Singular value decomposition of complexes

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Abstract

Singular value decompositions of matrices are widely used in numerical linear algebra with many applications. In this paper, we extend the notion of singular value decompositions to finite complexes of vector spaces. We provide two methods to compute them and present several applications.

1 Introduction

For a matrix $A \in \mathbb{R}^{m \times k}$, a singular value decomposition (SVD) of $A$ is

$$A = U \cdot \Sigma \cdot V^t$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{k \times k}$ are orthogonal and $\Sigma \in \mathbb{R}^{m \times k}$ is diagonal with nonnegative real numbers on the diagonal. The diagonal entries of $\Sigma$, say $\sigma_1 \geq \cdots \geq \sigma_{\min\{m,k\}} \geq 0$ are called the singular values of $A$ and the number of nonzero singular values is equal to the rank of $A$. Extensions to matrices in $\mathbb{C}^{m \times k}$ simply involve replacing orthogonal with unitary and transpose with Hermitian transpose (conjugate transpose). Singular value decomposition is used to solve many problems in numerical linear algebra such as pseudoinversion, least squares solving, and low-rank matrix approximation. For example, the Eckart-Young theorem [EY36] shows that for $r = 0, \ldots, \min\{m, k\} - 1$, $\sigma_{r+1}$ is the 2-norm distance between $A$ and the set of matrices of rank at most $r$. In fact,

$$A_r = U \cdot \text{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)_{m \times k} \cdot V^t$$
has rank at most \( r \) with \( \sigma_{r+1} = \|A - A_r\|_2 \) and solves

(1) \[
\min_{B \in \mathbb{R}^{m \times k}} \{ \|A - B\|_2 \mid \text{rank } B \leq r \}.
\]

A matrix \( A \in \mathbb{R}^{m \times k} \) defines a linear map \( A : \mathbb{R}^k \to \mathbb{R}^m \) via \( x \mapsto Ax \) denoted \( \mathbb{R}^m \xrightarrow{A} \mathbb{R}^k \).

Geometrically, the singular values of \( A \) are the lengths of the semi-axes of the ellipsoid arising as the image of the unit sphere under this map defined by \( A \).

Matrix multiplication simply corresponds to function composition. For example, if \( B \in \mathbb{R}^{\ell \times m} \), then \( B \circ A : \mathbb{R}^k \to \mathbb{R}^\ell \) is defined by \( x \mapsto BAx \) denoted \( \mathbb{R}^\ell \xleftarrow{B} \mathbb{R}^m \xrightarrow{A} \mathbb{R}^k \).

If \( B \circ A = 0 \), then this composition forms a complex denoted \( 0 \xleftarrow{B} \mathbb{R}^\ell \xrightarrow{A} \mathbb{R}^m \xrightarrow{A} \mathbb{R}^k \).

In general, a finite complex of finite-dimensional \( \mathbb{R} \)-vector spaces

\[
0 \xleftarrow{C_0} \xrightarrow{A_1} \xleftarrow{C_1} \xrightarrow{A_2} \ldots \xleftarrow{A_{n-1}} \xrightarrow{C_{n-1}} \xrightarrow{A_n} \xleftarrow{C_n} 0
\]

consists of vector spaces \( C_i \cong \mathbb{R}^{c_i} \) and differentials given by matrices \( A_i \) so that \( A_i \circ A_{i+1} = 0 \). We denote such a complex by \( C_\bullet \) and its \( i \)th homology group as

\[
H_i = H_i(C_\bullet) = \frac{\ker A_i}{\text{image } A_{i+1}}
\]

with \( h_i = \dim H_i \). Complexes are standard tools that occur in many areas of mathematics including differential equations, e.g. [AFW06, AFW10]. One of the reasons for developing a singular value decomposition of complexes is to compute the dimensions \( h_i \) efficiently and robustly via numerical methods when each \( A_i \) is only known approximately, say \( B_i \). For example, if the rank of each \( A_i \) is known, say \( r_i \), then each \( h_i \) can easily be computed via

\[
h_i = c_i - (r_i + r_{i+1}).
\]

One option would be to compute the singular value decomposition of each \( B_i \) in order to compute the rank of \( A_i \) since the singular value decomposition is an excellent rank-revealing numerical method. However, simply decomposing each \( B_i \) ignores the important information that the underlying matrices \( A_i \) form a complex.

The key point of this paper is that we can utilize information about the complex to provide more specific information that reflects the structure it imposes.
Theorem 1.1 (Singular value decomposition of complexes). Let $A_1, \ldots, A_n$ with $A_i \in \mathbb{R}^{c_i-1 \times c_i}$, $r_i = \text{rank } A_i$, and $h_i = c_i - (r_i + r_{i+1})$ be a sequence of matrices which define a complex $C_\bullet$, i.e., $A_i \circ A_{i+1} = 0$. Then, there exists sequences $U_0, \ldots, U_n$ and $\Sigma_1, \ldots, \Sigma_n$ of orthogonal and diagonal matrices, respectively, such that

$$U_t \circ A_i \circ U_i = \begin{pmatrix} r_i & r_{i+1} & h_i \\ r_{i-1} & 0 & 0 \\ \Sigma_i & 0 & 0 \\ h_{i-1} & 0 & 0 \end{pmatrix}$$

where all diagonal entries of $\Sigma_i$ are strictly positive. Moreover, if every $r_i > 0$ and at least one $h_i > 0$, then the orthogonal matrices $U_i$ can be chosen such that $\det U_i = 1$, i.e., each $U_i$ is a special orthogonal matrix.

The diagonal entries of $\Sigma_1, \ldots, \Sigma_n$ are the singular values of the complex which are described in Remark 4.3. Just as with matrices, singular value decomposition of complexes naturally extends to complexes involving entries with complex numbers by simply replacing orthogonal with unitary and transpose with Hermitian transpose (conjugate transpose). However, such an extension is not needed for the applications in this article.

We develop two methods that utilize the structure of the complex $C_\bullet$ to compute a singular value decomposition of $C_\bullet$. The successive projection method described in Algorithm 3.1 uses the orthogonal projection

$$P_{i-1}: C_{i-1} \rightarrow \ker A_{i-1}$$

together with the singular value decomposition of the matrix $P_{i-1} \circ A_i$. The second method, described in Algorithm 3.3, is based on using each Laplacian

$$\Delta_i = A_i^t \circ A_i + A_{i+1} \circ A_{i+1}^t.$$ 

Both of these methods can be applied to numerical approximations $B_i$ of $A_i$.

The organization of this paper is as follows. Section 2 proves Theorem 1.1 and collects a number of basic facts along with defining the pseudoinverse of a complex. Section 3 describes the algorithms mentioned above and illustrates them on an example. Section 4 considers projecting an arbitrary sequence of matrices onto a complex. Section 5 provides an application to computing Betti numbers of minimal free resolutions of graded modules over the polynomial ring $\mathbb{Q}[x_0, \ldots, x_n]$ which combines our method with ideas from [EMSS16].
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2 Basics

We first prove our main theorem on singular value decomposition of complexes.

Proof of Theorem 1.1. For convenience, let $A_0 = A_{n+1} = 0$ compliment the matrices $A_1, \ldots, A_n$ that describe the complex $C_*$. By the homomorphism theorem,

$$\text{(ker } A_i)^\perp \cong \text{image } A_i.$$ 

The singular value decomposition for a complex follows by applying singular value decomposition to this isomorphism and extending an orthonormal basis of these spaces to an orthonormal basis of $\mathbb{R}^{c_i-1}$ and $\mathbb{R}^{c_i}$. Since $\text{image } A_{i+1} \subset \text{ker } A_i$, we have an orthogonal direct sum

$$\text{(ker } A_i)^\perp \oplus \text{image } A_{i+1} \subset \mathbb{R}^{c_i}$$

with

$$H_i := ((\text{ker } A_i)^\perp \oplus \text{image } A_{i+1})^\perp = \text{ker } A_i \cap \text{image } A_{i+1}^\perp \cong \frac{\text{ker } A_i}{\text{image } A_{i+1}}.$$ 

With respect to these subspaces, we can decompose $A_i$ as

$$(\text{ker } A_i)^\perp \quad \text{image } A_{i+1} \quad H_i$$

Indeed, $A_i$ has no component mapping to $(\text{image } A_i)^\perp$, which explains six of the zero blocks, and $\text{ker } A_i = (\text{ker } A_i)^\perp\perp = \text{image } A_{i+1} \oplus H_i$ explains the remaining
two. Take $U_i$ to be the orthogonal matrix whose column vectors form the orthonormal basis of the spaces $(\ker A_i)^\perp$ and $\text{image } A_{i+1}$ induced from the singular value decomposition of $(\ker A_i)^\perp \to \text{image } A_i$ and $(\ker A_{i+1})^\perp \to \text{image } A_{i+1}$ extended by an orthogonal basis of $H_i$ in the decomposition $(\ker A_i)^\perp \oplus \text{image } A_{i+1} \oplus H_i = \mathbb{R}^c$.

The linear map $A_i$ has, in terms of these bases, the description $U_{i-1}^t \circ A_i \circ U_i$ which has the desired shape.

Finally, to achieve $\det U_i = 1$, we may, for $1 \leq k \leq r_i$, change signs of the $k^{th}$ column in $U_i$ and $(r_{i-1} + k)^{th}$ column of $U_{i-1}$ without changing the result of the conjugation. If $h_i > 0$, then changing the sign of any of the last $h_i$ columns of $U_i$ does not affect the result either. Thus, this gives us enough freedom to reach $\det U_i = 1$ for all $i = 0, \ldots, n$.

The singular values of a matrix $A$ are the square roots of the eigenvalues of $A^t \circ A$. The following generalizes this relationship to singular values of a complex and eigenvalues of the Laplacians.

**Corollary 2.1 (Repetition of eigenvalues).** Suppose that $A_1, \ldots A_n$ define a complex with $A_0 = A_{n+1} = 0$. Let $\Delta_i = A_i^t \circ A_i + A_{i+1} \circ A_{i+1}^t$ be the corresponding Laplacians. Then, using the orthonormal bases described by the $U_i$’s from Theorem 1.1, the Laplacians $\Delta_i$ have the shape

$$
\begin{pmatrix}
\Sigma_i^2 & 0 & 0 \\
0 & \Sigma_{i+1}^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

In particular,

1. $\ker \Delta_i = H_i$;

2. if $r_i = \text{rank } A_i$ and $\sigma_1^i \geq \sigma_2^i \geq \ldots \geq \sigma_{r_i}^i > 0$ are the singular values of $A_i$, then each $(\sigma_k^i)^2$ is an eigenvalue of both $\Delta_i$ and $\Delta_{i-1}$.

**Proof.** The structure of $\Delta_i$ follows immediately from the structure described in Theorem 1.1. The remaining assertions are immediate consequences. 

Let $A_i^+$ denote the Moore-Penrose pseudoinverse of the $A_i$. Thus, a singular value decomposition

$$
A_i = U_{i-1} \circ \begin{pmatrix} 0 & 0 & 0 \\ \Sigma_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ U_i^t \quad \text{yields} \quad A_i^+ = U_i \circ \begin{pmatrix} 0 & \Sigma_i^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ U_{i-1}^t.
$$
Proposition 2.2. Suppose that $A_1, \ldots, A_n$ define a complex with $A_0 = A_{n+1} = 0$. Then, $A_{i+1}^+ \circ A_i^+ = 0$ and

$$id_{\mathbb{R}^c} - (A_i^+ \circ A_i + A_{i+1} \circ A_{i+1}^+)$$

defines the orthogonal projection of $\mathbb{R}^c$ onto the homology $H_i$.

Proof. We know that $A_i^+ \circ A_i$ defines the projection onto $(\ker A_i)^\perp$ and $A_{i+1} \circ A_{i+1}^+$ defines the projection onto image $A_{i+1}$. The result follows immediately since these spaces are orthogonal and $H_i = ((\ker A_i) \oplus \text{image } A_{i+1})^\perp$. \qed

For a complex $C_\bullet$ with

$$0 \leftarrow \mathbb{R}^{c_0} \xrightarrow{A_1} \mathbb{R}^{c_1} \xrightarrow{A_2} \ldots \xrightarrow{A_n} \mathbb{R}^{c_n} \leftarrow 0,$$

the pseudoinverse complex, denoted $C_\bullet^+$, is

$$0 \longrightarrow \mathbb{R}^{c_0} \xrightarrow{A_i^+} \mathbb{R}^{c_1} \xrightarrow{A_2^+} \ldots \xrightarrow{A_n^+} \mathbb{R}^{c_n} \longrightarrow 0.$$

Remark 2.3. If the matrices $A_i$ have entries in a subfield $K \subset \mathbb{R}$, then the pseudoinverse complex is also defined over $K$. This follows since the pseudoinverse is uniquely determined by the Penrose relations [Pen55]:

$$A_i \circ A_i^+ \circ A_i = A_i, \quad A_i \circ A_i^+ = (A_i \circ A_i^+)^t,$$

$$A_i^+ \circ A_i \circ A_i^+ = A_i^+, \quad A_i^+ \circ A_i = (A_i^+ \circ A_i)^t,$$

which form an algebraic system of equations for the entries of $A_i^+$ with a unique solution whose coefficients are in $K$. In particular, this holds for $K = \mathbb{Q}$.

If the entries of the matrices are in the finite field $\mathbb{F}_q$, the pseudoinverse of $A_i$ is well defined over $\mathbb{F}_q$ with respect to the dot-product on $\mathbb{F}_q^{c_i}$ and $\mathbb{F}_q^{c_i-1}$ if

$$\ker A_i \cap (\ker A_i)^\perp = 0 \subset \mathbb{F}_q^{c_i} \quad \text{and} \quad \text{image } A_i \cap (\text{image } A_i)^\perp = 0 \subset \mathbb{F}_q^{c_i-1}.$$

We have implemented the computation of the pseudoinverse complex for double precision floating-point numbers $\mathbb{R}_{53}$, the rationals $\mathbb{Q}$, and finite fields $\mathbb{F}_q$ in our Macaulay2 package SVDComplexes.
3 Algorithms

We present two algorithms for computing a singular value decomposition of a complex followed by some examples.

Algorithm 3.1 (Successive projection method).

INPUT: Sequence $A_1, \ldots, A_n$ of real matrices forming a complex $C$. 
OUTPUT: Integers $r_1, \ldots, r_n$, orthogonal matrices $U_0, \ldots, U_n$ and diagonal matrices $\Sigma_1, \ldots, \Sigma_n$ forming a singular value decomposition of $C$.

1. Set $r_0 = 0$, $Q_0 = 0$, and $P_0 = \text{id}_{C_0}$.
2. For $i = 1, \ldots, n$
   a. Compute the $(c_{i-1} - r_{i-1}) \times c_i$ matrix $\tilde{A}_i = P_{i-1} \circ A_i$.
   b. Compute a singular value decomposition of $\tilde{A}_i$, say $\tilde{A}_i = \tilde{U}_i \circ \tilde{\Sigma}_i \circ \tilde{V}_i^t$.
   c. Set $r_i = \text{rank} \tilde{\Sigma}_i$.
   d. Decompose $\tilde{V}_i^t = \begin{pmatrix} Q_i \\ P_i \end{pmatrix}$ into submatrices consisting of the first $r_i$ and last $c_i - r_i$ rows of $\tilde{V}_i^t$.
   e. Compute $U_i^t = \begin{pmatrix} \tilde{U}_{i-1}^t \circ P_{i-1} \end{pmatrix}$.
   f. If $i = n$, set $U_n = \tilde{V}_n^t$ and then compute $\Sigma_1, \ldots, \Sigma_n$ satisfying (2).
3. Return $r_1, \ldots, r_n, U_0, \ldots, U_n, \Sigma_1, \ldots, \Sigma_n$.

Proof of correctness. By induction on $i$, we will see that $P_i$ defines the orthogonal projection $C_i \rightarrow \ker A_i$. Since $V_i^t$ is orthogonal, $\begin{pmatrix} Q_i \\ P_i \end{pmatrix} \circ \begin{pmatrix} Q_i^t & P_i^t \end{pmatrix} = \begin{pmatrix} \text{id}_{r_i} & 0 \\ 0 & \text{id}_{c_i - r_i} \end{pmatrix}$ where $\text{id}_k$ denotes a $k \times k$ identity matrix, we additionally conclude that $Q_i$ is the orthogonal projection $C_i \rightarrow (\ker A_i)^\perp$. This is trivially true for $A_0 = 0$. 

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For the induction step, image $A_i \subset \ker A_{i-1}$ implies that $Q_{i-1} \circ A_i = 0$. Hence, $A_i$ and $P_{i-1} \circ A_i = \tilde{A}_i$ have the same nonzero singular values. From

$$\tilde{A}_i = \tilde{U}_i \circ \tilde{\Sigma}_i \circ \tilde{V}_i^t \quad \text{and} \quad \tilde{V}_i^t = \begin{pmatrix} Q_i \\ P_i \end{pmatrix},$$

we see that $P_i$ defines the orthogonal projection $C_i \to \ker A_i$. Moreover,

$$U_{i-1}^t \circ A_i \circ U_i = \begin{pmatrix} Q_{i-1} \\ \tilde{U}_{i-1}^t \circ P_{i-1} \end{pmatrix} \circ A_i \circ \begin{pmatrix} Q_i^t \\ P_i^t \circ \tilde{U}_i \end{pmatrix} \circ \begin{pmatrix} 0 \\ \tilde{U}_{i-1}^t \circ \tilde{A}_i \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{U}_{i-1}^t \circ \tilde{U}_{i-1} \circ \tilde{\Sigma}_i \circ \begin{pmatrix} Q_i \\ P_i \end{pmatrix} \circ \begin{pmatrix} Q_i^t \\ P_i^t \circ \tilde{U}_i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\Sigma}_i \circ \begin{pmatrix} \text{id}_{\tilde{r}_i} & 0 \\ 0 & \text{id}_{c_i - \tilde{r}_i} \end{pmatrix} \end{pmatrix} \circ \begin{pmatrix} Q_i^t \\ P_i^t \circ \tilde{U}_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

since

$$\tilde{\Sigma}_i \circ \begin{pmatrix} 0 \\ \text{id}_{c_i - \tilde{r}_i} \end{pmatrix} = 0.$$

Hence, Algorithm 3.1 computes a singular value decomposition of $C_\bullet$. \hfill \Box

**Remark 3.2.** Algorithm 3.1 was presented using exact input data $A_1, \ldots, A_n$ for the complex $C_\bullet$ and exact computations. When using numerical approximations $B_1, \ldots, B_n$ for the matrices $A_1, \ldots, A_n$, this algorithm can be easily modified to use floating-point arithmetic to produce a good numerical approximation of a singular value decomposition for $C_\bullet$ provided that:

i) the approximations $B_1, \ldots, B_n$ of $A_1, \ldots, A_n$ are sufficiently accurate,

ii) the correct rank is identified in Step 2c, and

iii) floating-point arithmetic using sufficiently high precision is utilized.
We can alter Step 2c when using floating-point arithmetic to obtain more confidence in the correctness of the computation of $r_1, \ldots, r_n$. One natural approach is to start with two approximations $B_1, \ldots, B_n$ and $B_1', \ldots, B_n'$ in different precisions and determine $r_1, \ldots, r_n$ as the number of stable singular values, i.e., the singular values which have approximately the same value in both computations. Moreover, since orthogonal matrices have unit condition number, they maintain lengths so that the rank $r_i$ can be reliably computed in Step 2c for all $i = 1, \ldots, n$.

The second method for computing a singular value decomposition is based on using the Laplacians and Corollary 2.1 which generalizes the description of singular values of a matrix $A$ as the square roots of the eigenvalues of $A^t \circ A$.

**Algorithm 3.3** (Laplacian method).

**INPUT:** Sequence $A_1, \ldots, A_n$ of real matrices forming a complex $C_\bullet$.

**OUTPUT:** Integers $r_1, \ldots, r_n$, orthogonal matrices $U_0, \ldots, U_n$ and diagonal matrices $\Sigma_1, \ldots, \Sigma_n$ forming a singular value decomposition of $C_\bullet$.

1. Compute an eigendecomposition of $\Delta_0 = A_1 \circ A_1^t$, i.e., compute an orthogonal matrix $U_0 \in \mathbb{R}^{c_0 \times c_0}$ and diagonal matrix $D_0 \in \mathbb{R}^{c_0 \times c_0}$ where the diagonal entries are listed in decreasing order such that
   \[
   \Delta_0 = U_0 \circ D_0 \circ U_0^t.
   \]

2. Let $r_1$ be the number of nonzero diagonal entries of $D_0$, $\tilde{V}_1$ be the first $r_1$ columns of $U_0$, $\Sigma_1 \in \mathbb{R}^{r_1 \times r_1}$ be the diagonal matrix with $(\Sigma_1)_{jj} = \sqrt{(D_0)_{jj}}$, and $\tilde{U}_1 = A_1^t \circ \tilde{V}_1 \circ \Sigma_1^{-1}$.

3. For $i = 1, \ldots, n - 1$:
   a. Extend $\tilde{U}_i$ to an orthogonal matrix $U_i \in \mathbb{R}^{c_i \times c_i}$ forming a eigenbasis for $\Delta_i = A_{i-1}^t \circ A_{i-1} + A_i \circ A_i^t$ such that
      \[
      \Delta_i = U_i \circ D_i \circ U_i^t
      \]
      where $D_i \in \mathbb{R}^{c_i \times c_i}$ is a diagonal matrix of the form
      \[
      D_i = \begin{pmatrix}
      \Sigma_i^2 \\
      \Lambda_i
      \end{pmatrix}
      \]
      and the diagonal entries in $\Lambda_i$ are listed in decreasing order.
b. Let \( r_{i+1} \) be the number of nonzero diagonal entries of \( \Lambda_i \), \( \tilde{V}_{i+1} \) be the \( r_i + 1, \ldots, r_i + r_{i+1} \) columns of \( U_i \), \( \tilde{\Sigma}_{i+1} \in \mathbb{R}^{r_{i+1} \times r_{i+1}} \) be the diagonal matrix with \( (\tilde{\Sigma}_{i+1})_{jj} = \sqrt{(\Lambda_i)_{jj}} \), and \( \tilde{U}_{i+1} = A_{i+1}^t \circ \tilde{V}_{i+1} \circ \tilde{\Sigma}_{i+1}^{-1} \).

c. If \( i = n - 1 \), extend \( \tilde{U}_n \) to an orthogonal matrix \( U_n \in \mathbb{R}^{c_n \times c_n} \).

4. Return \( r_1, \ldots, r_n, U_0, \ldots, U_n, \) and \( \Sigma_1, \ldots, \Sigma_n \).

Proof of correctness. Since each \( \Delta_i \) is symmetric, it is diagonalizable, i.e., has an orthonormal basis consisting of eigenvectors. By construction, the first \( r_i \) columns of \( U_i \) form a basis for \((\ker A_i)^\perp\), the next \( r_i + 1 \) columns of \( U_i \) form a basis for image \( A_i \), and the last \( h_i = c_i - (r_i + r_{i+1}) \) columns of \( U_i \) form a basis for \( H_i \). By the homomorphism theorem, \((\ker A_i)^\perp \cong \text{image } A_i \) means that we can reuse \( r_i \) columns from \( U_{i-1} \) which form a basis for image \( A_i \) as the first \( r_i \) columns of \( U_i \) forming a basis for \((\ker A_i)^\perp\). This immediately yields that

\[
U_{i-1}^t \circ A_i \circ U_i = \begin{pmatrix}
0 & 0 & 0 \\
\Sigma_i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

thereby computing a singular value decomposition for \( C_* \).

Remark 3.4. If all of the eigenvalues of all of the Laplacians are distinct, then every eigenvector of length one is defined uniquely up to sign. Hence, one can compute eigendecompositions of the \( \Delta_i \)'s independently, e.g., using parallel computations. A singular value decomposition can then be computed by simply rearranging and changing signs on the eigenvectors as needed.

Remark 3.5. The comments in Remark 3.2 in reference to Algorithm 3.1 related to using numerical approximations hold for Algorithm 3.3 modulo identifying the correct rank in Steps 2 and 3b. The key aspect is to use sufficiently high precision to distinguish between small nonzero and zero eigenvalues due to the squaring of the singular values.

Example 3.6. We consider the complex

\[
0 \lla R^3 \lla A_1 \lla R^5 \lla A_2 \lla R^5 \lla A_3 \lla R^3 \lla 0
\]

where the matrices \( A_1, A_2, A_3 \), respectively, are

\[
\begin{pmatrix}
14 & -4 & 16 & 3 & -9 \\
14 & -5 & 20 & 9 & 1 \\
4 & 1 & -4 & -12 & -24
\end{pmatrix},
\begin{pmatrix}
-43 & -50 & -27 & -51 & 9 \\
12 & -24 & 36 & 0 & -12 \\
35 & 34 & 27 & 39 & -9 \\
-3 & -10 & 3 & -6 & -1 \\
-11 & -10 & -9 & -12 & 3
\end{pmatrix},
\begin{pmatrix}
-8 & -16 & -12 \\
-5 & -1 & -15 \\
-1 & 13 & -14 \\
12 & 12 & 28 \\
-1 & 25 & -24
\end{pmatrix}
\]
Hence, for $i = 1, 2, 3$, the matrices $\Sigma_i = U_{i-1}^T A_i U_i$ are

\[
\begin{pmatrix}
34.489 & 0 & 0 & 0 & 0 \\
0 & 28.714 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
45.993 & 0 & 0 \\
0 & 35.209 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

showing that $r_i = \text{rank } A_i = 2$ and $h_i = \dim H_i = 1$ for $i = 1, 2, 3$. In particular, the following diagram commutes:

\[
\begin{array}{c}
0 \xleftarrow{U_0} \mathbb{R}^3 \xrightarrow{A_1} \mathbb{R}^5 \xleftarrow{U_1} \mathbb{R}^5 \xrightarrow{A_2} \mathbb{R}^5 \xleftarrow{U_2} \mathbb{R}^3 \xrightarrow{A_3} \mathbb{R}^3 \xleftarrow{U_3} 0
\end{array}
\]

One can use this singular value decomposition to compute the pseudoinverse complex. For example, $A_1^+$, printed with 6 decimal places is

\[
\begin{pmatrix}
0.012191 & 0.011463 & 0.005043 \\
-0.003285 & -0.004260 & 0.001150 \\
0.013141 & 0.017040 & -0.004601 \\
0.001425 & 0.008366 & -0.014466 \\
-0.009815 & 0.002481 & -0.029152
\end{pmatrix}
\]

which is a numerical approximation of the exact matrix

\[
\begin{pmatrix}
5978/490373 & 5621/490373 & 2473/490373 \\
-1611/490373 & -2089/490373 & 564/490373 \\
6444/490373 & 8356/490373 & -2256/490373 \\
699/490373 & 8205/980746 & -14187/980746 \\
-4813/490373 & 2433/980746 & -28591/980746
\end{pmatrix}
\]
Moreover, this simple example is fairly stable against errors. For example, the algorithms predict the dimension of the homology groups correctly upon perturbing the entries of the matrices $A_i$ on the order of $\leq 10^{-3}$ and using a threshold of $10^{-2}$ to compute the ranks, e.g., see SVDComplexes.

**Example 3.7.** We compared Algorithms 3.1 and 3.3 for verifying the dimension of homology groups for randomly generated complexes of various sizes with known homology group dimensions. Table 1 compares the timings of these algorithms.

| $c_0$ | $c_1$ | $c_2$ | $c_3$ | $h_0$ | $h_1$ | $h_2$ | $h_3$ | Alg. 3.1 (sec) | Alg. 3.3 (sec) |
|------|------|------|------|------|------|------|------|--------------|--------------|
| 7    | 21   | 28   | 14   | 2    | 3    | 2    | 1    | 0.00211      | 0.0110       |
| 8    | 27   | 35   | 17   | 3    | 6    | 4    | 2    | 0.00225      | 0.0182       |
| 9    | 33   | 42   | 20   | 4    | 9    | 6    | 3    | 0.00254      | 0.0294       |
| 10   | 39   | 49   | 23   | 5    | 12   | 8    | 4    | 0.00291      | 0.0647       |
| 11   | 45   | 56   | 26   | 6    | 15   | 10   | 5    | 0.00355      | 0.1090       |
| 12   | 51   | 63   | 29   | 7    | 18   | 12   | 6    | 0.00442      | 0.1150       |

Table 1: Comparison of timings using Algorithms 3.1 and 3.3.

**Example 3.8.** We constructed a series of examples from Stanley-Reisner simplicial complexes of $N$ randomly chosen squarefree monomial ideals in a polynomial ring with $k$ variables. The results are summarized in Table 2.

## 4 Projection

One application of using the singular value decomposition of a complex is to compute the pseudoinverse complex as described in Section 2. The following projects a sequence of matrices onto a complex.

**Algorithm 4.1** (Projection to a complex).

**INPUT:** A sequence $B_1, \ldots, B_n$ of $c_{i-1} \times c_i$ matrices and a sequence $h_0, \ldots, h_n$ of desired dimension of homology groups.

**OUTPUT:** If possible, a sequence $A_1, \ldots, A_n$ of matrices which define a complex with desired homology.

1. Set $r_0 = 0$ and compute $r_1, \ldots, r_{n+1}$ from $h_i = c_i - (r_i + r_{i+1})$ recursively. If $r_i < 0$ or $r_i > \text{rank } B_i$ for some $i$ or $r_{n+1} \neq 0$, then return the error message: “The desired dimension of homology groups cannot be satisfied.”
2. Set $Q_0 = 0$ and $P_0 = \text{id}_{c_0}$.

3. For $i = 1, \ldots, n$
   a. Compute the $(c_{i-1} - r_{i-1}) \times c_i$ matrix $	ilde{B}_i = P_{i-1} \circ B_i$.
   b. Compute the singular value decomposition
      $$\tilde{B}_i = \tilde{U}_{i-1} \circ \tilde{\Sigma}_i \circ \tilde{V}_t^t.$$ 
   c. Compute
      $$\tilde{\Sigma}_i = \begin{pmatrix} r_i & r_{i+1} & h_i \\ r_{i-1} & 0 & 0 \\ h_{i-1} & \Sigma_i & 0 \end{pmatrix}$$
      as a block matrix where $\Sigma_i$ is a diagonal matrix whose entries are the largest $r_i$ singular values of $\tilde{B}_i$.
   d. Decompose
      $$\tilde{V}_i^t = \begin{pmatrix} Q_i \\ P_i \end{pmatrix}$$
      into submatrices consisting of the first $r_i$ and last $c_i - r_i$ rows of $\tilde{V}_i^t$. 

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**Table 2:** Comparison of timings using Algorithm 3.1.

| $k$ | $N$ | $c_0$ | $c_1$ | $c_2$ | ... | Alg 3.1 (sec) |
|-----|-----|-------|-------|-------|-----|--------------|
| 8   | 20  | 8     | 27    | 44    | 30  | 0.0019       |
| 9   | 21  | 9     | 35    | 74    | 85  | 0.0036       |
| 10  | 23  | 10    | 45    | 118   | 190 | 0.0198       |
| 11  | 26  | 11    | 55    | 165   | 326 | 0.2410       |
| 12  | 30  | 12    | 66    | 218   | 474 | 1.29         |
| 13  | 35  | 13    | 78    | 286   | 712 | 39.7         |
| 14  | 41  | 14    | 91    | 364   | 996 | 355.         |


e. Compute

\[ U_{i-1}^t = \left( Q_{i-1} \tilde{U}_{i-1}^t \circ P_{i-1} \right). \]

f. If \( i = n \), then set \( U_n = \tilde{V}_n^t. \)

4. Set \( A_i = U_{i-1} \circ \Sigma_i \circ U_i^t \) and return \( A_1, \ldots, A_n. \)

Proof of correctness. It is clear that the construction of the \( A_i \)'s yields a complex presented in the form of (2).

Example 4.2. In our package RandomComplexes, we have implemented several methods to produce complexes over the integers. The first function randomChainComplex takes as input sequences \( h_1, \ldots, h_n \) and \( r_1, \ldots, r_n \) of desired dimension of homology groups and ranks of the matrices, respectively. It uses the LLL algorithm [LLL82] to produce examples of desired moderate height. It runs fast for complexes with \( c_i \leq 100 \) but is slow for larger examples because of the use of the LLL-algorithm. Example 3.6 was produced this way.

For a given a complex, only allowing one homology group to change provides a description of its singular values. This is summarized in the following.

Remark 4.3. For a matrix \( A \in \mathbb{R}^{m \times k} \), \( \sigma_{r+1} \) is the distance between \( A \) and set of matrices of rank at most \( r \) via (1). Singular values of a complex have a similar description. In particular, if \( A_1, \ldots, A_n \) define a complex \( C_* \) with \( A_i \in \mathbb{R}^{c_i-1 \times c_i} \), \( r_i = \text{rank } A_i \), and \( A_0 = A_{n+1} = 0 \), then, for \( r = 0, \ldots, r_i - 1 \), the \( (r+1) \)st singular value of \( A_i \), namely \( \sigma_{r+i} \), is equal to

\[ \min_{B \in \mathbb{R}^{c_i-1 \times c_i}} \{ \| A_i - B \|_2 \mid \text{rank } B \leq r, A_{i-1} \circ B = 0, B \circ A_{i+1} = 0 \} \]

which one can view as the distance between the complex \( C_* \) and the set of complexes consisting of matrices \( A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n \) where \( \text{rank } B \leq r \). Moreover, one can solve (3) using a singular value decomposition for \( C_* \) with, say, orthogonal matrices \( U_0, \ldots, U_n \) and diagonal matrices \( \Sigma_1, \ldots, \Sigma_n \). In particular, with \( \Sigma_i = \text{diag}(\sigma_1^i, \ldots, \sigma_{r_i}^i)_{r_i \times r_i} \), the matrix

\[ B_r^i = U_{i-1} \circ \left( \begin{array}{ccc} 0 & 0 & 0 \\ \Lambda_r^i & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \circ U_i^t \]

solves (3) where \( \Lambda_r^i = \text{diag}(\sigma_1^i, \ldots, \sigma_{r_i}^i, 0, \ldots, 0)_{r_i \times r_i} \) and \( \sigma_{r+i} = \| A_i - B_r^i \|_2. \)
Measuring distances between two sequences of matrices where only one matrix is different can be viewed as equivalent to simply measuring the distance between the only differing matrices. Remark 4.3 uses this to describe the singular values of complex. Algorithm 4.1 also uses this idea via a greedy approach at each step in its construction. In general, one can construct equivalent norms on sequences of matrices by building from the norms of the individual entries of the matrices. Given a sequence of matrices $B_1, \ldots, B_n$ and constants $\alpha_i, \beta_i > 0$, then

$$\sum_{i=1}^{n} \alpha_i \|B_i\|_2 \quad \text{and} \quad \max \{ \beta_i \|B_i\|_2 \mid i = 1, \ldots, n \}$$

are easily seen to be equivalent norms. Therefore, given a norm on a sequence of matrices, we leave it as an open problem to compute the nearest complex with desired homology group dimensions to a given sequence of matrices.

5 Application to syzygies

We conclude with an application concerning the computation of Betti numbers in free resolutions. Let $S = K[x_0, \ldots, x_n]$ be the standard graded polynomial ring and $M$ a finitely generated graded $S$-module. Then, by Hilbert’s syzygy theorem, $M$ has a finite free resolution:

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_c \leftarrow 0$$

by free graded $S$-modules $F_i = \sum_j S(\ell_i - j)^{b_{ij}}$ of length $c \leq n + 1$. Here $S(\ell)$ denotes the free $S$-module with generator in degree $\ell$.

If we choose in each step a minimal number of homogenous generators, i.e., if $\varphi_i(F_i) \subset (x_0, \ldots, x_n)F_{i-1}$, then the free resolution is unique up to an isomorphism. In particular, the Betti numbers $b_{ij}$ of a minimal resolution are numerical invariants of $M$. On the other hand, for basic applications of free resolutions such as the computation of $\text{Ext}$ and $\text{Tor}$-groups, any resolution can be used.

Starting with a reduced Gröbner basis of the submodule $\varphi_1(F_1) \subset F_0$ there is, after some standard choices on orderings, a free resolution such that at each step the columns of $\varphi_{i+1}$ form a reduced Gröbner basis of $\ker \varphi_i$. This resolution is uniquely determined however, in most cases, highly nonminimal. An algorithm to compute this standard nonminimal resolution was developed in [EMSS16] which turned out to be much faster than the computation of a minimal resolution by previous methods.

The following forms the examples which we use as test cases.
**Proposition 5.1** (Graded Artinian Gorenstein Algebras). Let \( f \in \mathbb{Q}[x_0, \ldots, x_n] \) be a homogeneous polynomial of degree \( d \). In \( S = \mathbb{Q}[\partial_0, \ldots, \partial_n] \), consider the ideal \( I = \langle D \in S \mid D(f) = 0 \rangle \) of constant differential operators which annihilate \( f \). Then, \( A_f^\perp := S/I \) is an artinian Gorenstein Algebra with socle in degree \( d \).

For more information on this topic see, e.g., [RS00].

**Example 5.2.** Let \( f = \ell_1^4 + \ldots + \ell_{18}^4 \in \mathbb{Q}[x_0, \ldots, x_7] \) be the sum of \( 4^{th} \) powers of 18 sufficiently general chosen linear forms \( \ell_s \). The Betti numbers \( b_{ij} \) of the minimal resolution \( M = A_f^\perp \) as an \( S \)-module are zero outside the range \( i = 0, \ldots, 8, j = 0, \ldots, 4 \). In this range, they take the values:

| \( j \ \backslash \ i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------------------|---|---|---|---|---|---|---|---|---|
| 0                 | 1 | . | . | . | . | . | . | . | . |
| 1                 | . | 18 | 42 | . | . | . | . | . | . |
| 2                 | . | 10 | 63 | 288 | 420 | 288 | 63 | 10 | . |
| 3                 | . | . | . | . | . | . | 42 | 18 | . |
| 4                 | . | . | . | . | . | . | . | . | 1 |

which, for example, says that \( F_2 = S(-3)^{42} \oplus S(-4)^{63} \). We note that the symmetry of the table is a well-known consequence of the Gorenstein property.

On the other hand the Betti numbers of the uniquely determined nonminimal resolution are much larger:

| \( j \ \backslash \ i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------------------|---|---|---|---|---|---|---|---|---|
| 0                 | 1 | . | . | . | . | . | . | . | . |
| 1                 | . | 18 | 55 | 75 | 54 | 20 | 3 | . | . |
| 2                 | . | 23 | 145 | 390 | 580 | 515 | 273 | 80 | 10 |
| 3                 | . | 7 | 49 | 147 | 245 | 245 | 147 | 49 | 7 |
| 4                 | . | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

To deduce from this resolution the Betti numbers of the minimal resolution, we can use the formula

\[
b_{ij} = \dim \text{Tor}_i^S(M, \mathbb{Q})_{i+j}.\]

For example, to deduce \( b_{3,2} = 288 \), we have to show that the 5th constant strand of the nonminimal resolution

\[
0 \leftarrow \mathbb{Q}^1 \leftarrow \mathbb{Q}^{49} \leftarrow \mathbb{Q}^{390} \leftarrow \mathbb{Q}^{54} \leftarrow 0
\]

has homology only in one position.
The matrices defining the differential in the nonminimal resolution have polynomial entries whose coefficients in \( \mathbb{Q} \) can have very large height such that the computation of the homology of the strands becomes infeasible. There are two options, how we can get information about the minimal Betti numbers:

- Pick a prime number \( p \) which does not divide any numerator of the normalized reduced Gröbner basis and then reduce modulo \( p \) yielding a module \( M(p) \) with the same Hilbert function as \( M \). Moreover, for all but finitely many primes \( p \), the Betti numbers of \( M \) as an \( \mathbb{Q}[x_0, \ldots, x_n] \)-module and of \( M(p) \) as \( \mathbb{F}_p[x_0, \ldots, x_n] \)-module coincide.

- Pass from a normalized reduced Gröbner basis of \( \varphi_1(F_1) \subset F_0 \) to a floating-point approximation of the Gröbner basis. Since in the algorithm for the computation of the uniquely determined nonminimal resolution [EMSS16], the majority of ground field operations are multiplications, we can hope that this computation is numerically stable and that the singular value decompositions of the linear strands will detect the minimal Betti numbers correctly.

**Example 5.3.** We experimented with artinian graded Gorenstein algebras constructed from randomly chosen forms \( f \in \mathbb{Q}[x_0, \ldots, x_7] \) in 8 variables which were the sum of \( n \) 4th powers of linear forms where \( 11 \leq n \leq 20 \). This experiment showed that roughly 95% of the Betti table computed via floating-point arithmetic coincided with one computed over a finite field. The reason for this was that the current implementation uses only double precision floating-point computations which caused difficulty in detecting zero singular values correctly. This could be improved following Remark 3.2 using higher precision arithmetic.

We now consider a series of examples related to the famous Green’s conjecture on canonical curves which was proved in a landmark paper [Voi05] for generic curves. In \( S = \mathbb{Q}[x_0, \ldots, x_a, y_0, \ldots, y_b] \), consider the homogeneous ideal \( J_e \) generated by the \( 2 \times 2 \) minors of

\[
\begin{pmatrix}
  x_0 & x_1 & \cdots & x_{a-1} \\
  x_1 & x_2 & \cdots & x_a \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  y_0 & y_1 & \cdots & y_{b-1} \\
  y_1 & y_2 & \cdots & y_b \\
\end{pmatrix}
\]

together with the entries of the \((a-1) \times (b-1)\) matrix

\[
\begin{pmatrix}
  x_0 & x_1 & x_2 \\
  x_1 & x_2 & x_3 \\
  \vdots & \vdots & \vdots \\
  x_{a-2} & x_{a-1} & x_a \\
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & e_2 \\
  0 & -e_1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  y_0 & y_1 & \cdots & y_{b-2} \\
  y_1 & y_2 & \cdots & y_{b-1} \\
  y_2 & y_3 & \cdots & y_b \\
\end{pmatrix}
\]
for some parameters $e_1, e_2 \in \mathbb{Q}$. Then, by [ES18], $J_e$ is the homogeneous ideal of an arithmetically Gorenstein surface $X_e(a, b) \subset \mathbb{P}^{a+b+1}$ with trivial canonical bundle. Moreover, the generators of $J_e$ form a Gröbner basis. To verify the generic Green’s conjecture for curves of odd genus $g = 2a + 1$, it suffices to prove, for some values $e = (e_1, e_2) \in \mathbb{Q}^2$, that $X_e(a, a)$ has a “natural” Betti table, i.e., for each $k$ there is at most one pair $(i, j)$ with $i + j = k$ and $b_{ij}(X_e(a, a)) \neq 0$. For special values of $e = (e_1, e_2)$, e.g., $e = (0, -1)$, it is known that the resolution is not natural, see [ES18].

**Example 5.4.** For $a = b = 6$, our implementation computes the following Betti numbers for the nonminimal resolution: as

|   | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   |
| .  | 55  | 320 | 930 | 1688| 2060| 1728| 987 | 368 | 81  | 8   | 1   | .   |
| .  | .   | 39  | 280 | 906 | 1736| 2170| 1832| 1042| 384 | 83  | 8   | .   |
| .  | .   | .   | 1   | 8   | 28  | 56  | 70  | 56  | 28  | 8   | 1   | .   |

For $e = (2, -1)$ and $e = (0, -1)$, our implementation correctly computes the following Betti numbers, respectively, of the minimal resolutions:

|   | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   |
| .  | 55  | 320 | 891 | 1408| 1155| .   | .   | .   | .   | .   | .   | .   |
| .  | .   | .   | .   | .   | 1155| 1408| 891 | 320 | 55  | .   | .   | .   |
| .  | .   | .   | .   | .   | .   | .   | .   | .   | .   | 1   | .   | .   |

|   | 1   | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   | .   |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| .  | 55  | 320 | 900 | 1488| 1470| 720 | 315 | 80  | 9   | .   | .   | .   |
| .  | .   | 9   | 80  | 315 | 720 | 1470| 1488| 900 | 320 | 55  | .   | .   |
| .  | .   | .   | .   | .   | .   | .   | .   | .   | .   | 1   | .   | .   |

Each of these computations took several minutes and the results agree with those presented in [ES18] using exact methods which took several hours. To consider larger examples, more efficient algorithms and/or implementations for computing the singular value decomposition of a complex are needed.

**References**

[AFW06] D.N. Arnold, R.S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer. 15 (2006), 1–155.
[AFW10] _______. *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 2, 281–354.

[EMSS16] B. Eröcal, O. Motsak, F.-O. Schreyer, and A. Steenpaß, *Refined algorithms to compute syzygies*, J. Symbolic Comput. **74** (2016), 308–327.

[EY36] C. Eckart and G. Young, *The approximation of one matrix by another of lower rank*, Psychometrika **1** (1936), no. 3, 211–218.

[ES18] D. Eisenbud and F.-O. Schreyer, *Equation and syzygies of K3 carpets and union of scrolls*, preprint, arXiv:1804.08011 (2018).

[LLL82] A.K. Lenstra, H.W. Lenstra Jr., and L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann. **261** (1982), no. 4, 515–534.

[M2] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).

[Pen55] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413.

[RS00] K. Ranestad and F.-O. Schreyer, *Varieties of sums of powers*, J. Reine Angew. Math. **525** (2000), 147–181.

[Voi05] C. Voisin, *Green’s canonical syzygy conjecture for generic curves of odd genus*, Compos. Math. **141** (2005), no. 5, 1163–1190.
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