Quantum Circuit for Calculating Symmetrized Functions Via Grover-like Algorithm

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Abstract

In this paper, we give a quantum circuit that calculates symmetrized functions. Our algorithm applies the original Grover’s algorithm or a variant thereof such as AFGA (adaptive fixed point Grover’s algorithm). Our algorithm uses AFGA in conjunction with two new techniques we call “targeting two hypotheses” and “blind targeting”. Suppose AFGA drives the starting state $|s\rangle$ to the target state $|t\rangle$. When targeting two hypotheses, $|t\rangle$ is a superposition $a_0|0\rangle + a_1|1\rangle$ of two orthonormal states or hypotheses $|0\rangle$ and $|1\rangle$. When targeting blindly, the value of $\langle t|s \rangle$ is not known a priori.
1 Introduction

In this paper, we give a quantum circuit that calculates symmetrized functions (i.e., it calculates the right hand side of Eq. (21)).

Our algorithm utilizes the original Grover’s algorithm (see Ref. [1]) or any variant thereof, as long as it accomplishes the task of driving a starting state $|s\rangle$ towards a target state $|t\rangle$. However, we recommend to the users of our algorithm that they use a variant of Grover’s algorithm called AFGA (adaptive fixed point Grover’s algorithm) which was first proposed in Ref. [2].

A large portion of our algorithm for calculating symmetrized functions has been proposed before by Barenco et al in Ref. [3]. However, we make some important changes to their algorithm. One trivial difference between our work and that of Barenco et al is that our operators $V^{(A)}_1$ are different from the corresponding ones that Barenco et al use. A more important difference is that we combine their circuit with Grover’s algorithm (or variant thereof), which they don’t. Furthermore, we use Grover’s algorithm in conjunction with two new techniques that we call “targeting two hypotheses” and “blind targeting”. When targeting two hypotheses, $|t\rangle$ is a superposition $a_0|0\rangle + a_1|1\rangle$ of two orthonormal states or hypotheses $|0\rangle$ and $|1\rangle$. When targeting blindly, the value of $\langle t|s\rangle$ is not known a priori.

The technique of “targeting two hypotheses” can be used in conjunction with Grover’s algorithm or variants thereof to estimate (i.e., infer) the amplitude of one of many states in a superposition. An earlier technique by Brassard et al (Refs. [4, 5]) can also be used in conjunction with Grover’s algorithm to achieve the same goal of amplitude inference. However, our technique is very different from that of Brassard et al. They try to produce a ket $|x^n\rangle$, where the bit string $x^n$ encodes the amplitude that they are trying to infer. We, on the other hand, try to infer an amplitude $|a_1|$ by measuring the ratio $|a_1|/|a_0|$ and assuming we know $|a_0|$ a priori.

For more background information on the use of symmetrized functions in quantum information theory, we refer the reader to a recent review by Harrow, Ref. [6].

2 Notation and Preliminaries

In this section, we will review briefly some of the more unconventional notation used in this paper. For a more detailed discussion of Tucci’s notation, especially its more idiosyncratic aspects, see, for example, Ref. [7].

Let $\theta(S)$ stand for the truth function. It equals 1 when statement $S$ is true and 0 when it isn’t. The Kronecker delta function $\theta(a = b)$ will also be denoted by $\delta_a^b$ or $\delta(a, b)$. Given a set $A$, the indicator function for set $A$ is defined by $1_A(x) = \theta(x \in A)$.

We will sometimes use the following abbreviation for sets: \( \{ f(x) : \forall x \in S \} = \{ f(x) \}_{x \in S} \).

We will sometimes use the following abbreviation for Hermitian conjugates: $[x] + [h.c.] = x + x^\dagger$, and $[x][h.c.] = xx^\dagger$, where $x$ is some complicated expression that
we don’t want to write twice.

We will sometimes use the following abbreviation: \( \sum_{x}^{\text{num}} f(x) = \sum_{x}^{\text{num}} f(x) \), where \( f(x) \) is some complicated expression of \( x \) that we don’t want to write twice.

Let Bool = \{0, 1\}. For any \( b \in \text{Bool} \), let \( \bar{b} = 1 - b \).

Let \( \mathbb{C} \) stand for the complex numbers and \( \mathbb{R} \) for the real numbers. For integers \( a, b \) such that \( a \leq b \), let \( \{a..b\} = \{a, a + 1, \ldots, b\} = \{b..a\} \).

We will represent \( n \)-tuples or vectors with \( n \) components by \( x_n = x \).

Let \( \text{dec}() \) and \( \text{bin}() \) by \( \text{dec}(x_n) = \sum_{j=0}^{n-1} 2^j x_j \) and \( \text{bin}^n(\sum_{j=0}^{n-1} 2^j x_j) = x_n \).

We will use the term qu(d)it to refer to a quantum system that lives in a \( d \)-dimensional Hilbert space \( \mathbb{C}^d = \text{span}_{\mathbb{C}} \{|j\rangle : j = 0, 1, \ldots, d - 1\} \). Hence a qu(4)it has 4 possible independent states. A qubit is a qu(2)it. Systems (or horizontal wires in a quantum circuit) will be labelled by Greek letters. If \( \alpha \) lives in the Hilbert space \( (\mathbb{C}^d)^{\otimes n} \), we will say width(\( \alpha \)) = \( d^n \). For example, we’ll say width(\( \alpha \)) = 3\( 5 \) if wire \( \alpha \) carries 5 qu(3)its.

As is usual in the Physics literature, \( \sigma_X, \sigma_Y, \sigma_Z \) will denote the Pauli matrices.

\( H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \) will denote the 1 qubit Hadamard matrix. \( H^{\otimes n} \), the \( n \)-fold tensor product of \( H \), is the \( n \) qubits Hadamard matrix.

Define the number operator \( n \) and its complement \( \overline{n} \) by

\[
\begin{align*}
n = P_1 &= |1\rangle \langle 1|, & \quad \overline{n} = 1 - n &= P_0 = |0\rangle \langle 0|. \\
(1)
\end{align*}
\]

If we need to distinguish the number operator from an integer called \( n \), we will use \( n_{\text{op}} \) or \( \hat{n} \), or \( \underline{n} \) for the number operator.

The number operator just defined acts only on qubits. For qu(d)its, one can use instead

\[
P_b = |b\rangle \langle b|,
\]

where \( b \in \{0, 1, \ldots, d - 1\} \). For 2 qu(d)its, one can use

\[
P_{b_1 \otimes b_0} = P_{b_1}(\beta_1)P_{b_0}(\beta_0) = P_{b_1,b_0}(\beta_1, \beta_0),
\]

where \( b_1, b_0 \in \{0, 1, \ldots, d - 1\} \). Eq.(3) generalizes easily to an arbitrary number of qu(d)its.

We will often denote tensor products of kets vertically instead of horizontally. The horizontal and vertical notations will be related by the conventions:

\[
|a_{n-1}\rangle \otimes \ldots |a_1\rangle \otimes |a_0\rangle = |a_{n-1}, \ldots, a_1, a_0\rangle = |a_0\rangle \\
|a_1\rangle \\
|a_{n-1}\rangle.
\]

and
\[(|a_{n-1}, \ldots, a_1, a_0\rangle) = \langle a_0 | \langle a_1 | \ldots | a_{n-1} | \quad . \quad (5)\]

As usual for us, we will represent various types of controlled nots as follows:

\[\begin{array}{c}
\begin{array}{c}
\alpha \\
\beta
\end{array} = \sigma X (\beta)^n(\alpha)
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\alpha \\
\beta
\end{array} = \sigma X (\beta)^n(\alpha)
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\alpha_0 \\
\alpha_1
\end{array} = \sigma X (\beta)^n(\alpha_0)^n(\alpha_1) \quad (6)\end{array} \]

We will represent as follows a controlled \(U\), where the unitary operator \(U\) acts on \(\alpha\) and where \(\beta\) is the control:

\[\begin{array}{c}
\begin{array}{c}
U \\
\alpha
\end{array} = U(\alpha)^n(\beta) \quad . \quad (7)\end{array} \]

Note that \([U(\alpha)^n(\beta)][h.c.] = 1\) so controlled unitaries are themselves unitary.

We will use the following identity repeatedly throughout this paper. For any quantum systems \(\alpha\) and \(\beta\), any unitary operator \(U(\beta)\) and any projection operator \(\pi(\alpha)\) (i.e., \(\pi^2 = \pi\)), one has

\[U(\beta)\pi(\alpha) = (1 - \pi(\alpha)) + U(\beta)\pi(\alpha) . \quad (8)\]

We will denote ordered products of operators \(U_b\) as follows:

\[\prod_{b=0 \rightarrow 2} U_b = U_0 U_1 U_2 , \quad \prod_{b=2 \rightarrow 0} U_b = U_2 U_1 U_0 . \quad (9)\]

Suppose \(a, b \in \text{Bool}\) and \(x, y, \theta\) are real numbers. Note that \(\delta_a^0 = \pi\) and \(\delta_a^1 = a\). Furthermore, note that \(x^a y^\pi = xa + y\pi = x(a)\) where \(x(a) = x\) if \(a = 1\) and \(x(a) = y\) if \(a = 0\). If we let \(S = \sin \theta\) and \(C = \cos \theta\), then

\[\langle a | e^{-i\sigma Y \theta b} | 0 \rangle = \delta_a^0 \delta_b^0 + (C \delta_a^0 + S \delta_a^1) \delta_b^1 \quad (10)\]

\[= \frac{\delta_a^0 \delta_b^0}{\bar{a} \bar{b}} + (C \bar{a} + S a) b . \quad (11)\]

### 3 Permutation Circuits

For this section, we will assume that the reader has a rudimentary knowledge of permutations, as can be obtained from any first course in abstract algebra. In this section, we will attempt to connect that rudimentary knowledge of permutations with quantum computation. More specifically, we will show how to permute the qu(d)its of a multi-qu(d)it quantum state using a quantum circuit.
Given any finite set $S$, a permutation on set $S$ is a 1-1 onto map from $S$ to $S$. Define

$$ Sym(S) = \{ \sigma | \sigma \text{ is a permutation of set } S \} \quad (12) $$

The properties of $Sym(S)$ don’t depend on the nature of $S$, except for its cardinality $|S|$ (i.e., number of elements of $S$). Hence, we will often denote $Sym(S)$ by $Sym_{|S|}$.

If permutation $\sigma$ maps $x \in S$ to $\sigma(x) \in S$, we will often write $\sigma(x) = x^\sigma$. For example, if $\sigma$ maps 1 to 2, we will write $1^\sigma = 2$.

As usual, a permutation $\sigma$ will be represented by

$$ \left( \begin{array}{ccc} 1 & 2 & \cdots & n-1 \\ 1^\sigma & 2^\sigma & \cdots & (n-1)^\sigma \end{array} \right) = (1^\sigma, 2^\sigma, \ldots, (n-1)^\sigma) \quad (13) $$

For $\sigma \in Sym(S)$, and any set $A$, define

$$ A^\sigma = \{ a^\sigma : a \in A \}, \text{ where } a^\sigma = \begin{cases} a^\sigma & \text{if } a \in S \\ a & \text{if } a \notin S \end{cases}. \quad (14) $$

For example, if $S = \{1, 2, 3\}$ then $\{1, 2, 4\}^\sigma = \{1^\sigma, 2^\sigma, 4\}$.

If $\sigma \in Sym_n$ and $c^n = (c_{(n-1)}, \ldots, c_1, c_0) \in (S_2)^n$, define

$$ c^{n^\sigma} = (c_{(n-1)^\sigma}, \ldots, c_1^{\sigma}, c_0^{\sigma}) = (c_{(n-1)^\sigma}, \ldots, c_1^{\sigma}, c_0^{\sigma}) \quad (15) $$

For any permutation map $\sigma : S \rightarrow S$, one can define a matrix such that each of its columns has all entries equal to zero except for one single entry which equals 1. Also, the entry that is 1 is at a different position for each column. We will denote such a matrix (which is orthogonal and unitary) also by $\sigma$. Whether $\sigma$ stands for the map or the matrix will be clear from context, as in the following equation which uses $\sigma$ to stand for the matrix on its left side and the map on its right side:

$$ \begin{pmatrix} |a_0\rangle \\ |a_1\rangle \\ \vdots \\ |a_{n-1}\rangle \end{pmatrix} = \begin{pmatrix} |a_0^{\sigma}\rangle \\ |a_1^{\sigma}\rangle \\ \vdots \\ |a_{(n-1)^{\sigma}}\rangle \end{pmatrix} \quad (16) $$

Suppose $a^n = (a^{n-1}, a^{n-2}, \ldots, a^0) \in (S_2^n)$ and $\langle b^{n^\sigma} | a^n \rangle = \delta_{a^0}^{b^0}$ for all $a^n, b^n \in S_2^n$. If $|S_2| = d$, then we can assume without loss of generality that $S_2 = \{0..d-1\}$. Suppose $width(a^n) = d^n$. Let

$$ |\psi\rangle_{a^n} = \sum_{a^n} A(a^n) |a^n\rangle_{a^n}, \quad A(a^n) = \langle a^n | \psi \rangle. \quad (17) $$

Then
\begin{align}
\langle a_0 \rangle & = \langle a_0 \rangle, \\
\langle a_1 \rangle & = \langle a_1 \rangle, \\
\vdots & = \vdots, \\
\langle a_{n-1} \rangle & = \langle a_{(n-1)} \rangle,
\end{align}
\tag{18}

where \( \tau = \sigma^{-1} \). When \( \sigma \) is a permutation matrix, it’s unitary so \( \sigma^{-1} = \sigma^\dagger \).

Define
\[
\pi_{\text{Sym}_n} = \frac{1}{n!} \sum_{\sigma \in \text{Sym}_n} \sigma.
\tag{19}
\]

One finds that
\[
[\pi_{\text{Sym}_n}]^2 = \pi_{\text{Sym}_n}
\tag{20}
\]
so \( \pi_{\text{Sym}_n} \) is a projection operator. Furthermore, one finds that
\[
\langle a^n | \pi_{\text{Sym}_n} | \psi \rangle = \frac{1}{n!} \sum_{\sigma} A(a^{n\sigma}).
\tag{21}
\]

The goal of this paper is to find a quantum circuit that allows us to calculate \( |\langle a^n | \pi_{\text{Sym}_n} | \psi \rangle|^2 \) for some predetermined point \( a^n \in \{0..d-1\}^n \) and state \( |\psi\rangle_{\alpha^n} \), where \( \text{width}(\alpha^n) = d^n \).

As is well known, any permutation can be expressed as a product of transpositions (a.k.a. swaps). For quantum circuits, it is common to define a swap gate which acts as follows:

\[
\begin{array}{c}
\psi_1 \\
\hline
\psi_2 \\
\hline
\psi_3
\end{array}
\rightarrow
\begin{array}{c}
\psi_2 \\
\hline
\psi_1 \\
\hline
\psi_3
\end{array}.
\tag{22}
\]

In this example, the gate \( \text{swap}(1,2) \) is acting on 3 qu(d)its called 1,2,3. Clearly, \( [\begin{array}{c}
\psi_1 \\
\hline
\psi_2 \\
\hline
\psi_3
\end{array}]^2 = 1 \). One also finds that
\[
\begin{bmatrix}
\begin{array}{c}
\psi_1 \\
\hline
\psi_2 \\
\hline
\psi_3
\end{array}
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
\psi_2 \\
\hline
\psi_1 \\
\hline
\psi_3
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{c}
\psi_1 \\
\hline
\psi_2 \\
\hline
\psi_3
\end{array}
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
\psi_3 \\
\hline
\psi_1 \\
\hline
\psi_2
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{c}
\psi_1 \\
\hline
\psi_2 \\
\hline
\psi_3
\end{array}
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
\psi_2 \\
\hline
\psi_3 \\
\hline
\psi_1
\end{array}
\end{bmatrix}.
\tag{23}
\]

One can summarize these 3 identities by saying that the horizontal line with 3 arrow heads on it can be replaced by no arrow heads on it. At the same time, the horizontal line with 2 arrow heads on it can be replaced by 1 arrow head on it.

Note that the elements of \( \text{Sym}_3 \) in the so called dictionary order are

\[
\begin{bmatrix}
\begin{array}{c}
1 \\
\hline
2 \\
\hline
3
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{c}
1 \\
\hline
3 \\
\hline
2
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{c}
2 \\
\hline
1 \\
\hline
3
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{c}
2 \\
\hline
3 \\
\hline
1
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{c}
3 \\
\hline
1 \\
\hline
2
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{c}
3 \\
\hline
2 \\
\hline
1
\end{array}
\end{bmatrix}.
\tag{24}
\]
Note that the sum of the 6 elements of $\text{Sym}_3$ can be generated from a product of matrices which are themselves sums of permutation matrices, as follows:

\[
\left( \begin{array}{ccc}
\pi_1 & \pi_2 & \pi_3 \\
\pi_4 & \pi_5 & \pi_6
\end{array} \right) \left( \begin{array}{ccc}
\pi_1 & \pi_2 & \pi_3 \\
\pi_4 & \pi_5 & \pi_6
\end{array} \right) = \\
\sum_{\sigma \in \text{Sym}_3} \sigma .
\]

It’s fairly clear how to generalize the pattern of Eq. (25) to the case of $n$ qu(d)its and $\text{Sym}_n$, where $n$ is any integer greater than 1.

4 Decomposing a State Vector into 2 Orthogonal Projections

In this section, we will review a technique that we like to call “decomposing a state vector into orthogonal projections”. This technique is frequently used in quantum computation circuits, and will be used later on in this paper, inside more complicated circuits.

Suppose $\alpha$ is a qu(d)it and $\beta$ is a qubit. Let $\pi$ be a Hermitian projection operator (i.e., $\pi^\dagger = \pi$, $\pi^2 = \pi$) acting on $\alpha$, and let $\overline{\pi} = 1 - \pi$. Let $|\psi\rangle_\alpha$ be a state vector of qu(d)it $\alpha$. Applying identity Eq.(8) with $U = \sigma_X(\beta)$ yields:

\[
\sigma_X(\beta)^{\pi(\alpha)} |\psi\rangle_\alpha = \sigma_X(\beta)^{\pi(\alpha)} |\psi\rangle_\alpha = \pi(\alpha) |\psi\rangle_\alpha + \overline{\pi}(\alpha) |\psi\rangle_\alpha .
\]

One can say that the state vector $|\psi\rangle$ is “decomposed” by the circuit into two orthogonal projections $\pi |\psi\rangle$ and $\overline{\pi} |\psi\rangle$. Some examples of this decomposition are (1) when $\alpha$ is a qubit and $\pi(\alpha) = n(\alpha)$, (2) when $\alpha = (\alpha_1, \alpha_0)$ where $\alpha_0, \alpha_1$ are both qubits and $\pi(\alpha) = n(\alpha_0)\overline{\pi}(\alpha_1)$.

5 Labelling and Summing Unitaries

In this section, we will review a technique that we like to call “labelling and summing unitaries”. This technique is also frequently used in quantum computation circuits, and will be used later on in this paper, inside more complicated circuits.
Let \( \alpha \) be a qu(d)it for some \( d \geq 2 \). Let \( U \) be a \( d \) dimensional unitary matrix.

First we will consider the case that \( \beta \) is a qubit.

One finds that

\[
U(\alpha)^n(\beta) \left( \frac{\psi}{H(\beta)} \right)_{0} = \frac{1}{\sqrt{2}} \left( \left| \psi \right\rangle_{\alpha} + U \left| \psi \right\rangle_{\alpha} \right) \tag{27}
\]

and

\[
H(\beta) U(\alpha)^n(\beta) \left( \frac{\psi}{H(\beta)} \right)_{0} = \left( \frac{1+U}{2} \right) \left| \psi \right\rangle_{\alpha} + \left( \frac{1-U}{2} \right) \left| \psi \right\rangle_{\alpha} . \tag{28}
\]

One can say that the unitaries 1 and \( U \) are labelled by Eq.(27), and they are summed, in the coefficient of \( |0\rangle_\beta \), in Eq.(28).

So far we have considered \( \alpha \) to be a qu(d)it for arbitrary \( d \geq 2 \), but we have restricted \( \beta \) to be a qubit. Let’s next consider a \( \beta \) which has more than 2 independent states. For concreteness, suppose \( \beta \) is a qu(3)it. Let \( T^{(3)} \) be a 3 dimensional unitary matrix that satisfies

\[
T^{(3)} |0\rangle = \frac{1}{\sqrt{3}} \sum_{b=0}^{2} |b\rangle . \tag{29}
\]

Suppose \( U_2, U_1, U_0 \) are three 3-dimensional unitary matrices. Then Eq.(27) generalizes to

\[
\prod_{b=2 \to 0} \left\{ U_b(\alpha)^{P_b(\beta)} \right\} T^{(3)}(\beta) |0\rangle = \frac{1}{\sqrt{3}} \left( U_2 |\psi\rangle_{\alpha} + U_1 |\psi\rangle_{\alpha} + U_0 |\psi\rangle_{\alpha} \right), \tag{30}
\]

and Eq.(28) generalizes to

\[
T^{(3)\dagger}(\beta) \prod_{b=2 \to 0} \left\{ U_b(\alpha)^{P_b(\beta)} \right\} T^{(3)}(\beta) |0\rangle = \frac{1}{3}(U_2 + U_1 + U_0) |\psi\rangle_{\alpha} + \sum_{b=2,1} z_b |b\rangle_{\beta}, \tag{32}
\]

where \( \sum_{b=0}^{2} |z_b|^2 = 1 \) and \( \langle \psi_b | \psi_b \rangle = 1 \) for all \( b \). Note that one or more of the \( U_b \) can be equal to 1.
6 The operators $V_0^{(\lambda)}$ and $V_1^{(\lambda)}$

In the preceding Sec.5, we used operators $H(\beta)$ and $T^{(3)}(\beta)$ to "label" a set of unitary matrices $\{1, U\}$ and $\{U_2, U_1, U_0\}$, respectively. In this section, we will define new operators $V_0^{(\lambda)}$ and $V_1^{(\lambda)}$, where $\lambda = 1, 2, 3, \ldots$, that will be used in later circuits of this paper in a similar role, as "label producers" or "labellers" of a set of unitary matrices.

Throughout this section, let $\lambda \in \{1, 2, 3, \ldots\}$ and $m \in \{0, 1\}$.

For $\lambda = 4$ and $m = 0, 1$, define

$$V_1^{(4)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$ (34)

where

$$R_r^y = \exp(-i\sigma_Y\theta_r) \quad (35)$$

for row $r = 0, 1, 2, 3$. The angles $\{\theta_r : r = 0, 1, 2, 3\}$ for both $m = 0$ and $m = 1$ will be specified later on. $V_\lambda^{(\lambda)}$ for $\lambda$ other than 4 is defined by analogy to Eq. (34).

Below, we will use the shorthand notations

$$C_r = \cos(\theta_r), \quad S_r = \sin(\theta_r) \quad (36)$$

and

$$\{|1, 0^{\lambda-1}\rangle = |10^{\lambda-1}\rangle + |010^{\lambda-2}\rangle + |0210^{\lambda-3}\rangle + \ldots + |0^{\lambda-1}1\rangle \}. \quad (37)$$

Claim 1 If

$$S_0 = \sqrt{\frac{1}{5}} \quad S_1 = \sqrt{\frac{1}{4}} \quad S_2 = \sqrt{\frac{1}{3}} \quad S_3 = \sqrt{\frac{1}{2}} , \quad (38)$$

then $V_1^{(4)}$ maps

$$V_1^{(4)} : |0^4\rangle \mapsto \frac{1}{\sqrt{5}} [ |0^4\rangle + |1, 0^3\rangle ] . \quad (39)$$

From Eq. (39), it follows that for $b^4 \in \text{Bool}^4$, 

\end{document}
\[
\langle b^4 | V_1^{(4)} | 0^4 \rangle = \frac{1}{\sqrt{5}} \begin{cases} \\
\overline{b}_3\overline{b}_2\overline{b}_1b_0 \\
+ b_3b_2b_1\overline{b}_0 \\
+ b_3b_2\overline{b}_1b_0 \\
+ b_3\overline{b}_2b_1b_0 \\
+ \overline{b}_3\overline{b}_2b_1\overline{b}_0 \\
\end{cases} . \tag{40}
\]

**proof:** One has that

\[
A(b^4) = \begin{bmatrix} \\
\langle b_0 | \\
\langle b_1 | \\
\langle b_2 | \\
\langle b_3 |
\end{bmatrix} egin{bmatrix} \\
\langle 0 |_{b_0} \\
\langle 0 |_{b_1} \\
\langle 0 |_{b_2} \\
\langle 0 |_{b_3}
\end{bmatrix} \begin{bmatrix} \\
\exp\{-i\sigma_Y(\beta_0)\theta_0\} \\
\exp\{-i\sigma_Y(\beta_1)\theta_1P_0(\beta_0)\} \\
\exp\{-i\sigma_Y(\beta_2)\theta_2P_0(\beta_1P_0(\beta_0)\} \\
\exp\{-i\sigma_Y(\beta_3)\theta_3P_0(\beta_2P_0(\beta_1P_0(\beta_0)\}
\end{bmatrix} |0^4\rangle . \tag{41}
\]

\[
\begin{align*}
A(b^4) &= \\
&= \begin{cases} \\
S_0 \overline{b}_3\overline{b}_2\overline{b}_1b_0 \\
+ S_1C_0 \overline{b}_3b_2b_1\overline{b}_0 \\
+ S_2C_1C_0 \overline{b}_3b_2\overline{b}_1b_0 \\
+ S_3C_2C_1C_0 b_3b_2b_1\overline{b}_0 \\
+ C_3C_2C_1C_0 \overline{b}_3b_2b_1b_0 \\
\end{cases} . \tag{43}
\end{align*}
\]

It’s easy to convince oneself that the only non-vanishing matrix elements are those for which \(b^4\) has either (1) all 4 components equal to 0, or (2) a single component equal to 1 and the other 3 components equal to 0. Evaluating each of these possibilities separately, one finds

\[
A(b^4) = \begin{cases} \\
S_0 \overline{b}_3\overline{b}_2\overline{b}_1b_0 \\
+ S_1C_0 \overline{b}_3b_2b_1\overline{b}_0 \\
+ S_2C_1C_0 \overline{b}_3b_2\overline{b}_1b_0 \\
+ S_3C_2C_1C_0 b_3b_2b_1\overline{b}_0 \\
+ C_3C_2C_1C_0 \overline{b}_3b_2b_1b_0 \\
\end{cases} . \tag{43}
\]

Now one can plug into Eq.(43) the values of \(C_r\) and \(S_r\) given in the premise of our claim to show that the conclusion of our claim holds.

**QED**

**Claim 2** If \(C_r\) and \(S_r\) for \(r = 3, 2, 1, 0\) have the values given by Eqs.(38), then \(V_1^{(4)}\) maps

\[
V_1^{(4)} : \begin{cases} \\
|0\rangle \mapsto \frac{1}{\sqrt{5}} \left[-|0^4\rangle + |1,0^3\rangle \right] . \tag{44}
\end{cases}
\]

From Eq.(44) it follows that for \(b^4 \in \text{Bool}^4\),
$$\langle b^4 \mid V_1^{(4)} \rangle |0\rangle = \frac{1}{\sqrt{5}} \left\{ \begin{array}{l} \overline{b_3}b_2\overline{b_1}b_0 \\ \overline{b_3}b_2b_1\overline{b_0} \\ \overline{b_3}b_2\overline{b_1}b_0 \\ \overline{b_3}b_2b_1\overline{b_0} \\ -\overline{b_3}b_2\overline{b_1}b_0 \end{array} \right\}. \quad (45)$$

proof:

Eq. (43) is true in this case, but only if we replace $C_3 \to -S_3 = -\frac{1}{\sqrt{2}}$ and $S_3 \to C_3 = \frac{1}{\sqrt{2}}$.

QED

Claim 3 If

$$S_0 = \sqrt{\frac{1}{4}} \quad S_1 = \sqrt{\frac{1}{3}} \quad S_2 = \sqrt{\frac{1}{2}} \quad S_3 = 1 \quad C_0 = \sqrt{\frac{3}{4}} \quad C_1 = \sqrt{\frac{2}{3}} \quad C_2 = \sqrt{\frac{1}{2}} \quad C_3 = 0 \ , \quad (46)$$

then $V_0^{(4)}$ maps

$$V_0^{(4)} : |0^4\rangle \mapsto \frac{1}{\sqrt{4}} \{1,0^3\}. \quad (47)$$

From Eq. (47) it follows that for $b^4 \in \text{Bool}^4,$

$$\langle b^4 \mid V_0^{(4)} \rangle |0^4\rangle = \frac{1}{\sqrt{4}} \left\{ \begin{array}{l} \overline{b_3}b_2\overline{b_1}b_0 \\ +b_3b_2b_1\overline{b_0} \\ +b_3b_2\overline{b_1}b_0 \\ +b_3\overline{b_2}b_1\overline{b_0} \end{array} \right\}. \quad (48)$$

Note that $C_3 = 0, S_3 = 1$ means $\theta_3 = \pi/2$, and $e^{-i\sigma_Y \theta_3} = -i\sigma_Y$.

proof:

Plug into Eq. (43) the values of $C_r$ and $S_r$ given in the premise of our claim to show that the conclusion of our claim holds.

QED

Claim 4 $V_m^{(\lambda)}$ for $\lambda$ other than 4 satisfies claims analogous to Claims 1, 2, and 3.

proof: Obvious.

QED
7 Targeting Two Hypotheses

In this section, we will describe a simple trick that can sometimes be used when applying Grover’s original algorithm or some variant thereof like AFGA, as long as it drives a starting state $|s\rangle$ to a target state $|t\rangle$. Sometimes it is possible to arrange things so that the target state is a superposition $a_0 |0\rangle + a_1 |1\rangle$ of two orthonormal states $|0\rangle$ and $|1\rangle$, so that if we know $a_0$, we can infer $a_1$, a type of hypothesis testing with 2 hypotheses. If the target state were just proportional to say $|0\rangle$, then its component along $|0\rangle$ would be 1 after normalization so one wouldn’t be able to do any type of amplitude inference.

Suppose $z_0, z_1$ are complex numbers and $|\chi\rangle$ is an unnormalized state such that

\[ |z_0|^2 + |z_1|^2 + \langle \chi | \chi \rangle = 1 \ . \]

Define

\[ p = |z_0|^2 + |z_1|^2 , \ q = 1 - p \ , \]

and

\[ \hat{z}_0 = \frac{z_0}{\sqrt{p}} , \ \hat{z}_1 = \frac{z_1}{\sqrt{p}} . \]

Assume the states $\{ |\psi_j\rangle_{\mu} \}_{j=0,1}$ are orthonormal, the states $\{ |j\rangle_{\nu} \}_{j=0,1}$ are orthonormal, and the states $\{ |b\rangle_{\omega} \}_{b=0,1}$ are orthonormal.

We wish to do AFGA with the following starting state $|s\rangle_{\mu,\nu,\omega}$ and target state $|t\rangle_{\mu,\nu,\omega}$:

\[ |s\rangle_{\mu,\nu,\omega} = z_0 |\psi_0\rangle_{\mu} + z_1 |\psi_1\rangle_{\mu} + |\chi\rangle_{\mu,\nu,\omega} \]

and

\[ |t\rangle_{\mu,\nu,\omega} = \hat{z}_0 |\psi_0\rangle_{\mu} + \hat{z}_1 |\psi_1\rangle_{\mu} . \]

We will refer to $|0\rangle_{\nu}$ as the null hypothesis state, and to $|1\rangle_{\nu}$ as the alternative or rival hypothesis state.

From the previous definitions, one finds

\[ [ |t\rangle \langle t| ]_{\mu,\nu,\omega} |s\rangle_{\mu,\nu,\omega} = \sqrt{p} |t\rangle_{\mu,\nu,\omega} \]

\[ [ |0\rangle \langle 0| ]_{\omega} |s\rangle_{\mu,\nu,\omega} = \sqrt{p} |t\rangle_{\mu,\nu,\omega} \]

and
|t⟩ only appears in AFGA within the projection operator |t⟩ ⟨t|, and this projection operator always acts solely on the space spanned by |t⟩ and |s⟩. But |t⟩ ⟨t| and |0⟩ ⟨0| act identically on that space. Hence, for the purposes of AFGA, we can replace |t⟩ ⟨t| by |0⟩ ⟨0|. We will call |0⟩ the “sufficient” target state to distinguish it from the full target state |t⟩µ,ν,ω.

Recall that AFGA converges in order 1/|⟨t|s⟩| steps. From the definitions of |s⟩ and |t⟩, one finds

\[ \langle t|s \rangle = \sqrt{p}. \]  

(56)

Once system (µ, ν, ω) has been driven to the target state |t⟩µ,ν,ω, one can measure the subsystem ν while ignoring the subsystem (µ, ω). If we do so, the outcome of the measurements of ν can be predicted from the partial density matrix:

\[ \text{tr}_\mu,\omega \left\{ |t⟩ ⟨t|_{µ,ν,ω} \right\} = P(0) |0⟩_\nu ⟨0|_\nu + P(1) |1⟩_\nu ⟨1|_\nu, \]

(57)

where

\[ P(0) = |z_0|^2, \quad P(1) = |z_1|^2. \]  

(58)

Hence

\[ |z_1|^2 = \frac{P(1)}{P(0)} |z_0|^2. \]  

(59)

We see that |z_1| and |z_0| are proportional to each other, with a proportionality factor that can be calculated by measuring the subsystem ν multiple times. If we know |z_0|, we can use Eq. (59) to find |z_1|. More generally, if |z_j|^2 = f_j(θ) for j = 0, 1, and the functions f_j(θ) are known but the parameter θ isn’t, we can solve f_1(θ)/f_0(θ) = P(1)/P(0) for θ.

Eq. (59) only relates the magnitudes of z_0 and z_1. One can also measure the relative phase between z_0 and z_1 as follows. Let z_0(z_1)^* = z_0z_1 e^{iθ}. Before taking the final measurement of ν, apply a unitary transformation that maps |t⟩ given by Eq. (53) to |t’⟩ given by

\[ |t’⟩_{µ,ν,ω} = \left( \frac{\hat{z}_0 + \hat{z}_1}{\sqrt{2}} \right) |ψ_0⟩_µ |0⟩_ν + \left( \frac{\hat{z}_0 - \hat{z}_1}{\sqrt{2}} \right) |ψ_1⟩_µ |1⟩_ν. \]  

(60)

Then do as before, measure ν in the \{ |0⟩, |1⟩ \} basis while ignoring (µ, ω). If we do so, the outcome of the measurements of ν can be predicted from the partial density matrix:
\[
\text{tr}_{\mu,\omega} \left\{ |t'\rangle \langle t'|_{\mu,\nu,\omega} \right\} = P(+) |0\rangle \langle 0|_\nu + P(-) |1\rangle \langle 1|_\nu ,
\]

where

\[
P(\pm) = \frac{1}{2} |\hat{z}_0 \pm \hat{z}_1|^2
\]

and

\[
\frac{1}{2} [P(0) + P(1) \pm 2P(0)P(1) \cos \theta] .
\]

Hence,

\[
\cos \theta = \frac{P(+) - P(-)}{2P(0)P(1)} .
\]

**8 Blind Targeting**

At first sight, it seems that Grover-like algorithms and AFGA in particular require
to know \(| \langle t|s \rangle |\). In this section, we will describe a technique for bypassing that
onerous requirement.

For concreteness, we will assume in our discussion below that we are using
AFGA and that we are targeting two hypotheses, but the idea of this technique could
be carried over to other Grover-like algorithms in a fairly obvious way.

According to Eq.(56), when targeting two hypotheses, \(| \langle t|s \rangle |\) = \(\sqrt{p}\). Suppose
we guess-timate \(p\), and use that estimate and the AFGA formulas of Ref.[2] to cal-
culate the various rotation angles \(\alpha_j\) for \(j = 0, 1, \ldots, N_{Gro} - 1\), where \(N_{Gro}\) is the
number of Grover steps. Suppose \(N_{Gro}\) is large enough. Then, in the unlikely event
that our estimate of \(p\) is perfect, \(\hat{s}_j\) will converge to \(\hat{t}\) as \(j \to N_{Gro} - 1\). On the other
hand, if our estimate of \(p\) is not perfect but not too bad either, we expect that as
\(j \to N_{Gro} - 1\), the point \(\hat{s}_j\) will reach a steady state in which, as \(j\) increases, \(\hat{s}_j\) rotates
in a small circle in the neighborhood of \(\hat{t}\). After steady state is reached, all functions
of \(\hat{s}_j\) will vary periodically with \(j\).

Suppose we do AFGA with \(p\) fixed and with \(N_{Gro} = (N_{Gro})_0 + r\) Grover steps
where \(r = 0, 1, \ldots N_{tail} - 1\). Call each \(r\) a “tail run”, so \(p\) is the same for all \(N_{tail}\) tail
runs, but \(N_{Gro}\) varies for different tail runs. Suppose that steady state has already
been reached after \((N_{Gro})_0\) steps. For any quantity \(Q_r\) where \(r = 0, 1, \ldots N_{tail} - 1\), let
\(\langle Q \rangle_{LP}\) denote the outcome of passing the \(N_{tail}\) values of \(Q_r\) through a low pass filter
that takes out the AC components and leaves only the DC part. For example, \(\langle Q \rangle_{LP}\)
might equal \(\sum_r Q_r / N_{tail}\) or \([\max_r Q_r + \min_r Q_r]/2\). By applying the SEO of tail run
\(r\) to a quantum computer several times, each time ending with a measurement of the
quantum computer, we can obtain values \(P_r(0)\) and \(P_r(1)\) of \(P(0)\) and \(P(1)\) for tail
run \(r\). Then we can find \(\langle \sqrt{P(1)/P(0)} \rangle_{LP} = \langle |z_1|/|z_0| \rangle_{LP} \). But we also expect to
know \(|z_0|\), so we can use \(\langle |z_1|/|z_0| \rangle_{LP} |z_0|\) as an estimate of \(|z_1|\). This estimate of \(|z_1|\)
and the known value of $|z_0|$ yield a new estimate of $p = |z_1|^2 + |z_0|^2$, one that is much better than the first estimate we used. We can repeat the previous steps using this new estimate of $p$. Every time we repeat this process, we get a new estimate of $p$ that is better than our previous estimate. Call a “trial” each time we repeat the process of $N_{\text{tail}}$ tail runs. $p$ is fixed during a trial, but $p$ varies from trial to trial.

Appendix A describes a numerical experiment that we performed. The experiment provides some evidence that our blind targeting technique behaves as we say it does when used in conjunction with AFGA.

9 Quantum Circuit For Calculating $| \langle c^n | \pi_{\text{Sym}_n} | \psi \rangle |^2$

In this section, we will give the main quantum circuit of this paper, one that can be used to calculate $| \langle c^n | \pi_{\text{Sym}_n} | \psi \rangle |^2$ for some predetermined point $c^n \in \{0..d-1\}^n$ and state $|\psi\rangle_{\alpha_n}$, where $\text{width}(\alpha^n) = d^n$. Actually, in this paper, we will give two alternative methods for calculating $| \langle c^n | \pi_{\text{Sym}_n} | \psi \rangle |^2$. The method presented in this section will be called Method A. Appendix B presents an alternative method that will be called Method B.

We will assume that we know how to compile $|\psi\rangle_{\alpha_n}$ (i.e., that we can construct it starting from $|0^n\rangle_{\alpha_n}$ using a sequence of elementary operations. Elementary operations are operations that act on a few (usually 1, 2 or 3) qubits at a time, such as qubit rotations and CNOTS.) Multiplexor techniques for doing such compilations are discussed in Ref. [8]. If $n$ is very large, our algorithm will be useless unless such a compilation is of polynomial efficiency, meaning that its number of elementary operations grows as $\text{poly}(n)$.

For concreteness, henceforth we will use $n = 4$ in this section, but it will be obvious how to draw an analogous circuit for arbitrary $n$.

For $r = 4, 3, 2, 1$, define

$$Q^{(r)}(c^4) = | \langle c^4 | \pi_{\text{Sym}_r} (\alpha_{\leq r-1}) | \psi \rangle_{\alpha^4} |^2$$

where $\alpha_{\leq r-1} = (\alpha_{r-1}, \ldots, \alpha_1, \alpha_0)$. For instance,

$$Q^{(1)}(c^4) = | \langle c^4 | \psi \rangle |^2$$

and

$$Q^{(2)}(c^4) = | \langle c^4 | \pi_{\text{Sym}_2} (\alpha_1, \alpha_0) | \psi \rangle_{\alpha^4} |^2.$$
Define

\[ T(\alpha, \beta_0) = V_1^{(1)}(\beta_0) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \beta_0 \end{bmatrix} V_1^{(1)}(\beta_0)^\dagger, \]  

\( (68) \)

\[ T(\alpha, \beta_1) = V_1^{(2)}(\beta_1) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \beta_0 \end{bmatrix} V_1^{(2)}(\beta_1)^\dagger, \]  

\( (69) \)

\[ T(\alpha, \beta_2) = V_1^{(3)}(\beta_2) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \beta_0 \end{bmatrix} V_1^{(3)}(\beta_2)^\dagger, \]  

\( (70) \)

\[ T(\alpha, \beta) = \prod_{\ell=2}^{\infty} T(\alpha, \beta_\ell), \]  

\( (71) \)

\[ \pi(\alpha) = \prod_{j=0}^{3} P_{c_j}(\alpha_j) \]  

\( (72) \)

and

\[ \pi(\beta) = \begin{cases} P_0(\beta_0) \\ P_0(\beta_1)P_0(\beta_1) \\ P_0(\beta_2)P_0(\beta_2)P_0(\beta_2) \end{cases}. \]  

\( (73) \)

### 9.1 Method A

Method A for calculating \( Q^{(4)}(c^4) \) consists of applying the algorithm AFGA of Ref. [2] in the way that was described in Sec. 7, using the techniques of targeting two hypotheses and blind targeting. As in Sec. 7, when we apply AFGA in this section, we will use a sufficient target \( |0\rangle_\omega \). All that remains for us to do to fully specify our circuit for calculating \( Q^{(4)}(c^4) \) is to give a circuit for generating \( |s\rangle \).

A circuit for generating \( |s\rangle \) is given by Fig. 11. Fig. 11 is equivalent to saying that

\[ |s\rangle_{\mu,\nu,\omega} = \sigma_X(\omega) \pi(\beta)\pi(\alpha) \frac{1}{\sqrt{2}} \begin{bmatrix} T(\alpha, \beta) \begin{bmatrix} \psi_{\alpha}^\dagger \\ 0 \end{bmatrix}^\beta + \begin{bmatrix} \psi_{\alpha}^\dagger \\ 0 \end{bmatrix}^\beta \\ |1\rangle_{\mu_0} \\ |1\rangle_{\omega} \\ |1\rangle_{\gamma} \end{bmatrix}. \]  

\( (74) \)
Claim 5

\[ |s\rangle_{\mu,\nu,\omega} = z_1 |\psi_1\rangle_{\mu} + z_0 |\psi_0\rangle_{\mu} + |\chi\rangle_{\mu,\nu}, \quad (75) \]

for some unnormalized state \(|\chi\rangle_{\mu,\nu}\), where

\[
|\psi_1\rangle_{\mu} = \begin{bmatrix} |c^4\rangle_{\alpha} \\ |1\rangle_{\mu_0} \\ |0\rangle_{\beta_0} \\ |00\rangle_{\beta_1} \\ |000\rangle_{\beta_2} \\ |1\rangle_{\gamma} \end{bmatrix}, \quad |\psi_0\rangle_{\mu} = \begin{bmatrix} |c^4\rangle_{\alpha} \\ |0\rangle_{\mu_0} \\ |0\rangle_{\beta_0} \\ |00\rangle_{\beta_1} \\ |000\rangle_{\beta_2} \\ |0\rangle_{\gamma} \end{bmatrix}, \quad (76)
\]

\[
z_1 = \frac{1}{\sqrt{2}} \langle c^4 | \pi_{\text{Sym}_4} | \psi \rangle = \sqrt{\frac{Q(4)(c^4)}{2}}, \quad (77)\]
\[ z_0 = \frac{1}{\sqrt{2}} \langle c^4 | \psi \rangle = \sqrt{\frac{Q(1)(c^4)}{2}}, \quad (78) \]

\[ |z_1| = \left| \frac{P(1)}{P(0)} \right|. \quad (79) \]

**proof:**

Applying identity Eq. (8) with \( U = \sigma_X(\omega) \) yields:

\[ |s\rangle = \sigma_X(\omega)\pi(\beta)\pi(\alpha)|s'\rangle \]

\[ = \sigma_X(\omega)\pi(\beta)\pi(\alpha)|s'\rangle + |1\rangle_\omega. \quad (81) \]

Eq. (79) is just Eq. (59).

**QED**

In case \( \langle c^4 | \psi \rangle = 0 \), this procedure won’t yield \( Q(4)(c^4) \), but it can be patched up easily. Note that if we know how to compile \( |\psi\rangle_{c^4} \) with polynomial efficiency, then we also know how to compile \( |\psi'\rangle = swap(\alpha_0, \alpha_1)|\psi\rangle \) with polynomial efficiency. Furthermore,

\[ \langle c^4|\pi_{Sym_4}|\psi\rangle = \langle c^4|\pi_{Sym_4}|\psi'\rangle. \quad (82) \]

If \( \langle c^4|\psi'\rangle \neq 0 \), mission accomplished. Even if \( \langle c^4|\psi'\rangle = 0 \), as long as we can replace \( |\psi\rangle \) by some partially symmetrized version of it, call it \( |\psi_S\rangle \), such that \( \langle c^4|\psi_S\rangle \neq 0 \), we should be able to apply this method to get \( Q(4)(c^4) \).

## A Appendix: Numerical Experiment to Test Blind Targeting with AFGA

In this appendix, we will describe a numerical experiment that we conducted to test blind targeting with AFGA. The experiment is not a conclusive proof that blind targeting with AFGA always converges to the right answer, but it does provide some evidence that it often does.

Our algorithm for blind targeting is based on the following Bloch sphere picture. We will use the notation of Ref. [2]. Suppose we know the vector \( \hat{s}_0 \) but we don’t know that \( \hat{t} = \hat{z} \), so we don’t know the initial \( |\langle t|s \rangle| = \left| \cos(\frac{1}{2} \arccos(\hat{t} \cdot \hat{s}) \right| \). Suppose we guess-timate \( |\langle t|s \rangle| \), and use that estimate and the AFGA formulas of Ref. [2] to calculate the unit vector \( \hat{s}_j \) for \( j = 0, 1, \ldots, N_{Gro} - 1 \), where \( N_{Gro} \) is the number of Grover steps. Suppose \( N_{Gro} \) is large enough. Then, in the unlikely event that our estimate of \( |\langle t|s \rangle| \) is perfect, as \( j \to N_{Gro} - 1 \), the point \( \hat{s}_j \) will converge to \( \hat{t} = \hat{z} \). On the other hand, if our estimate of \( |\langle t|s \rangle| \) is not perfect but not too bad either,
we expect that as \( j \to N_{Gro} - 1 \), the point \( \hat{s}_j \) will reach a steady state in which, as \( j \) increases, \( \hat{s}_j \) rotates in a circle of constant latitude very close to the North Pole of the Bloch sphere.

If we pass through a low pass filter the values of \( \hat{s}_j \) after it reaches this steady state, we will get an estimate of the position of the North Pole. Using that estimate \( \hat{t}_{est} \) of the position of the North Pole and our assumed knowledge of \( \hat{s} \) allows us to get a new estimate of \( | \langle t|s \rangle | \), one that is much better than the first estimate we used. We can repeat the previous steps using this new estimate of \( | \langle t|s \rangle | \). Every time we repeat this process, we get a new estimate of \( | \langle t|s \rangle | \) that is better than our previous estimate.

To get some numerical evidence that this Bloch sphere picture argument applies, we wrote a new version of the .m files\(^1\) that were written to illustrate the AFGA algorithm of Ref.\(^2\) and were included with the arXiv distribution of that paper. The arXiv distribution of the present paper includes 3 new Octave .m files: `afga_blind.m`, `afga_step.m`, and `afga_rot.m`.

The files `afga_step.m` and `afga_rot.m` contain auxiliary functions called by the main file `afga_blind.m`. These 2 files are identical to the files with the same names that were included with Ref.\(^2\). Hence, we will say nothing more about them here.

The file `afga_blind.m` is a slight expansion of the file `afga.m` that was presented and explained in Ref.\(^2\). The first 7 non-comment lines of `afga_blind.m` instantiate the following 7 input parameters:

- \( g0\text{deg}= \gamma = \gamma_0 \) in degrees. Used only to calculate \( \hat{s}_0 \), which is assumed known, not to calculate the initial \( \langle t|s \rangle \), which is assumed a priori unknown.

- \( g0\text{est}\text{deg}= \gamma_0 \) in degrees. Used to get first estimate of \( \langle t|s \rangle \).

- \( \text{del}\_\lambda\text{deg}= \Delta \lambda \) in degrees

- \( \text{num}\_\text{steps}= N_{Gro} \) = number of Grover steps.

- \( \text{tail}\_\text{len}= N_{\text{tail}} = \text{tail length, number of tail runs. Low pass filtering is applied to points } j = N_{Gro} - N_{\text{tail}}, \ldots, N_{Gro} - 3, N_{Gro} - 2, N_{Gro} - 1 \text{ of each trial to get the estimate of } \langle t|s \rangle \text{ for the next trial.} \)

- \( \text{num}\_\text{trials}= \text{number of trials. } \gamma_0 \text{ remains constant during a trial, but changes from trial to trial.} \)

- \( \text{plotted}\_\text{trial}= \text{trial for which program will plot the time series } \hat{s}_j \text{ for } j = 0, 1, \ldots, N_{Gro} - 1. \)

---

\(^1\)Our .m files are written in the language of Octave. The Octave environment is a free, open source, partial clone of the MatLab environment. Octave .m files can usually be run in MatLab with zero or only minor modifications.
Each time `afga_blind.m` runs successfully, it outputs two files called `afga_blind.txt` and `afga_blind.svg`.

The output file `afga_blind.txt` is a text file. Its contents are very similar to the contents of the file `afga.txt` that is outputted by the program `afga.m` of Ref.[2]. The contents of an `afga.txt` file are thoroughly explained in Ref.[2]. From that, it’s very easy to understand the meaning of the contents of an `afga_blind.txt` file. An `afga_blind.txt` file contains the records of `num_trials` trials instead of just one trial like an `afga.txt` file does.

The output file `afga_blind.svg` is a picture of a plot, in .svg (scalable vector graphic) format. .svg files can be viewed with a web browser. They can be viewed and modified with, for example, the free, open source software program Inkscape. The plot in an `afga_blind.svg` file gives the 3 components of the unit vector $\hat{s}_j$ as a function of the Grover step $j$. The 7 input parameters just described are listed in a legend of the plot.

Here are some sample plots.

- We got Fig[2] with plotted_trial=0 (first trial) and with a $\gamma_0$ close to 90 degrees. Then we changed plotted_trial to 1 and got Fig[3].

- We got Fig[4] with plotted_trial=0 (first trial) and with a $\gamma_0$ close to 180 degrees. Then we changed plotted_trial to 4 and got Fig[5].

Further plots can be generated by the user using `afga_blind.m`. Note that $\gamma_0 = 180 - \epsilon$ degrees, where $0 < \epsilon << 1$, corresponds to the regime $|\langle t|s\rangle| << 1$ of the “hardest” problems. In that regime of hardest problems, we found that larger $N_{Gro}$ and larger $N_{tail}$ are required for convergence than in other regimes. Furthermore, in this regime the algorithm becomes very sensitive to various adjustable input parameters like $N_{Gro}$, $N_{tail}$, $\Delta \lambda$ and to the type of low pass filter we use. We used two types of low pass filters in the software. The user can test them both himself. One was the MMM filter; i.e., a “min-max-mean” filter that uses $[\max_r(\hat{s}_r) + \min_r(\hat{s}_r)]/2$. We found the MMM filter to be the more robust of the two filters we tried. The example plots presented in this section of the paper were all generated using the MMM filter. Further work will be required to determine how to choose adjustable input parameters and a low pass filter which are optimal, or nearly so, for this type of algorithm.
Figure 2: The 3 components of the unit vector $\hat{s}_j$ as a function of the Grover step $j$. Plot generated by `afgablind.m` with indicated inputs.

Figure 3: The 3 components of the unit vector $\hat{s}_j$ as a function of the Grover step $j$. Plot generated by `afgablind.m` with indicated inputs.
Figure 4: The 3 components of the unit vector $\hat{s}_j$ as a function of the Grover step $j$. Plot generated by `afga_blue.m` with indicated inputs.

Figure 5: The 3 components of the unit vector $\hat{s}_j$ as a function of the Grover step $j$. Plot generated by `afga_blue.m` with indicated inputs.
B Appendix: Method B of calculating $Q^{(4)}(c^4)$

In this appendix, we will present Method B, an alternative to the Method A that was presented in Sec.9.1. Both methods can be used to calculate $Q^{(4)}(c^4)$.

Figure 6: Method B circuit for generating $|s^{(4)}\rangle$ used in AFGA to calculate $|\langle c^4|\pi_{\text{Sym}_4}\psi\rangle|^2$

Unlike in Method A of calculating $Q^{(4)}(c^4)$, in Method B we will assume the restriction that $\langle c^4|\psi\rangle \geq 0$. See Appendix C for cases in which it is possible to sidestep this restriction.

In Method A, we applied TTH (Targeting Two Hypotheses) only once. In method B, we will apply TTH multiple times, for $k = 4, 3, 2$, each time applying it in the way that was described in Sec 7, together with blind targeting, and using a sufficient target $|0\rangle_\omega$. All that remains for us to do to fully specify our Method B circuit for calculating $Q^{(4)}(c^4)$ is to give a circuit for generating $|s^{(k)}\rangle$ for $k = 4, 3, 2$.

A circuit for generating $|s^{(4)}\rangle$ is given by Fig. 6. Note that in this circuit we do not use the qubit $\gamma$ that was used in method A. Define $\pi'(\beta)$ to be equal to the $\pi(\beta)$ defined by Eq. (73) but with the projector $P_0(\beta_{2,2})$ removed. In other words, “formally”,

$\pi'(\beta) = \pi(\beta) \setminus P_0(\beta_{2,2})$
\[
\pi' (\beta) = \pi (\beta) / P_0 (\beta; 2) .
\]  

Then Fig. 6 is equivalent to saying that

\[
|s\rangle^{(4)}_{\mu, \nu, \omega} = \sigma_X (\omega) \pi' (\beta) \pi (\alpha) \sigma_X (\mu_0) P_0 (\beta; 2) \cdot \begin{pmatrix}
T (\alpha, \beta) |\psi\rangle^{(4)}_{\alpha, \beta} \\
|0\rangle_{\mu_0} \\
|1\rangle_{\omega}
\end{pmatrix} .
\]  

Claim 6

\[
|s\rangle^{(4)}_{\mu, \nu, \omega} = \frac{z_1}{\sqrt{Q^{(4)} (c^4)}} |\psi_1\rangle_\mu + \frac{z_0}{\sqrt{Q^{(3)} (c^4)}} |\psi_0\rangle_\mu + \chi_{\mu, \nu},
\]  

for some unnormalized state \( |\chi\rangle_{\mu, \nu} \), where

\[
\begin{array}{c|c}
|\psi_1\rangle_\mu &=& \begin{pmatrix} c^4_\alpha \\ 0_{\mu_0} \\ 0_{\beta, 0} \\ 0_{\beta, 1} \\ 0_{00_{\beta, 2}} \end{pmatrix} \\
|\psi_0\rangle_\mu &=& \begin{pmatrix} c^4_\alpha \\ 1_{\mu_0} \\ 0_{\beta, 0} \\ 0_{\beta, 1} \end{pmatrix}
\end{array}
\]

\[
z_1 = \sqrt{Q^{(4)} (c^4)} \geq 0 ,
\]

\[
z_0 + z_1 = \frac{2}{4} \sqrt{Q^{(3)} (c^4)} ,
\]

\[
\frac{|z_0|}{|z_1|} = \sqrt{\frac{P(0)}{P(1)}},
\]

\[
sign (z_0) = \begin{cases}
+1 & \text{if } P(+) > P(-) \\
-1 & \text{otherwise}
\end{cases}
\]

**proof:**

According to Claims 1 and 2

\[
V_1^{(3)} |0\rangle = \frac{1}{\sqrt{4}} \begin{pmatrix}
|0\rangle + |1\rangle + |0\rangle + |0\rangle \\
|0\rangle + |0\rangle + |1\rangle + |0\rangle \\
|0\rangle + |0\rangle + |1\rangle + |1\rangle
\end{pmatrix} ,
\]  

and

\[
V_1^{(3)} |0\rangle = \frac{1}{\sqrt{4}} \begin{pmatrix}
|0\rangle - |1\rangle + |0\rangle + |0\rangle \\
|0\rangle + |0\rangle + |1\rangle + |0\rangle \\
|0\rangle + |0\rangle + |0\rangle + |1\rangle
\end{pmatrix} .
\]
Therefore, Figure 7 is true. Figs. 6 and 7 imply the two constraints given by Eqs. (87).

The two constraints given by Eqs. (88) are old news. They come directly from Eq. (59) and Eq. (64). Note that \( \cos \theta = \text{sign}(z_0) \) for the special case being considered in this claim, namely when \( z_1 \) is non-negative real and \( z_0 \) is either positive or negative real.

\[ \text{QED} \]

\[ \sqrt{Q^{(4)}(c^4)} = \frac{2}{\sqrt{3}} \sqrt{Q^{(1)}(c^4)} \prod_{k=4}^{2} \left[ 1 + \sigma^{(k)} \sqrt{\frac{P^{(0)}(0)}{P^{(1)}(1)}} \right] \]  

Figure 7: Two matrix elements of \( \beta_{3/2}^3 \).

After doing TTH with \( |s\rangle^{(4)} \), we are left knowing \( Q^{(4)}(c^4) \) in terms of \( Q^{(3)}(c^4) \). If we know \( Q^{(3)}(c^4) \), we can stop right there and we are done. Otherwise, we can do TTH again, this time with the \( |s\rangle^{(3)} \) given by Fig. 8. Again, we are left knowing \( Q^{(3)}(c^4) \) in terms of \( Q^{(2)}(c^4) \). If we know \( Q^{(2)}(c^4) \), we can stop right there and we are done. Otherwise, we can do TTH again, this time with the \( |s\rangle^{(2)} \) given by a circuit analogous to Figs. 6 and 8. Eventually we end up finding \( Q^{(4)}(c^4) \) in terms \( Q^{(1)}(c^4) \). We assume the latter is known.

Claim 7

\[ \sqrt{Q^{(4)}(c^4)} = \frac{2}{\sqrt{3}} \sqrt{Q^{(1)}(c^4)} \prod_{k=4}^{2} \left[ 1 + \sigma^{(k)} \sqrt{\frac{P^{(0)}(0)}{P^{(1)}(1)}} \right] \]  

25
\[ \sigma^{(k)} = \begin{cases} 
+1 & \text{if } P^{(k)}(+) > P^{(k)}(-) \\
-1 & \text{otherwise} 
\end{cases} \] (92)

**proof:** Follows from Claim 6 by analogy.

QED

Figure 8: Method B circuit for generating \(|s\rangle^{(3)}\) used in AFGA to calculate \(|\langle c^{(4)}|_{\text{Sym}}^\dagger|\psi\rangle|^2\)

### C Appendix: Linear Transform of Vector If Vector Not Normalized

Often, when calculating with a quantum computer the linear transform of a vector \(|\psi\rangle\), our algorithm works only if we assume that the vector \(|\psi\rangle\) has non-negative components in some basis, or is normalized in some norm, or both. The purpose of this appendix is to show that this restriction on \(|\psi\rangle\) does not imply a large reduction of generality of the algorithm. We will show that given some simple information about \(|\psi\rangle\), we can still use the restricted algorithm to find the linear transform of \(|\psi\rangle\), even if \(|\psi\rangle\) doesn’t satisfy the restrictions. The results of this appendix are very obvious but worth keeping in mind.
For any $z \in \mathbb{C}$, let $z_r, z_i$ be its real and imaginary parts respectively.

We wish to consider some finite set $S_x$ and two functions $f, f^- : S_x \to \mathbb{C}$ related by

$$f(x) = \sum_{x^- \in S_x} M(x, x^-) f^-(x^-)$$

(93)

where $M(x, x^-) \in \mathbb{C}$. Function $f$ will be referred to as the M-transform of function $f^-$.

**Claim 8** If one is given constants $a_r, a_i, b_r, b_i \in \mathbb{R}$ such that

$$a_r < f^-_r(x^-) < b_r, \quad a_i < f^-_i(x^-) < b_i$$

(94)

for all $x^-$, and one is given $\sum x^- M(x, x^-)$, then the M-transform of $f^-(x^-)$ can be calculated easily from the M-transform of functions $g_r(), g_i()$ which satisfy $0 \leq g_r(x^-), g_i(x^-) \leq 1$ for all $x^-$. 

**proof:** Define

$$L = \max(b_r - a_r, b_i - a_i)$$

(95)

and

$$g_r(x^-) = \frac{f^-_r(x^-) - a_r}{L}, \quad g_i(x^-) = \frac{f^-_i(x^-) - a_i}{L}.$$ 

(96)

Then

$$\sum_{x^-} M(x, x^-) \left[ \frac{f^-(x^-) - a}{L} \right] = \sum_{x^-} M(x, x^-) g_r(x^-) + i \sum_{x^-} M(x, x^-) g_i(x^-).$$

(97)

**QED**

**Claim 9** If one is given a constant $N > 0$, then the M-transform of $f^-(x^-)$ can be easily calculated from the M-transform of $f^-(x^-)/N$. For instance, $N$ might be $\sqrt{\sum_{x^-} |f^-(x^-)|^2}$. 

**proof:**

$$f(x) = N \sum_{x^- \in S_x} M(x, x^-) \frac{f^-(x^-)}{N}.$$ 

(98)

**QED**
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