CHARACTER SUMS OVER UNIONS OF INTERVALS

XUANCHENG SHAO

Abstract. Let $q$ be a cube-free positive integer and $\chi \pmod{q}$ be a non-principal Dirichlet character. Our main result is a Burgess-type estimate for $\sum_{n \in A} \chi(n)$, where $A \subset [1, q]$ is the union of $s$ disjoint intervals $I_1, \ldots, I_s$. We obtain a nontrivial estimate for the character sum over $A$ whenever $|A|s^{-1/2} > q^{1/4+\epsilon}$ and each interval $I_j$ ($1 \leq j \leq s$) has length $|I_j| > q^\epsilon$ for any $\epsilon > 0$. This follows from an improvement of a mean value Burgess-type estimate studied by Heath-Brown [6].

1. Introduction

Let $\chi \pmod{q}$ be a primitive Dirichlet character. Write

$$S(N; H) = \sum_{N < n \leq N+H} \chi(n).$$

The well-known estimates of Burgess [1, 2, 3] say that for any positive integer $r \geq 2$ and any positive real $\epsilon > 0$,

$$S(N; H) \ll_{\epsilon, r} q^{1/2+1/(2r)+\epsilon} H^{2r-2},$$

uniformly in $N$, providing that either $q$ is cube-free or $r \leq 3$. In particular, this gives a nontrivial estimate for $S(N; H)$ as long as $H > q^{1/4+\epsilon}$ for any $\epsilon > 0$. Under the assumption of the Generalized Riemann Hypothesis (GRH), nontrivial estimates for $S(N; H)$ can be obtained in the much wider region $H > q^\epsilon$ for any $\epsilon > 0$. However, Burgess’s bound remains the best unconditional result for around 50 years.

Our main purpose is to prove the following mean value Burgess-type estimate, which includes the Burgess bound as a special case.

**Theorem 1.1.** Let $r$ be a positive integer and $\epsilon > 0$ be real. Suppose that $\chi \pmod{q}$ is a primitive Dirichlet character, and let $H \leq q$ be a positive integer. Let $0 \leq N_1 < N_2 < \cdots < N_J < q$ be integers with the spacing condition

$$N_{j+1} - N_j \geq H.$$

Then

$$\sum_{j=1}^{J} \max_{h \leq H} |S(N_j; h)|^{2r} \ll_{\epsilon, r} q^{1/2+1/(2r)+\epsilon} H^{2r-2}$$

under any one of the following three conditions

(a) $r = 1$;
(b) \( r \leq 3 \) and \( H \geq q^{1/(2r)} \);
(c) \( q \) is cube-free and \( H \geq q^{1/(2r)} \).

This question has been investigated recently by Heath-Brown [6], where the following inequality is obtained:

\[
\sum_{j=1}^{J} \max_{h \leq H} |S(N_j; h)|^{2r} \ll_{\epsilon, r} q^{3/4+3/(4r)+\epsilon} H^{3r-3},
\]

under the same assumptions as those in Theorem 1.1. Heath-Brown attained \( 2r \) when \( r = 1 \), and asked the question of whether the exponent \( 3r \) in (2) can be replaced by \( 2r \). Theorem 1.1 is then an affirmative answer to this question.

This improvement from \( 3r \) to \( 2r \) has a major consequence. The left sides of the inequalities (2) and (1) have trivial bounds \( JH^{3r} \) and \( JH^{2r} \), respectively. Some simple algebra then reveals that (2) is nontrivial for large \( r \) when \( HJ^{1/3} > q^{1/4+\epsilon} \) for any \( \epsilon > 0 \), whereas (1) is nontrivial for large \( r \) when \( HJ^{1/2} > q^{1/4+\epsilon} \) for any \( \epsilon > 0 \).

Theorem 1.1 easily implies a Burgess-type bound for character sums over unions of intervals.

**Corollary 1.2.** Let \( \epsilon > 0 \) be real. Suppose that \( q \) is cube-free and let \( \chi \mod q \) be a primitive Dirichlet character. Let \( A \subset [1, q] \) be a union of \( s \) disjoint intervals \( I_1, \ldots, I_s \), each of which has length at least \( q^{\epsilon} \). Suppose that \( |A|s^{-1/2} > q^{1/4+\epsilon} \). Then there exists \( \delta = \delta(\epsilon) > 0 \) such that

\[
\sum_{n \in A} \chi(n) \ll \epsilon |A|q^{-\delta}.
\]

This problem of getting nontrivial bounds for character sums over unions of intervals has been considered before. Friedlander and Iwaniec [5] proved (3) but with an additional assumption that \( A \) is contained in a relatively short interval. The result of Chang [4] also gives (3) under a stronger hypothesis on \( |A| \) and \( s \). Chang’s method is different, though, and can be also used to treat character sums over generalized arithmetic progressions. Finally, note that Heath-Brown’s result (2) would lead to (3) assuming \( |A|s^{-2/3} > q^{1/4+\epsilon} \). Heuristically, square-root cancelation for the character sum is expected over the \( s \) intervals, and thus the set \( A \) can be thought of having an effective length \( |A|s^{-1/2} \). This heuristic suggests that Corollary 1.2 might be best possible (at least unconditionally) under current techniques, as any improvement may also lead to an improvement of Burgess’s inequality.

The proof of Theorem 1.1 follows the initial argument of Heath-Brown [6], where a variant of Burgess’s method was used to convert the problem to a diophantine problem. We refer the reader to Section 3 of [6] for this process. The improvement is a consequence of a better bound for this diophantine problem, as described below.

**Proposition 1.3.** Let \( \ell \) be a prime and \( S \) be a subset of \( \mathbb{F}_\ell \). Let \( n \) be a positive integer. Denote by \( N(\ell, S, n) \) the number of solutions to the congruence as \( -bt \equiv c \mod \ell \) with...
1 ≤ a, b ≤ n, \(|c| ≤ n\), and \(s, t \in S\). Then
\[
N(ℓ, S, n) ≪_ε ℓ^{-1}n^3|S|^2 + ℓ^εn^2|S|
\]
for any \(ε > 0\).

In Section 4 of [6], Heath-Brown obtained the bound
\[
N(ℓ, S, n) ≪ ℓ^{-1}n^3|S|^2 + |S|^2 + n^2|S|^{4/3}\log ℓ
\]
using the theory of lattices. The proof of Proposition 1.3 uses Fourier analysis instead. Note that, in the bound (4), the first term \(ℓ^{-1}n^3|S|^2\) is the expected main term, and the second term \(ℓ^εn^2|S|\) is also sharp (apart from the \(ℓ^ε\) factor). This can be seen by taking \(S\) to be a set of consecutive integers starting from 1.

Throughout the paper the implied constants in the \(O()\) and \(≪\) notations are always allowed to depend on \(r\) and \(ε\). The parameter \(ε\) appearing in different places are allowed to differ.

For a prime \(ℓ\) and a function \(f : \mathbb{F}_ℓ → \mathbb{R}\), we define its Fourier transform to be
\[
\hat{f}(r) = \sum_{x ∈ \mathbb{F}_ℓ} f(x)e(ℓxr)
\]
for \(r ∈ \mathbb{F}_ℓ\), where \(e(ℓxr) = \exp(2πixr/ℓ)\). For an \(L^1\) function \(f : \mathbb{R} → \mathbb{R}\), we define its Fourier transform to be
\[
\hat{f}(y) = \int_\mathbb{R} f(x)e(xy)dx,
\]
where \(e(xy) = \exp(2πixy)\).

The rest of this article is organized as follows. Heath-Brown’s argument, which converts the problem of bounding character sums to the problem of bounding the number of solutions to a certain congruence equation, is summarized in Section 2. Proposition 1.3 is proved in Section 3. Theorem 1.1 is then deduced from Proposition 1.3 in Section 4 and finally Corollary 1.2 is obtained in Section 5.

2. Preparations

In this section, we summarize Heath-Brown’s work, setting up the foundation for the proof of Theorem 1.1.

**Proposition 2.1.** Let \(r ≥ 2\) be a positive integer and \(ε > 0\) be real. Suppose that \(χ \pmod q\) is a primitive Dirichlet character, and let \(H ∈ (q^{1/(2r)}, q]\) be a positive integer. Assume that either \(q\) is cube-free or \(r ≤ 3\). Let \(0 ≤ N_1 < N_2 < \cdots < N_J < q\) be integers with the spacing condition
\[
N_{j+1} - N_j ≥ H.
\]
Let \(ℓ ∈ (q/H, 2q/H]\) be a prime. Let \(P\) be a parameter satisfying \(2HQ^{-1/(2r)} ≤ P ≪ HQ^{-1/(2r)}\). Then for some subset \(S ⊂ \mathbb{F}_ℓ\) with \(|S| = J\),
\[
\sum_{j=1}^{J} \max_{h ≤ H} |S(N_j; h)|^r ≪ q^{1/4 + 3/(4r) + ε}H^{r-2}(PJ^{1/2} + N(ℓ, S, 12P)^{1/2}),
\]
where $N(\ell, S, 12P)$ is defined as in the statement of Proposition 1.3.

Proof. We outline the arguments from [6]. Using a variant of Burgess’s method Heath-Brown obtained (equation (7) of [6])

$$\sum_{j=1}^{J} \max_{h \leq H} |S(N_j; h)|^r \ll q^{1/4+3/(4r)+\epsilon} H^{r-2} M^{1/2},$$

where $M$ is the number of tuples $(a_1, a_2, p_1, p_2, N_j, N_k)$ with $p_1, p_2$ primes in $(P, 2P]$, $0 \leq a_1 < p_1$, $0 \leq a_2 < p_2$, and

$$|(N_j - a_1q)/p_1 - (N_k - a_2q)/p_2| \leq H/P.$$

In the beginning of Section 4 of [6], Heath-Brown started to bound $M$ as follows. Split $M$ as $M_1 + M_2$, where $M_1$ counts solutions with $p_1 = p_2$ and $M_2$ counts those with $p_1 \neq p_2$. It can be easily seen that (equation (8) of [6])

$$M_1 \ll P^2 J.$$

To bound $M_2$, set $M_j = \lfloor N_j \ell/q \rfloor$ for each $1 \leq j \leq J$. The spacing condition on $N_j$ implies that

$$0 \leq M_1 < M_2 < \cdots < M_J < \ell.$$

Let $S = \{M_1, M_2, \cdots, M_J\} \subset \mathbb{Z}_\ell$. It is then shown that (the first displayed equation on page 11 of [6])

$$M_2 \leq \sum_{M_j, M_k} \# \{(p_1, p_2, m) : |m| \leq 12P, p_2 M_j - p_1 M_k \equiv m \pmod{\ell}\}.$$ 

The right side above is bounded above by $N(\ell, S, 12P)$. Hence

$$M = M_1 + M_2 \ll P^2 J + N(\ell, S, 12P).$$

Combining this with (5) completes the proof.

Heath-Brown’s subsequent estimates take advantage of the fact that $p_1$ and $p_2$ are primes; however, our method is insensitive to this.

We will also need the following Pólya-Vinogradov-type mean value estimate.

**Lemma 2.2.** Let $r$ be a positive integer and $\epsilon > 0$ be real. Suppose that $\chi \pmod{q}$ is a primitive Dirichlet character, and let $H \leq q$ be a positive integer. Let $0 \leq N_1 < N_2 < \cdots < N_J < q$ be integers with the spacing condition

$$N_{j+1} - N_j \geq H.$$

Then

$$\sum_{j=1}^{J} \max_{h \leq H} |S(N_j; h)|^2 \ll (\log q)^2$$
for all \( q \), and
\[
\sum_{j=1}^{J} \max_{h \leq H} |S(N_j; h)|^{2r} \ll q^r(qH^{r-1} + q^{1/2}H^{2r-1})
\]
under the assumption that either \( 2 \leq r \leq 3 \) or \( q \) is cube-free.

The \( r = 1 \) case is Lemma 4 in [6]. The proof there, however, works for arbitrary \( r \).

3. PROOF OF PROPOSITION 1.3

Recall that \( N(\ell, S, n) \) is the number of solutions to
\[
as - bt \equiv c \pmod{\ell}
\]
with \( 1 \leq a, b \leq n, |c| \leq n \), and \( s, t \in S \). If \( \ell \leq n \), then \( N(\ell, S, n) \) can be bounded easily as follows. For any fixed choice of \( a, b, s, t \), there are at most \( \lceil n/\ell \rceil \) choices for \( c \). Therefore
\[
N(\ell, S, n) \ll \ell^{-1}n^3|S|^2
\]
as desired. Henceforth assume that \( \ell > n \).

To facilitate the argument we introduce a smooth cutoff \( \phi : \mathbb{R} \to \mathbb{R}^+ \) satisfying the following properties:

1. \( \phi(x) \geq 0 \) for every \( x \in \mathbb{R} \) and \( \phi(x) \) is bounded away from 0 for \( x \in [-1, 1] \);
2. \( \hat{\phi} \) is supported in the interval \( [-1/10, 1/10] \) and \( |\hat{\phi}(y)| \leq 1 \) for every \( y \in \mathbb{R} \).

Such a function \( \phi \) can be easily constructed, for example, by taking \( \phi(x) = (\sin(tx)/tx)^2 \) for some appropriate \( t > 0 \).

We denote by \( S \) the characteristic function of the set \( S \). Since \( \phi(x) \gg 1 \) for \( x \in [-1, 1] \) and \( \phi(x) \geq 0 \) for all \( x \), we have the bound
\[
T \leq \sum_{1 \leq a, b \leq n} \sum_{s, t \in \mathbb{F}_\ell} \sum_{c \in \mathbb{Z}} S(s)S(t)\phi(c/n).
\]
Then by orthogonality of additive characters,
\[
T \leq \frac{1}{\ell} \sum_{1 \leq a, b \leq n} \sum_{s, t \in \mathbb{F}_\ell} \sum_{c \in \mathbb{Z}} S(s)S(t)\phi(c/n) \sum_{r \in \mathbb{F}_\ell} e_\ell(r(as - bt - c)).
\]
Changing the order of summation we get
\[
T \leq \frac{1}{\ell} \sum_{|r| \leq \ell/2} \sum_{1 \leq a, b \leq n} \hat{S}(ar)\hat{S}(-br) \sum_{c \in \mathbb{Z}} \phi(c/n)e_\ell(-cr).
\]
By Poisson summation,
\[
\sum_{c \in \mathbb{Z}} \phi(c/n)e_\ell(-cr) = n \sum_{k \in \mathbb{Z}} \hat{\phi} \left( n \left( k - \frac{r}{\ell} \right) \right).
\]
Since \( \hat{\phi} \) is compactly supported in \([-1/10, 1/10]\), the summand on the right above vanishes unless \( k = 0 \) and \( |r| \leq \ell/5n \). Hence

\[
T \leq \frac{n}{\ell} \sum_{|r| \leq \ell/5n} \sum_{1 \leq a, b \leq n} |\hat{S}(ar)\hat{S}(-br)\hat{\phi}(-nr/\ell)| \ll \frac{n^3|S|^2}{\ell} + \frac{n}{\ell} \sum_{0<|r| \leq \ell/5n} \sum_{1 \leq a, b \leq n} |\hat{S}(ar)\hat{S}(-br)|
\]

since \( |\hat{\phi}(y)| \leq 1 \) for all \( y \). It follows from the inequality

\[
2|\hat{S}(ar)\hat{S}(-br)| \leq |\hat{S}(ar)|^2 + |\hat{S}(-br)|^2
\]

that

\[
T \ll \frac{n^3|S|^2}{\ell} + \frac{n^2}{\ell} \sum_{0<|r| \leq \ell/5n} \sum_{1 \leq a \leq n} |\hat{S}(ar)|^2.
\]

Note that for any fixed nonzero \( s \in \mathbb{F}_\ell \), there are \( O(\ell^r) \) ways to write \( s \equiv ar \pmod{\ell} \) with \( 1 \leq |a| \leq n \) and \( 0 < |r| \leq \ell/5n \). Hence

\[
\sum_{0<|r| \leq \ell/5n} \sum_{1 \leq a \leq n} |\hat{S}(ar)|^2 \ll \ell^r \sum_{s \in \mathbb{F}_\ell} |\hat{S}(s)|^2 \ll \ell^{1+r}|S|
\]

by Parseval’s identity. This gives the desired bound

\[
T \ll \frac{n^3|S|^2}{\ell} + \ell^r n^2|S|,
\]

completing the proof of Proposition 1.3.

4. Proof of Theorem 1.1

It follows immediately from Propositions 2.1 and 1.3 that

\[
\sum_{j=1}^J \max_{h \leq H} |S(N_j; h)|^r \ll q^{1/4+3/(4r)+\epsilon} H^{r-2} \left(PJ^{1/2} + \frac{P^{3/2}J}{\ell^{1/2}}\right).
\]

Recall that \( P \sim Hq^{-1/2r} \) and \( \ell \sim q/H \). Hence

\[
(6) \quad \sum_{j=1}^J \max_{h \leq H} |S(N_j; h)|^r \ll q^{1/4+1/(4r)+\epsilon} H^{r-1} J^{1/2} + q^{-1/4+\epsilon} H^r J.
\]

Before proving Theorem 1.1 we remark that the bound above is already nontrivial when \( HJ^{1/2} > q^{1/4+\epsilon} \), and is thus sufficient to deduce Corollary 1.2. However, a little more work needs be done to get Theorem 1.1 as stated. The arguments here are analogous to those in Section 5 of [6].

We use induction on \( r \). The \( r = 1 \) case is simply Lemma 2.2. Now assume that \( r \geq 2 \). By a dyadic subdivision, we may assume that

\[
\max_{h \leq H} |S(N_j; h)| \sim V
\]
for some $V$. Hence from (6),

$$J^r V \ll q^{1/4 + 1/(4r) + \epsilon} H^{-1} J^{1/2} + q^{-1/4} H^r J.$$ 

Consider two cases depending on which term on the right side above dominates. If the first term dominates, then, upon moving $J^{1/2}$ to the left, we get

$$J^r V^2 \ll q^{1/2 + 1/(2r) + \epsilon} H^{2r-2},$$

as desired. If the second term dominates, then

$$J^r V \ll q^{-1/4 + \epsilon} H^r J,$$

and thus $V \ll q^{-1/(4r) + \epsilon/r} H$. Divide further into two cases.

If $H > q^{1/(2r-1)}$, then we may use the induction hypothesis with $r-1$ to deduce that

$$J^r V^{2(r-1)} \ll q^{1/2 + 1/(2(r-1)) + \epsilon} H^{2r-4}.$$ 

Hence

$$J^r V^{2r} \ll q^{1/2 + 1/(2r-1) + \epsilon} H^{2r-4} \cdot q^{-1/(2r) + 2\epsilon/r} H^2.$$ 

The desired bound follows from the inequality $1/2(r - 1) - 1/(2r) < 1/(2r)$ when $r \geq 2$.

If $H \leq q^{1/(2r-1)}$, then we use the conclusion of Lemma 2.2,

$$\sum_{j=1}^N \max_{h \leq H} \left| S(N_j; h) \right|^{2(r-1)} \ll q' (qH^{r-2} + q^{1/2} H^{2r-3}) \ll q^{1+\epsilon} H^{r-2},$$

to obtain

$$J^r V^{2r-2} \ll q^{1+\epsilon} H^{r-2}.$$ 

Hence

$$J^r V^{2r} \ll q^{1+\epsilon} H^{r-2} \cdot q^{-1/(2r) + 2\epsilon/r} H^2.$$ 

This again gives the desired bound using the assumption $H > q^{1/(2r)}$.

5. **Proof of Corollary 1.2**

For any $q^\epsilon \leq \ell \leq q/2$, let $A(\ell)$ be the union of those $I_j$ such that $\ell \leq |I_j| \leq 2\ell$. By a dyadic subdivision, it suffices to prove that

$$\sum_{n \in A}(n) \ll |A| q^{-\delta}$$

for any $q^\epsilon \leq \ell \leq q/2$. Assume that

$$A(\ell) = I'_1 \cup \cdots \cup I'_s,$$

where $I'_1, \cdots, I'_s$ ($J \leq s$) are disjoint intervals of length between $\ell$ and $2\ell$. Assume also that $\ell J \geq |A| q^{-\epsilon/2}$; otherwise the bound is trivial. Then

$$\ell J^{1/2} \geq |A| q^{-\epsilon/2} J^{-1/2} \geq |A| s^{-1/2} q^{-\epsilon/2} \geq q^{1/4+\epsilon/2}.$$ 

Write

$$I'_j = (N_j, N_j + L_j)$$
with $\ell \leq L_j \leq 2\ell$. Without loss of generality, assume that $0 \leq N_1 < \cdots < N_J < q$. By the disjointness of the intervals $I'_j$, we have the spacing condition

$$N_{j+2} - N_j \geq 2\ell.$$ 

By Theorem 1.1,

$$\sum_{j=1}^{J} |S(N_j; L_j)|^{2r} \leq \sum_{j=1}^{J} \max_{h \leq 2\ell} |S(N_j; h)|^{2r} + \sum_{j=1}^{J} \max_{h \leq 2\ell} |S(N_j; h)|^{2r} \ll q^{1/2+1/(2r)+\epsilon} \ell^{2r-2}$$

for sufficiently large $r$. It then follows from Hölder’s inequality that

$$\sum_{j=1}^{J} S(N_j; L_j) \ll J^{1-1/(2r)} q^{1/(4r)+1/(4r^2)+\epsilon} \ell^{1-1/r}.$$ 

A simple computation shows that the right side above is $O(\ell Jq^{-\delta})$ provided that

$$\ell J^{1/2} \gg q^{1/4+1/(4r)+\epsilon}.$$ 

This condition indeed holds by (7), if $r$ is chosen large enough. Henceforth

$$\sum_{j=1}^{J} S(N_j; L_j) \ll \ell Jq^{-\delta} \leq |A|q^{-\delta}$$

since $|A| \geq \ell J$, completing the proof of Corollary 1.2.

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