Comments on “Testing Conditional Independence of Discrete Distributions”

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Abstract
In this short note, we identify and address an error in the proof of Theorem 1.3 in Canonne et al. (2018), a recent breakthrough in conditional independence testing. After correcting the error, we show that the general sample complexity result established in Canonne et al. (2018) remains the same.

1 A Closer Look at Canonne et al. (2018)

Let \((X, Y, Z)\) follow a distribution \(p_{X,Y,Z}\) on a discrete domain \([\ell_1] \times [\ell_2] \times [n]\). Given \(m\) i.i.d. copies of \((X, Y, Z)\), the work of Canonne et al. (2018) is concerned with the problem of testing conditional independence between \(X\) and \(Y\) given \(Z\). Let \(\mathcal{P}\) denote the set of all discrete distributions over \([\ell_1] \times [\ell_2] \times [n]\) and \(\mathcal{P}_0 \subset \mathcal{P}\) denote the set of null distributions such that \(X \perp \!\!\!\perp Y \mid Z\). Consider the set of alternative distributions, denoted by \(\mathcal{P}_1(\varepsilon)\), which are \(\varepsilon\) far way from \(\mathcal{P}_0\) in the total variation distance:

\[
\mathcal{P}_1(\varepsilon) = \left\{ p \in \mathcal{P} : \inf_{q \in \mathcal{P}_0} \sup_{A \subseteq [\ell_1] \times [\ell_2] \times [n]} |p(A) - q(A)| \geq \varepsilon \right\}.
\]

Given these sets of distributions, the hypotheses of interest can be formulated as

\[
H_0 : p_{X,Y,Z} \in \mathcal{P}_0 \quad \text{versus} \quad H_1 : p_{X,Y,Z} \in \mathcal{P}_1(\varepsilon).
\]

For the above testing problem, Theorem 1.3 of Canonne et al. (2018) proves that there exists a test with sample complexity

\[
O\left(\max\left\{ \min\left\{ \frac{n^{7/8} \ell_1^{1/4} \ell_2^{1/4}}{\varepsilon}, \frac{n^{6/7} \ell_1^{2/3} \ell_2^{2/3}}{\varepsilon^{3/7}} \right\}, \frac{n^{3/4} \ell_1^{1/2} \ell_2^{1/2}}{\varepsilon}, \frac{n^{2/3} \ell_1^{2/3} \ell_2^{1/3}}{\varepsilon^{2/3}}, \frac{n^{1/2} \ell_1^{1/2} \ell_2^{1/2}}{\varepsilon^2} \right\}\right).
\]

(1)
This result is achieved under Poissonization where the sample size $M$ follows a Poisson random variable with parameter $m$. The proof of Theorem 1.3 relies on several innovative lemmas including Claim 2.2:

**Claim 2.2 of Canonne et al. (2018).** There exists an absolute constant $C > 0$ such that, for $X \sim \text{Poisson}(\lambda)$ and $a, b \geq 0$,

$$\text{Var} [X \sqrt{\min(X, a) \min(X, b)} \mathbb{1}(X \geq 4)] \leq C \mathbb{E} [X \sqrt{\min(X, a) \min(X, b)} \mathbb{1}(X \geq 4)].$$

Unfortunately, the upper bound in the claim turns out to be incorrect and therefore the proof of their Theorem 1.3 remains incomplete (confirmed via personal communication with one of the authors). We fix this error in Lemma 1 below, which is the main technical contribution of this work. To illustrate the error in Claim 2.2, suppose that the parameter $\lambda$ is sufficiently large such that $X$ is approximately $N(\lambda, \lambda)$ by the central limit theorem. Moreover, by taking $a$ and $b$ relatively small in comparison to $\lambda$, one can have an approximation

$$X \sqrt{\min(X, a) \min(X, b)} \mathbb{1}(X \geq 4) \approx N(\lambda, \lambda) \sqrt{ab}.$$ 

Hence the variance and the expectation in Claim 2.2 approximately become $\lambda ab$ and $\lambda \sqrt{ab}$, respectively, which contradicts the given inequality when $a, b \geq 1$.

**2 Main Result**

We now show that the upper bound in Claim 2.2 holds if we add an additional factor of $\max\{ab, \sqrt{ab}, \sqrt{ab}\}$ for $b \geq a \geq 0$ in the upper bound.

**Lemma 1 (Correction of Claim 2.2).** There exists an absolute constant $C > 0$ such that, for $X \sim \text{Poisson}(\lambda)$ and $b \geq a \geq 0$,

$$\text{Var} [X \sqrt{\min(X, a) \min(X, b)} \mathbb{1}(X \geq 4)]$$

$$\leq C \max\{ab, \sqrt{ab}, \sqrt{ab}\} \mathbb{E} [X \sqrt{\min(X, a) \min(X, b)} \mathbb{1}(X \geq 4)].$$

It is worth pointing out that one can improve the factor of $\max\{ab, \sqrt{ab}, \sqrt{ab}\}$ in the upper bound
with more effort. Nevertheless, the current form is sharp enough to reprove the sample complexity (1). We also note that in its application, \( a \) and \( b \) correspond to the domain sizes \( \ell_1 \geq 2 \) and \( \ell_2 \geq 2 \), respectively, in which case the additional factor becomes \( \max\{ab, \sqrt{ab}, \sqrt{ab}\} = ab \). Due to the non-linearity of \( X \sqrt{\min(X, a)} \min(X, b) \mathbb{1}(X \geq 4) \) in \( X \), the proof of Lemma 1 turns out to be non-trivial, requiring a delicate case-by-case analysis. The details can be found in Section 3.

Having provided a correction of Claim 2.2, we next reprove Theorem 1.3 of Canonne et al. (2018). There is essentially one place where Claim 2.2 is used in Canonne et al. (2018), more specifically in their Lemma 5.3. Once we establish Lemma 5.3, the rest of the proof remains the same. Before stating the result, we need to introduce some notation. For simplicity, we denote the conditional distribution of \( X \) and \( Y \) given \( Z = z \) by \( p_z \) and the product distribution of their marginals by \( q_z \). Define \( \varepsilon_z = \sup_{A \subseteq [n]} |p_z(A) - q_z(A)| \) and write \( \varepsilon'_z = \frac{\varepsilon_z}{\sqrt{4\ell_1\ell_2}} \). We further let \( \omega_z = \sqrt{\min(\sigma_z, \ell_1)} \min(\sigma_z, \ell_2) \). With this notation in place, we reprove Lemma 5.3 of Canonne et al. (2018) as follows.

**Lemma 2** (Lemma 5.3 of Canonne et al. (2018)). Consider \( D = \sum_{z \in [n]} \sigma_z \omega_z \varepsilon'^4_z \mathbb{1}(\sigma_z \geq 4) \) where \( \sigma_1, \ldots, \sigma_n \) are independent Poisson random variables with the corresponding parameters \( m\mathbb{P}(Z = 1), \ldots, m\mathbb{P}(Z = n) \). Then there exists a constant \( C > 0 \) such that \( \text{Var}[D] \leq C \mathbb{E}[D] \).

**Proof.** Using the independence of \( \sigma_1, \ldots, \sigma_n \), we have the identity

\[
\text{Var}[D] = \sum_{z \in [n]} \varepsilon'^4_z \text{Var}[\sigma_z \omega_z \mathbb{1}(\sigma_z \geq 4)].
\]

Moreover, Lemma 1 along with a trivial bound \( \varepsilon'_z \leq 1/\sqrt{4\ell_1\ell_2} \) yields

\[
\sum_{z \in [n]} \varepsilon'^4_z \text{Var}[\sigma_z \omega_z \mathbb{1}(\sigma_z \geq 4)] \leq C \sum_{z \in [n]} \varepsilon'^4_z \ell_1 \ell_2 \mathbb{E}[\sigma_z \omega_z \mathbb{1}(\sigma_z \geq 4)]
\]

\[
\leq C \sum_{z \in [n]} \varepsilon'^2_z \mathbb{E}[\sigma_z \omega_z \mathbb{1}(\sigma_z \geq 4)] = C \mathbb{E}[D].
\]

This completes the proof of Lemma 2, which in turn proves Theorem 1.3 of Canonne et al. (2018). \( \square \)
3 Proof of Lemma 1

It now remains to prove Lemma 1. We begin with the notation used in the proof.

**Notation.** We use \( C, C' \) to denote positive absolute constants whose values may vary from line to line. For constants \( x \) and \( y \), we write \( x \lesssim y \) (resp. \( x \gtrsim y \)) if there exists an absolute positive constant \( C \) (resp. \( C' \)) such that \( x \leq Cy \) (resp. \( x \geq C'y \)). \( a \wedge b \) indicates \( \min(a,b) \). For simplicity, we often write

\[
X_{a,b} := X \sqrt{(X \wedge a)(X \wedge b)} \mathbb{1}(X \geq 4).
\]

Our proof relies on two other claims in Canonne et al. (2018). We recall them here for completeness.

**Claim 2.1 of Canonne et al. (2018).** For \( X \sim \text{Poisson}(\lambda) \), the following inequality holds

\[
\text{Var}[X \mathbb{1}(X \geq 4)] \lesssim \mathbb{E}[X \mathbb{1}(X \geq 4)].
\]

(2)

**Claim 2.3 of Canonne et al. (2018)** For \( X \sim \text{Poisson}(\lambda) \) and integers \( a, b \geq 2 \), the following inequality holds

\[
\mathbb{E}[X \sqrt{(X \wedge a)(X \wedge b)} \mathbb{1}(X \geq 4)] \gtrsim (\lambda \sqrt{(\lambda \wedge a)(\lambda \wedge b)} \wedge \lambda^4).
\]

(3)

We split the proof into two parts depending on whether \( a \geq 2 \) or \( 0 \leq a < 2 \).

3.1 Case where \( a \geq 2 \)

We start with the first case where \( a \geq 2 \). In our analysis, it is more convenient to work with an alternative form of the variance of \( X_{a,b} \). First notice that for any random variable \( W \) and its i.i.d. copy \( W' \), the variance of \( W \) is equal to

\[
\text{Var}[W] = \frac{\mathbb{E}[(W - W')^2]}{2}.
\]
Let $Y$ be an i.i.d. copy of $X \sim \text{Poisson}(\lambda)$. Using the alternative formula of the variance, we have the identity given as

$$2\text{Var}[X_{a,b}] = \mathbb{E}[(X \sqrt{(X \wedge a)(X \wedge b)} \mathbb{1}(X \geq 4) - Y \sqrt{(Y \wedge a)(Y \wedge b)} \mathbb{1}(Y \geq 4))^2].$$

Equivalently, by letting $p$ be the probability mass function (pmf) of $\text{Poisson}(\lambda)$,

$$2\text{Var}[X_{a,b}] = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \left( x \sqrt{(x \wedge a)(x \wedge b)} - y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y) \tag{I}$$

$$+ \sum_{x=4}^{\infty} \sum_{y=0}^{3} \left( x \sqrt{(x \wedge a)(x \wedge b)} \right)^2 p(x)p(y) \tag{II}$$

$$+ \sum_{x=0}^{3} \sum_{y=4}^{\infty} \left( y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y), \tag{III}$$

where we use the fact that both $X_{a,b}$ and $Y_{a,b}$ are equal to zero when $x \leq 3$ and $y \leq 3$.

**Analysis of (II) and (III).** Let us start bounding the second term (II) as it is simpler to analyze than the first term (I). The third term can be handled similarly by symmetry. We proceed by considering the two cases (i) $\lambda \geq 1$ and (ii) $\lambda < 1$, separately.

- Suppose that $\lambda \geq 1$. Using the explicit form of the Poisson pmf and its fourth moment expression, we have

$$\sum_{x=4}^{\infty} \sum_{y=0}^{3} \left( x \sqrt{(x \wedge a)(x \wedge b)} \right)^2 p(x)p(y) \leq \sum_{x=4}^{\infty} x^4 p(x) \sum_{y=0}^{3} p(y)$$

$$\leq \mathbb{E}[X^4] \left( 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} \right) e^{-\lambda}$$

$$= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) \left( 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} \right) e^{-\lambda}.$$

We would like to verify that the above bound is less than or equal to $\mathbb{E}[X_{a,b}]$, up to a constant.
To this end, recall Claim 2.3 of Canonne et al. (2018) in (3):

$$
\mathbb{E}[X_{a,b}] \gtrsim (\lambda \sqrt{\lambda \land a)(\lambda \land b) \land \lambda^4}).
$$

Since we assume $a, b \geq 2$, we have $\sqrt{(\lambda \land a)(\lambda \land b) \geq 1}$ and so

$$
\mathbb{E}[X_{a,b}] \gtrsim (\lambda \land \lambda^4)
$$

$$
\gtrsim (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda)(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6})e^{-\lambda},
$$

where the last inequality uses the fact that the following function

$$
h(\lambda) := \frac{\lambda \land \lambda^4}{(\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda)(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6})e^{-\lambda}
$$

is bounded below by some positive constant (numerically $\approx 0.0119$) for all $\lambda \geq 1$. Therefore when $\lambda \geq 1$,

$$
\sum_{x=4}^{\infty} \sum_{y=0}^{3} \left(x \sqrt{(x \land a)(x \land b)}\right)^2 p(x)p(y) \lesssim \mathbb{E}[X_{a,b}].
$$

- Now consider the case where $\lambda < 1$. In this case, we observe $\mathbb{E}[X_{a,b}] \gtrsim \lambda^4$. Also note that

$$
\sum_{x=4}^{\infty} \sum_{y=0}^{3} \left(x \sqrt{(x \land a)(x \land b)}\right)^2 p(x)p(y)
$$

$$
\leq \sum_{x=4}^{\infty} \left(x \sqrt{(x \land a)(x \land b)}\right)^2 p(x)
$$

$$
\leq \sqrt{ab} \sum_{x=4}^{\infty} x^2 \sqrt{(x \land a)(x \land b)} \frac{\lambda^x e^{-\lambda}}{x!}
$$

$$
= \sqrt{ab} \lambda \sum_{x=4}^{\infty} \frac{x \sqrt{(x \land a)(x \land b)} \lambda^{x-1} e^{-\lambda}}{(x-1)!}
$$

$$
= \sqrt{ab} \lambda \sum_{x=3}^{\infty} (x+1) \sqrt{((x+1) \land a)((x+1) \land b)} \frac{\lambda^{x} e^{-\lambda}}{x!}
$$
\[
\begin{align*}
(iv) \quad & \leq 4\sqrt{ab\lambda} \sum_{x=3}^{\infty} x \sqrt{(x \wedge a)(x \wedge b)} \frac{\lambda^x e^{-\lambda}}{x!} \\
& = 4\sqrt{ab\lambda} \times 3 \sqrt{(3 \wedge a)(3 \wedge b)} \frac{\lambda^3 e^{-\lambda}}{3!} + 4\sqrt{ab\lambda} \sum_{x=4}^{\infty} x \sqrt{(x \wedge a)(x \wedge b)} \frac{\lambda^x e^{-\lambda}}{x!} \\
& \leq \sqrt{ab\lambda^4} + \sqrt{ab}\mathbb{E}[X_{a,b}],
\end{align*}
\]

where step (i) uses the fact \( p \) is a probability mass function, step (ii) uses the basic inequality 
\( \sqrt{(x \wedge a)(y \wedge b)} \leq \sqrt{ab} \), step (iii) follows by change of variables and step (iv) follows since \( x + 1 \leq 2x \) for all \( x \geq 3 \). Therefore we can conclude that

\[
\sum_{x=4}^{\infty} \sum_{y=0}^{3} \left( x \sqrt{(x \wedge a)(x \wedge b)} \right)^2 p(x)p(y) \lesssim \sqrt{ab}\mathbb{E}[X_{a,b}].
\]

Similarly,

\[
\sum_{x=0}^{3} \sum_{y=4}^{\infty} \left( y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y) \lesssim \sqrt{ab}\mathbb{E}[X_{a,b}].
\]

**Analysis of (I).** Now, we handle the first term (I) in the variance expansion:

\[
(I) = \sum_{x=4}^{\infty} \sum_{y=4}^{\infty} \left( x \sqrt{(x \wedge a)(x \wedge b)} - y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y).
\]

We separate the cases into three: (i) \( b \geq a \geq 4 \), (ii) \( 2 \leq a, b \leq 4 \) and (iii) \( 2 \leq a < 4 \leq b \) and proceed our analysis.

- We first assume that (i) \( b \geq a \geq 4 \) and decompose the above double summations into 9 terms:
starting with the first term $(I)_1$, 
\[
\sum_{x=4}^{a} \sum_{y=4}^{a} u_{x,y,a,b} = \sum_{x=4}^{a} \sum_{y=4}^{a} \left( x \sqrt{(x \land a)(x \land b)} - y \sqrt{(y \land a)(y \land b)} \right)^2 p(x)p(y) \\
= \sum_{x=4}^{a} \sum_{y=4}^{a} \left( x^2 - y^2 \right)^2 p(x)p(y) \\
= \sum_{x=4}^{a} \sum_{y=4}^{a} (x-y)^2 (x+y)^2 p(x)p(y) \\
\leq 4a^2 \sum_{x=4}^{a} \sum_{y=4}^{a} (x-y)^2 p(x)p(y) \\
\leq a^2 \text{Var}[X \mathbb{1}(X \geq 4)],
\]
where the last inequality follows by the fact that for i.i.d. random variables $X$ and $Y$, 
\[
\text{Var}[X \mathbb{1}(X \geq 4)] = \frac{1}{2} \mathbb{E}[(X \mathbb{1}(X \geq 4) - Y \mathbb{1}(Y \geq 4))^2] \\
= \frac{1}{2} \sum_{x=4}^{a} \sum_{y=0}^{3} (x-y)^2 p(x)p(y) + \frac{1}{2} \sum_{x=0}^{3} \sum_{y=4}^{a} x^2 p(x)p(y) + \frac{1}{2} \sum_{x=0}^{3} \sum_{y=4}^{a} y^2 p(x)p(y).
\]
For the second term $(I)_2$, 
\[
\sum_{x=4}^{a} \sum_{y=a+1}^{b} u_{x,y,a,b} = \sum_{x=4}^{a} \sum_{y=a+1}^{b} \left( x \sqrt{(x \land a)(x \land b)} - y \sqrt{(y \land a)(y \land b)} \right)^2 p(x)p(y) \\
= \sum_{x=4}^{a} \sum_{y=a+1}^{b} (x^2 - y\sqrt{ay})^2 p(x)p(y).
Since \( x \leq a < y \leq b \), we have \( y\sqrt{y}\sqrt{a} - x^2 \geq 0 \) and

\[
y\sqrt{y}\sqrt{a} - x^2 = x\sqrt{y}\sqrt{a} - x^2 - x\sqrt{y}\sqrt{a} + y\sqrt{y}\sqrt{a}
\]
\[
= x(\sqrt{y}\sqrt{a} - x) + \sqrt{y}\sqrt{a}(y - x)
\]
\[
\leq x(y - x) + \sqrt{y}\sqrt{a}(y - x)
\]
\[
= (x + \sqrt{y}\sqrt{a})(y - x),
\]

which implies

\[
(x^2 - y\sqrt{y}\sqrt{a})^2 \leq (x + \sqrt{y}\sqrt{a})^2(y - x)^2 \leq 4ab(y - x)^2.
\]

Using this inequality, we have

\[
\sum_{x=4}^{a} \sum_{y=a+1}^{b} u_{x,y,a,b} \lesssim ab\text{Var}[X \mathbb{1}(X \geq 4)].
\]

For the third term \((I)_3\), observe that

\[
\sum_{x=4}^{a} \sum_{y=b+1}^{\infty} u_{x,y,a,b} = \sum_{x=4}^{a} \sum_{y=b+1}^{\infty} \left( x\sqrt{(x \wedge a)(x \wedge b)} - y\sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y)
\]
\[
= \sum_{x=4}^{a} \sum_{y=b+1}^{\infty} (x^2 - y\sqrt{ab})^2 p(x)p(y).
\]

In this case, we have \( x \leq a \leq b < y \) and

\[
0 \leq y\sqrt{ab} - x^2
\]
\[
= x\sqrt{ab} - x^2 + y\sqrt{ab} - x\sqrt{ab}
\]
\[
= x(\sqrt{ab} - x) + \sqrt{ab}(y - x)
\]
\[
\leq (x + \sqrt{ab})(y - x),
\]

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where the last inequality holds since \( \sqrt{ab} < y \), which in turn implies that

\[
\sum_{x=1}^{a} \sum_{y=b+1}^{\infty} u_{x,y,a,b} \lesssim ab \text{Var}[X 1(X \geq 4)].
\]

By symmetry, the fourth term \((I)_{4}\) can be similarly analyzed as the second term. For the fifth term \((I)_{5}\),

\[
\sum_{x=a+1}^{b} \sum_{y=a+1}^{b} u_{x,y,a,b} = \sum_{x=a+1}^{b} \sum_{y=a+1}^{b} \left( x \sqrt{(x \wedge a)(x \wedge b)} - y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y)
\]

\[
= \sum_{x=a+1}^{b} \sum_{y=a+1}^{b} (x \sqrt{ax} - y \sqrt{by})^2 p(x)p(y)
\]

\[
= a \sum_{x=a+1}^{b} \sum_{y=a+1}^{b} (x \sqrt{x} - y \sqrt{y})^2 p(x)p(y).
\]

Without loss of generality, assume \( a < x \leq y \leq b \) and note that

\[
0 \leq y \sqrt{y} - x \sqrt{x}
\]

\[
= x \sqrt{b} - x \sqrt{x} + y \sqrt{b} - x \sqrt{b} + y \sqrt{y} - y \sqrt{b}
\]

\[
= x(\sqrt{b} - \sqrt{x}) + \sqrt{b}(y - x) + y(\sqrt{y} - \sqrt{b})
\]

\[
\leq y(\sqrt{b} - \sqrt{x}) + \sqrt{b}(y - x) + y(\sqrt{y} - \sqrt{b})
\]

\[
= y(\sqrt{y} - \sqrt{x}) + \sqrt{b}(y - x)
\]

\[
= y \frac{y - x}{\sqrt{y} + \sqrt{x}} + \sqrt{b}(y - x).
\]

Therefore we have

\[
(x \sqrt{x} - y \sqrt{y})^2 \leq \frac{2y^2}{(\sqrt{x} + \sqrt{y})^2}(x - y)^2 + 2b(x - y)^2
\]

\[
\leq 2y(x - y)^2 + 2b(x - y)^2,
\]

(4)
which implies along with condition $y \leq b$ that

$$\sum_{x=a+1}^{b} \sum_{y=a+1}^{b} u_{x,y,a,b} \lesssim ab \text{Var}[X \mathbb{1}(X \geq 4)].$$

For the sixth term $(I)_6$, it can be seen that

$$\sum_{x=a+1}^{b} \sum_{y=b+1}^{\infty} u_{x,y,a,b} = \sum_{x=a+1}^{b} \sum_{y=b+1}^{\infty} \left( x \sqrt{(x \wedge a)(x \wedge b)} - y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y)$$

$$= \sum_{x=a+1}^{b} \sum_{y=b+1}^{\infty} (x\sqrt{ax} - y\sqrt{ab})^2 p(x)p(y)$$

$$= a \sum_{x=a+1}^{b} \sum_{y=b+1}^{\infty} (x\sqrt{x} - y\sqrt{b})^2 p(x)p(y).$$

Since $a < x \leq b < y$, we have

$$0 < y\sqrt{b} - x\sqrt{x} = \sqrt{x} (\sqrt{x}\sqrt{b} - x) + \sqrt{b}(y-x)$$

$$\leq \sqrt{b}(\sqrt{x}\sqrt{b} - x) + \sqrt{b}(y-x) \leq 2\sqrt{b}(y-x),$$

which implies

$$\sum_{x=a+1}^{b} \sum_{y=b+1}^{\infty} u_{x,y,a,b} \lesssim ab \text{Var}[X \mathbb{1}(X \geq 4)].$$

The seventh term $(I)_7$ and the eighth term $(I)_8$ can be analyzed as the third term $(I)_3$ and the sixth term $(I)_6$, respectively. The final term $(I)_9$ is

$$\sum_{x=b+1}^{\infty} \sum_{y=b+1}^{\infty} u_{x,y,a,b} = \sum_{x=b+1}^{\infty} \sum_{y=b+1}^{\infty} \left( x \sqrt{(x \wedge a)(x \wedge b)} - y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y)$$

$$= \sum_{x=b+1}^{\infty} \sum_{y=b+1}^{\infty} (x\sqrt{ab} - y\sqrt{ab})^2 p(x)p(y)$$

$$= ab \sum_{x=b+1}^{\infty} \sum_{y=b+1}^{\infty} (x - y)^2 p(x)p(y)$$
In summary, we have established that

\[
\sum_{x=4}^{\infty} \sum_{y=4}^{\infty} u_{x,y,a,b} \lesssim ab \text{Var}[X \mathbb{1}(X \geq 4)] \quad \text{(i)}
\]

\[
\lesssim ab \mathbb{E}[X \sqrt{(X \wedge a)(X \wedge b)} \mathbb{1}(X \geq 4)],
\]

where step (i) uses Claim 2.1 of Canonne et al. (2018) in (2). Combining the pieces yields

\[
\text{Var}[X_{a,b}] \lesssim ab \mathbb{E}[X_{a,b}],
\]

as desired.

- We are only left with the cases where (ii) \(2 \leq a, b < 4\) and (iii) \(2 \leq a < 4 \leq b\), and only need to re-analyze the term (I). Since both \(x, y \geq 4\), it holds that \(\sqrt{(x \wedge a)(x \wedge b)} = \sqrt{ab}\) under case (i) \(2 \leq a, b < 4\), and therefore we have

\[
\sum_{x=4}^{\infty} \sum_{y=4}^{\infty} u_{x,y,a,b} = \sum_{x=4}^{\infty} \sum_{y=4}^{\infty} \left( x \sqrt{(x \wedge a)(x \wedge b)} - y \sqrt{(y \wedge a)(y \wedge b)} \right)^2 p(x)p(y)
\]

\[
= ab \sum_{x=4}^{\infty} \sum_{y=4}^{\infty} (x - y)^2 p(x)p(y)
\]

\[
\lesssim ab \text{Var}[X \mathbb{1}(X \geq 4)] \lesssim ab \mathbb{E}[X_{a,b}].
\]

Similarly, when case (ii) \(2 \leq a < 4 \leq b\) holds,

\[
\sum_{x=4}^{\infty} \sum_{y=4}^{\infty} u_{x,y,a,b} = a \sum_{x=4}^{\infty} \sum_{y=4}^{\infty} \left( x \sqrt{x \wedge b} - y \sqrt{y \wedge b} \right)^2 p(x)p(y)
\]

\[
= a \sum_{x=4}^{b} \sum_{y=4}^{b} v_{x,y,b} + a \sum_{x=b+1}^{\infty} \sum_{y=4}^{b} v_{x,y,b}
\]

\[
+ a \sum_{x=4}^{b} \sum_{y=b+1}^{\infty} v_{x,y,b} + a \sum_{x=b+1}^{\infty} \sum_{y=b+1}^{\infty} v_{x,y,b}
\]
Following a similar analysis in (4), the first term (I)$_1'$ in the decomposition satisfies

\[ a \sum_{x=4}^\infty \sum_{y=1}^b v_{x,y,b} \lesssim ab \sum_{x=4}^\infty \sum_{y=4}^b (x-y)^2 p(x)p(y) \lesssim abE[X_{a,b}]. \]

The second term (I)$_2'$ (and also the third term (I)$_3'$ by symmetry) satisfies

\[ a \sum_{x=b+1}^\infty \sum_{y=1}^b v_{x,y,b} = a \sum_{x=b+1}^\infty \sum_{y=1}^b (x\sqrt{b} - y\sqrt{b})^2 p(x)p(y) \lesssim abE[X_{a,b}], \]

where inequality (*) follows since it holds

\[ 0 \leq x\sqrt{b} - y\sqrt{b} = (x-y)\sqrt{b} + \sqrt{b}(\sqrt{y(b-y)} \leq 2\sqrt{b}(x-y) \quad \text{when } 4 \leq y \leq b < x. \]

For the fourth term (I)$_4'$, we have

\[ a \sum_{x=b+1}^\infty \sum_{y=b+1}^\infty v_{x,y,b} = ab \sum_{x=b+1}^\infty \sum_{y=b+1}^\infty (x-y)^2 p(x)p(y) \lesssim abE[X_{a,b}]. \]

Combining the pieces yields

\[ \sum_{x=4}^\infty \sum_{y=4}^\infty u_{x,y,a,b} \lesssim abE[X_{a,b}]. \]

This completes the proof of Lemma 1 when \( a \geq 2 \).
3.2 Case where $0 \leq a < 2$

Next we consider the remaining case where $0 \leq a < 2$. In this case along with $b \geq 2$, observe that $X_{a,b} = X\sqrt{a(X \land b)}1(X \geq 4)$. Consequently it follows that

$$\text{Var}[X_{a,b}] = a\text{Var}[X\sqrt{X \land b}1(X \geq 4)]$$
$$= \frac{a}{2}\text{Var}[X\sqrt{(X \land 2)(X \land b)}1(X \geq 4)]$$
$$\lesssim ab\mathbb{E}[X\sqrt{(X \land 2)(X \land b)}1(X \geq 4)],$$

where the last inequality uses the previously established result in Section 3.1 for $a = 2$ and $b \geq 2$. The desired result then follows by noting that

$$ab\mathbb{E}[X\sqrt{(X \land 2)(X \land b)}1(X \geq 4)] = \sqrt{ab}\mathbb{E}[X\sqrt{(X \land a)(X \land b)}1(X \geq 4)].$$

When $0 \leq a, b < 2$, Claim 2.1 of Canonne et al. (2018) in (2) yields

$$\text{Var}[X_{a,b}] = ab\mathbb{E}[X1(X \geq 4)]$$
$$\lesssim ab\mathbb{E}[X1(X \geq 4)]$$
$$\lesssim \sqrt{ab}\mathbb{E}[X\sqrt{(X \land a)(X \land b)}1(X \geq 4)].$$

Therefore the statement of Lemma 1 holds when $0 \leq a < 2$. Combining this with the previous result in Section 3.1 completes the proof of Lemma 1.

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Canonne, C. L., Diakonikolas, I., Kane, D. M., and Stewart, A. (2018). Testing Conditional Independence of Discrete Distributions. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, page 735–748, New York, NY, USA. Association for Computing Machinery. https://arxiv.org/pdf/1711.11560v2.pdf.