FUNDAMENTAL GROUP OF LOG TERMINAL $\mathbb{T}$-VARIETIES

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Abstract. In this article, we introduce an approach to study the fundamental group of a log terminal $\mathbb{T}$-variety. As applications, we prove the simply connectedness of the spectrum of the Cox ring of a complex Fano variety, we compute the fundamental group of a rational log terminal $\mathbb{T}$-varieties of complexity one, and we study the local fundamental group of log terminal $\mathbb{T}$-singularities with a good torus action and trivial GIT decomposition.







INTRODUCTION

We study the fundamental group of normal complex algebraic varieties endowed with an effective action of an algebraic torus $\mathbb{T} := (\mathbb{C}^*)^k$, these varieties are known as $\mathbb{T}$-varieties. The complexity of a $\mathbb{T}$-variety $X$ is defined to be $	ext{dim}(X) - k$. The $\mathbb{T}$-varieties of complexity zero are the classic toric varieties that can be described in terms of fans of polyhedral cones (see, e.g., [8, 10, 17]). In [17], [23] and [11] there are generalizations of such description to the case of $\mathbb{T}$-varieties of complexity one. Finally in [1, 2] the authors introduce the language of polyhedral divisors and divisorial fans to extend the theory of toric varieties to $\mathbb{T}$-varieties of arbitrary complexity.

In this paper, we are interested in the fundamental group of the underlying topological space of a complex $\mathbb{T}$-variety. The topology of toric varieties has been well studied (see, e.g., [8, 9, 12–14]). In the toric case, the fundamental group can be computed in terms of the defining fan of the toric variety (see [8, Theorem 12.1.10]). More precisely, the fundamental group of an affine toric variety is a free finitely generated abelian group, and the fundamental group of a toric variety is a finitely generated abelian group. In particular, a complex toric variety with a fixed point for the torus action is simply connected. In higher complexity, a $\mathbb{T}$-variety $X$

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is determined by a divisorial fan $\mathcal{S}$, which is a geometric and combinatorial object which depends on certain divisors on the Chow quotient of $X$ (see Definition 1.2) and polyhedra associated to such divisors in a fixed $\mathbb{Q}$-vector space. In the case that $X$ is affine with a good $T$-action (see Definition 1.1) we obtain the following result which generalize the toric case:

**Theorem 1.** Let $X$ be a complex affine log terminal variety with a good $T$-action and denote by $Y$ a resolution of singularities of its Chow quotient. Then the pushforward $\pi_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

First, let us point that the group $\pi_1(Y)$ is independent of the chosen resolution of singularities (see Remark 3.1) and that the log terminal condition in Theorem 1 cannot be weakened: For instance, a cone over an elliptic curve $C$ has a log canonical singularity at the vertex and trivial fundamental group, being contractible, but its Chow quotient is $C$ which has non-trivial fundamental group (see Remark 3.2). As a consequence, we obtain the following (see [5] for a definition of the Cox ring):

**Corollary 1.** Let $X$ be a complex Fano variety, and let $\overline{X}$ be the underlying topological space of the spectrum of the Cox ring of $X$. Then $\overline{X}$ is simply connected.

Then, we focus on the case of rational $T$-varieties $X$ with a one-dimensional Chow quotient, the so called $T$-varieties of complexity one. In Theorem 3.4, we give an explicit description of the fundamental group in terms of the defining divisorial fan $\mathcal{S}$ of $X$. For instance, we can use this description to characterize the fundamental group of an algebraic $C^*$-bundle over $\mathbb{P}^1$.

**Corollary 2.** Let $X \to \mathbb{P}^1$ be a $C^*$-bundle which is trivial outside $\{p_1, \ldots, p_r\}$. Then, we have that

$$\pi_1(X) \simeq \langle b_1, \ldots, b_r, t \mid b_1 \cdots b_r, [b_i, t], t^{e_i} b_i^{m_i} \text{ for } 1 \leq i \leq r \rangle,$$

where $\Delta_{p_i} = \{e_i/m_i\}$ is the polyhedral coefficient at $p_i$.

Finally, we study the local fundamental group of rational log terminal $T$-varieties with a good torus action. It is conjectured that the local fundamental group of a log terminal singularity is finite (see, e.g. [18]). Moreover, this conjecture is known for the algebraic fundamental group [24], and for toric singularities [8, Theorem 12.1.10]. We extend this latter result to the following context.

**Theorem 2.** Let $X$ be a log terminal $T$-variety with a good torus action and $x \in X$ the vertex. Assume that the GIT fan of $X$ has a unique maximal chamber and each fiber of $\pi : X \to Y$ over a codimension one point, contains a smooth point. Then, the local fundamental group at $x \in X$ is finite.

The paper is organized as follows: In Section 1, we introduce the combinatorial description of $T$-varieties via divisorial fans. In Section 2, we explain the general approach to compute $\pi_1(X(\mathcal{S}))$ by using the information of the divisorial fan $\mathcal{S}$. Section 3 is devoted to the applications: In subsection 3.1 we prove Theorem 1 and Corollary 1, in subsection 3.2 we describe the fundamental group of a rational log terminal $T$-variety of complexity one, and in subsection 3.3 we prove Theorem 2. Finally, in Section 4 we give explicit computations in the case of Du Val singularities.

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1. Basic Setup

In this section, we introduce the description of $T$-varieties in terms of divisorial fans due to Altmann, Hausen and Süss [1, 2], see [3] for a survey on known results about the geometry of $T$-varieties. The point of view here is to start with a variety $Y$ together with a combinatorial data called a divisorial fan on $Y$ and construct a $T$-variety $X$ whose normalized Chow quotient is $Y$. We start recalling the definition of Chow quotient:

**Definition 1.1.** Let $X$ be a normal affine variety. A **good $T$-action** on $X$ is an effective $T$-action on $X$ such that there exists a closed point $x \in X$ which is in the closure of any $T$-orbit. We shall call $x$ the **vertex point** of $X$.

**Definition 1.2.** Let $X$ be a $T$-variety embedded in a projective space $\mathbb{P}^N$, then there exists an open set of $X$ on which all the orbits have dimension $k$ and degree $d$, the Chow quotient of $X$ is the closure of the set of points corresponding to such orbits in $\text{Chow}^k_d(\mathbb{P}^N)$, the Chow variety parametrizing cycles of dimension $k$ and degree $d$ on $\mathbb{P}^N$. The isomorphism class of the Chow quotient is independent from the chosen embedding. The normalization of the Chow quotient of $X$ will be called the **normalized Chow quotient**.

Now we turn to introduce the language of **polyhedral divisors** and **divisorial fans**: Given $N$ a finitely generated free abelian group of rank $k$ we will denote by $M := \text{Hom}(N, \mathbb{Z})$ its dual and by $N_\mathbb{Q} := N \otimes \mathbb{Z} \mathbb{Q}$ and $M_\mathbb{Q} := M \otimes \mathbb{Z} \mathbb{Q}$ the associated $\mathbb{Q}$-vector spaces. We will denote by $T_N := \text{Spec} \mathbb{C}[M] \simeq (\mathbb{C}^*)^k$ the torus of $N$. For every convex polyhedron $\Delta \subseteq N_\mathbb{Q}$ one defines its **tail cone** as

$$\sigma(\Delta) := \{ v \in N_\mathbb{Q} \mid v + \Delta \subset \Delta \}.$$

A polyhedron $\Delta$ with tail cone $\sigma$ will be called a $\sigma$-**polyhedron**. The set of $\sigma$-polyhedra is denoted by $\text{Pol}(\sigma)$. Observe that $\text{Pol}(\sigma)$ endowed with the Minkowski addition rule is a semigroup.

We adopt the notation $\text{CaDiv}(Y)_\mathbb{Q}$ for the monoid of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors of a normal variety $Y$. A **polyhedral divisor** on $(Y, N)$ is a finite formal sum of the form

$$D := \sum_{\Delta \in \text{Pol}(\sigma)} \Delta_D \otimes D \in \text{Pol}(\sigma) \otimes \mathbb{Z}_{\geq 0} \text{CaDiv}(Y)_\mathbb{Q}$$

where the sum is taken over a finite set of prime $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, and $\Delta_D$ are convex $\sigma$-polyhedra in $N_\mathbb{Q}$. The common tail cone of the polyhedra $\Delta_D$ is called the **tail cone of $D$** and is denoted by $\sigma(D)$. The $\sigma(D)$-polyhedra $\Delta_D$ associated to the divisor $D$ will be called the **polyhedral coefficient** of $D$. We will also consider the enlarged monoid $\text{Pol}^+(\sigma) := \text{Pol}(\sigma) \cup \{ \emptyset \}$ with addition rule $\emptyset + \Delta := \emptyset$ for every $\Delta \in \text{Pol}^+(\sigma)$. The **locus** of a polyhedral divisor $D$ is defined as

$$\text{loc}(D) := Y - \bigcup_{\Delta_D = \emptyset} D$$

and we say that $D$ has **complete locus** whenever $\text{loc}(D) = Y$ meaning that there is no $\mathbb{Q}$-divisor $D \subset Y$ with coefficient $\emptyset$. The **support** of $D$ is

$$\text{supp}(D) := \text{loc}(D) \cap \bigcup_{\Delta_D \neq \emptyset} D$$

and the **trivial locus** of $D$ is the complement of the support of $D$ in the locus of $D$, and is denoted by $\text{triv}(D)$. 
Let $\mathcal{D}$ be a polyhedral divisor on $(Y,N)$ with tail cone $\sigma$. We have a natural homomorphism of monoids

$$\mathcal{D} : \sigma^\vee \to \text{CaDiv}(Y) \quad u \mapsto \mathcal{D}(u) := \sum_{v \in \Delta_D} \min_{\langle u,v \rangle} D(v).$$

Observe that $\mathcal{D}(u) + \mathcal{D}(u') \leq \mathcal{D}(u + u')$ holds for every $u, u' \in \sigma^\vee$ and that the support of any divisor $\mathcal{D}(u)$ is contained in the support of $\mathcal{D}$.

**Definition 1.3.** A $\mathbb{Q}$-divisor $\mathcal{D}$ is said to be semiample if it admits a base point free multiple and is said to be big if some multiple admits a section with affine complement. A polyhedral divisor $\mathcal{D}$ is said to be a proper polyhedral divisor if $\mathcal{D}(u)$ is semiample for every $u \in \sigma(\mathcal{D})^\vee$ and $\mathcal{D}(u)$ is big for $u \in \text{relint}(\sigma(\mathcal{D})^\vee)$. In order to shorten notation, we will say that a proper polyhedral divisor $\mathcal{D}$ is a pp-divisor.

We recall the relation between affine $\mathbb{T}$-varieties and pp-divisors. Given a pp-divisor $\mathcal{D}$ on $(Y,N)$ one defines the $M$-graded $\mathcal{O}_{\text{loc}}(\mathcal{D})$-algebra

$$\mathcal{A}(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\text{loc}}(\mathcal{D}(u)).$$

The $M$-grading induces an effective $\mathbb{T}_N$ action on both the relative spectrum $\tilde{X}(\mathcal{D})$ and the spectrum of global sections $X(\mathcal{D})$ of the above sheaf of algebras. The inclusion $\mathcal{O}_{\text{loc}}(\mathcal{D}) \to \mathcal{A}(\mathcal{D})$ induces a good quotient morphism $\pi : \tilde{X}(\mathcal{D}) \to \text{loc}(\mathcal{D})$.

The construction is summarized in the following diagram

$$
\begin{array}{ccc}
\tilde{X}(\mathcal{D}) & \xrightarrow{\phi} & X(\mathcal{D}) \\
\downarrow{=} & & \downarrow{\pi} \\
\text{loc}(\mathcal{D}) & & 
\end{array}
$$

where the natural morphism $r$ can be proved to be a $\mathbb{T}_N$-equivariant birational contraction. The main result in [1] states that every normal affine $\mathbb{T}$-variety arises in this way.

In what follows we will describe the gluing process of affine $\mathbb{T}$-varieties in terms of pp-divisors. Given two pp-divisors $\mathcal{D}$ and $\mathcal{D}'$ on $(Y,N)$ we write $\mathcal{D}' \subseteq \mathcal{D}$ if $\Delta_D' \subseteq \Delta_D$ for every $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D \subset Y$. If $\mathcal{D}' \subset \mathcal{D}$ then we have an induced morphism $X(\mathcal{D}') \to X(\mathcal{D})$ and we say that $\mathcal{D}'$ is a face of $\mathcal{D}$ if the induced morphism is an embedding, and we denote this relation by $\mathcal{D}' \leq \mathcal{D}$. If $\mathcal{D}'$ is a face of $\mathcal{D}$ then in particular we have that $\sigma(\mathcal{D}') \leq \sigma(\mathcal{D})$. The intersection of two pp-divisors $\mathcal{D}$ and $\mathcal{D}'$ is defined to be the polyhedral divisor $\mathcal{D} \cap \mathcal{D}' := \sum_{D} (\Delta_D \cap \Delta_{D'}) \otimes D$.

**Definition 1.4.** A set $S$ of pp-divisors is said to be a divisorial fan if it holds the following conditions:

- $S$ is finite,
- $S$ is closed under taking intersection,
- the intersection of any two pp-divisors of $S$ is a face of both.

Gluing the affine $\mathbb{T}$-varieties $X(\mathcal{D})$ and $X(\mathcal{D}')$ along the affine subvarieties $X(\mathcal{D} \cap \mathcal{D}')$ for every $\mathcal{D}$ and $\mathcal{D}'$ in $S$, we obtain a $\mathbb{T}$-variety $X(S)$ (see [3, Section 4.4] for details). Analogously for the $\mathbb{T}$-varieties $\tilde{X}(\mathcal{D})$ with $\mathcal{D} \in S$ we obtain a $\mathbb{T}$-variety.
The set \( \{ \sigma(D) : D \in S \} \) is the tail fan \( \Sigma(S) \) of \( S \). Observe that the tail fan \( \Sigma(S) \) is indeed a fan. We define the locus of \( S \) to be the set \( \text{loc}(S) := \bigcup_{D \in S} \text{loc}(D) \), the support of \( S \) to be the set \( \text{supp}(S) := \bigcup_{D \in S} \text{supp}(D) \), and the trivial locus of \( S \) to be \( \text{triv}(S) := \bigcap_{D \in S} \text{triv}(D) \).

When \( S \) is the only divisorial fan on \( Y \) we may denote \( \text{triv}(S) \) by \( Y_{triv} \).

2. General approach to compute the fundamental group

In this section, we explain an approach to compute the fundamental group of a complex log terminal \( T \)-variety \( X(S) \) using its defining divisorial fan \( S \). First, we recall some definitions and results from [19].

**Definition 2.1.** Consider a pp-divisor \( D \) on \((Y,N)\) with tail cone \( \sigma \), and a morphism \( \psi: Y' \to Y \), such that no irreducible component of \( \text{supp}(D) \) contains \( \psi(Y') \). The polyhedral pull back is

\[
\psi^*(D) = \sum_D \Delta_D \otimes \psi^*(D) \in \text{Pol}(\sigma) \otimes_{\mathbb{Z}_{\geq 0}} \text{CaDiv}(Y')_{\mathbb{Q}}.
\]

Observe that \( \psi^*(D) \) is a polyhedral divisor which may not be a pp-divisor, meaning that \( \psi^*(D) \) may not be a proper polyhedral divisor. Given a divisorial fan \( S = \{ D_i \mid i \in I \} \) on \((Y,N)\), its pull back is \( \psi^*(S) = \{ \psi^*(D_i) \mid i \in I \} \).

**Lemma 2.2.** Let \( S \) be a divisorial fan on \((Y,N)\). If \( \psi: Y' \to Y \) is a projective birational morphism, then \( \psi^*(S) \) is a divisorial fan. Moreover \( \psi \) induces a \( T_N \)-equivariant isomorphism \( X(\psi^*(S)) \cong X(S) \).

**Proof.** Without loss of generality we assume that all the pp-divisors of \( S \) have complete locus: in case it is not complete, replace \( Y \) by \( \text{Loc}(D) \) in the whole proof. Given a pp-divisor \( D \in S \) and an element \( u \in \sigma(D)^\vee \) we have

\[
\psi^*(D)(u) = \sum_D \min_{v \in \Delta_D} \langle u, v \rangle \psi^*(D) = \psi^* \left( \sum_D \min_{v \in \Delta_D} \langle u, v \rangle D \right) = \psi^*(D(u)).
\]

Therefore, \( \psi^*(D)(u) \) is semiample. Moreover, if \( u \in \text{relint}(\sigma(D)^\vee) \), we know that \( D(u) \) is big, and the pull-back of a big divisor with respect to a projective birational
map is again big, so we conclude that $\psi^*(D(u))$ is a big $\mathbb{Q}$-divisor. Thus, $\psi^*(D)$ is a pp-divisor.

Now we prove that $\psi$ induces a $\mathbb{T}$-equivariant isomorphism $X(\psi^*(D)) \simeq X(D)$. First of all observe that if $D$ is a $\mathbb{Q}$-Weil divisor on $Y$, and $f$ is a rational function then $\text{div}(f) + D \geq 0$ if and only if $\text{div}(f) + |D| \geq 0$. Now let $u \in \sigma(D)^\vee$ and let $m$ be the Cartier index of $D(u)$. Observe that we have

\begin{align*}
  f \in \Gamma(Y, O_Y(D(u))) & \iff f^m \in \Gamma(Y, O_Y(mD(u))) \\
  f^m \in \Gamma(Y, O_Y(m\psi^*(D(u)))) & \iff f \in \Gamma(Y', O_{Y'}(\psi^*D(u))),
\end{align*}

where the first and last equivalences are by the previous observation, while the second equivalence follows from the fact that $\psi$ is birational with connected fibers and the projection formula. So, there is a natural isomorphism of $M$-graded $O_Y$-algebras $A(\psi^*(D)) \simeq A(D)$, concluding the claim. The isomorphism $X(\psi^*(S)) \simeq X(S)$ follows from gluing the above $\mathbb{T}_N$-equivariant affine isomorphisms. □

**Remark 2.3.** The $\mathbb{T}$-varieties $\bar{X}(\psi^*(S))$ and $\bar{X}(S)$ are isomorphic if and only if the projective birational map $\psi: Y' \to Y$ is the identity (see, e.g., [19, Section 2]).

**Definition 2.4.** A pp-divisor $D$ on $(Y, N)$ is simple normal crossing if $Y$ is smooth and the support of $D$ is a divisor with simple normal crossing support. Analogously, a divisorial fan $S$ on $(Y, N)$ is simple normal crossing if all its pp-divisors $D \in S$ are simple normal crossing.

**Definition 2.5.** An algebraic variety $X$ is toroidal if for each point $x \in X$ there exists a formal neighborhood of $x$ on $X$ which is isomorphic to a formal neighborhood of a point in an affine toric variety.

The following lemma is proved in [19, Proposition 2.6].

**Lemma 2.6.** Let $S$ be a divisorial fan on $(Y, N)$. If the divisorial fan $S$ is simple normal crossing, then the $\mathbb{T}$-variety $\bar{X}(S)$ is toroidal.

**Definition 2.7.** Given a simple normal crossing pp-divisor $D$ on $(Y, N)$ we can write $D = \sum_D \Delta_D \otimes D$. Given the prime divisors $D_1, \ldots, D_r$ such that $\Delta_D \neq \sigma$, we define the strata of the prime divisors $D_1, \ldots, D_r$, to be the locally closed set

$$Z_{D_1,\ldots,D_r} = D_1 \cap \cdots \cap D_r - \bigcup_{\Delta_D \neq \sigma} D_1 \cap \cdots \cap D_r \cap D.$$

Observe that the trivial open set of $D$ is the strata of the empty set of prime divisors. The stratas of $D$ give a natural stratification of $Y$. We define a strata $Z$ of the divisorial fan $S$ to be a finite intersection of strata of the pp-divisors $D \in S$. Clearly, the divisorial fan $S$ define a natural stratification of $Y$.

**Remark 2.8.** In this remark we will describe a formal neighborhood of the preimage on $\bar{X}(S)$ of a strata $Z$ on $Y$ as defined in 2.7. First, we consider the case of a single proper polyhedral divisor $D$ on $(Y, N)$. By virtue of Lemma 2.2, we may assume that the projective variety $Y$ is smooth and the polyhedral divisor $D$ is simple normal crossing, therefore the strata $Z$ defined in 2.7 is indeed a simple normal crossing strata.
Consider $Y$ a smooth projective variety of dimension $n - k$ and $N$ a free finitely generated abelian group of rank $k$. Let $D$ be a simple normal crossing pp-divisor on $(Y, N)$ with $σ = σ(D)$ its tail cone. We write $D = \sum D_i \otimes D$, and denote by $Z := Z_{D_{1, \ldots, D_r}}$ the strata of the prime divisors $D_1, \ldots, D_r$. Then, we can describe a formal neighborhood of the fiber $π^{-1}(Z)$ as follows: Consider the finitely generated free abelian group $N' = \mathbb{Z}^r \times N$ and the cone

$$σ(\mathcal{D}, Z) := \langle (0, σ), (ε_1, Δ_{D_1}), \ldots, (ε_r, Δ_{D_r}) \rangle \subset N'_Q$$

where the $ε_i$’s give the canonical basis of $\mathbb{Z}^r$. Thus, the formal neighborhood of a closed point of $π^{-1}(Z)$ is isomorphic to that of a corresponding closed point of

$$X(σ(\mathcal{D}, Z)) \times Z.$$

Therefore, given a divisorial fan $\mathcal{S}$ on $(Y, N)$ and $Z \subseteq Y$ a strata of the divisorial fan, the formal neighborhood of a closed point of $π^{-1}(Z)$ for the good quotient $π: \tilde{X}(\mathcal{S}) \to Y$ is isomorphic to the formal neighborhood of a corresponding closed point of

$$X(Σ(\mathcal{S}, Z)) \times Z.$$

where $Σ(\mathcal{S}, Z)$ is the fan in $N'$ given by the cones $σ(\mathcal{D}, Z)$ for all $\mathcal{D} \in \mathcal{S}$. Hence, given an analytic tubular neighbourhood $W_Z$ of $Z$ (in the usual sense of manifolds [7, pag. 66]) the preimage $π^{-1}(W_Z)$ admits a natural structure of topological fibration with base $Z$ and fiber $X(Σ(\mathcal{S}, Z))$, given by the composition of the good quotient $π^{-1}(W_Z) \to W_Z$ and the retraction $W_Z \to Z$.

**Remark 2.9.** From the above description, we can see that the fiber over a closed point $y \in Z$ is isomorphic to the toric bouquet associated to the polyhedron

$$D_y = \sum_i Δ_{D_i} \subset N'_Q.$$

For the definition of toric bouquet, see [3, Section 2.2].

**Notation 2.10.** Let $σ$ be a rational polyhedral cone in $N_Q$. Denote by $N_σ$ the subgroup of $N$ generated by $σ \cap N$ and by $N(σ)$ the lattice quotient $N/N_σ$. By [8, Theorem 12.1.10], we know that

$$π_1(X(σ)) \simeq N(σ).$$

Observe that $N(σ)$ is a free finitely generated abelian group. Given a fan $Σ$ of polyhedral cones in $N_Q$, we denote by $N_Σ$ the semigroup generated by $\langle N_σ \mid σ ∈ Σ \rangle$, and by $N(Σ)$ the quotient $N/N_Σ$. By [8, Theorem 12.1.10], we know that

$$π_1(X(Σ)) \simeq N(Σ).$$

Observe that $N(Σ)$ is a finitely generated abelian group, however it may be not free. Given a basis $\{t_1, \ldots, t_k\}$ of the lattice $N$, a presentation for the fundamental group is the following

$$π_1(X(Σ)) \simeq \langle t_1, \ldots, t_k \mid R(Σ) \rangle,$$

where $R(Σ)$ is the set of monomials $t_1^{n_1} \cdots t_k^{n_k}$ where $(n_1, \ldots, n_k)$ runs over all the bases of the lattices $N_τ ↪ N$ for each $τ ∈ Σ$.

Using the notation of Remark 2.8, we can see that the vectors $(ε_i, 0)$ of $N'$ can be realized as loops on $Y$ which goes around the divisor $D_i$. Indeed, on a formal neighborhood of the generic point $η$ of $Z$, the variety $Y$ is analytically diffeomorphic
to $\mathbb{A}^r_\mathbb{C}$. We denote by $\{0\}_r$ the origin of the affine space $\mathbb{A}^r_\mathbb{C}$, and by $S^{2r-1}$ the sphere of elements of norm one in $\mathbb{A}^r_\mathbb{C}$.

**Construction 2.11.** Consider $S$ to be a divisorial fan on $(Y,N)$, such that the $T$-variety $X(S)$ has log terminal singularities. By Lemma 2.2, we can consider a resolution of singularities $\psi: Y' \to Y$, such that the pull back divisorial fan $\psi^*(S)$ is simple normal crossing. Then, by Lemma 2.6, we know that the $T_N$-variety $\tilde{X}(\psi^*(S))$ is toroidal. Since $\tilde{X}(\psi^*(S))$ has toroidal singularities it is log terminal (see, e.g. [8, Theorem 11.4.24]). Therefore by [22, Theorem 1.1], we know that the birational contraction $r: \tilde{X}(\psi^*(S)) \to X(S)$ induces an isomorphism of fundamental groups

$$r_*: \pi_1(\tilde{X}(\psi^*(S))) \simeq \pi_1(X(S)).$$

So we can assume, without loss of generality, that the divisorial fan of $X = X(S)$ is simple normal crossing and $\tilde{X} = X$. We now propose a procedure to describe the fundamental group of such a variety $X$. Let $V \subseteq Y$ be the trivial open set of the divisorial fan $S$, so that we have an isomorphism

$$\pi^{-1}(V) \simeq V \times X(\Sigma(S)),$$

where $X(\Sigma(S))$ is the general fiber of $\pi: X \to Y$, defined by the divisorial fan $\Sigma(S)$. By [8, Theorem 12.1.5] the inclusion $\pi^{-1}(V) \to X$ induces a surjection of fundamental groups. By [8, Theorem 12.1.10] the fundamental group of the toric variety $X_0$ is isomorphic to $N(\Sigma(S))$. Thus the above surjection becomes

$$\pi_1(V) \times N(\Sigma(S)) \to \pi_1(X).$$

Our task now is to describe the kernel of the above homomorphism. Given a codimension $r := \text{codim}(Z)$ strata $Z$ of the divisorial fan $S$, let $W_Z$ be a formal neighborhood of $Z$ and let $V_Z := W_Z \cup V$. We may assume without loss of generality that if $\overline{Z} \cap \overline{Z}' = \emptyset$ then $W_Z \cap W_{Z'} = \emptyset$, and if $\overline{Z} \supset \overline{Z}'$ then $W_Z \supset W_{Z'}$.

**Lemma 2.12.** Let $U \subseteq Y$ be an open subset, then $\pi_*: \pi_1(\pi^{-1}(U)) \to \pi_1(U)$ is surjective.

**Proof.** Let $V \subseteq Y$ as usual the open subset over which $\pi: X \to Y$ is trivial. We have a commutative diagram

$$\begin{diagram}
\pi_1(\pi^{-1}(U \cap V)) & \rto & \pi_1(\pi^{-1}(U)) \\
\uto{\pi_*} & \to & \uto{\pi_*} \\
\pi_1(U \cap V) & \rto & \pi_1(U)
\end{diagram}$$

where the non-labelled arrows are induced by the inclusion, and thus are surjections by [8, Theorem 12.1.5]. By the triviality of $\pi$ over $V$ we deduce that the left hand side pushforward is surjective (projection onto the first factor), so that the second pushforward must be surjective as well. \qed

**Lemma 2.13.** If $N(\Sigma(S,Z))$ is trivial and $U \subseteq W_Z$ is a tubular neighborhood of the non-empty intersection $U \cap Z$, then $\pi_*: \pi_1(\pi^{-1}(U)) \to \pi_1(U)$ is an isomorphism.

**Proof.** Let $\rho: U \to U \cap Z$ be a retraction whose fibers are isomorphic to an open toric subvariety $\mathbb{A}^r$. By Remark 2.8, the composition $\pi \circ \rho: \pi^{-1}(U) \to U \cap Z$ is a fibration with fiber an open toric subvariety of $X(\Sigma(S,Z))$ containing a fixed
point. Passing to the long exact sequence of homotopy groups and recalling the isomorphism \( \pi_1(X(\Sigma(S, Z)) \simeq N(\Sigma(S, Z)) \) we get the following exact sequence

\[
\cdots \rightarrow N(\Sigma(S, Z)) \rightarrow \pi_1(\pi^{-1}(W_Z)) \overset{\pi_* \rho_*}{\rightarrow} \pi_1(Z) \rightarrow 1
\]

In particular if \( N(\Sigma(S, Z)) \) is trivial then \( \pi_* \rho_* \) is an isomorphism, so that \( \pi_*: \pi_1(\pi^{-1}(U)) \rightarrow \pi_1(U \cap Z) \) is injective. We conclude by Lemma 2.12. \( \square \)

**Remark 2.14.** Let \( \rho: W_Z \rightarrow Z \) be a retraction whose fibers are isomorphic to \( \mathbb{A}^r \). By Remark 2.8, the composition \( \pi \circ \rho: \pi^{-1}(W_Z) \rightarrow Z \) is a fibration with fiber \( X(\Sigma(S, Z)) \). Moreover the fibers of the restriction of \( \rho \) to \( W_Z \cap V \) are isomorphic to \( \mathbb{A}^r \setminus \{0\}_r \), so that the fibers of the restriction of \( \pi \circ \rho \) to \( \pi^{-1}(W_Z \cap V) \) are isomorphic to \( (\mathbb{A}^r \setminus \{0\}_r) \times X(\Sigma(S)) \), which are homotopic to \( S^{2r-1} \times X(\Sigma(S)) \). We have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{A}^r \setminus \{0\}_r & \longrightarrow & W_Z \cap V \\
\rho \downarrow & & \downarrow \pi \\
(\mathbb{A}^r \setminus \{0\}_r) \times X(\Sigma(S)) & \longrightarrow & \pi^{-1}(W_Z \cap V) \\
\pi \downarrow & & \downarrow \pi_* \rho_* \\
X(\Sigma(S, Z)) & \longrightarrow & \pi^{-1}(W_Z) \\
\end{array}
\]

where the non-labelled arrows are inclusions. Passing to fundamental groups, recalling that \( S^{2r-1} \) is a strong deformation retract of \( \mathbb{A}^r \setminus \{0\}_r \), recalling the isomorphism \( \pi_1(X(\Sigma(S)) \simeq N(\Sigma(S)) \) and the isomorphism \( \pi_1(X(\Sigma(S, Z)) \simeq N(\Sigma(S, Z)) \) we obtain the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(S^{2r-1}) & \longrightarrow & \pi_1(W_Z \cap V) \\
\pi_* \downarrow & & \downarrow \pi_* \\
\pi_1(S^{2r-1}) \times N(\Sigma(S)) & \longrightarrow & \pi_1(\pi^{-1}(W_Z \cap V)) \\
\alpha_Z \downarrow & & \downarrow \beta_Z \\
N(\Sigma(S, Z)) & \longrightarrow & \pi_1(\pi^{-1}(W_Z)) \\
\end{array}
\]

(2.3)

Moreover each row in the above diagram is part of the long exact sequence of homotopy groups induced by a fibration, so that the last map of each row is a surjection. If \( r > 1 \) then \( \alpha_Z \) is the surjection \( N(\Sigma(S)) \rightarrow N(\Sigma(S, Z)) \) induced by the inclusion of lattices \( N_{\Sigma(S)} \hookrightarrow N_{\Sigma(S, Z)} \). If \( r = 1 \), the generator of \( \pi_1(S^1) \simeq \mathbb{Z} \) is the loop corresponding to \( (e_1, 0) \) in the notation of Remark 2.8. Observe that in both cases \( \alpha_Z \) is a homomorphism of abelian groups.

**Remark 2.15.** The following commutative diagram of inclusions

\[
\begin{array}{ccc}
\pi^{-1}(W_Z \cap V) & \longrightarrow & \pi^{-1}(W_Z) \\
\downarrow & & \downarrow \\
\pi^{-1}(V) & \longrightarrow & \pi^{-1}(V_Z)
\end{array}
\]
induces a commutative pushout diagram of fundamental groups by the Seifert-van Kampen theorem. Using the triviality of $\pi$ over $V$ we have a pushout diagram

$$
\pi_1(W_Z \cap V) \times N(\Sigma(S)) \xrightarrow{\beta_Z} \pi_1(\pi^{-1}(W_Z))
$$

(2.4)

$$
\pi_1(V) \times N(\Sigma(S)) \xrightarrow{\gamma_Z} \pi_1(\pi^{-1}(V_Z))
$$

where $\iota: W_Z \cap V \to V$ is the inclusion.

3. Applications

3.1. Simply connectedness of the spectrum of the Cox ring. The aim of this subsection is to use Construction 2.11 to prove Theorem 1 and Corollary 1.

Remark 3.1. Theorem 1 is independent of the choice of the resolution of singularities of the Chow quotient of $X$. Indeed, let $Y_1$ and $Y_2$ be two resolution of singularities of the Chow quotient of $X$. Then, using resolution of singularities we can find a common resolution $Y'$ of $Y_1$ and $Y_2$, so by [22, Theorem 1.1] we deduce that $\pi_1(Y') \simeq \pi_1(Y_2)$ and $\pi_1(Y') \simeq \pi_1(Y_1)$, concluding the claim.

Remark 3.2. We point out that the log terminal condition in Theorem 1 cannot be weakened: let $H = O_{\mathbb{P}^2}(1)|_C$ be an ample divisor on a plane elliptic curve $C \subset \mathbb{P}^2$, and let

$$
X := \text{Spec} \left( \oplus_{m \in \mathbb{Z}_{\geq 0}} H^0(Y, O_Y(mH)) \right)
$$

Then $X$ is a $\mathbb{T}$-variety of complexity one with an isolated log canonical singularity at the vertex, and $X$ is contractible. Therefore, we have that $\pi_1(X)$ is trivial, while its Chow quotient $C$ has non-trivial fundamental group.

Proof of Theorem 1. We use the notation of Construction 2.11. Since $X$ admits a good $\mathbb{T}$-action, there is a $\mathbb{T}$-equivariant isomorphism $X(D) \simeq X$, where $D$ is a pp-divisor on $(Y, N)$, the variety $Y$ is projective, and $\sigma(D) \subset N_Q$ is a full-dimensional cone (see, e.g., [19, Section 4]). Let $\psi: Y' \to Y$ be a resolution of singularities of $Y$ and let $\psi^*(D)$ be a pull back of the pp-divisor to $Y'$. By Remark 3.1, it suffices to prove that the good quotient $\pi: \tilde{X}(\psi^*(D)) \to Y'$ induces an isomorphism of the fundamental groups. Also by [22, Theorem 1.1] the birational contraction $r: \tilde{X}(\psi^*(D)) \to X(D)$ induces an isomorphism of fundamental groups. Thus from now on, without loss of generality, we will assume

$$
X = \tilde{X}(\psi^*(D))
$$

and $Y$ smooth. Observe that since $\sigma(D)$ is full-dimensional, the group $N(\sigma(D))$ is trivial. In particular, the group $N(\sigma(D, Z))$ is trivial for each strata $Z$ of $D$, so that

$$
N(\Sigma(S)) = N(\Sigma(S, Z)) = 0.
$$

Thus for every strata $Z$ of $\psi^*(D)$, the map $\pi_*: \pi_1(\pi^{-1}(W_Z)) \to \pi_1(W_Z)$ is an isomorphism by Lemma 2.12. By Remark 2.15 and the unicity of pushout, the pushforward

$$
\pi_*: \pi_1(\pi^{-1}(V_Z)) \to \pi_1(V_Z)
$$

is an isomorphism. Now, let

$$
V_{Z_1, \ldots, Z_k} := V_{Z_1} \cup \cdots \cup V_{Z_k} \quad W_{Z_1, \ldots, Z_k} := W_{Z_1} \cup \cdots \cup W_{Z_k}
$$
be the union of the open sets corresponding to \( k \) different strata. The following is a pushout diagram by the Seifert-van Kampen theorem.

\[
\begin{array}{ccc}
\pi_1(\pi^{-1}(W_Z \cap W_{Z_1,\ldots,z_k} \cap V)) & \rightarrow & \pi_1(\pi^{-1}(V)) \\
\downarrow & & \downarrow \\
\pi_1(\pi^{-1}(W_Z \cap W_{Z_1,\ldots,z_k})) & \rightarrow & \pi_1(\pi^{-1}(V_Z \cap V_{Z_1,\ldots,z_k}))
\end{array}
\]

If we apply \( \pi_* \) we get another pushout diagram by the same theorem. Moreover \( \pi_* \) is an isomorphism on the left hand side groups by Lemma 2.13, and is an isomorphism on the top-right group by the triviality of \( N(\Sigma(S)) \) and the fact that \( \pi^{-1}(V) \simeq V \times X(\Sigma(S)) \). By the unicity of pushouts we deduce that

\[
\pi_* : \pi_1(\pi^{-1}(V_Z \cap V_{Z_1,\ldots,z_k})) \rightarrow \pi_1(V_Z \cap V_{Z_1,\ldots,z_k})
\]

is an isomorphism as well. Now, we prove by induction on \( k \) that

\[
\pi_* : \pi_1(\pi^{-1}(V_{Z_1,\ldots,z_k})) \rightarrow \pi_1(V_{Z_1,\ldots,z_k})
\]

is an isomorphism. The case \( k = 1 \) is (3.1). If \( k \geq 2 \) we have a pushout diagram

\[
\begin{array}{ccc}
\pi_1(\pi^{-1}(V_{Z_1,\ldots,z_k-1} \cap V_{Z_k})) & \rightarrow & \pi_1(\pi^{-1}(V_{Z_k})) \\
\downarrow & & \downarrow \\
\pi_1(\pi^{-1}(V_{Z_1,\ldots,z_k-1})) & \rightarrow & \pi_1(\pi^{-1}(V_{Z_1,\ldots,z_k}))
\end{array}
\]

induced by the inclusions. By (3.1), the inductive hypothesis, the isomorphism (3.2), and the uniqueness of pushouts up to isomorphism, we conclude the claim. Since \( X \) can be covered by finitely many strata, the above argument shows that \( \pi_* : \pi_1(X) \rightarrow \pi_1(Y) \) is an isomorphism, proving the statement. \( \square \)

**Proof of Corollary 1.** By [21, 24], and [6, Corollary 1.3.2] it is known that Fano varieties are simply connected Mori dream spaces. Moreover, the singularities of the spectrum \( \overline{X} \) of the Cox ring of a Fano variety \( X \) are log terminal [15], and \( \overline{X} \) has a good action for the Picard torus [4]. Thus, we can apply Theorem 1, to deduce that \( \pi_1(\overline{X}) \simeq \pi_1(X) \simeq \{0\} \). \( \square \)

### 3.2. Rational log terminal \( T \)-varieties of complexity one

The aim of this section is to give a presentation of the possible fundamental groups of rational log terminal \( T \)-varieties of complexity one.

**Notation 3.3.** Consider a divisorial fan \( S \) on \((Y,N)\). For each \( D \in S \) and \( p \in Y \) denote by \( Q(D,p) \) a basis of the lattice \( N_\sigma(D,p) \subseteq \mathbb{Z} \times N \) and by

\[
B(D,p) := \left\{ \left( \frac{v_2}{v_1}, \ldots, \frac{v_{k+1}}{v_1} \right) \in \mathbb{Q}^{k+1} \mid (v_1, \ldots, v_{k+1}) \in Q(D,p) \right\}.
\]

Given a point \( v \in N_{\mathbb{Q}} \), we will denote by \( \mu(v) \) the smallest positive integer such that \( \mu(v)v \in N \). Observe that \( \mu(v) \leq v_1 \) for \( v \in B(D,p) \).

**Theorem 3.4.** Let \( S \) be a divisorial fan on \((\mathbb{P}^1,N)\), assume that \( X(S) \) has log terminal singularities and let \( \{p_1,\ldots,p_r\} \subseteq Y \) be the complement of the trivial locus of \( S \). Then \( \pi_1(X(S)) \) admits a presentation with generators

\[
b_1, \ldots, b_r, t_1, \ldots, t_k,
\]
where $k$ is the rank of the acting torus, and relations
\begin{align*}
b_1 \cdots b_r, \\
[t_i, t_j] &\quad \text{for any } i, j \in \{1, \ldots, k\} \\
[t_i, b_j] &\quad \text{for } i \in \{1, \ldots, k\} \text{ and } j \in \{1, \ldots, r\}, \\
\mathcal{R}(\Sigma(S)) &\quad \text{where } \Sigma(S) \text{ is the tail fan of } S, \\
t^\mu(v)p_{\mu(v)} &\quad \text{for each } j \in \{1, \ldots, r\}, v \in \mathcal{B}(D, p_j) \text{ and } D \in S.
\end{align*}

Proof. We use the notation of Construction 2.11. Let

\[ G(V) := \{b_1, \ldots, b_r, t_1, \ldots, t_k\} \]

and let

\[ \mathcal{R}(V) := \mathcal{R}(\Sigma(S)) \cup \{b_1 \ldots b_r\} \cup \{[t_i, t_j] \text{ for any } i, j\} \cup \{[t_i, b_j] \text{ for any } i, j\}. \]

By the triviality of $\pi$ over $V$, the formula (2.2) and the fact that $V = \mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$, we have that

\[ \pi_1(\pi^{-1}(V)) \simeq \langle G(V) \mid \mathcal{R}(V) \rangle. \]

Observe that the stratification of $\mathbb{P}^1$ induced by $S$ is given by the sets $V$ and $p_1, \ldots, p_r$. Moreover, for any $j \in \{1, \ldots, r\}$, by (2.3), with $r = 1$ and $Z = p_j$, there is a commutative diagram

\[
\begin{array}{ccc}
Z \times N(\Sigma(S)) & \xrightarrow{\alpha_j} & \pi_1(\pi^{-1}(W_{p_j} \cap V)) \\
\downarrow \alpha_j & & \downarrow \beta_j \\
N(\Sigma(S, p_j)) & \xrightarrow{\sim} & \pi_1(\pi^{-1}(W_{p_j}))
\end{array}
\]

whose horizontal arrows are isomorphisms and the vertical arrows are surjections. Recall that $N(\Sigma) := N/N_\Sigma$, as defined in Notation 2.10. The homomorphism $\alpha_j$ is induced by the toric embedding

\[ \mathbb{C}^* \times X(\Sigma(S)) \to X(\Sigma(S, p_j)), \]

which is induced by the fan inclusion $0 \times \Sigma(S) \hookrightarrow \Sigma(S, p_j)$. Thus, by [8, Theorem 12.1.10], $\alpha_j$ is induced by the inclusion of lattices $0 \times N_{\Sigma(S)} \hookrightarrow N_{\Sigma(S, p_j)}$ so that

\[ \ker(\alpha_j) = \frac{N_{\Sigma(S, p_j)}}{0 \times N_{\Sigma(S)}}. \]

The lattice $N_\Sigma$ is generated by the integral points of $\Sigma$. Thus $\ker(\alpha_j)$ is generated by a basis of lattice points of $N_{\Sigma(S, p_j)}$. If we denote by $b_j$ a generator of the $\mathbb{Z}$ group in the domain of $\alpha_j$, then the kernel of $\alpha_j$ is generated by the set

\[ \mathcal{B}_j := \{b_j^{\mu(v)}v \mid \text{for each } v \in \mathcal{B}(D, p_j) \text{ and } D \in S\} \]

because, by Notation 3.3, the map $v \mapsto (\mu(v), \mu(\mu(v))v)$ is a bijection between $\mathcal{B}(D, p_j)$ and a basis of $N_{\Sigma(D, p_j)}$. Thus we have a presentation

\[ \pi_1(\pi^{-1}(W_{p_j})) \simeq \langle b_j, t_1, \ldots, t_k \mid \mathcal{R}(\Sigma(S)), \mathcal{B}_j \rangle. \]
Observe that we have a pushout diagram

\[
\begin{array}{ccc}
Z \times N(\Sigma(S)) & \xrightarrow{\alpha_j} & N(\Sigma(S, p_j)) \\
\downarrow i \times \text{id} & & \downarrow \text{id} \\
\pi_1(V) \times N(\Sigma(S)) & \xrightarrow{\gamma_j} & \pi_1(\pi^{-1}(V_{p_j}))
\end{array}
\]

where the homomorphism \(i\) is the inclusion \(\langle b_j \rangle \hookrightarrow \langle b_1, \ldots, b_r \rangle\). Thus the kernel of \(\gamma_j\) is the smallest normal subgroup of \(\pi_1(V) \times N(\Sigma(S))\) containing \((i \times \text{id})(\ker(\alpha_j))\).

So, we obtain the following presentation

\[
\pi_1(\pi^{-1}(V_{p_j})) \simeq \langle G(V) | R(V), B_1, \ldots, B_r \rangle.
\]

By repeatedly applying the Seifert-van Kampen Theorem we conclude that the fundamental group of \(X\) is

\[
\pi_1(X) \simeq \langle G(V) | R(V), B_1, \ldots, B_r \rangle,
\]

proving the statement. \(\square\)

**Proof of Corollary 2.** In this case, the pp-divisor can be written as

\[
D = \sum_{i=1}^{r+1} \left\{ \frac{e_i}{m_i} \right\} \otimes p_i \in \text{Pol}(\{0\}) \otimes_{\mathbb{Z}_{\geq 0}} \text{CaDiv}(\mathbb{P}^1)_\mathbb{Q}.
\]

since the tail cone of \(D\) is \(\{0\}\) in \(N_\mathbb{Q} \simeq \mathbb{Q}\). Hence, we conclude by Theorem 3.4. Indeed, observe that in this case we have a single pp-divisor \(D\) and for every \(p_i\) we have that the lattice \(N_{\sigma(D, p_i)}\) is the lattice of \(\mathbb{Z} \times \mathbb{N}\) generated by

\[
\mathbb{Q}(D, p_i) = \left\{ (m_i, e_i) \right\},
\]

and therefore we have

\[
\mathcal{B}(D, p_i) = \left\{ \frac{e_i}{m_i} \right\}.
\]

\(\square\)

### 3.3. Finiteness of local fundamental group

In this subsection we study the finiteness of the local fundamental group of a log terminal \(T\)-singularity with a good torus action.

**Definition 3.5.** We say that the \(T\)-variety \(X\) has a trivial GIT decomposition if there exists exactly one GIT quotient of \(X\) of the expected dimension, and moreover this coincides with the Chow quotient of \(X\).

**Proof of Theorem 2.** The local fundamental group at the vertex is a local computation, so we may assume that the \(T\)-variety is affine. By [1, Theorem 1], we have a \(T\)-equivariantly isomorphism \(X \simeq X(D)\) for some pp-divisor \(D\) on \((Y, N)\), where \(Y\) is the Chow quotient of \(X\). Since we are assuming that the \(T\)-action on \(X(D)\) is good, we know that \(Y\) is projective and \(\sigma = \sigma(D) \subset N_\mathbb{Q}\) is a full-dimensional cone (see, e.g., [19, Section 4]). Moreover, by [19] we know that the trivial GIT decomposition implies that \(D(u)\) is an ample \(\mathbb{Q}\)-divisor for every \(u \in \text{relint}(\sigma^\vee)\). We will write

\[
D = \sum_{D \subseteq Y} \Delta_D \otimes D,
\]
where the sum runs over a finite set of \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisors on \(Y\) and \(\Delta_D\) are \(\sigma\)-polyhedra. For each vertex \(v \in \Delta_D\) we recall that \(\mu(v)\) is the smallest positive integer such that \(\mu(v) v\) is a lattice point of \(N_{\mathbb{Q}}\), by
\[
\mu_D := \max\{\mu(v) | v \text{ is a vertex of } \Delta_D\},
\]
and by
\[
B := \sum_{D \subseteq Y} \frac{\mu_D - 1}{\mu_D} D,
\]
where the sum runs over the same \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisors of \(D\). By [19, Theorem 4.9], we know that the pair \((Y, B)\) is a log Fano pair. We claim that \(Y\) has log terminal singularities. Indeed, the effective divisor \(B\) is \(\mathbb{Q}\)-Cartier and \(K_Y + B\) is \(\mathbb{Q}\)-Cartier, so we conclude that \(K_Y\) is \(\mathbb{Q}\)-Cartier, therefore \(Y\) is \(\mathbb{Q}\)-Gorenstein. Thus the klt property of \((Y, B)\) implies the log terminal property of \(Y\) (see, e.g., [16, Corollary 2.35]). Hence, by [22, Theorem 1.1] and [21] we conclude that \(\pi_1(Y') \simeq \pi_1(Y) \simeq \{0\}\) for every resolution of singularities \(\psi: Y' \to Y\).

Let \(\psi: Y' \to Y\) be a resolution of singularities such that \(\psi^*(D)\) is a pp-divisor with simple normal crossing support and let \(X(S) \to \tilde{X}(\psi^*(D))\) be a toroidal resolution of singularities, defined by a divisorial fan \(\mathcal{S}\) on \(Y'\). Denote by \(\phi: X(S) \to X(D)\) the projective birational morphism obtained by composition. Therefore, we have a quotient of smooth varieties \(\pi': X(S) \to Y'\). By the assumption that a fiber of \(\pi: X(D) \to Y\) over a codimension one point contains a smooth point, we can take the resolution of singularities which does not blow-up such smooth points, so the analogous assumption holds for the morphism \(\pi'\). Moreover, the general fiber of \(\pi'\) is isomorphic to \(X(\Sigma)\), where \(\Sigma\) is a simplicial refinement of \(\sigma \subset N_\mathbb{Q}\). Since \(\sigma\) is full-dimensional, then the rays of \(\Sigma\) span \(N_\mathbb{Q}\) so that, by [8, Theorem 12.1.10], the fundamental group of \(X(\Sigma)\) with the vertex removed is a finite group \(G(\Sigma)\). Hence, we can apply [20, Lemma 1.5], to conclude that we have an exact sequence
\[
G(\Sigma) \to \pi_1(\phi^{-1}(X(D) - \{x\})) \to \pi_1(Y') \to 1.
\]
Observe that by [22, Theorem 1.1] we have that \(\pi_1(Y') = \pi_1(Y) = 1\), so we conclude that \(\pi_1(\phi^{-1}(X(D) - \{x\}))\) is finite and thus \(\pi_1(X(D) - \{x\})\) is finite as well, again by [22, Theorem 1.1].

**Corollary 3.6.** With the same assumptions than Theorem 2. The inclusion of a general fiber of \(\pi: X \to Y\) induces a surjection of the local fundamental group at the vertex of the general fiber onto the local fundamental group at the vertex \(x \in X\).

**Proof.** Indeed, by the proof of the theorem we have a surjection
\[
G(\Sigma) \to \pi_1(\phi^{-1}(X(D) - \{x\}))
\]
and the latter group is isomorphic to \(\pi_1(X(D) - \{x\})\) by [22, Theorem 1.1].

**4. Examples**

It is know that a germ of surface Du Val singularity \(x \in X\), is a log terminal \(T\)-variety of complexity one with trivial fundamental group. Moreover, since Du Val singularities are quotients of \(\mathbb{C}^2\) by a binary polyhedral group \(G \subset SL_2(\mathbb{C})\), the fundamental group of the punctured neighborhood \(X \setminus \{x\}\) is isomorphic to \(G\). In this subsection, we recall an explicit computation of Du Val singularities as \(T\)-varieties and recover their fundamental group using Theorem 3.4.
Example 4.1. By [19, Corollary 5.6], we know that every quasi-homogeneous log-terminal surface singularity is isomorphic to the section ring of one of the following $\mathbb{Q}$-divisors on $\mathbb{P}^1$
\[
D = \left\{ \frac{e_1}{m_1} \right\} \otimes [0] + \left\{ \frac{e_2}{m_2} \right\} \otimes [1] + \left\{ \frac{e_3}{m_3} \right\} \otimes [\infty], \quad \text{and} \quad \frac{e_1}{m_1} + \frac{e_2}{m_2} + \frac{e_3}{m_3} > 0.
\]
Here, the triple $(m_1, m_2, m_3)$ is one of the platonic triples $(1, p, q), (2, 2, r), (2, 3, 3), (2, 3, 4)$ and $(2, 3, 5)$, where $p, q \geq 1$ and $r \geq 2$. Moreover, the contraction morphism $r: \tilde{X}(D) \to X(D)$ is contracting a curve onto the vertex $x$ of $X(D)$. Therefore, the map
\[
\pi: \tilde{X}(D) - r^{-1}(x) \to \mathbb{P}^1
\]
is a $\mathbb{C}^*$-bundle with at most three non-reduced fibers. Then we can apply Corollary 2 to give a presentation of the fundamental group of $\pi_1(X(D))$ in terms of generators and relations as follows
\[
\langle b_1, b_2, t \mid [b_1, t], [b_2, t], t^{e_1} b_1^{m_1}, t^{e_2} b_2^{m_2}, t^{e_3} b_3^{m_3} \rangle
\]
where $b = (b_1 b_2)^{-1}$.

Example 4.2. By [19, Theorem 5.7], we know that every quasi-homogeneous canonical surface singularity is isomorphic to the section ring of one of the following $\mathbb{Q}$-divisors on $\mathbb{P}^1$
\[
A_i: \left\{ \frac{i + 1}{i} \right\} \otimes [\infty] \quad i \geq 1.
\]
\[
D_i: \left\{ \frac{1}{2} \right\} \otimes [0] + \left\{ \frac{1}{i - 2} \right\} \otimes [1] + \left\{ \frac{-1}{2} \right\} \otimes [\infty] \quad i \geq 4.
\]
\[
E_i: \left\{ \frac{1}{3} \right\} \otimes [0] + \left\{ \frac{1}{i - 3} \right\} \otimes [1] + \left\{ \frac{-1}{2} \right\} \otimes [\infty] \quad i \in \{6, 7, 8\}.
\]
Remark that we correct some sign typos in the statement of [19, Theorem 5.7]. The proof therein remains valid without corrections. Proceeding as in Example 4.2, we can recover the local fundamental groups of DuVal singularities
\[
A_i: \langle b \mid b_i^{i+1} \rangle \quad i \geq 1.
\]
\[
D_i: \langle b_1, b_2 \mid b_2^{2} = b_1^{2-2} = (b_1 b_2)^2 \rangle \quad i \geq 4.
\]
\[
E_i: \langle b_1, b_2 \mid b_3^{i} = b_1^{2} = b_2^{2} \rangle \quad i \in \{6, 7, 8\}.
\]
More precisely, for the $E_8$ singularity we obtain the following presentation
\[
\langle b_1, b_2, t \mid [b_1, t], [b_2, t], t b_1^{3}, t b_2^{3}, t (b_1 b_2)^2 \rangle \simeq \langle b_1, b_2 \mid b_1^{3} = b_2^{3} = (b_1 b_2)^2 \rangle.
\]
The other computations are analogous.

Example 4.3. We give an example of a rational affine $\mathbb{T}$-variety of complexity one $X(D)$ with a good torus action and vertex $x \in X(D)$, such that the fundamental groups $\pi_1(X(D))$ and $\pi_1(X(D) - \{x\})$ are both trivial. Consider the cone
\[
\sigma = \langle (-1, 1), (1, 8) \rangle \subset N_\mathbb{Q} \simeq \mathbb{Q}^2
\]
and the proper polyhedral divisor given by
\[
D = \Delta_0 \otimes [0] + \Delta_1 \otimes [1] + \Delta_\infty \otimes [\infty],
\]
where

\[ \Delta_0 = \sigma + \left( \frac{2}{5}, \frac{1}{5} \right), \]
\[ \Delta_1 = \sigma + \left( \frac{1}{3}, \frac{1}{3} \right), \]
\[ \Delta_\infty = \sigma + \text{conv}((0,0), (1,0)), \]

where conv denotes the convex hull. By Theorem 1, we conclude that \( \pi_1(X(D)) \) is trivial. Moreover, we can apply Theorem 3.4 to see that \( \pi_1(X(D) - \{x\}) \) is isomorphic to

\[ \langle t_1, t_2, b_1, b_2 \mid [t_1, b_1], \ldots, [t_2, b_2], t_1^{-1}t_2, t_1t_2t_3b_1^5t_1t_2b_2^3, (b_1b_2)^{-1}t_1(b_1b_2)^{-1} \rangle. \]

Indeed, we have that

\[ \mathcal{Q}(D, \{0\}) = \{(0,-1,1), (0,11,8), (5,2,1)\}, \]
\[ \mathcal{Q}(D, \{1\}) = \{(0,-1,1), (0,11,8), (3,1,1)\}, \] and
\[ \mathcal{Q}(D, \{\infty\}) = \{(0,-1,1), (0,11,8), (1,0,0), (1,1,0)\}. \]

From the first and last two relations we obtain \( t_1 = t_2 = e, b_1^5 = b_2^3 = b_1b_2 = e \), so that \( b_1 = b_2 = e \).

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