Geometry of Random Cayley Graphs of Abelian Groups

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Abstract

Consider the random Cayley graph of a finite Abelian group $G$ with respect to $k$ generators chosen uniformly at random, with $1 \ll \log k \ll \log |G|$. Draw a vertex $U \sim \text{Unif}(G)$.

We show that the graph distance $\text{dist}(\text{id}, U)$ from the identity to $U$ concentrates at a particular value $M$, which is the minimal radius of a ball in $\mathbb{Z}^k$ of cardinality at least $|G|$, under mild conditions. In other words, the distance from the identity for all but $o(|G|)$ of the elements of $G$ lies in the interval $[M - o(M), M + o(M)]$. In the regime $k \gtrsim \log |G|$, we show that the diameter of the graph is also asymptotically $M$. In the spirit of a conjecture of Aldous and Diaconis [1], this $M$ depends only on $k$ and $|G|$, not on the algebraic structure of $G$.

Write $d(G)$ for the minimal size of a generating subset of $G$. We prove that the order of the spectral gap is $|G|^{-1/k}$ when $k - d(G) \asymp k$ and $|G|$ lies in a density-1 subset of $\mathbb{N}$ or when $k - 2d(G) \asymp k$. This extends, for Abelian groups, a celebrated result of Alon and Roichman [4].

The aforementioned results all hold with high probability over the random Cayley graph.

Keywords: typical distance, diameter, spectral gap, relaxation time, random Cayley graphs

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1 Introduction and Statement of Results

1.1 Brief Overview of Results and Notation

1.1.1 Brief Overview of Results

We analyse geometric properties of a Cayley graph of a finite group; the focus is on Abelian groups. The generators of this graph are chosen independently and uniformly at random. Precise definitions are given in §1.4.1; for now, let $G$ be a finite group, let $k$ be an integer (allowed to depend on $G$) and denote by $G_k$ the Cayley graph of $G$ with respect to $k$ independently and uniformly random generators. We consider values of $k$ with $1 \ll \log k \ll |G|$ for which $G_k$ is connected with high probability (abbreviated \whp), ie with probability tending to 1 as $|G|$ grows. For an Abelian group $G$, write $d(G)$ for the minimal size of a generating subset of $G$.

- Typical Distance. Draw $U \sim \text{Unif}(G)$. We show that the law of the graph distance between the identity and $U$ concentrates. The leading order term in this typical distance depends only on $k$ and $|G|$ when $1 \ll k \ll |G|/\log \log |G|$ and $k - d(G) \asymp k$ or $k \gg \log |G|$.

- Diameter. For $k \approx \log |G|$ under mild conditions on the group and $k \gg \log |G|$ for any Abelian group, we show that the diameter concentrates at the same value as the typical distance.

- Spectral Gap. For any $1 \ll k \lesssim \log |G|$ with $k - d(G) \approx k$, we determine the order of the spectral gap of the random walk on the random Cayley graph.

Introduced by Aldous and Diaconis [1], there has been a great deal of research into these random Cayley graphs. Motivation for this model and an overview of historical work is given in §1.3.

1.1.2 Notation and Terminology

Cayley graphs are either directed or undirected; we emphasise this by writing $G^+_k$ and $G^-_k$, respectively. When we write $G_k$ or $G_{\pm k}$, this means “either $G^+_k$ or $G^-_k$”, corresponding to the undirected, respectively directed, graphs with generators chosen independently and uniformly at random.

Conditional on being simple, $G^+_k$ is uniformly distributed over the set of all simple degree-$k$ Cayley graphs. Up to a slightly adjusted definition of simple for undirected Cayley graphs, our results hold with $G_k$ replaced by a uniformly chosen simple Cayley graph of degree $k$; see §1.4.2.

Our results are for sequences $(G_N)_{N \in \mathbb{N}}$ of finite groups with $|G_N| \to \infty$ as $N \to \infty$. For ease of presentation, we write statements like “let $G$ be a group” instead of “let $(G_N)_{N \in \mathbb{N}}$ be a sequence of groups”. Likewise, the quantities $d(G)$ and, of course, $k$ appearing in the statements all correspond to sequences, which need not be fixed (or bounded) unless we explicitly say otherwise. In the same vein, an event holds with high probability (abbreviated \whp) if its probability tends to 1.

We use standard asymptotic notation: “$\ll$” or “$\omega(\cdot)$” means “of smaller order”; “$\lesssim$” or $O(\cdot)$” means “of order at most”; “$\asymp$” means “of the same order”; “$\sim$” means “asymptotically equivalent”.

1.2 Statements of Main Results

1.2.1 Typical Distance

Our first result concerns typical distance in the random Cayley graph.

**Definition A.** For a group $G$, $k \in \mathbb{N}$ and $\beta \in (0, 1)$, define the $\beta$-typical distance $D_{G_k}(\beta)$ via

$$B_{G_k}(R) := \{x \in G \mid \text{dist}_{G_k}(\text{id}, x) \leq R\} \quad \text{and} \quad D_{G_k}(\beta) := \min\{R \geq 0 \mid |B_{G_k}(R)| \geq \beta |G|\}.$$

Informally, we show that the mass (in terms of number of vertices) concentrates at a thin ‘slice’, or ‘shell’, consisting of vertices at a distance $M \pm o(M)$ from the origin, with $M$ explicit.

Investigating this typical distance for $G_k$ when $k$ diverges with $|G|$ was suggested to us by Benjamini [5]. Previous work concentrated on fixed $k$, ie independent of $|G|$; see §1.3.

For an Abelian group $G$, write $d(G)$ for the minimal size of a generating subset of $G$ and

$$m_\ast(G) := \max\{\min_{j \in [d]} m_j \mid \ominus^d_{\mathbb{Z}} Z_{m_j} \text{ is a decomposition of } G\}.$$
More refined statements are given in Theorems 2.2, 3.2 and 4.2.

**Theorem A.** Let $G$ be an Abelian group. The following convergences are in probability as $|G| \to \infty$.

Consider $1 \ll k \ll \log |G|$; suppose that $k - d(G) \approx k$ and $d(G) \ll \log |G| / \log \log |G|$. Write $\mathcal{D}^+ := |G|^{1/k} / (2e)$ and $\mathcal{D}^- := |G|^{1/k} / e$. For all $\beta \in (0, 1)$, we have $\mathcal{D}^+_G(\beta) / \mathcal{D}^- \to^p 1$.

Consider $k \approx \lambda \log |G|$ with $\lambda \in (0, \infty)$; suppose that $d(G) \leq \frac{1}{4} \log |G| / \log \log |G|$ and $m_+(G) \gg 1$. There exists a constant $\alpha^+_\lambda \in (0, \infty)$ so that, for all $\beta \in (0, 1)$, we have $\mathcal{D}^+_G(\beta) / (\alpha^+\lambda k) \to^p 1$.

Consider $k \gg \log |G|$ with $\log k \ll \log |G|$; write $\rho := \log k / \log |G|$ so that $k = (\log |G|)^\rho$. (We allow $\rho \gg 1$.) For all $\beta \in (0, 1)$, we have $\mathcal{D}^+_G(\beta) / (\rho k \log |G|) \to^p 1$.

In all three cases, the implicit lower bound holds deterministically and for all Abelian groups.

**Remark A.1.** We establish the concentration of typical distance via three distinct approaches, in §2, §3 and §4. Conceptually, all involve sizes of lattice balls and drawing elements uniformly from balls. A precise statement for each approach is given, as is an outline of the proof. In summary, Theorem A is a direct consequence of Theorems 2.2, 3.2 and 4.2; see also Hypotheses A to C. △

**Remark A.2.** For smaller $k$, namely $1 \ll k \ll \sqrt{\log |G| / \log \log |G|}$, we can relax $k - d(G) \approx k$ to $k - d(G) \gg 1$. In order to generate the group, we certainly need $k \geq d(G)$, by definition. In many cases $k - d(G) \gg 1$ is necessary in order to generate the group whp, so this assumption can not be removed. For a characterisation of these cases and related discussion, see [13, Lemma 8.1]. △

**Remark A.3.** Interesting is how we prove this theorem. It is common in mixing time proofs to use geometric properties of the graph, such as expansion or distance properties. We do the opposite: we use mixing techniques to prove this geometric result. This is in the same spirit as [22]; see §1.3. △

### 1.2.2 Diameter

We can extend our proof to consider the **diameter**, i.e. the maximal distances between pairs of vertices in the graph, in the regime $k \gtrsim \log |G|$. For a graph $H$, denote by $\text{diam} H$ its diameter.

Our first diameter result gives concentration. A more refined statement is given in Theorem 5.1.

**Theorem B.** Let $G$ be an Abelian group. The following convergences are in probability as $|G| \to \infty$.

Consider $k \approx \lambda \log |G|$ with $\lambda \in (0, \infty)$; suppose that $d(G) \leq \frac{1}{4} \log |G| / \log \log |G|$ and $m_+(G) \gg 1$. Let $\alpha^+ \in (0, \infty)$ be the constant from Theorem A. We have $\text{diam} G^+_k / (\alpha^+ k) \to^p 1$.

Consider $k \gg \log |G|$ with $\log k \ll \log |G|$; write $\rho := \log k / \log |G|$ so that $k = (\log |G|)^\rho$. (We allow $\rho \gg 1$.) We have $\text{diam} G^+_k / (\rho k \log |G|) \to^p 1$. The upper bound holds for all groups.

In both cases, the implicit lower bound holds deterministically and for all Abelian groups.

**Remark B.** For any Cayley graph $H$ one has $\mathcal{D}^+_H(\frac{1}{2}) \leq \text{diam} H \leq 2 \mathcal{D}^+_H(\frac{1}{2}) + 1$. (Note that $(x_1, \ldots, x_\ell)$ is a path in $G(z)$ if and only if $(x_\ell, \ldots, x_1)$ is a path in $G(z^{-1})$ for any generators $z$.) So the typical distance and diameter are always equivalent up to constants. Theorem B gives conditions under which they are asymptotically equivalent whp for random Cayley graphs.

Combining Theorem A with [11, Theorem A] shows that $t_{\text{mix}}(G_k) \approx (\text{diam} G_k)^2 / k$ whp when $k - d(G) \approx k \gtrsim \log |G|$. One can also consider non-Abelian groups; see [12, Theorem E]. △

Our next diameter result shows, in a well-defined sense, that, amongst all groups, when $k - \log_2 |G| \approx k$ with $\log k \ll \log |G|$, the group $\mathbb{Z}_2^d$ gives rise to the largest typical diameter.

**Definition.** For two random sequences $\alpha := (\alpha_N)_{N \in \mathbb{N}}$ and $\beta := (\beta_N)_{N \in \mathbb{N}}$ of reals, we say that $\alpha \preceq \beta$ whp up to smaller order terms if there exist non-random sequences $(\gamma_N)_{N \in \mathbb{N}}$ and $(\delta_N)_{N \in \mathbb{N}}$ of reals with $\delta_N \to 0$ as $N \to \infty$ such that $(\alpha_N \leq (1 + \delta_N)\gamma_N)_{N \in \mathbb{N}}$ and $((1 - \delta_N)\gamma_N \leq \beta_N)_{N \in \mathbb{N}}$ both hold whp. We say that $\alpha \approx \beta$ whp if $\alpha \preceq \beta$ and $\beta \preceq \alpha$ whp up to smaller order terms.

We now define the candidate radius which we show is an upper bound for $\text{diam} G_k$ whp.
Definition C. Write $\mathcal{R}(k, n)$ for the minimal $R \in \mathbb{N}$ with $\binom{k}{n} \geq n$.

We now state our second diameter result. A more refined statement is given in Theorem 5.2.

Theorem C. Let $G$ be an arbitrary group. Suppose that $k - \log_2 |G| \geq k$ and $1 \ll \log k \ll \log |G|$. Then $\text{diam} \ G_k \leq \mathcal{R}(k, |G|)$ up to smaller order terms whp; further, if $H := \mathbb{Z}_2^k$, then $\text{diam} \ H_k \approx \mathcal{R}(k, 2^k) = \mathcal{R}(k, |H|)$ whp. (The limit is as $|G| \to \infty$.)

This gives a quantitative sense in which $\mathbb{Z}_2^k$ is the group giving rise to the largest diameter.

Corollary C. For all diverging $d$ and $n$ with $n \leq 2^d$, and all groups $G$ of size $n$, if $k - \log_2 n \approx k$ and $\log k \ll \log n$, then $\text{diam} \ G_k \leq \text{diam} \ H_k$ where $H := \mathbb{Z}_2^d$ up to smaller order terms whp.

Wilson [31, Conjecture 7] conjectures an analogous statement for mixing times. When restricted to nilpotent groups, we prove an extension of this conjecture in [11, Theorems C and D].

1.2.3 Spectral Gap

Our next result concerns the spectral gap and relaxation time of the random Cayley graph.

Definition D. Consider a reversible Markov chain with (real) eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$ of its transition matrix. The usual, respectively absolute, spectral gap is defined as

$$
\gamma := \min_{i \neq 1} \{1 - \lambda_i\} = 1 - \lambda_2, \quad \text{respectively} \quad \gamma_* := \min_{i \neq 1} \{1 - |\lambda_i|\} = 1 - \max\{|\lambda_2|, |\lambda_n|\};
$$

the usual, respectively absolute, relaxation time is defined as $t_{\text{rel}} := 1/\gamma$, respectively, $t_{\text{rel}}^* := 1/\gamma_*$. The spectral gap or relaxation time of a graph, is that of the simple random walk on the graph.

It is classical that under reversibility in continuous-time the spectral gap asymptotically determines the exponential rate of convergence to equilibrium, whereas in discrete-time it is determined by the absolute spectral gap; see [19, §12 and §20]. For a multiset $z = [z_1, \ldots, z_k]$ with $z_1, \ldots, z_k \in G$, write $G^+(z)$, respectively $G^-(z)$, for the undirected, respectively directed, Cayley graph with respect to generators $z_1, \ldots, z_k$. We do not require $k \to \infty$ as $|G| \to \infty$.

A more refined statement than the one given below is given in Theorem 6.1.

Theorem D. There exists a positive constant $c$ so that, for all Abelian groups $G$, all $k$ and all multisets of generators $z$ of size $k$, we have

$$
t_{\text{rel}}^*(G^-(z)) \geq t_{\text{rel}}(G^-(z)) \geq c|G|^{2/k}.
$$

For all $\delta > 0$, there exists a constant $C_\delta > 0$ so that, for all Abelian groups $G$, if $k \geq (2+\delta)d(G)$, then

$$
P\left(t_{\text{rel}}(G_k) \leq C_\delta |G|^{2/k}\right) \geq 1 - C_\delta 2^{-k/C_\delta}.
$$

Further, for all $\varepsilon \in (0, 1)$, there exists a density-$(1 - \varepsilon)$ subset $A \subseteq \mathbb{N}$ so that if $|G| \in A$ then the condition $k \geq (2 + \delta)d(G)$ can be relaxed to $k \geq (1 + \delta)d(G)$; the constants now also depend on $\varepsilon$.

The method of proof for this result is rather different to our previous results and also somewhat different to those used by others to study the spectral gap of random Cayley graphs; see §1.3.4.

1.3 Historic Overview

In this subsection, we give a fairly comprehensive account of previous work on distance metrics on and spectral gap of random Cayley graphs; we compare our results with existing ones. We also mention, where relevant, other results which we have proved in companion papers; see also §1.4.3.
1.3.1 Motivation: Random Cayley Graphs and Cutoff for Random Walks

In their seminal paper, Aldous and Diaconis [1, 2] considered random walks on random Cayley graphs. Diaconis [9] gave the following (paraphrased) motivation.

Erdős, when considering classes of mathematical objects, often combinatorial or graph theoretic, would often ask, “What does a typical object in this class ‘look like’?” If an object is chosen uniformly at random, are there natural properties which hold whp?

It is then natural to ask, “How does a typical random walk on a group behave?”

This lead Aldous and Diaconis [1, 2] to consider the set of all Cayley graphs of a given group G with k generators. Drawing such a Cayley graph uniformly at random corresponds to choosing generators $Z_1, ..., Z_k \sim \text{iid Unif}(G)$, conditional on giving rise to a simple graph; see §1.4.2. We study random walks in [11, 12], establishing cutoff and showing universal mixing bounds in different set-ups.

1.3.2 Universality: The Aldous–Diaconis Conjecture

Aldous and Diaconis [1, 2] made the following (informal) conjecture: regardless of the particular group G, provided $k \gg \log|G|$, the random walk on the random Cayley graph exhibits cutoff whp at a time which depends only on k and |G|. This was established for Abelian groups by Dou and Hildebrand [10, 16]; in [12], we provide a counter-example using unit upper triangular matrix groups. For more details, see our companion articles [11, 12] where we study cutoff extensively.

The point of the Aldous–Diaconis conjecture is that certain statistics should be “independent of the algebraic structure of the group”, i.e. only depend on G through |G|. The current article shows how very related statements to those above hold when “cutoff” is replaced by “typical distance”. Namely, we give conditions under which the typical distances concentrates on a value that depends only on k and |G|; see Theorems 2.2, 3.2 and 4.2.

1.3.3 Typical Distance and Diameter

Previous work on distance metrics (detailed below) had concentrated on the case where the number of generators k is a fixed number. The results establish (non-degenerate) limiting laws. This restricts the (sequences of) groups which can be studied; e.g., in order for it to be even possible to generate the group—never mind having independent, uniform generators do so whp—one needs $d(G) \leq k \asymp 1$. We discuss generation of groups further in [13, §8]; see in particular [13, §8.2] where we describe adaptations made in order to obtain connected graphs in the references given below.

Our results are in a different direction: for us, $k \to \infty$ as $|G| \to \infty$ and we establish concentration of the observables. This allows us to consider a much wider range of groups, in particular with $d(G)$ diverging with |G|. This line of enquiry was suggested to us by Benjamini [6].

Amir and Gurel-Gurevich [5] studied the diameter of the random Cayley graph of cyclic groups of prime order. They prove (for fixed k) that the diameter is order $|G|^{1/k}$; see [5, Theorems 1 and 2]. They conjecture that the diameter divided by $|G|^{1/k}$ converges in distribution to some non-degenerate distribution as $|G| \to \infty$; see [5, Conjecture 3].

Marklof and Strömbergsson [24] consider, as a consequence of a quite general framework, the diameter of the random Cayley graph of $\mathbb{Z}_n$ with respect to a fixed number k of random generators, for a random n, without any primality assumption. They derive distributional limits for the diameter, the average distance (defined with respect to various $L_p$ metrics) and the girth. They determine limit distributions for each of these, and in some cases derive explicit formulas.

Shapira and Zuck [30] build on the framework of Marklof and Strömbergsson [24], again for fixed k; they are able to consider non-random n, as well as Abelian groups of arbitrary (fixed) rank, instead of only cyclic groups. In particular, they verify the conjecture of Amir and Gurel-Gurevich [5, Conjecture 3]; they additionally work with average distance and girth.

Lubetzky and Peres [22] derive an analogous typical distance result for n-vertex, d-regular Ramanujan graphs: whp all but $o(n)$ of the vertices lie at a distance $\log_{d-1} n \pm \mathcal{O}(\log \log n)$; they establish this by proving cutoff for the non-backtracking random walk at time $\log_{d-1} n$.

Related work on the diameter of random Cayley graphs, including concentration of certain measures, can be found in [20, 29].
The Aldous–Diaconis conjecture for mixing can be transferred naturally to typical distance: the mass should concentrate at a distance $M$, where $M$ can be written as a function only of $k$ and $|G|$; i.e. there is concentration of mass at a distance independent of the algebraic structure of the group.

In [12, Theorem E] we consider typical distance analogously to this paper; there the underlying group is a non-Abelian matrix group. In contrast with the Abelian groups in Theorem A, the $M$ for these non-Abelian groups cannot be written as a function only of $k$ and $|G|$.

1.3.4 Spectral Gap

Hough [17, Theorem 1.1] showed that, for any prime $p$, the relaxation time of the random walk on any Cayley graph of $\mathbb{Z}_p$ with respect to an arbitrary set of $k$ generators is order at least $|\mathbb{Z}_p|^{2/k} = p^{2/k}$; provided that $k \leq \log p / \log \log p$. Using a different approach, we extend Hough’s result, removing the restrictions on $p$ and $k$ and considering general Abelian groups; see Theorem D.

This extends, in the Abelian set-up, a celebrated result of Alon and Roichman [4, Corollary 1], which asserts that, for any finite group $G$, the random Cayley graph with at least $C \log |G|$ random generators is whp an $\varepsilon$-expander, provided $C \varepsilon$ is a sufficiently large (in terms of $\varepsilon$). (A graph is an $\varepsilon$-expander if its isoperimetric constant is bounded below by $\varepsilon$; up to a reparametrisation, this is equivalent to the spectral gap of the graph being bounded below by $\varepsilon$.) There has been a considerable line of work building upon this general result of Alon and Roichman. (Pak [26] proves a similar result.) Their proof was simplified and extended, independently, by Loh and Schulman [21] and Landau and Russell [18]; both were able to replace $\log_2 |G|$ by $\log_2 D(G)$, where $D(G)$ is the sum of the dimensions of the irreducible representations of the group $G$; for Abelian groups $D(G) = |G|$. A ‘derandomised’ argument for Alon–Roichman is given by Chen, Moore and Russell [7]. Both [7, 18] use some Chernoff-type bounds on operator valued random variables.

Christofides and Markström [8] improve these further by using matrix martingales and proving a Hoeffding-type bound on operator valued random variables. They also improved the quantification for $C_\varepsilon$, showing that one may take $C_\varepsilon := 1 + c_\varepsilon$ with $c_\varepsilon \to 0$ as $\varepsilon \to 0$; this means that, whp, the graph is an $\varepsilon$-expander whenever $k \geq (1 + c_\varepsilon) \log D(G)$ and $c_\varepsilon \to 0$ as $\varepsilon \to 0$. They also generalise Alon–Roichman to random coset graphs. The proofs use tail bounds on the (random) eigenvalues.

Alon and Roichman [4, Theorem 2] also specifically consider Abelian groups. There they do a calculation directly in terms of the eigenvalues, rather than using a probabilistic tail bound.

In [11, Theorem E], we analyse for a nilpotent group $G$ the spectral gap of $G_k$ in the regime $k \asymp \log |G|$; we show that $G_k$ is an expander whp under a certain natural condition on $k$. In the special case of Abelian groups, this becomes $k - d(G) \asymp k$; the general condition is $k - d(G) \asymp k$ where $G$ is the direct product of the quotients in the lower central series of $G$. Hence in this set-up it extends Theorem D by removing the restriction that $|G|$ lies in a large-density subset of $\mathbb{N}$.

There are some fairly standard ways in which one can get bounds on the (usual) spectral gap of a Markov chain. The first is to look at the mixing time. For $\epsilon > 0$ and $\epsilon \in (0, \pi_{\min}^\epsilon)$, we have

$$t_{\text{mix}}(\varepsilon) \asymp t_{\text{rel}} \log(1/\varepsilon),$$

where $n$ is the size of the state space of the (reversible) Markov chain, $\pi_{\min}^\epsilon$ is the minimal value of the invariant distribution of the Markov chain and $c$ is a constant; see, eg, [19, Theorem 20.6 and Lemma 20.11]. Thus, if one can bound the mixing time at level $\pi_{\min}^\epsilon$ then one can bound the relaxation time. This method is used by Alon and Roichman [4] and Pak [26]; we use it in [11].

Another method is to obtain a tail estimate on the value of a random eigenvalue; one can then use the union bound to say that all (non-unitary) eigenvalues are at most some fixed value, which in turn lower bounds the spectral gap (ie upper bounds the relaxation time).

All these references consider the regime $k \asymp \log |G|$; our results also apply when $1 \ll k \ll \log |G|$. From a technical perspective, in order to obtain failure probability via a large deviation bound for a random eigenvector of $O(1/|G|)$, one needs $k \gtrsim \log |G|$. The purpose of this is to carry out a union bound over the $|G|$ eigenvalues; see, eg, [8]. Likewise, arguments that bound the $1/|G|^c$ mixing time, for some constant $c$, in terms of some generator getting picked once (cf [28]) cannot work unless $k \gtrsim \log |G|$. As such, to consider $1 \ll k \ll \log |G|$, a different approach is needed. We still use a union bound, but instead of asking for an error probability $O(1/|G|)$ for each eigenvalue, we group together eigenvalues according to a certain gcd and bound the error for each group.
1.4 Additional Remarks

1.4.1 Precise Definition of Cayley Graphs

Consider a finite group $G$. Let $Z$ be a multisubset of $G$. We consider geometric properties, namely through distance metrics and the spectral gap, of the Cayley graph of $(G, Z)$; we call $Z$ the generators. The undirected, respectively directed, Cayley graph of $G$ generated by $Z$, denoted $G^{-}(Z)$, respectively $G^{+}(Z)$, is the multigraph whose vertex set is $G$ and whose edge multiset is

$$\left\{ (g, g \cdot z) \mid g \in G, z \in Z \right\}, \text{ respectively } \left\{ (g, g \cdot z) \mid g \in G, z \in Z \right\}.$$

We focus attention on the random Cayley graph defined by choosing $Z_1, \ldots, Z_k \sim \text{iid Unif}(G)$; when this is the case, denote $G^+_k := G^+(Z)$ and $G^-_k := G^-(Z)$. While we do not assume that the Cayley graph is connected (i.e., $Z$ may not generate $G$), in the Abelian set-up the random Cayley graph $G_k$ is connected whp whenever $k - d(G) \gg 1$; see [13, Lemma 8.1].

The graph depends on the choice of $Z$. Sometimes it is convenient to emphasise this; we use a subscript, writing $P_{G(Z)}(\cdot)$ if the graph is generated by the group $G$ and multiset $z$. Mimosaically, $P_{G_k}(\cdot)$ stands for the random law $P_{G(Z)}(\cdot)$ where $Z = [Z_1, \ldots, Z_k]$ with $Z_1, \ldots, Z_k \sim \text{iid Unif}(G)$.

1.4.2 Typical and Simple Cayley Graphs

The directed Cayley graph $G^+(z)$ is simple if and only if no generator is picked twice, i.e., $z_i \neq z_j$ for all $i \neq j$. The undirected Cayley graph $G^-(z)$ is simple if in addition no generator is the inverse of any other, i.e., $z_i \neq z_i^{-1}$ for all $i, j \in [k]$. In particular, this means that no generator is of order 2, as any $s \in G$ of order 2 satisfies $s = s^{-1}$—this gives a multiedge between $g$ and $gs$ for each $g \in G$. Abusing terminology, we relax the definition of simple Cayley graphs to allow order 2 generators, i.e., remove the condition $z_i \neq z_i^{-1}$ for all $i$.

Given a group $G$ and an integer $k$, we are drawing the generators $Z_1, \ldots, Z_k$ independently and uniformly at random. It is not difficult to see that the probability of drawing a given multiset depends only on the number of repetitions in that multiset. Thus, conditional on being simple, $G_k$ is uniformly distributed on all simple degree-$k$ Cayley graphs. Since $k \ll \sqrt{|G|}$, the probability of simplicity tends to 1 as $|G| \to \infty$. So when we say that our results hold “whp (over $Z$)”, we could equivalently say that the result holds for almost all degree-$k$ simple Cayley graphs of $G$.

Our asymptotic evaluation does not depend on the particular choice of $Z$, so the statistics in question depend very weakly on the particular choice of generators for almost all choices. In many cases, the statistics depend only on $G$ via $|G|$ and $d(G)$. This is a strong sense of ‘universality’.

1.4.3 Overview of Random Cayley Graphs Project

This paper is one part of an extensive project on random Cayley graphs. There are three main articles [11, 12, 14] (including the current one [14]), a technical report [13] and a supplementary document [15]. Each main article is readable independently.

The main objective of the project is to establish cutoff for the random walk and determining whether this can be written in a way that, up to subleading order terms, depends only on $k$ and $|G|$; we also study universal mixing bounds, valid for all, or large classes of, groups. Separately, we study the distance of a uniformly chosen element from the identity, i.e., typical distance, and the diameter; the main objective is to show that these distances concentrate and to determine whether the value at which these distances concentrate depends only on $k$ and $|G|$.

[11] Cutoff phenomenon (and Aldous–Diaconis conjecture) for general Abelian groups; also, for nilpotent groups, expander graphs and comparison of mixing times with Abelian groups.
[14] Typical distance, diameter and spectral gap for general Abelian groups.
[12] Cutoff phenomenon and typical distance for upper triangular matrix groups.
[13] Additional results on cutoff and typical distance for general Abelian groups.
[15] Deferred technical results mainly regarding random walk on $Z$ and the volume of lattice balls.
1.4.4 Acknowledgements

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2 Typical Distance: \(1 \ll k \ll \log |G|\)

This section focusses on concentration of distances from the identity in the random Cayley graph of an Abelian group when \(1 \ll k \ll \log |G|\). (Subsequent sections deal with \(k \gtrsim \log |G|\).) The main result of the section is Theorem 2.2.

The outline of this section is as follows:

- §2.1 states precisely the main theorem of the section;
- §2.2 outlines the argument;
- §2.3 gives some crucial estimates on the size of lattice balls;
- §2.4 is devoted to the lower bound;
- §2.5 is devoted to the upper bound.

2.1 Precise Statement and Remarks

To start the section, we recall the typical distance statistic.

**Definition 2.1.** Let \(H\) be a graph and fix a vertex \(0 \in H\). For \(r \in \mathbb{N}\), write \(B_H(r)\) for the \(r\)-ball in the graph \(H\), i.e \(B_H(r) := \{h \in H \mid d_H(0, h) \leq r\}\), where \(d_H\) is the graph distance in \(H\). Define

\[
\mathcal{D}_H(\beta) := \min \{r \geq 0 \mid |B_H(r)| \geq \beta |H|\} \quad \text{for } \beta \in (0, 1).
\]

When considering sequences \((k_N, G_N)_{N \in \mathbb{N}}\) of integers and Abelian groups, abbreviate

\[
\mathcal{D}_N(\beta) := \mathcal{D}_{G_N((Z_1, \ldots, Z_{k_N}))}(\beta) \quad \text{where } Z_1, \ldots, Z_{k_N} \sim_{\text{iid}} \text{Unif}(G_N).
\]

Finally, considering such sequences, we define the candidate radius for the typical distance:

\[
\mathcal{D}_N^+: = k_N |G_N|^{1/k_N} / (2e) \quad \text{and } \quad \mathcal{D}_N^- := k_N |G_N|^{1/k_N} / e \quad \text{for each } N \in \mathbb{N}.
\]

As always, if we write \(\mathcal{D}_N\), then this is either \(\mathcal{D}_N^+\) or \(\mathcal{D}_N^-\) according to context.

We show that, whp over the graph (ie choice of \(Z\)), this statistic concentrates. The result will be valid for all Abelian groups, under some conditions on \(k\) in terms of \(G\). Further, the value at which the typical distance concentrates, which will be \(\mathcal{D}_N^+\) above, depends only on \(k\) and \(|G|\). This is in agreement with the spirit of the Aldous–Diaconis conjecture.

**Hypothesis A.** The sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis A if the following hold:

\[
\liminf_{N \to \infty} |G_N| = \infty, \quad \limsup_{N \to \infty} k_N / \log |G_N| = 0, \quad \liminf_{N \to \infty} (k_N - d(G_N)) = \infty
\]

and

\[
\frac{k_N - d(G_N) - 1}{k_N} \geq 5 \frac{k_N}{\log |G_N|} + 2 \frac{d(G_N) \log \log k_N}{\log |G_N|} \quad \text{for all } N \in \mathbb{N}.
\]

We study \(1 \ll k \ll \log |G|\) here. In Remark 2.3 below, we give some sufficient conditions for Hypothesis A to hold. Throughout the proofs, we drop the subscript-\(N\) from the notation, eg writing \(k\) or \(n = |G|\), considering sequences implicitly. Write \(D_k(\beta)\) for the \(\beta\)-typical distance of \(G_k\).

We now state the main theorem of this section.
Theorem 2.2. Let \((k_N)_{N \in \mathbb{N}}\) be a sequence of positive integers and \((G_N)_{N \in \mathbb{N}}\) a sequence of finite, Abelian groups; for each \(N \in \mathbb{N}\), define \(Z(1) := [Z_1, ..., Z_{k_N}]\) by drawing \(Z_1, ..., Z_{k_N} \sim \text{Unif}(G_N)\). Suppose that \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis A. Then, for all \(\beta \in (0, 1)\), we have
\[
D_N^\beta(\beta)/D_N^\beta \to^p 1 \quad \text{(in probability)} \quad \text{as } N \to \infty.
\]
Moreover, the implicit lower bound holds deterministically, i.e., for all choices of generators, and for all Abelian groups, i.e., Hypothesis A need not be satisfied—we just need lim sup\(_N k_N/\log|G_N| = 0\).

Remark 2.3. Write \(n := |G|\). Any of the following conditions imply Hypothesis A:
\[
\begin{align*}
1 < k & \leq \sqrt{\log n/\log \log \log n} \quad \text{and} \quad k - d \gg 1; \\
1 < k & \leq \sqrt{\log n} \quad \text{and} \quad k - d \gg \log \log k; \\
1 < k & \leq \log n/\log \log n \quad \text{and} \quad k - d \geq \delta k \quad \text{for some suitable} \quad \delta = o(1); \\
d & \leq \log n/\log \log n \quad \text{and} \quad k - d \geq k.
\end{align*}
\]

2.2 Outline of Proof

As remarked after the summarised statement (in Remark A.3), when considering properties of the random walk on a graph, such as the mixing time, geometric properties of the graph are often derived and used. In a reversal of this, we use knowledge about the mixing properties of a suitable random variable to derive a geometric result. We explain this in a little more detail now.

For the lower bound, for any Cayley graph \(G\) of an Abelian group of degree \(k\), (trivially) we have \(|B_G(R)| \leq |B_k(R)|\), where \(B_k(R)\) is the \(k\)-dimensional lattice ball of radius \(R\). If \(|B_k(R)| \ll n\), then immediately \(|B_G(R)| \ll n\), and so \(D_G(\beta) \geq R\) for all \(\beta \in (0, 1)\), asymptotically in \(n\).

Consider now the upper bound. We fix some target radius \(kL\) and draw \(W_1, ..., W_k \sim \text{Geom}(1/L)\) in the directed case. For the undirected case, we add to each \(W_i\) a uniform sign. It is well-known that the law of \(W := (W_1, ..., W_k)\) given \(\|W\|_1 = R\) is uniform on the \(L_1\) sphere of radius \(R\). Since the \(\|W\|_1 = \sum_1^k |W_i|\) is an iid sum, it concentrates around its mean, i.e., \(kL\), when \(kL \gg 1\). So this is roughly like drawing uniformly from a sphere of radius \(kL\), except that we have the added benefit that the coordinates \(W_1, ..., W_k\) are (unconditionally) independent.

We can then interpret \(W_i\) as the number of times which generator \(i\) is used in getting from the identity to \(W \cdot Z\). We show that \(W \cdot Z\) is well-mixed whp when \(kL\) is slightly larger than the target radius. Now, if the law of \(W \cdot Z\) is mixed in TV and \(\|W\|_1 \leq kL(1 + \delta)\) whp, then the law of \(W \cdot Z\) conditional on \(\|W\|_1 \leq kL(1 + \delta)\) is also mixed in TV. Thus, using the concentration of \(\|W\|_1\), we deduce that a proportion \(1 - o(1)\) of vertices \(x \in G\) can be written as \(x = w \cdot Z\) for some \(w\) with \(\|w\|_1 \geq kL\); this gives a path of length at most \(kL\) from the identity to \(x\).

We show this mixing estimate via a (modified) \(L_2\) argument, where \(W\) is conditioned to be ‘typical’, namely we define a set \(W\) and condition that \(W \in W\). The most important part is to bound the probability that two independent copies of \(W\) are equal conditional on both being in \(W\); this must be \(o(1/n)\). Since \(\|W\|_1\) concentrates and \(W\) is uniform on the sphere of this radius, we need to choose \(L\) so that the sphere of radius \(kL\) has volume slightly more than \(n\). In high dimensions—here we consider balls in \(k \gg 1\) dimensions—(discrete) spheres and balls are of asymptotically the same volume. Thus the desired radius coincides with that of the lower bound.

In an ideal world, we would directly sample \(W\) uniformly from a ball of radius \(kL\). However, the lack of independence between the coordinate causes difficulties, in particular in Lemma 2.13 below. We thus use this vector of geometrics as a proxy for the uniform distribution, but with the key property that the coordinates are independent.

2.3 Estimates on Sizes of Balls in \(\mathbb{Z}^k\)

We desire an \(R_0^+\) so that \(|B_k^\pm(R_0^+)| \approx n\), where \(B_k^\pm(R)\) is the lattice ball of radius \(R\), i.e.
\[
B_k^+(R) := \{w \in \mathbb{Z}^k \mid \|w\|_1 \leq R\} \quad \text{and} \quad B_k^-(R) := \{w \in \mathbb{Z}_+^k \mid \|w\|_1 \leq R\}.
\]
Definition 2.4. Set \( \omega := \max\{(\log k)^2, k/n^{1/2k}\} \). Note that \( 1 \ll \omega \ll k \ll \log n \). Define
\[
\mathcal{D}^k_0 := \inf\{ R \in \mathbb{N} \mid |B_k(R)| \geq ne^\omega \}.
\]

The following lemma controls the size of balls. Its proof is given in [15, §E]; see in particular [15, Lemmas E.2a and E.3a] where the index \( q \) corresponds to a type of \( L_q \) lattice balls; take \( q := 1 \) to recover the usual \( L_1 \) lattice balls here. Recall \( \mathcal{D}^k \) from Definition 2.1.

Lemma 2.5. Assume that \( 1 \ll k \ll \log n \). For all \( \xi \in (0,1) \), we have
\[
|\mathcal{R}_0 - \mathcal{D}|/\mathcal{D} \ll 1 \quad \text{and} \quad |B_k(\mathcal{D}(1 - \xi))| \ll n.
\]

2.4 Lower Bound on Typical Distance

From the results in §2.3, it is straightforward to deduce the lower bound in Theorem 2.2.

Proof of Lower Bound in Theorem 2.2. Let \( \xi \in (0,1) \) and set \( R := \mathcal{R}_0(1 - \xi) \). Since the underlying group is Abelian, applying Lemma 2.5, we have \( |B_k(R)| \leq |B_k(R)| \ll n \). Hence, for all \( \beta \in (0,1) \) and all \( Z \), we have \( \mathcal{D}_k(\beta) \geq R = \mathcal{R}_0(1 - \xi) \), asymptotically in \( n \).

2.5 Upper Bound on Typical Distance

The argument given here is in a similar vein to that of [11, §2.7]; there we analysed the mixing time of the random walk on the (random) Cayley graph. Let \( \epsilon > 0 \) and set \( L := (1 + 3\epsilon)\mathcal{R}_0/k \).

Draw \( W = (W_i)_i^k \sim \text{Geom}(1/L)^{\otimes k} \); later, we condition on \( \|W\|_1 \leq Lk \). Here the geometric random variables have support \( \{1, 2, \ldots\} \). Define \( \chi := (\chi_i)_i^k \) as follows: in the undirected case, \( \chi_i \sim \text{Unif}\{\pm 1\} \); in the directed case, \( \chi_i := 1 \) for all \( i \). Set \( S := (\chi W) \cdot Z \) where \( \chi W := (\chi_i W_i)_i^k \). Define \( W' \) and \( \chi' \) as independent copies of \( W \) and \( \chi \), respectively; set \( S' := (\chi' W') \cdot Z \).

In [11, §2.7], a key ingredient was conditioning that the auxiliary variable \( W \) was ‘typical’ in a precise sense. There we were interested in the law of the random walk; the introduction of typicality was a tool to study this, for establishing mixing bounds for the random walk. Here, somewhat in reverse, we can choose which random variable we study.

Definition 2.6. Abbreviate \( L_0 := L(1 - \log k/\sqrt{E}) \). Define
\[
\mathcal{W} := \{ w \in \mathbb{Z}_+^k \mid L_0 + 1 \leq \|w\|_1/k \leq L, \max_i w_i \leq 3L \log k \}.
\]

When \( W \) and \( W' \) are independent copies, write \( \text{typ} := \{ W, W' \in \mathcal{W} \} \).

Lemma 2.7 (Typicality). We have \( \mathbb{P}(W \in \mathcal{W}) \sim 1 \) and hence \( \mathbb{P}(\text{typ}) \sim 1 \).

Proof. We consider the three parts of typicality separately:
- the lower bound on \( \|W\|_1 \) holds with probability \( 1 - o(1) \) by Chebyshev’s inequality;
- the upper bound on \( \|W\|_1 \) holds with probability bounded away from 0 by Berry–Esseen;
- the upper bound on \( \max_i W_i \) holds with probability \( 1 - o(1) \) by the union bound.

We control the \( L_2 \) distance between \( S \) conditional on \( W \in \mathcal{W} \) and the uniform distribution.

Proposition 2.8. Suppose that Hypothesis A is satisfied. Then
\[
\mathbb{E}\left(\|P_{G_k}(S \in \cdot \mid W \in \mathcal{W}) - \text{Unif}(G)\|_2^2\right) = o(1),
\]
where we recall that \( P_{G_k}(\cdot) \) is the random law corresponding to the random Cayley graph \( G_k \).

We now have all the ingredients to prove the upper bound on typical distance.
Proof of Upper Bound in Theorem 2.2. Let $\overline{W}$ have the law of $W$ conditional on $W \in \mathcal{W}$. By Proposition 2.8, the $L_2$ distance between $\overline{S} := \overline{W} \cdot Z$ and $\text{Unif}(G)$ is $o(1)$ whp. Thus the support $\mathcal{S}$ of $\overline{S}$ is a proportion $1 - o(1)$ of the vertices whp. In particular, there is a path of length at most $Lk$ from id to all vertices in $\mathcal{S}$ whp, as $\|W\|_1 \leq Lk$ by definition of typicality. Hence $D_k(\beta) \leq Lk = (1 + 3\varepsilon)R_0$ whp. Applying Lemma 2.5 then gives $(D_k(\beta) - \mathcal{S})/ \mathcal{S} \leq 4\varepsilon$ whp.

The remainder of this subsection is devoted to proving Proposition 2.8. We have

$$\mathbb{E}(\|P_{G_k}(S \in \cdot | W \in \mathcal{W}) - \text{Unif}(G)\|_2^2) = n \mathbb{P}(S = S' | \text{typ}) - 1,$$

recalling that $\chi'$ and $W'$ are independent copies of $\chi$ and $W$, respectively, and $S' := (\chi'W') \cdot Z$.

First we control the probability that $\chi W = \chi' W'$; in this case we necessarily have $S = S'$.

**Lemma 2.9.** We have $\mathbb{P}(\chi W = \chi' W' | \text{typ}) = o(1/n)$.

**Proof.** Recall that $L_0 := L(1 - \log k/\sqrt{k})$. Consider the directed case first, ie $\chi = 1 = \chi'$. Then

$$\mathbb{P}(W = W', \text{typ}) \leq \sum_{w:|w|_1 \geq k(L_0 + 1)} \mathbb{P}(W = w = W')$$

$$= \sum_{w:|w|_1 \geq k(L_0 + 1)} \mathbb{P}(W' = w) \prod_{i=1}^k \mathbb{P}(W_i = w_i)$$

$$= \sum_{w:|w|_1 \geq k(L_0 + 1)} \mathbb{P}(W' = w) \prod_{i=1}^k L^{-1}(1 - L^{-1}w_i^{-1})$$

$$= \sum_{w:|w|_1 \geq k(L_0 + 1)} \mathbb{P}(W' = w) \cdot L^{-k}(1 - L^{-1}w_1^{-k})$$

$$\leq L^{-k}(1 - L^{-1})^{kL_0} = (L^{-1}(1 - L^{-1})^{L(1 - \log k/k)})^k$$

$$\leq (eL)^{-k} \exp(\sqrt{k \log k}) \leq n^{-1} e^{-3k/2},$$

with the final inequality using the fact that $L^+ \geq (1 + 2\varepsilon)n^{1/k}/e$, using Lemma 2.5. In the undirected case, we also need $\chi = \chi'$, which happens with probability $2^{-k}$, and is independent of $(W, W')$. Hence the same inequality holds with the event $(W = W')$ replaced by $(\chi W = \chi' W')$, recalling that $L^+ \approx \frac{1}{2} L^+$. Finally, $\mathbb{P}(\text{typ}) \approx 1$. Thus Bayes’s rule combined with the above calculation gives

$$\mathbb{P}(\chi W = \chi' W' | \text{typ}) \leq n^{-1} e^{-2k} \ll 1/n.$$

The following lemma describing the distribution of $v \cdot Z$ for a given $v \in \mathbb{Z}^k$ is crucial.

**Lemma 2.10.** For all $v \in \mathbb{Z}^k$ with $\gcd(v_1, ..., v_k, n) = \gamma$, we have $v \cdot Z \sim \text{Unif}(\gamma G)$.

We thus now need to control $|\gamma G|$.

**Lemma 2.11.** For all Abelian groups $G$ and all $\gamma \in \mathbb{N}$, we have

$$|G|/|\gamma G| \leq \gamma^{d(G)}.$$

These two lemmas were used in [11, §2.7]; see [11, Lemmas 2.11 and 2.12] for proofs. Define $V := \chi W - \chi' W'$ and $g := \gcd(V_1, ..., V_k, n)$.

**Corollary 2.12.** We have

$$n \mathbb{P}(V \cdot Z = 0, V \neq 0 | \text{typ}) \lesssim \mathbb{E}(g^d \mathbb{1}(V \neq 0) | \text{typ}).$$

**Proof.** The conditioning does not affect $Z$. The corollary follows from Lemmas 2.10 and 2.11.

**Lemma 2.13.** Given Hypothesis A, we have $\mathbb{E}(g^d \mathbb{1}(V \neq 0) | \text{typ}) = 1 + o(1)$.  

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Proof. Each coordinate of $V$ is unimodal and symmetric about 0. From this we can deduce that
\[ P(V_1 \in \gamma Z \mid V_1 \neq 0) \leq 1/\gamma, \]
as in [11, Lemma 2.14]. The probability of $V_1 = 0$ is roughly $1/(2L) \approx n^{-1/k}$; in particular, it is at most $3n^{-1/k}$. The coordinates are independent. Since $P(\text{typ}) \approx 1$, we thus have
\[ P(g = \gamma \mid \text{typ}) \lesssim (1/\gamma + 3/n^{1/k})^k. \]
By typicality, $g \leq 6L \log k \leq 3n^{1/k} \log k$. Hence, summing over $\gamma$, we obtain
\[ \mathbb{E}(g^d 1(V \neq 0)) \lesssim \sum_{\gamma=1}^{3n^{1/k} \log k} \gamma^d (1/\gamma + 3/n^{1/k})^k. \]
We handle almost exactly the same sum in [11, Corollary 2.15]. Hypothesis A here is designed precisely to control this sum; it is identical to [11, Hypothesis A]. There the $3/n^{1/k}$ part is replaced with $2/n^{1/k}$, but exactly the same arguments apply showing that the sum is $1 + o(1)$.

Proposition 2.8 now follows immediately from Lemmas 2.9 and 2.13 and Corollary 2.12.

Proof of Proposition 2.8. By Lemmas 2.9 and 2.13 and Corollary 2.12, we have
\[ n \mathbb{P}(S = S' \mid \text{typ}) \leq n \mathbb{P}(V = 0 \mid \text{typ}) + n \mathbb{P}(V \cdot Z = 0, V \neq 0 \mid \text{typ}) \]
\[ \leq n \mathbb{P}(V = 0 \mid \text{typ}) + \mathbb{E}(g^d 1(V \neq 0) \mid \text{typ}) = 1 + o(1). \]

3 Typical Distance: $k \approx \log |G|$

This section focuses on concentration of distances from the identity in the random Cayley graph of an Abelian group when $k \approx \log |G|$. (The previous section dealt with $1 \ll k \ll \log |G|$ and the next deal with $k \gg \log |G|$.) The main result of the section is Theorem 3.2; see also Hypothesis B.

The outline of this section is as follows:
- §3.1 states precisely the main theorem of the section;
- §3.2 outlines the argument;
- §3.3 gives some crucial estimates on the size of lattice balls;
- §3.4 is devoted to the lower bound;
- §3.5 is devoted to the upper bound under additional constraints;
- §3.6 describes how to relax these additional constraints;
- §3.7 describes an extension for $L_1$-type graph distances to $L_q$-type.

3.1 Precise Statement and Remarks

To start the section, we recall the typical distance statistic.

Definition 3.1. Let $H$ be a graph and fix a vertex $0 \in H$. For $r \in \mathbb{N}$, write $B_H(r)$ for the $r$-ball in the graph $H$, i.e. $B_H(r) := \{h \in H \mid d_H(0, h) \leq r\}$, where $d_H$ is the graph distance in $H$. Define
\[ D_H(\beta) := \min\{r \geq 0 \mid |B_H(r)| \geq \beta |H|\} \quad \text{for} \quad \beta \in (0, 1). \]

When considering sequences $(k_N, G_N)_{N \in \mathbb{N}}$ of integers and Abelian groups, abbreviate
\[ D_N(\beta) := D_{G_N(\{Z_1, \ldots, Z_{k_N}\})}(\beta) \quad \text{where} \quad Z_1, \ldots, Z_{k_N} \sim^{\text{iid}} \text{Unif}(G_N). \]
As always, if we write $D_N$, then this is either $D_N^+$ or $D_N^-$ according to context.

We show that, whp over the graph (ie choice of $Z$), this statistic concentrates. Here we consider $k \approx \lambda \log |G|$ for any $\lambda \in (0, \infty)$. The result holds for a large class of Abelian groups. Further, for these groups, the typical distance concentrates at $\alpha_{\lambda} k$ where $\alpha_{\lambda} \in (0, \infty)$ is a constant; so this depends only on $k$ and $|G|$. This is in agreement with the spirit of the Aldous–Diaconis conjecture.
Recall that any Abelian group can be decomposed as $\bigoplus_1^d \mathbb{Z}_{m_j}$ for some $d, m_1, ..., m_d \in \mathbb{N}$. For an Abelian group $G$, we define the dimension and minimal side-length, respectively, as follows:

$$
d(G) := \min \{d \in \mathbb{N} \mid \bigoplus_1^d \mathbb{Z}_{m_j} \text{ is a decomposition of } G\};$$

$$m_e(G) := \max \{\min_{i \in [d]} m_j \mid \bigoplus_1^d \mathbb{Z}_{m_j} \text{ is a decomposition of } G\}.$$

It can be shown that there is a decomposition which is optimal for both these statistics: there exist $d, m_1, ..., m_d \in \mathbb{N}$ so that $\bigoplus_1^d \mathbb{Z}_{m_j}$ is a decomposition of $G$ with $d = d(G)$ and $\min_{i \in [d]} m_j = m_e(G)$. From now on, we assume that we are always using such an optimal decomposition.

There are some conditions which the Abelian groups must satisfy.

**Hypothesis B.** The sequence $(k_N, G_N)_{N \in \mathbb{N}}$ satisfies Hypothesis B if

$$\lim_{N \to \infty} k_N = \infty, \quad \lim_{N \to \infty} k_N / \log |G_N| \in (0, \infty), \quad \liminf_{N \to \infty} m_e(G_N) = \infty$$

and $d(G_N) \leq \frac{1}{4} \log |G_N| / \log \log |G_N|$ for all $N \in \mathbb{N}$.

We are now ready to state the main theorem of this section.

**Theorem 3.2.** Let $(k_N)_{N \in \mathbb{N}}$ be a sequence of positive integers and $(G_N)_{N \in \mathbb{N}}$ a sequence of finite, Abelian groups; for each $N \in \mathbb{N}$, define $Z|_N := [Z_1, ..., Z_{kN}]$ by drawing $Z_1, ..., Z_{kN} \sim_{iid} \text{Unif}(G_N)$.

Suppose that $(k_N, G_N)_{N \in \mathbb{N}}$ satisfies Hypothesis B. Let $\lambda := \limsup_N k_N / \log |G_N|$. Then there exists a constant $\alpha_N^k \in (0, \infty)$ so that, for all $\beta \in (0, 1)$, we have

$$D_N^k(\beta) / (\alpha_N^k k_N) \to 1 \quad (\text{in probability}) \quad \text{as } N \to \infty.$$

Moreover, the implicit lower bound holds deterministically, ie for all choices of generators, and for all Abelian groups, ie Hypothesis B need not be satisfied—we just need $\lim\inf_N k_N / \log |G_N| \in (0, \infty)$.

For ease of presentation, in the proof we drop the $N$-subscripts.

**Remark 3.3.** In §3.7, we describe an extension from the usual $L_1$-type graph distances to $L_q$-type. An analogous concentration of typical distance is given. See Hypothesis B′ and Theorem 3.11. △

### 3.2 Outline of Proof

The outline here is very similar to that from before; see §2.2. In particular, the lower bound is exactly the same idea. For the upper bound, we were trying to bound the expectation of a $d$-th power of a gcd. Issues arose when $k$ became too large while $k - d$ is fairly small; see the proof of Lemma 2.13. This arose from the fact that we used the estimate

$$\mathbb{P}(V_1 \in \gamma \mathbb{Z}) \leq \mathbb{P}(V_1 \in \gamma \mathbb{Z} \mid V_1 \neq 0) + \mathbb{P}(V_1 = 0) \leq 1 / \gamma + 3 / n^{1/k}.$$

Once this was raised to the power $k$, the second term became an issue. We alleviate this by defining

$$\mathcal{I} := \{i \in [k] \mid V_i \neq 0\}$$

and studying $\mathbb{P}(V_i \in \gamma \mathbb{Z} \mid i \in \mathcal{I})$: the problematic term $3 / n^{1/k}$ then does not exist as we consider only non-zero coordinates of $V$. If $G = \bigoplus_1^d \mathbb{Z}_{m_j}$, then we are actually interested in $V_i \text{ mod } m_j$ for each $j$. Recall that $m_* = \min_j m_j$. ‘Typically’, one has $|V_i| \leq m_*$. We suppose initially that $m_*$ is large enough so that $\max_i |V_i| < m_*$ whp. Thus looking at $V_i = 0$ or $V_i \equiv 0 \text{ mod } m_j$ is no different.

For large $|\mathcal{I}|$, the gcd analysis goes through similarly to before. When $|\mathcal{I}|$ is small, eg smaller than $d$, it is more difficult to control; in this case, we use a fairly naive bound on the gcd, but control carefully the probability of realising such an $\mathcal{I}$. The case $\mathcal{I} = \emptyset$, which corresponds to $V = 0$, is handled by taking the lattice ball to be of large enough volume.

Previously we used a vector of geometries as a proxy for a uniform distribution on a ball. Here we are able to let $W$ be uniform on a ball. The coordinates are no longer independent, which makes the gcd analysis is slightly complicated. However, since we only consider $i$ with $V_i \neq 0$, this can be handled; see Lemma 3.9. This uniformity simplifies some other calculations somewhat.
3.3 Estimates on Sizes of Balls in $\mathbb{Z}^k$

We wish to determine the size of balls $B_k(R)$ when $k \asymp \log n$. In particular, we are interested in the growth when the volume is around $n$.

**Definition 3.4.** Define $M_\pm^+(k, N)$ to be the minimal integer $M$ satisfying $|B_k^\pm(M)| \geq N$.

**Lemma 3.5.** For all $\lambda \in (0, \infty)$, there exists a function $\omega \gg 1$ and a constant $\alpha^\pm$ so that, for all $\varepsilon \in (0, 1)$, if $k \sim \log n$, then $M_\pm^+ := M_\pm^+(k, \alpha^\varepsilon)$ satisfies

$$M_\pm^+ \sim \alpha^\pm k \sim \alpha^\pm \log n \quad \text{and} \quad |B_k^\pm(\alpha^\pm k(1-\varepsilon))| \ll n.$$ 

This will follow easily from the following auxiliary lemma controlling the size of lattice balls.

**Lemma 3.6.** There exists a strictly increasing, continuous function $c^\pm : (0, \infty) \to (0, \infty)$ so that, for all $a \in (0, \infty)$, we have

$$|B_k^\pm(ak)| = \exp(k(c^\pm(a) + o(1))).$$

**Proof.** The directed case follows immediately from Stirling’s approximation and the fact that

$$|B_k^\pm(ak)| = |\{b \in \mathbb{Z}_+^k : \sum_1^k b_i \leq ak\}| = \binom{ak+k}{k} = \binom{a+1+k}{k}.$$ 

Consider now the undirected case. Omit all floor and ceiling signs. By considering the number of coordinates which equal 0, we obtain

$$|B_k^-(ak)| = \sum_{i=0}^k A_i \quad \text{where} \quad A_i := A_i(k, a) := \binom{k}{i} 2^{-i} (k+i+ak).$$

Choose $i_* := i_*(k, a)$ that maximises $A_i$. Then $A_{i_*} \leq |B_k^-(ak)| \leq (k+1)A_{i_*}$. Observe that

$$\frac{A_{i+1}}{A_i} = \frac{(k-i)^2}{2(i+1)(k(1+a) - i)},$$

and hence one can determine $i_*$ as a function of $k$ and $a$, conclude that $i_*(a, k)/k$ converges as $k \to \infty$ and thus determine $c^\pm(a)$ in terms of the last limit. We omit the details. Knowing this limit allows us to plug this into the definition of $A_i$ and use Stirling’s approximation to get

$$A_{i_*} = \exp(k(c^-(a) + o(1))),$$

for some strictly increasing function $c^- : (0, \infty) \to (0, \infty)$. Since $k+1 = e^{o(k)}$, the claim follows. $$

From this lemma, Lemma 3.5 follows easily.

**Proof of Lemma 3.5.** Set $a := c^{-1}(1/\lambda)$. The upper bound is an immediate consequence of the continuity of $c$. The lower bound follows from the exponential growth rate. $$

3.4 Lower Bound on Typical Distance

From the results in §3.3, it is straightforward to deduce the lower bound in Theorem 3.2.

**Proof of Lower Bound in Theorem 3.2.** Let $\xi \in (0, 1)$ and set $R := \alpha^\pm k(1-\xi)$. Since the underlying group is Abelian, applying Lemma 3.5, we have $|B_k^\pm(R)| \leq |B_k^\pm(R^\pm k)| \ll n$. Hence, for all $\beta \in (0, 1)$ and all $Z$, we have $D_k^\pm(\beta) \geq R = \alpha^\pm k(1-\xi)$, asymptotically in $n$. $$

3.5 Upper Bound on Typical Distance Given $m_*(G) \gg k$

Define $M_\pm^\ast$, $\omega$ and $\alpha^\pm$ as in Definition 3.4 and Lemma 3.5. In this subsection we draw $W^\pm \sim \text{Unif}(B_k^\pm(M_\pm^\ast))$, i.e uniform on a ball of radius $M_\pm^\ast$. We show that $W^\pm \cdot Z$ is well-mixed on $G$, and hence its support contains almost all the vertices.
Proposition 3.7. Suppose that Hypothesis B is satisfied. Then
\[
\mathbb{E}(\|P_{G_k}(W^\pm \cdot Z \in \cdot) - \pi_G\|_2^2) = o(1),
\]

Given this proposition, the upper bound in Theorem 3.2 follows easily.

Proof of Upper Bound in Theorem 3.2 Given Proposition 3.7. If \(\|P_{G_k}(W^\pm \cdot Z \in \cdot) - \pi_G\|_2 \leq \varepsilon\), then the support \(S\) of \(W^\pm \cdot Z\) satisfies \(\pi_G(S^c) \leq \varepsilon\). Combined with Lemma 3.5 and Proposition 3.7, the upper bound in Theorem 3.2 follows.

The remainder of this subsection is devoted to proving Proposition 3.7. We tend to drop the ±-superscript from the notation, only writing + or - if there is ambiguity. Let \(W, W' \sim \text{Unif}(B_k(M_\ast))\) and let \(V := W - W'\). The standard \(L_2\) calculation gives
\[
\mathbb{E}(\|P_{G_k}(W \cdot Z \in \cdot) - \pi_G\|_2^2) = \mathbb{E}(n P(V \cdot Z = 0 \mid Z) - 1) = n P(V \cdot Z = 0) - 1.
\]

First, it is immediate that \(P(V = 0) = P(W = W') = |B_k(M_\ast)|^{-1} \leq n^{-1} e^{-o} \ll n^{-1}\). Now consider \(V \neq 0\). As in §2.5, it is key to analyse certain geds. In this section, we set
\[
g_j := \gcd(V_1, \ldots, V_k, m_j) \quad \text{for each} \quad j \in [d]; \quad \text{set} \quad g := \gcd(V_1, \ldots, V_k, n).
\]
The following lemma is equivalent to Lemma 2.10, rephrased slightly.

Lemma 3.8. Conditional on \(V\), we have \(V \cdot Z \sim \text{Unif}(\oplus^d g_jZ_{m_j})\).

For the remainder of this subsection, we assume that the minimal side-length \(m_\ast := m_\ast(G)\) satisfied \(m_\ast \gg k \approx M_\ast\). In the next subsection, we remove this assumption: we extend the proof to \(m_\ast \gg 1\), as in Hypothesis B. Given this, we have \(\max_{i \in [k]} |V_i| > \max_{j \in [d]} m_j\). Hence
\[
\mathcal{I} := \{i \in [k] \mid V_i \neq 0 \mod m_j \forall j \in [d]\} = \{i \in [k] \mid W_i \neq W'_i\}.
\]
To analyse the expected gcd, we breakdown according to the value of \(\mathcal{I}\).

Lemma 3.9. There exists a constant \(C\) so that, for all \(I \subseteq [k]\) with \(I \neq \emptyset\), we have
\[
n P(V \cdot Z = 0 \mid \mathcal{I} = I) \leq \mathbb{E}(g^d \mid \mathcal{I} = I) \leq \begin{cases}
C 2^d (2M_\ast)^{d-|I|+2} & \text{when } |I| \leq d + 1, \\
1 + \frac{5}{2} 2^d |I| & \text{when } |I| \geq d + 2.
\end{cases}
\]

Lemma 3.10. For all \(I \subseteq [k]\) with \(|I| \ll k\), we have \(P(\mathcal{I} = I) \leq e^{-o} n^{-1+o(1)}\). If \(I = \emptyset\), then the \(o(1)\) term may be taken to be 0.

Given these two lemmas, we have all the ingredients required to prove Proposition 3.7, from which we deduced the main theorem (Theorem 3.2). We defer the proofs of Lemmas 3.9 and 3.10 until after the proof of Proposition 3.7, which we give now.

Proof of Proposition 3.7. Here \(k \approx \lambda \log n\), \(M := M_\ast \approx \alpha k \approx \alpha \lambda \log n\) and \(d \leq \frac{1}{4} \log n / \log \log n\).

As noted previously, the standard \(L_2\) calculation gives
\[
\mathbb{E}(\|P_{G_k}(W \cdot Z \in \cdot) - \pi_G\|_2^2) = \mathbb{E}(n P(V \cdot Z = 0 \mid Z) - 1)
= n P(V \cdot Z = 0) - 1 = n \sum_{I \subseteq [k]} P(V \cdot Z = 0, \mathcal{I} = I) - 1.
\]

Consider \(I = \emptyset\). Then \(V \cdot Z = 0\) (for all \(Z\)). By Lemma 3.10, we have \(P(\mathcal{I} = \emptyset) \ll e^{-o}\). Thus
\[
n P(V \cdot Z = 0, \mathcal{I} = \emptyset) \leq e^{-o} = o(1).
\]

Consider \(I \subseteq [k]\) with \(1 \leq |I| \leq d + 1\). There are at most \((d + 1)(k)_{d+1} \leq k^{d+2}\) such sets \(I\). Since \(\log k = \log \log n + \log \lambda + o(1)\), we have \(k^{d+2} \leq n^{2/3}\). Applying Lemmas 3.9 and 3.10 gives
\[
n P(V \cdot Z = 0, \mathcal{I} = I) \leq C 2^d (3 \alpha \lambda \log n)^{d+2-|I|} n^{-1+o(1)} \leq k^{-d-2} n^{-1/4},
\]
noting that \( d \ll k \gg \log n \) and (so) \( 2^d = n^{o(1)} \). We now sum over all \( I \) with \( 1 \leq |I| \leq d + 1 \):

\[
n \sum_{1 \leq |I| \leq d+1} \mathbb{P}(V \cdot Z = 0, \mathcal{I} = I) \leq n^{-1/4} = o(1).
\]

Consider \( I \subseteq [k] \) with \( d + 2 \leq |I| \leq L := \frac{2}{3} \log n / \log \log n \); then \( L - 2d \gg 1 \). Similarly to above, there are at most \( L(\frac{1}{k}) \leq k^{d+1} \) such sets \( I \). Applying Lemmas 3.9 and 3.10 gives

\[
n \mathbb{P}(V \cdot Z = 0, \mathcal{I} = I) \leq n^{-1+o(1)} \leq k^{-L-1} n^{-1/4},
\]

noting that \( k^L \leq n^{2/3+o(1)} \). We now sum over all \( I \) with \( d + 2 \leq |I| \leq L \):

\[
n \sum_{d+2 \leq |I| \leq L} \mathbb{P}(V \cdot Z = 0, \mathcal{I} = I) \leq n^{-1/4} = o(1).
\]

Finally consider \( I \subseteq [k] \) with \( |I| \geq L \). Sum over these using Lemma 3.9:

\[
n \sum_{L \leq |I| \leq k} \mathbb{P}(V \cdot Z = 0, \mathcal{I} = I) \leq 1 + 5 \cdot (\frac{3}{2})^{2d-L} = 1 + o(1).
\]

Combining these four parts into a single sum, we deduce the result. \( \square \)

It remains to prove the auxiliary Lemmas 3.9 and 3.10.

**Proof of Lemma 3.9.** The first inequality is an immediate consequence of Lemma 3.8.

Note that \( g \leq 2M \), since \( \max_i |V_i| \leq 2M \). For \( \alpha, \beta \in \mathbb{Z} \), write \( \alpha \bot \beta \) if \( \alpha \) divides \( \beta \). Thus

\[
\mathbb{E}(g^d \mid \mathcal{I} = I) \leq \sum_{1 \leq |I| \leq d+1} \gamma^d \mathbb{P}(\gamma \mid V_i \forall i \in I \mid \mathcal{I} = I)
\]

For a set \( I \subseteq [k] \), write \( W_I := (W_i)_{i \in I} \) and \( W_{k\setminus I} := W_{[k]\setminus I} \). Consider conditioning on \( \mathcal{I} = I \). Let \( W_I' \) be given; since \( \mathcal{I} = I \), we have \( W_I = W_I' \). Let \( U \) have the distribution of \( W_I \) given \( W_I' \) and define \( U' \) analogously. Write \( D_i := D_i(\gamma) = \{ \gamma \mid (U_i - U_i') \} \). Then

\[
\mathbb{P}(\gamma \mid V_i \forall i \in I \mid \mathcal{I} = I, \|W_{k\setminus I}\|_1) = \mathbb{P}(D_i \forall i \in I).
\]

By exchangeability, it suffices to consider the case \( I = \{1, \ldots, \ell\} \). We then have

\[
\mathbb{P}(D_i \forall i \in I) = \mathbb{P}(D_i) \mathbb{P}(D_{\ell-1} \mid D_i) \cdots \mathbb{P}(D_1 \mid D_2, \ldots, D_\ell) = \prod_{i=1}^{\ell} \mathbb{P}(D_i \mid D_{i+1}, \ldots, D_\ell).
\]

For \( i \in [k] \), define \( M_i := M_i - \|W_{[1, \ldots, i]}\|_1 \) and \( M_i' \) analogously. Let \( i \in [\ell-1] \). Let \( (u_{i+1}, \ldots, u_\ell) \) and \((u_{i+1}', \ldots, u_\ell')\) be two vectors in the support of \( (U_{i+1}, \ldots, U_\ell) \). Then,

\[
\text{conditional on } (U_{i+1}, \ldots, U_\ell) = (u_{i+1}, \ldots, u_\ell) \text{ and } (U_{i+1}', \ldots, U_\ell') = (u_{i+1}', \ldots, u_\ell') \text{ we have } (U_1, \ldots, U_i) \sim \text{Unif}(B_i(R)) \text{ and } (U_1', \ldots, U_i') \sim \text{Unif}(B_i(R')) \text{ for some } R, R' \in \mathbb{R}.
\]

(Recall that the subscript in \( B_k \) denotes the dimension of the ball.)

In the case of undirected balls, the law of \( U_i - U_i' \) given this conditioning is symmetric and unimodal on \( \mathbb{Z} \setminus \{0\} \); see [27, Theorem 2.2]. It follows, as in the proof of Lemma 2.1.3, that

\[
\mathbb{P}(D_i^- \mid D_{i+1}, \ldots, D_\ell^-) \leq 1/\gamma.
\]

Further, this holds not just conditional on \( D_{i+1}^- \cap \cdots \cap D_\ell^- \), but conditional on any choice of \( (U_{i+1}, \ldots, U_i) \) and \( (U_{i+1}', \ldots, U_i') \) which satisfy \( D_{i+1} \cap \cdots \cap D_\ell \). By the same reasoning, \( \mathbb{P}(D_i^-) \leq 1/\gamma \).

Hence, for undirected balls,

\[
\mathbb{P}(D_i^- \forall i \in I) = \mathbb{P}(\gamma \mid V_i^- \forall i \in I \mid \mathcal{I} = I) \leq \gamma^{-|I|}.
\]

(The \(-\) superscript emphasises that this is for undirected balls.)

We now turn our attention to directed balls. In this case, \( U_i \) and \( U_i' \) are both unimodal, but with potentially different modes, if \( R \neq R' \). Instead of direct computation, we compare with the undirected case. Specifically, if \( U_i \) and \( U_i' \) have the same sign in the undirected case, then \( |V_i| = |U_i - U_i'| \) has the same law as in the directed case. The choice of sign is independent of
everything else; the two have the same sign with probability $\frac{1}{2}$. Hence, by conditioning on the specific values of $(U_{i+1}, \ldots, U_t)$ and $(U'_t, \ldots, U'_t)$, we obtain

$$1/\gamma \geq \mathbb{P}(D^-_t \mid D^-_{i+1}, \ldots, D^-_t) \geq \frac{1}{2} \mathbb{P}(D^+_t \mid D^+_{i+1}, \ldots, D^+_t).$$

For $\gamma = 2$, note that the probabilities are actually the same: this is because $x - y$ is even if and only if $|x| - |y|$ is even, since $x$ and $-x$ have the same parity.

From this we deduce, for both the undirected and directed cases, that

$$\mathbb{E}(p^d \mid |I| = \log n) \leq 1 + 2^{d-|I|} + \sum_{\gamma=3}^{2M} \gamma (2/\gamma) |I| = 1 + 2^{d-|I|} + 2^d \sum_{\gamma=3}^{2M} (\gamma/2)^{d-|I|}.$$ 

A case-by-case analysis, according to $d - |I|$, completes the proof. \hfill \square

**Proof of Lemma 3.10.** Recall from Definition 3.4 that $|B_k(M_\ast)| \geq ne^\omega$. Thus

$$\mathbb{P}(I = \emptyset) = \mathbb{P}(W = W') = |B_k(M_\ast)|^{-1} \leq n^{-1}e^{-\omega}.$$ 

Using the law of $W_I$ given $W'_I$ determined in the previous proof, we have

$$\mathbb{P}(W_I = W'_I) = \frac{\mathbb{P}(W = W')}{\mathbb{P}(W = W' \mid W_I = W'_I)} = \frac{|B_k(M_\ast)|^{-1}}{\mathbb{E}(|B_I(M_\ast) - \|W_I\|_1|^{-1})} \leq \frac{|B_{II}(M_\ast)|}{|B_k(M_\ast)|}.$$ 

It is a standard balls-in-bins combinatorial identity that

$$|B^+_{II}(R)| = \left| \{ b \in \mathbb{Z}^\ell_+ \mid \sum_1^\ell b_i \leq R \} \right| = \binom{\ell + R}{\ell}.$$ 

For the undirected case, we can choose a sign for each coordinate. Hence we see that

$$|B^+_{II}(R)| \leq |B^-_{II}(R)| = \left| \{ b \in \mathbb{Z}^\ell \mid \sum_1^\ell |b_i| \leq R \} \right| \leq 2^\ell \binom{R + \ell}{\ell}.$$ 

Abbreviate $M := M_\ast$ and $\ell := |I|$. It suffices to consider $I$ with $\ell \leq ck$, for an arbitrarily small positive constant $c$. From Lemma 3.5, we have $M \leq 2ak$. So

$$|B^\pm(M)| \leq 2^\ell \binom{\ell M + \ell}{\ell} \leq (2e(2ak/\ell + 1))^\ell \leq (8eca/\ell)^\ell,$$ 

with the last inequality requiring $2ak/\ell \geq 1$, which holds if $c$ is sufficiently small, as $\ell \leq ck$. Now, for $c$ sufficiently small, the map $\ell \mapsto (8eca/\ell)^\ell$ is increasing on $[1,ck]$. Hence

$$|B^\pm(M)| \leq (8eca/\ell)^\ell \leq (8eca/c)^ck \leq (8eca/c)^{2ck\log n}.$$ 

By taking $c$ sufficiently small, we can upper bound this by an arbitrarily small power of $n$. \hfill \square

### 3.6 Relaxing Condition on Minimal Side-Length to $m_\ast(G) \gg 1$

For the upper bound, we have been assuming that the minimal side-length $m_\ast(G)$ satisfies $m_\ast(G) \gg \log |G|$. (Recall that the lower bound had no conditions on $m_\ast(G)$.) We now describe how to relax this condition to $m_\ast(G) \gg 1$. We could go even further, with statements like “only a small number of $j$ in $G = \oplus_d^k \mathbb{Z}_{m_j}$ have $m_j \gg 1$.” Since we have no reason to believe our other conditions are optimal, we settle for the simpler $m_\ast(G) \gg 1$.

In this proof we consider both $L_1$ and $L_\infty$ balls. To distinguish these we use a superscript:

- $B_l^l(R)$ will be the $L_1$ ball in $\ell$ dimensions of radius $R$;
- $B^\infty_{II}(R)$ will be the $L_\infty$ ball in $\ell$ dimensions of radius $R$.

For a set $I \subseteq [k]$, recall that we write $W_I := (W_i)_{i \in I}$ and $W'_I := (W'_i)_{i \notin I}$.

We describe the adaptations for undirected graphs. The adaptations for directed graphs are completely analogous: simply replace appearances of $\mathbb{Z}^k$ with $\mathbb{Z}^k_+$ and $|W_I|$ with $W_I$. 

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Outline of Proof. The idea behind the proof is intuitive. Since $R \asymp k$, by symmetry we have $\mathbb{E}(|W_i|) \leq R/k \asymp 1$ for all $i$. Thus ‘almost all’ the coordinates should be smaller than any diverging function (these coordinates are good). Further, the contribution to the radius $||W||_1$ due to the bad coordinates should be small, ie $o(k)$. Roughly this allows us to replace $k$ with $\tilde{k} = k(1-o(1))$ and $R$ with $\tilde{R} = R(1-o(1))$. Choosing $R := \alpha_k/\log n \cdot (1+2\varepsilon)$ for $\varepsilon > 0$ then gives

$$\tilde{R} \geq \alpha_k/\log \tilde{k} \cdot (1+\varepsilon)$$

and hence $|B_k^{\tilde{R}}| \gg n$.

This was the key element in the proof previously; the remainder of the proof is as before.

We now proceed formally and rigorously.

Relaxing Minimal Side-Length Condition. Let $\varepsilon > 0$ and $\lambda := \lim k/\log n$. Set $R := \alpha\lambda k(1+2\varepsilon)$ and draw $W \sim \text{Unif}(B_{k,1}(R))$. Let $\nu$ satisfy $1 \ll \nu \ll n^{\alpha}(G)$. For $w \in \mathbb{Z}^k$, define

$$\mathcal{J}(w) := \{i \in [k] \, | \, |w_i| \leq \nu\}.$$

Call these coordinates good. By Markov’s inequality, clearly $|\mathbb{E}|k \setminus \mathcal{J}(W)| \lesssim 1/\nu = o(1)$ whp.

As always, we look at two independent realisations $W$ and $W'$. We then wish to look at coordinates $i \in [k]$ which are good for both $W$ and $W'$, ie in $\mathcal{J} := \mathcal{J}(W) \cap \mathcal{J}(W')$. We need to make sure that the contribution to the radius from the (abnormally large) bad coordinates is not too large. For $\delta > 0$ and $w \in \mathbb{Z}^k$, write $\mathcal{L}_\delta(w)$ for the collection of the $\lfloor 2\delta k \rfloor$-largest (in absolute value) coordinates of $w$. We then define typicality in the following way: for $\delta, \delta' > 0$, set

$$W := \{w \in \mathbb{Z}^k \mid |w|_1 \leq R, |\mathbb{E}|k \setminus \mathcal{J}(w)| \leq \delta k, |w_{\mathcal{L}_\delta(w)}| \leq \delta' k\}.$$

In particular now, if $w, w' \in \mathcal{W}$, then $|w_{\mathcal{J}(w) \cup \mathcal{J}(w')}|_1 \geq k - 2\delta' k$. It is not difficult to see that we can choose $\delta, \delta' = o(1)$ with $\mathbb{P}(W \in \mathcal{W}) = 1-o(1)$; we give justification at the end of the proof.

Consider now $W, W' \sim_{\text{ind}} \text{Unif}(B_{k,1}(R))$. We have the following conditional law:

$$W, W' \sim_{\text{ind}} \text{Unif}(B_k^R \cap \mathcal{B}_k^\nu) \quad \text{conditional on} \quad W_{\mathcal{J}} = w_{\mathcal{J}}, W'_{\mathcal{J}} = w'_{\mathcal{J}} \quad \text{and} \quad \mathcal{J} = \mathcal{J}.$$

where $\mathcal{J} := \mathcal{J}(W) \cup \mathcal{J}(W')$, $\tilde{R} := |\mathcal{J}|$ and $\tilde{R} := R - |w_{\mathcal{J}}|.$

Write typ := $\{W, W' \in \mathcal{W}\}$. On the event typ, given $\mathcal{J} = \mathcal{J}$ and $(W, W')$, we have

$$\tilde{R} \geq k(1-\delta) = k(1-o(1)) \quad \text{and} \quad \tilde{R} \geq R(1-\delta') = R(1-o(1)).$$

Since typ
ticality holds with probability $1-o(1)$, we have

$$|B_k^R \cap B_k^\nu| \gg n.$$

The remainder of the proof follows similarly to before. Formally, we define $\overline{W}$ and $\overline{W}'$ as follows:

$$\overline{W}_i := W_i \quad \text{and} \quad \overline{W}'_i := W'_i \quad \text{for} \quad i \in \mathcal{J};$$

$$\overline{W}_i := 0 \quad \text{and} \quad \overline{W}'_i := 0 \quad \text{for} \quad i \notin \mathcal{J}.$$
Remark. We believe that the typical distance should concentrate if $k \asymp \log |G|$ and $k - d \gg 1$ without any condition on like that on $m_*(G)$. However, without any such condition, we do have reason to believe that the value at which this concentration happens should depend on more than just $k$ and $|G|$—the algebraic structure of $G$ should be important. This exact phenomenon occurs when studying the mixing time of the random walk on the Cayley graph. See [11, Theorem A], in particular contrasting the case $k \asymp \log |G| \asymp d(G)$ with $1 \ll k \lesssim \log |G|$ and $d(G) \ll \log |G|$. \hfill \(\triangle\)

### 3.7 Typical Distances for $L_q$-Type Graph Distances

Graph distances in Cayley graphs have some special properties. Consider a collection $z = \{z_1, \ldots, z_k\}$ of generators and distances in the Cayley graph $G(z)$. For a path $\rho$ in $G(z)$, for each $i \in [k]$, write $\rho_{i,+}$ for the number of times $z_i$ is used, $\rho_{i,-}$ for the number of times $z_i^{-1}$ is used (if in the undirected case, otherwise $\rho_{i,-} = 0$) and $\rho_i := \rho_{i,+} - \rho_{i,-}$. The path connects the identity with $\rho \cdot z$. Then the length, in the usual graph distance, of $\rho$ is $||\rho||_1 := \sum_i (\rho_{i,+} + \rho_{i,-})$.

For any $q \in [1, \infty)$, define the $L_q$ graph distance of $\rho$ by $||\rho||_q := \left(\sum_i |\rho_{i,+}|^q + |\rho_{i,-}|^q\right)^{1/q}$. For the $L_\infty$ graph distance, define $||\rho||_\infty := \max_i (\rho_{i,+} + \rho_{i,-})$. (The usual graph distance is given by $q = 1$.)

For Abelian groups, clearly for any $q \in [1, \infty)$ an $L_q$ geodesic, is a path of minimal $L_q$ weight, will only use either $z_i$ or $z_i^{-1}$, not both (since the terms in the product can be reordered), i.e $\rho_{i,+}\rho_{i,-} = 0$ for all $i$. Thus $||\rho||_q^q = \sum_i |\rho_{i,\pm}|^q$. Similarly, any $L_\infty$ geodesic $\rho$ can be adjusted into a new path $\rho'$ with $\rho \cdot z = \rho' \cdot z$ and $||\rho'||_\infty = ||\rho||_\infty$ satisfying $\rho_{i,+}^\prime + \rho_{i,-}^\prime = 0$ for all $i$.

We define the $L_q$ typical distance $D_{G(z),q}$(⋅) analogously to $D_{G(z)}$(⋅), i.e $q = 1$ case.

**Hypothesis B'.** The sequence $(k_N, G_N)_{N \in \mathbb{N}}$ and $q \in [1, \infty]$ jointly satisfy Hypothesis B' if the following conditions hold (defining $k^{1/\infty} := 1$ for $k \in \mathbb{N}$):

\[
\lim_{N \to \infty} k_N = \infty, \quad \lim_{N \to \infty} k_N/\log |G_N| = 0 \quad \text{and} \quad \lim_{N \to \infty} k^{1/q}_N |G_N|^{1/k_N} / m_*(G_N) = 0;
\]

if $q \in (1, \infty)$ then additionally

\[
\frac{\log |G_N|}{\limsup_{N \to \infty} \log k_N} \leq \frac{1}{2} \quad \text{for undirected graphs},
\]

\[
\frac{1}{2} \quad \text{for directed graphs}.
\]

Finally we set up a little more notation. Make the following conditions hold:

\[
C_q : = 2\Gamma(1/q + 1)(qc)^{1/q}, \quad C_q^+ := \frac{1}{2} C_q^-, \quad \mathcal{O}_q^\pm(k, n) := k^{1/q} n^{1/k} / C_q^+,
\]

where the case $q = \infty$ is to be interpreted as the limit $q \to \infty$; e.g., $C_\infty = 2$ and $\mathcal{O}_\infty^+(k, n) = n^{1/k}$.

When these are sequences $(k_N, |G_N|)_{N \in \mathbb{N}}$, for $N \in \mathbb{N}$ and $q \in [1, \infty]$, write $D_{G_N}^\pm := \mathcal{D}_{G_N}^\pm((k_N, |G_N|))$.

Similarly, for a sequence $(G_N)_{N \in \mathbb{N}}$ of finite groups with corresponding multisubsets $(Z_N)_{N \in \mathbb{N}}$ of sizes $(k_N)_{N \in \mathbb{N}}$, for $N \in \mathbb{N}$, $\beta \in [0, 1]$ and $q \in [1, \infty]$, define $D_{G_N}^\pm := \mathcal{D}_{G_N}^\pm((Z_N)_{N \in \mathbb{N}})\beta$.

Using an extension of the methodology from this section (§3), along with analysis of $L_q$ lattice balls, we can prove the following theorem. We have already considered $q = 1$ and $k \asymp \log |G|$.

**Theorem 3.11.** Let $(k_N)_{N \in \mathbb{N}}$ be a sequence of positive integers and $(G_N)_{N \in \mathbb{N}}$ a sequence of finite, Abelian groups; for each $N \in \mathbb{N}$, define $Z_N := \{Z_1, \ldots, Z_{k_N}\}$ by drawing $Z_1, \ldots, Z_{k_N} \sim \text{Unif}(G_N)$.

Suppose that $(k_N, G_N)_{N \in \mathbb{N}}$ satisfies Hypothesis B'. Then, for all $\beta \in (0, 1)$, we have

\[
\mathcal{D}_{G_N}^\pm(\beta) / \mathcal{D}_{G_N}^\pm \to^{\text{P}} 1 \quad (\text{in probability}) \quad \text{as } N \to \infty.
\]

Moreover, the implicit lower bound holds for all choices of generators and for all Abelian groups, only requiring the conditions in Hypothesis B' which depend only on $(k_N, |G_N|)_{N \in \mathbb{N}}$ and $q$.

The arguments used to prove this theorem really are analogous to those used in this section (§3). The only real difference is that we have to look at lattice balls under an $L_q$ and in dimension $1 \ll k \ll \log n$, rather than $L_1$ and $k \asymp \log n$. Other than this, the remainder of the analysis, in particular the reduction to a gcd and the consideration of the set $\mathcal{I}$ of non-zero coordinates of $W$, is exactly the same. (Now $W$ is uniform on an $L_q$ ball of appropriate radius.) We do not give the details here; they can be found in [13, §7].

\[19\]
4 Typical Distance: $k \gg \log |G|$

This section focusses on concentration of distances from the identity in the random Cayley graph of an Abelian group when $k \gg \log |G|$. (The previous sections dealt with $1 \ll k \ll \log |G|$.)

The main result of the section is Theorem 4.2.

The outline of this section is as follows:
- §4.1 states precisely the main theorem of the section;
- §4.2 outlines the argument;
- §4.3 gives some crucial estimates on the size of lattice balls;
- §4.4 is devoted to the lower bound;
- §4.5 is devoted to the upper bound.

4.1 Precise Statement and Remarks

To start the section, we recall the typical distance statistic.

**Definition 4.1.** Let $H$ be a graph and fix a vertex $0 \in H$. For $r \in \mathbb{N}$, write $B_H(r)$ for the $r$-ball in the graph $H$, ie $B_H(r) := \{ h \in H \mid d_H(0, h) \leq r \}$, where $d_H$ is the graph distance in $H$. Define

$$D_H(\beta) := \min\{ r \geq 0 \mid |B_H(r)| \geq \beta |H| \} \quad \text{for} \quad \beta \in (0, 1).$$

When considering sequences $(k_N, G_N)_{N \in \mathbb{N}}$ of integers and Abelian groups, abbreviate

$$D_N(\beta) := D_{G_N([Z_1, ..., Z_N])}(\beta) \quad \text{where} \quad Z_1, ..., Z_{k_N} \sim_{\text{id}} \text{Unif}(G_N).$$

Finally, considering such sequences, we define the candidate radius for the typical distance:

$$\overline{\rho}_N := \frac{\log |G_N|}{\rho_N} \log k_N \quad \text{where} \quad \rho_N := \log k_N / \log \log |G_N| \quad \text{for each} \quad N \in \mathbb{N}.$$

As always, if we write $D_N$, then this is either $D_N^+$ or $D_N^-$ according to context. Up to subleading order, the typical distance will be the same for the undirected graphs as for the directed graphs.

We show that, whp over the graph (ie choice of $Z$), this statistic concentrates. Here we consider $k \gg \log |G|$. The result holds for all Abelian groups; in fact, the implicit upper bound is valid for all groups. Further, the typical distance concentrates at a distances which depends only on $k$ and $|G|$. This is in agreement with the spirit of the Aldous–Diaconis conjecture.

**Hypothesis C.** The sequence $(k_N, n_N)_{N \in \mathbb{N}}$ satisfies Hypothesis C if

$$\liminf_{N \to \infty} \frac{k_N}{\log n_N} = \infty \quad \text{and} \quad \liminf_{N \to \infty} \frac{\log k_N}{\log n_N} = 0.$$

**Theorem 4.2.** Let $(k_N)_{N \in \mathbb{N}}$ be a sequence of positive integers and $(G_N)_{N \in \mathbb{N}}$ a sequence of finite, Abelian groups; for each $N \in \mathbb{N}$, define $Z_{(N)} := [Z_1, ..., Z_{k_N}]$ by drawing $Z_1, ..., Z_{k_N} \sim_{\text{id}} \text{Unif}(G_N)$.

Suppose that $(k_N, |G_N|)_{N \in \mathbb{N}}$ satisfies Hypothesis C. Then, for all $\beta \in (0, 1)$, we have

$$D_N^+(\beta)/\overline{\rho}_N \to^p 1 \quad \text{(in probability)} \quad \text{as} \quad N \to \infty.$$

Moreover, the implicit lower bound holds deterministically, ie for all choices of generators, and the implicit upper bound holds for all groups, not just Abelian groups.

As always, for ease of presentation, in the proof we drop the $N$-subscripts.

4.2 Outline of Proof

When $k \gg \log |G|$, one can see that the typical distance statistic $D$ must satisfy $D \ll k$. By symmetry, the expected number of times a generator is used when drawing from a ball $B_k(R)$ is $o(1)$. The number of ways that precisely $R$ can be chosen is $\binom{k}{R}$. We choose $R$ with $\binom{k}{R} \approx |G|$. 


4.3 Estimates on Sizes of Balls in \( \mathbb{Z}^k \)

We consider balls and spheres in the \( L_1 \) and \( L_\infty \) senses: write \( B_{k,1}(\cdot) \), respectively \( S_{k,1}(\cdot) \), for the \( L_1 \) ball, respectively sphere, in \( \mathbb{Z}^k \); write \( B_{k,\infty}(1) \) for the \( L_\infty \) unit ball in \( \mathbb{Z}^k \).

**Lemma 4.3.** For all \( R \geq 0 \), we have
\[
|B_{k,1}^+(R)| \leq 2^R \binom{|R|+k}{|R|} \quad \text{and} \quad |S_{k,1}^+(R) \cap B_{k,\infty}^+(1)| \geq \binom{k}{|R|}.
\]
Furthermore, if \( R \ll k \), then
\[
2^R \binom{|R|+k}{|R|} = \exp(R \log(k/R) \cdot (1 + o(1))) = \binom{k}{|R|}.
\]
In particular, if \( k = (\log n)^\rho \gg \log n \) and \( \varepsilon > 0 \) is a constant, then
\[
|S_{k,1}^+(1 + \varepsilon) \frac{\log k}{\rho} \cap B_{k,\infty}^+(1)| \gg n.
\]

**Proof.** In the first display, the upper bound is proved in [15, Lemma E.2a]; the lower bound is the usual formula for the number of subsets of \(|k|\) of size \( R \). The second display is a simple application of Stirling’s approximation and asymptotics of the binary entropy function. The final display follows by combining the previous two and performing a simple calculation. \( \square \)

4.4 Lower Bound on Typical Distance

From the results in §4.3, it is straightforward to deduce the lower bound in Theorem 4.2.

**Proof of Lower Bound in Theorem 4.2.** Let \( \xi \in (0,1) \) and set \( R \coloneqq \overline{\mathbb{D}}(1-\xi) \). Since the underlying group is Abelian, applying Lemma 4.3, a simple calculation gives
\[
|B_{k}(R)| \leq |B_{k,1}(R)| \leq \exp(\mathfrak{D} \log(k/\mathfrak{D}) \cdot (1 - \frac{\varepsilon}{2})) \ll n.
\]
Hence, for all \( \beta \in (0,1) \) and all \( Z \), we have \( \mathcal{D}_k(\beta) \geq R = \overline{\mathbb{D}}(1-\xi) \), asymptotically in \( n \). \( \square \)

4.5 Upper Bound on Typical Distance

Lemma 4.3 gives a quantitative sense in which \( |B_{k,1}(R)| \approx |S_{k,1}(R) \cap B_{k,\infty}(1)| \geq \binom{k}{|R|} \); informally, this means that we do not really lose any volume by restricting to the sphere and requiring that each generator is used at most once. We show the upper bound for arbitrary groups.

**Proof of Upper Bound in Theorem 4.2.** Let \( \xi > 0 \) and set \( R \coloneqq \overline{\mathbb{D}}(1+\xi) \). Draw \( W, W' \sim_{\text{iid}} \text{Unif}(S_{k,1}(R) \cap B_{k,\infty}(1)) \). Define \( S \coloneqq \overline{Z}_{W_1} \overline{Z}_{W_2} \ldots \overline{Z}_{W_k} \) and \( S' \) similarly. We show that \( S \) is well-mixed whp (this time in the \( L_2 \) sense) to deduce the upper bound. By the standard \( L_2 \) calculation,
\[
\mathbb{E}(\|\mathbb{P}_{\mathbb{G}_k}(S \in \cdot) - \pi_G\|_2^2) = n \mathbb{P}(S' = S') - 1.
\]
If \( W \neq W' \), then there exists an \( i \in [k] \) so that \( W_i = 1 \) and \( W'_i = 0 \) or vice versa. By the uniformity and independence of the generators, \( S' S^{-1} \sim \text{Unif}(G) \) for all (not just Abelian) groups. Thus
\[
n \mathbb{P}(S = S') - 1 \leq n \mathbb{P}(W = W') = n |S_{k,1}(R) \cap B_{k,\infty}(1)|^{-1} \ll 1,
\]
using Lemma 4.3 for the final relation. This completes the proof. \( \square \)

**Remark.** This upper bound, in typical distance with \( k \gg \log |G| \), can be easily deduced from mixing results proved in the ’90s. Specifically, it was shown by Dou and Hildebrand [10, Theorem 1] that the mixing time for the usual random walk is upper bounded by \( \frac{1}{\rho+1} \log_k |G| \) for any group; Roichman [28, Theorems 1 and 2] subsequently gave a simpler proof, using an argument not that dissimilar from our proof above. The lower bound does not follow from mixing results, though.

There are a few reasons for including the proof above. Foremost is that we use the same argument in §5.2 to obtain universal bounds for \( k \) with \( k - \log_2 |G| \approx k \), not just \( k \gg \log |G| \). Additionally, we need to do most of the work for the lower bound anyway, and it demonstrates how easily our method adapts to this new regime. \( \triangle \)
5 Diameter

In this section we consider the diameter of the random Cayley graph. Our analysis is separated into two distinct sections.

§5.1 We show that the diameter concentrates for $k \gtrsim \log |G|$, and that the value at which it concentrates is the same as for typical distance.

§5.2 We show, for $k$ with $k - \log_2 |G| \asymp k$, that the group giving rise to the largest diameter (amongst all groups) is $\mathbb{Z}_2^d$.

5.1 Concentration for $k \gtrsim \log |G|$ 

Recall that in Theorem 3.2 we showed, in the regime $k \approx \log n$ and under some assumptions, that, up to subleading order terms, the typical distance concentrates at $\alpha k$, for some constant $\alpha$. The next theorem shows, in the same set-up, that the diameter does the same. The argument uses the typical distance result as a ‘black box’, then extending from this to diameter.

Theorem 5.1. Let $(k_N)_{N \in \mathbb{N}}$ be a sequence of positive integers and $(G_N)_{N \in \mathbb{N}}$ a sequence of finite, Abelian groups; for each $N \in \mathbb{N}$, define $Z_{(N)} := \{Z_1, \ldots, Z_k\}$ by drawing $Z_1, \ldots, Z_k \sim_{i.d.} \text{Unif}(G_N)$.

Suppose that $(k_N, G_N)_{N \in \mathbb{N}}$ satisfies either Hypotheses B or C. For $\lambda \in (0, \infty)$, let $\alpha^\pm_\lambda \in (0, \infty)$ be the constant from Theorem 3.2; for each $N \in \mathbb{N}$, write $p_N := \log k_N / \log \log |G_N|$, so that $k_N = (\log |G_N|)^{p_N}$. Then the following convergences in probability hold:

\[
\text{diam} G_N (Z_{(N)}) / (\alpha^{\pm}_\lambda k_N) \rightarrow P 1 \quad \text{when} \quad \lim k_N / \log |G_N| = \lambda \in (0, \infty);
\]

\[
\text{diam} G_N (Z_{(N)}) / (\frac{p_N}{\log k_N} |G_N|) \rightarrow P 1 \quad \text{when} \quad \lim k_N / \log |G_N| = \infty.
\]

Moreover, the implicit lower bound on the diameter holds deterministically, i.e. for all choices of generators, and for all Abelian groups, and, when $k \gg \log |G|$, the implicit upper bound holds for all groups, not just Abelian groups.

Remark. While we only state and prove the result for $k \gtrsim \log |G|$, the argument can be extended to allow $k \ll \log |G|$, provided $\log |G|/k$ diverges sufficiently slow. This requires a little more care; we do not explore the details here.

As always, we drop the $N$-subscripts in the proof, eg writing $\text{diam} G_k$ or $|G|$.

Proof of Theorem 5.1. Clearly $\text{diam} G_k = D_k(1) \geq D_k(\beta)$ for all $\beta \in [0, 1]$. Hence typical distance is trivially a lower bound on the diameter. It remains to consider the upper bound.

Assume first Hypothesis B, so $k \approx \lambda \log |G|$ for some $\lambda \in (0, \infty)$. Let $\epsilon \ll 1$, vanishing slowly and specified later. Define $\alpha := \alpha^\pm_\lambda$ as in Theorem 3.2. Let $A := \{Z_1, \ldots, Z_{(1-\epsilon)k}\}$ be the first $(1-\epsilon)k$ generators and $B := \{Z_{(1-\epsilon)k+1}, \ldots, Z_k\}$ be the remaining $\epsilon k$. By transitivity, it suffices to consider distances from the identity. The idea is to take $L$ steps using $A$ and then one more using $B$, where $L$ is the minimal radius of a ball in the $|A|$-dimensional lattice of volume at least $n e^\omega$, for some slowly diverging $\omega$. Write $M := \alpha k$. By Lemma 3.5, we have $L/M \approx 1 - \epsilon \approx 1$. The key point is that when $k \approx \log |G|$ replacing $k$ with $(1-\epsilon)k$ changes the typical distance by a factor $1 + o(1)$.

By Theorem 3.2, whp, $A$ is typical in the sense that the proportion of elements of the group which can be reached via a word of length at most $L$, using only the generators from $A$, is at least $1 - e^{-\nu}$, for some $\nu \gg 1$, independent of $\epsilon$.

Condition on $A$, and that it is typical; write $\mathbb{P}$ for the probability measure induced by this conditioning. Denote by $H$ the set of elements which can be reached in the above sense. (This is the vertex set of the ball of radius $L$ in $G(A)$.) Fix $x \in G$. Note that if $b \sim \text{Unif}(G)$, then

\[
\mathbb{P} (x \in b + H) \geq 1 - e^{-\nu} \quad \text{where} \quad b + H := \{b + h \mid h \in H\}.
\]

Furthermore, if $b, b' \sim \text{Unif}(G)$ are independent then the events $\{x \in b + H\}$ and $\{x \in b' + H\}$ are $\mathbb{P}$-independent; this is because we have conditioned on $A$, and so $H$ is a deterministic set.

Using the $\epsilon k$ generators from $B$, informally we get $\epsilon k$ Bernoulli trials to get to $x$ using $b + H$ for $b \in B$, and each trial has success probability $1 - o(1)$. Formally, write $R$ for the set of elements
reducible from the identity via a word of length at most \(L+1\) (ie the ‘range’); let \(b'\) be an arbitrary element of \(B\), so \(b' \sim \text{Unif}(G)\). (Recall that the conditioning makes \(H\) non-random.) Then
\[
\Pr(x \notin R) \leq \Pr(x \notin b+H) = \Pr(x \notin b+H \forall b \in B) = \Pr(x \notin b'+H)|B| \leq e^{-\nu \epsilon k}.
\]
Since \(\nu \to \infty\), we may choose \(\epsilon \to 0\) so that \(\nu \epsilon \to \infty\). Then, since \(k \sim \log n\), we have
\[
\Pr(R \neq G) = \Pr(\exists x \in G \text{ st } x \notin R) \leq n \Pr(x \notin R) \leq ne^{-\nu \epsilon k} = o(1).
\]
Averaging over \(A\) establishes an upper bound of \(\text{diam } G_k \leq L + 1\) whp, and \(L \leq M(1+\epsilon)\).

Finally consider Hypothesis C, so \(k \gg \log |G|\). Exactly the same argument holds here, using the typical distance to first get to almost all the elements and then one more step. Recall from Theorem 4.2 that the upper bound is valid for arbitrary groups.

### 5.2 Universal Bounds for \(k - \log_2 |G| \asymp k\)

In this subsection we show that the group \(Z_2^d\) gives rise to the random Cayley graph with the largest diameter when \(k - \log_2 |G| \asymp k\) whp, up to smaller order terms.

Recall that \(\mathcal{R}(k,n)\) is the minimal \(R \in \mathbb{N}\) with \((\binom{k}{R}) \geq n\).

**Theorem 5.2.** Let \((k_N)_{N \in \mathbb{N}}\) be a sequence of positive integers and \((G_N)_{N \in \mathbb{N}}\) a sequence of finite groups; for each \(N \in \mathbb{N}\), define \(Z_{(N)} := [Z_1, \ldots, Z_{k_N}]\) by drawing \(Z_1, \ldots, Z_{k_N} \sim \text{Unif}(G_N)\).

Suppose that \(\liminf_N (k_N - \log_2 |G_N|)/k_N > 0\) and \(\limsup_N \log k_N/\log |G_N| = 0\). Then
\[
\limsup_{N \to \infty} \text{diam } G_N(Z_{(N)})/\mathcal{R}(k_N, |G_N|) \leq 1 \quad \text{in probability.}
\]

**Proof.** From Lemma 4.3 and Theorem 5.1, when \(k \gg \log |G|\), the diameter concentrates at \(\mathcal{R}(k,|G|)\) when the underlying group is Abelian, and this is an upper bound for all groups.

Thus it remains to consider \(k\) with \(k - \log_2 |G| \asymp k\) and \(k \asymp \log |G|\). All that was required for the upper bound on typical distance when \(k \gg \log |G|\) was that \(\Pr(W = W') \ll 1/|G|\) where \(W, W' \sim \text{Unif}(S_{k,1}(D) \cap B_{k,\infty}(1))\) with \(D \equiv \overline{\mathcal{F}}(1+\xi)\), where \(\overline{\mathcal{F}}\) was the candidate typical distance radius and \(\xi > 0\) was a constant. We show that the analogous statement holds here.

Let \(\xi > 0\) be fixed and set \(R := \mathcal{R}(k,|G|)(1+\xi)\). Before proceeding, let us determine some estimates on \(\mathcal{R}\). Let \(h : (0,1) \to (0,1) : p \mapsto -p \log p - (1-p) \log(1-p)\) denote the binary entropy function (in nats). It is standard that Stirling’s approximation, like in Lemma 4.3, gives
\[
\binom{k}{R} = \exp(k h(r/k \cdot (1+o(1)))).
\]
Thus if \(k - \log_2 |G| \asymp k\), then we see that \(\mathcal{R}(k,|G|) \asymp k\). Further, the fact that the derivative of \(h\) is continuous and strictly positive on \((0,\frac{1}{2})\) gives \((\binom{k}{R}) \gg |G|\); hence \(P(W = W') \ll 1/|G|\).

This shows that the typical distance \(D_k(\beta) \leq \mathcal{R}(k,|G|)\) whp up to smaller order terms for all constants \(\beta \in (0,1)\). This is then converted from a statement about typical distance to one about the diameter via the same method as used previously (in §5.1), noting that \(\mathcal{R}(k,|G|) \asymp k\).

### 6 Spectral Gap

In this section, we calculate the spectral gap; see Theorem D. We first prove it for \(k \geq 3d(G)\).

In §6.4, we explain how to extend to \(k - d(G) \asymp k\) and then to \(k - d(G) \asymp k\) for a density-(1-\(\varepsilon\)) subset of values for \(|G|\). The lower bound holds deterministically, without any conditions.

#### 6.1 Precise Statement

For an Abelian group \(G\), we write \(d(G)\) for the minimal size of a generating set. It is convenient to phrase the statement in terms of the relaxation time, which is the inverse of the spectral gap.
\textbf{Theorem 6.1 (Spectral Gap).} First, there exists an absolute constant $c > 0$ so that, for all Abelian groups $G$ and all multisets $z$ of generators of size $k$, we have
\begin{equation}
t_{\text{rel}}(G^{-}(z)) \geq t_{\text{rel}}(G^{-}(z)) \geq c|G|^{2/k}. \tag{6.1a}
\end{equation}

Second, for all $\delta > 0$, there exist constants $c_{\delta}, C_{\delta} > 0$ so that, for all Abelian groups $G$, if $k \geq (2 + \delta)d(G)$ and $Z_{1}, ..., Z_{k} \sim \text{id Unif}(G)$, then
\begin{equation}
\mathbb{P}(t_{\text{rel}}^{*}(G_{k}^{-}) \leq C_{\delta}|G|^{2/k}) \geq 1 - C_{\delta}2^{-k/c_{\delta}}. \tag{6.1b}
\end{equation}

Furthermore, for all $\varepsilon \in (0, 1)$, there exists a subset $\mathcal{A} \subseteq \mathbb{N}$ of density at least $1 - \varepsilon$ so that if $|G| \in \mathcal{A}$ then, then condition $k \geq (2 + \delta)d(G)$ can be relaxed to $k \geq (1 + \delta)d(G)$ and (6.1b) still holds; the constant $C_{\delta}$ now also depends on $\varepsilon$, i.e., becomes $C_{\delta, \varepsilon}$, but $c_{\delta}$ need not be adjusted.

We prove this for the non-absolute spectral gap, i.e., $\min_{\lambda \neq 1}|1 - \lambda|$, where the minimum is over eigenvalues; the same proof also works for the absolute spectral gap, i.e., $\min_{\lambda \neq 1}|1 - |\lambda||$.

### 6.2 Lower Bound on Relaxation Time

In this subsection, we establish the lower bound in Theorem 6.1.

\textbf{Proof of Lower Bound in Theorem 6.1.} Write $n := |G|$. Abbreviate simple random walk by SRW. We may assume that $k \leq \log_{3}(\frac{1}{2}n)$, as otherwise (6.1a) indeed holds for some $c > 0$. Let $L := \lfloor \frac{1}{2}(\frac{1}{2}n)^{1/k} - 1 \rfloor$. By our assumption on $k$, we have $L \geq 1$. Consider the set
\begin{equation}
A := \{w \cdot Z \mid w \in \mathbb{Z}^{k} \text{ and } |w_{i}| \leq L \forall i = 1, ..., k \} \subseteq G. \tag{6.2}
\end{equation}

Clearly $|A| \leq (2L + 1)^{k} \leq \frac{1}{2}n$. Let $t \geq 0$, and let $(Y_{s})_{s \geq 0}$ be a continuous-time rate-1 SRW on $\mathbb{Z}$. Writing \(\tau_{A^{c}} := \inf\{s \geq 0 \mid S_{s} \not\in A\}\) for the exit time of $A$ by the SRW $S$, observe that
\begin{equation}
\mathbb{P}_{0}(\tau_{A^{c}} > t) \leq \mathbb{P}_{0}(\max_{s \in [0, t/k]} |Y_{s}| \leq L)^{k}, \tag{6.3}
\end{equation}

where $0 \in A$ is the identity of the group. It follows from Lemma 6.3 below that
\begin{equation}
\mathbb{P}_{0}(\max_{s \in [0, t/k]} |Y_{s}| \leq L) \geq \exp\left(-\frac{k}{2}\pi^{2}t^{2}/(L + 1)^{2}\right). \tag{6.4}
\end{equation}

Substituting this into (6.3) we get
\begin{equation}
\mathbb{P}_{0}(\tau_{A^{c}} > t) \geq \exp\left(-\frac{k}{2}\pi^{2}/(L + 1)^{2}\right). \tag{6.4}
\end{equation}

The minimal Dirichlet eigenvalue of a set $A$ is defined to be the minimal eigenvalue of minus the generator of the walk killed upon exiting $A$; we denote it by $\lambda_{A}$. For connected $A$, we show in Lemma 6.4 below that, for all $a \in A$, we have
\begin{equation}
-\frac{1}{t}\log\mathbb{P}_{a}(\tau_{A^{c}} > t) \to \lambda_{A} \quad \text{as } t \to \infty. \tag{6.4}
\end{equation}

From this and (6.4), it then follows that $\lambda_{A} \leq \lambda$ where
\[\lambda := \frac{1}{k}\pi^{2}/(L + 1)^{2} \leq \frac{\pi^{2}/((\frac{1}{2}n)^{1/k} + 1)^{2}}{2}.
\]

Since $|A| \leq \frac{1}{2}n$, applying [3, Corollary 3.34], we get
\begin{equation}
t_{\text{rel}} \geq (1 - \frac{1}{n}|A|)/\lambda \geq 1/(2\lambda). \tag{6.4}
\end{equation}

This concludes the proof of the lower bound in Theorem 6.1, namely (6.1a). \(\square\)
6.3 Upper Bound on Relaxation Time

In this subsection, we establish the upper bound in Theorem 6.1, namely (6.1b). We prove it for the usual spectral gap $t_{rel}$; the same proof applies to bound the absolute spectral gap $t^*_{rel}$. In particular, we bound the probability that $1 - \lambda_2$ is small; a completely analogous calculation can be used to bound the probability that $1 + \lambda_n$ is small. We only present the former calculation.

For ease of presentation, we assume first that $k \geq 3d(G)$. In §6.4, we explain how to relax this condition, to prove the complete theorem.

Proof of Upper Bound in Theorem 6.1. Decompose $G$ as $\bigoplus_d \mathbb{Z}_{m_d}$. An orthogonal basis of eigenvectors for $P$, the transition matrix of the corresponding discrete-time walk, is given by

$$(f_x \mid x \in G) \quad \text{where} \quad f_x(y) := \cos(2\pi \sum_{i=1}^d x_i y_i / m_i),$$

with corresponding eigenvalues given by

$$(\lambda_x \mid x \in G) \quad \text{where} \quad \lambda_x = \frac{1}{k} \sum_{i=1}^k \cos(2\pi (\bar{x} \cdot Z_i)),$$

where $\bar{x}_j = x_j / m_j$ for all $j = 1, \ldots, d$ and $\bar{x} \cdot Z_i = \sum_{j=1}^d x_j Z_{ij} / m_j$

is the standard inner-product on $\mathbb{R}^d$, where $Z_{ij}$ is the $j$-th coordinate of the $i$-th generator $Z_i$; here we identify $\bar{x}$ and $Z_i$ with elements of $\mathbb{R}^d$ in a natural manner.

Observe that $\lambda_0 = 1$. Our goal is to bound $\min_{x \in G \setminus \{0\}} \{1 - \lambda_x\}$ from below. For $\alpha \in \mathbb{R}$, let $\{\alpha\}$ be the unique number in $(-\frac{1}{2}, \frac{1}{2})$ so that $\alpha - \{\alpha\} \in \mathbb{Z}$. It follows from Lemma 6.5 below that

$$1 - \lambda_2 \geq \frac{2\pi^2}{\lambda^2} \sum_{i=1}^k \{\bar{x} \cdot Z_i\}^2. \quad (6.5)$$

For each $x \in G$, we make the following definitions:

$$g_j := g_j(x) := \gcd(x_j, m_j) \quad \text{for each} \ j \geq 1;$$

$$s_* := s_*(x) := \max\{m_j / g_j \mid j \in \{1, \ldots, d\}\};$$

$$A(s) := \{x \in G \mid s_*(x) = s\} \quad \text{for each} \ s \geq 1;$$

$$\phi(j) := |\{j' \in \{1, \ldots, j\} \mid \gcd(j, j') = 1\}| \quad \text{for each} \ j \geq 1.$$

From this, we claim that we are able to deduce, for $s \geq 2$, that

$$|A(s)| \leq (\sum_{j=1}^s \phi(j))^d \leq (1 + \sum_{j=2}^s (j - 1))^d \leq (\frac{1}{2} s^2)^d. \quad (6.6)$$

Indeed, $\phi(j) \leq j - 1$ for $j \geq 2$, and observe that

$$|\{a \in \{1, \ldots, m\} \mid \gcd(a, m) = r\}| = \phi(m / r);$$

hence, summing over the set of possible values for $m_j / g_j$, which by definition of $A(s)$ is $\{1, \ldots, s\}$, we have $|A(s)|^{1/d} \leq \sum_{j=1}^s \phi(j)$. We are then able to deduce the upper bound, i.e. (6.1b), from Proposition 6.2, which we state precisely below. Indeed, first write

$$p(s) := \max_{x : s_*(x) = s} \mathbb{P}(1 - \lambda_x \leq c_1 n^{-2/k}).$$

By (6.5) along with Proposition 6.2 and Lemma 6.5 (stated below), letting $c' := c_1 \cdot \frac{1}{2^6}$, we have

$$\sum_{x \in G \setminus \{0\}} \mathbb{P}(1 - \lambda_x \leq c_1 n^{-2/k}) \leq n \max_{s_2 \geq C_2 n^{1/k}} p(s) + \sum_{2 \leq s \leq C_2 n^{1/k}} |A(s)| p(s) \leq 2^{-k} + 2^{-4} \sum_{2 \leq s \geq 2} s^{2d}(2s)^{-9k/10} \lesssim 2^{-k};$$

where we have used $k \geq 3d$ and the fact that $s_*(x) > 1$ for all $x \neq 0$.

Modulo the proofs of the quoted results, i.e. Proposition 6.2 and Lemmas 6.3 to 6.5, this concludes the proof of the upper bound in Theorem 6.1, namely (6.1b).

It remains to state and prove the quoted results, i.e Proposition 6.2 and Lemmas 6.3 to 6.5.
Proposition 6.2. There exist absolute constants $c_1 \in (0, 1)$ and $C_2$ such that
\[
\mathbb{P}\left(\frac{1}{k} \sum_{i=1}^{k} (\bar{x} \cdot Z_i)^2 \leq c_1n^{-2/k}\right) \leq \begin{cases} 
s_*(x)^{-9k/10} & \text{where } s_*(x) \leq C_2n^{1/k}, \\
2^{-k/n} & \text{where } s_*(x) > C_2n^{1/k}.
\end{cases}
\] (6.7a)

Proof. Fix $x \in G$. First consider the case that $s_*(x) > C_2n^{1/k}$, ie (6.7b). Let $j := j(x)$ be a coordinate satisfying $s = m_j/g$. Denote $m := m_j(x)$ and $g := g_j(x)$. Observe that $x_jZ_i \sim \text{Unif}\{g, 2g, \ldots, m\}$ for each $i$. For each $i$, we have
\[
U_i := \bar{x}_iZ_i \sim \text{Unif}\{1/s, 2/s, \ldots, 1\}.
\] (6.8)

By averaging over $(a_i)_{i=1}^k$, where $a_i := \{\sum_{\ell \in \{1, \ldots, s\} \setminus j} x_i Z_{i(\ell)} / m_{\ell}\}$, recalling that $\{a\}$ is the unique number in $\{-\frac{1}{2}, \frac{1}{2}\}$ so that $a - \{a\} \in \mathbb{Z}$, it suffices to show that
\[
\max_{b_1, \ldots, b_k \in [-1/2, 1/2]} \mathbb{P}\left(\frac{1}{k} \sum_{i=1}^{k} (U_i + b_i)^2 \leq c_1n^{-2/k}\right) \leq 2^{-k/n}.
\] (6.9)

Replacing $c_1$ with $4c_1$ we may assume that $b_i \in \frac{1}{k}\mathbb{Z}$. Indeed, if
\[
|b_i - \ell/s| \leq 1/(2s), \quad \text{ie } |b_i - \ell/s| = \min\{|b_i - a| : a \in \{\frac{1}{k}(n-1)\}\},
\]
then $\{U_i + \ell/s\}^2 \leq 4\{U_i + b_i\}^2$. Hence
\[
\text{if } \frac{1}{k} \sum_{i=1}^{k} \{U_i + b_i\}^2 \leq c_1n^{-2/k}\quad \text{then } \frac{1}{k} \sum_{i=1}^{k} \{U_i + \ell/s\}^2 \leq 4c_1n^{-2/k}.
\]
In this case, $\{U_i + b_i\}$ has the same law as $\{U_i\}$. Hence it suffices to prove (6.9) for $b_1 = \cdots = b_k = 0$.

We now split $[0, 1/k]$ into $M := \lfloor 4n^{1/k} \rfloor$ consecutive intervals of equal length $J_1, \ldots, J_M$, where $J_1 := [0, \frac{1}{kM})$ and $J_t := (\frac{t}{kM}, \frac{t+1}{kM})$ for $t > 1$. Let $Y_t := \ell - 1$ if $\{U_i\} \in J_t$. Clearly, $\frac{1}{k}Y_t/M^2 \leq \frac{1}{k}Y_t^2/M^2 \leq \{U_i\}$.

This last claim follows by a simple counting argument: there are $M^k$ total assignments of the $Y_t$-s, but at most $L(k) := \left(\frac{11k}{10}\right)^{11k/10}$ of them satisfy $\frac{1}{k} \sum_{i=1}^{k} Y_i \leq \frac{1}{M}$, since $L(k)/M^k \leq 2^{-k/n}$.

We now prove the case $s_*(x) \leq C_2n^{1/k}$, ie (6.7a). By the same reasoning as for (6.9), it suffices to show that
\[
\max_{b_1, \ldots, b_k \in [-1/2, 1/2]} \mathbb{P}\left(\frac{1}{k} \sum_{i=1}^{k} (U_i + b_i)^2 \leq c_1n^{-2/k}\right) \leq s^{-9k/10}.
\] (6.10)

Regardless of $b_i$, there is at most one $a := a(b_i) \in \{1/s, 2/s, \ldots, 1\}$ such that $\{a + b_i\}^2 < (2s)^{-2}$, and hence by (6.8), for all $i$, we have
\[
\mathbb{P}\left(\{U_i + b_i\}^2 < (2s)^{-2}\right) \leq 1/s.
\]
If there is no such value $a(b_i)$, then set $a(b_i) := -1$.

If $\{U_i + b_i\}^2 \geq (2s)^{-2}$ for at least $q := k \cdot 4c_1s^2n^{-2/k}$ of the $i$-s, ie if
\[
\left|\{i \in \{1, \ldots, k\} : U_i \neq a(b_i)\}\right| \geq q,
\]
then $\frac{1}{k} \sum_{i=1}^{k} \{U_i + b_i\}^2 \geq c_1n^{-2/k}$, as desired. As $s \leq C_2n^{1/k}$, by taking $c_1$ sufficiently small in terms of $C_2$, we can make $q/k$ sufficiently small so that the following holds:
\[
\mathbb{P}\left(\{i \in \{1, \ldots, k\} : U_i \neq a(b_i)\}\right) < q \lesssim \left(\frac{k}{q}\right)s^q \lesssim s^{-9k/10}.
\]

We now state the auxiliary lemmas referenced above, ie Lemmas 6.3 to 6.5. These are technical results; their proofs are given in [15, §D].

Lemma 6.3. Let $\ell \in \mathbb{N}$ and $\tau := \inf\{s \geq 0 : \|Y_s\| = \ell\}$, where $(Y_s)_{s \geq 0}$ is a continuous-time rate-1 simple random walk on $\mathbb{Z}$. Let $\theta := \frac{1}{2}\pi/\ell$ and $\lambda := 1 - \cos\theta$. Then, for all $s \geq 0$, we have
\[
\mathbb{P}_0(\tau > s) \geq e^{-\lambda s} \geq \exp\left(-\frac{1}{8}s(\pi/\ell)^2\right).
\]
For a transition matrix $P$ and a set $A$, let $\lambda_A$ be the \textit{minimal Dirichlet eigenvalue}, defined to be the minimal eigenvalue of minus the generator of the chain killed upon exiting $A$, ie of

$$I_A - P_A \quad \text{where} \quad (I_A - P_A)(x, y) := 1(x, y \in A)(1(x = y) - P(x, y)).$$

Also, for a set $A$, write $\tau_{A^c}$ for the (first) exit time of this set by the chain.

**Lemma 6.4.** Consider a rate-$1$, continuous-time, reversible Markov chain with transition matrix $P$. Let $A$ be a connected set, and let $\lambda_A$ and $\tau_{A^c}$ be as above. Then, for all $a \in A$, we have

$$-\frac{1}{t} \log P_a(\tau_{A^c} > t) \to \lambda_A \quad \text{as} \ t \to \infty.$$

**Lemma 6.5.** For $\theta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we have

$$2(\pi \theta)^2 \geq 1 - \cos(2\pi \theta) \geq \frac{3}{4}(\pi \theta)^2.$$

**Remark.** Our proof gives an explicit form for $c$ in (6.1a). If $k \ll \log n$, then we get

$$t_{\text{rel}} \geq 2\pi^{-2}|G|^{2/k} \cdot (1 + o(1)).$$

Indeed, in this case, in the definition of the set $A$ in (6.2), we can take $L := \lfloor \frac{1}{2}(en)^{1/k} \rfloor$ for any $\varepsilon > 0$, making $|A|/|G|$ arbitrary small. One can improve the constant by replacing $A$ with

$$\{ w : Z \mid w \in \mathbb{Z}^k \text{ and } \sum_{i=1}^k |w_i|^2 \leq L(k, n) \},$$

where $L(k, n)$ is the maximal integer satisfying $|\{ w \in \mathbb{Z}^k \mid \sum_{i=1}^k |w_i|^2 \leq L(k, n) \}| \leq \frac{1}{2}n \quad \Delta$

### 6.4 Relaxing the Conditions on $k$

In this subsection, we explain how to relax the conditions on $k$. First we can relax from $k \geq 3d(G)$ to $k - 2d(G) \gg k$, valid for every group size $n = |G|.$

We now give conditions under which this can be relaxed to $k - d(G) \gg k$. If $G = \mathbb{Z}_p^d$ for a prime $p$, then one can relax this further to $k - d \gg d$, and even allow $k - d(G) \ll d(G)$, provided $p$ diverges. (In this case, the term $2^{-k}$ has to be replaced by another term which tends to zero at a slower rate as $k \to \infty$.) This follows from the fact that now we only need to consider (6.6) above with $s := p$ and we can replace (6.6) with $|A(p)| = p^d - 1$. So the condition $k - d(G) \gg k$ is sufficient when $G = \mathbb{Z}_p^d$ with $p$ prime.

We now show that if $|G|$ is ‘typical’ (in a precise sense), then the same condition is sufficient. In the proof above, in (6.6), we used the crude bound

$$|A(s)| \leq (\sum_{i \mid s} \phi(i))^d \leq (\frac{s}{d})^d.$$ 

Instead, recalling that we write $i \mid n$ to mean that $i$ divides $n$, we can use the improved bound

$$|A(s)| \leq (\sum_{i \mid s} i \mathbf{1}(i \mid n))^d.$$

In [15, Lemma F.7], we show that, for all $\varepsilon > 0$, there exists a constant $C_{\varepsilon}^*$ and a density-$(1 - \varepsilon)$ set $\mathbb{B}_\varepsilon \subseteq \mathbb{N}$ such that, for all $n \in \mathbb{B}_\varepsilon$ and all $2 \leq s \leq n$, we have

$$\sum_{i \mid s} i \mathbf{1}(i \mid n) \leq C_{\varepsilon} s (\log s)^2.$$

Using this to derive an improved bound on $|A(s)|$, and adjusting some of the constants in the proof in an appropriate manner, an inspection of the proof reveals that, for all $n \in \mathbb{B}_\varepsilon$ and all $2 \leq s \leq n$, we have

$$t_{\text{rel}}(G_k) \geq C_{\varepsilon} s n^{2/k} \leq e^{-k/C_{\varepsilon} s}.$$
7 Open Questions and Conjectures

We close the paper with some questions which are left open.

1: Typical Distance and Diameter for All Abelian Groups

In our typical distance theorem, there were some conditions on the group. We allowed any group with \( d(G) \ll \log |G|/\log \log k \) if \( 1 \ll k \ll \log |G| \), but once \( d(G) \) became larger than this or \( k \) became order \( \log |G| \), we had to impose conditions. We conjecture that these are artefacts of the proof.

**Conjecture 1.** Let \( G \) be an Abelian group. Suppose that \( 1 \ll k \lesssim \log |G| \) and \( k - d(G) \gg 1 \). Then the typical distance statistic concentrates at a value which depends only on \( k \) and \( G \), not the particular realisation of the generators. Further, if \( k \ll \log |G| \) and \( k - d(G) \ll k \), then it concentrates at a value which depends only on \( k \) and \( |G| \).

The claim when \( 1 \ll k \ll \log |G| \) and \( k - d(G) \approx k \) is a natural extension of Theorem 2.2. Further, if \( k \ll \sqrt{\log |G|}/\log \log \log |G| \), then \( k - d(G) \gg 1 \) is sufficient, by Hypothesis A. Once we relax to \( k - d(G) \gg 1 \), for larger \( k \), we still expect concentration of typical distance for all Abelian groups, but the value will likely depend on the specific group. Compare this with the occurrence of cutoff for the random walk on the random Cayley graph established in [11].

There are two levels on which concentration occurs: first, for a fixed graph \( G(z) \), one draws a \( U \sim \text{Unif}(G) \) and looks for concentration of \( \text{dist}(id, U) \) at some value, say \( f(z) \); second, one draws \( Z \) uniformly and looks for concentration of \( f(Z) \). The second is the meat of Conjecture 1.

Indeed, our lower bound on typical distance holds for all Abelian groups and all Cayley graphs with \( k \) generators, thus necessarily \( \mathbb{P}_{G(z)}(\text{dist}(id, U) \geq k|G|^{1/k}) = 1 - o(1) \) for all such Cayley graphs \( G(z) \). Additionally, our spectral gap estimate (Theorem D) says that the gap is order \( |G|^{-2/k} \) if \( k - 2d(G) \gg k \) (or when \( k - d(G) \approx k \) and \( |G| \) is ‘typical’) whp over uniform \( Z \).

Since \( u \mapsto \text{dist}(id, u) \) is a \( 1 \)-Lipschitz function, by Poincaré’s inequality \( \text{Var}_{G(z)}(\text{dist}(id, U)) \leq t_{\text{rel}}(G(z)) \). For all multisets \( z \) of size \( k \) satisfying the aforementioned spectral gap estimate from Theorem D (which holds whp for \( G_k \)), using our deterministic lower bound on the typical distance, we see that \( \text{dist}_{G(z)}(id, U) \) concentrates at some value \( f(z) \), which may depend on \( z \), by Chebyshev’s inequality. We conjecture that in fact \( f(Z) \) concentrates at some value \( \mathcal{D} \).

It is easy to see that the typical distance and diameter are always the same up to constants. We conjecture that the diameter of \( G_k \) concentrates whp whenever \( 1 \ll k \lesssim \log |G| \) and \( k - d(G) \gg 1 \). We leave open the question of finding conditions under which the diameter and typical distance are asymptotically equivalent whp.

2: Isoperimetry for Random Cayley Graphs

The isoperimetric, or Cheeger, constant of a finite \( d \)-regular graph \( G = (V, E) \) is defined as

\[
\Phi_* := \frac{1}{d} \min_{1 \leq |S| \leq \frac{|V|}{2}} \Phi(S) \quad \text{where} \quad \Phi(S) := \frac{1}{|S|}\left| \left\{ \{a, b\} \in E \mid a \in S, b \in S^c \right\} \right|.
\]

More generally, the isoperimetric constant is defined for Markov chains; see [19, §7.2]. For a given stochastic matrix \( P \), it is easy to see that the original chain \( P \), the time-reversal \( P^* \) and the additive symmetrisation \( \frac{1}{2}(P + P^*) \) all have the same isoperimetric profile. Thus the isoperimetric constant for a directed Cayley graphs is the same as that for the undirected version.

The following conjecture asserts that the Cheeger constant is, up to a constant factor, the same as that of the standard Cayley graph of \( \mathbb{Z}_L^k \) where \( L \) is such that \( n \asymp L^k \).

**Conjecture 2.** There exists a constant \( c \) so that, for all \( \varepsilon \in (0, 1) \), there exist constants \( n_\varepsilon \) and \( M_\varepsilon \) so that, for every finite group \( G \) of size at least \( n_\varepsilon \), when \( k \geq M_\varepsilon \), we have

\[
\mathbb{P}(\Phi_*(G_k) \leq c|G|^{-1/k}) \leq \varepsilon,
\]

where \( \Phi_*(G_k) \) is the Cheeger constant of a random Cayley graph with \( k \) generators.
By [23, Theorem 6.29], which regards expansion of general Cayley graphs, along with our upper bound on typical distance (and hence on diameter), we can prove this conjecture up to a factor $k$.

By the well-known discrete analogue of Cheeger’s inequality, discovered independently by multiple authors—see, for example, [19, Theorem 13.10]—we have $\frac{1}{2} \gamma \leq \Phi_* \leq \sqrt{2\gamma}$. Determining the correct order of $\Phi_*$ in our model remains an open problem. We conjecture that the correct order is given by $\sqrt{\gamma}$, i.e., order $|G|^{-1/k}$, using Theorem D for the order of the spectral gap.

The celebrated Alon–Roichman theorem states that the Cayley graph of any finite group $G$ is a $(1 - \varepsilon)$-expander (ie $\Phi_* \geq 1 - \varepsilon$) whp when $k \geq C_{\varepsilon} \log |G|$, for some constant $C_{\varepsilon}$; the best known upper bound on $C_{\varepsilon}$ is $O(1/\varepsilon^2)$. Naor [25, Theorem 1.2] refines this for Abelian groups: he showed that one can in fact bound $|\Phi(S) - 1| \leq \varepsilon \sqrt{\log |S|/\log |G|}$ for all $S$ with $1 \leq |S| \leq \frac{1}{2}|V|$ simultaneously, when $k/\log n \geq C/\varepsilon^2$, for a constant $C$.

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