The Forgotten Night: The Number Devil Explores Spherical Geometry

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(Dated: February 1, 2008)

This is a missing chapter from Hans Magnus Enzensberger's mathematical adventure The Number Devil (Henry Holt and Company, New York, 1997). In the book, a math-hating boy named Robert is visited in his dreams by the clever Number Devil, who teaches him to love all things numerical. However, we all forget our dreams from time to time. Here is one adventure that Enzensberger overlooked, where the Number Devil introduces Robert to geometry not-of-Euclid, great circles, parallel transport, the pendulum of Foucault, and the genius of Euler.

The next night, Robert found himself sitting in a strange square room with silver walls covered with rows upon rows of buttons and switches and little blinking lights. Across the room, the Number Devil sat perched upon a tall stool. He was peering closely at a column of dials set into the wall. Robert cleared his throat, and the Number Devil swiveled around on his stool.

"Hello, Robert," he said. "I have something quite extraordinary to show you today. How much do you know about geometry?"

"Oh, I don’t like geometry at all," sighed Robert. "Geometry is all pictures of circles and squares and parallel lines. You find how long something is or the area of a figure, and that’s it. Nothing interesting about it."

"How wrong you are," countered the Number Devil, with a distasteful twitch of his long black whiskers. "I’ll bet in school they only teach you Euclidean geometry."

"What’s Euclidean geometry?" Robert asked.

"The earliest and simplest form of the geometry of planes and three-dimensional space. The Greek mathematician who derived it, Euclid, was a pretty clever chap. He was able to come up with all kinds of theorems and proofs from just five simple, improvable rules. But Euclidean geometry is pretty basic stuff. I’m going to show you something much more interesting than just shapes on a piece of paper."

"But what other kind of geometry can there be?" demanded Robert.

"Why, non-Euclidean geometry, of course!" exclaimed the Number Devil, jumping off his stool. "Allow me to demonstrate."

He pushed a large greenish-blue button on the wall, and a panel slid back to reveal a blank whiteboard. Taking a piece of purple chalk from his pocket, the Number Devil began to draw furiously.

"Here is a soccer ball," he said, "something you probably see every day."

"Yes," said Robert.

"Can you tell me what shapes it is made of?" asked the Number Devil.

"It looks like pentagons and hexagons," said Robert. "The purple shapes have five sides, the white ones have six."

"Alright," said the Number Devil, "but look what happens when we try to flatten the soccer ball out." With a wave of his hand, the drawing of the soccer ball disappeared. He began scribbling furiously again.
“On a flat surface,” said the Number Devil, “suddenly the pentagons and the hexagons don’t fit together any-
more.”

“That can’t be right,” said Robert. “Those aren’t the same shapes.”

“The very same,” said the Number Devil.

“What’s the catch then?”

“The catch,” said the Number Devil, “is that on a soccer ball we are not dealing with Euclidean geometry. A soccer ball is a sphere, and shapes on a sphere behave very differently than shapes on a flat surface.”

“I think I see,” said Robert. “On a flat board, single interior angles from two hexagons and one pentagon don’t add up to a full 360 degrees. But they might on a sphere.”

“Exactly,” said the Number Devil.

“That seems easy enough,” said Robert. “What else is different from Euclidean geometry when you’re on a sphere?”

“I thought you’d never ask!” cried the Number Devil. He rushed excitedly to the wall covered in buttons, where he flicked three switches, turned a dial, and pushed a little lever up five notches. The panels of one of the walls began to slide away. Beyond them was a giant glass window through which Robert could see the vast reaches of space, with little pinpoint stars scattered throughout it. The view began to rotate slowly, and soon the giant blue-green globe of Earth was visible.

“The Earth,” said the Number Devil grandly. “Quite an important sphere to you and me, I should think.”

“Yes, I suppose,” said Robert. “But I don’t really think of it as a sphere when I’m living on it.”

“That’s because you’re so very small compared to the Earth,” said the Number Devil. “At very small distances along a sphere’s surface, Euclidean geometry makes a pretty good estimate for what is going on. But say, for example, you were in a plane traveling from New York to Paris”—here the Number Devil marked where the two cities were in chalk on the window—“oddly enough, you would fly north over Newfoundland to get from point A to point B.” He drew an upwards-curving dashed line connecting the two purple dots.

“But why would you do that?” Robert asked. “It looks like you’re going far out of the way. Why wouldn’t you just go across the Atlantic in a straight line?”

“Aha!” cried the Number Devil. “Therein lies the problem! There is no such thing as a straight line on a sphere. The closest thing to a straight line is a part of a Great Circle; that is, a circle, like the equator, that is cen-
tered on the Earth’s center. And, just like a straight line is the shortest distance between two points on a plane, a Great Circle segment is the shortest distance between two points on a sphere.”

“I think I understand,” said Robert. “If you were to take the equator and move it so that it went through New York and Paris, it would curve north up over Newfoundland.”

“You’ve got it!” said the Number Devil.

“That’s not so complicated,” said Robert.

“But there’s so much more!” said the Number Devil. “Say for example, you wanted to calculate the distance between the two dots at New York and Paris.”

“Well, if you know the radius of the Earth, you can find the distance around a Great Circle,” said Robert. “That will always be the same, won’t it? It’s just the circum-
ference of a circle with the same radius as the Earth.”

“Yes, that’s right,” said the Number Devil. “But the problem is finding how much of a Great Circle is being covered between those points.”

“Okay,” said Robert. “So you find the angle of that slice of the circle.”

“Yes,” said the Number Devil. “And there is a relatively simple way of finding it using the latitude and longitudes of the two cities, some trigonometry, and a lovely little device called the Polka Dot Product.”

“Alright,” said Robert. “Let’s see it!”

The Number Devil turned back to his whiteboard and wrote the following:

| City  | Latitude | Longitude |
|-------|----------|-----------|
| NYC   | 41° N    | 74° W     |
| Paris | 49° N    | 2° E      |

“Now, we can use these numbers to calculate the co-
ordinates of each city if it were located in x-y-z space.”

“Like the unit circle, except now in three dimensions!” said Robert.

“Exactly,” said the Number Devil. He erased the co-
ordinates and traced a large circle onto the board with two intersecting lines. “The line running North to South is the z-axis,” he explained. “It runs through that black dot—which is the very center of the Earth.”
“Okay,” said Robert, “and that line from side-to-side would be the equator.”

“Exactly,” said the Number Devil. He then took a red piece of chalk out of his pocket and handed it to Robert. “Mark a point on the circle, anywhere you like.”

“Okay,” said Robert, “there you go.”

“Now,” said the Number Devil, “that point is intersected by two lines—one of longitude and one of latitude. Do you know the definition of latitude, Robert?”

“Well, it must be the measure of an angle somewhere,” said Robert. “It’s in degrees.”

“Very good,” said the Number Devil. “Latitude is the angle between the equator and the line connecting your point to the center of the Earth. Let’s call it theta.”

“That would make the angle between that line and the North Pole 90 minus theta degrees” said Robert.

“Perfect!” declared the Number Devil.

“Okay,” said Robert. “We also know that the length of that line between the red dot and the center of the Earth is just the radius of the Earth. So if we draw in a triangle there, we could solve for some of the coordinates.”

“You couldn’t be more correct,” said the Number Devil. “Why don’t you go fill in as much as you can?”

“No problem,” said Robert.

Soon the diagram looked like this:

“Very good,” said the Number Devil. “I’ll make a mathematician out of you yet. Now, what do you think that orange segment can do for us?”

“That’s the height of the point!” cried Robert. “We’ve found one of the dimensions!”

“That’s right,” said the Number Devil. “The z-coordinate of that \((x, y, z)\) point is the sine of the latitude times the radius of the sphere. But in order to find the other coordinates, we’ll have to look at this slightly differently.” The Number Devil erased the whiteboard again, and drew another circle. “This time, we’re looking down from the North Pole,” he said.

“I see,” said Robert. “The circle around that latitude is concentric with the equator. And the x-axis looks like its covering what would be the Prime Meridian.”

“That’s right!” said the Number Devil. “Now, let’s put our friend longitude to use. Longitude is degrees from the prime meridian—let’s call it phi. So, draw in a triangle, Robert.”
"Wait half a second," said Robert. "This time the radius of the Earth doesn’t come into the picture. How are we supposed to find the lengths of the dimensions?"

"I’m so glad you asked," said the Number Devil. "Let’s think back to our first circle. Remember that pale blue line that connected the red point directly to the z-axis, making a right angle?"

"I think I see the connection," said Robert. "If we add the triangle to this diagram, the point connecting the North-South axis to the red dot is that same length, the radius of the Earth times the cosine of theta!"

"Brilliant," said the Number Devil. "Now fill everything."

"There," said Robert. "And that gives us the x- and y-coordinates too! The altitude of that triangle is along the y-axis and the base is along the x-axis. So here is what we know..."

A point with latitude and longitude of $(\theta, \phi)$ has three-dimensional coordinates:

$[R \cos(\theta) \cos(\phi), R \cos(\theta) \sin(\phi), R \sin(\theta)]$

"Lovely," said the Number Devil. "Now, given that the radius of the Earth is around 6378 kilometers, finding the coordinates of New York and Paris is just a matter of plugging-and-chugging."

"I can handle that," said Robert. He wrote out:

NYC: $\theta = 49^\circ$, $\phi = -74^\circ$

$x = 6378 \cos(41^\circ) \cos(-74^\circ) = 1327$ km
$y = 6378 \cos(41^\circ) \sin(-74^\circ) = -4627$ km
$z = 6378 \sin(41^\circ) = 4184$ km
→ $(1327, -4627, 4184)$

"Now for Paris," said Robert.

Paris: $\theta = 49^\circ$, $\phi = 3^\circ$

$x = 6378 \cos(49^\circ) \cos(3^\circ) = 4179$ km
$y = 6378 \cos(49^\circ) \sin(3^\circ) = 219$ km
$z = 6378 \sin(49^\circ) = 4814$ km
→ $(4179, 219, 4814)$

"So we have two sets of coordinates in three dimensions," said Robert, "but we still don’t know how to find the angle formed between the center of the sphere and those two points."

"Think about it this way," said the Number Devil, turning to the whiteboard. "We now have a triangle with these three points, and we know the coordinates of all three vertices. It’s no trouble at all to find one of the angles."

"I see," said Robert. "You can use the distance formula to calculate the lengths of the three sides, and then use the cosine formula to solve for alpha, the angle we need!"

"Perfect thinking, my boy," said the Number Devil. Robert grabbed a piece of chalk and wrote out:

$l_1 = \sqrt{N_1^2 + N_2^2 + N_3^2}$
$l_2 = \sqrt{P_1^2 + P_2^2 + P_3^2}$
$l_3 = \sqrt{(N_1 - P_1)^2 + (N_2 - P_2)^2 + (N_3 - P_3)^2}$

"Now," said the Number Devil, "you know the cosine formula, don’t you?"

"Sure," said Robert.

$l_3^2 = l_1^2 + l_2^2 - 2l_1l_2 \cos(\alpha)$
$\Rightarrow \cos(\alpha) = \frac{l_1^2 + l_2^2 - l_3^2}{2l_1l_2}$
“Now we just need to do a bit of substitution,” said the Number Devil.
“Okay,” said Robert. “The numerator of that fraction looks like it can be simplified quite a bit…”

\[
i_1^2 + i_2^2 - i_3^2 = N_1^2 + N_2^2 + N_3^2 + P_1^2 + P_2^2 + P_3^2
\]

\[
- (N_1 - P_1)^2 - (N_2 - P_2)^2 - (N_3 - P_3)^2
\]

\[
= N_1^2 + N_2^2 + N_3^2 + P_1^2 + P_2^2 + P_3^2
\]

\[
- (N_1^2 + N_2^2 + N_3^2 + P_1^2 + P_2^2 + P_3^2)
\]

\[
+ 2(N_1 P_1 + N_2 P_2 + N_3 P_3)
\]

\[
= 2(N_1 P_1 + N_2 P_2 + N_3 P_3)
\]

“That looks much prettier,” said Robert. “Now the cosine formula is just:”

\[
\cos(\alpha) = \frac{N_1 P_1 + N_2 P_2 + N_3 P_3}{i_1 i_2}
\]

“And that,” said the Number Devil, “is our friend the Polka Dot Product. It is defined as…”

\[
x_1 = (x_1, y_1, z_1), \quad x_2 = (x_2, y_2, z_2)
\]

\[
x_1 \cdot x_2 = x_1 x_2 + y_1 y_2 + z_1 z_2.
\]

“So therefore,” the Number Devil continued, “our cosine formula is just:”

\[
\cos(\alpha) = \frac{i_1 \cdot i_2}{i_1 i_2}
\]

“Is that elegant or what!” cried the Number Devil.
“It is pretty nice,” Robert admitted. “Now it’s no hassle at all to plug in our coordinates for New York City and Paris! We already know \(i_1\) and \(i_2\) are both just the radius of the Earth, or 6378 kilometers…”

\[
NYC = (1327, -4627, 4184)
\]

\[
Paris = (4179, 219, 4814)
\]

\[
\cos(\alpha) = \frac{1327 \cdot 4179 + (-4627 \cdot 219) + (4184 \cdot 4814)}{6378 \cdot 6378}
\]

\[
= 0.6065 \rightarrow \alpha = \cos^{-1}(0.6065) = 52.66^\circ
\]

“Finally!” cried Robert. “We found the angle. Now finding the distance between New York and Paris is as easy as pie!”

\[
\frac{52.66 \cdot 2\pi(6378)}{360} = 5862 \text{ km}!!
\]

“Phew!” sighed Robert. “That was a lot of work for one silly little distance. I’m almost ready for a nap.”
“But there are so many more interesting things to do with spheres!” the Number Devil exclaimed.

“Like what?” Robert yawned.
“Almost anything you want!” exclaimed the Number Devil.
“What about area?” asked Robert. “I bet the area of my backyard could be estimated as if it was on a flat plane, but that probably wouldn’t be accurate for points that are far apart.”
“You couldn’t be closer to the truth, Robert,” said the Number Devil proudly. “The area of a shape on a sphere goes back to our soccer ball problem. Remember how the corners of the hexagons and pentagons fit together on a sphere, but didn’t when they were laid out flat?”
“Yes,” said Robert.
“In the same way, the three angles of a flat triangle add up to 180 degrees, but if you had a triangle on a sphere, that wouldn’t necessarily be the case.”
“That makes sense,” said Robert. “But what do the angles have to do with the area of the triangle?”
“It turns out that the area of the figure is directly related to the difference between the sum of its angles, and the sum it should have if it were on a plane. So, for example, the area for a triangle on a sphere would be:”

\[
A = R^2(\alpha + \beta + \gamma - 180^\circ) \cdot \left(\frac{2\pi}{360}\right)
\]

“That seems simple enough,” said Robert. “But given only the latitudes and longitudes of the corners of the triangle, how can you find those angles?”
“That’s where it gets a bit tricky,” said the Number Devil. “We have to use both the Polka Dot Product and its friend, the Criss Cross Product.”
“I’m up for that!” said Robert. “I’d like to find the area of the Bermuda Triangle.”
“Fantastic!” said the Number Devil. “The three corners of the Bermuda Triangle are located in Bermuda (naturally), Melbourne, Florida and Puerto Rico. Their latitudes and longitudes are as follows…”

| Latitude | Longitude |
|----------|-----------|
| Florida  | 28° N, 81° W |
| Bermuda  | 32° N, 65° W |
| Puerto Rico | 18° N, 66° W |

“Now we just use the formula we used earlier to turn those into three-dimensional coordinates,” said Robert. “So they come out like this…”

\[
Florida = (881, -5562, 2994)
\]

\[
Bermuda = (2286, -4902, 3380)
\]

\[
Puerto Rico = (2467, -5541, 1971)
\]

“Okay,” said the Number Devil. “That’s all fine and dandy, but before we put those to use, I’m going to need to introduce you to our friend the Criss Cross Product. Now, the sides of a triangle on a sphere are all segments of Great Circles—that’s the definition of the triangle. So,
each corner of the triangle is the intersection of two Great Circles."

"And we’re trying to find the angle where those two Great Circles intersect!" said Robert. "That makes sense."

"Now," said the Number Devil. "This is a bit difficult to picture, but imagine if you drew two lines perpendicular to each of the Great Circles."

"I think I understand," said Robert. "The angle between the two perpendiculars would be congruent to the angle between the two circles!"

"That’s correct!" said the Number Devil. "So for a triangle on a sphere, there are three Great Circles, each passing through two corners of the triangle. Therefore, there are three perpendiculars to be found. And our friend, the Criss Cross Product, can give us the perpendicular to any plane!"

"And then once we have the three perpendiculars," said Robert, "we can just use the Polka Dot Product to find the angle between any two of them!"

"Spot on!" said the Number Devil. "Now, say we’re looking at the Great Circle containing our points in Florida and Bermuda, and our perpendicular exits the sphere at \((x, y, z)\). The Criss Cross Product is like the determinant of a matrix..."

\[
\mathbf{F} \times \mathbf{B} = \begin{vmatrix}
x & y & z \\
F_1 & F_2 & F_3 \\
B_1 & B_2 & B_3 \\
\end{vmatrix}
= x(F_2B_3 - B_2F_3) \\
+ y(F_3B_1 - B_3F_1) \\
+ z(F_1B_2 - B_1F_2)
\]

"But instead of computing the entire determinant like I have above," the Number Devil continued, "you just take the parts being multiplied by \(x\), \(y\) and \(z\), and put them into one three-dimensional coordinate!"

\[
x(F_2B_3 - B_2F_3) \\
y(F_3B_1 - B_3F_1) \\
z(F_1B_2 - B_1F_2)
\rightarrow (F_2B_3 - B_2F_3, F_3B_1 - B_3F_1, F_1B_2 - B_1F_2)
\]

"And, if you scale the length to be the radius of the Earth, you have the point where the perpendicular leaves the sphere!" cried the Number Devil.

"So you just calculate that point for each pair of points, and then take the Polka Dot Products of each pair of the perpendiculars to find each angle," said Robert. "That sounds like entirely too much algebra to me."

"It is an awful lot of computation," said the Number Devil. "Luckily, I have a nice little computer program that will do all the math for us, and spare us the possibility of making mistakes." He walked over to a panel in the wall covered with glowing buttons and punched a few rapidly. Robert heard three loud beeps, and suddenly rows of numbers appeared on the screen.

"These tell us the longitudes and latitudes of the points where the perpendiculars emerge from the sphere," the Number Devil explained.

\[
\begin{align*}
\text{FLORIDA} \times \text{PUERTO RICO} & \rightarrow (48^\circ \text{ N}, 45^\circ \text{ E}) \\
\text{PUERTO RICO} \times \text{BERMUDA} & \rightarrow (3^\circ \text{ N}, 157^\circ \text{ W}) \\
\text{BERMUDA} \times \text{FLORIDA} & \rightarrow (56^\circ \text{ S}, 43^\circ \text{ W}).
\end{align*}
\]

"And now with one more push of a button," said the Number Devil, "I can have the computer print out the angles given once the Polka Dot Products are taken!"

\[
\begin{align*}
\text{FLORIDA} & \rightarrow 52.8^\circ \\
\text{PUERTO RICO} & \rightarrow 54.8^\circ \\
\text{BERMUDA} & \rightarrow 74.1^\circ \\
\text{TOTAL} & \rightarrow 181.7^\circ.
\end{align*}
\]
“And look at that!” cried Robert. “They add up to more than 180 degrees.”

“See, I told you,” said the Number Devil. “Spheres are tricky blighters. Now, Robert, calculate the area. You don’t need a fancy computer for that.”

“Okay,” said Robert, turning back to the whiteboard. “Shouldn’t be any problem at all.”

$$6378^2 \cdot [181.7^\circ - 180^\circ] \cdot \left( \frac{2\pi}{360} \right) = 1,211,500 \text{ km}^2$$

“Now, just for the sake of comparison,” said the Number Devil, “let’s look at what the area of this triangle would be were it flat.”

“Alright,” said Robert. “We can find the Great Circle distances between Florida, Bermuda and Puerto Rico, just like we did for New York and Paris. That would give us…”

\[ a = \text{Florida – Puerto Rico: 1895 km} \]
\[ b = \text{Puerto Rico – Bermuda: 1562 km} \]
\[ c = \text{Bermuda – Florida: 1604 km} \]

“Very good,” said the Number Devil. “Now, I assume you know of the Heroic Formula?”

“Sure,” said Robert.

\[ A = \sqrt{s(s-a)(s-b)(s-c)}, \quad s = \frac{a+b+c}{2} \]

“And when we plug in our side lengths,” Robert continued, “we get the area to be 1,200,800 km². That’s around 10,700 km² extra real estate!”

“Unfortunately it’s mostly ocean,” said the Number Devil. “But see what a difference the sphere makes, even with relatively small triangles?”

“It is pretty neat,” Robert admitted.

“So, you wouldn’t mind if I were to show you another way to calculate the sum of the angles of a polygon on a sphere?” asked the Number Devil.

“Fine,” said Robert. “Although I am getting very sleepy.”

“Alright,” said the Number Devil. “Follow me, please.” He led Robert through a door that had suddenly appeared in the wall and down a long silvery corridor. At the end of the hall was yet another door, but this one opened out into the vast emptiness of space. In front of Robert was a bridge tethered to the vessel he had just left (which he now saw was a space station) that led to a small round asteroid.

The Number Devil then handed Robert a strange device from one of his pockets—it was a wheel with little notches marking degrees, like a protractor. The device was mounted on a perfectly frictionless axle, so even when the axle was twisted back and forth, the wheel would remain motionless.

“This is called a parallel transporter!” said the Number Devil. “Go stand at the very top of that asteroid, keeping the wheel held parallel with the ground.”

Robert obliged.

“Now, see where the zero degree marking is on the wheel?” asked the Number Devil. “Walk in that direction for three steps.”

“Again, Robert did as told. He was now almost standing sideways on the asteroid!”

“Make a left turn. The zero degree marker should still
be facing the direction you were walking. Say the angle you turned was of alpha degrees. Which marker on the wheel is lined up with your new direction of travel?"

"That would be 180 minus alpha!" said Robert.

"That’s right," said the Number Devil. "Now take three more steps and make another turn, this time of angle beta."

Robert had to look up to see the Number Devil, since his head was pointed almost toward him. "Now the wheel is reading 180 minus alpha, plus 180 minus beta!" said Robert.

"Fantastic!" cried the Number Devil. "So if you take a few more steps to end up back where you started, and turn once more with angle gamma to end up the way you were originally facing, your wheel would read"—and here the Number Devil drew out his favorite purple piece of chalk, and began writing in the air.

\[
\begin{align*}
(180^\circ - \alpha) + (180^\circ - \beta) + (180^\circ - \gamma) \\
= 540^\circ - (\alpha + \beta + \gamma) \\
= 360^\circ + 180^\circ - (\alpha + \beta + \gamma) \\
\rightarrow 180^\circ - (\alpha + \beta + \gamma)
\end{align*}
\]

"Fantastic!" cried Robert. "Just what we were looking for in order to find the area of the triangle! I bet if you were on a flat surface, the wheel would read exactly zero degrees when you completed the triangle."

"That’s correct—the angles add up to 180 degrees then!" said the Number Devil. "But the fun doesn’t stop here. One of the beautiful properties of spherical geometry is that you can calculate the area of any shape just from the interior angles, as long as its sides are segments of Great Circles. So if you walked around an \( N \)-sided polygon with interior angles \( \alpha_1, \alpha_2, \ldots, \alpha_N \) you would have an accumulated angle of:"

\[
\theta = (N - 2) \cdot 180^\circ - (\alpha_1 + \alpha_2 + \ldots + \alpha_N)
\]

"Brilliant!" cried Robert. "That’s the difference between the sum of all the polygon’s angles, and the sum it would have were it flat—180\( \times \)times two less than the number of sides."

"So," the Number Devil continued, "the area would just be:"

\[
A = R^2 \cdot (-\theta) \cdot \left( \frac{2\pi}{360} \right)
\]

"This remains true for an arbitrary smooth closed curve on the sphere, which is made of tiny Great Circle segments. Using your wheel device, you can still read off the accumulated angle after returning to your starting point, and this formula will still give you the area enclosed by that curve!" said the Number Devil.

"Brilliant!" cried Robert again.

"Even more brilliant is how they used to make these calculations in the old days, before they could travel to a conveniently sized asteroid," said the Number Devil. "In 1851, a scientist named Léon Foucault realized that any point on the Earth is rotating at a high speed around a circle of fixed latitude—a smooth curve of the type I was just talking about. So, if someone were to stand still, holding their wheel while the Earth rotated 360 degrees, they would accumulate an angle…"

\[
\theta = -\left( \frac{A}{R^2} \right) \cdot \left( \frac{360}{2\pi} \right)
\]

"I see," said Robert. "That’s just our area formula flipped around."

"It gets even more clever," said the Number Devil. "Foucault also calculated that the area enclosed by the circle of latitude \( \phi \) on which the person is sitting would equal:"

\[
A = 2\pi R^2 [1 - \sin(\phi)]
\]

"Now where on Earth did that come from?" cried Robert.

"It’s not all that complicated, actually, if you have a little bit of calcification" said the Number Devil. "It comes from the addition of all of the infinitesimally thin circles, of radius \( R \) times the cosine of the latitude, for all of the latitudes between latitude \( \phi \) and the North Pole."

"That makes my head hurt," Robert sighed.

"But here is the most beautiful part of all!" cried the Number Devil. "If you combine the two formulas I have written above, you get one elegant queen of a formula!"

"Alright," said Robert. "Let’s see it."

The Number Devil wrote out, very slowly and reverently:

\[
\theta = 360^\circ \sin(\phi)
\]

"Okay," said Robert. "So if you were in Paris, where \( \phi \) is 49 degrees, you would measure \( \theta \) as around 272°. And if you were at the North Pole, where \( \phi \) is 90°, \( \theta \) would be 360°, as the whole Earth rotates underneath you!"

"You’ve got it!" said the Number Devil. "However, we’re not quite done with our friend Foucault. Although we can use a device like our frictionless parallel transporter in your dreams, Robert, it would be impossible to actually build one for use on Earth."

Foucault had everything going against him, didn’t he," said Robert. "What did he do instead?"

"He used a very large, heavy pendulum," said the Number Devil. "To be specific, he hung a 27 kilogram weight from a 67 meter long wire from the dome of the Pantheon in Paris. This pendulum was able to swing back and forth, almost frictionlessly, for several hours."
“And the pendulum must have oscillated!” said Robert. “I think I’ve seen one of these at the Science Museum. It doesn’t just swing back and forth, it slowly rotates—you can even use it as a clock!”

“That’s right,” said the Number Devil. “Except if you were at the equator…”

“Well then it would just swing back and forth. But that wouldn’t be interesting at all,” said Robert.

“You’ve got it, my boy,” said the Number Devil. “Now, before I let you go back to sleep, I want you to take one more look at the soccer ball that started this lesson in spherical geometry. Notice that the pentagons and hexagons covering the soccer ball are all examples of Great Circle polygons, each covering some part of the surface of the sphere.”

“Yes,” said Robert, “and now I know that the interior angles of the pentagons add up to more than 540°, and those of the hexagons add up to more than 720°, and the difference is proportional to their area. What more is there to know?”

“Well,” said the Number Devil, “I want you to think about all of the polygons together, not one at a time. There is a beautiful formula, discovered by the great Swiss mathematician Leonhard Euler around 1740, that relates the number of vertices, faces and edges of such a polygonal covering of a sphere:

\[ V + F - E = 2 \]

Robert squinted and circled around the soccer ball, counting up the various features. “Yes, it seems to work,” he said. “There are twelve pentagons and twenty hexagons, and I count sixty vertices, thirty-two faces and ninety edges, and indeed sixty plus thirty-two minus ninety is exactly two. Phew! But what does this have to do with everything else we have talked about?”

“Well,” said the Number Devil, “let’s think about applying our angle formula to all of the faces at once. What is their total area?”

“The total area is four pi times the square of the radius.” said Robert. “That’s just a special case of the calcified formula where the angle phi is the south pole.”

“Wow!” said the Number Devil, pulling out his piece of purple chalk. “You have been paying attention! Now let’s write that total area as an angle sum, adding up all of the contributions from all of the faces:

\[
4\pi = \left( \frac{2\pi}{360} \right) \sum_{f=1}^{F} \left( \alpha_{f,1} + \alpha_{f,2} + \ldots + \alpha_{f,V_f} - 180(V_f - 2) \right)
\]

Robert frowned in concentration. “So, the radius-squared has canceled on both sides, and the number of vertices \( V_f \) belonging to face \( f \) is five for the pentagons, and six for the hexagons.”

“Right!” said the Number Devil. “But now let’s be really tricky, and notice that this sum is adding up every single angle on the soccer ball. So, we can reorganize it, to first add up every angle surrounding each vertex—instead of each face—and then add up the result over all vertices—instead of over all faces. Do you see what I’m getting at?”

Desperate to go back to sleep, Robert concentrated extra hard, and suddenly saw the pattern. “Yes,” He said, “I see it now! Each has vertex two hexagons and one pentagon around it, and the three angles must add
up to 360°. So the sum of all the angles on the soccer ball is just 360° times the total number of vertices $V$.

The Number Devil was impressed. “Wonderful! So now I can simplify the relation to:

$$4\pi = 2\pi V + 2\pi F - \pi (V_1 + V_2 + \ldots + V_F)$$

“Well,” said Robert, “that’s really looking a lot nicer. But what do we do with that last sum? Different faces have different numbers of vertices, so I don’t see how you can simplify it any further.”

“Oh,” said the Number Devil in a superior tone, “but the trick now is to realize that each polygon has the same number of edges as vertices: $E_f$ is the same as $V_f$. Now what’s special about edges?”

“Ummm…” hesitated Robert. “Well, an edge is where two faces meet.”

“Exactly!” said the Number Devil. “So when you add up the number of edges over all the faces, you are counting each edge exactly twice—once for each of its adjacent faces. The result of that last sum is therefore twice the total number of edges, $E$, and we finally arrive at:

$$4\pi = 2\pi (V + F - E)$$

“Wow!” exclaimed Robert. “If you divide both sides by twice pi you just get Euler’s formula.”

“Yes,” said the Number Devil, “and if you think about it, you will see that none of our steps actually used the fact that we were dealing with a soccer ball, so this formula works for any polygonal covering of the sphere. Also, since this formula depends only on the total number of vertices, faces and edges, we are free to distort the spherical polygon covering into more standard solid shapes, like pyramids and cubes, with straight edges and planar faces, without changing the formula. In fact, the soccer ball is just an icosahedron formed from 20 equilateral triangles, with its corners filed down to make the pentagons. That way it rolls better!

“You see also that the number ‘2’ relates to something very special—it comes from the area of the sphere. This is such an ingenious result that this ‘2’ is called the ‘genius of the sphere.’ If you were to cover other solids with polygons, you would get different amounts of genius. For example, if you tried this on a donut or inflatable inner tube, you would need to replace the ‘2’ by a ‘0’—which is logical given that those shapes look like zeros.”

“Aha!” said Robert. “Then if I glue two donuts together to make a figure eight, I bet you should replace the ‘0’ by an ‘8’ for that type of solid!”

“Er, no,” said the Number Devil. “Good guess, but actually you replace it by a ‘−2’. We’ll have to get a ‘handle’ on that on another night.

“And now, I think that it really is time to let you go back to sleep.”

“Finally,” said Robert. As he drifted off, Robert heard the Number Devil say, “And tomorrow we’ll learn about hyperbolic geometry and Klein bottles and Nikolai Ivanovich Lobachevsky…”

Robert awoke the next morning with his head spinning like the Earth, thinking that if he were to draw all possible lines connecting his nose, mouth, and eyes, he could distort his head into an tetrahedron with four faces, four vertices, and six edges.

THE END