NONLINEAR SCALAR FIELD EQUATION
WITH COMPETING NONLOCAL TERMS

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ABSTRACT. We find radial and nonradial solutions to the following nonlocal problem
\[-\Delta u + \omega u = (I_\alpha * F(u)) f(u) - (I_\beta * G(u)) g(u)\] in $\mathbb{R}^N$
under general assumptions, in the spirit of Berestycki and Lions, imposed on $f$ and $g$, where $N \geq 3$, $0 \leq \beta \leq \alpha < N$, $\omega \geq 0$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions with corresponding primitives $F, G$, and $I_\alpha, I_\beta$ are the Riesz potentials. If $\beta > 0$, then we deal with two competing nonlocal terms modelling attractive and repulsive interaction potentials.

1. INTRODUCTION

This paper mainly deals with the following problem
\[(1.1) \quad -\Delta u = (I_\alpha * F(u)) f(u) - (I_\beta * G(u)) g(u) \quad \text{in} \mathbb{R}^N,
\]
where $N \geq 3$, $0 \leq \beta \leq \alpha < N$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions with corresponding primitives $F(s) = \int_0^s f(t)dt$, $G(s) = \int_0^s g(t)dt$.

Moreover $I_\gamma : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential
\[I_\gamma(x) := \frac{\Gamma(N\frac{\gamma}{2})}{\Gamma(\frac{N}{2})\pi^{N/2}2^{\gamma}|x|^{N-\gamma}} \quad \text{for} x \in \mathbb{R}^N \setminus \{0\} \text{ and } \gamma \in (0, N),
\]
while we set $I_0 = \delta_0$, namely the identity for the convolution, where $\delta_0$ is the Dirac delta at zero.

If $N = 3$, $\alpha = 2$, $\beta = 0$, $F(s) = \frac{1}{\sqrt{2}}|s|^2$ and $G(s) = s$, then (1.1) is the well-known Choquard, or Choquard-Pekar equation
\[-\Delta u + u = (I_2 * |u|^2) u \quad \text{in} \mathbb{R}^N.
\]

This equation comes, for instance, from an approximation to the Hartree-Fock theory of a plasma [14,25] or it can be also derived in the framework of the Schrödinger-Newton system, (see [21]). A variational approach for this case was presented by Lieb [14] and Lions [16].

More generally, if $N \geq 3$, $F(s) = \frac{1}{\sqrt{p}}|s|^p$, for suitable $p$, $\alpha > 0$ and $G(s) = s$, then weak solutions to (1.1) can be obtained by means of critical points of the associated functional. If, for instance, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $\beta = 0$, then, according to the work of Moroz and Van Schaftingen [22], the Hardy-Littlewood-Sobolev inequality implies that $(I_\alpha * F(u)) F(u) \in L^1(\mathbb{R}^N)$ for $u \in H^1(\mathbb{R}^N)$. 

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Moreover the associated functional is well-defined and of class $C^1$ on $H^1(\mathbb{R}^N)$, and its critical points correspond to solutions to
\begin{equation}
-\Delta u + u = (I_\alpha \ast F(u))f(u) \quad \text{in } \mathbb{R}^N.
\end{equation}
A ground state solution and its properties were obtained in [22]. The same authors in [23] also studied the existence of solutions with a general nonlinearity $F$ in the spirit of the classical result of Berestycki and Lions [6], namely
\begin{equation}
|sf(s)| \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\beta}{N-2}}), \quad \lim_{s \to 0} F(s)/|s|^{\frac{N+\alpha}{N}} = \lim_{|s| \to +\infty} F(s)/|s|^{\frac{N+\alpha}{N-2}} = 0, \quad F(s_0) \neq 0,
\end{equation}
for some $s_0 \neq 0$ and $C > 0$, see also the survey [24] and the references therein. Note that, if $\alpha = 0$ in (1.2), since $I_0 \ast F(u) = F(u)$, (1.3) covers the Berestycki-Lions growth assumptions only for the nonnegative (attractive) nonlinearity $F^2(s) \geq 0$ of the corresponding energy functional (see (3.3) of [6]).

On the other hand, as for instance in the Hartree-Fock theory, the interaction potential could be also repulsive [5, 17], i.e. with $\beta > 0$ and a non-trivial $G(s) \geq 0$. Moreover problems similar to (1.1) may admit some local terms as well, see also [24] and the references therein.

Our aim is to investigate both nonlinear phenomena with both nonlocal terms ($0 < \beta \leq \alpha$) in (1.1), since, in the limiting case $\alpha = \beta = 0$, we can fully cover the Berestycki and Lions assumptions [6].

We impose the following assumptions on $f$ and $g$:

\begin{enumerate}[(H_1)]
\item there is a constant $C > 0$ and $p \in \left(\frac{2\beta}{N-2}, \frac{N+\beta}{N-2}\right)$ such that $|sf(s)| \leq C|s|^{\frac{N+\alpha}{N}}$ and $0 \leq g(s)s \leq C(|s|^p + |s|^{\frac{N+\beta}{N-2}})$ for $s \in \mathbb{R}$;
\item $\lim_{s \to 0} F(s)/|s|^{\frac{N+\alpha}{N}} = \lim_{|s| \to +\infty} F(s)/|s|^{\frac{N+\alpha}{N-2}} = 0$;
\item there is $s_0 > 0$ such that $F(s_0) \neq 0$; if $\alpha = \beta$, then we assume also $F(s_0) > G(s_0)$.
\end{enumerate}

Observe that, if $0 \leq \beta < \frac{N-2}{2}$, then, due to the continuity of $g$, we can take $p = 1 \in \left(\frac{2\beta}{N-2}, \frac{N+\beta}{N-2}\right)$.

We remark that these kinds of assumptions follow naturally from the local case, namely when $\alpha = \beta = 0$, and equation (1.1) becomes simply
\begin{equation}
-\Delta u = h(u) \quad \text{in } \mathbb{R}^N.
\end{equation}

This problem has been studied in [6] and [26, 27], under general assumptions. In particular, in [26, 27] Struwe considered a continuous and odd function $h : \mathbb{R} \to \mathbb{R}$ with primitive $H(s) = \int_0^s h(t) \, dt$ such that
\begin{enumerate}[(i)]
\item $-\infty \leq \limsup_{s \to 0} h(s)s/\sqrt{2} \leq 0$;
\item $-\infty \leq \limsup_{|s| \to +\infty} h(s)s/\sqrt{2} \leq 0$;
\item there is $s_0 > 0$ such that $H(s_0) > 0$.
\end{enumerate}

Observe that the above assumptions contain those in [6]. As usual, by the maximum principle, it is enough to solve (1.4) when $\limsup_{|s| \to +\infty} h(s)s/\sqrt{2} = 0$. Now, taking $F$ and $G$ even functions such that
\[ F^2(s) = \int_0^s \max\{h(t),0\} \, dt \quad \text{and} \quad G^2(s) = \int_0^s \max\{-h(t),0\} \, dt, \quad \text{for } s \geq 0, \]
we get $H(s) = F^2(s) - G^2(s)$ and, in the local case $\alpha = \beta = 0$, assumptions $(H_2)$ and $(H_3)$ are clearly satisfied. Moreover $F$ and $G$ satisfy the following condition
\begin{enumerate}[(H_1)']
\item there is a constant $C > 0$ such that $|F^2'(s)s| \leq C|s|^{\frac{2\beta}{N}}$ and $0 \leq (G^2)'(s)s \leq C(|s|^2 + |s|^{\frac{2\beta}{N-2}})$ for $s \in \mathbb{R}$.
\end{enumerate}
This is a slightly weaker variant of \((H_1)\), which is essentially designed for the nonlocal problem. In fact, with our argument, one can provide a different proof of the existence of a radial solution under assumptions (i)–(iii) from [26, 27].

Further progress on the Berestycki-Lions problem (1.4) has been made in [12, 18, 19]; see also the references therein.

We look for a weak solution \(u \in D^{1,2}(\mathbb{R}^N)\) to (1.1), i.e.
\[
\int_{\mathbb{R}^N} \nabla u \cdot \nabla \psi \, dx = \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) \psi \, dx - \int_{\mathbb{R}^N} (I_\beta * G(u)) g(u) \psi \, dx
\]
for any \(\psi \in C_0^\infty(\mathbb{R}^N)\), where \(D^{1,2}(\mathbb{R}^N)\) stands for the completion of \(C_0^\infty(\mathbb{R}^N)\) with respect to the norm \(\|\nabla \cdot \|_2\).

At least formally solutions of (1.1) are critical points of the functional \(\mathcal{I} : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}\) defined as
\[
\mathcal{I}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx + \int_{\mathbb{R}^N} (I_\beta * G(u)) G(u) \, dx,
\]
where \(D^{1,2}(\mathbb{R}^N)\). Since \(|F(s)| \leq C|s|^\frac{N+4}{2N}+\) for some constant \(C > 0\), we have that \((I_\alpha * F(u)) F(u) \in L^1(\mathbb{R}^N)\). On the other hand \((I_\beta * G(u)) G(u) \in L^1_{loc}(\mathbb{R}^N)\) and need not be integrable in \(\mathbb{R}^N\). Therefore \(\mathcal{I}\) may be infinite on a dense subset of \(D^{1,2}(\mathbb{R}^N)\) and, thus, cannot be Fréchet-differentiable.

We remark also that scaling properties of the problem play a crucial role, but, in our case, seem to be difficult to apply. Indeed, if \(\alpha \neq \beta\), then the nonlinear terms
\[
\int_{\mathbb{R}^N} (I_\alpha * F(u(\lambda \cdot))) F(u(\lambda \cdot)) \, dx = \lambda^{-(N+\alpha)} \int_{\mathbb{R}^N} (I_\beta * F(u)) F(u) \, dx
\]
\[
\int_{\mathbb{R}^N} (I_\beta * G(u(\lambda \cdot))) G(u(\lambda \cdot)) \, dx = \lambda^{-(N+\beta)} \int_{\mathbb{R}^N} (I_\beta * G(u)) G(u) \, dx
\]
have different scaling coefficients and, in particular, one cannot employ Lagrange multipliers as in [6], rescaling as in [26], or Pohozaev constraint approach as in [18, 19].

Moreover, to recover the lack of compactness due to the fact that we are working in the whole \(\mathbb{R}^N\), we start using the invariance of the functional \(\mathcal{I}\) with respect to the orthogonal group action \(O(N)\). Hence we may restrict our considerations to the subspace of radial function \(D^{1,2}_{O(N)}(\mathbb{R}^N)\), however \(\mathcal{I}|_{D^{1,2}_{O(N)}(\mathbb{R}^N)}\) still preserves the above difficulties and may be infinite.

In this setting, our main result reads as follows.

**Theorem 1.1.** Assume that \((H_1)–(H_3)\) hold. Then, there is a nontrivial and radial solution \(u \in D^{1,2}(\mathbb{R}^N)\) to (1.1) such that \((I_\beta * G(u)) G(u) \in L^1(\mathbb{R}^N)\).

Let us describe our variational approach. We observe that
\[(1.5) \quad \mathcal{F}(u) := \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx\]
is well-defined on \(D^{1,2}(\mathbb{R}^N)\), however \(\mathcal{I}\) may be infinite. Therefore we replace
\[(1.6) \quad \mathcal{G}(u) := \int_{\mathbb{R}^N} (I_\beta * G(u)) G(u) \, dx\]
with
\[(1.7) \quad \mathcal{G}_n(u) := \int_{\mathbb{R}^N} \varphi_n(x) (I_\beta * G(u)) G(u) \, dx,\]
where \(\{\varphi_n\}_{n \geq 1}\) is a sequence of \(C_0^\infty(\mathbb{R}^N)\) radial functions, decreasing with respect to the radius, such that, for every \(n \geq 1\), \(\varphi_n(x) = 1\) for \(x \in B_n\), \(\varphi_n(x) = 0\) for \(x \in \mathbb{R}^N \setminus B_{2n}\), \(0 \leq \varphi_n(x) \leq \)
$|x||\nabla \varphi_n(x)| \leq c$, and $\varphi_n(x) \leq \varphi_{n+1}(x)$ for $n \geq 1$ and $x \in \mathbb{R}^N \ (B_n$ stands for the ball of radius $n$ centred at $0$). Then $\mathcal{G}_n$ is well-defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and

$$I_n(u) := \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \mathcal{F}(u) + \mathcal{G}_n(u) \tag{1.8}$$

is of class $\mathcal{C}^1$.

The functional $I_n$ does not satisfy any variant of Ambrosetti-Rabinowitz condition [1], hence it is difficult to find a bounded Palais-Smale sequence on a positive level. Inspired by [10,11] we apply the variational method in [28, Theorem 2.8] to the functional

$$J_n := (\sigma, u) \in \mathbb{R} \times \mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N) \mapsto I_n(u(e^{\sigma \cdot})) \in \mathbb{R}.$$

We require a new nonlocal variant of the Brezis-Lieb Lemma for a general nonlinearity, see Lemma 2.1, and further compactness properties of $\mathcal{F}(u)$ on $\mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$ demonstrated in Section 2. Then, letting $n \to +\infty$, the careful analysis of the Mountain Pass levels provides a nontrivial radial solution to (1.1). This approach provides also an alternative proof of the existence of a radial solution in the local case considered in [26,27]. We would like to point out that, contrary to [6,26,27], we no longer use the uniform decay at infinity of radial functions from $\mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$ (see [6, Radial Lemma A.III]) and the compactness lemma due to Strauss [6, Lemma A.I].

Therefore more can be said in higher dimensions. Let $N \geq 4$, $N \neq 5$ and similarly as Bartsch and Willem in [3] (cf. [12,18,19]), let us fix $\tau \in \mathcal{O}(N)$ such that $\tau(x_1,x_2,x_3) = (x_2,x_1,x_3)$ for $x_1,x_2 \in \mathbb{R}^M$ and $x_3 \in \mathbb{R}^{N-2M}$, where $x = (x_1,x_2,x_3) \in \mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{N-2M}$ and $2 \leq M \leq N/2$, with $N-2M \neq 1$. We define

$$X_\tau := \{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u(x) = -u(\tau x) \ \text{for all} \ x \in \mathbb{R}^N \}.$$

Clearly, if $u \in X_\tau$ is radial, then $u = 0$. Hence $X_\tau$ does not contain nontrivial radial functions. Then let us consider $\mathcal{O} := \mathcal{O}(M) \times \mathcal{O}(M) \times \mathcal{O}(N-2M) \subset \mathcal{O}(N)$ acting isometrically on $\mathcal{D}^{1,2}(\mathbb{R}^N)$ with the subspace of invariant function denoted by $\mathcal{D}^{1,2}_{\mathcal{O}}(\mathbb{R}^N)$. Moreover our functionals are invariant under this action whenever $f$ and $g$ are odd or even.

Our result, in this setting, is

**Theorem 1.2.** Assume that (H1)–(H3) hold, $f$ and $g$ are odd, $N \geq 4$ and $N \neq 5$. Then, there is a nontrivial and nonradial solution $u \in \mathcal{D}^{1,2}_{\mathcal{O}}(\mathbb{R}^N) \cap X_\tau$ to (1.1) such that $(I_\alpha * G(u))G(u) \in L^1(\mathbb{R}^N)$.

Observe that in Theorem 1.1 and Theorem 1.2 we can take $G(s) = s$ and $\beta = 0$ and we obtain solutions in $H^1(\mathbb{R}^N)$ solving the Choquard problem (1.2). In fact, dealing with the operator $-\Delta u + u$, more general assumptions imposed on $F$ can be considered, which fully cover situation in [23].

Actually, our argument can be, quite easily, adapted to the following problem

$$-\Delta u + \omega u = (I_\alpha * F(u))f(u) - (I_\beta * G(u))g(u) \quad \text{in} \ \mathbb{R}^N, \tag{1.9}$$

where $\omega > 0$, assuming that

- (H1') there is a constant $C > 0$ and $p \in \left(\frac{2\beta}{N-2}, \frac{N+\beta}{N-2}\right)$ such that $|sf(s)| \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\beta}{N-2}})$ and $0 \leq g(s) \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\beta}{N-2}})$ for $s \in \mathbb{R}$;

- (H2') \( \lim_{s \to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = \lim_{s \to +\infty} \frac{F(s)}{|s|^{\frac{N+\beta}{N-2}}}; \)

- (H3') there is $s_0 > 0$ such that $F(s_0) \neq 0$; we assume also $F(s_0) > G(s_0)$, if $\alpha = \beta > 0$, and $F^2(s_0) > G^2(s_0) + \omega s_0^2$, if $\alpha = \beta = 0$.\]
Observe that the energy functional associated with (1.9) is given by

$$K_\omega(u) := I(u) + \omega \int_{\mathbb{R}^N} |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^N),$$

and may be also infinite due to the possible nonintegrable term \((I_\beta * G(u))G(u)\).

Our results for equation (1.9) read as follows.

**Theorem 1.3.** Assume that \((H'_1)-(H'_3)\) hold. Then, there is a nontrivial and radial solution \(u\) to (1.9) in \(H^1(\mathbb{R}^N)\) such that \((I_\beta * G(u))G(u) \in L^1(\mathbb{R}^N)\). Moreover, if \(f\) and \(g\) are odd, \(N \geq 4\) and \(N \neq 5\), there is also a nontrivial and nonradial solution \(v\) to (1.9) in \(H^1(\mathbb{R}^N) \cap X_\tau\) such that \((I_\beta * G(v))G(v) \in L^1(\mathbb{R}^N)\).

In particular, if

\[
F(s) := \frac{1}{\sqrt{q}} |s|^q \quad \text{with} \quad 1 < q < \frac{N + \alpha}{N - 2}, \quad \text{and} \quad G(s) := \sqrt{\frac{N - 2}{N - \beta}} |s|^\frac{N + \beta}{N - 2},
\]

then \(F\) and \(G\) satisfy \((H'_1)-(H'_3)\) if and only if \(\omega \in (0, \omega_0)\), where

\[
\omega_0 := \left\{ \begin{array}{ll}
\frac{2^* - 2q}{2^*(q-1)} \left( \frac{N(q-1)}{2q} \right)^\frac{2^*-2}{2q} & \text{if } \alpha = \beta = 0, \\
+\infty & \text{if } \alpha > 0.
\end{array} \right.
\]

Then, finally, we obtain the following corollary.

**Corollary 1.4.** Suppose that \(F\) and \(G\) are given by (1.10).

(a) For any \(\omega \in (0, \omega_0)\) there is a radially symmetric symmetric solution in \(H^1(\mathbb{R}^N)\) and a nonradial solution in \(H^1(\mathbb{R}^N) \cap X_\tau\) to (1.9).

(b) If \(\omega \notin (0, \omega_0)\), then (1.9) has only trivial finite energy solution.

Corollary 1.4 has been known only in the local case \(\alpha = \beta = 0\) and the problem appears in nonlinear optics as well as in the the study of Bose–Einstein condensates [9, 20]. Note that solutions exist only for \(0 < \omega < \omega_0 < +\infty\), see e.g. [6, 13, 19]. In the nonlocal case, for instance if \(N = 3\), \(q = 2\) and \(\alpha > \beta = 0\), we solve the nonlocal cubic-quintic problem of the nonlinear optics for all \(\omega > 0\), where \(I_\alpha\) is a nonlocal response function determined by the details of the physical process responsible for the nonlocality [8].

Through the paper we use the following notation.

We denote by \(\| \cdot \|_k\) the usual norm in \(L^k(\mathbb{R}^N)\), for \(k \geq 1\), and by \(B_R\) the ball centered in 0 with radius \(R > 0\) in \(\mathbb{R}^N\). Recall that \(2^* = \frac{2N}{N-2}\). Finally \(C\) is a generic positive constant which may vary from line to line.

2. Functional setting and compactness properties

We prove our results for \(\beta > 0\), the most difficult and fully nonlocal situation. Thus, from now on, we assume that \(0 < \beta \leq \alpha < N\) and \((H_1)-(H_3)\) hold, with \(p = 1\) when \(0 < \beta < \frac{N-2}{2}\). The proofs of the paper are simplified when \(\beta = 0\) or \(\alpha = \beta = 0\) and we skip these cases.

It is standard to see that the functional \(F : L^2(\mathbb{R}^N) \to \mathbb{R}\), defined in (1.5) is of class \(C^1\), cf. [23].

In order to control the convergence of \(F\), we need the following nonlocal variant of the Brezis-Lieb Lemma [7] for the general nonlinearity. Note that nonlocal variants for particular nonlinearities have already appeared in [4, Lemma 2.2], [22, Lemma 2.4]. The proofs of [4, 22] seem to be difficult to adapt to the general nonlinear term. We provide an independent proof for any continuous \(f\) satisfying \((H_1)\) and \((H_2)\).
**Lemma 2.1.** Let \( u_n \to u_0 \) in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \). Then
\[
\lim_n \left( \int_{\mathbb{R}^N} (I_\alpha \ast F(u_n)) f(u_n) u_n \,dx - \int_{\mathbb{R}^N} (I_\alpha \ast F(u_n - u_0)) f(u_n) u_n \,dx \right)
= \int_{\mathbb{R}^N} (I_\alpha \ast F(u_0)) f(u_0) u_0 \,dx.
\]

**Proof.** We claim that, passing to a subsequence, for any \( s \in [0,1] \),
\[
\lim_n \int_{\mathbb{R}^N} (I_\alpha \ast (f(u_n)u_n)) f(u_n - su_0) u_0 \,dx = \int_{\mathbb{R}^N} (I_\alpha \ast (f(u_0)u_0)) f(u_0 - su_0) u_0 \,dx.
\]

Let \( \varepsilon > 0 \) and \( \psi \in C_0^{\infty}(\mathbb{R}^N) \) such that \( \|u_0 - \psi\|_{2^*} < \varepsilon \). We have
\[
\left| \int_{\mathbb{R}^N} (I_\alpha \ast (f(u_n)u_n)) f(u_n - su_0) u_0 \,dx - \int_{\mathbb{R}^N} (I_\alpha \ast (f(u_0)u_0)) f(u_0 - su_0) u_0 \,dx \right|
\leq \left| \int_{\mathbb{R}^N} (I_\alpha \ast (f(u_n)u_n)) f(u_n - su_0)(u_0 - \psi) \,dx \right| \tag{A}
+ \left| \int_{\mathbb{R}^N} (I_\alpha \ast (f(u_n)u_n))(f(u_n - su_0) - f(u_0 - su_0)) \psi \,dx \right| \tag{B}
+ \left| \int_{\mathbb{R}^N} ((I_\alpha \ast (f(u_n)u_n)) - (I_\alpha \ast (f(u_0)u_0))) f(u_0 - su_0) \psi \,dx \right| \tag{C}
+ \left| \int_{\mathbb{R}^N} (I_\alpha \ast (f(u_0)u_0)) f(u_0 - su_0)(\psi - u_0) \,dx \right|. \tag{D}
\]

Since \( \{u_n\} \) is a bounded sequence in \( L^{2^*}(\mathbb{R}^N) \), we deduce by \( (H_1) \) that \( \{f(u_n)u_n\} \) is bounded in \( L^{\frac{2N}{N+2\alpha}}(\mathbb{R}^N) \). Moreover, by the continuity, we deduce that \( f(u_n)u_n \) converges to \( f(u_0)u_0 \) a.e. on \( \mathbb{R}^N \) along a subsequence. Therefore \( f(u_n)u_n \) tends weakly to \( f(u_0)u_0 \) in \( L^{\frac{2N}{N+2\alpha}}(\mathbb{R}^N) \). As the Riesz potential defines a linear and continuous map from \( L^{\frac{2N}{N+2\alpha}}(\mathbb{R}^N) \) to \( L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \), we obtain that \( I_\alpha \ast (f(u_n)u_n) \) tends weakly to \( I_\alpha \ast (f(u_0)u_0) \) in \( L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \). Moreover, since \( u_n - su_0 \) converges to \( u_0 - su_0 \) in \( L^q_{\text{loc}}(\mathbb{R}^N) \), for \( 1 \leq q < 2^* \), by \( (H_1) \) we infer that \( f(u_n - su_0) \) converges to \( f(u_0 - su_0) \) in \( L^q_{\text{loc}}(\mathbb{R}^N) \), for \( 1 \leq q < 2N/(\alpha + 2) \).

Then, by the Hardy–Littlewood–Sobolev inequality and since \( \{f(u_n - su_0)\} \) is bounded in \( L^{\frac{2N}{N+2\alpha}}(\mathbb{R}^N) \) we obtain
\[
(A) \leq C \|f(u_n)u_n\|_{\frac{2N}{N+2\alpha}} \|f(u_n - su_0)\|_{\frac{2N}{N+2\alpha}} \|u_0 - \psi\|_{2^*} \leq C\varepsilon
\]
and analogously, \( (D) \leq C\varepsilon \).

Moreover, denoting by \( K := \text{supp}(\psi) \), we have
\[
(B) \leq C \|f(u_n)u_n\|_{\frac{2N}{N+2\alpha}} \|f(u_n - su_0) - f(u_0 - su_0)\|_{\frac{2N(N+2\alpha+2)}{N(N+2\alpha+2)}(K)} \|\psi\|_{\frac{2N(N+2\alpha+2)}{N(N+2\alpha+2)}}(K) = o_n(1).
\]

Finally, also \( (C) = o_n(1) \), since \( f(u_0 - su_0)\psi \) belongs to \( L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \), namely the dual space of \( L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \).
Lemma 2.2. Let functions. Below we demonstrate the compactness properties in the following lemmas.

\[ \lim_{n} \left( \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n)) f(u_n) u_n dx - \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n - u_0)) f(u_n) u_n dx \right) \]

\[ = \int_{\mathbb{R}^N} (\phi_n(0) - \phi_n(1)) dx = - \int_{0}^{1} \left( \int_{\mathbb{R}^N} \phi_n'(s) dx \right) ds \]

\[ = \int_{0}^{1} \left( \int_{\mathbb{R}^N} (I_{\alpha} * (f(u_n - su_0)u_0)) f(u_n) u_n dx \right) ds \]

\[ = \int_{0}^{1} \left( \int_{\mathbb{R}^N} (I_{\alpha} * (f(u_n))) f(u_n - su_0) u_0 dx \right) ds \]

\[ = - \int_{0}^{1} \left( \int_{\mathbb{R}^N} \phi_0'(s) dx \right) ds = - \int_{\mathbb{R}^N} \left( \int_{0}^{1} \phi_0'(s) ds \right) dx \]

\[ = \int_{\mathbb{R}^N} (\phi_0(0) - \phi_0(1)) dx = \int_{\mathbb{R}^N} (I_{\alpha} * F(u_0)) f(u_0) u_0 dx. \]

Hence, by (2.1), taking into account the Lebesgue Dominated Convergence Theorem

\[ \lim_{n} \left( \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n)) f(u_n) u_n dx - \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n - u_0)) f(u_n) u_n dx \right) \]

\[ = \lim_{n} \int_{0}^{1} \left( \int_{\mathbb{R}^N} (I_{\alpha} * (f(u_n))) f(u_n - su_0) u_0 dx \right) ds \]

\[ = \int_{0}^{1} \left( \lim_{n} \int_{\mathbb{R}^N} (I_{\alpha} * (f(u_n))) f(u_n - su_0) u_0 dx \right) ds \]

\[ = \int_{0}^{1} \left( \int_{\mathbb{R}^N} (I_{\alpha} * (f(u_0))) f(u_0 - su_0) u_0 dx \right) ds \]

\[ = - \int_{0}^{1} \left( \int_{\mathbb{R}^N} \phi_0'(s) dx \right) ds = - \int_{\mathbb{R}^N} \left( \int_{0}^{1} \phi_0'(s) ds \right) dx \]

\[ = \int_{\mathbb{R}^N} (\phi_0(0) - \phi_0(1)) dx = \int_{\mathbb{R}^N} (I_{\alpha} * F(u_0)) f(u_0) u_0 dx. \]

\[ \square \]

Now, let \( \mathcal{O}' = \mathcal{O}(N) \), or \( \mathcal{O} = \mathcal{O}(M) \times \mathcal{O}(M) \times \mathcal{O}(N - 2M) \subset \mathcal{O}(N) \) provided that \( N \geq 4 \) and \( N \neq 5 \) with \( 2 \leq M \leq N/2 \) and \( N - 2M \neq 1 \). Let \( \mathcal{D}_{\mathcal{O}'}^{1,2}(\mathbb{R}^N) \) be the subspace of \( \mathcal{O}' \)-invariant functions. Below we demonstrate the compactness properties in the following lemmas.

Lemma 2.2. Let \( u_n \to u_0 \) in \( \mathcal{D}_{\mathcal{O}'}^{1,2}(\mathbb{R}^N) \). Then

\[ \lim_{n} \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n)) f(u_n) u_n dx = \int_{\mathbb{R}^N} (I_{\alpha} * F(u_0)) f(u_0) u_0 dx. \]

Proof. By Lemma 2.1, we conclude if we prove that

\[ \lim_{n} \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n - u_0)) f(u_n) u_n dx = 0. \]

Indeed, by the Hardy-Littlewood-Sobolev inequality and \( (H_1) \), we have

\[ \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n - u_0)) f(u_n) u_n dx \leq C \|F(u_n - u_0)\|_{2N/(N-\alpha)} \|f(u_n)u_n\|_{2N/(N-\alpha)} \leq C \|F(u_n - u_0)\|_{2N/(N+\alpha)} \|f(u_n)u_n\|_{2N/(N+\alpha)}. \]

The fact that \( \|F(u_n - u_0)\|_{2N/(N-\alpha)} \to 0 \) is a consequence of \( (H_2) \) and [19, Lemma A.1], where the symmetry \( \mathcal{O}' \) plays a crucial rôle.

\[ \square \]

Lemma 2.3. Let \( u_n \to u_0 \) in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \). Then, for any \( \psi \in C_c^\infty(\mathbb{R}^N) \),

\[ \lim_{n} \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n)) f(u_n) \psi dx = \int_{\mathbb{R}^N} (I_{\alpha} * F(u_0)) f(u_0) \psi dx. \]
Proof. Arguing as in the proof of Lemma 2.1 and passing to a subsequence, we have that \( f(u_n) \to f(u_0) \) in \( L^q_{\text{loc}}(\mathbb{R}^N) \), for \( 1 \leq q < 2N/(\alpha + 2) \), and \( \{ I_\alpha * F(u_n) \} \) is bounded in \( L^{2N/\alpha}(\mathbb{R}^N) \) and tends weakly to \( I_\alpha * F(u_0) \) in \( L^{2N/\alpha}(\mathbb{R}^N) \). Thus, since
\[
\left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) \psi dx - \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) \psi dx \right|
\leq \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) |f(u_n) - f(u_0)| |\psi| dx + \int_{\mathbb{R}^N} (I_\alpha * F(u_n) - I_\alpha * F(u_0)) f(u_0) \psi dx,
\]
using the same arguments as in (B) and (C) in the proof of Lemma 2.1, we get (2.2).

For what concerns the term with \( G \), at least formally, we define the functional \( \mathcal{G} \) as in (1.6). However, if in \( (H_1) \), \( p < \frac{N+\beta}{N-2} \), the situation is quite different from \( F \) and \( G \) need not be finite. Indeed, in such a case, let us consider the Banach spaces
\[ L^\mu(\Omega) + L^\nu(\Omega) := \{ u \in \mathcal{M}(\Omega) : u = u_1 + u_2, u_1 \in L^\mu(\Omega), u_2 \in L^\nu(\Omega) \}, \]
where \( 1 \leq \mu \leq \nu < +\infty \), \( \Omega \) is an arbitrary subset of \( \mathbb{R}^N \), and \( \mathcal{M}(\Omega) \) is the set of the real measurable functions defined on \( \Omega \), equipped with the norm
\[ ||u||_{\mu,\nu} := \inf_{u = u_1 + u_2} (||u_1||_{L^\mu(\Omega)} + ||u_2||_{L^\nu(\Omega)}) \]
(see e.g. [2, Section 2] for more details about these spaces).

Observe that if \( u \in D^{1,2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \), since \( |u|^p \in L^{2^*}(\mathbb{R}^N) \) and \( |u|^{N+\beta} \in L^{2N/\alpha}(\mathbb{R}^N) \), by [2, Proposition 2.3] and \( (H_1) \), we get
\begin{equation}
(2.3) \quad G(u) \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{2^*}(\mathbb{R}^N).
\end{equation}
Moreover, since
\[ I_\beta * G(u) \leq C(I_\beta * (|u|^p + |u|^{N+\beta})), \]
by [15, Inequality (9), page 107] and [2, Proposition 2.3] we have
\begin{equation}
(2.4) \quad I_\beta * G(u) \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*N}{N-2^*+2\beta}}(\mathbb{R}^N).
\end{equation}
However, this does not seem enough to assure that \( \mathcal{G}(u) < +\infty \) for any \( u \in D^{1,2}(\mathbb{R}^N) \), and so we need a different approach. We replace \( \mathcal{G}(u) \) with \( \mathcal{G}_\alpha(u) \) given by (1.7) together with the sequence \( \{ \varphi_n \} \) defined there.

We prove the following lemma.

Lemma 2.4. For every \( n \in \mathbb{N} \), \( \mathcal{G}_n \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R}) \).

Proof. We divide the proof in five steps.

Step 1: \( \mathcal{G}_n \) is well defined.

Observe that
\[ 0 \leq \mathcal{G}_n(u) \leq \int_{B_{2n}} (I_\beta * G(u)) G(u) \, dx \]
and, by (2.3) and (2.4), \( I_\beta * G(u) \in L^{\frac{2N}{N+\beta}}(B_{2n}) + L^{\frac{2^*N}{N-2^*+2\beta}}(B_{2n}) \subset L^{\frac{2N}{N+\beta}}(B_{2n}) \) and \( G(u) \in L^{\frac{2N}{N+\beta}}(B_{2n}) + L^{\frac{2^*N}{N-2^*+2\beta}}(B_{2n}) \). Thus, the Hölder inequality allows us to conclude.

Step 2: if \( \{ u_m \} \subset L^2(\mathbb{R}^N) \) and \( u_m \to u \) in \( L^2(\mathbb{R}^N) \), then, up to a subsequence, \( I_\beta * G(u_m) \to I_\beta * G(u) \) a.e. in \( \mathbb{R}^N \), as \( m \to +\infty \).

Since \( u_m \to u \) in \( L^2(\mathbb{R}^N) \), then, up to a subsequence, there exist \( \Omega_1 \subset \mathbb{R}^N \) with \( |\Omega_1| = 0 \) and \( w \in L^2(\mathbb{R}^N) \) such that \( |u_m| \leq w \) and \( u_m \to u \) in \( \mathbb{R}^N \setminus \Omega_1 \).

Since \( w^p + w^{\frac{N+\beta}{N-2}} \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{2^*}(\mathbb{R}^N) \), by [15, Inequality (9), page 107], we have that \( I_\beta *
\[
(w^p + w^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{(N-2)p-2}}(\mathbb{R}^N)
\]
and so, there exists \( \Omega_2 \subset \mathbb{R}^N \), with \( |\Omega_2| = 0 \), such that
\[
\frac{w^p(y) + w^{\frac{N+\beta}{N-2}}(y)}{|x-y|^{N-\beta}} \in L^1(\mathbb{R}^N), \quad \text{for all } x \in \mathbb{R}^N \setminus \Omega_2.
\]

Thus, if we fix \( x \in \mathbb{R}^N \setminus \Omega_2 \), we have that
\[
\frac{G(u_m(y))}{|x-y|^{N-\beta}} \to \frac{G(u(y))}{|x-y|^{N-\beta}}, \quad \text{for all } y \in \mathbb{R}^N \setminus \Omega_1
\]
and
\[
\frac{G(u_m(y))}{|x-y|^{N-\beta}} \leq C \frac{|u_m(y)|^p + |u_m(y)|^{\frac{N+\beta}{N-2}}}{|x-y|^{N-\beta}} \leq C \frac{w^p(y) + w^{\frac{N+\beta}{N-2}}(y)}{|x-y|^{N-\beta}} \in L^1(\mathbb{R}^N).
\]

Hence, by the Lebesgue Dominated Convergence Theorem we can conclude.

**Step 3: \( G_n \) is continuous.**

Let \( \{u_m\} \subset D^{1,2}(\mathbb{R}^N) \) be such that \( u_m \to u \) in \( D^{1,2}(\mathbb{R}^N) \) as \( m \to +\infty \). Up to a subsequence we have that \( u_m \to u \) in \( L^2(\mathbb{R}^N) \), \( u_m \to u \) a.e. in \( \mathbb{R}^N \), and there exists \( w \in L^2(\mathbb{R}^N) \) such that \( |u_m| \leq w \) a.e. in \( \mathbb{R}^N \). Thus, since \( G \) is continuous, \( G(u_m) \to G(u) \) a.e. in \( \mathbb{R}^N \) and, by Step 2, \( I_\beta * G(u_m) \to I_\beta * G(u) \) a.e. in \( \mathbb{R}^N \). Hence
\[
\varphi_n(x)(I_\beta * G(u_m))G(u_m) \to \varphi_n(x)(I_\beta * G(u))G(u) \text{ a.e. in } \mathbb{R}^N, \text{ as } m \to +\infty.
\]
Moreover,
\[
0 \leq \varphi_n(x)(I_\beta * G(u_m))G(u_m) \leq C \varphi_n(x)(I_\beta * (w^p + w^{\frac{N+\beta}{N-2}}))(w^p + w^{\frac{N+\beta}{N-2}}) \in L^1(\mathbb{R}^N)
\]
since, arguing as before, \( I_\beta * (w^p + w^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(B_{2n}) \) and \( w^p + w^{\frac{N+\beta}{N-2}} \in L^{\frac{2N}{N+\beta}}(B_{2n}) \). Thus, the Lebesgue Dominated Convergence Theorem allows us to conclude.

**Step 4: \( G_n \) is differentiable and, for any \( v \in D^{1,2}(\mathbb{R}^N) \),**
\[
G'_n(u)[v] = 2 \int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(u))g(u)v \, dx.
\]

First we prove that
\[
\left| \int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(u))g(u)v \, dx \right| < +\infty.
\]

Observe that
\[
\left| \int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(u))g(u)v \, dx \right| \leq \int_{B_{2n}} (I_\beta * G(u))|g(u)||v| \, dx,
\]
and, by assumptions on \( g \),
\[
|g(u)| \leq C(|u|^{p-1} + |u|^{\frac{N+\beta}{N-2}}) \in \begin{cases}
L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N), & \text{if } 0 < \beta < \frac{N-2}{2}, \\
L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{N-\beta}}(\mathbb{R}^N), & \text{if } \frac{N-2}{2} \leq \beta < N.
\end{cases}
\]

In any case we have that \( I_\beta * G(u) \in L^{\frac{2N}{N-\beta}}(B_{2n}) \) and, by (2.5), \( g(u) \in L^{\frac{2N}{N+\beta}}(B_{2n}) \). Thus, the Hölder inequality allows us to conclude.

Finally, arguing as before, we prove that the map
\[
v \in D^{1,2}(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(u))g(u)v \, dx
\]
is continuous and this implies also the claim.

**Step 5: \( G'_n \) is continuous.**

Let \( v \in D^{1,2}(\mathbb{R}^N) \), with \( \|\nabla v\|_2 \leq 1 \) and \( \{u_m\} \subset D^{1,2}(\mathbb{R}^N) \) be such that \( u_m \to u \) in \( D^{1,2}(\mathbb{R}^N) \) as
\( m \to +\infty \). Up to a subsequence we have that \( u_m \to u \) in \( L^{2^*}(\mathbb{R}^N) \), \( u_m \to u \) a.e. in \( \mathbb{R}^N \), and there exists \( w \in L^{2^*}(\mathbb{R}^N) \) such that \( |u_m| \leq w \) a.e. in \( \mathbb{R}^N \). Moreover
\[
\left| G_n'(u_m)[v] - G_n'(u)[v] \right| \leq \int_{B_{2n}} \left| (I_{\beta} * G(u_m))g(u_m) - (I_{\beta} * G(u))g(u) \right| v \, dx
\leq C \left( \int_{B_{2n}} \left| (I_{\beta} * G(u_m))g(u_m) - (I_{\beta} * G(u))g(u) \right|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}}.
\]

Using also Step 2, we have that \( (I_{\beta} * G(u_m))g(u_m) \to [I_{\beta} * G(u)]g(u) \) a.e. in \( \mathbb{R}^N \), and so, observing that, by \((H_1)\),
\[
0 \leq I_{\beta} * G(u_m) \leq CI_{\beta} * (w^p + w^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(B_{2n}),
\]
\[
0 \leq I_{\beta} * (|u|^p + |u|^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(B_{2n}),
\]
and
\[
|g(u_m)| \leq C(w^{p-1} + w^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{p+\beta}}(B_{2n}),
\]
\[
|g(u)| \leq C(|u|^{p-1} + |u|^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{p+\beta}}(B_{2n}),
\]
we can conclude by the Lebesgue Dominated Convergence Theorem.

Now we prove this further compactness result.

**Lemma 2.5.** Let \( u_n \rightharpoonup u_0 \) in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \). Then, for any \( \psi \in C_0^\infty(\mathbb{R}^N) \),
\[
\lim_n \int_{\mathbb{R}^N} \varphi_n(x)(I_{\beta} * G(u_n))g(u_n)\psi \, dx = \int_{\mathbb{R}^N} (I_{\beta} * G(u_0))g(u_0)\psi \, dx.
\]

**Proof.** Of course it is enough to show that
\[
\lim_n \int_{\text{supp}(\psi)} (I_{\beta} * G(u_n))g(u_n)\psi \, dx = \int_{\text{supp}(\psi)} (I_{\beta} * G(u_0))g(u_0)\psi \, dx,
\]
recalling that \( \text{supp}(\psi) \) is compact and then, for \( n \) large enough, \( \text{supp}(\psi) \subset B_n \).

Since \( u_n \rightharpoonup u_0 \) weakly in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \), up to a subsequence, \( u_n \to u_0 \) a.e. in \( \mathbb{R}^N \) and so \( G(u_n) \to G(u_0) \) a.e. in \( \mathbb{R}^N \), as \( n \to +\infty \).

Moreover \( \{G(u_n)\} \) is bounded in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N) \). Indeed, by the assumptions on \( g \), the definition of the norm in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N) \), and \([2, \text{Corollary 2.12}]\), we have
\[
\|G(u_n)\|_{L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N)} \leq C(\|u_n\|_{L^{2^*}(\mathbb{R}^N)}^p + \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{\frac{N+\beta}{N-2}}) \leq C.
\]

Thus, the reflexivity of \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N) \) (see \([2, \text{Corollary 2.11}]\)) implies that there exists \( \tilde{u} \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N) \) such that, up to a subsequence, \( G(u_n) \rightharpoonup \tilde{u} \) in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N) \). We claim that \( \tilde{u} = G(u_0) \).

Indeed, using a classical argument, (see e.g. \([7, \text{Exercise 3.4}]\)), the weak convergence \( G(u_n) \rightharpoonup \tilde{u} \) in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N) \) implies that there exists a sequence \( \{z_n\} \subset L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2N}{p+\beta}}(\mathbb{R}^N) \) such that, for all \( n \in \mathbb{N} \),
\[
z_n \in \text{conv} \left( \bigcup_{i=1}^n \{G(u_i)\} \right)
\]
and \( z_n \to \tilde{u} \) in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{p}}(\mathbb{R}^N) \). Thus, by [2, Proposition 2.8], up to a subsequence, we get that \( z_n \to \tilde{u} \) a.e. in \( \mathbb{R}^N \), that allows us to conclude.

About the sequence \( \{ I_\beta \ast G(u_n) \} \), since by (H1)

\[
0 \leq I_\beta \ast G(u_n) \leq C(I_\beta \ast |u_n|^p + I_\beta \ast |u_n|^{\frac{N+\beta}{N-\beta}}) \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{(N-2)p-2\beta}}(\mathbb{R}^N),
\]

using [2, Corollary 2.12], we have

\[
\| I_\beta \ast G(u_n) \|_{L^{\frac{2N}{N+\beta}}(\mathbb{R}^N)} \leq C\| I_\beta \ast |u_n|^{\frac{N+\beta}{N-\beta}} \|_{L^{\frac{2N}{N+\beta}}(\mathbb{R}^N)} + \| I_\beta \ast |u_n|^p \|_{L^{\frac{2}{(N-2)p-2\beta}}(\mathbb{R}^N)} \leq C(\| u_n \|_{L^{2\ast}} + \| u_n \|_{L^p}^p) \leq C.
\]

Moreover, observe that the linear functional

\[
w \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{p}}(\mathbb{R}^N) \mapsto I_\beta \ast w \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{(N-2)p-2\beta}}(\mathbb{R}^N)
\]

is continuous. Indeed, if \( w \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{p}}(\mathbb{R}^N) \), \( w = w_1 + w_2 \) with \( w_1 \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) \) and \( w_2 \in L^{\frac{2}{p}}(\mathbb{R}^N) \), by [15, Inequality (9), page 107] we get

\[
\| I_\beta \ast w \|_{L^{\frac{2N}{N+\beta}}(\mathbb{R}^N)} \leq \| I_\beta \ast w_1 \|_{L^{\frac{2N}{N+\beta}}(\mathbb{R}^N)} + \| I_\beta \ast w_2 \|_{L^{\frac{2}{(N-2)p-2\beta}}(\mathbb{R}^N)} \leq C(\| w_1 \|_{L^{2\ast}} + \| w_2 \|_{L^p})
\]

and, passing to the infimum on \( w_1 \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) \) and \( w_2 \in L^{\frac{2}{p}}(\mathbb{R}^N) \), we conclude.

This, combined with the weak convergence \( G(u_n) \rightharpoonup G(u_0) \) in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{p}}(\mathbb{R}^N) \), implies that \( I_\beta \ast G(u_n) \rightharpoonup I_\beta \ast G(u_0) \) in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{(N-2)p-2\beta}}(\mathbb{R}^N) \).

Hence, as done for \( f \) in Lemma 2.3, we have that

\[
\int_{\text{supp}(\psi)} (I_\beta \ast G(u_n))g(u_n)\psi \, dx - \int_{\text{supp}(\psi)} (I_\beta \ast G(u_0))g(u_0)\psi \, dx \leq \int_{\text{supp}(\psi)} (I_\beta \ast G(u_n))|g(u_n) - g(u_0)||\psi| \, dx + \int_{\mathbb{R}^N} (I_\beta \ast G(u_n) - I_\alpha \ast G(u_0))g(u_0)\psi \, dx.
\]

About the first integral, observe that the boundedness of \( \{ u_n \} \) in \( D^{1,2}(\mathbb{R}^N) \) implies also that \( u_n \to u_0 \) in \( L^1_{\text{loc}}(\mathbb{R}^N) \), for all \( 1 \leq \tau < 2^* \) and so, for any fixed \( 1 \leq \tau < 2^* \) and \( K \subset \subset \mathbb{R}^N \), up to a subsequence, there exists \( w_K \in L^{\tau}(K) \) such that \( |u_n| \leq w_K \) a.e. in \( K \). Thus, denoting for simplicity \( w := u_{\text{supp}(\psi)} \) and taking for instance

\[
\tau = \frac{N(N+2\beta+2)}{(N-2)(N+\beta)},
\]

by the assumptions on \( g \) we have

\[
g(u_n) \to g(u_0) \text{ a.e. in } \text{supp}(\psi),
\]

\[
|g(u_n)| \leq C(|u_n|^{p-1} + |u_n|^{\frac{\beta+2}{N-\beta}}) \leq C(w^{p-1} + w^{\frac{\beta+2}{N-\beta}}) \in L^{\frac{N(N+2\beta+2)}{(N+\beta)(\beta+2)}}(\text{supp}(\psi)),
\]

\[
|g(u_0)| \leq C(|u_0|^{p-1} + |u_0|^{\frac{\beta+2}{N-\beta}}) \in L^{\frac{N(N+2\beta+2)}{(N+\beta)(\beta+2)}}(\text{supp}(\psi)).
\]

Moreover, the boundedness of \( \{ I_\beta \ast G(u_n) \} \) in \( L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2}{(N-2)p-2\beta}}(\mathbb{R}^N) \) implies its boundedness in \( L^{\frac{2N}{N+\beta}}(\text{supp}(\psi)) + L^{\frac{2}{(N-2)p-2\beta}}(\text{supp}(\psi)) = L^{\frac{2N}{N+\beta}}(\text{supp}(\psi)).
\]

Thus, by the Hölder inequality and the Lebesgue Dominated Convergence Theorem, we have

\[
\int_{\text{supp}(\psi)} (I_\beta \ast G(u_n))|g(u_n) - g(u_0)||\psi| \, dx \leq C\left( \int_{\text{supp}(\psi)} |g(u_n) - g(u_0)|^{\frac{2N}{N+\beta}}|\psi|^{\frac{2N}{N+\beta}} \, dx \right)^{\frac{N+\beta}{2N}}.
\]
\[ \leq C \left( \int_{\operatorname{supp}(\psi)} |g(u_n) - g(u_0)| \frac{N(N+2\beta+2)}{(N+\beta)(\beta+2)} \, dx \right)^{(N+\beta)(\beta+2)} = o_n(1). \]

Finally the second integral goes to 0 due to the weak convergence \( I_\beta * G(u_n) \rightharpoonup I_\beta * G(u_0) \) in \( L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) \), since \( g(u_0)\psi \in L^{\frac{2N}{N+\beta}}(\operatorname{supp}(\psi)) \subset [L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{N-\beta}}(\mathbb{R}^N)]' \), being

\[
\int_{\operatorname{supp}(\psi)} |g(u_0)\psi|^{\frac{2N}{N+\beta}} \, dx \leq C \int_{\operatorname{supp}(\psi)} (|u_0|^{p-1} + |u_0|^{\frac{\beta+2}{N-\beta}} |\psi|^{\frac{2N}{N+\beta}}) \, dx
\leq C \left( \int_{\operatorname{supp}(\psi)} (|u_0|^{p-1} + |u_0|^{\frac{\beta+2}{N-\beta}}) \right)^{\frac{2(\beta+2)}{(N+\beta)(\beta+2)} < +\infty.}
\]

\[ \square \]

3. Proofs of our main results

Let \( X := D^{1,2}_{O(N)}(\mathbb{R}^N) \), or \( X := D^{1,2}_O(\mathbb{R}^N) \cap X_T \) provided that \( N \geq 4 \) and \( N \neq 5 \). As observed before, the functional \( I \) could be also \( +\infty \) on \( X \). To avoid this problem, for every \( n \geq 1 \), we introduce the truncated \( C^1 \)-functionals \( I_n : X \rightarrow \mathbb{R} \) defined by \((1.8)\).

The functionals \( I \) and \( I_n, \ n \geq 1 \), satisfy the geometrical assumptions of the Mountain Pass Theorem. Indeed, we prove the following lemma.

**Lemma 3.1.** We have:

(i) there exist \( \rho, c > 0 \) such that \( I(u) \geq c \) and, for every \( n \geq 1 \), \( I_n(u) \geq c \) for all \( u \in X \) such that \( \|\nabla u\|_2 = \rho \);

(ii) there exists \( v_0 \in X \) with \( \|\nabla v_0\|_2 > \rho \) such that \( I(v_0) < 0 \) and, for every \( n \geq 1 \), \( I_n(v_0) < 0 \).

**Proof.** We prove this lemma only for \( I_n \) since similar and easier arguments hold also for \( I \).

The positivity of \( G \) and \( \varphi_n \), \((H_1)\) and \((H_2)\), the Hardy-Littlewood-Sobolev and Sobolev inequalities imply

\[ I_n(u) \geq \|\nabla u\|_2^2 - C \int_{\mathbb{R}^N} \left( I_\alpha * |u|^{\frac{N+\alpha}{N-\beta}} \right) |u|^{\frac{N+\alpha}{N-\beta}} \, dx \geq \|\nabla u\|_2^2 - C \|u\|_2^{\frac{2(N+\alpha)}{N-\beta}} \geq \|\nabla u\|_2^2 - C \|\nabla u\|_2^{\frac{2(N+\alpha)}{N-\beta}}. \]

Since \( 2 < \frac{2(N+\alpha)}{N-\beta} \), we get (i).

Now let us prove (ii).

**Case** \( X = D^{1,2}_{C(\mathbb{R}^N)}(\mathbb{R}^N) \). Let \( w = s_0 \chi_{B_1} \), where \( s_0 \) is defined in \((H_3)\), then

\[ \mathcal{F}(w) = F^2(s_0) \int_{B_1 \times B_1} I_\alpha(x-y) \, dx \, dy > 0. \]

We take now \( \psi \in C_0^\infty(\mathbb{R}^N) \) radial, non-negative, non-increasing with respect to \( |x| \), and such that \( \psi(x) = s_0 \), for \( |x| \leq 1 \), and \( \psi(x) = 0 \), for \( |x| \geq \bar{r} \), with \( \bar{r} > 1 \). If \( \bar{r} \) is sufficiently close to 1, using the continuity of \( \mathcal{F} \) in \( L^2(\mathbb{R}^N) \), we get also

\[ \mathcal{F}(\psi) > 0. \]

We consider first the case \( \alpha > \beta \).

If we set \( \psi_\lambda(x) := \psi(x/\lambda) \), \( \lambda > 0 \) and since \( 0 \leq \varphi_n \leq 1 \) we have

\[ \int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(\psi_\lambda)) G(\psi_\lambda) \, dx \leq \lambda^{N+\beta} \int_{\mathbb{R}^N} (I_\beta * G(\psi)) G(\psi) \, dx < +\infty. \]

So we infer that

\[ I_n(\psi_\lambda) \leq \lambda^{N-2} \|\nabla \psi\|_2^2 - \lambda^{N+\alpha} \mathcal{F}(\psi) + \lambda^{N+\beta} G(\psi) \]

\[ \square \]
and we can conclude considering \( v_0 := \psi_\lambda \) with \( \lambda \) large enough, by (3.1).

We now study the case \( \alpha = \beta \).

If \( G(s_0) = 0 \), being, by \( (H_1) \), \( G \) non-decreasing on \( \mathbb{R}_+ \), then \( G(\psi(x)) = 0 \) in \( \mathbb{R}^N \) and so we can conclude easily as before.

If, instead, \( G(s_0) \neq 0 \), by \( (H_3) \) we can find \( \varepsilon > 0 \) sufficiently small such that \((1 - \varepsilon)F^2(s_0) > G^2(s_0) > 0 \). Moreover there exists \( \tilde{r} > 1 \) sufficiently close to 1 such that

\[
1 < \frac{\int_{B_{\tilde{r}} \times B_{\tilde{r}}} I_\alpha(x-y) \, dx \, dy}{\int_{B_1 \times B_1} I_\alpha(x-y) \, dx \, dy} < \frac{(1 - \varepsilon)F^2(s_0)}{G^2(s_0)}
\]

and, again by the continuity of \( F \) in \( L^2^* (\mathbb{R}^N) \),

\[
F(\psi) \geq (1 - \varepsilon)F^2(s_0) \int_{B_1 \times B_1} I_\alpha(x-y) \, dx \, dy > 0.
\]

Therefore, by the positivity of \( G \), we deduce that

\[
F(\tilde{\psi}) - G(\psi) \geq (1 - \varepsilon)F^2(s_0) \int_{B_1 \times B_1} I_\alpha(x-y) \, dx \, dy - G^2(s_0) \int_{B_r \times B_r} I_\alpha(x-y) \, dx \, dy > 0.
\]

Thus we get

\[
I_n(\psi_\lambda) \leq \lambda^{N-2} \| \nabla \psi \|^2_2 - \lambda^{N+\alpha}[F(\psi) - G(\psi)],
\]

we can conclude again considering \( v_0 := \psi_\lambda \) with \( \lambda \) large enough.

**Case** \( X = \mathcal{D}^{1,2}_{O}(\mathbb{R}^N) \cap X_\tau \).

We take \( \varepsilon \in (0, 1/2) \) and any odd and smooth function \( \eta : \mathbb{R} \to [-1, 1] \) such that \( \eta(s) = 1 \) for \( s \geq 1/2 \) and \( \eta(s) = 0 \) for \( s \leq 1/2 - \varepsilon \). Then we define \( \tilde{\psi}(x) = \eta(|x_1| - |x_2|)\psi(x) \) for \( x = (x_1, x_2, x_3) \in \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{N-2M} \), with the same \( \psi \) as before. Observe that \( \tilde{\psi} \in X \). Moreover, arguing as in the previous case, we can find \( \tilde{r} > 1 \), sufficiently close to 1, and \( \varepsilon > 0 \), sufficiently close to 0, such that, using the continuity of \( F \) in \( L^2^* (\mathbb{R}^N) \),

\[
F(\tilde{\psi}) \geq \frac{1}{2}F^2(s_0) \int_{B_{\tilde{r}} \times B_{\tilde{r}}} \cap \{ |x_1| \geq |x_2| + 1/2, |y_1| \geq |y_2| + 1/2 \} I_\alpha(x-y) \, dx \, dy > 0.
\]

Then we argue similarly as in case \( X = \mathcal{D}^{1,2}_{O}(\mathbb{R}^N) \).

Let

\[
\Gamma := \{ \gamma \in \mathcal{C}([0, 1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = v_0 \}
\]

and

\[
c_{I_n} := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I_n(\gamma(t)), \quad c_I := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I(\gamma(t)).
\]

Our aim is to find a sequence \( \{u_n\} \subset X \) such that \( I_n(u_n) = c_{I_n} \) and \( I'_n(u_n) \to 0 \), as \( n \to +\infty \).

However, due to the general assumptions on \( F \) and \( G \), it is not easy to prove the boundedness of such sequence. Therefore, inspired by [10, 11], we introduce the functional \( J : \mathbb{R} \times X \to \mathbb{R} \cup \{+\infty\} \)

\[
J(\sigma, u) := \mathcal{I}(u(e^{-\sigma \cdot})) = e^{(N-2)\sigma} \| \nabla u \|^2_2 - e^{(N+\alpha)\sigma} F(u) + e^{(N+\beta)\sigma} G(u),
\]

and, for every \( n \geq 1 \), the \( C^1 \)-functionals \( J_n : \mathbb{R} \times X \to \mathbb{R} \)

\[
J_n(\sigma, u) := \mathcal{I}_n(u(e^{-\sigma \cdot})) = e^{(N-2)\sigma} \| \nabla u \|^2_2 - e^{(N+\alpha)\sigma} F(u) + e^{(N+\beta)\sigma} \int_{\mathbb{R}^N} \varphi_n(e^{-\sigma}) (I_\beta \ast G(u)) G(u) \, dx.
\]

Since the functional \( J_n \) is non-autonomous, contrary to [10, 11], we have to deal with an additional term which appears in the calculus of the gradient of \( J_n \), see Proposition 3.2 below.
Let
\[ \Sigma := \{(\sigma, \gamma) \in C([0,1], \mathbb{R} \times X) : (\sigma(0), \gamma(0)) = (0, 0) \text{ and } (\sigma(1), \gamma(1)) = (0, v_0)\} \]
and
\[ c_{\mathcal{J}} := \inf_{(\sigma, \gamma) \in \Sigma} \sup_{t \in [0,1]} \mathcal{J}(\sigma(t), \gamma(t)), \quad c_{\mathcal{I}} := \inf_{(\sigma, \gamma) \in \Sigma} \sup_{t \in [0,1]} \mathcal{I}(\sigma(t), \gamma(t)). \]

As observed in [10, Lemma 4.1], using the relation, respectively, between \( \mathcal{I} \) and \( \mathcal{J} \) and \( \mathcal{I}_n \) and \( \mathcal{J}_n \), we have that
\[ c_{\mathcal{I}} = c_{\mathcal{J}}, \quad c_{\mathcal{I}_n} = c_{\mathcal{J}_n}. \]

Since, for any \( n \in \mathbb{N} \), \( \mathcal{J}_n \leq \mathcal{J}_{n+1} \leq \mathcal{J} \), we have that the sequence \( \{c_{\mathcal{J}_n}\} \) is increasing and bounded from above by \( c_{\mathcal{J}} \), and so there exists \( \bar{c} > 0 \) such that \( c_{\mathcal{J}_n} \to \bar{c} \), as \( n \to +\infty \).

**Proposition 3.2.** There is a sequence \( \{(\sigma_n, u_n)\} \) in \( \mathbb{R} \times X \) such that
\[ (i) \quad |\mathcal{J}_n(\sigma_n, u_n) - \bar{c}| = o_n(1); \]
\[ (ii) \quad |\sigma_n| = o_n(1); \]
\[ (iii) \quad \|\nabla \mathcal{J}_n(\sigma_n, u_n)\| = o_n(1); \]
\[ (iv) \quad \{u_n\} \text{ is bounded in } X. \]

**Proof.** In view of (3.2), for any \( n \geq 1 \) we find \( \gamma_{k,n} \in \Gamma \) such that
\[ \sup_{t \in [0,1]} \mathcal{I}_k(\gamma_{k,n}(t)) \leq c_{\mathcal{J}_n} + \frac{1}{n} \]
and, for sufficiently large \( k \),
\[ |c_{\mathcal{J}_n} - \bar{c}| \leq \frac{1}{n} \]
also holds. Therefore, passing to a subsequence with a diagonalization argument, we may assume that there exists \( \gamma_n \in \Gamma \) (hence \( (0, \gamma_n) \in \Sigma \)) such that
\[ \sup_{t \in [0,1]} \mathcal{J}_n(0, \gamma_n(t)) \leq c_{\mathcal{J}_n} + o_n(1) \quad \text{and} \quad |c_{\mathcal{J}_n} - \bar{c}| \leq o_n(1). \]

Thus, by [28, Theorem 2.8], for any \( n \geq 1 \) there is \( (\sigma_n, u_n) \in \mathbb{R} \times X \) such that (i)–(iii) hold. Since \( \mathcal{J}_n(\sigma_n, u_n) = \bar{c} + o_n(1) \) and \( \partial_\sigma \mathcal{J}_n(\sigma_n, u_n) = o_n(1) \), we have
\[ (1 - \frac{N-2}{N+\alpha}) e^{(N-2)\sigma} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left( 1 - \frac{N+\beta}{N+\alpha} \right) e^{(N+\beta)\sigma} \int_{\mathbb{R}^N} \varphi_n(e^{\sigma_n}x)(I_\beta * G(u_n))G(u_n) dx \]
\[ - \frac{1}{N+\alpha} e^{(N+\beta)\sigma} \int_{\mathbb{R}^N} (\nabla \varphi_n(e^{\sigma_n}x) \cdot e^{\sigma_n}x)(I_\beta * G(u_n))G(u_n) dx = \bar{c} + o_n(1). \]

Since the cut-off functions \( \varphi_n \) are decreasing with respect to the radius, we have that \( \nabla \varphi_n(x) \cdot x \leq 0 \), for any \( x \in \mathbb{R}^N \) and so, being \( \alpha \geq \beta \), we infer that \( \{u_n\} \) is a bounded sequence in \( X \). \( \square \)

We can now conclude the proof of our main theorems.

**Proof of Theorems 1.1 and 1.2.** Let \( \{(\sigma_n, u_n)\} \) in \( \mathbb{R} \times X \) be the sequence found in Proposition 3.2. Then there exists \( u_0 \in X \) such that \( u_n \rightharpoonup u_0 \) weakly in \( X \) and a.e. on \( \mathbb{R}^N \). By Lemma 2.3 and Lemma 2.5, for any \( \psi \in C_0^\infty(\mathbb{R}^N) \), we have that
\[ \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla \psi dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) \psi dx - \int_{\mathbb{R}^N} (I_\beta * G(u_0)) g(u_0) \psi dx. \]
So we have that \( u_0 \) is a weak solution of (1.1). We will prove that \( u_0 \neq 0 \).

Observe that, by Proposition 3.2, since \( \{u_n\} \) is bounded in \( X \) and \( \partial_n J_n(\sigma_n, u_n)[u_n] = o_n(1) \), we deduce that there exists \( C > 0 \) such that, for any \( n \geq 1 \),

\[
\int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(u_n))g(u_n)u_n \, dx \leq C.
\]

Therefore, by Fatou’s Lemma

\[
(3.3) \quad \int_{\mathbb{R}^N} (I_\beta * G(u_0))g(u_0)u_0 \, dx \leq \liminf_n \int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(u_n))g(u_n)u_n \, dx \leq C.
\]

For any \( m \geq 1 \), let

\[
\psi_m(x) = \begin{cases} 
1 & \text{if } |x| \leq m, \\
\frac{2m - |x|}{m} & \text{if } m \leq |x| \leq 2m, \\
0 & \text{if } |x| \geq 2m.
\end{cases}
\]

Observe that, for any \( m \geq 1 \), we have that \( \psi_m u_0 \) belongs to \( X \). Note that \( \psi_m u_0 \) has a compact support and \( \partial_n J_n(\sigma_m, u_m)[\psi_m u_0] = o_n(1) \). Therefore, arguing as in Lemma 2.3 and in Lemma 2.5, passing to the limit as \( n \to +\infty \), we have that for any \( m \geq 1 \)

\[
(3.4) \quad \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla (\psi_m u_0) \, dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_0))f(u_0)\psi_m u_0 \, dx - \int_{\mathbb{R}^N} (I_\beta * G(u_0))g(u_0)\psi_m u_0 \, dx.
\]

Being \( u_0 \in X \), we have

\[
\left| \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla (\psi_m u_0) \, dx - \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx \right|
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla u_0|^2 |\psi_m - 1| \, dx + \int_{\mathbb{R}^N} |\nabla u_0| |u_0| |\nabla \psi_m| \, dx
\]

\[
\leq \int_{B_m} |\nabla u_0|^2 \, dx + \left( \int_{A_m} |\nabla u_0|^2 \, dx \right)^{1/2} \left( \int_{A_m} |u_0|^2 \, dx \right)^{1/2} \left( \int_{A_m} |\nabla \psi_m|^N \, dx \right)^{1/N}
\]

\[
\leq \int_{B_m} |\nabla u_0|^2 \, dx + C \left( \int_{B_m} |\nabla u_0|^2 \, dx \right)^{1/2} \left( \int_{B_m} |u_0|^2 \, dx \right)^{1/2}
\]

\[
= o_m(1),
\]

where \( A_m := B_{2m} \setminus B_m \).

Moreover, observe that

\[
(I_\alpha * F(u_0))f(u_0)\psi_m u_0 \to (I_\alpha * F(u_0))f(u_0)u_0, \quad \text{a.e. in } \mathbb{R}^N, \text{ as } m \to +\infty,
\]

and

\[
|(I_\alpha * F(u_0))f(u_0)\psi_m u_0| \leq |(I_\alpha * F(u_0))f(u_0)u_0| \in L^1(\mathbb{R}^N).
\]

Thus, by the Dominated Convergence Theorem, we have that

\[
(3.6) \quad \lim_m \int_{\mathbb{R}^N} (I_\alpha * F(u_0))f(u_0)\psi_m u_0 \, dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_0))f(u_0)u_0 \, dx.
\]

Analogously, we have also that

\[
(I_\beta * G(u_0))g(u_0)\psi_m u_0 \to (I_\beta * G(u_0))g(u_0)u_0, \quad \text{a.e. in } \mathbb{R}^N, \text{ as } m \to +\infty,
\]

and, using (3.3),

\[
0 \leq (I_\beta * G(u_0))g(u_0)\psi_m u_0 \leq (I_\beta * G(u_0))g(u_0)u_0 \in L^1(\mathbb{R}^N).
\]
Again the Dominated Convergence Theorem implies
\begin{equation}
(3.7) \quad \lim_{m} \int_{\mathbb{R}^N} (I_{\beta} * G(u_0))(\psi_m u_0) \, dx = \int_{\mathbb{R}^N} (I_{\beta} * G(u_0)) \, g(u_0) u_0 \, dx.
\end{equation}

Therefore, by (3.4), (3.5), (3.6) and (3.7), we have
\[ \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx = \int_{\mathbb{R}^N} (I_{\alpha} * F(u_0)) f(u_0) u_0 \, dx - \int_{\mathbb{R}^N} (I_{\beta} * G(u_0)) g(u_0) u_0 \, dx. \]

By Lemma 2.2 and (3.3), since $\partial_u J_n(\sigma_n, u_n)[u_n] = o_n(1)$, we infer that
\[ \limsup_n \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \limsup_n \left[ \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n)) f(u_n) u_n \, dx - \int_{\mathbb{R}^N} \varphi_n(x)(I_{\beta} * G(u_n)) g(u_n) u_n \, dx \right] \]
\[ \leq \int_{\mathbb{R}^N} (I_{\alpha} * F(u_0)) f(u_0) u_0 \, dx - \int_{\mathbb{R}^N} (I_{\beta} * G(u_0)) g(u_0) u_0 \, dx = \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx. \]

This implies that $u_n \to u_0$ strongly in $X$. Thus, since $J_n(\sigma_n, u_n) \to I(u_0)$, we have that $I(u_0) = \tilde{c} > 0$ and so $u_0$ is a nontrivial weak solution of (1.1).

**Remark 3.3.** By the inspection of the proof of Theorems 1.1 and 1.2, we deduce that $c_{\mathcal{I}_n} = c_{\mathcal{J}_n}$ are attained, for any $n \geq 1$.

**Proof of Theorem 1.3.** The proof is a slight modification of our previous arguments. Here we just want to comment $(H_3')$. The change of assumption in the different cases is due to the scaling properties of the functional $\mathcal{K}_\omega$. Indeed, setting $u_\lambda(x) := u(x/\lambda)$, for $\lambda > 0$, when $\alpha = \beta$ we have
\[ \mathcal{K}_\omega(u_\lambda) = \lambda^{N-2} \|\nabla u\|^2 + \omega \lambda^N \|u\|^2 - \lambda^{N+\alpha} \left( F(u) - G(u) \right). \]

Thus, to show the Mountain Pass geometry, if $\alpha = \beta > 0$, we can proceed as in Lemma 3.1, but if $\alpha = \beta = 0$ (the local case), we need a stronger condition, namely we need to take into account the term $\omega s^2$ in order to show that $\mathcal{K}_\omega(u_\lambda) < 0$ for large $\lambda$ (see also [6]).

**Proof of Corollary 1.4.** Item (a) follows from Theorem 1.3. To prove (b), observe that, only in the local case $\alpha = \beta = 0$, $\omega_0$ is finite. Thus, in such a case, if $\omega \geq \omega_0$, then $\mathcal{F}^2(s) - \mathcal{G}^2(s) - \omega_0 s^2 \leq 0$ for $s \in \mathbb{R}$ and there are no nontrivial solutions (see e.g. [6]). If, instead, $\omega \leq 0$, similarly as in [23, Theorem 3], if $u \in H^1(\mathbb{R}^N)$ solves (1.9) with (1.10), then we obtain the following Pohozaev identity
\[ \|\nabla u\|^2 = -\omega \frac{N}{N-2} \|u\|^2 + \frac{N + \alpha}{q(N-2)} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^q) |u|^q \, dx - \int_{\mathbb{R}^N} (I_{\beta} * |u|^\frac{N+\alpha}{N-2}) |u|^\frac{N+\alpha}{N-2} \, dx \]
and, taking into account $\mathcal{K}'_\omega(u)[u] = 0$, i.e.
\[ \|\nabla u\|^2 = -\omega \|u\|^2 + \int_{\mathbb{R}^N} (I_{\alpha} * |u|^q) |u|^q \, dx - \int_{\mathbb{R}^N} (I_{\beta} * |u|^\frac{N+\alpha}{N-2}) |u|^\frac{N+\alpha}{N-2} \, dx, \]
we infer that $u = 0$.

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