A PERRON-TYPE THEOREM FOR NONAUTONOMOUS DIFFERENTIAL EQUATIONS WITH DIFFERENT GROWTH RATES

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Abstract. We show that if the Lyapunov exponents associated to a linear equation \( x' = A(t)x \) are equal to the given limits, then this asymptotic behavior can be reproduced by the solutions of the nonlinear equation \( x' = A(t)x + f(t, x) \) for any sufficiently small perturbation \( f \). We consider the linear equation with a very general nonuniform behavior which has different growth rates.

1. Introduction. We say that an increasing function \( \mu : [0, +\infty) \to [1, +\infty) \) is a growth rate if
\[
\mu(0) = 1 \quad \text{and} \quad \lim_{t \to +\infty} \mu(t) = +\infty.
\]
For example, \( e^{at} \), \( 1 + t \), \( t^a \), \( t^a \log(b + t) \) with \( a, b > 0 \) are growth rates. Given a growth rate \( \mu \), we can define the \( \mu \)-Lyapunov exponent associated to the linear equation
\[
x' = A(t)x
\]
by
\[
\chi(x_0) = \limsup_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)},
\]
where the function \( A : [0, +\infty) \to M_n(\mathbb{R}) \) is continuous with values in the set \( M_n(\mathbb{R}) \) of \( n \times n \) matrices and \( x(t) \) is the solution of equation (1) with \( x(0) = x_0 \) (with the convention that \( \log 0 = -\infty \)). The function \( \chi \) is a Lyapunov exponent. For the properties of the function \( \chi \), we refer to [1].

We will show that if all \( \mu \)-Lyapunov exponents of (1) are equal to the given limits, then the asymptotic exponential behaviour of the linear equation (1) can be reproduced by the solutions of the nonlinear equation
\[
x' = A(t)x + f(t, x)
\]
(2)

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for any sufficiently small perturbation \( f \). This means that for any solution \( x(t) \) of equation (2) the limit
\[
\chi = \lim_{t \to +\infty} \frac{\log \| x(t) \|}{\log \mu(t)}
\]
exists and coincides with a \( \mu \)-Lyapunov exponent of the linear equation (1). The required smallness of the perturbation is that
\[
\lim_{t \to +\infty} \mu_{\delta_1}(t) \int_t^{t+1} \nu_{\delta_2}(\tau) \sup_{x \neq 0} \frac{\| f(\tau, x) \|}{\| x \|} d\tau = 0
\]
for some \( \delta_1, \delta_2 > 0 \), where \( \mu, \nu \) are two given growth rates. When \( \mu(t) = \nu(t) = e^{\rho(t)} \), (3) becomes
\[
\lim_{t \to +\infty} e^{\delta_1 t} \int_t^{t+1} e^{\delta_2 \tau} \sup_{x \neq 0} \frac{\| f(\tau, x) \|}{\| x \|} d\tau = 0
\]
for some constant \( \delta_1, \delta_2 > 0 \), and \( \delta_1 \) is related to the Lyapunov exponent of equation (2), which is the given condition in [5]. More simply, when \( \mu(t) = \nu(t) = e^t \), (3) becomes
\[
\lim_{t \to +\infty} \int_t^{t+1} e^{\delta \tau} \sup_{x \neq 0} \frac{\| f(\tau, x) \|}{\| x \|} d\tau = 0
\]
which is the given condition in [7].

In the literature, the related problems are usually called “Perron-type theorem”, which is an important topic in the theory of differential equations and dynamical systems. When \( A(t) = A \) is constant, a related result can be found in Coppel’s book [11]. Earlier works are due to Perron [16], Lettenmeyer [14], Hartman and Wintner [12] for autonomous differential equations, and Coffman [10] for autonomous difference equations. Corresponding results on perturbations of autonomous delay equations were obtained by Pituk [17, 18] for values in \( \mathbb{C}^n \) and finite delay, and by Matsui et al [15] for values in a Banach space and infinite delay. For the nonautonomous case, we refer the reader to Barreira and Valls [6] for the recent results of difference equations. Especially, based on Lyapunov’s theory of regularity, Barreira and Valls considered the Perron-type theorem for differential equations with nonuniform exponential behavior in [5, 7] and the case for the perturbations of a linear cocycle in the context of ergodic theory [2].

Such problems are also very close to the theory of nonuniform exponential dichotomies, which was inspired both by the classical notion of exponential dichotomy and by the notion of nonuniformly hyperbolic trajectory introduced by Pesin [3], and have been developed in a systematic way by Barreira and Valls during the last several years, see [4] and the references therein. As explained by Barreira and Valls, in comparison to the notion of exponential dichotomies, nonuniform exponential dichotomy is a useful and weaker notion. A very general type of nonuniform exponential dichotomy, the so-called \( \mu, \nu \)-dichotomy, has been considered in [1, 8, 9, 13]. Especially the Perron-type theorem for nonautonomous difference equations with nonuniform behaviors was obtained in [19].

Compared with those results in the literature, the novelty of this work is that we establish the Perron-type theorem for nonautonomous differential equations with different growth rates in the uniform parts and nonuniform parts. More precisely, we consider the \( \mu, \nu \) nonuniform behaviour which is admitted in a large class of linear differential equation and this creates additional complications in the analysis. We refer the reader to [1] for related results on the so-called \( \mu, \nu \)-dichotomies.
The rest of the paper is organized as follows. In section 2, preliminaries and the assumptions are given. In section 3, we prove the main result. After evaluating some inequalities, we present the Perron-type theorem. Section 4 illustrates some applications and examples.

2. Preliminaries. Let \( T(t, s) \) be an evolution operator in the space \( \mathbb{C}^n \). More precisely, \( T(t, s) : \mathbb{C}^n \to \mathbb{C}^n \) is a linear operator for each \( t, s \geq 0 \) such that \( T(t, t) = \text{Id} \) for \( t \geq 0 \), and

\[
T(t, s)T(s, r) = T(t, r)
\]

for \( t, s, r \geq 0 \). Now we consider a growth rate \( \mu \) such that

\[
\limsup_{t \to +\infty} \frac{\log \|T(t, 0)x\|}{\log \mu(t)} < +\infty.
\]

(4)

The Lyapunov exponent \( \chi : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\} \) associated to the evolution operator \( T(t, s) \) is defined by

\[
\chi(x) = \limsup_{t \to +\infty} \frac{\log \|T(t, 0)x\|}{\log \mu(t)}
\]

with the convention that \( \log 0 = -\infty \). It follows from (4) that \( \chi \) never takes the value \( +\infty \). Using the same idea in theory of Lyapunov exponents [4, Section 1.1], we know that \( \chi \) can take at most \( n \) values in \( \mathbb{C}^n \setminus \{0\} \), say \(-\infty \leq \chi_1 < \cdots < \chi_p \) for some integer \( p \leq n \). Furthermore, for each \( i = 1, \ldots, p \), the set

\[
E_i = \{ x \in \mathbb{C}^n : \chi(x) \leq \chi_i \}
\]

is a linear subspace over \( \mathbb{C}^n \). Obviously, it has

\[
\{0\} = E_0 \subset E_1 \subset \cdots \subset E_p.
\]

We set \( k_i = \dim E_i - \dim E_{i-1} \).

Now we describe the assumptions in the paper. Roughly speaking, we assume that the evolution operator is in block form, with each block corresponding to a Lyapunov exponent. More precisely, we assume that

(H1) There exist a decomposition

\[
\mathbb{C}^n = F_1(t) \oplus F_2(t) \oplus \cdots \oplus F_p(t), \quad t \geq 0
\]

into subspaces of dimension \( \dim F_i(t) = k_i \) such that for each \( t, s \geq 0 \) and \( i = 1, \ldots, p \), there holds

\[
T(t, s)F_i(s) = F_i(t).
\]

Thus for a given number \( b \in \mathbb{R} \) that is not a Lyapunov exponent, when \( \chi_i < b < \chi_{i+1} \), there exists a decomposition

\[
\mathbb{C}^n = E(t) \oplus F(t),
\]

(5)

where

\[
E(t) = \bigoplus_{\chi_i < b} F_i(t) \quad \text{and} \quad F(t) = \bigoplus_{\chi_i > b} F_i(t).
\]

(H2) Take \( a < b < c \) such that the interval \([a, c]\) contains no Lyapunov exponent, and for a given constant \( \varepsilon > 0 \) there exists a constant \( K = K(\varepsilon) > 0 \) such that

\[
\|T(t, s)P(s)\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^\varepsilon, \quad t \geq s,
\]

(6)
Proof of Lemma 3.2. For each $t$ holds for all integers $s > 0$ that $d > \chi$ such that $\delta > 0$ for some $\epsilon > 0$, it follows from (6) that given $\epsilon > 0$ there exists an $N = N(\epsilon) > 0$ such that
\[ \|T(t,s)\| \leq K_{\epsilon}(s) \] for all sufficiently large $t$. Moreover, for every $t, s \geq 0$ we have
\[ P(t)T(t,s) = T(t,s)P(s), \quad Q(t)T(t,s) = T(t,s)Q(s). \]

3. Main result. The following is our main result. It claims that under sufficiently small perturbations, then the Lyapunov exponent of (2) coincides with some Lyapunov exponent of the unperturbed differential equation (1).

Theorem 3.1. Let $x$ be a solution of equation (2) such that
\[ \|f(t,x(t))\| \leq \gamma(t)\|x(t)\|, \quad t \geq 0 \] for some continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying
\[ \lim_{t \to +\infty} \left( \frac{\mu(t+1)}{\mu(t)} \right)^{2\chi-1} \left( \frac{\nu(t+1)}{\nu(t)} \right)^{d} \int_{t}^{t+1} \nu(\tau)\gamma(\tau)d\tau = 0 \] for some $\delta > 0$ and two growth rates $\mu, \nu$ are given in (H2). Then one of the following statements holds:
1. $x(t) = 0$ for all sufficiently large $t$; or
2. the limit
\[ \lim_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)} \] exists and coincides with a $\mu$-Lyapunov exponent of the linear equation (1).

Before presenting the proof of Theorem 3.1 we introduce several technical lemmas, and take $\epsilon = \delta/4$ in (H2).

Lemma 3.2. The inequality
\[ \|x(t)\| \leq N \|x(s)\| \left( \frac{\mu(t)}{\mu(s)} \right)^{d} \nu(s)^{\epsilon} \exp(N \int_{t}^{t} \nu(u)^{\epsilon}\gamma(u)du) \] holds for all $t \geq s$. Hence, there exists a $C > 0$ such that
\[ C^{-1} \left( \frac{\mu(k+1)}{\mu(k)} \right)^{-\epsilon} \nu(k+1)^{\epsilon} \|x(k+1)\| \leq \|x(t)\| \leq C \left( \frac{\mu(t)}{\mu(k)} \right)^{d} \nu(k)^{\epsilon} \|x(k)\| \] for all integers $k \geq s$ and all $k \leq t \leq k + 1$, where $k = [k + \eta]$ with $\eta \in [0,1]$. Proof of Lemma 3.2. For each $t \geq s \geq 0$, (2) has a solution $x(t)$ which can be written as
\[ x(t) = T(t,s)x(s) + \int_{s}^{t} T(t,\tau)f(\tau,x(\tau))d\tau. \] By taking $d > \chi$, it follows from (6) that given $\epsilon > 0$ there exists an $N = N(\epsilon) > 0$ such that
\[ \|T(t,s)\| \leq N \left( \frac{\mu(t)}{\mu(s)} \right)^{d} \nu(s)^{\epsilon}, \quad t \geq s. \]
Then by (13), it follows from (14) and (9) that
\[ \|x(t)\| \leq N \left( \frac{\mu(t)}{\mu(s)} \right)^d \nu(s)^\gamma \|x(s)\| + N \int_s^t \left( \frac{\mu(\tau)}{\mu(s)} \right)^d \nu(\tau)^\gamma \|x(\tau)\| d\tau \]
\[ = N \left( \frac{\mu(t)}{\mu(s)} \right)^d \nu(s)^\gamma \|x(s)\| + N \int_s^t \left( \frac{\mu(\tau)}{\mu(s)} \right)^d \nu(\tau)^\gamma \|x(\tau)\| d\tau,\]
and hence,
\[ \left( \frac{\mu(t)}{\mu(s)} \right)^{-d} \|x(t)\| \leq N\nu(s)^\gamma \|x(s)\| + N \int_s^t \left( \frac{\mu(\tau)}{\mu(s)} \right)^{-d} \|x(\tau)\| \nu(\tau)^\gamma \gamma(\tau) d\tau. \]
By Gronwall’s lemma, the above inequality leads to (11). By (10), we have
\[ G = \sup_{t \geq 0} \int_t^{t+1} \nu(\tau)^\gamma \gamma(\tau) d\tau < \infty. \]

Since \( \nu \) is increasing, it follows from (11) that inequality (12) holds for \( k \leq t \leq k + 1 \) with
\[ C = N \exp(NG). \]

This completes the proof of the lemma.

Now, let \( b \in \mathbb{R} \) be a number that is not a Lyapunov exponent. Let \( a < b < c \) be as in Section 2. We consider the norm
\[ \|x\|_t = \sup_{\tau \geq t} \left( \frac{\mu(\tau)}{\mu(t)} \right)^{-a} \|T(\tau, t) P(t)x\| + \sup_{\tau \leq t} \left( \frac{\mu(\tau)}{\mu(t)} \right)^{-c} \|T(\tau, t) Q(t)x\| \]
for each \( t \geq 0 \) and \( x \in \mathbb{C}^n \). Clearly, there holds
\[ \|x\|_t = \|P(t)x\|_t + \|Q(t)x\|_t, \quad \text{for } t \geq 0, \tag{15} \]
and it follows from (6) and (7) that
\[ \|x\| \leq \|x\|_t \leq 2K\nu(t)^\gamma \|x\|. \tag{16} \]

**Lemma 3.3.** We have
\[ \|T(t, s)P(s)x\|_t \leq \left( \frac{\mu(t)}{\mu(s)} \right)^a \|P(s)x\|_s \quad \text{for } t \geq s, \]
and
\[ \|T(t, s)Q(s)x\|_t \geq \left( \frac{\mu(t)}{\mu(s)} \right)^c \|Q(s)x\|_s \quad \text{for } t \geq s. \]

**Proof of Lemma 3.3.** For \( t \geq s \) we have
\[
\|T(t, s)P(s)x\|_t = \sup_{\tau \geq t} \left[ \|T(\tau, t)T(t, s)P(s)x\| \left( \frac{\mu(\tau)}{\mu(t)} \right)^{-a} \right] \\
= \sup_{\tau \geq t} \left[ \|T(\tau, s)P(s)x\| \left( \frac{\mu(\tau)}{\mu(s)} \right) \left( \frac{\mu(s)}{\mu(t)} \right)^{-a} \right] \\
\leq \left[ \frac{\mu(t)}{\mu(s)} \right]^a \sup_{\tau \geq s} \left[ \|T(\tau, s)P(s)x\| \left( \frac{\mu(\tau)}{\mu(s)} \right) \right] \\
\leq \left[ \frac{\mu(t)}{\mu(s)} \right]^a \|P(s)x\|_s. \tag{17}
\]
Similarly, for \( t \geq s \) we have
\[
\|T(t, s)Q(s)x\|_t = \sup_{\tau \leq t} \left( \|T(\tau, t)T(t, s)Q(s)x\| \left( \frac{\mu(\tau)}{\mu(t)} \right)^{-c} \right)
\]
\[
= \sup_{\tau \leq t} \left( \|T(\tau, s)Q(s)x\| \left( \frac{\mu(\tau)}{\mu(s)} \right)^{-c} \right)
\]
\[
= \left[ \frac{\mu(t)}{\mu(s)} \right]^c \sup_{\tau \leq t} \left( \|T(\tau, s)Q(s)x\| \left( \frac{\mu(\tau)}{\mu(s)} \right)^{-c} \right)
\]
\[
\geq \left[ \frac{\mu(t)}{\mu(s)} \right]^c \|Q(s)x\|_t.
\] (18)

So the proof of Lemma 3.3 is completed. \( \square \)

Next, we assume that \( x \) is a solution of equation (12). Using the decomposition in (5), we rewrite \( x(t) = x_P(t) + x_Q(t) \), where
\[
x_P(t) = P(t)x(t) \quad \text{and} \quad x_Q(t) = Q(t)x(t).
\]
By (13), we have
\[
x_P(t) = T(t, s)x_P(s) + \int_s^t T(t, \tau)P(\tau)f(\tau, x(\tau))d\tau,
\]
and
\[
x_Q(t) = T(t, s)x_Q(s) + \int_s^t T(t, \tau)Q(\tau)f(\tau, x(\tau))d\tau.
\]
Denote \( k = \overline{k} + \eta \) with \( \overline{k} \in \mathbb{N} \) and \( \eta \in [0, 1] \).

**Lemma 3.4.** One of the following alternatives holds:

1. either
\[
\limsup_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)} < b \quad (19)
\]
holds and for every \( \eta \in [0, 1] \) there holds
\[
\lim_{k \to +\infty} \frac{\|x_Q(k)\|_k}{\|x_P(k)\|_k} = 0; \quad (20)
\]

2. or
\[
\liminf_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)} > b \quad (21)
\]
holds and for every \( \eta \in [0, 1] \) there holds
\[
\lim_{k \to +\infty} \frac{\|x_P(k)\|_k}{\|x_Q(k)\|_k} = 0. \quad (22)
\]

**Proof of Lemma 3.4.** For \( t \geq k \) we have
\[
x_P(t) = T(t, k)P(k)x(k) + \int_k^t T(t, \tau)P(\tau)f(\tau, x(\tau))d\tau, \quad (23)
\]
and
\[
x_Q(t) = T(t, k)Q(k)x(k) + \int_k^t T(t, \tau)Q(\tau)f(\tau, x(\tau))d\tau. \quad (24)
\]
By (16) and (18), it follows from (24) that for \( t \geq k \),

\[
\|x_Q(t)\|_t \geq \|T(t,k)Q(k)x(k)\|_t - \left\| \int_k^t T(t,\tau)Q(\tau)f(\tau,x(\tau))d\tau \right\|_t \\
\geq \left( \frac{\mu(t)}{\mu(k)} \right)^c \|x_Q(k)\|_k - 2K\nu(t)^c \int_k^t \|T(t,\tau)Q(\tau)f(\tau,x(\tau))\|d\tau.
\]

Hence, by (8) and (14), we get

\[
\|x_Q(t)\|_t \geq \left( \frac{\mu(t)}{\mu(k)} \right)^c \|x_Q(k)\|_k - 2K^2N\nu(t)^c \int_k^t \left( \frac{\mu(t)}{\mu(\tau)} \right)^d \nu(\tau)^{2\varepsilon} \gamma(\tau) \|x(\tau)\|d\tau
\]

\[
= \left( \frac{\mu(t)}{\mu(k)} \right)^c \|x_Q(k)\|_k - 2K^2N \int_k^t \left( \frac{\mu(t)}{\mu(\tau)} \right)^d \nu(\tau)^{\varepsilon} \gamma(\tau) \|x(\tau)\|d\tau.
\]

On the other hand, by Lemma (3.2) and since \( \mu \) is increasing, for \( t \leq k + 1 \) we obtain

\[
\|x_Q(t)\|_t \geq \|x_Q(k)\|_k - 2K^2NC\|x(k)\|_k - 2K^2NC\|x(k)\|_k - 2K^2NC\|x(k)\|_k,
\]

where

\[
\gamma_k = \left( \frac{\mu(k+1)}{\mu(k)} \right)^{2d} \left( \frac{\nu(k+1)}{\nu(k)} \right)^\varepsilon \int_k^{k+1} \nu(\tau)^{1\varepsilon} \gamma(\tau)d\tau.
\]

By (15) and (16), we find that for \( k \leq t \leq k + 1 \) there holds

\[
\|x_Q(t)\|_t \geq \left( \frac{\mu(t)}{\mu(k)} \right)^c \|x_Q(k)\|_k - D_1\gamma_k(\|x_P(k)\|_k + \|x_Q(k)\|_k) \tag{25}
\]

for some constant \( D_1 > 0 \). For \( t \geq k \) it follows from (17) and (23) that

\[
\|x_P(t)\|_t \leq \left( \frac{\mu(t)}{\mu(k)} \right)^a \|x_P(k)\|_k + 2K^2C\|x(k)\|\gamma_k. \tag{26}
\]

By (26), for \( k \leq t \leq k + 1 \) we get

\[
\|x_P(t)\|_t \leq \left( \frac{\mu(t)}{\mu(k)} \right)^a \|x_P(k)\|_k + D_2\gamma_k(\|x_P(k)\|_k + \|x_Q(k)\|_k) \tag{27}
\]

for some constant \( D_2 > 0 \). Using inequalities (25) and (27) leads to

\[
\|x_Q(k+1)\|_{k+1} \geq \alpha_k\|x_Q(k)\|_k - D\gamma_k(\|x_P(k)\|_k + \|x_Q(k)\|_k), \tag{28}
\]

and

\[
\|x_P(k+1)\|_{k+1} \leq \beta_k\|x_P(k)\|_k + D\gamma_k(\|x_P(k)\|_k + \|x_Q(k)\|_k), \tag{29}
\]
for all integers $k \geq s$, where

$$D = D_1 + D_2, \quad \alpha_k = \left(\frac{\mu(k+1)}{\mu(k)}\right)^c \quad \text{and} \quad \beta_k = \left(\frac{\mu(k+1)}{\mu(k)}\right)^a.$$  \hspace{1em} (30)

Now, we claim that either

$$\|x_Q(k)\|_k \leq \|x_P(k)\|_k \quad \text{for all large } k, \hspace{1em} (31)$$

or

$$\|x_P(k)\|_k < \|x_Q(k)\|_k \quad \text{for all large } k. \hspace{1em} (32)$$

We shall show that if (31) fails, then (32) holds. Suppose that (31) does not hold. That is,

$$\|x_Q(k)\|_k > \|x_P(k)\|_k \quad \text{for infinitely many } k. \hspace{1em} (33)$$

By (33), there exists a $k_1 \geq 1$ arbitrarily large such that

$$\|x_P(k_1)\|_{k_1} < \|x_Q(k_1)\|_{k_1}.$$  \hspace{1em}

We show by mathematical induction on $k$ that

$$\|x_P(k)\|_k < \|x_Q(k)\|_k, \quad \text{for all } k \geq k_1,$$

provided that $k_1$ is sufficiently large, $k = k_1 + n, n = 0, 1, 2, \cdots$.

Assume that $\|x_P(k)\|_k < \|x_Q(k)\|_k$ for some $k \geq k_1$.

By (28) and (29), this implies that

$$\|x_Q(k+1)\|_{k+1} \geq (\alpha_k - 2D\gamma_k)\|x_Q(k)\|_k$$

and

$$\|x_P(k+1)\|_{k+1} \leq (\beta_k + 2D\gamma_k)\|x_Q(k)\|_k.$$

It follows from (10) that

$$\left(\frac{\mu(k+1)}{\mu(k)}\right)^{2\chi_P-\chi_1}\left(\frac{\nu(k+1)}{\nu(k)}\right)^{4\varepsilon} \int_k^{k+1} \nu(\tau)^{4\varepsilon}\gamma(\tau)d\tau \to 0$$

when $k \to \infty$. By taking $d$ sufficiently close to $\chi_P$, it implies that $\gamma_k/\alpha_k \to 0$ and $\gamma_k/\beta_k \to 0$ when $k \to \infty$. We take $k_1$ sufficiently large such that

$$\frac{\beta_{k_1} + 2D\gamma_{k_1}}{\alpha_{k_1} - 2D\gamma_{k_1}} < 1.$$  \hspace{1em}

Then we get

$$\frac{\beta_k + 2D\gamma_k}{\alpha_k - 2D\gamma_k} < 1 \quad \text{for } k \geq k_1,$$

and further have

$$\|x_P(k+1)\|_{k+1} \leq \frac{\beta_k + 2D\gamma_k}{\alpha_k - 2D\gamma_k} \|x_Q(k+1)\|_{k+1} < \|x_Q(k+1)\|_{k+1}.$$  \hspace{1em}

By induction, we see that $\|x_P(k)\|_k < \|x_Q(k)\|_k$ for all $k \geq k_1$. Thus, this indicates that if (31) fails, then (32) holds. As a consequence, we have the following two cases.

**Case 1.** Assume that (31) holds. We show that (19) and (20) hold. We note that $\|x_P(k)\|_k > 0$ for all large $k$, since otherwise (15) and (31) yield

$$\|x(k)\|_k = \|x_P(k)\|_k + \|x_Q(k)\|_k \leq 2\|x_P(k)\|_k = 0$$
for infinitely many \( k \), contradicting the hypothesis that \( \| x(t) \|_t \geq \| x(t) \| > 0 \) for all \( t \geq s \). Let us define

\[
J = \limsup_{k \to +\infty} \frac{\| x_Q(k) \|_k}{\| x_P(k) \|_k}.
\]

By (31), we have \( 0 \leq J \leq 1 \). It follows from (31) and (29) that for all large \( k \) there holds

\[
\| x_P(k+1) \|_{k+1} \leq (\beta_k + 2D\gamma_k) \| x_P(k) \|_k.
\]

Together with (28), for all large \( k \) it has

\[
\frac{\| x_Q(k+1) \|_{k+1}}{\| x_P(k+1) \|_{k+1}} \geq \frac{\alpha_k - D\gamma_k}{\beta_k + 2D\gamma_k} \frac{\| x_Q(k) \|_k}{\| x_P(k) \|_k} - \frac{D\gamma_k}{\beta_k + 2D\gamma_k}.
\]

Since \( \alpha_k/\beta_k \to +\infty \) when \( k \to +\infty \) (see (30)), taking \( \limsup \) on both sides, we see that \( J \geq +\infty \cdot J \). This implies that \( J = 0 \), and (20) holds.

Taking \( k_0 \) sufficiently large that \( \| x_Q(k) \|_k \leq \| x_P(k) \|_k \) holds for all \( k \geq k_0 \). By (29) and letting

\[
c_k = \frac{\gamma_k}{\beta_k} = \left( \frac{\mu(k+1)}{\mu(k)} \right)^{2d-a} \left( \frac{\nu(k+1)}{\nu(k)} \right)^a \int_k^{k+1} \nu(\tau)^{4c} \gamma(\tau) d\tau,
\]

we find that for \( k \geq k_0 \) there holds

\[
\| x_P(k+1) \|_{k+1} \leq \beta_k (1 + 2Dc_k) \| x_P(k) \|_k,
\]

which implies that

\[
\| x_P(k+1) \|_{k+1} \leq \| x_P(k_0) \|_{k_0} \prod_{j=k_0}^k (1 + 2Dc_j) \prod_{j=k_0}^k \beta_j
\]

\[
= \| x_P(k_0) \|_{k_0} \prod_{j=k_0}^k (1 + 2Dc_j) \left( \frac{\mu(k+1)}{\mu(k_0)} \right)^a
\]

for \( k \geq k_0 \). It follows from (10) and (34) that

\[
\frac{1}{k} \sum_{j=1}^k \log(1 + 2Dc_j) \leq \frac{2D}{k} \sum_{j=1}^k c_j \to 0,
\]

when \( k \to \infty \) uniformly on \( \eta \in [0,1] \). Moreover, for a given \( t \), one can always take \( \eta \in [0,1] \) such that \( t = k + 1 = \bar{k} + \eta + 1 \). Thus, it follows from (35) and the independence of the bound in (36) on \( \eta \) that

\[
\limsup_{t \to +\infty} \frac{\log \| x(t) \|}{\log \mu(t)} \leq a < b.
\]

This leads to (19).

**Case 2.** Assume that (32) holds. We show that both (21) and (22) hold. Let us define

\[
R = \limsup_{k \to +\infty} \frac{\| x_P(k) \|_k}{\| x_Q(k) \|_k}.
\]

By (32), we have \( 0 \leq R \leq 1 \). It follows from (32) and (28) that for all large \( k \) there holds

\[
\| x_Q(k+1) \|_{k+1} \geq (\alpha_k - 2D\gamma_k) \| x_P(k) \|_k.
\]
Together with (29), for all large $k$ it has
\[
\|x_P(k+1)\|_{k+1} \leq \frac{\beta_k + D\gamma_k}{\alpha_k - 2D\gamma_k} \cdot \|x_P(k)\|_k + \frac{D\gamma_k}{\alpha_k - 2D\gamma_k}.
\]
Since $\beta_k/\alpha_k \to 0$ when $k \to \infty$, taking $\lim sup$ on both sides, we get $R \leq 0 \cdot R$. This implies that $R = 0$, and (22) holds.

Choosing $k_0$ such that $\|x_P(k)\|_k < \|x_Q(k)\|_k$ for all $k \geq k_0$ ad letting
\[
d_k = \frac{\gamma_k}{\alpha_k} = \left(\frac{\mu(k+1)}{\mu(k)}\right)^{2d-\epsilon} \left(\frac{\nu(k+1)}{\nu(k)}\right)^{-\epsilon} \int_k^{k+1} \nu(\tau)^{4\epsilon} \gamma(\tau)d\tau,
\]
we find that for $k \geq k_0$ there holds
\[
\|x_Q(k+1)\|_{k+1} \geq \alpha_k(1 - 2Dd_k)\|x_Q(k)\|_k,
\]
and hence,
\[
\|x_Q(k)\|_k \geq \|x_Q(k_0)\|_{k_0} \prod_{j=k_0}^k (1 - 2Dd_j) \left(\frac{\mu(k+1)}{\mu(k)}\right)^c.
\]
It follows from (10) and (37) that
\[
-\frac{1}{k} \sum_{j=1}^k \log(1 - 2Dd_j) = \frac{1}{k} \sum_{j=1}^k \log \frac{1}{1 - 2Dd_j}
\leq \frac{1}{k} \sum_{j=1}^k \frac{2Dd_j}{1 - 2Dd_j} \to 0,
\]
when $k \to \infty$ uniformly on $\eta \in [0, 1]$. We then deduce that
\[
\liminf_{t \to \infty} \frac{\log \|x(t)\|}{\log \mu(t)} \geq c > b.
\]
Consequently, we arrive at (21).

**Proof of Theorem 3.1.** Let $x$ be a solution of equation (2) satisfying the hypotheses of the Theorem 3.1. If $x(t_1) = 0$ for some $t_1$, then it follows from (11) that $x(t) = 0$ for all $t \geq t_1$, and hence, the first alternative in the theorem holds. Now we assume that $x(t) \neq 0$ for all $t \geq s$. Let $\chi_1 < \cdots < \chi_p$ be the Lyapunov exponents of the evolution operator $T(t, s)$.

Take real numbers $b_j$ such that
\[
\chi_{j-1} < b_{j-1} < \chi_j
\]
and
\[
\chi_j < b_j < \chi_{j+1}.
\]
Take $b_0 < \chi_1$ when $\chi_1 \neq -\infty$ and $b_p > \chi_p$.

We write
\[
\Omega = \{x(t)|x' = A(t)x + f(t, x)\},
\]
\[
\Omega_{b_j} = \{x(t)|x' = A(t)x + f(t, x), \liminf_{t \to t_1} \frac{\log \|x(t)\|}{\log \mu(t)} > b_j\},
\]
\[
\Omega_{b_j}^- = \{x(t)|x' = A(t)x + f(t, x), \limsup_{t \to t_1} \frac{\log \|x(t)\|}{\log \mu(t)} < b_j\}.
\]
By Lemma 3.4 it is easy to see that
\[\Omega = \Omega_{b_j^+} \cup \Omega_{b_j^-} = \bigcup_j (\Omega_{b_{j-1}^+} \cap \Omega_{b_j^-})\]
for \(j \in \{1, 2, \ldots, p\}\). It is equivalent to say that for any solution of equation (2), there exists \(j \in \{1, \ldots, p\}\) such that
\[\limsup_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)} < b_j\]
and
\[\liminf_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)} > b_{j-1} - 1\]
Letting \(b_j \searrow \chi_j\) and \(b_{j-1} \nearrow \chi_j\), we find that
\[\lim_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)} = \chi_j\]
Consequently, the proof of Theorem 3.1 is completed.

4. Applications and example. In this section, we describe several applications of Theorem 3.1 and illustrate an example. In the following we suppose conditions (H1)-(H2) are hold. Firstly, we give a particular case of Theorem 3.1 formulated only for the usual exponential rates \(\mu(t) = e^t\) which is presented in [7].

**Theorem 4.1.** Let \(x\) be a solution of equation (2) such that condition (9) holds for some continuous function \(\gamma : \mathbb{R} \to \mathbb{R}\) satisfying
\[\lim_{t \to +\infty} e^{\delta(t)}\gamma(t) = 0\]
for some \(\delta > 0\). Then one of the following alternatives holds:
1. \(x(t) = 0\) for all sufficiently large \(t\); or
2. the limit
\[\lim_{t \to +\infty} \frac{1}{t} \log \|x(t)\|\]
exists and coincides with a Lyapunov exponent of the linear equation (1).

Secondly, we give another case of Theorem 3.1 formulated for the arbitrary growth rates \(\mu(t) = e^{\rho(t)}\) which is presented in [5].

**Theorem 4.2.** Let \(x\) be a solution of equation (2) such that condition (9) holds for some continuous function \(\gamma : \mathbb{R} \to \mathbb{R}\) satisfying
\[\lim_{t \to +\infty} e^{(2\chi_p - \chi_1 + \delta)[\rho(t+1) - \rho(t)]} \int_t^{t+1} e^{\delta \rho(\tau)}\gamma(\tau)d\tau = 0\]
for some \(\delta > 0\). Then one of the following alternatives holds:
1. \(x(t) = 0\) for all sufficiently large \(t\); or
2. the limit
\[\lim_{t \to +\infty} \frac{1}{\rho(t)} \log \|x(t)\|\]
exists and coincides with a Lyapunov exponent of the linear equation (1).
Thirdly, we consider the particular case of linear perturbations. Consider the linear equation
\[ x = A(t)x + B(t)x \]  \hspace{1cm} (40)
for some matrices \( B(t) \) with complex entries varying continuously with \( t \geq 0 \) which can be seen as a criterion for the persistence of the Lyapunov spectrum of a linear equation.

**Theorem 4.3.** If \( \|B(t)\| \leq \gamma(t) \) for some continuous function \( \gamma : \mathbb{R} \rightarrow \mathbb{R} \) satisfying (10) for some \( \delta > 0 \), then the \( \mu \)-Lyapunov exponents of equations (40) and (1) have the same values.

Next, we show that any solution \( x(t) \) of equation (2) satisfying the second alternative in Theorem 3.1 is essentially asymptotically tangent to the spaces \( F_i(t) \) with \( i \) as in \( \{1, \ldots, p\} \). We consider the decompositions
\[ C^n = E(t) \bigoplus F(t) \bigoplus F_i(t), \]
where
\[ E(t) = \bigoplus_{j < i} F_j(t) \quad \text{and} \quad F(t) = \bigoplus_{j > i} F_j(t) \]
for each \( t \geq 0 \). Also let \( P(t) \), \( Q(t) \) and \( R(t) \) be the projections associated to this decomposition.

**Theorem 4.4.** Let \( x \) be a solution of equation (2) such that condition (9) holds for some continuous function \( \gamma : \mathbb{R} \rightarrow \mathbb{R} \) satisfying (10) for some \( \delta > 0 \). If there exists \( i = \{1, \ldots, p\} \) such that
\[ \chi_i = \lim_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)}, \]
then there holds
\[ \lim_{t \to +\infty} \frac{\|P(t)x(t)\|}{\|R(t)x(t)\|} = 0, \]  \hspace{1cm} (41)
and
\[ \lim_{t \to +\infty} \frac{\|Q(t)x(t)\|}{\|R(t)x(t)\|} = 0. \]  \hspace{1cm} (42)

**Proof of Theorem 4.4.** We write
\[ x(t) = x_P(t) + x_Q(t) + x_R(t), \]
where
\[ x_P(t) = P(t)x(t), \quad x_Q(t) = Q(t)x(t) \quad \text{and} \quad x_R(t) = R(t)x(t). \]
Take \( b < \chi_i \) such that the interval \([b, \chi_i)\) contains no \( \mu \)-Lyapunov exponent of the evolution operator \( T(t, s) \). Then we get
\[ \lim_{t \to +\infty} \frac{\log \|x(t)\|}{\log \mu(t)} = \chi_i > b, \]
and for every \( \eta \in [0, 1] \) it follows from Lemma 3.4 that
\[ \lim_{k \to +\infty} \frac{\|x_P(k)\|}{\|x_Q(k) + x_R(k)\|} = 0. \]
Thus, we find
\[ \lim_{t \to +\infty} \frac{\|x_P(t)\|}{\|x_Q(t) + x_R(t)\|} = 0. \]  \hspace{1cm} (43)
Now we take \(c > \chi_t\) such that the interval \((\chi_t, c]\) contains no \(\mu\)-Lyapunov exponent of the evolution operator \(T(t, s)\). Then we get
\[
\lim_{t \to +\infty} \frac{\log ||x(t)||}{\log \mu(t)} = \chi_t > b,
\]
and for every \(\eta \in [0, 1]\) it follows from Lemma 3.3 that
\[
\lim_{k \to +\infty} \frac{||x_Q(k)||}{||x_P(k) + x_R(k)||_k} = 0.
\]
Thus, we find
\[
\lim_{t \to +\infty} \frac{||x_Q(t)||}{||x_P(t) + x_R(t)||_t} = 0. \tag{44}
\]
Given \(\delta > 0\), we take \(\eta \in (0, 1)\) such that \(\eta(1 + \eta)(1 - \eta^2)\eta^{-1} < \delta\). By (44) for all large \(t\) we have
\[
||x_Q(t)||_t \leq \eta||x_P(t) + x_R(t)||_t. \tag{45}
\]
Furthermore, for all large \(t\) the limit (43) implies that
\[
||x_P(t)||_t \leq \eta||x_Q(t) + x_R(t)||_t.
\]
By (45), we obtain
\[
||x_Q(t)||_t \leq \eta(1 + \eta)||x_R(t)||_t + \eta^2||x_Q(t)||_t,
\]
and further
\[
||x_Q(t)||_t \leq \eta(1 + \eta)(1 - \eta^2)^{-1}||x_R(t)||_t \leq \delta||x_R(t)||_t.
\]
Since \(\delta\) is arbitrary, this leads to equality (42). Similarly, reversing the roles of \(P\) and \(Q\) gives equality (41).

In the last, we demonstrate an example which will show the nonuniform \((\mu, \nu)\)-dichotomy.

**Example 4.5.** Given \(\varepsilon > 0\) and \(a < 0 < c\), we consider the system of differential equations in \(\mathbb{R}^2\) given by
\[
\begin{align*}
\dot{x} &= \left(\frac{\mu'(t)}{\mu(t)} + \frac{\nu'(t)}{2\nu(t)}(\cos t - 1) - \frac{\varepsilon}{2} \log \nu(t) \sin t\right)x, \\
\dot{y} &= \left(\frac{\mu'(t)}{\mu(t)} - \frac{\nu'(t)}{2\nu(t)}(\cos t - 1) + \frac{\varepsilon}{2} \log \nu(t) \sin t\right)y.
\end{align*} \tag{46}
\]
The evolution operator associated with this system is given by
\[
T(t, s)(x, y) = (X(t, s)x, Y(t, s)y),
\]
where
\[
X(t, s) = \left(\frac{\mu(t)}{\mu(s)}\right)^a e^{\frac{\varepsilon}{2} \log \nu(t)(\cos t - 1) - \frac{\varepsilon}{2} \log \nu(s)(\cos s - 1)},
\]
\[
Y(t, s) = \left(\frac{\mu(t)}{\mu(s)}\right)^c e^{-\frac{\varepsilon}{2} \log \nu(t)(\cos t - 1) + \frac{\varepsilon}{2} \log \nu(s)(\cos s - 1)}.
\]
One can easily verify that
\[
||T(t, s)P(s)|| = ||X(t, s)|| \leq \left(\frac{\mu(t)}{\mu(s)}\right)^a \nu^c(s),
\]
and
\[
||T(t, s)^{-1}Q(t)|| = ||Y(t, s)^{-1}|| \leq \left(\frac{\mu(t)}{\mu(s)}\right)^{-c} \nu^c(t).
\]
This indicates that equation (46) admits a nonuniform \((\mu, \nu)\)-dichotomy.
Moreover, if we take 
\[ t = 2k\pi \quad \text{and} \quad s = (2k - 1)\pi, \quad k \in \mathbb{N}, \]
then
\[ \|X(t,s)\| = \left( \frac{\mu(t)}{\mu(s)} \right)^{\alpha} \nu^\epsilon(s), \]
which ensures us that the nonuniform part can not removed when \( \epsilon > 0 \).

From the discussions mentioned above, it is clear to see that hypotheses (H1)-(H2) are satisfied in Example 4.5. Hence, Theorem 3.1 is applicable to the example.

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