HOLOMORPHIC EXTENSION OF EIGENFUNCTIONS

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Abstract. Let $X = G/K$ be a Riemannian symmetric space of non-compact type. We prove a theorem of holomorphic extension for eigenfunctions of the Laplace-Beltrami operator on $X$, by techniques from the theory of partial differential equations.
1. Introduction

Let $X$ be a Riemannian symmetric space of non-compact type. Then $X = G/K$, where $G$ is a connected semisimple Lie group and $K$ a maximal compact subgroup. We choose the group $G$ such that it is contained in a complexification $G_{\mathbb{C}}$, and we denote by $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ the complexification of $K$. The symmetric space $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ carries a natural complex structure, and it contains $X$ as a totally real submanifold.

We are interested in eigenfunctions of the Laplace-Beltrami operator $\Delta$ on $X$. Since this operator is elliptic and $G$-invariant, every eigenfunction admits a holomorphic extension to some open $G$-invariant neighborhood of $X$ in $X_{\mathbb{C}}$. The $G$-orbits in $X_{\mathbb{C}}$ are generally difficult to parametrize, but let us recall that a particular $G$-invariant open neighborhood $\Xi$ of $X$, for which the orbit structure is compellingly simple, has been proposed in [1]. It is commonly called the complex crown of $X$, and it has been thoroughly investigated in recent years. See for example [?, ?, 3, 4, 10, 11, 12]. In the present paper we show that every eigenfunction for $\Delta$ extends holomorphically to $\Xi$.

Our result generalizes a result from [11] that every joint eigenfunction for the full set of invariant differential operators on $X$ extends holomorphically to $\Xi$. The proof given in [11] invokes the Helgason conjecture, affirmed in [9] by micro-local analysis. Our proof is considerably simpler. The crucial step is an application of a theorem from the theory of analytic partial differential equations. This theorem asserts the existence of a holomorphic extension to solutions which are holomorphic on one side of a non-characteristic surface.

At the end of the paper a further generalization is given to functions on $G$, which are eigenfunctions for the Casimir operator and right-$K$-finite.

2. Notation

We denote by $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ the Lie algebra of $G$ and its Cartan decomposition. We choose a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by $\Sigma \subset \mathfrak{a}^*$ the corresponding system of restricted roots. The root spaces in $\mathfrak{g}$ are denoted by $\mathfrak{g}^\alpha$, where $\alpha \in \Sigma$, and by $\Sigma^+ \subset \Sigma$ we denote a positive system. The centralizer of $\mathfrak{a}$ in $K$ is $M = Z_K(\mathfrak{a})$, and the Weyl group is $W = {\mathcal{N}}_K(\mathfrak{a})/M$, where $N_K(\mathfrak{a})$ is the normalizer.

Recall the definition of the complex crown $\Xi$ of $X$. We set

$$\Omega := \{Y \in \mathfrak{a} \mid |\alpha(Y)| < \frac{\pi}{2}, \forall \alpha \in \Sigma\}.$$
Then
\[ \Xi := G \exp(i\Omega)K_C = \{ g \exp(iY) \cdot x_0 \mid g \in G, Y \in \Omega \} \subset X_C. \]
Here \( x_0 \) denotes the standard base point \( eK_C \) in \( X_C \).

3. Results

Lemma 3.1. The \( G \)-invariant crown \( \Xi \) is an open subset of \( X_C \). The surjective map
\[ \Phi : G \times \Omega \to \Xi, \quad (g, Y) \mapsto g \exp(iY) \cdot x_0 \]
is real analytic, and the topology of \( \Xi \) is identical to the quotient topology with respect to this map.

Let \( \Omega^+ \) be the intersection of \( \Omega \) with the positive open chamber in \( \mathfrak{a} \), and let \( \Xi' = \Phi(G \times \Omega^+) \). Then \( \Xi' \) is open and dense in \( \Xi \), and
\[ \Phi' : G/M \times \Omega^+ \to \Xi', \quad (gM, Y) \mapsto \Phi(g, Y) \]
is a diffeomorphism.

Proof. Apart from the statement about the topology of \( \Xi \), this can be found in [11], §4. For the topological statement we need to prove that a subset of \( \Xi \) is open if its preimage is open. It suffices to prove the following. Let \( z_n \to z \in \Xi \) be a converging sequence. Then there exists a subsequence of the form \( z_j = \Phi(g_j, Y_j) \) with converging sequences \( g_j \to g \in G \) and \( Y_j \to Y \in \Omega \). It follows from [11], Propositions 1 and 7, that there exist sequences \( g_n \) in \( G \), \( k_n \) in \( K \) and \( Y_n \) in \( \Omega \) such that \( z_n = \Phi(g_n k_n, Y_n) \), and such that \( g_n \) and \( k_n \exp(iY_n) \cdot x_0 \) both converge. By passing to a subsequence, we may assume that \( k_n \) converges, and since \( Y \mapsto \exp(iY) \cdot x_0 \) is a diffeomorphism of \( \Omega \) onto its image, it then follows that \( Y \) converges in \( \Omega \). \( \square \)

Theorem 3.2. Let \( f \in C^\infty(X) \) be an eigenfunction for \( \Delta \). Then \( f \) extends to a holomorphic function on \( \Xi \).

Proof. As \( \Delta \) is elliptic, the regularity theorem for elliptic differential operators (see [6], Theorem 7.5.1) implies that \( f \) is an analytic function. As such it has an extension to a holomorphic function on some open neighborhood \( U_0 \) of \( x_0 \) in \( \Xi \). It follows from the proof in [6], that \( U_0 \) can be chosen independently of \( f \), that is, every eigenfunction can be holomorphically extended to \( U_0 \) (the radius of convergence obtained in the proof depends only on the corresponding radii for the coefficients of the differential operator). In particular, it follows from the fact that \( \Delta \) is \( G \)-invariant, that \( L_g f \) extends to \( U_0 \) for all \( g \in G \). The union \( U \) of the
\[ G \text{-translated sets } L_{q-1}(U_0) \text{ is then a } G \text{-invariant open neighborhood of } X \text{ in } \Xi, \text{ to which } f \text{ extends.} \]

We now consider the open dense subset \( \Xi' \subset \Xi \) from Lemma 3.1. The intersection \( U \cap \Xi' \) is non-empty, open and \( G \)-invariant. Let \( Y_0 \in \Omega^+ \), and for \( r > 0 \) let \( B_r \) denote the open ball in \( a \) of radius \( r \), centered at \( Y_0 \). If \( B_r \subset \Omega^+ \), then we define an open set
\[ T_r = G \exp(iB_r) \cdot x_0 \subset \Xi', \]
which we regard as a \( G \)-invariant ‘circular tube’ in \( \Xi' \). We claim that if \( f \) extends holomorphically to a set containing some circular tube \( T_r \subset \Xi' \) centered at \( Y_0 \), then it extends to all circular tubes in \( \Xi' \) centered at \( Y_0 \). Since \( Y_0 \) was arbitrary, and since \( \Omega^+ \) is simply connected, it follows from this claim that \( f \) extends holomorphically from \( U \cap \Xi' \) to \( \Xi' \).

In order to establish the claim we use Theorem 9.4.7 of [8], due to Zerner [13]. We write \( \Delta_C \) for the extension of \( \Delta \) to a \( G_C \)-invariant holomorphic differential operator on \( X_C \). Obviously, the holomorphic extension that we seek will be an eigenfunction for \( \Delta_C \) on \( \Xi \). It follows from Lemma 3.1 that each circular tube \( T_r \), for which the closure is contained in \( \Xi' \), has real-analytic boundary \( \partial T_r \). In order to apply Zerner’s theorem it suffices to establish that \( \partial T_r \) is non-characteristic for \( \Delta_C \), for all such tubes. By \( G \)-invariance, it suffices to consider boundary points \( x \in \partial T_r \) with \( x \in \exp(i\Omega) \cdot x_0 \). Recall from [11], p. 207, that when \( x \in \exp(i\Omega) \cdot x_0 \) we have a complex-linear isomorphism
\[ (3.1) \quad p_C \ni Z \mapsto \tilde{Z}_x \in T_x \Xi, \]
where \( \tilde{Z} \) is the holomorphic vector field on \( X_C \) given by
\[ \tilde{Z}_x \varphi = L_Z \varphi(x) = \frac{d}{dz} \varphi(\exp(-zZ)x)|_{z=0}. \]
In this isomorphism the tangent space at \( x \) of the boundary \( \partial T_r \) will then be a real hyperplane given by an equation \( \text{Re } \zeta(Z) = 0 \) for some cotangent vector \( \zeta \in p_C^* \). Since the tube \( T_r \) is \( G \)-invariant, it follows that \( \text{Re } \zeta \) annihilates \( Z \) for all \( Z \in p \), so \( \zeta \) is purely imaginary on \( p \).

Let \( (X_j^\alpha)_{\alpha \in \Sigma^+, 1 \leq j \leq m_\alpha} \) together with \( Y_1, \ldots, Y_r \in a \) be an orthonormal basis for \( p \) such that \( X_j^\alpha \in p_\alpha := [g^\alpha + g^{-\alpha}] \cap p \). Here \( m_\alpha = \dim p_\alpha \) as usual. In the universal enveloping algebra we have
\[ \Delta = \sum_{\alpha \in \Sigma^+} \sum_{j=1}^{m_\alpha} (X_j^\alpha)^2 + \sum_{i=1}^{r} Y_i^2 \]
with respect to the right action, where functions on \( G/K \) are regarded as a right \( K \)-invariant function on \( G \). Observe that modulo \( \mathfrak{t} \),
\[ \text{Ad}(a)^{-1}(X_j^\alpha) = \cosh(\alpha(\log a))X_j^\alpha \]
for a ∈ A (see [11] p. 207), and hence
\[ R_{X_j^a}f(a) = -[\cosh \alpha(\log a)]^{-1}L_{X_j^a}f(a). \]

It follows that
\[ \Delta = \sum_{\alpha \in \Sigma^+} \sum_{j=1}^{m_{\alpha}} [\cosh \alpha(\log a)]^{-2}(X_j^a)^2 + \sum_{i=1}^{r} Y_i^2 \]
at z = a \cdot x_0 ∈ X, with respect to the left action.

By analytic continuation, the same equation holds as well for \( \Delta_C \) and \( a \in A_C \). In particular, at \( x = \exp(iY) \cdot x_0 \) we obtain
\[ \Delta_C = \sum_{\alpha \in \Sigma^+} \sum_{j=1}^{m_{\alpha}} [\cos \alpha(Y)]^{-2}[(\tilde{X}_j^\alpha x)]^2 + \sum_{i=1}^{r} (\tilde{Y}_i x)^2. \]

Note that the condition that \( x \) belongs to the crown precisely ensures that \( \cos \alpha(Y) \neq 0 \), so that the expression makes sense. As \( \zeta \) is purely imaginary on \( p \), it follows that all terms in the above sum are ≤ 0 when applied to \( \zeta \). Thus the principal symbol of \( \Delta_C \) is non-zero at \( \zeta \), and the boundary of \( T_r \) is non-characteristic. It follows that Zerner’s theorem can be applied, so that \( f \) extends holomorphically to \( \Xi' \).

For the extension to the full set \( \Xi \) we shall apply Bochner’s theorem (see [7], Theorem 2.5.10). From what we have seen so far, for all \( g \in G \) the function
\[ f_g : a \to \mathbb{C}, \ Y \mapsto f(g \exp(Y) \cdot x_0) \]
extends to a holomorphic function on a tubular neighborhood \( a + i\omega \) of \( a \) in \( a_C \), and also to \( a + i\Omega^+ \). For elements \( w \in N_K(a) \) we have
\[ f_g(\text{Ad}(w)Y) = f_{gw}(Y). \]

It follows that \( f_g \) extends to a holomorphic function on each Weyl conjugate of \( a + i\Omega^+ \), hence to \( a + i\Omega' \), where \( \Omega' = \bigcup \text{Ad}(w)(\Omega^+) \) is the set of regular elements in \( \Omega \). Now Bochner’s theorem implies that \( f_g \) extends to a holomorphic function on the tube over the convex hull of \( \omega \cup \Omega' \), that is, to \( a + i\Omega \). Furthermore, \( g \mapsto f_g \) is continuous into \( H(a + i\Omega) \) (with standard topology), since it is continuous into the space \( H(a + i(\omega \cup \Omega')) \) which by Bochner’s theorem is topologically isomorphic.

Recall that for all \( g, g' \in G \) and \( Y, Y' \in \Omega \) we have
\[ g \exp(iY) \cdot x_0 = g' \exp(iY') \cdot x_0 \]
if and only if there exists \( w \in N_K(a) \) and \( k \in Z_K(Y) \) with \( g' = gkw \) and \( Y' = \text{Ad}(w^{-1})Y \). It follows easily that by
\[ g \exp(iY) \cdot x_0 \mapsto f_g(Y) \]
we obtain a well-defined extension of $f$ on $\Xi$. The topological statement in the first part of Lemma 3.1 implies that this extension is continuous. Since the extension is holomorphic on $\Xi'$, it must be holomorphic everywhere. □

We list some easy consequences of the preceding theorem and its proof. From the Iwasawa decomposition $G = NAK$ associated to the positive system $\Sigma^+$, we obtain the familiar horospherical projection $x \mapsto H(x) \in a$, defined by $x \in N \exp H(x) \cdot x_0$ for $x \in X$. For each $\lambda \in \alpha_+^*$ the function $x \mapsto e^{\lambda(H(x))}$ on $X$ is an eigenfunction for $\Delta$, hence extends to a holomorphic function on $\Xi$. We obtain:

**Corollary 3.3.** The projection $H : X \to a$ extends to a holomorphic map $\Xi \to a_C$. Moreover, $\Xi \subset N_C A_C K_C$.

**Proof.** Let $h_\lambda(\cdot)$ denote the analytic continuation of $e^{\lambda(H(\cdot))}$. Since $h_{-\lambda}(\cdot) = h_\lambda(\cdot)^{-1}$ we conclude that $h_\lambda(\cdot) \neq 0$. As $\Xi$ is simply connected, the analytic continuation of $\lambda(H(\cdot))$ is obtained by taking logarithms, and the first statement follows. For the last statement, we note that once the Iwasawa $A$-component allows an analytic continuation, then so does the $N$-component. Indeed, knowing the $A$-component, we can determine the $N$-component of $x \in X$ from $\theta(x) x^{-1}$, where $\theta$ denotes the Cartan involution. □

The preceding corollary was obtained for classical groups in [10]. The general case follows from results established in [12], [2] and [4] with [5].

Let $\omega \subset \Omega$ be open, convex and $W$-invariant, and let $T_\omega \subset \Xi$ denote the open set $T_\omega = G \exp(i\omega) \cdot x_0$.

**Corollary 3.4.** Let $f \in C^\infty(X)$ and let $P$ be a non-trivial polynomial of one variable. If $P(\Delta) f$ extends to a holomorphic function on $T_\omega$, then so does $f$.

**Proof.** By treating the factors of $P$ successively we may assume that $P(\Delta) = \Delta - \lambda$. The proof of Theorem 3.2 can then be repeated. □

The following generalization is more far-reaching. We denote by $C \subset Z(\mathfrak{g})$ the Casimir element of $\mathfrak{g}$.

**Theorem 3.5.** Let $f \in C^\infty(G)$ be a right $K$-finite eigenfunction of $C$. Then $f$ extends to a holomorphic function on $\tilde{\Xi} := G \exp(i\hat{\Omega}) K_C \subset G_C$. 

Proof. Recall that $f$ being $K$-finite means that the translates $R_k f$ by $k \in K$ span a finite dimensional space, which is then a representation space for $K$. We may assume that it is irreducible, and then $f$ is an eigenfunction for the Casimir element $C_t$ of $\mathfrak{k}$, acting from the right. The operator $C + 2 C_t$ is elliptic, so it follows that $f$ is real analytic. The proof of Theorem 3.2 can now be repeated, with the following changes.

In Lemma 3.1 we replace the map $\Phi$ by $\tilde{\Phi} : G \times \Omega \times K \rightarrow \tilde{\Xi}$, $(g, Y, k) \mapsto g \exp(iY)k,$

and we define $\tilde{\Phi}'$ as before, but now on $(G \times \Omega^+ \times K_\mathbb{C})/M$, where $M$ acts on the first and last factor, from the right and left, respectively.

The $G$-invariant tubes $T_r \subset \Xi$ are replaced by their $G \times K_\mathbb{C}$-invariant preimages $\tilde{T}_r = T_r K_\mathbb{C} \subset \tilde{\Xi}$. The map $Z \mapsto L_Z$ from $p_\mathbb{C}$ onto $T_x \Xi$ in (3.1) is replaced by $Z \oplus U \mapsto L_Z + R_U$ from $g_\mathbb{C} = p_\mathbb{C} \oplus \mathfrak{k}_\mathbb{C}$ onto $T_x \tilde{\Xi}$. Here $x \in \exp(i\Omega)$. The cotangent vector $\zeta$, normal to $\tilde{T}_r$ at $x$ is zero on $p_\mathbb{C}$ and purely imaginary on $p$, by the same argument as before.

Since $f$ is a $C_t$-eigenfunction, the action of $C$ on it differs only by a constant from that of the operator $\Delta$ described in (3.2). Hence $\partial \tilde{T}_r$ is non-characteristic for $C$, and the application of Zerner’s theorem goes through. The rest of the argument is essentially unchanged. □

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