Path-Connected Components of Affine Schemes and Algebraic K-Theory

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Abstract

We introduce a functor $\mathcal{M} : \text{Alg} \times \text{Alg}^{\text{op}} \to \text{pro-Alg}$ constructed from representations of $\text{Hom}_{\text{Alg}}(A, B \otimes ?)$. As applications, the following items are introduced and studied: (i) Analogue of the functor $\pi_0$ for algebras and affine schemes. (ii) Cotype of Weibel’s concept of strict homotopization. (iii) A homotopy invariant intrinsic singular cohomology theory for affine schemes with cup product. (iv) Some extensions of $\text{Alg}$ that are enriched over idempotent semigroups. (v) Classifying homotopy pro-algebras for Cortiñas-Thom’s KK-groups and Weibel’s homotopy K-groups.

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1 Introduction

The main objective of this note is to introduce and study a specific bifunctor

$$\mathcal{M} : \text{Alg} \times \text{Alg}^{\text{op}} \to \text{pro-Alg}$$

on the category of (noncommutative and nonunital) algebras over any field $F$. For any two algebras $A$ and $B$, the pro-algebra $\mathcal{M}_{A,B}$ is defined to be the pro-object of $\text{Alg}$ representing the set-valued functor $\text{Hom}_{\text{Alg}}(A, B \otimes ?)$. We give an explicit construction of $\mathcal{M}_{A,B}$ associated with each generator-set of $A$ and each vector-space basis of $B$. It turns out that $\mathcal{M}_{A,B}$ may be regarded as the algebra of polynomial functions on the noncommutative affine ind-scheme of all morphisms $A \to B$. We consider some variants of $\mathcal{M}$ related to unital, commutative, and reduced cases. The behavior with respect to product and coproduct, and a specific exponential law for the functor $\mathcal{M}$ are considered. It is also shown that $\mathcal{M}$ preserves algebraic homotopy.

Using $\mathcal{M}$ and regarding it as a (dual) pure-algebraic version of the ordinary mapping space functor on topological spaces (see below), we introduce and study some new objects in homotopy theories of algebras and (noncommutative) affine schemes:

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(i) We introduce a functor $P = P_\mathcal{C}$ and some of its variants on $\text{Alg}$ that send any algebra $A$ to some specific subalgebra $P(A) \subseteq A$. The associated functor on affine schemes turns out to be an algebraic version of the usual path-connected component functor $\pi_0$ on topological spaces. Homotopy invariance, behavior with respect to direct sum and tensor product, and some other properties of $P$ are considered. Also, it is shown that in case $\text{char}(\mathbb{F}) = 0$, $P$ is isomorphic to the de Rham cohomology-group functor at degree 0 on the category of unital commutative algebras.

(ii) For any functor $F$ on an (admissible) subcategory of $\text{Alg}$ with values in an arbitrary category we define the costrict homotopization of $F$ to be a homotopy invariant functor $\lceil F \rceil$, with the same domain and target as $F$, satisfying a cotype of the universal property of Weibel’s strict homotopization [21]. It is shown that under some mild conditions on the domain and target categories, the functor $\lceil F \rceil$ exists. It turns out that $P = \lceil \text{id} \rceil$. Also, it became clear that the space of path-connected components of any affine scheme can be defined in three different fashions.

(iii) Using machinery of ordinary singular cohomology theory, with each functor $F$ on a specific subcategory of $\text{Alg}$ with values in abelian groups we associate two homotopy invariant homology theories $H^*_F, H_*^F$ for algebras. It is proved that $H^* F = [F]$ and thus the functors $H^*_F$ may be regarded as higher ordered costrict homotopizations of $F$. In case $F = \text{id}$ we have $H^*_F = H_0^*$, and we may consider the cup product for the homology theory. The associated theory on affine schemes is called intrinsic singular cohomology theory. A de Rham Theorem at degree zero is proved for that.

(iv) We construct an extension $\overline{\text{Alg}}$ (and some of its variants) of $\text{Alg}$. The objects of $\overline{\text{Alg}}$ are those of $\text{Alg}$ and the morphisms from an algebra $A$ to an algebra $B$ are (various classes of) pro-ideals of $\mathfrak{M}_{A,B}$. It is shown that $\overline{\text{Alg}}$ is an enriched category over partially ordered idempotent semigroups, by inclusion and intersection of pro-ideals. Also, the K-group functor $K_0$ can be extended to a version of $\overline{\text{Alg}}$.

(v) We consider a bivariant homology theory constructed as a pure-algebraic version of Cuntz’s interpretation [5] of the Kasparov bivariant K-theory of $C^*$-algebras [15]. It is shown that for every unital algebra $C$ the canonical group-morphism from $K_0(C)$ into the Weibel homotopy K-group $KH_0(C)$ [19] factors through $\Omega \Omega(C)$. Following a method introduced by Phillips [17] we prove the existence of classifying homotopy pro-algebras for $\Omega \Omega$-groups, Cortiñas-Thom’s KK-groups [4], and $KH_0$-groups. Thus, for instance, it is proved that there exists a pro-object $KH_0$ of the homotopy category $\text{Hot(Alg)}$ with a cocommutative cogroup structure such that

\[KH_0(B) \cong [KH_0(B)] \quad (B \in \text{Alg}).\]

In the following we explain some simple ideas from classical Topology that are behind the definitions of $\mathfrak{M}$ and the objects introduced in (i) and (iii).

Let $\text{Top}$ denote the category of topological spaces and $\text{Top}_k$ the subcategory of compact topological manifolds. Consider the bifunctor

$$M : \text{Top}_k^{\text{op}} \times \text{Top}_k \to \text{Top}$$

that associates with $(Y, X)$, the space of all (continuous) mappings from $Y$ into $X$, with compact-open topology. It is an elementary fact that any map $\phi : Z \times Y \to X$ defines a $Z$-parameterized family $\{\psi(z)\}_{z}$ in $M_{Y,X}$ given by

$$\psi : Z \to M_{Y,X} \quad \psi(z)(y) := \phi(z, y).$$
Identifying maps of the type $\phi$ with families of maps of the type $\psi$, we may define the space $M_{Y,X}$ as the parameter-space $M$ of a family $u : M \times Y \to X$ satisfying the following universal property: For any family $\phi : Z \times Y \to X$ with compact parameter space $Z$ there exists a unique map $\phi : Z \to M$ making the following diagram commutative:

\[
\begin{array}{ccc}
M \times Y & \xrightarrow{u} & X \\
\downarrow_{\phi \times \text{id}_Y} & & \downarrow_{\phi} \\
Z \times Y & \xrightarrow{\phi} & X
\end{array}
\]

Dualizing the above categorical definition of $M_{Y,X}$, we get the definition of $\mathfrak{M}_{A,B}$: For any two algebras $A$ and $B$, $\mathfrak{M}_{A,B}$ is defined to be an object of $\text{Alg}$ coming with a morphism $\Upsilon_{A,B} : A \to B \otimes \mathfrak{M}_{A,B}$ and satisfying the following universal property: For any algebra $C$ and every morphism $\phi : A \to B \otimes C$ there exists a unique morphism $\phi : \mathfrak{M}_{A,B} \to C$ making the following diagram commutative:

\[
\begin{array}{ccc}
B \otimes \mathfrak{M}_{A,B} & \xrightarrow{\Upsilon_{A,B}} & A \\
\downarrow_{\text{id}_B \otimes \phi} & & \downarrow_{\phi} \\
B \otimes C & \xrightarrow{} & C
\end{array}
\]

It will be shown that $\mathfrak{M}_{A,B}$ exists but not always as an algebra. Indeed, as in the general case $M_{Y,X}$ is a noncompact topological space, $\mathfrak{M}_{A,B}$ in general is a pro-object of $\text{Alg}$. Using the mentioned universal property it is easily verified that the assignment $(A,B) \mapsto \mathfrak{M}_{A,B}$ defines a bifunctor (Theorem 2.3). Also, it is easily checked that there is a canonical bijection between $\mathfrak{F}$-points of the noncommutative affine indscheme $S(\mathfrak{M}_{A,B})$ associated to $\mathfrak{M}_{A,B}$ (i.e. the pro-morphisms $\mathfrak{M}_{A,B} \to \mathfrak{F}$) and the morphisms $A \to B$ (Corollary 2.4). Thus, $S(\mathfrak{M}_{A,B})$ may be called noncommutative (or quantum) family of all the maps $S(B) \to S(A)$. For more details on the concept of quantum family see [18].

It is necessary to remark that the pro-algebras of the type $\mathfrak{M}_{A,B}$ and their properties have been known for a long time (see for instance [9]). But it seems that these objects have not been systematically studied.

Let $\pi_0 : \text{Top}_k \to \text{Top}_k$ denote the functor that associates with any space the set of its path-connected components endowed with quotient topology. For any $X \in \text{Top}_k$ we may identify $\mathcal{C}(\pi_0 X)$, the algebra of real-valued continuous functions on $\pi_0 X$, with the subalgebra of those functions $f \in \mathcal{C}(X)$ with the property that for any continuous curve $\gamma : [0,1] \to X$, $f \circ \gamma$ is constant. Thus if $X^{[0,1]} := M([0,1],X)$ denotes the path-space of $X$ then $f \in \mathcal{C}(X)$ belongs to $\mathcal{C}(\pi_0 X)$ iff the morphism

\[
\mathcal{C}(X) \to \mathcal{C}([0,1] \times X^{[0,1]}) \quad g \mapsto [(t,\gamma) \mapsto g(\gamma(t))]
\]

sends $f$ to a function of the form

\[
1 \otimes \tilde{f} \in \mathcal{C}([0,1]) \otimes \mathcal{C}(X^{[0,1]}) \subseteq \mathcal{C}([0,1] \times X^{[0,1]})
\]

for some $\tilde{f} \in \mathcal{C}(X^{[0,1]})$. 
Using the above formalism we may define the functor \( \pi_0 \) on the category of affine schemes: Let \( A \) be a unital commutative algebra regarded as the algebra of polynomial functions on the affine scheme \( S = S(A) \) over \( \mathbb{F} \). Denote by \( \mathfrak{M}^{cu} \), \( \mathfrak{Y}^{cu} \) the commutative unital versions of \( \mathfrak{M}, \mathfrak{Y} \) (Theorem 2.5). We let \( \mathfrak{P}(A) \) denote the subalgebra of those elements \( a \in A \) such that the image of \( a \) under the morphism

\[
\mathfrak{Y}^{cu}_{A, \mathbb{F}[x]} : A \to \mathbb{F}[x] \otimes \mathfrak{M}^{cu}_{A, \mathbb{F}[x]}
\]

is of the form \( 1 \otimes \bar{a} \) in each component of the pro-algebra \( \mathbb{F}[x] \otimes \mathfrak{M}^{cu}_{A, \mathbb{F}[x]} \) (Lemma 4.2). So, we have replaced the interval \([0, 1]\) with the affine line \( \mathbb{A}^1 = S(\mathbb{F}[x]) \). We define

\[
\pi_0 S := \mathfrak{S}(\mathfrak{P}(A)).
\]

A continuous version of the ordinary singular cohomology theory with values in \( \mathbb{R} \) may be simply defined as follows: Let \( \{ \Delta_n \}_{n} \) denote the standard cosimplicial space. For any space \( X \), the action of the cofunctor \( M(?, X) \) on \( \{ \Delta_n \}_{n} \) gives rise to the simplicial space \( \{ M(\Delta_n, X) \}_{n} \) and then to the cosimplicial vector space \( \{ C(M(\Delta_n, X)) \}_{n} \) that accidentally is also a cosimplicial algebra. The cohomology groups of the Moore cochain complex associated with that cosimplicial vector space may be called continuous singular cohomology groups of \( X \) [10].

A pure-algebraic version of the above formalism may be applied to affine schemes: Let \( \mathbb{F}[\Delta] := \{ \mathbb{F}[\Delta_n] \}_{n} \) denote the standard simplicial algebra [4, 7] (see [6] for the definition). For any affine scheme \( S = S(A) \) the action of \( \mathfrak{M}_{A, ?} \) on \( \mathbb{F}[\Delta] \) gives rise to the cosimplicial pro-algebra \( \mathfrak{M}_{A, \mathbb{F}[\Delta]} \). The cohomology groups of the Moore complex of \( \lim \mathfrak{M}_{A, \mathbb{F}[\Delta]} \) are called intrinsic singular cohomology groups of \( S \).

In the rest of this section we fix our notations. In § 2 we construct the functor \( \mathfrak{M} \) and its variants, and consider some basic properties of \( \mathfrak{M} \). In § 3 we review the classical notion of algebraic homotopy for pro- and ind-morphisms, and show that \( \mathfrak{M} \) is homotopy preserving. In § 4 we consider respectively the items (i)-(v).

**Notations & Conventions**

We denote by \textbf{Set}, \textbf{Ab}, \textbf{Chain}, \textbf{CChain} respectively the categories of sets, abelian groups, and chain and cochain complexes of abelian groups. Let \( \mathbf{C} \) be a category. The category of simplicial (resp. cosimplicial) objects of \( \mathbf{C} \) is denoted by \( \text{sim-\mathbf{C}} \) (resp. \( \text{cosim-\mathbf{C}} \)). The category of pro-objects (resp. ind-object) of \( \mathbf{C} \) (Appendix) is denoted by \( \text{pro-\mathbf{C}} \) (resp. \( \text{ind-\mathbf{C}} \)). An object of \( \text{pro-\mathbf{C}} \) (alternatively called a pro-object of \( \mathbf{C} \)) is an indexed family \( \{ C_i \}_{i \in I} \) of objects of \( \mathbf{C} \) over a directed set \( I \) together with the structural morphisms \( \alpha_{i'i} : C_{i'} \to C_i \) for \( i' \geq i \) which are compatible: \( \alpha_{ii} = \text{id}_{C_i} \) and \( \alpha_{ii'} = \alpha_{ii''} \alpha_{i''i'} \). For pro-objects \( C = (I, C_i, \alpha_{ii'}) \) and \( D = (J, D_j, \beta_{jj'}) \) of \( \mathbf{C} \) it is defined that

\[
\text{Hom}_{\text{pro-\mathbf{C}}}(C, D) := \lim_{i \to \infty} \lim_{j \to \infty} \text{Hom}_{\mathbf{C}}(C_i, D_j).
\]

(The structure of \( \text{Hom}_{\text{pro-\mathbf{C}}}(C, D) \) may be explained as follows: A represented morphism from \( C \) to \( D \) is distinguished by a function \( f : J \to I \) and a family \( \{ \phi_j : C_{f(j)} \to D_j \}_{j} \) of morphisms with the property that if \( j' \geq j \) then there exists \( i \geq f(j), f(j') \) such that \( \phi_j \alpha_{i'i} = \beta_{jj'} \phi_{j'} \alpha_{f(j')i} \). Two represented pro-morphisms \( (f, \phi_j) \) and \( (g, \psi_j) \) are equivalent if for every \( j \) there exists \( i \geq f(j), g(j) \) such that
$\phi_j \alpha_{f(j)i} = \psi_j \alpha_{g(j)i}$. Then, $\text{Hom}_{\text{pro-}C}(C, D)$ may be identified with the set of equivalence classes of represented pro-morphisms. We always collapse pro-pro- and ind-ind-objects to pro- and ind-objects, without further comment. We use freely the canonical embeddings

$$C \subset \text{pro-}C \quad \text{and} \quad C \subset \text{ind-}C.$$ 

We also freely use the canonical extensions of any functor $C \to D$ to the functors

$$\text{pro-}C \to \text{pro-}D, \quad \text{ind-}C \to \text{ind-}D, \quad \text{sim-}C \to \text{sim-}D.$$ 

Inverse limit $\lim \leftarrow$ (if exists) is considered as a functor from $\text{pro-}C$ to $C$. Recall that a functor $F : C \to \text{Set}$ is said to be pro-representable if there exist a pro-object $C = \{C_i\}$ of $C$ and a natural isomorphism $\Phi : \text{Hom}_{\text{pro-}C}(C, ?) \to F$. Note that $\Phi$ is exactly distinguished by the family $\{\phi_i \in F(C_i)\}$ where $\phi_i := \Phi(id_{C_i})$. We call

$$\{(C_i)_i, \{\phi_i\}_i\}$$

a pro-representation for $F$, and say that $F$ is pro-represented by $C$.

Throughout, we work over a fixed field $F$; all vector spaces and algebras are understood over $F$. $\text{Vec}$ denotes category of vector spaces. The symbol $\otimes$ without any subscript denotes $\otimes_F$. The sub- and super-scripts $\text{nc}, c, u, r, fg, fp$ stand for noncommutative, commutative, unital, reduced, finitely generated, finitely presented. From now on, the category of all algebras is denoted by $A_{\text{nc}}$. The category of unital algebras and unit preserving morphisms is denoted by $A_u$. The category $A_*$ is similarly considered for $* = c, cu, fg, fp, \ldots$. Thus, for instance, $A_{\text{cfg}}$ denotes the category of finitely generated commutative unital algebras and unit preserving morphisms. Tensor product, product, and coproduct of pro- and ind-algebras are defined componentwise. For any pro-algebra $A$ by a point of $A$ we mean a pro-morphism from $A$ into $F$. The set of all points and all nonzero points of $A$ are denoted by $\text{Put}(A)$ and $\text{Put}_{\neq 0}(A)$. The category of affine schemes (over $F$) as usual is defined by $\text{Aff} := A_{\text{cu}}^{\text{op}}$. The affine scheme associated with $A \in A_{\text{cu}}$ is denoted by $S(A)$. $M_n$ denotes the algebra of $n \times n$ matrixes with entries in $F$. We denote with the same symbol the functor $A_{\text{nc}} \to A_{\text{nc}}$ given by $A \mapsto M_n(A) := M_n \otimes A$.

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2 The Main Definition

Let $B, C, D$ be arbitrary categories and let $F : C \times D \to B$ be a functor. For objects $B \in B$ and $C \in C$ consider the functor

$$F_{B, C} := \text{Hom}_B(B, F(C, ?)) : D \to \text{Set}.$$ 

Let $D$ be a pro-object of $D$ and $\phi : B \to F(C, D)$ be a pro-morphism.
Lemma 2.1. $(D, \phi)$ is a pro-representation for $F_{B, C}$ iff the following universal property holds: For every pro-object $E$ of $D$ and every pro-morphism $\psi : B \to F(C, E)$ there exists a unique pro-morphism $\bar{\psi} : D \to E$ such that $\psi = F(id_C, \bar{\psi})\phi$.

The functor $F$ is said to be pro-$D$-representable if for every $B, C$, $F_{B, C}$ is pro-representable. For $D$ and $\phi$ as above, we use respectively the notations $\mathfrak{M}(B, C) = \mathfrak{M}_{B, C}$ and $\Upsilon(B, C) = \Upsilon_{B, C}$.

Let $f : B \to B'$ and $g : C' \to C$ be morphisms respectively in $B$ and $C$. The universal property of $(\mathfrak{M}_{B, C}, \Upsilon_{B, C})$ shows that there is a unique pro-morphism $\mathfrak{M}(f, g) : \mathfrak{M}_{B, C} \to \mathfrak{M}_{B', C'}$ satisfying the identity

$$F(g, id_{\mathfrak{M}_{B', C'}})\Upsilon_{B', C'}f = F(id_C, \mathfrak{M}(f, g))\Upsilon_{B, C}.$$

It is clear that $\mathfrak{M}(id_B, id_C) = id_{\mathfrak{M}(B, C)}$. For morphisms $f' : B' \to B''$, $g' : C'' \to C'$, again universal property of the involved objects shows that

$$\mathfrak{M}(f'f, gg') = \mathfrak{M}(f', g')\mathfrak{M}(f, g).$$

Ignoring some set-theoretical difficulties about the choice of a pro-representation $(\mathfrak{M}_{B, C}, \Upsilon_{B, C})$ for $F_{B, C}$, we have the following easily checked result.

**Proposition 2.2.** Suppose that the functor $F : C \times D \to B$ is pro-$D$-representable. Then, $\mathfrak{M}$ may be considered as a functor

$$\mathfrak{M} : B \times C^{\text{op}} \to \text{pro-D},$$

and $\Upsilon$ as a natural transformation, in the obvious way. For any $C \in C$, the functor $\mathfrak{M}(?, C) : \text{pro-B} \to \text{pro-D}$ is left-adjoint to the functor

$$F(C, ?) : \text{pro-D} \to \text{pro-B}.$$

Thus, for $B \in \text{pro-B}$, $C \in C$, and $D \in \text{pro-D}$, we have the following natural bijection:

$$\text{Hom}_{\text{pro-D}}(\mathfrak{M}(B, C), D) \cong \text{Hom}_{\text{pro-B}}(B, F(C, D)).$$

For $B \in B$, $C \in \text{ind-C}$, and $D \in D$, we have the following bijection:

$$\text{Hom}_{\text{pro-D}}(\mathfrak{M}(B, C), D) \cong \text{Hom}_{\text{ind-B}}(B, F(C, D)).$$
We now turn to our main construction in the category of algebras. Let $A, B$ be algebras. Suppose $\theta = \{\delta_a\}_{a \in G}$ is a family of finite linearly independent subsets $\delta_a$ of $B$ indexed by a set $G \subseteq A$ that generates $A$ as an algebra. Denote by $\mathfrak{M}_\theta$ the universal algebra in $A_{nc}$ generated by symbols $z_{a,v}$, where $a \in G$ and $v \in \delta_a$, subject to the condition that the assignment

$$a \mapsto \sum_{v \in \delta_a} v \otimes z_{a,v}$$

defines a morphism $\Upsilon_\theta : A \to B \otimes \mathfrak{M}_\theta$ in $A_{nc}$. The pair $(\mathfrak{M}_\theta, \Upsilon_\theta)$ has the following universal property: For every $C \in A_{nc}$ and any morphism $\varphi : A \to B \otimes C$ in $A_{nc}$ with the property that for every $a \in G$, $\varphi(a)$ is a linear combination of vectors in $\delta_a$, there is a unique morphism $\varphi : \mathfrak{M}_\theta \to C$ in $A_{nc}$ such that $\varphi = (\text{id}_B \otimes \varphi) \Upsilon_\theta$. We call $\varphi$ the morphism $\theta$-associated with $\varphi$.

Now, fix a vector basis $V$ for the underlying vector space of $B$, and let $\Theta = \Theta_{G,V}$ denote the set of all families $\theta = \{\delta_a\}_{a \in G}$ as above with $\delta_a \subseteq V$. For $\theta, \theta' \in \Theta$, write $\theta \subseteq \theta'$ if $\delta_a \subseteq \delta'_a$ for every $a \in G$. Then, $\subseteq$ makes $\Theta$ into a directed set. For $\theta \subseteq \theta'$ let $\phi_{\theta' \theta} : \mathfrak{M}_{\theta'} \to \mathfrak{M}_\theta$ denote the morphism $\theta'$-associated with $\Upsilon_\theta$. We have $\phi_{\theta' \theta} = \text{id}$ and if $\theta \subseteq \theta' \subseteq \theta''$ then $\phi_{\theta'' \theta} = \phi_{\theta' \theta} \phi_{\theta' \theta''}$. So, the data $\{\mathfrak{M}_\theta, \phi_{\theta' \theta}\}$ distinguish a pro-algebra $\mathfrak{M}_{nc,A,B}$ indexed by $\Theta$ and the family $\{\Upsilon_\theta\}$ defines a pro-morphism

$$\Upsilon_{nc,A,B} : A \to B \otimes \mathfrak{M}_{nc,A,B}.$$
is pro-$A_u$-representable. Also, for $* \in \{c, cr\}$ ($* \in \{cu, cur\}$) the functors
\[
\otimes : A_{nc} \times A \rightarrow A_{nc} \quad (\otimes : A_u \times A \rightarrow A_u)
\]
are pro-$A_*$-representable. We denote by $(\mathcal{M}^*_A, B, \mathcal{Y}_A, B)$ the pro-representations for these functors, constructed similar to the pair (2). Thus, for instance, for unital algebras $A, B$, $\mathcal{M}_{nc}^*(A, B)$ is an object in pro-$A_{nc}$ given by the inverse system $\{M^*_{\theta, \phi} \}$ indexed over $\Theta$ as above, where $M_{\theta}$ is the universal algebra in $A_{nc}$ generated by symbols $z_{a, v}$ subject to the condition that the assignment (1) defines a unit-preserving morphism from $A$ into $B \otimes M_{\theta}$. The following is a special case of Proposition 2.2.

**Theorem 2.3.** For $* \in \{nc, c, cr\}$, $\mathcal{M}^*$ may be considered as a functor
\[
\mathcal{M}^* : A_{nc} \times A_{nc}^\text{op} \rightarrow \text{pro-}A_*
\]
and $\mathcal{Y}^*$ as a natural transformation, in the obvious way. For any $B \in A_{nc}$, the functor
\[
\mathcal{M}^*(?, B) : \text{pro-}A_{nc} \rightarrow \text{pro-}A_*
\]
is left-adjoint to the functor
\[
B \otimes ? : \text{pro-}A_* \rightarrow \text{pro-}A_{nc}.
\]
For $A \in \text{pro-}A_{nc}$, $B \in A_{nc}$, and $C \in \text{pro-}A_*$, we have the natural bijection
\[
\text{Hom}_{\text{pro-}A_*}(\mathcal{M}^*(A, B), C) \cong \text{Hom}_{\text{pro-}A_{nc}}(A, B \otimes C),
\]
and for $A \in A_{nc}$, $B \in \text{ind-}A_{nc}$, and $C \in A_*$, the bijection
\[
\text{Hom}_{\text{pro-}A_*}(\mathcal{M}^*(A, B), C) \cong \text{Hom}_{\text{ind-}A_{nc}}(A, B \otimes C).
\]
The above statements hold also for $* \in \{u, cu, cur\}$ if $A_{nc}$ is replaced by $A_u$.

**Corollary 2.4.** For unital algebras $A, B$, we have the following identification:
\[
\text{Hom}_{A_u}(A, B) \cong \text{Pnt}_{\neq 0}(\mathcal{M}^*_A, B) \quad (* \in \{u, cu, cur\}).
\]
The similar statement holds for nonunital case.

**Proof.** It follows from Theorem 2.3 by putting $C = \mathbb{F}$. \hfill \Box

Using universality of $\mathcal{M}^*$ it is easily verified that
\[
\mathcal{M}^c = (\mathcal{M}^*_{A_{nc}})_{\text{com}}, \quad \mathcal{M}^u = (\mathcal{M}^*_{A_u})_{\text{com}}, \quad \mathcal{M}^{cr} = (\mathcal{M}^*_{A_c})_{\text{red}}, \quad \mathcal{M}^{cur} = (\mathcal{M}^*_{A_c})_{\text{red}},
\]
where $A \mapsto A_{\text{com}} (A_{nc} \rightarrow A_c)$ and $A \mapsto A_{\text{red}} (A_c \rightarrow A_{cr})$ denote commutativization and reduction functors. We have also canonical natural transformations
\[
\mathcal{M}^{nc} \rightarrow \mathcal{M}^c \rightarrow \mathcal{M}^{cr} \quad \text{and} \quad \mathcal{M}^u \rightarrow \mathcal{M}^{cu} \rightarrow \mathcal{M}^{cur}.
\]
(3)

For the relation between unital and nonunital cases we only mention the following result. Let $+ : A_{nc} \rightarrow A_u$ denote the unitization functor. It is well-known that for any algebra $A$ and any unital algebra $B$, every morphism $f : A \rightarrow B$ extends uniquely to a unit-preserving morphism $A^+ \rightarrow B$. (Indeed, this fact says that $+$ is a left-adjoint to the embedding $A_u \rightarrow A_{nc}$.) By abuse of notation, we denote this extension of $f$ by $f^+$. The same convention is applied to pro-morphisms.
Theorem 2.5. Let \( A \in \mathbf{A}_{nc}, B \in \mathbf{A}_n \). Then there is a canonical isomorphism
\[
(\mathcal{M}^{uc}_{A,B})^+ \cong \mathcal{M}^{uc}_{A^+,B^+}, \quad (\mathcal{M}^{uc}_{A,B})^+ \cong \mathcal{M}^{uc}_{A^+,B^+}.
\]

Proof. Let \( f := (id_B \otimes e)\gamma^{uc}_{A,B} \) where \( e : \mathcal{M}^{uc}_{A,B} \to \mathcal{M}^{uc}_{A,B} \) denotes the canonical embedding. We have \( f^+ : A^+ \to B \otimes \mathcal{M}^{uc}_{A,B} \) in pro-\( \mathbf{A}_n \). Let \( \bar{f}^+ : \mathcal{M}^{uc}_{A^+,B^+} \to \mathcal{M}^{uc}_{A,B} \) be the pro-morphism associated with \( f^+ \). Thus \( f := (id \otimes \bar{f}^+)\gamma^{uc}_{A,B} \). Let \( g := \gamma^{uc}_{A^+,B^+} \) where \( e \) this time shows the canonical embedding \( A \to A^+ \). Let \( \bar{g} : \mathcal{M}^{uc}_{A,B} \to \mathcal{M}^{uc}_{A^+,B} \) denote the pro-morphism associated with \( g \). Thus \( g = (id_B \otimes \bar{g})\gamma^{uc}_{A,B} \). We have \( \bar{g}^+ : \mathcal{M}^{uc}_{A,B} \to \mathcal{M}^{uc}_{A^+,B} \). It follows from universality of \( \mathcal{M}^{uc} \) and \( \mathcal{M}^{uc} \) that \( \bar{f}^+ \) and \( \bar{g}^+ \) are inverses of each other. The proof is complete. \( \square \)

From now on, when there is no confusion about \( * \in \{nc,c,cr,uc,cur\} \), we will use the short notations \( \mathcal{M}, \mathcal{Y} \) instead of \( \mathcal{M}^*, \mathcal{Y}^* \). In the following we consider some basic properties of \( \mathcal{M} \).

Theorem 2.6.

(i) \( \mathcal{M}^{uc}_{A,F} \cong A \) for \( A \in \mathbf{A}_{nc} \), and \( \mathcal{M}^{uc}_{B,F} \cong F \) for \( B \in \mathbf{A}_n \).

(ii) For any \( A \in \mathbf{A}_{nc} \) there is a canonical bijection \( \text{Pnt}(\mathcal{M}^{uc}_{A,F}) \cong \{ a \in A : a^2 = a \} \).

Similarly, for every \( n > 1 \) there is a canonical bijection between \( \text{Pnt}(\mathcal{M}^{uc}_{F^n,A}) \) and the set of \( n \)-tuples of pairwise orthogonal idempotents of \( A \).

(iii) For \( A \in \mathbf{A}_{nc} \) and \( n > 1 \), \( \mathcal{M}^{uc}_{A,F^n} \) is a coproduct of \( n \) copies of \( A \) in \( \mathbf{A}_{nc} \).

(iv) If \( A \in \mathbf{A}_{ncfg}, A' \in \mathbf{A}_{ncfp} \), then \( \mathcal{M}^{uc}_{A,B} \in \text{pro-} \mathbf{A}_{ncfg}, \mathcal{M}^{uc}_{A',B} \in \text{pro-} \mathbf{A}_{ncfp} \).

(v) If \( B \) is a finite-dimensional algebra then \( \mathcal{M}^{uc}_{A,B} \cong \varprojlim \mathcal{M}^{uc}_{A,B} \) for any algebra \( A \).

Proof. Straightforward. \( \square \)

Example 2.7. \( \mathcal{M}^{uc}_{F^n,M_n} \) is the universal algebra generated by the symbols \( \{ z_{ij} \}_{i,j=1}^n \) satisfying \( z_{ij} = \sum_{k=1}^n z_{ik} z_{kj} \). We have \( \gamma^{uc}_{F^n,M_n}(1) = \sum_{i,j=1}^n e_{ij} \otimes z_{ij} \) where \( \{e_{ij}\}_{i,j=1}^n \) denotes the canonical vector basis of \( M_n \).

Example 2.8. \( \mathcal{M}^{uc}(\mathbb{F}[x],[x^n]) \) is isomorphic to \( \mathcal{M}^{nc}(\mathbb{F}[x],[x^n]) \), the algebra of polynomials in noncommuting indeterminates \( x_1, \ldots, x_n \).

Example 2.9. Let \( A \in \mathbf{A}_{ncfg}, B \in \mathbf{A}_{nc} \). Suppose \( G \) is a finite generator-set for \( A \), and \( V \) is a vector basis for \( B \). For every finite subset \( S \) of \( V \), let \( \theta_S \in \Theta_{G,V} \) denote the family \( \{ \delta_a \}_{a \in A} \) where \( \delta_a = S \) for every \( a \in G \). Then \( \{ \theta_S \}_S \) is a cofinal subset of \( \Theta_{G,V} \), and hence the pro-algebra \( \mathcal{M}^{uc}_{A,B} \) is described also by the inverse system \( \{ \mathcal{M}\theta_S \}_S \) of algebras. Moreover, suppose that \( V \) is countable and \( V = \{ v_1, v_2, \ldots \} \), and let \( S_n := \{ v_1, \ldots, v_n \} \). Then, \( \mathcal{M}^{uc}_{A,B} \) is also described by the following nice inverse subsystem of \( \{ \mathcal{M}\theta_S \}_S \):

\[
\mathcal{M}\theta_{S_1} \leftarrow \cdots \leftarrow \mathcal{M}\theta_{S_n} \leftarrow \mathcal{M}\theta_{S_{n+1}} \leftarrow \cdots.
\]

Here, \( \mathcal{M}\theta_{S_n} \) is generated by \( \{ z_{a,i} : a \in G, i = 1, \ldots, n \} \), and \( \mathcal{M}^{uc}\theta_{S_{n+1}} \to \mathcal{M}^{uc}\theta_{S_n} \) is given by \( z_{a,i} \mapsto z_{a,i} \) for \( i = 1, \ldots, n \), and \( z_{a,n+1} \mapsto 0 \).
Let \( C \in \mathbf{A}_{\text{cfg}} \), and let \( G = \{g_1, \ldots, g_n\} \) be a finite ordered generator-set for \( C \). Let \( \phi \) be the morphism \( \mathbb{F}[x_1, \ldots, x_n] \to C \) given by \( x_i \mapsto g_i \). Denote by \( \mathcal{Z}(C, G) \) the zero-locus of \( \ker(\phi) \), i.e., the algebraic set in \( \mathbb{A}^n \) of \( n \)-tuples \((\lambda_1, \ldots, \lambda_n)\) under which the evaluation of every polynomial in \( \ker(\phi) \), is zero. It is clear that the assignment

\[(\lambda_1, \ldots, \lambda_n) \mapsto [g_1 \mapsto \lambda_1, \ldots, g_n \mapsto \lambda_n]\]
defines a bijection from \( \mathcal{Z}(C, G) \) onto \( \text{Pnt}_{\neq 0}(C) \). Thus, the choice of \( G \) gives rise to the structure \( \mathcal{Z}(C, G) \) of an algebraic set on \( \text{Pnt}_{\neq 0}(C) \). Similarly, if \( C = (C_i)_i \) be in \( \text{pro-} \mathbf{A}_{\text{cfg}} \) then any family \( G = \{G_i\}_i \), where \( G_i \) is a finite ordered generator-set for \( A_i \), gives rise to an algebraic ind-set structure \( \mathcal{Z}(C, G) := \{\mathcal{Z}(C_i, G_i)\}_i \) on the set

\[
\text{Pnt}_{\neq 0}(C) = \lim_{i \to \infty} \text{Pnt}_{\neq 0}(C_i).
\]

(We say that a set \( X \) has the structure of a ind-set \((X_i)\), if we have a distinguished bijection between \( X \) and \( \lim_{i \to \infty} X_i \), and for every \( i \), \( X_i \to \lim_{i \to \infty} X_i \) is injective.)

**Theorem 2.10.** Let \( A \in \mathbf{A}_{\text{cfg}} \) and let \( B \in \mathbf{A}_n \). Every finite ordered generator-set for \( A \) and any ordered vector basis for \( B \), give rise to a canonical algebraic ind-set structure on \( \text{Hom}_{\mathbf{A}_n}(A, B) \).

**Proof.** Consider the pro-algebra \( \mathcal{M}^{\text{cu}}_{A,B} \) with the model \( \{\mathcal{M}_S\}_S \) described in Example 2.9. \( \mathcal{M}_S \) is generated by elements \( z_{a,v} \) indexed by the set \( G \times S \). This set may be endowed the dictionary ordering induced by orderings on \( G \) and \( V \). Then, the method described above gives rise to an algebraic ind-set structure on \( \text{Pnt}_{\neq 0}(\mathcal{M}^{\text{cu}}_{A,B}) \). Then, the desired result follows from Corollary 2.4. \( \square \)

For an application of the structure given by the above theorem, see [14].

**Theorem 2.11.** Suppose \( \ast \) denotes coproduct in \( \mathbf{A}_{\text{nc}} \). There is an isomorphism:

\[
\mathcal{M}^{\text{nc}}(A, B \oplus B') \cong \mathcal{M}^{\text{nc}}(A, B) \ast \mathcal{M}^{\text{nc}}(A, B') \quad (A, B, B' \in \mathbf{A}_{\text{nc}})
\]

**Proof.** Let \( A \) be generated by \( G \), and \( V \subset B, V' \subset B' \) be vector bases. For \( \theta = \{\delta_a\} \in \Theta_{G, V} \) and \( \theta' = \{\delta'_a\} \in \Theta_{G, V'} \), let \( \theta \cup \theta' = \Theta_{G, V \cup V'} \) denote the family \( \{\delta_a \cup \delta'_a\} \) where \( V \cup V' \) is considered as a vector basis for \( B \oplus B' \). It is enough to prove that

\[
\mathcal{M}(\theta \cup \theta') = \mathcal{M}(\theta) \ast \mathcal{M}(\theta').
\]

We do that by checking that \( \mathcal{M}(\theta \cup \theta') \) has the required universal property. Let \( \phi_\theta := (p \otimes \text{id}) \Upsilon(\theta \cup \theta') \) where \( p : B \oplus B' \to B \) is the canonical projection. Then, \( \phi_\theta : \mathcal{M}(\theta) \to \mathcal{M}(\theta \cup \theta') \), the morphism \( \theta \)-associated with \( \phi_\theta \), plays the role of coproduct structural morphism. Similarly, the structural morphism \( \phi_{\theta'} : \mathcal{M}(\theta') \to \mathcal{M}(\theta \cup \theta') \) is defined. Suppose that \( \psi : \mathcal{M}(\theta) \to C, \psi' : \mathcal{M}(\theta') \to C \) are arbitrary morphisms, and let

\[
\varphi = [(\text{id}_B \otimes \psi) \Upsilon(\theta)] \oplus [(\text{id}_{B'} \otimes \psi') \Upsilon(\theta')].
\]

Then, \( \varphi_{\theta, \theta'} : \mathcal{M}(\theta \cup \theta') \to C \), the morphism \( (\theta \cup \theta') \)-associated with \( \varphi \), is the unique morphism satisfying \( \psi = \varphi_{\theta, \theta'} \phi_\theta \) and \( \psi' = \varphi_{\theta, \theta'} \phi_{\theta'} \). The proof is complete. \( \square \)
Theorem 2.12. (Exponential Law) There is a canonical isomorphism:
\[ \mathcal{M}^{nc}(A, B \otimes B') \cong \mathcal{M}^{nc}(\mathcal{M}^{nc}(A, B), B') \quad (A, B, B' \in A_{nc}). \]

Proof. Suppose \( G \subseteq A \) is a generator, and \( V \subseteq B, V' \subseteq B' \) are vector bases. For \( S \subseteq V, S' \subseteq V' \), denote by \( S \otimes S' \) the set \( \{ v \otimes v' : v \in S, v' \in S' \} \). For \( \theta = \{ \delta_a \} \) and \( \theta' = \{ \delta'_a \} \) respectively in \( \Theta_{G, V} \) and \( \Theta_{G, V'} \), let \( \theta \otimes \theta' \in \Theta_{G, V \otimes V'} \) denote \( \{ \delta_a \otimes \delta'_a \} \). We know that \( M \theta \) is generated by \( G \times \theta := \{ z_{a,v} : a \in G, v \in \delta_a \} \). Let \( \theta' | \theta \) in \( \Theta_{G \times \theta, V'} \), denote \( \{ u_{z_{a,v}} \} \) where \( u_{z_{a,v}} := \delta'_a \). The class \( \{ \theta \otimes \theta' \} \) is cofinal with \( \Theta_{G, V \otimes V'} \), and the class \( \{ \theta' | \theta \} \) is cofinal with \( \Theta_{G \times \theta, V'} \). Thus, the pro-algebras \( M_{A \otimes B} \) and \( M_{M \theta, B'} \) are described respectively by inverse systems \( \{ \mathcal{M}(\theta \otimes \theta') \} \) and \( \{ \mathcal{M}(\theta' | \theta) \} \). Now, the desired result follows from the canonical isomorphism of \( \mathcal{M}(\theta \otimes \theta') \) and \( \mathcal{M}(\theta' | \theta) \) which can be proved by using similar arguments as in [18, Theorem 2.10]. \( \square \)

The proof of the following result is similar to that of [18, Theorem 2.8] and omitted.

Theorem 2.13. There exists a canonical isomorphism:
\[ \mathcal{M}^{cu}(A \otimes A', B) \cong \mathcal{M}^{cu}(A, B) \otimes \mathcal{M}^{cu}(A', B) \quad (A, A' \in A_{cu}, B \in A_{cu}) \]

We denote by \( op : A_{nc} \rightarrow A_{cu} \) the functor that associates to any algebra the algebra with opposite multiplication.

Theorem 2.14. For any two algebras \( A, B \) we have the natural isomorphism
\[ op(\mathcal{M}^{nc}(A, B)) \cong \mathcal{M}^{nc}(op(A), op(B)). \]

Proof. It is easily seen from the construction of \( \mathcal{M}^{nc}_{A,B} \). \( \square \)

Proposition 2.15. For \( A, B, C \in A_{nc}/A_{u} \) and \( * \in \{ nc, c, cr \} \cup \{ cu, cur \} \) there is a natural pro-morphism
\[ \Phi_{A,B,C} : \mathcal{M}^{*}_{A,C} \rightarrow \mathcal{M}^{*}_{B,C} \otimes \mathcal{M}^{*}_{A,B} \]
such that
\[ (id_{\mathcal{M}^{*}_{C,D}} \otimes \Phi_{A,B,C})\Phi_{A,C,D} = (\Phi_{B,C,D} \otimes id_{\mathcal{M}^{*}_{A,B}})\Phi_{A,B,D} \quad (4) \]

Proof. By universality of \( \mathcal{M}^{*}_{A,C} \), there is a unique pro-morphism \( \Phi_{A,B,C} \) satisfying
\[ (id_{C} \otimes \Phi_{A,B,C})T_{A,C} = (T_{B,C} \otimes id_{\mathcal{M}^{*}_{A,B}})T_{A,B}. \]
Identity (4) follows from the universality of \( \mathcal{M}^{*}_{A,D} \). \( \square \)

3  Classical Algebraic Homotopy

A compatible relation on a category \( C \) is an equivalence relation \( \mathcal{R} \) on the class of morphisms of \( C \) such that for morphisms \( f, f' : C \rightarrow C' \) and \( g, g' : C' \rightarrow C'' \) if \( f \mathcal{R} f' \) and \( g \mathcal{R} g' \) then \( g \mathcal{R} g' f' \). Denote by \( \text{Hot}_{\mathcal{R}}(C) \) the category whose objects are those of \( C \) and whose hom-sets are
\[ [C, C']_{\mathcal{R}} = \text{Hom}_{\text{Hot}_{\mathcal{R}}(C)}(C, C') := \text{Hom}_{\mathcal{C}}(C, C') / \mathcal{R}. \]
Let \( \mathbf{A} \) be a subcategory of \( \mathbf{A}_{nc} \). We say that \( \mathbf{A} \) is admissible if \( \mathbf{A} \) is closed under polynomial extensions, and if for every algebra \( A \) in \( \mathbf{A} \) the canonical embedding \( e : A \to A[x] \) given by \( a \mapsto a \), the evaluation morphisms \( p_0, p_1 : A[x] \to A \) given respectively by \( x \mapsto 0, x \mapsto 1 \), and the morphism defined by \( x \mapsto 1 - x \) from \( A[x] \) onto \( A[x] \), belong to \( \mathbf{A} \). It is clear that \( \mathbf{A}_* \), for \( * \in \{ nc, c, cr, u, cu, cur \} \), is admissible.

Let \( \mathbf{A} \) be an admissible category of algebras. Two morphisms \( f, g : A \to B \) in \( \text{pro-} \mathbf{A} \) are said to be algebraic homotopic, denoted \( f \approx g \), if there is a morphism \( H : A \to B \otimes \mathbb{F}[x] \cong B[x] \) in \( \text{pro-} \mathbf{A} \), called elementary homotopy from \( f \) to \( g \), such that \( p_0H = f \) and \( p_1H = g \). Similarly, elementary homotopic morphisms in \( \text{ind-} \mathbf{A} \) are defined.

Two morphisms \( f, g : A \to B \) in \( \mathbf{A} \) are called algebraic homotopic, denoted \( f \sim g \), if there is a finite chain \( h_0, \ldots, h_n \) of morphisms in \( \mathbf{A} \) such that

\[
f = h_0 \approx h_1 \approx \ldots \approx h_n = g.
\]

It is easily verified that \( h \) is a compatible relation on \( \mathbf{A} \), and accordingly we have the homotopy category \( \text{Hot}(\mathbf{A}) = \text{Hot}_h(\mathbf{A}) \).

We say that two pro-morphisms \( f, g : A \to B \) are strongly homotopic, denoted \( f \# g \), if there is a finite chain \( h_0, \ldots, h_n \) of pro-morphisms such that (5) is satisfied. It is easily verified that \( \# \) is a compatible relation on \( \text{pro-} \mathbf{A} \), and accordingly we have the homotopy category \( \text{Hot}_\#(\text{pro-} \mathbf{A}) \). The compatible relation \( \# \) between ind-morphisms is defined similarly, and hence we have the category \( \text{Hot}_\#(\text{ind-} \mathbf{A}) \).

We say that two pro-morphisms \( f, g : A \to B \) are weakly homotopic, denoted \( f \simw g \), if their images in \( \text{pro-Hot}(\mathbf{A}) \) are equal. It is clear that \( \simw \) is a compatible relation on \( \text{pro-} \mathbf{A} \), and accordingly we have the homotopy category \( \text{Hot}_\simw(\text{pro-} \mathbf{A}) \). The homotopy category \( \text{Hot}_\simw(\text{ind-} \mathbf{A}) \) is defined similarly.

It is clear that \( \simw \) is coarser than \( \# \). Thus we have the following two sequences of functors all induced by \( id \) in the obvious way:

\[
\text{pro-} \mathbf{A} \to \text{Hot}_\#(\text{pro-} \mathbf{A}) \to \text{Hot}_\simw(\text{pro-} \mathbf{A}) \to \text{pro-Hot}(\mathbf{A})
\]
\[
\text{ind-} \mathbf{A} \to \text{Hot}_\#(\text{ind-} \mathbf{A}) \to \text{Hot}_\simw(\text{ind-} \mathbf{A}) \to \text{ind-Hot}(\mathbf{A})
\]

In each of the above two rows the first and second functors are full and the third one is faithful. It is customary (4, 7, 8, 9) to denote the hom-sets of the categories \( \text{pro-Hot}(\mathbf{A}) \) and \( \text{ind-Hot}(\mathbf{A}) \) just by \([A, B]\). It is easily verified that if \( A \in \text{pro-} \mathbf{A} \) and \( B \in \mathbf{A} \) (or if \( A \in \mathbf{A} \) and \( B \in \text{ind-} \mathbf{A} \)) then there are natural identifications

\[
[A, B]_h \equiv [A, B]_wh \equiv [A, B].
\]

Note that for pro-algebras \( A = (A_i) \) and \( B = (B_j) \), we have

\[
[A, B] = \lim_{i \to \infty} \lim_{j \to \infty} [A_i, B_j].
\]

Similarly, for ind-algebras \( A = (A_i) \) and \( B = (B_j) \), we have

\[
[A, B] = \lim_{i \to \infty} \lim_{j \to \infty} [A_i, B_j].
\]

We remark that in general for pro- or ind-algebras \( A, B \) the set \([A, B]\) is very bigger than \([A, B]_wh\).

The following simple lemma shows that for morphism \( f, g \) in \( \mathbf{A}_* \), where \( * \in \{ c, cr, u, cu, cur \} \), \( f \) and \( g \) are homotopic in \( \mathbf{A}_* \) iff \( f \) and \( g \) are homotopic in \( \mathbf{A}_{nc} \).
Lemma 3.1. Let $A, B$ be unital algebras, $f, g : A \to B$ be morphisms in $A_{nc}$, and $H$ be an elementary homotopy in $A_{nc}$ from $f$ to $g$. If $f \in A_{ua}$ then $H, g \in A_{ua}$.

Lemma 3.2. Let $A, B \in A_{nc}$ and $C \in $ pro-$A_{nc}$, and let $f, g : A \to B \otimes C$ be morphisms. Suppose that $\overline{f}, \overline{g} : M_{A,B} \to C$ denote the pro-morphisms associated with $f, g$. Then, $f \circ h \circ g$ iff $\overline{f} \circ \overline{h} \circ \overline{g}$.

Proof. If $H : A \to B \otimes C[x]$ is an elementary homotopy between $f, g$ then the pro-morphism $\overline{H} : M_{A,B} \to C[x]$ is an elementary homotopy between $\overline{f}, \overline{g}$. Conversely, if $G : M_{A,B} \to C[x]$ is an elementary homotopy between $\overline{f}, \overline{g}$ then $(\text{id}_B \otimes G)\overline{H}$ is an elementary homotopy between $f, g$.

Lemma 3.3. The analogue of the statement of Lemma 3.2 holds for $A, C \in A_{nc}$ and $B \in \text{ind}-A_{nc}$.

Theorem 3.4. For $A, B \in A_{nc}$ and $C \in $ pro-$A_{nc}$ ($A, C \in A_{nc}$ and $B \in \text{ind}-A_{nc}$), we have the following canonical identification:

$$[A, B \otimes C] \cong [M_{nc}(A, B), C]$$

Proof. It follows from Theorem 2.3, Identities (6), and Lemmas 3.2 and 3.3.

Theorem 3.5. The functor $M_{nc}$ preserves homotopy in the sense that for morphisms $f_0, f_1 : A \to A'$ and $g_0, g_1 : B \to B'$ in $A_{nc}$, if $f_0h_1$ and $g_0h_1$ then $M_{f_0\otimes g_0, h_1}$.

Proof. It follows from definitions of $M_{f_0, g_0}$, $M_{f_1, g_1}$ and Lemma 3.2.

Note that the above four results hold for all $M_{\ast}$.

A functor $F$ from an admissible category $A$ of algebras to an arbitrary category $C$ is said to be homotopy invariant if for any two morphism $f, g \in A$, $f \circ h \circ g$ implies $F(f) = F(g)$. We say that $F : A_{\text{aff}} \to C$ is $\mathbb{A}^1$-homotopy invariant if $F$ as the functor $A_{cu} \to C^\text{op}$, is homotopy invariant. The following lemma is very well-known.

Lemma 3.6. For any functor $F : A \to C$ the following statements are equivalent.

(i) $F$ is homotopy invariant.

(ii) For every algebra $A$ in $A$, $F(e) : F(A) \to F(A[x])$ is an isomorphism in $C$.

(iii) For every algebra $A$ in $A$, $F(p_0), F(p_1) : F(A[x]) \to F(A)$ are equal.

Proof. (i)⇒(ii): We show that $F(p_0)$ is the inverse of $F(e)$ in $C$: We have $p_0e = \text{id}_A$. Thus $F(p_0)F(e) = \text{id}_{F(A)}$. The morphism $A[x] \to A[x]\langle y \rangle$ given by $p(x) \mapsto p(xy)$ is an elementary homotopy between $p_0$ and $\text{id}_{A[x]}$. Thus $F(e)F(p_0) = \text{id}_{F(A[x])}$.

(ii)⇒(iii): We have $p_0e = p_1e$. Thus $F(p_0)F(e) = F(p_1)F(e)$. Since $F(e)$ is an isomorphism, we must have $F(p_0) = F(p_1)$. (iii)⇒(i) is trivial.
Recall that a differential graded commutative algebra (DGCA) is given by a pair $(A, d)$ where $A = \bigoplus_{n=0}^{\infty} A^n$ is a GA with a GC multiplication (i.e. $a_i a_j = (-1)^{|a_i|} a_j a_i$ for $a_i \in A^i, a_j \in A^j$) and where $d : A \to A$ is a GD (i.e. a $F$-linear map of degree 1 such that $d(a_i a_j) = d(a_i) a_j + (-1)^{|a_i|} a_j d(a_i)$ and $d d = 0$). It is well-known that for every $A \in A_{cu}$ there exists a DGCA $\Omega(A) = \bigoplus \Omega^n(A)$ with $\Omega^0(A) = A$ satisfying the following universal property: For every unital DGCA $(B, \delta)$, any arbitrary morphism $f : A \to B^0$ in $A_{cu}$ extends uniquely to a morphism $f : \Omega(A) \to B$ satisfying $f d = f \delta$. Indeed, $\Omega(A)$ is the exterior algebra associated to $A$-module $\Omega^1(A)$ and $d : A \to \Omega^1(A)$ is a derivation such that for every unital $A$-module $M$ and any derivation $\delta : A \to M$ there is a unique module homomorphism $\phi : \Omega^1(A) \to M$ satisfying $\delta = \phi d$. If $\mathbb{F}$ is real field and $A$ is the algebra of smooth real functions on a smooth compact manifold then $\Omega(A)$ is isomorphic to de Rham complex of differential forms on the manifold $\mathbb{M}$ Proposition 8.1. So, for any $A \in A_{cu}$ and its associated affine scheme $S = S(A)$ the cohomology groups of $\Omega(A)$ is denoted by $\mathcal{H}^n_{dR}(A)$ or $\mathcal{H}^n_{dR}(S)$ $(n \geq 0)$ and called de Rham cohomology groups of $A$ or $S$. Note that $\mathcal{H}^n_{dR}(S) := \bigoplus_{n=0}^{\infty} \mathcal{H}^n_{dR}(S)$ is a unital GCA with the multiplication induced by that of $\Omega(A)$. By universality of $\Omega$ we may consider the functors $\mathcal{H}^0_{dR} : \text{Aff}^{op} \to A_{cu}$, $\mathcal{H}^n_{dR} : \text{Aff}^{op} \to \text{Vec}$, $\mathcal{H}_{dR} : \text{Aff}^{op} \to A_u$.

It is shown that these functors are $A^1$-homotopy invariant provided that $\text{Char}(\mathbb{F}) = 0$: For $A \in A_{cu}$ let $p_0, p_1 : \Omega(A[x]) \to \Omega(A)$ denote the cochain maps induced by $p_0, p_1 : \Omega(A[x]) \to A$. By freeness of the variable $x$ we have a $A[x]$-module decomposition $\Omega^n(A[x]) = \bigoplus_{i=0}^{n} M_i$ such that $M_i$ is the $A[x]$-submodule generated by all elements $(d^k x) \omega_{n-i} \in \Omega^n(A[x])$ where $\omega_{n-i} \in \Omega^{n-i}(A)$ and where $d^k x$ denotes the (exterior) product of $k$ copies of $dx$ in $\Omega(A[x])$. Consider a linear map $\phi^n : \Omega^n(A[x]) \to \Omega^{n-1}(A)$ defined for $\alpha_i \in M_i$ by $\phi^n(\alpha_i) := 0$ if $i \neq 1$ and $\phi^n(\alpha_1) := \int_0^1 \alpha_1$. Here the formal integral is given by

$$\int_0^1 dx (x^k \omega) := \frac{1}{k+1} x^{k+1} \omega \bigg|_0^1 = \frac{1}{k+1} \omega \quad (\omega \in \Omega^{n-1}(A))$$

Then it can be checked that $(\phi^n)_n$ is a cochain homotopy between $p_0$ and $p_1$, and hence $\mathcal{H}^n_{dR}(p_0) = \mathcal{H}^n_{dR}(p_1)$.

4 Homotopy Invariant Subalgebras

The main aim of this section is to consider an analogue of the functor $\pi_0$ for algebras:

**Definition 4.1.** Let $A \in A_{nc}/A_u$. For $* \in \{nc, cr\}/\{u, cu, cur\}$, let

$$p_0, p_1 : A \to \lim_{\xi} \mathfrak{f}_A^{*}$$

be given by

$$p_0 := \lim_{\xi} \left[ (x \mapsto 0) \otimes \text{id} \right] Y_{A,F[x]}^* \quad \text{and} \quad p_1 := \lim_{\xi} \left[ (x \mapsto 1) \otimes \text{id} \right] Y_{A,F[x]}^*$$

We let the subalgebra $\mathfrak{g}^*_{\text{nc}}(A) \subseteq A$ be defined by

$$\mathfrak{g}^*_{\text{nc}}(A) := \{ a \in A : p_0(a) = p_1(a) \}.$$
We have the following useful lemma.

**Lemma 4.2.** With assumptions of Definition 4.1 suppose $A$ is generated by $G$ and $V$ is a vector basis for $F[x]$. For any $a \in A$ the following statements are equivalent.

(i) For every $\theta \in \Theta_{G,V}$, there exists $\hat{a}_\theta \in M^* \theta$ such that $\Upsilon^* \theta(a) = 1 \otimes \hat{a}_\theta$.

(ii) For every algebra $C \in \mathcal{A}_*$ and every morphism $\phi : A \to C[x]$ in $\mathcal{A}_{nc/\mathcal{A}_u}$, $\phi(a)$ is constant.

(iii) For every $\theta \in \Theta_{G,V}$, $((x \mapsto 0) \otimes \text{id}) \Upsilon^* \theta(a) = ((x \mapsto 1) \otimes \text{id}) \Upsilon^* \theta(a)$.

(iv) For every $C, \phi$ as in (ii), $(x \mapsto 0) \phi(a) = (x \mapsto 1) \phi(a)$.

(v) $a$ belongs to $\mathfrak{P}^*(A)$.

**Proof.** (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(iv) follow from the universal property of $M_{A,B}[x]$. (i)$\Rightarrow$(iii) and (iv)$\Rightarrow$(iii) are trivial. Also, (ii)$\Rightarrow$(i), (iv) and (iii)$\Leftrightarrow$(v) are trivial. If $\phi : A \to C[x]$ is a morphism then we can also consider the morphism $A \to (C[x])[y]$ given by $b \mapsto \phi(b)(xy)$. This shows that (iv)$\Rightarrow$(ii) is satisfied. \hfill \Box

It is easily verified that any morphism from $A$ to $B$ transforms $\mathfrak{P}^*(A)$ into $\mathfrak{P}^*(B)$. Thus we may consider the following subfunctor of $\text{id}_{\mathcal{A}_{nc/\mathcal{A}_u}}$:

$$\mathfrak{P}^* : \mathcal{A}_{nc/\mathcal{A}_u} \to \mathcal{A}_{nc/\mathcal{A}_u}.$$  

(We say that a functor $F : \mathcal{A} \to \mathcal{A}_*$ is a subfunctor of $\text{id}_{\mathcal{A}_*}$ if for any $A \in \mathcal{A}_*$, $F(A)$ is a subalgebra of $A$ (in unital cases, $F(A)$ is required to have the unit of $A$) and for any morphism $f : A \to B$, $F(f) = f|_{F(A)}$.) For any $A \in \mathcal{A}_{nc/\mathcal{A}_u}$, we have

$$\mathfrak{P}^{nc}(A) \subseteq \mathfrak{P}^*(A) \subseteq \mathfrak{P}^{cr}(A), \quad \mathfrak{P}^{cu}(A) \subseteq \mathfrak{P}^{cu}(A) \subseteq \mathfrak{P}^{cu}(A),$$

and for any unital algebra $A$, $\mathfrak{P}^{nc}(A) \subseteq \mathfrak{P}^u(A)$, $\mathfrak{P}^c(A) \subseteq \mathfrak{P}^{cu}(A)$, $\mathfrak{P}^{cr}(A) \subseteq \mathfrak{P}^{cu}(A)$.

**Theorem 4.3.** $\mathfrak{P}^* : \mathcal{A}_* \to \mathcal{A}_*$ for $* \in \{nc, c, cr, u, cu, cur\}$ is homotopy invariant.

**Proof.** For morphisms $f, g : A \to B$ in $\mathcal{A}_*$ suppose $H : A \to B[x]$ is an elementary homotopy in $\mathcal{A}_*$ from $f$ to $g$. For any $a \in \mathfrak{P}^*(A)$, $H(a)$ is constant and hence $f(a) = p_0 H(a) = p_1 H(a) = g(a)$. Thus $\mathfrak{P}^*(f) = \mathfrak{P}^*(g)$. The proof is complete. \hfill \Box

**Theorem 4.4.** For any algebra $A$, we have $\mathfrak{P}^{nc}(A) = 0$.

**Proof.** The morphism $A \to M_2(A)[x]$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} x$$

transforms any nonzero $a$ to a nonconstant polynomial. The proof is complete. \hfill \Box

It follows easily from Theorems 2.3 and 4.4 that for any algebra $A$, $\mathfrak{P}^u(A^+) = \mathbb{F}1$. 

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Theorem 4.5. For any $A \in \mathbf{A}_n$, we have $\mathfrak{P}^c(A) = \mathfrak{P}^{\text{cn}}(A)$ and $\mathfrak{P}^c(A) = \mathfrak{P}^{\text{cur}}(A)$.

Proof. We have $\mathfrak{P}^c(A) \subseteq \mathfrak{P}^{\text{cn}}(A)$. Let $a \in \mathfrak{P}^{\text{cn}}(A)$. For $C \in \mathbf{A}_c$ and $\phi : A \to C[x]$ it is easily verified that $\phi(1) \in C$ and $\phi(A) \subseteq \hat{C}[x]$ where $\hat{C} := \{ c \in C : \phi(1)c = c \}$. Thus $\phi : A \to \hat{C}[x]$ is a unital morphism and hence $\phi(a) \in \hat{C} \subseteq C$. So $\mathfrak{P}^{\text{cn}}(A) \subseteq \mathfrak{P}^c(A)$. The other case is similar. 

Theorem 4.6. For any algebra $A$ we have $\mathfrak{P}^c(A) = p^{-1}\mathfrak{P}^c(A_{\text{com}})$ where $p : A \to A_{\text{com}}$ denotes the canonical projection.

Proof. It follows from Lemma 4.2(ii) and the fact that any morphism $\phi : A \to C[x]$ where $\phi(1) = 0$. The proof is complete. 

In general, it is not easy to determine the subalgebras $\mathfrak{P}^*(A)$. We shall consider some other homotopy invariant subfunctors of $\text{id}_{\mathbf{A}_c}$ which are computable and may be applied for our proposes instead of $\mathfrak{P}^*$. In §5 we will show that $\mathfrak{P}^*$ is the largest subfunctor among all homotopy invariant subfunctors of $\text{id}_{\mathbf{A}_c}$.

Lemma 4.7. Let $C$ be an algebra and $p = \sum_{i=0}^n c_i x^i$ be a polynomial in $C[x]$.

(i) If $C$ is commutative and if $p^k = p$ for some $k \geq 2$ with the property $\text{Char}(\mathbb{F}) \nmid (k - 1)$, then $p$ is constant.

(ii) If $C$ is unital and has no nonzero nilpotent element and if $p$ is a root of a nonzero polynomial with coefficients in $\mathbb{F}$, then $p$ is constant.

Proof. (i) We have $c_0^k = c_0$ and $kc_0^{k-1}c_1 = c_1$. Thus $kc_0c_1 = c_0c_1$. This implies that $c_0c_1 = 0$ and therefore $c_1 = 0$. Analogously and respectively, it is proved that $c_2 = 0, \ldots, c_n = 0$. (ii) There are $\lambda_0, \ldots, \lambda_{k-1} \in \mathbb{F}$ with $k \geq 1$ such that $p^k + \lambda_{k-1}p^{k-1} + \cdots + \lambda_0 = 0$. Thus $c_n^k = 0$ and then, by the assumption, we have $c_n = 0$. The proof is complete. 

Proposition 4.8. Let $A$ be an algebra and let $a \in A$.

(i) If $a^k = a$ for some $k \geq 2$ with $\text{Char}(\mathbb{F}) \nmid (k - 1)$ then $a \in \mathfrak{P}^c(A)$.

(ii) If $a$ is an idempotent then $a \in \mathfrak{P}^c(A)$.

(iii) If $A$ is unital and $a$ is integral over $\mathbb{F}$ then $a \in \mathfrak{P}^{\text{cur}}(A)$.

Proof. (i) and (ii) follow from Lemma 4.7 and the definition of $\mathfrak{P}^c$ by the analogue of Lemma 4.2(ii). (iii) follows from (i). 

For any algebra $A$ and every $k \geq 2$ let $\mathfrak{P}_k(A) \subseteq A$ denote the subalgebra generated by all elements $a$ with the property $a^k = a$. Let also $\mathfrak{P}(A) \subseteq A$ denote the subalgebra generated by all elements $a$ such that $a^k = a$ for some $k \geq 2$. For any commutative unital algebra $A$ let $\text{Int}(A) \subseteq A$ denote integral closure of $\mathbb{F}$ in $A$. It is clear that $\mathfrak{P}_k(A) \subseteq \mathfrak{P}(A)$ and $\mathfrak{P}(A) \subseteq \text{Int}(A)$. We have also the subfunctors $\mathfrak{P}^c, \mathfrak{P}_k : \mathbf{A}_{\text{nc}} \to \mathbf{A}_{\text{nc}}$ and $\text{Int} : \mathbf{A}_{\text{cu}} \to \mathbf{A}_{\text{cu}}$.

Theorem 4.9.
Theorem 4.10. Let \( W, W' \) be vector spaces and \( W' \supseteq W \) be a vector subspace. Let \( x \in W' \otimes W \) be such that for every linear functional \( \alpha : W \to \mathbb{F} \) with \( \alpha|_{W_0} = 0 \) we have \((\text{id}_{W'} \otimes \alpha)(x) = 0\). Then \( x \) belongs to \( W' \otimes W_0 \).

Lemma 4.11. Suppose \( \mathbb{F} \) is algebraically closed. Let \( A \in \mathbb{A}_{\text{cur}} \), \( W \) be a vector space, and \( W_0 \subseteq W \) be a vector subspace. For any \( c \in A \otimes W \) if \( h \otimes \text{id}_W(c) \in W_0 \) for every \( h \in \text{Pnt}(A) \), then \( c \in A \otimes W_0 \).

Proof. Let \( Z \subseteq A^m \) be the zero-locus of a radical ideal \( I \subseteq \mathbb{F}[X_m] \) with \( A \cong \mathbb{F}[X_m]/I \). We know \( Z \) may be identified with \( \text{Pnt}_{\neq 0}(A) \) through \( \zeta \mapsto (p(X_m) \mapsto p(\zeta)) \), and by Hilbert’s Nullstellensatz, \( A \otimes W \) may be identified with vector space of all functions \( f : Z \to W \) such that \( f(\zeta) = \sum \zeta^i w_i \) for some polynomial \( \sum w_i X_m^i \) with \( w_i \in W \). So, suppose that \( f \) be such a representation for \( c \). By assumptions we have \( f(\zeta) \in W_0 \) for every \( \zeta \in Z \). Thus for any linear functional \( \alpha : W \to \mathbb{F} \) with \( \alpha|_{W_0} = 0 \) we have \( \text{id}_A \otimes \alpha(c) = \alpha \circ f = 0 \). Hence \( c \in A \otimes W_0 \).

We have the following corollary of Lemma 4.11.

Proposition 4.12. Suppose \( \mathbb{F} \) is algebraically closed and let \( F : \mathbb{A}_{\text{cur}} \to \mathbb{A}_{\text{cur}} \) be any subfunctor of \( \text{id}_{\mathbb{A}_{\text{cur}}} \). Then for every \( A, B \in \mathbb{A}_{\text{cur}} \), we have \( F(A \otimes B) = F(A) \otimes F(B) \).

Proof. Let \( c \in F(A \otimes B) \subseteq A \otimes B \). For any \( h \in \text{Pnt}(A) \), \( h \otimes \text{id}_B : A \otimes B \to B \) is a morphism in \( \mathbb{A}_{\text{cur}} \) and hence \( h \otimes \text{id}_B(c) \in F(B) \subseteq B \). Thus it follows from Lemma 4.11 that \( c \in A \otimes F(B) \). Similarly, \( c \in F(A) \otimes B \). Thus \( c \in F(A) \otimes F(B) \).

The following theorem follows directly from Proposition 4.12.

Theorem 4.13. Suppose \( \mathbb{F} \) is algebraically closed. Then the functors

\[ \mathbb{P}^c, \mathbb{P}^{c^*}, \mathbb{P}, \mathcal{I}_k \mathbb{P}, \text{Int} : \mathbb{A}_{\text{cur}} \to \mathbb{A}_{\text{cur}} \]

preserve tensor product.
For any affine scheme \( S = S(A) \) \((A \in \text{Aff})\) the affine scheme of path-connected components of \( S \) is defined to be the affine scheme

\[
\pi_0(S) := \mathcal{S}(\mathfrak{P}^{cu}(A)).
\]  

(7)

Thus \( \pi_0 \) may be considered as a \( \mathbb{A}^1 \)-homotopy invariant functor

\[
\pi_0 : \text{Aff} \to \text{Aff}.
\]

Recall that an affine group-scheme is a group object \( G \) in \( \text{Aff} \). Suppose that \( G = S(A) \). Then the group structure of \( G \) is induced by a Hopf algebra structure on \( A \). In case \( \mathbb{F} \) is algebraically closed and \( G \) is an affine variety, \( G \) is called affine group-variety. Note that in this case \( A \) is a finitely generated integral domain.

**Theorem 4.14.** For any affine group-variety \( G \), \( \pi_0(G) \) is an affine group-scheme.

*Proof.* It follows from Theorem 4.13.

For a discussion on path-connected components of algebraic groups see [13, Chapter 2]. The following important theorem and its proof have been kindly offered to me by Professor Cortiñas.

**Theorem 4.15.** Suppose that \( \text{Char}(\mathbb{F}) = 0 \). Then

\[
\pi_0(S) = \mathcal{H}_{\text{degR}}^0(S) \quad (S \in \text{Aff}).
\]

*Proof.* Let \( S = S(A) \). Consider the universal derivation \( d : A \to \Omega^1(A) \) as described in [13]. We must show that \( \ker(d) = \mathfrak{P}^{cu}(A) \). Let \( \bar{A} \) denote the commutative unital algebra with underlying vector space \( A \oplus \Omega^1(A) \) and multiplication given by \((a, \omega)(a', \omega') = (aa', a\omega + a'\omega)\) for \( a, a' \in A \) and \( \omega, \omega' \in \Omega^1(A) \). Then the assignment \( a \mapsto a + d(a)x \) defines a morphism from \( A \) into \( \bar{A}[x] \). Thus if \( a \in \mathfrak{P}^{cu}(A) \) then \( d(a) = 0 \). Hence \( \mathfrak{P}^{cu}(A) \subseteq \ker(d) \). Suppose that \( a \in \ker(d) \). Let \( \phi : A \to B[x] \) be an arbitrary unit preserving morphism for a commutative unital algebra \( B \). \( B[x] \) can be considered as a unital \( A \)-module through \( \phi \). Let \( D : B[x] \to B[x] \) denote the ordinary derivation given by \( D(\sum b_i x^i) = \sum i b_i x^{i-1} \). Then \( D\phi : A \to B[x] \) is a derivation into the \( A \)-module \( B[x] \). Thus \( D\phi(a) = 0 \). This means that \( \phi(a) \) must be a constant polynomial and hence \( a \in \mathfrak{P}^{cu}(A) \). The proof is complete.

## 5 Costrict homotopization

**Definition 5.1.** Let \( A \) be an admissible category of algebras and \( C \) be an arbitrary category. The costrict homotopization of a functor \( F : A \to C \) is defined to be a homotopy invariant functor \( [F] : A \to C \) coming with a natural transformation \( \alpha_F : F \to [F] \) such that the following universal property is satisfied: For any homotopy invariant functor \( G : A \to C \) and any natural transformation \( \beta : F \to G \) there is a unique natural transformation \( \underline{\beta} : [F] \to G \) such that \( \beta = \underline{\beta}\alpha_F \).

It is clear that if \( [F] \) exists then it is unique up to a natural isomorphism. Also, for any homotopy invariant functor \( F \) we have \( [F] = F \).

Note that Definition 5.1 is a simple generalization of Weibel’s concept of strict homotopization ([21][7]). It seems that this concept at the first time has been considered by Gersten [9].
Theorem 5.2. Suppose $\mathbf{C}$ has finite coequalizers. Then any functor $F : A \to \mathbf{C}$ has a costrict homotopization.

Proof. For any $A \in A$ let $\alpha_F(A) : F(A) \to [F](A)$ be a coequalizer of 

$$F(p_0), F(p_1) : F(A[x]) \to F(A)$$

in $\mathbf{C}$. For any morphism $f : A \to B$ in $A$, the coequalizer property of $[F](A)$ implies the existence of a unique morphism $\beta(F)[f]$ in $\mathbf{C}$ making the diagram

$$
\begin{array}{ccc}
F(A[x]) & \xrightarrow{F(p_0)} & F(A) \\
\downarrow F(f[x]) & & \downarrow [F](f) \\
F(B[x]) & \xrightarrow{F(p_0)} & F(B)
\end{array}
$$

commutative. Thus $[F]$ may be considered as a functor. Suppose that $H : A \to B[x]$ is an elementary homotopy from $f$ to $g$. We have

$$
\alpha_F(B)F(f) = \alpha_F(B)(F(p_0)F(H)) = (\alpha_F(B)(F(p_0)))F(H) = (\alpha_F(B)F(p_1))F(H) = \alpha_F(B)(F(p_1)F(H)) = \alpha_F(B)F(g)
$$

Thus $[F](f) = [F](g)$, and therefore $[F]$ is homotopy invariant. Suppose $G : A \to \mathbf{C}$ is homotopy invariant and $\beta : F \to G$ is a natural transformation. By Lemma 3.6(iii), $G(p_0) = G(p_1)$, and hence universality of the coequalizer $[F](A)$ implies the existence of a morphism $\beta(A)$ making the following diagram commutative:

$$
\begin{array}{ccc}
F(A[x]) & \xrightarrow{F(p_0)} & F(A) \\
\downarrow \beta(A[x]) & & \downarrow \beta(A) \\
G(A[x]) & \xrightarrow{G(p_0)} & G(A)
\end{array}
$$

It can be seen easily from the proof of Theorem 5.2 that for the identity functor $\text{id} : A \to A$ we have $[\text{id}] = 0$. (Indeed, it follows from the fact that for any $A \in A$ and every $a, b \in A$, $p_0(a + (b - a)x) = a$ and $p_1(a + (b - a)x) = b$.) Also, for any affine scheme $S = S(A)$ regarded as the functor

$$S = \text{Hom}_{\mathbf{A}_{cu}}(A, ?) : \mathbf{A}_{cu} \to \text{Set}
$$

we have

$$[S] = [A, ?].$$

A cotype for the concept of strict homotopization may be defined as follows.
**Definition 5.3.** Let $A$ be an admissible category of algebras and $C$ be an arbitrary category. The costrict homotopization of a functor $F : A \to C$ is defined to be a homotopy invariant functor $[F] : A \to C$ coming with a natural transformation $\alpha_F : [F] \to F$ such that the following universal property is satisfied: For any homotopy invariant functor $G : A \to C$ and any natural transformation $\beta : G \to F$ there is a unique natural transformation $\overline{\beta} : G \to [F]$ such that $\beta = \alpha_F \overline{\beta}$.

It is clear that if $[F]$ exists then it is unique up to a natural isomorphism. Also, for any homotopy invariant functor $F$ we have $[F] = F$.

The following theorem is one of the main results of this note.

**Theorem 5.4.** Suppose $C$ has finite equalizers and arbitrary inverse limits. Then any functor $F : A_* \to C$, where $* \in \{ nc, c, cr, u, cu, cur \}$, has a costrict homotopization.

**Proof.** Let $A, M, T$ denote $A_*, M^*, T^*$. For any $A \in A$, let $\alpha_F(A) : [F](A) \to F(A)$ denote the equalizer of the morphisms

$$p_0, p_1 : F(A) \to \lim_{x} F(A, F[x], x),$$

given (similarly to $p_0, p_1$ in Definition 4.1) by

$$\hat{p}_0 := \lim_{x} F((x \to 0) \otimes \text{id}) \mathcal{Y}_{A, F[x]} \text{ and } \hat{p}_1 := \lim_{x} F((x \to 1) \otimes \text{id}) \mathcal{Y}_{A, F[x]}.$$

For any $f : A \to B$ in $A$, by the universal property of the equalizer $[F](B)$, there is a unique morphism $[F](f)$ making the following diagram commutative:

$$\begin{array}{cccc}
[F](A) & \xrightarrow{\alpha_F(A)} & \lim_{x} F(A, F[x], x) \\
| & | \\
[F](f) & \downarrow & \downarrow \\
[F](B) & \xrightarrow{\alpha_F(B)} & \lim_{x} F(B, F[x], x)
\end{array}$$

Thus $[F]$ may be considered as a functor. Suppose that $H : A \to B[x]$ is an elementary homotopy from $f$ to $g$. Let $\overline{H} : \mathcal{Y}_{A, F[x]} \to B$ denote the associated pro-morphism of $H$. We have

$$f = \overline{H}((x \to 0) \otimes \text{id}) \mathcal{Y}_{A, F[x]} \text{ and } g = \overline{H}((x \to 1) \otimes \text{id}) \mathcal{Y}_{A, F[x]}.$$

It follows that,

$$\left(\lim_{x} F(f)\right) \alpha_F(A) = \left(\lim_{x} F(\overline{H})\right) \hat{p}_0 \alpha_F(A) = \left(\lim_{x} F(\overline{H})\right) \hat{p}_1 \alpha_F(A) = \left(\lim_{x} F(g)\right) \alpha_F(A).$$

This implies that $[F](f) = [F](g)$, and hence $[F]$ is homotopy invariant. Suppose $G : A \to C$ is homotopy invariant and $\beta : G \to F$ is a natural transformation. We
have the following commutative diagram in pro-$\mathbf{C}$:

\[
\begin{array}{c}
F(A) \\
\downarrow^{\beta(A)}
\end{array}
\quad
\begin{array}{c}
\overset{F(F[x])}{\Rightarrow}
\overset{F(x \mapsto 0)}{\Rightarrow}
\overset{F(x \mapsto 1)}{\Rightarrow}
G(A)
\end{array}
\quad
\begin{array}{c}
G(F[x] \otimes \mathcal{M}_{A,F[x]}) \\
\downarrow^{G(x \mapsto 0)}
\downarrow^{G(x \mapsto 1)}
\end{array}
\quad
\begin{array}{c}
\overset{G(F[x])}{\Rightarrow}
\overset{G(x \mapsto 0)}{\Rightarrow}
\overset{G(x \mapsto 1)}{\Rightarrow}
\end{array}
\quad
\begin{array}{c}
\beta(\mathcal{M}_{A,F[x]})
\end{array}
\]

On the other hand, by Lemma 3.6(iii), the pro-morphisms

\[
G(F[x] \otimes \mathcal{M}_{A,F[x]}) \\
\downarrow^{G(x \mapsto 0)}
\downarrow^{G(x \mapsto 1)}
\]

are (componentwise) equal. Hence we have

\[
\hat{p}_0 \beta(A) = \hat{p}_1 \beta(A).
\]

Now, it follows from the universal property of the equalizer $\lceil F \rceil (A)$ that there is a unique morphism $\overline{\beta}(A) : G(A) \to \lceil F \rceil (A)$ satisfying $\beta(A) = \alpha^F(A) \overline{\beta}(A)$.  

The following is seen directly from Definition 4.1 and the proof of Theorem 5.4.

**Theorem 5.5.** The functor $\mathcal{P}^* : A_* \to A_*$ is a costrict homotopization of the identity functor $id : A_* \to A_*$. Therefore, in a sense, $\mathcal{P}^*$ is the greatest homotopy invariant subfunctor of $id$.

More generally, the following result is also seen from the above theorem and the proof of Theorem 5.4.

**Theorem 5.6.** Let $C$ be a category with arbitrary limits and $F : A_* \to C$ be a functor that preserves arbitrary limits. Then $\lceil F \rceil = F \circ \mathcal{P}^*$.

As a direct corollary of the above theorem we have,

**Corollary 5.7.** Let $S$ denote an ordinary scheme over $F$ regarded as a Zariski-sheaf $A_{cu} \to \mathbf{Set}$ [6]. Then $\lceil S \rceil = S \circ \mathcal{P}^{cu}$.

In particular in case $S$ is an affine scheme regarded as the functor [8] we have

\[
\lceil S \rceil = \text{Hom}_{A_{cu}}(A, \mathcal{P}^{cu}(?)).
\]

Thus for any affine scheme $S$ we have defined three types $\pi_0(S), \lceil S \rceil, \lceil S \rceil$ of homotopizations given by [7], [9], [10].

**Remark 5.8.** It is clear that the concept of (co)strict homotopization can be defined for any functor $F : B \to C$ where $B$ is a category with a compatible relation as defined in [3] or more generally, for any category $B$ with a class of weak equivalences. Also it seems that the analogues of Theorems 5.2 and 5.4 can be stated for model categories $B$ with natural cylinder and path objects [2] §I.3 and §I.3a.
6 Intrinsic Singular Cohomology

Let $\Delta := \{n\}$ denote the category of finite ordinals $n := \{0 \leq \ldots < n\}$. The standard simplicial algebra $\mathbb{F}[\Delta] \in \text{sim-Ab}$ is given by the following assignments:

$$
n \mapsto \mathbb{F}[\Delta_n] := \mathbb{F}[x_0, \ldots, x_n]/\langle 1 - \sum_{i=0}^{n} x_i \rangle, \quad (\alpha: n \to m) \mapsto ([i] \mapsto \sum_{j \in \alpha^{-1}(i)} [x_j]).$$

For $* \in \{\text{nc, c, cr, u, cu, cur}\}$, consider the functors

$$\mathcal{G}^* := \mathcal{M}^*(?, \mathbb{F}[\Delta]) \quad \text{and} \quad \mathcal{G}^*_n := \lim \mathcal{M}^*(?, \mathbb{F}[\Delta])$$
given by:

$$\mathcal{G}^* : \text{nc-Ab} / \text{A}_u \to \text{cosim-pro-Ab} \quad \text{A} \mapsto (n \mapsto \mathcal{M}^*(A, \mathbb{F}[\Delta_n]))$$

$$\mathcal{G}^*_n : \text{nc-Ab} / \text{A}_u \to \text{cosim-Ab} \quad \text{A} \mapsto (n \mapsto \lim \mathcal{M}^*(A, \mathbb{F}[\Delta_n]))$$

Let $F : \text{A}_u \to \text{Ab}$ be an arbitrary functor. For any $A \in \text{nc-Ab} / \text{A}_u$, we have the cosimplicial abelian groups $F \mathcal{G}^*_n(A)$ and $\lim F \mathcal{G}^*(A)$. Let $\mathcal{N} : \text{cosim-Ab} \to \text{coChain}$ denote the normalization functor. We let (see [20] § 8.4):

$$\mathcal{H}^n_F(A) := H^n \mathcal{N} F \mathcal{G}^*_n(A) \cong \pi^n(F \mathcal{G}^*(A), 0)$$

$$\mathcal{H}^n_F(A) := H^n \lim F \mathcal{G}^*(A) \cong \pi^n(\lim F \mathcal{G}^*(A), 0)$$

It is clear that $\mathcal{H}^n_F$ and $\mathcal{H}^n_F$ may be considered as the functors

$$\mathcal{H}^n_F, \mathcal{H}^n_F : \text{nc-Ab} / \text{A}_u \to \text{Ab}.$$  

We have a canonical natural transformation $\mathcal{G}^* \to \mathcal{G}^*$ induced by the inverse limits. This, in turn, induces the canonical natural transformations

$$F \mathcal{G}^* \to \lim F \mathcal{G}^* \quad \text{and (hence)} \quad \mathcal{H}^n_F \to \mathcal{H}^n_F.$$

**Theorem 6.1.** The functors $\mathcal{H}^n_F$ and $\mathcal{H}^n_F$ are homotopy invariant.

The proof is an adaptation of the usual proof of homotopy invariance of singular cohomology of topological spaces by using prism operators. See [12] Theorem 2.10.

**Proof.** For every $0 \leq i \leq n$, let $\phi_i^\alpha : \mathbb{F}[\Delta_n] \otimes \mathbb{F}[x] \to \mathbb{F}[\Delta_{n+1}]$ be defined by $[x_j] \mapsto [x_j]$ if $j < i$; $[x_i] \mapsto [x_i + x_{i+1}]$; $[x_{j+1}] \mapsto [x_{j+1}]$ if $j > i$; and $x \mapsto [x_{i+1} + \cdots + x_{n+1}]$. For every algebra $B$ let

$$\Gamma_n : \mathcal{M}(B \otimes \mathbb{F}[x], \mathbb{F}[\Delta_n] \otimes \mathbb{F}[x]) \to \mathcal{M}(B, \mathbb{F}[\Delta_n])$$

be the unique pro-morphism satisfying

$$(\text{id}_{\mathbb{F}[\Delta_n]} \otimes \text{flip})(\mathcal{Y}_{B, \mathbb{F}[\Delta_n]} \otimes \text{id}_{\mathbb{F}[x]}) = (\text{id}_{\mathbb{F}[\Delta_n]} \otimes \text{id}_{\mathbb{F}[x]} \otimes \Gamma_n)(\mathcal{Y}_{B \otimes \mathbb{F}[x], \mathbb{F}[\Delta_n] \otimes \mathbb{F}[x]}).$$

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Suppose that $H : A \to B \otimes \mathbb{F}[x]$ is an elementary homotopy from $f$ to $g$. Let

$$\hat{\phi}^n_i : \lim_{\to n} \mathcal{M}(A, \mathbb{F}[\Delta_{n+1}]) \to \lim_{\to n} \mathcal{M}(B, \mathbb{F}[\Delta_n])$$

be given by

$$\hat{\phi}^n_i := \lim_{\to n} \mathcal{M}(H, \mathbb{F}[\Delta_n] \otimes \mathbb{F}[x]) \mathcal{M}(A, \phi^n_i),$$

and let the prism operator

$$P^n : F(\lim_{\to n} \mathcal{M}(A, \mathbb{F}[\Delta_{n+1}])) \to F(\lim_{\to n} \mathcal{M}(B, \mathbb{F}[\Delta_n]))$$

be given by $P^n := \sum_{i=0}^{n} (-1)^i F(\hat{\phi}^n_i)$. Then, it can be checked that $(P^n)_n$ is a cochain homotopy between $\mathcal{M}_F(\mathbb{S})$ and $\mathcal{M}_F(\mathbb{G})$, and hence $\overline{H}_F(f) = \overline{H}_F(g)$. Thus $\overline{H}_F$ is homotopy invariant. Homotopy invariance of $\overline{H}_F$ is obtained similarly.

**Theorem 6.2.** We have the following natural isomorphism of abelian groups:

$$\overline{H}_F^0(A) \cong [F](A) \quad (A \in \mathbf{A}_*)$$

**Proof.** It follows from the proof of Theorem 5.4 and definition of $\overline{H}_F(A)$. (Note that we are required to have $A \cong \mathcal{M}_{\mathbf{A}_*}[\Delta_0]$; thus $A$ must be in $\mathbf{A}_*$.)

So, the restricted functors

$$\overline{H}_F : \mathbf{A}_* \to \mathbf{Ab}$$

may be interpreted as a sort of higher costrict homotopizations of $F : \mathbf{A}_* \to \mathbf{Ab}$.

In case $F$ is the identity functor $\text{id} : \mathbf{A}_* \to \mathbf{A}_*$, we put

$$\overline{H}_*^0(A) := \overline{H}_F^0(A) = \overline{H}_F^0(A).$$

Thus we have defined the homotopy invariant functors

$$\overline{H}_{nc}^0, \overline{H}_{uc}^0, \overline{H}_{cr}^0 : \mathbf{A}_{nc} \to \mathbf{Vec} \quad \text{and} \quad \overline{H}_u^0, \overline{H}_{cu}^0, \overline{H}_{cur}^0 : \mathbf{A}_u \to \mathbf{Vec}.$$  

From (8) we have the natural transformations

$$\overline{H}_{nc}^0 \to \overline{H}_u^0 \to \overline{H}_{cr}^0 \quad \text{and} \quad \overline{H}_u^0 \to \overline{H}_{cu}^0 \to \overline{H}_{cur}^0.$$  

For any algebra $A$ it can be easily checked that

$$\overline{H}_{nc}^0(A) = \mathcal{P}_{nc}(A) = 0, \quad \overline{H}_u^0(A) = \mathcal{P}^c(\mathcal{A}_{\text{com}}), \quad \overline{H}_{cr}^0(A) = \mathcal{P}^{cr}(\mathcal{A}_{\text{com}})_{\text{red}}$$  

(11)

Analogous identities are satisfied for unital cases. By Theorem 4.5 we have

$$\overline{H}_u^0(A) = \overline{H}_{cu}^0(A), \quad \overline{H}_{cr}^0(A) = \overline{H}_{cur}^0(A) \quad (A \in \mathbf{A}_u).$$  

(12)

We now define a cup product between the cocycles:

Let the morphisms $\phi_{n,m}, \psi_{n,m}$ be given by

$$\phi_{n,m} : \mathbb{F}[\Delta_{n+m}] \to \mathbb{F}[\Delta_n], \quad [x_i] \mapsto [x_i] \text{ if } 0 \leq i \leq n \text{ and } [x_i] \mapsto 0 \text{ otherwise}$$

$$\psi_{n,m} : \mathbb{F}[\Delta_{n+m}] \to \mathbb{F}[\Delta_m], \quad [x_{n+i}] \mapsto [x_i] \text{ if } 0 \leq i \leq m \text{ and } [x_i] \mapsto 0 \text{ otherwise}$$  

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We let the cup product of the cochains
\[ c_n \in \lim_{\Delta_n} \mathcal{M}^*_{A,F[\Delta_n]} \quad \text{and} \quad c_m \in \lim_{\Delta_m} \mathcal{M}^*_{A,F[\Delta_m]} \]
be given by
\[ c_n \cup c_m := (\lim_{\Delta_n} \mathcal{M}^*_{A,\phi_{n,m}}(c_n)) (\lim_{\Delta_m} \mathcal{M}^*_{A,\psi_{n,m}}(c_m)) \in \lim_{\Delta_{n+m}} \mathcal{M}^*_{A,F[\Delta_{n+m}]} \]
It can be checked that \( \cup \) makes the graded vector space \( \oplus_{n=0}^{\infty} \lim_{\Delta_n} \mathcal{M}^*_{A,F[\Delta_n]} \) to a DGCA. Hence, \( \cup \) can be considered as a multiplication on
\[ \mathcal{H}_*(A) := \oplus_{n=0}^{\infty} \mathcal{H}^n_*(A), \]
making it to a GCA.

For any affine scheme \( S = S(A) \) \( (A \in \text{A}_u) \) the intrinsic singular cohomology algebra of \( S \) is defined to be the GCA given by
\[ \mathcal{H}_{\text{sing}}(S) = \oplus_{n=0}^{\infty} \mathcal{H}^n_{\text{sing}}(S) := \mathcal{H}_{\text{cu}}(A). \]
Thus we have defined a \( \text{A}^1 \)-homotopy invariant cohomology theory
\[ \mathcal{H}_{\text{sing}} : \text{Aff}^{\text{op}} \to \text{A}_u. \]

We end this section by the following result.

**Theorem 6.3. de Rham Theorem at degree zero.** Suppose \( \text{Char}(F) = 0 \). For any affine scheme \( S = S(A) \) \( (A \in \text{A}_u) \) the graded-algebra morphism \( \mathcal{H}_{\text{deR}}(S) \to \mathcal{H}_{\text{sing}}(S) \) induced by the universal property of de Rham complex is an isomorphism at degree zero:
\[ \mathcal{H}_{\text{deR}}^0(S) \cong \mathcal{H}_{\text{sing}}^0(S). \]

**Proof.** It follows from Theorem 4.15 and identities 11 and 12. \( \square \)

## 7 Generalized Morphisms Between Algebras

For any algebra \( A \) denote by \( \mathcal{J}(A) \) (resp. \( \mathcal{J}_l(A), \mathcal{J}_r(A) \)) the set of all ideals (resp. left, right ideals) of \( A \). Also, let \( \mathcal{J}_s(A) \subseteq \mathcal{J}(A) \) where \( s \in \{\text{prpr}, \text{fcodim}, \text{prm}, \text{max}\} \) denote respectively the subset of proper, proper with finite codimension, prime, and maximal ideals of \( A \). \( \mathcal{J} \) may be considered as a functor \( \text{A}_u^{\text{op}} \to \text{Set} \) in the obvious way.

Similarly, we have the functors \( \mathcal{J}_l, \mathcal{J}_r : \text{A}_u^{\text{op}} \to \text{Set}, \mathcal{J}_{\text{prpr}}, \mathcal{J}_{\text{fcodim}} : \text{A}_u^{\text{op}} \to \text{Set}, \) and \( \mathcal{J}_{\text{prm}} : \text{A}^{\text{op}}_{\text{cu}} \to \text{Set} \), and in case \( F \) is algebraically closed, the functor \( \mathcal{J}_{\text{max}} : \text{A}^{\text{op}}_{\text{cu}} \to \text{Set} \) as well. Thus by the canonical extension we have also the functors
\[ \mathcal{J}, \mathcal{J}_l, \mathcal{J}_r : \text{pro-A}^{\text{op}}_{\text{nc}} \to \text{ind-Set}, \mathcal{J}_{\text{prpr}}, \mathcal{J}_{\text{fcodim}} : \text{pro-A}^{\text{op}}_{\text{pr}} \to \text{ind-Set} \]
\[ \mathcal{J}_{\text{prm}} : \text{pro-A}^{\text{op}}_{\text{cu}} \to \text{ind-Set}, \mathcal{J}_{\text{max}} : \text{pro-A}^{\text{op}}_{\text{cu}} \to \text{ind-Set}. \]

For \( A \in \text{pro-A}_u \) we call any member of the set \( \lim \mathcal{J}(A) \) a pro-ideal of \( A \). We need some facts and definitions: Let \( A = (A_i, \phi_{i'}) \) be a pro-algebra. (i) Any pro-ideal \( T \) of \( A \) is a class \( [T_i] \) of the equivalence relation \( \sim \) on \( \cup \mathcal{J}(A_i) \) defined by
\[ (T_i \sim T'_i) \Leftrightarrow (\exists \ i'' \geq i', i : \phi_{i''}^{-1}(T_i) = \phi_{i'}^{-1}(T'_i)) \quad \text{for} \quad T_i \in \mathcal{J}(A_i), T_i' \in \mathcal{J}(A_i'). \]
(ii) For any pro-morphism \( f \) from \( A \) into an algebra \( B \) we let \( \ker(f) \) denote the pro-ideal \( \lim\{f(0)\}_i \) of \( A \) where \( 0 \) denotes the zero-ideal of \( B \). (Note that the notion of zero pro-ideal is meaningless.)

(iii) For pro-ideals \( T = [T_i], T' = [T'_i] \) of \( A \) we let the pro-ideal \( T \cap T' \) be defined by \( [\phi^{-1}_{i,i'}(T_i) \cap \phi^{-1}_{i,i'}(T'_i)] \) where \( i'' \) is an arbitrary index greater than \( i \) and \( i' \). It is easily verified that the intersection \( T \cap T' \) is well-defined. We write \( T \subseteq T' \) and say that \( T' \) includes \( T \) if there exists \( i'' \geq i, i' \) such that \( \phi^{-1}_{i,i'}(T_i) \subseteq \phi^{-1}_{i,i'}(T'_i) \). It can be checked that inclusion is a well-defined partial ordering on pro-ideals.

In (iv)-(vii) below suppose \( A = (A_i, \phi_{i,i'}) \) and \( B = (B_j, \psi_{j,j'}) \) are pro-algebras such that all structural morphisms \( \phi_{i,i'}, \psi_{j,j'} \) are surjective: (iv) If \( T_i \in \mathcal{I}(A_i) \) and \( T' \in \mathcal{I}(A'_i) \) represent the same pro-ideal \( T \) of \( A \), then

\[
A_i/T_i \cong A_i' / \phi^{-1}_{i,i'}(T_i) = A_i' / \phi^{-1}_{i,i'}(T_i') \cong A_i' / T_i' \quad \text{(for some } i'' \geq i, i').
\]

Thus we can define the associated quotient algebra up to isomorphism by \( A/T := A_i/T_i \). Note that we have a canonical quotient pro-morphism \( q_T : A \to A/T \) represented by the quotient morphism \( q_{T_i} : A_i \to A_i/T_i \). Also, \( T = \ker(q_T) \). (v) It is clear that if for every indexes \( i, i' \) the algebras \( A_i, A_i' \) are considered as ideals in themselves then \( [A_i] = [A_i] \). We denote the pro-ideal \([A_i] \) of \( A \) by \( A_{\text{id}} \). (vi) For pro-ideals \( T = [T_i], T' = [T'_i] \) of \( A \) we let the pro-ideal \( T + T' \) be defined by \( \phi^{-1}_{i,i'}(T_i) + \phi^{-1}_{i,i'}(T'_i) \) where \( i'' \) is an arbitrary index with \( i'' \geq i, i' \). The sum \( T + T' \) is well-defined. (vii) For pro-ideals \( T = [T_i], S = [S_j] \) of \( A, B \) we associate a pro-ideal of \( A \otimes B \) given by

\[
T \otimes B_{\text{id}} + A_{\text{id}} \otimes S := [T_i \otimes B_j + A_i \otimes S_j]
\]

It is easily checked that this is well-defined. \( (T \otimes S := [T_i \otimes S_j] \) is not well-defined.)

We define a category \( A_{nc} \) as follows: The objects of \( A_{nc} \) are those of \( A_{nc} \), and

\[
\text{Hom}_{A_{nc}}(A, B) := \lim_{\rightarrow} (\mathcal{M}_{A,B}^{nc}).
\]

The composition \( \circ : \text{Hom}_{A_{nc}}(B, C) \times \text{Hom}_{A_{nc}}(A, B) \to \text{Hom}_{A_{nc}}(A, C) \) is given by

\[
\lim_{\rightarrow} (\mathcal{M}_{B,C}^{nc}) \times \lim_{\rightarrow} (\mathcal{M}_{A,B}^{nc}) \to \lim_{\rightarrow} (\mathcal{M}_{B,C}^{nc} \otimes \mathcal{M}_{A,B}^{nc}) \to \lim_{\rightarrow} (\mathcal{M}_{A,C}^{nc})
\]

where the first map is given by

\[
(T, S) \mapsto T \otimes (\mathcal{M}_{A,B})_{\text{id}} + (\mathcal{M}_{B,C})_{\text{id}} \otimes S
\]

and the second map is \( \lim_{\rightarrow} (\Phi_{A,B,C}) \). Equivalently, we may let

\[
T \circ S := \ker[(q_T \otimes q_S)\Phi_{A,B,C}].
\]

Associativity of \( \circ \) follows from \( \{4\} \). For any morphism \( f : A \to B \) in \( A_{nc} \) denote by \( T \in \text{Pnt}(\mathcal{M}_{A,B}) \) the point associated with \( f \). By Corollary \( \{23\} \) the assignment \( f \mapsto \ker(T) \) defines an embedding of \( \text{Hom}_{A_{nc}}(A, B) \) into \( \text{Hom}_{A_{nc}}(A, B) \). We have

\[
\text{id}_{\mathcal{M}_{A,B}} = \left( \begin{array}{ccc}
\mathcal{M}_{A,B} & \Phi_{A,A,B} & \mathcal{M}_{A,B} \\
\Phi_{A,A,B} & \mathcal{M}_{A,A} & \mathcal{M}_{A,B} \\
\mathcal{M}_{A,B} & \mathcal{M}_{A,B} & \mathcal{M}_{A,B}
\end{array} \right)
\]
where \( \text{id}_A \in \text{Pnt}(\mathcal{M}_{A,B}) \) denotes the pro-morphism associated to \( \text{id}_A \). Thus for any pro-ideal \( T \) of \( \mathcal{M}_{A,B} \), \( T \circ \ker(\text{id}_A) = T \), and similarly, \( \ker(\text{id}_B) \circ T = T \). For morphisms \( f : A \to B \) and \( g : B \to C \) in \( \mathcal{A}_{nc} \) the identity \((\Phi \otimes \Phi) \Phi_{A,B,C} = \Phi g f \) implies that \( \ker(\Phi) \circ \ker(\Phi) = \ker(gf) \). Hence \( \mathcal{A}_{nc} \) is a category containing \( \mathcal{A}_{nc} \). Similarly, we define generalized categories of algebras listed below:

- For \( \ast \in \{ \ell,r \} \): \( \mathcal{A}_{nc}^\ast \supseteq \mathcal{A}_{nc} \) with
  \[
  \text{Obj}(\mathcal{A}_{nc}^\ast) := \text{Obj}(\mathcal{A}_{nc}) \quad \text{and} \quad \text{Hom}_{\mathcal{A}_{nc}^\ast}(A,B) := \lim_{\to} \mathcal{I}_\ast(\mathcal{M}^\ast_{A,B}).
  \]

- For \( \ast \in \{ \text{prpr}, \text{fcodim} \} \): \( \mathcal{A}_{nc}^\ast \supseteq \mathcal{A}_u \) with
  \[
  \text{Obj}(\mathcal{A}_{nc}^\ast) := \text{Obj}(\mathcal{A}_u) \quad \text{and} \quad \text{Hom}_{\mathcal{A}_{nc}^\ast}(A,B) := \lim_{\to} \mathcal{I}_\ast(\mathcal{M}^\ast_{A,B}).
  \]

- \( \mathcal{A}_u^{\text{prpr}} \supseteq \mathcal{A}_u \) with
  \[
  \text{Obj}(\mathcal{A}_u^{\text{prpr}}) := \text{Obj}(\mathcal{A}_u) \quad \text{and} \quad \text{Hom}_{\mathcal{A}_u^{\text{prpr}}}(A,B) := \lim_{\to} \mathcal{I}_\ast(\mathcal{M}^\ast_{A,B}).
  \]

- In case \( F \) is algebraically closed: \( \mathcal{A}_{\text{ufg}}^{\text{max}} \supseteq \mathcal{A}_{\text{ufg}} \) with
  \[
  \text{Obj}(\mathcal{A}_{\text{ufg}}^{\text{max}}) := \text{Obj}(\mathcal{A}_{\text{ufg}}) \quad \text{and} \quad \text{Hom}_{\mathcal{A}_{\text{ufg}}^{\text{max}}}(A,B) := \lim_{\to} \mathcal{I}_\ast(\mathcal{M}^\ast_{A,B}).
  \]

In the following theorem we record some additional properties of these categories.

**Theorem 7.1.**

(i) \( \mathcal{A}_{nc}^\ell \) and \( \mathcal{A}_{nc}^r \) are equivalent categories.

(ii) \( \mathcal{A}_u^{\text{prpr}} \supseteq \mathcal{A}_u^{\text{fcodim}} \). Also \( \mathcal{A}_u^{\text{prpr}} \) is equivalent to a subcategory of \( \mathcal{A}_{nc} \).

(iii) \( \mathcal{A}_u^{\text{prpr}} \) is equivalent to a subcategory of \( \mathcal{A}_u^{\text{prpr}} \).

(iv) \( \mathcal{A}_{\text{ufg}}^{\text{max}} \) and \( \mathcal{A}_{\text{ufg}} \) are equivalent (\( F \) is supposed to be algebraically closed).

**Proof.** (i) It follows from Theorem 2.14 that \( \text{op} : \mathcal{A}_{nc} \to \mathcal{A}_{nc} \) extends to an equivalence between \( \mathcal{A}_{nc}^\ell \) and \( \mathcal{A}_{nc}^r \) (ii) For every \( A,B \in \mathcal{A}_u \) let \( \psi_{A,B} : \mathcal{M}^\ast_{A,B} \to \mathcal{M}^\ast_{A,B} \) denote the pro-morphism satisfying \( \Phi_{A,B} = (\text{id}_B \otimes \psi_{A,B}) \Phi_{A,B} \). The functor given by \( A \to A \) and

\[
\lim_{\to} \mathcal{I}(\mathcal{M}^\ast_{A,B}) \supseteq \lim_{\to} \mathcal{I}_{\text{prpr}}(\mathcal{M}^\ast_{A,B}) \quad \lim_{\to} \mathcal{I}(\mathcal{M}^\ast_{A,B}) \quad \lim_{\to} \mathcal{I}(\mathcal{M}^\ast_{A,B})
\]

embeds \( \mathcal{A}_u^{\text{prpr}} \) into \( \mathcal{A}_{nc} \). The proof of (iii) is similar to that of (ii). (iv) We know that \( \mathcal{A}_{\text{ufg}} \subset \mathcal{A}_{\text{ufg}}^{\text{max}} \). Let \( A,B \in \mathcal{A}_{\text{ufg}} \). By Theorem 2.6(iii), all components of \( \mathcal{M}^\ast_{A,B} \) are in \( \mathcal{A}_{\text{ufg}} \). Thus if \( T \) is a maximal pro-ideal of \( \mathcal{M}^\ast_{A,B} \) then \( \mathcal{M}^\ast_{A,B}/T \cong F \) and hence the canonical quotient pro-morphism \( q : \mathcal{M}^\ast_{A,B} \to \mathcal{M}^\ast_{A,B}/T \) can be considered as a member of \( \text{Pnt} \neq 0(\mathcal{M}^\ast_{A,B}) \). The proof is complete.

We now consider some properties of the extended categories of algebras.
Theorem 7.2. Intersection (resp. sum) and inclusion of pro-ideals induce enriched category structure on $\mathcal{A}_{nc}$ by partially ordered idempotent semigroups.

Proof. It is clear that $\cap$ and $\subseteq$ make $\text{Hom}_{\mathcal{A}_{nc}}(A,B)$ into a partially ordered idempotent-semigroup. Suppose that $S = [S_i], S' = [S'_i]$ are pro-ideals of $\mathfrak{M}_{A,B}$ and $T = [T_j]$ be a pro-ideal of $\mathfrak{M}_{B,C}$. Without lost of generality assume $i = i'$. We have

$$(\mathfrak{M}_{B,C})_j \otimes (S_i \cap S'_i) + T_j \otimes (\mathfrak{M}_{A,B});$$

in the algebra $(\mathfrak{M}_{B,C})_j \otimes (\mathfrak{M}_{A,B}).$ It follows that $T \circ (S \cap S') = (T \circ S) \cap (T \circ S')$. Similarly, $(T \cap T') \circ S = (T \circ S) \cap (T' \circ S)$. Thus $\circ$ is a bihomomorphism with respect to the semigroup operation $\cap$. Also, it is clear that if $T \subseteq T', S \subseteq S'$ then $T \circ S \subseteq T' \circ S'$. The sum case is similar. \hfill \Box

Theorem 7.3. Suppose that $A$ and $B$ are isomorphic in $\mathcal{A}_{prpr}$. Then they are also isomorphic in $\mathcal{A}_u$.

We shall need the following elementary fact: Let $\phi : V \to W \otimes V', \psi : W \to V \otimes W'$ be morphisms in $\textbf{Vec}$. Suppose for some $x \in V', y \in W'$ we have that $(\psi \otimes \text{id})\phi(v) = v \otimes y \otimes x$ and $(\phi \otimes \text{id})\psi(w) = w \otimes x \otimes y$. Then there are isomorphisms $\hat{\phi} : V \to W$ and $\hat{\psi} : W \to V$ such that $\phi(v) = \hat{\phi}(v) \otimes x$ and $\psi(w) = \hat{\psi}(w) \otimes y$.

Proof. Let $S,T$ be proper pro-ideals respectively in $\mathfrak{M}^u_{A,B}, \mathfrak{M}^u_{B,A}$ such that

$$\ker[(q_T \otimes q_S)\Phi_{A,B,A}] = \ker(\bar{\text{id}}) \quad \ker[(q_S \otimes q_T)\Phi_{B,A,B}] = \ker(\bar{\text{id}})_B \quad (13)$$

Let $\phi := (\text{id}_B \otimes q_S)\Upsilon^u_{A,B}$ and $\psi := (\text{id}_A \otimes q_T)\Upsilon^u_{B,A}$. Since $\mathfrak{M}^u_{A,A}/\ker(\text{id}_A) \cong \mathbb{P}$, the left side of $(13)$ implies that image of $(q_T \otimes q_S)\Phi_{A,B,A}$ is the subalgebra generated by $1_{\mathfrak{M}^u_{B,A}/\mathfrak{M}^u_{B,A}}\otimes 1_{\mathfrak{M}^u_{A,B}/\mathfrak{M}^u_{A,B}}$. Thus, it follows from the identity

$$(\psi \otimes \text{id})\phi = [\text{id}_A \otimes ((q_T \otimes q_S)\Phi_{A,B,A})]\Upsilon^u_{A,A}$$

that $(\psi \otimes \text{id})\phi(a) = a \otimes 1_{\mathfrak{M}^u_{B,A}/\mathfrak{M}^u_{B,A}}\otimes 1_{\mathfrak{M}^u_{A,B}/\mathfrak{M}^u_{A,B}}$. Similarly, it is shown that $(\phi \otimes \text{id})\psi(b) = b \otimes 1_{\mathfrak{M}^u_{A,B}/\mathfrak{M}^u_{A,B}}\otimes 1_{\mathfrak{M}^u_{B,A}/\mathfrak{M}^u_{B,A}}$. Now, the desired result follows from the mentioned fact. \hfill \Box

Every morphism $f : A \to B$ in $\mathcal{A}_u$ induces a fundamental functor $\hat{f}$ between categories of unital left modules of $A$ and $B$ given by

$$\hat{f} : \text{Mod}(A) \to \text{Mod}(B) \quad \hat{f}(M) := B \otimes_A M$$

where $B$ is considered as a right $A$-module with module multiplication $b \cdot a := bf(a)$. The assignment $f \mapsto \hat{f}$ is simply extended to morphisms in $\mathcal{A}_u^{prpr}$ as follows: Let $S$ be a proper pro-ideal of $\mathfrak{M}^u_{A,B}$ and let $\phi : A \to B \otimes \mathfrak{M}_{A,B}/S$ be as in the proof of Theorem 7.3. $\hat{S}(M)$ is defined to be the $B$-submodule of $B \otimes \mathfrak{M}_{A,B}/S$ with the underlying vector space $\hat{\phi}(M)$. The action of $\hat{S}$ on morphisms of $\text{Mod}(A)$ is defined to be that of $\phi$. 27
Theorem 7.4. The K-group functor $K_0$ on $A_u$ extends canonically to a functor

$$K_0 : \tilde{A}_u^{\text{codim}} \to \text{Ab}.$$  

Proof. Let $S$ be a proper finite-codimensional pro-ideal of $\mathfrak{M}_{A,B}^u$. We show that $\hat{S}(M)$ is a finitely-generated projective $B$-module for any finitely-generated projective $A$-module $M$. Let $C = \mathfrak{M}_{A,B}^u/S, D = B \otimes C, N = D \otimes_A M$. We know that $C$ is a finite-dimensional algebra and $N$ as a $D$-module is finitely-generated and projective. Suppose that $\{c_i\}_{i=1}^k$ is a vector basis for $C$. If $\{x_j\}_{j=1}^n$ generate $N$ as a $D$-module then $\{(1_B \otimes c_i)x_j\}$ generates $N$ as a $B$-module. Thus $N$ is a finitely generated $B$-module. We know that there exists a $D$-module isomorphism $\Gamma : N \oplus N' \to \oplus_{i=1}^n D$ for some $n \geq 1$ and some $D$-module $N'$. It is clear that $\Gamma$ is also a $B$-module isomorphism. On the other hand, $D$ is a free $B$-module with $B$-basis $\{1_B \otimes c_i\}$. Thus $\oplus_{i=1}^n D$ is also a free $B$-module. So, $N$ as a $B$-module is projective. $K_0(S)$ is defined to be the group-morphism $K_0(A) \to K_0(B)$ induced by the assignment $M \mapsto \hat{S}(M)$ on finitely-generated projective modules. \hfill $\square$

Theorem 7.5. Products exist in $\tilde{A}_u^{\text{prism}}$ and (hence) coincide with products of $A_u$.

Proof. Let $B_1, B_2$ be unital algebras and let $p_1, p_2$ denote the canonical projections from $B_1 \oplus B_2$ respectively onto $B_1, B_2$. Let $S_1, S_2$ be morphisms in $\tilde{A}_u^{\text{prism}}$ from $A$ respectively to $B_1, B_2$. By a result similar to Theorem 2.1 we have the canonical isomorphism $\mathfrak{M}_{A,B_1 \oplus B_2}^{\text{cu}} \cong \mathfrak{M}_{A,B_1}^{\text{cu}} \otimes \mathfrak{M}_{A,B_2}^{\text{cu}}$. So, we may consider

$$I := S_1 \otimes (\mathfrak{M}_{A,B_2}^{\text{cu}})_{\text{ideal}} + (\mathfrak{M}_{A,B_1}^{\text{cu}})_{\text{ideal}} \otimes S_2$$

as a pro-ideal of $\mathfrak{M}_{A,B_1 \oplus B_2}^{\text{cu}}$. Then $I$ is the only morphism in $\tilde{A}_u^{\text{prism}}$ satisfying

$$S_1 := \ker(\mathfrak{F}_1) \circ I \quad \text{and} \quad S_2 := \ker(\mathfrak{F}_2) \circ I.$$  

\hfill $\square$

8 Classifying Algebras for Algebraic K-Groups

In this section we consider a bivariant K-theory $\Omega \Omega$ which is a pure-algebraic version of Cuntz’s interpretation \cite{Cuntz} of the Kasparov bivariant K-theory of C*-algebras \cite{Kasparov}. Using the functor $\mathfrak{M}_{\text{nc}}^{\text{cu}}$ and following a method introduced by Phillips \cite{Phillips} we prove the existence of classifying homotopy pro-algebras for $\Omega \Omega$, Cortiñas-Thom’s KK-groups \cite{CT}, and Weibel’s homotopy K-groups \cite{Weibel}.

In this section we denote by $\mathcal{R}$ the matrix in $M_2(F[x])$ given by

$$\mathcal{R} := \begin{pmatrix} 1 - x^2 & x^3 - 2x \\ x & 1 - x^2 \end{pmatrix}.$$  

Note that $\mathcal{R}$ is invertible. For a story about $\mathcal{R}$ in Algebraic Homotopy see \cite{AlgebraicHomotopy} §3.4. We begin with some well-known lemmas.

Lemma 8.1. For any $B \in A_{\text{nc}}$, the morphism $M_2(B) \to M_2(B[x])$ given by $M \mapsto \mathcal{R}^{-1}M\mathcal{R}$ is an elementary homotopy between $\text{id}_{M_2(B)}$ and the morphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$
Lemma 8.2. Let $\alpha, \beta : A \to B$ be morphisms in $\mathbf{A}_{nc}$. Then the morphisms

$$(a \mapsto \begin{pmatrix} \alpha(a) & 0 \\ 0 & \beta(a) \end{pmatrix}) \text{ and } (a \mapsto \begin{pmatrix} \beta(a) & 0 \\ 0 & \alpha(a) \end{pmatrix})$$

from $A$ into $M_2(B)$ are elementary homotopic.

Proof. It follows directly from Lemma 8.1. \hfill \Box

Lemma 8.3. With the assumptions of Lemma 8.2, suppose $B$ is an ideal of a unital algebra $C$, and there exists $c \in C$ such that $\beta(a) = c\alpha(a)c^{-1}$. Then the morphisms

$$(a \mapsto \begin{pmatrix} \alpha(a) & 0 \\ 0 & 0 \end{pmatrix}) \text{ and } (a \mapsto \begin{pmatrix} \beta(a) & 0 \\ 0 & 0 \end{pmatrix})$$

from $A$ into $M_2(B)$ are homotopic.

Proof. By Lemma 8.2 we have

$$(a \mapsto \begin{pmatrix} \alpha(a) & 0 \\ 0 & 0 \end{pmatrix}) \approx (a \mapsto \begin{pmatrix} 0 & 0 \\ 0 & \alpha(a) \end{pmatrix}).$$

Hence, the morphisms

$$a \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha(a) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha(a) \\ 0 \end{pmatrix}$$

$$a \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} \beta(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \beta(a) \end{pmatrix}$$

from $A$ into $M_2(B)$ are elementary homotopic. Now, applying Lemma 8.2 another time, we get the desired result. \hfill \Box

Lemma 8.4. Let $\alpha_i : A \to M_{k_i}(B)$ ($i = 1, \ldots, n$) be morphisms in $\mathbf{A}_{nc}$. Suppose $\sigma$ denotes a permutation of $\{1, \ldots, n\}$. Then the morphisms

$$a \mapsto \begin{pmatrix} \alpha_{\sigma(1)}(a) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_n(a) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$a \mapsto \begin{pmatrix} \alpha_{\sigma(1)}(a) & 0 & \cdots & 0 & 0 \\ 0 & \alpha_{\sigma(2)}(a) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{\sigma(n)}(a) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

from $A$ into $M_{2k}(B)$, where $k := \sum_i k_i$, are homotopic.

Proof. It follows directly from Lemma 8.3. \hfill \Box
For any algebra $B$, let $M\ast(B)$ denote the ind-algebra indexed over $\mathbb{N}$ with components $M_n(B)$ and structural morphisms $M_n(B) \to M_{n+1}(B)$ given by

\[ M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}. \] 

(14)

For any $m, n$, consider the morphism

\[ \Gamma_{m,n}: M_m(B) \oplus M_n(B) \to M_{m+n}(B) \quad (M, N) \mapsto \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}. \] 

(15)

Then it follows from Lemma 8.4 that the image of the family $\{\Gamma_{m,n}\}$ in ind-Hot($A_{nc}$) defines an ind-morphism

\[ \Gamma: M\ast(B) \oplus M\ast(B) \to M\ast(B). \]

(Note that $\{\Gamma_{m,n}\}$ does not define a morphism in ind-$A_{nc}$ except for $B = 0$.) It also follows from Lemma 8.4 that $M\ast(B)$ is an abelian monoid in ind-Hot($A_{nc}$) with comultiplication $\Gamma$ and null element given by the ind-morphism $0 \to M\ast(B)$. It is not hard to see that all the above statements hold if the algebra $B$ is replaced by any ind-algebra. Also, we may consider $M\ast$ as the wh preserving functor

\[ M\ast: \text{ind-}A_{nc} \to \text{ind-}A_{nc}. \]

The following result is obvious.

**Proposition 8.5.** For $A, B \in \text{ind-}A_{nc}$, the set $[A, M\ast(B)]$ has a canonical abelian monoid structure induced by $\Gamma$. Moreover, this structure is functorial in $A$ and $B$.

**Lemma 8.6.** Suppose there exists a morphism $f: A \to A$ such that $\tilde{f}h0$ where

\[ \tilde{f}: A \to M_2(A) \quad a \mapsto \begin{pmatrix} f(a) & 0 \\ 0 & a \end{pmatrix}. \]

Then the monoid $[A, M\ast(B)]$ is a group.

**Proof.** It is easily verified that for every ind-morphism $g: A \to M\ast(B)$, $g \circ f$ is an inverse for $g$ in $[A, M\ast(B)]$. \qed

Consider the coproduct $A \ast A$ in $A_{nc}$. For any $a \in A$, let $a_1, a_2$ denote the two copies of $a$ in $A \ast A$. The algebra $qA$, originally introduced by Cuntz, is defined to be the kernel of the codiagonal morphism $A \ast A \to A$ $(a_1, a_2 \mapsto a)$ \cite{14, §4.11}. The key property of $qA$ is that for any two morphisms $\alpha, \beta: A \to B$ if $I$ is an ideal of $B$ such that $\alpha(a) - \beta(a) \in I$, then the restriction to $qA$ of the morphism $A \ast A \to B$ given by $a_1 \mapsto \alpha(a)$ and $a_2 \mapsto \beta(a)$, takes its values in $I$, and thus may be regarded as a morphism $qA \to I$. We may consider $q$ as a functor $q: A_{nc} \to A_{nc}$ in the obvious way. Then, it is easily verified that $q$ is homotopy preserving. The proof of the next lemma is an adapted version of that of \cite[Proposition 1.15]{17}.

**Lemma 5.** $qA$ satisfies the assumption of Lemma 8.6.

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Proof. Consider the morphisms $\alpha, \beta : A \to M_2(A \star A)[x]$ given by

$$\alpha : a \mapsto \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad \beta : a \mapsto R^{-1} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} R.$$ 

It is easily verified that $\alpha(a) - \beta(a) \in M_2(qA)[x]$, and thus we have the canonical morphism $\varphi : qA \to M_2(qA)[x]$ induced by $\alpha, \beta$. Let $s : A \star A \to A \star A$ denote switch i.e. the morphism defined by $a_1 \mapsto a_2$ and $a_2 \mapsto a_1$. Put $f := |s|_{qA}$. Then $\varphi$ is an elementary homotopy from $\tilde{f}$ to 0. \hfill \square

The following result is a version of [5, Proposition 1.4].

**Theorem 8.8.** For any two algebras $A$ and $B$, the set

$$\Omega \Omega(A, B) := [qA, M_\bullet(B)]$$

has a canonical abelian group structure. Moreover, we may consider the following homotopy invariant functor in the obvious way:

$$\Omega \Omega : A_{nc}^{op} \times A_{nc} \to Ab.$$ 

**Proof.** It follows from Proposition 8.5 and Lemmas 8.6 and 8.7. \hfill \square

For any algebra $A$, let the evaluation morphisms $ev_1, ev_2 : qA \to A$ be defined respectively by $(a_1 \mapsto a, a_2 \mapsto 0)$ and $(a_1 \mapsto 0, a_2 \mapsto a)$.

**Remark 8.9.** It is proved by Cuntz that for any $C^*$-algebra $A$, $qA$ and $q^2A$ are analytically homotopic up to stabilization by $M_2$ [5, Theorem 1.6]. Using this result, the triple $(qA, ev_1, ev_2)$ can be regarded as an analogue of the triple $([0, 1], 0, 1)$, homology theories of $C^*$-algebras which are stable with respect to a tensor with compact operators. Unfortunately, it seems that there is no way to restate a purely-algebraic version of this result.

We let

$$\Omega(B) := \Omega \Omega(F, B).$$

Thus we have the following homotopy invariant functor:

$$\Omega : A_{nc} \to Ab.$$ 

Let $C \in A_n$. Denote by $\text{Idmp}_n(C)$ the set of idempotent matrices in $M_n(C)$. Then [14] makes the family $(\text{Idmp}_n(C))_{n \geq 1}$ into an ind-set; write $\text{Idmp}(C)$ for its direct limit. The group $\lim_{n \to \infty} GL_n(C)$ acts on $\text{Idmp}(C)$ by conjugation. Write $[\text{Idmp}(C)]$ for the set of conjugate classes. The morphisms [15] induce an abelian monoid structure on $[\text{Idmp}(C)]$. Its Grothendieck group is $K_0(C)$, the usual K-group of $C$. It is clear that any $M \in \text{Idmp}_n(C)$ is exactly distinguished by the morphism $\alpha_M : F \to M_n(C)$ given by $1 \mapsto M$. By Lemma 8.4 if the idempotents $M, M' \in \text{Idmp}_n(C)$ are conjugate then $\alpha_M, \alpha_M'$, considered canonically as morphisms into $M_{2n}(C)$, are homotopic. Thus the assignment $M \mapsto \alpha_M$ induces a natural monoid-morphism

$$[\text{Idmp}(C)] \to [F, M_\bullet(C)].$$ \hfill (16)
The morphism \( \text{ev}_1 : q\mathcal{F} \to \mathcal{F} \) induces the monoid-morphism

\[
[q\mathcal{F}, M_\bullet(C)] = \Omega(C).
\]

The composition of (16) and (17) gives rise to the group-morphism

\[
K_0(C) \to \Omega(C).
\]

We need the following Yoneda lemma.

**Lemma 8.10.** Let \( C \) be a category with finite coproducts and cofinal object. Let \( C \in \text{pro-}C \), and suppose that for every \( D \in C \) the set \( \text{Hom}_{\text{pro-}C}(C, D) \) has an abelian group structure such that these structures make \( \text{Hom}_{\text{pro-}C}(C, ?) \) to a functor from \( C \) to \( \text{Ab} \). Then \( C \) has a cocommutative cogroup structure in \( \text{pro-}C \) whose induces the group structures of \( \text{Hom}_{\text{pro-}C}(C, D) \) in the obvious way.

**Theorem 8.11.** For any algebra \( A \), there exists a pro-algebra \( \Omega A \) with a cocommutative cogroup structure as an object in \( \text{pro-Hot}(A_{\text{nc}}) \) such that the abelian groups \( [\Omega A, B] \) and \( \Omega(A, B) \) are naturally isomorphic for every algebra \( B \). In particular, there exists a pro-algebra \( \Omega \) with a natural abelian group-morphism

\[
[\Omega, B] \cong \Omega(B) \quad (B \in A_{\text{nc}}).
\]

**Proof.** By Theorem 3.4, for every algebra \( B \) we have

\[
\Omega\Omega(A, B) = [qA, M_\bullet(B)] = [qA, M_\bullet(\mathcal{F}) \otimes B] \cong [\mathcal{M}^\infty_{qA, M_\bullet(\mathcal{F})}, B].
\]

Thus it follows from Lemma 8.10 that \( \mathcal{M}^\infty_{qA, M_\bullet(\mathcal{F})} \) is the desired pro-algebra. Following Phillips \[17\], \( \Omega A \) may be called classifying pro-algebra.

For any algebra \( B \), we put

\[
M_{\infty}(B) := \lim_{n \to \infty} M_n(B) \quad \text{and} \quad M_{\infty}(B) := M_\bullet M_{\infty}(B).
\]

It is clear that we have the wh-preserving functor

\[
M_{\infty} : \text{ind-}A_{\text{nc}} \to \text{ind-}A_{\text{nc}}.
\]

As explained in \[4\] §4.1, the abelian monoid structure on \( M_{\infty}(B) \), may be induced also by the direct sum of infinite matrices in \( M_{\infty}(B) \). For any algebra \( B \), we denote by \( B[\Delta] \) the ind-algebra \( B \otimes \mathcal{F}[\Delta] \). For any simplicial set \( S \in \text{sim-Set} \), let

\[
\mathfrak{Z}(S, B) := \text{Hom}_{\text{sim-Set}}(S, B[\Delta])
\]

where \( B[\Delta] \), by the forgetful functor \( \text{sim-}A_{\text{nc}} \to \text{sim-Set} \), is considered as a simplicial set. Then \( \mathfrak{Z}(S, B) \) can be considered as an algebra with pointwise operations. Thus we have the functor

\[
\mathfrak{Z} : (\text{sim-Set})^{\text{op}} \times A_{\text{nc}} \to A_{\text{nc}}.
\]

We extend \( \mathfrak{Z} \) canonically to the functor

\[
\mathfrak{Z} : (\text{pro-sim-Set})^{\text{op}} \times A_{\text{nc}} \to \text{ind-}A_{\text{nc}}.
\]
and then again to the functor
\[ 3 : (\text{pro-sim-Set})^{\text{op}} \times \text{ind-}A_{nc} \to \text{ind-}A_{nc}. \]

Let \( \text{sd} : \text{sim-Set} \to \text{sim-Set} \) denote the simplicial subdivision functor \([10, \S III.4]\). For any \( S \in \text{sim-Set} \) let \( \text{sd}^\bullet S \in \text{pro-sim-Set} \) be given by the inverse system
\[ S = \text{sd}^0 S \leftarrow \text{sd}^1 S \leftarrow \cdots \leftarrow \text{sd}^n S \leftarrow \cdots \]
where the structural morphisms are given by a natural transformation \( \text{sd} \to \text{id} \) called last vertex map. Let \( \Delta^1 \) denote standard 1-simplex i.e. the simplicial set given by
\[ n \mapsto \text{Hom}_{\Delta}(n, 1). \]

For any ind-algebra \( B \) let \( B^{S^1} \) \(([4, \S 3.3])\) denote the kernel of ind-morphism
\[ 3(\text{sd}^\bullet \Delta^1, B) \to 3(\text{sd}^\bullet \partial \Delta^1, B) \]
induced by the canonical morphism \( \partial \Delta^1 \to \Delta^1 \) in \( \text{sim-Set} \). So we have the functor
\[ ?^{S^1} : \text{ind-}A_{nc} \to \text{ind-}A_{nc} \]
The \( n \) times iteration of this functor is denoted by \( ?^{S^n} \). It is not hard to see that for any two algebras \( A, B \) we have the following canonical isomorphisms of ind-algebras:
\[ (A \otimes B)^{S^n} \cong A^{S^n} \otimes B \quad (M_\infty(B))^{S^n} \cong M_\infty(B^{S^n}) \]
(When the ground ring is not a field, the proof of these latter identities follow from the nontrivial result \([11, \S 3.1.3]\).) In particular, we have
\[ (B[x])^{S^n} \cong B^{S^n}[x], \]
and hence the functor \( ?^{S^n} \) is homotopy preserving. By \([4, \S 3.3]\) the set
\[ [A, B^{S^n}] \quad (A \in A_{nc}, B \in \text{ind-}A_{nc}) \]
has a canonical group structure, functorial in \( A, B \), and abelian for \( n \geq 2 \).

By an extension \([4, \S 4.2.1]\) we mean a sequence
\[ C \xrightarrow{g} B \xrightarrow{f} A \quad (19) \]
of ind-morphisms between ind-algebras such that \( g = \ker(f) \) and \( f = \text{coker}(g) \). For \( B \in \text{ind-}A_{nc} \) the canonical morphism \( \Delta^0 \to \Delta^1 \) in \( \text{sim-Set} \) induces an ind-morphism
\[ 3(\text{sd}^\bullet \Delta^1, B) \to 3(\text{sd}^\bullet \Delta^0, B) \cong B. \]
Denote its kernel by \( \mathcal{P}(B) \). The canonical factorization
\[ \Delta^0 \to \partial \Delta^1 \to \Delta^1 \]
of $\Delta^0 \to \Delta^1$ gives us the so-called loop extension [4, §4.5]:

$$B^{S^1} \to \mathcal{P}(B) \to B. \quad (20)$$

For $V \in \textbf{Vec}$ let $T(V) := \bigoplus_{n=1}^{\infty} V \otimes^n$ denote the tensor algebra associated to $V$. Then

$$T : \text{ind-Vec} \to \text{ind-Anc}$$

is a left adjoint for the forgetful functor $\text{ind-Anc} \to \text{ind-Vec}$. For any ind-algebra $A$ let $\eta_A : T(A) \to A$ denote the adjoint of $\text{id}_A$ and put $J(A) := \ker \eta_A$. Let $\iota_A : J(A) \to T(A)$ denote the embedding. Then we have the following extension:

$$J(A) \xrightarrow{\iota_A} T(A) \xrightarrow{\eta_A} A.$$

It is not hard to see that $J$ can be considered as a homotopy preserving functor

$$J : \text{ind-Anc} \to \text{ind-Anc}.$$

Suppose the ind-morphism $h : A \to B$ in $\text{ind-Vec}$ is a splitting for extension [19] i.e. $fh = \text{id}_A$. Let $\gamma_h$ denote the adjoint of $h$. Then there is a unique morphism $\xi_h$ making the following diagram commutative:

$$\begin{array}{ccc}
J(A) & \xrightarrow{\iota_A} & T(A) \xrightarrow{\eta_A} A \\
\downarrow \xi_h & & \downarrow \eta_h & \downarrow \text{id} \\
C & \xrightarrow{g} & B & \xrightarrow{f} A
\end{array}$$

If $h' : A \to B$ is another splitting for [19] then by [4, Proposition 4.4.1] we have $\xi_h \sim \xi_{h'}$. So $\xi = \xi_h$ is just called a classifying map for [19]. For more details on the constructions of $J(A), \mathcal{P}(A), A^{S^n}$ see [4] or [8].

For any two algebras $A, B$ we let

$$kk^{(n)}(A, B) := [J^n(A), (\mathcal{M}_\infty(B))^{S^n}] = [J^n(A), \mathcal{M}_\infty(B^{S^n})]. \quad (21)$$

It is clear that $kk^{(n)}$ may be considered as a homotopy invariant functor

$$kk^{(n)} : \text{A_{nc}^{op}} \times \text{A_{nc}} \to \text{Ab}.$$

Let $f : A \to B$ be a morphism in $\text{A_{nc}}$. Then let $j(f) \in kk^{(1)}(A, B)$ be the composit

$$\begin{array}{ccc}
J(A) & \xrightarrow{j(f)} & J(B) \xrightarrow{\xi} J(\mathcal{M}_\infty(B)) \xrightarrow{\xi} \mathcal{M}_\infty(B^{S^1})
\end{array},$$

where the second arrow is induced by the canonical embedding $B \to \mathcal{M}_\infty(B)$ and where $\xi$ is a classifying map for the extension induced by [20]. Similarly, for any $n \geq 1$ and any homotopy class $[f] \in kk^{(n)}(A, B)$ of a ind-morphism $f$ let

$$\Phi^{n}_{A,B}([f]) \in kk^{(n+1)}(A, B)$$
denote the homotopy class of the ind-morphism defined by the composite
\[ J^{n+1}(A) \xrightarrow{J(f)} J((\mathcal{M}_\infty(B))^S) \xrightarrow{\xi} ((\mathcal{M}_\infty(B))^S)^{S^1} \cong (\mathcal{M}_\infty(B))^{S^{n+1}}. \]

Then, as it is explained in [4, §6.1], \( \Phi^n_{A,B} \) is a morphism in \( \text{Ab} \), and moreover, the assignment \( (A, B) \mapsto \Phi^n_{A,B} \) defines a natural transformation
\[ \Phi^n : kk^{(n)} \to kk^{(n+1)}. \]

Thus, we have the following direct system of abelian groups:
\[ kk^{(1)}(A, B) \xrightarrow{\Phi^1_{A,B}} kk^{(2)}(A, B) \xrightarrow{\Phi^2_{A,B}} \cdots. \]

The Cortiñas-Thom bivariant K-group is defined to be the abelian group
\[ kk(A, B) := \lim_{n \to \infty} kk^{(n)}(A, B). \]

It is clear that \( kk \) may be considered as a homotopy invariant functor
\[ kk : A^{\text{op}}_{\text{nc}} \times A_{\text{nc}} \to \text{Ab}. \]

It is proved that there is a triangulated category \( kk \) whose objects are those of \( A_{\text{nc}} \) and whose morphism-sets are \( kk(A, B) \). Also, the assignment \( f \mapsto j(f) \) defines a functor \( A_{\text{nc}} \to kk \) which is universal among all excisive, homotopy invariant and \( M_\infty \)-stable homology theories on \( A_{\text{nc}} \) [4, Theorem 6.6.2].

After the above long review of Cortiñas-Thom’s theory, we build a comparison morphism between \( \Omega(C) \) and \( kk(\mathbb{F}, C) \): For \( C \in A_{\text{nc}} \), consider the composite
\[ \Omega(C) = [q\mathbb{F}, \mathcal{M}_\bullet(C)] \xrightarrow{j} [q\mathbb{F}, M_\infty(C)] \xrightarrow{j} kk(q\mathbb{F}, M_\infty(C)) \cong kk(q\mathbb{F}, C) \quad (22) \]
where the first arrow is induced by the canonical ind-morphism \( M_\bullet(C) \to M_\infty(C) \). As it is noted in [4, §6.1] the sum operation in the \( kk \)-groups is the same as the operation induced by the abelian monoid \( M_\bullet \) in (21). Thus the composite (22) is a group-morphism. For every surjective morphism \( f \) in \( A_{\text{nc}} \) it follows from [4, Corollary 6.3.4] that \( j(f) \) has a right inverse in \( kk \). So, we have the following group-morphism induced by the inverse of \( j(\text{ev}_1 : q\mathbb{F} \to \mathbb{F}) \):
\[ kk(q\mathbb{F}, C) \to kk(\mathbb{F}, C) \quad (23) \]

Composition of (22) and (23) is the canonically defined group-morphism
\[ \Omega(C) \to kk(\mathbb{F}, C). \quad (24) \]

For Weibel’s homotopy algebraic K-theory \( KH \) we refer the reader to the original paper [19] or [21, 3]. By [4, Theorem 8.2.1], for any algebra \( C \), we have a natural isomorphism between abelian groups \( KH_0(C) \) and \( kk(\mathbb{F}, C) \). Thus, (24) gives rise to the natural group-morphism
\[ \Omega(C) \to KH_0(C). \quad (25) \]

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In case \( C \) is a unital algebra, it can be seen from the proof of [4, Theorem 8.2.1] that the canonical morphism (see [3, §5]),

\[
K_0(C) \to KH_0(C)
\]  

(26)
is equal to the composition of [18] and [25].

**Theorem 8.12.** Let \( C \) be a unital \( K_0 \)-regular algebra. Then (18) is injective and (25) is surjective.

**Proof.** The theorem follows from the fact that by [3, Proposition 5.2.3], (26) is an isomorphism.

We need the following elementary lemma.

**Lemma 8.13.** Let \( C = (C_i)_i \) be in pro-\( C \). Suppose that for every \( i \), \( C_i \) is a co-commutative cogroup object in \( C \) and all structural morphisms of \( C \) preserve the cogroup structures. Then \( C \) is a co-commutative cogroup in pro-\( C \) with comultiplication, counit, and coinverse induced by those of \( C_i \)’s in the obvious way.

**Theorem 8.14.** For any algebra \( A \), there exists an object \( \overline{kk}_A \in \text{pro-Hot}(\mathbf{A}_{nc}) \) with a co-commutative cogroup structure such that the abelian groups \( \overline{KK}_0, B \) and \( \underline{kk}(A, B) \) are naturally isomorphic for every algebra \( B \).

**Proof.** Let \( \overline{kk}_A^{(n)} = \mathfrak{F}^{nc}(J^n(A), \mathcal{M}_\infty(\mathbb{R}S^n)) \).

Then by Theorem 3.4 we have the natural bijection

\[
\underline{kk}^{(n)}(A, B) \cong [\overline{kk}_A^{(n)}, B].
\]  

(27)

Thus by Lemma 8.10, \( \overline{kk}_A^{(n)} \) has a co-commutative cogroup structure as an object in pro-\( \text{Hot}(\mathbf{A}_{nc}) \) that makes (27) into a natural isomorphism of groups. By the Yoneda Lemma there exists a morphism

\[
\alpha_n : \overline{kk}_A^{(n+1)} \to \overline{kk}_A^{(n)}
\]
in pro-\( \text{Hot}(\mathbf{A}_{nc}) \) that induces the natural transformation \( \Phi^n_{A,B} \). Thus, \( \alpha_n \) preserves the cogroup structures. Now, it follows from 8.12 that the inverse system

\[
(\overline{kk}_A^{(n)}, \alpha_n)_n
\]
defines the desired object \( \overline{kk}_A \).

Following Phillips [14], \( \overline{kk}_A \) may be called classifying homotopy pro-algebra. The following theorem is one of the main results of this note.

**Theorem 8.15.** There exists an object \( \overline{KH}_0 \) in pro-\( \text{Hot}(\mathbf{A}_{nc}) \) with a co-commutative cogroup structure such that the abelian groups \( \overline{KH}_0, B \) and \( KH_0(B) \) are naturally isomorphic for every algebra \( B \).

**Proof.** It follows directly from [4, Theorem 8.2.1] and Theorem 8.14.
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