Abstract. Continuing work begin in [19, 27], we interpret the Hurewicz homomorphism for Baker and Richter’s noncommutative complex cobordism spectrum $M\xi$ in terms of characteristic numbers (indexed by quasi-symmetric functions) for complex-oriented quasitoric manifolds, and show that automorphisms or cohomology operations on this representation are defined by the ‘renormalization’ Hopf algebra $N_*$ of formal diffeomorphisms at the origin of the noncommutative line, previously considered (over $\mathbb{Q}$) in quantum electrodynamics [3].

The resulting structure can be presented in purely algebraic terms, as a groupoid scheme over $\mathbb{Z}$ defined by a coaction of $N_*$ on the ring $N_*$ of noncommutative symmetric functions. We sketch some applications to symplectic toric manifolds, combinatorics of simplicial spheres, and statistical mechanics.

Introduction

In [1, 2] Andrew Baker and Birgit Richter defined a remarkable noncommutative analog $M\xi$ of the complex cobordism spectrum $MU$. The homology $H_*(MU, \mathbb{Z})$ is a free polynomial algebra with one generator in each even degree, while $H_*(M\xi, \mathbb{Z})$ is a free associative algebra with one generator in each even degree. The homology of $MU$ can be identified with the algebra of ‘characteristic numbers’ of complex-oriented manifolds (defined for example by the Chern-Weil theory of integrals of curvature forms), while $H_*(M\xi, \mathbb{Z})$ can be identified with invariants defined by quasi-symmetric functions of the line bundles associated to complex-oriented quasi-toric manifolds.

In the late 1960s, work of Landweber and Novikov led to an analysis of the cohomology operations in complex cobordism in terms of what has come to be understood as a ‘Hopf algebroid’ structure

$$MU_* MU \cong MU_* \otimes S_* ,$$

where $S_*$ is the commutative but noncocommutative Hopf algebra representing the group of formal diffeomorphisms of the line at the origin, known previously (over $\mathbb{Q}$) to combinatorialists as the Faa di Bruno algebra. An easy corollary identifies

$$H_*(MU \wedge MU, \mathbb{Z}) \cong H_*(MU, \mathbb{Z}) \otimes S_*$$
in terms of that algebra, and the ring of characteristic numbers.

It is fair to say that the noncommutative ring $M_\xi \ast M_\xi$ defining the cohomology operations in $M_\xi$ is not yet well-understood, but Baker and Richter show that it injects into $H_\ast(M_\xi \wedge M_\xi, \mathbb{Z})$, and that this injection becomes an isomorphism over $\mathbb{Q}$. The purpose of this note is to identify

$$H_\ast(M_\xi \wedge M_\xi, \mathbb{Z}) \cong H_\ast(M_\xi, \mathbb{Z}) \otimes N_*$$

in terms of a remarkable generalization $N_*$ of $S_*$, neither commutative nor cocommutative, already known (over $\mathbb{Q}$) to physicists [3, 4] as the ‘renormalization’ Hopf algebra of formal diffeomorphisms at the origin of the noncommutative line.

Since the work of Quillen it has become clear that it is useful to think of complex cobordism in terms of the moduli object for one-dimensional formal group laws. As a first step toward a deeper understanding of $M_\xi$, we suggest an interpretation of the algebra $H_\ast(CP^\infty \wedge M_\xi, \mathbb{Z})$ as functions on a noncommutative ‘grand canonical ensemble’ of characteristic numbers of symplectic quasitoric manifolds, with morphisms defined by noncommutative renormalization.

§I Some algebra

[This section reviews the basic properties of the algebra of symmetric functions (used in the theory of characteristic classes) following [21], and of the Hopf algebra representing the group of formal diffeomorphisms of the line at the origin used in cobordism theory [31]. It then summarizes some work of Brouder, Frabetti, Krattenthaler, and Schmitt [3, 4] on a (neither commutative nor cocommutative) Hopf algebra representing formal diffeomorphisms of the noncommutative line, used in renormalization theory.]

§1 The unique self-dual irreducible positive graded Hopf algebra

1.1 Let $\mathbb{Z}[x_*] = \mathbb{Z}_{k \geq 1}[x_k]$ be the graded polynomial algebra on generators $x_k$ of degree $|x_k| = 2$; then the subalgebra

$$\mathbb{Z}[e_*] = \mathbb{Z}_{k \geq 1}[e_k] := S_* \subset \mathbb{Z}[x_*]$$

of elementary symmetric ‘functions’ [21] (I §2.7 p 22) is generated by elements $e_k$ of degree $2k$, defined by the formal series

$$e(T) = \prod_{k \geq 1} (1 + x_k T) = \sum_{j \geq 0} e_j T^j \in S_*[[T]].$$

Similarly,

$$h(T) = e(-T)^{-1} = \prod_{k \geq 1} (1 - x_k T)^{-1} = \sum_{j \geq 0} h_j T^j \in S_*[[T]]$$
defines the ‘complete’ symmetric functions of degree $|h_k| = 2k$; thus $e_0 = h_0 = 1$, $e_1 = h_1$, etc. If $I = 1^{i_1}2^{i_2} \cdots r^{i_r}$ is an (unordered) partition of $|I| = \sum_{1 \leq k \leq r} ki_k$ into $r(I) = r$ parts, let

$$e_I := \prod_{1 \leq k \leq r} e_{i_k} \in S_{2|I|},$$

and similarly for $h_I$, etc. Elements of the form $e_I$ (or $h_I$ or $p_I$, see [21](I §6) and further below) provide bases for $S_*$.

The coproduct $\Delta_S : S_* \to S_* \otimes S_*$ defined by

$$\Delta e(T) = e(T) \otimes e(T)$$

(undecorated $\otimes$ signifies $\otimes Z$) and the antipode

$$\chi_S(e_n) = \sum_{|I| = n} (-1)^{r(I)} e_I$$

make $S_*$ into a binomial Hopf algebra. The group-valued Witt functor

$$A \mapsto (\text{sp } S)(A) = \text{Hom}_{\text{alg}}(S,A)$$

on commutative rings (ignoring the grading) sends a ring homomorphism $\alpha : S \to A$ to the (multiplicatively invertible) power series

$$\alpha(e)(T) = \sum_{k \geq 0} \alpha(e_k)T^k \in (1 + TA[[T]])^\times ;$$

the lost grading can be recovered from an action of the multiplicative group, as suggested below.

1.2 P Hall’s positive definite inner product [21](§1.4 p 63, 1.5 ex 25 p 91) on $S_*$ deﬁnes a canonical isomorphism with its dual Hopf algebra, rendering some applications to topology confusing. The (primitive) power sums

$$p(-T) := \sum_{k \geq 1} p_k T^{k-1} = e(T)^{-1} \cdot e'(T) \in S_*[[T]]$$

define an orthogonal basis $p_I$ for $S_* \otimes \mathbb{Q}$, while the monomial symmetric functions

$$m_I := \sum_{\sigma \in \Sigma_r} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(r)}^{i_r} \in S_{2|I|}$$

are dual to the $h_I$. The cohomology of the classifying space for stable complex vector bundles (see II 1.2) can be identiﬁed with $S_*$, with the Chern class $c_k \in H^{2k}(BU,\mathbb{Z})$ corresponding to $e_k$.

The operation which sends a stable vector bundle to its dual sends $e_k$ to $(-1)^k e_k$, while the Whitney sum inverse $V \mapsto -V$ interchanges $e_k$ and $(-1)^k h_k$, deﬁning a kind of adjoint involution $\omega : V \mapsto -V^*$. Hall’s duality $h_I \leftrightarrow m_I$ is a Hopf algebra involution on $S_*$. 

---

1sp is the canonical contravariant functor from a category to its opposite
2. Formal diffeomorphisms of the line at 0

2.1 When $A$ is a commutative algebra, the set 
\[ (\text{sp } S)(A) = \{ t(T) = \sum_{k \geq 0} t_k T^{k+1} \in A[[T]] \mid t_k \in A \} \]

is a noncommutative monoid under composition $t', t \mapsto (t' \circ t)(T) = t'(t(T))$ of formal power series. If $t_0$ is invertible, an elementary induction shows that $t$ has a unique two-sided compositional inverse $t^{-1}$ with $(t^{-1} \circ t)(T) = (t \circ t^{-1})(T) = T$. It follows that the localized polynomial algebra $S_* := \left( t^{-1}_{0}Z_{k \geq 0}[t_k] \right) := \mathbb{Z}[t^{\pm 1}_0] \otimes \tilde{S}_*$ has a noncocommutative coproduct which can be expressed as 
\[ (\Delta_{S*})(T) = (t \otimes 1)((1 \otimes t)(T)) \in (S_* \otimes S_*)[[T]] ; \]
the composition inverse defines an antipode $\chi_{S}$ on $S_*$ given explicitly by Lagrange’s reversion formula, defining a commutative but noncocommutative Hopf algebra. Enlarging $\tilde{S}_*$ to $S_*$ encodes the grading as a coaction of the Hopf algebra of the multiplicative groupscheme (with coproduct $\Delta t_0 = t_0 \otimes 0$). This can be suppressed by taking $t_0 = 1$.

The noncommutative dual algebra $S^*$ appears in algebraic topology as the analog of the Steenrod algebra for complex cobordism, where it is known as the Landweber-Novikov algebra. Quillen’s theorem interprets $S_*$ as the Hopf algebra representing the group of coordinate transformations of one-dimensional formal group laws.

The noncommutative groupscheme $\text{sp } S = \mathbb{G}_m \times \text{sp } \tilde{S}$ (i.e. of formal diffeomorphisms at the origin of the line over $\text{sp } \mathbb{Z}$) is closely related to the combinatorialists’ Faas di Bruno algebra: or, more precisely, to the analogous group of formal diffeomorphisms of the line over $\mathbb{Q}$, but now parametrized by Taylor-MacLaurin coordinates $t^{(n)}(0) = n! t^{-1}_{n-1}$.

2.2 There is a right action 
\[ \text{sp } S \times_{\text{sp } \mathbb{Z}} \text{sp } S \rightarrow \text{sp } S \]
of the noncommutative groupscheme $\text{sp } S$ on the commutative groupscheme $\text{sp } S$, defined by composition 
\[ e(T), t(T) \mapsto (e \circ t)(T) \]
of power series, i.e. by $\psi_{S}(e(T)) = (e \otimes 1)(t(1 \otimes T))$, representing a coaction 
\[ \psi_{S} : S_* \rightarrow S_* \otimes S_* \]
making $S_*$ into a Hopf $S_* -$ algebra comodule. This structure can be interpreted as defining a ‘right unit’ $\eta_R = \psi_S$ and a ‘left unit’ $\eta_L = 1_S \otimes 1$ for a (split) Hopf algebroid $S_*$ making $S_* S := (S_* S, S_* \otimes S_*) : S_* \Rightarrow S_* \otimes S_*$.
with coproduct
$$(S_* \otimes S_*) \to S_* \otimes (S_* \otimes S_*) \cong (S_* \otimes S_*) \otimes S_* (S_* \otimes S_*) .$$

These morphisms extend to define a cosimplicial (Amitsur) algebra
$$A_*^\bullet(S_* \otimes S_*) : 0 \to S_* \Rightarrow S_* \otimes S_* \Rightarrow S_* \otimes (S_* \otimes S_*) \ldots ,$$
a cobar construction or codescent complex $\text{16}(\S 5)$
$$A_*^\bullet C(B) : 0 \to C \Rightarrow B \Rightarrow B \otimes C \ldots$$
such that $D \otimes C A_*^\bullet C(B) \cong A_*^\bullet D(D \otimes C B)$, e.g.
$$A_*^\bullet(S_* \otimes S_*) \cong S_* \otimes A_*^\bullet(S_*) .$$

Such constructions underlie (for example) the Adams-Novikov spectral sequences of homotopy theory. In the dual language of schemes
$$\text{sp } A_*^\bullet(S_* \otimes S_*) \simeq [\text{sp } S/\text{sp } S]$$
represents a kind of pre-stack: the (untopologized) groupoid-valued functor on commutative algebras defined by the action of coordinate transformations on the symmetric functions.

§3. Noncommutative and quasisymmetric functions

3.1 The free associative graded cocommutative (but noncommutative) binomial Hopf algebra
$$N_* = \mathbb{Z}_{k \geq 1}\langle Z_k \rangle = \mathbb{Z}\langle Z_* \rangle$$
of noncommutative symmetric functions $\S 3$ (i.e. with coproduct $\Delta Z(T) = Z(T) \otimes Z(T), Z(T) = \sum_{k \geq 0} Z_k T^{k+1}$) is dual to a commutative but not cocommutative algebra
$$Q_* \subset \mathbb{Z}[x_*]$$
of quasisymmetric functions: an ordered partition
$$I = i_1 + \cdots + i_r$$
of the integer $|I|$ into $r$ nonempty parts (there are $2^{|I|}-1$ such things) defines a basis element $\S 4,\text{15}(\S 4)$
$$m_I = \sum_{0 < n_1 < \cdots < n_r} \prod_{1 \leq j \leq r} x_{n_j}^{i_j} \in Q_{|I|} \subset \mathbb{Z}[x_*]$$
dual to the basis element $Z_I = Z_{i_1} \cdots Z_{i_r} \in N_{|I|}$. We have
$$\chi N(Z_n) = \sum_{|I| = n} (-1)^{r(I)} Z_I$$
for the antipode of $N$. According to Ditters’ conjecture, $Q_*$ is a free commutative algebra; over $\mathbb{Q}$ it is polynomial, generated by elements of degree zero.

---

2 As in 2.1, the grading on $N_*$ can be encoded by adjoining a central unit $Z_0$ of degree zero.
indexed by Lyndon words of degree \( n \), with multiplicative structure defined by a certain shuffle product on the basis \( \{ m_1 \} \). The quasisymmetric functions have interesting relations with Koszul duality \( \textit{[24]}(\S 3) \) and number theory \( \textit{[6]}(\S 4.4) \), but these will not be pursued here. The surjection

\[
N_* \ni Z_k \mapsto e_k \in S_*
\]

of Hopf algebras defined by abelianization dualizes to a monomorphism \( S_* \to Q_* \).

3.2 Although composition of power series over noncommutative rings is \textbf{not} associative in general \( \textit{[2]} \), there is a remarkable generalization of the Hopf algebra of formal diffeomorphisms of the line in the noncommutative context. Brouder, Frabetti, and Krattenthaler \( \textit{[3]} \)(Thm 2.4) define a Hopf algebra \( N_* \) with coproduct

\[
(\Delta_N Z)(T) = \text{res}_{U=0} Z(U) \otimes Z(U - Z(T))^{-1} \in (N \otimes Z)([T])
\]
on generators of the underlying algebra \( N_* \) via a formal version

\[
\text{res}_{U=0} \left( \prod_{i \in Z} a_i U^i \right) := a_{-1}
\]
of Cauchy’s residue\( \textit{[3]} \) which elegantly simplifies proof of coassociativity: expanding the right-hand side, we have

\[
\text{res}_{U=0} U^{-1} Z(U) \otimes Z(1 - U^{-1} Z(T))^{-1} = \sum_{k \geq 0} \text{res}_{U=0} U^{-k-1} Z(U) \otimes Z(T)^k
\]

\[
= \sum_{k \geq 1} Z_{k-1} \otimes Z(T)^k = (Z \otimes 1)((1 \otimes Z)(T)),
\]
i.e.

\[
\Delta_N Z_k = \sum_{|I|+j=k} Z_j \otimes Z_I.
\]

Their Theorem 2.14 provides an explicit formula for the antipode \( \chi_N \) of the resulting Hopf algebra. Similarly, their \( \S 3 \) shows that the coproduct, regarded as an algebra homomorphism

\[
\Delta_N : N_* \to N_* \otimes N_*
\]
is compatible with the binomial coalgebra structure on \( N_* \), making it a Hopf algebra comodule over \( N_* \) \( \textit{[18]}(\text{III } \S 7) \). As in \( \S 2.2 \), this can be reformulated as the assertion that

\[
N_* N := (N_* N_* \otimes N_*) : N_* \Rightarrow N_* \otimes N_*
\]
is a (noncommutative, split) Hopf algebroid over \( Z \), with \( S_* S \) as its abelianization.

\( \text{\textsuperscript{3}} \)Here and throughout, variables such as \( T, U, \ldots \) will always be central, of cohomological degree two.
§4 A digression, on renormalization Hopf algebras

4.0 I owe Michiel Hazewinkel thanks for drawing attention to the Hopf algebra \(N_\ast\) and its applications in the formalization of quite classical quantum electrodynamics. It is a part of a wider literature (largely over \(\mathbb{Q}\)) expressed in terms of combinatorial Hopf algebras of trees, graphs, posets, and related structures, and to place it appropriately in that enormous field is not practical here; but its many possible generalizations deserve at least some mention. This section collects a few key ideas from work of Brouder and Schmitt on a general class of ‘renormalization bialgebras’ \(\mathcal{T}(\mathcal{T}(B))\) constructed functorially as cotensor algebras on a graded bialgebra \(B\). When \(B = \mathbb{Z}\) is the trivial bialgebra,

\[T(\mathbb{Z})^+ = \{\otimes^n e \mid n \geq 1\}\]

is the augmentation ideal of the (co)tensor algebra \(T(\mathbb{Z}) = \mathbb{Z}[e]\), identifying \(\mathbb{Z}_n\) with \((\otimes^n e) \in T^{2n}(T(\mathbb{Z}))\).

4.1 A certain partially ordered monoid \(C\) of ‘compositions’ – ordered partitions as in §3.1, but with \(\emptyset\) allowed as identity element – plays a basic role in [4]: concatenation, \(e \cdot g\).

\[(1 + 1) \ast (2 + 3) = (1 + 1 + 2 + 3)\]

is its multiplication operation. The iterated cotensor algebra \(T(T(\mathbb{Z}))^+)\) is shown to be naturally \(C\)-graded, and Brouder and Schmitt associate to \(\rho, \sigma \in C\), certain restriction and contraction coalgebra morphisms

\[\mathcal{T}_\sigma(T(T(\mathbb{Z}))^+) \to \mathcal{T}_{\sigma/\rho}(T(T(\mathbb{Z}))^+)\]

\[\mathcal{T}_\sigma(T(T(\mathbb{Z}))^+) \to \mathcal{T}_{\sigma'}(T(T(\mathbb{Z}))^+)\]

of a sort familiar (for example) in the study of Hopf algebras of trees and graphs. On generators \(u \in T(\mathcal{T}(\mathbb{Z}))\) with

\[\Delta u = \sum u'_i \otimes u''_i,\]

they define a new, ‘renormalization’ coproduct

\[\Delta u = \sum_{\sigma \leq \tau} u'_i |\sigma \otimes u''_i /\sigma,\]

which then extends to define a new bialgebra structure [3](§2.3 Th 1) on \(T(T(\mathbb{Z}))^+\).

4.2 Constructions of this sort are ubiquitous in work on combinatorial Hopf algebras, but vary in details, which are omitted here. If \(\varepsilon\) is the counit of \(B\), then elements of the form \((x) - \varepsilon(x)\) generate a two-sided ideal \((J) \subset T(T(\mathbb{Z}))^+)\), such that the quotient bialgebra \(T(T(\mathbb{Z}))^+)/(J)\) is a Hopf algebra [3](§5.1). This ideal is trivial for \(B = \mathbb{Z}\), and so recovers (in different notation) the construction of \(N_\ast\).
§II Some topology

§1 The Hurewicz homomorphism

This section reviews the theory of characteristic classes and numbers for complex cobordism in the language of §1. Its point is that the noncommutative cobordism spectrum $M\xi$ is not yet well-understood, but the associated theory of characteristic numbers is surprisingly accessible.

1.1 A topological group $G$, regarded as a topological category and hence as a simplicial space, has a canonical classifying space $BG$ for $G$-bundles as its topological realization. For example, the circle group $\mathbb{T} = U(1)$ has $B\mathbb{T} \simeq \mathbb{C}P^\infty$, with polynomial cohomology $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[c], |c| = 2$

generated by the Chern class of a complex line bundle. $\mathbb{T}$ being commutative, the product map $\mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is a group homomorphism, making $H^*(B\mathbb{T}, \mathbb{Z})$ a primitively generated Hopf algebra. Its dual $H_*(B\mathbb{T}, \mathbb{Z}) = \mathbb{Z}_{k\geq 1}[b(k)]$

is a divided power algebra, with generators satisfying $b(n) \cdot b(m) = (n, m) b(n+m)$.

A theorem of IM James identifies the stable homotopy type of the loopspace $\Omega \Sigma X$ (of the reduced suspension of a simply-connected space) with that of the free topological monoid

$$\Omega \Sigma X \sim \bigvee_{k \geq 1} X^\wedge k$$

generated by the pointed space $X$. [The free commutative topological monoid on $X$ is the infinite symmetric product $SP^\infty X$ of Thom and Dold.] For example, a level one projective representation of the loop group $LSU(2)$ defines a morphism

$$\Omega SU(2) \cong \Omega \Sigma S^2 \to \mathbb{C}P^\infty$$

of loop spaces, with an induced morphism

$$H_*(\Omega S^3, \mathbb{Z}) = \mathbb{Z}[\beta] \to \mathbb{Z}[b(s)] = H_*(\mathbb{C}P^\infty, \mathbb{Z})$$

of Hopf algebras sending $\beta \in \pi_2(\Omega S^3)$ to $b(1)$. [This notation is meant to suggest Boltzmann’s thermodynamic beta; see 1.3.2 below.]

Similarly, the homology of $\mathbb{C}P^\infty$ is torsion-free, so the Künneth theorem implies that

$$H_*(\Omega \Sigma \mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[Z_\ast] \cong \mathbb{N}_*$$

is isomorphic to the cofree cotensor coalgebra on $\tilde{H}_*(\mathbb{C}P^\infty, \mathbb{Z})$, with generator $b(k)$ corresponding to $Z_k$, defining a basis $Z_1$ as in I 3.1 and [1].
1.2 Block diagonal or Whitney sum composition

$$U(n) \times U(m) \to U(n+m)$$

defines a homotopy commutative- and -associative law on the monoid

$$\prod_{n \geq 0} BU(n),$$

whose group completion

$$\Omega B \left( \prod_{n \geq 0} BU(n) \right) \simeq \mathbb{Z} \times BU \supset 0 \times BU$$

has a classifying space for stable complex vector bundles as identity component. The maximal toruses $$\mathbb{T}^n \subset U(n)$$ then define Borel’s Hopf algebra isomorphism

$$H^*(BU, \mathbb{Z}) \cong S_*.$$

while the map $$BU(1) \to BU$$ defines a polynomial basis

$$H_{2k}(BU(1), \mathbb{Z}) \ni b_k \mapsto b_k \in H_{2k}(BU, \mathbb{Z})$$

with $$b(T) = \sum_{i \geq 0} b_k T^{k+1}$$ satisfying $$\Delta b(T) = b(T) \otimes b(T)$$. The associated primitives $$\tilde{p}_k$$ (as in I 1.2) then satisfy $$c_i \cdot \tilde{p}_k = \delta_{i,k}$$.

The universal complex vector bundle $$\xi_n \to BU(n)$$ defines the Thom spectrum

$$MU := \{ S^n \wedge \xi_m^+ = S^n \wedge MU(n) \to MU(n+m) = \xi_{n+m}^+ \}$$

representing the complex cobordism ring

$$MU_* = \pi_* MU = \lim_{k \to \infty} \pi_{*+k} MU(k)$$

of compact complex-oriented manifolds. Hurewicz’s ring homomorphism

$$\pi_* MU \to H_*(MU, \mathbb{Z}) \cong H_*(BU, \mathbb{Z}) \cong S_*$$

takes the homotopy groups of the composition

$$MU \simeq MU \wedge S^0 \to MU \wedge HZ$$

of morphisms of ring spectra defined by the identity map

$$[S^0 \to HZ] = 1 \in H^0(S^0, \mathbb{Z}).$$

(where $$S^0$$ is the sphere spectrum and $$HZ$$ is the integral Eilenberg - Mac Lane spectrum). This extends to a natural characteristic number homomorphism

$$h_{MU} : MU^*(X) \to H^*(X, S_*$$)

defined on multiplicative cohomology functors, familiar (over $$\mathbb{R}$$) from Chern-Weil theory.
1.3.1 A cobordism class \( \{X\} \in \text{MU}_{2k} \) can be interpreted as the equivalence class of a compact smooth submanifold \( X \subset \mathbb{R}^{2k+N}, \ N \gg 0 \), with complex normal bundle classified by \( \nu_X : X \to BU(N) \to BU \).

The Thom-Pontryagin collapse construction

\[
S^* \to \mathbb{R}^*/(\mathbb{R}^* - \nu(X)) = \nu(X)^+ \to \xi^+_*
\]

defines a map

\[
H_*(S^*) \xrightarrow{\nu_X^*} H_*(\xi^+_*) \to H_*(\text{MU}) \sim H_*(BU) \cong S_*
\]

sending the generator of \( H_*(S^*) \) to the Hurewicz image of \( \{X\} \). A dual construction sends \( \{X\} \) to the system of integrals of the tangential Chern classes \( c_I(\tau_X) \) with dim \( X = 2|I| \), regarded as a linear functional on \( S_* \). Hall duality equates this with the linear functional

\[
h_{\text{MU}}\{X\} : m_I \mapsto (-1)^{|I|} m_I(\nu_X) \cap [X] \in S_*
\]

defined by the monomial characteristic classes of the normal bundle.

1.3.2 Because \( \text{MU}_* \to H_*(\text{MU},\mathbb{Z}) \) is a homomorphism of torsion-free rings, it is convenient to work with their rationalizations. Quillen’s theorem identifies the completed Hopf algebra \( \text{MU}^*\mathbb{CP}^\infty \) with Lazard’s universal one-dimensional formal group law, and over \( \mathbb{Q} \) such group laws are classified by their logarithms. Mićenko’s theorem

\[
\log_{\text{MU}}(T) = \sum_{k \geq 1} \{\mathbb{CP}_{k-1}\} \frac{T^k}{k} \in \text{MU}^*_\mathbb{Q}[[T]]
\]

identifies this logarithm; on the other hand

\[
h_{\text{MU}} : \text{MU}^*(\mathbb{CP}^\infty) \to H^*(\mathbb{CP}^\infty, H_*(BU,\mathbb{Z})) \cong H^*(\mathbb{CP}^\infty, S_*)
\]

expresses the formal group law in terms of the exponential

\[
h_{\text{MU}}(X +_{\text{MU}} Y) = b(b^{-1}(X) + b^{-1}(Y)) \in S_*[[X,Y]]
\]

defined \( b(T) \) as above. Note that

\[
h_{\text{MU}}(\mathbb{CP}_n) = (n+1) \chi_S(b_n),
\]

see [13](28.5.012).

In [32] Ravenel and Wilson show that the canonical inclusions

\[
\beta_k = \{\mathbb{CP}_k \subset \mathbb{CP}^\infty\} \in \text{MU}_{2k}(\mathbb{CP}^\infty)
\]
generate the complex bordism of \( \mathbb{CP}^\infty \), modulo the relation

\[
\beta(X +_{\text{MU}} Y) = \beta(X) \cdot \beta(Y)
\]
(with \( \beta(T) = \sum_{k \geq 0} \beta_k T^{k+1} \)). Working rationally, it follows that

\[
\log(\beta \circ b)(T) = b(1)T
\]
and hence (recalling that $b_{(1)}$ is the image of $\beta$ as in 1.4)
\[
\beta(T) = \exp(\beta \log_{MU}(T)) \in (S^Q_\ast[\beta])[T].
\]
As suggested by Friedrich and McKay [11] (Prop 4.2) this resembles formally
the canonical partition function in statistical mechanics; see further in §III.

1.4 As in 1 2.1, the moduli object $[\text{sp } MU_\ast//\text{sp } S_\ast]$ for one-dimensional group
laws can thus be identified with the stack defined by the now-classical Hopf
algebroid
\[
MU_\ast MU : MU_\ast \Rightarrow MU_*MU;
\]
the Hurewicz or characteristic number map
\[
h_{MU} : [\text{sp } H_*MU//\text{sp } S_*] \subset [\text{sp } MU_\ast//\text{sp } S_*]
\]
then becomes the inclusion of the stratum of formal group laws of additive
type. It pulls back to an isomorphism over $\text{sp } Q \subset \text{sp } Z$.

§2 The Baker-Richter spectrum $M\xi$

2.1 Following 1.2, group completion of the monoidal map
\[
\coprod_{n \geq 0} BU(1)^{\times n} \to \coprod_{n \geq 0} BU(n)
\]
defines a morphism $BR : \Omega \Sigma BU(1) \to Z \times BU$ of $A_\infty$ spaces. Pulling back
the stable universal bundle $\xi$ over $0 \times BU$ defines Baker and Richter’s $A_\infty$
spectrum
\[
M\xi := \{ S^n \wedge BR^*\xi^+_{n+m} \to BR^*\xi^+_{n+m} \}
\]

Together with an abelianization morphism $M\xi \to MU$. A remarkable theorem [1](§7) shows that localization at a prime splits $M\xi$ as a wedge of
susensions of the Brown-Peterson spectrum, even though $M\xi$ is not an
$MU$-module spectrum; but it is complex-orientable, and possesses natural
Thom isomorphisms.

We will however make no use of this local structure here; our focus is the
ring-spectrum $M\xi \wedge HZ$, i.e. on the integral homology and cohomology
\[
H_*(M\xi, Z) \cong N_*, \ H^*(M\xi, Z) \cong Q_*
\]
and their relation to $M\xi$ through its Hurewicz homomorphism. Baker and
Richter show that
\[
M\xi_* = \pi_*M\xi \to H_*(M\xi, Z)
\]
is injective, and an isomorphism after rationalization. Following 1.3 and
[19], we can interpret the Hurewicz map as a noncommutative characteristic
number homomorphism
\[
h_{M\xi} : M\xi^*(Y) \to H^*(Y, H_*(M\xi, Z)) \cong H^*(Y, N_*) \cong \text{Hom}(Q_*, H^*(Y, Z)).
\]
Under Quillen’s conventions [30] we can assume that \( Y \) is a smooth manifold, and interpret an element \( \{X\} \in M^2(Y) \) as the cobordism class of a map \([\nu : X \to Y]\) between manifolds, with stable normal bundle

\[
v_X \cong \bigoplus L_i : X \to \bigvee_{n \geq 0} BU(1)^\times
\]
presented as a direct sum of stable complex line bundles. Then \( h_{M^2}\{X\} \) is represented by the linear functional

\[
m_I \mapsto (-1)^i \nu(m_I(c(L_*))) \in H^{2i}(Y, \mathbb{Z})
\]
on \( \mathbb{Q}_{2i} \) (where \( \nu \) is the covariant pushforward cohomology homomorphism induced by the complex-oriented map \( \nu \)).

**Remark** The right unit

\[
\eta_R : N_* \to N_* \otimes N_* \cong \text{Hom}(\mathbb{Q}_*, N_*)
\]
of I §2.3 can be defined on generators by

\[
\eta_R(Z_k)(m_I) = \sum_{i+j=k} m_I(Z_i) \cdot Z_j ,
\]
agreeing with the coproduct formula in I 3.2.

**2.2** The Euler class \( x_\xi \in M^2\mathbb{C}P^\infty \) provides a (noncentral [2]) coordinate for kind of noncommutative one-dimensional formal group structure on \( M^*\mathbb{C}P^\infty \). Its Hurewicz image

\[
h_{M^2}(x_\xi) = \sum_{k \geq 0} Z_k T^{k+1} = Z(T) \in H_*(M^2, \mathbb{Z})[[T]] \subset N_*[[T]]
\]
defines a (completed) Hopf coproduct

\[
\Delta_{\mathbb{C}P^\infty}(T) = \sum_{k \geq 0} Z_k \cdot (\chi_N Z)(T \otimes 1) + (\chi_N Z)(1 \otimes T))^{k+1} \in N_*[[T \otimes 1, 1 \otimes T]] ,
\]
cf. Zassenhaus’s noncommutative binomial theorem.

The Thom isomorphism \( M\xi_* M\xi \cong M\xi_* \Omega \Sigma \mathbb{C}P^\infty \) (together with a Künneth argument for torsion-free spaces) then defines a cosimplicial noncommutative algebra

\[
\mathcal{A}_{M^2\xi_*}^\bullet(M\xi_* M\xi) : 0 \to M\xi_* \Rightarrow M\xi_* M\xi \Rightarrow M\xi_* M\xi \otimes M\xi_* M\xi \Rightarrow M\xi_* M\xi \ldots
\]
as in I 1.2. In view of the remark above, the Hurewicz morphism

\[
M\xi \wedge M\xi \to (M\xi \wedge H\mathbb{Z}) \wedge_{H\mathbb{Z}} (M\xi \wedge H\mathbb{Z}) \simeq M\xi \wedge M\xi \wedge H\mathbb{Z}
\]
duces an injective homomorphism

\[
\mathcal{A}_{M^2\xi_*}^\bullet(M\xi_* M\xi) \to \mathcal{A}_{N_* M\xi_*}^\bullet(H_* M\xi \otimes H_* M\xi) \cong \mathcal{A}_{N_*}^\bullet(N_* \otimes N_*) ,
\]
of cosimplicial algebras defined by homotopy groups: dually, a morphism

\[
[\text{sp } N//\text{sp } N] \to \text{sp } \mathcal{A}_{M^2\xi_*}^\bullet(M\xi_* M\xi)
\]
of pre-stacks, which becomes an isomorphism over $\mathbb{Q}$. Thinking of commutative objects as a subclass of noncommutative ones defines an abelianization morphism

$$[\text{sp } S/\text{sp } S] \to [\text{sp } N/\text{sp } N].$$

It is tempting to imagine the descent object $A_{M\xi}(M\xi \land M\xi)$ as an approximation to a noncommutative analog of the sphere spectrum.

§III Examples and remarks

1 A $2n$-dimensional prequantized toric manifold [9, 12, 14, 27] is an omnioriented complex-oriented manifold (i.e., with a preferred decomposition of its stable tangent bundle as a sum of complex lines) defined by an effective action of a torus $\mathbb{T}^n$, together with an equivariant complex line bundle $L$ with connection, such that the curvature form $\omega(\nabla_L)$ is symplectic; it is roughly analogous to a compactified system of harmonic oscillators.

Such a manifold defines an element of $M\xi_{2n}\mathbb{C}P^\infty$ and thus has characteristic numbers in $H_*(\mathbb{C}P^\infty, \mathbb{N}_*) \cong \mathbb{N}_*[b]$]. Similarly toric varieties, considered as orbifolds [19, 20], have characteristic numbers in $\mathbb{N}_* \otimes \mathbb{Q}$, while Hamiltonian toric varieties have similar invariants in $M\xi_\ast \mathbb{C}P^\infty \otimes \mathbb{R} \cong M\xi_\ast \otimes \mathbb{R}[\beta]$. Projective toric varieties [13] provide examples; in particular, chemical reaction networks [8] with $k$ nodes and deficiency zero [27] (§1.3.2) have characteristic invariant $\sum_{1=k} Z_1$.

2 The equivariant cohomology of the moment-angle complex associated to a simplicial (e.g., Gorenstein [27] (§1.2.2)) complex can be identified with its Stanley-Reisner face ring, which thus has (quasi-symmetric) characteristic classes and numbers. This defines a homomorphism [19] from the (noncommutative) ring of simplicial spheres, with join as composition, to $\mathbb{N}_*$; the boundary of the $k$-simplex yields the example above.

3 Work in the theory of free, i.e., noncommutative, probability [28], [17] (Th 3.1), [29] suggests that

$$(\chi N Z)(-T) \in \mathbb{N}_*[T]$$

generalizes the cumulant generating function [10] [22] of classical statistical mechanics, along the lines suggested in II 1.3.2 above.
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