Generalized Sachs Form Factors and the Possibility of Their Measurement in Processes without and with Proton Spin Flip

M. V. Galynskii, a, * and R. E. Gerasimov b, c

a Joint Institute for Power and Nuclear Research—Sosny, National Academy of Sciences of Belarus, Minsk, 220109 Belarus
b Budker Institute of Nuclear Physics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, 630090 Russia
c Novosibirsk State University, Novosibirsk, 630090 Russia
* e-mail: galynski@sosny.bas-net.by
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The differential cross section for elastic electron–proton scattering has been calculated taking into account the two-photon exchange within the phenomenological description of the electromagnetic electron–proton interactions. The calculation is based on the consistent evaluation of the matrix elements of the proton current in a diagonal spin basis, which makes it possible to naturally obtain expressions for the generalized Sachs form factors. A new method has been proposed to independently measure these form factors in the elastic \( e\bar{p} \rightarrow e\bar{p} \) process in the case where the initial proton at rest is fully polarized along the direction of the motion of the final proton.

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INTRODUCTION

The study of electromagnetic proton form factors, which are important characteristics of this fundamental particle, allows better understanding of the structure of the proton and the properties of interactions between its constituent quarks. Experimental studies of the electric, \( G_E \), and magnetic, \( G_M \), proton form factors, the so-called Sachs form factors, have been performed since the mid-1950s [1, 2] in the elastic electron–proton scattering. In the case of unpolarized electrons and protons, all experimental data on the behavior of the proton form factors were obtained with the following Rosenbluth formula [1] for the differential cross section for the elastic \( e\bar{p} \rightarrow e\bar{p} \) scattering in the laboratory reference frame, where the initial proton is at rest:

\[
\sigma = \frac{d\sigma}{d\Omega_e} = \alpha^2 E_2 \cos^2(\theta_e/2) \frac{1}{4E_1 \sin^2(\theta_e/2)} \left[ \frac{G_E^2 + \frac{\tau_p}{\epsilon} G_M^2}{1 + \frac{\tau_p}{\epsilon}} \right]. \tag{1}
\]

Here, \( \alpha = 1/137 \) is the fine structure constant; \( E_1 \) and \( E_2 \) are the energies of the initial and final electrons, respectively; \( \theta_e \) is the electron scattering angle; \( \tau_p = Q^2/4M^2 \), where \( Q^2 = -q^2 = 4E_1E_2 \sin^2(\theta_e/2) \) is the square of the momentum transferred to the proton and \( M \) is the mass of the proton; and \( \epsilon = [1 + 2(1 + \tau_p) \tan^2(\theta_e/2)]^{-1} \) is the degree of linear polarization of a virtual photon [3–6] varying in the range 0 \( \leq \epsilon \leq 1 \). The Rosenbluth formula (1) is obtained in the one-photon exchange approximation with zero mass of the electron.

According to the Rosenbluth formula, the main contribution to the \( e\bar{p} \rightarrow e\bar{p} \) cross section at high \( Q^2 \) values comes from the term proportional to \( G_M^2 \), which reduces the accuracy of the separation of the \( G_E^2 \) contribution. For this reason, the use of the Rosenbluth formula to experimentally determine the form factors \( G_E \) and \( G_M \) gives significant uncertainties at \( Q^2 \geq 1 \text{ GeV}^2 \). The method of measurement of Sachs form factors based on Eq. (1) is called the Rosenbluth technique. The experimental dependence of \( G_E \) and \( G_M \) on \( Q^2 \) was determined using the Rosenbluth technique. Up to \( Q^2 = 6 \text{ GeV}^2 \), this dependence is well described by the dipole approximation

\[
G_E \approx G_M/\mu_p = G_p(Q^2) = (1 + Q^2/0.71)^{-2}, \tag{2}
\]

where \( \mu_p = 2.79 \) is the magnetic moment of the proton. In Eqs. (2) and (4), \( Q^2 \) is given in units of GeV\(^2\). The ratio \( R \equiv \mu_p G_E/G_M \) is \( R = 1 \).

Akhiezer and Rekalo [4] proposed a method for measuring the ratio of the Sachs form factors based on
polarization transfer from the longitudinally polarized initial electron to the final proton. This method involves the following expression obtained in [4] for the ratio of the form factors \( G_E \) and \( G_M \) in terms of the ratio of the transverse, \( P_t \), and longitudinal, \( P_l \), polarizations of the scattered proton:

\[
R = \frac{\mu_p G_E}{G_M} = -\frac{P_t E_1 + P_l E_2}{P_t} \tan \left( \frac{\theta_e}{2} \right).
\]  

(3)

High-precision experiments based on Eq. (3) were performed at the Thomas Jefferson National Accelerator Facility (JLab, United States) [7–12] for the range 0.5 GeV\(^2\) \( \leq Q^2 \leq 8.5\) GeV\(^2\). It appeared that the ratio \( R \) decreases linearly with increasing \( Q^2 \) in the range 0.5 GeV\(^2\) \( \leq Q^2 \leq 5.6\) GeV\(^2\) as

\[
R = 1 - 0.13(Q^2 - 0.04) = 1 - Q^2/8,
\]  

(4)

which indicates that \( G_E \) decreases more rapidly than \( G_M \).

Repeated, more accurate measurements of the ratio \( \mu G_E/G_M \) [11–13] only confirmed discrepancy with results obtained using the Rosenbluth technique. The current status of this problem was reviewed in detail in [14, 15].

To resolve the appearing contradiction, it was assumed that discrepancy in experiments could appear because the contribution of two-photon exchange was disregarded in the corresponding analysis, which initiated numerous theoretical [16–22] and experimental works [23, 24] (see also reviews [25, 26] and references therein).

The contribution of two-photon exchange to the cross section for elastic \( ep \rightarrow ep \) scattering has already been measured in three experiments. These are the experiment at the VEPP-3 storage ring in Novosibirsk [27], the EG5 CLAS experiment (JLab) [28], and the OLYMPUS experiment at the DORIS accelerator (DESY, Germany) [29]. The preliminary results of works [27–29] show that the inclusion of the contribution of two-photon exchange, as should be expected, can remove contradictions to \( Q^2 \leq 2–3\) GeV\(^2\).

In view of significant discrepancies between the measurements of the ratio of the Sachs form factors by two experimental methods, it would be very important to perform measurements by other independent methods. A new method for the experimental measurement of squares of the Sachs form factors was proposed in [30] in the one-photon approximation. In this method, the form factors \( G_E^2 \) and \( G_M^2 \) can be determined independently from each other from direct measurements of cross sections for the elastic \( ep \rightarrow ep \) process without and with proton spin flip in the case where the initial proton at rest is fully polarized along the direction of motion of the scattered proton. The aim of this work is to demonstrate that the method proposed in [30] is also applicable in the two-photon approximation and makes it possible to similarly measure the squares of absolute values of the generalized Sachs form factors \( |g_E|^2 \) and \( |g_M|^2 \). The proposed method appears to be applicable because the polarization structure of the cross sections for the \( ep \rightarrow ep \) process is the same in the one- and two-photon approximations. The calculation of the cross section in the two-photon approximation is performed using the method of calculation of the matrix elements of quantum electrodynamics (QED) processes in the diagonal spin basis [31], where the little Lorentz group [32, 33] common for two particles with different momenta is implemented. The diagonal spin basis allows naturally determining the generalized Sachs form factors at the stage of calculation of matrix elements of the proton current. In this case, the cross section for the \( ep \rightarrow ep \) process in an arbitrary reference frame contains only \( |g_E|^2 \) and \( |g_M|^2 \), i.e., “is diagonalized” in the language used in [20].

**DIAGONAL SPIN BASIS**

In the diagonal spin basis, spin 4-vectors \( s_1 \) and \( s_2 \) of the protons with the 4-momenta \( q_1 \) and \( q_2 \) (\( s_1 q_1 = s_2 q_2 = 0 \), \( s_1^2 = s_2^2 = -1 \), \( q_1^2 = q_2^2 = M^2 \)) or, correspondingly, with the 4-velocities \( v_1 = q_1/M \) and \( v_2 = q_2/M \) have the form [31]

\[
s_1 = -\frac{v_1 v_2}{v_1^2 - v_2}, \quad s_2 = \frac{v_1 v_2 v_2 - v_1}{\sqrt{(v_1 v_2)^2 - 1}}.
\]  

(5)

Spin 4-vectors (5) obviously do not change under transformations of the little Lorentz group common for particles with 4-momenta \( q_1 \) and \( q_2 \).

We consider spin vectors of the diagonal spin basis (5) in the laboratory reference frame. In the general case, the spin 4-vector \( s \) of a spin-1/2 particle with the 4-momentum \( q \) has the form

\[
s = (s_0, s), \quad s_0 = vc, \quad s = c + \frac{(ce)v}{1 + v_0},
\]  

(6)

where \( c \) is an arbitrarily directed three-dimensional unit (\( c^2 = 1 \)) vector called the spin projection axis.

In the laboratory reference frame, where \( q_1 = (M, 0) \) and \( q_2 = (q_{20}, q_z) \), the spin 4-vectors \( s_1 \) and \( s_2 \) of the diagonal spin basis (5) have the form

\[
s_1 = (0, n_2), \quad s_2 = (v_z n_2, v_{20} n_2), \quad n_2 = q_2/|q_2|.
\]  

(7)

This means that the spin projection axes \( e_1 \) and \( e_2 \) for the initial and final particles in the laboratory reference frame coincide with the direction of motion of the final particle

\[
e_1 = e_2 = n_2 = q_2/|q_2|.
\]  

(8)

For the system of two particles with different momenta \( q_1 = (q_{10}, q_1) \) (before interaction) and
\( q = (q_{20}, q_2) \) (after interaction), the possibility of the simultaneous projection of the spins on a single common direction in an arbitrary reference frame is determined by the three-dimensional vector \([32]\)
\[
a = q_2/q_{20} - q_1/q_{10}.
\] (9)

This result was obtained within the vector parameterization of the little Lorentz group \( L_{q_1,q_2} \), common for two particles with 4-momenta \( q_1 \) and \( q_2 \) \((L_{q_{20},q_3} = q_1, L_{q_{10},q_2} = q_2) \) \([32, 33]\).

The term “diagonal spin basis” is introduced because the three-dimensional vector \( a \) given by Eq. (9) is the difference of two vectors and is geometrically the diagonal of a parallelogram.

The coincidence of the little Lorentz groups for particles with the 4-momenta \( q_1 \) and \( q_2 \) in the diagonal spin basis specified by Eqs. (5) is responsible for a number of remarkable properties of this basis. In particular, in the diagonal spin basis given by Eqs. (5), the spin projection operators \( \sigma_1 \) and \( \sigma_2 \), as well as the raising and lowering spin operators \( \hat{\sigma}_1^\delta \) and \( \hat{\sigma}_2^\delta \), for the initial and final particles coincide with each other and have the form \([34, 35]\):
\[
\sigma = \sigma_1 = \sigma_2 = \gamma^\delta b_1 \delta_1 = \gamma^\delta b_2 \delta_2,
\] (10)
\[
\sigma^\delta = \sigma_1^\delta = \sigma_2^\delta = -1/2 \gamma^5 b_2^\delta,
\] (11)
\[
\sigma u^\delta(q) = \delta u^\delta(q), \quad \sigma^\delta u^\delta(q) = u^\delta(q). \quad (12)
\]

Here, \( u^\delta(q) = u^\delta(q, s_\lambda) \) are the bispinors of states of the particles \((i = 1, 2)\); \( \hat{\sigma} = a_i = \gamma^\mu \) is an arbitrary Dirac operator; \( \gamma^\mu \) and \( \gamma^5 \) are the Dirac matrices; and \( b_2^\delta = b_2 \pm i \delta b_2 \) are the circular 4-vectors, where \( \delta = \pm 1, b_2^{\delta=0} = 0, \) and \( b_2^{\delta=-1} = -2 \).

In Eqs. (10) and (11), to construct the spin operators, the following tetrad of orthonormalized 4-vectors \( b_A \) \((A = 0,1,2,3)\) is used:
\[
b_0 = q_2, b_1 = q_2, b_2 = q_2, b_3 = q_2, \quad (b_\mu) = \epsilon_{\mu
u\kappa\sigma} b_\nu b_\kappa g_\sigma^r / \rho,
\] (13)

where \( q_2 = q_2 + q_1, \quad q_2 = q_2 - q_1, \quad \epsilon_{\mu
u\kappa\sigma} \) is the Levi-Civita tensor \((\epsilon_{1230} = 1)\), \( r \) is the 4-momentum of a particle involved in the reaction different from \( q_1 \) and \( q_2 \), and \( \rho \) is determined from the normalization conditions \( b_0^2 = b_1^2 = b_2^2 = -b_3^2 = -1 \). The coincidence of the spin operators in the diagonal spin basis (5) makes it possible to separate in the covariant form the interactions without and with spin flip of particles involved in the reaction and, thereby, to trace the dynamics of the spin interaction.

**METHOD OF THE CALCULATION OF THE MATRIX ELEMENTS OF QED PROCESSES IN THE DIAGONAL SPIN BASIS**

The amplitudes of the QED processes in the scattering channel have the form
\[
M = \hat{u}(q) Q M u^\delta(q),
\] (14)

where \( u^\delta(q) = u^\delta(q, s) \) are the bispinors of the initial and final states of fermions satisfying the normalization condition \( u^\delta(q) u^{\delta*(q)} = 2M \), where \( q_i^2 = M^2 \) \((i = 1, 2)\), and \( Q_M \) is the interaction operator.

The matrix elements given by Eq. (14) can be reduced to the trace of the product of the operator:
\[
M = \text{Tr}(P_{21}^\delta Q_M), \quad P_{21}^\delta = u^\delta(q) Q_M u^\delta(q),
\] (15)

The operators \( P_{21}^\delta \) given by Eq. (16) can be determined by a number of methods \([35, 36]\). In the approach \([35]\) used in this work, in contrast to, e.g., \([36]\), the determination of \( P_{21}^\delta \) is reduced to finding the operators \( T_{21} \) and \( T_{12} \) such that
\[
u^\delta(q_2) = T_{21} u^\delta(q), \quad u^\delta(q_1) = T_{12} u^\delta(q_2),
\] (17)

which have the properties \( T_{21} = T_{21}^{-1} \) and \( T_{12} T_{12} = 1 \).

As a result, the operator \( P_{21}^\delta \) specified by Eq. (16) is obtained in the form
\[
P_{21}^\delta = u^\delta(q_3) u^\delta(q_4) T_{21} = \gamma^i T_{12} = T_{12} t_{12}^\delta,
\] (18)

where
\[
t_{12}^\delta = u^\delta(q_3) u^\delta(q_4) = 1/2(\hat{q}_1 + M)(1 - \delta \gamma^5 \hat{q}_1),
\] (19)

are the projective operators of states of particles with 4-momenta \( q_i \) and spin 4-vectors \( s_i \), \( q_i s_i = 0, s_i^2 = -1, \) \((i = 1, 2)\). The operator \( P_{21}^\delta \) given by Eq. (16) is the product of the operators \( \sigma^\delta \) \((11) \) and \( P_{21}^{\delta=-\delta} \) \((18)\):
\[
P_{21}^\delta = \sigma^\delta P_{21}^{\delta=-\delta} = \sigma^\delta t_{12}^\delta T_{12} = \sigma^\delta t_{12}^\delta t_{12}^\delta.
\] (20)

In the diagonal spin basis (5), \( T_{21} \) and \( T_{12} \) coincide with each other and have the form \([35]\):
\[
T_{21} = T_{12} = \hat{b}_0.
\]

As a result, the operators \( P_{21}^\delta \) \((16) \) are obtained in the form \([35]\):
\[
P_{21}^\delta = (\hat{q}_1 + M) \hat{b}_0 b_0^*/4, \quad P_{21}^{\delta=-\delta} = \delta(\hat{q}_1 + M) \hat{b}_0 b_0^*/2,
\] (21)

where \( b_0^* = b_0 = b_1 - i \delta b_2, b_0^{\delta=0} = -2, \) and \( b_0^* = b_0^{\delta=0} = 0 \).
The general structure of the dependence of the squares of the absolute values of the matrix elements (14) on the polarization of particles can be established in some cases from their general form using the symmetries of electromagnetic interactions. To demonstrate this, we rewrite the matrix element (14) in the most general form

\[ M(\delta_1, \delta_2) = M^{\delta_1, \delta_2} = \Phi(q_2)Q_{in} \delta(q_1). \]  

(23)

We introduce the polarization factors

\[ \omega_+ = (1 + \delta_1 \delta_2)/2, \quad \omega_- = (1 - \delta_1 \delta_2)/2. \]  

(24)

At \( \delta_{i2} = \pm 1 \), they have the properties

\[ \omega_\pm^2 = \omega_\pm, \quad \omega_+ \omega_- = 0. \]  

(25)

The matrix element given by Eq. (23) satisfies the relation

\[ M(\delta_1, \delta_2) = \omega_+ M^{\delta_1, \delta_2} + \omega_- M^{-\delta_1, \delta_2}. \]  

(26)

In view of the properties of polarization factors \( \omega_+ \) and \( \omega_- \) specified by Eqs. (24) at \( \delta_{i2} = \pm 1 \), we have

\[ |M(\delta_1, \delta_2)|^2 = |\omega_+ |M^{\delta_1, \delta_2}|^2 + |\omega_- |M^{-\delta_1, \delta_2}|^2. \]  

(27)

Because of the conservation of spatial, spin correlations in Eq. (27) for the \( e\bar{p} \rightarrow e\bar{p} \) process, where electrons are unpolarized, should be absent except for those contained in \( \omega_+ \) and \( \omega_- \). This means that \( |M^{\delta_1, \delta_2}|^2 \) are independent of \( \delta_1 \) and \( \delta_2 \), and the square of the absolute value of the matrix element given by Eq. (23) averaged and summed over polarizations has the form

\[ \overline{|M(\delta_1, \delta_2)|^2} = |M^{\uparrow\uparrow}|^2 + |M^{\downarrow\downarrow}|^2. \]  

(28)

**MATRIX ELEMENTS OF PROTON CURRENT IN THE ONE-PHOTON APPROXIMATION**

The matrix elements of the elastic process \( e\bar{p} \rightarrow e\bar{p} \)

\[ e(p_1) + p(q_1, s_1) \rightarrow e(p_2) + p(q_2, s_2) \]  

(29)

is the product of the electron and proton currents:

\[ M_{ep\rightarrow ep} = 4\pi \alpha T/q^2, \]  

(30)

\[ T = T^{\pm\delta, \delta} = (J_{ep})^\mu (J_{ep})^\mu. \]  

(31)

In the one-photon exchange approximation, the currents \( (J_{ep})^\mu \) and \( (J_{ep})^\mu \) have the form

\[ (J_{ep})^\mu = \phi(p_2)^\mu u(p_1), \]  

(32)

\[ (J_{ep})^\mu = \phi(q_2)^\mu u(q_1), \]  

(33)

\[ \Gamma_{\mu}^{\uparrow\uparrow} = F_{\mu}^\gamma \gamma_{\mu} - q_{\mu}^\gamma F_{\mu}^\gamma. \]  

(34)

Here, \( u(p_1) \) and \( u(q_1) \) are the bispinors of electrons and protons with 4-momenta \( p_1 \) and \( q_1 \), respectively, where \( p_1^2 = m^2 \) and \( q_1^2 = M^2 \), having the properties \( \pi(p)u(p) = 2m \) and \( \pi(q)u(q) = 2m \) \((i = 1, 2)\). \( F_1 \) and \( F_2 \) are the Dirac and Pauli proton form factors, respectively; \( q = q_2 - q_1 \) is the 4-momentum transferred to the proton; and \( s_1 \) and \( s_2 \) are the polarization 4-vectors of the initial and final protons, respectively.

The cross section for the \( e\bar{p} \rightarrow e\bar{p} \) process has the form

\[ \frac{d\sigma}{dT} = \frac{\pi \alpha^2 |T|^2}{4T^2 q^2}, \]  

(35)

where \( T^2 = (p_1 q_1)^2 - m^2 M^2 \) and \( q^2 = -q^2 \).

The matrix elements of the proton current (33) calculated in the diagonal spin basis (5) using Eqs. (15), (21), and (22) have the form [31, 35]

\[ (J_p^{\uparrow\uparrow})^\mu = 2M G_{\pi}^\mu (b_0)^\mu, \]  

(36)

\[ (J_p^{-\bar{\mu}})^\mu = -2\delta M \sqrt{\gamma_p} G_{\pi}^\mu (b_0)^\mu, \]  

(37)

where

\[ G_{\pi} = F_1 - \tau_p F_2, \quad G_{\pi} = F_1 + F_2 \]  

(38)

are the Sachs form factors. Consequently, the Sachs form factors \( G_{\pi} \) and \( G_{\pi} \) in the matrix elements of the proton current corresponding to transitions without, with, Eq. (36), and with, Eq. (37), proton spin flip in the diagonal spin basis are factorized. Because of this property, these form factors have a fundamental physical meaning as quantities determining the probabilities of transitions of the proton without and with spin flip in the case where the axes of the spin projections coincide and have the form of Eq. (9).

In the case of unpolarized electrons, the matrix elements of the proton current \( J_p^{\pm\delta, \delta} \) (33) reduce the squares of the absolute values of the amplitudes \( |T^{\pm\delta, \delta}|^2 \) to the trace of the product of operators:

\[ |T^{\pm\delta, \delta}|^2 = 2\text{Tr} \left( \tau^\mu \gamma^\nu \right) \left( J_{ep}^{\pm\delta, \delta} \right)^\mu \left( J_{ep}^{\pm\delta, \delta} \right)^\nu. \]  

(39)

Here, the asterisk \* means complex conjugation and \( \tau^\mu^* = 1/2(\hat{p}_i + m) \)

(40)

are the projective operators of the electron \((i = 1, 2)\). According to Eq. (27), the quantity \( |T|^2 \) determining the cross section (35) for the \( e\bar{p} \rightarrow e\bar{p} \) process has the form

\[ |T_{\delta, \delta}|^2 = \omega_+ |T^{\uparrow\uparrow}|^2 + \omega_- |T^{-\bar{\delta}}|^2. \]  

(41)

In the case of unpolarized particles, the spin-averaged square of the absolute value of the amplitude \( |T|^2 \) has the form

\[ \overline{|T|^2} = |T^{\uparrow\uparrow}|^2 + |T^{-\bar{\delta}}|^2. \]  

(42)
CROSS SECTION FOR THE $e\bar{p} \rightarrow e\bar{p}$ PROCESS IN THE ONE-PHOTON APPROXIMATION

In terms of the matrix elements of Eqs. (36) and (37), the calculation of $|T|^2$ (31) is reduced to the calculation of traces

$$|T^{+\delta,\delta^2}| = 4M^2G_{\mu}^2G_{\nu}^2G_{\lambda}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G_{\sigma}^2G_{\tau}^2G_{\rho}^2G}_{\lambda,(s,m^2,M^2)}^2$ is the Källén function.

The cross section given by Eq. (47) expressed in terms of the variables $s$, $t$, and $u$ coincides with Eq. (139.4) in [38].

We introduce the quantities

$$Y_3 = (p,q_3)^2 + q_3^2(q_3^2 - 4m^2), \quad (50)$$

where $(p,q_3)$ are the variables of the virtual photon, $q_3$ is the degree of linear polarization of the virtual photon $\epsilon$ in the laboratory reference frame, which was defined after Eq. (1).

MATRIX ELEMENTS OF THE PROTON CURRENT IN THE TWO-PHOTON APPROXIMATION

The matrix elements of the proton current in the two-photon exchange approximation have the form

$$(J_p)^{2\gamma} = \mu(q_2)\Gamma^{2\gamma}_\mu(q^2)\mu(q_1), \quad (54)$$

where $\Gamma^{2\gamma}_\mu(q^2)$ can be written in two equivalent representations [16–22]

$$\Gamma^{2\gamma}_\mu(q^2) = H_1\gamma_\mu + \frac{H_2}{4M}q_3^{(\gamma_\mu - \gamma_\mu)q_3} + \frac{\gamma_\mu(q_3)}{4M^2} H_3, \quad (55)$$

$$\Gamma^{2\gamma}_\mu(q^2) = H_1\gamma_\mu - \frac{(q_3)^2}{2M^2} H_2 + \frac{\gamma_\mu(q_3)}{4M^2} H_3. \quad (56)$$

Here, the complex proton form factors are denoted as $H_1, H_2, H_3$ in order to have the direct relation to the standard proton form factors $G_M, F_1, F_2$ in the Born approximation:

$$H_1^{(1)} = F_1, \quad H_2^{(1)} = F_2, \quad H_3^{(1)} = 0. \quad (57)$$

The matrix elements of the proton current (54) calculated with Eqs. (15), (21), (22), and (55) have the form

$$\bar{(J_p)^{\delta,\delta^2}_\mu} = 2M(H_1 - \gamma_\mu H_2 + \nu H_3)(b_0)_\mu, \quad (58)$$

$$\bar{(J_p)^{\delta,\delta^2}_\mu} = 2\Delta M\sqrt{\tau_\rho} \times (H_1 + H_2)(b_0)_\mu + \frac{p_0 b_2}{4M^2} H_3(\delta_\mu)_\mu. \quad (59)$$
The matrix elements of the proton current (54) calculated with Eqs. (15), (21), (22), and (56) have the form

\[ (J_{p}^{\delta \delta})_{\mu}^{2\gamma} = 2M (H_M - \tau H_2 + \nu H_3) (b_{0})_{\mu}, \]

\[ (J_{p}^{\delta \delta})_{\mu}^{2\gamma} = -2\delta M \sqrt{\tau_p} \left( H_M (b_0)_{\mu} + \frac{p \cdot q_{\mu}}{4M^2} H_3 (q_{+} )_{\mu} \right), \]

where

\[ \tau_1 = \frac{q^2}{4M^2} = 1 + \tau_p, \quad \tau_p = \frac{Q^2}{4M^2}, \quad \nu = \frac{p \cdot q_{\mu}}{4M^2} = s - u. \]

Comparing the matrix elements of the proton current (58) and (60), we obtain the following expression for the “generalized” electric form factor \( \mathcal{G}_E \) introduced in [19–21]:

\[ \mathcal{G}_E \equiv H_E + \nu H_3, \]

\[ H_E \equiv H_1 - \tau_1 H_2 = H_M - \tau_1 H_2, \]

\[ H_M \equiv H_1 + H_2. \]

In this case, the form factors \( H_E \) and \( H_M \) are naturally transformed to \( G_E \) and \( G_M \) in the case of one-photon exchange:

\[ H_E^{1\gamma} = H_1^{1\gamma} - \tau_1 H_2^{1\gamma} = H_M^{1\gamma} - \tau_1 H_2^{1\gamma} = F_1 - \tau_p F_2 = G_M (1 + \tau_p) F_1 \equiv G_E. \]

As a result, the matrix elements of (58) and (60) can be represented in the form

\[ (J_{p}^{\delta \delta})_{\mu}^{2\gamma} = 2M \mathcal{G}_E (b_{0})_{\mu}. \]

This expression is similar to Eq. (36) in the one-photon approximation and, therefore, \( \mathcal{G}_E \) can be called the generalized electric form factor of the proton. In the case of two-photon exchange, the form factor [19, 20]

\[ \mathcal{G}_M = H_M + \nu \nu H_3 = H_1 + H_2 + \nu \nu H_3 \]

serves as the generalized magnetic Sachs form factor. In terms of \( \mathcal{G}_M \), Eq. (61) has the form

\[ (J_{p}^{\delta \delta})_{\mu}^{2\gamma} = -2\delta M \sqrt{\tau_p} \left( \mathcal{G}_M - \nu \nu H_3 ) (b_{0})_{\mu} + \frac{p \cdot q_{\mu}}{4M^2} H_3 (q_{+} )_{\mu} \right). \]

We note that matrix elements of Eqs. (61) and (67) include a factor of \( \sqrt{\tau_p} \), which ensures the dominant contribution to the cross section for transitions of the proton with spin flip at \( \tau_p \gg 1 \); in the one-photon approximation, they are transformed to Eq. (37).

**CROSS SECTION FOR \( \vec{e}p \rightarrow \vec{e}p \) SCATTERING IN THE TWO-PHOTON APPROXIMATION**

Squares of the magnitudes of the amplitudes \( |T^{\delta \delta}|^2 \) (31) calculated by the general formula (39) in an arbitrary reference frame using the matrix elements of Eqs. (65) and (67) in the two-photon approximation in the case of unpolarized electrons have the form

\[ |T^{\delta \delta}|^2 = \frac{|\mathcal{G}_E|^2}{(1 + \tau_p)} Y_1, \]

\[ |T^{\delta \delta}|^2 = \frac{\tau_p Y_2}{(1 + \tau_p)} (|\mathcal{G}_M|^2 + \nu \nu \nu \nu (1 + \tau_p) |H_3||). \]

If the mass of the electron can be neglected, then

\[ \mathcal{E} = \mathcal{E}_1 = \mathcal{E}_2 = \frac{\nu^2 - \tau_p (1 + \tau_p)}{\nu^2 + \tau_p (1 + \tau_p)}, \]

As a result, Eq. (69) at \( m^2 = 0 \) has the form

\[ |T^{\delta \delta}|^2 = \frac{\tau_p Y_2}{(1 + \tau_p)} (|\mathcal{G}_M|^2 + \nu \nu (1 + \tau_p) |H_3||). \]

We present the useful relation between \( \mathcal{E}, \nu, \tau_p \)

\[ \nu^2 = \tau_p (1 + \tau_p) \frac{1 + \mathcal{E}}{1 - \mathcal{E}}, \]

which allows reducing the term containing \( |H_3|^2 \) in Eq. (71) to the form obtained in [20]

\[ |T^{\delta \delta}|^2 = \frac{\tau_p Y_2}{(1 + \tau_p)} (|\mathcal{G}_M|^2 + \nu \nu (1 + \mathcal{E}) |H_3||). \]

Since the contribution of the amplitude \( |H_3|^2 \) vanishes in the Born approximation and has the smallness order \( O(\alpha) \), the last terms in Eqs. (69) and (71) can be neglected. As a result, the cross section in the two-photon approximation is given by an expression similar to Eq. (46), where the Sachs form factors \( G_E \) and \( G_M \) are replaced by the generalized form factors \( \mathcal{G}_E \) and \( \mathcal{G}_M \):

\[ \frac{d\sigma_{\delta \delta}}{d|l|} = \frac{\pi \alpha^2}{4f^2 (1 + \tau_p)} \left( \omega_{\nu} |\mathcal{G}_E|^2 Y_1 + \omega_{\nu} \nu \nu (1 + \tau_p) |\mathcal{G}_M|^2 \right). \]

In the rest system of the initial proton,

\[ \frac{d\sigma_{\delta \delta}}{d\Omega_{\nu}} = \omega_{\nu} \nu \nu + \omega_{\nu} \nu \nu, \]

\[ \sigma_{\nu \nu} = \sigma_M (|\mathcal{G}_E|^2), \quad \sigma_{\nu \nu} = \sigma_M (|\mathcal{G}_M|^2), \]

where

\[ \sigma_M = \frac{\alpha^2 E_3 \cos^2 (\theta/2)}{4E_3 \sin^4 (\theta/2)} \frac{1}{1 + \tau_p}. \]

In Eq. (75), \( \delta_1 \) and \( \delta_2 \), appearing in \( \omega_{\nu} \), are doubled projections of the spins of the initial and final protons on the common \( \nu \) axis of spin projections (8), so that
−1 ≤ δ₁, δ₂ ≤ 1. According to Eq. (75), if electron–proton scattering occurs without proton spin flip (δ₁ = 1, δ₂ = 1), the contribution to the cross section comes from only the term containing |ге|² because the polarization factors ω₁ and ω₂ at |ге|² and |gm|² are unity (ω₁ = 1) and zero (ω₂ = 0). If scattering occurs with proton spin flip (δ₁ = 1, δ₂ = −1), the contribution to the cross section comes from only the term containing |gm|², because the polarization factors ω₁ and ω₂ at |ге|² and |gm|² are zero (ω₁ = 0) and unity (ω₂ = 1).

In the case of unpolarized electrons and protons, Eq. (75) gives the Rosenbluth cross section in the two-photon approximation denoted as \( \sigma_R = d\sigma/d\Omega_e \):

\[
\sigma_R = \sigma^{\uparrow\uparrow} + \sigma^{\uparrow\downarrow}. \tag{78}
\]

Consequently, the physical meaning of the separation into two terms in Eq. (78) containing only |ге|² and |gm|² is the sum of the cross sections for processes without and with proton spin flip when the initial proton at rest is fully polarized along the direction of motion of the final proton.

CONCLUSIONS

The differential cross section for elastic electron–proton scattering \( e\bar{p} \rightarrow e\bar{p} \), where the initial and final protons are polarized and have a common axis of spin projections has been calculated in an arbitrary reference frame taking into account two-photon exchange within the phenomenological description of the electromagnetic electron–proton interactions. The resulting expression (75) for the cross section in the laboratory reference frame can be used to measure the squares of absolute values of the generalized Sachs form factors |ге|² and |gm|² in processes without and with proton spin flip in the case where the initial proton at rest is fully polarized along the direction of motion of the final proton.

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