$, there exists some $Y \in \mathcal{V}$ so that $Y \subseteq U$.

The following definition is motivated by the $\Pi_\omega$ and $\Pi_k$ used in [6].

**Definition 17.** For a family $A \subseteq \wp(X)$, let $\Pi_A(X)$ denote the set of all $\pi$-networks consisting of sets from $A$.

Note that
- $\Pi_k$ is $\Pi_{\mathbb{N}}(X)$ and
- $\Pi_\omega$ is $\Pi_{[X]<\omega}$.

In Section 3 of [12], Li defined a version of $\pi$-network that is relevant to the full Vietoris and Fell topologies. These are called $\pi_V$- and $\pi_F$-networks. The definition below generalizes both of these.

**Definition 18.** Let $A$ be an ideal of closed subsets of $X$ and let $\mathcal{F}_A \subseteq A \times \mathcal{P}^<\omega_X$ be defined by the rule

$$\langle A, V_1, \ldots, V_n \rangle \in \mathcal{F}_A \iff (\forall j)[V_j \setminus A \neq \emptyset].$$

We say that a collection $\mathcal{V} \subseteq \mathcal{F}_A$ is a $\pi_{A,F}$-network of $X$ if, for each proper open subset $U$ of $X$, there exists $(A, V_1, \ldots, V_n) \in \mathcal{V}$ and a finite $F \subseteq X$ so that $F \cap V_j \neq \emptyset$ for each $j$, $A \subseteq U$, and $F \cap U = \emptyset$. Let $\Pi^F_{A}(X)$ denote the family of all $\pi_{A,F}$-networks of $X$.

**Lemma 12.** Suppose $B \subseteq A$ are ideals of closed sets in $X$. Then
- $\Pi_B(X) \subseteq \Pi_A(X)$ and
- $\Pi^F_B(X) \subseteq \Pi^F_A(X)$.

**Proof.** The first claim is clear.

Now suppose $\mathcal{V} \in \Pi^F_B(X)$ and let $U$ be open in $X$. Then there exists $(B, V_1, \ldots, V_n) \in \mathcal{V}$ and $F \in [X]^{<\omega}$ that satisfy the conditions in the definition. Since $B \subseteq A$, we see that $\mathcal{V} \in \Pi^F_A(X)$.

**Proposition 13.** Let $X$ be a topological space and $A$ be an ideal of closed subsets of $X$.

- $\mathcal{V} \in \Pi^F_{\mathbb{N}}(X)(X)$ if and only if $\mathcal{V}$ is a $\pi_F$-network,
- If $\mathcal{V} \in \Pi^F_{\mathbb{N}}(X)(X)$, then $\{\langle X \setminus A, V_1, \ldots, V_n \rangle : \langle A, V_1, \ldots, V_n \rangle \in \mathcal{V} \}$ is a $\pi_V$-network, and
- If $\mathcal{V}$ is a a $\pi_V$-network, then $\{\bigcap_{j=1}^n(X \setminus V_j), V_1, \ldots, V_n \} : \langle V_1, \ldots, V_n \rangle \in \mathcal{V} \} \in \Pi^F_{\mathbb{N}}(X)(X)$.

**Proof.** The first equivalence comes directly from the definitions in [12]. For the $\pi_V$-networks, first suppose that $\mathcal{V} \in \Pi^F_{\mathbb{N}}(X)(X)$. Let $U \subseteq X$ be a proper open set. Then there is $(A, V_1, \ldots, V_n) \in \mathcal{V}$ and $F \subseteq X$ finite so that $F \cap V_j \neq \emptyset$ (for each $j$), $A \subseteq U$, and $F \cap U = \emptyset$. Set $V_0 = X \setminus A$. Then since $(X \setminus A) \subseteq \bigcup_{j=0}^n V_j$,

$$\bigcap_{j=0}^n(X \setminus V_j) = X \setminus \left(\bigcup_{j=0}^n V_j\right) \subseteq A.$$ 

So $\{\langle X \setminus A, V_1, \ldots, V_n \rangle : \langle A, V_1, \ldots, V_n \rangle \in \mathcal{V} \}$ is a $\pi_V$-network.

Now suppose $\mathcal{V}$ is a $\pi_V$-network. Let $U \subseteq X$ be a proper open set. Then we can find $\langle V_1, \ldots, V_n \rangle \in \mathcal{V}$ and $F \subseteq X$ finite so that
- $F \cap V_j \neq \emptyset$ for $1 \leq j \leq n$,
- $\bigcap_{j=1}^n(X \setminus V_j) \subseteq U$, and
- $F \cap U = \emptyset$.

This is enough to show that $\{\langle \bigcap_{j=1}^n(X \setminus V_j), V_1, \ldots, V_n \rangle : \langle V_1, \ldots, V_n \rangle \in \mathcal{V} \} \in \Pi^F_{\mathbb{N}}(X)(X)$.
2.5. Selection Principles and Games. Though games of countable-length are more commonly studied, we extend our considerations to ordinal-length games for more generality. Ordinal-games have been introduced before (see [17]).

Definition 19. Given a set $A$ and another set $B$, we define the finite selection game $G_{\text{fin}}^\alpha(A,B)$ for $A$ and $B$ as follows:

\[
\begin{array}{c|c|c|c|c|c|c|c}
& A_0 & A_1 & A_2 & \ldots & A_\xi & \ldots \\
\hline
| & F_0 & F_1 & F_2 & \ldots & F_\xi & \ldots \\
\end{array}
\]

where $A_\xi \in A$ and $F_\xi \in [A_\xi]^{<\omega}$ for all $\xi < \alpha$. We declare Two the winner if $\bigcup\{F_\xi : \xi < \alpha\} \in B$. Otherwise, One wins. We let $G_{\text{fin}}^\alpha(A,B)$ denote $G_{\text{fin}}^\alpha(A,B)$.

Definition 20. Similarly, we define the single selection game $G_1^\alpha(A,B)$ as follows:

\[
\begin{array}{c|c|c|c|c|c|c|c}
& A_0 & A_1 & A_2 & \ldots & A_\xi & \ldots \\
\hline
| & x_0 & x_1 & x_2 & \ldots & x_\xi & \ldots \\
\end{array}
\]

where each $A_\xi \in A$ and $x_\xi \in A_\xi$. We declare Two the winner if $\{x_\xi : \xi \in \alpha\} \in B$. Otherwise, One wins. We let $G_1^\alpha(A,B)$ denote $G_1^\alpha(A,B)$.

For collections $A$ and $B$ and an ordinal $\alpha$, we refer to $G_1^\alpha(A,B)$ and $G_{\text{fin}}^\alpha(A,B)$ as selection games.

Definition 21. We define strategies of various strength below.

- A strategy for player One in $G_1^\alpha(A,B)$ is a function $\sigma : (\bigcup A)^{<\alpha} \rightarrow A$. A strategy $\sigma$ for One is called winning if whenever $x_\xi \in \sigma(\xi : \xi < \xi)$ for all $\xi < \alpha$, $\{x_\xi : \xi \in \alpha\} \notin B$. If player One has a winning strategy, we write $I \uparrow G_1^\alpha(A,B)$.
- A strategy for player Two in $G_1^\alpha(A,B)$ is a function $\tau : A^{<\alpha} \rightarrow \bigcup A$. A strategy $\tau$ for Two is winning if whenever $A_\xi \in A$ for all $\xi < \alpha$, $\{\tau(\alpha_0, \ldots, A_\xi) : \xi < \alpha\} \notin B$. If player Two has a winning strategy, we write $\Pi \uparrow G_1^\alpha(A,B)$.
- A predetermined strategy for One is a strategy which only considers the current turn number. We call this kind of strategy predetermined because One is not reacting to Two’s moves, they are just running through a pre-planned script. Formally it is a function $\sigma : \alpha \rightarrow A$. If One has a winning predetermined strategy, we write $\Pi \uparrow G_1^\alpha\text{pre}(A,B)$.
- A Markov strategy for Two is a strategy which only considers the most recent move of player One and the current turn number. Formally it is a function $\tau : A \times \alpha \rightarrow \bigcup A$. If Two has a winning Markov strategy, we write $\Pi \uparrow G_1^\alpha\text{mark}(A,B)$.
- If there is a single element $x_0 \in A$ so that the constant function with value $x_0$ is a winning strategy for One, we say that One has a constant winning strategy, denoted by $I \uparrow G_1^\alpha\text{cst}(A,B)$.

These definitions can be extended to $G_{\text{fin}}^\alpha(A,B)$ in the obvious way.

Definition 22. Two games $G_1$ and $G_2$ are said to be strategically dual provided that the following two statements hold:

- $I \uparrow G_1$ iff $\Pi \uparrow G_2$
- $I \uparrow G_2$ iff $\Pi \uparrow G_1$

Two games $G_1$ and $G_2$ are said to be Markov dual provided that the following two statements hold:

- $I \uparrow G_1\text{pre}$ iff $\Pi \uparrow G_2\text{mark}$
- $I \uparrow G_2\text{pre}$ iff $\Pi \uparrow G_1\text{mark}$
Two games $G_1$ and $G_2$ are said to be dual provided that they are both strategically dual and Markov dual.

**Definition 23.** The reader may be more familiar with selection principles than selection games. Let $A$ and $B$ be collections and $\alpha$ be an ordinal. The selection principle $S^\alpha_{pre}(A, B)$ for a space $X$ is the following property: Given any $\alpha$-length sequence $(A_\beta : \beta < \alpha)$ from $A$, there exists $(x_\beta : \beta < \alpha)$ with $x_\beta \in A_\beta$ for each $\beta < \alpha$ so that $(x_\beta : \beta < \alpha) \in B$. $S^\alpha_{fin}(A, B)$ is similarly defined, but with finite selections instead of single selections.

**Remark 3.** In general, $S^\alpha_{mark}(A, B)$ holds if and only if $\mathcal{I}_{pre} G^\alpha_{mark}(A, B)$ where $\square \in \{1, \text{fin}\}$. See [4, Prop. 15].

**Definition 24.** An even more fundamental type of selection is inspired by the Lindelöf property. Let $A$ and $B$ be collections. Then $(A^*_B)$ means that, for every $A \in A$, there exists $B \subseteq A$ so that $B \in B$. Scheepers calls this a Bar-Ilan selection principle in [16].

**Remark 4.** Let $A$ and $B$ be collections, and $\kappa$ be a cardinal. We let $B^\kappa = \{B \in B : |B| \leq \kappa\}$. Then One fails to have a constant strategy in $G^\kappa_A(A, B)$ if and only if $(A^*_B^\kappa)$ holds as shown in [4, Prop. 15].

**Remark 5.** Note the following relationship between strategies in selection games and other types of selection principles.

\[
\begin{align*}
\Pi \uparrow G^\kappa_{mark}(A, B) & \Rightarrow \Pi \uparrow G^\kappa_{mark}(A, B) \Rightarrow I \npre G^\kappa_{mark}(A, B) \Rightarrow I \npre G^\kappa_{mark}(A, B) \\
& \Downarrow \quad \Downarrow \\
S^\kappa_{mark}(A, B) & \Rightarrow (A^*_B^\kappa)
\end{align*}
\]

3. **Translation Theorems**

**Definition 25.** For two selection games $G$ and $H$, we say that $G \leq_H H$ if

- $\Pi \uparrow G \Rightarrow \Pi \uparrow H$,
- $\Pi \uparrow G \Rightarrow \Pi \uparrow H$,
- $I \npre G \Rightarrow I \npre H$, and
- $I \npre G \Rightarrow I \npre H$.  

Note that $\leq_H$ is transitive and that, if $G \leq_H H$ and $H \leq_H G$, then $G$ and $H$ are equivalent. In the case that $G \leq_H H$ and $H \leq_H G$, we write $G \equiv H$.

We start by recalling previous results about translating strategies.

**Theorem 14** (Theorem 12 of [3]). Let $A, B, C,$ and $D$ be collections and $\alpha$ be an ordinal. Suppose there are functions

- $\overrightarrow{T}_{1,\xi} : B \to A$ and
- $\overrightarrow{T}_{II,\xi} : (\bigcup A) \times B \to \bigcup B$

for each $\xi \in \alpha$, so that

- (Tr1) If $x \in \overrightarrow{T}_{1,\xi}(B)$, then $\overrightarrow{T}_{II,\xi}(x, B) \in B$
- (Tr2) If $x_\xi \in \overrightarrow{T}_{1,\xi}(B_{\xi})$ and $\{x_\xi : \xi \in \alpha\} \subseteq C$, then $\{\overrightarrow{T}_{II,\xi}(x_\xi, B_{\xi}) : \xi \in \alpha\} \subseteq D$.

Then $G^\alpha_\kappa(A, C) \leq_{II} G^\alpha_\kappa(B, D)$.  

Remark 6. Note that under the hypotheses of Theorem 14, we also have that $(A^\alpha) \implies (B^\beta)$. Set $A = \overrightarrow{T_{1,0}(B)}$, and then find $C \subseteq A$ so that $|C| \leq \alpha$ and $C \in \mathcal{C}$, say $C = \{x_\xi : \xi < \alpha\}$. Define $D = \{\overrightarrow{T_{1,\xi}(x_\xi, B)} : \xi < \alpha\}$. Then $|D| \leq \alpha$ and $D \in \mathcal{D}$.

Corollary 15 (Corollary 13 of [3]). Let $A$, $B$, $C$, and $D$ be collections and $\alpha$ be an ordinal. Suppose there is a map $\varphi : (\bigcup \mathcal{B}) \times \alpha \rightarrow (\bigcup \mathcal{A})$ so that

- for all $B \in \mathcal{B}$ and all $\xi < \alpha$, $\{\varphi(y, \xi) : y \in B\} \in \mathcal{A}$
- if $\{\varphi(y, \xi) : \xi < \alpha\} \in \mathcal{C}$, then $\{y_\xi : \xi < \alpha\} \in \mathcal{D}$

Then $G^\alpha_\mathcal{A}(A, C) \leq_\mathcal{II} G^\alpha_\mathcal{B}(B, D)$.

We now provide the finite selection counterpart to Theorem 14.

Theorem 16 (The Translation Theorem). Let $A$, $B$, $C$, and $D$ be collections and let $\alpha$ be an ordinal. Suppose there are functions

- $\overrightarrow{T_{1,\xi}} : B \rightarrow A$
- $\overrightarrow{T_{1,\xi}} : [\bigcup \mathcal{A}]^{<\omega} \times B \rightarrow [\bigcup \mathcal{B}]^{<\omega}$

defined. Since $G$ is a winning strategy for Two in $\alpha$, then $\{y_\xi : \xi < \alpha\} \in \mathcal{D}$

Then $G^\alpha_{\mathcal{A}}(A, C) \leq_\mathcal{II} G^\alpha_{\mathcal{B}}(B, D)$.

Proof. Suppose $\tau$ is a winning Markov strategy for Two in $G^\alpha_{\mathcal{A}}(A, C)$. Define a Markov strategy $t$ for Two in $G^\alpha_{\mathcal{B}}(B, D)$ by $t(B, \xi) = \overrightarrow{T_{1,\xi}}(\tau(B, \xi), B)$. First note that if $B \in \mathcal{B}$, then $\tau(\overrightarrow{T_{1,\xi}}(B), \xi) \in [\overrightarrow{T_{1,\xi}}(B)]^{<\omega}$, and so $t(B, \xi) \in [B]^{<\omega}$. So $t$ really is a Markov strategy. We now check that $t$ is a winning strategy. Let $\{B_\xi : \xi < \alpha\}$ be a sequence from $\mathcal{B}$. As $\tau$ was a winning Markov strategy for Two in $G^\alpha_{\mathcal{A}}(A, C)$, we have that $\bigcup_{\xi < \alpha} t(B_\xi, \xi) \in \mathcal{D}$. Hence, (P2) asserts that $\bigcup_{\xi < \alpha} t(B_\xi, \xi) \in \mathcal{D}$. So Two has a winning Markov strategy in $G^\alpha_{\mathcal{B}}(B, D)$.

Now suppose $\tau$ is a winning strategy for Two in $G^\alpha_{\mathcal{A}}(A, C)$. We will define a winning strategy $t$ for Two in $G^\alpha_{\mathcal{B}}(B, D)$ recursively.

Given $B_0 \in \mathcal{B}$, let $A_0 = \overrightarrow{T_{1,0}}(B_0)$ and $F_0 = \tau(A_0) \in [A_0]^{<\omega}$. Then define $t(B_0) = G_0 = \overrightarrow{T_{1,0}}(F_0, B_0)$. By (P1), we see that $t(B_0) \in [B_0]^{<\omega}$.

For a given $\beta < \alpha$, assume we have $\langle A_\xi : \xi < \beta\rangle$ coming from $\mathcal{A}$, $\langle B_\xi : \xi < \beta\rangle$ coming from $\mathcal{B}$, $\langle F_\xi : \xi < \beta\rangle$, and $\langle G_\xi : \xi < \beta\rangle$ all appropriately defined. Given $B_\beta \in \mathcal{B}$, let $A_\beta = \overrightarrow{T_{1,\beta}}(B_\beta)$ and $F_\beta = \tau(A_0, A_1, \ldots, A_\beta) \in [A_\beta]^{<\omega}$. Define $G_\beta = \overrightarrow{T_{1,\beta}}(F_\beta, B_\beta)$. Notice that $G_\beta \in [B_\beta]^{<\omega}$ by (P1). So we define $t(B_0, B_1, \ldots, B_\beta) = G_\beta$.

Now that $t$ is defined, we must show that it is winning. Suppose we have a complete run of the game which involves $\langle A_\xi : \xi < \alpha\rangle$, $\langle B_\xi : \xi < \alpha\rangle$, $\langle F_\xi : \xi < \alpha\rangle$, and $\langle G_\xi : \xi < \alpha\rangle$ appropriately defined. Since $\tau$ was assumed to be a winning strategy for Two in $G^\alpha_{\mathcal{A}}(A, C)$, $\bigcup_{\xi < \alpha} F_\xi \in \mathcal{C}$. Hence, (P2) guarantees that $\bigcup_{\xi < \alpha} G_\xi \in \mathcal{D}$. That is, $t$ is a winning strategy for Two in $G^\alpha_{\mathcal{B}}(B, D)$.

Suppose $\sigma$ is a winning strategy for One in $G^\alpha_{\mathcal{A}}(A, C)$. We define a strategy $s$ for One in $G^\alpha_{\mathcal{B}}(B, D)$ recursively as follows. Define $B_0 = \sigma(\emptyset)$ and $s(\emptyset) = A_0 = \overrightarrow{T_{1,\xi}}(\sigma(\emptyset))$. Then, for $F_\emptyset \in [A_0]^{<\omega}$, set $G_0 = \overrightarrow{T_{1,0}}(F_\emptyset, B_0)$. By (P1), $G_0 \in [B_0]^{<\omega}$.

Assume that, for $\xi < \alpha$, we have $\langle A_\eta : \eta < \xi\rangle$, $\langle B_\eta : \eta < \xi\rangle$, $\langle F_\eta : \eta < \xi\rangle$, and $\langle G_\eta : \eta < \xi\rangle$ so that $F_\eta \in [A_\eta]^{<\omega}$ and $G_\eta = \overrightarrow{T_{1,\eta}}(F_\eta, B_\eta)$ for all $\eta < \xi$. By (P1), $G_\eta \in [B_\eta]^{<\omega}$ for all $\eta < \xi$. Then
define $B_\xi = \sigma((G_\eta : \eta < \xi))$, $A_\xi = \overleftarrow{T}_1(\xi)(B_\xi)$ and $s((F_\eta : \eta < \xi)) = A_\xi$. Now we need only show that $s$ is a winning strategy for One in $G^\alpha_{\text{fin}}(A, C)$. Since $\sigma$ is a winning strategy for One in $G^\alpha_{\text{fin}}(B, D)$, we have that

$$\bigcup_{\xi < \alpha} G_\xi \notin D.$$ 

Since $F_\xi \in [A_\xi]^{<\omega}$ and $G_\xi = \overrightarrow{T}_{\Pi, \xi}(F_\xi, B_\xi)$, it must be the case that

$$\bigcup_{\xi < \alpha} F_\xi \notin C$$

by (P2).

Finally, suppose $\sigma$ is a pre-determined winning strategy for One in $G^\alpha_{\text{fin}}(B, D)$. Define $s : \alpha \to A$ by $s(\xi) = \overleftarrow{T}_1(\xi)(\sigma(\xi))$. To see that $s$ is a winning strategy for One in $G^\alpha_{\text{fin}}(A, C)$, consider $\langle F_\xi : \xi < \alpha \rangle$ where $F_\xi \in [s(\xi)]^{<\omega}$ for all $\xi < \alpha$. Let $G_\xi = \overrightarrow{T}_{\Pi, \xi}(F_\xi, \sigma(\xi))$ and notice that $G_\xi \in [\sigma(\xi)]^{<\omega}$ by (P1). Since $\sigma$ is a winning strategy for One in $G^\alpha_{\text{fin}}(B, D)$, we have that

$$\bigcup_{\xi < \alpha} G_\xi \notin D.$$ 

Since $F_\xi \in [s(\xi)]^{<\omega}$ and $G_\xi = \overrightarrow{T}_{\Pi, \xi}(F_\xi, \sigma(\xi))$, it must be the case that

$$\bigcup_{\xi < \alpha} F_\xi \notin C$$

by (P2).

\begin{corollary}
Let $A$, $B$, $C$, and $D$ be collections and $\alpha$ be an ordinal. Suppose there are functions

- $\overleftarrow{T}_1(\xi) : B \to A$ and
- $\overrightarrow{T}_{\Pi, \xi} : (\bigcup A) \times B \to \bigcup B$

for each $\xi < \alpha$ so that the following two properties hold.

(Ft1) If $x \in \overleftarrow{T}_1(\xi)(B)$, then $\overrightarrow{T}_{\Pi, \xi}(x, B) \in B$.

(Ft2) If $F_\xi \in \left[\overleftarrow{T}_1(\xi)(B)\right]^{<\omega}$ and $\bigcup_{\xi < \alpha} F_\xi \in C$, then $\bigcup_{\xi < \alpha} \overrightarrow{T}_{\Pi, \xi}(x, B_\xi) \in D$.

Then $G^\alpha_{\text{fin}}(A, C) \leq_{\Pi} G^\alpha_{\text{fin}}(B, D)$ and $G^\alpha_{\Pi}(A, C) \leq_{\Pi} G^\alpha_{\Pi}(B, D)$.

\end{corollary}

Proof. Suppose we have $\overleftarrow{T}_1(\xi)$ and $\overrightarrow{T}_{\Pi, \xi}$ as stated in the hypothesis. We define $\overleftarrow{S}_1(\xi) = \overleftarrow{T}_1(\xi)$ and $\overrightarrow{S}_{\Pi, \xi} : [\bigcup A]^{<\omega} \times B \to [\bigcup B]^{<\omega}$ by

$$\overrightarrow{S}_{\Pi, \xi}(\{x_1, \ldots, x_n\}, B) = \left\{ \overrightarrow{T}_{\Pi, \xi}(x_1, B), \ldots, \overrightarrow{T}_{\Pi, \xi}(x_n, B) \right\}.$$ 

Suppose $F \in \left[\overleftarrow{S}_1(\xi)(B)\right]^{<\omega}$, say $F = \{x_1, \ldots, x_n\}$. By (Ft1), each $\overrightarrow{T}_{\Pi, \xi}(x_j, B) \in B$, so

$$\overrightarrow{S}_{\Pi, \xi}(F, B) = \left\{ \overrightarrow{T}_{\Pi, \xi}(x_1, B), \ldots, \overrightarrow{T}_{\Pi, \xi}(x_n, B) \right\} \in [B]^{<\omega}.$$ 

This establishes (P1).
Now suppose $F_\xi \in \left[ \overrightarrow{S_{1,\xi}}(B_\xi) \right]^{<\omega}$ and $\bigcup_{\xi<\alpha} F_\xi \in \mathcal{C}$. Then $F_\xi \in \left[ \overrightarrow{T_{1,\xi}}(B_\xi) \right]^{<\omega}$, so

$$\bigcup_{\xi<\alpha} \overrightarrow{S_{1,\xi}}(F_\xi, B_\xi) = \bigcup_{\xi<\alpha} \left\{ \overrightarrow{T_{1,\xi}}(x, B_\xi) : x \in F_\xi \right\} \in \mathcal{D}$$

by (Ft2). This establishes (P2). So $\overrightarrow{S_{1,\xi}}$ and $\overrightarrow{S_{1,\xi}}$ witness that $G^\alpha_{\text{fin}}(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G^\alpha_{\text{fin}}(\mathcal{B}, \mathcal{D})$ by Theorem 16.

Now we need to establish that $G^\alpha_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G^\alpha_1(\mathcal{B}, \mathcal{D})$. First, notice that (Ft1) is identical to (Tr1). To see that (Tr2) is satisfied, let $x_\xi \in \overrightarrow{T_{1,\xi}}(B_\xi)$ for each $\xi < \alpha$ be so that $\{x_\xi : \xi < \alpha\} \in \mathcal{C}$. Let $\mathcal{F}_\xi = \{x_\xi\}$ and notice that $\mathcal{F}_\xi \in \left[ \bigcup \overrightarrow{T_{1,\xi}}(B_\xi) \right]^{<\omega}$ for each $\xi < \alpha$. Moreover,

$$\bigcup_{\xi<\alpha} \mathcal{F}_\xi = \{x_\xi : \xi < \alpha\} \in \mathcal{C}.$$

So, by (Ft2), we see that

$$\bigcup_{\xi<\alpha} \left\{ \overrightarrow{T_{1,\xi}}(x, B_\xi) : x \in \mathcal{F}_\xi \right\} = \left\{ \overrightarrow{T_{1,\xi}}(x_\xi, B_\xi) : \xi < \alpha \right\} \in \mathcal{D}.$$

That is, we can apply Theorem 14 to conclude that $G^\alpha_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G^\alpha_1(\mathcal{B}, \mathcal{D})$.

The following result is similar to Corollary 15.

**Corollary 18.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ be collections and $\alpha$ be an ordinal. Suppose there is a map $\varphi : [\bigcup \mathcal{B}] \times \alpha \to \bigcup \mathcal{A}$ so that the following two conditions hold.

- For all $B \in \mathcal{B}$ and all $\xi \in \alpha$, $\{\varphi(y, \xi) : y \in B\} \in \mathcal{A}$.
- If $G_\xi \in [B_\xi]^{<\omega}$ where $B_\xi \in \mathcal{B}$ for each $\xi < \alpha$ and $\bigcup_{\xi<\alpha} \varphi(G_\xi \times \{\xi\}) \in \mathcal{C}$, then $\bigcup_{\xi<\alpha} G_\xi \in \mathcal{D}$.

Then $G^\alpha_{\text{fin}}(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G^\alpha_{\text{fin}}(\mathcal{B}, \mathcal{D})$ and $G^\alpha_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G^\alpha_1(\mathcal{B}, \mathcal{D})$.

**Proof.** Define $\overrightarrow{T_{1,\xi}} : \mathcal{B} \to \mathcal{A}$ by

$$\overrightarrow{T_{1,\xi}}(B) = \varphi[B \times \{\xi\}] = \{\varphi(y, \xi) : y \in B\}.$$

Then $\overrightarrow{T_{1,\xi}}(B) \in \mathcal{A}$.

We now define $\overrightarrow{T_{1,\xi}} : [\bigcup \mathcal{A}] \times \mathcal{B} \to \bigcup \mathcal{B}$. Fix $B \in \mathcal{B}$ and, for $x \in \varphi[B \times \{\xi\}]$, choose $y_{x,\xi} \in B$ so that $\varphi(y_{x,\xi}, \xi) = x$. Then set $\overrightarrow{T_{1,\xi}}(x, B) = y_{x,\xi}$. For $x \notin \varphi[B \times \{\xi\}]$, set $\overrightarrow{T_{1,\xi}}(x, B)$ to be an arbitrary element of $\bigcup \mathcal{B}$. This guarantees that, if $x \in \overrightarrow{T_{1,\xi}}(B)$, then $\overrightarrow{T_{1,\xi}}(x, B) \in B$ which establishes (Ft1).

To check that (Ft2) holds, suppose $\mathcal{F}_\xi \in \left[ \overrightarrow{T_{1,\xi}}(B_\xi) \right]^{<\omega}$ and $\bigcup_{\xi<\alpha} \mathcal{F}_\xi \in \mathcal{C}$. Let

$$\mathcal{G}_\xi = \left\{ \overrightarrow{T_{1,\xi}}(x, B_\xi) : x \in \mathcal{F}_\xi \right\}$$

Notice that

$$x \in \mathcal{F}_\xi \implies x \in \overrightarrow{T_{1,\xi}}(B_\xi) \implies \mathcal{G}_\xi \in [B_\xi]^{<\omega}.$$  

Moreover, $\varphi[\mathcal{G}_\xi \times \{\xi\}] = \mathcal{F}_\xi$ since $\varphi(\overrightarrow{T_{1,\xi}}(x, B_\xi), \xi) = x$ for each $x \in \mathcal{F}_\xi$. That is, we have that

$$\bigcup_{\xi<\alpha} \varphi[\mathcal{G}_\xi \times \{\xi\}] = \bigcup_{\xi<\alpha} \mathcal{F}_\xi \in \mathcal{C}$$
which, by the hypotheses, allows us to conclude that

$$\bigcup_{\xi<\alpha} \left\{ T_{\mathcal{U}}(x, B_\xi) : x \in F_\xi \right\} = \bigcup_{\xi<\alpha} \mathcal{G}_\xi \in \mathcal{D}. $$

Hence, Corollary 17 applies and the proof is finished. 

Using these general translation theorems, we infer an easy to state, easy to apply, but less general version of the translation theorem. This version is the main tool we reference later to prove results about selection games on the Vietoris and Fell hyperspace topologies.

**Corollary 19.** Suppose that $A, B, C, D$ are collections so that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Additionally, suppose that there exists a bijection $\beta : \bigcup \mathcal{A} \to \bigcup \mathcal{B}$ with the following features:

- $A \in \mathcal{A} \iff \beta[A] \in \mathcal{B}$, and
- $C \in \mathcal{C} \iff \beta[C] \in \mathcal{D}$.

Let $\alpha$ be an ordinal. Then $G^\alpha_1(A, C) \equiv G^\alpha_1(B, D)$ and $G^\alpha_{\text{fin}}(A, C) \equiv G^\alpha_{\text{fin}}(B, D)$.

**Proof.** Note that, as $A \in \mathcal{A} \iff \beta[A] \in \mathcal{B}$, we also have $\beta^{-1}[B] \in \mathcal{A} \iff B \in \mathcal{B}$. Similarly, $\beta^{-1}[D] \in \mathcal{C} \iff D \in \mathcal{D}$. By symmetry, we need only show that $G^\alpha_1(A, C) \subseteq \mathcal{H} G^\alpha_1(B, D)$ and that $G^\alpha_{\text{fin}}(A, C) \subseteq \mathcal{H} G^\alpha_{\text{fin}}(B, D)$.

Define $\varphi : (\bigcup \mathcal{B}) \times \alpha \to \bigcup \mathcal{A}$ by the rule $\varphi(y, \xi) = \beta^{-1}(y)$. This is well-defined since $\beta$ is a bijection. Let $B \in \mathcal{B}$ and $\xi < \alpha$ be fixed. Then notice that

$$\{\varphi(y, \xi) : y \in B\} = \{\beta^{-1}(y) : y \in B\} = \beta^{-1}[B] \in \mathcal{A}. $$

Suppose $\varphi(y_\xi, \xi) : \xi < \alpha \in \mathcal{C}$. That is, $\{\beta^{-1}[y_\xi] : \xi < \alpha \} \subseteq \mathcal{C}$ and so $\{y_\xi : \xi < \alpha\} \subseteq \mathcal{D}$. Thus, by Corollary 15, $G^\alpha_1(A, C) \subseteq \mathcal{H} G^\alpha_1(B, D)$.

Suppose $\mathcal{G}_\xi \in [B_\xi]^{< \omega}$ and $\bigcup_{\xi<\alpha} \varphi[\mathcal{G}_\xi \times \{\xi\}] \in \mathcal{C}$. Notice that

$$\bigcup_{\xi<\alpha} \varphi[\mathcal{G}_\xi \times \{\xi\}] = \bigcup_{\xi<\alpha} \beta^{-1}[\mathcal{G}_\xi] = \beta^{-1} \left[ \bigcup_{\xi<\alpha} \mathcal{G}_\xi \right] \in \mathcal{C}, $$

so $\bigcup_{\xi<\alpha} \mathcal{G}_\xi \in \mathcal{D}$. Thus, by Corollary 18, $G^\alpha_{\text{fin}}(A, C) \subseteq \mathcal{H} G^\alpha_{\text{fin}}(B, D)$. 

**Remark 7.** Under the assumptions of Corollary 19, one can prove that $(A^\alpha) \iff (B^\alpha)$ for all $\alpha$ in a similar way to what was done above.

**4. Equivalences with Covers.**

**4.1. The Upper Topologies.**

**Lemma 20.** Let $\mathcal{A}$ be an ideal of closed subsets of $X$. $\mathcal{U} \in \mathcal{O}(X, \mathcal{A})$ if and only if $c.\mathcal{U} \in \mathcal{D}_{\mathcal{F}(X, A^+)}$.

**Proof.** Let $\mathcal{U} \in \mathcal{O}(X, \mathcal{A})$ and let $(X \setminus A)^+$ be an arbitrary open set in $\mathcal{F}(X, A^+)$. Then, since $\mathcal{U}$ is an $\mathcal{A}$-cover, there exists $U \in \mathcal{U}$ so that $A \subseteq U$. This is equivalent to $X \setminus U \subseteq X \setminus A$, which establishes that $c.\mathcal{U} \in \mathcal{D}_{\mathcal{F}(X, A^+)}$.

Now, suppose $\{X \setminus U : U \in \mathcal{U}\}$ is dense in the co-$\mathcal{A}$ hyperspace. As in the situation above, consider any $X \setminus A$ where $A \in \mathcal{A}$. There exists some $U \in \mathcal{U}$ so that $X \setminus U \subseteq X \setminus A$ by density, which means $A \subseteq U$. Hence, $\mathcal{U}$ is an $\mathcal{A}$-cover of $X$.

**Lemma 21.** Let $\mathcal{A}$ be an ideal of closed subsets of $X$ and $G \in \mathcal{F}(X)$. Then $\mathcal{U} \in \mathcal{O}(X, X \setminus G, \mathcal{A})$ if and only if $c.\mathcal{U} \in \mathcal{O}_{\mathcal{F}(X, A^+), G}$.

**Proof.** First suppose $\mathcal{U}$ is an $\mathcal{A}$-cover of $X \setminus G$ by open sets from $X$. Let $(X \setminus A)^+$ be an arbitrary open neighborhood of $G$ in $\mathcal{F}(X, A^+)$. Then $G \subseteq X \setminus A$ and so $A \subseteq X \setminus G$. Thus there exists $U \in \mathcal{U}$ so that $A \subseteq U$. So $(X \setminus U) \subseteq (X \setminus A)$. Now $X \setminus U \in c.\mathcal{U} \cap (X \setminus A)^+$. Therefore $c.\mathcal{U} \in \mathcal{O}_{\mathcal{F}(X, A^+), G}$.
Now, suppose \(c.\mathcal{U} \in \Omega_{F(X,A^+)} G\). Let \(A \in \mathcal{A}\) be so that \(A \subseteq X \setminus G\). Then \(G \subseteq X \setminus A\) and so \((X \setminus A)^+\) is an open neighborhood of \(G\). So there exists some \(U \in \mathcal{U}\) so that \(X \setminus U \subseteq X \setminus A\), which means \(A \subseteq U\). Hence, \(\mathcal{U}\) is a \(A\)-cover of \(X \setminus G\) by open sets from \(X\).

**Lemma 22.** Let \(A\) be an ideal of closed subsets of \(X\). \(\mathcal{U} \in \mathcal{O}^{gp}(X,A)\) if and only if \(c.\mathcal{U} \in D_{F(X,A^+)}^{gp}\).

**Proof.** First suppose \(\mathcal{U}\) is an \(A\)-cover of \(X\) and set \(D = c.\mathcal{U}\). Let \(\varphi : \mathcal{U} \to |\mathcal{U}|\) be a partition of \(\mathcal{U}\) as in the definition of \(\mathcal{U}\). Define \(\psi : D \to |\mathcal{U}|\) by \(\psi(F) = \varphi(X \setminus F)\). Notice that \(X \setminus U \in \psi^{-1}(\xi)\) if and only if \(U \in \varphi^{-1}(\xi)\). Let \((X \setminus A)^+\) be an arbitrary open set in \(F(X,A^+)\). Thus there exists a \(\xi \in |\mathcal{U}|\) so that for all \(\xi \geq \xi_0\), there is a \(U_\xi \in \varphi^{-1}(\xi)\) so that \(A \subseteq U_\xi\). So \(X \setminus U_\xi \in \psi^{-1}(\xi)\) and \((X \setminus U_\xi) \subseteq (X \setminus A)\). Therefore \(c.\mathcal{U} \in D_{F(X,A^+)}^{gp}\).

Now, suppose \(c.\mathcal{U} \in D_{F(X,A^+)}^{gp}\) and set \(D = c.\mathcal{U}\). Let \(\varphi : D \to |\mathcal{U}|\) be a partition of \(D\) as in the definition of \(\mathcal{U}\). Define \(\psi : D \to |\mathcal{U}|\) by \(\psi(U) = \varphi(X \setminus U)\). Notice that \(F \in \psi^{-1}(\xi)\) if and only if \(X \setminus U \in \psi^{-1}(\xi)\). Let \(V = (X \setminus A)^+\) be an arbitrary open set in \(F(X,A^+)\). Thus there exists a \(\xi \in |\mathcal{U}|\) so that \(A \subseteq \bigcup \varphi^{-1}(\xi)\). So \(\bigcap \varphi^{-1}(\xi) = X \setminus \bigcup \varphi^{-1}(\xi) \subseteq X \setminus A\).

Hence, \(\bigcap \varphi^{-1}(\xi) \in V\). Therefore \(c.\mathcal{U} \in D_{F(X,A^+)}^{gp}\).

**Lemma 23.** Let \(A\) be an ideal of closed subsets of \(X\). Then \(\mathcal{U} \in \mathcal{O}^{ugp}(X,A)\) if and only if \(c.\mathcal{U} \in D_{F(X,A^+)}^{ugp}\).

**Proof.** First suppose \(\mathcal{U}\) is a weakly \(A\)-cover of \(X\) and set \(D = c.\mathcal{U}\). Let \(\varphi : \mathcal{U} \to |\mathcal{U}|\) be a partition of \(\mathcal{U}\) as in the definition of weakly \(\mathcal{U}\). Define \(\psi : D \to |\mathcal{U}|\) by \(\psi(U) = \varphi(X \setminus U)\). Notice that \(F \in \psi^{-1}(\xi)\) if and only if \(X \setminus U \in \psi^{-1}(\xi)\). Let \(V = (X \setminus A)^+\) be an open set in \(F(X,A^+)\). So there is an \(\xi \in |\mathcal{U}|\) so that \(A \subseteq \bigcup \varphi^{-1}(\xi)\). So \(\bigcap \varphi^{-1}(\xi) = X \setminus \bigcup \varphi^{-1}(\xi) \subseteq X \setminus A\).

Hence, \(\bigcap \varphi^{-1}(\xi) \in V\). Therefore \(c.\mathcal{U} \in D_{F(X,A^+)}^{ugp}\).

**Lemma 24.** Fix a space \(X\) and a collection \(\mathcal{F} = \{(X \setminus A)^+ : A \in S\}\) of open subsets of \(F(X,A^+)\). Then \(\mathcal{F} \in \mathcal{O}_{F(X,A^+)}\) if and only if \(S \subseteq \Pi_A(X)\).

**Proof.** First suppose \(\mathcal{F} \in \mathcal{O}_{F(X,A^+)}\). Let \(U \subseteq X\) be open. Then there is an \(A \in S\) so that \(X \setminus U \in (X \setminus A)^+\). Thus \(X \setminus U \subseteq X \setminus A\) and so \(A \subseteq U\). Therefore, \(S \subseteq \Pi_A(X)\).

Now suppose that \(S \subseteq \Pi_A(X)\) and let \(F \in F(X,A^+)\). Then \(X \setminus F\) is open, so there is an \(A \in S\) so that \(A \subseteq X \setminus F\). Thus \(F \subseteq X \setminus A\) and so \(F \in (X \setminus A)^+\). Hence, \(\mathcal{F} \in \mathcal{O}_{F(X,A^+)}\).

**4.2. The Full Topologies.**

**Lemma 25.** For any space \(X\) and any ideal \(A\) consisting of closed sets, \(\mathcal{U} \in \mathcal{O}_F(X,A)\) if and only if \(c.\mathcal{U} \in D_{F(X,A)}\).

**Proof.** Suppose \(\mathcal{U}\) is a \(A\)-cover of \(X\). Let \([A;V_1,\ldots,V_n]\) be a non-empty basic open set in \(F(X,A)\). Then by Proposition 10, \(V_j \cap (X \setminus A) \neq \emptyset\) for all \(j\). Since \(\mathcal{U}\) is a \(A\)-cover of \(X\), we can find \(U \in \mathcal{U}\) and \(E \subseteq X\) finite so that \(A \subseteq U\), \(E \cap V_j \neq \emptyset\) for each \(j\), and \(E \cap U = \emptyset\). Set \(F = X \setminus U\) and notice
that \( F \in c.\mathcal{U}, F \subseteq X \setminus A \), and since \( E \cap (X \setminus A) = \emptyset, E \subseteq F \). Hence, \( \emptyset \neq E \cap V_j \subseteq F \cap V_j \) for each \( j \). That is, \( F \in [A; V_0, \ldots, V_n] \), so \( c.\mathcal{U} \) is dense.

Now suppose \( \mathcal{D} = c.\mathcal{U} \) is dense in \( \mathbb{F}(X, \mathcal{A}) \). Let \( A \in \mathcal{A} \) and \( V_1, V_2, \ldots, V_n \subseteq X \) be open so that \( V_j \cap (X \setminus A) \neq \emptyset \) for each \( j \). Notice that the basic open set \([A; V_1, \ldots, V_n]\) is non-empty, so since \( \mathcal{D} \) is dense, we can find \( F \in \mathcal{D} \cap [A; V_1, \ldots, V_n] \). Then \( F \subseteq X \setminus A \) and \( F \cap V_j \neq \emptyset \) for each \( j \). Immediately, we see that \( A \subseteq X \setminus F \). Set \( U = X \setminus F \) and notice that \( U \in \mathcal{U} \). Let \( x_j \in F \cap V_j \) and \( E = \{x_1, x_2, \ldots, x_n\} \). Clearly, \( E \cap V_j \neq \emptyset \) for each \( j \) and, since \( E \subseteq F, E \cap U = \emptyset \). That is, \( c.\mathcal{U} \) is an \( A_F \)-cover of \( X \). \( \square \)

**Lemma 26.** For any space \( X \), any closed set \( G \in \mathbb{F}(X) \), and any ideal \( \mathcal{A} \) consisting of closed sets, \( \mathcal{U} \in \mathcal{O}_F(X, X \setminus G, \mathcal{A}) \) if and only if \( c.\mathcal{U} \in \Omega_{\mathbb{F}(X, \mathcal{A})} \).

**Proof.** Suppose \( \mathcal{U} \in \mathcal{O}_F(X, X \setminus G, \mathcal{A}) \) and let \([A; V_1, \ldots, V_n]\) be a basic neighborhood of \( G \) in \( \mathbb{F}(X, \mathcal{A}) \). Since \( G \subseteq X \setminus A, A \subseteq X \setminus G \). Also \( V_j \cap G \neq \emptyset \) for each \( j \). Since \( \mathcal{U} \) is a \( A_F \)-cover of \( X \setminus G \), we can find \( U \in \mathcal{U} \) and \( E \subseteq X \) finite so that \( A \subseteq U, E \cap V_j \neq \emptyset \) for each \( j \), and \( E \cap U = \emptyset \). Set \( F = X \setminus U \) and notice that \( F \in c.\mathcal{U} \). Immediately, \( F \subseteq X \setminus A \). Moreover, since \( E \cap U = \emptyset \), we see that \( E \subseteq F \). Hence, for each \( j, x_j \neq E \cap V_j \subseteq F \cap V_j \). Therefore, \( F \in [A; V_1, \ldots, V_n] \), and so \( G \) is in the closure of \( c.\mathcal{U} \).

Now suppose \( c.\mathcal{U} \in \Omega_{\mathbb{F}(X, \mathcal{A})} \). Consider \( A \in \mathcal{A} \) with \( A \subseteq (X \setminus G) \) and \( V_1, \ldots, V_n \subseteq X \) open with \( V_j \cap (X \setminus (X \setminus G)) = V_j \cap G \neq \emptyset \) for all \( j \). Notice that \( G \in [A; V_1, \ldots, V_n] \). Thus we can find \( F \in c.\mathcal{U} \) so that both \( F \subseteq X \setminus A \) and \( F \cap V_j \neq \emptyset \) for each \( j \). Immediately, we see that \( A \subseteq X \setminus \mathcal{U} \). Set \( U = X \setminus F \) and notice that \( U \in \mathcal{U} \). For each \( j \), choose \( x_j \in F \cap V_j \). Then let \( E = \{x_1, \ldots, x_n\} \). Notice that \( E \) is a finite set with \( E \cap V_j \neq \emptyset \) for each \( j \). Moreover, \( E \subseteq F \) which means that \( E \cap U = \emptyset \). Therefore \( \mathcal{U} \in \mathcal{O}_F(X, X \setminus G, \mathcal{A}) \). \( \square \)

**Lemma 27.** For any space \( X \) and any ideal \( \mathcal{A} \) consisting of closed sets, \( \mathcal{U} \in \mathcal{O}_F^g(X, \mathcal{A}) \) if and only if \( c.\mathcal{U} \in \mathcal{D}_F^g(X, \mathcal{A}) \).

**Proof.** Suppose \( \mathcal{U} \in \mathcal{O}_F^g(X, \mathcal{A}) \). Let \( \varphi : \mathcal{U} \rightarrow |\mathcal{U}| \) be the partition of \( \mathcal{U} \) into finite sets as specified in the definition of \( \mathcal{O}_F^g(X, \mathcal{A}) \). Set \( \mathcal{D} = c.\mathcal{U} \), notice that \( |\mathcal{D}| = |\mathcal{U}| \), and define \( \psi : \mathcal{D} \rightarrow |\mathcal{D}| \) by \( \psi(F) = \varphi(X \setminus F) \). Notice that this partitions \( \mathcal{D} \) into finite sets and \( F \in \psi^{-1}(\xi) \) if and only if \( X \setminus F \in \varphi^{-1}(\xi) \). Let \([A; V_1, \ldots, V_n]\) be a non-empty basic open set in \( \mathbb{F}(X, \mathcal{A}) \). Then by Proposition 10, \( V_j \cap (X \setminus A) \neq \emptyset \) for all \( j \). Thus we can find a \( \xi_0 < |\mathcal{D}| \) so that for all \( \xi \geq \xi_0 \), there are \( U_\xi \in \varphi^{-1}(\xi) \) and \( E_\xi \subseteq X \) finite so that \( A \subseteq U_\xi, E_\xi \cap V_j \neq \emptyset \) for each \( j \), and \( E_\xi \cap U_\xi = \emptyset \). Set \( F_\xi = X \setminus U_\xi \) and notice that \( F_\xi \in \psi^{-1}(\xi) \), \( F_\xi \subseteq X \setminus A \), and since \( E_\xi \cap (X \setminus F_\xi) = \emptyset, E_\xi \subseteq F_\xi \). Hence, \( \emptyset = E_\xi \cap V_j \subseteq F_\xi \cap V_j \) for each \( j \). Therefore, \( F_\xi \in [A; V_0, \ldots, V_m] \), and so \( \mathcal{D} \in \mathcal{D}_F^g(X, \mathcal{A}) \). \( \square \)

**Lemma 28.** Let \( A \) be an ideal of closed subsets of \( X \). \( \mathcal{U} \in \mathcal{O}_F^{w^g}(X, \mathcal{A}) \) if and only if \( c.\mathcal{U} \in \mathcal{D}_F^{w^g}(X, \mathcal{A}) \).

**Proof.** Suppose \( \mathcal{U} \in \mathcal{O}_F^{w^g}(X, \mathcal{A}) \). Let \( \varphi : \mathcal{U} \rightarrow |\mathcal{U}| \) be the partition of \( \mathcal{U} \) into finite sets as specified in the definition of \( \mathcal{O}_F^{w^g}(X, \mathcal{A}) \). Set \( \mathcal{D} = c.\mathcal{U} \) and, noting that \( |\mathcal{D}| = |\mathcal{U}| \), define \( \psi : \mathcal{D} \rightarrow |\mathcal{U}| \) by \( \psi(F) = \varphi(X \setminus F) \). Notice that this partitions \( \mathcal{D} \) into finite sets and \( F \in \psi^{-1}(\xi) \) if and only if \( X \setminus F \in \varphi^{-1}(\xi) \). Let \([A; V_1, \ldots, V_n]\) be a non-empty basic open set in \( \mathbb{F}(X, \mathcal{A}) \). Then by Proposition
Proof. Suppose $\mathcal{Y} \in \Pi^\xi_F(X)$. Let $F \in \mathcal{F}(X)$ be arbitrary. Then we can find $\langle A, V_1, \ldots, V_n \rangle \in \mathcal{Y}$ and a finite set $E \subseteq X$ so that $E \cap V_j \neq \emptyset$ for each $j$, $A \subseteq X \setminus F$, and $E \cap (X \setminus F) = \emptyset$. Notice that $F \subseteq X \setminus A$ and that $\emptyset \neq E \cap V_j \subseteq F \cap V_j$ for each $j$. That is, $F \in [A; V_1, \ldots, V_n]$. Set $U = X \setminus F$ and let $x_j \in V_j \cap (X \setminus U)$ for each $j$ and let $E = \{x_1, \ldots, x_n\}$. Observe that $A \subseteq U$, $E \cap V_j \neq \emptyset$ for each $j$, and $E \cap U = \emptyset$. This finishes the proof. □

Lemma 29. Fix a space $X$ and an ideal $A \subseteq \mathcal{F}(X)$. Then $\mathcal{Y} \in \Pi^\xi_F(X)$ if and only if

$\{[A; V_1, \ldots, V_n] : \langle A, V_1, \ldots, V_n \rangle \in \mathcal{Y} \} \in \mathcal{O}_{\mathcal{F}(X,A)}$. 

Proof. This follows immediately from Proposition 13 and Lemma 29. □

Corollary 30. For any space $X$,

(i) $\mathcal{Y}$ is a $\pi^\alpha_F$-network if and only if $\{[K; V_1, \ldots, V_n] : \langle K, V_1, \ldots, V_n \rangle \in \mathcal{Y} \} \in \mathcal{F}(X,F)$,

(ii) $\mathcal{Y}$ is a $\pi^\alpha_V$-network on $X$, then $\{\bigcap_{i=1}^n (X \setminus V_j) ; V_1, \ldots, V_n \} : \langle V_1, \ldots, V_n \rangle \in \mathcal{Y} \} \in \mathcal{O}_{\mathcal{F}(X,V)}$, and

(iii) $\{[A; V_1, \ldots, V_n] : \langle A, V_1, \ldots, V_n \rangle \in \mathcal{Y} \} \in \mathcal{O}_{\mathcal{F}(X,V)}$, then $\{\langle X \setminus A, V_1, \ldots, V_n \rangle : \langle A, V_1, \ldots, V_n \rangle \in \mathcal{Y} \} \in \mathcal{O}_{\mathcal{F}(X,V)}$.

is a $\pi^\alpha_V$-network on $X$.

Proof. This follows immediately from Proposition 13 and Lemma 29. □

5. Applications of Cover Lemmas

5.1. The Upper Topologies.

Theorem 31. Fix a topological space $X$, $G \in \mathcal{F}(X)$, ideals $A$ and $B$ consisting of closed sets, $\alpha$ an ordinal, and a symbol $\square \in \{1, \infty\}$. Then

(i) $G^\alpha_{\square}((O(X,A), O(X,B)) = G^\alpha_{\square}(D_{\mathcal{F}(X,A^+), \mathcal{D}(X,B^+)})$,

(ii) $G^\alpha_{\square}((O(X,A), O(X,G,A)) = G^\alpha_{\square}(O_{\mathcal{F}(X,A^+,G), \Omega_{\mathcal{F}(X,B^+,G)})$,

(iii) $G^\alpha_{\square}((O(X,A), O^{gp}(X,B)) = G^\alpha_{\square}(D_{\mathcal{F}(X,A^+), \mathcal{D}^{gp}_{\mathcal{F}(X,B^+)})$,

(iv) $G^\alpha_{\square}((O(X,A), O^{gp}(X,B)) = G^\alpha_{\square}(D_{\mathcal{F}(X,A^+), \mathcal{D}^{gp}_{\mathcal{F}(X,B^+)})$, and

(v) if $B \subseteq A$, $G^\alpha_{\square}(\Pi_A(X), \Pi_B(X)) = G^\alpha_{\square}(D_{\mathcal{F}(X,A^+), \mathcal{O}_{\mathcal{F}(X,B^+)})$. 


Proof. Notice that $\bigcup \mathcal{O}(X, \mathcal{B}) = \bigcup \mathcal{O}(X, \mathcal{A}) = \mathcal{F}(X)$ and that $\bigcup \mathcal{D}(X, \mathcal{B}^+) = \bigcup \mathcal{D}(X, \mathcal{A}^+) = \mathcal{F}(X)$. Define $\beta_1 : \mathcal{F}(X) \to \mathcal{F}(X)$ by $\beta_1(U) = X \setminus U$. By Lemma 20, $\beta_1$ satisfies the conditions of Corollary 19 for $\mathcal{O}(X, \mathcal{A}), \mathcal{D}(X, \mathcal{A}^+)$, $\mathcal{O}(X, \mathcal{B})$, and $\mathcal{D}(X, \mathcal{B}^+)$. This proves (i).

Equivalences (ii), (iii), and (iv) follow in a similar way. The function $\beta_1$ is still used and we instead reference Lemmas 21, 22, and 23.

To prove part (v), we first note that $\bigcup \Pi_\mathcal{A}(X) = \mathcal{A}$, $\bigcup \Pi_\mathcal{B}(X) = \mathcal{B}$, $\bigcup \mathcal{O}_\mathcal{F}(X, \mathcal{A}^+) = \mathcal{F}(X)$, and $\bigcup \mathcal{O}_\mathcal{F}(X, \mathcal{B}^+) = \mathcal{F}(X)$. The fact that $\mathcal{F}(X)$ is an ideal of closed sets and we denote:\[ \beta_2(A) = (X \setminus A)^+ \].

We first show that $\beta_2$ is a bijection. Suppose $A_1, A_2 \in \mathcal{A}$ and $A_1 \neq A_2$. Without loss of generality, let $x \in A_1 \setminus A_2$. Then $\{x\} \in (X \setminus A_2)^+$ but $\{x\} \notin (X \setminus A_1)^+$. So $\beta_2(A_1) \neq \beta_2(A_2)$. This shows that $\beta_2$ is injective. The fact that $\beta_2$ is surjective follows from the fact that every open set in $\mathcal{F}(X)$ has the form $(X \setminus A)^+$, because $\mathcal{A}$ is an ideal of closed sets. The other hypotheses of Corollary 19 follow from Lemma 24.

Corollary 32. Fix a topological space $X$ and $G \in \mathcal{F}(X)$. Then

| Rothberger-Like Games |
|------------------------|
| $G_1(\mathcal{O}(X, \mathcal{K}(X)), \mathcal{O}(X, \mathcal{K}(X))) \equiv G_1(\mathcal{D}(X, \mathcal{F}^+), \mathcal{D}(X, \mathcal{F}^+))$ |
| $G_1(\mathcal{O}(X, [X]^{<\omega}), \mathcal{O}(X, [X]^{<\omega})) \equiv G_1(\mathcal{D}(X, \mathcal{Z}^+), \mathcal{D}(X, \mathcal{Z}^+))$ |
| $G_1(\mathcal{O}(X, \mathcal{K}(X)), \mathcal{O}(X, [X]^{<\omega})) \equiv G_1(\mathcal{D}(X, \mathcal{F}^+), \mathcal{D}(X, \mathcal{Z}^+))$ |

| Menger-Like Games |
|---------------------|
| $G_{\text{fin}}(\mathcal{O}(X, \mathcal{K}(X)), \mathcal{O}(X, \mathcal{K}(X))) \equiv G_{\text{fin}}(\mathcal{D}(X, \mathcal{F}^+), \mathcal{D}(X, \mathcal{F}^+))$ |
| $G_{\text{fin}}(\mathcal{O}(X, [X]^{<\omega}), \mathcal{O}(X, [X]^{<\omega})) \equiv G_{\text{fin}}(\mathcal{D}(X, \mathcal{Z}^+), \mathcal{D}(X, \mathcal{Z}^+))$ |
| $G_{\text{fin}}(\mathcal{O}(X, \mathcal{K}(X)), \mathcal{O}(X, [X]^{<\omega})) \equiv G_{\text{fin}}(\mathcal{D}(X, \mathcal{F}^+), \mathcal{D}(X, \mathcal{Z}^+))$ |

| Hurewicz-Like Games and Weakly Groupable Games |
|-----------------------------------------------|
| $G_1(\mathcal{O}(X, \mathcal{K}(X)), \mathcal{O}^{\text{op}}(X, \mathcal{K}(X))) \equiv G_1(\mathcal{D}(X, \mathcal{F}^+), \mathcal{D}(X, \mathcal{F}^+))$ |
| $G_1(\mathcal{O}(X, [X]^{<\omega}), \mathcal{O}^{\text{op}}(X, [X]^{<\omega})) \equiv G_1(\mathcal{D}(X, \mathcal{Z}^+), \mathcal{D}(X, \mathcal{Z}^+))$ |
| Equation                                                                 | Equivalent                                                                 |
|-------------------------------------------------------------------------|---------------------------------------------------------------------------|
| \( G_{\text{fin}}(\mathcal{O}(X), \mathcal{K}(X)), \mathcal{O}^{\text{gp}}(X, \mathcal{K}(X)) ) \equiv \ G_{\text{fin}}(\mathcal{D}_{F}(X,F^{+}), \mathcal{D}_{F}^{\text{gp}}(X,F^{+}) ) \) |
| \( G_{\text{fin}}(\mathcal{O}(X), [X]^{<\omega}), \mathcal{O}^{\text{gp}}(X, [X]^{<\omega}) ) \equiv \ G_{\text{fin}}(\mathcal{D}_{F}(X,Z^{+}), \mathcal{D}_{F}^{\text{gp}}(X,Z^{+}) ) \) |
| \( G_{1}(\mathcal{O}(X, \mathcal{K}(X)), \mathcal{O}^{\text{gp}}(X, \mathcal{K}(X))) \equiv \ G_{1}(\mathcal{D}_{F}(X,F^{+}), \mathcal{D}_{F}^{\text{gp}}(X,F^{+}) ) \) |
| \( G_{1}(\mathcal{O}(X, [X]^{<\omega}), \mathcal{O}^{\text{gp}}(X, [X]^{<\omega}) ) \equiv \ G_{1}(\mathcal{D}_{F}(X,Z^{+}), \mathcal{D}_{F}^{\text{gp}}(X,Z^{+}) ) \) |
| \( G_{\text{fin}}(\mathcal{O}(X), \mathcal{K}(X)), \mathcal{O}^{\text{gp}}(X, \mathcal{K}(X)) ) \equiv \ G_{\text{fin}}(\mathcal{D}_{F}(X,F^{+}), \mathcal{D}_{F}^{\text{gp}}(X,F^{+}) ) \) |
| \( G_{\text{fin}}(G_{1}(\mathcal{O}(X, [X]^{<\omega}), \mathcal{O}^{\text{gp}}(X, [X]^{<\omega}) )) \equiv \ G_{\text{fin}}(G_{1}(\mathcal{D}_{F}(X,Z^{+}), \mathcal{D}_{F}^{\text{gp}}(X,Z^{+}) ) \) |

**Notation.** For \( A \subseteq X \), let \( \mathcal{N}(A) \) be all open sets \( U \) so that \( A \subseteq U \) and let \( \mathcal{N}_{g} = \mathcal{N}([x]) \). Set \( \mathcal{N}[X] = \{ \mathcal{N}_{g} : x \in X \} \), and in general if \( A \) is a collection of subsets of \( X \), then \( \mathcal{N}[A] = \{ \mathcal{N}(A) : A \in A \} \). In the case when \( X \) and \( X' \) represent two topologies on the same underlying set, we will use the notation \( \mathcal{N}(X) \) to denote the collection of open sets relative to the topology according to \( X \) that contain \( A \).

The following corollary ties this discussion of upper-\( A \) topologies to selection games involving the space of continuous functions as witnessed by Theorems 25 and 26 of [3]. Recall that \( G_{1}(\mathcal{N}[A], \neg \mathcal{O}(X, A)) \) is a generalized version of the point-open game.

**Corollary 33.** Let \( \mathcal{A} \) be an ideal of closed subsets of a space \( X \). The games \( G_{1}(\mathcal{N}[A], \neg \mathcal{O}(X, A)) \) and \( G_{1}(\mathcal{G}_{E}(X,\mathcal{A}^{+}), \neg \mathcal{D}_{E}(X,\mathcal{A}^{+})) \) are equivalent.

**Proof.** The duality of \( G_{1}(\mathcal{N}[A], \neg \mathcal{O}(X, A)) \) and \( G_{1}(\mathcal{O}(X, A), \mathcal{O}(X, A)) \) is the conclusion of [3, Cor. 18] as a direct application of [4, Cor. 26]. Then note that \( G_{1}(\mathcal{O}(X, A), \mathcal{O}(X, A)) \) and \( G_{1}(\mathcal{D}_{E}(X,\mathcal{A}^{+}), \mathcal{D}_{E}(X,\mathcal{A}^{+})) \) are equivalent by Theorem 31. The duality between \( G_{1}(\mathcal{D}_{E}(X,\mathcal{A}^{+}), \mathcal{D}_{E}(X,\mathcal{A}^{+})) \) and \( G_{1}(\mathcal{G}_{E}(X,\mathcal{A}^{+}), \neg \mathcal{D}_{E}(X,\mathcal{A}^{+})) \) is a direct application of [4, Prop. 32]. \( \square \)

**Corollary 34.** For any space \( X \),

- the games \( G_{1}(\mathcal{N}[\mathcal{K}(X)], \neg \mathcal{O}(X, \mathcal{K}(X))) \) and \( G_{1}(\mathcal{G}_{E}(X,\mathcal{A}^{+}), \neg \mathcal{D}_{E}(X,\mathcal{A}^{+})) \) are equivalent, and

- the games \( G_{1}(\mathcal{N}[\mathcal{K}(X)], \neg \mathcal{O}(X, \mathcal{K}(X))) \) and \( G_{1}(\mathcal{G}_{E}(X,\mathcal{A}^{+}), \neg \mathcal{D}_{E}(X,\mathcal{A}^{+})) \) are equivalent.

**5.2. The Full Topologies.**

**Theorem 35.** Fix a topological space \( X, G \in \mathcal{F}(X) \), ideals \( \mathcal{A} \) and \( \mathcal{B} \) consisting of closed sets, \( \alpha \) an ordinal, and a symbol \( \square \in \{ 1, \text{fin} \} \). Then

\[
\begin{align*}
(i) \quad & G_{\square}(\mathcal{O}_{F}(X,A), \mathcal{O}_{F}(X,B)) \equiv G_{\square}(\mathcal{D}_{E}(X,A), \mathcal{D}_{E}(X,B)), \\
(ii) \quad & G_{\square}(\mathcal{O}_{F}(X,X \setminus G, A), \mathcal{O}_{F}(X \setminus G, B)) \equiv G_{\square}(\mathcal{O}(X,A,G), \mathcal{O}(X,B,G)), \\
(iii) \quad & G_{\square}(\mathcal{O}_{F}(X,A), \mathcal{O}_{F}^{\text{gp}}(X,B)) \equiv G_{\square}(\mathcal{D}_{E}(X,A), \mathcal{D}_{E}^{\text{gp}}(X,B)), \\
(iv) \quad & G_{\square}(\mathcal{O}_{F}(X,A), \mathcal{O}_{F}^{\text{gp}}(X,B)) \equiv G_{\square}(\mathcal{D}_{E}(X,A), \mathcal{D}_{E}^{\text{gp}}(X,B)), \\
(v) \quad & \text{if } B \subseteq A, \text{ then } G_{\square}(\mathcal{D}_{E}(X,A), \mathcal{O}(X,B)) \leq \text{II } G_{\square}(\mathcal{D}_{E}(X,A), \mathcal{O}(X,B)), \text{ and} \\
(vi) \quad & \text{if } A \subseteq B, \text{ then } G_{\square}(\mathcal{D}_{E}(X,A), \mathcal{O}(X,B)) \leq \text{II } G_{\square}(\mathcal{D}_{E}(X,A), \mathcal{O}(X,B)).
\end{align*}
\]

**Proof.** Notice that \( \bigcup \mathcal{O}(X,B) = \bigcup \mathcal{O}(X,A) = \mathcal{G}_{X} \) and that \( \bigcup \mathcal{D}_{E}(X,B) = \bigcup \mathcal{D}_{E}(X,A) = \mathcal{F}(X) \). Define \( \beta : \mathcal{G}_{X} \to \mathcal{F}(X) \) by \( \beta(U) = X \setminus U \). By Lemma 25, \( \beta_{1} \) satisfies the conditions of Corollary 19 for \( \mathcal{O}_{F}(X,A), \mathcal{D}_{E}(X,A), \mathcal{O}(X,B), \) and \( \mathcal{D}_{E}(X,B) \). This proves (i).

The equivalences (ii), (iii), and (iv) follow in a similar way using the same \( \beta \) but instead referencing Lemmas 26, 27, and 28.

To prove part (v), we first note that

- \( \bigcup \mathcal{O}_{E}(X,A) = \mathcal{G}_{A}, \bigcup \mathcal{O}_{E}(X,B) = \mathcal{G}_{B}, \)
We check that $\varphi$ satisfies the hypotheses of Corollary 18. Suppose $\mathcal{Y} \in \Pi^F_A(X)$ and $\xi \in \alpha$. Then by Lemma 29,

$$\{\varphi((A, V_1, \ldots, V_n), \xi) : (A, V_1, \ldots, V_n) \in \mathcal{Y}\} = \{[A; V_1, \ldots, V_n] : A \in \mathcal{Y}\}$$

is an open cover of $\mathbb{F}(X, A)$. Next suppose that $\mathcal{Y}_\xi \in \Pi^F_B(X)$ for each $\xi < \alpha$, that $\mathcal{G}_\xi$ is a finite subset of $\mathcal{Y}_\xi$, and that $\bigcup_{\xi < \alpha} \varphi[G_\xi \times \{\xi\}]$ is an open cover of $\mathbb{F}(X, B)$. We need to show that $\bigcup_{\xi < \alpha} \mathcal{G}_\xi \in \Pi^F_B(X)$. Write

$$\mathcal{G}_\xi = \{\langle A, k \rangle, V_1, \ldots, V_n(k), k \rangle : k \leq m(\xi)\}$$

Then $\{[A, k \rangle; V_1, \ldots, V_n(k), k \rangle : k \leq m(\xi) \text{ and } \xi \in \alpha\}$ is an open cover of $\mathbb{F}(X, B)$. So by Lemma 29, $\bigcup_{\xi < \alpha} \mathcal{G}_\xi \in \Pi^F_B(X)$. Thus $G^\alpha(\mathcal{O}_F(X, A), \mathcal{O}_F(X, B)) \leq G^\alpha(\Pi^F_A(X), \Pi^F_B(X))$.

Finally, we prove (vi). Define $\tilde{T}_{\Pi, \xi} : \mathcal{O}_F(X, A) \to \Pi^F_A(X)$ by

$$\tilde{T}_{\Pi, \xi}(\mathcal{Y}) = \{\langle A, V_1, \ldots, V_n \rangle, (\exists U \in \mathcal{Y})[[A; V_1, \ldots, V_n] \subseteq U]\}.$$

We first show that $\tilde{T}_{\Pi, \xi}(\mathcal{Y}) \in \Pi^F_A(X)$. Notice that $\mathcal{Y}$ can be refined to a cover $\mathcal{Y}'$ of basic open sets and that $\tilde{T}_{\Pi, \xi}(\mathcal{Y}')$ a subset of $\tilde{T}_{\Pi, \xi}(\mathcal{Y})$. By Lemma 29, $\tilde{T}_{\Pi, \xi}(\mathcal{Y}') \in \Pi^F_A(X)$. So $\tilde{T}_{\Pi, \xi}(\mathcal{Y})$ is, as well. Define $\tilde{T}_{\Pi, \xi} : \mathcal{A}_X \to \mathcal{O}_F(X, A) \to \mathcal{B}_X$ as follows. If $W = [A; V_1, \ldots, V_n]$ is a subset of some $U \in \mathcal{Y}$, choose $U_\mathcal{W} \in \mathcal{W}$ so that $W \subseteq U_\mathcal{W} \subseteq U$. Then

$$\tilde{T}_{\Pi, \xi}((A, V_1, \ldots, V_n), \mathcal{Y}) = U_{[A; V_1, \ldots, V_n], \mathcal{Y}}$$

if there is a $U \in \mathcal{Y}$ so that $[A; V_1, \ldots, V_n] \subseteq U$, and set $\tilde{T}_{\Pi, \xi}((A, V_1, \ldots, V_n), \mathcal{Y}) = \mathbb{F}(X)$ otherwise.

Since $A \subseteq B$, Lemma 11 implies that $\mathcal{A}_X \subseteq \mathcal{B}_X$. So $\tilde{T}_{\Pi, \xi}$ has the correct co-domain.

Now suppose $\mathcal{W} \in \mathcal{O}_F(X, A)$ and that $Y \in \tilde{T}_{\Pi, \xi}(\mathcal{W})$. Say $Y = \langle A, V_1, \ldots, V_n \rangle$. We define $W = [A; V_1, \ldots, V_n]$. Then, by the definition of $U_{W, \mathcal{Y}}$, $\tilde{T}_{\Pi, \xi}(Y, \mathcal{W}) = U_{W, \mathcal{Y}} \in \mathcal{Y}$.

Finally suppose $\mathcal{W}_\xi \in \mathcal{O}_F(X, A)$ and that $Y_{\xi, 1}, \ldots, Y_{\xi, n(\xi)} \in \tilde{T}_{\Pi, \xi}(\mathcal{W}_\xi)$ for $\xi < \alpha$. Say

$$Y_{\xi, k} = \langle A_{\xi, k}, V_{\xi, k, 1}, \ldots, V_{\xi, k, m(\xi, k)} \rangle.$$

Suppose $\{Y_{\xi, k} : \xi < \alpha \text{ and } k < n(\xi)\} \subseteq \Pi^F_B(X)$. Set $W_{\xi, k} = [A_{\xi, k}; V_{\xi, k, 1}, \ldots, V_{\xi, k, m(\xi, k)}]$. Notice that

$$\tilde{T}_{\Pi, \xi}(Y_{\xi, k}, \mathcal{W}_\xi) = U_{W_{\xi, k}, \mathcal{W}_\xi}.$$

By Lemma 29, we know that $\{W_{\xi, k} : \xi < \alpha \text{ and } k < n(\xi)\} \subseteq \mathcal{O}_F(X, B)$. Thus as the $W_{\xi, k} \subseteq U_{W_{\xi, k}, \mathcal{W}_\xi}$, $\{\tilde{T}_{\Pi, \xi}(Y_{\xi, k}, \mathcal{W}_\xi) : \xi < \alpha \text{ and } k < n(\xi)\} \subseteq \mathcal{O}_F(X, B)$ as well. Therefore Corollary 17 applies and $G^\alpha(\Pi^F_A(X), \Pi^F_B(X)) \leq G^\alpha(\mathcal{O}_F(X, A), \mathcal{O}_F(X, B))$. □

The following corollary generalizes the results from Sections 3, 4, and 5 of [12].

**Corollary 36.** Fix a topological space $X$ and $G \in \mathbb{F}(X)$. Then

| Rothberger-Like Games |
|-----------------------|
| $G_1(\mathcal{O}_F(X, \mathbb{K}(X)), \mathcal{O}_F(X, \mathbb{K}(X))) \equiv G_1(\mathcal{D}_F(X, \mathbb{K}), \mathcal{D}_F(X, \mathbb{K}))$ |
| $G_1(\mathcal{O}_F(X, \mathbb{F}(X)), \mathcal{O}_F(X, \mathbb{F}(X))) \equiv G_1(\mathcal{D}_F(X, \mathbb{V}), \mathcal{D}_F(X, \mathbb{V}))$ |
\[
\begin{align*}
G_1(O_F(X, F(X)), O_F(X, K(X))) &\equiv G_1(D_F(X, V), D_F(X, F)) \\
G_1(O_F(X, X \setminus G, K(X)), O_F(X, X \setminus G, K(X))) &\equiv G_1(\Omega_F(X, F), \Omega_F(X, F)) \\
G_1(O_F(X, X \setminus G, F(X)), O_F(X, X \setminus G, F(X))) &\equiv G_1(\Omega_F(X, V), \Omega_F(X, V)) \\
G_1(O_F(X, X \setminus G, K(X)), O_F(X, X \setminus G, K(X))) &\equiv G_1(\Omega_F(X, V), \Omega_F(X, V)) \\
G_1(\Pi^F_{K(X)}(X), \Pi^F_{K(X)}(X)) &\equiv G_1(O_F(X, F), O_F(X, F)) \\
G_1(\Pi^F_{F(X)}(X), \Pi^F_{F(X)}(X)) &\equiv G_1(O_F(X, V), O_F(X, V)) \\
\end{align*}
\]

**Menger-Like Games**

\[
\begin{align*}
G_{\text{fin}}(O_F(X, K(X)), O_F(X, K(X))) &\equiv G_{\text{fin}}(D_F(X, F), D_F(X, F)) \\
G_{\text{fin}}(O_F(X, F(X)), O_F(X, F(X))) &\equiv G_{\text{fin}}(D_F(X, V), D_F(X, F)) \\
G_{\text{fin}}(O_F(X, F(X)), O_F(X, F(X))) &\equiv G_{\text{fin}}(D_F(X, F), D_F(X, F)) \\
G_{\text{fin}}(O_F(X, X \setminus G, K(X)), O_F(X, X \setminus G, K(X))) &\equiv G_{\text{fin}}(\Omega_F(X, F), \Omega_F(X, F)) \\
G_{\text{fin}}(O_F(X, X \setminus G, F(X)), O_F(X, X \setminus G, F(X))) &\equiv G_{\text{fin}}(\Omega_F(X, V), \Omega_F(X, V)) \\
G_{\text{fin}}(O_F(X, X \setminus G, K(X)), O_F(X, X \setminus G, K(X))) &\equiv G_{\text{fin}}(\Omega_F(X, V), \Omega_F(X, V)) \\
G_{\text{fin}}(\Pi^F_{K(X)}(X), \Pi^F_{K(X)}(X)) &\equiv G_{\text{fin}}(O_F(X, F), O_F(X, F)) \\
G_{\text{fin}}(\Pi^F_{F(X)}(X), \Pi^F_{F(X)}(X)) &\equiv G_{\text{fin}}(O_F(X, V), O_F(X, V)) \\
\end{align*}
\]

**Hurewicz-Like Games and Weakly Groupable Games**

\[
\begin{align*}
G_1(O_F(X, K(X)), O^{gp}_F(X, K(X))) &\equiv G_1(D_F(X, F), D^{gp}_F(X, F)) \\
G_1(O_F(X, F(X)), O^{gp}_F(X, F(X))) &\equiv G_1(D_F(X, V), D^{gp}_F(X, V)) \\
G_{\text{fin}}(O_F(X, F(X)), O^{gp}_F(X, F(X))) &\equiv G_{\text{fin}}(D_F(X, F), D^{gp}_F(X, F)) \\
G_{\text{fin}}(O_F(X, F(X)), O^{gp}_F(X, F(X))) &\equiv G_{\text{fin}}(D_F(X, V), D^{gp}_F(X, V)) \\
G_1(O_F(X, K(X)), O^{wgp}_F(X, K(X))) &\equiv G_1(D_F(X, F), D^{wgp}_F(X, F)) \\
G_1(O_F(X, F(X)), O^{wgp}_F(X, F(X))) &\equiv G_1(D_F(X, V), D^{wgp}_F(X, V)) \\
G_{\text{fin}}(O_F(X, K(X)), O^{wgp}_F(X, K(X))) &\equiv G_{\text{fin}}(D_F(X, F), D^{wgp}_F(X, F)) \\
G_{\text{fin}}(O_F(X, F(X)), O^{wgp}_F(X, F(X))) &\equiv G_{\text{fin}}(D_F(X, V), D^{wgp}_F(X, V)) \\
\end{align*}
\]

5.3. **Tightness and Bar-Ilan Selection Principles.** Both [6] and [12] applied their analyses of the hyperspace topology to the tightness, set-tightness and T-tightness of the hyperspace. We recall the definitions of tightness and set-tightness now.

**Definition 26.** Suppose \(X\) is a topological space, \(x \in X\) and \(\kappa\) is a cardinal. Then the *tightness* of \(X\) at \(x\) is bounded above by \(\kappa\), written \(t(X, x) \leq \kappa\), if whenever \(x \in \overline{A} \setminus A\), there is a \(B \subseteq A\) so that \(|B| \leq \kappa\) and \(x \in \overline{B}\).

**Definition 27.** Suppose \(X\) is a topological space, \(x \in X\) and \(\kappa\) is a cardinal. Then the *set-tightness* of \(X\) at \(x\) is bounded above by \(\kappa\), written \(ts(X, x) \leq \kappa\), if whenever \(x \in \overline{A} \setminus A\), there is a \(B \subseteq A\) and a function \(\varphi : B \rightarrow \kappa\) so that \(x \in \overline{B}\), but \(x \notin \varphi^{-1}(\alpha)\) for any \(\alpha < \kappa\).
This kind of partition property for a blade inspires the next definition.

**Definition 28.** Let \( S \) be a collection, \( S \in S \), and \( \kappa \) be a regular cardinal. Then \( S \) is \( \kappa \)-breakable if there exists \( \varphi : S \rightarrow \kappa \) so that \( \varphi^{-1}(\xi) \notin S \) for each \( \xi < \kappa \). Let \( S^{\kappa_{br}} \) denote the class of \( \kappa \)-breakable elements of \( S \).

Note that \( t(X,x) \leq t_s(X,x) \) for any \( x \in X \). Also, set-tightness is essentially tightness, but modified by the \( \kappa \)-breakable definition. The countable-fan tightness game \( G(\Omega_{X,x}, \Omega_{X,x}) \) can describe countable tightness. Tightness for \( \kappa > \omega \) could be captured by \( G^\kappa(\Omega_{X,x}, \Omega_{X,x}) \). In a similar way, set-tightness is captured by \( G^\kappa(X, \Omega_{X,x}) \).

**Proposition 37.** Suppose \( X \) is a topological space, \( x \in X \), \( \lambda = |X| \), and \( \kappa \) is a cardinal. Then \( t_s(X,x) \leq \kappa \) if and only if \( I \mathrel{\not
subseteq}^{\text{const}} G^\kappa(X, \Omega_{X,x}) \).

**Proof.** First suppose that \( t_s(X,x) \leq \kappa \). Let \( A \) be the unique play of an arbitrary constant strategy for player One in the game. Note that \( x \in A \) and function \( \varphi : B \rightarrow \kappa \) so that \( x \notin \bar{\varphi}^{-1}(\alpha) \) for any \( \alpha < \kappa \). Enumerate \( B \) as \( \{ b_\xi : \xi < |B| \} \). Note that \( |B| \leq \lambda \). We construct a winning play of the game for player Two as follows. In round \( \xi < |B| \), player Two plays \( b_\xi \), and if necessary, in round \( \xi \geq |B| \), player Two plays \( b_0 \). Then at the end of the game, player Two has created \( B \), which we know is a winning play for Two.

Now suppose \( I \mathrel{\not
subseteq}^{\text{const}} G^\kappa(X, \Omega_{X,x}) \). Let \( A \subseteq X \) be so that \( x \in A \setminus A \). Then repeatedly playing \( A \) is a constant strategy in the game, so player Two has a winning counter play. Suppose Two wins against \( A \) with \( \langle x_\xi : \xi < \lambda \rangle \). Set \( B = \{ x_\xi : \xi < \lambda \} \). Then \( B \subseteq A \) and \( B \) is \( \kappa \)-breakable. Thus \( t_s(X,x) \leq \kappa \).

Since we are able to discuss tightness and set-tightness as the outcome of a constant strategy for a selection game, we can restate these notions as Bar-Ilan selection principles. Then the translation theorems apply and provide a quick proof of the connection between tightness properties on the hyperspace and Lindelöf-like properties on the ground space. To complete this analysis, we first prove equivalence lemmas for \( \Omega^{\kappa_{br}} \) that are similar to those in Section 4.

**Lemma 38.** Let \( A \) be an ideal of closed subsets of \( X \) and \( G \in F(X) \). \( U \in \mathcal{O}^{\kappa_{br}}(X, X \setminus G, A) \) if and only if \( c.U \in \Omega^{\kappa_{br}}_{F(X,A^+),G} \).

**Proof.** First suppose \( U \) is a \( \kappa \)-breakable \( A \)-cover of \( X \setminus G \) by open sets from \( X \). Suppose \( \varphi : U \rightarrow \kappa \) witnesses that \( U \) is \( \kappa \)-breakable. Define \( \psi : c.U \rightarrow \kappa \) by \( \psi(U) = \varphi(U) \). Note that by Lemma 21, since \( \varphi^{-1}(\xi) \) is not an \( A \)-cover of \( X \setminus G \) for all \( \xi < \kappa \), \( \psi^{-1}(\xi) \notin \Omega_{F(X,A^+),G} \). However, since \( U \) is an \( A \)-cover of \( X \setminus G \), \( c.U \in \Omega_{F(X,A^+),G} \). Therefore \( c.U \in \Omega^{\kappa_{br}}_{F(X,A^+),G} \).

The other direction is similar. \( \square \)

**Lemma 39.** Let \( A \) be an ideal of closed subsets of \( X \) and \( G \in F(X) \). \( U \in \mathcal{O}^{\kappa_{br}}_{F}(X, X \setminus G, A) \) if and only if \( c.U \in \Omega^{\kappa_{br}}_{F(X,A),G} \).

**Proof.** Use the partition provided by the definition of \( \kappa \)-breakability and appeal to Lemma 26. \( \square \)

**Proposition 40.** The following equivalences hold:

(i) Tightness in the hyperspace topologies can be captured with

\[
t(F(X,A^+),G) \leq \kappa \iff \left( \Omega_{F(X,A^+),G}, \Omega^\kappa_{F(X,A^+),G} \right) \iff \left( \mathcal{O}(X, X \setminus G, A), \mathcal{O}^\kappa(X, X \setminus G, A) \right)
\]

and

\[
t(F(X,A),G) \leq \kappa \iff \left( \Omega_{F(X,A),G}, \Omega^\kappa_{F(X,A),G} \right) \iff \left( \mathcal{O}_F(X, X \setminus G, A), \mathcal{O}^\kappa_F(X, X \setminus G, A) \right)
\]
(ii) Set-tightness in the hyperspace topologies can be captured with

\[ t_{s}(F(X,A),G) \leq \kappa \iff \left( \Omega_{\kappa}^{F(X,A),G} \right) \iff \left( \mathcal{O}(X,X \setminus G,A) \right) \]

and

\[ t_{s}(F(X,A),G) \leq \kappa \iff \left( \Omega_{\kappa}^{br,F(X,A),G} \right) \iff \left( \mathcal{O}^{br}(X,X \setminus G,A) \right) \]

Proof. For (i), let \( Y \) be a topological space and \( y \in Y \). First, suppose that \( t(Y,y) \leq \kappa \). Consider the game \( G_{\kappa}^{1}(\Omega_{Y,y},\Omega_{Y,y}) \). We show that One does not have a constant winning strategy for this game. Indeed, suppose \( A \subseteq X \) is any set so that \( y \in A \setminus \overline{A} \), which represents a constant strategy for One. By the tightness of \( Y \) at \( y \), we can find \( B = \{ b_\xi : \xi < \kappa \} \subseteq A \) so that \( y \in \overline{B} \). Let Two play \( b_\xi \) in the \( \xi^{th} \) inning and notice that Two wins.

On the other hand, if One doesn’t have a constant winning strategy in \( G_{\kappa}^{1}(\Omega_{Y,y},\Omega_{Y,y}) \), then we see that any \( A \subseteq Y \) with \( y \in \overline{A} \setminus A \) admits \( B \subseteq A \) with \( |B| \leq \kappa \) so that \( y \in \overline{B} \). That is, \( t(y,y) \leq \kappa \).

To finish this portion of the proposition, apply Remarks 4 and 7.

To prove (ii), apply Proposition 37, Remarks 4 and 7, and Lemmas 38 and 39.

6. Further Work

In Example 2, we demonstrated a cover which is in \( \mathcal{O}(\omega_{1},[\omega_{1}]^{\omega}) \), but not in \( \mathcal{O}^{gp}(\omega_{1},[\omega_{1}]^{\omega}) \). This cover, however, is not a countable cover. We do not currently know if there is an example separating \( \mathcal{O}(X,A) \) from \( \mathcal{O}^{gp}(X,A) \) which is countable. Based on the results of [11], finding such an example for \( \omega \)-covers would be equivalent to finding a space which is Hurewicz but one of its finite powers is not.

We demonstrated equivalences between games on a space \( X \) and games on the Fell/Vietoris hyperspaces of \( X \). Another common hyperspace construction creates the Pixley-Roy hyperspace (see [5] and [14]). Here we used the natural mapping of open sets to the complements to connect open sets in \( X \) to points in the hyperspace, and this lifted to various game equivalences. Is there such a connection between \( X \) and the Pixley-Roy hyperspace of \( X \)?

Can Proposition 5 be extended to include a broader variety of cover types like groupable covers and weakly groupable covers?

Finally, we did not exhaust the varieties of tightness covered by [6] and [12]. They connect T-tightness on the Fell/Vietoris hyperspace to a covering property of \( X \). We could not find a way to translate T-tightness into a more traditional selection principle, and so could not apply the translation result to recover this connection. Is there a way to characterize T-tightness that makes it fit into our framework?

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