WEAK COMMUTATION RELATIONS OF UNBOUNDED OPERATORS: NONLINEAR EXTENSIONS

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Abstract. We continue our analysis of the consequences of the commutation relation $[S, T] = I$, where $S$ and $T$ are two closable unbounded operators. The weak sense of this commutator is given in terms of the inner product of the Hilbert space $H$ where the operators act. We also consider what we call, adopting a physical terminology, a nonlinear extension of the above commutation relations.

1. Introduction

Let $A, B$ be two closed operators with dense domains, $D(A)$ and $D(B)$, in Hilbert space $H$. In [1] we have discussed some mathematical aspects connected to the formal commutation relation $[A, B] = I$. Since, as it is well known, $A$ and $B$ cannot be both bounded operators, a careful and rigorous analysis is needed. Thus, starting from the very beginning, we require that the identity $AB - BA = I$ holds, at least, on a dense domain $D$ of Hilbert space $H$. In other words, we assume that there exists a dense subspace $D$ of $H$ such that

\[ D \subset D(AB) \cap D(BA); \]
\[ AB\xi - BA\xi = \xi, \quad \forall \xi \in D, \]
\[ D \subset D(A) \cap D(B). \]

As we did in [1] we will suppose that

\[ D \subset D(A^*) \cap D(B^*). \]

(D.1), (D.2) and (D.3) then imply that the operators $S := A \upharpoonright D$ and $T := B \upharpoonright D$ are elements of the partial $*$-algebra $\mathcal{L}^I(D, H)$ and satisfy the equality

\[ \langle T\xi \mid S^\dagger \eta \rangle - \langle S\xi \mid T^\dagger \eta \rangle = \langle \xi \mid \eta \rangle, \quad \forall \xi, \eta \in D. \]

We recall that $\mathcal{L}^I(D, H)$ denotes the set of all (closable) linear operators $X$ such that $D(X) = D$, $D(X^*) \supseteq D$. The set $\mathcal{L}^I(D, H)$ is a partial $*$-algebra with respect to the usual sum $X_1 + X_2$, the scalar multiplication $\lambda X$, the involution $X \mapsto X^\dagger := X^*|D$ and the (weak) partial multiplication $X_1 \circ X_2 = X_1^\dagger * X_2$, defined whenever $X_2$ is a weak right multiplier of $X_1$ (we shall write $X_2 \in R^w(X_1)$ or $X_1 \in L^w(X_2)$), that is, whenever $X_2 D \subset D(X_1^\dagger)$ and $X_1^* D \subset D(X_2^*), [2]$. 

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Let $t \mapsto V(t)$, $t \geq 0$ be a semigroup of bounded operators in Hilbert space. We recall that $V$ is weakly (or, equivalently, strongly) continuous if
\[
\lim_{t \to t_0} \langle V(t)\xi | \eta \rangle = \langle V(t_0)\xi | \eta \rangle, \quad \forall \xi, \eta \in \mathcal{H}.
\]

A closed operator $X$ is the generator of $V(t)$ if
\[
D(X) = \left\{ \xi \in \mathcal{H} ; \exists \xi' \in \mathcal{H} : \lim_{t \to 0} \left\langle \frac{V(t) - \mathbb{I}}{t} \xi | \eta \right\rangle = \left\langle \xi' | \eta \right\rangle, \forall \eta \in \mathcal{H} \right\}
\]
and
\[
X\xi = \xi', \quad \forall \xi \in D(X).
\]

If $V(t)$ is a weakly continuous semigroup, then $V^*(t) := (V(t))^*$ is also a weakly continuous semigroup and if $X$ is the generator of $V(t)$, then $X^*$ is the generator of $V^*(t)$.

An operator $X_0 \in \mathcal{L}^1(\mathcal{D}, \mathcal{H})$ is the $\mathcal{D}$-generator of a semigroup $V(t)$ if $V(t)$ is generated by some closed extension $X$ of $X_0$ such that $X_0 \subset X \subset X_0^\dagger$.

In [1] we gave the following definition

**Definition 1.1.** Let $S,T \in \mathcal{L}^1(\mathcal{D}, \mathcal{H})$. We say that

(CR.1) the commutation relation $[S,T] = \mathbb{I}_\mathcal{D}$ is satisfied (in $\mathcal{L}^1(\mathcal{D}, \mathcal{H})$) if, whenever $S \circ T$ is well-defined, $T \circ S$ is well-defined too and $S \circ T - T \circ S = \mathbb{I}_\mathcal{D}$.

(CR.2) the commutation relation $[S,T] = \mathbb{I}_\mathcal{D}$ is satisfied in weak sense if
\[
\left\langle T\xi | S^\dagger \eta \right\rangle - \left\langle S\xi | T^\dagger \eta \right\rangle = \left\langle \xi | \eta \right\rangle, \quad \forall \xi, \eta \in \mathcal{D}.
\]

(CR.3) the commutation relation $[S,T] = \mathbb{I}_\mathcal{D}$ is satisfied in quasi-strong sense if $S$ is the $\mathcal{D}$-generator of a weakly continuous semigroups of bounded operators $V_S(\alpha)$ and
\[
\langle V_S(\alpha)T\xi | \eta \rangle - \langle V_S(\alpha)\xi | T^\dagger \eta \rangle = \alpha \langle V_S(\alpha)\xi | \eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}; \alpha \geq 0.
\]

(CR.4) the commutation relation $[S,T] = \mathbb{I}_\mathcal{D}$ is satisfied in strong sense if $S$ and $T$ are $\mathcal{D}$-generators of weakly continuous semigroups of bounded operators $V_S(\alpha), V_T(\beta)$, respectively, satisfying the generalized Weyl commutation relation
\[
V_S(\alpha)V_T(\beta) = e^{\alpha\beta}V_T(\beta)V_S(\alpha), \quad \forall \alpha, \beta \geq 0.
\]

As shown in [1], the following implications hold

(CR.4) ⇒ (CR.3) ⇒ (CR.2) ⇒ (CR.1).

Our analysis was motivated by the introduction, on a more physical side, of what have been called **pseudo-bosons**, arising from a particular deformation of the canonical commutation relations, see [3] for a recent review.
Later on, one of us (FB) has extended this notion to the so-called nonlinear pseudo-bosons, [4], in which the commutation rule $[S,T] = \mathbb{I}_D$ does not hold, in general, in any of the above meanings. Section 4 is dedicated to a mathematical treatment of this extension, while Sections 2 and 3 contain more results on the linear case.

2. Some consequences of (CR.3)

Assume that $S, T$ satisfy the commutation relation $[S, T] = \mathbb{I}_D$ in quasi-strong sense; i.e.

$$\langle V_S(\alpha)T^\dagger | \eta \rangle - \langle V_S(\alpha)\xi | T^\dagger \eta \rangle = \alpha \langle V_S(\alpha)\xi | \eta \rangle, \quad \forall \xi, \eta \in D; \alpha \geq 0.$$

If we take $\xi = \eta$ and apply the Cauchy-Schwarz inequality, we get, for every $z \in \mathbb{C}$ and $\alpha \geq 0$,

$$\alpha | \langle V_S(\alpha)\xi | \xi \rangle | \leq 2 \max\{\| (T-z)\xi \|, \| (T^\dagger - z)\xi \| \} \max\{\| V_S(\alpha)\xi \|, \| V_S(\alpha)^*\xi \| \}.$$

As an immediate consequence of (2.1), we get

**Proposition 2.1.** Let $S, T$ satisfy the commutation relation $[S, T] = \mathbb{I}_D$ in quasi-strong sense. Assume that $T = T^\dagger$. Then $\sigma_p(T) = \emptyset$.

**Proof.** If $\lambda$ is an eigenvalue of $T$, then the right hand side of (2.1) vanishes for $z = \overline{\lambda}$ and $\xi = \xi_0$ a corresponding eigenvector. Hence, $| \langle V_S(\alpha)\xi_0 | \xi_0 \rangle | = 0$. Taking the limit for $\alpha \to 0$, one gets $\| \xi_0 \|^2 = 0$. This is a contradiction. □

As we know, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\| V_S(\alpha) \| \leq Me^{\omega \alpha}$, for $\alpha \geq 0$.

Let us assume that $V_S$ is uniformly bounded, i.e. $\| V_S(\alpha) \| \leq M$ for every $\alpha \geq 0$, like it happens when $V_S$ is a semigroup of isometries or a semigroup of contractions. Then, by (2.1) it follows that

$$\lim_{\alpha \to \infty} | \langle V_S(\alpha)\xi | \xi \rangle | = 0, \quad \forall \xi \in D.$$

**Lemma 2.2.** Assume that $V_S$ is uniformly bounded. Then

$$\lim_{\alpha \to \infty} | \langle V_S(\alpha)\xi | \xi \rangle | = 0, \quad \forall \xi \in \mathcal{H}.$$

**Proof.** Let $\xi \in \mathcal{H}$ and $\{\xi_n\}$ a sequence in $\mathcal{D}$ converging to $\xi$. We have:

$$\| \langle V_S(\alpha)\xi | \xi \rangle - \langle V_S(\alpha)\xi_n | \xi_n \rangle \|$$

$$= | \langle V_S(\alpha)\xi | \xi \rangle - \langle V_S(\alpha)\xi_n | \xi_n \rangle + \langle V_S(\alpha)\xi | \xi_n \rangle - \langle V_S(\alpha)\xi_n | \xi_n \rangle |$$

$$\leq \| V_S(\alpha)\xi \| \| \xi - \xi_n \| + \| \xi - \xi_n \| \| V_S(\alpha)^*\xi_n \|$$

$$\leq M (\| \xi \| + \| \xi_n \|) \| \xi - \xi_n \|.$$

Hence

$$| \langle V_S(\alpha)\xi | \xi \rangle | \leq | \langle V_S(\alpha)\xi_n | \xi_n \rangle | + M (\| \xi \| + \| \xi_n \|) \| \xi - \xi_n \|. $$
Thus, by (2.2), we get
\[
\limsup_{\alpha \to \infty} |\langle V_S(\alpha)\xi | \xi \rangle| \leq M(\|\xi\| + \|\xi_n\|)\|\xi - \xi_n\|, \quad \forall n \in \mathbb{N}.
\]
This clearly implies that
\[
\lim_{\alpha \to \infty} |\langle V_S(\alpha)\xi | \xi \rangle| = 0.
\]
\[\square\]

**Theorem 2.3.** Assume that \( V_S \) is a semigroup of contractions (i.e., \( \|V_S(\alpha)\| \leq 1 \)), for every \( \alpha \geq 0 \). Then every eigenvalue of the generator \( X \supset S \) of \( V_S \) has negative real part.

**Proof.** Assume that \( \lambda \in \mathbb{C} \) is an eigenvalue of \( X \). Then, there exists \( \xi \in D(X) \setminus \{0\} \) such that \( X\xi = \lambda \xi \). The Hille-Yosida theorem then implies that \( V_S(\alpha)\xi = \lim_{\epsilon \to 0} e^{\alpha X(I - \epsilon X)^{-1}} \xi, \quad \forall \alpha \geq 0 \).

An easy computation shows that \( e^{\alpha X(I - \epsilon X)^{-1}} \xi = e^{\alpha \lambda(1 - \epsilon \lambda)^{-1}} \xi \to e^{\alpha \lambda} \xi \) as \( \epsilon \to 0 \). By Lemma 2.2 we conclude that \( \Re(\lambda) < 0 \).

\[\square\]

As a special case we obtain a result already proved by Miyamoto (under additional conditions), [5].

**Corollary 2.4.** Assume that the generator \( X \) of \( V_S \) has the form \( X = iH \) where \( H \) is a self-adjoint operator. Then \( \sigma_p(H) = \emptyset \).

### 3. Weyl Extensions

**Definition 3.1.** Let \( S, T \) be symmetric operators of \( \mathcal{L}(D, H) \). We say that \( \{S, T\} \) satisfy the weak Weyl commutation relation if there exists a self-adjoint extension \( H \) of \( S \) such that

- \((\text{ww}1)\ D(\overline{T}) \subset D(H)\);
- \((\text{ww}2)\ \langle e^{-itH} \xi | T\eta \rangle = \langle (T + t)\xi | e^{itH} \eta \rangle, \quad \forall \xi, \eta \in D, t \in \mathbb{R}\).

Then \( H \) is called the weak Weyl extension of \( S \) (with respect to \( T \)).

**Remark 3.2.** We note that we do not assume that \( e^{itH}D(\overline{T}) \subset D(\overline{T}) \).

**Remark 3.3.** From (ww2) it follows that \( e^{-itH} \xi \in D(T^*) \), for every \( \xi \in D, t \in \mathbb{R} \), and
\[
T^*e^{-itH} \xi = e^{-itH}(T + t)\xi, \quad \forall \xi \in D, t \in \mathbb{R}.
\]

**Proposition 3.4.** Let \( \{S, T\} \) satisfy the weak Weyl commutation relation and let \( H \) be the weak Weyl extension of \( S \). The following statements hold.

(i) Suppose that \( T \) is essentially self-adjoint. Then \( \{H, T\} \) satisfy the Weyl commutation relation, that is,
\[
e^{itH}e^{-isT} = e^{-its}e^{-isT}e^{itH}, \quad \forall s, t \in \mathbb{R}.
\]

(ii) If \( H \) is semibounded, then \( T \) is not essentially self-adjoint.
Proof. (i): By \((\text{ww}_1)\) and \((\text{ww}_2)\) it follows that
\[
\langle e^{-itH} \xi | \overline{T} \eta \rangle = \langle (\overline{T} + t) \xi | e^{itH} \eta \rangle, \quad \forall \xi, \eta \in D(\overline{T}); t \in \mathbb{R}.
\]
Then, by the functional calculus, we get
\[
\langle e^{-itH} \xi | e^{-isT} \eta \rangle = \langle e^{-is(\overline{T}+i)} \xi | e^{itH} \eta \rangle, \quad \forall \xi, \eta \in D; \forall s, t \in \mathbb{R}.
\]
This, in turn, easily implies (3.1).

(ii): Suppose that \(T\) essentially self-adjoint. Similarly to the proof of [6, Theorem 2.7], we have, by (i),
\[
\langle e^{-isT} \xi | H \eta \rangle = \langle H \xi | e^{isT} \eta \rangle - s \langle e^{-isT} \xi | \eta \rangle, \quad \forall \xi, \eta \in D(H); s, t \in \mathbb{R}.
\]
Put \(\eta = e^{-isT} \xi, \xi \in D, \|\xi\| = 1\). Then \(\eta \in D(\overline{T}) \subset D(H)\), by \((\text{ww}_1)\), and
\[
\langle e^{-isT} \xi | He^{-isT} \xi \rangle = \langle H \xi | \xi \rangle - s.
\]
Hence,
\[
\sup_{s,t \in \mathbb{R}} \langle e^{-isT} \xi | He^{-isT} \xi \rangle = +\infty,
\]
\[
\inf_{s,t \in \mathbb{R}} \langle e^{-isT} \xi | He^{-isT} \xi \rangle = -\infty.
\]
These equalities contradict the semiboundedness of \(H\).

Remark 3.5. A natural question is the following: When does there exist a semibounded Weyl extension of \(S\)? If \(S\) is a semibounded symmetric operator with finite deficiency indices, then any self-adjoint extension of \(S\) is bounded below, [7, Proposition X.3].

We now investigate the spectrum of \(T\).

Lemma 3.6. Suppose that \(H\) is bounded below. Then, for every \(\beta > 0\),
\[
e^{-\beta H} D(T) \subset D(T^*),
\]
\[
T^* e^{-\beta H} \xi = e^{-\beta H} (T - \beta i) \xi, \quad \forall \xi \in D(T).
\]

Proof. This is proved similarly to [6, Theorem 6.2].

Theorem 3.7. Suppose that \(H\) is semibounded and
\[
D^\infty(T^*) := \bigcap_{n \in \mathbb{N}} D(T^{*n}) \subset D(\overline{T}).
\]
Then \(\sigma(T) = \mathbb{C}\).

Proof. By [7, Theorem X.1] and (ii) of Proposition 3.4, \(\sigma(T) (= \sigma(\overline{T}))\) is one of the following sets
(a) \(\mathbb{C}\);
(b) \(\overline{\Pi}_+\), the closure of the upper half-plane \(\Pi_+ = \{z \in \mathbb{C}; \Im z > 0\}\);
(c) \(\overline{\Pi}_-\), the closure of the lower half-plane \(\Pi_- = \{z \in \mathbb{C}; \Im z < 0\}\).
Suppose that $\sigma(T) = \Pi_-$. For every $z \in \mathbb{C} \setminus \mathbb{R}$, we have
\[ \mathcal{H} = \text{Ker}(T^* - z) \oplus R(T - z). \]
Since $i \in \Pi_+ = \rho(T)$, we get $\text{Ker}(T^* + i) = \{0\}$. But $T$ is not self-adjoint (Proposition 3.4), hence $\text{Ker}(T^* - i) \neq \{0\}$; i.e. there exists a nonzero $\eta \in D(T^*)$ such that $T^*\eta = i\eta$. Then, $\eta \in D^\infty(T^\ast) \subset D(T)$, by (3.2). By Lemma 3.6,
\[ T^* e^{-\beta H} \eta = e^{-\beta H} (T - \beta i)\eta = (1 - \beta)ie^{-\beta H}\eta. \]
This implies that $\gamma := (1 - \beta)i \in \sigma_p(T^\ast)$. Since $\mathcal{H} = \text{Ker}(T^\ast - \gamma) \oplus R(T - \gamma)$, we obtain that $R(T - \gamma) \neq \mathcal{H}$. Thus $\gamma = (\beta - 1)i \in \sigma(T) = \sigma(T)$. Now, if we take $\beta > 1$, by the assumption, $(\beta - 1)i \in \Pi_+ = \rho(T)$. This is a contradiction. Therefore $\sigma(T) \neq \Pi_-$. In very similar way one can prove that $\sigma(T) \neq \Pi_+$. In conclusion, $\sigma(T) = \mathbb{C}$.

\[ \square \]

**Remark 3.8.** When does the inclusion $D^\infty(T^\ast) \subset D(T)$ hold? Let us consider the partial $O^*$-algebra $\mathfrak{M}_w(T)$ generated by $T$ described in [2, Section 2.6]. If $\mathfrak{M}_w(T)$ is essentially self-adjoint, then $D^\infty(T^\ast) \subset D(T)$. If $T \in \mathcal{L}^1(\mathcal{D})$, then $\mathfrak{M}_w(T)$ is nothing but the $O^*$-algebra $\mathcal{P}(T)$ of all polynomials in $T$; in this case if $\mathfrak{M}_w(T)$ is essentially self-adjoint, then $T$ is essentially self-adjoint. But we know [2, Example 2.6.28] that, in the case of partial $O^*$-algebras, the essential self-adjointness of $\mathfrak{M}_w(T)$ does not imply the essential self-adjointness of $T$.

### 4. A non-linear extension

In this section we will consider a generalization of condition (CR2) introduced in Section 1, to what could be called, borrowing a physical terminology adopted first in [4], a non-linear situation. We start considering two biorthogonal bases of the Hilbert space $\mathcal{H}$, both contained in $\mathcal{D}$, $\mathcal{F}_\varphi = \{\varphi_n \in \mathcal{D}, n \geq 0\}$ and $\mathcal{F}_\psi = \{\psi_n \in \mathcal{D}, n \geq 0\}$, i.e. two family of vectors of $\mathcal{D}$ such that the sets $\mathcal{D}_\varphi$ and $\mathcal{D}_\psi$ of their finite linear combinations are dense in $\mathcal{H}$, and such that $\langle \varphi_i | \psi_j \rangle = \delta_{i,j}$. We also consider a strictly increasing sequence of non negative numbers: $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots$. On $\mathcal{D}_\varphi$ and $\mathcal{D}_\psi$ we can introduce two operators, $a$ and $b^\dagger$:

\[ \mathcal{D}_\varphi \ni f = \sum_{k=0}^M c_k \varphi_k \mapsto a f = \sum_{k=1}^M c_k \sqrt{\epsilon_{k-1}} \varphi_{k-1}, \]

and

\[ \mathcal{D}_\psi \ni h = \sum_{k=0}^{M'} d_k \psi_k \mapsto b^\dagger h = \sum_{k=1}^{M'} d_k \sqrt{\epsilon_{k-1}} \psi_{k-1}. \]

It is possible to check that $\mathcal{D}_\psi \subseteq D(a^\dagger)$ and $\mathcal{D}_\varphi \subseteq D(b)$ and that, in particular

\[ a^\dagger \psi_k = \sqrt{\epsilon_{k+1}} \psi_{k+1}, \quad b \varphi_k = \sqrt{\epsilon_{k+1}} \varphi_{k+1}, \]
so that these operators act as \textit{generalized rising operators} on two different bases. Analogously, \(a\) and \(b^\dagger\) act as \textit{generalized lowering operators}, as we can deduce from the formulas above, which give, in particular,

\[
a \varphi_k = \sqrt{\epsilon_{k-1}} \varphi_{k-1}, \quad b^\dagger \psi_k = \sqrt{\epsilon_{k-1}} \psi_{k-1},
\]

if \(k \geq 1\), or zero if \(k = 0\). Taking now \(f \in D_\varphi\) and \(h \in D_\psi\) as above, we conclude that

\[
\langle b f \mid a^\dagger h \rangle - \langle a f \mid b^\dagger h \rangle = \sum_{l=0}^{\min(M,M')} (\epsilon_{l+1} - \epsilon_l) c_l \overline{d_l}.
\]

Let us now introduce an operator \(X\) satisfying

\[
X \psi_k = (\epsilon_{k+1} - \epsilon_k) \psi_k.
\]

This can be formally written as

\[
X = \sum_{l=0}^\infty (\epsilon_{l+1} - \epsilon_l) \psi_l \otimes \overline{\varphi_l},
\]

where \((\psi_l \otimes \overline{\varphi_l})\xi = \langle \xi \mid \varphi_l \rangle \psi_l\). Hence formula (4.1) can be re-written as

\[
\langle b f \mid a^\dagger h \rangle - \langle a f \mid b^\dagger h \rangle = \langle f \mid X h \rangle.
\]

Incidentally we observe that in the \textit{linear regime}, i.e. when \(\epsilon_l = l\), we recover (CR2).

In order to make meaningful the above formula (4.3) and proceed with our analysis, we need a better knowledge of operators of the form

\[
X = \sum_{k=0}^\infty \alpha_k (\psi_k \otimes \overline{\varphi_k})
\]

with \(\{\varphi_n\}\) and \(\{\psi_n\}\) two biorthogonal bases and \(\alpha_k \geq 0\), as above.

To simplify notations, we put \(R_k = \psi_k \otimes \overline{\varphi_k}\). This family of rank one operators enjoys the following easy properties:

\begin{enumerate}
  \item \(\|R_k\| \leq \|\varphi_k\| \|\psi_k\|\);
  \item \(R_k^* = L_k := \varphi_k \otimes \overline{\psi_k}\);
  \item \(R_k^2 = R_k\) and \(R_k R_m = 0\) if \(m \neq k\);
\end{enumerate}

In particular, (iii) implies that \(R_k\) is a nonselfadjoint projection (unless \(\varphi_k = \psi_k\)). Moreover

\[
\xi = \sum_{k=0}^\infty R_k \xi, \quad \forall \xi \in D.
\]

The previous equality implies that \(\{R_k\}\) is a resolution of the identity.

\textbf{Lemma 4.1.} Let \(\{\varphi_n\}\), \(\{\psi_n\}\) be two biorthogonal bases in \(D\) and let

\[
X = \sum_{k=0}^\infty \alpha_k (\psi_k \otimes \overline{\varphi_k}),
\]

with \(\{\alpha_n\}\) a sequence of positive real numbers. Then the following statements hold.
(1) \( D(X) \supset D_\psi \) and \( X_\psi \psi_k = \alpha_k \psi_k \), for every \( k \in \mathbb{N} \).
(2) \( D \subset D(X) \) if, and only if, for every \( \xi \in D \)
\[
\lim_{n \to \infty} \left\| \sum_{k=n+1}^{n+p} \alpha_k \psi_k \otimes \overline{\varphi_k} \right\| \xi = \lim_{n \to \infty} \left\| \sum_{k=n+1}^{n+p} \alpha_k \langle \xi | \varphi_k \rangle \psi_k \right\| = 0,
\]
for every \( p \in \mathbb{N}_0 \).
(3) \( D_\psi \subset D(X) \) if, and only if, for every \( l \in \mathbb{N} \)
\[
\lim_{n \to \infty} \left\| \sum_{k=n+1}^{n+p} \alpha_k \langle \varphi_l | \varphi_k \rangle \psi_k \right\| = 0,
\]
for every \( p \in \mathbb{N}_0 \).
(4) If \( D \subset D(X) \), \( X \) has an adjoint \( X^* \) and
\[
X^* = \sum_{k=0}^{\infty} \alpha_k (\varphi_k \otimes \overline{\psi_k}).
\]

**Proof.** We prove only the statement concerning \( X^* \). First, it is easy to check
that \( \sum_{k=0}^{\infty} \alpha_k (\varphi_k \otimes \overline{\psi_k}) \subset X^* \).
Conversely, let \( \eta \) be an arbitrary element of \( D(X^*) \). Then there exists \( \zeta \in \mathcal{H} \) such that
\[
\langle X \xi | \eta \rangle = \langle \xi | \zeta \rangle, \quad \forall \xi \in D(X).
\]
Since \( R_n \xi \in D(X) \) and \( XR_n \xi = \sum_{k=0}^{n} \alpha_k (\psi_k \otimes \overline{\varphi_k}) \xi \), for every \( \xi \in D(X) \)
and \( n \in \mathbb{N} \), we have
\[
\langle XR_n \xi | \eta \rangle = \langle R_n \xi | \zeta \rangle = \left\langle \sum_{k=0}^{n} (\psi_k \otimes \overline{\varphi_k}) \xi | \zeta \right\rangle = \langle \xi \left| \sum_{k=0}^{n} (\varphi_k \otimes \overline{\psi_k}) \zeta \right\rangle
\]
On the other hand,
\[
\langle XR_n \xi | \eta \rangle = \left\langle \sum_{k=1}^{n} \alpha_k (\psi_k \otimes \overline{\varphi_k}) \xi | \eta \right\rangle = \langle \xi \left| \sum_{k=1}^{n} \alpha_k (\varphi_k \otimes \overline{\psi_k}) \eta \right\rangle.
\]
Hence,
\[
\sum_{k=1}^{n} \alpha_k (\varphi_k \otimes \overline{\psi_k}) \eta = \sum_{k=1}^{n} (\varphi_k \otimes \overline{\psi_k}) \zeta.
\]
Moreover, since \( \sum_{k=1}^{\infty} (\varphi_k \otimes \overline{\psi_k}) = (\sum_{k=1}^{\infty} R_k)^* = I \), by a limiting procedure
in (4.6), it follows that \( \eta \) belongs to the domain of the operator \( \sum_{k=1}^{\infty} \alpha_k (\varphi_k \otimes \overline{\psi_k}) \) and \( \sum_{k=1}^{\infty} \alpha_k (\varphi_k \otimes \overline{\psi_k}) \eta = \zeta \).
\[
\square
\]
Of course, these conditions are clearly satisfied when \( \mathcal{F}_\varphi \) and \( \mathcal{F}_\psi \) collapse
into a single orthonormal set.

We are now ready to introduce the following definition:
**Definition 4.2.** Let $S, T \in \mathcal{L}^1(D, \mathcal{H})$ and $\{\varphi_n\}$ and $\{\psi_n\}$ two biorthogonal bases of $\mathcal{H}$, contained in $D$. We say that $S$ and $T$ satisfy the nonlinear CR.2 if, for all $\xi$ and $\eta$ in $D$,

$$
(4.7) \quad \left\langle T \xi \middle| S^\dagger \eta \right\rangle \rightarrow \left\langle S \xi \middle| T^\dagger \eta \right\rangle = \left\langle \xi \middle| X \eta \right\rangle,
$$

where $X$ is an operator of the form (4.5) with $D \subset D(X)$.

**Remark 4.3.** Let $\{\chi_n\}$ be an orthonormal basis in $D$ and $G$ a symmetric bounded operator with bounded inverse $G^{-1}$. Suppose that $GD = D$. Then, if we put $\varphi_k := G\chi_k$ and $\psi_k := G^{-1}\chi_k$, we obtain two biorthogonal bases of $\mathcal{H}$, still belonging to $D$. Under these assumptions, we get

$$(\psi_k \otimes \overline{\varphi_k})\xi = \langle \xi \mid \varphi_k \rangle \psi_k = \langle \xi \mid G\chi_k \rangle G^{-1}\chi_k = \langle G\xi \mid \chi_k \rangle G^{-1}\chi_k.$$ 

Hence $\psi_k \otimes \overline{\varphi_k} = G^{-1}(\chi_k \otimes \overline{\chi_k})G$. Thus if $S, T$ satisfy the non linear CR.2 with $X$ as in (4.4), the operators $K := G^{-1}TG$ and $H := G^{-1}SG$ satisfy

$$
\left\langle K \xi \middle| H^\dagger \eta \right\rangle \rightarrow \left\langle H \xi \middle| K^\dagger \eta \right\rangle = \left\langle \xi \mid Y \eta \right\rangle, \quad \forall \xi, \eta \in D,
$$

where

$$
(4.8) \quad Y = \sum_{k=0}^{\infty} \alpha_k(\chi_k \otimes \overline{\chi_k}).
$$

Therefore, $X = G^{-1}YG$, and $D(X) \supset D$ if, and only if,

$$
\sum_{k=0}^{\infty} \alpha_k^2 \left| \langle G\xi \mid \chi_k \rangle \right|^2 < \infty, \quad \forall \xi \in D.
$$

So, it is natural to consider, as a first step, this simpler situation, since this will also be relevant when the two sets are Riesz, non necessarily biorthogonal, bases\(^1\).

The operator $Y$, defined in (4.8) is bounded if, and only if, $\{\alpha_k\} \in l^\infty(\mathbb{N})$. Indeed, suppose first that $Y$ is bounded. Hence $\|Yf\| \leq M\|f\|$, for some positive $M$, for each $f \in \mathcal{B}$. Then, for all $k$, $|\alpha_k||\chi_k||Y\chi_k| \leq M||\chi_k||$. Therefore $|\alpha_k| \leq M$, for all $k$. Viceversa, let us assume that $|\alpha_k| \leq M$, for all $k$. Then, using the orthogonality of the $\chi_k$'s and the Parceval equality,

$$
\|Yf\|^2 = \sum_{k=0}^{\infty} |\alpha_k|^2||\langle \chi_k \mid f \rangle|^2 \leq M^2\|f\|^2.
$$

So $\|Y\| \leq M$, and $Y$ is bounded.

The spectrum $\sigma(Y)$ is also easily determined: in fact $\sigma(Y) = \sigma_p(Y) = \{\alpha_k; k \in \mathbb{N}\}$, where $\sigma_p(Y)$ denotes, as usual, the point spectrum of $Y$. We remark that in finite dimensional spaces every family of projections whose sum is the identity operator is similar to a family of orthogonal projections; so that the situation discussed above is the more general possible. For the

\(^1\)We recall that a Riesz basis is the image, via a bounded operator with bounded inverse, of an orthonormal basis.
infinite dimensional case, an analogous statement was obtained by Mackey [8, Theorem 55]: every nonselfadjoint resolution of the identity (i.e. a spectral measure on the Borel set of the plane or of the real line) is similar to a selfadjoint resolution of the identity.

The extension to \( X \) of the results outlined in Remark 4.3 is, under suitable assumptions, quite straightforward.

We have

**Proposition 4.4.** Let \( \mathcal{F}_\psi = \{ \psi_k \} \) and \( \mathcal{F}_\varphi = \{ \varphi_k \} \) be (possibly, non biorthogonal) Riesz bases for \( \mathcal{H} \) and let

\[
X = \sum_{k=0}^{\infty} \alpha_k (\psi_k \otimes \varphi_k), \quad \alpha_k \in \mathbb{R}^+.
\]

Then the following statements hold.

(i) \( X \) is bounded if and only if \( \{ \alpha_k \} \in l^\infty(\mathbb{N}). \) In particular \( \sum_{k=0}^{\infty} R_k = 1 \).

(ii) For every \( k \in \mathbb{N}, \psi_k \in D(X), \varphi_k \in D(X^*). \) and, if \( \mathcal{F}_\psi \) and \( \mathcal{F}_\varphi \) are biorthogonal,

\[
X \psi_k = \alpha_k \psi_k, \quad X^* \varphi_k = \alpha_k \varphi_k.
\]

(iii) If \( \mathcal{F}_\psi \) and \( \mathcal{F}_\varphi \) are biorthogonal, then \( \sigma(X) = \sigma_p(X) = \{ \alpha_k; k \in \mathbb{N} \}. \)

**Proof.** (i): By (1) of Lemma 4.1 it follows immediately that if \( X \) is bounded, then \( \{ \alpha_k \} \) is bounded too and \( \sup_k |\alpha_k| \leq \|X\|. \) On the other hand, if \( \{ \alpha_k \} \) is bounded, we have

\[
\left\| X \sum_k \beta_k \psi_k \right\| = \left\| \sum_k \alpha_k \beta_k \psi_k \right\| \leq \sup_k |\alpha_k| \|\beta_k \psi_k\|.
\]

The density of \( D_\psi \) in \( \mathcal{H} \) implies the statement.

(ii): The proof of this statement easily follows from the definition of \( X \) and from the biorthogonality of \( \mathcal{F}_\psi \) and \( \mathcal{F}_\varphi. \)

(iii): By (ii), the numbers \( \{ \alpha_k \} \) are eigenvalues of \( X. \) The spectrum \( \sigma(X) \) consists exactly of these eigenvalues. Indeed, if \( \lambda \in \mathbb{C} \setminus \{ \alpha_k; k \in \mathbb{N} \}, \) then defining

\[
Z = \sum_{k=0}^{\infty} \frac{1}{\alpha_k - \lambda} R_k,
\]

we obtain

\[
(X - \lambda I) Z \xi = (X - \lambda I) \sum_{k=0}^{\infty} \frac{1}{\alpha_k - \lambda} R_k \xi
\]

\[
= \sum_{m=0}^{\infty} (\alpha_m - \lambda) \sum_{k=0}^{\infty} \frac{1}{\alpha_k - \lambda} R_m R_k \xi
\]

\[
= \sum_{k=0}^{\infty} R_k \xi = \xi.
\]
By (i) $Z$ is bounded and has a continuous extension to $\mathcal{H}$. Thus $X - \lambda \mathbb{I}$ has a bounded inverse. \hfill \Box

A slightly weaker result can be deduced if we require, in $X$, that the two sets $\mathcal{F}_\psi$ and $\mathcal{F}_\varphi$ are biorthogonal but not necessarily complete in $\mathcal{H}$. With a similar argument as before we conclude that, if $X$ is bounded, then $\{\alpha_k\} \in l^\infty(\mathbb{N})$, but the vice-versa does not hold, in general. The Riesz-like nature of the sets is not important, here.

4.1. Consequences of Definition 4.2. We want to focus now on some consequences of Definition 4.2. In particular, we will show that if $S, T \in \mathcal{L}^\dagger(D, \mathcal{H})$ satisfy a nonlinear CR, then an interesting structure arises, under suitable assumptions.

First, we observe that, if $\varphi_k$ is such that both $T\varphi_k$ and $S\varphi_k$ belong to $D$, then

$$ST\varphi_k - TS\varphi_k = \alpha_k \varphi_k.$$ 

Indeed from (4.7) we deduce that $\langle (ST\varphi_k - TS\varphi_k) | \eta \rangle = \alpha_k \langle \varphi_k | \eta \rangle$ for all $\eta \in D$, so that the above result follows.

In a similar way we can prove that, if $\psi_k$ is such that both $T^\dagger \psi_k$ and $S^\dagger \psi_k$ belong to $D$, then $T^\dagger S^\dagger \psi_k - S^\dagger T^\dagger \psi_k = \alpha_k \psi_k.$

Raising and lowering operators play a crucial role in connections with bosons and their generalizations. Hence it is natural to check whether they can be introduced also in this settings. Indeed, a first simple raising and lowering property can be stated as follows: suppose $S\varphi_0 = 0$. Then $T\varphi_0 \neq 0$ and $T\varphi_0$ is an eigenvector of the (formal) operator $N_l := TS^\dagger$ with eigenvalue $\alpha_0$. Moreover, if $T\varphi_0 \in D$, then $\varphi_0$ is eigenvector of the (formal) operator $N_r := ST$ with the same eigenvalue, $\alpha_0$. For convenience we also put $N^\#_l := S^\dagger T^\star$ and $N^\#_r := T^\dagger S^\star$. Analogously, if $T^\dagger \psi_0 = 0$, then $S^\dagger \psi_0 \neq 0$ and $S^\dagger \psi_0$ is an eigenvector of $S^\dagger T^\star$ with eigenvalue $\alpha_0$. Moreover, if $S^\dagger \psi_0 \in D$, then $\psi_0$ is eigenvector of $T^\dagger S^\star$ with the same eigenvalue, $\alpha_0$.

A more interesting result is given by the following Proposition:

**Proposition 4.5.** Suppose that $S\varphi_0 = 0$ and that all the $\alpha_k$’s are different. Then the following statements are all equivalent.

1. $X^\dagger(T\varphi_n) = \alpha_{n+1}(T\varphi_n)$, for all $n \in \mathbb{N}$.
2. For every $n \in \mathbb{N}$, there exists $\gamma_n \in \mathbb{C}$ such that $T\varphi_n = \gamma_n \varphi_{n+1}$.
3. For every $n \in \mathbb{N}$, there exists $\beta_n \in \mathbb{C}$ such that $T^\dagger \psi_n = \beta_n \psi_{n-1}$, where $\beta_0 := 0$.

In this case $\beta_n = \frac{\gamma_n}{\alpha_{n-1}}$ for all $n \geq 1$.

**Proof.** Before starting, we observe that the relevant situation is when the $\gamma_n$’s are not zero: if they are zero the statements (1), (2) and (3) are trivially equivalent. Then, this is the situation we will consider here.
(1)⇒(2): By taking the scalar product of both sides of the equality in (1) with a generic \(\psi_l\), we deduce that
\[
(\alpha_{n+1} - \alpha_l) \langle T\varphi_n | \psi_l \rangle = 0,
\]
for all possible \(l\). Since the \(\alpha_k\)'s are different, if \(l \neq n + 1\), the vector \(T\varphi_n\) must be orthogonal to \(\psi_l\), \(\forall l \in \mathbb{N} \setminus \{n + 1\}\). Hence, due to the uniqueness of the biorthogonal basis \([9]\), \(T\varphi_n\) is necessarily proportional to \(\varphi_{n+1}\). Then (2) follows.

(2)⇒(3): For that we first observe that, using (2) and the biorthogonality condition \(\langle \varphi_n | \psi_m \rangle = \delta_{n,m}\), \(\langle T\varphi_n | \psi_l \rangle = \gamma_{l-1} \delta_{n,l-1}\). But, since \(\langle T\varphi_n | \psi_l \rangle = \langle \varphi_n | T^\dagger \psi_l \rangle\), we get, for \(l \geq 1\),
\[
\langle \varphi_n | (T^\dagger \psi_l - \gamma_{l-1} \psi_{l-1}) \rangle = 0, \quad \forall n \in \mathbb{N}.
\]
Then (3) follows from the completeness of \(F_\varphi\) with \(\beta_n = \overline{\gamma_{n-1}}\). Notice also that, if \(l = 0\), (3) is trivially true because of our assumption \(T^\dagger \psi_0 = 0\) and since \(\beta_0 = 0\).

(3)⇒(1): By (3) we get easily the equality \(T\varphi_n = \overline{\beta_{n+1}} \varphi_{n+1}\). Therefore, recalling the expression for \(X^*\), \(T\varphi_n \in D(X^*)\) and
\[
X^*(T\varphi_n) = \left(\sum_{k=0}^{\infty} \alpha_k (\varphi_k \otimes \overline{\psi_k})\right) \overline{\beta_{n+1}} \varphi_{n+1} = \alpha_{n+1} (T\varphi_n).
\]

\[\square\]

Analogous results can be proved for the operator \(S\). Indeed, we have

**Proposition 4.6.** Suppose that \(T^\dagger \psi_0 = 0\) and that all the \(\alpha_k\)'s are different. Then the following statements are all equivalent.

1. \(X(S^\dagger \psi_n) = \alpha_{n+1} (S^\dagger \psi_n)\) for all \(n \in \mathbb{N}\).
2. For every \(n \in \mathbb{N}\), there exists \(\gamma_n \in \mathbb{C}\) such that \(S^\dagger \psi_n = \gamma_n \psi_{n+1}\).
3. For every \(n \in \mathbb{N}\), there exists \(\beta_n \in \mathbb{C}\) such that \(S\varphi_n = \beta_n \varphi_{n+1}\), where \(\beta_0 := 0\).

In this case \(\beta_n = \overline{\gamma_{n-1}}\), for every \(n \geq 1\).

The proof is very similar to the previous one and will not be repeated here. These two propositions, together, have interesting consequences:

**Corollary 4.7.** Suppose that \(S\varphi_0 = 0\), \(T^\dagger \psi_0 = 0\), and that all the \(\alpha_k\)'s are different. Suppose also that \(X^*(T\varphi_n) = \alpha_{n+1}(T\varphi_n)\) and \(X(S^\dagger \psi_n) = \alpha_{n+1}(S^\dagger \psi_n)\), for all \(n \geq 0\). Then, for all \(n \geq 0\),

1. \(N_l \varphi_n = \gamma_{n-1} \gamma_n \varphi_n\) and \(N_l \varphi_n = \gamma_n \gamma_n \psi_n\);
2. \(N_l^\# \psi_n = \gamma_{n-1} \gamma_n \psi_n\) and \(N_l^\# \psi_n = \gamma_n \gamma_n \psi_n\);
3. \(\alpha_n = \gamma_n \gamma_n - \gamma_{n-1} \gamma_{n-1}\).
Proof. We will only prove here statement (3), since the others are easy consequences of the previous Propositions. First of all, from Propositions 4.5 and 4.6, we get
\[ \langle T\varphi_n | S^\dagger \psi_m \rangle = \gamma_n \tilde{\gamma}_n \delta_{n,m}. \]
Moreover
\[ \langle T\varphi_n | S^\dagger \psi_m \rangle = \langle \varphi_n | X\psi_m \rangle + \langle S\varphi_n | T^\dagger \psi_m \rangle = (\alpha_n + \gamma_{n-1} \tilde{\gamma}_{n-1}) \delta_{n,m}, \]
where we have used \( X^* \varphi_n = \alpha_n \varphi_n \). Hence (3) follows.
\[ \square \]

Remark 4.8. Since \( \alpha_n \geq 0 \) for all \( n \), then \( \gamma_n \tilde{\gamma}_n - \gamma_{n-1} \tilde{\gamma}_{n-1} \geq 0 \), for every \( n \geq 0 \). Also, since \( \gamma_{-1} = \tilde{\gamma}_{-1} = 0 \), we find that \( \alpha_0 = \gamma_0 \tilde{\gamma}_0 \), and that, for all \( n \geq 1 \),
\[ \alpha_n = \gamma_n \tilde{\gamma}_n - \sum_{k=0}^{n-1} \alpha_k, \]
which provides a relation between the \( \alpha_k \)'s and the \( \gamma_k \)'s, \( \tilde{\gamma}_k \)'s introduced previously.

We are now ready to show that Propositions 4.5 and 4.6 are somehow related:

**Proposition 4.9.** Suppose that \( S\varphi_0 = 0, T^*\psi_0 = 0 \), and that all the \( \alpha_k \)'s are different. Then \( T\varphi_n = \gamma_n \varphi_{n+1} \), for every \( n \in \mathbb{N} \), if, and only if, \( S\varphi_n = \tilde{\gamma}_n \varphi_{n-1} \), for every \( n \geq 1 \).

Proof. We use induction on \( n \). Let us first suppose that \( T\varphi_n = \gamma_n \varphi_{n+1} \), for all \( n \geq 0 \). Then, in particular, \( \varphi_1 = \frac{1}{\gamma_0} T\varphi_0 \).

Now, taking \( f \in \mathcal{F} \),
\[ \langle S\varphi_1 | f \rangle = \frac{1}{\gamma_0} \langle ST\varphi_0 | f \rangle = \frac{\alpha_0}{\gamma_0} \langle \varphi_0 | f \rangle = \langle \tilde{\gamma}_0 \varphi_0 | f \rangle, \]
so that, because of the arbitrariness of \( f \), \( S\varphi_1 = \tilde{\gamma}_0 \varphi_0 \).

Let us now assume that \( S\varphi_n = \tilde{\gamma}_n \varphi_{n-1} \). We want to check that \( S\varphi_{n+1} = \tilde{\gamma}_n \varphi_n \). In fact, from our hypothesis, we deduce that \( \varphi_{n+1} = \frac{1}{\gamma_n} T\varphi_n \). Therefore,
\[ S\varphi_{n+1} = \frac{1}{\gamma_n} ST\varphi_n = \frac{1}{\gamma_n} (X + T S) \varphi_n. \]
Now, recall that \( X\varphi_n = \alpha_n \varphi_n \). Moreover, using our induction hypothesis, we have \( T S \varphi_n = \tilde{\gamma}_n \varphi_{n-1} = \tilde{\gamma}_{n-1} \varphi_{n-1} \). Hence, by (4.9), we conclude that \( S\varphi_{n+1} = \tilde{\gamma}_n \varphi_n \).

The inverse implication can be proved in a similar way.

\[ \square \]

Adopting the standard notation for nonlinear coherent states we call \( \gamma_n! = \gamma_0 \gamma_1 \cdots \gamma_n \) and \( \tilde{\gamma}_n! = \tilde{\gamma}_0 \tilde{\gamma}_1 \cdots \tilde{\gamma}_n \). Iterating the formulas in the previous Proposition it is easy to find that
\[ \varphi_n = \frac{1}{\gamma_n!} T^n \varphi_0, \quad \psi_n = \frac{1}{\tilde{\gamma}_n!} (S^\dagger)^n \psi_0. \]
All the above formulas provide a rather natural interpretation of $T$, $S$ and their adjoints as lowering and rising operators with respect to two different bases, as usually deduced in pseudo-hermitian quantum mechanics, see [4] and references therein.

Again in connection with pseudo-hermitian quantum mechanics we can also raise the question of the existence of some intertwining operator. For that we introduce two operators, $S_\varphi$ and $S_\psi$, via their action on two generic vectors, $f \in D(S_\varphi)$ and $g \in D(S_\psi)$:

$$S_\varphi f = \sum_{k=0}^{\infty} \langle f | \varphi_k \rangle \varphi_k, \quad S_\psi g = \sum_{k=0}^{\infty} \langle g | \psi_k \rangle \psi_k.$$  

Of course, these operators are densely defined since, in particular, $S_\varphi \psi_k = \varphi_k$ and $S_\psi \varphi_k = \psi_k$, $\forall k$. They are positive and it is easy to check that $S_\varphi S_\psi \varphi_k = \varphi_k$ and $S_\psi S_\varphi \psi_k = \psi_k$, for all $k$. If, furthermore, they are bounded, then one is the inverse of the other: $S_\psi = S_\varphi^{-1}$. This is the case when $F_\varphi$ and $F_\psi$ are Riesz bases. Otherwise this is not ensured, in general. In this more general case the following weak intertwining relations can be deduced

$$N_l S_\varphi \psi_k = S_\varphi N_l^\# \psi_k, \quad N_r S_\varphi \psi_k = S_\varphi N_r^\# \psi_k,$$

and similarly

$$N_l^\# S_\psi \varphi_k = S_\psi N_l \varphi_k, \quad N_r^\# S_\psi \varphi_k = S_\psi N_r \varphi_k,$$

for all $k$. The existence of these relations is not surprising, since it is clearly related to the fact that, for instance, $N_l$ and $N_l^\#$ have equal eigenvalues.

4.2. Connections with the linear case. We end the paper by discussing some relations between the present situation, i.e. the nonlinear case, with the one discussed in [1] and in Sections 2 and 3. In particular, we will show that our previous results could be considered as special cases of the present settings.

The starting point is our Definition 4.2. In order to recover here similar results to those obtained in [1], we assume that a non zero vector $\Phi$ does exist in $\mathcal{D}$ which is annihilated by $S$, $S \Phi = 0$, and such that $T^k \Phi$ exists and is an eigenvector of $X^*$: $X^*(T^k \Phi) = \mu_k (T^k \Phi)$. Under these assumptions we can check that $T^k \Phi$ is an eigenvector of $S^* T$ with same eigenvalue $M_k := \mu_0 + \mu_1 + \cdots + \mu_k$, and that $T^{k+1} \Phi$ is an eigenvector of $T S^* T$ with the same eigenvalue, $M_k$. The proof, which can be given by induction on $k$, is easy and will not be given here. More interesting is, we believe, to notice that these assumptions are satisfied whenever we are in the situation briefly considered at the beginning of Section 4. In fact, in this case, it is enough to take $\Phi = \varphi_0$ and $T = b$. Hence, since $T^k \Phi = b^k \varphi_0 = \sqrt{\epsilon_k} \varphi_k$, using the explicit expression for $X$ we deduce that $X^*(T^k \Phi) = \alpha_k (T^k \Phi)$.

Secondly, we give the following result.
Proposition 4.10. Let us assume that $S, T \in \mathcal{L}(\mathcal{D}, \mathcal{H})$ satisfy Definition 4.2. Let $n \in \mathbb{N}$, $n \geq 1$, and $\xi$ a vector in $\mathcal{D}$ such that $T^k \xi \in \mathcal{D}$ for $k \leq n$. Then,

(i) $S \xi \in D((T^*k)k)$, for $k \leq n$

(ii) $X^*T^l \xi \in D((T^*m)^m)$, for all $l, m$ such that $l + m = k - 1$, $k \leq n$

(iii) the following equality holds

\begin{equation}
ST^k \xi - (T^*k)^k S \xi = \sum_{l=0}^{k-1} (T^*l)^{k-1-l} X^* T^l \xi, \quad \forall k \leq n.
\end{equation}

Proof. The proof is given by induction on $n$.

For $n = 1$ the statements follow immediately from Definition 4.2.

Let us assume that (i), (ii), (iii) hold for $n$ and let $k \leq n + 1$. If $k \leq n$, then the statements follow by the induction assumptions. Thus we need to prove it only for $k = n + 1$. Assume then that also $T^{n+1} \xi \in \mathcal{D}$. Then the vector $\xi' = T \xi \in \mathcal{D}$ satisfies $T^k \xi' \in \mathcal{D}$ for $k \leq n$. Thus the induction assumptions apply to $\xi'$. Therefore $ST \xi \in D((T^*k)k)$, for $k \leq n$; $X^*T^{l+1} \xi \in D((T^*m)^m)$, for all $l, m$ such that $l + m = k - 1$, $k \leq n$ We prove that equation (4.10) holds for $n + 1$. Indeed, we have

\begin{equation*}
ST^{n+1} \xi = ST^n (T \xi) = (T^*n)^n S (T \xi) + \sum_{l=0}^{n-1} (T^*l)^{n-1-l} X^* T^l (T \xi) =
\end{equation*}

\begin{equation*}
= (T^*n)^n (X^* \xi + T^*n S \xi) + \sum_{l=0}^{n-1} (T^*l)^{n-1-l} X^* T^{l+1} \xi,
\end{equation*}

from which formula (4.10) for $n + 1$ follows. \qed

Remarks:-- (1) Notice that, if $X = \mathbb{1}$, i.e. if $\alpha_k = 1$ for all $k$ in the definition of $X$, we recover Propositions 3.2 and 3.4 of [1]. In particular, (4.10) becomes $ST^k \xi - (T^*k)^k S \xi = k^k T^{k-1} \xi$.

(2) If, rather than this, we simply assume that $[X^*, T] \xi = 0$ for all $\xi \in \mathcal{D}$, and that $T^l (X^* \xi) \in \mathcal{D}$ for all $l$, the right-hand side of (4.10) becomes $k X^* T^{k-1} \xi$, which again, returns the previous result when $X = \mathbb{1}$.

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