AFPTAS results for common variants of bin packing: 
A new method to handle the small items

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Abstract

We consider two well-known natural variants of bin packing, and show that these packing problems admit asymptotic fully polynomial time approximation schemes (AFPTAS). In bin packing problems, a set of one-dimensional items of size at most 1 is to be assigned (packed) to subsets of sum at most 1 (bins). It has been known for a while that the most basic problem admits an AFPTAS. In this paper, we develop methods that allow to extend this result to other variants of bin packing. Specifically, the problems which we study in this paper, for which we design asymptotic fully polynomial time approximation schemes, are the following. The first problem is Bin packing with cardinality constraints, where a parameter $k$ is given, such that a bin may contain up to $k$ items. The goal is to minimize the number of bins used. The second problem is Bin packing with rejection, where every item has a rejection penalty associated with it. An item needs to be either packed to a bin or rejected, and the goal is to minimize the number of used bins plus the total rejection penalty of unpacked items. This resolves the complexity of two important variants of the bin packing problem. Our approximation schemes use a novel method for packing the small items. Specifically, we introduce the new notion of windows. A window is a space in which small items can be packed, and is based on the space left by large items in each configuration. The key point here is that the linear program does not assign small items into specific windows (located in specific bins), but only to types of windows. This new method is the core of the improved running times of our schemes over the running times of the previous results, which are only asymptotic polynomial time approximation schemes (APTAS).

1 Introduction

Classic bin packing [27, 11, 6, 7] is a natural and well studied problem which has applications in problems of computer storage, bandwidth allocation, stock cutting, transportation and many other important fields. In the basic variant of this problem, we are given $n$ items of size in $(0, 1]$ which need to be assigned to unit sized bins. Each bin may contain items of total size at most 1, and the goal is to minimize the number of bins used. Many variants of bin packing, coming from practical needs, were studied [10, 9]. Below, we mention several such variants that we address in this paper.

In various applications, such as storage of files on disks and scheduling of jobs on bounded capacity processors, a bin can contain only a limited number of items. This is the main motivation for the study of the following variant of the bin packing problem. The BIN PACKING PROBLEM WITH CARDINALITY CONSTRAINTS (BPCC) is defined as follows: The input consists of an integer $k \geq 1$ (called the cardinality bound) and a set of $n$ items $I = \{1, 2, \ldots, n\}$, of sizes $1 \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$. The goal is to partition the items into the minimum number of bins such that the total size of items in a bin is at most 1, and the

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number of items in a bin is at most \( k \). The problem was introduced as early as in the 1970’s by Krause, Shen and Schwetman \cite{23,24}, and studied in a sequence of papers \cite{4,22,2,15}.

In other applications, such as bandwidth allocation, or any application in storage that allows outsourcing, it is sometimes possible to refuse to pack an item. This rejection clearly needs to be compensated, and costs some given amount for each item. This amount is called the rejection penalty of the item. This situation can also occur in file servers where the items are files, the bins are disks and rejection penalty of a file is the cost of transferring it to be saved on an alternative media. In other applications the rejection penalty refers to the penalty caused by not serving a client. This motivates the following bin packing problem with rejection (BPR). The input to this problem consists of \( n \) items \( I = \{1, 2, \ldots, n\} \), where item \( i \) has a size \( s_i \in (0, 1] \) and rejection penalty \( r_i > 0 \), and the goal is to select a subset \( A \) of \( I \) (the set of accepted items, i.e., items to be packed) and a partition of \( A \) into subsets \( A_1, \ldots, A_z \), for some integer \( z \), such that \( \sum_{i \in A_j} s_i \leq 1 \) for all \( j = 1, 2, \ldots, z \), so that \( z + \sum_{i \in I \setminus A} r_i \) is minimized. The bin packing problem with rejection was introduced and studied by He and Dósa \cite{13}. Further work on this problem can be found in \cite{14,3}. Note that in this problem we assume without loss of generality that all sizes of items are strictly positive. If zero sized items exist, they could be all packed into one bin, changing the cost of the solution by 1 in the case that all other items are rejected by this solution and without any change in the cost otherwise.

Note that both problems studied in the paper are generalizations of classic bin packing. Classic bin packing is the special case of BPCC where \( k = n \), and the special case of BPR where all rejection penalties are infinite.

For an algorithm \( A \), we denote its cost by \( A \) as well. The cost of an optimal offline algorithm is denoted by \( \text{OPT} \). We define the asymptotic approximation ratio of an algorithm \( A \) is the infimum \( R \geq 1 \) such that for any input, \( A \leq R \cdot \text{OPT} + c \), where \( c \) is independent of the input. If we enforce \( c = 0 \), \( R \) is called the absolute approximation ratio. An asymptotic polynomial time approximation scheme is a family of approximation algorithms, such that for every \( \varepsilon > 0 \) the family contains a polynomial time algorithm with an asymptotic approximation ratio of \( 1 + \varepsilon \). We abbreviate asymptotic polynomial time approximation scheme by APTAS (also called an asymptotic PTAS). An asymptotic fully polynomial time approximation scheme (AFPTAS) is an APTAS whose time complexity is polynomial not only in the input size but also in \( \frac{1}{\varepsilon} \). If the scheme satisfies the definition above with \( c = 0 \), stronger results are obtained, namely, polynomial time approximation schemes and a fully polynomial approximation schemes, which are abbreviated as PTAS and FPTAS. Throughout the paper, we use \( \text{OPT} \) to denote the cost of an optimal solution for the original input, which is denoted by \( I \), and we use \( \text{APX} \) to denote the cost of the solution returned by our schemes. For an input \( J \) we use \( \text{OPT}(J) \) to denote the cost of an optimal solution for the input \( J \) (where \( J \) is typically an adapted input). Thus \( \text{OPT} = \text{OPT}(I) \).

Previous results. The classic bin packing problem is known to admit an APTAS \cite{12} and an AFPTAS \cite{21}. Moreover, BPCC and BPR are problems that are known to admit an APTAS \cite{4,14}. Specifically, the APTAS of Caprara, Kellerer and Pferschy \cite{4} generalizes the methods of Fernandez de la Vega and Lueker \cite{12} where items are rounded and grouped using linear grouping. The approach, that is used to deal with items that are too small to be rounded in this way, is not a greedy approach as in \cite{12} (which fails in this case, as is demonstrated in \cite{4}), but all possible packings of large items are enumerated, and for each such possibility, the small items are assigned to the already existing bins and to new bins via a linear program. This last approach, that is used also in \cite{14}, where it is combined with rounding of rejection penalties, results in an APTAS for each one of the problems BPCC and BPR. Clearly, these schemes have large running times due to enumeration. A different APTAS for BPR, with reduced running time, but that still uses some enumeration steps, is given in \cite{3}. Note that classic bin packing or any generalization of it
cannot be expected to admit an FPTAS or even a PTAS, since approximating it within an absolute factor smaller than $\frac{3}{2}$ is NP-hard (using a simple reduction from the Partition problem).

A problem which is dual to bin packing is the bin covering problem. In this problem the goal is to maximize the number of bins for which the total sizes of assigned items is at least 1. This problem is known to admit an APTAS (by Csirik, Johnson and Kenyon [8]) and an AFPTAS (by Jansen and Solis-Oba [19]).

The variant of this problem where a covered bin must in addition to the constraint on the total size of items assigned to it, must contain at least $k$ items, for a parameter $k$, we get the problem of bin covering with cardinality constraints. This last problem was considered in [16], and was shown to admit an AFPTAS as well. We note that even though bin packing problems are the dual problems of bin covering, the latter are maximization problems while the former are minimization problems, and the nature of the problems is very different. In particular, the methods of [16] as well as the methods of [8, 19] are not applicable for the problems considered in this paper, and hence we needed to develop the new methods considered here.

We briefly survey the previous approaches used in the literature for dealing with packing small items fractionally using a linear program. One approach is to see all small items as fluid, without distinguishing between different items. In this case, the small items completely lose their identities, and the conversion process of the fluid into items is performed in a later step, usually greedily. The other approach which allows to keep the identities of items simply allows to introduce constraints associated with each small item separately, and the solution of the mathematical program assigns each small item into a specific bin. The number of constraints is linear in the number of items. It can be seen that in the first approach there is no control whatsoever of the exact allocation of specific items to specific bins, while the second approach is rigid in the sense that all decisions must be made by the mathematical program. Examples for the first approach can be found as early as in the seminal work of Hochbaum and Shmoys [18] on scheduling uniformly related machines, in approximation schemes for other scheduling problems (e.g. [11]), and in previous work on bin packing related problems [19, 8]. Examples of the second approach appear in the previous work on the problems studied here [4, 14].

In this paper, we use a new and novel method of dealing with small items which is an intermediate way between these two extreme (previous) approaches. These items are packed using the (approximated) solution of a linear program. To keep the running time polynomial (in both the size of the input and $\varepsilon$), the linear program does not decide on the exact packing of these items, but only on the type of a bin that they should join, where a type of a bin is defined roughly according to the remaining size in it after the packing of (rounded) large items, and in BPCC, on the cardinality constraint as well.

In this paper, we design an AFPTAS for each of the two problems, BPCC and BPR. Studies of similar flavor was widely conducted for other variants of bin packing and it is an established direction of research, see e.g. [20, 25, 26]. The problems studied in this paper were open in the sense that no AFPTAS is known for them prior to this work.

We start the paper with Section 2 where we elaborate on the methods used in this paper. In Sections 3 and 4 we describe the asymptotic approximation schemes for BPCC and BPR, respectively, and prove their correctness. The approximation scheme for BPCC acts in two different ways according to the value of $k$. The case of small $k$ is easier, since in this case every bin has a relatively small number of items. This simpler scheme is presented first, and it allows the reader to get acquainted with the basic methods (but not with our new methods for dealing with small items). The other schemes presented in this paper, including the scheme for BPCC with large values of $k$, and the scheme for BPR, require much more advanced techniques. The methods that are employed, in order to obtain these two schemes, are related, but each scheme requires different specific ingredients, developed in this work, in order to solve the problem it is meant for. Due to space limitations, we present the easier case of Section 3 and the AFPTAS for BPR in the Appendix.
Note that asymptotic fully polynomial approximation schemes, unlike APTAS results, have practical running times especially if one uses a method of solving approximately the linear program which is faster than the ellipsoid method. Such techniques are available for packing problems. Hence, AFPTAS can actually be used to solve the problems they are developed for. Thus our contribution, where we settle the complexity of the problems studied here, is interesting not only from a theoretical point of view, but also from the practical viewpoint.

2 Methods

In this section we briefly describe the novel methods that are used in this paper, side by side with adaptations of well known methods, that are employed in this paper as well.

Linear grouping is a standard rounding method for bin packing algorithms. It was first presented by Fernandez de la Vega and Lueker [12]. The main idea of this method is to round the sizes of items into a small number of distinct sizes. Unlike rounding methods for scheduling [17], the output sizes resulting from the rounding must be representative sizes of real item sizes in the input. In fact, this last method of rounding, which rounds values from a given range (in our case, rejection penalties) to a closest number among a fixed sized set of values, such that no original value changes as a result by more than a factor of $1 + \varepsilon$, is used in the paper as well in Section B. Both methods are typically used for sizes that are large enough, where small sizes are treated separately. The resulting set of large items has a small number of sizes (usually, a function of $\varepsilon$). On this set of sizes, valid assignment configurations are defined, and a solution to a packing problem is described as a set of bins packed according to specific configurations.

In this paper, we introduce a new and novel method for treatment of small items. These are typically items whose size is below some threshold. Moreover, in Section B, items of small rejection penalty are seen as small as well. For the classic bin packing problem, the treatment of such items is relatively easy. After finding a solution, which is close enough to optimal solution for the large items, small items can be added greedily to the solution, using a simple packing heuristic such as Next Fit or First Fit. As demonstrated by Caprara, Kellerer and Pferschy [4], using this approach for BPCC leads to approximation algorithms of high approximation ratio. That is, finding an optimal packing of just the large items immediately leads to poor performance for the complete input, no matter how the small items are combined into the solution. In order to derive an APTAS, i.e., an algorithm whose approximation ratio is within a factor of $1 + \varepsilon$, but its running time is not necessarily polynomial in $\frac{1}{\varepsilon}$, it is possible to enumerate a large enough number of packings of the large items, and add the small items to each one of them in some way (possibly in an optimal fractional way). This results in a large number of potential outputs, the best of which is actually given as the output. On the other hand, if the goal is to design an AFPTAS, where the running time must be polynomial in $\frac{1}{\varepsilon}$, this approach cannot be successful. First, the number of packings of large items is exponential in $\frac{1}{\varepsilon}$, and second, if small items need to be added optimally and not greedily, it is unclear whether it is possible to handle small items efficiently since in their case, rounding of sizes is harmful. In summary, in order to obtain an AFPTAS, we need a method which allows to consider the small items in the linear program that seeks a solution for the large items, but this linear program cannot search for a complete and final solution for all items. We introduce the new notion of windows. A window is a space in which small items can be packed, and is based on the space left by large items in each configuration. The key point here is that the linear program does not assign small items into specific windows (located in specific bins), but to types of windows. Using rounding, we limit ourselves to a polynomial number of window types. After the linear program has been solved, the small items are assigned to specific windows based on the output. We show
that the items that cannot be assigned to windows (due to fractional solutions, due to rounding, or due to the fact that the windows reside in separate bins) can be packed separately into new bins (or possibly rejected, in BPR) with only a small increase in the cost. In order to allow the packing of almost all small items, in most cases studied here, their packing cannot just be done greedily given the windows reserved for them. A careful treatment that balances the load of small items in similarly packed bins, taking into account not only the total size of small items, but also their number, is required.

In order to find an AFPTAS and not an APTAS for each one of the problems, the linear program, that is associated with the problem on an adapted input (after rounding is applied, and our methods for handling small items are invoked), cannot be solved exactly in polynomial time (in $n$ and $\frac{1}{\varepsilon}$). Therefore, we need to solve it approximately. This is done via the column generation technique of Karmarkar and Karp [21]. The main idea is to find an approximate solution of the dual linear program. For that, we need to find a polynomial time separation oracle (possibly, an approximate one) for each one of the dual programs. This typically involves finding an approximate solution to a knapsack type problem. The exact problem results from the exact characteristics of the bin packing problem and the details of the linear program used to solve it. We develop the linear programs as well as the separation oracles for them in this paper. Clearly, each problem results in a different linear program, a different dual linear program, and a different separation oracle. These separation oracles are based on application of fully polynomial approximation schemes to variants of the knapsack problem.

3 An AFPTAS for BPCC

We fix a small value $\varepsilon < \frac{1}{2}$ such that $\frac{1}{\varepsilon}$ is an integer. Our AFPTAS acts according to one of two options, each of which is suitable for one case. In the first case $k \leq \frac{1}{\varepsilon^3}$ and in the second case $k > \frac{1}{\varepsilon^3}$. Recall that the set of items is denoted by $I$. We refer to an item by its index, so $I = \{1, 2, \ldots, n\}$. The first case can be found in the Appendix.

3.1 Second case: $k > \frac{1}{\varepsilon^3}$

We assume that $k < n$, otherwise there is no effective cardinality constraint, and an AFPTAS for the problem follows from the AFPTAS of Karmarkar and Karp [21]. In this case we partition the item set into large items, that is, items with size at least $\varepsilon$, and small items (all the non-large items). We denote by $L$ the set of large items, and by $S = I \setminus L$ the set of small items. We apply linear grouping to the large items (only). That is, we partition the large items into $\frac{1}{\varepsilon^3}$ classes $L_1, L_2, \ldots, L_{1/\varepsilon^3}$ such that $|L| = |L_1| \geq |L_2| \geq \cdots \geq |L_{1/\varepsilon^3}| = |L|\varepsilon^3$, and such that if there are two items $i, j$ with sizes $s_i > s_j$ and $i \in L_q$ and $j \in L_p$ then $q \leq p$. The two conditions uniquely define the allocation of large items into classes up to the allocation of equal-sized items. If $|L| < \frac{1}{\varepsilon^3}$, then instead of the above partition, each large item has its own set $L_i$ such that $L_1$ is an empty set, and for a large item $i$ we let $s'_i = s_j$ (i.e., we do not apply rounding in this case). We note that in both cases (where we use the original partition of the items or when $|L| < \frac{1}{\varepsilon^3}$) we have $|L_1| \leq 2|L|\varepsilon^3$.

Then, we round up the sizes of the items in $L_2, \ldots, L_{1/\varepsilon^3}$ as follows: For all values of $p = 2, 3, \ldots, 1/\varepsilon^3$ and for each item $i \in L_p$, we let $s'_i = \max_{j \in L_p} s_j$ to be the rounded-up size of item $i$. For a small item $i \in S$ we let $s'_i = s_i$. The rounded-up instance $I'$ consists of the set of items $I \setminus L_1$ where the size of item $i$ is $s'_i$ for all $i$ ($k$ remains unchanged). We also define $L' = L \setminus L_1$ to be the set of large items in the rounded-up instance. We have $\text{OPT}(I') \leq \text{OPT}$.
Given the rounded-up instance $I'$, we define a configuration of large items of a bin as a (possibly empty) set of at most $k$ items of $L'$ whose total (rounded-up) size is at most 1. We denote the set of all configurations of large items by $C$. We denote the set of item sizes in $L'$ by $H$. For each $v \in H$ we denote the number of items with size $v$ in $C$ by $n(v, C)$, and the number of items in $L'$ with size $v$ by $n(v)$ (where $n(v) = \lceil |L|/v \rceil$ or $n(v) = \lceil |L|/v^3 \rceil$, unless several classes are rounded to the same size).

We define the following set of windows that are actual main windows of at least one configuration by $C$.

$$\{w \in W : w_s(C) \text{ is defined} \}$$

where $w_s(C)$ is interpreted as an approximated available size for small items in a bin with configuration $C$, and $w_n(C)$ is interpreted as the number of small items that can be packed in a bin with configuration $C$ while still maintaining the cardinality constraint). More precisely, a main window is defined as follows. Assume that the total (rounded-up) size of the items in $C$ is $s'$. Let $w_s(C) = \left\lfloor \log_{1+\epsilon} \frac{1}{s_{\text{min}}} \right\rfloor + 1$ and that $s' = \left\lfloor \log_{1+\epsilon} \frac{1}{s_{\text{min}}} \right\rfloor + 1$, and $w_n(C)$ is the difference between $k$ and the number of large items in $C$ (i.e., $w_n(C) = k - \sum_{v \in H} n(v, C)$). Note that in any case, for a non-trivial input, $t$ can take at least three values. The main window of a configuration is a window (i.e., belongs to $W'$), but $W'$ may include windows that are not the main window of any configuration. We note that $|W'|$ is polynomial in the size of the input and in $1/\epsilon$, whereas $|C|$ may be exponential in $1/\epsilon$ (specifically, $|C| \leq (1/\epsilon + 1)^1/\epsilon$, since in configuration there are up to $1/\epsilon$ large items of $|H| \leq 1/\epsilon$ sizes). We denote the set of windows that are actual main windows of at least one configuration by $W'$. We first define a linear program that allows the usage of any window in $W$. After we obtain a solution to this linear program, we modify it so that it only uses windows of $W'$. We define a generalized configuration $\tilde{C}$ as a pair $\tilde{C} = (C, w = (w_s, w_n))$, for some configuration $C$ and some $w \in W$. A generalized configuration $\tilde{C}$ is a valid generalized configuration if $w \leq w(C)$. The set of all valid generalized configurations is denoted by $\tilde{C}$.

For $W \in W$ denote by $C(W)$ the set of generalized configurations $\tilde{C} = (C, w = (w_s, w_n))$, such that $w = W$. That is, $C(W) = \{\tilde{C} = (C, w) \in \tilde{C} : W = w\}$.

We next consider the following linear program. In this linear program we have a variable $x_{\tilde{C}}$ denoting the number of bins with generalized configuration $\tilde{C}$, and variables $y_{i,W}$ indicating if the small item $i$ is packed in a window of type $W$ (the exact instance of this window is not specified in the solution of the linear program).
\[
\begin{align*}
\min & \quad \sum_{\tilde{C} \in \tilde{C}} x_{\tilde{C}} \\
\text{s.t.} & \quad \sum_{\tilde{C} = (C, w) \in \tilde{C}} n(v, C) x_{\tilde{C}} \geq n(v) \quad \forall v \in H \\
& \quad \sum_{W \in W} Y_{i, W} \geq 1 \quad \forall i \in S \\
& \quad w_s \cdot \sum_{\tilde{C} \in C(W)} x_{\tilde{C}} \geq \sum_{i \in S} s_i' \cdot Y_{i, W} \quad \forall W = (w_s, w_n) \in W \\
& \quad w_n \cdot \sum_{\tilde{C} \in C(W)} x_{\tilde{C}} \geq \sum_{i \in S} Y_{i, W} \quad \forall W = (w_s, w_n) \in W \\
& \quad x_{\tilde{C}} \geq 0 \quad \forall \tilde{C} \in \tilde{C}, \\
& \quad Y_{i, W} \geq 0 \quad \forall W \in W, \forall i \in S.
\end{align*}
\]

Constraints (1) and (2) ensure that each item (large or small) of \( I' \) will be considered. The large items will be packed by the solution, and the small items would be assigned to some type of window (but not to a specific location). Constraints (3) ensure that the total size of the small items that we decide to pack in window of type \( W \) is not larger than the total available size in all the bins that are packed according to a generalized configuration, whose second component is a window of type \( W \). Similarly, the family of constraints (4) ensures that the total number of the small items that we decide to pack in a window of type \( W \) is not larger than the total number of small items that can be packed (in accord with the cardinality constraint) in all the bins whose generalized configuration of large items induces a window of type \( W \). The linear relaxation assumes, in particular, that small items can be assigned fractionally to windows, that is, the small items leave no gaps. We later show how to construct a valid allocation of small items, leaving a small enough number of small items, whose total size is small enough as well, unpacked. We further show how to deal with these unpacked items. We note that \( \text{OPT}(I') \) implies a feasible solution to the above linear program that has the cost \( \text{OPT}(I') \), since the packing of the small items clearly satisfies the constraints (3) and (4), and the packing of large items satisfies the constraints (1). Moreover, it implies a solution to the linear program in which all variables \( x_{\tilde{C}} \), that correspond to generalized configurations \( \tilde{C} = (C, w) \) for which \( w \) is not the main window of \( C \), are equal to zero, and all variables \( Y_{i, w} \), where \( w \not\in W' \) are equal to zero as well.

We invoke the column generation technique of Karmarkar and Karp \cite{KarmarkarKarp1982} as follows. The above linear program has an exponential number of variables and a polynomial number of constraints (neglecting the non-negativity constraints). Instead of solving the linear program we solve its dual program (that has a polynomial number of variables and an exponential number of constraints). The variables \( \alpha_v \) correspond to the item sizes in \( H \). The variables \( \beta_i \) correspond to the small items. The intuitive meaning of the variables can be seen as weights assigned to these items. For each \( W \in W \) we have a pair of dual variables \( \gamma_W \) and \( \delta_W \). Using these dual variables, the dual linear program is as follows.
\[
\max \sum_{v \in H} n(v)\alpha_v + \sum_{i \in S} \beta_i \\
\text{s.t.} \sum_{v \in H} n(v)\alpha_v + w_s\gamma_w + w_n\delta_w \leq 1 \quad \forall \tilde{C} = (C, w = (w_s, w_n)) \in \tilde{C} \\
\beta_i - s_i^*\gamma_W - \delta_W \leq 0 \quad \forall i \in S, \forall W \in \mathcal{W} \\
\alpha_v \geq 0 \quad \forall v \in H \\
\beta_i \geq 0 \quad \forall i \in S \\
\gamma_W, \delta_W \geq 0 \quad \forall W \in \mathcal{W}.
\]

(5)

(6)

First note that there is a polynomial number of constraints of type (6), and therefore we clearly have a polynomial time separation oracle for these constraints. If we would like to solve the above dual linear program (exactly) then using the ellipsoid method we need to establish the existence of a polynomial time separation oracle for the constraints (5). However, we are willing to settle on an approximated solution to this dual program. To be able to apply the ellipsoid algorithm, in order to solve the above dual problem within a factor of \(1 + \epsilon\), it suffices to show that there exists a polynomial time algorithm (polynomial in \(n, \frac{1}{\epsilon}\) and \(\log \frac{1}{\epsilon}\)) such that for a given solution \(a^* = (\alpha^*, \beta^*, \gamma^*, \delta^*)\) decides whether \(a^*\) is a feasible dual solution (approximately). That is, it either provides a generalized configuration \(\tilde{C} = (C, w = (w_s, w_n)) \in \tilde{C}\) for which \(\sum_{v \in H} n(v, C)\alpha^*_v + w_s\gamma^*_w + w_n\delta^*_w > 1\), or outputs that an approximate infeasibility evidence does not exist, that is, for all generalized configurations \(\tilde{C} = (C, w = (w_s, w_n)) \in \tilde{C}\), \(\sum_{v \in H} n(v, C)\alpha^*_v + w_s\gamma^*_w + w_n\delta^*_w \leq 1 + \epsilon\) holds. In such a case, \(\frac{a^*}{1+\epsilon}\) is a feasible dual solution that can be used. Such a configuration \(\tilde{C}\) can be found by the following procedure: For each \(W = (w_s, w_n) \in \mathcal{W}\) we look for a configuration \(C \in \mathcal{C}\) such that \((C, W)\) is a valid generalized configuration, and \(\sum_{v \in H} n(v, C)\alpha^*_v\) is maximized. If a configuration \(C\) that is indeed found, the generalized configuration, whose constraint is checked, is \((C, W)\). To find \(C\), we invoke an FPTAS for the KCC problem with the following input: The set of items is \(H\) where for each \(v \in H\) there is a volume \(\alpha^*_v\) and a size \(v\), the goal is to pack a multiset of the items, so that the total volume is maximized, under the following conditions. The multiset should consist of at most \(k - w_n\) items (taking the multiplicity into account, but an item can appear at most a given number of times). The total (rounded-up) size of the multiset should be smaller than \(1 - \frac{w_s}{1+\epsilon}\), unless \(w_s \leq s^*_{\min}\), where the total size should be at most 1 (in this case, the window leaves space only for items of size zero). Since the number of applications of the FPTAS for the KCC problem is polynomial (i.e., \(|\mathcal{W}|\)), this algorithm runs in polynomial time. If it finds a solution, that is, a configuration \(C\), with total volume greater than \(1 - w_s\gamma^*_W - w_n\delta^*_W\), we argue that \((C, W)\) is indeed a valid generalized configuration, and this implies that there exists a generalized configuration, whose dual constraint (5) is violated. By the definition of windows, the property \(w_s < s^*_{\min}\) is equivalent to \(w_s = \frac{s^*_{\min}}{1+\epsilon}\), which is the smallest size of window (and the smallest sized window forms a valid generalized configuration with any configuration, provided that the value of \(w_n\) is small enough). Since \(C\) has at most \(k - w_n\) items, the main window of \(C\) in this case is no smaller than \((w_s, w_n)\) and therefore, the generalized configuration \((C, W)\) is valid. If \(w_s \geq s^*_{\min}\), recall that the main window of \(C\), \((w_s(C), w_n(C))\) is chosen so that \(s^*(C) + w_s(C) \geq 1\), and that \(C\) is chosen by the algorithm for KCC so that \(s^*(C) < 1 - \frac{w_s}{1+\epsilon}\). We get \(1 - w_s(C) \leq s^*(C) < 1 - \frac{w_s}{1+\epsilon}\) and therefore \(w_s < (1 + \epsilon)w_s(C)\), i.e., \(w_s \leq w_s(C)\) (since the sizes of windows are integer powers of \(1 + \epsilon\)). Since \(C\) contains at most \(k - w_n\) items, we have \(w_n(C) \geq w_n\) and so we conclude that \(W \leq (w_s(C), w_n(C))\), and \((C, W)\) is a valid generalized configuration. Thus in this case
we found that this solution is a configuration whose constraint in the dual linear program is not satisfied, and we can continue with the application of the ellipsoid algorithm.

Otherwise, for any window $W = (w_s, w_n)$ and any configuration $C$ of total rounded-up size less than $1 - \frac{w_n}{1+\epsilon}$ (or at most 1, if $w_s < s_{\text{min}}'$), with at most $k - w_n$ items, the total volume is at most $(1 + \epsilon)(1 - w_s\gamma_W - w_n\delta_W) \leq (1 + \epsilon) - w_s\gamma_W - w_n\delta_W$. We prove that in this case, all the constraints of the dual linear program are satisfied by the solution $\frac{w_s}{1+\epsilon}$. Consider a valid generalized configuration $\tilde{C} = (C, (\tilde{w}_s, \tilde{w}_n))$. We have $(\tilde{w}_s, \tilde{w}_n) \leq (w_s(C), w_n(C))$, where $(w_s(C), w_n(C))$ is the main window of $C$. If $w_s(C) < s_{\text{min}}'$, then $\tilde{w}_s = w_s(C)$. Since $s'(C) \leq 1$ for any configuration, and $\tilde{w}_n \leq w_n(C)$, where $k - w_n(C)$ is the number of items in $C$, $C$ is a possible configuration to be used with the window $(\tilde{w}_s, \tilde{w}_n)$ in the application of the FPTAS for KCC. Assume next that $\tilde{w}_s < 1$, then when the FPTAS for KCC is applied on $W = (\tilde{w}_s, \tilde{w}_n)$, $C$ is a configuration that is taken into account for $W$ since $s'(C) < 1 - \frac{w_s(C)}{1+\epsilon} \leq 1 - \frac{w_s}{1+\epsilon}$, where the first inequality holds by definition of $w_s(C)$, and $C$ has at most $k - w_n(C) \leq k - \tilde{w}_n$ items. If $\tilde{w}_s = 1$ then $1 \geq w_s(C) \geq \tilde{w}_s = 1$, so $w_s(C) = 1$. A configuration $C_1$ that contains at least one large item satisfies $s'(C_1) \geq \epsilon$, so $s'(C_1) + \epsilon \geq 1 + \epsilon \geq \epsilon > 1$. Therefore if the main window of a configuration is of size 1, this configuration is empty. We therefore have that $C$ is an empty configuration, thus $s'(C) = 0$ and $\tilde{w}_n \leq w_n(C) = k$. This empty configuration $C$ is considered with any possible window.

We denote by $(x^*, Y^*)$ the solution to the primal linear program that we obtained. Since its cost is a $(1 + \epsilon)$ approximation for the optimal solution to the linear program, we conclude that $\sum_{C \in \mathbb{C}} x_{C}^* \leq (1 + \epsilon)\text{OPT}(I)$.

We modify the solution to the primal linear program, into a different feasible solution of the linear program, without increasing the objective function. We create a list of generalized configurations whose $x^*$ component is positive. From this list of generalized configurations, we find a list of windows that are the main window of at least one configuration induced by a generalized configuration in the list. This list of windows is a subset of $\mathbb{W}'$ defined above. We would like the solution to use only windows from $\mathbb{W}'$.

The new solution will have the property that any non-zero components of $x^*$, $x_{C_i}^*$ corresponds to a generalized configuration $\tilde{C} = (C, w)$, such that $w \in \mathbb{W}'$. We still allow generalized configurations $\tilde{C} = (C, w)$ where $w$ is not the main window of $C$, as long as $w \in \mathbb{W}'$. This is done in the following way. Given a window $w' \notin \mathbb{W}'$, we define $X_{w'} = \sum_{\tilde{C} \in \mathbb{W}} x_{\tilde{C}}^*$. The following is done in parallel for every generalized configuration $\tilde{C}' = (C', w')$, where $w' \notin \mathbb{W}$ and such that $x_{\tilde{C}'}^* > 0$, where the main window of $C$ is $w \geq w'$ (but $w' \notin w$). We let $\tilde{C} = (C, w)$. The windows allocated for small items need to be modified first, thus an amount of $\frac{x_{\tilde{C}}^*}{X_{w'}} Y_{i,w'}$ is transferred from $Y_{i,w'}$ to $Y_{i,w}$. We modify the values $x_{\tilde{C}}^*$ and $x_{\tilde{C}}^*$ as follows. We increase the value of $x_{\tilde{C}}^*$ by an additive factor of $x_{\tilde{C}}^*$ and let $x_{\tilde{C}}^* = 0$.

To show that the new vector $(x^*, Y^*)$ still gives a feasible solution of the same value of objective function, we consider the modifications. The sum of components of $x^*$ does not change at all in the above process, thus the value of the objective function is the same. Moreover, for every configuration $C$, the sum of components $x^*$, that correspond to generalized configurations whose configuration of large items is $C$, does not change. Thus the constraints (1) still hold. We next consider the constraint (2) for $i$, for a given small item $i \in S$. Since the sum of variables $Y_{i,W}$ does not change, this constraint still holds.

As for constraints (3) and (4), for a window $w \notin \mathbb{W}'$, the right hand side of each such constraint became zero. On the other hand, for windows in $\mathbb{W}'$, every increase in some variable $x_{\tilde{C}}^*$ for $\tilde{C} = (C, w = (w_s, w_n))$, that is originated on a decrease of $x_{\tilde{C}}^*$ for $\tilde{C}' = (C', w' = (w'_s, w'_n))$ is accompanied with an increase of $\frac{x_{\tilde{C}}^*}{X_{w'}} Y_{i,w'} = \frac{x_{\tilde{C}}^*}{X_{w'}} Y_{i,w'}$ in $Y_{i,w'}^*$ for every $i \in S$, that is, an increase of $\sum_{i \in S} \frac{x_{\tilde{C}}^*}{X_{w'}} s_i \cdot Y_{i,w'}^*$ in the right
hand size of the constraint (3) for \( w \), and an increase of \( w_s \cdot x_{C,w}^{*} \) in the left hand side. Since we have \( w_s \cdot X_{w} \geq w_s^{'} \cdot X_{w}^{'} \geq \sum_{i \in S} s_i \cdot Y_{i,w}^{*} \) before the modification occurs (since constraint (3) for the window \( w' \) holds for the solution before the modification), we get that the increase of the left hand side is no smaller than the increase in the right hand side. There is an increase of \( \sum_{i \in S} \frac{x_{C,i,w}^{*}}{n} \cdot Y_{i,w}^{*} \) in the right hand size of the constraint (4) for \( w \), and an increase of \( w_n \cdot x_{C,w}^{*} \) in the left hand side. Since we have \( w_n \cdot X_{w} \geq w_n^{'} \cdot X_{w}^{'} \geq \sum_{i \in S} Y_{i,w}^{*} \), we get that the increase of the left hand side is no smaller than the increase in the right hand side.

Now, we can delete the constraints of (3) and (4) that correspond to windows in \( W \setminus W' \). In the resulting linear program we consider a basic solution that is not worse than the solution we obtained above. Such a basic solution can be found in polynomial time. We denote this basic solution by \((x^{*}, Y^{*})\).

We apply several steps of rounding to obtain a feasible packing of the items into bins. We first round up \( x^{*} \). That is, denote by \( \hat{x} \) the vector such that \( \hat{x}_C = \lceil x_C^{*} \rceil \) for all \( C \in \hat{C} \). Moreover, each small item \( i \in S \) such that \((Y^{*}_{i,w})_{W \in W} \) is fractional, is packed using a dedicated bin. We modify the value of \( \hat{x}_C \) for \( \hat{C} \) that corresponds to an empty configuration \( C \), together with the window \((1,k)\), to reflect the additional bins that accommodate the small items that were previously packed fractionally. We modify the values \((Y^{*}_{i,w})_{W \in W} \) as follows. Let \( i \) be a large item which is packed into a new bin has \( \hat{Y}_{i,w} = 0 \) for all \( W \), except for \( W = (1,k) \) for which \( \hat{Y}_{i,w} = 1 \). For all other variables \( Y_{i,w} \) we define \( \hat{Y}_{i,w} = Y_{i,w}^{*} \). We next bound the increase in the cost due to this rounding.

**Lemma 1** \[ \sum_{C \in \hat{C}} \hat{x}_C \leq \sum_{C \in C} x_C^{*} + |H| + 2|W'|. \]

**Proof.** Consider now the primal linear program, where constraints (3) and (4) exist only for windows in \( W' \), the variables \( x_C \) exist only for generalized configurations \( C = (C,w) \) where \( w \in W' \), and the variables \( Y_{i,w} \) exists only for \( W \in W' \). The basic solution \((x^{*}, Y^{*})\) is a feasible solution for this linear program. In the primal linear program there are \(|H| + 2|W'| + |S|\) inequality constraints, and hence in a basic solution there are at most \(|H| + 2|W'| + |S|\) basic variables. For every \( i \in S \), there is at least one \( W' \) such that \( Y_{i,w'} \) is a basic variable, and therefore the number of basic variables from the \( x \) components and additional basic variables from the \( Y \) components is at most \(|H| + 2|W'|\). Hence the sum of the number of fractional components among the \( x_C \) variables, and the number of small items such that the vector \((Y^{*}_{i,w})_{W \in W} \) contains more than one non-zero component is at most \(|H| + 2|W'|\). This is an upper bound on the difference in the objective values of the two solutions and the claim follows.

Our scheme returns a solution that packs \( \hat{x}_C \) bins with configuration \( \hat{C} \). Each large item of the rounded-up instance is replaced by the corresponding item of \( I \). We clearly use at most \( \sum_{C \in \hat{C}} \hat{x}_C \) bins in this way. We next pack each item of \( L_1 \) by its own bin (if \( L_1 \) is non-empty). We denote the resulting solution by \( SOL_{large} \).

**Lemma 2** The cost of \( SOL_{large} \) is at most \( \sum_{C \in \hat{C}} \hat{x}_C + \varepsilon \cdot OPT \).

**Proof.** It suffices to show that \(|L_1| \leq \varepsilon \cdot OPT \). To see this last claim note that \(|L_1| \leq 2|L| \varepsilon^3 \) and each item in \( L \) has size at least \( \varepsilon \) and therefore \( OPT \geq |L| \varepsilon \), and therefore \(|L_1| \leq 2 \varepsilon^2 \cdot OPT \) and the claim follows since \( \varepsilon < 1/2 \).
We next consider the packing of the small items that are supposed to be packed (according to \( \hat{Y} \)) in bins with a window of type \( W = (w_s, w_n) \). Assume that there are \( X(W) \) such bins (i.e., \( X(W) = \sum_\mathcal{C} \hat{x}_\mathcal{C} \)). Denote by \( S(W) \) the set of small items that we decided to pack in bins with window \( W \) (for some of these items we will change this decision in the sequel). Then, by the feasibility of the linear program we conclude that \(|S(W)| \leq w_n X(W) \) and \( \sum_{i \in S(W)} s_i' \leq w_s X(W) \). We next show how to allocate almost all of \( S(W) \) to the \( X(W) \) bins with window \( W \) such that each such bin will contain at most \( k \) items (that is, at most \( w_n \) small items) and the total size of items in each such bin will be at most \( 1 + \frac{\epsilon w_s}{1+\epsilon} \).

To do so, we sort the items in \( S(W) \) according to non-increasing size (assume the sorted list of item indices is \( b_1 \geq b_2 \geq \ldots \geq b_{|S(W)|} \)). Then, allocate the items to the bins in a round-robin manner, so that bin \( j \) receives items of indices \( b_{j+p \cdot X(W)} \) for all integers \( p \geq 0 \) such that \( j + p \cdot X(W) \leq |S(W)| \). We call the allocation of items for a given value of \( p \) a round of allocations. If \( w_s = \frac{s_{\min}'}{1+\epsilon} \) then all items assigned to this type of window are of size \( 0 \), and the resulting allocation is valid in the sense that every bin contains items of total size at most \( 1 \), and at most \( k \) items per bin, and there is no need to adapt the packing of small items. We therefore assume \( w_s \geq s_{\min}' \). We claim that the last bin of index \( X(W) \) received at most \( \frac{1}{X(W)} \) fraction of the total size of the items, whose sum is equal to \( \sum_{i=1}^{|S(W)|} b_i \). To prove this, we artificially add at most \( X(W) - 1 \) items of size zero to the end of the list (these items are added just for the sake of the proof), and allocate them to the bins that previously did not receive an item in the last round of allocations, that is, bins \( r, \ldots, X(W) \) such that bin \( r - 1 < X(W) \) received the last item. If bin \( X(W) \) received the last item then no items are added. Now the total size of small items remained the same, but every bin got exactly one item in each round. Since the last bin received the smallest item in each round, the claim follows. On the other hand, we can apply the following process, at every time \( i < X(W) \), remove the first (largest) small item from bin \( i \). As a result, the round-robin assignment now starts from bin \( i + 1 \) and bin \( i \) becomes the bin that receives items last in every round, and thus by the previous proof, the total size of items assigned to it is at most \( \frac{1}{X(W)} \sum_{i=1}^{|S(W)|} b_i \) (since the total size of items does not increase in each step).

We create an intermediate solution \( SOL_{inter} \) by removing the largest small item from every bin and packing these removed items in separate bins in groups of \( \frac{1}{\epsilon} \) removed items per bin (note that such bin is feasible as the total size of \( \frac{1}{\epsilon} \) small items is at most \( 1 \) and \( k > \frac{1}{\epsilon} \) and hence the cardinality constraint is satisfied as well). The total cost of this intermediate solution is therefore at most \((1+\epsilon) \cdot ((1+2\epsilon) \cdot OPT + |H| + 2 |W'|)+1 \) (the last bin can contain less than \( \frac{1}{\epsilon} \) such removed items). We note that since we divide the items in \( S(W) \) equally (up to a difference of one item) to the \( X(W) \) bins, we conclude that after the removal of one item from each bin (or even before that), every such bin has at most \( k \) items (both large and small), and therefore all the bins satisfy the cardinality constraint. Moreover, the total size of small items assigned to such bin (after the removal of one item per bin) is at most \( w_s \) by the above argument regarding the total size of small items in a bin where the largest small item was removed.

The intermediate solution is infeasible because our definition of \( w_s \) is larger than the available space for small items in such bin. We create the final solution \( SOL_{final} \) as follows. Consider a bin such that the intermediate solution packs it large items according to configuration \( C \), and small items with total size at most \( w_s(C) \). For every bin, we do not change the packing of large items. As for the small items, we remove them from the bin and start packing the small items into this bin greedily in non-decreasing order of the item sizes, as long as the total size of items packed to the bin does not exceed \( 1 \). The first item that does not fit into the bin we pack in separate bins (each such separate bin will contain \( \frac{1}{\epsilon} \) such first items for different bins.
of $SOL_{inter})$. Similarly to the above argument these are feasible bins and they add an additive factor of $\varepsilon$ times the cost of $SOL_{inter}$ to the total cost of the packing (plus 1).

By the definition of windows, the actual space in a bin with window $(w_s, w_n)$, that is free for the use of small items, is at least of size $\frac{w_s}{1+\varepsilon}$. After the removal of the first item that does not fit into the space for small items, the remaining small items allocated to this bin have a total size of at most $w_s - \frac{w_s}{1+\varepsilon} = \varepsilon \frac{w_s}{1+\varepsilon}$. Since by definition, $\frac{w_s}{1+\varepsilon} < 1$, the small items that were assigned to a bin but cannot be packed (not including the first item that was packed in the previous step) are of total size at most $\varepsilon$. Similar considerations can be applied to the cardinality of these items. Since the unpacked items are the largest ones, the remaining unpacked items in a bin of $SOL_{inter}$ have cardinality of at most $\varepsilon w_n \leq \varepsilon k$. Therefore, we can pack the unpacked items of every $\frac{1}{\varepsilon} k$ bins of $SOL_{inter}$ using one additional bin. In this way we get our final solution $SOL_{final}$. We note that the cost of $SOL_{final}$ is at most $(1 + 2\varepsilon)$ times the cost of $SOL_{inter}$ plus two. Therefore the cost of $SOL_{final}$ is at most $(1 + 2\varepsilon) \cdot (1 + \varepsilon) \cdot (1 + 2\varepsilon) \cdot (OPT + |H| + 2|W'|) + 1) + 2 \leq (1 + 10\varepsilon)OPT + 5(|H| + |W'| + 1) \leq (1 + 10\varepsilon)OPT + 5\left(\frac{1}{\varepsilon^2} + \left(\frac{1}{\varepsilon^2} + 1\right)^{1/\varepsilon}\right) + 2$ where the last inequality holds by $\varepsilon < \frac{1}{2}$, $|H| \leq \frac{1}{\varepsilon}$ and $|W'| \leq |C| \leq \left(\frac{1}{\varepsilon^2} + 1\right)^{1/\varepsilon}$. Therefore, we have established the following theorem.

**Theorem 4** If $k \geq \frac{1}{\varepsilon^2}$, the above scheme is an AFPTAS for BPCC.

Since we covered both cases, we obtain the following.

**Theorem 5** The above scheme is an AFPTAS for BPCC.

**References**

[1] N. Alon, Y. Azar, G. J. Woeginger, and T. Yadid. Approximation schemes for scheduling on parallel machines. *Journal of Scheduling*, 1(1):55–66, 1998.

[2] L. Babel, B. Chen, H. Kellerer, and V. Kotov. Algorithms for on-line bin-packing problems with cardinality constraints. *Discrete Applied Mathematics*, 143(1-3):238–251, 2004.

[3] W. W. Bein, J. R. Correa, and X. Han. A fast asymptotic approximation scheme for bin packing with rejection. *Theoretical Computer Science*, 393(1-3):14–22, 2008.

[4] A. Caprara, H. Kellerer, and U. Pferschy. Approximation schemes for ordered vector packing problems. *Naval Research Logistics*, 92:58–69, 2003.

[5] A. Caprara, H. Kellerer, U. Pferschy, and D. Pisinger. Approximation algorithms for knapsack problems with cardinality constraints. *European Journal of Operational Research*, 123:333–345, 2000.

[6] E. G. Coffman, M. R. Garey, and D. S. Johnson. Approximation algorithms for bin packing: A survey. In D. Hochbaum, editor, *Approximation algorithms*. PWS Publishing Company, 1997.

[7] E. G. Coffman Jr. and J. Csirik. Performance guarantees for one-dimensional bin packing. In T. F. Gonzalez, editor, *Handbook of Approximation Algorithms and Metaheuristics*, chapter 32. Chapman & Hall/Crc, 2007. 18 pages.

[8] J. Csirik, D. S. Johnson, and C. Kenyon. Better approximation algorithms for bin covering. In *Proc.of the 12th Annual Symposium on Discrete Algorithms (SODA2001)*, pages 557–566, 2001.

[9] J. Csirik and J. Y.-T. Leung. Variable-sized bin packing and bin covering. In T. F. Gonzalez, editor, *Handbook of Approximation Algorithms and Metaheuristics*, chapter 34. Chapman & Hall/Crc, 2007. 11 pages.
[10] J. Csirik and J.-Y.-T. Leung. Variants of classical one-dimensional bin packing. In T. F. Gonzalez, editor, Handbook of Approximation Algorithms and Metaheuristics, chapter 33. Chapman & Hall/Crc, 2007. 13 pages.

[11] J. Csirik and G. J. Woeginger. On-line packing and covering problems. In A. Fiat and G. J. Woeginger, editors, Online Algorithms: The State of the Art, chapter 7, pages 147–177. Springer, 1998.

[12] W. Fernandez de la Vega and G. S. Lueker. Bin packing can be solved within $1+\varepsilon$ in linear time. Combinatorica, 1(4):349–355, 1981.

[13] G. Dósa and Y. He. Bin packing problems with rejection penalties and their dual problems. Information and Computation, 204(5):795–815, 2006.

[14] L. Epstein. Bin packing with rejection revisited. In Proc. of the 4th Workshop on Approximation and online Algorithms (WAOA2006), pages 146–159, 2006. Also in Algorithmica, to appear.

[15] L. Epstein. Online bin packing with cardinality constraints. SIAM Journal on Discrete Mathematics, 20(4):1015–1030, 2006.

[16] L. Epstein, Cs. Imreh, and A. Levin. Bin covering with cardinality constraints. Manuscript, 2007.

[17] D. S. Hochbaum and D. B. Shmoys. Using dual approximation algorithms for scheduling problems: theoretical and practical results. Journal of the ACM, 34(1):144–162, 1987.

[18] D. S. Hochbaum and D. B. Shmoys. A polynomial approximation scheme for scheduling on uniform processors: Using the dual approximation approach. SIAM Journal on Computing, 17(3):539–551, 1988.

[19] K. Jansen and R. Solis-Oba. An asymptotic fully polynomial time approximation scheme for bin covering. Theoretical Computer Science, 306(1-3):543–551, 2003.

[20] K. Jansen and R. van Stee. On strip packing with rotations. In Proc. of the 37th Annual ACM Symposium on Theory of Computing (STOC2005), pages 755–761, 2005.

[21] N. Karmarkar and R. M. Karp. An efficient approximation scheme for the one-dimensional bin-packing problem. In Proceedings of the 23rd Annual Symposium on Foundations of Computer Science (FOCS’82), pages 312–320, 1982.

[22] H. Kellerer and U. Pferschy. Cardinality constrained bin-packing problems. Annals of Operations Research, 92:335–348, 1999.

[23] K. L. Krause, V. Y. Shen, and H. D. Schwetman. Analysis of several task-scheduling algorithms for a model of multiprogramming computer systems. Journal of the ACM, 22(4):522–550, 1975.

[24] K. L. Krause, V. Y. Shen, and H. D. Schwetman. Errata: “Analysis of several task-scheduling algorithms for a model of multiprogramming computer systems”. Journal of the ACM, 24(3):527–527, 1977.

[25] F. D. Murgolo. An efficient approximation scheme for variable-sized bin packing. SIAM Journal on Computing, 16(1):149–161, 1987.

[26] H. Shachnai and O. Yehezkely. Fast asymptotic FPTAS for packing fragmentable items with costs. In Proc. of the 16th International Symposium on Fundamentals of Computation Theory, (FCT2007), pages 482–493, 2007.

[27] J. D. Ullman. The performance of a memory allocation algorithm. Technical Report 100, Princeton University, Princeton, NJ, 1971.
A First case of Section 3: $k \leq \frac{1}{\varepsilon^3}$

In this case we apply linear grouping for all items, and do not classify items into types. That is, we partition the items into $\frac{1}{\varepsilon^3}$ classes $L_1, L_2, \ldots, L_{1/\varepsilon^3}$ such that $[n\varepsilon^3] = |L_1| \geq |L_2| \geq \cdots \geq |L_{1/\varepsilon^3}| = [n\varepsilon^3]$ (note that this condition uniquely identifies the cardinality of each class), and such that if there are two items $i, j$ with sizes $s_i > s_j$ and $i \in L_q$ and $j \in L_p$, then $q \leq p$ (so $L_1$ receives the subset of largest items, and for $2 \leq p \leq \frac{1}{\varepsilon^3}$, $L_p$ receives the largest items from $I \setminus (L_1 \cup \cdots \cup L_{p-1})$). The two conditions uniquely define the allocation of items into classes up to the allocation of equal sized items.

Then, we round up the sizes of the items in $L_2, \ldots, L_{1/\varepsilon^3}$ as follows: For all values of $p$, $p = 2, 3, \ldots, 1/\varepsilon^3$, and for each item $i \in L_p$, we define $s_i' = \max_{j \in L_p} s_j$ to be the rounded-up size of item $i$. The rounded-up instance $I'$ consists of the set of items $I \setminus L_1$, where for every $i$, the size of item $i$ is $s_i'$ (the parameter $k$ remained unchanged). We next argue that $\text{OPT}(I') \leq \text{OPT}$. Given an optimal solution to $I$, $\text{OPT}$, we transform it into a solution to $I'$. We define a bijection from $I'$ to $I$, so that every item of $I'$ is mapped to an item of $I$ that is no smaller, and can take its place in the packing. Since $I'$ does not contain $L_1$, and since $|L_1| \leq |L_{i-1}|$ (in both $I$ and $I'$, since the size of sets is not influenced by the rounding), we map every item of $L_i$ (for all $i \geq 2$) in $I'$ to some item of $L_{i-1}$ in $I$. By our rounding, every item of $L_i$ in $I'$ is no larger than any item of $L_{i-1}$ in $I$.

Given the rounded-up instance $I'$, we let a configuration of a bin $C$ be a set of at most $k$ items of $I'$ whose total (rounded-up) size is at most 1. We denote the set of all configurations by $C$ (this set is not computed explicitly, and typically has an exponential size). We denote the set of item sizes in $I'$ by $H$. For each $v \in H$, we let $n(v, C)$ be the number of items with size $v$ in $C$, and we let $n(v)$ be the number of items in $I'$, with size $v$ (where $n(v) = \lceil n\varepsilon^3 \rceil$ or $n(v) = \lfloor n\varepsilon^3 \rfloor$), unless several classes are rounded to the same size, thus $|H| \leq \frac{1}{\varepsilon^3}$. We solve (approximately) the following linear program, where for each configuration $C$, there is a variable $x_C$ indicating the number of bins packed using configuration $C$.

\[
\begin{align*}
\min & \quad \sum_{C \in \mathcal{C}} x_C \\
\text{s.t.} & \quad \sum_{C \in \mathcal{C}} n(v, C)x_C \geq n(v) \quad \forall v \in H \\
& \quad x_C \geq 0 \quad \forall C \in \mathcal{C}.
\end{align*}
\]

We let $x^*$ be an approximate (within a factor of $1 + \varepsilon$) solution to this linear program, and further define $y_C = \lceil x_C^* \rceil$, for every configuration $C \in \mathcal{C}$. It can be seen that the vector $y$ is a feasible solution to the linear program (since $y_C \geq x_C$ for all $C$, it satisfies all the constraints). Our scheme returns a solution that packs $y_C$ bins with configuration $C$, in this solution, there are at least $n(v)$ slots for every $v \in H$. Note that some of these slots may remain empty, which happens in the case that the number of slots is strictly larger than $n(v)$. To get a solution for $I \setminus L_1$, each item of the rounded-up instance is replaced by the corresponding item of $I$. We clearly use at most $\sum_{C \in \mathcal{C}} y_C$ bins in this way.

To solve the above linear program approximately, we invoke the column generation technique of Karmarkar and Karp [21]. We next elaborate on this technique. The above linear program has an exponential number of variables and a polynomial number of constraints (neglecting the non-negativity constraints). Instead of solving the linear program, we solve its dual program (that has a polynomial number of variables and an exponential number of constraints). The variables $z_v$ correspond to the item sizes in $H$, their intuitive meaning can be seen as weights of these items.

\[
\begin{align*}
\max & \quad \sum_{v \in H} n(v)z_v \\
\text{s.t.} & \quad \sum_{v \in H} n(v, C)z_v \leq 1 \quad \forall C \in \mathcal{C} \\
& \quad z_v \geq 0 \quad \forall v \in H.
\end{align*}
\]

To be able to apply the ellipsoid algorithm, in order to solve the above dual problem within a factor of $1 + \varepsilon$, it suffices to show that there exists a polynomial time algorithm (polynomial in $n$ and $\frac{1}{\varepsilon}$) such that for a given solution $z^*$ (which is a vector of length $|H| \leq \frac{1}{\varepsilon^3}$), decides whether $z^*$ is a feasible dual solution (approximately). That is, it either
provides a configuration \( C \in C \) such that \( \sum_{v \in H} n(v, C) z_v^* > 1 \), or outputs that an approximate infeasibility evidence does not exist, that is, for all configurations \( C \in C \), \( \sum_{v \in H} n(v, C) z_v^* \leq 1 + \varepsilon \) holds. In such a case, \( \frac{z_v^*}{1+\varepsilon} \) is a feasible dual solution that can be used. Such a configuration \( C \) can be found using an FPTAS for the following knapsack problem with a maximum cardinality constraint (KCC): Given a set of item types \( H \), where each item type \( v \in H \) has a given multiplicity \( n(v) \), a volume \( z_v^* \) and a size \( v \), the goal is to pack a multiset of at most \( k \) items (taking the multiplicity, in which items are taken, into account, and letting the solution contain at most \( n(v) \) items of type \( v \)) and a total size of at most \( 1 \), so that the total volume is maximized. If the FPTAS to KCC finds a solution with a total volume greater than \( 1 \), then this solution is a configuration whose constraint in the dual linear program is violated, and we can continue with the application of the ellipsoid algorithm. Otherwise, the FPTAS to KCC finds a solution with a total volume of at most \( 1 \), since the FPTAS is an \( 1 + \varepsilon \) approximation, it means that no solution with a total volume larger than \( 1 + \varepsilon \) exists, and therefore, all the constraints of the dual linear program are satisfied by the solution \( \frac{z_v^*}{1+\varepsilon} \). To provide an FPTAS for KCC, note that one can replace an item with size \( v \) by \( n(v) \) copies of this item and then one can apply the FPTAS of Caprara et al. \([5]\) for the knapsack problem with cardinality constraints. The FPTAS of \([5]\) clearly has polynomial time in the size of its input, and \( \frac{1}{\varepsilon} \). Since the number of items that we give to this algorithm as input is at most \( n \), we can use this FPTAS and still let our scheme have polynomial running time.

Since the approximated separation oracle that we described above runs in polynomial time (polynomial in \( n \) and \( \frac{1}{\varepsilon} \)) we conclude that the approximated solution of the (primal) linear program \( x^* \) is obtained in polynomial time (again polynomial in \( n \) and \( \frac{1}{\varepsilon} \)). Since \( x^* \) is a solution of a linear program with an exponential number of variables, \( x^* \) is given in a compact representation, which is a list of non-zero components of the solution, together with their values. The set of items in \( I \setminus L_1 \) is packed according to the integral solution \( y \) as described above. It remains to pack \( L_1 \). To do so, we pack each item of \( L_1 \) in a separate (dedicated) bin. Note that there are \( |L_1| = \lfloor n \varepsilon^3 \rfloor \leq n \varepsilon^3 + 1 \) such bins, and since \( \text{OPT} \geq \frac{n}{\varepsilon} \geq n \varepsilon^2 \) we conclude that the number of additional bins (used to pack \( L_1 \)) is at most \( \varepsilon \text{OPT} + 1 \), so \( \text{APX} \leq \sum_{C \in C} y_C + |L_1| \leq \sum_{C \in C} y_C + \varepsilon \text{OPT} + 1 \). Therefore, it suffices to bound the cost, implied by \( y \), in terms of \( \text{OPT} \).

Instead of using an optimal solution to the linear program (whose value is a lower bound on \( \text{OPT}(I') \), since \( \text{OPT}(I') \) is a valid solution to the linear program), we use a \( 1 + \varepsilon \)-approximated solution, and this degrades the value of the returned solution within a factor of \( 1 + \varepsilon \) (i.e., \( \sum_{C \in C} x_C^* \leq (1 + \varepsilon) \cdot \text{OPT}(I') \)).

We next bound \( \sum_{C \in C} (y_C - x_C^*) \). Note that in the primal linear program there are at most \( \frac{1}{\varepsilon} \) constraints (not including non-negativity constraints), and hence in a basic solution (a property that we can always assume that \( x^* \) satisfies) there are at most \( \frac{1}{\varepsilon} \) positive components, and hence there are at most \( \frac{1}{\varepsilon} \) fractional components. Therefore, \( \sum_{C \in C} (y_C - x_C^*) \leq \frac{1}{\varepsilon} \).

Therefore, \( \text{APX} \leq \sum_{C \in C} y_C + \varepsilon \text{OPT} + 1 = \sum_{C \in C} x_C^* + \sum_{C \in C} (y_C - x_C^*) + \varepsilon \text{OPT} + 1 \leq (1 + \varepsilon) \cdot \text{OPT}(I') + \frac{1}{\varepsilon} + \varepsilon \text{OPT} + 1 \leq (1 + 2\varepsilon) \text{OPT} + \frac{1}{\varepsilon} + 1 \). Hence, we conclude the following theorem.

**Theorem 6** If \( k \leq \frac{1}{\varepsilon^3} \), the above scheme is an AFPTAS for BPCC.

## B An AFPTAS for BPR

In this section we use the similarity between the APTAS of Caprara et al. \([4]\) for BPCC and the APTAS of Epstein \([14]\) for BPR to develop our methods further. We obtain an AFPTAS for BPR using adaptations to the methods of the previous section.

Without loss of generality we assume that \( r_i \leq 1 \) for all \( i \). This is so as if there is an item with higher rejection penalty, then it is better to pack this item in a separate additional bin instead of rejecting it, and this situation is true for an item with a unit rejection penalty as well. Therefore, by changing the rejection penalty of such an item to 1, we do not change the optimal solution or a (reasonable) approximate solution.

Let \( 0 < \varepsilon \leq \frac{1}{4} \) be such that \( \frac{1}{\varepsilon^3} \) is an integer. An item \( j \) is large if both \( s_j \geq \varepsilon \) and \( r_j \geq \varepsilon \). All other items are small. We denote by \( L \) the set of large items, and by \( S \) the set of small items.
We perform rounding of the rejection penalties and the sizes of the large items (only). For \( i = 0, 1, \ldots, \Delta = \frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \), and every large item \( j \in L \), such that \( r_j \in [\varepsilon + i\varepsilon^2, \varepsilon + (i + 1)\varepsilon^2] \), we round down the rejection penalty \( r_j \) to \( \varepsilon + i\varepsilon^2 \). For a large item \( j \), denote the rounded rejection penalty of \( j \) by \( r'_j \), and for a small item \( j \) let \( r'_j = r_j \). Define \( I' \) to be the adapted input, \( ALG(I') \) be the cost of an arbitrary algorithm \( ALG \) on the input \( I' \), and let \( ALG'(I') \) be the cost of the same algorithm on the original items. Then, Epstein [14] showed the following:

**Lemma 7 (Lemma 1 in [14])** \( ALG'(I') \leq (1 + \varepsilon)ALG(I') \) and \( OPT(I') \leq OPT(I) \).

For \( i = 0, 1, \ldots, \Delta \), let \( L^i = \{ j_1, j_2, \ldots, j_n \} \) be the set of large items with rounded rejection penalty \( \varepsilon + i\varepsilon^2 \), such that \( s_{j_1} \geq s_{j_2} \geq \cdots \geq s_{j_n} \). For each set \( L^i \) such that \( |L^i| \geq \frac{1}{i\varepsilon^2} \), we perform linear grouping separately. That is, for values of \( i \) such that \( |L^i| \geq \frac{1}{i^2\varepsilon^3} \), we let \( m = \frac{1}{i\varepsilon^2} \) and we partition \( L^i \) into \( m \) classes \( L^1_i, \ldots, L^m_i \) such that \( |n_i\varepsilon^3| = |L^1_i| \geq |L^2_i| \geq \cdots \geq |L^m_i| = |n_i\varepsilon^3| \), and \( L^i_\sigma \) receives the largest items from \( L^i \setminus \{ L^1_i \cup \cdots \cup L^{\sigma - 1}_i \} \). For every \( i = 0, 1, \ldots, \Delta \) and \( j = 2, 3, \ldots, n_i \), we round up the size of the elements of \( L^i_j \) to the largest size of any element of \( L^i_j \). For item \( j \) of a set \( L^i_j \) (\( p \geq 2 \)), we denote by \( s'_j \) the rounded-up size of the item, which is defined to be equal to the maximum size of any item in \( L^i_j \). For items in \( L^1_i \) (for all \( i \)) we do not round the sizes, and we denote \( s'_j = s_j \) for all \( j \in L^1_i \). For values of \( i \) such that \( |L^i| < \frac{1}{i^2\varepsilon^3} \), each large item of \( L^i \) has its own set \( L^i_j \) such that \( L^i_j \) is an empty set, and for a large item \( j \in L^i \) we let \( s'_j = s_j \). We note that in both cases (i.e., for large and small cardinalities of \( L^i \)) we have \( |L^i_j| \leq 2\varepsilon^3 |L^i| \). For \( j \in S \) we also denote \( s'_j = s_j \). We denote by \( L_1 = \bigcup_{i=0}^{\Delta} L^i_1 \) and \( L' = L \setminus L_1 \). By the above, we have \( |L^i_1| \leq 2\varepsilon^3 |L^i| \). We consider the instance \( I'' \) consisting of the items in \( L'' \cup S \) with the (rounded-up) sizes function \( s' \) and the rounded rejection penalty function \( r' \). The items in \( L_1 \) are packed each in a separate bin. We have \( OPT(I'') \leq OPT(I'' \cup S) \), similarly to the previous sections. We next describe the packing of the items in \( I'' \).

Given the instance \( I'' \), we let a configuration of a bin \( B \) be a (possibly empty) set of items of \( I'' \) whose total (rounded-up) size is at most 1. We denote the set of all configurations by \( C \). Let \( H = \{(\sigma_1, \rho_1), \ldots, (\sigma_t, \rho_t)\} \) be the set of different types of large items where \( \sigma_i \) denotes the (rounded-up) size of an item with type \((\sigma_i, \rho_i)\), and \( \rho_j \) is its (rounded) rejection penalty. We have \(|H| \leq \frac{1}{\varepsilon} \cdot \Delta \leq \frac{1}{\varepsilon} \). For each \( v \in H \) we denote by \( n(v, C) \) the number of items with type \( v \) in \( C \), and we denote by \( n(v) \) the number of items in \( I'' \) with type \( v \). For a large item \( j \), we denote by \( type(j) \) the type of \( j \).

We denote the minimum size of an item by \( s_{\min} = \min_{s \in S} s'_i \) (recall that in BPR, it is assumed that sizes of items are strictly larger than 0), and as in the previous section, we let \( s'_i = \max\{\frac{1}{(1+\varepsilon)^t} t \in \mathbb{Z}, \frac{1}{(1+\varepsilon)^t} \leq s_{\min}\} \). The value \( log_{1+\varepsilon} s'_{\min} \) is polynomial in the size of the input. We define the following set \( \mathcal{W} = \{ \frac{1}{(1+\varepsilon)^t}, 0 \leq t \leq log_{1+\varepsilon} s'_{\min} + 1 \} \). A **window** is defined as a member of \( \mathcal{W} \). \( \mathcal{W} \) is also called the set of all possible windows. Then, \( |\mathcal{W}| \leq (log_{1+\varepsilon} s'_{\min} + 2) \). Since windows are scalars, they can be compared with respect to their size.

Note that each bin packed with large items according to a configuration \( C \) typically leaves space for small items. For a configuration \( C \) we denote the **main window of \( C \)** to be \( w(C) \), which can be seen as an approximation of the available size for small items in a bin with configuration \( C \). We define it as follows. Assume that the total (rounded-up) size of the items in \( C \) is \( s'(C) \). Then, \( w(C) = \frac{1}{(1+\varepsilon)^t} \) where \( t \) is the maximum integer such that \( 0 \leq t \leq \log_{1+\varepsilon} s'_{\min} + 1 \) and that \( s'(C) + \frac{1}{(1+\varepsilon)^t} \geq 1 \). The main window of a configuration is a window (i.e., belongs to \( \mathcal{W} \)), but \( \mathcal{W} \) may include windows that are not the main window of any configuration. We note that \( |\mathcal{W}| \) is polynomial in the input size and in \( \frac{1}{\varepsilon} \), whereas \( |C| \) may be exponential in \( \frac{1}{\varepsilon} \), specifically, \(|C| \leq (\frac{1}{\varepsilon} + 1)^{1/\varepsilon} \), since in configuration there are up to \( \frac{1}{\varepsilon} \) large items of \( |H| \leq \frac{1}{\varepsilon} \) types. We denote the set of windows that are actual main windows of at least one configuration by \( \mathcal{W}' \). Similarly to the previous section, we define a linear program that allows the usage of any window in \( \mathcal{W} \) and later modify the linear program and the solution to this linear program (that we obtain) to use only windows of \( \mathcal{W}' \).

We define a generalized configuration \( \tilde{C} \) as a pair \( \tilde{C} = (C, w = \frac{1}{(1+\varepsilon)^t}) \), for some configuration \( C \), and some \( w \in \mathcal{W} \). The generalized configuration \( \tilde{C} \) is valid if \( w \leq w(C) \). The set of all valid generalized configurations is denoted by \( \mathcal{C} \). For every \( W \in \mathcal{W} \) denote by \( C(W) \) the set of generalized configurations such that \( W \) is their window, i.e., \( C(W) = \{ \tilde{C} \in C : w = W \} \).

We next consider the following linear program. In this linear program we have a variable \( x_{\tilde{C}} \) denoting the number of bins with the generalized configuration \( \tilde{C} \), variables \( Y_{i,W} \) indicating if the small item \( i \) is packed in a window of type \( W \), and variables \( z_i \) indicating that item \( i \) is rejected (for both small and large items).
Constraints (7) and (8) ensure that each item (large or small) of generalized configuration that has this window is large enough. We note that total available space allocated to the windows of small items, that is, the number of bins that are packed according to a type in $H$, which all variables of the subset of rejected items) satisfies the constraints (9). Moreover, it implies a solution to the linear program in subset of rejected items) clearly satisfies the constraints (9), and the packing of large items (including the specification of the subset of rejected items) satisfies the constraints (7). Moreover, it implies a solution to the linear program in which all variables $x_{\hat{C}}$, that correspond to generalized configurations $\hat{C} = (C, w)$, for which $w$ is not the main window of $C$, are equal to zero, and all variables $Y_{i,w}$ where $w \notin \mathcal{W}'$ are equal to zero as well.

Once again (similarly to the AFPTAS for BPCC) we invoke the column generation technique of Karmarkar and Karp as follows. The above linear program has exponential number of variables and polynomial number of constraints (neglecting the non-negativity constraints). Instead of solving the linear program we solve its dual program (that has a polynomial number of variables and an exponential number of constraints). The variables $\alpha_v$, correspond to the item types in $H$, their intuitive meaning can be seen as weights of these items. The variables $\beta_i$ correspond to the small items, and their intuitive meaning can be seen as weights of these items. For each $W \in \mathcal{W}$ we have a dual variable $\gamma_W$. Using these dual variables, the dual linear program is as follows.

First note that there is a polynomial number of constraints of type (11), (12) and (13), and therefore we clearly have a polynomial time separation oracle for these constraints. If we would like to solve the above dual linear program (exactly) then using the ellipsoid method we need to establish the existence of a polynomial time separation oracle for the constraints (10). However, we are willing to settle on an approximated solution to this dual program. To be able to apply the ellipsoid algorithm, in order to solve the above dual problem within a factor of $1 + \varepsilon$, it suffices to show that there exists a polynomial time algorithm (polynomial in $n, \frac{1}{\varepsilon}$ and $\log \frac{1}{\varepsilon}$) such that for a given solution $a^* = (\alpha^*, \beta^*, \gamma^*)$ decides whether $a^*$ is a feasible dual solution (approximately).
That is, it either provides a generalized configuration \( \tilde{C} = (C, w) \in \tilde{C} \) for which \( \sum_{v \in H} n(v, C)\alpha_v^* + w\gamma_v^* > 1 \), or outputs that an approximate infeasibility evidence does not exist, that is, for all generalized configurations \( \tilde{C} = (C, w) \in \tilde{C} \), \( \sum_{v \in H} n(v, C)\alpha_v^* + w\gamma_v^* \leq 1 + \epsilon \) holds. In such a case, \( \frac{\alpha_v^*}{1+\epsilon} \) is a feasible solution that can be used.

Such a configuration \( \tilde{C} \) can be found by the following procedure: For each \( W \in \mathcal{W} \) we look for a configuration \( C \in \mathcal{C} \) such that \( (C, W) \) is a valid generalized configuration, and \( \sum_{v \in H} n(v, C)\alpha_v^* \) is maximized. If a configuration \( C \) that is indeed found, the generalized configuration, whose constraint is checked, is \( (C, W) \).

To find \( C \), we invoke an FPTAS for the standard knapsack problem with the following input: The set of items is \( H \) where for each \( v \in H \) there is a volume \( \alpha_v^* \) and a size \( v \), the goal is to pack a multiset of the items (an item can appear at most a given number of times), so that the total volume is maximized, under the condition that the total (rounded-up) size of the multiset should be smaller than \( 1 - \frac{W}{1+\epsilon} \), unless \( W < s_{min}', \) where the total size should be at most 1 (in this case, the window leaves space only for items of size zero). Since the number of applications of the FPTAS for the knapsack problem is polynomial (i.e., \( |\mathcal{W}| \)), this algorithm runs in polynomial time.

If it finds a solution, that is, a configuration \( C \), with total volume greater than \( 1 - W\gamma_W^* \), we argue that \( (C, W) \) is indeed a valid generalized configuration, and this implies that there exists a generalized configuration, whose dual constraint \( (10) \) is violated.

By the definition of windows, the property \( W < s_{min}' \) is equivalent to \( W = s_{min}' \frac{1+\epsilon}{1+\epsilon} \), which is the smallest size of window (which forms a valid generalized configuration with any configuration). If \( W \geq s_{min}' \), recall that the main window of \( C \), \( w(C) \) is chosen so that \( s'(C) + w(C) \geq 1 \), and that \( C \) is chosen by the algorithm for the knapsack problem, so that \( s'(C) < 1 - \frac{W}{1+\epsilon} \). We get \( 1 - w(C) \leq s'(C) < 1 - \frac{W}{1+\epsilon} \) and therefore \( W < (1+\epsilon)w(C) \), i.e., \( W \leq w(C) \) (since the sizes of windows are integer powers of \( 1+\epsilon \), so we conclude that \( (C, W) \) is a valid generalized configuration. Thus in this case we found that this solution is a configuration whose constraint in the dual linear program is not satisfied, and we can continue with the application of the ellipsoid algorithm.

Otherwise, for any window \( W \), any configuration \( C \) of total rounded-up size less than \( 1 - \frac{W}{1+\epsilon} \) (or at most 1, if \( W < s_{min}' \)), has a volume of at most \((1+\epsilon)(1 - W\gamma_W^*) \leq (1+\epsilon) - W\gamma_W^* \). We prove that in this case, all the constraints of the dual linear program are satisfied by the solution \( \frac{\alpha_v^*}{1+\epsilon} \). Consider a valid generalized configuration \( \tilde{C} = (C, \tilde{w}) \). We have \( \tilde{w} \leq w(C) \), where \( w(C) \) is the main window of \( C \). If \( w(C) < s_{min}' \), then \( \tilde{w} = w(C) \). Since \( s'(C) \leq 1 \) for any configuration, \( C \) is a possible configuration to be used with the window \( \tilde{w} \) in the application of the FPTAS for knapsack. Assume next that \( \tilde{w} > 1 \). Then when the FPTAS for knapsack is applied on \( W \), \( C \) is a configuration that is taken into account for \( W \) since \( s'(C) < 1 - \frac{w(C)}{1+\epsilon} \leq 1 - \frac{\tilde{w}}{1+\epsilon} \), where the first inequality holds by definition of \( w(C) \). If \( \tilde{w} = 1 \) then \( 1 \geq w(C) \geq \tilde{w} = 1 \), so \( w(C) = 1 \). A configuration \( C_1 \) that contains at least one large item satisfies \( s'(C_1) \geq \epsilon \), so \( s'(C_1) + \frac{1}{1+\epsilon} \geq \frac{1+\epsilon}{1+\epsilon} > 1 \). Therefore if the main window of a configuration is of size 1, this configuration is empty. We therefore have that \( C \) is an empty configuration, thus \( s'(C) = 0 \). This empty configuration is considered with any window.

We denote by \((X^*, Y^*, Z^*)\) the solution to the primal linear program that we obtained. Since its cost is a \((1 + \epsilon)\) approximation for the optimal solution to the linear program, we conclude that \( \sum_{\tilde{C} \in \tilde{C}} X_{\tilde{C}}^* + \sum_{i \in L \cup S} x_i^* z_i^* \leq (1 + \epsilon) \cdot \text{OPT}(P') \).

We modify the solution to the primal linear program, into a different feasible solution of the linear program, without increasing the objective function. We create a list of generalized configurations whose \( X^* \) component is positive. From this list of generalized configurations, we find a list of windows that are the main window of at least one configuration induced by a generalized configuration in the list. This list of windows is a subset of \( \mathcal{W}' \) defined above. We would like the solution to use only windows from \( \mathcal{W}' \).

We modify the \( X^* \) and \( Y^* \) components, while the \( Z^* \) components are not modified. The new solution will have the property that any non-zero components of \( X^* \), \( X_{\tilde{C}}^* \), corresponds to a generalized configuration \( \tilde{C} = (C, w) \), such that \( w \in \mathcal{W}' \). We still allow generalized configurations \( \tilde{C} = (C, w) \) where \( w \) is not the main window of \( C \), as long as \( w \in \mathcal{W} \). This is done in the following way. Given a window \( w' \notin \mathcal{W}' \), we define \( B_{w'} = \sum_{\tilde{C} \in \tilde{C}(w')} X_{\tilde{C}}^* \). The following is done in parallel for every generalized configuration \( \tilde{C}' = (C, w') \), where \( w' \notin \mathcal{W}' \) and such that \( X_{\tilde{C}'}^* > 0 \),
where the main window of $C$ is $w \geq w'$ (but $w' \neq w$). We let $\tilde{C} = (C, w)$. The windows allocated for small items need to be modified first, thus an amount of $X_{\tilde{C}, i, w'} = \frac{x_{C, i, w}}{B_{w'}}Y_{i, w'}$ is transferred from $Y_{i, w'}$ to $Y_{i, w}$. We modify the values $X_{\tilde{C}, i, w}$ and $X_{\tilde{C}}$ as follows. We increase the value of $X_{\tilde{C}, i, w'}$ by an additive factor of $X_{\tilde{C}, i, w}$ and let $X_{\tilde{C}, i, w} = 0$.

To show that the new vector $(X^*, Y^*, Z^*)$ still gives a feasible solution of the same value of objective function, we consider the modifications. The sum of components of $X^*$ does not change at all in the above process, thus the value of the objective function is the same. Moreover, for every configuration $C$, the sum of components $X^*$, that correspond to generalized configurations whose configuration of large items is $C$, does not change. Thus the constraints (7) still hold. We next consider the constraint (8) for $i$, for a given small item $i \in S$. Since the sum of variables $Y_{i,W}^*$ does not change, this constraint still holds.

As for constraints (9), for a window $w \in W'$, the right hand side of each such constraint became zero. On the other hand, for windows in $W'$, every increase in some variable $X_{\tilde{C}, i, w}$ for $C = (C, w)$, that is originated in a decrease of $X_{\tilde{C}, i, w}$ for $\tilde{C} = (C, w')$ is accompanied with an increase of $\frac{x_{C, i, w}}{B_{w'}}Y_{i,w'}$ in $Y_{i,w}$, for every $i \in S$, thus is, an increase of $\sum_{i \in S} \frac{x_{C, i, w}}{B_{w'}} s_{i,w'} Y_{i,w'}$ in the right hand size of the constraint (9) for $w$, and an increase of $w \cdot x_{\tilde{C}, i, w}$ in the left hand side. Since we have $w \cdot B_{w'} \geq w' \cdot B_{w'} \geq \sum_{i \in S} s_{i,w'} Y_{i, w'}$ before the modification occurs (since constraint (9) holds for the solution before the modification for the window $w'$), we get that the increase of the left hand side is no smaller than the increase in the right hand side.

Now, we can delete the constraints of (9) that correspond to windows in $W \setminus W'$. In the resulting linear program we consider a basic solution that is not worse than the solution we obtained above. Such a basic solution can be found in polynomial time. We denote this basic solution by $(X^*, Y^*, Z^*)$.

We apply several steps of rounding to obtain a feasible packing of the items into bins. We first round up $X^*$. That is, denote by $\hat{X}$ the vector such that $\hat{X}_C = \lceil X^*_C \rceil$ for all $C \in \hat{C}$. Moreover, each small item $i \in S$ such that $(Y_{i,W}^*)_w \in W$ is fractional, is packed using a dedicated bin. We modify the value of $\hat{X}_{\tilde{C}}$ for $\tilde{C}$ that corresponds to an empty configuration $C$, together with the window 1, to reflect the additional bins that accommodate the small items that were previously packed fractionally. We modify the values $(Y_{i,W}^*)_w \in W$ so that every item $i$ which is packed into a new bin has $\hat{Y}_{i,W} = 0$ for all $W$ except for $W = 1$, for which $\hat{Y}_{i,W} = 1$. For all other variables $Y_{i,W}$ we define $\hat{Y}_{i,W} = Y_{i,W}^*$.

We next bound the increase in the cost due to this rounding.

\textbf{Lemma 8}\quad $\sum_{C \in \tilde{C}} X_{\tilde{C}} \leq \sum_{C \in \tilde{C}} X^*_C + |H| + |W'|$.

\textbf{Proof.}\quad Consider now the primal linear program, where constraints (9) exist only for windows in $W'$, the variables $x_{\tilde{C}}$ exist only for generalized configurations $\tilde{C} = (C, w)$ where $w \in W'$, and the variables $Y_{i,w}$ exists only for $W \in W'$. The basic solution $(X^*, Y^*, Z^*)$ is a feasible solution for this linear program. In the primal linear program there are $|H| + |W'| + |S|$ inequality constraints, and hence in a basic solution there are at most $|H| + |W'| + |S|$ basic variables. For every $i \in S$, there is at least one variable associated with $i$ that is a basic variable. This variable is either $Z_0$ or $Y_{i,W'}$ for some $W'$, and therefore the number of basic variables from the $x$ components and additional basic variables from the $Y$ and $Z$ components is at most $|H| + |W'|$. Hence the sum of the number of fractional components among the $x_{\tilde{C}}$ variables, and the number of small items such that the vector $(Y_{i,W}^*)_w$ is non-integral is at most $|H| + |W'|$. This is an upper bound on the difference in the objective values of the two solutions and the claim follows. 

Our scheme returns a solution that packs $X_{\tilde{C}}$ bins with configuration $\tilde{C}$. Each large item of the rounded-up instance is replaced by the corresponding item of $I$. We clearly use at most $\sum_{C \in \tilde{C}} X_{\tilde{C}}$ bins in this way. We next pack each item of $L_1$ by its own bin (if $L_1$ is non-empty). We denote the resulting solution by $SOL_{large}$.

\textbf{Lemma 9}\quad The cost of $SOL_{large}$ is at most $\sum_{C \in \tilde{C}} X_{\tilde{C}} + \sum_{i \in I} r'_i \hat{Z}_i + \varepsilon OPT$.

\textbf{Proof.}\quad It suffices to show that $|L_1| \leq \varepsilon OPT$. To see this last claim note that $|L_1| \leq 2|L|\varepsilon^3$ and each item in $L$ has both
a size of at least $\varepsilon$ and a rejection cost of at least $\varepsilon$ and therefore $\text{OPT} \geq |L|\varepsilon$, and therefore $|L_1| \leq 2\varepsilon^2\text{OPT}$ and the claim follows since $\varepsilon < 1/2$.

**Corollary 10** The cost of $\text{SOL}_{\text{large}}$ is at most $(1 + 2\varepsilon)\text{OPT} + |H| + |W'|$.

We next consider the packing of the small items that are supposed to be packed (according to $\hat{Y}$) in bins with window $W$. Assume that there are $X(W)$ such bins (i.e., $X(W) = \sum_{C: \tilde{C} = (C, W)} \hat{X}_C$). Denote by $S(W)$ the set of small items that we decided to pack in bins with window $W$ (for some of these items we will change this decision in the sequel). Then, by the feasibility of the linear program we conclude that $\sum_{i \in S(W)} s'_i \leq W \cdot X(W)$. We allocate almost all $S(W)$ to the $X(W)$ bins with window $W$ such that the total size of the items in each such bin is at most $1 + \varepsilon W$, exactly as in the previous section. As in the previous section, we create an intermediate solution $\text{SOL}_{\text{inter}}$ by removing the largest small item from each such bin. Each removed item is small and therefore either its rejection penalty is at most $\varepsilon$ or its size is at most $\varepsilon$. We pack the removed small items with size at most $\varepsilon$ in new bins, packing $1/\varepsilon$ such items in a bin, except perhaps the last such bin, and the other removed items are rejected (incurring a rejection penalty of at most $\varepsilon$ for each such item). The total cost of this intermediate solution is therefore at most $(1 + \varepsilon) \cdot ((1 + 2\varepsilon)\text{OPT} + |H| + |W'|) + 1$ (the last bin can contain less than $1/\varepsilon$ such removed items). As in the previous section, after the largest small items is removed from each bin, the total size of small items assigned to such bin is at most $W$.

The intermediate solution is infeasible because our definition of $W$ is larger than the available space for small items in such bin. We create the final solution $\text{SOL}_{\text{final}}$ using the same process as in the previous section. That is, given a bin such that the intermediate solution packs to it large items according to configuration $C$, and small items with total size at most $w(C)$, we remove the small items and pack them back until the first item that causes an excess. The first items whose rejection penalties are smaller than $\varepsilon$ are rejected. The other ones are packed in separate bins (each such separate bin will contain $1/\varepsilon$ such first items for different bins of $\text{SOL}_{\text{inter}}$). Similarly to the above argument these are feasible bins and they add an additive factor of $\varepsilon$ times the cost of $\text{SOL}_{\text{inter}}$ to the total cost of the packing (plus 1). The remaining unpacked items in a bin of $\text{SOL}_{\text{inter}}$ have total size of at most $1 + \varepsilon W \leq \varepsilon$. Therefore, we can pack the unpacked items of $1/\varepsilon$ bins of $\text{SOL}_{\text{inter}}$ using one additional bin. In this way we get our final solution $\text{SOL}_{\text{final}}$.

We note that the cost of $\text{SOL}_{\text{final}}$ is at most $(1 + 2\varepsilon)$ times the cost of $\text{SOL}_{\text{inter}}$ plus one. Therefore the cost of $\text{SOL}_{\text{final}}$ is at most $(1 + 2\varepsilon) \cdot ((1 + \varepsilon) \cdot ((1 + 2\varepsilon)\text{OPT} + |H| + |W|) + 1) + 1 \leq (1 + 10\varepsilon)\text{OPT} + 4(|H| + |W|) + 1 \leq (1 + 10\varepsilon)\text{OPT} + 4\varepsilon + 4(\frac{1}{\varepsilon} + 1)^{3/\varepsilon} + 1$ where the last inequality holds by $\varepsilon < \frac{1}{3}$, $|H| \leq \frac{1}{\varepsilon}$ and $|W| \leq |C| \leq (\frac{1}{\varepsilon} + 1)^{3/\varepsilon}$. Therefore, we have established the following theorem.

**Theorem 11** The above scheme is an AFPTAS for BPR.