SIX-DIMENSIONAL SUPERMULTIPLETS FROM BUNDLES ON PROJECTIVE SPACES

FABIAN HAHNER, SIMONE NOJA, INGMAR SABERI, JOHANNES WALCHER

Mathematisches Institut der Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg, Deutschland

Ludwig-Maximilians-Universität München
Theresienstraße 37
80333 München, Deutschland

ABSTRACT. The projective variety of square-zero elements in the six-dimensional minimal supersymmetry algebra is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^3 \). We use this fact, together with the pure spinor superfield formalism, to study supermultiplets in six dimensions, starting from vector bundles on projective spaces. We classify all multiplets whose derived invariants for the supertranslation algebra form a line bundle over the nilpotence variety; one can think of such multiplets as being those whose holomorphic twists have rank one over Dolbeault forms on spacetime. In addition, we explicitly construct multiplets associated to natural higher-rank equivariant vector bundles, including the tangent and normal bundles as well as their duals. Among the multiplets constructed are the vector multiplet and hypermultiplet, the family of \( O(n) \)-multiplets, and the supergravity and gravitino multiplets. Along the way, we tackle various theoretical problems within the pure spinor superfield formalism. In particular, we give some general discussion about the relation of the projective nilpotence variety to multiplets and prove general results on short exact sequences and dualities of sheaves in the context of the pure spinor superfield formalism.

E-mail address: fahner@mathi.uni-heidelberg.de, noja@mathi.uni-heidelberg.de, i.saberi@physik.uni-muenchen.de, walcher@uni-heidelberg.de.
1. Introduction

The study of supermultiplets is an essential ingredient in the construction of interesting supersymmetric field theories. Typically, one characterizes a Lagrangian field theory by specifying its field content, together with the datum of a supersymmetric action functional. The field content consists of some collection of supermultiplets; if a superfield formalism is available, it is typically possible to write down manifestly supersymmetric action functionals in a compact and pleasing form as integrals over superspace.

Over the last fifty years, a variety of methods and techniques have been developed for the construction of supermultiplets and the development of superspace formalisms. One such approach is the pure spinor superfield formalism, the origins of which date back more than thirty years to the first papers by Nilsson [Nil86] and Howe [How91a; How91b]. The pure spinor approach was developed further, notably in the work of Berkovits on pure spinor methods in worldsheet string theory [Ber00, for example], and in papers by Cederwall and collaborators for field-theoretic applications. (See [Ced14] for a review, as well as references therein.) One motivating goal was to provide a formalism suitable for studying supersymmetric higher-derivative corrections and, more generally, classifying possible supersymmetric interactions. From a modern perspective, pure spinor superfields are suited to this purpose because they resolve supermultiplets freely over superspace, leading to simple models of the corresponding interactions: just for example, the action for ten-dimensional super Yang–Mills theory becomes a Chern–Simons action functional, whereas the action for perturbative eleven-dimensional supergravity can be written with only cubic and quartic terms [Ced10b; Ced10a].
In recent work [ESW21; Eag+21], the pure spinor superfield formalism was reinterpreted in modern mathematical language, and shown to work in the general setting of any super Lie algebra $\mathfrak{p}$ of “super Poincaré type.” (This is equivalent to the condition that $\mathfrak{p}$ admit a consistent $\mathbb{Z}$-grading with support in degrees zero, one, and two; the grading can be viewed as an action of an abelian one-dimensional Lie algebra by automorphisms, corresponding to rescaling or “engineering dimension” from the physics perspective.) Any such algebra has a subalgebra $\mathfrak{t} = \mathfrak{p}_{>0}$ of “supertranslations,” and there is a corresponding affine superspace modeled on the group $T = \exp(\mathfrak{t})$. (For some general context on supertranslation algebras, see [Cat+20].) The formalism then associates a supermultiplet on the body of $T$ to any equivariant sheaf on the affine scheme $\hat{Y}$ of square-zero elements in $\mathfrak{t}$ that is equivariant for both $\mathfrak{p}_0$ and rescalings. (In the physics literature, this scheme is often called either the nilpotence variety [ESW21] or the space of pure spinors, although it may not have anything to do with either the spin representation or its minimal orbit, which is the space of pure spinors in the sense of Cartan and Chevalley. With respect to the $\mathbb{Z}$-grading, it is simply the space of Maurer–Cartan elements in $\mathfrak{t}$.) In fact, the supermultiplet is naturally freely resolved over $T$, so that one automatically gets a superfield model.

Set up in this fashion, the pure spinor formalism works to produce multiplets in any dimension and with any amount of supersymmetry; it even works for non-standard examples of supersymmetry, such as the algebras that control residual supersymmetry in twisted theories. (In [SW21], the formalism was shown to commute with twisting in an appropriate sense, and was applied to swiftly and compactly compute the twists of ten- and eleven-dimensional supergravity multiplets, verifying the conjectural description from [CL16] in the case of type IIB supergravity and providing new results for eleven-dimensional supergravity and type IIA.) It is, of course, natural to ask whether all multiplets admit a description of this type.

This question was answered in [EHS22], in which a natural derived generalization of the pure spinor superfield formalism was given. The key observation is that the graded ring $R/I$ of functions on the affine scheme $\hat{Y}$ is the degree-zero Lie algebra cohomology of $\mathfrak{t}$ (after totalizing the bigrading). As such, it is natural to replace $\hat{Y} = \text{Spec} R/I$ by the derived model $\text{Spec} C^*(\mathfrak{t})$. [EHS22] showed that the pure spinor superfield formalism generalizes to a functor from equivariant $C^*(\mathfrak{t})$-modules to supermultiplets for $\mathfrak{p}$, and in fact provides an equivalence of categories. As such, every supermultiplet admits a pure spinor superfield description, and arises from a unique equivariant sheaf on the affine dg scheme $\text{Spec} C^*(\mathfrak{t})$, up to an appropriate notion of equivalence; the equivariant sheaf can be constructed by taking the derived $\mathfrak{t}$-invariants of the global sections of the multiplet. The equivalence even works when there are no supersymmetries at all. As explained in [EHS22], this result can be
thought of as a version of Kapranov’s formulation of Koszul duality [Kap91], which relates $D$-modules and $\Omega^\bullet$-modules on the same space, performed for the translation-invariant objects on a super Lie group; it is also related to Koszul duality for the graded Lie algebra, followed by an associated-bundle construction. (Yet another relation of pure spinors to Koszul duality occurs when viewing the functions on $\hat{Y}$ as a commutative algebra and considering the Koszul dual graded Lie algebra, as done in [MS04; GS09; Gál+16]; further work in this direction will appear in [CPS22].)

This result provides further justification for the program of studying $p$-multiplets and their interactions using the (derived) algebraic geometry of $C^\ast(t)$-modules. To get the best mileage out of the pure spinor technique, it is natural to start in a setting where the category of $C^\ast(t)$-modules, or at least the category of equivariant sheaves on $\hat{Y}$, is relatively easy to understand. A natural candidate is $N = (1,0)$ supersymmetry in six dimensions, where $\hat{Y}$ is the space of two-by-four matrices of rank one; as such, the corresponding projective variety $Y = \text{Proj} R/I$ is just $\mathbb{P}^1 \times \mathbb{P}^3$, sitting inside $\mathbb{P}^7$ via the Segre embedding. In this note we take another step forward in the pure spinor program by providing a detailed case study, as well as developing new methods using techniques from projective algebraic geometry. There is a great abundance of geometrically interesting equivariant vector bundles on $Y$. Taking the direct sum of the global sections of all twists of these bundles, we obtain graded equivariant modules over the ring of functions on the nilpotence variety, so that we can study the associated multiplets. In particular, we classify all multiplets originating from line bundles over $Y = \mathbb{P}^1 \times \mathbb{P}^3$; among others, this recovers the family of so-called “$\mathcal{O}(n)$-multiplets” studied in the literature [KNT17; KNT18; GR+98; LTM12], and encompasses the vector multiplet and its antifield muliplet, as well as the hypermultiplet. (These three examples have already been studied via the pure spinor superfield formalism in [CN08; Ced18].)

Roughly speaking, we provide a link between vector bundles on $Y$ and $p$-multiplets in two steps, by combining the connection between quasicoherent sheaves on $\text{Proj} R/I$ and graded $R/I$-modules (which is standard algebraic geometry) with the pure spinor superfield construction. Concretely, we convert a sheaf on $Y$ into a module by forming its graded module of global sections (i.e. by taking the sum of the global sections of all its twists). Equivariant vector bundles form a subcategory of equivariant quasi-coherent sheaves; conversely, one can assign a sheaf on $Y$ to each module, though it is important to note that the two operations are not inverses in general. Following the results on twisting pure spinor superfields in [SW21], we argue that modules whose associated sheaf is trivial correspond to multiplets that are perturbatively trivial in every twist.
In turn, the category of graded equivariant $R/I$-modules sits as a subcategory inside equivariant $C^\star(t)$-modules, which (by [EHS22]) is equivalent to the category of multiplets. We can summarize the situation with the following diagram:

$$
\begin{array}{c}
\text{LineBundles}^p_Y & \xleftarrow{\Gamma} & \text{QCo}h^p_Y & \xrightarrow{\sim} & \text{Mod}^p_Y & \xleftarrow{\sim} & \text{Mod}^p_{C^\star(t)} & \xrightarrow{A^\star} & \text{Mult}_p \\
\end{array}
$$

Since the inverse functor to the pure spinor superfield construction is given by taking derived $t$-invariants, classifying all multiplets associated to line bundles thus amounts to classifying all multiplets whose derived $t$-invariants are the graded global section module of a line bundle on the projective nilpotence variety. An alternative characterization of such multiplets is via their twists, which are necessarily holomorphic for minimal supersymmetry in six dimensions; in keeping with the results of [SW21], one expects that the holomorphic twist of such a multiplet is of rank one over Dolbeault forms on the spacetime, and we verify this below.

In addition to the classification, we translate duality theory for sheaves to the world of multiplets, extending the results of [Eag+21]. To this end, we study the Cohen–Macaulay property and prove that, in good situations, the antifield multiplet can be constructed using the dualizing sheaf on the projective nilpotence variety.

Moving on, we develop some general methods regarding short exact sequences of sheaves in the pure spinor superfield formalism. These can be used to tackle higher-rank bundles; generalizations would allow for the construction of the multiplet associated to any higher-rank bundle via a resolution into a chain complex of sums of line bundles, though we do not pursue this in detail here. Our results show that the multiplet associated to a nontrivial extension of two sheaves is a deformation of the direct sum of the multiplets associated to each sheaf by a further differential, and we study such deformations explicitly at the level of component-field presentations of various multiplets.

We then use the Euler exact sequence, as well as the normal and conormal bundle sequences, to explicitly construct the multiplets associated to the tangent bundle, the normal bundle, and their duals. Several of these multiplets are of obvious physical interest; in particular, we identify the supergravity multiplet with the conormal bundle, and the gravitino multiplet with the pullback of the tangent bundle to the ambient space.

**Acknowledgements.** We would like to give special thanks to R. Eager, C. Elliott, and B. Williams for numerous conversations and collaboration on closely related projects. We also thank I. Brunner, M. Cederwall, J. Palmkvist, for fruitful conversation. This work is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2181/1 — 390900948 (the Heidelberg STRUCTURES Excellence Cluster). I.S. is supported by the Free State of Bavaria.
2. Preliminaries

2.1. The pure spinor superfield formalism. We briefly review the pure spinor superfield formalism, as reinterpreted in [ESW21; Eag+21] and extended in [EHS22]; for a more traditional approach and an overview of the broader literature, the reader is referred to [Ced14] and references therein.

2.1.1. Supertranslation algebras. Let $p$ be a graded Lie algebra supported in degrees zero, one, and two, and let $t = p_{>0}$ be its positively graded subalgebra. Concretely, this means that $t_1$ and $t_2$ carry representations of $p_0$, and that the bracket $\text{Sym}^2(t_1) \to t_2$ is $p_0$-equivariant. The central example will be any super Poincaré algebra, for which $t_1$ consists of supersymmetries, $t_2$ of translations, and $p_0$ of Lorentz and $R$-symmetries. (Physically, the grading arises from the conformal weight, and should not be thought of as related to cohomological degree. In what follows, we will use round brackets to refer to shifts of the weight grading, as is typical in projective geometry; we reserve square brackets for shifts in cohomological degree.) In general, $p_0$ consists of (a subalgebra of) the automorphisms of $t$; all our constructions will be equivariant with respect to such automorphisms.

We consider the Lie algebra cochains of $t$, which are a bigraded commutative differential algebra with differential of degree $(1,0)$. (The first grading is by cohomological degree in the Chevalley–Eilenberg complex, which is just polynomial degree, and the second arises from the internal grading on $t$ defined above.) Since $t$ is a central extension of the form

\begin{equation}
0 \to t_2 \to t \to \Pi t_1 \to 0,
\end{equation}

there is a corresponding short exact sequence of bigraded cdga’s of the form

\begin{equation}
\mathbb{C} \to C^*(\Pi t_1) \to C^*(t) \to C^*(t_2) \to \mathbb{C}.
\end{equation}

The first arrow witnesses $C^*(t)$ as an algebra over the graded polynomial ring

\begin{equation}
R = C^*(\Pi t_1) = \text{Sym}^*(t_1^\vee[−1]),
\end{equation}

whose generators sit in bidegree $(1,−1)$ and are thus of even parity. In fact, after totalizing, $C^*(t)$ is the Koszul complex [Eis95; NR21] over $R$ defined by the $t_2$-valued set of equations

\begin{equation}
[Q,Q] = 0,
\end{equation}

where $Q \in t_1$.

These equations define a homogenous ideal $I \subset R$; the space

\begin{equation}
\hat{Y} = \text{Spec}R/I \subset \text{Spec}R = t_1[1]
\end{equation}


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associated to this ideal is called the nilpotence variety, and consists of the space of square-zero supercharges in $t$. (We emphasize, though, that we think of $\hat{Y}$ as an affine scheme.) This space has been studied extensively for physical supertranslation algebras in [ESW21; ES18]. Explicitly, we can expand $Q$ in a basis $Q = \lambda^\alpha Q_\alpha$ and identify $R = \mathbb{C}[\lambda^\alpha]$. Denoting the structure constants of the bracket by $\Gamma_{\alpha\beta}^{\mu}$, the ideal then takes the form

$$I = (\lambda^a \Gamma_{\alpha\beta}^{\mu} \lambda^\beta).$$

We further observe that, with respect to the totalized grading, $H^0(t) = R/I$. In general, though, $I$ is not a complete intersection, so that the Koszul complex of $I$ does not define a resolution of $R/I$. Rather, $C^*(t)$ should be thought of as a derived enhancement of the standard (affine) nilpotence variety $\hat{Y}$.

2.1.2. Constructing multiplets. As recalled in the introduction, the pure spinor superfield formalism is an equivalence of categories between $p$-multiplets and $p_0$-equivariant $C^*(t)$-modules; it restricts to $R/I$-modules, giving a systematic way to construct a class of supersymmetric multiplets by studying sheaves on the affine scheme $\hat{Y}$ of square-zero supersymmetries. Here, we give a rough overview, reviewing the aspects which are relevant to this work; for details, we refer to [Eag+21; EHS22]. In particular, we make use of the notion of a supermultiplet as defined in [Eag+21]; for us, a supermultiplet is a chain complex of affine super vector bundles on spacetime, equipped with a homotopy action of the super Poincaré algebra compatible with the affine structure.

We denote the supergroup of supertranslations by $T = \exp(t)$; its body is a vector space $\mathbb{R}^d$, thought of as an additive abelian group. There is a corresponding supergroup $P$, the analogue of the super Poincaré group, which is the semidirect product of $T$ with a group $P_0$ of its outer automorphisms. Recall that the algebra of free superfields is defined to be the supercommutative algebra of smooth functions on $T$:

$$\mathcal{C}^\infty(T) = \mathcal{C}^\infty(\mathbb{R}^d) \otimes \lambda^\alpha \Gamma_{\alpha\beta}^{\mu} = \mathcal{C}^\infty(\mathbb{R}^d) \otimes \mathbb{C}[\theta^\alpha].$$

(For a recent source for the relevant supergeometry, we refer to [Noj21].) We will write $x^\mu$ for coordinates on the even part of $T$ and $\theta^\alpha$ for anticommuting coordinates on the odd part. $\mathcal{C}^\infty(T)$ is the regular representation of $T$, and thus acquires two distinct actions of $t$ by derivations, namely by the left and right actions of $T$ on itself. Each extends to an action of $p$. We denote these by

$$R, L : p \longrightarrow \text{Vect}(T).$$
Now let \( \Gamma \) be a graded equivariant \( R/I \)-module. The pure spinor superfield associated to \( \Gamma \) is the cochain complex

\[
A^*(\Gamma) = (C^\infty(T) \otimes \Gamma, \mathcal{D} = \lambda^a R(Q_a)) .
\]

The super Poincaré algebra acts on \( A^*(\Gamma) \) on the left, endowing \( A^*(\Gamma) \) with the structure of a strict multiplet (see [Eag+21] for details). In particular, \( A^*(\Gamma) \) inherits a \( \mathbb{Z} \times \mathbb{Z}/2 \)-grading, where the integer grading is inherited from the weight grading on \( \Gamma \) and the \( \mathbb{Z}/2 \)-grading is inherited from the supermanifold structure on \( T \). In the resulting multiplet, the \( \mathbb{Z}/2 \)-grading determines the cohomological degree, while the \( \mathbb{Z}/2 \)-grading becomes the total (not the intrinsic) parity.

While \( A^*(\Gamma) \) is a strict multiplet, it is typically very large and does not resemble the standard component-field descriptions. These component-field descriptions arise by identifying a quasi-isomorphic subcomplex of \( A^*(\Gamma) \), choosing a retraction to this complex, and transferring all the relevant structures along this retraction via homotopy transfer.

A minimal component-field description can be canonically constructed as follows. In coordinates, the differential \( \mathcal{D} \) takes the form

\[
\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 = \lambda^a \frac{\partial}{\partial \theta^a} - \lambda^a \Gamma^\mu_{\alpha \beta} \theta^\beta \frac{\partial}{\partial x^\mu}.
\]

We can equip \( A^*(\Gamma) \) with a filtration such that the differential on \( \text{Gr} A^*(\Gamma) \) is \( \mathcal{D}_0 \). It is clear upon inspection that

\[
\text{Gr} A^*(\Gamma) = C^\infty(\mathbb{R}^d) \otimes (K^*(\Gamma), d_K),
\]

where \((K^*(\Gamma), d_K)\) is the Koszul complex of \( \Gamma \). This implies that the fields of the minimal multiplet take values in the Koszul homology of \( \Gamma \), which is bigraded by the cohomological degree and the weight grading on \( \Gamma \). (In our conventions, \( K^* \) is cohomologically, therefore nonpositively, graded.) The weight grading on \( H^*(K^*(\Gamma)) \) determines the cohomological degree of the corresponding multiplet, while the overall parity is determined by the cohomological degree in Koszul homology modulo two.

On general grounds, Koszul homology can be computed by considering a minimal equivariant free resolution \((L, d_L)\) of \( \Gamma \) in \( R \)-modules. Such a minimal free resolution is also bigraded, by cohomological degree and by the weight grading on \( R \). In our conventions, which are cohomological, \( L^* \) is nonpositively graded, and the differential \( d_L \) has cohomological degree one and weight zero. For convenience, we will often write this in the form

\[
L^* = \bigoplus L^{-k} = \bigoplus W_k \otimes R[k],
\]
where \( W_k \) is the finite-dimensional weighted \( p_0 \)-representation in which the generators of \( L^{-k} \) transform. There is then an isomorphism (after a shift) of bigraded \( p_0 \)-modules between the generators of \( L^\bullet \) and the Koszul homology. However, the bigradings do not agree naively: rather,

\[
H^{-k}(K^\bullet(\Gamma))_\ell \cong \left( L^{-k} \otimes_R \mathbb{C} \right)_{\ell+k} \Rightarrow H^{-k}(K^\bullet(\Gamma)) = W_k(k).
\]

This was established in [MSX12; KL09], and is a key insight, allowing one to quickly compute component fields by freely resolving \( \Gamma \) over \( R \). In [Eag+21] (see also [EHS22]) it was further shown that the resolution differential \( d_L \) encodes the \( p \)-module structure (as had been conjectured in [Ber02]).

It remains to recall how the structure of a multiplet is defined on the component fields. Choosing an equivariant set of representatives for Koszul homology, we consider a deformation retract of the form

\[
\begin{align*}
\left( \text{Gr} A^\bullet(\Gamma), D_0 \right) & \xrightarrow{p} \left( H^\bullet(\text{Gr} A^\bullet(\Gamma)), 0 \right).
\end{align*}
\]

Applying homotopy transfer to this data, we obtain a new multiplet with underlying complex \( H^\bullet(\text{Gr} A^\bullet(\Gamma)) \), new differential \( D' \) (obtained from \( D_1 \) via homotopy transfer), and supersymmetry action \( \rho' \) (obtained from \( L \)). The multiplet obtained in that way is minimal in the sense that its differential only contains differential operators of non-zero degree; we emphasize, however, that this characterization does not extend to the more general derived setting. By construction, this multiplet is quasi-isomorphic to \( A^\bullet(\Gamma) \), and is therefore perturbatively equivalent. We call this multiplet the component-field multiplet (or minimal multiplet) associated to \( \Gamma \), and denote it by \( \mu A^\bullet(\Gamma) \). When we want to refer to the underlying vector bundle of \( \mu A^\bullet(\Gamma) \) — which is the associated bundle of the bigraded Lorentz representation on \( H^\bullet(K^\bullet(\Gamma)) \) — we will write \( \mu A^\bullet(\Gamma)^\# \). This notation will in general apply to any multiplet and denote (a natural bigraded lift of) its underlying \( \mathbb{Z} \times \mathbb{Z}/2 \)-graded vector bundle, considered without the data of the differential and the module structure.

2.1.3. Computational techniques. In practice, there are various ways to extract information on \( \mu A^\bullet(\Gamma) \) from the algebraic input \( \Gamma \). The Hilbert series of \( \Gamma \), which is the graded dimension of the module viewed as a formal power series

\[
\text{grdim}(\Gamma) = \sum_{k=0}^{\infty} \dim(\Gamma_k) t^k \in \mathbb{Z}[[t]],
\]

2
can be resummed to the form

\[(2.16) \quad \text{grdim}(\Gamma) = \frac{P(t)}{(1-t)^n},\]

where \(n = \dim(t_1)\) is the number of variables in \(R\) and \(P\) is a polynomial with integer coefficients. The coefficients of \(P\) then describe the graded free \(R\)-module underlying the minimal resolution \((L, d_L)\), and thus record the Betti numbers of the graded bundle \(\mu A^*(\Gamma)^\#\). Concretely, we have that

\[(2.17) \quad \text{grdim}(\Gamma) = \sum_k (-1)^k \text{grdim}(L^k) = \sum_k (-1)^k \text{grdim}(W^k) \text{grdim}(R) = \frac{1}{(1-t)^n} \sum_k (-1)^k \text{grdim}(W^k).\]

The fact that \(L^*\) is minimal means that \(\text{wt}(W^k) \geq k\), so that \(t^k \mid \text{grdim}(W^k)\). Recalling the degree shift (2.13), we deduce that

\[(2.18) \quad P(t) = \sum_k (-1)^k \text{grdim}(W^k) = \text{grdim} \chi(W^*) = \sum_k (-t)^k \text{grdim} \left( H^{-k}(K^*(\Gamma)) \right).\]

Here \(\chi(\cdot)\) denotes the Euler characteristic of a cohomologically graded vector space. Of course, cancellations are possible, so that \(P(t)\) does not contain complete information about the Koszul homology. But in practice, one is often in a situation where no cancellations occur. In any case, if an acyclic resolution differential can be constructed, one has demonstrated that no further generators appear in \(L^*\).

This procedure can be improved in the \(p_0\)-equivariant setting. Since \(\Gamma\) is graded and equivariant, each weighted summand \(\Gamma_k\) is a finite-dimensional representation of \(p_0\). We can consider the equivariant Hilbert series as a formal power series in the representation ring of \(p_0\)\(^1\) by setting

\[(2.19) \quad \text{Hilb}(\Gamma) = \sum_{k=0}^\infty \Gamma_k t^k \in \text{Rep}(p_0)[[t]].\]

We can then rewrite the Hilbert series in the form

\[(2.20) \quad \text{Hilb}(\Gamma) = \left[ \sum_{d=0}^\infty \text{Sym}^d(t_1^\vee) t^d \right] \otimes \left[ \sum_{k,\ell} (-1)^\ell W^\ell_k t^k \right] = \text{Hilb}(R) \cdot \text{Hilb}(\chi(W^*)).\]

\(^1\)The representation ring of \(p_0\) is the free abelian group on the set of finite-dimensional irreducible \(p_0\)-representations, with the multiplication induced by the tensor product of representations.
Comparing coefficients order by order, one obtains a system of equations which allows to identify $\chi(W^*)$, and thus (at least in favorable cases) $W^*$ itself:

$$
\chi(W^*)_0 = \Gamma_0
\chi(W^*)_1 = \Gamma_1 - t_1^\gamma \otimes \chi(W^*)_0
\vdots
\chi(W^*)_k = \Gamma_k - \sum_{d=1}^{k} \text{Sym}^d(t_1^\gamma) \otimes \chi(W^*)_{k-d}.
$$

(2.21)

This technique is used frequently in the work of Cederwall and collaborators, and we will apply it in examples in what follows.

2.2. **Supersymmetry in six dimensions.** We review the concrete form of the above constructions for the six-dimensional $\mathcal{N} = (1,0)$ supertranslation algebra. We always work with complexified algebras and thus ignore signature. The translations are thus $t_2 = V = \mathbb{C}^6$, where $V$ denotes the vector representation of $\text{Spin}(6)$.

We denote the two chiral spinor representations of $\text{Spin}(6)$ by $S^\pm$. There is an exceptional isomorphism identifying $\mathfrak{so}(6) \cong \mathfrak{sl}_4$. Under this isomorphism, $S^+$ is identified with the fundamental representation of $\mathfrak{sl}_4$ and $S^- = (S^+)^\vee$ with the antifundamental, while $V$ is identified with the two-form. There are thus $\text{Spin}(6)$-equivariant isomorphisms

$$
\wedge^2(S^\pm) \cong V.
$$

(2.22)

The six-dimensional $\mathcal{N} = (1,0)$ supertranslation algebra takes the form

$$
t = V(-2) \otimes (S_+ \otimes U)(-1),
$$

(2.23)

where $U = (\mathbb{C}^2, \omega)$ is a symplectic vector space. The bracket takes the form

$$
[-,-] = \wedge \otimes \omega : (S_+ \otimes U) \otimes (S_+ \otimes U) \longrightarrow V,
$$

(2.24)

where the isomorphism (2.22) is used. The bracket is thus equivariant with respect to the obvious action of $\text{Sp}(1)$ on $U$, identifying the $R$-symmetry group as $\text{Sp}(1) \cong \text{SU}(2)$. (The relevant Lie algebra is thus $\mathfrak{sl}_2$.)

Explicitly, using a basis $Q^i_\alpha$, we can write

$$
[Q^i_\alpha, Q^j_\beta] = \epsilon^{ij}_\alpha \Gamma^\mu_{\alpha\beta} P_\mu.
$$

(2.25)

Here $i,j = 1,2$ are indices for $U$ and $\alpha, \beta = 1\ldots 4$ are indices for $S_+$. Identifying $R = \mathbb{C}[\lambda_i^a]$, the defining ideal of the nilpotence variety is spanned by the six quadratic equations

$$
I = (\lambda_i^a \Gamma^\mu_{\alpha\beta} \epsilon^{ij}_\alpha \lambda_j^\beta).
$$

(2.26)
These equations can be conveniently packaged as follows. Let us define the rank of a supercharge \( Q \in t_1 \) to be the rank of the associated linear map \((S_+)^\vee \to U\). From (2.24) it follows immediately that the square-zero supercharges are precisely those of rank one. In terms of coordinates this means that the ideal \( I \) is spanned by the \( 2 \times 2 \) minors of the matrix with entries \( \lambda_i^\alpha \),

\[
(2.27) \quad \begin{pmatrix}
\lambda_1^1 & \lambda_2^2 & \lambda_3^3 & \lambda_4^4 \\
\lambda_1^1 & \lambda_2^2 & \lambda_3^3 & \lambda_4^4
\end{pmatrix}.
\]

Accordingly, the nilpotence variety \( \hat{Y} = \text{Spec} R/I \) can be thought of as the space of rank one matrices inside \( M^{2 \times 4}(C) \). Its projective version \( Y = \text{Proj} R/I \) can be identified with the product of two projective spaces via the Segre embedding. In more detail, the square-zero supercharges are precisely those which can be written as

\[
(2.28) \quad Q = \xi \otimes r \quad \text{with} \quad \xi \in S_+, r \in U.
\]

Interpreting \([r_0 : r_1]\) and \([\xi_0 : \cdots : \xi_3]\) as homogeneous coordinates on \( \mathbb{P}^1 \) and \( \mathbb{P}^3 \) respectively identifies \( Y \) with the image of the Segre embedding

\[
(2.29) \quad \sigma : \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^7 \quad ([r_0 : r_1],[\xi_0 : \cdots : \xi_3]) \mapsto [r_0 \xi_0 : \cdots : r_1 \xi_3].
\]

We can thus explore supermultiplets in six dimensions using the algebraic geometry of projective spaces.

2.3. From sheaves on projective schemes to modules. As explained above, the pure spinor superfield formalism constructs a supersymmetric multiplet from a graded equivariant \( R/I \)-module. Clearly, these are closely related to sheaves of \( O_{\hat{Y}} \)-modules on \( \hat{Y} \): For any affine scheme \( X = \text{Spec} S \) there is an equivalence of categories between quasi-coherent sheaves of \( O_X \)-modules and \( S \)-modules. Explicitly, this equivalence is given by taking global sections

\[
(2.30) \quad \text{QCoh}_{O_X} \to \text{Mod}_S \quad \mathcal{F} \to \Gamma(\text{Spec}(S), \mathcal{F}),
\]

and conversely assigning

\[
(2.31) \quad \text{Mod}_S \to \text{QCoh}_{O_X} \quad M \to \bar{M},
\]

where \( \bar{M} \) is defined by the requirement \( \bar{M}(D_f) = M_f \) for all \( f \in S \). If \( S \) is graded, one can think of the grading as defining a \( \mathfrak{gl}_1 \)-action on \( \text{Spec} S \); it is then possible to define an equivalence between graded \( S \)-modules and quasicoherent sheaves of \( O_X \)-modules on \( X \) that are equivariant for rescalings.

\[\text{Here } D_f \subseteq \text{Spec } S \text{ denotes all prime ideals of } S \text{ not containing } f \text{ and } M_f \text{ the localization of } M \text{ at } f.\]
One can thus always think of the input to the (underived) pure spinor superfield formalism geometrically as a \((p_0 \oplus \mathfrak{gl}_1)\)-equivariant sheaf on the affine nilpotence variety. It is tempting to ask if one can picture the situation using the geometry of sheaves on \(Y = \text{Proj} R/I\). Here, the situation is geometrically compelling, but a bit less unequivocal. From a graded \(S\)-module \(M\), we can construct a quasi-coherent sheaf on \(\text{Proj} S\) by setting \(\tilde{M}(D_f) = (M_f)_0\). By the definition of the Proj-construction we have \(\tilde{S} = \mathcal{O}_{\text{Proj} S}\). The twisting sheaves are defined by

\[
\mathcal{O}_{\text{Proj} S}(n) = \tilde{S(n)}.
\]

For a sheaf \(\mathcal{F}\) on \(\text{Proj} S\), we define the associated \(S\)-module to be

\[
\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\text{Proj} S, \mathcal{F}(n)).
\]

We will call \(\Gamma_*(\mathcal{F})\) the graded global section module of \(\mathcal{F}\). In general, these assignments no longer give an equivalence of categories, but we can still use \(\Gamma_*(-)\) to construct large families of input data for the pure spinor superfield formalism from sheaves on the projective version of the nilpotence variety. This is in particular useful in the case of \(N=(1,0)\) supersymmetry in six dimensions, since—as we explained above—the projective version of the nilpotence variety can be identified with \(\mathbb{P}^1 \times \mathbb{P}^3\) and equivariant sheaves on this space are very well understood geometrically.

2.3.1. What the projective perspective misses. Contrary to the affine case, the functors \(\sim\) and \(\Gamma_*\) do not yield an equivalence of categories. While it is true that

\[
\Gamma_*(\mathcal{F}) \cong \mathcal{F}
\]

for any quasicoherent sheaf \(\mathcal{F}\), it can happen that \(\Gamma_*(\tilde{M})\) is not isomorphic to the original module \(M\). Let us restrict to the case where \(S = R\) is a polynomial ring and \(M\) is a finitely generated graded module. Consider the class \(\mathcal{C}\) of modules \(M\) such that \(M_n = 0\) for \(n\) large enough. One finds that these are precisely the modules which are in the kernel of \(\sim\). One has the following result:

**Proposition 2.1** ([Ser55]). Let \(M\) be a graded \(S\)-module. Then

\[
\tilde{M} = 0 \iff M \in \mathcal{C}.
\]

For the pure spinor superfield formalism, this means that multiplets corresponding to modules which are concentrated in finitely many degrees cannot be obtained from sheaves on the projective nilpotence variety. One such example is the free superfield \(A^*(\mathbb{C})\) which is constructed from the trivial module \(\mathbb{C}\) (thought of as the quotient of \(R\) by the maximal ideal corresponding to the origin). The corresponding sheaf on the affine nilpotence variety is the
A skyscraper sheaf with value \( \mathbb{C} \) at the origin; the associated sheaf on the projective nilpotence variety is trivial.

In general, such sheaves must have zero-dimensional support. The support of an equivariant sheaf must consist of a union of orbits of the \( P_0 \)-action; since we only consider sheaves that are equivariant for rescaling, the origin is the unique zero-dimensional orbit, so that any module in the kernel of \( \sim \) defines a sheaf supported entirely at the origin.

**Remark 2.2.** It is natural to wonder how conditions on the support of a sheaf translate into properties of the corresponding multiplet. An intuitive answer is suggested by the results of [SW21] on twisting in the pure spinor formalism. There, it was noted that deforming a super Poincaré-type algebra by a square-zero supercharge commutes with forming the pure spinor multiplet of the structure sheaf. When \( Y \) is smooth (as is the case here), only holomorphic twists are available, and the computations in [SW21] imply that the holomorphic twist of a given multiplet is freely generated over the Dolbeault complex on spacetime by the stalk of the corresponding sheaf at the holomorphic supercharge. We do not explain this in detail here, but will remark from time to time on the physical interpretations of our results that it suggests.

In keeping with Remark 2.2, we expect that multiplets corresponding to sheaves in the kernel of \( \sim \) are precisely those that are perturbatively trivial in every possible twist. We note that the free superfield falls into this class.

### 2.4. Some natural equivariant vector bundles.

In the bulk of this work we are going to consider various vector bundles over the nilpotence variety \( Y \cong \mathbb{P}^1 \times \mathbb{P}^3 \) and construct the associated multiplets using the pure spinor superfield formalism. For later reference and completeness, we now introduce the bundles that will appear later on.

#### 2.4.1. Line bundles.

The product space geometry of the nilpotence variety \( Y \cong \mathbb{P}^1 \times \mathbb{P}^3 \) makes it easy to describe all of its line bundles. Indeed, holomorphic line bundles are classified up to isomorphism by the Picard group \( \text{Pic}(Y) \cong H^1(Y, \mathcal{O}_Y^*) \), which can be easily computed using the exponential short exact sequence

\[
0 \longrightarrow \mathbb{Z}_{\mathbb{P}^1 \times \mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3} \longrightarrow \mathcal{O}^*_{\mathbb{P}^1 \times \mathbb{P}^3} \longrightarrow 0
\]

and its related long exact sequence in cohomology. In particular, one finds the isomorphism \( \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^3) \cong \mathbb{Z} \oplus \mathbb{Z} \), which tells that every line bundle on the product variety \( \mathbb{P}^1 \times \mathbb{P}^3 \) arises from line bundles defined on its factors \( \mathbb{P}^1 \) and \( \mathbb{P}^3 \). (Recall that \( \text{Pic}(\mathbb{P}^n) \cong \mathbb{Z} \) for any \( n \geq 1 \).) In
fact, given the structural projections

\[(2.37)\]

\[
\pi_1
\]
\[
\pi_3
\]

\[
P^1 \times P^3
\]

\[
P^1 \quad \pi_1 \quad P^3
\]

\[
P^3
\]

from \(P^1 \times P^3\) to its cartesian components, the line bundles on \(P^1 \times P^3\) are all given by the exterior tensor product of a pair line bundles defined over \(P^1\) and \(P^3\) respectively. In other words,

\[(2.38)\]

\[
O_{P^1 \times P^3}(n, m) := O_{P^1}(n) \boxtimes O_{P^3}(m) := \pi_1^* O_{P^1}(n) \boxtimes O_{P^1 \times P^3} \pi_3^* O_{P^3}(m), \quad (n, m) \in \mathbb{Z}^{ \oplus 2}.
\]

(The notation is standard.) Note that the generators of the Picard group \(\text{Pic}(P^1 \times P^3)\) are given by \(O_{P^1 \times P^3}(1, 0)\) and \(O_{P^1 \times P^3}(0, 1)\); the connecting (iso)morphism \(\delta_2 : \text{Pic}(P^1 \times P^3) \to \mathbb{Z} \oplus \mathbb{Z}\) thus carries \(O_{P^1 \times P^3}(n, m)\) to \((n, m)\), and tensor product yields an isomorphism (of \(O_{P^1 \times P^3}\)-modules)

\[(2.39)\]

\[
O_{P^1 \times P^3}(n, m) \otimes O_{P^1 \times P^3}(k, l) \cong O_{P^1 \times P^3}(n + k, m + l)
\]

for any \((n, m), (k, l) \in \mathbb{Z}^{ \oplus 2}\). We will often use the shorthand \(O(n, m) = O_{P^1 \times P^3}(n, m)\), since we focus on the example of six-dimensional minimal supersymmetry in this paper. Finally, we will denote a \(k\)-twisting sheaf for the nilpotence variety \(Y\) by

\[(2.40)\]

\[
O_Y(k) := O_{P^1 \times P^3}(k, k).
\]

\[2.4.2. \text{Tangent and Cotangent Bundles.}\] Similarly, tangent and cotangent bundles on a product variety can be reconstructed by the tangent and cotangent bundles of its Cartesian components. In fact, the tangent bundle of \(P^1 \times P^3\) is given by the exterior direct sum

\[(2.41)\]

\[
\mathcal{T}_{P^1 \times P^3} \cong \pi_1^* \mathcal{T}_{P^1} \oplus \pi_3^* \mathcal{T}_{P^3} =: \mathcal{T}_{P^1} \boxplus \mathcal{T}_{P^3}.
\]

Note that \(\mathcal{T}_{P^1}\) is a line bundle and one has \(\mathcal{T}_{P^1} \cong O_{P^1}(+2)\), while \(\mathcal{T}_{P^3}\) is an ample non-decomposable vector bundle of rank three. The tangent bundle on any projective space \(P^n\) sits in the Euler exact sequence

\[(2.42)\]

\[
0 \longrightarrow O_{P^n} \longrightarrow O_{P^n}(+1) \otimes V_{n+1} \longrightarrow \mathcal{T}_{P^n} \longrightarrow 0,
\]

where \(V_{n+1}\) is a \((n + 1)\)-dimensional complex vector space that carries the fundamental representation of \(\mathfrak{sl}_{n+1}\). The Euler exact sequence \((2.42)\) is a short exact sequence of \(\mathfrak{sl}_{n+1}\)-equivariant sheaves; this will play a role in \(\S 6\), when we will study the multiplet associated to the tangent bundle \(\mathcal{T}_Y\) of the nilpotence variety.
In a similar fashion, the cotangent bundle $\Omega^1_Y := \mathcal{H}om_{\mathcal{O}_{P^1 \times P^3}}(T_{P^1 \times P^3}, \mathcal{O}_{P^1 \times P^3})$ of the nilpotence variety $Y$ is given by the exterior direct sum

$$\Omega^1_{P^1 \times P^3} \cong \pi_1^* \Omega^1_{P^1} \oplus \pi_3^* \Omega^1_{P^3} = \Omega^1_{P^1} \oplus \Omega^1_{P^3},$$

(2.43)

where now $\Omega^1_{P^1} \cong \mathcal{O}_{P^1}(-2)$. Note that, taking the dual of the Euler sequence (2.42), one finds

$$0 \longrightarrow \Omega^1_{\mathcal{O}(n)} \longrightarrow \mathcal{O}(n-1) \otimes V^\vee_{n+1} \longrightarrow \mathcal{O}_{P^n} \longrightarrow 0,$$

(2.44)

which in turn describes the cotangent bundle on any projective space $\mathbb{P}^n$.

2.4.3. Normal and Conormal Bundles. Let us now consider $Y$ via its Segre embedding $\sigma : Y \hookrightarrow \mathbb{P}^7$. (We recall from §2 that this embedding is canonically associated to the datum of the supertranslation algebra $t$.) Having introduced the tangent bundle $T_Y$, one defines the normal bundle $N_{Y/P^7}$ of $Y$ in $\mathbb{P}^7$ to be quotient bundle $T_{\mathbb{P}^7}|_{Y} / T_Y$, where $T_{\mathbb{P}^7}|_{Y} := \sigma^* T_{\mathbb{P}^7}$. As such, the normal bundle sits in the exact sequence

$$0 \longrightarrow T_Y \xrightarrow{d\sigma} T_{\mathbb{P}^7}|_{Y} \longrightarrow N_{Y/P^7} \longrightarrow 0,$$

(2.45)

of vector bundles on $Y$, which will be referred to as the normal bundle exact sequence. Dualizing (2.45), one obtains the exact sequence defining the conormal bundle:

$$0 \longrightarrow N^\vee_{Y/P^7} \longrightarrow \Omega^1_{\mathcal{O}(n)} \xrightarrow{d\sigma^\vee} \Omega^1_Y \longrightarrow 0.$$

(2.46)

The conormal bundle $N^\vee_{Y/P^7}$ is thus the kernel of the morphism of vector bundles $d\sigma^\vee : \Omega^1_{\mathbb{P}^7}|_{Y} \rightarrow \Omega^1_Y$. Another characterization of the conormal bundle $N^\vee_{Y/P^7}$ is possible using the sheaf of ideals $j_Y$, which is defined as the kernel of the morphism of sheaves $\sigma^2 : \mathcal{O}_{\mathbb{P}^7} \rightarrow \sigma_+ \mathcal{O}_Y$. In fact, there is a natural isomorphism of vector bundles on $Y$ given by $\sigma^*(j_Y/j_Y^2) \cong N^\vee_{Y/P^7}$. In the following, since no confusion regarding the ambient space can arise, we will denote the normal and conormal bundles with respect to the Segre embedding by $N_Y$ and $N^\vee_Y$.

3. A FAMILY OF MULTIPLETS FROM LINE BUNDLES

3.1. General procedure. Let us now classify all multiplets associated to the infinite family of line bundles $\mathcal{O}(n,m)$. We will denote the multiplets by

$$\mu A^*(n,m) := \mu A^*(\Gamma_+((\mathcal{O}(n,m))).$$

(3.1)
As a first observation, we note that the construction exhibits the following symmetry under twists of line bundles:

\[
\Gamma_*(\mathcal{O}(n + k, m + k)) = \bigoplus_{d \in \mathbb{Z}} H^0(\mathcal{O}(n + k + d, m + k + d))
\]

\[
= \bigoplus_{d \in \mathbb{Z}} H^0(\mathcal{O}(n + d, m + d))(k)
\]

\[
= \Gamma_*(\mathcal{O}(n, m))(k).
\]

(3.2)

This implies that the multiplets \(\mu A^*(n, m)\) and \(\mu A^*(n + k, m + k)\) agree up to a total degree shift. Since the weight grading of a graded equivariant \(R/I\)-module becomes the cohomological grading of the corresponding multiplet, we have that

\[
\mu A^*(n + k, m + k) = \mu A^*\left[\Gamma_*(\mathcal{O}(n, m))(k)\right] = \mu A^*\left[\Gamma_*(\mathcal{O}(n, m))\right][k] = \mu A^*(n, m)[k].
\]

(3.3)

It is thus sufficient to consider the line bundles \(\mathcal{O}(n, 0)\) and \(\mathcal{O}(0, m)\) for \(n, m \geq 0\). (Equivalently, one could also consider the family \(\mathcal{O}(n, 0)\) for \(n \in \mathbb{Z}\).)

We will identify the field content of the multiplets using the technique sketched above in §2.1.3. We resum the equivariant Hilbert series, working in the ring of formal power series with coefficients in the representation ring of \(sl_2 \times sl_4\), and read off the equivariant structure of the minimal free resolution from its numerator.

Recall that \(\Gamma_*(\mathcal{O}(n, m))_d = \mathbb{C}[x_0, x_1]_{n+d} \otimes \mathbb{C}[y_0, \ldots, y_3]_{m+d}\). The monomials of degree \(d\) are the \(d\)-th symmetric power of the defining representation of the corresponding group of linear transformations, so that we have

\[
\Gamma_*(\mathcal{O}(n, m))_d = [n + d|m + d, 0, 0].
\]

(3.4)

in terms of Dynkin labels for \(sl_2 \times sl_4\). Thus the equivariant Hilbert series takes the form

\[
\text{Hilb}(n, m) := \text{Hilb}(\Gamma_*(n, m)) = \sum_{d=-\min(n,m)}^{\infty} [n + d|m + d, 0, 0] t^d.
\]

(3.5)

Following §2.1, we rewrite the Hilbert series using the identity

\[
\text{Hilb}(n, m) = \text{Hilb}(R) \cdot \text{Hilb}(\chi(W^*(n, m))),
\]

(3.6)

and solve for \(\chi(W^*(n, m))\). The equations (2.21) become

\[
\chi(W^*(n, m))_0 = [n|m, 0, 0]
\]

\[
\chi(W^*(n, m))_1 = [n + 1|m + 1, 0, 0] - [1|1, 0, 0] \otimes \chi(W^*(n, m))_0
\]

(3.7)

\[
\vdots
\]

\[
\chi(W^*(n, m))_k = [n + k|m + k, 0, 0] - \sum_{d=1}^{k} \text{Sym}^d([1|1, 0, 0]) \otimes \chi(W^*(n, m))_{k-d}.
\]
In what follows, we solve these equations case by case. We will often tabulate our results by writing a square table representing $H^*(K^*(\Gamma))$, or equivalently the bundle $\mu A^*(\Gamma)^\#$. In such tables, the (nonpositive) cohomological grading is on the horizontal axis and decreases to the right, while the weight grading on Koszul homology increases downward. As such, when reading the table as representing a multiplet, cohomological degree (ghost number) is on the vertical axis, while parity is determined by the position on the horizontal axis modulo two. (For further notes on this convention, see [EHS22].)

3.2. The bundles $O(n,0)$ for $n \geq 0$. We begin with the case of the bundles $O(n,0)$ for non-negative $n$. As we will see, these bundles include the vector multiplet, its antifield multiplet, and the hypermultiplet, as well as an infinite family of strict component-field multiplets associated to $O(n,0)$ with $n \geq 3$.

3.2.1. Computation of the Betti numbers. We specialize the Hilbert series (3.5) to the case at hand. At the level of the graded dimension,

\[(3.8) \quad \text{grdim}(n,0) = \sum_{d=0}^{\infty} (n + d + 1) \frac{(d + 3)(d + 2)(d + 1)}{6} t^d,\]

which can be rewritten as a derivative of a geometric series

\[(3.9) \quad \text{grdim}(n,0) = \frac{1}{6} \frac{\partial^3}{\partial t^3} t^{3-n} \frac{\partial}{\partial t} \sum_{d=0}^{\infty} t^{d+n+1} = \frac{1}{6} \frac{\partial^3}{\partial t^3} t^{3-n} \frac{\partial}{\partial t} t^{n+1} (1-t).\]

Performing the derivatives, the general result can be expressed in the following form.

\[(3.10) \quad \text{grdim}(n,0) = \frac{(n + 1) - 4nt + 6(n - 1)t^2 - 4(n - 2)t^3 + (n - 3)t^4}{(1-t)^8}\]

The coefficients of the numerator now correspond to the Betti numbers of the associated multiplet.

Let us write out these Betti numbers concretely for all $m$. There are three special cases when $m \in \{0,1,2\}$. For $m = 0$ one finds

\[(3.11) \quad \text{grdim} \mu A^*(0,0)^\# = \begin{bmatrix} 1 & - & - \\ - & 6 & 8 & 3 \end{bmatrix},\]

which corresponds to the vector multiplet. For $n = 1$, we obtain

\[(3.12) \quad \text{grdim} \mu A^*(1,0)^\# = \begin{bmatrix} 2 & 4 & - \\ - & - & 4 & 2 \end{bmatrix},\]

which corresponds to the hypermultiplet. For $n = 2$, the result reads

\[(3.13) \quad \text{grdim} \mu A^*(2,0)^\# = \begin{bmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{bmatrix},\]
which corresponds to the antifield multiplet of the vector multiplet. Finally, for \( n \geq 3 \), the resulting Betti numbers take the general form

\[
grdim \mu A^*(m, 0)^\# = \begin{bmatrix}
n + 1 & 4n & 6(n - 1) & 4(n - 2) & n - 3
\end{bmatrix}.
\]

3.2.2. *Equivariant decomposition.* The above recursive relations are easily solved, either by hand or with the help of a computer program such as LiE [LCL]. Let us again first consider the three special cases where \( n \in \{0, 1, 2\} \). For \( n = 0 \) we obtain

\[
W_0 = \{0|0,0,0\}
\]
\[
W_1 = 0
\]
\[
W_2 = \{0|0,1,0\}
\]
\[
W_3 = \{1|0,0,1\}
\]
\[
W_4 = \{2|0,0,0\}.
\]

Thus the resulting multiplet takes the form

\[
\mu A^*(0, 0)^\# = \begin{bmatrix}
\Omega^0 & - & - & - \\
- & \Omega^1 & \Omega^0 \otimes S_- & \Omega^0 \otimes \mathbb{C}^3
\end{bmatrix},
\]

where the three scalar fields live in the adjoint representation of the \( R \)-symmetry group. (Here and in the following tables showing the field content of multiplets \( \mathbb{C}^n \) will always denote the unique irreducible \( n \)-dimensional representation of \( sl_2 \).) This corresponds to the vector multiplet of six-dimensional \( \mathcal{N} = (1,0) \) supersymmetry. For \( n = 1 \), we find

\[
W_0 = \{1|0,0,0\}
\]
\[
W_1 = \{0|1,0,0\}
\]
\[
W_2 = 0
\]
\[
W_3 = \{0|0,0,1\}
\]
\[
W_4 = \{1|0,0,0\}.
\]

We can thus identify \( \mu A^*(1, 0) \) as the hypermultiplet

\[
\mu A^*(1, 0)^\# = \begin{bmatrix}
\Omega^0 \otimes \mathbb{C}^2 & S_+ \\
S_- & \Omega^0 \otimes \mathbb{C}^2
\end{bmatrix}
\]
For $n = 2$

\[
W_0 = [2|0,0,0] \\
W_1 = -[1|1,0,0] \\
W_2 = -[0|0,1,0] \\
W_3 = 0 \\
W_4 = -[0|0,0,0].
\]

(3.19)

The resulting multiplet $\mu A^*(2,0)$ is the antifield multiplet of the vector multiplet.

\[
\mu A^*(2,0)^\# = \begin{bmatrix}
\Omega^0 \otimes \mathbb{C}^3 & S_+ \otimes \mathbb{C}^2 & \Omega^1 \\
\Omega^0
\end{bmatrix}
\]

(3.20)

Finally for $n \geq 4$, the general form is

\[
W_0 = [n|0,0,0] \\
W_1 = -[n - 1|1,0,0] \\
W_2 = [n - 2|0,1,0] \\
W_3 = -[n - 3|0,0,1] \\
W_4 = [n - 4|0,0,0].
\]

(3.21)

Thus, $\mu A^*(n,0)$ for $n \geq 3$ are of the form

\[
\mu A^*(n,0)^\# = 
\begin{bmatrix}
\mathbb{C}^{n+1} & \mathbb{C}^n \otimes S_+ & \mathbb{C}^{n-1} \otimes \Lambda^2 S_+ & \mathbb{C}^{n-2} \otimes \Lambda^3 S_+ & \mathbb{C}^{n-3} \otimes \Lambda^4 S_+
\end{bmatrix}
\]

(3.22)

This family of multiplets was described in the physics literature under the name $O(n)$-multiplets [KNT17; KNT18; GR+98; LTM12].

3.2.3. Supersymmetry module structure and interpretation. Given our results so far, it is easy to give an explicit description of the module $\Gamma^*(O(n,0))$ as a cokernel of a map between free $R$-modules as well as to describe their minimal free resolutions in $R$-modules. The cases $n = 0$ and $n = 1$ were already discussed in [Eag+21]. For $n \geq 1$, we are looking for a map

\[
\phi_n : \mathbb{C}^n \otimes S_+ \otimes R \longrightarrow \mathbb{C}^{n+1} \otimes R
\]

(3.23)

which is linear in $\lambda_i^a$ and equivariant under $\mathfrak{sl}_2 \times \mathfrak{sl}_4$. Up to a non-zero constant prefactor, there is a unique such map which can be described in components by

\[
F \mapsto \lambda_{i_1...i_{n-1}}^a F_{i_1...i_{n-1}a}.
\]

(3.24)
Resolving $\Gamma_\ast(\mathcal{O}(n,0)) = \text{coker}(\varphi_n)$ one recovers the field content of the multiplets described above. In addition, the resolution differential encodes the part of the $p$-module structure acting by differential operators of degree zero (this is the $\text{Gr}(p)$-module structure induced from the $p$-module structure) [EHS22; Eag+21].

Let us describe the minimal free resolution and the module structure for the cases $n \geq 3$. This will provide an intuitive interpretation of $\mu A^\ast (n,0)$: it is a multiplet whose observables are generated by the degree-$n$ monomials in the observables of the $\mathcal{O}(1,0)$-multiplet, i.e. the hypermultiplet. One can thus imagine that the fields of the hypermultiplet map to the fields of the $\mathcal{O}(n,0)$ multiplet via a (ramified) $n$-fold covering, dual to the inclusion map on observables.

In components, the resolution differential

$$(d_L)_i : \mathbb{C}^{n-i} \otimes \wedge^i S_+ \otimes \mathbb{R} \rightarrow \mathbb{C}^{n-i+1} \otimes \wedge^{i-1} S_+ \otimes \mathbb{R} \quad i = 1 \ldots 4$$

is described by contracting along $S_+$ and symmetrizing along the $sl_2$-representation, for example

$$(d_L)_2 F_{i_1 \ldots i_{n-1} a} = \lambda_{(i_{n-1} \ldots i_2)}^{\beta} F_{i_1 \ldots i_{n-2}} \alpha \beta.$$ 

This translates into supersymmetry transformation rules of the form

$$\delta F_{i_1 \ldots i_m} = \epsilon_{(i_n \ldots i_{n-1})}^{\alpha} F_{i_1 \ldots i_{n-2}} \alpha \beta.$$ 

Recall that we identified the $\mu A^\ast (1,0)$ as the hypermultiplet. Let us denote the linear observables in physical fields of the hypermultiplet by $\phi_i$ and $\psi_a$. This suggests to identify the linear observables of the $\mathcal{O}(n,0)$-multiplet as polynomials of degree $n$ in the linear observables of the hypermultiplet, as follows:

$$F_{i_1 \ldots i_n} = \phi_{i_1} \cdots \phi_{i_n}$$

$$F_{i_1 \ldots i_{n-1} a} = \phi_{i_1} \cdots \phi_{i_{n-1}} \psi_a$$ 

$$(3.28)$$

$$
\vdots
$$

$$F_{i_1 \ldots i_{n-4} \alpha \beta \gamma \delta} = \phi_{i_1} \cdots \phi_{i_{n-4}} \psi_a \psi_\beta \psi_\gamma \psi_\delta.
$$

Further, recall that for the hypermultiplet the module structure of the supersymmetry algebra contains terms of the form

$$\delta \phi_i = \epsilon_{i}^{\alpha} \psi_a.$$ 

By the Leibniz rule, this precisely induces the supersymmetry transformations of the $\mathcal{O}(n,0)$-multiplet we recorded above. Thus, we can view, for $n \geq 3$, $\mu A^\ast (n,0)$ as consisting of polynomials of degree $n$ in the linear observables of $\mu A^\ast (1,0)$. Intuitively, this can be viewed
as a remnant of the statement $O(n,0) = O(1,0)^{\otimes n}$ after applying the pure spinor superfield formalism. We remark that a special case of this is already visible in the action for supersymmetric Yang–Mills theory coupled to hypermultiplets studied in [Ced18]. There, an action is written that reproduces the minimal coupling of the gauge sector to matter; the relevant term is cubic, containing two hypermultiplets and one gauge field. From our perspective, this makes use of the identification of the $O(2,0)$ multiplet both as the dual to the vector multiplet and as governing quadratic functionals on the hypermultiplet.

It is straightforward to compute the holomorphic twist of these multiplets uniformly for $n \geq 3$, and we sketch this briefly here. Following Remark 2.2, we expect to find that the twist is of rank one over Dolbeault forms on $\mathbb{C}^3$. Choosing a holomorphic supercharge fixes a complex structure on $\mathbb{R}^6$ and a polarization of the $R$-symmetry space (or equivalently a choice of Cartan subalgebra of $\mathfrak{s}l_2$.) We decompose

$$S_+ \cong \mathbb{C} \oplus V, \quad S_- \cong V^\vee \oplus \mathbb{C}$$

(3.30)

as $\mathfrak{s}l_3$-representations. Here $V = V_3$ is the three-dimensional fundamental representation of $\mathfrak{s}l_3$. Using the non-derivative supersymmetry transformations indicated above, we see that the complex of fields takes the following form:

$$\begin{array}{cccccccccccc}
\mathbb{C} & \mathbb{C} & V & V & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & \mathbb{C} & \mathbb{C} \\
\mathbb{C} & \mathbb{C} & V & V & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & \mathbb{C} & \mathbb{C} \\
\mathbb{C} & \mathbb{C} & V & V & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & \mathbb{C} & \mathbb{C} \\
\mathbb{C} & \mathbb{C} & V & V & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & V^\vee & \mathbb{C} & \mathbb{C} \\
\end{array}$$

Here, the vertical axis represents the $\mathfrak{s}l_2$-weight with respect to the fixed Cartan; we have drawn the diagram for $n = 4$, but the pattern is clear. For each $n$, the surviving fields are precisely isomorphic to Dolbeault forms on $\mathbb{C}^3$, and the remaining (derivative-dependent) components of the holomorphic supercharge generate the $\bar{\mathcal{D}}$ operator. Due to the twisting homomorphism, the twist naturally resolves holomorphic sections of $K^{n/2}$:

$$O(n,0)^Q \cong \Omega^0(\mathbb{C}^3) \otimes K^{n/2}.$$  

(3.32)

(This clearly generalizes the results for $n = 0, 1, \text{and } 2$, which are well-known.)
3.3. The bundles $\mathcal{O}(0,m)$ for $m \geq 0$.

3.3.1. Computation of the Betti numbers. The Hilbert series specializes to

\[
\text{Hilb}(0,m) = \sum_{d=0}^{\infty} (d+1) \frac{(m+d+3)(m+d+2)(m+d+1)}{6} t^d,
\]

which can be rewritten as

\[
\text{Hilb}(0,m) = \frac{1}{6 \partial t^1 \partial t^3} t^{m+3} t^1 - m \frac{1}{6 \partial t^3} t^m + 3 t - t.
\]

Again, we can bring the Hilbert series into a form such that we can read off the Betti numbers of the associated multiplet.

\[
\frac{1}{(1-t)^8} \left[ (1 + \frac{11}{6} m + m^2 + \frac{n^3}{6}) - (m^3 + 5m^2 + 6m)t - (\frac{10}{3} m^3 + 10m^2 - 4m - 3)t^3 + \left(\frac{2}{3} m^3 + 5m^2 - \frac{9}{2} m - 3\right)t^4 - (m^3 + m^2 - 2m)t^5 + \left(\frac{m^3}{6} - \frac{m}{6}\right)t^6 \right]
\]

It is immediate to see that for $m = 0$ we recover the result from above. Let us in addition give the Betti tables for some small values of $m$. For $m = 1$, we find

\[
\text{grdim} \mu A^*(0,1) = \begin{bmatrix} 4 & 12 & 12 & 4 \end{bmatrix}.
\]

For $n = 2$ one obtains

\[
\text{grdim} \mu A^*(0,2) = \begin{bmatrix} 10 & 40 & 65 & 56 & 28 & 8 & 1 \end{bmatrix}.
\]

3.3.2. Equivariant decomposition. Solving the equations (3.7) one finds the following representations appearing in $\mu A^*(0,m)$.

\[
\begin{align*}
W_0 &= [0|m,0,0] \\
W_1 &= [-1|m-1,1,0] \\
W_2 &= [0|m-2,2,0] + [2|m-1,0,1] \\
W_3 &= [-1|m-2,1,1] - [3|m-1,0,0] \\
W_4 &= [0|m-2,0,2] + [2|m-2,1,0] \\
W_5 &= [-1|m-2,0,1] \\
W_6 &= [0|m-2,0,0]
\end{align*}
\]

3.3.3. Presentation and equivariant resolution. We can describe the module $\Gamma_*(\mathcal{O}(0,1))$ explicitly as the cokernel of a map of free $R$-modules

\[
\psi_1 : (\wedge^2 S_+ \otimes \mathbb{C}^2) \otimes R \longrightarrow S_+ \otimes R.
\]
For degree reasons, the map should be linear in $\lambda$. It is easy to check that there is, up to non-zero constant prefactors, a unique such map explicitly given by

$$G \mapsto \lambda_i G^i_{[a \beta]} s^\beta.$$  

Here $s^\beta$ denotes a basis of $S_\ast$. The modules $\Gamma_\ast(O(0,m))$ are obtained by taking symmetric products. It can be checked explicitly (for example using a computer program such as Macaulay2 [GS]) that the minimal free resolutions of these modules reproduce the multiplets described above.

3.4. A classification result. The results above describe all multiplets for six-dimensional $\mathcal{N} = (1,0)$ supersymmetry which can be obtained from line bundles on the nilpotence variety $\mathbb{P}^1 \times \mathbb{P}^3$. Based on the equivalence of categories between multiplets and $\mathcal{C}^\ast(t)$-modules provided by [EHS22] this can be viewed as a classification result as follows. Given an $R/I$-module $\Gamma$, the derived $t$-invariants of the associated multiplet are concentrated in degree zero and we have

$$H^\ast(t, \mu A^\ast(\Gamma)) = \Gamma.$$  

Conversely, given a multiplet $(E,D,\rho)$ such that its derived $t$-invariants are concentrated in degree zero one can identify

$$\mu A^\ast(C^\ast(t, E)) \simeq (E,D,\rho).$$  

Therefore we obtain the following theorem.

**Theorem 3.1.** The above multiplets classify, up to quasi-isomorphism, all multiplets for six-dimensional $\mathcal{N} = (1,0)$ supersymmetry such that $H^\ast(t, E)$ is the graded global section module of a single line bundle on the projective nilpotence variety.

As remarked above (Remark 2.2), the interpretation of the input module as the Chevalley–Eilenberg cohomology with coefficients in the multiplet provides an interesting conceptual link to the twists of the multiplet involved. Twisting by a supercharge $Q$ takes invariants of the multiplet with respect to the abelian subalgebra spanned by that supercharge. The cohomology groups $H^\ast(t, E)$ define a sheaf on the nilpotence variety which, by the result of [SW21], encodes all the information on the twists of the original multiplet. In fact, one expects that the twist by a square-zero supercharge $Q \in Y$ is determined by the stalk of that sheaf at $Q$.

In our example, we see that—as the derived invariants of all the multiplets above are line bundles—the stalk at any point is isomorphic to $O_{Y,x}$. Our nilpotence variety $Y = \mathbb{P}^1 \times \mathbb{P}^3$ only has one stratum corresponding to the holomorphic twist. Correspondingly, as
we have seen above, the holomorphic twists of the above multiplets always have rank one over Dolbeault forms on $\mathbb{C}^3$.

This intuition makes many aspects of the physical behavior of the multiplets and their twists manifest. For example, we can take any of the above multiplets and dimensionally reduce to a four-dimensional $\mathcal{N} = 2$ multiplet. In this case, the nilpotence variety is reducible and has three different components, one of which is the image of the six-dimensional $\mathcal{N} = (1,0)$ nilpotence variety under the dimensional reduction map [ESW21]. The other two components correspond to the Donaldson–Witten twist, which does not descend from a square-zero supercharge in six dimensions. The Chevalley–Eilenberg cohomology with coefficients in the dimensionally reduced multiplets is obtained by pushing forward along the inclusion $Y_{6D} \hookrightarrow Y_{4D}$. Clearly, any supercharge corresponding to a Donaldson–Witten twist is outside of the support of the resulting sheaf, so that the respective stalks are trivial. Following Remark 2.2, one thus expects that the Donaldson–Witten twists of all multiplets arising by dimensional reduction are perturbatively trivial. We hope to give a more complete account of extensions of the methods developed in [SW21] to general multiplets in future work.

4. Antifield Multiplets and Duality

4.1. General observations. Given any multiplet $\mu A^*(\Gamma)$, one may form the dual (or antifield) multiplet $\mu A^*(\Gamma) \vee$ by dualizing the underlying vector bundle, the differential and the supersymmetry module structure (see [EHS22] for more information). Via the pure spinor superfield formalism, the operation of taking the antifield multiplet corresponds, in good cases, to taking the dualizing module of the input module $\Gamma$. This was already recognized in [Eag+21]; here we explore this direction further and link it to statements in terms of sheaves on the nilpotence variety.

4.1.1. Duality theory for modules and multiplets. Let us start by reviewing some facts from commutative algebra (see for example [Eis95]) and relate these to the pure spinor superfield formalism. Here the central object is the Cohen–Macaulay property for modules over a commutative ring.

**Definition 4.1.** Let $S$ be a commutative ring with unity and let $M$ be an $S$-module. We say that $M$ is Cohen–Macaulay if $\text{depth}_S(M) = \dim_S(M)$.

This definition implicitly refers to a choice of ideal in $S$ in the definition of depth. In our case, we are interested in modules over the polynomial ring $R$. Maximal ideals then
correspond to points $x \in \text{Spec} R$; there exists a unique maximal equivariant ideal in $R$, corresponding to the skyscraper sheaf at the origin, and consisting of all polynomials with zero constant term.

In this context, two equivalent characterizations of the Cohen–Macaulay property will be useful. The first one is in terms of the length of a minimal free resolution.\footnote{The length of a free resolution $L^* = (L^0 \leftarrow L^{-1} \leftarrow \cdots \leftarrow L^{-k} \leftarrow 0)$ is $k$.} The Auslander–Buchsbaum formula implies the following.

**Proposition 4.2.** An $R$-module $\Gamma$ is Cohen–Macaulay if the length of its minimal free resolution equals its codimension, i.e.

$$l_R(\Gamma) = n - \dim_R(\Gamma) = \text{codim}_R(\Gamma).$$

Equivalently we can characterize Cohen–Macaulay modules via their Ext-groups.

**Proposition 4.3.** An $R$-module $\Gamma$ of dimension $q$ is Cohen–Macaulay if and only if $\text{Ext}^k_R(\Gamma, R) = 0$ for all $k \neq \text{codim}_R(\Gamma) = n - q = r$.

In the context of the pure spinor superfield formalism we consider $R/I$-modules $\Gamma$ as input data, which can be canonically looked at as $R$-modules quotient map $R \to R/I$. Given a Cohen–Macaulay module $\Gamma$, the multiplet $\mu A^*(\Gamma)$ is described by the minimal free resolution of $\Gamma$ in $R$-modules. The antifield multiplet $\mu A^*(\Gamma)$ is described by the dual of that minimal free resolution, which is, by definition, a minimal free resolution of the dualizing module $\text{Ext}^r_R(\Gamma, R)$. Therefore we can identify for Cohen–Macaulay modules $\Gamma$,

$$\mu A^*(\Gamma) = \mu A^*(\text{Ext}^r_R(\Gamma, R)).$$

If $\Gamma$ is not Cohen–Macaulay, this is no longer true. Then the dual of the minimal free resolution of $\Gamma$ is no longer a resolution of a single module, but in fact a model for the dualizing complex of $\Gamma$. Its cohomology is the Ext-algebra $\text{Ext}^*_R(\Gamma, R)$.

**4.1.2. The relation to sheaves.** In this work we focus on modules which arise from sheaves on the nilpotence variety $Y$ via $\Gamma_*$. In this setting, we can link the above statements on duality to more geometric notions for sheaves on projective schemes.

Therefore, let us consider a Cohen–Macaulay projective scheme $\iota : X \hookrightarrow \mathbb{P}^n$ of codimension $r$. In this setting the dualizing sheaf of $X$ is a vector bundle, denoted by $\omega_X^o$. Explicitly, it can be defined in terms of the ambient projective space as

$$\omega_X^o = \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n}).$$

Let us further assume that $\Gamma_*(\mathcal{O}_X)$ is Cohen–Macaulay as an $R = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$-module. Then the following holds.
**Proposition 4.4.** Let $\mathcal{F} \in \text{Coh}(X)$ be a coherent sheaf on $X$ and $\Gamma_*(\mathcal{F})$ is its associated $R = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$-module. Then there exists a natural isomorphism

\begin{equation}
\text{Ext}_R^r(\Gamma_*(\mathcal{F}), R) \cong \Gamma_* \text{Hom}_{\mathbb{P}^n}(\mathcal{F}, \omega_X^\circ)(n+1),
\end{equation}

so that the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\Gamma_*} & \Gamma_*(\mathcal{F}) \\
\downarrow \text{Hom}_{\mathbb{P}^n}(\gamma^*, \omega_X^\circ) & & \downarrow \text{Ext}_R^r(-, R) \\
\text{Hom}_{\mathbb{P}^n}(\gamma^*, \omega_X^\circ) & \xrightarrow{\Gamma_*} & \text{Ext}_R^r(\Gamma_*(\mathcal{F}), R).
\end{array}
\]

**Proof.** Since shifting does not change the cohomology class, one has

\begin{equation}
\text{Ext}_R^r(\Gamma_*(\mathcal{F}), R) \cong \text{Ext}_R^r(\Gamma_*(\mathcal{F})(-n-1), R(-n-1))
\end{equation}

\begin{equation}
\cong \text{Ext}_R^r(\Gamma_*(\mathcal{F})(-n-1) \otimes_R \Gamma_*(\mathcal{O}_X), R(-n-1)),
\end{equation}

where the second isomorphism follows from the fact that the sheaf $\mathcal{F}$ is supported on $X$ and $\Gamma_*(\mathcal{O}_X) \cong \mathcal{O}_X$. By derived hom-tensor adjunction [Huy06] one has

\begin{equation}
\text{Ext}_R^r(\Gamma_*(\mathcal{F}), R) \cong \text{Hom}_R(\Gamma_*(\mathcal{F})(-n-1), \text{Ext}_R^r(\Gamma_*(\mathcal{O}_X), R(-n-1)))
\end{equation}

\begin{equation}
= \text{Hom}_R(\Gamma_*(\mathcal{F})(-n-1), \text{Ext}_R^r(\Gamma_*(\mathcal{O}_X), \Gamma_*(\omega_{\mathbb{P}^n}))),
\end{equation}

where we used that $\Gamma_*(\omega_{\mathbb{P}^n}) = R(-n-1)$ in the second step. Notice that, by assumption $\Gamma_*(\mathcal{O}_X)$ is Cohen–Macaulay as an $R$-module, hence the only non-zero Ext-module in the derived hom-tensor adjunction is indeed the dualizing module $\text{Ext}_R^r(\Gamma_*(\mathcal{O}_X), R(-n-1))$. Further, note that $\text{Hom}_R$ in (4.7) denotes graded morphisms of all degrees. For morphisms of degree zero, we have the adjunction [Vak; Har77]

\begin{equation}
\text{Hom}^\text{deg=0}_R(M, \Gamma_*(\mathcal{H})) = \text{Hom}_{\mathbb{P}^n}(\tilde{M}, \mathcal{H})
\end{equation}

between the functors $\Gamma_* : \text{QCoh}_{\mathcal{O}_X} \to \text{Mod}_R$ and $(\tilde{\cdot}) : \text{Mod}_R \to \text{QCoh}_{\mathcal{O}_X}$. Upon using $\Gamma_*(\mathcal{G}) = \mathcal{G}$, this implies that

\begin{equation}
\text{Hom}^\text{deg=0}_R(\Gamma_*(\mathcal{G}), \Gamma_*(\mathcal{H})) = \text{Hom}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H}).
\end{equation}

Shifting and summing on both sides, one reconstructs the graded morphisms:

\begin{equation}
\text{Hom}_R(\Gamma_*(\mathcal{G}), \Gamma_*(\mathcal{H})) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H}(k)) = \bigoplus_{k \in \mathbb{Z}} \Gamma \circ (\text{Hom}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H}(k))
\end{equation}

\begin{equation}
= \bigoplus_{k \in \mathbb{Z}} \Gamma \circ (\text{Hom}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H}) \otimes \mathcal{O}_{\mathbb{P}^n}(k))
\end{equation}

\begin{equation}
= \Gamma_*(\mathcal{H} \text{Hom}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H})).
\end{equation}
where we have used that \( \Gamma \circ \mathcal{H}om_{\mathcal{O}} = \text{Hom}_{\mathcal{P}} \), where \( \Gamma \) is the global section functor. Deriving the above functors, one gets a local-to-global spectral sequence, which in the case we are interested yields the isomorphism

\[
\text{Ext}_R^r(\Gamma_*(\mathcal{O}_X), \Gamma_*(\omega_{\mathcal{P}})) \cong \Gamma_*(\mathcal{E}xt_{\mathcal{P}}^r(i_* \mathcal{O}_X, \omega_{\mathcal{P}})).
\]

Plugging this into (4.7), we finally obtain

\[
\text{Ext}_R^r(\Gamma_*(\mathcal{F}), R) \cong \Gamma_*(\mathcal{H}om_{\mathcal{O}_{\mathcal{P}}}(i_* \mathcal{F}, \omega_X^c)(n + 1))
\]

which concludes the proof.

This result establishes that the dualizing module of \( \Gamma_*(\mathcal{F}) \) arises geometrically from the sheaf \( \mathcal{H}om_{\mathcal{O}_{\mathcal{P}}}(i_* \mathcal{F}, \omega_X^c)(n + 1) \). The above theorem, together with (4.2), implies the following corollary.

**Corollary 4.5.** If \( \Gamma_*(\mathcal{F}) \) is a Cohen–Macaulay \( R \)-module, then we have

\[
\mu A^*(\Gamma_*(\mathcal{F}))^\vee \cong \mu A^*(\Gamma_*(\mathcal{H}om_{\mathcal{O}_{\mathcal{P}}}(i_* \mathcal{F}, \omega_X^c))).
\]

It is possible to prove whether or not a sheaf \( \mathcal{F} \) gives rise to a Cohen–Macaulay module via \( \Gamma_* \) by studying its sheaf cohomology. In particular the following result holds [Kol13].

**Proposition 4.6.** Let \( X \) be a Cohen-Macaulay projective scheme and let \( \mathcal{L} \) be an ample line bundle on it. Given a coherent sheaf \( \mathcal{F} \) on \( X \), then \( \Gamma_*(\mathcal{F}) \) is a Cohen-Macaulay \( R \)-module if and only if \( H^i(X, \mathcal{F} \otimes \mathcal{L}^\otimes k) = 0 \) for any \( 0 < i < \text{dim}(X) \) for any \( k \in \mathbb{Z} \).

**4.2. Duality and line bundles.** As a case study for the general results above, let us consider the multiplets \( \mu A^*(n, 0) \) for \( n \in \mathbb{Z} \). For a start, recall that the field content of the multiplet \( \mu A^*(n, 0) \) takes values in the minimal free resolution of the \( R \)-module \( \Gamma_*(\mathcal{O}_Y(n, 0)) \). Therefore, given the results in the previous section, we can easily read off \( l_R(\Gamma_*(\mathcal{O}_Y(n, 0))) \):

\[
l_R(\Gamma_*(\mathcal{O}(n, 0))) = \begin{cases} 
3 & \text{for } n \in \{-1, 0, 1, 2, 3\} \\
4 & \text{for } n > 3 \\
6 & \text{for } n < -2.
\end{cases}
\]

Notice that all the modules \( \Gamma_*(\mathcal{O}_Y(n, 0)) \) come from line bundles supported on the nilpotence variety \( Y \cong \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7 \), which is of codimension 3 in \( \mathbb{P}^7 \). From this, we can infer the following lemma.

**Lemma 4.7.** The \( R \)-module \( \Gamma_*(\mathcal{O}(n, 0)) \) is Cohen–Macaulay if and only if \( n \in \{-1, 0, 1, 2, 3\} \).
Therefore, for \( n \) in this range, we have

\[
\mu \Lambda^*(n,0)^\vee \cong \mu \Lambda^*(\text{Ext}^3_R(\Gamma_*(\mathcal{O}(n,0)),R)).
\]

Lemma 4.7 can also be proved directly by studying the sheaf cohomology of the line bundles \( \mathcal{O}(n,m) \) and using Proposition 4.6. Indeed, we can choose \( \mathcal{L} = \mathcal{O}(1,1) \) as an ample line bundle and use the K"unneth theorem to verify that the middle cohomologies \( H^i(Y, \mathcal{O}(n+k,k)) \) vanish for \( i = 1, 2, 3 \) and for all \( k \) precisely when \( n \in \{-1, 0, 1, 2, 3\} \).

Furthermore, since \( Y = \mathbb{P}^1 \times \mathbb{P}^3 \), the dualizing sheaf can be described explicitly as the exterior tensor product of the respective dualizing sheaves on the factors. Explicitly,

\[
\omega_Y = \pi_1^* \omega_{\mathbb{P}^1} \otimes \pi_3^* \omega_{\mathbb{P}^3} = \mathcal{O}(-2,-4).
\]

Using this together with Theorem 4.4 gives

\[
\text{Ext}^3_R(\Gamma_*(\mathcal{O}(n,0)),R) = \Gamma_*(\mathcal{O}(n,0)^\vee \otimes \mathcal{O}(-2,-4))
\]

\[
= \Gamma_*(\mathcal{O}(-n-2,-4))
\]

\[
= \Gamma_*(\mathcal{O}(2-n,0))(4).
\]

Thus we obtain

\[
\mu \Lambda^*(2-n,0)[4] = \text{Mult} \left( \text{Ext}^3_R(\Gamma_*(\mathcal{O}(n,0)),R) \right).
\]

In the range where \( \Gamma_*(\mathcal{O}(n,0)) \) is Cohen–Macaulay, this implies

\[
\mu \Lambda^*(n,0)^\vee \cong \mu \Lambda^*(2-n,0)[4].
\]

This can be viewed as a remnant of Serre duality for line bundles on the multiplet side.

5. SHORT EXACT SEQUENCES

In this section, we discuss some general conclusions that can be drawn about short exact sequences of vector bundles in the context of the pure spinor superfield formalism, and then move on to study some concrete examples in our six-dimensional setting. The sections that follow will study the multiplets associated to the tangent and normal bundles and their duals, and apply these results in the context of natural short exact sequences in which those vector bundles appear. As such, we are motivated both by abstract considerations—having understood that the pure spinor construction is a functor, it is natural to ask about it not just on single objects, but on diagrams of objects—and, as throughout this paper, by concrete computational examples.

5.1. General observations. Let

\[
0 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma'' \longrightarrow 0
\]
be a short exact sequence of graded equivariant $R/I$-modules. Applying $A^\bullet$ is an exact functor; we thus obtain a short exact sequence of strict multiplets

$$0 \rightarrow A^\bullet(\Gamma') \rightarrow A^\bullet(\Gamma) \rightarrow A^\bullet(\Gamma'') \rightarrow 0.$$  

Up to perturbative equivalence, this is the end of the story. However, we are often interested in component-field descriptions, and therefore specifically in the minimal multiplets $\mu A^\bullet(\Gamma), \mu A^\bullet(\Gamma'),$ and $\mu A^\bullet(\Gamma'')$. To investigate the relationship at this level, we note the following: Each homogeneous degree $\Gamma_k$ is a finite-dimensional representation of $\mathfrak{sl}_2 \times \mathfrak{sl}_4$; restricting the above short exact sequence to degree $k$ gives a short exact sequence of $\mathfrak{sl}_2 \times \mathfrak{sl}_4$-representations.

$$0 \rightarrow \Gamma'_k \rightarrow \Gamma_k \rightarrow \Gamma''_k \rightarrow 0$$

Since $\mathfrak{sl}_2 \times \mathfrak{sl}_4$ is semisimple, all finite-dimensional representations are completely decomposable, and the sequence splits for all $k \in \mathbb{Z}$. We thus have

$$\Gamma_k \cong \Gamma'_k \oplus \Gamma''_k$$

as $\mathfrak{sl}_2 \times \mathfrak{sl}_4$-representations. This implies for the equivariant Hilbert series,

$$\text{Hilb}(\Gamma) = \text{Hilb}(\Gamma') + \text{Hilb}(\Gamma'')$$

and thus for the field content of the respective multiplets

$$\chi(W^\bullet_\Gamma) = \chi(W^\bullet_{\Gamma'}) + \chi(W^\bullet_{\Gamma''}).$$

In practical terms this means that the direct sum of $\mu A^\bullet(\Gamma')$ and $\mu A^\bullet(\Gamma'')$ admits a deformation to $\mu A^\bullet(\Gamma)$,

$$\mu A^\bullet(\Gamma) = [\mu A^\bullet(\Gamma') \oplus \mu A^\bullet(\Gamma'')]^{\text{Deform}} = [\mu A^\bullet(\Gamma' \oplus \Gamma'')]^{\text{Deform}}.$$  

Recall that the differential is given by the right action together with the module structure on $\Gamma$,

$$D = \lambda^a R(Q_a).$$

Thus, the deformation on the direct sum of the multiplets precisely corresponds to a deformation of the module structure on $\Gamma'$ such that

$$[\Gamma' \oplus \Gamma'']^{\text{Deform}} \cong \Gamma.$$  

This deformation of the module structure arises from the class of $\Gamma$ inside $\text{Ext}^1(\Gamma', \Gamma'')$. Even more explicitly, we notice that $\Gamma'$ sits inside $\Gamma$ as a submodule; therefore the deformation of
the module structure is characterized by a map

\[ R/I \times \Gamma'' \to \Gamma' \]

We can summarize these findings by the following lemma.

**Lemma 5.1.** Let \( 0 \to \Gamma' \to \Gamma \to \Gamma'' \to 0 \) be a short exact sequence of graded equivariant \( R/I \)-modules. Then the deformation of the module structure on \( \Gamma' \oplus \Gamma'' \) determined by this sequence induces a deformation on the respective multiplets

\[ \mu A^*(\Gamma) \cong [\mu A^*(\Gamma') \oplus \mu A^*(\Gamma'')] \text{Deform}. \]

In this work we often deal with short exact sequences of equivariant sheaves on \( Y \). Let

\[ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \]

be such a sequence. First, we observe that taking the tensor product with a line bundle keeps the sequence exact, thus we obtain short exact sequences

\[ 0 \to \mathcal{F}'(k) \to \mathcal{F}(k) \to \mathcal{F}''(k) \to 0 \]

for all \( k \in \mathbb{Z} \). Second a short exact sequence induces a long exact sequence in cohomology

\[ 0 \to H^0(\mathcal{F}'(k)) \to H^0(\mathcal{F}(k)) \to H^0(\mathcal{F}''(k)) \to H^1(\mathcal{F}'(k)) \to \ldots \]

Thus, if the map \( \delta \) vanishes (for example due to \( H^1(\mathcal{F}'(k)) \) being zero) for all \( k \), we obtain a short exact sequence on the global sections

\[ 0 \to H^0(\mathcal{F}'(k)) \to H^0(\mathcal{F}(k)) \to H^0(\mathcal{F}''(k)) \to 0, \]

and therefore a short exact sequence of graded equivariant \( R/I \)-modules

\[ 0 \to \Gamma_*(\mathcal{F}') \to \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{F}'') \to 0. \]

and we find ourselves in the situation described above.

**Remark 5.2.** It is worth recalling that extensions of sheaves are often interpreted as related to interactions or bound states in mathematical physics. Just for example, in topological string theory, \( B \)-branes are identified with coherent sheaves on the target space, which is typically a Calabi–Yau threefold. As emphasized in early work on the subject [Sha99; Dou01; Asp04, for example], a nontrivial extension sequence of the form

\[ 0 \to A \to B \to C \to 0 \]

indicates that \( B \) should be thought of as a bound state of the branes \( A \) and \( C \). (Making this interpretation precise led to the identification of the category of \( B \)-branes with the derived category of coherent sheaves.)
In our setting, as explained, the extension defines a deformation of the module structure, which in turn deforms the differential on the multiplet. Thinking in the context of the Batalin–Vilkovisky formalism, a deformation of the differential can in turn be thought of as a deformation of the quadratic part of the BV action. As such, the new differentials we consider on component fields can be interpreted, at least schematically, as (quadratic) supersymmetric interactions between the multiplets $\mu A^*(\Gamma')$ and $\mu A^*(\Gamma'')$, such that the deformed multiplet has derived supertranslation invariants $\Gamma$.

5.2. **Excursion: three-dimensional $N = 1$.** Let us illustrate the findings from above in the case of three-dimensional $N = 1$ supersymmetry. In three dimensions we have $\mathfrak{so}(3) \cong \mathfrak{sl}_2$. We denote the two-dimensional spinor representation by $S$ and the three-dimensional vector representation by $V$. The super Poincaré algebra takes the form

\begin{equation}
\mathfrak{p} = \mathfrak{sl}_2 \oplus S(-1) \oplus V(-2)
\end{equation}

with the bracket given on the odd elements by the isomorphism $\Gamma : \text{Sym}^2(S) \cong V$. Writing $R = \mathbb{C}[\lambda^a]$ for $a = 1, 2$, we can identify the defining ideal of the nilpotence variety by

\begin{equation}
I = ((\lambda_1^1)^2, \lambda_1^1\lambda_2^1, (\lambda_2^2)^2) = R_{\geq 2}.
\end{equation}

We are interested in the following short exact sequence

\begin{equation}
0 \longrightarrow S(-1) \longrightarrow R/R_{\geq 2} \longrightarrow R/R_{\geq 1} \longrightarrow 0,
\end{equation}

where the first map is given by sending a basis $s^a$ to the generators $\lambda^a$ and the second map is the obvious projection. Note that the module structure on the direct sum $R/R_{\geq 1} \oplus S(-1)$ is trivial; the deformation which makes it isomorphic to $R/I$ is simply given by

\begin{equation}
R/I \times R/R_{\geq 1} \longrightarrow S(-1) \quad (\lambda^a, 1) \mapsto s^a.
\end{equation}

Let us now study the associated multiplets. $\mu A^*(R/R_{\geq 2})$ is a free superfield and $\mu A^*(S(-1))$ is a free superfield with values in the spinor representation shifted to cohomological degree 1. Their direct sum is described as follows.

\begin{equation}
\mu A^*(S(-1))^\# \oplus \mu A^*(R/R_{\geq 1})^\# = 
\begin{bmatrix}
\mathbb{C} & S & \mathbb{C} \\
S & \Omega^1 \oplus \mathbb{C} & S
\end{bmatrix}
\end{equation}

Using the procedure described in [Eag+21], we find the following representatives for the component fields

\begin{equation}
\begin{bmatrix}
1 & \theta^a & \theta^1 \theta^2 \\
s^a & \theta^a s^\beta & \theta^1 \theta^2 s^a
\end{bmatrix}.
\end{equation}
Deforming the module structure by (5.20), induces a non-trivial differential. From
\begin{equation}
\mathcal{D} = \lambda^a \frac{\partial}{\partial \theta^a} + \lambda^a \theta^\beta \Gamma^\mu_{\alpha \beta} \frac{\partial}{\partial \chi^\mu},
\end{equation}
and using the representatives, it is easy to see that the direct sum of multiplets is deformed to
\begin{equation}
\begin{bmatrix}
\mathbb{C} & d & \mathbb{S} & \mathbb{C} \\
\mathbb{S} & \Omega^1 \oplus \mathbb{C} & \mathbb{S}
\end{bmatrix},
\end{equation}
where every arrow directed down and left is an identity morphism. On the other hand, the multiplet associated to \(R/R_{\geq 2}\) is the gauge multiplet.

\begin{equation}
\mu A^\star(\mathbb{R}/\mathbb{R}_{\geq 2}) = \begin{bmatrix}
\Omega^0 & d \\
\Omega^1 & \mathbb{S}
\end{bmatrix}
\end{equation}

It is immediate to see that the above deformation is quasi-isomorphic to this multiplet.

5.3. **The Euler sequence for \(\mathbb{P}^1\).** Let us now discuss a family of short exact sequences in six dimensions. Identifying \(\mathcal{T}_p \cong \mathcal{O}_{\mathbb{P}^1}(2)\), the Euler exact sequence for \(\mathbb{P}^1\) reads
\begin{equation}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0.
\end{equation}

Note that this is a sequence of equivariant sheaves and that \(\mathbb{C}^2\) carries the fundamental representation of \(\mathfrak{sl}_2\). Twisting by \(\mathcal{O}_{\mathbb{P}^1}(n)\) and pulling back along \(\pi_1\) we obtain a family of short exact sequences of equivariant sheaves on \(Y\).
\begin{equation}
0 \longrightarrow \mathcal{O}(n,0) \longrightarrow \mathcal{O}(n + 1,0) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(n + 2,0) \longrightarrow 0.
\end{equation}
Let us restrict to the case \(n \geq 0\) for the moment. Twisting by \(\mathcal{O}_Y(k) = \mathcal{O}(k,k)\) we obtain the sequences
\begin{equation}
0 \longrightarrow \mathcal{O}(n + k,k) \longrightarrow \mathcal{O}(n + k + 1,k) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(n + k + 2,k) \longrightarrow 0.
\end{equation}
The relevant first cohomology group is \(H^1(\mathcal{O}(n + k,k))\) which is easily seen to vanish for all \(k \in \mathbb{Z}\) by the Künneth theorem. Thus, we obtain for all \(n \geq 0\) a short exact sequence of graded equivariant \(R/I\)-modules,
\begin{equation}
0 \longrightarrow \Gamma_\star(\mathcal{O}(n,0)) \longrightarrow \Gamma_\star(\mathcal{O}(n + 1,0)) \otimes \mathbb{C}^2 \longrightarrow \Gamma_\star(\mathcal{O}(n + 2,0)) \longrightarrow 0.
\end{equation}
Let us study the associated multiplets.

5.3.1. \(n = 0\). Recall that \(\mu A^\star(0,0)\) is the vector multiplet, \(\mu A^\star(1,0)\) the hypermultiplet and \(\mu A^\star(2,0) = \mu A^\star(0,0)^\vee\) the antifield multiplet of the vector. Therefore, \(\mu A^\star(\mathcal{O}(2,0) \otimes \mathbb{C}^2) = \)
\( \mu A^*(O(2,0)) \otimes C^2 \) is a doublet of hypermultiplets with values in the fundamental representation of the \( R \)-symmetry \( sl_2 \). Let us arrange the direct sum as follows.

\[
(5.30) \quad \mu A^*(0,0)^\# \oplus \mu A^*(2,0)^\# = \begin{pmatrix}
C \oplus C^3 & S_+ \otimes C^2 & \Omega^1 \\
\Omega^1 & S_- \otimes C^2 & C \oplus C^3
\end{pmatrix}
\]

We can deform it by adding an acyclic differential relating the two one-forms, the Dirac operator for the fermions, and the Laplacian for the scalar fields.

\[
(5.31) \quad \left[ \mu A^*(0,0) \oplus \mu A^*(2,0) \right]^{\text{Deform}} = \begin{pmatrix}
C \oplus C^3 & S_+ \otimes C^2 & \delta & \Omega^1 \\
\Omega^1 & S_- \otimes C^2 & C \oplus C^3
\end{pmatrix}
\]

Taking cohomology with respect to the acyclic part of the differential (i.e. integrating out the auxiliary field) and recalling that for \( sl_2 \)-representations \( C^2 \otimes C^2 \equiv C \oplus C^3 \), we immediately see that we recover the hypermultiplet with values in \( C^2 \).

Interestingly there is another BV theory which can be formed out of \( \mu A^*(0,0) \) and \( \mu A^*(0,2) \). Adding both multiplets with an appropriate shift and deforming the resulting complex one obtains the BV theory of the vector multiplet. This was discussed in [Eag+21]. Denoting the vector multiplet by \( E \), these findings can be summarized by stating that the cotangent theory \( T^\vee[−1]E \) corresponds to the BV theory describing the vector multiplet, while the construction we presented above corresponds to \( (T^\vee[1]E)[−1] \) which is seen to be equivalent to the hypermultiplet. Let us finally remark that all these considerations are purely perturbative.

5.3.2. \( n = 1 \). Proceeding analogously, we can define a deformation on the direct sum of \( \mu A^*(1,0) \) and \( \mu A^*(3,0) \) that renders it quasi-isomorphic to \( \mu A^*(2,0) \otimes C^2 \):

\[
(5.32) \quad \left[ \mu A^*(1,0) \oplus \mu A^*(3,0) \right]^{\text{Deform}} = \begin{pmatrix}
C^2 \oplus C^4 & S_+ \otimes (C \oplus C^3) & \wedge^2 S_+ \otimes C^2 & S_- \\
S_- \otimes C^2 & \wedge S_+ & C^2
\end{pmatrix}
\]

\[
\simeq \mu A^*(2,0) \otimes C^2.
\]

Here we used the isomorphisms \( V \cong \wedge^2 S_+ \) and \( S_- \cong \wedge^3 S_+ \).
5.3.3. \( n = 2 \). Similarly, there is a deformation of \( \mu A^* (2, 0) \oplus \mu A^* (4, 0) \) giving \( \mu A^* (3, 0) \otimes \mathbb{C}^2 \).

\[
[\mu A^* (2, 0) \oplus \mu A^* (4, 0)]^{\text{Deform}} = \\
\begin{bmatrix}
\mathbb{C}^3 \oplus \mathbb{C}^5 & S_+ \otimes (\mathbb{C}^2 \oplus \mathbb{C}^4) & \wedge^2 S_+ \otimes (\mathbb{C} \oplus \mathbb{C}^3) & \wedge^3 S_+ \otimes \mathbb{C}^2 & \mathbb{C} \\
\end{bmatrix}
\]

\[
\cong \mu A^* (3, 0) \otimes \mathbb{C}^2.
\]

5.3.4. \( n \geq 3 \). For \( n \geq 3 \), we interpreted \( \mu A^* (n, 0) \) as receiving an \( n \)-fold covering map from the hypermultiplet, witnessed by the isomorphism between its observables and the subalgebra of hypermultiplet observables with polynomial degree divisible by \( n \). The short exact sequence gives a relation between \( \mu A^* (n + 1, 0) \otimes \mathbb{C}^2 \) and \( \mu A^* (n, 0) \oplus \mu A^* (n + 2, 0) \):

\[
\mu A^* (n, 0) \oplus \mu A^* (n + 2, 0) = \\
\begin{bmatrix}
\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+3} & (\mathbb{C}^n \oplus \mathbb{C}^{n+2}) \otimes S_+ & (\mathbb{C}^{n-1} \oplus \mathbb{C}^{n+1}) \otimes \wedge^2 S_+ & (\mathbb{C}^{n-2} \oplus \mathbb{C}^n) \otimes \wedge^3 S_+ & \mathbb{C}^{n-3} \oplus \mathbb{C}^{n-1} \\
\end{bmatrix}
\]

\[
\cong \mu A^* (n + 1, 0) \otimes \mathbb{C}^2.
\]

Here, identifying \( \mu A^* (n + 1) \otimes \mathbb{C}^2 \) just amounts to the decomposition rule \( \mathbb{C}^n \otimes \mathbb{C}^2 \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^{n-1} \). In other words, considering pairs of hypermultiplet observables of polynomial degree \( (n + 1) \) and regarding such pairs as transforming in the fundamental representation of the \( R \)-symmetry \( sl_2 \), we can either symmetrize with respect to the \( R \)-symmetry index (yielding \( \mu A^* (n + 2, 0) \)) or antisymmetrize to land in \( \mu A^* (n, 0) \). Note, however, that the polynomial degree of the observables involved does not change; the multiplet \( \mu A^* (n, 0) \), rather than its realization via a map from the hypermultiplet, is the fundamental object.

6. THE NORMAL BUNDLE EXACT SEQUENCE

In the last two sections, we use short exact sequences for geometric bundles on \( \mathbb{P}^1 \times \mathbb{P}^3 \) to extend our survey of multiplets to higher-rank bundles. In this section, we treat the tangent and normal bundles, using the defining short exact sequence that relates them; in the following section, we will consider the dual of this sequence and work out the multiplets involved explicitly.

As recalled above, the tangent bundle of the projectivized nilpotence variety \( \mathcal{T}_Y \), the restriction \( \mathcal{T}|_{\mathbb{P}^7} \) of the tangent bundle of the ambient \( \mathbb{P}^7 \) to \( Y \), and the normal bundle \( \mathcal{N}_Y \) sit in the normal bundle exact sequence

\[
0 \longrightarrow \mathcal{N}_Y \longrightarrow \mathcal{T}|_{\mathbb{P}^7} \longrightarrow \mathcal{T}_Y \longrightarrow 0.
\]
Since $H^1(\mathcal{Y}_k) = 0$ for all $k \in \mathbb{Z}$, this short exact sequence induces a short exact sequence on global sections. Thus, applying $\Gamma_*$, we obtain a short exact sequence of $R/I$-modules.

(6.2) $0 \longrightarrow \Gamma_*(\mathcal{Y}) \longrightarrow \Gamma_*(\mathcal{Y}_{7|Y}) \longrightarrow \Gamma_*(\mathcal{N}_Y) \longrightarrow 0$

We apply this short exact sequence to study the associated multiplets and their relations to one another. Again, we will find that there is a deformation of $\mu A^*(\mathcal{T}_Y) \oplus \mu A^*(\mathcal{N}_Y)$ which is quasi-isomorphic to $\mu A^*(\mathcal{Y}_{7|Y})$.

6.1. **Tangent bundle.**

6.1.1. **Cohomology and Hilbert Series.** Recall that, as seen above, the tangent bundle to the nilpotence variety is given by the exterior sum

(6.3) $\mathcal{T}_Y = \pi_1^*\mathcal{T}_{p1} \oplus \pi_3^*\mathcal{T}_{p3} = \mathcal{T}_{p1} \boxplus \mathcal{T}_{p3},$

where $\pi^*\mathcal{T}_{p1} \equiv \mathcal{O}_Y(2,0)$. Accordingly, the resulting multiplet will be given by a direct sum,

(6.4) $\mu A^*(\mathcal{T}_Y) = \mu A^*(\mathcal{O}_Y(2,0)) \oplus \mu A^*(\pi_3^*\mathcal{T}_{p3}).$

We have already identified $\mu A^*(\mathcal{O}_Y(2,0))$ as the antifield multiplet of the vector multiplet in §3.2, so we are left with studying $\mu A^*(\pi_3^*\mathcal{T}_{p3})$, which amounts to computing the zeroth cohomology of

(6.5) $\pi_3^*\mathcal{T}_{p3}(k) = \pi_1^*\mathcal{O}_{p1}(k) \oplus \pi_3^*\mathcal{T}_{p3}(k) = \mathcal{O}_{p1}(k) \boxplus \mathcal{T}_{p3}(k)$.

By the Künneth theorem, we have

(6.6) $H^0(\pi_3^*\mathcal{T}_{p3}(k)) = H^0(\mathcal{O}_{p1}(k)) \otimes H^0(\mathcal{T}_{p3}(k))$

which in turn reduces the problem to compute $H^0(\mathcal{T}_{p3}(k))$. Twisting the Euler exact sequence (2.42) by $\mathcal{O}_{p3}(k)$, we find

(6.7) $0 \longrightarrow \mathcal{O}_{p3}(k) \longrightarrow \mathcal{O}_{p3}(k+1) \otimes S_- \longrightarrow \mathcal{T}_{p3}(k) \longrightarrow 0,$

where $S_- \equiv \mathbb{C}^4$. The long cohomology sequence, for the relevant cases $k \geq 0$, reduces to the following short exact sequence

(6.8) $0 \longrightarrow H^0(\mathcal{O}_{p3}(k)) \longrightarrow H^0(\mathcal{O}_{p3}(k+1)) \otimes \mathbb{C}^4 \longrightarrow H^0(\mathcal{T}_{p3}(k)) \longrightarrow 0,$

since $H^1(\mathcal{O}_{p3}(k)) = 0$ for any $k \geq 0$. As a consequence, one has

(6.9) $h^0(\mathcal{T}_{p3}(k)) = 4h^0(\mathcal{O}_{p3}(k+1)) - h^0(\mathcal{O}_{p3}(k)) = 4 \left( \binom{k+4}{3} - \binom{k+3}{3} \right) = \frac{1}{2} (k+2)(k+3)(k+5).$
The resulting Hilbert series is easily resummed, giving
\[
\text{Hilb}(\pi^*_3 T^* P^3) = \sum_{k=0}^{\infty} \frac{1}{2}(k+1)(k+2)(k+3)(k+5)t^k
\]
(6.10)
\[
= \frac{15 - 48t + 54t^2 - 24t^3 + 3t^4}{(1-t)^8},
\]
such that the Betti numbers of the associated multiplet are
(6.11) \( \text{grdim} \mu A^*(T^* P^3) = \begin{bmatrix} 15 & 48 & 54 & 24 & 3 \end{bmatrix} \).  

6.1.2. **Equivariant Decomposition.** Since the Euler exact sequence splits, we find in terms of representations of \( \mathfrak{sl}_2 \times \mathfrak{sl}_4 \)
(6.12) \( H^0(T^* P^3(k)) = [0| k+1, 0, 0] \otimes S_- - [0|k, 0, 0] = [0|k+1, 0, 1] \),
and hence by (6.6)
(6.13)
\[
H^0((\pi^*_3 T^* P^3)(k)) = [k| k+1, 0, 1].
\]

Running our machinery, we obtain the representations
\[
W_0 = [0|0, 1, 0] \\
W_1 = -[1|0, 1, 1] - [1|1, 0, 0] \\
W_2 = [0|0, 1, 0] + [2|0, 0, 2] + [2|0, 1, 0] \\
W_3 = -[1|0, 0, 1] - [3|0, 0, 1] \\
W_4 = [2|0, 0, 0].
\]

Let us summarize the field content.
(6.15) \( \mu A^*(\pi^*_3 T^* P^3)^\# = \begin{bmatrix} \Omega^2 & S_- \otimes V \otimes C^2 & V \otimes \Omega_3^- \otimes C^3 \otimes V \otimes C^3 & (C^2 \otimes C^4) \otimes S_- \otimes C^3 \end{bmatrix} \)

6.2. **Restriction of \( T_{P^7}^* \) to the nilpotence variety.**

6.2.1. **Cohomology and Hilbert series.** Since restriction to a smooth subvariety is an exact functor, it is easy to describe the vector bundle \( T_{P^7}|_Y(k) \) as the quotient bundle sitting in the restriction of the ordinary \( k \)-twisted Euler exact sequence of the embedding space \( \mathbb{P}^7 \) of \( Y \), i.e.
(6.16) \[
0 \longrightarrow \mathcal{O}_Y(k, k) \longrightarrow \mathcal{O}_Y(k + 1, k + 1) \otimes [1|0, 0, 1] \longrightarrow T_{P^7}|_Y(k) \longrightarrow 0,
\]
where we have used that \( \mathcal{O}_{P^7}|_Y(k) \cong \mathcal{O}_Y(k, k) \). Observing that \( H^1(\mathcal{O}_Y(k, k)) \) vanishes for any \( k \geq 0 \), we find the short exact sequence in cohomology
(6.17) \[
0 \longrightarrow H^0(\mathcal{O}_Y(k, k)) \longrightarrow H^0(\mathcal{O}_Y(k + 1, k + 1) \otimes [1|0, 0, 1] \longrightarrow H^0(T_{P^7}|_Y(k)) \longrightarrow 0.
\]
This yields the formula
\[
h^0(\mathcal{T}_{p^7}|_Y(k)) = 8(k + 2) \binom{k + 4}{3} - (k + 1) \binom{k + 3}{3}
\]
(6.18)
\[
= \frac{4}{3} (k + 2)(k + 4)(k + 3)(k + 2) - \frac{1}{6} (k + 1)(k + 3)(k + 2)(k + 1)
\]
for the dimensions of the spaces of global sections of \(\mathcal{T}_{p^7}|_Y(k)\). Notice that this also accounts for the special case \(k = -1\), when \(H^0(\mathcal{T}_{p^7}|_Y(-1)) \cong H^0(\mathcal{O}_Y) \otimes [1|0,0,1] \cong [1|0,0,1]\). The Hilbert series of \(\mathcal{T}_{p^7}|_Y\) is found to be
\[
\text{Hilb}(\mathcal{T}_{p^7}|_Y) = \frac{8 - 48t^2 + 70t^3 - 32t^4 + 3t^5}{t(1 - t)^8}.
\]

6.2.2. Equivariant decomposition. Since (6.17) splits, we find on the level of representations
\[
H^0(\mathcal{T}_{p^7}|_Y(k)) = [k + 1|k + 1,1,0,0] \otimes [1|0,0,1] - [k|k,0,0].
\]
The associated representations appearing in \(\mu A^*(\mathcal{T}_{p^7}|_Y)\) are
\[
W_0 = [1|0,0,1], \quad W_1 = [-0|0,0,0], \quad W_2 = -[1|0,1,1] - [1|1,0,0],
\]
(6.21)
\[
W_3 = [0|0,0,2] + [2|0,0,2] + 2[0|0,1,0] + [2|0,1,0],
\]
\[
W_4 = -2[1|0,0,1] - [3|0,0,1], \quad W_5 = [2|0,0,0].
\]
Explicitly, the field content of \(\mu A^*(\mathcal{T}_{p^7}|_Y)\) is summarized in the array
\[
\mu A^*(\mathcal{T}_{p^7}|_Y)^\# = \begin{bmatrix}
S_+ \otimes \mathbb{C}^2 & \Omega^0 \\
S_+ \otimes \mathbb{C}^2 & \mathbb{C}^3 & \Omega^1 \otimes (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^3) & \Omega^2 \otimes (\mathbb{C} \oplus \mathbb{C}^3) & S_+ \otimes (\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^4) & \Omega^3 \otimes (\mathbb{C} \oplus \mathbb{C}^3) & \mathbb{C}^3
\end{bmatrix}
\]
This multiplet looks like a gravitino multiplet, containing a spin-3/2 (Rarita–Schwinger) field, but no metric or other degree of freedom corresponding to a particle of spin two.

6.3. Normal bundle.

6.3.1. Cohomology and Hilbert series. Using our results on the cohomology of the bundles \(\mathcal{T}_{p^7}|_Y(k)\) and \(\mathcal{T}_Y(k)\), it is easy to compute the cohomology of \(N_Y(k)\) by means of the twisted normal bundle exact sequence
\[
0 \longrightarrow \mathcal{T}_Y(k) \longrightarrow \mathcal{T}_{p^7}|_Y(k) \longrightarrow N_Y(k) \longrightarrow 0.
\]
Since \(H^1(\mathcal{T}_Y(k)) = 0\) for any \(k\), as can be seen upon using and Künneth theorem in combination with the twisted Euler exact sequence to evaluate the first cohomology group of \(\mathcal{T}_{p^3}(k)\),
then one finds a short exact sequence in cohomology

\[(6.24) \quad 0 \longrightarrow H^0(T_Y(k)) \longrightarrow H^0(T_{\mathbb{P}^7|Y}(k)) \longrightarrow H^0(N_Y(k)) \longrightarrow 0.\]

This implies that for \( k \geq -1 \) one finds the Betti numbers

\[(6.25) \quad h^0(N_Y(k)) = h^0(T_{\mathbb{P}^7|Y}(k)) - h^0(T_Y(k)).\]

Using our previous results for \( h^0(T_{\mathbb{P}^7|Y}(k)) \) and \( h^0(T_Y(k)) \) we can deduce

\[(6.26) \quad h^0(N_Y(k)) = \frac{1}{2}(k + 3)^2(k + 2)(k + 5).\]

Finally, resumming the Hilbert series yields

\[(6.27) \quad \text{Hilb}(N_Y) = \frac{8 - 19t + 8t^2 + 10t^3 - 8t^4 + t^5}{t(1-t)^8}.\]

6.3.2. Equivariant decomposition. For the equivariant decomposition, we identify

\[(6.28) \quad h^0(N_Y(k)) = [k + 2|k + 1, 0, 1].\]

Using this, we can find the representations appearing in \( \mu A^*(N_Y) \):

\[(6.29) \quad W_0 = [1|0, 0, 1], \quad W_1 = [-0|0, 0, 0] - [0|1, 0, 1] - [2|0, 0, 0],
\quad W_2 = [1|1, 0, 0], \quad W_3 = [0|0, 0, 2], \quad W_4 = [-1|0, 0, 1], \quad W_5 = [0|0, 0, 0].\]

Explicitly, the multiplet takes the following form:

\[(6.30) \quad \mu A^*(N_Y)^\# = \begin{bmatrix} S_- \otimes \mathbb{C}^2 & \mathbb{C} \oplus \Omega^2 \oplus \Omega^3 \oplus \mathbb{C}^3 & S_+ \otimes \mathbb{C}^2 \\ \Omega^3 & S_- \otimes \mathbb{C}^2 & \mathbb{C} \end{bmatrix}.\]

6.4. Deformation. As in the example of the Euler sequence for \( \mathbb{P}^1 \), we find that the field contents of the direct sums of the multiplets associated to tangent and normal bundle does not match the field content of \( \mu A^*(T_{\mathbb{P}^7|Y}) \), i.e.

\[(6.31) \quad \mu A^*(T_Y)^\# \oplus \mu A^*(N_Y)^\# \neq \mu A^*(T_{\mathbb{P}^7|Y})^\#.\]

This is again related to the fact that the normal exact sequence does not split as a sequence of \( R/I \)-modules. However, there is a deformation of the direct sum such that the resulting
The cotangent bundle, the conormal bundle, and the restriction of the cotangent bundle of the ambient $\mathbb{P}^7$ to the nilpotence variety sit in the conormal exact sequence, which is the dual of (6.1):

\[ 0 \rightarrow N_Y^\vee \rightarrow \Omega^1_{\mathbb{P}^7|Y} \rightarrow \Omega^1_Y \rightarrow 0. \]

In the same fashion as above, we now study the cohomology of these sheaves and their associated multiplets.

7. Cotangent bundle.

7.1. Cohomology and Hilbert series. As explained above, the cotangent bundle of the nilpotence variety $Y$ is given by the exterior sum

\[ \Omega^1_Y = \pi^*\Omega^1_{\mathbb{P}1} \oplus \pi^*\Omega^1_{\mathbb{P}3} = \Omega^1_{\mathbb{P}1} \oplus \Omega^1_{\mathbb{P}3}, \]

where $\Omega^1_{\mathbb{P}1} \equiv \mathcal{O}_{\mathbb{P}1}(-2)$. As a consequence, the associated multiplet is again a direct sum

\[ \mu A^*(\Omega^1_Y) = \mu A^*(\mathcal{O}_{Y}(-2, 0)) \oplus \mu A^*(\pi^*\Omega^1_{\mathbb{P}3}). \]

The multiplet $\mu A^*(\mathcal{O}_{Y}(-2, 0))$, arising from the cotangent bundle of $\mathbb{P}^1$, was already described in §3.3, therefore we are left with describing $\mu A^*(\pi^*\Omega^1_{\mathbb{P}3})$. To this end, we need to study the zeroth cohomology of

\[ \pi^*\Omega^1_{\mathbb{P}3}(k) = \pi^*\mathcal{O}_{\mathbb{P}1}(k) \oplus \pi^*\Omega^1_{\mathbb{P}3}(k) = \mathcal{O}_{\mathbb{P}1}(k) \otimes \Omega^1_{\mathbb{P}3}(k). \]

The Künneth theorem implies that

\[ H^0(\pi^*\Omega^1_{\mathbb{P}3}(k)) = H^0(\mathcal{O}_{\mathbb{P}1}(k)) \otimes H^0(\Omega^1_{\mathbb{P}3}(k)), \]

reducing the problem to compute the dimension of the zeroth cohomology of $\Omega^1_{\mathbb{P}3}(k)$. This can be obtained by Bott formulas [OSS80] or by explicitly studying the twist of the dual of the
Euler exact sequence for $\Omega^1_{\mathbb{P}^3}$,

\begin{equation}
(7.6) \quad 0 \longrightarrow \Omega^1_{\mathbb{P}^3}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(k-1) \otimes \mathbb{C}^4 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(k) \longrightarrow 0.
\end{equation}

In order to obtain a short exact sequence of modules, we have to check that the connection morphism vanishes. Clearly, if $k < 0$ then $H^0(\Omega^1_{\mathbb{P}^3}(k)) = 0$. If $k = 0$, then this corresponds to the Hodge number of $\mathbb{P}^3$ and in particular one finds $h^0(\Omega^1_{\mathbb{P}^3}) = 0 = h^{1,0}(\mathbb{P}^3)$. For $k = 1$ it is easy seen that the map \( \varphi_{k=1} : H^0(\mathcal{O}_{\mathbb{P}^3}) \otimes \mathbb{C}^4 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \),

given by $\mathbb{C}^4 \ni (c_0, \ldots, c_3) \mapsto \sum_{i=0}^3 c_i X_i$, for $(X_0, \ldots, X_1)$ global sections of $\mathcal{O}_{\mathbb{P}^3}(1)$ is an isomorphism and hence $H^0(\Omega^1_{\mathbb{P}^3}(1)) = 0$. On the other hand in the case $k > 1$ the map $\varphi_{k>1} : H^0(\mathcal{O}_{\mathbb{P}^3}) \otimes \mathbb{C}^4 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ is only surjective so that $H^1(\Omega^1_{\mathbb{P}^3}(k)) = 0$ and $\ker(\varphi_{k>1}) = H^0(\Omega^1_{\mathbb{P}^3})$.

If follows that for $k > 1$ one has

\begin{equation}
(7.8) \quad h^0(\Omega^1_{\mathbb{P}^3}(k)) = 4 \binom{k+2}{k-1} - \binom{k+3}{k} = \frac{1}{2} (k+2)(k+1)(k-1).
\end{equation}

In turn, this implies that

\begin{equation}
(7.9) \quad h^0(\pi^*_3 \Omega^1_{\mathbb{P}^3}(k)) = \frac{1}{2} (k+1)^2 (k+2)(k-1),
\end{equation}

and the related Hilbert series gives

\begin{equation}
(7.10) \quad \text{Hilb}(\pi^*_3 \Omega^1_{\mathbb{P}^3}) = t^2 \frac{18 - 64t + 89t^2 - 64t^3 + 28t^4 - 8t^5 + t^6}{(1-t)^8}.
\end{equation}

7.1.2. Equivariant decomposition. For the equivariant decomposition, we identify

\begin{equation}
(7.11) \quad H^0(\Omega^1_{\mathbb{P}^3}(k)) = [0|k-2,1,0],
\end{equation}

and find the following representations in $\mu \Lambda^*(\pi^*_3 \Omega^1_{\mathbb{P}^3})$.

\begin{align*}
W_0 &= [2|0,1,0] \\
W_1 &= [-1|0,0,1] - [1|1,1,0] - [3|0,0,1] \\
W_2 &= [0|0,0,0] + [0|0,2,0] + [0|1,0,1] + [2|0,0,0] + [2|1,0,1] + [4|0,0,0] \\
W_3 &= [-1|0,1,1] - [1|1,0,0] - [3|1,0,0] \\
W_4 &= [0|0,0,2] + [2|0,1,0] \\
W_5 &= [-1|0,0,1] \\
W_6 &= [0|0,0,0]
\end{align*}
The resulting multiplet takes the following form.

\[(7.13) \quad \mu A^*(\pi_3^* \Omega_1^3) \# = \]
\[
\begin{bmatrix}
\Omega^1 \otimes \mathbb{C}^3 & \mathbb{C}^2 \otimes V \otimes S_+ & \text{Sym}^2(V) \otimes \Omega^2 & \mathbb{C}^2 \otimes S_- \otimes V & \Omega^2 & \mathbb{C}^2 \otimes S_- \\
\mathbb{C}^4 \otimes S_- & \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \Omega^2 \otimes \mathbb{C}^5 & S_+ \otimes \mathbb{C}^4 & \Omega^1 \otimes \mathbb{C}^3 & \mathbb{C}^2 \otimes S_- & \mathbb{C}
\end{bmatrix}
\]

7.2. Restriction of $\Omega_{p^7}^1$ to the nilpotence variety.

7.2.1. Cohomology and Hilbert series. Dually to the case of $\mathcal{T}_{p^7|Y}$, the cohomology of $\Omega_{p^7|Y}^1$ and its twists is studied by restricting the dual of the Euler exact sequence for the ambient space $\mathbb{P}^7$ to $Y$. This gives

\[(7.14) \quad 0 \longrightarrow \Omega_{p^7|Y}^1(k) \longrightarrow \mathcal{O}_Y(k-1,k-1) \otimes \mathbb{C}^8 \longrightarrow \mathcal{O}_Y(k,k) \longrightarrow 0.\]

Studying the related long exact cohomology sequence, it is easy to see that if $k \leq 1$ then $H^0(\Omega_{p^7|Y}^1(k)) = 0$. In the remaining case, when $k > 1$, the space of global sections $H^0(\Omega_{p^7|Y}^1(k))$ is actually non-zero and the long cohomology sequence splits since the polynomial map

\[(7.15) \quad H^0(\mathcal{O}_Y(k-1,k-1)) \otimes \mathbb{C}^8 \xrightarrow{(X_i,Y_j)} H^0(\mathcal{O}_Y(k,k))\]

is surjective, so one gets the short exact sequence

\[(7.16) \quad 0 \rightarrow H^0(\Omega_{p^7|Y}^1(k)) \rightarrow H^0(\mathcal{O}_Y(k-1,k-1)) \otimes \mathbb{C}^8 \rightarrow H^0(\mathcal{O}_Y(k,k)) \rightarrow 0.\]

This says that

\[(7.17) \quad h^0(\Omega_{p^7|Y}^1(k)) = 8k \binom{k+2}{k-1} - (k+1) \binom{k+3}{k} \]

\[= \frac{4}{3} k(k+2)(k+1) - \frac{1}{6} (k+1)(k+3)(k+2)(k+1).\]

Finally, for future use, notice that for $k > 1$ one has $H^1(\Omega_{p^7|Y}^1(k)) = 0$, since it is mapped injectively into $H^1(\mathcal{O}_Y(k-1,k-1)) \otimes \mathbb{C}^8$ which is indeed zero.

The related Hilbert series of $\Omega_{p^7|Y}^1$ can be resummed easily to give

\[(7.18) \quad \text{Hilb}(\Omega_{p^7|Y}^1) = \frac{t^2 34 - 112t + 137t^2 - 80t^3 + 28t^4 - 8t^5 + t^6}{(1-t)^8}\]

7.2.2. Equivariant decomposition. In terms of representations, the sequence gives

\[(7.19) \quad H^0(\Omega_{p^7|Y}^1(k)) = [k-2][0,0] + [k][k-2,1,0] + [k-2][k-2,1,0].\]

Using these results, we can deduce the field content of $\mu A^*(\Omega_{p^7|Y}^1)$. 

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\[ W_0 = [0|0,1,0] + [2|0,0,0] + [2|0,1,0] \]
\[ W_1 = -2[1|0,0,1] - 2[1|1,1,0] - [3|0,0,1] \]
\[ W_2 = [0|0,0,0] + [0|2,0,0] + [0|1,0,1] + 2[2|0,0,0] + 2[2|1,0,1] + [4|0,0,0] \]
\[ W_3 = - [1|0,1,1] - [1|1,0,0] - 2[3|1,0,0] \]
\[ W_4 = [0|0,0,2] + [2|0,1,0] \]
\[ W_5 = - [1|0,0,1] \]
\[ W_6 = [0|0,0,0] \]

The resulting multiplet takes the following form.

\[
\begin{bmatrix}
\Omega^1 \otimes (\mathbb{C} \oplus \mathbb{C}^3) & (\mathbb{C}^2 \otimes V \otimes S_-)^{\otimes 2} & \text{Sym}^2(V) \otimes \Omega^2 & \mathbb{C}^2 \otimes V \otimes S_- & \text{Sym}^2(S_-) & S_- \otimes \mathbb{C}^2 & \mathbb{C}
\end{bmatrix}
\]

7.3. The conormal bundle and its supergravity multiplet.

7.3.1. Cohomology and Hilbert series. Having available the cohomology of the cotangent bundle and the restriction of \( \Omega^1_{p7} \) to the nilpotence variety \( Y \), one can study the conormal bundle and its related multiplet in a similar fashion as for the normal bundle above, i.e. by considering \( k \)-twists of the conormal exact sequence (2.46):

\[
0 \longrightarrow N^\vee_Y(k) \longrightarrow \Omega^1_{p7}(k) \longrightarrow \Omega^1_Y(k) \longrightarrow 0.
\]

The issue one faces following this approach is that the related long exact cohomology sequence starts with a four-term sequence in the relevant case \( k > 1 \)

\[
0 \rightarrow H^0(N^\vee_Y(k)) \rightarrow H^0(\Omega^1_{p7}(k)) \rightarrow H^0(\Omega^1_Y(k)) \rightarrow H^1(N^\vee_Y(k)) \rightarrow 0,
\]

and it is not completely trivial to establish the vanishing of the group \( H^1(N^\vee_Y(k)) \) for any \( k \geq 1 \). For the sake of exposition we have deferred this verification to Appendix A, which follows another route to describe the algebraic geometry of the normal and conormal bundle of \( Y \), reducing the problem to studying easy polynomial maps.

Using the above results, one computes

\[
h^0(N^\vee_Y(k)) = 8k \binom{k+2}{3} - 2k \binom{k+3}{3} - (k^2 - 1) \binom{k+2}{2} = \frac{1}{2} (k+1)(k-1)^2(k+2).
\]
The related Hilbert series is resummed to give

\[
\text{Hilb}(\mathcal{N}_Y^\vee) = \sum_{k=2}^{\infty} \frac{1}{2}(k+1)(k-1)^2(k+2)t^k = t^2 \frac{6 - 8t - 17t^2 + 40t^3 - 28t^4 + 8t^5 - t^6}{(1-t)^8}.
\]

7.3.2. **Equivariant decomposition.** In terms of representations the conormal exact sequence implies

\[
H^0(\mathcal{N}_Y^\vee(k)) = [k-2|k-2,1,0].
\]

We find the following field content for \(\mu A^*(\mathcal{N}_Y^\vee)\).

\[
W_0 = [0|0,1,0], \quad W_1 = [-1|0,0,1], \quad W_2 = [-0|0,2,0] + [2|0,0,0],
\]

\[
W_3 = [1|0,1,1], \quad W_4 = [-0|0,0,2] - [2|0,1,0], \quad W_5 = [-1|0,0,1], \quad W_6 = [0|0,0,0].
\]

In summary, the multiplet takes the form

\[
\mu A^*(\mathcal{N}_Y^\vee)^\# = \begin{bmatrix}
V & S_- \otimes \mathbb{C}^2 & \mathbb{C}^3 \\
\text{Sym}^2_0(V) & (V \otimes S_-)_{\frac{3}{2}} \otimes \mathbb{C}^2 & V \otimes \mathbb{C}^3 \oplus \Omega^2 & S_- \otimes \mathbb{C}^2 & \mathbb{C}
\end{bmatrix}
\]

This is precisely the field content of six-dimensional \(\mathcal{N} = (1,0)\) supergravity, presented as the “type-II Weyl multiplet” [LTM12].

7.4. **Deformation.** There is again a deformation of \(\mu A^*(\Omega^1_Y)^\# \oplus \mu A^*(\mathcal{N}_Y^\vee)^\#\) such that the result is quasi-isomorphic to \(\mu A^*(\Omega^1_{\mathbb{P}^7}/Y)^\#\):

\[
\mu A^*(\Omega^1_{\mathbb{P}^7}/Y)^\# \simeq [\mu A^*(\Omega^1_Y)^\# \oplus \mu A^*(\mathcal{N}_Y^\vee)^\#]^\text{Deform} =
\]

\[
\begin{bmatrix}
\Omega^1 \otimes \mathbb{C}^3 \\
\text{Sym}^2 S_+ \otimes V \\
V \\
\text{Sym}^2_0(V) \\
C^2 \otimes V \otimes S_- \\
\mathbb{C}^4 \otimes S_- \\
\mathbb{C}^2 \otimes (V \otimes S_+)^{\frac{3}{2}} \\
S_- \otimes \mathbb{C}^2 \\
\text{Sym}^2_0(V) \\
\mathbb{C}^2 \otimes (V \otimes S_-)^{\frac{3}{2}} \\
\Omega^1 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \\
\mathbb{C}^2 \otimes V \otimes S_- \\
\Omega^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5 \\
\text{Sym}^2_0(V) \\
\mathbb{C}^2 \otimes \mathbb{C}^3 \\
\text{Sym}^2(S_-) \\
S_- \otimes \mathbb{C}^2 \\
\Omega^3 \\
S_- \otimes \mathbb{C}^2 \\
\mathbb{C}
\end{bmatrix}
\]

APPENDIX A. **Equivariant vector bundles and representations**

This paper features different methods from algebraic geometry and representation theory, whose mutual relations might appear somewhat puzzling on a first reading. In order to help the reader make his way through the manuscript, we now spell out the general philosophy.
connecting representations of a certain symmetry group or Lie algebra and the space of
global sections of a certain related vector bundle.

For the sake of concreteness, we will work in the case which is relevant for this paper,
that of complex projective spaces \( \mathbb{P}^n \), but all of the following considerations hold true more
generally on a generic homogeneous space \( G/H \), for \( G \) a complex Lie group and \( H \subset G \) a
closed subgroup of \( G \).

We start by recalling that the complex projective space \( \mathbb{P}^n \) can indeed be realized as a
homogeneous space, being the quotient

(A.1) \[ \mathbb{P}^n \cong SL(n+1, \mathbb{C})/B^- \]

where \( B^- \) is the (negative) Borel — hence parabolic — subgroup of lower triangular matrices
in \( SL(n+1, \mathbb{C}) \), i.e. \( b \in B^- \) is of the form

\[
(A.2) \quad b = \begin{pmatrix}
* & 0 & 0 \\
* & * & 0 \\
* & * & * \\
\end{pmatrix}_{n+1}
\]

and the product of the elements on the diagonal elements is one.

In this context, an \( SL(n+1, \mathbb{C}) \)-equivariant vector bundle \( \pi : E \to \mathbb{P}^n \) on \( \mathbb{P}^n \) is a holomorphic
vector bundle that carries a \( SL(n+1, \mathbb{C}) \)-action on its total space \( E \) which is compatible with
the \( SL(n+1, \mathbb{C}) \)-action on the base space \( \mathbb{P}^n \). This means that we require the structure map
\( \pi : E \to \mathbb{P}^n \) to be \( SL(n+1, \mathbb{C}) \)-equivariant, i.e. \( \pi(g \cdot e) = g \cdot \pi(e) \) for all \( e \in E \) and \( g \in G \) and the
translation between fibers to be linear, i.e \( \ell_g : E_{[x]} \to E_{g[x]} \) given by \( e \mapsto g \cdot e \) is linear — and
hence an isomorphism of vector spaces for all \( e \in E \) and \( g \in G \).

It follows immediately from the given definition that the fiber \( E_{B^-} \) at the point corre-
spanding to the Borel subgroup \( B^- \) in \( \mathbb{P}^n \) is a \( B^- \)-module, and the map \( \rho : B^- \to Aut(E_{B^-}) \)
given by \( \rho(b) := \ell_b : E_{B^-} \to E_{B^-} \) is a representation of \( B^- \). Conversely, starting from a holo-
morphic representation \( \rho : B^- \to Aut(V) \) for some \( B^- \)-module \( V \), it is not hard to see that
the associated vector bundle \( \pi : SL(n+1, \mathbb{C}) \times \rho V \to \mathbb{P}^n \), with \( \pi((g,v)) := gB^- \) is indeed a
\( SL(n+1, \mathbb{C}) \)-equivariant vector bundle as defined above and that restricting the action of
\( B^- \) on the total space to the fiber lying over any point \( b \in B_\mathbb{P} \) one recovers the original rep-
sentation \( \rho \). Recall that here the total space \( SL(n+1, \mathbb{C}) \times \rho V \) is the quotient manifold
\( (SL(n+1, \mathbb{C}) \times V)/\sim \) by the relations \((g,v) \sim (gb^{-1}, \rho(h)g)\) for \( g \in SL(n+1, \mathbb{C}) \), \( b \in B^- \), endowed
with the quotient topology. These considerations suggest that there is a bijection

\[
\{ \text{holomorphic representations of } B^- \} \longleftrightarrow \{ \text{SL}(n+1, \mathbb{C})\text{-equivariant vector bundles on } \mathbb{P}^n \} \\
\rho : B^- \to Aut(V) \longleftrightarrow \pi : SL(n+1, \mathbb{C}) \times \rho V \to \mathbb{P}^n.
\]
This is in fact an equivalence of categories and it can be used to induce representations from the Borel subgroup $B^-$ to the whole group $SL(n + 1, \mathbb{C})$. The crucial point is that given an equivariant vector bundle $E_\rho := SL(n + 1, \mathbb{C}) \times _\rho V \to \mathbb{P}^n$ for a certain representation $\rho$ of $B^-$ and denoting with $\mathcal{E}_\rho$ the sheaf of its holomorphic sections, then the $SL(n + 1, \mathbb{C})$-action on the total space of $E_\rho$ induces a linear action on the (vector) space of global holomorphic sections $\Gamma(E_\rho) := H^0(\mathbb{P}^n, \mathcal{E}_\rho)$ given by

$$SL(n + 1, \mathbb{C}) \times H^0(\mathbb{P}^n, \mathcal{E}_\rho) \to H^0(\mathbb{P}^n, \mathcal{E}_\rho)$$

$$(g, s) \mapsto g \cdot s$$

where $(g \cdot s)(x) := g \cdot s(g^{-1}[x])$ for $g \in SL(n + 1, \mathbb{C})$, $s \in H^0(\mathbb{P}^n, \mathcal{E}_\rho)$ and $[x] \in \mathbb{P}^n$. More in general, by naturality and equivariance of $\rho$, one has a linear action of $SL(n + 1, \mathbb{C})$ on every cohomology group $R^q \Gamma(E)$ for $q = 0, \ldots, n$.

Notice that the action (A.3) does indeed defines a holomorphic representation

$$\varphi : SL(n + 1, \mathbb{C}) \to \text{Aut}(H^0(\mathbb{P}^n, \mathcal{E}_\rho))$$

of $SL(n + 1, \mathbb{C})$ on $H^0(\mathbb{P}^n, \mathcal{E})$, since $\mathbb{P}^n$ is compact and $E_\rho$ is a holomorphic vector bundle, and hence all its cohomology groups are finite-dimensional. This construction is heavily exploited in the paper, where the space of global sections of of a certain vector or line bundles on $\mathbb{P}^n$ will always be understood as carrying a certain representation of $SL(n + 1, \mathbb{C})$. Note that we take the liberty of moving freely from representation of $SL(n + 1, \mathbb{C})$ to representation of its Lie algebra $\mathfrak{sl}(n + 1, \mathbb{C})$ and viceversa, since $SL(n + 1, \mathbb{C})$ is simply connected for all $n$ and hence there is a one-to-one correspondence between representations of the Lie group and representations of its Lie algebra.

Now, recalling that $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, all of the $SL(n + 1, \mathbb{C})$-modules $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ for $k \in \mathbb{Z}$ corresponding to global sections of line bundles on $\mathbb{P}^n$ are easily identified as

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \cong \begin{cases} \text{Sym}^k(V_{n+1}^\vee) & k \geq 0, \\ 0 & k < 0, \end{cases}$$

where $V_{n+1}^\vee \cong (\mathbb{C}^{n+1})^\vee \cong \text{Span}_\mathbb{C}\{x_0, \ldots, x_n\}$ is in an obvious fashion an irreducible $SL(n + 1, \mathbb{C})$-module. Notice that $\text{Sym}^k(V_{n+1}^\vee) \cong \mathbb{C}[x_0, \ldots, x_n](k)$, the homogeneous polynomial of degree $k$. It follows that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \cong \text{Sym}^k(V_{n+1}^\vee)$ is as well an irreducible $SL(n + 1, \mathbb{C})$-module, corresponding to the irreducible representation of $A_n = \mathfrak{sl}(n + 1, \mathbb{C})$ given by

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \hookrightarrow [k, 0, \ldots, 0]_n$$
in terms of its Dynkin labels for the representations of the $A_n$-series. These findings are in agreement with Borel–Weil theorem, that adapted to the case of $\mathbb{P}^n \cong \text{SL}(n+1, \mathbb{C})/B^-$, constructs irreducible representations from the cohomology of line bundles $\pi_{\lambda_k} : \text{SL}(n+1, \mathbb{C}) \times \lambda_k \mathbb{C} \to \mathbb{P}^n$ associated to integral characters of $\lambda_k : B^- \to \mathbb{C}^*$ of $B^-$ - which corresponds to irreducible holomorphic representations since $B^-$ is solvable. Calling $\mathcal{L}_{\lambda_k}$ the sheaf of holomorphic sections of the equivariant line bundle associated to $\lambda_k$ one can see that $\mathcal{L}_{\lambda_k} \cong \mathcal{O}_{\mathbb{P}^n}(k)$ so that $\mathcal{O}_{\mathbb{P}^n}(k)$ is indeed $\text{SL}(n+1, \mathbb{C})$-equivariant and 

\[(A.7)\quad H^0(\mathbb{P}^n, \mathcal{L}_{\lambda_k}) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))\]

for any $k$. Indeed, the Borel–Weil theorem guarantees that $H^0(\mathbb{P}^n, \mathcal{L}_{\lambda_k})$ is an irreducible $\text{SL}(n+1, \mathbb{C})$-module of highest weight $k$ if and only $\lambda_k$ is a dominant weight: this is the case if and only if $k \geq 0$, that corresponds in fact to the irreducible representation $[k, 0, \ldots, 0]$.

Let us now pass to a higher rank case which is relevant for the present paper, i.e. that of the tangent bundle to a the projective space $\mathbb{P}^n$ seen as the homogeneous space $\text{SL}(n+1, \mathbb{C})/B^-$. Starting from the adjoint representation $\text{Ad}_{\text{SL}} : \text{SL}(n+1, \mathbb{C}) \to \text{Aut}(\text{sl}(n+1, \mathbb{C}))$ of $\text{SL}(n+1, \mathbb{C})$, one can consider the restriction to $B^- \subset \text{SL}(n+1, \mathbb{C})$, given by $\text{Ad}_{\text{SL}}|_{B^-} : B^- \to \text{Aut}(\text{sl}(n+1, \mathbb{C}))$. Denoting with $b^-$ the Lie algebra of $B^-$, it is easy to observe that

\[(A.8)\quad (\text{Ad}_{\text{SL}}|_{B^-}(b))X = (\text{Ad}_{\text{SL}}(b))X = (\text{Ad}_{B^-}(b))X \in b^-,
\]

for any $b \in B^-$ and $X \in b^-$, so that $b^-$ is an invariant subspace for the representation $\text{Ad}_{\text{SL}}|_{B^-} : B^- \to \text{Aut}(\text{sl}(n+1, \mathbb{C}))$, so that one gets the factor representation $\text{Ad}_{\text{SL}}^\perp : B^- \to \text{Aut}(\text{sl}(n+1, \mathbb{C})/b^-)$, and the following is a short exact sequence of $B^-$-equivariant modules

\[(A.9)\quad 0 \longrightarrow b^- \longrightarrow \text{sl}(n+1, \mathbb{C}) \longrightarrow \text{sl}(n+1, \mathbb{C})/b^- \longrightarrow 0.
\]

The associated vector bundle on $\mathbb{P}^n$ of the factor representation $\text{Ad}_{\text{SL}}^\perp$ is naturally isomorphic to the tangent bundle of $\mathbb{P}^n$, i.e. we have

\[(A.10)\quad \mathcal{T}_{\mathbb{P}^n} \cong \text{SL}(n+1, \mathbb{C}) \times_{\text{Ad}_{\text{SL}}^\perp} \mathfrak{g}/b^-,
\]

which makes $\mathcal{T}_{\mathbb{P}^n}$ into a $\text{SL}(n+1, \mathbb{C})$-equivariant vector bundle on $\mathbb{P}^n$. We recall that $\mathcal{T}_{\mathbb{P}^n}$ is very ample in the sense of Hartshorne so that its vector space of global sections is non-vanishing and as a consequence it is and irreducible $\text{SL}(n+1, \mathbb{C})$-module of dimension $(n + 1)^2 - 1$, which is identified with the adjoint representation, i.e.

\[(A.11)\quad H^0(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) \cong [1, 0, \ldots, 0, 1]_n
\]

in the Dynkin label notation for the $A_n$-series.
APPENDIX B. GEOMETRY OF THE NORMAL AND CONORMAL BUNDLES OF \(Y\)

In this appendix we study the geometry of the normal and conormal bundle of the nilpotence variety \(Y \cong \mathbb{P}^1 \times \mathbb{P}^3\) in more detail. We will follow a different approach compared to the one in the main text, aiming at finding resolutions for these vector bundles in line bundles, which make possible to easily evaluate their cohomology.

We start with the normal bundle, considering the following two short exact sequences:

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Y \\
& \quad \xrightarrow{i \oplus j} \mathcal{O}_Y(1,0)^\oplus 2 \oplus \mathcal{O}_Y(0,1)^\oplus 4 \\
& \quad \xrightarrow{} \mathcal{T}_Y \\
& \quad \longrightarrow 0 \quad \text{(B.1)}
\end{align*}
\]

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_Y \\
& \quad \xrightarrow{k} \mathcal{O}_Y(1,1)^\oplus 8 \\
& \quad \xrightarrow{} \mathcal{T}_{\mathbb{P}^7|Y} \\
& \quad \longrightarrow 0 \quad \text{(B.2)}
\end{align*}
\]

The first one comes from the Euler exact sequences for \(\mathbb{P}^1\) and \(\mathbb{P}^3\) respectively. The second sequence is the restriction of the Euler exact sequence for the ambient variety \(\mathbb{P}^7\) to \(Y\). These exact sequences fit in the following commutative diagram.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(q) & \xrightarrow{i} & \mathcal{O}_Y \oplus \mathcal{O}_Y & \xrightarrow{q} & \mathcal{O}_Y & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & \ker(f) & \xrightarrow{i \oplus j} & \mathcal{O}_Y(1,0)^\oplus 2 \oplus \mathcal{O}_Y(0,1)^\oplus 4 & \xrightarrow{f} & \mathcal{O}_Y(1,1)^\oplus 8 & \longrightarrow & Q \\
& & & & \downarrow & & \downarrow & & \downarrow & & \\
& & & & 0 & & \mathcal{T}_Y & \longrightarrow & \mathcal{T}_{\mathbb{P}^7|Y} & \longrightarrow & N_Y & \longrightarrow & 0 \\
& & & & & & \downarrow & & \downarrow & & \downarrow & & \\
& & & & & & 0 & & 0 & & 0 & & \\
\end{array}
\]

Working on the affine cones of \(C_a \mathbb{P}^1 \times C_a \mathbb{P}^3\) with coordinates \((u_0, u_1)\) and \((x_0, x_1, x_2, x_3)\) the above maps are defined as follows. The map \(i \oplus j : \mathcal{O}_Y \oplus \mathcal{O}_Y \to \mathcal{O}_Y(1,0)^\oplus 2 \oplus \mathcal{O}_Y(0,1)^\oplus 4\) is the multiplication by the following \(2 \times 6\) matrix

\[
[i \oplus j] = 
\begin{pmatrix}
  u_0 & 0 \\
  u_1 & 0 \\
  0 & x_0 \\
  0 & x_1 \\
  0 & x_2 \\
  0 & x_3 
\end{pmatrix}
\]
where frames in $\mathcal{O}_Y \oplus \mathcal{O}_Y$ are seen as column vectors. The map $k : \mathcal{O}_Y \to \mathcal{O}_Y(1,1)^{\oplus 8}$ is the multiplication by the following $1 \times 8$ matrix

$$
[k] = \begin{pmatrix}
  u_0 x_0 \\
  u_0 x_1 \\
  \vdots \\
  u_1 x_2 \\
  u_1 x_3
\end{pmatrix}
$$

(B.5)

The crucial map in the above diagram is the map $f : \mathcal{O}_Y(1,0)^{\oplus 2} \oplus \mathcal{O}_Y(0,1)^{\oplus 4} \to \mathcal{O}_Y(1,1)^{\oplus 8}$ which is given by the Jacobian of the Segre embedding. In coordinates it is described by the $8 \times 6$ matrix

$$
[f] = \begin{pmatrix}
  x_0 & 0 & u_0 & 0 & 0 & 0 \\
  x_1 & 0 & 0 & u_0 & 0 & 0 \\
  x_2 & 0 & 0 & 0 & u_0 & 0 \\
  x_0 & 0 & 0 & 0 & 0 & u_0 \\
  0 & x_1 & u_1 & 0 & 0 & 0 \\
  0 & u_1 & 0 & u_1 & 0 & 0 \\
  0 & u_1 & u_0 & 0 & u_1 & 0 \\
  0 & u_1 & u_0 & 0 & 0 & u_1
\end{pmatrix}
$$

(B.6)

It is easy to see that from the above expressions that $f \circ (i \oplus j) = k \oplus k$, hence by commutativity, one has $f \circ (i \oplus j) = k \circ q = k \oplus k$. This says that the map $q : \mathcal{O}_X \oplus \mathcal{O}_X \to \mathcal{O}_X$ corresponds to the sum of the elements in $\mathcal{O}_Y \oplus \mathcal{O}_Y$. In turn, this implies that $\ker(q) \cong \mathcal{O}_Y$ and the map $i : \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{O}_Y$ is simply given by $f \mapsto (f, -f)$. On the other hand, it is clear that $\ker(q) \cong \ker(f)$, so that also $\ker(f) \cong \mathcal{O}_Y$. The above diagram (B.3) thus reads

$$
\begin{array}{cccccccc}
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
\mathcal{O}_Y & \stackrel{i}{\longrightarrow} & \mathcal{O}_Y \oplus \mathcal{O}_Y & \stackrel{q}{\longrightarrow} & \mathcal{O}_Y & \longrightarrow & 0 \\
\cong & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y(1,0)^{\oplus 2} \oplus \mathcal{O}_Y(0,1)^{\oplus 4} & \stackrel{f}{\longrightarrow} & \mathcal{O}_Y(1,1)^{\oplus 8} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
\downarrow & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\mathcal{T}_Y & \quad & \quad & \quad & \quad & \quad & \mathcal{E}|_Y & \cong & \mathcal{N}_Y \\
\downarrow & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

(B.7)
This is enough to read out a resolution of the normal bundle. Namely, we have

\[(B.8) \quad 0 \longrightarrow \mathcal{O}_Y \overset{\beta}{\longrightarrow} \mathcal{O}_Y(1,0) \oplus \mathcal{O}_Y(0,1)^{\oplus 4} \overset{f}{\longrightarrow} \mathcal{O}_Y(1,1)^{\oplus 8} \overset{\epsilon}{\longrightarrow} \mathcal{N}_Y \longrightarrow 0,\]

where \(\beta := i \circ (i \oplus j)\) is injective as it is the composition of two injective morphisms. Also, notice that it can be explicitly verified that \(f \circ \beta = 0\). Finally, \(\epsilon\) is again the composition of two surjective morphisms, and thus it is surjective. Now, in order to study the cohomology of the normal bundle, together with all its twists, we break the previous resolution in two short exact sequences

\[(B.9) \quad 0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(1,0) \oplus \mathcal{O}_Y(0,1) \oplus \mathcal{O}_Y(0,1) \oplus \mathcal{O}_Y(1,1) \oplus \mathcal{O}_Y(0,1) \oplus \mathcal{N}_Y \longrightarrow 0,\]

\[(B.10) \quad 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_Y(1,1) \longrightarrow \mathcal{N}_Y \longrightarrow 0,\]

which implicitly define the sheaf \(\mathcal{M}\). The first one can be used to compute the cohomology of \(\mathcal{M}\): in particular one finds that \(H^1(\mathcal{M}(k)) = 0\) for any \(k\), so that the long exact cohomology sequence of the short exact sequence (B.10) gives

\[(B.11) \quad 0 \longrightarrow H^0(\mathcal{M}(k)) \longrightarrow H^0(\mathcal{O}_Y(k+1,k+1)^{\oplus 8}) \longrightarrow H^0(\mathcal{N}_Y(k)) \longrightarrow 0.\]

Computing \(H^0(\mathcal{M}(k))\) from (B.9), one gets

\[(B.12) \quad h^0(\mathcal{N}_Y) = \frac{1}{2} (k + 3)^2 (k + 2)(k + 5)\]

We now consider the conormal bundle. First it can be observed that the (B.8) is a resolution via vector bundles. This means that every map has constant rank and dualizing gives again an exact sequence

\[(B.13) \quad 0 \longrightarrow \mathcal{N}_Y^\vee \overset{\epsilon^\vee}{\longrightarrow} \mathcal{O}_Y(-1,-1)^{\oplus 8} \overset{f^\vee}{\longrightarrow} \mathcal{O}_Y(-1,0)^{\oplus 2} \oplus \mathcal{O}_Y(0,-1)^{\oplus 4} \overset{\beta^\vee}{\longrightarrow} \mathcal{O}_Y \longrightarrow 0.\]

Once again the (B.13) can be broken into two short exact sequences, namely

\[(B.14) \quad 0 \longrightarrow \mathcal{N}_Y^\vee \longrightarrow \mathcal{O}_Y(-1,-1)^{\oplus 8} \longrightarrow \mathcal{M}^\vee \longrightarrow 0,\]

\[(B.15) \quad 0 \longrightarrow \mathcal{M}^\vee \longrightarrow \mathcal{O}_Y(-1,0)^{\oplus 2} \oplus \mathcal{O}_Y(0,-1)^{\oplus 4} \longrightarrow \mathcal{O}_Y \longrightarrow 0.\]
We first need to study $\mathcal{M}^\vee$. We have the following commutative diagram.

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{N}_Y^\vee & \Omega_{p_Y}^1 & \Omega_Y^1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{N}_Y^\vee & \mathcal{O}_Y(-1,-1) & \mathcal{M}^\vee & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{O}_Y & \equiv & \mathcal{Q} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

Notice that $\mathcal{Q}$ is of rank 1, hence $\mathcal{Q} \cong \mathcal{O}_Y$, which shows that $\mathcal{M}^\vee$ is a (non split) extension

\[
\begin{array}{ccccc}
0 & \Omega_Y^1 & \mathcal{M}^\vee & \mathcal{O}_Y & 0
\end{array}
\]

Studying the cohomology of $\mathcal{M}^\vee(k)$ for $k \geq 1$ — when $\mathcal{N}_Y^\vee$ can have global sections —, one finds $H^1(\mathcal{M}^\vee(k)) = 0$ and

\[
h^0(\mathcal{M}^\vee(k)) = h^0(\Omega_Y^1(k)) + h^0(\mathcal{O}_Y(k,k)) = 2k \left( \binom{k+3}{3} + (k^2 - 1) \binom{k+2}{2} \right).
\]

Going back to the conormal bundle $\mathcal{N}_Y^\vee$ and looking at the resolution twisted by $\mathcal{O}_Y(k)$ and split into the above two exact sequences (B.14) and (B.15) one has

\[
\begin{array}{ccccccccc}
0 & \mathcal{N}_Y^\vee(k) & \pi^\vee & \mathcal{O}_Y(k-1,k-1)^{\oplus 8} & \mathcal{M}^\vee(k) & i & \mathcal{O}_Y(k-1,k)^{\oplus 2} \oplus \mathcal{O}_Y(k,k-1)^{\oplus 4} & \beta^\vee & \mathcal{O}_Y(k,k) & 0
\end{array}
\]

where $\tilde{f}^\vee$ is surjective and $i$ is injective. Then, the long exact cohomology sequence of (B.14) yields

\[
\begin{array}{ccccc}
0 & H^0(\mathcal{N}_Y^\vee) & H^0(\mathcal{O}_Y(k-1,k-1)^{\oplus 8}) & \tilde{f}^\vee & H^0(\mathcal{M}^\vee(k)) & H^1(\mathcal{N}_Y^\vee) & 0
\end{array}
\]

Hence in particular,

\[
H^1(\mathcal{N}_Y^\vee(k)) = \text{coker}(H^0(\mathcal{O}_Y(k-1,k-1)^{\oplus 8}) \tilde{f}^\vee H^0(\mathcal{M}^\vee(k))).
\]
so that
\[(B.22) \quad h^1(N_Y^\vee(k)) = h^0(M^\vee(k)) - 8h^0(\mathcal{O}_Y(k-1,k-1)) + \dim(\ker \tilde{f}^*). \]

Now, one has \(f^\vee = i \circ \tilde{f}^\vee\), but since \(i\) is injective, then \(\ker(f^\vee) = \ker(\tilde{f}^\vee)\), where
\[(B.23) \quad f^\vee : \mathcal{O}_Y(k-1,k-1) \otimes \mathcal{O}_Y \to \mathcal{O}_Y(k-1,k) \oplus \mathcal{O}_Y(k,k-1).\]

This implies that
\[(B.24) \quad h^1(N_Y^\vee(k)) = h^0(M^\vee(k)) - 8h^0(\mathcal{O}_Y(k-1,k-1)) + \dim(\ker(f^\vee)) = 2k \left(\frac{k+3}{3}\right) + \left(\frac{k^2}{2}\right) - 8k \left(\frac{k+3}{3}\right) + \dim(\ker(f^\vee)).\]

Notice that \(\dim(\ker(f^\vee)) = h^0(N_Y^\vee)\), hence the kernel of the maps yields the number of global sections of the conormal bundle. The map \(f^\vee\) can indeed be described explicitly, and it is given by the transpose of \(f\) introduced above for the normal bundle: it is a \(6 \times 8\) matrix given in the above affine cones coordinates
\[(B.25) \quad [f^\vee] = [f]^\top = \begin{pmatrix}
x_0 & x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 \\
u_0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\
0 & u_0 & 0 & 0 & u_1 & 0 & 0 & 0 \\
0 & 0 & u_0 & 0 & 0 & 0 & u_1 & 0 \\
0 & 0 & 0 & u_0 & 0 & 0 & 0 & u_1
\end{pmatrix}.\]

In the special case \(k = 1\) the long exact cohomology sequence reads
\[(B.26) \quad 0 \to H^0(N_Y^\vee(1)) \to H^0(\mathcal{O}_Y^\otimes 8) \to H^0(M^\vee(1)) \cong H^0(\mathcal{O}_Y(1,1)) \to H^1(N_Y^\vee(1)) \to 0.\]

Here the middle map is an isomorphism, corresponding to the multiplication by \(u_ix_j\). It follows that \(H^0(N_Y^\vee(1)) \cong 0 \cong H^1(N_Y^\vee(1))\). In the remaining case, for \(k \geq 2\), the kernel of the map \(f^\vee\) in cohomology can be evaluated directly, and it yields
\[(B.27) \quad h^0(N_Y^\vee(2)) = \ker(f^\vee) = 8k \left(\frac{k+2}{3}\right) - 2k \left(\frac{k+3}{3}\right) - \left(\frac{k^2}{2}\right) - 8k \left(\frac{k+3}{3}\right) + \dim(\ker(f^\vee)).\]

so that in particular it follows from (B.24) that \(H^1(N_Y^\vee) = 0.\)

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