A self-consistent solution in affine space with scalar field

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Abstract

Conformal connection of scalar field is shown to produce possible non-metricity in affine connection spaces. In case of self-consistent solution the non-metricity is a correction to background Riemannian structure with respect to gravitational constant and its magnitude may be essential in the early Universe.

1 Introduction

Riemannian structure of spacetime can be defined axiomatically. On the other hand, the idea to derive it from a more general setting is understandable. In this paper we address the following question: will spacetime possess a Riemannian structure starting from the simplest presumptions about external fields? One can assume the metric in self-consistent solution to be induced by the matter—a scalar field in the Universe. But its Riemannian structure need not be initially presumed, and the Lagrangian of the geometrical part is defined by the generalized curvature. Is the field in question conformal or not—it turns out to be the crucial point in this consideration and this is the main object of the paper. So, our basic assumption is that the matter defines the geometrical (not necessarily Riemannian) structure of spacetime [1].

2 Self consistent solution for a scalar field

Consider the full Lagrangian of the form

\[ L = -\frac{1}{2\kappa} R + L_{\text{ext}}, \]

where \( L_{\text{ext}} \) is the Lagrangian of matter fields, \( \kappa \) is the gravitational constant and \( R \) is the generalized curvature:

\[ R = g^{ik}(\Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik} \Gamma^m_{lm} - \Gamma^m_{il} \Gamma^l_{km}), \]

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where $\Gamma^i_{kl}$ is the generalized connection:

$$\Gamma^i_{kl} = \frac{1}{2} g^{ip} (g_{pk,l} + g_{pl,k} - g_{kl,p}) + T^i_{kl},$$

(3)

g^{ik}$ is the spacetime metric and $T^i_{pk}$ is a part of generalized connection that cannot be expressed via the metric tensor $g^{ik}$.

Self-consistent solutions are the solutions of Euler-Lagrange equations with respect to variables \(\{g^{ik}, \Gamma^i_{kl}, \text{matter fields}\}\). The reasons for non-metric structure of spacetime to occur worth more detailed study. We assume that the external fields are exhausted by a massive scalar field with conformal connection $\xi R$. The the generalized action reads:

$$S = \int_\Omega \left( -\frac{1}{2\kappa} R + \frac{1}{2} \partial_i \varphi \partial^i \varphi - \frac{1}{2} (m^2 - \xi R) \varphi^2 \right) \sqrt{-g} d\Omega. \quad (4)$$

Let us find the variation of each variable in (4). Introduce $\Lambda = -\frac{1}{2\kappa} + \frac{1}{2} \xi R \varphi^2$ and consider $\delta \Gamma^i_{kl}$.

$$\delta S_1 = -\int \{ \Lambda \left[ -\delta g^{ik} (\delta_i \delta^m_{kl} \Gamma^p_{lp} + \delta^m_{kl} \Gamma^p_{ip} - \delta_i \delta^m_{kl} \Gamma^p_{lp}) + g^{in} (\delta^m_{kl} \Lambda - \delta^m_{kl} \Lambda_{np} + \delta^m_{kl} \Lambda_{p}) \right] \delta \Gamma^i_{kl} \sqrt{-g} d\Omega. \quad (5)$$

The values $\delta g^{ik}$, $\delta \varphi$ and $\delta \Gamma^i_{kl}$ are independent on the equations of motion, therefore $\delta S_1 = 0$ and the expression in braces must vanish. It turns out that the affine connection is not accorded with the metric and we thus introduce the non-metricity tensor $T^i_{kl}$:

$$\Gamma^i_{kl} = \tilde{\Gamma}^i_{kl} + T^i_{kl},$$

where

$$\tilde{\Gamma}^i_{kl} = \frac{1}{2} g^{ip} (g_{pk,l} + g_{pl,k} + g_{kl,p}).$$

Then the term with $\tilde{\Gamma}^i_{kl}$ in (5) vanishes, and the remaining terms satisfy the following equations:

$$\Lambda \left[ -g^{nm} T^p_{ip} - \delta^m_{lp} g^{ik} T^m_{ik} + 2 g^{in} T^m_{il} \right] + g^{nm} \Lambda_{ip} - g^{np} \delta^m_{lp} \Lambda_{np} = 0. \quad (6)$$

Let

$$\alpha^m_{kl} = (\delta^m_{kl} \Lambda_{ip} - \delta^m_{lp} \Lambda_{kl}) / \Lambda.$$  

(7)

Then $2T^m_{kl} = \delta^m_{lp} T^p_{ip} + \delta^m_{np} g^{pq} T^m_{pq} + \alpha^m_{kl}$.

Multiplying (7) by $g^{kl}$, then by $\delta^m_{in}$ and summing up the results we get

$$T^i_{kl} = \frac{1}{2} a^i_{kl} - \frac{1}{6} (\delta^i_k a^p_{lp} + \delta^i_l a^p_{kp}). \quad (8)$$

2
So, we get
\[ \Gamma^i_{kl} = \tilde{\Gamma}^i_{kl} - \delta^i_k \Lambda, l / \Lambda. \]  
(9)
As a result, we find the Ricci tensor
\[ R_{ik} = \tilde{R}_{ik} + \tilde{\Gamma}^m_{im} \Lambda, m / \Lambda \]  
(10)
and the generalized curvature
\[ R = \tilde{R} + (g^{ik} \tilde{\Gamma}^m_{im} \Lambda, m - \tilde{\Gamma}^m_{im} \Lambda^i) \Lambda. \]  
(11)

The remaining equations read:
\[ \partial_i \varphi \partial_k \varphi - \frac{1}{2} g_{ik} (\partial_p \varphi \partial^p \varphi - m^2 \varphi^2) = \Lambda (R_{ik} - \frac{1}{2} g_{ik} R), \]
\[ \nabla_i \nabla^i \varphi + (m^2 - \xi R) \varphi = 0, \]
where \( \nabla_i \) stands for covariant derivative.

### 3 Non-metricity in self-consistent solutions

Recall that
\[ \Lambda = - \frac{1}{2 \kappa} + \frac{1}{2} \xi \varphi^2 = - \frac{1}{2 \kappa} (1 - \kappa \xi \varphi^2). \]
However,
\[ \Lambda, l / \Lambda \approx - \kappa \xi (\varphi^2)_{, l}. \]
since the gravitational constant is very small on time scales different from Plankean \( |\kappa \xi R \varphi^2 \ll 1| \). Therefore the torsion has the magnitude of the order of gravitational constant, thus the perturbation theory is applicable:
\[ \Gamma^i_{kl} \approx \tilde{\Gamma}^i_{kl} + \kappa \xi \delta^i_k (\varphi^2)_{, l} \]

On the other hand, suppose there is a domain where the matter is highly inhomogeneous and \( \varphi, l \) is large, that is, of order of the speed of the fields (for example, near black holes). Then the additional term will give a big impact (note that the series expansion with respect to \( \kappa \) still takes place as the expansion is carried out with respect to the field rather than to the speed) and \( |\tilde{\Gamma}^i_{kl}| \approx |\kappa \xi \delta^i_k (\varphi^2)_{, l}|. \) It worth mentioning that the impact of the scalar field to non-metricity is exclusively due to the conformal connection, and the self-consistent solution contains no non-metricity when \( \xi = 0 \).

If the metric in the Universe is conformally invariant \( g_{ik} = a^2(\tau) \eta_{ik} \) and \( \tilde{\Gamma}^i_{kl} = 4a_{i,l} / a \) (\( \tau \) is conformal time), then
\[ R_{ik} = \frac{4a_{i,k} - g_{ik} a_{l} a^l - g_{ik} a_{i} a - 2a a_{i,k}}{a^2} + \frac{a_{k} \Lambda_{, k} - 3a a_{i} \Lambda - g_{ik} \Lambda_{i} a^l}{\Lambda a} \]
In particular, for Friedmann metrics $a = a(t)$:

$$R = \tilde{R} - 6 a^\prime a \Lambda $$  

(\text{the derivatives are taken with respect to conformal time}). That is, even a uniform and isotropic space can possess non-metricity. For large values of non-metricity this can slow down the expansion of the Universe, since the main part of the energy will be wasted to non-metricity rather than to expansion \[2\]. It happens because the number of particles being created completely compensates the speed of expansions at early times \[3\]. As a result, the non-metricity can become very large for small $t$ ($nt \ll 1$).

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