Kolmogorov $n$-Widths of Function Classes Induced by a Non-Degenerate Differential Operator: A Convex Duality Approach

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Abstract

Let $P(D)$ be the differential operator induced by a polynomial $P$, and let $U_2^{[P]}$ be the class of multivariate periodic functions $f$ such that $\|P(D)(f)\|_2 \leq 1$. The problem of computing the asymptotic order of the Kolmogorov $n$-width $d_n(U_2^{[P]}, L_2)$ in the general case when $U_2^{[P]}$ is compactly embedded into $L_2$ has been open for a long time. In the present paper, we use convex analytical tools to solve it in the case when $P(D)$ is non-degenerate.

Keywords. asymptotic order · Kolmogorov $n$-widths · non-degenerate differential operator · convex duality

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1 Introduction

The aim of the present paper is to study Kolmogorov n-widths of classes of multivariate periodic functions induced by a differential operator. In order to describe the exact setting of the problem let us introduce some notation.

We first recall the notion of Kolmogorov n-widths [14,18]. Let \( \mathcal{X} \) be a normed space, let \( F \) be a nonempty subset of \( \mathcal{X} \) such that \( F = -F \), and let \( \mathcal{G}_n \) be the class of all vector subspaces of \( \mathcal{X} \) of dimension at most \( n \). The Kolmogorov n-width of \( F \) in \( \mathcal{X} \) is

\[
d_n(F, \mathcal{X}) = \inf_{G \in \mathcal{G}_n} \sup_{f \in F} \inf_{g \in G} \| f - g \|_\mathcal{X}.
\]

This notion quantifies the error of the best approximation to the elements of \( F \) by elements in a vector subspace of \( \mathcal{X} \) of dimension at most \( n \) [18,25,26].

In computational mathematics, the so-called \( \varepsilon \)-dimension \( n_\varepsilon(F, \mathcal{X}) \) is used to quantify the computational complexity. It is defined by

\[
n_\varepsilon(F, \mathcal{X}) = \inf \left\{ n \in \mathbb{N} \middle| \left( \exists G \in \mathcal{G}_n \right) \sup_{f \in F} \inf_{g \in G} \| f - g \|_\mathcal{X} \leq \varepsilon \right\}.
\]

This approximation characteristic is the inverse of \( d_n(F, \mathcal{X}) \) in the sense that the quantity \( n_\varepsilon(F, \mathcal{X}) \) is the smallest integer \( n_\varepsilon \) such that the approximation of \( F \) by a suitably chosen approximant \( n_\varepsilon \)-dimensional subspace \( G \) in \( \mathcal{X} \) gives an approximation error less than \( \varepsilon \). Recently, there has been strong interest in applications of Kolmogorov n-widths, and its dual Gelfand n-widths, to compressive sensing [3,10,11,19]. Kolmogorov n-widths and \( \varepsilon \)-dimensions of classes of functions with mixed smoothness have also been employed in recent high-dimensional approximation studies [5,9].

We consider functions on \( \mathbb{R}^d \) which are \( 2\pi \)-periodic in each variable as functions defined on \( \mathbb{T}^d = [-\pi, \pi]^d \). Denote by \( L_2(\mathbb{T}^d) \) the Hilbert space of square-integrable functions on \( \mathbb{T}^d \) equipped with the standard scalar product, i.e.,

\[
(\forall f \in L_2(\mathbb{T}^d))(\forall g \in L_2(\mathbb{T}^d)) \quad \langle f, g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)\overline{g(x)}dx,
\]

and by \( \mathcal{S}'(\mathbb{T}^d) \) the space of distributions on \( \mathbb{T}^d \). The norm of \( f \in L_2(\mathbb{T}^d) \) is \( \| f \|_2 = \sqrt{\langle f, f \rangle} \) and, given \( k \in \mathbb{Z}^d \), the \( k \)th Fourier coefficient of \( f \in L_2(\mathbb{T}^d) \) is \( \hat{f}(k) = \langle f, e^{i(k \cdot)} \rangle \). Every \( f \in \mathcal{S}'(\mathbb{T}^d) \) can be identified with the formal Fourier series

\[
f = \sum_{k \in \mathbb{Z}^d} \hat{f}(k)e^{i(k \cdot)},
\]

where the sequence \( (\hat{f}(k))_{k \in \mathbb{Z}^d} \) is a tempered sequence [22,26]. By Parseval’s identity, \( L_2(\mathbb{T}^d) \) is the subset of \( \mathcal{S}'(\mathbb{T}^d) \) of all distributions \( f \) for which

\[
\sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 < +\infty.
\]
Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and let $f \in \mathcal{S}'(\mathbb{T}^d)$. We set
\[
\mathbb{Z}_0^d(\alpha) = \left\{ (k_1, \ldots, k_d) \in \mathbb{Z}^d \mid (\forall j \in \{1, \ldots, d\}) \; \alpha_j \neq 0 \Rightarrow k_j \neq 0 \right\}.
\] (1.6)

As usual, we set $|\alpha| = \sum_{j=1}^d \alpha_j$ and, given $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we set $z^\alpha = \prod_{j=1}^d z_j^{\alpha_j}$. The $\alpha$th derivative of $f \in \mathcal{S}'(\mathbb{T}^d)$ is the distribution $f^{(\alpha)} \in \mathcal{S}'(\mathbb{T}^d)$ given through the identification
\[
f^{(\alpha)} = \sum_{k \in \mathbb{Z}_0^d(\alpha)} (ik)^\alpha \hat{f}(k)e^{i|k|}.
\] (1.7)

The differential operator $D^\alpha$ on $\mathcal{S}'(\mathbb{T}^d)$ is defined by $D^\alpha : f \mapsto (-i)^{|\alpha|} f^{(\alpha)}$. Now let $A \subset \mathbb{N}^d$ be a nonempty finite set, let $(c_\alpha)_{\alpha \in A}$ be nonzero real numbers, and define a polynomial by
\[
P : x \mapsto \sum_{\alpha \in A} c_\alpha x^\alpha.
\] (1.8)

The differential operator $P(D)$ on $\mathcal{S}'(\mathbb{T}^d)$ induced by $P$ is
\[
P(D) = \sum_{\alpha \in A} c_\alpha D^\alpha.
\] (1.9)

Set
\[
W_2^{[P]} = \left\{ f \in \mathcal{S}'(\mathbb{T}^d) \mid P(D)(f) \in L_2(\mathbb{T}^d) \right\},
\] (1.10)
denote the seminorm of $f \in W_2^{[P]}$ by
\[
\|f\|_{W_2^{[P]}} = \|P(D)(f)\|_2,
\] (1.11)
and let
\[
U_2^{[P]} = \left\{ f \in W_2^{[P]} \mid \|f\|_{W_2^{[P]}} \leq 1 \right\}.
\] (1.12)

The problem of computing asymptotic orders of $d_n(U_2^{[P]})$ in the general case when $W_2^{[P]}$ is compactly embedded into $L_2(\mathbb{T}^d)$ has been open for a long time; see, e.g., [24, Chapter III] for details. Our main contribution is to solve it for a non-degenerate differential operator $P(D)$ (see Definition 2.4).

Using convex-analytical tool, we establish the asymptotic order
\[
d_n(U_2^{[P]}) \asymp n^{-\varrho}(\log n)^{\nu},
\] (1.13)
where $\varrho$ and $\nu$ depend only on $P$.

The first exact values of $n$-widths of univariate Sobolev classes were obtained by Kolmogorov [14] (see also [15, pp. 186–189]). The problem of computing the asymptotic order of $d_n(U_2^{[P]})$ is directly related to hyperbolic crosses trigonometric approximations and to $n$-widths of classes multivariate periodic functions with a bounded mixed smoothness. This line of work was initiated by Babenko in [1, 2]. In particular, the asymptotic orders of $n$-widths in $L_2(\mathbb{T}^d)$ of these classes were established in [1]. Further work on asymptotic orders and hyperbolic cross approximation can be
found in [7] [8] [24] and recent developments in [16] [21] [23] [27]. In [6], the strong asymptotic order of $d_n(U_2^A, L_2(\mathbb{T}^d))$ was computed in the case when $U_2^A$ is the closed unit ball of the space $W_2^A$ of functions with several bounded mixed derivatives (see Subsection 4.4 for a precise definition).

The remainder of the paper is organized as follows. In Section 2 we provide as auxiliary results Jackson-type and Bernstein-type inequalities for trigonometric approximations of functions from $W_2^P$. We also characterize the compactness of $U_2^P$ in $L_2(\mathbb{N}^d)$ and the non-degenerateness of $P(D)$. In Section 3 we present the main result of the paper, namely the asymptotic order of $d_n(U_2^P, L_2(\mathbb{T}^d))$ in the case when $P(D)$ is non-degenerate. In Section 4 we derive norm equivalences relative to $\| \cdot \|_{W_2^P}$ and, based on them, we provide examples of $n$-widths $d_n(U_2^P, L_2(\mathbb{T}^d))$ for non-degenerate differential operators.

2 Preliminaries

2.1 Notation, standing assumption, and definitions

We set $\mathbb{N} = \{0, 1, \ldots, \}$, $\mathbb{N}^* = \{1, 2, \ldots, \}$, $\mathbb{R}_+ = [0, +\infty[$, and $\mathbb{R}_{++} = ]0, +\infty[$. Let $\Theta$ be an abstract set, and let $\phi$ and $\psi$ be functions from $\Theta$ to $\mathbb{R}$. Then we write

$$\left( \forall \theta \in \Theta \right) \phi(\theta) \asymp \psi(\theta)$$ (2.1)

if there exist $\gamma_1 \in \mathbb{R}_{++}$ and $\gamma_2 \in \mathbb{R}_{++}$ such that $\left( \forall \theta \in \Theta \right) \gamma_1 \phi(\theta) \leq \psi(\theta) \leq \gamma_2 \phi(\theta)$. For every $j \in \{1, \ldots, d\}$, $u^j$ denotes the $j$-th standard unit vector of $\mathbb{R}^d$ and

$$\mathcal{R}^j = \{ \lambda u^j \mid \lambda \in \mathbb{R}_{++} \}$$ (2.2)

the $j$-th standard strict ray.

**Definition 2.1** Let $B$ be a nonempty finite subset of $\mathbb{N}^d$. The convex hull $\text{conv}(B)$ of $B$ is the polyhedron spanned by $B$,

$$\Delta(B) = \left\{ \alpha \in B \mid \{ \lambda \alpha \mid \lambda \in [1, +\infty[ \} \cap \text{conv}(B) = \{ \alpha \} \right\},$$ (2.3)

and $\vartheta(B)$ is the set of vertices of $\text{conv}(\Delta(B))$. In addition,

$$\left( \forall t \in \mathbb{R}_+ \right) \Omega_B(t) = \left\{ k \in \mathbb{N}^d \mid \max_{\alpha \in B} k^\alpha \leq t \right\}.$$(2.4)

Throughout the paper, the convention $0^0$ is adopted and the following standing assumption is made.

**Assumption 2.2** $A$ is a nonempty finite subset of $\mathbb{N}^d$ and $(c_a)_{a \in A}$ are nonzero real numbers. We set

$$P : x \mapsto \sum_{a \in A} c_a x^a \text{ and } \tau = \inf_{k \in \mathbb{N}^d} |P(k)|.$$ (2.5)
Moreover, for every $t \in \mathbb{R}^+$, we set
\[ K(t) = \{ k \in \mathbb{Z}^d \mid |P(k)| \leq t \} \quad \text{and} \quad V(t) = \left\{ f \in \mathcal{S}'(\mathbb{T}^d) \mid f = \sum_{k \in K(t)} \hat{f}(k)e^{i(k \cdot \cdot)} \right\}. \tag{2.6} \]

**Remark 2.3** If $0 \in A$, then $0 \notin \partial(A)$ and $\Delta(\text{conv}(A)) = \Delta(A)$, so that $\partial(\text{conv}(A)) = \partial(A)$. Now suppose that $t \in ]\tau, +\infty[$. Then $K(t) \neq \emptyset$ and $\dim V(t) = \text{card} K(t)$, where $\text{card} K(t)$ denotes the cardinality of $K(t)$. In addition, if $\text{card} K(t) < +\infty$, then $V(t)$ is the space of trigonometric polynomials with frequencies in $K(t)$.

**Definition 2.4** The Newton diagram of $P$ is $\Delta(A)$ and the Newton polyhedron of $P$ is $\text{conv}(A)$. The intersection of $\text{conv}(A)$ with a supporting hyperplane of $\text{conv}(A)$ is a face of $\text{conv}(A)$; $\Sigma(A)$ is the set of intersections of $A$ with a face of $\text{conv}(A)$. The differential operator $P(D)$ is non-degenerate if $P$ and, for every $\sigma \in \Sigma(A)$, $P_\sigma : \mathbb{R}^d \to \mathbb{R}: x \mapsto \sum_{a \in \sigma} c_a x^a$ do not vanish outside the coordinate planes of $\mathbb{R}^d$, i.e.,
\[ (\forall x \in \mathbb{R}^d) \quad \left( \prod_{j=1}^d x_j \neq 0 \Rightarrow (\forall \sigma \in \Sigma(A)) \quad P(x)P_\sigma(x) \neq 0 \right). \tag{2.7} \]

**Remark 2.5** Suppose that $P$ is non-degenerate and let $\alpha \in \partial(A)$. Then it follows from (2.7) that all the components of $\alpha$ are even.

### 2.2 Trigonometric approximations

We first prove a Jackson-type inequality.

**Lemma 2.6** Let $t \in \mathbb{R}^+$ and define a linear operator $S_t : \mathcal{S}'(\mathbb{T}^d) \to \mathcal{S}'(\mathbb{T}^d)$ by
\[ (\forall f \in \mathcal{S}'(\mathbb{T}^d)) \quad S_t(f) = \sum_{k \in K(t)} \hat{f}(k)e^{i(k \cdot \cdot)}. \tag{2.8} \]

Let $f \in W_2^{[p]}$ and suppose that $t > \tau$. Then the distribution $f - S_t(f)$ represents a function in $L_2(\mathbb{T}^d)$ and
\[ \| f - S_t(f) \|_2 \leq t^{-1}\| f \|_{W_2^{[p]}}. \tag{2.9} \]

**Proof.** Set $g = f - S_t(f)$. Then $g \in \mathcal{S}'(\mathbb{T}^d)$. On the other hand, Parseval’s identity yields
\[ \| f \|_{W_2^{[p]}}^2 = \sum_{k \in \mathbb{Z}^d} |P(k)|^2 |\hat{f}(k)|^2. \tag{2.10} \]

Hence,
\[ \sum_{k \in \mathbb{Z}^d} |\hat{g}(k)|^2 = \sum_{k \in \mathbb{Z}^d \setminus K(t)} |\hat{f}(k)|^2 \leq \left( \sup_{k \in \mathbb{Z}^d \setminus K(t)} |P(k)|^{-2} \right) \sum_{k \in \mathbb{Z}^d \setminus K(t)} |P(k)|^2 |\hat{f}(k)|^2 \leq t^{-2}\| f \|_{W_2^{[p]}}^2, \tag{2.11} \]
which means that \( f - S_t(f) \) represents a function in \( L_2(\mathbb{T}^d) \) for which (2.9) holds.

**Corollary 2.7** Let \( t \in ]\tau, +\infty[ \). Then

\[
\sup_{f \in U_2^P} \inf_{g \in V(t)} \|f - g\|_2 \leq t^{-1}.
\]

(2.12)

Next, we prove a Bernstein-type inequality.

**Lemma 2.8** Let \( t \in ]\tau, +\infty[ \) and let \( f \in V(t) \cap L_2(\mathbb{T}^d) \). Then

\[
\|f\|_{W_2[P]} \leq t\|f\|_2.
\]

(2.13)

**Proof.** By (2.10), we have

\[
\|f\|_{W_2[P]}^2 = \sum_{k \in K(t)} |P(k)|^2|\hat{f}(k)|^2 \leq \left( \sup_{k \in K(t)} |P(k)|^2 \right) \sum_{k \in K(t)} |\hat{f}(k)|^2 \leq t^2\|f\|_2^2,
\]

(2.14)

which establishes (2.13).

2.3 Compactness and non-degenerateness

We start with a characterization of the compactness of the unit ball defined in (1.12).

**Lemma 2.9** The set \( U_2^P \) is a compact subset of \( L_2(\mathbb{T}^d) \) if and only if the following hold:

(i) For every \( t \in ]\tau, +\infty[ \), \( K(t) \) is finite.

(ii) \( \tau > 0 \).

**Proof.** To prove sufficiency, suppose that (i) and (ii) hold, and fix \( t \in ]\tau, +\infty[ \). By (11), \( V(t) \) is a set of trigonometric polynomials and, consequently, a subset of \( L_2(\mathbb{T}^d) \). In particular, using the notation (2.8), \( \forall f \in \mathcal{S}'(\mathbb{T}^d) \), \( S_t(f) \in L_2(\mathbb{T}^d) \). Hence, by Lemma 2.6

\[
\left( \forall f \in W_2^P \right) f = (f - S_t(f)) + S_t(f) \in L_2(\mathbb{T}^d).
\]

(2.15)

Thus, \( W_2^P \subset L_2(\mathbb{T}^d) \). On the other hand, (2.10) implies that \( U_2^P \) is a closed subset of \( L_2(\mathbb{T}^d) \). Therefore, \( U_2^P \) is compact in \( L_2(\mathbb{T}^d) \) if, for every \( \epsilon \in \mathbb{R}^+ \), it has a finite \( \epsilon \)-net in \( L_2(\mathbb{T}^d) \) or, equivalently, if the following following two conditions are satisfied:

(iii) For every \( \epsilon \in \mathbb{R}^+ \), there exists a finite-dimensional vector subspace \( G_\epsilon \) of \( L_2(\mathbb{T}^d) \) such that

\[
\sup_{f \in U_2^P} \inf_{g \in G_\epsilon} \|f - g\|_2 \leq \epsilon.
\]

(2.16)
(iv) \( U_2^{[p]} \) is bounded in \( L_2(\mathbb{T}^d) \).

It follows from (2.10) that [(iii) \rightarrow (iv)]. On the other hand, since \( \dim V(t) = \text{card} K(t) \), Corollary 2.7 yields [(i) \rightarrow (iii)]. To prove necessity, suppose that [(i)] does not hold. Then \( \dim V(t) = \text{card} K(t) = +\infty \) for some \( t \in \mathbb{R}_++ \). By Lemma 2.8, \( \tilde{U} = \{ f \in V(t) \cap L_2(\mathbb{T}^d) \mid \| f \|_2 \leq 1/t \} \) is a subset of \( U_2^{[p]} \) which is not compact in \( L_2(\mathbb{T}^d) \). If [(iii)] does not hold, then \( U_2^{[p]} \cap L_2(\mathbb{T}^d) \) is unbounded and, consequently, not compact in \( L_2(\mathbb{T}^d) \). \( \blacksquare \)

The following lemma characterizes the non-degenerateness of \( P(D) \).

**Lemma 2.10** \( P(D) \) is non-degenerate if and only if

\[
(\exists \gamma \in \mathbb{R}_+)(\forall x \in \mathbb{R}^d) \quad |P(x)| \geq \gamma \max_{a \in \Theta(A)} |x^a|. \quad (2.17)
\]

**Proof.** As proved in [12, 17], \( P(D) \) is non-degenerate if and only if

\[
(\exists \gamma_1 \in \mathbb{R}_+)(\forall x \in \mathbb{R}^d) \quad |P(x)| \geq \gamma_1 \sum_{a \in \Theta(A)} |x^a|. \quad (2.18)
\]

Hence, since there exist \( \gamma_2 \in \mathbb{R}_+ \) and \( \gamma_3 \in \mathbb{R}_+ \) such that

\[
(\forall x \in \mathbb{R}^d) \quad \gamma_2 \max_{a \in \Theta(A)} |x^a| \leq \sum_{a \in \Theta(A)} |x^a| \leq \gamma_3 \max_{a \in \Theta(A)} |x^a|, \quad (2.19)
\]

the proof is complete. \( \blacksquare \)

**Lemma 2.11** Let \( B \) be a nonempty finite subset of \( \mathbb{N}^d \) and let \( t \in \mathbb{R}_+ \). Then

\[
\Omega_b(t) = \left\{ k \in \mathbb{N}^d \mid \max_{a \in B} k^a \leq t \right\} \quad (2.20)
\]
is finite if and only if

\[
(\forall j \in \{1, \ldots, d\}) \quad B \cap \mathcal{R}^j \neq \emptyset. \quad (2.21)
\]

**Proof.** If (2.21) holds, then \( (\forall j \in \{1, \ldots, d\}) (\exists a_j \in \mathbb{R}_+) a_j t^j \in B \cap \mathcal{R}^j \). Hence, (2.4) implies that \( \Omega_b(t) \subset \bigcap_{j=1}^d \{ k \in \mathbb{N}^d \mid k_j \leq t^{1/a_j} \} \) and, therefore, \( \Omega_b(t) \) is bounded. Conversely, if (2.21) does not hold, then there exists \( j \in \{1, \ldots, d\} \) such that \( \{ m u^j \mid m \in \mathbb{N} \} \subset \Omega_b(t) \), which shows that \( \Omega_b(t) \) is unbounded. \( \blacksquare \)

**Theorem 2.12** Suppose that \( P(D) \) is non-degenerate. Then \( U_2^{[p]} \) is a compact subset of \( L_2(\mathbb{T}^d) \) if and only if (2.21) is satisfied and \( 0 \in A \).

**Proof.** Let us prove that there exists \( \gamma_1 \in \mathbb{R}_+ \) such that

\[
(\forall k \in \mathbb{Z}^d) \quad |P(k)| \leq \gamma_1 \max_{a \in \Theta(A)} |k^a|. \quad (2.22)
\]
Since there exists $\gamma_1 \in \mathbb{R}_{++}$ such that
\begin{equation}
(\forall k \in \mathbb{Z}^d) \quad |P(k)| \leq \gamma_1 \max_{a \in A} |k^a|,
\end{equation}
and since (2.22) trivially holds if there exists $j \in \{1, \ldots, d\}$ such that $k_j = 0$, it is enough to show that
\begin{equation}
(\forall \alpha \in A) \left( \forall k \in \mathbb{N}^d \right) \quad k^\alpha \leq \max_{\beta \in \Theta(A)} k^\beta,
\end{equation}
and a fortiori that
\begin{equation}
(\forall \alpha \in A) \left( \forall x \in \mathbb{R}^d \right) \quad \langle \alpha \mid x \rangle \leq \max_{\beta \in \Theta(A)} \langle \beta \mid x \rangle.
\end{equation}
Indeed, since $\alpha \in \text{conv}(\Theta(A))$, by Carathéodory’s theorem [20, Theorem 17.1], $\alpha$ is a convex combination of points $(\beta^j)_{1 \leq j \leq d+1}$ in $\Theta(B)$, say
\begin{equation}
\alpha = \sum_{j=1}^{d+1} \lambda_j \beta^j, \quad \text{where} \quad (\lambda_j)_{1 \leq j \leq d+1} \in \mathbb{R}^{d+1}_+ \quad \text{and} \quad \sum_{j=1}^{d+1} \lambda_j = 1.
\end{equation}
Therefore
\begin{equation}
(\forall x \in \mathbb{R}^d_+) \quad \langle \alpha \mid x \rangle = \sum_{j=1}^{d+1} \lambda_j \langle \beta_j \mid x \rangle \leq \sum_{j=1}^{d+1} \lambda_j \max_{\beta \in \Theta(A)} \langle \beta \mid x \rangle = \max_{\beta \in \Theta(A)} \langle \beta \mid x \rangle.
\end{equation}
Hence, Lemma 2.10 asserts that there exists $\gamma_2 \in \mathbb{R}_{++}$ such that
\begin{equation}
(\forall k \in \mathbb{Z}^d) \quad \gamma_2 \max_{a \in \Theta(A)} |k^a| \leq |P(k)| \leq \gamma_1 \max_{a \in \Theta(A)} |k^a|.
\end{equation}
Consequently, by Lemma 2.9, $U^P_2$ is a compact set in $L_2(\mathbb{T}^d)$ if and only if, for every $t \in \mathbb{R}_+$, $\Omega_A(t)$ is finite and
\begin{equation}
\inf_{k \in \mathbb{N}^d} \max_{a \in A} k^a > 0.
\end{equation}
In view of Lemma 2.11, the first condition is equivalent to (2.21) and the second to $0 \in A$. □

3 Main result

3.1 Convex-analytical results

Several important convex-analytical facts underly our analysis (see [4, 20] for background on convex analysis). We start with the following corollary.

**Corollary 3.1** Suppose that $P(D)$ is non-degenerate. Then $(\forall k \in \mathbb{Z}^d) \quad |P(k)| \asymp \max_{a \in \Theta(A)} |k^a|.$
Proof. Combine (2.28) and Lemma 2.10 □

Next, we investigate the geometry of our problem from the viewpoint of convex duality. Let $C$ be a subset of $\mathbb{R}^d$. Recall that the polar set of $C$ is

$$C^\circ = \{ x \in \mathbb{R}^d \mid (\forall \alpha \in C) \langle \alpha | x \rangle \leq 1 \},$$

(3.1)

and the indicator function of $C$ is

$$\iota_C : \mathbb{R}^d \to ]-\infty, +\infty] : x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise}. \end{cases}$$

(3.2)

Moreover, if $C$ is convex and $0 \in C$, the Minkowski gauge of $C$ is the lower semicontinuous convex function

$$m_C : \mathbb{R}^d \to ]-\infty, +\infty] : x \mapsto \inf \{ \xi \in \mathbb{R}^d_+ \mid x \in \xi C \}. \quad (3.3)$$

Finally, the domain of a function $\varphi : \mathbb{R}^d \to ]-\infty, +\infty]$ is

$$\text{dom } \varphi = \{ x \in \mathbb{R}^d \mid \varphi(x) < +\infty \}.$$

Lemma 3.2 Let $B$ be a nonempty finite subset of $\mathbb{R}^d_+$ such that

$$0 \in B \quad \text{and} \quad (\forall j \in \{1, \ldots, d\}) \quad B \cap \mathcal{R}^j \neq \emptyset. \quad (3.4)$$

Set $1 = (1, \ldots, 1) \in \mathbb{R}^d$, let $\mu(B)$ be the optimal value of the problem

$$\max_{x \in B^\circ} \sum_{j=1}^d x_j,$$

(3.5)

and set

$$\varrho(B) = \max\{ \rho \in \mathbb{R}^d_+ \mid \rho 1 \in \text{conv}(B) \}. \quad (3.6)$$

Then $\varrho(B) \in \mathbb{R}^d_+$ and $\mu(B) = 1/\varrho(B)$.

Proof. It follows from (3.4) that

$$\mathbb{R}^d_+ \cap B^\circ = \mathbb{R}^d_+ \cap \bigcap_{a \in B} \{ x \in \mathbb{R}^d \mid \langle x | a \rangle \leq 1 \}$$

(3.7)

is a nonempty compact set and hence (3.5) does have a solution. Now fix $j \in \{1, \ldots, d\}$. Then $(\exists a_j \in \mathbb{R}^d_+) a_j u^j \in B$. Hence $x^j = (1/a_j) u^j \in B^\circ$ and therefore $\mu(B) = \max_{x \in B^\circ} \langle x | 1 \rangle \geq \langle x^j | 1 \rangle = 1/a_j > 0$. Altogether $\mu(B) \in \mathbb{R}^d_+$. Likewise, (3.4) implies that $\varrho(B) \in \mathbb{R}^d_+$. Let us set $\varphi = m_{\text{conv}(B)}$ and $\psi = \iota_{\{1\}}$. Then it follows from (3.4) that $\text{dom } \varphi = \text{dom } m_{\text{conv}(B)} = \mathbb{R}^d_+$. Furthermore, the conjugate of $\varphi$ is $\varphi^* = \iota_{(\text{conv}(B))^\circ} = \iota_{B^\circ}$ [4 Propositions 14.12 and 7.14(vi)] and the conjugate of $\psi$ is $\psi^* = \langle \cdot | 1 \rangle$. 

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Hence, since \( 1 \in \text{int dom } \varphi = \mathbb{R}^d_+ \), \( \text{dom } \psi \cap \text{int dom } \varphi \neq \emptyset \) and the Fenchel duality formula \[4\] Proposition 15.13] yields

\[
\mu(B) = \max_{x \in B^\circ} \sum_{j=1}^d x_j \\
= - \min_{x \in \mathbb{R}^d} \langle -x, 1 \rangle \\
= - \min_{x \in \mathbb{R}^d} (t_{\mathcal{R}}(x) + \langle -x, 1 \rangle) \\
= - \min_{x \in \mathbb{R}^d} (\varphi^*(x) + \psi^*(-x)) \\
= \inf_{\alpha \in \mathbb{R}^d} (\varphi(\alpha) + \psi(\alpha)) \\
= \inf_{\alpha \in \mathbb{R}^d} (m_{\text{conv}(B)}(\alpha) + t_{\{1\}}(\alpha)) \\
= m_{\text{conv}(B)}(1) \\
= \inf \left\{ \xi \in \mathbb{R}_+^d \mid 1 \in \xi \text{conv}(B) \right\} \\
= \frac{1}{\sup \left\{ \rho \in \mathbb{R}_+^d \mid \rho 1 \in \text{conv}(B) \right\}}.
\]

(3.8)

We conclude that \( \mu(B) = 1/\mathcal{g}(B) \). \( \square \)

To illustrate the duality principles underlying Lemma 3.2 we consider two examples.

**Example 3.3** We consider the case when \( d = 2 \) and \( B = \{(6,0),(0,6),(4,4),(0,0)\} \) (see Figure 1). Then (3.4) is satisfied, \( \mu(B) = 1/4 \), and \( \mathcal{g}(B) = 4 \). The set of solutions to (3.5) is the set \( S \) represented by the solid red segment: \( S = \{ (x_1,x_2) \in [1/12,1/6]^2 \mid x_1 + x_2 = 1/4 \} \).

**Example 3.4** In this example we consider the case when \( B = \{(0,6),(2,4),(4,0),(0,0)\} \). Then (3.4) is satisfied, \( \mu(B) = 3/8 \), and \( \mathcal{g}(B) = 8/3 \). The set of solutions to (3.5) reduces to the singleton \( S = \{(1/4,1/8)\} \).

**Lemma 3.5** Let \( B \) be a nonempty finite subset of \( \mathbb{R}^d_+ \) and suppose that

\[
(\forall j \in \{1, \ldots, d\}) \quad B \cap \mathcal{R}^j \neq \emptyset.
\]

(3.9)

Let \( \mu(B) \) be the optimal value of the problem

\[
\text{maximize } \sum_{j=1}^d x_j, \quad (3.10)
\]

and let \( \nu(B) \) be the dimension of its set of solutions. Then \( \mu(B) \in \mathbb{R}_+^d \) and

\[
(\forall t \in [2, +\infty[) \quad \text{card } \Omega_B(t) \asymp t^{\mu(B)} (\log t)^{\nu(B)}.
\]

(3.11)
Figure 1: Graphical illustration of Example 3.3. In gray, the Newton polyhedron (top) and its polar (bottom). The dashed lines are the hyperplanes delimiting the polar set $B^\circ$ and the dotted line represents the optimal level curve of the objective function $x \mapsto \langle x \mid 1 \rangle$ in (3.5). The solid red segment depicts the solution set of (3.5).
Figure 2: Graphical illustration of Example 3.4. In gray, the Newton polyhedron (top) and its polar (bottom). The dashed lines are the hyperplanes delimiting the polar set $B^\circ$ and the dotted line represents the optimal level curve of the objective function $x \mapsto (x | 1)$ in (3.5). The red dot locates the unique solution to (3.5).
Proof. The fact that $\mu(B) \in \mathbb{R}_{++}$ was proved as in Lemma 3.2. Now fix $t \in [2, +\infty[$ and set $\Lambda_{B}(t) = \{ x \in \mathbb{R}_{+}^{d} \mid \max_{\alpha \in B} x^{\alpha} \leq t \}$. Then, as in the proof of Lemma 2.11 one can see that $\Lambda_{B}(t)$ is a bounded subset of $\mathbb{R}_{+}^{d}$. If we denote by $\text{vol}\Lambda_{B}(t)$ the volume of $\Lambda_{B}(t)$, then it follows from [6, Theorem 1] that
\[
\text{vol}\Lambda_{B}(t) \asymp t^{\mu(B)}(\log t)^{\nu(B)}.
\]
(3.12)
Furthermore, proceeding as in the proof of [6, Theorem 2], one shows that
\[
\text{card}\Omega_{B}(t) \asymp \text{vol}\Lambda_{B}(t).
\]
(3.13)
These asymptotic relations prove the claim. \[\square\]

3.2 Main result: asymptotic order of Kolmogorov $n$-width

Our main result can now be stated and proved.

**Theorem 3.6** Suppose that $P(D)$ is non-degenerate and that
\[
0 \in A \quad \text{and} \quad (\forall j \in \{1, \ldots, d\}) \quad A \cap \mathcal{R}^{j} \neq \emptyset.
\]
(3.14)
Let $\mu$ be the optimal value of the problem
\[
\text{maximize} \sum_{j=1}^{d} x_{j}, \quad \text{let } \nu \text{ be the dimension of its set of solutions, and set}
\]
\[
\varrho = \max\{ \rho \in \mathbb{R}_{+} \mid \rho \mathbf{1} \in \text{conv}(\vartheta(A)) \}.
\]
(3.16)
Then $\mu = 1/\varrho \in \mathbb{R}_{++}$ and, for $n$ sufficiently large,
\[
d_{U}(U_{2}^{[P]}, L_{2}(\mathbb{R}^{d})) \asymp n^{-\varrho} (\log n)^{\nu \varrho}.
\]
(3.17)
Equivalently, using (1.2), for $\varepsilon \in \mathbb{R}_{+}$ sufficiently small,
\[
n_{\varepsilon}(U_{2}^{[P]}, L_{2}(\mathbb{R}^{d})) \asymp \varepsilon^{-1/\varrho} (\log \varepsilon)^{\nu \varrho}.
\]
(3.18)
Proof. Since $A$ satisfies (3.14), so does $\vartheta(A)$. Hence the fact that $\mu = 1/\varrho \in \mathbb{R}_{++}$ follows from Lemma 3.2. We also note that the equivalence between (3.17) and (3.18) follows from (1.1) and (1.2). To show (3.17), set $\bar{t} = \max\{2, \tau\}$. Then we derive from Corollary 3.1 that
\[
(\forall t \in [\bar{t}, +\infty[) \quad \text{card}\Omega_{\vartheta(A)}(t) \asymp \text{card}\mathcal{K}(t).
\]
(3.19)
Applying Lemma 3.5 to $\vartheta(A)$ yields
\[
(\forall t \in [\bar{t}, +\infty[) \quad \dim V(t) = \text{card}\mathcal{K}(t) \asymp t^{1/\nu \varrho} (\log t)^{\nu \varrho}.
\]
(3.20)
Hence, for every \( n \in \mathbb{N} \) large enough, there exists \( t \in \mathbb{R}_{++} \) depending on \( n \) such that
\[
\gamma_1 \dim V(t) \leq \gamma_3 t^{1/\epsilon} (\log t)^\nu < n < \gamma_3 (t + 1)^{1/\epsilon} (\log(t + 1))^{\nu} \leq \gamma_2 \dim V(t + 1) \leq \gamma_4 t^{1/\epsilon} (\log t)^\nu,
\] (3.21)
where \( \gamma_1, \gamma_2, \gamma_3, \) and \( \gamma_4 \) are strictly positive real parameters that are independent from \( n \) and \( t \). Therefore,
\[
n \asymp t^{1/\epsilon} (\log t)^\nu.
\] (3.22)

or, equivalently,
\[
t^{-1} \asymp n^{-\epsilon} (\log n)^{\nu \epsilon}.
\] (3.23)

It therefore follows from (1.1) and Corollary 2.7 that
\[
d_n(U_2^{[p]}, L_2(\mathbb{T}^d)) \leq t^{-1} \asymp n^{-\epsilon} (\log n)^{\nu \epsilon},
\] (3.24)
which establishes the upper bound in (3.17). To establish the lower bound, let us recall from [25] that, for every \( n + 1 \)-dimensional vector subspace \( G_{n+1} \) of \( L_2(\mathbb{T}^d) \) and every \( \eta \in \mathbb{R}_{++} \), we have
\[
d_n(B_{n+1}(\eta), L_2(\mathbb{T}^d)) = \eta, \quad \text{where} \quad B_{n+1}(\eta) = \{ f \in G_{n+1} \mid \| f \|_{L_2(\mathbb{T}^d)} \leq \eta \}. \quad \text{(3.25)}
\]
Arguing as in (3.20)–(3.23), for \( n \in \mathbb{N} \) sufficiently large, there exists \( t \in \mathbb{R}_{++} \) such that
\[
\dim V(t) \geq \gamma_5 t^{1/\epsilon} (\log t)^\nu > n \geq \gamma_6 t^{1/\epsilon} (\log t)^\nu,
\] (3.26)
where \( \gamma_5 \in \mathbb{R}_{++} \) and \( \gamma_6 \in \mathbb{R}_{++} \) are independent from \( n \) and \( t \). Now set
\[
U(t) = \{ f \in V(t) \mid \| f \|_2 \leq t^{-1} \}. \quad \text{(3.27)}
\]
By Lemma 2.8, \( U(t) \subset U_2^{[p]} \). Consequently, it follows from (3.25)–(3.27) and (3.23) that
\[
d_n(U_2^{[p]}, L_2(\mathbb{T}^d)) \geq d_n(U(t), L_2(\mathbb{T}^d)) \geq t^{-1} \asymp n^{-\epsilon} (\log n)^{\nu \epsilon}, \quad \text{(3.28)}
\]
which concludes the proof of (3.17). Next, let us prove (3.18). Given a sufficiently small \( \epsilon \in \mathbb{R}_{++} \), take \( t \in \mathbb{R}_{++} \) such that \( 0 < t - 1 < \epsilon^{-1} \leq t \) and \( \dim V(t) > 1 \). From the above results, it can be seen that
\[
\dim V(t) - 1 \leq n_+(U_2^{[p]}, L_2(\mathbb{T}^d)) \leq \dim V(t) \quad \text{(3.29)}
\]
which, together with (3.20), proves (3.18). \( \blacksquare \)

**Remark 3.7** We have actually proven a bit more than Theorem 3.6. Namely, suppose that \( P(D) \) satisfies the conditions of compactness for \( U_2^{[p]} \) stated in Lemma 2.9 and, for every \( n \in \mathbb{N} \), let \( t(n) \) be the largest number such that \( \text{card} K(t(n)) \leq n \). Then, for \( n \) sufficiently large, we have
\[
d_n(U_2^{[p]}, L_2(\mathbb{T}^d)) \geq \frac{1}{t(n)}. \quad \text{(3.30)}
\]
4 Examples

We first establish norm equivalences and use them to provide examples of asymptotic orders of $d_n(U_2^{|P|}, L_2(\mathbb{R}^d))$ for non-degenerate and degenerate differential operators.

**Theorem 4.1** Suppose that $P(D)$ is non-degenerate and set

$$Q: x \mapsto \sum_{a \in \Theta(A)} x^a.$$  \hspace{1cm} (4.1)

Then

$$\forall f \in W_2^{|P|} \quad \|f\|_{W_2^{|P|}}^2 \asymp \|f\|_{W_2^{|Q|}}^2 \asymp \sum_{a \in \Theta(A)} \|D^a f\|_2^2 \asymp \max_{a \in \Theta(A)} \|D^a f\|_2^2. \hspace{1cm} (4.2)$$

Moreover, the seminorms in (4.2) are norms if and only if $0 \in A$.

**Proof.** Let $f \in W_2^{|P|}$. It is clear that

$$\sum_{a \in \Theta(A)} \|D^a f\|_2^2 \asymp \max_{a \in \Theta(A)} \|D^a f\|_2^2. \hspace{1cm} (4.3)$$

Parseval’s identity and Corollary 3.1 yield

$$\max_{a \in \Theta(A)} \|D^a f\|_2^2 = \max_{a \in \Theta(A)} \sum_{k \in \mathbb{Z}^d} |k|^{2a} |\hat{f}(k)|^2 \leq \sum_{k \in \mathbb{Z}^d} \left( \max_{a \in \Theta(A)} |k^a| \right)^2 |\hat{f}(k)|^2. \hspace{1cm} (4.4)$$

Now let $\mathbb{Z}^d(\alpha)_{a \in \Theta(A)}$ be a partition of $\mathbb{Z}^d$ such that

$$\max_{\beta \in \Theta(A)} |\beta^\alpha| = |\alpha^\alpha|, \quad k \in \mathbb{Z}^d(\alpha). \hspace{1cm} (4.5)$$

Then

$$\max_{a \in \Theta(A)} \|D^a f\|_2^2 = \max_{a \in \Theta(A)} \sum_{a' \in \Theta(A)} \sum_{k \in \mathbb{Z}^d(a')} |k|^{2a} |\hat{f}(k)|^2 \geq \sum_{a' \in \Theta(A)} \sum_{k \in \mathbb{Z}^d(a')} |k|^{2a'} |\hat{f}(k)|^2 \hspace{1cm} (4.6)$$

$$= \sum_{k \in \mathbb{Z}^d} \max_{a \in \Theta(A)} |k^a|^2 |\hat{f}(k)|^2.$$

Thus,

$$\max_{a \in \Theta(A)} \|D^a f\|_2^2 = \sum_{k \in \mathbb{Z}^d} \max_{a \in \Theta(A)} |k^a|^2 |\hat{f}(k)|^2. \hspace{1cm} (4.7)$$
Hence, appealing to Corollary 3.1 and (2.10), we obtain
\[
\max_{\alpha \in \vartheta(A)} \|D^\alpha f\|_2^2 \asymp \|f\|_{W_p^2}^2.
\] (4.8)

The relation
\[
\max_{\alpha \in \vartheta(A)} \|D^\alpha f\|_2^2 \asymp \|f\|_{W_q^2}^2
\] (4.9)
follows from the last seminorm equivalence and the identity \(\vartheta(\vartheta(A)) = \vartheta(A)\). Therefore, we derive from (4.2) that the seminorms in (4.2) are norms if and only if \(0 \in A\).

4.1 Isotropic Sobolev classes

Let \(s \in \mathbb{N}^\ast\). The isotropic Sobolev space \(H^s\) is the Hilbert space of functions \(f \in L_2(T^d)\) equipped with the norm
\[
\| \cdot \|_{H^s} : f \mapsto \sqrt{\|f\|_2^2 + \sum_{|\alpha| = s} \|f^{(\alpha)}\|_2^2}.
\] (4.10)

Consider
\[
P : x \mapsto 1 + \sum_{|\alpha| = s} x^\alpha = \sum_{\alpha \in A} x^\alpha,
\] (4.11)
where \(A = \{0\} \cup \{\alpha \in \mathbb{N}^d \mid |\alpha| = s\}\). If \(s\) is even, it follows directly from Lemma 2.10 that the differential operator \(P(D)\) is non-degenerate, and consequently, by Theorem 4.1 \(\| \cdot \|_{H^s}\) is equivalent to one of the norms appearing in (4.2) with \(\vartheta(A) = \{0\} \cup \{su^j \mid 1 \leq j \leq d\}\) and
\[
Q : x \mapsto 1 + \sum_{j=1}^d x_j^s.
\] (4.12)

Moreover, we have \(\varrho(A) = s/d\) and \(v(A) = 0\). Therefore, we retrieve from Theorem 3.6 the well-known result
\[
d_n(U^s, L_2(T^d)) = n^{-s/d},
\] (4.13)
where \(U^s\) denotes the closed unit ball in \(H^s\). This result is a direct generalization of the first result on \(n\)-widths established by Kolmogorov in \([14]\).

4.2 Anisotropic Sobolev classes

Given \(\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d\), the anisotropic Sobolev space \(H^\beta\) is the Hilbert space of functions \(f \in L_2\) equipped with the norm
\[
\| \cdot \|_{H^\beta}^2 : f \mapsto \sqrt{\|f\|_2^2 + \sum_{j=1}^d \|f^{(\beta_j u^j)}\|_2^2}.
\] (4.14)
Consider the polynomial

\[ P: x \mapsto 1 + \sum_{j=1}^{d} x^{\beta_j} = \sum_{\alpha \in A} x^\alpha, \]

where \( A = \{0\} \cup \{\beta_j u^j \mid 1 \leq j \leq d\} \). If the coordinates of \( \beta \) are even, the differential operator \( P(D) \) is non-degenerate. Consequently, by Theorem 4.1, \( \|\cdot\|_{H^\theta} \) is equivalent to one of the norms in (4.2) with \( \theta(A) = A \) and

\[ Q = P. \]

We have

\[ \varrho = \varrho(A) = \left( \sum_{j=1}^{d} 1/\beta_j \right)^{-1} \]

and \( \nu(A) = 0 \), and therefore, from Theorem 3.6 we retrieve the known result \([13]\)

\[ d_n(U^\beta, L_2(\mathbb{T}^d)) \asymp n^{-\varrho}, \]

where \( U^\beta \) denotes the unit ball in \( H^\beta \).

### 4.3 Classes of functions with a bounded mixed derivative

Let \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) with \( 0 < \alpha_1 = \cdots = \alpha_{v+1} < \alpha_{v+2} = \cdots = \alpha_d \) for some \( v \in \{0, \ldots, d-1\} \). Given a set \( e \subset \{1, \ldots, d\} \), let the vector \( \alpha(e) \in \mathbb{N}^d \) be defined by \( \alpha(e)_j = \alpha_j \) if \( j \in e \), and \( \alpha(e)_j = 0 \) otherwise (in particular, \( \alpha(\emptyset) = 0 \) and \( \alpha(\{1, \ldots, d\}) = \alpha \)). The space \( W^\alpha_2 \) is the Hilbert space of functions \( f \in L_2 \) equipped with the norm

\[ \|\cdot\|_{W^\alpha_2}: \|f\|_{W^\alpha_2}^2 = \sum_{e \subset \{1, \ldots, d\}} \|f(\alpha(e))\|_2^2. \]

Consider

\[ P: x \mapsto \sum_{e \subset \{1, \ldots, d\}} x^{\alpha(e)} = \sum_{\alpha \in A} x^\alpha, \]

where \( A = \{\alpha(e) \mid e \subset \{1, \ldots, d\}\} \). If the coordinates of \( \alpha \) are even, the differential operator \( P(D) \) is non-degenerate and hence, by Theorem 4.1, \( \|\cdot\|_{W^\alpha_2}^2 \) is equivalent to one of the norms in (4.2) with \( \theta(A) = A \) and \( Q = P \). We have \( \varrho(A) = \alpha_1 \) and \( \nu(A) = \nu \), and therefore, from Theorem 3.6 we recover the result proven in \([1]\), namely that for \( n \) sufficiently large

\[ d_n(U^\alpha_2, L_2(\mathbb{T}^d)) \asymp n^{-\alpha_1} (\log n)^{\nu_1}, \]

where \( U^\alpha_2 \) denotes the unit ball in \( W^\alpha_2 \). In the particular case when \( \alpha = \varrho 1 \), we have

\[ d_n(U_2^\varrho 1, L_2(\mathbb{T}^d)) \asymp n^{-\varrho} (\log n)^{(d-1)\varrho}. \]
4.4 Classes of functions with several bounded mixed derivatives

Suppose that (3.14) is satisfied. Let \( W_2^A \) be the Hilbert space of functions \( f \in L_2(\mathbb{T}^d) \) equipped with the norm

\[
\| \cdot \|_{W_2^A} : f \mapsto \sqrt{\sum_{a \in A} \|f(a)\|_2^2}.
\]

Notice that spaces \( H^s, H^r, \) and \( W_2^A \) are a particular cases of \( W_2^A \). Now consider

\[
P : x \mapsto \sum_{a \in A} x^a.
\]

If the coordinates of every \( \alpha \in \vartheta(A) \) are even, the differential operator \( P(D) \) is non-degenerate and it follows from Theorem 4.1 that \( \| \cdot \|_{W_2^A} \) is equivalent to one of the norms in (4.2). If \( \varrho = \varrho(\vartheta(A)) \) and \( \nu = \nu(\vartheta(A)) \), we again retrieve from Theorem 3.6 the result proven in [6], namely that for \( n \) sufficiently large

\[
d_n(U_2^A, L_2(\mathbb{T}^d)) \asymp n^{-\varrho}(\log n)^{\nu},
\]

where \( U_2^A \) denotes the unit ball in \( W_2^A \).

4.5 Classes of functions induced by a differential operator

We give two examples of spaces \( W_2^{[P]} \) with non-degenerate differential operator \( P(D) \) for \( d = 2 \). Consider the polynomials

\[
\begin{align*}
P_1 : x &\mapsto 8x_1^4 - 4x_1^3 - 3x_1^2 x_2 - 2x_1^2 x_2 - 4x_1 x_2 + 6x_2^2 - 4x_1 - 3x_2 + 13 \\
P_2 : x &\mapsto 6x_1^4 + x_1^3 x_2 - 6x_1^3 - 3x_1^2 x_2 + 5x_2^4 - 4x_2^3 + 3.
\end{align*}
\]

We have

\[
\begin{align*}
A_1 &= \{(4,0),(3,0),(2,1),(2,0),(1,1),(0,2),(1,0),(0,1),(0,0)\} \\
\vartheta(A_1) &= \{(4,0),(0,2),(0,0)\} \\
A_2 &= \{(6,0),(4,2),(5,0),(3,2),(0,4),(0,3),(0,0)\} \\
\vartheta(A_2) &= \{(6,0),(4,2),(0,4),(0,0)\}.
\end{align*}
\]

It is easy to verify that \( P_1(D) \) and \( P_2(D) \) are non-degenerate and that (5.14) holds. Moreover, \( \varrho(\vartheta(A_1)) = 4/3, \nu(\vartheta(A_1)) = 0, \varrho(\vartheta(A_2)) = 8/3, \) and \( \nu(\vartheta(A_2)) = 1 \). We derive from Theorem 3.6 that

\[
d_n(U^{[P_1]}, L_2(\mathbb{T}^2)) \asymp n^{-4/3},
\]

and

\[
d_n(U^{[P_2]}, L_2(\mathbb{T}^2)) \asymp n^{-8/3}(\log n)^{8/3}.
\]
Let us give an example of a degenerate differential operator. For

\[ P_3: x \mapsto x^4_1 - 2x^3_1x_2 + x^2_1x_2^2 + x^2_1 + x^2_2 + 1, \]  

(4.30)

the differential operator \( P_3(D) \) is degenerate, although \( P_3 \geq 1 \) on \( \mathbb{R}^2 \), and \( U^{[P_3]} \) is a compact set in \( L_2(\mathbb{T}^2) \). Therefore, we cannot compute \( d_n(U^{[P_3]}, L_2(\mathbb{T}^2)) \) by using Theorem 3.6. However, by a direct computation we get \( \text{card} K(t) \asymp t^{1/2} \log t \). Hence, (3.30) yields

\[ d_n(U^{[P_3]}, L_2(\mathbb{T}^2)) \asymp n^{-2} (\log n)^2. \]  

(4.31)

4.6 A conjecture

Suppose that \( U^{[P]}_2 \) is compact in \( L_2(\mathbb{T}^d) \). In view of Lemma 2.9, this is equivalent to the conditions:

(i) For every \( t \in \mathbb{R}_+ \), \( K(t) \) is finite.

(ii) \( \tau > 0 \).

As mentioned in (3.30), for every \( n \in \mathbb{N} \) sufficiently large, if \( t(n) \in \mathbb{R}_{++} \) is the maximal number such that \( \text{card} K(t(n)) \leq n \), then

\[ d_n(U^{[P]}_2, L_2(\mathbb{T}^d)) \asymp \frac{1}{t(n)}. \]  

(4.32)

This means that the problem of computing the asymptotic order of \( d_n(U^{[P]}_2, L_2(\mathbb{T}^d)) \) is equivalent to the problem of computing that of \( \text{card} K(t) \) when \( t \to +\infty \). Let us formulate it as the following conjecture.

**Conjecture 4.2** Suppose that, for every \( t \in \mathbb{R}_+ \), \( K(t) \) is finite (the condition \( \tau > 0 \) is not essential). Then there exist integers \( \alpha, \beta, \) and \( \nu \) such that \( 0 < \alpha \leq \beta, 0 \leq \nu < d \), and, for \( t \) large enough,

\[ \text{card} K(t) \asymp t^{\alpha/\beta} (\log t)^\nu. \]  

(4.33)

In view of 3.20, we know that the conjecture is true when \( P \) satisfies conditions (2.7) and (3.9).

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