Improving Linear State-Space Models with Additional Iterations

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Abstract: An estimated state-space model can possibly be improved by further iterations with estimation data. This contribution specifically studies if models obtained by subspace estimation can be improved by subsequent re-estimation of the $B$, $C$, and $D$ matrices (which involves linear estimation problems). Several tests are performed, which shows that it is generally advisable to do such further re-estimation steps using the maximum likelihood criterion. Stated more succinctly in terms of MATLAB® functions, ssest generally outperforms n4sid.

Keywords: Parameter estimation, State-space models, Subspace identification, Maximum Likelihood

1. INTRODUCTION

Linear state-space models are perhaps the most common model structure used in system identification. They can be estimated in several different ways. Among the most common techniques are so called subspace methods, such as MOESP, Verhaegen and Dewilde (1992), and N4SID, Overschee and Moor (1994). These were also developed to work with frequency domain data, McKelvey et al. (1996). Whichever way the model was obtained, an interesting question is if it can be improved in some way by further polishing using observed data.

In this contribution we investigate how further iterations on the $B$, $C$ and $D$ matrices can possibly improve the model quality and also whether it is worthwhile to reestimate the $A$ matrix.

We argue that such further estimation work is well motivated. The model properties are improved most of the time (but not always), and in some cases, the improvements are significant.

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2. THE STATE-SPACE MODEL

A linear state-space model in output error form (no noise model) is given by

\[ x(t + 1) = Ax(t) + Bu(t) \] \hspace{1cm} (1a)
\[ y(t) = Cx(t) + Du(t) + e(t) \] \hspace{1cm} (1b)

By assuming Gaussian noise distribution for $e$, the maximum likelihood estimate (MLE) of the matrices is given by

\[
\min_{A,B,C,D} \sum_{t=1}^{N} \|y(t) - \hat{C}\hat{x}(t) - Du(t)\|^2
\]

\[ \hat{x}(t + 1) = A\hat{x}(t) + Bu(t) \]

which can be rewritten as

\[
\min_{A,B,C,D} V(y, u, A, B, C, D)
\]

\[
V(y, u, A, B, C, D) = \sum_{t=1}^{N} \|y(t) - \hat{y}(t|A, B, C, D)\|^2
\]

\[
\hat{y}(t|A, B, C, D) = Du(t) + C \sum_{k=1}^{t} A^{t-k}Bu(k)
\]

We note that for fixed $A, C$ the prediction $\hat{y}$ is linear in $B, D$ and for fixed $A, B$ the prediction is linear in $C, D$.

Remark 1. The matrices $A, B, C, D$ may be parameterized by some parameter vector $\theta$. The estimation and minimization is carried out with respect to $\theta$. It will be assumed that any such parameterization is linear in $\theta$. 
3. REFINING THE ESTIMATE

With a given state-space model (1) and access to data \([y, u]\) from the system (either data that was used to estimate (1) or a fresh dataset), it is natural to ask if the maximum likelihood (ML) method in Eq. (2) can be used to refine the estimate.

\[ \hat{y}(t|A, B, C, D) \] is linear in \(B, D\) for fixed \(A, C\), and therefore reestimating \(B, D\) is a simple linear regression problem. Once this is done, one could fix \(A, B\) and reestimate \(C, D\) which again is a linear regression problem. Repeating that gives some simple \(B, C, D\) iterations, which is a way of minimizing (2a) for a fixed \(A\), which is a bilinear problem. See, for example, Eq. (10.68) in Ljung (1999). Simple experiments show that the \(B, C, D\) iterations typically lead to the same (possibly local) minimum of Eq. (2a) as applying the Gauss-Newton method to minimize Eq. (2a) for fixed \(A\).

The aforementioned \(B, C, D\) iterations are a block coordinate descent approach. It is faster than classical coordinate descent since in two steps all coefficients in \(B, C, D\) are updated and at each iteration more accurate values are used from previous iteration: updated \(C\) in \(B, D\) iteration and updated \(B\) in \(C, D\) iteration. The same approach is called separable least squares for a slightly different cost function appearing in identification of Hammerstein systems, Bai and Li (2004). The convergence of block coordinate descent is guaranteed when the objective function has a unique minimum in each coordinate block, see pages 261-262 in Luenberger and Ye (2008). This condition corresponds to the least squares problem in Eq. 2 not being rank deficient during each \(B, D\) and \(C, D\) iteration. This is frequently satisfied in practice per our experiments.

After the refinement of \(B, C, D\) is done, one can indeed fix these parameters and reestimate \(A\) using Eq. (2a) – or alternatively minimize the ML criterion with respect to all matrices simultaneously.

4. SISO, MISO, SIMO, AND MIMO

A typical implementation of subspace methods extract an estimate of the \(A, C\) matrices first, followed by a linear regression for the \(B\) and \(D\) matrices with fixed \(A, C\). The result is not necessarily a (local) optimum in terms of \(A, C\) for MIMO systems per the cost function Eq. (2a), and refinements are possible.

However, there are two special cases:

- **SISO and MISO systems**: The solution of the linear regression for \(B, D\) is also a (local) optimum in terms of \(C\). In other words, unless \(A\) is refined, further iterations for \(B, C, D\) matrices will not reduce the cost function.
- **SIMO systems**: The solution of the linear regression for \(B, D\) is not a local optimum in terms of \(C\). However, solving one additional linear regression problem for \(C, D\) matrices with fixed \(A, B\) corresponds to a (local) optimum with respect to the \(B\) matrix. After this additional linear regression, further iterations on \(B, C, D\) matrices are not needed unless \(A\) is changed.

To clarify the statements above we have the following results:

**Definition 1.** Two state-space realizations are input-output equivalent if their impulse responses (Markov parameters) are equal. State-space realization \((A, B, C, D)\) is input-output equivalent to \((A_0, B_0, C_0, D_0)\) if \(D_0 = D\) and \(CA'B = C_0A_0'B_0\) for all non-negative integers \(i\).

**Lemma 1.** Let \((A, B, C, D)\) be an order \(n\) state-space realization of a MISO linear system. Let \(C_0\) be any row vector of the same size as \(C\) such that \((A, C_0)\) is an observable pair. Then there exists matrices \(B_0\) and \(D_0\) such that the realization \((A, B_0, C_0, D_0)\) is input-output equivalent with \((A, B, C, D)\).

**Proof:** Trivially we have the choice \(D_0 = D\) and we will show that there exists a \(B_0\) such that the rest of the impulse response matrices coincide. Pick any matrix \(P\) such that \(CP = C_0\). Consider the construction \(T = \sum_{i=0}^{n-1} t_iA^i\) where the scalars \(t_i\) are selected such that

\[ C = C_0T = \sum_{i=0}^{n-1} t_iC_0A^i. \]  

(3)

Since the pair \((A, C_0)\) is observable the corresponding observability matrix has full rank which imply that the row vectors \(C_0A^i\) to the right in (3) are all linearly independent. Hence a solution exists. Note that by construction \(TA = AT\). Finally we have the identity

\[ CA'B = CPTA'B = C_0A'TB \]  

(4)

which directly shows that \(B_0 = TB\).

**Theorem 1.** Consider the case when the \(A\) matrix is fixed and we define \((A, B_0, C_0, D_0)\) to be a state-space realization which minimizes (2b), w.r.t. \(A, B, C\) with \(A\) fixed. Let \(C_0\) be any row vector such that \((A, C_0)\) is observable. Define \(B_0\) and \(D_0\) to be the minimizers to (2b) w.r.t. \(B\) and \(D\), when \(A\) and \(C_0\) are fixed. Then \(V(y, u, A, B_0, C_0, D_0) = V(y, u, A, B_0, C_0, D_0)\) since the loss function only depend on the input output properties and not on the specific realization. Since the minimization of \(V\) w.r.t. \(B\) and \(D\) only has a subset of the free parameters compared to the minimization w.r.t. \(B, C, D\) it follows that \(B'\) and \(D'\) are the minimizer to \(V\), i.e. \(B_0 = B'\) and \(D_0 = D'\).

**Proof:** By Lemma 1 we can conclude that for the realization \((A, B_0, C_0, D_0)\) there exists an input-output equivalent realization \((A, B', C_0, D')\) which imply \(V(y, u, A, B_0, C_0, D_0) = V(y, u, A, B', C_0, D')\) since the loss function only depend on the input output properties and not on the specific realization. Since the minimization of \(V\) w.r.t. \(B\) and \(D\) only has a subset of the free parameters compared to the minimization w.r.t. \(B, C, D\) it follows that \(B'\) and \(D'\) are the minimizer to \(V\), i.e. \(B_0 = B'\) and \(D_0 = D'\).

The results above can also be formulated and proved for SIMO systems.

Consider the following result.

**Theorem 2.** Given a matrix \(A \in \mathbb{R}^{n \times n}\) and assume there exists a vector \(v\) such that \((A, v)\) is observable. Then for a matrix \(B\) the following two statements are equivalent

(i) \(AB = BA\)

(ii) There exists \(b_i \in \mathbb{R}, i = 0, \ldots, n - 1\) such that \(B = \sum_{i=0}^{n-1} b_i A^i\).

**Proof:** If (ii) is true then (i) follows immediately. Assume (i) is true. Since \((A, v)\) is observable, the set \(\{A^i v\}_{i=0}^{n-1}\) forms a basis for \(\mathbb{R}^n\). Then it follows that
\[ Bv = \sum_{i=0}^{n-1} b_i A^i v \] for some scalars \( b_i \). Since \( B \) commutes with \( A \) by assumption it also commutes with all powers of \( A \). Hence for all integers \( r \geq 0 \)

\[ B(A^r v) = A^r Bv = A^r \sum_{i=0}^{n-1} b_i A^i v = \sum_{i=0}^{n-1} b_i A^i (A^r v) \] (5)

Since the set \( \{A^r v\}_{i=0}^{n-1} \) span \( \mathbb{R}^n \) we have shown that for any vector \( x \) we have \( Bx = (\sum_{i=0}^{n-1} b_i A^i)x \) which imply (ii).

The result in Theorem 2 shows a direct limitation for MIMO systems. The set of matrices that commutes with \( A \) is only an \( n \)-dimensional subspace of all \( n \times n \) matrices. Hence we can only fix \( n \) parameters in \( C_0 \), e.g. one row as shown in the proof of Lemma 1. The remaining rows in \( C_0 \) must be free parameters to optimize over. For MIMO systems it is thus beneficial to iterate between minimizing \( V \) w.r.t. \( C, D \) and \( B, D \) respectively until convergence.

Theorem 1 and Theorem 2 solely rely on the observability assumption on the \( (A, C) \) pair. One extra simplifying assumption can be made to illustrate the underlying structure of the least squares problem Eq. (2a). Assume \( A \) has distinct eigenvalues. Then \( A \) admits the decomposition \( A = P \Lambda P^{-1} \) with a diagonal matrix \( \Lambda \). Rewrite Eq. (2c) with \( B \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{1 \times n}, D \in \mathbb{R}^{1 \times n} \) exactly as:

\[ \bar{y}(t|A, B, C, D) = Du(t) + CP \sum_{k=1}^t \Lambda_{t-k} P^{-1} Bu(k) \]

Let \( \bar{C} = CP \in \mathbb{C}_1^{1 \times n}, \bar{B} = P^{-1}B \in \mathbb{C}_n^{n \times n} \) and \( L_{t-k} = [\Lambda_{t-k}^{1}, \Lambda_{t-k}^{2}, \ldots, \Lambda_{t-k}^{n}] \in \mathbb{C}_1^{1 \times n} \) where \( \Lambda_i \) for \( i = \{1, \ldots, n\} \) are the diagonal elements of \( \Lambda \). For real-valued \( A, B, \) and \( C \) matrices the columns of \( \bar{C} \) and the rows of \( \bar{B} \) come in conjugate-pairs. Since \( \Lambda_{t-k} \) is diagonal, \( CA_{t-k} = L_{t-k} \ diag(\bar{C}) \) where \( diag(\bar{C}) \) is the diagonal matrix with elements of the vector \( \bar{C} \) on the diagonals. Therefore:

\[ \bar{y}(t|A, B, C, D) = Du(t) + \sum_{k=1}^t L_{t-k} diag(\bar{C}) Bu(k) \]

This form shows that the eigenvalues of \( A \), contained in \( L_{t-k} \), form a set of basis functions for the linear regression problem for \( B \) (equivalently, \( \bar{B} \)). Fixed \( C \), and in turn \( diag(\bar{C}) \), is a scaling of the basis functions. Note that \( \bar{C} \in \mathbb{C}_1^{1 \times n} \) having a zero element is equivalent to \( (A, C) \) pair not being observable when \( A \) has distinct eigenvalues. When an element of \( \bar{C} \) is zero, the corresponding basis function is multiplied by zero and not utilized in regression.

A less formal view of Theorem 1 and its proof can also be seen from this form. Note that the unknowns in \( B, C \) can be combined into a single matrix \( X = diag(\bar{C})\bar{B} \) and the linear regression can be performed for \( X, D \) for fixed \( A \), where the rows of \( X \) are in conjugate-pairs if \( A, B, C \) are real-valued. \( diag(\bar{C}) \) is nonsingular when all elements of \( \bar{C} \) are nonzero. Any fixed choice of \( C \) that correspond to all nonzero elements in \( C = CV \) (i.e. \( (A, C) \) pair is observable when \( A \) has distinct eigenvalues) can be used to extract a \( B \) matrix estimate from \( X \), without affecting the cost function value observed in Eq. (2a).

The remarks made through distinct eigenvalues in \( A \) assumption can be relaxed by replacing the eigenvalue decomposition \( A = P \Lambda P^{-1} \) with Jordan matrix decomposition \( A = SJ S^{-1} \). \( J \) is block diagonal, where the blocks \( J_{i,j} \in \mathbb{C}_n^{n \times n} \) are elementary Jordan blocks. Then \( A_{t-k} = S J_{t-k} S^{-1} \) and \( J_{t-k} \) is again block diagonal with the blocks \( J_{i,j} \) as upper triangular Toeplitz matrices. This structure of the \( J_{t-k} \) matrix blocks can be exploited to again collect the unknowns in \( B, C \) into a single matrix. Similarly, which of the basis functions are utilized in the linear regression can be seen through (potentially a subset of) the elements in \( \bar{C} = CS \).

The optimality results for the \( B, C, D \) matrices presented in this section are also true for frequency-domain identification when a cost function of the following form is utilized:

\[ V(G, A, B, C, D) = \sum_{k} \| C(z_k I - A)^{-1} B + D - G_k \|_F^2 \] (6)

The cost function Eq. is typical for many estimation approaches including the subspace methods. Here \( z_k = e^{j \omega_k} \) is a point on the unit disk and \( G_k \) is a frequency response measurement corresponding to the \( k \)th frequency point \( \omega_k \) rad/s. It follows that since also this cost function only depends on the input-output properties of the state-space system the same argumentation as made for the time-domain case also holds here.

5. DISCRETE-TIME MODELS WITH TIME DOMAIN DATA

The following experiment was performed in MATLAB Release 2018a (2018):

(1) Generate 200 discrete-time systems randomly with \texttt{drss}. The systems are of order 7 with 4 inputs and 4 outputs. The feedthrough matrix \( D \) was set to zero.

(2) Simulate each system with a Gaussian white input with unit variance matrix. No attempt was made to tailor the input to the system properties. The noise-free data are denoted by \( z \). Add white Gaussian noise with unit covariance matrix to the output, giving the estimation data sequence \( z_n \).

(3) For each dataset, estimate a subspace model of order 7 without disturbance model with \texttt{n4sid}, giving \( mn = \text{n4sid}(zn,7,\text{'disturbancemodel'},\text{'none'}) \). Compute the error in that model compared to the true system by executing the commands:

\[ \text{en} = \text{pe}(z, mn) \]
\[ \text{mnn} = \text{norm}(\text{en}.y*\text{en}.y/N) \]

(4) Readjust the \( B, C \) parameters to data:

\[ \text{mpBC} = mn \]
\[ \text{mpBC}.\text{Structure}.A.\text{Free} = \text{zeros}(7, 7) \]
\[ \text{mpBC} = \text{ssest}(zn, mpBC) \]

(5) Compute the error of that model:

\[ \text{en} = \text{pe}(z, mpBC) \]
\[ \text{mnp} = \text{norm}(\text{en}.y*\text{en}.y/N) \]

(6) Readjust the \( A, B, C \) parameters of the model by further Maximum Likelihood iterations on \( mn \):

\[ \text{mp} = \text{ssest}(zn, mn) \]
\[ \text{en} = \text{pe}(z, mp) \]
\[ \text{mnp} = \text{norm}(\text{en}.y*\text{en}.y/N) \]

The results are summarized in the boxplot in Fig. 1. The medians of the error norms are
Fig. 1. Boxplots for the error norms in Section 5. From left to right: n4sid, mpBC, mp. The error norms were significantly above the plot limit for 9 systems for n4sid and mpBC.

Fig. 2. Plot of the error norm of the N4SID model (x-axis) vs the SSEST adjusted model (y-axis). 83% of the 200 points are below the equal-line.

Fig. 3. Boxplots for the error norms in Section 6. From left to right: n4sid, mpBC, mp.

6. CONTINUOUS-TIME MODELS WITH FREQUENCY DOMAIN DATA

Similar experiments were carried out in the frequency domain with continuous-time systems.

(1) Generate 200 seventh order continuous-time systems with 4 input and 4 outputs using $G = 	ext{rss}(7,4,4)$ possibly with feedthrough terms $D$. Compute their frequency response functions $G_f = \text{idfrd}(G,F_G)$ at 410 linearly spaced frequencies $F_G$. No attempts were made to adjust $F_G$ to the dynamics of $G$. Add 20% random, multiplicative noise to the response to form the frequency domain data $G_{fn}$.

(2) Estimate 7th order state-space models from the noisy frequency response functions, using n4sid and the $B,C$ adjusted models $mnBC$ and $A,B,C$ adjusted models $mp$ and their errors compared to the true system as in the previous section.

The medians of the errors were found to be:

$mn$: 48.768
$mpBC$: 44.191
$mp$: 31.485

The results are summarised in the boxplots in Fig. 3.

The error for $mn$ was larger than $mpBC$ in 83.5% of the cases. The error in $mnBC$ was larger than $mp$ in 74.5% of the cases and $mp$ outperformed $mn$ for 87% of the data sets.

A scatter plot for the error norms for the 200 tested systems is given in Fig. 4.

7. REAL DATA FROM FLEXIBLE STRUCTURES

Mechanical vibration testing was performed on a Volvo XC90 (2015) rear subframe structure. During the testing the subframe was equipped with 26 uniaxial and 10 triaxial accelerometers yielding a total of 56 measurement channels. A shaker was used to provide excitation at two different locations. Frequency data was obtained by employing a
Fig. 4. Plot of the error norm of the N4SID model (x-axis) vs. the B,C,D adjusted N4SID model (y-axis). 83.5% of the 200 points are below the equal-line.

Fig. 5. Magnitude of frequency response function and error magnitude of estimated model of order 40.

Fig. 6. Evolution of the cost function values for three optimization approaches. The values are normalized by the value attained by the initial model obtained from the subspace methods.

8. CONCLUSIONS

This contribution shows that it is useful to do further iterations on the B, C, and D state-space matrix estimates obtained by the subspace identification when the system has multiple outputs. This is tested with both simulated and real data. It is also recommended to adjust the A matrix.
estimate using maximum likelihood iterations. It leads to improvements in a majority of cases, but not always. One reason is that subspace identification algorithms contain several design variables (prediction horizons and prefilters) that can be difficult to choose in an optimal way. In a sense, maximum likelihood iterations (such as in \texttt{ssest}) perform such choices automatically.

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\textbf{REFERENCES}

Bai, E.W. and Li, D. (2004). Convergence of the iterative Hammerstein system identification algorithm. \textit{IEEE Transactions on Automatic Control}, 49(11), 1929–1940.

Gibanica, M. and Abrahamsson, T.J.S. (2017). Parameter Estimation and Uncertainty Quantification of a Sub-frame with Mass Loaded Bushings. In R. Bartherpe, R. Platz, I. Lopez, B. Moaveni, and C. Papadimitriou (eds.), \textit{Model Validation and Uncertainty Quantification, Volume 3}, Conference Proceedings of the Society for Experimental Mechanics Series, 61–76. Springer International Publishing. DOI: 10.1007/978-3-319-54858-6.7.

Ljung, L. (1999). \textit{System Identification - Theory for the User}. Prentice-Hall, Upper Saddle River, N.J., 2nd edition.

Ljung, L. (2003). Aspects and experiences of user choices in subspace identification methods. In S.W. P. van der Hof B. Wahlberg (ed.), \textit{Proc. 13th IFAC Symposium on System Identification}, 1802 –1807. Rotterdam, The Netherlands.

Luenberger, D.G. and Ye, Y. (2008). \textit{Linear and Nonlinear Programming}. Springer, 3rd edition.

MATLAB Release 2018a (2018). The MathWorks, Inc., Natick, Massachusetts, United States.

McKelvey, T., Akçay, H., and Ljung, L. (1996). Subspace-based multivariable system identification from frequency response data. \textit{IEEE Trans. on Automatic Control}, 41(7), 960–979.

Overschee, P.V. and Moor, B.D. (1994). \textit{NA4SID}: subspace algorithms for the identification of combined deterministic-stochastic systems . \textit{Automatica, (Special Issue)}, 30(1), 75–93.

Vakilzadeh, M.K., Yaghoubi, V., McKelvey, T., Abrahamsson, T., and Ljung, L. (2015). Experiment design for improved frequency domain subspace system identification of continuous-time systems. \textit{Proc. IFAC Symposium on System Identification, IFAC-PapersOnLine}, 48(28), 886–891.

Verhaegen, M. and Dewilde, P. (1992). The output-error state-space model identification class of algorithms. \textit{Int Journal of Control}, 56(5), 1187–1210.