Parametric vibrations of viscoelastic orthotropic cylindrical panels of variable thickness

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Abstract. In modern engineering and construction, thin-walled plates and shells of variable thickness, subjected to various static and dynamic loads, are widely used as structural elements. Advances in the technology of manufacturing thin-walled structural elements of any shape made it possible to produce structures with predetermined patterns of thickness variation. Calculations of strength, vibration and stability of such structures play an important role in design of modern apparatuses, machines and structures. The paper considers nonlinear vibrations of viscoelastic orthotropic cylindrical panels of variable thickness under periodic loads. The equation of motion for cylindrical panels is based on the Kirchhoff-Love hypothesis in a geometrically nonlinear statement. Using the Bubnov-Galerkin method, based on a polynomial approximation of deflections, the problem is reduced to the study of a system of ordinary integro-differential partial differential equations, where time is an independent variable. The solution to the resulting system is found by a numerical method based on the feature elimination in the Koltunov-Rzhanitsyn kernel used in the calculations. The behavior of a cylindrical panel with a wide range of changes in physico-mechanical and geometrical parameters is investigated.

1. Introduction

Many elements of modern designs can be classified as plates and shells. Such elements can have constant or variable thickness, various complex shapes and be subjected to both static and dynamic effects. With the aim of creating plates and shells of variable thickness, the use of composite materials is of considerable interest. The development of the nonlinear theory of plates and shells of variable thickness from a viscoelastic material is conditioned on their practical application.

There are a number of published studies on the strength and stability of plates and shells of constant thickness from composite materials. But until today, research on plates and shells of variable thickness from composite materials has not yet been presented in full, since the issue of vibrations and stability of plates and shells of variable thickness is one of the complex ones.

In [1], an estimate was given of the eigenfrequencies of doubly-curved shells of variable thickness.

The studies in [2] were devoted to the bending of composite plates of variable thickness. The upper and lower surfaces of the plate were symmetrical with respect to the middle plane.
Studies of free vibrations of composite shells of variable stiffness were given in [3]. There, the influence of fiber orientation inside and across the plates, as well as the geometry of the shell, was studied.

In [4], a method for modeling rectangular plates of variable thickness with cuts was proposed. The plate thickness was presented as a finite sum of one-dimensional functions.

The stability of composite inclined plates under the impact of compressive loads was studied in [5].

The research in [6] presents an optimization method for designing the shape and thickness of an orthotropic shell. Here, the problem of optimizing the shape and thickness in arbitrary form is formulated in a system with distributed parameters based on the variational method.

Using the displacement potential function, the bending of isotropic thick rectangular plates of variable thickness was studied in [7].

The study of free vibrations of cylindrical shells of variable thickness under various boundary conditions was the subject of [8]. There, the thickness varied linearly. Numerical results obtained in the paper were confirmed by the results of analytical relationships given in the previous studies.

In [9], the dynamic instability of composite plates of variable stiffness was studied under various parameters of the material and the geometry. The dynamic instability problem was solved using the Bolotin method.

Most problems in the theory of viscoelasticity lead to the solution of linear and nonlinear boundary value problems for systems of partial differential integro-differential equations, which can have a high order and variable coefficients [10]. Due to the complexity of integrating such equations, in most cases they are solved by numerical methods.

There are a number of published works in the literature devoted to solving viscoelastic problems by various numerical methods [11–17]. The effectiveness of the applied numerical methods depends on such factors as the time spent on solving the problem and the accuracy of the results.

Nonlinear parametric vibrations of viscoelastic isotropic plates and shallow shells of variable thickness were studied in [18, 19]. An analysis of the available literature showed that studies on nonlinear parametric vibrations of viscoelastic orthotropic plates and shells of variable thickness are almost non-existent. In this paper, nonlinear parametric oscillations of viscoelastic orthotropic cylindrical panels of variable thickness are investigated.

2. Methods

Consider a viscoelastic orthotropic cylindrical panel of variable thickness \( h = h(x) \) with sides \( a \) and \( b \), the radius of curvature of the middle surface \( R \) under the impact of a dynamic load \( P(t) = P_0 + P_1 \cos \Theta t \) (\( P_0, P_1 = \text{const} \); \( \Theta \) is the frequency of the external periodic load) along the generatrix. It is believed that the panel has an initial deflection \( w_0 \).

The mathematical model of the problem with respect to the transverse deflection \( w \) and displacements \( u, v \), under the corresponding boundary and initial conditions, is described by the following system of equations [20]

\[
\begin{align*}
& h B_{11} \left( 1 - \Gamma_{11} \right) \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x} \right] + \frac{1}{R} B_{12} \left( 1 - \Gamma_{12} \right) \frac{\partial^2 w}{\partial x} + \\
& + \left[ B_{12} \left( 1 - \Gamma_{12} \right) + 2B \left( 1 - \Gamma \right) \right] \left[ \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right] + 2B \left( 1 - \Gamma \right) \left[ \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial y^2} \right] + \\
& + \frac{\partial h}{\partial x} \left[ B_{11} \left( 1 - \Gamma_{11} \right) \left[ \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{12} \left( 1 - \Gamma_{12} \right) \left[ \frac{\partial w}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] - \frac{1}{R} B_{12} \left( 1 - \Gamma_{12} \right) v \right] - \rho h \frac{\partial^2 u}{\partial t^2} = 0,
\end{align*}
\]
\[
\begin{align*}
&h\left[ B_{22} \left( 1 - \Gamma_{22} \right) \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) - \frac{1}{R} B_{22} \left( 1 - \Gamma_{22} \right) \frac{\partial w}{\partial y} + \\
&+ \frac{\partial h}{\partial x} \left[ B_{21} \left( 1 - \Gamma_{21} \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) \right] + \\
&+ \frac{\partial^2 h}{\partial x^2} 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} \right) - \rho \frac{\partial^2 y}{\partial x^2} = 0, \\
&\frac{h^3}{12} \left[ B_{11} \left( 1 - \Gamma_{11}^* \right) \frac{\partial^4 (w - w_0)}{\partial y^4} + 8B \left( 1 - \Gamma^* \right) + B_{12} \left( 1 - \Gamma_{12}^* \right) + B_{21} \left( 1 - \Gamma_{21}^* \right) \right] \frac{\partial^4 (w - w_0)}{\partial x^2 \partial y^2} + \\
&+ B_{22} \left( 1 - \Gamma_{22}^* \right) \frac{\partial^4 (w - w_0)}{\partial y^2} \right] + \\
&+ \frac{1}{2} h^2 \frac{\partial h}{\partial x} \left[ B_{11} \left( 1 - \Gamma_{11}^* \right) \frac{\partial^3 (w - w_0)}{\partial y^3} + B_{12} \left( 1 - \Gamma_{12}^* \right) \frac{\partial^3 (w - w_0)}{\partial x \partial y^2} \right] + \\
&- h \left[ \frac{1}{R} B_{21} \left( 1 - \Gamma_{21}^* \right) \frac{\partial w}{\partial y} + \frac{1}{R} B_{22} \left( 1 - \Gamma_{22}^* \right) \frac{\partial v}{\partial y} \right] \frac{1}{R^2} B_{22} \left( 1 - \Gamma_{22}^* \right) w + \\
&+ \frac{1}{2R^2} B_{21} \left( 1 - \Gamma_{21}^* \right) \left( \frac{\partial w}{\partial y} \right)^2 + \frac{1}{2R} B_{22} \left( 1 - \Gamma_{22}^* \right) \left( \frac{\partial v}{\partial y} \right)^2 \right] \right] - \\
&- \frac{\partial w}{\partial x} \left[ h \left[ B_{11} \left( 1 - \Gamma_{11}^* \right) \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + \frac{1}{R} B_{12} \left( 1 - \Gamma_{12}^* \right) \frac{\partial w}{\partial x} + \\
&+ \left( B_{12} \left( 1 - \Gamma_{12}^* \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 v}{\partial x \partial y^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) \right] + \\
&+ \frac{\partial h}{\partial x} \left[ B_{11} \left( 1 - \Gamma_{11}^* \right) \left( \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + B_{12} \left( 1 - \Gamma_{12}^* \right) \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] \right] - \\
&- \frac{\partial^2 w}{\partial x^2} h \left[ B_{11} \left( 1 - \Gamma_{11}^* \right) \left( \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + B_{12} \left( 1 - \Gamma_{12}^* \right) \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] - \\
&- \frac{1}{R} B_{22} \left( 1 - \Gamma_{22}^* \right) w + \frac{1}{R} B_{21} \left( 1 - \Gamma_{21}^* \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial x \partial y^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) + \\
&+ \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y} \right) \right] + \frac{\partial h}{\partial x} B \left( 1 - \Gamma^* \right) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \right) + \\
&- \frac{\partial^2 w}{\partial y^2} h \left[ B_{21} \left( 1 - \Gamma_{21}^* \right) \left( \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + B_{22} \left( 1 - \Gamma_{22}^* \right) \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] + \frac{1}{R} B_{22} \left( 1 - \Gamma_{22}^* \right) w - (1)
\end{align*}
\]
\[ -4 \frac{\partial^2 w}{\partial x \partial y} h h \left( 1 - \Gamma \right) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) - P(t) \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} = q \]

Here, the weakly singular Koltunov - Rzhanitsyn kernel with three rheological parameters \((A, B, \alpha)\) of the form [21] is used as a relaxation kernel:

\[ \Gamma(t) = Ae^{-\beta t^{\alpha - 1}}, (0 < \alpha < 1) \]

The solution of the system of nonlinear partial integro-differential equations (1) under various boundary conditions and in the presence of weakly singular hereditary kernels presents significant mathematical difficulties. Therefore, to build a discrete model of the problem under consideration, the Bubnov-Galerkin method is used.

We approximate the complete and initial deflections \(w\) and \(w_0\), displacements \(u\), \(v\) in the resulting system (1) in the following form

\[ u(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} u_{nm}(t) \phi_{nm}(x, y), \quad v(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} v_{nm}(t) \phi_{nm}(x, y), \quad (2) \]

\[ w(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{nm}(t) \phi_{nm}(x, y), \quad w_0(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{0nm} \phi_{nm}(x, y) \]

where \( u_{nm} = u_{nm}(t), \quad v_{nm} = v_{nm}(t), \quad w_{nm} = w_{nm}(t) \) - are the unknown functions of time; \( \phi_{nm}(x, y) \), \( \psi_{nm}(x, y), \quad n = 1, 2, ..., N; \quad m = 1, 2, ..., M \) - are the coordinate functions satisfying the given boundary conditions of the problem.

Substituting (2) into the system of equations (1), applying the Bubnov-Galerkin method, and introducing the following dimensionless quantities

\[ \frac{u}{h_0}, \quad \frac{v}{h_0}, \quad \frac{w}{h_0}, \quad \frac{x}{h_0}, \quad \frac{y}{h_0}, \quad \lambda = \frac{a}{b}, \quad \delta = \frac{b}{h_0}, \quad k_y = \frac{b^2}{h_0}, \quad q \left( \frac{b}{h_0} \right), \quad \Theta, \quad \alpha \xi \]

and maintaining the previous notation, to determine the unknowns \( w_{nm} = w_{nm}(t), \quad u_{nm} = u_{nm}(t), \quad v_{nm} = v_{nm}(t) \), we obtain the following system of basic resolving nonlinear integro-differential equations:

\[ \sum_{n=1}^{N} \sum_{m=1}^{M} \alpha_{klmn} \tilde{u}_{nm} - \eta_1 \left\{ \sum_{n=1}^{N} \sum_{m=1}^{M} \left[ (1-\Gamma^*) I_{klmn}^1 + (1-\Gamma^*) I_{klmn}^2 \right] \tilde{u}_{nm} + \right. \]

\[ + \left. \left[ (1-\Gamma^*) I_{klmn}^3 + (1-\Gamma^*) I_{klmn}^4 \right] \tilde{u}_{nm} \right\} + \]

\[ + \sum_{n=1}^{N} \sum_{m=1}^{M} \left( [1-\Gamma^*] I_{ijkm} + [1-\Gamma^*] I_{ijkm} + [1-\Gamma^*] I_{ijkm} \right) \tilde{w}_{nm} w_{0nm} w_{0ij} \]
\[-\eta_3 \sum_{n,l=1}^{N} \sum_{m,j=1}^{M} w_{nm} \left[ (l-\Gamma_{11}^*)_1 \xi_{1kmijnj} + (l-\Gamma_{21}^*)_2 \xi_{2kmijnj} + \\
+ (l-\Gamma^*)_3 \xi_{3kmijnj} \right] + \left( l-\Gamma_{12}^* \right)_1 \xi_{4kmijnj} + \left( l-\Gamma^* \right)_2 \xi_{5kmijnj} + \left( l-\Gamma^* \right)_3 \xi_{7kmijnj} + \\
+ \sum_{n,l,r=1}^N \sum_{m,j=1}^M w_{nm} \left[ (l-\Gamma_{11}^*)_1 \gamma_{5kmijnjrs} + (l-\Gamma_{12}^*)_2 \gamma_{6kmijnjrs} + (l-\Gamma^*)_3 \gamma_{7kmijnjrs} + \\
+ (l-\Gamma_{21}^*)_1 \gamma_{8kmijnjrs} + (l-\Gamma^*)_2 \gamma_{9kmijnjrs} \right] = 0,
\]

where \( h_0 = h(0) = \text{const} \); 
\( P_{kmn}^2 = f_{5kmn} + f_{6kmn} + f_{7kmn} + f_{8kmn} + f_{9kmn} - 4\pi^2 \lambda^2 p_{kmn}^* \delta_0 \); 
\( \mu_{kmn} = \frac{2\pi^2 \lambda^2 p_{kmn}^*}{P_{kmn}^2} \delta_1 \); 
\( \delta_0 = \frac{P_0}{P_{cr}} \); 
\( \delta_1 = \frac{P_1}{P_{cr}} \); 
\( P_{cr} \) - is the static critical load; other coefficients are associated with coordinate functions and their derivatives.

3. Results and Discussion

To integrate the resulting system (3), a numerical method [17] was applied. Based on the algorithm for solving the problem, the software in the Delphi algorithmic language was compiled. The calculation results are reflected in the graphs in figures 1-3. The law of thickness variation is taken in the form: 
\( h = 1 + \alpha^* x \). As initial data, the following values were taken in calculations: 
\( \delta = 25 \); 
\( w_0 = 0.01 \); 
\( q = 0 \); 
\( \lambda = 1 \); 
\( k_y = 10 \); 
\( \alpha^* = 0.5 \); 
\( \delta_0 = 0.3 \); 
\( \delta_1 = 0.5 \); 
\( \Theta = 1.1 \).

Figure 1 shows the results of the thickness variation parameter \( \alpha^* \) effect on the behavior of a viscoelastic cylindrical panel.

**Figure 1.** Dependence of deflection on time at 
\( \alpha^* = 0 \) (1); \( 0.25 \) (2); \( 0.5 \) (3)

The results of the study show that an increase in the parameter \( \alpha^* \) leads to an increase in the amplitude of oscillations. Note that at the initial process of oscillations, the amplitudes slightly differ from each other.

Figure 2 shows the results of studying the panel behavior at various values of the curvature \( k_y \). An analysis of the results shows that an increase in this parameter leads to an increase in the amplitude of oscillations and a phase shift.
In figure 3, various curves correspond to the results obtained by various theories. Curve 1 corresponds to the elastic case, curve 2 to the results obtained with account for viscosity in shear directions only \( A = 0.05, A_{ij} = 0, i, j = 1,2 \), and curve 3 to the case when viscosity is taken into account in all directions \( A = A_{ij} = 0.05, i, j = 1,2 \).

The results obtained confirm the need to take into account the viscoelastic properties of the material not only in shear direction, but also in other directions.

4. Conclusion
1. A mathematical model, method and computer program for the study of parametric vibrations of viscoelastic orthotropic cylindrical panels of variable thickness were developed.
2. Based on the polynomial approximation of deflections and displacements, the parametric vibrations of viscoelastic orthotropic cylindrical panels of variable thickness were studied.
3. The effect on the amplitude-time characteristics of the change in the physicomechanical and geometrical parameters of the panel was estimated.
4. The proposed method and algorithm can be used for various types of thin-walled structures (plates, panels, shells) of variable thickness.
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