THE COHOMOLOGY OF THE MOD 2 STEENROD ALGEBRA
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ABSTRACT. A minimal resolution of the mod 2 Steenrod algebra in the range
$0 \leq s \leq 128, 0 \leq t \leq 200$, together with chain maps for each cocycle in that
range and for the squaring operation $Sq^0$ in the cohomology of the Steenrod
algebra.

This article describes the archived dataset [9], available for download at the NIRD
Research Data Archive https://archive.sigma2.no. Please refer to the dataset
and this article by their digital object identifier DOI:10.11582/2021.00077.

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1. INTRODUCTION

Let $A$ denote the classical mod 2 Steenrod algebra. This archive contains
(1) a minimal free resolution of $\mathbb{F}_2$ over $A$ in internal degrees $t \leq 200$ and
cohomological degrees $s \leq 128$,
(2) chain maps lifting each member in the resulting basis for $\text{Ext}_A^{s,t}(\mathbb{F}_2,\mathbb{F}_2)$ in
this range, and
(3) a chain map which gives the Hopf algebra squaring operation
$Sq^0 : \text{Ext}_A^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \rightarrow \text{Ext}_A^{s,2t}(\mathbb{F}_2,\mathbb{F}_2)$.

This document describes the files containing this information. The resolution
was produced by the first author’s software $\text{ext}$, version 1.9.3. This is contained in

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the file `ext.1.9.3.tar.gz`. The remaining contents of the top level directory are a copyright notice, a listing (`ls-1R.txt`) of the files herein, and a directory `A`. The latter has a subdirectory `S-200` which contains the data, and a subdirectory `src` containing the source code for a C program not included in `ext.1.9.3.tar.gz`. This program writes a script to create cocycles for each basis element in $[2]$ above.

The resolution together with the chain maps give $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ as an algebra. It is worth pointing out that the minimal resolution and the chain maps contain vastly more information than this. Much secondary and higher order structure is available from this data, as well. The discussion of Toda brackets (Massey products) below, and the data contained in the file `himults` are examples: the file `himults` contains information about products by elements of cohomological degree 1 obtained without use of chain maps, while the Toda brackets files are produced from the chain maps without reference to any null-homotopies.

The package `ext` is designed to produce minimal resolutions

$$0 \leftarrow M \stackrel{d_0}{\leftarrow} C_0 \stackrel{d_1}{\leftarrow} C_1 \leftarrow \cdots \leftarrow C_S$$

and

$$0 \leftarrow N \stackrel{d_0}{\leftarrow} D_0 \stackrel{d_1}{\leftarrow} D_1 \leftarrow \cdots \leftarrow D_S$$

for any finite $A$-modules $M$ and $N$, and to lift cocycles $x \in \text{Ext}^s(M, N)$ represented as $A$-homomorphisms $x : C_s \to \Sigma^s N$ to chain maps $\{C_{s+t} \to \Sigma^t D_t\}$. The exposition here is focused on the case $M = N = \mathbb{F}_2$, and the range $0 \leq s \leq 128$, $0 \leq t \leq 200$, but at various points it may be useful to remember the extra generality of the code.

2. The Resolution

Let us write

$$0 \leftarrow \mathbb{F}_2 \stackrel{d_0}{\leftarrow} C_0 \stackrel{d_1}{\leftarrow} C_1 \leftarrow \cdots \leftarrow C_{128}$$

for our minimal resolution. It is significant that minimality allows us to identify $\text{Hom}_A(C_s, \mathbb{F}_2) = \text{Hom}_A(C_s, \Sigma^i \mathbb{F}_2)$ with $\text{Ext}_A^s(\mathbb{F}_2, \mathbb{F}_2)$.

Since our resolution is limited to the range $0 \leq s \leq 128$ and $0 \leq t \leq 200$, it is simply an initial segment of a resolution. For brevity, we shall nonetheless call it “the resolution” and write Ext for that part of $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ which lies in this range. The resolution is described by the following files.

1. **Def.** This file defines the $A$-module $\mathbb{F}_2$ as the module which is 1-dimensional over $\mathbb{F}_2$ with its sole generator in degree 0.

2. **MAXFILT.** This contains the maximum cohomological degree, $S = 128$, through which the resolution is calculated.

3. **Shape.** This file describes the free $A$-modules $C_s$. Its first entry, 128, gives the maximum cohomological degree $S$. This is followed by the 129 integers $\dim_A(C_s)$ for $s = 0, 1, \ldots, 128$. This is followed by the internal degree of each of these generators, first for $C_0$, then for $C_1$, up to $C_{128}$. This data determines Ext as a bigraded $\mathbb{F}_2$-vector space.

We write $s_g$ or $s_g$ for the cocycle dual to the 0-indexed $g^{th}$ generator of $C_s$. Thus, $0_0 \in \text{Ext}^{0,0} = \text{Ext}_A^{0,0}(\mathbb{F}_2, \mathbb{F}_2)$ is the unit $1 : C_0 \to \mathbb{F}_2$ dual to the $A$-module generator of $C_0$, while $1_t \in \text{Ext}^{1,2t} = \text{Ext}_A^{1,2t}(\mathbb{F}_2, \mathbb{F}_2)$ is the “Hopf map” $h_t : C_1 \to \Sigma^2 \mathbb{F}_2$ dual to the $A$-module generator of $C_1$ that
$d_1$ sends to $Sq^2$. When we need to refer to the $A$-module generators of $C_s$, we shall write them as $s_g^*$ or $s \cdot g^*$.

(4) $\text{Diff.s}$ and $\text{hDiff.s}$ for each $s$, $0 \leq s \leq 128$. These files contain the differentials $d_s$ for $0 \leq s \leq 128$. The file $\text{Diff.s}$ contains the elements $d_s(s_g^*)$, for $g = 0, 1, \ldots$ in that order. Each is preceded by its internal degree. The file $\text{Diff.s}$ is written in a condensed format which takes the least space possible for the formats legible to the program $\text{ext}$. The files $\text{hDiff.s}$ are provided as “humanly readable Diff files”, and write the differentials using the Milnor basis. A couple of examples should suffice to show how to read them. The file $\text{hDiff.1}$ starts

\begin{verbatim}
8   200
1
1
0 1 1 i(1).
2
1
0 2 1 i(2).
4
1
0 4 2 i(4).
...
\end{verbatim}

which should be read as saying that the $A$-dimension of $C_1$ is 8, that the part of the resolution given here is complete through internal degree 200, and that

(a) generator $1^0_0$ lies in degree 1, and $d_1(1^0_0) = Sq^1(0^1_0)$,
(b) generator $1^0_1$ lies in degree 2, and $d_1(1^0_1) = Sq^2(0^2_0)$,
(c) generator $1^0_2$ lies in degree 4, and $d_1(1^0_2) = Sq^4(0^4_0)$, et cetera.

In the file $\text{hDiff.2}$, the fifth entry, i.e., the entry for $d_2(2^s_1)$, is

\begin{verbatim}
9
3
0 8 4 i(8)(2,2).
1 7 4 i(7)(4,1)(0,0,1).
3 1 1 i(1).
\end{verbatim}

This says that $2^s_1$ has internal degree 9, and that

\[
d_2(2^s_1) = (Sq^8 + Sq^{(2,2)})(1^0_0) + (Sq^7 + Sq^{(4,1)} + Sq^{(0,0,1)})(1^1_1) + Sq^1(1^2_2).
\]

Here, the initial 3 in the description of $d_2(2^s_1)$ says that $d_2(2^s_1)$ is the sum of three terms, and the subsequent lines describe those terms. The first line, “0 8 4 i(8)(2,2).”, tells us that the first term is a multiple of $1^0_0$.
with the degree 8 coefficient $Sq^8 + Sq^{(2,2)}$. The number 4 here is the $F_2$-dimension of $A$ in degree 8, and is used by the program \texttt{ext} to determine the amount of space which must be allocated. Note that elements of the Steenrod algebra are written in the Milnor basis, not the admissible basis.  

(5) \texttt{Ext-A-F2-F2-0-200.tex} and \texttt{Ext-A-F2-F2-0-200.pdf}. These figures are in the usual “Adams chart” format showing the resolution together with the action of the elements $h_0$, $h_1$, and $h_2$. The program \texttt{chart} included in \texttt{ext.1.9.3} allows one to make visualizations like this for any box $s_{lo} \leq s \leq s_{hi}$, $t_{lo} \leq t \leq t_{hi}$. It is also possible to simply use the \texttt{TikZ} command \texttt{clip} to extract desired sections from the TEX file we provide.

In these charts, each cocycle $s_g$ is represented by a filled circle with the sequence number, $g$, written to its left, as in Figures [1] to [3]. For example, in bidegree $(n,s) = (t-s,s) = (15,5)$ we see that $5_4 = h_1 \cdot 4_3$ and $5_5 = h_0 \cdot 4_4$. See also Figure [4] at the end of this document, which shows the full Adams chart in the range $t \leq 200$.

(6) \texttt{S-200.tex} and \texttt{S-200.pdf}. These give a stem-by-stem list of the results together with the products by the $h_i$.

(7) \texttt{Maxt} and \texttt{himults}. These are generated by the program \texttt{report} included in \texttt{ext.1.9.3} and are used by the \texttt{chart} and \texttt{vssumm} commands to create  

---

1The small “i” indicates that the notation is internal to the programs defining the Steenrod algebra. The main body of the code will compute minimal resolutions for any connected augmented $F_2$-algebra, and the other notations for coefficients are generic notations for bitstrings which are independent of the algebra (see Section [4] for some discussion of formats “x” and “s”).
the Adams charts and the TeX summary. The file Maxt records, for each cohomological degree $s$, the degree $t$ through which $\text{Diff.s}$ is complete. The file himults records the $h_i$ products using the observation cited in the first paragraph of Section 5 to extract this from the $\text{Diff.s}$ files. The format of each entry is $s \ g \ s0 \ g0 \ i$

meaning that the product $h_i \cdot (s0)_{g0}$ contains the term $s_g$. For example, in bidegree $(n, s) = (t - s, s) = (78, 8)$ we have three generators $8_{60}, 8_{61}$ and $8_{62}$, and the thirteen lines
Figure 3. $\text{Ext}^{s,n+s}_{A}(F_2,F_2)$, $32 \leq n \leq 48$, $0 \leq s \leq 28$
8 60 7 33 4
8 60 7 34 4
8 60 7 56 1
8 60 7 58 1
8 61 7 33 4
8 61 7 34 4
8 61 7 48 3
8 61 7 56 1
8 61 7 57 1
8 61 7 60 0
8 62 7 2 6
8 62 7 35 4
8 62 7 61 0

say that

\[ h_0 \cdot 7_{60} = 8_{61} \]
\[ h_0 \cdot 7_{61} = 8_{62} \]
\[ h_1 \cdot 7_{56} = 8_{60} + 8_{61} \]
\[ h_1 \cdot 7_{57} = 8_{61} \]
\[ h_1 \cdot 7_{58} = 8_{60} \]
\[ h_3 \cdot 7_{48} = 8_{61} \]
\[ h_4 \cdot 7_{33} = 8_{60} + 8_{61} \]
\[ h_4 \cdot 7_{34} = 8_{60} + 8_{61} \]
\[ h_4 \cdot 7_{35} = 8_{62} \]
\[ h_6 \cdot 7_2 = 8_{62} \]

while the other products \( h_i \cdot (s_0)_{g_0} \) in this bidegree are zero. This is used by chart to draw the lines representing \( h_0 \), \( h_1 \) and \( h_2 \) products.

3. Products

We compute products in Ext by composing chain maps. Suppose the classes \( x \in \text{Ext}_{\mathcal{A}}^{s_0,t_0}(N,P) \) and \( y \in \text{Ext}_{\mathcal{A}}^{s_1,t_1}(M,N) \) are represented by cocycles

\[ x : D_{s_0} \rightarrow \Sigma^{t_0} P \quad \text{and} \quad y : C_{s_1} \rightarrow \Sigma^{t_1} N. \]

Then \( \Sigma^{t_1} x \circ y_{s_0} \) is a cocycle representing the product \( xy \), where \( \{y_{s_1}\}_s \) is a chain map lifting \( y \). The chain map with components \( \Sigma^{t_1} x \circ y_{s_0} \) is a lift of this cocycle.

\[
\begin{align*}
M & \leftarrow C_0 & \cdots & \leftarrow C_{s_1} & \cdots & \leftarrow C_{s_0+s_1} & \cdots & \leftarrow C_{s+s_0+s_1} \\
& \downarrow y_0 & \downarrow y_1 & \downarrow y_0 & \downarrow y_1 & \downarrow y_1 & \downarrow y_1 & \downarrow y_1 & \downarrow y_1 \\
\Sigma^{t_1} N & \leftarrow \Sigma^{t_1} D_0 & \cdots & \leftarrow \Sigma^{t_1} D_{s_0} & \cdots & \leftarrow \Sigma^{t_1} D_{s+s_0} & \cdots & \leftarrow \Sigma^{t_1} E_s \\
& \downarrow \Sigma^{t_1} x & \downarrow \Sigma^{t_1} x_0 & \downarrow \Sigma^{t_1} x & \downarrow \Sigma^{t_1} x & \downarrow \Sigma^{t_1} x & \downarrow \Sigma^{t_1} x & \downarrow \Sigma^{t_1} x & \downarrow \Sigma^{t_1} x \\
& \Sigma^{t_0+t_1} P & \leftarrow \Sigma^{t_0+t_1} E_0 & \cdots & \leftarrow \Sigma^{t_0+t_1} E_s 
\end{align*}
\]

In our situation, \( M = N = P = \mathbb{F}_2 \) and \( E_s = D_s = C_s \). In this case, the composite \( \Sigma^{t_1} x \circ y_{s_0} \) is determined by recording those generators \((s_0 + s_1)_{g}^*\) of...
C_{s_0+s_1} that are mapped nontrivially. When the cocycle $x$ is $(s_0)_{g_0}$, dual to a generator $(s_0)_{g_0}^*$ of $C_{s_0} = D_{s_0}$, the list of such generators is the set of those whose image under $y_{s_0}$ contains a term $1 \cdot (s_0)_{g_0}^*$, since all other terms will be sent to $0 \in F_2$ by $(s_0)_{g_0}$. The program collect in the ext package gleans this information from the chain map files described in the next section and organizes it into the file A/S-200/all.products.

Each line in the file all.products has the form

$$s \ g \ (s_0 \ g_0 \ F2) \ s1 \_g1$$

which means that the chain map lifting the cocycle $(s_1)_{g_1}$, applied to the basis element $s^*_g$, contains the term $1 \cdot (s_0)_{g_0}^*$.

**Proposition 3.1.** $(s_0)_{g_0} \cdot (s_1)_{g_1}$ is the sum of all such $s_g$.

The file all.products is organized so that each paragraph lists, for a cocycle $s_g$, all such pairs $(s_0)_{g_0}, (s_1)_{g_1}$ with the $(s_0)_{g_0}$ in increasing order.

To compute $(s_0)_{g_0} \cdot (s_1)_{g_1}$ it is best to start by noting which $s_g$ span the bidegree containing the product, so as not to miss a term. For example, consider the products landing in $\text{Ext}^{7,17}$.

This bidegree is 2-dimensional, spanned by $7_{13}$ and $7_{14}$. The entries in the file all.products for these two cocycles are

$$7 \ 13 \ (0 \ 0 \ F2) \ 7_{13}$$
$$7 \ 13 \ (1 \ 1 \ F2) \ 6_{14}$$
$$7 \ 13 \ (1 \ 2 \ F2) \ 6_{13}$$
$$7 \ 13 \ (1 \ 3 \ F2) \ 6_{10}$$
$$7 \ 13 \ (2 \ 3 \ F2) \ 5_{13}$$
$$7 \ 13 \ (3 \ 9 \ F2) \ 4_{6}$$
$$7 \ 13 \ (4 \ 6 \ F2) \ 3_{9}$$
$$7 \ 13 \ (5 \ 13 \ F2) \ 2_{3}$$
$$7 \ 13 \ (6 \ 10 \ F2) \ 1_{3}$$
$$7 \ 13 \ (6 \ 13 \ F2) \ 1_{2}$$

$$7 \ 13 \ (6 \ 14 \ F2) \ 1_{1}$$

$$7 \ 14 \ (0 \ 0 \ F2) \ 7_{14}$$
$$7 \ 14 \ (1 \ 0 \ F2) \ 6_{15}$$
$$7 \ 14 \ (1 \ 3 \ F2) \ 6_{10}$$
$$7 \ 14 \ (2 \ 0 \ F2) \ 5_{17}$$
$$7 \ 14 \ (5 \ 17 \ F2) \ 2_{0}$$
$$7 \ 14 \ (6 \ 10 \ F2) \ 1_{3}$$
$$7 \ 14 \ (6 \ 15 \ F2) \ 1_{0}$$

This says that

$$7_{13} = 0_0 \cdot 7_{13} = 1_1 \cdot 6_{14} = 1_2 \cdot 6_{13} = 2_3 \cdot 5_{13} = 3_9 \cdot 4_6$$
$$7_{14} = 0_0 \cdot 7_{14} = 1_0 \cdot 6_{15} = 2_0 \cdot 5_{17}$$

$$7_{13} + 7_{14} = 1_4 \cdot 6_{10}.$$ 

Eliminating redundant cases and using the traditional notations $h_i = i_i$, $h_0^* = s_0$, $t = 6_{14}$, $n = 5_{13}$, $r = 6_{10}$, $c_1 = 3_9$, $f_0 = 4_6$, $x = 5_{17}$, these two paragraphs say that

---

2To be precise, the new program collect.JR, included in the directory S-200, sorts the entries in this manner. It will replace the old collect in the next release of the ext package. References to collect here should generally be interpreted to refer to the improved collect.JR.
the two elements $h_1 t = h_0^2 n = c_1 f_0$ and $h_0^2 x$ span $\mathrm{Ext}^{7, 7+37}$, that $h_3 x = h_1 t + h_0^2 x$, and that all other products landing in this bidegree are 0. The $h_0, h_1$ and $h_2$ products in this bidegree can be seen in the chart shown in Figure 3. (The original definitions of $f_0$ did not distinguish between $f_0$ and $f_0 + h_3 h_4$. We eliminate this ambiguity by defining $f_0 = Sq^1(c_0)$. It is shown in [7] that $Sq^1(c_0) = 4^{6}$.)

Remark 3.2. Cocycles $s_g$ whose paragraph in all.products has only a single entry, $0_0 \cdot s_g$, are clearly indecomposable. There are 1167 of these in the range calculated, lying in filtrations 1 through 65. However, to identify all the indecomposables, simple textual work is insufficient: we must calculate $m/m^2$, where $m$ is the maximal ideal of $\mathrm{Ext}$. For example $\mathrm{Ext}^{7, 7+133}$ is spanned by $7_{124}$, $7_{125}$ and $7_{126}$ with products $h_3 \cdot 6_{97} = 7_{124} + 7_{125}$ and $h_1 \cdot 6_{107} = 7_{126}$. Thus, $7_{124}$ and $7_{125}$ project to the same nonzero element of $m/m^2$. Similarly, $h_4 \cdot 8_{114} = 9_{178} + 9_{179}$ with each term indecomposable, but equal to one another modulo decomposables.

4. Chain maps

A cocycle $s_g$ of internal degree $t$ is an $A$-module homomorphism $s_g : C_s \longrightarrow \Sigma^t F_2$. Using $\mathrm{ext}$ this can be lifted to a chain map of bidegree $(s, t)$,

$$\{ C_{s+s_0} \longrightarrow \Sigma^t C_{s_0} \mid 0 \leq s_0 \leq s + s_0 \leq 128 \}.$$  

The data describing the cocycle $s_g$ and our chain map lifting the cocycle are in the subdirectory $s_g$ of the directory $S\rightarrow 200$ containing the resolution. The relevant files are as follows.

(1) maps and subdirectories $A/S\rightarrow 200/s_g$. In $A/S\rightarrow 200$, the file maps is a list of all the cocycles $s_g$. Each of these has a subdirectory $A/S\rightarrow 200/s_g$ which contains the cocycle’s definition, the chain map lifting it, and data derived from this.

In order to study various aspects of the resolution it may be useful to focus on a smaller set of maps. The package $\mathrm{ext}$ has utilities, such as collect, which operate on such lists. For example, the invocation

```
./collect somemaps someproducts
```

would create a file named someproducts containing all products by maps listed in somemaps.

(2) $s_g/Def$. This file contains the definition of the cocycle $s_g$. If the internal degree of $s_g$ is $t$, the file $s_g/Def$ will contain

```
s t F2 F2 s_g 1
```

$g$

```
1
0 0 1 x80
```

The reader who simply wants to use this data will not need the following discussion of cocycle definition files, but for completeness, here is the meaning of these entries.

Reading them in order, it says that $s_g : C_s \longrightarrow \Sigma^t F_2$ is stored in the subdirectory $s_g$ and maps exactly one generator of $C_s$ nontrivially, namely generator number $g$, sending it to the unique nontrivial element of $F_2$,
which is the $\mathbb{F}_2$ basis element numbered 0. The format of a general cochain $x : C_\ast \to \Sigma^t N$, where $C_\ast$ is a resolution of $M$, is

$\ast \to M \to N \to x \to n$

$g1$
$x(g1)$

$g2$
$x(g2)$

$\ldots$

$gn$
$x(gn)$

This specifies the values in $N$ of $x$ on the $n$ generators numbered $g_1$ through $g_n$. Generators of $C_\ast$ which are not mentioned are mapped to 0. This defines a cocycle iff it can be lifted to a chain map iff it can be lifted over the first stage, $d_1 : D_1 \to D_0$, of a resolution of $N$. Thus, the process of lifting can be used to check that a cochain $x$ is a cocycle. By minimality of the resolutions we produce, all cochains $x : C_\ast \to \Sigma^t \mathbb{F}_2$ are cocycles.

For example, if $N$ is 4-dimensional over $\mathbb{F}_2$, with basis elements 0 to 3 in degrees 0, 1, 1 and 2, respectively, and a cochain $x : C_5 \to \Sigma^{12} N$ had nonzero values $x(5_3) = 1 + 2$ and $x(5_6) = 3$, the cochain definition file $x/\text{Def}$ could be

5 12 M N x 2

3

2

1 0 1 x80
2 0 1 s0.

6

1

3 0 1 i(0).

Here, “0 1 x80”, “0 1 s0.” and “0 1 i(0).” are three different ways to write the unit 1 $\in A_0$. The initial “0 1” in each says that they describe an element of $A_0$, which is 1-dimensional over $\mathbb{F}_2$. The first, “x80”, uses hexadecimal notation to express the binary vector with a 1 in its first (and only) entry. The second, “s0.”, lists the sequence number, 0, of the coordinates whose entries are 1 rather than 0. The third, “i(0).”, writes the operation in Milnor basis form: $Sq^0 = 1 \in A_0$. We could (and usually do) write the same information in the form

5 12 M N x 2

3
2
1 0 1 x80
2 0 1 x80

6

1
3 0 1 x80

for simplicity and uniformity.

(3) \textit{s.g/Map} and \textit{s.g/Map.aug}. The \texttt{ext} package writes the chain map lifting the cocycle $s_g$ in the file \textit{s.g/Map}. This file is simply a list of entries of the form
\begin{align*}
s0 \cdot g0 & 0 \\
\end{align*}
meaning that the lift of $s_g$ sends $s0 \cdot g0*$ to 0, or
\begin{align*}
s0 \cdot g0 & y \\
\end{align*}
where $y$ is the representation, in the manner discussed in item (4) of Section 2, of the nonzero image of $s0 \cdot g0*$ under the chain map lifting $s_g$. The entries in the \textit{Map} file do not have to occur in any particular order. There is a program, \texttt{checkmap s}, which determines the elements, if any, mapping to filtration $\leq s$ which are not yet present in the \textit{Map} file. This has been used to verify the completeness of the data we present here.

The file \textit{s.g/Map.aug} is extracted from the \textit{Map} file by applying the augmentation $A \rightarrow \mathbb{F}_2$, i.e., by discarding all terms $a \cdot s^*_g$ in which $a \in A$ has positive degree. The information in this file is used by \texttt{collect} to compile the file \texttt{all.products} described in the previous section.

(4) \textit{s.g/brackets} and \textit{s.g/brackets.sym}. These are discussed in the next section.

5. Toda brackets

Products $h_i \cdot s_g$ can be calculated directly from the resolution without computing either of the chain maps lifting $h_i$ or $s_g$. Precisely, $h_i \cdot s_g$ is the sum of those $(s+1)_{g1}$ such that $d_{s+1}((s+1)_{g1})$ contains the term $Sq^{2i} \cdot s_{g}^*$. In a similar way, having computed the chain maps, we are now able to evaluate all Toda brackets of the form $\langle h_i, (s_0)_{g0}, (s_1)_{g1} \rangle$.

**Proposition 5.1.** If $h_i \cdot (s_0)_{g0} = 0$ and $(s_0)_{g0} \cdot (s_1)_{g1} = 0$ then the Toda bracket $\langle h_i, (s_0)_{g0}, (s_1)_{g1} \rangle$ contains the sum of all those $s_g$ such that the chain map lifting $(s_1)_{g1}$ applied to $s_g^*$ contains a term $Sq^{2i} \cdot (s_0)_{g0}^*$.

Conceptually, products by elements of cohomological degree 1 are visible in the resolution reduced modulo $\mathfrak{m}^2$, where $\mathfrak{m} \subset A$ is the augmentation ideal, while brackets with first entry of cohomological degree 1 are visible in the chain maps

\[\text{We take the point of view that secondary products with respect to composition should be called \textit{Toda brackets}, while secondary products in a DGA should be called \textit{Massey products}. By the usual device of considering the DGA of endomorphisms of a chain complex, the two are equivalent.}\]
reduced modulo \( m^2 \). This data is extracted from the Map file and placed in the files brackets and brackets.sym. The files contain the same information, but brackets.sym is easier for a human to read. Each entry in s1_g1/brackets.sym will have the form
\[
\text{s\_g in } < h_i, g_0, s1_g1 >
\]
Since the filtration \( s \) of the bracket is \( 1 + s_0 + s_1 - 1 = s_0 + s_1 \), we deduce that the middle entry is \( s_0,g_0 \) with \( s_0 = s - s_1 \).

As with products, it is important to remember that the value of the bracket is the sum of all the \( s_g \) which appear, so that it is necessary to survey all the possible terms in the relevant bidegree before reaching conclusions.

We also hold the point of view that it is sufficient to produce one element of the bracket, with the other elements being obvious from the known indeterminacy. We also note that the brackets.sym file is simply recording information about the chain map, and that it is the responsibility of the user to check whether the bracket is defined.

For example, the first few entries in 1.0/brackets.sym are
\[
\begin{align*}
2.23 & \text{ in } < h_7, 0, 1_0 > \\
2.8 & \text{ in } < h_4, 0, 1_0 > \\
2.1 & \text{ in } < h_0, 1, 1_0 > \\
2.5 & \text{ in } < h_3, 0, 1_0 > \\
3.3 & \text{ in } < h_3, 0, 1_0 > \\
3.3 & \text{ in } < h_1, 3, 1_0 >
\end{align*}
\]
Since \( h_0h_7 \neq 0 \) and \( h_0h_4 \neq 0 \), the first two brackets are not defined. The third says that \( h_3^2 = 2 \in \langle h_0, h_1, h_0 \rangle \), a familiar consequence of Hirsch’s formula \( y(x \cup_1 x) \in \langle x, y, x \rangle \). The next two are also not defined. Following that we find \( c_0 = 3 \in \langle h_1, h_2^2, h_0 \rangle \).

6. \( Sq^0 \)

In general, the cocommutative Hopf algebra Steenrod operations
\[
Sq^i : \text{Ext}^{s,t}_{A}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}^{s+i,2t}_{A}(\mathbb{F}_2, \mathbb{F}_2)
\]
are computationally quite expensive ([2 Problem Session] and [7]). However, the two extremes \( Sq^s : \text{Ext}^{s,t} \rightarrow \text{Ext}^{2s,2t} \) and \( Sq^0 : \text{Ext}^{s,t} \rightarrow \text{Ext}^{s,2t} \) are easily calculated using ext. The first is simply the squaring operation \( Sq^s(x) = x^2 \) for \( x \in \text{Ext}^{s,t} \), which we have already discussed. At the other extreme, in [17 Proposition 11.10], it is shown that the operation \( Sq^0 \) can be calculated by \( Sq^0([a_1] \ldots [a_n]) = [a_1^2] \ldots [a_n^2] \) in the cobar complex for the dual Steenrod algebra.

This implies that if \( \Phi_A \) is the double of the dual Steenrod algebra, in which the degrees of all the elements are doubled, then \( Sq^0 : \text{Ext}^{s,t}_{\Phi_A}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}^{s,2t}_{\Phi_A}(\mathbb{F}_2, \mathbb{F}_2) \) is induced by the degree-preserving Hopf algebra homomorphism \( F : \Phi_A \rightarrow A \), that sends \( \Phi \xi_i \) to \( \xi_i^2 \) for each \( i \geq 1 \). Dually, it is induced by the degree-preserving Hopf algebra homomorphism \( V : A \rightarrow \Phi_A \) that sends an “even” Milnor basis element \( Sq^{0(2r_1, \ldots, 2r_k)} \) to \( \Phi_Sq^{0(2r_1, \ldots, 2r_k)} \), and other Milnor basis elements to 0. Restricting along this homomorphism gives
\[
Sq^0 : \text{Ext}^{s,t}_{\Phi_A}(\mathbb{F}_2, \mathbb{F}_2) \cong \text{Ext}^{s,2t}_{\Phi_A}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}^{s,2t}_{A}(\mathbb{F}_2, \mathbb{F}_2).
\]
A slight modification of the computer code that calculates chain maps can compute this: a program startsq0 computes the restriction \( V_{s-1}(d(x)) \) for each generator
\[ x = s_0^* \text{ in the minimal } \mathcal{A}\text{-module resolution } (C_*, d) \text{ of } F_2, \text{ and the same program that}
\text{computes lifts for chain maps then solves for an element } V_*(x) \text{ satisfying } d(V_*(x)) = V_{*-1}(d(x)). \text{ We recover } S^0_q \text{ as } \text{Hom}_\mathcal{A}(V_*, F_2). \text{ This inductive calculation is begun}
\text{by setting } V_0(0_0^0) = 0_0^0, \text{ so that } S^0_q(1) = 1. \text{ (This discussion is quoted from our}
\text{proof of Proposition 11.26 in } [8].)

The files involved are

1. **dosq0** and **maps.sq0.** The first is a script to run the computation of the lifts, while the second tells which subdirectory contains the map \( S^0_q. \)

2. **S-200/Sq0/Map** and **S-200/Sq0/Map.aug.** This contains the data defining
the chain map \( V \) and its reduction modulo the augmentation ideal \( m \subset \mathcal{A}, \)
respectively. This latter defines the dual of the map \( S^0_q. \)

3. **all.sq0.** This contains the data in the **Sq0/Map.aug** file in the format of
the **all.products** file. For example, its entries

\[
\begin{align*}
0 & \ 0 ( \ 0 \ 0 \ F2) \ \text{Sq0} \\
1 & \ 1 ( \ 1 \ 0 \ F2) \ \text{Sq0} \\
1 & \ 2 ( \ 1 \ 1 \ F2) \ \text{Sq0} \\
& \ldots \\
3 & \ 9 ( \ 3 \ 3 \ F2) \ \text{Sq0} \\
3 & \ 19 ( \ 3 \ 9 \ F2) \ \text{Sq0} \\
3 & \ 34 ( \ 3 \ 19 \ F2) \ \text{Sq0} \\
3 & \ 55 ( \ 3 \ 34 \ F2) \ \text{Sq0} \\
& \ldots \\
4 & \ 16 ( \ 4 \ 5 \ F2) \ \text{Sq0} \\
4 & \ 19 ( \ 4 \ 6 \ F2) \ \text{Sq0}
\end{align*}
\]

show that

\[
\begin{align*}
S^0_q(1) & = S^0_q(0_0^0) = 0_0^0 = 1 \\
S^0_q(h_0) & = S^0_q(1_0^1) = 1_1 = h_1 \\
S^0_q(h_1) & = S^0_q(1_1^2) = 1_2 = h_2 \\
& \ldots \\
S^0_q(c_0) & = S^0_q(3_3^3) = 3_9 = c_1 \\
S^0_q(c_1) & = S^0_q(3_9^3) = 3_{19} = c_2 \\
S^0_q(c_2) & = S^0_q(3_{19}^3) = 3_{34} = c_3 \\
S^0_q(c_3) & = S^0_q(3_{34}^3) = 3_{55} = c_4 \\
& \ldots \\
S^0_q(e_0) & = S^0_q(4_5^4) = 4_{16} = e_1 \\
S^0_q(f_0) & = S^0_q(4_{16}^4) = 4_{19} = f_1.
\end{align*}
\]

Later we find

\[
\begin{align*}
6 & \ 102 ( \ 6 \ 33 \ F2) \ \text{Sq0} \\
6 & \ 103 ( \ 6 \ 32 \ F2) \ \text{Sq0} \\
6 & \ 103 ( \ 6 \ 33 \ F2) \ \text{Sq0} \\
6 & \ 104 ( \ 6 \ 32 \ F2) \ \text{Sq0}
\end{align*}
\]
which reminds us that these files are the dual, i.e., chain level, data. In Ext, these say
\[ Sq^0(A_0 + A'_0) = Sq^0(6_{32}) = 6_{103} + 6_{104}, \]
\[ Sq^0(A_0) = Sq^0(6_{33}) = 6_{102} + 6_{103}, \]
and hence
\[ Sq^0(A'_0) = Sq^0(6_{32} + 6_{33}) = 6_{102} + 6_{104}. \]

This information is used in Section 8 to organize Ext into “families” linked by \( Sq^0 \).

7. A canonical basis for \( \text{Ext}_{\lambda}(\mathbb{F}_2, \mathbb{F}_2) \)

The traditional, and by now familiar, notation for the elements of Ext starts in a systematic fashion, with the elements
\[ h_i, P^k h_1, P^k h_2, e_i, P^k e_0, d_i, P^k d_0, \]
but becomes somewhat chaotic as the calculations are extended into higher bidegrees. In the next section, we propose some ways of extending this notation in a methodical fashion, but they do not suffice to give names to elements of a basis for Ext even in the range we consider here, let alone for all of Ext.

In contrast, our \( s_g \) form a well-defined canonical basis for Ext, which we now describe. First, we totally order the terms \( Sq^R s_g^* \) of \( C_{s,t} \) by

\[ Sq^R s_g^* < Sq^R s_{g'}^* \]

iff

1. \( g < g' \), or
2. \( g = g' \) and \( Sq^R < Sq^R' \), where the Milnor basis elements \( Sq^R \) are given reverse lexicographic order: \( (r_1, r_2, \ldots) < (r'_1, r'_2, \ldots) \) iff for some \( k, r_k < r'_k \) and \( r_i = r'_i \) for all \( i > k \). Thus,

\[ (n) < (n-3, 1) < (n-6, 2) < \cdots \]
\[ < (n-7, 0, 1) < (n-10, 1, 1) < \cdots < (n-14, 0, 2) < (n-17, 1, 2) < \cdots \]
\[ < (n-15, 0, 0, 1) < (n-18, 1, 0, 1) < \cdots < (n-22, 0, 1, 1) < \cdots \]

Each nonzero element \( x \in C_{s,t} \) then has a leading term \( LT(x) \), which is the lowest term in \( x \).

In the totally ordered basis \( \{ Sq^R s_g^* \} \) of a given bidegree \( (s,t) \), the decomposable elements, those with \( \deg(Sq^R) > 0 \), form an initial segment which is followed by the generators \( s_g^* \) of bidegree \( (s,t) \).

We can now inductively define our canonical basis as follows. We start with the bases \( \{ 0 \} \) for \( C_0 \) and \( \{ \} \) for \( C_{s+1} \) with \( s > 0 \). We may inductively assume given the basis for \( C_{s-1} \) in degrees less than \( t \), and for \( C_{s-1} \) in degrees less than or equal to \( t \).

**Step 1:** Generating the image and kernel.
\( \text{Im}_{s,t} \) will be a totally ordered list of pairs \((x, dx)\) with the leading terms of the \( dx \) in strictly increasing order. \( \text{Ker}_{s,t} \) will be a list of terms \( x \). Both are initially empty.

Consider the terms \( Sq^R s_g^* \) in order. Let \( x = Sq^R s_g^* \) and compute \( dx = Sq^R d(s_g^*) \). Then, while \( dx \neq 0 \), if \( LT(dx) = LT(dy) \) for a pair \((y, dy) \in \text{Im}_{s,t}\), replace \( x \) by \( x - y \) and \( dx \) by \( dx - dy \). If not, add \((x, dx)\) to \( \text{Im}_{s,t} \) and proceed to the next decomposable term. If instead \( dx = 0 \), add \( x \) to the end of the list \( \text{Ker}_{s,t} \).
Note that the leading term of $dx$ will be increased each time we replace $dx$ by $dx - dy$ until $dx$ either becomes 0 or has a leading term not already found among the $dy$ in $\text{Im}_{s,t}$.

**Step 2:** Adding new generators.

We may inductively assume given $\text{Ker}_{s-1,t}$. For each $x \in \text{Ker}_{s-1,t}$, in order, let $c = x$. Then, while $LT(x) = LT(dy)$ for some pair $(y, dy) \in \text{Im}_{s,t}$, replace $x$ by $x - dy$. If this process terminates with $LT(x) \neq 0$, add a new generator $s^*_y$ with $d(s^*_y) = c$, then add a new pair $(z, x)$ to $\text{Im}_{s,t}$, where $z$ is the difference of $s^*_y$ and those $y$ whose images $dy$ were subtracted from $c$ to get the final $x$ with a new leading term. If the process terminates with $x = 0$, do nothing.

**Remark 7.1.** We could choose, at this second step, to let $d(s^*_y) = x$ and add the pair $(s^*_y, x)$ to $\text{Im}_{s,t}$. The $\text{ext}$ code prior to the year 2000 used that algorithm. Experience shows that the bases $s_y$ obtained from the algorithm described here have $h_i \cdot s_y$ monomial far more frequently than those produced by the old algorithm. The Wayne State Research Report [5] used the older algorithm. The first difference visible in Ext charts lies in bidegree $(9, 9 + 23)$. In the new algorithm, $h_1 \cdot s_3 = 9_4$ and $h_0 \cdot s_4 = 9_3$. In the older algorithm, $h_0 \cdot s_4 = 9_5$ also, but $h_1 \cdot s_3 = 9_4 + 9_5$.

The change alters the resolution much earlier, as can be seen by doing hand calculations in low degrees. In the old algorithm, $d(2^1_1) = Sq(01) \cdot 1^0_0 + Sq^2 \cdot 1^1_1$, while the new algorithm gives $d(2^1_1) = Sq^3 \cdot 1^0_0 + Sq^2 \cdot 1^1_1$.

8. Concordance

In this section, we present the relation between our $s_y$ basis and the notation used by other works on the cohomology of the Steenrod algebra. In the process, we make a natural extension to the traditional notation using $Sq^0$.

The existing names are based on Tangora’s calculation of the $E_\infty$ term of the May spectral sequence in [19] and on Chen’s Lambda algebra computation of $\text{Ext}^8 = \text{Ext}^s_4(\mathbb{F}_2, \mathbb{F}_2)$ for $s \leq 5$ in [10].

There is some indeterminacy in the translation between our names and both the May spectral sequence names and Chen’s Lambda algebra names.

In the case of the May spectral sequence, elements of the $E_\infty$-term of the May spectral sequence only determine elements of $\text{Ext}$ up to classes of higher May filtration. In Table 1, we indicate this by giving the indeterminacy of the May spectral sequence name in parentheses. For example, in bidegree $(5, 5+62)$, the May spectral sequence definition of $H_1(0)$ (written $H_1$ in [19]) defines the coset $5_{32} + (5_{33}, 5_{34})$. We denote this, in our table, by writing $32(33, 34)$ in the column giving the sequence number of the element.

The indeterminacy in relating Chen’s Lambda algebra element names to ours stems from the lack of a direct comparison between the two complexes. There are five such cases which we discuss in Remark 8.4. Except for $T_0$, which is beyond the range of Tangora’s May spectral sequence calculation, this indeterminacy is the same as that due to the May filtration. In the case of $T_0$, the indeterminacy is entirely due to the lack of a direct comparison between the Lambda algebra and our minimal resolution. For $T_0$, $Q_4(0)$ and $H_1(0)$, the indeterminacy could be eliminated if we knew certain $h_i$-multiples, as noted in Remark 8.4.

The preprint [14] of Isaksen-Wang-Xu studies the Adams spectral sequence through $t - s \leq 95$. They adopt some of Tangora’s notation for $\text{Ext}$, augmented
and regularized by the use of operators which they call $M$, $\Delta$ and $\Delta_1$. Their operator $Mx$ is principally defined to be $(g_2, h_{02}^3, x)$, though it is also sometimes used when this bracket is not defined. The operators $\Delta$ and $\Delta_1$ are given by products with non-permanent cycles $b_{03}^2$ and $b_{13}^1$, respectively, in the $E_2$-term of the May spectral sequence. This is analogous to Tangora’s definition of $Px = (b_{02})^2x$. They are precursors of Toda brackets discussed in Section 9 in the sense that, when the brackets are defined, they often compute them. In fact, each could be interpreted as two of three distinct brackets which must sum to zero by the Jacobi identity. We discuss these brackets in Section 9.

In a few bidegrees they encounter classes with no name under this system. They are given by products $Px = (b_{02})^2x$. They adopt the notation $x$ and $y$ for such a class if it is the unique such class in bidegree $(s, s + n)$. In bidegree $(10, 10 + 94)$ there are two such classes which they call $x_{94,10}$ and $y_{94,10}$.

8.1. Cohomological degrees up to 5. Recall the theorems of Wang and Palmieri:

**Theorem 8.1** ([20] and [18]). For $s < 4$, the homomorphism $Sq^0 : \text{Ext}^s \rightarrow \text{Ext}^s$ is injective. When $s = 4$, its kernel is $\langle h_0^4 \rangle$.

This makes the “$Sq^0$-families” $\{x, Sq^0(x), (Sq^0)^2(x), \ldots\}$ especially useful in low cohomological degrees.

**Remark 8.2.** Because $h_4^4 = 0$, $Sq^0 : \text{Ext}^5 \rightarrow \text{Ext}^5$ must send the nonzero elements $h_0^5$ and $h_0^5h_i, i \geq 4$, to 0. In the range we have calculated, the only other element in its kernel is $Ph_2$, reflecting $Sq^0(Ph_2) = h_3g = 0$.

Chen ([10]: Theorem 1.2 and Theorem 1.3) gives a complete description of $\text{Ext}^s$ for $s \leq 5$, building on the work of Adams [1], Wang [20] and Lin [15].

**Theorem 8.3** ([10]: Theorems 1.2 and 1.3). An $F_2$-base for the indecomposable elements in cohomological degrees $s \leq 5$ is as follows. In each, $i$ runs over all $i \geq 0$.

$\text{Ext}^1$: $h_i$.

$\text{Ext}^2$: $e_i$.

$\text{Ext}^3$: $f_i, g_{i+1}, p_i, D_2(i)$ and $p'_i$.

$\text{Ext}^4$: $ph_2, ph_2, n_i, x_i, D_1(i), H_1(i), Q_i(i), K_i, J_i, T_i, \nu_i, \nu_i', V_i, V_i'$ and $U_i$.

Note that $V_0'$ and $U_0$ are in the 252 and 260 stems, respectively, so are beyond the range of our computation. For each of the other families in these lists, at least the first member of the family lies in the range of our calculations. Chen adopts the notation $D_1(i)$, et cetera, for the $i$th member of a $Sq^0$-family in order to avoid double subscripts. We extend this practice into higher cohomological degrees as noted in the next section. In each family except $\{g_1 = g, g_2, \ldots\}$, the family starts with the $0^{th}$ element.

Since Chen’s list is (mostly) in terms of $Sq^0$-families, we need only consider the first member of each family in order to establish the relation between his classes and ours. Thirteen of the families start in a bidegree which is 1-dimensional over $F_2$, so that the correspondence is clear for them. The remaining families are $f_i$, $n_i$, $H_1(i)$, $Q_3(i)$ and $T_i$, and we consider each of these individually.

---

[4] Tangora [19] pp. 32 and 48 notes that this is not always equal to the periodicity operator $Px = \langle h_3, h_0^3, x \rangle$. For example see [19] Note 3 on p. 48.
Remark 8.4.

(1) It is long established practice to define \( f_0 = Sq^1(c_0) \), because that allows us to take advantage of \( H_\infty \) ring spectrum relations and differentials. We do not know whether Chen’s definition of \( f_0 \) in terms of the Lambda algebra equals this or \( f_0 + h_1^0 h_4 \).

(2) We choose \( n_0 \) to be the \( h_0 \)-annihilated class in bidegree \((5, 5 + 31)\). According to Chen’s unpublished preprint \[11\] Thm. 1.7 this agrees with his Lambda algebra definition of \( n_0 \).

(3) In two remaining cases, \( H_1(0) \) and \( Q_3(0) \), their May spectral sequence definition specifies a coset they must lie in. Chen’s Lambda algebra classes with these names lie in the specified cosets. The precise elements in these cosets are determined by the classes \( h_0 H_1(0) \), \( h_4 H_1(0) \) and \( h_3 Q_3(0) \). (That this suffices can be checked using all products.) These three products are shown to be zero in Chen’s \[11\] Thm. 1.7. Thus, the May spectral sequence indeterminacy for \( H_1(0) \) and \( Q_3(0) \) is the indeterminacy reported in Table \[11\] but Chen’s unpublished results allow the more precise correspondence \( H_1(0) = 5_{32}, Q_3(0) = 5_{39}, H_3(1) = 5_{81} \) and \( Q_3(1) = 5_{q1} \).

(4) The remaining case, \( T_0 \), has only Chen’s Lambda algebra definition from \[10\]. The precise indecomposable element in this bidegree is determined by the values of \( h_1 T_3 \) and \( h_4 T_0 \). As above, these suffice can be checked using all products, and both are shown to be zero in Chen’s \[11\] Thm. 1.7. The indeterminacy reported in Table \[7\] is the set of decomposables, but Chen’s unpublished results allow the more precise correspondence \( T_0 = 5_{33} \).

8.2. Cohomological degrees greater than 5. In higher cohomological degrees, names for the elements of Ext come from Tangora’s 1970 calculation \[19\] of the \( E_\infty \)-term of the May spectral sequence. His Appendix 1 lists its indecomposables in the range \( t - s \leq 70 \) (omitting classes of the form \( P^k a \)). In three cases, known hidden extensions between the associated graded and Ext allow us to ignore these elements. These are \( s = h_0 r, S_1 \in \langle h_1 x', h_0 R_1 \rangle \) and \( g_2^0 \in \langle h_1 B_21, h_0^2 B_4(0) \rangle \).

Some elements named by Tangora have \( Sq^0 \) which is decomposable. In those cases we keep Tangora’s (often unsubscripted) notation. Others appear to be the start of \( Sq^0 \)-families, with indecomposable members in the range of our calculation. In those cases, we add a subscript 0 or suffix (0) to Tangora’s name for the class, except for the \( g_i \) family, which starts with \( g_1 = g \).

For unsubscripted elements that start \( Sq^0 \)-families, like Tangora’s \( m \) or \( A \), we adopt the usual \( a_{i+1} = Sq^0(a_i) \) notation for the subsequent elements of the family. For subscripted elements like the \( B_i \) or \( H_i \) we use Chen’s suffix notation, \( Z(i + 1) = Sq^0(Z(i)) \), to avoid collision with other names used by Tangora. The suffix notation is also applied for \( C, C'' \) and \( G \), due to the prior presence of classes \( C_0 \) and \( G_0 \).

Collisions would occur because subscripts in Tangora’s subscripted capital letter classes do not indicate membership in a \( Sq^0 \)-family. In particular, the \( B_1 \) through \( B_5 \), the \( C \) and \( C_0 \), etc., are not related by \( Sq^0 \). Nor are \( q \) in the 32-stem and \( q_1 \) in the 64-stem related in this manner. This is solved by Chen’s \( Z(i) \) notation.

If there is indeterminacy in the \( s_q \) name for the first member of a \( Sq^0 \)-family, it is generally inherited by the subsequent members. \( X_2(1), E_1(1), C_0(1), G_{21}(1) \) and \( B_4(1) \) are exceptions: \( Sq^0 \) annihilates the indeterminacy in the descriptions of \( X_2(0), E_1(0), C_0(0), G_{21}(0) \) and \( B_4(0) \).
We have not listed all the indecomposable $s_g$ which can be described using the Adams periodicity operators, but we have included those appearing in Tangora’s list of indecomposables. Similarly, we have not listed all the indecomposable $s_g$ which can be described using the “Mahowald operator” $M(x) = \langle g_2, h_0^3, x \rangle$ discussed by Margolis–Priddy–Tangora [16] and Isaksen [13], but we have noted that seven classes $B_1$, $B_2$, $B_3$, $B_{21}$, $B_{22}$, $B_{23}$ and $G_{11}$ in Tangora’s list could be so described using the modified operator $M'(x) = \langle h_0, h_0^3g_2, x \rangle$. See Section 9 for further discussion of how to use our data to extend this into the full range of our calculation.

In a very few cases the indeterminacy reported in Table I is greater than the inherent indeterminacy in a May spectral sequence definition. These are the bidegrees $(t - s, s) = (63, 7)$, $(67, 9)$ and $(66, 10)$, where there are two indecomposables in the same bidegree, and possibly $(t - s, s) = (141, 5)$, where we do not know a May spectral sequence definition of the indecomposable class.

Table 1: Concordance between indecomposable $s_g$ and other notations. Elements $s_g$ for which we do not have a traditional name are omitted.

| $t - s$ | $s$ | $g$ | Tangora, Chen | Note |
|--------|-----|-----|---------------|------|
| 0      | 1   | 0   | $h_0$         | 1    |
| 1      | 1   | 1   | $h_1$         | 2    |
| 3      | 1   | 2   | $h_2$         | 2    |
| 7      | 1   | 3   | $h_3$         | 2    |
| 15     | 1   | 4   | $h_4$         | 2    |
| 31     | 1   | 5   | $h_5$         | 2    |
| 63     | 1   | 6   | $h_6$         | 2    |
| 127    | 1   | 7   | $h_7$         | 2    |
| 8      | 3   | 3   | $c_0$         | 1    |
| 19     | 3   | 9   | $c_1$         | 2    |
| 41     | 3   | 19  | $c_2$         | 2    |
| 85     | 3   | 34  | $c_3$         | 2    |
| 173    | 3   | 55  | $c_4$         | 2    |
| 14     | 4   | 3   | $d_0$         | 1    |
| 17     | 4   | 5   | $e_0$         | 1    |
| 18     | 4   | 6   | $f_0$         | 3    |
| 20     | 4   | 8   | $g_1$         | 2    |
| 32     | 4   | 13  | $d_1$         | 2    |
| 33     | 4   | 14  | $p_0$         | 2    |
| 38     | 4   | 16  | $e_1$         | 2    |
| 40     | 4   | 19  | $f_1$         | 2    |
| 44     | 4   | 22  | $g_2$         | 2    |
| 61     | 4   | 26  | $D_3(0)$      | 1    |
| 68     | 4   | 32  | $d_2$         | 2    |
| 69     | 4   | 33  | $p_0'$        | 1    |
Table 1: Concordance between indecomposable $s_g$ and other notations.

| $t - s$ | $s$ | $g$  | Tangora, Chen | Note |
|---------|-----|------|----------------|------|
| 70 4    | 34  | $p_1$| 2              |
| 80 4    | 40  | $e_2$| 2              |
| 84 4    | 44  | $f_2$| 2              |
| 92 4    | 48  | $g_3$| 2              |
| 126 4   | 53  | $D_3(1)$| 2          |
| 140 4   | 65  | $d_3$| 2              |
| 142 4   | 67  | $p_1'$| 2            |
| 144 4   | 69  | $p_2$| 2              |
| 164 4   | 79  | $e_3$| 2              |
| 172 4   | 84  | $f_3$| 2              |
| 188 4   | 89  | $g_4$| 2              |
| 9 5     | 1   | $P h_1$| 1            |
| 11 5    | 2   | $P h_2$| 1            |
| 31 5    | 13  | $n_0$| 4              |
| 37 5    | 17  | $x_0$| 1              |
| 52 5    | 30  | $D_1(0)$| 1          |
| 62 5    | 32(33, 34) | $H_1(0)$| 5        |
| 67 5    | 38  | $n_1$| 3              |
| 67 5    | 39(38) | $Q_3(0)$| 6          |
| 79 5    | 50  | $x_1$| 2              |
| 109 5   | 75  | $D_1(1)$| 2          |
| 125 5   | 77  | $K_0$| 1              |
| 128 5   | 80  | $J_0$| 1              |
| 129 5   | 81(82, 83) | $H_1(1)$| 2          |
| 139 5   | 90  | $n_2$| 2              |
| 139 5   | 91(90) | $Q_3(1)$| 2          |
| 141 5   | 93(94, 95) | $T_0$| 7              |
| 156 5   | 108 | $V_0$| 1              |
| 163 5   | 115 | $x_2$| 2              |
| 30 6    | 10  | $r_0$| 1              |
| 32 6    | 12  | $q$  | 1              |
| 36 6    | 14  | $t_0$| 1              |
| 38 6    | 16  | $y$  | 3              |
| 50 6    | 27  | $C(0)$| 1            |
| 54 6    | 30  | $G(0)$| 1            |
| 58 6    | 31  | $D_2$| 1              |
| 61 6    | 32 + 33 | $A_0'$| 8          |
| 61 6    | 33  | $A_0$| 8              |
| 64 6    | 38  | $A_0''$| 1           |
Table 1: Concordance between indecomposable $s_g$ and other notations.

| $t - s$ | $s$ | $g$ | Tangora, Chen | Note |
|---------|-----|-----|----------------|------|
| 66      | 6   | 40  | $r_1$          |      |
| 78      | 6   | 56  | $t_1$          |      |
| 106     | 6   | 87  | $C(1)$         | 2    |
| 114     | 6   | 92  | $G(1)$         | 2    |
| 128     | 6   | 102 | $A_1 + h_2K_0 = A_1' + h_0J_0$ | 9    |
| 134     | 6   | 110 | $A_1''$        | 2    |
| 138     | 6   | 115 | $r_2$          | 2    |
| 162     | 6   | 156 | $t_2$          | 2    |
| 16       | 7   | 3   | $P_{c_0}$      |      |
| 23       | 7   | 5   | $i$            |      |
| 26       | 7   | 6   | $j$            | 1    |
| 29       | 7   | 7   | $k$            | 1    |
| 32       | 7   | 10  | $\ell$        | 1    |
| 35       | 7   | 12  | $m_0$          | 1    |
| 46       | 7   | 20  | $B_1(0) = M'h_1$ | 1    |
| 48       | 7   | 22(23) | $B_2(0) = M'h_2$ | 10   |
| 57       | 7   | 27  | $Q_2(0)$       | 1    |
| 60       | 7   | 29  | $B_3 = M'h_4$  | 1    |
| 63       | 7   | 33(35) | $X_2(0)$       | 11   |
| 63       | 7   | 34(33,35) | $C'$         | 11   |
| 66       | 7   | 40(41) | $G_0(0)$     | 12   |
| 77       | 7   | 56  | $m_1 + h_16_{53}$ | 2    |
| 99       | 7   | 85  | $B_1(1)$       | 2    |
| 103      | 7   | 90(91) | $B_2(1)$     | 2    |
| 121      | 7   | 101 | $Q_2(1) + h_6D_2$ | 2    |
| 133      | 7   | 124 | $X_2(1) + h_36_{97} + h_16_{107}$ | 13   |
| 133      | 7   | 125 | $X_2(1) + h_16_{107}$ | 13   |
| 139      | 7   | 137(138) | $G_0(1)$ | 2    |
| 161      | 7   | 184 | $m_2 + h_26_{149} + h_2h_7n$ | 2    |
| 22       | 8   | 3   | $P_{d_0}$      | 1    |
| 25       | 8   | 5   | $P_{e_0}$      |      |
| 46       | 8   | 20  | $N$            | 1    |
| 62       | 8   | 32 + 33(34) | $E_1(0)$ | 14   |
| 62       | 8   | 33(34) | $C_0(0)$       | 14   |
| 68       | 8   | 43(44) | $G_{21}(0)$ | 15   |
| 69       | 8   | 46  | $PD_3(0)$      | 16   |
| 132      | 8   | 139 + 140 | $C_0(1) + h_2^26_{97}$ | 2    |
| 132      | 8   | 140 | $E_1(1)$       | 2    |
Table 1: Concordance between indecomposable \( s_g \) and other notations.

| \( t - s \) | \( s \) | \( g \) | Tangora, Chen | Note |
|---|---|---|---|---|
| 144 | 8 | 176 | \( G_{21}(1) + h_7 d_0 \) | 1 |
| 17 | 9 | 1 | \( P^2 h_1 \) | 1 |
| 19 | 9 | 2 | \( P^2 h_2 \) | 1 |
| 39 | 9 | 18 | \( u \) | 1 |
| 42 | 9 | 19 | \( v \) | 1 |
| 45 | 9 | 20 | \( w \) | 1 |
| 60 | 9 | 29(30) | \( B_4(0) \) | 17 |
| 61 | 9 | 31 | \( X_1 \) | 1 |
| 67 | 9 | 39(40) | \( C''(0) \) | 18 |
| 67 | 9 | 40(39) | \( X_3 \) | 18 |
| 67 | 9 | 39, 40 | \( C''(0), X_3 \) | 18 |
| 129 | 9 | 145 | \( B_4(1) + h_2 d_{118} \) | 2 |
| 143 | 9 | 197(199 + 200) | \( C''(1) \) | 2 |
| 41 | 10 | 14 | \( z \) | 1 |
| 53 | 10 | 18 | \( x' \) | 1 |
| 54 | 10 | 19(20) | \( R_1 \) | 19 |
| 56 | 10 | 22(21) | \( Q_1 \) | 20 |
| 59 | 10 | 24 | \( B_{21} = M'd_0 \) | 1 |
| 62 | 10 | 27(28, 29) | \( R \) | 21 |
| 62 | 10 | 28(29) | \( B_{22} \supset M'e_0 \) | 21 |
| 64 | 10 | 32(33) | \( q_1 \) | 22 |
| 65 | 10 | 34 | \( B_{23}(0) = M'g_1 \) | 1 |
| 66 | 10 | 35 + 36 | \( B_5 \) | 23 |
| 66 | 10 | 35 or 36 | \( D_2' \) | 23 |
| 66 | 10 | 36 | \( PD_2 \) | 16 |
| 69 | 10 | 40 | \( PA_0 \) | 1 |
| 140 | 10 | 196 + 197 | \( B_{23}(1) + h_1 d_{178} \) | 24 |
| 24 | 11 | 3 | \( P^2 e_0 \) | 1 |
| 34 | 11 | 7 | \( P j \) | 1 |
| 67 | 11 | 35(36) | \( C_{11} \) | 25 |
| 30 | 12 | 3 | \( P^2 d_0 \) | 1 |
| 33 | 12 | 5 | \( P^2 e_0 \) | 1 |
| 25 | 13 | 1 | \( P^3 h_1 \) | 1 |
| 27 | 13 | 2 | \( P^3 h_2 \) | 1 |
| 47 | 13 | 14 | \( Q \) | 26 |
| 47 | 13 | 14 + 15 | \( Q' = Q + Pu \) | 26 |
| 47 | 13 | 15 | \( Pu \) | 26 |
| 50 | 13 | 16 | \( Pv \) | 1 |
Table 1: Concordance between indecomposable \( s_g \) and other notations.

| \( t-s \) | \( s \) | \( g \) | Tangora, Chen | Note |
|---------|------|-----|--------------|------|
| 65      | 13   | 29(28) | \( R_2 \) |      |
| 68      | 13   | 30(31) | \( G_{11} = M'i \) | 28   |
| 69      | 13   | 32    | \( W_1 \) | 1    |
| 64      | 14   | 23    | \( PQ_1 \) | 16   |
| 70      | 17   | 26(27) | \( R'_1 \) | 29   |
| 69      | 18   | 20    | \( P^2x' \) (ill-defined) | 30   |
| 8k + 1  | 4k + 1 | 1   | \( P^k h_1 \) | 31   |
| 8k + 3  | 4k + 1 | 2   | \( P^k h_2 \) | 31   |
| 8k + 8  | 4k + 3 | 3   | \( P^k c_0 \) | 31   |
| 8k + 14 | 4k + 4 | 5   | \( P^k d_0 \) | 31   |
| 8k + 17 | 4k + 4 | 6   | \( P^k c_0 \) | 31   |
| 8k + 23 | 4k + 7 | 7   | \( P^k j \) (for k even) | 31   |
| 8k + 26 | 4k + 7 | 8   | \( P^k j \) (for k even) | 31   |
| 8k + 26 | 4k + 7 | 9   | \( P^k j \) (for k odd) | 31   |
| 8k + 39 | 4k + 9 | 10  | \( P^k u \) (for k even) | 31   |
| 8k + 39 | 4k + 9 | 10  | \( P^k u \) (for k odd) | 31   |
| 8k + 42 | 4k + 9 | 11  | \( P^k v \) (for k even) | 31   |
| 8k + 42 | 4k + 9 | 12  | \( P^k v \) (for k \( \equiv 1 \) mod 4) | 31   |
| 8k + 42 | 4k + 9 | 13  | \( P^k v \) (for k \( \equiv 3 \) mod 4) | 31   |

Notes:

1. There is a unique nonzero element in this bidegree.
2. This is \( Sq^1 \) of an element we have already identified.
3. We choose to let \( f_0 = Sq^1(c_0) \) and \( y = Sq^2(f_0) \), which are shown in [7] to be \( 4_6 \) and \( 6_{16} \), respectively.
4. This is the unique nonzero element in this bidegree whose \( h_0 \) multiple is 0.
5. The May spectral sequence definition of \( H_1 = H_1(0) \) has indeterminacy spanned by \( 5_{33} = h_1 D_3 \) and \( 5_{34} = h_0^3 h_2^5 \), so that \( H_1(0) \) must be the indecomposable \( 5_{32} \) modulo them. The Lambda algebra class which Chen defines as \( H_1(0) \) in [10] satisfies \( h_0 H_1(0) = 0 \) and \( h_4 H_1(0) = 0 \), according to Chen’s preprint [11, Thm. 1.7], eliminating the possible summands \( 5_{34} \) and \( 5_{33} \), respectively.
6. The May spectral sequence definition of \( Q_3 = Q_3(0) \) has indeterminacy spanned by \( 5_{38} = n_1 \), so that \( Q_3(0) \) must be \( 5_{39} \) modulo \( 5_{38} \). The Lambda algebra class which Chen defines as \( Q_3(0) \) in [10] satisfies \( h_3 Q_3(0) = 0 \), according to Chen’s preprint [11, Thm. 1.7], eliminating the possible summand \( 5_{38} \).
7. The indecomposable \( T_0 \) must be the indecomposable \( 5_{93} \) modulo the decomposables \( 5_{94} = h_7 d_0 \) and \( 5_{95} = h_3 d_5 \). If \( T_0 = 5_{93} + \alpha 5_{94} + \beta 5_{95} \), then \( h_1 T_0 = 6_{126} \) while \( h_4 T_0 = 36_{145} \). These products are both zero according to Chen’s preprint [11, Thm. 1.7], so that \( T_0 = 5_{93} \).
(8) Tangora \cite{19} shows $h_0A = h_2D_2$, hence $A = 6_{33}$. He also shows $h_0^2A' = 0$, so that $A' = 6_{32} + 6_{33}$. We write them as $A_0$ and $A_0'$, since $Sq^0$ is nonzero on both.

(9) $A_1 = Sq^0(A_0) = 6_{102} + 6_{103}$ and $A_1' = Sq^0(A_0') = 6_{102} + 6_{104}$, while $6_{103} = h_2K_0$ and $6_{104} = h_0J_0$.

(10) The May spectral sequence definition of $B_2$ has indeterminacy $7_{23} = h_0^2h_5e_0$, so that $B_2 \in \{7_{22}, 7_{22} + 7_{23}\}$. The value of $M'h_2$ is exactly the same set.

(11) Bidegree $(7, 7 + 63)$ is spanned by the two indecomposables $7_{33}$ and $7_{34}$ together with $7_{35} = h_0^3h_6$. We have $h_27_{33} = 0$ and $h_27_{34} = 8_{41}$. In the May spectral sequence there are indecomposables $C'$ and $X_2$. Tangora \cite{19} reports that $h_2C' = h_0G_0$ is nonzero and does not list $h_2X_2$ as a nonzero value. Granting that $h_2X_2 = 0$, it follows that $C' \equiv 7_{34}$ modulo $\{7_{33}, 7_{35}\}$ and $X_2 \equiv 7_{33}$ modulo $7_{35}$. Since $Sq^0(7_{33})$ is indecomposable, we write $X_2 = X_2(0)$. We do not know whether $Sq^0(C')$ is indecomposable; if it is we should set $C' = C'(0)$ and note that $C'(1) = X_2(1)$ modulo decomposables.

(12) Bidegree $(7, 7 + 66)$ is spanned by the indecomposable $7_{40}$ and $7_{41} = h_0r_1$. Hence, the indecomposable $G_0 = G_0(0)$ must be $7_{40}$ modulo $7_{41}$.

(13) Since $Sq^0(7_{34}) = 7_{124} + 7_{125} = h_36_{97}$, we see that $7_{124}$ and $7_{125}$ are congruent modulo decomposables, but are each indecomposable. Then $Sq^0(7_{33}) = 7_{125} + h_16_{107}$ shows that $7_{125} = Sq^0(7_{33}) + h_16_{107}$ and that $7_{124} = Sq^0(7_{41}) + h_16_{107} + h_36_{97}$.

(14) Bidegree $(8, 8 + 62)$ is spanned by the two indecomposables $8_{32}$ and $8_{33}$ together with $8_{34} = h_0^3h_5^2$. Tangora \cite{19} lists indecomposables $C_0$ and $E_1$, together with $h_0^3h_5^2$. He reports $h_1E_1 \neq 0$ and (implicitly) $h_1C_0 = 0$. He also reports $h_2C_0 \neq 0$ and (implicitly) $h_2E_1 \equiv 0$ modulo $h_0^3h_3D_2$. This implies $E_1 \equiv 8_{32} + 8_{33}$ modulo $8_{34}$ and $C_0 \equiv 8_{33}$ modulo $8_{34}$. We set $E_1(0) = E_1$ and $C_0(0) = C_0$ since $Sq^0(E_1) = 8_{140}$ and $Sq^0(C_0) = 8_{130} + 8_{140} + 8_{141} + 8_{142}$ are indecomposable.

(15) Bidegree $(8, 8 + 68)$ is spanned by the indecomposable $8_{43}$ and the decomposable $8_{44} = h_0h_3A_0'$. Hence, the indecomposable $G_{21}$ must be $8_{43}$ modulo $8_{44}$.

(16) This is the unique element in the bracket $(h_3, h_0^4, -)$.

(17) The May spectral sequence definition of $B_4 = B_4(0)$ has indeterminacy spanned by $9_{30} = h_0^3B_3$, so that $B_4(0)$ must be the indecomposable $9_{29}$ modulo $9_{30}$.

(18) Bidegree $(9, 9 + 67)$ is spanned by the two indecomposables $9_{39}$ and $9_{40}$. We have $h_09_{39} = 0$, $h_09_{40} = 10_{37}$, $h_09_{39} = 10_{41}$ and $h_02_{30} = 0$. In the May spectral sequence, there are indecomposables $C''$ and $X_3$. The relations reported by Tangora \cite{19} have $h_0X_3 \neq 0$ and $h_2C'' \neq 0$. It follows that $C'' \equiv 9_{39}$ modulo $9_{30}$ and $X_3 \equiv 9_{40}$ modulo $9_{39}$, but, of course, $C'' \neq X_3$. Since $Sq^0(C'')$ is indecomposable, we write $C'' = C''(0)$. We do not know whether $Sq^0(X_3)$ is indecomposable; if it is we should set $X_3 = X_3(0)$ and note that $C''(1) = X_3(1)$ modulo decomposables.

(19) Bidegree $(10, 10 + 54)$ is spanned by the indecomposable $10_{19}$ and the decomposable $10_{20} = h_0^2h_{5i}$. While Tangora’s calculation has this bidegree spanned by $h_0^2h_{5i}$ and $R_1$. Hence, $R_1$ must be $10_{19}$ modulo $10_{20}$. 

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(20) Bidegree (10, 10 + 56) is spanned by the indecomposable 10_{22} and the decomposable 10_{21} = q_1 t_0, while Tangora’s calculation has this bidegree spanned by gt and Q_1. Hence, Q_1 must be 10_{22} modulo 10_{21}.

(21) Bidegree (10, 10 + 62) is spanned by the decomposable 10_{29} = h_1 X_1 = PG together with the indecomposables 10_{27} and 10_{28}. In the May spectral sequence it is spanned by R, B_{22} and PG, in order of May filtration, so that B_{22} is defined modulo PG, while R is only defined modulo the other two. The relation h_0 B_{22} = d_0 B_2 holds in the May spectral sequence by [19]. Since h_0 10_{28} = d_0 B_2, while h_0 10_{27} is not divisible by d_0 in Ext, B_{22} must be 10_{28} modulo 10_{29}. Since R is linearly independent of B_{22} and PG, it must be 10_{27} modulo (10_{28}, 10_{29}).

(22) Tangora’s q_1 in (10, 10 + 64) is not Sq^0(g), which is instead the decomposable 6_{46} = Sq^0(6_{12}) = h_2 Q_3(0). The May spectral sequence definition of q_1 has indeterminacy 10_{33} = h_2 h_3 Q_2(0) = h_2 E_1(0) so that q_1 is 10_{32} modulo 10_{33}.

(23) Bidegree (10, 10 + 66) is spanned by the two indecomposables 10_{35} and 10_{36}. We have h_0 (10_{35} + 10_{36}) = h_1 B_{23}(0) = h_2 B_4(0), while h_3 10_{35} = h_4 10_{36} = 12_{22} = h_7 q_1. In the May spectral sequence, there are indecomposables D_2 and B_5. The relations reported by Tangora [19] include h_0 B_5 = h_1 B_{23} = h_2 B_4, which require B_5 = 10_{35} + 10_{36}. We then have D_2 = 10_{35} or 10_{36}. In any case (D_2, B_5) = (10_{35}, 10_{36}).

(24) B_{22}(1) = Sq^0(B_{23}(0)) = 10_{196} + 10_{197} + 10_{199} and 10_{199} = h_1 q_{178} = h_1 q_{179}. Both 10_{196} and 10_{197} are indecomposable.

(25) Bidegree (11, 11 + 67) is spanned by the indecomposable 11_{35} and 11_{36} = h_0^2 X_3 = h_0 h_3 B_4(0) = i g_2 = r_0 x_0. Hence, the indecomposable C_{11} must be 11_{35} modulo 11_{36}.

(26) Bidegree (13, 13 + 47) is spanned by the indecomposables 13_{14} and 13_{15}, while Tangora’s calculation has this bidegree spanned by Pu and Q with h_1 Q \neq 0. The brackets file shows that Pu = [h_3, h_0^8, u] = 13_{15}. Since h_1 13_{14} = h_1 13_{15}, we must have Q = 13_{14}. Tangora defines Q’ = Q + Pu.

(27) Bidegree (13, 13 + 65) is spanned by the indecomposable 13_{29} and the decomposable 13_{28} = g_1 w = r_0 m, while Tangora’s calculation has this bidegree spanned by gw and R_2. Hence, R_2 must be 13_{29} modulo 13_{28}.

(28) Bidegree (13, 13 + 68) is spanned by the indecomposable 13_{30} and the (highly) decomposable 13_{31} = h_0^6 G_{21} = h_0 h_5 d_0 i. Hence, the indecomposable G_{11} must be 13_{30} modulo 13_{31}. This coset is M’i since 13_{31} is in the indeterminacy of M’i.

(29) Bidegree (17, 17 + 65) is spanned by the indecomposable 17_{26} and the decomposable 17_{27} = d_0^2 v, while Tangora’s calculation has this bidegree spanned by Pc_0 w = d_0^2 v and R_1’. Hence, R_1’ must be 17_{26} modulo 17_{27}.

(30) Bidegree (18, 18 + 69) is spanned by the indecomposable 18_{20} together with 18_{31} = P(gz). Tangora writes P^2 x’ = [h_4, h_0^8, x’] for an indecomposable in this bidegree, but this is not defined as a Toda bracket, since h_0^8 x’ \neq 0.

(31) By Adams periodicity P^k h_1 = (4k + 1)_{11}, P^k h_2 = (4k + 1)_{12}, P^k c_0 = (4k + 3)_{13}, P^k d_0 = (4k + 4)_{14} and P^k e_0 = (4k + 4)_{15}. Likewise, P^k i = (4k + 7)_{16}, P^k j = (4k + 7)_{18} and P^k u = (4k + 9)_{19} for k even, P^k j = (4k + 7)_{17} and P^k u = (4k + 9)_{19} for k odd, P^k v = (4k + 9)_{16} for k \equiv 1 \mod 4 and P^k v = (4k + 9)_{17} for k \equiv 3 \mod 4.
9. Operators

In this section, we describe the files $\text{P}.\text{txt}$, $\text{P2}.\text{txt}$, $\text{P4}.\text{txt}$ and $\text{MM}.\text{txt}$, in which we collect the data needed to compute the Adams periodicity operators $P^k$ for $k = 1, 2, 4$, and the Mahowald operator $M$, then make some general remarks about such operators.

9.1. Adams and Mahowald operators. These are defined by

1. $P^1 x = \langle h_3, h_0^4, x \rangle$,
2. $P^2 x = \langle h_4, h_0^8, x \rangle$,
3. $P^4 x = \langle h_5, h_0^{16}, x \rangle$ and
4. $M x = \langle g_2, h_0^3, x \rangle$.

The Mahowald operator $M x = \langle g_2, h_0^3, x \rangle$ is not of the form that it is immediately evident in the brackets.sym files, but its variant $M'(x) = \langle h_0 g_2, h_0^2 g_2, x \rangle$ is. Both contain $\langle h_0 g_2, h_0^2 g_2, x \rangle$, when defined.

In Figure 4, the lighter gray region with $t > 200$ and $t - s \leq 200$ shows a range of bidegrees in which $\text{Ext}^s_t(F_2, F_2)$ is determined by Adams periodicity and our calculations for $t \leq 200$. The darker gray region shows bidegrees where further calculation would be required to determine these groups.

Recall from Proposition 5.1 that, if defined, the bracket $\langle h_i, (s_0)_{g_0}, (s_1)_{g_1} \rangle$ is the sum of those $s_g$ with $s = s_0 + s_1$ such that the file $s_1 g_1/\text{brackets}.\text{sym}$ contains a line $s_g \text{ in } < h_i, g_0, s_1 g_1 >$

Thus, the values of the operators $P$, $P^2$, $P^4$ and $M'$ are recognized by the presence of lines

1. $s_g \text{ in } < h_3, 0, s_1 g_1 >$
2. $s_g \text{ in } < h_4, 0, s_1 g_1 >$
3. $s_g \text{ in } < h_5, 0, s_1 g_1 >$
4. $s_g \text{ in } < h_0, 21, s_1 g_1 >$

with $s = s_1 + 4$, $s = s_1 + 8$, $s = s_1 + 16$ or $s = s_1 + 6$, respectively. For $M'$, we note that $6_{g_2} = h_0^2 g_2$.

The files $\text{P}.\text{txt}$, $\text{P2}.\text{txt}$, $\text{P4}.\text{txt}$ and $\text{MM}.\text{txt}$ collect this information from all the map files, together with information about products needed in determining the domain of definition and the indeterminacy. Each file starts with a short header describing the operator and the file’s organization, then has three sections:

(a): values of the brackets,
(b): nonzero products which obstruct existence of the bracket, and
(c): nonzero products which give the indeterminacy.

In general, the indeterminacy in a bracket $\langle a, b, c \rangle$ is $a(\text{Ext}) + (\text{Ext})c$. However, the brackets $P$, $P^2$, $P^4$ and $M'$ have indeterminacy $a(\text{Ext})$. For the periodicity operators, this is because $\text{Ext}^4_{12}$, $\text{Ext}^8_{24}$ and $\text{Ext}^{16}_{48}$ are zero, so that $(\text{Ext})c = 0$ in the relevant bidegree. For $M' = \langle h_0, h_0^2 g_2, - \rangle$, it is because $\text{Ext}^{6}_{51} = \langle h_0 h_5 d_0 \rangle$, so that $(\text{Ext})c$ is contained in $a(\text{Ext}) = h_0(\text{Ext})$ in the relevant bidegree.

For example, the file $\text{P}.\text{txt}$ starts

\% Adams operator $P$, in the range $t \leq 200$
\% (a) Brackets $P^1 x = < h_3, h_0^4, x >$
\% (b) Nonzero products $h_0^4 \star x$, obstructing existence.
% (c) Nonzero products $h_3 \ast y$, giving indeterminacy.

% (a) Brackets $P_x = \langle h_3, h_0^4, x \rangle$.

5_1 in $\langle h_3, 0, 1_1 \rangle$
5_2 in $\langle h_3, 0, 1_2 \rangle$
5_5 in $\langle h_3, 0, 1_3 \rangle$
6_1 in $\langle h_3, 0, 2_1 \rangle$

From the first three lines in part (a) we see that, if they are defined, $P_{h_1} = 5_1$, $P_{h_2} = 5_2$ and $P_{h_3} = 5_5 = h_0^4 h_4$, modulo their indeterminacy. To determine whether they are defined, we look at part (b), which starts

% (b) Nonzero products $h_0^4 \ast x$, obstructing existence.

| 5 | 0 ( 4 0 F2) 1_0 |
| 5 | 5 ( 4 0 F2) 1_4 |
| 5 | 14 ( 4 0 F2) 1_5 |
| 5 | 35 ( 4 0 F2) 1_6 |
| 5 | 79 ( 4 0 F2) 1_7 |

We see that there are no lines ending in $1_1$, $1_2$ or $1_3$, so that $h_0^4$ annihilates all three of these. Thus $P_{h_1}$, $P_{h_2}$ and $P_{h_3}$ are all defined, but $P_{h_0}$, $P_{h_4}$, ... are not.

Next, we consider the indeterminacy, which equals the $h_3$ multiples in the bidegree of the bracket. The bidegrees in question are $\text{Ext}^{5,5+9} = \langle 5_1 \rangle$, $\text{Ext}^{5,5+11} = \langle 5_2 \rangle$ and $\text{Ext}^{5,5+15} = \langle 5_4, 5_5 \rangle$. We look at part (c) to see the $h_3$ multiples among these. We find

% (c) Nonzero products $h_3 \ast y$, giving indeterminacy.

| 4 | 85 ( 1 3 F2) 3_54 |
| 5 | 11 ( 1 3 F2) 4_5 |
| 5 | 18 ( 1 3 F2) 4_12 |
| 5 | 19 ( 1 3 F2) 4_13 |

Since we do not find $5_1$, $5_2$, $5_4$ or $5_5$ among the $h_3$-multiples, the indeterminacy is 0.

For an example of a possible bracket which is, in fact, undefined, consider the entry

8_91 in $\langle h_3, 0, 4_{48} \rangle$

in part (a). This does not mean that $P(4_{48}) = 8_{91}$ because in part (b) we find that $h_0^4 \cdot 4_{48} = 8_{77} \neq 0$:

8 77 ( 4 0 F2) 4_48

Finally, for an example with nontrivial indeterminacy, in part (a) we find an entry

6_5 in $\langle h_3, 0, 2_5 \rangle$

so that $6_5 \in P(2_5)$, but in part (c) we find

6 5 ( 1 3 F2) 5_1
showing that $6_5 = h_3 \cdot 5_1$ is in the indeterminacy. Hence $P(2_5) = \{0, 6_5\}$. (We have $6_5 = h_3^2d_0 = h_3P h_1 = c_3^2$.)

9.2. **General remarks.** We finish this section with some observations about operators like those we have just considered. Defining $P_x = (b_0^2)^2 x$ in the May spectral sequence is justified by the differential $d_4(b_0^2) = h_0^4h_3$ ([19, Prop. 4.3]). However, this accounts for only part of the definition of a Massey product or Toda bracket. It is simplest to discuss this in terms of Massey products in a commutative differential graded algebra over $\mathbb{F}_2$ like the $E_r$ terms of the May spectral sequence.

Consider classes $a$, $b$ and $x$ satisfying $ab = bx = xa = 0$. The Jacobi identity says that

$$0 \in \langle a, b, x \rangle + \langle b, x, a \rangle + \langle x, a, b \rangle.$$ 

If we choose $A$, $U$ and $V$ such that $d(A) = ab$, $d(U) = bx$ and $d(V) = xa = ax$, then we have

1. $Ax + aU \in \langle a, b, x \rangle$,
2. $Ua + bV \in \langle b, x, a \rangle$ and
3. $Ax + bV \in \langle b, a, x \rangle = \langle x, a, b \rangle$.

Approximating the bracket $\langle a, b, x \rangle$ by $Ax$, in those cases where $Ax$ is a cycle, fails to distinguish between $\langle a, b, x \rangle$ and $\langle b, a, x \rangle = \langle x, a, b \rangle$. These differ by $\langle b, x, a \rangle = \langle a, x, b \rangle$.

This can lead to greater indeterminacy and to anomalies like Tangora’s observation [19, Note 3, p. 48] that $h_5i$ is annihilated by $h_0^3$, but $P(h_5i)$, if defined to be $(b_0^2)^2h_5i$, has $h_0^3 \cdot (b_0^2)^2h_5i \neq 0$. In fact, consulting $P$.txt we see that $P(h_5i) = P(8_26) = 0$ with zero indeterminacy.

By using only the precisely defined brackets we limit the indeterminacy and get the advantages of their good formal behavior.

Finally, let us point out two other operators which may be of use. They are

1. the complex Bott periodicity operator $v_1(x) = \langle h_0, h_1, x \rangle$, which acts on $h_1$-annihilated classes such as the unit in $\text{Ext}_{E(Q_0, Q_1)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_*ku$, and
2. the mod 2 complex Bott periodicity operator $v_1'(x) = \langle h_1, h_0, x \rangle$, which acts on $h_0$-annihilated classes.

In the universal example, the Adams spectral sequence for $\pi_*(S \cup_2 e^1 \cup_1 e^2)$, $v_1(0_0)$ is an $h_1$-annihilated class which supports an infinite $h_0$-tower, while $v_1'(0_0)$ is an $h_0$-annihilated class which supports $h_1^2$-multiplication. These are visible in the brackets.sym files in the form

```
s_g in < h0, 1, s1_g1 >
```

and

```
s_g in < h1, 0, s1_g1 >
```

respectively.

10. **Validity**

Several checks have been run to test the validity of the data.

1. After the resolution was computed, a separate computation was done to check that $d^2 = 0$.

---

5 An idealistic treatment would instead consider Toda brackets of chain maps, or Massey products in $\text{End}(C_\ast)$, which is homotopy commutative but not commutative.
(2) A check of the $\mathbb{F}_2$-dimension of the kernel and image at each step, to ensure that the image at each step has the same dimension as the kernel at the previous step. This checks exactness when combined with the check that $d^2 = 0$.

(3) A check that the Map files are complete. If this were not true, an element $s_g$ missing from $s_1g1/Map$ might not be reported in all.products as a term in a product $(s_0)_{g_0} \cdot (s_1)_{g_1}$ even when it belongs there.

(4) A check that the Map files do define chain maps, i.e., that the maps $m$ they specify do satisfy $dm = md$.

11. Machine processing of the data

Modern computer languages are adept at processing text. Nonetheless, the raw data which is used to produce all.products and brackets.sym is provided in the files $s_g/Map.aug$ and $s_g/brackets$, since this raw data may be easier to process by a computer program. Examples should suffice to make clear the translation.

The entry which is reported as

```
2 4 ( 1 3 \text{F2}) 1_0
```

in all.products is derived from the line in $1_0/Map.aug$ which says

```
2 \ 4 \ 3
```

as the rest of the data can be deduced from the map $1_0$.

Similarly, the entries

```
2_8 \ in \ < \ h4, \ 0, \ 1_0 >
2_1 \ in \ < \ h0, \ 1, \ 1_0 >
2_5 \ in \ < \ h3, \ 0, \ 1_0 >
```

in $1_0/brackets.sym$ are derived from the lines in $1_0/brackets$ which say

```
2 8 16 0
2 1 1 1
2 5 8 0
```

Here, the third entries in each line, 16, 1 and 8, are the internal degrees of the elements $h_4$, $h_0$ and $h_3$.

Finally, the entries in $Sq0/Map.aug$ have the form

```
2 1 0
2 3 1
2 5 2
```

meaning that $Sq^0(2_0)$ contains $2_1$, that $Sq^0(2_1)$ contains $2_3$, and that $Sq^0(2_2)$ contains $2_5$. 
Figure 4. $\text{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for $t \leq 200$.
References

[1] J. F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math. (2) 72 (1960), 20–104.
[2] Alejandro Adem, Jon Carlson, Stewart Priddy, and Peter Webb (eds.), Group representations: cohomology, group actions and topology, Proceedings of Symposia in Pure Mathematics, vol. 63, American Mathematical Society, Providence, RI, 1998.
[3] Robert R. Bruner, Calculation of large Ext modules, Computers in geometry and topology (Chicago, IL, 1986), Lecture Notes in Pure and Appl. Math., vol. 114, Dekker, New York, 1989, pp. 79–104.
[4] Robert R. Bruner, Ext in the nineties, Algebraic topology (Oaxtepec, 1991), Contemp. Math., vol. 146, Amer. Math. Soc., Providence, RI, 1993, pp. 71–90.
[5] Robert R. Bruner, The cohomology of the mod 2 Steenrod algebra: A computer calculation, Research Reports, vol. 37, Wayne State University, 1997.
[6] Robert R. Bruner, Some root invariants and Steenrod operations in Ext_{A}(F_2, F_2), Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), Contemp. Math., vol. 220, Amer. Math. Soc., Providence, RI, 1998, pp. 27–33.
[7] Robert Bruner, Christian Nassau, and Sean Tilson, Steenrod operations and A-module extensions, arXiv:1909.03115v3.
[8] Robert Bruner and John Rognes, The Adams Spectral Sequence for Topological Modular Forms, Mathematical Surveys and Monographs, vol. 253, American Mathematical Society, Providence, RI, 2021.
[9] Robert R. Bruner and John Rognes, The cohomology of the mod 2 Steenrod algebra (2021), https://doi.org/10.11582/2021.00077 [Dataset]. Norstore.
[10] Tai-Wei Chen, Determination of Ext^{5,*}_{A}(Z/2, Z/2), Topology Appl. 158 (2011), no. 5, 660–689.
[11] Tai-Wei Chen, The structure of decomposable elements in Ext^{6,*}_{A}(Z/2, Z/2) (2012), preprint.
[12] Daniel C. Isaksen, Stable stems, Mem. Amer. Math. Soc. 262 (2019), no. 1269, viii+159.
[13] Daniel C. Isaksen, The Mahowald operator in the cohomology of the Steenrod algebra (2020). arXiv:2001.01758.
[14] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu, More stable stems (2020). arXiv:2001.04511.
[15] Wen-Hsiung Lin, Ext^{4,*}_{A}(Z/2, Z/2) and Ext^{5,*}_{A}(Z/2, Z/2), Topology Appl. 155 (2008), no. 5, 459–496.
[16] Harvey Margolis, Stewart Priddy, and Martin Tangora, Another systematic phenomenon in the cohomology of the Steenrod algebra, Topology 10 (1970), 43–46.
[17] J. Peter May, A general algebraic approach to Steenrod operations, The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod’s Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970), Lecture Notes in Mathematics, Vol. 168, Springer, Berlin, 1970, pp. 153–231.
[18] J. H. Palmieri, The Lambda algebra and Sq^0, Proceedings of the School and Conference in Algebraic Topology, Geom. Topol. Monogr., vol. 11, Geom. Topol. Publ., Coventry, 2007, pp. 201–216.
[19] Martin C. Tangora, On the cohomology of the Steenrod algebra, Math. Z. 116 (1970), 18–64.
[20] John S. P. Wang, On the cohomology of the mod-2 Steenrod algebra and the non-existence of elements of Hopf invariant one, Illinois J. Math. 11 (1967), 480–490.

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