PERIODIC POINTS IN TOWERS OF FINITE FIELDS FOR POLYNOMIALS ASSOCIATED TO ALGEBRAIC GROUPS

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Abstract. We find the limiting proportion of periodic points in towers of finite fields for polynomial maps associated to algebraic groups, namely pure power maps \( \phi(z) = z^d \) and Chebyshev polynomials.

1. Introduction

We fix the following notation:
\[
\begin{align*}
\phi(z) & \quad \text{a polynomial.} \\
\phi^n(z) & \quad \text{the } n\text{th iterate of } \phi \text{ under composition; we take } \phi^0(z) = z. \\
\mathcal{O}_\phi(\alpha) & \quad \text{the (forward) orbit of a point } \alpha \text{ under } \phi; \text{ i.e. } \{\phi^n(\alpha) \mid n \geq 0\}. \\
\text{Per}(\phi, K) & \quad \text{the set of periodic points for } \phi \text{ in the field } K; \\
& \quad \text{i.e. } \{\alpha \in K \mid \phi^n(\alpha) = \alpha \text{ for some } n > 0\}.
\end{align*}
\]

When iterating a polynomial function \( \phi \) over a finite field, the orbit of any point \( \alpha \in \mathbb{F}_{p^n} \) is a finite set. That is, all points are preperiodic, meaning the orbit eventually enters a cycle. But many natural questions about the structure of orbits over finite fields remain:

1. Fix a finite field \( \mathbb{F}_{p^n} \) and look over all polynomials of fixed degree \( d \): On average are there “lots” of periodic points with relatively small tails leading into the cycles? Or do we expect few periodic points with long tails? (See Figures 1 and 2.)

2. Fix a polynomial: How does the proportion of periodic points in \( \mathbb{F}_p \) vary as \( p \to \infty \)?

3. Again fix a polynomial: How does the proportion of periodic points in \( \mathbb{F}_{p^n} \) vary as \( n \to \infty \)?

Recent work by Flynn and Garton [1] addresses the first question. Using combinatorial arguments, they bound the average number of periodic points over all polynomials of degree \( d \). For \( d \) large (that is, \( d \geq \sqrt{p^3} \)), their bound of \( \frac{5}{8} \sqrt{p^3} \) agrees with earlier heuristic arguments.

In her thesis [2], Madhu tackles the second question in the case \( \phi(z) = z^m + c \), using Galois-theoretic methods. With some restrictions on \( c \), she shows that for primes congruent to 1 modulo \( m \), the proportion of points in \( \mathbb{F}_p \) that are periodic points for \( \phi \) goes to zero as \( p \to \infty \).

In the current work, we focus on the third question in the special case that the polynomial map \( \phi(z) \) can be viewed as an endomorphism of an underlying algebraic group. This restriction makes the structure of the periodic points particularly...
Figure 1. Few periodic points: $\phi(z) = z^{11}$ on $\mathbb{F}_3^5$ has three fixed points and 240 strictly preperiodic points.

simple and is therefore a natural place to begin a more complete investigation of the question.

We will quickly see that in fact the naïve limit

$$\lim_{n \to \infty} \frac{\# \text{Per} (\phi, \mathbb{F}_{p^n})}{p^n}$$

does not exist in general, because the map $\phi$ acts as a permutation polynomial whenever $n$ is relatively prime to the multiplicative order of $p$ modulo the degree of $\phi$.

However, we are able to find limiting proportions along towers of finite fields $\mathbb{F}_{p^n}$ with suitable divisibility conditions on $n$. For example, we have the following two results for $q$ an odd prime. Similar results hold in the case $q = 2$ and for maps of composite degree.

**Theorem** (Theorems 4.5 and 5.7). Fix a prime $p$ and let $q$ be a different odd prime. Define $\delta = \text{ord}_q(p)$ and $\mu = v_q(p^\delta - 1) \geq 1$. Let $\phi(z) = z^q$, and let $T_q(z)$ be the $q^{th}$ Chebyshev polynomial. Then we have the following:

$$\lim_{n \to \infty} \frac{\# \text{Per} (\phi, \mathbb{F}_{p^n})}{p^n} = \frac{1}{q^\mu + \nu}, \text{ and}$$

$$\lim_{n \to \infty} \frac{\# \text{Per} (T_q, \mathbb{F}_{p^n})}{p^n} = \frac{q^\mu + \nu + 1}{2q^\mu + \nu}.$$
Figure 2. Lots of periodic points: $\phi(z) = z^3$ on $F_{5^4}$ has 209 periodic points and 416 strictly preperiodic points.

Haensch, Adriana Salerno, Lola Thompson, and Stephanie Treneer. Thanks also to Tom Tucker and Kalyani Madhu for help with pictures.
2. Polynomials associated to endomorphisms of algebraic groups

We first consider the multiplicative group $G_m$ where for a field $K$, the $K$-valued points are $G_m(K) = K^*$. The endomorphism ring of $G_m$ is $\mathbb{Z}$:

$$\mathbb{Z} \rightarrow \text{End}(G_m)$$

$$d \mapsto z^d.$$ 

So these pure power maps can be viewed as endomorphisms of an underlying group. Iteration of pure power maps is particularly easy to understand, as $\phi(z) = z^d$ means $\phi^n(z) = z^{dn}$.

Similarly, we consider the additive group $G_a$, whose underlying scheme is the affine line $\mathbb{A}^1$, which may be viewed as a quotient of $G_m$:

$$G_m/\{z = z^{-1}\} \rightarrow \mathbb{A}^1$$

$$z \mapsto z + z^{-1}.$$ 

Since the automorphism $z \mapsto z^{-1}$ commutes with the power map $\phi(z) = z^d$, the polynomial $\phi$ descends to an endomorphism of $\mathbb{A}^1$, which we denote $T_d$, the $d$th Chebyshev polynomial.

Taking as a definition the fact that $T_d(w) \in \mathbb{Z}[w]$ satisfies

$$T_d(z + z^{-1}) = z^d + z^{-d}, \quad (1)$$

one may prove existence and uniqueness of the Chebyshev polynomials along with a simple recursion

$$T_d(w) = \begin{cases} 
2 & d = 0 \\
w & d = 1 \\
wT_{d-1}(w) - T_{d-2}(w) & d \geq 2.
\end{cases} \quad (2)$$

A pleasant rule for composition of Chebyshev polynomials arises directly from the definition in (1):

$$T_d \circ T_e(w) = T_{de}(w) = T_e \circ T_d(w),$$

which in turn gives a simple form of iteration

$$T_d^n(w) = T_{dn}(w). \quad (3)$$

We refer the interested reader to \cite{3, Chapter 6} for more on the dynamics of pure power maps, Chebyshev polynomials, and other rational maps arising from algebraic groups, including proofs of some of the statements above.
3. Preliminaries

This section contains a few facts about valuations and periodic points over finite fields which will be useful in the sequel. Throughout this section, $p$ and $q$ represent distinct primes, $n$ is a positive integer, and we use the following additional notation:

- $v_q(n)$: $q$-adic valuation; i.e. if $n = q^d d$ with $q 
mid d$, then $v_q(n) = \nu$.
- $\delta = \text{ord}_q(p)$; i.e. the smallest positive integer such that $q \mid (p^\delta - 1)$.

Since our goal is ultimately to classify periodic points in finite fields, we need to be able to recognize which points are periodic as opposed to strictly preperiodic. Our first result says that any finite set that is forward invariant under $\phi$ contains only periodic points.

**Lemma 3.1.** Let $\phi(z) \in K[z]$ be a polynomial and let $S \subseteq K$ be finite. If $\phi(S) = S$, then $S \subseteq \text{Per}(\phi, K)$.

**Proof.** Fix $\alpha \in S$. For every $n > 0$ we have $\phi^n(S) = S$. Hence for every $n$, we can find $\beta_n \in S$ such that $\phi^n(\beta_n) = \alpha$.

Since $S$ is finite, for some $n > m > 0$, we must have $\beta_n = \beta_m$. But this means we have $\beta \in S$ such that $\phi^m(\beta) = \alpha$ and $\phi^n(\beta) = \alpha$, so $\phi^{n-m}(\alpha) = \alpha$ and $\alpha$ is periodic. \hfill \Box

The next three lemmas give us the tools to calculate the $q$-adic valuation of $p^{\delta d} - 1$ based on the valuations of $p^d - 1$ and $n$. These will be used to create the towers of finite fields for which we can calculate limiting proportions of periodic points. The results are different enough for $q = 2$ compared to odd primes that we break up the cases along those lines.

**Lemma 3.2.** Let $p$ and $q$ be distinct primes. Suppose $v_q(p^d - 1) = \mu \geq 1$ and $v_q(n) = 0$. Then $v_q(p^{\delta d} - 1) = \mu$.

**Proof.**

$$v_q(p^{\delta d} - 1) = v_q(p^d - 1) + v_q(p^{(n-1)d} + p^{(n-2)d} + \cdots + p^d + 1)$$

$= \mu + 0 = \mu$. \hfill \Box

**Lemma 3.3.** Let $p$ be an odd prime with $\max\{v_2(p - 1), v_2(p + 1)\} = \mu$. Let $v_2(n) = \nu \geq 1$. Then $v_2(p^\mu - 1) = \mu + \nu$.

**Proof.** We proceed by induction on $v_2(n)$. For every odd $d$, exactly one of $p^d - 1$, $p^d + 1$ is divisible by 4. (In particular, $\mu \geq 2$.) Similar to the proof of Lemma 3.2, we have

$$v_2(p^{2d} - 1) = v_2(p^d - 1) + v_2(p^d + 1)$$

$$= v_2(p - 1) + v_2(\text{odd number}) + v_2(p + 1) + v_2(\text{odd number})$$

$$= \mu + 1.$$  

Assume for all $n$ with $v_2(n) = \nu > 1$ we have $v_2(p^n - 1) = \mu + \nu > 1$, in which case $v_2(p^n + 1) = 1$. Consider some $n$ with $v_2(n) = \nu + 1$ and choose $d$ odd such that $n = 2^{\nu+1}d$. 


\[ v_2(p^n - 1) = v_2(p^{2^{v_2(d)}} - 1) = v_2(p^{2v_2(d)} - 1) + v_2(p^{2v_2(d)} + 1) = \mu + \nu + 1. \]

**Lemma 3.4.** Let \( q \) be an odd prime. Suppose \( v_q(p^d - 1) = \mu \geq 1 \) and \( v_q(n) = \nu \). Then \( v_q(p^{nd} - 1) = \mu + \nu \).

**Proof.** The result for \( \nu = 0 \) is exactly Lemma 3.2. Choose \( k \) so that \( p^d = 1 + kq^\mu \) (in particular \( q \nmid k \)). Since \( q \geq 3 \) and \( \mu \geq 1 \), we have \( q\mu \geq \mu + 2 \). Hence

\[ p^{qd} = (1 + kq^\mu)^d \equiv 1 + kq^{\mu+1} \pmod{q^{\mu+2}}. \]

The result then follows by a straightforward induction. \( \square \)

Our main results in Sections 4 and 5 will be stated for maps of prime degree \( q \).

The following Lemma shows that in fact the proportion of periodic points is identical for the maps of degree \( q \) and degree \( q^e \). We focus on the prime degree case for ease of exposition.

**Lemma 3.5.** Let \( \phi(z) = z^q \) and \( \psi(z) = z^{q^e} \). Then \( \text{Per}(\phi, \mathbb{F}_{p^n}) = \text{Per}(\psi, \mathbb{F}_{p^n}) \) for every \( n \). Similarly, \( \text{Per}(T_q, \mathbb{F}_{p^n}) = \text{Per}(T_{q^e}, \mathbb{F}_{p^n}) \).

**Proof.** Note that \( \phi^m(z) = z^{q^m} \) and \( \psi^m(z) = z^{q^em} \). So if \( \phi^m(\alpha) = \alpha \), then likewise \( \psi^m(\alpha) = \alpha \). On the other hand, if \( \psi^m(\alpha) = \alpha \), then \( \phi^m(\alpha) = \alpha \). Applying the iteration for Chebychev polynomials in \( [3] \) gives the result in that case as well. \( \square \)

4. Power maps

Throughout this section, we fix the polynomial

\[ \phi(z) = z^q, \]

for \( q \) prime. We also take \( p \) to be any prime different from \( q \). Our interest is in understanding the proportion of periodic points in \( \mathbb{F}_{p^n} \) as \( n \) grows. In particular, we consider the following limits.

**Definition 4.1.** We define the following proportions for integers \( \nu \geq 0 \). Recall that \( \delta \) is the multiplicative order of \( p \) modulo \( q \).

\[
P_\nu(\phi) = \lim_{\delta | n, v_q(n) = \nu} \frac{\# \text{Per}(\phi, \mathbb{F}_{p^n})}{p^n}.
\]

Since \( \delta = \text{ord}_q(p) \), we know that \( \delta < q \). So if \( n \) satisfies

\[ \delta | n \text{ and } v_q(n) = \nu, \]

then there is \( n' \) such that

\[ n = \delta n' \text{ and } v_q(n') = \nu. \]

We will implicitly use this fact later when applying Lemma 3.4.

We begin by classifying explicitly the periodic points of \( \phi \) in \( \mathbb{F}_{p^n} \).

**Lemma 4.2.** Let \( p^n - 1 = q^ed \) with \( q \nmid d \). Then

\[ \text{Per}(\phi, \mathbb{F}_{p^n}) = \{0\} \cup \{\alpha \in \mathbb{F}_{p^n} : \alpha^d = 1\}. \]
Proof. The defining equation for $\mathbb{F}_{p^n}$ is

$$z^{p^n} - z = z(z^d - 1)Q(z),$$

for some monic $Q(z) \in \mathbb{Z}[z]$. Clearly 0 is fixed by $\phi$. Since $q \nmid d$, the roots of $z^d - 1$ form a group of order prime to $q$. Hence $\phi(z) = z^q$ is a permutation of the group elements, and these roots are forward invariant under $\phi$. So we have

$$\{0\} \cup \{ \alpha \in \mathbb{F}_{p^n} : \alpha^d = 1 \} \subseteq \text{Per}(\phi, \mathbb{F}_{p^n}).$$

Now let $\alpha$ be a root of $Q(z)$; so in particular $\alpha^{q^d} = 1$ but $\alpha^d \neq 1$. Hence for some $1 \leq i \leq e$ and some $d' | d$, we have $\alpha^{q^d} = 1$. In other words, $\alpha^d$ has order dividing $d$ and is therefore a root of $z^d - 1$. Since roots of $z^d - 1$ are forward invariant under $\phi$, $\alpha$ is not periodic for $\phi$. □

Remark. We applied Lemma 4.2 to create the examples in Figures 1 and 2. Finding a value of $p^n - 1$ where, in the notation of the Lemma, $q^e$ is much smaller than $d$ gives “lots of periodic points.” Similarly, an example where $q^e$ gives few periodic points.

The following Proposition justifies our choice of limit in Definition 4.1 because the only interesting proportions of periodic points are those where $\text{ord}_q(p) = \delta | n$.

**Proposition 4.3.** If $\text{ord}_q(p) = \delta | n$, all points of $\mathbb{F}_{p^n}$ are periodic under $\phi$.

Proof. Since $\delta | n$, $q \nmid p^n - 1$. The result follows immediately from Lemma 4.2 □

We now prove our main results for pure power maps. The statement is slightly different depending on whether $q = 2$ or $q$ is an odd prime. The difference parallels exactly the difference between the valuation calculations in Lemmas 3.3 and 3.4.

**Theorem 4.4.** Let $v_2(p - 1) = \lambda$ and $\max\{v_2(p - 1), v_2(p + 1)\} = \mu$. Then for $\phi(z) = z^2$ we have

$$P_0(\phi) = \frac{1}{2^\lambda}, \quad \text{and}$$

$$P_\nu(\phi) = \frac{1}{2^{\nu+\nu}}, \quad \text{for } \nu \geq 1.$$

Proof. First consider $n$ odd. By Lemma 3.2 we may choose $d_n$ odd so that $p^n - 1 = 2^\lambda d_n$. By Lemma 4.2 the periodic points for $\phi$ in $\mathbb{F}_{p^n}$ are 0 and roots of $z^{d_n} - 1$. So there are $d_n + 1$ points in $\text{Per}(\phi, \mathbb{F}_{p^n})$. Then

$$P_0(\phi) = \lim_{n \to \infty} \frac{\# \text{Per}(\phi, \mathbb{F}_{p^n})}{p^n} = \lim_{d_n \to \infty} \frac{d_n + 1}{2^\lambda d_n + 1} = \lim_{d \to odd} \frac{d + 1}{2^\lambda d + 1} = \frac{1}{2^\lambda}.

Now let $v_2(n) = \nu \geq 1$. By Lemma 3.3 $p^n - 1 = 2^{\nu+\nu} d_n$ with $d_n$ odd. Again, the periodic points for $\phi$ in $\mathbb{F}_{p^n}$ are 0 and roots of $z^{d_n} - 1$. Hence

$$P_\nu(\phi) = \lim_{n \to \infty, v_2(n) = \nu} \frac{\# \text{Per}(\phi, \mathbb{F}_{p^n})}{p^n} = \lim_{d \to odd} \frac{d + 1}{2^{\nu+\nu} d + 1} = \frac{1}{2^{\nu+\nu}}. \quad \Box

In Tables 1, 2, we illustrate Theorem 4.4. The data were calculated using Sage 4.

**Theorem 4.5.** Let $q$ be an odd prime. We continue with the earlier notation: $\delta = \text{ord}_q(p)$ and $v_q(p^\mu - 1) = \mu$. For $\phi(z) = z^q$, we have

$$P_\nu(\phi) = \frac{1}{q^\mu+\nu}.$$
\[ \lambda = v_2(p - 1) \]

| \# Per\((z^2, \mathbb{F}_p)\) | \(p\) | \(3\) | \(5\) | \(41\) | \(17\) |
|-----------------|------|-----|-----|-----|-----|
| \(p\)           |      |     |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^2})\) |      |     |     |     |     |
| \(p^2\)         |      |     |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^3})\) |      |     |     |     |     |
| \(p^3\)         |      |     |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^5})\) |      |     |     |     |     |
| \(p^5\)         |      |     |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^7})\) |      |     |     |     |     |
| \(p^7\)         |      |     |     |     |     |
| \[\frac{1}{2^\lambda}\] |      | \(0.5\) | \(0.25\) | \(0.125\) | \(0.0625\) |

Table 1. \(\# \text{Per}(z^2, \mathbb{F}_{p^n})\) with \(n\) odd.

\[
\mu = \max \{v_2(p - 1), v_2(p + 1)\}
\]

| \# Per\((z^2, \mathbb{F}_{p^2})\) | \(p\) | \(3\) | \(7\) | \(17\) |
|-----------------|------|-----|-----|-----|
| \(p^2\)         |      |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^3})\) |      |     |     |     |
| \(p^3\)         |      |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^5})\) |      |     |     |     |
| \(p^5\)         |      |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^{10}})\) |      |     |     |     |
| \(p^{10}\)      |      |     |     |     |
| \# Per\((z^2, \mathbb{F}_{p^{14}})\) |      |     |     |     |
| \(p^{14}\)      |      |     |     |     |
| \[
\frac{1}{2^{\mu+1}}
\] |      | \(0.125\) | \(0.0625\) | \(0.03125\) |

Table 2. \(\# \text{Per}(z^2, \mathbb{F}_{p^n})\) with \(v_2(n) = 1\).

**Proof.** Recall that the limit for \(P_\nu(\phi)\) is taken over \(n\) such that \(\delta \mid n\) and \(v_\nu(n) = \nu\). By Lemma 3.4, for such \(n\) we have \(p^n - 1 = q^{\mu+\nu}d_n\) with \(q \mid d_n\), and by Lemma 4.2
the periodic points are 0 and roots of $z^{d_n} - 1$. So
\[
\lim_{n \to \infty} \frac{\# \text{Per}(\phi, \mathbb{F}_{p^n})}{p^n} = \lim_{n \to \infty} \frac{d_n + 1}{q^{\mu + \nu} d_n + 1} = \frac{1}{q^{\mu + \nu}}.
\]
\qed

Tables 3–4 illustrate Theorem 4.5 for the map $\phi(z) = z^3$. Again, the data were calculated using Sage \[4\].

| $p$ | $d$ | $\mu$ | $\delta$ = ord$_3(p)$ | $\mu = v_3(p^\delta - 1)$ |
|-----|-----|-------|------------------|------------------|
| $5$ | $19$ | $2$    | $2$             | $1$               |
| $53$ |      |       | $1$             | $2$               |

| $\frac{\# \text{Per}(z^3, \mathbb{F}_{p^\delta})}{p^\delta}$ | $\frac{\# \text{Per}(z^3, \mathbb{F}_{p^{2\delta}})}{p^{2\delta}}$ | $\frac{\# \text{Per}(z^3, \mathbb{F}_{p^{4\delta}})}{p^{4\delta}}$ |
|-------------------------------------------------------|-------------------------------------------------------|-------------------------------------------------------|
| $0.360000000$ | $0.157894737$ | $0.0373798505$ |
| $0.334400000$ | $0.113573407$ | $0.0370371591$ |
| $0.333350400$ | $0.11117932$ | $0.037037371$ |

| $\frac{1}{3^\mu}$ | $0.333333333$ | $0.111111111$ | $0.0370370370$ |

Table 3. $\frac{\# \text{Per}(z^3, \mathbb{F}_{p^n})}{p^n}$ with $v_3(n) = 0$.

We wish to extend our results to polynomials with composite degree. Lemma 3.5 takes care of prime power degree, so we are left to consider the case $\phi(z) = z^t$ for $t = q_1^{f_1} q_2^{f_2} \cdots q_r^{f_r}$ and $r \geq 2$. For each $1 \leq i \leq r$, let
\[
\delta_i = \text{ord}_{q_i}(p) \quad \text{and} \quad \mu_i = v_{q_i}(p^{\delta_i} - 1).
\]
We also define
\[
\Delta = \text{lcm}\{\delta_i\}_{1 \leq i \leq r}.
\]
An argument identical to the one in Proposition 5.5 shows that if $\gcd(\Delta, n) = 1$, then all points of $\mathbb{F}_{p^n}$ will be periodic. Unlike the case of prime degree, however, we need not require $\Delta \mid n$ to have a nontrivial ratio of periodic points.

In order to define the appropriate towers of fields, we need a bit more notation. For each nonempty subset $I \subseteq \{1, 2, \ldots, r\}$, let
\[
\delta_I = \text{lcm}\{\delta_i\}_{i \in I}.
\]
If $\delta_I = \delta_{I'}$, then $\delta_{I \cup I'} = \delta_I$ as well. Hence to a fixed value of $\delta$ we will associate the maximal subset $J \subseteq \{1, 2, \ldots, r\}$ such that $\delta_J \mid \delta$. Finally, given an integer $n$, we define an $r$-tuple of valuations
\[
v(n) = (v_{q_i}(n))_{1 \leq i \leq r}.
\]
| $p$ | $\delta = \text{ord}_3(p)$ | $\mu = v_3(p^\delta - 1)$ |
|-----|-----------------|------------------|
| 5   | 19              | 53               |
| 2   | 1               | 2                |
| 1   | 2               | 3                |

| $\frac{\# \text{Per}(z^3, F_{p^\delta})}{p^\delta}$ | $\frac{0.111168000}{0.0371774311}$ | $0.0123456791$ |
| $\frac{\# \text{Per}(z^3, F_{p^6\delta})}{p^6\delta}$ | $\frac{0.11111115}{0.0370370575}$ | $0.0123456790$ |
| $\frac{\# \text{Per}(z^3, F_{p^{12}\delta})}{p^{12}\delta}$ | $\frac{0.111111111}{0.0370370370}$ | $0.0123456790$ |
| $\frac{1}{3^{\mu+1}}$ | $0.111111111$ | $0.0370370370$ | $0.0123456790$ |

Table 4. $\frac{\# \text{Per}(z^3, F_{p^n})}{p^n}$ with $v_3(n) = 1$.

We now have the tools to define limiting proportions of periodic points along appropriate towers of finite fields. Define

$$P_{\delta, \nu}(\phi) = \lim_{\gcd(d_n, n) = \delta, v(n) = (\nu)} \frac{\# \text{Per}(\phi, F_{p^n})}{p^n}.$$

**Proposition 4.6.** Let $\phi(z) = z^t$ where $t = q_1^{f_1} q_2^{f_2} \ldots q_r^{f_r}$, with $q_i$ distinct odd primes for $1 \leq i \leq r$. Then for $J \subseteq \{1, 2, \ldots, r\}$ maximal with $\delta_J | \delta$,

$$P_{\delta, \nu}(\phi) = \prod_{j \in J} \frac{1}{q_j^{\mu_j + \nu_j}}.$$

**Remark.** If $\delta_i | \delta$, then the maximal set $J$ is empty, and we recover the fact that all points in $F_{p^n}$ are periodic in this case. This theorem also recovers our result in Theorem 4.5 when applied to the case $t = q$ for $q$ an odd prime.

**Proof.** Since $J$ is maximal such that $\delta_J | \gcd(\Delta, n)$, we have

$$p^n - 1 = d_n \prod_{j \in J} q_j^{e_j} \quad \text{with} \quad \gcd(t, d_n) = 1.$$

Lemma 3.4 shows that $e_j = v_{q_j}(p^n - 1) = \mu_j + \nu_j$ for each $j \in J$.

The proof of Lemma 4.2 extends easily to this case, and we have

$$\text{Per}(\phi, F_{p^n}) = \{0\} \cup \{\alpha \in F_{p^n} : \alpha^{d_n} = 1\}.$$

Hence

$$P_{\delta, \nu}(\phi) = \lim_{\gcd(d_n, n) = \delta, v(n) = (\nu)} \frac{\# \text{Per}(\phi, F_{p^n})}{p^n}.$$
In Tables 5–6 we use data from Sage [4] to illustrate Theorem 4.6 for the map \( \phi(z) = z^{15} \) over fields \( \mathbb{F}_{2^n} \). In the notation of the theorem, we have the following:

\[
q_1 = 3 \quad q_2 = 5 \quad p = 2 \\
\delta_1 = 2 \quad \delta_2 = 4 \quad \Delta = 4 \\
\mu_1 = v_3(2^2 - 1) = 1 \quad \mu_2 = v_5(2^4 - 1) = 1.
\]

The table contains values of \( n \) with \( \gcd(4, n) = \delta \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\delta & 1 & 2 & 4 \\
\hline
\text{\# Per}(z^{15}, \mathbb{F}_{2^3}) \quad 2^3 & 1.00000000 & 0.500000000 & 0.125000000 \\
\text{\# Per}(z^{15}, \mathbb{F}_{2^7}) \quad 2^7 & 1.00000000 & 0.333374023 & 0.0666666701 \\
\text{\# Per}(z^{15}, \mathbb{F}_{2^{11}}) \quad 2^{11} & 1.00000000 & 0.33333492 & 0.066666667 \\
\hline
\{q_j : j \in J\} & \emptyset & \{3\} & \{3, 5\} \\
\prod_{j \in J} \frac{1}{q_j^{\nu_j}} & 1 & 0.33333333 & 0.0666666666 \\
\hline
\end{array}
\]

Table 5. \( \text{\# Per}(z^{15}, \mathbb{F}_{2^n}) \quad 2^n \) with \( \nu = (v_3(n), v_5(n)) = (0, 0) \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\delta & 1 & 2 & 4 \\
\hline
\text{\# Per}(z^{15}, \mathbb{F}_{2^{13}}) \quad 2^{13} & 1.00000000 & 0.125000000 & 0.0224609375 \\
\text{\# Per}(z^{15}, \mathbb{F}_{2^{21}}) \quad 2^{21} & 1.00000000 & 0.111111111 & 0.0222222222 \\
\text{\# Per}(z^{15}, \mathbb{F}_{2^{33}}) \quad 2^{33} & 1.00000000 & 0.111111111 & 0.0222222222 \\
\hline
\{q_j : j \in J\} & \emptyset & \{3\} & \{3, 5\} \\
\prod_{j \in J} \frac{1}{q_j^{\nu_j + \nu_j}} & 1 & 0.111111111 & 0.0222222222 \\
\hline
\end{array}
\]

Table 6. \( \text{\# Per}(z^{15}, \mathbb{F}_{2^n}) \quad 2^n \) with \( \nu = (v_3(n), v_5(n)) = (1, 0) \).
Remark. A statement similar to Proposition 4.6 holds when \( t \) is even, though the bookkeeping is somewhat messier. One must apply the results in Lemma 3.3 with the exponent for 2 depending on \( \max\{v_2(p-1), v_2(p+1)\} \) and \( v_2(n) \). We leave the details to the interested reader.

5. Chebyshev Polynomials

Throughout this section, we consider \( T_q(z) \), the Chebyshev polynomial of prime degree \( q \). We take \( p \) to be any prime different from \( q \). The proportions of interest in this case run over slightly different towers of finite fields than in the power map case.

**Definition 5.1.** We define the following proportions for integers \( \nu \geq 0 \). Recall that \( \delta \) is the multiplicative order of \( p \) modulo \( q \).

\[
R_\nu(T_q) = \lim_{n \to \infty} \frac{\# \text{Per}(T_q, \mathbb{F}_{p^n})}{p^n}.
\]

We begin with an explicit classification of the periodic points of \( T_q \) in \( \mathbb{F}_{p^n} \). For any \( \omega \in \mathbb{F}_p \), we may solve a quadratic to find a nonzero \( \zeta \in \mathbb{F}_p \) such that \( \omega = \zeta + \zeta - 1 \).

**Lemma 5.2.** Consider some nonzero \( \zeta \in \mathbb{F}_p \) and an integer \( d \geq 0 \). Then \( \zeta + \zeta - 1 = \zeta^d + \zeta^{-d} \) if and only if \( \zeta = \zeta^d \) or \( \zeta = \zeta^{-d} \).

**Proof.**

\[
\begin{align*}
\zeta + \zeta^{-1} &= \zeta^d + \zeta^{-d} \\
\zeta^{2d} - \zeta^{d+1} - \zeta^{d-1} + 1 &= 0 \\
(\zeta^{d-1} - 1)(\zeta^{d+1} - 1) &= 0.
\end{align*}
\]

Since \( \zeta \neq 0 \), the first factor vanishes if and only if \( \zeta^d = \zeta \) and the second vanishes if and only if \( \zeta^d = 1/\zeta \). \( \Box \)

**Lemma 5.3.** Let \( \omega \in \mathbb{F}_p \). Then \( \omega \in \text{Per}(T_q, \mathbb{F}_p) \) if and only if \( \omega = \zeta + \zeta^{-1} \) where \( \zeta^d = 1 \) for some \( d \) relatively prime to \( q \).

**Proof.** Suppose \( \omega \in \mathbb{F}_p \) is periodic for \( T_q \), and choose \( \zeta \) so that \( \omega = \zeta + \zeta^{-1} \). Then \( T_q^n(\omega) = \omega \); that is,

\[
T_q^n(\zeta + \zeta^{-1}) = \zeta^{q^n} + \zeta^{-q^n} = \zeta + \zeta^{-1}.
\]

So by Lemma 5.2 \( \zeta^{q^n-1} = 1 \) or \( \zeta^{q^n+1} = 1 \).

Conversely, suppose there is \( d \) prime to \( q \) such that \( \zeta^d = 1 \), and let \( \varphi \) be the Euler totient function. Since \( d \mid (q^{\varphi(d)} - 1) \),

\[
\zeta^{q^{\varphi(d)}-1} = 1; \quad \text{that is,} \quad \zeta^{q^{\varphi(d)}} = \zeta.
\]

Hence \( \omega = \zeta + \zeta^{-1} \) is fixed by \( T_q^{\varphi(d)} \). \( \Box \)

We see that counting the periodic points for \( T_q(z) \) in \( \mathbb{F}_{p^n} \) reduces to counting \( \zeta \in \mathbb{F}_{p^n} \) such that \( \zeta + \zeta^{-1} \in \mathbb{F}_{p^n} \) and \( \zeta^d = 1 \) for some \( d \) prime to \( q \).

**Lemma 5.4.** Let \( \zeta \in \mathbb{F}_p \). Then \( \zeta + \zeta^{-1} \in \mathbb{F}_{p^n} \) if and only if \( 0 \neq \zeta \in \mathbb{F}_{p^n} \) or \( \zeta^{p^n+1} = 1 \).
Proof. We have \( \zeta + \zeta^{-1} \in \mathbb{F}_{p^n} \) if and only if it satisfies
\[
(\zeta + \zeta^{-1})^p = \zeta + \zeta^{-1} \\
\zeta^p + \zeta^{-p} = \zeta + \zeta^{-1}.
\]
So by Lemma 5.2 either \( \zeta = \zeta^p \) (i.e. \( \zeta \in \mathbb{F}_{p^n} \)) or \( 1/\zeta = \zeta^p \).

□

Once again, the classification of periodic points explains our choice of limit in Definition 5.1.

**Proposition 5.5.** If \( \text{ord}_q(p) = \delta \neq 2n \), then all points of \( \mathbb{F}_{p^n} \) are periodic under \( T_q \).

**Proof.** Given that
\[
q \nmid p^{2n} - 1,
\]
we conclude that \( q \nmid p^n + 1 \) and \( q \nmid p^n - 1 \).

By Lemma 5.4, every \( \omega \in \mathbb{F}_{p^n} \) can be written as \( \zeta + \zeta^{-1} \) for some \( \zeta \) with either \( \zeta^{p^n-1} = 1 \) or \( \zeta^{p^n+1} = 1 \). Since \( p^n - 1 \) and \( p^n + 1 \) are both prime to \( q \), the result follows from Lemma 5.4. □

We now prove our main results for Chebyshev polynomials. As in the case of pure power maps, the statements are slightly different in the case \( q = 2 \) versus \( q \) odd.

**Theorem 5.6.** Let \( \mu = \max \{v_2(p-1), v_2(p+1)\} \). Then
\[
R_\nu(T_2) = \frac{2^{\mu + \nu - 1} + 1}{2^{\mu + \nu + 1}}.
\]

**Proof.** Assume \( \omega \in \mathbb{F}_{p^n} \) is periodic for \( T_2 \). Then by Lemma 5.3, \( \omega = \zeta + \zeta^{-1} \), where \( \zeta^d = 1 \) for some odd \( d \). Since \( \zeta + \zeta^{-1} \in \mathbb{F}_{p^n} \), we apply Lemma 5.4 to conclude that \( \zeta^{p^n+1} = 1 \) or \( \zeta^{p^n-1} = 1 \).

First suppose \( v_2(n) = 0 \), so by Lemma 3.2, \( v_2(p-1) = v_2(p^n - 1) \). Then
\[
\begin{align*}
p^n - 1 &= 2^\mu d_1, \\
p^n + 1 &= 2^{\mu + 1} d_2;
\end{align*}
\]
where \( d_1 \) and \( d_2 \) are odd. Note that \( d_1 \) and \( d_2 \) are relatively prime since \( d_1 \mid (p^n + 1) \) and \( d_2 \mid (p^n - 1) \) or vice-versa, with both odd.

Similarly, if \( v_2(n) = \nu \geq 1 \), Lemma 3.3 shows that \( v_2(p^n - 1) = \mu + \nu \), so we have
\[
\begin{align*}
p^n - 1 &= 2^{\mu + \nu} d_1 \quad \text{and} \quad p^n + 1 = 2^\nu d_2;
\end{align*}
\]
where \( d_1 \) and \( d_2 \) are odd and relatively prime.

In either case, \( \zeta + \zeta^{-1} \) is periodic if and only if \( \zeta^{d_1} = 1 \) or \( \zeta^{d_2} = 1 \). Each such pair \( (\zeta, \zeta^{-1}) \) — including the pair \((1, 1)\) — corresponds to a periodic point for \( T_2 \). Therefore, we have \((d_1 + d_2)/2\) periodic points for \( T_2 \) in \( \mathbb{F}_{p^n} \).

Asymptotically, \( p^n + 1 \sim p^n - 1 \). That is,
\[
2^{\mu + \nu} d_1 \sim 2^\nu d_2 \quad \text{so} \quad 2^{\mu + \nu - 1} d_1 \sim d_2.
\]

Hence
\[
R_\nu(T_2) = \lim_{d \to \infty} \frac{\# \text{Per}(T_2, \mathbb{F}_{p^n})}{p^n} = \lim_{d \to \infty} \frac{(d + 2^{\mu + \nu - 1} d_2)/2}{2^\nu d_2 + 1} = \frac{2^{\mu + \nu - 1} + 1}{2^{\mu + \nu + 1}}.
\]

□

In Tables 7–8, we illustrate Theorem 5.6 using data from Sage [4].
\[\mu = \max\{v_2(p - 1), v_2(p + 1)\}\]

\[
\begin{array}{c|ccc}
\mu = \max\{v_2(p - 1), v_2(p + 1)\} & 3 & 7 & 17 \\
\hline
p & 2 & 3 & 4 \\
\hline
\# \text{Per}(T_2, \mathbb{F}_p) / p & 0.333333333 & 0.285714286 & 0.294117647 \\
\# \text{Per}(T_2, \mathbb{F}_{2^3}) / p^3 & 0.370370370 & 0.311953353 & 0.281294525 \\
\# \text{Per}(T_2, \mathbb{F}_{2^5}) / p^5 & 0.37485597 & 0.312488844 & 0.281250154 \\
\# \text{Per}(T_2, \mathbb{F}_{2^7}) / p^7 & 0.374942844 & 0.312499772 & 0.281250001 \\
\hline
\frac{2^{\mu - 1} + 1}{2^{\mu + 1}} & 0.375 & 0.3125 & 0.28125 \\
\end{array}
\]

Table 7. $\# \text{Per}(T_2, \mathbb{F}_{p^n}) / p^n$ with $n$ odd.

\[
\begin{array}{c|ccc}
\mu = \max\{v_2(p - 1), v_2(p + 1)\} & 3 & 7 & 17 \\
\hline
p & 2 & 3 & 4 \\
\hline
\# \text{Per}(T_2, \mathbb{F}_{2^2}) / p^2 & 0.333333333 & 0.285714286 & 0.266435986 \\
\# \text{Per}(T_2, \mathbb{F}_{2^6}) / p^6 & 0.312757202 & 0.281251859 & 0.265625010 \\
\# \text{Per}(T_2, \mathbb{F}_{2^{10}}) / p^{10} & 0.312503175 & 0.281250001 & 0.265625000 \\
\# \text{Per}(T_2, \mathbb{F}_{2^{14}}) / p^{14} & 0.312500039 & 0.281250000 & 0.265625000 \\
\hline
\frac{2^{\mu + 1} + 1}{2^{\mu + 2}} & 0.3125 & 0.28125 & 0.265625 \\
\end{array}
\]

Table 8. $\# \text{Per}(T_2, \mathbb{F}_{p^n}) / p^n$ with $v_2(n) = 1$.

**Theorem 5.7.** Let $q$ be an odd prime. Let $v_q(p^\delta - 1) = \mu \geq 1$. Then

\[R_\nu(T_q) = \frac{q^{\nu + \mu} + 1}{2q^{\nu + \mu}}.\]
Proof. Assume \( \omega \in \mathbb{F}_{p^n} \) is periodic for \( T_q \). Then by Lemma 5.3 \( \omega = \zeta + \zeta^{-1} \), where \( \zeta^d = 1 \) for some \( d \) prime to \( q \). Since \( \zeta + \zeta^{-1} \in \mathbb{F}_{p^n} \), we apply Lemma 5.4 to conclude that \( \zeta^{p^n+1} = 1 \) or \( \zeta^{p^n-1} = 1 \).

Since \( v_q(p^d - 1) = \mu \geq 1 \) and \( v_q(n) = \nu \), by Lemma 5.3 \( v_q(p^{2n} - 1) = \mu + \nu \geq 1 \). So

\[
q \mid p^{2n} - 1, \quad \text{which means that} \quad q \mid p^n - 1 \quad \text{or} \quad q \mid p^n + 1 \quad \text{but not both.}
\]

Therefore

\[
\begin{align*}
p^n - 1 &= q^\mu + \nu d_1 \\
p^n + 1 &= q^\mu + \nu d_2;
\end{align*}
\]

where \( q \nmid d_1 d_2 \).

Now, \( \zeta + \zeta^{-1} \) is periodic if and only if \( \zeta^{d_1} = 1 \) or \( \zeta^{d_2} = 1 \). Each such pair \((\zeta, \zeta^{-1})\) — including the pairs \((1, 1)\) and \((-1, -1)\) if \( p \) odd — corresponds to a periodic point for \( T_q \). So we have \( (d_1 + d_2)/2 \) periodic points for \( T_q \) in \( \mathbb{F}_{p^n} \).

Again, \( p^n + 1 \sim p^n - 1 \) meaning

\[
q^\mu + \nu d_1 \sim d_2.
\]

Hence

\[
R_\nu(T_q) = \lim_{n \to \infty, \nu(n) = \nu} \frac{\# \text{Per}(T_q, \mathbb{F}_{p^n})}{p^n} = \lim_{d \to \infty} \frac{(d + q^{\mu+\nu} d_2)/2}{q^{\mu+\nu} d_1 + 1} = \frac{q^{\mu+\nu} + 1}{2q^{\mu+\nu}}.
\]

\( \Box \)

Remark. Theorem 5.6 says that the proportion of periodic points in the appropriate towers for \( T_2 \) is something slightly more than \( 1/4 \), where the difference depends on the tower. Similarly, Theorem 5.7 says that for \( q \) an odd prime, the proportion is slightly greater than \( 1/2 \). We can understand these results a bit more intuitively in the following way.

Consider roots of the polynomials \( z^{p^n+1} - 1 \) and \( z^{p^n-1} - 1 \) over the field \( \overline{\mathbb{F}_p} \). Equation (6) shows that for one of the two equations, all roots \( \zeta \) yield a periodic point \( \zeta + \zeta^{-1} \) for \( T_q \). So we are guaranteed something close to \( p^n/2 \) periodic points from roots of one of the polynomials, and we pick up a few more from roots of the other polynomial. A similar explanation for \( T_2 \) can be derived from equation (5).

In Table 9 we illustrate Theorem 5.7 for \( T_3(z) \) over various finite fields. Note that for the choices of primes in the table, \( \delta \mid 2n \) for all integers \( n \).

Once again, we wish to extend our results to polynomials with composite degree. Lemma 5.4 takes care of prime power degree, so we are left to consider the case of the \( t \)th Chebyshev polynomial, \( T_t(z) \), for \( t = q_1^{f_1} q_2^{f_2} \cdots q_r^{f_r} \) and \( r \geq 2 \). We continue with the notation introduced at the end of Section 4 for each \( 1 \leq i \leq r \), let

\[
\delta_i = \text{ord}_{q_i}(p) \quad \text{and} \quad \mu_i = v_{q_i}(p^{\delta_i} - 1).
\]

We also define

\[
\Delta = \text{lcm}\{\delta_i\}_{1 \leq i \leq r}.
\]

The argument in Proposition 5.5 can be modified to show that if \( \gcd(\Delta, 2n) = 1 \), then all points of \( \mathbb{F}_{p^n} \) will be periodic. But as in Section 4, we need not require \( \Delta \mid 2n \) to have a nontrivial ratio of periodic points.

As before, for each \( n \in \mathbb{Z} \) we define an \( r \)-tuple of valuations

\[
v(n) = (v_{q_i}(n))_{1 \leq i \leq r}.
\]
| $p$ | $\delta = \text{ord}_3(p)$ | $\mu = v_3(p^\delta - 1)$ |
|-----|------------------|------------------|
|     | 5                | 19               |
|     | 19               | 53               |
|     | 2                | 1                |
|     | 1                | 2                |
|     | 1                | 3                |

| $\# \text{Per}(T_3, \mathbb{F}_p)$ | $\# \text{Per}(T_3, \mathbb{F}_{p^2})$ | $\# \text{Per}(T_3, \mathbb{F}_{p^4})$ |
|----------------------------------|----------------------------------|----------------------------------|
| $\frac{p}{2}$                    | $0.600000000$                    | $0.680000000$                    |
| $\frac{p^2}{2}$                  | $0.578947368$                    | $0.556786704$                    |
| $\frac{p^4}{2}$                  | $0.509433962$                    | $0.518689925$                    |

| $\frac{3^\mu + 1}{2 \cdot 3^\mu}$ | $0.666666667$ | $0.555555556$ | $0.518518519$ |

Table 9. $\frac{\# \text{Per}(T_3, \mathbb{F}_{p^n})}{p^n}$ with $v_3(n) = 0$.

We then define the ratios of interest:

$$R_{\delta, \nu}(T_t) = \lim_{n \to \infty} \frac{\# \text{Per}(T_t, \mathbb{F}_{p^n})}{p^n}.$$

**Theorem 5.8.** Let $t = q_1^{f_1} q_2^{f_2} \ldots q_r^{f_r}$, with $q_i$ distinct odd primes for $1 \leq i \leq r$. Then there are disjoint subsets $I, J \subseteq \{1, 2, \ldots, r\}$ such that

$$R_{\delta, \nu}(T_t) = \frac{Q_I + Q_J}{2Q_I Q_J},$$

where

$$Q_I = \prod_{i \in I} q_i^{\mu_i + \nu_i} \quad \text{and} \quad Q_J = \prod_{j \in J} q_j^{\mu_j + \nu_j}.$$

**Proof.** Take $J$ maximal with $\delta J \mid \delta$; then we know that $q_j \mid p^\delta - 1$ if and only if $j \in J$. Now define

$$I = \{1 \leq i \leq r: q_i \mid p^\delta + 1\}.$$  

Since the primes dividing $t$ are distinct odd primes, no $q_i$ divides both $p^\delta - 1$ and $p^\delta + 1$. Hence $I \cap J = \emptyset$.

Now consider any $n$ with $\gcd(\Delta, n) = \delta$. Clearly $q_j \mid p^n - 1$ if and only if $j \in J$. For any $i \in I$, we have

$$q_i \mid p^\delta + 1 \implies q_i \mid p^{2\delta} - 1 \implies q_i \mid p^{2n} - 1.$$

Since $i \not\in J$, $q_i \nmid p^n - 1$. Therefore $q_i \mid p^n + 1$. Furthermore, since $\gcd(\Delta, 2n) \mid 2\delta$, we have

$$q_i \mid p^{2n} - 1 \iff q_i \mid p^{2\delta} - 1 \iff i \in I \cup J.$$  

That is, $q_i \mid p^n + 1$ if and only if $i \in I$.

Therefore

$$p^n - 1 = d_1 \prod_{j \in J} q_j^{f_j} \quad \text{and} \quad p^n + 1 = d_2 \prod_{i \in I} q_i^{f_i},$$
with \( \gcd(t, d_1) = \gcd(t, d_2) = 1 \). Lemma 3.4, applied to \( n \) and \( 2n \) respectively, shows that 
\[
e_j = \mu_j + \nu_j \quad \text{for} \quad j \in J \quad \text{and} \quad e_i = \mu_i + \nu_i \quad \text{for} \quad i \in I.
\]
Lemma 5.3 extends easily to the case of composite degree, and we conclude that 
\( \omega \in \mathbb{F}_{p^n} \) is periodic for \( T_t \) if and only if 
\[
\omega = \zeta + \zeta^{-1} \quad \text{with} \quad \zeta^{d_1} = 1 \quad \text{or} \quad \zeta^{d_2} = 1.
\]
As before, we have 
\( \frac{d_1 + d_2}{2} \) periodic points for \( T_t \) in \( \mathbb{F}_{p^n} \).

Since \( p^n + 1 \sim p^n - 1 \), we have 
\[
d_1 \prod_{j \in J} q_j^{e_j} \sim d_2 \prod_{i \in I} q_i^{e_i}, \quad \text{meaning} \quad d_2 \sim d_1 \frac{Q_J}{Q_I}.
\]

We can now calculate the limit:
\[
R_{\delta, \nu}(T_t) = \lim_{n \to \infty} \frac{\# \text{Per}(T_t, \mathbb{F}_{p^n})}{p^n} = \lim_{d \to \infty} \frac{(d + d_1 \frac{Q_J}{Q_I})/2}{Q_J d + 1} = \frac{Q_I + Q_J}{2Q_I Q_J}.
\]

In Table 10, we use data from Sage [4] to illustrate Theorem 5.8 for the 15th Chebyshev polynomial over fields \( \mathbb{F}_{2^n} \). In the notation of the theorem, we have:
\[
q_1 = 3 \\
\delta_1 = 2 \\
\mu_1 = v_3(2^2 - 1) = 1
\]
\[
q_2 = 5 \\
\delta_2 = 4 \\
\mu_2 = v_5(2^4 - 1) = 1.
\]

Note that in the table, we restrict to values of \( n \) with \( \gcd(4, n) = \delta \).
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