FAST MÖBIUS INVERSION IN SEMIMODULAR LATTICES
AND U-LABELABLE POSETS

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Abstract. We consider the problem of fast zeta and Möbius transforms in
finite posets, particularly in lattices. It has previously been shown that for
a certain family of lattices, zeta and Möbius transforms can be computed in
$O(e)$ elementary arithmetic operations, where $e$ denotes the size of the covering
relation. We show that this family is exactly that of geometric lattices. We also
extend the algorithms so that they work in $e$ operations for all semimodular
lattices, including chains and divisor lattices. Finally, for both transforms, we
provide a more general algorithm that works in $e$ operations for all R-labelable
posets and their non-graded generalization, which we call U-labelable.

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1. Introduction

Fast methods for computing zeta and Möbius transforms in finite posets are im-
portant for many algorithms of combinatorial nature, such as graph coloring [1] and
fast Fourier transforms on inverse semigroups [8]. From an algebraic perspective,
these transforms are basis-changing isomorphisms analogous to the fast Fourier
transform [2]. From a computational perspective, the zeta transform is related to
ensemble computation, where one is required to compute several different products
of inputs, possibly sharing common subexpressions, and the challenge is to minimize
the number of elementary multiplications performed (see Garey and Johnson [4]).

Let $(P, \leq)$ be a finite partially ordered set, or poset, with $|P| = v$ elements, and
let $f: P \rightarrow A$ be a function to an abelian group $A$. The zeta transform of $f$ is the
function $g: P \rightarrow A$ such that, for all $y \in P$,

$$g(y) = \sum_{x \leq y} f(x).$$

For computational purposes, the function $f$ can be represented as a $v$-element
row vector $\vec{f} = (f(x))_{x \in P}$, and $g$ similarly. The transformation $\vec{f} \mapsto \vec{g}$ is linear, and
defined by the $v \times v$ matrix $\zeta$, where $\zeta_{xy} = 1$ if $x \leq y$, and 0 otherwise. The obvious
way to compute the zeta transform is to perform the matrix-vector multiplication
$\vec{g} = \vec{f}\zeta$, incurring $O(v^2)$ elementary additions. However, the special structure of
the transformation gives hope of performing it faster, in particular for posets of
some specific form.

For a given poset, we represent the computation from $f$ to its zeta transform $g$ as
a straight-line program (see Bürgisser et al. [3]), a sequence of elementary pairwise
arithmetic operations (additions and subtractions) to be executed in turn, without
loops or conditional statements. We seek to minimize the length of the program,
that is, the number of arithmetic operations performed.
The zeta transform is always invertible, since $\zeta$ is an upper triangular matrix with ones on the main diagonal, assuming the matrix lists the elements of $P$ in a linear order that is an extension of $P$. We also want to construct a short straight-line program for the inverse transform $\mu = \zeta^{-1}$, called the Möbius transform. The matrices $\zeta$ and $\mu$ are in fact representations of the zeta and Möbius functions of the poset. We refer to Stanley [11] for further discussion.

1.1. Measures of the complexity. We relate the program length to certain parameters characterizing the complexity of the poset: the number of elements or vertices ($v$), the number of join-irreducible elements ($n$), and the number of edges in the Hasse diagram of the poset ($e$).

In a lattice we always have $v - 1 \leq e \leq vn$. As an example where $e$ is large, consider the lattice of subsets of an $n$-element set. This lattice has $v = 2^n$ elements, of which $n$ are join-irreducible (the singletons), and $e = 2^{n-1}n = \Theta(vn)$ edges. Both zeta and Möbius transforms can be computed in $\Theta(e) = \Theta(vn)$ operations by Yates's construction [13]. As an example where $e$ is small, consider the $v$-element chain, which has $n = e = v - 1$, thus $vn = \Theta(e^2)$. In a chain both transforms are easily computed in $e$ operations.

The parameter $e$ also appears in a lower bound by Kennes [7]: for any lattice, any sequence of additions that computes the zeta transform has length at least $e$. Note that this lower bound does not apply to posets in general; as a counterexample, Björklund et al. [2, §6] exhibit a non-lattice bipartite poset whose zeta transform can be computed in $O(\sqrt{e})$ additions, using a construction by Valiant [12].

For any lattice, Björklund et al. [2] construct both programs with length $O(vn)$. This upper bound is always valid but possibly quite crude. A tighter length bound $O(e)$ applies for lattices fulfilling the condition that for any element $x$ and any join-irreducible element $i$ that is not below $x$, the join $x \lor i$ covers $x$. Examples of such lattices include the subset lattice, the partition lattice, and the lattice of subspaces of a finite vector space. For some lattices, the resulting program length is highly dependent upon the choice of certain details in the construction. As a simple example, with the $v$-element chain one choice leads to the optimal $O(e)$ additions, while another choice leads to $\Theta(e^2)$ additions. We seek here a better understanding of posets that admit fast computation of the zeta and Möbius transforms.

1.2. Our results. First we show that the aforementioned condition by Björklund et al. [2] is equivalent to the the lattice being geometric (semimodular and atomic). Thus in any geometric lattice both transforms can be done in $O(e)$ arithmetic operations. This result also provides an alternative “single-axiom” characterization of geometric lattices.

Second, we show that atomicity is not needed: in any semimodular lattice, the construction can be done in a manner that yields a program of length $e$, matching Kennes’s lower bound. The optimal straight-line programs for chains now follow as a special case, as well as for other non-atomic semimodular lattices such as the lattice of positive divisors of an integer.

1 The first inequality holds because every non-maximal element has at least one upward edge. The latter inequality follows by considering the upward edges of an arbitrary element $x$ and the spectrum map $\varphi$, as defined in Section 3. If both $y \gg x$ and $z \gg x$, then $\varphi(y) \setminus \varphi(x)$ and $\varphi(z) \setminus \varphi(x)$ must be disjoint, otherwise $y$ and $z$ would have two incomparable lower bounds. Hence $x$ has at most $n$ upward edges, and the whole lattice has $e \leq vn$. 
Third, for further generality, we show that if a poset admits a certain kind of edge labeling, then the transforms can be done in exactly $e$ arithmetic operations by following the edges in an order implied by their labels. The idea of such labelings goes back to Stanley [10], who studied edge labeling in the context of counting chains in a lattice; for bibliography see Stanley [11, Chapter 3]. The class of posets admitting zeta and Möbius transforms in $e$ operations now encompasses the family of $R$-labelable posets, and their non-graded generalization, which we will call $U$-labelable. Semimodular lattices are included as a special case, as well as lower semimodular lattices (by duality) and supersolvable lattices.

The new results are summarized in Table 1 in context with previous results.

| Poset family                  | Program length | Reference                  |
|------------------------------|----------------|----------------------------|
| all lattices                 | $O(v^2)$       | matrix-vector multiplication |
| subset lattices              | $O(vn)$        | Yates [13]                 |
| distributive lattices (zeta only) | $O(vn)$        | Parviainen and Koivisto [9] |
| all lattices                 | $O(vn)$        | Björklund et al. [2]       |
| lattices with a certain condition | $O(e)$         | Björklund et al. [2]       |
| geometric lattices           | $O(e)$         | new (§2)                   |
| semimodular lattices         | $O(e)$         | new (§3)                   |
| R-, U-labelable posets       | $O(e)$         | new (§4)                   |

Table 1. Program lengths for the zeta and Möbius transforms.

2. Preliminaries

For a general treatment on posets and lattices we refer to Grätzer [5] and Stanley [11]. In the present work a poset $P$ or a lattice $L$ is always finite. The covering relation is denoted by $\preceq$, and the set of covering pairs, or edges of the Hasse diagram, is $E = \{(x, y) : x \preceq y\}$. An element $x \in P$ is join-irreducible if it covers exactly one element, and $I$ is the set of join-irreducible elements. Poset size is characterized by three quantities: the number of elements $v = |P|$; the number of join-irreducible elements $n = |I|$; and the number of edges $e = |E|$.

Without loss of generality, we assume that the join-irreducible elements of a lattice are denoted by integers $I = [n]$. The spectrum map of an element $x \in L$ is $\varphi(x) = \{h \in I : h \leq x\}$, and for $i \leq n$, the prefix spectrum map is $\varphi_i(x) = \{h \in [i] : h \leq x\}$, with $\varphi_0(x) = \emptyset$.

In a lattice the minimum element is $\hat{0}$, and an element that covers $\hat{0}$ is an atom. A lattice is atomic if every element is a join of atoms. A lattice is (upper) semimodular, if for any two elements $x$ and $y$ such that $x$ covers $x \wedge y$, the join $x \vee y$ in turn covers $y$. The dual of a semimodular lattice is lower semimodular. An atomic semimodular lattice is geometric. Examples of geometric lattices include subset lattices and partition lattices; examples of non-atomic semimodular lattices include chains and divisor lattices (see Stanley [11, §3.3]).

Theorem 1.3 of Björklund et al. [2] shows that both zeta and Möbius transforms can be done in $O(e)$ arithmetic operations in any finite lattice $L$ fulfilling the following condition:

$$x \vee i \geq x \quad \text{for all } x \in L, \ i \in I, \ x \neq i.$$
The following theorem provides an alternative characterization.

**Theorem 1.** A finite lattice fulfills condition (1) if and only if it is geometric.

**Proof.** For the “if” direction, let $L$ be a finite geometric lattice, $x \in L$, and $i \in I$ such that $x \nleq i$. Since $L$ is atomic, $i$ is an atom, and $i \geq 0 = x \land i$. Now by semimodularity $x \lor i \geq x$.

For the “only if” direction, let $L$ be a finite lattice where (1) holds. For any join-irreducible $i$, choosing $x = 0$ shows that $i \geq 0$, thus $L$ is atomic.

Let then $x$ and $y$ be elements such that $y \rlhd x \land y = m$. Choose a join-irreducible element $i$ such that $i \leq y$ but $i \nleq x$. Now $y = m \lor i$, so $x \lor y = x \lor m \lor i = x \lor i$, which covers $x$ by condition (1). Thus $L$ is semimodular. □

3. Fast Möbius inversion in semimodular lattices

We shall next show that atomicity is unnecessary for fast zeta and Möbius transforms: semimodularity by itself suffices to ensure programs of length $e$.

3.1. Fast zeta transform. We recall the fast zeta transform algorithm of Björklund et al. [2, §3.2]. While the original algorithm is described by embedding the lattice $L$ into a set family $L$, for the sake of transparency we describe the algorithm directly in terms of lattice operations.

Algorithm 1 takes as input a representation of the lattice structure, and outputs a sequence of additions to be performed. Theorem 1.1 of Björklund et al. [2] shows that the resulting straight-line program indeed computes the zeta transform for any lattice, regardless of how $I$ is ordered. The outer loop beginning on line 4 is executed $n$ times, and the inner loop at most $|L| = v$ times, so at most $O(vn)$ additions are used; but the precise number may be much smaller, depending on lattice structure and the ordering of $I$. Consider a chain of $v$ elements, with $n = e = v - 1$. Bottom-up traversal incurs $e$ additions, which is the optimal number; each addition proceeds along an edge of the chain. In contrast, top-down traversal is suboptimal and incurs $\Theta(e^2)$ additions [2], many of which involve two distant elements that do not share an edge.

**Algorithm 1** Fast zeta transform

**Input:** Lattice $L$ with join-irreducibles $I = [n]$
**Output:** Straight-line program that, given $f$, computes its zeta transform $g$

```plaintext
1: for all $x \in L$ do
2:     print “$g(x) \leftarrow f(x)$” \{Initialization\}
3: end for
4: for $i = 1, 2, \ldots, n$ do
5:     for all $x \in L$ such that $x \nleq i$ do
6:         $y \leftarrow x \lor i$
7:         if $\varphi_{i-1}(x) = \varphi_{i-1}(y)$ then
8:             print “$g(y) \leftarrow g(y) + g(x)$” \{Addition\}
9:         end if
10: end for
11: end for
```
The observation that bottom-up traversal leads to fewer additions can be generalized to all semimodular lattices. The following theorem provides the optimal arrangement, whereby each addition corresponds to an edge of the lattice.

**Theorem 2.** In a semimodular lattice with \( e \) edges, the join-irreducible elements can be ordered so that Algorithm 1 generates exactly \( e \) additions.

**Proof.** Order the join-irreducible elements by increasing rank, breaking ties arbitrarily. Consider the situation when the condition on line 7 succeeds.

Since \( i \) is join-irreducible, there is a unique element \( k \) such that \( k \preceq i \). Since \( k \) has a rank strictly smaller than \( i \), it follows that \( \varphi_{i-1}(k) = \varphi(k) \). Now, because \( k \leq i \leq x \lor i = y \), we have

\[
\varphi(k) = \varphi_{i-1}(k) \subseteq \varphi_{i-1}(y) = \varphi_{i-1}(x) \subseteq \varphi(x),
\]

where the equality \( \varphi_{i-1}(y) = \varphi_{i-1}(x) \) is true due to the condition on line 7.

Since \( \varphi(k) \subseteq \varphi(x) \), it follows that \( k \leq x \). Thus since \( i \not\leq x \) we must have \( x \land i = k \), implying \( i \triangleright k = x \land i \), and then \( x \lor i \triangleright x \) by semimodularity.

We have seen that whenever an addition is generated on line 8, it involves elements \( x \) and \( y \) such that \( x \preceq y \), so we can associate the addition with an edge of the Hasse diagram. No two additions are associated with the same edge, so the number of additions is at most \( e \). Conversely, each edge is visited by the algorithm, since if \( x \preceq y \), then \( \varphi(x) \neq \varphi(y) \), and the edge is visited with the smallest \( i \) such that \( \varphi_{i}(x) \neq \varphi_{i}(y) \). Hence the algorithm performs exactly \( e \) additions. \( \square \)

Figure 1 illustrates how the zeta transform proceeds in a semimodular lattice that has 9 edges. Nine additions are performed in four phases, corresponding to the four join-irreducible elements of the lattice.

### 3.2. Fast Möbius transform

Next we shall show the corresponding result for the Möbius transform, which computes \( f \) from \( g \). Again we start with the algorithm by Björklund et al. [2, §3.6] expressed in terms of lattice operations.

Algorithm 2 is similar to the zeta transform algorithm, with a few crucial changes. The join-irreducible elements are visited in reverse order, the roles of \( f \) and \( g \) are inverted, and subtraction replaces the addition on line 8. Since the conditions on lines 5–7 are the same as before, the next theorem follows.

#### Algorithm 2 Fast Möbius transform

**Input:** Lattice \( L \) with join-irreducibles \( I = [n] \)

**Output:** Straight-line program that, given \( g \), computes its Möbius transform \( f \)

1: for all \( x \in L \) do
2:  print "\( f(x) \leftarrow g(x) \)" \{Initialization\}
3: end for
4: for \( i = n, n-1, \ldots, 1 \) do
5:  for all \( x \in L \) such that \( x \not\geq i \) do
6:    \( y \leftarrow x \lor i \)
7:    if \( \varphi_{i-1}(x) = \varphi_{i-1}(y) \) then
8:      print "\( f(y) \leftarrow f(y) - f(x) \)" \{Subtraction\}
9:    end if
10: end for
11: end for
Figure 1. Fast zeta transform in a semimodular lattice. In each phase the join-irreducible element \( i \) being considered is highlighted in blue. Red arrows indicate addition along edges. Thick black edges have been visited already, and dashed edges are yet unvisited.

| Straight-line program |
|-----------------------|
| **Phase 1**           |
| \[ g(1) \leftarrow g(1) + g(0) \] |
| \[ g(5) \leftarrow g(5) + g(2) \] |
| \[ g(6) \leftarrow g(6) + g(4) \] |
| **Phase 2**           |
| \[ g(2) \leftarrow g(2) + g(0) \] |
| \[ g(5) \leftarrow g(5) + g(1) \] |
| \[ g(6) \leftarrow g(6) + g(3) \] |
| **Phase 3**           |
| \[ g(3) \leftarrow g(3) + g(1) \] |
| \[ g(6) \leftarrow g(6) + g(5) \] |
| **Phase 4**           |
| \[ g(4) \leftarrow g(4) + g(2) \] |

Figure 2. Fast Möbius transform in a semimodular lattice. In each phase the join-irreducible element \( i \) being considered is highlighted in blue. Cyan arrows indicate subtraction along edges. Thick black edges have been visited already, and dashed edges are yet unvisited.

| Straight-line program |
|-----------------------|
| **Phase 4**           |
| \[ f(4) \leftarrow f(4) - f(2) \] |
| **Phase 3**           |
| \[ f(6) \leftarrow f(6) - f(5) \] |
| \[ f(3) \leftarrow f(3) - f(1) \] |
| **Phase 2**           |
| \[ f(6) \leftarrow f(6) - f(3) \] |
| \[ f(5) \leftarrow f(5) - f(1) \] |
| \[ f(2) \leftarrow f(2) - f(0) \] |
| **Phase 1**           |
| \[ f(6) \leftarrow f(6) - f(4) \] |
| \[ f(5) \leftarrow f(5) - f(2) \] |
| \[ f(1) \leftarrow f(1) - f(0) \] |
Theorem 3. In a semimodular lattice with \( e \) edges, the join-irreducible elements can be ordered so that Algorithm 2 generates exactly \( e \) subtractions.

Proof. Order and name the join-irreducible elements as 1, 2, \ldots, \( n \) by increasing rank, breaking ties arbitrarily. When the condition on line 7 succeeds, it holds that \( x \preceq y \), by the same reasoning as in Theorem 2. Thus the subtraction on line 8 can be associated with that edge, and total the number of subtractions performed equals the number of edges. \( \square \)

Figure 2 illustrates how the M"obius transform proceeds on a semimodular lattice. Note that join-irreducible elements are considered in the reverse order 4, 3, 2, 1.

4. Fast M"obius inversion in posets labelable with unique rising chains

Our next addition algorithm is constructed to proceed along the edges of a poset. By this we mean that the straight-line program consists of \( e \) lines of the form

\[
\begin{align*}
g(y) &\leftarrow g(y) + g(x),
\end{align*}
\]

one for each edge \((x, y)\), where \( x \preceq y \). The edges are visited in an order specified by an edge labeling \( \lambda : E \to \mathbb{Z} \). The algorithm requires the labels to be distinct so that the order is unambiguous. The inverse transform is obtained by reversing the order of operations and replacing additions with subtractions.

Algorithm 3 Add along edges

**Input:** Poset \( P \) with injective edge labeling \( \lambda : E \to \mathbb{Z} \)

**Output:** Straight-line program

1. for all \( x \in P \) do
2. \hspace{1em} print “\( g(x) \leftarrow f(x) \)” {Initialization}
3. end for
4. for all \((x, y) \in E\) in increasing order of \( \lambda \) do
5. \hspace{1em} print “\( g(y) \leftarrow g(y) + g(x) \)” {Addition}
6. end for

Algorithm 4 Subtract along edges

**Input:** Poset \( P \) with injective edge labeling \( \lambda : E \to \mathbb{Z} \)

**Output:** Straight-line program

1. for all \( x \in P \) do
2. \hspace{1em} print “\( f(x) \leftarrow g(x) \)” {Initialization}
3. end for
4. for all \((x, y) \in E\) in decreasing order of \( \lambda \) do
5. \hspace{1em} print “\( f(y) \leftarrow f(y) - f(x) \)” {Subtraction}
6. end for

Both Algorithms 3 and 4 generate exactly \( e \) operations by design, in contrast to Algorithms 1 and 2 which do so for semimodular lattices but not in general. We will next formulate a sufficient condition to ensure that Algorithm 3 computes the zeta transform (and that consequently Algorithm 4 computes the M"obius transform).
Consider two comparable elements \( x \leq y \) and a unrefinable chain \( C \) from \( x \) to \( y \),
\[
x = x_0 \preceq x_1 \preceq \cdots \preceq x_m = y.
\]
With a labeling \( \lambda : E \to \mathbb{Z} \), we say that \( C \) is rising if its labels are increasing,
\[
\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{m-1}, x_m).
\]
We allow here \( \lambda \) to be non-injective for compatibility with Stanley’s definition \cite{11, §3.14}. An edge labeling \( \lambda : E \to \mathbb{Z} \) is a U-labeling (labeling with unique rising chains) if, for each pair of elements \( x \leq y \), there is exactly one rising chain from \( x \) to \( y \).

Now if \( \lambda \) is injective and \( C \) is a rising chain from \( x \) to \( y \), then the inequalities in (2) are strict, and Algorithm 3 performs additions along \( C \) in the chain order. Thus the input term \( f(x) \) propagates along \( C \) to the output \( g(y) \). Conversely, if \( C \) is not rising, then some of the additions in the chain are performed out of the chain order, and \( f(x) \) does not propagate to \( g(y) \) along \( C \). Hence we have following sufficient condition.

**Proposition 4.** If \( \lambda \) is an injective U-labeling, then the straight-line programs from Algorithms 3 and 4 compute the zeta and Möbius transforms, respectively.

**Proof.** For each pair \( x \leq y \), the straight-line program from Algorithm 3 propagates \( f(x) \) up to \( g(y) \) exactly once, along the unique rising chain from \( x \) to \( y \). Thus for each element \( y \in P \), the output \( g(y) \) equals \( \sum_{x \leq y} f(x) \) as required. Hence the result is the zeta transform.

The program from Algorithm 4 consists of subtractions that undo the additions in reverse order, hence it performs the Möbius transform.

Below we show two different edge labelings on a poset. The labeling on the left has two rising chains from \( a \) to \( d \), so if additions are performed in this order, \( g(d) \) will incorrectly contain the term \( f(a) \) twice. On the right is a U-labeling, which leads to the correct zeta transform
\[
g(a) = f(a), \quad g(b) = f(a) + f(b), \quad g(c) = f(a) + f(c), \quad g(d) = f(a) + f(b) + f(c) + f(d).
\]

A poset that admits a U-labeling is U-labelable. A U-labeling is not necessarily injective as required by Algorithms 3 and 4 but an injective U-labeling can be produced as follows.

**Proposition 5.** If a poset has a U-labeling, then it also has an injective U-labeling.

**Proof.** Let \( P \) be a poset with a U-labeling \( \lambda \). Order the edges by \( \lambda \); among edges that have the same label, order by their first elements according to the poset order \( \leq \). Break any remaining ties arbitrarily. With the edges thus ordered, define an injective labeling \( \lambda' \) by assigning labels 1, 2, \ldots, \( e \) in order.
Consider now an unrefinable chain $C$. If $C$ is not rising under $\lambda$, then it contains three elements $s \prec t \prec u$ such that $\lambda(s, t) > \lambda(t, u)$. Then also $\lambda'(s, t) > \lambda'(t, u)$, and $C$ is not rising under $\lambda'$.

Conversely, if $C$ is rising under $\lambda$, then with any three consecutive elements $s \prec t \prec u$ in $C$, we have either $\lambda(s, t) < \lambda(t, u)$ or $\lambda(s, t) = \lambda(t, u)$. In either case, by construction, $\lambda'(s, t) < \lambda'(t, u)$. Hence $C$ is rising under $\lambda'$.

Since $\lambda$ and $\lambda'$ have the same rising chains, $\lambda'$ is also a U-labeling. □

By combining Propositions 4 and 5 we obtain our main result for U-labelable posets.

**Theorem 6.** In a U-labelable poset that has $e$ edges, the zeta transform can be computed in $e$ additions, and the Möbius transform in $e$ subtractions.

U-labelable posets are a generalization of R-labelable posets: an R-labelable poset is a graded U-labelable poset. Thus Theorem 6 provides fast zeta and Möbius transforms for all R-labelable posets, which include supersolvable lattices and semimodular lattices (see Stanley [11, §3.14]). For a semimodular lattice an R-labeling is obtained by naming the join-irreducible elements as $1, 2, \ldots, n$ in an order compatible with the lattice (for example, ordering by rank), and defining

$$\lambda(s, t) = \min\{i : s \lor i = t\}.$$ 

Lower semimodular lattices are U-labelable as well. More generally, duality preserves U-labelability: if $\lambda$ is a U-labeling for poset $P$, then $\lambda^*(s, t) = -\lambda(t, s)$ is a U-labeling for $P^*$. As an example, the figure below on the left shows an upper semimodular lattice with a U-labeling. On the right is its dual with the derived U-labeling, whereby Algorithm 3 computes the zeta transform in 9 additions. In comparison, Algorithm 1 can use up to 11 additions (depending on how $I$ is ordered).

The pentagon lattice, shown below on the left, is not graded but has a U-labeling, facilitating both transforms in 5 operations. The hexagon lattice (below on the right) cannot be U-labeled. This would require rising chains from $p$ to $r$ and from $q$ to $s$, implying that the chain $p \prec q \prec r \prec s$ is rising; but similarly $p \prec t \prec u \prec s$ would be rising, so there would be two rising chains from $p$ to $s$. 
In the hexagon it is impossible to compute the zeta transform in \( e \) pairwise operations (additions or subtractions); at the minimum, \( 7 = e + 1 \) operations are required. This can be seen by observing that at least four operations are required to compute the four outputs at \( q, r, t, \) and \( u \), and then enumerating the possibilities of computing the remaining output \( g(s) \).

The converse of Theorem 6 does not hold for posets in general. There are posets that are not U-labelable, but admit the zeta transform in \( e \) operations or less (for example the bipartite poset mentioned in the introduction, augmented with a hexagon on top). However, for lattices this seems to be an open question: if a lattice admits the zeta transform in \( e \) operations, is it necessarily U-labelable?

Another open question concerns whether there are any posets where the zeta transform requires more than \( 2e \) operations. The hardest known instance, in terms of \( e \), seems to be a lattice constructed from \( k \) parallel chains of \( k \) elements, adjoined with a common bottom and a common top, with a total of \( e = k(k+1) \) edges. If only addition is available, then Theorem 1 of Järvsalo et al. [6] implies that computing the zeta transform for this lattice requires \( 2e - O(1) \) additions.

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