Is the Universe Noise Sensitive?

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1 Noise sensitivity

Noise sensitivity is a notion related to probability and statistical physics that came up in my work with Itai Benjamini and Oded Schramm [3], which introduced this notion and mainly studies the model of percolation. A similar notion arose in the work of Tsirelson and Vershik [19], whose motivation came from mathematical quantum physics and the construction of “non-Fock spaces.” Noise sensitivity and the related notions of “non-classical stochastic processes” and “black noise” are further studied and applied to mathematical physics, theoretical computer science, social choice theory, and other areas, e.g., in [16, 11, 13, 15]. The notion of noise sensitivity applies both to classical and quantum stochastic models; see [22]. An implicit motivation for Tsirelson and Vershik’s paper was the idea that the Big Bang could be a natural occurrence of black noise. For an early high-energy physics non-Fock “toy model” see [20].

Noise sensitivity is related to some earlier works [10, 2, 8, 7], which study “harmonic analysis over the group $\mathbb{Z}/2$” of certain stochastic processes arising in combinatorics, computer science, and mathematical physics. Here, $\mathbb{Z}/2$ refers to the group of two elements.

Let me briefly describe the phenomenon of “noise sensitivity.” When you look at the spectral decomposition of various functions related to statistical physics models (like percolation) you discover that a substantial amount (or even most) of the “energy” is concentrated on eigenfunctions such that the eigenvalues are unbounded; namely, they depend on some parameter of the system that goes to infinity in the limit. The “primal” equivalent description asserts that these functions are extremely sensitive to small stochastic perturbation of the variables. For noise-sensitive models based on geometric lattice models, the eigenfunctions which support their “energy” are interesting geometric stochastic objects, leading to interesting scaling limits, and related to critical exponents.

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We refer (informally) to a stochastic process that can be regarded as the limit of stochastic functions \( f_n \) defined on ever finer lattice models as \( t \)-noise sensitive if in this representation the amount of energy on bounded eigenvalues is \( 1 - t \) of the total energy. If \( t = 0 \) we refer to the process as noise stable (or classical). The case where \( t = 1 \), that is, an (asymptotically) complete noise sensitivity, appears in various examples, some going back to [5], and it is forced in certain cases by symmetry [7, 8]. Noise sensitivity surprisingly occurs in (critical) percolation [3, 15, 14], where the scaling limit of the Fourier transform is supported (almost surely) on planar Cantor sets of dimension 3/4 [9]. Noise sensitivity also occurs for first-passage percolation [4], and the distribution of the largest eigenvalues of random matrices [18, 12]. (In some cases complete noise-sensitivity reflects “incorrect scaling,” but there are cases where noise sensitivity occurs at all scales.) Benjamini, Kalai, and Schramm showed [3, 4] that noise sensitivity necessarily emerges in very general circumstances.

Now, if you replace \( \mathbb{Z}/2 \) by a fixed group \( \Gamma \) like \( \mathbb{Z}/3 \), \( U(1) \), or \( SU(2) \) (or, more generally, consider products of a fixed graph or space), the basic notions and various results still extend, but there are some phenomena and new questions. (See, e.g., [6, 1].) Of interest is the study of “noise sensitivity” for harmonic analysis based on representations of a fixed non-Abelian group, as well as, more refined notions that take into accounts the type of representations that occur. It is also interesting to study spectral decomposition and noise sensitivity for probability distributions described by Potts and related models of interacting particles including analogous \( O(n) \)-models.

2 The universe

My very crude picture of the physicists’ view of the universe (taken mainly from popular accounts) in terms of particles corresponding to specific low-eigenvalue representations and some essentially pairwise correlations/interactions between them, corresponds to what we refer to as a “noise-stable” stochastic process. (The representations involved are of some fixed groups, be they \( U(1) \) for electromagnetism, or \( U(1) \times SU(2) \times SU(3) \) for the “standard model,” or larger but fixed groups for more general theories.) Recall from Section 1 that there are richer forms of stochastic processes where the picture is very different: much “energy” is concentrated on very “high” eigenvalues with eigenfunctions that correspond to “large” stochastic geometric objects.

Is it possible that our universe is \( t \)-noise sensitive for some \( t \), \( 0 < t < 1 \), that is, when described by a limit of discrete models does it have a substantial amount (a \( t \)-proportion) of “energy” on unbounded high eigenvalues? Such a possibility might be of no consequence for the noise-stable part describing the properties of particles and their interactions. Here are
some possible (naive) related questions:

- Is noise sensitivity related to unexplained notions of energy and mass, e.g., dark mass and dark energy?

- Is it indeed the case that the basic current models of particle physics are noise stable? (Or is there an internal inconsistency about their noise sensitivity?)

- Could noise sensitive models be of relevance regarding mathematical foundations of QED/QCD?

- Do noise-sensitive (black) stochastic perturbations of classical PDE appearing in physics have interesting or desirable properties? (Compare [21], Section 8.2.)

- Is noise sensitivity related to theories/ideas from physics on energy/mass not carried by particles?

- Suppose the universe is $t$-noise sensitive for some fraction $t$. Would this allow for string (or string-like) theory to exist in lower dimensions? In 3+1 dimensions?

It is important to point out that the definitions of the noise-sensitivity/noise-stability dichotomy require some presentation via i.i.d. variables. To make the questions about physics rigorous, extensions of the notion of noise sensitivity are required. (Otherwise, we can restrict our attention to special cases from physics where the original definitions apply.) Intuitively, for the general case, noise sensitivity describes a situation where a stochastic process cannot be described or well approximated by statistics on a bounded number of elements. Finding the right general mathematical formulation is interesting in its own right.

Of course, the main point is this: if noise stability is an implicit assumption made in current physics models for high-energy physics, and if noise sensitivity is a possibility, then this may enable us to move forward in problems where current models are insufficient. If noise stability is a law of physics or a (rather strong) consequence of current laws of physics, this is interesting as well.

References

[1] N. Alon, I. Dinur, E. Friedgut and B. Sudakov, Graph products, Fourier analysis and spectral techniques, Geom. Funct. Anal. 14 (2004), 913–940.

[2] N. Alon, G. Kalai, M. Ricklin and L. Stockmayer, Mobile users tracking and distributed job scheduling, in Proc. 33rd IEEE Annual Symposium on Foundations of Computer Science, 1992, pp. 334-343.
[3] I. Benjamini, G. Kalai, and O. Schramm, Noise sensitivity of Boolean functions and applications to percolation, *Publ. I.H.E.S.* 90 (1999), 5–43.

[4] I. Benjamini, G. Kalai, and O. Schramm, First-passage percolation has sublinear distance variance, *Ann. Probab.* 31 (2003), 1970–1978.

[5] M. Ben-Or and N. Linial, Collective coin flipping, in *Randomness and Computation* (S. Micali, ed.), New York, Academic Press, pp. 91–115, 1990.

[6] J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial, The influence of variables in product spaces, *Isr. J. Math.* 77 (1992), 55–64.

[7] J. Bourgain and G. Kalai, Influences of variables and threshold intervals under group symmetries, *Geom. Funct. Anal.* 7 (1997), 438-461.

[8] E. Friedgut and G. Kalai, Every monotone graph property has a sharp threshold, *Proc. American Mathematical Society* 124 (1996), 2993–3002.

[9] O. Schramm and S. Smirnov; G. Garban, C. Pete, and O. Schramm, works in progress.

[10] J. Kahn, G. Kalai and N. Linial, The influence of variables on Boolean functions, in *Proc. 29th Annual Symposium on Foundations of Computer Science*, pp. 68–80, 1988.

[11] G. Kalai, Noise sensitivity and chaos in social choice theory, preprint (2006).

[12] M. Ledoux, Deviation inequalities on largest eigenvalues, GAFA seminar notes, 2005.

[13] E. Mossel, R. O’Donnell and F. Oleszkiewicz, Noise stability of functions with low influence: Invariance and optimality, preprint (2005).

[14] O. Schramm, Conformally invariant scaling limit (an overview and collection of problems), Section 5, math.PR/0602151.

[15] O. Schramm and J. Steif, Quantitative noise sensitivity and exceptional times for percolation, math.PR/0504586.

[16] O. Schramm and B. Tsirelson, Trees, not cubes: Hypercontractivity, cosiness, and noise stability, *Electronic Communication in Probability* 4 (1999), 39–49.

[17] M. Talagrand, How much are increasing sets positively correlated?, *Combinatorica* 16 (1996), 243–258.

[18] C. A. Tracy and H. Widom, On orthogonal and symplectic matrix ensembles, *Comm. Math. Phys.* 177 (1996), 727-754.

[19] B. Tsirelson and A. Vershik, Examples of nonlinear continuous tensor products of measure spaces and non-Fock factorizations, *Rev. Math. Phys.* 10 (1998), 81–145.

[20] B. Tsirelson, A non-Fock fermion toy model, hep-th/9912031.

[21] B. Tsirelson, Scaling limit, noise, stability, in *Lectures on probability theory and statistics, (B. Tsirelson and W. Werner, eds.)*, pp.1–106, Lecture Notes in Math. 1840, Springer, Berlin, 2004.

[22] B. Tsirelson, Nonclassical stochastic flows and continuous products, *Prob. Surveys* 1 (2004), 173-298.