EXISTENCE OF UNCOUNTABLY MANY ASYMPTOTICALLY CONSTANT SOLUTIONS TO DISCRETE NONLINEAR THREE-DIMENSIONAL SYSTEM WITH $p$-LAPLACIAN

Magdalena Nockowska-Rosiak
Institute of Mathematics
Lodz University of Technology
Wolczanska 215, 90-924 Lodz, Poland

Piotr Hachuła*
Institute of Logistics and Warehousing
Estkowskiego 6, 61-755 Poznan, Poland

Ewa Schmeidel
Institute of Mathematics
University of Białystok
Ciolkowskiego 1M, 15-245 Białystok, Poland

Abstract. This work is devoted to the study of the existence of uncountably many asymptotically constant solutions to discrete nonlinear three-dimensional system with $p$-Laplacian.

1. Introduction and preliminaries. Equations involving the discrete $p$-Laplacian operator have been widely studied by many authors and several approaches, see for example [4], [5] and [6]. Agarwal, Perera and O’Regan obtain multiple positive solutions of singular discrete $p$-Laplacian problems via variational methods in [1]. Paper [2] is devoted to the study of the existence of at least one (non-zero) solution to a problem involving the discrete $p$-Laplacian. In [8], He by means of a fixed point theorem in a cone, studies the existence of positive solutions of $p$-Laplacian difference equations. In [9], Lee and Lee consider $p$-Laplacian systems with singular weights. By exploiting Amann type three solutions theorem for a singular system, the authors prove the existence, nonexistence, and multiplicity of positive solutions when nonlinear terms have a combined sublinear effect at infinity.

Three-dimensional systems were considered in [10], [13] and [14]. Here we continue such a study by considering the following three-dimensional system of difference equations with $p$-Laplacian operator

$$
\begin{align*}
\Delta(\Phi_p(x(n))) &= a(n)f(y(n - l)) \\
\Delta(y(n)) &= b(n)g(z(n - m)) \\
\Delta(z(n)) &= c(n)h(x(n - k))
\end{align*}
$$

(1)

where $n \geq n_0 = \max \{ k, l, m \} \in \mathbb{N}_0 = \{ 0, 1, 2, \ldots \}$ and $\Delta$ is the forward difference operator, $(a(n)), (b(n)), (c(n))$ are sequences of real numbers, $l, k, m$ are nonnegative

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* Corresponding author: Piotr Hachuła.
integers, functions $f, g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions. Moreover, for $p \in \mathbb{R}$, $p > 1$ the $p$-Laplacian operator $\Phi_p : \mathbb{R} \to \mathbb{R}$ is defined by formula

$$\Phi_p(t) = \begin{cases} |t|^{p-2}t, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0 \end{cases}.$$  \hfill (2)

In system (1), $(x(n))$, $(y(n))$ and $(z(n))$ are real sequences defined for $n \in \mathbb{N}$. Throughout this paper, $X(n)$ denotes a vector $[x(n), y(n), z(n)]^T$. By a solution of system (1) we mean a sequence $X = (X(n))$ which fulfils system (1) for $n \geq n_0$.

Notice that by taking $p = 2$ system (1) reduces to the system which was considered in [11] and [12].

For the elements of $\mathbb{R}^3$ the symbol $|\cdot|$ stands for the maximum norm. By $B$ we denote the Banach space of all bounded sequences in $\mathbb{R}^3$ with the supremum norm, i.e.,

$$B = \left\{ X : \mathbb{N} \to \mathbb{R}^3 : \|X\| = \sup_{n \in \mathbb{N}} |X(n)| < \infty \right\}.$$  

It is well known that $\Phi_p$ given by (2) is a homeomorphism on $\mathbb{R}$, with $(\Phi_p)^{-1} = \Phi_{p^*}$, where

$$\frac{1}{p} + \frac{1}{p^*} = 1 \quad \text{(or equivalently } p^* = \frac{p}{p-1}).$$

Under continuity of functions $f, g, h$ and assumptions

$$\sum_{n=0}^{\infty} |a(n)| < \infty, \quad \sum_{n=0}^{\infty} |b(n)| < \infty, \quad \sum_{n=0}^{\infty} |c(n)| < \infty,$$

we prove the sufficient conditions under which for any real constants $d_1, d_2, d_3$ there exists a solution to the considered system convergent to $[d_1, d_2, d_3]^T$. That connects the subject of this paper with so-called permanence of solutions, which is important for example in mathematical modelling of problems arising in biology.

The following theorems will be used in the sequel.

**Theorem 1.1** (Schauder’s Fixed Point Theorem, [3], p. 24). Let $M$ be a nonempty, compact and convex subset of a Banach space and let $T : M \to M$ be continuous. Then $T$ has a fixed point in $M$.

By $c_0$ we denote the Banach space of all convergent to zero sequences with the supremum norm.

**Theorem 1.2** ([7], p. 107). A set $A \subset c_0$ is relatively norm compact if and only if there is a sequence $\lambda = (\lambda(n)) \in c_0$ such that $|x(n)| \leq \lambda(n)$ for any $x = (x(n)) \in A$ and for any $n \in \mathbb{N}$.

2. Main results. In this section, we present conditions under which there exists asymptotically constant solution to system (1). In the first step we prove the following existence result.

**Theorem 2.1.** Assume that

(A$_1$) $p \in \mathbb{R}$, $p > 1$ and $k, l, m \in \mathbb{N}_0$,

(A$_2$) $f, g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions,

If

$$\sum_{n=0}^{\infty} |a(n)| < \infty, \quad \sum_{n=0}^{\infty} |b(n)| < \infty, \quad \sum_{n=0}^{\infty} |c(n)| < \infty,$$  \hfill (3)
then for any real constants \(d_1, d_2, d_3\) there exists a sequence \((X(n))\) which fulfils system (1) for sufficiently large \(n\) such that
\[
\lim_{n \to \infty} X(n) = [d_1, d_2, d_3]^T.
\]

**Proof.** Taking into consideration that \(\Phi_p : \mathbb{R} \to \mathbb{R}\) is a homeomorphism, then the considered system (1) by substitution \(\bar{x}(n) = \Phi_p(x(n))\) is equivalent to the following problem
\[
\begin{align*}
\Delta(\bar{x}(n)) &= a(n)f(y(n - l)) \\
\Delta(y(n)) &= b(n)g(z(n - m)) \\
\Delta(z(n)) &= c(n)(h \circ \Phi_p)(\bar{x}(n - k)).
\end{align*}
\]

Let \(d_1, d_2, d_3 \in \mathbb{R}, \delta > 0\). From (A2) there exist \(M_f, M_g, M_h > 1\) such that
\[
\begin{align*}
|f(t)| &\leq M_f, \quad t \in [d_2 - \delta, d_2 + \delta], \\
|g(t)| &\leq M_g, \quad t \in [d_3 - \delta, d_3 + \delta], \\
|h(t)| &\leq M_h, \quad t \in [\Phi_p(d_1) - \delta, \Phi_p(d_1) + \delta],
\end{align*}
\]
where \(\bar{h} := h \circ \Phi_p\). From (3) there exists \(n_1 \geq \max\{l, m, k\}\) such that
\[
\sum_{s=n_1}^{\infty} |a(s)| \leq \frac{\delta}{M_f}, \quad \sum_{s=n_1}^{\infty} |b(s)| \leq \frac{\delta}{M_g}, \quad \sum_{s=n_1}^{\infty} |c(s)| \leq \frac{\delta}{M_h}.
\]

Set \(D = [\Phi_p(d_1), d_2, d_3]^T\) and \(n_2 = n_1 + n_0\). We define
\[
\Omega = \left\{ X = (X(n)) \in B : (X(n)) = D, n < n_2, \left| \left(x(n) - \Phi_p(d_1)\right) \right| \leq M_f \sum_{s=n}^{\infty} |a(s)|, \right.
\]
\[
\left. |y(n) - d_2| \leq M_g \sum_{s=n}^{\infty} |b(s)|, \quad |z(n) - d_3| \leq M_h \sum_{s=n}^{\infty} |c(s)|, \quad n \geq n_2 \right\}.
\]

It is easy to see that \(\Omega\) is a closed and convex subset of Banach space \(B\). Taking into consideration that \(c_0\) is a closed subspace of space \(B\), to prove that \(\Omega\) is a compact subset of \(B\) we use the relative compactness criterion in \(c_0\). From (3) we get that
\[
\left( M_f \sum_{s=n}^{\infty} |a(s)| \right) \in c_0, \quad \left( M_g \sum_{s=n}^{\infty} |b(s)| \right) \in c_0, \quad \left( M_h \sum_{s=n}^{\infty} |c(s)| \right) \in c_0.
\]

From Theorem 1.2 and above we get that for any sequences \((X^k) = ([x^k, y^k, z^k]^T) \subset \Omega\) there exist subsequences \((x^{k_1} - \Phi_p(d_1)), (y^{k_1} - d_2), (z^{k_1} - d_3)\) which are convergent sequences in \(c_0\) to \(x, y, z \in c_0\), respectively. Hence, \((X^{k_1})\) is convergent to \(X = [x + \Phi_p(d_1), y + d_2, z + d_3]^T \in \Omega\).

We define \(F : \Omega \to B\), where \(F = [F_1, F_2, F_3]^T\) as follows
\[
(FX)(n) = \begin{bmatrix}
\Phi_p(d_1) - \sum_{s=n}^{\infty} a(s)f(y(s - l)) \\
d_2 - \sum_{s=n}^{\infty} b(s)g(z(s - m)) \\
d_3 - \sum_{s=n}^{\infty} c(s)h(x(s - k))
\end{bmatrix}
\]
for \(n \geq n_2\),

and \((FX)(n) = D\) for \(0 < n < n_2\). Firstly, we will show that \(F : \Omega \to \Omega\). For \(s \geq n_2\) and \(X \in \Omega\) we get from (9)
\[
|y(s - l) - d_2| \leq M_g \sum_{s=n}^{\infty} |b(s)| \leq \delta,
\]
and from (6)
\[ |f(y(s - l))| \leq M_f. \]

Hence,
\[ |(F_1X)(n) - \Phi_p(d_1)| \leq \sum_{s=n}^{\infty} |a(s)||f(y(s - l))| \leq M_f \sum_{s=n}^{\infty} |a(s)|. \]  

(10)

For \( s \geq n_2 \) and \( X \in \Omega \) we get from (9)
\[ |z(s - m) - d_3| \leq M_h \sum_{s=n}^{\infty} |c(s)| \leq \delta, \]

and from (7)
\[ |g(z(s - m))| \leq M_g. \]

Hence,
\[ |(F_2X)(n) - d_2| \leq \sum_{s=n}^{\infty} |b(s)||g(z(s - m))| \leq M_g \sum_{s=n}^{\infty} |b(s)|. \]  

(11)

For \( s \geq n_2 \) and \( X \in \Omega \) we get from (9)
\[ |x(s - k) - \Phi_p(d_1)| \leq M_f \sum_{s=n}^{\infty} |a(s)| \leq \delta, \]

and from (8)
\[ |\bar{h}(x(s - k))| \leq M_{\bar{h}}. \]

Hence,
\[ |(F_3X)(n) - d_3| \leq \sum_{s=n}^{\infty} |c(s)||\bar{h}(x(s - k))| \leq M_{\bar{h}} \sum_{s=n}^{\infty} |c(s)|. \]  

(12)

Therefore, \( F(\Omega) \subset \Omega \).

Now we prove the continuity of \( F \). Let \( X = (X(n)) = [x(n), y(n), z(n)]^T \in \Omega, \)
\( X_j = (X_j(n)) = [x_j(n), y_j(n), z_j(n)]^T \in \Omega \) such that \( X_j \to X \) as \( j \to \infty \). Let \( \varepsilon > 0 \).

The functions \( f, g, \bar{h} \) are uniformly continuous on any compact interval, so then there exists \( \delta_1 > 0 \) such that
\[ |f(u) - f(v)| \leq \delta_1, \quad |u - v| \leq \delta_1, \quad u, v \in [d_2 - \delta, d_2 + \delta], \]
\[ |g(u) - g(v)| \leq \delta_2, \quad |u - v| \leq \delta_2, \quad u, v \in [d_3 - \delta, d_3 + \delta], \]
\[ |\bar{h}(u) - \bar{h}(v)| \leq \delta_3, \quad |u - v| \leq \delta_3, \quad u, v \in [\Phi_p(d_1) - \delta, \Phi_p(d_1) + \delta], \]

(13) (14) (15)

where \( A := 1 + \sum_{s=0}^{\infty} |a(s)|, \ B := 1 + \sum_{s=0}^{\infty} |b(s)|, \ C := 1 + \sum_{s=0}^{\infty} |c(s)|. \)

From \( X_j \to X \) we get that there exists \( j_0 \in \mathbb{N}_0 \) such that
\[ |x_j(s - k) - x(s - k)| \leq \delta_1, \quad |y_j(s - l) - y(s - l)| \leq \delta_1, \quad |z_j(s - m) - z(s - m)| \leq \delta_1 \]
(16)

for any \( j \geq j_0, s \geq n_0. \) Then from (13), (16) we get for any \( n \geq n_2, j \geq j_0 \)
\[ |(F_1X_j)(n) - (F_1X)(n)| \leq \sum_{s=n}^{\infty} |a(s)||f(y_j(s - l)) + \sum_{s=n}^{\infty} |a(s)||f(y(s - l))| \leq \delta \sum_{s=n}^{\infty} |a(s)| \leq \varepsilon. \]
We get conditions $\|(F_2 X_j)(n) - (F_2 X)(n)\| \leq \varepsilon$, $\|(F_3 X_j)(n) - (F_3 X)(n)\| \leq \varepsilon$ for $n \geq n_2$, $j \geq j_0$ in an analogous way from (15). Because $(F X)(n) = (F X_j)(n)$ for $n < n_2$ and $j \in \mathbb{N}_0$ we get that for $j \geq j_0$

$$\|FX_j - FX\| \leq \varepsilon,$$

which means that $F$ is continuous on $\Omega$.

By Theorem 1.1, $F$ has the fixed point on $\Omega$, denoted by $\bar{X} = (\bar{X}(n)) = (\bar{x}(n), y(n), z(n))$. For $n \geq n_2$ it means that $(F_1 \bar{X})(n) = \bar{x}(n)$ and hence

$$\bar{x}(n) = \Phi_p(d_1) - \sum_{s=n}^{\infty} a(s)f(y(s-l)).$$

By applying the forward difference to the obtained equation we get

$$\Delta(\bar{x}(n)) = a(n)f(y(n-l)), \quad n \geq n_2.$$

By an analogous way we get for $n \geq n_2$

$$\Delta y(n) = b(n)g(z(n-m))$$

$$\Delta z(n) = c(n)h(\bar{x}(n-k)).$$

It means that $(\bar{X}(n))$ fulfills (5) for $n \geq n_2$. From (10), (11), (12) and (3) we get that

$$\lim_{n \to \infty} \bar{X}(n) = [\Phi_p(d_1), d_2, d_3]^T.$$ 

Since $\Phi_p$ is a homeomorphism on $\mathbb{R}$, then for any $n \geq n_2$ there exists $x(n)$ such that $\bar{x}(n) = \Phi_p(x(n))$. This implies that $X(n) = [x(n), y(n), z(n)]^T$ satisfies (1) for any $n \geq n_2$ with $\lim_{n \to \infty} X(n) = [d_1, d_2, d_3]^T$. $\square$

**Example 1.** Let us consider the following system:

$$\begin{cases}
\Delta(\Phi_p(x(n))) = \frac{(3^{n(2-p)}3^{1-p} - 1)}{2 \cdot 3^{n-3}} y(n-1) \\
\Delta y(n) = -2 \cdot \frac{3^{2n-1} - 1}{3^{n-2}} x^2(n-1) \\
\Delta z(n) = \frac{8}{81} \cdot \frac{1}{3^n} x(n-2)
\end{cases}$$

All assumptions of Theorem 2.1 are satisfied. It is easy to check that

$$X(n) = \left[\frac{1}{3^n} \cdot 2 + \frac{1}{3^n} \cdot 3 - \frac{1}{3^{2n}}\right]^T$$

is the solution of the above system having the property $\lim_{n \to \infty} X(n) = [0, 2, 3]^T$.

**Remark 1.** Assume that $(A_1)$, $(A_2)$ and (3) are satisfied. If $(A_3)$ $f, g, h$ are invertible, $(A_4)$ $a(n)b(n)c(n) \neq 0$ for any $n \in \mathbb{N}_0$,

then for any real constants $d_1, d_2, d_3$ there exists a solution $(X(n))$ to system (1) satisfying condition (4).

**Proof.** On virtue of Theorem 2.1, there exists a sequence $(X(n))$, satisfying condition (4), which fulfils system (1) for sufficiently large $n$. Since $(A_3)$ and $(A_4)$ hold, we can rewrite system (1) in the following form

$$\begin{cases}
y(n-l) = f^{-1}\left(\frac{\Phi_p(x(n+l)) - \Phi_p(x(n))}{a(n)}\right) \\
z(n-m) = g^{-1}\left(\frac{y(n+l) - y(n)}{b(n)}\right), \quad n \geq n_2, \\
x(n-k) = h^{-1}\left(\frac{z(n+l) - z(n)}{c(n)}\right)
\end{cases}$$
Using the above system we find $X(n_2 - 1), X(n_2 - 2), \ldots, X(0)$. Hence, sequence $(X(n))$ fulfills system (1) for $n \geq n_0$. \hfill \Box

Next, we present the necessary condition for existence of asymptotically constant solution to system (1).

**Theorem 2.2.** Assume that $(A_1)$ and $(A_2)$ are satisfied, and
\begin{equation}
(a(n)), (b(n)), (c(n)) \text{ are positive sequences.} \tag{17}
\end{equation}
If for real constants $d_1, d_2, d_3$ such that
$$f(d_2)g(d_3)h(d_1) \neq 0,$$
there exists a solution $(X(n))$ to system (1) satisfying condition (4), then (3) holds.

**Proof.** If $f(d_2) \neq 0$ then there exists $\varepsilon > 0$ such that $f(t)$ is of one sign, say positive, for $t \in [d_2 - \varepsilon, d_2 + \varepsilon] = U(d_2, \varepsilon)$. Set $L = \min \{f(t) : t \in U(d_2, \varepsilon)\}$. Let us take $n_3$ so large that $f(y(n - l)) \in U(d_2, \varepsilon)$ for $n \geq n_3$. Hence,
$$0 < L < f(y(n - l)). \tag{18}$$
From the first equation of system (1)
$$|\Phi_p(x(n)) - \Phi_p(x(n_3))| = |\sum_{n=n_3}^{n-1} a(n)f(y(n - l))|.$$
Since $\lim_{n \to \infty} |\Phi_p(x(n)) - \Phi_p(x(n_3))| = |\Phi_p(d_1) - \Phi_p(x(n_3))| < \infty$, we obtain $|\sum_{n=n_3}^{\infty} a(n)f(y(n - l))| < \infty$. This, (17) and (18) imply $\sum_{n=0}^{\infty} |a(n)| < \infty$. \hfill \Box

**REFERENCES**

[1] R. Agarwal, K. Perera and D. O’Regan, Multiple positive solutions of singular discrete $p$-Laplacian problems via variational methods, *Adv. Difference Equ.*, 2005 (2005), 93–99.

[2] G. Bisci and D. Repovš, *Existence of solutions for $p$-Laplacian discrete equations*, *Appl. Math. Comput.*, 242 (2014), 454–461.

[3] A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, Inc., Mineola, NY, 2006.

[4] A. Cabada, *Extremal solutions for the difference $\phi$-Laplacian problem with nonlinear functional boundary conditions*, *Comput. Math. Appl.*, 42 (2001), 593–601.

[5] A. Cabada and V. Otero-Espinar, *Existence and comparison results for difference $\phi$-Laplacian boundary value problems with lower and upper solutions in reverse order*, *J. Math. Anal. Appl.*, 267 (2002), 501–521.

[6] M. Cecchi, Z. Došlá and M. Marini, *Intermediate solutions for nonlinear difference equations with $p$-Laplacian*, *Adv. Stud. Pure Math.*, 53 (2009), 33–40.

[7] C. Costara and D. Popa, *Exercises in Functional Analysis*, Kluwer Academic Publisher Group, Dordrecht, 2003.

[8] Z. He, *On the existence of positive solutions of $p$-Laplacian difference equations*, *J. Comput. Appl. Math.*, 161 (2003), 193–201.

[9] E. Lee and Y. Lee, *A result on three solutions theorem and its application to $p$-Laplacian systems with singular weights*, *Boundary Value Problems*, 2012 (2012), 20pp.

[10] M. Migda, E. Schmeidel and M. Zdanowicz, *Existence of nonoscillatory solutions for system of neutral difference equations*, *Appl. Anal. Discrete Math.*, 9 (2015), 271–284.

[11] E. Schmeidel, *Boundedness of solutions of nonlinear three-dimensional difference systems with delays*, *Fasc. Math.*, 44 (2010), 107–113.

[12] E. Schmeidel, *Oscillation of nonlinear three-dimensional difference systems with delays*, *Math. Bohem.*, 135 (2010), 163–170.

[13] E. Schmeidel, *Properties of solutions of system of difference equations with neutral term*, *Funct. Differ. Equ.*, 18 (2011), 293–302.
[14] E. Thandapani and B. Ponnammal, Oscillatory properties of solutions of three dimensional difference systems, *Math. Comput. Modelling*, 42 (2005), 641–650.

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E-mail address: magdalena.nockowska@p.lodz.pl
E-mail address: piotr.hachula@gmail.com
E-mail address: eschmeidel@math.uwb.edu.pl