Supersymmetry and conformal Galilei algebras

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Abstract. Nonrelativistic $l$-conformal Galilei algebras and their supersymmetric extensions are considered. The construction presented in this article is different from usual based on supersymmetrization of constructed $l$-conformal Galilei algebras. We start instead from $N$-extended supersymmetric quantum mechanics, require in addition the conformal symmetry and construct the $l$-Galilei algebras.

The superconformal algebras obtained from the irreducible representations of one-dimensional SUSY are simple superalgebras given in the Kac classification. With enlarging the number $N$ of SUSY extensions different restrictions on conformal weight of fields appear.

1. Introduction
In recent years (super)conformal nonrelativistic algebras have been intensively studied. The interest is due to an appearance of this kind of symmetries in very different areas of physics and mathematics. The applications go from anisotropic critical systems [1] to conformal invariance of unitary Fermi gas [2], AdS/cold atom correspondence [3] and AdS/CFT duality in nonrelativistic systems [4].

A kinematical algebra is a Lie algebra spanned by the generators $P_i$ of spatial displacements, $P_t = H$ of time translations, $M_{ij}$ of rotations and $B_i$ of inertial transformations. Space and time reversals are automorphisms. Kinematical algebras associated with $d$ dimensional space have been classified by Bacry and Levy-Leblond in 1968 [5]. According to the classification they are divided in 2 classes, relativistic and nonrelativistic and in 3 subclasses each one. In the nonrelativistic class there are 2 types of algebras such that time translations and inertial transformations do not commute, the Newton-Hooke algebras and the Galilei algebras. Any nonrelativistic algebra admits only one nontrivial central extension.

A nonrelativistic conformal algebra contains a nonrelativistic kinematical algebra as a proper subalgebra. There is a countable class of finite-dimensional nonrelativistic conformal algebras [6] associated with the pair of numbers $(d, l)$, where $d$ is a dimension of space and $l$ is a positive integer or half-integer number. For each pair $(d, l)$ there is only one inequivalent nonrelativistic conformal algebra which is called in the literature $l$-conformal Galilei or nonrelativistic Galilei. The Newton-Hooke algebra is an example of $l = \frac{1}{2}$-conformal Galilei and the "Galilei" from Bacry and Levi-Leblond classification is $l = 1$-conformal Galilei in this sense.

The number $l$ measures the "non-relativiticity" of spacetime in the following way. $(\Delta t)^{2l}$ and $\Delta x$ transform with the same conformal factor. The inverse $z = \frac{1}{l}$ is called the dynamical exponent of the nonrelativistic $l$-conformal algebra.

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The $l$-conformal algebras for $l = \frac{1}{2}$ and $l = 1$ are largely studied together with their applications. The algebras with bigger numbers of $l$ are under active investigations in recent years, see, for example, [7] and [8].

If we consider $l$-conformal symmetries together with supersymmetry, we arrive at a class of superalgebras characterized by 3 numbers $(d, l, N)$, where $N$ is the number of supersymmetries. For some particular choices of $l$, $d$ and $N$ clear results are obtained. It is important to mention the Duval and Horvathy paper [9], where $l = \frac{1}{2}$-superconformal algebras are constructed for any $N$ and generalized for pseudo-supersymmetry governed by a pair $(N_+, N_-)$. In the paper by Fedoruk and Lukierski [10] supersymmetric $N = 4$ finite extensions are obtained for $d = 1, 2, 3, 4, 5$ and $l = \frac{3}{2}$ (this can be generalized for bigger number of $d$). For $N = 2$, any $d$ and $l$, the supersymmetric extensions and their dynamics are investigated in [12], [13] and [14].

In this article I describe the construction of D-module representations of nonrelativistic superconformal algebras from the representations of 1-dimensional $N$-extended supersymmetry. The initial dimension is related with time coordinate. The $d$ space dimensions can be added in a different way for nonrelativistic systems.

The plan of the paper is the following. In the 2nd Section a classification of 1-dimensional supersymmetric algebras is presented. The 3rd Section is devoted to superconformal models obtained from the multiplets of the $N$-extended supersymmetries in Section 2. In the 4th Section I present the extension of superconformal algebras to $l$-superconformal using the results of the previous Section, considering the cases $N = 1, 2, 4$.

2. $N$-extended supersymmetry

Let us consider representations of $N$-extended 1-dimensional supersymmetry algebra

$$\{Q_i, Q_j\} = \delta_{ij}H, \quad [H, Q_i] = 0. \quad (i, j = 1, \ldots, N) \quad (1)$$

The anti-commutational relations of (1) are similar to the relations for Clifford algebra generators $\Gamma_i$

$$\{\Gamma_i, \Gamma_j\} = 2\eta_{ij}, \quad (2)$$

where $\eta_{ij} = diag(+1, +1, \ldots, +1, -1, -1, \ldots, -1)$ is a diagonal matrix with $p$ positive and $q$ negative entries and $i, j = 1, 2, \ldots, (p + q)$. The difference is in the fact that the coefficient in the r.h.s. of (1) is only positive and that the constant 2 in (2) should be replaced by the operator $H$ in (1).

Therefore, for systematical construction of $N$-extended SUSY representations for any $N$, the matrix representations of Clifford algebras were used. These representations can be found in [15], [16]. We will use the term Gamma matrix for the generators of matrix representations of Clifford algebras.

From supersymmetry requirements (the supersymmetry generator should exchange bosons with fermions) the Gamma matrices used for the construction are of the form

$$\Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \pm\gamma_i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3)$$

where $1$ is $n \times n$ identity matrix, $0$ is $n \times n$ zero matrix and $\gamma_i$ are the Gamma matrices from $(p + q - 2)$ representation.

In the representation of (1) $H = 1_{(2n)}\partial_t$ and

$$Q_i = \begin{pmatrix} 0 & \gamma_i \\ \pm\gamma_i\partial_t & 0 \end{pmatrix}. \quad (4)$$
This representation acts on a multiplet with \( n \) bosonic and \( n \) fermionic fields. This multiplet is denoted as \((n,n)\) root multiplet. The mass dimensions of the fields in the multiplet are \( \lambda \) and \( \lambda + 1/2 \). The number of fields \( n \) for each number \( N \) of supersymmetries is presented in the table:

\[
\begin{array}{cccccc}
N = 1 & n = 1 & N = 9 & 16 & N = 17 & 256 \\
N = 2 & n = 2 & N = 10 & 32 & N = 18 & 512 \\
N = 3 & 4 & N = 11 & 64 & N = 19 & 1024 \\
N = 4 & 4 & N = 12 & 64 & N = 20 & 1024 \\
N = 5 & 8 & N = 13 & 128 & N = 21 & 2048 \\
N = 6 & 8 & N = 14 & 128 & N = 22 & 2048 \\
N = 7 & 8 & N = 15 & 128 & N = 23 & 2048 \\
N = 8 & 8 & N = 16 & 128 & N = 24 & 2048 \\
\end{array}
\]

The SUSY algebra representation (4) which acts on \((n,n)\) root multiplets can be changed via dressing transformations \( A \rightarrow A' = T^{-1}AT \). Here \( A = \{H,Q_i\} \) are any generators of (4) and \( T = \text{diag}(d/dt, \ldots, d/dt, 1, \ldots, 1) \) with some number \( m \) of derivatives \( d/dt \) in the diagonal. The new representation acts on a new multiplet with \((k,n,n-k)\) fields \((m = n-k)\) with mass dimensions \( \lambda \), \( \lambda + 1/2 \) and \( \lambda + 1 \). The string of numbers of fields with different mass dimensions is called field content of the multiplet and the number of mass dimensions is called the length of the multiplet.

For example, \((n,n)\) is a length-2 multiplet and \((k,n,n-k)\) is a length-3 multiplet.

The procedure can be continued with \( m = n, \ldots, 2n \) derivatives in \( T \) dressing operator and even with second derivatives in \( T \), but should be stopped if a singular operator in a new algebra \( \{A'\} \) appears. So all length-3 multiplets \((k,n,n-k)\), \( k = 1, 2, \ldots, n-1 \) can be constructed for any \( N \), while length-4 and length-5 are exceptional cases. The full classification and examples up to \( N = 10 \) are given in [15]. The results for \( N \leq 8 \) are given in the table below:

\[
\begin{array}{ccccccc}
| \text{length-2} | \text{length-3} | \text{length-4} | \\
| \text{N=1} | (1,1) | | | \\
| \text{N=2} | (2,2) | (1,2,1) | | \\
| \text{N=3} | (4,4) | (3,4,1), (2,4,2), (1,4,3) | | \\
| \text{N=4} | (4,4) | (3,4,1), (2,4,2), (1,4,3) | | \\
| \text{N=5} | (8,8) | (7,8,1), (6,8,2), (5,8,3), (4,8,4) | (1,5,7,3), (3,7,5,1) | \\
| | | (3,8,5), (2,8,6), (1,8,7) | (1,6,7,2), (2,7,6,1), (2,6,6,2) | \\
| | | | (1,7,7,1) | \\
| \text{N=6} | (8,8) | (7,8,1), (6,8,2), \ldots, (1,8,7) | (1,6,7,2), (2,7,6,1), (2,6,6,2) | \\
| | | | (1,7,7,1) | \\
| \text{N=7} | (8,8) | (7,8,1), (6,8,2), \ldots, (1,8,7) | | \\
| | | | (1,7,7,1) | \\
| \text{N=8} | (8,8) | (7,8,1), (6,8,2), \ldots, (1,8,7) | | \\
| | | | | \\
\end{array}
\]

Representations acting on different multiplets are non-equivalent in physical applications. We will call them, for example, as \( N = 4 \ (1,4,3) \) representation, to stress that it acts on the \((1,4,3)\) multiplet.

3. Superconformal algebras in one dimension

In this section the representations of superconformal algebras are obtained from \( N \)-SUSY representations adding conformal transformations and closing the algebras.

A superconformal algebra in one dimension contains \( 2N \) generators in the odd sector and the \( sl(2) \oplus R \) structure of generators in the even sector. \( R \) is called "\( R \)-symmetry".

The generators of \( sl(2) \) act on a component field with mass dimension \( \lambda \) as

\[
H = \frac{\partial}{\partial t}
\]
They form diagonal matrix operators acting on a multiplet of fields.

Let us remember simple superconformal algebras as abstract algebras.

- The \( N = 2 \) super-conformal algebra is \( A(1,0) = sl(2|1) \). Its R-symmetry is \( u(1) \).

- The \( N = 4 \) superconformal algebras are
  
  i) the \( A(3,1) = sl(4|2)/\mathbb{Z} \) superalgebra with 19 even generators organized as \( sl(2) \oplus sl(4) \oplus u(1) \);
  
  ii) the \( D(4,1) = osp(8|2) \) superalgebra with even sector given by \( sl(2) \oplus so(8) \) (31 even generators);
  
  iii) the \( D(2,2) = osp(4|4) \) superalgebra with 16 even generators in \( sl(2) \oplus so(3) \oplus sp(4) \);
  
  iv) the \( F(4) \) exceptional superalgebra with 24 even generators in \( sl(2) \oplus so(7) \).

We use the \( N \)-extended supersymmetry representations associated with the irreducible multiplets from (5) for constructing superconformal algebras.

The \( N = 2 \) supersymmetric multiplets are \( (2,2) = (x_1, x_2; \psi_1, \psi_2) \) and \( (1,2,1) = (x; \psi_1, \psi_2; g) \). The corresponding superalgebra \( sl(2|2) \) representations for \( (2,2) \) and \( (1,2,1) \) can be found in [17].

The \( N = 4 \) irreps are the root multiplet \( (4,4) \) and \( (1,4,3), (2,4,2), (3,4,1) \). Scaling dimensions of the component fields are \( \lambda, \lambda + \frac{1}{2}, \lambda + 1 \). Here we present the results from [17] for different multiplets.

The \( (4,4) \) root multiplet for \( \lambda = 0 \) together with the generators (6) give D-module representation of \( A(1,1) \) superalgebra.

For \( \lambda \neq 0 \) the SUSY representation corresponding to \( (4,4) \) gives \( D(2,1; \alpha) \) representation with \( \alpha \) and scaling dimension \( \lambda \) related as

\[
\alpha = -2\lambda
\]

Other results for the \( N = 4 \) superconformal representations are presented in

| \( N \) | multiplet | superalgebra | \( \lambda \) |
|------|-----------|--------------|-----------|
| \( N=2 \) | (2,2) | \( sl(2|1) \) | any \( \lambda \) |
| | (1,2,1) | \( sl(2|1) \) | any \( \lambda \) |
| \( N=4 \) | \( (4,4), (1,4,3), (2,4,2), (3,4,1) \) | \( A(1,1) \) \( D(2,1; \alpha) \) \( sl(2|2) \) | \( \lambda = 0 \) \( \lambda = -\frac{\alpha}{2} \) \( \lambda = \alpha \) \( \lambda = \frac{\alpha}{2} \) \( \lambda = 0 \neq 0 \) |
| | \( (4,4) \) | \( D(2,1; \alpha) \) | \( \lambda = -\alpha \) |
| | \( (1,4,3) \) | \( D(2,1; \alpha) \) | \( \lambda = \alpha \) |
| | \( (2,4,2) \) | \( sl(2|2) \) | any \( \lambda \neq 0 \) |
| | \( (3,4,1) \) | \( D(2,1; \alpha) \) | \( \lambda = -\alpha \) |
For $N = 8$ the irreducible multiplets are the $(8,8)$ root multiplet and $(1,8,7), (2,8,6), (3,8,5), (4,8,4), (5,8,3), (6,8,2), (7,8,1)$. For each multiplet physically nonequivalent D-module representations of supersymmetry are constructed in [15]. As in $N < 8$ cases, the $(6)_{sl(2)}$ generators have been added with the requirement to close new $N = 8$ superconformal algebras with finite number of generators. The $N = 8$ superconformal algebras are finite only for particular values of the conformal weight [18]. All irreps can be written as $(8,8,0)$ multiplets with $(8,8) \equiv (0,8,8)$ or $(8,8,0)$. The conformal weight $\lambda$ for all finite representations is restricted by

$$\lambda = \frac{1}{k-4}. \quad (8)$$

Therefore, a finite superconformal representation for the $(4,8,4)$ supermultiplet can not be obtained by this procedure.

The results are presented in the table below.

| $N$ | multiplet | superalgebra | $\lambda$ |
|-----|-----------|--------------|-----------|
| $N=8$ | $(8,8,0)$ | $osp(8|2)$ | $\lambda = 1/4$ |
|     | $(7,8,1)$ | $F(4)$       | $\lambda = 1/3$ |
|     | $(6,8,2)$ | $sl(4|2)$    | $\lambda = 1/2$ |
|     | $(5,8,3)$ | $osp(4|4)$   | $\lambda = 1$  |
|     | $(3,8,5)$ | $osp(4|4)$   | $\lambda = -1$ |
|     | $(2,8,6)$ | $sl(4|2)$    | $\lambda = -1/2$ |
|     | $(1,8,7)$ | $F(4)$       | $\lambda = -1/3$ |
|     | $(0,8,8)$ | $osp(8|2)$   | $\lambda = -1/4$ |
| $N=7$ | $(1,7,7,1)$ | $G(3)$        | $\lambda = -1/4$ |

Therefore, $N$-extended supersymmetry representations with different irreps induce superconformal simple superalgebra representations with $2N$ odd generators.

4. $l$ - superconformal Galilei

In this Section I consider superconformal nonrelativistic Galilei class of algebras associated with the triple $(d,l,N)$. The representations for $N = 2$ and $N = 4$ are presented for any $l$ and $d$.

4.1. General considerations

The $(d,l,N)$-superconformal Galilei algebra generators are:

(i) $sl(2)$ generators $H, D, K$;

(ii) $so(d)$ subalgebra of space rotations generators $M_{ij}$ (the $sl(2)$ generators commute with $so(d)$);

(iii) $N$-extended supersymmetry algebra (1) with odd generators $Q_a$ ($a = 1, \ldots, N$) and their superconformal partners $S_a$ ($a = 1, \ldots, N$);

(iv) $R$-symmetry bosonic generators $R_a$;

(v) Abelian subalgebra with generators $P_i^{(n)}$ ($i = 1, \ldots, d, n = 0, \ldots 2l$);

(vi) additional odd generators $X_{a,i}^{(n)}$ from the commutators $[Q_a, P_i^{(n)}]$;

(vii) additional even generators $J_i^{(n)}$ from $[R, P_i^{(n)}]$. 
The algebras for $N = 2$ are presented in [14]. These algebras are constructed for any $d$ and $l$.

As in superconformal case, the representations acting on different $N$-SUSY multiplets give rise to different superconformal Galilei algebras. For example, in $N = 2$ [14], different nonrelativistic Galilei algebras are constructed from the $(1,2,1)$ and $(2,2)$ superconformal representations.

The $(d,l,N)$-symmetry generators are acting on multiplets of bosonic and fermionic fields which depend on $t, x_1, \ldots, x_d$. The dependence on space coordinates are introduced by replacing

$$
\lambda \rightarrow \lambda + \frac{1}{2} \frac{\partial}{\partial x_i}
$$

in $D$ and $K$ generators of $sl(2)$.

The Abelian subalgebra generators are diagonal multiple of identity with entries $\frac{t^n}{x_i} \frac{\partial}{\partial x_i}$ for $P_i^{(n)}$. The $so(d)$ rotational symmetry is also presented also by diagonal matrices with differential operators $x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ in the entries.

4.2. $N = 2$

The $N = 2$ SUSY multiplets are $(2,2)$ and $(1,2,1)$.

The $(1,2,1)$ superconformal Galilei algebra $G_{(1,2,1)}$ coincides with the superalgebra realized on superspace by Masterov in [11].

The $(2,2)$ algebra $G_{(2,2)}$ has different (from the previous one) number of $J_i^{(n)}$ and $X_{i,a}^{(n)}$ generators.

The D-module representation of $G_{(2,2)}$ in terms of $(2 \times 2)$ matrix blocks is given by

$$
\begin{align}
H & = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}, \\
D & = \begin{pmatrix} -(t\partial_t + lx_i \partial_i + \lambda) & 0 \\ 0 & -(t\partial_t + lx_i \partial_i + \lambda + \frac{1}{2}) \end{pmatrix}, \\
K & = \begin{pmatrix} -(t^2 \partial_t + 2tx_i \partial_i + 2\lambda) & 0 \\ 0 & -(t^2 \partial_t + 2tx_i \partial_i + (2\lambda + 1)t) \end{pmatrix}, \\
R & = \begin{pmatrix} -2\sigma_A (\partial_t + lx_i \partial_i + \lambda) & 0 \\ 0 & [-2\sigma_A (\partial_t + lx_i \partial_i + \lambda) + 1] \end{pmatrix}, \\
Q_1 & = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix}, \\
Q_2 & = \begin{pmatrix} 0 & \sigma_A \\ -\sigma_A \partial_t & 0 \end{pmatrix}, \\
S_1 & = \begin{pmatrix} 0 & t \partial_t + 2lx_i \partial_i + 2\lambda \\ \partial_t & 0 \end{pmatrix}, \\
S_2 & = \begin{pmatrix} -\sigma_A (t\partial_t + 2lx_i \partial_i + 2\lambda) & 0 \\ 0 & t\sigma_A \end{pmatrix}, \\
P_i^{(n)} & = \begin{pmatrix} t^n \partial_t & 0 \\ 0 & t^n \partial_t \end{pmatrix}, \\
J_i^{(n)} & = \begin{pmatrix} \sigma_A t^n \partial_t & 0 \\ 0 & \sigma_A t^n \partial_t \end{pmatrix}, \\
X_{i,a}^{(n)} & = \begin{pmatrix} 0 & t^n \partial_t \\ t^n \partial_t & 0 \end{pmatrix}, \\
X_{i,a}^{(n)} & = \begin{pmatrix} 0 & \sigma_A t^n \partial_t \\ -\sigma_A t^n \partial_t & 0 \end{pmatrix}, \\
M_{ij} & = \begin{pmatrix} x_i \partial_j - x_j \partial_i \\ 0 \\ x_i \partial_j - x_j \partial_i \end{pmatrix},
\end{align}
$$

where the 0 block is understood as $(2 \times 2)$ zero matrix, $\sigma_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and all operators are multiplied by the $(2 \times 2)$ identity matrix. For example, $\partial_t \equiv \frac{\partial}{\partial t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also $\partial_i \equiv \frac{\partial}{\partial x_i}$ and the sum over repeating indexes is understood.

Let us present the (anti)commutation relations of $G_{(2,2)}$. 
The odd generators $X^i$ read as follows

$$[D, H] = H, \quad [D, K] = -K, \quad [H, K] = 2D,$$

$$\{Q_a, Q_b\} = 2\delta_{ab}H, \quad \{S_a, S_b\} = -2\delta_{ab}K, \quad \{Q_a, S_b\} = -2\delta_{ab}D + \sigma_A R$$

$$[D, Q_a] = \frac{1}{2}Q_a, \quad [K, Q_a] = S_a, \quad [R, Q_a] = -\sigma_A Q_b$$

$$[D, S_a] = -\frac{1}{2}S_a, \quad [H, S_a] = A_a, \quad [R, S_a] = -\sigma_A S_b$$

The even generators $P_i$ are obtained.

$$\{Q_a, P_i\} = nX^{(n-1)}_{a,i}, \quad \{S_a, P_i\} = (n - 2l)X^{(n)}_{a,i}, \quad [R, P_i] = 2lJ^{(n)}_i.$$ (12)

The odd generators $X^{(n)}_{a,i}$ (anti)commutators are

$$[H, X^{(n)}_{a,i}] = nX^{(n-1)}_{a,i}, \quad [D, X^{(n)}_{a,i}] = -(n + \frac{1}{2})X^{(n)}_{a,i},$$

$$[K, X^{(n)}_{a,i}] = -(n + 2l + 1)X^{(n+1)}_{a,i},$$

$$[R, X^{(n)}_{a,i}] = -(2l + 1)X^{(n+1)}_{2,i}, \quad [R, X^{(n)}_{2,i}] = (2l + q)X^{(n)}_{2,i}$$

$$\{Q_a, X^{(n)}_{b,j}\} = \delta_{ab}P^{(n)}_i - \epsilon_{ab}J^{(n)}_i, \quad \{S_a, X^{(n)}_{a,i}\} = \delta_{ab}P^{(n+1)}_i - \epsilon_{ab}J^{(n+1)}_i.$$ (14)

The even generators $J^{(n)}_i$ relations are

$$[H, J^{(n)}_i] = nJ^{(n-1)}_i, \quad [D, J^{(n)}_i] = -(n + l)J^{(n)}_i, \quad [K, J^{(n)}_i] = -(n + 2l)J^{(n+1)}_i$$

$$[Q_a, J^{(n)}_i] = -n\epsilon_{ab}X^{(n-1)}_{b,i}, \quad [S_a, J^{(n)}_i] = -(n - 2l)\epsilon_{ab}X^{(n)}_{a,i}, \quad [R, J^{(n)}_i] = -2lP^{(n)}_i.$$ (15)

The $so(d)$ commutators are

$$[M_{ij}, M_{kl}] = -\delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \delta_{il}M_{jk} + \delta_{jk}M_{il}$$

$$[P^{(n)}_k, M_{ij}] = \delta_{ik}P^{(n)}_j - \delta_{jk}P^{(n)}_i$$

$$[J^{(n)}_k, M_{ij}] = \delta_{ik}J^{(n)}_j - \delta_{jk}J^{(n)}_i$$

$$[X^{(n)}_{a,k}, M_{ij}] = \delta_{ik}X^{(n)}_{a,j} - \delta_{jk}X^{(n)}_{a,i}.$$ (16)

From the two $N = 2$ supersymmetric representations, two inequivalent $(d, l, 2)$-Galilei algebras are obtained.

### 4.3. $N = 4$

Here we consider only the representation associated to the $(4, 4)$ SUSY multiplet. The representations associated with the other $N = 4$ multiplets have similar properties.

For $N = 4$ supersymmetric irrep $(4, 4)$ the D-module representation of $\mathcal{G}_{(4,4)}$ is constructed similarly with $(11)$. In $\mathcal{G}_{(4,4)}$ the $(2 \times 2)$ 0-matrix blocks are substituted by $(4 \times 4)$ 0-matrix blocks, all operators are multiplied by $(4 \times 4)$ identity matrix, etc. The supersymmetry generators are

$$Q_a = \begin{pmatrix} 0 & \gamma_a \\ -\gamma_a \partial_t & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix},$$ (17)
where \( a = 1, 2, 3 \) and \( \gamma_a \gamma_b = -\delta_{ab}1 + \epsilon_{abc} \gamma_c \) are representation of \( su(2) \) by \((4 \times 4)\) real matrices. The superconformal partners are presented by

\[
S_a = tQ_a + 2(\lambda - lx_i \partial_i), \quad S_4 = tQ_4 - 2(\lambda - lx_i \partial_i) \left( \begin{array}{cc} 0 & 0 \\ \gamma_a & 0 \end{array} \right).
\]

(18)

The R-symmetry generators are

\[
R_a = \left( \begin{array}{cc} -2\gamma_a(\lambda x_i \partial_i) & 0 \\ 0 & \gamma_a(-1 + 2\lambda + 2lx_i \partial_i) \end{array} \right), \quad \bar{R}_a = \left( \begin{array}{cc} \gamma_a & 0 \\ 0 & 0 \end{array} \right).
\]

(19)

The Abelian subalgebra generators and the \( so(d) \) generators have the same form as in \( G_{(2,2)} \).

The odd generators \( X^{(n)}_{a,i} \) are of the form

\[
X^{(n)}_{a,i} = \left( \begin{array}{cc} 0 & 0 \\ -\gamma_a & 0 \end{array} \right) t^n \partial_i, \quad X_{4,i} = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) t^n \partial_i.
\]

(20)

The even generators induced by the Abelian subalgebra are of two types

\[
J^{(n)}_a = t^n \partial_L \left( \begin{array}{cc} \gamma_a & 0 \\ 0 & \gamma_a \end{array} \right), \quad W^{(n)}_a = t^n \partial_L \left( \begin{array}{cc} -\gamma_a & 0 \\ 0 & \gamma_a \end{array} \right)
\]

(21)

These generators are analogous to the \( G_{(2,2)} \) superalgebra generators. However, the \( G_{(4,4)} \) algebra does not close as a finite Lie algebra.

The non-vanishing (anti)commutators of the generators presented above close a non-linear W-algebra together with an \( L \) generator \( L = \left( \begin{array}{cc} \lambda + lx_i \partial_i & 0 \\ 0 & \lambda + lx_i \partial_i \end{array} \right) \). This W-algebra is at most quadratic in the generators. The nonvanishing relations are

\[
[D, H] = H, \quad [D, K] = -K, \quad [H, K] = 2D,
\]

\[
[D, Q_A] = \frac{1}{2} Q_A, \quad [K, Q_A] = S_A, \quad [H, S_A] = Q_A, \quad [D, S_A] = -\frac{1}{2} S_A,
\]

\[
Q_A Q_B = 2\delta_{AB}H, \quad Q_A S_B = -2\delta_{AB}K,
\]

\[
Q_4 S_4 = -2D, \quad Q_4 S_a = R_a, \quad Q_a S_4 = -R_a, \quad [Q_a, S_b] = -2\delta_{ab}D - \epsilon_{abc} (R_c + 4L \bar{R}_c)
\]

\[
[R_a, Q_A] = -Q_a, \quad [R_a, Q_b] = \delta_{ab}Q_4 + \epsilon_{abc}(1 - 4L)Q_c,
\]

\[
[R_a, S_A] = -S_a, \quad [R_a, S_b] = \delta_{ab}S_4 + \epsilon_{abc}(1 - 4L)S_c,
\]

\[
[R_a, Q_4] = Q_a, \quad [\bar{R}_a, Q_b] = -\delta_{ab}Q_4 + \epsilon_{abc}Q_c,
\]

\[
[R_a, S_4] = S_a, \quad [\bar{R}_a, S_b] = -\delta_{ab}S_4 + \epsilon_{abc}S_c,
\]

\[
[R_a, R_b] = \epsilon_{abc}[2(1 - 2L)R_c + 4L \bar{R}_c], \quad [R_a, \bar{R}_b] = -\epsilon_{abc}4L \bar{R}_c, \quad [\bar{R}_a, \bar{R}_b] = 2\epsilon_{abc} \bar{R}_c.
\]

(22)

These (anti)commutators correspond to the exceptional algebra \( D(2,1;\alpha) \) for a specific relation connecting \( \alpha \) and \( L \). Let us consider for simplicity \( d = 1 \). One should note that the generator \( L \) commutes with all the generators of \( D(2,1;\alpha) \). Therefore \( L \) entering (22) can be taken as a \( c \)-number, making this subalgebra a linear Lie superalgebra.

The remaining non-vanishing (anti)commutators are given by

\[
[H, P^{(n)}] = nP^{(n-1)}, \quad [D, P^{(n)}] = -(n - \ell)P^{(n)}, \quad [K, P^{(n)}] = -(n - 2\ell)P^{(n+1)},
\]

\[
[Q_A, P^{(n)}] = nP^{(n-1)}X_A, \quad [S_A, P^{(n)}] = (n - 2\ell)P^{(n)}X_A,
\]

\[
[R_a, P^{(n)}] = 2\ell P^{(n)}J_a,
\]

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\[ [D, X_A] = -\frac{1}{2} X_A, \quad [K, X_A] = -T X_A, \]
\[
\{Q_4, X_4\} = J, \quad \{Q_4, X_1\} = W_a, \quad \{Q_3, X_4\} = -W_a, \quad \{Q_3, X_1\} = \delta_{ab} J + \epsilon_{abc} W_c, \\
\{S_4, X_4\} = T J, \quad \{S_4, X_1\} = T W_a, \quad \{S_3, X_4\} = -T W_a, \quad \{S_3, X_1\} = \delta_a T J - \epsilon_{abc} T J_c, \\
\{R_a, X_4\} = -X_a, \quad [R_a, X_b] = \delta_{ab} X_4 + \epsilon_{abc}(1 - 4L) X_c, \\
\{R_a, X_1\} = X_a, \quad [R_a, X_b] = -\delta_{ab} X_4 + \epsilon_{abc} X_c, \\
\{Q_a, J_b\} = 2\epsilon_{abc} Q_c, \quad \{S_a, J_b\} = 2\epsilon_{abc} S_c, \quad \{R_a, J_b\} = 2\epsilon_{abc} R_c, \quad \{R_a, J_b\} = 2\epsilon_{abc} R_c, \\
\{X_a, J_b\} = 2\epsilon_{abc} X_c, \quad [J_a, J_b] = 2\epsilon_{abc} J_c, \\
\{Q_A, J\} = 2Q_A - 4Y_A, \quad \{S_A, J\} = 2S_A - 4TY_A, \quad \{X_A, J\} = 2X_A \tag{23} \]

and
\[ [D, Y_A] = \frac{1}{2} Y_A, \quad [K, Y_A] = TY_A, \]
\[
\{Q_4, Y_4\} = H, \quad \{Q_4, Y_1\} = -H J_a, \quad \{S_4, Y_1\} = -D + L + \frac{1}{4}(J - C), \\
\{S_a, Y_4\} = [D - L - \frac{1}{4}(J - C)] J_a, \\
\{R_a, Y_4\} = -Y_a, \quad [R_a, Y_4] = Y_a, \\
\{X_4, Y_4\} = C, \quad \{Q_4, Y_a\} = H J_a, \quad \{Q_4, Y_b\} = \delta_{ab} H - \epsilon_{abc} W_c H, \\
\{S_0, Y_a\} = [-D + L + \frac{1}{4}(J - C)] J_a, \\
\{S_a, Y_b\} = [-D + L + \frac{1}{4}(J - C)](\delta_{ab} - \epsilon_{abc} J_c), \\
\{R_a, Y_b\} = \delta_{ab} Y_0 - \epsilon_{abc} Y_c, \quad \{R_a, Y_b\} = -\delta_{ab} Y_0 + \epsilon_{abc} Y_c, \\
\{X_4, Y_a\} = J_a, \quad \{X_a, Y_b\} = \delta_{ab} C - \epsilon_{abc} J_c, \quad [J_a, Y_b] = 2\epsilon_{abc} Y_c, \\
\{H, T\} = C, \quad [D, T] = -T, \quad [K, T] = -T^2, \\
\{Q_A, T\} = X_A, \quad [S_A, T] = T X_A, \quad [P^{(n)}, L] = \ell P^{(n)}, \tag{24} \]

where \( W_a = J_a - 2\overline{R_a}, \) \( A = 1, 2, 3, 4, \) \( Y_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \) \( Y_a = \begin{pmatrix} 0 & \gamma_a \\ 0 & 0 \end{pmatrix}, \) \( C = 1_8, \) \( T = t \cdot 1_8. \) All other (anti)commutators are vanishing.

It seems that, for generic values of \( d \) and \( l, \) there exists an obstruction in obtaining finitely generated \( N \)-extended superconformal Galilei algebras of Lie type for \( N > 2. \) The possibility to get finitely generated Lie superalgebras for special values of \( d \) and \( l, \) beyond the cases of [9] and [10], is currently investigated.

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