Optical Hall conductivity in QHE systems

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Abstract. As a natural extension of the static Hall conductivity to ac regimes, the optical Hall conductivity $\sigma_{xy}(\omega)$ is studied theoretically in the THz region. We found that the Hall plateaus remain in the optical (THz) region when the disorder is not too large, although the quantization of the plateau height does not hold. We have also calculated $\sigma_{xy}(\omega)$ in the graphene QHE system to find that the Hall plateaus also remain with $\sigma_{xy}(\omega)$ reflecting the massless Dirac dispersion in graphene. We predict that the optical Hall conductivity should be measurable through an accurate detection of the Hall angle in the THz regime.

Introduction

While the quantum Hall effect is among the most remarkable static properties of two-dimensional electron systems at low temperatures in magnetic fields, recent experimental advances in optics in the THz region make spectroscopic measurements of the Hall angle possible in magnetic fields of a few tesla [1]. So a natural question we pose here is how the quantum Hall effect, a topological phenomenon, evolves into the optical Hall conductivity in the ac regime, especially in the THz region, which is the scale of the cyclotron energy.

Motivated by this, we have theoretically calculated the optical Hall conductivity $\sigma_{xy}(\omega)$, first for the standard quantum Hall system with the linear response formula, where the effect of disorder (short-range impurities) is taken into account with the self-consistent Born approximation. We shall show that the Hall plateaus do remain in the optical (THz) region when the disorder is not too large, although the quantization of the plateau height does not hold. We have then extended the calculation to the graphene QHE system, and found that the optical Hall conductivity $\sigma_{xy}(\omega)$ reflects the massless Dirac dispersion and the associated Landau level structure, for which the Hall plateaus again remain in the ac region. We predict that such phenomena should be measurable through an accurate detection of the Hall angle in the THz regime.

Optical Hall conductivity

To calculate the optical conductivity in the QHE system we utilized the Kubo formula,

$$\sigma_{xy}(\omega) = \frac{e^2}{4\pi} \int d\varepsilon f(\varepsilon) \frac{1}{\hbar\omega} \left\{ \text{Tr} \left[ j_x \text{Im} G(\varepsilon) j_y (G(\varepsilon + i\hbar\omega) - G^+(\varepsilon)) \right] ight. \right.$$ \left. - \left. \text{Tr} \left[ j_x (G^-(\varepsilon) - G^-(\varepsilon - i\hbar\omega)) j_y \text{Im} G(\varepsilon) \right] \right\}, \tag{1} \right.$$}

where $f(\varepsilon)$ is the Fermi distribution, and $G^\pm = G(\varepsilon \pm i\delta)$ are advanced and retarded Green’s functions. To include the effect of disorder, we adopt Green’s function $G$ with a self-energy.
\begin{equation}
\Sigma\alpha(\varepsilon) = \frac{\Gamma^2}{4} \sum_{n'} [\varepsilon - \varepsilon_{n'} - \Sigma\omega(\varepsilon)]^{-1} \text{ with } n \text{ the Landau index, in the self-consistent Born approximation (SCBA). The Landau level broadening is given by } \Gamma^2/4 = n_0 V_0^2 \text{ if we assume for simplicity short-ranged impurity potential, } V = \sum_i V_0 \delta(\mathbf{r} - \mathbf{r}_i)[2, 3].
\end{equation}

The current matrix elements between Landau levels in usual QHE systems such as GaAs are given as

\begin{align}
    j_x^{n,n'} &= i \sqrt{\frac{\hbar\omega_0}{2m^*}} \left[ \sqrt{n} \delta_{n-1,n'} - \sqrt{n+1} \delta_{n+1,n'} \right], \\
    j_y^{n,n'} &= \sqrt{\frac{\hbar\omega_0}{2m^*}} \left[ \sqrt{n} \delta_{n-1,n'} + \sqrt{n+1} \delta_{n+1,n'} \right].
\end{align}

We have numerically obtained the Green’s function and optical conductivity. In Fig.1 for \( \sigma_{xy}(\omega) \) for the usual QHE system, \( \omega = 0 \) cross section corresponds to the familiar static Hall conductivity, which is quantized into \( ne^2/\hbar \). If we go to the ac region, we can observe that the step structure of Hall conductance is preserved, although the plateau height is no longer quantized. The step structure is then disrupted when \( \omega \) reaches the cyclotron frequency around which the cyclotron resonance takes place for \( \sigma_{xy}(\omega) \).

To trace back the origin of the retained plateau structure, we can rewrite eqn.1 in terms of the eigenstates of the Hamiltonian as

\begin{equation}
\sigma_{xy}(\omega) = \sum_{\epsilon_a < E_f} \sum_{\epsilon_b \geq E_f} j_x^{ab} j_y^{ba} \frac{1}{\hbar\omega} \left( \frac{1}{\epsilon_b - \epsilon_a - \hbar\omega} - \frac{1}{\epsilon_b - \epsilon_a + \hbar\omega} \right) - j_y^{ab} j_x^{ba} \frac{1}{\hbar\omega} \left( \frac{1}{\epsilon_b - \epsilon_a} - \frac{1}{\epsilon_b - \epsilon_a + \hbar\omega} \right)
\end{equation}

This expression reduces to \( (ne^2/\hbar)[\omega_c^2/(\omega_c^2 - \omega^2)] \) in the clean limit, which shows a step structure in ac region with resonance at \( \omega = \omega_c \). A disorder acts to smear the steps. Thus,
Figure 2. Optical Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$ for the graphene QHE system where the disorder is $\Gamma = 0.1$. (a) $\sigma_{xy}(\varepsilon_F, \omega)$ against Fermi energy and frequency, (b) against Fermi energy with a fixed frequency $\omega = 0.25\hbar\omega_c$, and (c) against frequency with a fixed Fermi energy $\varepsilon_F = -0.5\hbar\omega_c$.

while the quantization of $\sigma_{xy}$ as protected topologically in the static case is destroyed for $\omega \neq 0$, plateaus remain when disorder is not too strong(Fig.1).

**Optical Hall conductivity in graphene**

We can extend the optical conductivity to graphene, which is the target of intensive research interests. In the case of graphene the current matrix elements between Landau levels in Eqn.(1) reflect the massless Dirac dispersion, and have an unusual selection rule, $|n| - |n'| = \pm 1$ in place of the ordinary $n - n' = \pm 1$. They are given as

\[
\begin{align*}
J_x^{n,n'} &= v_F C_n C_{n'} \left[ \text{sgn}(n)\delta_{|n|-1,|n'|} + \text{sgn}(n')\delta_{|n|,|n'|+1} \right], \\
J_y^{n,n'} &= iv_F C_n C_{n'} \left[ \text{sgn}(n)\delta_{|n|-1,|n'|} - \text{sgn}(n')\delta_{|n|+1,|n'|} \right].
\end{align*}
\]

(4)

where $C_n = 1(n = 0)$ or $1/\sqrt{2}$ (otherwise)[4].

We have obtained $\sigma_{xy}(\omega)$ for graphene in Fig.2. The result shows more complex Landau level structure, $\text{sgn}(n)\sqrt{n}\hbar\omega_C$ with $\omega_C = v_F\sqrt{2eB/\hbar}$, which again reflects the massless Dirac band. Due to the electron-hole symmetry, $\sigma_{xy}(\varepsilon_F, \omega)$ is odd in $\varepsilon_F$. On the $\omega$ axis a resonances appears around each allowed transition $|n| - |n'| = \pm 1$. Prior to these resonances, we again observe that a Hall step structure remains in ac region.

**Dependence on disorder**

We have finally examined how an increased disorder destroys the plateau structure in the optical Hall conductivity. The result in Fig.3(a) shows that, as we increase the disorder represented by $\Gamma$, the plateaus of static Hall conductivity $\sigma_{xy}(\omega = 0)$ become narrower because of level broadening $\sim \Gamma$, and disappears around $\Gamma \approx 0.5\hbar\omega_c$.

On the other hand, the optical Hall plateaus (Fig.3(b)) become narrower with $\Gamma$ as well, at smaller values of $\Gamma(\sim 0.4\hbar\omega_c)$ than that for the narrowing of the static plateau as expected since
Figure 3. (a) Static Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega = 0)$ and (b) optical Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega = 0.4\omega_c)$ against the disorder ($\Gamma$).

the ac Hall conductivity is not topologically protected, but the narrow plateaus in the optical Hall plateau persists up to $\Gamma \sim \hbar\omega_c$.

Summary
To summarize, we have shown that the optical Hall conductivity for the ordinary QHE system and graphene QHE system to find the Hall plateau structure remains even in ac region if the disorder is not too strong. We propose that these Hall step properties should be observable through accurate Hall angle measurements in THz spectroscopy. The Hall angle $\Theta$ is in fact directly connected to optical conductivity as $\Theta = \arg(t_-/t_+)$, where $t_\pm = 2n_0/(n_0 + n_s + (4\pi/c)\sigma_\pm)$ is the transmission coefficient with $\sigma_\pm = \sigma_{xx} \pm \sigma_{xy}$ being the optical conductivity for circularly polarized light [5]. As future problems we have to study the effects of long-range impurities and more generally the Anderson localization on the optical Hall conductivity.

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