THE HUREWICZ MAP IN MOTIVIC HOMOTOPY THEORY

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Abstract. For an $\mathbb{A}^1$-connected pointed simplicial sheaf $\mathcal{X}$ over a perfect field $k$, we prove that the Hurewicz map $\pi^A_1(\mathcal{X}) \to H^A_1(\mathcal{X})$ is surjective. We also observe that the Hurewicz map for $\mathbb{P}^1_k$ is the abelianisation map. In the course of proving this result, we also show that for any morphism $\phi$ of strongly $\mathbb{A}^1$-invariant sheaves of groups, the image and kernel of $\phi$ are also strongly $\mathbb{A}^1$-invariant.

1. Introduction

For a field $k$, let $Sm/k$ denote the category of smooth $k$-varieties with Nisnevich topology. Let $\Delta^{op} Sh(Sm/k)$ denote the category of simplicial sheaves on the category $Sm/k$. This category with its $\mathbb{A}^1$-model structure as defined in [6] is one of the main objects of study in $\mathbb{A}^1$-homotopy theory. For any pointed simplicial sheaf $\mathcal{X}$ in $\Delta^{op} Sh(Sm/k)$ one defines the $\mathbb{A}^1$-homotopy group sheaves, $\pi^A_i(\mathcal{X})$, to be the sheaves of simplicial homotopy groups of a fibrant replacement of $\mathcal{X}$ in the $\mathbb{A}^1$-model structure. Morel, in his foundational work in [5, Ch. 6] has defined, for every integer $i$, $\mathbb{A}^1$-homology groups $H^A_i(\mathcal{X})$ and canonical Hurewicz morphisms

$$\pi^A_i(\mathcal{X}) \to H^A_i(\mathcal{X})$$

The above maps are analogous to the Hurewicz map that we have in topology. In the topological setup, the Hurewicz morphism for $i = 1$ is known to be the abelianisation when the underlying space is connected. We will refer to this result as the Hurewicz theorem. Hurewicz theorem is expected in $\mathbb{A}^1$-homotopy theory (see [5, 6.36]), but not yet known. However the following theorem by Morel is the closest known result to the Hurewicz theorem.

Theorem 1.1. [5, 6.35] For a connected simplicial sheaf $\mathcal{X}$, the Hurewicz morphism

$$\pi^A_1(\mathcal{X}) \to H^A_1(\mathcal{X})$$

is a universal map to a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups.

Recall that a sheaf of groups $G$ is called strongly $\mathbb{A}^1$-invariant if for $i = 0, 1$ the maps

$$H^i(U, G) \to H^i(U \times \mathbb{A}^1, G)$$

are bijective for all $U$ in $Sm/k$. If $G$ is abelian, then it is called strictly $\mathbb{A}^1$-invariant if the above isomorphism holds for all $i \geq 0$. $\pi^A_1(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and $H^A_1(\mathcal{X})$ is known to be strictly $\mathbb{A}^1$-invariant (see [5, 6.1, 6.23]).

In topology, the surjectivity of the Hurewicz map is almost a direct consequence of the definitions. This is not the case in $\mathbb{A}^1$-homotopy theory. The main source of difficulty lies in the non-explicit nature of $\mathbb{A}^1$-fibrant replacements; non-explicit from the viewpoint of making explicit calculations. In this paper we prove this surjectivity by using Giraud’s theory of non-abelian cohomology.

Theorem 1.2. Let $k$ be a perfect field and $\mathcal{X}$ be a pointed simplicial sheaf on $Sm/k$ in the Nisnevich topology. Then the Hurewicz map $\pi^A_1(\mathcal{X}) \to H^A_1(\mathcal{X})$ is surjective.

The above theorem will be deduced from the following result, which is of independent interest.

Theorem 1.3. Let $k$ be a perfect field. Let $G$ be a strongly $\mathbb{A}^1$-invariant sheaf of groups on $Sm/k$ and $G \to H$ be an epimorphism. Then $H$ is strongly $\mathbb{A}^1$-invariant iff it is $\mathbb{A}^1$-invariant.

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Remark 1.4. If \( k \) is perfect field, a theorem of Morel [5, 5.46] says that any strongly \( \mathbb{A}^1 \)-invariant Nisnevich sheaf of abelian groups on \( \text{Sm}/k \) is also strictly \( \mathbb{A}^1 \)-invariant. Unfortunately it is not yet known if this statement holds for imperfect fields. This is the sole reason for assuming \( k \) to be perfect in Theorems 1.3 and 1.2. Also note that strongly \( \mathbb{A}^1 \)-invariant is a stronger notion than just \( \mathbb{A}^1 \)-invariant. In particular, there exists \( \mathbb{A}^1 \)-invariant sheaves which are not strongly \( \mathbb{A}^1 \)-invariant (see [7, Lemma 5.6]).

For a morphism of strongly \( \mathbb{A}^1 \)-invariant abelian sheaves over a perfect field, the kernel and image of the morphism are also strongly \( \mathbb{A}^1 \)-invariant. This result is a consequence of a nontrivial theorem of Morel (see [5, 6.24]) that the category of strongly \( \mathbb{A}^1 \)-invariant sheaves of abelian groups is an abelian category, as it is obtained as a heart of a \( t \)-structure. The theorem below, can be viewed as a generalization of this result for non-abelian strongly \( \mathbb{A}^1 \)-invariant sheaves. Moreover the proof of this generalization is completely different and is more direct in the sense that it does not appeal to the existence of \( t \)-structures.

Theorem 1.5. Let \( G \xrightarrow{\phi} H \) be a morphism of strongly \( \mathbb{A}^1 \)-invariant sheaves of groups. Then the image and the kernel of \( \phi \) are strongly \( \mathbb{A}^1 \)-invariant.

Proof. The image \( \text{Image}(\phi) \) is \( \mathbb{A}^1 \)-invariant, since it is a subsheaf of an \( \mathbb{A}^1 \)-invariant sheaf \( H \). Thus by Theorem 1.3 it is strongly \( \mathbb{A}^1 \)-invariant. The kernel \( K \) is strongly \( \mathbb{A}^1 \)-invariant as it fits in the following exact sequence
\[
1 \to K \to G \to \text{Image}(\phi) \to 1
\]
where the other two sheaves are strongly \( \mathbb{A}^1 \)-invariant. \( \square \)

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2. Preliminaries on Gerbes and Giraud’s non-abelian cohomology

Let \( \mathcal{C} \) be any small site (e.g. \( \text{Sm}/k \) with Nisnevich topology) and \( \Delta^\text{op}(\mathcal{C}) \) be the category of simplicial sheaves on \( \mathcal{C} \). The goal of this section is to recall the main results of Giraud on non-abelian cohomology. Everything in this section is a subset of [3]. We work with the following definition of a gerbe.

Definition 2.1. A simplicial sheaf \( \mathcal{X} \) on \( \mathcal{C} \) is called a gerbe if it is connected and if for any \( U \in \mathcal{C} \) and any \( x \in \mathcal{X}(U) \), the homotopy sheaves of groups \( \pi_i(\mathcal{X}_U, x) = 0 \) for all \( i \in \mathbb{N} \).

Given any simplicial sheaf \( \mathcal{X} \) (not necessarily a gerbe), one gets a category fibered in groupoids over \( \mathcal{C} \) defined by the fundamental groupoid construction: i.e. for every \( U \in \mathcal{C} \), the fiber category \( \mathcal{X}_U \) is the fundamental groupoid of the space \( \mathcal{X}(U) \), a category whose objects are elements of \( \mathcal{X}(U) \) and morphisms are paths up to homotopy. This category fibered in groupoids is in fact a gerbe in the sense of [4, 3.2] if \( \mathcal{X} \) is connected. If \( \mathcal{X} \) was a gerbe to start with, then it can be recovered, up to weak equivalence, using this category fibered in groupoids using the simplicial nerve construction. A gerbe \( \mathcal{X} \), is called neutral if it has a global section. In this case, by making a choice of a global section, one can define the fundamental group of \( \mathcal{X} \). Since \( \mathcal{X} \) is connected, a different choice gives a fundamental group which can be canonically identified with the previous one, modulo an inner automorphism.

This motivates the following definition by Giraud.

Definition 2.2. [3, 1.1.3] For two sheaves of groups \( F \) and \( G \), let \( \text{Isexc}(F,G) \) denote the set of isomorphisms from \( F \) onto \( G \) modulo the action of inner automorphisms of \( F \) (acting on the left) and the action of inner automorphisms of \( G \).

Consider the pre-stack whose objects over \( U \) are sheaves of groups over \( U \) (small w.r.t to a fixed universe) with morphisms between \( F \) and \( G \) defined as elements of \( \text{Isexc}(F,G) \). One can stackify this pre-stack (see [4]) and objects of this stack are called bands. In particular, every sheaf of groups defines a band. Since every band is represented locally by a sheaf of groups, all those concepts related to sheaves of groups which are local in nature (e.g. exact sequence, epimorphism, kernel, center)
also make sense for bands. It is a simple exercise to show that the center of a band is necessarily represented by a sheaf of groups. The 'fundamental group' of any gerbe $\mathcal{X}$ (neutral or not) is always defined as a band. For a band $L$, a gerbe banded by $L$ (or simply an $L$-gerbe) will mean a gerbe together with an isomorphism of $L$ with the band defined by $\mathcal{X}$. An equivalence of $L$-gerbes means an equivalence of the gerbes compatible with the given isomorphisms of their bands with $L$.

**Definition 2.3.** [3, 3.1.1] For a band $L$ on a site $C$, let $H^2(C, L)$ or simply $H^2(L)$ denote the equivalence class of $L$-gerbes. The subset represented by neutral classes in $H^2(C, L)$ is denoted by $H^2(C, L) \sim$ or simply $(H^2(L) \sim)$.

**Remark 2.4.** Note that $H^2(L) \sim$ is non-empty if and only if $L$ can be represented by a sheaf of groups (see [3, 3.2.4]), in which case it is a singleton set, as can be seen for e.g. by Theorem 2.8 stated below.

**Remark 2.5.** If a band $L$ is representable by a sheaf of abelian groups $A$, then $H^2(L)$ defined above is in canonical bijection with the $H^2(A)$ as defined by sheaf cohomology [3, 3.4].

Given an exact sequence of sheaves of groups

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$$

one has a long exact sequence

$$1 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C).$$

One of the goals of introducing the non-abelian $H^2$ is to extend this exact sequence on the right. We first note that $B$ acts on itself by inner automorphisms. Since $A$ is normal in $B$, this action also induces an action on $A$. Thus $B$ also acts on band$(A)$, where band$(A)$ denotes the band defined by $A$. This action factors through $C$.

We need the following definition to state the result extending the above long exact sequence to $H^2$:

**Definition 2.6.** [3, 4.2.3] For an epimorphism of bands $v : M \to N$, one defines a pointed set $O(v) := N(v)/R$

where

1. $N(v)$ is the set of all triples $(K, L, u)$ where $K$ is a gerbe with band $L$ and $u : L \to M$ is a monomorphism which makes the following sequence exact

$$1 \to L \xrightarrow{u} M \xrightarrow{\alpha} N \to 1.$$

2. $R$ is the equivalence relation defined by declaring $(K, L, u) \sim (K', L', u')$ if there exists a morphism of gerbes $\alpha : K \to K'$ such that the induced morphism $\alpha : L \to L'$ on bands makes the following diagram commute

$$\begin{array}{ccc}
L & \xrightarrow{u} & M \\
\alpha \downarrow & & \downarrow \\
L & \xrightarrow{u'} & M 
\end{array}$$

3. $O(v)'$ denotes the subset of $O(v)$ defined by all those $(K, L, u)$ where $K$ is a neutral $L$-gerbe.

Now one has the following general result.

**Theorem 2.7.** [3, 4.2.8, 4.2.10] Given an exact sequence of sheaves of groups $1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$, we have the following long exact sequence

$$1 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \xrightarrow{d} O(g) \to H^2(B) \to H^2(C).$$

where the map $d$ is as defined in [3, 4.7.2.4] and exactness of the sequence is defined similar to that in the case of pointed sets with subsets $O(g)'$ and $H^2(B)'$, $H^2(C)'$ playing the role of base points. When the action of $C$ on band$(A)$ is trivial, on has a canonical bijection $O(g) \cong H^2(A)$.

Another important result of Giraud we need is the following.
Theorem 2.8. [3, 3.3.3] Let $L$ be a band and $C$ be its center. Then one has a canonical action of $H^2(C)$ on $H^2(L)$ which is free and transitive.

This theorem, loosely speaking, says that the non-abelian cohomology set $H^2(L)$ is essentially all abelian as it “comes from” its center. Note however that if $H^2(L)$ has no class represented by a neutral gerbe, then there is no canonical bijection between $H^2(L)$ and $H^2(C)$.

The following is a direct consequence of the above theorem and the definition of

Lemma 2.9. Let $1 \to A \overset{g}{\to} B \overset{f}{\to} C \to 1$ be an exact sequence of sheaves of groups. Assume that $H^2(A)$ is trivial. Then $O(g)^{\prime} = O(g)$.

Proof. Let $(K, L, u)$ be any triple where $L$ is a band which fits in the exact sequence

$$
1 \to L \overset{u}{\to} \text{band}(B) \to \text{band}(C) \to 1
$$

and $K$ is an $L$-gerbe. To prove the theorem it is enough to show that $K$ is neutral. However we note that center $Z(L)$ coincides with $Z(A)$, the center of $A$. The result then directly follows from the above theorem. □

3. Strong $\mathbb{A}^1$-invariance of the center and applications

The goal of this section is to prove the theorems mentioned in the introduction. We start by proving the following.

Theorem 3.1. Let $G$ be a strongly $\mathbb{A}^1$-invariant sheaf of groups on $S\text{m}/k$. Then $Z(G)$, the center of $G$ is also strongly $\mathbb{A}^1$-invariant.

Proof. Since $G$ is strongly $\mathbb{A}^1$-invariant, $BG$ is $\mathbb{A}^1$-local. By choosing a simplicially fibrant model for $BG$ we may further assume, without loss of generality, that $BG$ is $\mathbb{A}^1$-fibrant. To show $Z(G)$ is strongly $\mathbb{A}^1$-invariant, we need to show $BZ(G)$ is $\mathbb{A}^1$-local. Let $BZ(G) \overset{u}{\to} \mathcal{X}$ be an $\mathbb{A}^1$-fibrant replacement (in the category of pointed spaces). Thus $u$ is a trivial $\mathbb{A}^1$-cofibration. To prove the theorem, it suffices to show that $u$ is a simplicial weak equivalence. Equivalently, it suffices to show that the map on sheaves of fundamental groups

$$
\pi_1(BZ(G)) \to \pi_1(\mathcal{X}) (=: H)
$$

is an isomorphism. Since $BG$ is $\mathbb{A}^1$-fibrant and $u$ is a trivial $\mathbb{A}^1$-cofibration, we have a factorization $h$ as below

$$
\begin{array}{ccc}
BZ(G) & \overset{u}{\to} & BG \\
& \downarrow \cong & \downarrow \exists h \\
\mathcal{X} & \overset{\exists h}{\to} & BG
\end{array}
$$

which gives a commutative diagram of the maps induces on the fundamental groups

$$
\begin{array}{ccc}
Z(G) & \overset{u}{\to} & G \\
& \downarrow \cong & \downarrow h_* \\
H & \overset{h_*}{\to} & BG
\end{array}
$$

In the above diagram, if we show that the image of $h_*$ is $Z(G)$, then it will follow that $Z(G)$ is a retract of the strongly $\mathbb{A}^1$-invariant sheaf $H$ and hence is itself strongly $\mathbb{A}^1$-invariant. Thus it suffices to show that image of $h_*$ is contained in $Z(G)$. This is equivalent to showing that for every smooth $k$-scheme $U$ and an element $g \in G(U)$, the map $h_{|U}$ is homotopic to the composite of

$$
\mathcal{X}_{|U} \overset{h_{|U}}{\to} BG_{|U} \overset{x \mapsto gxg^{-1}}{\to} BG_{|U}.
$$
But note that the base change functor from $\Delta^{op}(Sh(k)) \to \Delta^{op}(Sh(U))$ preserves trivial $\mathbb{A}^1$-cofibrations since it is a left quillen functor. Moreover it also preserves $\mathbb{A}^1$-fibrations. Thus

$$BZ(G)|_U \xrightarrow{\nu|_U} X|_U$$

is a trivial $\mathbb{A}^1$-cofibration. Since $Z(G)$ is the center of $G$, the map $BZ(G)|_U \xrightarrow{\nu|_U} BG|_U$ is homotopic (in fact equal) to the composite

$$BZ(G)|_U \to BG|_U \xrightarrow{x \mapsto gx^{-1}} BG|_U.$$

The proof now follows from commutative diagram below, using the fact that $BG|_U$ is a $\mathbb{A}^1$-local by Lemma 3.2.

$$\begin{array}{c}
BZ(G)|_U \xrightarrow{\nu|_U} BG|_U \xrightarrow{x \mapsto gx^{-1}} BG|_U \\
\downarrow \ h|_U \\
X|_U \end{array}$$

**Lemma 3.2.** Let $U \in Sm/k$. Then the restriction functor $\Delta^{op}(Sh(k)) \to \Delta^{op}(Sh(U))$ takes $\mathbb{A}^1$-fibrant objects to $\mathbb{A}^1$-local objects.

**Proof.** Let $Y \in \Delta^{op}(Sh(k))$ be an $\mathbb{A}^1$-fibrant object. Then $Y|_U$ has BG property, since BG property is defined in terms of Nisnevich distinguished triangles and every Nisnevich distinguished triangle in the category $Sm/U$ is also a Nisnevich distinguished triangle in $Sm/k$. Moreover, since $Y(V \times \mathbb{A}^1) \to Y(V)$ is a weak equivalence for every $V/U$. Thus by arguments given as in [5, A.6], $Y|_U$ is $\mathbb{A}^1$-local. □

**Proof of 1.3.** Let $K$ denote the kernel of the the epimorphism $G \to H$. Thus we have a short exact sequence of Nisnevich sheaves of groups

$$1 \to K \to G \to H \to 1.$$

**Step 1:** For every smooth $k$-scheme $U$, this gives us an exact sequence (see [3, 3.3.1]) of pointed cohomology sets

$$1 \to H^0(U, K) \to H^0(U, G) \to H^0(U, H) \to H^1(U, K) \to H^1(U, G) \to H^1(U, H).$$

Using functoriality of the above exact sequence in the case when $U$ is Hensel local, we deduce that the $\mathbb{A}^1$-invariance of $H$ implies (in fact is equivalent to) strong $\mathbb{A}^1$-invariance of $K$. By Theorem 3.1, $Z(K)$, the center of $K$ is strictly $\mathbb{A}^1$-invariant sheaf.

**Step 2:** To show strong $\mathbb{A}^1$-invariance of $H$, it is enough to show that for all henselian local essentially smooth schemes $U/k$, $H^1(U \times \mathbb{A}^1, H)$ is trivial. By [3, 4.7.2.4], we have an exact sequence of pointed sets

$$\to H^1(U \times \mathbb{A}^1, G) \to H^1(U \times \mathbb{A}^1, H) \to O(\phi).$$

where $\phi$ denotes restriction of $G \to H$ to the over-category $(Sm/k)/U \times \mathbb{A}^1$ which will be denoted by $Sm_k/U \times \mathbb{A}^1$ for simplicity and $O(\phi)$ is as defined in 2.6. It is enough to show $O(\phi)$ is trivial. This follows from Lemma 2.9 and the strict $\mathbb{A}^1$-invariance of $Z(K)$. □

Let $\mathcal{F}$ be a sheaf on $Sm/k$. For $U \in Sm/k$ we say $\alpha, \beta \in \mathcal{F}(U)$ are naive $\mathbb{A}^1$-homotopic if there exists a $\gamma \in \mathcal{F}(U \times \mathbb{A}^1)$ such that

$$\alpha = \sigma_0(\gamma) \quad \text{and} \quad \beta = \sigma_1(\gamma)$$

where $\sigma_0, \sigma_1 : U \to U \times \mathbb{A}^1$ are the sections defined by 0 and 1 respectively. As in [2, 2.9], let $S(\mathcal{F})$ denote the sheaf associated to the presheaf

$$U \mapsto \frac{\mathcal{F}(U)}{\sim}.$$
where \( \sim \) denotes the equivalence relation generated by naive \( \mathbb{A}^1 \)-homotopies. There is a canonical epimorphism \( F \to S(F) \). Let

\[
S^\infty(F) := \lim_{n \to \infty} S^n(F)
\]

The following lemma is straightforward to check.

**Lemma 3.3.** (see [2, 2.13]) The canonical morphism \( F \to S^\infty(F) \) is an epimorphism and is a universal map from \( F \) to an \( \mathbb{A}^1 \)-invariant sheaf. Moreover if \( F \) is a sheaf of groups, then so is \( S^\infty(F) \).

**Proof of Theorem 1.2.** By lemma 3.3, the map \( \pi_{A^1_1}(X) \xrightarrow{h} H_{A^1_1}(X) \) factors uniquely through \( \pi_{A^1_1}(X) \xrightarrow{s} S^\infty(\pi_{A^1_1}(X)^{ab}) \).

Since the map \( s \) is an epimorphism, Theorem 1.3 implies that \( S^\infty(\pi_{A^1_1}(X)^{ab}) \) is strongly \( A^1 \)-invariant sheaf of abelian groups. Thus \( s \) is a universal map to strictly \( A^1 \)-invariant sheaf of abelian groups. However by [5, 6.35], so is \( h \). Thus the induced map from \( S^\infty(\pi_{A^1_1}(X)^{ab}) \to H_{A^1_1}(X) \) must be an isomorphism. In particular \( h \) must be an epimorphism. \( \square \)

4. Hurewicz map for \( P^1_k \)

In this section we reserve the notation \( H \) to denote the Hurewicz map for \( P^1 \), i.e. \( \pi_1^A(P^1) \xrightarrow{H} H_{A^1_1}(P^1) \). The goal of this section is to prove the following propositions

**Proposition 4.1.** The kernel of the Hurewicz map for \( P^1_k \),

\[
\pi_1^A(P^1) \xrightarrow{H} H_1^A(P^1)
\]

is equal to the commutator subgroup of \( \pi_1^A(P^1) \).

As a consequence of the explicit computation of the Hurewicz map we obtain the following :

**Proposition 4.2.** The sequence of Nisnevich sheaves

\[
0 \to hK_2^{MW} \to K_2^{MW} \xrightarrow{\eta} K_1^{MW}
\]

is exact.

**Remark 4.3.** We do not know if there is an elementary way to prove the above proposition, using generator and relations. In particular we do not know if

\[
0 \to hK_n^{MW} \to K_n^{MW} \xrightarrow{\eta} K_{n-1}^{MW}
\]

is exact for every \( n \geq 1 \). However, as pointed out to us by O. Röndigs, it is possible that the above short exact sequence is induced by a cofiber sequence given in [1, Prop. 11].

The most difficult part of the computation in the above propositions is the universality of the Hurewicz map and the computation of \( \pi_1^A(P^1) \) itself, both of which has been elegantly done in [5, 6.35, 7.3]. We first restate Morel’s computation of \( \pi_1^A(P^1) \) as it will also help us to build notation for use in subsequent calculation.

Let \( F_{A^1_1}(1) := \pi_1^A(P^1) \). First we recall the following two maps defined by Morel:

1. A map \( \theta : \mathbb{G}_m \to F_{A^1_1}(1) \) which is a result of an \( A^1 \)-equivalence \( \mathbb{P}^1_k \sim \Sigma(\mathbb{G}_m) \).

2. A map \( K_2^{MW} \to F_{A^1_1}(1) \) which is a result of applying \( \pi_1 \) to the map \( \mathbb{A}^2 - 0 \to \mathbb{P}^1 \) and a theorem of Morel which shows \( K_2^{MW} \cong \pi_1^A(\mathbb{A}^2 - 0) \).

In what follows, we will freely use standard notation for denoting elements of \( K_1^{MW} \) used in [5, Chapter 3], e.g. \( \langle -1 \rangle, h, [U] \) etc.
Thus it is enough to show that
\[
1 \to K_2^{MW} \to F^{\lambda^1}(1) \xrightarrow{\gamma} \mathbb{G}_m \to 1
\]
is exact and is a central extension.

(i) For two units \( U, V \) in any field extension \( F \) of \( k \), the following hold
\[
\theta(U)\theta(V)^{-1} = [-U][-V]\theta(U^{-1}V^{-1})
\]
\[
\theta(U)^{-1}\theta(V) = [U^{-1}][-V]\theta(U^{-1}V).
\]

The following is the main calculation in the proof of Proposition 4.1.

**Lemma 4.5.** For any essentially smooth field extension \( F/k \), the commutator subgroup of \( F^{\lambda^1}(1)(F) \) is equal to \( hK_2^{MW}(F) \).

**Proof.** Since \( K_2^{MW}(F) \) is in the center of \( F^{\lambda^1}(1)(F) \), we have that the commutator subgroup
\[
[F^{\lambda^1}(1)(F), F^{\lambda^1}(1)(F)] = \langle \theta(U)\theta(V)\theta(U)^{-1}\theta(V)^{-1} | U, V \in F \rangle
\]
where the angle brackets on RHS denote subgroup generated by the elements within.
\[
\theta(U)\theta(V) = (-1)[U][V]\theta(UV) \quad \text{...(by [5, 7.31])}
\]
\[
\theta(V)\theta(U) = (-1)[U][V]\theta(UV) \quad \text{...(by [5, 7.31])}
\]
\[
\theta(U)\theta(V)\theta(U)^{-1}\theta(V)^{-1} = (-1)[U][V]\theta(UV) \cdot (- (-1)[V][U]\theta(UV))^{-1}
\]
\[
= (-1)([U][V] - [V][U]) \quad \text{...((\because K_2^{MW} is in the center)}
\]
\[
= h(h-1)(U)[V]...
\]
\[
\therefore h = 1 + (-1)
\]

Thus we have
\[
[F^{\lambda^1}(1)(F), F^{\lambda^1}(1)(F)] = \langle h(h-1)[U][V] | U, V \in F \rangle
\]
\[
= h(h-1)K_2^{MW}(F) \quad \text{...((\because h - 1 = -1) is a unit by [5, 3.5(4)])}
\]

**Proof of Proposition 4.1.** Recall that \( H \) is a universal map from \( \pi_1^A(\mathbb{P}^1) \) to a strongly \( A^1 \)-invariant sheaf of abelian groups. Thus it is enough to show that the abelianisation of \( \pi_1^A(\mathbb{P}^1) \) is strongly \( A^1 \)-invariant. \( \pi_1^A(\mathbb{P}^1)^{ab} \) is a quotient of a strongly \( A^1 \)-invariant sheaf therefore by Theorem 1.3, we need to show that \( \pi_1^A(\mathbb{P}^1)^{ab} \) is homotopy invariant. However by (4.4) and Lemma 4.5, \( \pi_1^A(\mathbb{P}^1)^{ab} \) fits in the following exact sequence
\[
0 \to \frac{K_2^{MW}}{hK_2^{MW}} \to \pi_1^A(\mathbb{P}^1)^{ab} \to \mathbb{G}_m \to 0.
\]

Thus it is enough to show that \( \frac{K_2^{MW}}{hK_2^{MW}} \) is \( A^1 \)-invariant or equivalently \( hK_2^{MW} \) is strongly \( A^1 \)-invariant. However this follows from [5, 3.32].

The following lemmas give explicit description of the Hurewicz morphism \( H \).

**Lemma 4.6.** There exists an isomorphism \( \phi : H_2^A(\mathbb{P}^1) \cong K_2^{MW} \) such that
\[
\phi \circ H([U][V]\theta(W)) = \eta [U^{-1}][V] + [W]
\]
where \( U, V, W \) are sections of \( \mathbb{G}_m \) on an object in \( \mathbb{G}_m \).
Proof. Since there is an $A^1$-equivalence $P^1_k \cong \Sigma \mathbb{G}_m$, we have

$$H^1_{\mathbb{A}^1}(P^1_k) \cong H^1_{\mathbb{A}^1}(\Sigma \mathbb{G}_m) \cong \tilde{H}^1_{\mathbb{A}^1}(\mathbb{G}_m)$$

where the second isomorphism is due to $A^1$-suspension theorem for homology [5, 6.30]. In the above statement $\mathbb{G}_m$ is considered as a sheaf of sets pointed by 1. By the definition of $A^1$-homology groups, $\tilde{H}^1_{\mathbb{A}^1}(\mathbb{G}_m)$ is the strictly $A^1$-invariant sheaf of abelian groups generated by the pointed sheaf $\mathbb{G}_m$.

By [5, 3.2], this must be isomorphic to $K_{MW}^1$. Now recall that we have the following commutative diagram

\[ \begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{\theta} & \pi_1(\Sigma \mathbb{G}_m) \\
U \mapsto (U) & & \pi_1^A(P^1_k) \\
\end{array} \]

The diagram commutes because all morphisms in this diagram are a result of some universal property. The commutativity of the diagram gives us the formula

(1) $\phi(H([U])) = [W]$  

Now, using the following equality proved in [5, 7.29(1)]

$$\theta(U^{-1})^{-1}\theta(U^{-1}V)\theta(V)^{-1} = [U][V]$$  

we get

$$\phi(H([U][V]\theta(W))) = \phi(H(\theta(U^{-1})^{-1}\theta(U^{-1}V)\theta(V)^{-1}\theta(W)))$$

$$= -[U^{-1}] + [U^{-1}V] - [V] + [W] \quad \text{...}(\text{using eqn}(1) \ \text{above})$$

$$= \eta[U^{-1}][V] + [W] \quad \text{...}(\text{using [5, 3.1(2)])}$$

□

To state the next lemma, recall that $K_{MW}^2$ is the free strongly $A^1$-invariant sheaf of abelian groups generated by the pointed sheaf of sets $\mathbb{G}_m \wedge \mathbb{G}_m$. Thus any automorphism of $\mathbb{G}_m \wedge \mathbb{G}_m$ gives rise to an automorphism of $K_{MW}^2$. In particular, the automorphism of $K_{MW}^2$ induced by

$$\begin{array}{c}
\mathbb{G}_m \wedge \mathbb{G}_m \rightarrow \mathbb{G}_m \wedge \mathbb{G}_m \\
(U, V) \mapsto (U^{-1}, V)
\end{array}$$

will be denoted by $\tau$.

**Lemma 4.7.** Let $\tau : K_{MW}^2 \rightarrow K_{MW}^2$ denote the automorphism which sends $[U][V] \mapsto [U^{-1}][V]$. Then the restriction of the Hurewicz map $H$ to $K_{MW}^2$ coincides with the composition

$$K_{MW}^2 \xrightarrow{\tau} K_{MW}^2 \xrightarrow{\eta} K_{MW}^1$$

**Proof.** This follows directly from the explicit formula for $H$ in the above lemma. □

**Proof of Proposition 4.2.** By universality of $H$, as noted before, the map $F^\mathbb{A}^1(1) \xrightarrow{\gamma} \mathbb{G}_m$ factors through $H$. Thus

$$\text{Ker}(H) \subset \text{Ker}(\gamma) = K_{MW}^1$$

$\text{Ker}(H)$ must coincide with the kernel of restriction of $H$ to $K_{MW}^2$. Now the proposition follows from Lemmas 4.5 and 4.7. □
References

[1] A. Ananyevskiy, O. Röndigs, P. Østvær; On very effective hermitian K-theory. Math. Z. 294 (2020), no. 3-4, 1021–1034.

[2] Chetan Balwe, Amit Hogadi and Anand Sawant; $A^1$-connected components of schemes. Adv. Math. 282 (2015), 335–361.

[3] Giraud, Jean; Cohomologie non abélienne. Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.

[4] G. Laumon, L. Moret-Bailly; Champs algébriques. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 39. Springer-Verlag, Berlin, 2000.

[5] F. Morel, $A^1$-algebraic topology over a field, LNM 2052, Springer, Heidelberg, 2012.

[6] F. Morel, V. Voevodsky, A1-homotopy theory of schemes. Inst. Hautes Études Sci. Publ. Math. No. 90 (1999), 45–143 (2001).

[7] U. Choudhury; Connectivity of motivic $H$-spaces. Algebraic and Geometric Topology 14 (2014) 37–55.