A pathwise regularization by noise phenomenon for the evolutionary $p$-Laplace equation

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Abstract. We study an evolutionary $p$-Laplace problem whose potential is subject to a translation in time. Provided the trajectory along which the potential is translated admits a sufficiently regular local time, we establish existence of solutions to the problem for singular potentials for which a priori bounds in classical approaches break down, thereby establishing a pathwise regularization by noise phenomena for this nonlinear problem.

1. Introduction

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $T > 0$ and $N \in \mathbb{N}$. We are interested in the evolutionary $p$-Laplace system

$$
\begin{align*}
\partial_t u - \operatorname{div} S(\nabla u) &= b(u) \quad \text{on } \Omega \times [0, T], \\
u|_{\Omega} &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
u(0, \cdot) &= u_0 \quad \text{on } \Omega,
\end{align*}
$$

(1.1)

where $S(\xi) = |\xi|^{p-2} \xi \in \mathbb{R}^{d \times N}$, $p \in (1, \infty)$ and $b : \mathbb{R}^N \to \mathbb{R}^N$.

The $p$-Laplace operator $\operatorname{div} S(\nabla u)$ is a prominent example of a maximal monotone operator. The famous theory of monotone operators traces back to the early works of Minty [32] and Browder [5]. It inspired many mathematicians to study well-posedness of monotone evolution equations and perturbations of it, see, e.g., [1,8,28,29,31,34,35]. However, if the potential $b(u)$ cannot be treated as a compact perturbation, well-posedness breaks down and solutions may blow up in finite time, see, e.g., [12,38].

In the following, we intend to study the effect of translations of the potential $b$ along so-called regularizing paths $w$, that is, we are interested in the problem

$$
\begin{align*}
\partial_t u - \operatorname{div} S(\nabla u) &= b(u - w) \quad \text{on } \Omega \times [0, T], \\
u|_{\Omega} &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
u(0, \cdot) &= u_0 \quad \text{on } \Omega,
\end{align*}
$$

(1.2)

From a physical point of view, translating the potential in time can be interpreted as uncertainty as to the location of the origin of the potential. By a “regularizing path”,
we will understand a continuous path $w$ that admits a sufficiently regular local time such that techniques from pathwise regularization by noise à la [7,16,27] become applicable. Let us briefly sketch some main ideas of these approaches, in particular, in the spirit of [27], which we will be able to employ also in the study of (1.2).

1.1. Pathwise regularization by noise in a nutshell

The starting point for investigations of this type usually consists in the study of averaging operators along a path $w$, given by

$$(T_{t}^{-w}b)(u) := \int_{0}^{t} b(u - w_{s})ds.$$ 

As was recognized already in [7], paths enjoying so-called $\rho$ irregularity [7, Definition 1.3], i.e., paths whose Fourier transform of the occupation measure decreases sufficiently rapidly, lead to a regularization effect in that the function $u \mapsto (T_{t}^{-w}b)(u)$ will enjoy higher regularity than $b$. The reason for this gain of regularity can be made clear as follows: Assuming $w$ to also admit a local time $L$, we can rewrite the averaging operator thanks to the occupation times formula as

$$\int_{0}^{t} b(u - w_{s})ds = \int_{\mathbb{R}^{N}} b(u - z)L_{t}(z)dz = (b \ast L_{t})(u).$$

If the Fourier transform of the occupation measure is decreasing rapidly in a certain quantifiable sense, the local time $L$ will enjoy some high quantifiable spatial regularity. By Young’s inequality (see, for example, (2.1) or [30] for a generalization to the Besov space setting), this implies that the regularity of $T_{t}^{-w}b$ will essentially increase by the regularity of $L_{t}$ with respect to the regularity of $b$. While the canonical paths for which this gain in regularity can be quantified are realizations of fractional Brownian motion [7,14,16], $\rho$ irregularity is in fact a typical property among Hölder continuous paths in the sense of prevalence [15].

Taking this observation of increased regularity of the averaging operator as a starting point, one can exploit this local gain of regularity to further study Riemann sum-type expressions of the form

$$I^{n}_{T} = \sum_{[s,t] \in \mathcal{P}^{n}([0,T])} (T_{s,t}^{-w}b)(u_{s})$$

for partitions $\mathcal{P}^{n}([0,T])$ of $[0,T]$ and a continuous function $u$. The tool that then ensures the convergence of such Riemann sums in the limit $|\mathcal{P}^{n}| \to 0$ is Gubinelli’s Sewing Lemma [11,24], also cited in the Appendix as Lemma 6.3. Let us stress at this point already that since the Sewing Lemma is formulated in a “Hölder-space setting”, convergence of $(I^{n})_{n}$ will, however, always require at least some Hölder regularity of the function $u$. Provided this is available, i.e., $u$ is sufficiently Hölder

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1We refer to the corresponding section in the Appendix for the definition of the local time of a path and related concepts.
regular and the averaging operator \((T^{-w}b)\) sufficiently regular in time and space, the Sewing Lemma ensures the convergence of \((T^n)\), as \(|P^n| \to 0\). Note that as this convergence eventually only requires information on the regularity of the averaging operator and not the regularity of \(b\), this construction (which can be shown to coincide with the classical Lebesgue integral for regular \(b\)) naturally extends the definition of the Lebesgue integral to irregular and potentially even distributional \(b\), i.e., one can define

\[
\mathcal{I}_T = \lim_{|P^n| \to 0} \sum_{[s,t] \in P^n([0,T])} (T^{-w}_{s,t} b)(u_s) =: \int_0^T b(u_s - w_s) ds
\]

Moreover, even in the case of a regular nonlinearity \(b\) where the above can be defined as a classical Lebesgue integral, this definition provides alternative a priori bounds through the Sewing Lemma of which we will crucially make use in our work.

1.2. Application to the p-Laplace equation with shifted potential

A first approach in studying (1.2) might consist in investigating the weak formulation

\[
\langle u_t - u_s, \varphi \rangle + \int_s^t \langle S(\nabla u_r), \nabla \varphi \rangle \, dr = \int_s^t \langle b(u_r - w_r), \varphi \rangle \, dr,
\]

where the right-hand side should be interpreted as

\[
\int_s^t \langle b(u_r - w_r), \varphi \rangle \, dr := \lim_{|P^n| \to 0} \sum_{[s',t'] \in P^n([s,t])} \langle (T^{-w}_{s',t'} b)(u_{s'}), \varphi \rangle.
\]

Note, however, that the classical monotone operator approach to this problem, working on the Gelfand triple \((W^{1,p}_0(\Omega) \cap L^2(\Omega), L^2(\Omega), (W^{1,p}_0(\Omega) \cap L^2(\Omega))')\), only yields a priori bounds that permit to conclude \(u \in C([0,T], L^2(\Omega))\). In particular, no additional Hölder regularity in time on this spatial regularity scale is obtained, meaning the Sewing argument can’t be closed making the right-hand side ill defined. To circumvent this problem, we employ a strong formulation to the problem, i.e., we strive for solutions \(u\) that satisfy

\[
\int_s^t b(u_r - w_r) \, dr := \lim_{|P^n| \to 0} \sum_{[s',t'] \in P^n([s,t])} (T^{-w}_{s',t'} b)(u_{s'}),
\]

understood as an equality in \(L^2(\Omega)\) and where for singular \(b\) the right-hand side is understood in the sense

\[
\int_s^t b(u_r - w_r) := \lim_{|P^n| \to 0} \sum_{[s',t'] \in P^n([s,t])} (T^{-w}_{s',t'} b)(u_{s'}),
\]

where the convergence on the right-hand side holds in \(L^2(\Omega)\), uniformly in time on \([0,T]\). Note that while such strong solutions naturally require higher regularity of
the initial condition, namely \( u_0 \in W_0^{1,p}(\Omega) \), they allow for further a priori bounds in \( C^{0,1/2}([0, T]; L^2(\Omega)) \) provided \( b \) is sufficiently regular (refer to Sect. 3). Having for regular \( b \) such a priori bounds at our disposal, we can then harness the regularizing effect of the averaging operator \( T^{-w} \) as discussed above to obtain a priori bounds that are robust even when considering singular potentials \( b \) (refer to Sect. 4). The so obtained new a priori bounds for singular \( b \) can then be used to obtain solutions to (1.3) using classical monotonicity arguments (refer to Sect. 5). In summary, this allows us to prove our main theorem:

**Theorem 1.1.** (Existence of robustified solution) Let \( d, N \in \mathbb{N}, \Omega \) a bounded Lipschitz domain in \( \mathbb{R}^d \), \( p > \frac{2d}{d+2} \) and \( u_0 \in W_0^{1,p}(\Omega) \). For \( r \in [1, \infty) \) and \( q \in [r, \infty) \) let \( b : \mathbb{R}^N \to \mathbb{R}^N \) satisfy \( b \in L^{2q}(\mathbb{R}^N) \). Suppose that \( w : [0, T] \to \mathbb{R}^N \) is continuous and admits a local time \( L \) which satisfies \( L \in C^{0,\gamma}([0, T]; W^{1,\gamma'}(\mathbb{R}^N)) \) for some \( \gamma \in (1/2, 1) \).

Then there exists a robustified solution

\[
\begin{align*}
  u \in \left\{ v \in L^\infty(0, T; L^2(\Omega) \cap W_0^{1,p}(\Omega)) \mid \div v, \text{ div} S(\nabla v) \in L^2([0, T] \times \Omega) \right\}
\end{align*}
\]

to (1.2) in the sense of Definition 2.2. Moreover, the following a priori bound is valid

\[
\begin{align*}
  \|u\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|\nabla u\|_{L^\infty(0, T; L^p(\Omega))}^2 + \|\partial_t u\|_{L^2([0, T] \times \Omega)}^2 + \|\div S(\nabla u)\|_{L^2([0, T] \times \Omega)}^2 \\
  \lesssim \|u_0\|_{L^2(\Omega)}^2 + \|\nabla u_0\|_{L^p(\Omega)}^2 + \|b\|_{L^{2q}(\mathbb{R}^N)}^4 \|L\|_{C^{0,\gamma}([0, T]; W^{1,\gamma'}(\mathbb{R}^N))}^2.
\end{align*}
\]

(1.5)

Suppose moreover that \( b \) satisfies the monotonicity condition, for all \( u, v \in \mathbb{R}^N \),

\[
\begin{align*}
  (b(u) - b(v)) \cdot (u - v) \leq 0,
\end{align*}
\]

(1.6)

then robustified solutions to (1.2) in the sense of Definition 2.2 are unique.

Let us illustrate this established regularization effect by means of a concrete example. A detailed verification of the claims of the example can be found in “Appendix 6”.

**Example 1.2.** Let \( K > 0 \) and define the potential

\[
\begin{align*}
  b(u) := -|u|^{p-1} u 1_{\{|u| \leq K\}}.
\end{align*}
\]

(1.7)

Let \( d \in \mathbb{N} \) and \( p > \frac{2d}{d+2} \). Then the following statements are true:

1. If \( \eta \leq -1 \), then (1.1), i.e., the problem without regularizing path does not have a weak solution for \( u_0 = 0 \).
2. Let \( \eta \in (-N/2, 0) \) and

\[
\begin{align*}
  H < \frac{1}{(2 - 4\eta) \vee N}
\end{align*}
\]

(1.8)

and \( w^H \) be a \( N \)-dimensional fractional Brownian motion with Hurst parameter \( H \). Then for almost any realization \( w^H(\omega) : [0, T] \to \mathbb{R}^N \) (1.2) has a robustified solution for any \( u_0 \in W_0^{1,p}(\Omega) \).
Essentially, (1) shows that in the unperturbed setting the origin is a singular state, which is due to the singularity of \( b \) in zero. In contrast to this, the presence of the highly oscillating path \( w^H(\omega) \) ensures that the solution \( u \) does not spend too much time in the singularity of \( b \). Condition (1.8) ensures that this effect is quantitatively sufficiently strong for the singularity not to obstruct existence theory, as formulated in our main theorem above. Overall, we can thus observe a regularization phenomenon in dimension \( N \geq 3 \). In particular, for \( N = 3 \) and \( \eta = -1 \), there is no weak solution to (1.1) for \( u_0 = 0 \), while (1.2) with \( u_0 = 0 \) admits a robustified solution for almost every realization \( w^H(\omega) \) of a fractional Brownian motion, provided \( H < 1/6 \).

**Remark 1.3.** Let us stress that we are employing a pathwise regularization by noise argument in the spirit of [27] relying on the study of local times and their regularities and not in the spirit of [7,16] based on the study of averaging operators. While results on the regularizing effect of averaging operators are slightly less optimal for realizations of fractional Brownian motion for example, they allow for a completely pathwise, i.e., analytical treatment. In contrast, arguing in the spirit of [7,16] would require to employ a tightness argument. Moreover, if the perturbing path is the realization of a stochastic process \( w \), note that as our proof employs a compactness argument, we lose measurability of our solution with respect to the probability space on which \( w \) is defined.

### 1.3. Remarks on the literature

Starting with the seminal work [7], taken up again by [16,17,27,36] pathwise regularization by noise has seen considerable developments in recent years. Areas in which such techniques have been successfully implemented include particle systems [26], distribution dependent SDEs [21,22], multiplicative SDEs [3,6,10,20,33] and perturbed SDEs [23]. An extension of pathwise regularization techniques to the two parameter setting with applications to regularization by noise for the stochastic wave equation was established in [4]. Pathwise regularization by noise for the multiplicative stochastic heat equations with spatial white noise was treated in [2,9].

### 1.4. Outline of the paper

In Sect. 2 we introduce basic notation and the notion of solution. Section 3 addresses quantified but implicit a priori bounds for strong solutions. In Sect. 4 we close the a priori bounds from Sect. 3. Lastly, Sect. 5 copes with the identification of the limit and a discussion on uniqueness. In Appendix 6 we verify Example 1.2, present results related to occupation measures, local times and sewing techniques and give details on the identification of limits for monotone equations.
2. Mathematical setup

Let $\Omega \subset \mathbb{R}^d$ for $d \geq 1$ be a bounded Lipschitz domain. For some given $T > 0$ we denote by $I := [0, T]$ the time interval and write $\Omega T := I \times \Omega$ for the time space cylinder. We write $f \lesssim g$ for two nonnegative quantities $f$ and $g$ if $f$ is bounded by $g$ up to a multiplicative constant. Accordingly, we define $\gtrsim$ and $\approx$. Moreover, we denote by $c$ a generic constant which can change its value from line to line. For $r \in [1, \infty]$, we denote by $r' = r/(r - 1)$ its Hölder conjugate. We do not distinguish between scalar, vector and matrix-valued functions.

2.1. Function spaces

As usual, for $q \in [1, \infty]$, let $L^q(\Omega)$ denote the Lebesgue space and $W^{1,q}(\Omega)$ the Sobolev space on the domain $\Omega$, respectively. Furthermore, we denote by $W_0^{1,q}(\Omega)$ the Sobolev space with zero boundary values. For $q < \infty$ it is the closure of $C^\infty_c(\Omega)$ (smooth functions with compact support) in the $W^{1,q}(\Omega)$-norm. Additionally, we denote by $W^{-1,q}(\Omega)$ the dual of $W_0^{1,q}(\Omega)$. We abbreviate function spaces on the domain by $L^p_x := L^p(\Omega)$ and on the full space by $L^p := L^p(\mathbb{R}^N)$ with suitable modifications for Sobolev norms. The inner product in $L^2_x$ is denoted by $(\cdot, \cdot)$, and duality pairings are written as $\langle \cdot, \cdot \rangle$.

For a Banach space $(X, \| \cdot \|_X)$ let $L^q(I; X)$ be the Bochner space of Bochner-measurable functions $u : I \to X$ satisfying $t \mapsto \|u(t)\|_X \in L^q(I)$. Moreover, $C(I; X)$ is the space of continuous functions with respect to the norm-topology. We also use $C^0,\alpha(I; X)$, $\alpha \in (0, 1]$, for the space of $\alpha$-Hölder continuous functions. For $u \in C^0,\alpha(I; X)$, we denote by $[u]_{C^0,\alpha(I; X)} := \sup_{s \neq t \in I} \frac{\|u_s - u_t\|_X}{|s - t|^{\alpha}}$ the corresponding semi-norm and by $\|u\| = \sup_{t \in I} \|u_t\|_X + [u]_{C^0,\alpha(I; X)}$ the corresponding norm. We abbreviate the notation $L^q_x(I; X)$ to $L^q(I; X)$ and $C_x(I; X)$ to $C(I; X)$. If a Banach space $(X, \| \cdot \|_X)$ embeds continuously into another Banach space $(Y, \| \cdot \|_Y)$, we write $X \hookrightarrow Y$. If the embedding is moreover compact, we write $X \hookrightarrow\hookrightarrow Y$. If a sequence $(u_n)_n \subset X$ converges to $u \in X$ weakly, respectively weakly star, in a Banach space $(X, \| \cdot \|_X)$, we write $u_n \rightharpoonup u$, respectively, $u_n \overset{*}{\rightharpoonup} u$.

For $s \in \mathbb{R}^+$ we further note by $H^s$ the space of Bessel potentials,

$$H^s := \{ f \in S' \mid \| f \|_{H^s} := \left\| \mathcal{F}^{-1}(1 + |\xi|^2)^s/2 \mathcal{F} f \right\|_{L^2_x} < \infty \}$$

Let us also recall a particular instance of Young’s convolution inequality adapted to our setting, i.e.,

$$\| f * g \|_{C^{0,1}} \lesssim \| f \|_{L^r} \| g \|_{W^{1,r'}}, \quad (2.1)$$

which is a consequence of $D_x (f * g) = f * (D_x g)$ and Young’s convolution inequality in the classical setting.
2.2. Solution concepts

In the following, let us discuss different notions of solutions to (1.2). We begin with the classical notions of weak and strong solutions before passing on to so-called robustified solutions that exploit the gain in regularity due to the regularizing path \( w \) as discussed in the introduction.

**Definition 2.1.** (*Classical*) A function \( u \) is called weak solution to (1.2) if

1. (Regularity) \( u \in C^t L^2_x \cap L^p_t W^{1,p}_{0,x} \), \( b(u - w) \in L^{p'}_t W^{-1,p'}_{0,x} \) and
2. (Tested equation) for all \( t \in I \) and \( \xi \in C^\infty_{c,x} \)

\[
\int_\Omega (u_t - u_0) \cdot \xi \, dx + \int_0^t \int_\Omega S(\nabla u) : \nabla \xi \, dx \, ds = \int_0^t \langle b(u - w), \xi \rangle_{W^{-1,p'} \times W^{1,p}_{0,x}} \, ds. \tag{2.2}
\]

A weak solution \( u \) is called strong solution to (1.2) if additionally

1. (Regularity) \( \partial_t u, \text{div} S(\nabla u), b(u - w) \in L^2_t L^2_x \) and
2. (Point-wise equation) for almost all \((t, x) \in \Omega_T\)

\[
\partial_t u - \text{div} S(\nabla u) = b(u - w). \tag{2.3}
\]

**Definition 2.2.** (*Robustified*) We call \( u \) as a solution to (1.2) in the robustified sense if

\[
u \in \left\{ v \in C^{0,1/2}_t L^2_x \cap L^\infty_t W^{1,p}_{0,x} \mid \partial_t v, \text{div} S(\nabla v) \in L^2_t L^2_x \right\}, \tag{2.4}
\]

and for any \( t \in I \) we have

\[
u_t - \nu_0 - \int_0^t \text{div} S(\nabla u_r) \, dr = (\mathcal{I} A^u)_{0,t} \tag{2.5}
\]

understood as an equality in \( L^2_x \) where \( \mathcal{I} A^u \) denotes the sewing\(^2\) of the germ

\[A^u_{s,t} = (b \ast L_{s,t})(u_s).\]

In this case, we have, in particular, also \( \partial_t \mathcal{I} A^u \in L^2_t L^2_x \).

The next two lemmata verify that the concept of robustified solutions coincides with classically defined strong solutions in the smooth setting.

**Lemma 2.3.** Let \( b \) be smooth and bounded and assume that \( w : [0, T] \to \mathbb{R}^N \) is measurable and admits a local time \( L \in C^{0,1/2+\epsilon}_t L^2_x \) for some \( \epsilon > 0 \). Then any strong solution \( u \in C^{0,1/2}_t L^2_x \) to (1.2) is a solution in the robustified sense.

\(^2\)Refer to Lemma 6.3 in the Appendix.
Proof. Since $u$ is a strong solution, it suffices to show that for any $s < t \in [0, T]$ we have

$$\langle \mathcal{I} A^u \rangle_{s,t} = \int_s^t b(u_r - w_r) dr$$

Remark first that $A^u$ does admit a sewing with values in $L_x^2$. Indeed,

$$\| (\delta A^u)_{s,t} \|_{L_x^2} \leq \| (b * L_{v,t})(u_s) - (b * L_{v,t})(u_v) \|_{L_x^2}$$

$$\leq \| b * L_{v,t} \|_{C^{0,1}} \| u_s - u_v \|_{L_x^2}$$

$$\leq \| b \|_{H^1} \| L \|_{C_t^{0,1/2+\epsilon} L_x^2} |t - s|^{1/2 + \epsilon} \quad \text{for some } \epsilon > 0,$n

meaning we may apply the Sewing Lemma 6.3. Moreover, we have

$$\left\| A^u_{s,t} - \int_s^t b(u_r - w_r) dr \right\|_{L_x^2} \leq \int_s^t \left\| b(u_r - w_r) - b(u_r - w_r) dr \right\|_{L_x^2}$$

$$\leq \| b \|_{C^{0,1}} \| u \|_{C_t^{0,1/2} L_x^2} \int_s^t \| r - s \|^{1/2} dr \leq \| b \|_{C^{0,1}} \| u \|_{C_t^{0,1/2} L_x^2} |t - s|^{3/2},$$

from which we conclude that

$$\left\| (\mathcal{I} A^u)_{s,t} - \int_s^t b(u_r - w_r) dr \right\|_{L_x^2} \leq A^u_{s,t} - (\mathcal{I} A^u)_{s,t} \|_{L_x^2} + \left\| A^u_{s,t} - \int_s^t b(u_r - w_r) dr \right\|_{L_x^2} = O(|t - s|^{1+\epsilon}).$$

Hence, the function

$$t \rightarrow (\mathcal{I} A^u)_{0,t} - \int_0^t b(u_r - w_r) dr$$

is constant. As it starts in zero, this concludes the claim. \qed

Lemma 2.4. Let $b$ be smooth and bounded and let $w : [0, T] \rightarrow \mathbb{R}^N$ be continuous with local time $L \in C_t^{0,1/2+\epsilon} L_x^2$ for some $\epsilon > 0$. Assume $u \in C_t^{0,1/2} L_x^2$, then

1. the sewing is weakly differentiable in time, i.e.,

$$t \mapsto (\mathcal{I} A^u)_t \in W^{1,2}(0, T; L_x^2),$$

and moreover, we find

$$\partial_t [(\mathcal{I} A^u)] : = \lim_{h \rightarrow 0} h^{-1} ((\mathcal{I} A^u)_{t+h} - (\mathcal{I} A^u)_t) = b(u_t - w_t),$$

where the convergence holds strongly in $L_x^2$.

2. If $u$ is moreover a robustified solution, then it is also a strong solution, that is for almost all $t \in I$

$$\partial_t u - \text{div} S(\nabla u) = b(u - w)$$

as an equation in $L_x^2$.
Proof. First, we will address (1). Recall that \((\mathcal{IA}^u)_{s,t}\) is constructed as the germ
\[
A_{s,t} = \int_s^t b(u_s - w_\tau) \, d\tau = (b \ast L_{s,t})(u_s).
\]
Additionally, we have as in the previous lemma
\[
\| (\mathcal{IA}^u)_{s,t} - A_{s,t} \|_{L^2_x} \lesssim \| b \|_{H^1} \| L \|_{C^{0,1/2+\epsilon}} L^2 \| u \|_{C^{0,1/2} L^2_x} |t - s|^{1+\epsilon}.
\]
As \(w\) is uniformly continuous due to the compactness of \([0, T]\), we find for any \(\epsilon > 0\) a \(\delta > 0\) such that for all \(s, t \in [0, T]\) such that \(|t - s| < \delta\) we have \(|w_s - w_t| < \epsilon\).

Thus, choosing \(h < \delta\), we have
\[
\left\| \frac{A_{t,t+h}}{h} - b(u_t - w_t) \right\|_{L^2_x} \leq \left\| h^{-1} \int_t^{t+h} b(u_t - w_\tau) - b(u_t - w_t) \, d\tau \right\|_{L^2_x} \lesssim \epsilon \| b \|_{C^{0,1}}.
\]
Overall, this implies
\[
\lim_{h \to 0} \left\| \frac{(\mathcal{IA}^u)_{t+h} - (\mathcal{IA}^u)_t}{h} - b(u_t - w_t) \right\|_{L^2_x} = 0,
\]
establishing the first claim. Next, we take a look at (2). By part (1) the right-hand side of (2.5) is weakly differentiable in \(L^2_x\) and due to the regularity assumptions \(\partial_t u \in L^2_x L^2_t\) and \(\text{div} S(\nabla u) \in L^2_x L^2_t\) also the left-hand side is differentiable. Therefore, we may rescale (2.5) by \(|t - s|^{-1}\) and pass to the limit \(t \to s\) to obtain
\[
\partial_t u - \text{div} S(\nabla u) = b(u - w).
\]

Remark 2.5. Note that the key difference between strong and robustified solutions lies in the fact that the latter can exploit the regularization property of the local time \(L\) associated with \(w\). In particular, provided this local time is sufficiently regular, the definition is meaningful even in instances in which \(b\) only enjoys distributional regularity as discussed in the introduction.

3. Classical a priori bounds for strong solutions

Global a priori bounds are generally inaccessible for singular potentials \(b\). However, if we restrict ourselves to regularized approximations of \(b\), i.e., we assume that there exists \((b_\epsilon)_{\epsilon \in (0,1)} \in C^{0,1}\) such that \(b_\epsilon \to b \in L^{2q}\), then for each \(\epsilon \in (0,1)\) the classical theory is applicable and existence of strong solutions to (1.2) for smooth \(b\) is a classical result. It can be found, e.g., in the books [18,31,39]. The objective of the present section is to trace the precise form of the a priori estimates that will be robustified in the next section.
**Theorem 3.1.** Let $p \in (1, \infty)$, $u_0 \in L^2_x \cap W^{1,p}_{0,x}$ and $b_\epsilon \in C^{0,1}$. Then there exists a unique solution $u^\epsilon$ solving
\[
\partial_t u^\epsilon - \text{div} S(\nabla u^\epsilon) = b_\epsilon(u^\epsilon - w) \quad \text{on } \Omega \times [0, T]
\]
\[
u^\epsilon |_{\Omega} = 0 \quad \text{on } \partial \Omega \times [0, T]
\]
\[
u^\epsilon(0, \cdot) = u_0 \quad \text{on } \Omega.
\]
in the strong sense of Definition 2.1. Moreover, the following a priori bounds are valid
\[
\sup_{t \in I} \| \nabla u^\epsilon_t \|^p_{L^p_x} + \int_0^T \| \partial_t u^\epsilon_t \|^2_{L^2_x} + \| \text{div} S(\nabla u^\epsilon_t) \|^2_{L^2_x} \, dt \lesssim \| \nabla u_0 \|^p_{L^p_x} + \int_0^T \int_\Omega |b_\epsilon(u^\epsilon_t(x) - w_t)|^2 \, dx \, dt,
\]
and
\[
\sup_{t \in I} \| u^\epsilon_t \|^2_{L^2_x} + \int_0^T \| \nabla u^\epsilon_t \|^p_{L^p_x} \, dt \lesssim \| u_0 \|^2_{L^2_x} + \int_0^T \int_\Omega |b_\epsilon(u^\epsilon_t(x) - w_t)|^2 \, dx \, dt.
\]

**Proof.** The existence of a unique strong solution $u^\epsilon$ to (3.1) is standard, see, e.g., [18, 31, 39], so let us just recall the main steps employed: In order to obtain existence, one first performs a Galerkin projection to the problem. The existence of solutions to the so obtained finite-dimensional problem is done through a fixed point theorem. Next, using monotonicity of the $p$-Laplace operator, one establishes a priori bounds uniformly along solutions to the projected problems. By Banach–Alaoglu, one extracts a weak-* convergent subsequence, whose limit one has to identify as a solution to the problem. Identifying the limit in the nonlinearity $b$ is done thanks to the Aubin–Lions Lemma 6.6. Identifying the limit in the $p$-Laplace operator is done with Minty’s Lemma 6.5. Finally, uniqueness is obtained by monotonicity of the $p$-Laplace operator. We will argue on the weak (3.3) and strong (3.2) energy estimates separately.

The weak energy estimate (3.3) naturally occurs when multiplying (3.1) by $u^\epsilon$. Integration in space and integration by parts imply
\[
\partial_t \left( \frac{1}{2} \| u^\epsilon_t \|^2_{L^2_x} \right) + \| \nabla u^\epsilon_t \|^p_{L^p_x} = \int_\Omega b_\epsilon(u^\epsilon_t(x) - w_t) \cdot u^\epsilon_t(x) \, dx.
\]

Integration in time together with Hölder’s and Young’s inequalities results in
\[
\frac{1}{2} \| u^\epsilon_t \|^2_{L^2_x} + \int_0^t \| \nabla u^\epsilon_s \|^p_{L^p_x} \, ds = \frac{1}{2} \| u_0 \|^2_{L^2_x} + \int_0^t \int_\Omega b_\epsilon(u^\epsilon_s(x) - w_s) \cdot u^\epsilon_s(x) \, dx \, ds \leq \frac{1}{2} \| u_0 \|^2_{L^2_x} + t \int_0^t \int_\Omega |b_\epsilon(u^\epsilon_x(x) - w_x)|^2 \, dx \, ds + \frac{1}{4t} \int_0^t \| u^\epsilon_s \|^2_{L^2_x} \, ds.
\]

Finally, take the supremum in $t$ over $(0, T)$ to conclude
\[
\sup_{t \in (0, T)} \frac{1}{4} \| u^\epsilon_t \|^2_{L^2_x} \leq \frac{1}{2} \| u_0 \|^2_{L^2_x} + T \int_0^T \int_\Omega |b_\epsilon(u^\epsilon_s(x) - w_s)|^2 \, dx \, ds.
\]
This and (3.4) establish the inequality (3.3).

The strong energy estimate follows from squaring both sides of (3.1) and integration in space and time

$$
\int_0^t \int_\Omega |\partial_t u^\varepsilon_x - \text{div} S(\nabla u^\varepsilon_x)|^2 \, dx \, ds = \int_0^t \int_\Omega |b^\varepsilon(u^\varepsilon_s(x) - w_s)|^2 \, dx \, ds.
$$

Note that, due to integration by parts,

$$
-2 \int_0^t \int_\Omega \partial_t u^\varepsilon_s \cdot \text{div} S(\nabla u^\varepsilon_s) \, dx \, ds = 2 \int_0^t \int_\Omega \partial_t \nabla u^\varepsilon_s : S(\nabla u^\varepsilon_s) \, dx \, ds
$$

$$
= \frac{2}{p} \int_0^t \partial_t \|\nabla u^\varepsilon_s\|_{L^p_x}^p \, ds = \frac{2}{p} \left(\|\nabla u^\varepsilon_t\|_{L^p_x}^p - \|\nabla u_0\|_{L^p_x}^p\right).
$$

Therefore, we obtain

$$
\frac{2}{p} \|\nabla u^\varepsilon_t\|_{L^p_x}^p + \frac{2}{p} \int_0^t \|\partial_t u^\varepsilon_s\|_{L^p_x}^p + \|\text{div} S(\nabla u^\varepsilon_s)\|_{L^2_x}^2 \, ds
$$

$$
= \frac{2}{p} \|\nabla u_0\|_{L^p_x}^p + \int_0^t \int_\Omega |b^\varepsilon(u^\varepsilon_s(x) - w_s)|^2 \, dx \, ds.
$$

The claim (3.2) immediately follows after taking the supremum in $t$ over $(0, T)$. □

We embed $W^{1,2}_t\,L^2_x \hookrightarrow C^{0,1/2}_t\,L^2_x$ as the Sewing Lemma is designed for the Hölder scale.

**Corollary 3.2.** Let the assumptions of Theorem 3.1 be satisfied. Then

$$
\|u^\varepsilon\|_{C^{0,1/2}_t\,L^2_x}^2 \lesssim \|u_0\|_{L^2_x}^2 + \|\nabla u_0\|_{L^p_x}^p + \int_0^T \int_\Omega |b^\varepsilon(u^\varepsilon_t(x) - w_t)|^2 \, dx \, dt. \quad (3.5)
$$

**Proof.** Due to (3.3), it holds

$$
\|u^\varepsilon\|_{L^\infty_T\,L^2_x}^2 \lesssim \|u_0\|_{L^2_x}^2 + \int_0^T \int_\Omega |b^\varepsilon(u^\varepsilon_t(x) - w_t)|^2 \, dx \, dt. \quad (3.6)
$$

The fundamental theorem of calculus and Hölder’s inequality reveal

$$
\|u^\varepsilon_t - u^\varepsilon_s\|_{L^2_x}^2 = \left\|\int_s^t \partial_t u^\varepsilon_r \, dr\right\|_{L^2_x}^2 \leq |t - s| \int_s^t \|\partial_t u^\varepsilon_r\|_{L^2_x}^2 \, dr.
$$

The estimate (3.2) bounds

$$
\left[u^\varepsilon\right]_{C^{0,1/2}_t\,L^2_x}^2 \lesssim \|\nabla u_0\|_{L^p_x}^p + \int_0^T \int_\Omega |b^\varepsilon(u^\varepsilon_t(x) - w_t)|^2 \, dx \, dt. \quad (3.7)
$$

Adding (3.6) and (3.7) verifies the claim. □
4. Robustified a priori bounds for the mollified problem

Given a singular potential $b \in L^{2q}$, we have seen in the previous section that for a mollification $b_\epsilon = b * \rho_\epsilon$, we obtain a unique strong solution $u_\epsilon$ to

$$
\partial_t u_\epsilon - \text{div} S(\nabla u_\epsilon) = b_\epsilon (u_\epsilon - w).
$$

(4.1)

In the following, we show that in harnessing the regularizing effect due to the perturbing path $w$, robust a priori bounds uniformly in $\epsilon > 0$ can be obtained. Toward this end, we first show that the right-hand side of (3.2), (3.3) and (3.5) can be robustified in the sense of the following identification.

**Lemma 4.1.** Let $r \in [1, \infty)$ and $\gamma > 1/2$. Suppose $w : [0, T] \to \mathbb{R}^N$ is continuous and admits a local time $L$ such that $L \in C^{0, \gamma}_{t} W^{1, r'}$ and $b \in L^{2q}$ for $q \in [r, \infty)$. Let $u_\epsilon$ be the unique strong solution to (3.1) of Theorem 3.1 associated to the mollification $b_\epsilon$ of $b$. Then for any $\epsilon > 0$ fixed, we have

$$
\int_0^T \int_{\Omega} |b_\epsilon(u_\epsilon - w_r)|^2 \, dx \, dr = (\mathcal{I} A_\epsilon)_{0,T}
$$

(4.2)

where $\mathcal{I}$ denotes the sewing of the germ

$$
A_\epsilon = \int_{\Omega} (b^2_\epsilon * L_{s,t})(u_\epsilon) \, dx,
$$

and where we used the shorthand notation $b^2_\epsilon(u) := |b_\epsilon(u)|^2$. Moreover, there holds the a priori bound

$$
|\mathcal{I} A_\epsilon| \lesssim \|b_\epsilon^2\|_{L^{2q}} \|L\|_{C^{0, \gamma}_{t} W^{1, r'}} (1 + \|u_\epsilon\|_{C^{0,1/2}_{t} L^2_x}) |t - s|^{\gamma}.
$$

**Proof.** Recall that by Corollary 3.2, we have $u_\epsilon \in C^{0,1/2}_{t} L^{2}_x$ for $\epsilon > 0$ fixed. The first part of the claim is established similarly to Lemma 2.3, the main difference being that we exploit the regularity gained from the local time $L$ in order to obtain the a priori bound in the second part of the claim. Let us start by remarking that for $\epsilon > 0$ fixed, $A_\epsilon$ does indeed admit a sewing as

$$
|\delta A^\epsilon_{s,r,t}| \leq \int_{\Omega} |(b^2_\epsilon * L_{r,t})(u^\epsilon_s) - (b^2_\epsilon * L_{r,t})(u^\epsilon_r)| \, dx
$$

$$
\leq \|b^2_\epsilon * L_{r,t}\|_{C^{0,1} L^1_x} \|u^\epsilon_r - u^\epsilon_s\|_{L^1_x}
$$

$$
\lesssim \|b^2_\epsilon\|_{L^q_x} \|L_{r,t}\|_{W^{1, r'}} \|u^\epsilon\|_{C^{0,1/2}_{t} L^2_x} |r - s|^{1/2}
$$

$$
\lesssim \|b\|_{L^{2q}} \|L\|_{C^{0, \gamma}_{t} W^{1, r'}} \|u^\epsilon\|_{C^{0,1/2}_{t} L^2_x} |r - s|^{1/2} |t - r|^{\gamma},
$$

where the sewing of the germ $A_{s,r,t}$ is defined as

$$
A_{s,r,t} = \int_{\Omega} (b^2_\epsilon * L_{s,r,t})(u_\epsilon) \, dx.
$$

(4.3)

**Proof.**}
where we exploited that due to continuity of $w$, $(L_t)_t$ is of compact support uniformly in $t \in [0, T]$ and thus $\|L_t\|_{W^{1,q'}} \lesssim \|L_t\|_{W^{1,r'}}$. Moreover, note that for $\epsilon > 0$ fixed, we have

$$A^\epsilon_{s,t} - \int_s^t \int_\Omega b^\epsilon_r(u^\epsilon_r - w_r) \,dx \,dr = \int_s^t \int_\Omega b^\epsilon_s(u^\epsilon_s - w_r) - b^\epsilon_r(u^\epsilon_r - w_r) \,dx \,dr$$

\[ \lesssim \left[ b^2_\epsilon \right]_{C^{0,1}} [u^\epsilon]_{C_t^{0,1/2} L^2_x} \int_s^t |r-s|^{1/2} \,dr \]

$$= \left[ b^2_\epsilon \right]_{C^{0,1}} [u^\epsilon]_{C_t^{0,1/2} L^2_x} |t-s|^{3/2}.$$

Similar to Lemma 2.3, we can thus conclude that indeed

$$\int_0^T \int_\Omega b^2_\epsilon(u^\epsilon_r - w_r) \,dx \,dr = (\mathcal{I} A^\epsilon)_0, \quad \text{for any } \epsilon > 0 \text{ fixed. Moreover, exploiting the a priori bounds that come with the Sewing Lemma 6.3, we infer}

$$|(\mathcal{I} A^\epsilon)_{s,t}| \leq |A^\epsilon_{s,t}| + |(\mathcal{I} A^\epsilon)_{s,t} - A^\epsilon_{s,t}| \lesssim \|b\|_{L^{2q}}^2 \|L\|_{C_t^{0,\gamma} W^{1,r'}} (1 + \|u^\epsilon\|_{C_t^{0,1/2} L^2_x}).$$

where we used that

$$|A^\epsilon_{s,t}| \lesssim \left\| b^2_\epsilon \ast L_{s,t} \right\|_{L^\infty} \lesssim \|b\|_{L^{2q}}^2 \|L\|_{C_t^{0,\gamma} L^{r'}} |t-s|^{\gamma},$$

which completes the proof. \hfill $\Box$

**Remark 4.2.** The sewing enables the local time to regularize the interplay of $b_\epsilon$ and $u^\epsilon$. In particular, the square $|b_\epsilon(u^\epsilon - w)|^2$ only acts on $b_\epsilon$ and not on $u^\epsilon$. Therefore, it is possible to work in the $L^1_x$ framework for $u^\epsilon$. Indeed, a short inspection of the proof of Lemma 4.1 shows that Hölder regularity of $u^\epsilon$ as a $L^1_x$-valued function is sufficient for the identification. Moreover, upon replacing $u^\epsilon$ by a generic function $v \in C^\alpha_t L^1_x$, the identification (4.2) holds provided $\alpha + \gamma > 1$. However, as our main application is the closing of the a priori bound in Corollary 4.3, we decided to formulate the result immediately on the $L^2(\Omega)$ scale.

**Corollary 4.3.** Let $r \in [1, \infty)$ and $\gamma > 1/2$. Suppose $w : [0, T] \to \mathbb{R}^N$ is continuous and admits a local time $L$ such that $L \in C_t^{0,\gamma} W^{1,r'}$ and $b \in L^{2q}$ for $q \in [r, \infty)$. Let $u^\epsilon$ be the unique solution to (3.1) of Theorem 3.1 associated to the mollification $b_\epsilon$ of $b$. Then we have the a priori bound

$$\left\| u^\epsilon \right\|_{C_t^{0,1/2} L^2_x}^2 \lesssim \|u_0\|_{L^2_x}^2 + \|\nabla u_0\|_{L^p_x}^p + \|b\|_{L^{2r}}^4 \|L\|_{C_t^{0,\gamma} W^{1,r'}}^2. \quad (4.3)$$

**Proof.** Plugging the a priori bound from Lemma 4.1 back into Corollary 3.2, we obtain

$$\left\| u^\epsilon \right\|_{C_t^{0,1/2} L^2_x}^2 \lesssim \|u_0\|_{L^2_x}^2 + \|\nabla u_0\|_{L^p_x}^p + \int_0^T \int_\Omega \left| b_\epsilon(u^\epsilon_t(x) - w_t) \right|^2 \,dx \,dt$$

\[ \lesssim \|u_0\|_{L^2_x}^2 + \|\nabla u_0\|_{L^p_x}^p + \|b\|_{L^{2r}}^2 \|L\|_{C_t^{0,\gamma} W^{1,r'}} (1 + \|u^\epsilon\|_{C_t^{0,1/2} L^2_x}). \]
An application of Lemma 6.7 with \( a = \|u^\varepsilon\|_{C_t^{0,1/2}L_x^2}, K = \|u_0\|^2_{L_x^2} + \|\nabla u_0\|^p_{L_x^p} + \|b\|^2_{L_x^{2q}} \|L\|_{C_t^{0,\gamma}W_x^{1,\gamma}} \) and \( C = \|b\|^2_{L_x^{2q}} \|L\|_{C_t^{0,\gamma}W_x^{1,\gamma}} \) results in
\[
\|u^\varepsilon\|^2_{C_t^{0,1/2}L_x^2} \lesssim \|u_0\|^2_{L_x^2} + \|\nabla u_0\|^p_{L_x^p} + \|b\|^4_{L_x^{2q}} \|L\|^2_{C_t^{0,\gamma}W_x^{1,\gamma}} \tag{4.4}
\]
uniformly in \( \varepsilon > 0 \). \( \square \)

**Remark 4.4.** (Refined a priori bounds) We want to stress that even stronger a priori bounds than (4.3) are available. Indeed, the a priori bounds derived in Theorem 3.1 carry over to the robustified formulation, i.e., uniformly in \( \varepsilon > 0 \) it holds
\[
\|u^\varepsilon\|^2_{L_t^\infty L_x^2} + \|\nabla u^\varepsilon\|^p_{L_t^\infty L_x^p} + \|\partial_t u^\varepsilon\|^2_{L_t^2 L_x^2} + \|\text{div} S(\nabla u^\varepsilon)\|^2_{L_t^2 L_x^2} \lesssim \|u_0\|^2_{L_x^2} + \|\nabla u_0\|^p_{L_x^p} + \|b\|^4_{L_x^{2q}} \|L\|^2_{C_t^{0,\gamma}W_x^{1,\gamma}}. \tag{4.5}
\]

By Lemma 2.4(1) we have \( b_\varepsilon(u^\varepsilon - w) = \partial_t \mathcal{I} A u^\varepsilon \), where \( A_{u^\varepsilon} = (b * L_{x,t})(u^\varepsilon) \) and consequently, since \( u^\varepsilon \) is a strong solution to (3.1) it also holds that
\[
\left\| \partial_t \mathcal{I} A u^\varepsilon \right\|^2_{L_t^2 L_x^2} = \|b_\varepsilon(u^\varepsilon - w)\|^2_{L_x^2 L_x^2} \lesssim \|u_0\|^2_{L_x^2} + \|\nabla u_0\|^p_{L_x^p} + \|b\|^4_{L_x^{2q}} \|L\|^2_{C_t^{0,\gamma}W_x^{1,\gamma}}. \tag{4.6}
\]

### 5. Passage to the limit and proof of Theorem 1.1

In the following, we show that exploiting the a priori bounds obtained in the previous section allows a passage to the limit on the level of the robustified formulation. Toward this end, we will use the monotonicity of the \( p \)-Laplace operator as well as suitable function space embeddings under the additional assumption \( p > \frac{2d}{d+2} \) given in Theorem 1.1.

So far, we constructed a family of solutions \((u^\varepsilon)\) that satisfies, for all almost all \((t, x) \in \Omega_T\),
\[
\partial_t u^\varepsilon - \text{div} S(\nabla u^\varepsilon) = \partial_t \mathcal{I} A u^\varepsilon. \tag{5.1}
\]

Additionally, the solutions obey the uniform bounds
\[
\begin{align*}
  u^\varepsilon &\in W_t^{1,2}L_x^2 \cap L_t^\infty W_0^{1,p}, \tag{5.2a} \\
  \text{div} S(\nabla u^\varepsilon) &\in L_t^2 L_x^2 \cap L_t^\infty W_x^{-1,p'}, \tag{5.2b} \\
  \mathcal{I} A u^\varepsilon &\in W_t^{1,2}L_x^2. \tag{5.2c}
\end{align*}
\]

Therefore, we can extract a subsequence (not relabeled) and limits
\[
(u, \xi, \eta) \in \left( W_t^{1,2}L_x^2 \cap L_t^\infty W_0^{1,p} \times L_t^2 L_x^2 \cap L_t^\infty W_x^{-1,p'} \times W_t^{1,2}L_x^2 \right) \tag{5.3}
\]
such that
\[ u^\varepsilon \to u \in W^{1,2}_t L^2_x, \quad u^\varepsilon \overset*{\to} u \in L^\infty_t W^{1,p}_{0,x}, \]
\[ \text{div } S(\nabla u^\varepsilon) \to \xi \in L^2_t L^2_x, \quad \text{div } S(\nabla u^\varepsilon) \overset*{\to} \xi \in L^\infty_t W^{-1,p}_x, \]
\[ \mathcal{I} A u^\varepsilon \to \eta \in W^{1,2}_t L^2_x. \] (5.4a, 5.4b, 5.4c)

In order to identify the limit dynamics, we test (5.1) with \( \xi \in L^2_t L^2_x \) and integrate in space and time. Using (5.4) we pass with \( \varepsilon \to 0 \) and find
\[ \int_0^T \int \Omega (\partial_t u^\varepsilon - \text{div } S(\nabla u^\varepsilon) - \partial_t \mathcal{I} A u^\varepsilon) \cdot \zeta \, dx \, dt = \int_0^T \int \Omega (\partial_t u - \xi - \partial_t \eta) \cdot \zeta \, dx \, dt, \]
\[ \text{i.e., the limit satisfies} \]
\[ \partial_t u - \xi - \partial_t \eta = 0. \] (5.5)

It remains to identify the nonlinear terms.

First, we discuss the sewing. Since \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^d \), the assumption \( p > \frac{2d}{d+2} \) allows us to use the Rellich–Kondratiev theorem and thus to conclude \( W^{1,p}_{0,x} \hookrightarrow L^2_x \) compactly. Therefore, applying the Aubin–Lions Lemma 6.6 in the form of (6.14) with \( X = W^{1,p}_{0,x} \cap L^2_x \) and \( B = Y = L^2_x \), yields \( L^\infty_t (W^{1,p}_{0,x} \cap L^2_x) \cap W^{1,2}_t L^2_x \hookrightarrow C_t L^2_x \). This allows to extract another subsequence (not relabeled) such that
\[ u^\varepsilon \to u \in C_t L^2_x. \] (5.6)

Crucially, the strong convergence (5.6) is sufficient to identify the nonlinear sewing.

**Lemma 5.1.** Let \( r \in [1, \infty), q \in [r, \infty) \) and \( \gamma > 1/2 \). Suppose \( w : [0, T] \to \mathbb{R}^N \) is continuous and admits a local time \( L \) such that \( L \in C^{0,\gamma}_t W^{1,r}_x \). Moreover, let \( b, b^\varepsilon \in L^{2q} \) and \( u, u^\varepsilon \in C^{1/2}_t L^2_x \) such that \( \|b - b^\varepsilon\|_{L^{2q}} \to 0 \) and \( \|u - u^\varepsilon\|_{C_t L^2_x} \to 0 \). Then, for
\[ A_{s,t}^u = (b * L_{s,t})(u), \quad A_{s,t}^{u^\varepsilon} = (b^\varepsilon * L_{s,t})(u^\varepsilon) \]
we have
\[ \|\mathcal{I} A^u - \mathcal{I} A^{u^\varepsilon}\|_{C^{0,\gamma}_t L^2_x} \to 0. \] (5.7)

**Proof.** Note that
\begin{align*}
\|A_{s,t}^u - A_{s,t}^{u^\varepsilon}\|_{L^2_x} & \leq \|(b - b^\varepsilon) * L_{s,t}(u^\varepsilon)\|_{L^2_x} + \|(b * L_{s,t})(u^\varepsilon) - (b * L_{s,t})(u)\|_{L^2_x} \\
& \lesssim \|b - b^\varepsilon\|_{L^{2r}} \|L_{s,t}\|_{L^{(2r)'}} + \|b\|_{L^{2r}} \|L_{s,t}\|_{W^{1,2r}'} \|u^\varepsilon - u\|_{C_t L^2_x} \\
& \lesssim \|b - b^\varepsilon\|_{L^{2q}} \|L_{s,t}\|_{L^{r'}} + \|b\|_{L^{2q}} \|L_{s,t}\|_{W^{1,r}'} \|u^\varepsilon - u\|_{C_t L^2_x} \\
& \lesssim \|t - s\|^{\gamma} \left( \|b - b^\varepsilon\|_{L^{2q}} \|L\|_{C^{0,\gamma}_t L^{r'}} + \|b\|_{L^{2q}} \|L\|_{C^{0,\gamma}_t W^{1,r}'} \|u^\varepsilon - u\|_{C_t L^2_x} \right)
\end{align*}
(5.8)
where we exploited again that due to the compact support of $L$, we have $\|L\|_{L^{(2,r)\prime}} \lesssim \|L\|_{L^{r'}}$ and similarly for the corresponding Sobolev scales. Moreover, we have accordingly
\[
\| (\delta A^e)_{s,r,t} \|_{L^2_x} = \| (b_\varepsilon \ast L_{r,t})(u_t^\varepsilon) - (b_\varepsilon \ast L_{r,t})(u_r^\varepsilon) \|_{L^2_x} \\
\lesssim \| b_\varepsilon \|_{L^{2q}} \| L_{r,t} \|_{W^{1,r'}} \| u_s^\varepsilon - u_r^\varepsilon \|_{L^2_x} \\
\lesssim \| b \|_{L^{2q}} \| L \|_{C^{0,0}_L W^{1,1}} \| u^\varepsilon \|_{C^{0,1/2}_L L^2_x} |t-s|^{\gamma+1/2}.
\]
(5.9)

The claim now follows from Lemma 6.4.

Lemma 5.1 allows to identify $\eta = TA^u$. Indeed, using the weak convergence (5.4c) and the strong convergence (5.7), it holds for $\zeta \in L^2_t L^2_x$
\[
\int_0^T \int_\Omega \eta \cdot \zeta \, dx \, dt = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega T A^u \cdot \zeta \, dx \, dt = \int_0^T \int_\Omega T A^u : \zeta \, dx \, dt.
\]

Next, we identify the limit of the monotone operator by means of Minty’s Lemma 6.5. Define $\Delta_p : X \to X^*$ by
\[
\langle \Delta_p v, \zeta \rangle_{X^* \times X} := -\int_0^T \int_\Omega S(\nabla v) : \nabla \zeta \, dx \, dt,
\]
(5.10)
where $X = L^2_t L^2_x \cap L^p_t W^{1,p}_{0,x}$ and $X^* = L^2_t L^2_x + L^{p'}_{t} W^{-1,p'}_{x}$ with norms
\[
\|v\|_X := \|v\|_{L^2_t L^2_x} + \|
abla v\|_{L^p_t L^p_x}, \\
\|v\|_{X^*} := \inf \{\|v_1\|_{L^2_t L^2_x} + \|v_2\|_{L^{p'}_t W^{-1,p'}_x} : v = v_1 + v_2\}.
\]

Notice that $L^p_t W^{1,p}_{0,x}$ is sufficient to define (5.10). However, if $\text{div} S(\nabla v) \in L^2_t L^2_x$ we can alternatively represent (5.10), due to integration by parts, by
\[
\langle \Delta_p v, \zeta \rangle_{X^* \times X} = \int_0^T \int_\Omega \text{div} S(\nabla v) \cdot \zeta \, dx \, dt.
\]
(5.11)
Hölder’s inequality shows
\[
\|\Delta_p v\|_{X^*} \leq \min \{\|\text{div} S(\nabla v)\|_{L^2_t L^2_x}, \|S(\nabla v)\|_{L^p_t L^{p'}_x}\} \leq \|v\|^{p-1}_{L^p_t W^{1,p}_{0,x}} \leq \|v\|^{p-1}_X.
\]
This establishes $\Delta_p : X \to X^*$. Notice that $\Delta_p$ is monotone and hemicontinuous.

Due to (5.4a) and (5.4b), we, in particular, have
\[
u^\varepsilon \rightharpoonup u \in X, \quad \Delta_p u^\varepsilon \rightharpoonup u_\varepsilon \in X^*,
\]
(5.12)
where $\langle u_\varepsilon, \zeta \rangle_{X^* \times X} := \int_0^T \int_\Omega u_\varepsilon \cdot \zeta \, dx \, dt.$
The strong convergence (5.6) of $u^\varepsilon$ and the weak convergence (5.4b) of $\text{div} S(\nabla u^\varepsilon)$ enable

$$
\langle \Delta_p u^\varepsilon, u^\varepsilon \rangle_{X^* \times X} = \int_0^T \int_\Omega \text{div} S(\nabla u^\varepsilon) \cdot u^\varepsilon \, dx \, dt \\
\to \int_0^T \int_\Omega \xi \cdot u \, dx \, dt = \langle t_\xi, u \rangle_{X^* \times X}.
$$

Now, Minty’s Lemma 6.5 implies $t_\xi = \Delta_p u$ and Riesz representation theorem verifies $\xi = \text{div} S(\nabla u)$.

The a priori bounds (4.3), (4.5) and (4.6) carry over to the limit by using weak lower semi continuity of the norms.

Overall, we have thus established the existence part of Theorem 1.1.

5.1. Uniqueness

In this subsection we verify uniqueness as claimed in Theorem 1.1. In fact, we will prove a stronger result.

Lemma 5.2. (Continuous dependence on initial data) Let $r \in [1, \infty]$ and $q \in [r, \infty)$. Moreover, let $b \in L^{2q}$ satisfy (1.6) and $w : [0, T] \to \mathbb{R}^N$ be continuous and admit a local time $L$ which satisfies $L \in C^{0, \gamma}_{t,W^1,r'}$ for some $\gamma \in (1/2, 1)$. Let $u, v$ be two robustified solutions to (1.2) started in $u_0, v_0 \in L^2_x$, respectively.

Then for all $t \in [0, T]$ it holds

$$
\|u_t - v_t\|_{L^2_x} \leq \|u_0 - v_0\|_{L^2_x}.
$$

Proof. Let $u, v$ be two robustified solutions to (1.2) in the sense of Definition 2.2 starting in $u_0, v_0 \in L^2_x$, respectively. In particular, they belong to the regularity class

$$
u, v \in \left\{ z \in C_t^{0,1/2}L^2_x \cap L^\infty_t W^{1,p}_{0,x} \mid \partial_t z, \text{div} S(\nabla z) \in L^2_t L^2_x \right\}
$$

and satisfy the system of equations, for all $t \in I$ and almost all $x \in \Omega$,

$$
\begin{align*}
\partial_t u_t - u_0 - \int_0^t \text{div} S(\nabla u_s) \, ds &= (\mathcal{I} A^u)_{0,t}, \\
\partial_t v_t - v_0 - \int_0^t \text{div} S(\nabla v_s) \, ds &= (\mathcal{I} A^v)_{0,t},
\end{align*}
$$

where $A^u_{s,t} = (b * L_{s,t})(u_s)$ and $A^v_{s,t} = (b * L_{s,t})(v_s)$, respectively.

Subtract (5.14b) from (5.14a), differentiate in time, multiply with $u_t - v_t$ and integrate in space to find

$$
\begin{align*}
(\partial_t u_t - \partial_t v_t, u_t - v_t) - (\text{div} S(\nabla u_t) - \text{div} S(\nabla v_t), u_t - v_t) \\
= \left( (\partial_t \mathcal{I} A^u),_t - (\partial_t \mathcal{I} A^v),_t, u_t - v_t \right).
\end{align*}
$$
Notice that

$$2 \left( \partial_t u_t - \partial_t v_t, u_t - v_t \right) = \partial_t \| u_t - v_t \|^2_{L^2}. \quad (5.16)$$

Moreover, due to the monotonicity of the $p$-Laplace operator,

$$-(\text{div}S(\nabla u_t) - \text{div}S(\nabla v_t), u_t - v_t) \geq 0. \quad (5.17)$$

Next, we integrate (5.15) in time and use (5.16) and (5.17)

$$\| u_t - v_t \|^2_{L^2} \leq \| u_0 - v_0 \|^2_{L^2} + 2 \int_0^t \left( \left( \partial_t \mathcal{I}A^u_s - \left( \partial_t \mathcal{I}A^v_s \right) \right), u_s - v_s \right) \, ds. \quad (5.18)$$

The estimate (5.13) immediately follows provided we can verify

$$\int_0^t \left( \left( \partial_t \mathcal{I}A^u_s - \left( \partial_t \mathcal{I}A^v_s \right) \right), u_s - v_s \right) \, ds \leq 0. \quad (5.19)$$

Since we cannot identify the time derivative of the sewing for non-smooth potentials, we will approximate the potential $b$ by a sequence of smooth potentials $(b^n)$ that preserve the monotonicity assumption (1.6). We will use Lemma 6.4 to justify the convergence of the approximations and Lemma 2.4(1) to identify the time derivative of the sewings for smooth potentials.

Let $b^n = (\rho^n * b)$, where $(\rho^n)_n$ is a sequence of nonnegative mollifiers. Notice that the monotonicity assumption on $b$, cf. (1.6), is preserved for $b^n$. Indeed,

$$(b^n(u) - b^n(v)) \cdot (u - v) = \int_{\mathbb{R}^N} \rho^n(z) (b(u - z) - b(v - z)) \cdot ((u - z) - (v - z)) \, dz \leq 0.$$  

Due to integration by parts

$$\int_0^t \left( \partial_t \mathcal{I}A^u_s - \partial_t \mathcal{I}A^v_s, u_s - v_s \right) \, ds$$

$$= (\mathcal{I}A^u_t - \mathcal{I}A^v_t, u_t - v_t) - \int_0^t \left( \mathcal{I}A^u_s - \mathcal{I}A^v_s, \partial_t u_s - \partial_t v_s \right) \, ds$$

$$= (\mathcal{I}A^u_t - \mathcal{I}A^n_t, u_t - v_t) - \int_0^t \left( \mathcal{I}A^n_s - \mathcal{I}A^v_s, \partial_t u_s - \partial_t v_s \right) \, ds$$

$$- (\mathcal{I}A^v_t - \mathcal{I}A^n_t, u_t - v_t) - \int_0^t \left( \mathcal{I}A^n_s - \mathcal{I}A^u_s, \partial_t u_s - \partial_t v_s \right) \, ds$$

$$+ (\mathcal{I}A^{n,u}_t - \mathcal{I}A^{n,v}_t, u_t - v_t) - \int_0^t \left( \mathcal{I}A^{n,u}_s - \mathcal{I}A^{n,v}_s, \partial_t u_s - \partial_t v_s \right) \, ds$$

$$=: R^n_u + R^n_v + R^n_{\text{smooth}}.$$
Here $A^{n,z}_{s,t} = (b^n \ast L_{s,t})(z_s), z \in \{u, v\}$. 

By Hölder’s inequality
\[ \sup_{t \in I} \left| R^n_{u} + R^n_{v} \right| \leq \left( \| \mathcal{I}A u - \mathcal{I}A^{n,u} \|_{L^\infty_t L^2_x} + \| \mathcal{I}A v - \mathcal{I}A^{n,v} \|_{L^\infty_t L^2_x} \right) \left( \| u - v \|_{L^\infty_t L^2_x} + \| \partial_t u - \partial_t v \|_{L^1_t L^2_x} \right). \]

Let $z \in \{u, v\}$. Similarly, to (5.8) and (5.9) it holds
\[ \left\| A^{n,z}_{s,t} - A^{z}_{s,t} \right\|_{L^2_x} \lesssim |t - s|^{\gamma} \left\| b - b^n \right\|_{L^{2q}} \left\| L \right\|_{C^0,\gamma L^r}^{0,0}, \]
\[ \left\| (\delta A^{n,z})_{srt} \right\|_{L^2_x} \lesssim \| b \|_{L^{2q}} \left\| L \right\|_{C^0,\gamma W^{1,r}}^{0,0} \left\| z \right\|_{C^{0,1/2}_t L^2_x} |t - s|^{\gamma+1/2}. \]

Therefore, we can apply Lemma 6.4 and obtain
\[ \left\| \mathcal{I}A^z - \mathcal{I}A^{n,z} \right\|_{C^0_t L^2_x} \to 0 \text{ as } n \to \infty. \]

Thus,
\[ \lim_{n \to \infty} \sup_{t \in I} \left| R^n_{u} + R^n_{v} \right| = 0. \] (5.20)

Reverting the partial integration, using Lemma 2.4(1) and the monotonicity of $b^n$
\[ R^n_{\text{smooth}} = \int_0^t \left( \partial_t \mathcal{I}A^n_{s,t} - \partial_t \mathcal{I}A^n_{s,t}, u_s - v_s \right) \, ds \]
\[ = \int_0^t \left( b^n (u_s - w_s) - b^n (v_s - w_s), u_s - v_s \right) \, ds \leq 0. \] (5.21)

Finally, (5.19) follows from (5.20) and (5.21) and the assertion is verified. \qed

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**Declarations**

**Conflict of interest** All the authors declare that they have no conflicts of interest.
6. Appendix

Verification of Example 1.2

For convenience we recall (1.7)

\[ b(u) = -|u|^{\eta-1} u 1_{|u| \leq K}, \]

where \( \eta \in (-N/2, 0) \) and \( K > 0 \).

Non-existence of weak solution

First, we verify that (1.1) does not have a weak solution started in 0 if \( \eta \leq -1 \). We proceed by contraposition. Assume that (1.1) does have a weak solution started in 0. Our goal is to use the energy equality (3.4) to derive a contradiction.

Due to the density of smooth functions in \( W^{-1,p'} \), we immediately find by (2.2)

\[ \| \partial_t u \|_{L_t^{p'} W_x^{-1,p'}} \leq \| S(\nabla u) \|_{L_t^{p'} L_x^{p'}} + \| b(u) \|_{L_t^{p'} W_x^{-1,p'}} \]

\[ = \| \nabla u \|_{L_t^{p-1} L_x^p} + \| b(u) \|_{L_t^{p'} W_x^{-1,p'}}. \]

Therefore, (2.2) implies for almost all \( s \in [0, T] \)

\[ \partial_t u_s - \text{div} S(\nabla u_s) = b(u_s) \quad \text{in} \quad W^{-1,p'} . \] (6.1)

Moreover, for all \( u \in L_t^p W_{0,x}^{1,p} \cap W_t^{-1,p'} L_x^{p'} \) it holds

\[ 2 \int_0^T \langle \partial_t u_s, u_s \rangle_{W^{-1,p'} \times W^{1,p}} \, ds = \| u_T \|_{L^2}^2 - \| u_0 \|_{L^2}^2. \] (6.2)

Since \( u \in L^p W_{0,x}^{1,p} \) we can test (6.1) with \( u_s \). Using (6.2) and \( u_0 = 0 \), we arrive at

\[ \| u_T \|_{L^2}^2 + 2 \int_0^T \| \nabla u_r \|_{L^p}^p \, dr + 2 \int_0^T \int_{\Omega} |u_r|^{\eta+1} 1_{|u_r| \leq K} \, dx \, dr = 0. \] (6.3)
Notice that the left-hand side of (6.3) contains only nonnegative terms. Therefore, each individual term needs to vanish. However,
\[ \int_0^T \| \nabla u_r \|^p_{L^p} \, dr = 0 \Rightarrow u \equiv 0, \tag{6.4} \]
which leads to the contradiction, since for \( \eta + 1 \leq 0 \),
\[ 0 = \int_0^T \int_\Omega |u_r|^{\eta+1} 1_{\{|u_r| \leq K\}} \, dx \, dr \geq T \, |\Omega|. \tag{6.5} \]

### Existence of robustified solution

Let \( H \) satisfy (1.8) and \( w^H \) be a fractional Brownian motion with Hurst parameter \( H \). We will check that the local time \( L \) associated to the fraction Brownian motion \( w^H \) is sufficiently regular for us to can apply Theorem 1.1.

We recall that a fractional Brownian motion with Hurst parameter \( H \) is a continuous centered Gaussian process whose covariance is given by
\[ \mathbb{E} \left[ w^H_s \otimes w^H_t \right] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right) \text{Id}. \]
where \( \text{Id} \) denotes the identity in \( \mathbb{R}^N \). The space–time regularity of local times associated to locally non-deterministic Gaussian processes has been addressed by Harang and Perkowksi in [27] (see also [25] for regularity results for local times associated with Volterra–Lévy processes). In particular, [27, Theorem 3.4] shows that if
\[ H \in (0, 1/N) \text{ and } \lambda < \frac{1}{2H} - \frac{N}{2}, \tag{6.6} \]
then almost any realization of the fractional Brownian motion \( w^H \) admits a local time \( L \) that satisfies \( L \in C^{0,\gamma}( [0, T]; H^\lambda(\mathbb{R}^N) \) for all \( \gamma \in [0, 1 - (\lambda + N/2)H) \).

Notice that due to (6.6)
\[ 1 - (\lambda + N/2)H \geq 1/2. \]
Therefore, if (6.6) is satisfied we can always choose \( \gamma > 1/2 \) for the temporal regularity. However, whether the spatial regularity is sufficient is to be determined.

If \( \eta \in (-N/4, 0) \) it holds \( b \in L^4(\mathbb{R}^N) \). Therefore, we can choose \( r = 2 \) implying \( r' = 2 \) and we have as a condition
\[ 1 \leq \lambda < \frac{1}{2H} - \frac{N}{2} \Rightarrow H < \frac{1}{N + 2}. \tag{6.7} \]

If \( \eta \in (-N/2, -N/4] \) we need to leave the Hilbert scale and work with Sobolev spaces. Let \( \epsilon \ll 1 \) and define \( r_\epsilon := -\frac{N-\epsilon}{2\eta} \). We observe that \( b \in L^{2r_\epsilon}(\mathbb{R}^N) \) for all \( \epsilon > 0 \). Moreover, \( r'_\epsilon = \frac{N-\epsilon}{N+2\eta-\epsilon} \to \frac{N}{N+2\eta} \) as \( \epsilon \to 0 \). Next, we transfer the
Hilbert scale $H^\lambda$ to the Sobolev scale $W^{1,q}$ by an application of Sobolev embeddings. Let $1 \leq \lambda < 1 + N/2$ and

$$1 - \frac{N}{q} \leq \lambda - \frac{N}{2} \iff q \leq \frac{2N}{2 + N - 2\lambda}.$$ 

Then $H^\lambda_{\text{loc}}(\mathbb{R}^N) \hookrightarrow W^{1,q}_{\text{loc}}(\mathbb{R}^N)$. In particular, if

$$\frac{2N}{2 + N - 2\lambda} > \frac{N}{N + 2\eta},$$

then there exists $\varepsilon > 0$ such that $H^\lambda_{\text{loc}}(\mathbb{R}^N) \hookrightarrow W^{1,q+\varepsilon}_{\text{loc}}(\mathbb{R}^N)$. Therefore, we find as conditions

$$1 \leq \lambda < 1 + \frac{N}{2}, \quad (6.8a)$$

$$1 \leq \lambda < \frac{1}{2H} - \frac{N}{2}, \quad (6.8b)$$

$$1 - \frac{N + 4\eta}{2} < \lambda. \quad (6.8c)$$

Since $1 - \frac{N+4\eta}{2} \to 1 + N/2$ as $\eta \to -N/2$ monotonically, system (6.8) has a solution $\lambda$ if

$$1 - \frac{N + 4\eta}{2} < \frac{1}{2H} - \frac{N}{2} \implies H < \frac{1}{2 - 4\eta}. \quad (6.9)$$

In summary, we have verified that fractional Brownian motions whose Hurst parameter satisfies (1.8) are sufficiently regularizing. The existence of a robustified solution follows by Theorem 1.1.

Local time and occupation times formula

We recall for the reader the basic concepts of occupation measures, local times and the occupation times formula. A comprehensive review paper on these topics is [19].

**Definition 6.1.** Let $w : [0, T] \to \mathbb{R}^N$ be a measurable path. Then the occupation measure at time $t \in [0, T]$ written $\mu^w_t$ is the Borel measure on $\mathbb{R}^d$ defined by

$$\mu^w_t(A) := \lambda(\{s \in [0, t] : w_s \in A\}), \quad A \in \mathcal{B}(\mathbb{R}^N),$$

where $\lambda$ denotes the standard Lebesgue measure.

The occupation measure thus measures how much time the process $w$ spends in certain Borel sets. Provided for any $t \in [0, T]$, the measure is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$, we call the corresponding Radon–Nikodym derivative local time of the process $w$: 

**Definition 6.2.** Let \( w : [0, T] \to \mathbb{R}^N \) be a measurable path. Assume that there exists a measurable function \( L^w_t : [0, T] \times \mathbb{R}^N \to \mathbb{R}_+ \) such that

\[
\mu^w_t(A) = \int_A L^w_t(z)dz,
\]

for any \( A \in \mathcal{B}(\mathbb{R}^N) \) and \( t \in [0, T] \). Then we call \( L^w_t \) local time of \( w \).

Note that by the definition of the occupation measure, we have for any bounded measurable function \( f : \mathbb{R}^N \to \mathbb{R} \) that

\[
\int_0^t f(w_s)ds = \int_{\mathbb{R}^N} f(z)\mu^w_t(dz).
\]

Equation (6.10) is called occupation times formula. Remark that, in particular, provided \( w \) admits a local time, we also have for any \( u \in \mathbb{R}^N \)

\[
\int_0^t f(u - w_s)ds = \int_{\mathbb{R}^N} f(u - z)\mu^w_t(dz) = \int_{\mathbb{R}^N} f(u - z)L^w_t(z)dz = (f \ast L^w_t)(u).
\]

Equation (6.11) is called occupation times formula.

**Sewing Lemma**

Let us recall the Sewing Lemma due to [24] (see also [11, Lemma 4.2]). Let \( E \) be a Banach space, \([0, T] \) a given interval. Let \( \Delta_n \) denote the \( n \)-th simplex of \([0, T] \), i.e., \( \Delta_n : \{(t_1, \ldots, t_n) | 0 \leq t_1 < t_2 \cdots < t_n \leq T\} \). For a function \( A : \Delta_2 \to E \) define the mapping \( \delta A : \Delta_3 \to E \) via

\[
(\delta A)_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}
\]

Provided \( A_{s,t} = 0 \) we say that for \( \alpha, \beta > 0 \) we have \( A \in C^{\alpha,\beta}_2(E) \) if \( \|A\|_{\alpha,\beta} < \infty \) where

\[
\|A\|_\alpha := \sup_{(s,t) \in \Delta_2} \frac{\|A_{s,t}\|_E}{|t-s|^{\alpha}}, \quad \|\delta A\|_\beta := \sup_{(s,u,t) \in \Delta_3} \frac{\|\delta A_{s,u,t}\|_E}{|t-s|^{\beta}}, \quad \|A\|_{\alpha,\beta} := \|A\|_\alpha + \|\delta A\|_\beta.
\]

In this case, we call \( A \) a germ. For a function \( f : [0, T] \to E \), we note \( f_{s,t} := f_t - f_s \). Moreover, if for any sequence \( (P^n([s, t]))_n \) of partitions of \([s, t] \) whose mesh size goes to zero, the quantity

\[
(I^n A)_{s,t} = \sum_{[u, v] \in P^n([s, t])} A_{u,v}
\]

converges to the same limit, we note

\[
(IA)_{s,t} := \lim_{n \to \infty} \sum_{[u, v] \in P^n([s, t])} A_{u,v}.
\]
Lemma 6.3. (Sewing) Let $0 < \alpha \leq 1 < \beta$. Then for any $A \in C^{\alpha,\beta}_{2}(E)$, $(\mathcal{I}A)$ is well defined and we say that the germ $A$ admits a sewing. Moreover, denoting $(\mathcal{I}A) := (\mathcal{I}A)_{0,t}$, we have $(\mathcal{I}A) \in C^{\alpha}([0,T],E)$ and $(\mathcal{I}A)_{0} = 0$ and for some constant $c > 0$ depending only on $\beta$ we have

$$\| (\mathcal{I}A)_{t} - (\mathcal{I}A)_{s} - A_{s,t}\|_{E} \leq c \| A \|_{\beta} |t - s|^{\beta}.$$ 

Proof. For readers unfamiliar with the Sewing Lemma, let us sketch the main argument why the Riemann-type sums (6.12) converge in the case of dyadic partitions. Let us thus consider $\mathcal{P}^{n}([0,t]) = \{t_{k} = \frac{k}{2^{n}}, k = 1, \ldots 2^{n} - 1\}$ and show that the sequence $(\mathcal{I}^{n}A)_{t}$ is Cauchy in $E$. Indeed, denoting by $u_{k}$ the midpoint of $t_{k}$ and $t_{k+1}$, remark that

$$(\mathcal{I}^{n}A)_{t} - (\mathcal{I}^{n+1}A)_{t} = \sum_{k}^{2^{n}} A_{t_{k},t_{k+1}} - \sum_{k}^{2^{n}} A_{t_{k},u_{k}} - \sum_{k}^{2^{n}} A_{u_{k},t_{k+1}} = \sum_{k}^{2^{n}} (\delta A)_{t_{k},u_{k},t_{k+1}}$$

Using the triangle inequality and the assumption $A \in C^{\alpha,\beta}_{2}$, we have

$$\| (\mathcal{I}^{n}A)_{t} - (\mathcal{I}^{n+1}A)_{t} \|_{E} \leq \| A \|_{\beta} 2^{n(1-\beta)}$$

meaning that for $\beta > 1$ we can obtain convergence of $(\mathcal{I}^{n}A)_{t}$ in $E$. \qed

Let us finally cite a result allowing to commute limits and sewings.

Lemma 6.4. (Lemma A.2 [13]) For $0 < \alpha \leq 1 < \beta$ and $E$ a Banach space, let $A \in C^{\alpha,\beta}_{2}(E)$ and $(A^{n})_{n} \subset C^{\alpha,\beta}_{2}(E)$ such that for some $R > 0$ $\sup_{n \in \mathbb{N}} \| \delta A^{n} \|_{\beta} \leq R$ and such that $\| A^{n} - A \|_{\alpha} \to 0$. Then

$$\| \mathcal{I}(A - A^{n}) \|_{\alpha} \to 0.$$ 

Some classical Lemmata from monotone operator theory

Suppose $u^{\epsilon}$ is some approximation to (1.1) (the solution to the Galerkin projected problem or the solution to the mollified problem (3.1), for example) for which uniform bounds in $L^{p}(0,T;W_{0}^{1,p}(\Omega))$ hold, meaning we find $u$ such that $u^{\epsilon} \rightharpoonup u$ along a subsequence in $L^{p}(0,T;W_{0}^{1,p}(\Omega))$. Moreover, as $\text{div}(\nabla u^{\epsilon})$ will be bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, we find a $\xi$ such that $\text{div}(\nabla u^{\epsilon}) \rightharpoonup \xi$ along a further subsequence in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. For the identification step to work, we need an argument ensuring that $\xi = \text{div}(\nabla u)$. This is precisely the content of Minty’s Lemma.

Lemma 6.5. (Minty’s Lemma/Monotonicity trick [39, p. 474]) Let $X$ be a real reflexive Banach space and $A : X \to X^{*}$ a monotone and hemicontinuous operator. Then, provided we have

$$u^{n} \rightharpoonup u \quad \text{in } X$$
$$Au^{n} \rightharpoonup \xi \quad \text{in } X^{*}$$
$$\langle Au^{n}, u^{n} \rangle_{X^{*} \times X} \to \langle \xi, u \rangle_{X^{*} \times X}$$
we may conclude $Au = \xi$.

Concerning the identification of the other nonlinear term, one proceeds differently. Indeed, for a generic nonlinearity $b : \mathbb{R}^N \to \mathbb{R}^N$, we can only hope for $b(u^\epsilon) \rightharpoonup b(u)$ provided that the convergence $u^\epsilon \rightharpoonup u$ holds in the strong topology of some function space. Toward this end, we require some refined a priori bounds and an associated compact embedding, given by the following classical result.

**Lemma 6.6.** (An Aubin–Lions Lemma [37, Corollary 5]) Given Banach spaces $X$, $B$ and $Y$ and assume $X \hookrightarrow \hookrightarrow B \hookrightarrow Y$. Additionally, let $q, r \in [1, \infty]$. Then

1. if $q < \infty$ and $s > (1/r - 1/q) \vee 0$

$$L^q(0, T; X) \cap W^{s,r}(0, T; Y) \hookrightarrow \hookrightarrow L^q(0, T; B), \quad (6.13)$$

2. if $q = \infty$ and $s > 1/r$

$$L^\infty(0, T; X) \cap W^{s,r}(0, T; Y) \hookrightarrow \hookrightarrow C(0, T; B). \quad (6.14)$$

An algebraic inequality

**Lemma 6.7.** Let $a, C, K \in \mathbb{R}$ such that $K \geq -C^2/4$. Additionally, assume that

$$a^2 \leq K + Ca. \quad (6.15)$$

Then

$$\frac{C}{2} - \sqrt{K + \frac{C^2}{4}} \leq a \leq \frac{C}{2} + \sqrt{K + \frac{C^2}{4}}. \quad (6.16)$$

**Proof.** Reordering (6.15) results in

$$a(a - C) \leq K.$$ 

Completing the square ensures

$$\left(a - \frac{C}{2}\right)^2 \leq K + \frac{C^2}{4}.$$ 

This implies

$$\left|a - \frac{C}{2}\right| \leq \sqrt{K + \frac{C^2}{4}}.$$ 

The assertion is verified. \qed

If $-K \leq C^2/4$, then the parabola $a \mapsto a(a - C)$ and the constant function $K$ do not intersect. Thus, (6.15) won’t be satisfied.
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