ON THE TOPOLOGY OF THE ZERO SETS OF MONOCHROMATIC RANDOM WAVES

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Abstract. This note concerns the topology of the connected components of the zero sets of monochromatic random waves on compact Riemannian manifolds without boundary. In [SW] it is shown that these are distributed according to a universal measure on the space of smooth topological types. We determine the support of this measure.

1. Introduction

For \( \ell \geq 0 \) and \( n \geq 2 \) let \( \mathcal{E}_\ell(S^n) \) denote the real linear space of (homogeneous) spherical harmonics of degree \( \ell \) in \( (n + 1) \) variables. These are eigenfunctions of the Laplacian \( \Delta_{S^n} \) on the sphere \( S^n \) endowed with the round metric sitting in \( \mathbb{R}^{n+1} \). With the corresponding \( L^2 \)-inner product

\[
\langle f, g \rangle = \int_{S^n} f(w)g(w) \, d\sigma(w),
\]

we get a Gaussian probability density \( P \) on \( \mathcal{E}_\ell(S^n) \). Namely,

\[
P(A) = \int_A e^{-\langle f,f \rangle} \, df,
\]

where \( df \) is the normalized Haar measure on \( \mathcal{E}_\ell(S^n) \) and \( A \subset \mathcal{E}_\ell(S^n) \).

We are interested in the zero set \( V(f) = f^{-1}(0) \) of a typical \( f \in (\mathcal{E}_\ell(S^n), P) \) as \( \ell \to \infty \). Let \( C(f) \) denote the connected components of \( V(f) \). Then, for almost all \( f \) these components are smooth, compact, \( (n-1) \)-dimensional manifolds. The distribution of topologies of \( V(f) \) is given by

\[
\mu_f := \frac{1}{|C(f)|} \sum_{c \in C(f)} \delta_{t(c)},
\]

where \( t(c) \) is the diffeomorphism type of \( c \), and \( \delta_\tau \) is the point measure at \( \tau \). If we denote these types by \( \overline{H}(n-1) \) (it is a countable discrete set), then clearly \( \mu_f \) is a probability measure on \( \overline{H}(n-1) \). Let \( H(n-1) \) denote the subset of \( \overline{H}(n-1) \) consisting of all those types that can be realized as embedded submanifolds of \( \mathbb{R}^n \).

Nazarov and Sodin [NS] have shown that for a typical \( f \in \mathcal{E}_\ell(S^n) \) and \( \ell \to +\infty \),

\[
|C(f)| \sim c_n \ell^n,
\]

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for some $c_n > 0$. Since $C(f)$ consists of many components, it makes sense to examine the behavior of $\mu_f$ as $\ell \to +\infty$. In the recent work [SW] it is shown that there is a probability measure $\mu_{\text{mono}}$ on $H(n-1)$ (denoted by $\mu_{C,n,1}$ in [SW]) such that for any $\varepsilon > 0$

\[ \mathbb{P} \{ f \in \mathcal{E}_\ell(S^n) : D(\mu_f, \mu_{\text{mono}}) > \varepsilon \} \to 0, \]

as $\ell \to +\infty$. Here the discrepancy $D$ between two measures $\mu$ and $\nu$ is given by

\[ D(\mu, \nu) = \sup \{ |\mu(F) - \nu(F)| : F \subset \tilde{H}(n-1), F \text{ is finite} \}. \]

(3)

In fact, the same is proved for Gaussian ensembles of monochromatic waves on any given compact Riemannian manifold $(M, g)$ of dimension $n$ with no boundary ([SW]).

If $(M, g)$ is a compact Riemannian manifold, a monochromatic random wave of energy $T$ is defined as the Gaussian ensemble of functions on $M$ given by

\[ f(x) = \sum_{T-\eta(T) \leq t_j \leq T} c_j \varphi_j(x). \]

Here, the functions $\varphi_j$ form an orthonormal basis of $L^2(M, g, \mathbb{R})$ and are eigenfunctions of the Laplacian $\Delta_g \varphi_j + t_j^2 \varphi_j = 0$. The coefficients $c_j$ are independent Gaussian random variables of mean 0 and variance 1. Also, $\eta(T) = o(T)$ and $\eta(T) \to +\infty$ as $T \to +\infty$.

The measure $\mu_{\text{mono}}$ is the universal distribution for the topologies of the zero set of a typical monochromatic wave. Our aim in this note is to identify the support of this measure.

**Theorem 1.** The support of $\mu_{\text{mono}}$ is equal to $H(n-1)$. That is, for all $c \in H(n-1)$,

\[ \mu_{\text{mono}}(c) > 0. \]

As shown in [SW], the following criterion, extending the condition $(\rho \, 4)$ of Sodin [Sod, (1.2.2)], suffices to establish the above theorem: Given $c \in H(n-1)$ find a trigonometric polynomial $f$ on $\mathbb{R}^n$ of the form

\[ f(x) = \sum_{\xi \in S^{n-1}} a_\xi e^{i\langle x, \xi \rangle}, \]

such that $f^{-1}(0)$ contains $c$ as one of its components. The sum in (4) is over a finite set of $\xi$'s, and the coefficients should satisfy $a_\xi = \overline{a_{-\xi}}$.

We note that for $n = 2$, $H(1)$ is a point and so the statement of Theorem 1 is trivial. For $n = 3$, $H(2) = \{0, 1, 2, \ldots\}$ with each $c \in H(2)$ being identified with its genus $g(c)$, and in this case (4) is verified in [SW] by deforming a carefully constructed $f$. For $n \geq 4$ the sets $H(n-1)$ are not known explicitly and we proceed here by more abstract and general arguments.

First, we give a number of criteria which are equivalent to (4) and which are closely connected to the underlying translation invariant Gaussian field on $\mathbb{R}^n$. The functions (4) satisfy

\[ (\Delta + 1)f(x) = 0 \quad \text{on } \mathbb{R}^n, \]

and the various criteria reflect this (these $f$'s being monochromatic!). We then apply some differential topology and Whitney’s approximation Theorem to realize $c$ as an
embedded real analytic submanifold of \( \mathbb{R}^n \). Then, following some of the techniques in [EP]] we find suitable approximations of \( f \) which satisfy (3) and whose zero set contains a diffeomorphic copy of \( c \). The construction of \( f \) hinges on the Lax-Malgrange Theorem and Thom’s isotopy Theorem.

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2. Proof of Theorem

Our interest is in the monochromatic Gaussian field on \( \mathbb{R}^n \) which is a special case of the band limited Gaussian fields considered in [SW], and which is fundamental in the proof of (2) above. For \( 0 \leq \alpha \leq 1 \), define the annulus \( A_\alpha = \{ \xi \in \mathbb{R}^n : \alpha \leq |\xi| \leq 1 \} \) and let \( \sigma_\alpha \) be the Haar measure on \( A_\alpha \) normalized so that \( \sigma_\alpha(A_\alpha) = 1 \). Using that the transformation \( \xi \mapsto -\xi \) preserves \( A_\alpha \) we choose a real valued orthonormal basis \( \{ \phi_j \}_{j=1}^\infty \) of \( L^2(A_\alpha, \sigma_\alpha) \) satisfying

\[
\phi_j(-\xi) = (-1)^{\eta_j} \phi_j(\xi), \quad \eta_j \in \{0, 1\}.
\]

(6)

The band limited Gaussian field \( H_{n,\alpha} \) is defined to be the random real valued functions \( f \) on \( \mathbb{R}^n \) given by

\[
f(x) = \sum_{j=1}^\infty b_j \hat{\phi}_j(x)
\]

(7)

where

\[
\hat{\phi}_j(x) = \int_{\mathbb{R}^n} \phi_j(\xi)e^{-i(x,\xi)}d\sigma_\alpha(\xi)
\]

(8)

and the \( b_j \)'s are identically distributed, independent, real valued, standard Gaussian variables. We note that the field \( H_{n,\alpha} \) does not depend on the choice of the orthonormal basis \( \{ \phi_j \} \).

The distributional identity \( \sum_{j=1}^\infty \phi_j(\xi)\phi_j(\eta) = \delta(\xi - \eta) \) on \( A_\alpha \) together with (6) lead to the explicit expression for the covariance function:

\[
\text{Cov}(x, y) = \mathbb{E}_{H_{n,\alpha}}(f(x)f(y)) = \int_{\mathbb{R}^n} e^{i(x-y,\xi)}d\sigma_\alpha(\xi).
\]

(9)

From (9), or directly from (7), it follows that almost all \( f \)'s in \( H_{n,\alpha} \) are analytic in \( x \) [AT]. For the monochromatic case \( \alpha = 1 \) we have

\[
\text{Cov}(x, y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{J_\nu(|x - y|)}{|x - y|^{\nu}},
\]

(10)

where to ease notation we have set

\[
\nu := \frac{n - 2}{2}.
\]

In this case there is also a natural choice of a basis for \( L^2(S^{n-1}, d\sigma) = L^2(A_1, \mu_1) \) given by spherical harmonics. Let \( \{ Y^\ell_m \}_{m=1}^{d_\ell} \) be a real valued basis for the space of spherical harmonics \( \mathcal{E}_\ell(S^{n-1}) \) of eigenvalue \( \ell(\ell + n - 2) \), where \( d_\ell = \dim \mathcal{E}_\ell(S^{n-1}) \). We compute the Fourier transforms for the elements of this basis.
\textbf{Proposition 2.} For every $\ell \geq 0$ and $m = 1, \ldots, d_\ell$, we have

$$\widehat{Y}_m^\ell(x) = (2\pi)^{-\frac{3}{2}} i^\ell Y_m^\ell \left( \frac{x}{|x|} \right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu}. \quad (11)$$

\textbf{Proof.} We give a proof using the theory of point pair invariants [Sel] which places such calculations in a general and conceptual setting. The sphere $S^{n-1}$ with its round metric is a rank 1 symmetric space and $\langle \dot{x}, \dot{y} \rangle$ for $\dot{x}, \dot{y} \in S^{n-1}$ is a point pair invariant (here $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$ restricted to $S^{n-1}$). Hence, by the theory of these pairs we know that for every function $h : \mathbb{R} \to \mathbb{C}$ we have

$$\int_{S^{n-1}} h(\langle \dot{x}, \dot{y} \rangle) Y(\dot{y}) \, d\sigma(\dot{y}) = \lambda_h(\ell) Y(\dot{x}), \quad (12)$$

where $Y$ is any spherical harmonic of degree $\ell$ and $\lambda_h(\ell)$ is the spherical transform. The latter can be computed explicitly using the zonal spherical function of degree $\ell$. Fix any $\dot{x} \in S^{n-1}$ and let $Z^\ell_{\dot{x}}$ be the unique spherical harmonic of degree $\ell$ which is rotationally invariant by motions of $S^{n-1}$ fixing $\dot{x}$ and so that $Z^\ell_{\dot{x}}(\dot{x}) = 1$. Then,

$$\lambda_h(\ell) = \int_{S^{n-1}} h(\langle \dot{x}, \dot{y} \rangle) Z^\ell_{\dot{x}}(\dot{y}) \, d\sigma(\dot{y}). \quad (13)$$

The function $Z^\ell_{\dot{x}}(\dot{y})$ may be expressed in terms of the Gegenbauer polynomials [GR, (8.930)] as

$$Z^\ell_{\dot{x}}(\dot{y}) = \frac{C^\nu_{\ell}(\langle \dot{x}, \dot{y} \rangle)}{C^\nu_1(1)}. \quad (14)$$

Now, for $x \in \mathbb{R}^n$,

$$\widehat{Y}_m^\ell(x) = \int_{S^{n-1}} h_x(\langle \frac{x}{|x|}, \dot{y} \rangle) Y_m^\ell(\dot{y}) \, d\sigma(\dot{y}),$$

where we have set $h_x(t) = e^{-i|x|t}$. Hence, by [12] we have

$$\widehat{Y}_m^\ell(x) = \lambda_{h_x}(\ell) Y_m^\ell(\frac{x}{|x|}),$$

with

$$\lambda_{h_x}(\ell) = \int_{S^{n-1}} e^{-i|x|\langle \frac{x}{|x|}, \dot{y} \rangle} Z^\ell_{\dot{x}}(\dot{y}) \, d\sigma(\dot{y}) = \frac{\text{vol}(S^{n-2})}{C^\nu_1(1)} \int_{-1}^1 e^{-it|x|} C^\nu_{\ell}(t) (1-t^2)^{\nu-\frac{1}{2}} \, dt. \quad (15)$$

The last term in (15) can be computed using [GR, (7.321)]. This gives

$$\lambda_{h_x}(\ell) = (2\pi)^{\frac{3}{2}} i^\ell \frac{J_{\ell+\nu}(|x|)}{|x|^\nu},$$

as desired. \qed

\textbf{Corollary 3.} The monochromatic Gaussian ensemble $H_{n,1}$ is given by random $f$’s of the form

$$f(x) = (2\pi)^{\frac{3}{2}} \sum_{\ell=0}^\infty \sum_{m=1}^{d_\ell} b_{\ell,m} Y_m^\ell \left( \frac{x}{|x|} \right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu},$$

where the $b_{\ell,m}$’s are i.i.d standard Gaussian variables.
The functions \( x \mapsto Y_m^\ell \left( \frac{x}{|x|} \right) \frac{J_{\ell+\nu}(|x|)}{|x|^{\nu}} \), \( x \mapsto e^{i(x,\xi)} \) with \(|\xi| = 1\), and those in (7) for which the series converges rapidly (e.g. for almost all \( f \) in \( H_{n,1} \)), all satisfy (5). We therefore introduce the space

\[
E_1 := \{ f : \mathbb{R}^n \to \mathbb{R} : f \in \text{Ker}(\Delta + 1) \}.
\]

In addition, consider the subspaces \( P_1 \) and \( T_1 \) of \( E_1 \) defined by

\[
P_1 := \text{span} \left\{ x \mapsto Y_m^\ell \left( \frac{x}{|x|} \right) \frac{J_{\ell+\nu}(|x|)}{|x|^{\nu}} : \ell \geq 0, \ m = 1, \ldots, d_\ell \right\},
\]

\[
T_1 := \text{span} \left\{ x \mapsto \frac{e^{i(x,\xi)} + e^{-i(x,\xi)}}{2}, \ x \mapsto \frac{e^{i(x,\xi)} - e^{-i(x,\xi)}}{2i} : |\xi| = 1 \right\}.
\]

**Proposition 4.** Let \( f \in E_1 \) and let \( K \subset \mathbb{R}^n \) be a compact set. Then, for any \( t \geq 0 \) and \( \varepsilon > 0 \) there are \( g \in P_1 \) and \( h \in T_1 \) such that

\[
\| f - g \|_{C^t(K)} < \varepsilon \quad \text{and} \quad \| f - h \|_{C^t(K)} < \varepsilon.
\]

That is, we can approximate \( f \) on compact subsets in the \( C^t \)-topology by elements of \( P_1 \) and \( T_1 \) respectively.

**Proof.** Let \( f \in E_1 \). Since \( f \) is analytic we can expand it in a rapidly convergent series in the \( Y_m^\ell \)'s. That is,

\[
f(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} a_{m,\ell}(|x|) Y_m^\ell \left( \frac{x}{|x|} \right).
\]

Moreover, for \( r > 0 \),

\[
\int_{S^{n-1}} |f(r\hat{x})|^2 d\sigma(\hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} |a_{m,\ell}(r)|^2.
\]

In polar coordinates, \((r, \theta) \in (0, +\infty) \times S^{n-1}\), the Laplace operator in \( \mathbb{R}^n \) is given by

\[
\Delta = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{n-1}},
\]

and hence for each \( \ell, m \) we have that

\[
r^2 a_{m,\ell}''(r) + (n-1) r a_{m,\ell}'(r) + (r^2 - \ell(\ell + n - 2)) a_{m,\ell}(r) = 0.
\]

where \( \ell \) is some positive integer. There are two linearly independent solutions to (17). One is \( r^{-\nu} J_{\ell+\nu}(r) \) and the other blows up as \( r \to 0 \). Since the left hand side of (16) is finite as \( r \to 0 \), it follows that the \( a_{m,\ell} \)'s cannot pick up any component of the blowing up solution. That is, for \( r \geq 0 \)

\[
a_{m,\ell}(r) = c_{\ell,m} \frac{J_{\ell+\nu}(r)}{r^{\nu}},
\]

for some \( c_{m,\ell} \in \mathbb{R} \). Hence,

\[
f(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} c_{\ell,m} Y_m^\ell \left( \frac{x}{|x|} \right) \frac{J_{\ell+\nu}(|x|)}{|x|^{\nu}}.
\]
Furthermore, this series converges absolutely and uniformly on compact subsets, as also do its derivatives. Thus, \( f \) can be approximated by members of \( P_1 \) as claimed, by simply truncating the series in (18).

To deduce the same for \( T_1 \) it suffices to approximate each fixed \( Y^f_m \left( \frac{x}{|x|} \right) \frac{f_\ell + \nu(|x|)}{|x|^n} \). To this end let \( \xi_1, -\xi_1, \xi_2, -\xi_2, \ldots, \xi_N, -\xi_N \) be a sequence of points in \( S^{n-1} \) which become equidistributed with respect to \( d\sigma \) as \( N \to \infty \). Then, as \( N \to \infty \),

\[
\frac{1}{2N} \sum_{j=1}^{N} \left( e^{-i(x,\xi_j)_{\cdot}} Y^f_m(\xi_j) + (-1)^{\ell} e^{i(x,\xi_j)_{\cdot}} Y^f_m(\xi_j) \right) \to \int_{S^{n-1}} e^{-i(x,\xi)_{\cdot}} Y^f_m(\xi) \, d\sigma(\xi). \tag{19}
\]

The proof follows since \((2\pi)^{\frac{n}{2}} t^\ell \frac{J_{\ell + \nu(|x|)}}{|x|^n} = \int_{S^{n-1}} e^{-i(x,\xi)_{\cdot}} Y^f_m(\xi) \, d\sigma(\xi)\). Indeed, the convergence in (19) is uniform over compact subsets in \( x \).

**Remark 1.** For \( \Omega \subset \mathbb{R}^n \) open, let \( E_1(\Omega) \) denote the eigenfunctions on \( \Omega \) satisfying \( \Delta f(x) + f(x) = 0 \) for \( x \in \Omega \). Any function \( g \) on \( \Omega \) which is a limit (uniform over compact subsets of \( \Omega \)) of members of \( E_1 \) must be in \( E_1(\Omega) \). While the converse is not true in general, note that if \( \Omega = B \) is a ball in \( \mathbb{R}^n \), then the proof of Proposition 4 shows that the uniform limits of members of \( E_1 \) (or \( P_1 \), or \( T_1 \)) on compact subsets in \( B \) is precisely \( E_1(B) \).

With these equivalent means of approximating functions by suitable members of \( H_{n,1} \) we are ready to prove Theorem 1. To verify the criterion following Theorem 1 we can use any function in \( E_1 \).

### 2.1. Proof of Theorem 1

By the discussion above it follows that given a representative \( c \) of a class \( t(c) \in H(n - 1) \), it suffices to find \( f \in E_1 \) for which \( C(f) \) contains a diffeomorphic copy of \( c \).

To begin the proof we claim that we may assume that \( c \) is real analytic. Indeed, if we start with \( \tilde{c} \) smooth, of the desired topological type, we may construct a tubular neighbourhood \( V_{\tilde{c}} \) of \( \tilde{c} \) and a smooth function

\[
H_{\tilde{c}} : V_{\tilde{c}} \to \mathbb{R} \quad \text{with} \quad \tilde{c} = H_{\tilde{c}}^{-1}(0).
\]

Note that without loss of generality we may assume that \( \inf_{x \in V_{\tilde{c}}} \|\nabla H_{\tilde{c}}(x)\| > 0 \). Fix any \( \epsilon > 0 \). We apply Thom’s isotopy Theorem [AR Thm 20.2] to obtain the existence of a constant \( \delta_\epsilon > 0 \) so that for any function \( F \) with \( \|F - H_{\tilde{c}}\|_{C^1(V_{\tilde{c}})} < \delta_\epsilon \) there exists \( \Psi_F : \mathbb{R}^n \to \mathbb{R}^n \) diffeomorphism with \( \|\Psi_F - Id\|_{C^0(\mathbb{R}^n)} < \epsilon \), \( \text{supp}(\Psi_F - Id) \subset V_{\tilde{c}} \), and

\[
\Psi_F(\tilde{c}) = F^{-1}(0) \cap V_{\tilde{c}}.
\]

To construct a suitable \( F \) we use Whitney’s approximation Theorem [Wh Lemma 6] which yields the existence of a real analytic approximation \( F : V_{\tilde{c}} \to \mathbb{R}^{mc} \) of \( H_{\tilde{c}} \) that satisfies \( \|F - H_{\tilde{c}}\|_{C^1(V_{\tilde{c}})} < \delta_\epsilon \). It follows that \( \tilde{c} \) is diffeomorphic to \( c := \Psi_F(\tilde{c}) \) and \( c \) is real analytic as desired.

By the Jordan-Brouwer Separation Theorem [L3], the hypersurface \( c \) separates \( \mathbb{R}^n \) into two connected components. We write \( A_c \) for the corresponding bounded component of \( \mathbb{R}^n \setminus c \). Let \( \lambda^2 \) be the first Dirichlet eigenvalue for the domain \( A_c \) and let \( h_\lambda \) be
the corresponding eigenfunction:
\[
\begin{cases}
(\Delta + \lambda^2)h_\lambda(x) = 0 & x \in \overline{A_c}, \\
h_\lambda(x) = 0 & x \in c.
\end{cases}
\]

Consider the rescaled function
\[
h(x) := h_\lambda(x/\lambda),
\]
defined on the rescaled domain \(\lambda A_c := \{x \in \mathbb{R}^n : x/\lambda \in A_c\}\). Since \((\Delta + 1)h = 0\) in \(\lambda A_c\), and \(\partial(\lambda A_c)\) is real analytic, \(h\) may be extended to some open set \(B_c \subset \mathbb{R}^n\) with \(\lambda A_c \subset B_c\) so that
\[
\begin{cases}
(\Delta + 1)h(x) = 0 & x \in B_c, \\
h(x) = 0 & x \in \lambda c,
\end{cases}
\]
where \(\lambda c\) is the rescaled hypersurface \(\lambda c := \{x \in \mathbb{R}^n : x/\lambda \in c\}\). Note that since \(h_\lambda\) is the first Dirichlet eigenfunction, then we know that there exists a tubular neighbourhood \(V_{\lambda c}\) of \(\lambda c\) on which \(\inf_{x \in V_{\lambda c}} \|\nabla h(x)\| > 0\) (see Lemma 3.1 in [BHM]). Without loss of generality assume that \(V_{\lambda c} \subset B_c\).

Given any \(\varepsilon > 0\) we apply Thom’s isotopy Theorem [AR, Thm 20.2] to obtain the existence of a constant \(\delta > 0\) so that for any function \(f\) with \(\|f - h\|_{C^1(V_{\lambda c})} < \delta\) there exists \(\Psi_f : \mathbb{R}^n \to \mathbb{R}^n\) diffeomorphism so that \(\|\Psi_f - Id\|_{C^0(\mathbb{R}^n)} < \varepsilon\), \(\text{supp}(\Psi_f - Id) \subset V_{\lambda c}\), and
\[
\Psi_f(\lambda c) = f^{-1}(0) \cap V_{\lambda c}.
\]
Since \(\mathbb{R}^n \setminus B_c\) has no compact components, Lax-Malgrange’s Theorem [Kr, p. 549] yields the existence of a global solution \(f : \mathbb{R}^n \to \mathbb{R}\) to the elliptic equation \((\Delta + 1)f = 0\) in \(\mathbb{R}^n\) with \(\|f - h\|_{C^1(B_c)} < \delta\).

We have then constructed a solution to \((\Delta + 1)f = 0\) in \(\mathbb{R}^n\), i.e. \(f \in E_1\), for which \(f^{-1}(0)\) contains a diffeomorphic copy of \(c\) (namely, \(\Psi_f(\lambda c)\)). This concludes the proof of the theorem.

We note that the problem of finding a solution to \((\Delta + 1)f = 0\) for which \(C(f)\) contains a diffeomorphic copy of \(c\) is related to the work [EP] of A. Enciso and D. Peralta-Salas. In [EP] the authors seek to find solutions to the problem \((\Delta - q)f = 0\) in \(\mathbb{R}^n\) so that \(C(f)\) contains a diffeomorphic copy of \(c\), where \(q\) is a nonnegative, real analytic, potential and \(c\) is a (possibly infinite) collection of compact or unbounded “tentacled” hypersurfaces. The construction of the solution \(f\) that we presented is based on the ideas used in [EP]. Since our setting and goals are simpler than theirs, the construction of \(f\) is much shorter and straightforward.
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