PARTNERS: FUNCTIONAL ANALYSIS AND TOPOLOGY

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INTRODUCTION

Functional analysis and topology were born in the first two decades of the twentieth century and each has greatly influenced the other. Identifying the dual space—the space of continuous linear functionals—of a normed space played an especially important role in the formative years of functional analysis. To further this endeavor, many new kinds (weak, strong, etc.) of convergence and compactness were introduced. Metric and general topological spaces evolved in order to provide a framework in which to treat these types of convergence. As general topology gestated, many concepts were greatly clarified and simplified. (For example, “continuous” meant transforming convergent sequences into convergent sequences until about 1935.) These clarifications led to the development of general topological vector spaces in the 1930’s.

BEGINNINGS

As set theory developed at the end of the nineteenth century, its paradoxes revealed that mathematics had a disturbingly shaky foundation. With the aim of placing set theory in particular and mathematics generally on a firmer logical pedestal, Hilbert and others looked to Euclidean geometry for a model. Until that time the objects of mathematical attention had been quite specific: real numbers, complex numbers, curves, surfaces. Something more general was sought this time. As Hilbert commented:

If among my points I consider some systems of things (e.g.,
the system of love, law, chimney sweeps . . . ) and then accept
only my complete axioms as the relationships between these
things, my theorems (e.g., the Pythagorean) are valid for
these things also.

In other words, ignorance of exactly what the objects were was mandatory. “Truth” was banished, replaced by “provability”. The new axiomatic spirit was to consider structures, arbitrary sets equipped with operations

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See Functional Analysis by G. Bachman and L. Narici, Dover, Mineola, New York, 2000, a reprint of the the 1966 Academic Press book of the same title. See also the invited contribution What is functional analysis? by the same author.

1He put it another way at a discussion with some mathematicians in the waiting room of the Berlin railway station. He said about his geometric axioms “One must be able at any time to replace points, lines and planes with tables, chairs and beer mugs.”
that obeyed certain rules. This formalist approach dominated the twentieth century, and is very much still with us.

Various extensions of *limit* and *continuity* to objects other than numbers or points has been with us since the 18th century but their rigorous study—what we might call early “functional analysis”, in the sense of analysis on sets whose members were functions—did not begin until around 1820. Convergence of a sequence of functions meant pointwise convergence. It was soon realized that imposing more uniformity conditions was helpful. Stokes and Seidel (1847–8), for example, discovered that trigonometric series converged with infinitely increasing slowness near a jump discontinuity and that the discontinuity cannot be enclosed in any interval in which the convergence is *von gleichem Grade* (uniformly convergent). Heine proved in 1870 that the Fourier series of a piecewise smooth $2\pi$-periodic function $f$ converges uniformly in any interval that does not contain a discontinuity of $f$; if $f$ is continuous, then its Fourier series converges uniformly and absolutely on every closed interval. In the presence of uniform convergence, certain attributes (notably continuity) of each term of a sequence persist to the limit and series can be integrated term by term. In 1883 Ascoli discovered the disturbing possibility of a sequence of continuous functions to possess a discontinuous (pointwise) limit. He found that this behavior disappeared if the sequence was *equicontinuous* \([1]\). These “uniform” concepts percolated into analysis generally. In the period 1890–1910, still other types of convergence of functions were considered such as *relative uniform* convergence and *weak* and *strong* convergence, the latter notions being from functional analysis in the modern sense of the term.

A comprehensive framework for these different kinds of convergence was evidently desirable. This forced the question: What do you need in order to talk about convergence? Clearly, a notion of nearness is vital. The first attempt was Fréchet’s metric space \([9]\), then there was Hausdorff’s topological space \([12]\). In the first application of this set-with-structure approach, Fréchet plucked what he deemed to be the essential properties of distance in the plane (mainly, just the triangle inequality) and used it to define the metric space. Were the axioms in use today his only choice as the distillate? Or did he experiment with weaker requirements? if so, more spaces are brought into the realm but the number of provable results diminishes. More or stronger conditions? Then there would be more and better theorems about fewer things. (Fréchet also introduced *norm* and the notation $\| \sup \cdot \|$ for it; the formal definition of normed *spaces* was not given until 1920–1922 by Banach, Hahn and Helly, however.) With the perspective of the past century, it is well-nigh incredible how much was deduced from such simple axioms; the same comment of course applies to topological spaces as well. These two structures alone vindicated faith in the axiomatic method, albeit with some degenerate cases of “axiomatics”—defining new things with no other motivation than to prove theorems about them.
Geometry and Duality

In the period 1890–1910, F. Riesz, and E. Schmidt introduced the language of Euclidean geometry ("orthogonal functions and families", "Pythagorean theorem", "space", "dimension", "triangle inequality", etc.) into Hilbert space. Using Lebesgue’s newly minted integral, Fréchet and Riesz commented in 1907 that the space $L^2[a,b]$ of square-integrable functions had a “geometry” analogous to that of “Hilbert space”, i.e., $\ell_2$.

In the same epoch the notions of “functional” (a numerical-valued function whose domain is a set of functions) and “operator” (a function whose domain and range are sets of functions) came into being. This led to the development of duality or topological duality, the study and use of the continuous dual $X'$ of all continuous linear functionals (or “forms”) on a topological vector space $X$. The following developments occurred in the period 1900–1918:

- (1903; cf. [2, pp. 218–227]) In the first formal attempt at describing the topological dual of a normed space, Hadamard seeks to characterize the continuous linear functionals on the sup-normed space $C[a,b]$ of continuous functions on $[a,b]$. Riesz magnificently completes Hadamard’s project in 1909; he shows that every continuous linear form $f$ on $C[a,b]$ may be written as a Stieltjes integral: $f(\cdot) = \int_{[a,b]} \cdot \, dg$, where $g$ is a function of bounded variation on $[a,b]$ whose total variation $V(g) = \|f\|$. In today’s language we say that $C'[a,b] = NBV[a,b]$, where $NBV[a,b]$ denotes the space of normalized functions of bounded variation on $[a,b]$ and = signifies surjective norm-isomorphism.

- (1907; cf. [2, p. 209]) Fréchet and Riesz demonstrate that a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ is self-dual: For each continuous linear form $f$ on $X$, there is a vector $x$ in $X$ such that $f(\cdot) = \langle \cdot, x \rangle$.

- (Riesz 1910; cf. [8, p. 286]) The continuous dual $L'_p[a,b]$ of the space $L^p[a,b]$ (1 < $p < \infty$) of $p$-th power integrable functions on $[a,b]$ is $L_q[a,b]$ where $1/p + 1/q = 1$. The analogous result for $\ell_p$ follows in 1913. (The discoveries about $L_p[a,b]$ led directly to the general notion of normed space.)

- (Riesz 1911, inspired by a boundedness notion of Hilbert’s; cf. [2, p. 209]) A linear functional $f$ is continuous if and only if $f$ is “bounded” in the sense that there is some $M$ such that $|f(x)|$ is at most $M\|x\|$ for all $x$.

- (Steinhaus 1918; cf. [8, p. 289]) $L'_1[a,b] = L_\infty[a,b]$, the space of measurable functions $f$ on $[a,b]$ such that, given $f$, there is some $M$ such that $|f(x)|$ is at most $M$ for almost all $x$ in $[a,b]$ (f is essentially bounded).

To further these investigations in duality, strong use was made of weak compactness, weak and strong convergence, relative uniform convergence, and complete continuity (mapping weakly convergent sequences into strongly
convergent ones, as Riesz originally used the term). Some cracks in the metric space approach to provide a common framework for the various kinds of convergence were visible almost immediately. Hausdorff’s remedy was a more general approach to “nearness”. Inspired by Hilbert’s axioms of open neighborhoods for the plane, he defined the general topological space in 1914 [12, Chapters 7–9].

1. FROM METRIC TO TOPOLOGICAL VECTOR SPACES

With the appearance of Banach’s book [3] in 1932, metric functional analysis (normed, Hilbert and Fréchet spaces) had come into its own. Its stature was elevated when Hilbert space proved to be a felicitous home for quantum mechanics. But even before 1930 it was known that pointwise convergence, convergence in measure and compact convergence eluded description by means of a norm. The treatment of these things in linear spaces had to await the introduction of locally convex spaces (von Neumann and Kolmogorov, 1935). It was time for topology to inspire functional analysis and it was progress in general topology throughout 1930–1940 that enabled the transition from metric linear spaces to topological vector spaces. With the locally convex space and von Neumann and Kolmogorov’s notion of bounded set (one which is contained in a sufficiently large scalar multiple of any neighborhood of 0), duality theory was transmogrified in the works of Mackey [14, 15], and Grothendieck [10, 11]. These changes led to Schwartz’s theory of distributions [18].

For further remarks on the developments during this formative period, see [7] and the historical remarks in [4].

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