Comment on "Dynamics of the Density of Quantized Vortex-Lines in Superfluid Turbulence"

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In the paper by Khomenko et al. [Phys. Rev. B 91, 180504 (2015)] the authors, analyzing numerically the steady counterflowing helium in inhomogeneous channel flow, concluded that the production term \( \mathcal{P} \) in the Vinen equation is proportional to \( |\mathbf{V}_{\text{ns}}|^3 \mathcal{L}^{1/2} \) (where \( \mathcal{L} \) is vortex line density and \( \mathbf{V}_{\text{ns}} \) is the counterflow velocity). In present comment we demonstrated that the procedure, implemented by the authors includes a number of questionable steps, such as a decomposition of velocity of line and interpretation of the flux term. Additionally, the overall strategy - extracting information on the temporal behavior from the stationary solution also remains questionable. Because of that the method of determination of the explicit shape of Vinen equation is very sensitive to the listed elements, the final conclusion of the authors cannot be considered as unambiguous.

I. INTRODUCTION.

The question of dynamics of the vortex line density (VLD) \( \mathcal{L}(\mathbf{r}, t) \) of the vortex tangle (VT) is one of the most sacramental problems in the theory of quantum turbulence. In fact the VLD is a crude characteristic of flow, however it is responsible for many (mainly hydrodynamic) phenomena in superfluid turbulence and knowing its exact dynamics is very important for an adequate interpretation of various experiments.

Long ago Vinen [1] suggested that the rate of change of VLD \( \partial \mathcal{L}(t)/\partial t \) can be described in terms of only quantity \( \mathcal{L}(t) \) itself (and also of other, external parameters, such as counterflow velocity \( \mathbf{V}_{\text{ns}} \) and temperature). He called this statement as a self-preservation assumption. The according balance equation for quantity \( \mathcal{L}(\mathbf{r}, t) \) reads:

\[
\frac{\partial \mathcal{L}(\mathbf{r}, t)}{\partial t} = \mathcal{P}(\mathbf{r}, t) - \mathcal{D}(\mathbf{r}, t).
\]  

(1)

Here \( \mathcal{P}(\mathbf{r}, t) \) is the so called production term, appeared from interaction with external counterflow, \( \mathcal{D}(\mathbf{r}, t) \) is the decay term. We imposed the spacial variable \( \mathbf{r} \), having in mind that we, after the authors of paper [2], will be discuss an inhomogeneous channel flow. The form of the decay term Vinen extracted with the use of some speculative interpretation of various experiments.

II. QUESTIONABLE STEPS

Equation for the length of the vortex-line segment. The starting point for the evolution of the length \( \delta \xi \) of the line element \( s(\xi) \) is the relation

\[
d\delta \xi/dt = (s' \cdot d\delta s/d\xi)\delta \xi.
\]  

(2)

This formula reflects the simple fact that the linear element changes its length due to different velocities \( \delta s \) at the ends of segment. The authors of paper [2] used the following form of this equation

\[
\frac{1}{\delta \xi} \frac{d\delta \xi}{dt} = \alpha V_{\text{ns}}(s, t) \cdot (s' \times s'').
\]  

(3)

In this form, originally proposed by Schwarz [3], formula (3) is valid only in the Local Induction Approximation. Moreover, it is supposed that \( V_{\text{ns}} \) is spatially constant (otherwise nonzero \( dV_{\text{ns}}(s, t)/d\xi \) changes the form of Eq. (3)). It is important for discussed work, where the full Biot-Savart law is applied.
Velocity decomposition. The authors decomposed expression for $V_{ns}(\text{Eqs. (4b), (4c) of paper [2]})$ in the following manner

$$V_{ns} = V_{\text{external}}^{ns} - V_{\text{LIA}}^{s}(s, t) - V_{nl}^{s}, \quad (4)$$

where $V_{\text{external}}^{ns}(s, t)$ is the external counterflow velocity, created, e.g., by heat load, $V_{\text{LIA}}^{s}(s, t)$ and $V_{nl}^{s}(s, t)$ are the local and nonlocal parts of superfluid velocity induced by the VT configuration. Then the authors used the decomposition 4 in the relation 3 to obtain an equation for the VLD evolution. At this stage the authors associated the LIA part $V_{\text{LIA}}^{s}(s, t)$ with the decay term $D(r, t)$, and nonlocal part $V_{nl}^{s}(s, t)$ with the production term $P(r, t)$. The reason of this action is unclear. Indeed, in the Schwarz’s formula 3 the quantity $V_{ns}$ is understood as an external relative velocity, and reflects the fact that the VT length (and, accordingly, its energy) grows due to external source. Eq. (6a) of paper 2 states that the production term works even when the $V_{\text{external}}^{ns}$ is absent. This decisively contradicts to the essence of the Feynman scenario of superfluid turbulence, and can be main reason for new form of the Vinen equation Therefore the presence of nonlocal quantity $V_{nl}^{s}(s, t)$ in the production term looks unmotivated. At the same time I agree that nonlocal velocity results in the growth of the VLD, but this effect should be evaluated in a different manner. It should be accounted by the use of starting Eq. 2. In this case the nonlocal part results in the stretching of lines (even when the mutual friction is absent, $\alpha = 0$), unlike the local part.

Vortex-line density flux. To take into account inhomogeneous flows, the authors introduced the vortex-line density flux $\nabla \cdot J(r, t)$ into the balance equation (see Eq. (3b) of paper 2). The authors associated it with the drift motion of the VT $V_{\text{slip}}$, It is defined in the middle part of Eq. (6c) of paper 2) and coincides with definition proposed by Schwarz (see 0, Eq. (21)). However in Schwarz’s paper the drift motion is associated with the full velocity $\alpha$, whereas the choice of $V_{\text{slip}}$ in paper 2 (right hand side of Eq. (6c) and equation (7) in their work) is not complete, the self-induced motion is missed. But this, missed self-induced motion generates "irreversible" flux of VLD $\mathcal{L}(r, t)$, realized by emission of vortex loops, (see, e.g., 7, 8, 9).

Furthermore, it is obviously that flux of quantity $\mathcal{L}$ must have the structure of sort $\mathcal{L}V$. As a matter of fact this structure presents in latent form in Eq. (6c) of paper 2, where VLD came from integration over $ds$. But later the closure version for flux ( Eq. (10)) was taken that it does not include VLD $\mathcal{L}$ at all. It is not motivated and looks strange.

Summarizing this part of our comments we would like to emphasize, that the adjustment of curves (6a)-(6c), extracted from numerical simulations, to the their closure counterparts (3a-3c, 1c, 3c) crucially is dependent on questionable steps, described above. Therefore the conclusion about new form of production term (3a) cannot be considered as definitely proven.

III. TEMPORAL VS SPACIAL

The second point concerns the principal idea of work 2, which the authors have stated in the abstract of their paper as: ”To overcome this difficulty we announce here an approach that employs an (steady) inhomogeneous channel flow which is excellently suitable to distinguish the implications of the various possible forms of the desired equation.” The applicability of such approach seems to be also questionable. Let me give the nearest and simplest counterexample. Take, for instance some quantity $\Psi(r, t)$. Consider three different equation

$$\dot{\Psi}(r, t) = \dot{\mathcal{F}}(\Psi(r, t)), \quad \Psi(r, t) = K\mathcal{F}(\Psi(r, t)), \quad (5)$$

and $\dot{\Psi}(r, t) = (\dot{\mathcal{F}}(\Psi(r, t)))^2,$

where $\dot{\mathcal{F}}$ is some operator acting on spatial variables, and $K$ is arbitrary number. In the steady case all three equations produce the same function $\Psi_{eq}(r, t)$, which satisfies equation $\dot{\mathcal{F}}(\Psi_{eq}(r, t)) = 0.$ At the same time, in the nonstationary situation the temporal evolution of all quantities $\Psi(r, t)$ (which are not equal to $\Psi_{eq}(r, t)$) will be absolutely different. This counterexample convinces that the way, chosen by the authors of paper 2 is not applicable. Or, any additional argumentation must be presented.

IV. STATIONARY SOLUTION

In fact, problems arise already at a stage of stationary solution. The point is that the equation for vortex line density derived in paper 2 and results on the distribution of quantity $\mathcal{L}(y)$ are inconsistent. Indeed, let’s take the stationary variant of equation for VLD offered in paper 2, (Eq. (3b)) with the “closure” terms (1c), (3a) (3c) and the coefficients $C_{\text{flux}}, C_{\text{prod}}, C_{\text{dec}}$, given in Table I.

$$-\frac{\alpha}{2\kappa}C_{\text{flux}} \frac{\partial^2 V_{ns}^2}{\partial y^2} = \frac{\alpha C_{\text{prod}}}{\kappa^2} \mathcal{L}^{1/2} - \alpha C_{\text{dec}} \mathcal{L}^2. \quad (6)$$

Let’s take further the velocity $V_{ns}(y)$ profile from Fig. 1 of paper 2. It is seen that with the good accuracy it can extrapolated by usual parabolic shape $(\rho/\rho_s V_{ns}(y)^2)$. The final equation is usual algebraic equation of forth order with respect to variable $\mathcal{L}^{1/2}$. The according solution leads to that with the good precision $\mathcal{L}(y) \approx (\alpha C_{\text{prod}}/C_{\text{dec}} \kappa^3)^{2/3} |V_{ns}(y)|^2$, which is very far from $\mathcal{L}(y)$, proposed in paper 2 (Fig. 1, Panels b). This inconsistency demonstrated that either the choice of the closure form is not correct, or something is missed.
V. VINEN EQUATION

The third issue, that I would like to discuss concerns the Vinen-type equation itself and its possible variants. In general, the Vinen equation (in any variant \((2a),(2b),(3a)\)) of production term is not valid. Indeed, assume that one changes velocity \(V_{ns}(s,t)\) instantaneously on the opposite. Since all the listed forms of the Vinen-type equation include the absolute value of relative velocity \(V_{ns}(s,t)\), then, from a formal point of view, nothing will happen. That is wrong, of course. The structure of the VT, mean curvature, anisotropy and polarization parameters will become reorganized. That implies the violation of the self-preservation assumption, and dynamics of the VLD \(L(t)\) depends on other, more subtle characteristics of the vortex structure, different from \(L(t)\).

Meantime, it is intuitively seems truthful, that for slow changes (both in space and in time) the self-preservation assumption takes place. Some justification of this assertion is that the resulting model describes well the experimental observations on the strong heat pulses propagation, which generate quantum vortices and interact with them (see review article [3]). It is understood that the equation \(\partial L(t)/\partial t = \mathcal{F}(L)\) can be used unless we are not interested in the special problems related to the fine structure of the VT.

To clarify situation, let’s consider a way of derivation of VE from the dynamics of vortex filaments in the local induction approximation. It is enough for illustration. Integrating Eq. (3) over \(\xi\) in volume \(\Omega\), we have, that in the counterflowing helium II VLD \(L(t)\) obeys equation [5, 10]) (cf. with Eqs. (6a), (6b) in [2])

\[
\frac{\partial L}{\partial t} = \frac{\alpha V_{ns}}{\Omega} \int (s^l \times s^l)\, d\xi - \frac{\alpha \beta}{\Omega} \int \langle |s|^3 \rangle\, d\xi . \tag{7}
\]

Notation are in [2]. Quantity \(L(t)\) is related to the first derivative of function \(s' (L(t)) \propto \int |s'| \, d\xi '\). The rate of change of \(L(t)\) includes quantities with a higher-order derivative \(s''\), namely \(\langle s'^l \times s'^l \rangle \) and \(\langle |s'|^3 \rangle\). In steady these quantities are related to VLD \(L\), as \(\langle s'^l \times s'^l \rangle \propto L^{1/2}\) and \(\langle |s'|^3 \rangle \propto c_2^2(T)\, L^{1/2}\) (\(c_1, c_2(T)\) is a temperature dependent parameters, introduced by Schwarz [6]). But in the nonstationary situation \(s''\) is a new independent variables, and one needs the new independent equation for it and for other quantities, related to curvature. This new equation, in turn, includes higher derivatives \(s'''\), \(s'^{IV}\) and so on. This infinite chain can be cut if, for some reasons, the higher-order derivatives relax faster, than the low-order derivatives, and take their "equilibrium" values (with respect to the moments of low order). Applying this speculations to equation (7) we are arrive at the the following bifurcation:

1. Time of relaxation of VLD \(L(t)\) is much larger than that for quantities with higher derivatives, and the latter have enough time to adjust to change of \(L(t)\), i.e. \(\langle s' \times s'' \rangle \propto L^{1/2}\). Then, the self-preservation assumption is valid, and generating term has a classical form \(P_v=\alpha V_{ns} L^{3/2}\).

2. Time of relaxation of VLD \(L(t)\) is of the same order as that for quantities with higher derivatives. Then, the self-preservation assumption is not valid, substitution of \(\langle s' \times s'' \rangle \propto L^{1/2}\) is inadmissible, and, there is no theoretical grounds to cut a chain. Thus, in general, no equation of type \(\partial L(t)/\partial t = \mathcal{F}(L, V_{ns}(t))\) exists! At the same time under some (unclear) conditions, and with the use of additional arguments (see. [1]) it is possible to write down the required equation. However in this case, the region of applicability of this equation is not clear, see my example above with the sudden inversion of the counterflow velocity. Therefore, Vinen equation in it classical form should be considered as a good approximation for applied, engineering problems (for instance, in study of propagation of large thermal pulses).

VI. CONCLUSION

Resuming, I would state that the final, sensational conclusion of the authors of [2], asserting that the production term \(P_v=\alpha \beta (V_{ns})^3 L^{1/2}\), cannot be considered as unambiguous. Undoubtedly, the authors raised interesting and actual question of inhomogenous quantum turbulence, but the change the form of the Vinen equation seems premature.

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