ON THE POSITIVE EIGENVALUES AND EIGENVECTORS OF A NON-NEGATIVE MATRIX

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1. Introduction

The purpose with the present note is to develop the general theory of positive eigenvalues and positive eigenvectors for a class of possibly infinite non-negative matrices which comprises, but is not restricted to the irreducible row-finite non-negative matrices. Specifically, we shall consider matrices $A = (A_{vw})$ over a countable index set $V$ such that

1) $A_{vw} \geq 0$ for all $v, w \in V$;
2) $\{w \in V : A_{vw} \neq 0\}$ is non-empty and finite for all $v \in V$, and
3) $A$ is coastal in the sense that for all $v \in V$ and for all sequences $w_1, w_2, w_3, \ldots$ in $V$ with $A_{w_kw_{k+1}} \neq 0$ for all $k$, there are $n, j \in \mathbb{N}$ such that $A_{w_nj} \neq 0$.

The choice of these conditions and the search for the positive eigenvalues and eigenvectors for matrices satisfying the conditions, is motivated by the search for the KMS weights for the gauge action on simple graph algebras, [Th].

Thanks to the relation to Markov chains, the literature involving positive eigenvalues and eigenvectors of non-negative matrices is enormous, but the focus is mostly on (sub-)stochastic and irreducible matrices, and the possible generalization of the results to more general non-negative matrices is largely ignored. Also there is a preference for the eigenvalue 1 which is not justified by our purpose. So, although much information can be obtained from the theory of Markov chains, at least in the irreducible case, it seems desirable to have a development of the theory which does not hinge on the probabilistic interpretations intrinsic to the Markov chain approach. It is not clear at first sight, that the theory can be successfully developed in the generality we need, since we must allow, for example, the adjacency matrices of directed graphs without loops, and even graphs for which the otherwise very weak assumptions from the survey [Sa] by Sawyer is not satisfied. It is the purpose of this note to demonstrate how the abstract theory of positive eigenvalues and eigenvectors can be developed for non-negative matrices satisfying the three conditions above. The approach is purely functional analytic and requires no knowledge of Markov chains.

The first section below, Section 2, contains some basic facts about the directed graph defined by the matrix and develops the relatively few tools needed to determine for which $\beta \in \mathbb{R}$ there is a non-zero non-negative vector $\xi_v$, $v \in V$, such that

$$\sum_{w \in V} A_{vw} \xi_w = e^{\beta} \xi_v$$

for all $v \in V$. It turns out that there is a multitude of possibilities, to a large extent determined by the graph of $A$, cf. Theorem 2.7. In fact, among the matrices

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satisfying the conditions above there are examples where the answer is 'no \( \beta \) at all', 'all \( \beta' \)', 'a unique \( \beta' \) and 'all \( \beta \) above and including a certain minimal \( \beta_0 \)'. In the irreducible case Theorem 2.7 follows from Perron-Frobenius theory when the matrix is finite, and from a result of Pruitt when it is not, [P].

The subsequent section, Section 3, contains an abstract description of the eigen-vectors. In the cases that resemble and to some extent actually correspond to a recurrent Markov chain, the positive eigenvector is unique up to scalar multiplication, and in the remaining cases, where the situation is analogous to a transient Markov chain, there is an integral representation, modeled on and generalizing the well-known Poisson-Martin integral representation of the harmonic functions of a countable state Markov chain of finite range, cf. Theorem 3.7.

I have tried to keep the note as self-contained as possible, and the tools and methods that are used are far from original. It is difficult to identify the genuine sources of the ideas that are in play, but my own sources have been the paper by Vere-Jones, [V], the survey by Sawyer, [Sa] and the books by Woess, [Wo], and Kitchens, [K], and for the convexity theory the book by Bratteli and Robinson, [BR].

2. Positive eigenvalues for non-negative matrices

2.1. The setting. Let \( V \) be a countable set and

\[
V \times V \ni (v,w) \mapsto A_{vw} \in [0,\infty)
\]

a non-negative matrix over \( V \). We can then consider the directed graph \( G_A \) with vertexes \( V \) such that there is an arrow from \( v \in V \) to \( w \in V \) if and only if \( A_{vw} \neq 0 \). Let \( E \) denote the set of edges in \( G_A \). When \( \mu \) is an edge, or more generally a finite path in \( G_A \), we denote by \( s(\mu) \) and \( r(\mu) \) its initial and terminal vertex, respectively. We assume that \( G_A \) is cofinal in the sense that every vertex can reach every infinite path in \( G_A \), i.e. when \( v \in V \) and \( e_0e_1e_2 \cdots \) is an infinite path in \( G_A \), consisting of the consecutive edges \( e_0,e_1,\text{etc.} \), there is a finite path \( \mu \) in \( G_A \) such that \( s(\mu) = v \) and \( r(\mu) = s(e_k) \) for some \( k \in \mathbb{N} \). We extend this terminology to \( A \) and say that \( A \) is cofinal when \( G_A \) is. The following will be standing assumptions in this paper:

i) There are no zero rows in \( A \), i.e. for all \( v \in V \) there is a \( w \in V \) such that \( A_{vw} \neq 0 \),

ii) \( A \) is row-finite in the sense that \( \{w \in V : A_{vw} \neq 0\} \) is finite for all \( v \in V \), and

iii) \( A \) is cofinal.

Note that i) means that \( G_A \) has no sinks. Note also that the assumptions i) and iii) are satisfied when \( A \) is irreducible in the usual sense, cf. e.g. [V] or [K].

Recall that a subset \( M \subseteq V \) of vertexes in \( G_A \) is hereditary when \( e \in E \), \( s(e) \in M \Rightarrow r(e) \in M \), and saturated when \( r(s^{-1}(v)) \subseteq M \Rightarrow v \in M \).

Lemma 2.1. \( G_A \) contains no non-trivial subset of vertexes which is both hereditary and saturated.

Proof. The argument is taken from the proof of Proposition 5.1 in [BPRS]. Assume that \( M \subseteq V \) is a non-trivial hereditary and saturated subset, and consider an element \( v \in V \setminus M \). Since \( G_A \) has no sinks and \( M \) is saturated there is an edge \( e_0 \in E \) such that \( s(e_0) = v \) and \( r(e_0) \notin M \). Repeating this argument we obtain an infinite path \( e_0e_1e_2 \cdots \) in \( G_A \) such that \( s(e_k) \notin M \) for all \( k \). However, since \( G_A \) is cofinal there is
an element $w \in M$ and a path in $G_A$ going from $w$ to $s(e_k)$ for some $k$. Since $M$ is
hereditary this is a contradiction. □

A vertex $v \in V$ is non-wandering when there is a finite path $\mu$ in $G_A$ such that $v = s(\mu) = r(\mu)$. We denote by $NW_A$ the set of non-wandering vertexes in $V$. The non-wandering subgraph of $G_A$ is the subgraph $G_A^{NW}$ consisting of the vertexes $NW_A$ and the edges emitted from any of its elements.

**Lemma 2.2.** $NW_A$ is a (possibly empty) hereditary subset of $V$ and the graph $G_A^{NW}$ is irreducible (i.e. for all $v, w \in NW_A$ there is a finite path $\mu$ in $G_A^{NW}$ such that $v = s(\mu)$, $r(\mu) = w$).

**Proof.** When $v \in NW_A$ there is an infinite path in $G_A$ which visits $v$ infinitely often.
So when $e \in E$ and $s(e) \in NW_A$ the cofinality of $G_A$ ensures that there is a path $\mu$ in $G_A$ connecting $r(e)$ to $s(e)$. Then $\mu e$ is a loop in $G_A$ containing $r(e)$, proving that $r(e) \in NW_A$, and hence that $NW_A$ is hereditary. The proof of the irreducibility of $G_A^{NW}$ is similar. □

2.2. The positive eigenvalues. We are here looking for non-negative eigenvectors corresponding to positive eigenvalues; that is, we seek non-zero maps $\xi : V \to [0, \infty)$ such that

$$
\sum_{w \in V} A_{vw} \xi_w = e^\beta \xi
$$

(2.1)

for all $v \in V$ and some $\beta \in \mathbb{R}$. We say then that $\xi$ is a positive $e^\beta$-eigenvector for $A$.

**Lemma 2.3.** Let $\xi$ be a positive $e^\beta$-eigenvector for $A$. Then $\xi_v > 0$ for all $v \in V$.

**Proof.** The set $\{v \in V : \xi_v = 0\}$ is hereditary and saturated. Since $\xi$ is not zero the set is not all of $V$, and it follows therefore from Lemma [2.1] that it must be empty. □

**Lemma 2.4.** Let $H$ be a non-empty hereditary subset of $V$ and $\eta : H \to [0, \infty)$ a function such that $\sum_{w \in V} A_{vw} \eta_w = e^\beta \eta_v$ for all $v \in H$. There is a unique positive $e^\beta$-eigenvector $\xi$ for $A$ such that $\xi_v = \eta_v$ for all $v \in H$.

**Proof.** Set $H_1 = \{v \in V : A_{vw} \neq 0 \Rightarrow w \in H\}$.

Then $H_1$ is hereditary, contains $H$ and there is a unique extension of $\eta$ to $H_1$ given by the condition that $e^\beta \eta_v = \sum_{w \in V} A_{vw} \eta_w$ for all $v \in H_1$. Continuing by induction we get a sequence $H \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ of subsets of $V$ and a unique extension $\xi$ of $\eta$ to $\bigcup_n H_n$. This completes the proof since $\bigcup_n H_n$ is hereditary and saturated, and hence equal to $V$ by Lemma [2.1] □

We define the matrices $A^n, n = 0, 1, 2, \ldots$, recursively such that $A^0 = I$, where $I$ is the identity matrix,

$$
I_{vw} = \begin{cases} 1 & \text{when } v = w \\
0 & \text{otherwise,} \end{cases}
$$

$A^1 = A$, and

$$
A^{n+1} = \sum_{u \in V} A_{vu} A^n_{uw}
$$

when $n \geq 1$. 

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Lemma 2.5. (Solidarity) Let $\beta \in \mathbb{R}$. Assume that there are vertexes $v_0, w_0 \in V$ such that $\sum_{n=0}^{\infty} A_{v_0w_0}^n e^{-n\beta} = \infty$. Then $\sum_{n=0}^{\infty} A_{vw_0}^n e^{-n\beta} = \infty$ for all vertexes $v \in V$.

Proof. Set

$$C = \left\{ v \in V : \sum_{n=0}^{\infty} A_{v_0w_0}^n e^{-n\beta} < \infty \right\}.$$

The equality

$$\sum_{u \in V} A_{vu} \sum_{n=0}^{N} A_{uw_0}^n e^{-n\beta} = e^{\beta} \sum_{n=0}^{N+1} A_{vw_0}^n e^{-n\beta} - e^{\beta} I_{vw_0}$$

shows that $C$ is hereditary and saturated. Since $v_0 \notin C$ it follows therefore from Lemma 2.1 that $C = \emptyset$. □

When $NW_A$ is not empty we take an element $v \in NW_A$ and set

$$\beta_0 = \log \left( \limsup_n \left( A_{vv}^n \right)^{\frac{1}{n}} \right)$$

with the convention that $\log \infty = \infty$. Since $G_A^{NW}$ is irreducible by Lemma 2.2, the value $\beta_0$ is independent of the choice of vertex $v \in NW_A$, and in fact

$$\beta_0 = \log \left( \limsup_n \left( A_{vv}^n \right)^{\frac{1}{n}} \right),$$

for all $v, w \in NW_A$.

Following [V] we introduce for $v, w \in V$ and $n = 0, 1, 2, \ldots$, the numbers $r_{vw}(n)$ such that $r_{vw}(0) = 0$, $r_{vw}(1) = A_{vw}$ and

$$r_{vw}(n+1) = \sum_{u \neq w} A_{vu} r_{uw}(n)$$

when $n \geq 1$.

Lemma 2.6. (Equation (4) in [V].) Assume that $NW_A \neq \emptyset$ and that $\beta > \beta_0$. Then

$$\sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta} = I_{vw} + \left( \sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta} \right) \left( \sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta} \right).$$

for all $v, w \in NW_A$.

Proof. By using the product rule for power series the stated equality follows from the observation that for $n \geq 1$, $A_{vw}^n = \sum_{s=1}^{n} r_{vw}(s) A_{vw}^{n-s}$.

□

Theorem 2.7. Let $\beta \in \mathbb{R}$.

1) Assume that the non-wandering part $NW_A$ is empty. There is a positive $e^\beta$-eigenvector for $A$ for all $\beta \in \mathbb{R}$.

2) Assume that the non-wandering part $NW_A$ is non-empty and finite. There is a positive $e^\beta$-eigenvector for $A$ if and only if $\beta = \beta_0$.

3) Assume that the non-wandering part $NW_A$ is non-empty and infinite. There is a positive $e^\beta$-eigenvector for $A$ if and only if $\beta \geq \beta_0$.
Proof. 1) Note that $\sum_{n=0}^{\infty} A^n_{wv} e^{-n\beta} = 1$ for all $w \in V$ since $NW_A = \emptyset$. It follows therefore from Lemma 2.5 that $\sum_{n=0}^{\infty} A^n_{wv} e^{-n\beta} < \infty$ for all $v, w \in V$ and all $\beta \in \mathbb{R}$. Fix $v_0 \in V$ and set

$$H_{v_0} = \{w \in V : A^l_{v_0w} \neq 0 \text{ for some } l \in \mathbb{N}\}.$$ 

Consider a vertex $v \in H_{v_0}$ and choose $l \in \mathbb{N}$ such that $A^l_{v_0v} \neq 0$. Then

$$A^l_{v_0v} \sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} \leq \sum_{n=0}^{\infty} A^{l+n}_{v_0w} e^{-n\beta}$$

$$= e^{l\beta} \sum_{n=0}^{\infty} A^n_{v_0w} e^{-n\beta} \leq e^{l\beta} \sum_{n=0}^{\infty} A^n_{v_0w} e^{-n\beta}. \tag{2.3}$$

It follows that

$$\sum_{n=0}^{\infty} A^n_{v_0w} e^{-n\beta} \leq \frac{e^{l\beta}}{A^{l}_{v_0w}} \sum_{n=0}^{\infty} A^n_{v_0w} e^{-n\beta} \tag{2.4}$$

for all $w \in H_{v_0}$. Since $H_{v_0}$ is infinite it follows from (2.4) that there is a sequence $\{w_k\}$ of distinct elements in $H_{v_0}$ such that the limit

$$\eta_v = \lim_{k \to \infty} \sum_{n=0}^{\infty} A^n_{v_0w_k} e^{-n\beta}$$

exists for all $v \in H_{v_0}$. By letting $N$ tend to $\infty$ in (2.2) we find that

$$\sum_{u \in V} A_{vu} \sum_{n=0}^{\infty} A^n_{uw} e^{-n\beta} = e^{\beta} \sum_{n=0}^{\infty} A^n_{uw} e^{-n\beta} - e^{\beta} I_{vw}. \tag{2.5}$$

It follows from (2.3) that $\sum_{u \in V} A_{vu} \eta_u = e^{\beta} \eta_v$ for all $v \in H_{v_0}$, and the existence of a positive $e^{\beta}$-eigenvector for $A$ follows then from Lemma 2.4.

2) Assume first that $\xi$ is a positive $e^{\beta}$-eigenvector. Since

$$A^n_{vw} \xi_v \leq \sum_{w \in V} A^n_{vw} \xi_w = e^{n\beta} \xi_v$$

when $v \in NW_A$, we see that

$$\lim_{n \to \infty} \sup_{v \in V} (A^n_{vw})^{\frac{1}{n}} = \lim_{n \to \infty} \sup_{v \in V} (A^n_{vw} \xi_v)^{\frac{1}{n}} \leq \lim_{n \to \infty} \sup_{v \in V} (e^{n\beta} \xi_v)^{\frac{1}{n}} = e^{\beta},$$

proving that $\beta \geq \beta_0$. Similarly, we find that

$$e^{n\beta} \xi_v \leq K \max_{w \in NW_A} A^n_{ew},$$

where $K = (\#NW_A)(\max_{w \in NW_A} \xi_w)$. There is therefore a vertex $w \in NW_A$ such that

$$e^{n\beta} \xi_v \leq KA^n_{ew}$$

for an increasing sequence $n_1 < n_2 < n_3 < \ldots$ of natural numbers. It follows that

$$e^{\beta} = \lim_{n \to \infty} (e^{n\beta} \xi_v)^{\frac{1}{n}} \leq \lim_{n \to \infty} (KA^n_{ew})^{\frac{1}{n}} = e^{\beta_0},$$

proving that $\beta \leq \beta_0$. To find a positive $e^{\beta_0}$-eigenvector, set

$$C = \left\{ \psi \in \mathbb{R}^{NW_A} : \psi_w \geq 0 \ \forall w \in NW_A, \sum_{v \in NW_A} \psi_v = 1 \right\}.$$
We can then define $f : C \to C$ by

$$f(\psi)_v = \frac{\sum_{w \in NW_A} A_{vw} \psi_w}{\sum_{u, w \in NW_A} A_{uw} \psi_w}.$$  

By Brouwer’s fixed point theorem there is an element $\phi \in C$ which is left invariant by $f$; viz. $f(\phi) = \phi$. Then

$$\sum_{w \in V} A_{vw} \phi_w = \sum_{w \in NW_A} A_{vw} \phi_w = e^\beta \phi_v$$

for all $v \in NW_A$ where $e^\beta = \sum_{u, w \in NW_A} A_{uw} \phi_w$. It follows from Lemma 2.2 and Lemma 2.4 that $\phi$ can be extended to a positive $e^\beta$-eigenvector for $A$, and then the first part of the argument shows that $\beta = \beta_0$.

3) Note that the first argument from the proof of 2) applies to show that there can only be a positive $e^\beta$-eigenvector if $\beta \geq \beta_0$. Let $\beta \geq \beta_0$ and assume first that $\sum_{n=0}^{\infty} A_{ww}^n e^{-\beta_0} < \infty$ for some $w \in NW_A$. Let $w' \in NW_A$. By Lemma 2.2 there is an $l \in N$ such that $A_{w'w}^l \neq 0$ and the inequality $A_{ww'}^n A_{w'w} \leq A_{ww'}^{n+l}$ implies that $\sum_{n=0}^{\infty} A_{ww'}^n e^{-\beta_0} < \infty$. It follows then from Lemma 2.5 that $\sum_{n=0}^{\infty} A_{ww'}^n e^{-\beta} < \infty$ for all $v \in V$ and all $w \in NW_A$. By taking $v_0 \in NW_A$ we can then use the same argument as in the proof of 1) to show that there is a positive $e^\beta$-eigenvector.

Assume then that $\sum_{n=0}^{\infty} A_{ww}^n e^{-\beta_0} = \infty$ for some $w \in NW_A$. Then $\beta = \beta_0$ and

$$\lim_{\beta \downarrow \beta_0} \sum_{n=0}^{\infty} A_{ww}^n e^{-\beta} = \infty$$

by Fatou’s lemma. Since

$$\sum_{n=0}^{\infty} A_{ww}^n e^{-\beta_0} = 1 + \left( \sum_{n=1}^{\infty} r_{ww}(n) e^{-\beta_0} \right) \left( \sum_{n=0}^{\infty} A_{ww}^n e^{-\beta} \right).$$

for all $\beta > \beta_0$ by Lemma 2.6 it follows that

$$\lim_{\beta \downarrow \beta_0} \sum_{n=1}^{\infty} r_{ww}(n) e^{-\beta_0} = 1.$$

By using Lebesgue’s monotone convergence theorem this leads to the conclusion that

$$\sum_{n=1}^{\infty} r_{ww}(n) e^{-\beta_0} = 1.$$  \hspace{1cm} (2.6)

Now note that

$$\sum_{u \in V} A_{vu} \left( \sum_{n=1}^{N} r_{uw}(n) e^{-\beta_0} \right)$$

$$= \sum_{n=1}^{N} \sum_{u \neq w} A_{vu} r_{uw}(n) e^{-\beta_0} + A_{vw} \sum_{n=1}^{N} r_{uw}(n) e^{-\beta_0}$$

$$= \sum_{n=1}^{N} r_{vw}(n + 1) e^{-\beta_0} + A_{vw} \sum_{n=1}^{N} r_{uw}(n) e^{-\beta_0}$$

$$= e^{\beta_0} \sum_{n=1}^{N+1} r_{vw}(n) e^{-\beta_0} + A_{vw} \left( \sum_{n=1}^{N} r_{uw}(n) e^{-\beta_0} - 1 \right).$$  \hspace{1cm} (2.7)
It follows from (2.7) and (2.6) that
\[ \{ v \in V : \sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta_0} < \infty \} \]
is both hereditary and saturated, and hence equal to \( V \) by Lemma 2.1 since it contains \( w \). By letting \( N \) tend to infinity in (2.7) we see that
\[ \xi_v = \sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta_0} \]
defines a positive \( e^{\beta_0} \)-eigenvector \( \xi \).

□

3. Positive eigenvectors for non-negative matrices

The problem of obtaining a description of the positive \( e^{\beta} \)-eigenvectors, whose existence can be decided from Theorem 2.7, may appear to be a waste generalization of the well-known and much studied problem of determining the harmonic functions of a countable state Markov chain. But in fact the theory of countable state Markov chains can in some cases solve the problem completely. This is because a general non-negative matrix \( A \), as the ones considered in the previous section, together with a given positive \( e^{\beta} \)-eigenvector \( \xi \), give rise to the matrix
\[ B_{vw} = e^{-\beta} \xi_v^{-1} \xi_w A_{vw} \]  
which besides being non-negative is also stochastic; meaning that
\[ \sum_{w \in V} B_{vw} = 1 \]
for all \( v \in V \). It follows that \( B \) is the transition matrix for a Markov chain on \( V \), cf. [Wo], and the association
\[ \eta \mapsto \eta \xi \]

is an affine bijection mapping the positive \( e^{\beta} \)-eigenvectors for \( A \) onto the harmonic functions for the Markov chain. In this way it is in some cases possible to determine all the positive \( e^{\beta} \)-eigenvectors for \( A \), in particular because the structure of the harmonic functions to a large extend is governed by the structure of the graph \( G_B \) which is the same as \( G_A \), and hence does not depend on \( \beta \). Unfortunately, in the theory of Markov chains it is almost always required that the matrix is irreducible; a restriction which is too strong for e.g. the applications to KMS weights on graph algebras which motivated the present study, and also unnecessary in the sense that the general theory can be easily extended to cover the cases we have dealt with so far. In fact, in the survey [Sa] by Sawyer, it is shown how to obtain the Poisson-Martin integral representation of the harmonic functions of a countable state Markov chain, assuming only that there is a vertex in the underlying graph from where all other vertexes can be reached. Compared to this the only novelty we offer here is fact that this assumption can be exchanged with the cofinality condition, which from the point of view of graph \( C^* \)-algebras is more natural. It is the purpose with this section to present such a generalization, and we shall take the opportunity to develop the theory from a purely functional analytic point of view. It may be that the reason
then either

Assume that there are vertexes

Proof. Assume that there are vertexes \( v, w \in V \) such that \( \sum_{n=0}^{\infty} A^n_{vw} e^{-\beta n} = \infty \) for all \( v, w \in V \), or

\( NW_A \neq \emptyset \), \( \beta = \beta_0 \) and \( \sum_{n=0}^{\infty} A^n_{vw} e^{-\beta_0 n} = \infty \) for all \( v \in V \) and all \( w \in NW_A \).

We must show that 2) holds. Note first that \( \sum_{n=0}^{\infty} A^n_{vw} e^{-\beta n} = \infty \) by Lemma 2.5 which in particular implies both that \( w \in NW_A \) and that \( \beta \leq \beta_0 \). But we assume that there is a positive \( e^\beta \)-eigenvector, so \( \beta \) can not be strictly smaller than \( \beta_0 \) by Theorem 2.7 whence \( \beta = \beta_0 \). Let \( w \in NW_A \). Since \( G^NW_A \) is irreducible by Lemma 2.2 there is an \( l \in \mathbb{N} \) such that \( A^l_{vw} > 0 \). Since \( A^n_{vw} A^l_{vw} \leq A^{n+l}_{vw} \), it follows that

\[
\sum_{n=0}^{\infty} A^n_{vw} e^{-\beta n} \geq e^{\beta l} \sum_{n=0}^{\infty} A^{n+l}_{vw} e^{-\beta n} \geq e^{\beta l} A^l_{vw} \sum_{n=0}^{\infty} A^n_{vw} e^{-\beta n} = \infty.
\]

Since \( w \in NW_A \) was arbitrary it follows from Lemma 2.5 that \( \sum_{n=0}^{\infty} A^n_{vw} e^{-\beta_0 n} = \infty \) for all \( v \in V \) and all \( w \in NW_A \).

We shall refer to the first alternative 1) in Lemma 3.1 as a transient case, and the second alternative 2) as a recurrent case, and we shall divide the considerations accordingly.

When the non-wandering part \( NW_A \) is empty, we are clearly in a transient case, regardless of which \( \beta \) we consider. When \( NW_A \) is non-empty and finite, there is only \( \beta_0 \) to consider by Theorem 2.7 and this is always a recurrent case when \( NW_A \) is finite. To see this, consider a positive \( e^{\beta_0} \)-eigenvector \( \xi \) for \( A \). Since

\[
\sum_{w \in NW_A} \sum_{n=0}^{\infty} A^n_{vw} e^{-\beta_0 n} \xi_w = \sum_{n=0}^{\infty} \sum_{w \in W} A^n_{vw} e^{-\beta_0 n} \xi_w = \sum_{n=0}^{\infty} \xi_v = \infty,
\]

when \( v \in NW_A \), it follows that \( \sum_{n=0}^{\infty} A^n_{vw} e^{-\beta_0 n} = \infty \) for at least one \( w \in NW_A \), proving that it is a recurrent case. Finally, when \( NW_A \) is infinite all cases with \( \beta > \beta_0 \) are transient while the case \( \beta = \beta_0 \) can be both recurrent and transient, cf. \( [V], [K] \).

3.1. The recurrent cases.

Lemma 3.2. (Lemma 4.1 in \( [V] \).) Assume that \( \xi : V \to [0, \infty) \) satisfies that

\[
\sum_{w \in V} A_{vw} \xi_w \leq e^{\beta} \xi_v
\]

for all \( v \). Assume that \( \xi_{v_0} \neq 0 \) for some vertex \( v_0 \in V \). It follows that

\[
\sum_{n=1}^{\infty} r_{v_0}(n) e^{-\beta n} \leq \frac{\xi_v}{\xi_{v_0}}
\]

for all \( v \).
Proof. We prove by induction that
\[
\sum_{n=1}^{N} r_{vw}(n) e^{-n \beta} \leq \frac{\xi_v}{\xi_{v_0}}
\]
for all \(N\) and all \(v\). To start the induction note that \(\xi_v \geq e^{-\beta} \sum_{w \in V} A_{vw} \xi_w \geq e^{-\beta} A_{v_0} \xi_{v_0} = \xi_{v_0} r_{v_0}(1) e^{-\beta}\). Assume then that (3.2) holds for all \(v\). It follows that
\[
\frac{\xi_v}{\xi_{v_0}} \geq e^{-\beta} \sum_{w \in V} A_{vw} \frac{\xi_w}{\xi_{v_0}} = e^{-\beta} \left( \sum_{w \neq v_0} A_{vw} \frac{\xi_w}{\xi_{v_0}} + A_{v_0} \right)
\]
\[
\geq e^{-\beta} \sum_{n=1}^{N} \sum_{w \neq v_0} A_{vw} r_{vw}(n) e^{-n \beta} + e^{-\beta} A_{v_0}
\]
\[
= \sum_{n=1}^{N} r_{v_0}(n+1) e^{-(n+1) \beta} + e^{-\beta} r_{v_0}(1) = \sum_{n=1}^{N+1} r_{v_0}(n) e^{-n \beta}
\]

Theorem 3.3. (The recurrent cases.) Assume \(NW_A \neq \emptyset\) and that \(\sum_{n=0}^{\infty} A_{v_0 v_0}^{n} e^{-n \beta_0} = \infty\) for some (and hence all) \(v_0 \in NW_A\). There is, up to scalar multiplication, a unique positive \(e^{\beta_0}\)-eigenvector \(\xi\) for \(A\). It is represented by
\[
\xi_v = \sum_{n=1}^{\infty} r_{v_0}(n) e^{-n \beta_0}.
\]

Proof. From the proof of Theorem 2.7.3 we see that \(\eta_v = \sum_{n=0}^{\infty} r_{v_0}(n) e^{-n \beta_0}\) is a positive \(e^{\beta_0}\)-eigenvector such that \(\eta_{v_0} = 1\). Let then \(\xi\) be a positive \(e^{\beta_0}\)-eigenvector such that \(\xi_{v_0} = 1\). We must show that \(\xi = \eta\). It follows from Lemma 3.2 that \(\xi_v \geq \eta_v\) for all \(v \in V\). By comparing this to the fact that
\[
e^{\beta_0} = \sum_{w \in V} A_{v_0 w}^{n} \eta_w = \sum_{w \in V} A_{v_0 w}^{n} \xi_w
\]
for all \(n \in \mathbb{N}\) we conclude that \(\xi_w = \eta_w\) for every vertex \(w \in V\) with the property that that \(A_{v_0 w}^{n} \neq 0\) for some \(n\). In particular, \(\xi\) and \(\eta\) agree on \(NW_A\) since \(v_0 \in NW_A\) and \(G_{A}^{NW}\) is irreducible by Lemma 2.2. As \(NW_A\) is also hereditary by the same lemma, it follows from Lemma 2.3 that \(\xi = \eta\).

3.2. The transient cases. In this section we consider the transient case, i.e. we assume that there is a positive \(e^{\beta}\)-eigenvector for \(A\) and that
\[
\sum_{n=0}^{\infty} A_{vw}^{n} e^{-n \beta} < \infty
\]
for all \(v, w \in V\).

We denote the set of positive \(e^{\beta}\)-eigenvectors for \(A\) by \(E(A, \beta)\). For a given vertex \(v \in V\) we set
\[
E(A, \beta)_v = \{ \xi \in E(A, \beta) : \xi_v = 1 \}.
\]

Equipped with the product topology \(\mathbb{R}^V\) is a locally convex real vector space, \(E(A, \beta) \cup \{0\}\) is a cone in \(\mathbb{R}^V\), and it follows from Lemma 2.3 that it is a closed cone in \(\mathbb{R}^V\). Furthermore, \(E(A, \beta)_v\) is a base for \(E(A, \beta) \cup \{0\}\), and we aim now
Lemma 3.4. \( E(A, \beta) \cup \{0\} \) is a lattice cone; i.e. every pair of elements \( \xi, \eta \in E(A, \beta) \cup \{0\} \) have a least upper bound \( \xi \vee \eta \in E(A, \beta) \cup \{0\} \) and a greatest lower bound \( \xi \wedge \eta \in E(A, \beta) \cup \{0\} \) for the order \( \geq \).

**Proof.** To find the greatest lower bound \( \xi \wedge \mu \) of \( \xi \) and \( \mu \), set \( \nu_v = \min\{\xi_v, \mu_v\} \). Then it follows by iteration that
\[
\sum_{w \in V} e^{-\beta A_{vw} \nu_w} \leq \sum_{w \in V} e^{-\beta A_{vw} \xi_w} = \xi_v \quad \text{and} \quad \sum_{w \in V} e^{-\beta A_{vw} \mu_w} = \mu_v,
\]
proving that
\[
\sum_{w \in V} e^{-\beta A_{vw} \nu_w} \leq \nu_v.
\]
It follows by iteration that
\[
\sum_{w \in V} e^{-(n+1)\beta A_{vw} \nu_w} \leq \sum_{w \in V} e^{-n\beta A_{vw} \nu_w} \leq \nu_v
\]
for all \( v \) and all \( n \). We can therefore consider the limit
\[
\varphi_v = \lim_{n \to \infty} \sum_{w \in V} e^{-n\beta A_{vw} \nu_w},
\]
and observe that \( \varphi \in E(A, \beta) \cup \{0\} \) while \( \psi \leq \mu, \psi \leq \xi \); i.e. \( \psi \) is a lower bound for \( \xi \) and \( \mu \) in \( E(A, \beta) \cup \{0\} \). To see that it is the greatest such, consider \( \varphi \in E(A, \beta) \cup \{0\} \) such that \( \varphi \leq \mu, \varphi \leq \xi \). Then \( \varphi \leq \nu \) and
\[
\varphi_v = \sum_{w \in V} e^{-n\beta A_{vw} \varphi_w} \leq \sum_{w \in V} e^{-n\beta A_{vw} \nu_w},
\]
for all \( v, n \), and hence \( \varphi \leq \psi \). This proves that \( \psi \) is the greatest lower bound for \( \xi \) and \( \mu \), i.e. \( \psi = \xi \wedge \mu \).

The least upper bound \( \xi \vee \mu \) is then given by
\[
\xi \vee \mu = \xi + \mu - \xi \wedge \mu.
\]
Indeed, \( \xi \vee \mu \) is clearly an upper bound and if \( \psi \) is another such, we find that \( \xi + \mu \leq (\psi + \mu) \wedge (\psi + \xi) = \psi + \xi \wedge \mu \) and hence \( \xi \vee \mu \leq \psi \). This shows that \( \xi \vee \mu \) is the least upper bound, as claimed. \( \square \)

Fix now a vertex \( v_0 \in V \). Set
\[
H_{v_0} = \{ w \in V : A_{vw}^n \neq 0 \text{ for some } n \in \mathbb{N} \},
\]
and note that \( H_{v_0} \) is a hereditary set of vertexes. For every \( v \in V \) we consider the function \( K_v : H_{v_0} \to [0, \infty] \) defined by
\[
K_v^\beta(w) = \frac{\sum_{n=0}^\infty A_{vw}^n e^{-n\beta}}{\sum_{n=0}^\infty A_{vw}^n e^{-n\beta}}.
\]

**Lemma 3.5.** Let \( H \subseteq V \) be a non-empty hereditary subset of vertexes. For each \( v \in V \) there is an \( m_v \in \mathbb{N} \) such that
\[
A_{vw}^l \neq 0, \quad l \geq m_v \implies w \in H.
\]
Proof. Define subsets \( H_i \subseteq V \) recursively such that \( H_0 = H \) and
\[
H_{n+1} = \{ v \in V : A_{vw} \neq 0 \Rightarrow w \in H_n \}.
\]
Then \( H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \) is a sequence of hereditary subsets, and the union \( \bigcup_n H_n \) is both hereditary and saturated. It is therefore all of \( V \) by Lemma 2.1.

When \( v \in H_k \) we can use \( m_v = k \).

Lemma 3.6. Let \( \beta \in \mathbb{R} \) and assume that \( \sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta} < \infty \) for all \( v, w \in V \). For every vertex \( v \in V \) there are positive numbers \( l_v, L_v \) and a finite set \( F_v \subseteq H_{v_0} \) such that \( K_v^\beta(w) \leq L_v \) for all \( w \in H_{v_0} \) and \( 0 < l_v \leq K_v^\beta(w) \) for all \( w \in H_{v_0} \setminus F_v \).

Proof. Consider first a vertex \( v \in H_{v_0} \). There is an \( l \in \mathbb{N} \) such that \( A_{vw}^l \neq 0 \). Set \( N_v = (A_{vw})^{-1}e^{l\beta} \) and note that the calculation (2.3) gives the upper bound

\[
K_v^\beta(w) \leq N_v
\]

for all \( w \in H_{v_0} \). Consider then a general vertex \( v \in V \). By Lemma 3.5 there is an \( m_v \in \mathbb{N} \) such that every path in \( G_A \) of length \( m_v \) emitted from \( v \) terminates in \( H_{v_0} \). Let \( S \subseteq H_{v_0} \) be the set of terminal vertexes of these paths. For \( w \in H_{v_0} \) we find that
\[
\sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta} = \sum_{n=0}^{m_v-1} A_{vw}^n e^{-n\beta} + \sum_{n=m_v}^{\infty} \sum_{s \in S} A_{vs}^n A_{sw} e^{-n\beta}
\]
\[
= \sum_{n=0}^{m_v-1} A_{vw}^n e^{-n\beta} + \sum_{s \in S} A_{vs}^{m_v} e^{-m_v\beta} K_s^\beta(w) \left( \sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta} \right)
\]
\[
\leq \sum_{n=0}^{m_v-1} A_{vw}^n e^{-n\beta} + \left( \sum_{s \in S} A_{vs}^{m_v} e^{-m_v\beta} N_s \right) \left( \sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta} \right)
\]
for all \( w \in H_{v_0} \). Let \( F \) be the set of vertexes \( w \in H_{v_0} \) with the property that \( A_{vw}^n \neq 0 \) for some \( n \leq m_v - 1 \). Note that \( F \) is finite. It follows that
\[
K_v^\beta(w) \leq L_v
\]
for all \( w \in H_{v_0} \) when we set
\[
L_v = \sum_{s \in S} A_{vs}^{m_v} e^{-m_v\beta} N_s + \max_{w \in F} \sum_{n=0}^{m_v-1} A_{vw}^n e^{-n\beta} \frac{\max_{w \in F} \sum_{n=0}^{m_v-1} A_{vw}^n e^{-n\beta}}{\sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta}}.
\]

To establish the existence of \( l_v \) and \( F_v \), assume for a contradiction that for all \( \epsilon > 0 \) there are infinitely many elements \( w \in H_{v_0} \) such that
\[
K_v^\beta(w) \leq \epsilon.
\]
We can then construct a sequence \( \{w_k\} \) of distinct elements in \( H_{v_0} \) such that
\[
K_v^\beta(w_k) \leq \frac{1}{k}
\]
for all \( k \). The calculation (2.5) shows that
\[
\sum_{w \in V} A_{w}^\beta K_v^\beta(w_k) = e^\beta K_v^\beta(w_k) - \left( \sum_{n=0}^{\infty} A_{vw_k}^n e^{-n\beta} \right)^{-1} I_{v'w_k}
\]
(3.6)
for all \( v' \in V \) and all \( k \in \mathbb{N} \). It follows from (3.6) that a condensation point 
\[
\xi = (\xi_u)_{u \in V} \text{ in } \prod_{u \in V} [0, L_u]
\]
of the sequence 
\[
(K^\beta_u(w))_{u \in V}, \ k \in \mathbb{N},
\]
is a positive \( e^\beta \)-eigenvector for \( A \) with \( \xi_0 = 1 \). But (3.5) implies that \( \xi_v = 0 \) which is impossible by Lemma 2.3. This contradiction shows that there must be an \( l_v > 0 \) and a finite set \( F_v \subseteq H_{v_0} \) such that \( l_v \leq K^\beta_v(w) \) for all \( w \in H_{v_0} \setminus F_v \).

\[ \square \]

It follows from Lemma 3.6 that \( K^\beta_v \) is a bounded function on \( H_{v_0} \), i.e. \( K^\beta_v \in l^\infty(H_{v_0}) \) for all \( v \in H_{v_0} \). We denote by \( 1_w \) the characteristic function of an element \( w \in H_{v_0} \). Let \( A_\beta \) be the \( C^* \)-subalgebra of \( l^\infty(H_{v_0}) \) generated by \( K^\beta_v, v \in V \), and the functions \( 1_w, w \in H_{v_0} \), and let \( B_\beta \) be the image of \( A_\beta \) in the quotient algebra \( l^\infty(H_{v_0})/c_0(H_{v_0}) \). Note that it follows from Lemma 3.6 that for every \( v \in V \) there is a finite subset \( F_v \subseteq H_{v_0} \) such that

\[
K^\beta_v + \sum_{w \in F_v} 1_w
\]
is invertible in \( l^\infty(H_{v_0}) \). Thus \( A_\beta \) and \( B_\beta \) are both unital \( C^* \)-algebras. Since they are also separable, the set \( X_\beta \) of characters of \( B_\beta \) is a compact metric space and \( B_\beta \) can be identified with \( C(X_\beta) \) via the Gelfand transform. In particular, for each \( v \in V \) the function \( w \mapsto K^\beta_v(w) \) is an element of \( A_\beta \) and its image in \( B_\beta \) is a continuous function on \( X_\beta \) which we also denote by \( K^\beta_v \). Let \( M(X_\beta) \) denote the set of Borel probability measures on \( X_\beta \). We aim to relate \( M(X_\beta) \) to \( E(A, \beta)_{v_0} \) and the set \( K_\beta \) of condensation points in \( \prod_{v \in V} [0, L_v] \) of the set \( \{ (K^\beta_v(w))_{v \in V} : w \in H_{v_0} \} \). In more detail, \( K_\beta \) consists of the elements \( k = (k_v)_{v \in V} \) of \( \mathbb{R}^V \) with the property that for any finite set \( F \subseteq V \) and any \( \epsilon > 0 \), the set

\[
\{ w \in H_{v_0} : |K^\beta_v(w) - k_v| \leq \epsilon, \ v \in F \}
\]
is infinite. \( K_\beta \) is a compact subset of \( \prod_{v \in V} [0, L_v] \subseteq \mathbb{R}^V \).

**Theorem 3.7.** Let \( \beta \in \mathbb{R} \). Assume that there is a positive \( e^\beta \)-eigenvector for \( A \) and that \( \sum_{n=0}^{\infty} A_{vu}^n e^{-n \beta} < \infty \) for all \( v, w \in V \). Then \( E(A, \beta)_{v_0} \) is a compact metrizable Choquet simplex identical with the closed convex hull \( \overline{\overline{K_\beta}} \) of \( K_\beta \) in \( \mathbb{R}^V \). Furthermore, there is a continuous affine surjection \( I : M(X_\beta) \to E(A, \beta)_{v_0} \) defined such that

\[
I(m)_v = \int_{X_\beta} K^\beta_u dm.
\]

**Proof.** Since \( E(A, \beta)_{v_0} \) is a base of \( E(A, \beta) \cup \{0\} \), which is a lattice cone by Lemma 3.4, we can conclude that \( E(A, \beta)_{v_0} \) is a compact Choquet simplex. We need only show that \( E(A, \beta)_{v_0} \) is compact in \( \mathbb{R}^V \), cf. e.g. [BR]. Note that \( I : M(X_\beta) \to \mathbb{R}^V \) is continuous by definition of the topologies. It follows from (3.6) that the identity

\[
\sum_{u \in V} A_{vu} K^\beta_u = e^\beta K^\beta_v
\]
holds in \( C(X_\beta) \) for all \( v \in V \) and it follows therefore by integration that \( I(m) \in E(A, \beta)_{v_0} \) for all \( m \in M(X_\beta) \). To complete the proof we need to show that

\[
I(M(X_\beta)) = \overline{\overline{K_\beta}} = E(A, \beta)_{v_0}.
\]

(3.7)
It follows from (3.6) that $\mathcal{K}_\beta \subseteq E(A, \beta)_{\nu_0}$. By construction the vertexes $V$, considered as evaluation maps on $A_\beta$, constitute a dense subset of the character space of $A_\beta$, and this implies that for any $x \in X_\beta$ there is a sequence $\{w_k\}$ of distinct elements of $H_{\nu_0}$ such that $\lim_{k \to \infty} K_\beta^v(w_k) = K_\beta^v(x)$ for all $v \in V$. It follows from this that $I$ sends the Dirac measure at $x$ to an element of $\mathcal{K}_\beta$ and since $M(X_\beta)$ is the closed convex hull of the Dirac measures, this implies that $I(M(X_\beta)) \subseteq \overline{co} \mathcal{K}_\beta$.

To obtain (3.7) it remains to show that $E(A, \beta)_{\nu_0} \subseteq I(M(X_\beta))$. For this we modify the argument from the proof of Theorem 4.1 in [Sa]. Let $\xi \in E(A, \beta)_{\nu_0}$. Fix an element $x \in X_\beta$ and let $\{w_k\}$ be a sequence of distinct elements in $V$ such that $\lim_{k \to \infty} K_\beta^v(w_k) = K_\beta^v(x)$ for all $v \in V$. Observe that since $K_\beta^v(x) > 0$ it follows that

$$\lim_{n \to \infty} nK_\beta^v(w_n) = \infty$$

(3.8)

for all $v \in V$. For each $n \in \mathbb{N}$, set

$$\xi_v^n = \min \{\xi_v, nK_\beta^v(w_n)\}.$$  

Then

$$\lim_{m \to \infty} e^{-m\beta} \sum_{w \in V} A_{vw}^m \xi_v^n = 0$$

(3.9)

for all $n, v$. To see this note that

$$\left( \sum_{j=0}^{\infty} A_{vw}^j e^{-j\beta} \right) e^{-m\beta} \sum_{w \in V} A_{vw}^m \xi_v^n \leq \left( \sum_{j=0}^{\infty} A_{vw}^j e^{-j\beta} \right) ne^{-m\beta} \sum_{w \in V} A_{vw}^m K_\beta^v(w_n)$$

$$= ne^{-m\beta} \sum_{w \in V} A_{vw}^m \sum_{j=0}^{\infty} A_{vw}^j e^{-j\beta} = n \sum_{j \geq m} \sum_{j \geq m} A_{vw}^j e^{-j\beta}.$$  

Hence (3.9) follows because $\sum_{j=0}^{\infty} A_{vw}^j e^{-j\beta} < \infty$. Set

$$k_n(v) = \xi_v^n - e^{-\beta} \sum_{w \in V} A_{vw} \xi_v^n.$$  

We claim that $k_n \geq 0$. To see this observe first that it follows from (3.6) that

$$e^{-\beta} \sum_{w \in V} A_{vw} K_\beta^v(w_n) \leq K_\beta^v(w_n).$$  

Combined with $e^{-\beta} \sum_{w \in V} A_{vw} \xi_v = \xi_v$ this implies that

$$e^{-\beta} \sum_{w \in V} A_{vw} \xi_v^n \leq \min \{\xi_v, nK_\beta^v(w_n)\} = \xi_v^n,$$

proving the claim. Since

$$\sum_{l=0}^{m} e^{-l\beta} \sum_{w \in V} A_{vw}^l k_n(w) = \xi_v^n - e^{-(m+1)\beta} \sum_{w \in V} A_{vw}^{m+1} \xi_v^n;$$

it follows from (3.9) that

$$\xi_v^n = \sum_{l=0}^{\infty} e^{-l\beta} \sum_{w \in V} A_{vw}^l k_n(w) = \sum_{w \in H_{\nu_0}} K_\beta^v(w)h_n(w)$$

(3.10)

when $v \in H_{\nu_0}$, where $h_n(w) = \sum_{l=0}^{\infty} e^{-l\beta} A_{vw}^l k_n(w)$. In particular, it follows from (3.10) that

$$\sum_{w \in H_{\nu_0}} h_n(w) = \sum_{w \in H_{\nu_0}} K_\beta^v(w)h_n(w) = \xi_v^n \leq \xi_{\nu_0} = 1.$$
We can therefore define a positive linear functional $\mu_n$ of norm $\leq 1$ on $\mathcal{A}_\beta$ such that
\[
\mu_n(g) = \sum_{w \in H_{v_0}} g(w) h_n(w).
\]
By compactness of the unit ball in the dual space of $\mathcal{A}_\beta$ there is a strictly increasing sequence $\{n_l\}$ in $\mathbb{N}$ and a positive linear functional $\mu$ on $\mathcal{A}_\beta$ such that
\[
\mu(g) = \lim_{l \to \infty} \mu_{n_l}(g)
\]
for all $g \in \mathcal{A}_\beta$. Since $\lim_{l \to \infty} \xi_{v_l} = \xi_v$ by (3.8), it follows from (3.10) that
\[
\mu(K_v^\beta) = \xi_v
\]
for all $v \in H_{v_0}$. For any fixed $w \in H_{v_0}$ we have that $\xi_{w_l} = \xi_w$ for all large $l$, and hence also that $k_{n_l}(w) = 0$ for all large $l$. It follows that $\lim_{l \to \infty} \mu_{n_l}(1_w) = 0$ for all $w \in H_{v_0}$, which shows that $\mu$ factors through $\mathcal{B}_\beta$. It follows therefore from (3.11) and the Riesz representation theorem that there is a Borel probability measure $m$ on $X_\beta$ such that $I(m)_v = \xi_v$ for all $v \in H_{v_0}$. By Lemma 2.4 this implies that $I(m) = \xi$. \qed

When $A$ is sub-stochastic, irreducible and $\beta = 0$, it is clear from the abstract characterization given in Theorem 7.13 of [Wo] that the spectrum of $\mathcal{A}_\beta$ is the Martin compactification of the associated Markov chain, cf. Definition 7.17 in [Wo], while $X_\beta$ is the Martin boundary. Assuming only that $A$ is irreducible the above theorem can be deduced from the Poisson-Martin representation for the Markov chain going with the stochastic matrix $B$ from (3.1). See also Theorem 6.4 of [MW] for a previous formulation of the above integral representation in a special case.

Let $\delta_x$ denote the Dirac measure at a point $x \in X_\beta$. Then
\[
I(\delta_x)_v = K_v^\beta(x)
\]
for all $v \in V$. Thus $I$ takes the extreme points in $M(X_\beta)$ to elements of the form $v \mapsto K_v^\beta(x)$ for some $x \in X_\beta$. By definition of $X_\beta$ there is a sequence $\{w_k\}$ of distinct elements in $H_{v_0}$ such that
\[
\lim_{k \to \infty} K_v^\beta(w_k) = K_v^\beta(x).
\]
Recall now that under a continuous affine surjection between compact convex sets, the pre-image of an extremal point is a closed face and therefore contains an extremal point. In this way we obtain from Theorem 3.7 the following characterization of the extreme points in $E(A, \beta)_{v_0}$.

**Corollary 3.8.** Let $\beta \in \mathbb{R}$. Assume that there is a positive $e^\beta$-eigenvector for $A$ and that $\sum_{n=0}^{\infty} A_{vw}^n e^{-n\beta} < \infty$ for all $v, w \in V$. Let $\xi$ be an extremal point of $E(A, \beta)_{v_0}$. There is a sequence $\{w_k\}$ of distinct elements in $H_{v_0}$ such that
\[
\xi_v = \lim_{k \to \infty} \frac{\sum_{n=0}^{\infty} A_{vw_k}^n e^{-n\beta}}{\sum_{n=0}^{\infty} A_{w_k}^n e^{-n\beta}}
\]
for all $v \in V$.

If we restrict the map $I$ of Theorem 3.7 to the elements of $M(X_\beta)$ that give measure one to the Borel subset of elements $x \in X_\beta$ for which $v \mapsto K_v^\beta(x)$ is extremal in $E(A, \beta)_{v_0}$, the resulting map is injective and in fact a bijection. This follows from Theorem 3.7 by using that in a compact Choquet simplex every element is the barycenter of a unique Borel probability measure supported on the extreme points, cf. e.g. Theorem 4.1.11 and Theorem 4.1.15 in [BR]. In this way we get from
Theorem 3.7 a generalization of the uniqueness properties of the Poisson-Martin representation for countable state Markov chains of finite range, cf. Theorem 7.53 in [Wo].

Applications of the results of this note to KMS weights on graph $C^*$-algebras can be found in [Th].

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