Estimation of the Lin-Yang bound of the least static energy of the Faddeev model

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Lin and Yang’s upper bound $E_Q \leq c Q^\frac{3}{4}$ of the least static energy $E_Q$ of the Faddeev model in a sector with a fixed Hopf index $Q$ is investigated. By constructing an explicit trial configuration for the Faddeev field $n$, a possible value of the coefficient $c$ is obtained numerically, which is much smaller than the value obtained quite recently by analytic discussions.

§1. Introduction

It is now well known that Faddeev’s $O(3)$ nonlinear $\sigma$ model (Faddeev model) possesses solitons of knot type.\textsuperscript{1–5)} This model concerns the real scalar fields

$n(x) = (n^1(x), n^2(x), n^3(x)) \quad (1.1)$

satisfying

$n^2(x) = n(x) \cdot n(x) = \sum_{a=1}^{3} n^a(x)n^a(x) = 1. \quad (1.2)$

The Lagrangian density of this model is given by

$\mathcal{L}_F(x) = c_2 l_2(x) + c_4 l_4(x), \quad (1.3)$

$l_2(x) = \partial_\mu n(x) \cdot \partial^\mu n(x), \quad (1.4)$

$l_4(x) = -\frac{1}{4} H_{\mu\nu}(x) H^{\mu\nu}(x), \quad (1.5)$

$H_{\mu\nu}(x) = n(x) \cdot [\partial_\mu n(x) \times \partial_\nu n(x)]$

$= \epsilon_{abc} n^a(x) \partial_\mu n^b(x) \partial_\nu n^c(x), \quad (1.6)$

where $c_2$ and $c_4$ are constants. Faddeev and Niemi discussed that $n(x)$ is intimately related to the low energy dynamics of the $SU(2)$ non-Abelian gauge field.\textsuperscript{6)}

The static energy functional $E_F[n]$ associated with $\mathcal{L}_F(x)$ is given by

$E_F[n] = \int dV[c_2 \epsilon_2(x) + c_4 \epsilon_4(x)], \quad (1.7)$

$\epsilon_2(x) = \sum_{a=1}^{3} \sum_{i=1}^{3} [\partial_i n^a(x)]^2, \quad (1.8)$

$\epsilon_4(x) = \frac{1}{4} \sum_{i,j=1}^{3} [H_{ij}(x)]^2, \quad (1.9)$
with \((x, y, z)\) being Cartesian coordinates. If we assume that \(n(x)\) satisfy the boundary condition

\[
n(x) = (0, 0, 1) \quad \text{at } |x| = \infty
\]

and that \(n(x)\) are regular for \(|x| < \infty\), we can regard \(n\) as a mapping from \(S^3\) to \(S^2\). Such mappings are classified by the topological number \(Q_H[n]\) called the Hopf index, which is also a functional of \(n\). Vakulenko and Kapitanski\(\text{i}^7\) found that the lower bound of \(E_F[n]\) is given by a constant multiple of \(Q_H[n]^{3/4}\). With the help of the best Sobolev inequality, Kundu and Rybakov found the following inequality for general \(n\):\(^8\)

\[
E_F[n] \geq K \sqrt{c_2 c_4} \left| Q_H[n] \right|^{3/4},
\]

\[
K = 2^{7/2} 3^{3/8} \pi^2 = 168.587.
\]

Then a configuration \(n(x)\) with \(Q_H[n] \neq 0\) is stable against collapsing into a trivial configuration. On the other hand, Lin and Yang\(^9\) have shown recently that the least energy in a sector with a fixed \(Q_H[n]\) is bounded from the above: defining \(E_Q\) by

\[
E_Q = \text{Min}\{E_F[n] \mid Q_H[n] = Q\},
\]

they showed that \(E_Q\) satisfies the inequality

\[
E_Q \leq C \sqrt{c_2 c_4} Q^{3/4},
\]

where \(C\) is a constant independent of \(Q\). This \(Q^{3/4}\) upper bound of the minimal energy is important because it ensures the stability of the configuration against collapsing into widely separated \(Q\) lumps each of which has the Hopf index 1. As for the value of \(C\), Lin and Yang did not mention. We note that Adam, Sánchez-Guillén, Vázques and Wereszczyński gave an analytic estimation of \(C\) quite recently.\(^{10}\)

For small \(|Q|\), the configuration considered in ref. 5) might be a candidate which makes the energy minimal. This configuration, however, contains the Hopf index \(Q\) explicitly and the derivative of the fields \((n_1, n_2, n_3)\) contains terms linear in \(Q\). Then the energy density contains \(Q^2\) and \(Q^4\) terms and hence, with an appropriate choice of the scale parameter, the minimal energy becomes proportional to \(Q^3\) for large \(Q\). This behavior is quite different from the \(Q^{3/4}\) behavior implied by the Lin-Yang theorem. In other words, it seems that the configuration other than the one considered in ref. 5) must be sought to describe the minimal energy state with large \(Q\). The purpose of this paper is to seek a possible value of \(C\). Since the functional \(E_F[n]\) should be minimized for the true solution of the field equation of the Faddeev model, \(E_Q\) is smaller than the value of \(E_F[n]\) for an arbitrary trial configuration \(n\) which has \(Q_H[n] = Q\). We calculate \(E_F[n]\) numerically for a special trial configuration whose \(E_F[n]\) is bounded above by \(|Q_H[n]|^{3/4}\).

This paper is organized as follows. After a brief explanation of the Hopf index and the static energy functional of the Faddeev model, we explain an example of the
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spectrum in a sector with fixed Hopf index in Sec. II. We thus observe that the $Q^{3/4}$ upper bound is realized only by rather special configurations. In Sec. III, according to the suggestion by Lin and Yang, we investigate a special configuration. It turns out that $E_Q$ of this configuration indeed has the bound $CQ^{3/4}$. We thus obtain an explicit possible value of $C$. The final section is devoted to summary. We compare our numerical result with the analytic bound of $C$ given in ref. 10) and find that the value obtained in this paper is much smaller.

§2. Preliminaries: Hopf index, static energy functional and example of spectrum in Hopf sector

2.1. Hopf index

To define the Hopf index, it is convenient to introduce real fields $\Phi_\alpha(x)$ ($\alpha = 1, 2, 3, 4$) satisfying

$$\sum_{\alpha=1}^{4} [\Phi_\alpha(x)]^2 = 1. \quad (2.1)$$

The complex fields $Z_1(x)$, $Z_2(x)$ and a column vector $Z(x)$ are defined by

$$Z_1(x) = \Phi_1(x) + i\Phi_2(x),$$
$$Z_2(x) = \Phi_3(x) + i\Phi_4(x),$$
$$Z(x) = \begin{pmatrix} Z_1(x) \\ Z_2(x) \end{pmatrix}. \quad (2.2)$$

If we define the fields $n_a$ ($a = 1, 2, 3$) by

$$n_a(x) = Z_\dagger(x)\sigma^a Z(x), \quad (2.3)$$

with $\sigma^a$ ($a = 1, 2, 3$) being Pauli matrices, $n$ is expressed as

$$n = \begin{pmatrix} u + u^* \\ -i(u - u^*) \\ |u|^2 - 1 \end{pmatrix}, \quad (2.4)$$

where the complex function $u(x)$ is defined by

$$u(x) = \begin{pmatrix} Z_1(x) \\ Z_2(x) \end{pmatrix}. \quad (2.5)$$

If we define the vector potential $A(x) = (A_1(x), A_2(x), A_3(x))$ by

$$A_i(x) = \frac{1}{i} \{ Z_\dagger(x)[\partial_i Z(x)] - [\partial_i Z_\dagger(x)]Z(x) \}, \quad (2.6)$$

we see that $H_{ij}(x)$ defined by

$$H_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x) \quad (2.7)$$
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coincides with \( \mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n}) \) of Eq. (1.4). The Hopf index \( Q_H[\mathbf{n}] \) is now defined by

\[
Q_H[\mathbf{n}] = \frac{1}{16\pi^2} \int dV \mathbf{A}(x) \cdot \mathbf{B}(x),
\]

(2.8)

\[
B_i(x) = \frac{1}{2} \epsilon_{ijk} H_{jk}(x).
\]

(2.9)

Although there is no local formula expressing \( \mathbf{A}(x) \) in terms of \( \mathbf{n}(x) \), it is known that the Hopf index is calculated solely in terms of \( \mathbf{n} \). Another formula for \( Q_H[\mathbf{n}] \) is

\[
Q_H[\mathbf{n}] = \frac{1}{12\pi^2} \int dV \epsilon_{\alpha\beta\gamma\delta} \phi_{\alpha} \frac{\partial(\phi_{\beta}, \phi_{\gamma}, \phi_{\delta})}{\partial(x,y,z)},
\]

(2.10)

where \( \epsilon_{\alpha\beta\gamma\delta} \) is the four-dimensional Levi-Civita symbol satisfying \( \epsilon_{1234} = 1 \). From this formula, we can see that the allowed values of \( Q_H[\mathbf{n}] \) for regular \( \Phi_{\alpha}(x) \) (\( \alpha = 1, 2, 3, 4 \)) are integers.

2.2. Static energy functional

In terms of \( u \) and \( u^* \), the energy densities \( \epsilon_2(x) \) and \( \epsilon_4(x) \) are expressed as

\[
\epsilon_2(x) = \frac{4}{(1 + |u|^2)^2} (\nabla u \cdot \nabla u^*),
\]

(2.11)

\[
\epsilon_4(x) = -2 \frac{(\nabla u \times \nabla u^*)^2}{(1 + |u|^2)^4}.
\]

(2.12)

Defining \( E_2[\mathbf{n}] \) and \( E_4[\mathbf{n}] \) by

\[
E_2[\mathbf{n}] = \int dV \epsilon_2(x),
\]

(2.13)

\[
E_4[\mathbf{n}] = \int dV \epsilon_4(x),
\]

(2.14)

we have

\[
E_F = c_2 E_2[\mathbf{n}] + c_4 E_4[\mathbf{n}].
\]

(2.15)

From the dimension analysis, the volume integrals \( E_2[\mathbf{n}] \) and \( E_4[\mathbf{n}] \) are proportional to a scale parameter \( \alpha \) and its inverse \( \alpha^{-1} \), respectively. Then we have \( E = \alpha \cdot c_2 D_2 + (1/\alpha) \cdot c_4 D_4 \), where \( D_2 \) and \( D_4 \) are independent of \( \alpha \). By fixing \( \alpha \) appropriately, the minimum of \( E \) is obtained as

\[
E_F = \sqrt{c_2 c_4 J}, \quad J = 2\sqrt{D_2 D_4}.
\]

(2.16)

If the volume \( V \) of the integral consists of some pieces \( V_1, V_2, \cdots \), we have

\[
E_F = \sqrt{c_2 c_4 (J_1 + J_2 + \cdots)},
\]

(2.17)

where \( J_1, J_2, \cdots \) are obtained by taking the scale parameters \( \alpha_1, \alpha_2, \cdots \) of \( V_1, V_2, \cdots \) appropriately.
2.3. Example of spectrum in Hopf sector

Lin-Yang theorem concerns the minimal energy in the sector of a fixed Hopf index. To understand what type of energy spectrum is possible in such a sector, we here briefly discuss the case of the Aratyn-Ferreira-Zimerman (AFZ) configurations.\(^{11}\) They are the exact solutions of the model whose Lagrangian density is equal to \(l_4(x)^{3/4}\), where \(l_4(x)\) is defined in Eq.(1.5). The AFZ configuration is defined by

\[
\begin{align*}
\psi_{m,n}(\xi,\phi) &= m\xi + n\phi, \\
\eta &= \frac{cosh - \sqrt{\frac{n^2 m^2}{m^2} + sinh^2 \eta}}{\sqrt{1 + \frac{m^2}{n^2} sinh^2 \eta - cosh \eta}}, \\
u(x) &= f_{m,n}(\eta)e^{-i\psi_{m,n}}.
\end{align*}
\]

where \(\eta, \xi, \phi\) are toroidal co-ordinates and \(m\) and \(n\) are integers. For the above configuration, \(Q_H[n]\) is equal to \(mn\).

Substituting the above \(u(x)\) in Eqs.(2·11) and (2·12) and choosing the scale parameter appearing in the definition of the toroidal co-ordinate appropriately, we find that the energy is given by

\[
E_{m,n} = 8\pi^2 \sqrt{c_2 c_4} \sqrt{|Q| (A(p) + |Q|B(p))} \equiv \sqrt{c_2 c_4} F(Q, p) \tag{2·21}
\]

where \(p\) is defined by

\[
p = \left| \frac{n}{m} \right|. \tag{2·25}
\]

For fixed \(Q\), the parameter \(p\) can take several values. In Fig.1, the distribution of \(F(Q, p)\) is shown.
If we denote the smallest $F(Q,p)$ in a sector with fixed $Q$ by $F_Q$, we obtain Fig. 2.

Lin-Yang theorem asserts that, for the true solutions of the Faddeev model, all the points in the $[F_Q/Q^{3/4}]$-$Q$ diagram should lie below a certain horizontal line. Of course, in the trial AFZ configurations considered here, this property is not attained. We now proceed to consider what kind of configurations leads to the $Q^{3/4}$ upper bound.

§3. $Q^{3/4}$ upper bounds

To realize the upper bound of $E_Q$ of the form $CQ^{3/4}$, we consider a map which is a combined map of $g : \mathbb{R}^3 \rightarrow S^3$, $h : S^3 \rightarrow S^2$ and $v : S^2 \rightarrow (S^2)'$ where $(S^2)'$ is another 2-sphere. We denote the ball in $\mathbb{R}^3$ centered at $x$ with the radius $\alpha$ as
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We assume that, for $x \in B_\alpha(0)$, the map $g_\alpha : \mathbb{R}^3 \to S^3$ is the stereographic projection $(x, y, z) \to (X_1, X_2, X_3, X_4)$ defined by

\[
\left\{
\begin{array}{ll}
X_i = \frac{4f_\alpha(r)x_i}{f_\alpha(r)^2 + 4r} & (i = 1, 2, 3), \\
X_4 = \frac{f_\alpha(r)^2 - 4}{f_\alpha(r)^2 + 4}.
\end{array}
\right.
\] (3.1)

Here $f_\alpha(r)$ inside and outside the ball $B_\alpha(0)$ is defined by

\[
f_\alpha(r) = \begin{cases} 
\frac{r}{\alpha - r} & r < \alpha, \\
\infty & r \geq \alpha.
\end{cases}
\] (3.2)

$h$ is the Hopf map $(X_1, X_2, X_3, X_4) \to (N_1, N_2, N_3)$ defined by

\[
u = \frac{N_1 + iN_2}{1 - N_3} = \frac{X_1 + iX_2}{X_3 + iX_4}.
\] (3.3)

Then $h \circ g_\alpha : (x, y, z) \to (N_1, N_2, N_3)$ maps $B_\alpha(0)$ to $S^2$ once and the Hopf index associated with $h$ is 1. We denote a point of $S^2$ by $(\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)$ (3.4) and a point of $(S^2)'$ by $(\sin \Theta' \cos \Phi', \sin \Theta' \sin \Phi', \cos \Theta')$. (3.5)

We assume that the degree of the mapping $v : S^2 \to (S^2)'$ is $n$. Then the Hopf index associated with the map $v \circ h$ is equal to $n^2$. (3.6)

We define the map $w(r) : \mathbb{R}^3 \to S^2$ by

\[
w(r) = \begin{cases} 
(v \circ h \circ g_\alpha)(x) & x \in B_\alpha(0) \\
(h \circ g_\beta)(x - x_i) & x \in B_\beta(x_i) (i = 1, \ldots, m) \\
(0, 0, 1) & \text{otherwise}.
\end{cases}
\] (3.7)

Here we are assuming that $m + 1$ balls are far apart from each other and do not intersect. Then the Hopf index associated with the map $w$ is given by

\[Q_H[w] = n^2 + m.\] (3.8)

Considering the cases $Q_H[w] > 0$, $m$ can be assumed to satisfy

\[0 \leq m < 2n + 1\] (3.9)

without loss of generality.

The map $v$ is specified by fixing $\Theta'$ and $\Phi'$ as functions of $\Theta$ and $\Phi$. It should be chosen so as to make the static energy as small as possible. We first consider the case

\[
\Theta' = \begin{cases} 
0 : & 0 \leq \Theta \leq \frac{\pi}{4}, \\
2k \left(\Theta - \frac{\pi}{4}\right) : & \frac{\pi}{4} < \Theta < \frac{3\pi}{4}, \\
k\pi : & \frac{3\pi}{4} \leq \Theta \leq \pi,
\end{cases}
\] (3.10)
and

\[ \Phi' = k\Phi, \]

(3.10)

where \( k \) and \( l \) are positive integers. For \( 0 \leq \Theta \leq \pi \), \( 0 \leq \Phi \leq 2\pi \), \( \Theta' \) and \( \Phi' \) range from 0 to \( k\pi \) and from 0 to \( 2l\pi \), respectively. Thus the degree of mapping of \( v \) in this case is \( kl \). Although it is possible to modify the above configuration so that \( \Theta' \) is a smooth function of \( \Theta \) even at \( \Theta = \frac{\pi}{4} \) and \( \frac{3\pi}{4} \), we make use of the above \( \Theta' \) for simplicity since we encounter no difficulty in the numerical analysis. We also note that, if the region \( \frac{\pi}{4} < \Theta < \frac{3\pi}{4} \) is replaced by \( a < \Theta < b \) with \( 0 < a < \frac{\pi}{4}, \frac{3\pi}{4} < b < \pi \), we would obtain a better result for the upper bound of the static energy.

Now, in the case \( \mathbf{x} \in B_a(0) \), we have

\[
\tan \Theta = \frac{2|Z_1||Z_2|}{1 - 2|Z_2|^2} = \frac{2\sqrt{R(1 - R)}}{1 - 2R}, \tag{3.11}
\]

\[
\tan \Phi = \frac{X_1X_4 - X_2X_3}{X_1X_3 - X_2X_4} = \frac{1 - S\tan \phi}{S + \tan \phi}, \tag{3.12}
\]

\[
R = 1 - j\sin^2 \theta, \tag{3.13}
\]

\[
S = r \left( \frac{1}{r - 2\alpha} + \frac{1}{3r - 2\alpha} \right) \cos \theta, \tag{3.14}
\]

\[
j = \left[ \frac{4r(\alpha - r)}{r^2 + 4(\alpha - r)^2} \right]^2, \tag{3.15}
\]

where \((r, \theta, \phi)\) are polar coordinates of \( \mathbf{x} \). We also have the formulae

\[
\epsilon'_2(\mathbf{x}) = (\nabla \Theta')^2 + \sin^2 \Theta'(\nabla \Phi')^2, \tag{3.16}
\]

\[
\epsilon'_4(\mathbf{x}) = \frac{1}{2} \sin^2 \Theta'(\nabla \Theta' \times \nabla \Phi')^2, \tag{3.17}
\]

\[
\sin^2 \Theta' = \frac{1}{2} \left( 1 - (-1)^k \cos 4k\Theta \right), \tag{3.18}
\]

\[
\nabla \Theta' = \begin{cases}
2k\nabla \Theta : \quad \frac{\pi}{4} \leq \Theta \leq \frac{3\pi}{4}, \\
0 : \quad \text{otherwise}.
\end{cases} \tag{3.19}
\]

Since \( \Theta \) and \( \Phi \) are expressed by \( R, S \) and \( \phi \), we also need

\[
J \equiv (\nabla R)^2|_{\alpha = 1} = \frac{1024\omega^2(1 - r)^2(2 - r)^2(2 - 3r)^2}{(5r^2 - 8r + 4)^2}, \tag{3.20}
\]

\[
K \equiv (\nabla S)^2|_{\alpha = 1} = \frac{16(5r^2 - 8r + 4)^2(1 - \omega)}{3r^2 - 8r + 4}^{4}, \tag{3.21}
\]

\[
L \equiv (\nabla R \cdot \nabla S)|_{\alpha = 1} = \frac{128r(1 - r)\omega\sqrt{1 - \omega}}{(3r^2 - 8r + 4)(5r^2 - 8r + 4)^4}, \tag{3.22}
\]

where \( \omega \) is defined by

\[
\omega = \sin^2 \theta. \tag{3.23}
\]

Noting that \( \frac{\pi}{4} < \Theta|_{\alpha = 1} < \frac{3\pi}{4} \) corresponds to

\[
c_1 \equiv \frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \left[ \frac{4r(1 - r)}{5r^2 - 8r + 4} \right]^2 \omega \equiv j_1(r)\omega < \frac{1}{2} + \frac{1}{2\sqrt{2}} \equiv c_2, \tag{3.24}
\]

\[
\frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \left( \frac{4r(1 - r)}{5r^2 - 8r + 4} \right)^2 \omega \equiv j_1(r)\omega < \frac{1}{2} + \frac{1}{2\sqrt{2}} \equiv c_2.
\]
it is convenient to define $\int dS$ by

$$
\int dS = \left( \int_{r_1}^{r_2} dr + \int_{r_3}^{r_4} dr \right)\int_{\frac{c_1}{\sqrt{\omega}}}^{1} \frac{d\omega}{\sqrt{\omega(1-\omega)}} + \int_{\frac{c_2}{\sqrt{\omega}}}^{\frac{c_1}{\sqrt{\omega}}} \frac{d\omega}{\sqrt{\omega(1-\omega)}},
$$

(3.25)

where $r_1, r_4$ and $r_2, r_3$ ($0 < r_1 < r_2 < r_3 < r_4 < 1$) are solutions of $j_1(r) = c_1$ and $j_1(r) = c_2$, respectively. Defining $X, Y$ and $Z$ by

$$
Y = \frac{K}{(S^2)_{\alpha=1} + 1} + \frac{1}{r^2\omega},
$$

(3.26)

$$
Z = \frac{JK - L^2}{(S^2)_{\alpha=1} + 1} + \frac{J}{r^2\omega},
$$

(3.27)

we obtain

$$
\int_{B_\alpha(0)} dV' = \alpha \cdot \left[ 8\pi k^2 f + \pi l^2 g(k) \right],
$$

(3.28)

$$
\int_{B_\alpha(0)} dV' = \frac{1}{\alpha} \cdot 2\pi k^2 l^2 h(k),
$$

(3.29)

where $f, g(k)$ and $h(k)$ are defined by

$$
f = \int dS \frac{r^2 \sqrt{J}}{[R(1-R)]}_{\alpha=1},
$$

(3.30)

$$
g(k) = \int dS \left[ 1 - (-1)^k \cos 4k\Theta \right] \left( r^2 \sqrt{Y} \right),
$$

(3.31)

$$
h(k) = \int dS \left[ 1 - (-1)^k \cos 4k\Theta \right] \frac{r^2 \sqrt{Z}}{[R(1-R)]}_{\alpha=1}.
$$

(3.32)

With an appropriate choice of the parameter $\alpha$, we find that the contribution to the static energy from the ball $B_\alpha(0)$ is given by

$$
E_{B_\alpha(0)} = \sqrt{c_2 c_4 D(k,l)},
$$

(3.33)

where $D(k,l)$ is defined by

$$
D(k,l) = 2\pi \sqrt{2kl} \sqrt{[8k^2 f + l^2 g(k)]h(k)}.
$$

(3.34)

Similarly, the contribution to the static energy from each of the balls $B_\beta(x_i)$ ($i = 1, \cdots, m$) is given by

$$
E_{B_\beta(x_i)} = \sqrt{c_2 c_4 D(1,1)}.
$$

(3.35)

The total static energy is now given by

$$
E_{k,l} = \sqrt{c_2 c_4 [D(k,l) + mD(1,1)]},
$$

(3.36)
while the Hopf index corresponding to the case considered here is given by
\[ Q_H = (kl)^2 + m. \] (3.37)
It is easy to see that \( g(k) \) and \( h(k) \) have \( k \)-independent upper bounds. Including \( f \), they are numerically calculated as
\[ f = 14.9, \] (3.38)
\[ g(k) < 2 \int dS r^2 \sqrt{\omega Y} = 19.3 \] (3.39)
\[ h(k) < 2 \int dS \frac{r^2 \sqrt{\omega Z}}{R(1 - R)} = 268.4. \] (3.40)
We also obtain
\[ g(1) = 9.1, \] (3.41)
\[ h(1) = 126.5, \] (3.42)
\[ D(1, 1) = 1130.5. \] (3.43)
From the upper bounds obtained above, we can discuss the upper bound of \( E_Q \). If we consider the configurations with \( k = 1 \), we have
\[ E_Q < \sqrt{c_2 c_4} \left\{ 2\pi \sqrt{2} k^3 \sqrt{8f + g(k)} \frac{h(k) + D(1, 1)}{k} \right\} \]
\[ < \sqrt{c_2 c_4} \left[ 1711 k^3 + 1131m \right], \] (3.44)
\[ Q = k^4 + m. \] (3.45)
Here \( m \) should be assumed to satisfy
\[ 0 \leq m < (k + 1)^4 - k^4. \] (3.46)
With the aid of the inequality
\[ ak^3 + bm \leq (a + 4b)(k^4 + m)^\frac{3}{4}, \quad (0 < a, b, k, \quad 0 \leq m < (k + 1)^4 - k^4) \] (3.47)
we find that \( E_Q \) is bounded as
\[ E_Q \leq 6233 \sqrt{c_2 c_4} Q^\frac{3}{4}. \] (3.48)
We have thus seen that the Lin and Yang bound is indeed realized in the configuration considered above. From this example, we conclude that the coefficient \( C \) in Lin-Yang inequality should satisfy
\[ C \leq 6233 \sqrt{c_2 c_4}. \] (3.49)

\[ \S 4. \] Summary

We have investigated the upper bound of the least static energy \( E_Q \) of the Faddeev model in a sector with the Hopf index \( Q \). By making use of a trial configuration, we have obtained the bound \( E_Q \leq 6233 \sqrt{c_2 c_4} Q^{3/4} \). Recently Adam,
Sánchez-Guillén, Vázques, and Wereszczynski\textsuperscript{10}) discussed the same problem and gave an analytic bound for $E_Q$:

$$E_Q \leq \frac{160}{3\sqrt{2}} (4^3 + 2) \sqrt{10\pi^4} \sqrt{c_2c_4} \frac{Q^{3/4}}{c_4} = 7.667 \cdot 10^5 \sqrt{c_2c_4} \frac{Q^{3/4}}{c_4}. \quad (4.1)$$

We see that our numerical method gives a smaller value for the coefficient of the Lin-Yang bound of $E_Q$.

Acknowledgements

The authors are grateful to Shinji Hamamoto, Takeshi Kurimoto, Hiroshi Kakuhata for kind interest and N. Sawado for discussions.

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