THE INVERSION HEIGHT OF THE FREE FIELD IS INFINITE

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Abstract. Let $X$ be a finite set with at least two elements, and let $k$ be any commutative field. We prove that the inversion height of the embedding $k(X) \hookrightarrow D$, where $D$ denotes the universal (skew) field of fractions of the free algebra $k(X)$, is infinite. Therefore, if $H$ denotes the free group on $X$, the inversion height of the embedding of the group algebra $k[H]$ into the Malcev-Neumann series ring is also infinite. This answer in the affirmative a question posed by Neumann in 1949 [27, p. 215].

We also give an infinite family of examples of non-isomorphic fields of fractions of $k(X)$ with infinite inversion height.

We show that the universal field of fractions of a crossed product of a commutative field by the universal enveloping algebra of a free Lie algebra is a field of fractions constructed by Cohn (and later by Lichtman). This extends a result by A. Lichtman.

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1. Introduction

Let $X$ be a set with $|X| \geq 2$, $H$ be the free group on $X$ and $k$ be a commutative field. Choose a total order on $H$ such that $(H, <)$ is an ordered group. Consider the Malcev-Neumann series ring $k((H, <))$ associated with the group ring $k[H]$. B.H. Neumann conjectured in [27, p. 215] that

$$(N) \text{ the inversion height of the embedding } k[H] \hookrightarrow k((H, <)) \text{ is infinite. Equivalently, in the (skew) subfield } E = E(X) \text{ of } k((H, <)) \text{ generated by } k[H] \text{ there exist elements which need an arbitrary large number of nested inversions to be constructed as a rational expression from elements of } k[H].$$

The field $E = E(X)$ can be characterized by its categorical properties. It was proved by Lewin [19] that it is the universal field of fractions of $k[H]$ and, hence, it is also the universal field of fractions of $k(X)$, the free algebra on $X$; because of that $E$ is usually named the free field on $X$. We recall that $k(X)$ can also be seen as the enveloping algebra of the free Lie algebra on $X$.

The interest on conjecture $(N)$ was renewed in [12] where the theory of Quasideterminants was developed. C. Reutenauer brilliantly proved in [29, Theorem 2.1] that the conjecture holds when $X$ is infinite and $k$ is a commutative field. As suggested in [29, Section 5.2], it was expected that $(N)$ should hold in general because a free algebra $R$ over a set of at least two elements contains many subalgebras $S$ that are isomorphic to a free algebra over an infinite (countable) set. The difficulty in settling the question with this approach was being able to choose a subalgebra $S$ such that the universal field of fractions of $S$ can be seen inside the one of $R = K(X)$ and that, in addition, the inversion height is preserved through the embedding. In this paper, we overcome this problem considering the more flexible structure of crossed product. More precisely, seeing $R$ as a crossed product of the subalgebra $S$ with something else we can produce, via Reutenauer’s result, elements in $E$ of arbitrary inversion height. Hence we give the final step to solve conjecture $(N)$.

Crossed products can be considered in the group context, in the context of Lie algebras or, unifying both settings, for Hopf algebras. They have proved to be specially suitable for induction-type arguments and also in the construction of quantum deformation of classical algebraic objects.

Throughout the paper, we give several constructions of elements in the free field $E$ of arbitrary inversion height, keeping in parallel the point of view of crossed...
products of Lie algebras and the one of group crossed products. In Section 4, we give the most elementary constructions to produce elements of arbitrary large inversion height. We use the ideas of an embedding due to Cohn [5] that allows to see the free algebra as an Ore differential extension of a free algebra on infinitely many variables. Such kind of extensions are the easiest example of crossed product of Lie algebras. Then we are able to give an elementary solution to conjecture (N) in Theorem 4.6.

On the group side, if $H$ is a free group, any onto group homomorphism $\varphi$ from $H$ to an infinite cyclic group allows to see $k[H]$ as a skew Laurent polynomial ring with coefficients on the group algebra over the free group Ker $\varphi$, again this is the easiest example of crossed product of groups. Such description of the group algebra allows us to give in Theorem 4.6 another elementary solution to conjecture (N).

In Section 5, we deeply use the theory of crossed product of groups to produce infinitely many non-isomorphic embeddings of the free algebra into division rings of infinite inversion height. Hence, the property of having infinite inversion height does not characterize the universal field of fractions.

In Sections 6 and 7, we develop some specific theory of crossed products for Lie algebras, and we give a construction of a field of fractions, as a subfield of a power series ring, for the crossed product of a field by a residually nilpotent Lie algebra with a $Q$-basis. In the case of a free Lie algebra $H$ or, more generally, when the crossed product is a fir, this gives a construction of the universal field of fractions. In Section 8, we use this theory to produce further examples of elements with arbitrary large inversion height into the free field. A different line of applications of this construction is given in Example 7.14 to the enveloping algebra of the free Poisson field, cf. [23].

In the case of an ordered group, the Malcev-Neumann series ring gives a very neat way to embed a crossed product of an arbitrary field by the group into a field. As mentioned before, when the group is free, this yields an embedding of the universal field of fractions of the crossed product in such power series ring. This was proved by Lewin in [19] using a deep result of Hughes on the uniqueness of some field of fractions [14].

On the Lie algebra side, a well known result of Cohn implies that any crossed product of a field by the universal enveloping algebra of a Lie algebra can be embedded into a field, cf. Proposition 6.5, which we call the canonical field of fractions. But an analog of the Malcev-Neumann series ring construction, possibly containing the canonical field of fractions, is missing in the setting of ordered Lie algebras. Our main results in Section 6 aim to fill this gap in the case of crossed products of residually nilpotent Lie algebras with a $Q$-basis. In our constructions, we follow and extend results and ideas due to Lichtman [21, 22].

As we have already mentioned, all our results on inversion height are based on Reutenauer’s ones. It seems an interesting and challenging question to extend Reutenauer’s results from commutative fields to arbitrary (skew) fields. We note that our approach to pass from the case of countable infinitely many variables to
the finite one does not use any commutativity and it works for general crossed products.

2. Preliminaries

We begin this section fixing some notions that will be used throughout the paper.

All rings are assumed to be associative and with 1. A morphism of rings \( \alpha : R \rightarrow S \) always preserves 1's, i.e. \( \alpha \) sends \( 1_R \) to \( 1_S \).

By an embedding \( \iota : R \hookrightarrow E \) we mean an injective morphism of rings where we identify \( R \) with its image in \( E \).

A domain is a nonzero ring \( R \) such that the product of any two nonzero elements is nonzero.

Following \[9\], a field \( E \) is a nonzero ring such that every nonzero element has an inverse, i.e. if \( x \in E \setminus \{0\} \) there exists \( x^{-1} \in E \) such that \( xx^{-1} = x^{-1}x = 1 \).

Note that domains and fields are not supposed to be commutative. In the literature, our concept of field is also known as division ring or skew field.

2.1. Skew polynomial rings and skew Laurent series. Let \( S \) be a ring and \( \alpha : S \rightarrow S \) an injective endomorphism of rings.

A (left) \( \alpha \)-derivation is an additive map \( \delta : S \rightarrow S \) such that \( \delta(ab) = \delta(a)b + \alpha(a)\delta(b) \).

We denote by \( S[\![x; \alpha, \delta]\!] \) the skew polynomial ring. It is a ring extension of \( S \) which is a free left \( S \)-module with basis \( \{1, x, \ldots, x^n, \ldots\} \), thus the elements can be uniquely written as

\[ a_0 + a_1x + \cdots + a_nx^n \text{ with } a_i \in S, \ n \in \mathbb{N}, \ a_n \neq 0, \]

and \( xa = \alpha(a)x + \delta(a) \) for all \( a \in S \). When \( \delta = 0 \), we write \( S[\![x; \alpha]\!] \) instead of \( S[\![x; \alpha, 0]\!] \), and when \( \alpha \) is the identity on \( S \), we write \( S[\![x; \delta]\!] \) instead of \( S[\![x; \alpha, \delta]\!] \).

If \( S \) is a domain, the ring \( S[\![x; \alpha, \delta]\!] \) is also a domain. If \( S \) is a left Ore domain, then \( S[\![x; \alpha, \delta]\!] \) is a left Ore domain. If \( S \) is a field, we denote its left Ore field of fractions by \( S((x; \alpha, \delta)) \) (respectively \( S((x; \alpha)) \), \( S((x; \delta)) \)).

When \( \delta = 0 \), we can consider the skew series ring \( S[\![x; \alpha]\!] \) which consists of all infinite series

\[ a_0 + a_1x + \cdots + a_nx^n + \cdots, \ a_n \in S \text{ for all } n \in \mathbb{N}, \]

with componentwise addition and multiplication based on the commutation rule

\[ xa = \alpha(a)x, \text{ for all } a \in S. \]

The set \( \{1, x, \ldots, x^n, \ldots\} \) is a left Ore set in \( S[\![x; \alpha]\!] \), and we denote its Ore localization by \( S((x; \alpha)) \). The elements of \( S((x; \alpha)) \) are of the form

\[ x^{-r} \sum_{n=0}^{\infty} a_nx^n \text{ with } r \in \mathbb{N}, \ a_n \in S \text{ for all } n. \]
If $S$ is a field, $S((x;\alpha))$ is a field that contains $S(x;\alpha)$. If $\alpha$ is bijective, the elements of $S((x;\alpha))$ can be written as \( \sum_{n \geq -r} a_n x^n \) with $r \in \mathbb{N}$ and $a_n \in S$ for all $n$.

When $\delta = 0$ and $\alpha$ is bijective, the subring of $S((x;\alpha))$ consisting of the polynomials of the form
\[
a_{-m} x^{-m} + a_{-m+1} x^{-m+1} + \cdots + a_0 + a_1 x + \cdots + a_n x^n,
\]
with $a_i \in S$, $m, n \in \mathbb{N}$, is called the skew Laurent polynomial ring and denoted by $S[x,x^{-1};\alpha]$. If $S$ is a left Ore domain, $S[x,x^{-1};\alpha]$ is also a left Ore domain. If $S$ is a field, the left Ore field of fractions is $S(x;\alpha)$.

When $\delta \neq 0$ and $\alpha$ is injective, we can also construct a similar ring of series (to understand its definition, notice that the relation $xa = \alpha(a)x + \delta(a)$ implies that $a x^{-1} = x^{-1}\alpha(a) + x^{-1}\delta(a)x^{-1}$). We introduce a new variable $y = x^{-1}$, and we consider the ring of series
\[
a_0 + ya_1 + \cdots + y^n a_n + \cdots \text{ with } a_n \in S \text{ for all } n \in \mathbb{N},
\]
(coefficients on the right) with componentwise addition and multiplication based on the commutation rule
\[
ay = ya(a) + y^2\alpha(y\alpha(a)) + \cdots + y^n\alpha(y^{n-1}(a)) + \cdots = \sum_{n \geq 1} y^n\alpha(y^{n-1}(a)),
\]
for each $a \in S$. This ring of series will be denoted by $S[[y;\alpha,\delta]]$. The set $\{1, y, \ldots, y^n, \ldots\}$ is a right Ore set and we denote by $S((y;\alpha,\delta))$ its Ore localization. So the elements of $S((y;\alpha,\delta))$ are of the form
\[
\left(\sum_{n=0}^{\infty} y^n a_n\right)y^{-r} \text{ with } r \in \mathbb{N}, \ a_n \in S \text{ for all } n \in \mathbb{N}.
\]

If $S$ is a field, then $S((y;\alpha,\delta))$ is a field.

From (2.1) it is easy to see that the assignment $x \mapsto y^{-1}$ induces an injective morphism of rings $S[x,\alpha,\delta] \to S[[y;\alpha,\delta]]$ which is the identity on $S$. The universal property of the Ore localization, allows to extend this embedding to an embedding of fields $S(x;\alpha,\delta) \to S((y;\alpha,\delta))$.

Finally, we observe that if $\alpha$ is an automorphism then the elements of $S((y;\alpha,\delta))$ can be written, in a unique way, in the form
\[
\sum_{n \geq l} a_n y^n \text{ with } l \in \mathbb{Z}, \ a_n \in S \text{ for all } n \in \mathbb{N}.
\]

\section*{2.2. Crossed products and Malcev-Neumann series.} Let $R$ be a ring, and let $G$ be a group. We define a crossed product $RG$ (of $R$ by $G$) as an associative ring which contains $R$ constructed in the following way. It is a free left $R$-module with basis $G$, a copy (as a set) of $G$. The elements in $RG$ are uniquely written as $\sum_{x \in G} a_x \bar{x}$ where only a finite number of $a_x \in R$ are nonzero. Multiplication is
determined by the two rules below:

Twisting. For \( x, y \in G \)
\[
\bar{x}\bar{y} = \tau(x, y)\bar{xy}
\]
where \( \tau: G \times G \rightarrow R^\times \) and \( R^\times \) denotes the group of units of \( R \).

Action. For \( x \in G \) and \( r \in R \)
\[
\bar{x}r = \sigma_r(x)
\]
where \( \sigma: G \rightarrow \text{Aut}(R) \), \( \text{Aut}(R) \) denotes the group of automorphisms of \( R \) and \( \sigma_r(x) \) denotes the image of \( r \) by \( \sigma(x) \). Hence if \( \sum_{x \in G} a_x \bar{x}, \sum_{x \in G} b_x \bar{x} \in RG \), then
\[
\sum_{x \in G} \left( \sum_{yz=x} a_y \sigma_r(y) b_z \tau(y, z) \right) \bar{x}.
\] (2.2)

We stress that neither \( \sigma \) nor \( \tau \) need to preserve any kind of structure.

If \( H \) is a subgroup of \( G \), then \( RH = \{ \eta \in RG \mid \text{supp} \eta \subseteq H \} \) is the naturally embedded sub-crossed product.

Crossed products do not have a natural basis. If \( d: G \rightarrow R^\times \) assigns to each \( x \in G \) a unit \( d_x \), then \( \tilde{G} = \{ \tilde{x} = d_x x \mid x \in G \} \) is another \( R \)-basis for \( RG \) which still exhibits the basic crossed product. After a change of basis if necessary, we will always suppose that \( 1_{RG} = 1 \).

A crucial property of crossed products is the following. If \( N \) is a normal subgroup of \( G \) then \( RG = RN \frac{G}{N} \), where the latter is some crossed product of the group \( G/N \) over the ring \( RN \).

If \( R \) is any ring and \( C \) denotes an infinite cyclic group then any crossed product \( RC \cong R[x, x^{-1}; \alpha] \) for a suitable ring automorphism \( \alpha: R \rightarrow R \) given by conjugation by \( x \).

We refer the reader to [28] for further details on crossed products. If \( k \) is a commutative field and \( G \) is an ordered group, then the construction of \( RG \) is a particular case of a Hopf algebra crossed product, see [26, Chapter 7].

We say that a group \( G \) is an orderable group if there exists a total order \( < \) on \( G \) which is compatible with the product defined on \( G \), that is, \( x < y \) implies that \( zx < zy \) and \( xz < yz \) for all \( x, y, z \in G \). In this event \( (G, <) \) is an ordered group.

Given a ring \( R \), an ordered group \( (G, <) \) and a crossed product group ring \( RG \), the Malcev-Neumann series ring \( R((G, <)) \) consists of the formal sums
\[
f = \sum_{x \in G} a_x \tilde{x},
\]
such that \( \text{supp} f = \{ x \in G \mid a_x \neq 0 \} \) is a well-ordered subset of \( G \), the sum is defined componentwise and the product is defined as in (2.2).

It was proved independently by A.I. Malcev [21] and B.H. Neumann [27] that if \( R \) is a field then \( R((G, <)) \) is also a field. Let \( f = \sum_{x \in G} a_x \tilde{x} \) be a nonzero series in \( R((G, <)) \). Set \( x_0 = \min\{ x \in G \mid x \in \text{supp} f \} \) and \( g = a_{x_0} \tilde{x}_0 - f \). Observe that \( \text{supp} g(a_{x_0} \tilde{x}_0)^{-1} \subseteq \{ x \in G \mid x > 1 \} \). As in [13, Corollary 14.23], it can be seen
that $\sum_{m \geq 0} (g(a_x x_0)^{-1})^m$ is a well-defined element in $R((G, <))$, that is, for each $x \in G$ the set $L_x = \{m \geq 0 \mid x \in \text{supp}(g(a_x x_0)^{-1})^m\}$ is finite. Then

$$f^{-1} = (a_x x_0)^{-1} \sum_{m \geq 0} (g(a_x x_0)^{-1})^m$$

2.3. Universal fields, matrix localization and the free field. See [9, Chapter 4] for the missing details. Let $R$ be a ring. An epic $R$-field is a morphism of rings $\iota: R \to E$ with $E$ a field which is rationally generated by the image of $\iota$. If $\iota$ is injective, it is called a field of fractions of $R$. It is known that epic $R$-fields (objects) together with specializations (morphisms) form a category. If there exists an initial object in this category it is called a universal field. If it exists, it is unique up to isomorphism.

Observe that an endomorphism $f: F \to F$ in the category of epic $R$-fields must be an automorphism of $R$-rings. In particular, epic $R$-fields are isomorphic if and only if they are isomorphic as $R$-rings.

Let $R$ be a ring and let $E$ be an epic $R$-field with morphism $\varphi: R \to E$. It was proved by Cohn that the set $P_E$ of all square matrices with entries in $R$ and such that its image via $\varphi$ is not invertible in $E$ form a prime matrix ideal of $R$ and the localization of $R$ at the set of all square matrices with entries in $R$ such that its image via $\varphi$ is invertible is a local ring, denoted by $R_{P_E}$, such that the canonical map $R_{P_E} \to E$ induces an isomorphism between the residue field of $R_{P_E}$ and $E$. Let us call $P_E$ the associated prime matrix ideal to the epic $R$-field $E$.

This correspondence between epic $R$-fields and prime matrix ideals of a ring $R$ is in fact bijective. If $P$ is a prime matrix ideal of $R$ then $R_P$ is a local ring, its residue field $E$ is an epic $R$-field and $P_E = P$.

Theorem 2.1. Let $R$ be a ring, and let $F_1$ and $F_2$ be epic $R$-fields with associated prime matrix ideals $P_1$ and $P_2$, respectively. Then the following statements are equivalent:

(i) There exists a specialization $F_1 \to F_2$.
(ii) $P_1 \subseteq P_2$.
(iii) The canonical localization homomorphism $R \to R_{P_1}$ factors through the canonical localization homomorphism $R \to R_{P_2}$.

In particular, if $P_2$ is a minimal prime matrix ideal, then $P_1 = P_2$ and $F_1$ is isomorphic to $F_2$.

Note also that the third statement in the theorem above implies that if $F_2$ is given by universal localization of $R$ (at a prime matrix ideal of $R$), then $F_2$ is isomorphic to $F_1$. Therefore one can deduce that the prime matrix ideal associated to $F_2$ is a minimal prime matrix ideal.

All prime matrix ideals contain the set of non-full matrices. The set $P$ of non-full matrices is a prime matrix ideal, hence the least prime matrix ideal, if and only if $R$ is a Sylvester domain and, in this case, $R_P$ is a field and hence, it is a universal field of fractions. A free algebra (or more generally a semifir) is a
Sylvester domain. The universal field of fractions of a free algebra is usually called a free field.

Let $G$ be a free group on a nonempty set $X$, $k$ a field and $kG$ a crossed product. Lewin proved that the universal field of fractions of $kG$ (and of $k(X)$) is the field of fractions of $kG$ inside $k((G, <))$ for any total order $<$ on $G$ such that $(G, <)$ is an ordered group, see [19] and the remark in [20] Section 2. An easier proof of this fact was given by C. Reutenauer [30] (or see [33]). Observe that if $N$ is a subgroup of $G$ (or $Y \subseteq X$), then the universal field of fractions of $kN$ (respectively $k(Y)$) is the field of fractions of $kN$ $(k(Y))$ inside $k((G, <))$.

3. Inversion height

Suppose that $\iota: R \hookrightarrow E$ is an embedding of a domain $R$ into a field $E$.

Set $E_{i}(-1) = \emptyset$, $E_{i}(0) = R$, and we define inductively for $n \geq 0$:

$$E_{i}(n + 1) = \text{subring of } E \text{ generated by } \{r, s^{-1} | r, s \in E_{i}(n), s \neq 0\}.$$  

Then $E_{i} = \bigcup_{n=0}^{\infty} E_{i}(n)$ is the field of fractions of $R$ inside $E$. That is, $E_{i}$ is the field rationally generated by $R$ inside $E$ or, equivalently, the intersection of all subfields of $E$ that contain $R$.

We define $h_{i}(R)$, the inversion height of $R$ (inside $E$), as $\infty$ if there is no $n \in \mathbb{N}$ such that $E_{i}(n)$ is a field. Otherwise,

$$h_{i}(R) = \min\{n \mid E_{i}(n) \text{ is a field}\}.$$  

Notice that if $h_{i}(R) = n$, then $E_{i}(m) = E_{i}(n)$ for all $m \geq n$.

Given an integer $n \geq 0$, we say that an element $f \in E$ has inversion height $n$ if $f \in E_{i}(n) \setminus E_{i}(n - 1)$, and we write $h_{i}(f) = n$. In other words, $h_{i}(f)$ says how many nested inversions are needed to express an element of $E_{i}$ from elements of $R$, and $h_{i}(R)$ is the supremum of all $h_{i}(f)$ with $f \in E_{i}$.

We now give some easy remarks that will be used throughout.

Remarks 3.1. Let $\iota: R \hookrightarrow E$ be an embedding of a domain $R$ in a field $E$.

(a) If $\kappa: E \hookrightarrow L$ is an embedding in a field $L$, then $\kappa_{i}$ is an embedding such that $E_{i}(n) = L_{\kappa_{i}}(n)$ for all $n \geq -1$. Therefore $E_{i} = L_{\kappa_{i}}$, $h_{i}(R) = h_{\kappa_{i}}(R)$, and $h_{i}(f) = h_{\kappa_{i}}(f)$ for all $f \in L_{\kappa_{i}}$.

(b) On the other hand, if $S$ is a subring of $R$ and we consider the embedding $\varepsilon = \iota|_{S}: S \hookrightarrow E$, then $E_{i}(n) \subseteq E_{i}(n)$, and thus $h_{i}(f) \leq h_{\varepsilon}(f)$ for all $f \in E_{i}$.

One of the problems when dealing with inversion height is the fact that we cannot be more accurate in Remarks 3.1(b). That is, we may know $h_{\varepsilon}(f)$ for some $f$ or even $h_{\varepsilon}(S)$, but usually it is not useful if we want to compute $h_{i}(f)$ or $h_{i}(R)$. Our key results on inversion height (Propositions 3.4 and 3.5) state that $h_{\varepsilon}(f) = h_{i}(f)$ in certain important cases.
Lemma 3.2. Let $k$ be a commutative field, and let $R$ be a $k$-algebra with a fixed embedding $i: R \hookrightarrow E$ into a field $E$. If $f \in E_\varepsilon$ satisfies that $h_\varepsilon(f) \leq m$, then there exists a finitely generated $k$-subalgebra $S$ of $R$ such that $f \in E_\varepsilon$ and $h_\varepsilon(f) \leq m$ where $\varepsilon = i|_S: S \to E$.

Proof. The proof is by induction on $m$. For $m = 0$ the claim is clear. Suppose that the claim is true for $m - 1 \geq 0$. Since $f \in E_\varepsilon(m)$, $f = \sum_{j=1}^n f_{i_1}\cdots f_{i_j}$ where, for each $i, j$, either $f_{i,j} \in E_\varepsilon(m - 1)$ or $f_{i,j}$ is the inverse of some nonzero element in $E_\varepsilon(m - 1)$.

The induction hypothesis implies that there exist $S_{i_1}, \ldots, S_{i_j}$ finitely generated $k$-subalgebras of $R$ such that $f_{i,j} \in E_{\varepsilon_{i,j}}$, where $\varepsilon_{i,j} = i|_{S_{i,j}}: S_{i,j} \to E$, and $h_{\varepsilon_{i,j}}(f_{i,j}) \leq m$. Let $S$ be the smallest subalgebra of $R$ containing $S_{i,j}$ for all $i, j$, and let $\varepsilon = i|_S: S \to E$. Then $f \in E_\varepsilon$, and $h_\varepsilon(f) \leq m$ because $E_{\varepsilon_{i,j}}(m) \subseteq E_\varepsilon(m)$. This proves the result. \hfill $\Box$

Lemma 3.3. Let $S$ be a domain with a fixed embedding $\varepsilon: S \hookrightarrow F$ into a field $F$. Let $\alpha: F \to F$ be a morphism of rings and $\delta: F \to F$ be an $\alpha$-derivation.

(i) If $\alpha(S) \subseteq S$ and $\delta(S) \subseteq S$, then

\[ \alpha(F_\varepsilon(n)) \subseteq F_\varepsilon(n) \quad \text{and} \quad \delta(F_\varepsilon(n)) \subseteq F_\varepsilon(n) \]

for all $n \geq 0$. Hence, $F_\varepsilon(n)((y; \alpha, \delta)) \leftrightarrow F_\varepsilon((y; \alpha, \delta))$ and $F_\varepsilon(n)((x; \alpha)) \leftrightarrow F_\varepsilon((x; \alpha))$.

(ii) If $\alpha(S) = S$, then $\alpha$ induces an automorphism of $F_\varepsilon(n)$ for each $n \geq 0$, and thus it induces an automorphism on $F_\varepsilon$.

Proof. (i) The hypothesis ensures that $\alpha(F_\varepsilon(0)) \subseteq F_\varepsilon(0)$ and $\delta(F_\varepsilon(0)) \subseteq F_\varepsilon(0)$. Since for each $f \in F \setminus \{0\}$, $\alpha(f^{-1}) = \alpha(f)^{-1}$ and $\delta(f^{-1}) = -\alpha(f)^{-1}\delta(f)f^{-1}$ cf. Lemma \[Q\], using the definition of $F_\varepsilon(n)$, it is easy to prove the first claim inductively.

The second claim follows from the first and the commutativity of the following diagram

\[
\begin{array}{ccc}
F_\varepsilon(n)[[y; \alpha, \delta]] & \xrightarrow{\eta} & F_\varepsilon([[y; \alpha, \delta]]) \\
\downarrow & & \downarrow \\
F_\varepsilon(n)((y; \alpha, \delta)) & \xrightarrow{\nu} & F_\varepsilon((y; \alpha, \delta))
\end{array}
\]

where the vertical arrows are given by the right Ore localization at the powers of $y$, $\eta$ is induced from $F_\varepsilon(n) \leftrightarrow F_\varepsilon$, and $\nu$ is given by the universal property of Ore localization. Similarly for $F_\varepsilon(n)((x; \alpha))$.

(ii) Assume that $\alpha: S \to S$ is an automorphism. We prove, by induction on $n$, that $\alpha: F_\varepsilon(n) \to F_\varepsilon(n)$ is an isomorphism for each $n \geq 0$. Our hypothesis ensures the case $n = 0$. Assume that $n > 0$ and $\alpha: F_\varepsilon(n - 1) \to F_\varepsilon(n - 1)$ is onto, hence an automorphism. As for any $r \in F_\varepsilon(n - 1) \setminus \{0\}$, $\alpha(r^{-1}) = \alpha(r)^{-1} \in F_\varepsilon(n)$ and $F_\varepsilon(n - 1) = \alpha(F_\varepsilon(n - 1))$, we deduce that all the ring generators of $F_\varepsilon(n)$ are in $\alpha(F_\varepsilon(n))$, which implies that $\alpha: F_\varepsilon(n) \to F_\varepsilon(n)$ is onto. \hfill $\Box$
Proposition 3.4. Let $S$ be a domain, let $\alpha : S \to S$ be an injective ring endomorphism, and let $\delta : S \to S$ be an $\alpha$-derivation. Suppose that $\varepsilon : S \rightarrow F$ is a field of fractions of $S$, that $\alpha$ and $\delta$ extend to $F$ and that
\[
\alpha(F_s(n) \setminus F_s(n-1)) \subseteq F_s(n) \setminus F_s(n-1), \tag{3.1}
\]
for each integer $n \geq 0$. Let $E = F(x; \alpha, \delta)$, and let $\iota : R \hookrightarrow E$ be the natural embedding of $R$ in $E$. Consider the field of skew Laurent series $F((y; \alpha, \delta))$. Then

(i) For each $a \geq 0$, $E_a(n) \subseteq F_s(n)((y; \alpha, \delta))$.

(ii) Let $f \in F$. If $h_\varepsilon(f) = n$, then $h_\iota(f) = n$.

(iii) $h_\iota(R) \geq h_\varepsilon(S)$.

Proof. To simplify the notation, let $L_n = F_s(n)((y; \alpha, \delta))$ for each $n \geq 0$. By Lemma 3.3(i), we may consider $L_n$ as a subring of $F((y; \alpha, \delta))$.

(i) We proceed by induction on $n$. For $n = 0$, observe that $E_\varepsilon(0) = S[x; \alpha, \delta]$.

Given $f = a_0 + a_1x + \cdots + a_nx^n \in S[x; \alpha, \delta]$, it can be expressed as $(a_0y^n + a_1y^{n-1} + \cdots + a_n)y^n$. Now $a_0y^n, \ldots, a_n \in S[\alpha, \delta]$ by (2.9). Thus $E_\varepsilon(0) = S[x; \alpha, \delta] \subseteq \alpha(\varepsilon)\delta(\varepsilon) = F_s$. Suppose that the result holds for $n \geq 0$. Let $f \in E_s(n) \setminus \{0\}$. Express $f$ as an element in $L_n$, $f = (\sum_{m=0}^{n} y^m a_m) y^{-r}$. Suppose that $m_0$ is the first natural such that $a_{m_0} \neq 0$. Then $f$ can be written as $y^{m_0}(1 - \sum_{m \geq 1} y^m b_m) a_{m_0} y^{-r}$.

where $b_m = -a_{m+m_0} a_{m_0}^{-1}$. Hence
\[
f^{-1} = y^r a_{m_0}^{-1} \left( \sum_{s=0}^{m_0} \left( \sum_{m \geq 1} y^m b_m \right)^s \right) y^{-m_0}. \tag{3.2}
\]

Observe that for each $s \geq 0$, the terms from $(\sum_{m \geq 0} y^m b_m)^t$ with $t > s$ do not contribute to the coefficient of $y^s$, and the coefficient of $y^s$ belongs to $F_s(n+1)$ by Lemma 3.3(i). Hence $\sum_{s=0}^{m_0} \left( \sum_{m \geq 1} y^m b_m \right)^s \in F_s(n+1) || y; \alpha, \delta ||$. Now it is easy to prove that $f^{-1} \in L_{n+1}$.

Since $E_s(n) \subseteq E_\varepsilon(n+1)$, we have shown that the generators of $E_s(n+1)$ are contained in the ring $L_{n+1}$. Therefore $E_s(n+1) \subseteq L_{n+1}$, as desired.

(ii) If $S$ is a field, the result is clear. So suppose that $S$ is not a field and let $f \in F$ with $f \in F_s(n+1) \setminus F_s(n)$ for some $n \geq 0$. If $f \in L_n$, i.e. $f = (\sum_{m \geq 0} y^m a_m) y^{-r}$ with $a_m \in F_s(n)$, then $fy^r = (\sum_{m \geq 0} y^m a_m) y^{-r}$. On the other hand $fy^r$ is a series of the form $y^{r} \alpha'(f) + \sum_{m \geq 1} y^{r+m} b_m$. Since $a_m \in F_s(n)$ for all $m \geq 0$ this is a contradiction because $\alpha (S) \subseteq F_s(n+1) \setminus F_s(n)$ by the hypothesis (3.1).

(iii) follows from (ii).

Note that if $\alpha$ is an automorphism, then 3.3 in Proposition 3.4 holds.

Proposition 3.5. Let $S$ be a domain, $\alpha : S \rightarrow S$ be an automorphism and $R = S[x, x^{-1}; \alpha]$. Suppose that $\varepsilon : S \rightarrow F$ is a field of fractions of $S$ and that $\alpha$ extends to $F$. Let $E = F(x; \alpha)$ and $\iota : R \rightarrow E$ be the natural embedding of $R$ in $E$. Consider the field of skew Laurent series $F((x; \alpha))$. Then
For each integer \( m \geq 0 \), \( E_*(n) \subseteq F_*(n)((x; \alpha)) \).

Let \( f \in F \). If \( h_*(f) = n \), then \( h_n(f) = n \).

Let \( \eta \) be the embedding of the free algebra \( \epsilon \) in \( \eta \).

Proof. Consider \( L_n = F_*(n)((x; \alpha)) \) as a subring of \( F((x; \alpha)) \). Then proceed as in the proof of Proposition \([3\text{.}3] \).

4. Two solutions

We shall use the following notation. Let \( A \) be an \( n \times n \) matrix with entries over a ring. Let \( i, j, p, q \in \{1, \ldots, n\} \). By \( A^{ij} \) we denote the matrix obtained from \( A \) by deleting the \( i \)-th row and the \( j \)-th column. By \( r_p^i \) we mean the row vector obtained from the \( p \)-th row of \( A \) deleting the \( j \)-th entry. And by \( s_q^j \) we denote the column vector obtained from the \( q \)-th column of \( A \) by deleting the \( i \)-th entry.

Let \( k \) be a commutative field and \( X \) a set. Let \( A = (x_{ij}) \) be an \( n \times n \) matrix with entries over the free \( k \)-algebra \( k(X) \). We say that \( A \) is a generic matrix (over \( k(X) \)) if the \( x_{ij} \)'s are distinct variables in \( X \). If \( \epsilon : k(X) \hookrightarrow E \) is the universal field of fractions of \( k(X) \), then such a generic matrix is invertible over \( E \). Moreover the \((j, i)\)-th entry of \( A^{-1} \in M_n(E) \) is the inverse of

\[
|A|_{ij} = x_{ij} - r_p^i(A^{ij})^{-1}s_q^j.
\]

The element \(|A|_{ij}\) is known as the \((i, j)\)-th quasideterminant of \( A \).

Theorem 4.1. (C. Reutenauer [29 Theorem 2.1]) Let \( k \) be a commutative field and let \( X \) be a finite set of cardinality at least \( n^2 \), where \( 1 \leq n < \infty \). Let \( \epsilon : k(X) \hookrightarrow E \) be the embedding of the free algebra \( k(X) \) in its universal field of fractions \( E \). Let \( A \) be an \( n \times n \) generic matrix. If \( f \) is an entry of \( A^{-1} \in M_n(E) \), then \( h_n(f) = n \).

To adapt this result to our purposes, we note the following Corollary.

Corollary 4.2. Let \( k \) be a commutative field, let \( Z \) be an infinite set and let \( N \) be the free group on \( Z \). Let \( \epsilon : k[N] \hookrightarrow F \) be the universal field of fractions of the group algebra \( k(N) \), and \( \epsilon = \epsilon_{k(Z)} : k(Z) \hookrightarrow F \). Then \( h_{\epsilon'}(k[N]) = h_{\epsilon'}(k(Z)) = \infty \).

Indeed, if \( A_n \) is an \( n \times n \) generic matrix and \( f \) is an entry of \( A_n^{-1} \in M_n(F) \), then \( h_n(f) = n \) and \( h_{\epsilon'}(f) = n - 1 \).

Proof. First of all notice that since \( k(Z) \subseteq k[N] \),

\[
E_{\epsilon'}(m) \subseteq E_{\epsilon'}(m) \subseteq E_{\epsilon'}(m + 1) \subseteq E_{\epsilon'}(m + 1) \tag{4.1}
\]

for each integer \( m \geq 0 \). Thus if \( h_{\epsilon'}(k(Z)) = \infty \), then \( h_{\epsilon'}(k[N]) = \infty \).

Let \( A_n \) be an \( n \times n \) generic matrix. Recall that if \( Y \) is a subset of \( Z \) and \( \eta = \epsilon_{k(Y)} : k(Y) \hookrightarrow F \), then \( F_{\eta} \) is the universal field of fractions of \( k(Y) \). Thus if \( Y \) is any finite subset of \( Z \) that contains the entries of \( A_n \) and \( f \) is an entry of \( A_n^{-1} \), then \( h_{\epsilon'}(f) = n \) by Theorem 4.1. Now Lemma \([3\text{.}2] \) implies that \( h_{\epsilon'}(f) = n \), and \((4.1) \) that \( h_{\epsilon'}(f) \geq n - 1 \).

Since \( Z \) is an infinite set, there exist \( n \times n \) generic matrices \( A_n \) for each natural \( n \geq 1 \) and therefore \( h_{\epsilon'}(k(Z)) \) is not finite by the foregoing.
We prove that \( h_\varepsilon(f) \leq n - 1 \) by induction on \( n \geq 1 \). If \( n = 1 \), the result follows because \( f \in \mathbb{Z} \) and therefore \( f^{-1} \in \mathbb{N} \). Suppose the claim holds for \( n \geq 1 \). Consider an \( (n + 1) \times (n + 1) \) generic matrix \( A_{n+1} = (x_{ij}) \). Then \( f \) is the \((j,i)\)-th entry of \( A_{n+1}^{-1} \). Thus \( f = \left( x_{ij} - r_i^j (A_{n+1}^{-1})^{-1} s_i^j \right)^{-1} \) for some \( i,j \). Since \( A_{n+1}^{-1} \) is an \( n \times n \) generic matrix, the induction hypothesis implies that if \( g \) is any entry of \((A_{n+1}^{-1})^{-1}\) then \( h_\varepsilon(g) \leq n - 1 \). Therefore \( h_\varepsilon(f) \leq n \).

4.1. First solution. If \( x, y \) are two elements of a ring, we denote by \([x,y]\) the element \([x,y] = xy - yx\).

We are interested in extending derivations to certain localizations of \( R \). We recall the following easy and well-known formula which implies that such extensions, if they exist, are unique.

**Lemma 4.3.** Let \( R \) be a ring, and let \( \delta: R \to R \) be a derivation. If \( r \in R \) is invertible, then \( \delta(r^{-1}) = -r^{-1}\delta(r)r^{-1} \). Hence, if \( R \to D \) is a ring extension such that \( D \) is a field of fractions of \( R \) and \( \delta, \delta' \in \text{Der}(D) \) are such that \( \delta(r) = \delta'(r) \), for any \( r \in R \), then \( \delta = \delta' \).

In the next lemma, we show that derivations can be extended to matrix localizations provided the set \( \Phi \) we localize at is upper multiplicative, that is, \( 1 \in \Phi \) and whenever \( A,B \in \Phi \), then \((\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}) \in \Phi \) for any matrix \( C \) of appropriate size. The result, at least for fields of fractions of Sylvester domains, is well known and the proof for the general case follows the same pattern. However we include the proof for completeness’ sake.

Recall that if \( R \) is a ring, \( \delta: R \to R \) is a derivation if and only if the map \( R \to M_2(R) \) given by \( r \mapsto \begin{pmatrix} r & \delta(r) \\ 0 & r \end{pmatrix} \), for any \( r \in R \), is a ring homomorphism.

For the proof of the next result it is useful to keep in mind the following explicit description of an isomorphism between \( M_{2n}(S) \) and \( M_n(M_2(S)) \) for any natural number \( n \) and any given ring \( S \). The elements of \( M_n(M_2(S)) \) are matrices of the form

\[
A = \begin{pmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{pmatrix}
\]

where \( A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in M_2(S) \) for each \( i,j \in \{1,\ldots,n\} \). The map \( \rho_n: M_n(M_2(S)) \to M_{2n}(S) \) defined by

\[
\begin{pmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{pmatrix}
\mapsto
\begin{pmatrix}
a_{11} & a_{1n} & b_{11} & \cdots & b_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{n1} & a_{nn} & b_{n1} & \cdots & b_{nn} \\
\cdots & \ddots & c_{11} & \cdots & d_{1n} \\
\cdots & \ddots & \cdots & \ddots & \vdots \\
c_{n1} & \cdots & c_{nn} & \cdots & d_{nn}
\end{pmatrix}
\]
is an isomorphism of rings.

**Lemma 4.4.** Let $R$ be a ring, $\Phi$ an upper multiplicative set of square matrices over $R$, and let $R \to R_\Phi$, $a \mapsto \hat{a}$, be the matrix localization of $R$ at $\Phi$. Then any derivation $\delta: R \to R$, $a \mapsto a^\delta$, extends to a unique derivation of $R_\Phi$.

In particular, if $R \hookrightarrow D$ is the universal field of fractions of a Sylvester domain $R$, then any derivation in $R$ can be uniquely extended to $D$.

**Proof.** We suppose that $R \to R_\Phi$ is given by $a \mapsto \hat{a}$. For each matrix $A = (a_{ij}) \in M_n(R)$, denote by $\hat{A} = (\hat{a}_{ij}) \in M_n(R_\Phi)$ and by $A^\delta = (a_{ij}^\delta) \in M_n(R)$.

For each natural $n$, consider the map $\psi_n: M_n(R) \to M_{2n}(R_\Phi)$ given by $A \mapsto \left( \hat{A} \hat{A}^\delta \hat{A}^{-1} \right)$. Since $\delta$ is a derivation, $\psi_n$ is a morphism of rings. For each $n \times n$ matrix $A \in \Phi$, the matrix $\psi_n A$ is invertible in $M_{2n}(R_\Phi)$. Indeed, since $\hat{A}$ is invertible in $R_\Phi$ by definition, the matrix $\left( \hat{A}^{-1} - \hat{A}^{-1}\hat{A}^\delta \hat{A}^{-1} \right)$ (4.2) is the inverse of $\psi_n A$. Thus the image of any $n \times n$ matrix in $\Phi$ by the morphism $\rho^{-1}_n \psi_n: M_n(R) \to M_n(M_{2n}(R_\Phi))$ is invertible. Note that if $A = (a_{ij})$, then

$$\rho^{-1}_n \psi_n A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

(4.3)

where $A_{ij} = \left( \hat{a}_{ij} \hat{a}_{ij}^\delta \hat{a}_{ij} \right)$. Hence, the morphism $R \to M_2(R_\Phi)$, $a \mapsto \left( \hat{a} \hat{a}^\delta \hat{a} \right)$ is $\Phi$-inverting, and there exists a unique morphism $R_\Phi \to M_2(R_\Phi)$ making the diagram commutative. Note that $R_\Phi$ is the $\Phi$-rational closure of $R$ in $R_\Phi$ [10, Theorem 7.12]. Thus, for any element $x \in R_\Phi$, there is some $A \in \Phi$ such that $x$ is an entry of the inverse matrix of $\hat{A}$. Looking at (4.2) and (4.3), we see that the image of $x \in M_2(R_\Phi)$ is of the form $\left( \begin{smallmatrix} x & x^\Delta \\ 0 & x \end{smallmatrix} \right)$ for some $x^\Delta \in R_\Phi$. Hence $\Delta: R_\Phi \to R_\Phi$, $x \mapsto x^\Delta$, is a derivation extending $\delta$, as desired.

For the last part, it is known that if $R$ is Sylvester domain, then its universal field of fractions is of the form $R_\Phi$ where $\Phi$ is the set of all full matrices over $R$. Note that $\Phi$ is upper multiplicative because it is the set of matrices over $R$ that become invertible in its universal field of fractions.

Next result is based on the ideas of [8], where a particular kind of embedding of a free algebra of infinite countable rank into free algebra of rank two is given.
Theorem 4.5. Let \( k \) be a commutative field and \( k(x, y_1, \ldots, y_n) \) be the free algebra with \( n \geq 1 \). Let \( i : k(x, y_1, \ldots, y_n) \hookrightarrow E \) be the universal field of fractions of \( k(x, y_1, \ldots, y_n) \). Then \( h_i(k(x, y_1, \ldots, y_n)) = \infty \). Moreover, if

\[
A_m = \begin{pmatrix}
\omega_0 & \omega_m & \cdots & \omega_{m^2-m} \\
\omega_1 & \omega_{m+1} & \cdots & \omega_{m^2-m+1} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{m-1} & \omega_{2m-1} & \cdots & \omega_{m^2-1}
\end{pmatrix},
\]

where \( \omega_0 = y_1 \), \( \omega_i = [x, \ldots [x, x, y_1] \cdots] \) with \( i \) factors \( x \), and \( f \) is an entry of \( A_m^{-1} \in M_m(E) \), then \( h_i(f) = m \).

Proof. Set \( Z = \{z_0, z_1, \ldots, z_m, \ldots\} \), \( S = k(Z) \), \( R = k[x, y_1, \ldots, y_n] \) and \( \varepsilon : S \hookrightarrow F \) the universal field of fractions of \( S \).

Proceeding as in \( \cite{8} \) Lemma 2.1 or using Lemma 4.4 it can be seen that there exists a derivation \( \delta : S \to S \) such that \( \delta(z_i) = z_{i+1} \), for each \( i \in \mathbb{N} \), and that it can be extended to a unique derivation of \( F \).

Express each integer \( i \geq 0 \) (uniquely) as \( i = rn + j \) with \( 0 \leq j \leq n - 1 \). As in \( \cite{8} \) Theorem 2.2, one can prove that there is an embedding \( \beta_0 : S \to R \) defined by

\[
\beta_0(z_i) = \begin{cases} 
y_{i+1} & \text{for } 0 \leq i \leq n - 1 \\
y_{r} \cdot [x, \ldots [x, x, y_{j+1}] \cdots] & \text{for } n - 1 < i.
\end{cases}
\]

which is honest (and 1-inert). Thus \( \beta_0 \) can be extended to a morphism of rings \( \beta : F \to E \). Again as in \( \cite{8} \), identifying \( S \) and \( F \) with their images via \( \beta_0 \), we get that \( R = S[x; \delta] \) and that \( E = F(x; \delta) \). Since \( h_i(S) = \infty \) by Corollary 4.2, also \( h_i(R) = \infty \) by Proposition 3.4(iii).

Now let \( f \) be an entry of the inverse of \( A_m \). Note that the matrix \( A_m \) is (the image of) a generic matrix over \( S \). Thus Corollary 4.2 says that \( h_i(f) = m \). Therefore \( h_i(f) = m \) by Proposition 3.4(ii). \( \square \)

4.2. Second solution.

Theorem 4.6. Let \( k \) be a commutative field, \( X = \{x, y_1, \ldots, y_n\} \) be a finite set with \( n \geq 1 \), and \( H \) be the free group on \( X \). Let \( i' : k[H] \hookrightarrow E \) be the universal field of fractions of \( k[H] \), and \( \epsilon = i'|_{k\langle X \rangle} : k\langle X \rangle \hookrightarrow E. \) Then \( h_i(k\langle X \rangle) = h_{i'}(k[H]) = \infty \). Moreover, if

\[
A_m = \begin{pmatrix}
z_0 & z_m & \cdots & z_{m^2-m} \\
z_1 & z_{m+1} & \cdots & z_{m^2-m+1} \\
\vdots & \vdots & \ddots & \vdots \\
z_{m-1} & z_{2m-1} & \cdots & z_{m^2-1}
\end{pmatrix}
\quad \text{where } z_i = x^i y_1 x^{-i},
\]

and \( f \) is an entry of \( A_m^{-1} \in M_m(E) \), then \( h_i(f) = m \) and \( h_{i'}(f) = m - 1. \)
Proof. Fix an order on \( H \) such that \((H, <)\) is an ordered group. We identify \( E \) with the field of fractions of \( k[H] \) inside \( L = k((H, <)) \).

Let \( C = \langle e \rangle \) be the infinite cyclic group. Consider the morphism of groups \( \varphi : H \to C \) given by \( x \mapsto c \) and \( y_j \mapsto 1 \) for \( 1 \leq j \leq n \). Let \( N = \ker \varphi \). Thus \( H \) is the extension of \( N \) by the infinite cyclic group generated by \( x \). It is well known that \( N \) is a free group with basis the infinite set \( Z = \{ x^j y_j x^{-1} \mid 1 \leq j \leq n, i \in \mathbb{Z} \} \), see for example [10] Section 36.

Let \( \varepsilon' = \iota_{\langle k[N] \rangle} : k[N] \to F \) and \( \varepsilon = \iota_{\langle k(Z) \rangle} : k(Z) \to F \) be the universal field of fractions of \( k[N] \) and \( k(Z) \) respectively, where we identify \( F \) with \( E_\varepsilon = E_{\varepsilon'} \), the subfield rationally generated by \( k[N] \) inside \( E \).

Let \( \alpha : E \to E \) be the automorphism of \( E \) given by \( f \mapsto xf x^{-1} \) for all \( f \in E \). Notice that \( \alpha \) restricts to an automorphism of \( k[N] \) and also to an automorphism of \( k(Z) \). Then \( \alpha \) can be extended to an automorphism of \( F \) by Lemma 3.3(ii). Notice also that \( k[H] = k[N][x, x^{-1}; \alpha] \), cf. [21]. Let \( \iota_Z = \iota_{\langle k(Z) \rangle} : k(Z) \to F \).

Observe that \( F \) is contained in \( k((N, <)) \subseteq L \). Since \( n_1 x^{r_1} = n_2 x^{r_2} \) for \( n_1, n_2 \in N \) and \( r_1, r_2 \in \mathbb{Z} \) if and only if \( n_1 = n_2 \) and \( r_1 = r_2 \), the powers of \( x \) are \( k((N, <)) \)-linearly independent. In particular the powers of \( x \) are \( F \)-linearly independent. Therefore \( \Upsilon : F[x, x^{-1}; \alpha] \to E \) and, by the universal properties of the Ore localization, \( E = F(x; \alpha) \).

Note that the entries of \( A_m \) belong to \( Z \). Let \( f \in F \), be one of the entries of \( A_m^{-1} \). By Corollary 4.2, \( h_v(f) = m - 1 \). Now, if we set \( S = k[N] \) and \( R = k[H] = k[N][x, x^{-1}; \alpha] \), Proposition 3.3(ii) implies that \( h_v(f) = m - 1 \).

Similarly, by Corollary 4.2, \( h_v(f) = m \). Now, if we set \( S = k(Z) \) and \( R = k(Z)[x, x^{-1}; \alpha] \), Proposition 3.3(ii) implies that \( h_v(f) = m \).

Since \( k(X) \subseteq k(Z)[x, x^{-1}; \alpha] \subseteq k[H] \) we obtain that
\[
E_v(m - 1) \subseteq E_v(m - 1) \subseteq E_v(m) \subseteq E_v(m),
E_v(m - 1) \subseteq E_v(m - 1) \subseteq E_v(m) \subseteq E_v(m).
\]
The first expression says that \( m \leq h_v(f) \), and the second one \( h_v(f) \leq m \). Therefore \( h_v(f) = m \).

Since \( m \) is any natural number \( \geq 1 \), we obtain that there exist elements \( f \in E \) with any prescribed inversion height \( m \geq 1 \). Therefore \( h_v(k(X)) = h_v(k[H]) = \infty \). \( \square \)

5. Other embeddings of infinite inversion height

Let \( S \) be a ring, \( G \) a group and \( SG \) a crossed product (determined by maps \( \sigma \) and \( v \) as in section 2.2). Let \( \varepsilon : S \to F \) be an epimorphism of rings such that the automorphism \( \sigma(S) \in \text{Aut}(R) \) can be extended to an automorphism of \( F \) for every \( x \in G \). It is easy to prove, for example as in [31] Lemma 4], that there exists a crossed product \( FG \) with an embedding \( \kappa : SG \to FG \) with \( \kappa_S = \varepsilon \) and \( \kappa(\bar{x}) = \bar{x} \).

If \( \varepsilon : S \to F \) is a field of fractions, then it is easy to prove that \( \varepsilon, \varepsilon_n : S \to F_n(n) \) and \( F_n(n) \to F \) are epimorphisms of rings for each \( n \). Suppose now that
we are in the situation of the foregoing paragraph. By Lemma 3.3(ii), $\sigma(x)$ can be extended to $F_\varepsilon(n)$ for each $x \in G$ and $n \geq 0$. Thus we obtain the embeddings

$$SG \hookrightarrow F_\varepsilon(n)G \hookrightarrow FG$$

for each $n \geq 0$. If, moreover, $(G, <)$ is an ordered group, we get the embeddings of Malcev-Neumann series rings

$$S((G, <)) \hookrightarrow F_\varepsilon(n)((G, <)) \hookrightarrow F((G, <))$$

for each $n \geq 0$.

Next result is a general version for Malcev-Neumann series of Proposition 3.4

**Theorem 5.1.** Let $S$ be a domain with a field of fractions $\varepsilon: S \hookrightarrow F$. Let $(G, <)$ be an ordered group. Consider a crossed product $SG$ such that it can be extended to a crossed product $FG$. Let $E = F((G, <))$ be the associated Malcev-Neumann series ring and $\iota: SG \hookrightarrow E$ be the natural embedding. Then

(i) $E_\varepsilon(n) \subseteq \mathcal{L}_n = F_\varepsilon(n)((G, <))$ for each integer $n \geq 0$.

(ii) Let $f \in F$. If $h_\varepsilon(f) = n$, then $h_\varepsilon(f) = n$.

(iii) $h_\varepsilon(SG) \geq h_\varepsilon(S)$.

**Proof.** We prove (i) by induction on $n$. For $n = 0$ the result is clear because $E_\varepsilon(0) = SG \subseteq \mathcal{L}_0$. So suppose that (i) holds for $n \geq 0$, and we must prove it for $n + 1$.

By the definition of $E_\varepsilon(n + 1)$, and the fact that $\mathcal{L}_{n+1}$ is a ring, it suffices to prove that if $f \in E_\varepsilon(n) \setminus \{0\}$ then $f^{-1} \in \mathcal{L}_{n+1}$. By induction hypothesis, $f \in \mathcal{L}_n$. Suppose that $f = \sum_{x \in G} a_x \bar{x}$ with $a_x \in F_\varepsilon(n)$. Let $x_0 = \min\{x \in G \mid x \in \text{supp } f\}$. Then,

$$f^{-1} = (a_{x_0} \bar{x_0})^{-1} \sum_{m \geq 0} (g(a_{x_0} \bar{x_0})^{-1})^m,$$

where $g = a_{x_0} \bar{x_0} - f \in \mathcal{L}_n$. Note that $(a_{x_0} \bar{x_0})^{-1} = \bar{x_0}^{-1}a_{x_0}^{-1} = \bar{x_0}^{-1}a_{x_0}^{-1} \bar{x_0} \bar{x_0}^{-1} = \sigma(x_0)^{-1}(a_{x_0}^{-1}) \bar{x_0}^{-1} \in \mathcal{L}_{n+1}$, and thus $g(a_{x_0} \bar{x_0})^{-1} \in \mathcal{L}_{n+1}$. Since $\mathcal{L}_{n+1}$ is a ring, $(g(a_{x_0} \bar{x_0})^{-1})^m \in \mathcal{L}_{n+1}$ for each $m \geq 0$. By (2.2) the series $\sum_{m \geq 0} (g(a_{x_0} \bar{x_0})^{-1})^m$ is well defined in $E$. Hence, for each $x \in G$, the coefficient of $\bar{x}$ in $\sum_{m \geq 0} (g(a_{x_0} \bar{x_0})^{-1})^m$ is an element of $F_\varepsilon(n + 1)$, i.e. $\sum_{m \geq 0} (g(a_{x_0} \bar{x_0})^{-1})^m \in \mathcal{L}_{n+1}$. Therefore

$$f^{-1} = (a_{x_0} \bar{x_0})^{-1} \sum_{m \geq 0} (g(a_{x_0} \bar{x_0})^{-1})^m \in \mathcal{L}_{n+1}.$$

(ii) If $S$ is a field, the result is clear. So suppose that $S$ is not a field. Let $f \in F_\varepsilon(n + 1) \setminus F_\varepsilon(n)$ for some integer $n \geq 0$. Since $S \subseteq SG$, clearly $f \in F_\varepsilon(n + 1) \subseteq E_\varepsilon(n + 1)$. Suppose that $h_\varepsilon(f) \leq n$, that is, $f \in E_\varepsilon(n) \subseteq \mathcal{L}_n$. By (i), $f = \sum_{x \in G} a_x \bar{x}$ with $a_x \in F_\varepsilon(n)$. Observe that two series $\sum_{x \in G} b_x \bar{x}$, $\sum_{x \in G} c_x \bar{x} \in E$,
where \( b_x, c_x \in F \) for each \( x \in G \), are equal if and only if \( b_x = c_x \) for each \( x \in G \). Hence \( f = a_1 \in F_x(n) \), a contradiction. Hence \( h_i(f) = n + 1 \).

(iii) Follows from (ii). \( \Box \)

If \( G \) is a group and \( x, y \in G \), by \((x, y)\) we denote the commutator \((x, y) = x^{-1}y^{-1}xy\).

It is well known that a torsion-free nilpotent group is orderable. Also, the free product of orderable groups is orderable. Hence, if we are given a set of torsion-free nilpotent groups \( \{G_i\}_{i \in I} \), the free product \(*_{i \in I} G_i\) is an orderable group.

**Corollary 5.2.** Let \( k \) be a commutative field, \( I \) be a set of cardinality at least two and \( \{G_i\}_{i \in I} \) be a set of torsion-free nilpotent groups. Set \( G = *_{i \in I} G_i \), and suppose that \((G, \prec)\) is an ordered group. Let \( k[G] \) be the group ring and \( \iota: k[G] \hookrightarrow E = k((G, \prec)) \) be the natural embedding in its Malcev-Neumann series ring. Then \( h_i(k[G]) = \infty \). Indeed, let \( x \in G_i \setminus \{1\} \) and \( y \in G_j \setminus \{1\} \) with \( i \neq j \). If \( f \) is any entry of the inverse of the \( n \times n \) matrix

\[
A_n = \begin{pmatrix}
(x, y) & (x, y^2) & \cdots & (x, y^n) \\
(x^2, y) & (x^2, y^2) & \cdots & (x^2, y^n) \\
\vdots & \vdots & \ddots & \vdots \\
(x^n, y) & (x^n, y^2) & \cdots & (x^n, y^n)
\end{pmatrix},
\]

then \( h_i(f) = n \).

In particular, if \( X \) is a set of cardinality at least two and \( G \) is the free group on \( X \), then the universal field of fractions \( \iota' : k[G] \hookrightarrow F \) and \( \iota : k(X) \hookrightarrow F \) are of infinite inversion height. Indeed, let \( x, y \in X \) be different elements, if \( f \) is any entry of the inverse of the \( n \times n \) matrix,

\[
A_n = \begin{pmatrix}
(x, y) & (x, y^2) & \cdots & (x, y^n) \\
(x^2, y) & (x^2, y^2) & \cdots & (x^2, y^n) \\
\vdots & \vdots & \ddots & \vdots \\
(x^n, y) & (x^n, y^2) & \cdots & (x^n, y^n)
\end{pmatrix},
\]

then \( h_i(f) = n - 1 \) and \( h_i(f) = n \).

**Proof.** Consider \( \bigoplus_{i \in I} G_i \), the subgroup of the cartesian product \( \prod_{i \in I} G_i \) consisting of all \((x_i)_{i \in I} \in \prod_{i \in I} G_i \) such that \( x_i = 1 \) for almost all \( i \in I \).

For each \( i \in I \), let \( \pi_i : G_i \rightarrow \bigoplus_{i \in I} G_i \) be the canonical inclusion and let \( \pi : \ast_{i \in I} G_i \rightarrow \bigoplus_{i \in I} G_i \) be the unique morphism of groups such that \( \pi_{G_i} = \pi_i \). Set \( N = \ker \pi \), then \( N \) is a free group. Since the cardinality of \( I \) is at least two, and each \( G_i \) is an infinite group for each \( i \), \( N \) is not finitely generated. Indeed, if we fix a total order \( \prec \) on \( I \), then \( N \) is the free group on the nontrivial elements of the set

\[
\big\{ (x_{i_1}, x_{i_2}, \ldots, x_{i_{r+1}}, \ldots, x_{i_s}) \mid x_{i_j} \in G_{i_j}, \ i_1 \prec i_2 \prec \cdots \prec i_s \in I \big\}.
\]
Hence $G$ is the extension of the free group $N$ by the group $G/N \cong \bigoplus_{i \in I} G_i$. Recall that since $G/N$ is locally nilpotent, any crossed product $F \bar{G}$, with $F$ a field, is an Ore domain.

If $\varepsilon = \iota_{k[N]}: k[N] \rightarrow E_{\varepsilon}$, then $\varepsilon$ is the universal field of fractions of $k[N]$. Any automorphism of $k[N]$ can be extended to $E_{\varepsilon}$ by [10, Corollary 7.5.16]. Hence the crossed product $k[N] \bar{G}$ extends to $E_{\varepsilon} \bar{G}$. Another way of proving this extension can be found in [21, Proposition 2.5(1)].

If for each $\alpha \in \frac{\mathbb{Q}}{\mathbb{Z}}$, we pick a coset representative $x_{\alpha} \in G$, then the set \( \{x_{\alpha}\}_{\alpha \in \frac{\mathbb{Q}}{\mathbb{Z}}} \) is linearly independent over $E_{\varepsilon}$ (in fact over $k((N, <))$). Thus $E_{\varepsilon} \bar{G} \rightarrow E$. Hence $k[G] \rightarrow E_{\varepsilon} \bar{G} \rightarrow E$. The crossed product $E_{\varepsilon} \bar{G} \rightarrow E$ is an Ore domain and $E_{\varepsilon} \bar{G} \rightarrow E_{\varepsilon}$ is the Ore field of fractions of $E_{\varepsilon} \bar{G}$. By Theorem 5.1(iii), $h_{\varepsilon}(k[G]) = h_{\varepsilon}(k[N]) \geq h_{\varepsilon}(k[N])$, and $h_{\varepsilon}(k[N]) = \infty$ by Corollary 4.2. Also, by Theorem 5.1(ii) and Corollary 4.2, $h_{\varepsilon}(f) = n$.

The fact that $h_{\varepsilon}(X) = h_{\varepsilon}(G[N]) = \infty$ follows from the fact that a free group is a free product of infinite cyclic groups, and because we can identify the free field inside the Malcev-Neumann power series ring cf. [22,3] That $h_{\varepsilon}(f) = n$ and $h_{\varepsilon}(f) = n-1$ follows from Corollary 4.2.

\[ \Box \]

**Proposition 5.3.** Let $k$ be a commutative field. For each finite set $X$ with $|X| \geq 2$, there exist infinite non-isomorphic fields of fractions $\iota: k(X) \rightarrow D$ such that $h_{\varepsilon}(X) = \infty$.

**Proof.** Step 1: We define a poly-orderable group $\Gamma_r$ for each integer $r \geq 1$.

We follow the notation in [11, Chapter 1]. Let $r \geq 1$. Let $Y$ be the connected graph with vertex set $VY = \mathbb{Z}$, edge set $EY = \{e_i \mid i \in \mathbb{Z}\}$ and incidence functions $\bar{i}(e_i) = i$ and $\bar{\pi}(e_i) = i + 1$, i.e.

\[ \ldots \bar{e}_{i-1} \bar{i} \bar{e}_i \bar{i+1} \bar{e}_{i+1} \ldots \]

Let $(G(\ ), Y)$ be the graph of groups

\[ \ldots \rightarrow G(\bar{i}) \xrightarrow{G(\bar{e}_i)} G(\bar{i+1}) \xrightarrow{G(\bar{e}_{i+1})} \ldots \]

where $G(\bar{i})$ is the free abelian group on $\{T_i, T_{i+1}, \ldots, T_{i+r}\}$ and $G(\bar{e}_i)$ is the free abelian group on $\{T_{i+1}, \ldots, T_{i+r}\}$ for each $i \in \mathbb{Z}$. Let $N_r$ be the fundamental group of $(G(\ ), Y)$, i.e. $N_r = \pi(G(\ ), Y, Y_0)$ with $Y_0 = Y$. Then, by definition,

\[ N_r = \left\{ T_i, \ i \in \mathbb{Z} \mid \begin{array}{l} T_i T_{i+1} = T_{i+1} T_i \\ T_{i+2} T_{i+1} = T_{i+1} T_{i+2} \\ T_i T_{i+r} = T_{i+r} T_i \end{array} \right\}. \tag{5.1} \]

Also $N_r$ can be seen as

\[ \cdots G(\bar{i}+1) G(\bar{i}) G(\bar{i}+2) \cdots \tag{5.2} \]

Consider the morphism of groups $\theta: N_r \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ defined by $\theta(T_i) = f_i$ where $f_i$ is the sequence $(x_n)_{n \in \mathbb{Z}}$ with $x_i = 1$ and $x_n = 0$ for $n \neq i$. It is easy to deduce from (5.1) that $\theta$ is well defined. Let $L_r = \ker \theta$. Observe that $\theta(G(\bar{i}))$ is injective.
for each $i \in \mathbb{Z}$. Hence $L_r$ is a free group by \[11\] Proposition 7.10. Moreover, $L_r$ is not commutative because for example $T_0T_{r+1}^{-1}T_0^{-1}T_{r+1}^{-1}$ and $T_{2r+3}^{-1}T_{2r-1}^{-1}T_{2r+3}^{-1}T_{2r+1}^{-1}$ belong to $L_r$, but they do not commute as can be deduced from \[5.2\]. In a similar way, it can be seen that $L_r$ is not finitely generated.

Define now $\Gamma_r = N_r \rtimes C$, where $C = \langle S \rangle$ is the infinite cyclic group, and $C$ acts on $N_r$ as $T_i \mapsto T_{i+1}$, i.e. $S^iT_iS^{-i} = T_{i+1}$. Hence $\Gamma_r$ has the subnormal series

$$1 < L_r < N_r < \Gamma_r,$$

with $\Gamma_r/N_r = C$ infinite cyclic, $N_r/L_r \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ a torsion-free abelian group and $L_r$ a noncommutative free group. Hence all factors are orderable groups.

**Step 2:** We construct a field of fractions $\delta_r : k[\Gamma_r] \hookrightarrow E^r$ with $h_{\delta_r}(k[\Gamma_r]) = \infty$ for each integer $r \geq 1$.

Let $r \geq 1$. Consider $\beta_r : k[L_r] \hookrightarrow C_r$ the universal field of fractions of the free group algebra $k[L_r]$. Consider $k[N_r]$ as a crossed product $k[L_r]\frac{N_r}{L_r}$. Any automorphism of $k[L_r]$ can be extended to an automorphism of $C_r$ by \[10\] Corollary 7.5.16. Hence we can consider a crossed product $C_r\frac{N_r}{L_r}$ that contains $k[L_r]\frac{N_r}{L_r}$ in the natural way. Since $N_r/L_r$ is a torsion-free abelian group, $C_r\frac{N_r}{L_r}$ is an Ore domain. Let $\gamma_r : k[N_r] \hookrightarrow C_r\frac{N_r}{L_r} \hookrightarrow D_r$ where $D_r$ is the Ore field of fractions of $C_r\frac{N_r}{L_r}$. The group ring $k[\Gamma_r]$ can be seen as a skew Laurent polynomial ring $k[N_r]\langle S, S^{-1}; \alpha \rangle$ where $\alpha$ is given by left conjugation by $S$ cf. \[5.2\]. Observe that conjugation by $S$ induces an automorphism on $L_r$, thus on $k[L_r]$ and on $C_r$. Therefore it can be extended to an automorphism of $C_r\frac{N_r}{L_r}$. Since $C_r\frac{N_r}{L_r}$ is an Ore domain and conjugation by $S$ gives an automorphism of $C_r\frac{N_r}{L_r}$, it can be extended to $D_r$. Hence

$$k[\Gamma_r] = k[N_r]\langle S, S^{-1}; \alpha \rangle \hookrightarrow C_r\frac{N_r}{L_r}\langle S, S^{-1}; \alpha \rangle \hookrightarrow D_r[\langle S, S^{-1}; \alpha \rangle].$$

Let $E^r$ be the Ore field of fractions of $D_r[\langle S, S^{-1}; \alpha \rangle]$, and $\delta_r : k[\Gamma_r] \hookrightarrow E^r$ be the natural embedding. Observe that it is a field of fractions of $k[\Gamma_r]$.

For $C_r\frac{N_r}{L_r}$ and $D_r[\langle S, S^{-1}; \alpha \rangle]$ are Ore domains, we can think of $E^r$ and $D_r$ as embedded in $D_r[\langle S, \alpha \rangle]$ and $C_r[\langle \frac{N_r}{L_r}, \alpha \rangle]$ for a certain order $\prec$ of $N_r/L_r$, respectively. Now by Proposition \[5.3(iii), h_{\delta_r}(k[\Gamma_r]) \geq h_{\gamma_r}(k[N_r])$. By Theorem \[5.1(iii), h_{\alpha_r}(k[N_r]) \geq h_{\gamma_r}(k[N_r])$. By Corollary \[5.4, h_{\delta_r}(k[\Gamma_r]) = \infty$. Therefore $h_{\delta_r}(k[\Gamma_r]) = \infty$.

**Step 3:** For each pair of integers $1 \leq r < s$, the free algebra $k \langle X_0, X_1, \ldots, X_r \rangle$ embeds in $k[\Gamma_s]$ via $X_i \mapsto T_i^s S$.

Let $K = k(t)$ be the field of fractions of the polynomial ring $k[t]$. Let $\alpha_s : K \rightarrow K$ be the morphism of rings given by $\alpha_s(t) = t^{s+1}$. Consider the skew polynomial ring $R_s = K[x; \alpha_s]$ and let $F_s$ be the Ore field of fractions of $R_s$. Then $\{1, t, \ldots, t^s\}$ are right linearly independent over $k(t^{s+1}) = \alpha_s(K)$. Then $\{x, tx, \ldots, t^sx\}$ are right linearly independent over $R_s$ \[17\] Lemma 9.19. By Jategaonkar’s Lemma \[17\] Lemma 9.21, there is an embedding of rings $k \langle X_0, \ldots, X_r \rangle \hookrightarrow R_s \hookrightarrow F_s$ sending $X_i \mapsto t^i x$. Consider now the morphism of groups $\varepsilon : \Gamma_r \rightarrow F_s^\times$ defined
by $\varepsilon(T_i) = t^{(s+1)i}$ and $\varepsilon(S) = x$. Since $\varepsilon(T_0(S)) = t^rx$ for $1 \leq i \leq r$, the set 
\{S, T_0S, \ldots, T_0^rS\} is a basis of a free monoid inside $\Gamma_r$. Therefore we obtain the 
embedding $k\langle X_0, \ldots, X_r \rangle \hookrightarrow k[\Gamma_s]$ of $k$-algebras defined by $X_i \mapsto T_0^iS$.

Step 4: For each pair of integers $1 \leq r \leq s$, there is a field of fractions $\iota_{rs}: k\langle X_0, X_1, \ldots, X_r \rangle \hookrightarrow E^s$, defined by $\iota_{rs}(X_i) = T_0^iS$, of infinite inversion height.

Let $\iota_{rs}$ be the embedding of Step 3 composed with $\delta_s$. Note that $S = \iota_{rs}(X_0)$ and $T_0 = \iota(X_0)i(X_0)^{-1}$. Since $\Gamma_s$ is generated by $S$ and $T_0$, $k[\Gamma_s] \subseteq E^s(2)$.

Therefore $E^s$ is generated, as a field, by the image of $\iota_{rs}$ and $h_{rs}(k(X_0, \ldots, X_r)) \geq h_{rs}(k[\Gamma_s]) = \infty$.

Step 5: The fields of fractions $\iota_{rs}: k\langle X_0, X_1, \ldots, X_r \rangle \hookrightarrow E^s$ and $\iota_{rs'}: k\langle X_0, X_1, \ldots, X_r \rangle \hookrightarrow E^{s'}$ are not isomorphic for $s \neq s'$.

Let $1 \leq r \leq s < s'$ be integers. First of all observe that there does not exist an isomorphism of groups $\Gamma_s \rightarrow \Gamma_{s'}$ with $S \rightarrow S$ and $T_0 \rightarrow T_0$.

If there is an isomorphism of rings $\eta_{ss'}: E^s \rightarrow E^{s'}$ such that $\iota_{rs'} = \eta_{ss'}\iota_{rs}$, then

$$\eta_{ss'}(S) = \eta_{ss'}(\iota_{rs}(X_0)) = \iota_{rs'}(X_0) = S,$$

$$\eta_{ss'}(T_0) = \eta_{ss'}(\iota_{rs}(X_1)\iota_{rs}(X_0)^{-1}) = \iota_{rs'}(X_1)\iota_{rs'}(X_0)^{-1} = T_0.$$

Hence the restriction of $\eta_{ss'}: \Gamma_s \rightarrow \Gamma_{s'}$ gives an isomorphism of groups sending $S \rightarrow S$ and $T_0 \rightarrow T_0$, a contradiction. $\square$

**Corollary 5.4.** Let $k$ be a commutative field and $Z = \{z_1, z_2, \ldots\}$ be an infinite countable set. Then the free algebra $k(Z)$ has infinite non-isomorphic fields of fractions $s: k(Z) \rightarrow D$ such that $h_s(k(Z)) = \infty$.

**Proof.** Follows from [13, Proposition 2.3] and Proposition 5.3. $\square$

6. **Crossed products of a ring by a universal enveloping algebra**

Throughout this section, $k$ will denote a commutative field.

Let $L$ be a Lie $k$-algebra. We will denote by $U(L)$ its universal enveloping algebra. Suppose that $R$ is a $k$-algebra, and let $\text{Der}_k(R)$ denote the set of $k$-linear derivations of $R$. A $k$-algebra $S$ containing $R$ is called crossed product of $R$ by $U(L)$ (and written $R \ast U(L)$) provided that there is a $k$-linear embedding $\cdot\hookrightarrow L \rightarrow S$, $x \mapsto \bar{x}$, such that:

(i) $S$ has the additive structure of $R \otimes_k U(L)$.

(ii) There exist a $k$-linear map (called action) $\delta: L \rightarrow \text{Der}_k(R)$, $x \mapsto \delta_x$, and a $k$-bilinear antisymmetric map (called twisting) $t: L \times L \rightarrow R$, $(x, y) \mapsto t(x, y)$ such that the following two conditions hold:

\begin{align}
    xa &= a\bar{x} + \delta_a(x) & \text{for all } x \in L \text{ and } a \in R, \\
    xy - y\bar{x} &= [x, y] + t(x, y) & \text{for all } x, y \in L.
\end{align}

(6.1) (6.2)
Crossed products for Lie Algebras were introduced in [25, 1.7.12] and in [5].

Let $\mathcal{C}$ be a $k$-linear independent subset of $L$. Suppose that we have defined a total order $<_{\mathcal{C}}$ in $\mathcal{C}$. The standard monomials in $\mathcal{C}$ is the subset of $R \ast U(L)$ consisting on the monomials of the form $x_1 x_2 \cdots x_m$ with $m \geq 0$, $x_i \in \mathcal{C}$ and $x_1 \leq x_2 \leq \cdots \leq x_m$ where we understand that the identity element in $U(L)$ is the standard monomial corresponding to $m = 0$. A standard monomial that is the product of $m$ elements of $\mathcal{C}$ has degree $-m$.

Let $\mathcal{B} = \{x_i \mid i \in I\}$ be a totally ordered basis of $L$. The Poincaré-Birkhoff-Witt (PBW) Theorem states that the standard monomials in $\mathcal{B}$ form a $k$-basis of $U(L)$. Thus (i) above is equivalent to the fact that $R \ast U(L)$ is a free left $R$-module with basis the standard monomials in $\mathcal{B}$.

One of the most important properties of crossed products is the following result which is [2, Lemma 1.1]. We will need how the identification (6.3) is made, thus we sketch the proof of [2, Lemma 1.1].

**Lemma 6.1.** If $H$ is an ideal of $L$, then

$$R \ast U(L) = (R \ast U(H)) \ast U(L/H).$$

**(Proof.** Set $T = R \ast U(H)$. Let $W$ be a subspace of $L$ with $L = H \oplus W$ and let $\sigma \colon L/H \rightarrow W$ be a $k$-vector space isomorphism. Let $\mathcal{D}$ be an ordered basis for $L/H$ and let $\mathcal{C}$ be one for $H$. Then $\mathcal{C} \cup \{\sigma(d) \mid d \in \mathcal{D}\}$ is an ordered basis for $L$ with the elements of $\mathcal{C}$ coming first. Then $R \ast U(L)$ has the additive structure of $T \ast U(L/H)$ by the PBW-theorem. Let $-$ denote the composition of $\sigma$ followed by $-$. Then, for each $x \in L/H$ and $t \in T$, we have that $\zeta(t) = t\bar{x} - \bar{x} t \in T$. Thus we get a $k$-linear map $\zeta \colon L/H \rightarrow \text{Der}_k(T)$, $x \mapsto \zeta_x$. Also it is not very difficult to see that, for each $x, y \in L/H$, $s(x, y) = \bar{x} \bar{y} - \bar{y} \bar{x} - [\bar{x}, \bar{y}] \in T$. We thus define the $k$-linear map $s \colon L/H \times L/H \rightarrow T$, $(x, y) \mapsto s(x, y)$. \)

For a given embedding of rings $R \hookrightarrow D$, we will be interested in extending the crossed product structure of $R \ast U(L)$ to $D \ast U(L)$ in the natural way. In order to do that we need to be precise on the conditions that $\delta$ and $t$ must satisfy. This is explained in the next lemma which can be seen as a corollary of [25, Theorem 7.1.10].

**Lemma 6.2.** Let $R$ be a $k$-algebra, and let $L$ be a Lie $k$-algebra. Suppose that there exist a $k$-linear map $\delta \colon L \rightarrow \text{Der}_k(R)$, $x \mapsto \delta_x$, and a $k$-bilinear antisymmetric map $t \colon L \times L \rightarrow R$, $(x, y) \mapsto t(x, y)$. They define a crossed product $R \ast U(L)$ if and only if $\delta$ and $t$ satisfy the following relations:

(i) $\delta_x(t(y, z)) + \delta_y(t(z, x)) + \delta_z(t(x, y)) + t(x, [y, z]) + t(y, [z, x]) + t(z, [x, y]) = 0$.

(ii) $[\delta_x, \delta_y] = [\delta_x, t(x, y)]$ where $\delta_t(x, y)$ denotes the $k$-derivation of $R$ defined by $a \mapsto [t(x, y), a] = t(x, y)a - at(x, y)$ for all $a \in R$.

Moreover, $R \ast U(L)$ can be constructed as the $k$-coproduct of $R$ with $T(L)$, the $k$-tensor algebra over $L$, modulo the two-sided ideal $\mathcal{I}$ generated by the set

$$\{xa - ax - \delta_x(a), \quad xy - yx - [x, y] - t(x, y) \mid \text{for any } x, y \in L \text{ and } a \in R\},$$
and it is free as a right and as a left $R$-module. More precisely, if $B = \{e_j \mid j \in J\}$ is a fixed ordered basis for $L$, then the set $G$ of standard monomials on $B$ is a basis of $R \ast U(L)$ as a right and as a left $R$-module; and if, for any $m \geq 0$, $G_m \subseteq G$ denotes the set of standard monomials of degree at most $m$, then $\sum_{x \in G_m} xR = \sum_{x \in G} Rx$. □

Remark 6.3. Let $f: R \hookrightarrow D$ be an extension of $k$-algebras, and let $L$ be a Lie $k$-algebra such that there exists a crossed product $R \ast U(L)$. To extend the crossed product structure to a crossed product $D \ast U(L)$ in such a way there is a ring inclusion $\tilde{f}: R \ast U(L) \hookrightarrow D \ast U(L)$ extending $f$ and such that $\tilde{f}(x) = x$, for any $x \in L$, one has:

(1) to make sure that the standard monomials are left $D$-independent;
(2) to extend the action $\delta_R$ to a $k$-linear map $\delta_D: L \rightarrow \text{Der}_k(D)$ in such a way that, for any $r \in R$, $\delta_R(x)(r) = \delta_D(x)(f(r));$
(3) to make sure that condition (ii) in Lemma 6.2 is satisfied.

Notice that the twisting must be the same for both crossed products, so that it is not necessary to verify condition (i) in Lemma 6.2.

Usually, we will be working with ring embeddings such that the derivations over $R$ extend in a unique way to $D$ (as in Lemma 4.4), so that condition (2) above will be automatically satisfied. Hence, only conditions (1) and (3) above need to be verified. □

The existence of a PBW-basis for $R \ast U(L)$, asserted in Lemma 6.2, gives a structure of filtered ring to $R \ast U(L)$ by setting, for any $m \geq 0$, $F_m$ to be the $R$-subbimodule of $R \ast U(L)$ generated by the monomials of degree at most $m$. By the definition of crossed product and Lemma 6.2, the associated graded ring is a polynomial ring over $R$ in the commuting variables given by the basis of the Lie algebra $L$. For further quoting we summarize this in the next Lemma.

Lemma 6.4. Let $R$ be a $k$-algebra, and let $L$ be a Lie $k$-algebra. Suppose that there exists a crossed product $R \ast U(L)$. Fix $B$ to be a basis of $L$, then $\text{gr}(R \ast U(L)) \cong R[B]$, that is, a polynomial algebra over $R$ in the commuting variables $B$. □

In the foregoing lemma, if $R$ is a field, then $\text{gr}(R \ast U(L))$ is an Ore domain, which implies that $R \ast U(L)$ embeds in a field with some good properties. This is expressed more precisely in the next proposition.

Proposition 6.5. Let $L$ be a Lie $k$-algebra and $K$ be a field with $k$ as a central subfield. For each crossed product $K \ast U(L)$, there is a canonically constructed field of fractions

$K \ast U(L) \hookrightarrow \mathcal{O}(K \ast U(L))$.

Suppose that $N$ is a subalgebra of $L$. The following properties are satisfied:
Proof. By Lemma 6.4, $\text{gr}(K \ast U(L))$ is a free left $R$-module with basis the free monoid on the set $\{1\}$. Thus we have just proved that $K \ast U(L)$ is a free left $R$-algebra.

Now we turn our attention to crossed products where the underlying Lie algebra is free.

**Lemma 6.6.** Let $R$ be a $k$-algebra. Let $H$ be the free Lie algebra on a set $X$. If $R \ast U(H)$ is a crossed product then, for each $x \in X$, there exists a $k$-derivation $\partial_x : R \to R$ such that $R \ast U(H) \cong \prod_{x \in X} R[x; \partial_x]$ the ring coproduct over $R$.

In particular, if $R = K$ is a field then $K \ast U(H)$ is a fir.

**Proof.** Consider the Lie $k$-algebra structure of $R \ast U(H)$ where the Lie product is given by $[a, b] = ab - ba$ for all $a, b \in R \ast U(H)$. Consider the morphism of Lie $k$-algebras $\hat{\cdot} : H \to K \ast U(H)$ which sends each $x \in X$ to $\hat{x}$. Thus $\hat{z} \hat{w} - \hat{w} \hat{z} = [\hat{z}, \hat{w}]$ for all $z, w \in H$.

By induction on the length of the Lie words on $X$ and then extending by linearity to $H$, it is not difficult to see that for each $z \in H$,

$$\hat{z} = \hat{a} + b_z$$

for some $b_z \in R$. (6.4)

It is known that $U(H)$ is $k\langle X \rangle$, the free $k$-algebra on the set $X$. Thus $R \ast U(H)$ is a free left $R$-module with basis the free monoid on the set $\{\hat{x} \mid x \in X\}$. By (6.4), it follows that $R \ast U(H)$ is a free left $R$-module with basis the free monoid on the set $\{\hat{x} \mid x \in X\}$. Thus $R \ast U(H)$ has the same additive structure as $R \otimes_k k\langle X \rangle = R \otimes_k U(H)$.

Also from (6.4), it follows that

$$\hat{z}a = a\hat{z} + \partial_z(a)$$

for each $a \in R$ and $z \in H$. (6.5)

where $\partial_z \in \text{Der}_k(R)$ and is given by $a \mapsto \partial_z(a) + [b_z, a]$. Thus we have just proved that $R \ast U(H)$ can be thought as a crossed product with trivial twisting.
From \((6.5)\), we deduce that, for each \(x \in X\), there exists a morphism of \(R\)-rings

\[
\varphi_x : R[x; \partial_x] \rightarrow R * U(H)
\]

which sends \(x \mapsto \bar{x}\). Consider now the unique morphism of \(R\)-rings

\[
\varphi : \prod_{x \in X} R[x; \partial_x] \rightarrow R * U(H)
\]

extending all \(\varphi_x\). Proceeding as in \([3\) Section 4]\) it is possible to prove that the free monoid on \(X\) is a right and left \(R\)-basis of \(\prod_{x \in X} R[x; \partial_x]\). Thus \(\varphi\) is an isomorphism.

The statement when \(R\) is a field follows from \([7, 36]\). \(\square\)

Hence for a free Lie algebra \(H\) and crossed product \(K * U(H)\), Lemma \(6.6\) implies the existence of the universal field of fractions of \(K * U(H)\) and Proposition \(5.5\)\) the existence of \(K * U(H) \hookrightarrow \mathcal{D}(K * U(H))\). We will show in the next section that both fields of fractions are in fact the same. Next two results will be useful in proving this assertion, for their proof it is important to keep in mind the results quoted in the section \(2.3\).

The statement of the next lemma is a slight generalization of \([20\), Lemma 1\] while the proof remains essentially the same. In Proposition \(6.8\) it will be helpful in recognizing isomorphic fields of fractions.

**Lemma 6.7.** Let \(R\) be a ring. Let \(F\) and \(L\) be epic \(R\)-fields, and \(\rho\) an \(R\)-specialization from \(F\) to \(L\). Suppose that \(S\) is a subring of \(F\) contained in the domain of \(\rho\). Denote by \(F_S\) and \(L_{\rho(S)}\) the subfields of \(F\) and \(L\) generated by \(S\) and \(\rho(S)\), respectively, and consider their induced structure of \(S\)-fields. If \(L_{\rho(S)}\) is an \(S\)-field with a minimal prime matrix ideal, then \(F_S\) is an \(S\)-field of fractions contained in the domain of \(\rho\), and so \(\rho\) maps \(F_S\) isomorphically onto \(L_{\rho(S)}\).

**Proof.** Let \(F_0\) be the domain of \(\rho\). Then \(F\) and \(L\) are \(F_0\)-fields, via the inclusion \(F_0 \hookrightarrow F\) and via \(\rho : F_0 \rightarrow L\), hence \(\rho\) is an \(F_0\)-specialization. Let \(\Sigma\) be the set of matrices over \(S\) that become invertible over \(L_{\rho(S)}\). Recall that there exists a unique \(S\)-ring homomorphism \(g : S_\Sigma \rightarrow L_{\rho(S)}\), and this morphism happens to be onto (cf. section \(2.3\)).

Each matrix of \(\Sigma\) is invertible over \(F_0\) because it is invertible over its residue class field \(F_0/\ker \rho \cong L\), thus it is also invertible over \(F\). The matrices of \(\Sigma\) are also invertible over \(F_S\) because when considered as endomorphisms of finite dimensional vector spaces over \(F_S\) they are injective. By the universal property of the localization, there exists a unique morphism of \(S\)-rings \(f : S_\Sigma \rightarrow F\) such that \(f(S_\Sigma) \subseteq F_0 \cap F_S\). Therefore we may consider the morphism of \(S\) rings \(\rho \circ f : S_\Sigma \rightarrow L_{\rho(S)}\); the uniqueness of \(g\) implies that \(g = \rho \circ f\). Since \(g\) is onto, we can deduce that \(f\) induces an onto \(S\)-morphism from a subring of \(F_S\) to \(L_{\rho(S)}\). Therefore such \(S\)-morphism is a specialization from \(F_S\) to \(L_{\rho(S)}\) and, by the minimality of the prime matrix ideal of \(L_{\rho(S)}\), it must be an isomorphism between \(F_S\) and \(L_{\rho(S)}\). Therefore, the image of \(f\) is exactly \(F_S\), i.e. \(F_S\) is contained in \(F_0\). \(\square\)
The following proposition is a generalization of [22, Lemma 3.1] to crossed products.

**Proposition 6.8.** Let $K$ be a field with $k$ as a central subfield. Let $H$ be a Lie $k$-algebra and let $N$ be an ideal of $H$. Consider a crossed product $K \ast U(H)$. Suppose that the following two conditions are satisfied:

1. $K \ast U(H)$ has a universal field of fractions $K \ast U(H) \hookrightarrow E$.
2. $R = K \ast U(N)$ has a prime matrix ideal $\mathcal{P}$ whose localization $R_\mathcal{P}$ is a field of fractions of $R$.

Then $K \ast U(H) = R \ast U(H/N)$ can be extended to a crossed product structure $R_\mathcal{P} \ast U(H/N)$, the embedding $K \ast U(H) \hookrightarrow E$ can be extended to $R_\mathcal{P} \ast U(H/N) \hookrightarrow E$ and this embedding is the universal field of fractions of $R_\mathcal{P} \ast U(H/N)$.

**Proof.** First note that since $R_\mathcal{P}$ is a field of fractions, $\mathcal{P}$ is a minimal prime matrix ideal (cf. [22, §2.3]).

We view $K \ast U(H)$ as $R \ast U(H/N)$. By Lemma 4.3 for each $x \in H/N$, the $k$-derivation $\delta_x$ of $R$ can be extended to $R_\mathcal{P}$. We denote this extension again by $\delta_x$.

We want to construct a crossed product $R_\mathcal{P} \ast U(H/N)$. For that we see that the conditions of Lemma 6.2 are satisfied. The first one is clearly satisfied because it is an equality in $R$. For the second one, we have to verify the equality of two $k$-derivations of $R_\mathcal{P}$. Since this equality holds in $R$, it also holds over $R_\mathcal{P}$ because of Lemma 4.3.

Let $B$ be a basis of $H/N$. By Proposition 6.3, $R_\mathcal{P} \ast U(H/N)$ has a field of fractions $R_\mathcal{P} \ast U(H/N) \hookrightarrow D = \Omega(R_\mathcal{P} \ast U(H/N))$. Clearly the restriction $K \ast U(H) \hookrightarrow D$ is a field of fractions of $K \ast U(H)$. Thus there exists a $K \ast U(H)$-specialization $\rho$ from $E$ to $D$. By Lemma 6.1, $\rho$ gives by restriction an isomorphism between the subfield $E_N$ of $E$ generated by $R$ and $R_\mathcal{P}$. Moreover, the standard monomials on $B$ are linearly independent over $E_N$ in $E$, because their images via $\rho$ are linearly independent over $R_\mathcal{P}$ in $D$. Thus the subring of $E$ generated by $E_N$ and \{ $x \mid x \in H/N$ \} is a crossed product isomorphic to $R_\mathcal{P} \ast U(H/N)$, because $\rho^{-1} \circ \delta$ and $\rho^{-1} \circ \tau$ induce and action and twisting, respectively, for this subring provided $\delta$ and $\tau$ are the action and the twisting of $R_\mathcal{P} \ast U(H/N)$. Thus $R_\mathcal{P} \ast U(H/N) \hookrightarrow E$ is a field of fractions of $R_\mathcal{P} \ast U(H/N)$. To prove that it is the universal field of fractions, observe that any $R_\mathcal{P} \ast U(H/N)$-field is a $(K \ast U(H))$-field that contains the field $R_\mathcal{P}$. By Lemma 6.4, there exists a $(K \ast U(H))$-specialization from $E$ that contains $R_\mathcal{P}$, and thus, arguing as above, it also contains $R_\mathcal{P} \ast U(H/N)$. Hence, such specialization is also an $(R_\mathcal{P} \ast U(H/N))$-specialization.

Next corollary is relatively easy but it gives an idea of how weak is the structure of crossed product.

**Corollary 6.9.** For each field $K$ with $k$ as a central subfield and each Lie $k$-algebra $L$, there exists a field $D$ that contains $K$ and a crossed product $D \ast U(L)$ that has a universal field of fractions.
Proof. Let $X$ be a set of generators of $L$. Let $H$ be the free Lie $k$-algebra on $X$. Consider the morphism of Lie algebras $H \to L$ that is the identity on $X$, and let $N$ be the kernel of this morphism. Note that $L \cong H/N$.

Consider a crossed product $K \ast U(H)$. Set $R = K \ast U(N)$. Then $K \ast U(H) = R \ast U(L)$. By Lemma 6.6 $K \ast U(H)$ has a universal field of fractions $E$. Since $N$ is also a free Lie $k$-algebra, $R$ is a fir by Lemma 6.6. Thus $R$ has a universal field of fractions and it is of the form $R_P$ where $P$ is the prime matrix ideal consisting of the nonfull matrices over $R$. By Proposition 6.8 there is a crossed product $R_P \ast U(L)$ and $R_P \ast U(L) \hookrightarrow E$ is its universal field of fractions. □

Let $G$ be a group, and fix an isomorphism $G \cong H/N$ where $H$ is a free group and $N$ is a normal subgroup (hence, it is a free group) of $H$. Consider an ordering of $H$. For any field $K$ consider the group algebra $K[H]$. It was proved in [31 Proposition 2.5], that the crossed product structure $K[H] = K[N] \ast H/N$ can be extended to $K((N)) \ast H/N$ and this is a subring of the Malcev-Neumann series field $K((H))$. This result combined with the fact that the universal field of fractions of $K[H]$ can be seen as a subring of $K((H))$, allows us to prove a result analogous to Corollary 6.9 for the case of groups. That is, for any field $K$ and any group $G$ there is a field $D$ containing $K$ and a crossed product $D \ast G$ that has a universal field of fractions.

7. A field of fractions of a crossed product of a residually nilpotent Lie algebra.

Throughout this section, $k$ will denote a commutative field.

In this section we present a ring of series introduced by A. I. Lichtman in [22]. This ring of series $K((H))$ is constructed from a crossed product $K \ast U(H)$ of a field $K$ by $U(H)$ where $H$ is a residually nilpotent Lie algebra satisfying the Q-condition (see Section 7.2). It will play the role of the Malcev-Neumann series ring $K((G))$ constructed from a crossed product $KG$ of a field $K$ by an ordered group $G$.

We will give a detailed exposition of the construction of the ring of series for some reasons. First, we expect to clarify and generalize in some aspects the one given in [22]. Secondly, in Theorem 7.13 we prove that, for a free Lie algebra $H$, this power series ring contains the universal field of fractions of $K \ast U(H)$, this is an extension of [22 Theorem 1]. Having in mind the analogy between free groups and free Lie algebras, this result can be seen as a counterpart of Lewin’s Theorem [19]. Moreover, as an application, we will produce further examples of elements with arbitrary inversion height.

The construction is divided in two parts. In Section 7.1 we construct a ring of series for a crossed product of a ring $R$ by $U(L)$ where $L$ is a nilpotent Lie algebra. In Section 7.2 we give the general construction using the preceding case. The main idea of such construction is presented in the following argument, which is a generalization of [22 Section 4].
Let $L$ be a Lie $k$-algebra, $R$ a $k$-algebra and $R \ast U(L)$ a crossed product.

Suppose that the center of $L$, $Z(L) = \{ x \in L \mid [L, x] = 0 \}$, is not zero. Fix a nonzero element $u \in Z(L)$. The $k$-subspace $N = ku$ is an ideal of $L$ and $[L, N] = 0$. Note that $R \ast U(N)$, the $k$-subalgebra of $R \ast U(L)$ generated by $R$ and $u$, is a skew polynomial ring $R[u; \delta_u]$.

Let $x \in L$, then

$$\bar{x}u - u\bar{x} = [x, u] + t(x, u) = t(x, u) \in R \quad (7.1)$$

Thus the restriction of the inner derivation of $R \ast U(L)$ determined by $\bar{x}$ induces a $k$-derivation $R[u; \delta_u] \to R[u; \delta_u]$, $f \mapsto \bar{x}f - f\bar{x}$. Notice that it extends $\delta_x : R \to R$, thus we will denote the extension again by $\delta_x : R[u; \delta_u] \to R[u; \delta_u]$.

Introduce the new variable $z = (u)^{-1}$, and let $R((z; \delta_u))$ be the power series ring described in Section 2.1. Observe that if we want to extend $\delta_x$ to $R((z; \delta_u))$, we have to define $\delta_x(z) = -z\delta_x(u)z$, and therefore

$$\delta_x(z) = \sum_{i=1}^{\infty} (-1)^i \delta_x^i(u)z^{i+1}. \quad (7.2)$$

**Lemma 7.1.** For each $x \in L$, the derivation $\delta_x : R[u; \delta_u] \to R[u; \delta_u]$, $f \mapsto \bar{x}f - f\bar{x}$ can be extended to a derivation $R((z; \delta_u)) \to R((z; \delta_u))$, $\sum_n a_nz^n \mapsto \sum_n \delta_x(a_nz^n) = \sum_n (\delta_x(a_n)z^n + a_n\delta_x(z^n))$, where $\delta_x(z)$ is defined as in (7.2).

**Proof.** Set $S = R(z ; za = az - z\delta_u(a)z, a \in R)$. So that, the $k$-algebra $S$ is isomorphic to $R \coprod k[z]$ modulo the two-sided ideal generated by $\{ za = az - z\delta_u(a)z, a \in R \}$. Let $\varepsilon_1 : k[z] \to S$ and let $\varepsilon_2 : R \to S$ be the induced $k$-algebra homomorphism. By the universal property of the coproduct, there is a ring homomorphism $\phi : S \to R[[z; \delta_u]]$ such that, for any $n \geq 1$, $\phi(\varepsilon_1(z^n)) = z^n$ and, for any $a \in R$, $\phi(\varepsilon_2(a)) = a$. Therefore $\varepsilon_1$ and $\varepsilon_2$ are injective homomorphisms. To ease the notation, we just identify $R$ and $k[z]$ with their image in $S$ without making any reference to the embeddings $\varepsilon_1$ and $\varepsilon_2$.

The existence of $\phi$ also implies that the powers of $z$ are right and left $R$ independent in $S$. In addition, by Proposition 4.1, $S$ is generated by the powers of $z$. Since, by the definition of $S$, $zR \subseteq Sz$, for any $n \geq 1$ the ideal $Sz^n$ is two-sided.

Fix $s \in S$ and $n \geq 0$, then $s = a_0 + a_1z + \cdots + a_nz^n + r_n$ where $a_i \in R$ for $i \in \{0, \ldots, n\}$ and $r_n \in Sz^{n+1}$. Let $\pi_n : R[[z; \delta_u]] \to R[[z; \delta_u]]/R[[z; \delta_u]]z^n$ denote the canonical projection. Since $\pi_n \circ \phi(s) = a_0 + a_1z + \cdots + a_nz^n$, $a_0, \ldots, a_n$ are uniquely determined. This implies that $\pi_n \circ \phi$ induces an isomorphism $S/Sz^n \cong R[[z; \delta_u]]/R[[z; \delta_u]]z^n$. Since $\lim_n R[[z; \delta_u]]/R[[z; \delta_u]]z^n \cong R[[z; \delta_u]]$ we conclude that the completion of $S$ with respect to the topology induced by the two-sided ideals $\{ Sz^n \}$ is isomorphic to $R[[z; \delta_u]]$ and that the isomorphism $\phi : S \to R[[z; \delta_u]]$ is defined by $\phi(s) = \sum_{i=0}^{\infty} a_i z^i$. Finally, since $\bigcap_{n \geq 1} Sz^n = \{0\}$, $S$ embeds in
$R[[z; \delta_u]]$ and this embedding sends $z \mapsto z$, $a \mapsto a$ for all $a \in R$ and $za \mapsto za = \sum_{i \geq 1} (-1)^{i-1} \delta_u^{i-1}(a)z^i$.

Now we are ready to prove that $\delta_x$ extends to $R((z; \delta_u))$. As a first step, we claim that $\delta_x$ can be extended to $S$ by setting

$$\delta_x(z) = -z\delta_x(\bar{u})z.$$

To prove the claim we must show that there is a morphism of $k$-algebras $\Phi : S \to T_2(S)$, $f \mapsto (f, f)$, where $T_2(S)$ is the ring of $2 \times 2$ upper triangular matrices over $S$.

There is a morphism of $k$-algebras $\Phi_1 : k[z] \to T_2(S)$ given by $\Phi_1(p(z)) = \left( \begin{smallmatrix} p(z) & 0 \\ 0 & \overline{p(z)} \end{smallmatrix} \right)$ for any $p(z) \in k[z]$. There is also a morphism of $k$-algebras $\Phi_2 : R \to T_2(S)$ given by $\Phi_2(a) = \left( \begin{smallmatrix} a & 0 \\ 0 & \overline{a} \end{smallmatrix} \right)$ for any $a \in R$. By the universal property of the coproduct, there is a unique algebra homomorphism $\Phi_3 : R \prod k[z] \to T_2(S)$ such that $\Phi_3(z) = \Phi_1(z)$ and such that, for any $a \in R$, $\Phi_3(a) = \Phi_2(a)$.

We show that, for any $a \in R$, $za - az + z\delta_u(a)z \in \text{Ker } \Phi_3$. This is equivalent to the matrix equality

$$\begin{pmatrix} z & \delta_x(z) \\ 0 & -z \end{pmatrix} \begin{pmatrix} a & \delta_u(a) \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} a & \delta_u(a) \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} z & \delta_x(z) \\ 0 & -z \end{pmatrix},$$

which yields

$$za \delta_x(z) + \delta_u(a)z = \begin{pmatrix} z & \delta_x(z) \\ 0 & -z \end{pmatrix} \begin{pmatrix} a & \delta_u(a) \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} za \delta_x(z) \\ \delta_u(a)z \end{pmatrix} = \begin{pmatrix} \delta_u(a)z \delta_u(a)z + \delta_x(z)z + z\overline{\delta_u(a)z} + \delta_u(a)z \delta_u(a)z \end{pmatrix}.$$

Hence $za - az + z\delta_u(a)z \in \text{Ker } \Phi_3$ if and only if the equality

$$z\delta_x(z) + \delta_u(a)z = \begin{pmatrix} z & \delta_x(z) \\ 0 & -z \end{pmatrix} \begin{pmatrix} a & \delta_u(a) \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} za \delta_x(z) \\ \delta_u(a)z \end{pmatrix} = \begin{pmatrix} \delta_u(a)z \delta_u(a)z + \delta_x(z)z + z\overline{\delta_u(a)z} + \delta_u(a)z \delta_u(a)z \end{pmatrix}.$$

After substituting $\delta_x(z) = -z\delta_x(\bar{u})z$, the right hand side of the equality (*) equals to

$$-az\delta_x(\bar{u})z + \delta_u(a)z + z\delta_u(a)z - z\delta_x(z)z = \delta_x(z)\delta_u(a)z.$$

Now, the left hand side of (*) is

$$z\delta_u(a) + \delta_x(z)u = \delta_u(a)z - z\delta_u(\overline{\delta_u(a)z})z - z\delta_x(a)z$$

$$= \delta_u(a)z - z\delta_u(a)z + z\delta_u(a)z + z\delta_u(z)z \delta_u(a)z$$

$$= \delta_u(a)z - z\delta_u(a)z + z\delta_u(a)z + z\delta_u(a)z + z\delta_u(a)z + z\delta_u(a)z.$$

After eliminating equal terms on both sides of (*), we see that it holds if and only if

$$-z\delta_u(\overline{\delta_u(a)z})z - z\delta_x(z)z = -z\delta_x(z)\delta_u(a)z.$$

Equivalently,

$$-z[\delta_x(z), \delta_u(a)]z = -z\delta_x(z)\delta_u(a)z = 0.$$

This last equality holds because by Lemma 6.2(ii),

$$[\delta_x(z), \delta_u(a)] = [t(x, u), a] = [\delta_x(\bar{u}), a], \text{ for all } a \in R.$$
There exists a crossed product structure \( R \to T_2(S) \) which must be a morphism of \( k \)-algebras. This finishes the proof of the claim.

The embedding \( S \hookrightarrow R[[z; \delta_u]] \) induces a morphism \( T_2(S) \to T_2(R[[z; \delta_u]]) \). The completion of \( T_2(S) \) with respect to the ideals \( \{ T_2(S)Z^n \}_{n \geq 1} \), where \( Z = (\delta_z) \), is \( T_2(R)[Z; \Delta_u] \), where the derivation \( \Delta_u \) is given by \( \Delta_u(\begin{array}{c} a \\ b \end{array}) = (\delta_u(a) \delta_u(b)) \) for all \( (a \ b) \in T_2(R) \). Recall that \( T_2(R)[Z; \Delta_u] \) is canonically isomorphic to \( T_2(R[[z; \delta_u]]) \). We will use this identification in what follows.

Note that the morphism

\[
\varphi : S \xrightarrow{\delta_z} T_2(S) \to T_2(R[[z; \delta_u]]), \quad f \mapsto \left( \begin{array}{c} f \delta_z(f) \\ 0 \end{array} \right) \mapsto \left( \begin{array}{c} f \delta_z(f) \\ 0 \end{array} \right)
\]

satisfies that \( \varphi(S)S^* \subseteq T_2(R)Z^n \) and thus induces morphisms \( \varphi_n : S^{\otimes n} \to T_2(R)^{\otimes n} \) such that for all \( n \geq m \) the diagram

\[
\begin{array}{ccc}
S^{\otimes n} & \xrightarrow{\varphi_n} & T_2(R)^{\otimes n} \\
\downarrow \varphi_{\otimes n} & & \downarrow \varphi_{\otimes n} \\
S^{\otimes m} & \xrightarrow{\varphi_m} & T_2(R)^{\otimes m}
\end{array}
\]

is commutative. Therefore there exists a morphism of \( k \)-algebras

\[
R[[z; \delta_u]] \cong S \to T_2(R[[z; \delta_u]]), \quad \sum a_i z^i \mapsto \left( \sum_{i \geq 0} a_i z^i \right)
\]

\[
\sum_{i \geq 0} \delta_x(a_i z^i) \to \sum_{i \geq 0} \delta_x(a_i z^i).
\]

where \( \delta_x \) is the composition \( S \xrightarrow{\delta_z} S \to R[[z; \delta_u]] \). Thus the derivation \( \delta_x : S \to S \) extends to \( \delta_x : R[[z; \delta_u]] \to R[[z; \delta_u]] \) as

\[
\delta_x \left( \sum_{i \geq 0} a_i z^i \right) = \sum_{i \geq 0} \delta_x(a_i z^i).
\]

Since \( R((z; \delta_u)) \) is the left Ore localization of \( R[[z; \delta_u]] \) at the set \( \{ 1, z, \ldots, z^n, \ldots \} \), \( \delta_x \) also extends to a derivation of \( R((z; \delta_u)) \) in a unique way (cf. Lemma 1.3).

Since \( \bar{u} = z^{-1} \), the equality \( za = az - z \delta_u(a)z \) implies that \( \bar{u}a = a\bar{u} + \delta_u(a) \) for each \( a \in R \); hence \( R[\bar{u}; \delta_u] \hookrightarrow R((z; \delta_u)) \). Also, as \( \delta_x(z^{-1}) = -z^{-1} \delta_x(z)z^{-1} \), \( \delta_x(z^{-1}) = \delta_x(\bar{u}) \). So that the derivation \( \delta_x \) has the properties claimed in the statement. \( \square \)

**Corollary 7.2.** There exists a crossed product structure \( R((z; \delta_u)) \ast U \left( \frac{L}{N} \right) \) such that

\[
R \ast U(L) = R[\bar{u}; \delta_u] \ast U \left( \frac{L}{N} \right) \hookrightarrow R((z; \delta_u)) \ast U \left( \frac{L}{N} \right).
\]

**Proof.** By the proof of Lemma 6.1, we know that, for each \( w \in L/N \), there exists \( x \in L \) such that the \( k \)-derivation \( \delta_u : R[\bar{u}; \delta_u] \to R[\bar{u}; \delta_u] \) given by the definition of \( R[\bar{u}; \delta_u] \ast U(L/N) \) coincides with \( \delta_x : R[\bar{u}; \delta_u] \to R[\bar{u}; \delta_u] \). We extend it to a
Observe that if there exist an ordinal \( \nu \) enough to choose \( L \) is that of a nilpotent Lie algebra. Indeed, if \( \nu \) was arbitrary, both derivations coincide on \( f \) and we obtain our result. \( \square \)

7.1. The case of hypercentral Lie algebras. A Lie \( k \)-algebra \( L \) is hypercentral if there exist an ordinal \( \nu \) and a chain of ideals \( \{ L_\mu \}_{\mu \leq \nu} \) of \( L \) that satisfy the following conditions:

(i) \( L_0 = 0, \ L_\nu = L \).

(ii) \( L_\mu \subseteq L_{\mu+1} \) for all \( 0 \leq \mu < \nu \).

(iii) \( L_\mu = \bigcup_{\mu < \mu'} L_\mu \) for all limit ordinals \( \mu' \leq \nu \).

(iv) \( [L, L_{\mu+1}] \subseteq L_\mu \) for all \( \mu < \nu \), or equivalently, \( L_{\mu+1}/L_\mu \) is contained in the center of \( L/L_\mu \).

We will say that \( \{ L_\mu \}_{\mu \leq \nu} \) is an hypercentral series of \( L \).

For our purposes, the most important example of hypercentral Lie algebra is that of a nilpotent Lie algebra. Indeed, if \( L \) is a nilpotent Lie \( k \)-algebra, it is enough to choose \( L_0 = 0, L_1 = Z(L) \), and for \( i \geq 1, L_{i+1}/L_i = Z(L/L_i) \). It is not difficult to prove that any hypercentral Lie algebra is locally nilpotent.

Fix a hypercentral Lie \( k \)-algebra \( L \) together with a hypercentral series \( \{ L_\mu \}_{\mu \leq \nu} \) of \( L \). Let \( R \) be a \( k \)-algebra and consider a crossed product \( R \circledast U(L) \).

For each \( 0 \leq \mu < \nu \), we pick in \( L_{\mu+1} \) a set of elements \( B_\mu \) which gives a basis of \( L_{\mu+1}/L_\mu \), and we endow \( B_\mu \) with a well-ordered set structure. Set \( B = \bigcup_{\mu < \nu} B_\mu \). Observe that \( B \) is a basis of \( L \). Then we order \( B \) extending the ordering in each \( B_\mu \) in the following way: given \( u_1 \in B_\mu_1 \) and \( u_2 \in B_\mu_2 \) we set

\[
   u_1 < u_2 \quad \text{iff} \quad \begin{cases} 
   
   \mu_1 < \mu_2, \text{ or } \\
   \mu_1 = \mu_2 \text{ and } u_1 \text{ is smaller than } u_2 \text{ in } B_{\mu_1}.
   
   \end{cases}
\]
Then \( (\mathcal{B}, <) \) is a well-ordered set. Thus, we can suppose that there exists an ordinal \( \varepsilon \) such that \( \mathcal{B} = \{ u_\gamma \}_{0 \leq \gamma < \varepsilon} \) and \( u_\gamma \leq u_\eta \) if and only if \( \gamma \leq \eta \).

For each \( 0 \leq \beta \leq \varepsilon \), set \( N_\beta \) as the \( k \)-subspace of \( \mathcal{L} \) generated by \( \{ u_\gamma \mid \gamma < \beta \} \). By convention, \( N_0 = 0 \). Observe that \( N_\beta \) is an ideal of \( \mathcal{L} \), hence a Lie subalgebra of \( \mathcal{L} \), and that
\[
[L, u_\beta] \subseteq N_\beta.
\]

By transfinite induction, we construct a ring of series \( R((N_\beta)) \) and a crossed product \( R((N_\beta)) \ast U(L/N_\beta) \), for each \( \beta \leq \varepsilon \), such that the following properties are satisfied for \( \gamma < \beta \leq \varepsilon \):

1. \( R((N_\gamma)) \hookrightarrow R((N_\beta)) \)
2. \( R \ast U(L) \hookrightarrow R((N_\gamma)) \ast U(L/N_\gamma) \hookrightarrow R((N_\beta)) \ast U(L/N_\beta) \) extending the embedding of (a) in the natural way.

We define \( R((N_0)) = R \). Let \( 0 < \beta \) be an ordinal and suppose that we have defined \( R((N_\gamma)) \) for all \( \gamma < \beta \) such that conditions (a) and (b) are satisfied. Suppose first that \( \beta \) is not a limit ordinal, thus \( \beta = \gamma + 1 \) for some ordinal \( \gamma \).

Set \( R_\gamma = R((N_\gamma)) \) and \( T_\gamma = R_\gamma[u_\gamma; \delta_{u_\gamma}] \). Introduce a new variable \( z_\gamma = (u_\gamma)^{-1} \).

Define \( R((N_\beta)) = R_\gamma((z_\gamma; \delta_{u_\gamma})) \). By Corollary [7.2] there exists a crossed product structure \( R((N_\beta)) \ast U(L/N_\beta) \) such that
\[
R \ast U(L) \hookrightarrow R((N_\gamma)) \ast U(L/N_\gamma) \hookrightarrow R((N_\beta)) \ast U(L/N_\beta).
\]

Thus conditions (a) and (b) are satisfied.

Suppose now that \( \beta \) is a limit ordinal. Define \( R((N_\beta)) = \bigcup_{\gamma < \beta} R((N_\gamma)) \). Set
\[
M = \lim_{\gamma < \beta} R((N_\gamma)) \ast U(L/N_\gamma).
\]

We want to prove that \( M \) has a natural crossed product structure of \( R((N_\beta)) \) by \( U(L/N_\beta) \). We show that \( M \) satisfies conditions (i) and (ii) in the definition of a crossed product. For that it is helpful to have in mind the proof of Lemma [6.1].

Consider \( \mathcal{S} = \{ u_\alpha \}_{0 \leq \alpha < \varepsilon} \subseteq L \). Let \( \mathcal{M} \) be the set of standard monomials on \( \mathcal{S} \). Abusing notation, we may suppose that \( \mathcal{M} \subseteq R((N_\gamma)) \ast U(L/N_\gamma) \) for all \( \gamma < \beta \), and the embedding \( R((N_\gamma)) \ast U(L/N_\gamma) \hookrightarrow R((N_\beta)) \ast U(L/N_\beta) \) can be seen as the identity on \( \mathcal{M} \) for \( \gamma_1 < \gamma_2 < \beta \).

Let \( f \in M \). There exists \( \gamma < \beta \) such that \( f \) is a finite sum of the form \( \sum_{m \in \mathcal{M}} f_m m \) with \( f_m \in R((N_\gamma)) \) for all \( m \in \mathcal{M} \). Moreover, given \( f_m, \ldots, f_m \in R((N_\delta)) \), there exists \( \gamma < \beta \) such that \( f_{m_1}, \ldots, f_{m_n} \in R((N_\gamma)) \), and \( \sum_{i=1}^n f_{m_i} m_i = 0 \) implies that \( f_{m_1} = \cdots = f_{m_n} = 0 \). Hence \( M \) has the additive structure of \( R((N_\delta)) \otimes U(L/N_\delta) \).

For each \( \gamma < \beta \), we identify the subspace of \( L \) generated by \( \mathcal{S} \) with a subspace of \( L/N_\gamma \) in the natural way. Let \( x \in L \) be any \( k \)-linear combination of elements in \( \mathcal{S} \). For each \( \gamma < \beta \), the crossed product structure \( R((N_\gamma)) \ast U(L/N_\gamma) \) defines a derivation \( \delta_{x, \gamma} : R((N_\gamma)) \rightarrow R((N_\gamma)) \). Moreover, if \( \gamma_1 < \gamma_2 < \beta \), since \( R((N_\gamma_1)) \subseteq R((N_{\gamma_2})) \) and \( R((N_{\gamma_2})) \ast U(L/N_{\gamma_2}) \hookrightarrow R((N_{\gamma_2})) \ast U(L/N_{\gamma_2}) \) we have that \( \delta_{x, \gamma_1} \) equals \( \delta_{x, \gamma_2} \) on \( R((N_{\gamma_2})). \) Set \( \delta_x : R((N_\beta)) \rightarrow R((N_\beta)) \) in the natural way, that is, for each \( f \in R((N_\beta)) \) there exists \( \gamma < \beta \) such that \( f \in R((N_\gamma)) \), then we
set $\delta_x(f) = \delta_x \gamma(f)$. Then $\delta: L/N_\beta \to \text{Der}_k(R((N_\beta)))$, $x \mapsto \delta_x$, is defined, where we are identifying the subspace of $L$ generated by $S$ and $L/N_\beta$ in the natural way. Let $x, y \in L/N_\beta$ and $f \in R((N_\beta))$. The equality $\bar{x}f = f\bar{x} + \delta_x(f)$ holds because $f \in R((N_\gamma))$ for some $\gamma < \beta$. Let $t: L \times L \to R((N_\beta))$ be given by the crossed product structure of $(R \ast U(N_\beta)) \ast U(L/N_\beta)$ s.t. $R \ast U(L)$. Hence, in particular $t(x, y) \in R \ast U(N_\beta)$. Then $\bar{xy} - \bar{x}\bar{y} = [\bar{x}, \bar{y}] + t(x, y)$. Thus conditions (6.1) and (6.2) are satisfied. Therefore $M = R((N_\beta)) \ast U(L/N_\beta)$ and $R \ast U(L) \hookrightarrow R((N_\gamma)) \ast U(L/N_\gamma) \hookrightarrow R((N_\beta)) \ast U(L/N_\beta)$ for $\gamma < \beta$.

We then define $R(L) = R((N_\beta))$.

**Remarks 7.3.** (a) The ring $R((L))$ depends on the order $< $ in (7.4) of the basis $\{u_\alpha\}_{0 \leq \gamma < \epsilon}$ of $L$ obtained from the hypercentral series $\{L_\mu\}_{\mu \leq \nu}$ of $L$. The same hypercentral series $\{L_\mu\}_{\mu \leq \nu}$ can give rise to different rings of series $R((L))$ because $R((L))$ depends on the basis $B_\mu$ and the different well-ordered set structures that each $B_\mu$ can be given. Also, different hypercentral series can give rise to the same ring of series $R((L))$ if we choose the same basis $\{u_\alpha\}_{0 \leq \gamma < \epsilon}$ and the same order $< $ obtained as in (7.4).

(b) By construction, if $\beta < \epsilon$, then

$$R((L)) = R((N_\beta))/((L/N_\beta)) \tag{7.5}$$

where we are identifying the ordered set $\{u_\alpha\}_{\beta \leq \alpha < \epsilon}$ with an ordered basis of the hypercentral Lie algebra $L/N_\beta$ in the natural way.

(c) If $R$ is a domain then $R((L))$ is also a domain.

(d) If $L'$ is a subalgebra of $L$ with a basis $B' \subseteq B$, where we understand that the order of $B'$ is inherited from the one in $B$, then $R((L')) \hookrightarrow R((L))$ in the natural way. Indeed, if we define $N'_\beta = \{u_\gamma \mid u_\gamma \in B', \gamma < \beta\}$, then $R((N'_\beta)) \subseteq R((N_\beta))$ in the natural way for each $0 \leq \beta < \epsilon$. $\square$

Now we want to define the so called least element map $\ell: R((L)) \to R$. Let $f \in R((L))$. Let $\beta_1$ be the least ordinal such that $f \in R((N_{\beta_1}))$. Note that $\beta_1$ is not a limit ordinal. If $\beta_1 = 0$, i.e. $f \in R$, we define $\ell(f) = f$. Suppose $\beta_1 \neq 0$. Thus there exists an ordinal $\gamma_1$ such that $\beta_1 = \gamma_1 + 1$. By construction, $R((N_{\beta_1})) = R((N_{\gamma_1}))(z_{\gamma_1}; \delta_{\gamma_2})$. Hence $f$ is a series in $z_{\gamma_1}$ with coefficients in $R((N_{\gamma_1}))$. Let $f_1 \in R((N_{\gamma_1}))$ be the coefficient of the least element in $\text{supp} f$ as a series in $z_{\gamma_1}$. Let $\beta_2$ be the least ordinal such that $f_1 \in R((N_{\beta_2}))$. If $\beta_2 = 0$, we define $\ell(f) = f_1$. If $\beta_2 \neq 0$, there exists an ordinal $\gamma_2$ such that $\beta_2 = \gamma_2 + 1$. By construction, $R((N_{\beta_2})) = R((N_{\gamma_2}))(z_{\gamma_2}; \delta_{\gamma_3})$. Thus $f_1$ is a series in $z_{\gamma_2}$ with coefficients in $R((N_{\gamma_2}))$. Let $f_2 \in R((N_{\gamma_2}))$ be the coefficient of the least element in $\text{supp} f_1$ as a series in $z_{\gamma_2}$. Let $\beta_3$ be the least ordinal such that $f_2 \in R((N_{\beta_3}))$. If $\beta_3 = 0$, we define $\ell(f) = f_2$. If $\beta_3 \neq 0$, there exists an ordinal $\gamma_3, \ldots$

Continuing in this way we obtain a descending chain of nonlimit ordinals

$$\beta_1 = \gamma_1 + 1 > \beta_2 = \gamma_2 + 1 > \beta_3 = \gamma_3 + 1 > \ldots$$

Note that if $\beta_t \neq 0$, then $\beta_t = \gamma_t + 1$ and $f_t \in R((N_{\gamma_t}))(z_{\gamma_t}; \delta_{\gamma_{t+1}})$ and $\beta_{t+1}$ is defined. Hence, since the set of ordinals $\{\beta \mid 0 \leq \beta < \epsilon\}$ is a well ordered set, there
exists a natural $n$ such that $\beta_n = 0$. We define $\ell(f) = f_{n-1}$. We say that $\ell(f)$ is the least element of $f$.

We collect some properties of the least element map in the following Lemma.

**Lemma 7.4.** Let $\ell: R((L)) \to R$ be the least element map. Let $f, g \in R((L))$. The following hold true:

(i) $\ell(f) = f$ if, and only if, $f \in R$.
(ii) $\ell(f) = 0$ if, and only if, $f = 0$.
(iii) Let $L'$ be a subalgebra of $L$ with a basis $B' \subseteq B$, where we understand that the order of $B'$ is inherited from the one in $B$. If $f \in R((L'))$, then the least element of $f$ viewed as an element of $R((L'))$ coincides with $\ell(f)$.
(iv) If $R$ is a domain, then $\ell(fg) = \ell(f)\ell(g)$.
(v) If $\ell(f)$ is invertible in $R$, then $f$ is invertible in $R((L))$. If $R$ is a domain, the converse is true.

**Proof.** (i) and (ii) follow easily from the construction.

(iii) follows by construction, defining $N'_\beta$ as in Remarks 7.3(d) and by induction on $\beta$.

We prove (iv) by induction on the least ordinal $\beta$ such that $f, g \in R((N_\beta))$. Observe that $\beta$ is not a limit ordinal. If $\beta = 0$, the result is clear by (i). Suppose that $\beta > 0$ and the result is true for $\gamma < \beta$. As $\beta = \gamma + 1$, $f, g \in R((N_\beta)) = R((N_\gamma))((z_\gamma; \delta_{u_\gamma}))$. If both $f, g \in R((N_\gamma))((z_\gamma; \delta_{u_\gamma})) \setminus R((N_\gamma))$. Then $f_1g_1 = (fg)_1 \in R((N_\gamma))$ because of the way series in one indeterminate are multiplied and the fact that $R((N_\gamma))$ is a domain (since $R$ is). Now observe that, by construction, $\ell(f_1) = \ell(f)$, $\ell(g_1) = \ell(g)$ and $\ell(fg) = \ell((fg)_1)$. Thus applying the induction hypothesis

$$\ell(fg) = \ell((fg)_1) = \ell(f_1g_1) = \ell(f_1)\ell(g_1) = \ell(f)\ell(g).$$

If $f \in R((N_\gamma))((z_\gamma; \delta_{u_\gamma})) \setminus R((N_\gamma))$ but $g \in R((N_\gamma))$, then $(fg)_1 = f_1g_1 \in R((N_\gamma))$.

Using the induction hypothesis,

$$\ell(fg) = \ell((fg)_1) = \ell(f_1g) = \ell(f_1)\ell(g) = \ell(f)\ell(g).$$

The remaining case is done analogously.

(v) Suppose that $\ell(f)$ is invertible. Set $f_0 = f$. By definition of $\ell(f)$, there exists a descending chain of nonlimit ordinals

$$\beta_1 = \gamma_1 + 1 > \beta_2 = \gamma_2 + 1 > \cdots > \beta_{n-1} = \gamma_{n-1} + 1 > \beta_n = 0$$

such that $f_{i-1} \in R((N_{\gamma_{i-1}}))((z_{\gamma_i}; \delta_{u_{\gamma_i}}))$ and $f_i$ is the coefficient of the least element in $\text{supp}(f_{i-1})$ as a series in $z_{\gamma_i}$, and $\ell(f) = f_{n-1}$. The fact that $f_{n-1}$ is invertible in $R \subseteq R((N_{\gamma_{n-1}}))$ implies that $f_{n-2}$ is invertible in $R((N_{\gamma_{n-2}}))((z_{\gamma_{n-1}}; \delta_{u_{\gamma_{n-1}}})) \subset R((N_{\gamma_{n-2}}))$. Hence $f_{n-3}$ is invertible in $R((N_{\gamma_{n-2}}))((z_{\gamma_{n-2}}; \delta_{u_{\gamma_{n-2}}})) \cdots$ Continuing in this way, we get that $f_1 \in R((N_{\gamma_2}))((z_{\gamma_2}; \delta_{u_{\gamma_2}})) \subseteq R((N_{\gamma_1}))$ is invertible, and therefore $f = f_0 \in R((N_{\gamma_1}))((z_{\gamma_1}; \delta_{u_{\gamma_1}}))$ is invertible.

Suppose now that $R$ is a domain and that $f$ is invertible. Applying (i) and (iv), we get $\ell(f^{-1})\ell(f) = \ell(f^{-1})f = 1 = \ell(1) = \ell(ff^{-1}) = \ell(f)\ell(f^{-1})$. $\square$
As a first outcome, we obtain a slight generalization of [22 Section 5].

**Corollary 7.5.** Let $L$ be an hypercentral Lie $k$-algebra. Let $K$ be a field with $k$ as a central subfield. Any crossed product $K \ast U(L)$ is an Ore domain and $K((L))$ is a field that contains the Ore field of fractions of $K \ast U(L)$.

**Proof.** Any hypercentral Lie $k$-algebra is locally nilpotent. Thus $K \ast U(L)$ is locally an iterated skew polynomial ring $K[x_1; \delta_1] \cdots [x_n; \delta_n]$, which is an Ore domain. We have already seen that $K \ast U(L) \hookrightarrow K((L))$. Now $K((L))$ is a field by Lemma 7.2(v). By the universal property of the Ore localization, the Ore field of fractions of $K \ast U(L)$ is contained in $K((L))$. \hfill \Box

### 7.2. The residually nilpotent case

Let $H$ be a Lie $k$-algebra. We say that $H$ is residually nilpotent if $H$ has a descending sequence of ideals

$$H = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_i \supseteq H_{i+1} \supseteq \cdots$$

with $[H, H_i] \subseteq H_{i+1}$ for all $i$, and such that $\bigcap_{i \geq 1} H_i = 0$. In this event, we call $\{H_i\}_{i \geq 1}$ an RN-series of $H$. The RN-series $\{H_i\}_{i \geq 1}$ satisfies the Q-condition if, for each $i$, there exists a set of elements $C_i$ of $H_i$ which gives a basis of $H_i/H_{i+1}$ such that $C = \bigcup_{i=1}^\infty C_i$ is a basis of $H$. We also say that $C$ is a Q-basis of $H$.

Given a Q-basis $C$, a canonical ordering of $C$ is an ordering $< < of $C$ obtained as we are going to see next. First we give an (arbitrary) well ordered set structure to $C_i$ for each $i \geq 1$. Then we order $C$ extending the order in each $C_i$ in the following way: given $u_1 \in C_{i_1}$ and $u_2 \in C_{i_2}$ we set

$$u_1 < u_2 \iff \left\{ \begin{array}{l} i_1 > i_2, \text{ or } \vspace{1mm} \\
 \quad i_1 = i_2 \text{ and } u_{i_1} \text{ is smaller than } u_{i_2} \text{ in } C_{i_1}. \end{array} \right.$$  

(7.7)

Notice that there may exist infinite canonical orderings of $C$.

We remark that $(C, <)$ need not be a well-ordered set, but $\bigcup_{i=1}^m C_i$ can be seen as a well ordered basis of $H/H_{m+1}$ for any $m$ under the obvious identification.

Not all residually nilpotent Lie $k$-algebras have a Q-basis. Important examples of residually nilpotent Lie $k$-algebras with Q-basis are the following.

**Examples 7.6.** (1) Suppose that $H$ is a nilpotent Lie $k$-algebra. Let $H = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n+1} = 0$ be an RN-series. If $C_i$ is a set of elements of $H_i$ which gives a basis of $H_i/H_{i+1}$, then clearly $C = \bigcup_{i=1}^n C_i$ is a Q-basis of $H$.

(2) Suppose that $H$ is a graded Lie $k$-algebra, that is, there exists a sequence $\{N_i\}_{i \geq 1}$ of subspaces of $H$ such that $H = \bigoplus_{i=1}^\infty N_i$ and $[N_i, N_j] \subseteq N_{i+j}$ for all $i, j \geq 1$. If we now define $H_i = \bigoplus_{j \geq 1} N_{i+j}$, and $C_i$ as any basis of $N_i$ for each $i \geq 1$, then it is easy to see that $\{H_i\}_{i=1}^\infty$ is an RN-series and $C = \bigcup_{i=1}^\infty C_i$ a Q-basis of $H$.

Examples of these algebras are the Lie algebras arising from torsion-free nilpotent and residually torsion-free nilpotent groups using the lower central series (of the groups), and the graded Lie algebras that appear in [22 Examples A,B,C,D]. \hfill \Box
Fix a residually nilpotent Lie k-algebra $H$ with an RN-series $\{H_i\}_{i=1}^{\infty}$ that has a Q-basis $C = \bigcup_{i=1}^{\infty} C_i$ and a canonical ordering of $C$.

Note that for each $n > m \geq 1$, $H/H_m$ and $H_m/H_n$ are nilpotent and hence hypercentral Lie k-algebras. Moreover

$$H_m/H_n = 0 < H_m/H_{m+1} < \cdots < H_n/H_{n+1} < H/H_n$$

is a chain of ideals of $H/H_n$ with $[H/H_m, H/H_n] \subseteq H_{m+1}/H_m$, and

$$H_n/H_n = 0 < H_{n+1}/H_n < \cdots < H/H_n$$

is a chain of ideals of $H_m/H_n$ with $[H_m/H_{n+1}, H_{n+1}/H_n] \subseteq H_{m+1}/H_m$.

For $1 \leq i \leq m-1$, let $B_{m,i}$ be the basis of $H_i/H_m \cong H_i/H_{i+1}$ obtained via the natural identification with $C_i$. Set $B_m = \bigcup_{i=1}^{m-1} B_{m,i}$ be the basis of $H/H_m$ with the well order inherited from $\bigcup_{i=1}^{m-1} C_i$.

Let $R$ be a k-algebra and consider a crossed product $R * U(H)$. For each $m \geq 1$, set $R_m = R * U(H_m)$. Then, with each basis $B_m$ fixed, we can construct the embedding $R_m * U(H/H_m) \hookrightarrow R_m((H/H_m))$. If $n > m$, since $R_n = R_n * U(H_m/H_n)$ and $R_n((H/H_n)) = R_n((H_m/H_n))(H/H_m)$, we obtain the commutativity of the following diagram

$$R_m * U(H/H_m) \xrightarrow{R_m((H/H_m))} R_m((H/H_m)) = R_m * U(H_m/H_n)((H/H_m)) \quad (7.8)$$

$$R_n * U(H/H_n) \xrightarrow{R_n((H/H_n))} R_n((H/H_n)) = R_n((H_m/H_n))(H/H_m)$$

It allows us to define

$$R((H)) = \lim_{m \to \infty} R_m((H/H_m)).$$

For each $n > m \geq 1$, let $\ell_{m}: R_m((H/H_m)) \to R_m$ be the least element map, and let $t_{m}: R_m \to R_{m+1}$ be the least element map of $R_{m+1}((H_m/H_{m+1}))$ restricted to $R_m$ (or equivalently, the restriction of $\ell_{m+1}$ to $R_m$ by Lemma 7.4(iii)).

The commutativity of the diagram

$$R_m((H/H_m)) \xrightarrow{\ell_{m}} R_m \quad (7.9)$$

$$R_{m+1}((H/H_{m+1})) \xrightarrow{\ell_{m+1}} R_{m+1}$$

follows from (7.8).

Note that, because of Lemma 7.4(i), each $t_m$ is the identity on $R_{m+1} \subseteq R_m$, and hence on $R$.

We claim that if $f \in R * U(L)$, there exists $m \geq 1$ such that $\ell_m(f) \in R$. Indeed, we may express $f = \sum_{i=1}^{n} a_i m_i$ where each $a_i \in R$ and each $m_i$ is a
standard monomial in the set \( C \). Thus there exists \( m \geq 1 \) such that \( f \) is an \( R \)-linear combination of the standard monomials in \( \bigcup_{i=1}^{m-1} C_i \). Now, by definition of \( \ell_m: R_m((H/H_m)) \to R_m \), it follows that \( \ell_m(f) \in R \), and the claim is proved.

Let now \( f \in R((H)) \). There exists \( m \geq 1 \) such that \( f \in R_m((H/H_m)) \). By the claim and the commutativity of (7.9), there exists \( m_0 \) such that \( \ell_m(f) \in R \). The commutativity of (7.9) and the fact that \( \ell_\ell \) is the identity on \( R \) for each \( \ell \) implies that \( \ell_n(f) = \ell_m(f) \) for all \( n \geq m_0 \). Thus we have a well defined map \( \ell: R((H)) \to R \) where for each \( f \in R((H)) \), \( \ell(f) = \ell_m(f) \) where \( m_0 \) is any natural such that \( \ell_m(f) \in R \). The map \( \ell: R((H)) \to R \) is called the least element map of \( R((H)) \), and \( \ell(f) \) the least element of \( f \in R((H)) \).

**Lemma 7.7.** The least element map \( \ell: R((H)) \to R \) satisfies the properties (i)-(v) in Lemma 7.4.

**Proof.** (i) and (ii) are clear from the construction.

(iii) Define \( H'_m = H_m \cap H' \) and \( R'_m = R_m U(H'_m) \). Then \( R((H')) = \lim_{m \to \infty} R'_m((H'/H'_m)) \) and the result follows from Lemma 7.4(iii).

(iv) Let \( f, g \in R((H)) \). There exists \( m \geq 1 \) such that \( f, g \in R_m((H/H_m)) \) and \( \ell_m(f), \ell_m(g) \in R \). Now apply Lemma 7.4(iv).

(v) if \( \ell(f) \) is invertible, then \( \ell_m(f) \) is invertible for some \( m \) such that \( f \in R_m((H/H_m)) \). By Lemma 7.4(v), \( f \) is invertible in \( R_m((H/H_m)) \), and therefore in \( R((H)) \). If \( R \) is a domain, then so is \( R_m \) for all \( m \). Now apply Lemma 7.4(v).

From all this, we obtain the extension of [22, Theorem 2] to crossed products \( K \ast U(H) \). More precisely, it follows from Lemma 7.4(v).

**Corollary 7.8.** Let \( H \) be a residually nilpotent Lie \( k \)-algebra \( H \) with an RN-series \( \{H_i\}_{i=1}^\infty \) that has a \( Q \)-basis \( C = \bigcup_{i=1}^\infty C_i \). Let \( K \) be a field with \( k \) as a central subfield. For any crossed product \( K \ast U(H) \) and canonical ordering, the ring of series \( K((H)) \) is a field that contains \( K \ast U(H) \).  

The subfield of \( K((H)) \) generated by \( K \ast U(H) \) will be denoted by \( K(H) \).

7.3. **Main results.** The next result gives a condition that ensures when two fields of fractions of a crossed product are isomorphic. It is the generalization of [23, Section 6, Corollary] to crossed products. Although weaker, it should be seen as a similar result to [13, Theorem].

**Theorem 7.9.** Let \( H \) be a residually nilpotent Lie \( k \)-algebra with an RN-series \( \{H_i\}_{i=1}^\infty \) that has a \( Q \)-basis \( C = \bigcup_{i=1}^\infty C_i \). Let \( K \) be a field with \( k \) as a central subfield. Consider a crossed product \( K \ast U(H) \) and suppose that it has a field of fractions \( K \ast U(H) \hookrightarrow D \). For each \( m \geq 1 \), denote by \( D_m \) the subfield of \( D \) generated by \( K \ast U(H_m) \). Assume that, for each \( m \geq 1 \), the standard monomials in \( \bigcup_{i=1}^{m-1} C_i \) are linearly independent over \( D_m \). Then \( K(H) \) and \( D \) are isomorphic fields of fractions of \( K \ast U(H) \).

**Proof.** For each \( m \geq 1 \), set \( R_m = K \ast U(H_m) \) and consider \( K \ast U(H) \) as \( R_m \ast U(H/H_m) \). Fix \( x \in H/H_m \), the derivation \( \delta_x \) of \( R_m \) extends to \( D \) as the inner
Corollary 7.11. Let $H$ be a Lie $k$-algebra. Let $K$ be a field containing $k$ as a central subfield. Consider a crossed product $K \ast U(H)$. Suppose that $N$ is an ideal of $H$ such that both $N$ and $H/N$ are residually nilpotent and they both have RN-series with Q-basis. Then the natural embedding $K \ast U(H) \hookrightarrow K(N)(\frac{H}{N})$ gives a field of fractions of $K \ast U(H)$ isomorphic to $K \ast U(H) \hookrightarrow \mathfrak{D}(K \ast U(H))$. 

Theorem 7.10. The field $K(H)$ does not depend on the RN-series with a Q-basis chosen, nor on the Q-basis $\mathcal{C}$ chosen nor on the canonical ordering of $\mathcal{C}$, chosen. In fact $K \ast U(H) \hookrightarrow K(H)$ and $K \ast U(H) \hookrightarrow \mathfrak{D}(K \ast U(H))$ (cf. Proposition 6.5) are isomorphic fields of fractions.

Proof. First note that the construction of $\mathfrak{D}(K \ast U(H))$ does not depend on the RN-series with a Q-basis chosen, nor on the Q-basis $\mathcal{C}$ nor on the canonical ordering of $\mathcal{C}$, see [9] Theorem 2.6.5 or [21].

Let $\{H_i\}_{i=1}^{\infty}$ be an RN-series with a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} C_i$ and set a canonical ordering of $\mathcal{C}$.

For each $m \geq 1$, $\bigcup_{i=1}^{m-1} C_i$ is a set of elements in $H$ which give a basis of $H/H_m$. By Proposition (6.5(ii)), the standard monomials in $\bigcup_{i=1}^{m-1} C_i$ are linearly independent over $\mathfrak{D}(K \ast U(H_m))$. Hence $K \ast U(H) \hookrightarrow K(H)$ and $K \ast U(H) \hookrightarrow \mathfrak{D}(K \ast U(H))$ are isomorphic fields of fractions of $K \ast U(H)$ by Theorem 7.9. □

The next result should be seen as a weaker version of [15] Theorem, along the lines of [31] Proposition 2.5(3)(ii).

Corollary 7.11. Let $H$ be a Lie $k$-algebra. Let $K$ be a field containing $k$ as a central subfield. Consider a crossed product $K \ast U(H)$. Suppose that $N$ is an ideal of $H$ such that both $N$ and $H/N$ are residually nilpotent and they both have RN-series with Q-basis. Then the natural embedding $K \ast U(H) \hookrightarrow K(N)(\frac{H}{N})$ gives a field of fractions of $K \ast U(H)$ isomorphic to $K \ast U(H) \hookrightarrow \mathfrak{D}(K \ast U(H))$. 

Proof. First note that the construction of $\mathfrak{D}(K \ast U(H))$ does not depend on the RN-series with a Q-basis chosen, nor on the Q-basis $\mathcal{C}$ nor on the canonical ordering of $\mathcal{C}$, see [9] Theorem 2.6.5 or [21].

Let $\{H_i\}_{i=1}^{\infty}$ be an RN-series with a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} C_i$ and set a canonical ordering of $\mathcal{C}$.

For each $m \geq 1$, $\bigcup_{i=1}^{m-1} C_i$ is a set of elements in $H$ which give a basis of $H/H_m$. By Proposition (6.5(ii)), the standard monomials in $\bigcup_{i=1}^{m-1} C_i$ are linearly independent over $\mathfrak{D}(K \ast U(H_m))$. Hence $K \ast U(H) \hookrightarrow K(H)$ and $K \ast U(H) \hookrightarrow \mathfrak{D}(K \ast U(H))$ are isomorphic fields of fractions of $K \ast U(H)$ by Theorem 7.9. □
Moreover, if $H$ is residually nilpotent with an RN-series that has a $Q$-basis, then $K \star U(H) \hookrightarrow K(H)$ and $K \star U(H) \hookrightarrow K(N)(\underline{H}/\underline{N})$ are isomorphic fields of fractions.

**Proof.** By Proposition 6.5(iii), we have $\mathcal{D}(K \star U(N)) \ast U(H/N) \subseteq \mathcal{D}(K \star U(H))$. Now Theorem 7.9 and again Proposition 6.5 imply that $\mathcal{D}(\mathcal{D}(K \star U(N)) \ast U(H/N)) \cong \mathcal{D}(K \star U(H))$. By Theorem 7.10 $K(N)$ and $\mathcal{D}(K \ast U(N))$ are isomorphic fields of fractions of $K \ast U(N)$. Hence $\mathcal{D}(K(N) \ast U(\underline{N})) \cong \mathcal{D}(K \star U(H))$. Again by Theorem 7.10 $K(N)(\underline{H}/\underline{N}) \cong \mathcal{D}(K \star U(H))$ as fields of fractions of $K \ast U(H)$.

If $K(H)$ exists, then Theorem 7.10 implies that $K(H) \cong \mathcal{D}(K \ast U(H)) \cong K(N)(\underline{H}/\underline{N})$. □

We showed in Lemma 6.6 that $K \ast U(H)$, the crossed product of a field $K$ by $U(H)$ where $H$ is a free Lie $k$-algebra, is a fir. Thus it has a universal field of fractions. We are going to prove that $K \ast U(H) \hookrightarrow K(H)$ and $K \ast U(H) \hookrightarrow \mathcal{D}(K \ast U(H))$ are both the universal field of fractions. This result was already known for $U(H)$ [22, Theorem 1], where the proof relies on the existence of some specialization (see [22, Lemma 3.1]). The techniques for the construction of such specialization do not work for crossed products. In our proof, the role of [22, Lemma 3.1] is played by Proposition 6.5.

**Remark 7.12.** Let $H$ be a free Lie $k$-algebra. Then $H$ is graded. Indeed $H = \bigoplus_{i \geq 1} N_i$ where each $N_i$ is the subspace generated by the Lie monomials of degree $i$. Then $H_i = \bigoplus_{i \geq 1} N_i$ is the $i$-th term of the lower central series of $H$. Let $\mathcal{C}_i$ be a basis of $N_i$ for $i \geq 1$. Therefore we are in the situation of Examples 7.2 and we can deduce that $U_{\mathcal{C}_i} H$ is a $Q$-basis of the residually nilpotent algebra $H$.

**Theorem 7.13.** Let $H$ be a free Lie $k$-algebra, $K$ a field with $k$ as a central subfield and consider $K \ast U(H)$. Then $K \ast U(H) \hookrightarrow K(H)$ and $K \ast U(H) \hookrightarrow \mathcal{D}(K \ast U(H))$ coincide with the universal field of fractions of $K \ast U(H)$.

**Proof.** Denote by $K \ast U(H) \hookrightarrow E$ the universal field of fractions of $K \ast U(H)$. We follow the notation of Remark 7.12.

It is known that any subalgebra of a free Lie algebra is a free Lie algebra. Thus, for each $m \geq 1$, $K \ast U(H_m)$ is a fir and therefore it has a universal field of fractions $K \ast U(H_m) \hookrightarrow R_p$, which by Lemma 6.6 is a universal localization at the prime matrix ideal $P_m$. Now, by Proposition 6.5 $K \ast U(H) \hookrightarrow R_{P_m} \ast U(H/H_m) \hookrightarrow E$. Hence the conditions of Theorem 7.9 are satisfied. Thus we can deduce that $K \ast U(H) \hookrightarrow E$ and $K \ast U(H) \hookrightarrow K(H)$ are isomorphic fields of fractions. By Theorem 7.10 $K \ast U(H) \hookrightarrow \mathcal{D}(K \ast U(H))$ is also isomorphic to the universal field of fractions of $K \ast U(H)$. □

For the missing details and definitions in the next example, the reader is referred to [23] and the references therein.

**Example 7.14.** Let $Q = P(x_1, \ldots, x_n)$ be the free Poisson field over $k$ in the variables $x_1, \ldots, x_n$ and let $Q^e$ be its universal enveloping algebra.
In [23, Theorem 1], it is proved that $Q^e$ satisfies the weak algorithm for a certain filtration of $Q^e$. Thus $Q^e$ is a free ideal ring and, therefore, it has a universal field of fractions. Although not stated explicitly, it is also proved in [23] Proposition 1, Corollary 1 that $Q^e$ is in fact a crossed product $K \ast U(H)$ of a commutative field $K$ over $U(H)$, the universal enveloping algebra of the free Lie algebra $H$ on $x_1, \ldots, x_n$. Indeed, by [23] Proposition 1, the morphism given in [6] Theorem 5 is in fact an isomorphism by a basis argument, and thus $Q^e$ is a crossed product as stated. Then, by Theorem 8.1, $Q^e \hookrightarrow \mathcal{D}(Q^e)$ and $Q^e \hookrightarrow K(H)$ are isomorphisms by a basis argument, and thus $Q^e$ is a crossed product as stated. Then, by Theorem 8.1, $Q^e \hookrightarrow \mathcal{D}(Q^e)$ and $Q^e \hookrightarrow K(H)$ are the universal field of fractions of $Q^e$. We stress that it cannot be deduced from the results in [22] that these embeddings are the universal field of fractions of $Q^e$. □

We remark on passing that if $R$ is an ordered $k$-algebra with positive cone $P(R)$ (for unexplained terminology see for example [9, Section 9.6]), and $H$ is a residually nilpotent Lie $k$-algebra with a Q-basis, then $R((H))$ is an ordered ring for any crossed product $R \ast U(H)$. In particular, if $R = K$ is a field, $K((H))$, $K(H)$ and $\mathcal{D}(K \ast U(H))$ are ordered fields. Indeed, if $\ell: R((H)) \to R$ is the least element map, then $\mathcal{P} = \{ f \in R((H)) \mid \ell(f) \in P(R) \}$ is a positive cone for $R((H))$. Clearly, $\mathcal{P} \cap -\mathcal{P} = \emptyset$ and $\mathcal{P} \cup -\mathcal{P} = R((H)) \setminus \{ 0 \}$. Moreover $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$ by Lemma 7.7 iv), and it is not difficult to prove that $\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}.$

8. Inversion height: the point of view of crossed products of Lie algebras.

Let $R$ be a $k$-algebra with a field of fractions $\varepsilon: R \hookrightarrow D$. Let $H$ be a residually nilpotent Lie $k$-algebra with an RN-series $\{ H_n \}_{n=1}^\infty$ that has a Q-basis $\mathcal{C} = \bigcup_{n=1}^\infty \mathcal{C}_n$. Consider a crossed product $R \ast U(H)$ and suppose that it can be extended to a crossed product structure $D \ast U(H)$. Then, by Remark 6.3 and Lemma 4.3, we can consider the crossed product $D_\varepsilon(n) \ast U(H)$ for each $n \geq 0$. Moreover,

$$R \ast U(H) \hookrightarrow D_\varepsilon(n) \ast U(H) \hookrightarrow D_\varepsilon(n + 1) \ast U(H) \hookrightarrow D \ast U(H).$$

Consider the embedding

$$\iota: R \ast U(H) \hookrightarrow L_n = D_\varepsilon(n)((H)) \hookrightarrow L_{n+1} = D_\varepsilon(n + 1)((H)) \hookrightarrow D((H)) = E.$$ 

Note that if $f \in D_\varepsilon(n)((H))$, then the least element map $\ell: D((H)) \to D$ is such that $\ell(f) \in D_\varepsilon(n)$. With this notation, we can prove an analogous result to Theorem 5.4.1.

**Theorem 8.1.** The following hold true

(i) $E_i((n)) \subseteq L_n$ for each integer $n \geq 0$.

(ii) Let $f \in D$. If $h_\varepsilon(f) = n$, then $h_\varepsilon(f) = n$.

(iii) $h_\varepsilon(R \ast U(H)) \geq h_\varepsilon(R)$.

**Proof.** (i) We proceed by induction on $n$. For $n = 0$, the result holds since $R \ast U(H) \hookrightarrow R((H))$. Suppose that the result holds for $n \geq 0$. Let $0 \neq f \in E_i((n)) \subseteq D_\varepsilon(n)((H))$. Consider the least element map $\ell: D_\varepsilon(n)((H)) \to D_\varepsilon(n)$.

For each $m \geq 1$, set $R_m = D_\varepsilon(n) \ast U(H_m)$, $S_m = D_\varepsilon(n + 1) \ast U(H_m)$ and consider $\ell_m: R_m((H/H_m)) \to R_m$. 
Corollary 8.2. Let $I$ be a set of cardinality at least two and let $\{H_i\}_{i \in I}$ be a set of Lie $k$-algebras. Set $H$ to be the free product of such algebras, that is, $H = \prod_{i \in I} H_i$. Consider the direct sum $\bigoplus_{i \in I} H_i$. For each $i \in I$, let $\pi_i : H_i \to \bigoplus_{i \in I} H_i$ be the canonical inclusion. Let $\pi : \prod_{i \in I} H_i \to \bigoplus_{i \in I} H_i$ be the unique morphism of Lie $k$-algebras such that $\pi|_{H_i} = \pi_i$. The subalgebra $\ker \pi$ is called the cartesian subalgebra of the free product $H$.

By $[x, y]_n$, we denote the product $[\ldots [[x, y], y], \ldots, y]$ with $n$ factors $y$.

**Corollary 8.2.** Let $I$ be a set of cardinality at least two and let $\{H_i\}_{i \in I}$ be a set of nilpotent Lie $k$-algebras. Set $H = \prod_{i \in I} H_i$. Let $U(H)$ be the universal enveloping algebra of $H$ and consider the embedding $\iota : U(H) \hookrightarrow \mathcal{D}(K \ast U(H))$. Then $h_i(U(H)) = \infty$. Indeed, let $x \in H_i \setminus \{0\}$ and $y \in H_j \setminus \{0\}$ with $i \neq j$. If $f$ is any entry of the inverse of the $n \times n$ matrix

$$A_n = 
\begin{pmatrix}
[x, y], x & [x, y], x & \cdots & [x, y], x \\
[x, y], y & [x, y], y & \cdots & [x, y], y \\
\vdots & \cdots & \ddots & \cdots \\
[x, y], y, x & [x, y], y, x & \cdots & [x, y], y, x \\
\end{pmatrix}
$$

then $h_i(f) = n$.

In particular, if $X$ is a set of cardinality at least two and $k(X)$ is the free $k$-algebra on $X$, then the universal field of fractions $\iota : k(X) \hookrightarrow F$ is of infinite
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Indeed, let \( x, y \in X \) be different elements. If \( f \) is any entry of the inverse of the \( n \times n \) matrix,

\[
A_n = \begin{pmatrix}
[x, y], x & [x, y], x & \cdots & [x, y], x \\
[x, y], y & [x, y], y & \cdots & [x, y], y \\
\vdots & \vdots & \ddots & \vdots \\
[x, y], n & [x, y], n & \cdots & [x, y], n
\end{pmatrix}
\]

then \( h_\epsilon(f) = n \).

**Proof.** Let \( N \) be the cartesian subalgebra of \( H \). By [4, Theorem 4.10.5], \( N \) is a free Lie \( k \)-algebra on an infinite set \( Y \), and thus \( U(N) \) is a free \( k \)-algebra on \( Y \). Moreover, it is not difficult to see that \( H/N \cong \bigoplus_{i \in I} H_i \) is a residually nilpotent Lie \( k \)-algebra with an RN-series that has a Q-basis. By Corollary 7.11, \( U(H) \hookrightarrow \mathfrak{D}(K + U(H)) \) can be seen as \( U(H) \hookrightarrow k(N)(\frac{H}{N}) \hookrightarrow k(N)(\frac{H}{N}) \). By Theorem 4.11 and Theorem 7.13, \( \epsilon: U(N) \hookrightarrow K(N) \) is of infinite inversion height. By Theorem 8.1, \( h_\epsilon(U(H)) = \infty \).

Moreover, using [4, Section 4.10], \( Y \) can be chosen to contain the elements \([x, y], x\). By Theorem 7.11 for each entry \( f \) of \( A_n^{-1} \), \( h_\epsilon(f) = n \). Applying Theorem 8.1, we obtain that \( h_\epsilon(f) = n \).

When \( H \) is the free Lie algebra on a set \( X \), put \( I = X \). Then \( H \) is the free product of the abelian (and hence nilpotent) Lie \( k \)-algebras generated by each \( x \in X \). Now apply the foregoing, and note that \( \mathfrak{D}(U(H)) \) is the universal field of fractions of \( U(H) \), by Theorem 7.13. \( \square \)

We remark that the statement of Corollary 8.2 works for any set \( \{H_i\}_{i \in I} \) of residually nilpotent Lie \( k \)-algebras with a Q-basis because they induce a natural RN-series with a Q-basis in \( \bigoplus_{i \in I} H_i \). Also, it is known that the free product of residually nilpotent Lie algebras is a residually nilpotent Lie algebra, see for example [4, p.175]. On the other hand, we do not know whether there exists an RN-series of the free product with a Q-basis.

Note that by choosing different elements (or changing the basis) of \( N \), other elements of prescribed inversion height \( n \) can be found.

Another way of obtaining the second part of Corollary 8.2 is the following. By [1], if \( N \neq H \) is an ideal of the free (not commutative) Lie algebra \( H \), then \( N \) is a free Lie algebra not finitely generated. Thus, choosing \( N \) such that \( H/N \) is nilpotent, we get elements of inversion height \( n \) for any \( n^2 \) different free generators of \( N \).

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