AN GENERAL INTEGRAL INEQUALITY FOR CONVEX Functions AND APPLICATIONS

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Abstract. In this paper, we establish new general inequality for convex functions. Then we apply this inequality to obtain the midpoint, trapezoid and averaged midpoint-trapezoid integral inequality. Also, some applications for special means of real numbers are provided.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers and \( a, b \in I \), with \( a < b \). the following double inequality is well known in the literature as the Hermite-Hadamard inequality:

\[
\frac{f(a + b)}{2} \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) whose derivative \( f' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\), i.e., \( \|f'\|_{\infty} = \sup_{t \in (a, b)} |f'(t)| < \infty \). Then the following inequality

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a + b}{2})^2}{(b - a)^2} \right] (b - a) \|f'\|_{\infty}
\]

holds. This result is known in the literature as the Ostrowski inequality[6].

In the realm of real functions of real variable, convex functions constitute a conspicuous body both because they are frequently encountered in practical applications, and because they satisfy a number of useful inequalities and theorems [see, [1]-[3], [5]]. The most important of the inequalities is of course the defining one which states that a real function \( f(x) \) defined on a real-numbers interval \( I = [a, b] \) is convex if, for any three elements \( x_1, x, x_2 \) of \( I \) such that \( x_1 \leq x \leq x_2 \),

\[
f(x) \leq f(x_1)\left[(x_2 - x)/(x_2 - x_1)\right] + f(x_2)\left[(x - x_1)/(x_2 - x_1)\right].
\]

Graphically, this means that the point \( \{x, f(x)\} \) never falls above the straight line segment connecting the points \( \{x_1, f(x_1)\} \) and \( \{x_2, f(x_2)\} \).

Definition 1 ([2]). Let \( f : [a, b] \to \mathbb{R} \) be a given function. We say that \( f \) is an even function with respect to the point \( t_0 = \frac{a + b}{2} \) if \( f(a + b - t) = f(t) \) for \( t \in [a, b] \). We say that \( f \) is an odd function with respect to the point \( t_0 = \frac{a + b}{2} \) if \( f(a + b - t) = -f(t) \) for \( t \in [a, b] \).

2000 Mathematics Subject Classification. 26D15, 41A55, 26D10 .

Key words and phrases. convex function, Ostrowski inequality and special means.

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Here, we use the term even(odd) function for a given \( f : [a, b] \to \mathbb{R} \) if \( f \) is even(odd) with respect to the point \( t_0 = \frac{a+b}{2} \). We know that each function \( f : [a, b] \to \mathbb{R} \) can be represented as a sum of one even and one odd function,

\[
f(t) = f_1(t) + f_2(t)
\]

where

\[
f_1(t) = \frac{f(t) + f(a + b - t)}{2}
\]

is an even function and

\[
f_2(t) = \frac{f(t) - f(a + b - t)}{2}
\]

is an odd function.

In this article, our work is motivated by the works of N. Ujevic [7] and Z. Liu [4]. We obtain new general integral inequality for convex functions. Finally, new error bounds for the midpoint, trapezoid and other are obtained. Some applications for special means of real numbers are also provided.

2. Main Results

In order to prove our main results, we need the following identities:

**Lemma 1.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be twice differentiable function on \( I^o \) with \( f'' \in L_1[a, b] \), then

\[
2 \int_a^b f(t) \, dt - (\beta - \alpha) \left[ f(x) + f(a + b - x) \right]
\]

\[
+ (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2 (a - \alpha) f(a) - 2 (b - \beta) f(b)
\]

\[
(2.1)
\]

\[
+ (\beta - \alpha) \left[ (x - \frac{3\alpha + \beta}{4}) f'(x) + (a + b - x - \frac{\alpha + 3\beta}{4}) f'(a + b - x) \right]
\]

\[
= \int_a^b k(a, b, t) f''(t) \, dt
\]

where

\[
k(a, b, t) := \begin{cases} 
(t - \alpha)^2, & a \leq t < x \\
(t - \frac{a+\beta}{2})^2, & x \leq t \leq a + b - x \quad \text{with} \quad a \leq \alpha < \beta \leq b \\
(t - \beta)^2, & a + b - x < t \leq b
\end{cases}
\]

(2.2)

for any \( x \in [a, \frac{a+b}{2}] \).
Proof. It suffices to note that

$$I = \int_{a}^{b} k(a, b, t) f''(t) dt$$

$$= \int_{a}^{x} (t - \alpha)^2 f''(t) dt + \int_{x}^{a+b-x} \left( t - \frac{\alpha + \beta}{2} \right)^2 f''(t) dt + \int_{a+b-x}^{b} (t - \beta)^2 f''(t) dt$$

$$= I_1 + I_2 + I_3.$$ 

By integration by parts, we have the following identity

$$I_1 = \int_{a}^{x} (t - \alpha)^2 f''(t) dt$$

$$= (x - \alpha)^2 f'(x) - (a - \alpha)^2 f'(a) + 2 (a - \alpha) f(a) - 2 (x - \alpha) f(x) + 2 \int_{a}^{x} f(t) dt.$$ 

Similarly, we observe that

$$I_2 = \int_{x}^{a+b-x} \left( t - \frac{\alpha + \beta}{2} \right)^2 f''(t) dt$$

$$= \left( a + b - x - \frac{\alpha + \beta}{2} \right)^2 f'(a + b - x) - \left( x - \frac{\alpha + \beta}{2} \right)^2 f'(x)$$

$$+ 2 \left( x - \frac{\alpha + \beta}{2} \right) f(x) - 2 \left( a + b - x - \frac{\alpha + \beta}{2} \right) f(a + b - x) + 2 \int_{x}^{a+b-x} f(t) dt$$

and

$$I_3 = \int_{a+b-x}^{b} (t - \beta)^2 f''(t) dt$$

$$= (b - \beta)^2 f'(b) - (a + b - x - \beta)^2 f'(a + b - x)$$

$$+ 2 (a + b - x - \beta) f(a + b - x) - 2 (b - \beta) f(b) + 2 \int_{a+b-x}^{b} f(t) dt.$$
Thus, we can write
\[ I = I_1 + I_2 + I_3 \]
\[ = (\beta - \alpha) \left[ (x - \frac{3\alpha + \beta}{4}) f'(x) + (a + b - x - \frac{\alpha + 3\beta}{4}) f'(a + b - x) \right] \]
\[ - (\beta - \alpha) \left[ f(x) + f(a + b - x) \right] + (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) \]
\[ + 2(a - \alpha) f(a) - 2(b - \beta) f(b) + 2 \int_a^b f(t) dt \]
which gives the required identity (2.1).

**Corollary 1.** Under the assumptions Lemma 5 with \( \alpha = a, \beta = b \), we have the following identity:
\[ 2 \int_a^b f(t) dt - (b - a) [f(x) + f(a + b - x)] + (b - a) (x - \frac{3a + b}{4}) [f'(x) - f'(a + b - x)] \]
\[ = \int_a^b k_1(a, b, t) f''(t) dt \]
where
\[ k_1(a, b, t) = \begin{cases} 
(t - a)^2, & a \leq t < x \\
(t - \frac{a + b}{2})^2, & x \leq t \leq a + b - x \\
(t - b)^2, & a + b - x < t \leq b 
\end{cases} \]
for any \( x \in [a, \frac{a + b}{2}] \).

The proof of the Corollary 1 is proved by Liu in [4]. Hence, our results in Lemma 5 are generalizations of the corresponding results of Liu [4].

**Corollary 2.** Under the assumptions Lemma 5 with \( \alpha = \beta = \frac{a + b}{2} \), we have the following identity:
\[ 2 \int_a^b f(t) dt + \frac{(b - a)^2}{4} [f'(b) - f'(a)] - (b - a) [f(a) + f(b)] = \int_a^b \left( t - \frac{a + b}{2} \right)^2 f''(t) dt. \]

Let us show that the kernel \( k(a, b, t) \) defined by (2.2) is an even function if \( \alpha + \beta = a + b \). Indeed, for \( t \in [a, x] \) we have
\[ k(a, b, a + b - t) = (a + b - t - a)^2 = (t - \beta)^2 = k(a, b, t). \]
For \( t \in [x, a + b - x] \) we have
\[ k(a, b, a + b - t) = (a + b - t - \frac{\alpha + \beta}{2})^2 = \left( t - \frac{\alpha + \beta}{2} \right)^2 = k(a, b, t). \]
For $t \in (a + b - x, b]$ we have
\[ k(a, b, a + b - t) = (a + b - t - \beta)^2 = (t - \alpha)^2 = k(a, b, t). \]
Hence, $k(a, b, t)$ is an even function.

Now, by using the above lemma, we prove our main theorems:

**Theorem 1.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^2$ such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $f'$ is a convex on $[a, b]$ then the following inequality holds:

\[
\left| 2 \int_a^b f(t) dt - (\beta - \alpha) [f(x) + f(a + b - x)] 
+ (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2(a - \alpha) f(a) - 2(b - \beta) f(b)
\right| \\
+ (\beta - \alpha) \left[ (x - \frac{3\alpha + \beta}{4}) f''(x) + (a + b - x - \frac{\alpha + \beta}{4}) f'(a + b - x) \right] \\
\leq \|k\|_\infty \left[ f'(a) + f'(b) - 2f' \left( \frac{a + b}{2} \right) \right].
\]

**Proof.** From Lemma 1 we get,

\[
\left| 2 \int_a^b f(t) dt - (\beta - \alpha) [f(x) + f(a + b - x)] 
+ (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2(a - \alpha) f(a) - 2(b - \beta) f(b)
\right| \\
+ (\beta - \alpha) \left[ (x - \frac{3\alpha + \beta}{4}) f''(x) + (a + b - x - \frac{\alpha + \beta}{4}) f'(a + b - x) \right] \\
\leq \int_a^b |k(a, b, t)| |f''(t)| dt.
\]

Let us consider the following notations
\[ f_1''(t) = \frac{f''(t) + f''(a + b - t)}{2}, \quad f_2''(t) = \frac{f''(t) - f''(a + b - t)}{2}, \]
then we have $f''(t) = f_1''(t) + f_2''(t)$ and $k(a, b, t) f_2''(t)$ is an odd function while $|f_2''(t)|$ and $k(a, b, t) f_1''(t)$ are even functions. Thus, we obtain
\[
\int_a^b k(a, b, t) f''(t) dt = \int_a^b k(a, b, t) [f_1''(t) + f_2''(t)] dt \\
= \int_a^b k(a, b, t) f_1''(t) dt.
\]
Using (2.5) in (2.4), we obtain (2.3) which completes the proof. \(\square\)

**Theorem 2.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be twice differentiable function on \( I \) such that \( f'' \in L_1[a, b] \) where \( a, b \in I, a < b \). If \( |f''| \) is a convex on \( [a, b] \) then the following inequality holds:

\[
2 \int_a^b f(t) dt - (\beta - \alpha) [f(x) + f(a + b - x)] + (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2 (a - \alpha) f(a) - 2 (b - \beta) f(b) + (\beta - \alpha) \left[ (x - \frac{3\alpha + \beta}{4}) f'(x) + (a + b - x - \frac{\alpha + 3\beta}{4}) f'(a + b - x) \right] \leq \frac{b - a}{4} \|k\|_{\infty} \left[ |f''(a)| + |f''(b)| + \left| f'' \left( \frac{a + b}{2} \right) \right| \right].
\]
Proof. By similar computation the proof of Theorem 1 we get

$$
\begin{align*}
&\left| 2 \int_a^b f(t) dt - (\beta - \alpha) \left[ f(x) + f(a + b - x) \right] \\
&+ (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2 (a - \alpha) f(a) - 2 (b - \beta) f(b) \\
&+ (\beta - \alpha) \left[ (x - \frac{3a + \beta}{4}) f'(x) + (a + b - x - \frac{\alpha + 3\beta}{4}) f'(a + b - x) \right] \right| \\
&\leq \|k\|_\infty \int_a^b \left[ |f''(t)| + |f''(a + b - t)| \right] dt.
\end{align*}
$$

(2.7)

Since $|f''|$ is a convex on $[a, b]$, by Hermite-Hadamard’s integral inequality we have

$$
\int_a^b \left[ |f''(t)| + |f''(a + b - t)| \right] dt \leq \frac{b - a}{4} \left[ |f''(a)| + |f''(b)| + \left| f''\left(\frac{a + b}{2}\right)\right| \right].
$$

(2.8)

Therefore, using (2.8) in (2.7), we obtain (2.6) which completes the proof. □

3. Applications to Quadrature Formulas

In this section we point out some particular inequalities which generalize some classical results such as: trapezoid inequality, Ostrowski’s inequality, midpoint inequality and others.

Proposition 1. Under the assumptions Theorem 1 we have

$$
\begin{align*}
&\left| \int_a^b f(t) dt - \frac{b - a}{2} \left[ f(x) + f(a + b - x) \right] + \frac{b - a}{2} (x - \frac{3a + b}{4}) \left[ f'(x) - f'(a + b - x) \right] \right| \\
&\leq \frac{1}{3} \left[ (x - a)^3 + \left( \frac{a + b}{2} - x \right)^3 \right] \left[ f'(a) + f'(b) - 2 f'\left(\frac{a + b}{2}\right) \right].
\end{align*}
$$

(3.1)

Proof. If we choose $\alpha = a, \beta = b$ in (2.2), then we obtain $\|k\|_\infty = \frac{2}{3} \left[ (x - a)^3 + \left( \frac{a + b}{2} - x \right)^3 \right]$. Thus, from the inequality (2.3) it follows that (3.1) holds. □

Remark 1. If we put $x = \frac{a + b}{2}$ in (3.1), we get the "midpoint inequality":

$$
\begin{align*}
&\left| \frac{1}{b - a} \int_a^b f(t) dt - f\left(\frac{a + b}{2}\right) \right| \leq \frac{(b - a)^2}{24} \left[ f'(a) + f'(b) - 2 f'\left(\frac{a + b}{2}\right) \right].
\end{align*}
$$

(3.2)
Proposition 2. Under the assumptions Theorem \[1\] we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} + \frac{(b-a)}{8} [f'(b) - f'(a)] \right|
\]

(3.3)

\[\leq \frac{(b-a)^2}{48} \left[ f'(a) + f'(b) - 2 f' \left( \frac{a+b}{2} \right) \right].\]

Proof. If we choose \(\alpha = \beta = \frac{a+b}{2}\) in \(2.2\), then we obtain \(\|k\|_{\infty} = \frac{(b-a)^3}{24}\). Thus, from the inequality \(2.3\) it follows that \(3.3\) holds.

\[\Box\]

Remark 2. If in \(3.3\) we put \(f'(b) = f'(a)\), we get the ”trapezoid inequality”:

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^2}{24} \left[ f'(b) - f' \left( \frac{a+b}{2} \right) \right].
\]

Another particular integral inequality with many applications is the following one:

Proposition 3. Under the assumptions Theorem \[2\] we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} + \frac{(b-a)}{8} [f'(b) - f'(a)] \right|
\]

(3.5)

\[\leq \frac{(b-a)^3}{192} \left[ f''(a) + |f''(b)| + \left| f'' \left( \frac{a+b}{2} \right) \right| \right].\]

Proof. If we choose \(\alpha = \beta = \frac{a+b}{2}\) in \(2.2\), then we obtain \(\|k\|_{\infty} = \frac{(b-a)^3}{24}\). Thus, from the inequality \(2.3\) it follows that \(3.5\) holds.

\[\Box\]

Remark 3. It is clear that the best estimation we can have in \(3.5\) is for \(f'(b) = f'(a)\) and \(|f''(a)| + |f''(b)| = \left| f'' \left( \frac{a+b}{2} \right) \right|\) getting the ”trapezoid inequality”:

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^3}{192} \left[ f''(a) + |f''(b)| + \left| f'' \left( \frac{a+b}{2} \right) \right| \right].
\]

4. Applications for special means

Recall the following means:

(a) The arithmetic mean

\[A = A(a,b) := \frac{a+b}{2}, \ a, b \geq 0;\]

(b) The geometric mean

\[G = G(a,b) := \sqrt{ab}, \ a, b \geq 0;\]

(c) The harmonic mean

\[H = H(a,b) := \frac{2ab}{a+b}, \ a, b > 0;\]
(d) The logarithmic mean
\[
L = L(a, b) := \begin{cases} 
  a & \text{if } a = b \\
  \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b
\end{cases}, \quad a, b > 0;
\]

(e) The identric mean
\[
I = I(a, b) := \begin{cases} 
  a & \text{if } a = b \\
  \frac{1}{e} \left( \frac{b^p}{a^p} \right)^{1/p} & \text{if } a \neq b
\end{cases}, \quad a, b > 0;
\]

(f) The \(p\)-logarithmic mean:
\[
L_p = L_p(a, b) := \begin{cases} 
  \left[ \frac{(b^{p+1} - a^{p+1})}{(p+1)(b-a)} \right]^{1/p} & \text{if } a \neq b \\
  a & \text{if } a = b
\end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad a, b > 0.
\]

It is also known that \(L_p\) is monotonically nondecreasing in \(p \in \mathbb{R}\) with \(L_{-1} := L\) and \(L_0 := I\). The following simple relationships are known in the literature
\[
H \leq G \leq L \leq I \leq A.
\]

Now, using the results of Section 3, some new inequalities is derived for the above means.

**Proposition 4.** Let \(p > 1\) and \(0 \leq a < b\). Then we have the inequality:
\[
|L_p(a, b) - A_p(a, b)| \leq p \frac{(b - a)^2}{12} \left[ A(a^{p-1}, b^{p-1}) - A^{p-1}(a, b) \right].
\]

**Proof.** The assertion follows from (3.2) applied for \(f(x) = x^p\), \(x \in [a, b]\). We omitted the details. \(\square\)

**Proposition 5.** Let \(0 \leq a < b\). Then we have the inequality:
\[
|A^{-1}(a, b) - L^{-1}(a, b)| \leq \frac{(b - a)^2}{12} \left[ H^{-1}(a^2, b^2) - A^{-2}(a, b) \right].
\]

**Proof.** The assertion follows from (3.2) applied for \(f(x) = -\frac{1}{x}\), \(x \in [a, b]\). \(\square\)

**Proposition 6.** Let \(p > 1\) and \(0 \leq a < b\). Then we have the inequality:
\[
\left| L_p(a, b) - A^p(a, b) + (p-1) \frac{(b-a)^2}{8} L_p^{-2}(a, b) \right| \leq p \frac{(b-a)^2}{24} \left[ A(a^{p-1}, b^{p-1}) - A^{p-1}(a, b) \right].
\]

**Proof.** The assertion follows from (3.3) applied for \(f(x) = x^p\), \(x \in [a, b]\). \(\square\)

**Proposition 7.** Let \(0 \leq a < b\). Then we have the inequality:
\[
\left| \ln[I(a, b)G(a, b)] + \frac{(b-a)^2}{8} G^{-2}(a, b) \right| \leq \frac{(b-a)^3}{96} \left[ H^{-1}(a^2, b^2) + \frac{1}{2} A^{-2}(a, b) \right].
\]

**Proof.** The assertion follows from (3.5) applied for \(f(x) = -\ln x\), \(x \in [a, b]\). \(\square\)
Proposition 8. Let $0 \leq a < b$. Then we have the inequality:

$$|L^{-1}(a, b) - H^{-1}(a, b)| \leq \frac{(b - a)^3}{48} \left[ H^{-1}(a^3, b^3) + \frac{1}{2} A^{-3}(a, b) \right].$$

Proof. The assertion follows from (3.6) applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$. □

References

[1] P. Cerone, S.S. Dragomir and J. Roumeliotis, *An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications*, RGMIA Research Report Collection, V.U.T., 1(1998), 33-39.

[2] P. Cerone and S.S. Dragomir, *Trapezoidal type rules from an inequalities point of view*, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press N.Y. (2000).

[3] S.S. Dragomir and N. S. Barnett, *An Ostrowski type inequality for mappings whose second derivatives are bounded and applications*, RGMIA Research Report Collection, V.U.T., 1(1999), 67-76.

[4] Z. Liu, *Some companions of an Ostrowski type inequality and application*, J. Inequal. in Pure and Appl. Math, 10(2), 2009, Art. 52, 12 pp.

[5] G.V. Milovanovic and J.E. Pecaric, *On generalizations of the inequality of A. Ostrowski and related applications*, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz., No. 544-576 (1976), 155–158.

[6] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert*, Comment. Math. Helv. 10(1938), 226-227.

[7] N. Ujević, *An integral inequality for convex functions and applications in numerical integration*, Appl. Math. E-Notes, 5(2005), 253-260.

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