CRITICAL VISCOSITY EXPONENT FOR FLUIDS: WHAT HAPPENED TO THE HIGHER LOOPS

Palash Das and Jayanta.K.Bhattacharjee
Department of Theoretical Physics
Indian Association for the Cultivation of Science
Jadavpur, Calcutta 700 032, India

Abstract

We arrange the loopwise perturbation theory for the critical viscosity exponent $x_\eta$, which happens to be very small, as a power series in $x_\eta$ itself and argue that the effect of loops beyond two is negligible. We claim that the critical viscosity exponent should be very closely approximated by $x_\eta = \frac{8}{15\pi^2}(1 + \frac{8}{3\pi^2}) \simeq 0.0685$.

PACS number(s):64.60Ht
Critical exponents, amplitude ratios and scaling functions were issues of considerable importance three decades ago. Sophisticated calculations and experiments were carried out which clearly established the correctness of the various theoretical models (Landau-Ginzburg equations for statics and the various models of dynamics [1-4] introduced by Hohenberg and Halperin, Kawasaki, Ferrell etc). Basically, the exponents could be classified into two types: i) large exponents i.e. exponents of O(1) and ii) small exponents i.e. exponents of O(0.1) or even smaller. It is the small exponents where the most crucial confrontation between theory and experiment can occur. That is why even after three decades the small exponents remain an interesting issue. In static critical phenomena [5] the small exponents are associated with the critical correlation function at the transition point (\(\eta\), the anomalous dimension exponent) and specific heat (\(\alpha\), the specific heat exponent), the specific at constant volume for the liquid-gas transition and the specific heat at constant pressure for the superfluid transition of \(He^4\), while in the critical dynamics the small exponent is associated with the shear viscosity. Accurate determination [6] of \(\alpha\) for the superfluid transition and comparison with very detailed calculation [7] confirm the theoretical expectation. For the shear viscosity exponent \(z_\eta\), the recent measurements [8] in the space shuttle have yielded an accurate value, namely \(z_\eta = 0.0690 \pm 0.0006\). The theoretical self consistent 2-loop calculation in D=3 of Hao yields \(z_\eta = 0.066 \pm 0.002\), amazingly close to the experimental value. This raises the immediate question: what happened to the higher loops? The one loop answer is 20% away from the experimental answer, the two loop calculation produces the 20% enhancement almost entirely and so what happens to the infinite number of loops that have been left out. This is the question that we address in the paper and provide the insight into why the higher loops happen to be unimportant.

In a liquid-gas system near the critical point or a binary liquid mixture near the critical mixing point, the order parameter \(\phi\) is the density (concentration) difference and relaxes when disturbed from equilibrium according to the Langevin equation

\[
\frac{\partial \phi(\vec{k})}{\partial t} = -\Gamma k^2 (k^2 + \kappa^2) \phi(\vec{k}) + N(\vec{k}) \tag{1}
\]

where \(\phi(\vec{k})\) is the Fourier transformation of the D-dimensional field \(\phi(x_1, \ldots x_D)\). In the relaxation rate the factor \(k^2\) indicates that \(\phi(\vec{x})\) is conserved. \(\Gamma\) is the Onsagar coefficient and the diffusion constant is \(D = \frac{\Gamma}{\chi}\), where \(\chi\) is the susceptibility. Near the critical point the susceptibility is \(\chi = (k^2 + \kappa^2)^{-1}\) with \(\kappa = \xi^{-1}\), the inverse correlation length which diverges near \(T = T_c\) as \(\xi \propto |T - T_c|^{-\nu}\). The term \(N\) is a stochastic forcing that comes from the short wavelength modes. Fluctuation dissipation holds and the correlation of \(N\) is related the usual way to the dissipation.

In a fluid the density (concentrations) fluctuations will be affected by the velocity fluctuations and the effect of the velocity is to advect the concentra-
this implies writing the non-linear term as one field at zeroth order and the other at first order. For Eq.(2)
and it is easy to see that the non-linear terms in Eq.(2) yields a term of
The fields being stochastic in nature, the effect of the non-linear terms in
to a one loop result. The two loop results come from all the pairings of 3, the
where
Note $v_\alpha$ and $N_\alpha$ are solenoidal. However Eqs.(2) and (3) do not conserve the
and similarly for Eq.(4). This is exactly equivalent
to a one loop result. The two loop results come from all the pairings of 3, the
The fact that the velocity fluctuations affect the concentration means that we need to know the velocity fluctuations. The equation of motion (for small fluctuation) is Navier-Stokes equation

$$\frac{\partial v_\alpha(\vec{k})}{\partial t} = -\eta k^2 v_\alpha(\vec{k}) + N_\alpha^v(\vec{k})$$ (3)

Where $T_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$, the projection operator. The effect of the non-linear terms in Eqs. (4) and (5) is to renormalize the Onsager co-efficient $\Gamma$ and the shear viscosity $\eta$. Dropping the non-linear terms, we get the zeroth order solution

$$\phi^{(0)}(\vec{k}, t) = \int e^{-\Gamma k^2 (k^2 + \kappa^2)(t-t')} N(t') dt'$$ (5)

and

$$v^{(0)}_\alpha(\vec{k}, t) = \int e^{-\eta k^2 (t-t')} N_\alpha(t') dt'$$ (6)

The first order solution is easily seen to be

$$\phi^{(1)}(\vec{k}, t) = -i \int e^{-\Gamma k^2 (k^2 + \kappa^2)(t-t')} i k_\alpha \sum_{\vec{p}} v^{(0)}_\alpha(\vec{p}, t') \phi^{(0)}(\vec{k} - \vec{p}, t') dt'$$ (7)

and

$$v^{(1)}_\alpha(\vec{k}, t) = -i \int e^{-\eta k^2 (t-t')} p^2 p_\beta \phi^{(0)}(\vec{p}, t') \phi^{(0)}(\vec{k} - \vec{p}, t') dt'$$ (8)

The fields being stochastic in nature, the effect of the non-linear terms in
Eq.(2) and Eq.(4) are to be understood as averaged over the noise terms and it is easy to see that the non-linear terms in Eq.(2) yields a term of the form $-k^2(k^2 + \kappa^2) \int \Gamma^{(R)}(\vec{k}, t - t') \phi^{(0)}(\vec{k}, t') dt'$ and those in Eq.(4) give

$-k^2 \int \eta^{(R)}(\vec{k}, t - t') v^{(0)}_\alpha(\vec{k}, t') dt'$, when we split the quadratically non-linear term as one field at zeroth order and the other at first order. For Eq.(2) this implies writing the non-linear term as $ik_\alpha[< \sum_{\vec{p}} v^{(0)}_\alpha(\vec{p}) \phi^{(1)}(\vec{k} - \vec{p}) > + < \sum_{\vec{p}} v^{(1)}_\alpha(\vec{p}) \phi^{(0)}(\vec{k} - \vec{p}) >]$ and similarly for Eq.(4). This is exactly equivalent to a one loop result. The two loop results come from all the pairings of 3, the three loop from the pairings of 5 and so on. The Fourier transforms $\Gamma^{R}(\vec{k}, \omega)$ and $\eta^{R}(\vec{k}, \omega)$ of $\Gamma^{R}(\vec{k}, t - t')$ and $\eta^{R}(\vec{k}, t - t')$ are the renormalized Onsager co-efficient and the shear viscosity respectively.
The renormalized transport coefficients $\Gamma^{(R)}(k, \omega)$ and $\eta^{(R)}(k, \omega)$ diverge at the critical point in the zero frequency, zero wavelength limit and dominates the molecular contributions. From now on we will refer to these as $\Gamma(k, \omega)$ and $\eta(k, \omega)$. A little algebra shows that at one loop, we get the standard results ($\eta(k)k^2 \gg \Gamma(k)k^2(k^2 + \kappa^2)$)

$$\Gamma(k, \kappa) = \frac{1}{C_3} \int \frac{d^3 p}{(p^2 + \kappa^2)} \frac{\sin^2 \theta}{\eta\bar{k}(\bar{k} - \bar{p})(\bar{k} - \bar{p})^2}$$

(9)

and

$$\eta(k, \kappa) = \frac{1}{4C_3} \int \frac{d^3 p}{(p^2 + \kappa^2)(p^2 + \kappa^2)} \frac{\Gamma(p) + \Gamma(p')\Gamma(p'' \kappa^2)}{p^2(p^2 - p'^2)^2 \sin^2 \theta}$$

(10)

where $p' = \bar{k} - \bar{p}$.

We now introduce the scaling behavior (long wavelength divergence at the critical point) at $\kappa = 0$ as

$$\Gamma(k) = \Gamma_0 k^{-1+x_\eta}$$

(11)

$$\eta(k) = \eta_0 k^{-x_\eta} k^2$$

(12)

consistent with Eqs.(9) and (10), where $x_\eta$ is the exponent that is yet unknown. Working at $\kappa = 0$ in Eq.(9), we find

$$\eta_0 \Gamma_0 = \frac{\pi^2}{8} + O(x_\eta)$$

(13)

We anticipate at this stage that $x_\eta$ is very small and are going to use it as a small parameter in setting up our calculation. Our main observation is that a loopwise expansion can be cast as an expansion in powers of $x_\eta$ for the quantity $\eta_0 \Gamma_0$. We can get yet another expansion for $\eta_0 \Gamma_0$ by using Eq.(10) at $\kappa = 0$. The integral has a long wavelength divergence at $x_\eta = 0$ and this leads to the evaluation of the integral as a pole in $x_\eta$. This yields

$$\eta_0 \Gamma_0 = \frac{1}{15x_\eta} + O(1)$$

(14)

Combing Eqs(13) and (14),

$$x_\eta = \frac{8}{15\pi^2}$$

(15)

to the lowest order.

We now observe that the perturbation theory for $\Gamma$ and $\eta$ can be expressed through diagrams as shown in Fig.1 and 2, with a wavy line denoting the velocity field (propagator or correlator as the case may be) and a solid line the density field.

For higher loops, self energy insertion [13-15] are not shown separately. They are handled by a finite frequency evaluation of a lower loop and yields
insignificant corrections. The important graphs are the vertex correction varieties [16] that are shown in Fig.1 and 2. If one compare a one loop and a two loop graph, we note that compared to the one loop graph, the two loop graph has two time zones, one dominated by viscosity relaxation, the other lacking any viscosity contribution. This means an additional factor of \((\eta_0 \Gamma_0)^{-1}\) every time a loop increases. Now, in addition we note that for every loop the viscosity graphs diverge logarithmically if \(x_\eta = 0\) and has a pole for small \(x_\eta\). We simply need to evaluate this pole in a manner very similar to the dimensional regularization scheme in field theory. From Fig.1, there emerges

\[
\eta_0 \Gamma_0 = J_1 + \frac{1}{\eta_0 \Gamma_0} J_2 + \frac{1}{(\eta_0 \Gamma_0)^2} J_3 + \ldots \quad \quad (16)
\]

while from Fig.2, we get

\[
\eta_0 \Gamma_0 = \frac{I_1}{x_\eta} + \frac{I_2}{\eta_0 \Gamma_0 x_\eta} + \frac{I_3}{(\eta_0 \Gamma_0)^2 x_\eta} + \ldots \quad \quad (17)
\]

where \(I_n\) and \(J_n\) are integrals of which the one loop parts, \(I_1\) and \(J_1\). Fig.1a and 2b are shown in Eqs.(9) and (10). The integrals corresponding to the two loop (Fig.1b and 2b)

\[
k^2 I_2 = \frac{1}{2} \int \frac{d^3p}{C_3} \int \frac{d^3q}{C_3} T_{\alpha\beta}(k) q_{\beta} p_{\alpha} T_{\mu\nu}(k-p-q) q_{\nu} [p^2 - (\vec{k} - \vec{p})^2] [q^2 - (\vec{k} - \vec{q})^2] \quad \quad (18)
\]

\[
J_2 = \int \frac{d^3p}{C_3} \int \frac{d^3q}{C_3} \left[ (\vec{k} - \vec{p} - \vec{q})^2 - (\vec{k} - \vec{p})^2 \right] \left[ (\vec{k} - \vec{p} - \vec{q})^2 - (\vec{k} - \vec{q})^2 \right] \\
\times \frac{k_{\alpha} T_{\alpha\beta}(p)(\vec{k} - \vec{q})_{\beta} k_{\mu} T_{\mu\nu}(q)(\vec{k} - \vec{p})_{\nu}}{p^2 q^2 (\vec{k} - \vec{p})^3 (\vec{k} - \vec{q})^3 (\vec{p} - \vec{q})^3} \quad \quad (19)
\]

Using Eq.(14) to substitute for \(\eta_0 \Gamma_0\) in Eqs.(16) and (17), we end up with

\[
\eta_0 \Gamma_0 = J_1 + 15 x_\eta J_2 + (15 x_\eta)^2 J_3 + \ldots \quad \quad (20)
\]

The important fact is that \(15 x_\eta J_2 \ll 1\) and this trend continues through higher loops. This fixes \(\eta_0 \Gamma_0\).

Turning now to the diagrams of Fig.2, they lead to (using Eq.(14) repeatedly)

\[
\eta_0 \Gamma_0 = \frac{I_1}{x_\eta} + 15 I_2 + x_\eta (15)^2 I_3 + \ldots \\
= \frac{I_1}{x_\eta} [1 + 15 x_\eta \frac{I_2}{I_1} + x_\eta (15)^2 x_\eta \frac{I_3}{I_1} + \ldots] \\
= \frac{1}{15 x_\eta} [1 + \frac{8}{\pi^2} + x_\eta \frac{8}{\pi^2} \frac{15 J_3}{I_1} + x_\eta^2 \frac{8}{\pi^2} \frac{15^2 I_4}{I_1} + \ldots] \quad \quad (21)
\]
leading to the ordering in $x_\eta$. The calculation of $I_2$ yields $\frac{1}{3}$ and hence to two loop order

$$x_\eta = \frac{8}{15\pi^2}(1 + \frac{8}{3\pi^2}) \simeq 0.0685 \quad (22)$$

The reason why $I_2$ is smaller than $I_1$ has to do with the projection factors which yield zeroes in the integrand. The large number of zeroes and their distributions in the three loop integral leads to $\frac{I_2}{I_1}$ being significantly smaller than $\frac{1}{10}$. The additional factor of $x_\eta$ now makes the three loop contribution negligible. The important point is that for an $n$-loop integral $I_n$, the projection factor produce sufficient cancellation that $15^{n-2}I_n$ is always of $O(1)$ and that ensures that higher loops produce insignificant corrections when $x_\eta \ll 1$.

The generic form of the three loop graph of Fig.2c involves a few different time ordering, all of which are shown in Fig.3. A typical contribution $I^{(1)}_3$ coming from the last two graphs of Fig.(3) is

$$k^2 I^{(1)}_3 = - \int \frac{d^3p d^3q d^3r}{C_3 C_3 C_3} \frac{[(\vec{k} - \vec{p})^2 - p^2][[(\vec{k} - \vec{r})^2 - r^2]}{(\vec{k} - \vec{p})^2(\vec{k} - \vec{q})^2}
\times \frac{p_\alpha T_{\alpha\beta}(k)r_\beta p_\mu T_{\mu\nu}(p - q)p_\nu}{(\vec{k} - \vec{r})^2[p^3 + |\vec{p} - \vec{k}|^3][q^3 + |\vec{q} - \vec{k}|^3]}
\times \frac{q_\gamma T_{\gamma\lambda}(q - r)q_\lambda}{r^3 + |\vec{k} - \vec{r}|^3]}(\vec{k} - \vec{p} - \vec{q})^2(\vec{k} - \vec{q} - \vec{r})^2
\times \left[ p^2 q^2 - (\vec{p} \cdot \vec{q})^2 \right] \left[ q^2 r^2 - (\vec{q} \cdot \vec{r})^2 \right] \frac{p^D q^D r^D}{p^D q^D r^D}
(23)$$

The evaluation of $I_3$ has to be in the limit of $k \to 0$. This allows us to drop ‘$k$’ from all the the terms after a factor of $k^2$ has been extracted from the integral. We now carry out the following steps in a D-dimensional space for generality

a) expand the number in powers of $k^2$ and keep the first term (this is proportional to $k^2$) and set $k=0$ everywhere else.

b) do an angular average over the directions of $\vec{k}$.

In a D-dimensional space, $I^{(1)}_3$ after some long algebra reduces to

$$I^{(1)}_3 = - \frac{1}{4D(D + 2)} \int d^Dp d^Dq d^Dr \frac{[D(\vec{p} \cdot \vec{r})^2 - p^2 r^2]}{r^4(q - \vec{r})^4(\vec{p} - \vec{q})^4}
\times \left[ p^2 q^2 - (\vec{p} \cdot \vec{q})^2 \right] \left[ q^2 r^2 - (\vec{q} \cdot \vec{r})^2 \right] \frac{p^D q^D r^D}{p^D q^D r^D}
(24)$$

It is the factor $(\vec{p} \cdot \vec{r})^2 - p^2 r^2$ which is qualitatively new. The two loop integral $I_2$ did not have such a factor. The characteristic feature of this factor is that in the absence of the quite indirect additional appearance of the angle between $\vec{p}$ and $\vec{r}$ because of the term $|\vec{q} - \vec{r}|^4$ and $|\vec{p} - \vec{q}|^4$ in the denominator, the averaging over the directions of $\vec{r}$ (or $\vec{p}$) would make $I^{(1)}_3$ identically zero.
In practice, this effect makes it unusually small compared to \( I_2 \) or \( I_1 \), which do not have such a factor. If we look at the higher loops, each additional loop brings in a factor of this type and that is the reason behind the successive diminishing of each of these integrals.

A numerical evaluation yields \( I^{(1)}_3 \simeq -(\frac{1}{15})^2 \) which makes the point that I wanted to make. The correction from the three loop graphs are down by an order of \( x_\eta \), and this effect persists to higher orders. This is the reason why the two loop calculation of the viscosity exponent gives an answer surprisingly close to the experimental value.

In closing we would like to mention that we have used a gaussian free energy in this calculation. There is a quartic part in the free energy which is responsible for the anomalous dimension \( \eta \). The correction coming from this is once again largest at the loop level when it first appears. Higher loop graphs involving the four point vertex give a much smaller contribution once again because of the frequent zeroes in the integrand at higher loops. The net result is that from one to two loops there is a substantial change in \( x_\eta \), but thereafter the contribution of the higher loops are ordered by \( x_\eta \) itself and with the integrals themselves quite small, the small value of \( x_\eta \) ensures that the higher loop effects are small.

References

[1] K.Kawasaki, Ann Phys (N.Y) 61, 1 (1970)
[2] T.Ohta and K.Kawasaki, Prog.Theor.Phys. 55, 1384 (1976)
[3] R.A.Ferrell , Phys.Rev.Lett. 24, 1169 (1970)
[4] E.D.Siggia, B.I.Halperin and P.C.Hohenberg , Phys.Rev. B13, 2110 (1976)
[5] See e.g. J.V.Sengers and J.M.H. Levelt Sengers, Ann.Rev.Phys.Chem. 37, 189 (1986)
[6] C.Bagnuls and C.Bervillier, Phys.Rev.B32, 7209 (1989) Also see V.Dohm, J.Low.Temp.Phys. 69 51 (1987) and Schloms and V.Dohm Euro.Phys.Lett. 3, 413 (1987)
[7] J.A.Lipa, D.R.Swanson, J.A.Nisen, T.C.P.Chui and U.E.Israelson, Phys.Rev.Lett 76, 944 (1996)
[8] R.F.Berg, M.R.Moldover and G.A.Zimmerli, Phys.Rev.Lett bf 82, 920 (1999)
[9] R.F.Berg, M.R.Moldover and G.A.Zimmerli, Phys.Rev. E60, 920 (1999)
[10] H.Hao, Ph.D thesis, University of Maryland (1991)
[11] R. Folk and G. Moser, Phys. Rev. E 57, 683 (1998) and see also Phys. Rev. E 57, 705 (1998)

[12] G. Flossmann, R. Folk and G. Moser, Phys. Rev. E 60, 779 (1999) and see also Int. J. Thermophysics 22, 89 (2001)

[13] J. K. Bhattacharjee and R. A. Ferrell, Phys. Lett. 27A, 290 (1980)

[14] J. K. Bhattacharjee and R. A. Ferrell, Phys. Rev. A 23, 1511 (1981)

[15] J. K. Bhattacharjee and R. A. Ferrell, Phys. Rev. A 27, 1544 (1983)

[16] P. Das and J. K. Bhattacharjee, Phys. Rev. E 63, 020202(R) (2001)
FIGURE – CAPTIONS

**Fig. 1**: One and two loop diagrams for the density relaxation rate. The solid lines stand for density fluctuation and the wavy lines for velocity fluctuation.

**Fig. 2**: One and two loop diagrams for the viscosity. The solid lines stand for density fluctuation and the wavy lines for velocity fluctuation.

**Fig. 3**: Three loop diagrams of the vertex correction variety for the viscosity showing all possible time orderings. Propagators appear with an arrow and correlators with an open circle.