Intersection number and stability of some inscribable graphs

Jinsong Liu¹,² · Ze Zhou¹,³

Received: 29 August 2015 / Accepted: 3 May 2016 / Published online: 11 May 2016
© Springer Science+Business Media Dordrecht 2016

Abstract A planar graph is inscribable if it is combinatorial equivalent to the skeleton of an inscribed polyhedron in the unit sphere $S^2$. Giving an inscribable graph, in its combinatorial equivalent class if we could also find a polyhedron inscribed in each convex surface sufficiently close to the unit sphere $S^2$, then we call such an inscribable graph a stable one. By combining the Teichmüller theory of packings with differential topology method, in this paper we shall investigate the stability of some inscribable graphs.

Keywords Inscribable graph · Stability · Intersection number · Circle pattern

Mathematics Subject Classification (2000) 51M10 · 51M20 · 51M15 · 52C26

1 Introduction

In graph theory, a connected graph $G$ is said to be planar if it can be embedded in the unit sphere $S^2$. For a connected planar graph, it is said to be $k$-connected if it has more than $k$ vertices and remains connected whenever fewer than $k$ vertices are removed. In fact, given any polyhedron $P$ in Euclidean 3-space $\mathbb{R}^3$, the skeleton of $P$, denoted by $G(P)$, is then a 3-connected graph. A surprise result of Steinitz [26] assert that the converse is also valid. That is, every 3-connected planar graph can be represented as the skeleton of a convex polyhedron.

Therefore, it is of interest to consider the geometric representation of combinatorial objects. For example, as an extension of Steinitz’s theorem, the circle pattern theorem [16, 25]...
implies that every 3-connected planar graph may be represented as a convex polyhedron in such a way that all of its edges are tangent to the same unit sphere. More generally, according to Schramm’s Midscribability theorem [22], if \( G \) is a polyhedral graph (i.e. the skeleton of a convex polyhedron) and \( K \) is any compact convex surface, it is possible to find a polyhedral representation of \( G \) in which all edges are tangent to \( K \).

Note that a planar graph is called inscribable if it can be realized as the skeleton of the convex hull of a set of finite points lying over the unit sphere. In the book [23], the Swiss mathematician Jakob Steiner asked for a combinatorial characterization of those inscribable graphs. To be specific, in which cases does a polyhedral graph can be combinatorially equivalent to the skeleton of a convex polyhedron inscribed in the sphere?

In contrast to the harmony scenery manifested by Midscribability theorem, this seems to be a rather intractable problem. In fact, it’s almost a hundred years later when Steinitz [24] found an example of “non-inscribable” graph in 1927. Whereafter, more and more non-inscribable graphs are discovered. For instance, the polyhedral graph of the following singly-truncated cube is exactly the simplest non-inscribable one (Fig. 1).

Recall the Klein model of the hyperbolic 3-space \( \mathbb{H}^3 \). In this model, \( \mathbb{H}^3 \) is identified with the interior of the unit ball \( \mathbb{B}^3 \subset \mathbb{R}^3 \subset \mathbb{R}P^3 \). In addition, a geodesic in \( \mathbb{H}^3 \) corresponds to the intersection of a straight line of \( \mathbb{R}P^3 \) with \( \mathbb{B}^3 \), and a totally geodesic plane in \( \mathbb{H}^3 \) corresponds to the intersection of a plane of \( \mathbb{R}P^3 \) with \( \mathbb{B}^3 \). Then a convex body in \( \mathbb{H}^3 \) is represented by a convex body in \( \mathbb{B}^3 \). Moreover, one could regard an inscribed polyhedron as an ideal hyperbolic polyhedron. In view of such an observation, Rivin [18] then completely resolved Steiner’s problem by an investigation of the geometry of ideal hyperbolic polyhedra.

For a polyhedral graph \( G \), let \( G^* \) denote its dual graph. We call a set of edges \( \Gamma = \{e_1, e_2, \ldots, e_k\} \subset G \) a prismatic circuit, if the dual edges \( \{e_1^*, e_2^*, \ldots, e_k^*\} \) form a simple closed curve in the dual graph \( G^* \) and does not bound a face in \( G^* \). Rivin’s theorem [18] is then stated as follows.

**Theorem 1.1** A polyhedral graph \( G = G(V, E) \) is of inscribable type if and only if there exists a weight \( w \) assigned to its edges set \( E \) such that:

(W1) For each edge \( e \in E \), \( 0 < w(e) \leq 1/2 \).
(W2) For each vertex \( v \), the total weights of all edges incident to \( v \) is equal to 1.
(W3) For each prismatic circuit \( \gamma \subset E \), the total weights of all edges in \( \gamma \) is strictly greater than 1.
Note that the condition (W2) is equivalent that the sum of the weights of edges bounding a face in the dual graph $G^*$ is equal to 1.

Based on this theorem, Dillencourt and Smith [9] then presented several sufficient graph-theoretic conditions for inscribability. As an example, it implies that every 4-connected polyhedral graph is of inscribable type. In addition, for any given polyhedral graph $G$, Hodgson et al. [13] indicate that there always exists a polynomial time algorithm (in the number of vertices) to decide whether it is inscribable. These consequences are really elegant. But the sphere case seems a little too special in view of generality. And few results are known in regard of the problem of “$K$-inscribability” (i.e. find a convex polyhedron with a given combinatoric having its vertices on a given convex surface $K$). The result concerning polyhedra inscribed in a quadric by Danciger et al. [8] is a significant breakthrough in this direction.

Furthermore, a “sphere” in the real physical world often doesn’t mean a standard sphere in mathematic sense. It then seems significant to go a step further to consider the stability problem of inscribable graphs. Namely, given any convex surface $S \subset \mathbb{R}^3$ sufficiently close to the unit sphere $S^2$, for an inscribable graph $G$, is there always a polyhedron $P_{G,S}$ inscribed in $S$ with skeleton combinatorially equivalent to $G$?

In what follows, to formulate the above question as a mathematic one, let’s introduce some notions which will depict the exact meaning of “sufficiently close”.

Suppose that $S_1 : \mathcal{C} \overset{f_1}{\rightarrow} \mathbb{R}^3$, $S_2 : \mathcal{C} \overset{f_2}{\rightarrow} \mathbb{R}^3$ are two $C^k$ embeddings of the Riemann sphere in the 3-dimensional Euclidean space $\mathbb{R}^3$. Given $\epsilon > 0$, we say $S_1$, $S_2$ are $\epsilon$-$C^k$-close to each other, if the $C^k$-norm of every coordinate component of $f_1 - f_2$ is less than $\epsilon$. For example, if two embedding of the Riemann sphere are $C^3$-close to each other, it follows from the elementary surface theory that the images of $S_1$, $S_2$ and their curvatures will be close to each other (see [4]). Particularly, if $S : \mathcal{C} \overset{f}{\rightarrow} \mathbb{R}^3$ is an embedding sphere which is $\epsilon$-$C^3$-close to the unit sphere $S^2$ for some sufficiently small $\epsilon > 0$, then the surface $S$ is both strictly convex and sufficiently round.

For any given inscribable graph $G$, suppose that there exists an $\epsilon > 0$ such that: for any surface $S(\epsilon)$ which is $\epsilon$-$C^k$-close to the unit sphere $S^2$, there is always a polyhedron $P_{G,S(\epsilon)}$ inscribed in $S(\epsilon)$ with skeleton combinatorially equivalent to $G$. Then we say $G$ is $C^k$-stable. Recalling Rivin’s result (Theorem 1.1), the problem on how to characterize an inscribable graph is equivalent to solve a system of linear inequalities. But this theorem gives no information about polyhedra inscribed in convex surfaces aside from the sphere. The stability problem of inscribable graphs may not be a trivial task.

Now let $P = P(\mathcal{V}, \mathcal{E}, \mathcal{F}) \subset \mathbb{R}^3$ be any given convex polyhedron. For every vertex $v \in \mathcal{V}$, we cut a small pyramid from $P$ by a plane which is sufficiently near to $v$ and transversal to every edge $e \in \mathcal{E}$ emanating from $v$. Then we obtain a new polyhedron $P_\diamond$, called the truncated polyhedron of $P$. Denote by $G(P_\diamond)$ the skeleton of $P_\diamond$. Let $G(P_\diamond) = (\mathcal{V}_\diamond, \mathcal{E}_\diamond, \mathcal{F}_\diamond)$. It is not hard to see

$$|\mathcal{V}_\diamond| = 2|\mathcal{E}|, |\mathcal{E}_\diamond| = 3|\mathcal{E}|, |\mathcal{F}_\diamond| = |\mathcal{V}| + |\mathcal{F}|.$$

For $G(P_\diamond) = (\mathcal{V}_\diamond, \mathcal{E}_\diamond, \mathcal{F}_\diamond)$, we call $e_\diamond \in \mathcal{E}_\diamond$ an ordinary edge if $e_\diamond$ actually corresponds to an edge $e \in \mathcal{E}$ in the polyhedron $P$. Other edges of $\mathcal{E}_\diamond \setminus \mathcal{E}$ are called special edges. Similarly, we can define the ordinary faces and the special faces of $\mathcal{F}_\diamond$. Obviously, each special face of $\mathcal{F}_\diamond$ corresponds to a vertex of $\mathcal{V}$.

**Remark 1.2** In the above process, the reason why we need to chose the cutting planes “sufficiently” near to the vertexes is to make the new facets (special faces) of $P_\diamond$ don’t intersect each other.
In this paper we shall prove

**Theorem 1.3** Let \( P, P_\Diamond \) and \( G(P_\Diamond) \) be as above. Assume that the degree \( d(v) \) of each vertex \( v \in V \) is odd. Then the graph \( G(P_\Diamond) \) is inscribable and \( C^1 \)-stable.

In addition, for a polyhedral graph \( G(P) = (V, E, \mathcal{F}) \), we can construct a new graph \( G_+(P) \). More precisely, for every edge \( e \in E \), we associate it with a vertex \( v_e \). Whenever two different edges \( e_1, e_2 \in E \) both belong to a common face \( f \in \mathcal{F} \) and meet at a same vertex \( v \in V \), we then connect an edge from \( v_{e_1} \) to \( v_{e_1} \). Thus we obtain a new graph \( G_+(P) \) associated to \( P \), which is called the **rectified graph** of the polyhedron \( P \) (Figs. 2, 3).

For the rectified graph \( G_+(P) = (V_+, E_+, \mathcal{F}_+) \), obviously we have

\[
|V_+| = |E|, \quad |E_+| = 2|E|, \quad |\mathcal{F}_+| = |V| + |\mathcal{F}|.
\]

By means of circle packings, there are other relations between the skeleton \( G(P) \), the dual graph \( G^*(P) \) of \( G(P) \) and the rectified graph \( G_+(P) \). For example, according to the simultaneous circle packing theorem proved by Bobenko and Springborn [7], \( G(P) \) and \( G^*(P) \) can be simultaneously realized by two circle packings \( P \) and \( P^* \) such that the tangent circle pair of \( P \) corresponding to the edge in \( G(P) \) and the tangent circle pair of \( P^* \) corresponding to the dual of this edge in \( G^*(P) \) are always orthogonal to each other at the same point in the unit sphere. Consider the convex hull of all these orthogonal points, we then get a polyhedron inscribed in the sphere with skeleton \( G_+(P) \). Furthermore, we shall prove
Theorem 1.4 Let $P, G_+(P)$ be as above. If $d(v)$ is odd for any vertex $v \in \mathcal{V}$, then $G_+(P)$ is inscribable and $C^3$-stable.

Given an oriented plane $\mathbb{F}$, it can be considered as the boundary of an affine half space $H^+$. For any compact strictly convex surface $K$ with $K \not\subseteq H^+$, the intersection $H^+ \cap K$ is either empty, or a point, or a topological disk. In the last case we call it a $K$-disk, and its boundary (in $K$) a $K$-circle. We recall that a planar graph $G$ is $K$-inscribable if there exists a polyhedron $P_G$ inscribed in $K$ with skeleton combinatorially equivalent to the graph $G$.

In terms of the above conventions, to prove Theorem 1.4 is equivalent to prove that there exists $\epsilon > 0$ such that $G_+(P)$ is $S(\epsilon)$-inscribable provided that the embedded surface $S(\epsilon)$ is $\epsilon$-$C^3$-close to the unit sphere $S^2$. Recall that $G_+(P) = (\mathcal{V}_+, \mathcal{E}_+, \mathcal{F}_+)$ and $G^*(P) = (\mathcal{V}_+, \mathcal{E}_+, \mathcal{F}_+)$. To acquire such a polyhedron, we need to find the vertices set $\mathcal{V}_+$ such that: (1) they correspond to the tangent points of a $S(\epsilon)$-circle packing realizing the graph $G^*(P)$, where $G^*(P)$ is the dual graph of the skeleton of $P$; (2) if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \mathcal{E}$ are incident to a same vertex $v \in \mathcal{V}_+$, then the corresponding points $v_{\varepsilon_1}, v_{\varepsilon_2}, \ldots, v_{\varepsilon_k} \in \mathcal{V}_+$ locate in a same plane.

Hence it’s necessary to prove that the intersection of these two configuration spaces is non-empty. By combining the intersection number theory from differential topology with a homotopy technique, we shall obtain the desired result. Similarly, Theorem 1.3 could be deduced by means of considering the intersection of two kinds of configuration spaces as well. Furthermore, to complete both the proof of Theorem 1.3 and Theorem 1.4, we shall depend on a very key proposition (Proposition 6.4) concerning the transversality of the above configuration spaces.

We now briefly describe how this paper is organized. In the preliminary section we briefly give an introduction to transversality theory and intersection number theory, which will play an important role throughout this paper. In Sect. 2 we study the Teichmüller theory of packings, which characterizes the configuration space of $K$-circle packings. Section 3 gives a description of several configuration spaces, which builds the basic framework of our proofs. Section 4 is devoted to the proof of Theorem 1.4. The next section provides a geometric insight into the tangent space of another configuration. With the help of this method, we demonstrate the consequence of transversality which leads to a proof of Theorem 1.3. Furthermore, we complete some details on the computation of intersection number used in Sect. 4.

Notational conventions

Through this paper, for any given set $A$ we use the notation $|A|$ to denote the cardinality of $A$.

2 Preliminaries

In this section, we will introduce several definitions and notations from differential topology, especially transversality and intersection number. Please refer to [10, 12] for background on these notions.

First of all, assume that $M, N$ are two oriented smooth manifolds, and assume that $S \subset N$ is a submanifold.

Definition 2.1 Suppose that $f : M \to N$ is a $C^1$ map. Given $A \subset M$, we say $f$ is transverse to $S$ along $A$, denoted by $f \pitchfork_A S$, if

$$\text{Im}(df_x) + T_{f(x)}S = T_{f(x)}N,$$
whenever \( x \in A \cap f^{-1}(S) \). That is, the tangent space to \( N \) at \( f(x) \) is spanned by the tangent space to \( S \) at \( f(x) \) and the image of the tangent space to \( M \) at \( x \). When \( A = M \), we simply denote \( f \cap S \).

Let \( S \subset N \) be a closed submanifold such that \( \dim M + \dim S = \dim N \). Suppose that \( \Lambda \subset M \) is an open subset with compact closure \( \bar{\Lambda} \subset M \). Given a continuous map \( f : M \to N \) such that \( f(\partial \Lambda) \cap S = \emptyset \), where \( \partial \Lambda = \bar{\Lambda} \setminus \Lambda \), we will define a topological invariant \( I(f, \Lambda, S) \), called the intersection number between \( f \) and \( S \) in \( \Lambda \).

If \( f \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N) \) such that \( f \cap S \), then \( \Lambda \cap f^{-1}(S) \) consists of finite regular points. Let \( x \in \Lambda \cap f^{-1}(S) \). Put \( y = f(x) \). The \( \text{sgn}(f, S)_x \) at \( x \) is \( +1 \), if the isomorphism \( T_x f : M_x \to N_y \) preserves orientation. If \( T_x f \) reverses orientation then \( x \) has negative type, we write \( \text{sgn}(f, S)_x = -1 \).

**Definition 2.2** If \( \Lambda \cap f^{-1}(S) = \{x_1, x_2, \ldots, x_m \} \), then we define the intersection number between \( f \) and \( S \) in \( \Lambda \) to be

\[
I(f, \Lambda, S) := \sum_{j=1}^{m} \text{sgn}(f, S)_{x_j}.
\]

The proof of the following proposition is in the same style as that of the homotopy invariance of Brouwer degree. Please refer to [10, 12] or Milnor’s book [17] for its complete proof.

**Proposition 2.3** Suppose that \( f_i \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N) \), \( f_i \cap S \) and \( f_i(\partial \Lambda) \cap S = \emptyset \), \( i = 0, 1 \). If there exists a homotopy

\[
H \in C^0(I \times \bar{\Lambda}, N)
\]

such that \( H(0, \cdot) = f_0(\cdot), \ H(1, \cdot) = f_1(\cdot), \) and \( H(I \times \partial \Lambda) \cap S = \emptyset \), then

\[
I(f_0, \Lambda, S) = I(f_1, \Lambda, S).
\]

The next lemma, which helps us to define the intersection number for general mappings, is a consequence of Sard’s theorem [10, 12].

**Lemma 2.4** For any \( f \in C^0(\bar{\Lambda}, N) \) with \( f(\partial \Lambda) \cap S = \emptyset \), there exists \( g \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N) \) and \( H \in C^0(I \times \bar{\Lambda}, N) \) such that

1. \( g \cap S \);
2. \( H(0, \cdot) = f(\cdot), \ H(1, \cdot) = g(\cdot) \);
3. \( H(I \times \partial \Lambda) \cap S = \emptyset \).

We are now ready to define the intersection numbers of general continuous mappings. Suppose that \( f \in C^0(\bar{\Lambda}, N) \) with \( f(\partial \Lambda) \cap S = \emptyset \).

**Definition 2.5** We can define the intersection number

\[
I(f, \Lambda, S) = I(g, \Lambda, S).
\]

where \( g \) is given as Lemma 2.4.

From Proposition 2.3, it follows that the intersection number \( I(f, \Lambda, S) \) is well-defined. Furthermore, we have the following result on the homotopy invariance property of the intersection numbers.

For \( i = 0, 1 \), suppose that \( f_i \in C^0(\bar{\Lambda}, N) \) such that \( f_i(\partial \Lambda) \cap S = \emptyset \).
Theorem 2.6 If there exists $H \in C^0(I \times \hat{\Lambda}, N)$ such that

1. $H(0, \cdot) = f_0(\cdot)$, $H(1, \cdot) = f_1(\cdot)$,
2. $H(I \times \partial \Lambda) \cap S = \emptyset$,

then we have $I(f_0, \Lambda, S) = I(f_1, \Lambda, S)$.

In particular, it immediately follows from the definition that:

Theorem 2.7 If $I(f, \Lambda, S) \neq 0$, then we have $\Lambda \cap f^{-1}(S) \neq \emptyset$.

3 Teichmüller theory of packings

Given a compact strictly convex surface $K$, in this section we shall introduce the Teichmüller theory of $K$-circle packings with the same contact graph.

Roughly speaking, a $K$-circle (or $K$-disk) packing $P$ is a configuration of $K$-circles $\{C_v : v \in V\}$ (or disks $\{D_v : v \in V\}$) with specified patterns of tangency. The contact graph (or nerve) of $P$ is a graph $G_P$, whose vertex set is $V$ and an edge appears if and only if the corresponding $K$-circles (or $K$-disks) touch.

Given a planar graph $G = G(V, E)$, let’s fix a vertex $v_0 \in V$ and three ordered edges $e_1, e_2, e_3 \in E$ emanating from $v_0$. We call the 4-tuple $\theta = \{v_0, e_1, e_2, e_3\}$ a combinatorial frame associated to the graph $G$. Suppose $P = \{C_v\}$ is a $K$-circle packing with the contact graph $G_P = G$. Denoting by $p_1, p_2, p_3$ the three tangent points corresponding to the edges $e_1, e_2, e_3$, we call $P$ a normalized $K$-circle packing with mark $\mathcal{M} = \{\theta, p_1, p_2, p_3\}$.

For each component strictly convex surface $K$, without loss of generality, we now assume it lies below the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ and is tangent to this plane at the point $N = (0, 0, 1)$. The point $N$ is regarded as the “North Pole” of $K$. Let $h : K \to \hat{C} = \mathbb{C} \cup \{\infty\}$ denote the stereographic projections with $h(N) = \infty$. Since $h$ can be extended to a diffeomorphism between $K$ and $\hat{C}$, we then endow $\partial K$ with a complex structure by pulling back the standard complex structure of $\hat{C}$. Hence, up to conformal equivalence, we can identify $K$ with the Riemann sphere $\hat{\mathbb{C}}$.

Given a convex polyhedron $P = P(V, E, \mathcal{F}) \subset \mathbb{R}^3$, we recall that $G^*(P)$ is the dual graph of the skeleton of $P$. Denote $G^*(P) = (V, E)$. Let us fix a disk packing $P_0 = \{D_0(v)\}_{v \in V}$ on the unit sphere $S^2(\subseteq \hat{\mathbb{C}})$ with the contact graph $G^*(P)$. Denote $\hat{\mathbb{C}} \setminus \cup_{v \in V} D_0(v) = \{I_1, I_2, \ldots, I_m\}$. For each component $I \in \{I_1, I_2, \ldots, I_m\}$, we call it an open interstice. Evidently, $I$ is a topological polygon. The region $I$ has only finitely many boundary components. And each boundary component is a piecewise smooth curve formed by finitely many circular arcs or circles. Each (maximal) circular arc or circle on the boundary $\partial I$ belongs to a unique circle in the disk packing $P_0$, and therefore is marked by an element of $V$. The region $I$, together with a marking of the circular arcs or circles on its boundary by elements of $V$ is called an interstice of $P_0$.

For each interstice $I$ of $P_0$, we can define a conformal polygon as pairs $h : I \to \hat{\mathbb{C}}$, where $h$ is a quasiconformal embedding. For details on quasiconformal mappings, please refer to Ahlfors’ book [1]. The conformal polygons are considered as analogs of the conformal quadrangal.

Denote $\partial I = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, where $\{\gamma_j\}_{1 \leq j \leq n}$ is a marking of the circular arcs or circles on its boundary. We say two such quasiconformal embeddings $h_1, h_2 : I \to \hat{\mathbb{C}}$ are Teichmüller equivalent, if the composition mapping $h_2 \circ (h_1)^{-1} : h_1(I) \to h_2(I)$ is isotopic to a conformal homeomorphism $f : h_1(I) \to h_2(I)$ such that for each side $\gamma_j \subset \partial I$, $f$ maps $h_1(\gamma_j)$ onto $h_2(\gamma_j)$. 
Definition 3.1 The Teichmüller space of $I$, denoted by $T_I$, is the space of all equivalence classes of quasiconformal embeddings $h : I \rightarrow \hat{\mathbb{C}}$.

Remark 3.2 If the interstice $I$ is $k$-sided, it follows from the classical Teichmüller theory that $T_I$ is diffeomorphic to the Euclidean space $\mathbb{R}^{k-3}$. Moreover, $T_I$ could be endowed with the Teichmüller metric, which reflects the distortion of conformal structures. See e.g [15].

Denote $T_{G^+(P)} = \prod_{i=1}^m T_{I_i}$, where $\{I_1, I_2, \ldots, I_m\}$ are all interstices of the circle packing $P_0$. Due to Remark 3.2, we easily verify that

$$T_{G^+(P)} \cong \mathbb{R}^{2|\mathcal{E}| - 3|\mathcal{V}|}$$

Moreover, as far as the $K$-circle packings concerned, the authors [14] have established the following result, which will be used in this paper as well. It’s proof is a combination of the methods due to Schramm [21] and Rodin and Sullivan [19].

Lemma 3.3 Let $K, P, G^+(P)$ and $T_{G^+(P)}$ be as above. Suppose $p_1, p_2, p_3$ are three distinct points in $K$. For any

$$[\tau] = ([\tau_1], [\tau_2], \ldots, [\tau_m]) \in T_{G^+(P)},$$

there exists a unique $K$-circle packing $P^*_K([\tau])$ realizing the dual graph $G^+(P)$ with mark $\{\tau, p_1, p_2, p_3\}$. Moreover, it’s interstice corresponding to $I_i$ is endowed with the given conformal structure $[\tau_i]$, $1 \leq i \leq m$.

4 Configuration spaces

Recall that $P = P(\mathcal{V}, \mathcal{E}, \mathcal{F}) \subset \mathbb{R}^3$ is the given convex polyhedron and $P_\circ$ is the corresponding truncated polyhedron. To prove the main theorems, in this section we will construct, step by step, two configurations spaces $Z_{oc}, Z(P_\circ)$ associated to $P, P_\circ$ respectively.

In view of analytic geometry, we know that each oriented $\mathbb{F} \subset \mathbb{R}^3$ can be defined as

$$\mathbb{F} = \{(x, y, u) : Ax + By + Cu + D = 0 \} \quad (A^2 + B^2 + C^2 = 1).$$

Hence each $\mathbb{F}$ is uniquely determined by the exterior unit normal vector and the intercept. In other words, it could be depicted by a point in $S^2 \times \mathbb{R}$.

Let $Z_{\mathcal{F}}$ denote the space $(S^2 \times \mathbb{R})^{\mathcal{F}}$. Namely, a point $z_{\mathcal{F}} \in Z_{\mathcal{F}}$ gives a choice of an oriented plane for each $f \in \mathcal{F}$. $Z_{\mathcal{F}}$ will be called the $\mathcal{F}$-configuration space, and a point $z_{\mathcal{F}} \in Z_{\mathcal{F}}$ will be called a $\mathcal{F}$-configuration. For each $\mathcal{F}$-configuration $z_{\mathcal{F}} \in Z_{\mathcal{F}}$, we denote by $z_{\mathcal{F}}(f)$ the oriented plane corresponding to the face $f \in \mathcal{F}$.

For any $e \in \mathcal{E}$, there are $f_1, f_2 \in \mathcal{F}$ such that $f_1 \cap f_2 = e$. Let $Z_{\mathcal{F}e} \subset Z_{\mathcal{F}}$ be the set of $\mathcal{F}$-configurations $z_{\mathcal{F}}$ such that $z_{\mathcal{F}}(f_1)$ is parallel to $z_{\mathcal{F}}(f_2)$. Moreover, let $Z_{\mathcal{F}R} \subset Z_{\mathcal{F}}$ be the set of $\mathcal{F}$-configurations $z_{\mathcal{F}}$ such that the intersection $z_{\mathcal{F}}(f_{i_1}) \cap z_{\mathcal{F}}(f_{i_2}) \cap z_{\mathcal{F}}(f_{i_3})$ contains more than one points for at least one triple $\{i_1, i_2, i_3\} \subset \{1, 2, \ldots, |\mathcal{F}|\}$. Evidently, both $Z_{\mathcal{F}e}$ and $Z_{\mathcal{F}R}$ are closed in $Z_{\mathcal{F}}$, which implies that

$$Z_{\mathcal{F}0} = Z_{\mathcal{F}} \setminus (\cup_{e \in \mathcal{E}} Z_{\mathcal{F}e} \cup Z_{\mathcal{F}R})$$

is open in $Z_{\mathcal{F}}$. Hence it’s a manifold with the same dimension as $Z_{\mathcal{F}}$.
Let $Z$ denote the space $Z_{\mathcal{F} O} \times \mathbb{R}^{|\mathcal{E}|} \times \mathbb{R}^{|\mathcal{E}|}$. Namely, a point $z \in Z$ gives a choice of an oriented plane for each $f \in \mathcal{F}$, and a choice of two points corresponding to the vertices $v_1, v_2 \in V$ in the line $z_{\mathcal{F}}(f_1) \cap z_{\mathcal{F}}(f_2)$, where $f_1 \cap f_2 = v_1 v_2 = e \in \mathcal{E}$. Similarly, we call $Z$ the configuration space. In addition, a point $z \in Z$ will be called a configuration.

For a configuration $z \in Z$, here and hereafter we simply denote by $z(f)$ the oriented plane corresponding to the face $f \in \mathcal{F}$. Moreover, if $f_1 \cap f_2 = e \in \mathcal{E}$, then we denote by $z(ve)$ the point in $z(f_1) \cap z(f_2)$ corresponding to the vertex $v \in V$.

Now let $Z_{oc} \subset Z$ denote the set of configurations $z$ such that $z(ve_1), z(ve_2), z(ve_3)$ are not collinear whenever $e_1, e_2, e_3$ are three distinct edges incident to the same vertex $v \in V$. Obviously, $Z_{oc}$ is open in $Z$. Hence, $Z_{oc}$ is a manifold with the same dimension as $Z$. More precisely,

$$
dim Z_{oc} = \dim Z = 3|\mathcal{F}| + 2|\mathcal{E}|. \tag{2}$$

For any $v \in V$, suppose that $e_1, e_2, \ldots, e_{d(v)}$ are all edges of $P$ incident to the vertex $v$, where $d(v)$ is the degree of $v$. Denote by $Z_v \subset Z_{oc}$ the set of configurations $z$ such that $z(ve_1), z(ve_2), \ldots, z(ve_{d(v)})$ belong to the same plane. Define

$$Z(P_v) = \cap_{v \in V} Z_v.$$ 

In some cases, a configuration $z \in Z(P_v)$ would correspond to a polyhedron in $\mathbb{R}^3$ combinatorially equivalent to $P_v$. However, it’s worth pointing out that there do exist configurations corresponding to other intricate geometric patterns as well. For instance, when a configuration $z \in Z(P_v)$ satisfies $z(ve) = z(v'e)$ for every $e \in \mathcal{E}$, then it will correspond to a polyhedra combinatorially equivalent to $\mathbb{G}_+(P)$. Aside from these complexity, we have:

**Lemma 4.1** $Z(P_v)$ is a closed submanifold of $Z_{oc}$ with dimension $\dim Z(P_v) = 3|\mathcal{E}| + 6$.

**Proof** As above, let $e_1, e_2, \ldots, e_{d(v)}$ be the edges of the polyhedron $P$ emanating from $v$. For each $i = 1, 2, \ldots, d(v)$, denote $z(ve_i) = (x_i, y_i, u_i)$.

Consider the matrix

$$\begin{pmatrix}
  x_1 & y_1 & u_1 & 1 \\
  x_2 & y_2 & u_2 & 1 \\
  x_3 & y_3 & u_3 & 1 \\
  \vdots & \vdots & \vdots & \vdots \\
  x_{d(v)} & y_{d(v)} & u_{d(v)} & 1
\end{pmatrix}$$

Then $z(ve_1), z(ve_2), \ldots, z(ve_{d(v)})$ belong to the same plane if and only if the rank of the above matrix is less than 4. Equivalently, the determinant

$$R(z(ve_1), z(ve_2), z(ve_3), z(ve_4)) = \begin{pmatrix}
  x_{i_1} & y_{i_1} & u_{i_1} & 1 \\
  x_{i_2} & y_{i_2} & u_{i_2} & 1 \\
  x_{i_3} & y_{i_3} & u_{i_3} & 1 \\
  x_{i_4} & y_{i_4} & u_{i_4} & 1
\end{pmatrix} = 0.$$ 

for each subset $\{i_1, i_2, i_3, i_4\} \subset \{1, 2, \ldots, d(v)\}$.

In view of the definition of the space $Z_{oc}$, it follows that $z(ve_{j_1}), z(ve_{j_2}), z(ve_{j_3})$ aren’t collinear for any three different subscripts $\{j_1, j_2, j_3\} \subset \{1, 2, \ldots, d(v)\}$. That implies that 0 is a regular value of the smooth function $R(z(ve_{i_1}), z(ve_{i_2}), z(ve_{i_3}), z(ve_{i_4}))$. Owing to the regular value theorem [12], $Z(P_v)$ is then a closed submanifold of $Z_{oc}$. Moreover, we have

$$\dim Z(P_v) = 3|\mathcal{F}| + 2|\mathcal{E}| - \left(\sum_{v \in V} d(v) - 3\right) = 3|\mathcal{F}| + 2|\mathcal{E}| - (2|\mathcal{E}| - 3|\mathcal{V}|) = 3|\mathcal{E}| + 6, \tag{3}$$

\[\square\]
5 Homotopy, intersection numbers and proof of Theorem 1.4

Let $K$ be a given compact strictly convex surface. Choose a combinatorial frame $\mathcal{F}$ for $G^*(P)$ and three different points $p_1, p_2, p_3$ in $K$. For each $[\tau] \in T_{G^*(P)}$, from Lemma 3.3, it follows that there is a unique normalized $K$-circle packing $\mathcal{P}_K([\tau])$ realizing the graph $G^*(P)$ with the mark $\mathcal{M} = \{ \mathcal{F}, p_1, p_2, p_3 \}$.

Note that $G^*(P) = (V, E, F)$ is the dual graph of the polyhedral graph $G(P) = (V, E, \mathcal{F})$. For any $f \in \mathcal{F}$, then $f^* \in V$. For the $K$-circle packing $\mathcal{P}_K([\tau])$, denote by $\mathbb{F}(f^*)$ the oriented plane corresponding to the vertex $f^* \in V$.

Moreover, by Eqs. (1)–(3), an elementary calculation gives

$$\text{dim} T_{G^*(P)} + \text{dim} Z(P_0) = 3|\mathcal{F}| + 2|\mathcal{E}| = \text{dim} Z_{oc}$$

(4)

It reminds us of the intersection number theory. In order to apply this tool, it’s necessary to find a proper compact set $\Lambda \subset T_{G^*(P)}$ such that $f_{K, \mathcal{M}}(\partial \Lambda) \cap Z(P_0) = \emptyset$.

Given $\epsilon > 0$, we denote by $B(S^2, \epsilon)$ the set of compact convex surfaces which are $\epsilon$-$C^3$-close to the unit sphere $S^2$.

**Lemma 5.1** Assume that $d(v)$ is odd for every $v \in V$. Then there exists $\epsilon > 0$ such that: for any $K \in B(S^2, \epsilon)$, there is a compact set $\Lambda \subset T_{G^*(P)}$ such that $f_{K, \mathcal{M}}(\partial \Lambda) \cap Z(P_0) = \emptyset$.

**Proof** Recall that the space $T_{G^*(P)}$ could be endowed with the Teichmüller metric, which reflects the distortion of the conformal structures of the interstices (see [15]). Our aim is to prove that the conformal structures of the interstices associated with those $K$-inscribed polyhedra couldn’t be terribly distorted.

When a compact convex surface $K$ is sufficiently $\epsilon$-$C^3$-close to the unit sphere $S^2$, it is not hard to see that the conformal structures of the interstices associated with the $K$-inscribed polyhedron are not terribly distorted if and only if the conformal structures in $S^2$-case are not terribly distorted. In other words, the above lemma will be deduced if we could prove the existence of $\Lambda$ such that $f_{K, \mathcal{M}}(\partial \Lambda) \cap Z(P_0) = \emptyset$ for $K = S^2$.

To simplify notations, let $H_0 = f_{S^2, \mathcal{M}}$. Suppose a configuration $z \in H_0(T_{G^*(P)}) \cap Z(P_0)$. According to the definition, $z$ satisfies that:

- There exists a circle packing $\mathcal{P}^* = \{ C_{f^*} : f \in \mathcal{F} \}$ realizing $G^*(P)$ with each circle $C_{f^*}$ belongs to $z(f)$.
- For each $v \in V$, let $e_1, e_2, \ldots, e_{d(v)} \in \mathcal{E}$ be all the edges of $P$ emanating from $v$. Let $p_{e_1^*}, p_{e_2^*}, \ldots, p_{e_{d(v)}^*}$ be the tangent points of the packing $\mathcal{P}^*$ corresponding to $e_1, e_2, \ldots, e_{d(v)} \in \mathcal{E}$. Then $p_{e_1^*}, p_{e_2^*}, \ldots, p_{e_{d(v)}^*}$ are contained in a common plane, which implies that they are contained in a common circle $C_v$.

For $i \in \{1, 2, \ldots, d(v)\}$, let $\theta_i$ be the dihedral angle between $C_v$ and $C_{f_i^*}$. Since $d(v)$ is odd, a simple computation shows that (Fig. 4)
Let $P = \{C_v : v \in V\}$. Then $P$ is a circle packing realizing $G(P)$. Moreover, Eq. (5) implies that $P$ together with $P^*$ satisfies the simultaneous circle packing condition [7]. As mentioned in the introduction section, consider the convex hull of all the tangents points, we then obtain a inscribed polyhedron with skeleton combinatorial equivalent to $G_+(P)$. Furthermore, it is not hard to see that the lines $l_{e_1^*}, l_{e_2^*}, \ldots, l_{e_{d(v)}^*}$ intersect at the center of the circle $C_v$, where each $l_{e_i^*}$ is the common tangent line of the circles $\{C_{f_i^*}, C_{f_{i+1}^*}\}$ in the packing $P^*$.

Now we assume, by contradiction, that there is not such a compact set $\Lambda_1$. That means there exist configuration sequence $z_n \in H_0(T_{G^*(P)}) \cap Z(P_0)$ such that the conformal structure of one of the interstice of the corresponding packing sequence $P_n^*$ will be terribly distorted. To be specific, for $P_n^*$, one of the the following two cases occurs:

- As $n \to \infty$, there exists $f^* \in V$, such that the corresponding circles $\{C_n, f^*\}$ in the packings $P_n^*$ tends to a point;
- For some $v^*$, as $n \to \infty$, the distance of two non-adjacent arcs of the interstice $l_{v^*,n}$ of the packings $P_n^*$ tends to zero.

In the first case, suppose that there exists at least one circle tending to a point. Note that any three circles with disjoint interiors can not meet at a common point. Therefore, all circles in the packing sequence $P_n^*$ will degenerate to points, except for at most two circles. It contradicts to our normalization conditions. We thus rule out the first possibility.

Now we turn to the second case. Due to the above analysis, we shall have simultaneous circle packing sequence $P_n$ for $P_n^*$ and inscribed polyhedron $P_n$. Hence the tangent lines of the packings $P_n^*$ will separate the non-adjacent arcs. On the other hand, we have known that
the sizes of all circles in $\mathcal{P}_n^*$ have positive infimum. These facts tell us that the distance of such non-adjacent arcs can’t tend to zero, which rules out the second possibility. 

**Remark 5.2** It’s worth pointing out that Eq. (5) wouldn’t hold any more if $d(v)$ is even for some $v \in \mathcal{V}$. In fact, this seems to be the main obstruction on why we couldn’t extend Theorems 1.3 and 1.4 to more general cases.

Assume that $K \in B(S^2, \epsilon)$. If we could prove $I(f_{K, M}, \Lambda, Z(P_\circ)) \neq 0$, then Theorem 2.7 implies that $f_{K, M}^{-1}(Z(P_\circ)) \cap \Lambda \neq \emptyset$, which proves Theorem 1.4. Recalling Theorem 2.6, to determine the intersection numbers, let’s use a homotopy method.

Note that $K$ is a given compact strictly convex surface. Without loss of generality, we assume that its diameter is larger than 1. Furthermore, assume that the unit sphere $S^2$ is internally tangent to $K$ at the point $N = (0, 0, 1)$. Then $N = (0, 0, 1)$ could be considered as the common “North Pole” of $S^2$ and $K$. Let $h_0, h_1$ be the “stereographic projections” for $S^2, K$ respectively. Define a one parameter family of closed surfaces by 

$$\{s \cdot h_1^{-1}(z) + (1 - s) \cdot h_0^{-1}(z) : z \in \hat{C}\}.$$ 

For each $s \in [0, 1]$, the above set is a compact strictly convex surface in $\mathbb{R}^3$. Denote it by $K_s$. Then $\{K_s\}_{0 \leq s \leq 1}$ is a family of compact strictly convex surface joining $S^2$ and $K$. Similarly, we can endow the smooth convex surface $K_s$ with the complex structure $\hat{C}$ for each $s \in [0, 1]$ by the “stereographic projection”. Moreover, with the help of Lemma 3.3, we could construct a mapping 

$$H_s = f_{K_s, M} : T_{G^*(P)} \to Z_{oc},$$

which is a homotopy from $H_0$ to $f_{K, M}$. Furthermore, if $K \in B(S^2, \epsilon)$, from Lemma 5.1 it follows that there exists $\Lambda \subset T_{G^*(P)}$ such that $H_s(\partial \Lambda) \cap Z(P_\circ) = \emptyset$ for all $s \in [0, 1]$. Furthermore, we conclude that:

**Theorem 5.3** Suppose that $d(v)$ is odd for each $v \in \mathcal{V}$. Given any $K \in B(S^2, \epsilon)$, then $I(f_{K, M}, \Lambda, Z(P_\circ)) = 1$.

**Proof** Due to Theorem 2.6, it remains to calculate $I(H_0, \Lambda, Z(P_\circ))$. From the following Proposition 5.4, we have $I(H_0, \Lambda, Z(P_\circ)) = 1$. It thus completes the proof. 

**Proposition 5.4** Suppose that $d(v)$ is odd for any $v \in \mathcal{V}$. Then $I(H_0, \Lambda, Z(P_\circ)) = 1$.

The proof of this result is postponed to the next section.

Up to now, we have developed all the necessary results for our purpose. We are ready to prove one of the main results of this paper.

**Proof of Theorem 1.4** As pointed out, it is an immediate consequence of Theorem 2.7 and Theorem 5.3, which completes the proof. 

**6 Transversality and proof of theorem 1.3**

It remains to prove Theorem 1.3 and Proposition 5.4. To reach this goal, we shall make use of transversality theory and a homotopy method.

Let’s employ a consequence concerning the Teichmüller theory of circle patterns. Recall that $G^*(P)$ is the dual graph of the skeleton of $P$. In [11], He-Liu have proved the following theorem:
Theorem 6.1 Suppose that a weight function \( w : E \to [0, \pi/2] \) satisfies the following two conditions:

(i) If three distinct edges \( e_i^*, e_j^*, e_k^* \) form a simple closed loop in \( G^*(P) \), then \( w(e_i^*) + w(e_j^*) + w(e_k^*) < \pi \).

(ii) If four distinct edges \( e_i^*, e_j^*, e_k^*, e_l^* \) form a simple closed loop in \( G^*(P) \), then \( w(e_i^*) + w(e_j^*) + w(e_k^*) + w(e_l^*) < 2\pi \).

For any

\[
[\tau] \in \{[\tau_1], [\tau_2], \ldots, [\tau_n]\} \in T_G^*(P),
\]

there exists a unique normalized circle pattern \( \mathcal{P}^*_w([\tau]) \) on the unit sphere with contact graph \( G^*(P) \) and dihedral angle \( w(e^*) : e^* \in E \). Moreover, the corresponding interstices of \( \mathcal{P}^*_w([\tau]) \) are endowed with the given complex structure \([\tau_i], 1 \leq i \leq n\).

Let \( W \) be the set of weight functions that satisfy the above conditions (i) and (ii). Theorem 6.1 implies that we can define, for each \( w \in W \), a mapping \( f_{w, W} : T_G^*(P) \to Z_{oc} \) via associating every \([\tau] \in T_G^*(P)\) with the unique normalize circle pattern realizing the complex structure \([\tau]\). More precisely, we define \( f_{w, W}(\tau) \) as \( z \), where \( z \) is the unique configuration such that: (1) \( z(f) \) (view it as an oriented plane) contains the circle \( C_{f^*}; \) (2) \( z(v_1 e) \), \( z(v_2 e) \) are the two intersection points of \( C_{f^*} \) corresponding the vertices \( v_1, v_2 \in \mathcal{V} \), where \( f_1 \cap f_2 = e \).

Denoting \( w_0 = (0, 0, \ldots, 0) \) and \( w_s = sw + (1 - s)w_0 \), \( s \in [0, 1] \), then \( f_{w_s, W} \) is a homotopy from \( f_{w_0, W} \) to \( f_{w, W} \). Furthermore, suppose that we have chosen \( w \in W \) sufficiently close to \( w_0 \). By using a similar argument as in Lemma 5.1, we deduce that there exists a compact subset \( \Lambda \subset T_G^*(P) \) such that \( f_{w, W}(\partial \Lambda) \cap Z(P_\o) = \emptyset \) for all \( s \in [0, 1] \).

In order to calculate \( I(f_{w, W}, \Lambda, Z(P_\o)) \), it seems necessary to investigate the transversality between \( f_{w, W} \) and \( Z(P_\o) \). We thus need the following Andreev’s theorem [2, 3, 20], which provide us a geometric insight into the tangent space of \( Z(P_\o) \). Denote by \( E_\o \) the edges set of \( P_\o \). Then we have

Theorem 6.2 Let \( P_\o \) be a trivalent polyhedron in \( \mathbb{R}^3 \) with a weight function \( w_\o : E_\o \to (0, \pi/2] \) attached to its edge set. There is a compact hyperbolic polyhedron \( Q_\o \) combinatorially equivalent to \( P_\o \) with the dihedral angle \( \theta(e_\o) \) equal to \( w(e_\o) \) if and only if the following conditions hold:

(1) If three distinct edges \( e_{ij^*}, e_{oj^*}, e_{ok^*} \) meet at a vertex, then \( w(e_{ij^*}) + w(e_{oj^*}) + w(e_{ok^*}) > \pi \).

(2) If \( \{e_{ij^*}, e_{oj^*}, e_{ok^*}\} \) is a prismatic 3-circuit, then \( w(e_{ij^*}) + w(e_{oj^*}) + w(e_{ok^*}) < \pi \).

(3) If \( \{e_{ij^*}, e_{oj^*}, e_{ok^*}, e_{ol^*}\} \) is a prismatic 4-circuit, then \( w(e_{ij^*}) + w(e_{oj^*}) + w(e_{ok^*}) + w(e_{ol^*}) < 2\pi \).

Furthermore, this polyhedron is unique up to hyperbolic isometries of \( \mathbb{H}^3 \).

Recall the Klein model of the hyperbolic 3-space \( \mathbb{H}^3 \). In that model, \( \mathbb{H}^3 \) is identified with the interior of the unit ball \( \mathbb{B}^3 \subset \mathbb{R}^3 \subset \mathbb{H}^3 \). Furthermore, in terms of Bao-Bonahon [5], a hyperideal polyhedron \( Q_{hi} \) is defined to be a compact convex polyhedron in \( \mathbb{H}^3 \) whose vertices locate outside of the closed unit ball \( \mathbb{B}^3 \) and whose edges all meet \( \mathbb{B}^3 \).

Observe that the truncated polyhedron \( P_\o \) is a trivalent polyhedron if and only if \( G^*(P_\o) \) is a triangular graph, where \( G^*(P_\o) \) is the dual graph of the skeleton of the polyhedron \( P_\o \). Recall the definition of prismatic circuits given in Sect. 1. By either circle pattern theorem [16, 25] or hyperideal polyhedra theorem [5], we have:
Lemma 6.3 Let $P_\circ$ be a trivalent polyhedron in $\mathbb{R}^3$ with a weight function $w_\circ : \mathcal{E}_\circ \rightarrow [0, \pi/2]$ attached to its edges set. There is a compact hyperideal polyhedra $Q_{hi}$ combinatorially equivalent to $P_\circ$ with the dihedral angle of $e_\circ$ equal to $w(e_\circ)$ if and only if the following conditions hold:

1. If three distinct edges $e_{oi}, e_{oj}, e_{ok}$ meet at a vertex, then $w(e_{oi}) + w(e_{oj}) + w(e_{ok}) = \pi$.
2. If $\{e_{oi}, e_{oj}, e_{ok}\}$ is a prismatic 3-circuit, then $w(e_{oi}) + w(e_{oj}) + w(e_{ok}) < \pi$.
3. If $\{e_{oi}, e_{oj}, e_{ok}, e_{ol}\}$ is a prismatic 4-circuit, then $w(e_{oi}) + w(e_{oj}) + w(e_{ok}) + w(e_{ol}) < 2\pi$.

This polyhedron is unique up to an element of $PO(3, 1)$, where the group $PO(3, 1)$ consists of those projective transformations of $\mathbb{RP}^3$ which respect the unit sphere $S^2 \subset \mathbb{R}^3 \subset \mathbb{RP}^3$.

Furthermore, a vertex is located on the unit sphere if and only if the equality holds in Condition 1b for this vertex.

Recall that the skeleton of the truncated polyhedron is denoted by $G(P_\circ) = (V_\circ, \mathcal{E}_\circ, \mathcal{F}_\circ)$. We call $e_\circ \in \mathcal{E}_\circ$ an ordinary edge if $e_\circ$ actually corresponds to an edge $e \in \mathcal{E}$ in the polyhedron $P$. Other edges of $\mathcal{E}_\circ \setminus \mathcal{E}$ are called special edges. Without leading to ambiguity, here and hereafter we shall not distinguish an ordinary edge $e \in \mathcal{E}$ with its corresponding edge in $\mathcal{E}_\circ$.

Similarly, we can define the ordinary faces and the special faces of $\mathcal{F}_\circ$. Obviously, each special face of $\mathcal{F}_\circ$ corresponds to a vertex of $\mathcal{V}$.

By using the above two lemmas, we shall have the following result.

Proposition 6.4 Suppose that $d(v)$ is odd for any $v \in \mathcal{V}$. Then $f_{w, \mathcal{M}} \cap Z(P_\circ)$.

Proof If $z = f_{w, \mathcal{M}} ([\tau]) \in f_{w, \mathcal{M}} (T_{G^+(P)}) \cap Z(P_\circ)$, then the configuration $z$ is combinatorially equivalent to the truncated polyhedron $P_\circ$.

From Theorems 1.1, 6.2 and Lemma 6.3, it follows that there exists an injection

$$\Psi : PO(3, 1) \times U \rightarrow Z(P_\circ),$$

where $U$ is the relatively open convex set of $(0, \pi/2)^{3|\mathcal{E}|}$ defined by the constraint conditions (2) and (3). Moreover, an elementary computation shows that the map $\Psi$ is differentiable. Note that

$$dim PO(3, 1) + dim U = 6 + 3|\mathcal{E}| = dim Z(P_\circ).$$

The injectivity then tells us that there exist $(m_1, w_{o1}) \in PO(3, 1) \times U$ such that $z = \Psi(m_1, w_{o1})$ and the pushing map

$$\Psi^+_z : T_{m_1} PO(3, 1) \times T_{w_{o1}} U \rightarrow T_z Z(P_\circ)$$

is a linear isomorphism.

For any ordinary edge $e$, denote by $v_1 e, v_2 e$ the two end points of the edge $e$ in the truncated polyhedral (corresponding to the vertices $v_1, v_2 \in \mathcal{V}$). Moreover, for $i = 1, 2$, we define the defect curvature $k(v_i e)$ at the vertex $v_i e$ to be

$$k(v_i e) = \pi - (w(e) + w(v_{i,1}) + w(v_{i,2})),$$

where $e, v_{i,1}, v_{i,2}$ are the three distinct edges incident to the vertex $v_i e$ in the truncated polyhedron $P_\circ$. Note that the tangent space $T_{w_{o1}} U$ is expanded by the vectors

$$\left\{ \frac{\partial}{\partial w(e_{o1})}, \frac{\partial}{\partial w(e_{o2})}, \cdots, \frac{\partial}{\partial w(e_{o3|\mathcal{E}|})} \right\}.$$
When \( d(v) \) is odd for each \( v \in \mathcal{V} \), it’s not hard to deduce that this tangent space is equivalent to the \( \mathbb{R} \)-linear space expanded by
\[
\left\{ \frac{\partial}{\partial w(e_1)} \cdot \frac{\partial}{\partial k(v_1 e_1)} \cdot \frac{\partial}{\partial k(v_2 e_1)} \cdot \cdots \cdot \frac{\partial}{\partial w(e_i)} \cdot \frac{\partial}{\partial k(v_1 e_i)} \cdot \frac{\partial}{\partial k(v_2 e_i)} \right\},
\]
where \( \{e_1, e_2, \ldots, e_{|E|}\} \) are all ordinary edges of the polyhedron \( P_o \).

Since \( \Psi_\mathcal{S} : T_mP O(3, 1) \times T_{w_1}U \to T_zZ(P_o) \) is a linear isomorphism, we can identify
\( PO(3, 1) \) with the space of all marks \( \mathcal{M} = \{\emptyset, p_1, p_2, p_3\} \).

Any special face of \( \mathcal{F}_\mathcal{O} \) corresponding to the vertex \( v \in \mathcal{V} \) is a \( d(v) \)-sided polygon. We fix \( d(v) - 3 \) diagonal lines emanating from the same vertex of this special face. Using these diagonal lines serving as bending lines, from Theorem 1.1 it follows that there is a family of ideal hyperbolic convex polyhedra \( z(\theta_1, \theta_2, \ldots, \theta_{|E|-3}|\mathcal{V}|) \), where \( 0 \leq \theta_j \leq \pi, 1 \leq j \leq 2|E| - 3|\mathcal{V}| \), are exterior dihedral angles. Please refer to Part I & II of the book [6]. It implies that the tangent map
\[
df_{w,\mathcal{M}} : T_{[\tau]}\mathcal{T}_{G^*(P)} \to T_zZ_{oc}
\]
is differentiable and injective.

Now we assume that \( t \in df_{w,\mathcal{M}}(T_{[\tau]}\mathcal{T}_{G^*(P)}) \cap T_zZ(P_o) \) is a tangent vector. Then \( t \in T_zZ(P_o) \) corresponds to an infinitesimal change of the dihedral angle \( w(e_j) \) of some ordinary edge \( e_j \), or the defect curvature \( k(v_i e_j) \) of some vertex \( v_i e_j \in \mathcal{V}_o \), or the mark \( \mathcal{M} = \{\emptyset, p_1, p_2, p_3\} \). On the other hand, \( t \in df_{w,\mathcal{M}}(T_{[\tau]}\mathcal{T}_{G^*(P)}) \), the mark and the dihedral angles of the ordinary edges never change. Furthermore, due to the definition, the vertices of \( z(\theta_1, \theta_2, \ldots, \theta_{|E|-3}|\mathcal{V}|) \) keep locating on \( S^2 \). Note that a non-trivial change on defect curvature \( k(v_i e_j) \) means a deviation from \( \partial\mathbb{B}^3 = S^2 \). Hence \( t = 0 \), which thus completes the proof.

**Corollary 6.5** \( I(H_0, \Lambda, Z(P_o)) = I(\mathcal{F}_{w,\mathcal{M}}, \Lambda, Z(P_o)) = 1 \)

**Proof** Owing to the rigidity of ideal hyperbolic polyhedra [18], there exists only one point in the intersection \( f_{w,\mathcal{M}}(T_{G^*(P)}) \cap Z(P_o) \). By Proposition 6.4, we thus show that \( I(\mathcal{F}_{w,\mathcal{M}}, \Lambda, Z(P_o)) = 1 \). In view of Theorem 2.6, the corollary is complete. □

In the end, let’s prove Theorem 1.3, which is another main result of this paper.

**Proof of Theorem 1.3** Without loss of generality, we assume that the unit sphere \( S^2 \) is contained in the interior of the convex surface \( K \). We shall construct a new mapping \( f_{w,K,\mathcal{M}} : G^*(P) \to Z_{oc} \) associated to the data \( w \) and \( K \).

As mentioned above, for any \( [\tau] \in G^*(P) \), from Theorem 6.1 it follows that there exists a unique normalize circle pattern \( P^*_w([\tau]) \) on \( S^2 \) realizes the data \( w \) and \( [\tau] \) with the contact graph \( G^*(P) \). Denote \( P^*_w([\tau]) = \{C^*_j : f \in \mathcal{F}\} \) and let \( \mathbb{F}_f \) be the oriented plane where the circle \( C^*_j \) belongs to. Now let’s define \( f_{w,K,\mathcal{M}}([\tau]) = z \) such that \( z(f) = \mathbb{F}_f \) and \( z(v_1, e), z(v_2, e) \) are exactly the two intersection points in \( \mathbb{F}_{f_1} \cap \mathbb{F}_{f_2} \cap K \) whenever \( f_1 \cap f_2 = e \).

We thus construct the mapping \( f_{w,K,\mathcal{M}} : G^*(P) \to Z_{oc} \). Denote that \( f_{w,\mathcal{S},\mathcal{M}} = f_{w,\mathcal{M}} \). By Proposition 6.4, we have
\[
f_{w,\mathcal{M}} \cap Z(P_o)
\]
Since transversality is stable under a slight \( C^1 \)-perturbation, we prove the statement of this theorem. □
7 Additional comment

Given a compact convex surface $K \subset \mathbb{R}^3$. For any $x \in \mathbb{R}^3$ which locates outside the convex body bounded by $K$, let $O_x$ be the set of points in $K$ which are visible from $x$. That is, the set of $p \in K$ such that the ray $\overrightarrow{xp}$ and $\mathbf{n}_p$ form an angle $\theta_p \in [0, \pi/2]$, where $\mathbf{n}_p$ is the inner normal vector of $K$ at $p$. Clearly, $O_x$ is a topological disk. Therefore, the boundary of $O_x$ is a circle.

For this kind of circles, by using an analogous argument as Lemma 3.3, it is not hard to establish the Teichmüller theory. Moreover, combining this result with the intersection number technique, we could demonstrate that the dual of the two classes of graphs mentioned in Theorem 1.3 and Theorem 1.4 are of stable “circumscribable” type.

Acknowledgments We would like to thank Dr. Peng for the help of Fig. 3.

References

1. Ahlfors, L.V.: Lectures on Quasiconformal Mappings, vol. 10. AMS Bookstore, New York (1966)
2. Andreev, E.M.: On convex polyhedra in Lobachevskii spaces. Mat. Sb. Nov. 123, 445–478 (1970)
3. Andreev, E.M.: On convex polyhedra of finite volume in Lobachevskii space. Mat. Sb. Nov. 12, 255–259 (1970)
4. Banchoff, T., Lovett, S.: Differential Geometry of Curves and Surfaces. A K Peters Ltd, Natick, MA (2010)
5. Bao, X., Bonahon, F.: Hyperideal polyhedra in hyperbolic 3-space. Bull. Soc. Math. Fr. 130, 457–491 (2002)
6. Richard, D. C., David, E., Albert, M.: Fundamentals of Hyperbolic Geometry: Selected Expositions. London Mathematical Society Lecture Note Series, 328. Cambridge University Press, Cambridge (2006)
7. Bobenko, A.I., Springborn, B.A.: Variational principles for circle patterns and Koebes theorem. Trans. Am. Math. Soc. 356(2), 659–689 (2004)
8. Danciger, J., Maloni, S., Schlenker, J.M.: Polyhedra Inscribed in a Quadric. arXiv preprint arXiv:1410.3774, (2014)
9. Dillencourt, M.B., Smith, W.D.: Graph-theoretical conditions for inscribability and Delaunay realizability. Discrete Math. 161, 63–77 (1996)
10. Guillemin, V., Pollack, A.: Differential Topology, vol. 370. AMS, Englewood Cliffs (2010)
11. He, Z., Liu, J.: On the Teichmüller theory of circle patterns. Trans. Am. Math. Soc. 365, 6517–6541 (2013)
12. Hirsch, M.: Differential Topology. Graduate Texts in Mathematics, vol. 3. Springer, New York (1976)
13. Hodgson, C.D., Rivin, I., Smith, W.D.: A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere. Bull. Am. Math. Soc. 27, 246–251 (1992)
14. Liu, J., Zhou, Z.: How many cages midcribe an egg. Invent. Math. 203, 655–673 (2016)
15. Lehto, O., Virtanen, K.I.: Quasiconformal Mappings in the Plane. Springer, New York (1973)
16. Marden, A., Rodin, B.: On Thurston’s formulation and proof of Andreev’s theorem. In: Computational Methods and Function Theory, pp. 103–115 (1990)
17. Milnor, J.W.: Topology from the Differentiable Viewpoint. Princeton University Press, Princeton (1997)
18. Rivin, I.: A characterization of ideal polyhedra in hyperbolic 3-space. Ann. Math. 143, 51–70 (1996)
19. Rodin, B., Sullivan, D.: The convergence of circle packings to the Riemann mapping. J. Differ. Geom. 26, 349–360 (1987)
20. Roeder, R.K.W., Hubbard, J.H., Dunbar, W.D.: Andreev’s theorem on hyperbolic polyhedra. Ann. Inst. Fourier (Grenoble) 57, 825–882 (2007)
21. Schramm, O.: Existence and uniqueness of packings with specified combinatorics. Isr. J. Math. 73, 321–341 (1991)
22. Schramm, O.: How to cage an egg. Invent. Math 107, 543–560 (1992)
23. Steiner, J.: Systematisch Entwicklung der Abhängigkeit Geometrischer Gestalten von Einander, mit Bercksichtigung der Arbeiten alter und neuer Geometer ber Porismen, Projections-Methoden, Geometrie der Lage, etc. Erster Theil (1832)
24. Steinitz, E.: Über isoperimetrische Probleme bei konvexen Polyedern. J. für Die Reine Angew. Math. 159, 133–143 (1928)
25. Thurston, W.P.: Three-dimensional geometry and topology. In: Levy, Silvio (ed.) Princeton Mathematical Series, 35, vol. 1. Princeton University Press, Princeton (1997)
26. Ziegler, G.M.: Lectures on Polytopes. Graduate Texts in Mathematics, vol. 152. Springer, New York (1995)