Revisiting Elementary Denotational Semantics

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Abstract. Operational semantics have been enormously successful, in large part due to its flexibility and simplicity, but they are not compositional. Denotational semantics, on the other hand, are compositional but the lattice-theoretic models are complex and difficult to scale to large languages. However, there are elementary models of the λ-calculus that are much less complex: by Coppo, Dezani-Ciancaglini, and Salle (1979), Engeler (1981), and Plotkin (1993).

This paper takes first steps toward answering the question: can elementary models be good for the day-to-day work of language specification, mechanization, and compiler correctness? The elementary models in the literature are simple, but they are not as intuitive as they could be. To remedy this, we create a new model that represents functions literally as finite graphs. Regarding mechanization, we give the first machine-checked proof of soundness and completeness of an elementary model with respect to an operational semantics. Regarding compiler correctness, we define a polyvariant inliner for the call-by-value λ-calculus and prove that its output is contextually equivalent to its input. Toward scaling elementary models to larger languages, we formulate our semantics in a monadic style, give a semantics for System F with general recursion, and mechanize the proof of type soundness.

Keywords: denotational semantics, intersection types, set-theoretic models, mechanized metatheory

1 Introduction

This paper revisits elementary models of the λ-calculus [11, 17, 23, 51] with an eye towards determining whether they are a suitable choice for modern programming language semantics. That is, are they good for the day-to-day work of language specification, mechanization, and compiler correctness? The author hypothesizes that the answer is yes because these models satisfy three important properties.

compositional The semantics are defined by structural recursion on syntax [72].
extensional The semantics specify externally observable behavior [53].
elementary The semantics use quite simple mathematics [51].

Compositionality enables proof by structural induction on the syntax, which simplifies proofs of properties such as type soundness and compiler correctness.
Extensionality is beneficial because, ultimately, a language specification must be extensional; intensional semantics require the circuitous step of erasing internal behavior. The use of only elementary mathematics, that is, mathematics familiar to undergraduates in computer science, is beneficial because language specifications should be readable by all computer scientists and because the size of a language’s metatheory depends on the complexity of the mathematics.

Historically, the two main approaches to specifying programming languages have been denotational semantics\cite{56, 58} and operational semantics\cite{16, 24, 39, 42, 49}. Denotational semantics are compositional and extensional but the standard lattice-based models are not elementary. Their mathematical complexity is evident in the size of mechanized definitions of the \(\lambda\)-calculus: Benton et al.\cite{12} build a model in 11,000 LOC and Dockins\cite{20} in 54,104 LOC (though it is difficult to determine how many of those LOC is strictly necessary).

Operational semantics are elementary but neither extensional nor compositional. A mechanized definition of the \(\lambda\)-calculus using operational techniques is under 100 LOC. Small-step semantics are intensional in that the input-output behavior of a program is a by-product of a sequence of transitions. Big-step semantics are intensional in that the value of a \(\lambda\) abstraction is a syntactic object, a closure\cite{38}. The lack of compositionality in operational semantics imposes significant costs when reasoning about programs; sophisticated techniques such as logical relations\cite{48, 64} and simulations\cite{4, 44} are often necessary. For example, the correctness proofs for the CompCert C compiler made extensive use of simulations but sometimes resorted to translation validation in cases where verification was too difficult or expensive\cite{40, 65}. Likewise, the logical relations necessary to handle modern languages are daunting in their complexity\cite{5, 33}.

But what if there were denotational semantics that were also elementary? In fact, in the 1970’s and 1980’s several groups of researchers discovered elementary models for the untyped \(\lambda\)-calculus. Plotkin\cite{50, 51} and Engeler\cite{23} discovered elementary models based on two insights:

1. it suffices to use a finite approximation of a function’s graph when passing it to another function, and
2. self application can be handled by allowing larger approximations to be used when a smaller approximation is expected.

Coppo et al.\cite{17} discovered type-theoretic models for the \(\lambda\)-calculus based on two insights:

1. that the behavior of \(\lambda\) abstractions can be completely characterized by intersection types, and
2. self application can be handled using subtyping.

Plotkin\cite{51} shows how the elementary and type-theoretic models are closely related. The above pairs of insights are really the same insights.

**Contributions** This paper makes several technical contributions that begin to answer the question of whether elementary models are a good choice for programming language specification, mechanization, and compiler correctness. All
of the results in this paper are mechanized in Isabelle and are available in the Archive of Formal Proof [60].

1. To yield a more intuitive model for the CBV λ-calculus, we construct a domain that approximates functions literally by their finite graphs (Section 3). The mechanization is under 100 LOC.

2. We give a type-theoretic version of this model based on intersection types (50 LOC) and prove that the two are isomorphic (Section 4).

3. We give the first mechanized proofs of soundness and completeness for a elementary model with respect to operational semantics. We also mechanize soundness with respect to contextual equivalence (Section 5).

4. We show how compositionality can be beneficial by proving correctness for a compiler optimization pass that performs inlining in under 100 LOC (Section 6).

5. Toward scaling to more language features, we formulate the semantics in a monadic style (Section 7) and we define a semantics for a language with first-class parametric polymorphism and general recursion and mechanize its proof of semantic type soundness (Section 8).

We begin with a review of the three elementary semantics of Plotkin [51], Engeler [23], and Coppo et al. [17] (Section 2). We discuss related work in Section 9 and conclude in Section 10.

2 Background on Elementary Semantics

*But f is a function; an infinite object. What does it mean to “compute” with an “finite” argument? In this case it means most simply that h(f) is determined by asking of f finitely many questions: f(m₀), f(m₁), ..., f(m_k−1).*

—Scott [59]

We review elementary semantics in the setting of a call-by-value (CBV) untyped λ-calculus extended with integer arithmetic. The syntax of this λ-calculus is defined in Figure 1. We write \( n \) for integers, \( e₁ \oplus e₂ \) for arithmetic operations, \( x \) for variables, \( λx. e \) for abstraction, \( e₁ \ e₂ \) for application, and \( \text{if} e₁ \text{then} e₂ \text{else} e₃ \) for conditionals.
\[ \mathcal{E}_\rho[\lambda x. e] = \{(D, D') | D' \subseteq \mathcal{E}_\rho[e_\rho(x:=D)] \} \]

\[ \mathcal{E}_\rho[e_1 \dots e_n] = \bigcup \{ D' | \exists D. (D, D') \in \mathcal{E}_\rho[e_1] \land D \subseteq \mathcal{E}_\rho[e_2] \} \]

\[ \mathcal{E}_\rho[\lambda x. e] = \{(D, D') | D' \subseteq \mathcal{E}_\rho[e_\rho(x:=D)] \} \]

\[ \mathcal{E}_\rho[e_1 \dots e_n] = \bigcup \{ D' | \exists D. (D, D') \in \mathcal{E}_\rho[e_1] \land D \subseteq \mathcal{E}_\rho[e_2] \} \]

\[ \mathcal{E}_\rho[x] = \rho(x) \]

\[ \mathcal{E}_\rho[n] = \{ n \} \]

\[ \mathcal{E}_\rho[e_1 \oplus e_2] = \{ n_1 \oplus n_2 | n_1 \in \mathcal{E}_\rho[e_1] \land n_2 \in \mathcal{E}_\rho[e_2] \} \]

\[ \mathcal{E}_X[\text{if } e_1 \text{ then } e_2 \text{ else } e_3] = \left\{ v \left| \exists n. n \in \mathcal{E}_X[e_1] \land (n \neq 0 \implies v \in \mathcal{E}_X[e_2]) \land (n = 0 \implies v \in \mathcal{E}_X[e_3]) \right. \right\} \]

**Fig. 2.** Two elementary semantics for CBV \(\lambda\)-calculus, \(\mathcal{E}_\rho\) using Plotkin’s model and \(\mathcal{E}_\rho\) using Engeler’s. The common parts are parameterized, i.e., \(\mathcal{E}_X\) where \(X \in \{P, E\}\).

### 2.1 Set-Theoretic Models

The domains of Plotkin [50, 51] and Engeler [23] are \(\mathcal{P}(\aleph_0)\) and \(\mathcal{P}(\aleph_0)\), where \(\aleph_0\) and \(\aleph_0\) and inductively defined by the following recursive equations.

\[ \mathcal{D}_P = \mathbb{Z} + \mathcal{P}_f(\mathcal{D}_P) \times \mathcal{P}_f(\mathcal{D}_P) \]

\[ \mathcal{D}_E = \mathbb{Z} + \mathcal{P}_f(\mathcal{D}_E) \times \mathcal{D}_E \]

We let \(d\) range over elements of \(\mathcal{D}_P\) or \(\mathcal{D}_E\), and \(D\) ranges over finite sets of them. For Plotkin, an element \((D, D') \in \mathcal{P}(\mathcal{D}_P)\) represents a single input-output entry in the graph of a function. For first-order functions over integers, \(D\) and \(D'\) are just singletons. For higher-order functions, \(D\) and \(D'\) are finite subsets of a function’s graph. It turns out that finite sets in output position are not necessary. One of Plotkin’s entries \((D, \{d'_1, \ldots, d'_n\})\) is instead represented by multiple entries \((D, d'_1), \ldots, (D, d'_n)\) in Engeler’s model.

The wonderful thing about \(\mathcal{P}(\mathcal{D}_P)\) and \(\mathcal{P}(\mathcal{D}_E)\) is their simplicity. Their construction does not require advanced techniques such as inverse limits [56, 63, 70]. Both \(\mathcal{D}_P\) and \(\mathcal{D}_E\) are straightforward to define as algebraic datatypes in proof assistants such as Isabelle or Coq. In Isabelle, the `fset` library provides finite sets, but one could also use lists at the cost of a few extra lemmas.

Plotkin [50, 51] used his domain to give a semantics to the \(\lambda\beta\)-calculus whereas Engeler [23] used his to give semantics to combinatory logic (the S and K combinators). In this paper, we are instead concerned with a CBV \(\lambda\)-calculus. To make for a clear comparison with our work, we adapt their semantics to CBV \(\lambda\)-calculus, defining \(\mathcal{E}_P\) and \(\mathcal{E}_E\) in Figure 2. We conjecture that these two semantics are equivalent to our own.

Now to explain the two semantics in Figure 2. As usual, we write \(\rho(x:=d)\) for the map that sends \(x\) to \(d\) and any other variable \(y\) to \(\rho(y)\). In \(\mathcal{E}_P\), the meaning of an abstraction \(\lambda x. e\) is the set of all input-output entries \((D, D')\) such that \(D'\) is a subset of the meaning of the body \(e\) in a context where \(x:=D\). Similarly,
in \( \mathcal{E}_E \), the meaning of an abstraction \( \lambda x. e \) is the set of all input-output entries \((D, d')\) such that \(d'\) is an element in meaning of the \(e\) with \(x := D\). Regarding the meaning of function application \((e_1 e_2)\), \( \mathcal{E}_E \) collects up all of the outputs \(d'\) from the entries \((D, d')\) in the meaning of \(e_1\) whenever \(D\) an finite approximation of the argument \(e_2\). A finite approximation is good enough because, during a terminating call to a higher-order function, only a finite number of calls will be made to its argument. For a non-terminating call, the semantics assigns the meaning \(\emptyset\). The use of subset in \( D \subseteq \mathcal{E}_E[e_2]\rho \) is critically important, as it enables self application and thereby general recursion via the \(Y\) combinator. The \( \mathcal{E}_P \) semantics for function application is slightly more complex because the outputs \(D'\) must be flattened to produce something in \( \mathcal{P}(\mathcal{D}_P) \) and not \( \mathcal{P}(\mathcal{P}(\mathcal{D}_P)) \). Note that in both of these semantics, the environment \(\rho\) maps variables to finite sets, which either encode an integer \(n\) with a singleton \(\{n\}\) or encode a finite approximation of a function \(\{(D_1, D'_1), \ldots, (D_n, D'_n)\}\).

### 2.2 Type-Theoretic Models

Coppo et al. [17] showed that a type system based on intersection types can characterize the behavior of \(\lambda\) terms, in particular, showing that their type system induced the same equalities as the \(\mathcal{P}(\omega)\) model of Scott [57]. Their work led to a long line of research on filter models based on intersection types for many different \(\lambda\)-calculi [8, 18, 31]. Barendregt et al. [11] give a detailed survey of this work. Here we review an intersection type system that characterizes CBV \(\lambda\)-calculus [8, 18, 54].

Figure 3 defines the syntax for types \(A, B, C \in \mathcal{T}\), a subtyping relation \(A <: B\), and a type system \(\Gamma \vdash e : A\). We write \(\top\) for the “top” of all function types, written \(\nu\) in the literature [22]. The \(\lambda\)-calculus we study here includes integers and arithmetic, so we have added a singleton type \(n\) for every integer. We define a filter model \(\mathcal{E}_C\) in terms of the type system as follows. The domain is \(\mathcal{P}(\mathcal{T})\).

\[
\mathcal{E}_C[e][\Gamma] = \{A \mid \Gamma \vdash e : A\}
\]

The name filter comes from topology and order theory, and refers to a set that is upward closed and closed under finite intersection. These two properties are satisfied by fiat in intersection type systems because of the subsumption and \(\wedge\)-introduction rules.

Alessi et al. [6] show, for many variations of intersection type systems, that typing is preserved by both reduction and expansion, that is

**Preservation under Reduction** If \(\Gamma \vdash e : A\) and \(e \rightarrow e'\), then \(\Gamma \vdash e' : A\).

**Preservation under Expansion** If \(\Gamma \vdash e' : A\) and \(e \rightarrow e'\), then \(\Gamma \vdash e : A\).

While type systems generally preserve types under reduction, preserving under expansion is unusual and is what enables intersection type systems to completely characterize the behavior of a program. In terms of the filter model, meaning is invariant under reduction: \(e \rightarrow e'\) implies \(\mathcal{E}_C[e] = \mathcal{E}_C[e']\).

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1. Really the \(Z\) combinator because we are in a call-by-value setting.
VI

Types

\[ A, B, C \in \mathbb{T} ::= n \mid A \rightarrow B \mid A \land B \mid \top \]

Subtyping

\[
\begin{align*}
A \rightarrow B &<: \top & A &: A \land A & A \land B &: A & A \land B &: B \\
(A \rightarrow B) \land (A \rightarrow C) &<: A \rightarrow (B \land C) \\
A &: B & B &: C & A &: A' & B &: B' & A' &: A & B &: B'
& A \land B &: A' \land B' & A &: A \rightarrow A' \rightarrow B'
\end{align*}
\]

Typing

\[
\begin{align*}
\Gamma \vdash n : n \\
\Gamma \vdash e_1 : n_1 & \quad \Gamma \vdash e_2 : n_2 \\
\Gamma, x : A \vdash e : B \\
\Gamma \vdash M : A & \quad \Gamma \vdash M : B \\
\Gamma \vdash M : A & \vdash \lambda x. e : A \rightarrow B \\
\Gamma \vdash e_1 : A & \vdash e_2 : A \\
\Gamma \vdash e_1 : A & \vdash e_2 : B \\
\Gamma \vdash e_1 : n & \vdash e_2 : B \\
\Gamma \vdash e_1 : n & \vdash e_2 : B \\
\Gamma \vdash e_3 : B & \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : B \\
\Gamma \vdash e_3 : B & \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : B
\end{align*}
\]

Fig. 3. An intersection type system that characterizes the CBV \(\lambda\)-calculus.

3 A Straightforward Elementary Semantics

We wish to use elementary semantics for the specification of real programming languages, so it is important that the semantics be as intuitive as possible. To pick some nits, the placement of finite sets in the domains of Plotkin and Engeler is unintuitive. For example, one might naively think that the meaning of \(\lambda x. x + 1\) would include an input-output entry such as \((3, 4)\). But in Plotkin’s model we instead have \((\{3\}, \{4\})\) and in Engeler’s model we have \((\{3\}, 4)\). However, there is a domain from the semantic subtyping literature that places the finite sets where one would expect \(^{27}\).

\[ D = \mathbb{Z} + \mathcal{P}_f(D \times D) \]

The idea is straightforward: a function is represented by a finite approximation of its graph. To our knowledge, this domain has never been used to give meaning to programs, only to types.

So the domain for our elementary semantics is \(\mathcal{P}(D)\). Let \(t\) (for table) range over \(\mathcal{P}_f(D \times D)\). We define the semantics \(E\) in Figure 4 to take an expression and an environment and return a set of elements. Thanks to the change in domain, the environment \(\rho\) is simply a partial map from variables to elements (not sets of elements).

The meaning of a \(\lambda x. e\) is the set of all finite graphs \(t\) such that for every input-output entry \((d, d') \in t\), the output element \(d'\) is in the meaning of the body
Proposition 1 (Downward closed aka. Subsumption).

1. If $d \in E[e] \rho$, $d' \sqsubseteq d$, and $f x(e) \vdash \rho \sqsubseteq \rho'$, then $d' \in E[e] \rho'$.
2. If $d \in E[e] \rho$, $d' \sqsubseteq d$, then $d' \in E[e] \rho$.

Proof. The proof of part (1) is by induction on $e$. Part (2) follows from part (1).

Fig. 4. A new elementary semantics for CBV $\lambda$-calculus.
The proof of Proposition 1 is interesting in that it influenced our definition of the semantics. Regarding variables, the semantics for a variable $x$ includes not just the element $\rho(x)$ but also all elements below $\rho(x)$ (see Figure 4). If instead we had defined $E[x]_\rho = \rho(x)$, then the case for variables in the proof of Proposition 1 would break. The same can be said regarding $d \sqsubseteq d'$ in function application.

Next we consider finite unions, that is, we define a join operator on $\mathbb{D}$.

$$n \sqcup n = n \quad t \sqcup t' = t \sqcup t' \quad d \sqcup d$$

Of course the join is not always defined, e.g., there is no join of the integers 0 and 1. The join is indeed the least upper bound of $\sqsubseteq$.

**Proposition 2 (Join is the least upper bound of $\sqsubseteq$).**

1. $v_1 \sqsubseteq v_1 \sqcup v_2$, $v_2 \sqsubseteq v_1 \sqcup v_2$, and
2. If $v_1 \sqsubseteq v_3$ and $v_2 \sqsubseteq v_3$, then $v_1 \sqcup v_2 \sqsubseteq v_3$.

We write $\overline{v}$ for syntactic values that are closed:

$$\overline{v} ::= n \mid \lambda x. e \quad \text{where } \text{fv}(\lambda x. e) = \emptyset$$

The following lemma says that, if we have two elements $d_1, d_2$ in the meaning of a syntactic value $\overline{v}$, then their join is too.

**Lemma 1 (Closed Under Join on Values).** If $d_1 \in E[\overline{v}]_\rho$ and $d_2 \in E[\overline{v}]_\rho$, then $d_1 \sqcup d_2 \in E[\overline{v}]_\rho$.

### 3.2 Is this semantics fully abstract?

No. Recall that a denotational semantics is *fully abstract* if contextual equivalence implies denotational equivalence. Many denotational semantics are not fully abstract, and neither is our elementary semantics. The upshot is that one cannot use the inequality of two programs denotations to prove that they actually behave differently. The counter example is the usual one: parallel-or. The parallel-or function, $\text{por}$, takes two thunks, and returns true if either of them returns true. It cannot be implemented in a sequential language because inside $\text{por}$, either $f$ or $g$ must be called first. If the first thunk goes into an infinite loop, then the second one will never be evaluated.

However, the domain $P(\mathbb{D})$ contains a semantics for $\text{por}$ as follows. For clarity, we add $\text{unit}$ and pairs to the language and encode true as 1 and false as 0.

$$\text{POR} = \{((t_1, t_2), 1) \mid (\text{unit}, 1) \in t_1 \lor (\text{unit}, 1) \in t_2\}$$

$$\cup\{((t_1, t_2), 0) \mid (\text{unit}, 0) \in t_1 \land (\text{unit}, 0) \in t_2\}$$

This denotation will return true if either argument to $\text{POR}$ returns true.
Now to use POR to show that contextual equivalence does not imply denotational equivalence. We first define two test functions \( T_i \) for \( i \in \{0, 1\} \) as follows. Let \( \Omega \) be the divergent combinator \((\lambda x. xx) (\lambda x. xx)\).

\[
T_i = \lambda f. \begin{cases} 
  f (\lambda x. 1, \lambda x. \Omega) & \text{if } i = 0 \\
  f (\lambda x. \Omega, \lambda x. 1) & \text{then } i = 1 \\
  f (\lambda x. 0, \lambda x. 0) & \text{then } \Omega \\
  \text{else } \Omega \\
  \text{else } \Omega 
\end{cases}
\]

We have that \( T_0 \simeq T_1 \), but \( E[\llbracket T_0 \rrbracket] \neq E[\llbracket T_1 \rrbracket] \). To see why \( T_0 \simeq T_1 \), consider the possibilities for the input. They could be given a function that 1) always diverges, 2) forces the first thunk, 3) forces the second thunk, or 4) forces neither and always returns the same thing which could be a) zero, b) a non-zero integer, or c) a function. In case 1), both \( T_0 \) and \( T_1 \) diverge in the first call to \( f \). In case 2), depending on the result of the first call to \( f \), both \( T_0 \) and \( T_1 \) either get stuck, take the else branch of the first if and diverge, or take the then branch and diverge in the second call to \( f \). In case 3), both \( T_0 \) and \( T_1 \) diverge on the first call to \( f \). In case 4a) both \( T_0 \) and \( T_1 \) take the else branch of the first if and diverge. In case 4b) both \( T_0 \) and \( T_1 \) take the then branch of all three if’s and diverge. In case 4c), both \( T_0 \) and \( T_1 \) get stuck after the first call to \( f \). So we have \( T_0 \simeq T_1 \), but it remains to show \( E[\llbracket T_0 \rrbracket] \neq E[\llbracket T_1 \rrbracket] \). We have that \((POR, 0) \in E[\llbracket T_0 \rrbracket]\) but \((POR, 0) \notin E[\llbracket T_1 \rrbracket]\).

We note that Ronchi Della Rocca and Paolini \[54\] proved full abstraction for a filter model for the CBV \( \lambda \)-calculus at the cost of some added complexity.

### 3.3 Example of a Recursive Function

To provide a concrete example of our semantics, we show the semantics of the factorial function, implemented using the \( Z \) combinator (the \( Y \) combinator for strict languages). The main idea for how our semantics gives meaning to recursive functions is that of a Matryoshka doll: it nests ever-smaller versions of it’s graph inside of them.

Recall the \( Z \) combinator:

\[
M \equiv \lambda x. f (\lambda v. (x x) v) \quad Z \equiv \lambda f. M \ M
\]

The factorial function is defined as follows, with a parameter \( r \) for calling itself recursively. We give names (\( F \) and \( H \)) to the \( \lambda \) abstractions because we shall define tables for each of them.

\[
F \equiv \lambda n. \begin{cases} 
  1 & \text{if } n = 0 \\
  n \times r (n - 1) & \text{else}
\end{cases} \quad H \equiv \lambda r. F \quad \text{fact} \equiv Z \ H
\]

We begin with the tables for \( F \); we define a function \( F_i \) that gives the table for just one input \( n \); it simply maps \( n \) to \( n \) factorial.

\[
F_i(n) = \{(n, n!)\}
\]
The tables for $H$ map a factorial table for $n - 1$ to a table for $n$. We invite the reader to check that $H_t(n) \in E[H] \emptyset$ for any $n$.

$$H_t(n) = \{(\emptyset, F_t(0)), (F_t(0), F_t(1)), \ldots, (F_t(n-1), F_t(n))\}$$

Next we come to the most important part: describing the tables for $M$. Recall that $M$ is applied to itself; this is where the analogy to Matryoshka dolls comes in. We define $M_t$ by recursion on $n$. Each $M_t(n)$ extends the smaller version of itself, $M_t(n-1)$, with one more entry that maps the smaller version to the table for factorial of $n-1$.

$$M_t(0) = \emptyset$$

$$M_t(n) = M_t(n-1) \cup \{(M_t(n-1), F_t(n-1))\}$$

The tables $M_t$ are meanings for $M$ because $M_t(n+1) \in E[M](f:=H_t(k))$ for any $n \leq k$. The application of $M$ to itself is OK, $F_t(n) \in E[M][M](f:=H_t(k))$, because $(M_t(n), F_t(n)) \in M_t(n+1)$, $M_t(n) \sqsubseteq M_t(n+1)$, and $F_t(n) \sqsubseteq F_t(n)$.

To finish things up, the tables for $Z$ map the $H_t$’s to the factorial tables.

$$Z_t(n) = \{(H_t(n), F_t(n))\}$$

Then we have $Z_t(n) \in E[Z] \emptyset$ for all $n$. So $F_t(n) \in E[Z][H] \emptyset$ for any $n$ so we conclude that $n! \in E[fact] n \emptyset$.

4 The Elementary Semantics as a Type System

In this section we present a type system based on intersection types, similar to the one in Section 2.2, and mechanize the proof that it is isomorphic to our elementary semantics. The grammar of types is the following and consists of two non-terminals, one for types of functions and one for types in general.

$$F, G, H ::= A \rightarrow B \mid F \land G \mid \top$$

$$A, B, C \in T ::= n \mid F$$

Again $n$ is a singleton integer type. The intersection type $G \land H$ is for functions that have type $F$ and $G$. All functions have type $\top$. The use of two non-terminals enables the restriction of the intersection types to not include singleton integer types, which would either be trivial (e.g. $1 \land 1$) or garbage (e.g. $0 \land 1$).

We define subtyping $<$: and type equivalence $\approx$ in Figure 4, but with fewer rules than in Figure 3. We omit the following rules, i.e., the subtyping rule for functions and for distributing intersections through functions, because they were not needed for the isomorphism with our elementary semantics, and therefore not needed in the proofs of correctness (Section 5).

$$A' < A \quad B < B' \quad (C \rightarrow A) \land (C \rightarrow B) < (C \rightarrow (A \land B))$$
Fig. 5. Our subtyping relation on intersection types.

The set of types $\mathbb{T}$ is isomorphic to the set $\mathbb{D}$ defined in Section 3. The singleton types are isomorphic to integers. The function, intersection, and $\top$ types taken together are isomorphic to function tables. Each entry in a table corresponds to a function type. The following two functions, $\text{typof}$ and $\text{eltof}$, witness the isomorphism. Strictly speaking, the isomorphism is between $\mathbb{D}$ and $\mathbb{T}/\approx$, so elements of $\mathbb{D}$ are a canonical form for types.

\[
\begin{align*}
\text{typof} : \mathbb{D} &\rightarrow \mathbb{T} \\
\text{eltof} : \mathbb{T} &\rightarrow \mathbb{D} \\
\text{typof}(n) &= n \\
\text{eltof}(n) &= \{n\} \\
\text{typof}(t) &= \bigwedge_{(d,d') \in t} \text{typof}(d) \rightarrow \text{typof}(d') \\
\text{eltof}(F) &= \text{tabof}(F) \\
\text{tabof}(A \rightarrow B) &= \{(\text{eltof}(A), \text{eltof}(B))\} \\
\text{tabof}(A \land B) &= \text{eltof}(A) \cup \text{eltof}(B) \\
\text{tabof}(\top) &= \emptyset
\end{align*}
\]

The subtyping relation of Figure 5 is the inverse of the $\sqsubseteq$ ordering on $\mathbb{D}$.

**Proposition 3 (Subtyping is inverse of $\sqsubseteq$ and is related to $\in$).**

1. If $A < B$, then $\text{eltof}(B) \sqsubseteq \text{eltof}(A)$.
2. If $d \sqsubseteq d'$, then $\text{typof}(d') < : \text{typof}(d)$.
3. If $(d,d') \in t$, then $\text{typof}(t) < : \text{typof}(d) \rightarrow \text{typof}(d')$.

**Proposition 4 (Types and Elements are Isomorphic).**

$\text{eltof}(\text{typof}(d)) = d$ and $\text{typof}(\text{eltof}(A)) \approx A$

Figure 6 defines our elementary semantics as a type system. The type system is the same as the one in Figure 3 except that it replaces $\land$-introduction:

\[
\begin{array}{c}
\Gamma \vdash e : A \\
\Gamma \vdash e : B \\
\hline
\Gamma \vdash e : A \land B
\end{array}
\]

with a more specialized rule that requires $e$ to be a $\lambda$ abstraction.
Theorem 1 (Equivalence of type system and elementary semantics).
1. $d \in E[\epsilon] \rho$ implies $\text{typof}(\rho) \vdash e : \text{typof}(d)$ and
2. $\Gamma \vdash e : A$ implies $\text{eltof}(A) \in E[\epsilon] \text{eltof}(\Gamma)$.

where $\text{typof}(\rho)(x) = \text{typof}(\rho(x))$ and $\text{eltof}(\Gamma)(x) = \text{eltof}(\Gamma(x))$.

5 Mechanized Correctness of Elementary Semantics

We prove that our elementary semantics for CBV $\lambda$-calculus is equivalent to the standard operational semantics on programs. Section 5.1 proves one direction of the equivalence and Section 5.2 proves the other. Then to justify the use of the elementary semantics in compiler optimizations, we prove in Section 5.3 that it is sound with respect to contextual equivalence.

5.1 Sound with respect to operational semantics

In this section we prove that the elementary semantics is sound with respect to the relational semantics of Kahn [38], written $\rho \vdash e \Rightarrow v$. For each lemma and theorem we give a proof sketch that includes which other lemmas or theorems were needed. For the full details we refer the reader to the Isabelle mechanization.

We relate $D$ to sets of syntactic values $V$ of Kahn [38] with the following logical relation. Logical relations are usually type-indexed, not element-indexed, but our domain elements are isomorphic to types.

$$G : D \rightarrow \mathcal{P}(V)$$

$$G(n) = \{n\}$$

$$G(t) = \{ \langle \lambda x. e, \rho \rangle \mid \forall (d_1, d_2) \in t. \forall v_1 \in G(d_1) \implies \exists v_2. G(x:=v_1) \vdash e \Rightarrow v_2 \land v_2 \in G(d_2) \}$$

Lemma 2 ($G$ is downward closed). If $v \in G(d)$ and $d' \subseteq d$, then $v \in G(d')$.

Proof. The proof is a straightforward induction on $d$.

We relate the two environments with the inductively defined predicate $G(\rho, g)$.

$$G(\emptyset, \emptyset) \quad G(\rho, g) \quad G(\rho(x:=d), g(x:=v))$$

Lemma 3. If $G(\rho, g)$, then $g(x) \in G(\rho(x))$
Proof. The proof is by induction on the derivation of \( G(\rho, \varphi) \).

**Lemma 4.** If \( d \in \mathcal{E}[e] \rho \) and \( G(\rho, \varphi) \), then \( \varphi \vdash e \Rightarrow v \) and \( v \in \mathcal{G}(d) \) for some \( v \).

Proof. The proof is by induction on \( \varphi \). The cases for integers and arithmetic operations are straightforward. The case for variables uses Lemmas 2 and 3. The cases for lambda abstraction and application use Lemma 2.

**Theorem 2 (Sound wrt. op. sem.).** (aka. Adequacy)

If \( \mathcal{E}[e] \emptyset = \mathcal{E}[n] \emptyset \), then \( e \Downarrow n \).

Proof. From the premise we have \( n \in \mathcal{E}[e] \emptyset \). Also we immediately have \( \mathcal{G}(\emptyset, \emptyset) \). So by Lemma 4 we have \( \emptyset \vdash e \Rightarrow v \) and \( v \in \mathcal{G}(n) \) for some \( v \). So \( v = n \) and therefore \( e \Downarrow n \).

This proof of soundness can also be viewed as a proof of implementation correctness. The operational semantics, being operational, can be viewed as a kind of implementation, and in this light, the above proof is an example of how convenient it can be to prove (one direction of) implementation correctness using the elementary semantics as the specification.

### 5.2 Complete with respect to operational semantics

In this section we prove that the elementary semantics is complete with respect to the small-step semantics for the CBV \( \lambda \)-calculus \[47\]. The proof strategy is adapted from work by Alessi et al. \[6\] on intersection types. We need to show that if \( e \rightarrow^* n \), then \( \mathcal{E}[e] \emptyset = \mathcal{E}[n] \emptyset \). The meaning of the last expression in the reduction sequence is equal to \( \mathcal{E}[n] \emptyset \), so if we could just walk this backwards, one step at a time, we would have our result. That is, we need to show that if \( e \rightarrow e' \), then \( \mathcal{E}[e] \rho = \mathcal{E}[e'] \rho \). We can decompose the equality into \( \mathcal{E}[e] \rho \subseteq \mathcal{E}[e'] \rho \) and \( \mathcal{E}[e'] \rho \subseteq \mathcal{E}[e] \rho \). The forward direction is equivalent to proving type preservation for our intersection type system, which is straightforward. Let us focus on the backward direction and the case of \( \beta \) reduction.

Consider the following example reduction:

\[
(\lambda x. (x \ 1) + (x \ 2)) \ (\lambda y. \ldots) \rightarrow ((\lambda y. \ldots) \ 1) + ((\lambda y. \ldots) \ 2)
\]

where \( e = (\lambda x. \ldots) \ (\lambda y. \ldots) \) and \( e' = ((\lambda y. \ldots) \ 1) + ((\lambda y. \ldots) \ 2) \). For some arbitrary \( d \), we can assume that \( d \in \mathcal{E}_{e'} \emptyset \) and need to show \( d \in \mathcal{E}_e \emptyset \). From \( d \in \mathcal{E}_{e'} \emptyset \) we know there must have been some tables \( t_1 \) and \( t_2 \) such that \( t_1 \in \mathcal{E}_{\lambda y. \ldots} \emptyset \) and \( t_2 \in \mathcal{E}_{\lambda y. \ldots} \emptyset \), but we only know for sure that \( 1 \) is in the domain of \( t_1 \) and \( 2 \) is in the domain of \( t_2 \). Perhaps \( t_1 = \{(1, 7)\} \) and \( t_2 = \{(2, 0)\} \).

However, to obtain \( d \in \mathcal{E}_e \emptyset \) we need a single table \( t_3 \) that can be bound to variable \( x \), and has both 1 and 2 in its domain so that the applications \( (x \ 1) \) and \( (x \ 2) \) make sense. Fortunately, we can simply combine the two tables (Lemma 1).

\[
t_3 = t_1 \cup t_2 = \{(1, 7), (2, 0)\}
\]  

(1)
In general, we also need a lemma about reverse substitution. Recall the rule for $\beta$ reduction

$$(\lambda x. e_1) \, v \rightarrow e_1[v/x]$$

Then we need a lemma that says, for any $d_1$, there is some $d_2 \in \mathcal{E}[v]0$ such that

$$d_1 \in \mathcal{E}[e_1[v/x]]0 \implies d_1 \in \mathcal{E}[e_1]0(x:=d_2)$$

Generalizing this so that it can be proved by induction gives us Lemma 5 below.

We proceed with the formal development of the completeness proof. We define

$$x. \rho$$

for $\rho$ equivalence of environments, written $\rho \approx \rho'$, and note that $\rho \approx \rho'$ implies $\rho \subseteq \rho'$.

$$\rho \approx \rho' \equiv \forall x. \rho(x) = \rho'(x)$$

**Lemma 5 (Reverse substitution preserves meaning).** If $d \in \mathcal{E}[e[\tau/y]]\rho$, then $d \in \mathcal{E}[e]d' \rho'$, $d' \in \mathcal{E}[\tau]0$, and $\rho' \approx \rho(x:=d')$ for some $\rho', d'$.

**Proof.** The proof is by induction on $e' = e[\tau/y]$. But before considering the cases for $e'$, we first consider whether or not $e = y$. The proof uses Propositions 1 and 3 and Lemma 1.

**Lemma 6 (Reverse reduction preserves meaning).**

1. If $e \rightarrow e'$, then $\mathcal{E}[e'] \rho \subseteq \mathcal{E}[e] \rho$.
2. If $e \rightarrow^* e'$, then $\mathcal{E}[e'] \rho \subseteq \mathcal{E}[e] \rho$.

**Proof.**

1. The proof is by induction on the derivation of $e \rightarrow e'$. All of the cases are straightforward except for $\beta$ reduction. In that case we have $(\lambda x. e_1) \, \tau \rightarrow e_1[\tau/x]$. Fix an arbitrary $d$ and assume $d \in \mathcal{E}[e_1[\tau/x]] \rho$. We need to show that $d \in \mathcal{E}[(\lambda x. e_1) \, \tau] \rho$. By Lemma 5 and the assumption there exist $d'$ and $\rho'$ such that $d \in \mathcal{E}[e_1] \rho'$, $d' \in \mathcal{E}[\tau]0$, and $\rho' \approx \rho(x:=d')$. Then we have $d \in \mathcal{E}[e_1] \rho(x:=d')$ by Proposition 1 noting that $\rho' \subseteq \rho(x:=d')$. Therefore we have $(d', d') \in \mathcal{E}[(\lambda x. e_1) \, \tau] \rho$. Also, we have $d' \in \mathcal{E}[\tau] \rho$ by another use of Proposition 1 noting that $\emptyset \subseteq \rho$. With these two facts, we conclude that $d \in \mathcal{E}[(\lambda x. e_1) \, \tau] \rho$.
2. The proof is by induction on the derivation of $e \rightarrow^* e'$. The base case is trivial and the induction step follows immediately from part (1).

Next we prove the forward direction, that reduction preserves meaning. The proof follows the usual pattern for preservation of a type system. However, we shall continue to use the denotational semantics here.

**Lemma 7 (Substitution preserves meaning).** If $d \in \mathcal{E}[e] \rho'$, $d' \in \mathcal{E}[\tau]0$, and $\rho' \approx \rho(x:=d')$, then $d \in \mathcal{E}[e[v/x]]$

**Proof.** The proof is by induction on $e$. The case for variables uses Proposition 1. The case for $\lambda$ uses Proposition 1.
Lemma 8 (Reduction preserves meaning).
1. If \( e \rightarrow e' \), then \( \mathcal{E}[e] \rho \subseteq \mathcal{E}[e'] \rho \).
2. If \( e \rightarrow^* e' \), then \( \mathcal{E}[e] \rho \subseteq \mathcal{E}[e'] \rho \).

Proof.
1. The proof is by induction on \( e \rightarrow e' \). Most of the cases are straightforward.
   The case for \( \beta \) uses Proposition 1 and Lemma 7.
2. The proof is by induction on the derivation of \( e \rightarrow^* e' \), using part (1) in the induction step.

Corollary 1 (Meaning is invariant under reduction).
If \( e \rightarrow^* e' \), then \( \mathcal{E}[e] = \mathcal{E}[e'] \).

Proof. The two directions are proved by Lemma 6 and 8.

Theorem 3 (Complete wrt. op. sem.). If \( e \Downarrow n \), then \( \mathcal{E}[e] \emptyset = \mathcal{E}[n] \emptyset \).

Proof. From the premise we have \( e \rightarrow^* n \), from which we conclude by use of Lemma 1.

The completeness theorem is rather important for the elementary semantics. It says that \( \mathcal{E} \) gives the right meaning to all the terminating programs in the CBV \( \lambda \)-calculus.

We also prove that \( \mathcal{E} \) gives the right meaning to diverging programs. That is, \( \mathcal{E} \) maps diverging programs to \( \emptyset \). We write \( e \uparrow \) when \( e \) diverges.

Proposition 5 (Diverging programs have empty meaning).
If \( e \uparrow \) then \( \mathcal{E}[e] \emptyset = \emptyset \).

Proof. Towards a contradiction, suppose \( \mathcal{E}[e] \emptyset \neq \emptyset \). So \( d \in \mathcal{E}[e] \emptyset \) for some \( d \). We have \( \mathcal{G}(\emptyset, \emptyset) \), so \( \emptyset \vdash e \Rightarrow v \) by Lemma 1. Thus, we also have \( e \rightarrow^* v \). Then from \( e \uparrow \), we have \( v \rightarrow e' \) for some \( e' \), but that is impossible because \( v \) is a value and so cannot further reduce.

In contrast, syntactic values have non-empty meaning.

Proposition 6 (Syntactic values have non-empty meaning). \( \mathcal{E}[\overline{\pi}] \rho \neq \emptyset \).

Proof. The proof is by case analysis on \( \overline{\pi} \).

5.3 Sound with respect to contextual equivalence

We would like to use our elementary semantics to justify compiler optimizations, which replace sub-expressions within a program with other sub-expressions (that are hopefully more efficient). Two sub-expressions are contextual equivalent, defined below, when replacing one with the other does not change the behavior of the program. We define contexts \( C \) with the following grammar.

\[
C ::= C \oplus e \mid e \oplus C \mid \lambda x. C \mid C e \mid e C
\]

contexts

\[
e \simeq e' \equiv \forall C. \text{FV}(C[e]) = \text{FV}(C[e']) = \emptyset \implies C[e] \Downarrow \iff C[e'] \Downarrow \text{ctx. equivalence}
\]
The correctness property that we are after is that denotational equality should imply contextual equivalence. A common way to prove this is to show that the denotational semantics is a congruence and then use soundness and completeness of the semantics for programs [29]. Indeed, we take that approach.

**Lemma 9 (\(E\) is a congruence).** For any context \(C\), if \(E[e] = E[e']\), then \(E[C[e]] = E[C[e']]\).

**Proof.** The proof is a straightforward induction on \(C\).

**Theorem 4 (Sound wrt. Contextual Equivalence).**
If \(E[e] = E[e']\), then \(C[e] \Downarrow\) iff \(C[e'] \Downarrow\) for any closing context \(C\).

**Proof.** We discuss one direction of the iff, that \(C[e] \Downarrow\) implies \(C[e'] \Downarrow\). The other direction is similar. From the premise, congruence gives us \(E[C[e]] = E[C[e']]\). From \(C[e] \Downarrow\) we have \(C[e] \rightarrow^* \tau\) for some \(\tau\). Therefore we have \(E[C[e]]|\emptyset = E[\tau]|\emptyset\) by completeness (Theorem 3). Then we also have \(E[C[e']]|\emptyset = E[\tau]\). So by Proposition \(6\) we have \(v \in E[\tau]\) for some \(v\), and therefore \(v \in E[C[e']]|\emptyset\). We conclude that \(\emptyset \vdash C[e'] \Rightarrow w\) for some \(w\) by soundness (Lemma \(9\)).

## 6 Mechanized Correctness of an Optimizer

We turn to address the question of whether the declarative semantics is useful. Our first case study is proving the correctness of a compiler optimization pass: constant folding and function inlining. The setting is still the untyped \(\lambda\)-calculus extended with integers and arithmetic. Figure 7 defines the optimizer as a function \(O\) that maps an expression and a counter to an expression. One of the challenging problems in creating a good inliner is determining when to stop. Here we use a counter that limits inlining to a fixed depth \(k\). A real compiler would use a smarter heuristic but it would employ the same program transformations.

The third equation in the definition of \(O\) performs constant folding. For an arithmetic operation \(e_1 \oplus e_2\), it recursively optimizes \(e_1\) and \(e_2\). If the results are integers, then it performs the arithmetic. Otherwise it outputs an arithmetic expression. The fourth equation optimizes the body of a \(\lambda\) abstraction. The fifth equation, for function application, is the most interesting. If \(e_1\) optimizes to a \(\lambda\) abstraction and \(e_2\) optimizes to a syntactic value \(v_2\), then we perform inlining by substituting \(v_2\) for parameter \(x\) in the body of the function. We then optimize the result of the substitution, making this a rather aggressive polyvariant optimizer [10, 35, 66, 67]. The counter is decremented on this recursive call to ensure termination.

We turn to proving the optimizer correct with respect to the declarative semantics. The proof is pleasantly straightforward!

**Lemma 10 (Optimizer Preserves Denotations).** \(E(O[e]|k) = E[e]\)

**Proof.** The proof is by induction on the termination metric for \(O\), which is the lexicographic ordering of \(k\) then the size of \(e\). All the cases are straightforward to prove because reduction preserves meaning (Lemma \(1\)) and because meaning is a congruence (Lemma \(9\)).
\[ O[x]k = x \]
\[ O[n]k = n \]
\[ O[e_1 \oplus e_2]k = \begin{cases} n_1 \oplus n_2 & \text{if } O[e_1]k = n_1 \text{ and } O[e_2]k = n_2 \\ [O[e_1]k \oplus O[e_2]k] & \text{otherwise} \end{cases} \]
\[ O[\lambda x. e]k = \lambda x. O[e]k \]
\[ O[e_1 \cdot e_2]k = \begin{cases} O[e[v_2/x]](k-1) & \text{if } k \geq 1 \text{ and } O[e_1]k = \lambda x. e \\ [O[e_1]k O[e_2]k] & \text{otherwise} \end{cases} \]

Fig. 7. A compiler optimization pass that folds and propagates constants and inlines function calls.

The mechanized proof of Lemma 10 is under 30 lines!

**Theorem 5 (Correctness of the Optimizer).** \( e \simeq O[e]k \)

**Proof.** The proof follows immediately from the above Lemma 10 and soundness with respect to contextual equivalence (Theorem 4).

### 7 Elementary Semantics in a Non-determinism Monad

Toward making the elementary semantics easier to scale to larger languages, we show how to hide the set-valued aspect of the denotation function \( E[e] \) behind a non-determinism monad \( 69 \). Recall that a non-deterministic computation returns not just one result, but a set of results. Non-deterministic choice is provided by the bind operation. It chooses, one at a time, an element \( a \) from a sub-computation \( m \), and proceeds with another sub-computation \( f \) that depends on \( a \), then collect all the results into a set. The following is the definition of bind together with short-hand notation.

\[
\text{bind : } \mathcal{P}(\alpha) \rightarrow (\alpha \rightarrow \mathcal{P}(\beta)) \rightarrow \mathcal{P}(\beta)
\]
\[
\text{bind } m \ f = \{ b \mid \exists a \in m \land b \in f(a) \}
\]
\[
X \leftarrow m_1; m_2 \equiv \text{bind } m_1 (\lambda X. m_2)
\]

Backtracking is provided by the way the zero operation interacts with bind. The zero operation simply says to “fail” or “abort” by returning an empty set.

\[
\text{zero : } \mathcal{P}(\alpha) \quad \text{zero} = \emptyset
\]

As usual, the monad also provides a return operation to inject a result into the monad, in this case producing a (singleton) set.

\[
\text{return : } \alpha \rightarrow \mathcal{P}(\alpha) \quad \text{return } a = \{a\} \]
With these monad operations in hand, we create some auxiliary functions that are needed for the interpreter. The following mapM function applies a function $f$ to each element of a finite set, producing a finite set, all within the non-determinism monad. (We write $\uplus$ for the union of two sets that have no elements in common.)

$$\text{mapM} : P(f) \to (\alpha \to P(\beta)) \to P(P_f(\beta))$$

$$\text{mapM} \ 0 \ f = \text{return} \ 0$$

$$\text{mapM} \ (\mathcal{a} \uplus \mathcal{as}) \ f = b \leftarrow f(a); bs \leftarrow \text{mapM as} \ f; \text{return} \ \{b\} \cup bs$$

The next auxiliary function makes sure that the semantics is downward closed with respect to $\sqsubseteq$. The function \text{down} $d$ chooses an arbitrary element $d'$ and returns it if $d' \sqsubseteq d$ and otherwise backtracks to pick another element.

$$\text{down} : \mathcal{D} \to P(\mathcal{D})$$

$$\text{down} \ d = d' \leftarrow \mathcal{D}; \text{if } d' \sqsubseteq d \text{ then return } d' \text{ else zero}$$

The non-deterministic interpreter for the CBV $\lambda$-calculus is defined in Figure 8. Let us focus on the cases for $\lambda$ abstractions and function application. The cases for integers, arithmetic, and conditionals look just as they would for any interpreter in monadic style. For $E[\llbracket\lambda x. e\rrbracket]\rho$, we non-deterministically choose a finite domain $ds$ and then map over it to produce the function’s table. For each input $d$, we interpret the body $e$ in an environment extended with $x$ bound to $d$. This produces the output $d'$, which we pair with $d$ to form one entry in the table. For application $E[\llbracket e_1 e_2\rrbracket]\rho$, we interpret $e_1$ and $e_2$ to the elements $d_1$ and $d_2$, then check whether $d_1$ is a function table. If it is, we non-deterministically select an entry $(d, d')$ from the table and see if the argument $d_2$ matches the parameter $d$. If so we return $\text{down} d'$, otherwise we backtrack and try another entry in the table or possibly backtrack even further and try another table altogether!

8 \ Elementary Semantics and Soundness for System F

This section serves two purposes: it demonstrates that our elementary semantics is straightforward to extend to a typed language with first-class polymorphism, and it demonstrates how to use the elementary semantics to prove type soundness. Our setting is the polymorphic $\lambda$-calculus (System F) extended with general recursion ($\text{fix}$). The idea is that this section is a generalization of Milner’s “well-typed expressions do not go wrong”.

8.1 Static Semantics

The syntax and type system of this language is defined in Figure 9. We represent type variables using DeBruijn indices, with $\forall$ and $\Lambda$ acting as implicit binding forms. The shift operation $\uparrow^k_c(A)$ increases the DeBruijn indices greater or equal to $c$ in type $A$ by $k$ and the substitution operation $[i \mapsto B]A$ replaces DeBruijn index $i$ within type $A$ with type $B$. [17]
\[
\begin{align*}
E[n] & \rho = \text{return } n \\
E[e_1 \oplus e_2] & \rho = d_1 \leftarrow E[e_1] \rho; \ d_2 \leftarrow E[e_2] \rho; \\
\text{case } (d_1, d_2) \text{ of } (n_1, n_2) & \Rightarrow \text{return } n_1 \oplus n_2 \\
E[x] & \rho = \text{down } \rho(x) \\
E[\lambda x. e] & \rho = ds \leftarrow \mathcal{P}_f (\mathbb{D}); \ \text{mapM } ds \ (\lambda d. d' \leftarrow E[e] \rho(x:=d); \ \text{return } (d, d')) \\
E[e_1 e_2] & \rho = d_1 \leftarrow E[e_1] \rho; \ d_2 \leftarrow E[e_2] \rho; \\
\text{case } d_1 \text{ of } \\
& \quad t \Rightarrow (d_1, d_2) \leftarrow t; \ \text{if } d_1 \sqsubseteq d_2 \text{ then down } d' \text{ else zero} \\
& \quad \_ \Rightarrow \text{zero} \\
E[\text{if } e_1 \text{ then } e_2 \text{ else } e_3] & \rho = d_1 \leftarrow E[e_1] \rho; \\
\text{case } d_1 \text{ of } \\
& \quad n \Rightarrow \text{if } n \neq 0 \text{ then } E[e_2] \rho \text{ else } E[e_3] \\
& \quad \_ \Rightarrow \text{zero}
\end{align*}
\]

Fig. 8. Elementary semantics for CBV \(\lambda\)-calculus as a non-deterministic interpreter.

\[
\begin{align*}
i, j & \in \mathbb{N} \\
A, B, C & ::= \text{int } | \ A \rightarrow B \ | \ \forall A \ | \ i \\
e & ::= n \ | \ e \oplus e \ | \ x \ | \ \lambda x : A . e \ | \ e_1 e_2 \ | \ A e \ | \ e[A] \ | \ \text{fix} x : A . e \\
\Gamma & \vdash n : \text{int} \\
\Gamma & \vdash x : \text{lookup}(\Gamma, x) \\
\text{extend}(\Gamma, x, A) & \vdash e : B \\
\Gamma & \vdash \lambda x : A . e : A \rightarrow B \\
\Gamma & \vdash \text{fix} x : A \rightarrow B . e : A \rightarrow B \\
\Gamma & \vdash e : A \rightarrow B \quad \Gamma & \vdash e' : A \\
\text{tyExtend}(\Gamma) & \vdash e : A \\
\Gamma & \vdash A e : \forall A \\
\Gamma & \vdash e[B] : [0 \rightarrow B]A
\end{align*}
\]

Fig. 9. Syntax and type system of System F extended with general recursion (via fix).
Type environments and their operations deserve some explanation in the way they handle type variables. A type environment $\Gamma$ is a pair consisting of 1) a mapping for term variables and 2) a natural number representing the number of enclosing type-variable binders. The mapping is from variables names to a pair of a) the variable’s type and b) the number of enclosing type-variable binders at the variable’s point of definition. We define the operations $\text{extend}$, $\text{tyExtend}$, and $\text{lookup}$ in the following way.

$$\text{extend}(\Gamma, x, A) \equiv (\text{fst}(\Gamma)(x) := (A, \text{snd}(\Gamma)), \text{snd}(\Gamma))$$

$$\text{tyExtend}(\Gamma) \equiv (\text{fst}(\Gamma), \text{snd}(\Gamma) + 1)$$

$$\text{lookup}(\Gamma, x) \equiv \sum_{k-j=0}^{k} (A) \quad \text{where} \quad (A, j) = \text{fst}(\Gamma)(x) \quad \text{and} \quad k = \text{snd}(\Gamma)$$

Thus, the $\text{lookup}$ operation properly transports the type of a variable from its point of definition to its occurrence by shifting its type variables the appropriate amount. We say that a type environment $\Gamma$ is well-formed if $\text{snd}(\text{fst}(\Gamma)(x)) \leq \text{snd}(\Gamma)$ for all $x \in \text{dom}(\text{fst}(\Gamma))$.

### 8.2 Denotational Semantics

The domain differs from that of the $\lambda$-calculus in two respects. First, we add a wrong element to represent a runtime type error, just like Milner [43]. Second, we add an element to represent type abstraction. From a runtime point of view, a type abstraction $\Lambda e$ simply produces a thunk. That is, it delays the execution of expression $e$ until the point of type application (aka. instantiation). A thunk contains an optional element: if the expression $e$ does not terminate, then the thunk contains none, otherwise the thunk contains the result of $e$.

$$o ::= \text{none} \mid \text{some}(d) \quad \text{optional elements}$$

$$t ::= \{(d_1, d'_1), \ldots , (d_n, d'_n)\} \quad \text{tables}$$

$$d ::= n \mid t \mid \text{thunk}(o) \mid \text{wrong} \quad \text{elements}$$

Building on the non-determinism monad (Section 7), we add support for the short-circuiting due to errors with the following alternative form of bind.

$$X ::= E_1; E_2 \equiv X \leftarrow E_1; \text{if } X = \text{wrong} \text{ then return wrong else } E_2$$

We define an auxiliary function $\text{apply}$ to give the semantics of function application. The main difference with respect to Figure 8 is returning wrong when $e_1$ produces a result that is not a function table. We also replace uses of $\leftarrow$ with $\leftarrow$ to short-circuit the computation in case wrong is produced by a sub-computation.

$$\text{apply}(D_1, D_2) = d_1 := D_1; d_2 := D_2;$$

- case $d_1$ of $t$ $\Rightarrow$ $(d, d') \leftarrow t$; if $d \subseteq d_2$ then down $d'$ else zero
- $\_ \Rightarrow \text{return wrong}$

An elementary semantics for System F extended with $\text{fix}$ is defined in Figure 10. Regarding integers and arithmetic, the only difference is that arithmetic
\[
\begin{align*}
\mathcal{E}[n] \rho &= \text{return } n \\
\mathcal{E}[e_1 \oplus e_2] \rho &= d_1 := \mathcal{E}[e_1] \rho; \ d_2 := \mathcal{E}[e_2] \rho; \\
\text{case } (d_1, d_2) \text{ of } (n_1, n_2) &\Rightarrow \text{return } n_1 \oplus n_2 \ | \ _ &\Rightarrow \text{return wrong} \\
\mathcal{E}[x] \rho &= \text{down } \rho(x) \\
\mathcal{E}[\lambda x. A. e] \rho &= ds \leftarrow \mathcal{P}(D); \ \text{mapM } ds \ (\lambda d. d' := \mathcal{E}[e] \rho(x := d); \text{return } (d, d')) \\
\mathcal{E}[e_1. e_2] \rho &= \text{apply}(\mathcal{E}[e_1] \rho, \mathcal{E}[e_2] \rho) \\
\mathcal{E}[\text{fix } x. A. e] \rho &= k \leftarrow \mathbb{N}; \ \text{iterate}(k, x, \mathcal{E}[e], \rho) \\
\text{where } \text{iterate}(0, x, L, \rho) &= \text{zero} \\
\text{iterate}(k + 1, x, L, \rho) &= d := \text{iterate}(k, x, L, \rho); \ L \rho(x := d) \\
\mathcal{E}[\Lambda e] \rho &= \text{if } \mathcal{E}[e] \rho = \emptyset \ \text{then return thunk}\text{(none)} \\
\text{else } d := \mathcal{E}[e] \rho; \ \text{return thunk}\text{(some}(d)) \\
\mathcal{E}[e[A]] \rho &= x := \mathcal{E}[e] \rho; \ \text{case } x \text{ of} \\
\text{thunk}(\text{none}) &\Rightarrow \text{zero} \\
| \ \text{thunk}(\text{some}(d)) &\Rightarrow \text{down } d \\
| \ _ &\Rightarrow \text{return wrong}
\end{align*}
\]

Fig. 10. An elementary semantics for System F with general recursion.

operations return \text{wrong} when an input in not an integer. Regarding \lambda abstraction, the semantics is the same as for the untyped \lambda-calculus (Figure 5).

To give meaning to \((\text{fix } x. A. e)\) we form its ascending Kleene chain but never take its supremum. In other words, we define an auxiliary function \text{iterate} that starts with no tables (\emptyset) and then iteratively feeds the function to itself some finite number of times, produces ever-larger sets of tables that better approximate the function. The parameter \(L\) is the meaning function \(\mathcal{E}\) partially applied to the expression \(e\). We fully apply \(L\) inside of \text{iterate}, binding \(x\) to the previous approximations.

The declarative semantics of polymorphism is straightforward. The meaning of a type abstraction \(\Lambda e\) is \text{thunk}(\text{some}(d)) if \(e\) evaluates to \(d\). The meaning of \(\Lambda e\) is \text{thunk}(\text{none}) if \(e\) diverges. The meaning of a type application \(e[A]\) is to force the thunk, that is, it is any element below \(d\) if \(e\) evaluates to \text{thunk}(\text{some}(d)). On the other hand, if \(e\) evaluates to \text{thunk}(\text{none}), then the type application diverges. Finally, if \(e\) evaluates to something other than a thunk, the result is \text{wrong}.

One strength of this elementary semantics is that it enables the use of monads to implicitly handle error propagation, which is important for scaling up to large language specifications [9, 15, 46].

8.3 Semantics of Types

We define types in the domain \(\mathcal{P}(D)\) with the meaning function \(T\) defined in Figure 11. The meaning of \text{int} is the integers. To handle type variables, \(T\) has a second parameter \(\eta\) that maps each DeBruijn index to a set of elements, i.e., the
meaning of a type variable. So the meaning of index \( i \) is basically \( \eta_i \). The cleanup function serves to make sure that \( T \) is downward closed and that it does not include wrong. (Putting the use of cleanup in \( T[i] \) instead of \( T[\forall A] \) makes for a slightly simpler definition of well-typed environments, which is defined below.)

We write \(|\eta|\) for the length of a sequence \( \eta \). The meaning of a function type \( A \rightarrow B \) is all the finite tables \( t \) in which those entries with input in \( T[A] \eta \) have output in \( T[B] \eta \).

Last but not least, the meaning of a universal type \( \forall A \) includes thunk(some\((d)\)) whenever \( d \) is in the meaning of \( A \) but with \( \eta \) extended with an arbitrary set of elements.

We write \( \vdash \rho, \eta : \Gamma \) to say that environments \( \rho \) and \( \eta \) are well-typed according to \( \Gamma \) and define it inductively as follows.

\[
\begin{align*}
\vdash & \emptyset, \emptyset : \text{empty} \\
\vdash & \rho, \eta : \Gamma \quad v \in T[A]\eta \\
\vdash & \rho(x:=v), \eta : \text{extend}(\Gamma, x, A) \\
\vdash & \rho, \forall \eta : \text{tyExtend}(\Gamma)
\end{align*}
\]

### 8.4 Type Soundness

We prove type soundness for System F, that is, if a program is well-typed and has type \( A \), then its meaning (in a well-typed environment) is a subset of the meaning of its type \( A \).

**Theorem 1 (Semantic Soundness).**

If \( \Gamma \vdash e : A \) and \( \vdash \rho, \eta : \Gamma \), then \( E[e] \rho \subseteq T[A]\eta \).

We use four small lemmas. The first corresponds to Milner’s Proposition 1 [43].

**Lemma 11 (\( T \) is downward closed).**

If \( d \in T[A]\eta \) and \( d' \subseteq d \), then \( d' \in T[A]\eta \).

**Proof.** A straightforward induction on \( A \)

**Lemma 12 (wrong not in \( T \)).** For any \( A \) and \( \eta \), wrong \( \notin T[A]\eta \).

**Proof.** A straightforward induction on \( A \).
Lemma 13. If $\vdash \rho: \Gamma$, then $\Gamma$ is a well-formed type environment.

Proof. This is proved by induction on the derivation of $\vdash \rho: \Gamma$.

Lemma 14. $T[A](\eta_1 \eta_3) = T[\uparrow^{\eta_1}_{\eta_3} (A)](\eta_1 \eta_2 \eta_3)$

Proof. This lemma is proved by induction on $A$.

Next we prove one or two lemmas for each language feature. Regarding term variables, we show that variable lookup is sound in a well-typed environment.

Lemma 15 (Variable Lookup).
If $\vdash \rho, \eta: \Gamma$ and $x \in \text{dom}(\Gamma)$, then $\rho(x) \in T[\text{lookup}(\Gamma, x)]\eta$

Proof. The proof is by induction on the derivation of $\vdash \rho, \eta: \Gamma$. The first two cases are straightforward but the third case requires some work and uses lemmas 13 and 14.

The following lemma proves that function application is sound, similar to Milner’s Proposition 2.

Lemma 16 (Application cannot go wrong).
If $D \subseteq T[A \rightarrow B]\eta$ and $D' \subseteq T[A]\eta$, then $\text{apply}(D, D') \subseteq T[B]\eta$.

Proof. The proof is direct and uses lemmas 12 and 11.

When it comes to polymorphism, our proof necessarily differs considerably from Milner’s, as we must deal with first-class polymorphism. So instead of Milner’s Proposition 4 (about type substitution), we instead have a Compositionality Lemma analogous to what you would find in a proof of Parametricity [62]. We need this lemma because the typing rule for type application is expressed in terms of type substitution ($e[B]$ has type $[0 \mapsto \rightarrow[B]]A$) but the meaning of $\forall A$ is expressed in terms of extending the environment $\eta$.

Lemma 17 (Compositionality).

$$T[A](\eta_1 D \eta_2) = T[\uparrow^{\eta_1}_{\eta_2} (A)](\eta_1 \eta_2)$$

where $D = T[B](\eta_1 \eta_2)$.

Proof. The proof is by induction on $A$. All of the cases were straightforward except for $A = \forall A'$. In that case we use the induction hypothesis to show the following and then apply Lemma 14.

$$T[A'](D \eta_1 D_B \eta_2) = T((\uparrow^{\eta_1}_{\eta_2} (B)) A') (D \eta_1 \eta_2)$$

where $D_B = T[\uparrow^{\eta_1}_{\eta_2} (B)](\eta_1 \eta_2)$

The last lemma proves that iterate produces tables of the appropriate type.

Lemma 18 (Iterate cannot go wrong). If

- $d \in \text{iterate}(k, x, L, \rho)$ and
- for any $d'$, $d' \in T[A \rightarrow B]\eta$ implies $L \rho(x:=d') \eta \subseteq T[A \rightarrow B]\eta$,
then $d \in T[A\rightarrow B]\eta$.

**Proof.** This is straightforward to prove by induction on $k$.

We proceed to the main theorem, that well-typed programs cannot go wrong.

**Theorem 1 (Semantic Soundness).**
If $\Gamma \vdash e : A$ and $\vdash \rho, \eta : \Gamma$, then $E[e]_\rho \subseteq T[A]_\eta$.

**Proof.** The proof is by induction on the derivation of $\Gamma \vdash e : A$. The case for variables uses Lemmas 11 and 15. The case for application uses Lemma 16. The case for `fix` applies Lemma 18 using the induction hypothesis to establish the second premise of that lemma. The case for type application uses Lemma 17.

### 8.5 Comparison to Syntactic Type Safety

The predominant approach to proving type safety of a programming language is via *progress and preservation* [30, 41, 47, 73] over a small-step operational semantics. Recall that the Progress Lemma says that a well-typed program can either reduce or it is a value, but it is never stuck, which would correspond to a runtime type error. The strength of this syntactic approach is that the semantics does not need to explicitly talk about runtime type errors, but nevertheless they are ruled out by the Progress Lemma.

In comparison, the semantic soundness approach that we used in this section relies on using an explicit *wrong* element to distinguish between programs that diverge versus programs that encounter a runtime type error. The downside of this approach is that the author of the semantics could mix up *wrong* and *zero* (divergence) and the Semantic Soundness (Theorem 1) would still hold. However, a simple auditing of the semantics can catch this kind of mistake. Also, we plan to investigate whether techniques such as step-indexing [9, 46, 61] could be used to distinguish divergence from *wrong*.

### 8.6 From Type Soundness to Parametricity

An exciting direction for future research is to use the elementary semantics for proving Parametricity and using it to construct Free Theorems, replacing the frame models in the work of Wadler [68]. The idea would be to to adapt $T$, our unary relation on $D$, into $V$, a binary relation on $D$. We would define the following logical relation $R$ in terms of $V$ and the semantics $E$.

$$R[A]\eta = \{(e_1, e_2) \mid \exists v_1 v_2. v_1 \in E[e_1]\emptyset \land v_2 \in E[e_1]\emptyset \land (v_1, v_2) \in V[A]\eta\}$$

Then the Parametricity Theorem could be formulated as:

If $\Gamma \vdash e : A$ and $\vdash \rho, \eta : \Gamma$, then $(e, e) \in R[A]_\eta$.

The proof would likely be similar to our proof of Semantic Soundness.
9 Related Work

Intersection Type Systems The type system view of our elementary semantics (Section 4) is a variant of the intersection type system invented by Coppo et al. [17] to study the untyped $\lambda$-calculus. Researchers have studied numerous properties and variations of the intersection type system [6, 8, 18, 19, 31]. Our mechanized proof of completeness with respect to operational semantics (Section 5.2) is based on a (non-mechanized) proof by Alessi et al. [6].

By making subtle changes to the subtyping relation it is possible to capture alternate semantics [7] such as call-by-value [22, 54] or lazy evaluation [2]. Barendregt et al. [11] give a thorough survey of these type systems. Our $\top$ type corresponds to the type $\nu$ of Egidi et al. [22] and Alessi et al. [6]. Our subtyping relation is rather minimal, omitting the usual rules for function subtyping and the distributive rule for intersections and function types. Our study of singleton integer types within an intersection type system appears to be a novel combination.

Intersection type systems have played a role in the full abstraction problem for the lazy $\lambda$-calculus, in the guise of domain logics [1, 2, 37]. The problem of inhabitation for intersection type systems has seen recent progress [21] and applications to example-directed synthesis [26].

Other Semantics There are many other approaches to programming language semantics that we have not discussed, from axiomatic semantics [25, 32] to games [3, 34], event structures [71], and traces [36, 53]. Our function tables can be viewed as finitary versions of the tree models for SPCF [13, 14], a language with exceptions, and we are interested in seeing whether our model might be fully abstract in that setting.

10 Conclusions and Future Work

In this paper we present an elementary semantics for a CBV $\lambda$-calculus that represents a $\lambda$ abstraction with an infinite set of finite tables. We give a mechanized proof that this semantics is correct with respect to the operational semantics of the CBV $\lambda$-calculus and we present two case studies that begin to demonstrate that this semantics is useful. We leverage the compositionality of our semantics in a proof of correctness for a compiler optimization. We extend the semantics to handle parametric polymorphism and prove type soundness, i.e., well-typed programs cannot go wrong.

Of course, we have just scratched the surface in investigating how well elementary semantics scales to full programming languages. We invite the reader to help us explore elementary semantics for mutable state, exceptions, continuations, recursive types, dependent types, objects, threads, shared memory, and low-level languages, to name just a few. Regarding applications, there is plenty to try regarding proofs of program correctness and compiler correctness. For your next programming language project, give elementary semantics a try!
