

Z/2-Godeaux surfaces

Eduardo Dias, Carlos Rito

Abstract

We compute explicit equations for all (universal coverings of) Godeaux surfaces with torsion group Z/2. We show that their moduli space is irreducible and unirational of dimension 8.

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1 Introduction

Let $S$ be a smooth minimal complex algebraic surface. Its topological invariants are the geometric genus $p_g$, the irregularity $q$ and the self-intersection $K^2$ of a canonical divisor. The holomorphic Euler characteristic is $\chi = 1 + p_g - q$. Gieseker [Gie77] has shown that for each pair $(\chi, K^2)$ and $S$ of general type, there exists a coarse moduli space $\mathcal{M}_{\chi,K^2}$ that is a quasi-projective variety. Naturally geometers want to understand which of these families are non-empty, and then if possible to classify them. It is frustrating that this has not been achieved even for the first case in the list, the one with $\chi = K^2 = 1$.

For these surfaces $p_g = q = 0$, and they are known to exist since Godeaux’ construction in 1931 [God31]. Nowadays surfaces of general type with $p_g = q = 0$, $K^2 = 1$ are called numerical Godeaux surfaces. Miyaoka [Miy76] showed that the order of their torsion group is at most 5, and Reid [Rei78] excluded the case $(\mathbb{Z}/2)^2$, so their possible torsion groups are $\mathbb{Z}/n$ with $1 \leq n \leq 5$. Reid constructed the moduli space for the cases $n = 5, 4, 3$, and it follows from his work that the topological fundamental group coincides with the torsion group for $n = 5, 4$. Coughlan and Urzúa [CU18] showed that the same happens for $n = 3$. In those three cases the moduli space is irreducible of dimension 8.

Coughlan [Cou16] has obtained a family of $\mathbb{Z}/2$-Godeaux surfaces (i.e. with torsion group $\mathbb{Z}/2$) depending on 8 parameters. More recently, we have studied with Urzúa [DRU20] all possible degenerations of $\mathbb{Z}/2$-Godeaux surfaces into stable surfaces with one Wahl singularity, which produces many boundary divisors of dimension 7 in the KSBA compactification of the moduli space of these surfaces. This is done by means of abstract constructions (i.e. showing the existence of particular singular surfaces with no obstructions in deformations), and computational constructions based on Coughlan’s family. We have ended up proving in [DRU20] that Coughlan’s family is at most 7 dimensional.

For the case of Godeaux surfaces with trivial torsion, we know the examples due to Barlow [Bar85], Craighero-Gattazzo [CG94] (see also [RTU17]), and Lee-Park type of constructions (cf. [LP07]). Catanese and LeBrun [CL97] proved that the Barlow surface belongs to an irreducible component of dimension 8, and Catanese and Pignatelli [CP00] proved the same for the Craighero-Gattazzo surface.
Besides the work of several other authors, these cases $n = 2, 1$ are still open (at the time of submitting this paper, Schreyer and Stenger put out a preprint [SS20] claiming the construction of an 8-dimensional family of simply connected Godeaux surfaces, but without obtaining a full classification).

Catanese and Debarre [CD89] showed that the étale double covers of $\mathbb{Z}/2$-Godeaux surfaces have hyperelliptic canonical curve and birational bicanonical map onto an octic in $\mathbb{P}^3$, and they did a general study of its canonical ring. That octic is given by the determinant of a certain matrix $\alpha$.

In this paper we continue their work. Using an idea from Miles Reid [Re90], we get more precise information about $\alpha$ by looking first to its restriction to the case of the canonical curve, then extending to the surface. Then we give an algorithm for the computation of all such matrices, from which we obtain equations for the étale double covers of all $\mathbb{Z}/2$-Godeaux surfaces. We show that their moduli space is irreducible of dimension 8, which implies that the topological fundamental group of $\mathbb{Z}/2$-Godeaux surfaces is also $\mathbb{Z}/2$.

We note that our method is not brute force computation: for the main algorithm, the calculations used only 32 MB of RAM memory, and took 85 seconds on a low-end computer.

Recently two special $\mathbb{Z}/2$-Godeaux surfaces have appeared in the literature: a $(\mathbb{Z}/3)^2$-quotient of a fake projective plane, constructed by Borisov and Fatighenti [BF20], which has 4 cusp singularities; a degree 6 quotient of the so-called Cartwright-Steger surface, given by Borisov-Yeung [BY20], which has 3 cusp singularities and a certain configuration of rational curves. As an exercise, we give the coordinates of these surfaces in our family.

All computations are implemented with Magma [BCP97], and can be found in some arXiv ancillary files. In particular, using the files 5_Verifications_alpha_i_c_j.txt one can choose any surface in the family and compute its invariants and singular set.

At an advanced stage of the review process, F. Catanese informed us about some overlapping with his work, compare our Proposition 3 with [CCO94].

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2 Results from Catanese-Debarre

We collect here some results from the paper [CD89] that will be used throughout the text.

Let $S$ be the étale double cover of a numerical Godeaux surface with torsion group $\mathbb{Z}/2$, and denote the corresponding involution by $\sigma$. The invariants of $S$ are $K^2 = 2$, $p_g = 1$, $q = 0$. Define the canonical ring of $S$ as

$$R = \bigoplus_{n=0}^{\infty} H^0(S, nK_S),$$
and let $\mathcal{A} = \mathbb{C}[x, y_1, y_2, y_3]$ be the $\mathbb{C}$-graded algebra with
\[
\deg(x) = 1, \quad \deg(y_i) = 2.
\]

The involution $\sigma$ acts on $R$, the canonical ring of $S$, splitting it into eigenspaces $R = R^+ \oplus R^-$. Denoting the Godeaux surface by $T$, we have
\[
R^+ \cong \bigoplus_{n \geq 0} H^0(T, nK_T), \quad R^- \cong \bigoplus_{n \geq 0} H^0(T, nK_T + \eta),
\]
where $\eta \in \text{Pic}(T)$ is a 2-torsion element. Furthermore, by Riemann-Roch,
\[
\dim R^+_n = \dim R^-_n = 1 + \binom{n}{2}, \quad \text{for } m \geq 2.
\]

Throughout the paper we denote the set of generators of $R$ by
\[
x^2, y_2 \in R^+_2, \quad x_1, x_3, z_3, z_4 \in R^+_3, \quad x^3, y_2, z_1, z_2 \in R^-_3, \quad t \in R^-_4.
\]

Notice that the vector space $R^+_4$ is 7 dimensional and contains
\[
\{x^4, x^2y_2, y_1^2, y_1y_3, y_2^2, y_3^2, xz_1, xz_2\}.
\]

Then from [CD89, Lemma 4.5], and possibly doing a change of variables, there is a (unique) relation that can be written as
\[
Q^+(y_1, y_2, y_3) + \lambda xz_1.
\]

We have the following.

(1) The bicanonical map of $S$ is a birational morphism and its canonical curve is hyperelliptic (see [CD89, Proposition 1.1, Theorem 6.1]).

(2) $\mathcal{R}$ is a Cohen-Macaulay $\mathcal{A}$-module, which implies that $\mathcal{R}$ admits a length one free resolution of $\mathcal{A}$-modules that can be written as
\[
0 \longrightarrow \mathcal{A}(-8) \oplus \mathcal{A}(-5)^4 \oplus \mathcal{A}(-4) \longrightarrow \mathcal{A} \longrightarrow \mathcal{R} \longrightarrow 0,
\]
where $\alpha$ is a matrix with homogeneous entries in $\mathcal{A}$ (see [CD89, Proposition 4.2, Theorem 4.3]).

(3) The matrix $\alpha$ can be chosen symmetric of the form
\[
\alpha = \begin{pmatrix}
x^2 G & xq_1 & xq_2 & xq_3 & xq_4 & Q \\
xq_1 & a_{11} & a_{12} & a_{13} & a_{14} & x \\
xq_2 & a_{12} & a_{22} & a_{23} & a_{24} & 0 \\
xq_3 & a_{13} & a_{23} & a_{33} & a_{34} & 0 \\
xq_4 & a_{14} & a_{24} & a_{34} & a_{44} & 0 \\
Q & x & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where $G, q_i, a_{ij}$ are of degrees 3, 2, 1 in $(y_0 = x^2, y_1, y_2, y_3)$, respectively. The $3 \times 3$-minors of $(a_{ij})$ are in the ideal $(x^2, Q)$, and det$(a_{ij})$ is in $(x^2, Q^2)$. (See [CD89], proof of Theorem 4.6, case 4.)
The matrix $\alpha$ satisfies the following rank condition:

(RC) For each cofactor $\beta_{ij}$ of $\alpha$ there exist $l_{ij}^k \in A$ such that

$$\beta_{ij} = \sum_{k=1}^{6} l_{ij}^k \beta_{1k}$$

(see [CD89, Theorem 4.6]).

Conversely, for any matrix $\alpha$ belonging to an open subset of the set of matrices as in (3) and (4), it is possible to define a ring structure on the $A$-module $R$ which $\alpha$ defines. The surface $X = \text{Proj}(R)$ is the canonical model of a minimal surface $S$ with $K^2 = 2$, $p_g = 1$, $q = 0$ for which the bicanonical map is birational onto an octic in $\mathbb{P}^3$ with equation $\det(\alpha)$. (See [CD89], end of Section 4.)

Furthermore, from [Cat84]:

The equations of $S$ are given by

$$v_i v_j = \sum_{k=1}^{6} l_{ij}^k v_k, \quad i, j = 2, \ldots, 6,$$

$$\sum_{j=1}^{6} \alpha_{ij} v_j, \quad i = 1, \ldots, 6,$$

with

$$v_1 = 1, v_2 = z_1, v_3 = z_2, v_4 = z_3, v_5 = z_4, v_6 = t.$$

### 3 The matrix $\alpha$

**Proposition 1.** The matrix $\alpha$ can be written in the form

$$\alpha = \begin{pmatrix}
  x^2 G^- & xq_1^- & xq_2^- & xq_3^- & xq_4^- & Q^+ \\
  xq_1^+ & a_{11} & a_{12} & a_{13} & a_{14} & x \\
  xq_2^+ & a_{12} & a_{22} & a_{23} & a_{24} & 0 \\
  xq_3^+ & a_{13} & a_{23} & a_{33} & a_{34} & 0 \\
  xq_4^+ & a_{14} & a_{24} & a_{34} & a_{44} & 0 \\
  Q^+ & x & 0 & 0 & 0 & 0
\end{pmatrix}$$

with

$$Q = y_1^2 - y_2^2 - d^2 y_3^2,$$

and where the superscript signs mean $\sigma$-invariant (+) or $\sigma$-anti-invariant (−).

**Proof.** The bicanonical map of $S$ sends its (hyperelliptic) canonical curve $C$ onto the plane conic $Q = 0$, which is contained in the octic surface $\det(\alpha) = 0$ in $\mathbb{P}^3$. Suppose that this conic is a double line. Then $C = 2D + Z$, where $Z$ is supported on a union of $(-2)$-curves. These curves must be preserved by the Godeaux involution $\sigma$, giving rise to curves $C' = 2D' + Z'$ in the Godeaux surface $S/\sigma$, with $Z'$ also a union of $(-2)$-curves. This contradicts the fact...
Therefore $Q$ is either a smooth conic or the union of two distinct lines, so there is a change of variables that allow us to write $Q = y_1^2 - y_2^2 - d^2 y_3^2$, for some constant $d$ (we note that we could consider only the two cases $d = 1$ and $d = 0$, but we prefer to keep the modulus $d$ hoping to find a family of surfaces such that one can see the smooth conic degenerating to the singular one).

Using Riemann-Roch and the local basis of $\mathcal{R}$, one sees that there are two $\sigma$-invariant relations of degree 5 and two anti-invariant ones. Since $x, z_1, z_2, t$ are anti-invariant, $z_3, z_4$ are invariant, and we are assuming $Q$ invariant, the relations

$$\alpha \cdot (1, z_1, \ldots, z_4, t)^T = 0$$

from Section 2 (6) imply that the superscript signs must be as claimed.

**Lemma 2.** The matrix $\alpha|_{x=0}$ cannot be of the type

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & Q \\
0 & y_1 + dy_3 & 0 & y_2 & 0 & 0 \\
0 & 0 & y_1 + dy_3 & 0 & y_2 & 0 \\
0 & y_2 & 0 & y_1 - dy_3 & 0 & 0 \\
0 & 0 & y_2 & 0 & y_1 - dy_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

with $Q = y_1^2 - y_2^2 - d^2 y_3^2$.

**Proof.** Since $\beta_{1k}|_{x=0} = 0$ for $k = 1, \ldots, 5$, in this case the rank condition (RC) is

$$(\beta_{ij} = t_{ij}^6 |_{x=0})_{|x=0}.$$  

We compute (see the arXiv ancillary file Lemma2.txt) the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & y_1 - dy_3 & 0 & -y_2 & 0 & 0 \\
0 & 0 & y_1 - dy_3 & 0 & -y_2 & 0 \\
0 & -y_2 & 0 & y_1 + dy_3 & 0 & 0 \\
0 & 0 & -y_2 & 0 & y_1 + dy_3 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

Now it follows from Section 2 (6) that the equations of the effective canonical divisor of the corresponding surface $S$ contain the polynomial $t^2 = 0$, thus giving a double curve. As in the proof of Proposition 1, this is a contradiction.

Using this result (see Case 3) below), we show the following.

**Proposition 3.** The matrix $\alpha|_{x=0}$ can be written as

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & Q \\
d^2 y_3 & y_1 & y_2 & 0 & 0 & 0 \\
y_1 & y_3 & 0 & y_2 & 0 & 0 \\
y_2 & y_3 & 0 & -y_3 & -y_1 & 0 \\
0 & 0 & y_2 & y_1 & -d^2 y_3 & 0 \\
Q & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

with $Q = y_1^2 - y_2^2 - d^2 y_3^2$.  

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Proof. From Proposition 1 there exist constants $r_i$ such that the matrix $\alpha|_{x=0}$ can be written as

$$M := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & Q \\
0 & m_1 & m_2 & r_1 y_2 & r_2 y_2 & 0 \\
0 & m_2 & m_3 & r_3 y_2 & r_4 y_2 & 0 \\
0 & r_1 y_2 & r_3 y_2 & m_4 & m_5 & 0 \\
0 & r_2 y_2 & r_4 y_2 & m_5 & m_6 & 0 \\
Q & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

with $m_i = a_i y_1 + b_i y_3$. Denote its rows, columns by $l_i, c_i$, respectively.

If $m_2 \neq 0$, then we can assume $m_3 \neq 0$, by possibly doing the elementary operations $l_1 \rightarrow l_1 + \gamma l_2$, $c_3 \rightarrow c_3 + \gamma c_2$, for some constant $\gamma$. Now operations of the type $l_2 \rightarrow l_2 + \beta l_3$, $c_2 \rightarrow c_2 + \beta c_3$ take us to one of the cases $m_2 = 0$, $m_2 \propto y_1$ or $m_2 \propto y_3$, where here the notation $a \propto b$ means that $a = \tau b$ for some constant $\tau \neq 0$.

Suppose that $m_2 \propto y_1$. If $m_3 \propto y_1$, we can go to the case $m_2 = 0$ (by doing elementary operations over rows and columns). If not, we can take $m_3 \propto y_3$. Analogously if $m_2 \propto y_3$, we can obtain the case $m_2 = 0$ or $m_3 \propto y_1$. Now by multiplying $l_3, c_3$ and $l_2, c_2$ by constants, we can assume that $m_2 = 0$, or $m_2 = y_1$ and $m_3 = y_3$, or $m_2 = y_3$ and $m_3 = y_1$.

Let $N$ be the $4 \times 4$ central matrix of $M$ and $D := \det(N)$. One can check that the coefficient of $y_3^2$ in $D$ is $(r_1 r_4 - r_2 r_3)^2$. Since $D$ is a multiple of $Q^2$ (from Section 2 (3)), we must have $r_1 r_4 - r_2 r_3 \neq 0$. Then elementary operations over the rows $l_4, l_5$ and the columns $c_4, c_5$ allow us to assume that $r_1 = r_4 = 1$ and $r_2 = r_3 = 0$.

Summing up, we have three possible cases:

1) $m_2 = y_1$, $m_3 = y_3$, $r_1 = r_4 = 1$, $r_2 = r_3 = 0$;

2) $m_2 = y_3$, $m_3 = y_1$, $r_1 = r_4 = 1$, $r_2 = r_3 = 0$;

3) $m_2 = 0$, $r_1 = r_4 = 1$, $r_2 = r_3 = 0$.

Notice that the rank condition (RC) implies that each cofactor $C_{ij}$ of $N$ is divisible by the quadric $Q$. We show below that this is enough to conclude the proof.

The arXiv ancillary file Proposition3.txt contains the computation of the cofactors that appear in the following three cases.

Case 1)
We have

$$-C_{1,3}/y_2 = a_5 y_1^2 + (b_5 + a_6) y_1 y_3 - y_2^2 + b_3 y_3^2,$$

$$C_{1,4}/y_2 = a_4 y_1^2 + (b_4 + a_5) y_1 y_3 + b_3 y_3^2,$$

$$C_{2,3}/y_2 = (a_1 a_5 + a_6) y_1^2 + (a_1 b_5 + b_1 a_5 + b_6) y_1 y_3 + b_1 b_5 y_3^2.$$ 

By comparing coefficients with $Q$, the only possibility is that the first one is equal to $Q$, and the other two are zero. This implies

$$a_1 = 0, b_1 = d^2, a_4 = 0, b_4 = -1, a_5 = 1, b_5 = 0, a_6 = 0, b_6 = -d^2.$$
Case 2)
We have
\[
-C_{1,3}/y_2 = a_6y_1^2 + (a_5 + b_6)y_1y_3 - y_2^2 + b_3y_3^2,
\]
\[
C_{1,4}/y_2 = a_5y_1^2 + (a_4 + b_5)y_1y_3 + b_4y_3^2,
\]
\[
-C_{2,3}/y_2 = a_1a_4y_1^2 + (a_1b_4 + b_1a_4 + a_5)y_1y_3 - y_2^2 + (b_1b_4 + b_5)y_3^2.
\]
The only possibility is that the second one is zero, and the other two are equal to \( Q \). This implies that \( d \neq 0 \) and
\[
-a_1 = d^{-2}, b_1 = 0, a_4 = d^2, b_4 = 0, a_5 = 0, b_5 = -d^2, a_6 = 1, b_6 = 0.
\]
Now let
\[
P := \text{Diag} \left( 1, r^3, r^3d^{-2}, -rd^{-2}, -r, 1 \right)
\]
with \( r^4 + d^2 = 0 \). The change of variables
\[
(y_1, y_2, y_3) \mapsto \left( -r^2y_3, y_2, -r^2d^{-2}y_1 \right)
\]
sends the matrix \( M \) above to the product \( PMP^T \).

Case 3)
In this case we have
\[
-C_{12}/y_2 = y_2(a_5y_1 + b_5y_3),
\]
\[
-C_{13}/y_2 = (a_3a_6y_1^2 + (b_3a_6 + a_3b_6)y_1y_3 - y_2^2 + b_3b_6y_3^2),
\]
\[
-C_{24}/y_2 = (a_1a_4y_1^2 + (b_1a_4 + a_1b_4)y_1y_3 - y_2^2 + b_1b_4y_3^2).
\]
This implies \( a_5 = b_5 = b_3a_6 + a_3b_6 = b_1a_4 + a_1b_4 = 0, a_3a_6 = a_1a_4 = 1 \) and \( b_3b_6 = b_1b_4 = -d^2 \). Then \( a_1a_3 \neq 0 \) and we can assume \( a_1 = a_3 = 1 \). This way we obtain 4 matrices which, by changing \( y_3 \) to \( -y_3 \), reduce to

\[
M_j := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & Q \\
0 & y_1 + dy_3 & 0 & 0 & 0 & 0 \\
0 & 0 & y_1 - (-1)^jdy_3 & 0 & y_2 & 0 \\
0 & y_2 & 0 & y_1 - dy_3 & 0 & 0 \\
0 & 0 & y_2 & 0 & y_1 + (-1)^jdy_3 & 0 \\
Q & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

with \( j = 1, 2 \).

From Lemma 2 only the matrix \( M_2 \) with \( d \neq 0 \) can correspond to a Godeaux surface. Let

\[
R := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & i & d/2 & 0 & 0 & 0 \\
0 & -i/d & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & -i/2 & 1/d & 0 \\
0 & 0 & 0 & id/2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

with \( i^2 = -1 \). The product \( RM_2R^T \) shows that \( M_2 \) is equivalent to a matrix of the type \( M \) above.

Now we introduce a new degree 2 variable \( y_4 \) negative for the involution \( \sigma \).
Remark 5. The restrictions with $i, q$ as above the superscript signs mean $\sigma$-invariant or $\sigma$-anti-invariant.)

Theorem 4. The matrix $\alpha$ can be written as

$$\alpha_j = \begin{pmatrix} x^2G^- & xq_1^- & xq_2^- & xq_3^- & xq_4^- & Q \\ xq_1 & y_4 & y_1 & y_2 & 0 & x \\ xq_2 & y_1 & y_3 & cx^2 & y_2 & 0 \\ xq_3 & y_2 & cx^2 & -y_3 & y_1 & 0 \\ xq_4 & 0 & y_2 & y_1 & -y_4 & 0 \\ Q & x & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$Q = y_1^2 - y_2^2 - y_3y_4, \quad G, q_i \text{ polynomials of degree 3, 2 in } (y_0 = x^2, y_1, y_2, y_3, y_4),$$

respectively, and

$$y_4 = d^2y_3 \quad (j = 1) \quad \text{or} \quad y_1 = -\frac{1}{2d}y_3 \quad (j = 2) \quad \text{or} \quad y_3 = 0 \quad (j = 3).$$

(As above the superscript signs mean $\sigma$-invariant or $\sigma$-anti-invariant.)

Moreover, we can assume $c = 1$ or $c = 0$.

Proof. (of Theorem 4) We want to extend the matrix (1) from Proposition 3 by adding polynomials divisible by $x$. This must respect the signs given in Proposition 3 hence concerning the entries of order 2, we can only add multiples of $x^2$ to the $\sigma$-invariant ones. We get the matrix

$$\alpha = \begin{pmatrix} x^2G^- & xq_1^- & xq_2^- & xq_3^- & xq_4^- & Q \\ xq_1 & d^2y_3 & y_1 & y_2 + c_1x^2 & c_2x^2 & c_5x \\ xq_2 & y_1 & y_3 & c_3x^2 & -y_3 & y_1 \\ xq_3 & y_2 + c_1x^2 & c_3x^2 & y_1 & -d^2y_3 & 0 \\ xq_4 & c_2x^2 & y_2 + c_4x^2 & 0 & 0 & 0 \\ Q & c_5x & c_6x & 0 & 0 & 0 \end{pmatrix}.$$

We know that det$(\alpha)$ defines an irreducible surface in $\mathbb{P}^3$, thus $c_5 = c_6 = 0$ is impossible. If $c_6 = 0$, we can take $c_5 = 1$ from the change of variable $x \rightarrow x/c_5$.

Then elementary operations using the last row and column give us $c_1 = c_4$ and $c_2 = 0$. We can assume $c_4 = 0$ by doing $y_2 \rightarrow y_2 - c_4x^2$ and the result follows.

Now assume that $c_6 \neq 0$. We consider 3 cases. (The computational details are available in the arXiv ancillary file Theorem4.txt.)

Case 1: $d^2 \neq (c_5/c_6)^2, c_5 \neq 0$

Let

$$a := \frac{c_2^2 + d^2c_6^2}{c_3^2 - d^2c_6^2}, \quad b := \frac{2c_5c_6}{c_3^2 - d^2c_6^2}$$

and

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & r & -d^2rc_6/c_5 & 0 & 0 \\ 0 & -rc_6/c_5 & r & 0 & 0 \\ 0 & 0 & 0 & r & rc_6/c_5 \\ 0 & 0 & 0 & d^2rc_6/c_5 & r \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
Let $\alpha^2 = \frac{c^2_3}{c^2_3 - d^2 c^2_6}$.

The product $P\alpha P^T$ is a matrix of the type

$$
\begin{pmatrix}
\begin{bmatrix}
x^2 & xq_1 & xq_2 & xq_3 & Q
\end{bmatrix} & d^2 Y_3 & Y_1 & y_2 + c_1 x^2 & c_2 x^2 & c_3 x \\
xq_1 & Y_3 & Y_1 & y_2 + c_1 x^2 & c_2 x^2 & y_2 + c_4 x^2 & 0 \\
xq_2 & Y_1 & Y_3 & c_4 x^2 & -y_3 & Y_1 & 0 \\
xq_3 & c_4 x^2 & y_2 + c_4 x^2 & Y_1 & -d^2 Y_3 & 0 \\
Q & c_3 x & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{pmatrix}
$$

with

$$
\begin{pmatrix}
Y_1 \\
Y_3
\end{pmatrix} = \begin{pmatrix} a & -d^2 b \\ -b & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \end{pmatrix}.
$$

Notice that the determinant of this $2 \times 2$ matrix is $a^2 - d^2 b^2 = 1$.

Since $c_4 = c_5/r \neq 0$, we can proceed as before to get $c_5 = 1, c_1 = c_2 = c_4 = 0$.

Finally from

$$Y_1^2 - y_2^2 - d^2 Y_3^2 = y_1^2 - y_2^2 - d^2 y_3^2$$

we see that the matrix $P\alpha P^T$ is in the form of the matrix $\alpha_1$ above.

Case 2: $d^2 \neq (c_5/c_6)^2, c_5 = 0$

Let

$$P := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & 0 & 0 & 0 \\
0 & 1/d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/d & 0 \\
0 & 0 & 0 & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The product $P\alpha P^T$ gives us a matrix of the type $\alpha$ with $c_6 = 0$. We proceed as above to get $\alpha_1$.

Case 3: $d^2 = (c_5/c_6)^2$

Since $c_6 \neq 0$, we can take $c_6 = 1, c_5 = \pm d$ and, as above, $c_1 = c_3 = c_4 = 0$. By looking to $P\alpha P^T$ with $P := \text{Diag}(1, -1, 1, -1, 1, 1)$, we see that we can consider $c_5 = d$.

Let

$$P' := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -d & 0 & 0 & 0 \\
0 & 0 & 0 & d & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The product $P'\alpha P'^T$ gives us a matrix of the type $\alpha_2$ if $d \neq 0$, or of the type $\alpha_3$ if $d = 0$.

Finally, the assertion about $c$: Assuming $c \neq 0$, take $s$ such that $c = s^4$. We can assume $c = 1$ by taking the product $P\alpha P^T$ with

$$P := \text{Diag}(1, s, 1/s, 1/s, s, 1)$$
followed by the changes
\[ x \mapsto x/s, \quad y_5 \mapsto s^2 y_3, \quad y_4 \mapsto y_4/s^2, \]
and
\[ d \mapsto d/(s^2) \quad (j = 1) \quad \text{or} \quad d \mapsto s^2 d \quad (j = 2). \]

4 Computation of the equations

Recall from Theorem 4 that \( G \) is a degree 3 anti-invariant polynomial on the variables \( (x^2, y_1, y_2, y_3) \). \( q_1, q_2 \) are anti-invariant of degree 2, and \( q_3, q_4 \) are invariant of degree 2. We write
\[
G = g_1 x^4 y_1 + g_2 x^4 y_3 + g_3 x^2 y_1 y_2 + g_4 x^2 y_2 y_3 + g_5 y_1^2 + g_6 y_1^2 y_2 + g_7 y_1^2 y_3 + g_8 y_2^2 y_3 + g_9 y_3^3,
\]
where \( q_1 = b_1 x^2 y_1 + b_2 x^2 y_3 + b_3 y_1 y_2 + b_4 y_2 y_3, \)
\( q_2 = b_5 x^2 y_1 + b_6 x^2 y_3 + b_7 y_1 y_2 + b_8 y_2 y_3, \)
\( q_3 = b_9 x^4 + b_{10} x^2 y_2 + b_{11} y_1^2 + b_{12} y_1 y_3 + b_{13} y_2^2 + b_{14} y_3^2, \)
\( q_4 = b_{15} x^4 + b_{16} x^2 y_2 + b_{17} y_1^2 + b_{18} y_1 y_3 + b_{19} y_2^2 + b_{20} y_3^2. \)

Our goal is to compute the set of parameters \( g_1, \ldots, g_{10} \) and \( b_1, \ldots, b_{20} \) such that the matrices \( \alpha_i \) from Theorem 4 satisfy the rank condition (RC). We note that the equalities \( \beta_{ij} = \sum_{k=1}^6 l_{ij}^k \beta_{1k} \) hold in the polynomial ring \( \mathcal{A} \), so comparison of coefficients is valid here.

By doing elementary operations over the rows and columns of the matrix \( \alpha_i \), we can assume that 8 of the \( b_i \) are zero. For instance in the case of \( \alpha_1 \):

- We remove the monomials \( x^2 y_2, y_1^2, y_3^2 \) from \( q_4 \)
  (using the lines/columns \( l_3, l_4/c_3, c_4 \));
- We remove the monomials \( y_1^2 y_3, y_3^2 \) from \( q_3 \)
  (using the lines/columns \( l_4, l_5/c_4, c_5 \));
- We remove the monomials \( x^2 y_1, x^2 y_3 \) from \( q_1 \)
  (using the line/column \( l_6/c_6 \)).

The idea for the computations is the following: we write the polynomials \( l_{ij}^k \) as a sum with coefficients \( r_m \) and monomials in \( \mathcal{A} = \mathbb{C}[x, y_1, y_2, y_3] \) of the right degree and eigenspace, i.e. according to the (RC)
\[
\beta_{ij} - \sum_{k=1}^6 l_{ij}^k \beta_{1k}.
\]
Then we need to determine values for the parameters \( d, g_p, b_r, r_m \) such that the coefficients of the polynomials from (RC) vanish. After this the equations of the surfaces \( S \) follow from Section 2 (6).

The polynomials \( l_{ij}^k \) depend on 371 parameters, and the cofactors \( \beta_{i,j} \) depend on \( d, g_1, \ldots, g_{10} \) and twelve of \( b_1, \ldots, b_{20} \). We have a huge system of 876 coefficients depending on 371 + 23 = 394 parameters, but the parameters \( r_m \) appear with degree 1. We have developed an algorithm for this problem.
4.1 The algorithm

The idea is to eliminate all parameters $r_m$, then to eliminate some of the remaining parameters to obtain a space of dimension 8. We have tried to use the Magma function `LinearElimination` to eliminate the $r_m$, but we did not get an optimal result. We need a linear elimination function that gives us more control on the elimination process.

We define a Magma function `LinElim` that eliminates always using in first place the simplest equations, i.e. the ones containing the smallest possible number of parameters $r_m$:

**Input:** $f$ a sequence of polynomials, $g$ a subset of $f$, $var$ a sequence of variables, and $n \in \mathbb{N}$.

**Process:** It checks the elements of $g$ one-by-one and whenever it finds one that is of the form $r - h$ with $r \in var$, $h$ not depending on $r$, and involving at most $n$ variables that are in $var$, it eliminates $r$ from the sequences $f$ and $g$. It also adds the pair $[r, h]$ to a list that we call `dependencies`.

**Output:** The new list $f$, and the `dependencies`.

We can now start.

1. We work on $R[x, y_1, y_2, z_1, z_2, z_3, z_4, t]$ with $R$ a polynomial ring of rank 394 (the parameters). Recall that the involution $\sigma$ is $[-x, -y_1, y_2, -y_3, -z_1, -z_2, z_3, z_4, -t]$.

2. We define the matrix $\alpha_j$, with $G, q_i$ depending on the parameters $g_m, b_n$.

3. We define the polynomials $l_{ij}^k$ depending on parameters $r_p$. Notice that these must be chosen with the right degree and $\sigma$ sign.

4. We write the polynomials that define the rank condition and a sequence $f$ containing their coefficients. Our goal is to determine the $r_p$ in terms of the $g_m, b_n, d$ such that these polynomial coefficients vanish.

5. We now use the Magma function `LinElim` defined above to solve the system of equations $f = 0$. We expect to eliminate all parameters except some of the $g_i$ and $b_j$ (which are in the first 23 variables of the ring $R$). So, the function is used for the remaining 371 parameters. The input here is $(f, g := f, var := [r_1, \ldots, r_{371}], n := 1)$.

6. Now by examining the new list $f$ computed in the previous step, we see that there is a set $g \subset f$ of polynomials containing only parameters $g_i$ or $b_j$, and such that some of these parameters can also be eliminated. So we run `LinElim`(f, g, var, 22) now with $var = [g_1, \ldots, g_{23}, b_1, \ldots, b_{12}]$.

7. We repeat step (5) with $n = 2$, followed by step (6), then step (5) with $n = 3$, etc. The system is solved when the output $f$ is empty.

8. The output list `dependencies` keeps track of all eliminations that occurred above. Its elements are of the form $[a, h]$ with $a$ one of the $r_m, g_i, b_j$ that were eliminated. We use these data to compute the matrix $\alpha_j$ depending on a smaller number of parameters. We do the same for the $l_{ij}^k$. 

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(9) Using these new $a_j$ and $\ell_{ij}$, we compute the equations given by Section 2 (6), on the variables $x, y_1, y_2, y_3, z_1, z_2, z_3, z_4, t$.

The arXiv ancillary file 1_TheAlgorithm_alpha_1_c_1.pdf contains a Magma implementation of this algorithm for the case $a_1$ with $c = 1$ of Theorem 3.

We get a set of equations that depend on 9 of the parameters $d, g_j, b_j$ and some of the $r_m$. But the equations $F_j = 0$ of degree $\leq 5$ do not depend on the $r_m$, and we check that, on the equations of degree $> 5$, the $r_m$ appear as $r_m G_m$ with $G_m$ in the ideal $< F_j >$. Thus we can take $r_m = 0$ (the $F_j = 0$ can be used to eliminate all the $r_m$) (see the arXiv ancillary file 3_RemovingTheRi_alpha_1_c_1.txt). This gives the final equations depending on exactly 9 of the parameters $d, g_j, b_j$.

These computations used only 32 MB of RAM memory, and took 85 seconds on a low-end computer.

The computations for the case $j = 1, c = 1$ of Theorem 4 (here we can take $y_4 = d y_3$ instead of $y_4 = d^2 y_3$) give that the matrix $a_1, c = 1$ is given by:

$$G = \begin{pmatrix} (-2b_b b_d + 2b_b b_d + 4b_d d + 2b_b b_1 - 2b_b b_1 - 4d b_1) x^4 y_1 + \\
(-2b_b b_d + 2b_b b_d + 2b_b b_d + b_b b_2 + b_d d + d g_2 + 2 d b_2 + b_d b_1) x^4 y_3 + \\
(-2b_b b_d + 2b_d d - 2b_b b_1 + 2b_b d + 2b_d + 2 b_2 + 2 b_1) x^2 y_1 y_3 + \\
(-2b_b b_d + b_b b_2 - 2b_b b_d + 2b_b b_d - b_d b_2 + b_b b_1 + 2 d b_2 - 2 d b_1) x^2 y_2 y_3 + \\
(2 b_d d - 2 b_d d) y_2^2 + \\
(b_b b_1 + b_d d + b_b b_2 + 2 d b_2 - 2 b_b b_1) y_1 y_3^2 + \\
(-2 b_b d + 4 b_d d + 4 b_2 - 2 b_1) y_1 y_3^2 + \\
(b_b b_2 - 2 b_b d + b_b b_2 + b_b b_1) y_1 y_3^2 + \\
g y_2 y_3^2 + \\
(-b_d d + 2 b_d d + b_b b_2 + d g_2 + 2 d b_2) y_3^2, \end{pmatrix}$$

$q_1 = b_2 y_2 y_3$,  
$q_2 = (b_b - b_b - 2b) x^2 y_1 + (b_b b_2 + b_1) x^2 y_3 + b_2 y_1 y_2 + b_3 y_2 y_3$,  
$q_3 = (-b_b b_1 + b_1) x^4 + b_2 x^2 y_2 + b_2 y_2^2$,  
$q_4 = (b_b d + d^2 + b_2) x^4 + b_1 x y_1 y_3 + b_2 y_3^2$,  
$Q = y_1^2 - y_2^2 - d y_3^2$.

The computations for the other cases $a_i, c = j$ are given in the files 1_TheAlgorithm_alpha_i_c_j.txt.

5 The moduli space

Denote by $\mathcal{M}$ the moduli space of numerical Godeaux surfaces with torsion group $\mathbb{Z}/2$, and let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ corresponding to the matrix $a_i$ with $c = j$ of Theorem 4.

Theorem 6. The space $\mathcal{M}_1$ is dominated by an open dense subset of the 8-dimensional weighted projective space $P(1, 1, 1, 2, 2, 3, 4, 4)$. We have $\mathcal{M}_2 = \mathcal{M}_3 = \emptyset$. The spaces $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ are at most 7-dimensional, and are contained in the closure of $\mathcal{M}_1$. 

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Proof. The computations of Section 4 give a matrix $\alpha_1, c = 1$ whose entries are polynomials on the variables $y_0 = x^2, y_1, y_2, y_3$ with coefficients depending on 9 parameters $b_5, b_9, b_8, b_6, d, b_2, b_{11}, g_9, b_{12}$. Each determinant $D := \det(\alpha_1)$ gives an octic surface in $\mathbb{P}^3$ which, for general values of the parameters, is the image of $Y$ by its bicanonical map, where $Y$ is the surface that has also been computed by the algorithm. We check that, for any nonzero constant $u$, we have

$$D = D(y_0/u, y_1, y_2, y_3/u, u b_5, u^2 b_6, u^2 b_9, u^2 b_{11}, u^3 g_9, u^4 b_{12})$$

(see the arXiv ancillary file $7_{ItIsWeightedProjSpace_alpha_1_c_1.txt}$).

Therefore, the octics corresponding to $(b_5, b_9, b_8, d, b_2, b_{11}, g_9, b_{12})$ and

$$(u b_5, u^2 b_6, u^2 b_9, u^2 b_{11}, u^3 g_9, u^4 b_{12})$$

are identified by the change of variables $(y_0/u, y_1, y_2, y_3/u)$. This implies that the above family of octic surfaces is parametrized by $\mathbb{P}(1, 1, 2, 2, 2, 2, 3, 3, 4, 4)$. In a similar way we get that the spaces $M_1^0, M_2^1, M_3^1$ are dominated by weighted projective spaces of dimensions 8, 8, 7, respectively (see the corresponding arXiv ancillary files).

It is easy to check that for $c = 0$ the determinant of the matrix $\alpha_1$ is a square, hence $M_3^0 = \emptyset$. The computations for the case $\alpha_2, c = 0$ give equations that contain the point $(0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : 0)$. Since this would be a base point for the tricanonical map, this is not possible for surfaces with $p_g = 1, q = 0, K^2 = 2$. Thus $M_0^0 = \emptyset$.

The space $M_1^0$ depends on parameters $b_5, b_9, b_8, b_6, d, b_{22}, b_{11}, g_9, b_{12}$, with weights 1, 2, 2, 2, 3, 3, 4, 4, respectively. The corresponding family of octic surfaces in $\mathbb{P}^3$ is given by a single polynomial $D_{b_5, b_9, b_8, d, b_2, b_{11}, g_9, b_{12}}(y_0, y_1, y_2, y_3)$, of degree 8 on the $y_i$. We want to know if there are two such octics that are projectively equivalent, i.e. if there are parameters $b_5', b_9', b_8', d', b_{2}', b_{11}', g_9', b_{12}'$ and a change of variables $Y'$ such that the coefficients of the polynomial

$$F(y_0, y_1, y_2, y_3) := D_{b_5, b_9, b_8, d, b_2, b_{11}, g_9, b_{12}}(Y_0, Y_1, Y_2, Y_3)$$

$$- D_{b_5', b_9', b_8', d', b_{2}', b_{11}', g_9', b_{12}'}(y_0, y_1, y_2, y_3) = 0 \quad (2)$$

vanish. Our computer was able to output results only for fixed values of $b_5, b_9, b_8, d, b_2, b_{11}, g_9, b_{12}$. By analyzing these results we were able to find a 1-dimensional change of variables $Y_i := y_i(k)$ and polynomials $b_5', b_9', b_8', d', b_{2}', b_{11}', g_9', b_{12}'$ on the parameters $k$, $b_5, b_9, b_8, d, b_{2}, b_{11}, g_9, b_{12}$ such that equation (2) holds (see the arxiv ancillary file $8_{Dimension_alpha_1_c_0.txt}$). This implies that the dimension of $M_1^0$ is at most 7. Analogously, we show that also the dimension of $M_2^1$ is at most 7 (see the file $8_{Dimension_alpha_2_c_1.txt}$).

For the space $M_3^1$ we have found octic surfaces that are not projectively equivalent to a surface in the family (see the file $8_{Dimension_alpha_1_c_1.txt}$), which suggests that the dimension of $M_3^1$ is 8. In fact, by the results of Kuranishi [Kur65] and Wavrik [Wav69] (as explained in [Cat83]), the number of moduli of each $\mathbb{Z}/2$-Godeaux surface is at least 8. Since all the $M_i^j$ are of dimension $\leq 7$ except for $i = j = 1$, then the dimension of $M_1^1$ is 8 and the spaces $M_0^1, M_3^1, M_4^1$ must be contained in the closure of $M_1^1$.

\[ \square \]

Corollary 7. The moduli space of numerical Godeaux surfaces with torsion group $\mathbb{Z}/2$ is irreducible and unirational of dimension 8. The topological fundamental group of these surfaces is also $\mathbb{Z}/2$. 

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Proof. The first part is immediate from Theorem \ref{thm:1}. For the second part it suffices to note that there exist \(\mathbb{Z}/2\)-Godeaux surfaces with topological fundamental group \(\mathbb{Z}/2\), see \cite{Bar84}.

\section{Two special surfaces}

Borisov and Fatighenti \cite{BF20} give the equations of a surface \(X\) with an action of \(\mathbb{Z}/3\) such that the surface \(Y := X/(\mathbb{Z}/3)\) is an étale double covering of a \(\mathbb{Z}/2\)-Godeaux surface with 4 cusps, which in turn is a \((\mathbb{Z}/3)^2\)-quotient of a fake projective plane.

The surface \(X\) is embedded in \(\mathbb{P}^7\) by its bicanonical map, and the action of \(\mathbb{Z}/3\) is

\[
(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7) \mapsto (x_0 : x_2 : x_3 : x_1 : x_5 : x_6 : x_4 : x_7).
\]

The map given by \((x_0 : x_1 + x_2 + x_3 : x_4 + x_5 + x_6 : x_7)\) sends \(X\) to the bicanonical image of \(Y\), an octic surface in \(\mathbb{P}^1\). Magma gives the equation of this surface, and with computations similar to the ones in the arXiv ancillary file 8_Dimension_alpha_1_c_1.txt, we get its coordinates in our family (case \(\alpha_1, c = 1\)):

\[
(b_5, b_9, b_6, b_8, d, b_{11}, g_9, b_{12}) = (36r + 36, 64, -360r + 1752, 360r + 4392, -30r - 366, -10176r - 45504, 20976r + 78960, 238008r + 1635576, -383328r + 867744),
\]

with \(r = \sqrt{-15}\).

Now with computations analogous to the ones in the file 5_Verifications_alpha_1_c_1.txt, one can see that the surface \(Y\) is as claimed.

Borisov and Yeung \cite{BY20} give the equations of a \(\mathbb{Z}/3\)-quotient \(Z\) of the Cartwright-Steger surface, and they show that \(Z\) is an étale double covering of a \(\mathbb{Z}/2\)-Godeaux surface. The surface \(Z\) has 6 cusp singularities and contains 3 disjoint \((-3)\)-curves. Proceeding as above, we find the image of \(Z\) by its bicanonical map, and then we compute its coordinates in our family (case \(\alpha_1, c = 1\)):

\[
(b_5, b_9, b_6, b_8, d, b_{11}, g_9, b_{12}) = (-60, 40, -120, -302, 9, 252, 360, 15903, 648).
\]

Again with computations analogous to the ones in the file 5_Verifications_alpha_1_c_1.txt, one can check the invariants of \(Z\) and its singularities.

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Eduardo Dias  
Departamento de Matemática  
Faculdade de Ciências da Universidade do Porto  
Rua do Campo Alegre 687  
4169-007 Porto, Portugal  
www.fc.up.pt, eduardo.dias@fc.up.pt

Carlos Rito  
Universidade de Trás-os-Montes e Alto Douro, UTAD  
Quinta de Prados  
5000-801 Vila Real, Portugal  
www.utad.pt, crito@utad.pt