Arbitrary Phase Rotation of the Marked State Cannot Be Used for Grover’s Quantum Search Algorithm

LONG GuiLu, ZHANG WeiLin, LI YanSong and NIU Li
Department of Physics, Tsinghua University, Beijing 100084, China
(Received April 30, 1999)

Abstract A misunderstanding that an arbitrary phase rotation of the marked state together with the inversion about average operation can be used to construct a (less efficient) quantum search algorithm is cleared. The \( \pi \) rotation of the phase of the marked state is not only the choice for efficiency, but also vital in Grover’s quantum search algorithm. The results also show that Grover’s quantum search algorithm is robust.

PACS numbers: 03.67.-a, 03.67.Lx
Key words: quantum searching, quantum computing, Grover algorithm

Grover’s quantum search algorithm is one of the most important development in quantum computation.\(^1\) It achieves quadratic speedup in searching a marked state in an unordered list over classical search algorithms. As the algorithm involves only simple operations, it is easy to implement in experiments. By now, it has been realized in NMR quantum computers.\(^2,3\) Benett et al.\(^4\) have shown that no quantum algorithm can solve the search problem in less than \( O(\sqrt{N}) \) steps. Boyer et al.\(^5\) have given analytical expressions for the amplitude of the states in Grover’s search algorithm and tight bounds. Zalka\(^6\) has improved this tight bounds and showed that Grover’s algorithm is optimal. Zalka also proposed\(^7\) an improvement on Grover’s algorithm. In another development, Biron et al.\(^8\) generalized Grover’s algorithm to an arbitrarily distributed initial state. Pati\(^9\) recast the algorithm in geometric language and studied the bounds on the algorithm.

In each iteration of Grover’s search algorithm, there are two steps: (i) a selective inversion of the amplitude of the marked state, which is a phase rotation of \( \pi \) of the marked state; (ii) an inversion about the average of the amplitudes of all basis states. This second step can be realized by two Hadamard–Walsh transformations and a rotation of \( \pi \) of all the basis states different from \( |0\rangle \). Grover’s search algorithm is a series of rotations in an SU(2) space spanned by \( |n_0\rangle \), the marked state and \( |c\rangle = (1/\sqrt{N-1}) \sum_{n \neq n_0} |n\rangle \). Each iteration rotates the state vector of the quantum computer system an angle \( \psi = 2 \arcsin(1/\sqrt{N}) \) towards the \( |n_0\rangle \) basis of the SU(2) space.\(^10\) Grover further showed\(^11\) that the Hadamard–Walsh transformation can be replaced by almost any unitary transformation. The inversions of the amplitudes can be instead rotated by arbitrary phases.\(^11\) It is believed that\(^7,11\) if one rotates the phases of the states arbitrarily, the resulting transformation is still a rotation of the state vector of the quantum computer towards the \( |n_0\rangle \) basis in the SU(2) space. But the angle of rotation is smaller than \( \psi \). From the consideration of efficiency, the phase rotation of \( \pi \) should be adopted. This fact has been used to the advantage by Zalka recently\(^7\) to improve the efficiency of the quantum search algorithm. According to the proposal, the inversion of the amplitude of the marked state in step one is replaced by a rotation through an angle between 0 and \( \pi \) to produce a smaller angle of SU(2) rotation towards the end of a quantum search calculation so that the amplitude of the marked state in the computer system state vector is exactly 1.

In this letter, we show by explicit construction that the above concept is actually wrong. When the rotation of the phase of the marked state is not \( \pi \), one cannot simply construct a
quantum search algorithm at all. Suppose that the initial state of the quantum computer is
\[ |\phi\rangle = B_0|n_0\rangle + A_0 \frac{1}{\sqrt{N-1}} \sum_{n \neq n_0} |n\rangle. \]
(1)
The modified quantum search algorithm now consists of the two steps: (i) \( |n_0\rangle \rightarrow e^{i\theta}|n_0\rangle \); (ii) an inversion about the average operation \( D \), whose matrix elements are
\[ D_{ij} = \begin{cases} 
\frac{2}{N}, & i \neq j, \\
\frac{2}{N} - 1, & i = j.
\end{cases} \]
(2)
After each iteration of the modified Grover's quantum search, the state vector still has the form (1). The recurrent formula for the amplitudes are
\[ B_{j+1} = - \frac{N - 2}{N} e^{i\theta} B_j + \frac{2\sqrt{N-1}}{N} A_j, \quad A_{j+1} = \frac{2\sqrt{N-2}}{N} e^{i\theta} B_j + \frac{N - 2}{N} A_j. \]
(3)
Denoting \( \cos \psi = (N - 2)/N \), \( \sin \psi = 2\sqrt{N - 1}/N \), we can rewrite the recurrent relation in a matrix form
\[ \begin{pmatrix} B_{j+1} \\
A_{j+1} \end{pmatrix} = \begin{pmatrix} - \cos \psi e^{i\theta} & \sin \psi \\
\sin \psi e^{i\theta} & \cos \psi \end{pmatrix} \begin{pmatrix} B_j \\
A_j \end{pmatrix}. \]
(4)
It is not difficult to diagonalize the transformation matrix. The eigenvalues are
\[ \lambda_{1,2} = e^{i\gamma_{1,2}}, \]
with
\[ \sin \gamma_{1,2} = -\frac{\sin \theta \cos \psi \pm 2\sqrt{1 - \cos^2 \psi \sin^2 \theta \sin(\theta/2)}}{2}, \]
(5)
It is worth while pointing out that the two eigenphases satisfy \( \gamma_1 + \gamma_2 = \pi + \theta \). The corresponding normalized eigenvectors are the column vectors of the matrix \( U \),
\[ U = \begin{pmatrix} \sin \psi / \sqrt{2(1 - \cos \psi \cos \gamma_2)} \\
(cos \psi e^{i\theta} + e^{i\gamma_1}) / \sqrt{2(1 - \cos \psi \cos \gamma_2)} \end{pmatrix} \begin{pmatrix} - \cos \psi + e^{i\gamma_2} / \sqrt{2(1 - \cos \psi \cos \gamma_2)} \\
\sin \psi e^{i\theta} / \sqrt{2(1 - \cos \psi \cos \gamma_2)} \end{pmatrix}. \]
(7)
This \( U \) matrix is unitary and diagonalizes the transformation matrix in Eq. (4), that is \( U^{-1}TU \) is diagonal. The amplitude of the marked state after \( j + 1 \) iterations is
\[ B_{j+1} = \frac{\sin \psi}{2(1 - \cos \psi \cos \gamma_2)} e^{i(j+1)\gamma_1} [\sin \psi B_0 + (\cos \psi e^{-i\theta} + e^{-i\gamma_1}) A_0] \]
\[ + \frac{-\cos \psi + e^{i\gamma_2}}{2(1 - \cos \psi \gamma_2)} e^{i(j+1)\cos \gamma_2} [(- \cos \psi + e^{-i\gamma_2}) B_0 + \sin \psi e^{-i\theta} A_0]. \]
(8)
When \( \theta = \pi \) and \( B_0 = \sqrt{1/N}, A_0 = \sqrt{(N - 1)/N} \), we recover the original Grover's quantum search algorithm, and \( B_{j+1} = \sin((j + 1 + 1/2)\psi) \) as given by Boyer et al.[5]
To see the effect of the rotation angle \( \theta \) on the quantum search algorithm, we plot the norm \( |B_{j+1}| \) with respect to \( \theta \). As examples, we draw \( |B_4| \) in Fig. 1, and \( |B_7| \) in Fig. 2. For simplicity, \( N = 100, B_0 = \sqrt{1/N} \) and \( A_0 = \sqrt{(N - 1)/N} \). From these studies, we see the following points:
(i) As \( j \) increases, \( |B_3| \) increases too for small \( j \) values for \( \theta = \pi \). When \( \theta = \pi \), Grover's original quantum search algorithm is working.
(ii) For other values of \( \theta \) between 0 and \( 2\pi \), the dependence of \( |B_{j+1}| \) on \( \theta \) is not monotonic. There are oscillations. There are peaks and valleys in the values of \( |B| \) for a given \( j \). What is more, when \( j \) changes, the positions of these peaks and valleys change too. In other words, at a given \( \theta \) value, \( |B_{j+1}| \) does not always increase when \( j \) increases. For instance, when \( j = 3 \), there is only one peak for \( \theta \) between 0 and \( \pi \); whereas for \( j = 6 \), there are three peaks.
This is contrary to the common expectation that for small number of iterations, $|B_{j+1}|$ should monotonically increase, though not as big as the standard Grover's quantum search algorithm.

(iii) For a $\theta$ different from $\pi$, even one increases the number of iterations, the norm of the amplitude of the marked state cannot reach 1. There is a limit at which the norm of the amplitude can reach. In Figs 3 and 4, we plot the $|B_{j+1}|$ versus $j$ for $\theta = \pi/4$ and $\theta = \pi/3$ respectively. The behavior is quite interesting. For $\theta = \pi/4$, there are rapid irregular oscillations in the norm. In particular, the maximum height is only about 0.15. The minimum is not zero, it is about 0.07. For $\theta = \pi/3$, the plot can be seen as three lines at an interval of three points. Again, the maximum height is small, only about 0.18. The norm of the amplitude is in a range between 0.06 and 0.18. Even if one increases the number of iterations, the norm cannot be increased any further. In this case, we have plotted $j$ up to 100, which is equal to the number of items in the unsorted system.

(iv) In the vicinity of $\pi$, the algorithm still works, though the height of the norm cannot reach 1. But it can still reach a considerably large value. As shown in Fig. 5, when $\theta = \pi/1.1$, the algorithm can still work, with a maximum $|B_{j+1}|$ of 0.8. This shows that Grover's quantum search algorithm is robust with respect to $\theta$ at $\pi$. This is important as an imperfect gate operation may lead to a phase rotation not exactly equal to $\pi$. Grover's quantum search algorithm has a good tolerance on the phase rotating angle near $\pi$. A small deviation from $\pi$ will not destroy the algorithm.

Fig. 1. $|B_4|$ versus $\theta$.

Fig. 2. $|B_7|$ versus $\theta$.

Fig. 3. $|B_{j+1}|$ versus $j$ for $\theta = \pi/4$.

Fig. 4. $|B_{j+1}|$ versus $j$ for $\theta = \pi/3$. 
To summarize, we see that $\theta = \pi$ is not only a requirement for efficiency, but also a necessary condition for the algorithm. At this angle, the algorithm is also robust. To achieve a smaller increase in the marked state amplitude (or a smaller rotation towards the marked state basis in the SU(2) space), one has to resort to more complicated modifications to Grover's quantum search algorithm.

**Acknowledgments**

Encouragement from Prof. CHEN HaoMing is gratefully acknowledged. We thank Prof. Grover for helpful email discussions regarding Grover's quantum search algorithms and bringing our attention to new references on the algorithm. Helpful discussions with Prof SUN ChangPu are gratefully acknowledged.

**References**

[1] L.K. Grover, Phys. Rev. Lett. 79 (1997) 325.
[2] I.L. Chuang, N. Gershfeld and M. Kubinec, Phys. Rev. Lett. 80 (1998) 3408.
[3] J.A. Jones, M. Mosca and R.H. Hansen, Nature 393 (1998) 344.
[4] C. Bennett et al., Lanl-eprint/quant-ph/9701001, also in SIAM journal on Computing.
[5] M. Boyer, G. Brassard, P. Høyer and A. Tapp, Lanl-eprint/quant-ph9605034, also in Fortsch. Phys. 46 (1998) 493.
[6] C. Zalka, Lanl-eprint/quant-ph/9711070.
[7] C. Zalka, Lanl-eprint/quant-ph/9902049.
[8] D. Biron et al., Lanl-eprint/quant-ph/9801066.
[9] A. Kumar Pati, Lanl-eprint/quant-ph/9807067.
[10] YU SiXia and SUN ChangPu, Lanl-eprint/quant-ph/990375.
[11] L.K. Grover, Phys. Rev. Lett. 80 (1998) 4329.