Power of Supersymmetry in D-particle Dynamics

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Abstract

A new systematic method is developed to study to what extent the symmetry requirements alone, above all the invariance under 16 supersymmetries (SUSY), determine the completely off-shell effective action $\Gamma$ of a D-particle, i.e. without imposing any restrictions on its position $r^m(\tau)$ and spin $\theta^\alpha(\tau)$. Our method consists of (i) writing down the proper closure relations for general SUSY transformations $\delta_\epsilon$ (which necessarily involves $\Gamma$ itself) together with the invariance condition $\delta_\epsilon \Gamma = 0$ (ii) and solving this coupled system of functional differential equations for $\delta_\epsilon$ and $\Gamma$ simultaneously, modulo field redefinitions, in a consistent derivative expansion scheme. Our analysis is facilitated by a novel classification scheme introduced for the terms in $\Gamma$. At order 2 and 4, although no assumption is made on the underlying theory, we reproduce the effective action previously obtained at the tree and the 1 loop level in Matrix theory respectively (modulo two constants), together with the quantum-corrected SUSY transformations which close properly. This constitutes a complete unambiguous proof of off-shell non-renormalization theorems.

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1 Introduction

One of the most remarkable discoveries with far-reaching consequences in recent years in string/M theory is the gauge/gravity correspondence, the prototype of which was uncovered by Maldacena [1] [2] in the form of AdS/CFT duality. Although it is certain that this phenomenon must be deeply related to the so called s-t duality of string theory, i.e. the modular duality between the open and the closed string channels, it is equally certain that its validity hinges crucially on the existence of supersymmetry: Whereas the whole tower of massive string modes is needed at least in one of the channels for the generic s-t duality, the miracle of gauge/gravity correspondence in question is that it holds without such massive modes and this can only be possible by supersymmetric cancellations [4].

More recently, the gauge/gravity relation has been extended to include the correspondence between the massive modes of the closed string in a Penrose limit of $AdS_5 \times S^5$ spacetime and a special class of gauge-invariant composite operators in $\mathcal{N} = 4$ super Yang-Mills theory [3]. Although the significance of supersymmetry seems less apparent in this extension, the fact that the Penrose limit preserves the maximal supersymmetry of the original spacetime strongly suggests that its role is still of considerable importance.

In this paper, we focus on the effective dynamics of a D-particle (in interaction with a large number of coincident source D-particles), a rare example in which one can explicitly study the details of a type of gauge/gravity correspondence. As is well-known, the dynamics of a system of D-particles is efficiently described by Matrix theory for M theory [5] [6] [7] and strong evidence has been accumulated that quantum effects in Matrix theory reproduce the tree-level dynamics of the 11-dimensional supergravity compactified along a light-like circle $\sim [21]$. Particularly impressive is the agreement in the case of multi-body scattering [10, 11], which probes the non-linear structure of the 11-dimensional supergravity.

It has been suspected that behind such remarkable successes lie the high degrees of symmetries of the system, in particular the maximally implemented supersymmetry. Indeed a number of investigations have been performed [22, 23, 24, 26, 25, 27], which strongly indicated that supersymmetry, together with a few other symmetries, is powerful enough to fix the form of the effective action completely up to two constants at least at low orders in derivative expansions. Since the D-particle dynamics is physically non-trivial starting at order$^1$ 4, this evinces a rather surprising fact that global symmetries can be so restrictive as to dictate even the dynamics of the system.

$^1$The concept of order will be precisely defined in the next section.
However, as we have emphasized previously (see Sec. 3 of [29]), existing analyses have several unsatisfactory features and are not complete\(^2\). The essential shortcoming is that such analyses made use of the eikonal approximation, or equivalently the tree-type on-shell condition, which sets all but \(r_m, \dot{r}_m\) and \(\theta_\alpha\) to zero; higher derivatives are simply neglected. This is not justified for the following two related reasons: (i) Since the derivatives can be moved around by integration by parts, naive eikonal approximation is logically inconsistent. Derivative expansion must be organized by a concept unaffected by the freedom of adding total derivatives, which requires retention of previously discarded terms. (ii) As we shall demonstrate in our analysis, fully consistent treatment involves expressions which would vanish by the quantum-corrected on-shell condition, which can only be obtained by off-shell computations.

Consequently, the only consistent and unambiguous procedure is to deal with the trajectory \(r_m(\tau)\) and the spin degrees of freedom \(\theta_\alpha(\tau)\) with arbitrary time dependence. Based on this consideration, we have performed, in a series of papers, fully off-shell analysis of the D-particle system with emphasis on the role of supersymmetry. After deriving the relevant Ward identity [28], we computed the off-shell effective action and the SUSY transformations at order 4 [29], including all the fermionic contributions for the first time, proved that, given SUSY transformations, the Ward identity uniquely fixes the effective action at this order [30], and finally extended this demonstration to all orders in perturbation theory [31]. These investigations were performed in the context of Matrix theory. Although exceptionally powerful nature of the supersymmetry even at the off-shell level was revealed in these works, this was not sufficient to claim that SUSY determines the dynamics. One must be able to show that it determines the SUSY transformations as well as the effective action simultaneously in a self-consistent manner without any knowledge of the underlying theory apart from its symmetries.

In this paper, we complete our program for such a demonstration. The basic idea is to fully utilize the proper ‘off-shell’ closure relations\(^3\) that must be satisfied by the SUSY transformations \(\delta_\epsilon\), in addition to the SUSY Ward identity for the effective action \(\Gamma\). Since the expressions \(\delta \Gamma / \delta r_m\) and \(\delta \Gamma / \delta \theta_\alpha\), which would vanish on-shell, appear in the closure relations, we must deal with a system of coupled non-linear functional differential equations for \(\delta_\epsilon\) and \(\Gamma\). This will be solved in a consistent derivative expansion with a new efficient classification scheme for various terms and with a careful analysis of how to fix the ambiguity of adding total derivatives. After a rather long analysis, with a considerable use of various complicated Fierz identities, the following results are obtained:

\(^2\)Below we discuss only the most important points. Further remarks are provided in Sec. 2.3.

\(^3\)We elucidate what we exactly mean by ‘off-shell’ SUSY carefully in Sec. 2.
1. There exists a frame (i.e. the choice of the definitions of the fields) in which the effective action and the SUSY transformations at order 2 take the tree-level forms.

2. At order 4, the effective action is determined completely, modulo two constants, which in an appropriate frame coincides with the one obtained in the eikonal-type analysis\[24\] and with the explicit 1-loop result in Matrix theory [29].

3. SUSY transformations in relevant orders are determined uniquely in a chosen frame and are shown to satisfy proper closure relations.

It is clear that the results 1 and 2 can be interpreted as complete proofs of non-renormalization theorems in the context of Matrix theory. The result 3 has never been obtained before.

Due to the large amount and to the intricate nature of the works performed in this study, the exposition in this paper has become somewhat long even after many omissions of the calculational details. However, once the basic ideas and methods described in Sec. 2 and in Sec. 4.1 are understood the rest of the manipulations are conceptually straightforward to follow.

The organization of the rest of the article is as follows: We begin in Sec. 2 by describing our basic formalism. The symmetry requirements are explained, including what we exactly mean by off-shell supersymmetry, and basic equations are written down together with our expansion scheme. Then, some salient features of our formalism in comparison with previous works are clarified. The actual analysis begins in Sec. 3, starting at order 2. First the SUSY transformation laws are simplified by appropriate field redefinitions and a use of a part of the closure relations. Then, after introducing a crucial concept of independent basis, the Ward identity is solved and the effective action is fully determined. The description of our main effort, namely the analysis at order 4, is given in Sec. 4. In Sec. 4.1, we introduce an efficient classification scheme called “E-type - D-type separation method”, which at the same time greatly reduces the amount of work and allows us to read off the SUSY transformation laws. Using this method, we analyze the effective action in Sec. 4.2 ∼ 4.3. Subsequently, the SUSY transformation laws at this order are obtained in Sec. 4.4 and their closure relations are studied in Sec. 4.5 and 4.6. Finally, in Section 5, we summarize our results and indicate some directions for further study.

Two appendices are provided for some technical details. In Appendix A, we describe the analysis of certain special fermionic transformations, called ‘null transformations’, which is needed to justify our scheme used in Section 4. In Appendix B, we display the

\[4\text{As we shall explain in Sec. 4.3.6, this agreement does not however mean that a naive eikonal analysis is justified.}\]
SUSY transformation laws obtained in Sec. 4 which are too space-filling to be presented in the main text.

2 Basic Formalism

2.1 Formulation of symmetry requirements for the effective action

A D-particle in 10 dimensions in Euclidean formulation is described by the 9-component position vector \( r^m(\tau) \) and the 16-component Majorana-Weyl spinor \( \theta_{\alpha}(\tau) \) representing the spin state, with \( \tau \) the Euclidean time. As was already emphasized, their dependence on \( \tau \) will be taken to be completely arbitrary throughout. The dynamics is assumed to be governed by some effective action of the form

\[
\Gamma[r, \theta, g] = \int d\tau L(r, \theta, g),
\]

where \( g \) is a coupling constant\(^5\). We assign the mass dimensions \(-1, 3, 1, \frac{3}{2}\) to \( \tau, g, r^m, \theta_{\alpha} \) respectively. Thus, \( L \) is taken to be a local expression of dimension 1. Terms composing \( L \) are classified according to the order, defined as the number of time derivatives plus half the number of \( \theta \)'s involved. This notion will be used to organize a consistent derivative expansion.

We will require that \( \Gamma \) be invariant under (i) \( SO(9) \) rotations, (ii) C-P-T transformations and (iii) 16 supersymmetry transformations. \( SO(9) \) rotations act on \( r^m \) and \( \theta_{\alpha} \) in the usual way. C-P-T transformation properties are defined to conform to those valid in the Matrix theory. P and T are separately violated due to the Weyl nature of the spinor \( \theta_{\alpha} \) and we only impose invariance under C and CPT. Under the C-transformation, \( r^m \rightarrow -r^m \), while \( \theta_{\alpha} \) are unchanged. On the other hand, CPT-transformation does not transform the fields but flips the sign of the time-derivative and effects \( i \rightarrow -i \) as it is anti-unitary. Together with the requirement of hermiticity, C-P-T invariance of \( L \) can be summarized as the following simple rule [30]:

- In constructing \( L \), use \( i^{1+m+n} \theta^{(m)} \gamma^{i_1 i_2 \ldots i_k} \theta^{(n)} \) as fermionic building block, where \( \theta^{(m)} \equiv \partial^m \theta \) and \( \gamma^{i_1 i_2 \ldots i_k} \) are the antisymmetrized products of \( SO(9) \) \( \gamma \)-matrices.

Demand also that the number of \( r^m \), the number of \( \gamma^m \) and the ‘order’ be all even.

Now we come to the main focus of our attention, the invariance under 16 supersymmetries. This must be formulated and explained with care for several reasons.

\(^5\)As we can easily recover its dependence from the dimensional analysis, we will set \( g = 1 \).
1. As is well known, there is as yet no formulation of 16 supersymmetries with off-shell closure: Commutator\(^6\) of SUSY transformations yields translation only up to terms which vanish upon the use of the equations of motion. As we do not wish to impose such on-shell conditions, we must allow for these additional terms in the closure relations. An obvious complication is that, as they must involve \(\delta \Gamma / \delta r^m(\tau)\) or \(\delta \Gamma / \delta \theta_\alpha(\tau)\), they depend on the effective action itself which we wish to determine.

2. Sometimes, this lack of off-shell closure is rephrased as the statement that for such a system “supersymmetry exists only on shell”. This statement is both true and false, depending on what one means by supersymmetry. If one insists that supersymmetry must act between equal numbers of bosonic and fermionic fields, then the statement is obviously correct; imposition of the on-shell condition is indeed necessary to achieve this equality. This, however, does not mean that there is no fermionic symmetry off the mass shell. A prime example is the super Yang-Mills theory in 10 dimensions, from which the Matrix theory can be obtained by dimensional reduction. For such a theory the action is invariant under so-called supersymmetry transformations without any use of the equations of motion. A purist would carefully call it “a symmetry which becomes the supersymmetry on-shell”. It is precisely this type of off-shell global fermionic symmetry that we will impose on the effective action. Having clarified its meaning, we shall hereafter simply refer to it as supersymmetry, following common usage.

3. Since we are dealing with the most general effective action without assuming the knowledge of the underlying theory, we must consider also the most general forms for our SUSY transformation laws \(\delta_r r_m\) and \(\delta_r \theta_\alpha\). They are to be restricted only by the generalized closure relations explained above, \(SO(9)\) and CPT symmetries, and dimensional considerations.

4. We must allow arbitrary field redefinitions of the type which do not change the physical S-matrix.

5. The actual analysis will be performed on the effective Lagrangian \(\mathcal{L}\). Therefore we must always allow for the freedom of adding total derivatives. This means that a naive approximation scheme, such as the often-used eikonal approximation, where fields with more than a fixed number of derivatives are set to zero is not consistent. On the other hand, the notion of ‘order’ is stable against such additions. Although

\(^6\)Global spinor parameter \(\epsilon_\alpha\) is understood to be included in the transformation.
we call it a “derivative expansion”, what we will employ throughout is the expansion with respect to this quantity.

2.2 Basic equations and expansion scheme

We are now ready to write down our basic equations which embody the scheme explained above. We express the supersymmetry transformations, their closure relations and the invariance of the effective action under them in the following manner:

\[ \delta \epsilon_\alpha \theta^\beta = T_{\alpha \beta} \epsilon^\beta, \quad (2.2) \]

\[ \delta \epsilon_r^m = \Omega_{m \beta} \epsilon^\beta, \quad (2.3) \]

\[ [\delta_\epsilon, \delta_{\lambda}] \theta^\alpha = -2(\epsilon \lambda) \dot{\theta}^\alpha + A_{\alpha \beta \gamma \delta} \frac{\delta \Gamma}{\delta \theta^\delta} \epsilon_\beta \lambda^\gamma + B_{\alpha \beta \gamma n} \frac{\delta \Gamma}{\delta r^\gamma} \epsilon_\beta \lambda^\gamma, \quad (2.4) \]

\[ [\delta_\epsilon, \delta_{\lambda}] r^m = -2(\epsilon \lambda) \dot{r}^m + C_{m \beta \gamma \delta} \frac{\delta \Gamma}{\delta \theta^\delta} \epsilon_\beta \lambda^\gamma + D_{m \beta \gamma n} \frac{\delta \Gamma}{\delta r^\gamma} \epsilon_\beta \lambda^\gamma, \quad (2.5) \]

\[ \delta_\epsilon \Gamma = \int d\tau \delta_\epsilon L = 0. \quad (2.6) \]

\[ T, \Omega, A, B, C, D \] and \( L \) are as yet unknown local functions of \( \{ r^m(\tau), \theta^\alpha(\tau) \} \) and their derivatives. In what follows, \( A \sim D \) will be referred to as off-shell coefficients. In the context of Matrix theory, the equation (2.6) represents an invariance of the quantum effective action under quantum-corrected effective SUSY transformations, hence it is often referred to as the SUSY Ward identity or simply the Ward identity. In the closure relations (2.4) and (2.5), we have written out the expressions \( \delta \Gamma / \delta r^m \) and \( \delta \Gamma / \delta \theta^\alpha \), which vanish on shell, explicitly. We are not, however, excluding the possibility\(^8\) that \( A \sim D \) may contain additional dependence on functional derivatives of \( \Gamma \). Apart from the symmetry and dimensional requirements, the only assumption we shall make is that \( \Gamma \) starts at order 2. The prime question is to what extent the unknown quantities, in particular \( \Gamma \) and \( \delta_\epsilon \), can be determined just from these relations, up to field redefinitions.

Let us express the basic set of equations introduced above in a slightly more explicit fashion. By using the definitions (2.2) and (2.3), the left-hand-sides (LHS) of (2.4) \sim (2.6) become

\[ [\delta_\epsilon, \delta_{\lambda}] \theta^\alpha(\tau) = \int ds \left[ \left( \Omega_{\alpha \beta}(s) \frac{\delta T^\gamma}{\delta r^\gamma}(s) - T_{\delta \beta}(s) \frac{\delta T^\gamma}{\delta \theta^\gamma}(s) \right) \epsilon_\beta \lambda^\gamma \right] \quad (2.7) \]

\(^7\)Here and hereafter, the dot signifies differentiation with respect to the Euclidean time \( \tau \) and we will use \( v^m \) and \( a^m \) to denote \( \dot{r}^m \) and \( \ddot{r}^m \) respectively. Contractions of the spinor indices are often suppressed, so that \((\epsilon \lambda)\) stands for \( \epsilon_\beta \lambda^\beta \), etc.

\(^8\)Judging from the Matrix theory calculations, this is highly unlikely.
\[ [\delta \epsilon, \delta \lambda] r_m = \int \mathrm{d}s \left[ \left( -\Omega_m(s) \frac{\delta \Omega_m(\tau)}{\delta r_n(s)} + T_{\alpha\beta}(s) \frac{\delta \Omega_m(\tau)}{\delta \theta^\alpha(s)} \right) + (\beta \leftrightarrow \gamma) \right] \epsilon_\beta \lambda_\gamma, \]  
\[ \delta \epsilon \Gamma = \int \mathrm{d}\tau \left( \Omega_m(\tau) \frac{\delta \Gamma}{\delta r_m(\tau)} - T_{\alpha\beta}(\tau) \frac{\delta \Gamma}{\delta \theta^\alpha(\tau)} \right) \epsilon_\beta. \]

This makes it clear that what we are dealing with is a set of coupled non-linear functional equations, which are in general extremely hard to solve. Nevertheless, by the systematic use of the derivative expansion and a novel classification scheme for the terms in \( \Gamma \), to be described in detail later, one can analyze them to get concrete results at low orders.

Now let us explain our scheme of expansion of various quantities with respect to order, needed for the analysis up to order 4.

First, the effective action, the order of which must be even from CPT symmetry, is expanded as

\[ \Gamma = \Gamma^{(2)} + \Gamma^{(4)}, \]

where the superscripts in parentheses refer to their orders. They are further expanded according to the number of \( \theta \)'s as

\[ \Gamma^{(2)} = \Gamma^{\theta^2} + \Gamma^{\theta^2 \theta^2} + \Gamma^{\theta^4}, \]
\[ \Gamma^{(4)} = \Gamma^{\theta^4} + \Gamma^{\theta^2 \theta^2} + \Gamma^{\theta^2 \theta^4} + \Gamma^{\theta^6} + \Gamma^{\theta^8}. \]

On the right-hand-sides (RHS), the superscript indicates the schematic structure of each term in a self-explanatory manner.

Next, consider the SUSY transformation laws. A quick examination of the closure relations tells us that \( \Omega_m \) and \( T_{\alpha\beta} \) start from order \( 1/2 \) and order 1 respectively. Also, the order of \( \Gamma \) being even, their orders must go up by 2 units. Thus, we have the expansion

\[ T_{\alpha\beta} = T^{(1)}_{\alpha\beta} + T^{(3)}_{\alpha\beta}, \]
\[ T^{(1)}_{\alpha\beta} = T^{\theta}_{\alpha\beta} + T^{\theta^2}_{\alpha\beta}, \]
\[ T^{(3)}_{\alpha\beta} = T^{\theta^3}_{\alpha\beta} + T^{\theta^2 \theta^2}_{\alpha\beta} + T^{\theta \theta^4}_{\alpha\beta} + T^{\theta^6}_{\alpha\beta}, \]
\[ \Omega_m = \Omega^{(1/2)}_m + \Omega^{(5/2)}_m, \]
\[ \Omega^{(1/2)}_m = \Omega^\theta_m, \]
\[ \Omega^{(5/2)}_m = \Omega^{\theta^2 \theta}_{m} + \Omega^{\theta^6}_{m} + \Omega^{\theta^8}_{m}. \]

Finally, consider the expansion of the off-shell coefficients. Again by a simple analysis of the basic equations, we find that \( A, B, C, D \) must start at orders \( 0, \frac{3}{2}, \frac{3}{2}, 1 \) respectively,
and go up again by 2 units. Hence their expansions become

\[ A_{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta}^{(0)} + A_{\alpha\beta\gamma\delta}^{(2)}, \]
\[ A_{\alpha\beta\gamma\delta}^{(0)} = A_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta}, \]  \hspace{1cm} (2.19)
\[ A_{\alpha\beta\gamma\delta}^{(2)} = A_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta} + A_{\alpha\beta\gamma\delta}^{2}, \]
\[ A_{\alpha\beta\gamma\delta}^{(0)} = A_0^{\alpha\beta\gamma\delta}, \] \hspace{1cm} (2.20)
\[ A_{\alpha\beta\gamma\delta}^{(2)} = A_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta} + A_{\alpha\beta\gamma\delta}^{\partial^2} + A_{\alpha\beta\gamma\delta}^{\partial^{2\theta}}, \]
\[ A_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta} = A_0^{\alpha\beta\gamma\delta}, \] \hspace{1cm} (2.21)

In sections 3 and 4, we substitute these expansions into our basic equations, identify independent structures to produce more explicit set of equations and solve them.

### 2.3 Comparison with previous approaches

Before we begin the analysis of our basic equations, it should be helpful to make a comparison of our framework with previous works in the literature and clarify what are new and/or improved in our approach. As we have already mentioned the essential shortcomings of the eikonal-type approximation employed in existing literature in the introduction, below we wish to make a little more explicit comparison with the work by Paban et al. [22] and the one by Hyun et al. [25], which are most closely related to the present study.

In Section 3, we shall give a complete proof of the non-renormalization theorem for the effective action at order 2, which was discussed in [22]. The arguments presented in [22] were incomplete in several respects: (i) It was assumed that by field redefinition the effective Lagrangian can be brought to the form \( f(r)\nu^2 \) in the basis where the SUSY transformation laws take the simple tree-level form without any corrections. As we shall see in Section 3.2, the field redefinitions which can be used at order 2 are actually so restricted that it is not possible to make both the effective action and the SUSY transformation laws simple at the same time. (ii) The \( \Gamma^{\theta^4} \) term allowed in the effective action was neglected from the beginning. It requires some arguments to show that this can be eliminated. (iii) The work [22] also discussed the determination of the \( \Gamma^{\theta^8} \) structure at order 4, which will be dealt with in Sec. 4.3.4 and 4.3.5. While conditions weaker than what SUSY requires were used in [22], we shall deal with the genuine conditions dictated by SUSY.

In Section 4, we will determine the effective action at order 4, which was studied in [25]. (i) As the authors of [25] employed the eikonal approximation, they unduly
neglected the higher derivative terms that should be kept for consistent analysis. (ii) As we shall explain in Sec. 4.2, these higher derivative terms can actually be removed by appropriate field redefinitions\(^9\). This fortunate fact does not however justify their treatment completely since in analyzing the Ward identity they again discarded higher derivative terms arbitrarily. (iii) Furthermore, during the course of the analysis, they replaced an arbitrary spinor \(\epsilon_\alpha\) by a special structure \((\theta^i_\gamma)_\alpha\) to simplify the analysis. As a result the resultant equations provide only necessary conditions. In contrast, we shall deal with the full set of constraints dictated by the SUSY Ward identity. (iv) Finally, they only analyzed the Ward identity and did not clarify the nature of the fermionic transformations. Our analysis will determine the complete form of these transformations and by analyzing the closure relations we shall prove that they do qualify as SUSY transformations.

Having spelled out the various new features of our work in advance, we now describe the essential part of the analysis.

3 Analysis at Order 2

3.1 Strategy

We start our analysis from order 2. At this order, various simplifications occur and the analysis is essentially straightforward.

The first simplification is that, by a simple counting of the order, the off-shell coefficient functions \(B, C, D\) can be shown to vanish at this order and we only need to keep \(A\). Thus, the basic equations (2.2), (2.3), (2.7), (2.8) and (2.9) become

\[
\delta \theta_\alpha = T^{(1)}_{\alpha \beta} \epsilon_\beta, \quad (3.1)
\]

\[
\delta r_m = \Omega^{(1/2)}_{m \beta} \epsilon_\beta, \quad (3.2)
\]

\[
\int ds \left[ \left( \Omega^{(1/2)}_{m \beta}(s) \frac{\delta T^{(1)}_{\alpha \gamma}(\tau)}{\delta r_n(s)} - T^{(1)}_{\delta \beta}(s) \frac{\delta T^{(1)}_{\alpha \gamma}(\tau)}{\delta \theta_\delta(s)} \right) + (\beta \leftrightarrow \gamma) \right] = -2 \delta_{\beta \gamma} \theta_\alpha + A^0_{\alpha \beta \gamma \delta} \frac{\delta \Gamma^{(2)}}{\delta \theta_\delta}, \quad (3.3)
\]

\[
\int ds \left[ -\Omega^{(1/2)}_{m \beta}(s) \frac{\delta \Omega^{(1/2)}_{m \gamma}(\tau)}{\delta r_n(s)} + T^{(1)}_{\alpha \beta}(s) \frac{\delta \Omega^{(1/2)}_{m \gamma}(\tau)}{\delta \theta_\delta(s)} \right] + (\beta \leftrightarrow \gamma) = -2 \delta_{\beta \gamma} \dot{r}_m, \quad (3.4)
\]

\[
\int d\tau \left( \Omega^{(1/2)}_{m \beta}(\tau) \frac{\delta \Gamma^{(2)}}{\delta r_m(\tau)} - T^{(1)}_{\alpha \beta}(\tau) \frac{\delta \Gamma^{(2)}}{\delta \theta_\alpha(\tau)} \right) = 0, \quad (3.5)
\]

\(^9\)This fact was first recognized by Okawa \[32\]
where in the last three equations we have removed the arbitrary spinors $\epsilon_\beta$ and $\lambda_\gamma$.

These equations will be solved in the following steps:

1. First we write down the most general form of the SUSY transformation laws compatible with the symmetry requirements.

2. Next, by utilizing the freedom of field redefinitions, we further simplify the form of the SUSY transformations and study the restrictions from the closure relation on $r_m$. This will reduce $\delta_\epsilon$ to be of the simple tree-level form.

3. We then write down the most general expressions for the effective action, and determine its form from the Ward identity (3.5).

4. Finally, we solve the closure relation (3.3) on $\theta_\alpha$ to determine $A^{(0)}_{\alpha\beta\gamma\delta}$.

These steps are rather easy to perform due to several simplifying features that occur at this order: Allowed structures for various quantities are limited and it is not difficult to enumerate them. In addition, as the number of spinors is small, we need not use complicated Fierz rearrangement identities in solving the Ward identity.

We now exhibit some details of the above procedures in the remainder of this section.

### 3.2 The SUSY transformation laws and the closure relation

We begin by writing down the general form of the SUSY transformation laws. As we have already described in Sec. 2, $\Omega_{m\alpha}$ at this order is composed of terms of $O(\theta)$, while $T_{\alpha\beta}$ consists of terms of $O(\partial)$ and $O(\theta^2)$. The most general $SO(9)$ covariant such structures are given by

\[
\begin{align*}
\Omega_{m\beta}^{(1/2)} &= i(\gamma_{\beta\gamma}^m \Omega_1^0 + r_m f_{\beta\gamma}^m \Omega_2^0 + r_m \delta_{\beta\gamma} \Omega_3^0 + r^m \gamma_{\beta\gamma}^m \Omega_4^0) \theta_\gamma, \\
T_{\alpha\beta}^{(1)} &= i(\psi_{\alpha\beta}(r \cdot v) T_1^0 + \psi_{\alpha\beta} T_2^0 + \delta_{\alpha\beta}(r \cdot v) T_3^0 + \gamma_{\alpha\beta}^m r_m v_n T_4^0 + T_{\alpha\beta\sigma\rho}^\theta \theta_\sigma \theta_\rho).
\end{align*}
\]  

(3.6)  

Here $\Omega_i^\theta$, $T_i^\theta$ ($i = 1 \sim 4$) are functions of $r(\tau) \equiv \sqrt{r^m(\tau)r^m(\tau)}$ only and $T_{\alpha\beta\sigma\rho}^\theta$ is composed of $r^m(\tau)$ and $\gamma$-matrices. The details of the structure of $T_{\alpha\beta\sigma\rho}^\theta$ will not be needed in our analysis.

Some of the terms written above are actually forbidden by C-symmetry. The rule is that $\Omega_{m\beta}$ and $T_{\alpha\beta}$ must contain even and odd number of $r^m$ (and its derivatives) respectively, since the tree level SUSY transformations enjoy this property and C-preserving
quantum corrections cannot change it. This reduces the allowed structures down to
\[ \Omega^{(1/2)}_{m\beta} = i(\gamma^m_{\beta\gamma} + r_m \theta_{\beta\gamma}), \]  
\[ T^{(1)}_{\alpha\beta} = i(\theta_{\alpha\beta}(r \cdot v) + \theta_{\alpha\beta} T^2_{\alpha\beta} + T^{\theta^2}_{\alpha\beta\sigma\rho} \theta_{\sigma\rho}). \]  

Now we can further simplify these transformation laws by the use of field redefinitions. The most general field redefinitions that do not change the order are of the form
\[ \tilde{r}_m(r, \theta) = r_m Z_1(r), \quad \tilde{\theta}_\alpha(r, \theta) = \theta_\alpha Z_2(r), \]  
where \(Z_i(r)\) are functions of \(r(\tau)\) only. They must satisfy the conditions
\[ Z_i(r) \to 1 \quad \text{as} \quad r \to \infty, \]  
in order that the transformations do not change the S-matrix. As we have two arbitrary functions \(Z_i(r)\), we may “gauge-fix” two functions of \(r\). It is not difficult to check that indeed we can set \(\Omega^0 = 1\) and \(\Omega^2 = 0\). This choice reduces the transformation laws to
\[ \Omega^{(1/2)}_{m\beta} = i\gamma^m_{\beta\gamma} \theta_{\gamma}, \]  
\[ T^{(1)}_{\alpha\beta} = i(\theta_{\alpha\beta}(r \cdot v) T^1_{\alpha\beta} + \theta_{\alpha\beta} T^2_{\alpha\beta} + T^{\theta^2}_{\alpha\beta\sigma\rho} \theta_{\sigma\rho}), \]  
where we have omitted the tilde for simplicity.

Having simplified the form of the transformation laws as much as possible, let us substitute (3.12) and (3.13) into the closure relation (3.4) on \(r_m\). The \(O(\theta^0)\) and \(O(\theta^2)\) parts of the closure relation give
\[ -2 T^2_2 v_m \delta_{\beta\gamma} - 2 T^1_1 v_m (r \cdot v) \delta_{\beta\gamma} = -2 \delta_{\beta\gamma} v_m, \]  
\[ - \left( T^{\theta^2}_{\alpha\gamma\sigma\rho} \gamma^m_{\alpha\beta} + T^{\theta^2}_{\alpha\beta\sigma\rho} \gamma^m_{\alpha\gamma} \right) = 0. \]  
They are easily solved and we get
\[ T^2_1 = 0, \quad T^2_2 = 1, \quad T^{\theta^2}_{\alpha\beta\sigma\rho} = 0. \]  
Thus the SUSY transformation laws finally become
\[ \Omega^{(1/2)}_{m\beta} = i\gamma^m_{\beta\gamma} \theta_{\gamma}, \]  
\[ T^{(1)}_{\alpha\beta} = i\theta_{\alpha\beta}. \]  
What we have shown is that there exists a frame of fields (or a gauge) in which the SUSY transformation laws at order 2 take precisely the tree-level form.

\(^{10}\)A possible term of the form \(\theta_{\beta\gamma} f_{\beta\alpha}\) in \(\tilde{\theta}_\alpha\) is forbidden by C-symmetry.
3.3 Determination of the effective action from the Ward identity

Now we move on to the analysis of the Ward identity for the effective action $\Gamma^{(2)}$. As we have already used up the freedom of field redefinitions, we must deal with the most general form of $\Gamma^{(2)}$. By substituting the expansion (2.11) of $\Gamma^{(2)}$ and the above SUSY transformation laws (3.17), (3.18) into the equation (3.5) and collecting terms with the same number of $\theta$'s, the Ward identity can be split into the following three equations:

$$\int d\tau \left( i\gamma^m_{\beta\gamma}\theta_\gamma(\tau) \frac{\delta \Gamma^{\partial^2}}{\delta r_m(\tau)} - i\delta_{\alpha\beta}(\tau) \frac{\delta \Gamma^{\partial^4}}{\delta \theta_\alpha(\tau)} \right) = 0, \quad (3.19)$$

$$\int d\tau \left( i\gamma^m_{\beta\gamma}\theta_\gamma(\tau) \frac{\delta \Gamma^{\partial^2}}{\delta r_m(\tau)} - i\delta_{\alpha\beta}(\tau) \frac{\delta \Gamma^{\theta^4}}{\delta \theta_\alpha(\tau)} \right) = 0, \quad (3.20)$$

$$\int d\tau i\gamma^m_{\beta\gamma}\theta_\gamma(\tau) \frac{\delta \Gamma^{\theta^4}}{\delta r_m(\tau)} = 0. \quad (3.21)$$

As is characteristic of any Ward identity, these equations are of global integrated form and due to the inherent total derivative ambiguities it is non-trivial to extract the information on the local quantities such as $\delta \Gamma^{\partial^2}/\delta r_m(\tau)$ etc. that we wish to obtain.

This difficulty can however be overcome by the following consideration. First, consider the possible algebraically independent structures at a fixed order with a definite number of $\theta$'s and denote them by $\{\tilde{e}_A\}$. The number $N$ of such structures is obviously finite. In general, there are certain number, say $n$, of linear combinations $\sum_A g^A(r) \tilde{e}_A$, ($i = 1 \sim n$) which are actually total derivatives. Thus, we can choose among $\{\tilde{e}_A\}$ what we shall call an independent basis $\{e_a\}_{(a=1\sim N-n)}$ for which the following properties hold:

- $e_a$'s are algebraically independent.

- The set $\{e_a\}$ is such that $\sum g_a(r)e_a$ cannot be a total derivative for any choice of $g_a$'s.

This is equivalent to the property

$$\int d\tau \sum_a g_a(r)e_a = 0 \quad \Rightarrow \quad g_a(r) = 0. \quad (3.22)$$

Clearly the choice of such a set $\{e_a\}$ is not unique, but once we fix one independent basis and stick to it, we can unambiguously obtain local equations from an integrated equation using (3.22). We must of course be very careful to check that a chosen set $\{e_a\}$ really satisfies these properties. For structures involving more than 4 $\theta$'s, even the algebraic independence can be highly non-trivial due to the existence of often formidable Fierz
identities. It is important to note that once we find an independent basis, (3.22) holds for any subset of it since it is a special case with some of the $g_a$'s already set to zero. Hereafter, we shall say that a set of terms are “independent” whenever the property (3.22) holds.

Using this notion of “independence”, we now solve the Ward identities. It is convenient to first prove $\Gamma^{\theta^4} = 0$. Of the various possible structures for $\Gamma^{\theta^4}$, the following actually vanish by the Fierz identities:

\[(\theta\gamma_{mn}\theta)(\theta\gamma_{mn}\theta), \quad (\theta\gamma_{mnk}\theta)(\theta\gamma_{mnk}\theta), \quad (\theta\gamma_{mn}\theta)(\theta\gamma_{mnk}\theta)r_k.\]

Furthermore, by using another Fierz identity, the structure $(\theta\gamma_{mnk}\theta)r_k(\theta\gamma_{mnl}\theta)r_l$ can be expressed in terms of the one shown just below. In this way, the only possible structure for $\Gamma^{\theta^4}$ is

\[\Gamma^{\theta^4} = \int d\tau F^{\theta^4}(\theta\gamma^{an}\theta)(\theta\gamma^{ak}\theta)r_n r_k, \quad (3.23)\]

where $F^{\theta^4}$ is a function only of $r(\tau)$. Now substitute this into the Ward identity (3.21) and contract it with an arbitrary spinor $\epsilon_\beta$. This gives

\[\int d\tau \left( 2(\epsilon\gamma^m\theta)(\theta\gamma^{am}\theta)(\theta\gamma^{am}\theta)r_n F^{\theta^4} + r_m(\epsilon\gamma^m\theta)(\theta\gamma^{am}\theta)(\theta\gamma^{ak}\theta)r_n r_k \frac{dF^{\theta^4}}{dr} \right) = 0. \quad (3.24)\]

It can be checked that the integrand does not vanish by any of the Fierz identities and these two terms form an independent basis. Thus we must set $F^{\theta^4} = 0$ and hence $\Gamma^{\theta^4} = 0$.

With $\Gamma^{\theta^4}$ eliminated, the most general form of $\Gamma^{(2)}$ can be written as

\[\Gamma^{(2)} = \Gamma^{\theta^2} + \Gamma^{\theta \theta^2}, \quad (3.25)\]

\[\Gamma^{\theta^2} \equiv \int d\tau \left( v^2 F^{\theta^2}_1 + (r \cdot v)^2 F^{\theta^2}_2 \right), \quad (3.26)\]

\[\Gamma^{\theta \theta^2} \equiv \int d\tau \left( r_i v_j (\theta\gamma^{ij}\theta) F^{\theta \theta^2}_1 + \dot{\theta} F^{\theta \theta^2}_2 \right), \quad (3.27)\]

where $F^{\theta^2}_i$ and $F^{\theta \theta^2}_i$ ($i = 1, 2$) are functions of $r(\tau)$ only. Here we have already discarded terms forbidden by C-symmetry and those which can be eliminated by integration by parts.

As the next step, we analyze the Ward identity at $O(\epsilon \partial \theta^3)$. By substituting the expression (3.27) into (3.20), we get

\[\int d\tau \left( -\frac{i r_i r_j v_k (\epsilon\gamma^i\theta)(\theta\gamma^{jk}\theta)}{r} \frac{dF^{\theta \theta^2}_1}{dr} + \frac{i r_i (\dot{\theta})(\epsilon\gamma^i\theta)}{r} \frac{dF^{\theta \theta^2}_2}{dr} + i v_i (\epsilon\gamma^j\theta)(\theta\gamma^{ij}\theta) F^{\theta \theta^2}_1 - i r_i (\epsilon\gamma^j\theta)(\theta\gamma^{ij}\theta) F^{\theta \theta^2}_1 \right) = 0. \quad (3.28)\]
The terms in the integrand are already independent and hence we must have
\[ F_1^{\theta^2} = 0, \quad F_2^{\theta^2} = c_1, \quad (3.29) \]
where \( c_1 \) is a numerical constant.

Now we come to the analysis of the last Ward identity (3.19). In this case, it turns out that the expression we get by the direct substitution of (3.26), (3.27) and (3.29) contains dependent terms and we must perform an integration by parts. In this way, (3.19) can be brought to the form
\[
\int d\tau \left( i \left( \frac{2}{r} \frac{dF_1^{\theta^2}}{dr} - F_2^{\theta^2} \right) v_i (r \cdot v) (e_i^j \theta) + i \left( 2F_1^{\theta^2} - 2c_1 \right) a_i (e_i^j \theta) - \frac{i v^2 r_i (e_i^j \theta)}{r} \frac{dF_1^{\theta^2}}{dr} \right. \\
\left. - i r_i (r \cdot v)^2 (e_i^j \theta) \frac{dF_2^{\theta^2}}{dr} - 2 i r_i (r \cdot v) (e_i^j \dot{\theta}) F_2^{\theta^2} \right) = 0, \quad (3.30)
\]
where the structures are now all independent. Thus their coefficients must separately vanish and we get
\[
F_1^{\theta^2} = c_1, \quad F_2^{\theta^2} = 0. \quad (3.31)
\]

Combining all the results so far obtained, we find that the effective action must be of the form
\[
\Gamma^{(2)} = \int d\tau c_1 \left( v^2 + (\theta \dot{\theta}) \right). \quad (3.32)
\]
Since \( c_1 \) is simply a normalization constant, we will set it to 1/2.

What remains to be done is the examination of the closure relation (3.3) on \( \theta_\alpha \). By using the SUSY transformation laws (3.17), (3.18) and the form of the effective action (3.32), we easily see that the closure relation fixes the off-shell coefficient \( A_{0\alpha\beta\gamma\delta} \) to be
\[
A_{0\alpha\beta\gamma\delta} = \gamma_{\beta\gamma} \gamma_{\alpha\delta}^m + \delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\gamma} \delta_{\beta\delta}. \quad (3.33)
\]

### 3.4 Summary of the results at order 2

Let us pause to summarize the results found at order 2. What we have shown is that at this order the symmetry requirements are powerful enough to fix the effective action and the SUSY transformation laws completely in such a manner that the proper closure relations are fulfilled. In an appropriate frame, they take the simple tree-level form
\[
\Gamma^{(2)} = \int d\tau \frac{1}{2} \left( v^2 + (\theta \dot{\theta}) \right), \quad (3.34)
\]
\[
\Omega_{m\beta}^{(1/2)} = i \gamma_{m}^{\beta} \gamma_{\gamma}, \quad (3.35)
\]
\[
T_{\alpha\beta}^{(1)} = i \delta_{\alpha\beta}. \quad (3.36)
\]
The analysis was completely non-perturbative and it can be interpreted as an unambiguous proof of a non-renormalization theorem in the context of Matrix theory for M theory.

4 Analysis at Order 4

The analysis at order 4 is considerably more involved due to a vast number of possible structures and to the need of often formidable Fierz identities. We shall overcome the essential part of this difficulty by devising a novel classification scheme for various terms that occur in the effective action. The basic idea is to separate, within a given order, the type of terms which occur in the naive eikonal approximation and the rest containing more derivatives. Combined with judicious field redefinitions and the use of the notion of “independent basis” already described, we can reduce the amount of analysis considerably to be able to solve our basic equations \( (2.2) \sim (2.6) \) completely. This method, to be described in detail below, has a further advantage that we can obtain the SUSY transformation laws rather easily.

4.1 Scheme of the analysis

Since the actual process of solving the basic equations is somewhat complicated, we spell out, in this subsection, the essence of our scheme of analysis.

E-type - D-type separation and simplification of the effective action

First, we classify each term that may occur in the effective action into \( E \)-type and \( D \)-type, defined as follows:

- **E-type:** An expression involving \( r_m, v_m \) and \( \theta_\alpha \) only will be called of eikonal- or E-type.
- **D-type:** An expression containing higher derivatives, such as \( a_m, \dot{\theta}_\alpha \) etc., will be called of derivative- or D-type.

Using this terminology, the effective Lagrangian \( \mathcal{L}^{(4)} \) at order 4 can be written as a sum of an E-type part \( \bar{\mathcal{L}}^{(4)} \) and the rest forming a D-type part in the following way:

\[
\mathcal{L}^{(4)} \simeq \bar{\mathcal{L}}^{(4)} + a_m X_m - \Psi_\alpha \dot{\theta}_\alpha .
\]
Here, the symbol \( \simeq \) signifies equality up to a total derivative, and \( X_m \) and \( \Psi_\alpha \) are arbitrary expressions of order 2 and 5/2 respectively. It should be clear that the D-type part can always be brought to the form above by adding appropriate total derivatives. Obviously this E-D separation is not unique: An E-type term in \( \mathcal{L}^{(4)} \) containing \( v_m \) can be rewritten, by “integration by parts”, into sum of E-type and D-type terms. As we shall explicitly demonstrate in Sec. 4.2, this ambiguity can be completely eliminated by first fixing a complete basis for \( \mathcal{L}^{(4)} \) and then choosing among them an independent basis for \( \bar{\mathcal{L}}^{(4)} \), \( X_m \) and \( \Psi_\alpha \). Here we suppose that such a basis has been chosen.

Now we make use of the observation by Okawa \[32\] that the D-type terms in \( \mathcal{L}^{(4)} \) can be removed by the following field redefinitions applied to the Lagrangian \( \mathcal{L}^{(2)} \) at order 2:

\[
\begin{align*}
    r_m &\rightarrow r_m + X_m, \\
    \theta_\alpha &\rightarrow \theta_\alpha + \Psi_\alpha.
\end{align*}
\]

Indeed, one can easily check that, up to total derivatives, the extra terms produced from \( \mathcal{L}^{(2)} \) through these field redefinitions cancel the D-type terms of \( \mathcal{L}^{(4)} \). Thus, \( \mathcal{L}^{(4)} \) can be brought to a form consisting only of E-type terms. We will schematically write \( \bar{\mathcal{L}}^{(4)} \) as \( \bar{\mathcal{L}}^{(4)} = \sum_i f_i(r) e_i \), where \( \{e_i\} \) is a basis of E-type terms and \( f_i(r) \) are the coefficient functions.

**Procedure for the analysis of the Ward Identity**

Next we will examine the Ward identity at order 4. As we have already made use of field redefinitions, we must deal with the most general form of the SUSY transformations. Denoting such transformation at order 0 and 2 by \( \delta^{(0)} \) and \( \delta^{(2)} \) respectively, the Ward identity is expressed as

\[
0 \simeq \delta^{(0)} \bar{\mathcal{L}}^{(4)} + \delta^{(2)} \mathcal{L}^{(2)}. \tag{4.4}
\]

Consider the first term. Since \( \bar{\mathcal{L}}^{(4)} \) has terms containing \( v_m \), the action of \( \delta^{(0)} \) on \( v_m \) produces terms with one \( \dot{\theta} \), which are of D-type. Hence, \( \delta^{(0)} \bar{\mathcal{L}}^{(4)} \) is of the structure \( \bar{E}[f] + \bar{D}[f] \), where \( \bar{E}[f] \) and \( \bar{D}[f] \) denote schematically the E-type and the D-type terms respectively, which depend on the coefficient functions \( f_i \). If the terms in \( \bar{E}[f] \) are not all independent, we rewrite the non-independent terms as much as possible into D-type terms using integration by parts. After this manipulation, we get

\[
\delta^{(0)} \bar{\mathcal{L}}^{(4)} \simeq E[f] + D[f], \tag{4.5}
\]

where \( E[f] \) here contains independent structures only. Furthermore, it is important to recognize that the terms composing \( D[f] \) are actually of special type. As they are produced...
either from the variation of $v_m$, as already explained, or from partial integration of E-type terms, they can only contain one $\dot{\theta}_\alpha$ or one $a^m$. Thus, $D[f]$ must be of the form

$$D[f] = a^m E_m[f] + E_\alpha[f] \dot{\theta}_\alpha,$$

(4.6)

where $E_m[f]$ and $E_\alpha[f]$ are schematic expressions for bosonic and fermionic E-type terms respectively, which are functions of $f_i$.

Now consider the second term of the Ward identity (4.4), namely $\delta^{(2)} L^{(2)}$. Due to the form of $L^{(2)}$, it can be brought to the form

$$\delta^{(2)} L^{(2)} \simeq -a_m \delta^{(2)} r_m[h] + \delta^{(2)} \theta_\alpha \dot{\theta}_\alpha[h],$$

(4.7)

where $h = \{h_k\}$ collectively denotes the coefficient functions for the structures that can appear in the SUSY transformation laws. Evidently, $\delta^{(2)} L^{(2)}$ consists only of D-type terms, which we denote by $\{\bar{d}_i\}$. Recalling that $\delta^{(2)} r_m$ and $\delta^{(2)} \theta_\alpha$ are still arbitrary, this set contains the special type of D-type terms composing $D[f]$ above.

Now an important question is whether the set $\{\bar{d}_i\}$ forms an independent basis for D-type terms. The answer would be “no” if there exists some SUSY transformation $\Delta^{(2)}$, referred to as “null” transformation, for which the RHS of (4.7) becomes a total derivative, i.e.

$$-a_m \Delta^{(2)} r_m + \Delta^{(2)} \theta_\alpha \dot{\theta}_\alpha = \frac{dG}{d\tau},$$

(4.8)

for some $G$. A detailed investigation of this equation, summarized in Appendix A, shows that although such null transformations exist, they cannot satisfy the proper SUSY closure relations and therefore should be excluded. This proves that the set $\{\bar{d}_i\}$ forms an independent basis for D-type terms.

Combining the results (4.4), (4.5), (4.6) and (4.7), the Ward identity can be written as

$$E[f] + a^m E_m[f] + E_\alpha[f] \dot{\theta}_\alpha - a_m \delta^{(2)} r_m[h] + \delta^{(2)} \theta_\alpha \dot{\theta}_\alpha[h] \simeq 0,$$

(4.9)

and, as it is expressed in terms of independent basis, it leads to the following set of local equations:

$$E[f] = 0,$$

(4.10)

$$E_m[f] - \delta^{(2)} r_m[h] = 0,$$

(4.11)

$$E_\alpha[f] + \delta^{(2)} \theta_\alpha[h] = 0.$$
The first equation imposes relations among $f_i$’s and, as we shall see, determines the form of the effective action. The second and the third equations, on the other hand, will enable us to express $h_k$ in terms of $f_i$, thereby determining the form of the SUSY transformations directly. This feature is extremely useful since it spares us of enumerating all possible SUSY transformations, a task of considerable complexity.

Analysis of the Closure relations

Finally, using the effective action and the transformation laws thus obtained, we examine the closure equations to prove that these transformation laws truly qualify as those of supersymmetry. This type of analysis has never been performed before and it at the same time determines the form of the off-shell coefficient functions $A \sim D$ completely.

In what follows, we will describe in some detail how the procedures sketched above are actually executed.

4.2 General form of the effective action

First, we must write down the most general form of the effective action $\Gamma^{(4)}$. Performing appropriate integration by parts to the expressions already obtained in our previous work \[30\] to bring it to the “standard form” \[4\], we can write it as

$$\Gamma^{(4)} = \Gamma^{\partial^4} + \Gamma^{\partial^3 \theta^2} + \Gamma^{\partial^2 \theta^2} + \Gamma^{\partial \theta^6} + \Gamma^\theta,$$

$$\Gamma^{\partial^4} = \int d\tau \left( f_1^4 v^4 + F_2^4 (r \cdot a)^2 + F_3^4 (r \cdot v) (v \cdot a) + F_4^4 (r \cdot a)^2 + F_5^4 (r \cdot v)^2 (r \cdot a) + F_6^4 a^2 \right),$$

$$\Gamma^{\partial^3 \theta^2} = \int d\tau \left( f_1^3 \theta \gamma^{ij} \theta v^2 r_j (\theta \gamma^{ij} \theta) + F_2^3 \theta \gamma^{ij} \theta (r \cdot v)^2 (\hat{\theta} \theta) + F_3^3 \theta \gamma^{ij} \theta (r \cdot a) (\hat{\theta} \theta) + F_4^3 \theta \gamma^{ij} \theta (\hat{\theta} \theta)^2 + F_5^3 \theta \gamma^{ij} \theta (r \cdot v) r_j (\theta \gamma^{ij} \theta) v_i + F_6^3 \theta \gamma^{ij} \theta (r \cdot a) r_j (\theta \gamma^{ij} \theta) v_i + F_7^3 \theta \gamma^{ij} \theta (\hat{\theta} \theta) + F_8^3 \theta \gamma^{ij} \theta a_i + F_9^3 \theta \gamma^{ij} \theta (r \cdot v) r_j (\theta \gamma^{ij} \theta) a_i + F_{10}^3 \theta \gamma^{ij} \theta (r \cdot a) r_j (\theta \gamma^{ij} \theta) a_i + F_{11}^3 \theta \gamma^{ij} \theta (v_i a_j) \right),$$

$$\Gamma^{\partial^2 \theta^2} = \int d\tau \left( f_2^2 \theta \gamma^{ij} \theta v^2 r_j (\theta \gamma^{ij} \theta) + F_3^2 \theta \gamma^{ij} \theta (r \cdot v)^2 (\hat{\theta} \theta) + F_4^2 \theta \gamma^{ij} \theta (r \cdot a) (\hat{\theta} \theta) + F_5^2 \theta \gamma^{ij} \theta (\hat{\theta} \theta)^2 + F_6^2 \theta \gamma^{ij} \theta (r \cdot v) r_j (\theta \gamma^{ij} \theta) v_i + F_7^2 \theta \gamma^{ij} \theta (r \cdot a) r_j (\theta \gamma^{ij} \theta) v_i + F_8^2 \theta \gamma^{ij} \theta (\hat{\theta} \theta) + F_9^2 \theta \gamma^{ij} \theta a_i + F_{10}^2 \theta \gamma^{ij} \theta (r \cdot v) r_j (\theta \gamma^{ij} \theta) a_i + F_{11}^2 \theta \gamma^{ij} \theta (r \cdot a) r_j (\theta \gamma^{ij} \theta) a_i + F_{12}^2 \theta \gamma^{ij} \theta (v_i a_j) \right).$$
\[ \Gamma^{\partial^4 \theta^4} = \int \mathrm{d} \tau \left( f_1^{\partial^4 \theta^4} v^2 r_i r_j (\theta \gamma^{ij} \theta) (\theta \gamma^{jk} \theta) + f_2^{\partial^4 \theta^4} v_i v_j (\theta \gamma^{ik} \theta) (\theta \gamma^{jk} \theta) \right. \\
+ f_3^{\partial^4 \theta^4} r_i r_j v_k v_l (\theta \gamma^{ik} \theta) (\theta \gamma^{jl} \theta) + F_4^{\partial^4 \theta^4} r_i r_k (r \cdot a) (\theta \gamma^{ij} \theta) (\theta \gamma^{kj} \theta) \\
+ F_5^{\partial^4 \theta^4} r_k a_i (\theta \gamma^{ij} \theta) (\theta \gamma^{j} \theta) + F_6^{\partial^4 \theta^4} r_j v_k (\theta \gamma^{ik} \theta) (\theta \gamma^{j} \theta) \\
+ F_7^{\partial^4 \theta^4} (\theta \gamma^{ij} \theta)^2 + F_8^{\partial^4 \theta^4} r_i v_j (\theta \gamma^{ij} \theta) (\theta \gamma^{jk} \theta) + F_9^{\partial^4 \theta^4} r_j v_k (\theta \gamma^{ij} \theta) (\theta \gamma^{ik} \theta) \\
+ F_{10}^{\partial^4 \theta^4} r_j r_k (r \cdot v) (\theta \gamma^{ij} \theta) (\theta \gamma^{ik} \theta) + F_{11}^{\partial^4 \theta^4} (\theta \gamma^{i} \theta)^2 \\
+ F_{12}^{\partial^4 \theta^4} r_i r_j (\theta \gamma^{i} \theta) (\theta \gamma^{j} \theta) + F_{13}^{\partial^4 \theta^4} r_j r_k (\theta \gamma^{ij} \theta) (\theta \gamma^{ik} \theta) \bigg), \tag{4.16} \]

\[ \Gamma^{\partial^6 \theta^6} = \int \mathrm{d} \tau \left( f_1^{\partial^6 \theta^6} r_i v_j (\theta \gamma^{il} \theta) (\theta \gamma^{jk} \theta) (\theta \gamma^{ki} \theta) (\theta \gamma^{lj} \theta) + f_2^{\partial^6 \theta^6} r_i r_j v_k v_l (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\theta \gamma^{ln} \theta) \right. \\
+ f_3^{\partial^6 \theta^6} r_i r_j r_k v_l (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\theta \gamma^{ln} \theta) \\
+ f_4^{\partial^6 \theta^6} r_i r_j r_k r_l (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\theta \gamma^{ln} \theta) \bigg), \tag{4.17} \]

\[ \Gamma^{\theta^8} = \int \mathrm{d} \tau \left( f_1^{\theta^8} (\theta \gamma^{ij} \theta) (\theta \gamma^{kl} \theta) (\theta \gamma^{il} \theta) (\theta \gamma^{jk} \theta) \\
+ f_2^{\theta^8} r_i r_j (\theta \gamma^{ik} \theta) (\theta \gamma^{jl} \theta) + f_3^{\theta^8} r_i r_j r_k v_l (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\theta \gamma^{ln} \theta) \bigg), \tag{4.18} \]

where the coefficients \( f_i \)'s (for E-type terms) and \( F_i \)'s (for D-type terms) are functions of \( r(\tau) \) only.

As already explained, we can remove all the D-type terms contained in this expression by appropriate field redefinitions. Consider for example the case of \( \Gamma^{\partial^4 \theta^4} \), where all but the first term are of D-type. It is easy to verify that they can be removed by the following field redefinitions applied to \( \mathcal{L}^{(2)} \):

\[ r^m \rightarrow \tilde{r}^m = r^m + F_{\partial^4 \theta^4 \theta}^1 (r \cdot a) r^m + F_{\partial^4 \theta^4 \theta}^3 (r \cdot v) v^m \\
+ F_{\partial^4 \theta^4 \theta}^4 (r \cdot v^2) r^m + F_{\partial^4 \theta^4 \theta}^5 (r \cdot v)^2 r^m + F_{\partial^4 \theta^4 \theta}^6 (r \cdot a^m). \tag{4.19} \]

In a similar manner, all the D-type terms in \( \Gamma^{\partial^2 \theta^2} \), \( \Gamma^{\partial^4 \theta^4} \), \( \Gamma^{\partial^6 \theta^6} \) can be eliminated. The simplified effective action, consisting only of E-type terms, contains terms of the form

\[ \Gamma^{\partial^4 \theta^4} = \int \mathrm{d} \tau f_1^{\partial^4 \theta^4} v^4, \tag{4.20} \]

\[ \Gamma^{\partial^6 \theta^6} = \int \mathrm{d} \tau f_1^{\partial^6 \theta^6} v^2 r_j v_i (\theta \gamma^{ij} \theta), \tag{4.21} \]

\[ \Gamma^{\partial^8 \theta^8} = \int \mathrm{d} \tau \left( f_1^{\partial^8 \theta^8} v^2 r_i r_j (\theta \gamma^{ik} \theta) (\theta \gamma^{jk} \theta) + f_2^{\partial^8 \theta^8} v_i v_j (\theta \gamma^{ik} \theta) (\theta \gamma^{jk} \theta) \\
+ f_3^{\partial^8 \theta^8} r_i r_j v_k v_l (\theta \gamma^{ik} \theta) (\theta \gamma^{jl} \theta) \right) \tag{4.22} \]
\[
\Gamma^{\partial \theta_6} = \int d\tau \left( f_1^{\partial \theta_6} r_i v_j (\partial \gamma^i \theta) (\partial \gamma^j \theta) (\partial \gamma^k \theta) \\
+ f_2^{\partial \theta_6} r_i r_j r_k v_1 (\partial \gamma^i m \theta) (\partial \gamma^j m \theta) (\partial \gamma^k \theta) \right),
\]
(4.23)

\[
\Gamma^{\theta_8} = \int d\tau \left( f_1^{\theta_8} (\partial \gamma^i \theta) (\partial \gamma^j \theta) (\partial \gamma^l \theta) (\partial \gamma^k \theta) \\
+ f_2^{\theta_8} r_i r_j (\partial \gamma^i m \theta) (\partial \gamma^j m \theta) (\partial \gamma^l \theta) (\partial \gamma^k \theta) \\
+ f_3^{\theta_8} r_i r_j r_k r_l (\partial \gamma^i m \theta) (\partial \gamma^j m \theta) (\partial \gamma^k \theta) (\partial \gamma^l \theta) \right). 
\]
(4.24)

4.3 E-type part of the Ward identities and determination of the effective action

Our next task is the analysis of the Ward identity (2.9). By substituting the expansions (2.12), (2.15), (2.18) and the results (3.31), (3.17), (3.18) obtained at order 2, and subsequently classifying terms by the number of \(\theta\)'s, the Ward identity can be split into the following five equations:

\(O(\partial^4 \theta)\):
\[
\int d\tau \left( -\Omega^{\partial^2 \theta} a^m \epsilon^a \gamma^m \theta + T^{\partial^3 \theta} \epsilon \delta \Gamma^{\partial^4 \theta} \right) = 0, 
\]
(4.25)

\(O(\partial^3 \theta^3)\):
\[
\int d\tau \left( -\Omega^{\partial^3 \theta^3} a^m \epsilon^a \gamma^m \theta + T^{\partial^2 \theta^2} \epsilon \delta \Gamma^{\partial^4 \theta} \right) = 0, 
\]
(4.26)

\(O(\partial^2 \theta^5)\):
\[
\int d\tau \left( -\Omega^{\theta^5} a^m \epsilon^a \gamma^m \theta + T^{\partial^4 \theta^4} \epsilon \delta \Gamma^{\partial^4 \theta} \right) = 0, 
\]
(4.27)

\(O(\partial^2 \theta^7)\):
\[
\int d\tau \left( -\Omega^{\theta^7} a^m \epsilon^a \gamma^m \theta + T^{\partial^6 \theta^6} \epsilon \delta \Gamma^{\partial^4 \theta} \right) = 0, 
\]
(4.28)

\(O(\partial \theta^9)\):
\[
\int d\tau \left( -\Omega^{\partial \theta^9} a^m \epsilon^a \gamma^m \theta + T^{\partial^8 \theta^8} \epsilon \delta \Gamma^{\partial^4 \theta} \right) = 0. 
\]
(4.29)

By the E-type - D-type separation procedure explained previously, we decompose each of these equations into 3 types of local equations (4.10) ∼ (4.12). In the rest of this subsection, we solve the purely E-type equations of the type (4.10) to determine the coefficient functions \(f_i\)'s. The other two types of equations, which will fix the SUSY transformation laws, will be studied later.
4.3.1 Analysis at $\mathcal{O}(\partial^4 \epsilon \theta)$

We begin our analysis by looking at the part with one power of $\theta_\alpha$, i.e. at $\mathcal{O}(\partial^4 \epsilon \theta)$. The relevant $E$-type terms are produced by the last two terms in the Ward identity (4.25). When we substitute the explicit form of the effective action (4.20) and (4.21), the resultant $E$-type terms turned out to be not all independent. Thus we have to add appropriate total derivative terms. In this way, we obtain

$$
-i (\epsilon^m \theta) \frac{\delta \Gamma^{\partial^4}}{\delta r^m} + i (\delta \epsilon) \frac{\delta \Gamma^{\partial^3 \theta^2}}{\delta \theta_\alpha} \simeq 
$$

$$
- i \left( -2 f_1^{\partial^3 \theta^2} + \frac{1}{r} \frac{d f_1^{\partial^4}}{dr} \right) v^4 r_i (\epsilon^i \theta) + 2 i G_1^{\partial^3 \theta^2} v^2 a_i (\epsilon^i \theta) 
$$

$$
+ 4 i G_1^{\partial^3 \theta^2} v_i (v \cdot a) (\epsilon^i \theta) - i \left( 4 f_1^{\partial^4} - 2 G_1^{\partial^3 \theta^2} \right) v^2 v_i (\epsilon^i \theta),
$$

(4.30)

where $G_1^{\partial^3 \theta^2}(r)$ is given by

$$
G_1^{\partial^3 \theta^2}(r) \equiv \int_r^\infty f_1^{\partial^3 \theta^2}(r')dr'.
$$

(4.31)

Although it is expressed as an integral, it is actually a local expression. On the RHS of (4.30), the first term is the only $E$-type term and hence it must vanish by itself. This gives the relation

$$
f_1^{\partial^3 \theta^2} = \frac{1}{2r} \frac{d f_1^{\partial^4}}{dr},
$$

(4.32)

which gives a direct connection between $\Gamma^{\partial^4}$ and $\Gamma^{\partial^3 \theta^2}$.

4.3.2 Analysis at $\mathcal{O}(\partial^3 \epsilon \theta^3)$

In an entirely similar manner, $E$-type terms at this order are produced by the last two terms of (4.26), and after adding appropriate total derivatives, we arrive at the following
expression consisting of 3 independent E-type terms and 7 independent D-type terms,

\[-i(\epsilon \gamma^m \theta) \frac{\delta \Gamma^{\theta^2 \theta^4}}{\delta r^n} + i(\overline{\psi} \epsilon) \frac{\delta \Gamma^{\theta^2 \theta^4}}{\delta \theta_\alpha} \simeq \]

\[+ 4 i v^2 f_1^{\theta^2 \theta^4} r_j r_k v_i (\epsilon \gamma^{kl} \theta) (\theta \gamma^{ij} \theta) - iv^2 \left( 4 f_2^{\theta^2 \theta^4} - f_1^{\theta^2 \theta^4} - 4 G_1^{\theta^2 \theta^4} \right) v_j (\epsilon \gamma^i \theta) (\theta \gamma^{ij} \theta) \]

\[-iv^2 r_i r_j v_k \begin{pmatrix} 4 f_1^{\theta^2 \theta^4} + 4 f_3^{\theta^2 \theta^4} - \frac{1}{r} \frac{df_1^{\theta^2 \theta^4}}{dr} \end{pmatrix} (\epsilon \gamma^i \theta) (\theta \gamma^{jk} \theta) \]

\[+ 8 i G_1^{\theta^2 \theta^4} r_j (v \cdot a) (\epsilon \gamma^i \theta) (\theta \gamma^{ij} \theta) - 4i G_3^{\theta^2 \theta^4} r_j v_k a_i (\epsilon \gamma^i \theta) (\theta \gamma^{ij} \theta) \]

\[-4i G_3^{\theta^2 \theta^4} r_j v_i a_k (\epsilon \gamma^i \theta) (\theta \gamma^{jk} \theta) - iv^2 \left( f_1^{\theta^2 \theta^4} - 4 G_1^{\theta^2 \theta^4} \right) r_j (\epsilon \gamma^i \theta) (\theta \gamma^{ij} \theta) \]

\[-i \left( -2 f_1^{\theta^2 \theta^4} + 4 G_3^{\theta^2 \theta^4} \right) r_j v_k (\epsilon \gamma^i \theta) (\theta \gamma^{jk} \theta) + 8 i v^2 G_1^{\theta^2 \theta^4} r_j (\epsilon \gamma^i \theta) (\theta \gamma^{ij} \theta) \]

\[-8 i G_3^{\theta^2 \theta^4} r_j v_i v_k (\epsilon \gamma^i \theta) (\theta \gamma^{jk} \theta) \],

(4.33)

where we have defined

\[G_1^{\theta^2 \theta^4}(r) \equiv \int^r r' f_1^{\theta^2 \theta^4}(r') \, dr', \quad G_3^{\theta^2 \theta^4}(r) \equiv \int^r r' f_3^{\theta^2 \theta^4}(r') \, dr'. \quad (4.34)\]

It can be checked that the E-type structures, the first 3 terms on the RHS, cannot be related by Fierz identities and are independent. Setting them separately to zero, we obtain

\[f_1^{\theta^2 \theta^4} = 0, \quad (4.35)\]

\[f_2^{\theta^2 \theta^4} = \frac{f_1^{\theta^2 \theta^4}}{4}, \quad (4.36)\]

\[f_3^{\theta^2 \theta^4} = \frac{1}{4} \frac{df_1^{\theta^2 \theta^4}}{dr}. \quad (4.37)\]

Evidently, these relations determine \(\Gamma^{\theta^2 \theta^4}\) in terms of \(f_1^{\theta^2 \theta^4}\), which in turn has already been related to \(f_1^{\theta^2 \theta^4}\).

### 4.3.3 Analysis at \(O(\partial^2 \epsilon \theta^5)\)

Beginning at this order with 6 spinors, our task becomes much more difficult, since, in addition to adding total derivatives, we must find and apply judicious Fierz identities in order to bring the relevant E-type terms to completely independent expressions. The last
two terms of the Ward identity (4.27) yields

\[-i(\epsilon^\alpha \gamma^m) \frac{\delta \Gamma_{\theta^i \bar{\theta}^j}}{\delta r^m} + i(\bar{\phi} \gamma^a) \frac{\delta \Gamma_{\bar{\theta}^a \theta^i}}{\delta \phi} \sim \]

\[-2 f_{2} r_{1} r_{2} r_{3} v_{1} (r \cdot v) (\epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j) + 2 f_{1} r_{1} r_{2} r_{3} v_{1} (\epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j) \]

\[+ 2 v^2 f_{2} r_{1} r_{2} r_{3} v_{1} (\epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j) - 2 f_{1} r_{1} r_{2} r_{3} s \epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j) \]

\[+ 4 f_{2} r_{1} r_{2} r_{3} v_{1} (\epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j) \]

\[+ \left( 2 f_{1} + \frac{1}{2} f_{2} \right) r_{1} r_{2} r_{3} v_{1} (\epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j) \]

\[+ \left( f_{1} + \frac{1}{2} f_{2} \right) r_{1} r_{2} r_{3} v_{1} (\epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j) \]

\[+ 4 f_{2} r_{1} r_{2} r_{3} v_{1} (\epsilon^i \gamma^j \theta) (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^j), \tag{4.38} \]

where we have set \(f_{1} \gamma^a \gamma^b\) to zero according to (4.35). The E-type terms in this expression are not (even algebraically) independent and we must make use of various Fierz identities\(^{11}\) as well as integration by parts to reduce them to independent forms. Since these Fierz identities and the results generated by their applications at intermediate steps are too space-filling to be displayed here, we shall only sketch the reduction procedure.

First, we rewrite the last term on the RHS of (4.38) by the use of several 5-free-index type Fierz identities. The results so obtained are further reduced by using the 3-free-index Fierz identities of the following form:

\[(\epsilon^i \gamma^a \theta) (\theta \gamma^a \theta^i) = (\epsilon^i \gamma^a \theta) (\theta \gamma^a \theta^i) - 2(\epsilon^j \gamma^k \theta) (\theta \gamma^a \theta^j) + 2(\epsilon^j \gamma^k \theta) (\theta \gamma^a \theta^j) \delta_{ij} \]

\[+ 2(\epsilon^i \gamma^k \theta) (\theta \gamma^a \theta^j) - 2(\epsilon^j \gamma^k \theta) (\theta \gamma^a \theta^j) - (\epsilon^i \gamma^k \theta) (\theta \gamma^a \theta^j) \delta_{ij}, \tag{4.39} \]

\[(\epsilon^i \gamma^a \theta) (\theta \gamma^a \theta^i) = (\epsilon^i \gamma^a \theta) (\theta \gamma^a \theta^i) + (\epsilon^a \gamma^i \theta) (\theta \gamma^a \theta^i) + (\epsilon^a \gamma^i \theta) (\theta \gamma^a \theta^i) - (\epsilon^i \gamma^a \theta) (\theta \gamma^a \theta^i) \delta_{ij}, \tag{4.40} \]

\[(\theta \gamma^a \theta^i) (\theta \gamma^a \theta^i) = (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^i) + (\theta \gamma^a \theta^i) (\theta \gamma^a \theta^i), \tag{4.41} \]

\(^{11}\)These Fierz identities, many of which are quite complicated, are generated using an efficient algorithm described in the Appendix A of [20]. In particular, the ones involving several different spinors and/or with large numbers of free-indices (\textit{i.e.} uncontracted indices) can be extremely complicated. For example, the longest five-free-index identity consists of 109 terms.
4.3.4 Analysis at \( O(\partial \epsilon \theta^7) \)

We now come to the structure with 8 spinors. The relevant E-type terms are produced by the second and the third terms of the Ward identity (4.28). The procedure for reducing we finally obtain the following completely independent form for the E-type terms:

\[
\left( -i(\epsilon \gamma^m \theta) \frac{\delta \Gamma^{\partial \theta^4}}{\delta r^m} + i(\epsilon \gamma^i \theta) \right) \left( 4f_1^{\partial \theta^6} - \frac{4r^2 f_2^{\partial \theta^6}}{5} - \frac{4G_2^{\partial \theta^6}}{5} \right) r_{i_1} v_{i_2} v_{i_3} (\epsilon \gamma^{i_2} \theta) (\theta \gamma^{a_1} \partial \theta^1) (\theta \gamma^{a_1} \partial \theta^3) \\
+ \left( 2f_1^{\partial \theta^6} - \frac{4r^2 f_2^{\partial \theta^6}}{5} - \frac{2G_2^{\partial \theta^6}}{5} \right) r_{i_1} v_{i_2} v_{i_3} (\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1} \partial \theta^1) (\theta \gamma^{a_1} \partial \theta^3) \\
+ \left( 4f_1^{\partial \theta^6} - \frac{4r^2 f_2^{\partial \theta^6}}{5} - \frac{4G_2^{\partial \theta^6}}{5} \right) r_{i_1} v_{i_2} v_{i_3} (\epsilon \gamma^{a_1} \partial \theta^1) (\theta \gamma^{a_1} \partial \theta^3) (\theta \gamma^{a_1} \partial \theta^3) \\
+ \left( -4r^2 f_2^{\partial \theta^6} + 2f_3^{\partial \theta^4} - 8G_2^{\partial \theta^6} \right) r_{i_1} v_{i_2} v_{i_3} (\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1} \partial \theta^1) (\theta \gamma^{a_1} \partial \theta^3) \\
+ \left( 6f_1^{\partial \theta^6} - \frac{6r^2 f_2^{\partial \theta^6}}{5} - \frac{6G_2^{\partial \theta^6}}{5} \right) r_{i_1} v_{i_2} v_{i_3} (\epsilon \theta) (\theta \gamma^{a_1} \partial \theta^1) (\theta \gamma^{a_1} \partial \theta^3) \\
+ \left( -4f_1^{\partial \theta^6} - \frac{6r^2 f_2^{\partial \theta^6}}{5} - \frac{16G_2^{\partial \theta^6}}{5} \right) r_{i_1} v_{i_2} v_{i_3} (\epsilon \gamma^{i_1} \theta) (\theta \gamma^{a_1} \partial \theta^1) (\theta \gamma^{a_1} \partial \theta^3) \\
+ \left( 18f_2^{\partial \theta^6} + \frac{1}{r^2 \frac{d f_2^{\partial \theta^4}}{dr}} \right) r_{i_1} r_{i_2} r_{i_3} v_{i_4} v_{i_5} (\epsilon \gamma^{i_1} \theta) (\theta \gamma^{i_2} \partial \theta^1) (\theta \gamma^{i_3} \partial \theta^3). \tag{4.42}
\]

Setting the coefficient of each term to zero and making some rearrangements, we obtain the relations

\[
f_2^{\partial \theta^6} = - \frac{1}{18r} \frac{d f_2^{\partial \theta^4}}{dr}, \tag{4.43}
\]

\[
f_1^{\partial \theta^6} = \frac{G_2^{\partial \theta^6}}{5} - \frac{r}{90} \frac{d f_3^{\partial \theta^4}}{dr}, \quad f_2^{\partial \theta^4} = 4G_2^{\partial \theta^6} - \frac{r}{9} \frac{d f_3^{\partial \theta^4}}{dr}, \quad \frac{d f_2^{\partial \theta^4}}{dr} = 4r G_2^{\partial \theta^6} - \frac{r^2}{9} \frac{d f_3^{\partial \theta^4}}{dr}. \tag{4.44}
\]

The first of these relations, (4.43), coincides with the one previously obtained in (25) in the eikonal approximation. The other 3 relations are new. Although we shall not elaborate on it, they can be used to fix the dependence on \( r \) of various coefficients without resort to the analysis of terms with higher number of \( \theta \)’s.

4.3.4 Analysis at \( O(\partial \epsilon \theta^7) \)

At this stage, the resultant terms become algebraically independent. To make them truly independent in the sense defined before, we must add total derivative terms. In this way, we finally obtain the following completely independent form for the E-type terms:
these terms to independent ones parallels the one at $O(\partial^2 \epsilon \theta^5)$. Uses of intricate five-free-index and other types of Fierz identities together with integration by parts leads to the following independent E-type terms:

\[
\left(-i(\epsilon \gamma^m \theta) \frac{\partial \Gamma^{\theta \theta \theta \theta}}{\partial r^m} + i(\epsilon \epsilon) \frac{\partial \Gamma^{\theta \theta \theta \theta}}{\partial \theta}\right) \mid \text{E-type term} \approx \\
-\frac{24 r^2}{5} f_3^{\theta \theta \theta} + f_2^{\theta \theta \theta} - \frac{32 G_3^{\theta \theta \theta}}{5} r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1 i_3} \theta) (\theta \gamma^{a_2 i_2} \theta) (\theta \gamma^{a_2 i_2} \theta) \\
+ \left(-4 f_2^{\theta \theta \theta} - \frac{192 r^2}{25} f_3^{\theta \theta \theta} - \frac{1}{r} \frac{d f_1^{\theta \theta \theta}}{dr} - \frac{96 G_3^{\theta \theta \theta}}{25}\right) r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{i_1} \theta) (\theta \gamma^{a_1 i_2} \theta) (\theta \gamma^{a_1 i_2} \theta) (\theta \gamma^{a_2 i_3} \theta) \\
+ \left(-4 f_2^{\theta \theta \theta} - \frac{8 r^2}{5} f_3^{\theta \theta \theta} - 184 G_3^{\theta \theta \theta} \right) r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1 i_1} \theta) (\theta \gamma^{a_2 i_2} \theta) (\theta \gamma^{a_2 i_2} \theta) \\
+ \left(-8 f_1^{\theta \theta \theta} + \frac{4 r^4}{25} f_3^{\theta \theta \theta} + \frac{12 r^2 G_3^{\theta \theta \theta}}{25} + \frac{12 \tilde{G}_3^{\theta \theta \theta}}{25} + \frac{4 \bar{G} H_3^{\theta \theta \theta}}{25}\right) r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{a_1 a_2 i_1} \theta) (\theta \gamma^{a_1 a_3} \theta) (\theta \gamma^{a_2 a_4} \theta) (\theta \gamma^{a_3 a_1} \theta) \\
+ \left(-16 f_1^{\theta \theta \theta} - \frac{24 r^2}{25} f_3^{\theta \theta \theta} + f_1^{\theta \theta \theta} - \frac{72 r^2}{25} G_3^{\theta \theta \theta} - \frac{72 \tilde{G}_3^{\theta \theta \theta}}{25} - \frac{24 G_3^{\theta \theta \theta}}{25}\right) r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_2 a_3} \theta) (\theta \gamma^{a_3 a_1} \theta) \\
+ \left(-4 f_2^{\theta \theta \theta} + \frac{32 r^2}{25} f_3^{\theta \theta \theta} + \frac{16 G_3^{\theta \theta \theta}}{25}\right) r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{a_1 i_1 i_3} \theta) (\theta \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_2 a_3} \theta) (\theta \gamma^{a_3 a_1} \theta) \\
+ \left(-4 f_2^{\theta \theta \theta} + \frac{8 r^2}{5} f_3^{\theta \theta \theta} + \frac{24 G_3^{\theta \theta \theta}}{5}\right) r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{a_1 a_2 i_3} \theta) (\theta \gamma^{a_1 i_1} \theta) (\theta \gamma^{a_2 a_3} \theta) (\theta \gamma^{a_3 i_2} \theta) \\
+ \left(4 f_2^{\theta \theta \theta} + \frac{8 r^2}{5} f_3^{\theta \theta \theta} - 2 f_2^{\theta \theta \theta} + 8 G_3^{\theta \theta \theta}\right) r_{i_1} r_{i_2} v_{i_3} (\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_2 i_1} \theta) (\theta \gamma^{i_2 i_3} \theta) \\
+ \left(56 f_3^{\theta \theta \theta} + \frac{1 d f_3^{\theta \theta \theta}}{dr}\right) r_{i_1} r_{i_2} r_{i_3} r_{i_4} v_{i_5} (\epsilon \gamma^{i_1} \theta) (\theta \gamma^{a_1 i_2} \theta) (\theta \gamma^{a_1 i_3} \theta) (\theta \gamma^{i_1 i_2} \theta), \tag{4.45}\end{array}
\]

where

\[
G_i^{\theta \theta \theta}(r) \equiv \int r' f_i^{\theta \theta \theta}(r') dr', \quad \tilde{G}_i^{\theta \theta \theta}(r) \equiv \int r' G_i^{\theta \theta \theta}(r') dr', \quad \bar{G}_i^{\theta \theta \theta}(r) \equiv \int r'^3 f_i^{\theta \theta \theta}(r') dr'. \tag{4.46}
\]

Setting the coefficients to zero and rearranging the resultant equations, we get

\[
f_2^{\theta \theta \theta} = \frac{56 r^2}{13} f_3^{\theta \theta \theta}, \tag{4.47}\]
\[
\frac{d f_1^{\theta \theta \theta}}{dr} = -\frac{112 r^3}{13} f_3^{\theta \theta \theta}, \tag{4.48}\]
\[
\frac{df^{9}_{2}}{dr} = -56 r f^{9}_{3}, \quad G^{9}_{3} = - \frac{r^{2} f^{9}_{3}}{13}, \quad f^{9}_{4} = \frac{10 r^{4} f^{9}_{3} + 39 \tilde{G}^{9}_{3} + 13 \tilde{G}^{9}_{3}}{650},
\]
\[
f^{9}_{2} = \frac{4 r^{2} f^{9}_{3}}{13}, \quad f^{9}_{1} = \frac{4 \left(80 r^{4} f^{9}_{3} + 325 G^{9}_{2} + 312 \tilde{G}^{9}_{3} + 104 \tilde{G}^{9}_{3}\right)}{325},
\]
\[(4.49)\]
\[(4.50)\]

4.3.5 Analysis at \( \mathcal{O}(\epsilon \theta) \)

Finally, we are left with the structures with 10 spinors. These structures without any derivatives have already been studied by Paban et al. [22]. However, as they examined equations weaker than the actual invariance conditions, our results will give stronger constraints\(^{12}\). Starting from the Ward identity (4.29), we rewrite the terms with five free indecies by using five-free-index Fierz identities. Then the Ward identity can be brought to the form

\[
- i (\epsilon \gamma^{m} \theta) \frac{\delta \Gamma^{9}_{i}}{\delta r^{m}} \simeq
\left(4 i f^{9}_{3} + \frac{4 i}{15} r \frac{df^{9}_{3}}{dr}\right) r_{i_{1}} r_{i_{2}} r_{i_{3}} (\epsilon \gamma^{a_{1}a_{2}} \theta) (\theta \gamma^{a_{1}a_{2}} \theta) (\theta \gamma^{a_{3}a_{2}} \theta) (\theta \gamma^{a_{3}a_{2}} \theta)
\]
\[
+ \left(-i \frac{df^{9}_{2}}{dr} + \frac{4 i}{15} r \frac{df^{9}_{3}}{dr}\right) r_{i_{1}} r_{i_{2}} r_{i_{3}} (\epsilon \gamma^{i_{1}i_{2}} \theta) (\theta \gamma^{a_{1}a_{2}} \theta) (\theta \gamma^{a_{1}i_{2}} \theta) (\theta \gamma^{a_{3}a_{2}} \theta) (\theta \gamma^{a_{3}i_{2}} \theta)
\]
\[
+ \left(-i \frac{df^{9}_{1}}{dr} + \frac{2 i}{195} r^{3} \frac{df^{9}_{3}}{dr}\right) r_{i_{1}} (\epsilon \gamma^{i_{1}} \theta) (\theta \gamma^{a_{1}a_{2}} \theta) (\theta \gamma^{a_{1}i_{2}} \theta) (\theta \gamma^{a_{3}i_{2}} \theta) (\theta \gamma^{a_{3}i_{2}} \theta)
\]
\[
+ \left(2 i f^{9}_{2} + \frac{8 i}{195} r^{3} \frac{df^{9}_{3}}{dr}\right) r_{i_{1}} (\epsilon \gamma^{a_{1}} \theta) (\theta \gamma^{a_{1}a_{2}} \theta) (\theta \gamma^{a_{1}a_{3}} \theta) (\theta \gamma^{a_{3}a_{4}} \theta) (\theta \gamma^{a_{3}i_{1}} \theta).
\]
\[(4.51)\]

Setting the coefficient of each term to zero and rearranging the resultant equations, we get,

\[
f^{9}_{2} = \frac{-4 r^{3} \frac{df^{9}_{3}}{dr}}{195},
\]
\[
f^{9}_{3} = \frac{-r \frac{df^{9}_{3}}{dr}}{15},
\]
\[
\frac{df^{9}_{1}}{dr} = \frac{2 r^{4} \frac{df^{9}_{3}}{dr}}{195},
\]
\[
\frac{df^{9}_{2}}{dr} = \frac{4 r^{2} \frac{df^{9}_{3}}{dr}}{15}.
\]
\[(4.52)\]
\[(4.53)\]
\[(4.54)\]
\[(4.55)\]

\(^{12}\)However, the extra solutions allowed by the weaker conditions do not satisfy physical requirements and their effective action agrees with our result (4.61).
4.3.6 Determination of the effective action

Having found all the relations imposed by the Ward identities, we now combine them to determine the coefficient functions \( f_i \). First, by solving the differential equation (4.53), one finds

\[ f_3^{\theta_8} \propto \frac{1}{r^{15}}. \]  

(4.56)

Now we put this result into (4.32), (4.36), (4.37), (4.43), (4.47), (4.48), (4.52) and (4.54), none of which contains integrated coefficients \( G_i \)’s. There can be two choices of the physical boundary condition for solving these set of differential equations, depending on one’s view. One choice would be to require that \( \mathcal{L}^{(4)} \) should be finite as \( r \to \infty \). A slightly stronger alternative is that \( \mathcal{L}^{(4)} \) should not only be finite but should vanish in the above limit. If we adopt the former, we get

\[
\begin{align*}
    f_1^{\theta_4} &= b + \frac{c}{r^7}, \\
    f_1^{\theta^3 \theta^2} &= -\frac{7}{2} c r^9, \\
    f_1^{\theta^2 \theta^4} &= 0, \\
    f_2^{\theta^2 \theta^4} &= -\frac{7}{8} c r^9, \\
    f_3^{\theta^2 \theta^4} &= \frac{63}{8} c r^{11}, \\
    f_1^{\theta^6} &= \frac{7}{8} c r^{11}, \\
    f_2^{\theta^6} &= \frac{77}{16} c r^{13}, \\
    f_3^{\theta^6} &= \frac{143}{128} c r^{15},
\end{align*}
\]

where \( b \) and \( c \) are finite constants, while the latter stronger condition sets \( b \) to zero. At first sight, the presence of a term like \( bv^4 \) appears to violate the cluster property. This is certainly correct if such a term is generated by some interactions of the underlying theory. Actually, in the case of Matrix theory a simple dimensional analysis tells us that the coefficient \( b \) must be proportional to \( g^{-14/3} \), where \( g \) is the gauge coupling constant, and hence could only be of non-perturbative origin. However, since we do not make any assumption about the underlying theory, one may simply accept such a term as describing a self-interaction of a D-particle. As the rest of our analysis is not affected by the presence of \( b \), we will keep it.

Let us briefly compare our results (4.57) ∼ (4.61) with those obtained previously by various authors. We should distinguish two categories:

- General analysis without assuming underlying theory: This type of analysis at order 4 was initiated by Paban et al. [22] for the \( \mathcal{O}(\theta^8) \) part which does not contain any derivatives and later extended by Hyun et al. [25] to the full structures containing
all the allowed powers of $\theta$. In spite of the fact that these analyses were incomplete in several senses, as already explained before, our complete fully off-shell results agree precisely with those obtained in [25]. This can be 'explained' by our method of E-type - D-type separation. Although careful analysis of independent basis was crucial, the E-type part of the equations turned out to be essentially the same as those in [25]. From the point of view of eikonal approximation, however, this is largely a coincidence: By adding total derivatives, the structures of the E-type part could have been different.

- Explicit calculation in Matrix theory: Various authors performed explicit 1-loop calculation of the effective action with or without $\theta$'s in the eikonal approximation [4, 5, 15, 19, 17]. At the off-shell level, some partial results were reported in [12, 34] and finally the full 1-loop result, including all the fermionic terms, was obtained in [29], which agreed with all the previous results where comparisons could be made. Due to the different 'frame' adopted, the result of [29] is superficially different from the one obtained here, but we have checked that after appropriate field redefinitions they agree completely provided that we take

$$b = 0, \quad c = -\frac{15}{16}. \quad (4.62)$$

Before we turn to the determination of the SUSY transformations, we should make a remark. As an alert reader may have noticed already, the set of all the relations imposed on the coefficient functions forms an over-determined system. It can be checked that our solutions obtained using a part of these relations do satisfy all the rest of the equations, as they should.

### 4.4 D-type part of the Ward identities and determination of the SUSY transformation laws

Having determined the effective action from the E-type part of the Ward identities, we now solve the remaining D-type part of the identities to obtain the form of the SUSY transformation laws.

We start with the analysis of the part containing one power of $\theta$. Using the D-type terms left in (4.30), the Ward identities of the type (4.11) and (4.12) at $O(\partial^4 \theta)$ are given (in a combined form) by

$$- Q_i^{a} a^m e_{a}^{m} + T_{a}^{\alpha} \epsilon_{a}^{\alpha} \hat{\theta}_{a} + 2 i G_{1}^{\alpha_{i}^{a}} v^{2} a_{i} (\epsilon_{\gamma_{i}^{a}} \theta) + 4 G_{1}^{a_{i}^{a}} v_{i} (v \cdot a) (\epsilon_{\gamma_{i}^{a}} \theta) - i \left(4 f_{1}^{\alpha_{i}^{a}} - 2 G_{1}^{\alpha_{i}^{a}} \right) v^{2} v_{i} (\epsilon_{\gamma_{i}^{a}} \hat{\theta}) \simeq 0. \quad (4.63)$$
By substituting the results (4.57) and (4.58) into the above equation, and reading off the coefficients of $a^m$ and $\theta_\alpha$, we immediately obtain

\[
\Omega_{\beta \gamma}^{\partial^2 \theta} \epsilon_\beta = 2 i \left( b + \frac{c}{r^2} \right) v_i v_m (\epsilon r^i \theta) + i \left( b + \frac{c}{r^2} \right) v^2 (\epsilon r^m \theta), \tag{4.64}
\]

\[
T_{\alpha \beta}^{\partial^2 \theta} \epsilon_\beta = 3 i \left( b + \frac{c}{r^2} \right) v^2 v_i (\epsilon r^i \theta). \tag{4.65}
\]

Likewise, using the D-type terms in (4.33) and the knowledge of the coefficients (4.58) and (4.59), we get, from the Ward identity at $O(\partial^3 \theta)$,

\[
\Omega_{\beta \gamma}^{\partial^3 \theta} \epsilon_\beta = \frac{7 i c r_i v_j (\epsilon r^m \theta)(\theta \gamma^i \theta)}{2 r^9} + \frac{7 i c r_i v_j (\epsilon r^j \theta)(\theta \gamma^i \theta)}{2 r^9}, \tag{4.66}
\]

\[
T_{\alpha \beta}^{\partial^3 \theta} \epsilon_\beta = \frac{7 i c v^2 r_i (\theta \gamma^i \theta)(\epsilon r^j \theta)}{2 r^9} + \frac{7 i c r_i v_j v_k (\theta \gamma^i \theta)(\epsilon r^j \theta)}{2 r^9} - \frac{7 i c r_i v_j v_k (\epsilon r^j \theta)(\theta \gamma^i \theta)}{r^9}. \tag{4.67}
\]

The procedures to get the SUSY transformation laws at $O(\partial^3 \theta)$ and $O(\partial \theta^7)$ are entirely similar. The results, which are rather involved, are recorded in Appendix B.

### 4.5 Closure relations on $\theta_\alpha$

The final step of our endeavor is to show that the transformation laws obtained above are bonafide those of supersymmetry, \textit{i.e.} they satisfy the proper closure relations (2.4) and (2.5). In the course of this demonstration, we will be able to fix the off-shell coefficients $A \sim D$ completely.

In this subsection, we study the closure relation (2.4) on $\theta_\alpha$. To this end, we substitute into (2.4) the explicit expression (2.7) for the LHS, the expansions (2.12), (2.15), (2.18) and the results obtained at order 2, namely (3.34), (3.17) and (3.18). Collecting terms with the same number of $\theta$'s, the closure relation can then be split into the following four equations:

\[
O(\partial^3 \theta):
\]

\[
\left\{ i(\gamma^m \theta)_{\beta}^{\partial \gamma} \frac{\delta T_{\alpha \gamma}^{\partial^3 \theta}}{\delta r^m} - i \phi_{\beta \delta} \frac{\delta T_{\alpha \gamma}^{\partial^2 \theta^2}}{\delta \theta_\delta} + i \delta_{\gamma}^{\partial \theta} \right\} + (\beta \leftrightarrow \gamma) = A_{\alpha \beta \gamma \delta}^{0} \frac{\delta T_{\partial^3 \theta}}{\delta \theta_\delta} + A_{\alpha \beta \gamma \delta}^{\partial^2 \theta} \delta \theta_\delta - B_{\alpha \beta \gamma m} a^m, \tag{4.68}
\]

30
\( O(\partial^2 \theta^3): \)
\[
\left\{ i(\gamma^m \theta)_\beta \frac{\delta T^{\alpha_\gamma^4}}{\delta r^m} - i\tilde{\psi}_\beta \frac{\delta T^{\alpha_\gamma}}{\delta \theta_\delta} + i\tilde{\psi}_m \gamma_\alpha \beta \right\} + (\beta \leftrightarrow \gamma) = A^0_{\alpha \beta \gamma \delta} \frac{\delta \Gamma^{\partial^2 \theta^4}}{\delta \theta_\delta} + A^{\partial \theta^4}_{\alpha \beta \gamma \delta} \dot{\theta}_\delta - B^{\partial^4 \theta}_{\alpha \beta \gamma \gamma} a^n, \tag{4.69}
\]

\( O(\partial^5 \theta): \)
\[
\left\{ i(\gamma^m \theta)_\beta \frac{\delta T^{\partial \theta^4}}{\delta r^m} - i\tilde{\psi}_\beta \frac{\delta T^{\partial \theta^6}}{\delta \theta_\delta} + i\tilde{\psi}_m \gamma_\alpha \beta \right\} + (\beta \leftrightarrow \gamma) = A^0_{\alpha \beta \gamma \delta} \frac{\delta \Gamma^{\partial \theta^6}}{\delta \theta_\delta} + A^{\partial \theta^4}_{\alpha \beta \gamma \delta} \dot{\theta}_\delta, \tag{4.70}
\]

\( O(\theta^7): \)
\[
i(\gamma^m \theta)_\beta \frac{\delta T^{\theta^6}}{\delta r^m} + (\beta \leftrightarrow \gamma) = A^0_{\alpha \beta \gamma \delta} \frac{\delta \Gamma^{\theta^8}}{\delta \theta_\delta}, \tag{4.71}
\]

where \( A^0_{\alpha \beta \gamma \delta} \) is already given in \( \text{(3.33)} \). For later convenience we have stripped off the arbitrary spinors \( \epsilon_\beta \) and \( \lambda_\gamma \). Below, we will examine the consistency of each of these relations and determine the form of the remaining off-shell coefficients.

### 4.5.1 Analysis at \( O(\partial^3 \theta) \)

First, we analyze the closure relation at \( O(\partial^3 \theta) \). Substituting the SUSY transformation laws \( \text{(1.64), (1.65) and (1.67)} \) into the LHS of \( \text{(4.68)} \) and contracting with arbitrary spinors \( \epsilon_\beta \lambda_\gamma \psi_\alpha \) from left, we get

LHS of \( \text{(4.68)} \) = \(- \frac{7 c v^2 r_i v_j (\epsilon \gamma^i \theta) (\lambda \gamma^j \psi)}{r^9} - \frac{7 c v^2 r_i v_j (\epsilon \gamma^i \psi) (\lambda \gamma^j \theta)}{r^9} + \frac{7 c v^2 r_i v_j (\epsilon \gamma^i \theta) (\lambda \gamma^j \psi)}{r^9} \)
\[
+ \frac{7 c v^2 r_i v_j (\epsilon \gamma^i \psi) (\lambda \gamma^j \theta)}{r^9} - \frac{7 c v^2 r_i v_j (\epsilon \gamma^i \psi) (\lambda \gamma^j \theta)}{r^9}
+ \frac{7 c v^2 r_i v_j (\epsilon \gamma^i \psi) (\lambda \gamma^j \theta)}{r^9} + \frac{14 c v^2 r_i v_j (\epsilon \lambda) (\psi \gamma^i \theta)}{r^9}
+ 2 \left( b + \frac{c}{r^7} \right) (v \cdot a) (\epsilon \gamma^a \theta) (\lambda \gamma^a \psi) - 2 \left( b + \frac{c}{r^7} \right) v^2 (\epsilon \gamma^a \theta) (\lambda \gamma^a \psi)
- 2 \left( b + \frac{c}{r^7} \right) (v \cdot a) (\epsilon \gamma^a \psi) (\lambda \gamma^a \theta) + 2 \left( b + \frac{c}{r^7} \right) v^2 (\epsilon \gamma^a \psi) (\lambda \gamma^a \theta)
+ 2 \left( b + \frac{c}{r^7} \right) v_i a_{i_2} (\epsilon \gamma^{i_2} \theta) (\lambda \gamma^{i_1} \psi) - 2 \left( b + \frac{c}{r^7} \right) v_{i_1} a_{i_2} (\epsilon \gamma^{i_2} \psi) (\lambda \gamma^{i_1} \theta)
+ 2 \left( b + \frac{c}{r^7} \right) v_{i_1} a_{i_2} (\epsilon \gamma^{i_1} \theta) (\lambda \gamma^{i_2} \psi) - 4 \left( b + \frac{c}{r^7} \right) v_{i_1} v_{i_2} (\epsilon \gamma^{i_1} \psi) (\lambda \gamma^{i_2} \theta)
- 2 \left( b + \frac{c}{r^7} \right) v_{i_1} a_{i_2} (\epsilon \gamma^{i_1} \psi) (\lambda \gamma^{i_2} \theta) + 4 \left( b + \frac{c}{r^7} \right) v_{i_1} v_{i_2} (\epsilon \gamma^{i_1} \psi) (\lambda \gamma^{i_2} \theta). \tag{4.72}
\)
Note that the first 7 terms are of E-type and the rest are of D-type. Turning to the RHS of (4.68), the first term can be easily computed using the explicit expression \( \Gamma^{\partial^3 \theta^2} = \int \frac{d \tau}{7 c \psi} v^2 r_i v_j (\theta \gamma^{ij} \theta) / (2 r^9) \). The result is

\[
\psi_{\alpha} \epsilon_{\beta} \lambda_{\gamma} A^{0}_{\alpha \beta \gamma \delta} \frac{\delta \Gamma^{\partial^3 \theta^2}}{\delta \theta^\delta} = -7 c v^2 r_i v_j (\lambda \psi) (\epsilon \gamma^{ij} \theta) + 7 c v^2 r_i v_j (\lambda \psi) (\lambda \gamma^{ij} \theta) \frac{1}{r^9} - 7 c v^2 r_i v_j (\epsilon \gamma^{ij} \lambda) (\psi \gamma^i \theta) + 7 c v^2 r_i v_j (\epsilon \gamma^{ij} \lambda) (\lambda \gamma^i \theta) \frac{1}{r^9} + 7 c v^2 r_i v_j (\epsilon \lambda) (\psi \gamma^{ij} \theta) + 7 c v^2 r_i v_j (\epsilon \lambda) (\lambda \gamma^{ij} \theta) \frac{1}{r^9}.
\]

(4.73)

Since the remaining two terms on the RHS of (4.68) are both of D-type, E-type terms in (4.72) and (4.73) must cancel for consistency. Though it is not self-evident, we can show, with the help of Fierz identities, that they do cancel each other.

We are thus left with purely D-type terms on both sides, and we can easily read off \( A^{\partial^2}_{\alpha \beta \gamma \delta} \) and \( B^{\partial \theta}_{\alpha \beta \gamma \delta} \) from this relation:

\[
A^{\partial^2}_{\alpha \beta \gamma \delta} = 2 \left( b + \frac{c}{r^7} \right) v^2 \gamma_{\alpha}^{i} \gamma_{\beta}^{j} \gamma_{\gamma}^{i} \gamma_{\delta}^{j} - 2 \left( b + \frac{c}{r^7} \right) v^2 \gamma_{\alpha}^{j} \gamma_{\beta}^{i} \gamma_{\gamma}^{i} \gamma_{\delta}^{j} \\
+ 4 \left( b + \frac{c}{r^7} \right) v_i v_j \gamma_{\beta}^{i} \gamma_{\gamma}^{j} \gamma_{\gamma}^{i} - 4 \left( b + \frac{c}{r^7} \right) v_i v_j \gamma_{\beta}^{j} \gamma_{\gamma}^{i} \gamma_{\delta}^{j},
\]

(4.74)

\[
B^{\partial \theta}_{\alpha \beta \gamma \delta} = 2 \left( b + \frac{c}{r^7} \right) v_m \gamma_{\alpha}^{i} \gamma_{\beta}^{j} (\theta \gamma^{i})_{\gamma} - 2 \left( b + \frac{c}{r^7} \right) v_m \gamma_{\alpha}^{j} \gamma_{\beta}^{i} (\theta \gamma^{i})_{\gamma} \\
- 2 \left( b + \frac{c}{r^7} \right) v_i \gamma_{\alpha}^{i} \gamma_{\gamma}^{j} (\theta \gamma^m)_{\beta} + 2 \left( b + \frac{c}{r^7} \right) v_i \gamma_{\alpha}^{j} \gamma_{\gamma}^{i} (\theta \gamma^m)_{\gamma} \\
+ 2 \left( b + \frac{c}{r^7} \right) v_i \gamma_{\alpha}^{m} \gamma_{\gamma}^{i} (\theta \gamma^{i})_{\gamma} - 2 \left( b + \frac{c}{r^7} \right) v_i \gamma_{\alpha}^{i} \gamma_{\gamma}^{m} (\theta \gamma^{i})_{\beta}.
\]

(4.75)

This completes the analysis at \( O(\partial^3 \theta) \).

### 4.5.2 Analysis at \( O(\partial^2 \theta^3) \)

Although the procedure is entirely similar as above, the amount of computations needed at \( O(\partial^2 \theta^3) \) increases considerably. For example, the number of E-type terms in the relation (4.69) is 269 on the LHS and 12 on the RHS and we must show that they precisely cancel. By using the explicit representation of \( SO(9) \) \( \gamma \)-matrices and with the aid of Mathematica, we have checked that they indeed cancel. Once E-type terms are cancelled, determination of the form of \( A^{\partial^2 \theta}_{\alpha \beta \gamma \delta} \) and \( B^{\partial \theta}_{\alpha \beta \gamma \delta} \) from the remaining D-type terms is not difficult. The
results are

\[ A_{\alpha i}^{\theta^2} = \]

\[-7cr_i v_j (\partial_\gamma \theta^{ik}) \gamma^j \delta \gamma^k_{\alpha \beta} + 7cr_i v_j (\partial_\gamma \theta^{ik}) \gamma^j \delta \gamma^k_{\alpha \gamma} - 7cr_i v_j (\partial_\gamma \theta^{ik}) \gamma^j \delta \gamma^k_{\beta \gamma} \]

\[-2r^9 + \frac{7cr_i v_j (\partial_\gamma \theta^{ik}) \gamma^j \delta \gamma^k_{\alpha \beta}}{2r^9} + \frac{7cr_i v_j (\partial_\gamma \theta^{ik}) \gamma^j \delta \gamma^k_{\alpha \gamma}}{2r^9} - \frac{7cr_i v_j (\partial_\gamma \theta^{ik}) \gamma^j \delta \gamma^k_{\beta \gamma}}{2r^9} \]

\[-\frac{2r^9}{4.76}, \]

\[ B_{\alpha i}^{\theta^2} = \]

\[ + \frac{7cr_i (\partial_\gamma \theta^{im}) \gamma^j \alpha \beta (\theta_\gamma \beta)}{2r^9} - \frac{7cr_i (\partial_\gamma \theta^{im}) \gamma^j \alpha \gamma (\theta_\gamma \beta)}{2r^9} - \frac{7cr_i (\partial_\gamma \theta^{im}) \gamma^j \alpha \beta (\theta_\gamma \beta)}{2r^9} + \frac{7cr_i (\partial_\gamma \theta^{im}) \gamma^j \alpha \gamma (\theta_\gamma \beta)}{2r^9}. \]

\[ (4.77) \]

4.5.3 Analysis at \(O(\partial \theta^5)\)

The situation is quite analogous to the one just described, except that it is even more involved. We have found that 393 and 35 E-type terms on the LHS and RHS, respectively, of the relation \(4.70\) cancel exactly and the relation among the remaining D-type terms fixes the form of \(A_{\alpha i}^{\theta^4}\) uniquely. Unfortunately, the result consists of 152 terms, which is too space-consuming to be displayed in this paper.

4.5.4 Analysis at \(O(\theta^7)\)

The final closure relation \(4.70\) left to be examined consists only of E-type terms and does not contain any of the off-shell coefficients. Thus it serves as a consistency check of our SUSY transformation laws. The LHS of \(4.70\) has 85 E-type terms while its RHS has 35. In a manner similar to the previous analyses, we have checked that these E-type terms match precisely.

This completes the analysis of the closure relation on \(\theta_\alpha\).

4.6 Closure relations on \(r^m\)

To finish up our rather long exploration, we examine the closure relation \(4.70\) on \(r^m\).
By using the explicit expression (2.8), the expansions (2.12), (2.15), (2.18) and the results (3.34), (3.17), (3.18) obtained at order 2, we can decompose the closure relation (2.5) into the following 4 equations:

**O(∂³):**

\[
\left\{ i\gamma_\alpha^\gamma T_{\gamma\beta}^{\partial^3} + \gamma_\alpha^m T_{\gamma\beta}^{\partial^3} \right\} + (\alpha \leftrightarrow \beta) = -D_{\alpha\beta\gamma}^\delta a^\gamma, \tag{4.78}
\]

**O(∂²θ²):**

\[
\left\{ i(\gamma^{\theta})_\alpha \frac{\delta\Omega_m^{\partial^3}}{\delta \gamma^n} + i\gamma_\alpha^m T_{\gamma\beta}^{\partial^2\theta^2} \right\} + (\alpha \leftrightarrow \beta) = C_{\alpha\beta\gamma\delta}^{\partial^3} \theta^\delta - D_{\alpha\beta\gamma}^{\partial^2} a^\gamma, \tag{4.79}
\]

**O(∂θ⁴):**

\[
\left\{ i(\gamma^{\theta})_\alpha \frac{\delta\Omega_m^{\partial^3}}{\delta \gamma^n} + i\gamma_\alpha^m T_{\gamma\beta}^{\partial^4} \right\} + (\alpha \leftrightarrow \beta) = C_{\alpha\beta\gamma\delta}^{\partial^3} \theta^\delta, \tag{4.80}
\]

**O(θ⁶):**

\[
\left\{ i(\gamma^{\theta})_\alpha \frac{\delta\Omega_m^{\partial^5}}{\delta \gamma^n} + i\gamma_\alpha^m T_{\gamma\beta}^{\partial^6} \right\} + (\alpha \leftrightarrow \beta) = 0. \tag{4.81}
\]

At **O(∂³)**, substituting the SUSY transformation laws (4.64), (4.65) into (4.78), one can easily find

\[
D_{\alpha\beta\gamma}^\delta a^\gamma = 0. \tag{4.82}
\]

Similarly, at **O(∂²θ²)**, substituting (4.64), (4.66), (4.67) into (4.79), we obtain

\[
D_{\alpha\beta\gamma}^{\partial^2} a^\gamma = 0, \tag{4.83}
\]

and

\[
C_{\alpha\beta\gamma\delta}^{\partial^3} = -2 \left(b + \frac{c}{r^i}\right) v_m \gamma^{\delta \gamma} (\theta \gamma)i_\gamma + 2 \left(b + \frac{c}{r^i}\right) v_m \gamma^{i \delta \gamma} (\theta \gamma)_\gamma + 2 \left(b + \frac{c}{r^i}\right) v_i \gamma^{i \delta \gamma} (\theta \gamma)_\gamma - 2 \left(b + \frac{c}{r^i}\right) v_i \gamma^{i \delta \gamma} (\theta \gamma)_\gamma + 2 \left(b + \frac{c}{r^i}\right) v_i \gamma^{i \delta \gamma} (\theta \gamma)_\gamma. \tag{4.84}
\]

Beginning at **O(∂θ⁴)**, we need various Fierz identities. Substituting (4.66), (B.1) and (B.2) into (4.80), we get 175 E-type terms and 4 D-type terms on the LHS, while we do not have any E-type terms on the RHS. Thus the E-type terms on the LHS should vanish.
by themselves. As before, with the help of Mathematica, we can show that they indeed do. From the relations among the remaining D-type terms, we read off the \( C^{\theta^3}_{\alpha \beta \gamma} \) as

\[
C^{\theta^3}_{m \beta \gamma} = -\frac{7 c r_i (\theta \gamma^m \theta) \gamma^j \beta \delta (\theta \gamma^j)_\gamma}{2 r^9} + \frac{7 c r_i (\theta \gamma^m \theta) \gamma^i \delta \gamma (\theta \gamma^i)_\beta}{2 r^9} + \frac{7 c r_i (\theta \gamma^i \theta) \gamma^j \delta \gamma (\theta \gamma^j)m_\beta}{2 r^9} - \frac{7 c r_i (\theta \gamma^i \theta) \gamma^i \beta \delta (\theta \gamma^m)_\gamma}{2 r^9}.
\] (4.85)

Finally at \( O(\theta^6) \), the relevant closure relation (4.81) does not contain any off-shell coefficients and hence it only provides a consistency check. Calculating the LHS of (4.81) using (B.1) and (B.3), we get 97 E-type terms. It can be shown that these terms cancel out due to Fierz identities.

5 Summary and Discussions

In this paper we have developed an efficient unambiguous scheme to analyze the SUSY Ward identity for the effective action, the SUSY transformations and their closure relations to clarify the role of maximal supersymmetry in the dynamics of a D-particle. Our analysis is valid for completely off-shell configurations and assumes no knowledge of the underlying theory.

We found that the effective actions at order 2 and at order 4 are completely determined, up to two numerical constants, by the symmetry requirements alone. In the context of Matrix theory for M theory, this provides a complete unambiguous proof of off-shell non-renormalization theorems.

Moreover, in contrast to previous investigations, we have been able to determine the SUSY transformations uniquely and proved that they satisfy the proper closure relations. This includes the determination of the off-shell coefficient functions appearing in the closure relation as well. As far as the system under consideration is concerned, we believe that our analysis has fully elucidated the power of the symmetries, in particular the supersymmetry.

A natural extension of this work would be the generalization to higher orders in the derivative expansion. For example, let us consider the effective action at order 6. The purely bosonic part at 1-loop was computed in [33] in Matrix theory and a crude analysis without assuming such an underlying theory has been attempted in [23]. To perform a complete analysis, we need to examine the relevant Ward identity, which schematically is
of the form

$$\delta^{(0)}_\epsilon \Gamma^{(6)} + \delta^{(2)}_\epsilon \Gamma^{(4)} + \delta^{(4)}_\epsilon \Gamma^{(2)} = 0, \quad (5.1)$$

where $\Gamma^{(6)}$ is the effective action at order 6 and $\delta^{(4)}_\epsilon$ is the SUSY transformation at order 4. Since we now have the explicit form of $\delta^{(2)}_\epsilon$ and $\Gamma^{(4)}$, the second term on the LHS can be computed. Further, similarly to the case of $\Gamma^{(4)}$, $\Gamma^{(6)}$ can be brought to the form

$$\Gamma^{(6)} = \int d\tau \left( \bar{L}^{(6)} + a_m X^{(4)}_m - \Psi^{(9/2)}_\alpha \dot{\theta}_\alpha \right), \quad (5.2)$$

where $\bar{L}^{(6)}$ denotes the purely E-type part. This means that, by the use of E-type - D-type separation method developed in this paper, it should be possible to determine $\Gamma^{(6)}$ and $\delta^{(4)}_\epsilon$. Moreover, since $\delta^{(2)}_\epsilon \Gamma^{(4)}$ part acts as an ‘inhomogeneous term’ in the relevant equations, even the normalization of $\Gamma^{(6)}$ is expected to be fixed by that of $\Gamma^{(4)}$. This is a new situation starting at this order. The actual calculation would require considerable effort, however.

Another important direction into which to extend our work is to apply our scheme to the multi-body system. Although performed in the eikonal approximation, an explicit calculation in Matrix theory revealed [10, 11] that even the non-linear part of the 11-dimensional supergravity interactions are correctly encoded in Matrix theory. It is extremely important to clarify to what extent this feature is due to supersymmetry. Again, practically this requires a vast amount of work mainly because the number of possible terms in various quantities increases significantly compared to the two-body case.

We hope that progress on these issues can be made in future investigations.

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Appendix A: Null transformations and their closure relations

In this appendix, we study the null transformations $\Delta^{(2)}_{\epsilon} r_m$, which are the solutions of the equation

$$-a_m \Delta^{(2)}_{\epsilon} r_m + (\Delta^{(2)}_{\epsilon} \theta_\alpha) \hat{\theta}_\alpha = \frac{dG(\tau)}{d\tau},$$  \hspace{1cm} (A.1)

for some $G$, and clarify how they affect the closure relations.

Enumeration of null transformations

An efficient algorithm for finding solutions to (A.1) is to write down the most general form of $\Delta^{(2)}_{\epsilon} \theta_\alpha$ and see if $(\Delta^{(2)}_{\epsilon} \theta_\alpha) \hat{\theta}_\alpha$ can be rewritten completely into the form $a_m X_m$ by integration by parts. When that is possible, we get a solution by setting $\Delta^{(2)}_{\epsilon} r_m = X_m$.

We now enumerate all possible solutions. Since $\Delta^{(2)}_{\epsilon} \theta_\alpha$ is of order 3, apart from $\epsilon_\alpha$, it may contain derivatives of $r_m$ up to $\dot{a}_m$.

1. First, consider the case $\Delta^{(2)}_{\epsilon} \theta_\alpha = \dot{a}_m X_{ma}$. By integration by parts, we can rewrite $\dot{a}_m X_{ma} \hat{\theta}_\alpha$ into $-a_m \partial_\tau (X_{ma} \hat{\theta}_\alpha)$. So there is always a solution.

2. Next consider the case where $\Delta^{(2)}_{\epsilon} \theta_\alpha = a_m Y_{ma}$. Then $\Delta^{(2)}_{\epsilon} r_m = Y_{ma} \hat{\theta}_\alpha$ always gives a solution (with $G = 0$).

3. The remaining case is the one in which $\Delta^{(2)}_{\epsilon} \theta_\alpha$ does not contain $a_m$. There are two possibilities:

   1. One possibility of rewriting $(\Delta^{(2)}_{\epsilon} \theta_\alpha) \hat{\theta}_\alpha$ entirely into the form $a_m X_m$ occurs when $\Delta^{(2)}_{\epsilon} \theta_\alpha$ consists of $v_m$ only, since then integration by parts always produces a factor of $a_m$. Since the order of $\Delta^{(2)}_{\epsilon} \theta_\alpha$ is 3, we must use three $v_m$‘s and $\epsilon_\alpha$. The only possibility is $\Delta^{(2)}_{\epsilon} \theta_\alpha = k_1 v_2 (\epsilon \epsilon_\alpha)$ where $k_1$ is a numerical constant. Then, by performing integration by parts, we find

      $$\Delta^{(2)}_{\epsilon} r_m = -2k_1 v_2 (\epsilon \epsilon_\alpha) + k v^2 (\epsilon \epsilon_\alpha) \theta_\alpha. \hspace{1cm} (A.2)$$

   2. The second possibility is when $\Delta^{(2)}_{\epsilon} r_m$ consists of $\theta$ only, since after integrating $a^2 \Delta^{(2)}_{\epsilon} r_m$ by parts the terms containing $\dot{\theta}$ may be canceled by $(\Delta^{(2)}_{\epsilon} \theta_\alpha) \hat{\theta}_\alpha$. Taking into account its order and C-symmetry requirement, the only possibility is

      $$\Delta^{(2)}_{\epsilon} r_m = k_2 (\epsilon \epsilon_\alpha) (\epsilon \gamma m, \theta) (\epsilon \gamma a n, \theta), \hspace{1cm} (A.3)$$
where $k_2$ is a numerical constant. Performing integration by parts, we find that the following $\Delta^{(2)} \theta_\alpha$ gives a solution:

$$
\Delta^{(2)} \theta_\alpha = -k_2 (\epsilon^n)_\alpha (\theta \gamma^{am} \theta) (\theta \gamma^{an} \theta) v_m - 2 k_2 (\epsilon \gamma^n \theta) (\theta \gamma^{an} \theta) (\theta \gamma^{am})_\alpha v_m
$$

$$
- 2 k_2 (\epsilon \gamma^n \theta) (\theta \gamma^{an} \theta) (\theta \gamma^{an}) v_m.
$$

(A.4)

Summarizing, there are 4 types of solutions:

(i) $\Delta^{(2)} \theta_\alpha = \dot{a}_m X_{ma,\beta} \epsilon_\beta$,  
$$
\Delta^{(2)} r_m = -\partial_r (X_{ma,\beta} \epsilon_\beta \dot{\theta}_\alpha), \quad G = a_m X_{ma,\beta} \epsilon_\beta \dot{\theta}_\alpha, \tag{A.5}
$$

(ii) $\Delta^{(2)} \theta_\alpha = a_m Y_{ma,\beta} \epsilon_\beta$,  
$$
\Delta^{(2)} r_m = Y_{ma,\beta} \dot{\theta}_\alpha, \quad G = 0, \tag{A.6}
$$

(iii) $\Delta^{(2)} \theta_\alpha = k_1 v^2 (\phi \epsilon)_\alpha$,  
$$
\Delta^{(2)} r_m = -2 k_1 v_m \epsilon \phi \theta - k_1 v^2 \epsilon_m \theta, \quad G = k_1 v^2 \epsilon \phi \theta, \tag{A.7}
$$

(iv) $\Delta^{(2)} \theta_\alpha = -k_2 (\epsilon \gamma^n)_\alpha (\theta \gamma^{am} \theta) (\theta \gamma^{an} \theta) v_m - 2 k_2 (\epsilon \gamma^n \theta) (\theta \gamma^{an} \theta) (\theta \gamma^{am})_\alpha v_m
$$

$$
- 2 k_2 (\epsilon \gamma^n \theta) (\theta \gamma^{an} \theta) (\theta \gamma^{an}) v_m, \tag{A.8}
$$

$$
\Delta^{(2)} r_m = k_2 (\epsilon \gamma^n \theta) (\theta \gamma^{an} \theta), \quad G = k_2 (\epsilon \gamma^n \theta) (\theta \gamma^{an} \theta) v_m. \tag{A.9}
$$

(A.10)

(A.11)

(A.12)

(A.13)

(A.14)

(A.15)

(A.16)

**Examination of the closure relation**

Now we study how these null transformations affect the closure relations. Let $\delta_\epsilon = \delta^{(0)}_\epsilon + \delta^{(2)}_\epsilon$ be a SUSY transformation which already satisfies the proper closure relations. Then, an addition of $\Delta^{(2)} \lambda^{(2)}$ produces, at order 2, a contribution to the commutator

$$
\left[ \left[ \delta^{(0)}_\epsilon, \Delta^{(2)} \lambda \right] - (\epsilon \leftrightarrow \lambda) \right] \left( \begin{array}{c} r_m \\ \dot{\theta}_\alpha \end{array} \right).
$$

(A.17)

We shall examine if this is of an appropriate form for proper closure relations to be maintained, for each of the 4 solutions above.

(i) $X_{ma,\beta}$ is an arbitrary structure of order 0. Combined with the restriction from C-symmetry, the only possible structure for $X_{ma,\beta}$ is

$$
X_{ma,\beta} = X_1 \gamma_{a,\beta}^m. \tag{A.18}
$$

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where $X_1$ is a function of $r(\tau)$ only. Then, the SUSY transformation laws become

$$
\Delta^{(2)}_\epsilon r^m = -\frac{(r \cdot v) (e \gamma^m \hat{\theta})}{r} \frac{dX_1}{dr} - X_1 (e \gamma^m \hat{\theta}), \quad (A.19)
$$

$$
\Delta^{(2)}_\epsilon \theta_\alpha = X_1 (\gamma^i \epsilon)_\alpha \hat{a}_i, \quad (A.20)
$$

and the additional contribution to the closure relation for $r^m$ takes the form

$$
\left( [\delta^{(0)}_\epsilon, \Delta^{(2)}_\chi] - (\epsilon \leftrightarrow \lambda) \right) r^m = \frac{2i (\epsilon \lambda) (r \cdot v) a_m}{r} \frac{dX_1}{dr} + 4i X_1 (\epsilon \lambda) \hat{a}_m. \quad (A.21)
$$

While the first term, proportional to $a^m$, only modifies the form of the off-shell coefficient $D_{\alpha\beta\gamma\delta}$, the second term containing $\hat{a}^m$ cannot be absorbed into any of the coefficient functions and hence spoils the proper closure relation. Thus, the solution (i) does not qualify as proper SUSY transformation laws.

(ii) $Y_{\alpha\beta}$ is an arbitrary structure of order 1. Combined with the restriction from C-symmetry, the possible structures for $Y_{\alpha\beta}$ are

$$
Y_{\alpha\beta} = Y_1 (r \cdot v) \gamma^m_{\alpha\beta} + Y_2 \gamma^{mnl}_{\alpha\beta} r_l v_n + Y^{\theta_2}_{\alpha\beta\rho\sigma} \theta_\rho \theta_\sigma. \quad (A.22)
$$

Here $Y_i, Y^{\theta_2}_{\alpha\beta\rho\sigma}$ are functions of $r(\tau)$ only and the most general form of $Y^{\theta_2}_{\alpha\beta\rho\sigma}$ is

$$
Y^{\theta_2}_{\alpha\beta\rho\sigma} = Y_1 \gamma^i_{\alpha\beta} \gamma^m_{\rho\sigma} + Y_2 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_3 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_4 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_5 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_6 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_7 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_8 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_9 \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_{10} \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_{11} \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma} + Y_{12} \gamma^i_{\alpha\beta} \gamma^{mij}_{\rho\sigma}, \quad (A.23)
$$

where $Y_i^{\theta_2}$ ($i = 1 \sim 12$) are functions of $r(\tau)$ only. SUSY transformation laws then become

$$
\Delta^{(2)}_\epsilon r^m = Y_1 (r \cdot v) (e \gamma^m \hat{\theta}) - Y_2 r_i v_j (e \gamma^{mij} \hat{\theta}) + Y^{\theta_2}_{\alpha\beta\rho\sigma} \epsilon_{\beta\rho\sigma} \theta_\alpha \theta_\rho, \quad (A.24)
$$

$$
\Delta^{(2)}_\epsilon \theta^\alpha = Y_1 a_i (r \cdot v) (\gamma^i \epsilon)_\alpha + Y_2 r_k v_j a_i (\gamma^{ij} \epsilon)_\alpha + Y^{\theta_2}_{\alpha\beta\rho\sigma} \epsilon_{\beta\rho\sigma} \theta_\alpha \theta_\rho a_m. \quad (A.25)
$$

Now we note that there exists a field redefinition of the form

$$
\tilde{r}^m = r^m, \quad \tilde{\theta}_\alpha = \theta_\alpha + i Y_1 (r \cdot v) \hat{\theta}_\alpha. \quad (A.26)
$$

which preserves the form of the effective action. It, however, changes the form of the transformation laws as

$$
\delta^{(2)}_\epsilon \tilde{r}^m = \delta^{(2)}_\epsilon r^m - Y_1 (r \cdot v) (e \gamma^m \hat{\theta}), \quad (A.27)
$$

$$
\delta^{(2)}_\epsilon \tilde{\theta}_\alpha = \delta^{(2)}_\epsilon \theta_\alpha - Y_1 a_i (r \cdot v) (\gamma^i \epsilon)_\alpha. \quad (A.28)
$$
Thus, by using this field redefinition, we can always set $Y_1 = 0$. With this choice, the additional contribution to the closure relation for $\theta_\alpha$ becomes

\[
\left( [\delta^{(0)}_\epsilon, \Delta^{(2)}_\lambda] - (\epsilon \leftrightarrow \lambda) \right) \theta_\alpha = i Y_2 r_j v_k (\epsilon \gamma^i \tilde{\theta}) (\lambda \gamma^{ijk})_\alpha + i Y_2 r_j v_k (\epsilon \gamma^{ijk} \tilde{\theta}) (\lambda \gamma^i)_\alpha \\
- i Y_2 r_j v_k (\lambda \gamma^i \tilde{\theta}) (\epsilon \gamma^{ijk})_\alpha - i Y_2 r_j v_k (\lambda \gamma^{ijk} \tilde{\theta}) (\epsilon \gamma^i)_\alpha \\
- i (\epsilon \gamma^m \tilde{\theta}) Y^{\beta_\sigma}_{\alpha \beta \lambda \sigma} \lambda \beta \theta_\rho \theta_\sigma + i (\lambda \gamma^m \tilde{\theta}) Y^{\alpha_\sigma}_{\alpha \beta \lambda \sigma} \epsilon \beta \theta_\rho \theta_\sigma \\
- i Y^{\alpha_\sigma}_{\alpha \beta \lambda \sigma} \lambda \beta \theta_\rho \theta_\sigma (\gamma^m \epsilon)_\alpha + i Y^{\alpha_\sigma}_{\alpha \beta \lambda \sigma} \epsilon \beta \theta_\rho \theta_\sigma (\gamma^m \lambda)_\alpha \\
+ \text{terms with } a_m \text{ and } \dot{\theta}_\alpha. \tag{A.29}
\]

As before, the terms with $a_m$ and $\dot{\theta}_\alpha$ only produce changes in the off-shell coefficient $A_{\alpha \beta \gamma \delta}, B_{\alpha \beta \gamma \sigma}$. On the other hand, the first 8 terms, which do not vanish by any use of the Fierz identities, contain $\dot{a}_m$ and cannot be absorbed by the off-shell coefficients. Thus, the solution (ii) does not lead to proper SUSY transformation laws.

(iii) By using the SUSY transformation laws \ref{eq:A.11} and \ref{eq:A.12}, we can easily compute the extra term produced in the closure relation on $r^m$ to be $-8 k_1 (\epsilon \lambda) \psi^2 v_m$. Proper closure relation cannot contain such a term and hence the case (iii) is also excluded.

(iv) Finally we come to the case (iv). The additional terms produced in the closure relation on $r^m$ take the form

\[
8 i k_2 \psi^2 (\epsilon \gamma^i \theta) (\lambda \gamma^j \theta) (\psi \gamma^{ij} \theta) - 4 i k_2 v_i v_j (\epsilon \gamma^{ikl} \theta) (\lambda \gamma^k \theta) (\psi \gamma^{ij} \theta) \\
+ 4 i k_2 v_i v_j (\epsilon \gamma^k \theta) (\lambda \gamma^{ikl} \theta) (\psi \gamma^{ij} \theta) + 4 i k_2 \psi^2 (\epsilon \gamma^i \theta) (\lambda \gamma^j \psi) (\theta \gamma^{ij} \theta) \\
+ 4 i k_2 \psi^2 (\epsilon \gamma^i \psi) (\lambda \gamma^j \theta) (\theta \gamma^{ij} \theta) + 8 i k_2 v_i v_j (\epsilon \lambda) (\psi \gamma^k \theta) (\theta \gamma^{ij} \theta) \\
- 2 i k_2 v_i v_j (\epsilon \gamma^{ikl} \theta) (\lambda \gamma^k \psi) (\theta \gamma^{ij} \theta) - 2 i k_2 v_i v_j (\epsilon \gamma^{ikl} \psi) (\lambda \gamma^k \theta) (\theta \gamma^{ij} \theta) \\
+ 2 i k_2 v_i v_j (\epsilon \gamma^k \theta) (\lambda \gamma^{ikl} \psi) (\theta \gamma^{ij} \theta) + 2 i k_2 v_i v_j (\epsilon \gamma^k \psi) (\lambda \gamma^{ikl} \theta) (\theta \gamma^{ij} \theta). \tag{A.30}
\]

They are of unallowed form and it is not difficult to show that they do not vanish by any of the Fierz identities. Thus we must discard this final possibility.

This demonstrates that there are no acceptable null transformations and hence the D-type basis is independent. This in turn is responsible for the uniqueness of the the SUSY transformation laws determined by the SUSY Ward identities.
Appendix B: SUSY transformation laws

In Sec. 4.3, we recorded $O(\theta^0)$ and $O(\theta^2)$ parts of the SUSY transformation laws. In this appendix, we display the remaining $O(\theta^4)$ and $O(\theta^6)$ parts.

$O(\theta^4)$ part:

\[
\Omega_{\theta^4}^{\theta^4} = \frac{7i c (e_\gamma^i \theta) (\theta_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta)}{16 r^9} + \frac{91 i c r_i r_j (e_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} + \frac{119 i c r_i r_j (e_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{20 r^{11}} + \frac{2513 i c r_i r_j (e_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta)}{640 r^{11}} - \frac{119 i c r_i r_j (e_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} + \frac{2513 i c r_i r_j (e_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta)}{640 r^{11}} - \frac{189 i c r_i r_j (e_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} + \frac{2289 i c r_i r_j (e_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta)}{640 r^{11}} - \frac{4641 i c r_i r_j (e_\gamma^{j\theta}) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{640 r^{11}}.
\]

$T_{\alpha\beta}^{\theta^4}$

\[
T_{\alpha\beta}^{\theta^4} = -\frac{119 i c r_i r_j r_k (\theta_\gamma^{l_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} + \frac{49 i c r_i r_j r_k (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} - \frac{21 i c (r \cdot v) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{16 r^{11}} + \frac{49 i c r_i r_j r_k (\theta_\gamma^{l_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} + \frac{49 i c r_i r_j r_k (\theta_\gamma^{l_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} - \frac{189 i c r_i r_j r_k (\theta_\gamma^{l_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{80 r^{11}} + \frac{2289 i c r_i r_j r_k (\theta_\gamma^{l_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{640 r^{11}} - \frac{2513 i c r_i r_j r_k (\theta_\gamma^{l_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta) (\theta_\gamma^{j_{\theta}}\theta)}{640 r^{11}}.
\]
\[\begin{align*} \frac{-2793 i c r_{ij} v_k (\epsilon \gamma^{ijl} \theta) (\theta \gamma^{lm} \theta) (\theta \gamma^{im} \theta) \alpha}{320 r^{11}} & + \frac{2513 i c r_{ij} v_k (\epsilon \gamma^{kl} \theta) (\theta \gamma^{lm} \theta) (\theta \gamma^{im} \theta) \alpha}{320 r^{11}} + \frac{91 i c r_{ij} (r \cdot v) (\epsilon \gamma^{ij} \theta) (\theta \gamma^{lm} \theta) (\theta \gamma^{im} \theta) \alpha}{40 r^{11}} \\, \text{etc.} \end{align*}\]

\[O(\theta^6) \text{ part:}\]

\[T^{\theta^6}_{\alpha \beta} = \frac{27 i c r_{ij} (\theta \gamma^{il} \theta) (\theta \gamma^{jk} \theta) (\theta \gamma^{kl} \theta) (\epsilon \gamma^{lm} \theta) \alpha}{40 r^{11}} + \frac{143 i c r_{ij} r_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\epsilon \gamma^{lm} \theta) \alpha}{40 r^{11}} + \frac{11 i c r_{ij} r_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{mn} \theta) (\epsilon \gamma^{kl} \theta) \alpha}{40 r^{11}} + \frac{99 i c r_{ij} r_k (\epsilon \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\theta \gamma^{ln} \theta) \alpha}{40 r^{11}} \, \text{etc.} \]
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