Abelian Sandpile Model: a Conformal Field Theory Point of View

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March 23, 2022

Abstract

In this paper we derive the scaling fields in \( c = -2 \) conformal field theory associated with weakly allowed clusters in abelian sandpile model and show a direct relation between the two models.

Keywords: conformal field theory, self organized criticality, Sandpile model

1 Introduction

There exists some phenomena which naturally show power law behavior, that is, without fine tuning any parameter, the system shows behavior similar to the critical point, in contrast with the usual critical phenomena, where you should fine tune an external parameter like temperature to arrive at the critical point. These kind of phenomena are said to have self organized criticality [1]. Sandpile [1, 2], surface growth [3] and river networks [4] are a few examples of such phenomena.

The concept of SOC was first introduced by Bak, Tang and Wiesenfeld [1]. While many other models have been found after that, still the abelian sandpile model (ASM) is one of the simplest and most studied models. Despite its simplicity, it shows all the features that a self organized critical phenomena ought to present, so huge amount of work has been done on this model [5-18]. Many analytic results have been derived, for example the probability of different heights and many specific clusters are calculated explicitly [6, 17]. Also the relation of this model to other known models has been discussed. First of all, there is a connection between ASM and spanning trees [7]. Then other models such as dense polymers, Scheidegger’s model of river networks and \( q \to 0 \) limit of \( q \)-state Potts models arise [19].

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These statistical models, show conformal symmetry while they are at their critical point. So it is natural to look for a conformal field theory (CFT) which corresponds to these systems. The CFT associated with these models is suggested to be $c = -2$, which belongs to a specific group of CFT’s, known as logarithmic conformal field theories (LCFT’s). In LCFT’s, where correlation functions may have logarithmic terms in contrast with the ordinary CFT’s, there exist pairs of fields with the same conformal weight, which mix under conformal transformations\cite{20, 21}.

Maheiu and Ruelle \cite{10} have found a way to relate ASM to $c = -2$ theory. They have found some operators in the LCFT model which correspond to different clusters in ASM. But the correspondence is shown only through correlation function and it is not clear why one should take these operator. In this paper we’ll address this question and find a direct way to derive the operators from the action of $c = -2$, and hence connect ASM to $c = -2$ directly. The paper is organized as follows: in section 2 we will briefly introduce the model and some analytical results including the result obtained by \cite{10}. In the 3rd section we will derive the previously mentioned operators directly from the action of $c = -2$ model.

2 Abelian Sandpile Model

The Abelian Sandpile Model is defined on a square lattice of the size $L \times M$. To each site $i$, a height variable $h_i$ is assigned. Height of sand at each site can take one of the values form the set $\{1, 2, 3, 4\}$. So the total number of different allowed configuration is equal to $4^{L M}$. The dynamic of the system is defined as follows: at each step a random site $i$ is selected and a grain of sand is added to that site. If the new height of sand becomes more than four, the column of sand is called to be unstable and topples, that is, four grains will leave the site and each of them will be added to one of the neighbors. So the total number of sands is conserved during the toppling process except at the boundaries where one or more grains leave the system.

The toppling process can be stated in another way, which will be more appropriate. If the site $i$ becomes unstable, $h_j$ will be decreased by amount of $\Delta_{ij}$, that is $h_j \rightarrow h_j - \Delta_{ij}$ where

$$\Delta_{ij} = \begin{cases} 
4, & i = j; \\
-1, & i, j \text{ are neighbors}; \\
0, & \text{otherwise}. 
\end{cases}$$ \hspace{1cm} (1)

The matrix $\Delta_{ij}$ is called toppling matrix.

After a while the system reaches a steady state, in which it shows SOC. It has been shown that in this state, the number of different configurations the system accepts is $\det \Delta$ and the probability of all of them are the same. These configurations are named recurrent configurations in contrast with the transient ones which can only appear in the first steps of evolution, where the system has not yet reached the steady state. Note that the number of recurrent configurations is fairly smaller than the total number of configuration as $\det \Delta \sim 3.2^{L M}$. Determining whether a given configuration is recurrent or transient is a relatively hard question, though there exist tests to answer this question. The first observation is that some forbidden subconfigurations exist, that is if in a given configuration you find one
of these subconfigurations then it could not be recurrent. The simplest example of these subconfigurations is two neighboring height one sites.

One of the most important analytic results derived so far, is the probability of finding some specific clusters in the system while it is in the steady state. Dhar and Majumdar \[6\] have found a subtle way to calculate such probabilities for the clusters named weakly allowed clusters (WAC’s). WAC’s are those clusters which by decreasing the height of any of its sites by one, it becomes a forbidden subconfiguration. The simplest one is a single site with height equal to one. They have proved that the number of recurrent configuration with a particular WAC is equal to the number of recurrent configurations in another sandpile model with a modified toppling rule. In fact there are several different ways to modify the toppling rule. One of them is to remove all the connections of the WAC to the rest of the system but one as shown in figure 1 for two simplest clusters known as $S_0$ and $S_1$ respectively\[^1\]. As the grains of sand are not allowed to flow through the disconnected bonds, both the condition for instability and the toppling matrix should be modified. The new toppling matrix is given by $\Delta' = \Delta + B$, where for each disconnected bond running from site $i$ to site $j$, $B_{ij} = B_{ji} = 1$ and if $n$ bonds have been cut from site $i$ we should set $B_{ii} = -n$. As it is clear, the matrix $B$ is nonzero only in the vicinity of the cluster we are dealing with. So, $B$ is somehow local.

There are some other ways to modify the lattice and the toppling matrix. One of them which is more appropriate for our future uses, is to disconnect the cluster completely from other sites of lattices. This scheme is better because it preserves the symmetries of the clusters manifestly. The rules for modification of the toppling matrix is more or less the same as before, the only thing you should take into account is that $B_{ii}$ should be set to $-3$ for the sites in the cluster.

Using these methods, the probability of finding different WAC’s has been found analyti-

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\[^1\]If you are considering several disconnected pieces of WAC’s, you should do the same for each piece.
cally [10]. To do this, one should compute both \( \det \Delta \) and \( \det \Delta' \). Then the probability of finding the cluster \( S \) would be

\[
P(S) = \frac{\det \Delta'}{\det \Delta} = \det(1 + G B_S),
\]

where \( G = \Delta^{-1} \), and \( \Delta'_S \) and \( B_S \) are the matrices associated with the cluster \( S \). As an example, the probability of finding a site with height one \( (S_0) \) is found to be \( P(1) = 2(1 - 2/(2\pi))/\pi^2 \approx 0.074 \). These analytic results have been confirmed by different simulations [22, 23, 24] though it seems there are some small disagreements [25]. Additionally, by taking two clusters of this kind, placing far away from each other, one is able to compute correlations between these clusters. Taking two clusters to be \( S_0 \), separating with the distance \( r \), one finds that the dominant term in correlation functions at large distances, is proportional to \( r^{-4} \) and is independent of angular distance of the two sites. For other clusters, the radial dependance is again like \( r^{-4} \), but you’ll have angular dependence.

On the other hand, one is interested in the continuum limit of the problem, that is, finding a field theory that describes the same model. As ASM is shown to be equivalent to spanning trees and hence to \( q \to 0 \) of \( q \)-state Potts model, the most appropriate filed theory seems to be \( c = -2 \) conformal field theory. Different features of the \( c = -2 \) model is described in many articles [26]. It has the simple Gaussian action

\[
S = \frac{1}{\pi} \int \partial \theta \bar{\partial} \bar{\theta},
\]

where \( \theta \) and \( \bar{\theta} \) are complex grassman variables. Comparing correlation functions, Mahieu and Ruelle [10] have shown that one can assign a suitable operator in \( c = -2 \) theory to each WAC in sandpile model. A table of operators versus clusters for some simple WAC’s can be found in [10]. Actually, they have extended ASM to a model with dissipation in the following way: the threshold beyond which the column of sand becomes unstable is increased to \( x (x > 4) \) and during toppling process \( x \) sands are removed from the unstable site, but only four of them is added to the neighbors. So, \( x - 4 \) grains of sand are dissipated during each toppling. As a consequence, the action of \( c = -2 \) theory should be modified to

\[
S = \frac{1}{\pi} \int \left( \partial \theta \bar{\partial} \bar{\theta} + \frac{m^2}{4} \theta \bar{\theta} \right),
\]

where \( m^2 \) is equal to \( x - 4 \).

The operators assigned to different WAC’s are very similar, in fact all of them are of the form

\[
\phi_S(z) = - \left\{ A : \partial \theta \bar{\partial} \bar{\theta} + \bar{\partial} \bar{\partial} \theta : + B_1 : \partial \theta \bar{\partial} \bar{\theta} + \bar{\partial} \bar{\partial} \theta : + i B_2 : \partial \theta \bar{\partial} \bar{\theta} - \bar{\partial} \bar{\partial} \theta : + C P(S) \frac{m^2}{2\pi} ; \theta \bar{\theta} : \right\}.
\]

with \( A, B_1, B_2 \) and \( C \) to be determined by comparing with the results of Dhar and Majumdar method. Also \( P(S) \) is the probability of finding the cluster \( S \). The parameter \( C \) is present only in the massive theory and due to Mahieu and Ruelle it is very striking that \( C \) takes only integer values and is equal to number of sites of the cluster \( S \).
In the next section, we will show a more direct way to find these operators. Also the connection between ASM and \( c = -2 \) model will be established more precisely.

## 3 Connection Between ASM and \( c = -2 \) Theory

The first step to connect the two theories could be considering their action and partition functions. The \( c = -2 \) action is given by equation (3), but in the case of ASM we have not yet introduce the action. This can be done by considering that in the steady state there are \( \det \Delta \) different configurations with the same probability. So, the partition function of the model is just equal to \( \det \Delta \). One can write the determinant of an arbitrary matrix in terms of Gaussian integration on grassman variables, namely:

\[
\det \Delta = \int \prod d\theta_i \prod d\bar{\theta}_j \exp \left( \sum \theta_i \Delta_{ij} \bar{\theta}_j \right).
\]  

(6)

As the matrix \( \Delta \) is the discrete version of Laplacian operator, it is clear that the above action leads to the action (3) in continuum limit. This was noted before by Ivashkevich [27].

Now we turn on to scaling fields assigned to WAC’s. The starting point is equation (2), where the probability of finding the cluster \( S \) is given. Again, the determinant of \( \Delta' \) can be written in terms of Gaussian integrals, so the probability \( P(S) \) turns out to be

\[
P(S) = \frac{\det \Delta'_S}{\det \Delta} = \frac{\int d\theta_i d\bar{\theta}_j \exp \left( \sum \theta_i \Delta'_{ij} \bar{\theta}_j \right)}{\int d\theta_i d\bar{\theta}_j \exp \left( \sum \theta_i \Delta_{ij} \bar{\theta}_j \right)} = \frac{\int d\theta_i d\bar{\theta}_j \exp \left( \sum \theta_i \Delta_{ij} \bar{\theta}_j + \theta_i B_{ij} \bar{\theta}_j \right)}{\int d\theta_i d\bar{\theta}_j \exp \left( \sum \theta_i \Delta_{ij} \bar{\theta}_j \right)}.
\]  

(7)

Looking carefully at the above relation, one is able to derive the proper discrete field for the cluster \( S \):

\[
\varphi_S = \exp \left( \sum \theta_i B_{ij} \bar{\theta}_j \right).
\]  

(8)

The next step is to translate this field to continuum language. This translation should be done very carefully. Using the definition of the matrix \( B \), the continuum version of the term inside the exponential is calculated easily. For the clusters shown in figure 1, we have

\[
\sum \theta_i B_{ij} \bar{\theta}_j |_{S=S_0} \propto \partial \theta \bar{\partial} \bar{\theta} + \bar{\partial} \theta \partial \bar{\theta},
\]

\[
\sum \theta_i B_{ij} \bar{\theta}_j |_{S=S_1} \propto 3(\partial \theta \bar{\partial} \bar{\theta} + \bar{\partial} \theta \partial \bar{\theta}) - (\partial \theta \bar{\partial} \bar{\theta} + \bar{\partial} \theta \partial \bar{\theta}) + \ldots.
\]  

(9)

The exponential can be expanded in powers of \( \theta \) and \( \bar{\theta} \), but since these variables are grassmans, the series ends at quadratic term. So, the operators we are interested in, are just the above fields, apart from an unimportant unity term in the series.

The operator derived here for \( S_0 \) is in complete agreement with the one derived in [10], you should just put the probability of finding height one to complete the proportionality. The field obtained for the cluster \( S_1 \), however, is a little bit different, but not quite far away. The corresponding operator in [10] is

\[
\varphi_{S_1} \simeq 0.020143(\partial \theta \bar{\partial} \bar{\theta} + \bar{\partial} \theta \partial \bar{\theta}) - 0.006190(\partial \theta \bar{\partial} \bar{\theta} + \bar{\partial} \theta \partial \bar{\theta}),
\]  

(10)
and the ratio of the two coefficients is about 3.25. Finding other operators corresponding to
other clusters, one observes the same deviations, that is, although the ratios between different
coefficient are not just the same as the one derived by [10], but they are very close to them.
This shows that we are on the right path.

To derive better result, one should do the process of continuation with more care. This
means that we should first expand the exponential in terms of $\theta$ and $\bar{\theta}$ before going from
discrete to continuum. Let’s consider a two point correlation function. Using Dhar and
Majumdar method, the height correlation of two WAC’s have the following form

$$P(S^o, S^p) = \det(1 + G B)$$

Expansion of the exponentials leads to several terms whose expectation values can be obtained
using Wick theorem. We categorize the contractions in the following way: firstly, one may
contract all the grassman variables in each of the clusters with themselves; secondly, only
two connections are established between the two clusters$^2$ and so on. With every connection
between the two WAC’s, we will have a long range Green function. This means that the
terms with more long range connections fall off more rapidly. As we are interested in the
leading terms of the correlations function, only the first two terms could be considered

$$P(S^o, S^p) = \langle \left(1 + \theta_i B^o_{ij} \bar{\theta}_j + \ldots \right) \left(1 + \theta_k B^p_{kl} \bar{\theta}_l + \ldots \right) \rangle$$

$$= P(S^o) P(S^p) + \langle \langle \left(1 + \theta_i B^o_{ij} \bar{\theta}_j + \ldots \right) \left(1 + \theta_k B^p_{kl} \bar{\theta}_l + \ldots \right) \rangle \rangle,$$  \hspace{1cm} (12)

where $\langle \langle \cdot \cdot \cdot \rangle \rangle$ indicates that only the contractions with two long connections should be consid-
ered. The first term, which comes from inter-contractions only, is simply equal to product of
probabilities of the two clusters. However, in the second term one finds nontrivial correlations,
and hence could be considered to contain the scaling operators we are looking for.

The process of derivation of the operator associated with a given cluster can be summa-
rized as follows. First we should expand the exponential $\exp(\theta_i B_{ij} \bar{\theta}_j)$. Then contract all the
variables in any possible way, leaving one $\theta$ and one $\bar{\theta}$ uncontracted. As an example if we
have a term in the exponential of the form $\theta \bar{\theta} \theta \bar{\theta}$ we should transform it to

$$\theta_1 \bar{\theta}_2 \theta_3 \bar{\theta}_4 \rightarrow \theta_1 \bar{\theta}_2 G_{34} - \theta_1 \bar{\theta}_4 G_{32} + \theta_3 \bar{\theta}_4 G_{12} - \theta_3 \bar{\theta}_2 G_{14}. \hspace{1cm} (13)$$

Here, $G_{ij}$ is the Green function between the two sites $i$ and $j$. The last step is to state the
obtained expression in terms of $\theta$, $\bar{\theta}$ at the origin and their derivatives.

Doing all these together one is able to derive the desired scaling field. Though the pro-
cedure is straightforward, it is very long and cumbersome. But fortunately, the method of
expansion has a nice graphic representation as follows. First we put a point for every site
involved in the matrix $B$. The contraction between $\theta_i$ and $\bar{\theta}_j$ is represented with an arrow

$^2$The number of connections should be even, because if you take it to be odd, odd number of $\theta$ and $\bar{\theta}$ remain
in each cluster and so the contribution of such term is zero.
pointing from $i$ to $j$, if $i = j$ then we draw a loop. For the two sites whose variables are not
contracted, one draws a dashed line from one to the other. Then for every line pointing from
$i$ to $j$, one puts the corresponding Green function, $G_{ij}$ multiplied by $B_{ij}$. For the dashed line,
one simply puts $B_{ij}$. The last rule is to put a minus sign for every loop with even number
of lines. For example for the cluster $S_0$ two of the possible graphs and the corresponding
expressions are shown in figure 2.

This equivalency makes the job more tractable. We draw all of the possible graphs and
write the equivalent terms in the series. Then, By expanding $\theta_i$ and $\theta_j$ in terms of $\theta$, $\bar{\theta}$ and
their derivatives, we arrive at the scaling field of the specific cluster. The result for the cluster
$S_0$ is

$$\phi_{S_0}(z) = -\frac{2(\pi - 2)}{\pi^2} \partial \theta \bar{\theta} \hat{\partial} \bar{\theta} \hat{\partial} ;$$

(14)

which, apart from a factor of $\pi$, is the same as the one derived in [10]. The $1/\pi$ factor, comes
in the very same way as it appears in the action [5].

The result can be confirmed by doing it in another way. If we contract the remaining $\bar{\theta}_i$
and $\theta_j$, in the series (to produce an additional $G_{ij}$), then the series will be equal to $P(S)$, or
equivalently $\det(1 + BG)$. So, the term proportional to $G_{ij}$ in the expression for $\det(1 + BG)$
is simply the coefficient of $\theta_i \bar{\theta}_j$ in the above series. This method is much faster, specially if
done numerically. For other clusters like $S_1$, the corresponding scaling field is derived by the
latter method and the results are in agreement with [10].

There remain some points to be clarified. First, we should note that any correlation
function in $c = -2$ theory is zero unless you have the zero mode $\theta \bar{\theta}$ in the correlation. So
all the correlations like $\langle \phi_{S_0} \phi_{S_0} \rangle$ should be computed as $\langle \phi_{S_0} \phi_{S_0} \theta \bar{\theta} \rangle$. This has a physical
interpretation. The theory could not be defined on the whole plane, as the boundaries play
an important role in ASM. In fact we should put mass term on the boundaries so that the
grains of sand have the opportunity to leave the system. But if we would like to have the
whole plane, it would be enough to put a field $\theta \bar{\theta}$ at infinity.

The other point is about the factor $C$ in equation [5]. By the method we derive the
scaling field, it is quite natural that this factor should be an integer and the integer is just
the number of sites in the cluster. Because in the massive theory, the matrix $B$ is a little bit changed: $B_{ii}$ is set to $1 - x$ instead of $-3$. So, for each site of the cluster we will have an additional term proportional to $m^2 \bar{\theta} \theta$, which is multiplied by the probability of the cluster when the contraction of other variables is done.

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