Confluent primary fields in the conformal field theory

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Abstract
For any complex simple Lie algebra, we generalize the primary fields in the Wess–Zumino–Novikov–Witten conformal field theory for a case with irregular singularities. We refer to these generalized primary fields as confluent primary fields. We present the screening currents Ward identity, a recursion rule for computing the expectation values of the products of confluent primary fields. In the case of \( sl_2 \), the expectation values of the products of confluent primary fields are integral formulas of solutions to confluent Knizhnik–Zamolodchikov (KZ) equations given in Jimbo \textit{et al} (2008 \textit{J. Phys. A: Math. Theor.} 41 175205).

By computing the operator product expansion of the energy–momentum tensor \( T(z) \) and the confluent primary fields, we obtain new differential operators. Moreover, in the case of \( sl_2 \), these differential operators are the same as those of the confluent KZ equations (Jimbo \textit{et al} 2008).

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1. Introduction
In the last 25 years, many mathematicians and physicists have contributed to the study and development of the two-dimensional conformal field theory (CFT). This theory finds many applications in statistical physics and string theory. It also finds applications in many fields of mathematics, such as representation theory, integrable systems and topology.

In the Wess–Zumino–Novikov–Witten (WZNW) model, the Knizhnik–Zamolodchikov (KZ) equation plays a key role. The KZ equation is satisfied by the correlation functions of the model \([12]\) and a system of linear partial differential equations with regular singularities. Accordingly, solutions of the KZ equation are expressed in terms of integral representations of hypergeometric functions in several variables \([16]\). Furthermore, the KZ equation can be viewed as a quantization of the Schlesinger equation; this equation defines the isomonodromy deformation of linear differential equations with regular singularities \([8, 14, 17]\).
The extensions of the KZ equations to irregular singularities have been considered for the following cases. Generalized KZ equations with Poincaré rank 1 at infinity were first obtained for \(\mathfrak{sl}_2\) in [3] and later for any simple Lie algebra in [7]. For an arbitrary Poincaré rank, confluent KZ equations for \(\mathfrak{sl}_2\) were presented in [10]. The authors have presented confluent KZ equations for \(\mathfrak{sl}_N\) with Poincaré rank 2 at infinity in [13]. In all the above-mentioned cases, the solutions to the equations are represented in terms of integral representations of hypergeometric functions of the confluent type.

Integral representations of solutions to the KZ equations have also been constructed using free field realizations and Wakimoto modules [2, 5, 9, 11, 19]. Free field realizations provide a clear understanding of the WZNW CFT and also provide valuable insights on representation theory and quantum field theory. The solutions to confluent KZ equations, however, have not been constructed using free field realizations.

In this paper, using free field realizations, we generalize primary fields in the WZNW CFT for a case with irregular singularities for any simple Lie algebra \(\mathfrak{g}\). Hereafter, we refer to these generalized primary fields as confluent primary fields. We present the screening currents Ward identity, a recursion rule for computing the expectation values of the products of confluent primary fields. As seen in section 5, the integral representations for the expectation values of the products of confluent primary fields are multi-variable confluent hypergeometric functions. In the case of \(\mathfrak{sl}_2\), these integral representations are equivalent to the solutions of confluent KZ equations [10] obtained through the confluent process from solutions of the KZ equations.

Furthermore, we compute the operator product expansion (OPE) of the energy–momentum tensor \(T(z)\) and the confluent primary fields. Consequently, new differential operators corresponding to the Virasoro operators \(L_{-1}, L_0, \ldots, L_{r-1}\) appear. Note that in the case of the KZ equations, essentially all Virasoro operators correspond to the differential operator \(\partial/\partial z\). Moreover, in the case of \(\mathfrak{sl}_2\), these differential operators are the same as those of the confluent KZ equations [10].

In addition to free field realizations and OPE, we use the truncated Lie algebra \(\mathfrak{g}(r) = \mathfrak{g}[t]/t^{r+1}\mathfrak{g}[t]\), where the nonnegative integer \(r\) corresponds to the Poincaré rank, and a confluent Verma module of \(\mathfrak{g}(r)\). Confluent Verma modules have been defined in [10], and these are natural generalizations of standard Verma modules. In addition, the confluent Verma modules correspond to the non-highest weight representations of affine Lie algebras in [6].

The remainder of this paper is organized as follows. In section 2, we introduce confluent Verma modules and recapitulate Wakimoto realization of the affine Lie algebra, following [1, 2, 18]. In section 3, we define confluent primary fields and we show that the set consisting of these fields is a \(\mathfrak{g}(r)\)-module. In section 4, we compute the OPE of the energy–momentum tensor \(T(z)\) and a confluent primary field. In the final section, we present integral representations of hypergeometric functions of the confluent type from the WZNW CFT. Moreover, we write down those integral representations for the case of \(\mathfrak{sl}_2\), and we see that they coincide with solutions of the confluent KZ equations for \(\mathfrak{sl}_2\) [10].

2. Preliminary

Let \(\mathfrak{g}\) be a complex simple Lie algebra with the Cartan subalgebra \(\mathfrak{h}\) and let \(\Delta\) and \(\Delta_+\) be the set of roots and positive roots, respectively. We denote the Chevalley generators by \(e_i, f_i\) and \(h_i\) and the simple roots by \(\alpha_i\) (\(i = 1, \ldots, l = \text{rank} \mathfrak{g}\)). Let \(A = (a_{ij})\) be the Cartan matrix of \(\mathfrak{g}\). The Cartan matrix is realized as \(a_{ij} = (\nu_i, \alpha_j)\), where \(\nu_i = 2\alpha_i/\alpha_i^2\) is the coroot and \((,\)\) is the symmetric bilinear form.
2.1. The confluent (highest weight) Verma modules in the case of $g = \mathfrak{sl}_2$ have been introduced in [10]. In this section, we generalize them to the case of a complex simple Lie algebra.

To describe an irregular singularity, we use the truncated Lie algebra $g(r) = g[t]/t^{r+1}g[t]$ for $r \in \mathbb{Z}_{\geq 0}$. We denote $x \otimes t^r$ by $x[r]$. Let $b = \bigoplus_{\alpha \in \Delta_1} \mathbb{C} e_\alpha \oplus \mathfrak{h}$ and $b(r) = b[t]/t^{r+1}b[t]$. For an $(r+1)$-tuple of weights $\lambda = (\lambda_0, \ldots, \lambda_{r-1}, \lambda_r)$ with a regular element $\lambda_r$, we define a one-dimensional $b(r)$-module $C_{v\lambda}$ by

\[
e_{\alpha}[p]v_\lambda = 0, \quad h[p]v_\lambda = \lambda_p(h)v_\lambda \quad (0 \leq p \leq r, \ h \in \mathfrak{h}).
\]

(2.1)

We denote $\lambda_p(h_i)$ by $\lambda^i_p$. The parameters $\lambda^i_1, \ldots, \lambda^i_r$ are the new variables corresponding to the irregular singularities.

Consider the induced module $M(\lambda) = \text{Ind}_{b(r)}^{g(r)} C_{v\lambda}$, (2.2) hereafter called the confluent Verma module.

2.2. Following [1, 2, 9, 18], we recall free field realizations for simple Lie algebras. Let $\beta_\alpha(z)$ and $\gamma_\alpha(z)$ ($\alpha \in \Delta_+$) be boson operators with conformal weights 1 and 0 that satisfy the canonical OPE

\[
\beta_\alpha(z) \gamma_\beta(w) = \frac{\delta_{\alpha,\beta}}{z-w} + \cdots,
\]

(2.3)

where the dots denote the terms that are regular at $z = w$. We also introduce a free boson $\psi(z)$ taking value in the Cartan subalgebra and $\psi_i(z) = (v_i, \psi(z))$ ($i = 1, \ldots, l$) with the OPE

\[
\psi_j(z) \psi_i(w) = \frac{\langle v_i, v_j \rangle}{\kappa} \log(z-w) + \cdots.
\]

(2.4)

Note that we have

\[
\left( \lambda, \frac{\partial^m \psi(z)}{m!} \right) \left( \mu, \frac{\partial^n \psi(w)}{n!} \right) = \frac{\langle \lambda, \mu \rangle}{\kappa} \left( \frac{m+n}{m} \right)^{m+1} \frac{1}{m + n} \left( \frac{z-w}{m+n} \right)^{m+n} + \cdots,
\]

(2.5)

for $m, n \in \mathbb{Z}_{\geq 0}, m + n \neq 0$.

We recall the definition of the currents $E_i(z)$, $H_i(z)$ and $F_i(z)$.

Let $V(\lambda)$ be the Verma module of $g$ with the highest weight vector $|\lambda\rangle$ and $V(\lambda)^*$, the dual module of $V(\lambda)$ generated by $\langle \lambda |$ with $\langle \lambda | e_\alpha = 0$ and $\langle \lambda | h_i = \lambda^i_i$. The bilinear form $\langle , \rangle$ is defined from $\langle \lambda | \lambda \rangle = 1$.

We realize the elements of the algebra $g$ in terms of those in the polynomial ring $\mathbb{C}[x^\alpha]$ with positive roots $\alpha \in \Delta_+$ as differential operators. The differential operator $J \frac{\partial^m}{\partial x^m}, x, \lambda$ corresponding to an element $J$ in $g$ is defined by the following right action:

\[
J \left( \frac{\partial}{\partial x}, x, \lambda \right) \langle \lambda | Z = \langle \lambda | J Z,
\]

(2.6)

where $Z = \exp \left( \sum_{\alpha \in \Delta_+} x^\alpha e_\alpha \right)$.

The differential operators $E_i$, $H_i$ and $F_i$ corresponding to the generators $e_i$, $h_i$ and $f_i (i = 1, \ldots, l = \text{rank } g)$ have the following form:

\[
E_i = \sum_{\alpha \in \Delta_+} E^\alpha_i(x) \frac{\partial}{\partial x^\alpha},
\]

(2.7)
\[ H_i = \sum_{\alpha \in \Delta_+} H_{i\alpha}^a(x) \frac{\partial}{\partial x^\alpha} + \lambda^i, \]  
(2.8)

\[ F_i = \sum_{\alpha \in \Delta_+} F_{i\alpha}^a(x) \frac{\partial}{\partial x^\alpha} + \lambda^i x^\alpha, \]  
(2.9)

where \( X^a_i(x) (X = E, H, F) \) are polynomials in \( \mathbb{C}[x^a] \). These operators give the highest weight representation of \( g \) on \( \mathbb{C}[x^a] \). We call this the differential realization of \( g \).

In the differential realization, the highest weight vector \( |\lambda\rangle \) is \( 1 \in \mathbb{C}[x^a] \). For an ordered set \( I = \{ \alpha_i, \ldots, \alpha_n \} \) of simple roots \( \alpha_i, \ldots, \alpha_n \), the vectors \( P^I_{\lambda} \cdot 1 \) form the basis of the descendants of 1. This basis \( P^I_{\lambda} \) is expressed by the expectation value

\[ P^I_{\lambda} = \langle \lambda|Z \prod_{k=1}^n f_{i_k}|\lambda\rangle, \]  
(2.10)

because, by definition,

\[ F_{i_1} \cdots F_{i_n} \langle \lambda|Z = \langle \lambda|Z f_{i_1} \cdots f_{i_n}. \]  
(2.11)

For \( i = 1, \ldots, l \), let the currents be defined by

\[ E_i(z) = \sum_{\alpha \in \Delta_+} : E_{i\alpha}^a(\gamma(z)) \beta_\alpha(z) :, \]  
(2.12)

\[ H_i(z) = \sum_{\alpha \in \Delta_+} : H_{i\alpha}^a(\gamma(z)) \beta_\alpha(z) : + a_i(z), \]  
(2.13)

\[ F_i(z) = \sum_{\alpha \in \Delta_+} : F_{i\alpha}^a(\gamma(z)) \beta_\alpha(z) : + \gamma^{\alpha}(z) a_i(z) + r_i \beta^\alpha(z), \]  
(2.14)

where \( : \cdot : \) stands for the normal ordering and for \( X = E, H, F, X^a_i(\gamma(z)) (i = 1, \ldots, l) \) are obtained by replacing \( x^\alpha \) with \( \gamma^\alpha(z) \) in \( X^a_i(x) \), while \( a_i(z) = \kappa \partial \phi_i(z) \). The currents \( E_i(z), H_i(z) \) and \( F_i(z) \) satisfy the following OPEs:

\[ H_i(z) H_j(w) = \frac{k(v_i, v_j)}{(z-w)^2} + \cdots, \]  
(2.15)

\[ H_i(z) E_j(w) = \frac{a_{ij} - E_j(w)}{z-w} + \cdots, \]  
(2.16)

\[ H_i(z) F_j(w) = -\frac{a_{ij}}{z-w} F_j(w) + \cdots, \]  
(2.17)

\[ E_i(z) F_j(w) = \frac{k \delta_{i,j}}{(z-w)^2} + \frac{\delta_{i,j}}{z-w} H_i(w) + \cdots; \]  
(2.18)

these give the level \( k \) Wakimoto realization. The coefficients \( r_i \) of \( \beta^\alpha \) in (2.14) are also determined (see [1] and [2], for example).

2.3.

We also introduce differential operators for the screening currents. Let \( S_a \) be defined by

\[ S_a \left( \frac{\partial}{\partial x^a}, x \right) |\lambda\rangle = |\lambda| e_a Z \]  
(2.19)
with
\[ S_a \left( \frac{\partial}{\partial x}, x \right) = \sum_{\beta \in \Delta_\alpha} S^\beta_j(x) \frac{\partial}{\partial x^\beta}, \] (2.20)
for some polynomials \( S^\beta_j(x) \in \mathbb{C}[x]. \)

The energy–momentum tensor \( T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) is realized as
\[ T(z) = \sum_{a \in \Delta_\alpha} : \partial y^a(z) : + \sum_{i=1}^l \frac{\beta_i}{2} \partial \phi_i(z) \partial \phi_i(z) - \rho_i \partial^2 \phi_i(z) :, \] (2.21)
where \( \rho = \frac{1}{2} \sum_{a \in \Delta_\alpha} \beta_a. \) This is also realized by the Sugawara construction [15]
\[ T(z) = \frac{1}{2\kappa} \sum_{i=1}^l H_i(z) H_i(z) + \sum_{a \in \Delta_\alpha} \frac{\alpha^2}{2} (E_a(z) F_a(z) + F_a(z) E_a(z)) :, \] (2.22)
The operators \( L_n (n \in \mathbb{Z}) \) generate the Virasoro algebra.

Let us introduce the screening currents
\[ s_i(z) = S_i(z) : e^{-\alpha_i \phi(z)} :, \] (2.23)
where \( S_i(z) = \sum_{\beta \in \Delta_\alpha} S^\beta_i(\gamma(z)) \beta_{\beta}(z) : \) and the screening operators are given as
\[ Q_i = \oint s_i(z) \, dz. \] (2.24)
Then, we have the following proposition (see [1, 2, 18], for example).

**Proposition 2.1.** The products \( E_i(z)s_j(w) \) and \( H_i(z)s_j(w) \) are regular at \( z = w \), and the products of \( F_i(z)s_j(w) \) and \( T(z)s_j(w) \) are given as follows:
\[ F_i(z)s_j(w) = \kappa \delta_{ij} \frac{\partial}{\partial w} \left( \frac{1}{z - w} : e^{-\alpha_i \phi(w)} :, \right) + \cdots, \] (2.25)
\[ T(z)s_j(w) = \frac{\partial}{\partial w} \left( \frac{1}{z - w} s_j(w) \right) + \cdots. \] (2.26)

The above proposition implies that the screening operators \( Q_i \) commute with the currents and the energy–momentum tensor \( T(z) \).

2.4.

We consider the differential realization corresponding to the confluent Verma module. Let \( M(\lambda) \) be a confluent Verma module of \( g_r \), with weights \( (\lambda_0, \lambda_1, \ldots, \lambda_r) \) and \( M(\lambda)^* \) the dual of \( M(\lambda) \). We replace \( Z \) in the regular case with \( Z = \exp \left( \sum_{i=0}^r \sum_{\alpha \in \Delta_+} x_\alpha e_\alpha[i] \right) \). In the same manner as the regular case, we define the differential realization of \( g_r \) on \( \mathbb{C}[x_\alpha^i] \) \((\alpha \in \Delta_+), \ 0 \leq i \leq r \) as
\[ J \left( \frac{\partial}{\partial x}, x, \lambda \right) | \lambda \rangle = \langle \lambda | J \] (2.27)
for \( J \in \mathfrak{g}_r \). We also introduce an irregular version of \( S_a \) as
\[ S_a[p] | \lambda \rangle | Z = \langle \lambda | e_a[p] | Z (p = 0, \ldots, r). \] (2.28)
The differential operators \( X[p] (X = E_i, H_i, F_i) \) corresponding to \( x[p] \in \mathfrak{g}_r \) \((x = e_i, h_i, f_i) \) and \( S_a[p] \) are given by
for some polynomials $X_{\alpha}^{\beta}[p](x)$ that are obtained by replacing monomials $x^{\beta_1} \cdots x^{\beta_m}$ ($\beta_1, \ldots, \beta_m \in \Delta_+$) in $X_{\alpha}^{\beta}(x)$ with

$$\sum_{j_1 + \cdots + j_p = q} x^{\beta_1}_{j_1} \cdots x^{\beta_m}_{j_m}. \tag{2.34}$$

For an ordered set $I = \{(\alpha_{i_1}, k_{i_1}), \ldots, (\alpha_{i_n}, k_{i_n})\}$ ($\alpha_{i_1}, \ldots, \alpha_{i_n} \in \Delta_+$, $0 \leq k_{i_1}, \ldots, k_{i_n} \leq r$), we define a polynomial $P_{\lambda}^{I}(x)$ in $\mathbb{C}[x_\alpha]$ as

$$P_{\lambda}^{I}(x) = \langle \lambda | Z \prod_{j=1}^{m} f_{\alpha_j} [k_j] | \lambda \rangle. \tag{2.35}$$

Note that we have

$$P_{\lambda}^{I}(x) = F_{i_1}[k_{i_1}] \cdots F_{i_n}[k_{i_n}] \cdot 1, \tag{2.36}$$

because, by definition,

$$F_{i_1}[k_{i_1}] \cdots F_{i_n}[k_{i_n}] \langle \lambda | Z f_{\alpha_1} [k_{i_1}] \cdots f_{\alpha_n} [k_{i_n}] \rangle = \langle \lambda | Z f_{\alpha_1} [k_{i_1}] \cdots f_{\alpha_n} [k_{i_n}] \rangle \tag{2.37}$$

holds.

### 3. Confluent primary field

In this section, we introduce confluent primary fields; these are natural generalizations of primary fields with irregular singularities in the WZNW CFT.

#### 3.1.

For an $(r + 1)$-tuple of weights $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$ with a regular element $\lambda_r$, we set

$$v_{\lambda}(z) =: \exp \left( \sum_{i=0}^{r} \frac{\partial^i \varphi(z)}{i!} \right), \tag{3.1}$$

and we define a vector space $V$ over $\mathbb{C}$ generated by

$$\{ P_{\lambda}^{I}(\gamma(z)) v_{\lambda}(z) \}, \tag{3.2}$$

where $P_{\lambda}^{I}(\gamma(z))$ is a polynomial of bosons $\partial^i \gamma^\alpha(z)$ ($\alpha \in \Delta_+$, $0 \leq i \leq r$) that are obtained by replacing $x_\alpha^{\beta}$ in the polynomial $P_{\lambda}^{I}(x)$ for an ordered set $I = \{(\alpha_{i_1}, k_{i_1}), \ldots, (\alpha_{i_n}, k_{i_n})\}$.
From (2.1) we define the action of $X[n]$, where $X(z) = \sum_{n \in \mathbb{Z}} X[n] z^{-n-1}$, on the element $\Phi(w)$ as

$$X[n] \Phi(w) = \oint \frac{dz}{2\pi i} (z-w)^n X(z) \Phi(w). \quad (3.3)$$

Since a polynomial $P^i_j(\gamma(z))$ consists of $\partial^i \gamma^a(z)$ ($a \in \Delta_+$, $0 \leq i \leq r$) and $v_i(z)$ consists of $\partial^i \phi(z)$ ($0 \leq i \leq r$), the OPE between $X(z)$ and $\Phi(w)$ and the definition (3.3) induce

$$X[n] \Phi(w) = 0 \quad (n > r). \quad (3.4)$$

We call elements $\Phi(z) \in \mathcal{P}$ confluent primary fields.

By (3.3), the loop algebra $\mathfrak{gl} \otimes \mathbb{C}[t, t^{-1}]$ acts on the space of confluent primary fields $\mathcal{P}$, and the Lie subalgebra $\mathfrak{gl} \otimes t^{r+1} \mathbb{C}[r]$ annihilates the vectors $\Phi(z)$ of $\mathcal{P}$ because of (3.4). Hence, the vector space $\mathcal{P}$ generates a non-highest weight representation in [4] and [6]. Moreover, we have the next proposition.

**Proposition 3.1.** The vector space of confluent primary fields $\mathcal{P}$ is a $\mathfrak{gl}(r)$-module with the highest weight vector $v_i(z)$ such that

$$e_i[p] v_i(z) = 0, \quad h_i[p] v_i(z) = \lambda^i_p v_i(z) \quad (1 \leq i \leq l, \ 1 \leq p \leq r), \quad (3.5)$$

and

$$f_i[k_i] \cdots f_i[k_i] v_i(z) = P^i_k(\gamma(z)) v_i(z), \quad (3.6)$$

where $I = \{(\alpha_i, k_i), \ldots, (\alpha_i, k_i)\} (\alpha_i, \ldots, \alpha_i \in \Delta_+, 0 \leq k_i, \ldots, k_i \leq r)$.

**Proof.** Since the current $E_i(t)$ does not depend on free bosons $\phi(t)$, the OPE of $E_i(t)$ and $: \exp \left( \sum_{i=0}^r \lambda_i \frac{\partial^i \gamma(z)}{t^i} \right) :$ does not have singular parts. Hence, $e_i[p]$ annihilates the element $v_i(z)$.

Computing the OPE of $a_i(t)$ and $: \exp \left( \sum_{i=0}^r \lambda_i \frac{\partial^i \gamma(z)}{t^i} \right) :$, we observe that the elements $h_i[p] (1 \leq i \leq l, 0 \leq p \leq r)$ act as $\lambda^i_p$ on $v_i(z)$.

We now prove (3.6). By definition, we have

$$f_j[p] : \exp \left( \sum_{i=0}^r \lambda_i \frac{\partial^i \gamma(z)}{t^i} \right) : = \oint \frac{dz}{2\pi \sqrt{-1}} (t-z)^p F_{\lambda_i}(t) : \exp \left( \sum_{i=0}^r \lambda_i \frac{\partial^i \gamma(z)}{t^i} \right) : \quad (3.7)$$

From (2.14), we only need to compute the OPE of $\gamma^a(t)a_j(t)$. Hence, the right-hand side of (3.7) equals

$$\oint \frac{dz}{2\pi \sqrt{-1}} (t-z)^p : \sum_{i=0}^r \lambda_i \frac{\partial^i \gamma(z)}{(t-z)^{i+1}} \gamma^a(t) \exp \left( \sum_{i=0}^r \lambda_i \frac{\partial^i \gamma(z)}{t^i} \right) : \quad (3.8)$$

By taking the Taylor expansion of $\gamma^a(t)$ at $z$, (3.8) equals

$$\sum_{i=p}^r \lambda_i \frac{\partial^i-p \gamma^a(z)}{(t-p)!} : \exp \left( \sum_{i=0}^r \lambda_i \frac{\partial^i \gamma(z)}{t^i} \right) : \quad (3.9)$$
Therefore, recalling (2.31), we obtain
\[ f_j[p] : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) = P_{\lambda}^{(q_{\lambda},p)}(\gamma(z)) : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) :. \] (3.10)

For a polynomial \( g(x) \in \mathbb{C}[x] \), we can verify
\[ f_j[p]g(\gamma(z)) : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) : = (F_j[p]g)(\gamma(z)) : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) :. \] (3.11)
in the same manner as above. Therefore, we obtain

\[ f_{i_1}[k_1] \cdots f_{i_n}[k_n] : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) : = \langle \lambda | Z f_{i_1}[k_1] \cdots f_{i_n}[k_n] \rangle (\gamma(z)) : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) : = P^{\prime}_{\lambda}(\gamma(z)) : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) :.
\]

\[ \square \]

4. Operator product expansion

In this section, we compute the OPE of the energy–momentum tensor and confluent primary fields. For \( k = 0, 1, \ldots, r-1 \), let \( \overline{D}_k \) be an endomorphism of \( \mathcal{P} \) defined by
\[ \overline{D}_k \left( \langle \lambda | Z \prod_{j=1}^{n} f_{a_j}[\lambda] \rangle (\gamma(z)) : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) : \right) = \langle \lambda | d_k(z) \prod_{j=1}^{n} f_{a_j}[\lambda] \rangle (\gamma(z)) D_k \left\{ : \exp \left( \sum_{i=0}^{r} \frac{\lambda_i}{i!} \frac{\partial_i \varphi(z)}{i!} \right) : \right\}, \] (4.1)

where \( d_k \) is the derivation given by \( d_k(x[p]) = px[p+k] \) (\( x[p] \in \mathfrak{g}(r) \)) and \( D_k \) is the differential operator given by \( D_k = \sum_{j=1}^{r} \sum_{p=1}^{r-k} p\lambda_{p+1} \partial/\partial \lambda_p \). We note that in the case of \( sl_2 \), the derivations \( d_k + D_k \) \( (k = 0, 1, \ldots, r-1) \) acting on the confluent Verma module are the same as those of the confluent KZ equation [10].

**Proposition 4.1.** The operator product expansion of the energy–momentum tensor \( T(z) \) and the confluent primary field \( \Phi(w) \in \mathcal{P} \) is given as follows:
\[ T(z) \Phi(w) = \sum_{k=0}^{r-1} \frac{1}{(z-w)^{k+2}} \overline{D}_k \Phi(w) + \frac{1}{z-w} \partial_w \Phi(w) \]
\[ + \frac{1}{2\kappa} \left( \sum_{p=0}^{r} \frac{\lambda_p}{(z-w)^{p+1}} \right)^2 \Phi(w) + \frac{1}{\kappa} \sum_{p=0}^{r} \frac{(p+1)(p+\lambda_p)}{(z-w)^{p+2}} \Phi(w) + \cdots. \] (4.2)

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Proof. Using the OPE between the bosons, we obtain
\[
T(z)\Phi(w) = \sum_{p=0}^r \frac{\lambda_p}{(z-w)^{p+1}} :\partial_z \psi(z)\Phi(w) : + \frac{1}{2\kappa} \left( \sum_{p=0}^r \frac{\lambda_p}{(z-w)^{p+1}} \right)^2 \Phi(w) + \frac{1}{\kappa} \sum_{p=0}^r \frac{(p+1)\lambda_p}{(z-w)^{p+2}} \Phi(w) 
- \sum_{a \in \Delta_p} \sum_{p=0}^r \frac{e_a[p]}{(z-w)^{p+1}} :\partial_z \gamma^a(z)\Phi(w) : + \ldots.
\]

Taking the Taylor expansion at \( z = w \) in the above, we obtain
\[
T(z)\Phi(w) = \sum_{k=0}^r \frac{1}{(z-w)^{k+1}} \left( \sum_{p=0}^r \frac{\lambda_{p+k}}{p!} :\partial_w^{p+1} \psi(w)\Phi(w) : \right) 
- \sum_{a \in \Delta_p} \sum_{p=0}^r \frac{e_a[p+1]}{p!} :\partial_w^{p+1} \gamma^a(w)\Phi(w) : 
+ \frac{1}{2\kappa} \left( \sum_{p=0}^r \frac{\lambda_p}{(z-w)^{p+1}} \right)^2 \Phi(w) + \frac{1}{\kappa} \sum_{p=0}^r \frac{(p+1)(\rho, \lambda_p)}{(z-w)^{p+2}} \Phi(w) + \ldots.
\]

On the other hand, we have
\[
\partial_w\Phi(w) = \sum_{p=0}^r \frac{\lambda_p}{p!} :\partial_w^{p+1} \psi(w)\Phi(w) : - \sum_{a \in \Delta_p} \sum_{p=0}^r \frac{e_a[p]}{p!} :\partial_w^{p+1} \gamma^a(w)\Phi(w) :-
\]
and
\[
D_k\Phi(w) = \sum_{p=0}^r \frac{\lambda_{p+k}}{p!} :\partial_w^{p+1} \psi(w)\Phi(w) : 
- \sum_{a \in \Delta_p} \sum_{p=0}^r \frac{e_a[p+k+1]}{p!} :\partial_w^{p+1} \gamma^a(w)\Phi(w) : (k = 0, \ldots, r-1).
\]

This completes the proof. \( \square \)

Corollary 4.2. For \( n \in \mathbb{Z} \), we have
\[
[L_n, \Phi(w)] = u^{n+1} \partial_w \Phi(w) + \sum_{k=0}^{r-1} \frac{(n+1)!}{(n-k)!} u^{n-k} D_k \Phi(w) 
+ \frac{1}{2\kappa} \sum_{k=0}^{2r} \sum_{p=0}^r (\lambda_p, \lambda_q) \frac{(n+1)!}{(n-k)!} u^{n-k} \Phi(w) 
+ \frac{1}{\kappa} \sum_{k=0}^{r} (k+1)(\rho, \lambda_k) \frac{(n+1)!}{(n-k)!} u^{n-k} \Phi(w). \tag{4.3}
\]

Here, for \( k \geq n+1 \), we set \( \frac{(n+1)!}{(n-k)!} = 0 \).

We note that when \( n = -1 \), relation (4.3) reduces to
\[
[L_{-1}, \Phi(w)] = \partial_w \Phi(w). \tag{4.4}
\]
5. Integral representation

In this section, following [1, 2, 18], we compute an expectation value of the composition of confluent primary fields multiplied by the screening operators. In the case of $g = \mathfrak{sl}_2$, we see that the integral representations derived from confluent primary fields coincide with solutions to the confluent KZ equations for $\mathfrak{sl}_2$ [10].

5.1.

For $r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0}$, let $\lambda^{(a)}$ be $(r_a + 1)$-tuple weights $((\lambda^{(a)})_0, \ldots, (\lambda^{(a)})_{r_a})$ with a regular element $\gamma^{(a)}$. $P_a(v)$ denotes a polynomial $[(\lambda^{(a)})Z|v_a]$ for $v_a \in M(\lambda^{(a)})$ and $P_a(\gamma(z_a))$ a polynomial of $\partial^i\gamma^a(z_a)$ $i = 0, \ldots, r_a$, $a \in \Delta$, obtained by replacing $x^a$ with $\partial^i\gamma^a(z_a)/i!$ in $P_a(v_a)$. An integral representation of a hypergeometric function of the confluent type is given by an expectation value

$$\left\langle \int \prod_{i=1}^m dt_i : e^{-\alpha_i\psi(t_i)} : S_i(t_i) \prod_{a=1}^n P_a(\gamma(z_a)) : \exp \left( \sum_{i=0}^{r_a} (\lambda^{(a)})_i \frac{\partial^i\psi(z_a)}{i!} \right) : \right\rangle, \quad (5.1)$$

where $i$ is an element in $[1, \ldots, l = \text{rank } g]$, $\alpha_i$ $(i = 1, \ldots, m)$ are simple roots and $: e^{-\alpha_i\psi(t_i)} : S_i(t_i)$ are the screening currents defined in (2.23).

Let us calculate the $\psi$ field correlation and the $\beta\gamma$ correlation separately. First, we compute the $\psi$ field correlation

$$\Psi(t, z, \lambda) = \left\{ \prod_{i=1}^m : e^{-\alpha_i\psi(t_i)} : \prod_{a=1}^n : \exp \left( \sum_{i=0}^{r_a} (\lambda^{(a)})_i \frac{\partial^i\psi(z_a)}{i!} \right) : \right\}. \quad (5.2)$$

Recalling the OPE of $\partial^i\psi(z)$ and $\partial^j\psi(w)$ (2.5), we obtain

$$\Psi(t, z, \lambda) = \prod_{1 \leq a < b \leq n} \left( z_a - z_b \right)^{(\lambda^{(a)})_0(\lambda^{(b)})_0}/\kappa \times \exp \left( \sum_{0 \leq p < q, p+q \neq 0}^m \frac{(\lambda^{(a)})_p(\lambda^{(b)})_q}{\kappa} \frac{(-1)^{p+1}}{p+q} \frac{1}{(z_a - z_b)^{p+q}} \right) \times \prod_{1 \leq i < j \leq m} \left( t_i - t_j \right)^{\epsilon_{ij}} \prod_{i=1}^m \prod_{a=1}^n \left( t_i - z_a \right)^{-\epsilon_{i1}(\lambda^{(a)})_1}/\kappa \times \exp \left( \sum_{0 \leq p \neq 0}^m \frac{\alpha_i(\lambda^{(a)})_p}{\kappa} \frac{1}{p(t_i - z_a)^p} \right) \right\}. \quad (5.3)$$

5.2.

Next, we compute the $\beta\gamma$ correlation

$$\omega = \left\langle \prod_{i=1}^m S_i(t_i) \prod_{a=1}^n P_a(\gamma(z_a)) \right\rangle \quad (5.3)$$
using the OPEs

\[ S_\alpha(z)S_\beta(w) = \frac{1}{z - w} [S_\alpha, S_\beta](w) + \cdots, \]  
\[ S_\alpha(z)P_a(\gamma(w)) = \sum_{p=0}^{r_a} \frac{1}{(z - w)^{p+1}} (S_\alpha[p] P_a)(\gamma(w)) + \cdots, \]

where \( \alpha, \beta \in \Delta^+, [S_\alpha, S_\beta](w) \) is obtained by replacing \( x_\alpha \) with \( \gamma_\alpha(w) \) and \( \partial_x \) with \( \beta_\alpha(w) \) in the differential operator \( [S_\alpha, S_\beta] \), and \( (S_\alpha[p] P_a)(\gamma(w)) \) is obtained by replacing \( x_\alpha \) with \( \partial_i \gamma_\alpha(w)/i! \) in the polynomial \( S_\alpha[p] P_a \).

Then, we obtain the screening currents Ward identity

\[ \omega = \left\langle S_1(t_1) \cdots S_m(t_m) \prod_{a=1}^{n} P_a(\gamma(z_a)) \right\rangle = \sum_{\sigma \in S_m} \sigma \left( \sum_{(I_1, \ldots, I_n) \in Y} \prod_{a=1}^{n} P(a, I_a) \right), \]

(5.7)

where \( S_m \) is the set of all permutations of \( \{1, \ldots, m\} \) and \( Y \), a set of \((I_1, \ldots, I_n)\) such that \( I_f = I_1 < \cdots < I_k \) and \( I_1 \cup I_2 \cup \cdots \cup I_n = \{1, \ldots, m\} \) is a disjoint union, and for \( I_a = \{i_1, \ldots, i_k\} \),

\[ P(a, I_a) = \sum_{p_1, \ldots, p_k = 0} [S[p_1] \cdots S[p_k] P_a] \left( \prod_{(I_1, \ldots, I_n) \in Y} \prod_{a=1}^{n} P(a, I_a) \right), \]

(5.8)

where \( \left[f(x)\right] \) represents the constant term of \( f(x) \in \mathbb{C}[x] \).
For a polynomial $P_a = \langle \lambda(a) | Z[f[q_1] \cdots f[q_s]] | \lambda(a) \rangle$, from the definition of the differential realization (2.28), we have

$$S[p_1] \cdots S[p_k] P_a = \langle \lambda(a) | e[p_k] \cdots e[p_1] Z[f[q_1] \cdots f[q_s]] | \lambda(a) \rangle.$$  (5.9)

Since the constant term of this polynomial is given by the value at $x_i = 0$, that is, $Z = 1$, we obtain

$$[S[p_1] \cdots S[p_k] P_a] = \langle \lambda(a) | e[p_k] \cdots e[p_1] f[q_1] \cdots f[q_s]] | \lambda(a) \rangle.$$  (5.10)

Therefore, an element $u$ in $M = M(\lambda(1)) \otimes \cdots \otimes M(\lambda(n))$ is determined by the relation of the integral representation and the pairing

$$\int \prod_{i=1}^m dt_i \Psi(t, z, \lambda) \omega = \langle u^* | v_1 \otimes \cdots \otimes v_n \rangle,$$  (5.11)

where the element $\langle u^* \rangle$ in the dual module $M^*$ is the dual vector corresponding to $u$ and $v_a \in M(\lambda(a))$ ($a = 1, \ldots, n$), and this element $u$ coincides with the integral representation for the solution of the confluent KZ equation for $\mathfrak{sl}_2$.

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References

[1] Awata H 1992 Screening currents Ward identity and integral formulas for the WZNW correlation functions Prog. Theor. Phys. Suppl. 110 303–19
[2] Awata H, Tsuchiya A and Yamada Y 1991 Integral formulas for the WZNW correlation functions Nucl. Phys. B 365 680–96
[3] Babujian H M and Kitaev A V 1998 Generalized Knizhnik–Zamolodchikov equations and isomonodromy quantization of the equations integrable via the inverse scattering transform: Maxwell–Bloch system J. Math. Phys. 39 2499–506
[4] Fedorov R 2010 Irregular Wakimoto modules and the Casimir connection Sel. Math. New Ser. (arXiv:0812.4472v2 [math.RT])
[5] Feigin B and Frenkel E 1990 Affine Kac–Moody algebras and semi-infinite flag manifolds Commun. Math. Phys. 128 161–89
[6] Feigin B, Frenkel E and Toledano Laredo V 2010 Gaudin models with irregular singularities Adv. Math. 223 873–948
[7] Felder G, Markov Y, Tarasov V and Varchenko A 2000 Differential equations compatible with KZ equations Math. Phys. Anal. Geom. 3 139–77
[8] Harnad J 1996 Quantum isomonodromic deformations and the Knizhnik–Zamolodchikov equations Symmetries and Integrability of Difference Equations (Estérel, PQ, 1994) (CRM Proc. Lecture Notes vol 9) (Providence, RI: American Mathematical Society) pp 155–61 (arXiv:hep-th/9406078v2)
[9] Ito K and Komata S 1991 Feigin-Fuchs representations of arbitrary affine Lie algebras Mod. Phys. Lett. A 6 581–9
[10] Jimbo M, Nagoya H and Sun J 2008 Remarks on the confluent KZ equation for $\mathfrak{sl}_2$ and quantum Painlevé equations J. Phys. A: Math. Theor. 41 175205
[11] Kuroki G 1991 Fock space representations of affine Lie algebras and integral representations in Wess–Zumino–Witten models Commun. Math. Phys. 142 511–42
[12] Knizhnik V G and Zamolodchikov A B 1984 Current algebra and Wess–Zumino model in two dimensions Nucl. Phys. B 247 83
[13] Nagoya H and Sun J 2010 Confluent KZ equations for $\mathfrak{sl}_N$ with Poincaré rank 2 at infinity arXiv:1002.2273v2 [math-ph]
[14] Reshetikhin N 1992 The Knizhnik–Zamolodchikov system as a deformation of the isomonodromy problem
Lett. Math. Phys. 26 167–77
[15] Sugawara H 1968 A field theory of currents Phys. Rev. 170 1659
[16] Schechtman V V and Varchenko A 1990 Hypergeometric solutions of Knizhnik–Zamolodchikov equations Lett.
Math. Phys. 20 279–83
[17] Takasaki K 1998 Gaudin model, KZ equation and an isomonodromic problem on the torus Lett. Math.
Phys. 44 143–56
[18] Yamada Y 2006 Introduction to the Conformal Field Theory (Japan: Baifukan) (in Japanese)
[19] Wakimoto M 1986 Fock representation of the algebra $A_1^{(1)}$ Commun. Math. Phys. 104 605–9