Global existence and asymptotics for the modified two-dimensional Schrödinger equation in the critical regime

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Abstract
We study the asymptotic behaviour of the modified two-dimensional Schrödinger equation
\((D_t - F(D))u = \lambda |u|^2 u\) in the critical regime, where \(\lambda \in \mathbb{C}\) with \(3\lambda \geq 0\) and \(F(\xi)\) is a second order constant coefficients elliptic symbol. For any smooth initial datum of size \(\varepsilon \ll 1\), we prove that the solution is global-in-time, combining the vector fields method and a semiclassical analysis method introduced by Delort. Moreover, we present the pointwise decay estimates and the large time asymptotic formulas of the solution.

Keywords: Schrödinger equation, semiclassical analysis, modified scattering

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1. Introduction

We consider the following modified critical nonlinear Schrödinger equation
\[(D_t - F(D))u = \lambda |u|^2 u, \quad x \in \mathbb{R}^n\] (1.1)
where \(D_t = \frac{\partial}{\partial t}\), \(D = \nabla\), \(\lambda \in \mathbb{C}\) and \(F(D)\) is defined via its real symbol, i.e.
\[F(D)u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} F(\xi)(F(D)(\xi)) d\xi,\]

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where $F$ is the Fourier transform with respect to $x$ variables. The Cauchy problem (1.1) is critical because the best time decay one can expect for the solution of the linear equation $(D_t - F(D))u = 0$ with smooth, decaying Cauchy data is $\|u(t, \cdot)\|_{L^\infty} = O(t^{-n/2})$, so that the nonlinearity will satisfy $\|u^2 u(t, \cdot)\|_{L^2} \leq C t^{-1} \|u(t, \cdot)\|_{L^2}$, with a time factor $t^{-1}$ just at the limit of integrability. Schrödinger equations with modified dispersion contain many important equations from the fields of physics. Typical models are the Schrödinger equation $(F(\xi) = \xi^2)$, KDV equation $(F(\xi) = \xi^3)$, Klein–Gordon equation $(F(\xi) = \sqrt{1 + \xi^2}$) and Benjamin–Ono equation $(F(\xi) = \xi |\xi|)$ etc. Over the past decades, the local smoothing properties, the dispersive estimates and the well-posedness of the modified Schrödinger equations have been studied extensively, see e.g. [2–4, 6, 13, 17, 22–24] etc. Then we want to study the global existence and asymptotics of solutions to the modified Schrödinger equation (1.1), provided that the initial data is small and spatially localised.

When $F(\xi) = \frac{1}{2} |\xi|^2$, it is well known that equation (1.1) represents the classical critical nonlinear Schrödinger equation

$$i \partial_t u + \frac{1}{2} \Delta u = \lambda |u|^2 u. (1.2)$$

For $\lambda \in \mathbb{R}, n = 1, 2, 3$, Hayashi and Naumkin [16] proved the asymptotics for the small solution $u(t)$: There exists $W(x) \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$u(t, x) = \frac{1}{(it)^{n/2}} e^{i W(x)/t} W \left( \frac{x}{t} \right) \chi_{\mathbb{R}^n} W(x)^2 \log t + O_L^{1/2} \left( t^{-n/2-\beta} \right), \beta > 0,$$

as $t \to \infty$. One note that this is not linear scattering, but rather a modified linear scattering. The idea is to apply the operator $F U(-t)$ to the equation (1.2) and use the factorisation technique of the Schrödinger operator to derive the ODE for $F U(-t) u(t)$:

$$i \partial_t F U(-t) u = \lambda t^{-1} |F U(-t) u(t)|^2 F U(-t) u + R(t, \xi), (1.3)$$

where $U(t) = e^{i \Delta t}$ and $\|R(t, \xi)\|_{L^\infty} = O(t^{-1-\beta})$ is integrable as $t \to \infty$. By a standard ODE argument, they deduce from (1.3) the asymptotic formula for $F U(-t) u(t)$ and then in the solution $u(t)$.

For the asymptotics of the classical critical nonlinear Schrödinger equation (1.2) in one-dimension with $\lambda \in \mathbb{R}$, there are four alternate approaches. In the paper [7], Deift and Zhou established the asymptotics of the solution for large data in the defocusing case. The proof is based on the complete integrability of (1.2) and the inverse scattering techniques. By assuming the higher regularity of the initial data, Lindblad and Soffer [28] derived an asymptotic equation in the physical space along the rays and obtained the asymptotic formula of the solution. Inspired by the space-time resonances method that introduced by Germain et al [11, 12], Kato and Pusateri [21] refined Hayashi–Naumkin’s method and obtained the same result as [16]. In the paper [19], Ifrim and Tataru combined these two approaches and developed the wave packet test method. By measuring the decay of the solution along the rays, they obtained the asymptotic formulas of the solution both in physical space and frequency space.

In this paper, we consider the global existence and asymptotics of the solutions to the modified Schrödinger equation (1.1). Since $F(\xi)$ has no explicit expression, the methods used in [7, 16, 19, 21, 28] can not be applied directly, and we have to develop a new approach. The second author of this paper first studied the asymptotic behaviour of the solution to the modified one-dimensional Schrödinger equation. Assuming that $F(\xi)$ satisfies certain elliptic assumption, Zhang [35] obtained the asymptotics of the solution for small initial data, combining the vector fields method and a semiclassical analysis method introduced by Delort [9]. In this paper,
we consider the two-dimensional case and assume that \( F(\xi) \) satisfies the following elliptic assumptions:

\[
F(\xi) = F_1(\xi_1) + F_2(\xi_2), \quad \xi = (\xi_1, \xi_2) \rightarrow F(\xi) \in \mathbb{R}
\]  

(1.4)

is a smooth function defined on \( \mathbb{R}^2 \), satisfying

\[
F_k(\xi_k) \in C^\infty(\mathbb{R}), \quad 0 < c_k \leq F_k'(\xi_k) \leq d_k, \quad \forall \xi_k \in \mathbb{R},
\]

(1.5)

for some positive constants \( c_k, d_k, k = 1, 2 \). Here we assume the specific form (1.4), since we cannot ensure that the approximate Leibniz rule (1.25)–(1.27) hold when \( \partial_{\xi_1} \partial_{\xi_2} F(\xi) \neq 0 \) (see remark 2.2). We assume also that \( F_k(\xi_k) \) has an expansion

\[
F_k(\xi_k) = c^2_k \xi_k^2 + c^1_k \xi_k + c^0_k + c^{-1}_k \xi_k^{-1} + c^{-2}_k \xi_k^{-2} + \cdots
\]

(1.6)

when \( \xi_k \) goes to \( \pm \infty \), where \( c^2_k > 0, k = 1, 2 \). In particular, we can choose \( F(\xi) = |\xi|^2 \).

Therefore the asymptotic formulas established in this paper can be seen as a generalisation of [7, 16, 19, 21, 28].

To state our result precisely, we now give some notations. For \( \psi \in L^1(\mathbb{R}^2) \), the Fourier transform of \( \psi \) is represented as \( \hat{\psi}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} \psi(x) e^{-ix \cdot \xi} dx \). \([A, B] \) denotes the commutator \( AB - BA \) and \( (x) = \sqrt{1 + |x|^2} \). Different positive constants we denote by the same letter \( C \). We write \( f \lesssim g \) when \( f \leq Cg \). \( S(\mathbb{R}^2) \) and \( H^s(\mathbb{R}^2) \) denote the usual Schwartz and Sobolev space respectively. \( L^p(\mathbb{R}^2) \) denotes the usual Lebesgue space with the norm \( \| \phi \|_{L^p} = (\int_{\mathbb{R}^2} |\phi(x)|^p dx)^{1/p} \) if \( 1 \leq p < \infty \) and \( \| \phi \|_{L^\infty} = \text{ess} \sup \{ |\phi(x)|; x \in \mathbb{R}^2 \} \). Moreover, we denote \( \| (\phi(x), \psi(x)) \|_{L^2} = (\int_{\mathbb{R}^2} |\phi(x)|^2 dx)^{1/2} + (\int_{\mathbb{R}^2} |\psi(x)|^2 dx)^{1/2} \).

Throughout the paper, \( F(\xi) = F_1(\xi_1) + F_2(\xi_2) \) denotes the second order constant coefficients classical elliptic symbol, satisfying (1.4)–(1.6). Denote by \( M \) the propagation set \( M = \{(x, \xi) : x + F'(\xi) = 0\} \). It is well-known that, by the vector field method, the truncation of the solutions away from the propagation set \( M \) will decay rapidly. Since \( F'_k \) is strictly increasing, there are smooth strictly concave functions \( \phi_k : \mathbb{R} \rightarrow \mathbb{R} \) such that \( x_k + F'_k(d\phi_k(x_k)) = 0, k = 1, 2 \), where we write \( d\phi_k(x_k) = d\phi_k(x_k)/dx \). We then define \( d\phi_k(x_k) \) to be \( 1 \) for simplicity. Set \( \phi_0(x) = (\phi_1(x_1), \phi_2(x_2)) \), then the propagation set \( M \) can be written as \( M = \{(x, \xi) : \xi - d\phi(x) = 0\} \). Therefore one can extract the main part of semiclatical pseudo-differential operators by developing the symbol at \( \xi = d\phi(x) \). In particular, by the definition of \( F(\xi) \) and \( \phi(x) \), we have

\[
x + F'(d\phi(x)) = (x_1 + F'_1(d\phi_1(x_1)), x_2 + F'_2(d\phi_2(x_2))) = 0,
\]

(1.7)

which will be used frequently from now on.

Next we state our results.

**Theorem 1.1.** Assume that \( \lambda \in \mathbb{C} \) with \( \Re \lambda \geq 0 \), and the initial datum \( u_0 \in H^2 \), \( (x_k + F'_k(D))^2 u_0 \in L^2, k = 1, 2 \), satisfying

\[
\| u_0 \|_{H^2} + \| (x_k + F'_k(D))^2 u_0 \|_{L^2} \leq 1.
\]

(1.8)

Then there exists \( \varepsilon_0 > 0 \), which is independent of \( u_0 \), such that for all \( \varepsilon \in (0, \varepsilon_0) \), the Cauchy problem

\[
\begin{cases}
(D_t - F(D)) u = \lambda |u| u, & t > 1, x \in \mathbb{R}^2 \\
u(x, 1) = \varepsilon u_0(x),
\end{cases}
\]

(1.9)

admits a unique global solution \( u \in C([1, \infty); L^2) \), satisfying the pointwise estimate

\[
\| u(t, \cdot) \|_{L^\infty} \lesssim \varepsilon t^{-1},
\]

(1.10)
for all $t > 1$.

**Remark 1.1.** The fact that the initial datum is given at time $t = 1$ does not have deep meaning: it is simply more convenient when performing estimates, since the $L^\infty$ decay of $\frac{1}{t}$ given by the linear part of the equation is not integrable at $t = 0$.

In order to investigate the large time asymptotic behaviour of the solution, the following theorem plays an important role.

**Theorem 1.2.** Suppose that $u_0$ satisfies (1.8), $u$ is the solution obtained in theorem 1.1 and $v_A$ is the function defined by (1.22) and (1.31). Let

$$\Phi(t,x) = \int_1^t s^{-1} |v_A(s,x)| \, ds. \quad (1.11)$$

Then there exists $\varepsilon_0 > 0$, which is independent of $u_0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a unique complex function $z_+(x) \in L^2_x \cap L^\infty_x$ such that

$$v_A(t,x) \exp(-i(w(x) t + \lambda \Phi(t,x))) - z_+(x) = O_L^\infty \left(t^{-1/2 + \varepsilon}\right) \cap O_L^2 \left(t^{-1 + \varepsilon}\right),$$

holds as $t \to \infty$, where $w(x) = x \cdot d\phi(x) + F(d\phi(x))$.

By theorem 1.2, we obtain the following asymptotics of the solution.

**Theorem 1.3.** Suppose that $\exists \lambda = 0$, $u_0$ satisfies (1.8) and $u$ is the solution obtained in theorem 1.1. Then there exists $\varepsilon_0 > 0$, which is independent of $u_0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, the followings hold:

(a) Let $z_+(x)$ be the asymptotic function obtained in theorem 1.2 and

$$\phi_+(x) = \int_1^\infty s^{-1} \left(|v_A(s,x)| - |z_+(x)|\right) \, ds. \quad (1.12)$$

The asymptotic formula

$$u(t,x) = \frac{1}{t} e^{-i(w(x) t + \lambda \Phi(t,x))} z_+ \left(\frac{x}{t}\right) + O_L^\infty \left(t^{-3/2 + \varepsilon}\right) \cap O_L^2 \left(t^{-1 + \varepsilon}\right) \quad (1.13)$$

holds as $t \to \infty$.

(b) Suppose also that $F^{''}_k$ are constants, i.e. $F_k(\xi_k) = c_k^2 \xi_k^2 + c_k^0$, $k = 1, 2$. Then the following modified linear scattering formula holds:

$$\lim_{t \to \infty} \|u(t,x) - e^{i\lambda \phi_+(\frac{x}{t}) + i|z_+(\frac{x}{t})| |\xi_1|} e^{iF(D)j} u_+\|_{L^2_x} = 0 \quad (1.14)$$

where

$$u_+(x) := \frac{1}{i \sqrt{c_1 c_2}^2} e^{-i \sum_{i=1}^2 \frac{x_i}{2 c_i} z_+ \left(\frac{x_1}{2 c_1}, \frac{x_2}{2 c_2}\right)} \quad (1.15)$$

**Remark 1.2.** The assumption that $F^{''}_k$ are constants in theorem 1.3 (b) ensures that the integral (4.11) converges in $L^2_x(\mathbb{R}^2)$ as $t \to \infty$. In fact, when $F^{''}_k$ are not constants, there is no apparent decay in $\eta$, and it is difficult to show that (4.11) converges as $t \to \infty$, even in the weak $L^2_x(\mathbb{R}^2)$ topology.

When $\exists \lambda > 0$, (1.2) represents the standard dissipative nonlinear Schrödinger equation. It models the evolution of pulses propagating through optical fibres and in the field of optical fibre...
engineering (see e.g. [1]). In the paper [32], Shimomura first treated the dissipative nonlinear Schrödinger equation (1.2) and established the asymptotic formula of the small solutions for \( n = 1, 2, 3 \). He proved that the dissipative effects are visible not only in the phase correction of the asymptotic profile but also in the decay rate of the solution. In fact, the uniform norm of the solution decays like

\[
\| u(t,x) \|_{L^\infty} \lesssim (t \log t)^{-n/2},
\]

which decays faster than the free solution. After the work of Shimomura [32], there are many papers devoted to studying the decay estimates and the asymptotics of the solutions to the dissipative nonlinear Schrödinger equation, see e.g. [14, 15, 18, 25, 26, 29–31] and references therein.

Inspired by the work of Shimomura [32], we also study the influence of the dissipative effects on the decay estimates and the asymptotics of the solutions to the modified two-dimensional Schrödinger equation. Our results are the following.

**Theorem 1.4.** Suppose that \( \Im \lambda > 0, u_0 \) satisfies (1.8) and \( u \) is the solution obtained in theorem 1.1. Then there exists \( \varepsilon_0 > 0 \), which is independent of \( u_0 \), such that for all \( \varepsilon \in (0, \varepsilon_0) \), the followings hold:

1. Let \( z_+(x) \) be the asymptotic function obtained in theorem 1.2 and

\[
\psi_+(x) = \Im \lambda \int_1^\infty s^{-1} \left( |v_+(s,x)| e^{2\Im \lambda \phi(s,x)} - |z_+(x)| \right) ds, \tag{1.16}
\]

\[
S(t,x) = \frac{1}{3\Im \lambda} \log \left( 1 + \Im \lambda |z_+(x)| \log |t + \psi_+(x)| \right). \tag{1.17}
\]

The asymptotic formula

\[
u(t,x) = \frac{1}{t} e^{i(\psi(t,x) + iS(t,x))} z_+ \left( \frac{x}{t} \right) + O_{L^\infty} \left( t^{-3/2+C}\right) \cap O_{L^2} \left( t^{-1+C}\right) \tag{1.18}\]

holds as \( t \to \infty \).

2. Suppose also that \( F_k^{(i)} \) are constants, then the following modified linear scattering formula holds:

\[
\lim_{t \to \infty} \| u(t,x) - e^{i\lambda S(t,x)} e^{F_k^{(i)}(t)} u_+(x) \|_{L^2} = 0, \tag{1.19}
\]

where \( u_+ \) is the function defined in (1.15).

3. If \( u_0 \neq 0 \), then the following limit exists

\[
\lim_{t \to \infty} (t \log t) \| u(t,x) \|_{L^\infty} = \frac{1}{3\Im \lambda}. \tag{1.20}
\]

**Remark 1.3.** Note that the limit (1.20) is independent of the initial value \( u_0 \). This property was first established in [5] for a class of nonvanishing initial data.

**Remark 1.4.** By the definition of \( S(t,x) \), we can write the modification factor \( e^{i\lambda S(t,x)} \) explicitly:

\[
e^{i\lambda S(t,x)} = \frac{\exp \left( \frac{\Im \lambda}{3\Im \lambda} \log \left( 1 + \Im \lambda |z_+(x)| \log t + \psi_+(x) \right) \right)}{1 + \Im \lambda |z_+(x)| \log t + \psi_+(x)}. \tag{1.21}
\]

We briefly sketch the strategy used to derive our main results. We adopt the semiclassical analysis method introduced by Delort [9], see also [29, 33, 35] which are closer to the problem.
we are considering. We refer also to the method of testing with wave packets introduced by Ifrim and Tataru [19, 20], which is similar to the semiclassical analysis method, in particular in terms of the cut-off to near the propagation set $M$.

We make first a semiclassical change of variables

$$u(t, x) = \frac{1}{t} v\left(\frac{x}{t}, \frac{\xi}{t}\right)$$

(1.22)

for some new unknown function $v$, that allows to rewrite the equation (1.9) as

$$(D_t - G^w_h (x \cdot \xi + F(\xi))) v = \lambda h |v| v,$$

(1.23)

where the semiclassical parameter $h = \frac{1}{\tau}$, and the Weyl quantisation of a symbol $a$ is given by

$$G^w_h (a) u(x) = \frac{1}{(2\pi h)^{3/2}} \int_{\mathbb{R}^3} e^{i(x-y) \cdot \xi / 2} a \left(\frac{x+y}{2}, \xi\right) u(y) \, dy \, d\xi.$$  

The semiclassical change of variables technique was first introduced by Delort in [8] and was later used by Lindblad and Soffer in [27, 28] to study asymptotics of critical equation.

We remark that the operator

$$\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) \quad \text{with} \quad \mathcal{L}_k = \frac{1}{h} G^w_h (x_k + F^*_k(\xi_k)), \quad k = 1, 2.$$  

(1.24)

commutes exactly to the linear part of the equation (1.23) and an approximate Leibniz rule holds for the action of $\mathcal{L}$ on $|v| v$. Actually, by (1.5) and (1.7), the quotient $e_k(x, \xi) = \frac{\pi + F_k^*(\xi_k)}{\xi_k - d\phi_k(x_k)}$ is smooth and $|e_k(x, \xi)|$ stays between two positive constants. Consequently, one may write, using symbolic calculus for semiclassical operators

$$\mathcal{L}_k = \frac{1}{h} G^w_h (e_k (\xi_k - d\phi_k(x_k))) = G^w_h (e_k) \left[ \frac{1}{h} G^w_h (\xi_k - d\phi_k(x_k)) \right] + G^w_h (r_k), k = 1, 2$$

(1.25)

with some other symbol $r_k$. If one makes act the main contribution on the right-hand side of the above equality on $|v| v$, one gets ($D_k = \frac{\partial}{\partial x_k}$)

$$G^w_h (e_k) \left[ \left( D_k - \frac{1}{h} d\phi_k(x_k) \right) (|v| v) \right]$$

$$= G^w_h (e_k) \left[ \frac{3}{2} |v| \left( D_k - \frac{1}{h} d\phi_k(x_k) \right) v - \frac{1}{2} |v|^{-1} v^2 \left( D_k - \frac{1}{h} d\phi_k(x_k) \right) v \right].$$

(1.26)

that is a quantity whose $L^2(\mathbb{R}^3)$ norm may be bounded by $C |v|_{L^\infty} \|L v\|_{L^2}$ (if one re-expresses on the right-hand side ($D_k - \frac{1}{h} d\phi_k(x_k)$) $v$ from $L_k v$). In other words, $\mathcal{L}$ obeys a Leibniz rule when acting on $|v| v$. Repeating this procedure, one sees that the nonlinearity $|v| v$ not only obey a Leibnitz rule for $\mathcal{L}$ but also for the operator $L^2 = : (L^2_1, L^2_2)$ (see lemma 3.2)

$$\|L^2 (|v| v)\|_{L^2} \lesssim \|v\|_{L^\infty} \left( \|L^2 v\|_{L^2} + \|L^2 v\|_{L^2} \right).$$

(1.27)

Applying the operator $L^2$ to (1.23), then using (1.27), one obtains an energy inequality of the form

$$\|L^2 v(\tau, \cdot)\|_{L^2} \lesssim \sum_{k=1}^2 \| (x_k + F^*_k(D))^2 u_0 \|_{L^2} + \int_0^\tau \|v(\tau, \cdot)\|_{L^\infty} \left( \|v(\tau, \cdot)\|_{L^2} + \|L^2 v(\tau, \cdot)\|_{L^2} \right) \frac{d\tau}{\tau}.$$  

(1.28)
If one has an \textit{a priori} estimate $\|v(\tau, \cdot)\|_{L^\infty} = O(\varepsilon)$, Gronwall lemma provides for the left-hand side of (1.28) a $O(\varepsilon^2)$ bound.

On the other hand, one can establish from an \textit{a priori} $\|L^2 v(t, \cdot)\|_{L^2} = O(\varepsilon^2)$ an $L^\infty$ estimate for $v$, by deducing from (1.23) an ODE satisfied by $v$. Actually, if one develops the symbol $x \cdot \xi + F(\xi)$ at $\xi = d \phi(x)$, one gets, using that $\nabla_\xi (x \cdot \xi + F(\xi)) |_{\xi = d \phi(x)} = 0$ (see (1.7)),

$$x \cdot \xi + F(\xi) = w(x) + \sum_{k=1}^{2} \int_{0}^{1} F''_k(\theta \xi_k + (1 - \theta) d \phi_k(x_k))(1 - \theta) d \theta (\xi_k - d \phi_k(x_k))^2,$$

(1.29) where $w(x) = x \cdot d \phi(x) + F(d \phi(x))$. Taking the Weyl quantisation and using the symbol calculus, one deduces from (1.23) an ODE for $v$

$$D_t v = w(x) v + \lambda h |v| v + h^2 \sum_{k=1}^{2} G_{h}^k(b_k) \circ L^2_k v,$$

(1.30) where $b_k, k = 1, 2$ are some other symbols. Assume for a while that one can deduce from the \textit{a priori} estimate $\|L^2 v(t, \cdot)\|_{L^2} = O(\varepsilon^2)$ that $\|h^2 \sum_{k=1}^{2} G_{h}^k(b_k) L^2_k v\|_{L^\infty}$ is time integrable, one derives from the ODE (1.30) a uniform $L^\infty$ control of $v$. Putting together these $L^2$ and $L^\infty$ estimates and performing a bootstrap argument, one finally shows that (1.23) has global solutions and determines their asymptotic behaviour.

To estimate $\|h^2 \sum_{k=1}^{2} G_{h}^k(b_k) L^2_k v\|_{L^\infty}$ one would be tempted to use the semiclassical Sobolev inequality. Note however that this would lead to an energy norm of $v$ containing the spatial derivative of order three, and we cannot recover this norm from the equation (1.23) via the standard energy method due to the lack of the regularity of $|v|v$. We instead use the operators whose symbols are localised in a neighbourhood of $M := \{(x, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 : x + F'(\xi) = 0\}$ of size $O(\sqrt{h})$. In that way, we can apply proposition 2.3 to pass uniform norms of the remainders to the $L^3$ norm. More precisely, we set

$$v_\Lambda = G_{h}^\Lambda \left( \gamma \frac{x + F'(\xi)}{\sqrt{h}} \right) v,$$

(1.31) where $\gamma \in C^\infty_c(\mathbb{R}^2)$ satisfying $\gamma = 1$ in a neighbourhood of zero. Applying $G_{h}^\Lambda \left( \gamma \frac{x + F'(\xi)}{\sqrt{h}} \right)$ to the equation (1.23), one obtains the ODE for $v_\Lambda$

$$D_t v_\Lambda = w(x) v_\Lambda + \lambda h |v| v_\Lambda + R(v),$$

(1.32) where the remainder

$$R(v) = \left[ D_t - G_{h}^\Lambda (x \cdot \xi + F(\xi)) \right] G_{h}^\Lambda \left( \gamma \frac{x + F'(\xi)}{\sqrt{h}} \right) v + G_{h}^\Lambda (x \cdot \xi + F(\xi) - w(x)) v_\Lambda - \lambda h G_{h}^\Lambda \left( 1 - \gamma \frac{x + F'(\xi)}{\sqrt{h}} \right) (|v|v + \lambda h (|v|v - |v_\Lambda|v_\Lambda)).$$

To derive from (1.32) and the \textit{a priori} estimate of the form $\|L^2 v(t, \cdot)\|_{L^2} = O(\varepsilon^2)$ a uniform $L^\infty$ control of $v$, we need to show that $R(v)$ is a time integrable error term and $v_\Lambda := v - v_\Lambda$ decays faster than the main part $v_\Lambda$. The proof of these two estimates will strongly use the semiclassical pseudo-differential calculus and constitutes the most technical part of the paper.

The framework of this paper is organised as follows. In section 2, we will present the definitions and properties of Semiclassical pseudo-differential operators. In section 3, we use the Strichartz estimate and the semiclassical analysis method to prove theorem 1.1. In section 4, we
prove theorems 1.2–1.4 by deducing the long-time behavior of solutions from the associated ODE dynamics. Finally, in the appendix, we give the proof of lemma 3.2.

2. Semiclassical pseudo-differential operators

In order to prove an $L^\infty$ estimate on $u$ we need to reformulate the starting problem (1.9) in terms of an ODE satisfied by a new function $v$ obtained from $u$, and this will strongly use the semiclassical pseudo-differential calculus. In this section, we introduce this semiclassical environment, defining classes of symbols and operators we shall use and several useful properties. A general reference is chapter 7 of the book of Dimassi and Sjöstrand [10] or chapter 4 of the book of Zworski [36].

**Definition 2.1.** An order function on $\mathbb{R}^2 \times \mathbb{R}^2$ is a smooth map from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}_+$ $(x, \xi) \to M(x, \xi)$ such that there are $N_0 \in \mathbb{N}$ and $C > 0$,

$$M(y, \eta) \leq C|y|^N_0 (\xi - \eta)^N_0 M(x, \xi),$$

holds for any $(x, \xi), (y, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$.

**Definition 2.2.** Let $M$ be an order function on $\mathbb{R}^2 \times \mathbb{R}^2, \delta \geq 0$. One denotes by $S_M(M)$ the space of smooth functions $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, 1) \to \mathbb{C}$ satisfying for any $\alpha, \beta \in \mathbb{N}^2$

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha, \beta} M(x, \xi) h^{-|\alpha| - |\beta|} \delta.$$

**Remark 2.1.** In the rest of this paper we shall not indicate explicitly the dependence of symbols in $h$.

Recall that $d\phi_k(x_k), k = 1, 2$ are the functions defined in (1.7). Let

$$e_k(x, \xi) = \frac{x_k + F_k'(\xi_k)}{\xi_k - d\phi_k(x_k)}, \quad \tilde{e}_k(x, \xi) = \frac{\xi_k - d\phi_k(x_k)}{x_k + F_k'(\xi_k)}, \quad k = 1, 2. \quad (2.2)$$

By definition 2.2, we have the following lemma.

**Lemma 2.1.** Assume that $e_k(x, \xi), \tilde{e}_k(x, \xi), d^2\phi_k(x_k), k = 1, 2$ are the functions defined in (1.7) and (2.2). Then we have $e_k(x, \xi), \tilde{e}_k(x, \xi), d^2\phi_k(x_k) \in S_0(1), k = 1, 2$.

**Proof.** Taking a derivative of $x_k + F_k'(d\phi_k(x_k)) = 0$, one gets by using (1.5)

$$|d^2\phi_k(x_k)| = \left| - \frac{1}{F_k''(d\phi_k(x_k))} \right| \lesssim 1.$$  

Moreover, by induction and (1.5), one gets $|d^j\phi_k(x_k)| \lesssim 1, \forall j \geq 2$, i.e. $d^2\phi_k(x_k) \in S_0(1)$. On the other hand, since $x_k + F_k'(d\phi_k(x_k)) = 0$, one can rewrite $e_k$ as following

$$e_k(x, \xi) = \frac{F_k'(\xi_k) - F_k'(d\phi_k(x_k))}{\xi_k - d\phi_k(x_k)} = \int_0^1 F_k''(t\xi_k + (1 - t)d\phi_k(x_k)) dt, \quad k = 1, 2. \quad (2.3)$$

Using $d^2\phi_k(x_k) \in S_0(1)$ and (2.3) we can easily check that $e_k \in S_0(1), k = 1, 2$. Similarly, we can get $\tilde{e}_k \in S_0(1), k = 1, 2$ and omit the details. \qed

**Remark 2.2.** Lemma 2.1 is the key to ensuring that the approximate Leibnitz rule (1.25)–(1.27) hold, and it is here that specific form (1.4) needs to be assumed. In fact, if $\partial_{\xi_k}\partial_{\xi_1} F(\xi) \neq 0$, then

$$e_1(x, \xi) = \frac{x_1 + F_1'(\xi_1)}{\xi_1 - d\phi_1(x)} = \frac{F_1'(\xi_1) - F_1'(d\phi_1(x))}{\xi_1 - d\phi_1(x)}.$$  

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Let $M$ be an order function on $\mathbb{R}^2 \times \mathbb{R}^2$, $\delta \geq 0$. From (2.6) where the Moyal product of the symbols is defined by

$$a \ast b(x, \xi) = \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i \pi (a + b)} a(x + z, \xi + \zeta) b(x + y, \xi + \eta) dyd\eta dzd\zeta. \quad (2.5)$$

It is often useful to derive an asymptotic expansion for $a \ast b$, as it allows easier computations than the integral formula (2.5). For the one-dimensional case, we refer to the appendix of [33].

**Proposition 2.3.** Let $M$ be an order function on $\mathbb{R}^2 \times \mathbb{R}^2$, $\delta \geq 0$. For any $a, b \in S_\delta(M)$

$$a \ast b(x, \xi) = ab(x, \xi) + \frac{i\hbar}{2} \sum_{j=1}^{2} \left| \frac{\partial a(x, \xi)}{\partial \xi_j} \frac{\partial b(x, \xi)}{\partial \xi_j} \right| + \mu_{ab},$$

where

$$\mu_{ab} = -\frac{\hbar^2}{2} \sum_{\alpha = (\alpha_1, \alpha_2)} |\alpha| \left(1 - |\alpha| \right)^{\frac{1}{4}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i \pi (a + b)} \left( \int_{0}^{1} \partial^{\alpha_1} \partial^{\alpha_2} a(x + tz, \xi + t\zeta) dzd\zeta \right) \right).$$

**Proof.** From (2.5) and Taylor’s formula

$$a(x + z, \xi + \zeta) = a(x, \xi) + \sum_{\alpha = (\alpha_1, \alpha_2)} \partial^{\alpha_1} \partial^{\alpha_2} a(x, \xi) \zeta^{\alpha_1} \zeta^{\alpha_2}$$

$$|\alpha| = 1$$

$$+ \sum_{\alpha = (\alpha_1, \alpha_2)} |\alpha| \left(1 - |\alpha| \right)^{\frac{1}{4}} \int_{0}^{1} \partial^{\alpha_1} \partial^{\alpha_2} a(x + tz, \xi + t\zeta) \zeta^{\alpha_1} \zeta^{\alpha_2} (1 - t) dz.$$
we have
\[ a \hat{g} b (x, \xi) = \frac{1}{(\pi h)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i \hat{\mathfrak{F}}(\eta - \gamma, \zeta)} a(x, \xi) b(x + y, \xi + \eta) \, dy \, dz \, d\zeta + \frac{1}{(\pi h)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i \hat{\mathfrak{F}}(\eta - \gamma, \zeta)} \sum_{\alpha = (\alpha_1, \alpha_2)} | \alpha | \, dy \, dz \, d\zeta \]
\[ | \alpha | = 1 \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} a(x, \xi) b(x + y, \xi + \eta) \hat{\mathfrak{F}}(\eta - \gamma, \zeta) \]
\[ + \frac{1}{(\pi h)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i \hat{\mathfrak{F}}(\eta - \gamma, \zeta)} \sum_{\alpha = (\alpha_1, \alpha_2)} | \alpha | \, dy \, dz \, d\zeta \]
\[ | \alpha | = 2 \frac{2}{\alpha_1} \int_{\mathbb{R}^3} \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} a(x + \eta, \xi + \eta) \hat{\mathfrak{F}}(\eta - \gamma, \zeta) \]
\[ \times (1 - t) dt \, db (x + y, \xi + \eta) \, dy \, dz \, d\zeta \]
\[ =: I_1 + I_2 + I_3. \]

By direct computation, we get
\[ I_1 = a(x, \xi) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(x + y, \xi + \eta) \hat{\delta}_0(y) \hat{\delta}_0(\eta) \, dy \, d\eta = a(x, \xi) b(x, \xi), \]
where \( \hat{\delta}_0 \) is the Dirac function. Integrate \( I_2 \) by parts via the identity
\[ \hat{\mathfrak{F}}(\eta - \gamma, \zeta) = \frac{h}{2i} (1) | \alpha_1 | \partial_\eta^{\alpha_1} \partial_\gamma^{\alpha_2} e^{i \hat{\mathfrak{F}}(\eta - \gamma, \zeta)}, \quad | \alpha_1 | + | \alpha_2 | = 1 \]
to get
\[ I_2 = \frac{1}{(\pi h)^3} \sum_{\alpha = (\alpha_1, \alpha_2)} | \alpha | \]
\[ | \alpha | = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{h}{2i} (1) | \alpha_1 | \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} a(x, \xi) \]
\[ \times \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} b(x + y, \xi + \eta) \, dy \, dz \, d\zeta \]
\[ = \frac{h}{2} \sum_{\alpha = (\alpha_1, \alpha_2)} | \alpha | \]
\[ | \alpha | = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{h}{2i} (1) | \alpha_1 | \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} a(x, \xi) \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} b(x, \xi) = \frac{i h}{2} \sum_{j=1}^2 \left| \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} a \right| \]

The same calculation shows that \( I_3 \) is given by the right-hand side of (2.6). This completes the proof of proposition 2.3.

In particular, from proposition 2.3, we have that
\[ (x_k + F_k' (\xi_k))^2 = (x_k + F_k' (\xi_k)) \hat{\mathfrak{F}}(x_k + F_k' (\xi_k)), \quad k = 1, 2. \]

Taking the Weyl quantisation, then using proposition 2.2, we see that
\[ \mathcal{L}_k^2 = \frac{1}{h^2} G_h \left( (x_k + F_k' (\xi_k))^2 \right), \quad k = 1, 2. \]
which will be used frequently in the rest of the paper without reference.

We will use the following boundedness of the Weyl quantisation.

**Proposition 2.4 (Theorem 7.11 in [10]).** Let \( a \in S_b(1), \delta \in [0, \frac{1}{2}], \) then

\[
\| G_h^n(a) \|_{L^2(L^2)} \leq C, \text{ for all } h \in (0, 1].
\]

A key role in this paper will be played by symbols \( a \) verifying (2.1) with \( M(x, \xi) = \left(1 + \frac{x + \sqrt{h} \xi}{\sqrt{h}}\right)^{\frac{\alpha}{2}} \), for \( N \in \mathbb{N} \). Although \( M(x, \xi) \) is no longer an order function because of the term \( h^\frac{\alpha}{2} \), it is useful to keep the notation \( a \in S_b(M) \) whenever \( a, M \) verify (2.1).

**Proposition 2.5.** Let \( a \in S_b(\left(1 + \frac{x + \sqrt{h} \xi}{\sqrt{h}}\right)^{\frac{\alpha}{2}}), \delta \in [0, \frac{1}{2}], \) then

\[
\| G_h^n(a) \|_{L^2(L^\infty)} \lesssim h^{-\frac{1}{2}}, \text{ for all } h \in (0, 1].
\]

**Proof.** By the definition of the operator \( G_h^n(a) \) in (2.4), we have

\[
G_h^n(a)f(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(\xi \cdot y)} \xi a\left(\frac{x + \sqrt{h} y}{2}, \sqrt{h} \xi\right)f\left(\sqrt{h} y\right) dy d\xi
\]

\[
= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} h^{-1}f\left(\frac{\eta}{\sqrt{h}}\right) d\eta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(\xi + \eta \cdot y)} \xi + i\eta \gamma a\left(\frac{x + \sqrt{h} y}{2}, \sqrt{h} \xi\right) dy d\xi.
\]

(2.8)

Since \( a \in S_b(\left(1 + \frac{x + \sqrt{h} \xi}{\sqrt{h}}\right)^{\frac{\alpha}{2}}) \), we have that for \( \forall \alpha, \beta \in \mathbb{N}^2 \)

\[
|\partial_\alpha \partial_\beta a\left(\frac{x + \sqrt{h} y}{2}, \sqrt{h} \xi\right)| \lesssim h^{\left(\frac{\alpha}{2} - \delta \right)|\alpha| + |\beta|} \left(\frac{x + \sqrt{h} y}{2} + F\left(\frac{\sqrt{h} \xi}{\sqrt{h}}\right)\right)^{-2}
\]

\[
\lesssim \left(\frac{x + \sqrt{h} y}{\sqrt{h}} + F\left(\frac{\sqrt{h} \xi}{\sqrt{h}}\right)\right)^{-2}.
\]

Integrating (2.8) by parts via the identity

\[
\left(1 - i \frac{\xi - \eta}{\sqrt{h}} \cdot \partial_\xi \right)^3 \left(1 + i \frac{\xi - \eta}{\sqrt{h}} \cdot \partial_\xi \right)^3 e^{i(\xi + \eta \cdot y) \xi + i\eta \gamma} = e^{i(\xi - \eta) \xi + i\eta \gamma}
\]

and using (2.9), we get

\[
|G_h^n(a)f(x)| \lesssim h^{-1} \int_{\mathbb{R}^2} \left|f\left(\frac{\eta}{\sqrt{h}}\right)\right| d\eta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\frac{x + \sqrt{h} y}{\sqrt{h}} - y\right)^{-3} (\xi - \eta)^{-3}
\]

\[
\times \left(\frac{x + \sqrt{h} y}{\sqrt{h}} + F\left(\frac{\sqrt{h} \xi}{\sqrt{h}}\right)\right)^{-2} d\gamma.
\]

(2.10)

On the other hand, by Young’s inequality, we have
\[
\left\| \int (\xi - \eta)^{-3} \left\langle \frac{x + \sqrt{\xi}}{\sqrt{h}} + F'\left(\sqrt{\xi}\right) \right\rangle \right\|_{L^2_k}^{-2} \lesssim \left\| (\eta)^{-3} \left\langle \frac{x + \sqrt{\xi}}{\sqrt{h}} + F'\left(\sqrt{h}\xi\right) \right\rangle \right\|_{L^2_k}^{-2} \lesssim 1, \tag{2.11}
\]
where the last inequality holds since by (1.5) and (1.7)
\[
\left\| \left\langle \frac{x + \sqrt{\xi}}{\sqrt{h}} + F'\left(\sqrt{\xi}\right) \right\rangle \right\|_{L^2_k}^{-2} = \left\| \left\langle F'\left(\sqrt{h}\xi\right) - F'\left(\sqrt{\xi}\right) \right\rangle \right\|_{L^2_k}^{-2} \lesssim \left\| \xi - \frac{d\phi}{\sqrt{h}} \left(\frac{x + \sqrt{\xi}}{\sqrt{h}}\right) \right\|_{L^2_k}^{-2} \lesssim 1.
\]
Applying Cauchy–Schwarz’s inequality to (2.10) and using (2.11), we obtain
\[
\left\| G^0_h(a)(\xi) \right\| \lesssim h^{-1} \left\| \int \left(\eta \frac{\eta}{\sqrt{h}}\right) \int_{\mathbb{R}^2} \left\langle x - \eta \right\rangle^{3} \, dy \right\| \lesssim h^{-1/2} \left\| f \right\|_{L^2}.
\]
This completes the proof of proposition 2.5.

**Proposition 2.6.** Let \( h \in [0, 1] \) and \( a(\xi) \) be a smooth function that satisfies \( |\partial^\alpha \xi a(\xi)| \lesssim C_\alpha (\xi)^{-2-|\alpha|} \) for any \( \alpha \in \mathbb{N}^2 \). Then
\[
\left\| G^0_h\left( a\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \right) \right\|_{L^2(L^\infty)} = O\left(h^{-\frac{1}{2}}\right), \quad \left\| G^0_h\left( a\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \right) \right\|_{L^2(L^2)} = O(1).
\]

**Proof.** Since \( a\left(\frac{x + F'(\xi)}{\sqrt{h}}\right) \in S^1_k\left(\frac{x + F'(\xi)}{\sqrt{h}}\right)^{-2} \cap S^2_k(1) \), from propositions 2.4–2.5, we have the estimates in proposition 2.6.

The rest of this section is devoted to establishing some technical results in the semiclassical framework. More precisely, lemmas 2.2–2.4 will be often recalled to compute the composition of the symbols. While lemmas 2.5–2.7 deal with the boundedness of the operators and the commutators of the symbols, which will be used to prove lemmas 3.6 and 3.7.

**Lemma 2.2.** Suppose that \( a_k(x, \xi) \in S_0(1), \ 1 \leq k \leq 2 \). There exist symbols \( b_k(x, \xi), \ c_k(x, \xi), \ d_k(x, \xi) \in S_0(1), \ 1 \leq k \leq 2 \) such that
\[
a_k(x, \xi) (x_k + F'_k(\xi_3)) = a_k(x, \xi) (x_k + F'_k(\xi_3)) + h b_k(x, \xi), \tag{12.12}
\]
and
\[
a_k(x, \xi) (x_k + F'_k(\xi_3)) = a_k(x, \xi) (x_k + F'_k(\xi_3)) + h c_k(x, \xi) + h d_k(x, \xi), \tag{12.13}
\]
for \( k = 1, 2 \). As a corollary, we have the following estimates for the commutators
\[
\left\| [L_k, G^0_k(a_k(x, \xi))] v \right\|_{L^2_k} \lesssim \left\| v \right\|_{L^2_k}, \quad \left\| \left[ L^2_k, G^0_k(a_k(x, \xi)) \right] v \right\|_{L^2_k} \lesssim \left\| L_k v \right\|_{L^2_k} + \left\| v \right\|_{L^2_k}, \ \ k = 1, 2. \tag{12.14}
\]

**Proof.** We give only the proof for \( k = 1 \); the case for \( k = 2 \) can be treated similarly. An application of proposition 2.3 yields
\[
a_k(x, \xi) (x_k + F'_k(\xi_3)) = a_1(x, \xi) (x_k + F'_k(\xi_3)) + \frac{i h}{2} H_k(x, \xi) + p_k(x, \xi) F''_k(\xi_3),
\]

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where $H_1(x, \xi) = \partial_x a_1(x, \xi) F'_1(\xi) - \partial_\xi a_1(x, \xi) \in S_0(1)$ by (1.5). For (2.12), it remains to show that $\rho^{\mu_{(x, \xi)}(x, F'_1(\xi))} \in h^2 S_0(1)$. By $\int_{\mathbb{R}^2} e^{-\frac{\pi}{h^2} \xi^2} dy = (\pi h)^2 \delta_0(\xi)$, we get

\[
\rho^{\mu_{(x, \xi)}(x, F'_1(\xi))} = \frac{h^2}{4 \pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\pi}{h^2} (\eta \cdot \zeta)} \frac{1}{1 + 4|\eta|^2} e^{2|\eta|^2 z} (1 - 2i \eta \cdot \xi) \, d\eta d\zeta = h^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\xi)^{-3} (\eta)^{-3} \, d\eta d\zeta \lesssim h^2.
\]

Using the identity

\[
\left( \frac{1}{1 + 4|\eta|^2} \right)^3 \left( \frac{1 - 2i \eta \cdot \xi}{1 + 4|\eta|^2} \right)^3 e^{2|\eta|^2 z} = e^{2|\eta|^2 z}
\]

to integrate by parts, we obtain

\[
|\rho^{\mu_{(x, \xi)}(x, F'_1(\xi))}| \lesssim h^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\xi)^{-3} (\eta)^{-3} \, d\eta d\zeta \lesssim h^2.
\]

Similarly, we have

\[
|\partial_\eta a_1 \rho^{\mu_{(x, \xi)}(x, F'_1(\xi))}| \leq C_{a, \beta} h^2 \quad \forall a, \beta \in \mathbb{N}_2,
\]

i.e. $\rho^{\mu_{(x, \xi)}(x, F'_1(\xi))} \in h^2 S_0(1)$. This completes the proof of (2.12). Now we proceed to prove (2.13). By proposition 2.3

\[
a_1(x, \xi) F'_1(\xi) = a_1(x, \xi) (x_1 + F'_1(\xi))^2 + i h H_2(x, \xi) (x_1 + F'_1(\xi)) + \rho^{\mu_{(x, \xi)}(x, F'_1(\xi))} = a_1(x, \xi) \xi (x_1 + F'_1(\xi)) + \rho^{\mu_{(x, \xi)}(x, F'_1(\xi))},
\]

where $H_2(x, \xi) = \partial_x a_1(x, \xi) F''_1(\xi) - \partial_\xi a_1(x, \xi) \in S_0(1)$ by (1.5). Using (2.12), we can write, for some symbol $a(x, \xi) \in S_0(1)$

\[
ih H_2(x, \xi) (x_1 + F'_1(\xi)) = ih H_2(x, \xi) \xi (x_1 + F'_1(\xi)) + h^2 a(x, \xi).
\]

Therefore, for (2.13) it suffices to show that $\rho^{\mu_{(x, \xi)}(x, F'_1(\xi))} \in h^2 S_0(1)$ by (2.6). By (6), we have

\[
\rho^{\mu_{(x, \xi)}(x, F'_1(\xi))} = -\frac{h^2}{2} \left( \frac{1}{\pi h} \right)^3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\pi}{h^2} (\eta \cdot \zeta)} \frac{1}{1 + 4|\eta|^2} e^{2|\eta|^2 z} (1 - 2i \eta \cdot \xi) \, d\eta d\zeta \\
\times \left[ \left( F''_1(\xi + \eta_1) \right)^2 + (x_1 + y_1 + F'_1(\xi_1 + \eta_1)) \right] d\eta_1 d\zeta \\
+ \frac{h^2}{\pi h^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\pi}{h^2} (\eta \cdot \zeta)} \frac{1}{1 + 4|\eta|^2} \, d\eta_1 d\zeta \\
\times \partial_\xi \partial_\eta a_1 (x_1 + F'_1(\xi_1 + \eta_1)) \, d\eta_1 d\zeta \\
- \frac{h^2}{2} \left( \frac{1}{\pi h} \right)^3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\pi}{h^2} (\eta \cdot \zeta)} \frac{1}{1 + 4|\eta|^2} e^{2|\eta|^2 z} (1 - 2i \eta \cdot \xi) \, d\eta d\zeta.
\]

It is easy to see that the right-hand side of (2.19) belongs to the symbol class $h^2 S_0(1)$ by the similar argument as in the proof of (2.16) and (2.17), except the term
\[ R(x, \xi) = -\frac{\hbar^2}{2} \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^3} e^{\frac{i}{\hbar} (\eta \cdot x - \xi \cdot y)} \int_0^1 \partial_x^3 a_1 (x + t\eta, \xi + t\zeta) \left(1 - t\right) dt \]
\[ \times \left(x_1 + y_1 + F'_1 (\xi_1 + \eta_1) \right) F''''_1 (\xi_1 + \eta_1) \, dyd\eta dzd\zeta. \]

By mean value theorem
\[ x_1 + y_1 + F'_1 (\xi_1 + \eta_1) = x_1 + F'_1 (\xi_1) + \eta_1 \int_0^1 F''''_1 (\xi_1 + s\eta_1) \, ds + y_1, \]
we decompose \( R(x, \xi) \) accordingly into three parts
\[ R(x, \xi) = A(x, \xi) (x_1 + F'_1 (\xi_1)) + R_2 (x, \xi) + R_3 (x, \xi). \]
Integrate \( R_2 \) via the identity \( \eta_1 e^{\frac{i}{\hbar} (\eta \cdot x - \xi \cdot y)} = \frac{\hbar}{2} \partial_x e^{\frac{i}{\hbar} (\eta \cdot x - \xi \cdot y)} \) to get
\[ R_2 (x, \xi) = \frac{\hbar^3}{4i} \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{i}{\hbar} (\eta \cdot x - \xi \cdot y)} \int_0^1 \partial_x^3 a_1 (x + t\eta, \xi + t\zeta) \left(1 - t\right) dt \]
\[ \times F''''_1 (\xi_1 + \eta_1) \int_0^1 F''''_1 (\xi_1 + s\eta_1) \, dsdyd\eta dzd\zeta, \]
which belongs to the symbol class \( \hbar^3 S_0(1) \) by the similar argument as in the proof of (2.16) and (2.17). Moreover, a similar argument shows that \( A(x, \xi), R_3(x, \xi) \in \hbar^2 S_0(1) \); so that by (2.12)
\[ A(x, \xi) (x_1 + F'_1 (\xi_1)) + R_2 (x, \xi) + R_3 (x, \xi) = \hbar \tilde{a}(x, \xi) \tilde{z} (x_1 + F'_1 (\xi_1)) + \hbar^2 \tilde{b}(x, \xi), \]
for some symbols \( \tilde{a}(x, \xi), \tilde{b}(x, \xi) \in S_0(1) \). This finishes the proof of (2.13).

Similarly, by calculating \( (x_1 + F'_k (x_1)) \tilde{z} a_k (x, \xi) \) and \( (x_1 + F'_k (x_1))^2 \tilde{z} a_k (x, \xi) \), we get
\[ a_k (x, \xi) (x_1 + F'_k (x_1)) \tilde{z} a_k (x, \xi) = (x_1 + F'_k (x_1)) \tilde{z} a_k (x, \xi) + \hbar \tilde{b}_k (x, \xi), \]
\[ a_k (x, \xi) (x_1 + F'_k (x_1))^2 \tilde{z} a_k (x, \xi) + \hbar \tilde{a}_k (x, \xi) \tilde{z} (x_1 + F'_k (x_1)) \]
\[ + \hbar^2 \tilde{d}_k (x, \xi), \]
for some symbols \( \tilde{b}_k (x, \xi), \tilde{c}_k (x, \xi), \tilde{d}_k (x, \xi) \in S_0(1) \). Combining (2.12) and (2.13) and (2.20) and (2.21), we obtain
\[ (x_1 + F'_k (\xi_1)) \tilde{z} a_k - a_k \tilde{z} (x_1 + F'_k (\xi_1)) = \hbar \left( b_k - \tilde{b}_k \right), \]
\[ (x_1 + F'_k (\xi_1))^2 \tilde{z} a_k - a_k \tilde{z} (x_1 + F'_k (\xi_1))^2 = \hbar \left[ c_k \tilde{z} (x_1 + F'_k (\xi_1)) - \tilde{c}_k (x_1 + F'_k (\xi_1)) \right] + \hbar^2 \left( d_k - \tilde{d}_k \right). \]
Taking the Weyl quantisation, recalling the definition (1.24), it follows that
\[ [\mathcal{L}_k, G_k^v (a_k)] v = G_k^v \left( b_k - \tilde{b}_k \right) v, \]
\[ [\mathcal{L}_k^v, G_k^v (a_k)] v = G_k^v (c_k) \circ \mathcal{L}_k v - G_k^v (\tilde{c}_k) \circ \mathcal{L}_k v + G_k^v \left( d_k - \tilde{d}_k \right) v, \]
for \( k = 1, 2 \). An application of proposition 2.4 then yields the desired estimates (2.14).

Using the same argument as in the proof of lemma 2.2, we get the following lemma easily and omit the details.

\[ \square \]
Lemma 2.3. Suppose that $a_k(x, \xi) \in S_0(1)$, $1 \leq k \leq 2$. There exist symbols $b_k(x, \xi), c_k(x, \xi), d_k(x, \xi) \in S_0(1)$, $1 \leq k \leq 2$ such that

$$a_k(x, \xi)(\xi_k - d\phi_k(x_k)) = a_k(x, \xi)\hat{z}(\xi_k - d\phi_k(x_k)) + h\hat{b}_k,$$

and

$$a_k(x, \xi)(\xi_k - d\phi_k(x_k))^2 = a_k(x, \xi)\hat{z}(\xi_k - d\phi_k(x_k))^2 + h\hat{c}_k\hat{z}(\xi_k - d\phi_k(x_k)) + h^2\hat{d}_k,$$

for $k = 1, 2$.

Lemma 2.4. Suppose that $\Gamma(\xi)$ is a smooth function that satisfies $\Gamma \equiv 0$ in a neighbourhood of zero and $|\partial^{\alpha_0} \Gamma(\xi)| \leq C_\alpha \langle \xi \rangle^{-2-|\alpha|}$ for any $\alpha \in \mathbb{N}^2$. There exist symbols $a_0(x, \xi), b_0(x, \xi) \in S_1(\frac{\langle x + F'(\xi) \rangle}{\sqrt{\eta}})^{-2}$, $1 \leq k, j \leq 2$, such that

$$\Gamma \left( \frac{x + F'(\xi)}{\sqrt{\eta}} \right) (x_k + F_k(\xi_k))^2 = \left( \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\eta}} \right) \hat{z}(x_k + F_k(\xi_k)) \right)^2 + \sum_{j=1}^{2} (ha_0(x, \xi)\hat{z}(x_j + F'_j(\xi_j)) + h\hat{b}_0(x, \xi)),$$

for $k = 1, 2$.

Proof. We consider only the case $k = 1$; the case $k = 2$ follows from a similar argument. An application of proposition 2.3 yields

$$\left( \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\eta}} \right) \hat{z}(x_1 + F'_1(\xi_1)) \right)^2 = \left( \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\eta}} \right) (x_1 + F'_1(\xi_1)) \right)^2 - \frac{h}{4} a_0(x, \xi)\hat{z}(x_1 + F'_1(\xi_1)),

$$

where

$$a_0(x, \xi) = \frac{1}{(\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\hat{z}(x + z - y - \zeta)}{\sqrt{\eta}}} \eta e^{\frac{x + tz + F'(\xi + \zeta)}{\sqrt{\eta}}} \left( \partial^2_{\xi_1} \Gamma \left( \frac{x + tz + F'(\xi + \zeta)}{\sqrt{\eta}} \right) \right) (1 - t) \, dz \, dy \, dx \, dw = \frac{1}{(\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\hat{z}(x + \eta)}{\sqrt{\eta}}} \eta e^{\frac{x + tz + F'(\xi)}{\sqrt{\eta}}} \left( \partial^2_{\xi_1} \Gamma \left( \frac{x + tz + F'(\xi)}{\sqrt{\eta}} \right) \right) (1 - t) \, dz \, dy \, dx \, dw.$$
By making a change of variables and then integrating by parts, we get
\[
|a_0(x, \xi)| = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( 1 - \frac{2\imath z \cdot \partial_\eta}{1 + 4|\xi|^2} \right)^5 \left( 1 - \frac{2\imath \eta \cdot \partial_z}{1 + 4|\eta|^2} \right)^5 e^{2\imath \eta \cdot z} \\
\times \int_0^1 \left( \partial_\xi^2 \Gamma \right) \left( \frac{x + F'(\xi) + \imath t z}{\sqrt{\imath}} + tz \right) (1 - t) dt F_1''' \left( \xi_1 + \sqrt{\imath} \eta \right) d\eta dz \\
\lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle z \rangle^{-5} \langle \eta \rangle^{-5} \int_0^1 \left\langle \frac{x + F'(\xi)}{\sqrt{\imath}} + tz \right\rangle^{2} (1 - t) dtd\eta dz \\
\lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle z \rangle^{-3} \langle \eta \rangle^{-5} \left\langle \frac{x + F'(\xi)}{\sqrt{\imath}} \right\rangle^{2} d\eta dz \lesssim \left\langle \frac{x + F'(\xi)}{\sqrt{\imath}} \right\rangle^{-2},
\]
(2.27)
where in the second inequality we used the elementary inequality
\[
\langle x + y \rangle^{-2} \lesssim \langle x \rangle^2 \langle y \rangle^{-2}, \quad \forall x, y \in \mathbb{R}^2.
\]
Similarly, we have
\[
|\partial_\xi^2 \partial_\xi \beta a_0(x, \xi)| \leq C_{\alpha, \beta} h^{-\frac{\beta}{2}(|\alpha| + |\beta|)} \left\langle \frac{x + F'(\xi)}{\sqrt{\imath}} \right\rangle^{-2} \forall \alpha, \beta \in \mathbb{N}^2,
\]
(2.28)
i.e. \(a_0(x, \xi) \in S_{\frac{1}{4}} \left( \left\langle \frac{x + F'(\xi)}{\sqrt{\imath}} \right\rangle^{-2} \right)\). This finishes the proof of (2.26).

Next, we deal with the first term on the right-hand side of (2.24). Setting \(\Gamma(\xi) = \Gamma(x)\xi_1\), we deduce from proposition 2.3 that
\[
\left( \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\imath}} \right) (x_1 + F'_1(\xi_1)) \right) \zeta (x_1 + F'_1(\xi_1)) \\
= \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\imath}} \right) (x_1 + F'_1(\xi_1))^2 - \frac{\hbar^2}{4(\sqrt{\imath})} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\imath \eta \cdot (-y - \zeta)} \\
\times \int_0^1 \left( \partial_\xi^2 \Gamma \right) \left( \frac{x + tz + F'(\xi + \imath \zeta)}{\sqrt{\imath}} \right) (1 - t) dt F_1''''(\xi_1 + \eta) d\eta d\zeta \\
= \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\imath}} \right) (x_1 + F'_1(\xi_1))^2 - \frac{\hbar^2}{4(\sqrt{\imath})} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\imath \eta \cdot (-y - \zeta)} \int_0^1 \left( \partial_\xi^2 \Gamma \right) \left( \frac{x + tz + F'(\xi)}{\sqrt{\imath}} \right) (1 - t) dt \\
\times F_1''''(\xi_1 + \eta) d\eta d\zeta.
\]
(2.29)
Assume for a while that we have proved
\[
\frac{\hbar^2}{(\sqrt{\imath})^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\imath \eta \cdot \zeta} \int_0^1 \left( \partial_\xi^2 \Gamma \right) \left( \frac{x + tz + F'(\xi)}{\sqrt{\imath}} \right) (1 - t) dt F_1''''(\xi_1 + \eta) d\eta d\zeta \\
= \sum_{j=1}^2 \left( \hbar \tilde{a}_j (x, \xi) \zeta (x_1 + F'_j(\xi_1)) + \hbar^2 \tilde{b}_j (x, \xi) \right),
\]
(2.30)
for some symbols \(\tilde{a}_j(x, \xi), \tilde{b}_j(x, \xi) \in S_{\frac{1}{4}} \left( \left\langle \frac{x + F'(\xi)}{\sqrt{\imath}} \right\rangle^{-2} \right)\), \(j = 1, 2\). Then lemma 2.4 follows by substituting (2.26), (2.29) and (2.30) into (2.24).
In what follows, we prove (2.30). Let \( \Gamma'(\xi) = \frac{\partial_t^2 \Gamma(\xi)}{\partial t^2} \), \( j = 1, 2 \), satisfying \(|\partial_x^\alpha \Gamma(\xi)| \leq C_\alpha (\xi)^{-2-|\alpha|}\) for any \( \alpha \in \mathbb{N}^2 \). Noting that \( \partial_x^\alpha \Gamma(\xi) = \sum_{j=1}^{2} \xi_j \Gamma_j(\xi) \), we have

\[
\frac{h^2}{(\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x+y}{h}} \int_0^1 \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} \right) (1-t) dF_t''(\xi_1+\eta_1) d\eta_1 dz
\]

\[
= \frac{h}{(\pi h)^2} \sum_{j=1}^{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x+y}{h}} \int_0^1 \Gamma_j \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} \right) (1-t) dF_t''(\xi_1+\eta_1) d\eta_1 dz (x_j+F'_j(<\xi>))
\]

\[
+ i \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x+y}{h}} \int_0^1 \Gamma_j \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} \right) (1-t) r dF_t''(\xi_1+\eta_1) d\eta_1 dz (x_j+F'_j(<\xi>))
\]

\[
= i \frac{h}{(\pi h)^2} \sum_{j=1}^{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x+y}{h}} \int_0^1 \Gamma_j \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} \right) (1-t) r dF_t''(\xi_1+\eta_1) d\eta_1 dz (x_j+F'_j(<\xi>))
\]

\[
= h \sum_{j=1}^{2} \left[ A_j(x,\xi)(x_j+F'_j(<\xi>)) + h B_j(x,\xi) \right]. \tag{2.31}
\]

By a similar argument as that used to derive (2.26), we get that

\[
A_j(x,\xi) \in S_1^2 \left( \left< \frac{x+F'(<\xi>)}{\sqrt{t}} \right> \right)^{-2}, \quad B_j(x,\xi) \in S_1^4 \left( \left< \frac{x+F'(<\xi>)}{\sqrt{t}} \right> \right)^{-2}, \quad k = 1, 2.
\]

This proves (2.30), from which lemma 2.4 follows. \( \square \)

**Lemma 2.5.** Suppose that \( \Gamma'(\xi) \in C_0^\infty(\mathbb{R}^2) \) is a smooth function that satisfies \( \Gamma \equiv 0 \) in a neighbourhood of zero and \( a(x,\xi) \in S_0(1) \). Then we have the following estimates

\[
\left\| G'' \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} a(x,\xi) \right) v \right\|_{L^2_0} \lesssim h^{1/2} \left( \|L^2v\|_{L^2_0} + \|L^2v\|_{L^2_0} + \|v\|_{L^2_0} \right),
\]

\[
\left\| G'' \left( \Gamma \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} a(x,\xi) \right) v \right\|_{L^2_0} \lesssim h \left( \|L^2v\|_{L^2_0} + \|L^2v\|_{L^2_0} + \|v\|_{L^2_0} \right),
\]

where \( L^2 = (L^2_1, L^2_2) \) with \( L_k, \ k = 1, 2 \) defined by (1.24).

**Proof.** Assume for a while that we have proved: there exist smooth functions \( \tilde{\Gamma}_j(\xi) \in C_0^\infty(\mathbb{R}^2) \) that satisfies \( \tilde{\Gamma}_j \equiv 0 \) in a neighbourhood of zero, and symbols \( a_j(x,\xi) \in S_0(1), \ b_{jk}(x,\xi), \ c(x,\xi) \in S_4^2 \left( \left< \frac{x+F'(<\xi>)}{\sqrt{t}} \right> \right)^2, \ 1 \leq j, k \leq 3, \ 1 \leq k \leq 2 \) such that

\[
\Gamma' \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} a(x,\xi) \right) = \frac{1}{h} \sum_{j=1}^{2} \sum_{k=1}^{2} \tilde{\Gamma}_j \left( \frac{x+tc+F'(<\xi>)}{\sqrt{t}} \right) \tilde{a}(x_k+F'_k(<\xi>)) \tilde{a}_j
\]

\[+ \sum_{j=1}^{2} \sum_{k=1}^{2} b_{jk}(x_k+F'_k(<\xi>)) \tilde{a}_j + h c. \tag{2.32}
\]
Taking the Weyl quantisation in (2.32), we get
\[ G_{\hbar}^w \left( \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) a(x, \xi) \right) v = h \sum_{j=1}^{3} \sum_{k=1}^{2} G_{\hbar}^w \left( \tilde{\Gamma}_j \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) \right) \circ \mathcal{L}_k^w \circ G_{\hbar}^w (a_j) v + h \mathcal{L}_k^w \circ G_{\hbar}^w (a_j) v \]
\[ + h \sum_{j=1}^{3} \sum_{k=1}^{2} \mathcal{L}_k (b_{jk}) \circ \mathcal{L}_k \circ G_{\hbar}^w (a_j) v + h G_{\hbar}^w (c) v. \]

Therefore, using propositions 2.4 and 2.5, we estimate
\[ \| G_{\hbar}^w \left( \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) a(x, \xi) \right) v \|_{L^\infty} \leq h^{1/2} \sum_{j=1}^{3} \sum_{k=1}^{2} \left( \| \mathcal{L}_k^w \circ G_{\hbar}^w (a_j) v \|_{L^2} + \| \mathcal{L}_k \circ G_{\hbar}^w (a_j) v \|_{L^2} \right) + h^{1/2} \| v \|_{L^2}, \]
\[ \| G_{\hbar}^w \left( \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) a(x, \xi) \right) v \|_{L^2} \leq h \sum_{j=1}^{3} \sum_{k=1}^{2} \left( \| \mathcal{L}_k^w \circ G_{\hbar}^w (a_j) v \|_{L^2} + \| \mathcal{L}_k \circ G_{\hbar}^w (a_j) v \|_{L^2} \right) + h \| v \|_{L^2}, \]
which together with (2.14) yields the desired estimates in lemma 2.5.

We now prove (2.32). By proposition 2.3, we have
\[ \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) z \alpha(x, \xi) \]
\[ = \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) a(x, \xi) + \frac{i \sqrt{\hbar}}{2} \sum_{j=1}^{2} \left( \partial_{\xi} \Gamma \right) \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) E_k (x, \xi) + \Gamma \left( \frac{i x}{\sqrt{\hbar}} \right) \alpha(x, \xi), \tag{2.33} \]
where \( E_k (x, \xi) = \partial_{\xi} a(x, \xi) - \partial_{\xi_j} a(x, \xi) F_{ij}'(\xi_k) \in S_0(1) \) by (1.5). Using proposition 2.3 again, we have, for some symbols \( H_{kj}(x, \xi) \in S_0(1) \), \( 1 \leq i, j \leq 2 \),
\[ \left( \partial_{\xi_i} \Gamma \right) \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) z \alpha_k (x, \xi) \]
\[ = \left( \partial_{\xi_i} \Gamma \right) \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) E_k (x, \xi) + \frac{i \sqrt{\hbar}}{2} \sum_{j=1}^{2} \left( \partial_{\xi_i} \partial_{\xi_j} \Gamma \right) \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) H_{kj} (x, \xi) \]
\[ + \frac{r \left( \partial_{\xi_i} \Gamma \right) \left( \frac{i x}{\sqrt{\hbar}} \right)}{2} H_{kj} (x, \xi). \tag{2.34} \]
Substitution of (2.34) into (2.33) yields
\[ \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) a(x, \xi) = \Gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) z \alpha(x, \xi) - \frac{i \sqrt{\hbar}}{2} \sum_{j=1}^{2} \left( \partial_{\xi_j} \Gamma \right) \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) z \alpha_k (x, \xi) \]
\[ - \frac{h}{4} \sum_{j=1}^{2} \sum_{k=1}^{2} \left( \partial_{\xi_j} \partial_{\xi_k} \Gamma \right) \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) H_{kj} (x, \xi) - \Gamma \left( \frac{i x}{\sqrt{\hbar}} \right) z \alpha(x, \xi) \]
\[ + \frac{i \sqrt{\hbar}}{2} \sum_{k=1}^{2} \left( \partial_{\xi_i} \Gamma \right) \left( \frac{i x}{\sqrt{\hbar}} \right) H_{kj} (x, \xi). \tag{2.35} \]
Since $\Gamma \equiv 0$ in a neighbourhood of zero and $H_{ij} \in S_0(1)$, we have

$$\left( \partial_{\xi}^i \partial_{\xi}^j \Gamma \right) \frac{x + F^{i}(\xi)}{\sqrt{h}} H_{ij}(x, \xi) \in \mathcal{S}_{\frac{1}{2}} \left( \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right)^{-2} \right), \quad 1 \leq k, j \leq 2. \quad (2.36)$$

Moreover, by the definition in (2.6) and the similar argument as in the proof of (2.27) and (2.28), we get

$$r \left( \frac{\xi + F^{i}(\xi)}{\sqrt{h}} \right) \mathcal{S}_{\frac{1}{2}} \left( \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right)^{-2} \right), \quad k = 1, 2. \quad (2.37)$$

Substitution of (2.36) and (2.37) into (2.35) yields

$$\Gamma \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right) a(x, \xi) = \sum_{j=1}^{3} \Gamma_{j} \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right) \mathcal{S}_{\frac{1}{2}} \left( \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right)^{-2} \right)$$

for some smooth functions $\Gamma_{j}(\xi) \in C_{2}^{\infty} \left( \mathbb{R}^{2} \right)$ that satisfies $\Gamma_{j} \equiv 0$ in a neighbourhood of zero, and symbols $a_{i}(x, \xi) \in S_{0}(1)$, $c_{i}(x, \xi) \in S_{\frac{1}{2}} \left( \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right)^{-2} \right)$.

Let $\Gamma_{j}(\xi) = \frac{\Gamma(\xi)}{|\xi|^{j}}, j = 1, 2$, satisfying $|\partial_{\xi}^{\alpha} \Gamma_{j}(\xi)| \leq C_{\alpha} |\xi|^{-2-|\alpha|}$ for any $\alpha \in \mathbb{N}^{2}$. By lemma 2.4, there are symbols $b_{j}(x, \xi), c_{j}(x, \xi) \in S_{\frac{1}{2}} \left( \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right)^{-2} \right)$ such that

$$\Gamma_{j} \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right) = \frac{1}{h} \sum_{k=1}^{2} \tilde{\Gamma}_{j} \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right) (x_{k} + \tilde{F}_{k}(\xi))^{2} \quad (2.39)$$

$$= \frac{1}{h} \sum_{k=1}^{2} \left( \tilde{\Gamma}_{j} \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right) \left( x_{k} + \tilde{F}_{k}(\xi) \right) \right) (x_{k} + \tilde{F}_{k}(\xi)) + \sum_{k=1}^{2} b_{jk} (x_{k} + \tilde{F}_{k}(\xi)) + h c_{j}.$$

Substitution of (2.39) into (2.38) then gives (2.32), from which lemma 2.5 follows.

**Lemma 2.6.** Suppose that $\gamma(\xi) \in C_{2}^{\infty} \left( \mathbb{R}^{2} \right)$, satisfying $\gamma \equiv 1$ in a neighbourhood of zero. There exist symbols $a_{k}(x, \xi), b_{k}(x, \xi) \in S_{\frac{1}{2}} \left( \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right)^{-2} \right), k = 1, 2$ such that

$$r(x + F(\xi)) \mathcal{S}_{\frac{1}{2}} \left( \left( \frac{x + F^{i}(\xi)}{\sqrt{h}} \right)^{-2} \right) = \sum_{k=1}^{2} (ha_{k}(x, \xi)) (x_{k} + \tilde{F}_{k}(\xi)) + h^{2} b_{k}(x, \xi).$$

**Proof.** For simplicity, we use the notation $\gamma_{ij}(\xi) = \partial_{\xi}^{i} \partial_{\xi}^{j} \gamma(\xi), \gamma_{i0}(\xi) = \partial_{\xi}^{i} \gamma(\xi), 1 \leq i, j \leq 2$. By proposition 2.3
where $\hat{\delta}_0$ is the Dirac function. Similarly, we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x + y + F^\prime (\xi + \eta)}{\sqrt{h}}} \gamma_{12} \left( \frac{x + y + F^\prime (\xi + \eta)}{\sqrt{h}} \right) F'_{\tilde{h}} (\xi_k + t\zeta_k) (1 - t) \, dy \, dz \, d\zeta = \frac{1}{2} (\pi h)^4 \gamma_{12} \left( \frac{x + F^\prime (\xi)}{\sqrt{h}} \right) F'_{\tilde{h}} (\xi_k),$$

(2.42)

Substitution of (2.41) and (2.42) into (2.40) yields

$$r(x, \xi + F(\xi)) \gamma \left( \frac{1}{2} \frac{1}{\sqrt{h}} \right) = r(x, \xi + F(\xi))$$

$$= \frac{h}{4} (\pi h)^4 \sum_{k=1}^{2} \left[ - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x + y + F^\prime (\xi)}{\sqrt{h}}} \gamma_{12} \left( \frac{x + y + F^\prime (\xi)}{\sqrt{h}} \right) dy \, dz \, d\zeta \right]$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x + y + F^\prime (\xi)}{\sqrt{h}}} \gamma_{12} \left( \frac{x + y + F^\prime (\xi)}{\sqrt{h}} \right) (1 - t) \, dy \, dz \, d\zeta$$

$$= \frac{h}{4} (\pi h)^4 \sum_{k=1}^{2} \left( A_k (x, \xi) + B_k (x, \xi) \right),$$

(2.43)

where

$$A_k = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x + y + F^\prime (\xi)}{\sqrt{h}}} \left( F_{\tilde{h}}' (\xi_k + t\zeta_k) - F_{\tilde{h}}' (\xi_k) \right) (1 - t) \, dy \, dz \, d\zeta,$$

$$B_k = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{x + y + F^\prime (\xi)}{\sqrt{h}}} \gamma_{12} \left( \frac{x + y + F^\prime (\xi)}{\sqrt{h}} \right) (1 - t) \, dy \, dz \, d\zeta.$$
For $A_k$, using the mean value theorem and the identity $\zeta e^{-\frac{x^2}{2}} = -\frac{1}{2x} \frac{\partial}{\partial x} e^{-\frac{x^2}{2}}$ to integrate by parts, we get

$$\frac{h}{4} \frac{1}{(\pi h)^2} A_k(x, \xi) = \frac{ih}{8 (\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{x^2}{2}} \int_0^1 \int_0^1 F''''_k(\xi_k + \sigma t \xi) t(1-t) \, ds \, dr \times \gamma_{\xi k} \left( \frac{x + y + F'(\xi)}{\sqrt{n}} \right) dy d\xi.$$ 

Using the same method as that used to derive (2.30), we can write, for some symbols $a_{kj}(x, \xi)$, $b_{kj}(x, \xi) \in S^2 \left( \left( \frac{x + F'(\xi)}{\sqrt{n}} \right)^{-2} \right)$, $1 \leq j \leq 2$

$$\frac{h}{4} \frac{1}{(\pi h)^2} A_k(x, \xi) = \sum_{j=1}^{2} \left( h a_{kj}(x, \xi) \frac{\partial}{\partial x} (x_j + F'_k(\xi_j)) + h^2 b_{kj}(x, \xi) \right). \quad (2.44)$$

For $B_k$, we have a similar representation like (2.44). Lemma 2.6 then follows by substituting the representation of $A_k$, $B_k$ into (2.43).

**Lemma 2.7.** Suppose that $\gamma(\xi) \in C^\infty_0(\mathbb{R}^2)$, satisfying $\gamma \equiv 1$ in a neighbourhood of zero. There exist symbols $a_k(x, \xi), b_k(x, \xi) \in S^2 \left( \left( \frac{x + F'(\xi)}{\sqrt{n}} \right)^{-2} \right)$, $k = 1, 2$ such that

$$r_w(x) \gamma \left( \frac{x + F'(\xi)}{\sqrt{n}} \right) - r_w(x) \gamma \left( \frac{x + F'(\xi)}{\sqrt{n}} \right) = \sum_{k=1}^{2} \left( h a_k(x, \xi) \frac{\partial}{\partial x} (x_k + F'_k(\xi_k)) + h^2 b_k(x, \xi) \right),$$

where $w(x) = x \cdot d\phi(x) + F(d\phi(x))$.

**Proof.** Set $\Gamma'_k(x, \xi) = \partial_{\xi_k} \left( \gamma \left( \frac{x + F'(\xi)}{\sqrt{n}} \right) \right), k = 1, 2, j \in \mathbb{N}$. Since

$$w(x) = x \cdot d\phi(x) + F(d\phi(x)) = \sum_{k=1}^{2} (x_k d\phi_k(x_k) + F'_k(x_k)), $$

we have by (1.7)

$$\partial_{\xi_k} w(x) = d\phi_k(x_k) + (x_k + F'_k(d\phi_k(x_k))) d^2\phi_k(x_k) = d\phi_k(x_k); \quad (2.45)$$

so that

$$\partial_{\xi_k} \partial_{\xi_k} w(x) = 0, \quad \partial_{\xi_k} \partial_{\xi_k} w(x) = d^2\phi_k(x_k), \quad k = 1, 2. \quad (2.46)$$

It then follows from proposition 2.3 that
\[ r^\nu(s)\gamma\left(\frac{\nu^\nu(x)}{\eta}\right) - r^\nu\left(\frac{\nu^\nu(x)}{\eta}\right) s(x) \]
\[
= \frac{h^2}{4} \frac{1}{(\pi h)^2} \sum_{k=1}^{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\pi}{\eta} y \cdot \zeta} \left[ - \int_0^1 d^2 \phi_k(x_k + t\zeta_k)(1-t)dt \times \Gamma^2_k(x_\nu + \xi_\nu + \eta) + \int_0^1 \Gamma^2_k(x_\nu + \xi_\nu + \eta + \xi)(1-t)d\xi d^2 \phi_k(x_k + y_k)dyd\zeta \right] \\
= \frac{h^2}{4} \frac{1}{(\pi h)^2} \sum_{k=1}^{2} - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^1 d^2 \phi_k(x_k + t\zeta_k)(1-t)dt \Gamma^2_k(x_\nu + \xi_\nu + \eta + \xi)d\xi dy dz \\
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{\pi}{\eta} y \cdot \zeta} \int_0^1 \Gamma^2_k(x_\nu + \xi + \zeta)(1-t)d\zeta d^2 \phi_k(x_k + y_k)dyd\zeta \\
= \frac{h^2}{4} \frac{1}{(\pi h)^2} \sum_{k=1}^{2} (A_k(x, \xi) + B_k(x, \xi)), \tag{2.47}
\]

where

\[ A_k = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{\pi}{\eta} y \cdot \zeta} \left( d^2 \phi_k(x_k + t\zeta_k) - d^2 \phi_k(x_k)(1-t)d\Gamma^2_k(x_\nu + \xi_\nu + \eta + \xi)dyd\zeta , \right. \]

\[ B_k = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{\pi}{\eta} y \cdot \zeta} \int_0^1 \Gamma^2_k(x_\nu + \xi + \zeta)(1-t) dt \left( d^2 \phi_k(x_k + y_k) - d^2 \phi_k(x_k) \right)dyd\zeta. \]

For \(B_k\), using the mean value theorem and the identity \(y_k e^{-\frac{\pi}{\eta} y \cdot \zeta} = -\frac{\partial \zeta_k}{\partial \zeta_k} e^{-\frac{\pi}{\eta} y \cdot \zeta}\) to integrate by part, we obtain

\[
\frac{h^2}{4} \frac{1}{(\pi h)^2} B_k(x, \xi) = \frac{1}{(\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{\pi}{\eta} y \cdot \zeta} \int_0^1 h^3 \Gamma^2_k(x_\nu + \xi + \zeta)(1-t)dt dy d\zeta, \\
\]

where

\[
h^3 \Gamma^2_k(x_\nu, \xi) = h^2 \left( \partial_{\xi_k} \gamma \right) \left( \frac{x + F^\nu(\xi)}{\sqrt{h}} \right) \left( F^\nu_k(\xi_k) \right)^3 + 3h^2 \left( \partial_{\xi_k} \gamma \right) \left( \frac{x + F^\nu(\xi)}{\sqrt{h}} \right) F^\nu_k(\xi_k) F^\nu_k(\xi_k) + h^2 \left( \partial_{\xi_k} \gamma \right) \left( \frac{x + F^\nu(\xi)}{\sqrt{h}} \right) F^\nu_k(\xi_k), k = 1, 2. \]

Using the same method as that used to derive (2.30), we can write, for some symbols \(a_{ij}(x, \xi), b_{ij}(x, \xi) \in S^i_2\left( \left( \frac{x + F^\nu(\xi)}{\sqrt{h}} \right) \right)^2, 1 \leq j \leq 2\)

\[
\frac{h^2}{4} \frac{1}{(\pi h)^2} B_k(x, \xi) = \sum_{j=1}^{2} \left( ha_{ij}(x, \xi) \frac{\partial}{\partial \xi_j} \left( x_j + F^\nu_j(\xi) \right) + h^2 b_{ij}(x, \xi) \right). \tag{2.48}
\]

For \(A_k\), we have a similar representation like (2.48). Lemma 2.7 then follows by substituting the representation of \(A_k, B_k\) into (2.47).
3. Proof of theorem 1.1

This section is devoted to proving theorem 1.1. It is organised into two subsections. In the first one, we apply the contraction argument and the Strichartz estimate to prove the global existence and uniqueness of the solution to (1.9). In the second one, we prove the decay estimate (1.10), combining the semiclassical analysis method and the ODE argument.

3.1. Proof of the global existence and uniqueness

Using the classical energy estimate method, we can obtain the following lemma easily and omit the details.

**Lemma 3.1.** Suppose that $\Im \lambda \geq 0$ and $u$ is a strong $L^2$ solution of (1.9) on the time interval $[1, T]$ with $T > 1$, then we have

$$\|u(t, \cdot)\|_{L^2} \leq \varepsilon \|u_0\|_{L^2}, \quad t \in [1, T].$$

**Theorem 3.1.** For any $\varepsilon \in (0, 1]$ and any initial datum $u_0 \in H^2$, $(x_k + F_k'(D))^2 u_0 \in L^2$, $k = 1, 2$ satisfying

$$\|u_0\|_{h^2}^2 + \sum_{k=1}^{2} \| (x_k + F_k'(D))^2 u_0 \|_{L^2} \leq 1,$$

(3.1)

the Cauchy problem (1.9) has a unique global solution $u \in C([1, \infty); L^2) \cap L^4_{loc}([1, \infty); L^\infty)$. Moreover, there exist absolute constants $T_0, C_1 > 0$ such that

$$\|u(t, \cdot)\|_{L^\infty} \leq C_1 \varepsilon,$$

(3.2)

for all $t \in (1, T_0)$.

**Proof.** Since the proof is classical, we only give a brief here. Considering the linear inhomogeneous system,

$$(D_t - F(D)) u = f, \quad u(1) = \varepsilon u_0,$$

(3.3)

we have,

$$u = e^{F(D) t \varepsilon u_0} + \int_1^t e^{F(D) (t-s) f(s)} ds.$$

By direct computation, we have

$$\|e^{F(D) t \varepsilon u_0}\|_{L^2} = ||f||_{L^2}$$

(3.4)

and

$$|F^{-1} \left( e^{F(D)} \right) (x) |$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{iF(x) + ix} e^{-\varepsilon |x|^2} d\xi$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{iF(x) + i \frac{1}{\varepsilon} \eta \cdot \xi} e^{-\varepsilon |\eta|^2} d\eta$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i|\eta|^2} \left( \frac{1 - i \left( \sqrt{F'} \left( \frac{\eta}{\sqrt{\varepsilon}} \right) + \frac{\eta}{\sqrt{\varepsilon}} \right) \cdot \partial_\eta}{1 + \sqrt{F'} \left( \frac{\eta}{\sqrt{\varepsilon}} \right) + \frac{\eta}{\sqrt{\varepsilon}}^2} \right)^3 e^{iF(x) + i \frac{1}{\varepsilon} \eta \cdot \xi} d\eta.$$
\begin{equation}
\lesssim \frac{1}{t} \int_{\mathbb{R}^3} \left\langle \sqrt{F'} \left( \frac{\eta}{\sqrt{t}} \right) + \frac{x}{\sqrt{t}} \right\rangle^{-3} \, d\eta \lesssim \frac{1}{t},
\end{equation}

where the last inequality holds since by (1.5) and (1.7):

\[ \left| \sqrt{F'} \left( \frac{\eta}{\sqrt{t}} \right) + \frac{x}{\sqrt{t}} \right| = \sqrt{F'} \left( \frac{\eta}{\sqrt{t}} \right) - F' \left( \frac{d\phi}{d\eta} \left( \frac{1}{\sqrt{t}} \right) \right) \gtrsim |\eta - \sqrt{td}\phi \left( \frac{x}{\sqrt{t}} \right)|. \]

The energy estimate (3.4) and the dispersive estimate (3.5) imply the Strichartz estimate for the solution \( u \) (3.3) (see the theorem of Keel and Tao [34]):

\[ \|u\|_{L_t^\infty L_x^2 \cap L_t^5 L_x^3} \lesssim C \varepsilon \|u_0\|_{L^2} + C \|f\|_{L_t^3 L_x^1}. \]

Choosing that \( f = \lambda |u|u \), and applying Hölder’s inequality, we have

\[ \|u\|_{L_t^\infty L_x^2 \cap L_t^6 L_x^3} \lesssim C \varepsilon \|u_0\|_{L^2} + C \sqrt{1 - T} \|u\|_{L_t^3 L_x^1}. \]

Using the contraction principle in the space \( LH^\infty([1, T^*]; L_x^2) \cap L^5([1, T^*]; L_x^3) \) provided \( 1 < T^* < 1 + (2\varepsilon)^{-2} \), we obtain that the Cauchy problem (1.9) has a unique local solution. Since \( \|u(t, \cdot)\|_{L^2} \leq \varepsilon \|u_0\|_{L^2} \) by lemma 3.1, this local solution can be extended to \( [0, \infty) \) with \( u \in LH^\infty([1, T]; L_x^2) \cap L^5([1, T]; L_x^3) \) for all \( T > 1 \). Moreover, it is easy to check that \( u \in C([0, T]; L^2) \) for all \( T > 0 \), and omit the details.

It remains to prove the inequality (3.2). Notice that

\[ \partial_t^2 u = e^{-iDt} \partial_t^2 u_0 + i\lambda \int_0^t e^{-iF(t-s)} \partial_t^2 (|u||u|) \, ds, \quad k = 1, 2 \]

and \( |\partial_t^2 (|u||u|)| \leq |u||\Delta u| + |\nabla u|^2 \) (see (A.6)). Using successively Strichartz estimate (3.6) and Hölder’s inequality, we get

\[ \|\Delta u\|_{L_t^\infty L_x^2 \cap L_t^6 L_x^3} \leq C \varepsilon \|u_0\|_{HF} + C \|u \Delta u\|_{L_t^5 L_x^3} + C \|\nabla u\|^2_{L_t^5 L_x^3} \lesssim C \varepsilon \|u_0\|_{HF} + C \sqrt{1 - T} \|u\|_{L_t^3 L_x^1} \|\Delta u\|_{L_t^3 L_x^1}, \]

where we also used Gagliardo–Nirenberg’s inequality \( \|\nabla u\|^2 \lesssim \|u\|^2 \|\Delta u\| \). Choosing \( T_0 > 1 \) sufficiently approaches 1, the above inequality implies that

\[ \|\Delta u\|_{L_t^\infty([1, T]; L^2) \cap L_t^5([1, T]; L^3)} \lesssim 2C \varepsilon \|u_0\|_{HF}, \quad \forall t \in (1, T_0). \]

This together with lemma 3.1 and Sobolev’s embedding \( H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \) yields the desired estimate (3.2).

\[ \square \]

3.2. Proof of the global decay estimate

The goal of this subsection is to derive the decay estimate (1.10), thus completing the proof of theorem 1.1.

Let \( u \) be the solution of (1.9) given by theorem 1.3. We make first a semiclassical change of variables

\[ u(t, x) = hv(t, hx), \quad h = \frac{1}{t}, \]

that allows rewriting the equation (1.9) as

\[ (D_t - G^w_h (x \cdot \xi + F(\xi))) v = \lambda h |v| v. \]
By direct calculation, we have
\[
\|u(t, \cdot)\|_{L^2} = \|v(t, \cdot)\|_{L^2}, \quad \|u(t, \cdot)\|_{L^\infty} = \sqrt{h}\|v(t, \cdot)\|_{L^\infty}.
\] (3.10)
Moreover, we have that (recall that $\mathcal{L}_k$ are the operators defined in (1.24))
\[
h(\mathcal{L}_k^2v)(t, hx) = (x_k + tF_k(D))^2 u(t, x) \quad k = 1, 2;
\]
so that
\[
\|\mathcal{L}_k^2v(t, x)\|_{L^2} = \sum_{k=1}^2 \| (x_k + tF_k(D))^2 u \|_{L^2}.
\] (3.11)
By (3.10), the decay estimate (1.10) is equivalent to
\[
\|v(t, x)\|_{L^\infty} \lesssim \varepsilon, \quad t > 1.
\] (3.12)
To prove (3.12), we decompose $v = v_\Lambda + v_{\Lambda^c}$ with
\[
v_\Lambda = G_\alpha^\Gamma (\Gamma(x, \xi)) v,
\]
where $\Gamma(x, \xi) = \gamma(\frac{x + tF_\alpha(D)}{\varepsilon})$, $\gamma \in C_0^\infty(\mathbb{R}^2)$ satisfying $\gamma \equiv 1$ in a neighborhood of zero.

We will use the following Sobolev type inequality. For the convenience of the reader, we give the proof in the appendix.

**Lemma 3.2.** Assume $v : [1, T] \times \mathbb{R}^2 \to \mathbb{C}, T > 1$, there exists a positive constant $C_2$ independent of $T$ and $v$ such that for all $t \in [1, T]$ and $v$
\[
\|\mathcal{L}_k^2(|v|v)\|_{L^2} \leq C_2 \|v\|_{L^\infty} \left(\|v\|_{L^2} + \|\mathcal{L}_2^2v\|_{L^2}\right),
\] (3.13)
and
\[
\|\mathcal{L}_v\|_{L^2} \leq C_2 \left(\|v\|_{L^2} + \|\mathcal{L}_2^2v\|_{L^2}\right),
\] (3.14)
provided the right-hand sides are finite.

The rest of this subsection is organised as follows. In lemma 3.3, we show that $v_{\Lambda^c}$ decays faster than $v$ and (3.12) is reduced to prove
\[
\|v_{\Lambda^c}(t, x)\|_{L^\infty} \lesssim \varepsilon, \quad t > 1.
\] To derive this, we apply $G_\alpha^\Gamma (\Gamma(x, \xi))$ to (3.9) and deduce an ODE for $v_\Lambda$:
\[
D_t v_\Lambda = w(x)v_\Lambda + \lambda^{-1}|v_\Lambda||v_\Lambda + R(v),
\] (3.15)
where $w(x) = x \cdot d\phi(x) + F(d\phi(x))$ and
\[
R(v) = [D_t - G_\alpha^\Gamma(x \cdot \xi + F(\xi)), G_\alpha^\Gamma(\Gamma)]v + G_\alpha^\Gamma(x \cdot \xi + F(\xi) - w(x))v_\Lambda - \lambda^{-1}G_\alpha^\Gamma(1 - \Gamma)(|v||v_\Lambda + \lambda^{-1}(|v||v - |v_\Lambda||v_\Lambda|).\] (3.16)
Then we establish the decay estimates of $R(v)$ in lemmas 3.4–3.7. Finally, at the end of this subsection, we derive the desired $L^\infty$ estimate for $v_\Lambda$ and then in the solution $u$, combining the ODE and the bootstrap argument. This together with theorem 1.3 finishes the proof of theorem 1.1.

**Lemma 3.3.** Suppose $u$ is a solution of (1.9) given by theorem 3.1 and $v$ is defined by (3.8), there is a constant $C_3 > 0$ such that for all $t > 1$,
\[
\|v_\Lambda(t, x)\|_{L^\infty} \leq C_3t^{-1/2} \left(\|v\|_{L^2} + \|\mathcal{L}_2^2v\|_{L^2}\right),
\] (3.17)
\[
\|v_{\Lambda^c}(t, x)\|_{L^2} \leq C_3t^{-1} \left(\|v\|_{L^2} + \|\mathcal{L}_2^2v\|_{L^2}\right).
\] (3.18)
Proof. Set $\Gamma_{-2}(\xi) = \frac{1 - \gamma(\xi)}{|\xi|^2}$, satisfying $|\partial_\xi^2 \Gamma_{-2}(\xi)| \leq C_\alpha |\xi|^{-2-|\alpha|}$ for any $\alpha \in \mathbb{N}^2$. By lemma 2.4, there are symbols $a_{ij}(x, \xi)$, $b_{ij}(x, \xi) \in S_\frac{1}{2} \left( \left( \frac{\xi + F'(\xi)}{\sqrt{h}} \right)^{-2} \right)$, $1 \leq k, j \leq 2$ such that

$$1 - \gamma \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) = \frac{1}{h} \sum_{k=1}^{2} \Gamma_{-2} \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) (x_k + F_k(\xi_j))^2$$

$$= \frac{1}{h} \sum_{k=1}^{2} \left[ \Gamma_{-2} \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) (x_k + F_k(\xi_j)) \right] + \sum_{j=1}^{2} \left( h a_{ij} (x_j + F_j(\xi_j)) + h^2 b_{ij} \right).$$

Taking the Weyl quantisation, then using proposition 2.2, one obtains

$$v_{\mathcal{A}'}(t, x) = \frac{h}{t} \sum_{k=1}^{2} G^\hbar_n \left( \Gamma_{-2} \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) \right) \circ L^2_v + \sum_{j=1}^{2} \left( G^\hbar_n (a_{ij}) \circ L_v + G^\hbar_n (b_{ij}) \right) v.$$

By propositions 2.5 and 2.6, one has

$$\|v_{\mathcal{A}'}(t, x)\|_{L^2_v} \lesssim h^{1/2} \left( \|L^2_v\|_{L^2_v} + \|L_v\|_{L^2_v} + \|v\|_{L^2_v} \right).$$

On the other hand, since $S_\frac{1}{2} \left( \left( \frac{\xi + F'(\xi)}{\sqrt{h}} \right)^{-2} \right) \subset S_\frac{1}{2}(1)$, it follows from proposition 2.4 that

$$\|v_{\mathcal{A}'}(t, x)\|_{L^2_v} \lesssim h \left( \|L^2_v\|_{L^2_v} + \|L_v\|_{L^2_v} + \|v\|_{L^2_v} \right).$$

Substitution of (3.14) into (3.19) and (3.20) yields the desired estimates in lemma 3.3. □

Lemmas 3.4–3.7 are devoted to prove the decay estimate of $R(v)$ in (3.16).

Lemma 3.4. With the same assumptions as in lemma 3.3, the following inequalities hold for all $t > 1$

$$\|G^\hbar_n (1 - \Gamma)(|v|v)\|_{L^\infty_v} \lesssim \gamma \left( \|v\|_{L^\infty_v} + \|L_v\|_{L^\infty_v} + \|L^2_v\|_{L^2_v} \right),$$

$$\|G^\hbar_n (1 - \Gamma)(|v|v)\|_{L^2_v} \lesssim \gamma \left( \|v\|_{L^2_v} + \|L^2_v\|_{L^2_v} \right).$$

Proof. Using the same method as that used to derive (3.17) and (3.18), one gets

$$\|G^\hbar_n (1 - \Gamma)(|v|v)\|_{L^\infty_v} \lesssim \gamma^{-1/2} \left( \|v\|_{L^2_v} + \|L^2_v\|_{L^2_v} \right),$$

$$\|G^\hbar_n (1 - \Gamma)(|v|v)\|_{L^2_v} \lesssim \gamma^{-1} \left( \|v\|_{L^2_v} + \|L^2_v\|_{L^2_v} \right),$$

which together with the inequality (3.13) yields the desired estimates in lemma 3.4. □

Similarly, we can obtain the following lemma easily and omit the details.

Lemma 3.5. With the same assumptions as in lemma 3.3, the following inequalities hold for all $t > 1$

$$\|v|v - v_{\mathcal{A}}v_{\mathcal{A}}\|_{L^\infty_v} \lesssim \gamma \left( \|v\|_{L^\infty_v} + \|L_v\|_{L^\infty_v} \right) \left( \|v\|_{L^2_v} \right),$$

$$\|v|v - v_{\mathcal{A}}v_{\mathcal{A}}\|_{L^2_v} \lesssim \gamma^{-1} \left( \|v\|_{L^2_v} \right).$$
Lemma 3.6. With the same assumptions as in lemma 3.3, the following inequalities hold for all $t > 1$

$$\| D_t - G_h^m(x \cdot \xi + F(\xi)), G_h^m(\Gamma) \|_{L^\infty} \lesssim t^{-3/2} \left( \| v \|_{L^2} + \| \mathcal{C}^2 v \|_{L^2} \right),$$

and

$$\| D_t - G_h^m(x \cdot \xi + F(\xi)), G_h^m(\Gamma) \|_{L^2} \lesssim t^{-2} \left( \| v \|_{L^2} + \| \mathcal{C}^2 v \|_{L^2} \right).$$

Proof. First we start by calculating $[D_t, G_h^m(\Gamma)] = D_t G_h^m(\Gamma) - G_h^m(\Gamma) D_t$. Since $h = r^{-1}$, by direct computation, we have, under the notation $\gamma_k(\xi) = \partial_{\xi^k} \gamma(\xi)$,

$$D_t G_h^m(\Gamma) v = \frac{1}{i} \partial_t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i (x-y) \cdot \xi} \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) v(t,y) dy dx$$

$$= G_h^m(\Gamma) D_t v - \frac{it}{2\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i (x-y) \cdot \xi} \xi \gamma \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) v(t,y) dy dx$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i (x-y) \cdot \xi} \gamma \gamma_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) \left( \frac{x_k + y_k}{2} + F_k'(\xi_k) \right) \sqrt{\frac{h}{2}} v(t,y) dy dx$$

$$= -2hi G_h^m(\Gamma) v + \frac{1}{(2\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i (x-y) \cdot \xi} \gamma_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) (x_k + F_k'(\xi_k)) v(t,y) dy dx$$

$$- \frac{i\sqrt{h}}{2} \sum_{k=1}^2 G_h^m \left( \gamma_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) (x_k + F_k'(\xi_k)) \right) v + G_h^m(\Gamma) D_t v. \quad (3.21)$$

Moreover, using the identity $(x-y) \cdot \xi e^{i (x-y) \cdot \xi} = \frac{h}{i} \sum_{k=1}^2 \partial_{\xi^k} e^{i (x-y) \cdot \xi} \xi_k$ to integrate by parts, we get

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i (x-y) \cdot \xi} \gamma_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) (x_k + F_k'(\xi_k)) v(t,y) dy dx$$

$$= \frac{2hi}{(2\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i (x-y) \cdot \xi} \xi_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) v(t,y) dy dx$$

$$+ \frac{1}{(2\pi h)^2} \sum_{k=1}^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i (x-y) \cdot \xi} \xi_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) \xi_k F_k''(\xi_k) i \sqrt{h} v(t,y) dy dx$$

$$= 2hi G_h^m \left( \gamma \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) \right) v + i\sqrt{h} \sum_{k=1}^2 G_h^m \left( \gamma_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) \xi_k F_k''(\xi_k) \right) v. \quad (3.22)$$

Substitution of (3.22) into (3.21) yields

$$[D_t, G_h^m(\Gamma)] v = i\sqrt{h} \sum_{k=1}^2 G_h^m \left( \gamma_k \left( \frac{x+y + F'(\xi)}{\sqrt{h}} \right) \left( \xi_k F_k''(\xi_k) - \frac{x_k + F_k'(\xi_k)}{2} \right) \right) v. \quad (3.23)$$
On the other hand, an application of proposition 2.3 gives

\[
\begin{align*}
[G^w_h (x \cdot \xi + F(\xi)), G^w_h (\Gamma)] v &= i \sqrt{h} \sum_{k=1}^{2} G^w_h \left( \gamma_k \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) (\xi_k F_k''(\xi_k) - (x_k + F_k'(\xi_k))) \right) v + G^w_h (r) v, \\
\end{align*}
\]

(3.24)

where \( r =: r(\xi + F(\xi)) \gamma(\frac{\xi + F'(\xi)}{\sqrt{h}}) - r^\gamma(\frac{2(\xi + F'(\xi))}{\sqrt{h}}) \). Combining (3.23) and (3.24), then using lemma 2.6, we have for some symbols \( a_k(x, \xi), b_k(x, \xi) \in S_{k} \left( \left( \frac{x + F'(\xi)}{\sqrt{h}} \right)^{-2} \right), 1 \leq k \leq 2 \)

\[
\begin{align*}
[D_t - G^w_h (x \cdot \xi + F(\xi)), G^w_h (\Gamma)] v &= i \sqrt{h} \sum_{k=1}^{2} G^w_h \left( \gamma_k \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) (x_k + F_k'(\xi_k)) \right) v - G^w_h (r) v \\
&= h \sum_{k=1}^{2} G^w_h \left( \Gamma^k \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) \right) v + h^2 \sum_{k=1}^{2} (G^w_h (a_k) \circ L v + G^w_h (b_k) v),
\end{align*}
\]

where \( \Gamma^k(\xi) =: \frac{\gamma(\xi, \xi_k)}{2} \in C^\infty_0 (\mathbb{R}^2) \), satisfying \( \Gamma^k \equiv 0 \) in a neighbourhood of zero. By propositions 2.4–2.6 and lemma 2.5, we find the estimates

\[
\begin{align*}
\| [D_t - G^w_h (x \cdot \xi + F(\xi)), G^w_h (\Gamma)] v \|_{L^\infty} &\lesssim h^{3/2} \left( \| L^2 v \|_{L^2} + \| L v \|_{L^2} + \| v \|_{L^2} \right), \\
\| [D_t - G^w_h (x \cdot \xi + F(\xi)), G^w_h (\Gamma)] v \|_{L^2} &\lesssim h^{2} \left( \| L^2 v \|_{L^2} + \| L v \|_{L^2} + \| v \|_{L^2} \right),
\end{align*}
\]

which together with (3.14) yields the desired estimates in lemma 3.6.

**Lemma 3.7.** With the same assumptions as in lemma 3.3, the following inequalities hold for all \( t > 1 \)

\[
\begin{align*}
\| G^w_h (x \cdot \xi + F(\xi) - w(x)) v \|_{L^\infty} &\lesssim t^{-3/2} \left( \| v \|_{L^2} + \| L^2 v \|_{L^2} \right), \\
\| G^w_h (x \cdot \xi + F(\xi) - w(x)) v \|_{L^2} &\lesssim t^{-2} \left( \| v \|_{L^2} + \| L^2 v \|_{L^2} \right),
\end{align*}
\]

(3.25)

(3.26)

**Proof.** From proposition 2.3 and (2.45) and (2.46), we have

\[
\begin{align*}
(x \cdot \xi + F(\xi) - w(x)) \gamma \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) &= \gamma \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) (x \cdot \xi + F(\xi) - w(x)) \\
&+ i \sqrt{h} \sum_{k=1}^{2} \gamma_k \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) [F''_k(\xi_k)(\xi_k - d(\xi_k) - (x_k + F'_k(\xi_k))] \\
&+ (\gamma(\xi + F(\xi)) \gamma \left( \frac{x + F'(\xi)}{\sqrt{h}} \right) - \gamma(\frac{x + F'(\xi)}{\sqrt{h}}) \gamma(\xi + F(\xi)) - (\gamma(\xi) \gamma(\frac{x + F'(\xi)}{\sqrt{h}}) - \gamma(\frac{x + F'(\xi)}{\sqrt{h}}) \gamma(\xi)),
\end{align*}
\]

where \( \gamma_k(\xi) \) denotes \( \partial_{\xi_k} \gamma(\xi) \). By lemmas 2.6–2.7, there are symbols \( a_k(x, \xi), b_k(x, \xi) \in S_{k} \left( \left( \frac{x + F'(\xi)}{\sqrt{h}} \right)^{-2} \right) \) such that
\( \langle x \cdot \xi + F(\xi) - w(x) \rangle \geq \frac{1}{2} \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) \]
\[= \gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) \langle x \cdot \xi + F(\xi) - w(x) \rangle + \hbar \sum_{k=1}^{2} \Gamma_k \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) E_k(x, \xi) \]
\[+ \sum_{k=1}^{2} (h_a k(x, \xi) (x_k + F_k'(\xi_k))) + h^2 b_k(x, \xi), \quad (3.27)\]

where \( E_k(x, \xi) := F_k''(\xi_k) \hat{e}_k(x, \xi) - 1 \in S_0(1) \) by (1.5) and lemma 2.1 and \( \Gamma_k(x, \xi) := i \gamma_k(\xi_k) \xi_k \in C_{0}^{\infty}(\mathbb{R}^2) \), satisfying \( \Gamma_k \equiv 0 \) in a neighbourhood of zero.

We now deal with the first term on the right-hand side of (3.27). Since \( \nabla \xi (x \cdot \xi + F(\xi)) |_{\xi = d_{\phi}(x)} = 0 \), one would get
\[x \cdot \xi + F(\xi) \]
\[= x \cdot d\phi(x) + F(d\phi(x)) + \sum_{k=1}^{2} (\xi_k - d\phi_k(x_k))^2 \int_{0}^{1} F_k''(s\xi_k + (1-s)d\phi_k(x_k)) (1-s) ds \]
\[= w(x) + \sum_{k=1}^{2} \tilde{b}_k(x, \xi) (x_k + F_k'(\xi_k))^2, \]

where \( w(x) = x \cdot d\phi(x) + F(d\phi(x)) \) and
\[\tilde{b}_k(x, \xi) = \frac{d^2}{dx^2} \left( x \cdot \xi \right) \int_{0}^{1} F_k''(s\xi_k + (1-s)d\phi_k(x_k)) (1-s) ds, \quad k = 1, 2.\]

Since \( \tilde{b}_k(x, \xi) \in S_0(1) \) by (1.5) and lemma 2.1, it follows from (2.13) that
\[x \cdot \xi + F(\xi) - w(x) = \sum_{k=1}^{2} \left( \tilde{b}_k(x, \xi) \xi_k + F_k'(\xi_k)ight)^2 + h \xi_k (x, \xi) \xi_k + F_k'(\xi_k) + h^2 d_k(x, \xi), \quad (3.28)\]

for some symbols \( c_k(x, \xi), d_k(x, \xi) \in S_0(1) \).

Substituting (3.28) into (3.27), then taking the Weyl quantisation, we get
\[G_{\hbar}^n (x \cdot \xi + F(\xi) - w(x)) \nu_{\Lambda} \]
\[= \hbar^2 G_{\hbar}^n \left( \gamma \left( \frac{x + F'(\xi)}{\sqrt{\hbar}} \right) \right) \circ \sum_{k=1}^{2} \left[ G_{\hbar}^n \left( \tilde{b}_k \right) \circ L_k^2 v + G_{\hbar}^n \left( c_k \right) \circ L_k v + G_{\hbar}^n \left( d_k \right) v \right] \]
\[+ \hbar^2 \sum_{k=1}^{2} \left[ G_{\hbar}^n \left( \Gamma_k \right) \circ L_k \right] (x, \xi) + \hbar^2 v \sum_{k=1}^{2} \left[ G_{\hbar}^n \left( a_k \right) \circ L_k v + G_{\hbar}^n \left( b_k \right) v \right]. \]

Therefore, using propositions 2.4–2.6 and lemma 2.5, we estimate
\[\| G_{\hbar}^n (x \cdot \xi + F(\xi) - w(x)) \nu_{\Lambda} \|_{L^\infty_{\mathbb{R}}} \lesssim t^{-3/2} (\| v \|_{L^2} + \| L v \|_{L^2} + \| L^2 v \|_{L^2}), \]
\[\| G_{\hbar}^n (x \cdot \xi + F(\xi) - w(x)) \nu_{\Lambda} \|_{L^2_{\mathbb{R}}} \lesssim t^{-2} (\| v \|_{L^2} + \| L v \|_{L^2} + \| L^2 v \|_{L^2}), \]

which together with the Sobolev type inequality (3.14) yields the desired estimates in lemma 3.7.
Proof. [Proof of the decay estimate (1.10)]. It follows from lemmas 3.4–3.7 that there exists a constant $C_4 > 0$ such that
\[
\|R(v)\|_{L^\infty_x} \leq C_4 t^{-3/2} \left(1 + \|v\|_{L^\infty_x} + \|v_A\|_{L^\infty_x}\right) \left(\|v\|_{L^2_t} + \|L^2 v\|_{L^2_t}\right),
\] (3.29)
and
\[
\|R(v)\|_{L^2_t} \leq C_4 t^{-2} \left(1 + \|v\|_{L^\infty_x} + \|v_A\|_{L^\infty_x}\right) \left(\|v\|_{L^2_t} + \|L^2 v\|_{L^2_t}\right).
\] (3.30)
Let
\[
A = \max\{C_1, C_2, 24C_4, C_5\},
\] (3.31)
\[
\varepsilon_0 = \min\left\{\frac{1}{6A}, \frac{1}{32A|\lambda|^2C_2}\right\},
\] (3.32)
where $C_1, C_2, C_3, C_4, C_5$ are the constants in (3.2), lemmas 3.2 and 3.3, (3.29), (3.40), respectively.

In what follows, we assume that $0 < \varepsilon < \varepsilon_0$ and $\nu$ satisfies the following bootstrap assumption on $t \in (1, T_1)$:
\[
\|\nu(t, x)\|_{L^\infty_x} \leq 4A\varepsilon.
\] (3.33)
From (3.2), (3.10) and (3.31), we see that $T_1 > 1$.

Claim 3.1. With the preceding notations, $\|L^2 v(t, \cdot)\|_{L^2} \leq 2\varepsilon^4|\lambda|^2 C_2^2$.

Proof. We notice the fundamental commutation property
\[
[D_t - G^n_k(x \cdot \xi + F(\xi)), L^2] = 0,
\]
that follows by direct computation (one can also see that more easily going back to the non-semiclassical coordinates). Applying the operator $L^2$ to the equation (3.9), we get
\[
(D_t - G^n_k(x \cdot \xi + F(\xi))) L^2 v = \mu^{-1} L^2 (|v| v).
\] (3.34)
Since $G^n_k(x \cdot \xi + F(\xi))$ is self-adjoint on $L^2$ by proposition 2.1, it follows from lemmas 3.1 and 3.2, the Sobolev type inequality (3.13) and the bootstrap assumption (3.33) that,
\[
\frac{d}{dt}\|L^2 v\|_{L^2} \leq \frac{2C_2|\lambda|\|v\|_{L^\infty_x}}{t} \left(\|v\|_{L^2_t} + \|L^2 v\|_{L^2_t}\right)
\] \[
\leq \frac{8A|\lambda|^2 C_2\varepsilon}{t} \left(\varepsilon + \|L^2 v\|_{L^2_t}\right).
\] (3.35)
By (3.11) and (3.1)
\[
\|L^2 v(1, \cdot)\|_{L^2} = \varepsilon \sum_{k=1}^{2} \|\left(x_k + F'_k(D)\right)^2 u_0\|_{L^2} \leq \varepsilon,
\]
we get after integrating the inequality (3.35) from 1 to $t$
\[
\varepsilon + \|L^2 v(t, \cdot)\|_{L^2} \leq 2\varepsilon + \int_1^t \frac{8A|\lambda|^2C_2\varepsilon}{s} \left(\varepsilon + \|L^2 v(s, \cdot)\|_{L^2}\right) ds.
\]
From Gronwall’s inequality, we have the desired estimate in claim 3.1. 

\[
\Box
\]
Lemma 3.3 and claim 3.1 imply
\[
\|v_{A_t}(t,x)\|_{L^\infty_x} \leq C_1 t^{-1/2} \left( \varepsilon + 2 \varepsilon t^{8A_t|\bar{c}|} \right) \leq 4 C_3 \varepsilon \leq 1 + 8A_{t} C_3 \varepsilon. \tag{3.36}
\]

This together with the bootstrap assumption (3.33) and (3.31) yields
\[
\|v_{A_t}(t,x)\|_{L^\infty_x} \leq \|v(t,x)\|_{L^\infty_x} + \|v_{A_t}(t,x)\|_{L^\infty_x} \leq 4 A \varepsilon + C_4 \leq 8 A \varepsilon. \tag{3.37}
\]

Combining (3.33), (3.37) and claim 3.1, we can upgrade the estimates of \( R(v) \) in (3.29) and (3.30) to
\[
\|R(v)\|_{L^\infty_x} \leq 6 C_4 \varepsilon t^{-3/2 + 8A_{t} |\bar{c}|}, \quad \|R(v)\|_{L^2_x} \leq 6 C_4 \varepsilon t^{-2 + 8A_{t} |\bar{c}|}. \tag{3.38}
\]

\textbf{Claim 3.2.} With the preceding notations, \( \|v_{A_t}(t,\cdot)\|_{L^\infty_x} \leq 2 A \varepsilon. \)

\textbf{Proof.} Multiplying the equation (3.15) by \( v_{A_t} \), then taking Imaginary part, we get
\[
\partial_t |v_{A_t}|^2 = -3 \Re \{ R(v) v_{A_t} \},
\]
which implies
\[
\partial_t |v_{A_t}| \leq \frac{1}{2} |R(v)|.
\]

Integrating the above inequality from 1 to \( t \), then using (3.38) and (3.31) and (3.32), we obtain
\[
\|v_{A_t}(t,x)\|_{L^\infty_x} \leq \|v_{A_t}(1,x)\|_{L^\infty_x} + \frac{1}{2} \int_1^t \|R(v)\|_{L^\infty_x} \, ds \\
\leq C_1 \varepsilon + 3 C_4 \varepsilon \int_1^t s^{-3/2 + 8A_{t} |\bar{c}|} \, ds \\
\leq C_1 \varepsilon + \frac{6 C_4 \varepsilon}{1 - 16 A_{t} |\bar{c}|} \leq 2 A \varepsilon, \tag{3.39}
\]
where the second inequality holds since by proposition 2.5, there exists a absolute constant \( C_5 > 0 \)
\[
\|v_{A_t}(1,x)\|_{L^\infty_x} = \|G_{\eta}^\infty \left( \gamma \left( \frac{x + F'(\xi)}{\sqrt{\eta}} \right) \right) \|_{\hat{L}^\infty_x} \leq C_5 \|v\|_{L^2} \leq C_5 \varepsilon. \tag{3.40}
\]

The estimates in (3.36) and claim 3.2 imply
\[
\|v(t,x)\|_{L^\infty_x} \leq \|v_{A_t}(t,x)\|_{L^\infty_x} + \|v_{A_t}(t,x)\|_{L^\infty_x} \leq 2 A \varepsilon + C_4 \|v\|_{L^2} \leq 3 A \varepsilon. \tag{3.41}
\]

A standard continuation argument then yields that \( T_1 = \infty \) and the estimates in claims 3.1, 3.2 and (3.41) hold for all \( t > 1 \). Estimate (3.41) together with (3.10) yields the desired time decay estimate (1.10), from which theorem 1.1 follows. \( \square \)

\textbf{4. Proof of theorems 1.2–1.4}

In this section, we prove theorems 1.2–1.4 by deducing the long-time behaviour of solutions from the associated ODE dynamics. It is divided into three subsections.
4.1. Proof of theorem 1.2

The proof is inspired by section 4 of Shimomura [32]. By the definition of $\Phi(t,x)$ in (1.11) and Claim 3.2, we have that
\[
\|\Phi(t,x)\|_{L^\infty_t L^\infty_x} \leq \int_1^t s^{-1} \|v_\Lambda(t,x)\|_{L^\infty_x} \, ds \leq 2A \varepsilon \log t, \quad t > 1.
\] (4.1)

Let
\[
z(t,x) = v_\Lambda(t,x) e^{-i(w(x)t + \lambda \Phi(t,x))}, \quad t > 1.
\] (4.2)

From the equations (3.15) and (4.2), we have that
\[
\partial_t z(t,x) = iR(v) e^{-i(w(x)t + \lambda \Phi(t,x))},
\]
so that for all $t_2 > t_1 > 1$
\[
z(t_2,x) - z(t_1,x) = i \int_{t_1}^{t_2} R(v) e^{-i(w(x)s + \lambda \Phi(s,x))} \, ds.
\]

Estimates (3.38) and (4.1) imply
\[
\|z(t_2,x) - z(t_1,x)\|_{L^\infty_x} \lesssim \varepsilon \int_{t_1}^{t_2} \|R(v)\|_{L^\infty_x} e^{3\lambda|\Phi(s,x)|} \, ds \\
\lesssim \varepsilon \int_{t_1}^{t_2} s^{-3/2} + C_6 \varepsilon + 2\lambda A \varepsilon \, ds \\
\lesssim \varepsilon t_1^{-1/2} + C_6 \varepsilon + 2\lambda A \varepsilon,
\]
where $C_6 = 8A|\lambda|C_2$. A similar argument shows that
\[
\|z(t_2,x) - z(t_1,x)\|_{L^2_x} \lesssim \varepsilon t_1^{-1 + 2\lambda A \varepsilon}, \quad \text{for all } t_2 > t_1 > 1.
\]

Therefore, there exists $z_+(x) \in L^\infty_t \cap L^2_x$ such that
\[
\|z(t,x) - z_+(x)\|_{L^\infty_x} \lesssim \varepsilon t_1^{-1/2 + C_6 \varepsilon + 2\lambda A \varepsilon}, \quad \|z(t,x) - z_+(x)\|_{L^2_x} \lesssim \varepsilon t_1^{-1 + C_6 \varepsilon + 2\lambda A \varepsilon},
\] (4.3)

from which theorem 1.2 follows.

4.2. Proof of theorem 1.3

Since $\Im \lambda = 0$ and $\Phi(t,x)$ is a real-valued function, it follows from (4.2) that $|v_\Lambda(t,x)| = |z(t,x)|$. Recalling the definitions of $\Phi(t,x)$ and $\phi_+(x)$ in (1.11) and (1.11), it follows that
\[
\phi_+(x) + |z_+(x)| \log t - \Phi(t,x) = \int_1^t s^{-1} (|z(x,s)| - |z_+(x)|) \, ds.
\] (4.4)

Applying (4.3) to (4.4), we find the estimate
\[
\|\phi_+(x) + |z_+(x)| \log t - \Phi(t,x)\|_{L^\infty_x} \lesssim t^{-1/2 + C_6 \varepsilon}.
\] (4.5)

Estimates (4.3) and (4.5) imply
\[\|z_+(x) e^{i(w(x) + \lambda \phi_+(x) + \lambda |z_+(x)| \log t)} - v_\lambda\|_{L^\infty} \lesssim \|z_+(x) e^{i(w(x) + \lambda \phi_+(x) + \lambda |z_+(x)| \log t)} - z_+(x) e^{i(w(x) + \lambda \Phi(x, \lambda))}\|_{L^\infty}
+ \|z_+(x) e^{i(w(x) + \lambda \Phi(x, \lambda))} - v_{\lambda}(t, x)\|_{L^\infty} \lesssim \|z_+(x)\|_{L^\infty} \|\phi_+(x) + |z_+(x)| \log t - \Phi(t, x)\|_{L^\infty} + \|z_+(x) - z(t, x)\|_{L^\infty} \lesssim t^{-1/2 + C_\varepsilon}.\]  

(4.6)

On the other hand, from Lemma 3.3 and claim 3.1
\[\|v_{\lambda}(t, x)\|_{L^\infty} \lesssim t^{-1/2 + C_\varepsilon}, \quad \|v_{\lambda}(t, x)\|_{L^2} \lesssim t^{-1 + C_\varepsilon}, \quad t > 1.\]  

(4.7)

we obtain, by substituting (4.7) into (4.6)
\[\|v(t, x) - z_+(x) e^{i(w(x) + \lambda \phi_+(x) + \lambda |z_+(x)| \log t)}\|_{L^\infty} \lesssim t^{-1/2 + C_\varepsilon}, \quad t > 1.\]  

(4.8)

In the same manner, we get
\[\|v(t, x) - z_+(x) e^{i(w(x) + \lambda \phi_+(x) + \lambda |z_+(x)| \log t)}\|_{L^2} \lesssim t^{-1 + C_\varepsilon}, \quad t > 1.\]  

(4.9)

The asymptotic formula (1.13) then follows from the estimates (3.10), (4.8) and (4.9).

It remains to derive the modified linear scattering formula (1.14). From the asymptotic formula (1.13), we have
\[e^{-iF(D)\xi} e^{-i\lambda \phi_+(\xi) + i|\xi| \log t}) u(t, x) = e^{-iF(D)\xi} e^{-i\lambda \phi_+(\xi) + i|\xi| \log t}) u(t, x) = \frac{1}{(2\pi)^2} \int e^{i(\xi - t\xi)} \xi e^{-iF(\xi)\frac{1}{t} + i|\xi| \log t}) d\xi + O_{L^2}(t^{-1 + C_\varepsilon}),\]  

(4.10)

as \(t \to \infty\). Making a change of variables then using (1.29), we obtain
\[e^{-iF(D)\xi} e^{-i\lambda \phi_+(\xi) + i|\xi| \log t}) u(t, x) = \frac{1}{(2\pi)^2} \int e^{i(\xi - t\xi)} \xi e^{-iF(\xi)\frac{1}{t} + i|\xi| \log t}) d\xi + O_{L^2}(t^{-1 + C_\varepsilon}),\]  

(4.11)

Since \(F_k(\xi_k) = c_k^2 \xi_k^2 + c_k^1 \xi_k + c_k^0\), we have \(d\phi_k(y_k) = -\frac{y_k + c_k^1}{2c_k^2}, k = 1, 2\). Therefore
\[e^{-iF(D)\xi} e^{-i\lambda \phi_+(\xi) + i|\xi| \log t}) u(t, x) = \frac{1}{(2\pi)^2} \int e^{i(\xi - t\xi)} \xi e^{-iF(\xi)\frac{1}{t} + i|\xi| \log t}) d\xi + O_{L^2}(t^{-1 + C_\varepsilon}),\]  

(4.12)

where in the last step we used \(\int_{-\infty}^{+\infty} e^{-i\xi \eta} d\eta = \sqrt{\pi} e^{-i\xi^2/4}.\) Letting \(t \to \infty\) in (4.12) and using the dominated convergence theorem, we get
\[\lim_{t \to \infty} e^{-iF(D)\xi} e^{-i\lambda \phi_+(\xi) + i|\xi| \log t}) u(t, x) = \frac{1}{(2\pi)^2} \int e^{i(\xi - t\xi)} \xi e^{-iF(\xi)\frac{1}{t} + i|\xi| \log t}) d\xi = u_+(x),\]  

from which the modified scattering formula (1.14) follows.
4.3. Proof of theorem 1.4

We start by deriving the asymptotic formula (4.14) for $\Phi(t,x)$, since (4.5) does not hold in the case $\Im \lambda > 0$. Note that by the definition of $\Phi(t,x)$ in (1.11) and (4.2)

$$\partial_t \Phi(t,x) = i\lambda \Phi(t,x),$$

so that

$$\partial_t e^{\Im \lambda \Phi(t,x)} = \Im \lambda e^{\Im \lambda \Phi(t,x)}, \quad t > 1;$$

Integrating the above equation form 1 to $t$, we get

$$e^{\Im \lambda \Phi(t,x)} = 1 + \Im \lambda \int_1^t e^{\Im \lambda \Phi(s,x)} |z(s,x)| ds, \quad t > 1.$$

Recalling the definition of $\psi_+(x)$ in (1.16), it follows that

$$e^{\Im \lambda \Phi(t,x)} - (1 + \Im \lambda |z_+(x)| \log t - \psi_+(x)) = -\Im \lambda \int_t^\infty s^{-1} (|z(s,x)| - |z_+(x)|) ds. \quad (4.13)$$

Applying (4.3), we find the estimate

$$\|e^{\Im \lambda \Phi(t,x)} - (1 + \Im \lambda |z_+(x)| \log t - \psi_+(x))\|_{L^\infty} \lesssim e t^{-1/2} + C_0 e + 2 \Im \lambda \varepsilon. \quad (4.14)$$

In particular, we have that

$$1 + \Im \lambda |z_+(x)| \log t + \psi_+(x) \geq \frac{1}{2} \quad (4.15)$$

provided that $\varepsilon > 0$ is sufficiently small.

We now prove the asymptotic formula (1.18). By triangle inequality

$$\|e^{i(w(s)+AS(t,x))} z_+(x) - v_A(t,x)\|_{L^\infty} \leq \|e^{iw(s)}\| \left( e^{iAS(t,x)} - e^{i\Phi(t,x)} \right) z_+ + \|e^{i\Phi(t,x)}\|_{L^\infty} \quad (4.16)$$

where we used $|ae^{iw(s)}| = e^{-\Im \lambda \Phi(t,x)} \leq 1$. By the definition of $S(t,x)$ in (1.17), (4.15) and the estimate (4.14), we have

$$\|e^{iAS(t,x)} - e^{-\Im \lambda \Phi(t,x)}\|_{L^\infty} \leq \|1 + 1 + \Im \lambda |z_+(x)| \log t + \psi_+(x)\|_{L^\infty} \lesssim e f^{-1/2} + C_0 e + 2 \Im \lambda \varepsilon, \quad (4.17)$$

A similar argument shows that

$$\|\left( e^{iAS(t,x)} - e^{i\Phi(t,x)} \right) e^{-\Im \lambda \Phi(t,x)}\|_{L^\infty} \lesssim \|S(t,x) - \Phi(t,x)\|_{L^\infty} \lesssim \|e^{iAS(t,x)} - e^{i\Phi(t,x)}\|_{L^\infty} \lesssim \|1 + 1 + \Im \lambda |z_+(x)| \log t + \psi_+(x)\|_{L^\infty} - \|e^{\Im \lambda \Phi(t,x)}\|_{L^\infty} \leq f^{-1/2} + C_0 e + 2 \Im \lambda \varepsilon \quad (4.18)$$

Substituting the estimates (4.3), (4.17) and (4.18) into (4.16), and using (4.7), we get

$$\|e^{i(w(s)+AS(t,x))} z_+(x) - v(t,x)\|_{L^\infty} \lesssim f^{-1/2} + C_0 e + 2 \Im \lambda \varepsilon, \quad t > 1. \quad (4.19)$$
In the same manner, we get
\[
\|e^{i(\psi(x)+\lambda S(x))}z_+(x) - v(t,x)\|_{L^2} \lesssim t^{-1+3\lambda \varepsilon+2\lambda^2 \lambda \varepsilon}, \quad t > 1. \tag{4.20}
\]
The asymptotic formula (1.18) then follows from the estimates (3.10), (4.19) and (4.20).

Using the same method as that used to derive (1.14), we obtain the modified linear scattering formula (1.19) easily and omit the details.

It remains to prove the limit (1.20). By (3.10), it is equivalent to proving that
\[
\lim_{t \to \infty} t \log t \|v(t,x)\|_{L^\infty} = \frac{1}{3\lambda}. \tag{4.21}
\]
By the definition of \(\psi_+(x)\) in (1.16) and the estimate (4.3), we have that
\[
\|\psi_+(x)\|_{L^\infty} \lesssim \int_1^\infty s^{-1} \|z(s,x) - z_+(x)\|_{L^\infty} \, ds \lesssim \varepsilon \int_1^\infty s^{-2+6\lambda \varepsilon+2\lambda^2 \lambda \varepsilon} \, ds \lesssim \varepsilon.
\]
Thus we have \(1 + \psi_+(x) > 0\), provided that \(\varepsilon > 0\) is sufficient small. It then follows from the asymptotic formulas (4.19) and (1.21) that
\[
\log t \|v(t,x)\| \leq \frac{\|z_+(x)\| \log t}{1 + 3\lambda \|z_+(x)\| \log t + \psi_+(x)} + O \left( t^{-1/2+6\lambda \varepsilon} \log t \right) \leq \frac{1}{3\lambda} + O \left( t^{-1/2+6\lambda \varepsilon} \log t \right),
\]
which implies
\[
\limsup_{t \to \infty} \log t \|v(t,x)\|_{L^\infty} \leq \frac{1}{3\lambda}.
\]
Therefore, for (4.21) it suffices to prove that
\[
\liminf_{t \to \infty} \log t \|v(t,x)\|_{L^\infty} \geq \frac{1}{3\lambda}. \tag{4.22}
\]
Assume for a while that we have proved

**Claim 4.1.** If the limit function \(z_+(x)\) in (4.3) satisfies \(z_+(x) = 0\) for a.e. \(x \in \mathbb{R}^2\), then we must have \(u_0 = 0\).

Since \(u_0 \neq 0\), there exists \(x_0 \in \mathbb{R}^2\) such that \(\|z_+(x)\|_{L^\infty} \geq \|z_+(x_0)\| > 0\). Therefore, using (1.21) and (4.19), we estimate
\[
\log t \|v(t,x)\|_{L^\infty} \geq \frac{\log t \|z_+(x_0)\|}{1 + 3\lambda \|z_+(x_0)\| \log t + \psi_+(x_0)} + O \left( t^{-1/2+6\lambda \varepsilon} \log t \right).
\]
Letting \(t \to \infty\), we get the limit (4.22), from which the desired limit (1.20) follows.

**Proof of claim 4.1.** Since \(z_+ = 0\), it follows from (4.2) and (4.3) that
\[
\|v_\Lambda (t,x)\|_{L^\infty} \lesssim \|z(t,x)\|_{L^\infty} \lesssim t^{-1/2+6\lambda \varepsilon}, \quad \forall t > 1
\]
which together with (4.7) and (3.10) implies
\[
\|u(t,x)\|_{L^\infty} \lesssim t^{-3/2+6\lambda \varepsilon}, \quad \forall t > 1. \tag{4.23}
\]
On the other hand, since \(z_+ = 0\) implies \(u_+ = 0\) (see (1.15)), it follows from the equation (1.9), the asymptotic formula (1.19) and Duhamel’s formula that
\[
u(t,x) = \lambda \int_t^\infty e^{iD(t-x)} (|u|u)(x) \, ds.
\]
Using successively Strichartz’s estimate, Hölder’s inequality and (4.23) to get
\[ \|u(t,x)\|_{L^\infty_t(L^2)} \lesssim \int_t^\infty \|u(t,x)\|_{L^\infty_t(L^2)} \|u(x)\|_{L^\infty} ds \]
\[ \lesssim \|u(t,x)\|_{L^\infty_t(L^2)} T^{-1/2} + C\epsilon. \]
Choosing \( T > 1 \) sufficiently large, we deduce that \( \|u(t,x)\|_{L^\infty_t(L^2)} = 0 \), which together with the uniqueness of solutions implies \( u \equiv 0 \). This finishes the proof of claim 4.1. \qed

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix

This appendix is devoted to the proof of lemma 3.2.

Proof of lemma 3.2. We start by recalling that (see lemma 2.1)
\[ e_k(x,\xi) = \frac{x_k + F_k'(\xi_k)}{\xi_k - d\phi_k(x_k)} \in S_0(1), \quad \tilde{e}_k(x,\xi) = \frac{\xi_k - d\phi_k(x_k)}{x_k + F_k'(\xi_k)} \in S_0(1). \]
Using (2.13) and (2.23), we can write, for some symbols \( c_{ij} \), \( d_{ij} \in S_0(1) \), \( 1 \leq k, j \leq 2 \)
\[ (x_k + F_k'(\xi_k))^2 = e_k^2(x,\xi) (\xi_k - d\phi_k(x_k))^2 \]
\[ = e_k^2(x,\xi) \xi_k (\xi_k - d\phi_k(x_k))^2 + hc_{k1} (x,\xi) \xi_k (x_k + F_k'(\xi_k)) + h^2 d_{k1} (x,\xi), \quad (A.1) \]
\[ (\xi_k - d\phi_k(x_k))^2 = \tilde{e}_k^2(x,\xi) (x_k + F_k'(\xi_k))^2 \]
\[ = \tilde{e}_k^2(x,\xi) \xi_k (x_k + F_k'(\xi_k))^2 + hc_{k2} (x,\xi) \xi_k (\xi_k - d\phi_k(x_k)) + h^2 d_{k2} (x,\xi). \quad (A.2) \]
Moreover, using (2.22), we can write
\[ x_k + F_k'(\xi_k) = e_k(x,\xi) (\xi_k - d\phi_k(x_k)) = e_k(x,\xi) \xi_k (\xi_k - d\phi_k(x_k)) + h b_k(x,\xi), \quad k = 1, 2, \quad (A.3) \]
for some symbols \( b_k(x,\xi) \in S_0(1), k = 1, 2 \). Substituting (A.3) into (A.1), then taking the Weyl quantisation, we get
\[ L_k^2 = \frac{1}{h^2} G_k^k(e_k^2) \circ G_k^k(\xi_k (\xi_k - d\phi_k(x_k))^2) + G_k^k(c_{k1}) \circ G_k^k(e_k) \circ \frac{1}{h} G_k^k(\xi_k - d\phi_k(x_k)) \]
\[ + G_k^k(c_{k1}) \circ G_k^k(b_k) + G_k^k(d_{k1}), \quad k = 1, 2. \quad (A.4) \]
Therefore, using proposition 2.4 we estimate
\[
\left\| L_h^2 \right\|_{L^2} \lesssim \left\| \frac{1}{h^2} G_h^\omega \left( (\xi_k - d\phi_k(x_k))^2 \right) \right\|_{L^2} \\
+ \left\| \frac{1}{h} G_h^\omega (\xi_k - d\phi_k(x_k)) \right\|_{L^2} + \left\| v \right\|_{L^\infty} \left\| v \right\|_{L^2}.
\]  

(A.5)

Next, we consider the estimate of the right-hand side of (A.5). Since

\[
\partial_{x_k} (|v|) = \frac{3}{2} |v| \partial_{x_k} v + \frac{1}{2} |v|^{-1} v \partial_{x_k} (|v|^2) + \frac{3}{2} |v|^{-1} \partial_{x_k} v \Re(\partial_{x_k} v \overline{v}) \\
- \frac{1}{2} |v|^{-1} v \partial_{x_k} (|v|^2) \Re(\partial_{x_k} v \overline{v}) + |v|^{-1} |v| \partial_{x_k} v^2, \quad k = 1, 2.
\]  

(A.6)

we have, by direct calculation

\[
G_h^\omega \left( (\xi_k - d\phi_k(x_k))^2 \right) (|v|) \\
= \left[ -h^2 \partial_{x_k}^2 + \left( (d\phi_k(x_k))^2 + \text{hid}^2 \phi_k(x_k) \right) + 2hi d\phi_k(x_k) \partial_{x_k} \right] (|v|) \\
= -\frac{3}{2} h^2 |v| \partial_{x_k} v - \frac{1}{2} h^2 |v|^{-1} v \partial_{x_k} v - \frac{3}{2} h^2 |v|^{-1} \partial_{x_k} v \Re(\partial_{x_k} v \overline{v}) \\
+ \frac{1}{2} h^2 |v|^{-1} v \partial_{x_k} (|v|^2) - h^2 |v|^{-1} |v| \partial_{x_k} v^2 + \left( (d\phi_k(x_k))^2 + \text{hid}^2 \phi_k(x_k) \right) (|v|) \\
+ 2hi d\phi_k(x_k) \left( \frac{3}{2} |v| \partial_{x_k} v + \frac{1}{2} |v|^{-1} v \partial_{x_k} v \right)
\]  

\[
= \frac{3}{2} |v| \left( -h^2 \partial_{x_k}^2 + \left( (d\phi_k(x_k))^2 + \text{hid}^2 \phi_k(x_k) \right) + 2hi d\phi_k(x_k) \partial_{x_k} \right) v \\
+ \frac{1}{2} |v|^{-1} v \left( -h^2 \partial_{x_k}^2 + ((d\phi_k(x_k))^2 + \text{hid}^2 \phi_k(x_k)) + 2hi d\phi_k(x_k) \partial_{x_k} \right) v \\
- \frac{3}{2} |v|^{-1} (h \partial_{x_k} v - id\phi_k(x_k)v) \Re((h \partial_{x_k} v - id\phi_k(x_k)v) \overline{v}) \\
+ \frac{1}{2} |v|^{-1} h \partial_{x_k} v - id\phi_k(x_k)v \Re((h \partial_{x_k} v - id\phi_k(x_k)v) \overline{v}) \\
- \frac{1}{2} |v|^{-1} h \partial_{x_k} v - id\phi_k(x_k)v \partial_{x_k} v^2, \quad k = 1, 2.
\]

Substituting \( h \partial_{x_k} - id\phi_k(x_k) = iG_h^\omega (\xi_k - d\phi_k(x_k)) \) into the above expression, we get

\[
G_h^\omega \left( (\xi_k - d\phi_k(x_k))^2 \right) (|v|) \\
= \frac{3}{2} |v| G_h^\omega ((\xi_k - d\phi_k(x_k))^2) v + \frac{1}{2} |v|^{-1} v \mathcal{G}_h^\omega ((\xi_k - d\phi_k(x_k))^2) v \\
+ \frac{3}{2} |v|^{-1} iG_h^\omega (\xi_k - d\phi_k(x_k)) v \Im(G_h^\omega (\xi_k - d\phi_k(x_k)) (v) \overline{\psi}) \\
- \frac{1}{2} |v|^{-1} v \mathcal{G}_h^\omega ((\xi_k - d\phi_k(x_k))^2) v \Im(G_h^\omega (\xi_k - d\phi_k(x_k)) (v) \overline{\psi}) \\
- \frac{1}{2} |v|^{-1} v \mathcal{G}_h^\omega ((\xi_k - d\phi_k(x_k))^2) v^2, \quad k = 1, 2.
\]  

(A.7)

By proposition 2.4, we have, for \( k = 1, 2 \)

\[
\left\| \frac{1}{h^2} G_h^\omega \left( (\xi_k - d\phi_k(x_k))^2 \right) \right\|_{L^2} \\
\lesssim \left\| v \right\|_{L^\infty} \left\| \frac{1}{h^2} G_h^\omega (\xi_k - d\phi_k(x_k))^2 \right\|_{L^2} + \left\| \frac{1}{h} G_h^\omega (\xi_k - d\phi_k(x_k)) \right\|_{L^2}^2.
\]
Substitution of the above inequality into (A.5) gives, for $k = 1, 2$
\[
\| L^2_k (|v| v) \|_{L^2} + \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) (|v| v) \|_{L^2} \leq \| v \|_{L^\infty} \left( \| v \|_{L^2} + \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} \right) + \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} + \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) (|v| v) \|_{L^2}. \quad (A.8)
\]

We now estimate the right-hand side of (A.8). Notice that
\[
\partial_x e^{-i \phi_k (x_k)} v = ie^{-i \phi_k (x_k)} \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v,
\]
and
\[
\partial_x e^{-i \phi_k (x_k)} v = -e^{-i \phi_k (x_k)} \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v;
\]
so that by Gagliardo–Nirenberg’s inequality
\[
\| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} = \| \partial_x e^{-i \phi_k (x_k)} v \|_{L^2}^2 \leq \| e^{-i \phi_k (x_k)} v \|_{L^2} \| \partial_x e^{-i \phi_k (x_k)} (|v| v) \|_{L^2}^2 \leq \| v \|_{L^\infty} \| \partial_x e^{-i \phi_k (x_k)} (|v| v) \|_{L^2} \leq C (\eta) \| v \|_{L^\infty} \| v \|_{L^2} + \eta \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) (|v| v) \|_{L^2}. \quad (A.11)
\]

By a similar argument and Young’s inequality, we find the estimate
\[
\| L^2_k (|v| v) \|_{L^2} \leq \| v \|_{L^\infty} \left( \| v \|_{L^2} + \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} \right), \quad k = 1, 2. \quad (A.13)
\]

Taking the Weyl quantisation in (A.2), then using propositions 2.4 and 2.2, we obtain
\[
\| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} \leq \| L^2_k v \|_{L^2} + \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} + \| v \|_{L^2}. \quad (A.14)
\]

Moreover, it follows from (A.9) and (A.10), Gagliardo–Nirenberg and Young’s inequality that
\[
\| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} \leq \| e^{-i \phi_k (x_k)} v \|_{L^2} \| \partial_x e^{-i \phi_k (x_k)} v \|_{L^2} \leq \| v \|_{L^2} \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} + \eta \| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2}, \quad (A.15)
\]

Combining (A.14) and (A.15), choosing $\eta > 0$ sufficiently small, we obtain
\[
\| \frac{1}{\hbar^2} G_h^n \left( (\xi_k - d \phi_k (x_k))^2 \right) v \|_{L^2} \leq \| L^2_k v \|_{L^2} + \| v \|_{L^2}, \quad k = 1, 2. \quad (A.16)
\]

The inequality (3.13) is now an immediate consequence of (A.13) and (A.16).
Finally, we prove (3.14). Taking the Weyl quantisation in (A.3), then using propositions 2.4 and 2.2, we obtain
\[ \|L^k v\|_{L^2} \leq \frac{1}{\hbar} \bigg\| G_w^k (\xi_k - d\phi_k(x_k)) v\bigg\|_{L^2} + \|v\|_{L^2}, \quad k = 1, 2. \tag{A.17} \]
Substitution of (A.15) and (A.16) into (A.17) yields the desired estimate (3.14). This completes the proof of lemma 3.2.

\[ \square \]

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