ON THE VIRIAL THEOREM
FOR NONHOLONOMIC LAGRANGIAN SYSTEMS

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Abstract. A generalization of the virial theorem to nonholonomic Lagrangian systems is given. We will first establish the theorem in terms of Lagrange multipliers and later on in terms of the nonholonomic bracket.

1. Introduction. The virial theorem was established by Clausius in 1870 and since then the range of applicability has been increasing almost continuously till our days. It was stated by Clausius in the form The mean vis viva of the system is equal to its virial where the vis viva integral is the total kinetic energy of the system and the Latin word virias was used by Clausius to denote the scalar quantity represented in terms of the forces $F_i$ acting on the system as $\frac{1}{2}\langle\sum_i F_i \cdot r_i\rangle$, and it was shown to be one half of the averaged potential energy of the system.

The virial theorem has been applied in dynamical and thermodynamical systems, it can also be formulated to deal with relativistic (in the sense of special relativity) systems, it is applicable to systems with velocity dependent forces and viscous systems. Even though it provides less information than the equations of motion, it is simpler to apply and then it can provide information concerning systems whose complete analysis may defy description. For instance, in astronomy, the virial theorem finds applications in the theory of dust and gas of interstellar space as well as cosmological considerations of the universe as a whole and in other discussions concerning the stability of clusters of galaxies. For an excellent historical account one can see [7].

In Newtonian mechanics, for one particle of mass $m$ under the action of a force $F$ the virial function introduced by Clausius is $G(x, \dot{x}) = m \dot{x} \cdot \dot{x}$, and one can show using Newton's second law $F = m \ddot{x}$ that $dG/dt = m \dot{x} \cdot \dot{x} + x \cdot F$, and when integrating this expression between $t = 0$ and $t = \tau$, dividing by the total time interval $\tau$ and taking the limit of $\tau$ going to infinity we find that if the possible values of $G$ are bounded then $\langle\langle 2T(\dot{x}) + x \cdot F \rangle\rangle = 0$. In the particular case of a conservative force, $F = -\nabla V$, $\langle\langle 2T(\dot{x}) - x \cdot \nabla V \rangle\rangle = 0$. Moreover, when the potential $V$ is homogeneous of degree $k$, Euler’s theorem of homogeneous functions implies that $x \cdot \nabla V = kV$, and therefore, $\langle\langle 2T(\dot{x}) - kV(x) \rangle\rangle = 0$, i.e. if $E$ is the total energy,

$$\langle\langle T(\dot{x}) \rangle\rangle = \frac{kE}{k + 2}, \quad \langle\langle V(x) \rangle\rangle = \frac{2E}{k + 2}.$$
However, when non-holonomic constraints are present the forces are \( F = -\nabla V + F_c \) where \( F_c \) are the constraint forces due to the nonholonomic constraints. Hence a similar calculation shows that \( \langle 2 T(\dot{x}) - \mathbf{x} \cdot \nabla V \rangle + \langle F_c \cdot \mathbf{x} \rangle = 0 \). Assuming ideal constraints, so that the energy \( E \) is conserved, and in the particular case of a homogeneous potential of degree \( k \) we get
\[
\langle T(\dot{x}) \rangle = \frac{k}{k+2} E - \frac{1}{k+2} \langle F_c \cdot \mathbf{x} \rangle
\]
\[
\langle V(\mathbf{x}) \rangle = \frac{2}{k+2} E + \frac{1}{k+2} \langle F_c \cdot \mathbf{x} \rangle.
\]

The problem of extending the virial theorem to nonholonomic mechanics has been considered by Seeger [13] and more recently by Papastavridis [11, 12]. The purpose of this paper is to analyze and generalize such results by using the appropriate differential geometric tools of Geometric Mechanics. We will follow the results in [3, 5] in which the virial theorem is understood in terms of the temporal mean value of the Poisson bracket of the energy and a virial function. The extension of such results to the framework of mechanics in Lie algebroids was developed in [4].

We will first use the standard description of the nonholonomic systems in terms of Lagrange multipliers, using D’Alembert principle, and later on we will pose the problem in the more modern language of the distributional approach in which the Lagrange multipliers are eliminated by considering the appropriate manifolds. This will allow us to express the theorem in terms of the nonholonomic bracket in a similar way to the holonomic case. We will finally give a description in terms of quasivelocities.

2. Nonholonomic Lagrangian systems.} We consider an \( n \)-dimensional manifold \( Q \), and its tangent bundle \( TQ \to Q \). We also consider a set of linear constraints which defines a vector subbundle \( D \subset TQ \) of rank \( r \), and which is called the constraint submanifold. The admissible velocities are the elements of \( D \), and a curve is said to be admissible if it takes values in \( D \). From the annihilator \( D^\circ \subset T^*Q \) of \( D \), i.e. the set of linear 1-forms vanishing on the elements of \( D \), we construct the set \( \widehat{D}^\circ \subset T^*(TQ) \) defined by \( \widehat{D}^\circ = \{ \alpha \circ T\tau_Q \in T^*(TQ) \mid \alpha \in D^\circ \} \). It is a vector bundle over \( TQ \), whose fibre at a point \( v \in TQ \), such that \( \tau_Q(v) = q \), is more explicitly described as
\[
\widehat{D}^\circ_v = \{ \lambda_v \in T^*_q(TQ) \mid \text{ there exists } \alpha_q \in D^\circ_q \text{ such that } \lambda_v = \alpha_q \circ T_v\tau_Q \}.
\]

Given a Lagrangian function \( L \in C^\infty(TQ) \), we consider the nonholonomic system defined by the Lagrangian \( L \) and the linear constraints given by \( D \). The evolution of the nonholonomic system is determined by the Lagrange–d’Alembert principle, which states that the dynamics of the system is given by the integral curves (with initial condition in \( D \)) of the vector fields \( \Gamma \in \mathfrak{x}(TQ) \) tangent to \( D \) satisfying the second-order condition and the Lagrange–d’Alembert equation (see for instance [10])
\[
(i_T\omega_L - dE_L)|_D \in \text{Sec}(\widehat{D}^\circ).
\]

In this expression \( \omega_L \) is the Cartan 2-form associated with \( L \), defined by \( \omega_L = -d\theta_L \) with \( \theta_L = S^*(dL) \) the Cartan 1-form, and \( S \) being the vertical endomorphism. This equation above means that at every point of \( D \) the 1-form \( i_T\omega_L - dE_L \) takes value in the codistribution \( \widehat{D}^\circ \). This value is the reaction force exerted by nonholonomic constraints, the constraint forces.

From now on we assume that \( L \) is a regular Lagrangian, which means one of the three equivalent conditions: i) the fibre derivative (Legendre transformation) \( F_L : TQ \to T^*Q \) is a local diffeomorphism, ii) the Lagrange 2-form \( \omega_L \) is a symplectic form, iii) its fibre Hessian \( F^2L = G^L : TQ \to T^*Q \otimes T^*Q \) is everywhere a nondegenerate bilinear form. Given \( u, v, w \in T_qQ \), the fibre Hessian of the Lagrangian can also be expressed as \( G^L_u(v, w) = \omega_L(\tilde{v}, \tilde{w}) \),
that where \(\gamma\) is a virial function on \(TQ\), we have

\[
\gamma(t) \text{ be a solution of the constrained dynamics, } \Gamma \circ \gamma = \frac{d\gamma}{dt}.
\]

Then, \(\gamma^* L_{\Gamma} G = \frac{d}{dt} \gamma^* G\). From the definition of the time average we have

\[
\langle \langle L_{\Gamma} G \rangle \rangle = \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \frac{d}{dt} \gamma^* G \, dt = \lim_{\tau \to +\infty} \frac{\gamma^* G(\tau) - \gamma^* G(0)}{\tau}.
\]

Since \(\gamma^* G\) is bounded we conclude that the limit is zero.

As an immediate consequence we have the following result.

**Theorem 2.1.** Under the conditions stated above, if \(G\) is a virial function on \(TQ\), we have that

\[
\langle \langle \Gamma_L(G) + \lambda_A Z^A(G) \rangle \rangle = 0,
\]
In local coordinates, taking in account (2.5) we have
\[
\left\langle v_i \frac{\partial G}{\partial x^i} + W^{ij} \left( \frac{\partial L}{\partial x^j} - v^k \frac{\partial^2 L}{\partial x^k \partial v^j} + \lambda_A \omega^A_j \right) \frac{\partial G}{\partial v^i} \right\rangle = 0 \quad (2.8)
\]

In the applications, the virial function \( G \) is generally chosen as the Hamiltonian function associated to a vector field which generates a 1-parameter group of transformation of interest (for instance the dilation group in the case \( Q = \mathbb{R}^n \)) and we pretend to write the consequences of the virial theorem directly in terms of such a vector field.

**Theorem 2.2.** Let \( X^c \) be the complete lift of a vector field \( X \) on \( Q \), and \( G = \langle \theta_L, X^c \rangle \) be the virial function. Then,
\[
\langle X^c(L) + \lambda_A \omega^A(X) \rangle = 0. \quad (2.9)
\]

**Proof.** Since the solution \( \Gamma = \Gamma_L + \lambda_A Z^A \) is a sode vector field we can rewrite Lagrange-D’Alembert equations in the form
\[
\mathcal{L}_T \theta_L = dL + \lambda_A \omega^A,
\]
where \( \mathcal{L}_T \) denotes Lie derivative. Contracting with \( X^c \) we obtain:
\[
\Gamma(G) = \mathcal{L}_T(\langle \theta_L, X^c \rangle) = \langle \mathcal{L}_T \theta_L, X^c \rangle + \langle \theta_L, \mathcal{L}_T X^c \rangle = dL(X^c) + \lambda_A \omega^A(X),
\]
where we have taken into account that \( \mathcal{L}_T X^c \) is vertical and \( \theta_L \) is semibasic. It follows from Theorem 2.1 that \( \langle \Gamma(G) \rangle = 0 \), i.e. \( \langle X^c(L) + \lambda_A \omega^A(X) \rangle \) vanishes.

We remark that, if \( X \) is a section of the vector bundle \( D \) then the above Virial Theorem reduces to the simpler form \( \langle X^c(L) \rangle = 0 \).

**Example: nonholonomic harmonic oscillator.** To illustrate the theory we will consider the nonholonomic dynamical system known as the nonholonomic harmonic oscillator.

Consider an isotropic harmonic oscillator moving in \( Q = \mathbb{R}^3 \) with coordinates \((x_1, x_2, x_3)\). The Lagrangian function is
\[
L = T - V = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).
\]

We have \( \omega_L = dx_1 \wedge dx_2 + dx_3 \wedge d\dot{x}_2 + dx_3 \wedge d\dot{x}_3 \) and \( dE_L = \dot{x}_1 dx_1 + \dot{x}_2 dx_2 + \dot{x}_3 dx_3 + x_1 dx_1 + x_2 dx_2 + x_3 dx_3 \), so the unconstrained dynamics is the well-known dynamics described by the vector field
\[
\Gamma_L = \dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{x}_3 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial \dot{x}_1} - x_2 \frac{\partial}{\partial \dot{x}_2} - x_3 \frac{\partial}{\partial \dot{x}_3}.
\]

We constraint the motion of the particle by introducing the nonholonomic constraint
\[
\phi = \dot{x}_3 - x_2 \dot{x}_1 = 0.
\]
The constraint submanifold is given by
\[
D = \{(x_1, x_2, x_3; \dot{x}_1, \dot{x}_2, \dot{x}_3) \in TQ \mid \dot{x}_3 = x_2 \dot{x}_1 \}.
\]
Applying Lagrange-D’Alembert’s principle we find
\[
\Gamma = \Gamma_L + \dot{x}_1 \dot{x}_2 - x_1 x_2 + x_3 \left( \frac{\partial}{\partial \dot{x}_3} - x_2 \frac{\partial}{\partial x_1} \right).
\]

We consider the dilation vector field
\[
X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}
\]
and we apply the virial theorem. The virial function is \( G = x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3 \) and we get
\[
\left\langle 2T - 2V + \frac{\dot{x}_1 \dot{x}_2}{1 + x_2^2} (x_3 - x_2) \right\rangle = 0,
\]
and taking into account conservation of the energy we finally get
\[ \langle T \rangle = \frac{E}{2} + \frac{1}{4} \left( \frac{x_1 x_2 - x_1 x_2 + x_3}{1 + x_2^2} (x_3 - x_1 x_2) \right), \]
\[ \langle V \rangle = \frac{E}{2} + \frac{1}{4} \left( \frac{x_1 x_2 - x_1 x_2 + x_3}{1 + x_2^2} (x_3 - x_1 x_2) \right). \]

3. The distributional approach. In the above treatment it is not explicit that the different objects have to be defined in the constraint distribution. Moreover, in the unconstrained case, the virial theorem, as well as many other interesting results, are stated in terms of the Poisson bracket associated to the symplectic form. In what follows we will study how the theory can be developed only in terms of objects intrinsically defined in \( D \) and in terms of the nonholonomic bracket.

**Regularity.** As we said above, the nonholonomic system \((L, D)\) is said to be regular if there is a unique solution to Lagrange–d’Alembert equation. In the present context ‘uniqueness’ must be understood as follows: two SODE solutions are considered equal if they coincide when restricted to \( D \).

There are several equivalent ways to ensure regularity of the constrained system. We define the subbundle \( T^D D \subset TD \to D \) by
\[ T^D D = \{ V \in TD \mid T\tau_Q(V) \in D \}. \]  
We also consider the restriction \( G_L^{L,D} \) of the fibre Hessian \( G_L \) to the distribution \( D \).

**Theorem 3.1.** The following properties are equivalent:
1. The constrained Lagrangian system \((L, D)\) is regular,
2. \( \text{Ker} G_L^{L,D} = \{ 0 \} \),
3. \( TT_Q|_D = T^D D \oplus (T^D D)^\perp \).

where \((T^D D)^\perp\) denotes the orthogonal complement of \( T^D D \) with respect to the symplectic form \( \omega_L \).

For the proof, see for instance [8] and references there in.

**Remark.** In the case of a constrained mechanical system \( L = T_y - V \), the tensor \( G_L \) is given by \( G^L_u(b, c) = g_{\tau(a)}(b, c) \), so that it is positive definite at every point. Thus the restriction to any subbundle \( D \) is also positive definite and hence regular. Thus, nonholonomic mechanical systems are always regular.

The manifold \( T^D D \) has a double vector bundle structure over \( D \) with the projections \( \tau_{\tau_Q}|_{\tau_{\tau_Q}D} \) and \( \tau = T\tau_Q|_{\tau_{\tau_Q}D} \). The rank of \( T^D D \) is \( 2 \text{rank} D \). By a SODE in \( D \) we mean a section \( \Gamma \) of the vector bundle \( T^D D \) such that \( T\tau_Q(\Gamma(v)) = v \) for every \( v \in D \). It follows that a SODE in \( D \) can be extended (non uniquely) to a SODE on \( TQ \) which is tangent to \( D \) and conversely, a SODE vector field on \( TQ \) which is tangent to \( D \) restricts to a SODE in \( D \).

**Projection to \( T^D D \).** As a consequence of the above theorem we get that if the constrained system \((L, D)\) is regular we can obtain the constrained dynamics by projection of the free dynamics according to the decomposition given in item 3. Let us denote by \( \bar{P} \) and \( \bar{Q} \) the complementary projectors defined by the decomposition \( T_a TQ = T^D a \oplus (T^D a)^\perp \) for \( a \in D \), that is,
\[ \bar{P}_a : T_a TQ \to T^D a \quad \text{and} \quad \bar{Q}_a : T_a TQ \to (T^D a)^\perp, \quad \text{for all} \ a \in D. \]

Then, we have the following result.

**Theorem 3.2.** Let \((L, D)\) be a regular constrained Lagrangian system and let \( \Gamma_L \) be the solution of the free dynamics, i.e., \( i_{\Gamma_L} \omega_L = dE_L \). Then the solution of the constrained dynamics is the SODE \( \Gamma \) obtained by projection \( \Gamma = \bar{P}(\Gamma_L|_D) \).
For the proof, see [8] and references there in.

The distributional approach. A second consequence of Theorem 3.1 is that we can write Lagrange-d’Alembert equations as symplectic equations entirely in terms of objects in the manifold \( T^D \). Indeed, since \((L, D)\) is regular, from item (3) we have that \( T^D \) is a symplectic subbundle of \((TTQ, \omega_L)\). Hence the restriction \( \omega^{L, D} \) of \( \omega_L \) to \( T^D \) is a symplectic form on the vector bundle \( T^D \) (i.e., it is a regular skew-symmetric bilinear form). We denote by \( dE_L \) the restriction of \( dE \) to \( T^D \). Then, taking the restriction of Lagrange-d’Alembert equations to \( T^D \), we get the following equation

\[
i_{\Gamma} \omega^{L, D} = dE_L,
\]

which uniquely determines the section \( \Gamma \). Indeed, the unique solution \( \Gamma \) of the above equations is the solution of Lagrange-d’Alembert equations: if we denote by \( \lambda \) the constraint force, we have for every \( u \in T_{\Gamma} \) that

\[
\omega_L(\Gamma(a), u) - \langle dE_L(a), u \rangle = \langle \lambda(a), T\tau(u) \rangle = 0,
\]

where we have taken into account that \( T\tau(u) \in D \) and \( \lambda(a) \in D^\circ \).

This approach, the so-called distributional approach, was initiated by Bocharov and Vinogradov [15] and further developed by Śniatycki and coworkers [1, 9, 14].

The nonholonomic bracket. The symplectic section \( \omega^{L, D} \) allows us to define an almost-Poisson bracket on \( D \) which is known as the nonholonomic bracket. We recall that an almost-Poisson bracket on a manifold \( P \) is a bracket \{\cdot, \cdot\} of functions on \( P \) which is \( \mathbb{R} \)-bilinear, skew-symmetric, a derivation in each argument with respect to the usual product of functions but it does not necessarily satisfy the Jacobi identity.

For every function \( f \in C^\infty(D) \) we consider the restriction \( \tilde{d}f \) to \( T^D \) of its differential \( df \). Since \( \omega \) is regular, we have a unique section \( \tilde{X}_f \in \text{Sec}(T^D) \) such that \( \tilde{X}_f \) is another extension of \( f \) to \( TQ \).

Definition 3.3. The nonholonomic bracket of two functions \( f, g \in C^\infty(D) \) is the function \( \{f, g\}_{nh} \in C^\infty(D) \) given by

\[
\{f, g\}_{nh} = \omega^{L, D}(\tilde{X}_f, \tilde{X}_g).
\]

Alternatively, the nonholonomic bracket can be defined as follows. We first notice that if \( \tilde{f} \in C^\infty(TQ) \) is an extension to \( TQ \) of \( f \) then \( \tilde{X}_f = P(X_f|D) \). Let \( f, g \) be two smooth functions on \( D \) and take arbitrary extensions \( \tilde{f}, \tilde{g} \) to \( TQ \). Let \( X_f \) and \( X_g \) the associated Hamiltonian vector fields

\[
i_{X_f} \omega_L = \tilde{d}f \quad \text{and} \quad i_{X_g} \omega_L = \tilde{d}g.
\]

Then the nonholonomic bracket of \( f \) and \( g \) is

\[
\{f, g\}_{nh} = \omega_L(P(X_f|D), P(X_g|D)).
\]

Indeed, the result follows by noticing that if \( \tilde{f} \) is another extension of \( f \), then \( (X_f - X_f)|_D \) is a section of \((T^D)^\perp \), and therefore the result does not depend on the choice of extensions.

In what follows, the nonholonomic bracket of two functions on \( TQ \) should be understood as the nonholonomic bracket of their restrictions to \( D \).

As a consequence of the above we have the following result, which is fundamental for our purposes.

Theorem 3.4. If \( f, g \in C^\infty(D) \) then

\[
\{f, g\}_{nh} = \tilde{X}_g f = -\tilde{X}_f g.
\]

Moreover, for any function \( f \in C^\infty(D) \) we have

\[
\tilde{f} = \{f, E_L\}_{nh}.
\]
Theorem 3.5. For any virial function $G$ we have that

$$\langle\{G, E_L\}_{nh}\rangle = 0.$$  \hspace{1cm} (3.7)

In this way the nonholonomic virial theorem can be expressed in a similar to the holonomic virial theorem with the only difference that the bracket is the nonholonomic bracket instead of the Poisson bracket.

In particular, if $X$ is a section of $\mathcal{D}$, i.e. a vector field on $Q$ taking values in $\mathcal{D}$, then for $G = \langle \theta_L, X^c \rangle$ a simple calculation shows that $\{G, E_L\}_{nh} = \mathcal{L}_G - \mathcal{L}_X \cdot L|_\mathcal{D}$, so that $\langle\mathcal{L}_X \cdot L\rangle = 0$.

4. Quasivelocities. In nonholonomic mechanics the use of quasivelocities is highly convenient \cite{6}. Consider a local basis \{\mathcal{X}_\alpha\} of vector fields spanning the distribution $\mathcal{D}$ and complete with a family of vector fields \{\mathcal{X}_A\} to a local basis \{\mathcal{X}_\alpha, \mathcal{X}_A\} of $\mathcal{X}(Q)$. Taking a local coordinate system $(x^i)$ on the manifold $Q$ we have that

$$\mathcal{X}_\alpha = \rho^i_\alpha \partial / \partial x^i,$$  \hspace{1cm} (4.1)

for some local functions $\rho^i_\alpha \in C^\infty(Q)$. The brackets of the vector fields in such a basis are $[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\alpha_{\beta\gamma} \mathcal{X}_\gamma$, where the functions $C^\alpha_{\beta\gamma} \in C^\infty(Q)$ are the so called Hammel's transpositional symbols, which are determined by

$$\rho^i_\alpha \partial \rho^k_\beta / \partial x^i - \rho^i_\beta \partial \rho^k_\alpha / \partial x^i = \rho^k_\gamma C^\alpha_{\beta\gamma}.$$  \hspace{1cm} (4.2)

Associated to this choice of coordinates in $Q$ and the local basis of vector fields in $Q$ there is a coordinate system $(x^i, y^\alpha)$ in $TTQ$ where $y^\alpha$ are the coordinates of a vector in the basis \{\mathcal{X}_\alpha\}. For vector fields on $TTQ$ we have a local basis \{\mathcal{X}_\alpha, \mathcal{V}_\alpha\} given by

$$\mathcal{X}_\alpha = \rho^i_\alpha \partial / \partial x^i, \hspace{1cm} \mathcal{V}_\alpha = \partial / \partial y^\alpha.$$  \hspace{1cm} (4.3)

Notice that we have denoted with the same symbol $\mathcal{X}_\alpha$ the local vector fields on $Q$ and on $TTQ$ which have the same coordinate expression. The dual basis will be denoted by \{$\chi^\alpha, \upsilon^\alpha$\}, and it is related to the differential of the coordinates by means of $dx^i = \rho^i_\alpha \chi^\alpha$, $dy^\alpha = \upsilon^\alpha$. Notice that $\chi^A = \omega^A$ are the constraint 1-forms that we used in section 2.

The local expressions for the Lagrangian energy and the Cartan 2-form are \cite{4, 6}

$$E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L$$  \hspace{1cm} (4.4)

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \chi^\alpha \wedge \upsilon^\beta + \frac{1}{2} \left( \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \rho^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta} \rho^i_\alpha + \frac{\partial L}{\partial y^\gamma} C^\alpha_{\beta\gamma} \right) \chi^\alpha \wedge \chi^\beta.$$  \hspace{1cm} (4.5)

In the coordinates $(x^i, y^\alpha) = (x^i, y^a, y^A)$ on $TTQ$ the equations for $\mathcal{D}$, i.e. the constraints, are simply $y^A = 0$, or in other words, $(x^i, y^a)$ are coordinates for $\mathcal{D}$. In what respect to the decomposition $TTQ|_\mathcal{D} = T^\mathcal{D} \mathcal{D} \oplus (T^\mathcal{D})^\perp$ we have that \{\mathcal{X}_\alpha, \mathcal{V}_\alpha\} is a local basis of $T^\mathcal{D}$,
and a simple calculation shows that a local basis \( \{Y_A, Z_A\} \) of sections of \((\mathcal{T}^D)\perp\) is given by
\[
Z_A = Y_A - Q_A^\alpha Y_\alpha, \quad Y_A = X_A - Q_A^\alpha X_\alpha + C^{\alpha\nu}(M_{\alpha\beta} - M_{\alpha\beta}Q_{\alpha}^\nu)Y_\nu,
\]
where \( Q_A^\alpha = W_{A\beta}C^{\alpha\beta} \) and \( C^{\alpha\beta} \) are the components of the inverse of the matrix \( C_{ab} = \frac{\partial^2 L}{\partial y^a \partial y^b}(x^i, y^c, y^A = 0) \), and \( M_{\alpha\beta} = \omega_L(X_\alpha, X_\beta) \). Therefore the expression of the projector onto \((\mathcal{T}^D)\perp\) is \( \mathcal{Q} = Z_A \otimes V^A + Y_A \otimes X^A \).

For the constrained dynamics, we look for a section \( \Gamma \) of \( \mathcal{T}^D \), so that it is of the form \( \Gamma = g^aX_a + f^aY_a \). Assuming a regular constrained system, from the local expression (4.5) of the Cartan 2-form and the local expression (4.4) of the energy function, we get that \( g^a = y^a \) and the functions \( f^a \) are the solution of the linear equations
\[
\frac{\partial^2 L}{\partial y^b \partial y^a}f^b + \frac{\partial^2 L}{\partial x^i \partial y^a}\rho_{ib}y^b + \frac{\partial L}{\partial y^c}C_{ab}^c y^b - \rho_{ia}\frac{\partial L}{\partial x^i} = 0,
\]
where all the partial derivatives must be taken at \( y^A = 0 \).

The differential equations for the integral curves of \( \Gamma \), i.e. Lagrange-d’Alembert differential equations, are in quasivelocities
\[
\dot{x}^i = \rho_{ia}y^a, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) + \frac{\partial L}{\partial y^c}C_{ab}^c y^b - \rho_{ia}\frac{\partial L}{\partial x^i} = 0, \quad y^A = 0.
\]

Finally, the contraction of \( i_1\omega_L - dE_L \) with \( X_A \) just gives the value of the Lagrange multipliers \( \lambda_A = \langle i_1\omega_L - dE_L, X_A \rangle|_{y^A=0} \), i.e. the components of the constraint forces \( \lambda = \lambda_A X^A \).

In what respect to the virial theorem, if \( X \) is a section of \( \mathcal{D} \) with local expression \( X = X^aX_a \), then
\[
\left\langle \rho_a^a \frac{\partial L}{\partial x^i} + \left[ \rho_b^a \frac{\partial X^a}{\partial x^j} + C_{ab}^c X^d \right] y^b \frac{\partial L}{\partial y^c} + C_{ab}^c X^d y^b \frac{\partial L}{\partial y^A} \right\rangle = 0.
\]
where all partial derivatives must be taken at \( y^A = 0 \).

**Example: the nonholonomic harmonic oscillator.** Consider again an isotropic harmonic oscillator in \( Q = \mathbb{R}^3 \), with Lagrangian function
\[
L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2),
\]
subjected to the nonholonomic constraint \( \phi = \dot{x}_3 - x_2\dot{x}_1 = 0 \).

As a basis \( \{X_1, X_2\} \) of sections of \( \mathcal{D} \) we can take,
\[
X_1 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2},
\]
which we can complete with the vector field \( X_3 = \frac{\partial}{\partial x_3} \). The only non-vanishing bracket is \([X_1, X_2] = -X_3 \), so that \( C_{21}^3 = -C_{12}^3 = 1 \), and for other indices \( C_{\alpha\beta}^\gamma = 0 \).

The associated quasivelocities are related to the velocities by
\[
(y_1, y_2, y_3) = (\dot{x}_1, \dot{x}_2, \dot{x}_3 - x_2\dot{x}_1),
\]
and substituting in the Lagrangian we get
\[
L(x_1, x_2, x_3, y_1, y_2, y_3) = \frac{1}{2}(y_1^2 + y_2^2 + (y_3 + x_2y_1)^2) - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).
\]
Taking the vector field \( X = x_1X_1 + x_2X_2 \) so that
\[
X^c = x_1X_1 + x_2X_2 + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + (y_2x_1 - x_2y_1)\frac{\partial}{\partial y_3},
\]
and applying the virial theorem we get $\langle\langle X^c(L) \rangle\rangle = 0$ which, after taking into account that $y_3 = 0$, reads

$$\langle\langle 2T - X(V) + x_1x_2y_1y_2 \rangle\rangle = 0,$$

or equivalently

$$\langle\langle 2T - 2V + x_1x_2y_1y_2 + x_3(x_3 - x_1x_2) \rangle\rangle = 0.$$

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