Abstract—In this work, we study distributed sketching methods for large scale regression problems. We leverage multiple randomized sketches for reducing the problem dimensions as well as preserving privacy and improving straggler resilience in asynchronous distributed systems. We derive novel approximation guarantees for classical sketching methods and analyze the accuracy of parameter averaging for distributed sketches. We consider random matrices including Gaussian, randomized Hadamard, uniform sampling and leverage score sampling in the distributed setting. Moreover, we propose a hybrid approach combining sampling and fast random projections for better computational efficiency. We illustrate the performance of distributed sketches in a serverless computing platform with large scale experiments.

I. INTRODUCTION

We investigate distributed sketching methods in regression problems. In particular, we study parameter averaging for variance reduction. Averaging enables asynchronous distributed computation in large scale problems since it is not required to wait for all workers to finish. One can average the available parameters and obtain an approximate result.

We consider the well-known linear regression problem in the regime where there is more data samples than the number of features. Linear regression problems are commonplace in many different disciplines. Being able to solve large scale linear regression problems efficiently is crucial for many applications. In this paper, the setting we consider is a distributed system with \( q \) workers that work in parallel. The idea of applying randomized sketches to linear regression and other optimization problems has been extensively studied in the recent literature by works including \([3], [14], [17], [5], [7], [10]\). In this work, we investigate averaging the solutions of sketched sub-problems. This setting was studied by \([14]\). In addition, we also consider regression problems where the number of data samples is less than the dimensionality and investigate the properties of averaging such problems.

We note that in distributed sketching, all of the computation nodes are independent and identically distributed copies of each other. Therefore this offers an extremely resilient computing model where node failures, straggler workers as well as additions of new nodes are readily handled. An alternative to averaging that would offer similar benefits is the asynchronous SGD algorithm \([12]\). To compare, one can expect convergence in higher number of updates with asynchronous SGD and convergence guarantees may depend on the properties of the input data such as the condition number whereas in distributed sketching, the theoretical guarantees we obtain do not depend on the condition number \([1]\).

In order to illustrate straggler resilience, we have implemented distributed sketching for AWS Lambda, which is a serverless computing platform. In serverless computing, one does not need to manage servers but instead has access to a high number of serverless compute functions each with limited resources. If one can divide their problem into many sub-problems with no communication in-between, serverless computing platforms can offer many advantages over server-based computing platforms due to their agility, scalability, and pricing model. Given that in this work we are interested in solving large scale problems with number of data samples much higher than the dimensionality, it suits to use serverless computing in solving these types of problems using model averaging. We discuss how the averaging methods scale and perform for different large scale datasets on AWS Lambda \([4]\).

Data privacy is an increasingly important issue in cloud computing, one that has been studied in recent works including \([17], [13], [16]\). Distributed sketching comes with the benefit of privacy preservation. To clarify, let us consider a setting where the master node computes sketched data \( S_kA, S_kb, k = 1, \ldots, q \) locally where \( S_k \in \mathbb{R}^{m \times n} \) are the sketching matrices and \( A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n \) are the data matrix and the output vector, respectively. The master node sends only the sketched data to worker nodes for computational efficiency, as well as preserving data privacy. In particular, the mutual information between \( S_kA \) and \( A \) per entry scales as \( O(m/n) \), which can be controlled by choosing small sketch dimensions \( m \).

We study various sketching methods in the context of averaging including Gaussian sketch, randomized orthonormal systems based sketch, uniform sampling, and leverage score sampling. In addition, we discuss a hybrid sketching approach and illustrate its performance through numerical examples.

A. Related Work and Main Contributions

The work \([14]\) investigates model averaging for regression from optimization and statistical perspectives. Our work studies sketched model averaging from the optimization perspective. The most relevant result of \([14]\), using our notation, can be stated as follows. Setting the sketch dimension \( m = O(\mu d(\log d) / \epsilon) \) for uniform sampling (where \( \mu \) is row coherence) and \( m = \tilde{O}(d/\epsilon) \) (with \( \tilde{O} \) concealing logarithmic factors) for other sketches, the inequality \( f(\bar{x}) - f(x^*) \leq (\epsilon/q + \epsilon^2) f(x^*) \) holds with high probability. According to this
result, for large \( q \), the cost of the averaged solution \( f(\bar{x}) \) will be less than \( \epsilon^2 f(x^*) \). We prove in this work that for Gaussian sketch, \( \mathbb{E}[f(\bar{x})] \) will converge to the optimal cost \( f(x^*) \) as \( q \) increases. We identify the exact expected error for a given number of workers \( q \) for Gaussian sketch.

We show that the expected difference between the costs for the averaged solution and the optimal solution has two components, namely variance and bias squared (see Lemma 2). This result implies that for Gaussian sketch, which we prove to be unbiased (see Lemma 1), the number of workers required for a given error \( \epsilon \) scales with \( 1/\epsilon \). For the Hogwild algorithm [12], which is also asynchronous, but addresses a more general class of problems, the number of iterations required for error \( \epsilon \) scales with \( \log(1/\epsilon)/\epsilon \) and also depends on the input data.

We derive upper bounds for the estimator biases for additional sketching matrices including randomized orthonormal systems (ROS) based sketch, uniform sampling, and leverage score sampling sketches. Analysis of the bias term is critical in understanding how close to the optimal solution we can hope to get and how bias depends on the sketch dimension \( m \).

II. PRELIMINARIES

We consider the linear least squares problem given by \( x^* = \arg\min_x \|Ax - b\|^2 \), \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \). We assume a distributed setting with \( q \) workers running in parallel and a central node (i.e. master node) that forms the averaged solution \( \bar{x} \):

\[
\bar{x} := \frac{1}{q} \sum_{k=1}^{q} \hat{x}_k, \quad \text{where} \quad \hat{x}_k = \arg\min_x \|S_k(Ax - b)\|^2_2,
\]

(1)

where \( \hat{x}_k \) is the output of the \( k \)th worker. The matrices \( S_k \in \mathbb{R}^{m \times n} \) are independent random sketch matrices that satisfy \( \mathbb{E}[S_k^T S_k] = I_n \). The algorithm is listed in Algorithm 1.

Algorithm 1: Distributed sketching for least squares.

**Input:** Data matrix \( A \in \mathbb{R}^{n \times d} \), target vector \( b \in \mathbb{R}^n \).

**Workers** \( k = 1, \ldots, q \) in parallel:

1. Sample \( S_k \in \mathbb{R}^{m \times n} \).
2. Compute sketched data \( S_kA \) and \( S_kb \).
3. Solve \( \hat{x}_k = \arg\min_x \|S_k(Ax - b)\|^2_2 \), send to master.

**Master:** return \( \bar{x} = \frac{1}{q} \sum_{k=1}^{q} \hat{x}_k \).

The expected difference between the costs for the averaged solution \( f(\bar{x}) \) and the optimal solution \( f(x^*) \) is equal to:

\[
\mathbb{E}[f(\bar{x})] - f(x^*) = \mathbb{E}[\|A(\bar{x} - x^*) + Ax^* - b\|_2^2] - f(x^*) = \mathbb{E}[\|A(\bar{x} - x^*)\|^2_2 + \|Ax^* - b\|^2_2] - f(x^*) = \mathbb{E}[\|A(\bar{x} - x^*)\|^2_2],
\]

(2)

where the second line follows from the orthogonality property of the optimal least squares solution \( x^* \) given by the normal equations \( A^T (Ax^* - b) = 0 \).

Throughout the text, the notation we use for the SVD of \( A \in \mathbb{R}^{n \times d} \) is \( A = U\Sigma V^T \). Whenever \( n \geq d \), we assume \( A \) has full column rank, hence the dimensions for \( U, \Sigma, V \) are as follows: \( U \in \mathbb{R}^{n \times d} \), \( \Sigma \in \mathbb{R}^{d \times d} \), and \( V^T \in \mathbb{R}^{d \times d} \). We use \( u_i^T \) to refer to the \( i \)th row of \( U \).

III. ERROR OF A SINGLE GAUSSIAN SKETCH

We first obtain a characterization of the expected prediction error of a single sketched least squares solution.

**Lemma 1.** For the Gaussian sketch, the estimator \( \hat{x}_k \) satisfies

\[
\mathbb{E}[\|A(\hat{x}_k - x^*)\|^2_2] = \mathbb{E}[f(\hat{x}_k)] - f(x^*) = f(x^*) \frac{d}{m - d - 1},
\]

which is valid for \( m \geq d + 1 \).

To the best of our knowledge, this result is novel in the theory of sketching. Existing results (see e.g. [7], [14], [15]) characterize a high probability upper bound on the error, whereas the above is a sharp and exact formula for the expected squared norm of the error.

**Proof.** Suppose that the matrix \( A \) is full column rank. Then, for \( m \geq d \), the matrix \( A^T S_k^T S_k A \) follows a Wishart distribution, and is invertible with probability one. Conditioned on the invertibility of \( A^T S_k^T S_k A \), we have

\[
\hat{x}_k = (A^T S_k^T S_k A)^{-1} A^T S_k^T S_k b
\]

\[
= (A^T S_k S_k A)(A^T S_k^T S_k A)^{-1} A^T S_k^T S_k b
\]

\[
= x^* + (A^T S_k^T S_k A)^{-1} A^T S_k^T S_k b,
\]

where we have defined \( b^\perp = b - Ax^* \). Note that \( S_k A \) and \( S_k b^\perp \) are independent random matrices since they are Gaussian and uncorrelated as a result of the normal equations \( A^T b^\perp = 0 \). Conditioned on the realization of the matrix \( S_k A \) and the event \( A^T S_k^T S_k A \succ 0 \), a simple covariance calculation shows that

\[
\hat{x}_k \sim \mathcal{N}(x^*, \frac{1}{m} f(x^*)(A^T S_k^T S_k A)^{-1}).
\]

Multiplying with the data matrix \( A \) on the left yields the distribution of the prediction error, conditioned on \( S_k A \)

\[
A(\hat{x}_k - x^*) \sim \mathcal{N}(0, \frac{1}{m} f(x^*) A(A^T S_k^T S_k A)^{-1} A^T).
\]

Then we can compute the conditional expectation of the squared norm of the error

\[
\mathbb{E}[\|A(\hat{x}_k - x^*)\|^2_2 | S_k A] = \frac{f(x^*)}{m} \mathbb{E}[\text{tr}(A(A^T S_k^T S_k A)^{-1} A^T)].
\]

Next we recall that the expected inverse of the Wishart matrix \( A^T S_k^T S_k A \) satisfies (see, e.g., [6])

\[
\mathbb{E}[(A^T S_k^T S_k A)^{-1}] = A^T A \frac{m}{m - d - 1}.
\]

Plugging in the previous result, using the tower property of expectations and noting that \( \text{tr}(A(A^T)^{-1} A) = \text{rank} A = d \), we obtain the claimed result.

**Theorem 1 (Cost Approximation).** Let \( S_k, k = 1, \ldots, q \) be Gaussian sketches, then Alg. 7 runs in time \( O(md^2) \) and

\[
\mathbb{E}[f(\bar{x})] - f(x^*) = \frac{1}{q} \frac{d}{m - d - 1}.
\]
given that $A^T S_k^T S_k A > 0$ for all $k = 1, ..., q$. Equivalently,
\[
\frac{f(\bar{x}) - f(x^*)}{f(x^*)} \leq \frac{\epsilon}{q}
\]

holds with probability at least $(1 - e^{-c_1 m})^q \left(1 - \frac{d}{\epsilon m - d - 1}\right)$.

**Proof of Theorem 1.** Because the Gaussian sketch estimator is unbiased (i.e., $\mathbb{E}[\hat{x}_k] = x^*$), Lemma 3 reduces to
\[
\mathbb{E}[f(\bar{x})] - f(x^*) = 1 - \epsilon \mathbb{E}[\|A\hat{x}_1 - Ax^*\|_2^2].
\]
By Lemma 2, the error of the averaged solution conditioned on the events that $E_k = A^T S_k^T S_k A > 0$, $\forall i = 1, ..., q$ can exactly be written as
\[
\mathbb{E}[\|A(\bar{x}_i - x^*)\|_2^2 | E_1 \cap \ldots \cap E_q] = \frac{d}{q a m - d - 1} f(x^*).
\]

Using Markov’s inequality, it follows that
\[
P(\|A(\bar{x}_i - x^*)\|_2^2 \geq a | E_1 \cap \ldots \cap E_q) \leq \frac{d}{qa m - d - 1} f(x^*).
\]

The LHS can be rewritten as
\[
\text{LHS} = \frac{P(\|A(\bar{x}_i - x^*)\|_2^2 \geq a \cap (\bigcap_{k=1}^q E_k))}{P(\bigcap_{k=1}^q E_k)}
\]
\[
\geq P(\|A(\bar{x}_i - x^*)\|_2^2 \geq a) \frac{P(\bigcap_{k=1}^q E_k)}{P(\bigcap_{k=1}^q E_k) - 1}
\]
\[
= P(\|A(\bar{x}_i - x^*)\|_2^2 \geq a) + P(E_1)^q - 1
\]
\[
= \frac{P(E_1)^q - 1}{P(E_1)^q}
\]

where we have used the identity $P(A \cap B) \geq P(A) + P(B) - 1$ in [3] and the independence of the events $E_k$ in [4]. It follows
\[
P(\|A(\bar{x}_i - x^*)\|_2^2 \leq a) \geq P(E_1)^q \left(1 - \frac{1}{q a m - d - 1} f(x^*)\right)
\]

Setting $a = f(x^*)^2 \gamma^2$ and plugging in $P(E_1) \geq 1 - e^{-c_1 m}$ where $c_1$ is a constant [14], we obtain:
\[
P\left(\frac{\|A(\bar{x}_i - x^*)\|_2^2}{f(x^*)} \leq \frac{\epsilon}{q}\right) \geq (1 - e^{-c_1 m})^q \left(1 - \frac{d}{\epsilon m - d - 1}\right)
\]

\[\Box\]

**A. Privacy Preserving Property**

We now digress from the convergence properties of Gaussian sketch to state the following result on privacy. Let us assume the data matrix $A$ is randomly sampled from a distribution with finite variance $\gamma^2$ as in [16] and [10]. Then, the mutual information per symbol between $S_k A$ (data seen by worker $k$) and $A$ is upper bounded by,
\[
\frac{I(S_k A; A)}{md} \leq \frac{m}{n} \log(2\pi e) \gamma^2.
\]

We note that as $n \to \infty$ for fixed $m$, the mutual information per matrix entry approaches zero. It suffices to choose $m$ proportional to $d$ as stated in Theorem 1.

**IV. OTHER SKETCHING MATRICES**

In this section, we consider three additional sketching matrices, namely, randomized orthonormal systems (ROS) based sketching, uniform sampling, and leverage score sampling. For each of these sketches, we present upper bounds on the norm of the bias. In particular, the results of this section focus on the bias bounds for a single output $\hat{x}_k$, and the way these results are related to the averaged solution $\bar{x}$ is through Lemma 2. Lemma 2 expresses the expected objective value difference in terms of bias and variance of a single estimator. It is possible to obtain high probability bounds for $(f(\bar{x}) - f(x^*)) / f(x^*)$ for these other sketches based on the bias bounds given in this section, using an argument similar to the one given in the proof of Theorem 1. This approach would involve defining an event that bounds the error for the single sketch estimator. We skip the details in this version of the work.

**Lemma 2.** For any i.i.d. sketching matrices $S_k$, $k = 1, ..., q$, the expected objective value difference can be decomposed as
\[
\mathbb{E}[f(\bar{x}_i) - f(x^*)] = \frac{1}{q} \mathbb{E} \left[\|A\hat{x}_1 - Ax^*\|_2^2\right] + \frac{q - 1}{q} \mathbb{E}[Ax^*_1] - Ax^*\|_2^2,
\]
where $\hat{x}_1$ is the solution returned by worker $k = 1$ (in fact it could be any of the worker outputs $\hat{x}_k$ as they are all statistically identical).

**Lemma 3.** Let $z := U^T S_k^T S_k b^\perp$ and $Q := (U^T S_k^T S_k U)^{-1}$ where $b^\perp = b - Ax^*$. For $S_k$ any sketch matrix with $\mathbb{E}[S_k^T S_k] = I_n$ and assuming $(1 - \epsilon)I_d \preceq Q \preceq (1 + \epsilon)I_d$, the norm of the bias for a single sketch is upper bounded as:
\[
\|\mathbb{E}[^{\hat{x}_k} - Ax^*\|_2 \leq \sqrt{4 \epsilon \mathbb{E}[\|\|_2^2].
\]

**A. Randomized Orthonormal Systems (ROS) based Sketches**

We consider the sketching matrix given by $S = PHD$ where $P$ is for sampling $m$ rows out of $n$, $H$ is the $(n \times n)$-dimensional Hadamard matrix, and $D$ is a diagonal matrix with diagonal entries sampled from the Rademacher distribution. Randomized Hadamard based sketch has the advantage of lower computational complexity ($O(n \log n)$) over Gaussian sketch ($O(n^3)$).

**Lemma 4.** Let $S_k$ be the ROS sketch and $z := U^T S_k^T S_k b^\perp$. Then we have
\[
\mathbb{E}[\|z\|_2^2] \leq \frac{d}{m} \left(1 - 2 \min\{\|\hat{u}\|_2^2\} f(x^*)\right),
\]
\[
\mathbb{E}[Ax^*_k] - Ax^*\|_2 \leq \sqrt{4 \epsilon \frac{d}{m} f(x^*)}.
\]

**B. Uniform Sampling**

In uniform sampling, each row of $S$ consists of a single 1 and $(n - 1)$ 0’s and the position of 1 in every row is distributed according to uniform distribution. Then, the 1’s in $S$ are scaled so that $\mathbb{E}[S^T S] = I_n$. We note that the bias
for uniform sampling is different when it is done with or without replacement. Lemma 5 gives bounds for both cases and verifies that the bias of \( \varepsilon_k \) is less if sampling is done without replacement.

**Lemma 5.** Let \( S_k \) be the sketching matrix for uniform sampling and \( z = U^T S_k^T S_k b^\perp \). Then we have

\[
\mathbb{E}[\|z\|_2^2] \leq \frac{n}{m} f(x^*) \max_i \|\tilde{u}_i\|_2^2, \tag{10}
\]

\[
\mathbb{E}[\|z\|_2^2] \leq \frac{n-m}{m} f(x^*) \max_i \|\tilde{u}_i\|_2^2, \tag{11}
\]

for sampling with and without replacement, respectively. The norm of the bias satisfies

\[
\|\mathbb{E}[A\varepsilon_k] - Ax^*\|_2 \leq \sqrt{4\frac{n}{m} f(x^*) \max_i \|\tilde{u}_i\|_2^2}, \tag{12}
\]

\[
\|\mathbb{E}[A\varepsilon_k] - Ax^*\|_2 \leq \sqrt{4\frac{n-m}{m} f(x^*) \max_i \|\tilde{u}_i\|_2^2}, \tag{13}
\]

for sampling with and without replacement, respectively.

**C. Leverage Score Sampling**

Row leverage scores of \( A = U\Sigma V^T \) are given by \( \ell_i = \|\tilde{u}_i\|_2^2 \) for \( i = 1, \ldots, n \) where \( \tilde{u}_i \) denotes the \( i \)th row of \( U \). There is only one nonzero element in every row of the sketching matrix \( S \) and the probability that the \( j \)th entry of \( s_i \) is nonzero is proportional to the leverage score \( \|\tilde{u}_j\|_2^2 \). That is, \( \mathbb{P}[s_{ij} \neq 0] = \frac{\|\tilde{u}_i\|_2^2}{\sum_{j=1}^n \|\tilde{u}_j\|_2^2} \). The denominator is equal to \( d \) because it is equal to the Frobenius norm of \( U \) and the columns of \( U \) are normalized. So, \( \mathbb{P}[s_{ij} \neq 0] = \frac{1}{d} \|\tilde{u}_i\|_2^2 = \frac{1}{d} \ell_i \).

**Lemma 6.** Let \( S_k \) be the leverage score sampling based sketch and \( z = U^T S_k^T S_k b^\perp \). Then we have

\[
\mathbb{E}[\|z\|_2^2] \leq \frac{d}{m} f(x^*), \tag{14}
\]

\[
\|\mathbb{E}[A\varepsilon_k] - Ax^*\|_2 \leq \sqrt{4\frac{d}{m} f(x^*)}. \tag{15}
\]

**D. Hybrid Sketch**

In a distributed computing setting, the amount of data that can be fit into the memory of a worker and the size of the largest problem that can be solved by that worker often do not match. The hybrid sketch idea is motivated by this mismatch and it is basically a sequentially concatenated sketching scheme where we first perform uniform sampling with dimension \( m' \) and then sketch the data using another sketch preferably with better convergence properties (say, Gaussian) with dimension \( m \). In other words, the worker reads \( m' \) rows of the data matrix \( A \) and sketches this down to \( m \) rows. If \( m' = m \), hybrid sketch reduces to sampling; if \( m' = n \), it reduces to Gaussian sketch; hence hybrid sketch can be thought of as middle ground between these. We note that this scheme does not take privacy into account as workers are assumed to have access to the original data.

We present experimental results in the numerical results section showing the practicality of the hybrid sketch idea. For the experiments involving very large-scale datasets, we have used Sparse Johnson-Lindenstrauss Transform (SJLT) \([8]\) as the second sketching method in the hybrid sketch due to its low computational complexity.

**V. DISTRIBUTED SKETCHING FOR LEAST-NORM PROBLEMS**

Now we consider the high dimensional case where \( n < d \) and applying the sketching matrix from the right, i.e., on the features. Let us define the minimum norm solution

\[
x^* = \arg \min_x \|x\|_2^2 \quad \text{s.t.} \quad Ax = b. \tag{16}
\]

The solution to the above problem is given by \( x^* = A^T (AA^T)^{-1} b \) when the matrix \( A \) is full row rank. We will assume that the full row rank condition holds in the sequel. Let us denote the optimal value of the minimum norm objective as \( f(x^*) = \|x^*\|_2^2 = b^T (AA^T)^{-1} b \). Then we consider the approximate solution given by \( \hat{x}_k = S_k^T \hat{\varepsilon}_k \) and \( \hat{\varepsilon}_k \) is given by

\[
\hat{\varepsilon}_k = \arg \min_{\varepsilon_k} \|\varepsilon_k\|_2^2 \quad \text{s.t.} \quad A S_k^T \varepsilon_k = b, \tag{17}
\]

where \( S_k \in \mathbb{R}^{m \times d} \) and \( z \in \mathbb{R}^n \). The averaged solution is computed as \( \bar{x} = \frac{1}{q} \sum_{k=1}^q \hat{x}_k \).

Now we consider sketching matrices that are i.i.d. Gaussian.

**Lemma 7.** For the Gaussian sketch, the estimator \( \hat{x}_k \) satisfies

\[
\mathbb{E}[\|\hat{x}_k - x^*\|_2^2] = \frac{d-n}{m-n-1} f(x^*),
\]

which is valid for \( m > n + 1 \).

**Proof.** It follows that conditioned on \( A S_k^T \), we have

\[
\hat{x}_k \sim \mathcal{N}(x^*, P_{\text{Null}(A)} (A S_k^T (A S_k^T S_k A^T)^{-1} b)_{22}).
\]

Noting that \( \mathbb{E}[A S_k^T S_k A^T]^{-1} = A A^T \frac{m}{m-n-1} \), taking the expectation and noting that \( \text{tr}(P_{\text{Null}(A)}) = d - n \), we obtain

\[
\mathbb{E}[\|\hat{x}_k - x^*\|_2^2] = \frac{d-n}{m-n-1} b^T (AA^T)^{-1} b = \frac{d-n}{m-n-1} f(x^*).
\]

An exact formula for averaging multiple sketches that parallels Theorem 1 can be obtained in a similar fashion. We omit the details.

**VI. NUMERICAL RESULTS**

We have implemented our distributed sketching methods for AWS Lambda in Python using Pywren [4], which is a framework for serverless computing. The setting is a centralized computing model where a single master node collects and averages the outputs of the \( q \) worker nodes. Most of the figures in this section plot approximation error against time or the number of averaged outputs where we refer to \( (f(\bar{x}) - f(x^*)) / f(x^*) \) by approximation error.
A. Airline Dataset

We have conducted experiments in the publicly available Airline dataset [9]. This dataset contains information on domestic USA flights between the years 1987-2008. We are interested in predicting whether there is going to be a departure delay or not, based on information about the flights. The dataset contains information for around 120 million flights with each flight having 29 attributes. More precisely, we are interested in predicting whether \( \text{DepDelay} > 15 \) minutes using the attributes Month, DayofMonth, DayOfWeek, CRSDepTime, CRSElapsedTime, Dest, Origin, and Distance. Most of these attributes are categorical and we have used dummy coding to convert these categorical attributes into binary representations. The size of the input matrix \( A \), after converting categorical features into binary representations, becomes \((1.21 \times 10^8) \times 774\).

We have solved the linear least squares problem on the entire airline dataset: minimize \( \left\| Ax - b \right\|_2^2 \) using \( q \) workers on AWS Lambda. The output \( b \) for the plots a and b in Fig. 1 is a vector of binary variables indicating delay. The output \( b \) (planted) for the plots c and d in Fig. 1 is generated via \( b = Ax_{\text{truth}} + \epsilon \) where \( x_{\text{truth}} \) is the underlying solution and \( \epsilon \) is random Gaussian noise distributed as \( \mathcal{N}(0,0.01I) \). Fig. 1 shows that sampling followed by SJLT leads to a lower error because it has less bias than sampling.

Note that it makes sense to choose \( m \) and \( m' \) as large as the available resources allow because for larger \( m \) and \( m' \), the convergence is faster. Based on the run times given in the caption of Fig. 1 we see that workers take longer to finish their tasks if SJLT is involved. Decreasing \( m' \) will help reduce this processing time at the expense of error performance.

In the experiments shown in Fig. 1 the worker nodes perform sketching during run time. For a privacy sensitive dataset, the master node would sketch the dataset as a first step and then send sketched data to workers. In particular, the mutual information bound given by [8] evaluates to the following for the airline dataset (taking \( \gamma = 1 \), \( m = 5 \times 10^5 \)):

\[
\frac{I(S_k;A;A)}{nd} \leq \frac{m}{n} \log(2\pi e\gamma^2) = 1.17 \times 10^{-2}.
\]

Choosing a smaller \( m \) for better a privacy bound does not in fact lead to higher bias in the case of Gaussian sketch as shown in Lemma 1.

B. Image Dataset: Extended MNIST

The experiments of this subsection are performed on the image dataset EMNIST (extended MNIST) [2]. We have used the “bymerge” split of EMNIST, which has 700K training and 115K test images. In this dataset, the dimensions of the images are \( 28 \times 28 \) and there are 47 classes in total (letters and digits).

Fig. 2 shows the cost and test accuracy plots when we solve the least squares problem on EMNIST-bymerge dataset using the model averaging method. Because this is a classification problem, we have one-hot encoded the labels. Fig. 2 demonstrates that SJLT is able to drive the cost down more and increase the accuracy more than uniform sampling.

C. Performance on Large-Scale Synthetic Datasets

This subsection contains the experiments performed on randomly generated large-scale data to illustrate scalability of the methods. Plots in Fig. 3 show the approximation error as a function of time, where the problem dimensions are as follows:

\[ \begin{align*}
A & \in \mathbb{R}^{10^5 \times 10^5} \quad \text{for plot a and} \quad A \in \mathbb{R}^{2(10^2) \times 2(10^2)} \quad \text{for plot b.}
\end{align*} \]

These data matrices take up 75 GB and 298 GB, respectively. The data used in these experiments were randomly generated from the student’s t-distribution with degrees of freedom of 1.5 for plot a and 1.7 for plot b. The output vector \( b \) was computed according to \( b = Ax_{\text{truth}} + \epsilon \) where \( \epsilon \in \mathbb{R}^n \) is i.i.d. noise distributed as \( \mathcal{N}(0,0.1I_n) \). Other parameters used in the experiments are \( m = 10^4, m' = 10^5 \) for plot a, and \( m = 8 \times 10^3, m' = 8 \times 10^4 \) for plot b. We observe that both plots in Fig. 3 reveal similar trends where the hybrid approach leads to a lower approximation error but takes longer due to the additional processing required for SJLT.

D. Numerical Results for Right Sketch: \( n < d \) Case

Fig. 4 shows the approximation error as a function of averaged outputs in solving the least norm problem for two different datasets. The dataset for Fig. 4(a) is randomly generated with dimensions \( A \in \mathbb{R}^{50 \times 10000} \). We observe that Gaussian sketch outperforms uniform sampling in terms of the approximation error. Furthermore, Fig. 4(a) verifies that if we apply the hybrid approach of first sampling and then using
Gaussian sketch, the performance is between the extreme ends of only sampling and only using Gaussian sketch. Moreover, Fig. 4(b) shows the results for the same experiment on a subset of the airline dataset where we have included the pairwise interactions as features which makes this an underdetermined linear system. Originally, we had 774 features for this dataset, if we include all \( x_i x_j \) terms as features, we would have a total of 299925 features, most of which are zero for all samples. We have excluded the all-zero columns from this extended matrix to obtain the final dimensions 2000 \( \times \) 11406.

![Fig. 4](image.png)

Fig. 4. Averaging for least norm problems. Plot (a): The parameters are \( n = 50, d = 1000, m = 200, m' = 500 \). Plot (b): Least norm averaging applied to a subset of the airline dataset. The parameters are \( n = 2000, d = 11406, m = 4000, m' = 8000 \). For this plot, the features include the pairwise interactions in addition to the original features.

VII. CONCLUSION

In this work, we have studied averaging sketched solutions for linear least squares problems for both \( n > d \) and \( n < d \). We have discussed distributed sketching methods from the perspectives of convergence, bias, privacy, and (serverless) computing framework. Our results and numerical experiments suggest that distributed sketching methods offer a competitive straggler-resilient solution for solving large scale linear least squares problems for distributed systems.

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We give below the proofs for the rest of the lemmas stated in this paper.

**Proof of Lemma 2** The expectation of the difference between \( f(\bar{x}) \) and \( f(x^*) \) is given by:

\[
E[f(\bar{x})] - f(x^*) = E[\|A(\bar{x} - x^*)\|^2] \\
= E \left[ \left\| \frac{1}{q} \sum_{k=1}^{q} (A\hat{x}_k - Ax^*) \right\|^2 \right] \\
= \frac{1}{q^2} E \left[ \sum_{k=1}^{q} \sum_{l=1}^{q} (A\hat{x}_k - Ax^*, A\hat{x}_l - Ax^*) \right] \\
= \frac{1}{q} \sum_{k=1}^{q} E \left[ \|A\hat{x}_k - Ax^*\|^2 \right] + \frac{1}{q^2} \sum_{k \neq l, 1 \leq k, l \leq q} E \left[ \langle A\hat{x}_k - Ax^*, A\hat{x}_l - Ax^* \rangle \right] \\
= \frac{1}{q} E \left[ \|A\hat{x}_1 - Ax^*\|^2 \right] + \frac{q^2 - q}{q^2} E \left[ \langle A\hat{x}_1 - Ax^*, A\hat{x}_2 - Ax^* \rangle \right] \\
= \frac{1}{q} E \left[ \|A\hat{x}_1 - Ax^*\|^2 \right] + \frac{q - 1}{q} E[A\hat{x}_1 - Ax^*]^T E[A\hat{x}_2 - Ax^*] \\
= \frac{1}{q} E \left[ \|A\hat{x}_1 - Ax^*\|^2 \right] + \frac{q - 1}{q} \|E[A\hat{x}_1] - Ax^*\|^2,
\]

where the first line follows as in (2).

**Proof of Lemma 3** Assuming \( A^T S_k^T S_k A \) is invertible (we will take the probability of this event into account later), the norm of the bias for a single sketch can be expanded as follows:

\[
\|E[A\hat{x}_k] - Ax^*\|_2 = \|E[A(A^T S_k^T S_k A)^{-1} A^T S_k^T S_k b - Ax^*]\|_2 \\
= \|E[A(A^T S_k^T S_k A)^{-1} A^T S_k^T S_k (Ax^* + b^\perp)] - Ax^*\|_2 \\
= \|E[A(A^T S_k^T S_k A)^{-1} A^T S_k^T S_k b^\perp]\|_2 \\
= \|UE[(U^T S_k^T S_k U)^{-1} U^T S_k^T S_k b^\perp]\|_2 \\
= \|E[(U^T S_k^T S_k U)^{-1} U^T S_k^T S_k b^\perp]\|_2 \\
= \|E[Qz]\|_2,
\]

where we define \( Q := (U^T S_k^T S_k U)^{-1} \) and \( z := U^T S_k^T S_k b^\perp \). The term \( \|E[Qz]\|_2 \) can be upper bounded as follows:

\[
\|E[Qz]\|_2^2 = E[Qz]^T E[Qz] \\
= E_{S_k} (Qz)^T E_{S_k} [Q' z'] \\
= E_{S_k} E_{S_k} [z^T QQ' z'] \\
= \frac{1}{2} E_{S_k} E_{S_k} [(z + z')^T QQ' (z + z') - z^T QQ' z - z'^T QQ' z'] \\
\leq \frac{1}{2} E_{S_k} E_{S_k} [(\|z + z'\|^2 (1 + \epsilon)^2 - \|z\|^2 (1 - \epsilon)^2) - \|z'\|^2 (1 + \epsilon)^2] \\
= E_{S_k} E_{S_k} [(\|z\|^2 + (1 + \epsilon)^2) (\|z'\|^2 + (1 + \epsilon)^2) - (1 + \epsilon)^2 (\|z'\|^2)] \\
= 2\epsilon E_{S_k} E_{S_k} [\|z\|^2] + 2\epsilon E_{S_k} E_{S_k} [\|z'\|^2] + (1 + \epsilon)^2 E_{S_k} E_{S_k} [z^T z'] \\
= 4\epsilon E[\|z\|^2] + (1 + \epsilon)^2 \|E[z]\|^2,
\]

where the inequality in the fifth line follows from the assumption \((1 - \epsilon)I_d \preceq Q \preceq (1 + \epsilon)I_d\) and some simple bounds for the minimum and maximum eigenvalues of the product of two positive definite matrices. Furthermore, the expectation of \( z \) is equal to zero because \( E[z] = E[U^T S_k^T S_k b^\perp] = U^T E[S_k^T S_k] b^\perp = U^T b^\perp = 0 \). Hence we have

\[
\|E[A\hat{x}_k] - Ax^*\|_2 \leq \sqrt{4\epsilon E[\|z\|^2]}.
\]
Proof of Lemma 4: For the randomized Hadamard sketch (ROS), the term \( \mathbb{E}[\|z\|^2_2] \) can be expanded as follows:

\[
\mathbb{E}[\|z\|^2_2] = \mathbb{E} \left[ b^\top \frac{1}{m} \sum_{i=1}^{m} s_i s_i^T U U^T \frac{1}{m} \sum_{j=1}^{m} s_j s_j^T b \right]
\]
\[
= \mathbb{E} \left[ \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} b_i^\top s_i s_i^T U U^T s_j s_j^T b \right]
\]
\[
= \frac{1}{m^2} \sum_{1 \leq i = j \leq m} b_i^\top \mathbb{E} [s_i s_i^T U U^T s_i s_i^T] b_i + \frac{1}{m^2} \sum_{i \neq j, 1 \leq i, j \leq m} b_i^\top \mathbb{E} [s_i s_i^T] U U^T \mathbb{E} [s_j s_j^T] b_j
\]
\[
= \frac{1}{m^2} \sum_{i=1}^{m} b_i^\top \mathbb{E} [s_i s_i^T U U^T s_i s_i^T] b_i + \frac{1}{m^2} \sum_{i \neq j, 1 \leq i, j \leq m} b_i^\top I_n U U^T I_n b_j
\]
\[
= \frac{1}{m} b_i^\top \mathbb{E} [s_i s_i^T U U^T s_i s_i^T] b_i,
\]

where we have used the independence of \( s_i \) and \( s_j \), \( i \neq j \) in going from the second line to the third line. This is true because of the assumption that the matrix \( P \) corresponds to sampling with replacement.

\[
b_i^\top \mathbb{E} [s_i s_i^T U U^T s_i s_i^T] b_i = \mathbb{E} [(s_i^T U U^T s_i)(s_i^T b_i b_i^\top s_i)]
\]
\[
= \mathbb{E} [(s_i^T U U^T s_i)(b_i^\top s_i)^2]
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} [(h_i^T D U U^T D h_i)(b_i^\top D h_i)^2],
\]

where the row vector \( h_i^T \) corresponds to the \( i \)'th row of the Hadamard matrix \( H \). We also note that the expectation in the last line is with respect to the randomness of \( D \).

Let us define \( r \) to be the column vector containing the diagonal entries of the diagonal matrix \( D \), that is, \( r := [D_{11}, D_{22}, ..., D_{nn}]^T \). Then, the vector \( D h_i \) is equivalent to \( \text{Diag}(h_i) r \) where \( \text{Diag}(h_i) \) is the diagonal matrix with the entries of \( h_i \) on its diagonal.

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} [(h_i^T D U U^T D h_i)(b_i^\top D h_i)^2] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} [(r^T \text{Diag}(h_i) U U^T \text{Diag}(h_i) r)(b_i^\top \text{Diag}(h_i) r)^2]
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} [b_i^\top \text{Diag}(h_i) r (r^T P r)(r^T \text{Diag}(h_i) r)^2]
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} b_i^\top \text{Diag}(h_i) \mathbb{E} [r (r^T P r)(r^T \text{Diag}(h_i) r)] \text{Diag}(h_i) b_i,
\]

where we have defined \( P := \text{Diag}(h_i) U U^T \text{Diag}(h_i) \). It follows that \( \mathbb{E}[r (r^T P r) r^T] = 2P - 2\text{Diag}(P) + \text{tr}(P) I_n \). Here, \( \text{Diag}(P) \) is used to refer to the diagonal matrix with the diagonal entries of \( P \) as its diagonal.

The trace of \( P \) can be easily computed using the cyclic property of matrix trace as \( \text{tr}(P) = \text{tr}(\text{Diag}(h_i) U U^T \text{Diag}(h_i)) = \text{tr}(U^T \text{Diag}(h_i) \text{Diag}(h_i) U) = \text{tr}(U^T U) = \text{tr}(I_d) = d \).

We note that the term \( \text{Diag}(P) \) can be simplified as \( \text{Diag}(P)_{jj} = \|\tilde{u}_j\|_2^2 \) where \( \tilde{u}_j \) is the \( j \)'th row of \( U \). This leads to

\[
b_i^\top \text{diag}(P) b_i = \sum_{j=1}^{n} (b_j^\top \tilde{u}_j)^2 \|\tilde{u}_j\|_2^2
\]
\[
\geq \sum_{j=1}^{n} (b_j^\top \tilde{u}_j)^2 \text{min}_i \|\tilde{u}_i\|_2^2
\]
\[
= \|b_i^\top \|_2^2 \text{min}_i \|\tilde{u}_i\|_2^2.
\]
Going back to $\mathbb{E}[\|z\|^2]$, 

$$
\mathbb{E}[\|z\|^2] = \frac{1}{mn} b^T n \text{Diag}(h_i) (2P - 2 \text{Diag}(P) + \text{tr}(P) I_n) \text{Diag}(h_i) b^\perp
$$

$$
= \frac{d}{m} \|b^\perp\|^2 - \frac{2}{m} b^T \text{Diag}(P) b^\perp
$$

$$
\leq \frac{d}{m} \|b^\perp\|^2 - \frac{2}{m} \min_i \|\tilde{u}_i\|^2
$$

$$
= \frac{1}{m} \|b^\perp\|^2 (d - 2 \min_i \|\tilde{u}_i\|^2)
$$

$$
= \frac{d}{m} \left( 1 - \frac{2 \min_i \|\tilde{u}_i\|^2}{d} \right) f(x^*).
$$

\[\square\]

**Proof of Lemma** Note that $\mathbb{E}[S_k^T S_k] = \frac{1}{m} \sum_{i=1}^m s_i s_i^T = I_n$ where $s_i \in \mathbb{R}^n$ is the column vector corresponding to the $i$th row of $S_k$ scaled by $1/\sqrt{m}$. Note that because of this particular scaling, $\mathbb{E}[s_i s_i^T] = I_n$ holds. For uniform sampling with replacement, we have

$$
\mathbb{E}[\|z\|^2] = \frac{1}{m^2} \mathbb{E} \left[ b^T \sum_{i=1}^m s_i s_i^T U U^T \sum_{j=1}^m s_j s_j^T b^\perp \right]
$$

$$
= \frac{1}{m^2} \mathbb{E} \left[ \sum_{i=1}^m \sum_{j=1}^m b^T s_i s_i^T U U^T s_j s_j^T b^\perp \right]
$$

$$
= \frac{1}{m^2} \sum_{1 \leq i=j \leq m} b^T \mathbb{E}[s_i s_i^T U U^T s_j s_j^T] b^\perp + \frac{1}{m^2} \sum_{i \neq j, 1 \leq i,j \leq m} b^T \mathbb{E}[s_i s_i^T] U U^T \mathbb{E}[s_j s_j^T] b^\perp
$$

$$
= \frac{1}{m} b^T \mathbb{E}[s_i s_i^T U U^T s_i s_i^T] b^\perp
$$

$$
= \frac{1}{m} b^T \frac{1}{n^2} \sum_{i=1}^n e_i e_i^T U U^T e_i e_i^T b^\perp
$$

$$
= \frac{n}{m} b^T \sum_{i=1}^n \|\tilde{u}_i\|^2 e_i e_i^T b^\perp
$$

$$
= \frac{n}{m} b^T \text{Diag}(\|\tilde{u}_i\|^2) b^\perp
$$

$$
= \frac{n}{m} \sum_{i=1}^n b_i^2 \|\tilde{u}_i\|^2
$$

$$
\leq \frac{n}{m} \sum_{i=1}^n b_i^2 \max_j \|\tilde{u}_j\|^2
$$

$$
= \frac{n}{m} f(x^*) \max_i \|\tilde{u}_i\|^2.
$$

For uniform sampling without replacement, the rows $s_i$ and $s_j$ are not independent which can be seen by noting that given $s_i$, we know that $s_j$ will have its nonzero entry at a different place than $s_i$. Hence, differently from uniform sampling with replacement, the following term will not be zero:

$$
\frac{1}{m^2} \sum_{i \neq j, 1 \leq i,j \leq m} b^T \mathbb{E}[s_i s_i^T U U^T s_j s_j^T] b^\perp
$$

$$
= \frac{m^2 - m}{m^2} b^T \mathbb{E}[s_i s_i^T U U^T s_2 s_2^T] b^\perp
$$

$$
= \frac{m}{m} b^T \frac{1}{n^2-n} \sum_{i \neq j, 1 \leq i,j \leq n} e_i e_i^T U U^T e_j e_j^T b^\perp
$$

$$
= \frac{m}{m} \frac{n}{n-1} b^T \sum_{i \neq j, 1 \leq i,j \leq n} e_i \tilde{u}_i e_j^T \tilde{u}_j e_j^T b^\perp
$$
\[
= \frac{m - 1}{m} \frac{n}{n - 1} b_i^T (U U^T - \text{Diag}(\|\tilde{u}_i\|_2^2)) b_i^T
\]
\[
= \frac{m - 1}{m} \frac{n}{n - 1} \left(0 - \sum_{i=1}^{n} b_i^T \|\tilde{u}_i\|_2^2\right)
\]
\[
= - \frac{m - 1}{m} \frac{n}{n - 1} \sum_{i=1}^{n} b_i^T \|\tilde{u}_i\|_2^2.
\]

It follows that for uniform sampling without replacement, we obtain
\[
\mathbb{E}[\|z\|_2^2] = \left(\frac{n}{m} - \frac{m - 1}{m} \frac{n}{n - 1}\right) \sum_{i=1}^{n} b_i^T \|\tilde{u}_i\|_2^2
\]
\[
= \frac{n - m}{m - 1} \sum_{i=1}^{n} b_i^T \|\tilde{u}_i\|_2^2
\]
\[
\leq \frac{n - m}{m - 1} f(x^*) \max_i \|\tilde{u}_i\|_2^2.
\]

**Proof of Lemma 6** We consider leverage score sampling with replacement. The rows \(s_i, s_j \ i \neq j\) are independent because sampling is with replacement. For leverage score sampling, the term \(\mathbb{E}[\|z\|_2^2]\) is upper bounded as follows:
\[
\mathbb{E}[\|z\|_2^2] = \frac{1}{m^2} \mathbb{E}\left[b_i^T \sum_{i=1}^{m} s_i s_i^T U U^T \sum_{j=1}^{m} s_j s_j^T b_i^T\right]
\]
\[
= \frac{1}{m^2} \mathbb{E}\left[\sum_{i=1}^{m} \sum_{j=1}^{m} b_i^T s_i s_i^T U U^T s_j s_j^T b_i^T\right]
\]
\[
= \frac{1}{m^2} \sum_{1 \leq i, j \leq m} b_i^T \mathbb{E}[s_i s_i^T U U^T s_j s_j^T] b_i^T + \frac{1}{m^2} \sum_{i \neq j, 1 \leq i, j \leq m} b_i^T \mathbb{E}[s_i s_i^T] U U^T \mathbb{E}[s_j s_j^T] b_i^T
\]
\[
= \frac{1}{m} b_i^T \mathbb{E}[s_i s_i^T] U U^T b_i^T + \frac{m^2 - m}{m^2} b_i^T \mathbb{E}[s_i s_i^T] U U^T \mathbb{E}[s_i s_i^T] b_i^T
\]
\[
= \frac{1}{m} b_i^T \sum_{1 \leq i \leq m} \ell_i d \ell_i d e_i e_i^T U U^T d e_i e_i^T b_i^T + \frac{m^2 - m}{m^2} b_i^T I_n U U^T I_n b_i^T
\]
\[
= \frac{1}{m} b_i^T \sum_{1 \leq i \leq m} d \ell_i \ell_i d e_i e_i^T b_i^T
\]
\[
= \frac{d}{m} b_i^T \sum_{1 \leq i \leq m} e_i e_i^T b_i^T
\]
\[
= \frac{d}{m} \|b_i\|_2^2
\]
\[
= \frac{d}{m} f(x^*).
\]