Dynamics of Massive Shell ejected in a Supernova Explosion

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Abstract

An expanding shell is accelerated outward by radiation from the remnant star and slowed by ram pressure and accretion as it plows into the interstellar medium. We set up the general relativistic equations of motion for such a shell. In the non-relativistic limit, they reduce to the standard Ostriker-Gunn equations. Furthermore, it is shown that the motion equation can be integrated in general, reducing in this way the study of the dynamics of a shell to a set of four coupled first order differential equations, solvable up to quadrature.

Subject headings: Shells, dynamics, explosion.

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1. INTRODUCTION

The study of the dynamics of a supernova remnant, a plerion, has been greatly developed since the works of Ostriker and Gunn (1971), where they modeled the dynamics of this remnant as a thin shell moving in a background which inside has a radiating mass and outside is dust. Supposing spherical symmetry, they started with the Newton’s second law for describing the acceleration of the shell. The force terms were introduced essentially by hand. This is the equation of motion:

\[ M_s \frac{d^2 R}{dt^2} = -G \left( \frac{M_N + \frac{1}{2} M_S}{R^2} \right) M_S + 4 \pi R^2 \left( P_c - P_{IS} \right), \]  

where \( M_S \) is the mass of the shell, \( M_N \) the mass of the neutron star, \( R \) is the radius of the shell, \( P_c \) is the pressure in the cavity, related to the radiation emitted by the star, and \( P_{IS} \) is the pressure due to the contact with the interstellar medium, defined as \( P_{IS} = \rho_{IS} \left( \frac{dR}{dt} \right)^2 \), a friction term known as the ”snow plow” effect (Oort 1946). The first term represents the gravitational and self-gravitational force, the second a driving force due to the radiation and the third a decelerating one due to the interaction with the medium.

The dynamical study is completed with an equation for the change of the mass of the shell (usually due only to the dust that the shell collects while moving in the medium) and finally an equation for the change of the shell’s internal energy, proportional to the radiation emitted by the star.

This system of equations has been analysed in many ways. Chevalier (1977) and Ostriker & Mc Kee (1988) have analysed the kinetic equation of the shell to show the different stages of the evolution when each of the terms in turn becomes dominant, while Sato (1988a) and Sato & Yamada (1991) have applied this set to the study of the SN1987A in particular. Most of the studies use numerical methods to obtain the change in the luminosity or in the mass with respect to the change in the radius of the remnant.

2. SHELLS IN GENERAL RELATIVITY

On the other hand, the study of the dynamics of a shell separating two backgrounds in the context of General Relativity has developed in a powerful and direct formalism since the firsts works of Werner Israel (1966), and has been applied to cosmology, mainly to inflation (Berezin1987), and to modeling the dynamics of the border between two regions in different states, like bubbles in water, or between two given spaces (Sato 1988b). Nevertheless, it has not been widely used in the study of the dynamics of the plerion.

In the context of General Relativity the equation of motion for the shell is obtained from Einstein’s equations and from the junction conditions for the metric tensor and its first derivative. We will work in spherically symmetric spaces, described by

\[ ds^2 = -e^{2\psi} f dv^2 - 2 b e^{\psi} dr dv + r^2 d\Omega^2, \]

where \( f \) and \( \psi \) are in general functions of the null coordinate \( v \) and of the radius \( r \), and \( b = \pm 1 \) depending whether the null coordinate is advanced or retarded. The kinetic equation for the shell in this case is the following, with our conventions following MTW (Misner, Thorne, & Wheeler 1973),

\[
\left[ n_\alpha \frac{\delta u^\alpha}{\delta \tau} \right] = \frac{M}{R^2} + 8 \pi p, \]  

\[
\sigma(n_\alpha \frac{\delta u^\alpha}{\delta \tau}) = 2 \left\{\left[ T_{\alpha \beta} n^\alpha n^\beta \right] + \frac{p (F_+ + F_-)}{R} \right\}, \]
where $n_\alpha$ is the 4-vector normal to the shell, $u^\alpha$ is the 4-velocity vector, $\tau$ is the proper time, $M = 4 \pi R^2 \sigma$ is the proper mass of the shell, $\sigma$ being the surface energy density of the shell, and $R$ is the radius. A function in square brackets stands for the difference of that function in the region outside, denoted by $+$, with the region inside the shell, denoted by $-$, and evaluated on the shell. Finally $p$ is the pressure of the shell, and $F_\pm$ is given by

$$F_\pm = \sqrt{f_\pm + \dot{R}^2},$$

and dot $\dot{}$ stands for differentiation with respect to the proper time, $\tau$. In the general relativistic context the dynamical study is completed by the conservation equation, which is an equation for the rate of change of the proper mass

$$\dot{M} + p \dot{A} = 4 \pi R^2 \left[ T_{\alpha \beta} u^\alpha n^\beta \right],$$

where $A = 4 \pi R^2$, is the area of the shell. For a derivation of the kinetic equations in some particular cases, see for instance (de la Cruz & Israel 1967).

3. INTEGRATION OF THE MOTION EQUATIONS

In the present work we want to show that the formalism of shells in General Relativity gives not only a motion equation more general than the Newtonian one with tiny correction whose measure is "beyond the present technology", but first: includes all the possible terms which generate the motion of the shell, no more and no less. Second: even though there are terms which in the Newtonian case seem to be highly dissipative, like friction or radiation terms, we will show that all these terms can be described within the formalism of General Relativity. And Third: we will show that the motion equation, in general, allows a first integral, which reduces the problem to a first order dynamical set of equations. We also present the Newtonian limits of this integral of the motion.

For the line element given by equation (2), we have that the 4-velocity and the normal vector, already orthonormalized, at the shell are given by

$$u^\alpha = (\dot{v}, \dot{R}, 0, 0), n_\alpha = b e^\psi (-\dot{R}, \dot{v}, 0, 0)$$

where $\dot{v}$ is related to $\dot{R}$ as follows

$$\dot{v} = -b \dot{R} + \frac{F}{f e^\psi}.$$  

The motion equation (3) is then rewritten as

$$\left[ \frac{\ddot{R} + B}{F} \right] = \frac{M}{R^2} + 8 \pi p,$$

where

$$B = \frac{f_r}{2} + \psi_r F + \frac{b}{2} f_v e^\psi \dot{v}^2.$$  

Now, from the definition of $F$, equation (5), we see that

$$\frac{\dot{R} \dddot{R}}{F} = \dddot{F} - \frac{\dot{f}}{2F},$$
so, after some manipulations, the motion equation (9) can be expressed as

\[
\frac{d}{d\tau} ([F] + \frac{M}{R}) - \frac{\dot{M}}{R} + \left[ \frac{2B\dot{R} - \dot{\hat{f}}}{2F} \right] - 8\pi \dot{R} p = 0.
\]  

(12)

The equation for the rate of change of the mass, equation (6), in the spherical case can be expressed as

\[
\frac{\dot{M}}{R} = \left[ -\frac{f,\nu}{2} + \psi, F\dot{R} \right] - 8\pi \dot{R} p,
\]  

(13)

Substituting this last equation in the motion equation (12), we obtain that

\[
\frac{d}{d\tau} ([F] + \frac{M}{R}) + \left[ \frac{2B\dot{R} - \dot{\hat{f}}}{2F} \right] + \frac{1}{2} f,\psi \dot{\psi}^2 - \psi, r F\dot{R} = 0.
\]  

(14)

Finally, using the formula for \( \dot{v} \), equation (8), can be shown that the expression inside the squared brackets is zero!, so

\[
\frac{d}{d\tau} ([F] + \frac{M}{R}) = 0,
\]  

(15)

which implies that

\[
[F] + \frac{M}{R} = 0,
\]  

(16)

the value of the integration constant, \( C \), is zero, as can be shown substituting back \([F] + \frac{M}{R} = C\) in the second motion equation (11). In this way we have shown that equation (16) is a first integral of the general motion equation of a thin shell with spherical symmetry.

It proves convenient to introduce a function \( m = m(u, r) \) such that

\[
f_\pm = 1 - \frac{2m_\pm}{r},
\]  

(17)

then, using equation (3), it is easy to show that

\[
F_+^2 - F_-^2 = -\frac{2m}{R},
\]  

(18)

where the function \( m \) is defined as \( m = m_+ - m_- \), and in some cases can be identified with the gravitational mass\(^1\). Combining the first integral equation (16), with equation (18), can

\(^1\)In the cases studied by Israel and de la Cruz or Chase, \( m_\pm = m_{1,2} - e_{1,2}^2 \frac{2}{r} \), where \( m_{1,2} \) and \( e_{1,2} \) stand for the gravitational masses and electrical charges in both sides of the shell. They defined \( m = m_2 - m_1 = constant \) as the gravitational mass of the shell and identified this quantity as the conserved total energy of the shell.
be obtain an expression for $F$ on either side of the shell in terms of the proper mass and the gravitational mass

$$F_{\pm} = \frac{m}{M} \mp \frac{M}{2R}. \quad (19)$$

The squared of this last equation, recalling equation (17), results in

$$\dot{R}^2 = \left(\frac{m}{M}\right)^2 - 1 + \frac{m_+ + m_-}{R} + \frac{M^2}{4R^2}, \quad (20)$$

which is the motion equation obtain by Lake (1979) from the Lanczos equation.

4. CONCLUSIONS

This result of being able to integrate the motion equation in general is quite remarkable since it is valid for any matter distribution on either side of the shell, as well as for any type of shell. Furthermore, the integration is valid for any way in which the shell interacts with the two backgrounds, the only requirements have been spherical symmetry, the fact that the backgrounds satisfy the Einstein’s equations, and the conditions of continuity on the shell.

To have a better understanding of the meaning of the first integral of the motion equation, it might help to see its form in the Newtonian limit. For this purpose it proves better to take the form of the motion equation given by equation (13), considering the cases when the masses and the velocity, $\dot{R}$, are smaller than the unit and making a Taylor expansion we obtain

$$m \approx M + \frac{1}{2} M \dot{R}^2 - \frac{M (m_+ + \frac{M}{2})}{R}. \quad (21)$$

The above equation simply states that the sum of the rest energy, the kinetic energy, the mutual potential energy and the gravitational self-energy of the shell, respectively in the rhs, is not constant, but varies according with the quantity $m$.

In this way, the dynamics of a shell moving between any two backgrounds, with spherical symmetry reduces to a set of first order equations, namely, the radial motion equation (20), the balance equation for the proper mass, equation (13), the motion equation for the null coordinate, equation (8), a state equation for the shell, giving a relation between the pressure $p$ and the energy density, $\sigma$, and for specifying the kind of backgrounds between which the shell moves, we have the Einstein’s equations, which for this spherical case are (Barrabes & Israel 1991)

$$m_v = 4 \pi r^2 T_{v,v}; m_r = -4 \pi r^2 T_{v,r}; \psi_r = 4 \pi r T_{rr}, \quad (22)$$

where $T_{\mu\nu}$ is the stress energy tensor, specified by the distribution of matter and energy in the chosen background space.

Despite the above characteristic concerning the dynamics of massive shells, it will be worth while to write down conveniently the second order motion equation in order to obtain the Newtonian limit, and show how equation (11) is recovered. After a direct calculation starting from equation (21), we obtain:

$$M \ddot{R} = -\frac{M (m_- + \frac{M}{2} F_-)}{R^2} + \frac{\dot{m}_- M}{R \dot{R}} + \frac{\dot{m} F_-}{R} - \frac{\dot{M} F_- F_+}{\dot{R}}. \quad (23)$$
Note that the first term in the rhs is nothing else than the relativistic generalization of the gravitational interaction, and the remaining terms are related with the variation of $m_-$, $m$ and $M$. In fact, we have obtained the relativistic version of the second order equation analysed by Ostriker and Gunn (1971) in studying the dynamics of remnants of supernovas modelled by thin spherical shells. To show in a clear way the role played by the three last terms of the above equation, let us consider that the shell separates two Vaidya spacetimes: the interior filled with outgoing radiation and the exterior with ingoing radiation. Thus, after taking the Newtonian limit of equation (23), we arrive to the following expression:

$$M \ddot{R} \approx -\frac{M (m_- + \frac{M}{2})}{R^2} + 8\pi R p + 4\pi R^2 (q_- - q_+)$$  \hspace{1cm} (24)$$

where $q_-$ and $q_+$ are the energy density of the radiation at each side of the shell and $p$ is the pressure of the shell. Therefore, we have recovered, in this simple model, the terms that describe the effect of the pressure of radiation inside the cavity and the snow plow effect (compare the last term of this equation with the correspondent of the Ostriker’s equation). Also, we have an extra term related to the pressure of the shell, since Ostriker has apparently considered pressure-less shell. However, a more realistic model is realized in considering the exterior spacetime characterized by dust, like in the Friedman-Walker universe, in order to taken into account more properly the interstellar medium outside the shell.

We want to stress the fact that the second order equation, equation (24) obtained from the first order one, equation (23) obtained from the Einstein’s equations, equations (3, 4) but has a more tractable form and the different terms are easier to identify.

Finally, if instead of the proper time of the shell, $\tau$, is used the time of an exterior or an interior observer, $v_+, v_-$, and then consider the radius of the shell as a function of $v$, $R = R(v)$, so that $\dot{R} = R_\dot{v} \ddot{v}$, after some manipulations can be shown that the motion equation for the radius of the shell is given by

$$R_{v\pm} = e^{\psi \pm} A [b A + \eta (\frac{m_g}{M} + \frac{M}{2R})]$$  \hspace{1cm} (25)$$

with $\eta = \pm$, for the two possible solutions to the algebraic equation and with

$$A^2 = (\frac{m_g}{M})^2 - 1 + (\frac{m_+ + m_-}{R}) + \frac{M^2}{4R^2}.$$  \hspace{1cm} (26)$$

Usually a problem could be posed as follows: given the two backgrounds separated by the shell, i.e., two spherically symmetric solutions to the Einstein’s equations and a state equation for the matter in the shell, determine $R, v_+, v_-$ and $M$ from the motion equations which are all first order. In this way we have a set of four first order coupled differential equations for 4 unknown functions, so the problem is solvable up to quadrature.

Nevertheless the analytical integration might prove to be very hard except for the most simple cases, so numerical methods are still needed. In a future work (Núñez & Oliveira 1994) we will present the analysis of this set of equations in various specific backgrounds.

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REFERENCES

Barrabès, C., & Israel, W. 1991, Phys. Rev D, 43, 1129

Berezin, V. A., Kuzmin, V. A., & Tkachev, I. I. 1987, Phys. Rev. D, 36, 2919

Chase, J. E, 1972, Ph. D thesis, University of Alberta, Canada.

Chevalier, R. A. 1977, Astroph. and Space Science Lib., 66, proceedings (Supernovae, ed. Schramm D, Reidel Pub. Co.)

De la Cruz, V., & Israel, W. 1967, Il Nuovo Cimento, LI A, 3, 744

Israel, W. 1966, Il Nuovo Cimento, 44 B, 1

——. 1967, 48 B, 463

Lake, K. 1979, Phys. Rev. D, 19, 2847

Misner, C. W., Thorne, K. S., & Wheeler, J. A. 1973, Gravitation, (S. Fco. W. H. Freemann and Co.)

Oort, J. H. 1946, MNRA, 106, 159

Ostriker, J. P., & Gunn, J. E. 1971, ApJ, 164, L95

Ostriker, J. P., & McKee, C. F. 1988, Rev. Mod. Phys., 60, 1

Sato, H. 1988a, Prog. of Theo. Phys., 80, 96

Sato, H. 1988b, Prog. of Theo. Phys., 80, 96

Sato, H., & Yamada, Y. 1991, Prog. of Theo. Phys., 85, 541

Núñez, D., & de Oliveira, H. P., in preparation