On the Cubic Polynomial Slice $\text{Per}_1(e^{2\pi i \frac{p}{q}})$

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Abstract

We prove that every parabolic component in the cubic polynomial slice $\text{Per}_1(e^{2\pi i \frac{p}{q}})$ is a Jordan domain. We also show that the central components of its connected locus are copies of the Julia set of the quadratic polynomial $P_{\frac{p}{q}} (z) = e^{2\pi i \frac{p}{q}} z + z^2$.

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1 Introduction

Consider the space of unitary cubic polynomials fixing the origin 0:

$$f_{\lambda,a}(z) = \lambda z + az^2 + z^3, \ (\lambda, a) \in \mathbb{C}^2.$$  \hfill (1)

Let $J_{\lambda,a}$ denote the Julia set of $f_{\lambda,a}$. The connected locus is defined to be

$$C = \{(\lambda, a) \in \mathbb{C}^2; J_{\lambda,a} \text{ is connected}\} = \{(\lambda, a) \in \mathbb{C}^2; \text{both critical points of } f_{\lambda,a} \text{ do not escape to } \infty\}.$$  

By fixing $\lambda$, one gets the slice

$$\text{Per}_1(\lambda) = \{f_{\lambda,a}; a \in \mathbb{C}\} \cong \mathbb{C}$$

and its corresponding connected locus $C_{\lambda} = \{a \in \mathbb{C}; J_{\lambda,a} \text{ is connected}\}$. If $|\lambda|<1$ resp. $\lambda = e^{2\pi i \frac{p}{q}}$ with $p, q$ being coprime positive integers, 0 is an attracting resp. parabolic fixed point. By classical results in holomorphic dynamics, 0 attracts at least one of $c_1, c_2$, the two finite critical points of $f_{a,\lambda}$. So it is natural to consider the locus:

$$\mathcal{H}^\lambda = \{a \in \mathbb{C}; c_1, c_2 \text{ are contained in Fatou components and are attracted by } 0\}.$$  \hfill (2)

Following Milnor in [12], $a \in \mathcal{H}^\lambda$ are divided into four different types:

(A) Adjacent: $c_1, c_2$ are contained in the same Fatou component.

(B) Bitransitive: $c_1, c_2$ are contained in different Fatou components but in the same attracting cycle of Fatou components.

(C) Capture: one of $c_1, c_2$ does not belong to the attracting cycle of Fatou components but eventually hits the cycle.

(D) Disjoint: $c_1, c_2$ belong to different attracting (parabolic) cycles of Fatou components.

When $|\lambda|<1$, 0 is always a (super-)attracting fixed point. Type (B) and (D) do not exist in $\mathcal{H}^\lambda$. Thus $\mathcal{H}^\lambda$ is decomposed into $\bigcup_{n \geq 0} \mathcal{H}^\lambda_n$, where $\mathcal{H}^\lambda_0$ is the collection of type (A) parameters, $\mathcal{H}^\lambda_n$ with $n \geq 1$ is the collection of type (C) parameters of depth $n$. By the famous MSS J-stability theorem [16], it is easy to see that $\mathcal{H}^\lambda_n$ are open for $n \geq 0$. The global picture of $C_{\lambda}$ is well understood: when $\lambda = 0$, $C_{\lambda}$ is the union of the central hyperbolic component $\mathcal{H}^\lambda_0$ (the component of $\mathcal{H}^\lambda_0$ containing $a = 0$) and the limbs attached at $\partial \mathcal{H}^\lambda_0$, [13]. Moreover the boundary of any component of $\mathcal{H}^\lambda_0$ is a
Jordan curve, cf. [5], [13]. For $|\lambda| < 1$, $C_\lambda \setminus \mathcal{H}\lambda$ are dynamically quasiconformal equivalent, [17]. As a direct consequence the results in $C_0$ hold for all $C_\lambda$ with $|\lambda| < 1$.

In this article we investigate $C_\lambda$ when $\lambda = e^{\frac{2\pi i}{q}}$. Now $z = 0$ becomes a parabolic fixed point with rotation number $\frac{p}{q}$. Again by MSS J-stability theorem, it is not hard to show that collections of Type (A) (B) (C) are open. The case $\lambda = 1$ is studied in [20] and parallel results as [13] have been obtained. For general $\frac{p}{q}$, global description of $C_\lambda$ is more complicated mainly for two reasons:

- parameters of Type (B) appear;
- there are more than one parameter such that $z = 0$ becomes degenerate parabolic, or equivalently, are of Type (D).

Compare to the case $\lambda = 1$, where there is no Type (B) component, and $a = 0$ is the only Type (D) parameter.

Figure 1: $C_\lambda$ when $\lambda = 1$ on the left and $e^{\frac{2\pi i}{3}}$ on the right. Different types of parameters in $\mathcal{H}\lambda$ are marked out. Type (A) (B) and (C) are in green, $0, a_0, a_1, a_2$ are of Type (D). Black parts are copies of the Mandelbrot set or the parabolic Mandelbrot set, correspond respectively to polynomial-like or parabolic-like maps [10].

Definition. Let $\lambda = e^{\frac{2\pi i}{q}}$. The parameters $a \in C_\lambda$ such that $z = 0$ becomes parabolic degenerate (Type (D)) are called double parabolic, and denote by $A_{p/q}$ the collection of them. A connected component of Type (A) (B) or (C) is called a parabolic component.

In the same spirit, we have decomposition $(\mathcal{H}\lambda = \bigcup_{n \geq 0} \mathcal{H}_n) \cup A_{p/q}$, where $\mathcal{H}_0^\lambda$ is the collection of type (A) and (B) parameters, $\mathcal{H}_n$ with $n \geq 1$ is the collection of depth $n$ type (C) parameters.
All $\mathcal{H}_\lambda^a$ are open.

Let $f$ be a polynomial with connected Julia set and $z$ a (pre-)periodic point. It is a classical result by Yoccoz that $z$ admits an external ray landing (cf. [8]). The collection of all the angles of external rays landing at $z$ is called the portrait at $z$. For family (1), the Böttcher coordinate $\phi_{\lambda,a}^\infty$ at $\infty$ depends analytically on $(\lambda,a)$ by taking the normalization $\phi_{\lambda,a}^\infty(z) = z + o(1)$. So there is no ambiguity of angles when $(\lambda,a)$ varies.

Our first result concerns with the description of $A_{p/q}$:

**Proposition I.** $\# A_{p/q} = q$. The different $a_m \in A_{p/q}$ ($0 \leq m \leq q-1$) are characterized by the portrait at $z = 0$. Moreover each $a_m$ admits 4 parameter external rays landing, cutting $\text{Per}_1(\lambda)$ with respect to the portrait at $z = 0$. Moreover, these four rays are unique among all rays with rational angle.

We also describe the relative position between adjacent and bitransitive components:

**Proposition II.** $\mathcal{H}_\lambda$ has exactly 2 adjacent components $B_0, B_q$ symmetric with respect to $a \mapsto -a$ and $q-1$ bitransitive components $B_1, \ldots, B_{q-1}$. These components are characterized by the portrait at $z = 0$. Moreover,

$$B_m \cap A_{p/q} = a_m \cup a_{m-1}, \quad 1 \leq m \leq q-1; \quad B_0 \cap A_{p/q} = a_0, \quad B_q \cap A_{p/q} = a_{q-1}.$$  

**Definition.** By Proposition I and II, at every $a_m \in A_{p/q}$, four external rays land and cut $\mathbb{C}$ into four open regions, two of which do not intersect $\bigcup_{i=0}^{q} B_i$. These two regions, denoted by $W^\pm(a_m)$, are called double parabolic wakes.

Our construction of $a_m$ and $B_m$ is based on the technique of pinching deformation developed by Cui-Tan [3]. Its advantage is that the combinatorial information is explicitly prescribed. See a different approach using transversality applied to the same problem of quadratic rational maps [1].

Let us mention that the uniqueness of the 4 external rays in Proposition I is not obvious: this is the cubic analogue to the ray landing problem at parabolic parameters in the Mandelbrot set, which has been solved by Douady-Hubbard [8]. See also [9] for an easier presentation. Essentially, the "Tour de valse" argument should work, but also should be more delicate since the fixed point is parabolic degenerate. Instead, here we use a "ray counting" argument to avoid the complicated analysis in [8].

Next we investigate local connectivity of $C_\lambda$. For $a \in C_\lambda \setminus \mathcal{H}_\lambda$, denote by $c_+ (a)$ the critical point attracted by 0 and $c_- (a)$ the other one not. The approach here is the dynamical-parameter puzzles: one first finds infinitely ringed puzzles around critical point or critical value in the dynamical plan, which allows us to apply Yoccoz’s Theorem to give the dichotomy: either the puzzles shrink to a point, or the intersection of the puzzle is a quadratic copy of Julia set; then one passes this dichotomy to puzzles in the parameter space through parametrisation and holomorphic motion. More precisely, under the following assumption on $f_a$

**Assumption** ($\Diamond$), $a \in C_\lambda$ is neither adjacent, bitransitive, capture, Misiurewicz parabolic nor double parabolic, and is not contained in any double parabolic wake.

we prove the following Key Lemma:
**Key Lemma.** If $a$ satisfies Assumption $(\diamond)$, then there is an admissible graph infinitely ringing $c_-(a)$.

Therefore by Yoccoz’s Theorem

$$a \in C_\lambda \setminus \left[ \left( \bigcup_{n \geq 0} \mathcal{H}_n^\lambda \right) \cup \left( \bigcup_{m=0}^{q-1} \mathcal{W}^+(a_m) \cup \mathcal{W}^-(a_m) \right) \right] .$$

is divided into four cases (recall that $\mathcal{H}_\lambda = (\bigcup_{n \geq 0} \mathcal{H}_n^\lambda) \cup \mathcal{A}_{p/q}$):

1. Double parabolic: $a \in \mathcal{A}_{p/q}$.

2. Misiurewicz parabolic: $\exists n \geq 1$ s.t. $f_{na}^n(c_-(a)) = 0$.

3. Renormalisable (in the sense of Douady-Hubbard [4]): there exists $\mathcal{U}' \subset \subset U \setminus \mathcal{P}_0$ surrounding $c_-(a)$ and $k \geq 1$ such that $f_{\mathcal{U}', U'}^k : \mathcal{U}' \rightarrow U$ is quadratic like, and the copy of the quadratic Julia set is connected.

4. Non renormalisable.

Using Shishikura’s trick, one passes the dichotomy of dynamical puzzles surrounding $c_-(a)$ to parameter puzzles surrounding $a$:

**Theorem I.** If $a$ satisfies Assumption $(\diamond)$, i.e. is of case 3 and 4, then there is a dynamically defined nest of puzzles $\{\mathcal{P}_n(a)\}$ surrounding $a$, such that $\bigcap_n \mathcal{P}_n(a) = \{a\}$ if $a$ is non renormalisable; $\bigcap_n \mathcal{P}_n(a) = M_a$ is a copy of the Mandelbrot set containing $a$ if $a$ is renormalisable. In particular, $C_\lambda$ is locally connected at $a$ for a non renormalisable.

In general, we do not get local connectivity for a renormalisable, since essentially this needs the MLC conjecture to be true. However for certain a renormalisable (for instance $a$ is the cusp of the main cardioid or is a tip of $M_a$), we can separate $M_a$ from $C_\lambda$ by two external rays landing at $a$. Hence $C_\lambda$ is also locally connected at such $a$. If we are in case 1 or 2, then the parameter puzzles we construct no longer surround $a$, but have $a$ on their boundaries. But still we can show that a decreasing sequence of such puzzles shrink to $a$, by first showing that the corresponding decreasing dynamical puzzles shrink to a certain preimage of $z = 0$, and passing this shrinking property to parameter puzzles via holomorphic motion. For $a$ in double parabolic wakes, we only investigate local connectivity of $\partial \mathcal{U}_n$ for $\mathcal{U}_n$ a connected component of $\mathcal{H}_n^\lambda$. Here we use a holomorphic motion argument instead of puzzles. But it should be possible to prove with puzzles.

To summarize, we obtain

**Theorem II.** The boundary of every component of $\bigcup_{n \geq 0} \mathcal{H}_n^\lambda$ is a Jordan curve.

Theorem [II] actually gives more than local connectivity: the shrinking property of decreasing parameter puzzle pieces gives landing parameter external rays, which provide clean cuts of $C_\lambda$. This enables us to give global descriptions of $C_\lambda$. For every fixed $0 \leq m \leq q$, let $\mathcal{H}^m$ be the closure of the union of $B^m$ and all the capture components linked to it. $\mathcal{H}^m$ is called a ”central component”. For instance, the big green connected patches in Figure [I] are central components. (compare to Definition [6.4.3] which is slightly different).
Theorem III. $C_\lambda$ is decomposed into the union of all the centre components and the limbs attached to them at end points.

Theorem IV. $\mathcal{H}^m$ is a copy of a part of the Julia set of $P_{p/q}(z) = e^{2\pi i \frac{p}{q}} z + z^2$. (See a precise statement in Theorem 6.4.9).

The paper is organized as follows:

- In Section 2 we recall some known results of parameter external rays, and the description of $C_0$ given by [13].
- In Section 3, we first prove the existence of Type (A) (B) components in 3.1 and prove that there are exactly $q$ Type (D) parameters in 3.2. Then we parametrize Type (A) (B) components in 3.3 and prove their uniqueness (Corollary 3.4.7). Proposition II is somehow a direct consequence from this and is proved in 3.5. At last we prove Proposition I (Corollary 3.6.4 and 3.6.5).
- Section 4 is mainly devoted to the proof of the Key Lemma (Lemma 4.3.8) and the construction of admissible dynamical graphs and puzzles.
- In Section 5, we first prove local connectivity of $\partial U_n$ for any capture component $U_n$ in double parabolic wakes (Corollary 5.1.5). Then we construct parameter puzzles and emphasize that dynamical graphs moves holomorphically when parameter varies in parameter puzzles (Lemma 5.3.2 and 5.3.3), which will be used in the last section.
- Finally Section 6 is devoted to the proof of Theorem I (Corollary 6.2.6, Proposition 6.3.1 and 6.3.3), Theorem II, Theorem III (Theorem 6.4.8) and Theorem IV (Theorem 6.4.9).

2 Preparations

2.1 Families with marked out critical points

Consider the following two families

$$g_{\lambda,c}(z) = \lambda z \left(1 - \frac{1 + 1/c}{2} z + \frac{1/c}{3} z^2\right), \ (\lambda, c) \in (\mathbb{C}^*)^2$$

$$\hat{g}_{\lambda,s}(z) = \lambda z \left(1 - \frac{s + 1/s}{2} z + \frac{1}{3} z^2\right), \ (\lambda, s) \in (\mathbb{C}^*)^2$$

The advantage of $g_{\lambda,c}$ (or $\hat{g}_{\lambda,s}$) is that the two critical points $1, c$ (resp. $s, 1/s$) are marked out. Fix $\lambda$-slice, denote by $\hat{C}_\lambda$ $\hat{C}_\lambda$ respectively the connected locus for these two families. The attracting locus $\hat{H}^\lambda, \hat{H}^\lambda$ are defined likewise as in [2]. For $\lambda \neq 0$, the relation between $f_{\lambda,a}$, $g_{\lambda,c}$ and $\hat{g}_{\lambda,s}$ is given by

Lemma 2.1.1. $\hat{g}_{\lambda,s}$ is conjugate to $f_{\lambda,a}$ by $z \mapsto \sqrt{3/\lambda} \cdot z$ with $a = \sigma(s) = -\sqrt{3\lambda} \cdot \frac{s + 1/s}{2}$ and is conjugate to $g_{\lambda,c}$ by $z \mapsto \frac{1}{s} \cdot z$ with $c = \iota(s) = s^2$. 

6
2.2 Parameter external rays in $\text{Per}_1(\lambda)$, $|\lambda| \leq 1$

As mentioned before, the Böttcher coordinate $\phi_{\lambda,a}^\infty$ of $f_{\lambda,a}$ at $\infty$ depends analytically on $(\lambda,a)$ by taking the normalisation $\phi_{\lambda,a}^\infty(\infty) = z + o(1)$. Therefore dynamical external rays $R_{\lambda,a}^\infty(t)$ are well-defined (at least near $\infty$). Now suppose $|\lambda| \leq 1$. It is a classical result (cf. [19]) that $\mathcal{C}_\lambda$ is a full continuum containing $\pm \sqrt{3}\lambda$. Thus one can define analytically the two critical points of $f_{\lambda,a}$ for $a \in \mathbb{C} \setminus \mathcal{C}_\lambda$: $c_{\lambda,a} = \frac{-a \pm \sqrt{a^2 - 3\lambda}}{3}$ such that $c_{\lambda,a}^+ \in K_{\lambda,a}$ and $c_{\lambda,a}^- \in \mathbb{C} \setminus K_{\lambda,a}$. Set $v_{\lambda,a} = f_a(c_{\pm}(a))$. One can parametrize $\mathbb{C} \setminus \mathcal{C}_\lambda$ by looking at the position of $v_{\lambda,a}$ in the Böttcher coordinate:

**Proposition 2.2.1** ([19]). Let $|\lambda| \leq 1$. The mapping $\Phi^\lambda_\infty : \mathbb{C} \setminus \mathcal{C}_\lambda \to \mathbb{C} \setminus \overline{D}$ defined by $a \mapsto \phi_{\lambda}^\infty(v_{\lambda,a})$ is a degree 3 covering.

A parameter external ray with angle $t$ is therefore defined to be a connected component of $(\Phi^\lambda_\infty)^{-1}([re^{2\pi it}; r>1])$. In most cases, we denote a parameter external ray by $R_{\lambda,a}^\infty$ without precising which component of $(\Phi^\lambda_\infty)^{-1}([re^{2\pi it}; r>1])$.) When we say "$R_{\lambda,a}^\infty$ lands at $a_0 \in \mathbb{C}_\lambda$", it means that one of the three components of $(\Phi^\lambda_\infty)^{-1}([re^{2\pi it}; r>1])$ accumulates at $a_0$.

**Remark 2.2.2.** Let us just mention that in [19] the proposition above is stated only for $\lambda = e^{2\pi i\theta}$ with $\theta$ of Brjuno type. However the proof there works without any change for all $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$.

Proposition 2.2.3 can be passed to family ([4]) by Lemma 2.1.1.

**Proposition 2.2.3.** For $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$, $\mathbb{C}^* \setminus \hat{\mathcal{C}}_\lambda$ has exactly two connected components ($\tau : s \mapsto 1/s$)

$$\hat{\mathcal{H}}_\infty = \{ s \in \mathbb{C}^* ; \hat{g}_s^n(s) \to \infty \text{ as } n \to \infty \}$$

$$\tau \hat{\mathcal{H}}_\infty = \{ s \in \mathbb{C}^* ; \hat{g}_s^n(1/s) \to \infty \text{ as } n \to \infty \}$$

which are punctured neighborhoods of $\infty, 0$ respectively and are homeomorphic to punctured disk. Moreover the mapping $\Phi^\infty : \hat{\mathcal{H}}_{\infty,0} \to \mathbb{C} \setminus \overline{D}$ given by $\Phi^\infty(s) = \phi_{\infty}(\hat{g}_{\sigma(s)}(s))$ is a degree 3 covering, where $\hat{g}_{\sigma(z)}(s) = \phi_{\infty}(s)$. When we say "$R_{\infty,0}$ lands at $a_0 \in \mathbb{C}_{\infty,0}$" , it means that one of the three components of $\Phi^\infty(s)$ accumulates at $a_0$.

By Lemma 2.1.1 we can write $\mathbb{C}^* \setminus \hat{\mathcal{C}}_\lambda = \hat{\mathcal{H}}_{\infty}^\lambda \cup \tau \hat{\mathcal{H}}_{\infty}^\lambda$, where $\tau : c \mapsto \frac{1}{c}$ and

$$\hat{\mathcal{H}}_{\infty}^\lambda = \{ c \in \mathbb{C}^* ; g_{\lambda,c}^n(c) \to \infty \text{ as } n \to \infty \}$$

$$\tau \hat{\mathcal{H}}_{\infty}^\lambda = \{ c \in \mathbb{C}^* ; g_{\lambda,c}^n(1) \to \infty \text{ as } n \to \infty \}$$

by noticing that $cg_{\lambda,c}(\frac{1}{c}) = g_{\lambda,1/c}$. These two escaping regions are simply connected.

**Lemma 2.2.4.** Let $\hat{U} \subset \hat{\mathcal{H}}^\lambda$ be an open component. Then $\hat{U}$ is simply connected and $\partial \hat{U} \subset \partial \hat{\mathcal{C}}_\lambda$. In particular, if $\partial \hat{U}$ is locally connected, then it is a Jordan curve.

**Proof.** Simply connectivity can be shown easily by applying maximum principle to $g_{\lambda,c}^n(c)$ or $g_{\lambda,c}^n(1)$; for $\partial \hat{U} \subset \partial \hat{\mathcal{C}}_\lambda$, see [20] Lem. 2.1.5]. We prove the last statement. Suppose $\partial \hat{U}$ is locally connected, then any conformal representation $\Psi : \mathbb{D} \to \hat{U}$ can be extended continuously and surjectively to the boundary. Moreover $\Psi : \partial \mathbb{D} \to \partial \mathcal{H}$ is injective: if not, then there exists a $a \in \partial \mathcal{H}$ accessible by two rays $\Psi(re^{it_1}), \Psi(re^{it_2})$ which bound a simply connected region containing part of $\partial \hat{U}$. This contradicts $\partial \hat{U} \subset \hat{\mathcal{C}}_\lambda$. 

\[\]
Lemma 2.2.6. Let $|\lambda| \leq 1$, $t \in \mathbb{Q}/\mathbb{Z}$. Then $R_\infty^\lambda(t)$ land at some $a_0 \in \mathcal{C}_\lambda$ which is geometrically finite.

Proof. It suffices to prove the accumulation set of $R_\infty^\lambda(t)$ is finite, since the accumulation set is connected. It will be convenient to work in family (4) since the critical points are marked out by Proposition 2.2.3 external rays with angle $t$, denoted by $R_\infty^\lambda(t)$, in $\mathcal{H}_\infty$ are well-defined. Suppose $s_0$ is accumulated by $R_\infty^\lambda(t)$. Moreover if $\lambda = e^{\pi i / q}$ suppose $s_0$ is not double parabolic (it is not hard to see that $\#A_{p/q} < \infty$). Suppose $R_\infty^\lambda(s_0) \lambda$ (the dynamical external ray of $\hat{g}_{\lambda,s_0}$) lands at $x_{\lambda,s_0}$.

- If $x_{\lambda,s_0}$ is repelling periodic pre-periodic or if $\lambda = e^{\frac{2\pi i p}{q}}$ and $x_{\lambda,s_0}$ is pre-periodic to $z = 0$, then by stability of repelling Koenigs coordinate or repelling Fatou coordinate one concludes that $x_{\lambda,s_0} = \hat{v}(s_0) := \hat{g}_{\lambda,s_0}(s_0)$. Therefore $s_0$ satisfies a non-trivial algebraic equation: $\hat{g}^k(\hat{v}(s_0)) = \hat{g}^l(\hat{v}(s_0))$ which has only finitely many solutions.

- If $x_{\lambda,s_0}$ is parabolic pre-periodic and when $\lambda = e^{\frac{2\pi i p}{q}}$ it is not pre-periodic to 0. Then there exists $l \geq 0, k \geq 1$ (only depend on $t$) such that $\hat{g}^k(\hat{v}(s_0)) = \hat{g}^l(\hat{v}(s_0))$. By the snail lemma, $(s_0, x_{\lambda,s_0})$ verifies

$$\hat{g}^{k+l}(z) = \hat{g}^l(z), \ (\hat{g}^{k+l})'(\hat{g}^l(z)) = 1.$$  \hspace{1cm} (5)

defines a non-trivial algebraic variety (not equal to $\mathbb{C}^* \times \mathbb{C}$) and hence consists of only finitely many irreducible components of dimension 1 and 0 (i.e. points). We claim that when $\lambda \neq e^{\frac{2\pi i p}{q}}$, there are no dimension 1 component; when $\lambda \neq e^{\frac{2\pi i p}{q}}$, the only dimension 1 component is $z = 0$. Indeed, suppose there is another such component $X$. Then $X$ is unbounded (goes to the boundary of $\mathbb{C}^* \times \mathbb{C}$). Consider the projections $\pi_1 : X \rightarrow \mathbb{C}, \pi(s,z) = s$ and $\pi_2 : X \rightarrow \mathbb{C}, \pi(s,z) = z$. Then at least one of $\pi_i(X), i = 1, 2$ is unbounded. If $\pi_1(X)$ is unbounded, let $(s_n, z_n) \subset X$ with $s_n \rightarrow \infty$ or 0, then $\lambda$ has to be $e^{\frac{2\pi i p}{q}}$ and for all $n$ large enough, $z_n = 0$, so $X$ has to be in $X$ with $z_n \rightarrow \infty$. If $\pi_2(X)$ is unbounded, let $(s_n, z_n) \subset X$ with $z_n \rightarrow \infty$. If $(s_n)$ also tends to $\infty$ or 0, then we are in the precedent case; if not, then for all $n$, the basin at infinity of $\hat{g}_{\lambda,s_0}$ contains a common neighborhood of $\infty$. Hence for $n$ large enough, $z_n$ escapes to $\infty$, contradicting the assumption that $(s_n, z_n)$ verifies (5).

To conclude, the above analysis shows that the accumulation set of $R_0(t)$ is finite and the possible accumulations are geometrically finite maps.

For our purpose, we extract from the lemma above the case of $\lambda = e^{2\pi i \frac{p}{q}}$:

Lemma 2.2.7. Let $\lambda = e^{2\pi i \frac{p}{q}}$, $t \in \mathbb{Q}/\mathbb{Z}$. Then $R_\infty^\lambda(t)$ lands at some $a_0$. In the dynamical plan of $f_{a_0}$, $R_\infty^\lambda(a_0)$ lands at a (pre-)periodic point $x(a_0)$. Moreover

- If $x(a_0)$ is repelling, then $x(a_0)$ is the free critical value.

- If $a_0 \notin A_{p/q}$, and $x(a_0)$ is in the inverse orbit of 0, then $x(a_0)$ is the free critical value.
• If \( a_0 = a_m \in \mathcal{A}_{p/q} \) and \( t \) is not pre-periodic to the portrait at \( z = 0 \) of \( f_{\lambda,a_0} \), then \( \mathcal{R}_{\lambda,a}^\infty(t) \) does not land at \( a_0 \).

**Lemma 2.2.8.** Suppose \( |\lambda| \leq 1 \). Let \( a_0 \) be a Misiurewicz parameter or parabolic Misiurewicz parameter, \( t \in \mathbb{Q}/\mathbb{Z} \). Then \( \mathcal{R}_{\lambda,a_0}^\infty(t) \) lands at one of the critical values of \( f_{\lambda,a} \) if and only if \( \mathcal{R}_{\lambda,a}^\infty(t) \) lands at \( a_0 \).

**Proof.** First of all suppose the contrary that it is parabolic (pre-)periodic beyond degree 4 holomorphic covering ramified at 0. Suppose \( \lambda \in \mathbb{Z} \), i.e. the two critical points are not the same. This is clear when \( |\lambda| < 1 \) or \( \lambda = e^{2\pi i \frac{k}{q}} \) since 0 attracts at least one critical point. If \( \lambda \) is Siegel, then the boundary of the Siegel disk is contained in the accumulation set of critical orbits; if \( \lambda \) is Cremer, then the Julia set is not locally connected. Therefore \( c_{\lambda,a}^\pm \) can be analytically defined near \( a_0 \). Then one uses stability of repelling Koenigs coordinate or repelling Fatou coordinate to conclude the proof. □

**Lemma 2.2.9.** Let \( t \in \mathbb{Q}/\mathbb{Z} \).

• Suppose \( |\lambda| \leq 1 \) but \( \lambda \neq e^{2\pi i \frac{k}{q}} \). Let \( \mathcal{O} \subset \mathbb{C} \) be the set of parameters \( a \) such that \( \mathcal{R}_{\lambda,a}^\infty(t) \) lands at a repelling (pre-)periodic point, Then \( \mathbb{C} \setminus \bigcup_{k \geq 1} \mathcal{R}_{\lambda,a}^\infty(3^k t) \) \( \subset \mathcal{O} \). Moreover \( \bigcup_{k \geq 0} \mathcal{R}_{\lambda,a}^\infty(3^k t) \) admits a dynamical holomorphic motion for \( a \) in any component of \( \mathcal{O} \).

• Suppose \( \lambda = e^{2\pi i \frac{k}{q}} \). Let \( \mathcal{O} \subset \mathbb{C} \) be the set of parameters \( a \) such that \( \mathcal{R}_{\lambda,a}^\infty(t) \) lands at a repelling (pre-)periodic point or the inverse orbit of 0. Then \( (\mathbb{C} \setminus \bigcup_{k \geq 1} \mathcal{R}_{\lambda,a}^\infty(3^k t)) \setminus \mathcal{A}_{p/q} \subset \mathcal{O} \). Moreover \( \bigcup_{k \geq 0} \mathcal{R}_{\lambda,a}^\infty(3^k t) \) admits a dynamical holomorphic motion for \( a \) in any component of \( \mathcal{O} \setminus \mathcal{A}_{p/q} \) (\( \mathcal{A}_{p/q} \) is the set of double parabolic parameters).

**Proof.** For the first point, see [13] Lem. 3.8]. For the second point, for \( a \in \mathbb{C} \setminus \bigcup_{k \geq 1} \mathcal{R}_{\lambda,a}^\infty(3^k t) \), \( \mathcal{R}_{\lambda,a}^\infty(3^k t) \) land for \( k \geq 0 \) since they do not crash on the free critical point. It suffices to prove that the landing point \( x(a) \) of \( \mathcal{R}_{\lambda,a}^\infty(t) \) is either repelling pre-periodic or in the inverse orbit of 0. Suppose the contrary that it is parabolic (pre-)periodic beyond \( z = 0 \), then the portrait at \( z = 0 \) of \( f_{\lambda,a} \) does not intersect \( \{3^k t\}_{k \geq 0} \). Since the portrait at \( z = 0 \) is stable by the stability of repelling Fatou coordinate at \( z = 0 \) (notice that \( a \notin \mathcal{A}_{p/q} \), so for \( a' \) in a neighborhood of \( a \), the portrait at \( z = 0 \) of \( f_{\lambda,a'} \) does not intersect \( \{3^k t\}_{k \geq 0} \). However on the other hand, from the second point in the proof of Lemma 2.2.6 the set of parabolic parameter with a given period is discrete. So for \( a' \) in a punctured neighborhood of \( a \), \( x(a') \) is either repelling or sent to \( z = 0 \) with portrait containing the cycle in \( \{3^k t\}_{k \geq 0} \). So the second case is impossible. The first case contradicts the maximum principle for \( a \mapsto 1/(f^n)(x(a)) \) for \( n \) large enough. □

### 2.3 Results for \( \mathcal{C}_0 \)

When \( \lambda = 0 \), 0 is a supper-attracting fixed point, \( -\frac{2n}{3} \) is the free critical point. \( \mathcal{C}_0 \) has a unique hyperbolic component \( \mathcal{H}_0^\infty \) containing \( a = 0 \). (The upper index means \( \lambda = 0 \), the lower means it is of depth 0, i.e. \( -\frac{2n}{3} \) is in the immediate basin of 0). Similarly as \( \mathcal{C}_0 \), one can parametrize \( \mathcal{H}_0^\infty \):

**Proposition 2.3.1 ([13]).** The mapping \( \Phi_0^0 : \mathcal{H}_0^\infty \rightarrow \mathbb{D} \) given by \( \Phi_0^0(a) = \phi_{0,a}^0(-\frac{2n}{3}) \) is a degree 4 holomorphic covering ramified at 0.

The internal ray of angle \( t \) in \( \mathcal{H}_0^\infty \) is defined to be a connected component of

\[
(\Phi_0^0)^{-1}(\{re^{2\pi it} : r \in (0,1)\}).
\]
Remark 2.3.2. It is easy to verify by symmetric that
\[ \Phi_0^0(-a) = \Phi_0^0(a), \Phi_0^0(\tau) = \overline{\Phi_0^0(a)}; \quad \Phi_\infty^0(-a) = -\Phi_\infty^0(a), \Phi_\infty^0(\tau) = \overline{\Phi_\infty^0(a)} \]

Remark 2.3.3. By Proposition 2.3.1 and 2.2.1 there are 4 (resp. 3) internal (resp. external) rays associated to a given angle. Let \( S = \{ x + iy, x \geq 0, y > 0 \} \). These four rays are contained in four sectors \( S, iS, -S, -iS \) respectively. Hence we precise a ray contained in a sector by adding this index, for example \( \mathcal{R}_0^0S(t) \) (resp. \( \mathcal{R}_\infty^0S(t) \)) denotes the internal (resp. external) ray with angle \( t \) contained in \( S \). Nevertheless in most time we omit for simplicity this index if the sector to which this ray belong is clear or if there is no need to precise, according to the context.

The main results in [13] can be summarized as the following proposition and theorem:

Proposition 2.3.4. For all \( t \in \mathbb{R}/\mathbb{Z} \), \( \mathcal{R}_0^0(t) \) lands at some \( a(t) \in \partial \mathcal{H}_a^0 \). In the dynamical plan, \( \mathcal{R}_{a(t)}^0(t) \) lands at a parabolic \( k \)-periodic point if and only if \( t \cdot \frac{k}{2} \) is \( k \)-periodic under multiplication by \( 2 \).

Theorem 2.3.5. For all \( n \geq 0 \), \( \partial \mathcal{H}_n^0 \) is a union of Jordan curve. If \( a_0 \in \partial \mathcal{H}_n^0 \) is renormalizable, then there are exactly two external rays landing at it; otherwise there is only one.

If \( a_0 \in \partial \mathcal{H}_n^0 \) is renormalizable, then the two corresponding rays landing at it separate \( \mathbb{C} \) into two connected components \( U_{a_0} \) and \( W_{a_0} \) with \( \mathcal{H}_n^0 \subset U_{a_0} \) and \( W_{a_0} \) containing a copy of Mandelbrot set rooted at \( a_0 \). For \( a \in \partial \mathcal{H}_n^0 \), define limbs by \( \mathcal{L}_a := W_a \cap C_{a} \) if \( a \) is renormalisable; otherwise \( \mathcal{L}_a \) is unique. Denote this cycle by \( \mathcal{L}_a \).

Let us give three lemmas which will be used later in the proof of Proposition 3.6.1

Lemma 2.3.6. Let \( a_1, a_2 \in \partial \mathcal{H}_n^0 \) be two different renormalizable parameters. By Theorem 2.3.5, suppose that \( \mathcal{R}_{a_1}^0(t_i), \mathcal{R}_{a_2}^0(t_i') \) land at \( a_i \), \( i = 1, 2 \). Then \( \{ t_1, t_1' \} \neq \{ t_2, t_2' \} \).

Proof. Suppose the contrary that \( \{ t_1, t_1' \} = \{ t_2, t_2' \} \). Then the portrait of the co-critical point \( \tilde{c}_{a_i} \) will be the same, i.e. the angles of the two external rays bounding \( \tilde{c}_{a_i} \) equal those for \( \tilde{c}_{a_1} \). On the other hand, since the parametrisation of \( \mathcal{H}_n^0 \) defined by \( a \mapsto \phi_n^0(a) \) is an isomorphism, we conclude that the angles of parameter rays landing at \( a_i \) are different. Thus in the dynamical plan, the portrait at \( \tilde{c}_{a_i} \) should also be different, a contradiction. \( \square \)

Lemma 2.3.7. There are exactly \( 4q \) external rays whose angles are of rotation number \( p/q \) (under multiplication by \( 3 \)) land at \( \partial \mathcal{H}_0^0 \).

Proof. By Theorem A.2, the periodic cycle of rotation number \( p/q \) under multiplication by \( 2 \) is unique. Denote this cycle by \( T_{\frac{p}{q}} \).

Claim. Let \( t \in T_{\frac{p}{q}} \) such that \( \frac{t+1}{2} \) is not \( q \)-periodic. Then \( 1 - t \in T_{1 - \frac{p}{q}} \) and \( \frac{1-t+1}{2} \) is \( q \)-periodic.

proof of the claim. Let \( t = \frac{m}{2^n - 1} \). By hypothesis \( 2^q \cdot \frac{t+1}{2} \neq \frac{t+1}{2} \), which by an elementary computation is equivalent to \( \frac{m}{2^n} \neq \frac{1}{2^n} \), i.e. \( m \) is even. Hence \( 1 - t = \frac{2^n - 1 - m}{2^n - 1} \) where \( m = 2^q - 1 - m \) is odd. Hence \( \frac{1-t+1}{2} \) is \( q \)-periodic. \( \square \)
Let $a(t)$ be the landing point of $R^0_{a}(t)$. Suppose there are $k$ elements in $T_k$ such that $R^0_{a(t)}(t)$ lands at a parabolic periodic point, then from Proposition 2.3.4 and the claim we see that there are $q - k$ elements in $T_q$ such that $R^0_{a(1-t)}(1-t)$ lands at a parabolic periodic point. Moreover, if we name these $k$ elements by $t_1, \ldots, t_k$, $\tilde{t}_1, \ldots, \tilde{t}_{q-k}$, the claim tells us that $T_{p/q} = \{t_1, \ldots, t_k, 1-t_1, \ldots, 1-t_{q-k}\}$. Denote by $a(t_i)$ the landing point of $R^0_{a}(t_i)$ for $1 \leq i \leq k$ and $a(\tilde{t}_i)$ the landing point of $R^{0-t_i}_{0}(\tilde{t}_i)$ for $1 \leq i \leq q - k$. In the dynamical plan, let $(x_i)_{1 \leq i \leq q}$ be the landing point of $R^0_{a(t)}(t_i)$ and $R^0_{a(t)}(\tilde{t}_i)$ respectively. Then $(x_i), (\tilde{x}_i)$ are all parabolic periodic points with rotation number $p/q$. By Theorem 2.3.5 for any $t \in T_{p/q}$, there are exactly two parameter external rays $R_{\infty}(\eta), R_{\infty}(\eta')$ landing at $a(t)$ with $\eta, \eta' \in \bigcup_k \Theta_k$. Therefore we have found $2q$ external rays landing at parameters on $\partial H_0$ in the right-half plane. By symmetric, there are in total $4q$ rays.

Lemma 2.3.8. Suppose $t \in \mathbb{Q}/\mathbb{Z}$ has rotation number $p/q$ under multiplication by 3. Suppose that $a_0$ the landing point of $R^\infty_0(t)$ is a parabolic parameter, i.e. $R^\infty_{a_0}(t)$ lands at a parabolic periodic point. Then there are exactly two external rays $R^\infty_{0}(t^+), R^\infty_{0}(t^-)$ landing at $a_0$ (t is one of $t^+, t^-$). Moreover, $R^\infty_{a_0}(t^+), R^\infty_{a_0}(t^-)$ bound the critical value $v_{0,a_0}$, separating it from the immediate basin of $t$.

Proof. Let $P^0_n(a_0)$ be the puzzle piece ($a_0 \in P^0_n(a_0)$) constructed in [13]. Since $a_0$ is renormalisable, by [13] Prop. 3.26, $M_{a_0} := \bigcap_n P^0_n(a_0)$ is a copy of Mandelbrot set and let $\chi : M_{a_0} \to M$ be the homeomorphism. Let $\theta^+_{n}$ (resp. $\theta^-_{n}$) be the smallest/largest angle of the external rays involved in $P^0_n(a_0)$.

Step 1. For $n$ large enough, $3^k \theta^\pm_n = \theta^\pm_{n-k}$.

By [13] Lem. 3.17, Prop. 3.22, for $n$ large enough, there is a natural homeomorphism preserving equipotentials and rays between $\partial P^0_n(a_0)$ and $\partial P^0_{n-k}$. Thus $\theta^\pm_n$ are also the largest/smallest angle involved in $\partial P^0_{n-k}$. Notice that $R^\infty_{a_0}(\theta^\pm_n)$ land at the same hyperbolic component since adjacent capture components have disjoint boundaries (cf. [13] Lem. 1). Thus the two corresponding dynamical rays $R^\infty_{a_0}(\theta^\pm_n)$ land at the boundary of some Fatou component $U$. Let $x^\pm$ be their landing points and $y^\pm$ their intersection with the equipotential in $\partial P^0_{n-k}$. Then $y^+, y^-$ are linked by a curve $L$ consisting of two segments of $R^\infty_{a_0}(\theta^\pm_n)$ linking $y^\pm$ to $x^\pm$ and a segment contained in $\overline{U} \cap \partial P^0_{n-k}$ linking $x^+, x^-$. Suppose the contrary that $3^k \theta^\pm_n = \theta^\pm_{n-k}$. Then there is a segment of equipotential $\gamma$ with angles between $(\theta^-_n, \theta^+_n)$ linking $f^k_{a_0}(y^+), f^k_{a_0}(y^-)$. Thus $\gamma \cup f^k_{a_0}(L)$ bounds a simply connected domain $P$ which do not contain the free critical value of $f_{a_0}$. Let $Q$ be the component of $f^k_{a_0}(P)$ whose boundary contains $L$. Then $f^k_{a_0} : Q \to P$ is injective. Hence $\partial Q$ contains the segment of equipotential consisting of angles in $\Theta_k \setminus \theta^+_n \Theta^+_k$, which in particular contains the angle 0. While $R^\infty_{a_0}(0)$ is fixed by $f_{a_0}$, this leads to a contradiction.

Step 2. Two external rays land at $\chi^{-1}\frac{1}{4}$.

Since $\overline{P^0_{n,k}} \subset P^0_{n-k}$, we obtain a sequence of decreasing/increasing angles $\{\theta^\pm_{n+k}\}_{k \geq 0}$ for some $n_0$ large enough. Let $l \to \infty$ we get two limits $\theta^\pm$ which satisfy $3^k \theta^\pm = \theta^\pm$ by Step 1. Let $a^\pm$ be the landing point of $R_{a_0,\infty}(\theta^\pm)$. Clearly $a^\pm \in M_{a_0}$. We claim that $a^+ = a^- = \chi^{-1}\frac{1}{4}$. We prove for $a^+$ and the left one will be the same. If not, then $R^\infty_{a_0}(t^+)$ land at a repelling fixed point for $f^k_{a_0}$. By [13] Lem. 2.24, this point is the free critical value of $f_{a_0}$. Let $P_{c^+} = z^2 + c^+$ be the corresponding quadratic polynomial to which $f^k_{a_0} : P^0_{n-k}$ is conjugate. Then we have $P_{c^+}(c^+) = c^+$, i.e., $c^+ = 0$. While $c^+ \in \partial M$, a contradiction. Thus we have proved that $R^0_{a_0}(\theta^\pm)$ land at $\chi^{-1}\frac{1}{4}$. 11
Step 3. Two external rays land at \( \chi^{-1}(c) \) for \( c \neq \frac{1}{2} \) a parabolic parameter on the cardioid.

Let \( a = \chi^{-1}(c) \) and \( R_{0,a}^\infty(t) \), \( R_{0,a}^\infty(3^k t) \), ..., \( R_{0,a}^\infty(3^{s-1}k t) \) be the cycle of external rays landing at the parabolic fixed point of \( f_{0,a}^k \). Notice that \( c \) is satellite, hence it is the intersection of the closure of the main hyperbolic component \( H_0 \) and the closure of the periodic hyperbolic component \( H_1 \) attached at \( c \). Thus there are two pinching paths \( \gamma_0 \subset \chi^{-1}(H_0) \) and \( \gamma_1 \subset \chi^{-1}(H_1) \) converging to \( f_{0,a} \) such that

1. if \( a' \in \gamma_0 \), then \( \{ R_{0,a'}^\infty(3^k t) \}_{t \geq 0} \) land at a repelling \( s \)-periodic cycle of \( f_{0,a'}^k \).

2. if \( a' \in \gamma_1 \), then \( \{ R_{0,a'}^\infty(3^k t) \}_{t \geq 0} \) land at a common repelling fixed point of \( f_{0,a'}^k \).

Since landing at a repelling periodic cycle is an open property, there must exist two external rays among \( \{ R_{a}^\infty(3^k t) \}_{t \geq 0} \) landing at \( a \).

**Step 4.** The two rays landing are unique. The proof of [13 Thm. 3] can be adapted.

**Step 5.** \( a \) is on the cardioid of \( M_{a_0} \).

Let \( k \) be the period of renormalisation of \( a_0 \). Since the cycle \( t, 3t, ..., 3^{s-1} t \) has rotation number \( p/q \) under multiplication by \( 3 \), then so does the cycle \( t, 3^k t, ..., 3^{sk} t \) under multiplication by \( 3^k \). By [13 Prop. 3.22], \( f_{0,a_0}^k : P_{0,n}^a \to P_{0,n-k}^a \) is hybrid conjugate to a quadratic polynomial \( P_c(z) = z^2 + c \) for \( n \) large enough, where \( P_{0,n}^a \) is the dynamical puzzle piece containing the free critical value of \( f_{0,a_0} \). Thus the cycle of the parabolic periodic point of \( P_c \) has a cycle of access, which also admits a rotation number. This cycle of access is homotopic to a cycle of external rays with the same rotation number. Since by Theorem [A.2] there is only one cycle of angles for a given rotation number under multiplication by \( 2 \), this implies that the parabolic periodic point of \( P_c \) is actually fixed, hence \( c \) is on the cardioid, so is \( a_0 \).

From Step 1 and 3 we see that \( R_{0,a_0}^\infty(\theta^+) \), \( R_{0,a_0}^\infty(\theta^-) \) bound a sector containing \( v_{0,a_0} \), separating it from \( z = 0 \).

**Corollary 2.3.9.** Under the same hypothesis and notations of Lemma 2.3.8, the interval \( (\theta^-, \theta^+) \) does not contain any angle in \( \{ 3^k \theta^+ ; k \geq 0 \} \cup \{ 3^k \theta^- ; k \geq 0 \} \).

**Proof.** It suffices to prove for the case when \( a_0 \) is the root of \( M_{a_0} \).

First we prove that in the dynamical plan of \( f_{0,a_0} \), the wake \( W \) bounded by \( R_{0,a_0}^\infty(\theta^\pm) \) (that is also the wake containing the critical value \( v_{0,a_0} \)) does not contain the critical point \( \frac{-2a}{3} \). Let \( \tilde{W} \) be the wake bounding \( -\frac{2a}{3} \) and bounded by \( R_{0,a_0}^\infty(\tilde{\theta}^\pm) \). Suppose the contrary, then we have \( \tilde{W} \subset W \). Hence \( f_{0,a_0}^k|_{W \setminus \tilde{W}} \) is injective and proper, where \( k \) is the period of the parabolic cycle.

Since \( 3^{k-1} \theta^\pm = \tilde{\theta}^\pm \), we conclude that \( f_{0,a_0}^k(W \setminus \tilde{W}) \subset \tilde{W} \). However on the other hand since \( \tilde{W} \subset W \), the wake bounded by \( R_{0,a_0}^\infty(3^{k-1} \theta^\pm) \) contains the wake bounded by \( R_{0,a_0}^\infty(3^{k-1} \tilde{\theta}^\pm) \). Thus \( f_{0,a_0}(f_{0,a_0}^{-1}(W \setminus \tilde{W})) \) will not intersect \( W \), a contradiction.

Next we prove that \( W \) contains no other point in the parabolic periodic cycle. Suppose the contrary, then \( W \) contains another wake \( W' \) bounded by some \( R_{0,a_0}^\infty(3^l \theta^\pm) \) with \( 1 \leq l \leq k - 2 \). Thus \( f_{0,a_0}^l(W) = W' \). Apply Denjoy-Wolff theorem we obtain \( f_{0,a_0}^l(W) \) converges to \( \infty \), a contradiction.

\[ \square \]
3 Description of $\mathcal{H}^\lambda$, $\lambda = e^{2\pi ip/q}$

3.1 Existence of adjacent and bitransitive components

In this subsection we prove existence of adjacent and bitransitive components satisfying given combinatorics type. The main tool is pinching deformation.

When $(\lambda, a) \in (0,1) \times (-\sqrt{3\lambda}, \sqrt{3\lambda})$, $f_{\lambda,a} = f_{\lambda,a}$. Therefore the maximal linearization domain $\Omega_{\lambda,a}$ is symmetric with respect to $x-$axis. Hence $\Omega_{\lambda,a}$ intersects $\mathbb{R}^+$ at a unique point $z_0$. Denote by $c_{\lambda,a}^+$ (resp. $c_{\lambda,a}^-$) the critical point of $f_{\lambda,a}$ contained in upper-half (resp. lower-half) plane, then by symmetricity, $c_{\lambda,a}^\pm \in \partial \Omega_{\lambda,a}$. Set $\tilde{\Omega}_{\lambda,a} = f_{\lambda,a}(\Omega_{\lambda,a})$, $z'_0 = J_{\lambda,a} \cap \mathbb{R}^+$, $l_0 = [z_0, z'_0]$. Then

$$f_{\lambda,a}^{-1}(\Omega_{\lambda,a} \cup l_0) = \overline{(\Omega_{\lambda,a} \cup l_0 \cup (\Omega_{\lambda,a}^+ \cup l_+)) \cup (\Omega_{\lambda,a}^- \cup l_-)},$$

where $\Omega_{\lambda,a}^\pm$ (resp. $l_\pm$) is the preimage of $f_{\lambda,a}^{-1}(\Omega_{\lambda,a})$ (resp. $l_0$) contained in the upper/lower-half plane and $\Omega_{\lambda,a}^\pm \cap \Omega_{\lambda,a} = c_{\lambda,a}^\pm$. Therefore $B_{\lambda,a}(0) \setminus f_{\lambda,a}^{-1}(\Omega_{\lambda,a} \cup l_0)$ has three connected components $D_+, D_0, D_-$ (written in cycle order) mapped 1:1 to $B_{\lambda,a}(0) \setminus (\Omega_{\lambda,a} \cup [z_0, z'_0])$ by $f_{\lambda,a}$, where $\tilde{z}_0 = \partial \tilde{\Omega}_{\lambda,a} \cap \mathbb{R}^+$.

Let $\phi_{\lambda,a} : \tilde{\Omega}_{\lambda,a} \to \mathbb{D}$ be the Koenigs coordinate at 0 of $f_{\lambda,a}$ normalised by $\phi_{\lambda,a}(c_{\lambda,a}^+) = 1$. Take a branch of log such that log(1) = 0. Set $\varphi_{\lambda,a} = -\log \phi_{\lambda,a}$. Then clearly $\varphi_{\lambda,a}(\partial \Omega_{\lambda,a}) = i(-2\pi, 0]$. Define $I_\lambda(a) = \text{Im}(-\varphi_{\lambda,a}(c_{\lambda,a}^-))$. Then by definition $I$ is strictly positive. It is not hard to prove the following lemma by quasiconformal deformation:

**Lemma 3.1.1.** $I_\lambda : (-\sqrt{3}, \sqrt{3}) \to (0, 2\pi)$ is strictly decreasing with $\lim_{a \to \sqrt{3}} I_\lambda(a) = 0$ and $\lim_{a \to -\sqrt{3}} I_\lambda(a) = 2\pi$. 

13
Let $L_{p/q}^0$ be the line passing 0 and $q \log \lambda + 2\pi (q-p)i$. For $k = 0, 1, \ldots, q-1$, let $L_{p/q}^k = L_{p/q}^0 - k \log \lambda \pmod{2\pi i}$ be the line intersecting $i(-2\pi, 0)$. The open Strip $S$ bounded by $L_{p/q}^0, L_{p/q}^0 - 2\pi i$ is divided into $q$ sub-strips by $L_{p/q}^k$. Let $S_k$ be the open sub-strip whose upper boundary is $L_{p/q}^k$. Let $\tilde{L}_{p/q}^k$ be the central line of $S_k$ and suppose that it intersects $iR$ at $i\tilde{y}_k$. Notice that $L_{p/q}^k, \tilde{L}_{p/q}^k, \tilde{y}_k$ do not depend on $(\lambda, a)$. For any $0 \leq m \leq q$, fix $a_m \in (-\sqrt{3}, \sqrt{3})$ such that $-I_\lambda(a) \notin \{y_0, \ldots, y_{q-1}\}$ and the interval $(-I_\lambda(a), 0)$ contains exactly $m$ elements in $\{y_0, \ldots, y_{q-1}\}$. Such $a_m$ necessarily exists by Lemma 3.1.1. For each $k$ pick an open strips $\tilde{S}_k$ centered at $\tilde{L}_{p/q}^k$ such that $\forall k, \phi_{\lambda, a_m}(c_{\lambda, a_m}) \notin \tilde{S}_k$. Then $\{S_k\}_{0 \leq k \leq q-1}$ defines a non-separating multi-annulus $\mathcal{A}$ in the quotient space of $f_{\lambda, a_m}$ by setting

$$\mathcal{A} = \pi \phi^{-1}_{\lambda, a_m}(\exp(-\bigcup_{k=0}^{q-1} \tilde{S}_k))$$

which by Theorem B.8 gives a converging pinching path $\psi_t \circ f_{\lambda, a} \circ \psi_t^{-1}$. Here we choose a different normalisation for $\psi_t$ by setting $\psi_t(0) = 0$ and $\psi_t(z) = z + o(1)$ at $\infty$. Therefore for $t \geq 0$ there exists $\lambda_t \in \mathbb{D}$ such that $\psi_t \circ f_{\lambda, a} \circ \psi_t^{-1} \in \text{Per}_1(\lambda_t)$ and the resulting limit $\psi_\infty \circ f_{\lambda, a} \circ \psi_\infty^{-1} \in \text{Per}_1(e^{2\pi i p/q})$ is of adjacent or bitransitive type.

Now we give a more detailed description for the pinching limit in terms of the $q$ external rays landing at 0. Recall that for every $0 \leq m \leq q$ we chose properly a parameter $a_m \in (-\sqrt{3}, \sqrt{3})$ such that $0 \leq m \leq q$ and construct a non-separating annulus $\mathcal{A}$ for $f_{\lambda, a_m}$. Notice that the 0-level skeleton for $\mathcal{A}$ is a $q$-cycle of rays with rotation number $p/q$ starting from 0 and landing at $\partial B_{\lambda, a}^\times(0)$. Denote by
\{x_i(a_m)\} this corresponding landing cycle. Then there is a cycle of external rays with rotation number \(p/q\) landing at \(x_i(a_m)\). Denote by \(\{\theta_i(a_m)\}\) the corresponding angles for these external rays, then these angles form a \(q\)-cycle with rotation number \(p/q\) under multiplication by 3. We claim that \(m \mapsto \{\theta_i(a_m)\}\) is injective. Indeed, \(m\) is the number of points in \(\{x_i(a_m)\}\) contained in the segment of counter-clockwise direction between \(z_+, z_-\), where \(z_+, z_-\) are the landing points of \(l_+, l_-\). Since \(z_+, z_-\) are preimages of \(f_{\lambda,a_m}^{-1}(0)\) other than 0, \(R_{\lambda,a_m}(\frac{1}{3}), R_{\lambda,a_m}(\frac{2}{3})\) lands at \(z_+, z_-\) respectively. Hence \(m\) is also the number of angles in \(\{\theta_i(a_m)\}\) between \((\frac{1}{3}, \frac{2}{3})\). Thus \(m \mapsto \{\theta_i(a_m)\}\) is injective. Since the pinching deformation preserves external rays, we get \(q + 1\) parameters in \(\text{Per}_{1}(e^{2\pi ip/q})\) which are contained respectively in \(q + 1\) different parabolic components.

![Figure 3: An illustration for pinching bands of depth 0 in \(B_{\lambda,a}^*\) for \(\frac{p}{q} = \frac{2}{3}\). The resulting pinching limit is in an 1-component.](image)

**Remark 3.1.2.** By Theorem A.2 there are exactly \(q + 1\) cycles of rotation number \(p/q\) under multiplication by 3. Hence \(\{\theta_i(a_m)\}\) actually does not depend on the choice of \(a_m\): it is completely characterized by the number of angles between \((\frac{1}{3}, \frac{2}{3})\). Hence we denote such cycle by \(\Theta_m\).

**Definition 3.1.3.** Let \(0 \leq m \leq q\). We call an adjacent or bitransitive component of the family \(f_a\) a **m-component** if in which all the parameters have the external rays with angles in \(\Theta_m\) landing at 0 in the dynamical plan. In particular, an adjacent component is either a 0-component or a \(q\)-component.

**Definition 3.1.4.** Let \(0 \leq m \leq q - 1\). For the family \(g_{\lambda,c}\), an adjacent or bitransitive component \(\mathcal{B}\) is called a **m-component** if there are \(m\) repelling axis between 1 and \(c\) in the counterclockwise direction. In particular, an adjacent component is a 0-component.

From the discussion above, we see that \(m\)-component exists with every \(0 \leq m \leq q\) for the family \(f_{\lambda,a}\) and with every \(0 \leq m \leq q - 1\) for the family \(g_{\lambda,c}\).
3.2 Double parabolic parameters

Let \( \lambda = e^{2\pi i \frac{p}{q}} \). By Fatou-Leau Theorem we have Taylor expansions near 0:

\[
\begin{align*}
\lambda^q, \alpha(z) &= z + A_{p/q}(a)z^{q+1} + O(z^{q+2}), \\
\lambda^q, \beta(z) &= z + C_{p/q}(c)z^{q+1} + O(z^{q+2}).
\end{align*}
\]

**Definition 3.2.3.** We say that \( f_{\lambda,a} \) (resp. \( g_{\lambda,c} \)) is double parabolic if \( A_{p/q}(a) = 0 \) (resp. \( C_{p/q}(c) = 0 \)). For the family \( f_{\lambda,a} \), denote by \( A_{p/q} \) the collection of these parameters.

In this subsection we will show that there are exactly \( q \) double parabolic parameters for families \( f_{\lambda,a} \) and \( g_{\lambda,c} \).

**Lemma 3.2.2** ([2]). \( C_{p/q}(\frac{1}{c}) \) is a polynomial in \( c \) of degree \( q \).

**Proof.** Since \( C_{p/q}(c) \) is a polynomial in \( \frac{1}{c} \), \( C_{p/q}(\frac{1}{c}) \) is a polynomial in \( c \). When \( c \) tends to 0, the map \( g_c(z) = c^{-1}g_{\lambda,c}(cz) \) converges uniformly on compact subsets of \( \mathbb{C} \) to \( P_\lambda(z) = \lambda z(1-\frac{z}{2}) \). Let \( P_\lambda(z) = z + C_0z^q + 1 + O(z^{q+2}) \). Then \( C_0 \neq 0 \) because \( P_\lambda \) has only one parabolic basin. While from (7) we see that \( g_c(z) = z + c^qC_{p/q}(c)z^{q+1} + O(z^{q+2}) \). Therefore \( c^qC_{p/q}(c) \) converges to \( C_0 \) when \( c \to 0 \). Hence \( C_{p/q}(\frac{1}{c}) \) has degree \( q \).

By the above lemma and the relation between \( f_{\lambda,a}, g_{\lambda,c} \) given by Lemma 2.1.1, it suffices to find \( q \) different double parabolic parameters for the family \( f_{\lambda,a} \). Similarly we construct such parameters by pinching deformation.

Consider the family \( f_{\lambda,a} \) for \( 0 < \lambda < 1 \). Recall in Subsection 3.1 the construction of lines \( L_{p/q,k} \), the corresponding strips \( S_k \) and the corresponding central lines \( L_{p/q,k} \). \( 0 \leq k \leq q-1 \). Recall that \( L_{p/q,k} \) intersect \( \mathbb{R} \) at \( y_k \). For \( 0 \leq k \leq q-1 \), choose \( \bar{a}_m \in (-\sqrt{3}, \sqrt{3}) \) such that \( \varphi_{\lambda,a}(\bar{a}_m) = \bar{y}_m \). Notice that \( L_{p/q,k} \) cut \( S_k \) into two substrips \( S_k^+, S_k^- \). Let \( (L_{p/q,k})^+, (L_{p/q,k})^- \) be their central line. For each \( k \) pick narrow strip \( S_k^+ \), \( S_k^- \) centred at \( (L_{p/q,k})^+, (L_{p/q,k})^- \) respectively such that \( \varphi_{\lambda,\bar{a}}(c_\lambda^{\pm}a_m) \notin \bar{S}_k^+ \cup \bar{S}_k^- \).

Define a non-separating annulus for \( f_{\lambda,\bar{a}_m} : 
\mathcal{A} = \pi \phi_{\lambda,\bar{a}_m}^{-1}(\exp(-\bigcup_{k=0}^{q-1} \bar{S}_k^+ \cup \bar{S}_k^-))
\)

The corresponding pinching limit yields a double parabolic parameter in \( \text{Per}_1(e^{2\pi i \frac{p}{q}}) \). Moreover these parameters can be characterized by the two \( q \)-cycles of external rays landing at 0 in the corresponding dynamical plan. For every \( 0 \leq m \leq q-1 \), we obtain a double parabolic parameter associated with two cycles of angles \( \Theta_m, \Theta_{m+1} \) respectively (recall in Remark 3.1.2 for the definition of \( \Theta_m \)). The external rays with these angles land at 0.

**Definition 3.2.3.** Such double parabolic parameter is called \( m \)-type and is denoted by \( a_m \).

**Lemma 3.2.4.** A \( m \)-component for \( f_a \) can only have double parabolic parameters of type \( m, m-1 \) on its boundary.
Proof. Suppose the contrary that on the boundary there is a double parabolic parameter $a_0$ of type $n, n \neq m, m - 1$. Then the cycle with angle $\Theta_m$ will land a repelling $q$-cycle of $f_{a_0}$. Since the landing property is stable (cf. [5]), we conclude that for all $a$ in this $m$-component, $R^\infty_{X_m}(t)$ with $t \in \Theta_m$ lands at a repelling periodic point, a contradiction since these rays should land at $z = 0$. 

3.3 Parametrization of parabolic components

In this subsection always fix $\lambda = e^{2\pi i \frac{\ell}{7}}$. We parametrize $m$-components $\tilde{B}_m$ of family (3) by locating the free critical value in the immediate basins of the quadratic model $P_\lambda(z) = \lambda z + z^2$.

The critical point of $P_\lambda$ is $-\frac{1}{2}$. Denote by $B_\lambda^m(0)$ the immediate basin of $P_\lambda$ containing $-\frac{1}{2}$. There are exactly $q$ immediate basins attached at 0 in the cyclic order $B_\lambda^0(0), \ldots, B_\lambda^{q-1}(0)$. Let $\Omega^0_\lambda$ be a maximal admissible petal of $P_\lambda|_{B_\lambda^m(0)}$. Let $\phi : \Omega^0_\lambda \to \mathbb{H}$ (H is the right half plan) be the Fatou coordinate normalised by $\phi(-\frac{1}{2}) = 0$. Let $1 \leq m \leq q$. Define $\Omega^0_{\lambda,c,m} := P_\lambda^m(\Omega^0_\lambda,0)$. For any $0 \leq k \leq q - 1$, $n \in \mathbb{Z}$ such that $np + k = 0$ (mod $q$), define $\Omega^m_k$ and $\Omega^n_k$ as in (29). For simplicity we will omit index $\lambda$ for all terms related to family $g_c := g_{\lambda,c}$.

Now consider family $g_c$. Let $\Omega^m_{c,0} \subset B^m_{\lambda,0}(0)$ be the standard maximal petal of $g^m_{\lambda}|_{B^m_{\lambda,0}(0)}$. For $1 \leq m \leq q - 1$, let $\Omega^m_{c,s} := g^m_{\lambda}(\Omega^m_{c,0})$ where $1 \leq s \leq q - 1$ is the smallest integer such that $sp = m$ (mod $q$). For any $0 \leq k \leq q - 1$, $n \in \mathbb{Z}$ such that $np + k = 0$ (mod $q$), $\Omega^m_{c,k}$ is defined as in (29). The degree of $g^m_{\lambda}|_{B^m_{\lambda,0}(0)}$ is 3 if $m = 0$ and is 4 if $1 \leq m \leq q - 1$. In the latter case, denote by $1, cr_1, cr_2$ respectively the three critical points of $g^m_{\lambda}|_{B^m_{\lambda,0}(0)}$; denote by $c, cr_1, cr_2$ respectively the three critical points of $g^m_{\lambda}|_{B^m_{\lambda,0}(0)}$. Notice that $g^m_{\lambda}(cr_1,2) = c, g^m_{\lambda}^{-1}(cr_1,2) = 1$.

We introduce the following loci in order to distinguish which critical point is on the maximal petal:

$$
\tilde{\mathcal{D}}_0 = \{ c \in \tilde{B}_m; 1 \in \partial \Omega^0_{c,0} \text{ but } c \not\in \partial \Omega^0_{c,0} \}
$$

$$
\tilde{\mathcal{I}}_0 = \{ c \in \tilde{B}_m; 1, c \in \partial \Omega^0_{c,0} \}
$$

$$
\tilde{\mathcal{D}}_m = \{ c \in \tilde{B}_m; 1 \in \partial \Omega^0_{c,0} \text{ but } cr_1, cr_2 \not\in \partial \Omega^0_{c,0} \}, \text{ for } 1 \leq m \leq q - 1
$$

$$
\tilde{\mathcal{I}}_m = \{ c \in \tilde{B}_m; 1 \text{ and one of } cr_1, cr_2 \in \partial \Omega^0_{c,0} \}, \text{ for } 1 \leq m \leq q - 1
$$

Let $\hat{\mathcal{D}}_m = \tilde{B}_m \setminus (\tilde{\mathcal{I}}_m \cup \tilde{\mathcal{D}}_m)$. Clearly for $c \in \hat{\mathcal{D}}_m$, $1 \not\in \partial \Omega^0_{c,0}$. In particular, if $1 \leq m \leq q$, then $1 \not\in \partial \Omega^0_{c,0}$, hence one of $cr_1, cr_2 \in \partial \Omega^0_{c,0}$, and $\Omega^0_{c,m} \subset B^m_{\lambda,m}(0)$ is a maximal petal for $g^m_{\lambda}|_{B^m_{\lambda,m}(0)}$, having $c$ on its boundary but not containing $cr_1, cr_2$.

$$
\hat{\mathcal{D}}_0 = \{ c \in \hat{B}_m; c \in \partial \Omega^0_{c,0} \text{ but } 1 \not\in \partial \Omega^0_{c,0} \}
$$

$$
\hat{\mathcal{D}}_m \subset \{ c \in \hat{B}_m; c \in \partial \Omega^0_{c,m} \text{ but } cr_1, cr_2 \not\in \partial \Omega^0_{c,m} \}, \text{ for } 1 \leq m \leq q - 1
$$

Remark 3.3.1. By Proposition $\square$ $\tilde{\mathcal{D}}_m, \hat{\mathcal{D}}_m$ are open for $0 \leq m \leq q - 1$.

Lemma 3.3.2. Let $c \in \tilde{B}_m$ and $\hat{\phi}_c$ any Fatou coordinate for $g^m_{\lambda}|_{B^m_{\lambda,0}(0)}$. For $1 \leq m \leq q - 1$, if $g^m_{\lambda}(cr_1,2), g^m_{\lambda}(1)$ are contained $\Omega^0_{c,0}$ and $\hat{\phi}_c(cr_1,2) = \hat{\phi}_c(1)$, then $cr_1 = cr_2$; for $m = 0$, $g^m_{\lambda}(c), g^m_{\lambda}(1)$ are contained $\Omega^0_{c,0}$ and $\hat{\phi}_c(c) = \hat{\phi}_c(1)$, then $1 = c$. 

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Proof. We only do the proof for $1 \leq m \leq q - 1$, the case $m = 0$ is similar. By hypothesis, since $\tilde{\phi}_c$ is injective on $\Omega^0_{c,0}$, we have $g^q_c(0,1) = g^q_c(1)$. If $cr_1, cr_2, 1$ are not distinct, then $g^q_c(1)$ has at least 5 preimages counting multiplicity, contradicting $deg(g^q_c|_{B^c_{c,0}(0)}) = 4$. 

For $0 \leq m \leq q - 1$ and $c \in \hat{\mathcal{D}}_m$ (resp. $c \in \hat{\mathcal{D}}_0$), let $\phi_c : \hat{\Omega}^m_{c,0} \longrightarrow \hat{\mathbb{H}}$ be the Fatou coordinate of $g^q_c|_{B^c_{c,0}(0)}$ normalised by $\phi_c(1) = 0$ (resp. $\phi_c(c) = 0$). Define $h_c : \Omega^m_{c,0} \longrightarrow \Omega^0_{c,0}$ by $h_c = \phi^{-1} \circ \phi_c$. Pull back $h_c$ by $g_c$ and $P_\lambda$ until we reach the critical value $g_c(c)$ (resp. $g_c(1)$):

\[
\begin{array}{c}
\Omega^n_{c,m+p} \\
\downarrow h_c
\end{array} \xrightarrow{g_c} \begin{array}{c}
\Omega^{n-1}_{c,m+2p} \\
\downarrow h_c
\end{array} \xrightarrow{g_c} \ldots \xrightarrow{g_c} \begin{array}{c}
\Omega^1_{c,q-p} \\
\downarrow h_c
\end{array} \xrightarrow{g_c} \begin{array}{c}
\Omega^0_{c,0}
\end{array}
\]  

where $n_c$ is the smallest integer such that $\Omega^n_{c,m+p}$ contains $g_c(c)$ (resp. $g_c(1)$). Moreover at each step $h_c$ is conformal.

Similarly, for $1 \leq m \leq q - 1$ and $c \in \hat{\mathcal{D}}_m$, let $\tilde{\phi}_c : \tilde{\Omega}^m_{c,m} \longrightarrow \hat{\mathbb{H}}$ be the Fatou coordinate of $g^q_c|_{B^c_{c,m}(0)}$ normalised by $\tilde{\phi}_c(c) = 0$, where $\hat{\mathbb{H}} = \tilde{\phi}_c(\mathbb{C}_{c,m})$. Notice that $\tilde{\Omega}^m_{c,m}$ is an admissible maximal petal for $g^q_c|_{B^c_{c,m}(0)}$, hence so is $\Omega^m_{c,0} = \phi^{-1}(\tilde{\mathbb{H}})$ for $P^1_\lambda|_{B^c_{c,m}(0)}$. Define $\tilde{h}_c : \tilde{\Omega}^m_{c,m} = : \tilde{\Omega}^m_{c,m} \longrightarrow \tilde{\Omega}^0_{c,0}$ by $\tilde{h}_c = \phi^{-1} \circ \tilde{\phi}_c$. Pull back $\tilde{h}_c$ by $g_c$ and $P_\lambda$ until we reach the critical value $g_c(1)$:

\[
\begin{array}{c}
\tilde{\Omega}^{n}_{c,2p-m} \\
\downarrow \tilde{h}_c
\end{array} \xrightarrow{g_c} \begin{array}{c}
\tilde{\Omega}^{n-1}_{c,2p-m} \\
\downarrow \tilde{h}_c
\end{array} \xrightarrow{g_c} \ldots \xrightarrow{g_c} \begin{array}{c}
\tilde{\Omega}^1_{c,q-p} \\
\downarrow \tilde{h}_c
\end{array} \xrightarrow{g_c} \begin{array}{c}
\tilde{\Omega}^0_{c,0}
\end{array}
\]

Define four holomorphic mappings

\[
\begin{align*}
\Phi_0^{adj} &: \mathcal{D}_0 \longrightarrow B^*_p(0), \ c \mapsto h_c(g_c(c)) \\
\tilde{\Phi}_0^{adj} &: \hat{\mathcal{D}}_0 \longrightarrow B^*_p(0), \ c \mapsto h_c(g_c(1)) \\
\Phi_m^{bit} &: \mathcal{D}_m \longrightarrow \mathcal{B}^*_p(0) \setminus \Omega^s_{m+p} \setminus \{0\}, \ c \mapsto h_c(g_c(c)) \\
\tilde{\Phi}_m^{bit} &: \hat{\mathcal{D}}_m \longrightarrow \hat{\mathcal{B}}^*_p(0) \setminus \Omega^\tilde{s}_{m+p} \setminus \{0\}, \ c \mapsto \tilde{h}_c(g_c(1))
\end{align*}
\]

where $2 \leq s \leq q$, $0 \leq l \leq q - 1$ are the unique integers such that $-sp + m + p = 0$ (mod $q$) and $lp + p - m = 0$ (mod $q$).

**Lemma 3.3.3.** Let $\Phi$ be one of the four holomorphic maps in \{12\}. Then $\Phi$ satisfies the following property: let $(c_n) \subset \mathcal{B}_m$ be a sequence contained in the domain of definition of $\Phi$ which converges to $\partial \mathcal{B}_m$ (resp. $\mathcal{F}_m$), then $\Phi(c_n)$ converges to the boundary of the corresponding immediate basin (resp. $\partial \Omega^{-1}_p \setminus \{0\}, \partial \Omega^{-s}_{m+p} \setminus \{0\}$ or $\partial \Omega^\tilde{s}_{m+p} \setminus \{0\}$).

**Proof.** We only do the proof for $\Phi_m^{bit}$, the others are similar. First we justify that $\Phi_m^{bit}(\hat{\mathcal{D}}_m) \subset B^*_p(0) \setminus \Omega^s_{m+p}$. Since $\Phi_m^{bit}$ is open, it suffices to prove that its image does not intersect $\Omega^s_{m+p}$.
If not, then there exists $c \in \mathcal{D}_m$ such that $g_c(c) \in \Omega_{c, \alpha}^{-s}$, hence $g_c^q(cr_1) = g_c^q(cr_2) \in \Omega_{c, \alpha}^{-q}$. Let $\Lambda = g_c^{-q}(\Omega_{c, \alpha}^{-q}) \cap B_{c,0}(0)$. Then $g^q : \Lambda \to \Omega_{c, \alpha}^{-q}$ is of degree 4. Since $g_c$ is a polynomial, each component of $\Lambda$ is simply connected. Applying Riemann-Hurwitz formula one easily sees that $\Lambda$ should contain 3 critical points of $g_c^q$. However $1 \not\in \Lambda$ since $1 \in \partial D^0_{c,0}$, so $g_c^q(1) \in \partial \Omega_{c, \alpha}^{-q}$.

Next we verify properness of $\Phi_m^{bit}$. Clearly $\partial \mathcal{D}_m \subset \mathcal{I}_m \cup \partial \mathcal{B}_m$. Let $(c_n) \subset \mathcal{D}_m$ be a sequence converging to some $c_0 \in \partial \mathcal{D}_m$. If $c_0 \in \mathcal{I}_m$, then by Corollary D.2 $g_{c_n}^k(c_n)$ is compactly contained in $\Omega_{c_n,0}^0$ for $n$ large enough, where $k$ is the smallest integer such that $g_{c_n}^k(c_n) \in B_{c_n,0}(0)$. Moreover $h_{c_n}(g_{c_n}^k(c_n))$ converges to $\partial \Omega_{c_n,0}^0 \setminus \{0\}$ since $\Re\{\phi_{c_n}(g_{c_n}^k(c_n)) - \phi_{c_n}(g_{c_n}^q(1))\}$ converges to 0 and $\Im\{\phi_{c_n}(g_{c_n}^k(c_n)) - \phi_{c_n}(g_{c_n}^q(1))\}$ remains bounded. (In fact we have $h_{c_n}(g_{c_n}^k(c_n)) \to 0$ in $\partial \Omega_{c_n,0}^0 \setminus \{0\}$ with $\phi(0) = I_m(c_0)$). This implies that $\Phi_m^{bit}(c_n) = h_{c_n}(c_n) = P^{-k}h_{c_n}(g_{c_n}^k(c_n))$ converges to $\partial \Omega_{c_n,0}^0 \setminus \{0\}$.

So let $c_0 \in \partial \mathcal{B}_m$. We want to prove that $\Phi_m^{bit}(c_n)$ converges to $\partial B_{c_0,0}^m(0)$. Suppose the contrary that, up to taking a subsequence, $\Phi_m^{bit}(c_n)$ converges to $z_0 \in B_{c_0,0}^m(0)$. Clearly $\{h_{c_n}(\Omega_{c_n,0}^0)\}^{-1}$ is a normal family on $\Omega_{c_0,0}^0$. Up to taking a subsequence, suppose $h_{c_n}$ converges uniformly to $\psi$. We claim that $\psi(\Omega_0^0)$ does not intersect $J_{c_0}$. Indeed, if not, then there exists a repelling periodic point $x_{c_0} \in J_{c_0} \cap \psi(\Omega_0^0)$ (since the repelling cycles are dense in $J_{c_0}$). Notice that $x_{c_0}$ moves holomorphically for $c$ in a small neighborhood of $c_0$, which gives a repelling periodic point $x_c$ of $g_c$. However for $n$ large enough, $x_c \in h_{c_n}^{-1}(\Omega_0^0) \subset B_{c_n,0}^*$, a contradiction. Therefore $\psi(\Omega_0^0)$ and $g_{c_n}^s(1)$ are contained in the same Fatou component of $g_{c_0}$. On the other hand, take $s \geq 0$ such that $P^{-s}_\lambda(z_0) \in \Omega_{c_0,0}^0$, then by diagram (10) $g_{c_n}^s(c_n) \in \Omega_{c_n,0}^0$ and $h_{c_n}(g_{c_n}^s(c_n))$ converges to $P^{-s}_\lambda(z_0)$. Therefore

$$\psi(P^{-s}_\lambda(z_0)) = \lim_{n \to \infty} h_{c_n}^{-1}(h_{c_n}(g_{c_n}^s(c_n))) = g_{c_0}^s(0) = \psi(0)^0$$

which means that $1, c_0$ are eventually attracted by the same Fatou component, contradicts $c_0 \in \partial \mathcal{B}_m$.

**Proposition 3.3.4.** $\Phi_m^{bit}, \tilde{\Phi}_m^{bit}$ in (12) are conformal, i.e. injective and surjective.

**Proof.** We only do for $\Phi_m^{bit}$, the other is similar. By Lemma 3.3.3 it suffices to verify injectivity. Suppose $\Phi_m^{bit}(c_1) = \Phi_m^{bit}(c_2)$. Starting from the conjugacy $g_{c_2} \circ \phi_{c_2}^{-1} \circ \phi_{c_1} = \phi_{c_2}^{-1} \circ \phi_{c_1} \circ g_{c_1}$ on $\Omega_{c_1,0}^0$, lift $\phi_{c_2}^{-1} \circ \phi_{c_1}$ by $g_{c_1}, g_{c_2}$ to $\Omega_{c_1,k}^n \subset B_{c_1,k}^*$ (recall definition of $\Omega_{c_1,k}^n$ at the beginning of 3.3). Denote by $\varphi$ the lifting of $\phi_{c_2}^{-1} \circ \phi_{c_1}$ on $g_{c_1}$. The only possible issue that might stop the lifting is when $k = m+p$ and $g_{c_1}(c_1) \in \Omega_{c_1,1}^n$. However $\Phi_m^{bit}(c_1) = \Phi_m^{bit}(c_2)$ implies $\varphi(g_{c_1}(c_1)) = g_{c_2}(c_2)$, which ensures that the lifting is still valid (also notice that $g_{c_1}|_{B_{c_1,m}} g_{c_2}|_{B_{c_2,m}}$ are of degree 2). So $\varphi$ is extended to $\bigcup B_{c_1,i,k}$ and hence to all filled-in Julia set $K_{c_1}$.

On the other hand, take a connected open set $\mathcal{V} \subset \mathcal{B}_m$ linking $c_1, c_2$, on which the Böttcher coordinate at $\infty$ depends analytically on $c$. Therefore for $c \in \mathcal{V}$, $\psi_c := (\phi_c^{-\infty})^{-1} \circ \phi_c^{\infty}$ is a dynamical holomorphic motion on $C \setminus K_c$, and can be quasiconformally extend to $\mathbb{C}$ (Slodkowski’s theorem). In particular, $\psi_{c_2}$ conjugates $g_{c_1}$ to $g_{c_2}$ on $\mathcal{V} \setminus K_{c_1}$, coincides with $\varphi$ on $\partial K_{c_1}$. Applying Rickman’s lemma, the global conjugacy defined by sewing $\varphi$ and $\psi_{c_2}$ is conformal, i.e. identity, so $c_1 = c_2$.

The injectivity for $\Phi_0^{adj}, \tilde{\Phi}_0^{adj}$ is more subtle:

**Proposition 3.3.5.** Let $c_1, c_2 \in \mathcal{D}_0$ be such that $\Phi_0^{adj}(c_1) = \Phi_0^{adj}(c_2)$.


1. If $\Phi_0^{adj}(c_1) \in \overline{\Omega_{c_p}^{-1}}$, then $c_1 = c_2$.

2. If not, then for $i = 1, 2$, $(g_e^{-1})^{-1}(\Omega_{c_i}^{-1})$ has two connected components, one of which has the figure of a filled eight, denoted by $H_{c_i}$. $H_{c_i}$ cuts $B_{e_i, 0}^*$ into two connected components $\Delta_{c_i}^+, \Delta_{c_i}^-$ (+ is on the right-hand side of −). If $c_i \in \Delta_{c_i}^+$ for $i = 1, 2$ or $c_i \in \Delta_{c_i}^-$ for $i = 1, 2$, then $c_1 = c_2$.

The same result holds for $\Phi_0^{adj}$.

Proof. We do the proof for the second point. The first is similar. Without loss of generality suppose $c_i \in \Delta_{c_i}^-$ for $i = 1, 2$. Let us what we did at the beginning of the proof of Proposition 3.3.4 lift $\varphi = \phi_{e_i}^{-1} \circ \phi_{c_i}$ by $g_{c_1}, g_{c_2}$. Similarly, when we lift $\varphi$ to $\Omega_{c_i, 0}^n$ with $n_0$ the smallest integer such that $g_{c_i}(c_1) \in \Omega_{c_i}^{n_0}$, it is not clear whether $\varphi$ can be lifted once more to $\Omega_{c_i, 0}^{n+1}$, where $n_0 = k_0 q$ for some $k_0 \geq 1$. Notice that for $i = 1, 2, 1 \leq k \leq k_0 + 1$, $\Omega_{c_i, 0}^{kq}$ is simply connected with piecewise smooth boundary intersecting $\partial B_{e_i, 0}^*$ at $2^k$ points, and $\Omega_{c_i, 0}^{kq} \setminus \Omega_{c_i, 0}^{(k-1)q}$ has $2^{k-1}$ connected components $D_{e_0 \cdots e_{k-2}}^i$ with

$$
\epsilon_j = \begin{cases} 0, & \text{if } g_{c_i}^{[q]}(D_{e_0 \cdots e_{k-2}}^i \cap \Omega_{c_i, 0}^{kq}) \subset \Delta_{c_i}^+ \\ 1, & \text{if } g_{c_i}^{[q]}(D_{e_0 \cdots e_{k-2}}^i \cap \Omega_{c_i, 0}^{kq}) \subset \Delta_{c_i}^- \end{cases}
$$

Since $\Phi_0^{adj}(c_1) = \Phi_0^{adj}(c_2)$, there exists $\tilde{c}_0 \cdots \tilde{c}_k - 2$ such that $g_{c_i}(c_i) \in D_{e_0 \cdots e_{k-2}}^i$. So $c_i \in D_{\tilde{c}_0 \cdots \tilde{c}_k - 2}^i$. But by hypothesis $c_i \in \Delta_{c_i}^-$, so $\delta_i = 0$ for $i = 1, 2$. Therefore we can extend $\varphi$ to $\Omega_{c_i, 0}^{(k+1)q}$ by assigning injectively $D_{e_0 \cdots e_{k-1}}^i$ onto $D_{e_0 \cdots e_{k-1}}^i$. Notice that $\Omega_{c_i, 0}^{(k+1)q}$ contain the two critical points and the two cocritical points of $g_{c_i}^{[q]}|_{B_{e_i, 0}^*}$, so the lifting process is valid for all $k \geq k_0 + 1$, so $\varphi$ can be extended to $B_{e_i, 0}^*$, and hence to the filled-in Julia set $K_{c_i}$. Now apply the same strategy as in the second paragraph of the proof of Proposition 3.3.4 we conclude that $c_1 = c_2$.

3.4 Describing the special locus $\tilde{I}_m$

Recall the definition of $\tilde{I}_m$ in (8). For $m = 0, c \in \tilde{I}_0$, define $I_0(c) = \mathfrak{M}(\phi_e(c) - \phi_e(1))$; for $1 \leq m \leq q - 1$, $c \in \tilde{I}_m$, define $I_m(c) = \mathfrak{M}(\phi_e(c r) - \phi_e(1))$ where $c r = c r_1$ or $c r_2$, since $\phi_e(c r_1) = \phi_e(c r_2)$.

Lemma 3.4.1. For every $m$-component $\tilde{B}_m$, $I_m^{-1}(0) \neq \emptyset$.

Proof. Let $\Phi$ be the corresponding mapping in (12) defined on $\tilde{D}_m$. Clearly $P^n(\frac{1}{\Phi(N)} \cap B_{m+p}^*) \in \partial \Phi(N) \cap B_{m+p}^*(0)$. Thus by Lemma 3.3.3 there exists a sequence of parameters $(c_k) \subset N$ converging to $c_0 \in \tilde{I}_m$ such that $\Phi(c_k)$ converges to $P^n(\frac{1}{\Phi(N)}$. Recall that $\Phi(c) = h_e(g_e(c))$, thus $P^n(\Phi(c) = h_e(g_e^{-1}(g_e^{-1}(c)))$ converges to $P^n(\Phi(c) = h_e(g_e^{-1}(c)))$, which implies that $\phi_{c_k}(g_e^{-1}(c)) - \phi_{c_k}(g_e^{-1}(c)) \to 0$ if $m = 0$ and $\phi_{c_k}(g_e^{-1}(c r_2)) - \phi_{c_k}(g_e^{-1}(c r_2)) \to 0$ if $1 \leq m \leq q - 1$. Thus at $c_0$ we have respectively $\phi_{c_k}(c) = \phi_{c_k}(1)$ and $\phi_{c_k}(c r_2) = \phi_{c_k}(1)$. By Lemma 3.4.2 $c_0 = 1$ resp. $c r_1, c r_2 = 1$, i.e. $I_m(c_0) = 0$.

Corollary 3.4.2. For family $g_e$ the 0-component is unique. It contains $c = 1$ and is symmetric with respect to $\tau: c \mapsto 1 / c$.

Proof. By Lemma 3.4.1 $I_m^{-1}(0) \subset \tilde{B}_0$, while by Lemma 3.3.2 $I_0^{-1}(0) = 1$. Thus $\tilde{B}_0$ is unique. Since $\tau(\tilde{B}_0)$ is also a 0-component, hence $\tau(\tilde{B}_0) = \tilde{B}_0$. 

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Lemma 3.4.3. Let \(0 \leq m \leq q - 1\), \(\tilde{B}_m\) be a \(m\)-component, \(\tilde{I}_m\) be as in \([8]\). Then \(I_m : \tilde{I}_m \to (-\infty, +\infty)\) is injective.

Proof. For \(1 \leq m \leq q - 1\), adapt the proof of Proposition 3.3.4 for \(m = 0\) adapt that of Proposition 3.3.5.

Lemma 3.4.4. If there exists \(c_0 \in \tilde{I}_m\) such that \(t_0 := I_m(c_0) > 0\) (resp. \(< 0\)), then for any \(t \geq 0\) (resp. \(< 0\)) there is \(c_t \in \tilde{I}_m\) with \(I_m(c_t) = t\). Moreover \(I_m^{-1}((-\infty, 0)), I_m^{-1}((0, +\infty))\) are curves parametrized by \(I_m^{-1}\).

Proof. This can be shown by quasiconformal deformation.

Lemma 3.4.5. Let \((c_k) \subset \tilde{I}_m\) be a sequence of parameters. If

1. \(I_m(c_k) \to \pm \infty\), then \(c_k\) converges to a double parabolic parameter;
2. \(I_m(c_k) \to 0\), then \(c_k\) converges to \(I_m^{-1}(0)\).

Proof. Let \(c_0\) be any accumulation point of \(c_k\).

1. If \(I_m(c_k) \to \pm \infty\), then clearly \(c_0 \in \partial \tilde{B}_m\). If \(c_0\) is not double parabolic, then only one critical point is in the immediate basin, say 1. One can then apply Proposition D.3 which implies that for \(c\) near \(c_0\), 1 is always on the boundary of the maximal petal of \(g_m^k\) while the other critical points of \(g_m^k\) are not, contradicting the definition of \(\tilde{I}_m\).

2. If \(I_m(c_k) \to 0\). It suffices to prove that \(c_0 \in \tilde{B}_m\), since then by taking limit in \(k\), we get \(I_m(c_0) = 0\). For the same reason as above, if \(c_0\) is not double parabolic, then \(c_0 \notin \partial \tilde{B}_m\) and \(c_0 \in \tilde{B}_m\). So it remains to show that \(c_0\) is not double parabolic. Suppose the contrary. Without loss of generality we assume that \(I_m(c_k) > 0\). Then by the first point and Lemma 3.4.4, \(I_m^{-1}((0, +\infty))\) is a curve separating \(\tilde{B}_m\) into two simply connected components (\(\tilde{B}_m\) is simply connected, Lemma 2.2.4). Let \(N\) be the one not containing \(I_m^{-1}(0)\). This \(N\) exists since \(I_m^{-1}(0)\) is a single point by Lemma 3.4.3. Then \(N \setminus \tilde{I}_m\) is \(\mathcal{D}_m\) or \(\tilde{D}_m\). But recall that \(\partial \mathcal{D}_m \cap \tilde{B}_m = \partial \tilde{D}_m \cap \tilde{B}_m = \tilde{I}_m\), so \(I_m^{-1}(0) \in \partial (N \setminus \tilde{I}_m)\), a contradiction.

Proposition 3.4.6. For \(0 \leq m \leq q - 1\), \(\tilde{I}_m\) is a curve parametrized by \(I_m^{-1} : (-\infty, +\infty) \to \tilde{I}_m\). Moreover, when \(q > 1\), \(\tilde{I}_m\) has two different end points; when \(q = 1\), \(\tilde{I}_0\) has only one end point \(-1\).

Proof. First we prove that \(I_m^{-1}\) is a parametrisation. By Lemma 3.4.4 and 3.4.5, it suffices to prove that \(I_m^{-1}((-\infty, 0))\) and \(I_m^{-1}((0, +\infty))\) are not empty. Consider the case \(m = 0\): by Corollary 3.4.2 \(\tilde{B}_0\) is unique and symmetric with respect to \(\tau : c \mapsto 1/c\). Moreover it is easy to see that \(\tau(\tilde{D}_0) = \tilde{D}_0\), which are non empty. If both \(I_0^{-1}((-\infty, 0)), I_0^{-1}((0, +\infty))\) are empty, then one of \(\tilde{D}_0, \tilde{D}_0\) is empty, a contradiction. So suppose \(I_0^{-1}((-\infty, 0)) \neq \emptyset\), hence \(\tau(I_0^{-1}((-\infty, 0))) = I_0^{-1}((0, +\infty)) \neq \emptyset\).

Next we prove for \(1 \leq m \leq q - 1\). By Proposition 3.3.4 \(\Phi_m^* : \tilde{D}_m \to B_{m+1} \setminus \{0\}\) is conformal, so for any \(z \in \partial \Omega \setminus \{0\}\), there exists \(c_n \to c_0 \in \tilde{I}_m\) such that \(\Phi_m^*(c_n) \to z\). Thus \(I_m(c_0)\) can take arbitrary value in \(\mathbb{R}\) by choosing properly \(z\).

Now we investigate the end points of \(\tilde{I}_m\). First consider the case \(m = 0\). Notice that \(\tilde{I}_0\) is symmetric with respect to \(\tau : c \mapsto 1/c\). If \(q = 0\), it is easy to see that there is only one double
parabolic parameter $-1$, hence by Lemma 3.4.5 the end point of $I_0$ is $-1$. If $q>1$, we prove that $I_0$ ends at two different points. Suppose the contrary, then $I_0$ must land at $-1$ since $I_0 = \tau(I_0)$.

By Lemma 2.1.1 $\tau^{-1}(I_0)$ is a curve symmetric with respect to $z \mapsto -z$, starting from $\sqrt{3e^{2\pi i/3}}$ and ending at $-\sqrt{3e^{2\pi i/3}}$. Moreover $0 = \tau^{-1}(-1)$ is a double parabolic parameter for the family $f_a$. By Lemma 3.2.4 0 is of type 0 and $q-1$, i.e. $q = 1$, a contradiction.

Now we consider the case $1 \leq m \leq q-1$. If not, then $I_m$ is a simple closed curve. We claim that $I_m$ must separate 0, $\infty$. Suppose not, then $I_m$ will bound a simply connected region $\mathcal{O} \subset B_m$, since by Proposition 2.2.3 and Lemma 2.1.1 there are only two connected components $\tau H_\infty, H_\infty$ of $\mathbb{C} \setminus \mathbb{C}_\lambda$, which are punctured neighborhoods of 0 and $\infty$ respectively. Thus $\mathcal{O}$ is either $D_m$ or $\bar{D}_m$ and $\Phi^\text{bit}_m$ or $\Phi^\text{bit}_m$ is well-defined on $\mathcal{O}$. But this contradicts Lemma 3.3.3 since $\partial \mathcal{O} = I_m$. So $I_m$ is a closed curve separating 0, $\infty$. Now we claim that $B_m$ is invariant under $\tau : c \mapsto 1/c$. Suppose not, then $\tau(B_m) \cap B_m = \emptyset$. In particular, $\tau(I_m) \cap I_m = \emptyset$ and both their closures separate 0, $\infty$. Therefore $\mathbb{C} \setminus (I_m \cup \tau(I_m))$ has a connected component $\mathcal{V}$ which do not intersect $\tau H_\infty, H_\infty$. By MSS-J-stability theorem, $g_c$ is stable on $\mathcal{V}$, hence $I_m, \tau(I_m)$ are in fact in the same parabolic component $B_m$, a contradiction. Since $B_m$ is invariant under $\tau$, so is $B_m$. Write $\partial B_m = c_0 \cup K_0 \cup K_\infty$, where $c_0$ is the end point of $I_m$, $K_0$ is the connected component of $\partial B_m \setminus \{c_0\}$ contained in the bounded component of $\mathbb{C} \setminus I_m$ and $K_\infty$ the one contained in the unbounded component of $\mathbb{C} \setminus I_m$. Clearly $K_0 \subset \partial H_\infty \setminus \partial H_\infty$ and $K_\infty \subset \partial H_\infty \setminus \partial H_\infty$. By the relation $\frac{1}{z}g_c(\frac{z}{c}) = g_1(\frac{z}{c})$ we conclude that $\tau(K_0) = K_\infty$ and $\tau(K_\infty) = K_0$, hence $\tau(c_0) = c_0$, $c_0 = \pm 1$, while $1 \in B_0$, so $c_0 = -1$. Thus by Lemma 2.1.1 $\tau^{-1}(-1) = 0$ and $\tau^{-1}(I_m)$ is a closed curve separating $\mathbb{C}$ into two connected components, each of which intersects $\sigma(H_\infty \cup \tau H_\infty)$, the complementary of the connected locus for the family $f_a$. But this contradicts Proposition 2.2.3 which implies that $\sigma(H_\infty \cup \tau H_0)$ has only one connected component.

**Corollary 3.4.7.** For $0 \leq m \leq q-1$, the $m$-component for the family $g_c$ is unique.

**Proof.** The case $m = 0$ has already been justified in Corollary 3.4.2. We prove for $m \geq 1$. For every $0 \leq k \leq q-1$ pick a $k$-component. By the above proposition and Lemma 3.2.4 $\mathcal{Z} = \bigcup_{k=1}^{q-1} I_k$ is a simple closed curve surrounding 0. If for some $m \geq 1$ there exists another $m$-component $B'_m$ with corresponding $I'_m$, then by the above proposition, $\mathbb{C} \setminus (\mathcal{Z} \cup I'_m)$ has a component which do not intersect $H_\infty, \tau H_\infty$ while it intersects $\partial \mathbb{C}_\lambda$, a contradiction.

### 3.5 Parametrizations transferred for family $f_a$

The first thing to do here is to find a dynamically defined curve $I \subset \text{Per}_1(e^{2\pi i/3})$ symmetric with respect to $a \mapsto -a$ linking $-\sqrt{3\lambda}, \sqrt{3\lambda}$, so that on $\mathbb{C} \setminus I$ we can define the two critical points of $f_a$ such that they vary analytically for $a \in \mathbb{C} \setminus I$.

$\tau^{-1}(I_{-1})$ is a "good" candidate, but it is not necessarily symmetric with respect to $a \mapsto -a$. In order to solve this problem, let us first notice that if $q$ is odd, then $c = -1$ is double parabolic if $q$ is even, then $c = -1 \in B_2^\ast$. Let $\rho = \text{Re}(\phi_{-1}(g_{\frac{q}{2}}))(-1)$, then $0 < \rho < 1$. Let $\Lambda \subseteq \Omega_0 \subset B_0^\ast(0)$ be the petal of $P = P_\lambda$ of level $\rho$. Since

$$\Phi^\text{bit}_{-1} : B_2^\ast(0) \setminus \Omega_0 \rightarrow \Omega_0^\ast, c \mapsto h_c(g_c(c))$$
is an isomorphism (Lemma 3.3.3), \( (\Phi^{\beta\gamma})^{-1}(P-\frac{3}{2}+1(\partial\Lambda)) \) is a curve linking the two double parabolic parameters on \( \partial B_{\frac{3}{2}} \). Let \( \tilde{\gamma} \) be the subcurve of it linking the double parabolic parameter on \( \partial B_{\frac{3}{2}} \cap \partial B_{\frac{3}{2}-1} \) and \( c = -1 \).

Let \( \mathcal{Z} = \bigcup_{m=0}^q \mathcal{I}_m \) if \( q \) is odd or \( \mathcal{Z} = \bigcup_{m=0}^q \bigcup_{m=0}^q \mathcal{I}_m \cup \tilde{\gamma} \) if \( q \) even. Then \( \mathcal{Z} \) be the closed subcurve linking \( c = 1 \) and \( c = -1 \). Now \( \nu^{-1}(\mathcal{Z}) \) (in the s-plane, recall in \( \Phi \)) has two connected components. Take the one containing \( s = -1 \) and denote its image under \( \mathcal{G} \). Set \( \mathcal{I} = \mathcal{G} \cup -\mathcal{G} \). Thus \( \mathcal{I} \) is a curve passing \( a = 0 \), symmetric under \( a \mapsto -a \). Set \( \mathcal{I}_m := \mathcal{I} \cap B_m \). To see that

**Remark 3.5.1.** When \( q \) is even and \( m = \frac{q}{2} \), we have \( \mathcal{I}_m \cap \sigma^{-1}(\mathcal{I}_m) = \emptyset \). We need to prove that \( \omega \sigma^{-1}(\mathcal{I}_m) \cap B_m \cap \mathcal{I}_m = \emptyset \). Indeed, by construction \( \omega \sigma^{-1}(\mathcal{I}_m) \cap B_m = \tilde{\gamma} \cup \tau \tilde{\gamma} \) (\( \tau : c \mapsto 1/c \)) and \( \tilde{\gamma} \cap \mathcal{I}_m = \emptyset \). To see that \( \tau \tilde{\gamma} \cap \mathcal{I}_m = \emptyset \), it suffices to justify that for \( c \in \mathcal{I} \), \( \Re(\phi_{c}(g^2(g^2(1)) - \phi_{c}(g^2(c)))) < 0 \). This is clear since \( g^2(c) \in \Omega_{c,0}^{\infty} \), while \( 1 \in \partial \Omega_{c,0}^{\infty} \).

Let \( a \mapsto \sqrt{a^2 - 3\lambda} \) be the inverse branch defined on \( \mathbb{C} \setminus \mathcal{I} \) such that \( (a - \sqrt{a^2 - 3\lambda}) \to 0 \) as \( |a| \to +\infty \). Define \( c_\pm(a) = \frac{-a \pm \sqrt{a^2 - 3\lambda}}{3} \) and let \( v_\pm(a) = f_a(c_\pm(a)) \).

**Proof of Proposition II.** It is just a summary of what we have obtained for the family \( f_a \):

- For every \( 0 \leq m \leq q \), there is a unique \( m \)-component \( B_m \) (Definition 3.1.3). This is direct from Corollary 3.4.7 and the relation between families \( g_c \) and \( f_a \) (Lemma 2.1.1). Moreover, since \( C_\lambda \) is symmetric under \( a \mapsto -a \), we have \( B_m = -B_{q-m} \).

- For every \( 0 \leq m \leq q - 1 \), there is a unique double parabolic parameter \( a_m \) of \( m \)-type (Definition 3.2.3). Moreover \( a_m \in \partial B_m \cap \partial B_{m+1} \) (Lemma 3.2.4).

- The special curve \( \mathcal{I} \) defined above passes \( \bigcup_{m=0}^q B_m \) in the following order:

  \[ B_0, a_0, B_1, a_1, B_2, \ldots, B_{q-2}, a_{q-2}, B_{q-1}, a_{q-1}, B_q. \]

Moreover for \( a \in \mathbb{C} \setminus \mathcal{I} \), \( c_+ (a) \) is always on the boundary of the maximal petal for \( f_0 \big|_{B_{a_0}^{+}(0)} \).

\( \square \)
Figure 4: The curves $\mathcal{I}$ for $C_\lambda$ with $\lambda = e^{\frac{2\pi i}{3}}$ and $e^{\frac{2\pi i}{4}}$.

The parametrization of $m$-components for the family $g_c$ (recall (12)) can be transferred by $\sigma^{-1}$ to the family $f_a$. More precisely:

- For $m = 0$, define $\Psi_{00}^{-}\, : \, B_m \backslash \mathcal{I}_0 \longrightarrow B_p^*(0)$ by $\Psi_{00}^{-} = \Phi_{00}^{-} \circ \iota \circ \sigma^{-}$, where $\sigma^{-}$ the inverse branch such that $\iota \circ \sigma^{-} (B_m \backslash \mathcal{I}_0) = \tilde{D}_0$.

- For $1 \leq m < \left\lfloor \frac{q}{2} \right\rfloor$, define $\mathcal{D}_m = \sigma \iota^{-1}(\tilde{D}_m)$ and $\hat{D}_m = \sigma \iota^{-1}(\hat{D}_m)$, where $\iota^{-1}$ is the inverse branch such that $\sigma \iota^{-1}(\hat{B}_m) = B_m$. Define $\Psi_{m}^{-} : D_m \longrightarrow B_{\tilde{m}}(0) \backslash \Omega_{\tilde{m}}^{-}$ by $\Psi_{m}^{-} = \Phi_{m}^{-} \circ \iota \circ \sigma^{-}$ and $\hat{\Psi}_{m}^{-} : \hat{D}_m \longrightarrow B_{\tilde{m}}^*(0) \backslash \Omega_{\tilde{m}}^{-}$ by $\hat{\Psi}_{m}^{-} = \hat{\Phi}_{m}^{-} \circ \iota \circ \sigma^{-}$, where $\sigma^{-}$ is the inverse branch such that $\iota \circ \sigma^{-} (D_m) = \hat{B}_m$.

- For $m = \frac{q}{2}$, $B_m$ is divided by $\sigma \iota^{-1}(\hat{I}_m)$ into two components. Let $D_m$ be the one that is contained in a connected component of $B_m \backslash \mathcal{I}_m$ (Remark 3.5.1). Define similarly $\Psi_{m}^{-}$ on $D_m$.

From Proposition 3.3.4 we have

**Proposition 3.5.2.** For $1 \leq m < \left\lfloor \frac{q}{2} \right\rfloor$, $\Psi_{m}^{-}$, $\hat{\Psi}_{m}^{-}$ are isomorphisms. For $m = \frac{q+1}{2}$, $\Psi_{m}^{-}$ is an isomorphism.

Capture components can also be parametrized by locating $v_-(a)$:

**Proposition 3.5.3.** Let $U_k$ be a capture component of depth $k \geq 1$. For $a \in U_k$, suppose $f_a^k(U_a) = B_{a,l}^*(0)$, where $U_a$ is the Fatou component containing $c_-(a)$. Then $\Psi_{U_k} : U_k \longrightarrow B_{a,l}^*(0)$ defined
by $a \mapsto h_a(f^k(c_-(a)))$ is an isomorphism, where $h_a$ is the conjugating map between $f^k|B^*_a(0)$ and $P^q|B^*_a(0)$.

Proof. The proof is exactly the same as Proposition 3.3.4.

**Proposition 3.5.4.** For $m = 0$, $(\Psi_0^{adj})^{-1}(\Omega_p^{-1})$ is a topological disk whose boundary is a piecewise smooth closed curve passing $\sqrt{3}\lambda, 2\sqrt{\lambda}$. Let $D_0, \bar{D}_0$ be the two connected components of $(B_0 \setminus I_0) \setminus \Psi_0^{adj}(\Omega_p^{-1})$. Then $\Psi_0^{adj}|D_0, \Psi_0^{adj}|\bar{D}_0$ are isomorphisms with image $B^*_p(0) \setminus \Omega_p^{-1}$. See Figure 7.

Proof. First of all $\Psi_0^{adj}(\Omega_p^{-1})$ is not empty. Indeed, there is a holomorphic motion induced by $\phi_1^{-1} \circ \phi_{a_0}$ of $\Omega_p^{-1}$ in a small neighborhood of $a_0 = 2\sqrt{\lambda}$. Notice that if $a_0 = 2\sqrt{\lambda}$, then $v_-(a_0) = 0$. Let $z_h \in \partial \Omega_{a_0}$ be such that $\phi_{a_0}(z_h) = 1/h \in \mathbb{R}$. Apply Rouché's Theorem to $F(a, h) = v_-(a) - f^{1-q}_a \phi_1^{-1} \phi_{a_0}(z_h)$ for $h$ near 0 (take $f^{1-q}_a$ to be the inverse branch of $f^{q-1}_a|_{\partial \Omega_{a_0}}$), there is a sequence of $a_n \in B_m \setminus I_0$ converging to $2\sqrt{\lambda}$ such that $v_-(a_n) \in \Omega_p^{-1}$, i.e. $a_n \in (\Psi_0^{adj})^{-1}(\partial \Omega_p^{-1})$ and $a_n \to 2\sqrt{\lambda}$. So $\Psi_0^{adj}(\Omega_p^{-1}) \neq \emptyset$.

Next we investigate the end points of $\partial((\Psi_0^{adj})^{-1}(\Omega_p^{-1}))$. Notice that by properness of $\Psi_0^{adj}$ (Lemma 3.3.3) and stability of Fatou coordinate,

$$\partial((\Psi_0^{adj})^{-1}(\Omega_p^{-1})) \subset \{0, 2\sqrt{\lambda}\} \cup (\Psi_0^{adj})^{-1}(\partial \Omega_p^{-1}) \cup I_0.$$  

By Proposition 3.3.5 and a quasiconformal deformation argument, $\partial((\Psi_0^{adj})^{-1}(\partial \Omega_p^{-1})$ is the closure of the union of two curves $\gamma_1, \gamma_2$ having $2\sqrt{\lambda}$ as a common end point. We parametrize $\gamma_1, \gamma_2$ by $I(a) = \mathfrak{I}(\phi_1(-a))$. By the above analysis, we see that as $|I_a| \to \infty, a \to 2\sqrt{\lambda}$. We want to prove that as $|I_a| \to 0, a \to \sqrt{3}\lambda$. By Lemma 3.3.2, it suffices to show that $a$ do not accumulate at $\partial B_0$. Suppose the contrary. If $a \in \gamma_1$ accumulates at $a_0 \in \partial B_0 \setminus A_{p/q}$, then by stability of Fatou coordinate, $a_0 = 2\sqrt{\lambda}$, then $\gamma_1$ surround a topological disk $U$ such that $\Psi_0^{adj}(U) \cap \partial B_0 = \{2\sqrt{\lambda}\}$. But $\Psi_0^{adj}(\partial U \setminus \{2\sqrt{\lambda}\}) = \Psi_0^{adj}(\gamma_1)$ is a semi-arc of $\partial \Omega_p^{-1}$, which does not separate $B^*_p(0)$, a contradiction. So it remains to exclude the case where both $\gamma_1, \gamma_2$ land at $a_0$ as $|I(a)| \to \infty$. Suppose we have this. Then $\gamma_1 \cup \gamma_2$ bounds a topological disk $V$ such that $V \setminus I_0$ is sent conformally onto $\Omega_p^{-1}$ by $\Psi_0^{adj}$ (Proposition 3.3.5). Since $\partial V \cup I_0$ is locally connected, $(\Psi_0^{adj})^{-1}|_{\Omega_p^{-1}}$ can be continuously extended to $\partial \Omega_p^{-1}$ and in particular $(\Psi_0^{adj})^{-1}(0) = 2\sqrt{\lambda}$. On the other hand, by Lemma 3.3.3 when $a \in B_0 \setminus I_0$ tends to $a_0$, $\Psi_0^{adj}(a) \to \partial B_0$. Hence $(\Psi_0^{adj})^{-1}(0) = a_0$, a contradiction.

The rest of the proposition is immediate by Lemma 3.3.3 and Proposition 3.3.5.
Figure 5: A zoom of $\mathcal{B}_0$ in $C_{1/4}$. The union of curves in red, blue and orange is equipotentials and external rays of level 0,1,2 in $D_0$. $W = (\Psi_{0}^{adj})^{-1}(\Omega_{p}^{-1})$.

Figure 6: Equipotentials in $D_1 \subset C_{1/4}$ and $D_2 \subset C_{1/4}$.

3.6 Landing properties at double parabolic parameters

Proposition 3.6.1. Let $a_m \in \mathcal{A}_{p/q}$ be the double parabolic parameter of $m$-type. Then among all the external rays with angles in $\bigcup_k \Theta_k$, there are exactly four landing at $a_m$.

Proof. First we prove the existence of four such rays. For $0 \leq m \leq q$, let $\mathcal{O}_m \subset (\mathbb{C} \setminus \mathcal{A}_{p/q})$ be the set of parameters such that the dynamical external rays with angles $\Theta_m$ (recall in Remark 3.1.2)
are at least four parameter external rays $R_{\infty}(t_{1,2})$ with $t_{1,2} \in \Theta_m$ and two rays $R_{\infty}(t'_{1,2})$ with $t'_{1,2} \in \Theta_{m+1}$ landing at the double parabolic parameter of $m$-type. Moreover both $R_{\infty}(t_{1,2})$ and $R_{\infty}(t'_{1,2})$ separate $B_m, B_{m+1}$.

The harder part is the uniqueness. Since the parametrisation $\Phi_{\infty}$ is of degree 3, there are in total $Q := 3q(q + 1)$ parameter external rays whose angles belong to $\bigcup_k \Theta_k$. So in order to prove uniqueness, it suffices to find $Q - 4q$ rays not landing at $A_{p/q}$ among these $Q$ rays. By Lemma 2.3.7 in $\text{Per}_1(0)$ there are $Q - 4q$ rays landing at $C_0 \setminus H_{\theta_0}^0$. Let $A$ be the set of landing points of these $Q - 4q$ rays. Write $A = \text{Mis} \cup \text{Par}$ where "Mis" (resp. "Par") means that $a_0$ is MiSierewicz (resp. parabolic).

Claim. Let $a_1, a_2 \notin H$ be two different geometrically finite parameters. let $a'_1, a'_2 \in \text{Per}_1(e^{2\pi i \frac{p}{q}})$ be their pinching limit. If $a'_1, a'_2 \notin A_{p/q}$, then $a'_1 \neq a'_2$.

Admitting the claim, we finish the proof of the proposition:

- for $a_0 \in \text{Mis}$, its pinching limit $a'_0 \in \text{Per}_1(e^{2\pi i \frac{p}{q}})$ is also MiSierewicz (since $a_0 \notin \partial H_0^0$ and $v_{0,a}$ is periodic, $v_{0,a_0}$ does not belong to any skeleton). Recall that the pinching deformation preserves external rays, thus the portrait at $v_{0,a_0}$ is the same as that of $v_{a'_0}$. Suppose there are $r$ external rays among the $Q - 4q$ rays landing at $a_0$. Then by Lemma 2.2.8 there are also $r$ external rays with angles in $\bigcup_k \Theta_k$ landing at $a'_0$.

- If $a_0 \in \text{Par}$, then for the same reason its pinching limit $a'_0 \in \text{Per}_1(e^{2\pi i \frac{p}{q}})$ is also parabolic. We want to prove that there exist two external rays with angles in $\bigcup_k \Theta_k$ landing at $a'_0$. Let $\theta_+, \theta_-$ be the two angles given in Lemma 2.3.8. Since pinching preserves external rays, $R_{a'_0}^\infty(\theta_+), R_{a'_0}^\infty(\theta_-)$ land at the parabolic periodic point, bounding the critical value $v_{a'_0}$. By a simple plumbing surgery (see [3]), for any neighborhood $U$ of $a_0$, there exists $a'_1 \in U$ such that $R_{a'_1}^\infty(t_+), R_{a'_1}^\infty(t_-)$ land at the same repelling periodic point and bound $v_-(a'_1)$. By Lemma 2.2.9 there is at least one ray $R_{\infty}(\theta)$ with $\theta \in \bigcup_k \Theta_k$ landing at $a'_0$. Suppose the contrary that this is the only ray landing at $a'_0$. Then the holomorphic motion given by Lemma 2.2.9 implies that $R_{a'_0}^\infty(\theta_+), R_{a'_0}^\infty(\theta_-)$ land at the same repelling periodic point and bound $v_-(a'_1)$ when $a$ is close to $R_{\infty}(\theta)$. Since $a \in R_{\infty}(t)$ is equivalent to $v_-(a) \in R_{\infty}(t)$ (definition of parameter external rays), therefore $\theta \in (t_-, t_+)$. This contradicts Corollary 2.3.9.

To conclude, notice that by Lemma 2.3.8 the $Q - 4q$ rays in $\text{Per}_1(0)$ are decomposed into

$$\left( \bigcup_{a_0 \in \text{Mis}} \bigcup_{i=0}^r R_{\infty}^0(s_i) \right) \cup \left( \bigcup_{a_0 \in \text{Par}} \left( R_{\infty}^0(t) \cup R_{\infty}^0(t') \right) \right).$$

While by the above discussion and the claim, we obtain $Q - 4q$ external rays with angles in $\bigcup_m \Theta_m$. This finishes the proof.

Proof of the claim. Suppose the contrary that we have $a'_1 = a'_2$. Notice that in the dynamical plans of $a_1, a_2$, their critical values $v_{0,a_1}, v_{0,a_2}$ are bounded respectively by wakes $W_1, W_2$ attached at the boundary of the immediate basin of 0. Since the pinching deformation preserves external rays, we conclude that the angles of the two rays defining $W_1$ are the same to those defining $W_2$. By Lemma 2.3.6 this implies that $a_1, a_2$ belongs to the same wake (in the parameter plan) attached
at \( \partial \mathcal{H}_0 \), which in particular is contained in a quadrant. Thus the angles of external rays landing at them are distinct, since the parametrisation \( \Phi^0 : \mathbb{C} \setminus \mathbb{C}_0 \to \mathbb{C} \setminus \mathbb{R} \) is injective on each quadrant. Thus the pinching limits \( a_1', a_2' \) are distinct, a contradiction.

**Definition 3.6.2.** Let \( a_m \in \mathcal{A}_{p/q} \) be of type \( m \). The four rays landing at \( a_m \) is separated by \( I \) into two groups \( \alpha_m^+, \beta_m^+ \) and \( \alpha_m^-, \beta_m^- \) with \( \alpha_m^\pm \in \Theta_m, \beta_m^\pm \in \Theta_{m+1} \). They bound 2 open regions separated by \( I \). These two regions are called **double parabolic wakes** attached to \( a_m \), denoted by \( \mathcal{W}^\pm(a_m) \) respectively.

![Figure 7: An illustration for \( C_{1/3} \).](image)

Denote by \( S_m \) the connected component of \( V \) containing \( B_m \), where

\[
V := \mathbb{C} \setminus \bigcup_{m=0}^{q-1} (\mathcal{R}_\infty(\alpha_m^+) \cup \mathcal{R}_\infty(\alpha_m^-) \cup \mathcal{R}_\infty(\beta_m^+) \cup \mathcal{R}_\infty(\beta_m^-))
\]  

(13)

**Definition 3.6.3.** For \( 1 \leq m \leq \lfloor \frac{q}{2} \rfloor \), denote by \( S_m^+ \) the connected component of \( S_m \setminus I_m \) intersecting \( \mathcal{D}_m \), \( S_m^- \) the other component. For \( m = 0 \), set \( S_0^+ = S_m \).

Now as a corollary of Proposition 3.6.1, we can give a description of the portrait at the parabolic fixed point 0 for the family \( f_a \):

**Corollary 3.6.4.** Let \( a \in \mathbb{C} \setminus \mathcal{A}_{p/q} \). If \( a \in \mathcal{W}^\pm(a_m) \), the portrait of \( f_a \) at the parabolic fixed point 0 is \( \Theta_m, \Theta_{m+1} \); if \( a \in S_m \), the portrait is \( \Theta_m \).
Proof. First we prove that when we go through $I$ in the direction $B_0, ..., B_4$, $a_m^\pm$ is on the left-hand side of $\beta_m^\pm$. Suppose the contrary. Then By Proposition 3.6.1 there exists $a \in W^\pm(a_m) \cap C_\lambda$ such that $a$ and $B_m$ (resp. $B_{m+1}$) are contained in the same connected component of 

$$\mathbb{C} \setminus \bigcup_{\theta \in \Theta_{m+1}} \mathcal{R}_\infty(\theta), \text{ resp. } \mathbb{C} \setminus \bigcup_{\theta \in \Theta_m} \mathcal{R}_\infty(\theta),$$

where $\hat{\Theta}_k = (\bigcup_j \Theta_j) \setminus \Theta_k$. Since the portrait at $z = 0$ for $a$ in $B_k$ is $\Theta_k$, Lemma 2.2.9 implies that the portrait at $z = 0$ of $f_a$ can not contain $\Theta_m, \Theta_{m+1}$. But it can neither contain other $\Theta_k$ with rotation number $p/q$ for the same reason. But $z = 0$ should at least admits a cycle of landing rays with rotation number $p/q$, a contradiction.

Let $O_m^\pm \subset W^\pm(a_m)$ be the collection of parameters whose portrait at $z = 0$ is exactly $\Theta_m, \Theta_{m+1}$. Then clearly $O_m^\pm$ is open. By the above analysis, $O_m^\pm \neq \emptyset$ and 

$$\overline{\mathcal{R}_\infty(a_m^\pm) \cup \mathcal{R}_\infty(\beta_m^\pm)} \subset \partial O_m^\pm.$$

Suppose the inclusion above is strict. Then for any $a_0 \in \partial O_m^\pm \setminus \overline{\mathcal{R}_\infty(a_m^\pm) \cup \mathcal{R}_\infty(\beta_m^\pm)}$, there exists $t_{a_0} \in \Theta_m \cup \Theta_{m+1}$ such that $\mathcal{R}_a^{\infty}(t_{a_0})$ crashes on $c_-(a_0)$. So $a_0 \in \mathcal{R}_\infty(3t_{a_0})$. By stretching external rays we can show that for all $a \in \mathcal{R}_\infty(3t_{a_0})$, $\mathcal{R}_a^{\infty}(t_{a_0})$ crashes on $c_-(a)$ and also $a \in \partial O_m^\pm$. Therefore $\mathcal{R}_\infty(3t_{a_0})$ must land at $a_m$. By Proposition 3.6.1, $3t_{a_0}$ is one of $a_m^\pm, \beta_m^\pm$, a contradiction. Hence $O_m^\pm = W^\pm(a_m)$.

Let $\hat{O}_m \subset \mathbb{C}$ be the collection of parameters whose portrait at $z = 0$ is exactly $\Theta_m$. By a similar argument as above we can show that $\hat{O}_m = S_m$. 

\[\square\]

Corollary 3.6.5. For any $m$, there are no other external rays with rational angles landing at $a_m \in A_{p/q}$.

Proof. By Lemma 2.2.7 it suffices to treat the case $3^n t = a_m^+$ for some $n \geq 1$. Since $t$ is rational, the three external rays with angle $t$ in $\text{Per}_1(0)$ land at 3 different parameters $a_1, a_2, a_3$. Now we prove that $a_1, a_2, a_3$ are all Misiurewicz parameters. Suppose the contrary that $a_1$ is a parabolic parameter, then by hypothesis on $t$, $\mathcal{R}_a^{\infty}(a_m^\pm)$ lands at the parabolic point of $f_{0,a_1}$. Then by stability of rays landing at repelling point, there exists $\theta \in \Theta_m$ such that $\mathcal{R}_a^{\infty}(\theta)$ lands at $a_1$. This contradicts Lemma 3.6.1. Notice that by the claim in Proposition 3.6.1, the corresponding pinching limits (clearly not double parabolic) $a_1', a_2', a_3' \in \text{Per}_1(e^{2\pi i \frac{\ell}{2}})$ are distinct. By Lemma 6.4.1 they admit in total three external rays with angle $t$ landing. 

\[\square\]

4 Dynamical graphs and puzzles

In this section we first aim to investigate the local connectivity of Julia set for the family $\text{Per}_1(\lambda)$ for $\lambda = e^{2\pi i \frac{\ell}{2}}$. Since the Julia set of a geometrically finite rational map is always locally connected (I), it remains to study the case where $f_s \in C_\lambda \setminus \mathcal{H}$ and $f_s$ is not Misiurewicz parabolic. In the sequel we always fix such an $f_s$ and omit the index $a$. The critical point in the parabolic basin of 0 will be denoted by $c_+$, the free critical point by $c_-$. Let $B_0^c(0)$ be the immediate basin containing $c_+$ and $B_1^c(0), ..., B_{q-1}^c(0)$ the other immediate basins in cycle order. Let $\{R^\infty(\theta_i); \theta_i \in \Theta\}$ be a cycle of external rays landing at 0 and $R^\infty(\theta_0^\pm)$ be the two external rays landing at 0 and bounding
Let $\Delta^\pm \mathbb{F}$ component of $f$ that $f$.

**Proof.**

Let $\Theta$ be the conclusion stated in the proposition. Let $r_0 > 1$. Define the graph of depth $n$ by

$$Y_0 = \left( \bigcup_{i=1}^{q} (f^i(\partial \Omega) \cup R^\infty(\theta_i)) \right) \cup E^\infty(r_0), \quad Y_n = f^{-n}(Y_0).$$

A puzzle piece $Q_n$ of depth $n$ is a connected component of $\mathbb{C} \setminus Y_n$ intersecting the Julia set. Let $Q_n^\pm$ be the puzzle piece whose boundary intersects $R^\infty(\theta_\pm)$ respectively.

**Proposition 4.1.1.** Let $X^\pm = \bigcap_{k \geq 0} Q_{kq}^\pm$. Then either $X^\pm = \{0\}$, or it is a continuum and $f^q : X^\pm \rightarrow X^\pm$ is a degree two ramified covering. Moreover in the second case, there are two cycles of external rays landing at 0 with angle cycles $\Theta, \Theta'$ and there exists $\zeta^\pm \in \Theta'$ such that $\gamma^\pm = R^\infty(\theta^+_0) \cup R^\infty(\zeta^\pm) \cup \{0\}$ separates $X^\pm \setminus \{0\}$ from $B^*_0(0) \setminus \{0\}$.

**Proof.**

We only prove for $X = X_+$ and the other is similar. Set $\theta = \theta^+_0$. First notice that since $f^q(Q_{(k+1)q}) = Q_{kq}$, hence $f^q(X) = X$. Suppose that $X \neq \{0\}$, then there exists an isomorphism $\phi : \mathbb{C} \setminus X \rightarrow \mathbb{C} \setminus \mathbb{E}$. Take a small enlargement $U$ of $X$ with $U$ simply connected and open, such that $f^q : U' \rightarrow U$ has degree $d \geq 2$ and $U \setminus X$ does not contain $f^q(c_{-})$, where $U'$ is the connected component of $f^{-q}(U)$ containing $X$. Hence $f^{-q}(X) \cap U' = X$. Let $W = \phi(U)$ and $W' = \phi(U')$.

Then the map $F = \phi \circ f \circ \phi^{-1}$ is a ramified covering of degree $d$. By Schwarz reflection principle, $F$ extends holomorphically to a neighborhood of $S^1$. Let $g = F|_{S^1}$. Adapting the same strategy in the proof of [14] Prop. 2.4, we can show that $g : S^1 \rightarrow S^1$ is a degree $d$ covering and has $d - 1$ fixed points. Since in the dynamical plane of $f$, there is an access $\delta \in \mathbb{C} \setminus X$ to 0 fixed by $f^q$ (for example one may take $\delta$ to be one of the external rays landing at 0), then $\phi(\delta)$ lands at $S^1$ and gives a fixed point of $g$ (see [7]). Thus $d = 2$.

Now we prove that if $X \neq \{0\}$, there is another cycle of external rays landing at 0 and satisfying the conclusion stated in the proposition. Let $R^\infty(\theta_k)$ be the external ray landing at $\partial B^*_0(0) \setminus \{0\}$ involved in $\partial Q_{kq}$. Then $3^q\theta_{k+1} = \theta_k$ since $f^q(R^\infty(\theta_{k+1})) = R^\infty(\theta_k)$. Clearly $\theta_k$ has a limit $\theta'$ when $k \rightarrow \infty$ since $\theta_k$ is monotone. Thus $3^q\theta = \theta$ and $R^\infty(\theta')$ enters every puzzle piece $Q_{kq}$. This implies that $R^\infty(\theta')$ lands at a fixed point $x \in X$ of $f^q$. Notice that $R^\infty(\theta')$ is also fixed by $f^q$, thus $x$ corresponds to the unique fixed point of $g$, i.e. $x = 0$. Since $X \neq \{0\}$ and is contained in every $Q_{kq}$, there exists $\eta$ between $\theta, \theta_k$ for all $k$ such that $R^q(\eta)$ landing at $x' \in X$ with $x \neq 0$. Hence $\theta' \neq \theta$. Clearly $\gamma$ separates $X^\pm \setminus \{0\}$ from $B^*_0(0) \setminus \{0\}$. \[ \square \]

We get immediately

**Corollary 4.1.2.** $\partial B^*_0(0)$ is locally connected at 0 and at its inverse orbit.

**4.2 Wakes attached to $\partial B^*_0(0)$**

Let $T_0 = \left( \bigcup_{i=1}^{q} (f^i(\partial \Omega) \cup R^\infty(\theta_i)) \right) \cup \{0\}$ and $T_n$ the connected component of $f^{-n}(T_0)$ containing 0. Let $\Delta^\pm$ be the two unbounded connected components of $S_0 \setminus T_0$, such that $R^\infty(\theta^*_\pm) \subset \partial \Delta^\pm$. 30
Hence any $z \in \partial B^*_n(0)$ which is not in the inverse orbit of 0 has a unique dyadic representation $(\epsilon_n)_{n \geq 0}$ encoding its orbit position under $f^n$. More precisely, $\epsilon_n = 1$ if $f^{nq}(z) \in \Delta_+$, $\epsilon_n = 0$ if $f^{nq}(z) \in \Delta_-$. For $z$ in the inverse orbit of 0, we take the convention that $\epsilon_n$ is zero for all but finitely many $n$. Then the following mapping is well defined:

$$\kappa : \partial B^*_n(0) \rightarrow S^1, \ z \mapsto (\epsilon_n)_{n \geq 0}$$

For $i \geq 1$, $\kappa$ is naturally extended to $\partial B^*_n(0)$ by $\kappa(z) = \kappa(w)$ where $w \in \partial B^*_n(0)$ is the first iterated image of $z$. Notice that there are exactly $q$ external rays involved in $T_n$ at any $u \in \bigcup_i \partial B^*_n(0)$ which is in the inverse orbit of 0. Suppose $u \neq 0$, then among these $q$ rays there are two of them bounding an open region separating all the other $q - 2$ rays with $\bigcup_i \partial B^*_n(0)$. Let $U(u)$ be the closure of this region. Now $\kappa$ is extended to $\mathbb{C} \setminus \bigcup_i \partial B^*_n(0)$ in the following way: if $z$ belongs to some $U(u)$, then $\kappa(z) := \kappa(u)$ (which is dyadic); if $z \in T_n$, then $\kappa(z) := 0$; else, for every $n \geq 0$, let $U_n$ be the component of $\mathbb{C} \setminus T_n$ containing $z$, then set $z \in U(z) := \bigcap_n U_n$ and define $\kappa(z) = \kappa(u)$ for any $u \in (\bigcup_i \partial B^*_n(0)) \cap U(z)$. Notice that in the second case $\kappa(z)$ does not depend on the choice of $u$. $U(z)$ is called a wake attached to $\partial B^*_n(0)$. Define the corresponding limb by $L(z) = U(z) \cap K_f$. Denote by $U(t)$ resp. $L(t)$ the union of $U(z)$ resp. $L(z)$ with $\kappa(z) = t$.

**Remark 4.2.1.** By construction, $\mathbb{C} = \bigcup_i B^*_n(0) \cup \bigcup_t U(t)$, $K_f = \bigcup_i B^*_n(0) \cup \bigcup_t L(t)$.

**Lemma 4.2.2.** $L(0) = 0$ if and only if $f$ does not belong to wakes attached at double parabolic parameters.

**Proof.** If $L(0) = 0$ but $f$ belongs to some wake attached at a double parabolic parameter. Then by Corollary 3.6.4 there are two cycles of external rays landing at 0. Suppose the corresponding cycle of angles are $\Theta, \Theta'$ respectively, then there exists $\theta \in \Theta, \theta' \in \Theta'$ such that $R^\infty(\theta), R^\infty(\theta')$ bound a open sector $S$ not intersecting any immediate basin at 0, hence $S \cap K_f \subset L(0)$, contradicting $L(0) = 0$.

If $f$ does not belong to wakes at double parabolic parameters, then one may deduce by the same argument in the second part of the proof of Proposition 4.1.1 that $L(0) = 0$. \qed

### 4.3 Finding infinitely ringed puzzle pieces around $c_-$

**Theorem 4.3.1 (Yoccoz).** Let $f : U' \rightarrow U$ be a quadratic rational-like map and $x_0$ be its unique critical point. Let $x \in K_f$. For any admissible graph $\Gamma$ that rings $x_0$ and rings infinitely $x$, we have the following alternative:

- **if the tableau of $x_0$ is k-periodic, then $f^k : P_{l+k}(x_0) \rightarrow P_l(x_0)$ is quadratic-like for $l$ large enough.** $\text{Imp}(x)$ is either $x$ or a conformal copy of $\text{Imp}(x_0)$, depending on whether the forward orbit of $x$ intersects $\text{Imp}(x_0)$.
- **if the tableau of $x_0$ is not periodic, then $\text{Imp}(x) = x$.**

**Observation.** If $\Gamma$ rings infinitely $x_0$, then $\Gamma \cap K_f$ contains no parabolic cycle. Indeed, since a parabolic cycle (say period $k$) must attract a critical point, we may suppose that $f^{kn}(x_0)$ are close to each other when $n$ large enough. Fix such $n$, pick $n_i$ large enough such that $f^{nk}(x_0)$ is ringed at depth $n_i$, then $P_{n_i}(f^{nk}(x_0))$ is compactly contained in $f^{kn}(P_{n_i}(f^{nk}(x_0))) = P_{n_i-kN}(f^{n+kN}(x_0))$ for $N$ large enough. In particular $\Gamma \cap K_f$ does not contain the parabolic cycle.
Thus the construction of "jigsawed" internal rays for \( P \) dynamics on the immediate basins of \( n \). Define the union of internal rays of depth \( \lambda \)

\[
\eta = \bigcup_{i=1}^{q} P^{i}(\partial H), \quad \eta_{n} = P^{-n}(\eta),
\]

Define the union of internal rays of depth \( n \) by

\[
\rho_{\lambda} = \bigcup_{i=1}^{q} P^{i}(R(\lambda) \setminus H), \quad \rho_{n} = P^{-n}(\rho),
\]

Now return to \( f \). We abuse the notations of \( S, B_{f}(0) \), etc.. Suppose \( c_{\pm} \in S_{m} \). Since the dynamics on the immediate basins of \( f \) is equivalent to that of \( P_{\lambda} \), \( E_{0} \) in (15) and \( R_{0} \) in (16) can be transferred to the immediate basins of \( f \). Define similarly \( E_{n}, R_{n} \) for \( f \) to be the \( n \)-th preimage. Let \( \theta \in \Theta_{\pm} \), then the landing point \( x_{\theta} \) of \( R(\theta) \) is pre-periodic, hence there is at least an external ray \( R^{\infty}(t) \) landing at it. For each \( x_{\theta} \) we choose a \( R^{\infty}(t) \) and denote by \( T \) the collection of these \( t \) such that \( 3t \subset T \).

Pick \( r \leq 1 \). Define the graph of depth \( n \) by

\[
X_{n} = E_{n} \cup R_{n} \cup \bigcup_{i \in T} R^{\infty}(3^{i}t) \cup E^{\infty}(r),
\]

Notice that if \( m = 0 \), the graph above can also be written as

\[
X_{0} = \bigcup_{i=1}^{q} f^{i}(\partial \Omega) \cup \bigcup_{i=1}^{q} \bigcup_{\theta \in \Theta_{\pm}} f^{i}(R(\theta) \setminus \Omega) \cup \bigcup_{i \in T} R^{\infty}(3^{i}t) \cup E^{\infty}(r)
\]

Recall that \( \Omega \subset B_{f}(0) \) is the maximal petal for the return map \( f^{\eta}|_{B_{f}(0)} \).
Figure 8: An admissible graph for \( f \in Per_1(e^{\frac{2\pi i}{3}}) \) with \( m = 1 \) and \( \theta \). The graph in red is \( X^1_0 \), the graph in blue is the first pre-image of \( X^1_0 \) that is not contained in \( X^1_0 \). For this graph, \( c_- \) is ringed at depth 0.

**Lemma 4.3.2.** If \( z \in (K_f \setminus \bigcup B_i^*(0)) \cap S_m \) satisfies \( \kappa(z) \in \left[ \frac{1}{4}, \frac{1}{2} \right) \cup \left( \frac{1}{2}, \frac{3}{4} \right] \), then \( z \) is ringed at depth 0 by one of the graphs (17) for all \( k \) large enough.

**Proof.** By construction, the annulus \( P^0 \setminus P^1 \) of depth 0 is non-degenerated if \( P^1 \) is a piece whose boundary contains two of the internal rays \( R^{d}(2j\theta_\pm + \frac{1}{2}) \), \( j = 0, \ldots, k-2 \). The lemma follows by noticing that

\[
\theta_+ + \frac{1}{2} < 2\theta_+ + \frac{1}{2} < \ldots < 2^{k-2}\theta_+ + \frac{1}{2}
\]

\[
\theta_- + \frac{1}{2} > 2\theta_- + \frac{1}{2} > \ldots > 2^{k-2}\theta_- + \frac{1}{2}
\]

and \( 2^{k-2}\theta_+ + \frac{1}{2} > \frac{3}{4} \) tends to \( \frac{3}{4} \) as \( k \to \infty \); \( 2^{k-2}\theta_- + \frac{1}{2} < \frac{1}{4} \) tends to \( \frac{1}{4} \) as \( k \to \infty \).

**Lemma 4.3.3.** Suppose that \( t_0 \neq \frac{1}{2} \). Then the critical value \( v_- = f(c_-) \in S_{m'p} \) and \( f^q(c) \in L(2t_0) \cap S_m \). Moreover if \( z \in L(t_0) \cap S_m \) such that \( f(z) \) is not in the puzzle piece (defined by (17)) of depth 0 containing \( v_- \), then \( z \) is ringed at depth 0.

**Proof.** Suppose \( v_- \in S_{m'} \) and let \( L'(t_0) \cap S_{m'} \) be the limb containing \( v_- \). Notice that \( f^{-1}(L'(t_0)) \) has only two connected components, one containing \( c_- \). If \( m = 0 \), then \( m' = p \), otherwise
Lemma 4.3.4. Suppose \( z \in K_f \) is not in the inverse orbit of 0. If \( f^n(z) \) eventually hits \( \partial B^*_0(0) \). Then \( z \) is infinitely ringed by one of the graphs (18) for all \( k \) large enough.

Proof. The proof goes the same as in [14, Lem. 6.1]. The orbit of \( z \) must satisfy one of the following conditions, and the result follows by Lemma 4.3.2.

- If there exists a subsequence \( n_j \) such that \( 2^{n_j} \cdot \kappa(z) \to \frac{1}{4} \) resp. \( \frac{3}{4} \), then \( z \) is infinitely ringed by (17) defined by \( \theta_- \) resp. \( \theta_+ \).

- Otherwise \( z \) is ringed by (17) defined by \( \theta_+ \) and \( \theta_- \).

Lemma 4.3.5. Suppose \( z \in K_f \) is not in the inverse orbit of 0 and not in any basin of 0. If \( z \) satisfies \( f^n(z) \notin L(t_0) \) for \( n \) large enough, then \( z \) is infinitely ringed by one of the graphs (18) for all \( k \) large enough.

Proof. The proof goes the same as in [14, Lem. 6.1] if \( f^n(z) \) avoid \( L(\frac{1}{2}) \) for \( n \) large enough. If \( f^n(z) \) meets infinitely many times \( L(\frac{1}{2}) \), then we treat by two subcases:

- If \( f^n(z) \in L(\frac{1}{2}) \) for \( n \) large enough. Let \( U \) be a connected component of \( f^{-j}(L(\frac{1}{2})) \) (\( 1 \leq j \leq q \)) contained in \( L(\frac{1}{2}) \cap S_0 \) that intersects infinitely the orbit of \( z \). Notice that \( L(\frac{1}{2}) \cap S_0 \) is divided into \( q - 1 \) connected components by \( q - 2 \) external rays landing at \( x_0 \), the first preimage of 0 (other than 0) on \( \partial B^*_0(0) \). Suppose \( U \) belongs to one of these \( q - 1 \) components \( V \), which is bounded by \( R^\infty(\eta_1), R^\infty(\eta_2) \). Then \( U \subsetneq V \) and the angles of external rays involved in \( \partial U \) are between \( \eta_1, \eta_2 \). Recall graph (14) and its corresponding puzzle pieces \( Q_n \). Let the \( Q_n^\pm \) be the two puzzle pieces of depth \( n \) at \( x_0 \). Since \( L(0) = 0 \), all the angles of external rays involved in \( Q_n^\pm \) converge to \( \eta_1 \) or \( \eta_2 \). Thus if we take \( k \) large enough in \( \theta_+ = \frac{\pi k^{1/2}}{2} \), \( U \) will be included by a puzzle piece \( P^\pm_1 \) of depth 1 defined by graph (18) such that \( P^\pm_1 \) is compactly contained in some puzzle piece of depth 0.

- There is a subsequence such that \( f^n(z) \notin L(\frac{1}{2}) \). By hypothesis, \( z \) returns infinitely many times to \( L(\frac{1}{2}) \cap S_0 \). Thus \( \{ f^n(z); n \geq 0 \} \cap (L(\frac{1}{4}) \cap S_0)^c \) will meet infinitely \( (L(\frac{1}{2}) \cup L(\frac{3}{4})) \cap S_0 \). By Lemma 4.3.2 if \( f^n(z) \) is contained in \( (L(\frac{1}{2}) \cup L(\frac{3}{4})) \cap S_0 \), then it is ringed at depth 0.

Thus in both cases, we have found a graph that rings infinitely \( z \).
Proposition 4.3.6. Suppose \( t_0 \neq \frac{1}{2} \) and that \( z \in K_f \) returns infinitely many times to \( L(c) = L(t_0) \cap S_m \) but never hits \( \partial B_m(0) \). Then for all \( k \) large enough, there exists one of the two graphs \([17]\) that rings infinitely \( z \).

Proof. For simplicity we only do the proof for the case \( m = 0 \). The case \( m \neq 0 \) will be similar. By hypothesis there is a subsequence \( \{ f^{n_i}(z) \}_i \subset L(t_0) \). Take any \( w \in \{ f^{n_i}(z) \}_i \).

- \( t_0 \in \left[ \frac{1}{4}, \frac{1}{2} \right] \). Then by Lemma 4.3.2 \( w \) is ringed at depth 0 by graph (18) defined by \( \theta_- \) for \( k \) large enough.

- \( t_0 \in \left( \frac{1}{5}, \frac{1}{4} \right) \). By Lemma 4.3.3 the limb containing the critical value \( v_- \) is \( L(2t_0) \cap S_p \) with \( 2t_0 \in \left( \frac{1}{4}, \frac{1}{2} \right) \). Take \( k \) large enough and the corresponding graph \( X_0^+ \) such that \( L(2t_0) \cap S_p \) is bounded by the puzzle piece of depth 0 whose boundary intersects \( f(R^0(2k^{-2} \theta_+)), f(R^0(2k^{-1} \theta_+)) \). Thus by Lemma 4.3.3 any \( w \) satisfying

\[
\kappa(f(w)) \in (0, 2k^{-2} \theta_+) \cup (2k^{-1} \theta_+, 1)
\]

is ringed by graph (18) defined by \( \theta_+ \) at depth 0. Hold the same \( k \) and take \( \theta_- = \frac{1}{2k-1} \). If \( w \) satisfies

\[
\kappa(f(w)) \in [2k^{-2} \theta_+, \frac{1}{2} + \theta_-),
\]

then by Lemma 4.3.2 \( f^j(f(w)) \) is ringed at depth 0 by graph (18) defined by \( \theta_- \), where \( j \leq q - 1 \) is the smallest integer such that \( f^j(f(w)) \in S_0 \). Now suppose that \( w \) satisfies

\[
\kappa(f(w)) \in [\frac{1}{2} + \theta_-, 2k^{-1} \theta_+],
\]

Take \( k' \) slightly smaller than \( k \) such that \( \theta'_- = \frac{1}{2k'} - \frac{1}{2} \) satisfies \( 2t_0 < 1 + 2k'^{-1} \theta'_- < \frac{1}{2} + \theta_- \). (adjust \( k, k' \) to be larger if necessary). Thus we can apply Lemma 4.3.3 to graph (18) defined by \( \theta'_- \) and hence \( w \) is ringed at depth 0. To conclude, we have proved that for any \( w \), there exists a graph (18) defined by \( \theta_+, \theta_- \) or \( \theta'_- \) (not depending on \( w \)) that rings \( w \) at some depth less than \( q \).

- \( t_0 \in \left( \frac{1}{m+1}, \frac{1}{m+2} \right], n \geq 3 \). The strategy is quite similar as above. In this case we have \( 2t_0 \in \left( \frac{1}{m+1}, \frac{1}{m+2} \right] \). For the same reason, \( w \) is ringed by graph defined by \( \theta_+ \) for \( k \) large if \( \kappa(f(w)) \in (0, 2k^{-n} \theta_+) \cup (2k^{-n+1} \theta_+, 1) \). Take \( \epsilon > 0 \) small enough such that \( 2n^{-2}(2k^{-n+1} \theta_+ - \epsilon) > \frac{1}{2} + \theta_- \). Thus for \( w \) satisfying

\[
\kappa(f(w)) \in [2k^{-n} \theta_+, 2k^{-n+1} \theta_+ - \epsilon)
\]

and \( j \) the smallest integer such that \( f^j(f(w)) \in S_0 \), we have \( \kappa(f^N(w)) \in \left[ \frac{1}{4}, \frac{1}{2} + \theta_- \right) \), where \( N = (n-2)q + j + 1 \). Thus \( f^N(w) \) is ringed at depth 0 by graph defined by \( \theta_- \). If \( w \) satisfies

\[
\kappa(f(w)) \in [2k^{-n+1} \theta_+ - \epsilon, 2k^{-n+1} \theta_+)
\]

and \( j \leq q - 1 \) is the smallest integer such that \( f^j(f(w)) \in S_0 \), then

\[
\kappa(f^{N'}(w)) \in [2k^{-2} \theta_+ - 2n^{-2} \epsilon, 2k^{-2} \theta_+) \subset \left[ \frac{1}{4}, \frac{1}{2} + \theta_- \right)
\]

(take \( \epsilon \) smaller if necessary), where \( N' = (n-3)q + j + 1 \). Hence \( f^{N'}(w) \) is ringed at depth 0 by graph defined by \( \theta_- \).

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Lemma 4.3.10. $t_0 \in (\frac{1}{2}, 1)$. The argument is symmetric to the case $t_0 \in (0, \frac{1}{2})$, which is already handled above.

\[\square\]

Lemma 4.3.7. Let $t_0 = \frac{1}{2}$ and $z$ satisfy the hypothesis in Proposition 4.3.6. Then $z$ is infinitely ringed by some graph $(18)$.  

Proof. The prove goes exactly the same as Lemma 4.3.5.  

To summarize all the cases above, Lemma 4.3.4, 4.3.5, 4.3.7 and Proposition 4.3.6 gives

Proposition 4.3.8. Suppose $L(0) = 0$. Let $z \in K_f$ not in the inverse orbit of 0 and not in any basin of 0, then for all $k$ large enough, $z$ is infinitely ringed by one of the graphs $(17)$.  

Applying Yoccoz’s Theorem to $v_- = f(c_-), f = f_a$, we get

Theorem 4.3.9. Let $U \subset C$, be a parabolic component of adjacent, bitransitif or capture type that is not in any $W^\pm(a_m)$. Suppose $a \in \partial U$, $f^n(c_-(a)) \neq 0, \forall n \geq 1$ and $a \notin A_{p/q}$. Then there exists a graph $(17)$ and a sequence of non-degenerated annuli $A_{n_i}^a, i \geq 0$, such that

1. $A_{n_i}^a = P_{n_i}^{a,v} \setminus P_{n_i+1}^{a,v}$, $i \geq 1$, where $P_{n}^{a,v}$ is the puzzle piece of depth $n$ containing $v_-(a)$.
2. $f^i_{n_i-n_0} : A_{n_i}^a \rightarrow A_{n_0}^a$ is a non-ramified covering.
3. either $\sum_i \text{mod}(A_{n_i}^a) = \infty$ or there exists $k \geq 1$ such that $f_k^a : P_{m+k}^{a,v} \rightarrow P_m^{a,v}$ is quadratic-like for all $m$ large enough and $\bigcap P_{n}^{a,v}$ is the filled Julia set of the renormalized map $f_a^k$.

The following lemma allows us to apply Yoccoz’s Theorem to other points of the Julia set.

Lemma 4.3.10. Let $L(t_n)$ be the limb containing $f^n(c_-), n \geq 0$. If

- $t_n$ is not dyadic for $n$ large enough,
- or there exists $n_1, n_2$ such that $t_{n_1} \in \left[\frac{1}{4}, \frac{1}{2}\right), t_{n_2} \in \left(\frac{1}{2}, \frac{3}{4}\right]$,

then there exists $N$ such that $c_-$ is ringed at depth $N$ for both graphs $(17)$ for all $k$ large enough.

Proof. The proof for the first case is just a repeat of [13 Lem. 6.2]. The second case is deduced directly from Lemma 4.3.2.  

\[\square\]

Theorem 4.3.11. Let $f$ satisfy Assumption (\circ). For all $n \geq 0$, $f^{-n}(B_{m}^*(0))$ is locally connected.

Proof. Let $z \in B_{m}^*(0)$. Corollary 4.1.2 treat the case when $z$ is in the inverse orbit of 0. For other $z$, suppose first that $c_- \in L(t_n)$ satisfies one of the hypothesis in Lemma 4.3.2 then Lemma 4.3.4 allows us to use Yoccoz’s Theorem to get the dichotomy: if the intersection of the puzzle pieces containing $z$ shrink to $z$, then we are done; if the intersection is a quadratic copy, then this copy is separated from $B_{m}^*(0)$ by two external rays landing at $z$. Let $W$ be the open region bounded by the two rays containing the quadratic copy, then $(P_n(z) \setminus W)_n$ form a connected basis of $z$. For details of this part, see the proof of [13 Thm. 1].

The only case left not covered by Lemma 4.3.10 is that $t_n$ is dyadic for all $n$, but either $t_n \notin \left[\frac{1}{4}, \frac{1}{2}\right)$ for all $n$ or $t_n \notin \left(\frac{1}{2}, \frac{3}{4}\right]$ for all $n$. For example if it is the first alternative, we treat it furthermore by two subcases.
• $t_n = \frac{1}{2}$ for all $n$. By Lemma 4.3.7 we may suppose \[ \Box \] defined by $\theta_+$ rings $c_-$ for all $k$ large enough. By the proof of Lemma 4.3.4, $z$ might not be infinitely ringed by the graph of $\theta_+$ only when $\kappa(z)$ is recurrent to $\frac{1}{2}$. However, in this case there exists $k$ large enough such that the graph with $\theta_+$ does not contain $z$ for all $n$. There is $N$ such that the puzzle piece $P_0$ containing $w \in \partial B^*m(0)$ with $\kappa(w) = \frac{1}{2}$ contains $f^N(z)$. Then $\overline{P_0}$ does not contain forward orbit of $c_-$. Hence by the shrinking lemma, the pieces around $f^N(z)$ shrink to $f^N(z)$.

• $t_n$ is not always $\frac{1}{2}$. Without loss of generality, suppose $t_m = \frac{3}{4}$. Then by Lemma 4.3.2 $c_-$ is ringed at depth 0 by graph of $\theta_+$. Similarly as the above case we can apply the shrinking lemma for $\kappa(z)$ recurrent to $\frac{1}{4}$.

\[ \square \]

4.4 Wakes attached to end points

Suppose $f$ satisfy Assumption (\S). Set $B_0 := \bigcup_m B_m^*(0)$ Define $B_n$ to be the union of Fatou components not in $B_{n-1}$ but attached to $\partial B_{n-1}$ at a preimage of 0. Set $B = \bigcup_n B_n$. For each component $U \subset B_n$, $\overline{U}$ is eventually, firstly and bijectively sent to some $\overline{B}_n^*(0)$ by a certain iteration of $f$. So any $z \in \partial U$ is associated to an angle $\omega_n$. Recall that $B_n^*(0) \subset S_m$, so we associate $U$ with $\epsilon_n = m$. Therefore we can use the following sequence to encode the position of $z$:

\[(\epsilon_0; \omega_0); (\epsilon_1; \omega_1); \ldots; (\epsilon_n; \omega_n)\] (19)

where $\omega_0, \ldots \omega_{n-1}$ are dyadic, $\omega_n \in \mathbb{R}/\mathbb{Z}$; $0 \leq \epsilon_k \leq q - 1$ is chosen so that $f^k(z) \in S_{\epsilon_k}$.

Define wakes $W_U(\omega)$ and limbs $L_U(\omega)$ attached to $\partial U$ similarly as [4.2] Let $W$ be the open region bounded by the two external rays separating $U$ with $B_0$. Likewise we have $W = U \bigcup W_{U}(\omega)$, $K_{f} \cap W = U \bigcup_{\omega \neq 0} L_{U}(\omega)$. Notice that the wakes and limbs defined here include those in [4.2] $L_{B_n^*(0)}(\omega) = L(\omega) \cap S_m$.

Proposition 4.4.1. Suppose $\omega$ is not dyadic. Let $\{z\} = L_{U}(\omega) \cap \partial U$. If $L_{U}(\omega)$ is non trivial, then either $z$ is in the inverse orbit of $c_-$ or $z$ is (pre)-periodic. Moreover, if $L_{U}(\omega)$ is non trivial, then there are exactly two external rays landing at $z$ separating $L_{U}(\omega)$ with $U$, otherwise there is only one.

Proof. Suppose $L_{U}(\omega)$ is non trivial and $z$ not in the inverse orbit of $c_-$. Then $L_{U}(\omega)$ is sent injectively to the critical limb by some $f^N$ (since the width of $W_{U}(\omega)$ is zero and it is multiple by 3 by every iteration of $f$). So we may suppose $L_{U}(\omega)$ is the critical limb. Then $U = B_n^*(0)$, since for otherwise $f^n(L_{f(\omega)}(f(z)))$ will never contain $L_{U}(\omega)$. Hence $z$ is periodic.

Since in the proof of Theorem 4.3.11 we see that the pieces shrink to $z$ if the limb is trivial, so it is direct that there is only one external ray landing at $z$ if furthermore $z$ does not hit $c_-$. Existence of two external rays landing at $z$ if the tableau of $z$ is (pre)-periodic comes from the shrinking property of $P_n \setminus W$, where $W$ is as in the beginning of the proof of Theorem 4.3.11. For the proof of the uniqueness of these two rays, see [14, Lem. 7.2].

More generally, consider $z \in B \setminus \bigcup_n \overline{B}_n$, and suppose $z_k \in \bigcup_n \overline{B}_n$ converging to $z$. Then for all $n$, there exists $z_k$ such that $z \in \overline{B}_n$. Otherwise, there exists $N$ such that for $k \geq N$, $z_k$ belongs to the closure of some Fatou component in $B_{N+1}$ attached at some $\partial U \subset B_N$. Clearly $z_k$ converges to some $L_U(t)$ with $z_k \notin L_U(t)$ since $z \in B \setminus \bigcup_n \overline{B}_n$. Then $L_U(t)$ must be non trivial since $z$ is not
the root of \( L_U(t) \). By Proposition \[4.4.1\], \( L_U(t) \) is separated from all other limbs by two external rays, so \( f_k \) can not converge to \( z \). Therefore there exists a unique sequence of Fatou components \( U_n \subset B_n \) such that \( z \in \bigcap_n L_{U_n}(\omega_n) \) with \( \omega_n \) dyadic. Thus any \( z \in \partial B \setminus \{0\} \) can be represented by a unique infinite sequence (if \( z \in \partial B_n \), add zeros after \( (\epsilon_n, \omega_n) \)):
\[
[z] := [(\epsilon_0, \omega_0); (\epsilon_1, \omega_1); \ldots; (\epsilon_n, \omega_n); \ldots].
\]
where \( \omega_k \in (0, 1), 0 \leq \epsilon_k \leq q - 1 \). The coordinate for \( f(z) \) is
\[
[f(z)] = \begin{cases}
[(\epsilon_0 + p, 2\omega_0); (\epsilon_1, \omega_1); \ldots; (\epsilon_n, \omega_n); \ldots], & \text{if } \epsilon_0 = 0, \omega_0 \neq \frac{1}{2} \\
[(\epsilon_1, \omega_1); (\epsilon_2, \omega_2); \ldots; (\epsilon_n, \omega_n); \ldots], & \text{if } \epsilon_0 = 0, \omega_0 = \frac{1}{2} \\
[(\epsilon_0 + p, \omega_0); (\epsilon_1, \omega_1); \ldots; (\epsilon_n, \omega_n); \ldots], & \text{if } \epsilon_0 \neq 0
\end{cases}
\quad (20)
\]

**Lemma 4.4.2.** \( z \in B \) has at least 2 preimages in \( B \).

**Proof.** This is clear for \( z \in \overline{B_n} \). Take \( z \in B \setminus \bigcup_n \overline{B_n} \), then \( z \) is the limit of some \( (z_n) \) with \( z_n \in \overline{B_{N_n}} \). Each \( z_n \) has two preimages \( x_n, y_n \), which, up to taking a subsequence, have different limits \( x, y \). Clearly \( f(x) = f(y) = z \). \( \square \)

**Remark 4.4.3.** The construction of the sequence \( [(\epsilon_0, \omega_0); (\epsilon_0, \omega_1); \ldots; (\epsilon_n, \omega_n); \ldots] \) is also valid for the quadratic model \( P_{p/q}(z) = e^{2\pi i \frac{p}{q}} z + z^2 \). Moreover the following map is bijective:
\[
\Xi : J_{p/q} \setminus \{0\} \longrightarrow [\mathbb{Z}/q\mathbb{Z} \times (0, 1)]^\mathbb{N}, \ z \mapsto [z].
\quad (21)
\]

Indeed, the surjectivity is clear; the injectivity is also clear for \( [z] \) finite. Notice that \( J_{p/q} \) can be regarded as a pinching limit of the Julia set of \( \lambda z + z^2 \) with \( |\lambda| < 1 \) and the pinching map \( \phi \) is injective beyond the skeletons, while the skeletons are \( \phi^{-1}(y) \) with \( y \) preimage of 0 (cf. Theorem \[B.8\]). Thus if \([z]\) is infinite, then clearly \( \phi^{-1}(z) \) does not intersect any skeleton, hence \( \# \phi^{-1}(z) = 1 \).

**Lemma 4.4.4.** Suppose \( f \) is non renormalisable. If \( v_- = f(c_-) \in B \setminus \bigcup_n \overline{B_n} \), then there is only one external ray landing at \( v_- \).

**Proof.** The existence of external ray landing at \( v_- \) comes from the shrinking property of puzzles pieces around it given by Theorem \[4.3.9\]. Suppose the contrary that there are two rays landing. Let \( W \) be the open region separated from \( B \) by these two rays. Then there exists \( N \) such that \( W \) is sent injectively onto \( f^N(W) \) with \( c_- \in f^N(W) \). But this is impossible, since by Lemma \( f^{-1}(v_-) \) has two preimages in \( B \), so \( c_- \) must belong to \( B \) because \( f \) is degree 3. \( \square \)

## 5 Passing to parameter plan

In this section we pass the dynamical combinatorics to parameter ones. To simplify the redaction, from now on we will mainly work in the “fundamental region” \( S_m^+ \) for \( 0 \leq m \leq \left\lfloor \frac{q}{2} \right\rfloor \) (recall Definition \[3.6.3\].
5.1 Parameter equipotentials and rays

For $m = 0$, define parameter equipotentials and union of internal rays in $\mathcal{D}_0$ and $\tilde{\mathcal{D}}_0$ by

$$\mathcal{E}_n^0 = (\Psi_0^{-ad})^{-1}(E_n^0 \cap B_2^*(0)), \quad \mathcal{R}_n^0 = (\Psi_0^{-ad})^{-1}(R_n^0 \cap B_2^*(0)), \quad n \geq 0. \quad (22)$$

For $1 \leq m \leq \lceil \frac{d}{2} \rceil$, define parameter equipotentials and union of internal rays in $\mathcal{D}_m$ resp. $\tilde{\mathcal{D}}_m$ by

$$\mathcal{E}_n^m = (\Psi_m^{-bit})^{-1}(E_n^m \cap B_2^*(0)), \quad \mathcal{R}_n^m = (\Psi_m^{-bit})^{-1}(R_n^m \cap B_2^*(0)), \quad n \geq 0. \quad (23)$$

Let $k \geq 1$, $U_k \subset \tilde{S}_m$ be a capture component. Suppose for $a \in U_k$, $c_a$ (a) firstly hits $B_2^*(0)$ after $k$ iterations. Define the parameter equipotentials and internal rays by

$$\mathcal{E}_{\mathcal{U}_k}^m = (\Psi_{\mathcal{U}_k})^{-1}(E_n^m \cap B_2^*(0)), \quad \mathcal{R}_{\mathcal{U}_k}^m = (\Psi_{\mathcal{U}_k})^{-1}(R_n^m \cap B_2^*(0)), \quad n \geq 0. \quad (24)$$

Next we investigate landing properties for equipotentials in $\mathcal{B}_m$. We only state the result for $\mathcal{E}_n^m$ in $\mathcal{D}_m$ since it will be the same for $\tilde{\mathcal{E}}_n^m$.

**Proposition 5.1.1.** Let $n, k \geq 0$. Let $1 \leq l \leq q$ be such that $m = lp \pmod q$. Then $\mathcal{E}_n^m \cap \partial \mathcal{D}_m$ consists of finitely many points. Denote by $\mathcal{E}_n^m / \sim$ the quotient space of $\mathcal{E}_n^m \cap \partial \mathcal{D}_m$ by

- gluing the double parabolic parameter on $\partial \mathcal{B}_0$ and $2\sqrt{\lambda}$ if $m = 0$;
- gluing the two parabolic parameters on $\partial \mathcal{B}_m$ if $1 \leq m \leq \frac{q}{2}$.

Then $\mathcal{E}_n^m / \sim$ is homeomorphic to $E_n^m \cap B_m^*$. In particular $\mathcal{E}_n^m = \mathcal{E}_n^{m+1} = \cdots = \mathcal{E}_n^{(k+1)q-1}$ and $(\mathcal{E}_n^m \cap \partial \mathcal{D}_m) \subseteq (\mathcal{E}_n^{(k+1)q} \cap \partial \mathcal{D}_m)$. Moreover for $a \in \mathcal{E}_n^m \cap \partial \mathcal{D}_m$ not double parabolic, $f_a^{(k+1)q-1}(v_-(a)) = 0$.

**Proof.** The proof has no difference to the case $p/q = 1$. See [20, Prop. 3.2.10].

By the above proposition, we see that $\mathcal{D}_m$ is cut ”binarily” by $E_{kq}^m$ into $2^k$ pieces if $m = 0$ and $2^{k+1}$ pieces otherwise. For a double parabolic parameter $b \in \partial \mathcal{B}_m$, let $\{a_k\}_{k \geq 0}$ be the sequence such that $a_k \in \partial \mathcal{B}_m$ be the closest summit to $b$ among $E_{kq}^m \cap \partial \mathcal{B}_m$. By Lemma 2.2.8 there are $q$ parameter external rays landing at $a_k$ and $q$ dynamical rays with the same angles landing at $v_-(a_k)$. The angles are preimages of 0 under multiplication by 3. Denote by $T_k$ the set of these angles.

**Lemma 5.1.2.** $3^k T_{k+1} = T_k$. Moreover all angles in $T_k$ converges to $\alpha$, the angle of the external ray $\mathcal{R}_\infty(\alpha)$ landing at $b$ with $\mathcal{R}_\infty(\alpha) \subset \partial \mathcal{S}_m$.

**Proof.** We only do the proof for $m = 0$ to illustrate the idea. In this case, $b = a_0$ and hence $\theta = \beta_0^+$ (Definition 3.6.2). For $n \geq 0$, let $U_k$ be the component bounded by $\mathcal{R}_\infty(\alpha_0^+) \cup E_{kq}^0 \cup \mathcal{R}_\infty(t_k)$, where $\mathcal{R}_\infty(t_k)$ is the external ray closest to $\mathcal{R}_\infty(\beta_0^+)$ landing at $a_k \in E_{kq}^0 \cap \partial \mathcal{B}_0$, the point closest to $a_0$ (in the binary sense). By Corollary 3.6.5 external rays $\mathcal{R}_\infty(t)$ with $3^k t \in \Theta_0$ do not land at $a_0$. Hence

$$U_k \setminus \bigcup_{\{t : 3^k t \in \Theta_0\}} \mathcal{R}_\infty(t)$$

contains $\mathcal{W}^+(a_0)$, and denote by $\hat{U}_k$ the component containing it. Notice that for $a \in \hat{U}_{k-1}$, there is a dynamical holomorphic motion of $\bigcup_{t \in T_k} \mathcal{R}_a(\hat{U}_k \setminus \mathcal{R}_\infty(\alpha_{k-1}))$. Therefore $T_k$ is exactly the set of angle
of external rays landing at \( f_{a}^{-k_{a}}(0) \cap \partial B_{a,m}^{*}(0) \) which is closest to \( \beta_{0}^{+} \). By the proof of Proposition 4.1.1 we see that \( T_{k} \) satisfies \( 3^{q}T_{k+1} = T_{k} \) tends to \( \beta_{0}^{+} \) as \( k \to +\infty \).

\[ \square \]

**Proposition 5.1.3.** Let \( \mathcal{U}_{k} \) be a capture component. Then \( \partial \mathcal{U}_{k} \) contains no double parabolic parameter.

**Proof.** Suppose the contrary that there exists \( b \in A_{p/q} \cap \partial \mathcal{U}_{k} \). Consider respectively two cases:

- \( \mathcal{U}_{k} \) is contained in some \( S_{m} \). Since \( \partial \mathcal{U}_{k} \subset \partial S_{\lambda} \) (Lemma 2.2.4), there is an external ray contained in \( S_{m} \) accumulating to \( \partial \mathcal{U}_{k} \). This contradicts Lemma 5.1.2.

- \( \mathcal{U}_{k} \) is contained in some double parabolic wake. Consider equation of \( z, f_{a}^{q}(z) = z \). When \( a = b, 0 \) is a multiple solution of order \( 2q + 1 \), while for \( a \not= b \) near \( b, 0 \) is has order \( q + 1 \). All the other solution besides \( 0 \) of \( f_{b}^{q}(z) = z \) gives a repelling periodic cycle for \( b \), which admits a holomorphic motion for \( a \) near \( b \). Thus there are only \( q \) non-zero solutions left of \( f_{b}^{q}(z) = z \) for \( a \not= b \) (\( a \) near \( b \)) which do not come from the holomorphic motion. Among these \( q \) solutions, there is at least a repelling cycle \( C \) of period dividing \( q \) for \( f_{a} \). Now suppose furthermore \( a \in \mathcal{U}_{k} \). Let \( \Theta \) be the set of angle of external rays for \( f_{a} \) landing at \( C \). Notice that \( \Theta \) does not depend on \( a \) since in \( \mathcal{U}_{k} \), there is a holomorphic motion of \( B_{a,\infty}(\infty) \) induced by Böttcher coordinate. \( \Theta \) can not be \( \Theta_{m}, \Theta_{m-1} \) since \( C \) is not 0. Hence for \( f_{a} \), the external rays with angles in \( \Theta \) land at a repelling cycle, which in turn gives a repelling cycle for \( f_{a} \) by holomorphic motion. This contradicts how we choose \( C \).

\[ \square \]

**Corollary 5.1.4.** \( \mathcal{E}^{\mathcal{U}_{k}}_{0} \) is homeomorphic to \( \Psi_{k}(\mathcal{E}^{\mathcal{U}_{k}}_{0}) \). Moreover \( a \in \mathcal{E}^{\mathcal{U}_{k}}_{0} \cap \partial \mathcal{U}_{k} \) is Misiurewicz parabolic.

**Corollary 5.1.5.** Let \( \mathcal{U}_{k} \) be a capture component contained in some double parabolic wake \( \mathcal{W}_{\infty}(a_{0,m}) \), then for \( a_{0} \in \partial \mathcal{U}_{k}, f_{a_{0}}^{-k_{a}}(c_{-}(a_{0})) \) is on the boundary of some \( B^{*}_{a_{0,m}} \). Moreover \( \partial \mathcal{U}_{k} \) is a Jordan curve.

**Proof.** By Proposition 5.1.3, there is an open neighborhood of \( \mathcal{U}_{k} \), on which the union of immediate basins \( \bigcup \mathcal{B}_{a_{0,m}}(0) \) admits an holomorphic motion \( h_{a} \) induced by Fatou coordinate. One can choose the base point \( a_{0} \) of \( h_{a} \) in \( \mathcal{U}_{k} \). By \( \lambda \)-lemma, the motion is extended to \( \bigcup \mathcal{B}_{a_{0,m}}(0) \). For \( a \in \mathcal{U}_{k} \), by definition of capture component there exists \( m \) such that \( f_{a_{0}}^{k_{a}}(c_{-}(a)) \in \bigcup \mathcal{B}_{a_{0,m}}(0) \). Let \( a \) tend to \( a_{0} \), we get \( f_{a_{0}}^{k_{a}}(c_{-}(a_{0})) \in \partial \mathcal{B}_{a_{0,m}}^{*} \) since \( \mathcal{B}_{a_{0,m}}^{*} \) moves holomorphically.

Define \( H : \partial \mathcal{U}_{k} \longrightarrow \partial \mathcal{B}_{a_{0,m}}^{*} \) by \( a \rightarrow h_{a}^{-1}(f_{a_{0}}^{k_{a}}(c_{-}(a))) \). \( H \) is locally regular, hence \( \partial \mathcal{U}_{k} \) is locally connected since \( \partial \mathcal{B}_{a_{0,m}}^{*} \) is (Julia set of \( f_{a_{0}}^{k_{a}} \)) is locally connected since it is geometrically finite [18]. \( \partial \mathcal{U}_{k} \) is then a Jordan curve by Lemma 2.2.4.

\[ \square \]

From Proposition 5.1.1 and 5.1.3 we see that \( \mathcal{R}_{m}^{0}, \mathcal{R}_{n}^{\mathcal{U}_{k}} \) can not land at double parabolic parameters. We have the following landing property of internal rays (the proof is similar to that for external rays) :

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Lemma 5.1.6. Let \( 0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( U_k \) be a capture component not in double parabolic wake. Then any component of \( R_m^n \) resp. \( R_{m+k}^n \) land at some \( a_0 \in \partial S_m \) resp. \( \partial U_k \) which is neither Misiurewicz parabolic nor double parabolic. The landing points of \( f_{a_0}^{-k}(R_m^n) \) are (pre-)periodic to the same cycle, where \( R_m^n \) is just \( R_0^n \) in [16]. Moreover if this cycle is repelling, then some component of \( f_{a_0}^{-k}(R_m^n) \) will land at \( v_-(a_0) \).

We also have the analogue to Lemma 6.4.1

Lemma 5.1.7. Suppose \( a_0 \in C \backslash \bigcup_m W^{\pm}(a_m) \) is a Misiurewicz parameter. Then \( a_0 \) is the landing point of some \( R_m^n \) or \( R_{m+k}^n \), if and only if \( v_-(a_0) \) is the landing point of some \( f_{a_0}^{-k}(R_m^n) \).

5.2 Dynamical objects move holomorphically

Definition 5.2.1. Let \( A_0 \subset S^1 \) be finite and satisfy \( 3A_0 = A_0 \), \( A_n \) be the \( n \)-th preimage of \( A_0 \) under multiplication by 3. A parameter object of depth \( n \geq 0 \) is the intersection of one of the following four sets and \( S_m^+ \).

\[
\bigcup_{\eta \in A_n} R_\infty(\eta), \bigcup_{k=1}^n \left( \bigcup_{U_k \subseteq H_{k-1}} \varepsilon_{4-k}^{U_k} \right) \cup \bigcup_{k=1}^n \left( \bigcup_{U_k \subseteq H_{k}} \frac{R_m}{n+k} \right) \cup \bigcup_{k=1}^n \bigcup_{n+k \in H_k} \frac{R_m}{n+k}, E_\infty(r^{1/3^n}).
\] (25)

Let \( O_n \) be a parameter object of depth \( n \). Let \( O \) be a connected component of \( S_m^+ \backslash O_n \) intersecting \( \partial C \). Let \( a \in O \).

When \( O_n = \bigcup_{\eta \in A_n} R_\infty(\eta) \) or \( E_\infty(r^{1/3^n}) \), \( \bigcup_{\eta \in A_{n+1}} R_\infty(\eta) \) resp. \( E_\infty(r^{1/3^{n+1}}) \) does not contain \( c_-(a) \). Hence each external ray \( R_\infty(\eta) \) is well defined and lands at \( J_a \), resp. \( E_\infty(r^{1/3^{n+1}}) \) is homeomorphic to \( S^1 \). Moreover there is a dynamical holomorphic motion of \( \bigcup_{\eta \in A_{n+1}} R_\infty(\eta) \) resp. \( E_\infty(r^{1/3^{n+1}}) \) induced by Böttcher coordinate \( \phi_\infty \).

Now let \( O_n = \bigcup_{k=1}^n \left( \bigcup_{U_k \subseteq H_k} \varepsilon_{4-k}^{U_k} \right) \cup \bigcup_{k=1}^n \bigcup_{n+k \in H_k} \frac{R_m}{n+k} \). We want to find a dynamical holomorphic motion for equipotentials in \( B_a(0) \). Recall that \( \Omega_a \) is the maximal petal contained in \( B_a^+(0) \) and \( \tilde{\Omega}_a \) is the \( (a) \)-th preimage of \( \Omega_a \) contained in \( B_a^+(0) \). Then by definition of \( E_\infty \), \( E_\infty^{a} \), \( f_a^{-1}(E_0^a) \) does not contain \( c_-(a) \), where \( E_0^a := \bigcup_{l=1}^q f_a^l(\partial H_a) \). Hence there is a dynamical holomorphic motion of \( E_\infty : f_a^{n+1}(E_0^a) \) induced by Fatou coordinate and pulling back by \( f_a \).

Now let \( O_n = \bigcup_{k=1}^n \left( \bigcup_{U_k \subseteq H_k} \frac{R_m}{n+k} \right) \cup \bigcup_{k=1}^n \bigcup_{n+k \in H_k} \frac{R_m}{n+k} \). Notice that here the construction of internal rays is more subtle, since \( c_-(a) \) might be in \( B_a^+(0) \), as a consequence we can not identify completely the dynamics in \( B_a^+(0) \) with the quadratic model (but still partially, up to certain depth, see below the rewritten diagram of [10]). The idea is to pull back the beginning of the internal rays so long as they do not contain \( v_-(a) \). Similar to the quadratic model, one can define internal rays \( R_a(\theta) \subset B_a^+(0) \) with \( \theta \in \Theta \) where \( \Theta := \{ \theta : \exists \eta \text{ s.t. } 2\theta = 2\eta \} \) and \( \theta = \frac{1}{2^{n+1}} \). Define similarly \( R_0^a \) to \( R_0^n \) in [16]. There is also a dynamical holomorphic motion of \( R_{n+1}^a := f_a^{-n-1}(R_0^a) \). See [20, 3.1] for more details.

\[
\begin{align*}
\Omega_{a,m+p}^{n} & \xrightarrow{f_a} \Omega_{a,m+2}^{n-1} \xrightarrow{f_a} \Omega_{a,q-p}^{n} \xrightarrow{f_a} \Omega_{a,0}^\theta \\
\Omega_{a,m+p}^{n} & \xrightarrow{P_\lambda} \Omega_{a,m+2}^{n-1} \xrightarrow{P_\lambda} \Omega_{a,q-p}^{n} \xrightarrow{P_\lambda} \Omega_{a,0}^\theta \\
\end{align*}
\] (26)
where \( \phi_a : \Omega_{a,0}^n \to \mathbb{H} \) be the Fatou coordinate of \( f_a^3|_{B_{a,0}(0)} \) normalised by \( \phi_a(c_+(a)) = 0, n_a \) is the smallest integer such that \( \Omega_{a,m+p}^n \) contains \( v_-(a) \).

**Definition 5.2.2.** Let \( a \in \hat{\mathcal{O}}_m \). The dynamical objects \( \mathcal{O}_n^a \) of depth \( n \) corresponding to the four parameter ones are

\[
\bigcup_{\eta \in A_{n+1}} \mathcal{R}_n^a(\eta), \ f_a^{-n-1}(\mathcal{E}_0^n), \ f_a^{-n-1}(R^n_0), \ \mathcal{E}_n^\infty(r^{1/3^n+1})
\]

(27)

To conclude from the discussion in this subsection:

**Proposition 5.2.3.** Let \( \mathcal{O}_n \) be a parameter object of depth \( n \), \( a,a_0 \in \hat{\mathcal{O}}_n \). Let \( \mathcal{O}_{n+1}^a, \mathcal{O}_{n+1}^{a_0} \) be the corresponding dynamical object of depth \( n + 1 \). There exists a dynamical holomorphic motion \( L_n : \mathcal{O}_n \times \mathcal{O}_{n+1}^{a_0} \to \mathbb{C} \) with \( L_n(a,\mathcal{O}_{n+1}^{a_0}) = \mathcal{O}_n^a \). Moreover, if \( \mathcal{O}_{n+1}^a \) is the first one or the third one in \([27]\), \( \mathcal{O}_{n+1}^a \cap J_a \) is repelling.

### 5.3 Parameter graphs and puzzles

**Parameter graphs \( \mathcal{Y}_n \) and puzzles \( \mathcal{Q}_n \) in \( \hat{S}_m^+ \)**

Fix some \( r > 1 \), for each \( n \geq 0 \) define the graph adapted for parameters of Misiurewicz parabolic type.

\[
\mathcal{Y}_n = \bigcup_{k=1}^{n} \left( \bigcup_{U_k \subset \mathcal{H}_k \cap \mathcal{S}_m^+} \mathcal{E}_{n-k}^{U_k} \right) \cup \left( \bigcup_{t \in T_n} \mathcal{R}_{\infty}(t) \right) \cup \mathcal{E}_m \cup (\mathcal{E}_\infty(r^{1/3^n}) \cap \mathcal{S}_m^+)
\]

where

\[
T_n = \{ t \; 3^{n+l}t \in \Theta_m \} \cap [\alpha_{m-1}^+, \beta_{m+1}^+] \quad 0 \leq l \leq q - 1 \text{ s.t. } l p + m = 0 \quad (\text{mod } q)
\]

By Proposition 5.1.1 Corollary 5.1.4 the landing points of equipotentials in parabolic components are Misiurewicz parabolic. By Lemma 5.4.1 these points are also landing points of external rays with angles in \( T_n \). Hence \( \mathcal{Y}_n \) is connected, and every connected component of \( \mathcal{S}_m^+ \setminus \mathcal{Y}_n \) is simply connected. We call such a connected component \( \mathcal{Q} \) a puzzle piece associated to \( \mathcal{Y}_n \) if it is bounded and \( \partial \mathcal{Q} \cap \mathcal{E}_\infty(r^{1/3^n}) \neq \emptyset \). We denote it by \( \mathcal{Q}_n \) in the sequel. By construction, every \( \mathcal{Q}_{n+1} \) is contained in a unique puzzle piece \( \mathcal{Q}_n \).

**Parameter graphs \( \mathcal{X}_n \) and puzzles \( \mathcal{P}_n(a_0) \) in \( \mathcal{S}_m^+ \)**

Next we define the graph adapted for parameters which are not of Misiurewicz parabolic type. Let \( a_0 \in \mathcal{S}_m^+ \) such that \( f_{a_0} \) satisfies Assumption (\( \diamond \)). By Theorem 4.3.9 there is a graph \([17]\) infinitely ringing \( v_-(a_0) \). Recall that this graph is associated to the angle \( \theta_1 = \frac{1}{2\pi} \) of an internal ray. Let \( H \) be the collection of angles of all external rays in \([17]\) of depth 0. For each \( n \geq 0 \) consider

\[
\mathcal{X}_n = \bigcup_{k=1}^{n} \left( \bigcup_{U_k \subset \mathcal{H}_k \cap \mathcal{S}_m^+} \mathcal{E}_{n-k}^{U_k} \cup \mathcal{R}_{n-k}^{U_k} \right) \cup \left( \bigcup_{\eta \in A_n} \mathcal{R}_{\infty}(\eta) \right) \cup \mathcal{E}_m \cup (\mathcal{E}_\infty(r^{1/3^n}) \cap \mathcal{S}_m^+)
\]

where

\[
A_n = \{ \eta; \ 3^n \eta \in H \} \cap [\alpha_{m-1}^+, \beta_{m+1}^+].
\]
We call a connected component $P$ of $S^+_m \setminus X_n$ a puzzle piece associated to $X_n$ if it is bounded and $\partial P \cap E_\infty (r^{1/3^n}) \neq \emptyset$. We denote it by $P_n$ in the sequel. Clearly every $P_{n+1}$ is contained in a unique $P_n$. Define $P_n(a_0)$ to be the puzzle piece containing $a_0$. This is well-defined since $a_0 \notin X_n$, for otherwise some external ray $R_\infty^\infty (\eta)$ or internal ray will land at a parabolic pre-periodic point or $v_-(a_0)$, contradicting with the construction of the dynamical graph for $f_{a_0}$ (see the observation at the beginning of 4.3).

**Lemma 5.3.1.** Let $n \geq 1$. Any $a \in X_n \cap \partial H \cap P_0(a_0)$ is a Misiurewicz parameter. It is the landing point of an internal ray $R \subset X_n$ and an external ray $R_\infty (t) \subset X_n$. In particular $X_n$ is connected.

**Proof.** By Proposition 5.2.3, landing points of $R_0^a$ are repelling, since those of $R_0^\infty (\eta)$ are. Therefore by Lemma 5.1.6, $f_a$ is Misiurewicz. Also each landing point is landed by some external ray $R_\infty (t)$ involved in the admissible graph of $f_{a_0}$. By Lemma 6.4.1, $a$ is the landing point of $R_\infty (t)$. Hence $X_n$ is connected.

**Dynamics graphs defined up to certain depth**

From the discussion in Subsection 5.2, we see that for $a$ satisfying the condition in Proposition 5.2.3, $E_0^a, R_0^a$ are well defined and homeomorphic to the corresponding objects in the quadratic model (15), (16). Set $E_n^a = f_{a_0}^{-n}(E_n^a), R_n^a = f_{a_0}^{-n}(R_n^a)$. Define

$$Y_n^a = E_n^a \cup \left( \bigcup_{t \in T_n} \overline{R_\eta^\infty (3^n t)} \right) \cup E_\infty^\infty (r_n).$$

$$X_n^a = E_n^a \cup R_n^a \cup \left( \bigcup_{\eta \in A_n} \overline{R_\eta^\infty (3^n \eta)} \right) \cup E_\infty^\infty (r_n).$$

Proposition 5.2.3 gives immediately

**Lemma 5.3.2.** Let $a_0 \in Q_n$. Then for $0 \leq k \leq n+1$ there exists a dynamical holomorphic motion $L_k : Q_n \times Y_n^a \to \mathbb{C}$ with base point $a_0$ such that $L_k(a, Y_n^a) = Y_n^a$.

**Lemma 5.3.3.** Let $a_0 \in S^+_m$ such that $f_{a_0}$ satisfies the Assumption (♦). For $0 \leq k \leq n+1$, there is a dynamical holomorphic motion $L_k : \mathcal{P}_n(a_0) \times X_n^a \to \mathbb{C}$ with base point $a_0$ such that $L_k(a, X_n^a) = X_n^a$.

**6 Local connectivity of $\partial H^\lambda_n$, $\lambda = e^{2\pi i \frac{p}{q}}$**

By Corollary 5.1.3 it remains to justify the local connectivity for parabolic component not in double parabolic wakes.

**6.1 Misiurewicz parabolic and double parabolic case**

**Proposition 6.1.1.** Let $\mathcal{U} \subset S^+_m$ be a parabolic component. Then $\partial \mathcal{U}$ is locally connected at all Misiurewicz parabolic parameters.
Proof. Let \( a_0 \in \partial \mathcal{H}_k \) be Misiurewicz parabolic. By Lemma 6.4.1, let \( t_1^0, t_2^0, \ldots, t_q^0 \) be the \( q \) angles such that \( R_{\infty}(t_i^0) \) lands at \( a_0 \). Then \( R_{\infty}(t_0) \) is part of \( \mathcal{V}_n \) for all \( n \geq N \), where \( N \) is some fixed integer. Therefore there are \( q+1 \) adjacent puzzle pieces of depth \( n \) \( Q_n^0, Q_n^1, \ldots, Q_n^q \) attached at \( a_0 \). Clearly \( Q_{n+1}^i \subset Q_n^i \). Now we claim that for \( a \in Q_n^i, v_-(a) \in (Q_n^i)^i \). Indeed, for any \( \epsilon > 0 \), there exists \( n'/n \) large enough and \( t_0' \) such that \( 3^nt_0' = 1, R_{\infty}(t_0') \cap Q_n^i \neq \emptyset \) and \( |t_0' - t_0| < \epsilon \). Hence for \( a' \in R_{\infty}(t_0') \cap Q_n^i, v_-(a') \in (Q_n^i)^i \). By Lemma 5.3.2, \( Y_n^a \) moves holomorphically for \( a \in Q_n^i \). Notice that \( v_-(a) \) also moves holomorphically and it does not belong to \( Y_n^a \) when \( a \in Q_n^i \), we deduce that \( v_-(a) \in (Q_n^i)^i \) since \( v_-(a') \in (Q_n^i)^i \). Now if \( a \in \bigcap_n Q_n^i \), then \( a \) is in \( \mathcal{C}_\lambda \) but does not belong to any parabolic component since \( a \) is excluded by every equipotentials in \( \mathcal{H}_\infty \) and \( \mathcal{H} \). Moreover \( f_n^a(-\epsilon-a) \neq 0, \forall n \geq 1 \) by definition of \( Q_n^i \). Therefore by Proposition 4.1.1, \( \bigcap_{m \geq 1} (Q_m^a)^i = \emptyset \). But on the other hand \( v_-(a) \in \bigcap_n (Q_n^a)^\pm \), a contradiction. Hence \( \bigcap_n Q_n^a = \emptyset \).

This means that if we set

\[
\mathcal{O}_n := \bigcup_{i=0}^{q} Q_n^i \cup \bigcup_{i=0}^{q-1} R_{\infty}(t_0) \ast (0, r^{1/3n}) \cup \{a_0\},
\]

then \( (\mathcal{O}_n \cap \partial \mathcal{U})_n \) form a basis of connected neighborhood of \( a_0 \).

Notice that \( a_0 = a_m \in A_{p/q} \) is not covered by Proposition 6.1.1. But still we can verify the local connectivity with a similar argument.

**Proposition 6.1.2.** Let \( Q_n^\pm \subset S_n \) be the unique two puzzle pieces of depth \( n \) containing \( a_m \) on their boundaries. Without loss of generality suppose that their boundaries all contain \( R_{\infty}(a_m^\ast) \ast (0, r^{1/3n}) \). Then \( \bigcap_n Q_n^\pm = \emptyset \).

**Proof.** We only prove for the sequence \( \{Q_n^+\}_{n} \), the case left is similar. Suppose the contrary \( \bigcap_n Q_n^+ \neq \emptyset \). Recall that in Lemma 5.1.2 there is a sequence of external rays \( R_{\infty}(t_n) \) landing at \( \partial B_n \) with \( t_k \) in the preimage of \( \Theta_m \) and \( t_n \) converging monotonously to \( \beta_n^+ \) as \( n \to \infty \). In particular \( R_{\infty}(t_n) \ast (0, r^{1/3n}) \subset \partial \mathcal{Q}_n^+ \). Therefore for \( a \in \mathcal{Q}_n^+ \), the critical value \( v_-(a) \) is contained in \( \mathcal{Q}_n^a \), the puzzle piece whose boundary contains \( R_n^\ast(a_m^\ast) \ast (0, r^{1/3n}) \). Now if \( a \in \bigcap_n \mathcal{Q}_n^+ \), then \( a \in \mathcal{C}_\lambda \) is not Misiurewicz parabolic and not in any parabolic component. This contradicts Proposition 4.1.1 which tells us that \( \bigcap_n Q_n^a = \emptyset \).

### 6.2 Non renormalizable case

In this subsection, we always fix some \( a_0 \in S_n^+ \) such that \( f_{a_0} \) satisfies the Assumption (\( \diamond \)). Consider the corresponding graphs \( X_n \) and puzzle pieces \( P_n(a_0) \).

We give the relation between para-puzzles and dynamical puzzles:

**Lemma 6.2.1.** Let \( n \geq 0 \). The mapping \( H_n : \mathcal{P}_n(a_0) \cap X_{n+1} \to P_{n_0}^0 \cap X_{n+1}^0 \) defined by \( H_n(a) = (L_{n+1})^{-1}(v_-(a)) \) is injective. Moreover there exists \( N \geq 0 \) such that \( \forall n \geq N, H_n \) is surjective.

**Proof.** First we prove that \( H_n \) is well-defined. By definition of \( X_n \), we have \( v_-(a) \in P_{n_0}^0 \cap X_{n+1}^0 \). Consider the holomorphic motion starting at \( a : L_{n+1}(a', z) := L_{n+1}(a', a_n^0(z)) \). By continuity there exists a disk \( B(a, r) \) on which \( v_-(a') \) is surrounded by \( L_{n+1}(a', a_n^0(r)) \). Pick any \( a' \) on \( \partial B(a, r) \), if \( a' \notin \partial P_n(a_0) \), then for any \( a'' \) in a small disk \( B(a', r') \), \( v_-(a'') \) is surrounded by \( L_{n+1}(a', a_n^0(r')) \). Hence we can extend this property step by step until we reach \( \partial P_n(a_0) \).
particular \( H_n(a) \in \mathcal{P}^{a_0,v}_n \). By Lemma 5.3.3 \( \partial \mathcal{P}^{a_0,v}_n \) moves holomorphically when \( a \in \mathcal{P}_n(a_0) \), hence \( H_n(a) \in X^0_{n+1} \).

Next we verify injectivity. The injectivity is clear when \( a \) belongs to parameter external rays and equipotentials since in the dynamical plan, angles of external rays and equipotentials are preserved by \( L_{n+1} \). We will only prove injectivity for \( a \) belongs to parameter internal rays. The proof for internal equipotentials is similar. Suppose there are two distinct parameters \( a, a' \) such that \( H_n(a) = H_n(a') \). Then clearly \( a, a' \) belong to different components \( \mathcal{U}, \mathcal{U}' \). Set \( a \in \mathcal{R}_U, a' \in \mathcal{R}_{U'} \).

- First suppose that the landing point \( b, b' \) of \( \mathcal{R}_U, \mathcal{R}_{U'} \) do not coincide. Consider the external rays \( \mathcal{R}_\infty(s), \mathcal{R}_\infty(s') \) involved in \( \chi_{n+1} \) landing at \( b, b' \) respectively. Then the two rays \( H_n(\mathcal{R}_\infty(s)), H_n(\mathcal{R}_\infty(s')) \) land at a common point \( x(a_0) \) since \( H_n(a) = H_n(a') \). But this is impossible since \( f_{a_0} \) is injective near the forward orbit of \( x(a_0) \).

- Next if \( b = b' \). There are two possibilities: \( H_n(\mathcal{R}_\infty(s)) = H_n(\mathcal{R}_\infty(s')) \) or they are two different rays landing at a common point \( x(a_0) \). While in the graph \( X^0_{n+1} \), only one internal ray lands at \( x(a_0), \) so the second is impossible. Then there exists two other internal rays \( \mathcal{R}_{U_1} \subset \mathcal{U}, \mathcal{R}_{U_2} \subset \mathcal{U}' \) landing at \( b, b' \) respectively which have the same image under \( H_n \). Then \( b \neq b' \), for otherwise one finds a loop in \( \mathcal{U}_1 \cup \mathcal{U}_2 \) surrounding points in \( H_\infty \). We repeat the argument above to \( b, b' \) and get a contradiction.

Finally we verify surjectivity. First we prove

**Claim.** \( \partial \mathcal{P}_n(a_0) \cap (\mathcal{R}_\infty(a^+_m) \cup \mathcal{R}_\infty(\beta^+_m)) = \emptyset \) for \( n \) large enough.

**Proof of the claim.** Suppose for example \( \partial \mathcal{P}_0(a_0) \) contains parts of \( \beta^+_m \) (the proof proceeds the same way if it is \( a^+_m \)). Recall in Lemma 5.1.2 there is a sequence of external rays \( \mathcal{R}_\infty(s_n) \) landing at \( a_n \in \partial \mathcal{B}_m \), \( s_n \to \beta^+_m \) and \( a_n \) Misiurewicz parabolic. For \( n \) large enough, let \( \mathcal{R}_n \subset \mathcal{B}_m \cap \chi_n \cap \mathcal{P}_0(a_0) \) be a sequence of internal rays landing at \( b_n \in \partial \mathcal{B}_m \). By Lemma 5.3.1 there exists \( \mathcal{R}_\infty(t_n) \subset \chi_n \) landing at \( b_n \) with \( |t_n - \beta^+_m| < |s_n - \beta^+_m| \). Thus if \( \partial \mathcal{P}_n(a_0) \cap \mathcal{R}_\infty(\beta^+_m) \neq \emptyset \), then \( a_0 \in Q^+_n \), where \( Q^+_n \) as defined in Proposition 6.1.2. Recall there we have shown that \( \bigcap Q^+ \neq \emptyset \). Hence \( \partial \mathcal{P}_n(a_0) \cap \mathcal{R}_\infty(\beta^+_m) = \emptyset \) for \( n \) large enough.

\( H_n \) is therefore surjective on the part of external equipotential and external rays by its injectivity and the claim. So we verify surjectivity for internal rays and for internal equipotentials will be similar. Let \( z_0 \in X^0_{n+1} \cap \mathcal{B}_m(a_0) \cap \mathcal{P}^{a_0,v}_n \). Suppose that \( z_0 \) is on some connected component \( \zeta \) of \( f^{-k}(R_{a_0}) \). Let \( x_0 \) be the landing point of \( \zeta \), then there is an external ray \( R^\infty_{a_0}(t) \subset X^0_{n+1} \) landing at \( x_0 \). By Lemma 5.3.1 \( R^\infty(t) \) lands at a Misiurewicz parameter \( a' \) which is accessible by some internal ray \( \mathcal{R}_U \) (Lemma 5.1.7). Clearly \( H_n(\mathcal{R}_U) = \zeta \).

**Corollary 6.2.2.** Let \( a \in \mathcal{P}_{n-1}(a_0) \). Let \( C^a_n \) be the puzzle boundary by \( L_n(\partial \mathcal{P}^{a_0,v}_n) \). Then \( a \notin \mathcal{P}_n(a_0) \) if and only if \( v_-(a) \notin C^a_n \).

**Proof.** If \( a \notin \mathcal{P}_n(a_0) \), then take a simple path \( a_t \subset \mathcal{P}_{n-1}(a_0) \) connecting \( a_0, a \) such that \( a_t \cap \mathcal{P}_n(a_0) \) only contains one point. Thus Lemma 6.2.1 ensures that once \( a_t \) goes out of \( \mathcal{P}_n(a_0) \), \( f_{a_t}(c_-(a_t)) \) will never enter again \( C^a_n \). Conversely if \( a \in \mathcal{P}_n(a_0) \), then clearly \( v_-(a) \in C^a_n \) since \( \partial \mathcal{P}^{a_0,v}_n \) moves holomorphically and \( v_-(a) \) does not intersect \( X^0_n \).

**Corollary 6.2.3.** For \( n \) large enough, if \( \mathcal{P}^{a_0,v}_{n+1} \subset \mathcal{P}^{a_0,v}_n \), then \( \mathcal{P}_{n+1}(a_0) \subset \mathcal{P}_n(a_0) \). Moreover \( H_n : \partial \mathcal{P}_{n+1}(a_0) \to \partial \mathcal{P}^{a_0,v}_{n+1} \) is a homeomorphism.
Proof. By Lemma 6.2.1, $H_0^{-1}(\partial P_{n+1}^{\chi}) \subset P_n(a_0)$ bounds a puzzle piece $\mathcal{P}$. So it suffices to prove that $a_0 \in \partial \mathcal{P}$. Suppose not, then there is $a' \in \partial \mathcal{P}$ but $a' \notin \overline{P_{n+1}(a_0)}$. By Corollary 6.2.2, $v_-(a') \notin C_{n+1}$, a contradiction.

**Corollary 6.2.4.** Let $U \subset S_m^+$ be a capture component. If $a_0 \in \partial U$ or $\partial B_m$ is non renormalisable, then $v_-(a_0)$ eventually hit the boundary of immediate basins.

**Proof.** By Corollary 6.2.3, $\partial P_{n+1}^{\omega, v}$ intersects a Fatou component $U$ (preimage to some immediate basin) for all $n$. Hence by Theorem 4.3.9, $\bigcap_{n=0}^\infty \partial P_{n+1}^{\omega, v}$ intersects $\partial U$ at $v_-(a_0)$.

By Theorem 4.3.9, there exists graph (17), such that in the dynamical plane of $f_{a_0}$ one has a sequence of non-degenerated annuli $A_n^{a_0} := P_{n+1}^{\omega, v} \setminus \overline{P_n^{\omega, v}}$. The above corollary implies that in the parameter plane, $A_n^{a_0} := P_n(a_0) \setminus \overline{P_{n+1}(a_0)}$ is also non-degenerated for $i$ large enough. Applying Shishikura’s trick (15), we can get the estimate between the moduli of the para-annuli and dynamical annuli:

**Lemma 6.2.5.** There exists $K > 1$ such that for $i$ large enough

$$\frac{1}{K} \text{mod}(A_n^{a_0}) \leq \text{mod}(A_{n+1}^{a_0}) \leq K \text{mod}(A_n^{a_0}).$$

**Corollary 6.2.6.** $C_\lambda$ is locally connected at non renormalisable parameters.

**Proof.** This follows immediately from the above lemma and Grötzsch’s inequality.

### 6.3 Renormalizable case

In this subsection, we always fix some $a_0 \in S_m^+$ such that $f_{a_0}$ satisfies the Assumption $(\mathfrak{G})$ and is renormalisable. Recall that a set $M' \subset \mathbb{C}$ is called a copy of the Mandelbrot set (cf. [4]) $M$ if there exists $k \geq 1$ and a homeomorphism $\chi : M' \to M$ such that $f_a$ is $k$–renormalisable and $f_a^k$ is quasiconformally conjugated to $z^2 + \chi(a)$.

**Proposition 6.3.1.** $M_{a_0} := \bigcap_{n=0}^\infty \mathcal{P}_n(a_0)$ is a copy of the Mandelbrot set. Moreover, there are exactly 2 external rays landing at the cusp $\chi^{-1}(\frac{1}{4})$, separating $M_{a_0}$ with $B_m$. In particular, if $a_0 \in \partial B_m$, then $a_0 = \chi^{-1}(\frac{1}{4})$, $\partial B_m$ is locally connected at $a_0$.

**Proof.** For the first statement, see [13, Prop. 3.26]. As for the second statement, the proof for the existence is similar to Step 1, Step 2 in Lemma 2.3.8 for the uniqueness, see the proof of [13, Thm. 3]

**Remark 6.3.2.** By the above proposition, the wake $W(a)$ and limb $L(a)$ can be defined in the classical sense.

**Proposition 6.3.3.** Let $U \subset S_m^+$ be a capture component. Suppose $a_0 \in \partial U$ is renormalisable. Then $a_0$ is a tip of $M_{a_0}$, with $M_{a_0} \cap B_m = a_1$. Moreover there are exactly two external rays landing at $a_0$, separating $U$ with $M_{a_0}$. In particular, $\partial U$ is locally connected at $a_0$.

**Proof.** See [13, Prop. 4.15, Thm. 4].

**Proof of Theorem 6.3.** Let $U$ be an open connected component of $H^\lambda$. Corollary 5.1.5 solves already the case where $U$ is contained in a double parabolic wake. So suppose $U$ is in the complement. From the discussion in 6.1, 6.2 and 6.3 we see that $\partial U$ is locally connected. It is then a Jordan curve by Lemma 2.2.4.
6.4 Global descriptions

Lemma 6.4.1. Let $a_0 \in C_{\lambda}$ be Misiurewicz parabolic. Suppose moreover that $a_0$ is not in any double parabolic wake. Then there are $q$ external rays landing at it, each two adjacent rays bound a parabolic component.

Proof. One can prove the existence of parabolic component attached to $a_0$ by showing that there are parameter equpotentials rays landing at $a_0$. To prove the existence of landing rays, one uses holomorphic motion and Rouché’s Theorem, similar to the proof of Proposition 3.5.4.

Let $a_0 \in \partial B_m$ be Misiurewicz parabolic. By Lemma 6.4.1, there are $q$ external rays (whose angles are preimages of angles in $\Theta_m$) landing at $a_0$ each two adjacent rays separate a capture component attached at $a_0$ with $B_m$. Applying a similar argument as in the proof of Lemma 5.1.2, one can show that when a sequence of Misiurewicz parabolic parameters $a_n \in \partial B_m$ converges to $a_0$, the angles of external rays landing at $a_n$ converges to the biggest or smallest angle among the angles of the $q$ external rays landing at $a_0$. Therefore there are exactly $q$ external rays landing at $a_0$ and exactly $q-1$ capture components attached at $a_0$. A similar result holds for the capture components attached at $\partial B_m$, also for the next ”generation” of capture components attached, and so on.

Definition 6.4.2. Let $0 \leq m \leq \lfloor \frac{q}{2} \rfloor$. A capture component $U_1 \subset \mathcal{S}^+_m$ is of level 1, if it is attached to $\partial B_m$ at a Misiurewicz parabolic parameter. A capture component $U_0$ is of level $n \geq 2$ if it is not of level $n-1$, and is attached to some $\partial U_{n-1}$ at a Misiurewicz parabolic parameter with $U_{n-1}$ of level $n-1$. The point $r_n = \partial U_n \cap \partial U_{n-1}$ is called the root of $U_n$. The open region bounded by the two external rays containing $U_n$ is called the wake of $U_n$, denoted by $\mathcal{W}_{U_n}$. The corresponding limb is defined to be $L_{U_n} = \overline{\mathcal{W}_{U_n}} \cap C_{\lambda}$.

Definition 6.4.3. Set $\mathcal{H}^m_0 = B_m \cap \mathcal{S}^+_m \subset \mathcal{H}^m$ to be the union of all level $n \geq 1$ capture components, $\mathcal{H}^m := \bigcup_{n \geq 0} \mathcal{H}^m_n$ is called a central component.

By Proposition 6.3.3, there are no renormalisable parameter on $\partial U_n$ for $n \geq 1$. By Corollary 6.2.4, if $a \in \partial U_n$, $c(a)$ eventually hits the boundary of immediate basins for $a \in U_n$. Recall the parametrization (19), define the parametrisation of $a_n \in \partial U_n \setminus \{r_n\}$ to be

$$[[a]] := [c(a_n)] = [(m, \omega_0); (e_1, \omega_1); \ldots; (e_n, \omega_n)].$$

(28)

where $\omega_0, \ldots, \omega_{n-1}$ are dyadic, $\omega_n \in (0, 1); 0 \leq e_k \leq q-1$ is chosen so that $f^k_{a_n}(c(a_n)) \in S_{a, e_k}$ $(S_{a, e_k}$ is the sector $S_{e_k}$ in 1.4). If $a \in \partial B_m \setminus A_{p/q}$, define $[[a]]$ to be $[r_a]$ if $a$ is renormalisable, where $r_a \in \partial B_{a,m}(0) \setminus \{0\}$ is the parabolic periodic point (Proposition 6.3.1); otherwise $[[a]] := [c(a_n)]$.

Lemma 6.4.4. For $a_0 \in \partial U_n$ or $\partial B_m$ non renormalisable, there is a unique external ray landing at it.

Proof. This comes from Proposition 4.4.1 and the homeomorphism between para-puzzles and dynamical puzzles, cf. Corollary 6.2.3.

Lemma 6.4.5. We have decompositions

$$\mathcal{S}^+_m \cap C_{\lambda} = (\overline{B_m \cap \mathcal{S}^+_m}) \cup \left( \bigcup_{U \in \mathcal{H}^m} L_{U} \cup \bigcup_{a \in \partial B_m \cap \mathcal{S}^+_m \text{parabolic}} L(a) \right)$$

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\[ \mathcal{L}_{t_n} = \mathcal{U}_n \cup \bigcup_{\mathcal{U}_{n+1} \subset \mathcal{H}_{n+1}} \mathcal{L}_{t_{n+1}}. \]

As a direct consequence, \( \mathcal{H}_n \setminus \mathcal{A}_{p/q} = \bigcup_{t_n \in \mathcal{H}_n} \mathcal{U}_n. \)

**Proof.** This is a direct consequence of the above lemma. Let us do the proof for \( S_m^+ \cap C_\lambda \). For any \( n \geq 0 \), consider all the Misiurewicz parabolic parameters of depth \( n \) (that is, \( f_n^m(v,(a)) = 0 \)) on \( \partial \mathcal{B}_m \cap S_m^+ \) and the \( q \) unique landing external rays. These rays together with \( \partial \mathcal{B}_m \) separate \( S_m^+ \) into several open sectors of depth \( n \). Take \( b \in \mathcal{C}_{p/q} \cap (\mathcal{U}_n)^c \cap S_m^+ \) not in any \( \mathcal{L}_{t_n} \). Let \( \mathcal{S}_n(b) \) be the sector of depth \( n \) containing \( b \). Let \( \mathcal{R}_\infty(t_n), \mathcal{R}_\infty(t'_n) \) be the two external rays bounding \( \mathcal{S}_n(b) \) and \( a_n, a'_n \) their landing point respectively. Clearly \( a_n, a'_n \) converges to some \( a \in \partial \mathcal{B}_m \) since \( \partial \mathcal{B}_m \) is a Jordan curve. Then \( a \) must be renormalisable. If not, then by the previous lemma, \( t_n, t'_n \) converge to the same angle, which implies that \( b = a \), a contradiction since we take \( b \notin \mathcal{B}_m \). \( \square \)

**Proposition 6.4.6.** Let \( a_0 \in \mathcal{H}_m \setminus \bigcup_{n \geq 0} \mathcal{H}_n \) not be double parabolic. Then \( a_0 \) is contained in an infinite sequence of limbs, i.e. there exist \( U_n \subset \mathcal{H}_n \) such that \( a_0 \in \bigcap_n \mathcal{L}(U_n) \). Moreover \( a_0 \) can not be Misiurewicz parabolic. If \( a_0 \) is non renormalisable, there is only one external ray landing at \( a_0 \); if it is renormalisable, then it is the cusp of \( M_{a_0} \) and exactly two external rays land at it.

**Proof.** The existence of \( U_n \) is just a direct consequence of Lemma [6.4.5]. Next we prove that \( a_0 \) is not Misiurewicz parabolic. Suppose the contrary that \( f_n^\infty(v,(a_0)) = 0 \). Let \( a_n \in \mathcal{U}_{N_{\lambda}} \subset \mathcal{H}_{N_{\lambda}} \) be a sequence converging to \( a_0 \) and suppose \( f_{a_n}^\infty(v,(a_n)) = 0 \). Clearly \( M_n \geq n \). Fix \( N' \gg N \), set \( N'' := M_{N'} - N \). Let \( \tilde{B}_{N''} \) be the connected component containing \( 0 \) of \( f_{a_n}^\infty((\bigcup_{i} \tilde{B}_{m_i}^*)(0)) \). Notice that there exists a neighborhood \( V \) of \( a_0 \) on which there is a dynamical holomorphic motion \( h_n \) of \( \tilde{B}_{N''} \). By construction, \( f_n^\infty(v,(a_n)) \) is bounded by a wake attached at a preimage of \( 0 \) (not equal to \( 0 \)) on \( \partial(h_n(\tilde{B}_{N''})) \). Moreover the angles of the two external rays \( R^\infty_{a_n}(t_1), R^\infty_{a_n}(t_2) \) determining this wake do not depend on \( n \). Shrink \( V \) if necessary so that there is a holomorphic motion of \( R^\infty_{a_0}(t_1) \cup R^\infty_{a_0}(t_2) \). This implies that \( f_{a_n}^\infty(v,(a_n)) \) does not converge to \( 0 \) as \( n \to \infty \), contradicting \( a_n \to a_0 \).

So \( a_0 \) is either non renormalisable or renormalisable. Suppose the first case. The existence of external ray landing comes from the shrinking property of parameter puzzles around \( a_0 \) (Theorem 4.3.9 and Lemma 6.2.5). The uniqueness comes from Lemma 6.2.1 and Lemma 4.4.4. Next suppose \( a_0 \) is renormalisable. By Proposition 6.3.1 \( M_{a_0} \) is separated from \( \mathcal{H} \) by two external rays landing at the cusp. While \( a_0 \in \mathcal{H}_m \), it has to be the cusp. \( \square \)

We conclude that \( \partial \mathcal{H}_m \) is combinatorially rigid:

**Corollary 6.4.7.** Let \( a_0 \in \mathcal{H}_m \setminus \bigcup_{n \geq 0} \mathcal{H}_n \) not be double parabolic. Let \( U_n \) be the sequence of parabolic component associated to \( a_0 \). Then \( \bigcap_n \mathcal{L}(U_n) \cap \mathcal{H}_m = \{a_0\} \).

**Proof.** This comes from the shrinking property of puzzle pieces \( \mathcal{P}_n(a) \) when \( a \) is non renormalisable; for \( a \) renormalisable, it comes from the shrinking property of \( \mathcal{P}_n(a) \setminus \mathcal{W}(a) \). \( \square \)

**Theorem 6.4.8.** We have the decomposition

\[ S_m^+ \cap C_\lambda = \mathcal{H}_m \cap S_m^+ \cup \bigcup_{a \in \partial \mathcal{H}_m} \mathcal{L}(a). \]
Proof. Here $\mathcal{H}$ is parallel to $\overline{B_m}$ in Lemma 6.4.5 and instead of considering sectors defined by rays landing at $\partial B_m$, one should consider sectors defined by rays landing at $\partial \mathcal{H}_m$. One then concludes by using the landing property given in Proposition 6.4.6.

By Proposition 6.4.6 every $a \in \partial \mathcal{H}_m \setminus \bigcup_n \partial \mathcal{H}_n^m$ can be represented by an infinite sequence

$$[[a]] := [(m, \omega_0); (\varepsilon_1, \omega_1); (\varepsilon_2, \omega_2); \ldots; (\varepsilon_n, \omega_n), \ldots]$$

with $\omega_k$ dyadic, $0 \leq \varepsilon_k \leq q - 1$ are chosen so that $f_n^k(c_\pm(a_n)) \in S_{a, \pm \epsilon}$. Compare with Remark 4.4.3 the way we encode $J_{p/q} \setminus \{0\}$, where $J_{p/q}$ is the Julia set of the quadratic model $P_{p/q}$.

**Theorem 6.4.9.** The mapping defined by

$$\partial \mathcal{H}_m \cap S_m^+ \longrightarrow (J_{p/q} \setminus \{0\}) \cap S_m, \ a \mapsto \Xi^{-1}([[a]])$$

is a homeomorphism, where $\Xi^{-1}$ is defined in [27].

Proof. Clearly the mapping is surjective and is injective on $\partial \mathcal{H}_m \cap \mathcal{H}_n^m$ for all $n$. The injectivity when $a \in \partial \mathcal{H}_m \setminus \bigcup_n \partial \mathcal{H}_n^m$ comes from the combinatorial rigidity (Corollary 6.4.7) i.e. the mapping $a \mapsto [[a]]$ is injective.

It remains to verify continuity. $\Xi^{-1}([[a]])$ is bi-continuous on $\mathcal{H}_n^m$ since the boundary of any parabolic component is a Jordan curve. So suppose $[[a]]$ is infinite. Let $a_n \in \partial \mathcal{H}_m \cap S_m^+$ converge to $a$. Then there exists a subsequence $a_{n_j}$ and a subsequence $(N_j) \subset \mathbb{N}$ such that $a_{n_j} \in L(U_{N_j})$, where $(U_n)_n$ is the sequence of capture components defining $a$ (Corollary 6.4.7). Thus $\Xi^{-1}([[a_{n_j}]])$ is contained in the corresponding limb with root $\Xi^{-1}([[r_{N_j}]])$ in the filled in Julia set $K_{p/q}$. Since the sequence of these limbs in $K_{p/q}$ converges to $\Xi^{-1}([[a]])$, we get that $\Xi^{-1}([[a_{n_j}]]) \rightarrow \Xi^{-1}([[a]])$. The inverse continuity is similar, one uses Corollary 6.4.7.

**Appendices**

**A Rotation number**

**Definition A.1.** Let $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$ be the $d$-fold covering map, i.e. $f(\theta) = d\theta$. A periodic cycle $\Theta = \{\theta_0, \ldots, \theta_q-1\}$ has rotation number $\frac{p}{q}$ if, $\forall i, f(\theta_i) = \theta_{i+p \mod q}$.

**Theorem A.2 ([10]).** The $d$-fold mapping $\theta \mapsto d\theta$ has $\left(\frac{d+q-2}{q}\right)$ periodic cycles of rotation number $p/q$.

**Lemma A.3.** If $\Theta$ has rotation number $p/q$, then $\tilde{\Theta} = \{\tilde{\theta}_0, \ldots, \tilde{\theta}_{q-1}\}$ has rotation number $1 - \frac{p}{q}$, where $\tilde{\theta}_i = 1 - \theta_{q-i-1}$.

Proof. By definition, it suffices to prove that $f(\tilde{\theta}_i) = \tilde{\theta}_{i+p \mod q}$:

$$f(\tilde{\theta}_i) = d(1 - \theta_{q-i-1}) = -d\theta_{q-i-1} = -\theta_{q-i-1+p \mod q} = -\theta_{q-i-1+q \mod q} = \tilde{\theta}_{i+p \mod q}.$$
B Pinching deformation

This part is a resume of the theory of pinching deformation developed by Cui-Tan in \cite{3}.

Let $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map. Denote by $\mathcal{F}_f, \mathcal{J}_f$ the Fatou, Julia set respectively. Denote by $\text{Crit}_f$ the set of critical points of $f$. Define the post-critical set of $f$ by

$$\text{Post}_f := \bigcup_{n > 0} f^n(\text{Crit}_f).$$

$f$ is called geometrically finite if the accumulation points of $\text{Post}_f$ is finite. Define $\tilde{\mathcal{R}}_f \subset \mathcal{F}_f$ as follows:

1. $z$ does not belong to supper-attracting basins,
2. $\#(\{f^n(z); n \geq 0\}) = +\infty$ and $\{f^n(z); n \geq 0\} \cap \text{Post}_f = \emptyset$.

**Definition B.1.** The quotient space $\mathcal{R}_f$ of $f$ is defined as the set $\tilde{\mathcal{R}}_f$ quotient the grand orbit equivalence relation $\sim$, i.e. $z_1 \sim z_2$ if and only if $\exists m, n \geq 0, f^m(z_1) = f^n(z_2)$. Denote by $\pi_f: \tilde{\mathcal{R}}_f \rightarrow \mathcal{R}_f$ the natural projection.

**Remark B.2.** $\mathcal{R}_f$ has only finitely many components, each of them is either an at least 1-punctured torus, corresponding to an attracting basin, or an at least 1-punctured infinite cylinder, corresponding to a parabolic basin. Refer to \cite{11} for details.

In the sequel we always assume that $\tilde{\mathcal{R}}_f \neq \emptyset$.

**Lemma B.3.** Let $\gamma$ be a Jordan curve. Then either each component of $\pi_f^{-1}(\gamma)$ is a Jordan curve or each component of $\pi_f^{-1}(\gamma)$ is an eventually periodic arc. Moreover, every eventually periodic arc $\beta$ lands at both ends.

**Definition B.4.** Let $\beta$ be a $k$-periodic arc and $b \in \beta$. The limit point of the forward (resp. backward) orbit of $b$ under $f^k$ is called the attracting end (resp. repelling end). Notice that the two ends might coincide if both of them are parabolic.

**Definition B.5.** A multi-annulus $\mathcal{A} \subset \mathcal{R}_f$ is a finite disjoint union of annuli whose boundaries are pairwise disjoint simple closed curves such that each component of $\pi_f^{-1}(e(\mathcal{A}))$ is an eventually periodic arc, where $e(\mathcal{A})$ is the union of the equators of the annuli in $\mathcal{A}$. A multi-annulus $\mathcal{A}$ is called non-separating if for any choice of finitely many components of $\pi_f^{-1}(e(\mathcal{A}))$, the closure of their union (denote by $K$) does not separate $\mathcal{J}_f$, i.e. there is only one component of $\hat{\mathbb{C}} \setminus K$ intersecting $\mathcal{J}_f$.

**Definition B.6.** A component $B$ of $\pi_f^{-1}(\mathcal{A})$ is called a band. Notice that $B$ is a topological open disk bounded by two eventually periodic arcs, and hence itself is also eventually periodic and lands at both ends. The core arc $\beta$ of $B$ is the lift of the equator of $\pi_f(B)$ to $B$, whose two (or maybe one) end points are exactly those of $B$. A band $B$ (resp. its core arc $\beta$) is of level $n$ ($n \geq 0$) if $n$ is the smallest integer such that $f^n(B)$ (resp. $f^n(\beta)$) is periodic. A skeleton of level $n$ is a connected component of $\bigcup_{\text{level } n} B$. The fill-in of a skeleton $S$, denote by $\hat{S}$, is the union of $S$ with all its complement components disjoint from $\mathcal{J}_f$. $\hat{S}$ is called a fill-in skeleton.
For \( r > 1 \), denote by \( \mathcal{A}(r) := \{ z; \frac{1}{r} < |z| < r \} \). For \( t \geq 0 \), let \( w_{t,r} : \mathcal{A}(r) \to \mathcal{A}(r^{1+t}) \) be the pinching model defined as in [3], 5.1. Let \( \mathcal{A}' = \cup A_i \) be a non-separating multi-annulus. Let \( \chi_i : A_i \to \mathcal{A}(r_i) \) be a conformal representation and \( \mu_i \) be the Beltrami differential of \( w_{t,r} \circ \chi_i \). Set \( \mu_t = \mu_{i,t} \) on \( A_i \) and \( \mu_t = 0 \) elsewhere. Define \( \tilde{\mu}_t \) to be the pullback of \( \mu_t \) under \( \pi_f \), i.e. \( \tilde{\mu}_t = \pi_f \mu_t \) on \( \tilde{\mathcal{A}}_f \) and \( \tilde{\mu}_t = 0 \) elsewhere. Integrate \( \mu_t \) with a choice of normalization (not depending on \( t \)) by fixing 3 distinct points in \( \text{Post}_f \), we get a quasiconformal mapping \( \phi_t \).

**Definition B.7.** The path \( f_t = \phi_t \circ f \circ \phi_t^{-1} \) (\( t \geq 0 \)) is called a pinching path starting from \( f \) supported on \( \mathcal{A}' \).

The following result in [3] affirms that the pinching path is converging:

**Theorem B.8.** Let \( f \) be a geometrically finite rational map. Let \( f_t = \phi_t \circ f \circ \phi_t^{-1} \) be a pinching path starting from \( f \) supported on a non-separating multi-annulus. Then the following properties hold:

1. \( f_t \) converges uniformly to a geometrically finite rational map \( g \) as \( t \to \infty \).
2. \( \phi_t \) converges uniformly to a continuous surjective map \( \varphi \) with \( \varphi(J_f) = J_g \).
3. For each fill-in skeleton \( \tilde{S}, \varphi(\tilde{S}) \) is a parabolic (pre-)periodic point. Moreover \( \varphi \) is injective in the complement of the union of all fill-in skeletons.

## C Admissible petals

Let \( R : \mathbb{C} \to \mathbb{C} \) be a rational map with \( R(0) = 0, R'(0) = e^{2\pi i \frac{p}{q}} \). Write the Taylor expansion near 0: \( R^q(z) = z + \omega(a) z^{q+1} + o(z^{q+1}) \). There exactly \( v \) cycles of immediate basins with rotation number \( p/q \) around 0. Let \( B_0^*(0), \ldots, B_{q-1}^*(0) \) be one of such cycle, written in cycle order. On every \( B_m^*(0) \) there is a unique Fatou coordinate up to translation \( \phi_k : B_m^*(0) \to \mathbb{C} \) semi-conjugating \( R \) to \( z \mapsto z + 1 \).

**Definition C.1.** An admissible petal \( P^* \subset B_m^*(0) \) is a petal such that the boundary of \( \phi_k(P) \) is a smooth curve \( \gamma \) (well-defined up to translation) intersecting every horizontal line \( y = b \) at exactly one point. We say that \( P^* \) is standard if \( \gamma \) is a vertical line. \( P^* \) is called maximal if \( \partial P^* \) contains at least one critical point of \( R^q|_{B_m^*(0)} \). Clearly once \( \gamma \) is chosen, the corresponding maximal petal \( P^* \) is unique.

**Lemma C.2.** If \( P^* \subset B_m^*(0) \) is an admissible petal, then so is \( R^n(P^*) \) for any \( n \geq 0 \). Moreover \( R^n(P^*) \) are also associated to \( \gamma \).

Let \( P^*_m := P \subset B_m^*(0) \) be a maximal admissible petal (we omit the index of associated curve \( \gamma \)). For any \( 0 \leq k \leq q-1, n \in \mathbb{Z} \) such that \( np + k = m \) (mod \( q \)), \( P_k^* \) is defined to be the (unique) connected component of \( R^{-n}(P_m^*) \) satisfying \( P_k^* \subset B_m^*(0) \) and \( 0 \in \partial P_k^* \). See the sequence of mappings below, where the bar means \( (\text{mod } q) \):

\[
P_m^* \to_{R_{-np}} R_{-p} \to_{R_{-m-p}} R_{-m} \to_{R_{-m+p}} R_{-m-1} \to_{R_{-m+p}} \cdots \to_{R_{-m+p}} R_{-n} \to_{R_{-m+n+p}} \]

**D Holomorphic dependence of Fatou coordinate**

Let \( R_a : \mathbb{C} \to \mathbb{C} \) be an analytic family (parametrized by \( a \in \Lambda \subset \mathbb{C} \), \( \Lambda \) open) with \( R_a(0) = 0, R_a'(0) = e^{2\pi i \frac{p}{q}} \). Write the Taylor expansion near 0: \( R_a^q(z) = z + \omega(a) z^{q+1} + o(z^{q+1}) \). In this
section we always suppose that at some \( a_0, \omega(a_0) \neq 0 \). Then for every attracting (resp. repelling) axis of \( R_{a_0} \), there exists

- a small neighborhood \( D_{a_0} \) of \( a_0 \); a topological disk \( V_{att} = \{ x + iy; x \geq c - b|y| \} \) (resp. \( V_{rep} = \{ x + iy; x \leq c - b|y| \} \), where the constants \( b, c \) are positive and do not depend on \( a \); a family of attracting (resp. repelling) petals \(( P_a )_{a \in B(a_0)}\)

- a family of Fatou coordinates \(( \phi_a )_{a \in D_{a_0}}\) of \( R_a \) such that \( \phi_a : P_a \to V_{att} \) (resp. \( V_{rep} \)) is conformal and \( \phi_a^{-1} \) is analytic in \( a \).

An immediate consequence from this is that \( h_a = \phi_a^{-1} \circ \phi_{a_0} \) (parametrized by \( a \in U_{a_0} \)) defines a dynamical holomorphic motion of \( P_{a_0} \) such that \( h_a(\bar{P}_{a_0}) = \bar{P}_a \).

Next we prove that \( h_a \) can be extended to "sub-petal" arbitrarily close to the maximal petal \( \Omega_{a_0} \) of \( R_{a_0} \). Suppose that \( \phi_{a_0} : \Omega_{a_0} \to \mathbb{H}_\rho \) is conformal, where \( \mathbb{H}_\rho = \{ z; \Re(z) > \rho \} \). For any \( \delta > 0 \), define \( \Omega_{a_0}^\delta = \phi_{a_0}^{-1}(\mathbb{H}_{\rho + \delta}) \). Notice that there exists \( k \geq 1 \) such that \( h_a \) is well-defined on \( \bar{P}_a := R_{a_0}^k(\Omega_{a_0}^\delta) \). Set \( \bar{P}_a := h_a(\bar{P}_{a_0}) \).

**Lemma D.1.** Let \( W_a \) be the connected component of \( R_{a_0}^{-kq}(\bar{P}_a) \) containing \( \bar{P}_a \). Then for a close enough to \( a_0 \), \( R_{a_0}^{kq} : W_a \to P_a \) is conformal (injective, surjective).

**Proof.** Let \( a_n \to a_0 \). First we prove that

\[
\forall \varepsilon > 0, \exists N \text{ such that } \forall n \geq N, W_{\varepsilon_{a_n}} \subset \Omega_{a_0}^\delta
\]

where for two sets \( A, B \subset \mathbb{C} \), \( A \subset_{\varepsilon} B \) means that \( A \) is contained in the \( \varepsilon \)-neighborhood of \( B \). Suppose the contrary, then up to taking a subsequence we may suppose that

\[
\exists \varepsilon_0 > 0, \text{ a sequence } x_n \in W_{\varepsilon_0} \text{ with } x_n \to x_0, \text{ such that } \text{dist}(x_n, \Omega_{a_0}^\delta) > \varepsilon_0.
\]

Clearly \( R_{a_0}^{kq}(x_0) \in \bar{P}_{a_0} \) since \( R_{a_0}^{kq}(x_n) \in \bar{P}_{a_0} \). For every \( n \) take a path \( \gamma_n \subset W_{\varepsilon_n} \) connecting \( 0, x_n \) such that \( \gamma \cap \bar{P}_{a_0} \) does not depend on \( n \). This is possible since \( \bar{P}_{a_0} \) moves holomorphically. Moreover since the Hausdorff topology on the space of compact sets is sequentially compact, we may suppose that \( \gamma_n \) converges to a connected compact set \( \gamma_0 \) containing \( 0 \) and \( x_0 \). This implies that \( \text{dist}(x_0, \Omega_{a_0}^\delta) = 0 \), contradicting (30). Hence for a close enough to \( a_0 \), \( W_a \) contains no critical points of \( R_{a_0}^{kq} \) and the lemma is proven.

Pull back \( h_a \) by \( R_{a_0}^{kq}, R_{a_0}^{kq} \) and apply \( \lambda \)-Lemma we get immediately:

**Corollary D.2.** For any \( \delta > 0 \), the holomorphic motion \( h_a : \bar{P}_{a_0} \to \bar{P}_a \) can be dynamically extended to \( \Omega_{a_0}^\delta \) when \( a \) is close enough to \( a_0 \).

**Proposition D.3.** Suppose for \( a \) in a neighborhood of \( a_0 \), \( R_a^q \) has \( l \) (does not depend on \( a \)) distinct critical points \( c_1(a),...,c_l(a) \) which vary analytically on \( a \). Suppose moreover that the immediate basin of \( R_a \) is always simply connected. If \( c_1(a_0) \) is the unique critical point on \( \partial \Omega_{a_0} \), then for a close enough to \( a_0 \), \( c_1(a) \) is also the unique critical point on \( \partial \Omega_{a_0} \).

**Proof.** Clearly for a close enough to \( a_0 \), \( c_1(a) \) is in the immediate basin. Indeed, first take a path \( \gamma_{a_0} \) in the immediate basin, linking \( c_1(a_0) \) and some \( z_{a_0} \) in a small petal \( P_{a_0} \). One can connected
it with $h_a(z_{a_0}) \in P_a = h_a(P_{a_0})$ by a compact path $\gamma_a$ which is very close to $\gamma_{a_0}$. Hence for a very close to $a_0$, $\gamma_a$ is attracted by $z = 0$.

Hence we can normalise $\phi_a$ by taking $\phi_a(c_1(a)) = 0$. By Corollary 2.2, $\phi_a^{-1} : \mathbb{H}_1 \to \hat{\Omega}_a$ is conformal such that $R_a(c_1(a)) \in \partial \Omega_a$. Now we prove that for a close enough to $a_0$, the component of $(R_a)^{-1}(\Omega_a)$ containing $\Omega_a$, denoted by $\Omega_a$, contains no critical point of $R_a(\Omega_a)$. Suppose the contrary, then there is a sequence $a_n \to a_0$ such that, without loss of generality, $c_2(a_n) \in \Omega_{a_n}$. Up to taking a subsequence, $\Omega_{a_n}$ has a limit $\Omega^*$ in the Hausdorff topology. Moreover $\Omega^*$ is compact, connected and $\Omega^*$ is the closure of the union of several components of $(R_a)^{-1}(\Omega_{a_0})$. Clearly $\Omega_{a_0} \subset \Omega^*$. Suppose $c_1(a)$ has multiplicity $m$, then besides $\Omega_a$, there are $m$ distinct components of $(R_a)^{-1}(\Omega_{a})$ attached at $c_1(a)$ since the basin is simply connected. Denote by $U_{a,1}, \ldots, U_{a,m}$ these components. Then up to taking subsequences, $\overline{U_{a,n,i}}$ admits a limit $U_{i}^*$ containing $\overline{U_{a_0,i}}$. Since $\Omega_{a_n} \cap U_{a,n,i} = \emptyset$, so $\Omega^* \cap U_{i}^* \subset \partial \Omega^*$, $\Omega^* = \Omega_{a_0}$. Hence $c_2(a_0) \in \partial \Omega_{a_0}$, a contradiction. 

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