AN APPROXIMATION FOR THE rp-PROCESS

FELIX REMBGES, CHRISTIAN FREIBURGHUS, THOMAS RAUSCHER, AND FRIEDRICH-KARL THIELEMANN

In the present investigation we present an approximation scheme that leads to accuracy of more than 15% for the energy generation in hot hydrogen burning from $10^8 - 1.5 \times 10^9$ K, which covers the whole range of all presently known astrophysical sites. It is based on the concept of slowly varying hydrogen and helium abundances and assumes a kind of local steady flow by requiring that all reactions entering and leaving a nucleus add up to a zero flux. This scheme can adapt itself automatically and covers low-temperature regimes, characterized by a steady flow of reactions, as well as high-temperature regimes where a $(p, \gamma) - (\gamma, p)$-equilibrium is established, while $\beta^+$-decays or $(\alpha, p)$-reactions feed the population of the next isotonic line of nuclei.

In addition to a gain of a factor of 15 in computational speed over a full-network calculation and energy generation accurate to more than 15% this scheme also allows the correct prediction of individual isotopic abundances. Thus, it delivers all features of a full network at a highly reduced cost and can easily be implemented in hydrocalculations.

Subject headings: novae, cataclysmic variables — nuclear reactions, nucleosynthesis, abundances — X-ray: bursts

1. INTRODUCTION

Close binary stellar systems can exchange mass when at least one of the stars fills its Roche lobe. After a critical mass $\Delta M$ of unburned transferred matter is accumulated on the surface of the accreting star, ignition sets in, typically under degenerate conditions when the accreting object is a white dwarf or neutron star. Degenerate conditions for which the pressure is not a function of temperature prevent temperature adjustment via pressure increase and expansion and cause a thermonuclear runaway and explosive burning. For white dwarfs, a layer of $10^{-12} M_\odot$ forms, before pycnonuclear ignition of hydrogen burning sets in (see, e.g., for white dwarfs, a layer of $10^{-12} M_\odot$ and cause a thermonuclear runaway and explosive burning. For white dwarfs, a layer of $10^{-12} M_\odot$ and cause a thermonuclear runaway and explosive burning.

The burning takes 100–1000 s before the partially burned hydrogen envelope is ejected.

X-ray bursts (for an observational overview, see Lewin, van Paradijs, & Taam 1993) involve accreting neutron stars with an unburned, hydrogen-rich surface layer. The critical size of the hydrogen layer before ignition is as small as $10^{-12} M_\odot$. Temperatures of the order of $(1-2) \times 10^{8}$ K and densities of $\approx 10^6-10^7$ g cm$^{-3}$ are attained (see e.g., Wallace & Woosley 1981; Ayasli & Joss 1982; Hanawa, Sugimoto, & Hashimoto 1983; Woosley, Taam, & Weaver 1986; Taam et al. 1993; Taam, Woosley, & Lamb 1996). This explosive burning with rise times of about 1–10 s leads to the release of $10^{39}-10^{40}$ ergs. Many of the observed features and general characteristics are understood; however, there is still a lack of a quantitative understanding of the detailed observational data. Another aspect is that the explosion energies are smaller than the gravitational binding energy of the accreted hydrogen envelope. An evenly distributed explosion energy would not unbind and eject matter. It remains to be seen whether an interesting amount of matter can escape the neutron star. Super-Eddington X-ray bursts (Taam et al. 1996) are the best candidates for this behavior.

A description of hot (explosive) hydrogen burning has been given by Arnould et al. (1980), Wallace & Woosley (1981), Ayasli & Joss (1982), Hanawa et al. (1983), Wiescher et al. (1986), van Wormer et al. (1994), and Thielemann et al. (1994) (for Thorne-Zytkow objects, see Biel le 1991, 1994; Cannon et al. 1992; and Cannon 1993; but see the recent results in Fryer, Benz, & Herant 1996 that raise doubts about their existence). The burning is described by proton captures, $\beta^+$-decays, and possibly $\alpha$-induced reactions on unstable proton-rich nuclei, usually referred to as the rapid proton capture process, or rp-process. Cross sections can be obtained from either the best available application of present experimental knowledge, e.g., the determination of resonance properties from mirror nuclei and transfer and/or charge exchange reaction studies, or actual cross section measurements, like those for $^{13}\text{N}(p, \gamma)^{14}\text{O}$, the first reaction cross section analyzed with a radioactive ion beam facility (see Champagne & Wiescher 1992; and references therein).

There exist two major motivations in nuclear astrophysics: (1) to understand the reaction flow to a necessary degree, in
order to predict the correct energy generation required for hydrodynamic, astrophysical studies, and (2) to predict a detailed isotopic composition that helps to understand the contribution of the process in question to nucleosynthesis in general. Our main motivation in this paper is the first; to provide a fast energy-generation network as a tool for nova and X-ray burst studies. This might be underlined by the fact that novae and X-ray bursts seem not to be major contributors to nucleosynthesis because of the small ejected masses involved and by the question whether gravitational binding can be overcome at all. (However, novae can be important for nuclei like \(^{16}\text{O}\), \(^{22}\text{Ne}\), \(^{26}\text{Al}\), and even Si and S, and super-Eddington X-ray bursts [Taam et al. 1996] will be able to eject some matter, probably containing some light \(p\)-process elements.) It has become apparent that our efficient approximation and energy-generation scheme can also be used to predict abundances accurately. The nucleosynthesis for nuclei above Kr is, however, discussed in a second paper (Schatz et al. 1997).

In order to understand energy generation correctly, we have to be able to understand the main reaction fluxes. In the past we performed a series of \(rp\)-process studies (Wiescher et al. 1986, to Ar; van Wormer et al. 1994, to Kr). The reaction rate predictions were based on resonance and direct-capture contributions for proton-rich nuclei, making use of the most recent experimental data. Several \((p, \gamma)\)-reaction rates below mass \(A = 44\) have been recalcualted by Herndl et al. (1995) in the framework of a shell model description for the level structure of the compound nucleus. A preliminary analysis of the major aspects has been given by Thielemann et al. (1994). In § 2 we will present this in more detail and discuss the constraints that an approximation scheme has to fulfill in the whole range of temperatures occurring in explosive hydrogen-burning environments, i.e., \(10^7-1.5 \times 10^8\) K. The approximation scheme will be presented in § 3 and its application and comparison with full-network calculations in § 4, followed by concluding remarks in § 5.

2. REACTION FLOWS IN EXPLOSIVE HYDROGEN BURNING

At low temperatures, the \(rp\)-process is dominated by cycles of two successive proton captures, starting out from an even-even nucleus, a \(\beta^+\)-decay, a further proton capture (into an even-\(Z\) nucleus), another \(\beta^+\)-decay, and a final \((p, \alpha)\)-reaction close to stability (similar to the hot CNO). The final closing of the cycle via a \((p, \alpha)\)-reaction occurs because proton-capture \(Q\)-values increase with increasing neutron number \(N\) in an isotopic chain for a given \(Z\), while \(\alpha\)-capture \(Q\)-values or thresholds show a very weak dependence on \(N\). Thus, a cycle closure occurs at the most proton-rich compound nucleus, which has a lower \(\alpha\) than proton threshold. These are the even-\(Z\) and especially well-bound \(^{16}\alpha\) nuclei with isospin \(T_z = (N - Z)/2 = 0\), like \(^{16}\text{O}\), \(^{20}\text{Ne}\), \(^{24}\text{Mg}\), \(^{28}\text{Si}\), \(^{32}\text{S}\), \(^{36}\text{Ar}\), and \(^{40}\text{Ca}\). Therefore, \((p, \alpha)\)-reactions can operate on odd-\(Z\) targets with \(T_z = \pm \frac{1}{2}\), \(^{15}\text{N}\), \(^{19}\text{F}\), \(^{21}\text{Na}\), \(^{27}\text{Al}\), \(^{31}\text{P}\), \(^{35}\text{Cl}\), and \(^{39}\text{K}\). The hydrogen-burning cycles are connected at these nuclei because of possible competition between the \((p, \gamma)\) and \((p, \alpha)\)-reaction. An exception to the rule is the Ne isotopic chain, in which the transition happens already for the compound nucleus \(^{18}\text{Ne}\). Thus, \(^{18}\text{F}(p, \alpha)^{15}\text{O}\) is a possible reaction, bypassing the OF(\(\text{Ne}\)) cycle. Therefore, only the extended CN(\(\text{O}\)), NeNa, MgAl, S\(\beta\)P, S\(\beta\)Cl, and other such cycles exist. This is displayed in Figure 1.

The progress of burning toward heavier nuclei depends on the leakage ratio \((p, \gamma)/(p, \alpha)\) into the next cycle, which makes a good experimental determination of these reactions important, which is possible as we deal here with stable targets. On the other hand, the excitation energies in the corresponding compound nuclei are of the order of 8.5–12 MeV and make statistical model approaches also reliable (see, e.g., the experiments for \(^{31}\text{P}(p, \gamma)\), and \(^{35}\text{Cl}(p, \gamma)\) by Iliadis et al. 1993, 1994; Ross et al. 1995; and the later discussion).

Increasing temperatures allow Coulomb barriers to overcome and the cycles to be extended to more proton-rich nuclei, which permit additional leakage via proton captures, competing with long \(\beta\)-decays. The cycles open first at the last \(\beta\)-decay before the \((p, \alpha)\)-reaction via a competing proton capture, i.e., at the even-\(Z\) \(T_z = -\frac{1}{2}\) nuclei like \(^{23}\text{Mg}\), \(^{25}\text{Si}\), \(^{31}\text{S}\), \(^{35}\text{Ar}\), and \(^{39}\text{Ca}\), excluding \(^{15}\text{O}\) (because \(^{16}\text{F}\) is particle unstable) and \(^{19}\text{Ne}\), which is bypassed by the stronger \(^{18}\text{F}(p, \alpha)\)-rate. These nuclei are only one unit away from stability and have small decay \(Q\)-values and long half-lives. The situation is illustrated in Figure 2, which shows the possible breakout points and the \(Q\)-values of the proton-capture reactions. Since the reaction proceeds toward the proton drip line and into an odd-odd nucleus,
the capture $Q$-values of these $(p, \gamma)$-reactions are small (generally less than 2 MeV). This implies a too-small density of excited states for employing statistical model cross sections (see the discussion in the following paragraphs). At about $3 \times 10^8$ K, essentially all such cycles are open, and a complete rp-pattern of proton captures and $\beta$-decays is established, with the exception of the extended CNO cycle. The process time is determined by the sum of $\beta$-decay and proton-capture timescales, $\tau_p$ and $\tau_{\beta p}$, along the rp-path, which generally does not extend to more proton-rich even-$Z$ nuclei than $T_z = -1$. The reaction timescales are dominated by the $T_z = -\frac{1}{2}$ even-$Z$ nuclei close to stability, listed above.

At $4 \times 10^8$ K, the CNO cycle breaks open via $^{15}\text{O}(x, \gamma)^{18}\text{Ne}$. When followed by a proton capture and $(x, p)$-reaction on $^{18}\text{Ne}$, $^{14}\text{O}(x, p)^{17}\text{F}$ can also lead to a successful breakout. Because of higher energies, Coulomb barriers can be more easily overcome by heavier projectiles (like $a$ captures on CNO targets), and proton captures can occur further from stability. However, in all cases, the decay of $T_z = -1$ even-$Z$ nuclei [and, at high densities, the competition with $(x, p)$-reactions] is a prominent part of the reaction path. Alpha-reactions are not important for $Z > 20$, because of increasing Coulomb barriers and decreasing $Q$-values. For heavier nuclei beyond Ca, the reaction pattern seems complicated. However, temperatures approaching $10^9$ K cause strong photodisintegrations for proton-capture $Q$-values smaller than $25kT \approx 3$ MeV, which is comparable to $Q$-values found for even-$Z$ nuclei. $(25-30kT \geq Q$ is a rough criterion used to find out whether photodisintegrations can counterbalance capture reactions and lead to an equilibrium situation.) Thus, reactions breaking out of the cycles discussed above (up to the proton drip line) come into a $(p, \gamma)-(\gamma, p)$-equilibrium along isotonic lines with equal $N$. Under such circumstances, the nuclear connection boils down to the necessary knowledge of nuclear masses and $\beta$-decay half-lives, which is similar to the $r$-process's experiencing an $(n, \gamma)-(\gamma, n)$-equilibrium in isotopic chains (see Cowan, Thielemann, & Truran 1991; Kratz et al. 1993; Meyer 1994).

The general situation is explained in Figure 3, which contains all possible reactions from the original cycles over proton captures breaking out of the cycles, the accompanying $\beta$-decays, and $(x, p)$-reactions bridging waiting points with long decay half-lives. By comparing them with Figure 2, which also lists the $Q$-values involved, Figures 4a and 4b give an indication of the minimum temperatures required to permit statistical model calculations for proton- and $x$-induced reactions. The reason is that a high density of compound nuclear states is necessary for statistical model calculations to become reliable. The energy of the Gamow peak, where the integrand in a thermonuclear cross section integral (cross section times Boltzmann distribution) has a maximum, increases with increasing temperature. Because higher level densities are encountered at higher excitation energies corresponding to higher temperatures of the Boltzmann distribution, a critical temperature exists beyond which such calculations are reliable (for more details, see Rauscher, Thielemann, & Kratz 1996). We can analyze the expected behavior: reactions 4j, 4k, and 4l in Figure 3 have reaction $Q$-values less than $\approx 3$ MeV, and for conditions close to $10^9$ K, we have $25kT \approx Q$, which permits a $(p, \gamma)-(\gamma, p)$-equilibrium. Thus, although the cross sections are highly uncertain and statistical model approaches are not valid (see Fig. 4a), only the $Q$-values, i.e., masses, are needed for these nuclei. Reactions 4b, 4c, 4d, and 4e [(p, $\gamma$) and (p, $x$) reactions] have $Q$-values in excess of 5 MeV and, in most cases, densities of a high enough level to ensure the use of statistical model cross sections for the appropriate temperature regime. Reaction 4f, an $(x, p)$-reaction, provides also a sufficiently high-level density to apply statistical model cross sections [see Fig. 4b for $^{18}\text{Ne}(x, p)$, see Görres & Wiescher 1995]. Reaction 4g is crucial for the breakout, a statistical model approach is clearly not permitted, and a $(p, \gamma)-(\gamma, p)$-equilibrium is not valid at the lower appropriate temperatures of about $3-4 \times 10^8$ K. Thus, all reactions of type 4g ask strongly for experimental determination, possibly with radioactive ion beams. Reactions 4h, 4i, and all other connecting $\beta$-decays are required, preferably from experiments but otherwise from theoretical quasi-particle random phase approximation predictions.

Beyond Ca and Ti, this scheme simplifies into one dominated by proton captures and $\beta$-decays, with the proton captures in a $(p, \gamma)-(\gamma, p)$-equilibrium. The proton drip line develops a zig-zag shape and can separate two even-$Z$ proton-stable nuclei on the same isotope by an unstable odd-$Z$ nucleus. Görres, Wiescher, & Thielemann (1995) have shown that such gaps can be bridged by two proton-capture reactions, proceeding through the unstable nucleus, similar to the $3\alpha$-reaction passing through $^8\text{Be}$. This can extend the $(p, \gamma)-(\gamma, p)$-equilibrium beyond the drip line to proton-stable “peninsulas,” thus avoiding long $\beta$-decay half-lives. While $x$-induced reactions are not important for nuclei beyond Ca and Ti, small $Q$-values for $A > 56$ can permit $(\gamma, x)$-photodisintegrations and cause a backflow, decreasing the buildup of heavy nuclei.
After having analyzed the reaction patterns in the previous discussion, we now have to implement these findings into an approximation that can handle and describe all of the patterns. Considering the hydrogen and helium abundances to be slowly varying, and assuming a kind of local steady flow by requiring that all reactions entering and leaving a nucleus add up to a zero flux, we will devise a scheme that can adapt itself to the whole range of conditions of interest in hot hydrogen burning. It can automatically cover situations at low temperatures, characterized by...
a steady flow through all connected hot CNO-type cycles and branchings feeding the next cycle, to high-temperature regimes in which a \((p, \gamma)\)-\((\gamma, p)\)-equilibrium is established, while \(\beta^-\)decays or \((x, p)\)-reactions feed the population of the next isotonic line. The details are discussed in the following section.

3. APPROXIMATION SCHEME

3.1. Reaction Rates and Quasi-Decay Constants

The thermonuclear reaction rates used in our calculations have been discussed in detail by van Wormer et al. (1994; see the appendix therein). The temporal change of the isotopic abundances \(Y_i = X_i/A_i\) can be calculated by all depleting producing reactions

\[
\dot{Y}_i = \sum_j i\lambda_i^j Y_j + \sum_{j,k} i\alpha_{j,k} Y_j Y_k + \sum_{j,k,l} i\alpha_{j,k,l} Y_j Y_k Y_l .
\]  

(1)

The first term in equation (1) includes all one-particle reactions (decays or photodisintegrations) of nuclei \(j\) producing \((i \neq j)\) or destroying \((i = j)\) nucleus \(i\). The \(\alpha_i\) are either decay constants or effective temperature-dependent decay constants in the case of photodisintegrations. The second sum represents all two-particle reactions between nuclei \(j, k\) and \(\alpha_{j,k} = \gamma_{j,k} \rho N_A^{j}(\langle \sigma v \rangle_{j,k})\). Similarly, the third term presents three-particle processes with \(\alpha_{j,k,l} = \gamma_{j,k,l} \rho^2 N_A^{j}(\langle \sigma v \rangle_{j,k,l})\) (Fowler, Caughlan, & Zimmermann 1967). The coefficients \(\gamma\) include the statistical factors for avoiding double counting of reactions of identical particles.

Van Wormer et al. (1994) investigated numerically which reactions must be considered in the rp-process. Apart from the temperature-insensitive \(\beta^-\)-decays, the only one-particle reactions that must be especially considered at high temperatures are photodisintegrations. The two-body reactions dominating the rp-process are \((p, \gamma)\), \((\gamma, p)\), \((x, p)\)-reactions. Three-body reactions can be neglected in explosive hydrogen burning, with the exception of the \(3\alpha\)-reaction, which plays a central role in this context. Taking into account individual reactions, equation (1) can be rewritten as

\[
\dot{Y}_i = \sum_j i\lambda_i^j Y_j + \sum_j i\lambda_i^{p,j} Y_j + \sum_j \rho N_A^{j}(\langle \sigma v \rangle_{j,p}) Y_p \rho N_A^{j}(\langle \sigma v \rangle_{j,a}) Y_a . \tag{2}
\]

If the proton and helium abundances do not change greatly during relevant rp-process timescales, we are able to describe proton captures and \((x, p)\)-reactions with quasi-decay constants, provided that temperature and density do not change greatly during the process timescale of interest:

\[
i\lambda_i^{p,j} = \rho N_A^{j}(\langle \sigma v \rangle_{j,p}) \gamma_{j,p} = \text{const} , \tag{3a}
\]

\[
i\lambda_i^{p,a} = \rho N_A^{j}(\langle \sigma v \rangle_{j,p}) \gamma_{j,a} = \text{const} , \tag{3b}
\]

\[
i\lambda_i^{x,j} = \rho N_A^{j}(\langle \sigma v \rangle_{j,a}) \gamma_{j,a} = \text{const} , \tag{3c}
\]

\[
i\lambda_i^{x,p} = \rho N_A^{j}(\langle \sigma v \rangle_{j,a}) \gamma_{j,a} = \text{const} . \tag{3d}
\]

Thus, the system of nonlinear differential equations can be written in a linearized form

\[
\dot{Y}_i = \sum_j i\lambda_i^j Y_j , \tag{4}
\]

where \(x\) represents one of the following reactions: \((\gamma, p)\), \((p, \gamma)\), \((x, p)\), \((p, x)\), \((x, \gamma)\), \((\gamma, x)\), \((x, \gamma)\), \((\beta^-, v)\). When we choose \(Y_0 = \{Y_1, \ldots, Y_n, \ldots, Y_n\}\) to represent the initial abundance

\[
\dot{Y}_i = A Y , \quad A(N \times N) - \text{matrix} . \tag{5}
\]

Such systems of linear differential equations with constant coefficients can be solved by Jordan-normalizing matrix \(A\). Its transformation,

\[
J = S A S^{-1} , \quad S e G L(n, R) , \tag{6}
\]

yields the \(N\)-dimensional set of eigenfunctions

\[
u_{i,j}(t) = e^{\lambda_i^j t} u_{i,j} . \tag{7}
\]

Here \(\lambda_i^j\) represents eigenvalue \(i\) with degeneracy \(m\), \(u_{i,j}\) eigenvector \(j\) to eigenvalue \(i\) \((1 \leq j \leq m)\) and \(p_{i,j}(t)\) a polynomial of degree \((j - 1)\). However, the transformation \(u_{i,j} = S v_{i,j}\) are in general difficult to survey and to handle. Furthermore, the elements of \(A\) become slowly varying functions when the abundances of hydrogen and helium change slightly over a certain process timescale. This is, for instance, the case when we consider hot hydrogen burning. Only numeric simulations will be feasible for these two reasons.

We will show in the following subsection how certain sequences of a thermonuclear reaction network are regarded as quasi-nuclei. In such sequences, a steady flow materializes, and the ratios between the different abundances are constant and determined only by the values of the quasi-decay constants. A procedure of this kind provides us with a system of equations with a smaller dimension, which can be solved numerically.

3.2. The Approximation

In § 2, we discussed the burning cycles that have the same structure as the hot CNO cycle (see also Figs. 1–3). It is well understood how a steady state equilibrium is established in hot CNO-like reaction chains at temperatures \(T_9 \approx 0.2\) and on timescales governed by the longest \(\beta^-\)decay in the cycle. Then the abundance \(Y_{i,j}\) of nucleus \(i\) in a cycle \((j)\) can be calculated from the equilibrium condition

\[
Y_{i,j} \equiv \text{const} . \tag{8}
\]

This condition can be used for simplifying the system of equations that determines the thermonuclear reaction flows. However, an approximation of this kind is no longer justified when we consider timescales shorter than the half-lives of the \(\beta^-\)-unstable nuclei or temperatures greater than \(T_9 = 1\), when photodisintegrations become so important that a \((p, \gamma)\)-\((\gamma, p)\)-equilibrium rather than a steady flow is attained (see Fig. 3). Then, proton-induced reactions can be associated with large quasi-decay constants and, in comparison with \(\beta^-\)decays, small half-lives. Thus, all the abundances of isotonic nuclei that are connected by proton-capture reactions can be considered in a quasi-static equilibrium on timescales exceeding the half-lives associated with the quasi-decay constants. Such sequences shall be now defined as quasi-nuclei.

In the following paragraphs, the abundances of nuclei in proton-capture sequences shall be calculated under the assumption of a quasi-static equilibrium. Consider the \(j^{th}\) of \(m\) sequences that consists of \(n - 1\) proton captures between \(n\) nuclei (see Fig. 5b; where two proton capture sequences are shown with \(n = 4\) and \(n = 5\)). Each nucleus \((i)\) in a sequence \((j)\) is involved in \(k + 2\) reactions, including proton captures and photodisintegrations. Furthermore, we assume for simplicity that the abundances of helium and hydrogen stay constant over the timescale of interest and

1. The parameterized reaction rates can be found, e.g., in the database REACLIB, located at http://csa5.lbl.gov/~fchw/astro.html/.
that nucleus \((i, j)\) is not fed by reactions from another chain, \(j\). With the equilibrium condition (8), the temporal change of the abundance of nucleus \((i, j)\) can be written as

\[
\dot{Y}_{t,j} = \lambda_{t-1,j}^p Y_{t-1,j} - \sum_k \lambda_{t,k,j}^\gamma Y_{t,k} + \lambda_{t+1,j}^p Y_{t+1,j} = 0 .
\] (9)

The terms summed over \(k\) are leaks out of chain \((j)\) via nucleus \((i, j)\). The extreme cases of flux equilibria, like steady flows on the one hand \((\lambda^{p,\gamma} \gg \lambda^{\gamma,p}, \lambda^{p,p})\) and \((p, \gamma)-(\gamma, p)\)-equilibria on the other hand \((\lambda^{p,\gamma}, \lambda^{\gamma,p} \gg \lambda^{p,p})\), can be discussed on the basis of this equation. Here the general case shall be discussed.

For a chain of \(n\) nuclei, \(n - 1\), such equations permit us to relate the abundance \(Y_{t-1,j}\) to \(Y_{t,j}\) starting with a “known” \(Y_{n,j}\), the abundance of the last nucleus in a sequence. Thus, we obtain for each sequence \((j)\) an \((n - 1)\)-dimensional system of linear equations that allows us to express each abundance \(Y_{t,j}\) in terms of \(Y_{n,j}\):

\[
Y_{t,j} = v_{t,j} (\lambda_{t,j}^1 \ldots \lambda_{t,j}^n) Y_{n,j} .
\] (10)

The \(v_{t,j}\) depend only on the constant or slowly varying quasi-decay constants \(\lambda_{t,j}\). The abundance of a sequence can be treated as the abundance of a quasi-nucleus by \(Y_j = \sum_i Y_{i,j}\). When \(Y_j\) is already determined at a point in time \(t = t_s\), each abundance \(Y_{t,j}\) can be written as

\[
Y_{t,j}(t) = \frac{v_{t,j}}{v_j} Y_j(t) ,
\] (11)

with \(v_j = \sum_i v_{t,j}\). Hence, when the abundances of the quasi-nuclei \(Y_j\) are given at a time \(t\), the abundances \(Y_{t,j}(t)\) in each sequence can be determined simultaneously.

However, we also have to take care of incoming fluxes to a sequence \((j)\) not yet considered in equation (9). As a consequence, the coefficients \(v_{i,j}\) are no longer constant because they become explicitly abundance dependent for the following reason: in contrast to the outgoing fluxes, the incoming fluxes to a sequence \(j\) are not proportional to the abundances \(Y_{t,j}\) of the nuclei in this sequence. Therefore, abundances can no longer be determined simultaneously.

Incoming fluxes can nevertheless be considered as long as the abundances do not change too fast during a time step. We can add to equation (9) all incoming fluxes from nuclei \((l)\) of different chains \((m)\) proportional to their abundances \(Y_{l,m}(t_{s-1})\), \(m \neq j\) (when no time \(t\) is indicated, the quantities are given at \(t = t_s\)),

\[
\dot{Y}_{t,j} = -Y_{t,j} \sum_k \lambda_{t,k,j}^\gamma Y_{t,k} + \lambda_{t+1,j}^p Y_{t+1,j} + \lambda_{t-1,j}^p Y_{t-1,j} + \sum_{l,m \neq j} \lambda_{l,m}^\gamma Y_{l,m(t_{s-1})} = 0 .
\] (12)

This can be solved for \(Y_{t-1,j}\), which leads to

\[
Y_{t-1,j} = \frac{1}{\lambda_{p-1,j}^\gamma} \left( Y_{t,j} \sum_k \lambda_{t,k,j}^\gamma Y_{t+1,j} - \sum_{l,m \neq j} \lambda_{l,m}^\gamma Y_{l,m(t_{s-1})} \right) - \frac{1}{\lambda_{p-1,j}^\gamma} \sum_{l,m \neq j} \lambda_{l,m}^\gamma Y_{l,m(t_{s-1})} ,
\] (13)

or, more generally,

\[
Y_{t,j} = v_{t,j} Y_{n,j} - w_{t,j} Y_{n,j}(t_{s-1}) .
\] (14)
When a slowly varying function \( v'(t_{n-1}) \) is given so that we can approximate \( Y_{n,j} \) by

\[
Y_{n,j}(t_n) \approx \frac{1}{v'(t_{n-1})} Y_{n,j}(t_n),
\]

we can extend \( w_{l,j} \) in equation (14) as

\[
Y_{l,j} = w_{l,j} Y_{n,j} - \frac{v'_j}{v_j} w_{l,j} Y_{n,j}
\]

and finally obtain

\[
Y_{l,j} = v'_j Y_{n,j},
\]

\[
v'_j = v_{i,j} - \frac{v'_j}{v_j} w_{l,j},
\]

with \( v'_j = \sum_i v'_{i,j} \). Setting \( v'_{i,j}(t_0 = 0) = v_{i,j}(t_0 = 0) \) as initial values, the existence of the functions \( v'_j \) is guaranteed.

So far, we have made two approximations: (1) including reactions within a chain and leaking out of a chain described by equation (9). In that case, all abundances could be described simultaneously; and (2) when also considering incoming fluxes (see eq. [12]), we become dependent on abundances in other chains. Approximating their abundances at \( t = t_c \) by abundances at \( t = t_c - 1 \) permits us to stay within the same approximation scheme. But we have to be aware that this is valid only for slowly varying abundances or equivalently small time steps.

We obtained this approximation under the assumption that the abundances of hydrogen and helium stay constant. \( Y_p \) and \( Y_{\alpha} \) are slowly varying functions in realistic environments. We can adjust for the change of \( Y_p \) and \( Y_{\alpha} \) and obtain their new values applying relation (11):

\[
\dot{Y}_p = \sum_{l,j} (\lambda_{i,j}^{p} - \lambda_{i,j}^{\alpha} - \lambda_{i,j}^{n} - Y_p^{p}) Y_{l,j}(t_n),
\]

\[
\dot{Y}_{\alpha} = \sum_{l,j} (\lambda_{i,j}^{\alpha} - \lambda_{i,j}^{p} - \lambda_{i,j}^{n} - \lambda_{i,j}^{\alpha}) Y_{l,j}(t_n).
\]

As a consequence, the quasi-decay constants must be recalculated after each time step, making use of the updated proton and \( \alpha \) abundances considering them constant for one time step.

Thus, it is possible to express simultaneously every abundance \( Y_{l,j} \) in a sequence with the sequence abundance \( Y_l \).

With an approximation scheme of this kind, we reduced our full network described by \( \sum_{m=1}^{m} m(j) \) equations to a system of \( m \) equations. In the following, we want to check the accuracy of such an approximation.

4. RESULTS

The calculations were done with a reaction network that consists mainly of proton-capture sequences, each starting out from a \( T_9 = 0 \) nucleus. The number of nuclei in a sequence is determined by the condition that the quasi-decay constant \( \lambda_{i,j}^{p} \) of nucleus \( i \) into nucleus \( (i + 1) \) is larger than a certain \( \lambda_{\min} \). When this condition is fulfilled, nucleus \( (i + 1) \) is considered to be in the sequence. For the following calculations, \( \lambda_{\min} \) was set to \( 10^3 \) s\(^{-1} \) because we wanted to reproduce the energy generation of explosive hydrogen burning on timescales \( \tau = \lambda_{\min}^{-1} > 10^{-2} - 10^{-1} \) s.

As a consequence, a sequence contains not more than five nuclei for temperatures \( T_9 \leq 1.5 \) and densities \( \rho \leq 10^6 \) g cm\(^{-3} \).

In addition to the proton-capture sequences, \( T_9 = \frac{1}{2} \) nuclei had to be considered in the network. They are the most neutron-rich nuclei of the hot CNO-type cycles, in which after a proton capture, the \( \alpha \)-threshold in the compound nucleus is lower than the proton threshold and \( (p, \alpha) \)-reactions can occur that compete with the still-possible \( (p, \gamma) \)-transitions (see Fig. 5b). Since the \( (p, \alpha) \)- and \( (p, \gamma) \)-reactions occur on the same timescale, these \( T_9 = \frac{1}{2} \) nuclei cannot be affiliated to a proton-capture chain but have to be considered separately.

Between \( ^{12}\text{C} \) and \( ^{52}\text{Mn} \), the network can be composed of proton-capture reaction sequences and \( T_9 = \frac{1}{2} \) nuclei with one exception. The transition occurs already at compound nucleus \( ^{19}\text{Ne} \) in the Ne isotopic chain. A \( (p, \alpha) \)-reaction is possible on the \( T_9 = 0 \) nucleus \( ^{18}\text{F} \), and a proton-capture reaction sequence starts out from \( ^{19}\text{Ne} \) with isospin \( T_9 = -\frac{1}{2} \).

The low \( \alpha \)-threshold of \( ^{19}\text{Ne} \), which caused an exception to the rule and prevented an OFNe cycle, permits on the other hand the \( ^{15}\text{O}(\alpha, \gamma) \)-reaction and triggers a breakout from CNO cycles. The network described so far is shown in Figures 5a and 5b.

At temperatures exceeding \( T_9 = 0.8 \) and \( \rho \geq 10^6 \) g cm\(^{-3} \), the network must be completed with the \( 3\gamma \)-reaction. Furthermore, it is very likely that fluxes up to \( ^{72}\text{Kr} \) may influence energy generation. In the region between \( ^{51}\text{Mn} \) and \( ^{72}\text{Kr} \), the \( \beta^+ \)-reactions for the rp-process-relevant nuclei are, with the exception of \( ^{55}\text{Ni} \), \( ^{64}\text{Ge} \), and \( ^{72}\text{Kr} \), considerably smaller than the typical timescales of novae and X-ray bursts. Consequently, only the above-mentioned nuclei were considered explicitly in the approximation scheme. The approximated abundances of \( ^{53}\text{Ni} \) and these three waiting-point nuclei are compared with the full-network calculation results in Figures 6a and 6b.

The nuclei used in the approximation scheme are listed in Table 1 according to their affiliation to a certain proton-capture reaction chain. Nuclei that had to be considered separately (\( p, ^{4}\text{He} \), \( T_9 = \frac{1}{2} \) nuclei, \( ^{55}\text{Ni} \), \( ^{64}\text{Ge} \), and \( ^{72}\text{Kr} \)) are given their own sequence number.

Approximating the network according to § 3.2, we calculated the energy-generation rate \( \epsilon \) and the integrated energy generation \( \epsilon \) for nova-like conditions \( (T_9 = 0.2, \rho = 10^5 \) g cm\(^{-3} \) \) and X-ray–burst peak conditions \( (T_9 = 1.5, \rho = 10^6 \) g cm\(^{-3} \) \) and compared the results with full-size network calculations that take into account each nucleus separately. In each case we assumed a solar abundance distribution for the accreted matter. The results for low temperatures are shown in Figures 7a–7c and those for X-ray–burst conditions in Figures 8a–8c. Considering the timescale important for novae (100–1000 s) and X-ray bursts (1–10 s), we see that the approximation scheme devised in § 3 always agrees to a high degree for the astrophysical applications considered here. The overall results show only deviations smaller than 15% or even less (down to 5%) at a gain in computational speed by a factor of 15. Several calculations at intermediate temperature and densities, i.e., \( 10^4 < \rho < 10^6 \), \( 0.2 < T_9 < 1.5 \) confirmed the above-mentioned efficiency and accuracy of the approximation scheme.

Such an approximation would not be very useful if it was valid only for constant temperatures and densities. In order
to check its accuracy for varying temperatures and densities, we assumed after peak conditions were obtained in the ignition an exponentially decreasing temperature and an adiabatically expanding accretion layer with a radiation-dominated equation of state ($\gamma = 4/3$):

$$ T = T_0 e^{-\nu \tau}, $$

$$ \rho = \rho_0 \left( \frac{T}{T_0} \right)^{4/3}. $$

Using typical peak temperatures $T_0$, densities $\rho_0$, and timescales $\tau$ for X-ray bursts and novae, we again calculated the energy-generation rate (Figs. 9 and 10). The results show that the approximation is valid for changing temperatures and densities as long as they are in the range of $10^4 \leq \rho \leq 10^6$, $0.1 \leq T_0 \leq 1.5$. No significant change in computational speed was observed.

The differences at timescales smaller than $t = 10^{-2} \text{ s}$ are due to the very fast reaction flows in the proton-capture reaction chains: They occur in full-scale network
that it also permits the prediction of isotopic abundances of 15% while gaining a factor of about 15 in computational speed. The approximation scheme discussed in this paper is therefore well suited for realistic hydrocalculations of novae or X-ray bursts and other possible sites of explosive hydrogen burning. The additional advantage of this method is that it also permits the prediction of isotopic abundances with similar accuracy and can thus even replace full-network calculations for this purpose (see Fig. 11).

We were also able to determine key nuclear properties, which directly enters into the precision of calculations for explosive hydrogen burning, its energy generation, and abundance determination. We have identified the even-Z $T_e = -\frac{1}{2}$ nuclei like $^{22}$Mg, $^{27}$Si, $^{31}$S, $^{35}$Ar, and $^{39}$Ca as essential targets for proton captures that are in competition

TABLE 1

| Sequence Number | Nuclei |
|-----------------|--------|
| 1               | $p$    |
| 2               | $^{4}$He |
| 3               | $^{12}$C, $^{13}$N, $^{14}$O |
| 4               | $^{14}$N, $^{15}$O |
| 5               | $^{18}$O |
| 6               | $^{19}$Ne, $^{20}$Na, $^{21}$Mg |
| 7               | $^{16}$O, $^{17}$F, $^{18}$Ne |
| 8               | $^{20}$Ne, $^{21}$Na, $^{22}$Mg, $^{23}$Al, $^{24}$Si |
| 9               | $^{22}$Na, $^{23}$Mg, $^{24}$Al, $^{28}$Si |
| 10              | $^{23}$Na |
| 11              | $^{24}$Mg, $^{25}$Al, $^{26}$Si, $^{27}$P, $^{28}$S |
| 12              | $^{26}$Al, $^{27}$Si, $^{28}$P, $^{29}$S |
| 13              | $^{27}$Al |
| 14              | $^{28}$Si, $^{29}$P, $^{30}$S, $^{31}$Cl, $^{32}$Ar |
| 15              | $^{30}$P, $^{31}$S, $^{32}$Cl, $^{33}$Ar |
| 16              | $^{34}$Al |
| 17              | $^{32}$S, $^{33}$Cl, $^{34}$Ar, $^{35}$K, $^{36}$Ca |
| 18              | $^{34}$Cl, $^{35}$Ar, $^{36}$K, $^{37}$Ca |
| 19              | $^{35}$Cl |
| 20              | $^{36}$Ar, $^{37}$K, $^{38}$Ca, $^{39}$Sc, $^{40}$Ti |
| 21              | $^{38}$K, $^{39}$Ca, $^{40}$Sc, $^{41}$Ti |
| 22              | $^{39}$K |
| 23              | $^{40}$Ca, $^{41}$Sc, $^{42}$Ti, $^{43}$V, $^{44}$Cr |
| 24              | $^{42}$Sc, $^{43}$Ti, $^{44}$V, $^{45}$Cr |
| 25              | $^{43}$Sc |
| 26              | $^{44}$Ti, $^{45}$V, $^{46}$Cr, $^{47}$Mn, $^{48}$Fe |
| 27              | $^{46}$V, $^{47}$Cr, $^{48}$Mn, $^{49}$Fe |
| 28              | $^{47}$V |
| 29              | $^{48}$Cr, $^{49}$Mn, $^{50}$Fe, $^{51}$Co, $^{52}$Ni |
| 30              | $^{50}$Mn, $^{51}$Fe, $^{52}$Co, $^{53}$Ni |
| 31              | $^{51}$Mn |
| 32              | $^{55}$Ni |
| 33              | $^{64}$Ge |
| 34              | $^{72}$Kr |

**NOTE**—The nuclei used in the approximation scheme are listed according to their affiliation to a certain proton capture reaction chain. Nuclei that had to be considered separately ($p$, $^{4}$He, $T_e = \frac{1}{2}$ nuclei, $^{15}$N, $^{40}$Ge, and $^{72}$Kr) are given their own sequence number.
Fig. 8a — (a) Same as Fig. 7a, but for X-ray burst conditions; (b) same as Fig. 7b, but for X-ray burst conditions; (c) same as Fig. 7c, but for X-ray burst conditions.

Fig. 9.—Same as Fig. 7a, but for varying density and temperature starting with novae peak conditions. The approximation is valid as long as the temperature does not fall considerably below \( T_9 = 0.1 \).

Fig. 10.—Same as Fig. 9, but for X-ray–burst peak condition.
small $\alpha$-capture $Q$-values around $A = 80$ seems to be predicted with large variations among different mass models, and the even-even $N = Z$ nuclei between $A = 68$ and 100 play a dominant role.

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