MONOPOLES AND CONFINING STRINGS IN QCD

M.N. Chernodub\textsuperscript{a}, F.V. Gubarev\textsuperscript{a,b}, M.I. Polikarpov\textsuperscript{a}, V.I. Zakharov\textsuperscript{b}

\textsuperscript{a} Institute of Theoretical and Experimental Physics, B.Cheremushkinskaya 25, Moscow, 117259, Russia

\textsuperscript{b} Max-Planck Institut für Physik, Föhringer Ring 6, 80805 München, Germany

Abstract

We review the recent results in the physics of the magnetic monopoles in gluodynamics and a dual formulation of non-Abelian theories, relevant to the physics of the confinement. It occurs that the dual gluon is a U(1) gauge boson, despite of the fact that usual gluons are non-Abelian. The effective infrared Lagrangian for gluodynamics is suggested which leads to the Casimir scaling of the string tension for quarks in various representations. We also show that the results of the calculations in lattice gauge theories confirm our theoretical predictions.

1 Introduction

Magnetic monopoles were introduced theoretically by Dirac about 70 years ago \cite{1}. By definition, the radial magnetic field of the monopole is similar to the electric field of a point-like charge:

\[ \vec{H}_{\text{Coul}}^\text{Coul} = \frac{Q_m\vec{r}}{4\pi r^3}, \]  

(1)

where \( Q_m \) is the magnetic charge. However, the field (1) by itself is inconsistent with the Maxwell equation \( \text{div}\vec{H} = 0 \). To ensure the conservation of the magnetic flux Dirac postulated, therefore, that the magnetic field is transported to the monopole from infinity through a string:

\[ \vec{H} = \vec{H}_{\text{Coul}}^\text{Coul} + \vec{H}_{\text{Str}}^\text{Str}, \quad \vec{H}_{\text{Str}}^\text{Str} = -Q\hat{z}\delta(x)\delta(y)\theta(z), \]  

(2)

where the Dirac string is directed along the negative z axis and \( \hat{z} \) is the unit vector. The corresponding monopole gauge potential has a simple form in the spherical coordinates:
\[ A_r = A_\theta = 0, \quad A_\phi = \frac{Q_m (1 + \cos \theta)}{4\pi} \frac{r}{r \sin \theta}. \]  

Thus, monopoles cannot be introduced without a string attached to it and, at first sight, any similarity between the electric and magnetic charges is lost. The only way to nevertheless maintain the similarity between the two types of charges is to ensure that the Dirac string is in fact unobservable. Thus, Dirac derived a few constraints on the theory which should be satisfied to make the monopoles effectively particle-like, not string-like objects. Theoretically, the Dirac monopole and the Dirac strings are fascinating subjects, for review see, e.g., [2, 3]. Still, they look exotic objects with infinite energy and have never been observed.

The saga of the monopoles took a new turn with formulation of the (dual) superconductor model of confinement [4]. It is well known that the color states are not observable as free particles since the potential between two static quarks grows linearly with the distance \( r \) at large \( r \):

\[ \lim_{r \to \infty} V_{q\bar{q}}(r) = \sigma \cdot r \]

Note that the potential \( V_{q\bar{q}} \) is defined in a gauge invariant way in terms of the Wilson loop \( W_C \) (for a pedagogical discussion and further references see, e.g., Ref. [5]). Namely,

\[ V_{q\bar{q}} = - \lim_{T \to \infty} \frac{1}{T} \log < W_{\Gamma_0} >, \]  

where

\[ W_C = Tr P \exp \left( i \oint_C A_\mu dx_\mu \right) \]  

and \( A_\mu \) is the non-Abelian gauge potential while \( \Gamma_0 \) is a rectangular contour \( r \times T \).

Searching for an analogy to this spectacular phenomenon of confinement, one can notice that if magnetic monopoles, as test particles would be placed into a superconductor the heavy monopole potential would grow linearly as well. Indeed the magnetic field cannot penetrate the superconductor and would stream into an Abrikosov vortex [6]. The superconductor model of confinement (for review see, e.g., [4, 5]) assumes that the color electric field of the heavy quarks is organized into a similar tube-like structure in the vacuum state of the gluodynamics. For the model to realize, one needs condensation of the magnetic monopoles in the QCD vacuum.

Although the model looks very appealing and, in a way, no alternative to it has ever been found, it poses quite a few questions which seem very difficult to answer. First of all, we consider now a non-Abelian theory while the (dual) superconductor model of the confinement copies the electrodynamics, that is a \( U(1) \) theory. Moreover, there are no stable classical monopole solutions to the pure Yang-Mills equations with a finite energy [8] and the condensation of the monopoles is difficult to visualize.

The scope of this review does not allow even to mention the milestones on the road to the resolution of the puzzles brought by the model [1]. To make the story very short, the monopoles were copiously observed and the dual superconductor model of confinement
has been confirmed, for review see, e.g., [9]. However, there was a price to be paid. Just to mention a few points: the lattice regularization and numerical methods seem crucial, as well as the choice of a particular gauge.

The need for the lattice regularization in the ultraviolet might look most surprising since the motivation to introduce the monopoles was to explain the confinement, i.e. the basic feature of the physics in the infrared. However, there are good reasons for this. Indeed, the Dirac strings attached to the monopoles are infinitely thin in the continuum limit. Therefore, the Dirac constraints on the strings can be checked only within a particular ultraviolet regularization. Moreover, the gauge potentials associated with the monopoles are singular both at the string and at small \( r \), see, e.g., Eq. (3). In other words the monopoles, also in non-Abelian theories represent topological defects and the potentials are to be regularized in the ultraviolet.

Moreover, the use of the numerical simulations is not only due to the complexity of the vacuum state but, even more so, due to the lack of understanding of a single monopole configuration. Indeed, classically only the monopoles with the minimal magnetic charge (in the units of the Dirac quantization condition) are stable and have an infinite action [8]. The “lattice monopoles” observed in the numerical simulations would correspond to unstable classical filed configurations and cannot be, therefore, understood quasiclassically. In this respect the lattice monopoles in non-Abelian theories are very different from instantons in case of the same non-Abelian theories or Dirac monopoles in the compact \( U(1) \) [10].

The use of the specific gauges also appears as a precondition for the success of the dual superconductor model. Partly, this can be understood theoretically. Indeed, it was realized long time ago [11] that the monopoles can emerge as effective dynamical degrees of freedom only if one fixes gauge up to a remaining \( U(1) \) symmetry. However, the numerical simulations indicate strongly that various ways of fixing the remaining \( U(1) \) are not equivalent to each other at all as far as the phenomenology is concerned. The so-called Maximal Abelian gauge turns to be most successful [12, 9].

Schematically, the idea of the monopole condensation was formulated to explain the basic feature of the QCD as we know it in the real world, that is confinement. The idea was brought to the form suitable for the lattice simulations and dramatically confirmed through such simulations. However, the monopoles observed on the lattice look quite specific just for the lattice formulation and we need now to address anew the continuum theory to incorporate the lessons learnt through the simulations. One can say that the pendulum is swinging now in the opposite direction: from the lattice formulation towards the continuum theory. Broadly speaking, this review is summarizing the recent developments in this direction. While the review is based mostly on the original papers in Ref [13], it is worth emphasizing that there are other papers devoted to the same or related topics, see, e.g., [14, 15, 16]. Thus, it appears quite a common trend and we believe that work in the direction of incorporating lessons from the lattice studies into the continuum theory would continue in the future.

More specifically, we will concentrate on the question how to bridge the \( U(1) \) nature of the monopoles with the observation that they play a key role in the dynamics of non-
Abelian theories. This double-face nature of the monopoles is revealed most straightforwardly through study of the heavy monopole potential \( V_{m\bar{m}}(r) \), or the so called ’t Hooft loop \[17\]. The ’t Hooft loop allows to introduce a pair of (infinitely heavy) external monopoles, like the Wilson loop \[3\] allows to introduce a pair of external (infinitely heavy) quarks. It is worth emphasizing already at this early point that the monopoles introduced via the ’t Hooft loop have the minimal magnetic charge \( Q_m = 1 \) in the units of the Dirac quantization condition while the lattice monopoles living in the vacuum have \( Q_m = 2 \). In particular, the Dirac string corresponding to the \( Q_m = 1 \) monopoles is not visible to the gluons but would be visible to quarks while the Dirac string attached to the \( Q_m = 2 \) monopoles is not visible both to particles in the adjoint and fundamental representations of the color group.

The ’t Hooft loop operator is most easily formulated on the lattice \[18\]. The monopoles are understood then as the end-points of the Dirac strings which in turn are defined as piercing negative plaquettes (see Section 3.2 for a detailed discussion). In the language of the continuum theory it would be natural to define the ’t Hooft loop in terms of the dual gluon field \( B_\mu \):

\[
H_C \equiv \exp \left( i \frac{2\pi}{g} \int_C B_\mu dx_\mu \right),
\]

(6)

Then the heavy monopole potential is defined similarly to (4):

\[
V_{m\bar{m}} = - \lim_{T \to \infty} \frac{1}{T} \log \langle H_{r_0} \rangle.
\]

(7)

The central point is of course, how to relate the “dual gluon” to the fields entering the standard Lagrangian of the non-Abelian gauge theory. And this is one of the central issues which we will try to clarify here following \[13\].

Generally, a dual gauge boson is understood as a field interacting with the magnetic current \( j_m \):

\[
L_{\text{int}} = Q_mB \cdot j_m.
\]

(8)

One of our basic points is that the field \( B \) should be treated as fundamental in gluodynamics since the external monopoles introduced via the ’t Hooft loop are point like in the continuum limit. A well known example of introduction of a dual gauge boson is the Zwanziger Lagrangian describing electrodynamics with both electric and magnetic point like charges \[19\]. Formally one introduces two vector fields, the “standard” photon \( A_\mu \) and the dual photon \( B_\mu \), so that \( L_{\text{int}} = Q_e A \cdot j_e + Q_m B \cdot j_m \). However, the number of the degrees of freedom is not changed since there is a constraint that the field strength tensor constructed on the potential \( A \) coincides with the dual field strength tensor constructed on the potential \( B_\mu \). More precisely:

\[
m_\mu F_{\mu\nu}(B) = m_\mu^* F_{\mu\nu}(A)
\]

(9)
where $m_\mu$ is an arbitrary space-like vector. The choice of the vector $m_\mu$ is a kind of new gauge freedom. Physically, one can visualize $m_\mu$ as the vector along which all Dirac strings are directed.

In case of the gluodynamics, the dual gluon is still a $U(1)$ gauge boson despite of the fact the direct gluons $A^a$ are in an adjoint representation \[13\]. For simplicity we concentrate on the $SU(2)$ case. Then the generalization of (9) looks as:

$$m_\mu F_{\mu\nu}(B) = m_\mu^*(n^a F^a_{\mu\nu}(A))$$

where $F^a_{\mu\nu}$ is the non-Abelian field strength tensor and $n^a$ is an arbitrary vector in the color space. Again, the choice of $n^a$ is a matter of gauge fixing and the physical results do not depend on this choice. One can visualize the vector $n^a$ as the direction in the color space of the magnetic fields transported along the Dirac strings attached to the monopoles.

One can derive a Zwanziger-type Lagrangian which ensures the validity of the constraint (10), see Section 4.3. This formulation allows then to make predictions for the heavy monopole potential (7). In particular, at short distances the potential can be derived from the first principles:

$$V_{m\bar{m}}(R) = -1 \cdot \frac{Q_m^2}{4\pi R}, \quad Q_m \cdot g = 2\pi$$

where $g$ is the standard coupling entering the non-Abelian Lagrangian. To appreciate the meaning of (11) it is instructive to compare it with the standard prediction for the heavy quark potential:

$$V_{q\bar{q}}(R) = -\frac{3}{4} \cdot \frac{g^2}{4\pi R}. \quad (12)$$

Note that the overall coefficient in front of the potential, that is $3/4$, reflects the non-Abelian nature of the quarks which belong to a fundamental representation. There is no such factor in case of the monopole potential (11). Thus, monopoles in $SU(2)$ gluodynamics are Abelian!

On the other hand, the non-Abelian nature of the theory gets manifested in the running of the coupling. In the both cases (12) and (14) the quantum corrections result in $g^2 \to g^2(R)$. Although the final result is similar, the derivation is actually different. Namely, in case of the radiative corrections to (12) one deals of course with a usual field theory. In case of (14), performing the regularization procedure one should address anew the role of the Dirac string attached to the heavy monopoles and check that this procedure is respecting the Dirac constraints. This is one of most subtle and exciting points and we are going to discuss it in details in Sections 4.2, 4.4. It is worth emphasizing that there appearing first direct measurements of the $V_{m\bar{m}}$ on the lattice [20] so that the prediction (11) can be checked through measurements [13].

The Abelian nature of the monopoles poses another serious problem which is the theoretical understanding of the so called Casimir scaling [21]. First, let us remind the reader what the Casimir scaling is. One can measure the heavy quark potential not only for the sources in the fundamental representation, as we discussed so far, but in any other
representation as well. At very large distances the potentials are expected to depend crucially on the representation. For example, if the quarks have isospin \( T = 1 \) then at large distances the confining string connecting the quarks should break into two mesons and the potential, respectively, is expected to flatten out. However at all the distances available for the lattice measurements so far an approximate equation for \( V_{qq} \) holds \([22], [23]\):

\[
V_T(R) \approx -T(T+1) \frac{\alpha_s}{\pi R} + T(T+1) \sigma R,
\]

where the string tension \( \sigma \) does not depend on the isospin \( T \). The Casimir scaling, at least at first sight, makes Abelian dominance very questionable. Indeed, in, say, \( T=1 \) representation one quark is neutral with respect to any singled out \( U(1) \) subgroup of the \( SU(2) \) and this quark should have escaped any interaction, in bald violation of (13).

We will argue \([13]\) that in the approach where the choice of the \( U(1) \) subgroup is a matter of the gauge fixing, see above, the problem of the Casimir scaling can be settled. Within this approach the heavy quark potential is:

\[
V_T(R) \approx -T(T+1) \frac{\alpha_s}{\pi R} + T(T+1) \sigma_T(m_H, m_V) R,
\]

where \( m_H, m_V \) are the parameters of the dual model. The Casimir scaling corresponds then to a particular limit, \( m_H/m_V \gg 1 \). In Section 4.6 we will discuss this problem in detail, along with other related phenomenological issues.

In Section 5 we show that the results of the numerical calculations in lattice gauge theories confirm the dual superconductor model of the gluodynamic vacuum. Moreover the recent study of the confining string structure and monopole structure support the theoretical ideas given in this review.

The review can be subdivided into two parts: Sections 2–3 contain the introductory remarks and main definitions, the advanced topics are discussed in Sections 4–5.

2 Monopoles as Classical Solutions

In this section, which is pure pedagogical in nature we summarize the facts known about the magnetic monopoles viewed as solutions to the classical equations of motion. We consider both electrodynamics and gluodynamics.

2.1 Dirac Veto, Dirac Quantization Condition

As is mentioned in the Introduction, there is a number of constraints imposed on the theory to ensure that the Dirac string does not produce any physical effect. First, there is the Dirac veto which forbids any direct interaction with the string. The condition is not trivial at all, in fact. Indeed, let us consider perturbation theory and the scattering of a charged particle off the string. In perturbative approach one uses the basis of plane waves. Which implies that the particles can be found at any space-point and their wave
functions overlap with the string. Which is in violation of the Dirac veto. Thus, a simple
minded Born approximation can be misleading. Of course, this consideration alone does
not yet allow to judge whether this overlap is significant numerically. But a little bit
more involved estimates do indicate that it is important and the perturbation theory
can and does bring wrong results. We shall examine these issues in much more detail in
Sections 4.2, 4.4 devoted to the radiative corrections and here just state that a consistent
treatment of the Dirac string is always non-perturbative in nature because of the Dirac
veto. This simple observation allows to resolve many theoretical puzzles.

The best known constraint on the Dirac string seems to be the Dirac quantization
condition which ensures absence of the Aharonov-Bohm effect for the electrons scattered
off the string:

$$Q_e \oint_{\text{string}} A \cdot dx = Q_e Q_m = 2\pi k,$$

(15)

where $Q_e$ is the electric charge of the electron and $k$ is an integer number. Note that
as a result of the quantization condition the interaction of the monopoles is described
by a large coupling constant if the electrons interact weakly and vice versa. Also, the
interaction of an electron with the monopole is always governed by a coupling of order
unit and is never weak. This is another reason for the use of non-perturbative methods
in the theory of monopoles, which is of course always a challenge.

### 2.2 Energy of the Dirac String

Unlike the quantization condition (15), the energy of the Dirac string is rarely discussed
in the literature. It is worth to emphasize, therefore, that naively the energy of the string
is infinite in the ultraviolet.

Indeed, let us estimate the energy of the string:

$$\epsilon_{\text{string}} \sim \int (H^{\text{Str}})^2 d^3r \sim \frac{(\text{Length})}{Q^2_e(\text{Area})} \sim Q_e^{-2} \Lambda_{UV}^2(\text{Length}).$$

(16)

We accounted here for the fact that the magnetic flux is quantized (see above),

$$(\text{flux}) \equiv \int H \cdot ds = \frac{2\pi k}{Q_e}$$

and tended the cross section of the string denoted by (Area) to zero at the end of the
calculation. Thus, we substituted (Area)$^{-1}$ by $\Lambda_{UV}^2$.

The radial part of the magnetic field is also associated with an infinite energy:

$$\epsilon_{\text{rad}} \sim \int (H^{\text{Coul}})^2 d^3r \sim \frac{1}{r_0} \sim \Lambda_{UV}.$$  

(17)

Note that this ultraviolet divergence is linear, i.e. weaker than the divergence due to the
string, see Eq. (16).
The infinite magnetic field of the string may have more subtle manifestations as well. Consider interaction of two magnetic monopoles with magnetic charge $\pm Q_m$ placed at distance $R$ from each other. Then, by the analogy with the case of two electric charges, we would like to have the following expression for the interaction energy:

$$\epsilon_{\text{int}} = \int H_1^{\text{Coul}} H_2^{\text{Coul}} d^3r = -\frac{Q_m^2}{4\pi R}.$$  

Note, however, that if we substitute the sum of the radial and string fields for $H_1^{\text{Coul}}$, we have an extra term in the interaction energy:

$$\tilde{\epsilon}_{\text{int}} = \int (H_1^{\text{Coul}} H_2^{\text{Str}} + H_1^{\text{Str}} H_2^{\text{Coul}}) d^3r = +2 \frac{Q_m^2}{4\pi R}.$$  

In other words, the account of the string field would flip the sign of the interaction energy! This contribution, although looks absolutely finite, is of course a manifestation of the singular nature of the string magnetic field, $|H^{\text{Str}}| \sim (\text{Flux})/(\text{Area})$. Note that the integral in (19) does not depend on the shape of the string.

To maintain the unphysical nature of the Dirac string we should use a regularization scheme which would allow to get rid of these singularities.

### 2.3 Lattice Regularization in Abelian Theories

Since the monopoles are naively having divergent energy (or action) in the ultraviolet, the regularization is a crucial issue. Moreover, we would like to follow the lattice formulation since the monopoles are observed on the lattice. And for a good reason, as we shall immediately see.

Consider first the $U(1)$ case. As is emphasized in Ref. [10], the lattice formulation implies that a Dirac string which produces no Aharonov-Bohm scattering costs no action as well. The reason is very simple. The lattice action is written originally in terms of the contour integrals like (15) rather than field strength $F_{\mu\nu}$:

$$S = \sum_p \text{Re} \exp\{iQ_e \oint_{\partial p} A_\mu dx^\mu\},$$  

where the sum is taken over all the plaquettes $p$. Thus, the condition (15) means absence, in the lattice formulation, of both the Aharonov-Bohm effect and the quadratic divergence (16). Moreover, it is straightforward to see that the interference term (19) also vanishes.

Later, we will also discuss Dirac strings which correspond to negative plaquettes in the lattice formulation. Such strings are associated with the fundamental monopoles with $Q_m = 1$ in units of the Dirac quantization condition discussed above. The energy of such Dirac strings is infinite in the continuum limit, in agreement with the naive estimate (16). The interference term (19), however, disappears in the lattice formulation in this case as well. Moreover, this observation is crucial to understand the heavy monopole potential in the continuum limit.
The radial field, $H^{Coul}$ may also cause problems with infinite energy, see (17). The lattice regularization is not much specific in that case, however. The role of $r_0$ is simply played by the lattice spacing $a$. Thus, the probability to find a monopole on the lattice is suppressed by the action as:

$$e^{-S} \sim \exp(-const \cdot Q_e^{-2} L/a),$$

where $L$ is the length of the monopole trajectory, and the $Q_e^{-2}$ factor appears because of the Dirac quantization condition (15) which relates the magnetic charge $Q_m$ to the inverse electric charge.

Although the Eq. (21), at first sight, rules out monopoles as physically significant excitations, the fate of the monopoles in the $U(1)$ case depends in fact on the value of the charge $Q_e$. The point is that the entropy factor, or the number of various trajectories with the same length $L$ grows also exponentially with the length of the monopole trajectory:

$$(\text{Entropy}) \sim \exp(+const' \cdot L/a),$$

where the $const'$ is a pure geometric factor. As a result for $Q_e \sim 1$ there is a phase transition corresponding to the condensation of the monopoles. This phase transition, which is well studied on the lattice, is the first and striking example of importance of the UV regularization in the non-perturbative sector.

### 2.4 Classification of Monopoles in non-Abelian Theories

From now on, we will discuss monopoles in unbroken non-Abelian gauge theories, having in mind primarily gluodynamics, i.e. quantum chromodynamics without dynamical quarks. Moreover, for the sake of simplicity we will consider only the $SU(2)$ gauge group.

A natural starting point to consider monopoles in non-Abelian theories is their classification. There are actually a few approaches to the monopole classification and it is important to realize both similarities and differences between them.

**The dynamical, or $U(1)$ classification.** Within this approach [24], one looks for monopole-like solutions of the classical Yang-Mills equations. Where by the “monopole-like” solutions one understands potentials which fall off as $1/r$ at large $r$, see Eq. (3). The basic finding is that there are no specific non-Abelian solutions and all the monopoles can be viewed as Abelian-like embedded into the $SU(2)$ group. Moreover, using the gauge invariance one can always choose the corresponding $U(1)$ group as, say, the rotation group around the third direction in the color space. According to this classification, the monopoles are characterized by their charge with respect to a $U(1)$ which are integer numbers:

$$|Q_m| = 0, 1, 2, \ldots$$

**The topological, or $Z_2$ classification.** The $Z_2$ classification [25] is based entirely on topological arguments. Namely, independent types of monopoles are enumerated by considering the first homotopy group of the gauge group. The $SU(2)$ gauge group is trivial since $\pi_1(SU(2)) = 0$, while in case of $SO(3)$,

$$\pi_1(SU(2)/Z_2) = Z_2,$$
and there exists a single non-trivial topological monopole. We will denote the magnetic 
charge of such monopoles as $|Q_m| = 1$. Note, however, that the charges $Q_m = \pm 1$ are 
indistinguishable in fact. As for the charges $Q_m = 2$ they are equivalent, from this point 
of view, to no magnetic charge at all.

The topological classification (24) is readily understood as a classification of the Dirac 
strings whose end points represent the monopoles. Then there is only one non-trivial 
string, that is the one for which Eq. (15) is satisfied for gluons but not for quarks. 
Namely, because the $U(1)$ charge associated with gluons is twice as large as that of the 
quarks we may have 
\[ \exp \{ ig \oint A_\mu dx^\mu \} = -1 \] 
and such a string is still not visible for the isospin-one particles. On the other hand, 
the standard plaquette action is based on the phase factor evaluated for particles in the 
fundamental representation. Which means, in turn, that the Dirac string is piercing the 
negative plaquettes.

2.5 $Z_2$ Monopoles

In principle, the $U(1)$ and $Z_2$ classifications are different. Indeed, while the $U(1)$ classi-

fication allows for any integer charge, the $Z_2$ classification leaves space only for a single 
non-trivial charge: 
\[ Q_m = 0, 1. \] 

The reconciliation of the two classifications is that the $U(1)$ solutions with $|Q_m| \geq 2$ are 
in fact unstable because of the presence of massless charged vector particles (gluons) [8]. 
The instability of the solutions implies that even if an external source with $|Q_m| \geq 2$ were 
introduced into the vacuum state of the gluodynamics, charged gluons would fall onto the 
monopole center because of the strong magnetic interactions. Moreover, one can imagine 
that as result of this instability the charged fields $A^{\pm}$ are build up as well.

In a somewhat related way, one can demonstrate the apparent irrelevancy of the $|Q_m| = 2$ monopoles by producing an explicit non-Abelian field configuration which looks as a 
$|Q_m| = 2$ monopole in its Abelian part but has no $SU(2)$ action at all [13]. This field 
configuration is a Dirac string with open ends, which correspond to a monopole-anti-
monopole pair separated by distance $R$. Note that the Abelian flux is still transported 
along the Dirac string and is still conserved for the radial field. What is lost, however, is 
the relation between the Abelian flux and action. In the Abelian case non-vanishing flux 
means non-vanishing magnetic field and non-vanishing action since the action density is 
simply $H^2$. Now the action is $(F_\mu^a)^2$ and the Abelian part of the $F^3_{\mu\nu}$ can be canceled by 
the commutator term.

The simplest example illustrating the cancellation between the Abelian component 
and the commutator term in $F^a_{\mu\nu}$ is produced by the potential:

\[ A^a_\mu = -\frac{2}{r^2} \epsilon^{abc} r^b \sigma^a \frac{r^c}{2} , \]
where $\sigma^a$ are the Pauli matrices and $r^a$ is the radius vector. One can readily check by a direct calculation that the corresponding non-Abelian field-strength tensor vanishes identically (some care should be exercised to analyze the singularity at $r = 0$). Another way to convince oneself that the configuration (27) has zero action is to observe that

$$A^a_\mu = i(\Omega^0)^\dagger \partial_\mu \Omega^0,$$  
where

$$\Omega^0 = i\sigma^a n^a,$$ and $n^a$ is the unit vector looking from the center of the “monopole” to the observation point.

We pause here to note that the potential (27) does not exhibit any Dirac string. The reason is that we have not gauge rotated it to the form suitable for the $U(1)$ classification of the monopoles. The absence of the Dirac string may look appealing at first sight. On the other hand, a shortcoming of using an explicitly non-Abelian field, like (27) is that one cannot add up in this case the monopole fields if the monopoles are situated at different space points. Thus, we are not using such forms in this review although one cannot rule out that in future some progress can be made to overcome the difficulty with adding up the monopoles written in the form similar to (27).

Coming back to the problem of writing a filed configuration which in its Abelian part looks as a monopole-antimonopole pair ($|Q_m| = 2$) we will use formulation with a Dirac string. Algebraically the problem becomes more complicated and here we reproduce only the final answer [13]. Namely, such a configuration is generated from the vacuum by the following gauge rotation matrix:

$$\Omega = \begin{pmatrix} e^{i\varphi} \sqrt{A_D} & \sqrt{1 - A_D} \\ -\sqrt{1 - A_D} & e^{-i\varphi} \sqrt{A_D} \end{pmatrix},$$

where $\varphi$ is the angle of rotation around the axis connecting the monopoles and $A_D$ is the $U(1)$ potential representing pure Abelian monopole pair:

$$A_\mu dx_\mu = \frac{1}{2} \left( \frac{z^0}{r^+} - \frac{z^-}{r^-} \right) d\varphi \equiv A_D(z, \rho) d\varphi,$$

where $z^\pm = z \pm R/2$, $\rho^2 = x^2 + y^2$, $r^2 = z^2_\pm + \rho^2$. Note that the action associated with the Dirac string is considered in this case zero, in accordance with the lattice version of the theory (for details see [13]).

In this example, the monopoles with $|Q_m| = 2$ are a kind of a pure gauge field configurations carrying no action.

It is somewhat more difficult to visualize dynamically the equivalence of the $Q_m = \pm 1$ monopoles, also implied by the $Z_2$ classification. The mechanism mixing the $Q_m = \pm 1$ solutions seems to be the following. Imagine that we start with, say, $Q_m = +1$ solution. Then a Dirac string carrying the flux corresponding to the $Q_m = -2$ can be superimposed on this solution. It is important at this point that such a Dirac string costs no action (or
energy). Then the radial magnetic field can also change its direction since this does not contradict the flux conservation any longer. In a related language, one could say that the \(|Q_m| = 2\) monopoles are condensed in the vacuum and as a result the magnetic charge can be changed freely by two units.

As far as interaction of two \(|Q_m| = 1\) monopoles is concerned, one might expect that they would interact as a monopole-antimonopole pair rather than as a monopole-monopole configuration. Indeed, the monopole and antimonopole attract each other and thus represent the lowest energy state of the system.

2.6 Conclusions # 1

Thus, the physics of the monopoles in non-Abelian theories in the classical approximation turns very simple.

Namely, there exist only monopoles with \(|Q_m| = 1 \equiv 2\pi/g\) where \(g\) is the coupling constant of the non-Abelian \(SU(2)\) theory. The monopoles are infinitely heavy and their interaction is Abelian like:

\[
V_{m\bar{m}} = -\frac{Q_m^2}{4\pi R} = -\frac{\pi}{g^2 R},
\]

where \(R\) is the separation between the monopoles.

Clearly enough, this first approximation falls far beyond an adequate description of the empirical data on the monopoles, see the Introduction. Thus, we are invited to go into more advanced approaches which we would try to introduce step by step.

3 Lattice Monopoles, Introductory Remarks

3.1 Dirac Strings on the Lattice, Non-Abelian Case

The Dirac strings remain essentially Abelian objects in case of non-Abelian theories as well. A novel feature is that the field transported along the string can be arbitrarily oriented in the color space. Which is a kind of a new gauge freedom.

In more detail, the general one-plaquette action of \(SU(2)\) lattice gauge theory (LGT) can be represented as:

\[
S_{\text{lat}}(U) = \beta \sum_p S_p \left( 1 - \frac{1}{2} Tr U[\partial p] \right),
\]

where \(\beta = 4/g^2\), \(g\) is the bare coupling, \(\partial p\) is the boundary of an elementary plaquette \(p\), the sum is taken over all \(p\), \(U[\partial p]\) is the ordered product of link variables \(U_i\) along \(\partial p\). In particular, if \(S_p(x) = x\) then (32) is the standard Wilson action. The exponent of the lattice field strength tensor \(F_p\) is defined in terms of \(U[\partial p]\):

\[
U[\partial p] = e^{iF_p} = \cos \left[ \frac{1}{2} |F_p| \right] + i r^a n^a \sin \left[ \frac{1}{2} |F_p| \right],
\]

12
where $\tilde{F} = F^a \cdot \tau^a / 2$, $|F| = \sqrt{F^a F^a}$ and we define $n_p^a = F_p^a / |F_p|$ for $|F_p| \neq 0$, while $n_p^a$ is an arbitrary unit vector for $|F_p| = 0$.

The lattice action (32) depends only on $\cos \left( \frac{|F_p|}{2} \right)$. Therefore the action of the SU(2) LGT possesses not only the usual gauge symmetry, but allows also for the gauge transformations which shift the field strength tensor by $4\pi k$, $|F_p| \rightarrow |F_p| + 4\pi k$, $k \in \mathbb{Z}$:

$$e^{i\tilde{F}_p} = \exp \{i |F_p| n_p \} = \exp \{i(|F_p| + 4\pi) n_p \} = \exp \{i(F_p^a + 4\pi n_p^a)\tau^a / 2 \}.$$  (34)

Thus, the symmetry inherent to the lattice formulation can be represented as:

$$F_p^a \rightarrow F_p^a + 4\pi n_p^a, \quad \tilde{F}_p \times \tilde{n}_p = 0, \quad n_p^2 = 1.$$  (35)

The symmetry (35) is absent in the conventional continuum limit, $\int (F_{\mu\nu})^2 d^4x$. If $n_p \neq 0$ then one can say that a Dirac string is piercing the plaquette. Similar to the Abelian case, such a Dirac string costs no action.

Note that in the continuum limit $n_p^a$ becomes a singular two-dimensional structure $*\Sigma_{\mu\nu}^a$ which is representing the Dirac string world sheet.

### 3.2 The ’t Hooft Loop on the Lattice

So far we discussed invisible Dirac strings. As we know, the Dirac strings corresponding to the fundamental monopoles with $Q_m = 1$ have an infinite action, or energy. In the lattice formulation they correspond to the plaquettes with the phase factor $\exp(i\pi\tau)$. This infinite energy precludes the fundamental monopoles from being dynamical objects present in the QCD vacuum. However, they can be introduced as external objects. The infinite energy of the corresponding Dirac string then does not matter, the same as with the infinite self-energy of heavy quarks introduced via the Wilson loop. The fundamental monopoles can be introduced on the lattice via the ’t Hooft loop [17]. In a simplified way, the monopoles are visualized as end-points of the corresponding Dirac strings which in turn are defined as piercing negative plaquettes. The trick to introduce the negative plaquettes on the lattice is to formally change the sign of the coupling $g^2$ on a manifold of plaquettes. Then these plaquettes become negative in the limit $|g^2| \rightarrow 0$.

Proceeding to more detailed definitions, the ’t Hooft loop is formulated [18] in terms of the action

$$S(\beta, -\beta) = \beta \sum_{p \in M} \text{Tr} \ U_p - \beta \sum_{p \in M} \text{Tr} \ U_p,$$  (36)

where $M$ is a manifold which is dual to the surface spanned on the monopole world-line $j$. Introducing the corresponding partition function, $Z(\beta, -\beta)$ and considering a planar rectangular $T \times R$, $T \gg R$ contour $j$ one can define

$$V_{m\bar{m}}(R) \equiv -\frac{1}{T} \ln \frac{Z(\beta, -\beta)}{Z(\beta, \beta)}.$$  (37)

By the analogy with expectation value of the Wilson loop the quantity $V_{m\bar{m}}(R)$ is referred to as monopole-antimonopole, or heavy monopole potential. The potential $V_{m\bar{m}}(R)$, in a way, is the same fundamental quantity as the heavy-quark potential $V_{QQ}$ and its understanding within the fundamental QCD would be of great importance.
3.3 Lattice Monopoles in the Abelian Projection of $SU(2)$ Gluodynamics

The $SU(2)$ gauge fields $U_l$ on the lattice are defined by $SU(2)$ matrices attached to the links $l$. These lattice fields are related to the continuum $SU(2)$ fields $\hat{A}$: $U_{x,\mu} = e^{i a x_\mu \hat{A}_x}$, here $a$ is the lattice spacing. Under the gauge transformation, the field $U_l$ transforms as $U_{x,\mu}^{\Omega} = \Omega_x U_{x,\mu} \Omega_{x+\mu}$, the matrices of the gauge transformation are attached to the sites $x$ of the lattice.

The Maximal Abelian gauge is defined on the lattice by the following condition [26]:

$$\max_{\Omega} R[U_l^\Omega], \quad R[U_l] = \sum_l Tr[\sigma_3 U_l^+ \sigma_3 U_l], \quad l = \{x, \mu\}. \quad (38)$$

This gauge condition corresponds to an Abelian gauge, since $R$ is invariant under the gauge transformations defined by the matrices:

$$\Omega_{U(1)}(x) = \begin{pmatrix} e^{i a(x)} & 0 \\ 0 & e^{-i a(x)} \end{pmatrix}, \quad \alpha \in [0, 2\pi). \quad (39)$$

Let us parameterize the link matrix $U$ in the standard way

$$U_l = \begin{pmatrix} \cos \varphi_l e^{i \theta_l} & \sin \varphi_l e^{i \chi_l} \\ -\sin \varphi_l e^{-i \chi_l} & \cos \varphi_l e^{-i \theta_l} \end{pmatrix}, \quad (40)$$

where $\theta, \chi \in [-\pi, +\pi)$ and $\varphi \in [0, \pi)$. In this parameterization,

$$R[U_l] = \sum_l \cos 2\varphi_l. \quad (41)$$

Thus, the maximization of $R$ corresponds to the maximization of the diagonal elements of the link matrix $(33)$. Under the $U(1)$ gauge transformations, $(39)$, the components of the gauge field $(40)$ are transformed as

$$\theta_{x,\mu} \rightarrow \theta_{x,\mu} + \alpha_x - \alpha_{x+\mu}, \quad \chi_{x,\mu} \rightarrow \chi_{x,\mu} + \alpha_x + \alpha_{x+\mu}, \quad \varphi_{x,\mu} \rightarrow \varphi_{x,\mu}. \quad (42)$$

Therefore, in the MaA the gauge, the field $\theta$ is the $U(1)$ gauge field, the field $\chi$ is the Abelian charge 2 vector matter field, the field $\varphi$ is the non-charged vector matter field.

A configuration of Abelian gauge fields $\theta_l$ can contain monopoles. The position of the monopoles is defined by the lattice analogue of the Gauss theorem. Consider the elementary three-dimensional cube $C$ (Figure 1(a)) on the lattice.

The Abelian magnetic flux $\vec{H}$ through the surface of the cube $C$ is given by the formula

$$m = \frac{1}{2\pi} \sum_{P \in \partial C} \bar{\theta}_P, \quad (43)$$

where $\bar{\theta}_P$ is the magnetic field defined as follows. Consider the plaquette angle $\theta_P = \theta_1 + \theta_2 - \theta_3 - \theta_4 \equiv \text{d}\theta$, the $\theta_i$’s are attached to the links $i$ which form the boundary of
the plaquette $P$, Figure 1(b). The definition of $\bar{\theta}_P$ is $\bar{\theta}_P = \theta_P + 2\pi k$, where the integer $k$ is such that $-\pi < \theta_P \leq \pi$. The restriction of $\theta_P$ to the interval $(-\pi, \pi]$ is natural since the Abelian action for the compact fields $\theta_l$ is a periodic function of $\bar{\theta}_P$. Equation (43) is the lattice analogue of the continuum formula $m = \oint \vec{H} \, d\vec{S}$. Due to the compactness of the lattice field $\theta (-\pi < \theta \leq \pi)$ there exist singularities (Dirac strings), and therefore, $\text{div} \vec{H} \neq 0$.

The magnetic charge $m$ defined by eq. (43) has the following properties:

1. $m$ is quantized: $m = 0, \pm 1, \pm 2$;

2. If $m \neq 0$, then there exists a magnetic current $j$. This current is attached to the link dual to cube $C$.

3. Monopole currents $j$ are conserved: $\delta j = 0$, the currents form closed loops on the $4D$ lattice.

We discuss in details the properties of the lattice monopoles in gluodynamics in Section 3.

### 3.4 Dirac Strings and Singular Gauge Transformations

If we wish to build up the continuum theory in a way reproducing the basic features of the lattice formulation, we need first of all to ensure the invisibility of the Dirac string. One way is to put the energy of the Dirac strings to zero “by hands”. A more educated way is to allow for singular gauge transformations which would correspond to introduction of the Dirac strings. Let us start with the compact electrodynamics (cQED).
Under the gauge transformation, \( A_\mu \rightarrow A_\mu + \partial_\mu \alpha \), the field strength tensor transforms as: 
\[
F_{\mu \nu} \rightarrow F_{\mu \nu} + \partial_\mu \alpha \partial_\nu \alpha.
\]
For a singular gauge transformation \([\partial_\mu, \partial_\nu] \alpha \neq 0\). Moreover, for a sufficiently general singular gauge parameters \( \alpha \) the commutator of derivatives is proportional to the 2D \( \delta \) function on some closed surface: 
\[
[\partial_\mu, \partial_\nu] \alpha \propto *\Sigma_{\mu \nu}(\alpha).
\]
If \( x \) are the 4D coordinates of the surface, \( \sigma_\alpha (\alpha = 1, 2) \) are 2D coordinates on the world-sheet, then the general representation of the surface is:
\[
\Sigma_{\mu \nu} = \int d^2 \sigma \sqrt{g} t_{\mu \nu}(\sigma) \, \delta^{(4)}(x - \tilde{x}(\sigma)), \quad g(\sigma) = \text{Det}[\partial_\alpha \tilde{x}_\mu \partial_\beta \tilde{x}_\nu],
\]
\[
t_{\mu \nu}(\sigma) = \frac{1}{\sqrt{g}} \epsilon^{\alpha \beta} \partial_\alpha \tilde{x}_\mu \partial_\beta \tilde{x}_\nu, \quad t_{\mu \nu}^2 = 2.
\]
Now, we can write down a partition function of electrodynamics which is invariant under the singular gauge transformations:
\[
Z = \int DA_\mu D\Sigma \exp \left\{ -\frac{1}{2e^2} \int [F_{\mu \nu} + 2\pi^*\Sigma_{\mu \nu}]^2 \right\}.
\] (46)

Of course \( \int D\Sigma \) is only formal notation for the summation over all surfaces in 4D as far as we do not specify the measure. On the other hand, this summation is well defined on the lattice and (46) can be viewed as the partition function of cQED with the Villain type of the action written in the continuous notations, if \( D\Sigma \) in (46) includes summation over closed and not closed surfaces.

Proceeding now to the non-Abelian case, we note again that the symmetry (35) is absent in the conventional continuum action, \( \int (F_{\mu \nu}^a)^2 d^4x \). To bring the continuum theory into agreement with the lattice formulation, we need a generalization of (46). In the continuum limit \( n^a_\mu \) becomes a singular two-dimensional structure \( *\Sigma_{\mu \nu}^a = \frac{1}{2} \varepsilon_{\mu \nu \lambda \rho} \Sigma^a_{\lambda \rho} \), which is a generalization of the Dirac strings in the compact electrodynamics and which transforms in the adjoint representation of the gauge group:
\[
\Sigma_{\mu \nu}^a = \int d^2 \sigma \sqrt{g} t_{\mu \nu}^a(\sigma) \, \delta^{(4)}(x - \tilde{x}(\sigma)).
\] (47)

The surface (17) need not to be closed. Moreover, the second equality in (35) requires that
\[
\tilde{t}_{\mu \nu}(\sigma) \times *\tilde{F}_{\mu \nu}(\tilde{x}) = 0,
\] (48)
where the continuum field strength tensor \( \tilde{F}_{\mu \nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - i[\tilde{A}_\mu, \tilde{A}_\nu] \). Eq. (48) determines the color structure of \( t_{\mu \nu}^a \):
\[
t_{\mu \nu}^a(\sigma) = t_{\mu \nu}(\sigma) \, n^a(\sigma), \quad n^a(\sigma) = (t \cdot *F^a) \left( (t \cdot *F^b)^2 \right)^{-1/2},
\] (49)
where \( (t \cdot F^a) \equiv t_{\mu \nu}(\sigma) F^a_{\mu \nu}(\tilde{x}) \) and \( n^a \) is normalized as \( \tilde{n}^2 = 1 \). On the set of points where \( (t \cdot F^a) = 0 \) the direction of \( n^a(\sigma) \) is arbitrary.

Now, the continuum analog of the lattice symmetry (35) is:
\[
F_{\mu \nu}^a \rightarrow F_{\mu \nu}^a + 4\pi^*\Sigma_{\mu \nu}^a,
\] (50)
The action of $SU(2)$ gluodynamics which possesses the additional symmetry (50) can be formally represented as:

$$Z = \int \mathcal{D}A \exp\{-S(F)\},$$  

(51)

$$S(F) = -\log \int \mathcal{D}\Sigma \exp\{-\frac{1}{4g^2} \int d^4x \left[F^a_{\mu\nu} + 4\pi^* \Sigma^a_{\mu\nu}\right]^2\},$$  

(52)

where the integration is over all possible surfaces,

$$\Sigma^a_{\mu\nu} = \int d^2\sigma_{\mu\nu} n^a(\sigma) \delta^{(4)}(x - \tilde{x}(\sigma)).$$  

(53)

The expressions (51,52) are only formal since it is impossible to separate rigorously the measure $\mathcal{D}\Sigma$ from the gauge degrees of freedom in $\mathcal{D}A$. Nevertheless, the Eq. (51,52) is a good starting point since it reproduces the basic symmetries of the theory. Indeed, the action (52) is invariant under smooth $SU(2)$ gauge transformations since vector $n^a$ transforms in the same way as $F^a_{\mu\nu}$ does. It is also invariant under transformations (50) which correspond to the lattice symmetry relations (35).

Note also that for self-intersecting surface $\Sigma_{\mu\nu}$, Eq. (55), the world-sheet vector field $n^a(\sigma)$ is generally multi-valued as function of $\tilde{x}$. Furthermore, for the non-orientable surfaces the field $n^a(\sigma)$ cannot be defined smoothly everywhere on $\Sigma$. To avoid these complications we consider only the orientable surfaces without self-intersections. This reservation is specific for Dirac strings in the non-Abelian case.

### 3.5 The ’t Hooft Loop in the Continuum Limit

After the experience with working out the continuum limit of the Dirac string world sheet in the preceding subsection, there is no difficulty to figure out that a continuum analog of the ’t Hooft loop looks as [13]:

$$H(\Sigma_C) = \exp\left\{ \frac{1}{4g^2} \int d^4x \left[\left(F^a_{\mu\nu}\right)^2 - \left(F^a_{\mu\nu} + 2\pi^* \Sigma^a_{C\mu\nu}\right)^2\right]\right\},$$  

(54)

$$\Sigma^a_{C\mu\nu} = \int d^2\sigma_{\mu\nu} n^a(\sigma) \delta^{(4)}(x - \tilde{x}(\sigma)), $$

(55)

where the surface $\Sigma^a_C$ spanned on the contour $C$ is assumed to be non-intersecting. The unit three-dimensional vector field $n^a(\sigma)$, $\vec{n}^2 = 1$ is defined on the world-sheet of the Dirac string. For analogous definitions see (49). Therefore $n^a(\sigma)$ is not an independent variable, it is completely determined by the components of the field strength tensor $F^a_{\mu\nu}$. It can be shown [13] that the Eq. (54)-(49) define the correct ’t Hooft loop operator the expectation value of which depends only on the contour $C$, not on a particular position of the surface $\Sigma_C$. 

17
3.6 Conclusions #2

The main lesson from the lattice formulation which we learnt so far is that to match the lattice and continuum formulations of the gluodynamics one should allow for singular gauge transformations. In this way one can hope to reproduce the compactness of the $U(1)$ subgroups of the $SU(2)$ group and the non-observability of the Dirac strings which is crucial for introduction of the monopoles.

4 Lagrangian Approach to the Dual Gluodynamics

In this section we are going to derive some important results concerning monopoles and their interaction in gluodynamics, understood as a continuum limit of the lattice formulation. At first we discuss the Zwanziger receipt of introduction of the dual photons (Subsection 4.1). In Subsection 4.3 we introduce a Zwanziger–type Lagrangian for gluodynamics. The central point here is the match of ordinary gluons in an adjoint representation with a single dual gluon which is an $U(1)$ Abelian gauge boson. In Subsections 4.2, 4.4 we use the Lagrangian formulation to derive the short distance behavior of the heavy monopole potential, with one-loop quantum corrections included. In Subsections 4.5, 4.6 we discuss the phenomenological Lagrangians describing the monopole condensation. The central point here is a derivation of the Casimir scaling.

4.1 Photons, Direct and Dual

Now, that we know that magnetic monopoles are relevant degrees of freedom on the lattice, at least in some gauges, we would like to have a continuum version of gluodynamics which would allow both for color and magnetic charges. We begin, however, with a review of the corresponding construction in case of the electrodynamics, that is of the Zwanziger Lagrangian [19].

We have already mentioned some basic features of the Zwanziger approach in the Introduction and here would like to directly proceed to the Lagrangian:

$$L_{Zw}(A, B) = \frac{1}{2}(m \cdot [\partial \wedge A])^2 + \frac{1}{2}(m \cdot [\partial \wedge B])^2 +$$

$$+ \frac{i}{2}(m \cdot [\partial \wedge A])(m \cdot *[\partial \wedge B]) - \frac{i}{2}(m \cdot [\partial \wedge B])(m \cdot *[\partial \wedge A]) + i j_e \cdot A + i j_m \cdot B,$$

where $j_e, j_m$ are electric and magnetic currents, respectively, $m_\mu$ is a constant vector, $m^2 = 1$ and

$$[A \wedge B]_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu , \quad (m \cdot [A \wedge B])_\mu = m_\nu [A \wedge B]_{\nu\mu} ,$$

$$*[A \wedge B]_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} [A \wedge B]_{\lambda\rho} .$$

At first sight, we have introduced two different vector fields, $A$ and $B$, to describe interaction with electric and magnetic charges, respectively. If it were so, however, we would
have solved a wrong problem because we need to have a single photon interacting both with electric and magnetic charges. And this is what is achieved by the construct (56). Indeed, the action (56) is not diagonal in the $A$, $B$ fields and one can convince oneself that the form of the $A$, $B$ interference terms in (56) is such that the field strength tensors constructed on the potentials $A$ and $B$ are in fact related to each other:

$$m_{\mu} F_{\mu\nu} (A) = m_{\mu}^{*} F_{\mu\nu} (B), \tag{57}$$

where $*F$ denotes the dual tensor. In perturbation theory, Eq. (57) ensures that there are only two physical degrees of freedom corresponding to the transverse photons which can be described either in terms of the potential $A$ or $B$. Topological excitations, however, can be different in terms of $A$ and $B$.

The physical content of (56) is revealed by the propagators for the fields $A$, $B$. In the $\alpha$-gauge one can derive:

$$\langle A_\mu A_\nu \rangle = \langle B_\mu B_\nu \rangle = \frac{1}{k^2} \left( \delta_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2} \right), \tag{58}$$

$$\langle A_\mu B_\nu \rangle = - \langle B_\mu A_\nu \rangle = \frac{i}{k^2 (k \cdot m)} [m \wedge k]_{\mu\nu}. \tag{59}$$

The propagators should reproduce, as usual, the classical solutions. And indeed, the $\langle AA \rangle$, $\langle BB \rangle$ propagators describe the Coulomb-like interaction of two electric charges and magnetic monopoles, respectively. While the $\langle AB \rangle$ propagator reproduces interaction of the magnetic field of a monopole with a moving electric charge. The appearance of the poles in $(k \cdot m)$ is a manifestation of the Dirac strings.

To summarize, the Zwanziger Lagrangian in electrodynamics [19] reproduces the classical interaction of monopoles and charges. Upon the quantization, it describes the correct number of the degrees of freedom associated with the photon.

### 4.2 Monopole Charge Renormalization in Abelian Theory

We will now demonstrate that the quantum field theory with the Zwanziger Lagrangian is not selfconsistent, and the knowledge of propagators (58) is not enough to get the correct answers. To demonstrate these difficulties we consider the simplest theory containing light electrons, photons, and magnetically charged particles which are heavy and thus we can neglect their contributions to the vacuum polarization.

Consider the renormalization of the monopole charge due to the vacuum polarization induced by the electron loop, see the review [3] and references therein. As is emphasized in Ref. [13], the crucial point is that only loops with insertion of two external (i.e., monopole) fields can be considered despite of the fact that there is no perturbative expansion in $g \cdot e$. Indeed, considering more insertions makes the graphs infrared sensitive, with no possibility for $\ln \Lambda_{UV}$ to emerge.

Then, the evaluation of, say, first radiative correction to the propagator $\langle B_\mu B_\nu \rangle$ in the Zwanziger formalism (58) seems very straightforward and reduces to taking a product of
two \langle AB \rangle \text{ propagators and inserting in between the standard polarization operator of two}
electromagnetic currents, see Fig. 1(a). The result is \[3\]:
\[
\langle B_\mu B_\nu \rangle (k) = \frac{\delta_{\mu\nu}}{k^2} (1 - L) + \frac{1}{(k \cdot m)^2} (\delta_{\mu\nu} - m_\mu m_\nu) L,
\]
(61)
where
\[
L = \frac{\alpha_{el}}{6} \ln \Lambda_{UV}^2 / k^2
\]
and we have neglected the electron mass so that the infrared cut-off is provided, in the
logarithmic approximation, by the momentum \(k\).

At first sight, there is nothing disturbing about the result (61). Indeed, we have
a renormalization of the original propagator which is to be absorbed into the running
coupling, and a new structure with the factor \((k \cdot m)^{-2}\) in front which is non-vanishing,
however, only on the Dirac string. The latter term would correspond to the renormalization
of the Dirac-string self-energy which we do not follow in any case since it is included
into self-energy of the external monopoles. What is, actually, disturbing is that according
to (61) the magnetic coupling would run exactly the same as the electric charge (see Fig.
1(b)),
\[
\langle A_\mu A_\nu \rangle (k) = (1 - L) \frac{\delta_{\mu\nu}}{k^2},
\]
(62)
violating the Dirac quantization condition.

The origin of the trouble is not difficult to figure out. Indeed, using the propagator \(\langle AB \rangle\) while evaluating the radiative corrections is equivalent, of course, to using the full
potential corresponding to the Dirac monopole \(A_D^\text{cl}\). Then, switching on the interaction
with electrons would bring terms like \(A_D^\text{cl} \bar{\psi} \gamma \psi\). Since \(A_D^\text{cl}\) includes the potential of the
string, electrons do interact with the Dirac sheet and we are violating the Dirac veto
which forbids any direct interaction with the string.

Let us demonstrate quantitatively that, indeed, it is the interaction with the Dirac
string that changes the sign of the radiative correction. This can be done in fact in an

Figure 2: (a) Renormalization of the \(A_\mu\) gauge field propagator. (b) Renormalization of
the \(B_\mu\) gauge field propagator.
amusingly simple way. First, let us note that it is much simpler to remove the string if one works in terms of the field strength tensor, not the potential. Indeed, we have $H = H_{\text{Coul}} + H_{\text{Str}}$ while in terms of the potential $A_\mu$ any separation of the string would be ambiguous.

We discussed already that the account of the string field would flip the sign of the interaction energy, see eq. (19). On the other hand the duality principle (18) implies that at the classical level we should write:

$$H_1 \cdot H_2 \equiv H_{1\text{Coul}} \cdot H_{2\text{Coul}}. \quad (63)$$

However, the electrons in the diagram shown in Fig. 1(a) interact with the full potential $A_\mu^{(D)}$ and therefore the first radiative correction would bring the product of the total $H_1 \cdot H_2$ which includes also the string contribution. Indeed, the result in the log approximation would be as follows:

$$\delta(H_1 \cdot H_2) = L(H_{1\text{Str}} + H_{1\text{Coul}}) \cdot (H_{2\text{Str}} + H_{2\text{Coul}}) = -LH_{1\text{Coul}} \cdot H_{2\text{Coul}}, \quad (64)$$

where at the last step we have used the observation (19).

Now, it is clear how we could ameliorate the situation. Namely, to keep the Dirac string unphysical we should remove the string field from the expression (64) which arises automatically if we use the propagators (58) following from the Zwanziger Lagrangian. Thus, one should change $H_1 \cdot H_2$ in the expression (64) to $H_{1\text{Coul}} \cdot H_{2\text{Coul}}$. The justification is that we should remove the effect of the string field from any observable and consider this as a constraint on the ultraviolet regularization. The constraint is satisfied by the lattice regularization. The corrected propagator $\langle B_\mu B_\nu \rangle$ at the one-loop level is becoming then:

$$\langle B_\mu B_\nu \rangle(k) = \frac{\delta_{\mu\nu}}{k^2}(1 + L). \quad (65)$$

In particular, there is no Dirac string self-energy renormalization.

The most important conclusion is that (65) does satisfy the Dirac quantization condition also for the running coupling (compare eq. (62) and eq. (65)).

It might worth emphasizing that the running of the magnetic coupling became possible only because we do not have any longer the Maxwell equation $\text{div} \ B = 0$ on the operator level. Indeed, we have a source of the dual photon (or gluon) which is mixed up with the ordinary photon (gluon). Similarly, for the electric field we have $\text{div} \ E = \rho_{\text{el}}$. Classically, for a point-like charge $\rho_{\text{el}} = Q_{\text{e}}\delta^{(3)}(r)$ and the coupling is not running of course. However, on the quantum level the fact that $\text{div} \ E \neq 0$ as far as the operator equations are concerned gets revealed through the running of the electrical coupling. Now the story repeats itself, in the dual formulation, for the magnetic coupling. In Subsection 4.4 we demonstrate that in the non-Abelian case the Dirac quantization condition is also valid for the running coupling.

---

1 At this point we assume in fact that $\Lambda_{\text{UV}}$ is larger than the inverse size of the string, which is convenient for our purposes. Other limiting procedures could be considered as well, however.
4.3 Dual Gluon as an Abelian Gauge Boson

Now, if we try to extend the Lagrangian approach to the dual gluons, we immediately come to a paradoxical conclusion that the dual field, if any, is Abelian. Indeed, we have already emphasized that monopoles associated with, say, $SU(N)$ gauge group are classified according to $U(1)^{N-1}$ subgroups and can be realized as pure Abelian objects. Thus, there is no place for a non-Abelian dual gluon because the monopoles do not constitute representations of the non-Abelian group.

From this simple observation we immediately conclude that the dual field $B$ is an Abelian gauge field. Therefore in the Lagrangian a la Zwanziger the dual field $B$ can be mixed up only with one of the fields $A^a$. Moreover, which field $A^a$ is being mixed up is matter of a gauge fixing. The Lagrangian realizing these ideas looks as (for more detail see [13]):

$$L_{\text{dual}}(A^a, B) = \frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}(m \cdot [\partial \wedge B] - i \ast F^a_{\nu}n^a)^2 + i j_m B + i j_e^a A^a. \quad (66)$$

Now, a few remarks concerning the Lagrangian (66) are in order:

(a) First, if the magnetic current is vanishing, $j_m = 0$ then the integration over the field $B$ reproduces the standard Lagrangian of the gluodynamics (we are omitting the gauge-fixing and ghost terms).

(b) As far as the quantization is concerned, the Lagrangian (66) reproduces the correct degrees of freedom of the free gluons. Indeed, in the limit $g \to 0$ and for $n^a = \delta^{a3}$ the Lagrangian (66) becomes:

$$L_{\text{dual}}(A^a, B) = \frac{1}{4}(\partial \wedge A^1)^2 + \frac{1}{4}(\partial \wedge A^2)^2 + i j_m B + i j_e^a A^a +$$

$$+ \frac{1}{2}[m \cdot (\partial \wedge A^3)]^2 + \frac{1}{2}[m \cdot (\partial \wedge B)]^2 +$$

$$+ i \frac{1}{2}[m \cdot (\partial \wedge A^3)][m \cdot (\partial \wedge B)] - i \frac{1}{2}[m \cdot (\partial \wedge A^3)][m \cdot (\partial \wedge B)], \quad (67)$$

which is essentially the Zwanziger Lagrangian (56) plus non-interacting non-diagonal gluons. Quantization at this point is the same as in the case of a single photon.

(c) The emergence of the vector $n^a$ in the Lagrangian (66) is of crucial importance. The point is that the origin of the vector $n^a$ goes back to choosing the color orientation of the monopoles. As is emphasized above the monopole solutions are Abelian in nature which means, in particular, that they can be rotated to any direction in the color space by gauge transformations. Thus, picking up a particular $n^a$ is nothing else but using the gauge fixing freedom. Therefore, we can either average over the directions of $n^a$ or fix $n^a$ but evaluate only gauge invariant quantities, like the Wilson loop.

(d) The $Z_2$ nature of the monopoles is manifested in the freedom of changing $n^a \to -n^a$, $B_\mu \to -B_\mu$. Indeed, under such transformation a monopole with the charge $Q_m = +1$ is transformed into a monopole with $Q_m = -1$ and vice versa. In the language we used above such a transformation corresponds to adding a Dirac string with a double...
magnetic flux. We see that the averaging over $\pm n^a$ is a part of the overall averaging over all possible embedding of the $U(1)$ into the $SU(2)$ gauge group.

(e) In its generality, the Lagrangian (66) obviously possesses $SU(2) \times U(1)$ gauge invariance:

\[
SU(2) : \quad F_{\mu\nu} \rightarrow \Omega^{-1} F_{\mu\nu} \Omega, \quad U(1) : \quad B_\mu \rightarrow B_\mu + \partial_\mu \alpha .
\] (68)

All the features (a)-(e) indicate that the Lagrangian (66) is a reasonable choice to describe the dual gluodynamics. Now, we would like to put the construction (66) on further tests. An obvious check is the evaluation of the quantum corrections to the heavy monopole potential $V_{\tilde{m}m}$ at short distances. Indeed, it can be shown, [13], that the radiative corrections result in the standard running of the non-Abelian constant $g^2$.

4.4 The ’t Hooft Loop and the Dual Gluon

We have already derived a continuum analog of the ’t Hooft loop, see Eq.(54). Now, using the notion of the dual gluon field we will turn to a more detailed analysis of the heavy monopole potential.

For a rectangular contour $T \times R$ we can write for the surface $\Sigma^a$:

\[
\Sigma^a_j = n^a \frac{1}{(m\partial)}[m \wedge j]
\] (69)

which is a particular solution to $\partial \Sigma = j$. It is obvious that the current $j$ will serve now as a source of the dual gluon $B$. Upon substitution of (69) the expectation value $H(j)$ is represented as:

\[
H(j) = \frac{1}{Z} \int D A \int D n \exp \left\{ -\frac{1}{4g^2} \int d^4x \left( F^a + 2\pi n^a \frac{1}{(m\partial)}[m \wedge j] \right)^2 \right\} ,
\] (70)

Note that Eq. (59) extends the definition of the field $n^a$ into the entire space–time while only its value of $n^a$ on the string $\Sigma_j$ does matter, while a particular way of extending $n^a(\sigma) \rightarrow n^a(x)$ is irrelevant. Moreover, the path integral in (70) is to be performed with a constraint:

\[
\varepsilon^{abc} n^b(F^c \cdot \frac{1}{(m\partial)}[m \wedge j]) = 0, \quad a = 1, 2, 3 ,
\] (71)

which is a consequence of the constraint (49). The constraint equation (71) may be implemented by an additional field $\chi^a$:

\[
H(j) = \frac{1}{Z} \int D n D A D \chi \exp \left\{ -\frac{1}{4g^2} \int d^4x \left( F^a + 2\pi n^a \frac{1}{(m\partial)}[m \wedge j] \right)^2 + i \frac{2\pi}{g} \int jC \right\} ,
\] (72)
\[ C_\mu = C_\mu (\chi, n, F) = \frac{1}{(m\partial)} \varepsilon^{abc} \chi^a n^b (m \cdot * F^c)_\mu. \] (73)

It is worth emphasizing that the field \( n^a(x) \) is a kind of a fake variable in the representation (72). The path integral is clearly independent on \( n^a \) away from the string (69), but at the same time \( n^a(x) \) for \( x \in \Sigma_j \) is determined through (71).

Coming back to the evaluation of the potential \( V_{mm} \), on the classical level we tend the coupling \( g \to 0 \). In this limit and for \( n^a = \delta^a \) the Lagrangian (66) reduces to:

\[ L^0 = \frac{1}{4} [\partial \wedge A]^2 + \frac{1}{2} (m \cdot [\partial \wedge B - i * \partial \wedge A^3])^2, \] (74)

which essentially coincides with that of Zwanziger (56). In particular, to the lowest order in \( g^2 \) the monopole-antimonopole interaction is given by a single \( B \)-field exchange. Using Eq. (58) one immediately obtains:

\[ H(j) = \exp \left\{ -\frac{1}{2} \left( \frac{2\pi}{g} \right)^2 \int j_\mu \Delta^{-1} j_\mu \right\} \sim \exp \left\{ T \frac{4\pi^2}{g^2} \Delta^{-1}_3(R) \right\} \] (75)

\[ V_{mm}(R) = -\frac{1}{g^2} \cdot \frac{\pi}{R}, \] (76)

Which is the expected result of course.

Having full Lagrangian (66) we can consider the radiative corrections to (76). The calculations similar to that of Section 4.2 give the result (13):

\[ V_{mm}(R) = -\frac{\pi}{g^2(R)} \frac{1}{R}, \] (77)

which is valid of course at small distances. Eq. (77) makes manifest that monopoles in gluodynamics unify Abelian and non-Abelian features. Namely, the overall coefficient, \( \pi/g^2 \) is the same as in Abelian theory, while the running of the coupling \( g^2(R) \) is the same as in the non-Abelian theory. The \( U(1) \) normalization is based on the classification of the monopoles in non-Abelian theory (see, e.g., [2] and references therein). The running of the coupling reflects the general rule that the effect of the fluctuations at short distances can be absorbed into the renormalization of the coupling. Eq. (77) also show that the Dirac quantization condition is valid at the perturbative level in non-Abelian theory.

In order to derive (77) we have to neglect the contribution of the Dirac strings. We should subtract the contribution of the strings, even if it is not vanishing within perturbation theory. Essentially, the rule is that the product \( \mathbf{H}_1 \cdot \mathbf{H}_2 \) should not include the piece \( \mathbf{H}^{\text{Coul}} \cdot \mathbf{H}^{\text{Str}} \).

To justify this rule we have to go actually beyond the Lagrangian (66) and consider a regularization procedure which takes into account the Dirac strings. It is worth emphasizing that within the lattice regularization the term \( \mathbf{H}^{\text{Coul}} \cdot \mathbf{H}^{\text{Str}} \) is indeed absent. Let us recall the reader that in the lattice regularization the Dirac string pierces negative plaquettes. This is true in the limit \( g^2 \to 0 \), or \( a \to 0 \). If one looks for small deviation from \( U_p = -1 \), then the action does not contain terms, which are linear in perturbations. Note that this will be not true for the expansion around arbitrary \( U_p \neq \pm 1 \). This result
means in turn that in the continuum limit there is no term \( H_{\text{Coul}} \cdot H_{\text{Str}} \). Thus, our naive removal of the effects of the Dirac string from the radiative corrections, is justified by the lattice regularization.

The prediction of the monopole potential at large distances is also possible [13] if we use the Abelian dominance model. The existing numerical data [21] do not contradict the predictions of Ref. [13]. Higher statistics is needed, however, to crucially check the theory.

4.5 Phenomenological Lagrangians

We now switch to the physics of large distances, or the dual superconductor model of the confinement, see the Introduction.

Here, the basic lesson brought about by the lattice simulations is that the monopoles condense (see reviews [9] and Section 5). In Subsection 2.3 we have sketched theoretical arguments [10] for the monopole condensation within the compact \( U(1) \). In case of the non-Abelian theory, a simple-minded generalization of these arguments is as follows. Let us begin with a lattice formulation and a small coupling \( g_{SU(2)}^2 \). Let us furthermore integrate, a la Wilson, over small-scale fluctuations and go in this way to a larger-size lattice. The only effect of this is the rescaling of the coupling \( g_{SU(2)}^2 \) according to the rules of the renorm-group. However, we could expect that this procedure would not work further once we reach \( g_{SU(2)}^2 \sim 1 \). Indeed, the same non-Abelian coupling \( g_{SU(2)}^2 \) governs the physics associated with any \( U(1) \) subgroup. And we know that for \( g_{U(1)}^2 \sim 1 \) there is a phase transition due to the monopole condensation. Thus, we expect the monopole condensation in non-Abelian theory as well, unless something else happens at smaller \( g_{SU(2)}^2 \).

In the continuum limit and in the field theoretical language, one describes usually the monopole condensation within a (dual) Abelian Higgs model (AHM), see, e.g. reviews [7, 3, 9] and references therein. The action of the model looks as:

\[
S = \int d^4x \left\{ \frac{1}{4Q_m^2} F_{\mu\nu}^2 + \frac{1}{2} |(\partial - iB)|^2 + \frac{1}{4} \lambda (|\Phi|^2 - \eta^2)^2 \right\}
\]

(78)

where \( Q_m \) is the magnetic charge, \( B_\mu \) is an Abelian gauge field, \( F_{\mu\nu} \) is the corresponding field-strength tensor, \( F_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu \), and \( \lambda, \eta \) are constants. The magnetically charged scalar field \( \Phi \) condenses in the vacuum, \( \langle \Phi \rangle = \eta \), and the physical vector and scalar particles are massive, \( m_V^2 = Q_m^2 \eta^2, m_H^2 = 2\lambda \eta^2 \).

The corresponding equations of motion possess an Abrikosov-type string solution. An open string with electrically charged particles at the end points is thought to mimic a confined \( Q\bar{Q} \) pair in QCD. The string tension is a function of the parameters of the theory, that is \( m_V, m_H \).

There is one subtle point about the dual superconductor model which we would like to emphasize. Namely, if we turn back to the simple arguments of the Subsection 2.3 which explain the monopole condensation, we can easily visualize that, indeed, the monopole condensate is non-vanishing, that is \( \langle \Phi \rangle = \eta \neq 0 \). Moreover, it is then consequence of
the gauge invariant interaction that $m_V^2 \neq 0$ as well. However, it is not easy to understand at all which parameter is related to the Higgs mass, $m_H$. One allows usually for arbitrary $m_V, m_H$ which are treated as fit parameters to reproduce the lattice data. Moreover, the most recent fits to the transverse structure of the confining string are close to the Bogomolny limit, $m_V = m_H$ see, e.g., [27] and Subsection 5.2. Of course, there is an intrinsic uncertainty in all the fitting procedure since one using the classical solutions to (78) to fit the results of the lattice simulations which account fully for the quantum effects.

From purely theoretical point of view another fitting procedure would be more logical. Namely, at large distances it is logical to assume the London limit since the monopole condensation in the compact $U(1)$ does correspond to this limit. As for the finite Higgs mass $m_H$ it should be produced by account for the non-Abelian gluons which are to be manifested at short distances. However, there is no theoretical papers which would produce an explicit procedure of this type and we cannot include any detailed discussion of such an approach into this review. Although, as we shall see in the next Subsection, further exploration of this approach seems urgent.

### 4.6 Casimir Scaling

As we already mentioned in the Introduction, a direct use of the Abelian models (78) is in contradiction with the Casimir scaling. Indeed, whatever $U(1)$ subgroup we choose there are quarks neutral with respect to this subgroup once the (color) isospin of the quarks is integer. However, after the experience with evaluation of the heavy monopole potential (see above) this puzzle does not look very difficult to solve. Namely, we can use a $U(1)$ language for dual gluons. However, we should understand the choice of a particular $U(1)$ as a gauge fixing procedure and evaluate gauge-invariant quantities.

Thus, combining the idea of introducing an effective Higgs field $\Phi$ with fundamental Lagrangian (63) we come to the following effective theory of $SU(2)$ gluodynamics:

$$ Z = \int \mathcal{D}A \mathcal{D}B \mathcal{D}\Phi \mathcal{D}n \exp\left\{ -\int d^4x L_{\text{eff}} \right\}, $$

$$ L_{\text{eff}} = \frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2} \left[ \left( m \cdot [\partial \wedge B - i^* F^a n^a - \varepsilon^{abc} \chi^a n^b F^c] \right)_{\mu} \right] + \frac{1}{2} \left( \partial + i \frac{4\pi}{g} B \right) \Phi^2 + V(|\Phi|), $$

where

$$V(|\Phi|) = \lambda \left( |\Phi|^2 - \eta^2 \right)^2,$$

and $\lambda$ and $\eta$ are phenomenological constants. Of course, the vacuum expectation value of the Higgs, or monopole field is of order $\Lambda_{QCD}$. The physical assumption behind (79) is that the effective size of the monopoles with $|Q_m| = 2$ is in fact numerically small, although generically it is of order $\Lambda_{QCD}$ (see also [28]).
So far we discussed only embedding of the $U(1)$ into $SU(2)$. As far as the physics of large distances is concerned, there is a subtle point, how to define the $U(1)$ dynamically. As is mentioned above the $U(1)$ arises in lattice simulations through gauge fixing. The monopole properties depend on the particular choice of $U(1)$. The Maximal Abelian projection turns most successful as far as the interaction of the heavy quarks in the fundamental representation is concerned [9]. Thus, we will assume the use of this gauge. Moreover, let us fix the orientation of the dual $U(1)$ group in the color space as the rotations around the third axis. Then knowing the classical, Abrikosov-type solution allows to find directly the string tension, $\sigma_{Ab}(2T_3)$ as a function of the third component of the isospin of the quarks, $T_3$ and of the parameters of the Lagrangian(79). The role of the Wilson loop is to ensure that the $Q\bar{Q}$ system is in the state with the total isospin $T = 0$. As a result, the string tension measured via the Wilson loop is given by:

$$\sigma_T = \frac{1}{2T + 1} \sum_{T_3 = -T}^{T} \sigma_{Ab}(2T_3),$$

where $\sigma_{Ab}$ denotes the Abrikosov string tension, calculated in pure Abelian Higgs model with and for external charges of the magnitude $\pm 2T_3$ while the overall factor $(2T + 1)^{-1}$ is due to the normalization of the wave function. Note that quantum mechanically Eq (82) is similar, say, to expressing the hyperfine splitting for a state with a given total spin in terms of the spin-spin interaction of the constituents.

In the London limit, $m_H \gg m_V$, the string tension $\sigma_{Ab}(2T_3) \sim T_3^2$ and we reproduce the Casimir scaling:

$$\sigma_T = (\text{const}) \frac{1}{2T + 1} \sum_{T_3 = -T}^{T} T_3^2 = (\text{const}) \frac{1}{3} T(T + 1).$$

In the Bogomolny limit, $m_H = m_V$, $\sigma_{Ab}(2T_3) \sim |T_3|$ and $\sigma_T \sim T(T + 1)/(2T + 1)$.

As is mentioned above, the structure of the confining string for the quarks in the fundamental representation is best described in the Bogomolny limit, see Subsection 5.2 and Ref. [27]. As is mentioned in the preceding Subsection, a possible way out of the difficulty could be account for the non-Abelian gluons at short distances. Indeed, the string tension is sensitive to much larger distances than the string structure in the transverse direction. At large distances, where the picture of the monopole condensation applies, one expects the London limit to hold (see the preceding Subsection). At smaller distances the charged gluons could become important.

We conclude this subsection with a remark, that from the theoretical point of view, the most important limitation in the use of (81) is that it can be consistently treated on the classical level only. The difficulty to extend it to the quantum level is due to the Dirac strings. Indeed, perturbatively the Dirac veto is violated for virtual particles (see Subsections 4.2, 4.4). When the monopoles condense the Dirac strings are filling the whole of the vacuum and there are no known ways to rectify the perturbation theory. In case of the Abelian Higgs model, it is even more convenient to use the dual language when the Dirac strings are attached to the electric charges. Then, to respect the Dirac veto one should
impose the condition that the Higgs field vanishes along a line connecting charges [29]. The static Abrikosov-Nielsen-Olesen string satisfies this constraint [3]. However, there is no known complete set of solutions satisfying this boundary condition.

4.7 Conclusions # 3

The use of the Zwanziger Lagrangian is a convenient means to describe interaction of photon with both magnetic and electric charges. Formally, one introduces two potentials, \( A_\mu, B_\mu \). However, because of the condition \( F_{\mu\nu}(A) = \ast F_{\mu\nu}(B) \) there is a single photon in fact. One can get rid of one of the potentials. But then the Dirac string emerges explicitly and the formulation becomes non-local.

Another point, very important from the point of view of the phenomenological applications, is that in case of the gluodynamics the dual gluon (that is the field \( B \)) is still Abelian like although the “direct” gluons (that is the fields \( A^a \)) are in an adjoint representation of the non-Abelian group.

An important conclusion concerning the phenomenological Lagrangians is that observation of the Casimir scaling imposes the London limit, as far as one fits the lattice data with classical solutions. To avoid contradictions with fitting the structure of the confining string one should assume then that the applicability of the AHM is limited at short distances by existence of the non-Abelian gluons, while the parameterization in terms of a finite Higgs mass, \( m_H \) can be misleading. However, at present time there exists no explicit realization of this idea.

5 Monopoles in Lattice Gluodynamics

In this section we give the brief review of the numerical results obtained for monopoles on the lattice. We pay the special attention to two new topics: structure of the confining string and structure of monopoles in the Abelian projection. Most of the results are obtained for \( SU(2) \) lattice gauge theory for the maximal Abelian projection. The main definitions are given in Section 3.3.

5.1 Properties of Abelian Monopoles in \( SU(2) \) Lattice Gluodynamics

At first we list the main properties of the Abelian monopoles in \( SU(2) \) lattice gluodynamics in the maximal Abelian projection [20], the reader can find many important details in reviews [1].

1. In the confinement phase the monopole currents form a dense cluster, we call it infrared (IR) cluster, and there is a number of small mutually disjoint clusters, ”ultraviolet” (UV) clusters. In the deconfinement phase the monopole currents are dilute. In Figure 3 we demonstrate these facts showing the abelian monopole currents for the confinement (a) and the deconfinement (b) phases. The IR cluster percolates [30] and has a nontrivial
fractal dimension [31], $D_f > 1$. The properties of UV clusters differs much from those of the IR cluster, it can be shown that the IR monopole cluster is responsible for the confinement of quarks [32].

2. The $SU(2)$ string tension is well reproduced by the contribution of the abelian fields and/or abelian monopole currents [12]. This property is called monopole (Abelian) dominance. While the Abelian dominance is an almost trivial property of the system [33], the monopole dominance indicates that Abelian monopoles are important degrees of freedom for the infrared physics.

3. The monopole condensate is nonzero in the confinement phase of $SU(2)$ gluodynamics and it vanishes at the critical temperature corresponding to the deconfinement phase transition, thus it plays the role of the order parameter [34]. We can treat this fact as a justification of the dual superconductor model of the QCD vacuum, the monopoles playing the (dual) role of the Cooper pairs.

4. The monopoles are correlated with the density of $SU(2)$ action. The total action of $SU(2)$ fields is correlated with the total length of the monopole currents [35], so there exists a global correlation. The Abelian monopoles in the MaA projection are also locally correlated with the non-Abelian action density [36]. This fact shows that monopoles are some physical objects (not the artifacts of the singular gauge transformation), since by

Figure 3: The abelian monopole currents for the confinement (a) ($\beta = 2.4$, $10^4$ lattice) and the deconfinement (b) phases ($\beta = 2.8$, $12^3 \cdot 4$ lattice). All monopole currents are closed (conserved) due to the periodic boundary conditions.
definition we call the object physical if it carries the action.

5. The correlations of the monopole currents, the electric currents, the topological charge density and the action density was found in Refs. [37].

6. In Refs. [38] the effective Lagrangian for monopoles was reconstructed from numerical data for monopole currents for $SU(2)$ gluodynamics in the Maximal Abelian gauge. It occurs that this Lagrangian corresponds to the Abelian Higgs model, the monopoles are condensed and the classical string tension of the Abrikosov-Nielsen-Olesen string describes well the quantum string tension of $SU(2)$ gluodynamics. It means that the monopole degrees of freedom are important for the description of the gluodynamics at large distances.

5.2 Anatomy of the Confining String

The authors of Ref. [39] presented the results of the numerical study of the confining string in $SU(2)$ lattice gluodynamics. They measured the expectation value of the electric field, $\vec{E}$, and the expectation value of the monopole current, $\vec{k}$, near the line connecting the test quark–antiquark pair. For the test quarks placed at the $z$ axis, only the $z$ component, $E_z(\rho)$, of $\vec{E}$ and the azimuthal component, $k_\theta(\rho)$, of $\vec{k}$ were found to be non-vanishing (here $\rho = \sqrt{x^2 + y^2}$ is the distance from the center of the flux tube and the azimuthal angle $\theta$ is defined as usual: $\tan \theta = y/x$). It occurs [27] that these data can be perfectly described by the solution of the classical equations of motion for the Abelian Higgs model (88). In Fig. 1 by solid lines we show $E_z$ and $k_\theta$ for the Abrikosov string of the Abelian Higgs model (78). The discretized version of the classical equations of motion is used in order to imitate the coarseness of the lattice used in numerical simulations [39].

The parameters of the model corresponding to the best fit of the numerical data are:

$$Q_m/2\pi = 0.9519 \pm 0.0041,$$

Figure 4: The fit of the lattice data by the classical solution of the discretized version of the Abelian Higgs model (78).
\[ \begin{align*}
m_V &= 0.4522 \pm 0.0206 = (1.0351 \pm 0.0472) \text{ GeV}, \\
m_H &= 0.4747 \pm 0.0600 = (1.0866 \pm 0.1373) \text{ GeV}.
\end{align*} \]

It appears that vector and scalar masses are equal to each other within the errors, thus we are at the so-called Bogomolny limit of the model. The Abrikosov string tension turns out to be

\[ \sqrt{\sigma} = 0.1808 \pm 0.0213 \approx (414.8 \pm 48.9) \text{ MeV} \approx 0.94 \sqrt{\sigma_{SU(2)}}, \]

here \( \sigma_{SU(2)} \) is the string tension of \( SU(2) \) gluodynamics. To get the dimensional quantities we have used \( \sqrt{\sigma_{SU(2)}} = 440 \text{ Mev} \).

### 5.3 Anatomy of \( SU(2) \) Monopole

At small values of the bare charge (at large values of \( \beta \)) the compact electrodynamics is in the deconfinement phase, and gluodynamics is in the confinement phase. On the other hand, at large values of \( \beta \) in the maximal Abelian projection \( \cos \phi_l \) (we use the parameterization (40)) is close to unity (due to (38) and (41)), and the gluodynamic plaquette action \( \frac{1}{2} tr U_P \) is close to the action of cQED, \( \cos \theta_P \). Why the monopoles are not condensed in cQED and are condensed in gluodynamics if the actions of both theories are close to each other? It occurs that monopoles in gluodynamics have nontrivial structure and even near the continuum limit they differ much from the Abelian monopoles. The action of monopoles in gluodynamics is smaller than the action of monopoles in cQED. In Refs. [40] it was found that the action of the non-diagonal gluons, \( S_{\text{off}} \), on the plaquettes near the monopole is negative, and the full non Abelian action, \( S_{SU(2)} = S_{\text{off}} + S_{\text{Abel}} \), is smaller than the Abelian part of the action. Thus the action of monopoles in gluodynamics is smaller than that in Abelian theory and this is the possible explanation of the fact why the Abelian monopoles in gluodynamics are condensed at any value of the bare coupling.

As we discussed in Section 2.5 there exists the pure gauge field (27), (28) which Abelian (\( A_3^\mu \)) component contains Dirac monopole. The lattice analogue of this field has zero non-Abelian action since the action of the Dirac string is zero on the lattice. And this configuration presumably corresponds [13] to the fields inside Abelian monopoles in gluodynamics. At large distances due to quantum fluctuations the fields around monopole may have a finite non-Abelian action density.

Below we present the recent results [11] of numerical calculations in lattice \( SU(2) \) gauge theory which confirm the above discussion (see that paper for details). In Fig. 5 we show the dependence of \( \tilde{S}_{SU(2)} = < S_{SU(2)}^{\text{mon}} > \) on the half of the lattice spacing, \( a/2 \), in fermi\(^2\). The explanation of the scale of the horizontal axis is: \( < S_{SU(2)}^{\text{mon}} > \) is measured on the plaquettes which are faces of the cube dual to monopole, thus in a sense we are measuring the average field strength on the distance \( a/2 \) from the monopole center.

The circles on Fig. 5 correspond to the calculation which take into account all monopoles, the squares correspond to monopoles taken from the percolating cluster (IR cluster). The

\(^2\)We find the correspondence between the bare charge and lattice spacing fixing the value of the string tension \( \sigma = 440 \text{ MeV} \) and using the numerical data [42] for the string tension in lattice units, \( \sigma \cdot a^2 \).
Figure 5: The dependence of $\bar{S}^{SU(2)}$ on $a/2$, the calculations are performed on the lattices up to $30^4$.

results of the analogous calculation of the Abelian action near the monopole, $\bar{S}^{Abel} = < S_{mon}^{Abel} >$, are presented in Fig. 6.

The results which follow from Figs. 5, 6 can be summarized as follows:

- $\bar{S}^{SU(2)}$ at the center of the monopole belonging to the IR cluster is compatible with zero (at least it decreases when we approach the center). Thus at the center of monopole the fields are close to our prediction, eq. (28). The small value of $\bar{S}^{SU(2)}$ is the reason why IR monopole clusters percolate. Note that the percolation reflects the existence of the monopole condensate.

- $\bar{S}^{Abel}$ for monopoles belonging to IR and UV clusters is approximately the same. Thus there is no difference in the Abelian part of the monopole fields in IR and UV clusters.

- $\bar{S}^{Abel}$ increases when we approach the center of the monopole. The closer we are to the center (the larger $\beta$) the larger lattice we have to use since the properties of the IR monopole cluster are strongly affected by the finite volume effects [32]. This technical difficulty prevents us to decide whether the Abelian action density diverges or finite at the center of the monopole.

Note that these facts (and also Figs. 5, 6) are given for the action densities in lattice units, if some action density is constant in lattice units, it corresponds to $1/a^4$ (Coulombic) behavior of this density in the continuum limit.
5.4 Conclusions # 4

The results of numerical experiments in $SU(2)$ lattice gauge theory confirm the validity of the dual superconductor model of the gluodynamic vacuum. The monopoles are condensed in the confinement phase, and already at the classical level are responsible for 94% of the $SU(2)$ string tension. The structure of monopoles in gluodynamics is nontrivial, the non-diagonal gluons reduce the monopole action.

Acknowledgments

The authors are thankful to V.G. Bornyakov, T. Suzuki and A.I. Veselov for valuable discussions. M.N.Ch. and M.I.P. acknowledge the kind hospitality of the staff of the Max-Planck Institut für Physik (München), where the work was initiated. Work of M.N.C., F.V.G. and M.I.P. was partially supported by grants RFBR 99-01230a, RFBR 01-02-17456, INTAS 96-370, JSPS Grant in Aid for Scientific Research (B) (Grant No. 10440073 and No. 11695029) and CRDF award RP1-2103.

References

[1] P.A.M. Dirac, *Proc. Roy. Soc. A* 133 (1931) 60.

[2] S. Coleman, in *The Unity of the Fundamental Interactions*, Erice lectures (1981), ed. A. Zichichi (Plenum, London, 1983), p. 21; J. Preskill, *Ann. Rev. Nucl. Part. Sci.* 34 (1984) 461.

[3] M. Blagojevic and P. Senjanovic, *Phys. Rep.* 157 (1988) 233.
[4] Y. Nambu, *Phys. Rev.* **D10** (1974) 4662;  
G. 't Hooft, in "High Energy Physics", Proc. EPS Intern. Conf., ed A. Zichichi, Editrici Compositori, (1976);  
S. Mandelstam, *Phys. Rep.* **23** (1976) 245.

[5] N. Brambilla and A. Vairo, "Quark confinement and the hadron spectrum", hep-ph/9904330.

[6] A.A. Abrikosov, *Sov. Phys. JETP* **5** (1957) 1174;  
H.B. Nielsen and P. Olesen, *Nucl. Phys.* **B61** (1973) 45.

[7] M. Baker, J.S. Ball and F. Zachariasen, *Phys. Rep.* **209** (1991) 73; *Phys. Rev.* **D51** (1968) 1995.

[8] R.A. Brandt and F. Neri, *Nucl. Phys.* **B161** (1979) 253.

[9] T. Suzuki, *Nucl.Phys.Proc.Suppl.* **30** (1993) 176;  
M.I. Polikarpov, *Nucl.Phys.Proc.Suppl.* **53** (1997) 134;  
M.N. Chernodub and M.I. Polikarpov, in "Confinement, Duality and Non-perturbative Aspects of QCD", p.387, Ed. by Pierre van Baal, Plenum Press, 1998, hep-th/9710203;  
M.N. Chernodub, F.V. Gubarev, M.I. Polikarpov and A.I. Veselov, *Prog. Theor. Phys. Suppl.* **131** (1998) 309;  
H. Suganuma, H. Ichie, A. Tanaka and K. Amemiya, *Prog. Theor. Phys. Suppl.* **131** (1998) 559;  
A. Di Giacomo, *Prog. Theor. Phys. Suppl.* **131** (1998) 161;  
R.W. Haymaker, *Phys. Rep.* **315** (1999) 153.

[10] A.M. Polyakov, *Phys. Lett.* **B59** (1975) 82.

[11] G. 't Hooft, *Nucl. Phys.* **B190** (1981) 455.

[12] T. Suzuki and I. Yotsuyanagi, *Phys. Rev.* **D42** (1990) 4257;  
S. Hioki *et. al.*, *Phys. Lett.* **B272** (1991) 326, *Erratum-ibid.* **B281** (1992) 416;  
H. Shiba and T. Suzuki, *Phys. Lett.* **B333** (1994) 461;  
J.D. Stack, S.D. Neiman and R.J. Wensley, *Phys. Rev.* **D50** (1994) 3399;  
G.S. Bali, V. Bornyakov, M. Muller-Preussker and K. Schilling, *Phys. Rev.* **D54** (1996) 2863.

[13] M.N. Chernodub, F.V. Gubarev, M.I. Polikarpov and V.I. Zakharov, *Nucl. Phys.* **B592** (2000) 107; “Towards Abelian-like formulation of the dual gluodynamics”, hep-th/0007135, to appear in *Nucl. Phys.*; “On the heavy monopole potential in gluodynamics”, hep-ph/0101012.

[14] H. Ichie and H. Suganuma, "Dual Higgs theory for color confinement in quantum chromodynamics", hep-lat/9906005.
[15] K.I. Kondo and T. Shinohara, Phys. Lett. B491 (2000) 263; "Renormalizable Abelian projected effective gauge theory derived from quantum chromodynamics"; hep-th/0005125.

[16] J. Frohlich and P.A. Marchetti, “An order parameter reconciling Abelian and center dominance in SU(2) Yang-Mills theory”; hep-th/0011246.

[17] G. ’t Hooft, Nucl. Phys. B138 (1978) 1.

[18] A. Ukawa, P.Windley and A.Guth, Phys. Rev. D21 (1980) 1013; M. Srednicki and L. Susskind, Nucl. Phys. B179 (1980) 239.

[19] D. Zwanziger, Phys. Rev. D3 (1971) 343.

[20] C. Hoelbling, C. Rebbi and V.A. Rubakov, Phys. Rev. D63 (2001) 034506 P.de Forcrand, M. D’Elia and M. Pepe, “A study of the ’t Hooft loop in SU(2) Yang-Mills theory”; hep-lat/0007034.

[21] L. Del Debbio, M. Faber, J. Greensite and S. Olejnik, Phys. Rev. D53 (1996) 5891; Nucl.Phys.Proc.Suppl. 53 (1997) 141.

[22] G.S. Bali Phys. Rev. D62 (2000) 114503.

[23] V.I. Shevchenko and Yu. A. Simonov, Phys.Rev.Lett. 85 (2000) 1811.

[24] P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B125 (1977) 1.

[25] E. Lubkin, Ann. Phys. 23 (1963) 233.

[26] A. S. Kronfeld, M. L. Laursen, G. Schierholz and U. J. Wiese, Phys. Lett. B198 (1987) 516; A. S. Kronfeld, G. Schierholz and U. J. Wiese, Nucl. Phys. B293 (1987) 461.

[27] F.V. Gubarev, E.M. Ilgenfritz, M.I. Polikarpov and T. Suzuki, Phys. Lett. B468 (1999) 134.

[28] S.J. Huber, M. Reuter and M.G. Schmidt, Phys. Lett. B462 (1999) 158; E.V. Shuryak, "Scales and phases of nonperturbative QCD"; hep-ph/9911244.

[29] F.V. Gubarev, M.I. Polikarpov and V.I. Zakharov, Phys. Lett. B438 (1998) 147; Mod. Phys. Lett. A14 (1999) 2039.

[30] T. L. Ivanenko, A. V. Pochinsky and M. I. Polikarpov, Phys. Lett. B252 (1990) 631.

[31] T. L. Ivanenko, A. V. Pochinsky and M. I. Polikarpov, Phys. Lett. B302 (1993) 458.

[32] S. Kitahara, Y. Matsubara and T. Suzuki, Prog.Theor.Phys. 93 (1995) 1; A. Hart and M. Teper, Phys. Rev. D58 (1998) 014504.
[33] M. C. Ogilvie, *Phys. Rev.* **D59** (1999) 074505;  
M. Faber, J. Greensite and S. Olejnik, *JHEP* **9901** (1999) 008;  
J. Ambjorn and J. Greensite, *JHEP* **9805** (1998) 004.

[34] M.N. Chernodub, M.I. Polikarpov and A.I. Veselov, *Phys. Lett.* **B399** (1997) 267;  
L. Del Debbio, A. Di Giacomo, G. Paffuti and P. Pieri, *Phys. Lett.* **B355** (1995) 255;  
N. Nakamura, V.Bornyakov, S. Ejiri, S. Kitahara, Y. Matsubara and T. Suzuki,  
*Nucl. Phys. Proc. Suppl.* **53** (1997) 512;  
M. N. Chernodub, M. I. Polikarpov and A. I. Veselov, *JETP Lett.* **69** (1999) 174.

[35] H. Shiba and T. Suzuki, *Phys. Lett.* **B351** (1995) 519.

[36] B.L.G. Bakker, M.N. Chernodub and M.I. Polikarpov, *Phys. Rev. Lett.* **80** (1998) 30;  
B. L. Bakker, M. N. Chernodub, M. I. Polikarpov and A. I. Veselov, *Phys. Lett.* **B449** (1999) 267.

[37] M. Feurstein, H. Markum and S. Thurner, *Phys. Lett.* **B396** (1997) 203;  
E.–M. Ilgenfritz, H. Markum, M. Müller–Preussker and S. Thurner,  
*Nucl. Phys. Proc. Suppl.* **73** (1999) 560;  
S. Thurner, H. Markum, E.–M. Ilgenfritz and M. Müller–Preussker, “Gauge invariant properties of abelian monopoles”, [hep-lat/9809153](http://arxiv.org/abs/hep-lat/9809153);  
E.–M. Ilgenfritz, H. Markum, M. Müller–Preussker, W. Sakuler and S. Thurner,  
*Progr. Theor. Phys. Suppl.* **131** (1998) 353;  
V. Bornyakov and G. Schierholz, *Phys. Lett.* **B384** (1996) 190;  
M. N. Chernodub, F. V. Gubarev and M. I. Polikarpov, *JETP Lett.* **69** (1999) 169.

[38] S. Kato, N. Nakamura, T. Suzuki and S. Kitahara, *Nucl. Phys.* **B520** (1998) 323;  
S. Fujimoto, S. Kato and T. Suzuki, *Phys. Lett.* **B476** (2000) 437;  
M. N. Chernodub, S. Fujimoto, S. Kato, M. Murata, M. I. Polikarpov and T. Suzuki,  
*Phys. Rev.* **D62** (2000) 094506.

[39] K. Schilling, G. S. Bali, and C. Schlichter, *Prog. Theor. Phys. Suppl.* **131** (1998) 645;  
*Nucl. Phys. Proc. Suppl.* **63** (1998) 519;  
*Nucl. Phys. Proc. Suppl.* **73** (1999) 638.

[40] H. Suganuma, H. Ichie, A. Tanaka and K. Amemiya, *Prog. Theor. Phys. Suppl.* **131** (1998) 559;  
H. Ichie and H. Suganuma, *Nucl. Phys.* **B574** (2000) 70.

[41] V. G. Bornyakov, M. N. Chernodub, F. V. Gubarev, M. I. Polikarpov, T. Suzuki,  
A. I. Veselov and V.I. Zakharov, ”Anatomy of the lattice magnetic monopoles”, [hep-lat/0103032](http://arxiv.org/abs/hep-lat/0103032).

[42] J. Fingberg, U. M. Heller and F. Karsch, *Nucl. Phys.* **B392** (1993) 493.