Classifying Markets up to Isomorphism

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Abstract

We define a notion of isomorphism for financial markets in both discrete and continuous time. We classify complete one-period markets. We define an invariant of continuous time complete markets which we call the absolute market price of risk. This invariant plays a role analogous to the curvature in Riemannian geometry. We classify markets when this invariant takes a simple form. We show that in general markets with non-trivial automorphism groups admit mutual fund theorems and prove a number of such theorems.

Introduction

Two financial markets should be considered equivalent if there is a bijective correspondence between the investment strategies in each market which preserves both the costs and the payoff distributions of these strategies.

This intuition allows us to define formal notions of isomorphism for financial markets. This raises numerous interesting questions. Can we define invariants of financial markets? More ambitiously, can classify financial markets up to isomorphism?

This question is of interest in its own right, and from a practical point of view classification theorems allow one to simplify proofs using “without loss of generality arguments”. For example, we will prove a classification result for continuous time markets that shows that the multi-dimensional portfolio optimization problem considered by Merton in [13] (and similar problems considered by many authors) can be solved without loss of generality in a market containing only two assets: a risk free bond, and an asset following arithmetic Brownian motion.

Further motivation is provided by our Theorem 1.4. This shows that trading strategies found by solving convex optimization problems can be taken to be invariant under the automorphisms of the market. Hence if we can show that a market has a large number of automorphisms we can assume that optimal investment strategies must take a specific form without considering the details of the investment problem. This idea was first exploited in [11] to give a new, and more powerful, proof of the classical one- and two-mutual-fund theorems for the markets considered by Markowitz. We will generalize these ideas to other
markets, proving a number of classification theorems and corresponding mutual fund theorems.

In Section 1 we begin by defining a one period market and the isomorphisms of such a market. This allows us to prove the general mutual fund theorem Theorem 1.3 discussed above. The notion of isomorphism used in this paper is ostensibly rather different from the notion of isomorphism used in [1]. The former is based on mod 0 isomorphisms of probability spaces and latter on linear transformations of the space of portfolios. The main task of Section 1, therefore, is to relate these two notions of isomorphism. To this end we define the notion of a finite dimensional linear market and show how the two possible notions of isomorphism are related for such markets. We illustrate with the example of the Markowitz model and give a short proof of the classical two-mutual-fund theorem based on invariance arguments. Similar invariance arguments were already used to prove this theorem in [1]; we repeat the arguments here to see how they can be understood as a particular case of Theorem 1.4.

In Section 2 we consider complete one period markets. We obtain a full classification of such markets. The essential tool required to do this is the disintegration of measure studied by Rokhlin in [17]. We show that the full classification can be greatly simplified by allowing investors the additional flexibility of betting in a “casino” after the results of their trades in a more conventional market are known. A casino is the market on probability space $[0,1]$ where the $Q$ measure is equal to the $P$ measure. Once a casino is introduced, complete markets are always symmetric and this allows us to prove a mutual fund theorem for convex optimization problems in complete markets. We will also see that this mutual fund theorem still applies when the casino is not available despite the lack of market symmetry.

In Section 2.2 we show that our classification also allows us to prove a mutual fund theorem for “monotonic” investment problems in complete markets. These are problems where all market players have monotonic preferences: they agree that making money is good and losing money is bad. For example, monotonicity is one of the less controversial axioms for a risk measure considered in [3]. In their celebrated empirical studies, Kahenman and Tversky [11] found that agents do not behave in a manner consistent with convex utility functions. This may be explained by the agents not being rational, having limited liability or simply not being risk-averse. However, these agents still have monotonic preferences and so the results of this section can be applied to such agents.

In Section 3 we consider continuous time markets. We first define multi-period markets and their isomorphisms. We show that they are equivalent to our definitions in the one period case.

Next we specialise to considering complete continuous time markets and ask if we can define invariants of such markets. In fact, we seek something more precise. It is clear that one can easily define invariants of a complete market, for example the cumulative distribution function of the Radon-Nikodym derivative $\frac{dP}{dQ}$. However, such an invariant may be extremely difficult to compute. We therefore seek “local invariants”. These are invariants which can be readily computed from the coefficients of a defining SDE by nothing more complex
than simple algebraic operations and differentiation.

The situation is analogous to the question of finding invariants of a Riemannian manifold. The diameter of a manifold is one possible invariant, but it is typically very hard to compute. By contrast the Riemannian curvature of a manifold is readily computed and so provides a more useful invariant.

We will define one such invariant which we call the absolute market price of risk. We will give a complete classification of markets where the absolute market price of risk is a sufficiently well-behaved deterministic function. A corollary of our classification is that the $d$-dimensional markets studied by Merton in [13] are all isomorphic to linear markets and are determined up to isomorphism by two real parameters: the risk-free rate and the absolute market price of risk.

We will also obtain a mutual fund theorem for such markets. This essentially shows that a solution of any invariant investment problem whose solutions lie in a convex set can be found by continuous time trading in the risk free asset and a single mutual fund. We note that a set of a single point is always convex, so this includes the case of all invariant problems with a unique solution. This justifies the common practice of simplifying a continuous time market model to the one dimensional case of a stock following geometric Brownian motion and a risk free asset when considering topics such as optimal pension investment.

Finally, let us make some historical notes. The classical two-mutual-fund theorem is due to Merton and was proved in [14]. It was generalized using invariance arguments in [1]. The idea of defining a notion of isomorphism between mathematical objects is, of course, a familiar idea in mathematics and has been formalized as “category theory” in [4]. Von Neumann began the study of isomorphisms of probability space in [19], this work was completed in [17] which gives complete classifications of “standard” probability spaces and their homomorphisms.

1 Finite dimensional linear markets

In this section we will give a coordinate free definition of a one period financial market and relate this to the elementary, coordinate based approach of defining a market in $n$-assets using a probability distribution on $\mathbb{R}^n$. We will illustrate with the example of the Markowitz model. We will use this to demonstrate the relationship between invariant investment strategies and mutual fund theorems.

We begin by recalling a number of definitions from [17] for morphisms between probability spaces.

**Definition 1.1.** Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be two probability spaces. A map $\phi : \Omega_1 \to \Omega_2$ is called a homomorphism if $\phi$ is measurable and if $\mathbb{P}_1(\phi^{-1}U) = \mathbb{P}_2(U)$ for all $U \in \mathcal{F}_2$. A homomorphism $\phi$ is called an isomorphism if it is bijective and its inverse is a homomorphism. We call $\phi$ a mod 0 isomorphism if there a subspaces $\Omega'_1 \subseteq \Omega_1$ and $\Omega'_2 \subseteq \Omega_2$ both of full measure such that $\phi$ restricted to $\Omega_1$ is an isomorphism to $\Omega_2$. 

3
From the point of view of probability theory, two probability spaces should be considered as equivalent if they are mod 0 isomorphic.

**Definition 1.2.** A one period financial market \(((\Omega,\mathcal{F},\mathbb{P}),c)\) consists of: a probability space \((\Omega,\mathcal{F},\mathbb{P})\); a function \(c : L^0(\Omega;\mathbb{R}) \to \mathbb{R} \cup \{\pm\infty\}\). We will call \(c^{-1}(\mathbb{R} \cup \{-\infty\})\) the domain of \(c\), denoted \(\text{dom} \; c\).

We interpret a real random variable \(X\) on \(\Omega\) as an investment strategy with payoff \(X(\omega)\) in scenario \(\omega \in \Omega\). \(c(X)\) denotes the up front cost of strategy \(X\) and is equal to \(\infty\) if one cannot pursue a strategy. A strategy with \(c(X) = -\infty\) results in liabilities so bad that the market is willing to pay arbitrarily large incentives to encourage someone to take these liabilities on. A typical investment strategy is the purchase of an asset or of a portfolio of assets which are then sold at a final time \(T\). In this case \(c(X)\) would be the cost of purchasing the asset. However, one can also model a commitment to pursue a continuous time trading strategy as yielding a single payoff at the final time \(T\) and our definition of a market is flexible enough to include such strategies.

This definition is deliberately minimal. To obtain interesting markets one would typically want to impose additional conditions, such as that the market should be arbitrage free. This condition can be expressed as: for random variables \(X\), if \(X \geq 0\) and \(X \not= 0\) then \(c(X) > 0\).

In this section we will be interested only on one period markets so we will refer to them simply as markets.

**Definition 1.3.** An isomorphism of markets \(((\Omega_1,\mathcal{F}_1,\mathbb{P}_1),c_1)\) and \(((\Omega_2,\mathcal{F}_2,\mathbb{P}_2),c_2)\) is a mod 0 isomorphism \(\phi : \Omega_1 \to \Omega_2\) satisfying \(c_2(X) = c_1(X \circ \phi)\) for all \(X \in L^0(\Omega_2;\mathbb{R})\). Note that the inverse mapping of an isomorphism is well defined mod 0 and is automatically an isomorphism. An automorphism of a market is an isomorphism from a market to itself.

In finance, optimal investment problems are typically convex optimization problems (see for example [5] and [15]). We therefore expect the solution of such problems will be convex sets. We also expect that if these sets are financially meaningful, they will be invariant under the automorphism group of the market.

**Theorem 1.4.** Let \(G\) be a group with a left-invariant probability measure \(G\). This means that for all measurable sets \(A \subseteq G\) and elements \(h \in G\), \(G(A) = G(hA)\). Let \(V\) be a Banach space and let \(\rho : G \to \text{Aut} V\) be a measurable group homomorphism where \(\text{Aut} V\) is the group of linear isometries of \(V\). We think of \(\rho\) as defining an action of \(G\) on \(V\) on the left, given by \(gv = \rho(g)v\).

Let \(S\) be a non-empty \(G\)-invariant convex subset of \(V\). Then \(S\) contains a \(G\)-invariant element.

If \(G\) is a finite group, we only need require that \(V\) is a vector space and \(G\) acts by linear automorphisms.

**Remark 1.5.** For financial applications, we we may take \(G\) to be a subgroup of the automorphism group of the market which admits a left invariant density.
and $\rho$ to be the standard action of $G$ on $L^1(\Omega; \mathbb{R})$. This will allow us to simplify invariant convex optimization problems by restricting attention to invariant investment strategies.

**Proof.** Given $h \in G$, define $\phi_h : G \to G$ by left multiplication, so $\phi_h(g) = hg$. Let $A$ be a measurable set and let $1_A$ denote the indicator function of $A$ then

$$1_A \circ \phi_h = 1_{h^{-1} \cdot A}.$$  

We deduce that

$$\mathbb{E}(X \circ \phi_h) = \mathbb{E}(X)$$  

if $X$ is an indicator function of a set, and hence this holds for all integrable random variables $X$.

By assumption $S$ is non-empty, so we may choose an element $s' \in S$. We define a random variable $X : G \to V$ by

$$X(g) = \rho(g)s'.$$  

Because $G$ acts by isometries on $V$, $\|X(g)\| = \|\rho(g)s'\| = \|s'\|$ for all $g$. Hence by the dominated convergence theorem we may define an element $s$ by

$$s := \mathbb{E}_G(X).$$  

By the convexity of $X$, $s \in X$. Given $h \in G$, we now compute that

$$s = \mathbb{E}_G(X) = \mathbb{E}_G(X \circ \phi_h) \quad \text{using (1)}$$

$$= \mathbb{E}_G(\rho(h)1_{h^{-1} \cdot A}) \quad \text{using (2)}$$

$$= \mathbb{E}_G(\rho(h)\rho(g)s') \quad \text{as } \rho \text{ is a homomorphism}$$

$$= \rho(h)\mathbb{E}_G(\rho(g)s') \quad \text{by linearity of expectation}$$

$$= \rho(h)s \quad \text{by (2) and (3)}.$$  

So $s$ is invariant under $G$.

If $G$ is finite, the expectation is a finite sum so we do not need the dominated convergence theorem.

We will see a number of applications of this general result throughout this paper. In this section we will use this result to prove the classical two-mutual-fund theorem of [14]. A similar argument was used in [1] to prove the classical two-mutual-fund theorem but the notion of isomorphism was different. Before proving the two-mutual-fund theorem we will show how the notion of isomorphism in [1] relates to our new definition. We will do this by defining a general notion of a “finite dimensional linear market” and giving a classification result for such markets and their isomorphisms.

**Definition 1.6.** A one period financial market $M = ((\Omega, \mathcal{F}, \mathbb{P}), c)$ is separated if there is a subset $\Omega \subset \Omega$ of full measure such that for any distinct $\omega_1, \omega_2 \in \Omega$ there exists $X \in \text{dom} \ c$ with $X(\omega_1) \neq X(\omega_2)$.  

5
A one period financial market is linear if \( \text{dom} \, c \) is a linear subspace of \( L^0(\Omega; \mathbb{R}) \) and \( c \) is linear on \( \text{dom} \, c \). The dimension of a linear market is the dimension of \( \text{dom} \, c \).

On a linear market, we may define a map \( \pi \) from \( \Omega \) to \( (\text{dom} \, c)^* \), the dual space of \( \text{dom} \, c \), by
\[
\pi(\omega)(X) = X(\omega)
\]
for \( X \in \text{dom} \, c \) and \( \omega \in \Omega \). One checks that \( \pi(\omega)(\alpha X_1 + X_2) = (\alpha X_1 + X_2)(\omega) = \alpha X_1(\omega) + X_2(\omega) = \alpha \pi(X_1) + \pi(X_2) \), so \( \pi(\omega) \in (\text{dom} \, c)^* \) as claimed. The map \( \pi \) induces a measure on \( (\text{dom} \, c)^* \), denoted \( d_M \), which we call the distribution of the market. If \( M \) is separated, then \( \pi \) will be a mod 0 isomorphism.

Financially, a market is linear if all traded assets can be bought and sold in unlimited quantities at a fixed price per unit. A market is separated if the probability space contains no information other than that captured by asset prices.

A finite dimensional real vector space has a natural topology defined by the requirement that linear isomorphisms to \( \mathbb{R}^n \) are homeomorphisms. We would like to require that the measure \( d_M \) is in some sense compatible with this topology. To be precise we recall the following definition:

**Definition 1.7.** (see [9]) A regular probability measure is a probability measure arising as the Lebesgue extension of a Borel probability measure on a topological space.

We would like to be able to ensure that \( d_M \) is a regular probability measure. To do this we will require an additional condition on the probability space \( (\Omega, \mathcal{F}, P) \).

**Definition 1.8.** (see [17] and [9]) A probability space \( (\Omega, \mathcal{F}, P) \) is standard if it is isomorphic mod 0 to either: the Lebesgue measure on \([0, 1] \); a probability space on a finite or countable number of atoms; a convex combination of both.

The study of standard probability spaces was started by [19]. Although it may appear to be a highly restrictive condition, it is in fact a very mild assumption. Itô summarised the situation in [9] as “all probability spaces appearing in practical applications are standard”. We note a number of important examples that justify this claim. All regular probability measures on a complete separable metric space are standard. This includes all regular measures on \( \mathbb{R}^n \) and the Wiener measure on \( C^0[0, \infty) \). Finite and countable products of standard spaces are standard. A non-null measurable subset of a standard probability space becomes a standard probability space when endowed with the conditional measure. For proofs of these assertions see [17] or [9].

**Lemma 1.9.** If \( M \) is a finite dimensional linear market based on a standard probability space, then \( d_M \in \mathbb{P}((\text{dom} \, c)^*) \) where \( \mathbb{P}(S) \) denotes the set of regular probability measures on \( S \).
Proof. We recall that a perfect probability measure is a complete probability measure, \( \mu \) on a set \( S \) such that for every measurable map \( f : S \to \mathbb{R} \) the image measure is a regular measure on \( \mathbb{R} \). Lemma 2.4.3. of [9] proves that all standard probability spaces are perfect. Let \( S \) be a perfect probability space and let \( V \) be a finite dimensional real vector space, then Exercise 3.1(iii) of [9] shows that any measurable map \( f : S \to V \) will induce a regular measure on \( V \). Thus it suffices to show that \( \pi \) defined by \( \Box \) is measurable.

Choose a basis \( \{ X_i \} \) for \( \text{dom} \ c \). Define a map \( X : \Omega \to \mathbb{R}^n \) by requiring that the \( i \)-th component of \( X(\omega) \) is given by \( X(\omega)_i = X_i(\omega) \). This map is measurable since each \( X_i \) is measurable. Define a map \( X^{**} : (\text{dom} \ c)^* \to \mathbb{R}^n \) by requiring that the \( i \)-th component of \( X^{**}(f) \) is given by \( X^{**}(f)_i = f(X_i) \). \( (X^{**})^{-1} \) is a linear isomorphism and so is measurable by the definition of the topology on \( \text{dom} \ c \). Since \( \pi = (X^{**})^{-1} \circ X \), \( \pi \) is measurable.

**Definition 1.10.** A regular probability measure on a finite dimensional vector space, \( V \), is said to be non-degenerate if for any \( X, Y \in V^* \), \( X = Y \) almost everywhere implies \( X = Y \). Degenerate probability measures arise when the measure is concentrated on a vector subspace.

We write \( \text{Vec}(M) \) for the set of triples \((V, d, c)\) with \( V \) a finite dimensional vector space, \( d \in \mathbb{P}(V) \) with \( d \) non-degenerate and \( c \in V \). \( \text{Vec}(M) \) is equipped with a notion of isomorphism given by linear isomorphisms preserving \( d \) and \( c \).

**Theorem 1.11** (Equivalence of vector space and probabilistic categories of market). Write \( \text{Fin}(M) \) for the set consisting of separated finite dimensional linear markets whose probability space is standard. For any element \( M \) of \( \text{Fin}(M) \) define

\[
\text{Vec}(M) = ((\text{dom} \ c)^*, d_M, c).
\]

\( \text{Vec}(M) \) lies in \( \text{Vec}(M) \) and the map \( \text{Vec} : \text{Fin}(M) \to \text{Vec}(M) \) defines a bijection on isomorphism classes. Moreover the maps \( \text{Vec} \) and \( \text{Fin} \) described in the proof define an equivalence of categories (see [4] for the a definition of categories and their equivalence).

**Proof.** We must first show that \( \text{Vec}(M) \) lies in \( \text{Vec}(M) \).

We have already seen in Lemma [14] that \( d_M \) is regular.

We must also show that \( d_M \) is non-degenerate. Given \( X \in \text{dom} \ c \) we may define a linear functional \( X^{**} \in (\text{dom} \ c)^{**} \) by \( X^{**}(f) = f(X) \). Double duality is an isomorphism, so given distinct \( X, Y \in (\text{dom} \ c)^{**} \) we may find distinct \( X, Y \in (\text{dom} \ c) \) with \( X^{**} = X \) and \( Y^{**} = Y \). For any \( Z \in (\text{dom} \ c) \), \( Z^{**} \circ \pi = Z \).

This completes the proof that \( \text{Vec}(M) \) lies in \( \text{Vec}(M) \).

We now define an additional map, also denoted \( \text{Vec} \), which sends isomorphisms between elements of \( \text{Vec}(M) \) to isomorphisms between elements of \( \text{Fin}(M) \).

Given a market isomorphism \( T \) between two such markets \( M_i = ((\Omega_i, \mathcal{F}_i, \mathbb{P}_i), c_i) \in \text{Fin}(M) \) we define \( T^* : \text{dom} \ c_2 \to \text{dom} \ c_1 \) by

\[
T^*(f) = f \circ T
\]
We define $\text{Vec}(T) = T^{**} : (\text{dom } c_1)^* \rightarrow (\text{dom } c_2)^*$ to be the ordinary vector space dual of $T^*$.

In the opposite direction we define a map $\text{Fin} : \text{VecM} \rightarrow \text{FinM}$ by

$$\text{Fin}((V, d, c)) = ((V, \mathcal{F}, d), \mathcal{L})$$

where $\mathcal{F}$ is the Lebesgue sigma algebra on $V$ and the map $\mathcal{L} : L^0(V; \mathbb{R}) \rightarrow \mathbb{R}$ satisfies

$$\mathcal{L}(X) = \begin{cases} X(c) & \text{if } X \text{ is equal to a linear map almost everywhere} \\ \infty & \text{otherwise.} \end{cases}$$

It is not immediately obvious that $\text{Fin}((V, d, c))$ defined in this way is an element of $\text{FinM}$. We first note that the probability space underlying $\text{Fin}((V, d, c))$ is standard since regular distribution on a real vector space always defines a standard probability distribution. Since all elements of $\text{VecM}$ have non-degenerate distributions, $\text{dom } \mathcal{L} \subset L^0(V; \mathbb{R})$ is equal to $V^*$ (rather than a non-trivial quotient space of $V^*$ by equivalence almost everywhere). The dual space of a finite dimensional vector space separates the points of the vector space, so $\text{Fin}((V, d, c))$ is separated. It is now clear that $\text{Fin}((V, d, c))$ lies in $\text{FinM}$.

Since $\text{dom } \mathcal{L} = V^*$, we have $(\text{dom } \mathcal{L})^* = V^{**}$. Hence the composition $\text{Vec} \circ \text{Fin}$ is given by double duality of vector spaces. In particular $\text{Vec} \circ \text{Fin}(V, d, c)$ is isomorphic to $(V, d, c)$. We define a mapping on isomorphisms, also called $\text{Fin}$, by $\text{Fin}(T) = T$ for any isomorphism $T$ between elements of $\text{VecM}$. The composition $\text{Vec} \circ \text{Fin}$ acting on isomorphisms is given by double duality of linear transformations.

We also note that $\text{Fin} \circ \text{Vec}(M)$ is isomorphic to $M$ with the isomorphism given by $\pi$ defined in [4].

If the reader is familiar with category theory, they will see that $\text{Vec}$ and $\text{Fin}$ define an equivalence of the categories $\text{FinM}$ and $\text{VecM}$. Indeed we used the same name for the mapping $\text{Vec}$ defined on markets and the mapping $\text{Vec}$ defined on market isomorphisms precisely to make this clear. Elementary category theory now implies that $\text{Vec}$ is a bijection on isomorphism classes.

For the benefit of readers unfamiliar with category theory we fill in the remaining details.

Suppose that $M_1$ and $M_2$ are two markets and $T : M_1 \rightarrow M_2$ is an isomorphism. Then $\text{Vec}(T)$ defines an isomorphism between $\text{Vec}(M_1)$ and $\text{Vec}(M_2)$. So $\text{Vec}$ induces a well-defined map on equivalence classes.

Suppose that $\text{Vec}(M_1)$ and $\text{Vec}(M_2)$ are isomorphic via some isomorphism, $T$. Then $\text{Fin}(T)$ defines an isomorphism between $\text{Fin} \circ \text{Vec}(M_1)$ and $\text{Fin} \circ \text{Vec}(M_2)$. These are isomorphic via $\pi^{-1}$ to $M_1$ and $M_2$ respectively. Hence $M_1$ and $M_2$ are isomorphic. So $\text{Vec}$ defines an injective map on equivalence classes.

Suppose that $(V, d, c) \in \text{Vec}(M_1)$ then $\text{Vec} \circ \text{Fin}(V, d, c)$ is isomorphic to $(V, d, c)$. Hence $\text{Vec}$ defines a surjective map on equivalence classes.

Let $M = ((\Omega, \mathcal{F}, \mathbb{P}), c) \in \text{FinM}$ be a market. A basis $\{X_i\}$ ($i = 1 \ldots n$) for $\text{dom } c$ can be thought of as the prices of $n$ preferred assets. Any investment...
strategy \( X \in \text{dom } c \) can be obtained by taking a portfolio of these \( n \) assets. Let us write \( \{ e_i \} \) for the dual basis of \( \{ X_i \} \). Using this dual basis we obtain an isomorphism from \((\text{dom } c)^* \rightarrow \mathbb{R}^n\). Applying this isomorphism to \( d_M \), we obtain a distribution on \( \mathbb{R}^n \) which is the distribution of asset prices. Applying the same isomorphism to \( c \) we obtain an element \( c \) of \( \mathbb{R}^n \). The \( i \)-th component of this vector corresponds to the price of asset \( i \).

Conversely given a non-degenerate distribution of asset prices on \( \mathbb{R}^n \) and a vector \( c \in \mathbb{R}^n \) we may apply \( \text{Fin} \) to obtain an element of \( \text{Fin}M \).

This discussion shows how to relate our abstract definition of a one period market to the more concrete approach of considering the distribution of asset prices. The significance of Theorem 1.11 is that it shows the notion of equivalence of markets obtained by treating all portfolios as equally valid investment strategies is the same as the notion of equivalence given in Definition 1.3. The advantage of Definition 1.3 is that it can be applied equally well to infinite markets and non-linear markets.

We now apply this general theory to the case of assets following a multivariate normal distribution, as considered by Markowitz [12]. Let \( g_\mu \) be the multivariate normal distribution with mean \( \mu \in \mathbb{R}^n \) and covariance matrix given by the identity \( \text{id}_n \). We say that a market is Gaussian if it is isomorphic to a market on \( \mathbb{R}^n \) with density \( g_\mu \). Trivially any Gaussian market is isomorphic to a market of the form \( \text{Fin}(\mathbb{R}^n, g_\mu, c) \) for some \( \mu, c \in \mathbb{R}^n \). Let \( \{ e_i \} \) be the standard basis for \( \mathbb{R}^n \). Since isometries of \( \mathbb{R}^n \) preserve the Gaussian measure, we may apply a rotation so that \( \mu \) lies in the span of \( e_1 \) and \( c \) lies in the span of \( e_1 \) and \( e_2 \). This shows that any Gaussian market can be written in the form

\[
\text{Fin}(\mathbb{R}^n, g_\alpha e_1, \beta e_1 + \gamma e_2), \quad \alpha, \beta, \gamma \in \mathbb{R}.
\] (5)

We now have the following classification theorem.

**Theorem 1.12** (Classification of Markowitz markets). Let \( M \in \text{Fin}M \) be a market and suppose that \( \{ X_i \} \) is a basis for \( \text{dom } c \) given by assets following a multivariate normal distribution. Then \( M \) is Gaussian, and hence is isomorphic to a market of the form (5). This theorem is essentially a restatement of the main classification result of [1] in the language of one period markets.

**Proof.** Let \( \text{Cov} : \text{dom } c \times \text{dom } c \rightarrow \mathbb{R} \) be given by the covariance. This is a non-degenerate symmetric bilinear form and hence defines an inner product on \( \text{dom } c \). All real inner product spaces of dimension \( n \) are isomorphic to the standard Euclidean space \( \mathbb{R}^n \), hence we can find a second basis \( \{ Y_i \} \) for \( \text{dom } c \) with covariance matrix \( \text{id}_n \). The distribution of these assets will still be a multivariate normal distribution but now with covariance matrix \( \text{id}_n \). This shows that the market is Gaussian. \( \square \)

**Corollary 1.13.** All invariant investment strategies \( X \in \text{dom } c \) in a Gaussian market lie in a two dimensional vector subspace of \( \text{dom } c \).
Proof. It suffices to prove the result for markets of the form (5). Let \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) be the linear transformation given by the matrix

\[
\phi_{ij} = \begin{cases} 
1 & i = j \text{ and } i, j \leq 2 \\
-1 & i = j \text{ and } i, j > 2 \\
0 & \text{otherwise}
\end{cases}
\]

\( \phi \) defines an automorphism of any market of the form (5). Any invariant investment strategy must be invariant under \( \phi^* \). \( \phi^* \) has the same matrix representation as \( \phi \) when written with respect to the standard dual basis \( \{ e^*_i \} \) for \((\mathbb{R}^n)^*\). If \( X \) is an invariant investment strategy its components \((X)_i\) written with respect to this basis satisfy \((X)_i = 0 \) for \( i > 2 \).

Corollary 1.14. (Two-mutual-fund theorem [14]) Suppose we have \( n \) assets of a given cost whose payoffs follow a multivariate normal distribution. We wish to find the portfolio of assets with minimum variance but with a given expected payoff \( C_1 \) and cost \( C_2 \). There are two portfolios \( X_1 \) and \( X_2 \) independent of \( C_1 \) and \( C_2 \) such that we can solve these mean–variance optimization problems for any \( C_1 \) and \( C_2 \) simply by considering linear combinations of \( X_1 \) and \( X_2 \). The portfolios \( X_1 \) and \( X_2 \) are the two “mutual funds” that give this theorem its name.

Proof. Take \( G = \{ 1, \phi \} \), where \( \phi \) is the linear transformation defined in the proof of Corollary 1.13. Equip \( G \) with the probability measure that gives each element a measure of \( \frac{1}{2} \). The result follows Theorem 1.4.

We remark that Corollary 1.13 is a much stronger result than the classical two-mutual-fund theorem. The paper [1] gives numerous concrete examples of financially interesting results arising from invariance arguments other than just the two-mutual-fund-theorem.

We also remark that the concrete isomorphism found in Theorem 1.12 makes it extremely easy to solve the classical mean-variance optimization problem directly, thereby recovering the full set of results found in [14]. This approach is pursued in [1].

2 One period complete markets

Definition 2.1. A one period market \( M = ((\Omega, \mathcal{F}, P), c) \) is complete if there exists a measure \( Q \) on \( \Omega \) equivalent to \( P \), and \( C > 0 \) such that

\[
c(X) = \begin{cases} 
C(\mathbb{E}_Q(X^+) + \mathbb{E}_Q(X^-)) & \text{one of } \mathbb{E}_Q(X^\pm) \text{ is finite} \\
\infty & \text{otherwise}
\end{cases}
\] (6)

In this formula \( X^+ \) and \( X^- \) denote the positive and negative parts of the random variable \( X \). We note that \( c(1) = C \), so we interpret \((C - 1)\) as a deterministic interest rate.
Example 2.2. Let $I$ be the market given by taking the $\mathbb{P}$ and $\mathbb{Q}$ measure to both be equal to the Lebesgue measure on $[0, 1)$ and with cost of the constant function with value 1, equal to 1. In this market prices are given by expectations, so we call $I$ a casino.

Given a complete market $M$ we may define a new complete market $M \times I$ by taking the product measures for both the $\mathbb{P}$ and the $\mathbb{Q}$ measures and taking the constant $C$ to be that given by the market $M$.

From a financial point of view the market $M \times I$ represents the market obtained by considering investment strategies where one first invests in the market $M$ and then places a bet at the casino. This assertion requires us to define multi-period markets to justify it rigorously, so we postpone this detail until Appendix A.

In applications it is not unreasonable to assume that there is a casino available should a trader wish to use it. So classifying complete markets of the form $M \times I$ should be just as useful in practice as a full classification. The theorem below gives a classification for markets of this form.

Theorem 2.3 (Classification of complete markets up to a casino). Let $M$ be a complete market on a standard probability space. Then $M \times I$ is isomorphic to $\tilde{M} \times I$, where $\tilde{M}$ is the market with probability space given by $\tilde{\Omega} = [0, 1]$ equipped with the Lebesgue measure and with pricing function

$$\tilde{c}(X) = C \int_0^1 F^{-1}_{\mathbb{Q}} d\mathbb{P} X(x) dx.$$  

Here $F^{-1}_{\mathbb{Q}}$ is the inverse distribution function of $d\mathbb{Q} d\mathbb{P}$ on $M$.

The first step toward proving this is to observe that we may recover $\mathbb{Q}$ from $c$ by observing that for any measurable set $A \subset \Omega$ we have

$$Q(A) = E_Q(1_A) = \frac{c(1_A)}{c(1)}.$$  

It follows that two one period complete markets $((\Omega_i, \mathcal{F}_i, \mathbb{P}_i), c_i)$ ($i = 1, 2$) are isomorphic if and only if (a) there is a mod 0 isomorphism for the $\mathbb{P}_i$ measures which is also a mod 0 isomorphism for the $\mathbb{Q}_i$ measures; and (b) the cost of the constant function with value 1 is equal in both markets.

There may be more than just 2 measures on the market which are of financial interest. A trader with views about the market represented by a measure $\mathbb{P}$ may be constrained by a risk manager or regulator with different views about the market. These can be represented by alternative measures. Let us state a classification result similar to Theorem 2.3 that applies to this situation.

Theorem 2.4 (Classification of complete markets with multiple views). Let $I$ denote the interval $[0, 1)$ with the Lebesgue measure. We suppose that $\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_n$ are equivalent probability measures on $(\Omega, \mathcal{F})$. We assume $\mathbb{P}_0$ is standard. Then there is a unique Lebesgue measure $\mathbb{P}'_0$ on $\Omega' = (0, \infty)^n$ such that $\mathbb{P}_0 \times I$ and
\(P'_0 \times I\) are mod 0 isomorphic via an isomorphism which also acts as a mod 0 isomorphism between the measures \(P_i \times I\) and \(P'_i \times I\) where \(P'_i\) is the Lebesgue measure given by

\[P'_i(A) = \int_{(0, \infty)^n} \omega_i 1_A(\omega) \, d\mu.\]

In this formula, \(A\) is a measurable set, \(1_A\) is the indicator function \(A\) and \(\omega_i\) is the \(i\)-th coordinate function on \(\mathbb{R}^n\). Note that we must have \(E_{P'_i}(\omega_i) = 1\) for these \(P'_i\) to be probability measures.

We will prove this result in Section 2.1 we note the following financial implication.

**Corollary 2.5** (Convex mutual fund theorem for complete markets). We use the notation of Theorem 2.4. Let \(A\) be a non-empty convex subset of the space of \(P_0\)-integrable random variables on \(\Omega\). Suppose that \(A\) is also invariant under mod 0 isomorphisms that preserve all the \(P_i\). Then \(A\) contains an element which can be written as a function of the Radon-Nikodym derivatives \(\frac{dP'_i}{dP_0}\). For example, \(A\) might arise as the optimal investment strategies in a convex optimization problem with a cost constraint and risk management constraints imposed by a number of regulators and risk managers given in terms of the \(P_i\).

**Proof.** We have the obvious inclusion \(\iota : L^1_{P_0}(\Omega) \to L^1_{P_0}(\Omega \times I)\). Any element of \(L^1_{P_0}(\Omega \times I)\) which can be written as a function of the Radon-Nikodym derivatives \(\frac{dP'_i}{dP_0}\) must lie in the image of \(\iota\). Hence it suffices to prove that \(\iota A\) contains an element which can be written as a function of these Radon-Nikodym derivatives.

By Theorem 2.4 we may assume without loss of generality that the market \(\Omega \times I\) is given by \(\Omega' \times I = (0, \infty)^n \times I\) and \(P'_i\) described in Theorem 2.4. In this case the Radon-Nikodym derivatives are given by the coordinate functions \(\omega_i\).

Let \(G = S^1 \cong \mathbb{R}/\mathbb{Z}\) with measure given by the quotient measure. Since each element of \(\mathbb{R}/\mathbb{Z}\) has a unique representative on \([0, 1)\), \(G\) is strictly isomorphic to \([0, 1)\) as a probability space. Hence we may define an action of \(G\) on any product space \(X \times I\) by using the action on the right hand side of the product. We can apply Theorem 1.4 with this choice of \(G\) and taking as \(\iota A\) as the convex set. The result now follows.

For example, a special case of the result above is the problem of expected utility optimisation in a complete market subject to a single cost constraint for a concave, increasing utility function. In this case it is well-known that the optimal investment has a payoff function given as a function of the Radon-Nikodym derivative (see [6]).

### 2.1 Full classification of complete markets

We will now state and prove a full classification theorem which does not involve products with \(I\) and which implies Theorems 2.4 and 2.3. Readers interested in the financial implications rather than the proofs may wish to skip to Section 2.2.
Our result will follow from the theory of disintegration of measure described in [17]. Let us first summarise the results we shall need.

Write \( S \) for the set of mod 0 isomorphism classes of standard probability spaces. We call \( S \) the moduli space of standard probability spaces.

Given \( m \in S \), we define \( m_0 \) to be the measure of the continuous component of \( m \) (or zero if it has no continuous component) and we define \( m_i \) for \( i > 0 \) to be the measure of the \( i \)-th largest atom in our probability space (or 0 if there less than \( i \) atoms). Thus we have identified a correspondence between \( S \) and sets of numbers \( m_i \) \( (i \in \mathbb{N}) \) which satisfy

\[
\begin{align*}
m_i &\in [0, 1] \\
m_i &\geq m_{i+1} \quad i \in \mathbb{N}^+ \\
m_0 &= 1 - \sum_{i=1}^{\infty} m_i. 
\end{align*}
\]

We give \( S \) the topology induced by thinking of it as a subset of \( \mathbb{R}^\infty \) in this way. Thus we may talk about measurable maps to \( S \), or \( S \) valued random variables.

The theory of disintegration of measure tells us that for a complete market \( M \) based on a standard probability space, there is a \( \mu_M \)-almost-surely unique measurable function

\[
m_M : (0, \infty) \to S
\]

with \( m_M(x) \) given by the mod 0 isomorphism class of the \( \mathbb{P} \) conditional measure conditioned on the value of \( \frac{dQ}{dP} = x \). We will show that this map \( m_M \) together with the measure \( \mu_M \) will allow us to classify markets. Let us review sufficient features of the theory of disintegration of measures to see why this is true.

**Definition 2.6.** Let \( \{S_\alpha\} \) be a countable collection of subsets of a set \( S \). We write \( \zeta(\{S_\alpha\}) \) for the collection of sets of the form

\[
\bigcap_{i=1}^{\infty} S'_\alpha, \quad (S'_\alpha = S_\alpha \text{ or } S'_\alpha = S \setminus S_\alpha).
\]

These sets are disjoint and cover \( S \) so they define a decomposition of \( S \) called the *decomposition generated by \( \{S_\alpha\} \).* A decomposition of a measurable set \( S \) generated by a countable collection of measurable sets is called a *measurable decomposition.* Here we are using the terminology of [17] p5 and p26. These decompositions are called *separable decompositions* in [9].

We say that two measurable decompositions \( \zeta \) and \( \zeta' \) of probability spaces \( \Omega \) and \( \Omega' \) are mod 0 isomorphic if there is a mod 0 isomorphism of \( \Omega \) mapping the elements of \( \zeta \) to the elements of \( \zeta' \).

Given a decomposition \( \zeta \) of a probability space \( \Omega \) we may define a projection map, \( \pi_\zeta : \Omega \to \zeta \) by sending a point \( \omega \) to the element of \( \zeta \) containing \( \omega \). This projection map induces a measure \( \mu_\zeta \) on \( \zeta \). Rokhlin refers to the resulting measurable space as the quotient space \( \Omega/\zeta \) (see p4 of [17]).

13
**Definition 2.7.** Let $\zeta$ be a decomposition of a standard probability space $\Omega$. Let $\mu_C$ be a set of measures defined indexed by $C \in \zeta$. We say that $\mu_C$ is **canonical with respect to** $\zeta$ if

(i) $\mu_C$ is a standard probability space for $\mu_\zeta$-almost-all $C \in \zeta$.

(ii) If $A$ is a measurable subset of $\Omega$ then

(a) The set $A \cap C$ is $\mu_C$ measurable for $\mu_\zeta$-almost-all $C$;

(b) $\mu_C(A \cap C)$ defines a $\mu_\zeta$-measurable function acting on $C \in \zeta$;

(c) the measure $A$ can be recovered by integrating over $\zeta$, i.e.

\[
\mu(A) = \int_\zeta \mu_C(A \cap C) \, d\mu_\zeta.
\]

This definition is simply a translation of the definition on p25 of [17] into our notation. We note that what we call a standard probability space, Rohklin calls a Lebesgue space. The equivalence of these notions is given on p 20 of [17].

We may now state two theorems, both due to Rohklin

**Theorem 2.8.** Let $\Omega$ be a standard probability space. There exists a set of measures $\mu_C$ canonical with respect to $\zeta$ if and only if $\zeta$ is a measurable decomposition ([17] p26). Moreover, $\mu_C$ is defined essentially uniquely: if $\mu_C$ and $\mu_C'$ are both canonical for $\zeta$ then $\mu_C$ is mod 0 isomorphic to $\mu_C'$ for $\mu_\zeta$-almost-all $C$ ([17] p25).

**Theorem 2.9.** Let $\Omega$ be a standard probability space and $\zeta$ a measurable decomposition. Let $m_\zeta : \zeta \to S$ be given by mapping the measure $\mu_C$ to the element of $S$ corresponding to its isomorphism class. Then $m_\zeta$ is $\mu_\zeta$ measurable. Two decompositions $\zeta$ and $\zeta'$ are mod 0 isomorphic if and only $\mu_\zeta$ and $\mu_{\zeta'}$ are mod 0 isomorphic via a map sending $m_\zeta$ to $m_{\zeta'}$ ([17] p40).

Finally, Theorem 3.3.1 of [9] tells us that if $X$ is a real random variable, and if we define $\zeta$ to be the set of sets of the form $X^{-1}(x)$ then $\zeta$ is a measurable decomposition. When we apply Theorem [17] to the level sets of a random variable $\zeta$, the measure $\mu_{X^{-1}(x)}$ on the level set $X^{-1}(x)$ for $x \in \mathbb{R}$ is called the **conditional probability measure**, conditioned on $X = x$ (see [9] Section 3.5). Note that in this case the projection map sending the level set $X^{-1}(x)$ to $x$ defines a mod 0 isomorphism between $\zeta$ with measure $\mu_\zeta$ and the probability measure on $\mathbb{R}$ induced by $X$.

This completes our summary of the theory of disintegration measures. We now wish to state and prove a classification of complete markets on standard probability spaces.

**Definition 2.10.** Let $\text{Measures}(n)$ be the set consisting of pairs $(\mu, m)$ where:

(i) $\mu$ is a regular probability measure on $(0, \infty)^n$ satisfying

\[ \mathbb{E}_\mu(\omega_i) = 1 \]

for the $i$th coordinate function $\omega_i$ on $\mathbb{R}^n$;
(ii) \( m \) is a \( S \) valued \( \mu \) random variable.

**Theorem 2.11** (Generalised classification of complete markets). Standard probability spaces \((\Omega, \mathcal{F}, \mathbb{P}_0)\) equipped with \( n \)-additional equivalent measures \( \mathbb{P}_1, \ldots, \mathbb{P}_n \) are classified up to joint \( \mathbb{P}_0 \)-isomorphism by elements \((\mu_q, m_q) \in \text{Measures}(n)\). Here \( \mu_q \) is the measure on \((0, \infty)^n\) induced by the \( \mathbb{R}^n \) vector valued function \( q \) with \( i \)-th component given by the Radon-Nikodym derivative \( \frac{dq}{d\mu} \). \( m_q \) is the function \( m_\zeta \) supplied by Theorem 2.9 for the decomposition induced by \( q \).

Moreover, for each \( M \in \text{Measures}(n) \) we can construct a concrete representative for each isomorphism class, which we denote by \( \Omega(M) \). The details of the construction are stated in the proof.

**Proof.** It is clear that \((\mu_q, m_q) \in \text{Measures}(n)\) defines an invariant of \( \Omega \).

Given a pair \( M = (\mu, m) \in \text{Measures}(n) \), let us see how to define \( \Omega(M) \) with \((\mu_q, m_q) = M \).

Let \( \omega_0 \) be the probability space \([0, 1]\). For \( i > 0 \), let \( a_i \) be a probability space consisting of a single atom. We take as probability space

\[
\Omega(M) = (0, \infty)^n \times (\bigcup_{i=0}^\infty a_i).
\]

This has a measure we will denote by \((\mu \times \lambda)\) induced by taking the standard construction of product measures and measures on disjoint unions and then obtaining the Lebesgue extension. Using our concrete realisation of \( S \), given in (7), we define the components \( m_i \) of the function \( m \) for \( i \in \mathbb{N} \cup \{ \infty \} \). Let \( \pi_1 : \Omega_M \to (0, \infty)^n \) denote the projection onto the \((0, \infty)^n\) component. We then obtain measurable functions \( m_i \circ \pi_1 \) defined on \( \Omega \). Given a Lebesgue measurable subset \( A \) of \( \Omega \) we define a measure \( \mathbb{P}_0(A) \) by

\[
\mathbb{P}_0(A) := \int_{\Omega} \sum_{i=0}^\infty (m_i \circ \pi_1) \cdot 1_{A \cap ((0, \infty)^n \times a_i)} \, d(\mu \times \lambda)
= \int_{(0, \infty)^n} \sum_{i=0}^\infty (m_i \circ \pi_1) \int_{a_i} 1_{A \cap ((0, \infty)^n \times a_i)} \, d(\mu \times \lambda|_{a_i})
= \int_{(0, \infty)^n} \sum_{i=0}^\infty m_i \mathbb{P}_{a_i}(A \cap \pi_1^{-1}(\omega) \cap a_i) \, d\mu
= \int_{(0, \infty)^n} \mathbb{P}_m(A \cap \pi_1^{-1}(\omega)) \, d\mu. \tag{8}
\]

Let \( \zeta \) be the decomposition of \( \Omega(M) \) given by the pre-images \( \pi_1^{-1}(\omega) \) for \( \omega \in (0, \infty)^n \). For \( \omega \in (0, \infty)^n \), let \( \mu_{\pi_1^{-1}(\omega)} \) be the measure \( m(\omega) \). We observe that \( m(\omega) \) is canonical with respect to \( \zeta \). We explicitly check the requirements given in Definition 2.9 Property (i) follows since \( \pi_1^{-1}(\omega) \) is always standard. Similarly property (ii) (a) follows since \( A \cap \pi_1^{-1}(\omega) \) is always measurable. Property (ii) (b) follows from Fubini’s theorem as used in the derivation of equation (8) above. Property (ii) (c) is given by (8) itself.
For $1 \leq i \leq n$, we define measures $P_{i,M}$ by

$$P_{i,M}(A) = \omega_i \mathbb{E}_\mu(\pi_1 \cdot 1_A)$$

(9)

where $\omega_i$ is the $i$th coordinate function on $(0, \infty)^n$ as before. This is an equivalent probability measure to $P_0$ since $\omega_i$ is positive and has $P_0$ expectation of 1.

We see that $\Omega(M)$ equipped with these measures satisfies $(\mu_q, m_q) = M$.

Suppose $\Omega$ is a probability space with $n$ additional equivalent measures $P_i$. Let $M = (\mu_q, m_q)$. By Theorem 2.9 we can find a mod 0 isomorphism, $\phi$, from $\Omega$ to $\Omega_M$ equipped with measure $P_0$ which also sends $q$ to $\pi_1$ for each $i$. It follows from (9) that $\phi$ must be a $P_i$-isomorphism too.

Proof of Theorem 2.4. Let $S$ be a standard probability space and $\zeta$ a decomposition of $S$. Let $T$ be another standard probability space. We write $\zeta \bigstar T$ for the decomposition of $S \times T$ given by taking the product of elements of $\zeta$ with $T$. Given a set of measures $\mu_C$ on $\zeta$ we write $\mu_C \times \mu_T$ for the product measures. It is clear that if $\mu_C$ is canonical with respect to $\zeta$ then $\mu_C$ is canonical with respect to $\zeta \bigstar T$. Thus the conditional measures of $dP_i/dP_0$ on $\Omega \times I$ are all given by products with the standard measure on $I$. Hence $(m_{\Omega \times I})_0 = 1 \mu_{\Omega \times I}$-almost-everywhere.

On the other hand, taking the product of a $\Omega$ with $I$ does not affect the distribution $\mu_\Omega$. So if we take $\Omega'$ to be the space defined in the statement of Theorem 2.4 we will have that the invariants of $\Omega' \times I$ are equal to the invariants of $\Omega \times I$. The result now follows from Theorem 2.11.

Proof of Theorem 2.5. Pick a Lebesgue measure $\mathbb{P}$ on $(0, \infty)$ with

$$\mathbb{E}_\mathbb{P}(\omega_1) = 1.$$

The coordinate function $\omega_1$ on $(0, \infty)$ is just the identity. Define a measure $\mathbb{Q}$ by requiring that the Radon-Nikodym derivative is $\frac{d\mathbb{Q}}{d\mathbb{P}} = \omega_1 = \text{id}$.

Let $F$ be the distribution function of this measure and $F^{-1} : [0, 1] \rightarrow (0, \infty)$ its inverse distribution function. We equip the interval $[0, 1]$ with the Lebesgue measure $\mathbb{P}'$ and a measure $\mathbb{Q}'$ given by requiring that the Radon-Nikodym derivative $\frac{d\mathbb{Q}'}{d\mathbb{P}'} = F^{-1}$.

If we can find a simultaneous mod 0 isomorphism between the measures $(\mathbb{P}, \mathbb{Q})$ on $M \times I$ and $(\mathbb{P}', \mathbb{Q}')$ on $M \times I$ we see that Theorem 2.5 follows from Theorem 2.4. We take $\mathbb{P}_0 = \mathbb{P}$ and $\mathbb{P}_1 = \mathbb{Q}$ when applying Theorem 2.4.

We will now find the required isomorphism.

In what follows, if $X$ is a set with measure $\mu$ we will write $X_\mu$ to emphasize the measure on $X$.

Let $0 \leq p_1 \leq p_2 \leq 1$.

Suppose that $p_1$ and $p_2$ are the two ends of a connected component of $\text{Im} F$ then $F$ is continuous between $p_1$ and $p_2$ and so $F$ defines a mod 0 isomorphism between $[F^{-1}(p_1), F^{-1}(p_2)]_\mathbb{P}$ and $[p_1, p_2]_\mathbb{P}'$. So $(F^{-1}|_{[p_1, p_2]_\mathbb{P}}) \times I$ is mod 0 isomorphic to $(p_1, p_2)_{\mathbb{P}'} \times I$ via $F \times \text{id}$. This isomorphism maps the random
variable $\omega_1 = \text{id}$ to $F_{\mathcal{X}}^{-1}$. Hence it is also a mod 0 isomorphism for the measures $\mathbb{Q}$ and $\mathbb{Q}'$.

Suppose that $p_1$ and $p_2$ are the two ends of a connected component of $[0, 1] \setminus \text{Im } F$. $(F^{-1}[p_1, p_2])_P$ is mod 0 isomorphic to the atom $\{F^{-1}(p_1)\}_P$ with mass $(p_2 - p_1)$. So $(F^{-1}[p_1, p_2])_{\mathbb{Q}_0} \times I$ is mod 0 isomorphic to $[p_1, p_2]_{\mathbb{Q'}} \times I$. The $\mathbb{Q}$-measure on the atom $\{F^{-1}(p_1)\}$ is equal to $F^{-1}(p_1)$, which is equal to $F^{-1}(p)$ for all $p_1 \leq p \leq p_2$. Hence $(F^{-1}[p_1, p_2])_{\mathbb{Q}_0} \times I$ is simultaneously mod 0 isomorphic to $[p_1, p_2]_{\mathbb{Q'}} \times I$.

We may therefore cover $[0, 1] \times I$ a countable set of disjoint intervals the form $[p_1, p_2] \times I$ which are simultaneously $P/\mathbb{Q} \mod 0$ isomorphic to $(F^{-1}[p_1, p_2]) \times I$.

We may therefore combine these mod 0 isomorphisms on intervals to obtain the desired mod 0 isomorphism for the $P$ and $\mathbb{Q}$ measures.

2.2 Non-convex problems and rearrangement

We will see in this section that Theorem 2.4 allows us to identify a mutual fund theorem that applies to optimization in complete markets when we assume that the problem is “monotonic” rather than convex.

We have in mind applications to behavioural economics based on the observations of Kahneman and Tversky in [11]. For examples of applications of Kahneman and Tversky’s ideas to mathematical finance and risk management, see, for example, [10], the review [20], and [2] which contains numerous further references.

It has been observed in this literature (see for example [8]) that the solution to optimal investment in complete markets involving S-shaped utility functions can be obtained by considering monotonic functions of the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$. The aim of this section is to show how these results arise from general monotonicity properties, automorphism invariance and our classification theorems. We take the opportunity to show how these results can be generalized to situations where there are more than two measures $P$ and $\mathbb{Q}$, for example, to the case where risk managers and traders have different beliefs about the future evolution of the market.

Given two random variables $X$, $Y$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we write

$$d^\mathbb{P}(X) \preceq d^\mathbb{P}(Y)$$

if $F_X(k) := \mathbb{P}(X \leq k) \geq \mathbb{P}(Y \leq k) =: F_Y(k)$ for all $k$. The notation $d^\mathbb{P}(X)$ is intended to suggest “the $\mathbb{P}$-distribution of $X$”. Given a third random variable $Z$ we write

$$d^\mathbb{P}(X \mid Z) \preceq d^\mathbb{P}(Y \mid Z)$$

if $\mathbb{P}(X \leq k \mid Z) \geq \mathbb{P}(Y \leq k \mid Z)$ almost surely for all $k$.

We suppose that market participants such as traders and risk managers impose some form of relation $\preceq'$ on random variables to express their preferences between different investment opportunities. One might reasonably expect that

$$X \preceq Y \implies X \preceq' Y.$$  (10)
If this condition holds, we will say that $\preceq'$ is increasing. We say that $\preceq'$ is decreasing if the reversed relation is increasing. We say that a relation on random variables is monotonic if it is either increasing or decreasing. We say that the sign of a monotonic relation is $1$ if it increasing or $-1$ if it is decreasing.

**Definition 2.12** (Rearrangement). Let $m$ be a Lebesgue probability measure on $(0, \infty)$. Let $F_m$ denote the cumulative distribution function of $m$. Write $x, y$ for the coordinate functions on $(0, \infty) \times [0, 1)$. Define $U_m : (0, \infty) \times [0, 1) \to [0, 1]$ by

$$U_m(\omega) = (1 - y(\omega)) \lim_{x' \to x(\omega)} F_m(x') + y(\omega) \lim_{x' \to x(\omega)} F_m(x').$$

$U_m$ is well-defined since $F_m$ is càdlàg. We write $P_m$ for the product measure on $(0, \infty) \times [0, 1)$. If $E_{P_m}(x(\omega)) = 1$ \hspace{1cm} (11)

then $x$ is the Radon-Nikodym derivative of an equivalent measure we call $Q_m$. Given $X \in L^0_{P_m}((0, \infty) \times [0, 1); \mathbb{R})$ we define the increasing and decreasing rearrangements of $X$ by

$$R^+_m(X) = F_X^{-1}(U_m), \hspace{1cm} R^-_m(X) = -F^{-1}_X(U_m)$$

respectively, where $F_X^{-1}$ is the $P_m$ inverse distribution function of $X$.

We collect together the key properties of rearrangement in a single lemma.

**Lemma 2.13.** Let $m$ be a Lebesgue measure on $(0, \infty)$ satisfying condition (11). Then $U_m$ is a uniformly distributed random variable. Let $X$ be a random variable in $X \in L^0_{P_m}((0, \infty) \times [0, 1); \mathbb{R})$.

The $P_m$ distribution is left fixed by rearrangement of $X$. The $Q_m$ distributions are increased or decreased according to whether one applies the increasing or decreasing rearrangement. Symbolically:

$$d^{P_m}(X) = d^{P_m}(R^+_m(X)), \hspace{1cm} (12)$$

$$d^{Q_m}(X) \leq d^{R^+_m}(R^+_m(X)), \hspace{1cm} (13)$$

$$d^{Q_m}(X) \geq d^{R^-_m}(R^-_m(X)). \hspace{1cm} (14)$$

In addition:

$$d^{Q_m}(\omega) < d^{Q_m}(\omega') \implies R^+_m(\pm X(\omega)) \leq R^+_m(\pm X(\omega')), \hspace{1cm} (15)$$

$$d^{P_m}(X) \leq d^{P_m}(Y) \implies d^{Q_m}(R^+_m(X)) \leq d^{Q_m}(R^+_m(Y)), \hspace{1cm} (16)$$

$F^{Q_m}_{R^+_m}$ is continuous at $x$ $\implies R^+_m(X)(x, y) = R^+_m(X)(x, y_1) = R^+_m(X)(x, y_2)$ $\forall y_1, y_2$. \hspace{1cm} (17)
Proof. Pick \( z \in (0, 1) \). Since \( F_m \) is an increasing function, we can find \( x_0 \in (0, \infty) \) with \( \lim_{x' \to x_0^-} F_m(x') \leq z \leq \lim_{x' \to x_0^+} F_m(x') \). Hence we can find \( y_0 \) with \( U_m(x_0, y_0) = z \). Since \( F_m \) is increasing, we deduce that

\[
\mathbb{P}_m(U_m(\omega) \leq z) = \mathbb{P}_m(x(\omega) < x_0 \text{ or } (x(\omega) = x_0 \text{ and } y(\omega) \leq y_0))
\]

\[
= \mathbb{P}_m(x(\omega) < x_0) + \mathbb{P}_m(x(\omega) = x_0)\mathbb{P}_m(y(\omega) \leq y_0)
\]

\[
= \lim_{x \to x_0^-} F_m(x) + y(\lim_{x \to x_0^+} F_m(x) - \lim_{x \to x_0^-} F_m(x))
\]

\[
= z. \tag{18}
\]

We deduce first that \( U_m \) is measurable since its sublevel sets are measurable. We then deduce that \( U_m \) is a uniform random variable as \( \text{(15)} \) is the defining property of uniform random variables.

Property \( \text{(12)} \) of rearrangement follows immediately from the fact that \( U_m \) is uniform and from the definition of rearrangement.

We note that for \( \alpha \in (0, 1) \),

\[
\inf \{ x \in \mathbb{R} \mid F_X(x) \geq \alpha \} \leq k \implies F_X(k) \geq \alpha.
\]

So from the definition of rearrangement

\[
\mathbb{P}(R_m^+(X)(\omega) \leq k) = \mathbb{P}(F_m^{-1}(U_m(\omega)) \leq k)
\]

\[
= \mathbb{P}(\inf \{ x \in \mathbb{R} \mid F_X(x) \geq U_m(\omega) \} \leq k)
\]

\[
\leq \mathbb{P}(F_X(k) \geq U_m(\omega))
\]

\[
= F_X(k). \tag{19}
\]

The last step uses \( \text{(18)} \). We have established \( \text{(13)} \). Property \( \text{(14)} \) is now obvious.

From the definition of \( U_m \), if \( x(\omega) \leq x(\omega') \) then \( U_m(\omega) \leq U_m(\omega') \). \( F_X \) is increasing and \( x \) is equal to the Radon-Nikodym derivative \( \frac{dF_m}{d\mathbb{P}_m} \). Hence \( \text{(15)} \) follows.

From the definition of \( U_m \), \( U_m(x, y) \) is independent of \( y \) when \( F_m \) is continuous at \( x \). Hence \( R_m^+(X)(x, y) \) is also independent of \( y \). Note that \( F_m = F_m^{\text{rel}} \).

This establishes \( \text{(17)} \).

To establish \( \text{(16)} \) let us suppose \( d^{\mathbb{P}_m}(X) \leq d^{\mathbb{P}_m}(Y) \). This means that

\[
F_X(k) \geq F_Y(k) \quad \forall k \in \mathbb{R}
\]

where \( F_X \) and \( F_Y \) are the \( \mathbb{P}_m \) measure distribution functions of \( X \) and \( Y \). Hence

\[
F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \forall p \in [0, 1]. \tag{19}
\]

We then find

\[
\mathbb{Q}(R_m^+(X) \leq k) = \mathbb{E}_m(x1_{(R_m^+(X) \leq k)})
\]

\[
= \mathbb{E}_m(x1_{(F_m^{-1}(U_m) \leq k)})
\]

\[
\geq \mathbb{E}_m(x1_{(F_m^{-1}(U_m) \leq k)}) \quad \text{by } \text{(19)}
\]

\[
= \mathbb{Q}(R_m^+(Y) \leq k).
\]

So \( d^{\mathbb{Q}_m}(R_m^+(X)) \leq d^{\mathbb{Q}_m}(R_m^+(Y)) \) as claimed. \( \square \)
Our next theorem will show that the notion of rearrangement can be generalized to situations when there are more than two probability measures under consideration.

**Theorem 2.14** (Monotone mutual fund theorem for complete markets). Let $(\Omega, \mathcal{F}, \mathbb{P}_0)$ be a standard probability space equipped with $n$ equivalent measures $\mathbb{P}_i$ ($1 \leq i \leq n$). Let $I = [0, 1)$. Let $\preceq_i$ ($1 \leq i \leq n$) be monotonic relations on the set of probability distributions on $\mathbb{R}$. Write $\text{sign}_i$ for the sign of $\preceq_i$. There exists a mapping $R : L^0(\Omega \times I) \to L^0(\Omega \times I)$, which we will call rearrangement, with the following properties.

(i) Rearrangements do not change $\mathbb{P}_0$ distributions:

$$d\mathbb{P}_0(X) = d\mathbb{P}_0(R(X)).$$

(ii) Rearrangements increase or decrease $\mathbb{P}_i$ distributions according to the sign of $\preceq_i$:

$$d\mathbb{P}_i((\text{sign}_i)X) \preceq_i d\mathbb{P}_i((\text{sign}_i)R(X)), \quad 1 \leq i \leq n.$$

(iii) Let $q$ denote the vector of $n$ Radon-Nikodym derivatives $\frac{d\mathbb{P}_i}{d\mathbb{P}_0}$. Define $\succeq_i$ on $\mathbb{R}^n$ by $x \succeq_i y$ if $(\text{sign}_i)x_i \leq (\text{sign}_i)y_i$ for all components $i$, and hence define $\prec$ on $\mathbb{R}^n$. Then $R(X)$ satisfies

$$R(X)(\omega) \preceq_i R(X)(\omega') \quad \text{if} \quad q(\omega) \prec q(\omega').$$

**Proof.** By Theorem 2.4, we only need consider the case when $\Omega = (0, \infty)^n$ equipped with a measure $\mu$ satisfying

$$E_\mu(x_i) = 1$$

for each coordinate function $x_i$.

Given an integer $j$, $1 \leq j \leq n$, we define a random $n - 1$ vector $\hat{q}_j(\omega)$ to be all the components of $q$ except the $j$th. We write $\hat{\mu}_j$ for the measure induced on $(0, \infty)^{n-1}$ by $\hat{q}_j$. We write $q_j$ for the $j$th component of $q$, and write $\mu_j$ for the measure on $(0, \infty)$ induced by $q_j$.

Given a random variable $X$ on $(0, \infty)^n \times [0, 1)$ and a value $Q \in (0, \infty)^{n-1}$ we may define $X_{j,Q} : (0, \infty) \times [0, 1) \to \mathbb{R}$ by

$$X_{j,Q}(x, y) = X(Q \oplus_j x, y),$$

where $Q \oplus_j x$ is the vector obtained by inserting a new component with value $x$ at the $j$th index of the vector $Q$. $X_{j,Q}$ is $\hat{\mu}_j$-almost surely measurable.

Let $y$ denote the final coordinate function on $(0, \infty)^n \times [0, 1)$. We define *conditional rearrangements* $R^+_j$ and $R^-_j$ as follows

$$R^+_j(X)(\omega) := R^\pm_{\mu_j}(X_{j,\hat{q}_j}(\omega)) (q_j(\omega), y(\omega)).$$
We define $R_j = R_j^+$ if sign $j = 1$, and $R_j = R_j^-$ otherwise. Since $X_j, \hat{q}$ is $\hat{\mu}_j$-almost surely measurable, $R_j^\pm$ is well-defined mod 0.

We need to check that $R_j^\pm$ is measurable. We note that

$$F_{X_j, \hat{q}_j(\omega)}^{-1}(p) = \inf\{z \in \mathbb{R} | F_{X_j, \hat{q}_j(\omega)}(z) \geq p\}$$

$$= \inf\{z \in \mathbb{Q} | F_{X_j, \hat{q}_j(\omega)}(z) \geq p\}$$

using the monotonicity of distribution functions. Define

$$f(z, \omega, p) = \begin{cases} 
  z & F_{X_j, \hat{q}_j(\omega)}(z) \geq p \\
  \infty & \text{otherwise}
\end{cases}$$

It is obvious from chasing through the definitions that $f$ is measurable. The infimum of a countable sequence of measurable functions is measurable. Hence $F_{X_j, \hat{q}_j(\omega)}^{-1}(p)$ is measurable as a function of the pair $(\omega, p)$. By definition

$$R_{\mu_j(X_j, \hat{q}_j(\omega))}(x, y) = F_{X_j, \hat{q}_j(\omega)}^{-1}(U_{\mu_j}(x, y)),$$

so this quantity is measurable as a function of $(\omega, x, y)$. The measurability of $R_j^\pm(X)$ is now immediate.

We inductively define $R^*_0(X) = X$ and $R^*_j(X) = R_j(R^*_{j-1}(X))$ for $1 \leq j \leq n$.

We define $R(X) = R^*_n(X)$.

Let us suppose as induction hypothesis that for some $1 \leq i < n$

$$d^{\hat{q}_j}(X) = d^{\hat{q}_j}(R_j^*(X)) \quad \text{if } i = 0 \text{ or } i > j$$

$$d^{\hat{q}_j}((\text{sign } j)X) \preceq d^{\hat{q}_j}(R_j^*((\text{sign } j)X)) \quad \text{otherwise.}$$

We may then apply equations (12), (13), (14) and (16) to find

$$d^{\hat{q}_j}(X \mid \hat{q}_j+1) = d^{\hat{q}_j}(R_{j+1}^*(X) \mid \hat{q}_j+1) \quad \text{if } i = 0 \text{ or } i > j + 1$$

$$d^{\hat{q}_j}(R_{j+1}^*((\text{sign } j)X) \mid \hat{q}_j+1) \preceq d^{\hat{q}_j}(R_{j+1}^*((\text{sign } j)X) \mid \hat{q}_j+1) \quad \text{otherwise}$$

(21)

Applying Lemma 2.16 below, we may deduce from equations (21) that our induction hypothesis (20) will also hold when $j \to j + 1$. We deduce that (20) holds for $0 \leq j \leq n$. This establishes properties (i) and (ii) of $R(X)$.

For each $i \ (0 \leq i \leq n)$, define a partial order $\preceq_i$ on $\mathbb{R}^n$ by

$$x \preceq_i y \iff \begin{cases} 
  (\text{sign } j)x_j \leq (\text{sign } j)y_j & 1 \leq j \leq i \\
  x_j = y_j & i < j \leq n.
\end{cases}$$

We suppose as induction hypothesis that for some $1 \leq i \leq n - 1$,

$$R^*_{i-1}(X)(\omega) \leq R^*_{i-1}(X)(\omega') \quad \text{if } q(\omega) \prec_i q(\omega').$$

(22)
Write \( q^a(\omega) \) for the vector containing the first \((i-1)\) components of \( q(\omega) \), \( q^b(\omega) \) for the \(i\)th component of \( q(\omega) \) and \( q^c(\omega) \) for the remaining components. So \( q(\omega) = q^a(\omega) \oplus q^b(\omega) \oplus q^c(\omega) \).

Suppose that \( q(\omega) \prec_{i+1} q(\omega') \) then \( q^a(\omega) \leq q^a(\omega') \), \( q^b(\omega) \leq q^b(\omega') \), \( q^c(\omega) = q^c(\omega') \). We also have either

(a) \( q^a(\omega) \prec q^a(\omega') \) and \( q^b(\omega) = q^b(\omega') \)

(b) \( q^a(\omega) = q^a(\omega') \) and \( q^b(\omega) < q^b(\omega') \)

(c) \( q^a(\omega) \prec q^a(\omega') \) and \( q^b(\omega) < q^b(\omega') \)

In case (a) our induction hypothesis \( \circledast \) tells us that

\[
R^*_{i-1}(X)(\omega) \leq R^*_{i-1}(X)(\omega').
\]

Hence by property \( \circledast \) of rearrangement

\[
R^*_i(X)(\omega) = R_i(R^*_{i-1}(X))(\omega) \leq R_i(R^*_{i-1}(X))(\omega') = R^*_i(X)(\omega').
\]

In case (b), we may apply \( \circledast \) to the rearrangement \( R_i \) of the random variable \( R^*_{i-1}(X) \) to find that \( R^*_i(X)(\omega) \leq R^*_i(X)(\omega') \). In case (c) we apply our results for case (a) and case (b) in succession and use the transitivity of \( \leq \) to again find that \( R^*_i(X)(\omega) \leq R^*_i(X)(\omega') \). Thus \( \circledast \) remains true when we change \((i-1) \rightarrow i\).

The induction hypothesis \( \circledast \) is trivially true when \( i = 1 \), so claim (iii) follows.

This theorem gives a general structural theorem about optimal investments in complete markets containing a casino. So long as the optimality criterion and any pricing or risk constraints are monotonic in some measures \( \mathbb{P}_i \), we can restrict our attention to strategies that lie in the image of \( R \). We interpret this as a mutual fund theorem since it says that for a general class of optimization problems we can safely restrict attention to a subset of the random variables available in the market.

The assumption that there is a casino can be dropped in many cases since, as one might intuitively expect, one often doesn’t take any real advantage of the casino. This is formalized in the next corollary.

**Corollary 2.15.** Let \( (\Omega, \mathcal{F}, \mathbb{P}_i) \) \( (1 \leq i \leq n) \) be as in the previous Theorem \( \circledast \) We can find a map \( \hat{R} : L^0(\Omega) \rightarrow L^0(\Omega) \) which shares properties (i), (ii) and (iii) described in Theorem \( \circledast \) so long as either:

(a) \( \mathbb{P}_0 \) is atomless and \( n = 1 \),

(b) for some \( j \), the distribution of \( \frac{d\mathbb{P}_i}{d\mathbb{P}_0} \) conditioned on the value of all the other Radon–Nikodym derivatives is almost surely continuous. Note that the theory of conditional distributions detailed in \( \circledast \) ensures that this conditional distribution exists. In this case \( \hat{R} \) can be assumed to depend only on the value of \( q \).
Proof. Let \( X \in L^0(\Omega) \). We define \( \tilde{X} \in L^0(\Omega \times [0,1]) \) by \( \tilde{X}(\omega, y) = X(\omega) \). This will satisfy \( dP_i(X) = dP_i(\tilde{X}) \) for all \( i \).

Consider case (b) of our claim. By property (17) of rearrangement, \( R_j \), and hence \( R \), only depends upon \( q \). So we may write \( R(\tilde{X}) = \tilde{X}(q) \) for some \( \tilde{X} \). We define \( \tilde{R}(X) = \tilde{X}(q) \), and it will satisfy all the desired properties.

Now consider case (a) of our claim. Let us write \( \{x_n\} \) for the countable set of discontinuities of \( F_{q_1} \). We define a set \( \Delta_n \) by

\[
\Delta_n := (q_1)^{-1}(x_n).
\]

Since the probability space is standard and atomless, there is a mod 0 isomorphism \( \phi_n \) from the set \( \Delta_n \) to the set

\[
\{x_n\} \times I.
\]

We write \( \Delta = \bigcup \Delta_n \). Property (17) tells us that the rearrangement \( R(\tilde{X})(\omega, y) \) only depends upon \( y \) if \( x \in \Omega \setminus \Delta \). So we may define a function \( \hat{X} \) on \( (0,\infty) \setminus \{x_n\} \) by \( \hat{X}(q_1) = R(\tilde{X}) \) on \( \Omega \setminus \Delta \). We now define

\[
\tilde{R}(X)(\omega) = \begin{cases} 
\hat{X}(q_1(\omega)) & \omega \in \Omega \setminus \Delta \\
R(\tilde{X})(\phi(X)) & \text{otherwise}.
\end{cases}
\]

Since each \( \phi_n \) is a mod 0 isomorphism on \( \Delta_n \) and preserves the Radon-Nikodym derivatives, we see that

\[
dP_i(\tilde{R}(X)) = dP_i(R(\tilde{X}))
\]

for \( i = 0,1 \). The result follows. \( \square \)

We finish by proving the following lemma which was used during the proof of Theorem 2.14.

**Lemma 2.16.** If \((\Omega, F, \mathbb{P})\) is a probability space, \( X \) and \( Y \) are real random variables and \( Z \) is an \( \mathbb{R}^k \) random variable satisfying

\[
d\mathbb{P}(X \mid Z) \leq d\mathbb{P}(Y \mid Z)
\]

then \( d\mathbb{P}(X) \leq d\mathbb{P}(Y) \).

**Proof.**

\[
\mathbb{P}(X \leq k) = \int_{\mathbb{R}^k} \mathbb{P}(X \leq k \mid Z) \, dZ \leq \int_{\mathbb{R}^k} \mathbb{P}(Y \leq k \mid Z) \, dZ = \mathbb{P}(Y \leq k).
\]

\( \square \)
3 Continuous Time Markets

Let us extend our definitions of markets to the multi-period setting.

**Definition 3.1.** A multi-period market consists of

(i) A filtered probability space \((\Omega, \mathcal{F}_t, \mathbb{P})\) where \(t \in \mathcal{T} \subseteq [0, T]\) for some index set \(\mathcal{T}\) containing both 0 and \(T\). We write \(\mathcal{F} = \mathcal{F}_T\). We require \(\mathcal{F}_0 = \{\emptyset, \Omega\}\).

(ii) For each \(X \in L^0(\Omega; \mathbb{R})\), an \(\mathcal{F}_t\) adapted process \(c_t(X)\) defined for \(t \in \mathcal{T} \setminus T\).

Random variables \(X \in L^0(\Omega, \mathcal{F}_T; \mathbb{R})\) are interpreted as contracts which have payoff \(X\) at time \(T\). The cost of this contract at time \(t\) is \(c_t(X)\).

We note that this is deliberately bare-bones definition of a market. In practice would want to impose additional conditions on the \(c_t\). For example, one would normally wish to forbid arbitrage and to impose “the usual conditions” on the filtered probability space.

**Definition 3.2.** A filtration isomorphism of filtered spaces \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) where \(t \in \mathcal{T}\) for some index set \(\mathcal{T}\) is a mod 0 isomorphism for \(\mathcal{F}\) which is also a mod 0 isomorphism for each \(\mathcal{F}_t\). An isomorphism of multi-period markets is a filtration isomorphism that preserves the cost functions.

Given a one period market \(((\Omega, \mathcal{F}, \mathbb{P}), c)\) we can trivially define a filtration \(\mathcal{F}_0 = \{\emptyset, \Omega\}\), \(\mathcal{F}_1 = \mathcal{F}\) indexed by \(\{0, 1\}\) and we may define \(c_0 = c\). Hence we can define a multi-period market in a canonical fashion from a one-period market. The notion of isomorphism is preserved. In this sense, our definition of multi-period markets and their isomorphisms is a generalization of the corresponding notions for one-period market.

**Definition 3.3** (Exchange market). Let \((\Omega, \mathcal{F}_t, \mathbb{P})\) be an \(n\)-dimensional Wiener space, that is the probability space generated by the \(n\)-dimensional Brownian motion \(W_t\). Let \(X_t\) be an \(n\)-dimensional stochastic processes defined by a stochastic differential equations of the form

\[
\frac{dX_t}{dt} = \mu(X_t, t) dt + \sigma(X_t, t) dW_t.
\]

Here \(\mu\) is an \(\mathbb{R}^n\)-vector valued function and \(\sigma\) is an invertible-matrix valued function. We assume the coefficients \(\mu\) and \(\sigma\) are sufficiently well-behaved for the solution of the equation to be well-defined on \([0, T]\). The components, \(X^i_t\), of the vector \(X_t\) are intended to model the prices of \(n\)-assets.

The exchange market for (23) with risk free rate \(r\) over a time period \([0, T]\) is given by defining \(c_t\) for \(t \in [0, T]\) by

\[
c_t(X) = \begin{cases} 
\alpha_0 e^{-r(T-t)} + \sum_{i=1}^{n} \alpha_i X^i_t & \text{if } X = \alpha_0 + \sum_{i=1}^{n} \alpha_i X^i_T \\
\infty & \text{otherwise.}
\end{cases}
\]

This is well-defined so long as we assume that \(X^i_T\) are linearly independent random variables. This will be the case in all situations of interest (see Remark 3.13 later).
The market defined above is called an exchange market because it models the basic assets that can be purchased directly on an exchange, but does not take into account the possibility of replicating payoffs via hedging. The next definition does take this into account.

**Definition 3.4** (Superhedging market). The *superhedging market* for \((23)\) with risk free rate \(r\) over a time period \([0, T]\) is given by defining \(c_t(X)\) to be the infimum of the cost at time \(t\) of self-financing trading strategies that superhedge \(X\). See [7] for a definition of a self-financing trading strategy. A self-financing trading strategy superhedges \(X \in L^0(\Omega; \mathcal{F}_T)\) if the final payoff of the strategy is always greater than or equal to \(X\).

Thus the superhedging market represents the effective market of derivatives that a trader can achieve given the exchange market.

**Definition 3.5.** A continuous time market \((\Omega, \mathcal{F}_t, \mathbb{P}), c_t)\) on \([0, T]\) is called a *continuous time complete market with risk free rate \(r\)* if there exists a measure \(Q\) equivalent to \(\mathbb{P}\) with

\[
c_t(X) = e^{-r(T-t)} \mathbb{E}_Q(X | \mathcal{F}_t)
\]

for \(Q\)-integrable random variables \(X\) and equal to \(\infty\) otherwise. We follow our usual conventions on expectations to allow \(-\infty\) when the positive part of an expectation is finite and the negative part is infinite.

There is a standard construction of such a \(Q\)-measure from the SDE \((23)\) of Definition \(3.3\) subject to sufficient regularity assumptions on the coefficients. We review the construction as we will require the formulae later.

We assume that further to the assumptions of Definition \(3.3\) we may define a process \(Z_t\) by

\[
Z_t = \int_0^t (\sigma^{-1}(rX_s - \mu)) \cdot dW_s
\]

(25)

where \(\cdot\) denotes the usual inner product of vectors. We have suppressed the parameters \((X_s, s)\) of the functions \(\sigma_t\) and \(\mu_t\) to keep our expressions readable, and will do this throughout this section. It is immediate from this definition that \(Z_t\) is a continuous local martingale. We then define \(q_t\) to be the Doléans-Dade exponential of \(Z_t\), i.e.

\[
q_t = \exp \left( Z_t - \frac{1}{2} \langle Z, Z \rangle_t \right)
\]

(26)

so that \(q\) is a positive process and a local \(\mathbb{P}\)-martingale. We further assume that \(q_t\) is a \(\mathbb{P}\)-martingale.

In these circumstances we may apply Girsanov’s theorem to find that the measure \(Q\) defined by

\[
Q(A) = \mathbb{E}_\mathbb{P}(q_T A)
\]

(27)

for a measurable set \(A \subset \Omega\) is an equivalent measure to \(\mathbb{P}\) and satisfies

\[
q_t = \mathbb{E}_\mathbb{P}(q_T | \mathcal{F}_t).
\]
Furthermore if we define a vector valued process $W^Q$ by

$$W^Q_t = W_t - [W, Z]_t$$

then this will be a $Q$-Martingale. Since the quadratic covariation is invariant under a change to an equivalent measure, $W^Q$ will be a standard $n$-dimensional $Q$-Brownian motion. We now rewrite (23) in terms of these $W^Q_t$.

\[
dX_t = \mu(X_t, t) \, dt + \sigma(X_t, t) \, dW^Q_t + d[\sigma(X, t)W, s]
\]

We have used the definition (25) to get from the first line to the second. Now we consider the discounted process $e^{-rt}X_t$ it will satisfy

\[
d(e^{-rt}X)_t = e^{-rt}dX_t - re^{-rt}X_t \, dt
\]

So long as $X$ is absolutely $Q$-integrable, the discounted price process is a martingale, hence

$$X_t = e^{-r(T-t)}E_Q(X_T). \quad (28)$$

We have constructed a complete market with prices given by the measure $Q$. These coincide with the prices given by the original SDE (23).

**Definition 3.6.** The complete market associated with the SDE (23) and with risk free rate $r$ and time horizon $T$ is the complete market determined by defining $Q$ via (27), where $q_t$ is given by (26) and $Z_t$ is given by (25). As noted in the text, one must make a number of assumptions on the SDE for this market to be well-defined.

The well-known theory of hedging in markets defined by (23) and described in [7] tells us that the superhedging market associated with the SDE (23) is equal to the complete market associated with the SDE. This statement is subject to the assumptions required to define the complete market.

We differ slightly in our presentation from [7] in that they discuss replication and we consider superhedging. This is why we are willing to ascribe a cost of $-\infty$ to some $X \in L^0(\Omega, \mathcal{F}_T)$ whereas if one insists on replication, $X$ must be absolutely integrable. The definition of the superhedging market associated to a given market can be applied equally well to incomplete markets where there will be a more meaningful difference between replication and superhedging. This is why we prefer to think in terms of superhedging, and in this we are influenced by the presentation of [16].

**Example 3.7.** Given a vector $X$ let us write $\text{diag}(X)$ for the diagonal matrix with entries given by the elements of $X$. $n$-dimensional geometric Brownian motion is the SDE given by (23) with

$$\mu(X, t) = \text{diag}(X_t)\tilde{\mu}$$
\[ \sigma(X_t, t) = \text{diag}(X_t) \tilde{\sigma} \]

for a fixed vector \( \tilde{\mu} \) called the drift vector and a fixed invertible, symmetric matrix \( \tilde{\sigma} \) called the covariance matrix. The complete market associated with \( n \)-dimensional geometric Brownian motion is called the Black–Scholes–Merton market.

**Example 3.8.** A *Bachelier market with time dependent drift* is a complete market given by the SDE (23) with

\[ \mu(X_t, t) = r X_t + \tilde{\mu}(t) \]

\[ \sigma(X_t, t) = \tilde{\sigma} \]

for a deterministic vector valued function \( \tilde{\mu}(t) \) called the drift vector, and a fixed invertible, symmetric matrix \( \tilde{\sigma} \) called the covariance matrix. Note that the form of \( \mu \) is determined by requiring that the discounted price process \( e^{-rt}X_t \) follows arithmetic Brownian motion.

Our aim in this section will be to define easily computed invariants of complete continuous time markets and to classify those markets whose higher-order invariants vanish.

As a first step, let us ignore the cost functions on our markets and give some basic definitions and results on the mod 0 filtration isomorphism classes of filtered probability spaces.

**Definition 3.9.** Let \( X_t \) be an adapted process on a filtered probability space with measure \( \mathbb{P} \). We define the *drift* of \( X_t \), if it exists, to be the stochastic process given by

\[ \text{drift}_\mathbb{P}(X)_t = \lim_{h \to 0} \mathbb{E} \left( \frac{X_{t+h} - X_t}{h} \right). \]

We note that this definition is manifestly invariant under filtration isomorphisms (see [1] for a formal discussion of the precise meaning of an invariant definition).

Suppose that \( X_t \) is a continuous stochastic process satisfying the stochastic differential equation

\[ X_t = X_0 + \int_0^t \mu(X_s, s) \, ds + \int_0^t \sigma(X_s, s) \, dW_s \]

for continuous functions \( \mu \) and \( \sigma \) and a Brownian motion \( W_t \), then it follows from the fundamental theorem of calculus that

\[ \text{drift}_\mathbb{P}(X)_t = \mu(X_t, s). \]

Thus our definition of drift should cause no confusion with the usual definition of the drift term of a stochastic differential equation. The advantage of our definition is its invariance.
Given two adapted stochastic processes $X_t$ and $Y_t$ we may compute their quadratic covariation $[X,Y]_t$ (if this exists). The definition of quadratic covariation is again manifestly invariant. If we define two filtered spaces to be mod 0 equivalent if they admit mod-0-isomorphic equivalent measures, then quadratic covariation is in fact invariant under mod 0 equivalence.

**Definition 3.10.** The $n$-dimensional *Wiener space* on $[0,T]$ is the filtered probability space generated by $n$ independent standard Brownian motions on $[0,T]$. A filtered probability space is called a Wiener space if it is filtration isomorphic to an $n$-dimensional Wiener space.

An important observation is that martingales and local martingales are preserved under filtration isomorphisms. We recall the Martingale Representation Theorem, a result about Wiener spaces we will use heavily.

**Theorem 3.11** (Martingale Representation Theorem). Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be the $n$-dimensional Wiener space defined for $t \in [0,T]$. Any martingale $X_t$ can be written as

$$X_t = X_0 + \sum_{i=1}^{n} \int_0^t \alpha_i^s dW_i^s$$

for some predictable processes $\alpha_i^t$. If the martingale is square integrable, this representation is unique.

**Corollary 3.12.** The dimension of a Wiener space is invariantly defined.

**Proof.** Suppose for a contradiction that $n$ dimensional Wiener space, $\Omega_n$, is isomorphic to $m$ dimensional Wiener space with $m > n$. Using this isomorphism we may find $m$ independent standard Brownian motions on $\Omega_n$, $\tilde{W}_j^t$ ($1 \leq j \leq m$). By the Martingale representation theorem, there are unique, predictable processes $\alpha_i^j$ ($1 \leq i \leq n$, $1 \leq i \leq m$) such that

$$\tilde{W}_j^t = \int_0^t \sum_{a=1}^{n} \alpha_{aj}^s dW_i^a.$$

Let $\alpha_t$ be the $n \times m$ matrix with components $\alpha^{ij}$ and let $\text{id}_m$ denote the identity matrix of dimension $m$. We compute the quadratic covariation matrix of each side in the above expression to obtain

$$\text{id}_m = (\alpha_t)(\alpha_t)^\top.$$

Since $\alpha_t$ has rank less than or equal to $n$, and $\text{id}_m$ has rank $m$ we obtain the desired contradiction.

**Remark 3.13.** We mentioned earlier that we require the components of $X_T$ to be linearly independent random variables in order for the exchange market associated with (23) to be well-defined. We have from (28) that subject to sufficient assumptions on the SDE, we can define a random variable $q_T$ such that

$$X_t = e^{-r(T-t)}\mathbb{E}_P(q_T X_T \mid \mathcal{F}_t).$$
It follows that if the components of \( \mathbf{X}_T \) were linearly dependent, the components of \( \mathbf{X}_t \) would be too with the same coefficients. This would then imply that the quadratic covariation matrix of \( \mathbf{X} \) at time \( t \) would have rank less than \( n \). However, from (23) we see that the quadratic covariation matrix of \( \mathbf{X} \) is given by \( \mathbf{\sigma}^T \mathbf{\sigma} \) which has rank \( n \). So the components of \( \mathbf{X}_T \) will be linearly independent whenever we can define the complete market associated with (23).

We now wish to define invariants of continuous time complete markets. We want these invariants to be easy to calculate in that they should simply involve algebraic manipulation and possibly differentiation of the coefficients of (23). We call such invariants local invariants.

To find such an invariant we first note that there we have the following invariantly defined process on any continuous time complete market

\[
Q_t := E_P \left( \frac{dQ}{dP} \bigg| F_t \right).
\]

For the superhedging market described in Definition 3.4, this will coincide with the function \( q_t \) defined by (26). \( Q_t \) is invariantly defined but is not a local invariant. Intuitively this is because the distribution of \( Q_t \) depends upon the distribution of \( Q \) at earlier times, and one can easily use this intuition to give examples of markets based on diffusion processes where the coefficients of the diffusion and their derivatives are equal at time \( t \) but where the \( Q_t \) differ.

**Definition 3.14.** The absolute market price of risk (AMPR) of a continuous time complete market is given by

\[
\text{AMPR}_t = \sqrt{\mathbf{\sigma}^{-1} \left( r \mathbf{X}_t - \mathbf{\mu} \right)}.
\]

(We will see in a moment that this is a square root of a non-negative quantity.)

**Lemma 3.15.** For the complete market associated with the SDE (23) we have

\[
\text{AMPR}_t = | \mathbf{\sigma}^{-1} (r \mathbf{X}_t - \mathbf{\mu}) |\]

where \(| \cdot |\) denotes the Euclidean norm of a vector. This shows that \( \text{AMPR}_t \) is a local invariant and moreover that in the 1-dimensional Black–Scholes–Merton case

\[
\text{AMPR}_t = \left| \frac{r - \tilde{\mu}}{\tilde{\sigma}} \right|
\]

This justifies the name absolute market price of risk.

For all continuous time markets

\[
\text{drift}_P(Q_t) = 0.
\]

Moreover

\[
-2 \mathbf{Q}_t^2 \text{drift}_P(\log Q_t) = [Q, Q]_t \geq 0.
\]

This shows that our definition \( \text{AMPR}_t \) does indeed express AMPR square root of a non-negative quantity.
Proof. We begin by considering the statements about general continuous time markets.

\[ 1 = \mathbb{E}_Q(1 \mid \mathcal{F}_t) = \mathbb{E}_P \left( \frac{dQ}{dP} \mid \mathcal{F}_t \right) = \mathbb{E}_P(Q_t) = \mathbb{E}_P(\mid Q_t \mid) \]

It follows that \( Q_t \) is a \( P \)-martingale and hence has \( P \)-drift equal to 0.

By Itô’s lemma,

\[ d(\log Q)_t = \frac{1}{Q} dQ_t - \frac{1}{2} Q_t^2 [Q, Q]_t dt \]

We can read off the drift:

\[ \text{drift}_P(\log Q)_t = -\frac{1}{2} Q_t^2 [Q, Q]_t. \]

We now specialise to complete markets associated with SDEs.

By equation (26) we see that

\[ \log Q_t = \int_0^t \sigma^{-1}(rX_s - \mu) dZ_s. \]

(32)

Recall that \( Z_t \) is a local martingale, it has drift zero. From the definition of \( Z_t \)

we compute that

\[ [Z, Z]_t = \int_0^t |\sigma^{-1}(rX_s - \mu)|^2 dZ_s. \]

So taking drifts of either side of (32) we find

\[ \text{drift}_P(\log Q)_t = -\frac{1}{2} |\sigma^{-1}(rX_s - \mu)|^2. \]

\[ \square \]

Remark 3.16. Given that \( Q_t \) is an invariant, one might ask whether we need a formula as complex as (29) to define a local invariant. To see why we use such a complex formula, note that (30) shows that the drift of \( Q_t \) is not a useful invariant. Equation (31) shows that \( Q_t \) can be calculated from AMPR\(_t\) and the quadratic variation of \( Q_t \). Since \( Q_t \) is not a local invariant, but AMPR\(_t\) is a local invariant, we conclude that the quadratic variation cannot be a local invariant.

Now we have found one local invariant, it is easy to find more local invariants by taking drifts of previously discovered local invariants and their quadratic covariations. This should make it reasonably simple in practice to determine whether two complete markets associated with SDEs are isomorphic.

To test the usefulness of our invariants we prove the following.
Theorem 3.17 (The test case). Let $M$ be a continuous time complete market with risk free rate $r$, time period $T$ based on a Wiener space of dimension $n$ and with $\text{AMPR}$ given by

$$\text{AMPR}_t = A(t) \geq 0$$

for a bounded measurable function of time $A(t)$. Suppose that the process $Q_t$ is continuous. In these circumstances $M$ is isomorphic to the Bachelier market of Example 3.8 with $\tilde{\mu} = A(t)e_1$ $\tilde{\sigma} = \text{id}_n$ and $X_0 = 0$. We recall that $\{e_i\}$ is the standard basis for $\mathbb{R}^i$ and $\text{id}_n$ is the identity matrix. We will call markets of this form canonical Bachelier markets.

Proof. Given such a complete market, we define

$$\tilde{Z}_t = \log Q_t + \frac{1}{2} \int_0^t A(s)^2 \, ds.$$  \hfill (33)

Taking the drift of this expression we find

$$\text{drift}_P \tilde{Z}_t = \text{drift}_P (\log Q)_t + \frac{1}{2} A(t)^2 = -\frac{1}{2} \text{AMPR}_t^2 + \frac{1}{2} A(t)^2 = 0.$$  

So $\tilde{Z}_t$ is a continuous local martingale. We now define

$$W^1_t = - \int_0^t \frac{1}{A(s)} \, d\tilde{Z}_s.$$  \hfill (34)

$W^1_t$ is a continuous local martingale by our assumptions on $A(t)$. We compute its quadratic variation

$$[W^1, W^1]_t = \int_0^t \frac{1}{A(s)^2} [\tilde{Z}, \tilde{Z}]_s \, ds, \quad \text{by (34),}$$

$$= \int_0^t \frac{1}{A(s)^2} [\log Q, \log Q]_s \, ds, \quad \text{by (33),}$$

$$= \int_0^t \frac{1}{Q^2} A(s)^2 [Q, Q]_s \, ds, \quad \text{by Itô’s Lemma,}$$

$$= \int_0^t - \frac{2}{A(s)^2} \text{drift}_P (\log Q)_s \, ds, \quad \text{by (31),}$$

$$= \int_0^t ds, \quad \text{by (29)}$$

$$= t.$$  

It follows by Lévy’s characterisation of Brownian motion that $W^1_t$ is Brownian motion.
We may now find additional Brownian motions $\tilde{W}_t^i$ for $2 \leq i \leq n$ such that the vector process $\tilde{W}_t$ with components $\tilde{W}_t^i$ is a standard $n$-dimensional Brownian motion.

To see this we use the fact that $\Omega$ is assumed to be an $n$-dimensional Wiener space so admits an $n$-dimensional standard Brownian motion $\hat{W}_t$. Using the martingale representation theorem, we may write $\tilde{W}_1^i = \int_0^t \alpha_s \cdot d\hat{W}_s$ for a predictable vector process $\alpha_t$ of norm 1. Given a vector $v \in \mathbb{R}^n$ of norm 1 we define a number $i_k$ for each $2 \leq k \leq n$ by

$$i_k = \inf \{ i \mid \dim (v, e_1, e_2, \ldots, e_i) \geq k \}.$$

Then \{v, e_{i_2}, e_{i_3}, \ldots, e_{i_n}\} is an basis of $\mathbb{R}^n$. Applying the Gram–Schmidt process to this basis yields an orthonormal basis $\{\alpha_t^j\}$. We now define

$$\tilde{W}_t^i = \int_0^t \alpha_s^i \cdot dW_s.$$

The process $\tilde{W}_t$ is a continuous semi-martingale and its quadratic covariation matrix has components

$$[\tilde{W}_t^i, \tilde{W}_t^j]_t = \int_0^t \alpha_s^i \cdot \alpha_s^j = t \delta^i_j.$$

Hence by Lévy’s characterisation this is indeed $n$-dimensional Brownian motion.

We now define a stochastic processes $X_t$ by

$$dX_t = (rX_t + A(t)e_1)dt + d\tilde{W}_t.$$

Here we use our the boundedness and measurability of $A$ to ensure existence and uniqueness of the solution to this SDE. The continuous time market associated to (35) has $Z_t$ given by formula (25) so

$$Z_t = -\int_0^t A(s) d\tilde{W}_s^1.$$

In particular

$$[Z, Z]_t = \int_0^t A(s)^2 dt.$$

So the equation (26) becomes

$$\log(q_t) = Z_T - \frac{1}{2} \int_0^t A(s)^2 dt.$$ 

So we find

$$d(\log q_t) = dZ_T - \frac{1}{2} A(t)^2 dt$$

$$= -A(t) d\tilde{W}_t^1 - \frac{1}{2} A(t)^2 dt$$

by (36).
On the other hand we compute that
\[
d(log Q_t) = d\tilde{Z}_t - \frac{1}{2}A(t)^2 \, dt \tag{39}
\]
\[
= -A(t)d\tilde{W}_t^1 - \frac{1}{2}A(t)^2 \, dt \tag{40}
\]
by (33) (39).

Since we also have \( q_0 = Q_0 = 1 \), we see that \( Q_t = q_t \).

Prices in \( M \) are, by definition, given by
\[
c_t(X) = \mathbb{E}(e^{-r(T-t)}QX \mid F_t) = \mathbb{E}(e^{-r(T-t)}Q_tX_t).
\]
Prices in the complete market associated with \( \tilde{Q} \) are given by the same formulae with \( Q \) replaced by \( q \). Hence the costs are same in both markets, showing that we have identified a market isomorphism.

We have called Theorem 3.17 “the test case” as it is an analogous result to the theorem in differential geometry that a Riemannian manifold with vanishing curvature is flat. This latter result is called “the test case” in [18].

It follows from Lemma 3.15 that AMPR\(_t\) is constant in Black–Scholes–Merton models, and so these are isomorphic to canonical Bachelier markets. In general, stochastic control problems are hard to solve explicitly. Our definition of the Black–Scholes–Merton model involves the ugly term \( \text{diag}(X_t) \) so it seems a non-trivial model. It therefore seems surprising at first that Merton was able to solve portfolio optimization problems in the Black–Scholes–Merton model [13]. Our result shows that the Black–Scholes–Merton model is simply a canonical Bachelier model in disguise, and this explains its surprising tractability. Note that the assumption of a continuous time model and the theory of delta hedging plays an essential role in constructing this isomorphism. Discrete time versions of the Black–Scholes–Merton model cannot be expected to be tractable. Note also that our approach shows that the problems such as those considered by Merton in [13] are best tackled by solving the problem in the canonical Bachelier market.

A financially significant consequence of our classification is the following.

**Theorem 3.18 (Continuous time one-mutual-fund theorem).** Let \( M \) be a complete continuous time market with continuous \( q_t \) and with deterministic, bounded absolute market price of risk. Let \( X_i^t \) for \( 1 \leq i \leq n \) be a collection of square integrable stochastic processes representing \( n \) basic assets, then there exists \( n \) predictable real valued processes \( \alpha_i^t \) such that any invariant, non-empty, convex set of martingales contains an element which can be replicated by a continuous time trading strategy using only the asset and the portfolio consisting of \( \alpha_i^t \) units of asset \( X_i^t \). We note that a convex set of martingales can be interpreted as a convex set of self-financing trading strategies or as a convex set of derivative securities.

In complete markets arising from SDEs of the form (23) which also have a deterministic absolute market price of risk, we may take the portfolio \( \alpha \) with components \( \alpha_i^t \) to be given by the vector
\[
(\sigma\sigma^T)^{-1}(\nu X_t - \mu).
\]
Proof. Without loss of generality our market is a canonical Bachelier market. Let \( A \) be an invariant convex set of martingales. Let \( Y \) be an element of \( A \). By the martingale representation theorem

\[
Y_t = Y_0 + \sum_{i=1}^{n} \int_0^t a_i^s \, dW_s^i
\]

for some predictable processes \( a_i^s \). By invariance of \( A \) we see that

\[
Y_t = Y_0 + \int_0^t a_1^s \, dW_s^1 - \sum_{i=2}^{n} \int_0^t a_i^s \, dW_s^i
\]

is also in \( A \), as flipping the signs of the Brownian motions \( W_k^i \) for \( 2 \leq k \leq n \) induces an isomorphism of the canonical Bachelier model.

By convexity of \( A \)

\[
Y_t = Y_0 + \int_0^t a_1^s \, dW_s^1
\]

lies in \( A \). Hence by the theory of [7], the martingale \( Y_t \) can be replicated using a predictable self-financing trading strategy using only the asset \( W_1^1 \) and the risk free asset. A second application of the martingale representation theorem shows that the asset \( W_1^1 \) may itself be replicated by the a trading strategy using only the assets \( X_i^1 \). The hedging portfolio obtained in this way gives rise to the portfolio referred to in the statement of the theorem.

We wish to compute this portfolio explicitly in the case of markets of the form \( \text{Form 2.3} \).

We may read off from (25) and (34) that

\[
\tilde{W}_1^t = -\int_0^t 1 \frac{1}{A(s)} \sigma^{-1} (rX_s - \mu) \cdot dW_s.
\]

From (23) we may write

\[
\tilde{W}_1^t = -\int_0^t \frac{1}{A(s)} \sigma^{-1} (rX_s - \mu) \cdot (\sigma^{-1} (dX_s - \mu ds))
\]

\[
= -\int_0^t \frac{1}{A(s)} (\sigma \sigma^T)^{-1} (rX_s - \mu) \cdot (dX_s - \mu ds)
\]

We can now read off that the portfolio of risky assets one should hold in order to replicate \( W_1^1 \) is proportional to

\[
(\sigma \sigma^T)^{-1} (rX_s - \mu).
\]

\( \square \)
This result explains the general form of the solution to the portfolio optimization problem studied in [13]. However, it goes considerably beyond this.

As an example, let us suppose we wish to create a new life-course dependent product such as a pension or annuity. We model the life-course events using a probability model independent of our market model and we use the Black–Scholes–Merton model to model the market. We price the products using some form of convex optimization: perhaps indifference pricing or some form of robust optimization. We expect that the end result will be a market invariant convex optimisation problem of some sort. We immediately deduce that any optimal investments will be of the form predicted above.

We call this result a one-mutual-fund theorem because it shows that a fund manager can create a single mutual fund that can be used to implement these trading strategies. A key difference between our result and the classical one-mutual-fund theorem is that an investor will need to trade in our mutual fund in continuous time.

A The product of a market and a casino

In Section 2 we asserted that the market obtained by taking the product of a complete market $M$ with a casino $I$ can be interpreted as a multi-period market where one first invests in $M$ and then invests in $I$. At the time we had not defined multi-period markets so we could not justify this statement. The purpose of this appendix is to provide the necessary justification.

We mirror the definition of a self-financing strategy in discrete time described in [6] so that it can be used in our more general setting. The definition in [6] restricts us to finite dimensional linear markets.

**Definition A.1.** Let $M = (\Omega, \{\mathcal{F}_t\}_{t \in \mathcal{T}}, \mathbb{P})$ be a multi-period market with $\mathcal{T} = \{t_0, t_1, \ldots, t_k\}$. A self-financing trading strategy is a process $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$ which satisfies:

(i) (predictability) $\xi_i$ is a $L^0(\Omega, \mathcal{F}_i, \mathbb{R})$ valued random variable;

(ii) and (self-financing) $c_{i-1}(\xi_i - \xi_{i-1}) \leq 0$

for $1 \leq i \leq k$. The payoff of a self-financing trading strategy $\xi$ is $\xi_k$ and the cost is $c_0(\xi_1)$.

We can associate a one period financial market to a $k$-step market by taking $M = ((\Omega, \mathcal{F}_k, \mathbb{P}), c)$ with cost function $\tilde{c}$ defined by

$$\tilde{c}(X) = \inf_{\xi \in T_X} c_0(\xi_1),$$

where $T_X = \{\xi : \xi_k = X \text{ and } \xi \text{ is a self-financing trading strategy}\}$. (41)

If $\xi_k = X$ we say that $\xi$ replicates $X$.

Our definition of predictability may seem incorrect at first glance since the corresponding definition in [6] talks about $\mathbb{R}^{d+1}$ valued processes which are
assumed to be $\mathcal{F}_{i-1}$ measurable rather than $\mathcal{F}_i$ measurable. However, this is resolved by noticing that our $\xi_i$ do not represent quantities of assets, they instead represent the payoff of the market value of the investment at the next time period.

The inequality in our definition of self-financing may also look suspect, but one can simply assume that any excess cash is donated to charity. It is required because our approach allows infinitely negative costs. We remark that the convex analysis approach to defining markets in [16] has influenced this aspect of our work.

We now define the filtrations implied in our discussion of casinos. Let $M = ((\Omega, \mathcal{F}, \mathbb{P}), c)$ be a complete market with equivalent measure $\mathbb{Q}$ and $c(1) = C$. Define a filtration on $\Omega \times [0, 1]$ by $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} \times [0, 1]$, $\mathcal{F}_2 = \mathcal{F}_{\Omega \times [0,1]}$. We define

\[
    c_1(X) = \mathbb{E}_\mathbb{Q}(X|\mathcal{F}_1), \quad X \in L^1_\mathbb{Q}(\mathcal{F}_2),
\]

\[
    c_0(X) = C\mathbb{E}_\mathbb{Q}(X), \quad X \in L^1_\mathbb{Q}(\mathcal{F}_1).
\]

Fubini’s theorem ensures these are well-defined and have the required measurability properties.

We define the market obtained by investing first in $M$ and then in the casino $\mathcal{F}$ to be precisely this 2-step market. Our assertion that the one-period market $M \times I$ is equivalent to this market can now be formally stated as saying that the one-period market $M \times I$ and the one period market derived from the 2-step market are isomorphic. Their probability spaces are trivially isomorphic, so it is simply a question of confirming that they have the same cost function.

Given an $\mathcal{F}_2$ measurable random variable $X$ we may define

\[
    \xi_2 = X, \quad \xi_1 = \mathbb{E}_\mathbb{Q}(X|\mathcal{F}_1).
\]

It is immediate from the definitions of $\xi$ and $c$ that $\xi = (\xi_1, \xi_2)$ is a self-financing trading strategy which replicates the payoff $X$ at a cost of $C\mathbb{E}_\mathbb{Q}(X)$. Hence by (41), $\tilde{c}(X) \leq C\mathbb{E}_\mathbb{Q}(X)$.

Suppose that we have some other self-financing trading strategy, $\xi = (\xi_1, \xi_2)$ which replicates the payoff $X$. We have $\xi_2 = X$ so by the self-financing condition, $\xi_1 \geq c_1(\xi_2) = c_1(X) = \mathbb{E}_\mathbb{Q}(X, \mathcal{F}_\infty)$. Taking expectations of this inequality and using monotonicity of expectation and the tower property, we find $c_0(\xi_1) = C\mathbb{E}_\mathbb{Q}(\xi_1) \geq C\mathbb{E}_\mathbb{Q}(X)$. Hence by (41), $\tilde{c}(X) \geq C\mathbb{E}_\mathbb{Q}(X)$.

We conclude that $\tilde{c}(X) = C\mathbb{E}_\mathbb{Q}(X)$. This is, of course, the cost in the market $M \times I$ and so our claim is proved.

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