Henstock-Orlicz space and its dense space

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Abstract. The motivation of the article is to introduce Henstock-Orlicz space with non absolute integrable functions. We prove $C^\infty_0$ is dense in the Henstock-Orlicz space, which is not dense in the classical Orlicz space.

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1. Introduction and Preliminaries

The Lebesgue integration theory was developed in the year 1904 by Hendri Lebesgue. This theory and the function $x^p$ in the definition of $L^p$ space inspired by Z.W. Birnbaum and W. Orlicz to proposed a generalized space of $L^p$, so called later Orlicz space. This space was later developed by Orlicz himself.

The fundamental properties of Orlicz space with Lebesgue measure found in [7]. In [9] the theory of Orlicz space is in more generalized version with Young functions and the underlying measure. The basic ideas of the proofs of the theorems of Orlicz space are analogues of the basic results on $L^p$-space.

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In [12], authors present a translation invariant subspace \( L_1(R^n) \cap L_\phi(R^n) \) to be dense in Orlicz space \( L_\phi(R^n) \) by overcoming difficulties as \( C_0^\infty(R^n) \) is dense in \( L^p(R^n) \) but not generally dense in \( L_\phi(R^n) \). On the other hand in [2] authors proved that \( C_0^\infty(R^n) \) is dense in Orlicz-Sobolev space. Based on the articles [2, 10, 6, 4, 11], we introduce Henstock-Orlicz or (H-Orlicz) space with the concept of Henstock integrable functions in place of Lebesgue integrable functions. Recalling measurable Henstock is Lebesgue, so our space is equivalent to classical Orlicz space. H-Orlicz space is motivated by Orlicz space. We note some important difference as \( C_0^\infty(R^n) \) is dense in H-Orlicz space.

Henstock integral was first developed by R. Henstock and J. Kurzweil independently during 1957-1958 from Riemann integral with the concept of tagged partitions and gauge functions. Henstock integral is a kind of non absolute integral and contain Lebesgue integral (see [10]).

We denote \( H(I) \) be the space of all Henstock integrable functions defined on \( I \). In [10, 4, 11] shows that \( H(I) \) is a vector space under the usual operations of pointwise addition and scalar multiplication. The norm defined on \( H(I) \) by Alexiewicz as \(||f|| = \sup\{|\int_a^b f| : a \leq t \leq b\} \). In [5], Gill and Zachary defined two new classes of Banach space of Henstock integrable functions. Throughout our work we assume an abstract measure space \((\mathcal{B}, \Sigma, \mu_\infty)\), where \( \mathcal{B} \) is some set of points and \( \Sigma \) is an \( \sigma \)-algebra of its subsets on which a \( \sigma \)-additive function \( \mu_\infty : \Sigma \to \mathbb{R}^+ \) is given. We consider only real functions in H-Orlicz class \( \mathcal{H}^{-\theta}(\mu_\infty) \). We assume that our measure \( \mu_\infty \) have the finite subset property. That is \( E \in \Sigma, \mu(E) > 0 \) implies \( F \in \Sigma, F \subset E \) and \( 0 < \mu_\infty(F) < \infty \).

This give us

\[
\mu_\infty(E) = \begin{cases} 
0, & \text{if } E = \phi, \\
+\infty, & \text{if } E \neq \phi 
\end{cases}
\]

Otherwise it does not restrict the generality of our assumption.

**Definition 1.1.** A function \( f : [a, b] \to \mathbb{R} \) is Henstock integrable if \( A \in \mathbb{R} \) and \( \epsilon > 0 \) there exists a gauge \( \delta : [a, b] \to \mathbb{R} \) such that for each tagged partition \( (P, (c_k)_{k=1}^n) \) that is \( \delta(x) \)-fine,

\[
|R(f, P) - A| < \epsilon.
\]

Or A function \( f : [a, b] \to \mathbb{R} \) is Henstock integrable if there exists a function \( F : [a, b] \to \mathbb{R} \) and for every \( \epsilon > 0 \) there is a function \( \delta(t) > 0 \) such that for any \( \delta \)-fine partition
D = \{[u, v], t\} of [a, b], we have
\[\left\| \sum [f(t)(v - u) - F(u, v)] \right\| < \epsilon,\]
where the sum \(\sum\) is understood to be over \(D = \{(u, v), t\}\) and \(F(u, v) = F(v) - F(u)\).
We write \(H\int_{I_0} f = F(I_0)\).

**Definition 1.2.** [9] A function \(\theta : \mathbb{R} \to \mathbb{R}^+\) is said to be Young function, so that \(\theta(x) = \theta(-x), \theta(0) = 0, \theta(x) \to \infty\) as \(x \to \infty\) but \(\theta(x_0) = +\infty\) for some \(x_0 \in \mathbb{R}\) is permitted.

**Definition 1.3.** Let \(m : \mathbb{R}^+ \to \mathbb{R}^+\) be non decreasing right continuous and non negative function satisfying
\[m(0) = 0, \text{ and } \lim_{t \to \infty} m(t) = \infty.\]
A function \(\theta : \mathbb{R} \to \mathbb{R}\) is called an \(N\)-function if there is a function \('m'\) satisfying the above sense that
\[\theta(u) = \int_0^{|u|} m(t)dt.\]
Evidently, \(\theta\) is an \(N\)-function if it is continuous, convex, even satisfies
\[\lim_{u \to \infty} \frac{\theta(u)}{u} = \infty \text{ and } \lim_{u \to 0} \frac{\theta(u)}{u} = 0.\]
For example \(\theta_p(x) = x^p; \ p > 1\).

**Definition 1.4.** An \(N\)-function \(\theta\) is said to satisfy \(\Delta_2\)-condition if there is a \(k > 0\) so that \(\theta(2x) \leq k\theta(x)\) for large values of \(x\).

**Theorem 1.1.** If \(f : \mathcal{B} \to \mathbb{R}\) is a measurable function with gauge function \(\delta : I_0 \subseteq \mathcal{B} \to \mathbb{R}\) then \(\theta(f) : \mathcal{B} \to \mathbb{R}^+\) is Henstock integrable.

**Proof.** Let \(f : I_0 \subseteq \mathcal{B} \to \mathbb{R}^+\) be measurable integral, then for all \(\epsilon > 0\) there exists a \(\delta : I_0 \to \mathbb{R}\) such that
\[\left| \sum_{(I, w) \in \pi} \sum_{(I', w') \in \pi'} [f(w) - f(w')] \lambda(I \cap I') \right| < \epsilon\]
for all partitions \(\pi\) and \(\pi'\) of \(I_0\) finer than \(\delta\).
As \(\theta\) is Young function. So, \(\theta(x) \to \infty\) as \(x \to \infty\). Our claim is \(\theta(f)\) is Henstock integrable.
Since, Young function by definition, is an extended real Borel function. So, \( \theta(f) \) is measurable. If \( \pi \) and \( \pi' \) are both partitions of the same interval of \( I_0 \), then for any sub interval \( I \) of \( I_0 \) we can write

\[
\lambda(I) = \sum_{(I',w') \in \pi'} \lambda(I \cap I').
\]

So,

\[
\sum_{(I,w) \in \pi} \theta(f)(w)\lambda(I) = \sum_{(I,w) \in \pi} \sum_{(I',w') \in \pi'} \theta(f)(w)\lambda(I \cap I').
\]

That is

\[
\left| \sum_{(I,w) \in \pi} \theta(f)(w)\lambda(I) - \sum_{(I',w') \in \pi'} \theta(f)(w)\lambda(I) \right| < \epsilon
\]

So, \( \theta(f) \) is Henstock integrable.

2. Structure of H-Orlicz class

**Definition 2.1.** Let \( \mathcal{H}^{-\theta}(\mu_\infty) \) be the set of all \( f : B \to \mathbb{R} \) measurable integral for \( \Sigma \) such that \( \int_B \theta(|f|)d\mu_\infty \) is Henstock integral.

**Theorem 2.1.** The space \( \mathcal{H}^{-\theta}(\mu_\infty) \) is absolutely convex. That is if \( f, g \in \mathcal{H}^{-\theta}(\mu_\infty) \) and \( \alpha, \beta \) are scalars such that \( |\alpha|+|\beta| \leq 1 \) then \( \alpha f + \beta g \in \mathcal{H}^{-\theta}(\mu_\infty) \). Also \( h \in \mathcal{H}^{-\theta}(\mu_\infty), |f| \leq |h|, f \) is Henstock integrable, then \( f \in \mathcal{H}^{-\theta}(\mu_\infty) \).

**Proof.** Let \( f, g \in \mathcal{H}^{-\theta}(\mu_\infty) \). Then by the monotonicity and convexity of \( \theta \), we get \( 0 < \gamma = |\alpha|+|\beta| \leq 1 \)

Now,

\[
\theta(|\alpha f + \beta g|) \leq \theta(|\alpha||f| + |\beta||g|) \leq \gamma \theta \left( \frac{|\alpha|}{\gamma} |f| + \frac{|\beta|}{\gamma} |g| \right) \leq |\alpha|\theta(|f|) + |\beta|\theta(|g|).
\]

The right hand side is Henstock integrable. So, \( \alpha f + \beta g \in \mathcal{H}^{-\theta}(\mu_\infty) \).

For second part \( |f| \leq |h| \). This implies \( |h-\epsilon| \leq |f| \leq |h| \).

As \( h \in \mathcal{H}^{-\theta}(\mu_\infty) \) gives \( \theta(|h|) \) is Henstock integrable. Also, \( \theta(|h-\epsilon|) \) is Henstock integrable. So, by Lemma 9 \[10\] \( f \in \mathcal{H}^{-\theta}(\mu_\infty) \).

**Theorem 2.2.** The space \( \mathcal{H}^{-\theta}(\mu_\infty) \) is a linear space if and only if \( \theta \) satisfies \( \Delta_2 \) condition.
Proof. For linearity, it is sufficient to verify that for each \( f \in \mathcal{H}^{-\theta}(\mu_\infty) \), \( nf \in \mathcal{H}^{-\theta}(\mu_\infty) \) for any integer \( n \) and hence also for each \( \alpha > 0, \alpha f \in \mathcal{H}^{-\theta}(\mu_\infty) \).

Then \( \alpha f + \beta f = \gamma \left( \frac{\alpha}{\gamma} f + \frac{\beta}{\gamma} g \right) \in \mathcal{H}^{-\theta}(\mu_\infty) \), \( \gamma = |\alpha| + |\beta| > 0 \) for any \( f, g \in \mathcal{H}^{-\theta}(\mu_\infty) \).

If \( \theta \) is \( \Delta_2 \)-regular, then \( \mu_\infty(\mathcal{B}) = +\infty \) and \( \theta(2|f|) \leq k\theta(|f|), \ k > 0 \) implies \( 2f \in \mathcal{H}^{-\theta}(\mu_\infty) \) with \( f \).

If \( \mu_\infty(\mathcal{B}) < \infty \) then \( \theta(2x) \leq k\theta(x) \) for \( x \geq x_0 \geq 0 \).

Let

\[
 f_1 = \begin{cases} 
 f, & \text{if } |f| \leq x_0, \\
 0, & \text{otherwise}
\end{cases}
\]

We set \( f_2 = f - f_1 \) that is \( f = f_1 + f_2 \) and

\[
\theta(2|f|) = \theta(2|f_1 + f_2|) \leq \theta(2|f_1|) + \theta(2|f_2|) \\
\leq \theta(2|f_1|) + k\theta(|f_2|).
\]

Therefore \( \int_B \theta(2|f|)d\mu_\infty \) is Henstock integrable as right side is Henstock integrable.

So, \( 2f \in \mathcal{H}^{-\theta}(\mu_\infty) \). That is \( nf \in \mathcal{H}^{-\theta}(\mu_\infty) \) for any integer \( n \)

Therefore \( \mathcal{H}^{-\theta}(\mu_\infty) \) is linear when \( \theta \in \Delta_2 \).

For converse, let \( E \in \Sigma \) be a set of positive measure on which \( \mu_\infty \) is diffuse and that \( \theta \) not belong in \( \Delta_2 \).

We construct \( f \in \mathcal{H}^{-\theta}(\mu_\infty) \) such that \( 2f \notin \mathcal{H}^{-\theta}(\mu_\infty) \).

If \( 0 < \alpha < \mu_\infty(E) < \infty \). Then by hypothesis on \( \mu_\infty \), there is \( F \subset E, \ F \in \Sigma \) with \( \mu(F) = \alpha < \infty \).

We construct a function supported by \( F \) to satisfy our assumption. Let \( \theta(\mathbb{R}) \subset \mathbb{R}^+ \) since \( \theta \) does not satisfies \( \Delta_2 \) condition.

There exists a sequence \( x_j \geq j \) such that \( \theta(2x_j) > n\theta(x_j), \ j \geq 1 \).

Let \( n_0 \) be an integer such that

\[
\sum_{n \geq n_0} \frac{1}{n^2} < \infty
\]

and \( \theta(x_n) \geq 1 \) for all \( n \geq n_0 \).

This is possible by diffuseness of \( \mu_\infty \) on \( F \) there is a measurable \( F_0 \subset F \) such that

\[
\mu_\infty(F_0) = \sum_{n \geq n_0} \frac{1}{n^2} < \infty.
\]
Similarly we can find a set $D_1 \in \Sigma, D_1 \subset F_0$ such that $\mu_\infty(D_1) = \frac{1}{n^2}$.

Since $\mu_\infty(F_0 - D) > 0$, we can again find $D_2 \in \Sigma$ and $D_2 \subset F_0 - D_1$ such that $\mu_\infty(D_2) = \frac{1}{(n_0 + 1)^2}$.

Then for disjoint sets $D_n \in \Sigma, \mu_\infty(D_n) = \frac{1}{(n_0 + n - 1)^2}, n \geq 1$.

Let $F_k \subset D_k, F_k \in \Sigma$, be chosen such that $\mu_\infty(F_k) = \frac{\mu_\infty(D_k)}{\theta(x_k)}$.

Let

$$f = \sum_{j=1}^{n} \theta(x_j) \mu_\infty(F_j) = \int_{I_0} f.$$  

Then $f \in \mathcal{H}^{-\theta}(\mu_\infty)$ (Remarks 5 of [10]). However

$$\int_{\mathcal{B}} \theta(2f)d\mu_\infty = \sum_{j=1}^{n} \theta(2x_j) \mu_\infty(F_j) \geq \sum_{n \geq n_0} j \theta(x_j) \mu_\infty(F_j) = \sum_{n \geq n_0} \frac{1}{j} = +\infty.$$

So, $2f \notin \mathcal{H}^{-\theta}(\mu_\infty)$.

So, our conclusion $\mathcal{H}^{-\theta}(\mu_\infty)$ is a class of scalar function and it is linear if and only if it is closed under multiplication by positive number.

**Example 2.1.** Let $\mathcal{B} = \{[i - 1, i] : i \in \mathbb{N}\}$. Let $\Sigma$ be power set of $\mathcal{B}$ and $\mu_\infty$ be positive measure as $\mu_\infty(i) = 1; i > 1$ let $\theta(x) = e^{x^2} - 1$ then $\theta(x)$ is $N$-function such that $\theta \notin \Delta_2$ then $\mathcal{H}^{-\theta}(\mu_\infty)$ is linear space.

**Proof.** By definition $\theta(x)$ is $N$-function and it is not satisfies $\Delta_2$ condition. We will see $\mathcal{H}^{-\theta}(\mu_\infty)$ is linear space.

If $f \in \mathcal{H}^{-\theta}$ then $\int_{\mathcal{B}} \theta(f)d\mu_\infty = \sum_{n=1}^{\infty} (e^{[f]^2} - 1) < \infty$.

So, $\int_{\mathcal{B}} \theta(f)d\mu_\infty$ is bounded and Henstock integrable.

Let $K > 0$ be bound so, $(e^{[f]^2} - 1) \leq K$ implies $e^{[f]^2} \leq 1 + K$.

Therefore $\int_{\mathcal{B}} (\theta(f))d\mu_\infty = \sum_{n=1}^{\infty} (e^{4[f]^2} - 1)$.

Therefore

$$\int_{\mathcal{B}} \theta(2f)d\mu_\infty \leq [(K + 2)(K + 1)^2 + 1] \int_{\mathcal{B}} \theta(f)d\mu_\infty.$$  

Right side is Henstock integrable. So, $2f \in \mathcal{H}^{-\theta}(\mu_\infty)$ and the space is linear.
Proposition 2.3. Let $(B, \Sigma, \mu_\infty)$ be finite measure space. Then $H(I) = \bigcup \{ \mathcal{H}^{-\theta}(\mu_\infty) : \theta \text{ ranges over all } N\text{-function.} \}$

Proof. Since $\theta$ is convex, $\theta(x) \geq ax + b$ for some constant $a, b$.

For each $f \in \mathcal{H}^{-\theta}(\mu_\infty)$, we have $\int_B (a|f| + b) d\mu_\infty \leq \int_B \theta(|f|) d\mu_\infty$.

As $\int_B \theta(|f|) d\mu_\infty$ is Henstock integrable. Therefore by [6], we have $f$ is Henstock integrable, so $f \in H(I)$. Since this hold for all such convex function. Therefore $\bigcup \mathcal{H}^{-\theta}(\mu_\infty) \subset H(I)$.

If $f \in H(I)$, By Proposition 7.3 [10], we get $|f|$ is Henstock-integrable. So, $\theta(|f|)$ is Henstock integrable.

Therefore $f \in \mathcal{H}^{-\theta}(\mu_\infty)$ so, $H(I) \subset \bigcup \mathcal{H}^{-\theta}(\mu_\infty)$ and hence the result. □

Corollary 2.4. $L_1(\mu_\infty) = \bigcup \mathcal{H}^{-\theta}(\mu_\infty)$.

3. H-Orlicz Space

Definition 3.1. Let $\mathcal{H}^{-\theta}(\mu_\infty)$ be the set on an arbitrary measure space $(B, \Sigma, \mu_\infty)$, then the space $\mathcal{H}^\theta(\mu_\infty)$ of all measurable integrable function $f : B \to \mathbb{R}$ such that $\alpha f \in \mathcal{H}^{-\theta}(\mu_\infty)$ for some $\alpha > 0$ is called H-Orlicz space. We define the H-Orlicz space as follows:

$$\mathcal{H}^\theta(\mu_\infty) = \left\{ f : B \to \mathbb{R} \text{ measurable} : \int_B \theta(\alpha f) d\mu_\infty \in H(I) \text{ for some } \alpha > 0 \right\}.$$ 

Proposition 3.1. $\mathcal{H}^\theta(\mu_\infty)$ is a linear space.

Proof. Let $f_i \in \mathcal{H}^\theta(\mu_\infty)$, $i = 1, 2$. Then there exist $\alpha_i$ such that by definition $\alpha_i f_i \in \mathcal{H}^{-\theta}(\mu_\infty)$.

Let $\alpha = \min(\alpha_1, \alpha_2)$. Then for $\alpha > 0$.

$$\int_B \theta\left(\frac{\alpha}{2}(f_1 + f_2)\right) d\mu_\infty \leq \frac{1}{2} \left[ \int_B \theta(\alpha_1 f_1) d\mu_\infty + \int_B \theta(\alpha_2 f_2) d\mu_\infty \right].$$

Right hand side is Henstock-integrable. So, $f_1 + f_2 \in \mathcal{H}^\theta(\mu_\infty)$ as $\frac{\alpha}{2} > 0$.

If $f = f_1 = f_2$ implies $2f \in \mathcal{H}^\theta(\mu_\infty)$. So, $nf \in \mathcal{H}^\theta(\mu_\infty)$ for all integer $n > 1$.

Therefore $\beta f \in \mathcal{H}^\theta(\mu_\infty)$ for any scalar $\beta$. So, $\mathcal{H}^\theta(\mu_\infty)$ is a linear space. □
Definition 3.2. We define the norm on \( \mathcal{H}^{\theta}(\mu_{\infty}) \) as follows:

\[
H_{\theta}(f) = \inf \left\{ a > 0 : \int_{B} \theta \left( \frac{f}{a} \right) d\mu_{\infty} \in H(I) \right\}.
\]

Theorem 3.2. The space \( (\mathcal{H}^{\theta}, ||.||_{H_{\theta}}) \) is normed linear space.

Proof. (a) If \( f = 0 \) a.e., then \( H_{\theta}(f) = 0 \)

Conversely, if \( H_{\theta}(f) = 0 \).

Let \( |f| > 0 \), on a set of positive measure, if possible then there exists a number \( \delta > 0 \) such that

\[
A = \{ p : |f(p)| \geq \delta \}
\]

satisfies \( \mu_{\infty}(A) > 0 \).

Now

\[
\theta(n\delta)\mu_{\infty}(A) = \int_{A} \theta(n\delta) d\mu_{\infty}
\]

\[
\leq \int_{A} \theta(nf) d\mu_{\infty}
\]

\[
\leq \int_{B} \theta(nf) d\mu_{\infty}, \ n \geq 1.
\]

Since \( \mu_{\infty}(A) > 0 \) and \( \theta(n\delta)d\mu_{\infty} \to \infty \) as \( n \to \infty \). This is not possible so, \( \mu_{\infty}(A) = 0 \).

Thus \( f \equiv 0 \) a.e. so, \( H_{\theta}(f) = 0 \) if and only if \( f = 0 \) a.e.

(b) \( H_{\theta}(\alpha f) = |\alpha| H_{\theta}(f), \ \alpha \in \mathbb{R} \) or \( \alpha \in \mathbb{C} \).

Let \( \alpha \neq 0 \).

\[
H_{\theta}(\alpha f) = \inf \left\{ k > 0 : \int_{I \subset B} \theta \left( \frac{\alpha f}{k} \right) d\mu_{\infty} \in H(I) \right\}
\]

\[
= |\alpha| \inf \left\{ \frac{k}{|\alpha|} > 0 : H \int_{B} \theta \left( \frac{f}{k} \right) d\mu_{\infty} \right\}
\]

\[
= |\alpha| H_{\theta}(f).
\]

(c) \( H_{\theta}(f + g) \leq H_{\theta}(f) + H_{\theta}(g) \).

Let \( f, g \in H_{\theta}() \), \( b > 0 \). Then \( H \int_{B} \theta \left( \frac{f+g}{b} \right) d\mu_{\infty} = H \int_{B} \theta \left( \frac{f}{b} \right) d\mu_{\infty} + H \int_{b} \theta \left( \frac{g}{b} \right) d\mu_{\infty} \).

By convexity of \( \theta \),

\[
H \int_{B} \theta \left( \frac{f+g}{b} \right) d\mu_{\infty} \leq H \int_{B} \theta \left( \frac{f}{b} \right) d\mu_{\infty} + H \int_{B} \theta \left( \frac{g}{b} \right) d\mu_{\infty}.
\]
So, by infimum property
\[
\inf \left\{ \int_B \theta(f + g) d\mu_\infty \right\} \geq \inf \left\{ \int_B \theta(f) d\mu_\infty \right\} + \inf \left\{ \int_B \theta(g) d\mu_\infty \right\}.
\]
So,
\[
\inf \left\{ b > 0, \int_B \theta \left( \frac{f + g}{b} \right) d\mu_\infty \right\} \\
\leq \inf \left\{ b > 0, \int_B \theta \left( \frac{f}{b} \right) d\mu_\infty \right\} + \inf \left\{ b > 0, \int_B \theta \left( \frac{g}{b} \right) d\mu_\infty \right\}.
\]
Therefore, \( H_\theta(f + g) \leq H_\theta(f) + H_\theta(g) \). Hence, \( (\mathcal{H}^\theta, ||.||_{H_\theta}) \) is a normed linear space. □

**Lemma 3.3.** For \( \mu_\infty(B) < \infty \). Assume \( \mu_\infty \) is bounded, then \( \mathcal{H}^\theta \subset L_1(\mu_\infty) \) where the inclusion from \( \mathcal{H}^\theta \) to \((L_1, ||.||_1)\) is continuous.

**Proof.** We have for \( u > 0 \) and \( v \geq 0 \) such that \( \theta(s) \geq us - v \) for all \( s \geq 0 \) implies \( us \leq \theta(s) + v \).

Let \( f \in \mathcal{H}^\theta(\mu_\infty) \), then for small enough \( a > 0 \)
\[
a \int_B |f| d\mu_\infty \leq \frac{1}{u} \int_B [\theta(af) + v] d\mu_\infty \\
= \frac{1}{u} \int_B \theta(af) d\mu_\infty + \frac{v \mu_\infty(B)}{u} \\
< \infty.
\]
Therefore \( f \in L_1(\mu_\infty) \), so \( \mathcal{H}^\theta(\mu_\infty) \subset L_1(\mu_\infty) \).

For the inclusion is continuous we need to prove \( c||f||_1 \leq ||f||_{H_\theta} \) for all \( f \in \mathcal{H}^\theta \).

As \( ua \int_B |f| d\mu_\infty \leq \int_B \theta(af) d\mu_\infty + v \mu_\infty(B) \)

If \( a = \frac{1}{c||f||_1} \)
\[
\int_B \frac{||f||_1}{c||f||_1} d\mu_\infty - v \mu_\infty(B) \leq \int_B \theta(\frac{|f|}{c||f||_1}) d\mu_\infty
\]
Implies \( \frac{a}{c} - v \mu_\infty(B) \leq \int_B (\frac{|f|}{c||f||_1}) d\mu_\infty \).

In Particular, \( \int_B \theta(\frac{|f|}{c||f||_1}) d\mu_\infty \geq 1 \)
Implies \( \int_B \theta(\frac{c}{|f|}) d\mu_\infty \geq c||f||_1. \)

Therefore, \( \inf \{ c > 0 : H_\theta(c) \geq c||f||_1 \} \).

So, \( c||f||_1 \leq ||f||_{H_\theta} \). □

**Corollary 3.4.** \( \mathcal{H}^\theta \subset H(I) \)
Lemma 3.5. Let \((f_n)_{n \geq 1}\) be a sequence in \(H^\theta(\mu_\infty)\). Then the followings are equivalent:

(i) \(\lim_{n \to \infty} ||f_n||_{H^\theta} = 0\)

(ii) For all \(a > 0\), \(\lim_{n \to \infty} \sup H \int_B \theta(a f_n) d\mu_\infty \leq 1\)

(iii) For all \(a > 0\), \(\lim_{n \to \infty} H \int_B \theta(a f_n) d\mu_\infty = 0\)

Proof. (i) implies (ii) \(\lim_{n \to \infty} ||f_n||_{H^\theta} = 0\)

\(\implies \lim_{n \to \infty} \inf \{a > 0 : H \int_B \theta(\frac{f_n}{a}) d\mu_\infty\} = 0\)

\(\implies \inf \{a > 0 : \lim_{n \to \infty} H \int_B \theta(\frac{f_n}{a}) d\mu_\infty\} = 0\).

Therefore, \(\lim_{n \to \infty} \sup H \int_B \theta(a f_n) d\mu_\infty \leq 1\).

(ii) implies (i) \(\lim_{n \to \infty} \sup H \int_B \theta(a f_n) d\mu_\infty \leq 1\)

\(\implies \inf \{a > 0 : \lim_{n \to \infty} H \int_B \theta(\frac{f_n}{a}) d\mu_\infty\} = 0\)

\(\implies \lim_{n \to \infty} ||f_n||_{H^\theta} = 0\)

(iii) implies (ii) Since \(\theta\) is convex and \(\theta(0) = 0\) for all \(x \geq 0\) and \(0 < \epsilon \leq 1\)

\(\theta(x) = \theta(1 - \epsilon)0 + \epsilon \theta(\frac{x}{\epsilon})\).

Thus \(\theta(x) \leq \epsilon \theta(\frac{x}{\epsilon})\), \(x \geq 0\), \(0 < \epsilon \leq 1\).

Now for all \(a > 0\)

\(\lim_{n \to \infty} \sup H \int_B \theta(a f_n) \leq 1\)

\(\implies \lim_{n \to \infty} \int_B \theta(a f_n) d\mu_\infty = 0\) for all \(a > 0\). \(\square\)

Theorem 3.6. \((H^\theta(\mu_\infty), ||.||_{H^\theta})\) is Banach space.

Proof. Let \(\{f_n : n \geq 1\}\) be a Cauchy sequence in \(H^\theta(\mu_\infty)\).

We find a countable partition \((E_K : k \geq 1)\) of measurable subset of \(\mathcal{B}\) such that \(\mu_\infty(E_K) < \infty\) for all \(k \geq 1\).

Now, in restriction \(\mu_\infty K(.) = \mu_\infty(E_K \cap .)\).

As \((f_n)_{n \geq 1}\) is a Cauchy sequence in \(H^\theta(E_K, \mu_\infty K)\). Using Lemma \(3.3\) \((f_n : n \geq 1)\) is a Cauchy sequence in \(L_1(\mu_\infty)\).

As \(L_1(\mu_\infty)\) is complete, it is convergent in \(L_1(\mu_\infty)\), then there exists a subsequence which converges \(\mu_\infty K\) a.e. pointwise to \(f\) in \(E_K\).

Now \((f_{n_k})_{k \geq 1}\) converges \(\mu_\infty\) a.e. pointwise to \(f\) on the whole space.

Let \(a > 0\) then there exists integer \(N_a\) and Lemma \(3.5\) \(\int_B \theta(a(f_m - f_n)) d\mu_\infty \leq 1\) for all
By Fatou’s Lemma
\[
\int_B \theta(a(f_m - f_n))d\mu_{\infty} \leq \lim_{K \to \infty} \inf \int_B \theta(a(f_m - f_n))d\mu_{\infty}.
\]
Using \([8]\), we get
\[
\int_B \theta(a(f_m - f_n))d\mu_{\infty} \leq \lim_{K \to \infty} \inf \int_B \theta(a(f_m - f_n))d\mu_{\infty} < \infty.
\]
Therefore \(\int_B \theta(a(f_m - f_n))d\mu_{\infty}\) is Henstock integrable.

So, \(f_m - f \in H^\theta\). As \(f_m \in H^\theta\) so, \(f \in H^\theta\). So, it is a Banach space. \(\square\)

**Corollary 3.7.** \((H^\theta(\mu_\infty),(\cdot)_{H^\theta})\) is a Banach space if \(H(I) = L_1(\mu_\infty)\).

**Theorem 3.8.** The space \((H^\theta(\mu_\infty),(\cdot)_{H^\theta})\) is a symmetric space.

**Proof.** If \(f\) is bounded measurable function, then
\[
H^{-\theta}(f) = \int_B \theta(|f|)d\mu_{\infty}
\]
\[
= \int_B \eta_{|f|}d\mu_{\infty}
\]
\[
= \int_B \mu_{\infty}\{\theta(|f|) > x\}d\mu_{\infty}
\]
\[
= \int_B \mu_{\infty}\{|f| > \theta^{-1}(x)\}dx
\]
\[
= \int_B \mu_{\infty}\{|f| > y\}\theta(y)
\]
\[
= \int_B \eta_{|f|}d\theta.
\]

Let \(f \in H^\theta(\mu_\infty)\).

Let \(g\) be equimeasurable to \(f\), that is \(\eta_{|f|} = \eta_{|g|}\). Then we have
\[
H^{-\theta}(f) = \int_B \eta_{|f|}d\theta
\]
\[
= \int_B \eta_{|g|}d\theta
\]
\[
= H^{-\theta}(g).
\]

Therefore \(g \in H^\theta(\mu_\infty)\) so, \(||f||_{H^\theta} = ||g||_{H^\theta}\). Hence, this space is symmetric. It is easily checked it is a normed ideal lattice. \(\square\)
Lemma 3.9. For any bounded measurable function $g$ on $B$, $\|g\|_\theta = 0$ if and only if $g = 0$ a.e.

Proof. As $\|g\|_\theta = 0$ if and only if $\inf\{a > 0 : H \int_B \theta(ag)d\mu_\infty u\} = 0$.

This gives

$$\|g\|_\theta = 0 \text{ if and only if } H \int_B \theta(ag)d\mu_\infty = 0 \text{ for all } a > 0$$

if and only if $\theta(ag) = 0$ a.e. for all $a > 0$

if and only if $g = 0 \mu_\infty$ a.e.

\[ \square \]

Lemma 3.10. If $0 < \|f\|_\theta < \infty$ then $H \int_{I \subset B} \theta \left( \frac{f}{\|f\|_\theta} \right) \leq A$, where $A = \min \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_K \right)$, $D_K$ is a sequence of tagged partition.

Proof. Let $\|f\|_\theta > 0$. Then $\int_B \theta \left( \frac{f}{\|f\|_\theta} \right) d\mu_\infty$ is Henstock integrable. By Theorem 1.6 [10], for every $K$ there exists a gauge $\delta_K$ on $I$ such that $D_1, D_2 \leq \delta_K$, then

$$\left| S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_1 \right) - S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_2 \right) \right| < \frac{1}{K}.$$

For each $K$, let $D_K < \delta_K$. If $K > j$, we have

$$\left| S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_K \right) - S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_j \right) \right| < \frac{1}{j}.$$

Thus $\left\{ S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_K \right) \right\}$ is a Cauchy sequence in $\mathbb{R}$.

Let $A = \min S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_K \right)$, then $\left| S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_K \right) - A \right| \leq \frac{1}{K}$ for all $K$.

Let $\epsilon > 0$ there exists natural number 'n' such that $\frac{1}{n} < \frac{\epsilon}{2}$.

Assume $D < \delta_n$. Then

$$\left| S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D \right) - A \right|$$

$$\leq S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D \right) - S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_n \right) \right| + \left| S \left( \theta \left( \frac{f}{\|f\|_\theta} \right), D_n \right) - A \right|$$

$$< \frac{1}{n} + \frac{1}{n} < \epsilon.$$

So, $\int_I \theta \left( \frac{f}{\|f\|_\theta} \right) d\mu_\infty \leq A$. \[ \square \]
Theorem 3.11. For all \( f \in \mathcal{H}^\theta \), \( g \in \mathcal{H}^\Phi \), where \( \Phi \) is complementary function of \( \theta \), then

\[
\int_B |fg| d\mu_\infty \leq ||f||_\theta ||g||_\Phi.
\]

Proof. If \( ||f||_\theta = 0 \), \( ||g||_\Phi = 0 \).

Then by Lemma 3.9 gives the result.

Let \( 0 < ||f||_\theta \), \( 0 < ||g||_\Phi \). By Young’s inequality we get

\[
st \leq \theta(s) + \Phi(t); \ s, t \geq 0.
\]

Let \( s = \frac{f}{||f||_\theta} \), \( t = \frac{g}{||g||_\Phi} \).

\[
H \int_B \frac{f}{||f||_\theta} \frac{g}{||g||_\Phi} d\mu_\infty \leq H \int_B \theta \left( \frac{f}{||f||_\theta} \right) d\mu_\infty + H \int_B \Phi \left( \frac{g}{||g||_\Phi} \right) d\mu_\infty \leq A + A.
\]

Therefore \( \int_B fg d\mu_\infty \leq 2A||f||_\theta ||g||_\Phi \).

In particular \( A = \frac{1}{2} \) gives \( \int_\Omega |fg| d\mu \leq ||f||_\theta ||g||_\Phi \).

\[\square\]

4. Denseness of \( C_0^\infty \)

Lemma 4.1. Let \( \zeta \) be a \( C_0^\infty \) function satisfying \( \zeta \geq 0 \) and \( \int \zeta(t) dt = 1 \). Define \( \zeta_k \), \( k = 1, 2, \ldots \) by \( \zeta_k(t) = k\zeta(kt) \). Let \( \theta \) be an \( N \)-function and \( f \in \mathcal{H}^\theta \), then the convolution \( \zeta_k f \in \mathcal{H}^\theta \) and \( ||\zeta_k f - f||_{H_\theta} \to 0 \) as \( k \to 0 \).

Proof. Let \( \phi \) be the complementary \( N \)-function to \( \theta \) and let \( g \in \mathcal{H}^\phi \) with \( ||g||_{H_\phi} = 1 \).

Then

\[
\int |\zeta_k t(k) - f(x)| |f(x)| \leq \int \left\{ \int |f(x - t) - f(x)| |g(x)| dx \right\} \zeta_k(t) dt
\]

\[
\leq \int ||f_t - f||_\zeta \zeta_k(t) dt,
\]

where \( f_t(x) = f(x - t) \).

Hence

\[
||\zeta_k f - f||_{H_\theta} \leq \int ||f_t - f||_\theta \zeta_k(t) dt
\]

\[
= \int ||f_t^k - f||_\theta \zeta(t) dt.
\]

Since, \( f \in \mathcal{H}^\theta \), \( \zeta \) is compact so, for every \( \epsilon > 0 \) there exists \( k \) sufficiently large such that

\[
\int ||f_t^k - f||_\theta \zeta(t) dt \leq \epsilon \int \zeta(t) dt
\]

\[= \epsilon.\]
Therefore \( ||\zeta_k^* f - f||_{H^\theta} \to 0 \) as \( k \to \infty \).

\[ \square \]

**Theorem 4.2.** If \( B = E^n, C_0^\infty(E^n) \) is dense in \( H^\theta(E^n, \mu_\infty) \).

**Proof.** Let \( s \in H^\theta \) be compact. Let us assume a sequence of \( C_0^\infty \) functions \( R(x) \) such that \( R(x) = 1 \) for \( x \leq \mathcal{R} \) and \( R(x) = 0 \) for \( x \geq 2\mathcal{R} \).

Clearly \( R.s \) is compact function. So, \( R.s \to s \in H^\theta \).

Let \( \zeta_k \) be the sequence defined in Lemma 4.1 as

\[
\zeta_k(x) = s^* \zeta_k(x) = \int s(t) \zeta_k(x-t) dt.
\]

As derivative of distribution function is again distribution function page 33 [1] and Lemma 5.7 of [3] shows that \( D^\alpha s \in H^\theta \).

So, for all \( \alpha, |\alpha| \leq m \) and \( \zeta_k \in C_0^\infty \).

We have

\[
D^\alpha s_k(x) = D^\alpha \int s(t) \zeta_k(x-t) dt
= \int s(t) D^\alpha \zeta_k(x-t) dt
= \int D^\alpha s(t) \zeta_k(x-t) dt.
\]

So, \( ||D^\alpha s_k - D^\alpha s||_{H^\theta} = ||D^\alpha s^* \zeta_k - D^\alpha s||_{H^\theta} \to 0 \) as \( k \to \infty \).

Therefore, \( C_0^\infty(E^n) \) is dense in \( H^\theta(E^n, \mu_\infty) \). \( \square \)

**References**

[1] M.A-Algwaiz *Theory of Distributions*, Pure and Applied Mathematics, Monograph, Marcel Dekker, Inc, (1992).

[2] Thomas K. Donaldson, Neil S. Trudinger, Orlicz Sobolev space and Imbedding Theorems, J. Funct. Anal. 8(1971) 52–75.

[3] J.J. Duistermaat, J. A. C Kolk *Distribution: Theory and application*, Berlin Heidelberg, Newyork, Springer (2006).

[4] Russell A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Mathematics Vol 4, AMS, 1994
[5] T. L. Gill, W. W. Zachary  *Functional Analysis and The Feynman Operator Calculus*, Springer International Publishing Switzerland, 2006

[6] Ralph Henstock *The General Theory of Integration*, Oxford mathematical monographs, Clarendo Press, Oxford (1991).

[7] M.A. Krasnosel’skii, Ya. B. Rutickii, *Convex functions and Orlicz Spaces*, P. Noordhoff Ltd, Groningen (1961).

[8] L. Paxton, A Sequential approaches to the Henstock Integral, arXiv:1609.05454v1 (2016).

[9] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Vol 146 of Pure and Applied Mathematics , Marcel Dekker, Inc (1991).

[10] C. Swartz, *Introduction to Gauge Integral*, World Scientific Pub. Co. (2001).

[11] B. S. Thomson, *Theory of Integral*, Classical Real Analysis. Com (2008).

[12] Truong Van Thung, Some collections of functions Dense in an Orlicz Space, Acta Mathematica. Vietnamica. 25(2)(2000) 195–208.