Focusing properties and focal shift of vortex Hermite-cosh-Gaussian beams

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Abstract
In this work, we investigate the focusing properties of a vortex Hermite-cosh-Gaussian beam (vHChGB) passing through a converging lens system. The analytical propagation equation as well as the beam width expression of a focused vHChGB is derived based on the Huygens-Fresnel diffraction principle. From the obtained formulae, the effects of the Gaussian Fresnel number \( N_F \) and the initial beam parameters, namely the beam order, the decentered parameter \( b \), and the vortex charge number \( m \), on the structure of the light intensity distribution and the beam spot size in the focal region are analyzed numerically. It is shown that the focal shift, which is determined from the minimum beam spot width criterion, is strongly dependent on the Fresnel number \( N_F \). And it is affected by the parameter \( b \) and the vortex charge \( m \) but is nearly insensitive to the beam order \( n \). For a fixed value of \( N_F \), the focal shift is smaller when \( b \) or \( m \) is larger. The focal shift decreases monotonously with the increase of \( N_F \) until it vanishes asymptotically for large values of \( N_F \). The obtained results may be useful for the applications of vHChGB in beam shaping and beam focusing.

Keywords Vortex Hermite-cosh-Gaussian beam · Focusing properties · Focal shift · Fresnel number

1 Introduction
In the last few years, the beam focusing has drawn the attention of laser researchers because of its theoretical and practical interests. It is known that the point of maximum intensity in the propagated field for a focused laser beam does not coincide with the geometric focus but is shifted toward the focusing lens. This effect which is referred to as focal shift is crucial due to the need for accurate determination of the actual focal plane in practical applications. The focal shift has been firstly discovered for the fundamental Gaussian beam by Li and Wolf (Li and Wolf 1981, 1982), afterwards, the phenomenon has
been investigated for various types of laser beams as Laguerre–Gaussian beams (Carter and Aburdene 1987), Bessel-Gaussian beams (Lü and Huang 1994), flat-topped beams (Borghi et al. 1998), partially coherent conical Bessel-Gauss beams (Hricha et al. 2003), hyperbolic-cosine-Gaussian beams (Hricha and Belafhal 2005a), the axisymmetric Bessel-modulated Gaussian beam (Hricha and Belafhal 2005b), Hermite–Gaussian beams (Lu and Penga 2003), Hermite–Cosh-Gaussian beams (Mei et al. 2005) and vortex cosine-hyperbolic Gaussian beams (Hricha et al. 2021a, 2022) and so on. Recently, a generalized form of the hollow vortex Gaussian beam named the vortex Hermite-cosh-Gaussian beam (vHChGB) has been introduced and its propagation properties have been investigated by our research group (Hricha et al. 2021b). The vHChGB can be generated from a standard Hermite-cosh-Gaussian beam (HChGB) passing through a spiral phase plate (Oemrawsingh et al. 2004; Kotlyar et al. 2005). The vHChGB profile has zero intensity at the center and a spirally wave-front phase which create the orbital angular momentum (OAM) (Gao et al. 2000; Allen et al. 1999; Simpson et al. 1997). In the initial plane, its intensity distribution depends on three key parameters, namely the beam order \( n \), the decentered parameter \( b \), and the vortex charge number \( m \). By choosing adequately the beam parameters values, a vHChGB can be reduced to many known hollow beams, e.g., the vortex-Gaussian beam (Zhou et al. 2013), vortex Hermite-Gaussian beam (vHGB) (Kotlyar et al. 2015; Monin and Ustinov 2018), and vortex-cosh-Gaussian beam (vChGB) (Hricha et al. 2020, 2021c), etc. In practical applications, the vHChGB can be used as an optical trap or a light spanner for micro-particles thanks to the OAM of the light (Simpson et al. 1997). More recently, the propagation properties of a vHChGB in various optical systems have been reported (Hricha et al. 2021d, 2021e, 2021f; Halba et al. 2021; Lazrek et al. 2022).

In the present paper, we investigate the focusing properties and the focal shift of a vHChGB passing through a converging lens system. In the rest of the paper, the intensity distributions of a vHChGB at the initial plane are illustrated as a function of the beam parameters in Sect. 2. Then the propagation equation of vHChGB focused by a thin lens system as well as the beam spot width of the focused vHChGB is derived in detail in Sects. 3 and 4, respectively. The characteristics of the beam intensity distribution in the focal region and the focal shift are discussed with numerical examples as a function of the Fresnel Number \( N_F \) and the beam parameters in Sect. 5. Finally, the main results are outlined in the conclusion.

\section{Field distribution of a vHChGB at the initial plane}

In the Cartesian coordinates system, the \( z \)-axis is taken to be the propagation direction. The electric field of a symmetrical vHChGB at the source plane \( z=0 \) takes the form of Hricha et al. 2021b

\[ E_n(x_0, y_0, z = 0) = (x_0 + iy_0)^m E_n(x_0, z = 0) E_n(y_0, z = 0), n = 1, 2 \ldots \]

where

\[ E_n(u, z = 0) = H_n\left(\frac{\sqrt{2} u}{\alpha_0}\right) \cosh\left(\frac{b}{\alpha_0} u\right) \exp\left(-\frac{u^2}{\alpha_0^2}\right), \]

\[ n = 1, 2 \ldots \]
with \( u = x_0 \) or \( y_0 \) and \((x, y)\) is the transverse Cartesian coordinates at the source plane. \( n \) is the mode index associated with the Hermite polynomial \( H_n(.) \) and is called the beam order. \( \cosh(.) \) is the hyperbolic-cosine function, \( \omega_0 \) is the waist size of the Gaussian part, \( b \) is the decentered parameter associated with the cosh part, and the integer parameter \( m \) is an integer that denotes the topological charge of the vortex.

If \( n = 0 \), Eq. (1a) will reduce to be the vortex-cosh-Gaussian beam (vChGB) (Hricha et al. 2020), while for \( b = 0 \) or \( m = 0 \), one will obtain the vortex Hermite–Gaussian beam (Kotlyar et al. 2015; Monin and Ustinov 2018) and HChGB (Caspersen and Tovar 1998; Tovar and Caspersons 1998; Belafhal and Ibnchaikh 2000; Hricha and Belafhal 2004, 2005c; Ibnchaikh et al. 2001; Yaalou et al. 2020), respectively. The special case \( n = b = 0 \) describes the hollow vortex-Gaussian beam (Zhou et al. 2013).

Equation (1a) can be written as

\[
E_n(x_0, y_0, z = 0) = \frac{1}{4} \exp \left( \frac{b^2}{2} \right) H_n \left( \frac{\sqrt{2}}{\omega_0} x_0 \right) H_n \left( \frac{\sqrt{2}}{\omega_0} y_0 \right) \\
\times \left\{ \exp \left[ - \left( \frac{x_0}{\omega_0} - \frac{b}{2} \right)^2 \right] + \exp \left[ - \left( \frac{x_0}{\omega_0} + \frac{b}{2} \right)^2 \right] \right\} \\
\times \left\{ \exp \left[ - \left( \frac{y_0}{\omega_0} - \frac{b}{2} \right)^2 \right] + \exp \left[ - \left( \frac{y_0}{\omega_0} + \frac{b}{2} \right)^2 \right] \right\} (x_0 + iy_0)^m.
\]

(2)

This means that a vHChGB can be obtained in practice by a superposition of four Hermite-decentered Gaussian modes of order \( n \) with the same vortex charge \( m \).

![Fig. 1](image-url) The intensity distribution of a vHChGB at the source plane with \( \omega_0 = 1 \text{ mm} \) and \( m = 1 \) for different values of \( n \) and \( b \)
Typical intensity distribution patterns of a vHChGB at the source plane are illustrated in Figs. 1 and 2 for different values of $n$, $b$, and $m$. For convenience, in all the following numerical calculations, the waist width $\omega_0$ is set to be 1 mm. The plots of Figs. 1 and 2 show that the vHChGB is hollow dark-like, with a central dark region surrounded by an array spot structure. The beam profile is mirror-symmetric, and the intensity distribution can either have a multi-spots pattern or a four-spot pattern depending on the value of the parameter $b$. For a small $b$ ($b < 1$), the beam exhibits $4n$ lobes with the four main lobes
located at the square vertices of the beam spot (see Fig. 1). While for a large $b$, the beam is four-petal like for an arbitrary value of $n$. In addition, one can observe an elongation of the main lobes which becomes stronger as $m$ is larger (see Fig. 2).

### 3 Focusing of a vHChGB by a thin lens system

Let us consider an incident vHChGB passing through an aplanatic lens of focal length $f$, as schematized in Fig. 3. Assuming that the initial beam is placed at the lens plane ($z=0$), then the $ABCD$ transfer matrix of the optical system reads as

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix} 1 - \frac{s f}{s f} & \frac{s f}{s f} \\
-\frac{1}{f} & 1
\end{pmatrix},
$$

where $s = \frac{z}{f}$ is the normalized propagation distance, with $z$ is the distance from the lens plane to the output plane. $A$, $B$, $C$, and $D$ are the matrix elements associated with the lens system.

Within the paraxial approximation, the propagation of a vHChGB through the optical system obeys the Huygens Fresnel diffraction integral, which can be expressed as (Collins 1970)

$$
E(x, y, z) = i \frac{N_F}{\omega_0^2 s} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_n(x_0, y_0, 0) \exp \left\{ -\frac{i \pi N_F}{\omega_0^2 s} \left[ (1-s)(x_0^2 + y_0^2) - 2(x x_0 + y y_0) + (x^2 + y^2) \right] \right\} dx_0 dy_0,
$$

where $E(x, y, z)$ is the field at the receiver plane $z$, and $N_F = \frac{\alpha \omega_0^2}{2f}$ is the Gaussian Fresnel number. In Eq. (4), an unimportant propagation term was omitted for convenience.

Substituting from Eqs. (1a) into Eq. (4), and recalling the binomial formula

$$
(x_0 + iy_0)^m = \sum_{l=0}^{m} C_m^l x_0^l (iy_0)^{m-l},
$$

where

$$
C_m^l = \frac{m!}{l!(m-l)!},
$$

and after some algebraic arrangements, Eq. (4) can be expressed as

![Fig. 3 Schematic illustration of the focusing lens system](image-url)
\[ E(x, y, z) = \frac{i N_F}{\omega_0^2 s} \exp \left[ -i \frac{\pi N_F}{\omega_0^2 s} (x^2 + y^2) \right] \sum_{l=0}^{m} C_l^m (i)^{m-l} A_\ell(x) A_{m-l}(y), \] 

(6a)

where

\[ A_q(t) = \int_{-\infty}^{+\infty} u_0^q H_n \left( \frac{\sqrt{2} u_0}{\omega_0} \right) \cosh \left( b \frac{u_0}{\omega_0} \right) \exp \left\{ - \left( \frac{1}{\omega_0^2} + \frac{i \pi N_F (1 - s)}{\omega_0^2 s} \right) u_0^2 + \frac{i 2 \pi N_F}{\omega_0^2 s} t u_0 \right\} du_0. \] 

(6b)

with \( t = x \) or \( y \), \( q = l \) or \( m-l \).

Now, by using the definition of \( \cosh \) function, Eq. (6b) can be written as

\[ A_q(t) = \frac{1}{2} \left[ B_q^+(t) + B_q^-(t) \right], \] 

(7a)

where

\[ B_q^\pm(t) = \int_{-\infty}^{+\infty} u_0^q H_n \left( \frac{\sqrt{2} u_0}{\omega_0} \right) \exp \left( -\eta u_0^2 + \frac{i 2 \pi N_F}{\omega_0^2 s} t \pm \frac{b}{\omega_0} \right) du_0. \] 

(7b)

with

\[ \eta = \frac{1}{\omega_0^2} + \frac{i \pi N_F (1 - s)}{\omega_0^2 s}. \] 

(7c)

Recalling the following formulas (Gradshteyn and Ryzhik 1994; Belafhal et al. 2020)

\[ H_n(x) = \sum_{p=0}^{[n/2]} \frac{(-1)^p n!}{p! (n-2p)!} (2x)^{n-2p}, \] 

(8)

and

\[ \int_{-\infty}^{+\infty} x^m \exp \left( -px^2 + 2qx \right) dx = \sqrt{\frac{\pi}{p}} \exp \left( \frac{q^2}{p} \right) \left( \frac{1}{2i \sqrt{p}} \right)^m H_m \left( \frac{i q}{\sqrt{p}} \right), \quad \text{Re}(p) > 0. \] 

(9)

and after carrying on the tedious integral calculations, Eq. (7b) turns out to be

\[ B_q^\pm(t) = \sqrt{\frac{\pi}{\eta}} \left( \frac{1}{2i \sqrt{\eta}} \right)^q \sum_{p=0}^{[n/2]} \frac{(-1)^p n!}{p! (n-2p)!} \left( \frac{\sqrt{2}}{\omega_0 b \sqrt{\eta}} \right)^{n-2p} \left( f^\pm(t) H_{q+n-2p} \left( \frac{i \pi N_F}{\omega_0^2 s} t \pm \frac{b}{2 \omega_0} \right) \right), \] 

(10a)

where

\[ f^\pm(t) = \exp \left\{ \frac{1}{4 \eta} \left( \frac{i 2 \pi N_F}{\omega_0^2 s} t \pm \frac{b}{\omega_0} \right)^2 \right\}. \] 

(10b)

Substituting from Eqs. (10a) and (7a) into Eq. (6a), one obtains
where

\begin{equation}
F_{n,l}^{\pm}(u) = \sum_{p=0}^{\infty} \frac{n!}{p!(n-2p)!} \left( \frac{\beta}{2s} \right)^p \exp \left( \pm \frac{i\pi b N_F}{\beta} u \right) H_{l+n-2p} \left( \frac{1}{2\sqrt{s\beta}} \left( \pm ibs - 2\pi N_F u \right) \right),
\end{equation}

and

\begin{equation}
F_{m-l}^{\pm}(v) = \exp \left( \pm \frac{i\pi b N_F}{\beta} v \right) H_{m-l} \left( \frac{1}{2\sqrt{s\beta}} \left( \pm ibs - 2\pi N_F v \right) \right).
\end{equation}

Equation (11a) is the closed-form expression of a vHChGB passing through a thin lens system, which is a convenient way to study the propagation characteristics of the focused beam. From Eq. (11a) one can distinguish many special cases, which are described as follows:

(i) When \( m = 0 \), Eq. (11a) will give the field expression of a focused vChGB, which can be expressed as

\begin{equation}
E(u, v, s) = \frac{i\pi N_F}{2m+2} \left( \frac{s}{\beta} \right)^n \exp \left\{ \frac{sb^2}{2\beta} - \frac{i\pi N_F(1 - i\pi N_F)}{\beta} (u^2 + v^2) \right\}
\end{equation}

\begin{equation}
\times \sum_{l=0}^{m} C_l^m i^{-l} \left[ F_l^{+}(u) + F_l^{-}(u) \right] \left[ F_{m-l}^{+}(v) + F_{m-l}^{-}(v) \right],
\end{equation}

where

\begin{equation}
F_l^{\pm}(u) = \exp \left( \frac{\pm i\pi b N_F}{\beta} u \right) H_l \left( \frac{1}{2\sqrt{s\beta}} \left( \pm ibs - 2\pi N_F u \right) \right),
\end{equation}

and

\begin{equation}
F_{m-l}^{\pm}(v) = \exp \left( \frac{\pm i\pi b N_F}{\beta} v \right) H_{m-l} \left( \frac{1}{2\sqrt{s\beta}} \left( \pm ibs - 2\pi N_F v \right) \right).
\end{equation}

Equation (12a) is consistent with the result obtained in Ref. (Hricha et al. 2022). (ii) In the limiting case \( m = 0 \), i.e., in the absence of the vortex, Eq. (11a) reduces to

\begin{equation}
E(u, v, s) = \frac{i\pi N_F}{\beta} \exp \left\{ \frac{s}{2\beta} b^2 + Q(u^2 + v^2) \right\} \cosh(Su) \cosh(Sv),
\end{equation}

where

\begin{equation}
S = \frac{i\pi b N_F}{\beta},
\end{equation}

\( (12b) \)

\( (12c) \)
and
\[ Q = -\frac{i\pi N_F (1 - i\pi N_F)}{\beta}. \] (13c)

Equation (13a) is the field expression of the focused ChGB. This result is consistent with Eq. (7a) of Ref. (Hricha and Belafhal 2005a).

(iii) If one takes \( m = n = 0 \), Eq. (11a) will be simplified to
\[ E(u, v, s) = \frac{i\pi N_F}{\beta} \exp \left\{ Q(u^2 + v^2) \right\}, \] (14)

which is the propagation formula of the focused Gaussian beam (Li and Wolf 1981, 1982).

The irradiance \( I(u, v, s) \) of the focused vHChGB is defined as the squared modulus of the field,
\[ I(u, v, s) = |E(u, v, s)|^2. \] (15)

By substituting from Eq. (11a) into Eq. (15), and after some algebraic operations we obtain the irradiance of the focused vHChGB, which can be expressed as
\[
I(u, v, s) = I_a(s) \exp \left\{ -2 \left( \frac{\pi N_F}{s} \right)^2 (u^2 + v^2) \right\} \times \sum_{l=0}^{m} C_l^m (i)^{-l} \times \sum_{\kappa=0}^{m} C_k^m (-i)^{-\kappa} \times \left[ G_{n,(l,\kappa)}^{-}(u) + G_{n,(l,\kappa)}^{+}(u) \right] \left[ G_{n,(m-l,m-\kappa)}^{-}(v) + G_{n,(m-l,m-\kappa)}^{+}(v) \right],
\] (16a)

where
\[
G_{n,(l,\kappa)}^{\pm}(u) = \sum_{p=0}^{\left\lfloor \frac{l}{2} \right\rfloor} \frac{n!}{p!(n-2p)!} \sum_{r=0}^{\left\lfloor \frac{\kappa}{2} \right\rfloor} \frac{n!}{r!(n-2r)!} \left( \frac{\beta}{2s} \right)^{r+p} \times H_{l+n-2p}(\delta(u \mp \alpha s)) \left\{ \exp \left( \pm 2S_1 u \right) H_{k+n-2r}(\delta^*(u \mp \alpha s)) \right\} + \exp \left( \pm 2S_2 u \right) H_{k+n-2r}(\delta^*(u \mp \alpha s)) \right\},
\] (16b)

where
\[
I_a(s) = \frac{(\pi/4)^2(s)^{m+2n}(2)^{2n-2m}}{\left[ \left( \frac{s}{N_F} \right)^2 + (\pi(s-1))^2 \right] \left[ (s)^2 + (\pi N_F(s-1)) \right]^{\frac{\pi}{2} + n}} \times \exp \left\{ b^2 \left[ 1 + \left( \frac{\pi N_F}{s} (s-1) \right)^2 \right] \right\},
\] (16c)

with \( S_1, S_2, \delta, \text{ and } \alpha \) are given, respectively by
The on-axis intensity of the focused beam is obtained by substituting \((u, v) = (0, 0)\) into Eq. (16a), so one obtains

\[
I(s) = I_a(s) \times \sum_{l=0}^{m} C_l^m(i)^{-l} \times \sum_{k=0}^{m} C_k^m(-i)^{-k} \times \left[ G_{n,(l,k)}^-(s) + G_{n,(l,k)}^+(s) \right] \left[ G_{n,(m-l,m-k)}^-(s) + G_{n,(m-l,m-k)}^+(s) \right],
\]

where

\[
G_{n,(l,s)}^\pm (s) = \sum_{p=0}^{l} \frac{n!}{p!(n-2p)!} \sum_{\tau=0}^{\frac{l}{2}} \frac{n!}{\tau!(n-2\tau)!} \left( \frac{\beta}{2s} \right)^{\tau+p} \times H_{l+n-2p}(\delta(\mp as)) \times H_{k+n-2\tau}(\delta^*(\mp as)) + H_{k+n-2\tau}(\delta^*(\mp as)).
\]

It is readily seen from Eq. (17a) that the on-axis intensity of the focused vHChG beam is generally not always zero, i.e., there may exist some special cases (i.e., for particular values for \(n, b,\) and \(m\)) in which the focused field may exhibit a central bright intensity. From our numerical results (see Sect. 4), it is found that when \(m=2,\) and \(n=1, 3,\) the focused vHChGB has a central peak intensity.

### 4 Beam with the focused vHChGB

To further analyze the propagation properties of a focused vHChGB, it is useful to investigate the beam spot width evolution (in the sense of root mean square width or rms) within the focus region.

As it is known, the rms width for a general laser beam can be defined from the second-order moment intensity as (Siegman 1990; Weber 1992)
\[ W_\sigma = \left( \frac{4}{P_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma^2 |E(x, y, z)|^2 \, dx \, dy \right)^{1/2}, \]  

(18a)

where \( \sigma \) denotes either \( x \) or \( y \) transverse coordinate, and \( P_0 \) is the total power of the beam that is given by

\[ P_0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |E(x_0, y_0, z = 0)|^2 \, dx_0 \, dy_0. \]  

(18b)

Because of the symmetry of the vHChGB, the second-order moments in the \( x \)-direction and \( y \)-direction are identical, so in the following, only the calculation steps for the \( x \)-direction will be presented. For a vHChGB, the \( P_0 \) expression can be developed as

\[ P_0 = \sum_{l=0}^{m} C_l^m A_l(b)A_{m-l}(b), \]  

(19a)

where the typical integral \( A_R(b) \) is given as

\[ A_R(b) = \int_{-\infty}^{+\infty} u_0^{2R} H_n^2 \left( \frac{\sqrt{2} u_0}{\omega_0} \right) \cosh^2 \left( b \frac{u_0}{\omega_0} \right) \exp \left( -\frac{2 u_0^2}{\omega_0^2} \right) \, du_0. \]  

(19b)

By using the Eqs. (8) and (9), and after straightforward integral calculations, \( A_R(b) \) can be expressed as

\[
A_R(b) = \frac{1}{2} \sqrt{\pi} \left( \frac{1}{2i} \right)^{2R} \left( \frac{w_0}{\sqrt{2}} \right)^{2R+1} \sum_{p=0}^{\left\lceil \frac{n}{2} \right\rceil} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\left\lceil \frac{n}{2} \right\rceil} \frac{n!(-1)^n}{q!(n-2q)!} \exp \left( \frac{b^2}{2} \right) H_{2R+2n-2p-2q} \left( \frac{ib}{\sqrt{2}} \right) + H_{2R+2n-2p-2q} \left( 0 \right). \]  

(19c)

As can be seen, it is too complicated to get the beam width directly from Eq. (11a). Fortunately, one can that by using the ABCD law of the second-order moments, which is given as (Weber 1992)

\[ \langle x^2 \rangle = A^2 \langle x_0^2 \rangle + 2AB \langle x_0 \theta \rangle + B^2 \langle \theta^2 \rangle, \]  

(20)

where \( x \) and \( x_0 \) are the transverse coordinates in the output plane \( z \) and input plane \( z=0 \), respectively, and the second-order intensity moments at the input plane \( z=0 \), are given by Weber (1992)

\[ \langle x_0^2 \rangle = \frac{1}{P_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_0^2 |E(x_0, y_0, z = 0)|^2 \, dx_0 \, dy_0, \]  

(21a)
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\[ \langle x_0 \theta_0 \rangle = \frac{1}{2ikP_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_0 E(x_0, y_0, z = 0) \frac{\partial E^* (x_0, y_0, z = 0)}{\partial x_0} dx_0 dy_0 + c.c. \tag{21b} \]

and

\[ \langle \theta_0^2 \rangle = -\frac{1}{P_0 k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x_0, y_0, z = 0) \frac{\partial^2 E^* (x_0, y_0, z = 0)}{\partial x_0^2} dx_0 dy_0 + c.c. \tag{21c} \]

As the initial field is real valued, one can state then from Eq. (21b) that \( \langle x_0 \theta_0 \rangle = 0 \).

For the lens system, Eq. (20) reduces to

\[ \langle x^2 \rangle = (s - 1)^2 \langle x_0^2 \rangle + \frac{w_0^4}{\lambda^2 N_F^2} s^2 \langle \theta_0^2 \rangle. \tag{22} \]

Substituting from Eq. (1) into Eq. (21a), and after some algebraic manipulations, one obtains

\[ \langle x_0^2 \rangle = \frac{1}{P_0} \sum_{l=0}^{m} C_l^m A_{l+1}(b) A_{m-l}(b). \tag{23} \]

Substituting from Eq. (1) into Eq. (21c) and using the same calculation technique as above, then after algebraic operations, the result can be expressed as

\[ \langle \theta_0^2 \rangle = -\frac{1}{P_0 k^2} [Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + Q_{10}], \tag{24} \]

where

\[ Q_1 = m(m - 1) \sum_{l=0}^{m-2} C_l^m \left[ A_{l+1}(b) A_{m-l-2}(b) - A_l(b) A_{m-l-1}(b) \right], \tag{25a} \]

\[ Q_2 = \left( \frac{2\sqrt{2}}{\omega_0} \right)^2 n(n - 1) \sum_{l=0}^{m} C_l^m X_l(b) A_{m-l}(b), \tag{25b} \]

\[ Q_3 = \left( \frac{b}{w_0} \right)^2 P_0, \tag{25c} \]

\[ Q_4 = \frac{2P_0}{w_0^2} \left[ \frac{2}{w_0^2} \langle x_0^2 \rangle - 1 \right], \tag{25d} \]

\[ Q_5 = \left( \frac{4\sqrt{2}}{\omega_0} \right) nm \sum_{l=0}^{m-1} C_l^{m-1} \xi_l(b) A_{m-l-1}(b), \tag{25e} \]
\[ Q_6 = \left( \frac{2mb}{w_0} \right)^{m-1} \sum_{l=0}^{m-1} C_{l}^{m-1} \xi_l(b) A_{m-l-1}(b), \]  
(25f)

\[ Q_7 = \left( \frac{-4m}{w_0^2} \right)^{m-1} \sum_{l=0}^{m-1} C_{l+1}^{m-1} A_{l+1}(b) A_{m-l-1}(b), \]  
(25g)

\[ Q_8 = \left( \frac{-2}{\omega_0^2} \right) \left( \frac{4\sqrt{2}}{\omega_0} \right) n \sum_{l=0}^{m} C_{l}^{m} B_l(b) A_{m-l}(b), \]  
(25h)

\[ Q_9 = \left( \frac{-4}{w_0^2} \right) \left( \frac{b}{w_0} \right) \sum_{l=0}^{m} C_{l}^{m} \xi_l(b) A_{m-l}(b), \]  
(25i)

and

\[ Q_{10} = \left( \frac{b}{\omega_0} \right) \left( \frac{4\sqrt{2}}{\omega_0} \right) n \sum_{l=0}^{m} C_{l}^{m} G_l(b) A_{m-l}(b), \]  
(25j)

where \( A_R(b) \) is given by Eq. (19e) and \( \xi_R(b), \xi_R(b) X_R(b) \) and \( G_R(b) \) are indicated by

\[ \xi_R(b) = \frac{1}{2} \sqrt{\pi} \left( \frac{1}{2i} \right)^{2R+1} \left( \frac{w_0}{\sqrt{2}} \right)^{2R+2} \sum_{p=0}^{\frac{\varepsilon}{2}} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\frac{\varepsilon}{2}} \frac{n!(-1)^n}{q!(n-2q)!} \]  
(25k)

\[ \times \exp \left( \frac{b^2}{2} \right) H_{2R+2n-2p-2q+1} \left( \frac{ib}{\sqrt{2}} \right), \]

\[ X_R(b) = \frac{1}{2} \sqrt{\pi} \left( \frac{1}{2i} \right)^{2R} \left( \frac{w_0}{\sqrt{2}} \right)^{2R+2} \sum_{p=0}^{\frac{\varepsilon}{2}} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\frac{\varepsilon}{2}} (-1)^n(n-2)\]  
(25l)

\[ \left[ \exp \left( \frac{b^2}{2} \right) H_{2R+2n-2p-2q-2} \left( \frac{ib}{\sqrt{2}} \right) + H_{2R+2n-2p-2q-2}(0) \right], \]

\[ \xi_R(b) = \frac{1}{4} \sqrt{\pi} \left( \frac{1}{2i} \right)^{2R} \left( \frac{w_0}{\sqrt{2}} \right)^{2R+2} \sum_{p=0}^{\frac{\varepsilon}{2}} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\frac{\varepsilon}{2}} (-1)^n(n-1)\]  
(25m)

\[ \left[ \exp \left( \frac{b^2}{2} \right) H_{2R+2n-2p-2q-2} \left( \frac{ib}{\sqrt{2}} \right) + H_{2R+2n-2p-2q-2}(0) \right], \]

and
Hence, the propagation equation of the beam width for a focused vHChGB is obtained directly by substituting Eqs. (23–25) into Eq. (22). The obtained formula indicates that the beam width of a focused vHChGB depends explicitly on the initial beam parameters and the Fresnel number, so it will be convenient to investigate the beam evolution within the focus region.

5 Numerical calculations and analysis

Based on the analytical formulas obtained above, numerous numerical examples are performed to illustrate in detail the focusing properties as well a focal shift behavior of a vHChGB in the focus region with different beam parameters conditions. The calculation parameters are set to be $\lambda=632.8$ nm and $\omega_0=1$ mm, and $f=50$ mm. As previously indicated, the initial vHChGB can have two types of profile depending on the magnitude of $b$ (see Fig. 1 in Sect. 2), thus in the following the two-beam configurations are investigated separately.

Figure 4 displays the irradiance distribution (contour lines) of a focused vHChGB at different propagation distances.

The plots show an asymmetrical evolution of the irradiance around the geometric focus plane ($z=f$). one can see that the beam energy is focused slightly just before the lens focus. This means that the actual focus of the beam is shifted toward the lens. In addition, one can see that nearby the focus the irradiance distribution for a vHChGB with $m=1$ is fan blades-like; the beam profile consists of $4n$ main blade spots and a hole intensity at the beam center. The direction of rotation of the light blades changes from anticlockwise to clockwise as the beam passes beyond the actual focal plane.

The calculation parameters used in Fig. 5 are the same as those in Fig. 4 except that the vortex charge is $m=2$.

It can be seen from this figure that when $n=2$, the beam profile is quite similar to that obtained for $m=1$ in Fig. 4. While for a value of small $b$ and odd value of $n$ ($n=1, 3$), the output field exhibits a central bright lobe (see Fig. 6). One can also notice that in far-field a VHChGB with a large value of $b$ has four main lobes surrounding several faint secondary lobes.

The beam spot width of the focused vHChGB is calculated from Eq. (22) and its variation against the propagation distance under different parameter conditions is shown in Fig. 7, from which it can be seen that the beam spot size reaches a minimum at a certain plane $z_m$ just before the lens focus. In addition, one can note that when $b$ has a large value the output beam has almost the same width at the real focus even if $n$ varies (see the bottom row in Fig. 7), and in this case, the focal shift is slightly affected by the change of $n$. When $m=1, 2,$ and $4$, one can see that the real focus plane is more shifted when $m$ is smaller.

The focal shift behavior versus the initial beam parameters is sown in Fig. 8. It can be seen that the focal shift takes always a negative value, and decreases monotonously with
the increase of $N_F$ until it conceals out asymptotically for larger values of $N_F$. Furthermore, one can easily note that for a fixed $N_F$, the focal shift is smaller when $b$ is larger, and it is nearly insensitive to the beam order value, this confirms the result obtained in Fig. 7.

### 6 Conclusion

Based on the Huygens-Fresnel diffraction integral, we have derived the analytical expression for a vHChGB focused by a thin lens system. The closed-form expression of the beam width is obtained by using the second-moment intensity method. The numerical examples show that the intensity distribution of a focused vHChGB depends on the Fresnel number $N_F$ and the initial beam parameters. It is found that the focal shift, which is determined by using the minimum beam width criterion, is strongly dependent on the Fresnel number $N_F$ and the decentered parameter $b$. And it is slightly altered by the change of the vortex charge $m$, but it is nearly insensitive to the beam order $n$. For a fixed value of $N_F$, the focal shift is smaller when the parameter $b$ or $m$ is larger.
Fig. 5 the parameters are the same as Fig. 4, except $m = 2$

Fig. 6 Contour lines of irradiance distribution of vHChGB with $m = 2$ at the real focal plane $z_m$ with $w_0 = 1 \text{ mm}$, $m = 2$, $N_F = 1$ for different values of $b$ and $n$
The focal shift decreases monotonously with the increase of $N_F$ until it vanishes asymptotically for large $N_F$. The obtained results may be useful for the applications of the vHChGB in beam shaping and beam focusing.

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**Declarations**

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