Attainable subspaces and the bang-bang property of time optimal controls for heat equations

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Abstract

In this paper, we study two subjects on internally controlled heat equations with time varying potentials: the attainable subspaces and the bang-bang property for some time optimal control problems. We present some equivalent characterizations on the attainable subspaces, and provide a sufficient conditions to ensure the bang-bang property. Both the above-mentioned characterizations and the sufficient condition are closely related to some function spaces consisting of some solutions to the adjoint equations. It seems for us that the existing ways to derive the bang-bang property for heat equations with time-invariant potentials (see, for instance, [4], [7], [16] and [26]) do not work for the case where the potentials are time-varying. We provide another way to approach it in the current paper.

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1 Introduction

Let Ω ⊂ R^d, d ≥ 1, be a bounded domain with a C^2 boundary ∂Ω. Write ω ⊂ Ω for an open and non-empty subset with its characteristic function χ_ω. Consider the controlled
heat equation:

\[
\begin{aligned}
&\begin{cases}
y_t - \Delta y + ay = \chi_\omega u & \text{in } \Omega \times \mathbb{R}^+, \\
y = 0 & \text{on } \partial \Omega \times \mathbb{R}^+, \\
y(x, 0) = y_0(x) & \text{in } \Omega,
\end{cases}
\end{aligned}
\tag{1.1}
\]

where \( a \in L^\infty(\Omega \times \mathbb{R}^+) \), \( y_0 \in L^2(\Omega) \) and \( u \in L^p(\mathbb{R}^+; L^2(\Omega)) \), with \( 1 < p \leq \infty \). We will treat the solution of Equation (1.1) as a function from \( \mathbb{R}^+ \) to \( L^2(\Omega) \), and denote it by \( y(\cdot; y_0, u) \). When \( \hat{u} \in L^p(0, T; L^2(\Omega)) \) for some \( T > 0 \), we use \( y(\cdot; y_0, \hat{u}) \) to stand for the solution of Equation (1.1), where \( u = \hat{u} \) over \((0, T)\) and \( u = 0 \) over \((T, \infty)\).

Throughout the paper, \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) stand for the usual norm and inner product in \( L^2(\Omega) \); \( \| \cdot \|_\omega \) and \( \langle \cdot, \cdot \rangle_\omega \) denote the usual norm and inner product in \( L^2(\omega) \). Given \( T > 0 \) and \( z \in L^2(\Omega) \), write \( \varphi(\cdot; T, z) \) for the solution to the adjoint equation:

\[
\begin{aligned}
&\begin{cases}
\varphi_t + \Delta \varphi - a\varphi = 0 & \text{in } \Omega \times (0, T), \\
\varphi = 0 & \text{on } \partial \Omega \times (0, T),
\end{cases}
\end{aligned}
\tag{1.2}
\]

with the initial condition \( \varphi(T) = z \) over \( \Omega \).

This paper studies two subjects on internally controlled equation (1.1): the attainable subspaces and the bang-bang property of some time optimal control problems. These subjects are related to the spaces \( Y_{T,q} \) (with \( T > 0 \) and \( 1 \leq q < \infty \)), which are defined by

\[
Y_{T,q} = \overline{X_{T,q} \| \cdot \|_{L^q(0,T;L^2(\omega))}},
\tag{1.3}
\]

endowed with the norm:

\[
\| \cdot \|_{Y_{T,q}} \triangleq \| \cdot \|_{L^q(0,T;L^2(\omega))},
\tag{1.4}
\]

where \( X_{T,q} = \{ \chi_\omega \varphi(\cdot; T, z) \mid z \in L^2(\Omega) \} \) endowed with the \( L^q(0,T;L^2(\omega)) \)-norm. We start with introducing the attainable subspaces. The attainable subspaces of (1.1) at time \( T > 0 \) are defined by

\[
A_{T,p} = \{ y(T; 0, u) \mid u \in L^p(0,T;L^2(\Omega)) \}, \quad 1 < p \leq \infty,
\tag{1.5}
\]

endowed with the norms:

\[
\| y_T \|_{A_{T,p}} \triangleq \inf \{ \| u \|_{L^p(0,T;L^2(\Omega))} \mid y(T; 0, u) = y_T \}, \quad y_T \in A_{T,p}.
\tag{1.6}
\]

We next introduce the following time optimal control problem \( (TP)_{y_0}^{M,p} \):

\[
T_p(M, y_0) \triangleq \inf_{u \in \mathcal{U}^{M,p}} \{ t \mid y(t; y_0, u) = 0 \},
\tag{1.7}
\]

where \( y_0 \in L^2(\Omega) \setminus \{0\} \), \( M > 0 \), \( 1 < p \leq \infty \) and

\[
\mathcal{U}^{M,p} \triangleq \{ v : \mathbb{R}^+ \to L^2(\Omega) \mid \| v \|_{L^p(\mathbb{R}^+;L^2(\Omega))} \leq M \}.
\]
In Problem $\{(TP)^{M,p}_{y_0}\}$, $u^* \in U^{M,p}$ is called an optimal control if $y(T_p(M, y_0); y_0, u^*) = 0$; while $\hat{u} \in U^{M,p}$ is called an admissible control if $y(T; y_0, \hat{u}) = 0$ for some $T > 0$.

**Definition 1.1.** Problem $\{(TP)^{M,p}_{y_0}\}$ has the bang-bang property if any optimal control $u^*$ verifies that $\|\chi_\omega u^*\|_{L^p(0, T_p(M, y_0); L^2(\Omega))} = M$ and $\|\chi_\omega u^*(t)\| \neq 0$ for a.e. $t \in (0, T_p(M, y_0))$, when $1 < p < \infty$; while $\|\chi_\omega u^*(t)\| = M$ for a.e. $t \in (0, T_\infty(M, y_0))$, when $p = \infty$.

**Remark 1.1.** We agree that when $\{(TP)^{M,p}_{y_0}\}$ has no any optimal control, it does not hold the bang-bang property.

Our studies on $\{(TP)^{M,p}_{y_0}\}$ are connected with the norm optimal control problem $\{(NP)^{T,p}_{y_0}\}$.

$$N_p(T, y_0) \triangleq \inf \{ \|u\|_{L^p(0, T; L^2(\Omega))} \mid y(T; y_0, u) = 0 \}, \quad T > 0. \quad \text{(1.8)}$$

In Problem $\{(NP)^{T,p}_{y_0}\}$, $u^*$ is called an optimal control if $\|u^*\|_{L^p(0, T; L^2(\Omega))} = N_p(T, y_0)$ and $y(T; y_0, u^*) = 0$; while $\hat{u} \in L^p(0, T; L^2(\Omega))$ is called an admissible control if $y(T; y_0, \hat{u}) = 0$.

**Definition 1.2.** Problem $\{(NP)^{T,p}_{y_0}\}$ has the bang-bang property if any optimal control $u^*$ satisfies that $\|\chi_\omega u^*\|_{L^p(0, T; L^2(\Omega))} = N_p(T, y_0)$ and $\|\chi_\omega u^*(t)\| \neq 0$ for a.e. $t \in (0, T)$, when $1 < p < \infty$; while $\|\chi_\omega u^*(t)\| = N_p(T, y_0)$ for a.e. $t \in (0, T)$, when $p = \infty$.

We treat $N_p(\cdot, y_0)$ as a function of $T$. It is proved that the limit of $N_p(T, y_0)$, as $T$ goes to $\infty$, exists (see Lemma 4.2). Hence, we can let

$$\tilde{N}_p(y_0) \triangleq \lim_{T \to \infty} N_p(T, y_0). \quad \text{(1.9)}$$

To ensure the bang-bang property for $\{(TP)^{M,p}_{y_0}\}$, we impose the following condition on the space $Y_{T,q}$ (with $q$ the conjugate exponent of $p$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$):

$$Y_{T,q} = Z_{T,q} \quad \text{for each} \quad T > 0, \quad \text{(1.10)}$$

where

$$Z_{T,q} = \{ \chi_\omega \psi \in L^q(0, T; L^2(\omega)) \mid \psi \in C([0, T]; L^2(\Omega)) \text{ solves Equation (1.2)} \}. \quad \text{(1.11)}$$

The main results obtained in this paper are as follows.

**Theorem 1.1.** Let $T > 0$. Let $p \in (1, \infty]$ and $q$ be the conjugate exponent of $p$. (i) When $1 < p \leq \infty$, there is a linear isomorphism $G_p$ from $A_{T,p}$ to $Y^*_{T,q}$ (i.e., $G_p$ is linear, one to one and preserves the norms); (ii) When $1 < q < \infty$, there is an isomorphism $H_q : Y_{T,q} \to A_{T,p}$ defined by

$$H_q(\xi) = \begin{cases} y(T; 0, u\xi), & \text{if } \xi \neq 0, \\ 0, & \text{if } \xi = 0, \end{cases} \quad \text{(1.12)}$$
where
\[
    u_\xi(x,t) = \begin{cases} 
        \|\xi\|_{L^q(0,T;L^2(\omega))}^{2-q} : \|\xi(t)\|_{L^\infty}^{q-2} : \xi(x,t), & (x,t) \in \omega \times (0,T), \\
        0, & (x,t) \in (\Omega \setminus \omega) \times (0,T). 
    \end{cases} \tag{1.13}
\]

**Theorem 1.2.** Let \( y_0 \in L^2(\Omega) \setminus \{0\} \). Suppose that (1.10) holds. Then \( (TP)^{M,p}_{y_0} \) has the bang-bang property if and only if \( M > \tilde{N}_p(y_0) \), where \( \tilde{N}_p(y_0) \) is given by (1.9).

**Remark 1.2.** (i) It is proved that \( \|\xi(t)\|_\omega \neq 0 \) for each \( t \in [0,T) \), when \( \xi \in Y_{T,q} \setminus \{0\} \) (see Lemma 2.1). Hence, \( u_\xi \) in (1.13) is well-defined; (ii) \( H_q \) is nonlinear except for the case that \( q = 2 \); (iii) It is proved that (1.10) holds for the case where \( a(x,t) = a_1(x) + a_2(t) \) in \( \Omega \times \mathbb{R}^+ \), with \( a_1 \in L^\infty(\Omega) \), \( a_2 \in L^\infty(\mathbb{R}^+) \) (see Proposition 4.2). Unfortunately, we don’t know if it holds when \( a = a(x,t) \) in \( \Omega \times \mathbb{R}^+ \); (iv) It is worth mentioning that when \( y_0 \in L^2(\Omega) \setminus \{0\} \), \( (TP)^{M,p}_{y_0} \) has optimal controls if and only if \( M > \tilde{N}_p(y_0) \) (see Proposition 4.1).

The attainable subspaces play important roles in the studies of control problems governed by Equation (1.1) (see, for instance, [29] where the connection of attainable subspaces and the stabilization for some periodic evolution system are provided). To our surprise, the studies on the attainable subspaces of internally controlled heat equations are quite limited from the past publications. In [18], the author provided a way to characterize the elements of a subspace of \( A_{T,2} \), via a Riesz basis (see Remarks after Theorem 2 on page 530 in [18]). The method used there is borrowed from [23] and [11], where the elements of a subspace of the controlled wave equation (without the geometric condition imposed on the control region) are explicitly expressed via a Riesz basis. In [29] (see also [30]), the authors presented some properties of attainable subspaces for some \( T \)-periodic evolution systems. Those properties gives the connection of the space \( \bigcup_{T>0} A_{t,\infty} \) and the spaces \( A_{kT,\infty}, k \in \mathbb{N} \). The observations presented in Theorem 1.1 seem to be new. From these observations, we can see that the structure of the attainable subspace \( A_{T,p} \) is very complicated, since \( Y_{T,q} \) is the completion of the function space \( X_{T,q} \) under the norm of \( L^q(0,T;L^2(\omega)) \).

The bang-bang property is one of the most important properties of time optimal control problems, from which one can derive the uniqueness of the optimal control (see [4] and [26]) and the equivalence of the minimal time and norm controls (see [7], [31] and [28]). The bang-bang property was first built up in [5] for \( (TP)^{M,\infty}_{y_0} \) where \( \omega = \Omega \) and \( a \) is time-invariant. When \( p \in (1,\infty), \omega \subset \subset \Omega \) and \( a \) is time-invariant, the bang-bang property of \( (TP)^{M,p}_{y_0} \) was studied in [7]. It was first realized in [16] (partially inspired by the work [24]) that the bang-bang property of \( (TP)^{M,\infty}_{y_0} \), where \( w \subset \subset \Omega \) and \( a \) is time-invariant, can be derived from the E-controllability: For each \( T > 0 \), each measurable subset \( E \subset (0,T) \) of positive measure and each \( y_0 \in L^2(\Omega) \), there is a control
\[ u \in L^\infty(0, T; L^2(\Omega)) \text{ with } \|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\Omega, \omega, T, E) \|y_0\| \text{ s.t. } y(T; y_0, \chi_E u) = 0 \] (where \( \chi_E \) is the characteristic function of \( E \)). In fact, once the E-controllability holds, one can easily prove the bang-bang property by contradiction, through using the E-controllability and the time-invariance of the system. The E-controllability was first built up for the case where \( a = 0 \) (see \cite{26}), and then was extended to the case where \( a \) is time-varying (see \cite{19} and \cite{21}). Here, we would like to mention that when \( \omega \subset\subset \Omega \), the bang-bang property for some time-invariant semilinear heat equations was first built up in \cite{21}, via a very smart way. However, we are not able to use the methods in \cite{7} and \cite{16} (see also \cite{26}) to derive the bang-bang property of \((TP)^{M,p}_{y_0}\). Our Theorem 1.2 provides the sufficient (1.10) to ensure the bang-bang property for the time-varying case. This theorem, along with Proposition 4.2, implies the bang-bang property for the above-mentioned special case.

About works on the time optimal control problems, we would like to mention the papers \cite{1, 2, 3, 4, 5, 8, 9, 10, 12, 13, 14, 15, 16, 19, 21, 22, 24, 25, 26, 27, 31, 32, 33, 34, 35} and the references therein.

The rest of the paper is organized as follows: Section 2 proves Theorem 1.1. Section 3 presents some properties on \((NP)^{T,p}_{y_0}\). Section 4 provides the proof of Theorem 1.2.

## 2 Attainable subspaces

The aim of this section is to prove Theorem 1.1. We start with proving its first part.

**Proof of the part (i) of Theorem 1.1.** First of all, from equations (1.1) and (1.2), one can easily check that

\[ \int_0^T \langle v(t), \chi_\omega \varphi(t; T, z) \rangle \, dt = \langle y(T; 0, v), z \rangle \text{ for all } z \in L^2(\Omega), v \in L^p(0, T; L^2(\Omega)). \] (2.1)

Let \( y_T \in A_{T,p} \). Then \( y_T = y(T; 0, \hat{u}) \) for some \( \hat{u} \in L^p(0, T; L^2(\Omega)) \). Define \( \mathcal{F}_{y_T,q} : X_{T,q} \rightarrow \mathbb{R} \) by setting

\[ \mathcal{F}_{y_T,q}(\chi_\omega \varphi(\cdot; T, z)) = \int_0^T \langle \hat{u}(t), \chi_\omega \varphi(t; T, z) \rangle \, dt \text{ for each } z \in L^2(\Omega). \] (2.2)

From (2.1) and (2.2), one can easily check that \( \mathcal{F}_{y_T,q} \) is well-defined and linear. Meanwhile, using the Hölder’s inequality to the right side of (2.2), we see that \( \mathcal{F}_{y_T,q} \) is bounded. Thus \( \mathcal{F}_{y_T,q} \in X_{T,q}^* \). Since \( X_{T,q}^* = Y_{T,q}^* \) (see (1.3)), we have

\[ \mathcal{F}_{y_T,q} \in Y_{T,q}^*. \] (2.3)
Define $G_p : A_{T,p} \rightarrow Y^*_{T,q}$ by setting
\[
G_p(y_T) = \mathcal{F}_{y_{T,q}} \text{ for each } y_T \in A_{T,p}. \tag{2.4}
\]
Clearly, $G_p$ is linear. From (2.2) and (2.1), one can easily verify that $G_p$ is injective.

We now prove that $G_p$ is surjective. Let $i : Y^*_{T,q} \rightarrow L^q(0, T; L^2(\omega))$ be the embedding map and $i^* : L^p(0, T; L^2(\omega)) \rightarrow Y^*_{T,q}$ be the adjoint operator of $i$. We claim that
\[
\text{Range}(i^*) = Y^*_{T,q}, \quad \text{i.e., } i^* \text{ is surjective.} \tag{2.5}
\]
By the Hahn-Banach theorem, for each $\mathcal{F} \in Y^*_{T,q}$, there is a $\tilde{\mathcal{F}} \in \left( L^q(0, T; L^2(\omega)) \right)^*$ s.t.
\[
\tilde{\mathcal{F}}(\xi) = \mathcal{F}(\xi) \text{ for each } \xi \in Y_{T,q} \tag{2.6}
\]
and
\[
\|\tilde{\mathcal{F}}\|_{L(L^q(0, T; L^2(\omega)); \mathbb{R})} = \|\mathcal{F}\|_{Y^*_{T,q}}.
\]
According to the Riesz representation theorem, there is a $\tilde{\psi} \in L^p(0, T; L^2(\omega))$ s.t.
\[
\tilde{\mathcal{F}}(\psi) = \int_0^T \langle \tilde{\psi}(t), \psi(t) \rangle_{\omega} \, dt \text{ for each } \psi \in L^q(0, T; L^2(\omega)). \tag{2.7}
\]
Because $X_{T,q} \subset Y_{T,q}$, it follows from (2.6) and (2.7) that
\[
\mathcal{F}(\chi_\omega \varphi(:, T, z)) = \int_0^T \langle \hat{\psi}(t), \chi_\omega \varphi(t; T, z) \rangle_{\omega} \, dt \text{ for each } z \in L^2(\Omega).
\]
Thus, it holds that
\[
\langle i^*(\tilde{\psi}), \chi_\omega \varphi(:, T, z) \rangle_{Y^*_{T,q}, Y_{T,q}} = \langle \hat{\psi}, \chi_\omega \varphi(:, T, z) \rangle_{L^p(0, T; L^2(\omega)), L^q(0, T; L^2(\omega))} = \int_0^T \langle \hat{\psi}(t), \chi_\omega \varphi(t; T, z) \rangle_{\omega} \, dt = \mathcal{F}(\chi_\omega \varphi(:, T, z)) \text{ for each } z \in L^2(\Omega).
\]
This, along with (1.3), yields that $i^*(\tilde{\psi}) = \mathcal{F}$, which leads to (2.5).

By (2.5), for each $\mathcal{F} \in Y^*_{T,q}$, we can find a $\psi \in L^p(0, T; L^2(\omega))$ s.t. $i^*(\psi) = \mathcal{F}$. We extend $\psi$ over $\Omega \times (0, T)$ by setting it to be 0 on $(\Omega \setminus \omega) \times (0, T)$, and denote the extension by $\tilde{\psi}$. Then $\tilde{\psi} \in L^p(0, T; L^2(\Omega))$ and $\tilde{\mathcal{F}} = \mathcal{F}_{\tilde{\mathcal{F}}, q}(\chi_\omega \varphi(:, T, z)) = \int_0^T \langle \tilde{\psi}(t), \chi_\omega \varphi(t; T, z) \rangle \, dt $
\[
= \int_0^T \langle \psi(t), \chi_\omega \varphi(t; T, z) \rangle_{\omega} \, dt \text{ for each } z \in L^2(\Omega) \tag{2.8}
\]
On the other hand, since \( i^*(v) = F \) and
\[
\langle i^*(v), \chi_\omega \varphi(\cdot; T, z) \rangle_{Y_{T,q}^*, Y_{T,q}} = \langle v, i(\chi_\omega \varphi(\cdot; T, z)) \rangle_{L^p(0,T; L^2(\omega)), L^q(0,T; L^2(\omega))}
\]
\[
= \langle v, \chi_\omega \varphi(\cdot; T, z) \rangle_{L^p(0,T; L^2(\omega)), L^q(0,T; L^2(\omega))}
\]
\[
= \int_0^T \langle v(t), \chi_\omega \varphi(t; T, z) \rangle_\omega \, dt \quad \text{for each } z \in L^2(\Omega),
\]
we have
\[
F(\chi_\omega \varphi(\cdot; T, z)) = \int_0^T \langle v(t), \chi_\omega \varphi(t; T, z) \rangle_\omega \, dt \quad \text{for each } z \in L^2(\Omega).
\]
This, along with (2.8), (1.3) and (2.9), yields \( F = F_{\tilde{y}_{T,q}} \) and \( G_p(\tilde{y}_T) = F \). Hence, \( G_p \) is surjective.

Finally, we show that
\[
\|y_T\|_{A_{T,p}} = \|F_{y_{T,q}}\|_{Y_{T,q}^*} \quad \text{(i.e., } \|y_T\|_{A_{T,p}} = \|G_p(y_T)\|_{Y_{T,q}^*}, \text{ for each } y_T \in A_{T,p}. \tag{2.10}\)
\]
Let \( y_T \in A_{T,p} \). Arbitrarily take a \( \hat{u} \in L^p(0,T; L^2(\Omega)) \) such that \( y_T = y(T; 0, \hat{u}) \). From (2.2) and (1.3), it follows that
\[
\langle F_{y_{T,q}}, \xi \rangle_{Y_{T,q}^*, Y_{T,q}} = \int_0^T \langle \hat{u}(t), \xi(t) \rangle_\omega \, dt \quad \text{for each } \xi \in Y_{T,q}. \tag{2.11}\]
Hence, it holds that
\[
\|F_{y_{T,q}}\|_{Y_{T,q}^*} \leq \|\hat{u}\|_{L^p(0,T; L^2(\omega))} \leq \|\hat{u}\|_{L^p(0,T; L^2(\Omega))},
\]
which, as well as (1.6), leads to
\[
\|F_{y_{T,q}}\|_{Y_{T,q}^*} \leq \inf \{ \|u\|_{L^p(0,T; L^2(\omega))} \mid y(T; 0, u) = y_T \} = \|y_T\|_{A_{T,p}}. \tag{2.12}\]
Conversely, we fix a \( \tilde{v} \in L^p(0,T; L^2(\Omega)) \) s.t. \( y(T; 0, \tilde{v}) = y_T \). It follows from (2.11) that
\[
\left| \int_0^T \langle \hat{v}(t), \xi(t) \rangle_\omega \, dt \right| \leq \|F_{y_{T,q}}\|_{Y_{T,q}^*} \cdot \|\xi\|_{Y_{T,q}} \quad \text{for each } \xi \in Y_{T,q}. \tag{2.13}\]
Define \( G^\circ : Y_{T,q} \to \mathbb{R} \) by
\[
G^\circ(\xi) = \int_0^T \langle \hat{v}(t), \xi(t) \rangle_\omega \, dt \quad \text{for each } \xi \in Y_{T,q}. \tag{2.14}\]
By (2.14) and (2.13), \( G^0 \in Y^*_{T,q} \) and \( \| G^0 \|_{Y^*_{T,q}} \leq \| F_{yT,q} \|_{Y^*_{T,q}} \). Then, by the Hahn-Banach theorem, the Riesz representation theorem and (2.14), there is a \( v \in L^p(0, T; L^2(\Omega)) \) s.t.

\[
\| v \|_{L^p(0, T; L^2(\Omega))} \leq \| F_{yT,q} \|_{Y^*_{T,q}} \tag{2.15}
\]

and

\[
\int_0^T \langle \hat{v}(t), \xi(t) \rangle \omega \ dt = \int_0^T \langle v(t), \xi(t) \rangle \omega \ dt \quad \text{for each } \xi \in Y_{T,q}.
\tag{2.16}
\]

Since \( X_{T,q} \subset Y_{T,q} \), we have from (2.16) that

\[
\int_0^T \langle \hat{v}(t), \chi \varphi(t; T, z) \rangle \ dt = \int_0^T \langle \hat{v}(t), \chi \varphi(t; T, z) \rangle \ dt \quad \text{for each } z \in L^2(\Omega),
\tag{2.17}
\]

where \( \hat{v} \) is the extension of \( v \) over \( \Omega \times (0, T) \) such that \( \hat{v} = 0 \) over \( \Omega \setminus \omega \times (0, T) \). Since \( y(T; 0, \hat{v}) = y_T \), one can easily check, by using (2.17) and (2.1), that \( y_T = y(T; 0, \hat{v}) \). This, along with (1.6) and (2.15), leads to

\[
\| y_T \|_{A_{T,q}} \leq \| \hat{v} \|_{L^p(0, T; L^2(\Omega))} = \| v \|_{L^p(0, T; L^2(\omega))} \leq \| F_{yT,q} \|_{Y^*_{T,q}}.
\tag{2.18}
\]

Now, (2.10) follows from (2.12) and (2.18). This completes the proof of the part (i) of Theorem 1.1.

\[\blacksquare\]

To prove the part (ii) of Theorem 1.1, we need to present some properties on \( Y_{T,q} \).

**Lemma 2.1.** Let \( 1 \leq q < \infty \). (i) \( Y_{T,q} \) consists of all such functions \( \chi \varphi \in L^q(0, T; L^2(\omega)) \) that \( \varphi \in C([0, T]; L^2(\Omega)) \) solves Equation (1.2), and \( \chi \varphi = \lim_{n \to \infty} \chi \varphi(\cdot; T, z_n) \) for some sequence \( \{z_n\} \subset L^2(\Omega) \), where the limit is taken in \( L^q(0, T; L^2(\omega)) \); (ii) When \( \xi \in Y_{T,q} \setminus \{0\} \), it holds that \( \| \xi(t) \|_\omega \neq 0 \) for each \( t \in [0, T) \).

**Proof.** (i) Let \( \xi \in Y_{T,q} \). By (1.3), there is a sequence \( \{z_n\} \subset L^2(\Omega) \) such that

\[
\chi \varphi(\cdot; T, z_n) \to \xi \quad \text{strongly in } L^q(0, T; L^2(\omega)).
\tag{2.19}
\]

In particular, \( \{\chi \varphi(\cdot; T, z_n)\} \) is bounded in \( L^2(0, T; L^2(\omega)) \). Let \( \{T_k\} \subset (0, T) \) such that \( T_k \nearrow T \) (i.e., \( T_k \) strictly monotonically converges to \( T \) from the left). Given a \( k \in \mathbb{N} \), by the observability estimate (see, for instance, [6]),

\[
\| \varphi(T_{k+1}; T, z_n) \| \leq C(k) \| \chi \varphi(\cdot; T, z_n) \|_{L^1(T_{k+1}; L^2(\omega))} \leq C(k) \| \chi \varphi(\cdot; T, z_n) \|_{L^q(0, T; L^2(\omega))} \leq C(k) \quad \text{for all } n \in \mathbb{N},
\tag{2.20}
\]

where \( C(k) \) stands for a positive constant depending on \( k \) but independent of \( n \), which may vary in different contexts. Arbitrarily take two subsequences \( \{\varphi(\cdot; T, z_{n_1})\} \) and
\{\varphi(\cdot; T, z_{n_1})\}$ from \{\varphi(\cdot; T, z_n)\}. By (2.20) and the properties of heat equations, there are two subsequences of \{\varphi(\cdot; T, z_{n_1})\} and \{\varphi(\cdot; T, z_{n_2})\} respectively, denoted in the same way, such that

$$
\varphi(\cdot; T, z_{n_1}) \to \hat{\varphi}_{k,1}(\cdot); \varphi(\cdot; T, z_{n_2}) \to \hat{\varphi}_{k,2}(\cdot) \text{ strongly in } C([0, T_k]; L^2(\Omega)),
$$

where \(\hat{\varphi}_{k,1}\) and \(\hat{\varphi}_{k,2}\) solve equation (1.2) (with \(T\) being replaced by \(T_k\)). These, along with (2.19), yield that

$$
\chi_\omega \hat{\varphi}_{k,1}(t) = \chi_\omega \hat{\varphi}_{k,2}(t) = \xi(t) \text{ for a.e. } t \in [0, T_k].
$$

Then by the unique continuation estimate for heat equations built up in [21] (see also [20]), we have

$$
\hat{\varphi}_k \triangleq \hat{\varphi}_{k,1} = \hat{\varphi}_{k,2} \text{ over } [0, T_k].
$$

Hence, it holds that

$$
\varphi(\cdot; T, z_n) \to \hat{\varphi}_k(\cdot) \text{ in } C([0, T_k]; L^2(\Omega)); \chi_\omega \hat{\varphi}_k = \xi \text{ over } (0, T_k).
$$

(2.21)

Since \(k\) in the above was arbitrarily taken from \(\mathbb{N}\), it follows from (2.21) that

$$
\hat{\varphi}_k = \hat{\varphi}_{k+l}; \chi_\omega \hat{\varphi}_k = \xi \text{ over } [0, T_k] \text{ for all } k, l \in \mathbb{N}.
$$

(2.22)

We now define the function \(\hat{\varphi}\) over \(\Omega \times [0, T]\) by setting

$$
\hat{\varphi} = \hat{\varphi}_k \text{ over } [0, T_k], \ k = 1, 2, \ldots.
$$

Then by (2.22), \(\hat{\varphi}\) is well defined; \(\hat{\varphi} \in C([0, T); L^2(\Omega))\) solves Equation (1.2); \(\xi = \chi_\omega \hat{\varphi}\). Clearly, \(\chi_\omega \hat{\varphi}\) is the limit of \(\chi_\omega \varphi(\cdot; T, z_n)\) in \(L^q(0, T; L^2(\omega))\) (see (2.19)). Thus, we have proved \((i)\).

\((ii)\) Let \(\xi \in Y_{T,q} \setminus \{0\}\). By \((i)\), there is a function \(\varphi \in C([0, T); L^2(\Omega))\), with \(\chi_\omega \varphi \in L^q(0, T; L^2(\omega))\), solving Equation (1.2), such that \(\xi = \chi_\omega \varphi\). Since \(\xi \neq 0\) in \(Y_{T,q}\), it holds that \(\varphi \neq 0\) in \(L^q(0, T; L^2(\omega))\). Then by the unique continuation estimate in [21] (see also [19], [20]), it follows that \(\|\chi_\omega \varphi(t)\| \neq 0\) for each \(t \in [0, T]\). This completes the proof. 

The proof of the part \((ii)\) of Theorem 1.1 needs help from the following norm optimal control problem \((NP)_{y_T,p}\):

$$
\inf \left\{ \|u\|_{L^p(0,T;L^2(\Omega))} \mid y(T;0,u) = y_T \right\},
$$

(2.23)

where \(p \in (1, \infty]\) and \(y_T \in A_{T,p}\). The optimal control and the admissible control to this problem can be defined by a very similar way as those for \((NP)_{y_0,p}\) (see Section 1). This problem is related to the variational problem \((JP)_{y_T,q}\):

$$
\inf_{\xi \in Y_{T,q}} J_{y_T,q}(\xi) \triangleq \inf_{\xi \in Y_{T,q}} \left( \frac{1}{2} \|\xi\|_{L^q(0,T;L^2(\omega))}^2 - \mathcal{F}_{y_T,q}(\xi) \right), \xi \in Y_{T,q},
$$

(2.24)

where \(q\) is the conjugate exponent of \(p\) and \(\mathcal{F}_{y_T,q}\) is given by (2.2) (see also (2.3)).
Lemma 2.2. Let $p \in (1, \infty)$ and $q$ be the conjugate exponent of $p$. (i) When $y_T \in A_{T,q} \setminus \{0\}$, it holds that zero (the origin of $Y_{T,q}$) is not a minimizer of $J_{y_T,q}$; $J_{y_T,q}$ has a unique minimizer $\chi_\omega \widehat{\varphi}$ in $Y_{T,q}$, where $\widehat{\varphi} \in C([0,T);L^2(\Omega)) \cap L^q(0,T;L^2(\omega))$ solves Equation (1.2); $(NP)_{y_T,q}$ has a unique optimal control $\hat{u}_{y_T,q}$ given by

$$\hat{u}_{y_T,q}(t) = \|\chi_\omega \widehat{\varphi}\|_{L^q(0,T;L^2(\omega))}^{2-q} \|\chi_\omega \widehat{\varphi}(t)\|^{q-2} \cdot \chi_\omega \widehat{\varphi}(t), \ t \in (0,T); \quad (2.25)$$

(ii) If $y_T = 0$ in $A_{T,q}$, then zero is the unique minimizer of $J_{0,q}$ and the unique optimal control to $(NP)_{0,p}$ is the null control.

Proof. (i) Write $y_T = y(T;0,u_T)$ for some $u_T \in L^p(0,T;L^2(\Omega))$. By contradiction, we suppose that zero was a minimizer. Since $X_{T,q} \subset Y_{T,q}$ (see (1.3)), we would have

$$0 \leq \frac{J_{y_T,q}(\varepsilon \varphi(;0,T,z))}{\varepsilon} \quad \text{for all } \varepsilon > 0 \text{ and } z \in L^2(\Omega).$$

This, along with (2.24), (2.2) and (2.1), yields that $<y_T,z> = 0$ for all $z \in L^2(\Omega)$, which contradicts the fact that $y_T \neq 0$.

Since $1 < q < \infty$, $L^q(0,T;L^2(\omega))$ is reflexible. Thus, $Y_{T,q}$, as a closed subspace of $L^q(0,T;L^2(\omega))$, is also reflexible. Meanwhile, one can directly check that $J_{y_T,q}(\cdot)$ is strictly convex and coercive in $Y_{T,q}$. Hence, $J_{y_T,q}$ has a unique minimizer. Furthermore, it follows from Lemma 2.1 that this minimizer can be expressed by $\chi_\omega \widehat{\varphi} \in L^q(0,T;L^2(\omega))$, where $\widehat{\varphi} \in C([0,T);L^2(\Omega))$ solves Equation (1.2) and verifies $\chi_\omega \widehat{\varphi}(t) \neq 0$ for all $t \in [0,T]$.

Since $F_{y_T,q} \in Y_{T,q}^*$, one can easily derive from (2.24) the following Euler-Lagrange equation associated with the minimizer $\chi_\omega \widehat{\varphi}$:

$$\int_0^T \langle \hat{u}_{y_T,q}(t), \xi(t) \rangle_\omega \ dt - F_{y_T,q}(\xi) = 0 \quad \text{for each } \xi \in Y_{T,q}, \quad (2.26)$$

where $\hat{u}_{y_T,q}$ is defined by (2.25). From (2.26) and (2.11), it follows that

$$\int_0^T \langle \hat{u}_{y_T,q}(t), \xi(t) \rangle_\omega \ dt - \int_0^T \langle v(t), \xi(t) \rangle_\omega \ dt = 0 \quad \text{for each } \xi \in Y_{T,q}, \quad (2.27)$$

when $v$ is an admissible control to $(NP)_{y_T,q}$. This, as well as (2.1), in particular, implies

$$\langle y(T;0,\hat{u}_{y_T,q}), z \rangle = \langle y_T, z \rangle \quad \text{for all } z \in L^2(\Omega),$$

which leads to

$$y(T;0,\hat{u}_{y_T,q}) = y_T. \quad (2.28)$$

On the other hand, it follows from (2.25) that

$$\|\hat{u}_{y_T,q}\|_{L^p(0,T;L^2(\omega))} = \|\chi_\omega \widehat{\varphi}\|_{L^q(0,T;L^2(\omega))}. \quad (2.29)$$
By (2.25), (2.27), with \( \xi = \chi_\omega \hat{\varphi} \), and (2.29), for each admissible control \( v \) to \( (NP)_{y_{T,P}} \), we see
\[
\left( \| \hat{u}_{y_{T,P}} \|_{L^p(0,T;L^2(\Omega))} \right)^2 = \int_0^T \langle \hat{u}_{y_{T,P}}(t), \chi_\omega \hat{\varphi}(t) \rangle \, dt = \int_0^T \langle v(t), \chi_\omega \hat{\varphi}(t) \rangle \, dt \\
\leq \| v \|_{L^p(0,T;L^2(\Omega))} \cdot \| \chi_\omega \hat{\varphi} \|_{L^2(0,T;L^2(\Omega))} \\
= \| v \|_{L^p(0,T;L^2(\Omega))} \cdot \| \hat{u}_{y_{T,P}} \|_{L^p(0,T;L^2(\Omega))}.
\] (2.30)

Hence, \( \| \hat{u}_{y_{T,P}} \|_{L^p(0,T;L^2(\Omega))} \leq \| v \|_{L^p(0,T;L^2(\Omega))} \), when \( v \) is an admissible control to \( (NP)_{y_{T,P}} \). From this and (2.28), \( \hat{u}_{y_{T,P}} \) is an optimal control to \( (NP)_{y_{T,P}} \). The uniqueness of the optimal control to \( (NP)_{y_{T,P}} \) follows from the uniform convexity of \( L^p(0,T;L^2(\Omega)) \) (with \( 1 < p < \infty \)) immediately.

(ii) Its proof is trivial. This completes the proof.
\[
\square
\]

**Lemma 2.3.** Let \( \xi \in Y_{T,q} \setminus \{0\} \) with \( q \in (1,\infty) \). Then (i) \( u_\xi \) (given by (1.13)) is the optimal control to \( (NP)_{y_{T,\xi,q}} \) where \( y_{T,\xi,q} \triangleq y(T;0,u_\xi) \); (ii) \( \xi \) is the minimizer of \( J_{y_{T,\xi,q}} \).

**Proof.** (i) Given an admissible control \( v \) to \( (NP)_{y_{T,\xi,q}} \), it follows from (2.11) that
\[
F_{y_{T,\xi,q}}(\eta) = \int_0^T \langle u_\xi(t), \eta(t) \rangle \omega \, dt = \int_0^T \langle v(t), \eta(t) \rangle \omega \, dt \text{ for each } \eta \in Y_{T,q}.
\] (2.31)

From (1.13), we have
\[
\| u_\xi \|_{L^p(0,T;L^2(\Omega))} = \| \xi \|_{L^q(0,T;L^2(\omega))}.
\] (2.32)

Taking \( \eta = \xi \) in the second equality of (2.31), using (1.13), (2.32) and the Hölder inequality, we get
\[
\| u_\xi \|_{L^p(0,T;L^2(\Omega))} \leq \| v \|_{L^p(0,T;L^2(\Omega))}.
\]

Hence, \( u_\xi \) is the optimal control to \( (NP)_{y_{T,\xi,q}} \).

(ii) By (2.24), the first equality of (2.31) and (1.13), after some simple computations involving the Cauchy-Schwartz and the Hölder inequalities, one can get that \( J_{y_{T,\xi,q}}(\xi) \leq J_{y_{T,\xi,q}}(\eta) \) for all \( \eta \in Y_{T,q} \), i.e, \( \xi \) is the minimizer of \( J_{y_{T,\xi,q}} \). This completes the proof.
\[
\square
\]

**Remark 2.1.** Unfortunately, we don’t know how to get the similar results in Lemma 2.2 and Lemma 2.3 for the case where \( p = \infty \).

Now we continue the proof of Theorem 1.1.

**Proof of the part (ii) of Theorem 1.1.** Let \( H_q \) be defined by (1.12). We first show that \( H_q \) is injective. Let \( \xi \neq \eta \) in \( Y_{T,q} \). In the case that both \( \xi \) and \( \eta \) are not zero, we suppose by contradiction that \( H_q(\xi) = H_q(\eta) \). Then, \( y_{T,\xi,q} \triangleq y(T;0,u_\xi) = y(T;0,u_\eta) \triangleq y_{T,q} \). By
Lemma 2.3, both $\xi$ and $\eta$ are the unique minimizer of $J_{y_T,\xi,q}$. Thus $\xi = \eta$ which leads to a contradiction. Hence, $H^p_\xi(\xi) \neq H^p_\eta(\eta)$ when $\xi \neq \eta$ in $Y_{T,q} \setminus \{0\}$. In the case where $\xi \neq 0$ and $\eta = 0$, it suffices to show that $H^q_\xi(\xi) \neq 0$. By contradiction, we suppose that $0 = H^q_\xi(\xi)$. By (1.12), we have $y(T; 0, u_\xi) = 0$, where $u_\xi$ is given by (1.13). According to Lemma 2.3, $u_\xi$ is the optimal control to $(NP)_{0,p}$. This, along with (ii) of Lemma 2.2, yields that $u_\xi = 0$ in $L^p(0, T; L^2(\Omega))$. However, it follows from Lemma 2.1, as well as (1.13), that $\|u_\xi(t)\|_\omega \neq 0$ when $t \in [0, T)$. This leads to a contradiction. In summary, we conclude that $H^q_\xi$ is injective.

We next show that $H^q_\xi$ is surjective. Given $y_T \in A_{T,p} \setminus \{0\}$, let $\xi$ be the minimizer of $J_{y_T,q}$ in $Y_{T,q}$. By Lemma 2.2, $u_\xi$ (given by (1.13)) is the optimal control to $(NP)_{y_T,p}$. Hence, $H^q_\xi(\xi) = y(T; 0, u_\xi) = y_T$. This, along with the fact that $H^q_\xi(0) = 0$, indicates that $H^q_\xi$ is surjective.

Finally, we show that $H^q_\xi$ preserves the norms. Given $\xi \in Y_{T,q} \setminus \{0\}$, it holds that $H^q_\xi(\xi) = y(T; 0, u_\xi) \triangleq y_T,\xi$. Since $u_\xi$ is the optimal control to $(NP)_{y_T,\xi,p}$ (see Lemma 2.3), we derive from (1.6) that

$$
\|H^q_\xi(\xi)\|_{A_{T,p}} = \|y_T,\xi\|_{A_{T,p}} = \|u_\xi\|_{L^p(0,T;L^2(\Omega))}.
$$

which, together with (2.32) and (1.4), leads to $\|H^q_\xi(\xi)\|_{A_{T,p}} = \|\xi\|_{Y_{T,q}}$. This completes the proof of the part (ii) of Theorem 1.1.

\[\square\]

3 Some properties on $N_p(T, y_0)$

This section presents some properties on $N_p(T, y_0)$ (given by (1.8)). These properties will be used in the proof of Theorem 1.2. We focus on the case where $y_0 \neq 0$, since $N_p(\cdot, 0) \equiv 0$.

Lemma 3.1. Let $p \in (1, \infty]$ and $q$ be the conjugate exponent of $p$. Then

$$
N_p(T, y_0) = \sup_{z \in L^2(\Omega) \setminus \{0\}} \langle y(T; y_0, 0), z \rangle \|z\|_{\chi\omega \Phi(\cdot; T, z)} \|\chi\omega \Phi(\cdot; T, z)\|_{L^q(0,T;L^2(\Omega))} \text{ for all } T > 0, y_0 \in L^2(\Omega) \setminus \{0\}. \quad (3.1)
$$

Proof. Let $y_0 \in L^2(\Omega) \setminus \{0\}$ and $T > 0$. Write $y_T \triangleq -y(T; y_0, 0)$. From the $L^\infty$-null controllability (see [6] or [21]), it follows that $y_T \in A_{T,p}$. Clearly, $y(T; y_0, u) = 0$ if and only if $y(T; 0, u) = y_T$. These, along with (1.8) and (1.6), yields that

$$
N_p(T, y_0) = \inf \left\{ \|u\|_{L^p(0,T;L^2(\Omega))} \mid y(T; y_0, u) = 0 \right\}
= \inf \left\{ \|u\|_{L^p(0,T;L^2(\Omega))} \mid y(T; 0, u) = y_T \right\}
= \|y_T\|_{A_{T,p}}. \quad (3.2)
$$
Let \( \hat{u} \in L^p(0, T; L^2(\Omega)) \) be such that \( y(T; 0, \hat{u}) = y_T \). By (2.2) and (2.1), it follows that
\[
\mathcal{F}_{y_T,q}(\chi_\omega \varphi(\cdot; T, z)) = \langle y_T, z \rangle \quad \text{for all } z \in L^2(\Omega).
\]
This, combined with (1.4) and (1.3), yields that
\[
\|\mathcal{F}_{y_T,q}\|_{Y_{T,q}^*} = \sup_{z \in L^2(\Omega) \setminus \{0\}} \frac{\langle y(T; y_0, 0), z \rangle}{\|\chi_\omega \varphi(\cdot; T, z)\|_{L^q(0,T;L^2(\Omega))}}. \tag{3.3}
\]
By (3.2), (2.10) and (3.3), we are led to (3.1). This completes the proof.

The studies on \( N_p(T, y_0) \) are closely related to the variational problem \((JP)^{T,q}_{y_0}\):
\[
V_q(T, y_0) \triangleq \inf_{\chi_\omega \varphi \in Y_{T,q}} J_{y_0}^{T,q}(\chi_\omega \varphi) \triangleq \frac{1}{2} \left( \| \chi_\omega \varphi \|_{L^q(0,T;L^2(\Omega))} \right)^2 + \langle y_0, \varphi(0) \rangle. \tag{3.4}
\]
By \((i)\) of Lemma 2.1, \( J_{y_0}^{T,q} \) is well-defined over \( Y_{T,q} \).

**Lemma 3.2.** Let \( p \in (1, \infty) \) and \( q \) be the conjugate exponent of \( p \). Then
\[
V_q(T, y_0) = -\frac{1}{2} N_p(T, y_0)^2 \quad \text{for all } T > 0 \text{ and } y_0 \in L^2(\Omega) \setminus \{0\}. \tag{3.5}
\]
**Proof.** We first prove that
\[
V_q(T, y_0) \geq -\frac{1}{2} N_p(T, y_0)^2 \quad \text{for all } T > 0 \text{ and } y_0 \in L^2(\Omega) \setminus \{0\}. \tag{3.6}
\]
From the unique continuation estimate of heat equations (see, for instance, [21], [19]), it follows that \( \chi_\omega \varphi(t; T, z) \neq 0 \), when \( z \in L^2(\Omega) \setminus \{0\} \) and \( t \in [0, T) \). This, along with (3.4), indicates that
\[
J_{y_0}^{T,q}(\chi_\omega \varphi(\cdot; T, z)) = \frac{1}{2} \left[ \| \chi_\omega \varphi(\cdot; T, z) \|_{L^q(0,T;L^2(\Omega))} + \langle y_0, \varphi(0; T, z) \rangle \right]^2 \tag{3.7}
\]
\[
\geq -\frac{1}{2} \left[ \frac{\langle y(T; y_0, 0), z \rangle}{\| \chi_\omega \varphi(\cdot; T, z) \|_{L^q(0,T;L^2(\Omega))}} \right]^2 \quad \text{for each } z \in L^2(\Omega) \setminus \{0\}.
\]
Meanwhile, it follows from Lemma 3.1 that
\[
N_p(T, y_0) \geq \frac{\langle y(T; y_0, 0), z \rangle}{\| \chi_\omega \varphi(\cdot; T, z) \|_{L^q(0,T;L^2(\Omega))}} \quad \text{for each } z \in L^2(\Omega) \setminus \{0\} \tag{3.8}
\]
By (3.9), we find that
\[-N_p(T, y_0) \leq \frac{\langle y(T; y_0, 0), z \rangle}{\|\chi_\omega \varphi(\cdot; T, z)\|_{L^2(0, T; L^2(\Omega))}} \text{ for all } z \in L^2(\Omega) \setminus \{0\}. \tag{3.10}\]

From (3.8) and (3.10), it follows that
\[-N_p(T, y_0) \leq \frac{\langle y(T; y_0, 0), z \rangle}{\|\chi_\omega \varphi(\cdot; T, z)\|_{L^2(0, T; L^2(\Omega))}} \leq N_p(T, y_0) \text{ for all } z \in L^2(\Omega) \setminus \{0\}. \]

Hence,
\[\sup_{z \in L^2(\Omega) \setminus \{0\}} \left\{\left[\frac{\langle y(T; y_0, 0), z \rangle}{\|\chi_\omega \varphi(\cdot; T, z)\|_{L^2(0, T; L^2(\Omega))}}\right]^2\right\} \leq N_p(T, y_0)^2. \tag{3.11}\]

From (3.7) and (3.11), one can easily check that
\[\inf_{z \in L^2(\Omega) \setminus \{0\}} J^T, q_{y_0}(\chi_\omega \varphi(\cdot; T, z)) \geq -\frac{1}{2} N_p(T, y_0)^2. \tag{3.12}\]

By the same method used to prove the part (i) of Lemma 2.2, we can easily check that 0 is not the minimizer of \(J^T, q_{y_0}\). This, along with (3.4) and (1.3), yields that
\[V_q(T, y_0) = \inf_{z \in L^2(\Omega) \setminus \{0\}} J^T, q_{y_0}(\chi_\omega \varphi(\cdot; T, z)) \text{ for all } y_0 \in L^2(\Omega) \setminus \{0\} \text{ and } T > 0. \tag{3.13}\]

From (3.12) and (3.13), we are led to (3.6).

We next show that
\[V_q(T, y_0) \leq -\frac{1}{2} N_p(T, y_0)^2. \tag{3.14}\]

Clearly, \(N_p(T, y_0) > 0\) since \(y_0 \neq 0\). By (3.9), given \(\varepsilon \in (0, N_p(T, y_0))\), there is a \(z_\varepsilon \in L^2(\Omega) \setminus \{0\}\) such that
\[\frac{\langle y_0, \varphi(0; T, z_\varepsilon) \rangle}{\|\chi_\omega \varphi(\cdot; T, z_\varepsilon)\|_{L^2(0, T; L^2(\Omega))}} \leq -N_p(T, y_0) + \varepsilon. \tag{3.15}\]
Then, it follows from (3.4) and (3.15) that for each \( \lambda \geq 0 \)
\[
J_{y_0}^{T,p}(\chi \omega \varphi(\cdot; T, \lambda \varepsilon)) \leq \frac{1}{2} \left[ \lambda \| \chi \omega \varphi(\cdot; T, \lambda \varepsilon) \|_{L^q(0,T;L^2(\Omega))} - (N_p(T,y_0) - \varepsilon) \right]^2 - \frac{1}{2} (N_p(T,y_0) - \varepsilon)^2.
\]

By taking the infimum for \( \lambda \in \mathbb{R}^+ \) on the both sides of the above inequality, we find that
\[
\inf_{\lambda \in \mathbb{R}^+} J_{y_0}^{T,p}(\chi \omega \varphi(\cdot; T, \lambda \varepsilon)) \leq -\frac{1}{2} (N_p(T,y_0) - \varepsilon)^2 \text{ for each } \varepsilon \in (0, N_p(T,y_0)),
\]
which, together with (3.13), yields that
\[
V_q(T,y_0) \leq -\frac{1}{2} (N_p(T,y_0) - \varepsilon)^2 \text{ for each } \varepsilon \in (0, N_p(T,y_0)).
\]

Sending \( \varepsilon \to 0 \) in the above inequality leads to (3.14).

Finally, (3.5) follows from (3.6) and (3.14). This completes the proof. \( \square \)

**Lemma 3.3.** Let \( p \in (1,\infty] \) and \( q \) be the conjugate exponent of \( p \). Let \( T > 0 \) and \( y_0 \in L^2(\Omega) \setminus \{0\} \). Write \( y_T \triangleq -y(T;y_0,0) \). Then (i) \( J_{y_T,q}(\cdot) = J_{y_0}^{T,q}(\cdot) \) over \( Y_{T,q} \); (ii) Problems \( (NP)_{y_T,p} \) and \( (NP)^{T,p}_{y_0} \) have the same optimal controls. (Here, \( J_{y_T,q} \) and \( (NP)_{y_T,p} \) are defined by (2.24) and (2.23) respectively.)

**Proof.** (i) By the \( L^\infty \)-null controllability (see, for instance, [6] or [21]), one can show that \( y_T \in A_{T,q} \). Thus, \( y_T = y(T;0,\hat{u}) \) for some \( \hat{u} \in L^p(0,T;L^2(\Omega)) \). This, together with (2.2) and (2.1), indicates that for each \( z \in L^2(\Omega) \),
\[
F_{y_T,q}(\chi \omega \varphi(\cdot; T, z)) = \int_0^T \langle \hat{u}(t), \chi \omega \varphi(t; T, z) \rangle \, dt = \langle y_T, z \rangle = -\langle y_0, \varphi(0; T, z) \rangle.
\]

From this, as well as the definitions of \( J_{y_T,q} \) and \( J_{y_0}^{T,q} \) (see (2.24) and (3.4) respectively), Lemma 2.1, (2.3) and (1.3), one can easily get that \( J_{y_T,q}(\cdot) = J_{y_0}^{T,q}(\cdot) \) over \( Y_{T,q} \).

(ii) The proof is trivial. This completes the proof. \( \square \)

**Lemma 3.4.** Let \( p \in (1,\infty] \). Let \( T > 0 \) and \( y_0 \in L^2(\Omega) \setminus \{0\} \). Then (i) \( (NP)^{T,p}_{y_0} \) holds the bang-bang property; (ii) \( (NP)^{T,p}_{y_0} \) has a unique optimal control.

**Proof.** When \( p = \infty \), the results in (i) and (ii) have been proved in [19, Theorem 3.1].

Suppose that \( p \in (1,\infty] \). Since \( y_0 \neq 0 \), it follows from the backward uniqueness and the \( L^\infty \)-null controllability of heat equations that \( 0 \neq y_T \equiv -y(T;y_0,0) \in A_{T,p} \). Then (i) and (ii) follow from Lemma 3.3, Definition 1.2, (i) of Lemma 2.2 and (ii) of Lemma 2.1. This completes the proof. \( \square \)
Lemma 3.5. Let $q \in (1, \infty)$. Let $T > 0$ and $y_0 \in L^2(\Omega) \setminus \{0\}$. Then (i) $0$ is not a minimizer of $J^{T,q}_{y_0}$; (ii) $J^{T,q}_{y_0}$ has a unique minimizer $\chi_0 \bar{\varphi}$ in $Y_{T,q}$; (iii) it holds that

$$V_q(T, y_0) = -\frac{1}{2} \| \chi_0 \bar{\varphi} \|_{L^q(0,T;L^2(\Omega))}^2,$$

(3.17)

where $V_q(T, y_0)$ is given by (3.4).

Proof. Let $p$ be the conjugate exponent of $q$. Write $y_T \triangleq -y(T, y_0, 0)$. Clearly, $0 \neq y_T \in A_{T,p}$. Then, according to Lemma 3.3 and (i) of Lemma 2.2, $J^{T,q}_{y_0}$ has a unique minimizer $\chi_0 \bar{\varphi}$ in $Y_{T,q}$. By Lemma 2.1 and (3.4), one can easily check the following Euler-Lagrange equation associated with $\chi_0 \bar{\varphi}$:

$$\langle y_0, \varphi(0) \rangle + \int_0^T \langle \dot{u}(t), \chi_0 \varphi(t) \rangle \, dt = 0 \quad \text{for all} \quad \chi_0 \varphi \in Y_{T,q},$$

(3.18)

where

$$\dot{u}(t) \triangleq \| \chi_0 \bar{\varphi} \|_{L^q(0,T;L^2(\Omega))}^{2-q} \cdot \| \chi_0 \bar{\varphi} \|_{L^q(0,T;L^2(\Omega))}^{q-2} \cdot \chi_0 \bar{\varphi}(t), \quad t \in (0, T).$$

Taking $\varphi = \bar{\varphi}$ in (3.18) gives

$$\langle y_0, \bar{\varphi}(0) \rangle + \left( \| \chi_0 \bar{\varphi} \|_{L^q(0,T;L^2(\Omega))} \right)^2 = 0.$$

This, along with (3.4), leads to (3.17) and completes the proof.

Remark 3.1. Since $L^1(0,T; L^2(\omega))$ is not reflexive and its norm is not strictly convex, the studies on the functional $J^{T,1}_{y_0}$ is much more complicated. In the rest of this section, we will show that the functional $J^{T,1}_{y_0}$ is strictly convex in $Y_{T,1}$. This is not obvious (see the last paragraph on Page 2940 in [31]). Unfortunately, we do not know if $J^{T,1}_{y_0}$ has a minimizer, in general. (At least, we do not know how to prove it.) We will show the existence of the minimizer for this functional under the assumption (1.10).

Lemma 3.6. Let $T > 0$ and $y_0 \in L^2(\Omega) \setminus \{0\}$. Then (i) The functional $J^{T,1}_{y_0}$ is strictly convex in $Y_{T,1}$. Consequently, the minimizer of $J^{T,1}_{y_0}$, if exists, is unique; (ii) Zero is not the minimizer of $J^{T,1}_{y_0}$.

Proof. (i) By contradiction, suppose that $J^{T,1}_{y_0}$ was not strictly convex in $Y_{T,1}$. Then there would be two distinct $\chi_0 \varphi_1$ and $\chi_0 \varphi_2$ in $Y_{T,1}$ and a $\lambda \in (0, 1)$ such that

$$\left( \int_0^T \| \chi_0 \varphi_\lambda \| \, dt \right)^2 = (1-\lambda) \left( \int_0^T \| \chi_0 \varphi_1 \| \, dt \right)^2 + \lambda \left( \int_0^T \| \chi_0 \varphi_2 \| \, dt \right)^2,$$

(3.19)
where $\varphi_\lambda \equiv (1 - \lambda)\varphi_1 + \lambda\varphi_2$. We first prove that

$$\|\chi_\omega \varphi_1(t)\| \neq 0, \|\chi_\omega \varphi_2(t)\| \neq 0 \text{ for each } t \in [0, T). \quad (3.20)$$

In fact, if it was not true, then we could suppose, without loss of generality, that $\chi_\omega \varphi_1(t_0) = 0$ for some $t_0 \in [0, T)$. Since both $\varphi_1$ and $\varphi_2$ solve equation (1.2) (see Lemma 2.1), it follows by the unique continuation estimate of heat equations (see, for instance, [21]) that $\varphi_1 \equiv 0$ over $[0, T]$. Consequently, $\varphi_\lambda = \lambda \varphi_2$, which, as well as (3.19), yields

$$\lambda \left( \int_0^T \|\chi_\omega \varphi_2\| \, dt \right)^2 = \left( \int_0^T \|\chi_\omega \varphi_2\| \, dt \right)^2.$$  

Because $\lambda \in (0, 1)$, the above equality implies that $\|\chi_\omega \varphi_2(\cdot)\| = 0$ over $(0, T)$. This, along with the unique continuation of heat equations, gives that $\varphi_2 \equiv 0$ over $[0, T]$, which contradicts with the fact that $\chi_\omega \varphi_1 \neq \chi_\omega \varphi_2$. Hence, (3.20) holds.

Two observations are given in order: First, it is clear that

$$\left( \int_0^T \|\chi_\omega \varphi_2\| \, dt \right)^2 \leq \left( (1 - \lambda) \int_0^T \|\chi_\omega \varphi_1\| \, dt + \lambda \int_0^T \|\chi_\omega \varphi_2\| \, dt \right)^2. \quad (3.21)$$

Since $\varphi_1, \varphi_2, \varphi_\lambda \in C([0, T); L^2(\Omega))$ (see Lemma 2.1) and because

$$\|(1 - \lambda)\chi_\omega \varphi_1(t) + \lambda \chi_\omega \varphi_2(t)\| \leq (1 - \lambda)\|\chi_\omega \varphi_1(t)\| + \lambda \|\chi_\omega \varphi_2(t)\| \text{ for each } t \in [0, T), \quad (3.22)$$

the equality in (3.21) holds if and only if the equality in (3.22) holds for each $t \in [0, T)$. On the other hand, the equality in (3.22) holds for each $t \in [0, T)$ if and only if for each $t \in [0, T)$, there is a $d(t) > 0$ such that

$$\chi_\omega \varphi_1(t) = d(t) \chi_\omega \varphi_2(t) \text{ in } L^2(\Omega). \quad (3.23)$$

Thus, the equality in (3.21) holds if and only if (3.23) stands. Second, it is obvious that

$$\left( (1 - \lambda) \int_0^T \|\chi_\omega \varphi_1\| \, dt + \lambda \int_0^T \|\chi_\omega \varphi_2\| \, dt \right)^2 \leq (1 - \lambda) \left( \int_0^T \|\chi_\omega \varphi_1\| \, dt \right)^2 + \lambda \left( \int_0^T \|\chi_\omega \varphi_2\| \, dt \right)^2 \quad (3.24)$$

and the equality in (3.24) holds if and only if

$$\int_0^T \|\chi_\omega \varphi_1\| \, dt = \int_0^T \|\chi_\omega \varphi_2\| \, dt. \quad (3.25)$$
By (3.19), we see that the equalities in both (3.21) and (3.24) hold respectively. Hence, we have both (3.23) and (3.25). Since \( \| \chi_\omega \varphi_2(t) \| \neq 0 \) for each \( t \in [0, T) \) (see (3.20)), we derive from (3.23) that \( d(t) = \| \chi_\omega \varphi_1(t) \| / \| \chi_\omega \varphi_2(t) \| \) for each \( t \in [0, T) \). This, along with the fact that \( \varphi_1, \varphi_2 \in C([0, T); L^2(\Omega)) \), indicates that \( d(\cdot) \in C([0, T); \mathbb{R}^+) \). By making use of (3.23) again, we find that

\[
\int_0^T d(t) \| \chi_\omega \varphi_2(t) \| \, dt = \int_0^T \| \chi_\omega \varphi_1 \| \, dt. \tag{3.26}
\]

Applying the mean value theorem of integral to the left side of (3.26), we get that there is a \( \hat{t} \in (0, T) \) such that

\[
\int_0^T d(t) \| \chi_\omega \varphi_2(t) \| \, dt = d(\hat{t}) \int_0^T \| \chi_\omega \varphi_2 \| \, dt. \tag{3.27}
\]

From (3.25), (3.26) and (3.27), it follows that \( d(\hat{t}) = 1 \). This, as well as (3.23), leads to \( \chi_\omega \varphi_1(\hat{t}) = \chi_\omega \varphi_2(\hat{t}) \) in \( L^2(\Omega) \), which, together with the unique continuation for heat equations, yields that \( \varphi_1 = \varphi_2 \) over \( [0, T] \). This leads to a contradiction. Hence, \( J_{T,1}^{T,1} \) is strictly convex in \( Y_{T,1} \).

(ii) The proof follows from the same way used to prove the part (i) of Lemma 2.2. This completes the proof.

\[
\square
\]

The proof of the existence for the minimizer to \( J_{T,1}^{T,1} \) (under the assumption (1.10)), as well as of Theorem 1.2, needs the help of the following preliminaries. Let

\[
\beta(t, T) \triangleq \sup_{z \in L^2(\Omega) \setminus \{0\}} \frac{\| \varphi(t; T, z) \|}{\| \chi_\omega \varphi(\cdot; T, z) \|_{L^1(t, T; L^2(\Omega))}}, \quad T > 0, \ t \in [0, T). \tag{3.28}
\]

The term on the right hand side of (3.28) is well-defined because of the unique continuation for heat equations. From Proposition 3.2 in [6], we can derive the following estimate:

\[
\beta(t, T) \leq C_1(T, t), \quad \text{for all } T > 0, \ t \in [0, T). \tag{3.29}
\]

Here,

\[
C_1(T, t) \triangleq \exp \left[ \left( 1 + \frac{1}{T-t} \right) \widehat{C}_0 \right], \quad T > 0, \ t \in [0, T), \tag{3.30}
\]

where \( \widehat{C}_0 > 0 \) depends only on \( \Omega, \omega \) and \( \| a \|_\infty \) which is the \( L^\infty(\Omega \times \mathbb{R}^+) \)-norm of \( a \). The proof of (3.29) will be given in Appendix, for sake of the completeness of the paper.
Lemma 3.7. Let $T > 0$ and $y_0 \in L^2(\Omega \setminus \{0\}$. Suppose that (1.10) holds. Then $J_{y_0}^{T,1}$ has a minimizer $\chi_\omega \tilde{\varphi}$ in $Y_{T,1}$. Furthermore, it holds that

$$V_1(T, y_0) = -\frac{1}{2} \left( \int_0^T \|\chi_\omega \varphi\| \, dt \right)^2,$$  \hspace{1cm} (3.31)

where $V_1(T, y_0)$ is given by (3.4).

Proof. We start with proving the coercivity of $J_{y_0}^{T,1}$. By Lemma 2.1, and by using the standard density argument, one can easily derive from (3.28) and (3.29) that

$$\|\varphi(t)\| \leq C_1(T, t) \int_t^T \|\chi_\omega \varphi\| \, ds \text{ for all } T > 0, t \in [0, T) \text{ and } \chi_\omega \varphi \in Y_{T,1}. \hspace{1cm} (3.32)$$

From (3.32), we see that

$$\langle y_0, \varphi(0) \rangle \geq -C_1(T, 0)^2\|y_0\|^2 - \frac{1}{4} \left( \int_0^T \|\chi_\omega \varphi\| \, dt \right)^2 \text{ for each } \chi_\omega \varphi \in Y_{T,1}.$$

This, along with (3.4) and (1.4), indicates that

$$J_{y_0}^{T,1}(\chi_\omega \varphi) \geq \frac{1}{4}\|\chi_\omega \varphi\|_{Y_{T,1}}^2 - C_1(T, 0)^2\|y_0\|^2 \text{ for each } \chi_\omega \varphi \in Y_{T,1},$$

which leads to the coercivity of $J_{y_0}^{T,1}$.

We next write $\{\chi_\omega \varphi_n\}$ for a minimizing sequence of $J_{y_0}^{T,1}$. By the coercivity of $J_{y_0}^{T,1}$, there is a positive constant $C$ independent of $n$ such that

$$\int_0^T \|\chi_\omega \varphi_n\| \, dt \leq C \text{ for all } n \in \mathbb{N}. \hspace{1cm} (3.33)$$

Let $\{T_k\} \subset (0, T)$ be such that $T_k \not\to T$. By (3.32) and (3.33), it holds that

$$\|\varphi_n(T_k)\| \leq C_1(T, T_k) \int_{T_k}^T \|\chi_\omega \varphi_n\| \, dt \leq CC_1(T, T_k) \Delta C(k), \forall n, k \in \mathbb{N} \hspace{1cm} (3.34)$$

Let $k = 2$ in (3.34). By properties of heat equations, there are a $z_1 \in L^2(\Omega)$ and a subsequence $\{\varphi_{n_l}\}$ of $\{\varphi_n\}$ such that

$$\varphi_{n_l}(. \cdot) \to \varphi(. \cdot ; z_1, T) \text{ strongly in } C([0, T_1]; L^2(\Omega)), \text{ as } l \to \infty.$$

Let $k = 3$ in (3.34). By properties of heat equations, we can find a $z_2 \in L^2(\Omega)$ and a subsequence $\{\varphi_{n_s}\}$ of $\{\varphi_n\}$ such that

$$\varphi_{n_s}(. \cdot) \to \varphi(. \cdot ; z_2, T_2) \text{ strongly in } C([0, T_2]; L^2(\Omega)), \text{ as } s \to \infty.$$

19
Continuing this procedure with respect to \( k \), and then using the diagonal law, we find a subsequence of \( \{ \varphi_n \} \), still denoted in the same way, and a sequence \( \{ z_k \} \) in \( L^2(\Omega) \) such that for each \( k \geq 2 \),

\[
\varphi_n(\cdot) \to \varphi(\cdot; z_k, T_k) \text{ strongly in } C([0, T_k]; L^2(\Omega)), \quad \text{as } n \to \infty.
\] (3.35)

From (3.35), we see that

\[
\varphi(t; z_k, T_k) = \varphi(t; z_{k+j}, T_{k+j}) \quad \text{for all } t \in [0, T_k], \; k = 2, 3, \ldots, \; j = 1, 2, \ldots.
\] (3.36)

Now, we define a function \( \hat{\varphi} \) over \([0, T)\) by setting

\[
\hat{\varphi}(t) = \varphi(t; z_k, T_k), \quad t \in [0, T_k], \; k = 2, 3, \ldots.
\] (3.37)

From this and (3.36), \( \hat{\varphi} \) is well-defined. Then by (3.35) and (3.37), we see that

\[
\hat{\varphi} \in C([0, T); L^2(\Omega)) \text{ solves Equation (1.2)}
\] (3.38)

and

\[
\chi_\omega \varphi_n \to \chi_\omega \hat{\varphi} \text{ strongly in } L^1(0, T_k; L^2(\omega)) \quad \text{for each } k.
\] (3.39)

From (3.39) and (3.33), we find that for each \( k \in \mathbb{N} \),

\[
\int_0^{T_k} \| \chi_\omega \hat{\varphi} \| \, dt \leq \liminf_{n \to \infty} \int_0^{T_k} \| \chi_\omega \varphi_n \| \, dt \leq \liminf_{n \to \infty} \int_0^{T} \| \chi_\omega \varphi_n \| \, dt \leq C.
\]

This implies

\[
\int_0^{T} \| \chi_\omega \hat{\varphi} \| \, dt \leq \liminf_{n \to \infty} \int_0^{T_k} \| \chi_\omega \varphi_n \| \, dt \leq C.
\] (3.40)

From (1.11), (3.38) and (3.40), it follows that \( \chi_\omega \hat{\varphi} \in Z_{T,1} \). This, along with the assumption (1.10), indicates that

\[
\chi_\omega \hat{\varphi} \in Y_{T,1}.
\] (3.41)

From (3.35) and (3.37), we, in particular, have that \( \varphi_n(0) \to \hat{\varphi}(0) \) strongly in \( L^2(\Omega) \). This, together with (3.4) and (3.40), yields

\[
J_{y_0}^{T,1}(\chi_\omega \hat{\varphi}) \leq \liminf_{n \to \infty} J_{y_0}^{T,1}(\chi_\omega \varphi_n).
\] (3.42)

From (3.42) and (3.41), we see that \( \chi_\omega \hat{\varphi} \) is the minimizer of \( J_{y_0}^{T,1}(\cdot) \).

Finally, we prove (3.31). The Euler-Lagrange equation associated with \( \chi_\omega \hat{\varphi} \) reads:

\[
\langle y_0, \varphi(0) \rangle + \int_0^T \langle \dot{u}(t), \chi_\omega \varphi(t) \rangle \, dt = 0, \quad \chi_\omega \varphi \in Y_{T,1},
\] (3.43)
where
\[ \hat{u}(t) = \int_0^T \| \chi_\omega \hat{\varphi} \| \, ds \cdot \frac{\chi_\omega \hat{\varphi}(t)}{\| \chi_\omega \hat{\varphi} \|}, \quad t \in [0, T). \]

Letting \( \varphi = \hat{\varphi} \) in (3.43), we get
\[ \langle y_0, \hat{\varphi}(0) \rangle + \left( \int_0^T \| \chi_\omega \hat{\varphi} \| \, dt \right)^2 = 0. \]

This, along with (3.4), leads to (3.31) and completes the proof. \( \square \)

4 The bang-bang property for \((TP)_{y_0}^{M,p}\)

This section is mainly devoted to the proof of Theorem 1.2. Our strategy is as follows. We first show that \((TP)_{y_0}^{M,p}\) has the bang-bang property if and only if \(M = N_p(T, y_0)\) for some \(T > 0\); then prove that the function \(N_p(\cdot, y_0)\) is strictly monotonically decreasing and continuous from \((0, \infty)\) onto \((\hat{N}_p(y_0), \infty)\); finally, through utilizing the bang-bang property of \((NP)_{y_0}^{T,p}\) (see Lemma 3.4), derive the bang-bang property for \((TP)_{y_0}^{M,p}\) for any \(M > \hat{N}_p(y_0)\). To show the left continuity of \(N_p(\cdot, y_0)\), we need the assumption (1.10).

**Lemma 4.1.** Let \(y_0 \in L^2(\Omega) \setminus \{0\}\). Then \((TP)_{y_0}^{M,p}\), with \(M > 0\), has the bang-bang property if and only if \(M = N_p(T, y_0)\) for some \(T > 0\).

**Proof.** First we suppose that \(M = N_p(T, y_0)\) for some \(T > 0\). Let \(u_1\) be the optimal control to \((NP)_{y_0}^{T,p}\). (The existence of the optimal control is ensured by Lemma 3.4.) One can easily check that \(u_1\) is an admissible control to \((TP)_{y_0}^{M,p}\). This, along with the definition of \(T_p(M, y_0)\) (see (1.7)), yields that
\[ T_p(M, y_0) \leq T. \] \( (4.1) \)

Meanwhile, since \((TP)_{y_0}^{M,p}\) has admissible controls, one can use the standard way to show that \((TP)_{y_0}^{M,p}\) has optimal controls (see for instance, [4], or the proof of Lemma 3.2 in [22]). Arbitrarily take an optimal control \(u_2\) to \((TP)_{y_0}^{M,p}\). Clearly,
\[ \| u_2 \|_{L^p(\mathbb{R}^+, L^2(\Omega))} \leq M = N_p(T, y_0) \quad \text{and} \quad y(T_p(M, y_0); y_0, u_2) = 0. \] \( (4.2) \)

Let \(\tilde{u}_2 \in L^p(0, T; L^2(\Omega))\) be such that \(\tilde{u}_2 = u_2\) over \([0, T_p(M, y_0)]\) and \(\tilde{u}_2 = 0\) over \([T_p(M, y_0), T]\). From (4.2) and (4.1), one can easily verify that \(\tilde{u}_2\) is an optimal control to \((NP)_{y_0}^{T,p}\). Since \(\tilde{u}_2(t) = 0\) for a.e. \(t \in [T_p(M, y_0), T]\), it follows from the bang-bang property of \((NP)_{y_0}^{T,p}\) (see Lemma 3.4) that \(T_p(M, y_0) = T\) and \(u_2 = \tilde{u}_2\) over \((0, T)\). These,
along with the bang-bang property of \((NP)^{T,y_0}_{T,y_0}\) (see Definition 1.2), lead to the bang-bang property of \((TP)^{M,p}_{y_0}\) (see Definition 1.1).

Conversely, we suppose that for some \(M > 0\), \((TP)^{M,p}_{y_0}\) has the bang-bang property. It suffices to show
\[
M = N_p(T_p(M, y_0), y_0). \tag{4.3}
\]
From Remark 1.1, \((TP)^{M,p}_{y_0}\) has an optimal control \(u_3\), which is clearly an admissible control to \((NP)^{T_p(M,y_0),p}_{T_p(M,y_0)}\). Thus, it holds that
\[
N_p(T_p(M, y_0), y_0) \leq \|u_3\|_{L^p(0,T_p(M,y_0);L^2(\Omega))} \leq M. \tag{4.4}
\]
Let \(u_4\) be the optimal control to \((NP)^{T_p(M,y_0),p}_{T_p(M,y_0)}\). Then
\[
y(T_p(M, y_0); y_0, u_4) = 0 \quad \text{and} \quad \|u_4\|_{L^p(0,T_p(M,y_0);L^2(\Omega))} = N_p(T_p(M, y_0), y_0). \tag{4.5}
\]
We extend \(u_4\) over \(\mathbb{R}^+\) by setting it to be 0 over \([T_p(M, y_0), \infty)\), and denote the extension by \(\hat{u}_4\). Then, from (4.5) and (4.4), we see that \(\hat{u}_4\) is an optimal control to \((TP)^{M,p}_{y_0}\). By the bang-bang property of \((TP)^{M,p}_{y_0}\) (see Definition 1.1), we find that
\[
\|\chi \omega u_4\|_{L^p(0,T_p(M,y_0);L^2(\Omega))} = \|\chi \omega \hat{u}_4\|_{L^p(0,T_p(M,y_0);L^2(\Omega))} = M,
\]
which, along with (4.5) and (4.4), leads to (4.3). This completes the proof.

\[\square\]

**Lemma 4.2.** Let \(y_0 \in L^2(\Omega) \setminus \{0\}\). (i) The function \(N_p(\cdot, y_0)\) is strictly monotonically decreasing and right-continuous over \((0, \infty)\). Moreover, it holds that
\[
\lim_{T \to 0^+} N_p(T, y_0) = \infty \tag{4.6}
\]
and
\[
\lim_{T \to +\infty} N_p(T, y_0) = \bar{N}_p(y_0) \in [0, \infty), \tag{4.7}
\]
where \(\bar{N}_p(y_0)\) is given by (1.9); (ii) Suppose that (1.10) holds. Then the function \(N_p(\cdot, y_0)\) is left-continuous from \((0, \infty)\) onto \((\bar{N}_p(y_0), \infty)\).

**Proof.** (i) We start with showing the strictly monotonicity of \(N_p(\cdot, y_0)\). Let \(0 < T_1 < T_2\). Let \(u_1\) be the optimal control to \((NP)^{T_1,y_0}_{T_0}\). We extend \(u_1\) over \((0, T_2)\) by setting it to be 0 over \((T_1, T_2)\) and denote the extension by \(u_2\). It is clear that
\[
y(T_2; y_0, u_2) = 0. \tag{4.8}
\]
Hence, \(u_2\) is an admissible control to \((NP)^{T_2,y_0}_{T_0}\). Therefore, it holds that
\[
N_p(T_1, y_0) = \|u_1\|_{L^p(0,T_1;L^2(\Omega))} = \|u_2\|_{L^p(0,T_2;L^2(\Omega))} \geq N_p(T_2, y_0). \tag{4.9}
\]
We claim that \( N_p(T_1, y_0) > N_p(T_2, y_0) \). By contradiction, we suppose that it did not hold. Then by (4.9), we would have \( N_p(T_1, y_0) = N_p(T_2, y_0) \). Thus,

\[
\|u_2\|_{L^p(0,T_2;L^2(\Omega))} = \|u_1\|_{L^p(0,T_1;L^2(\Omega))} = N_p(T_1, y_0) = N_p(T_2, y_0).
\]

This, together with (4.8), shows that \( u_2 \) is an optimal control to \((NP)^{T_2,p}_{y_0}\). By the bang-bang property of \((NP)^{T_2,p}_{y_0}\) (see Lemma 3.4), we have that \( \|\chi_{\omega} u_2(t)\| \neq 0 \) for a.e. \( t \in (0,T_2) \) (see Definition 1.2). This contradicts with the fact that \( u_2 = 0 \) over \((T_1,T_2)\). Hence, \( N_p(\cdot,y_0) \) is strictly monotonically decreasing.

Next, we show the right-continuity of \( N_p(\cdot,y_0) \). Arbitrarily fix a \( \hat{T} \in (0,\infty) \). Let \( \{T_n\} \subset (\hat{T},\hat{T}+1) \) be such that \( T_n \searrow \hat{T} \). Then by the monotonicity of \( N_p(\cdot,y_0) \), there is a \( \hat{M} \in (0,\infty) \) such that \( N_p(T_n, y_0) \nearrow \hat{M} \). \((4.10)\)

It suffices to show

\[
\hat{M} = N_p(\hat{T}, y_0).
\] \((4.11)\)

Seeking for a contradiction, we suppose that (4.11) did not hold. Then by the monotonicity of \( N_p(\cdot,y_0) \), we would have

\[
\hat{M} < N_p(\hat{T}, y_0).
\] \((4.12)\)

Let \( u_n \) be the optimal control to \((NP)^{T_n,p}_{y_0}\). We extend \( u_n \) over \((0,\hat{T}+1)\) by setting it to be 0 over \((T_n,\hat{T}+1)\), and denote the extension by \( \hat{u}_n \). Then one can easily check that

\[
\|\hat{u}_n\|_{L^p(0,\hat{T}+1;L^2(\Omega))} = N_p(T_n, y_0) \leq \hat{M} \quad \text{and} \quad y(T_n; y_0, \hat{u}_n) = 0.
\] \((4.13)\)

Thus, we can extract a subsequence from \( \{\hat{u}_n\} \), still denoted in the same way, such that for some \( \hat{u} \in L^p(0,\hat{T}+1;L^2(\Omega)) \),

\[
\hat{u}_n \rightharpoonup \hat{u} \quad \text{weakly star in} \quad L^p(0,\hat{T}+1;L^2(\Omega)).
\] \((4.14)\)

This, along with (4.13) and (4.12), yields

\[
\|\hat{u}\|_{L^p(0,\hat{T}+1;L^2(\Omega))} \leq \liminf_{n \to \infty} \|u_n\|_{L^p(0,\hat{T}+1;L^2(\Omega))} \leq \hat{M} < N_p(\hat{T}, y_0).
\] \((4.15)\)

Meanwhile, by (4.14) and the equations satisfied by \( y(\cdot;y_0, \hat{u}_n) \) and \( y(\cdot;y_0, \hat{u}) \over (0,\hat{T}+1) \), using the standard argument involving the Ascoli-Arzelà theorem, we can get a subsequence of \( \{y(\cdot;y_0, \hat{u}_n)\} \), denoted in the same way, such that

\[
y(\cdot;y_0, \hat{u}_n) \to y(\cdot;y_0, \hat{u}) \quad \text{in} \quad C([0,\hat{T}+1];L^2(\Omega)).
\]
This, together with (4.15) and the second equality in (4.13), indicates that
\[
\|y(\hat{T}; y_0, \hat{u})\| \leq \|y(\hat{T}; y_0, \hat{u}) - y(T_n; y_0, \hat{u})\| + \|y(T_n; y_0, \hat{u}) - y(T_n; y_0, \hat{u}_n)\| + \|y(T_n; y_0, \hat{u}_n)\| \to 0,
\]
i.e., \(y(\hat{T}; y_0, \hat{u}) = 0\). Thus, \(\hat{u}\) is an admissible control to \((NP)_{y_0}^{T,n}\), which yields
\[
\|\hat{u}\|_{L^p(0, \hat{T}; L^q(\Omega))} \geq N_p(\hat{T}, y_0).
\]
This contradicts with (4.15). Hence, \(N_p(\cdot, y_0)\) is right continuous over \((0, +\infty)\).

Finally, we show (4.7) and (4.6). Since \(N_p(T, y_0) > 0\) for each \(T > 0\) (notice that \(y_0 \neq 0\)), (4.7) follows from the monotonicity of \(N_p(\cdot, y_0)\) at once. To prove (4.6), we suppose, by contradiction that it did not hold. Then there would be a sequence \(\{T_n\} \subset (0, 1)\) such that \(T_n \searrow 0\) and \(N_p(T_n, y_0) \not\to \tilde{N} \in (0, \infty)\). Let \(u_n\) be the optimal control to \((NP)_{y_0}^{T_n,n}\). Then
\[
\|u_n\|_{L^p(0, T_n; L^q(\Omega))} = N_p(T_n, y_0) \leq \tilde{N} \quad \text{for all } n \in \mathbb{N}^+
\]
and
\[
0 = y(T_n; y_0, u_n) = \Phi(T_n, 0)y_0 + \int_0^{T_n} \Phi(T_n, s)\chi_\omega u_n(s)\, ds,
\]
where \(\{\Phi(t, s) \mid 0 \leq s \leq t < \infty\}\) is the evolution system generated by \(\Delta - aI\) (see Chapter 5 in [17]). By (4.16), we have
\[
\left\| \int_0^{T_n} \Phi(T_n, s)\chi_\omega u_n(s)\, ds \right\| \leq \sup_{0 \leq s \leq t \leq 1} \|\Phi(t, s)\|_{\mathcal{L}(L^q(\Omega))} \tilde{N} \cdot T_n^{1 - \frac{a}{2}} \to 0.
\]
This, along with (4.17), yields
\[
0 = \lim_{n \to \infty} \Phi(T_n, 0)y_0 = y_0 \neq 0,
\]
which leads to a contradiction. Hence, (4.6) holds. This completes the proof of the part (i).

(ii) Arbitrarily fix a \(\hat{T} \in (0, \infty)\). Let \(\{T_n\} \subset [\hat{T}/2, \hat{T}]\) be such that \(T_n \uparrow \hat{T}\). By the monotonicity of \(N_p(\cdot, y_0)\), it suffices to show that on a subsequence of \(\{T_n\}\), denoted in the same way,
\[
N_p(T_n, y_0) \to N_p(\hat{T}, y_0) \text{ as } n \to \infty.
\]
By Lemmas 3.5, 3.6, 3.7 and Lemma 2.1, the functional \(J_{y_0}^{T_n,q}\) has a unique non-zero minimizer \(\chi_\omega \psi_n\) (on \(Y_{T_n,q}\)), where \(\psi_n \in C([0, T_n]; L^2(\Omega)) \cap L^q(0, T_n; L^2(\omega))\) solves Equation
(1.2) with $T$ being replaced by $T_n$. From (3.5), (3.17) and (3.31) (see Lemmas 3.2, 3.5 and 3.7 respectively), it holds that

$$0 < \| \chi_n \psi_n \|_{L^n(0,T_n;L^2(\Omega))} = N_p(T_n, y_0) \quad \text{for all } n \in \mathbb{N}. \quad (4.19)$$

Since $T_n < \hat{T}$, it follows from $(i)$ of Lemma 4.2 that $N_p(T_n, y_0) \leq N_p(\hat{T}/2, y_0)$ for all $n \in \mathbb{N}$. This, as well as (4.19), yields that

$$\| \chi_n \psi_n \|_{L^n(0,T_n;L^2(\Omega))} \leq N_p(\hat{T}/2, y_0) \quad \text{for all } n \in \mathbb{N}. \quad (4.20)$$

We extend $\psi_n$ over $[0, \hat{T})$ by setting it to be zero over $[T_n, \hat{T})$, and denote the extension by $\tilde{\psi}_n$. Then by (3.32), (3.30) and (4.20), one has

$$\| \tilde{\psi}_n(T_2) \| = \| \psi_n(T_2) \| \leq C_1(T_n, T_2) \int_{T_2}^{T_n} \| \chi_n \psi_n \| \, ds$$

$$\leq C_1(T_n, T_2)(T_n - T_2)^{1 - \frac{1}{q}} \| \chi_n \psi_n \|_{L^n(0,T_n;L^2(\Omega))}$$

$$\leq \exp \left[ C_0 (1 + 1/(T_3 - T_2)) \right] (\hat{T} - T_2)^{1 - \frac{1}{q}} N_p(\hat{T}/2, y_0)$$

$$\equiv C(T_2, T_3, \hat{T}) N_p(\hat{T}/2, y_0) \quad \text{for each } n \geq 3. \quad (4.21)$$

By (4.21) and the properties of heat equations, there are a subsequence $\{ \tilde{\psi}_{n_l} \}$ of $\{ \tilde{\psi}_n \}$ and a $z_1 \in L^2(\Omega)$ such that

$$\tilde{\psi}_{n_l}(T_1) \to z_1 \quad \text{strongly in } L^2(\Omega), \quad \text{as } l \to \infty$$

and

$$\tilde{\psi}_{n_l}(\cdot) \to \varphi(\cdot; z_1, T_1) \quad \text{strongly in } C([0, T_1]; L^2(\Omega)), \quad \text{as } l \to \infty,$$

where $\varphi(\cdot; z_1, T_1)$ is the solution of Equation (1.2) (where $T = T_1$), with $\varphi(T_1) = z_1$. With respect to $\tilde{\psi}_{n_l}(T_3)$, we can have a similar estimate as (4.21). Thus, we can take a subsequence $\{ \tilde{\psi}_{n_{ls}} \}$ from $\{ \tilde{\psi}_{n_l} \}$ and get a $z_2 \in L^2(\Omega)$ such that

$$\tilde{\psi}_{n_{ls}}(\cdot) \to \varphi(\cdot; z_2, T_2) \quad \text{strongly in } C([0, T_2]; L^2(\Omega)), \quad \text{as } s \to \infty.$$

Continuing this procedure and making use of the diagonal law, we can get a sequence $\{ z_k \}$ in $L^2(\Omega)$ and a subsequence of $\{ \psi_n \}$, still denoted in the same way, such that

$$\tilde{\psi}_n(\cdot) \to \varphi(\cdot; z_k, T_k) \quad \text{strongly in } C([0, T_k]; L^2(\Omega)) \quad \text{for each } k \in \mathbb{N}. \quad (4.22)$$

This implies that

$$\varphi(t; z_k, T_k) = \varphi(t; z_{k+j}, T_{k+j}) \quad \text{for all } t \in [0, T_k], \ k = 1, 2, \ldots, \ j = 1, 2, \ldots. \quad (4.23)$$
We construct a function \( \psi \) over \([0, T]\) by setting
\[
\psi(t) = \varphi(t; z_k, T_k), \quad t \in [0, T_k], \quad k = 1, 2, \ldots.
\] (4.24)
By (4.24) and (4.23), \( \psi \) is a well-defined function over \([0, T]\). From (4.22) and (4.24), it follows that
\[
\psi \in C([0, \hat{T}); L^2(\Omega)) \text{ solves Equation (1.2), where } T = \hat{T};
\] (4.25)
\[
\psi_n(0) \to \psi(0) \text{ strongly in } L^2(\Omega)
\] (4.26)
and
\[
\chi_\omega \tilde{\psi}_n \to \chi_\omega \psi \text{ strongly in } L^q(0, T_k; L^2(\Omega)) \quad \text{for each } k.
\] (4.27)
From (4.27), we have
\[
\|\chi_\omega \tilde{\psi}_n\|_{L^q(0, T_n; L^2(\Omega))} = \lim inf_{n \to \infty} \|\chi_\omega \tilde{\psi}_n\|_{L^q(0, T_k; L^2(\Omega))} \leq \lim inf_{n \to \infty} \|\chi_\omega \psi_n\|_{L^q(0, T_n; L^2(\Omega))}, \quad \forall k \in \mathbb{N}^+,
\] which, along with (4.20), yields that
\[
\|\chi_\omega \tilde{\psi}\|_{L^q(0, \hat{T}; L^2(\Omega))} \leq \lim inf_{n \to \infty} \|\chi_\omega \psi_n\|_{L^q(0, T_n; L^2(\Omega))} \leq \mathcal{N}_p(\hat{T}/2, y_0).
\] (4.28)
From (1.11), (4.25), (4.28) and (1.10), we see that
\[
\chi_\omega \psi \in Z_{\hat{T}, q} = Y_{\hat{T}, q}.
\] (4.29)
By (3.4), (4.29), (4.28) and (4.26), one can easily verify that
\[
J_{y_0}^{\hat{T}, q}(\chi_\omega \psi) \leq \lim inf_{n \to \infty} J_{y_0}^{T_n, q}(\chi_\omega \psi_n) = \lim inf_{n \to \infty} V_q(T_n, y_0).
\]
This, along with (3.5) (see Lemma 3.2), indicates that
\[
J_{y_0}^{\hat{T}, q}(\chi_\omega \psi) \leq \lim inf_{n \to \infty} -\frac{1}{2} \mathcal{N}_p(T_n, y_0)^2.
\] (4.30)
By (3.5), (4.29), (3.4) and (4.30), we see that
\[
-\frac{1}{2} \mathcal{N}_p(\hat{T}, y_0)^2 = V_q(\hat{T}, y_0) \leq J_{y_0}^{\hat{T}, q}(\chi_\omega \psi) \leq \lim inf_{n \to \infty} -\frac{1}{2} \mathcal{N}_p(T_n, y_0)^2,
\]
from which, it follows that
\[
\lim sup_{n \to \infty} \mathcal{N}_p(T_n, y_0) \leq \mathcal{N}_p(\hat{T}, y_0).
\] (4.31)
On the other hand, since \( \mathcal{N}_p(\cdot, y_0) \) is decreasing and \( T_n < \hat{T} \) for all \( n \), it holds that
\[
\lim inf_{n \to \infty} \mathcal{N}_p(T_n, y_0) \geq \mathcal{N}_p(\hat{T}, y_0).
\] (4.32)
Now, (4.18) follows from (4.31) and (4.32) at once. This completes the proof. \( \square \)
With the aid of the part (i) of Lemma 4.2, we can prove the following existence result on optimal controls to Problem \((TP)^{M,p}_{y_0}\).

**Proposition 4.1.** Let \(y_0 \in L^2(\Omega) \setminus \{0\}\). Then problem \((TP)^{M,p}_{y_0}\), with \(M > 0\), has optimal controls iff \(M \in (\tilde{N}_p(y_0), \infty)\) where \(\tilde{N}_p(y_0)\) is given by (1.9).

**Proof.** First we suppose that \(M \in (\tilde{N}_p(y_0), \infty)\). Then by (4.7) and the monotonicity of \(N_p(\cdot, y_0)\) (see the part (i) of Lemma 4.2), there is a \(T_1 \in (0, \infty)\) such that \(N_p(T_1, y_0) < M\). Let \(u_1\) be the optimal control to \((NP)_{y_0}^{T_1,p}\). (The existence of optimal controls is ensured by Lemma 3.4). Then we have

\[\|u_1\|_{L^p(0,T_1;L^2(\Omega))} = N_p(T_1, y_0) < M \text{ and } y(T_1; y_0, u_1) = 0.\]

From these, \(u_1\) is an admissible control to \((TP)^{M,p}_{y_0}\). By the standard arguments (see, for instance, the proof of Lemma 3.2 in [22]), we can get the existence of optimal controls to \((TP)^{M,p}_{y_0}\).

Conversely, we assume that \(M \leq \tilde{N}_p(y_0)\). Seeking for a contradiction, we suppose that \((TP)^{M,p}_{y_0}\) did have an optimal control \(\bar{u}\) in this case. Then we would have that

\[\|\bar{u}\|_{L^p(0,T_p(M,y_0);L^2(\Omega))} \leq M \quad (4.33)\]

and

\[y(T_p(M, y_0); y_0, \bar{u}) = 0. \quad (4.34)\]

By (4.34), \(\bar{u}\) is an admissible control to \((NP)_{y_0}^{T_p(M,y_0),p}\). Then by (4.33) and the optimality of \(N_p(T_p(M,y_0), y_0)\), it holds that \(N_p(T_p(M,y_0), y_0) \leq M\). This, along with the strict monotonicity of \(N_p(\cdot, y_0)\) (see Lemma 4.2), yields that \(M \geq N_p(T_p(M,y_0), y_0) > \tilde{N}_p(y_0)\), which leads to a contradiction. This completes the proof. \(\square\)

Now we prove Theorem 1.2.

**Proof of Theorem 1.2.** When \(M \leq \tilde{N}_p(y_0)\), it follows from Proposition 4.1 that \((TP)^{M,p}_{y_0}\) has no any optimal control. Hence, it has no bang-bang property (see Remark 1.1). Conversely, if \(M > \tilde{N}_p(y_0)\), then by Lemma 4.2, there is a unique \(\tilde{T} \in (0, \infty)\) such that \(M = N_p(\tilde{T}, y_0)\). According to Lemma 4.1, \((TP)^{M,p}_{y_0}\) has the bang-bang property. This completes the proof of Theorem 1.2. \(\square\)

Finally, we will show that the condition (1.10) holds for some cases.

**Proposition 4.2.** Suppose that \(a \in L^\infty(\Omega \times \mathbb{R}^+)\) verifies \(a(x,t) = a_1(x) + a_2(t)\) in \(\Omega \times \mathbb{R}^+\), with \(a_1 \in L^\infty(\Omega)\) and \(a_2 \in L^\infty(\mathbb{R}^+)\). Then \(Y_{T,q} = Z_{T,q}\) for all \(T > 0\) and \(q \in [1,\infty)\).
Proof. It suffices to show that

\[ Z_{T,q} \subset Y_{T,q}, \quad \text{when } T > 0 \text{ and } q \in [1, \infty). \]  

(4.35)

Let \( T > 0 \) and \( q \in [1, \infty) \) be arbitrarily given. Observe that \( \psi \in C([0, T); L^2(\Omega)) \cap L^q(0, T; L^2(\Omega)) \) solves the equation:

\[
\begin{cases}
\partial_t \psi(x, t) + \Delta \psi(x, t) - (a_1(x) + a_2(t))\psi(x, t) = 0 & \text{in } \Omega \times (0, T), \\
\psi(x, t) = 0 & \text{on } \partial \Omega \times (0, T)
\end{cases}
\]

(4.36)

if and only if \( \varphi \in C([0, T]; L^2(\Omega)) \cap L^q(0, T; L^2(\Omega)) \) solves

\[
\begin{cases}
\partial_t \varphi(x, t) + \Delta \varphi(x, t) - a_1(x)\varphi(x, t) = 0 & \text{in } \Omega \times (0, T), \\
\varphi(x, t) = 0 & \text{on } \partial \Omega \times (0, T),
\end{cases}
\]

(4.37)

where the function \( \varphi \) is defined by

\[
\varphi(x, t) = \exp \left[ \int_t^T a_2(\tau) d\tau \right] \psi(x, t), \quad (x, t) \in \Omega \times (0, T).
\]

(4.38)

Given \( \chi_\omega \hat{\psi} \in Z_{T,q} \), let \( \hat{\varphi} \) be given by (4.38) where \( \psi = \hat{\psi} \). Let \( \{T_k\} \subset (0, T) \) be such that \( T_k \nearrow T \). Write \( \varphi_k \) for the solution of Equation (4.37) with the initial condition \( \varphi_k(T) = \hat{\varphi}(T_k) \) (which belongs to \( L^2(\Omega) \)). Let \( \psi_k \) be given by (4.38) where \( \varphi = \varphi_k \). Then, \( \psi_k \in C([0, T]; L^2(\Omega)) \) solves (4.36). We claim that

\[
\chi_\omega \psi_k \rightharpoonup \chi_\omega \hat{\psi} \quad \text{strongly in } \quad L^q(0, T; L^2(\omega)).
\]

(4.39)

When (4.39) is proved, we get from Lemma 2.1 that \( \chi_\omega \hat{\psi} \in Y_{T,q} \), which leads to (4.35).

The remainder is to show (4.39). Clearly, (4.39) is equivalent to

\[
\chi_\omega \varphi_k \rightharpoonup \chi_\omega \hat{\varphi} \quad \text{strongly in } \quad L^q(0, T; L^2(\omega)).
\]

(4.40)

Let \( \tilde{\varphi} \) satisfy

\[
\begin{cases}
\partial_t \tilde{\varphi}(x, t) + \Delta \tilde{\varphi}(x, t) - a_1(x)\tilde{\varphi}(x, t) = 0 & \text{in } \Omega \times (-T, T), \\
\tilde{\varphi}(x, t) = 0 & \text{on } \partial \Omega \times (-T, T)
\end{cases}
\]

(4.41)

and

\[
\tilde{\varphi}(x, t) = \hat{\varphi}(x, t), \quad (x, t) \in \Omega \times (0, T).
\]

(4.42)

It is clear that

\[
\tilde{\varphi} \in C([-T, T); L^2(\Omega)) \cap L^q(-T, T; L^2(\Omega)).
\]

(4.43)
Because the equations satisfied by $\tilde{\varphi}$ and $\varphi_k$ are time-invariant, one can easily check that
\begin{equation}
\varphi_k(t) = \tilde{\varphi}(t - (T - T_k)), \text{ when } t \in (0, T). \tag{4.44}
\end{equation}

By (4.43), we see that given $\varepsilon > 0$, there are two positive constants $\delta(\varepsilon)$ and $\eta(\varepsilon) = \eta(\varepsilon, \delta(\varepsilon))$ such that
\begin{equation}
\|\chi_\omega \tilde{\varphi}\|_{L^q(a,b;L^2(\omega))} \leq \varepsilon, \text{ when } (a, b) \subset (-T, T), |a - b| \leq \delta(\varepsilon) \tag{4.45}
\end{equation}
and
\begin{equation}
\|\tilde{\varphi}(a) - \tilde{\varphi}(b)\| \leq \varepsilon, \text{ when } (a, b) \subset [-T, T - \delta(\varepsilon)], |a - b| \leq \eta(\varepsilon). \tag{4.46}
\end{equation}

Let $k_0 = k_0(\varepsilon)$ verify that
\begin{equation}
0 < T - T_k \leq \eta(\varepsilon), \text{ when } k \geq k_0. \tag{4.47}
\end{equation}

From (4.42) and (4.44), it follows that
\begin{align*}
\|\chi_\omega \varphi_k - \chi_\omega \tilde{\varphi}\|_{L^q(0,T;L^2(\omega))} &= \|\chi_\omega \left(\tilde{\varphi}(\cdot - (T - T_k)) - \tilde{\varphi}(\cdot)\right)\|_{L^q(0,T;L^2(\omega))} \\
&\leq \|\chi_\omega \left(\tilde{\varphi}(\cdot - (T - T_k)) - \tilde{\varphi}(\cdot)\right)\|_{L^q(0,T-\delta(\varepsilon);L^2(\omega))} \\
&\quad + \|\chi_\omega \tilde{\varphi}(\cdot - (T - T_k))\|_{L^q(T-\delta(\varepsilon),T;L^2(\omega))} + \|\chi_\omega \tilde{\varphi}\|_{L^q(T-\delta(\varepsilon),T;L^2(\omega))}.
\end{align*}

This, along with (4.46), (4.47) and (4.45), yields that
\begin{equation}
\|\chi_\omega \varphi_k - \chi_\omega \tilde{\varphi}\|_{L^q(0,T;L^2(\omega))} \leq (T - \delta(\varepsilon))^\frac{1}{2} \varepsilon + 2\varepsilon \leq (T^\frac{1}{2} + 2)\varepsilon, \text{ when } k \geq k_0,
\end{equation}
which leads to (4.40), as well as (4.39). This completes the proof. \qed

**Remark 4.1.** The idea to show (4.40) in the above proof is borrowed from [31] (see the proof of (3.8) on pages 2955-2957 in [31]).

By Theorem 1.2 and Proposition 4.2, we have the following consequence:

**Corollary 4.1.** Let $y_0 \in L^2(\Omega) \setminus \{0\}$. Suppose that $a \in L^\infty(\Omega \times \mathbb{R}^+)$ verifies $a(x,t) = a_1(x) + a_2(t)$ in $\Omega \times \mathbb{R}^+$, with $a_1 \in L^\infty(\Omega)$ and $a_2 \in L^\infty(\mathbb{R}^+)$. Then $(TP)_{y_0}^{M,p}$ has the bang-bang property if and only if $M \in (\hat{N}_p(y_0), \infty)$, where $\hat{N}_p(y_0)$ is given by (1.9).

## 5 Appendix

**The proof of (3.29).** By the observability estimate for heat equations (see [6, Proposition 3.2]) and by (3.28), we have
\begin{equation}
\beta(t,T) \leq \exp \left[C_0 \left(1 + \frac{1}{T-t} + (T-t) + ((T-t)^{\frac{1}{2}} + (T-t))\|a\|_\infty + \|a\|_{\infty}^\frac{2}{3}\right)\right], \tag{5.1}
\end{equation}
where \( C_0 = C_0(\Omega, \omega) > 0 \) depends only on \( \Omega \) and \( \omega \). Let \( y_0 \in L^2(\Omega) \setminus \{0\} \). Define
\[
N_\infty(T, t, y_0) \triangleq \inf \left\{ \| u \|_{L^\infty(t,T;L^2(\Omega))} \left| \right. \Phi(T, t)y_0 + \int_t^T \Phi(T, s)\chi_\omega u(s) \, ds = 0 \right\}, \quad T > 0, \, t \in [0, T).
\]
Here \( \{\Phi(t, s) \mid 0 \leq s \leq t < +\infty\} \) is the evolution system generated by \( \Delta - aI \) (see Chapter 5 in [17]). By the same way to prove Lemma 3.1, we can obtain
\[
N_\infty(T, t, y_0) = \sup_{z \in L^2(\Omega) \setminus \{0\}} \frac{\langle \Phi(T, t)y_0, z \rangle}{\| \chi_\omega \Phi(T, \cdot)^* z \|_{L^1(t,T;L^2(\Omega))}}. \tag{5.2}
\]
From (3.28) and (5.2), it follows that
\[
\beta(t, T) = \sup_{z \in L^2(\Omega) \setminus \{0\}} \frac{\| \Phi(t, T, z) \|}{\| \chi_\omega \Phi(t, z) \|_{L^1(t,T;L^2(\Omega))}} \tag{5.3}
\]
By the same way to show the monotonicity of \( N_p(\cdot, y_0) \) (see the proof of the part (i) of Lemma 4.2), we can verify that for each \( t \geq 0 \) and \( y_0 \in L^2(\Omega) \setminus \{0\} \), \( N_\infty(\cdot, t, y_0) \) is monotonically decreasing over \((t, \infty)\). This, along with (5.3), yields that when \( t \leq 0 \), \( \beta(t, \cdot) \) is monotonically decreasing on \((t, \infty)\). When \( T - t < 1 \), (3.29) follows from (5.1) directly. When \( T - t \geq 1 \), we have \( T \geq t + 1 \). By the monotonicity of \( \beta(t, \cdot) \), we have \( \beta(t, T) \leq \beta(t, t + 1) \). This, along with (5.1), yields \( \beta(t, T) \leq \exp(\tilde{C}_0) \), where \( \tilde{C}_0 \) depends only on \( \Omega, \omega \) and \( \|a\|_\infty \). Hence, (3.29) holds. This completes the proof. \( \square \)

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