Bounding the inefficiency of compromise*

Ioannis Caragiannis  Panagiotis Kanellopoulos  Alexandros A. Voudouris
University of Patras & CTI “Diophantus”

Abstract

Social networks on the Internet have seen an enormous growth recently and play a crucial role in different aspects of today’s life. They have facilitated information dissemination in ways that have been beneficial for their users but they are often used strategically in order to spread information that only serves the objectives of particular users. These properties have inspired a revision of classical opinion formation models from sociology using game-theoretic notions and tools. We follow the same modeling approach, focusing on scenarios where the opinion expressed by each user is a compromise between her internal belief and the opinions of a small number of neighbors among her social acquaintances. We formulate simple games that capture this behavior and quantify the inefficiency of equilibria using the well-known notion of the price of anarchy. Our results indicate that compromise comes at a cost that strongly depends on the neighborhood size.

1 Introduction

Opinion formation has been the subject of much research in sociology, economics, physics, and epidemiology. Due to the widespread adoption of the Internet and the subsequent blossoming of social networks, it has recently attracted the interest of researchers in AI (e.g., see [3, 16, 17]) and CS at large (e.g., see [6, 14, 15]) as well.

An influential model that captures the adoption of opinions in a social context has been proposed by Friedkin and Johnsen [10]. According to this, the opinion an individual expresses for an issue follows by an averaging between her internal belief and the opinions expressed by her social acquaintances. Very recently, Bindel et al. [6] show that this behavior can be explained through a game-theoretic lens: averaging between the internal belief of an individual and the opinions in her social circle is simply a strategy that minimizes an implicit cost for the individual.

Bindel et al. [6] use a quadratic function to define this cost, equal to the total squared distance of the opinion expressed by the individual from her belief and the opinions expressed in her social circle. In a sense, this behavior leads to opinions that follow the majority of her social acquaintances. Bindel et al. [6] consider a static snapshot of the social network. In contrast, Bhawalkar et al. [4] implicitly assume that the opinion of an individual depends on a small number of acquaintances only, her neighbors. So, in their model, opinion formation co-evolves with the neighborhood for each individual: her neighborhood consists of those who have opinions similar to her belief. Then, the opinion expressed is assumed to minimize the same cost function used by Bindel et al. [6], taking into account the neighborhood instead of the whole social circle.

*This work was partially supported by Caratheodory research grant E.114 from the University of Patras and by a PhD scholarship from the Onassis Foundation.
We follow the co-evolutionary model of [4], but we deviate from their cost definition and instead consider individuals that seek to *compromise* with their neighbors. Hence, we assume that each individual aims to minimize the *maximum* distance of her expressed opinion from her belief and each of her neighbors’ opinion. Like [4], we assume that opinion formation co-evolves with the social network. The neighborhood of each individual consists of the $k$ players with the closest opinions to her belief. Naturally, these modeling decisions lead to the definition of strategic games, which we call *$k$-compromising opinion formation* (or, simply, *$k$-COF*) games. Each individual is a (cost-minimizing) player with the opinion expressed as her strategy.

**Technical contribution.** We study questions related to the existence, computational complexity, and quality of equilibria in *$k$*-COF games. We show that there exist simple 1-COF games that do not admit pure Nash equilibria and, furthermore, that even in games where equilibria exist, their quality may be suboptimal, i.e., the *price of stability* (defined in [1]) is strictly greater than 1. We also show that there is a representation of each 1-COF game as a directed acyclic graph, in which every pure Nash equilibrium corresponds to a path between two designated nodes. Hence, the problems of computing the best or worst pure Nash equilibrium (or even of computing whether such an equilibrium exists) are equivalent to simple path computations that can be performed in polynomial time. These results appear in Section 4. In Sections 5 and 6 we present upper and lower bounds on the *price of anarchy* (introduced in [13]) of *$k$*-COF games that suggest a linear dependence on $k$. Our upper bound on the price of anarchy exploits, in a non-trivial way, linear programming duality in order to lower-bound the optimal social cost. For the fundamental case of 1-COF games, we obtain a tight bound of 3.

**Related work.** DeGroot [8] proposed a framework that models the opinion formation process, where each individual updates her opinion based on a weighted averaging procedure. Subsequently, Friedkin and Johnsen [10] refined the model by assuming that each individual has a private belief and expresses a (possibly different) public opinion that depends on her belief and the opinions of people to whom she has social ties. More recently, Bindel et al. [6] studied this model and proved that, for the setting where beliefs and opinions are in $[0, 1]$, the repeated averaging process leads to an opinion vector that can be thought of as the unique equilibrium in a corresponding opinion formation game.

Deviating from the assumption that opinions depend on the whole social circle, Bhawalkar et al. [4] consider co-evolutionary opinion formation games, where as opinions evolve so does the neighborhood of each person. This model is conceptually similar to previous ones that have been studied by Hegselmann and Krause [11] and Holme and Newman [12]. Both Bindel et al. [6] and Bhawalkar et al. [4] show constant bounds on the price of anarchy of the games they study. In contrast, the modified cost function we use in order to model compromise yields considerably higher price of anarchy.

A series of recent papers from the EconCS community consider discrete models with binary opinions. Chierichetti et al. [7] consider discrete preference games, where beliefs and opinions are binary and study questions related to the price of stability. For these games, Auletta et al. [2] characterize the social networks where the belief of the minority can emerge as the opinion of the majority. Auletta et al. [3] generalize discrete preference games so that players are not only interested in agreeing with their neighbors and more complex constraints can be used to represent the players’ preferences. Bilò et al. [5] extend co-evolutionary formation games to the discrete setting. Other models assume that opinion updates depend on the entire social circle of each individual, who consults a small random subset of social acquaintances; see the recent paper by Fotakis et al. [9] and the survey of Mossel and Tamuz [14].
In spite of the extensive related literature in many different disciplines, we believe that our model captures the tendency to compromise more accurately.

2 Preliminaries

A compromising opinion formation game defined by the \( k \) nearest neighbors (henceforth, called \( k \)-COF game) is played by a set of \( n \) players whose beliefs lie on a line. Let \( s = (s_1, s_2, \ldots, s_n) \) be the vector containing the players’ beliefs such that \( s_i \leq s_{i+1} \) for each \( i \in \{n-1\} \). Let \( z = (z_1, z_2, \ldots, z_n) \) be a vector containing the (deterministic or randomized) opinions expressed by the players; these opinions define a state of the game. We denote by \( z_{-i} \) the opinion vector obtained by removing \( z_i \) from \( z \). In an attempt to simplify notation, we omit \( k \) from all relevant definitions.

Given vector \( z \) (or a realization of it in case \( z \) contains randomized opinions), we define the neighborhood \( N_i(z, s) \) of player \( i \) to be the set of \( k \) players whose opinions are the closest to the belief of player \( i \) breaking ties arbitrarily (but consistently). For each player \( i \), we define \( I_i(z, s) \) as the shortest interval of the real line that includes the following points: the belief \( s_i \), the opinion \( z_i \), and the opinion \( z_j \) for each player \( j \in N_i(z, s) \). Furthermore, let \( \ell_i(z, s) \) and \( r_i(z, s) \) be the players with the leftmost and rightmost point in \( I_i(z, s) \), respectively. For example, \( \ell_i(z, s) \) can be equal to either player \( i \) or some player \( j \in N_i(z, s) \), depending on whether the leftmost point of \( I_i(z, s) \) is \( s_i \), \( z_i \), or \( z_j \). To simplify the notation, we will frequently use \( \ell(i) \) and \( r(i) \) instead of \( \ell_i(z, s) \) and \( r_i(z, s) \) when \( z \) and \( s \) are clear from the context. In the following, we present the relevant definitions for the case of possibly randomized opinion vectors; clearly, these can be simplified whenever \( z \) consists entirely of deterministic opinions.

Given a \( k \)-COF game with belief vector \( s \) and the opinion vector \( z \), the cost of player \( i \) is defined as

\[
\mathbb{E}[\text{cost}_i(z, s)] = \mathbb{E}\left[ \max_{j \in N_i(z, s)} \left\{ |z_i - s_i|, |z_j - z_i| \right\} \right]
\]

\[
= \mathbb{E}\left[ \max \left\{ |z_i - s_i|, |z_{\ell_i(z, s)} - z_i|, |z_i - z_{r_i(z, s)}| \right\} \right].
\]

(1)

For the special case of 1-COF games, we denote by \( \sigma_i(z, s) \) (or \( \sigma(i) \) when \( z \) and \( s \) are clear from context) the player (other than \( i \)) whose opinion is closest to the belief \( s_i \) of player \( i \). In this case, the cost of player \( i \) can be simplified as

\[
\mathbb{E}[\text{cost}_i(z, s)] = \mathbb{E}\left[ \max \left\{ |z_i - s_i|, |z_{\sigma_i(z, s)} - z_i| \right\} \right].
\]

(2)

We say that an opinion vector \( z \) consisting entirely of deterministic opinions is a pure Nash equilibrium if no player \( i \) has any incentive to unilaterally deviate to a deterministic opinion \( z_i' \) in order to decrease her cost, i.e.,

\[
\text{cost}_i(z, s) \leq \text{cost}_i((z_i', z_{-i}), s),
\]

where by \((z_i', z_{-i})\) we denote the opinion vector in which player \( i \) chooses the opinion \( z_i' \) and all other players choose the opinions they have according to vector \( z \). Similarly, a possibly randomized opinion vector \( z \) is a mixed Nash equilibrium if for any player \( i \) and any deviating deterministic opinion \( z_i' \) we have

\[
\mathbb{E}[\text{cost}_i(z, s)] \leq \mathbb{E}_{z_{-i}}[\text{cost}_i((z_i', z_{-i}), s)].
\]

3
The social cost of the opinion vector \( z \) is defined as the total cost experienced by all players, i.e.,

\[
E[SC(z, s)] = \sum_{i=1}^{n} E[\text{cost}_i(z, s)].
\]

Let \( z^* \) be a deterministic opinion vector that minimizes the social cost; we will refer to it as the optimal opinion vector. The price of anarchy (PoA) of a \( k \)-COF game over pure Nash equilibria is defined as the ratio between the social cost of its worst (in terms of the social cost) pure Nash equilibrium and the optimal social cost, i.e.,

\[
\text{PoA} = \sup_{z \in \text{PNE}} \frac{SC(z, s)}{SC(z^*, s)},
\]

where PNE denotes the set of pure Nash equilibria. The price of stability (PoS) over pure Nash equilibria is defined as the ratio between the social cost of the best pure equilibrium (in terms of social cost) and the optimal social cost, i.e.,

\[
\text{PoS} = \inf_{z \in \text{PNE}} \frac{SC(z, s)}{SC(z^*, s)}.
\]

Similarly, the price of anarchy and the price of stability over mixed Nash equilibria are defined as

\[
\text{PoA} = \sup_{z \in \text{MNE}} \frac{E[SC(z, s)]}{SC(z^*, s)}, \quad \text{PoS} = \inf_{z \in \text{MNE}} \frac{E[SC(z, s)]}{SC(z^*, s)}
\]

where MNE denotes the set of mixed Nash equilibria.

3 Some properties about equilibria

We begin by proving some interesting properties of pure Nash equilibria; these will be useful in the following. The first one is obvious due to the definition of the cost function.

**Lemma 1.** In any pure Nash equilibrium \( z \) of a \( k \)-COF game with belief vector \( s \), the opinion of any player \( i \) lies in the middle of the interval \( I_i(z, s) \).

The next lemma allows us to argue about the order of player opinions in a pure Nash equilibrium \( z \).

**Lemma 2.** In any pure Nash equilibrium \( z \) of a \( k \)-COF game with belief vector \( s \), it holds that \( z_i \leq z_{i+1} \) for any \( i \in [n-1] \).

**Proof.** For the sake of contradiction, let us assume that \( z_{i+1} < z_i \) for a pair of players \( i \) and \( i+1 \). Then, it cannot be the case that the leftmost endpoint of the interval \( I_i(z, s) \) of player \( i \) is at the left of (or coincides with) the leftmost endpoint of interval \( I_{i+1}(z, s) \) of player \( i+1 \) and the rightmost endpoint of \( I_i(z, s) \) is at the left of (or coincides with) the rightmost endpoint of \( I_{i+1}(z, s) \). In other words, it cannot be the case that \( \min\{s_i, z_{\ell(i)}\} \leq \min\{s_{i+1}, z_{\ell(i+1)}\} \) and \( \max\{s_i, z_{r(i)}\} \leq \max\{s_{i+1}, z_{r(i+1)}\} \) hold simultaneously. Since, by Lemma 1, points \( z_i \) and \( z_{i+1} \) lie in the middle of the corresponding intervals, we would have \( z_i \leq z_{i+1} \), contradicting our assumption.

So, at least one of the two inequalities between the interval endpoints above must not hold. In the following, we assume that \( \min\{s_i, z_{\ell(i)}\} > \min\{s_{i+1}, z_{\ell(i+1)}\} \) (the case where \( \max\{s_i, z_{r(i)}\} > \max\{s_{i+1}, z_{r(i+1)}\} \) is symmetric). Our assumption implies that \( z_{\ell(i+1)} < s_i \leq s_{i+1} \), and, subsequently,
Lemma 2. If \( n \) players are in an interval \( I_i(z,s) \), then for any \( i \leq k \) players different than \( i \), \( s_i \) has opinions at distance strictly less than \( s_i - z_{(i+1)} \) from belief \( s_i \). Also, if \( i \) is the unique player with opinions at distance strictly less than \( s_i - z_{(i+1)} \) from belief \( s_i \), then all other players are at distance strictly less than \( s_i - z_{(i+1)} \) from belief \( s_i \). The lemma follows trivially by the definition of the neighborhood.

Figure 1: An example of the argument used in the proof of Lemma 2.

that \( z_{(i+1)} < z_{(i)} \). Hence, the opinion at the leftmost endpoint of interval \( I_{i+1}(z,s) \) cannot belong to player \( i + 1 \), i.e., \( \ell(i + 1) \neq i + 1 \); see also Figure 1.

Since \( \ell(i + 1) \) does not belong to \( I_i(z,s) \), there are at least \( k \) players different than \( i + 1 \) and \( i \) that have opinions at distance strictly less than \( s_{i+1} - z_{(i+1)} \) from belief \( s_i \). All these players are also at distance strictly less than \( s_{i+1} - z_{(i+1)} \) from belief \( s_{i+1} \). This contradicts the fact that the opinion of player \( \ell(i + 1) \) is among the \( k \) closest opinions to \( s_{i+1} \).

Furthermore, we can also specify the range of neighborhoods and opinions in a pure Nash equilibrium.

Lemma 3. Let \( z \) be a pure Nash equilibrium of a \( k \)-COF game with belief vector \( s \). Then, for each player \( i \), there exists \( j \) with \( i - k \leq j \leq i \) such that \( I_i(z,s) \) is the shortest interval that contains the opinions \( z_j, z_{j+1}, \ldots, z_{j+k} \) and belief \( s_i \).

Proof. If \( I_i(z,s) \) consists of a single point, the lemma follows trivially by the definition of the neighborhood and Lemma 2 since at least \( k + 1 \) consecutive players including \( i \) should have opinions in \( I_i(z,s) \). Also, by Lemma 2 the lemma is true if there is at most one opinion in each of the left and the right boundary of \( I_i(z,s) \); in this case, there are exactly \( k + 1 \) consecutive players including player \( i \) with opinions in \( I_i(z,s) \).

In the following, we handle the subtleties that may arise due to tie-breaking at the boundaries of \( I_i(z,s) \). Let \( Y_L \) and \( Y_R \) be the set of players with opinions at the leftmost and the rightmost point of \( I_i(z,s) \), respectively. From Lemma 1 player \( i \) belongs neither to \( Y_L \) nor to \( Y_R \). Now consider the following set of players: the \( |Y_L \cap N_i(z,s)| \) players with highest indices from \( Y_L \), the \( |Y_R \cap N_i(z,s)| \) players with lowest indices from \( Y_R \) and all players with opinions that lie strictly in \( I_i(z,s) \). Due to the definition of \( N_i(z,s) \) and by Lemma 2 there are \( k + 1 \) players in this set, including player \( i \), with consecutive indices.

In the following, we assume that \( N_i(z,s) \cup \{i\} \) consists of \( k + 1 \) players with consecutive indices. This does not affect the cost of player \( i \) at equilibrium in the proofs of our upper bounds (since, by Lemma 3 the interval defined is exactly the same). Our lower bound constructions are defined carefully so that no ties appear at all (and, hence, \( N_i(z,s) \cup \{i\} \) is uniquely defined as a set of \( k + 1 \) consecutive players anyway).

Lemma 4. Let \( z \) be a pure Nash equilibrium of a \( k \)-COF game with belief vector \( s \). Then, for each player \( i \), it holds that \( s_{\ell(i)} \leq z_i \leq s_{r(i)} \).

Proof. For the sake of contradiction, let us assume that \( s_{\ell(i)} \leq s_{r(i)} < z_i \) for some player \( i \) (the case where \( z_i \) lies at the left of \( s_{\ell(i)} \) is symmetric). Since \( z_i \) is at the middle of \( I_i(z,s) \), it holds that \( z_{r(i)} > z_i \). Also, since \( z_{r(i)} > z_i > s_{r(i)} \) and because \( z_{r(i)} \) is in the middle of \( I_{r(i)}(z,s) \), it holds that \( z_{r(r(i))} > z_{r(i)} \) and, by Lemma 2 \( r(r(i)) > r(i) \).

We now claim that \( \ell(i) \notin N_{r(i)}(z,s) \). Assume otherwise that \( \ell(i) \in N_{r(i)}(z,s) \). By definition, \( r(r(i)) \in N_{r(i)}(z,s) \). Then, Lemma 2 implies that any player \( j \), different than \( r(i) \), with \( \ell(i) < j < r(r(i)) \) is also in \( N_{r(i)}(z,s) \). Hence, \( N_{r(i)}(z,s) \) contains at least the \( k - 1 \) players in \( N_i(z,s) \setminus \{r(i)\} \), as well as players \( i \)
and \( r(\ell(i)) \). This, however, contradicts the fact that \( |N_{\ell(i)}(z, s)| = k \). Therefore, player \( \ell(i) \) is not among the \( k \) nearest neighbors of \( r(i) \); see also Figure 2.

\[ \cdots - I_{\ell(i)}(z, s) \cdots \]

Figure 2: An example of the argument used in the proof of Lemma 4.

So, we obtain that \( z_{\ell(r(i))} - s_{\ell(r(i))} > z_{r(i)} - s_{r(i)} > z_{\ell(i)} - z_i = z_i - \min\{s_i, z_{\ell(i)}\} \). Hence, we obtain that \( z_{\ell(r(i))} - \min\{s_i, z_{\ell(i)}\} \geq s_{\ell(i)} - z_{\ell(i)} \). If \( z_{\ell(i)} > s_{\ell(i)} \), then since, by Lemma 2, \( z_{\ell(i)} \leq z_{\ell(r(i))} \) and \( r(r(i)) \in N_{r(i)}(z, s) \), we obtain that also \( \ell(i) \in N_{r(i)}(z, s) \); a contradiction. So, we obtain that \( z_{\ell(r(i))} - s_{\ell(i)} > s_{\ell(i)} - z_{\ell(i)} \geq 0 \), and, again, we obtain a contradiction to the fact that \( \ell(i) \notin N_{r(i)}(z, s) \).

4 Existence, complexity, and quality of equilibria

In this section, we focus entirely on 1-COF games. We first warm up with a negative statement: pure Nash equilibria may not exist (Theorem 6). Interestingly, deciding whether a game admits pure Nash equilibria or not can be solved in polynomial time (Theorem 7). We conclude the section by showing that even the best equilibrium can be inefficient. Inefficiency of (worst) equilibria will then be the subject of the upcoming two sections. We begin with a technical lemma; recall that \( \sigma(i) \) denotes the player (other than \( i \)) with closest opinion to \( s_i \).

**Lemma 5.** Consider a 1-COF game and any three consecutive players \( a, b, c \) with beliefs \( s_a \leq s_b \leq s_c \), respectively. For any pure Nash equilibrium \( z \) where \( \sigma(a) = b, \sigma(b) = c \) and \( \sigma(c) = b \), it must hold that \( s_b \geq \frac{3s_a + 5s_c}{8} \), while for any equilibrium \( z \) where \( \sigma(a) = b, \sigma(b) = a \) and \( \sigma(c) = b \), it must hold that \( s_b \leq \frac{5s_a + 3s_c}{8} \).

**Proof.** It suffices to prove the first case; the second case is symmetric. By Lemma 4 the opinion \( z_a \) lies between beliefs \( s_a \) and \( s_b \) while \( z_b \) and \( z_c \) lie between \( s_b \) and \( s_c \). Since \( \sigma(b) = c \) and \( \sigma(c) = b \), by Lemma 1 it holds that \( z_b = (s_b + s_c)/2 \) and \( z_c = (z_b + s_c)/2 \) which yields \( z_b = s_b + \frac{s_a - 3s_b}{3} \) and \( z_c = s_b + \frac{2(s_a - s_b)}{3} \). Hence, we obtain that \( z_b - s_b = \frac{2(s_a - s_b)}{3} \). Similarly, since \( \sigma(a) = b \), it holds that \( z_a = s_a + \frac{s_b}{2} = \frac{3s_a + 2s_b + s_c}{6} \) and, therefore, we obtain that \( s_b - z_a = \frac{-3s_a + 4s_b - s_c}{6} \). Since \( \sigma(b) = c \), this implies that \( z_c - s_b \leq s_b - z_a \) and the lemma follows.

The construction in the proof of the next statement is inspired by 4.

**Theorem 6.** There exists a 1-COF game with no pure Nash equilibria.

**Proof.** Consider a 1-COF game with three players having the belief vector \( s = (0, 1, 2) \). By Lemma 2 we have that \( z_1 \leq z_2 \leq z_3 \), and, hence, it must be \( \sigma(a) = \sigma(c) = b \), while \( \sigma(b) \in \{a, c\} \). The proof follows by observing that \( s \) does not satisfy the requirements in Lemma 5.

We continue to present a polynomial-time algorithm that determines whether a 1-COF game admits pure Nash equilibria, and, in case it does, allows us to compute the best and worst pure Nash equilibrium with
with respect to the social cost. We do so by establishing a correspondence between pure Nash equilibria and source-sink paths in a suitably defined directed acyclic graph.

Assume that we are given neighborhood information according to which each player \( i \) has either player \( i - 1 \) or player \( i + 1 \) as neighbor. From Lemma \( 3 \), such a neighborhood structure is necessary in a pure Nash equilibrium. We claim that this information is enough in order to decide whether there is a consistent opinion vector that is a pure Nash equilibrium or not. All we have to do is to use Lemma \( 1 \) and obtain \( n \) equations that relate the opinion of each player to her belief and her neighbor’s opinion. These equations have a unique solution which can then be verified whether it indeed satisfies the neighborhood conditions or not. So, the main idea of our algorithm is to cleverly search among all possible neighborhood structures that are not excluded by Lemma \( 3 \) for one that defines a pure Nash equilibrium.

For integers \( 1 \leq a \leq b < c \leq n \), let us define the segment \( C(a, b, c) \) to be the set of players \( \{a, a + 1, \ldots, c\} \) together with the following neighborhood information for them: \( \sigma(p) = p + 1 \) for \( p = a, \ldots, b \) and \( \sigma(p) = p - 1 \) for \( p = b + 1, \ldots, c \). See Figure 3. It can be easily seen that the neighborhood information for all players at a pure Nash equilibrium can always be decomposed into disjoint segments. Importantly, given the neighborhood information in segment \( C(a, b, c) \) and the beliefs of its players, the opinions they could have in any pure Nash equilibrium that contains this segment is uniquely defined using Lemma \( 1 \). In particular, the opinions of the players within a segment \( C(a, b, c) \) are computed as follows. First, we set \( z_b = s_b + \frac{s_{b + 1} - s_b}{3} \) and \( z_{b + 1} = s_b + \frac{2(s_{b + 1} - s_b)}{3} \). Then, we set \( z_p = \frac{s_p + z_{p + 1}}{2} \) if \( a \leq p < b \), and \( z_p = \frac{s_p + z_{p - 1}}{2} \) if \( b < p \leq c \).

We remark that the opinion vector implied by a segment is not necessarily consistent to the given neighborhood structure. So, we call segment \( C(a, b, c) \) legit if \( a \neq 2 \), \( c \neq n - 1 \) (so that it can be part of a decomposition) and the uniquely defined opinions are consistent to the neighborhood information of the segment, i.e., if \( |z_{\sigma(p)} - s_p| \leq |z_p - s_p| \) for any pair of players \( p, p' \) (with \( p \neq p' \)) in \( C(a, b, c) \).

A decomposition of neighborhood information for all players will consist of consecutive segments \( C(a_1, b_1, c_1) \), \( C(a_2, b_2, c_2) \), ..., \( C(a_t, b_t, c_t) \) so that \( a_1 = 1 \), \( c_t = n \), \( a_\ell = c_{\ell - 1} + 1 \) for \( \ell = 2, \ldots, t \). Such a decomposition will yield a pure Nash equilibrium if it consists of legit segments and, furthermore, the uniquely defined opinions of players in consecutive segments is consistent to the neighborhood information in both of them.

Now, consider the directed graph \( G \) that has two special nodes designated as the source and the sink, and a node for each legit segment \( C(a, b, c) \). Note that \( G \) has \( O(n^3) \) nodes. The source node is connected to all segment nodes \( C(1, b, c) \) while all segment nodes \( C(a, b, n) \) are connected to the sink. An edge from segment node \( C(a, b, c) \) to segment node \( C(a', b', c') \) exists if \( a' = c + 1 \) and the uniquely defined opinions of players in the two segments are consistent to the neighborhood information in both of them, i.e., \( |z_{c - 1} - s_c| \leq |z_{c' + 1} - s_{c'}| \) and \( |z_{c + 2} - s_{c + 1}| \leq |z_c - s_{c + 1}| \). By the definition of segments and of its edges, \( G \) is acyclic.

Based on the discussion above, there is a bijection between pure Nash equilibria and source-sink paths in \( G \). In addition, we can assign a weight to each node of \( G \) that is equal to the total cost of the players in the corresponding segment, i.e.,

\[
\text{weight}(C(a, b, c)) = \sum_{a \leq p \leq c} |z_p - s_p|.
\]

Figure 3: Graphical representation of a segment \( C(a, b, c) \).
Then, the total weight of a source-sink path \( P \) is equal to the social cost of the corresponding pure Nash equilibrium, i.e,

\[
SC(z, s) = \sum_{C(a,b,c) \in P} \text{weight}(C(a,b,c)).
\]

Hence, standard algorithms for computing shortest or longest paths in directed acyclic graphs can be used not only to detect whether a pure Nash equilibrium exists, but also to compute the equilibrium of best or worst social cost.

**Theorem 7.** Given a 1-COF game, deciding whether a pure Nash equilibrium exists can be done in polynomial time. Furthermore, computing a pure Nash equilibrium of highest or lowest social cost can be done in polynomial time as well.

**Example 1.** Consider a 1-COF game with four players with belief vector \( s = (0, 3, 4, 7) \). According to the discussion above, there are 10 segments of the form \( C(a,b,c) \) with \( 1 \leq a \leq b < c \leq 4 \), but it can be shown that only 3 of them are legit; these are \( C(1,1,2) \), \( C(3,3,4) \), and \( C(1,2,4) \). For example, segment \( C(1,1,4) \) corresponds to the opinion vector \( (1,2,3,5) \) which is not consistent to the neighborhood information \( \sigma(2) = 1 \) in the segment. The resulting directed acyclic graph \( G \), appears in Figure 4 and implies that there exist two pure Nash equilibria for this 1-COF game, namely the opinion vectors \( (1,2,5,6) \) and \( (5/3, 10/3, 11/3, 16/3) \).

![Figure 4: The directed acyclic graph \( G \) for Example 1](image)

We now present another application of Lemma 5

**Theorem 8.** The price of stability of 1-COF games is at least \( 17/15 \).

**Proof.** We use the following 1-COF game with six players and belief vector

\[
s = (0, 5 - 3\lambda, 8, 15, 18 + 3\lambda, 23),
\]

where \( \lambda \in (0, 1/4) \).

Consider the opinion vector

\[
\tilde{z} = (3 - \lambda, 6 - 2\lambda, 7 - 6\lambda, 16 + 6\lambda, 17 + 2\lambda, 20 + \lambda).
\]

It can be easily seen that it has social cost \( SC(\tilde{z}, s) = 10 + 12\lambda \). Clearly, \( SC(z^*, s) \leq 10 + 12\lambda \) for the optimal opinion vector \( z^* \).

Now, consider the opinion vector

\[
z = \left( \frac{5 - 3\lambda}{3}, \frac{10 - 6\lambda}{3}, \frac{31}{3}, \frac{38}{3}, \frac{59 + 6\lambda}{3}, \frac{64 + 3\lambda}{3} \right)
\]
with social cost $SC(z, s) = 34/3 - 4\lambda$. It is not hard to verify (by showing, as Lemma 1 requires, that each opinion lies in the middle of its player’s interval) that $z$ is a pure Nash equilibrium; we argue that this equilibrium is unique. By observing the belief vector $s$, we remark that for each $s_j$, with $j \in \{2, 5\}$, it holds that $\frac{5s_j - 3s_j + \frac{5}{8}}{s} < s_j < \frac{3s_j - 3s_j + \frac{5}{8}}{s}$. Hence, by Lemma 5 there cannot be a pure Nash equilibrium where both $\sigma(j - 1) = j$ and $\sigma(j + 1) = j$ for any $j \in \{2, 5\}$. This observation, together with Lemma 2 implies that $\sigma(1) = 2$, $\sigma(3) = 4$, $\sigma(4) = 3$ and $\sigma(6) = 5$ in any equilibrium. This leaves only $\sigma(2) \in \{1, 3\}$ and $\sigma(5) \in \{4, 6\}$ undefined. Consider an equilibrium $\sigma'$ with $\sigma(2) = 3$; the case $\sigma(5) = 4$ is symmetric. Since $\sigma(3) = 4$, Lemma 4 implies that $z'_3 > s_3$ and, hence, $z'_3 - s_2 > 3 + 3\lambda$. Since $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(5)' = s_1 + s'_2$, Lemma 4 implies that $z'_2 > s_2$ and we obtain that $z'_1 > \frac{5 - 3\lambda}{2}$ and, hence, $s_2 - z'_1 < \frac{5 - 3\lambda}{2}$. Since $z'_3 - s_2 > s_2 - z'_1$, we obtain a contradiction to our assumption that $\sigma(2) = 3$. So, it must hold that $\sigma(2) = 1$ (and, respectively, $\sigma(5) = 6$) which implies that $z$ is the unique pure Nash equilibrium.

We conclude that the price of stability is lower-bounded by

$$\frac{SC(z, s)}{SC(z^*, s)} = \frac{34/3 - 4\lambda}{10 + 12\lambda},$$

and the theorem follows by taking $\lambda$ to be arbitrarily close to 0. \hfill \Box

5 Upper bounds on the price of anarchy

In the proof of the upper bound on the price of anarchy of $k$-COF games, we relate the social cost of any deterministic opinion vector, including the optimal one, to a quantity that depends only on the beliefs of the players and can be thought of as the cost of the truthful opinion vector (in which the opinion of every player is equal to her belief).

Consider an $n$-player $k$-COF game with belief vector $s = (s_1, \ldots, s_n)$. For player $i$, we denote by $\ell^*(i)$ and $r^*(i)$ the integers in $[n]$ that are such that $\ell^*(i) \leq i \leq r^*(i)$, $r^*(i) - \ell^*(i) = k$, and $|s_{r^*(i)} - s_{\ell^*(i)}|$ is minimized. The proof of the next lemma exploits linear programming and duality.

**Lemma 9.** Consider a $k$-COF game and let $s = (s_1, \ldots, s_n)$ denote the belief vector and $z$ be any deterministic opinion vector. Then, $SC(z, s) \geq \frac{1}{2(k+1)} \sum_{i=1}^{n} |s_{r^*(i)} - s_{\ell^*(i)}|$.

**Proof.** Consider any deterministic opinion vector $z$ and let $\pi$ be a permutation of the players so that $z_{\pi(j)} \leq z_{\pi(j+1)}$ for each $j \in [n - 1]$. We refer to player $\pi(j)$ as the player with rank $j$. For each player $i$, we will identify an effective neighborhood $F_i(z, s)$ that consists of $k + 1$ players with consecutive ranks and includes player $i$. Define $\tilde{\ell}(i)$ and $\tilde{\ell}(i)$ to be the players in $F_i(z, s)$ with the lowest and highest belief, respectively. The effective neighborhood will be defined in such a way that it satisfies the properties $cost_i(z, s) \geq z_{\tilde{\ell}(i)} - z_i$ and $cost_i(z, s) \geq z_i - z_{\tilde{\ell}(i)}$.

Let $N_i(z, s)$ denote the neighborhood of player $i$, i.e., the set of players (not including $i$) with the $k$ closest opinions to the belief $s_i$ of player $i$. Let $J_i(z, s)$ be the smallest contiguous interval containing all opinions of players in $N_i(z, s)$ and let $D_i(z, s)$ be the set of players with opinions in $J_i(z, s)$. Clearly, $|D_i(z, s)| \geq k + 1$. We define $F_i(z, s)$ to be a subset of $D_i(z, s)$ that consists of $k + 1$ players with consecutive ranks including player $i$.

Let $\ell'(i)$ and $r'(i)$ be the players in $N_i(z, s)$ with the leftmost and rightmost opinion. In order to show that the definition of $F_i(z, s)$ satisfies the two desired inequalities, we distinguish between three different cases depending on the location of opinion $z_i$ among the players in $N_i(z, s) \cup \{i\}$. 

9
• **Case I:** Player \( i \) has neither the leftmost nor the rightmost opinion in \( N_i(z,s) \cup \{ i \} \), i.e., \( z_{\ell(i)} < z_i < z_{r(i)} \). In this case, \( J_i(z,s) = [z_{\ell(i)}, z_{r(i)}] \). Then, the definition of \( N_i(z,s) \) implies that \( \text{cost}_i(z,s) \geq z_{r(i)} - z_i \) and \( \text{cost}_i(z,s) \geq z_i - z_{\ell(i)} \). Hence, \( \text{cost}_i(z,s) \geq |z_j - z_i| \) for every \( z_j \in J_i(z,s) \) or \( j \in D_i(z,s) \) and, subsequently, for each \( j \in F_i(z,s) \). This implies the two desired inequalities \( \text{cost}_i(z,s) \geq z_{r(i)} - z_i \) and \( \text{cost}_i(z,s) \geq z_i - z_{\ell(i)} \).

• **Case II:** Player \( i \) has the leftmost opinion, i.e., \( z_i \leq z_{\ell(i)} \). Then, \( J_i(z,s) = [z_i, z_{r(i)}] \). Now, the definition of \( N_i(z,s) \) implies that \( \text{cost}_i(z,s) \geq z_{r(i)} - z_i \) and, hence, \( \text{cost}_i(z,s) \geq |z_j - z_i| \) for every \( z_j \in J_i(z,s) \) or \( j \in D_i(z,s) \) and, subsequently, for each \( j \in F_i(z,s) \). Again, this implies the two desired inequalities.

• **Case III:** Player \( i \) has the rightmost opinion, i.e., \( z_i \geq z_{r(i)} \). Then, \( J_i(z,s) = [z_i, z_{\ell(i)}] \). Now, the definition of \( N_i(z,s) \) implies that \( \text{cost}_i(z,s) \geq z_i - z_{\ell(i)} \) and, hence, \( \text{cost}_i(z,s) \geq |z_j - z_i| \) for every \( z_j \in J_i(z,s) \) or \( j \in D_i(z,s) \) and, subsequently, for each \( j \in F_i(z,s) \). Again, the two desired inequalities follow.

Setting the variable \( t_i \) equal to \( \text{cost}_i(z,s) \) for \( i \in [n] \), the discussion above and the fact that \( \text{cost}_i(z,s) \geq |s_i - z_i| \) imply that the opinion vector \( z \) together with \( t = (t_1, \ldots, t_n) \) is a feasible solution to the following linear program:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in [n]} t_i \\
\text{subject to} & \quad t_i + z_i \geq s_i, \forall i \in [n] \\
& \quad t_i - z_i \geq -s_i, \forall i \in [n] \\
& \quad t_i + z_i - z_{\ell(i)} \geq 0, \forall i \in [n] \text{ such that } \ell(i) \neq i \\
& \quad t_i + z_{\ell(i)} - z_i \geq 0, \forall i \in [n] \text{ such that } \ell(i) \neq i \\
& \quad t_i, z_i \geq 0, \forall i \in [n]
\end{align*}
\]

Using the dual variables \( \alpha_i, \beta_i, \gamma_i, \text{ and } \delta_i \) associated with the four constraints of the above LP, we obtain its dual LP:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i \in [n]} s_i \alpha_i - \sum_{i \in [n]} s_i \beta_i \\
\text{subject to} & \quad \alpha_i + \beta_i + \gamma_i \cdot 1\{\ell(i) \neq i\} + \delta_i \cdot 1\{\ell(i) \neq i\} \leq 1, \forall i \in [n] \\
& \quad \alpha_i - \beta_i + \gamma_i \cdot 1\{\ell(i) \neq i\} - \delta_i \cdot 1\{\ell(i) \neq i\} - \sum_{j \neq i: \ell(j) = i} \gamma_j + \sum_{j \neq i: \ell(j) = i} \delta_j \leq 0, \forall i \in [n] \\
& \quad \alpha_i, \beta_i, \gamma_i, \delta_i \geq 0
\end{align*}
\]

We will show that the solution defined as

\[
\begin{align*}
\alpha_i &= \frac{|\{j \in [n] : \ell(j) = i\}|}{2(k + 1)}, \\
\beta_i &= \frac{|\{j \in [n] : \ell(j) = i\}|}{2(k + 1)}
\end{align*}
\]

10
\[
\gamma_i = \delta_i = \frac{1}{2(k + 1)},
\]
is a feasible dual solution. Indeed, to see why the first dual constraint is satisfied, first observe that player \(i\) belongs to at most \(2k + 1\) different effective neighborhoods. Hence, player \(i\) can have the lowest or highest belief among the players in the effective neighborhood of at most \(2k + 1\) players (implying that \(\alpha_i + \beta_i \leq 1 - \frac{1}{2(k + 1)}\)) when \(\tilde{r}(i) = i\) or \(\tilde{\ell}(i) = i\) and of at most \(2k\) players (implying that \(\alpha_i + \beta_i \leq 1 - \frac{1}{k + 1}\)) when \(\tilde{r}(i) \neq i\) and \(\tilde{\ell}(i) \neq i\). The first constraint follows.

It remains to show that the second constraint is satisfied as well (with equality). We do so by distinguishing between three cases:

- When \(\tilde{r}(i) \neq i\) and \(\tilde{\ell}(i) \neq i\), the dual solution guarantees that \(\alpha_i = \sum_{j \neq i : \tilde{r}(j) = i} \gamma_j\) and the term \(\alpha_i\) in the left-hand side of the second constraint cancels out with the sum of \(\gamma\)'s. Similarly, \(\beta_i = \sum_{j \neq i : \tilde{\ell}(j) = i} \delta_j\) and the term \(\beta_i\) cancels out with the sum of \(\delta\)'s. Also, the terms \(\gamma_i\) and \(\delta_i\) are both equal to \(\frac{1}{2(k + 1)}\) and cancel out as well.

- When \(\tilde{r}(i) = i\) (then, clearly, \(\tilde{\ell}(i) \neq i\)), we have that \(\alpha_i = \delta_i \cdot 1\{\tilde{\ell}(i) \neq i\}\) + \(\sum_{j \neq i : \tilde{r}(j) = i} \gamma_j\) and \(\beta_i = \sum_{j \neq i : \tilde{\ell}(j) = i} \delta_j\) and the second constraint is satisfied with equality.

- Finally, when \(\tilde{\ell}(i) = i\) (now, it is \(\tilde{r}(i) \neq i\)), we have that \(\alpha_i = \sum_{j \neq i : \tilde{r}(j) = i} \gamma_j\) and \(\beta_i = \gamma_i \cdot 1\{\tilde{r}(i) \neq i\}\) + \(\sum_{j \neq i : \tilde{\ell}(j) = i} \delta_j\) and the second constraint is satisfied with equality.

So, the social cost of the solution \(z\) is lower-bounded by the objective value of the primal LP which, by duality, is lower-bounded by the objective value of the dual LP. Hence

\[
\begin{align*}
\text{SC}(z, s) &\geq \sum_{i \in [n]} s_i \alpha_i - \sum_{i \in [n]} s_i \beta_i \\
&= \frac{1}{2(k + 1)} \left( \sum_{i \in [n]} |\{j \in [n] : \tilde{r}(j) = i\}| s_i - \sum_{i \in [n]} |\{j \in [n] : \tilde{\ell}(j) = i\}| s_i \right) \\
&= \frac{1}{2(k + 1)} \sum_{i \in [n]} (s_{\tilde{r}(i)} - s_{\tilde{\ell}(i)}) \\
&= \frac{1}{2(k + 1)} \sum_{i \in [n]} |s_{\tilde{r}(i)} - s_{\tilde{\ell}(i)}|,
\end{align*}
\]

The last equality follows since \(s_{\tilde{r}(i)} \geq s_{\tilde{\ell}(i)}\), by the definition of \(\tilde{r}(i)\) and \(\tilde{\ell}(i)\).

Note that for each player \(i\), there are at least \(k + 1\) beliefs of different players with values in \([s_{\tilde{\ell}(i)}, s_{\tilde{r}(i)}]\), including player \(i\). By the definition of \(\ell^*(i)\) and \(r^*(i)\) for each player \(i\), the above inequality yields

\[
\text{SC}(z, s) \geq \frac{1}{2(k + 1)} \sum_{i \in [n]} |s_{r^*(i)} - s_{\ell^*(i)}|,
\]
as desired. \(\square\)

We are now ready to prove our upper bound on the price of anarchy for \(k\)-COF games.

**Theorem 10.** The price of anarchy of \(k\)-COF games over pure Nash equilibria is at most \(4(k + 1)\).
Proof. Let \( z = (z_1, \ldots, z_n) \) be an equilibrium and \( z^* = (z_1^*, \ldots, z_n^*) \) be the opinion vector that minimizes the social cost. By Lemma 9 we have

\[
\text{SC}(z^*, s) \geq \frac{1}{2(k + 1)} \sum_{i=1}^n |s_{r^*}(i) - s_{l^*}(i)|.
\]

(3)

Now, consider any pure Nash equilibrium \( z \) of the game. We will show that

\[
\text{SC}(z, s) \leq 2 \sum_{i=1}^n |s_{r^*}(i) - s_{l^*}(i)|.
\]

(4)

The theorem will then follow by (3) and (4).

We focus on a single player \( i \) and assume that \( z_i \geq s_i \) (the case \( z_i \leq s_i \) is symmetric). We will show that \( \text{cost}_i(z, s) \leq 2(s_{r^*}(i) - s_{l^*}(i)) \). Then, (4) will follow by summing over all players.

Recall that \( \ell(i) \) and \( r(i) \) denote the players in \( N_i(z, s) \cup \{i\} \) with leftmost and rightmost point in \( I_i(z, s) \). First, observe that if \( z_{r(i)} = z_i \), the assumption \( z_i \geq s_i \) implies that all players in the neighborhood of player \( i \) have opinions at \( s_i \) (since \( z_i \) is in the middle of interval \( I_i(z, s) \) at equilibrium). In this case, \( \text{cost}_i(z, s) = 0 \) and the desired inequality holds trivially. So, in the following, we assume that \( r(i) > i \) and \( z_{r(i)} > z_i \), i.e., \( z_{r(i)} \) is at the right of \( z_i \) which in turn is at the right of \( s_i \). We distinguish between two main cases, having two subcases each.

Case I: \( r(i) > r^*(i) \) and \( \ell(i) > \ell^*(i) \). Since \( z_{r(i)} \) is at the right of \( (of \text{coincides with}) \ s_i \) and \( \ell^*(i) \) does not belong to the neighborhood of player \( i \) (while player \( r(i) \) does), \( z_{\ell^*(i)} \) is at the left of \( s_i \) and, furthermore, \( z_{r(i)} - s_i \leq s_i - z_{\ell^*(i)} \) or, equivalently,

\[
z_{r(i)} \leq 2s_i - z_{\ell^*(i)}.
\]

(5)

This yields

\[
\text{cost}_i(z, s) = z_{r(i)} - z_i \leq 2s_i - z_{\ell^*(i)} - z_i.
\]

(6)

These inequalities will be useful in several places of the proof for this case below.

If \( z_{\ell^*(i)} \geq s_{l^*(i)} \) then, since \( r^*(i) \geq i \) and \( z_i \geq s_i \), inequality (6) becomes \( \text{cost}_i(z, s) \leq s_i - s_{\ell^*(i)} \leq s_{r^*}(i) - s_{l^*}(i) \) and the desired inequality follows. So, in the following, we assume that \( z_{\ell^*(i)} < s_{\ell^*(i)} \{strictly} at \ the left \ of \ s_{\ell^*(i)}. Hence, \( \ell^*(i) \) has her leftmost neighbor with \( z_{\ell(\ell^*(i))} \) and

\[
z_{\ell(\ell^*(i))} = \frac{z_{\ell^*(i)} + \max\{s_{\ell^*(i)}, z_{r(\ell^*(i))}\}}{2}.
\]

(7)

Since \( r^*(i) - \ell^*(i) = k \) and \( \ell(\ell^*(i)) < \ell^*(i) \), players \( r^*(i) \) and \( \ell(\ell^*(i)) \) cannot belong in the same neighborhood and \( r^*(i) \) does not belong to the neighborhood of \( \ell^*(i) \). Hence, \( s_{\ell^*(i)} - z_{\ell(\ell^*(i))} \leq z_{r^*(i)} - s_{l^*(i)} \) or, equivalently

\[
z_{\ell(\ell^*(i))} \geq 2s_{l^*(i)} - z_{r^*(i)} \geq 2s_{l^*(i)} - 2s_i + z_{\ell^*(i)}.
\]

(8)

where the second inequality follows by (5) using \( z_{r^*(i)} \leq z_{r(i)} \).

We now further distinguish between two cases, depending on whether player \( i \) belongs to the neighborhood of player \( \ell^*(i) \) or not.
• **Case I.1:** $i \in N_{\ell^*(i)}(z, s)$. See Figure 5. Then,

$$\max \{s_{\ell^*(i)}, z_{r(\ell^*(i))}\} \geq z_{r(\ell^*(i))} \geq z_i. \quad \text{(9)}$$

Using (8) and (9), (7) yields

$$z_{\ell^*(i)} \geq \frac{z_{\ell(\ell^*(i))} + z_{r(\ell^*(i))}}{2} \geq s_{\ell^*(i)} - s_i + \frac{z_{\ell^*(i)}}{2} + \frac{z_i}{2},$$

which implies that $z_{\ell^*(i)} \geq 2s_{\ell^*(i)} - 2s_i + z_i$. Now, (6) becomes

$$\text{cost}_i(z, s) \leq 4s_i - 2s_{\ell^*(i)} - 2z_i \leq 2s_i - 2s_{\ell^*(i)} \leq 2(s_{r^*(i)} - s_{\ell^*(i)})$$

as desired. The second inequality follows since $z_i \geq s_i$ and the last one follows since $r^*(i) \geq i$.

![Figure 5: An example of Case I.1.](image)

• **Case I.2:** $i \notin N_{\ell^*(i)}(z, s)$. See Figure 6. Then $s_{\ell^*(i)} - z_{\ell(\ell^*(i))} \leq z_i - s_{\ell^*(i)}$, which implies that $z_{\ell(\ell^*(i))} \geq 2s_{\ell^*(i)} - z_i$. Using this inequality together with the fact that $\max \{s_{\ell^*(i)}, z_{r(\ell^*(i))}\} \geq s_{\ell^*(i)}$, (7) yields

$$z_{\ell^*(i)} \geq \frac{3s_{\ell^*(i)} - z_i}{2}$$

and (6) becomes

$$\text{cost}_i(z, s) \leq 2s_i - \frac{3}{2}s_{\ell^*(i)} - \frac{z_i}{2} \leq \frac{3}{2}s_i - \frac{3}{2}s_{\ell^*(i)} \leq 2(s_{r^*(i)} - s_{\ell^*(i)}),$$

as desired. The second last inequality follows since $z_i \geq s_i$ and the last one follows since $r^*(i) \geq i$.

![Figure 6: An example of Case I.2. Note that $z_i \notin I_{\ell^*(i)}(z, s)$ as $i \notin N_{\ell^*(i)}(z, s)$.](image)

**Case II:** $r(i) \leq r^*(i)$ and $\ell(i) \leq \ell^*(i)$. Since $z_i$ is in the middle of the interval $I_i(z, s)$ and $z_{r(i)}$ is the rightmost opinion in $I_i(z, s)$, we have $z_i = \frac{\min\{s_i, z_{\ell(i)}\} + z_{r(i)}}{2} \leq \frac{z_{\ell(i)} + z_{r(i)}}{2} \leq \frac{z_{\ell^*(i)} + z_{r^*(i)}}{2}$. Since $s_i \leq z_i$, we get

$$z_{\ell^*(i)} \geq 2s_i - z_{r^*(i)}. \quad \text{(10)}$$

13
We also have

\[ \text{cost}_i(z, s) = z_r(i) - z_i \leq z_r^*(i) - z_i. \quad (11) \]

If \( z_r^*(i) \leq s_r^*(i) \) then, since \( s_r^*(i) \leq s_i \leq z_i \), (11) yields \( \text{cost}_i(z, s) \leq s_r^*(i) - s_i \leq s_r^*(i) - s_{\ell^*}(i) \), which is even stronger than the desired inequality. So, in the following we assume that \( z_r^*(i) > s_r^*(i) \), i.e., \( z_r^*(i) \) is at the right of \( s_r^*(i) \). Since \( z_r^*(i) \) is in the middle of the interval \( I_r^*(i)(z, s) \), there is a player \( r'(r^*(i)) \neq r^*(i) \) that belongs to the neighborhood of player \( r^*(i) \) and

\[ z_r^*(i) = \frac{\min\{s_r^*(i), z_{\ell^*(i)}\} + z_{r^*(i)}}{2}. \quad (12) \]

Moreover, player \( \ell^*(i) \) does not belong to the neighborhood of player \( r^*(i) \). Hence, \( z_{r'(r^*(i))} - s_r^*(i) \leq s_r^*(i) - z_{\ell^*(i)} \) which, together with (10), yields that

\[ z_{r'(r^*(i))} \leq 2s_r^*(i) - z_{\ell^*(i)} \leq 2s_r^*(i) - 2s_i + z_r^*(i). \quad (13) \]

We now further distinguish between two cases, depending on whether player \( i \) belongs to the neighborhood of player \( r^*(i) \) or not.

- **Case II.1**: \( i \in N_r^*(i)(z, s) \). See Figure 7. Then, using the fact that \( \min\{s_r^*(i), z_{\ell^*(i)}\} \leq z_{\ell^*(i)} \leq z_i \) and inequality (13), equation (12) becomes \( z_r^*(i) \leq z_i + 2s_r^*(i) - 2s_i + z_r^*(i) \) and, equivalently, \( z_r^*(i) \leq z_i + 2s_r^*(i) - 2s_i \). Hence, (13) yields

\[ \text{cost}_i(z, s) \leq 2s_r^*(i) - 2s_i \leq 2(s_r^*(i) - s_{\ell^*(i)}), \]

as desired. The last inequality follows since \( \ell^*(i) \leq i \).

- **Case II.2**: \( i \notin N_r^*(i)(z, s) \). See Figure 8. Since \( i \) does not belong to the neighborhood of player \( r^*(i) \) but player \( r'(r^*(i)) \) does, we have that \( z_{r'(r^*(i))} - s_r^*(i) \leq s_r^*(i) - z_i \) or, equivalently, \( z_{r'(r^*(i))} \leq 2s_r^*(i) - z_i \). Then (12) becomes \( z_r^*(i) \leq \frac{3s_r^*(i) - z_i}{2} \) and (11) yields

\[ \text{cost}_i(z, s) \leq \frac{3}{2}(s_r^*(i) - z_i) \leq \frac{3}{2}(s_r^*(i) - s_{\ell^*(i)}), \]

which is even stronger than the desired inequality. The last inequality follows since \( z_i \geq s_i \) and \( \ell^*(i) \leq i \).

The proof is complete.

For the case of 1-COF games we can prove a stronger statement using a similar proof roadmap but simpler (and shorter) arguments. We denote by \( \eta(i) \) the player (other than \( i \)) that minimizes the distance \( |s_i - s_{\eta(i)}| \); note that \( \eta(i) \in \{i - 1, i + 1\} \).
Lemma 11. Consider a 1-COF game and let \( s = (s_1, \ldots, s_n) \) denote the belief vector and \( z \) be any deterministic opinion vector. Then, \( SC(z, s) \geq \frac{1}{3} \sum_{i=1}^{n} |s_i - s_{\eta(i)}| \).

Proof. We call a player \( i \) with \( z_i \not\in [s_{i-1}, s_{i+1}] \) a kangaroo player and associate the quantity excess\(_i\) with her. If \( z_i \in [s_j, s_{j+1}] \) for some \( j > i \), we say that the players in the set \( C_i = \{i + 1, \ldots, j\} \) are covered by player \( i \) and define excess\(_i\) = \( z_i - s_j \). Otherwise, if \( z_i \in [s_{j-1}, s_j] \) for some \( j < i \), we say that the players in the set \( C_i = \{j, \ldots, i-1\} \) are covered by player \( i \) and define excess\(_i\) = \( s_j - z_i \).

Let \( K \) be the set of kangaroo players and \( C \) the set of players that are covered by a kangaroo; these need not be disjoint. We now partition the players not in \( K \cup C \) into the set \( L \) of large players \( i \) with cost\(_i(z, s) \geq \frac{1}{3}(|s_i - s_{\eta(i)}|) \) and the set \( S \) that contains the remaining small players. See Figure 9.

![Figure 9: Kangaroos, covered, large, and small players: 1 ∈ K, 2 ∈ K ∩ C, 3 ∈ C, 4 ∈ S, and 5 ∈ L.](image)

We proceed to prove five useful claims.

Claim 12. Let \( i \in S \) such that \( \sigma(i) \in K \). Then, cost\(_i(z, s) + \text{excess}_{\sigma(i)} \geq \frac{1}{3}|s_i - s_{\eta(i)}| \).

Proof. We assume that \( \sigma(i) > i \) (the other case is symmetric). If \( z_{\sigma(i)} > s_{\sigma(i)} \), then cost\(_i(z, s) = \max\{|s_i - z_i|, |z_i - z_{\sigma(i)}|\} \geq \frac{1}{2}(z_{\sigma(i)} - s_i) > \frac{1}{2}(s_{\sigma(i)} - s_i) \geq \frac{1}{3}|s_i - s_{\eta(i)}| \), which contradicts the fact that \( i \) is a small player. Hence, \( z_{\sigma(i)} \in [s_i, s_{\sigma(i)}] \), otherwise player \( i \) would be covered. Let \( j \) be the player with the leftmost belief that is covered by player \( \sigma(i) \). Then, excess\(_{\sigma(i)} = s_j - z_{\sigma(i)} \). We have cost\(_i(z, s) + \text{excess}_{\sigma(i)} = \max\{|s_i - z_i|, |z_i - z_{\sigma(i)}|\} + s_j - z_{\sigma(i)} \geq \frac{1}{2}(z_{\sigma(i)} - s_i) + \frac{1}{2}(s_j - z_{\sigma(i)}) \geq \frac{1}{2}(s_j - s_i) \geq \frac{1}{3}|s_i - s_{\eta(i)}| \).

Claim 13. Let \( i \in S \) such that \( \sigma(i) \in L \) or \( \sigma(i) \in C \setminus K \). Then, cost\(_i(z, s) + \cos\sigma(i)(z, s) \geq \frac{1}{3}(|s_i - s_{\eta(i)}|) \).

Proof. We assume that \( \sigma(i) > i \) (the other case is symmetric). If \( z_{\sigma(i)} > s_{\sigma(i)} \), then cost\(_i(z, s) = \max\{|s_i - z_i|, |z_i - z_{\sigma(i)}|\} \geq \frac{1}{2}(z_{\sigma(i)} - s_i) > \frac{1}{2}(s_{\sigma(i)} - s_i) \geq \frac{1}{3}|s_i - s_{\eta(i)}| \), which contradicts the fact that \( i \) is a small player. Hence, \( z_{\sigma(i)} \in [s_i, s_{\sigma(i)}] \), otherwise player \( i \) would be covered. Then, cost\(_i(z, s) + \cos\sigma(i)(z, s) = \max\{|s_i - z_i|, |z_i - z_{\sigma(i)}|\} \geq \frac{1}{2}(z_{\sigma(i)} - z_i) + \frac{1}{2}(z_{\sigma(i)} - z_{\sigma(i)}) \geq \frac{1}{2}(s_{\sigma(i)} - z_i) + s_{\sigma(i)} - z_{\sigma(i)} \geq \frac{1}{3}(s_{\sigma(i)} - s_i) \).

Claim 14. Let \( i \in K \). Then, cost\(_i(z, s) - \text{excess}_i \geq \frac{1}{3}(|s_i - s_{\eta(i)}| + \sum_{j \in C_i} |s_j - s_{\eta(j)}|) \).

15
Proof. We assume that $z_i > s_i$ (the other case is symmetric). Let \( \ell \) be the player with the rightmost belief that is covered by \( i \). Then, \( \text{excess}_i = z_i - s_\ell \). We have \( \text{cost}_i(z, s) - \text{excess}_i = \max \{|s_i - z_i|, |z_i - z_{\sigma(i)}|\} - (z_i - s_\ell) \geq s_\ell - s_i = \sum_{j=1}^{i-1} (s_j + 1 - s_j) \geq \frac{1}{3} |s_i - s_{\eta(i)}| + \sum_{j \in C_i} |s_j - s_{\eta(j)}| \).  

Let \( N(S) \) denote the set of players \( j \) that are neighbors of players in \( S \) (i.e., \( j \in N(S) \) when \( \sigma(i) = j \) for some player \( i \in S \)).

Claim 15. \( N(S) \) does not contain small players.

Proof. Assume otherwise that for some player \( i \in S \), \( \sigma(i) \) also belongs to \( S \). Without loss of generality \( \sigma(i) > i \). If \( z_{\sigma(i)} \geq s_{\sigma(i)} \), then \( \text{cost}_i(z, s) \geq \frac{1}{2} |z_{\sigma(i)} - s_i| \geq \frac{1}{2} |s_{\sigma(i)} - s_i| \geq \frac{1}{2} |s_i - s_{\eta(i)}| \) contradicting the fact that \( i \in S \). So, \( z_{\sigma(i)} < s_{\sigma(i)} \). Also, \( z_{\sigma(i)} \geq s_i \) (since neither \( i \) is covered nor \( \sigma(i) \) is kangaroo). Since \( \sigma(i) \) is small, \( s_{\sigma(i)} - z_{\sigma(i)} < \frac{1}{3} |s_{\sigma(i)} - s_{\eta(\sigma(i))}| \leq \frac{1}{9} (s_{\sigma(i)} - s_i) \), i.e., \( z_{\sigma(i)} > \frac{2}{3} s_{\sigma(i)} + \frac{1}{3} s_i \). Hence, \( \text{cost}_i(z, s) \geq \frac{1}{2} (z_{\sigma(i)} - s_i) > \frac{1}{2} (s_{\sigma(i)} - s_i) \), which contradicts \( i \in S \).

Claim 16. For every two players \( i, i' \in S \), \( \sigma(i) \neq \sigma(i') \).

Proof. Assume otherwise and let \( \sigma(i) = \sigma(i') = j \) with \( i < i' \). If \( z_j \notin \{s_i, s_{i'}\} \), then the cost of either \( i \) or \( i' \) is at least \( \frac{1}{3} (s_{i'} - s_i) \), contradicting the fact that both players are small. Hence, \( z_j \in \{s_i, s_{i'}\} \). Notice that \( s_j \in \{s_i, s_{i'}\} \) as well, otherwise either \( i \) or \( i' \) would be covered by \( j \). Now the fact that \( i \) and \( i' \) are small implies that \( \text{cost}_i(z, s) + \text{cost}_{i'}(z, s) < \frac{1}{3} |s_i - s_{\eta(i)}| + \frac{1}{3} |s_{i'} - s_{\eta(i')}| \leq \frac{1}{3} (s_j - s_i) + \frac{1}{3} (s_{i'} - s_j) = \frac{1}{3} (s_{i'} - s_i) \). On the other hand, \( \text{cost}_i(z, s) + \text{cost}_{i'}(z, s) \geq \frac{1}{2} (z_j - s_i) + \frac{1}{2} (s_{i'} - s_j) = \frac{1}{2} (s_{i'} - s_i) \), a contradiction.

We now have

\[
\text{SC}(z, s) = \sum_{i=1}^{n} \text{cost}_i(z, s) \\
\geq \sum_{i \in S: \sigma(i) \in K} \left( \text{cost}_i(z, s) + \text{excess}_{\sigma(i)} \right) + \sum_{i \in S: \sigma(i) \in L \cup (C \setminus K)} \left( \text{cost}_i(z, s) + \text{cost}_{\sigma(i)}(z, s) \right) \\
+ \sum_{i \in K} \left( \text{cost}_i(z, s) - \text{excess}_i \right) + \sum_{i \in L \setminus N(S)} \text{cost}_i(z, s) \\
\geq \frac{1}{3} \sum_{i \in S: \sigma(i) \in K} |s_i - s_{\eta(i)}| + \frac{1}{3} \sum_{i \in S: \sigma(i) \in L \cup (C \setminus K)} \left( |s_i - s_{\eta(i)}| + |s_{\sigma(i)} - s_{\eta(\sigma(i))}| \right) \\
+ \frac{1}{3} \sum_{i \in K} |s_i - s_{\eta(i)}| + \sum_{j \in C_i} |s_j - s_{\eta(j)}| + \frac{1}{3} \sum_{i \in L \setminus N(S)} |s_i - s_{\eta(i)}| \\
\geq \frac{1}{3} \sum_{i=1}^{n} |s_i - s_{\eta(i)}|,
\]

as desired. The first inequality follows by the classification of the players and due to Claims 15 and 16. The second one follows by Claims 12, 13 and 14 and by the definition of large players. The last one follows since the players enumerated in the first two sums at its left cover the whole set \( S \) (by Claim 15). \( \square \)

We are ready to present our first upper bound on the price of anarchy for 1-COF games.

Theorem 17. The price of anarchy of 1-COF games over pure Nash equilibria is at most 3.
Proof. Let us consider a 1-COF game with \( n \) players and belief vector \( s \). Let \( z^* \) be an optimal opinion vector and recall that \( \eta(i) \) is the player that minimizes the distance \( |s_i - s_\eta(i)| \). By Lemma 11 we have

\[
    SC(z^*, s) \geq \frac{1}{3} \sum_{i=1}^{n} |s_i - s_\eta(i)|.
\]

(14)

Now, consider any pure Nash equilibrium \( z \) of the game. We will show that

\[
    SC(z, s) \leq \sum_{i=1}^{n} |s_i - s_\eta(i)|.
\]

(15)

The theorem will then follow by (14) and (15).

In particular, we will show that \( \text{cost}_i(z, s) \leq |s_i - s_{\eta(i)}| \) for each player \( i \). Let us assume that \( \eta(i) = i-1 \); the case \( \eta(i) = i + 1 \) is symmetric. Define \( \sigma(i) \) to be the neighbor of player \( i \) in state \( z \). We distinguish between four cases.

- **Case I:** \( \sigma(i) = i - 1 \). By Lemma 4 we have \( s_{i-1} \leq z_i \leq s_i \). Then, clearly, \( \text{cost}_i(z, s) = |s_i - z_i| \leq |s_i - s_{i-1}| \) as desired.

- **Case II:** \( \sigma(i) = i + 1 \) and \( \sigma(i - 1) = i \). By Lemmas 2 and 4 we have \( s_{i-1} \leq z_{i-1} \leq s_i \leq z_i \).

Since player \( i \) has player \( i + 1 \) as neighbor, we have \( |z_{i+1} - s_i| \leq |s_i - z_{i-1}| \). Hence, \( \text{cost}_i(z, s) = |z_i - s_i| \leq |z_{i+1} - s_i| \leq |s_i - z_{i-1}| \leq |s_i - s_{i-1}| \).

- **Case III:** \( \sigma(i) = i + 1, \sigma(i - 1) = i - 2, \) and \( \text{cost}_i(z, s) \leq \text{cost}_{i-1}(z, s) \). By the definition of \( \sigma(\cdot) \) and Lemma 2 we have \( z_{i-2} \leq z_{i-1} \leq s_{i-1} \leq s_i \leq z_{i+1} \). We have

\[
    \text{cost}_i(z, s) \leq 2 \text{cost}_{i-1}(z, s) - \text{cost}_i(z, s)
\]

\[
    = |s_{i-1} - z_{i-2}| - |z_i - s_i|
\]

\[
    \leq |z_i - s_{i-1}| - |z_i - s_i|
\]

\[
    = |s_i - s_{i-1}|.
\]

The second inequality follows since player \( i - 2 \) (instead of \( i \)) is the neighbor of player \( i - 1 \).

- **Case IV:** \( \sigma(i) = i + 1, \sigma(i - 1) = i - 2, \) and \( \text{cost}_i(z, s) > \text{cost}_{i-1}(z, s) \).

\[
    \text{cost}_i(z, s) < 2 \text{cost}_i(z, s) - \text{cost}_{i-1}(z, s)
\]

\[
    = |z_{i+1} - s_i| - |s_{i-1} - z_{i-1}|
\]

\[
    \leq |s_i - z_{i-1}| - |s_{i-1} - z_{i-1}|
\]

\[
    = |s_i - s_{i-1}|.
\]

The second inequality follows since player \( i + 1 \) (instead of \( i - 1 \)) is the neighbor of player \( i \).

This completes the proof. \( \square \)
6 Lower bounds on the price of anarchy

We begin by considering 1-COF games. First, we will show a lower bound on the price of anarchy over pure Nash equilibria.

**Theorem 18.** The price of anarchy of 1-COF games over pure Nash equilibria is at least 3.

**Proof.** Let \( \lambda \in (0, 1) \) and consider a 1-COF game with 6 players and belief vector \( s = (-10 - \lambda, -10 - \lambda, -2 - \lambda, 2 + \lambda, 10 + \lambda, 10 + \lambda) \). This game is depicted in Figure 10(a). We can show that the opinion vector \( z = (-10 - \lambda, -10 - \lambda, -6 - \lambda, 6 + \lambda, 10 + \lambda, 10 + \lambda) \) is a pure Nash equilibrium with social cost \( SC(z, s) = 8 \). The first two players suffer zero cost as they follow each other and their opinions coincide with their beliefs; the same holds also for the last two players. For the third player, it is \( \sigma(3) \in \{1, 2\} \) since \( |z_1 - s_3| = |z_2 - s_3| = 8 < |z_4 - s_3| = 8 + 2\lambda \) and \( z_3 \) is in the middle of the interval \([-10 - \lambda, -2 - \lambda] \); hence, \( \text{cost}_3(z, s) = 4 \). Similarly, we have \( \sigma(4) \in \{5, 6\} \), \( z_4 \) lies in the middle of the interval \([2 + \lambda, 10 + \lambda] \) and \( \text{cost}_4(z, s) = 4 \). Hence, \( z \) is indeed a pure Nash equilibrium.

Now, consider the opinion vector

\[
\tilde{z} = \left( -10 - \lambda, -10 - \lambda, -\frac{2 - \lambda}{3}, \frac{2 + \lambda}{3}, 10 + \lambda, 10 + \lambda \right)
\]

which yields a social cost of \( SC(\tilde{z}, s) = \frac{8 + 4\lambda}{3} \); here, again, the first and last two players have zero cost, but players 3 and 4 now each have cost \( \frac{4 + 2\lambda}{3} \) since they follow each other. The optimal social cost is upper bounded by \( SC(\tilde{z}) \) and, hence, the price of anarchy is at least

\[
\frac{SC(z, s)}{SC(\tilde{z}, s)} = \frac{3}{1 + \lambda/2},
\]

and the theorem follows by setting \( \lambda \) arbitrarily close to 0.

Our next theorem gives a lower bound on the price of anarchy over mixed Nash equilibria for 1-COF games; we remark that this lower bound is greater than the upper bound of Theorem 17 for the price of anarchy over pure Nash equilibria.

**Theorem 19.** The price of anarchy of 1-COF games over mixed Nash equilibria is at least 6.

**Proof.** Consider again the 1-COF game depicted in Figure 10(a) with 6 players and belief vector \( s = (-10 - \lambda, -10 - \lambda, -2 - \lambda, 2 + \lambda, 10 + \lambda, 10 + \lambda) \), where \( \lambda \in (0, 1) \). To simplify the following discussion,
we will refer to the first two players as the $L$ players, the third player as player $\ell$, the fourth player as player $r$, and the last two players as the $R$ players.

Let $z$ be a randomized opinion vector according to which $z_i = s_i$ for every $i \in L \cup R$, $z_\ell$ is chosen equiprobably from $\{-6 - \lambda, -6 + 3\lambda\}$, and $z_r$ is chosen equiprobably from $\{6 + \lambda, 6 - 3\lambda\}$. Since $E[\text{cost}_\ell(z, s)] = E[\text{cost}_r(z, s)] = 8 - \lambda$, we have that $E[\text{SC}(z, s)] = 16 - 2\lambda$. In the following, we will prove that $z$ is a mixed Nash equilibrium. First, observe that all players in sets $L$ and $R$ have no incentive to deviate since they follow each other and have zero cost. We will now argue about player $\ell$; due to symmetry, our findings will apply to player $r$ as well.

First, observe that $\sigma(\ell) \in L$ whenever $z_r = 6 + \lambda$, and $\sigma(\ell) = r$ whenever $z_r = 6 - 3\lambda$; each of these events occurs with probability $1/2$. Consider a deterministic deviating opinion $y$ for player $\ell$. We will show that $E[\text{cost}_\ell(z, s)] \leq E_{z_{-\ell}}[\text{cost}_\ell(y, z_{-\ell}, s)]$ for any $y$, which implies that player $\ell$ has no incentive to deviate from the randomized opinion $z_\ell$. Indeed, we have that
\[
E_{z_{-\ell}}[\text{cost}_\ell((y, z_{-\ell}), s)] = \frac{1}{2}\max\{|\lambda - 2 - \lambda - y|, |y + 10 + \lambda|\} + \frac{1}{2}\max\{|\lambda - 2 - \lambda - y|, |6 - 3\lambda - y|\} \\
\geq \frac{1}{2}(y + 10 + \lambda) + \frac{1}{2}(6 - 3\lambda - y) \\
= 8 - \lambda,
\]
where the inequality holds since $\max\{|a|, |b|\} \geq a$ for any $a$ and $b$. Hence, player $\ell$ has no incentive to deviate from her strategy in $z$, and neither has player $r$ due to symmetry. Therefore, $z$ is a mixed Nash equilibrium.

Now, consider the opinion vector
\[
\tilde{z} = \left(-10 - \lambda, -10 - \lambda, -\frac{2 - \lambda}{3}, \frac{2 + \lambda}{3}, 10 + \lambda, 10 + \lambda\right)
\]
which, as in Theorem 18 yields a social cost of $\text{SC}(\tilde{z}, s) = \frac{8 + 4\lambda}{3}$. Hence, the optimal social cost is upper bounded by $\text{SC}(\tilde{z}, s)$, and the price of anarchy over mixed equilibria is at least
\[
\frac{E[\text{SC}(z, s)]}{\text{SC}(\tilde{z}, s)} = 3 \frac{16 - 2\lambda}{8 + 4\lambda},
\]
and the theorem follows by setting $\lambda$ arbitrarily close to 0. \hfill \square

We will now present lower bounds for $k$-COF games, with $k \geq 2$. We start with the case of pure Nash equilibria and continue with the harder case of mixed equilibria. As in the case of 1-COF games, a particular game will be used in order to derive the lower bounds both for pure and mixed Nash equilibria.

**Theorem 20.** The price of anarchy of $k$-COF games over pure Nash equilibria is at least $k + 1$ for $k \geq 3$, and at least $18/5$ for $k = 2$.

**Proof.** Let $\lambda \in (0, 1)$ and consider a $k$-COF game with $3k + 3$ players that are partitioned into the following five sets. The first set $L$ consists of $k + 1$ players with $s_i = -16 - 2\lambda$ for any $i \in L$, the second set consists of a single player $\ell$ with $s_\ell = -4 - \lambda$, the third set $M$ has $k - 1$ players with $s_i = 0$ for any $i \in M$, the fourth set is a single player $r$ with $s_r = 4 + \lambda$, and the last set $R$ consists of $k + 1$ players with $s_i = 16 + 2\lambda$ for any $i \in R$. This instance is depicted in Figure 10(b).

Let $z$ be the following opinion vector: $z_i = -16 - 2\lambda$ for any $i \in L$, $z_\ell = -8 - \lambda$, $z_i = 0$ for any $i \in M$, $z_r = 8 + \lambda$, and $z_i = 16 + 2\lambda$ for any $i \in R$. It is not hard to verify that this opinion vector is a pure
Nash equilibrium with social cost \( SC(z, s) = (8 + \lambda)(k + 1) \). First, observe that all players in sets \( L \) and \( R \) have zero cost, and, hence, have no incentive to deviate to another opinion. Furthermore, no player \( i \in M \) has no incentive to deviate either since \( z_i \) lies in the middle of the interval \([-8 - \lambda, 8 + \lambda]\) which is defined by the opinions of players \( \ell \) and \( r \) who, together with the remaining players of \( M \), constitute the neighborhood \( N_i(z, s) \) of player \( i \). The cost experienced by such a player \( i \) is \( 8 + \lambda \). Finally, the neighborhood \( N_i(z, s) \) of player \( \ell \) consists of all players in \( M \) (who have opinions that are closest to \( s_\ell \)) and some player \( i \in L \); note that player \( r \) does not belong in \( N_i(z, s) \) since \( z_r - s_r = 12 + 2\lambda > 12 - \lambda = s_\ell - z_i \) for all \( i \in L \). Hence, player \( \ell \) has no incentive to deviate to another opinion since \( z_\ell \) lies in the middle of the interval \([-16 - 2\lambda, 0]\) and she experiences cost equal to \( 8 + \lambda \). Due to symmetry, player \( r \) does not have incentive to deviate as well. Hence, \( z \) is indeed a pure Nash equilibrium with \( SC(z, s) = (8 + \lambda)(k + 1) \).

We now present an opinion vector \( \tilde{z} \) with social cost \( SC(\tilde{z}, s) = 8 + 2\lambda \) for \( k \geq 3 \) and \( cost(\tilde{z}, s) = \frac{5}{3}(4 + \lambda) \) for \( k = 2 \). In particular, for \( k \geq 3 \), \( \tilde{z} \) is defined as follows: \( \tilde{z}_i = -16 - 2\lambda \) for any \( i \in L \), \( \tilde{z}_\ell = -\frac{1}{2}(4 + \lambda) \), \( \tilde{z}_r = 0 \) for any \( i \in M \), and \( \tilde{z}_i = 8 + 2\lambda \) for any \( i \in R \). Observe that all players in \( L \), \( M \), and \( R \) have zero cost, while players \( \ell \) and \( r \) have cost equal to \( 4 + \lambda \) each. For \( k = 2 \), \( \tilde{z} \) is defined as follows: \( \tilde{z}_i = -16 - 2\lambda \) for any \( i \in L \), \( \tilde{z}_\ell = -\frac{1}{2}(4 + \lambda) \), \( \tilde{z}_r = 0 \) for any \( i \in M \), and \( \tilde{z}_i = 16 + 2\lambda \) for any \( i \in R \). Again, all players in \( L \) and \( R \) have zero cost. However, players \( \ell \) and \( r \) now each have cost \( \frac{8}{3}(4 + \lambda) \) and the unique player in \( M \) has cost \( \frac{1}{3}(4 + \lambda) \).

Clearly, since \( SC(\tilde{z}, s) \) is an upper bound on the optimal social cost, we conclude that the price of anarchy over pure Nash equilibria is at least \( \frac{8 + \lambda(k + 1)}{8 + 2\lambda} \) for \( k \geq 3 \) and \( \frac{9(8 + \lambda)}{8(4 + \lambda)} \) for \( k = 2 \), and the theorem follows by setting \( \lambda \) arbitrarily close to 0.

**Theorem 21.** The price of anarchy of \( k \)-COF games over mixed Nash equilibria is at least \( k + 2 \) for \( k \geq 3 \), and at least \( 24/5 \) for \( k = 2 \).

**Proof.** As in the proof of Theorem 20 let \( \lambda \in (0, 1) \) and consider the \( k \)-COF game depicted in Figure 10(b) game with \( 3k + 3 \) players that form 5 sets. Again, the first set \( L \) consists of \( k + 1 \) players where \( s_i = -16 - 2\lambda \) for all \( i \in L \), the second set consists of a single player \( \ell \) with \( s_\ell = -4 - \lambda \), the third set \( M \) has \( k - 1 \) players with \( s_i = 0 \) for all \( i \in M \), the fourth set is a single player \( r \) with \( s_r = 4 + \lambda \), and the last set \( R \) consists of \( k + 1 \) players with \( s_i = 16 + 2\lambda \) for all \( i \in R \).

Consider the following (randomized) opinion vector \( z \): \( z_i = s_i \) for every \( i \in L \cup M \cup R \), while \( z_\ell \) is chosen uniformly at random among \([-8 - \lambda, -8 + 3\lambda]\) and \( z_r \) is chosen uniformly at random among \([8 - 3\lambda, 8 + \lambda]\). We will show that the opinion vector \( z \) is a mixed Nash equilibrium with \( \mathbb{E}[SC(z, s)] = 8k + 16 - \lambda \).

First, observe that the players in sets \( L \) and \( R \) constitute local neighborhoods, that is, \( N_i(z, s) = L \setminus \{ i \} \) for any player \( i \in L \), and \( N_i(z, s) = R \setminus \{ i \} \) for any player \( i \in R \). Hence, all these players have zero cost and no incentive to deviate.

Next, let us focus on a player \( i \in M \). Clearly, the neighborhood of player \( i \) consists of the remaining \( k - 2 \) players in \( M \) as well as players \( \ell \) and \( r \). The expected cost of player \( i \) in \( z \) is \( \mathbb{E}[cost_i(z, s)] = \frac{2}{3}(8 + \lambda) + \frac{1}{3}(8 - 3\lambda) = 8 \) since at least one of players \( \ell \) and \( r \) is at distance \( 8 + \lambda \) with probability \( 3/4 \) and both of them are at distance \( 8 - 3\lambda \) with probability \( 1/2 \). Hence, these \( k - 1 \) players contribute \( 8(k - 1) \) to the expected social cost of \( z \). We now argue that if player \( i \in M \) deviates to a deterministic opinion \( y \), her expected cost does not decrease. Clearly, if \( y \geq 3\lambda \), then this trivially holds as the expected cost of \( i \) is at least \( y - z_\ell \) which is at least \( y + 8 - 3\lambda \); the case where \( y \leq -3\lambda \) is symmetric. Hence, it suffices to consider the case where \( |y| < 3\lambda \). The expected cost of \( i \) when deviating to \( y \) is

\[
\mathbb{E}_{z_{-i}}[cost_i((y, z_{-i}), s)] = \frac{1}{4} \max\{8 + \lambda - y, y + 8 + \lambda\} + \frac{1}{4} \max\{8 + \lambda - y, y + 8 - 3\lambda\}
\]
where the inequality holds since $\max\{a, b\} \geq a$ for any $a$ and $b$.

Now, let us examine player $r$; the case of player $\ell$ is symmetric. Observe that the $k - 1$ players in $M$ always belong to the neighborhood $N_r(z, s)$ of player $r$ and it remains to argue about the identity of the last player in $N_r(z, s)$. Whenever $z_\ell = -8 + 3\lambda$, then $\ell \in N_r(z, s)$, otherwise, if $z_\ell = -8 - \lambda$, one of the players in set $R$ belongs to $N_r(z, s)$. The expected cost of player $r$ is $E[\text{cost}_r(z, s)] = 1/4(8 + \lambda) + 1/4(8 + 5\lambda) + 1/4(16 - 2\lambda) + 1/4(16 - 6\lambda) = 12 - \lambda/2$, and, hence, players $\ell$ and $r$ contribute $24 - \lambda$ to the expected social cost of $z$. It remains to show that player $r$ cannot decrease her expected cost by deviating to another opinion $y$. The expected cost of player $r$ when deviating to $y$ is

$$E_{z \rightarrow r}[\text{cost}_r((y, z_{-r}), s)] = \frac{1}{2}\max\{|16 + 2\lambda - y|, |y|\} + \frac{1}{2}(|y + 8 - 3\lambda|)$$

$$\geq \frac{1}{2}(16 + 2\lambda - y) + \frac{1}{2}(y + 8 - 3\lambda)$$

$$= 12 - \lambda/2,$$

where the inequality holds since $\max\{|a|, |b|\} \geq a$ for any $a$ and $b$. Hence, we conclude that $z$ is a mixed Nash equilibrium with expected social cost $E[\text{SC}(z, s)] = 8k + 16 - \lambda$.

As in the proof of Theorem 20 there exists an opinion vector $\tilde{z}$ with social cost $\text{SC}(\tilde{z}, s) = 8 + 2\lambda$ for $k \geq 3$ and $\text{SC}(\tilde{z}, s) = \frac{2}{3}(4 + \lambda)$ for $k = 2$. Since $\text{SC}(\tilde{z}, s)$ is an upper bound on the optimal social cost, we have that the price of anarchy over mixed equilibria is at least $\frac{8k + 16 - \lambda}{8 + 2\lambda}$ for $k \geq 3$ and $\frac{3(32 - \lambda)}{5(4 + \lambda)}$ for $k = 2$, and the theorem follows, again by setting $\lambda$ arbitrarily close to 0.

## 7 Conclusion and open problems

We have introduced the class of compromising opinion formation games. Our findings indicate that the quality of their equilibria grows linearly with the neighborhood size $k$. Still, there exists a gap between our lower and upper bounds for $k \geq 2$ and closing this gap is a challenging technical task. Furthermore, we know that mixed equilibria are strictly worse for 1-COF games but we have been unable to prove upper bounds on their price of anarchy. Is their price of anarchy still linear? Also, whether the price of stability depends on $k$ or is at most a (small) constant is an interesting and wide open question.

Another natural question is about the complexity of pure Nash equilibria in $k$-COF games for $k \geq 2$. We conjecture that there exists a polynomial time algorithm for computing them, but finding such an algorithm remains elusive. Similarly, what is the complexity of computing an optimal opinion vector (even for $k = 1$)?

Finally, our modeling assumption that the number of neighbors is the same for all players is rather restrictive. Extending our results to the general case of different neighborhood size per player deserves investigation.
References

[1] E. Anshelevich, A. Dasgupta, J. M. Kleinberg, É. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. *SIAM Journal on Computing*, 38(4):1602–1623, 2008.

[2] V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, and G. Persiano. Minority becomes majority in social networks. In *Proceedings of the 11th International Conference on Web and Internet Economics (WINE)*, pages 74–88, 2015.

[3] V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, and G. Persiano. Generalized discrete preference games. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 53–59, 2016.

[4] K. Bhawalkar, S. Gollapudi, and K. Munagala. Coevolutionary opinion formation games. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC)*, pages 41–50, 2013.

[5] V. Bilò, A. Fanelli, and L. Moscardelli. Opinion formation games with dynamic social influences. In *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*, pages 444–458, 2016.

[6] D. Bindel, J. M. Kleinberg, and S. Oren. How bad is forming your own opinion? *Games and Economic Behavior*, 92:248–265, 2015.

[7] F. Chierichetti, J. M. Kleinberg, and S. Oren. On discrete preferences and coordination. In *Proceedings of the 14th ACM Conference on Electronic Commerce (EC)*, pages 233–250, 2013.

[8] M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974.

[9] D. Fotakis, D. Palyvos-Giannas, and S. Skoulakis. Opinion dynamics with local interactions. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 279–285, 2016.

[10] N. E. Friedkin and E. C. Johnsen. Social influence and opinions. *Journal of Mathematical Sociology*, 15(3-4):193–205, 1990.

[11] R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3), 2002.

[12] P. Holme and M. E. J. Newman. Nonequilibrium phase transition in the coevolution of networks and opinions. *Physical Review E*, 74, 2006.

[13] E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. In *Proceedings of the 16th Annual Conference on Theoretical Aspects of Computer Science (STACS)*, pages 404–413, 1999.

[14] E. Mossel and O. Tamuz. Opinion exchange dynamics. *arXiv: 1401.4770*, 2014.

[15] A. Olshevsky and J. N. Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM Journal on Control and Optimization*, 48(1):33–55, 2009.
[16] N. Schwind, K. Inoue, G. Bourgne, S. Konieczny, and P. Marquis. Belief revision games. In Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI), pages 1590–1596, 2015.

[17] A. Tsang and K. Larson. Opinion dynamics of skeptical agents. In Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), pages 277–284, 2014.