EXTREMUM ESTIMATES OF THE $L^1$-NORM OF WEIGHS
FOR EIGENVALUE PROBLEMS OF VIBRATING STRING
EQUATIONS BASED ON CRITICAL EQUATIONS

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(Communicated by Susanna Terracini)

Abstract. The present paper is concerned with the extremal problem of the
$L^1$-norm of the weights for non-left-definite eigenvalue problems of vibrating
string equations with separated boundary conditions. Applying the critical
equations of the weights, the infimum is obtained in terms of the given eigen-
value and the parameter in boundary conditions.

1. Introduction. The present paper is concerned with Lyapunov-type inequalities
for the boundary value problem associated to the vibrating string equations

$$- y''(x) = \lambda w(x)y(x), \quad x \in [0,1]$$

and the boundary condition

$$y(0) = 0 = y(1) - hy'(1), \quad 0 < h < 1,$$

where $0 \neq \lambda \in \mathbb{R}$, $w \in L^1[0,1]$. Here, $w$ is allowed to change its sign on $[0,1]$ in the
meaning of that

$$\text{mes}\{x : w(x) > 0, \quad x \in [0,1]\} > 0 \quad \text{and} \quad \text{mes}\{x : w(x) < 0, \quad x \in [0,1]\} > 0,$$

$\lambda$ may be positive or negative. Denote by $\sigma(w)$ the spectrum of (1) and (2). Set

$$\Omega(\lambda) = \left\{ w \in L^1[0,1] : \lambda \in \sigma(w) \cap \mathbb{R}, \quad \int_0^1 |w| > 0 \right\}.$$ (3)

The expression of the infimum function $E(\lambda, h)$ of $\|w\|_1$ is given if (1) and (2) has
a nontrivial solution, that is

$$\int_0^1 |w| \geq E(\lambda, h)$$

2020 Mathematics Subject Classification. Primary: 34B24; Secondary: 34L15.

Key words and phrases. Lyapunov-type inequality, extremum, vibrating string equations, critical
equations, eigenvalue.

The first author is supported by the NSF of China grants 11771253, 11971262 and 61977043.
The second author is supported by NSF of the Shandong Province grants ZR2019MA038,
ZR2019MAO50 and ZR2017MA049.

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and the inequality is sharp in the meaning of that
\[ E(\lambda, h) = \inf \{ ||w||_1 : w \in \Omega(\lambda) \} . \] (5)
Clearly, the above result is a generalization of the classical Lyapunov inequality, which states that if \(-y''(x) = w(x)y(x), y(0) = 0 = y(1)\) has a nontrivial solution, then
\[ \int_0^1 |w| > 4. \]
But the situation in the present paper is essentially different from the classical one. From the spectral theory point of view, the eigenvalue problem
\[-y''(x) = \mu w(x)y(x), \ x \in [0, 1], \ y(0) = 0 = y(1), \] (6)
is left-definite, but the problem (1) and (2) is non-left-definite. This fact results in that all eigenvalues of (6) are positive and there exists a negative eigenvalue of (1) and (2) if \(w(x) > 0, \ a.e. \ x \in [0, 1] \).
Lyapunov’s inequality was raised by Lyapunov almost 160 years ago, and the first proof was given by Borg in [1]. Because of its importance in many fields, Lyapunov-type inequalities have been generalized to many other types of differential (or difference) equations (or systems), and the reader is referred to the summary book [13].
The topic of this paper is related to the extremal problems in spectral theory, and the similar study for classical Sturm-Liouville eigenvalue problems goes back to the famous problem of Lagrange(see [4]). Krein in his classical work [11] investigated the maximum and the minimum of the frequencies of the harmonic oscillation of the string under unit tension and fixed at its end points. There are a lot of such research works in this direction (see [7, 10, 14, 17, 18]).
Indeed there are closed relations between Lyapunov’s inequalities and the extremal problems in spectral theory. In the recent works [5, 15], the extremal problems of \(L^1\)-norm for the potentials in Sturm-Liouville eigenvalue problems were solved by using the improved Lyapunov’s inequalities.
In the present paper we will set up the Lyapunov-type inequality (4) for (1) and (2) by solving the extremal problem (5). But the method used here is different from existing ones, see [5, 6, 15] in which Mercer theorem plays an important role based on the positivity of the Green function of
\[-y'' = 0, \ y(0) = 0 = y(1). \]
But Mercer theorem can’t be used to solve this problem because of the non-positivity of Green function of
\[-y'' = 0, \ y(0) = 0 = y(1) - hy'(1), \ h \in (0, 1). \]
Another method to solve the extremal problem is to use the critical equations in \(L^p[0, 1]\) for \(p > 1\), which were early used by the authors in [16] and [19] for Sturm-Liouville operators, and recently were set up in [8] for elliptic operators, where the existence and uniqueness of the critical point were proved if one eigenvalue is given. We attempt to apply the critical equation of weights (see (11)) to solve the extremal problem (5). Generally, the critical equations are difficult to solve directly because of its nonlinearity. We will carefully analyze the properties of the critical points to make clear the limitation of \(L^p\)-norm of the critical points as \(p \to 1\).
Following this section, we first solve the extremal problem (5) in Theorem 2.7 for a special case in Section 2 based on the critical equation (11), where some
extremum estimates of weights based on critical equations 3

properties of the critical points are presented in this section. The main result is given by Theorem 3.2 in Section 3, which gives the answer to the extremal problem (5).

2. The critical equation method for the case \( \lambda < 0 \) and \( w \geq 0 \). In this section, we will solve the extremal problem (5) for the case

\[
\lambda < 0, \; w(x) \geq 0, \; \text{a.e. } x \in [0, 1]
\]  

(7)

by using the critical equations of \( w \) in \( L^p[0, 1] \) for \( p > 1 \), see (11).

To begin with we indicate that there must exist a negative eigenvalue of

\[-y'' = \mu wy, \; y(0) = 0 = y(1) - hy'(1)\]  

under the condition that \( w(x) \geq 0, \; \text{a.e. } x \in [0, 1] \) and \( h \in (0, 1) \).

**Lemma 2.1.** Suppose that \( w \in L^1[0, 1], \; w \geq 0, \; \text{a.e. } x \in [0, 1] \) and \( \|w\|_1 > 0 \). Then (8) has exactly one negative eigenvalue if and only if \( h \in (0, 1) \).

**Proof.** Consider the eigenvalue problem

\[-y'' = -\rho^2 y, \; y(0) = 0 = y(1) - hy'(1), \; \rho > 0.\]  

(9)

It has two linearly independent solutions \( \exp(\rho x) \) and \( \exp(-\rho x) \). Then \(-\rho^2\) is a negative eigenvalue of (9) if and only if

\[e^\rho - e^{-\rho} - h\rho(e^\rho + e^{-\rho}) = 0.\]

We claim that the mapping \( h: \; \rho \in (0, \infty) \to h(\rho) \in (0, 1) \) defined as

\[h(\rho) = \frac{e^{2\rho} - 1}{\rho(e^{2\rho} + 1)}\]

is bijective. Clearly, \( h(0+) := \lim_{\rho \downarrow 0} h(\rho) = 1 \), \( h(\infty) := \lim_{\rho \uparrow \infty} h(\rho) = 0 \), hence it is sufficient to prove \( h'(\rho) < 0, \; \rho \in (0, \infty) \). This fact can be proved by a series of calculations as follows,

\[h'(\rho) = \frac{-e^{4\rho} + 4\rho e^{2\rho} + 1}{\rho^2(e^{2\rho} + 1)^2} =: \frac{f(\rho)}{\rho^2(e^{2\rho} + 1)^2},\]

\[f(0+) = 0, \; f'(\rho) = 4e^{2\rho}(1 + 2\rho - e^{2\rho}) =: 4e^{2\rho}g(\rho),\]

\[g(0+) = 0, \; g'(\rho) = 2 - e^{2\rho} < 0, \; \rho \in (0, \infty).\]

Therefore, the conclusion of Lemma 2.1 is true for (9).

Since the number of negative eigenvalues of (8) is the same as that of (9), the conclusion of Lemma 2.1 is valid.

Now, we turn to estimate the minimum of \( \|w\|_p^p = \int_0^1 w^p \) in \( L^p[0, 1](p > 1) \) based on the critical equations. Recall the definition of \( \Omega(\lambda) \). For \( p > 1 \), set

\[\Omega_p(\lambda) = \{ w \in L^p[0, 1] : \; w \in \Omega(\lambda), \; w \geq 0, \; \text{a.e. } x \in [0, 1] \}.\]  

(10)

By Pontryagin method, one sees that if \( w \) is a critical point, say \( w_p \) of \( \{ \int_0^1 w^p : \; w \in \Omega_p(\lambda) \} \), then \( w_p^{p-1}(x) = \phi_p^p(x) \) and \( \phi_p \) satisfies

\[-\phi_p'' = \lambda w_p\phi_p = \lambda \phi_p^{p+1}, \; \phi_p(0) = 0 = \phi_p(1) - h\phi_p'(1).\]  

(11)

The existence and uniqueness of the critical point \( w_p \) or \( \phi_p \) of (11) are guaranteed by the results in [16] and [8, 9]. In fact, for \( p > 1 \), \( L^p[0, 1] \) is a reflexive space and we can get \( \Omega_p(\lambda) \) is a closed set, hence Eberlein–Smulian theorem (see [2, p.163])
ensures that there is a sequence \( \{w_n, n = 1, 2, \cdots \} \subset \Omega_p(\lambda) \) weakly convergent to \( w_p \in \Omega_p(\lambda) \), and
\[
\|w_n\|_p^p \to \|w_p\|_p^p = \inf \left\{ \int_0^1 w^p : w \in \Omega_p(\lambda) \right\}.
\]

One can prove that the \( w_p \) is the critical point of (11) and the uniqueness follows from the properties of convex sets and convex functions. For more details, the reader is referred to [8, The proof of Theorem 1].

By the definition of \( \Omega_p(\lambda) \), we know \( w_p \neq 0 \) or \( \phi_p \neq 0 \) a.e. \( x \in [0, 1] \). Therefore, without loss of generality, we can choose \( \phi_p \) such that \( \phi_p(0) > 0 \). It follows from Lemma 2.1 that \( \lambda < 0 \) is the first eigenvalue and hence the corresponding eigenfunction \( \phi_p \) has no zero on \( (0, 1] \). As a result \( \phi_p(x) > 0, \phi_p'(x) > 0, \phi_p''(x) \geq 0 \) on \( (0, 1] \).

**Lemma 2.2.** \( \|w_p\|_p \) is non-decreasing for \( p \in (1, \infty) \) and the set \( \{\|w_p\|_p : p \in (1, \infty)\} \) has positive lower bound.

**Proof.** Since \( \int_0^1 w_p^p \) is the the minimum of \( \{\int_0^1 w^p : w \in \Omega_p(\lambda)\} \), for \( 1 < p < q \), it follows from \( L^q[0, 1] \subset L^p[0, 1] \) that \( \int_0^1 w_p^p \leq \int_0^1 w_q^q \). Hence
\[
\|w_p\|_p = \left( \int_0^1 w_p^p \right)^{1/p} \leq \left( \int_0^1 w_q^q \right)^{1/p} \leq \left( \int_0^1 w_q^q \right)^{1/q} = \|w_q\|_q,
\]
by Hölder inequality or Jensen’s inequality. Note that \( \phi_p'(x) > 0 \) on \( [0, 1] \). Set
\[
u(x) = \phi_p(x)/\phi_p'(x).
\]
Then \( u \) satisfies \( u(0) = 0, u(1) = h, u(x) > 0 \) on \( (0, 1] \) and
\[
u'(x) = 1 + \lambda w_p(x)u^2(x), \ x \in [0, 1].
\]
Clearly \( \nu'(x) \leq 1 \), and hence \( u(x) \leq x \). Therefore, it follows from
\[
-\lambda \int_0^1 w_p(x)u^2(x)dx = 1 - u(1) = 1 - h
\]
that \( \int_0^1 w_p \geq (1 - h)/|\lambda| \), and hence
\[
\|w_p\|_p = \left( \int_0^1 w_p^p \right)^{1/p} \geq \int_0^1 w_p > (1 - h)/|\lambda|,
\]
which implies that \( \{\|w_p\|_p : p \in (1, \infty)\} \) has positive lower bound.

Applying the boundedness of \( \{\int_0^1 w_p^p : p \in (1, 2]\} \), one can prove the following lemma.

**Lemma 2.3.** The set \( \{\phi_p(x), \phi_p'(x) : x \in [0, 1], p \in (1, 2]\} \) is bounded.

**Proof.** Since \( \phi_p'(x) \geq \phi_p'(0) \) and
\[
\phi_p(x) = \int_0^x \phi_p'(t)dt \geq x\phi_p'(0),
\]
one sees that the set \( \{\phi_p'(0) : p \in (1, 2]\} \) is bounded by
\[
\int_0^1 w_p^p(x)dx = \int_0^1 \phi_p^2/(p-1)(x)dx \geq (\phi_p'(0))^{2p/(p-1)} \frac{p-1}{3p-1}
\]
Multiplying two sides of the critical equation (11) by \( \phi_p'(x) \) and integrating over \([0, x]\) give the first integral

\[
(\phi_p'(x))^2 = \frac{(p-1)|\lambda|}{p} w_p^n(x) + (\phi_p'(0))^2. \tag{12}
\]

Multiplying two sides of the critical equation (11) by \( \phi_p(x) \) and integrating by parts on \([0, 1]\) yield that

\[
h(\phi_p'(1))^2 = \int_0^1 (\phi_p'(x))^2 dx + |\lambda| \int_0^1 w_p(x) \phi_p^2(x) dx. \tag{13}
\]

Here, \( \phi_p(1) = h\phi_p'(1) \) is used. Inputting (12) to (13) yields that

\[
h(\phi_p'(1))^2 = \frac{2p-1}{p} |\lambda| \int_0^1 w_p^n(x) dx + (\phi_p'(0))^2,
\]

and hence \( \{\phi_p'(1) : p \in (1, 2]\} \) is bounded from the boundedness of \( \{\int_0^1 w_p^n : p \in (1, 2]\} \) and \( \{\phi_p'(0) : p \in (1, 2]\} \). Then it follows from \( \phi_p'(x) \leq \phi_p'(1) \) that \( \{\phi_p(x), \phi_p'(x) : x \in [0, 1], p \in (1, 2]\} \) is bounded. \( \square \)

The following result indicates that \( \phi_p'(x) \) has a positive lower bound in a neighborhood of the right end point 1.

**Lemma 2.4.** There exist \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) such that

\[ \phi_p'(x) \geq \varepsilon_0, \forall x \in [1 - \delta_0, 1], \forall p \in (1, 2]. \]

**Proof.** For otherwise, there exist \( x_n \in [1 - 1/n, 1], p_n \in (1, 2] \) such that \( \phi_p'(x_n) < 1/n (n \geq 1) \), which implies that \( \phi_p'(x_n) < 1/n \) for \( x \in [0, x_n] \), and hence

\[
\phi_{p_n}(x) = \int_0^x \phi_{p_n}'(t) dt < \frac{x_n}{n}, \quad x \in [0, x_n].
\]

On the other hand, for \( x \in [x_n, 1] \), since \( \phi_p'(x) \) is bounded on \([0, 1]\) for \( p \in (1, 2] \), say \( \phi_p'(x) \leq M' \), one sees that

\[
\phi_{p_n}(x) = \phi_{p_n}(x_n) + \int_{x_n}^x \phi_{p_n}'(t) dt < \frac{x_n}{n} + (1 - x_n) M' < \frac{1 + M'}{n}.
\]

The above fact implies that \( \phi_{p_n}(x) \) converges to 0 uniformly on \([0, 1]\), and hence

\[
\int_0^1 w_{p_n}^{p_n}(x) dx = \int_0^1 \phi_{p_n}'(x)^2 dx \to 0
\]

as \( n \to \infty \), which contradicts the the positive lower boundness of \( \{||w_p||_p : p \in (1, 2]\} \) in Lemma 2.2. \( \square \)

**Lemma 2.5.** Let \( x_p \) be the unique root of the equation \( \phi_p(x) - 1 = 0 \). Then there exists sufficient small \( \eta > 0 \) such that the equation \( \phi_p(x) - 1 = 0 \) has a solution \( x_p \in (0, 1) \) for \( p \in (1, 1 + \eta) \), and it holds that \( x_p \to 1 \) as \( p \to 1+ \).

**Proof.** In fact, if there exist \( p_n \to 1 \) as \( n \to \infty \) such that \( \phi_{p_n}(x) \leq 1 \) on \([0, 1]\) for \( n > 1 \), then for \( x \in [1 - \delta_0, 1] \),

\[
\phi_{p_n}(x) = \phi_{p_n}(1) - \int_x^1 \phi_{p_n}'(t) dt \leq 1 - \varepsilon_0 (1 - x),
\]

as \( n \to \infty \), which contradicts the the positive lower boundness of \( \{||w_p||_p : p \in (1, 2]\} \) in Lemma 2.4. \( \square \)
where \(\varepsilon_0\) and \(\delta_0\) are defined as in Lemma 2.4, and hence \(\phi_{p_n}^{2p_n/(p_n-1)}(x) \to 0\) as \(n \to \infty\). By the monotonicity of \(\phi_{p_n}(x)\), one sees that \(\phi_{p_n}^{2p_n/(p_n-1)}(x) \to 0\) as \(n \to \infty\) on \([0, 1]\). Since \(0 \leq \phi_{p_n}^{2p_n/(p_n-1)}(x) \leq 1\), by the dominant convergence theorem we have

\[
\int_0^1 w_{p_n}^p(x)dx = \int_0^1 \phi_{p_n}^{2p_n/(p_n-1)}(x)dx \to 0
\]
as \(n \to \infty\), which contradicts the positive lower boundedness of \(\{\int_0^1 w_p^p : p \in (1, 2]\) in Lemma 2.2. This implies the existence of \(x_n \in \phi_{p_n}\) in Lemma 2.4. Then for \(x \in [1 - \delta/2, 1]\)

\[
\phi_{p_n}(x) = \phi_{p_n}(1 - \delta) + \int_{1-\delta}^x \phi_{p_n}'(t)dt \geq 1 + \varepsilon_0(x + \delta - 1) \geq 1 + \varepsilon_0\delta/2,
\]
and hence

\[
\int_0^1 w_{p_n}^p(x)dx \geq \int_{1-\delta/2}^{1-\delta/2} w_{p_n}^p(x)dx \geq \int_{1-\delta/2}^{1-\delta/2} (1 + \varepsilon_0\delta/2)^{2p_n/(p_n-1)}dx \to \infty,
\]
which contradicts the boundedness of \(\{\int_0^1 w_p^p : p \in (1, 2]\) in Lemma 2.2 again. \(\square\)

Applying Lemma 2.5, one can prove that the limitation of \(w_p\) should be a Dirac-Delta distribution at \(x = 1\).

**Lemma 2.6.** Let \(w_p\) be the critical point of (11). Then \(w_p(x) \to 0\) as \(p \to 1+\) for \(x \in [0, 1]\). Furthermore, it holds that

\[
|\lambda| \lim_{p \to 1+} \int_0^1 w_p^p = 1/h - 1, \quad \|w_p\|_p \geq \frac{1 - h}{|\lambda|h}. \quad (14)
\]

**Proof.** Let \(\phi_p\) or \(w_p\) be the critical point. For \(x \in [1 - \delta_0, 1]\), since \(x_p \to 1\) as \(p \to 1+\), there exists sufficiently small \(\eta > 0\) such that \(x + \eta < x_p\) for \(p \in (1, 1 + \eta)\). As a result

\[
\phi_p(x) = \phi_p(x_p) - \int_x^{x_p} \phi_p'(t)dt \leq 1 - \varepsilon_0(x_p - x) \leq 1 - \eta \varepsilon_0,
\]
where \(\delta_0, \varepsilon_0\) and \(x_p\) are defined as in Lemma 2.4 and 2.5, respectively. And hence \(w_p(x) = \phi_p^{2/(p-1)}(x) \to 0\) as \(p \to 1+\). It follows from the monotonicity of \(\phi_p\) that \(w_p(t)\) converges to 0 as \(p \to 1+\) uniformly on \([0, x]\) for every \(x \in (0, 1)\).

It follows from \(-\phi_p''(x) = \lambda w_p(x)\phi_p(x), \phi_p(0) = 0 = \phi_p(1) - h\phi_p'(1)\) that

\[
\phi_p(x) = \phi_p'(0)x - \lambda \int_0^x (x - t)\phi_p^{p-1}(t)dt,
\]
and hence

\[
\phi_p(1) = \phi_p'(0) - \lambda \int_0^1 (1 - t)\phi_p^{p-1}(t)dt.
\]
Since \(w_p(x) \to 0\) as \(p \to 1+\) uniformly on every compact subinterval of \([0, 1]\), we have

\[
\lim_{p \to 1} \int_0^1 (1 - t)\phi_p^{p+1}(t)dt = \lim_{p \to 1} \int_0^1 (1 - t)w_p^{(p+1)/2}(t)dt = 0,
\]
and hence $\phi_p(1) - \phi_p'(0) \to 0$ as $p \to 1^+$. On the other hand

$$\phi_p(1) - 1 = \phi_p(1) - \phi_p(x_p) = \int_{x_p}^1 \phi_p'(t)dt \leq M'(1 - x_p) \to 0$$

as $p \to 1$, and hence $\phi_p(1), \phi_p'(0) \to 1^+$ as $p \to 1^+$, where $M'$ is the bound of $\phi_p'$ in Lemma 2.4. Replacing $\phi_p'(1)$ in (13) with $\phi_p(1)/h$ we have

$$-\frac{1}{h} \phi_p^2(1) + \int_0^1 |\phi_p'(x)|^2dx = \lambda \int_0^1 w_p^p(x)dx. \tag{15}$$

Integrating two sides of the first integral (12) over $[0,1]$ one sees that

$$\int_0^1 |\phi_p'(x)|^2dx = \frac{(p-1)|\lambda|}{p} \int_0^1 w_p^p(x)dx + (\phi_p'(0))^2.$$

Taking the righthand term of the above equality into (15) we get

$$\frac{(2p-1)|\lambda|}{p} \int_0^1 w_p^p(x)dx = \frac{1}{h} \phi_p^2(1) - (\phi_p'(0))^2.$$

Letting $p \to 1^+$ one finds that

$$|\lambda| \int_0^1 w_p^p(x)dx \to \frac{1}{h} - 1.$$

This together with the monotonicity of $\|w_p\|_p$ in Lemma 2.2 means that

$$\|w_p\|_p \geq \frac{1 - h}{|\lambda|h},$$

and hence (14) is valid.  

We now turn to minimize the set \{$\int_0^1 w : 0 \leq w \in \Omega(\lambda)$\} in the following result.

**Theorem 2.7.** Let $\lambda < 0$ be given and $\Omega(\lambda)$ be defined as in (3). Then

$$\inf \left\{ \int_0^1 w : w \in \Omega(\lambda), w \geq 0, \text{ a.e. } x \in [0,1] \right\} = \frac{1 - h}{h|\lambda|}. \tag{16}$$

**Proof.** We first prove that for all $0 \leq w \in \Omega(\lambda)$,

$$\int_0^1 w \geq \frac{1 - h}{h|\lambda|}. \tag{17}$$

For fixed $0 \leq w \in \Omega(\lambda)$, set $A_n = \{x \in [0,1] : |w(x)| \leq n\}$ and

$$w_n(x) = \chi(A_n)(x)w(x), \hspace{1cm} x \in [0,1],$$

where $\chi(A_n)$ is the indicative function of $A_n$. Clearly $w_n \to w$ in $L^1[0,1]$ as $n \to \infty$. It follows from the continuity of eigenvalues on the weights that $\lambda_n(w_n) \to \lambda$ as $n \to \infty$. Note that $w_n \in L^p[0,1]$ for $p > 1$, and hence Lemma 2.6 gives that for fixed $n$,

$$|\lambda_1(w_n)| \int_0^1 w_n^p(x)dx \geq |\lambda_1(w_n)| \int_0^1 w_p^p(x)dx = \frac{1 - h}{h} + o(1), p \to 1^+, \tag{17}$$

where $w_p$ is the critical point of (11) with $\lambda$ replaced by $\lambda_1(w_n)$. Choosing $p_n = 1 + 1/n$, we claim that $w_n^{p_n} \to w$ in $L^1[0,1]$ as $n \to \infty$. Since

$$\int_0^1 |w_n^{p_n}(x) - w(x)|dx \leq \int_0^1 |w_n^{p_n}(x) - w_n(x)|dx + \int_0^1 |w_n(x) - w(x)|dx,$$
we only need to prove that
\[ \int_0^1 w_n(x) \sqrt{w_n(x)} - 1 \, dx \to 0, \quad n \to \infty. \]

From the definition of \( w_n \), one sees from \( \sqrt{\lambda} < 2 \) that
\[ w_n(x) \sqrt{w_n(x)} - 1 \leq w(x), \quad x \in [0, 1]. \]
Therefore, it follows from the dominant convergence theorem that
\[ \int_0^1 w_n(x) \sqrt{w_n(x)} - 1 \, dx \to 0, \quad n \to \infty. \]

Let \( p = p_n \) in (17) and let \( n \to \infty \), one sees that
\[ \int_0^1 w(x) \, dx \geq \frac{1 - h}{\| \lambda \|}. \]

Now we prove that the inequality is sharp. In fact, there exists a sequence \( w_n \in \Omega(\lambda) \) such that \( |\lambda|h \int_0^1 w_n \to 1 - h \).

Set \( w(x) = \frac{1}{\sqrt{\lambda}} \delta(x - 1) \), where \( \delta(\cdot - a) \) is the Dirac distribution at a point \( a \in (0, 1] \). One can verify that (1) and (2) has a nontrivial solution \( \phi(x) = x \).

Remark that the term \( y'(1) \) appeared in the boundary condition \( y(1) - hy'(1) = 0 \) should be understood in the meaning of \( y'(1+) \). Since the Dirac distribution can be approximated by the functions of \( L^1[0, 1] \) in the weak* topology of \( C[0, 1] \), and eigenvalues are continuous on the weak* topology, see [19, Theorem 4.1 (ii)]. This proves that the constant \( \frac{1 - h}{\| \lambda \|} \) is indeed the minimum. Here, for details of Dirac distribution and weak* topology, the reader is referred to [3, 12].

\[ \square \]

3. The solution of the extremal problem. In this section we will give the answer to the extremal problem (5). We need the following lemma to prove the main result in this section. Set \( w_+ := w \vee 0 \) and \( w_- := w_+ - w \). And let \( w_+ \neq 0 \) denote that \( \| w_+ \|_1 > 0 \) or \( \text{mes}\{w > 0\} > 0 \).

**Lemma 3.1.** Consider the problem (1) and (2) with \( w \in L^1[0, 1] \).

(i) Suppose that \( w_+ \neq 0 \) and \( h \in (0, 1) \). If (1) and (2) with \( \lambda < 0 \) has nontrivial solution \( \phi(x) \neq 0 \) on \( (0, 1) \), then
\[ \int_0^1 |w| \geq \frac{1 - h}{h|\lambda|}. \] (18)

(ii) If \( h \geq 1 \) and (1) and (2) with \( \lambda > 0 \) has nontrivial solution on \( (0, 1) \), then \( 0 < \mu_1(|w|) \leq \lambda \), where \( \mu_1(|w|) \) is the first eigenvalue of
\[ -y'' = \mu_1|w|y, \quad y(0) = y(1) - hy'(1). \] (19)

**Proof.** (i) We first prove that if (1) and (2) has a nontrivial solution, then there exists \( 0 > \mu_1(w_+) \geq \lambda \) such that the problem
\[ -y'' = \mu_1(w_+)w_+y, \quad y(0) = y(1) - hy'(1) \] (20)
has a nontrivial solution. Clearly \( (\lambda, \phi) \) is an eigen-pair of
\[ -y'' + \lambda w_-y = \gamma w_+y, \quad y(0) = y(1) - hy'(1). \] (21)
Since \( \phi \) has no zero on \( (0, 1) \), it follows from the spectral theory of regular eigenvalue problem that \( \gamma_1(w_+) = \lambda \). Since \( \lambda < 0 \), one sees from (20) and (21) that \( 0 >
\[ \mu_1(w_+) \geq \gamma_1(w_+) = \lambda \] by Min-max principle. From now on, the case is just the one studied in Section 2, and hence we have
\[ \int_0^1 |w| \geq \int_0^1 w_+ \geq \frac{1 - h}{h|\mu_1(w_+)|} \geq \frac{1 - h}{h|\lambda|}. \]

(ii) First of all \( w_+ \neq 0 \) by Lemma 2.1. For otherwise the eigenvalue problem
\[ -y'' = \mu w_-y, \quad y(0) = y(1) - hy'(1) \]
has a negative eigenvalue \( -\lambda \), which contradicts Lemma 2.1 since \( h \geq 1 \). Consider the eigenvalue problem (21). Since \( \lambda > 0 \) is its eigenvalue, one sees that \( 0 < \gamma_1(w_+) \leq \lambda \), and hence \( 0 \leq \mu_1(|w|) \leq \mu_1(w_+) \leq \gamma_1(w_+) \leq \lambda \) by Min-max principle again.

The main result of this paper is given in the following theorem.

**Theorem 3.2.** If (1) and (2) has non-trivial solution for \( 0 \neq \lambda \in \mathbb{R} \), \( w \in L^1[0,1] \) and \( h \in (0,1) \), then
\[ E(\lambda, h) = \frac{1}{|\lambda|} \min \left\{ \frac{1 - h}{h}, \frac{4}{1 - h} \right\}. \] (22)

**Proof.** Without loss of generality we suppose that \( \lambda > 0 \). For otherwise one can replace \( w \) by \(-w\). Let \( \phi(x) \) be a non-trivial solution of (1) and (2) and \( x_0 = 0 < x_1 < \cdots < x_n < 1 \) be its all zeros, \( n \geq 0 \).

If \( n = 0 \), or \( \phi(x) \) has no zero on \((0,1]\), then one can prove that \( w_- \neq 0 \). In fact, if \( w_- = 0 \), then \( w(x) \geq 0 \) a.e. \( x \in [0,1] \), and hence the eigenvalue problem
\[ -y'' = \mu wy, \quad y(0) = y(1) - hy'(1), \]
has an eigenvalue \( \mu = \lambda > 0 \) with the corresponding eigenfunction \( \phi(x) \) having no zero on \((0,1]\). Since the first eigenvalue must be negative by Lemma 2.1, \( \lambda \) is not the first one, and hence its corresponding eigenfunction \( \phi(x) \) should have at least one zero on \((0,1]\), which is a contradiction. Set \( \bar{\lambda} = -\lambda < 0 \) and \( \bar{w} = -w \). Then the problem
\[ -y'' = \bar{\lambda} \bar{w}y, \quad y(0) = y(1) - h\bar{y}'(1), \]
has nontrivial solution with \( \bar{\lambda} < 0 \) and \( \bar{w}_+ \neq 0 \). It follows from Lemma 3.1 that
\[ \int_0^1 |w| \geq \frac{1 - h}{\lambda h}. \] (23)

If \( n \geq 1 \), then the classical Lyapunov inequality yields that
\[ \int_{x_{j-1}}^{x_j} |w| > \frac{4}{\lambda(x_j - x_{j-1})}, 1 \leq j \leq n. \] (24)

On the interval \([x_n,1]\), \( \phi \) satisfies
\[ -y'' = \lambda wy, \quad y(x_n) = 0 = y(1) - hy'(1). \]
The above eigenvalue problem is equivalent to the problem
\[ -u'' = \tilde{\lambda} \tilde{w}u, \quad u(0) = 0 = u(1) - \tilde{h}u'(1), \quad \tilde{w}(t) = w(x) \]
with \( \tilde{\lambda} = (1 - x_n)^2 \lambda \) and \( \tilde{h} = h/(1 - x_n) \) by the transformation
\[ t = \frac{x - x_n}{1 - x_n}, \quad u(t) = y(x), \quad x \in [x_n,1], \quad t \in [0,1]. \]
If \( \hat{h} \in (0, 1) \), then the case is just the former one since \( \hat{\phi}(t) = \phi(x) \) has no zero on \((0, 1]\), and hence we have
\[
\int_0^1 |\hat{\omega}(t)| dt \geq \frac{1 - \hat{h}}{\lambda} h = \frac{1 - x_n - h}{\lambda h(1 - x_n)^2},
\]
and hence
\[
\int_{x_n}^1 |w(x)| dx = (1 - x_n) \int_0^1 |\hat{\omega}(t)| dt \geq \frac{1 - x_n - h}{\lambda h(1 - x_n)}.
\]
This inequality together with (24) yields that
\[
\int_0^1 |w(x)| dx = (1 - x_n) \int_0^1 |\hat{\omega}(t)| dt \geq \frac{1 - x_n - h}{\lambda h(1 - x_n)}.
\]
by the harmonic inequality. Note that \( \hat{h} < 1 \), or \( x_n \in (0, 1 - h) \). It is easy to verify that the function
\[
f(x) = \frac{4}{x} + \frac{1 - x - h}{h(1 - x)}, \quad x \in (0, 1 - h]
\]
takes its minimum \( f_{\text{min}} = 4/(1 - h) \) at \( x = 1 - h \), and hence we have
\[
\int_0^1 |w| > \frac{4}{\lambda(1 - h)}. \tag{25}
\]
If \( \hat{h} \geq 1 \), or \( h \geq 1 - x_n \), then \( 0 < \mu_1(|\hat{\omega}|) \leq \hat{\lambda} = (1 - x_n)^2 \lambda \) by (ii) of Lemma 3.1, where \( \mu_1(|\hat{\omega}|) \) is the first eigenvalue of
\[
-\hat{u}'' = \mu |\hat{\omega}| u, \quad u(0) = 0 = u(1) - \hat{h}u'(1).
\]
Let \( G(t, s) \) be the Green function of
\[
-\hat{u}'' = 0, \quad u(0) = 0 = u(1) - \hat{h}u'(1).
\]
Then
\[
g(t) := G(t, t) = \frac{1 - x_n}{h - 1 + x_n} \left( t + \frac{h}{1 - x_n} - 1 \right), \quad t \in [0, 1].
\]
Clearly \( g(t) > 0 \) on \((0, 1)\) and \( g_{\text{max}} = h/(h - 1 + x_n) \). It follows from Mercer Theorem, see Theorem 2.1 and Lemma 2.2 in [5], that
\[
\int_0^1 g(t) |\hat{\omega}(t)| dt = \sum_{j=1}^{\infty} \frac{1}{\mu_1(|\hat{\omega}|)} > \frac{1}{\hat{\lambda}} = \frac{1}{(1 - x_n)^2 \lambda},
\]
and hence
\[
\int_{x_n}^1 |w(x)| dx = (1 - x_n) \int_0^1 |\hat{\omega}(t)| dt > \frac{h - 1 + x_n}{\lambda h(1 - x_n)}.
\]
This inequality together with (24) yields that
\[
\int_0^1 |w| > \sum_{j=1}^{n} \frac{4}{\lambda(x_j - x_{j-1})} + \frac{h - 1 + x_n}{\lambda h(1 - x_n)}.
\]
The harmonic inequality yields that
\[
\int_0^1 |w| > \frac{4n^2}{\lambda x_n} + \frac{h - 1 + x_n}{\lambda h(1 - x_n)} = \frac{1}{\lambda} \left( \frac{4n^2}{x_n} + \frac{1}{1 - x_n} - \frac{1}{h} \right)
\]
\[
\geq \frac{1}{\lambda} \left( \frac{4}{x_n} + \frac{1}{1 - x_n} - \frac{1}{h} \right).
\]
Set \( f(x) = 4/x + 1/(1 - x) - 1/h \) for \( x \in [1 - h, 1] \). Then
\[
f_{\text{min}} = \begin{cases} 
9 - 1/h, & h \in [1/3, 1]; \\
4/(1 - h), & h \in (0, 1/3],
\end{cases}
\]
and hence we have
\[
\int_0^1 |w| > 1 \lambda \begin{cases} 
9 - 1/h, & h \in [1/3, 1]; \\
4/(1 - h), & h \in (0, 1/3].
\end{cases}
\]
(26)

It follows from (23), (25) and (26) that
\[
\int_0^1 |w| \geq 1 \lambda \min \left\{ \frac{1 - h}{h}, \frac{4}{1 - h}, f_{\text{min}} \right\}.
\]
(27)

Since \( 9 - 1/h > (1 - h)/h \) as \( h \in [1/3, 1) \), the above equality is reduced to
\[
\int_0^1 |w| \geq 1 \lambda \max \left\{ \frac{1 - h}{h}, \frac{4}{1 - h} \right\}
\]
by resetting the sign of \( \lambda \), which means
\[
E(\lambda, h) \geq 1 \lambda \min \left\{ \frac{1 - h}{h}, \frac{4}{1 - h} \right\}.
\]
(28)

To prove the “=” holds one can verify that for
\[
w(x) = \frac{4}{\lambda(1 - h)} \delta \left( x - \frac{1 - h}{2} \right),
\]
the problem has a nontrivial solution
\[
\phi(x) = \begin{cases} 
x, & x \in [0, (1 - h)/2], \\
1 - h - x, & x \in [(1 - h)/2, 1]
\end{cases}
\]
for the case \( 4/(1 - h) < (1 - h)/h \). The case of \( 4/(1 - h) \geq (1 - h)/h \) has been proved in Theorem 2.7. This completes the proof of Theorem 3.2.

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Received for publication November 2019.

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