Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation

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Abstract

We prove the existence and uniqueness of a local solution to the periodic renormalized $\Phi_3^4$ model of stochastic quantisation using the method of controlled distributions introduced recently by Imkeller, Gubinelli and Perkowski (“Paraproducts, rough paths and controlled distributions”, arXiv:1210.2684).

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1 Introduction

We study here the following Cauchy problem

\[
\begin{align*}
\partial_t u &= \Delta_{T^3} u - u^3 + \xi \\
\quad u(0, x) &= u^0(x) \quad x \in T^3
\end{align*}
\]  

(1)

where \( \xi \) is a space-time with noise such that \( \int_{T^3} \xi(x)dx = 0 \) i.e. it is a centered Gaussian space-time distribution such that

\[ E[\xi(s, x)\xi(t, y)] = \delta(t-s)\delta(x-y) \]

and \( u : \mathbb{R}_+ \times T^3 \to \mathbb{R} \) is a space-time distribution which is continuous in time. We write this equation in its mild formulation

\[ u = P_t u^0 - \int_0^t P_{t-s}(u_s)^3ds + X_t \]  

(2)

where \( P_t = e^{t\Delta} \) is the Heat flow and \( X_t = \int_0^t P_{t-s}\xi_s ds \) is a the solution of the linear equation :

\[ \partial_t X_t = \Delta_{T^3} X_t + \xi; \quad X_0 = 0. \]  

(3)

Moreover \( X \) is a Gaussian process and as we see below \( X \in C([0,T], C^{-1/2-\varepsilon}(T^3)) \) for every \( \varepsilon > 0 \) with \( C^\alpha = B^\alpha_{\infty,\infty} \) is the Besov-Hölder space. The main difficulty of the equation (1) comes from the fact that for any fixed time \( t \) the space regularity of the solution \( u(t, x) \) cannot be better than the one of \( X_t \). If we measure spatial regularity in the scale of Hölder spaces \( C^\alpha \) we should expect that \( u^3 \) is not well defined. A natural approach to give a well defined meaning to the equation would consist in regularizing the noise in \( \xi^{\varepsilon} = \xi * \rho^\varepsilon \) with \( \rho^\varepsilon = \varepsilon^{-3}\rho(\varepsilon) \) a smooth kernel and taking the limit of the solution \( u^{\varepsilon} \) of the approximate equation

\[ \partial_t u^{\varepsilon} = \Delta u^{\varepsilon} - (u^{\varepsilon})^3 + \xi^{\varepsilon}. \]  

(4)

Since the non-linear term diverges when \( \varepsilon \) goes to zero, an a priori estimate for the wanted solution is hard to find. To overcome this problem we have to focus on the following modified equation

\[ \partial_t u^{\varepsilon} = \Delta u^{\varepsilon} - (u^{\varepsilon})^3 - C^{\varepsilon} u^{\varepsilon} + \xi^{\varepsilon}. \]  

(5)

where \( C^{\varepsilon} > 0 \) is a renormalization constant which diverges when \( \varepsilon \) goes to 0. We will show that we have to take \( C^{\varepsilon} \sim \frac{2}{\varepsilon} + b \log(\varepsilon) + c \) to obtain a non trivial limit for \( (u^{\varepsilon})^2 - C^{\varepsilon} \).

Therefore this paper aims at giving a meaning of the equation (2) and at obtaining a (local in time) solution. The method developed here uses some ideas of [9] where the author deals with the KPZ equation. More precisely we use the partial series expansion of the solution to define the reminder term using the notion of paracontrolled distributions introduced in [7]. A solution of this equation has already been constructed in the remarkable paper of Hairer [8] where the author shows the convergence of the solution of the mollified equation (5).

The stochastic quantization problem has been studied since the eighties in theoretical physics (see for example [11] and [12] In [2] and the references about it in [8]).
From a mathematical point of view, several articles deals with the 2-dimensional case. Weak probabilistic solutions where find by Jona-Lasinio and and Mitter in [11] and [12]. Some other probabilistic results are obtain thanks to non perturbative methods by Bertini, Jona-Lasinio and Parrinello in [2]. In [3] Da Prato and Debusche found a strong (in the probabilistic sense) formulation for this 2d problem.

In a recent paper, Hairer [8] gives a fixed point solution to the 3-dimensional case thanks to his theory of regularity structures. Like the theory of paracontrolled distributions, Hairer’s theory of regularity structures is a generalization of rough path theory. Hairer gets his result by giving a generalization of the notion of pointwise Hölder regularity. With this extended notion, it is possible to work on a more abstract space where the solutions are constructed thanks to a fixed point argument, and then project the abstract solution into a space of distributions via a reconstruction map. The regularity structures approach is quite general and can treat more singular models.

In the approach of the paracontrolled distribution developed in [7] by Gubinelli, Imkeller and Perkowski, on the other hand, it is the notion of controlled path which is generalized. This allows us to give a reasonable notion of product of distributions. Since all the problems treated by the theory of paracontrolled distributions can be solved by using the theory of regularity structures, asking whether or not the opposite is true is a legitimate (and reasonable) question. The following theorems are a piece of the answer.

We will proceed in two steps. In an analytic part we will extend the flow of the regular equation,

\[
\partial u_t = \Delta u_t - u^3 + 3au + 9bu + \xi
\]

with \((a, b) \in \mathbb{R}^2\) and \(\xi \in C([0, T], C^0(\mathbb{T}^3))\) to the situation of more irregular driving noise \(\xi\). More precisely we will prove that the solution \(u\) is a continuous function of \((u^0, R_{a,b}^\xi X)\) with

\[
R_{a,b}^\xi X = (X, X^2 - a, I(X^3 - 3aX), \pi_0(I(X^3 - 3aX), X), \pi_0(I(X^2 - a), (X^2 - a)) - b - \varphi, \pi_0(I(X^3 - 3aX), (X^2 - a)) - 3bX - 3\varphi X, \varphi)
\]

where \(X_t = \int_0^t P_{t-s} \xi ds\), \(\pi_0(., .)\) denotes the reminder term of the paraproduct decomposition given in (2.3) and \(I(f)_t = \int_0^t P_{t-s} f ds\). This extension is given in the following theorem.

**Theorem 1.1.** Let \(F : C^1(\mathbb{T}^3) \times C(\mathbb{R}^+, C^0(\mathbb{T}^3)) \times \mathbb{R} \times \mathbb{R} \to C(\mathbb{R}^+, C^1(\mathbb{T}^3))\) the flow of the equation

\[
\begin{cases}
\partial_t u_t = \Delta u_t - u^3 + 3au + 9bu + \xi, & t \in [0, T_C(u^0, X, (a, b))] \\
\partial_t u_t = 0, & t \geq T_C(u^0, X, (a, b)) \\
u(0, x) = u^0(x) \in C^1(\mathbb{T}^3)
\end{cases}
\]

where \(\xi \in C(\mathbb{R}^+, C^0(\mathbb{T}^3))\) and \(T_C(u^0, \xi, (a, b))\) is a time such that the the equation holds for \(t \leq T_C\). Now let \(z \in (1/2, 2/3),\) then there exists a Polish space \(\mathcal{X}\), called the space of rough distribution, \(\hat{T}_C : C^{-z} \times \mathcal{X} \to \mathbb{R}^+\) a lower semi-continuous function and \(\hat{F} : C^{-z} \times \mathcal{X} \to C(\mathbb{R}^+, C^{-z}(\mathbb{T}^3))\) continuous in \((u^0, X) \in C^{-z}(\mathbb{T}^3) \times \mathcal{X}\) such that \((\hat{F}, \hat{T})\) extends \((F, T)\) in the following sense:

\[
T_C(u^0, \xi, (a, b)) \geq \hat{T}_C(u^0, R_{a,b}^\xi X)
\]

and

\[
F(u^0, \xi, a, b)(t) = \hat{F}(u^0, R_{a,b}^\xi X)(t), \text{ for all } t \leq \hat{T}_C(u^0, R_{a,b}^\xi X)
\]

for all \((u^0, \xi, \varphi) \in C^1(\mathbb{T}^3) \times C(\mathbb{R}^+, C^0(\mathbb{T}^3)) \times C^\infty([0, T]), (a, b) \in \mathbb{R}^2\) with \(X_t = \int_0^t ds P_{t-s} \xi\) and where \(R_{a,b}^\xi\) is given in the equation (6).
In a second part we obtain a probabilistic estimate for the stationary Ornstein Uhlenbeck (O.U.) process which is the solution if the linear equation (3) and this allows us to construct the rough distribution in this case.

**Theorem 1.2.** Let $X$ be the stationary (O.U.) process and $X^\varepsilon$ a space mollification of $X$. There exists two constants $C_1^\varepsilon, C_2^\varepsilon \to \varepsilon \to 0 + \infty$ and a function $\varphi^\varepsilon \in C^\infty(\mathbb{R}^+)$ such that $R_{C_1^\varepsilon, C_2^\varepsilon}^\varepsilon X^\varepsilon$ converge in $L^p(\Omega, \mathcal{X})$ to some $\mathcal{X} \in \mathcal{X}$. Furthermore the first component of $\mathcal{X}$ is $X$.

In the setting, the Corollary below follows immediately.

**Corollary 1.3.** Let $\xi$ a space time white noise, and $\xi^\varepsilon$ is a space mollification of $\xi$ such that :

$$\xi^\varepsilon = \sum_{k \neq 0} f(\varepsilon k) \hat{\xi}(k)e_k$$

with $f$ a smooth radial function with compact support satisfying $f(0) = 0$, let $X$ the stationary (O.U.) process associated to $\xi$, $X$ the element of $\mathcal{X}$ given in the Theorem (1.2) and $u^0 \in C^{-z}$ for $z \in (1/2, 2/3)$ then if $u^\varepsilon$ is the solution of the mollified equation :

$$\begin{cases}
\partial_t u^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon_t)^3 + 3C_1^\varepsilon u_t + 9C_2^\varepsilon u^\varepsilon_t + \xi^\varepsilon_t, & t \in [0, T^\varepsilon[ \\
\partial_t u^\varepsilon_t = 0, & t \geq T^\varepsilon \\
u(0, x) = (u^0)^\varepsilon(x)
\end{cases}$$

We have the following convergence :

$$\lim_{\varepsilon} u^\varepsilon = \tilde{F}(u^0, X)$$

where the limit is understood in the probability sense in the space $C(\mathbb{R}^+, C^{-z})$.

The proofs of those two theorems are almost independent, but we need the existence and the properties of the rough distribution, specified in the Definition 2.9, to prove the first theorem.

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**Plan of the paper.** It is the aim of Section 2 to introduce the notion spaces of paracontrolled distributions where the renormalized equation will be solved. In Section 3 we prove that for a small time the application associated to the renormalized equation is a contraction, which, by a fixed point argument, gives the existence and uniqueness of the solution, but also the continuity with respect to the rough distribution and the initial condition. The last Section 4 is devoted to the existence of the rough distribution for the (O.U.) process.

## 2 Paracontrolled distributions

### 2.1 Besov spaces and paradifferential calculus

The results given in this Subsection can be found in [1] and [7]. Let us start by recalling the definition of Besov spaces via the Littelwood-Paley projectors.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions such that
1. The support of \( \chi \) is contained in a ball and the support of \( \theta \) is contained in an annulus;
2. \( \chi(\xi) + \sum_{j \geq 0} \theta(2^{-j} \xi) = 1 \) for all \( \xi \in \mathbb{R}^d \);
3. \( \text{supp}(\chi) \cap \text{supp}(\theta(2^{-j} \cdot)) = \emptyset \) for \( i \geq 1 \) and \( \text{supp}(\theta(2^{-j} \cdot)) \cap \text{supp}(\theta(2^{-j} \cdot)) = \emptyset \) when \( |i-j| > 1 \).

For the existence of \( \chi \) and \( \theta \) see [1], Proposition 2.10. The Littlewood-Paley blocks are defined as

\[
\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u) \quad \text{and} \quad j \geq 0, \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j} \cdot) \mathcal{F} u).
\]

We define the Besov space of distribution by

\[
B^s_{p,q} = \left\{ u \in S'(\mathbb{R}^d); \quad \|u\|_{B^s_{p,q}}^q = \sum_{j \geq -1} 2^{jsq} \|\Delta_j u\|_{L^p}^q < +\infty \right\}.
\]

In the sequel we will deal with the special case of \( C^\alpha := B^\alpha_{\infty,\infty} \) and write \( \|u\|_\alpha = \|u\|_{B^\alpha_{\infty,\infty}} \). We hold the following result for the convergence of localized series in the Besov spaces, which will prove itself useful.

**Proposition 2.1.** Let \((p,q,s) \in [1, +\infty]^2 \times \mathbb{R}, B \) a ball in \( \mathbb{R}^d \) and \((u_j)_{j \geq 1} \) a sequence of functions such that \( \text{supp}(u_j) \) is contained in \( 2^j B \) moreover we assume that

\[
\Xi_{p,q,s} = \left\{ \|(2^j \cdot) u_j\|_{L^p} \right\}_{j \geq 1} < +\infty
\]

then \( u = \sum_{j \geq 1} u_j \in B^s_{p,q} \) and \( \|u\|_{B^s_{p,q}} \lesssim \Xi_{p,q,s} \).

The trick to manipulate stochastic objects is to deal with Besov spaces with finite indexes and then go back to space \( C^\alpha \). For that we have the following useful Besov embedding.

**Proposition 2.2.** Let \( 1 \leq p_1 \leq p_2 \leq +\infty \) and \( 1 \leq q_1 \leq q_2 \leq +\infty \). For all \( s \in \mathbb{R} \) the space \( B^s_{p_1,q_1} \) is continuously embedded in \( B^{s-d \left( \frac{1}{p_1} - \frac{1}{p_2} \right)}_{p_2,q_2} \), in particular we have \( \|u\|_{C^{\frac{d}{p}}} \lesssim \|u\|_{B^s_{p,q}} \).

Taking \( f \in C^\alpha \) and \( g \in C^\beta \) we can formally decompose the product as

\[
f g = \pi_<(f,g) + \pi_0(f,g) + \pi_>(f,g)
\]

with

\[
\pi_<(f,g) = \pi_>(g,f) = \sum_{j \geq -1} \sum_{i < j} \Delta_i f \Delta_j g; \quad \pi_0(f,g) = \sum_{j \geq -1} \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.
\]

With these notations the following results hold.

**Proposition 2.3.** Let \( \alpha, \beta \in \mathbb{R} \)

\[
\bullet \quad \|\pi_<(f,g)\|_{C^\beta} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} \quad \text{for } f \in L^\infty \text{ and } g \in C^3
\]
\[
\bullet \quad \|\pi_>(f,g)\|_{C^{\alpha + \beta}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} \quad \text{for } \beta < 0, f \in C^\alpha \text{ and } g \in C^3
\]
\[
\bullet \quad \|\pi_0(f,g)\|_{C^{\alpha + \beta}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} \quad \text{for } \alpha + \beta > 0 \text{ and } f \in C^\alpha \text{ and } g \in C^3
\]

On of the key result of [7] is a commutation result for the operator \( \pi_< \) and \( \pi_0 \).
Proposition 2.4. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha < 1$, $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$ then

$$R(f, x, y) = \pi_0(\pi_<(f, x), y) - f\pi_0(x, y)$$

is well-defined when $f \in C^\alpha$, $x \in C^\beta$ and $y \in C^\gamma$ and more precisely

$$||R(f, x, y)||_{\alpha+\beta+\gamma} \lesssim ||f||_\alpha||x||_\beta||y||_\gamma$$

We finish this Section by describing the action of the Heat flow on the Besov spaces and a commutation property with the paraproduct. See the appendix for a proof.

Lemma 2.5. Let $\theta \geq 0$ and $\alpha \in \mathbb{R}$ then the following inequality holds

$$||P_t f||_{\alpha+2\theta} \lesssim \frac{1}{t^\theta}||f||_\alpha, \quad ||(P_t^{-s} - 1) f||_{\alpha-2\varepsilon} \lesssim |t-s|^\varepsilon ||f||_\alpha$$

for $f \in C^\alpha$. Moreover if $\alpha < 1$ and $\beta \in \mathbb{R}$ we have

$$||P_t \pi_<(f, g) - \pi_<(f, P_t g)||_{\alpha+\beta+2\theta} \lesssim \frac{1}{t^\theta}||f||_\alpha||g||_\beta$$

for all $g \in C^\beta$.

In the following, we will extensively use some space-time function spaces. Let us introduce the notation

Notation 2.6.

$$C^\beta_T = C([0, T], C^\beta)$$

For $f \in C^\beta_T$ we introduce the norm

$$||f||_\beta = \sup_{t \in [0, T]} ||f_t||_{C^\beta} = \sup_{t \in [0, T]} ||f_t||_\beta$$

and by

$$C^{\alpha, \beta}_T := C^\alpha([0, T], C^\beta(T^3)).$$

Furthermore, we endow this space with the following distance

$$d_{\alpha, \beta}(f, g) = \sup_{t \neq s \in [0, T]} \frac{||(f - g)_t - (f - g)_s||_\beta}{|t-s|^\alpha} + \sup_{t \in [0, T]} ||f_t - g_t||_\beta.$$

2.2 Renormalized equation and rough distribution

Let us focus on the mild formulation of the equation (1)

$$u = \Psi + X + I(u^3) = X + \Phi$$

(7)

where we remind the notation $I(f)(t) = -\int_0^t P_{t-s} f_s ds$, $X = -I(\xi)$ and $\Psi_t = P_t u^0$ for $u^0 \in C^{-\varepsilon}(T^3)$. We can see that a solution $u$ must have at least the same regularity as $X$. Yet thanks to the definition of $I$, as $\xi \in C([0, T], C^{-5/2-\varepsilon})$, for all $\varepsilon > 0$, we have $X \in C([0, T], C^{-1/2-\varepsilon})$. But in that case the non-linear term $u^3$ is not well-defined, as there is no universal notion for the product of distributions. A first idea is to proceed by regularization of $X$, such that
products of the regularized quantities are well-defined, and then try to pass to the limit. Let us recall that the stationary O.U process is defined by the fact that \((\hat{X}_t(k))_{t \in \mathbb{R}, k \in \mathbb{Z}^3}\) is a centered Gaussian process with covariance function given by

\[
E \left[ \hat{X}_t(k) \hat{X}_s(k') \right] = \delta_{k+k'=0} \frac{e^{-|k|^2|t-s|}}{|k|^2}
\]

and \(\hat{X}_t(0) = 0\). Let \(X_t^\varepsilon = \int_0^t P_{t-s} \xi \, ds\) more precisely \(\hat{\xi}^\varepsilon = f(\varepsilon k) \hat{\xi}(k)\) where \(f\) is a smooth radial function with bounded support such that \(f(0) = 1\). Then we have the following approximated equation

\[
\Phi^\varepsilon = \Psi^\varepsilon + I((X^\varepsilon)^3) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon (X^\varepsilon)^2) + I((\Phi^\varepsilon)^3)
\]

for \(\Phi^\varepsilon = I((u^\varepsilon)^3) + \Psi^\varepsilon\) which is well-posed. Then an easy computation gives for \((X^\varepsilon)^2\)

\[
E \left[ (X_t^\varepsilon)^2 \right] = \sum_{k \in \mathbb{Z}^3} \sum_{k_1+k_2=k} f(\varepsilon k_1)f(\varepsilon k_2) \frac{1}{|k_1|^2} \delta_{k_1+k_2=0}
\]

\[
= \sum_{k \in \mathbb{Z}^3} \frac{|f(\varepsilon k)|^2}{|k|^2} \sim_0 \frac{1}{\varepsilon} \int_{\mathbb{R}^3} f(x)(1 + |x|)^{-2} dx
\]

and there is no hope to obtain a finite limit for this term when \(\varepsilon\) goes to zero. This difficulty has to be solved by subtracting to the original equation these problematic contributions. In order to do so consistently we will introduce a renormalized product. Formally we would like to define

\[
X^{\Diamond 2} = X^2 - E[X^2]
\]

and show that it is well-defined and that \(X^{\Diamond 2} \in C^{-1-\delta}_t\) for \(\delta > 0\). More precisely we will defined

\[
(X^\varepsilon)^{\Diamond 2} = (X^\varepsilon)^2 - E[(X^\varepsilon)^2]
\]

and we will show that it converges to some finite limit. The same phenomenon happens for \(X^3\) and other terms, and we have to renormalize them too. This is the meaning of Theorem 1.2. We remind the notation of that theorem

**Notation 2.7.** Let \(C_1^\varepsilon\) and \(C_2^\varepsilon\) two positives constants (to be specified later). We denote by

\[
(X^\varepsilon)^{\Diamond 2} := (X^\varepsilon)^2 - C_1^\varepsilon
\]

\[
I((X^\varepsilon)^{\Diamond 3}) := I((X^\varepsilon)^3) - 3C_1^\varepsilon X^\varepsilon
\]

\[
\pi_0(0)(I((X^\varepsilon)^{\Diamond 2}), (X^\varepsilon)^{\Diamond 2}) = I((X^\varepsilon)^{\Diamond 2}) (X^\varepsilon)^{\Diamond 2} - C_2^\varepsilon
\]

\[
\pi_0(0)(I((X^\varepsilon)^{\Diamond 3}), (X^\varepsilon)^{\Diamond 2}) = I((X^\varepsilon)^{\Diamond 3}) (X^\varepsilon)^{\Diamond 2} - 3C_2^\varepsilon X^\varepsilon.
\]

**Remark 2.8.** In all the sequel the symbol \(\Diamond\) does not stand for the usual Wick product, also it looks like it, but for renormalized product, where we have subtracted only the diverging quantity in the expression of the stochastic processes. It can be seen as a product between the usual one and the Wick one. When in Section 4 we use the usual Wick product (see [10] for its definition and its properties) we use the usual notation \(::\).

To include such considerations and notations in the approximated equation, we need to add a renormalized term

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\[ \Phi^\varepsilon = \Psi^\varepsilon + I((X^\varepsilon)^3) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon X^\varepsilon) + I((\Phi^\varepsilon)^3) - C^\varepsilon I(\Phi^\varepsilon + X^\varepsilon) \]

\[ = \Psi^\varepsilon + I((X^\varepsilon)^3 - 3C^\varepsilon_{1} X^\varepsilon) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon ((X^\varepsilon)^2 - C^\varepsilon_{1}) - 3C^\varepsilon_{2}(\Phi^\varepsilon + X^\varepsilon)) + I((\Phi^\varepsilon)^3) \]

with \( C^\varepsilon = 3(C^\varepsilon_{1} - 3C^\varepsilon_{2}) \). Then the approximated equation is given by

\[ \Phi^\varepsilon = \Psi^\varepsilon + I((X^\varepsilon)^3) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon ((X^\varepsilon)^2 - C^\varepsilon_{1})) + 9C^\varepsilon_{2} I(\Phi^\varepsilon + X^\varepsilon) \]

where

\[ I(\Phi^\varepsilon ((X^\varepsilon)^2 - C^\varepsilon_{1})) := 3I((\Phi^\varepsilon - X^\varepsilon)^2) + 9C^\varepsilon_{2} I(\Phi^\varepsilon + X^\varepsilon) \]

Then our goal is obtain a uniform bound for the solution \( \Phi^\varepsilon \). For that we proceed in two steps

1. In a first analytic step we build an abstract fix point equation which allows us to extend continuously the flow of the regular equation given by

\[ \left\{ \begin{array}{l}
\Phi = I(X^3 - 3aX) + 3I(\Phi^2 X) + 3 \left\{ I((X^2 - a)) - 3bI(\Phi + X) \right\} + I(\Phi^3) + \Psi \\
(X, \Psi, (a, b)) \in C^1_T([T^3]) \times C^1_T([T^3]) \times \mathbb{R}^2
\end{array} \right. \]

(9)

to a space \( \mathcal{X} \) of a more rough signal \( X \) which satisfies some algebraic and analytic assumptions (see (2.9) for the exact definition of \( \mathcal{X} \)).

2. In a second probabilistic step we show that the stationary (O.U) process can be enhanced in a canonical way in an element \( \mathcal{X} \) of \( \mathcal{A}' \).

We will give the exact definition of the space \( \mathcal{X} \)

**Definition 2.9.** Let \( T > 1, \nu, \rho > 0 \). We denote by \( \overline{C}^{\nu, \rho}_T \) the closure of the set of smooth functions \( C^\infty([0, T], \mathbb{R}) \) by the semi-norm :

\[ ||\varphi||_{\nu, \rho} = \sup_{t \in [0, T]} t^\nu |\varphi_t| + \sup_{s, t \in [0, T], t \neq s} \frac{s^\rho |\varphi_t - \varphi_s|}{|t - s|^\rho}. \]

For \( 0 < 4\delta' < \delta \) we define the normed space \( \mathcal{W}_{T,K} \)

\[ \mathcal{W}_{T,K} = C^\delta_{T} \times C^{\delta', 1/2 - \delta}_{T} \times C^{\delta', 1/2 - \delta}_{T} \times C^{\delta', -\delta}_{T} \times C^\delta_{T} \times C^{\delta', -\delta}_{T} \times C^{\delta', 1/2 - \delta}_{T} \times \overline{C}^{\nu, \rho}_T \]

with \( K = (\delta, \delta', \nu, \rho) \) equipped with the product topology. For \( (X, \varphi) \in C([0, T], C([T^3])) \times C^\infty([0, T]), \), and \( (a, b) \in \mathbb{R}^2 \) we define \( R_{a,b}^\varepsilon X \in \mathcal{W}_{T,K} \) by

\[ R_{a,b}^\varepsilon X = (X, X^2 - a, I(X^3 - 3aX), \pi_0(I(X^3 - 3aX), X), \pi_0(I(X^2 - a), (X^2 - a)) - b - \varphi, \pi_0(I(X^3 - 3aX), (X^2 - a)) - 3bX - 3\varphi X, \varphi). \]

The space of the rough distribution \( \mathcal{X}_{T,K} \) is defined as the closure of the set

\[ \{ R_{a,b}^\varepsilon X, (X, \varphi) \in C([0, T], C([T^3])) \times C^\infty([0, T]), (a, b) \in \mathbb{R}^2 \} \]

in \( \mathcal{W}_{T,K} \). For \( X \in \mathcal{X} \) we denote its components by

\[ X = (X, X^{\varphi, 2}, I(X^{X^3}), \pi_0(I(X^{X^3}, X), \pi_0(I(X^{X^2}, X^{\varphi, 2}) - \varphi X, \pi_0(I(X^{X^3}, X^{\varphi, 2}) - 3\varphi X X, \varphi X)). \]
For two rough distributions $X \in \mathcal{X}_{T,K}$ and $Y \in \mathcal{X}_{T,K}$ we introduce the distance :
\[
d_{T,K}(Y, X) = d_{T,-1/2-\delta}(Y, X) + d_{T,-1-\delta}(Y^{02}, X^{02}) + d_{T,1/2-\delta}(I(Y^{03}), I(X^{03}))
\]
\[
+ d_{T,-1/2-\delta}(\pi_0(I(Y^{03}), Y), \pi_0(I(X^{03}), X))
\]
\[
+ d_{T,-1-\delta}(\pi_0(I(Y^{02}), Y^{02}) - \varphi^Y, \pi_0(I(X^{03}), X^{02}) - \varphi^X)
\]
\[
+ d_{T,-1-\delta}(\pi_0(I(Y^{03}) \otimes Y^{02}) - 3\varphi^Y, \pi_0(I(X^{03}) \otimes X^{02}) - 3\varphi^X)
\]
\[
+ \|\varphi^Y - \varphi^X\|_{\nu, \rho}.
\]
with $K = (\delta, \delta', \rho, \nu) \in [0,1]^4$ and we denote by $||X||_{T,K} = d_{T,K}(X, 0)$.

**Remark 2.10.** As we see in the Section (4), the term $\pi_0(I((X^\varepsilon)^2), (X^\varepsilon)^2) - C_2^\varepsilon$ where $X^\varepsilon$ is a mollification of the O.U process does not converge in the space $C_T^{\delta',-\delta}$. On the other hand it converge in a explosive norm and more precisely there exit a function $\varphi^\varepsilon \in C^\infty([0,T])$ such that
\[
\varphi^\varepsilon \rightarrow \varepsilon^0 \varphi \text{ in } C_T^{\delta',0} \text{ and } \pi_0(I((X^\varepsilon)^2), (X^\varepsilon)^2) - C_2^\varepsilon - \varphi^\varepsilon \text{ converge in } C_T^{\delta',-\delta} \text{ for all } 0 < \delta' < \delta/4.
\]

For $X \in \mathcal{X}$ we can obviously construct $I(X^{02}) \otimes X^{02}$ using the Bony paraproduct in the following way
\[
I(X^{02}) \otimes X^{02} = \pi_<(I(X^{02}), X^{02}) + \pi_(I(X^{02}), X^{02}) + \pi_0(I(X^{02}), X^{02})
\]
and a similar definition for $I(X^{03}) \otimes X^{02}$. In the sequel we might abusively denote $X$ by $X$ if there is no confusion, and the rough path terminology we denote the other components of $X$ by the area components of $X$.

### 2.3 Paracontrolled distributions and fixed point equation

The aim of this Section is to define a suitable space in which it is possible to formulate an fix point for the eventual limit of the mollified solution, to be more precise let $X \in \mathcal{X}$ then we know that there exist $X^\varepsilon \in C^{1}(\mathbb{T}^3)$, $a^\varepsilon, b^\varepsilon \in \mathbb{R}$ and $\varphi^\varepsilon \in C^\infty([0,T])$ such that $\lim_{\varepsilon \rightarrow 0} R^{a^\varepsilon, b^\varepsilon}_{\varepsilon, \varphi^\varepsilon} X^\varepsilon = X$.

Let us focus more intently on the regular equation given by :
\[
\Phi^\varepsilon = I((X^\varepsilon)^3 - 3a^\varepsilon X^\varepsilon) + 3 \left\{I(\Phi^\varepsilon((X^\varepsilon)^2 - a^\varepsilon)) - 3b^\varepsilon I(X^\varepsilon + \Phi^\varepsilon)\right\} + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + I((\Phi^\varepsilon)^3)
\]
where we have omitted temporarily the dependence on the initial condition. If we assume that $\Phi^\varepsilon$ converge to some $\Phi$ in $C^{1/2-\delta}$ we see that the regularity of $X$ is not sufficient to define $I(\Phi^2 X) := \lim_{\varepsilon \rightarrow 0} I((\Phi^\varepsilon)^2 X^\varepsilon)$ and $I(\Phi \otimes X^{02}) := \lim_{\varepsilon \rightarrow 0} I(\Phi^\varepsilon(X^\varepsilon - a^\varepsilon)) + 3b^\varepsilon I(X^\varepsilon + \Phi^\varepsilon)$. To bypass this problem we remark that
\[
\Phi^\varepsilon = I((X^\varepsilon)^3 - 3a^\varepsilon X^\varepsilon) + I(\pi_<(\Phi^\varepsilon, (X^\varepsilon)^2 - a^\varepsilon)) + (\Phi^\varepsilon)^2
\]
then if we impose the convergence of $(\Phi^\varepsilon)^2$ to some $\Phi^2$ in $C^{3/2-\delta}$ we see that the limit $\Phi$ should satisfy the following relation
\[
\Phi = I(X^{03}) + I(\pi_<(\Phi, X^{02})) + \Phi^2.
\]
This is the missing ingredient which allows to construct the quantity $I(\Phi^2 X)$ and $I(\Phi \otimes X^{02})$ and to solve the equation
\[
\Phi = X^{02} + I(\Phi^2 X) + I(\Phi \otimes X^{02}) + \Phi^2
\]
Notation 2.11. Let us introduce some useful notations for the sequel

$$B_>(f, g) = I(\pi_>(f, g)), \quad B_0(f, g) = I(\pi_0(f, g)) \quad \text{and} \quad B_<(f, g) = I(\pi_<(f, g)).$$

As we observed in the beginning of this Section to deal with the difficulty of defining the products of distributions, we use the notion of controlled distribution introduced in [7].

Definition 2.12. Let $\mathcal{X} \in \mathcal{X}$ and $z \in (1/2, 2/3)$. We said that a distribution $\Phi \in \mathcal{C}_T^{-z}$ is controlled by $\mathcal{X}$ if

$$\Phi = I(X^{\otimes 3}) + B_<(\Phi', X^{\otimes 2}) + \Phi^\sharp$$

such that

$$||\Phi^\sharp||_{\ast, 1, L,T} = \sup_{t \in [0,T]} \left( t^{1/3+\delta} ||\Phi^\sharp_t||_{1+\delta} + t^{1/4+\frac{\gamma}{2}} ||\Phi^\sharp_t||_{1/2+\gamma} + t^{\frac{\gamma}{2}} ||\Phi^\sharp_t||_{\kappa} \right)$$

$$+ \sup_{(s,t) \in [0,T]^2} s^{\frac{\gamma + a}{2}} \frac{||\Phi^\sharp_t - \Phi^\sharp_s||_{a-2b}}{|t-s|^b} < +\infty$$

and

$$||\Phi'||_{\ast, 2, L,T} = \sup_{(s,t) \in [0,T]^2} s^{\frac{\gamma + c}{2}} \frac{||\Phi'_t - \Phi'_s||_{c-2d}}{|t-s|^d} + \sup_{t \in [0,T]} t^{\frac{\gamma}{2}} ||\Phi'_t||_{\eta} < +\infty$$

with $L := (\delta, \gamma, \kappa, a, b, c, d, \eta) \in [0,1]^8$, $z \in (1/2, 2/3)$ and $2d \leq c, 2b \leq a$. Let us denote by $\mathcal{D}^L_{T,X}$ the space of controlled distributions, endowed with the following metric

$$d_{L,T}(\Phi_1, \Phi_2) = ||\Phi'_1 - \Phi'_2||_{\ast, T} + ||\Phi^\sharp_t - \Phi^\sharp_s||_{\ast, T}$$

for $\Phi_1, \Phi_2 \in \mathcal{D}^L_X$ and the quantity

$$||\Phi||_{\ast, T,L} = ||\Phi||_{\mathcal{D}^L_{T,X}} = d_{L,T}(\Phi, I(X^{\otimes 3})).$$

We notice that the distance and the metric introduced in this last definition do not depend on $\mathcal{X}$. More generally for $\Phi \in \mathcal{D}^L_{T_1,X}$ and $\Psi \in \mathcal{D}^L_{T_2,Y}$ we denote by $d_{\min(L,G),\min(T_1, T_2)}(\Phi, \Psi)$ the same quantity. We claim that if $\Phi \in \mathcal{D}^L_X$ for a suitable choice of $L$ then we are able to define $I(\Phi \otimes X^{\otimes 2})$ and $I(\Phi^2 X)$ modulo the use of $\mathcal{X}$.

Let us decompose the end of this Section into two parts, namely we show that $I(\Phi \otimes X^{\otimes 2})$ and $I(\Phi^2 X)$ are well-defined when $\Phi$ is a controlled distribution. We also have to prove that when $\Phi$ is a controlled distribution, $\Psi + I(X^{\otimes 3}) + 3I(\Phi^2 X) + 3I(\Phi \otimes X^{\otimes 2}) + I(\Phi^3)$ is also a controlled distribution. All those verifications being made, the only remaining point will be to show that we can apply a fixed point argument to find a solution to the renormalized equation. This is the aim of Section 3.

2.4 Decomposition of $I(\Phi^2 X)$

Let $\mathcal{X} \in \mathcal{X}$ and $\Phi \in \mathcal{D}^L_{X,T}$, a quick computation gives:

$$I(\Phi^2 X) = I(I(X^{\otimes 3})^2 X) + I((\theta^2)^2 X) + 2I(\theta^2 I(X^{\otimes 3}) X)$$

with

$$\theta^2 = B_<(\Phi', X^{\otimes 2}) + \Phi^\sharp.$$
Using the fact that $\Phi \in \mathcal{D}_{X,T}^L$ and that $I(X^{\odot 3}) \in C_{T}^{1/2-\delta}$ we can see that the two terms $I((\theta^z)^2)X$ and $I(\theta^2I(X^{\odot 3})X)$ are well defined. Let us focus on the term $I(X^{\odot 3})^2X$ which is at this stage is not well understood, then a paraproduct decomposition of this term give us that

\[ I(X^{\odot 3})^2X = 2\pi_0(\pi_<(I(X^{\odot 3}), I(X^{\odot 3})), X) + \pi_0(\pi_0(I(X^{\odot 3}), I(X^{\odot 3})), X) + \pi_<(I(X^{\odot 3})^2, X) + \pi_>(I(X^{\odot 3})^2, X) \]

We remark that only the first term of this expansion is not well understood and to overcome this problem we use the Proposition (2.4), indeed we know that

\[ R(I(X^{\odot 3}), I(X^{\odot 3}), X) = \pi_0(\pi_<(I(X^{\odot 3}), I(X^{\odot 3})), X) - I(X^{\odot 3})\pi_0(I(X^{\odot 3}), X) \]

is well defined and it lies in the space $C_{T}^{1/2-\delta}$ due to the fact that $X \in \mathcal{X}$

**Remark 2.13.** We remark that the ”extension” of the term $I(\Phi^2X)$ is a functional of ”$(\Phi, X) \in \mathcal{D}_{X,T}^L \times \mathcal{X}$” and then we use sometimes the notation $I(\Phi^2X)[\Phi, X]$ to underline this fact.

**Proposition 2.14.** Let $z \in (1/2, 2/3)$, $\Phi \in \mathcal{D}_{X,T}^L$ and assume that $X \in \mathcal{X}$. Then the quantity $I(\Phi^2X)[\Phi, X]$ is well-defined via the following expansion

\[ I(\Phi^2X)[\Phi, X] := I(I(X^{\odot 3})^2X) + I((\theta^z)^2X) + 2I(\theta^2I(X^{\odot 3})X) \]

with

\[ \theta^z = B_<(\Phi', X^2) + \Phi^z \]

and

\[ I(X^{\odot 3})^2X := \pi_0(\pi_<(I(X^{\odot 3}), I(X^{\odot 3})), X) + 2\pi_<(\pi_<(I(X^{\odot 3}), I(X^{\odot 3})), X) + 2\pi_>(\pi_<(I(X^{\odot 3}), I(X^{\odot 3})), X) + 2I(X^{\odot 3})\pi_0(I(X^{\odot 3}), X) + R(I(X^{\odot 3}), I(X^{\odot 3}), X) \]

where

\[ R(I(X^{\odot 3}), I(X^{\odot 3}), X) = \pi_0(\pi_<(I(X^{\odot 3}), I(X^{\odot 3})), X) - I(X^{\odot 3})\pi_0(I(X^{\odot 3}), I(X^{\odot 3})) \]

is well-defined by the Proposition 2.4. And there exists a choice of $L$ such that the following bound holds

\[ ||I(\Phi^2X)[\Phi, X]||_{*1,T} \lesssim T^\theta \left(||\Phi||_{\mathcal{D}_{X}^L} + 1\right)^2 (1 + ||X||_{T, \nu, \rho, \delta, \delta})^3 \]

for $\theta > 0$ and $\delta, \delta', \rho, \nu > 0$ small enough depending on $L$ and $z$. Moreover if $X \in C_{T}^{1}(\mathbb{T}^3)$ then

\[ I(\Phi^2X)[\Phi, R_{u,\nu}X] = I(\Phi^2X) \]

**Proof.** By a simple computation it is easy to see that

\[ ||B_<(\Phi', X^{\odot 2})(t)||_{k} \lesssim \int_{0}^{t} ds(t-s)^{-(\kappa+1+r)/2} ||\Phi'||_{k} ||X^{\odot 2}_{s}||_{-1-r} \lesssim_{r,\kappa} T^{1/2-r/2-r/2-z/2} ||\Phi'||_{*,2,T} \leq |X^{\odot 2}_{-1-r}| \]

for $r, \kappa > 0$ small enough and $1/2 < z < 2/3$. A similar computation gives

\[ ||B_<(\Phi', X^{\odot 2})(t)||_{1/2+\gamma} \lesssim \int_{0}^{t} ds(t-s)^{-(3/2+\gamma+r)/2} ||\Phi'||_{\kappa} ||X^{\odot 2}_{s}||_{-1-r} \]

\[ \lesssim_{r,\gamma,\zeta} ||\Phi'||_{*,2,T} \leq |X^{\odot 2}_{-1-r}| \int_{0}^{t} ds(t-s)^{-(3/2+\gamma+r)/2} \leq |t^{1/4-(\gamma+\kappa+r)/2}||\Phi'||_{*,2,L,T} \leq |X^{\odot 2}_{-1-r}| \]

for $r, \kappa > 0$ small enough and $1/2 < z < 2/3$. A similar computation gives
for $\gamma, r, \kappa > 0$ small enough. Using this bound we can deduce that

$$
\|I((\theta^2)X(t))\|_{1+\delta} \lesssim \int_0^t ds(t - s)^{-\frac{3}{2+\delta + \beta}/2}\|s^2 X_s\|_{-1/2-\beta}
$$

$$
\lesssim_{\beta, \delta} \int_0^t ds(t - s)^{-\frac{3}{2+\delta + \beta}/2}\|s^2 \|_{1+\gamma} \|X_s\|_{-1/2-\beta}
$$

$$
\lesssim_{L, z} \|\Phi\|^{2}_{*, L,T}(\|X^{\delta/2}\|_{1-r} + \|X\|_{-1/2-\beta} + 1)^2
$$

$$
\times \int_0^t ds(t - s)^{-\frac{3}{2+\delta + \beta}/2} s^{-(1/2+\kappa+\gamma+2\delta)/2}
$$

$$
\lesssim_{L, z} t^{-(\delta+\kappa+\gamma+2\delta)} \|\Phi\|^{2}_{*, L,T}(\|X^{\delta/2}\|_{1-r} + \|X\|_{-1/2-\beta} + 1)^2
$$

for $\gamma, \beta, \delta > 0$ small enough and $2/3 > z > 1/2$. Hence we obtain that

$$
\sup_{t \in [0,T]} t^{(1+\delta + z)/2} \|I((\theta^2)X(t))\|_{1+\delta} \lesssim_{L, z} T^\theta_1 \|\Phi\|^{2}_{*, L,T}(\|X^{\delta/2}\|_{1-r} + \|X\|_{-1/2-\beta} + 1)^2
$$

for some $\theta_1 > 0$ depending on $L$ and $z$. The same type of computation gives

$$
\sup_{t \in [0,T]} t^{(\kappa + z)/2} \|I((\theta^2)X(t))\|_{\kappa} \lesssim_{L, z} T^\theta_2 \|\Phi\|^{2}_{*, L,T}(\|X^{\delta/2}\|_{1-r} + \|X\|_{-1/2-\beta} + 1)^2
$$

and

$$
\sup_{t \in [0,T]} t^{(1/2+\gamma + z)/2} \|I((\theta^2)X(t))\|_{1/2+\gamma} \lesssim_{L, z} T^\theta_3 \|\Phi\|^{2}_{*, L,T}(\|X^{\delta/2}\|_{1-r} + \|X\|_{-1/2-\beta} + 1)^2
$$

with $\theta_2$ and $\theta_3$ two non negative constants depending only on $L$ and $z$. To complete our study for this term, we have also

$$
\|I((\theta^2)X(t)) - I((\theta^2)X(s))\|_{a-2b} \lesssim I^1_{st} + I^2_{st}
$$

with

$$
I^1_{st} = \left\| \int_0^s du(P_{t-u} - P_{s-u})(\theta^2_u)X_u \right\|_{a-2b}, \quad I^2_{st} = \left\| \int_s^t duP_{t-u}(\theta^2_u)X_u \right\|_{a-2b}.
$$

Let us begin by bounding $I^1$:

$$
I^1_{st} \lesssim (t-s)^b \int_0^s du \|P_{s-u}(\theta^2_u)X_u\|_a
$$

$$
\lesssim (t-s)^b \int_0^s du(s-u)^{-(1/2+a+\beta)} \|s^2 X_u\|_{-1/2-\beta}
$$

$$
\lesssim T^\theta_4 |t - s|^b \|\Phi\|^{2}_{*, L,T}(\|X^{\delta/2}\|_{1-r} + \|X\|_{-1/2-\beta} + 1)^2
$$

with $\theta_4 > 0$ depending on $L$ and $z$. Let us focus on the bound for $I^2$,

$$
I^2_{st} \lesssim \int_s^t (t-u)^{-(1/2+a-2b+\beta)/2} \|s^2 X_u\|_{-1/2-\beta}
$$

$$
\lesssim_{L, z} \|\Phi\|^{2}_{*, L,T}(\|X^{\delta/2}\|_{1-r} + \|X\|_{-1/2-\beta} + 1)^2 \int_s^t du(t-u)^{-(1/2+a-2b+\beta)/2} u^{-(1/2+\kappa+\gamma+2\delta)/2}
$$

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and
\[ \int_s^t du(t - u)^{-\left(1/2 + a - 2b + \beta\right)/2} u^{-\left(1/2 + \kappa + \gamma + 2z\right)/2} = (t - s)^{3/4 - (a - 2b + \beta)} \int_0^1 dx (1 - x)^{-(a - 2b + \beta)} (s + x(t - s))^{-\left(1/2 + \kappa + \gamma + 2z\right)/2} \lesssim I, \kappa, \gamma, a, b \geq 0 \]

Then using the fact \( z < 1 \) and choosing \( I, \kappa, \gamma, b > 0 \) small enough we can deduce that
\[ \int_s^t du(t - u)^{-\left(1/2 + a - 2b + \beta\right)/2} u^{-\left(1/2 + \kappa + \gamma + 2z\right)/2} \lesssim T^{\theta_3} (t - s)^b s^{-(z + a)/2} \]
with \( \theta_3 > 0 \). This gives the needed bound for \( I_2 \). Finally we have
\[ \sup_{(s, t) \in [0, T]} s^{(z + a)/2} \parallel (I((\theta^2)X)(t) - I((\theta^2)X)(s)) \parallel a - 2b \parallel T^{\theta_3} \parallel \Phi \parallel^2_{L,T} (\|X\|^{2\circ}_{-1 - \rho} + \|X\|_{-1/2 - \beta} + 1)^2 \]
hence
\[ \|I((\theta^2)X)\|_{L, T} \lesssim T^{\theta} \|\Phi\|^2_{L,T} (\|X\|^{2\circ}_{-1 - \rho} + \|X\|_{-1/2 - \beta} + 1)^2. \]
The bound for \( \|I(\theta^2 I(X^3))\|_{L, T} \) can be obtained by a similar way and then, according to the hypothesis given on the area \( I(X^3)X \) and the decomposition of \( I(I(X^3)^2X) \), we obtain easily from the Proposition 2.4 and the Proposition 2.3 that
\[ \|I(I(X^3)^2)\circ X\|_{L, T} \lesssim T^\theta (1 + \|\pi_0(I(X^3), X)\|_{\delta, -1/2 - \rho} + \|I(X^3)\|_{\delta, -1/2 - \rho} + \|X\|_{\delta, -1/2 - \rho})^3 \]
for \( 3\rho < \delta \) small enough, which gives the wanted result.

### 2.5 Decomposition of \( I(\Phi \diamond X^{2\circ}) \)

Let us apply the controlled structure to the mollified equation. As in that case the equation is well-posed, we have
\[ \tilde{\Gamma}(\Phi^\varepsilon) = \Phi^\varepsilon \text{ where } \tilde{\Gamma}(\Phi^\varepsilon) = I((X^\varepsilon)^{2\circ}) + 3I(\Phi^\varepsilon(X^\varepsilon)^{2\circ}) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + I((\Phi^\varepsilon)^3) \]
with of course \( \Phi^\varepsilon \) controlled by \( X^\varepsilon \)
\[ \Phi^\varepsilon = I((X^\varepsilon)^{2\circ}) + B_{<}(\Phi^\varepsilon') + (X^\varepsilon)^{2\circ} = (\Phi^\varepsilon)^2. \]

By a direct computation we also have
\[ I(\Phi^\varepsilon(X^\varepsilon)^{2\circ}) = B_{<}(\Phi^\varepsilon, (X^\varepsilon)^{2\circ}) + B_0(I((X^\varepsilon)^{2\circ}, (X^\varepsilon)^{2\circ}) + B_0(B_{<}(\Phi^\varepsilon'), (X^\varepsilon)^{2\circ}), (X^\varepsilon)^{2\circ}) + B_0((\Phi^\varepsilon)^2, (X^\varepsilon)^{2\circ}) + B_{>}(\Phi^\varepsilon, (X^\varepsilon)^{2\circ}). \]

Indeed, thanks to the Bony paraproduct, the first and the last terms in the r.h.s are well defined. The only problem is to define \( B_0(.) \). By an analysis of the regularity, the structure of controlled
distribution for $\tilde{\Gamma}^\varepsilon(\Phi^\varepsilon)$ appears, and we have $\tilde{\Gamma}^\varepsilon(\Phi^\varepsilon)' = 3\Phi^\varepsilon$ hence $(\Phi^\varepsilon)' = 3\Phi^\varepsilon$. Furthermore, $B_0(I((X^\varepsilon)^{03}), (X^\varepsilon)^{02})$ does not converge, and we need to renormalize it by subtracting $3C^\varepsilon_2 I(X^\varepsilon)$. We have to deal with the (ill-defined) diagonal term.

$X^\varepsilon,0(\Phi^\varepsilon)(t) = B_0(B_<(\Phi^\varepsilon)'), (X^\varepsilon)^{02}, (X^\varepsilon)^{02})(t) = \int_0^t ds P_{t-s} \pi_0 \int_0^s d\sigma P_{s-\sigma} \pi_<(\Phi^\varepsilon)'(\sigma), (X^\varepsilon)^{02}(\sigma), (X^\varepsilon)^{02}(\sigma))$

Thanks to the properties of the paraproduct, we decompose this term in the following way

$$(X^\varepsilon)^{0,0}(\Phi^\varepsilon)'(t) = \int_0^t ds P_{t-s}(\Phi^\varepsilon)'_s \pi_0(I((X^\varepsilon)^{02}(s), (X^\varepsilon)^{02})(s))$$

$$+ \int_0^t ds P_{t-s} \int_0^s d\sigma ((\Phi^\varepsilon)'_s - (\Phi^\varepsilon)'_s) \pi_0((X^\varepsilon)^{02}, (X^\varepsilon)^{02})$$

$$+ \int_0^t ds P_{t-s} \int_0^s d\sigma \pi_0(R_{s-\sigma}^1((\Phi^\varepsilon)'_s, (X^\varepsilon)^{02}, (X^\varepsilon)^{02}))$$

$$+ \int_0^t ds P_{t-s} \int_0^s R^2((\Phi^\varepsilon)'_s, (X^\varepsilon)^{02}, (X^\varepsilon)^{02}))$$

$$\equiv \sum_{i=1} X^{\varepsilon,0,1}(t)$$

with

$$R_{s-\sigma}^1(f, g) = P_{s-\sigma} \pi_<(f, g) - \pi_<(f, P_{s-\sigma}g), \quad R^2(f, g, h) = \pi_0(\pi_<(f, g), h) - f \pi_0(g, h)$$

Here again, to have a convergent quantity we need to renormalize $(X^\varepsilon)^{0,1}$ by $C^\varepsilon_2 \int_0^t P_{t-s}(\Phi^\varepsilon)'_s = C^\varepsilon_2 I(\Phi^\varepsilon)$. Hence, the approximated equation must be

$$\tilde{\Gamma}(\Phi^\varepsilon) = \Phi^\varepsilon$$

with

$$\Gamma^\varepsilon(\Phi^\varepsilon) = \tilde{\Gamma}^\varepsilon(\Phi^\varepsilon) + 9C^\varepsilon_2(\Phi^\varepsilon + X^\varepsilon).$$

The same computation holds for the renormalized equation, and we have

$$I(\Phi^\varepsilon X^\varepsilon) = B_< I(X^\varepsilon), X^\varepsilon) + B_{0,0}(I(X^\varepsilon), X^\varepsilon) + B_{0,0}(B_< I(\Phi^\varepsilon), X^\varepsilon) + B_{0,0}(B'_< I(\Phi^\varepsilon), X^\varepsilon) + B_{0,0}(B_{<,0} I(\Phi^\varepsilon), X^\varepsilon).$$

Indeed, thanks to the Bony paraproduct, the first and the last terms in the r.h.s are well-defined. The only problem is to define $B_{0,0}(,)$. The term in $\Phi^\varepsilon X^\varepsilon$ is also well-defined as $\Phi^\varepsilon \in C^{1+\delta}_T$. The term $B_{0,0}(I(X^\varepsilon), X^\varepsilon)$ is also well-defined by Definition 2.9. So we only have to deal with the diagonal term

$$X^{\varepsilon}(\Phi^\varepsilon)(t) = B_{0,0}(B_< I(\Phi^\varepsilon), X^\varepsilon))(t) = \int_0^t ds P_{t-s} \pi_{0,0} \left( \int_0^s d\sigma P_{s-\sigma} \pi_<(\Phi^\varepsilon)'_s, X^\varepsilon)^{02}(s), X^\varepsilon)^{02}(s) \right)$$

Thanks to the properties of the paraproduct, we decompose this term in the following way

$$X^{\varepsilon}(\Phi^\varepsilon)(t) = \int_0^t ds P_{t-s} \pi_{0,0}(I(X^\varepsilon)^{02}(s), X^\varepsilon)^{02}(s) + \int_0^t ds P_{t-s} \int_0^s d\sigma (\Phi^\varepsilon)'_s \pi_0(X^\varepsilon)^{02}(s), P_{s-\sigma}X^\varepsilon)^{02}(s)$$

$$+ \int_0^t ds P_{t-s} \int_0^s d\sigma P_{s-\sigma}^1(\Phi^\varepsilon)'_s, X^\varepsilon)^{02}(s), X^\varepsilon)^{02}(s)) + \int_0^t ds P_{t-s} \int_0^s R^2(\Phi^\varepsilon)'_s, P_{s-\sigma}X^\varepsilon)^{02}(s), X^\varepsilon)^{02}(s))$$

$$\equiv \sum_{i=1} X^{\varepsilon,0,1}(t)$$
where

\[ R^1_{s, \sigma}(f, g) = P_{s - \sigma} \pi_{<}(f, g) - \pi_{<}(f, P_{s - \sigma}g), \quad R^2(f, g, h) = \pi_0(\pi_{<}(f, g), h) - f \pi_0(g, h) \]

and \( f, g, h \) are distributions lying in the suitable Besov spaces for \( R^1 \) and \( R^2 \) to be defined. Before starting to bound the term \( X^\Phi \), let us give a useful lemma to deal with the explosive Hölder type norm

**Lemma 2.15.** Let \( f \) a space time distribution such that \( \sup_{t \in [0, T]} t^{(r+z)/2} ||f_t||_r < +\infty \) then the following bound holds

\[
\sup_{s, t \in [0, T]} \frac{||I(f)(t) - I(f)(s)||_{a-2b}}{|t-s|^b} \lesssim_{b, a, z, r} T^{\theta} \sup_{t \in [0, T]} t^{(r+z)/2} ||f_t||_r
\]

with \( a + z < 2, \ z + r < 2, \ a - r < 2, \ 0 < a, b < 1 \) and \( \theta > 0 \) is a constant depending only on \( a, r, b, z \).

**Proof.** By a simple computation we have

\[
I(f)(t) - I(f)(s) = I^1_{st} + I^2_{st}
\]

with

\[
I^1_{st} = (P_{t-s} - 1) \int_0^s du P_{s-u} f_u \quad \text{and} \quad I^2_{st} = \int_s^t du P_{t-u} f_u.
\]

Using the lemma (2.5) the following bound holds

\[
||I^1_{st}||_{a-2b} \lesssim |t-s|^b \int_0^t du (t-u)^{-(a-r)/2} u^{-(r+z)/2} \sup_{t \in [0, T]} t^{(r+z)/2} ||f_t||_r < +\infty.
\]

To handle the second we use Hölder inequality,

\[
||I^2_{st}||_{a-2b} \lesssim |t-s|^b \left( \int_0^1 du (t-u)^{-(a-2b-r)/(2(1-b))} u^{-(r+z)/(2(1-b))} \right)^{1-b} \sup_{t \in [0, T]} t^{(r+z)/2} ||f_t||_r < +\infty
\]

which ends the proof. \( \square \)

The following proposition gives us the regularity for our terms.

**Proposition 2.16.** Assume that \( X \in \mathcal{X} \) then there exists a choice of \( L \) such that for all \( z \in (1/2, 2/3) \) the following bound holds

\[
||X^\phi(\mathcal{F}^s) ||_{\ast, 1, T} \lesssim T^\theta (1 + ||X||_{T, K})^2 ||\mathcal{F}^s||_{\ast, 2, T}
\]

with \( K \in [0, 1]^4, \theta > 0 \) are two small parameters depending only on \( L \) and \( z \).

**Proof.** We begin by estimate the first term of the expansion (2.5)

\[
||X_{\ast, 1}^\phi(\mathcal{F}^s)(t)||_{1+\delta} \lesssim \int_0^t ds (t-s)^{-(1+\delta+\eta/2)/2} ||\mathcal{F}^s_{\sigma_0, \delta}(I(X^{\mu_2}(s), X^{\mu_2}) - \varphi_{\sigma} X^{\mu}) + (\sup_{\sigma \in [0, T]} \sigma^{\mu} ||\varphi_{\sigma} X^{\mu}||_{\sigma}) \int_0^t ds (t-s)^{-(1+\delta-\eta)/2} ||\mathcal{F}^s_{\sigma}||_{\eta}
\]

\[
\lesssim ||\mathcal{F}^s||_{\ast, 2, T} (I(X^{\mu_2}), X^{\mu_2}) - \varphi_{\sigma} X^{\mu} ||_{\eta/2} + \sup_{\sigma \in [0, T]} \sigma^{\mu} ||\varphi_{\sigma} X^{\mu}||_{\eta} + 1 \]

\[
\times \int_0^t ds (t-s)^{-(1+\delta+\eta/2)/2} s^{-(\eta+\nu+z)/2} \lesssim_{\beta, L} T^\theta \sup_{\sigma \in [0, T]} ||\mathcal{F}^s||_{\ast, 2, T} (\pi_{0, \delta}(I(X^{\mu_2}), X^{\mu_2}) - \varphi_{\sigma} X^{\mu} ||_{\eta/2} + \sup_{\sigma \in [0, T]} \sigma^{\mu} ||\varphi_{\sigma} X^{\mu}||_{\eta} + 1)
\]

\[
\lesssim_{\beta, L} T^\theta ||\mathcal{F}^s||_{\ast, 2, T} (\pi_{0, \delta}(I(X^{\mu_2}), X^{\mu_2}) - \varphi_{\sigma} X^{\mu} ||_{\eta/2} + \sup_{\sigma \in [0, T]} \sigma^{\mu} ||\varphi_{\sigma} X^{\mu}||_{\eta} + 1)
\]
for $\nu, \eta, \delta > 0$ small enough and with $\theta_1 > 0$ depending on $L$. Hence

$$
\sup_{t \in [0, T]} (1+t^{\nu+\delta})^{\frac{1}{2}} \|X^{\Phi'}(t)\|_{1+\delta} \lesssim_{L, z} T^{\theta_1} \left( \| \pi_0 (I(X^{\phi_2}), X^{\phi_2}) - \varphi X \|_{-\eta/2} + \sup_{\sigma \in [0, T]} \sigma^\nu |\varphi^X_\sigma| + 1 \right)
$$

Let us focus on the second term. We have

$$
\|X^{\phi_2}(\Phi')(t)\|_{1+\delta} \lesssim \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma \left| (\Phi' - \Phi)_\sigma \pi_0 (P_{\sigma - \sigma} X^{\phi_2}_\sigma, X^{\phi_2}_\sigma) \right|_{\beta}
$$

$$
\lesssim \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma (s-\sigma)^{-2+\rho/2} \| \Phi' - \Phi \|_{c-2d} \| X^{\phi_2}_\sigma \|_{1-\rho}
$$

$$
\lesssim \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma (s-\sigma)^{-1+\rho/2} \| \Phi' - \Phi \|_{c-2d} \| X^{\phi_2}_\sigma \|_{1-\rho}
$$

$$
\lesssim \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma (s-\sigma)^{-2+\rho/2} \| \Phi' - \Phi \|_{c-2d} \| X^{\phi_2}_\sigma \|_{1-\rho}
$$

$$
\lesssim \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma (s-\sigma)^{-1+\rho/2} \| \Phi' - \Phi \|_{c-2d} \| X^{\phi_2}_\sigma \|_{1-\rho}
$$

for $\beta = \min(c-2d, \rho) \geq 0$ and all $c, d, \rho > 0$ small enough, $z < 1$ and $\theta_2 > 0$ is a constant depending only on $L$ and $z$. Using the Lemma 2.5 we see

$$
\| R_{\sigma - \sigma}^1 (\Phi'_\sigma, X^{\phi_2}_\sigma) \|_{1+2\beta} \lesssim (s-\sigma)^{-(2+3\beta-\eta)/2} \| \Phi'_\sigma \|_{\theta} \| X^{\phi_2}_\sigma \|_{1-\beta}
$$

for all $\beta > 0, \beta < \eta/3$ small enough. By a straightforward computation we have

$$
\|X^{\phi_3}(\Phi')(t)\|_{1+\delta} \lesssim \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma \left| \pi_0 (R_{\sigma - \sigma}^1 (\Phi'_\sigma, X^{\phi_2}_\sigma), X^{\phi_2}_\sigma) \right|_{\beta}
$$

$$
\lesssim \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma |R_{\sigma - \sigma}^1 (\Phi'_\sigma, X^{\phi_2}_\sigma)\|_{1+2\beta} \| X^{\phi_2}_\sigma \|_{1-\beta}
$$

$$
\lesssim \| X^{\phi_2}_\sigma \|_{C_{T}^{-1-\beta}}^{2} \| \Phi' \|_{1+2\beta} T \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma (s-\sigma)^{-2+\rho/2} \| \Phi' - \Phi \|_{c-2d} \| X^{\phi_2}_\sigma \|_{1-\rho}
$$

$$
\lesssim \| X^{\phi_2}_\sigma \|_{C_{T}^{-1-\beta}}^{2} \| \Phi' \|_{1+2\beta} T \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma (s-\sigma)^{-2+\rho/2} \| \Phi' - \Phi \|_{c-2d} \| X^{\phi_2}_\sigma \|_{1-\rho}
$$

where $\theta_3 > 0$ is a constant depending on $L$ and $z$, $0 < \beta < \eta/3$ small enough and $z < 1$. To treat the last term it is sufficient to use the commutation result given in the Proposition (2.4), indeed we have

$$
\| R^2 (\Phi'_\sigma, P_{\sigma - \sigma} X^{\phi_2}_\sigma, X^{\phi_2}_\sigma) \|_{\eta-3\beta} \lesssim \| \Phi' \|_{\theta} s^{-(\eta+z)/2} (s-\sigma)^{-(2-\beta)/2} \| X^{\phi_2}_\sigma \|_{1-\beta} \| \Phi' \|_{1+2\beta}
$$

for $0 < \beta < \eta/3$ small enough and then

$$
\|X^{\phi_4}(\Phi')(t)\|_{1+\delta} \lesssim \| X^{\phi_2}_\sigma \|_{C_{T}^{-1-\beta}}^{2} \| \Phi' \|_{1+2\beta} T \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma s^{-(\eta+z)/2} (s-\sigma)^{-(2-\beta)/2}
$$

$$
\lesssim \| X^{\phi_2}_\sigma \|_{C_{T}^{-1-\beta}}^{2} \| \Phi' \|_{1+2\beta} T \int_0^t ds (t-s)^{-1+\delta/2} \int_0^s d\sigma s^{-(\eta+z)/2} (s-\sigma)^{-(2-\beta)/2}
$$

$$
\lesssim T^{\theta_3} \| X^{\phi_2}_\sigma \|_{C_{T}^{-1-\beta}}^{2} \| \Phi' \|_{1+2\beta}
$$

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for $\theta_4 > 0$ depending on $L$ and $z < 1$ and $\beta, \eta, \delta > 0$ small enough. Binding all these bounds together we conclude that

$$
\sup_{t \in [0,T]} f^1(1 + \delta)^{1/2} \|X^0(\Phi')(t)\|_{1+\delta} \lesssim_L T^\theta (1 + \|X^0\|_{C^{1-\rho}_T} + \|\pi_0(I(X^0), X^0) - \varphi^X\|_{C^{1-\rho}_T} + \sup_{t \in [0,T]} \|\varphi^X(t)\|^2\|\Phi'\|_{*,2,T})
$$

for $\theta > 0$ depending on $L$ and $z$. The same arguments gives

$$
\sup_{t \in [0,T]} f^1(1 + \delta)^{1/2} \|X^0(\Phi')(t)\|_{1+\delta} \lesssim_L T^\theta (1 + \|X^0\|_{C^{1-\rho}_T} + \|\pi_0(I(X^0), X^0) - \varphi^X\|_{C^{1-\rho}_T} + \sup_{t \in [0,T]} \|\varphi^X(t)\|^2\|\Phi'\|_{*,2,T})
$$

and

$$
\sup_{t \in [0,T]} f^1(1 + \delta)^{1/2} \|X^0(\Phi')(t)\|_{1+\delta} \lesssim_L T^\theta (1 + \|X^0\|_{C^{1-\rho}_T} + \|\pi_0(I(X^0), X^0) - \varphi^X\|_{C^{1-\rho}_T} + \sup_{t \in [0,T]} \|\varphi^X(t)\|^2\|\Phi'\|_{*,2,T})
$$

To obtain the needed bound we still need to estimate the following quantity

$$
\sup_{(s,t) \in [0,T]^2} s^{\frac{\mu}{2+b}} \|X^0(\Phi')(t) - X^0(\Phi')(s)\|_{a-2b} / |t - s|^b.
$$

To deal with is we use the fact that $X^0(\Phi') = I(f^i)$ with

$$
f^1(s) = \Phi' \pi_0(I(X^0)(s), X^0_s), \quad f^2(s) = \int_0^s d\sigma(\Phi' - \Phi') \pi_0(X^0_s, P_{s-\sigma} X^0_s)
$$

and

$$
f^3(s) = \int_0^s d\sigma \pi_0(R^\sigma_{s-\sigma}(\Phi', X^0_s), X^0_s), \quad f^4(s) = \int_0^s R^2(\Phi', P_{s-\sigma} X^0_s, X^0_s).
$$

By a easy computation we have

$$
\|f^1(t)\|_{\eta/2} \lesssim \eta s^{-(\eta + z)/2} \|\Phi'\|_{*,2,T} (1 + \|\pi_0(I(X^0), X^0) - \varphi^X\|_{-\eta/4} + \sup_{t \in [0,T]} \|\varphi^X(t)\|^2\|\Phi'\|_{*,2,T})^2
$$

$$
\|f^2(t)\|_{-d} \lesssim \|\Phi'\|_{*,2,T} \|X^0\|_{-1-d/4} \int_0^s d\sigma (s-\sigma)^{-1+d/2} s^{-\eta/2} \lesssim z_{c,d} s^{d/2-\eta/2} \|\Phi'\|_{*,2,T} \|X^0\|_{-1-d/4}
$$

$$
\|f^3(t)\|_{2\eta/3} \lesssim \|\Phi'\|_{*,2,T} \|X^0\|_{-1-\eta/9} \int_0^s ds (s-\sigma)^{-1+\eta/3} s^{-(\eta + z)/2} \lesssim s^{-(\eta + 9z)/2} \|\Phi'\|_{*,2,T} \|X^0\|_{-1-\eta/9}
$$

with $\nu > 0$ depending only on $L$, and a similar bound for $f^4$ which allows us to conclude by the Lemma (2.15) that we have

$$
\sup_{(s,t) \in [0,T]^2} s^{\frac{\mu}{2+b}} \|X^0(\Phi')(t) - X^0(\Phi')(s)\|_{a-2b} / |t - s|^b \lesssim T^\theta \|\Phi'\|_{*,2,T} \|X^0\|^2_{-1-\rho}
$$

for some $\rho > 0$, $\theta > 0$ and $\eta, c, d > 0$ small enough and $z \in (1/2, 2/3)$. \qed
We are now able to give the meaning of $I(\Phi \otimes X^{\otimes 2})$ for $\Phi \in \mathcal{D}^L_{\mathbb{R}}$.

**Corollary 2.17.** Assume that $X \in \mathcal{X}$ and let $\Phi \in \mathcal{D}^L_{\mathbb{R}}$ then for $z \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and for a suitable choice of $L$ the term $I(\Phi \otimes X^{\otimes 2})[\Phi, X]$ is defined via the following expansion

$$I(\Phi \otimes X^{\otimes 2})[\Phi, X] := B_<(\Phi, X^{\otimes 2}) + B_>(\Phi, X^{\otimes 2}) + B_{0,\delta}(I(X^{\otimes 3}), X^{\otimes 2}) + X^{\otimes}(\Phi') + B_0(\Phi^\dagger, X^{\otimes 2})$$

And we have the following bound

$$||B_0(\Phi^\dagger, X^{\otimes 2})||_{*,1,T} + ||B_>(\Phi, X^{\otimes 2})||_{*,1,T} \lesssim T^\theta||\Phi||_{*,T}||X^{\otimes 2}||_{C^{-1-\rho}}$$

for some $\theta, \rho > 0$ being a non-negative constant depending on $L$ and $z$. Moreover if $a, b \in \mathbb{R}, X \in C^1_T(T^3)$ and $\varphi \in C^\infty([0, T])$ then we have that

$$I(\Phi \otimes X^{\otimes 2})[\Phi, R^e_{a,b} X] = I(\Phi (X^2 - a)) + 3bI(X + \Phi)$$

for every $\Phi \in \mathcal{D}R^e_{a,b} X$.

**Proof.** We remark that all the term in the definition of $I(\Phi \otimes X^{\otimes 2})$ are well-defined due to the Proposition 2.16 and the definition of the paraproduct, and we also notice that

$$||B_0(\Phi^\dagger, X^{\otimes 2})||_{1+\delta} \lesssim \int_0^t ds(t - s)^{-(1+\delta/2)/2}||\Phi^\dagger||_{1+\delta}||X^{\otimes 2}||_{-1-\delta/2}$$

$$\lesssim ||\Phi^\dagger||_{*,1,T}||X^{\otimes 2}||_{C^{-1-\delta/2}} \int_0^t ds(t - s)^{-(1+\delta/2)/2}s^{-(1+\delta+z)/2}$$

$$\lesssim s^{-(3/2\delta+z)/2}||\Phi^\dagger||_{*,1,T}||X^{\otimes 2}||_{C^{-1-\delta/2}}$$

which gives easily

$$\sup_{t \in [0,T]} t^{(1+\delta+z)/2}||B_0(\Phi^\dagger, X^{\otimes 2})(t)||_{1+\delta} \lesssim T^{1/2-\delta}||\Phi^\dagger||_{*,1,T}||X^{\otimes 2}||_{C^{-1-\delta/2}}$$

for $\delta < 1/2$. By a similar computation we obtain that there exists $\theta > 0$ depending on $L$ and $z$ such that

$$\sup_{t \in [0,T]} t^{(1/2+\gamma+z)/2}||B_0(\Phi^\dagger, X^{\otimes 2})(t)||_{1/2+\gamma} + \sup_{t \in [0,T]} t^{(\kappa+z)/2}||B_0(\Phi^\dagger, X^{\otimes 2})(t)||_{\kappa} \lesssim T^\theta||\Phi^\dagger||_{*,1,T}||X^{\otimes 2}||_{C^{-1-\delta/2}}.$$

To obtain the needed bound for this term we still need to estimate the Hölder type norm for it. We remark that

$$||\pi_0(\Phi^\dagger_s, X^{\otimes 2}_s)||_{\delta/2} \lesssim s^{-(1+\delta+z)/2}||\Phi^\dagger_s||_{1+\delta}||X^{\otimes 2}_s||_{-1-\delta/2}$$

and then as usual we decompose the norm in the following way

$$B_0(\Phi^\dagger_t, X^{\otimes 2})(t) - B_0(\Phi^\dagger_s, X^{\otimes 2}_s) = I^1_{st} + I^2_{st}$$

with

$$I^1_{st} = (P_{t-s} - 1) \int_0^t du P_{t-u} \pi_0(\Phi^\dagger_u, X^{\otimes 2}_u), \quad I^2_{st} = \int_s^t du P_{t-u} \pi_0(\Phi^\dagger_u, X^{\otimes 2}_u).$$

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A straightforward computation gives

\[ ||I_{st}||_{a-2b} \lesssim ||\Phi^3||_{1,T}||X^{\Omega^2}||_{C^{1-\delta/2}_T} |t - s|^b \int_0^t du(t-u)^{-\frac{a-\delta}{2}}u^{-\frac{1}{2}(1+\delta+z)/2}\]

\[ \lesssim T^{(1-a-\delta/2-z)/2} |t - s|^b ||\Phi^3||_{1,T}||X^{\Omega^2}||_{C^{1-\delta/2}_T}.\]

For \( I^2 \) we use Hölder inequality which gives

\[ ||I_{st}||_{a-2b} \lesssim |t - s|^b ||\Phi^3||_{1,T}||X^{\Omega^2}||_{C^{1-\delta/2}_T} (\int_0^t du(t-u)^{-a/2(1+\delta)} u^{-\frac{1}{2}(1+\delta+z)})^{1-b}\]

\[ \lesssim T^{(1-a-\delta/2-z)/2} |t - s|^b ||\Phi^3||_{1,T}||X^{\Omega^2}||_{C^{1-\delta/2}_T}\]

for \( a, \delta > 0 \) small enough and \( z < 1 \). We have obtained that

\[ ||B_0(\Phi^3, X^{\Omega^2})||_{1,T} \lesssim T^\theta ||\Phi^3||_{1,T}||X^{\Omega^2}||_{C^{1-\delta/2}_T}\]

for some \( \theta > 0 \) depending on \( L \) and \( z \). The bound for the term \( B_{>}(\Phi, X^{\Omega^2}) \) is obtained by a similar argument and this ends the proof. \( \square \)

**Remark 2.18.** When there are no ambiguity we use the notation \( I(\Phi \hat{\otimes} X^{\Omega^2}) \) instead of \( I(\Phi \hat{\otimes} X^{\Omega^2})[\Phi, X] \).

### 3 Fixed point procedure

Using the analysis of \( I(\Phi \hat{\otimes} X^{\Omega^2}) \) and \( I(\Phi^2 X) \) developed in the previous Section, we can now show that the equation

\[ \Phi = I(X^{\Omega^3}) + 3I(\Phi \hat{\otimes} X^{\Omega^2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi \]

admits a unique solution \( \Phi \in D^L_X \) for a suitable choice of \( L \) and \( z \in (1/2, 2/3) \) via the fixed point method. We also show that if \( u^\varepsilon \) is the solution of the regularized equation and \( \Phi^\varepsilon \) is such that \( u^\varepsilon = X^\varepsilon + \Phi^\varepsilon \) then \( d(\Phi^\varepsilon, \Phi) \) goes to 0 as \( \varepsilon \). Hence, by the convergence of \( X^\varepsilon \) to \( X \) we have the convergence of \( u^\varepsilon \) to \( u = \Phi + X \). Let us begin by giving our fixed point result.

**Theorem 3.1.** Assume that \( X \in X \) and \( u^0 \in C^{-z}(T^3) \) with \( z \in (1/2, 2/3) \). Let \( \Phi \in D^L_X \) and \( \Psi = Pu^0 \) then we define the application \( \Gamma : D^L_X, T \rightarrow C^{-z}(T^3) \) by

\[ \Gamma(\Phi) = I(X^{\Omega^3}) + 3I(\Phi \hat{\otimes} X^{\Omega^2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi \]

where \( I(\Phi \hat{\otimes} X^{\Omega^2}) \) and \( I(\Phi^2 X) \) are given by the Corollary (2.17) and the Proposition (2.14). Then \( \Gamma(\Phi) \in D^L_X \) for a suitable choice of \( L \) and it satisfies the following bound

\[ ||\Gamma(\Phi)||_{1,T} \lesssim (T^\theta ||\Phi||_{1,T})^\gamma (1 + ||X||_{T,K} + ||u^0||_{-z})^3. \]  

(13)

Moreover for \( \Phi_1, \Phi_2 \in D^L_X \) the following bound hold

\[ d_{T,L}(\Gamma(\Phi_1), \Gamma(\Phi_2)) \lesssim T^\theta d_{T,L}(\Phi_1, \Phi_2) (||\Phi_1||_{1,T,K} + ||\Phi_2||_{1,T,K} + ||u^0||_{-z})^3. \]

(14)

for some \( \theta > 0 \) and \( K \in [0,1]^8 \) depending on \( L \) and \( z \). We can conclude that for this choice of \( L \) there exist \( T > 0 \) and a unique \( \Phi \in D^L_X, T \) such that

\[ \Phi = \Gamma(\Phi) = I(X^{\Omega^3}) + 3I(\Phi^2 X^{\Omega^2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi. \]

(15)
Proof. By the the Corollary (2.17) and the Proposition (2.14) we see that $\Gamma(\Phi)$ has the needed algebraic structure of the controlled distribution more precisely

\[ \Gamma(\Phi)^2 = 3\Phi, \quad \Gamma(\Phi)^2 = 3B_>(\Phi, X^{02}) + X^{\Diamond}(\Phi') + 3B_0(\Phi^5, X^{02}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi \]

and $\Gamma(\Phi) \in \mathcal{C}_T$. To show that $\Gamma(\Phi) \in \mathcal{D}_\mathcal{C}^2$ and obtain the first bound it remains to estimate $\|\Phi\|_{*, 2, L, T}$ and $\|\Gamma(\Phi)^2\|_{*, 1, L, T}$. A straightforward computation gives

\[ \|\Phi_t\|_{\eta} \lesssim \|I(X^{03})(t)\|_{\eta} + \|B_<(\Phi', X^{02})(t)\|_{\eta} + \|\Phi_t^2\|_{\eta} \]

\[ \lesssim \|I(X^{03})\|_{\eta} + \|\Phi\|_{*, 2, T}\|X^{02}\|_{-1, \eta} \int_0^t ds (t - s)^{-1 + 2\eta/2} s^{-(\eta + z)/2} + t^{-(\kappa + z)} \|\phi^c\|_{*, 1, T} \]

\[ \lesssim (\|\Phi\|_{*, L, T} + 1)(\|X^{02}\|_{-1, \eta} + \|I(X^{03})\|_{\eta} + 1)t^{\min(1, 1/2, -3\eta/2, -(\kappa + z)/2)}. \]

Then for $0 < \eta < \kappa$ and $\eta < 1/2$ and $z \in (1/2, 2/3)$ small enough we see that

\[ \sup_{t \in [0, T]} t^{(\eta + z)/2} \|\Phi_t\|_{\eta} \lesssim T^{\kappa - \eta}(\|\Phi\|_{*, T} + 1)(\|X^{02}\|_{-1, \eta} + \|I(X^{03})\|_{c_T^+} + 1). \]

We focus on the explosive Hölder type norm for this term, indeed a quick computation gives

\[ \|\Phi_t - \Phi_s\|_{c, -2d} \lesssim \|I(X^{03})(t) - I(X^{03})(s)\|_{c, -2d} + \|B_<(\Phi', X^{02})(t) - B_<(\Phi', X^{02})(s)\|_{c, -2d} + \|\Phi_t^2 - \Phi_s^2\|_{c, -2d}. \]

Let us estimate the first term in the right hand side. Using the regularity for $I(X^{03})$ we obtain that for $d > 0$ small enough and $c < 1/2$

\[ \|I(X^{03})(t) - I(X^{03})(s)\|_{c, -2d} \lesssim |t - s|^d \|I(X^{03})\|_{d, c, -2d}. \]

Then we notice that the increment appearing in second term has the following representation

\[ B_<(\Phi', X^{02}) = I(f) \]

with $f = \pi_<(\Phi', X^{02})$. To treat this term it is sufficient to notice that

\[ \|f_t\|_{-1, \delta} \lesssim \|\Phi_t\|_{\eta}\|X^{02}\|_{-1, \delta} \lesssim t^{-(\eta + z)/2}\|\Phi\|_{*, L, T}\|X^{02}\|_{-1, \delta} \]

and then a usual argument gives

\[ \|B_<(\Phi', X^{02})(t) - B_<(\Phi', X^{02})(s)\|_{c, -2d} \lesssim T^\theta |t - s|^d t^{-(\kappa + z)/2}\|\Phi\|_{*, L, T}\|X^{02}\|_{-1, \delta} \]

for some $\theta > 0$ and $c, \delta > 0$. For the last term we use that

\[ \|\Phi_t^2 - \Phi_s^2\|_{c, -2d} \lesssim |t - s|^d t^{-(\eta + z)/2}\|\Phi\|_{*, L, T} \lesssim T^{b - d + a - c}|t - s|^d t^{-(\kappa + z)/2}\|\Phi\|_{*, L, T} \]

for $c - 2d < a - 2b, d < b$ and then $c < a$ which gives:

\[ \sup_{s, t \in [0, T]} s^{-(\kappa + z)/2}|\Phi_t^2 - \Phi_s^2|_{c, -2d} \lesssim T^{\theta}(1 + \|I(X^{03})\|_{d, c, -2d} + \|X^{02}\|_{-1, \delta})(1 + \|\Phi\|_{*, L, T}). \]

Hence the following bound holds

\[ \|\Gamma(\Phi)^2\|_{*, 2, L, T} \lesssim T^{\theta}(1 + \|I(X^{03})\|_{d, c, -2d} + \|X^{02}\|_{-1, \delta})(1 + \|\Phi\|_{*, T}). \]
We need to estimate the remaining term $\Gamma(\Phi)^3$. Due to the propositions (2.14), (2.16) and the corollary (2.17) it only remains to estimate the following terms $I(\Phi^3)$ and $\Psi$. In fact a simple computation gives

$$||\Psi||_{*,1,L,T} \lesssim ||u^0||_{-z}.$$  

Let us focus to the term $I(\Phi^3)$. We notice that

$$||I(\Phi^3)(t)||_{1+\delta} \lesssim \int_0^t ds (t-s)^{-1+\delta-\eta/2} s^{-\eta+1-2/3} ||\Phi||_{*,T}^3 ||X^{\delta/2}||_{-1-\rho} + 1^3$$

for $\delta, \kappa > 0$ small enough and $z < 2/3$ and we obtain the existence of some $\theta > 0$ such that

$$\sup_{t \in [0,T]} \int_0^t ds (t-s)^{1+\delta+\gamma/2} ||\Phi||_{*,T}^3 (1 + ||X^{\delta/2}||_{-1-\rho})^3$$

A similar argument gives

$$\sup_{t \in [0,T]} \int_0^t ds (t-s)^{1+\delta+\gamma/2} ||\Phi||_{*,T}^3 (1 + ||X^{\delta/2}||_{-1-\rho})^3$$

Let us remark that

$$||\Phi^3||_{\eta} \lesssim t^{-2(1/2+\gamma/2)} ||\Phi||_{*,L,T}^3 (1 + ||X^{\delta/2}||_{-1-\rho})^3$$

and then as usual to deal with the Hölder norms we begin by writing the following decomposition

$$||I(\Phi^3)(t) - I(\Phi^3)(s)||_{c-2d} \lesssim I_{st}^1 + I_{st}^2$$

with

$$I_{st}^1 = (P_{t-s} - 1) \int_0^s du P_{s-u} \Phi^3_u,$$

$$I_{st}^2 = \int_s^t du P_{t-u} \Phi^3_u.$$  

For $I^1$ is suffice to observe that

$$||I_{st}^1||_{c-2d} \lesssim |t-s|^d \int_0^s du (s-u)^{c/(c-2d)} u^{-3/2} ||\Phi||_{*,T}^3 (1 + ||X^{\delta/2}||_{-1-\rho})^3$$

for $\eta, c > 0$ small enough, $z < 2/3$. To obtain the second bound we use the Hölder inequality and then

$$||I_{st}^2||_{c-2d} \lesssim |t-s|^d \left( \int_s^t du ||P_{t-u} \Phi^3_u||_{c-2d} \right)^{1-d}$$

$$\lesssim |t-s|^d \left( \int_s^t du (t-u)^{c/(c-2d)} u^{-3/2} ||\Phi||_{*,T}^3 (1 + ||X^{\delta/2}||_{-1-\rho})^3 \right)^{1-d}$$

$$\lesssim |t-s|^d T^{1-(c-2d)/2} ||\Phi||_{*,T}^3 (1 + ||X^{\delta/2}||_{-1-\rho})^3$$

for $c, \eta, d > 0$ small enough and $z < 2/3$. We can conclude that there exists $\theta > 0$ such that

$$\sup_{s,t} s^{(c+2)/2} \frac{||I(\Phi^3)(t) - I(\Phi^3)(s)||_{c-2d}}{|t-s|^d} \lesssim T^\theta ||\Phi||_{*,T}^3 (1 + ||X^{\delta/2}||_{-1-\rho})^3$$
and then we obtain all needed bounds for the remaining term and we can state that
\[ ||\Gamma(\Phi)^2||_{\ast, 2, L,T} \lesssim (T^0||\Phi||_{\ast, T} + 1)^3(1 + ||X||_{T,K} + ||u^0||_{-z})^3 \]
for some \( K \in [0, 1]^4 \) depending on \( L \) and this gives the first bound (13). The second estimate (14) is obtained by the same manner.

Due to the bound (13) for \( T_1 > T > 0 \) small enough, there exists \( R_T > 0 \) such that
\[ B_{R_T} := \{ \Phi \in D_{X,T}^L; ||\Phi||_{\ast, T} \leq R_T \} \]
is invariant by the map \( \Gamma \). The bound (14) tells us that \( \Gamma \) is a contraction on \( B_{R_{T_2}} \) for \( 0 < T_2 < T_1 \) small enough. Then by the usual fixed point theorem there exists \( \Phi \in D_{X,T_2}^L \) such that \( \Gamma(\Phi) = \Phi \). The uniqueness is obtained by a standard argument. \( \square \)

A quick adaptation of the last proof gives a better result (see for example [6] and the continuity result theorem). In fact the flow is continuous with respect to the rough distribution \( X \) and with respect to the initial condition \( \psi \) (or \( u^0 \)).

**Proposition 3.2.** Let \( X \) and \( Y \) two rough distributions such that \( ||X||_{T,K}, ||Y||_{T,K} \leq R \), \( z \in (-2/3, -1/2) \), \( u_X^0 \) and \( u_Y^0 \) two initial conditions and \( \Phi^X \in D_{X,X}^L \) and \( \Phi^Y \in D_{Y,Y}^L \) the two unique solutions of the equations associating to \( X \) and \( Y \), and \( T_X \) and \( T_Y \) there respective living times. For \( T^* = \inf T_X, T_Y \) the following bound hold
\[ ||\Phi^X - \Phi^Y||_{C([0,T],C^{-z}(\mathbb{R}^3))} \lesssim d_{T,L}(\Phi^X, \Phi^Y) \lesssim_R d_{T,K}(X, Y) + ||u_X^0 - u_Y^0||_{-z} \]
for every \( T \leq T^* \), where \( d \) is defined in Definition 2.12 and \( d \) is defined in Definition 2.9.

Hence, using this result and combining it with the convergence Theorem (4.3), we have this second corollary, where the convergence of the approximated equation is proved.

**Corollary 3.3.** Let \( z \in (1/2, 2/3) \), \( u^0 \in C^{-z} \) and denote \( u^\varepsilon \) the unique solutions (with life times \( T^\varepsilon \) ) of the equation
\[ \partial_t u^\varepsilon - \Delta u^\varepsilon - (u^\varepsilon)^3 + C^\varepsilon u^\varepsilon + \xi^\varepsilon \]
where \( \xi^\varepsilon \) is a mollification of the space-time white noise \( \xi \) and \( C^\varepsilon = 3(C_1^\varepsilon - 3C_2^\varepsilon) \) with \( (C_1^\varepsilon) \) and \( C_2^\varepsilon \) are the constant given by the Definition 4.2. Let us introduce \( u = X + \Phi \) where \( \Phi \) is the local solution with life-time \( T > 0 \) for the fixed point equation given in the Theorem 3.1. Then we have the following convergence result
\[ \mathbb{P}(d_{T^*, L}(\Phi^\varepsilon, \Phi) > \lambda) \rightarrow_{\varepsilon \rightarrow 0} 0 \]
for all \( \lambda > 0 \) with \( T^* = \inf(T, T^\varepsilon) \) and \( \Phi^\varepsilon = u^\varepsilon - X^\varepsilon \in D_{X^\varepsilon,T} \).

### 4 Renormalization and construction of the rough distribution

To end the proof of existence and uniqueness for the renormalized equation, we need to prove that the O.U. process associated to the white noise can be extend to a rough distribution of \( \mathcal{X} \). (see Definition 2.9). As explained above, to define the appropriate process we proceed
by regularization and renormalization. Let us take a a smooth radial function \( f \) with compact support and such that \( f(0) = 1 \). We regularize \( X \) in the following way

\[
X_t^\varepsilon = \sum_{k \neq 0} f(\varepsilon k) \hat{X}_t(k) e_k
\]

and then we show that we can choose two divergent constants \( C_1^\varepsilon, C_2^\varepsilon \in \mathbb{R}^+ \) and a smooth function \( \varphi^\varepsilon \) such that \( R_{C_1^\varepsilon}^\varepsilon C_2^\varepsilon X^\varepsilon := X^\varepsilon \) converge in \( \mathcal{X} \). As it has been noticed in the previous Sections, without a renormalization procedure there is no finite limit for such a process.

**Notation 4.1.** Let \( k_1, \ldots, k_n \in \mathbb{Z}^3 \) we denote by \( k_{1,\ldots,n} = \sum_{i=1}^n k_i \), and for a function \( f \) we denote by \( \delta f \) the increment of the function given by \( \delta f_{st} = f_t - f_s \)

**Definition 4.2.** Let

\[
C_1^\varepsilon = \mathbb{E} \left[ (X^\varepsilon)^2 \right]
\]

and

\[
C_2^\varepsilon = 2 \sum_{k_1 \neq 0, k_2 \neq 0} \frac{|f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{1,2}|^2)}.
\]

Notice that thanks to the definition of the Littlewood-Paley blocs, we can also choose to write \( C_2^\varepsilon \) as

\[
C_2^\varepsilon = 2 \sum_{|i-j| \leq 1 k_1 \neq 0, k_2 \neq 0} \theta(2^{-|k_{1,2}|}) \theta(2^{-j}|k_{1,2}|) \frac{f(\varepsilon k_1) f(\varepsilon k_2)}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{1,2}|^2)}.
\]

Let us define the following renormalized quantities

\[
(X^\varepsilon)^{\otimes 2} := (X^\varepsilon)^2 - C_1^\varepsilon
\]

\[
I((X^\varepsilon)^{\otimes 3}) := I((X^\varepsilon)^3 - 3C_1^\varepsilon X^\varepsilon)
\]

\[
\pi_0 (I((X^\varepsilon)^{\otimes 2}), (X^\varepsilon)^{\otimes 2}) = \pi_0 (I((X^\varepsilon)^{\otimes 2}), (X^\varepsilon)^{\otimes 2}) - C_2^\varepsilon
\]

\[
\pi_0 (I((X^\varepsilon)^{\otimes 3}), (X^\varepsilon)^{\otimes 2}) = \pi_0 (I((X^\varepsilon)^{\otimes 3}), (X^\varepsilon)^{\otimes 2}) - 3C_2^\varepsilon X^\varepsilon.
\]

Then the following theorem holds.

**Theorem 4.3.** For \( T > 0 \), there exists a deterministic sequence \( \varphi^\varepsilon : [0, T] \rightarrow \mathbb{R} \), a deterministic distribution \( \varphi : [0, T] \rightarrow \mathbb{R} \) such that for all \( \delta, \delta', \nu > 0 \) small enough we have

\[
\| \varphi \|_{\nu, \rho, T} = \sup_t t^\nu |\varphi_t| + \sup_{t \neq s} s^\nu |\varphi_t - \varphi_s|^{\rho} < +\infty
\]

and the sequence \( \varphi^\varepsilon \) converges to \( \varphi \) for that norm, that is

\[
\| \varphi^\varepsilon - \varphi \|_{1, \ast, T} \rightarrow 0.
\]

Furthermore there exists some stochastic processes

\[
X^{\otimes 2} \in \mathcal{C}([0, T], \mathcal{C}^{1-\delta})
\]

\[
I(X^{\otimes 3}) \in \mathcal{C}^\delta([0, T], \mathcal{C}^{1/2-\delta-2\delta'})
\]
\[ \pi_0(I(X^{\varphi^3}), X) \in C^{\rho}([0, T], C^{-\delta - 2\delta'}) \]

\[ \pi_{00}(I(X^{\varphi^2}), X^{\varphi^2}) - \varphi \in C^{\rho}([0, T], C^{-\delta - 2\delta'}) \]

\[ \pi_{00}(I(X^{\varphi^3}), X^{\varphi^2}) - 3 \varphi X \in C^{\rho}([0, T], C^{-1/2 - \delta - 2\delta'}). \]

Moreover each component of the sequence \( X^\varepsilon \) converges respectively to the corresponding component of the rough distribution \( X \) in the good topology, that is for all \( \delta, \delta' > 0 \) small enough, and all \( p > 1 \),

\[ X^\varepsilon \to X \in L^p(\Omega, C^{\rho}([0, T], C^{-1 - \delta - 3\delta' - 3/2p})) \]  \hspace{1cm} (17)

\[ (X^\varepsilon)^{\varphi^2} \to X^{\varphi^2} \text{ in } L^p(\Omega, C^{\rho}([0, T], C^{-1 - \delta - 3\delta' - 3/2p})) \]  \hspace{1cm} (18)

\[ I((X^\varepsilon)^{\varphi^3}) \to I(X^{\varphi^3}) \text{ in } L^p(\Omega, C^{\rho}([0, T], C^{1/2 - \delta - 3\delta' - 3/2p})) \]  \hspace{1cm} (19)

\[ \pi_0(I((X^\varepsilon)^{\varphi^3}), X^\varepsilon) \to \pi_0(I(X^{\varphi^3}), X) \text{ in } L^p(\Omega, C^{\rho}([0, T], C^{-\delta - 3\delta' - 3/2p})) \]  \hspace{1cm} (20)

\[ \pi_{00}(I((X^\varepsilon)^{\varphi^2}), (X^\varepsilon)^{\varphi^2} - \varphi \to \pi_{00}(I(X^{\varphi^2}), X^{\varphi^2}) - \varphi \text{ in } L^p(\Omega, C^{\rho}([0, T], C^{-\delta - 3\delta' - 3/2p})) \]  \hspace{1cm} (21)

\[ \pi_{00}(I((X^\varepsilon)^{\varphi^3}), (X^\varepsilon)^{\varphi^2} - 3 \varphi X \to \pi_{00}(I(X^{\varphi^3}), X^{\varphi^2}) \text{ in } L^p(\Omega, C^{\rho}([0, T], C^{-1 - 2 - \delta - 3\delta' - 3/2p})) \]  \hspace{1cm} (22)

**Remark 4.4.** Thanks to the proof below (especially in Subsection 4.5 and 4.6) we have the following expressions for \( \varphi^\varepsilon \) and \( \varphi \).

\[ \varphi_t^\varepsilon = - \sum_{|i-j| \leq 1} \sum_{k_1 \neq 0, k_2 \neq 0} \frac{\theta(2^{-i}|k_1|)|\theta(2^{-j}|k_2|)|f(\varepsilon k_1)f(\varepsilon k_2)|}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)} \exp(-t(|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)) \]

and

\[ \varphi_t = - \sum_{|i-j| \leq 1} \sum_{k_1 \neq 0, k_2 \neq 0} \frac{|\theta(2^{-i}|k_1|)|\theta(2^{-j}|k_2|)}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)} \exp(-t(|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)). \]

We split the proof of this theorem according to the various components. We start by the convergence of \( X^\varepsilon \) to \( X \). Then we also give a full proof for \( X^{\varphi^2} \). For the other components we only prove the crucial estimates.

### 4.1 Convergence for \( X \)

We start by an easy computation for the convergence of \( X \)

**Proof of (17).** By a quick computation we have that

\[ \delta(X - X^\varepsilon)_{st} = \sum_k (f(\varepsilon k) - 1) \delta \hat{X}_{st}(k)e_k \]

and then

\[ \mathbb{E} [\Delta_{\theta} \delta(X - X^\varepsilon)_{st}]^2 = 2 \sum_{k \neq 0, |k| \sim 2^\theta} |f(\varepsilon k) - 1|^2 \frac{1 - e^{-|k|^2|t-s|}}{|k|^2} \lesssim_{h, \rho} c(\varepsilon)2^{(1+2h+\rho)|t-s|} \]

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for $h, \rho > 0$ small enough, and $c(\varepsilon) = \sum_{k \neq 0} |k|^{-3-\rho} |f(\varepsilon k) - 1|^2$. The Gaussian Hypercontractivity gives

$$
\mathbb{E} \left[ |\Delta_q \delta(X - X^\varepsilon)_{st}|_{L^p}^{2} \right] \lesssim_p \int_{\mathbb{R}^3} \mathbb{E} \left[ |\Delta_q \delta(X - X^\varepsilon)_{st}(x)|^2 \right]^{1/p} \, dx \lesssim_{\rho, h} c(\varepsilon)^p |t - s|^{hp/2}.2^{hp/2(2h+\rho+1)}.
$$

for $p > 1$. We obtain that

$$
\mathbb{E} \left[ |\delta(X - X^\varepsilon)_{st}|_{L^{p-1/2-\rho-h}}^{2} \right] \lesssim c(\varepsilon)^{hp/2}
$$

Using the Besov embedding (Proposition 2.2) we get

$$
\mathbb{E} \left[ |\delta(X - X^\varepsilon)_{st}|_{C^{-1/2-\rho-h-3/p}} \right] \lesssim c(\varepsilon)^{p/2} |t - s|^{hp/2}
$$

and by the standard Garsia-Rodemich-Rumsey Lemma (see [4]) we finally obtain :

$$
\mathbb{E} \left[ |X - X^\varepsilon|_{C^{h-\theta}((0,T],C^{-1/2-\rho-h-3/p})} \right] \lesssim c(\varepsilon)^p
$$

for all $h > \theta > 0$, $\rho > 0$ small enough and $p > 1$. Moreover we have $X_0 = X_0^\varepsilon = 0$ and then using the fact that $c(\varepsilon) \to \varepsilon \to 0$ we obtain that

$$
\lim_{\varepsilon \to 0} \|X^\varepsilon - X\|_{L^p(\Omega,C_T^{h-\theta}(-1/2-\delta-3/p))} = 0
$$

for all $0 < \delta' < \delta/3$ and $T > 0$. \hfill \square

### 4.2 Renormalization for $X^2$

To prove the theorem for $X^{\otimes^2}$ we first prove the following estimate, and we use the Garsia-Rodemich-Rumsey lemma to conclude.

**Proposition 4.5.** Let $p > 1$, $\theta > 0$ small enough, then the following bound holds

$$
\sup_{\varepsilon} \mathbb{E} \left[ |\Delta_q \delta(X^\varepsilon_{st})_{\otimes^2} |_{L^{2p}}^{2p} \right] \lesssim_{p, \theta} |t - s|^{2p\theta+2(p+1)}
$$

and

$$
\mathbb{E} \left[ |\Delta_q \left( \delta(X^\varepsilon_{st})_{\otimes^2} - \delta(X^\varepsilon'_{st})_{\otimes^2} \right) |_{L^{2p}}^{2p} \right] \lesssim_{p, \theta} C(\varepsilon, \varepsilon') |t - s|^{2p\theta+2(p+1)}
$$

with $C(\varepsilon, \varepsilon') \to 0$ when $|\varepsilon - \varepsilon'| \to 0$.

**Proof.** By a straightforward computation we have

$$
\text{Var}(\Delta_q((X^\varepsilon_{st} - X^\varepsilon_{st}')(X^\varepsilon_{st}'))) = \sum_{k, k' \in \mathbb{Z}^3} \theta(2^{-q}k)\theta(2^{-q}k') \sum_{k_1, k_2, k_1', k_2'} f(\varepsilon k_1) f(\varepsilon k_2) f(\varepsilon k_1') f(\varepsilon k_2') \times (I_{st}^1 + I_{st}^2) e_{k e_{k'}}
$$

where $(e_k)$ denotes the Fourier basis of $L^2(\mathbb{T}^3)$ and

$$
I_{st}^1 = \mathbb{E} \left[ (\hat{X}_t(k_1) - \hat{X}_s(k_1))(\hat{X}_t(k_1') - \hat{X}_s(k_1')) \right] \mathbb{E} \left[ \hat{X}_s(k_2) \bar{X}_s(k_2') \right] = 2\delta_{k_1=k_1'} \delta_{k_2=k_2'} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2}
$$

and

$$
I_{st}^2 = \mathbb{E} \left[ (\hat{X}_t(k_1) - \hat{X}_s(k_1))(\bar{X}_t(k_1') - \bar{X}_s(k_1')) \right] \mathbb{E} \left[ \hat{X}_s(k_2) \bar{X}_s(k_2') \right] = 2\delta_{k_1=k_1'} \delta_{k_2=k_2'} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2}
$$

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\[ I_{st}^2 = \mathbb{E}\left[ (\hat{\mathcal{X}}_t(k_1) - \hat{\mathcal{X}}_s(k_1)) \overline{\mathcal{X}_s(k_2)} \right] \mathbb{E}\left[ (\mathcal{X}_t(k'_1) - \mathcal{X}_s(k'_1)) \mathcal{X}_s(k_2) \right] \]
\[ = \delta_{k_1=k'_1} \delta_{k_2=k_2} \frac{(1 - e^{-|k_1|^2|t-s|})(1 - e^{-|k_2|^2|t-s|})}{|k_1|^2|k_2|^2}. \]

Injecting these two identities in the equation (23) we obtain that
\[ \text{Var}(\Delta_\rho((X^\varepsilon_t - X^\varepsilon_s)X^\varepsilon_s)) \lesssim \sum_{|k|\sim 2^q, k_{12} = k} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2} \frac{(1 - e^{-|k_1|^2|t-s|})(1 - e^{-|k_2|^2|t-s|})}{|k_1|^2|k_2|^2} \]
\[ \lesssim \sum_{|k|\sim 2^q, k_{12} = k} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2}. \]

We have
\[ \sum_{|k|\sim 2^q, k_{12} = k, |k_1| \leq |k_2|} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2} \lesssim |t-s|^\theta \left\{ \sum_{|k|\sim 2^q, k_{12} = k} \frac{|k_1|^{-2+2\theta} |k_2|^{-2}}{|k_1| \leq |k_2|} + \sum_{|k|\sim 2^q, k_{12} = k} \frac{|k_1|^{-2+2\theta} |k_2|^{-2}}{|k_1| \geq |k_2|} \right\} \]
\[ \lesssim |t-s|^\theta 2q(1+2\theta) \left( \sum_{k_1} |k_1|^{-3-2\theta} + \sum_{k_1} |k_2|^{-3-2\theta} \right) < +\infty \]

and then by the Gaussian hypercontractivity we have
\[ \mathbb{E}\left[ ||\Delta_\rho((X^\varepsilon_t)_{st})^2||_{L^{2p}}^p \right] = \int_{T^q} (\text{Var} (\delta(X^\varepsilon_{st}) (\xi)))^p d\xi \lesssim |t-s|^{\rho \theta 2q(1+2\theta)}. \]

For the second assertion we see that the computation of the beginning gives
\[ \text{Var}(\Delta_\rho((X^\varepsilon_t - X^\varepsilon_s)X^\varepsilon_s) - (X^\varepsilon_t - X^\varepsilon_s)X^\varepsilon_s)) \lesssim |t-s|^\theta 2q(1+3\theta) C(\varepsilon, \varepsilon') \]
where
\[ C(\varepsilon, \varepsilon') = \sum_{k_{12} = k} \left( |f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2 - \delta(f(\varepsilon' k_1)|^2 |f(\varepsilon' k_2)|^2) |k|^{-3-\theta} |k_1|^{-3-2\theta} \rightarrow |\varepsilon - \varepsilon'| \rightarrow 0 \right) \]
by the dominated convergence theorem. Once again the Gaussian hypercontractivity gives us the needed bound.

Using the Besov embedding 2.2 combined with the standard Garsia-Rodemich-Rumsey lemma (see [4]) the following convergence result holds.

**Proposition 4.6.** Let \( \theta, \delta, \rho > 0 \) small enough such that \( \rho < \theta/2 \) and \( p > 1 \) then the following bound hold
\[ \mathbb{E}\left[ ||(X^\varepsilon_t)_{st} - (X^\varepsilon_s)_{st}||_{L^{2p}} \right] \lesssim_{\theta, \rho, \delta} C(\varepsilon, \varepsilon')^p \]
and due to the fact that \( (X^\varepsilon_0)_{st} = 0 \) and \( (X^\varepsilon_0)_{st} = 0 \) we see that the sequence \( (X^\varepsilon)_{st} \) converges in \( L^{2p}(\Omega, C^{\theta/2-\rho}([0,T], C^{-1-\delta/(2p)-\delta-3\theta})) \) to random field noted by \( X^\varepsilon \).
4.3 Renormalization for $I(X^3)$

As the computations are quite similar, we only prove the equivalent of the $L^2$ estimate in proposition (4.5). Furthermore we only prove it for a fixed $t$ and not for an increment.

Proof of (19). By a simple computation we have that

$$ I((X^3_t)^{\circ 3}) = \sum_{k \in \mathbb{Z}^3} \left( \int_0^t \mathcal{F}((X^3_t)^{\circ 3})(k) e^{-|k|^2|t-s|} ds \right) e_k $$

and then

$$ \mathbb{E} \left[ |\Delta q I((X^3_t)^{\circ 3})|^2 \right] = 6 \sum_{k \in \mathbb{Z}^3, k_{123} = k} |\theta(2^{-q}k)|^2 \prod_{i=1,\ldots,3} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_1|^2+|k_2|^2+|k_3|^2|s-\sigma|-|k|^2(t-s)|t-\sigma|} $$

$$ = \sum_k |\theta(2^{-q}k)|^2 \Xi^{\varepsilon,1}(k), $$

where

$$ \Xi^{\varepsilon,1}(k) = \sum_{k_{123} = k, k_i \neq 0} \prod_{i=1,\ldots,3} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_1|^2+|k_2|^2+|k_3|^2|s-\sigma|-|k|^2(t-s)|t-\sigma|} $$

$$ \lesssim \sum_{k_{123} = k} \frac{1}{|k_1|^2|k_2|^2|k_3|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_1|^2+|k_2|^2+|k_3|^2|s-\sigma|-|k|^2(t-s)} $$

$$ \lesssim_T \frac{1}{|k|^2-\rho} \sum_{k_{123} = k} \frac{1}{|k_1|^{4-\rho}|k_2|^2|k_3|^2} \lesssim_T \frac{1}{|k|^{4-\rho}} \sum_{k_2} |k_2|^{-3-\rho}. $$

We have used that

$$ \int_0^t ds \int_0^s d\sigma e^{-|k_1|^2+|k_2|^2+|k_3|^2|s-\sigma|-|k|^2(t-s)|t-\sigma|} \lesssim_T \frac{1}{|k_1|^{2-\rho}|k_2|^2} \int_0^t ds \int_0^s d\sigma |s-\sigma|^{-1+\rho/2} $$

for $\rho > 0$ small enough. Using again the Gaussian hypercontractivity we have

$$ \mathbb{E} \left[ ||\Delta q I((X^3_t)^{\circ 3})||_{L^{2p}}^{2p} \right] \lesssim 2^{-2pq(1/2-\rho)} $$

and then the Besov embedding gives

$$ \sup_{t \in [0,T] \in \epsilon} \mathbb{E} \left[ ||I((X^3_t)^{\circ 3})||_{1/2-\rho-3/3}^{1/2-\rho} \right] < +\infty. $$

The same computation gives

$$ \sup_{t \in [0,T] \in \epsilon} \mathbb{E} \left[ ||I((X^3_t)^{\circ 3}) - I((X^3_t)^{\circ 3})||_{1/2-\rho-3/3}^{2p} \right] \rightarrow |\epsilon' - \epsilon| \rightarrow 0 0 $$

and this gives the needed convergence. \qed

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4.4 Renormalization for $\pi_0(I(X^{03}), X)$

Here, we only prove the $L^2$ estimate for the term $I(X^{03})X$ instead of $\pi_0(I(X^{03}), X)$ since the computations in the two cases are essentially similar. We remark that in that case we do not need a renormalization.

**Proof of (20).** We have the following representation formula

$$E \left[ |\Delta_q (I((X^e)^{03}X^e)) (t)|^2 \right] = \sum_k |\theta(2^{-q}k)|^2 (6I_1^e(t)(k) + 18I_2^e(t)(k) + 18I_3^e(t)(k))$$

with

$$I_1^e(t)(k) = 2 \sum_{k_{1234} = k} \prod_{i=1, \ldots, 4} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)^2s-\sigma]}$$

$$\lesssim \sum_{k_{1234} = k} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)^2s-\sigma]}$$

$$= I_{11}^e(t)(k) + I_{12}^e(t)(k)$$

and

$$I_{11}^e(t)(k) \lesssim \sum_{k_{1234} = k} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)^2s-\sigma]}$$

$$\lesssim \frac{1}{|k_1|^2} \sum_{k_{2,3,1,\text{max}} = |k_1|, |k_1| \leq |k_4|} \frac{1}{|k_2|^{4-\rho}|k_2|^2|k_3|^2|k_{123}|^2|k_1|^2} \lesssim |k|^{-2}$$

for $\rho > 0$ small enough. Hence we obtain the needed result for $I_1^e$. We can treat the second term by a similar computation, indeed

$$I_{12}^e(t)(k) \lesssim \sum_{k_{1234} = k} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)^2s-\sigma]}$$

$$\lesssim |k|^{-2+\rho} \sum_{k_{2,3,4}} |k_2|^{-3-\rho}|k_2|^{-3-\rho}|k_3|^{-3-\rho} \lesssim |k|^{-2+\rho}$$

with $\rho > 0$ small enough; this gives the bound for $I_1^e$. More precisely we have $I_1^e(t)(k) \lesssim |k|^{-2+\rho}$

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for $\rho > 0$ small enough. Let us focus on the second term $I_2^s(t)(k)$ which is given by

$$I_2^s(t)(k) = \sum_{k_{12}=k} \prod_{k_1, k_3, k_4}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^t d\sigma e^{-(|k_1|^2 + |k_2|^2)(s - \sigma) - (|k - k_3|^2 + |k_3|^2)|t-s| - (|k_4|^2 + |k-k_4|^2)|t-\sigma|}$$

$$\lesssim \sum_{k_{12}=k} \frac{1}{\max_{k_3, k_4} |k_1|} \sum_{k_3} \frac{1}{|k_3|^2} \frac{1}{|k_3|^2} \int_0^t d\sigma e^{-(|k_3|^2 + |k-k_3|^2)|t-s|} \lesssim_{\rho, T} \sum_{k_3 \neq 0, k} \frac{1}{|k_3|^{3-3\rho}}$$

and we obtain the bound for $I_2^s$. We notice that

$$I_3^s(t)(k) = \sum_{k_{12}=k} \prod_{k_1, k_3, k_4}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^t d\sigma e^{-(|k_1|^2 + |k_2|^2)(s-\sigma) - (|k-k_3|^2 + |k_3|^2)|t-s| - (|k_4|^2 + |k-k_4|^2)|t-\sigma|}$$

$$= I_2^s(t)(k)$$

We have

$$\sup_{t \in [0,T], \varepsilon} \mathbb{E} \left[ |\Delta_q \left( \mathcal{I}((X^\varepsilon)^{03} X^\varepsilon) (t) \right) |^2 \right] \lesssim_{\rho, T} 2^{q(1+\rho)}$$

which is the wanted bound. \qed

### 4.5 Renormalization for $\pi_0(I(X^{02}), X^{02})$

We only prove the crucial estimate for a renormalization of $\pi_0(I((X^\varepsilon)^{02}, (X^\varepsilon)^{02}))$. We recall that since all the other terms of the product $I((X^\varepsilon)^{02}, (X^\varepsilon)^{02})$ are well-defined and converge to a limit with a good regularity, only this term need to be checked.

**Proof of (21).** Let us begin by giving the computation for the first term. Indeed a chaos decomposition gives

$$-\pi_0(I((X^\varepsilon)^{02})(t), (X^\varepsilon)^{02}) =$$

$$= \sum_{k \in \mathbb{Z}^3} \sum_{|i-j| \leq 1} \sum_{k_{1234}=k} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|)$$

$$\times \int_0^t d\sigma e^{-(|k_{12}|^2 + |i-j|)|t-s|} : \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) \hat{X}_s^\varepsilon(k_3) \hat{X}_s^\varepsilon(k_4) : e_k$$

$$+ 4 \sum_{k \in \mathbb{Z}^3} \sum_{|i-j| \leq 1} \sum_{k_{123}=k} \theta(2^{-i}(|k_{12}|) \theta(2^{-j}|k_{2(-3)}|)|f(\varepsilon k_2)|^2 \int_0^t d\sigma e^{-(|k_{12}|^2 + |i-j|)|t-s|} |k_2|^{-2} : X_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_3) : e_k$$

$$+ 2 \sum_{|i-j| \leq 1} \sum_{k_{12}} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{12}|)|f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2 \frac{1 - e^{-(|k_1|^2 + |k_2|^2 + |k_{12}|^2)t}}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{12}|^2)}$$

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where \( \cdot \cdot \cdot \) denotes the usual Gaussian Wick product. Let us focus on the last term

\[
A^\varepsilon(t) = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} |\theta(2^{-i}k_{12})||\theta(2^{-j}k_{12})||f(\varepsilon k_1)|^2||f(\varepsilon k_2)|^2 \frac{1 - e^{-((|k_1|^2 + |k_2|^2 + |k_{12}|^2)t)} }{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{12}|^2)} = C^\varepsilon_2 + I^\varepsilon_3(t)
\]

where \( I^\varepsilon_3 \) is defined below. Moreover is not difficult to see that

\[
\lim_{\varepsilon \to 0} C^\varepsilon_2 = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} \theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{12}|) |k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{12}|^2) = +\infty.
\]

To obtain the needed convergence we have to estimate the following term

\[
I^\varepsilon_1(t) = \sum_{k \in \mathbb{Z}^4} \sum_{|i-j| \leq 1} \sum_{k_{1234} = k} \theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|) \int_0^t ds e^{-|k_{12}|^2|t-s|} : \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) X_i^\varepsilon(k_3) X_t(k_4) : \varepsilon_k
\]

\[
I^\varepsilon_2(t) = \sum_{k \in \mathbb{Z}^4} \sum_{|i-j| \leq 1} \sum_{k_{13} = k, k_2} \theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{2(-3)}|) |f(\varepsilon k_2)|^2 \int_0^t ds e^{-|k_{12}|^2 + |k_2|^2|t-s|} |k_2|^{-2} X_s^\varepsilon(k_1) X_t^\varepsilon(k_3) e_k
\]

and

\[
I^\varepsilon_3(t) = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} \theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{12}|) |f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2 e^{-|(k_1)^2 + |k_2|^2 + |k_{12}|^2|t} \frac{1}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{12}|^2)}
\]

We notice that for the deterministic part we have the following bound

\[
I^\varepsilon_3(t) \lesssim t^{-\rho} \sum_{k_1, k_2, |k_1| \leq |k_2|} \frac{1}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{12}|^2)^{1+\rho}} \lesssim t^{-\rho} \sum_{k_1, k_2, |k_1| \leq |k_2|} |k_2|^{-4-2\rho} |k_1|^{-2} \lesssim t^{-\rho}
\]

and then the dominated convergence gives for \( \rho > 0 \)

\[
\sup_{t \in [0, T]} t^\rho |I^\varepsilon_3(t) - I_3(t)| \to \varepsilon \to 0 0
\]

with

\[
I_3(t) = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} \theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{12}|) \frac{e^{-((|k_1|^2 + |k_2|^2 + |k_{12}|^2)t)} }{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{12}|^2)}
\]

and this gives the bound for the deterministic part. Let us focus on \( I^\varepsilon_1(t) \) and \( I^\varepsilon_2(t) \). A simple
computation gives

\[
\mathbb{E} \left[ \Delta_q I_k^2(t) \right] = 2 \sum_{k \in \mathbb{Z}^d} \sum_{i \sim j \sim i' \sim j'} I_{1234=k} \cdot \sum \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{12}|) \theta(2^{-j'}|k_{34}|) \prod_{l=1}^{4} \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
\times \int_0^t \int_0^t dsd\sigma e^{-|k_{12}|^2((t-s)+(t-\sigma))-(|k_1|^2+|k_2|^2)|s-\sigma|} \\
+ 2 \sum_{k \in \mathbb{Z}^d} \sum_{i \sim j \sim i' \sim j'} I_{1234=k} \cdot \sum \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{12}|) \theta(2^{-j'}|k_{34}|) \prod_{l=1}^{4} \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
\times \int_0^t \int_0^t dsd\sigma e^{-(|k_1|^2+|k_2|^2)|t-s|-(|k_{34}|^2+|k_3|^2+|k_4|^2)|t-\sigma|} \\
+ 2 \sum_{k \in \mathbb{Z}^d} \sum_{i \sim j \sim i' \sim j'} I_{1234=k} \cdot \sum \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{12}|) \theta(2^{-j'}|k_{34}|) \prod_{l=1}^{4} \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
\times \int_0^t \int_0^t dsd\sigma e^{-(|k_1|^2+|k_2|^2)|t-s|-(|k_{14}|^2+|k_4|^2)|t-\sigma|-|k_1|^2|s-\sigma|}
\equiv 3 \sum_{j=1}^3 I_{1,j}(t).
\]

Let us begin by treating the first term. As usual by symmetry we have

\[
I_{1,1}^*(t) \lesssim \sum_{k \in \mathbb{Z}^d} \sum_{q \leq i} \sum_{k_{1234}=k} \theta(2^{-q}|k|) \theta(2^{-i}|k_{12}|) \prod_{l=1}^{4} \frac{|k_l|^{-2}}{k_l} \int_0^t \int_0^t dsd\sigma e^{-|k_{12}|^2((t-s)+(t-\sigma))-(|k_1|^2+|k_2|^2)|s-\sigma|} \\
+ \sum_{k \in \mathbb{Z}^d} \sum_{q \leq i} \sum_{k_{1234}=k} \theta(2^{-q}|k|) \theta(2^{-i}|k_{12}|) \prod_{l=1}^{4} \frac{|k_l|^{-2}}{k_l} \int_0^t \int_0^t dsd\sigma e^{-|k_{12}|^2((t-s)+(t-\sigma))-(|k_1|^2+|k_2|^2)|s-\sigma|} \\
\equiv A_1^*(t) + A_2^*(t).
\]

We notice that if \( \max_{i=1,\ldots,4} |k_i| = |k_1| \) then \( |k| \lesssim |k_1| \), then

\[
A_1^*(t) \lesssim \sum_{k \in \mathbb{Z}^d} |k|^{-1+2\eta} \theta(2^{-q}|k|) \sum_{k_{1234}=k} \prod_{l=1}^{4} \frac{|k_l|^{-3-\eta/3}}{|k_1|^{-3-\eta/3}} |k_{34}|^{-3-\eta/3} \sum_{q \leq i} 2^{-i(2-\eta)} \lesssim t^{\eta} 2^{3\eta
\}

where we have used

\[
\int_0^t \int_0^t dsd\sigma e^{-|k_{12}|^2((t-s)+(t-\sigma))-(|k_1|^2+|k_2|^2)|s-\sigma|} \lesssim t^\eta \frac{1}{|k_2|^{2-\eta} |k_{12}|^{2-\eta}}.
\]

By a similar argument we have

\[
A_2^*(t) \lesssim \sum_{k \in \mathbb{Z}^d} |k|^{-1+4\eta} \theta(2^{-q}|k|) \sum_{k_{1234}=k} \prod_{l=1}^{4} \frac{|k_l|^{-3-\eta}}{|k_1|^{-3-\eta}} |k_{34}|^{-3-\eta} \sum_{q \leq i} 2^{-i(2-\eta)} \lesssim t^{\eta} 2^{5\eta
\}

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and then \( \sup_r I_{1,1}(t) \lesssim t^{n/2^{5\eta}} \). Let us treat the second term \( I_{1,2}^r(t) \). we have

\[
I_{1,2}^r(t) \lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \leq i-j} \sum_{k_1, k_2, \ldots, k_4 \text{max}(1, \ldots, 4, k_1) = \ldots = k_2} \theta(2^{-q}\|k\|) \theta(2^{-q}\|k_{12}\|) \theta(2^{-j}\|k_{14}\|) \prod_{i=l}^{4} |k_i|^{-2} \\
\times \int_0^t \int_0^t ds dt \epsilon_{k_1, k_2, \ldots, k_4} e^{-|\sum_{i=1}^{4} k_i|^2|t-s| - (|k_{12}|^2 + |k_{13}|^2 + |k_{14}|^2)\|t-s\| - \sum_{i=1}^{4} |k_i|^2} \\
\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \leq j} \sum_{k_1, k_2, \ldots, k_4 \text{max}(1, \ldots, 4, k_1) = \ldots = k_2} \theta(2^{-q}\|k\|) \theta(2^{-i}\|k_{12}\|) \theta(2^{-j}\|k_{14}\|) |k_{12}|^{-2} |k_{13}|^{-2} |k_{14}|^{-2} \|t-s\|^{-\eta} \\
\lesssim t^{n/2^{3\eta}} \sum_{q \leq j} \sum_{l} |l|^{-3-\eta} \lesssim t^{n/2^{5\eta}}.
\]

We have to treat the last term in the fourth chaos. A similar computation gives

\[
I_{1,3}^r(t) \lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \leq i-j} \sum_{k_1, k_2, \ldots, k_4 \text{max}(1, \ldots, 4, k_1) = \ldots = k_2} \theta(2^{-i}\|k_{12}\|) \theta(2^{-j}\|k_{14}\|) \theta(2^{-i}\|k_{14}\|) \theta(2^{-j}\|k_{14}\|) \prod_{i=l}^{4} |k_i|^{-2} \\
\times \int_0^t \int_0^t ds dt \epsilon_{k_1, k_2, \ldots, k_4} e^{-|\sum_{i=1}^{4} k_i|^2|t-s| - (|k_{12}|^2 + |k_{13}|^2 + |k_{14}|^2)\|t-s\|} \\
\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \leq j} \sum_{k_1, k_2, \ldots, k_4 \text{max}(1, \ldots, 4, k_1) = \ldots = k_2} \theta(2^{-i}\|k_{12}\|) \theta(2^{-j}\|k_{14}\|) \theta(2^{-i}\|k_{14}\|) \theta(2^{-j}\|k_{14}\|) \prod_{i=l}^{4} \frac{1}{|k_i|^2} \\
\times \int_0^t \int_0^t ds dt \epsilon_{k_1, k_2, \ldots, k_4} e^{-|k_{12}|^2|t-s| - |k_{14}|^2|t-s|} \\
\lesssim t^{n/2^{3\eta}} \sum_{k \in \mathbb{Z}^3} \sum_{k_1, k_2, \ldots, k_4 \text{max}(1, \ldots, 4, k_1) = \ldots = k_2} \theta(2^{-q}\|k\|) \prod_{k_1, k_2, \ldots, k_4 \text{max}(1, \ldots, 4, k_1) = \ldots = k_2} \frac{1}{|k_1|^2 |k_2|^{-4\eta} |k_3|^2 |k_4|^2}.
\]

We still need to bound the sum

\[
\sum_{k \in \mathbb{Z}^3} \frac{1}{|k_1|^2 |k_2|^{-4\eta} |k_3|^2 |k_4|^2}
\]

for that we notice that when \( |k_3| \leq |k_2| \) we can use the bound

\[
\frac{1}{|k_1|^2 |k_2|^{-4\eta} |k_3|^2 |k_4|^2} \lesssim |k_1|^{-1+4\eta} |k_1|^{-3-\eta} |k_3|^{-3-\eta} |k_4|^{-3-\eta}
\]

and in the case \( |k_2| \leq |k_3| \) we can use that

\[
\frac{1}{|k_1|^2 |k_2|^{-4\eta} |k_3|^2 |k_4|^2} \lesssim |k_1|^{-1+4\eta} |k_1|^{-2} |k_2|^{-4+4\eta} |k_3|^{-1+4\eta} |k_4|^{-2} \lesssim |k_1|^{-1+4\eta} |k_1|^{-3-\eta} |k_2|^{-3-\eta} |k_4|^{-3-\eta}
\]

where we have used that \( |k_4| \leq |k_2| \) and then we can conclude that \( \sup_r I_{1,3}^r(t) \lesssim t^{n/2^{5\eta}} \). This gives the needed bound for the term lying in the chaos of order four; in fact, we have

\[
\sup_r \mathbb{E} \left[ |\Delta_q I_{1,3}^r(t)|^2 \right] \lesssim t^{n/2^{5\eta}}.
\]

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Let us focus on the term lying in second chaos.

\[
\mathbb{E} \left[ |\Delta_q I_2(t) |^2 \right] = \sum_{k \in \mathbb{Z}^3} \sum_{q \leq i \leq j, q \leq i' \leq j'} \sum_{k_1 = k, k_2, k_4} \theta(2^{-i} |k_{12}|) \theta(2^j |k_{2(-3)}|) \theta(2^{-j'} |k_{4(-3)}|) \prod_{i=1}^{4} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \\
\times \int_0^t \int_0^t ds \, ds \, e^{-((k_{12}^2 + |k_2|^2)|t-s| - (|k_{14}^2 + |k_4|^2)|t-\sigma|--\varepsilon k_i^2|s-\sigma|} \\
+ \sum_{k \in \mathbb{Z}^3} \sum_{q \leq i \leq j, q \leq i' \leq j'} \sum_{k_1 = k, k_2, k_4} \theta(2^{-i} |k_{12}|) \theta(2^j |k_{2(-3)}|) \theta(2^{-j'} |k_{34}|) \theta(2^{-j'} |k_{4(-3)}|) \prod_{i=1}^{4} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \\
\times \int_0^t \int_0^t ds \, ds \, e^{-((k_{12}^2 + |k_2|^2)|t-s| - (|k_{34}^2 + |k_4|^2)|t-\sigma|--\varepsilon k_i^2|s-\sigma|} \\
\equiv I_{2,1}^q(t) + I_{2,2}^q(t).
\]

We treat these two terms separately. In fact, by symmetry, we have

\[
I_{2,1}^q(t) \lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \leq i \leq j, q \leq i' \leq j'} \sum_{k_1 = k, k_2, k_4, |k_1| \leq |k_3|} \theta(2^{-i} |k|) \theta(2^{-j} |k_{12}|) \theta(2^j |k_{2(-3)}|) \theta(2^{-j'} |k_{14}|) \theta(2^{-j'} |k_{4(-3)}|) \\
\times \prod_{i=1}^{4} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t \int_0^t ds \, ds \, e^{-((k_{12}^2 + |k_2|^2)|t-s| - (|k_{14}^2 + |k_4|^2)|t-\sigma|--\varepsilon k_i^2|s-\sigma|} \\
\lesssim t^{\nu} \sum_{|k| \sim q} |k|^{-1+\eta} \sum_{q \leq i \leq i'} \theta(2^{-i} |k_{12}|) \theta(2^{-j'} |k_{14}|) \sum_{k_1, k_2, k_4} |k_1|^{-3-\eta} |k_2|^{-3-\eta} |k_3|^{-3-\eta} |k_{34}|^{-1+2\eta} |k_{14}|^{-1+2\eta} \\
\lesssim 2^{\nu (2+\eta)} \sum_{q \leq i \leq i'} 2^{-((i+i')(1-2\eta))} \lesssim t^{\eta} 2^{3\eta
}
\]

which gives the first bound. The second term has a similar bound, indeed

\[
I_{2,2}^q(t) \lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \leq i \leq j, q \leq i' \leq j'} \sum_{k_1 = k, k_2, k_4, |k_1| \leq |k_3|} \theta(2^{-i} |k_{12}|) \theta(2^{-j} |k_{34}|) \prod_{i=1}^{4} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \\
\times \int_0^t \int_0^t ds \, ds \, e^{-((k_{12}^2 + |k_2|^2)|t-s| - (|k_{34}^2 + |k_4|^2)|t-\sigma|--\varepsilon k_i^2|s-\sigma|} \lesssim t^{\eta} 2^{3\eta
}
\]

which ends the proof. \(\square\)

### 4.6 Renormalization for \(\pi_0(I(X^{\circ3}), X^{\circ2})\)

Here again we only give the crucial bound, but for \(I(X^{\circ3}) \cap X^{\circ2}\) instead of \(\pi_0(I(X^{\circ3}), X^{\circ2})\).

**Proposition 4.7.** For all \(T > 0\), \(t \in [0, T]\), \(\delta, \delta' > 0\) and all \(\nu > 0\) small enough, there exists two constants and \(C > 0\) depending on \(T, \delta, \delta'\) and \(\nu\) such that for all \(q \geq -1,

\[
\mathbb{E}[\varepsilon^q \varepsilon^{\delta}|\Delta_q I(I((X^{\circ3}_t)(X^{\circ2}_t) - 3C_T^2 X^{\circ2}_t)|^2] \leq Cq^\delta 2^{q(1+\nu)}.
\]

**Proof.** Thanks to a straightforward computation we have

\[
-I((X^{\circ3}_t)(X^{\circ2}_t) = I^{(1)}_t + I^{(2)}_t + I^{(3)}_t
\]

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where

\[
I^{(1)}_t = \sum_{k \neq 0} e_k \sum_{k_{12345} = k, k_i \neq 0} \int_0^t ds e^{-|k_1+k_2+k_3|^2|t-s|} \cdot \hat{X}_\varepsilon^\varepsilon(k_1) \hat{X}_\varepsilon^\varepsilon(k_2) \hat{X}_\varepsilon^\varepsilon(k_3) \hat{X}_\varepsilon^\varepsilon(k_4) \hat{X}_\varepsilon^\varepsilon(k_5):
\]

\[
I^{(2)}_t = 6 \sum_{k \neq 0} e_k \sum_{k_{12} = k, k_i \neq 0} \int_0^t ds e^{-|k_1+k_2+k_3|^2|t-s|} e^{-|k_3|^2|t-s|} \frac{|k_3|^2}{|k_3|^2} f(\varepsilon k_3)^2 \cdot \hat{X}_\varepsilon^\varepsilon(k_1) \hat{X}_\varepsilon^\varepsilon(k_2) \hat{X}_\varepsilon^\varepsilon(k_4):
\]

and

\[
I^{(3)}_t = 6 \sum_{k \neq 0} e_k \int_0^t ds \sum_{k_{12} \neq 0} f(\varepsilon k_1)^2 f(\varepsilon k_2)^2 e^{-|(k+k_1+k_2)^2+k_1^2+k_2^2|t-s|} \hat{X}_\varepsilon^\varepsilon(k):
\]

Hence,

\[
- (I((X_t^\varepsilon)^{\varepsilon3})(X_t^\varepsilon)^{\varepsilon2} - 3C_2^\varepsilon X_t^\varepsilon) = I((X_t^\varepsilon)^{\varepsilon3})(X_t^\varepsilon)^{\varepsilon2}I^{(3)}_t
\]

\[
+ (I^{(3)}_t - \tilde{I}^{(3)}_t) + (\tilde{I}^{(3)}_t - 3\tilde{C}_2(t) X_t^\varepsilon) + 3(C_2^\varepsilon - \tilde{C}_2(t)) X_t^\varepsilon
\]

where we remind that

\[
C_2^\varepsilon = \sum_{k_{12} \neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)}
\]

and where we have defined

\[
\tilde{I}^{(3)}_t = 6 \sum_{k \neq 0} e_k \int_0^t ds \sum_{k_{12} \neq 0} f(\varepsilon k_1)^2 f(\varepsilon k_2)^2 e^{-|(k+k_1+k_2)^2+k_1^2+k_2^2|t-s|} \hat{X}_\varepsilon^\varepsilon(k):
\]

and

\[
\tilde{C}_2^\varepsilon = 2 \int_0^t ds \sum_{k_{12} \neq 0} f(\varepsilon k_1)^2 f(\varepsilon k_2)^2 e^{-|(k+k_1+k_2)^2+k_1^2+k_2^2|t-s|}.
\]

Hence for \( q \geq -1, \)

\[
\mathbb{E}[|\Delta_q(I((X_t^\varepsilon)^{\varepsilon3})(X_t^\varepsilon)^{\varepsilon2} - 3C_2^\varepsilon X_t^\varepsilon)|^2] \lesssim \mathbb{E}[|\Delta_q(I^{(1)}_t)|^2] + \mathbb{E}[|\Delta_q(I^{(2)}_t)|^2] + \mathbb{E}[|\Delta_q(I^{(3)}_t) - \tilde{I}^{(3)}_t)|^2]
\]

\[
+ \mathbb{E}[|\Delta_q(I^{(3)}_t) - \tilde{C}_2(t) X_t^\varepsilon)|^2] + |C_2^\varepsilon - \tilde{C}_2(t)|^2 \mathbb{E}[|\Delta_q X_t^\varepsilon|^2].
\]

**Terms in the first chaos.** Let us first deal with the “deterministic” part, here \( C_2^\varepsilon - \tilde{C}_2(t). \) An obvious computation gives for all \( \delta' > 0, |C_2^\varepsilon - \tilde{C}_2(t)|^2 \lesssim 1/t^{\delta'} \). Furthermore, \( \mathbb{E}[|\Delta_q X_t^\varepsilon|^2] \lesssim 2^q \), hence for all \( \delta' > 0, \)

\[
|C_2^\varepsilon - \tilde{C}_2(t)|^2 \mathbb{E}[|\Delta_q X_t^\varepsilon|^2] \lesssim 2^q /t^{\delta'}
\]

Let us deal with \( \mathbb{E}[|\Delta_q(I^{(3)}_t) - \tilde{I}^{(3)}_t)|^2]. \) For \( k \neq 0 \) we define

\[
a_k(t-s) = \sum_{k_{12} \neq 0} f(\varepsilon k_1)^2 f(\varepsilon k_2)^2 e^{-|(k+k_1+k_2)^2+k_1^2+k_2^2|t-s}|k_1|^2 |k_2|^2
\]

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such that
\[ \mathbb{E}[|\hat{\Delta}_q(I_t^{(3)} - \hat{I}_t^{(3)})|^2] \]
\[ = \mathbb{E} \left[ \int_0^t \sum_k \theta(2^{-q}k) e_k a_k(t-s)(\hat{X}^*_t(k) - \hat{X}_t(k))^2 \right] \]
\[ = \int_{[0,t]^2} dsdt \sum_{k \neq 0} e_k e_k^* \theta(2^{-q}k) \theta(2^{-q}k) a_k(t-s) a_k(t-s) \mathbb{E}[(\hat{X}^*_t(k) - \hat{X}_t(k))(\hat{X}^*_t(k) - \hat{X}_t(k))] \]

But
\[ \mathbb{E}[(\hat{X}^*_t(k) - \hat{X}_t(k))(\hat{X}^*_t(k) - \hat{X}_t(k))] = \delta_{k=-k} \frac{f(\epsilon k)^2}{|k|^2} (e^{-(s-\overline{\sigma})|k|^2} - e^{-(t-\overline{\sigma})|k|^2} - e^{-(t-s)|k|^2} + 1) \]
\[ \lesssim \delta_{k=-k} \frac{f(\epsilon k)^2}{|k|^2} |k|^{2\eta} |t-s|^\eta |t-\overline{\sigma}|^{\eta/2}. \]

Hence
\[ \mathbb{E}[|\hat{\Delta}_q(I_t^{(3)} - \hat{I}_t^{(3)})|^2] \lesssim \sum_{k \neq 0} \theta(2^{-q}k)^2 \frac{f(\epsilon k)^2}{|k|^{2(1-n)}} \left( \int_0^t ds |t-s|^\eta a_k(|t-s|) \right)^2 \]
and
\[ \int_0^t ds |t-s|^\eta a_k(|t-s|) = \sum_{k_1 \neq 0 \atop k_2 \neq 0} \int_0^t ds |t-s|^\eta e^{-(k_1+k_2)^2|s-k_1|^2+|k_1|^2+|k_2|^2|t-s|} f(\epsilon k_1)^2 f(\epsilon k_2)^2 \frac{|k|^2 |k|^2}{|k_1|^2 |k_2|^2} \]
\[ \lesssim \sum_{k_1 \neq 0 \atop k_2 \neq 0} |k_1|^{-3-\eta'} |k_2|^{-3-\eta'} \int_0^t ds |t-s|^{-1+(n/2-\eta')} \lesssim t^{\eta/2-n'} \]
for \( \eta/2 - \eta' > 0 \). Then we have
\[ \mathbb{E}[|\hat{\Delta}_q(I_t^{(3)} - \hat{I}_t^{(3)})|^2] \lesssim 2^{(1+2n)} t^{\eta-2\eta'}. \]

We have furthermore
\[ \mathbb{E}[|\hat{\Delta}_q(I_t^{(3)} - C_2^* X_t^*)|^2] = \sum_{k \neq 0} \frac{f(\epsilon k)^2}{|k|^2} \theta(2^{-q}k)^2 b_k(t)^2 \]
with
\[ b_k(t) = \int_0^t \sum_{k_1 \neq 0 \atop k_2 \neq 0} \frac{f(\epsilon k_1)^2 f(\epsilon k_2)^2}{|k_1|^2 |k_2|^2} e^{-(k_1^2+k_2^2)|t-s|} \left\{ e^{-|k_1+k_2|^2|t-s|} - e^{-|k_1+k_2|^2|t-s|} \right\}. \]

Using that
\[ |e^{-|k_1+k_2|^2|t-s|} - e^{-|k_1+k_2|^2|t-s|}| \lesssim |t-s|^\eta |k|^\eta (|k| + \max\{|k_1|, |k_2|\})^\eta \]
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we have the following bound
\[
b_k(t) \lesssim \int_0^t \sum_{k_1 \neq 0, k_2 \neq 0} |k_1|^{-3-\eta'} |k_2|^{-3-\eta''} |k|^{\eta} (|k| + \max\{|k_1|, |k_2|\})^\eta |t - s|^{-1+(\eta-\eta'/2-\eta''/2)}.
\]
We can suppose that \(\max\{|k_1|, |k_2|\} = |k_1|\) as the expression is symmetric in \(k_1, k_2\), then if \(|k| > |k_1|\),
\[
b_k(t) \lesssim t^{(\eta-\eta'/2-\eta''/2)} |k|^{2\eta}
\]
for \(\eta - \eta'/2 - \eta''/2 > 0\). Furthermore if \(|k_1| > |k|\), and \(\eta' > \eta\) then
\[
b_k(t) \lesssim t^{(\eta-\eta'/2-\eta''/2)} |k|^{\eta} \sum_{k_1 \neq 0, k_2 \neq 0} |k_1|^{-3-(\eta'-\eta)} |k_2|^{-3-\eta''} \lesssim t^{(\eta-\eta'/2-\eta''/2)} |k|^{\eta}.
\]
Hence, there exists \(\delta > 0\) and \(\nu > 0\) such that
\[
\mathbb{E}[|\Delta_\theta(I_t^{(3)} - 3C_2^\tau X_\tau^\xi)|^2] \lesssim t^{2(1+\nu)\eta}.
\]

**Terms in the third chaos.** Let us define \(c_{k_1,k_2}(t-s) = \sum_{k_3 \neq 0} \frac{f(\varepsilon k_3)^2}{|k_3|^2} e^{-(|k_1+k_2+k_3|^2 + |k_3|^2)(t-s)}\) such that
\[
I_t^{(2)} = 6 \sum_{k \neq 0, k_1 \neq 0, k_{124} = k} c_k \int_0^t ds c_{k_1,k_2}(t-s) : \hat{X}_s(k_1) \hat{X}_s(k_2) \hat{X}_s^\xi(k_4) :
\]
But for all suitable variables we have
\[
\mathbb{E}[\hat{X}_s^\xi(k_1) \hat{X}_s^\xi(k_2) \hat{X}_s^\xi(k_4) :: \hat{X}_s^\xi(k_1) \hat{X}_s^\xi(k_2) \hat{X}_s^\xi(k_4)] =
2 \delta_{k_1=-\bar{k}_1} \frac{f(\varepsilon k_1)^2}{|k_1|^2} \delta_{k_2=-\bar{k}_2} \frac{f(\varepsilon k_2)^2}{|k_2|^2} \delta_{k_3=-\bar{k}_3} \frac{f(\varepsilon k_3)^2}{|k_3|^2} e^{-(|k_1+k_2|^2 + |k_3|^2)(s-\bar{s})}
+ 2 \delta_{k_1=-\bar{k}_1} \frac{f(\varepsilon k_1)^2}{|k_1|^2} \delta_{k_2=-\bar{k}_2} \frac{f(\varepsilon k_2)^2}{|k_2|^2} \delta_{k_3=-\bar{k}_3} \frac{f(\varepsilon k_3)^2}{|k_3|^2} e^{-|k_1|^2 s-\bar{s}} e^{-(|k_3|^2)(t-\bar{\tau})} e^{-(|k_2|^2)(t-\bar{s})}
\times e^{-|k_1|^2 |s-\bar{s}|} e^{-(|k_3|^2)|t-\bar{\tau}|} e^{-(|k_2|^2)|t-\bar{s}|}.
\]
and by an easy computation the following holds \(\mathbb{E}[|\Delta_\theta(I_t^{(2)})|^2] = E_t^{2,1} + E_t^{2,2}\) with
\[
E_t^{2,1} = 2 \int_0^t ds \int_0^s d\bar{s} \sum_{k, k_1 \neq 0, k_{124} = k} \theta(2-\eta) \frac{f(\varepsilon k_1)^2}{|k_1|^2} c_{k_1,k_2}(t-s) c_{k_1,k_2}(t-\bar{s}) e^{-(|k_1|^2 + |k_3|^2)(s-\bar{s})}
\]
and
\[
E_t^{2,1} = 2 \int_0^t ds \int_0^s d\bar{s} \sum_{k \neq 0, k_1 \neq 0, k_{124} = k} \theta(2-\eta) \frac{f(\varepsilon k_1)^2}{|k_1|^2} c_{k_1,k_2}(t-s) c_{k_1,k_4}(t-\bar{s}) e^{-|k_3|^2 |s-\bar{s}|} e^{-|k_4|^2 |t-\bar{\tau}|} e^{-|k_2|^2 |t-\bar{s}|}.
\]
In $E_{t}^{2,1}$, we have a symmetry in $k_1, k_2$, hence we can assume that $|k_1| \geq |k_2|$. Furthermore, we have $c_{k_1,k_2} (t-s) \lesssim |t-s|^{-\frac{1}{2q^2}}$ and $c_{k_1,k_2} (t-\bar{s}) \lesssim |s-\bar{s}|^{-\frac{1}{2q^2}}$. If we assume that $|k_1| \geq |k_4|$ and that $\eta''/2 - \eta > 0$, then

$$E_{t}^{2,1} \lesssim \int_{0}^{t} ds \int_{0}^{s} d\bar{s} |t-s|^{-\frac{1}{2q^2}} |s-\bar{s}|^{-1+(\eta'/2-\eta)} \sum_{k \neq 0, k \neq 0, k_{124} = k} \theta(2^{-q}k)^2 \frac{1}{|k_1|^{1-\eta} |k_2|^2 |k_4|^2}$$

$$\lesssim t^\delta \sum_{k \neq 0, k \neq 0, k_{124} = k} \theta(2^{-q}k)^2 \frac{1}{|k_2|^{-3-\frac{\eta''}{2}} |k_4|^{-3-\frac{\eta''}{2}}} \lesssim t^\delta 2^q(\eta'')$$

for $\eta'' > \eta'$. When $|k_4| \geq |k_1|$ it is almost the same computation.

In $E_{t}^{2,2}$, we can assume that $|k_2| \geq |k_4|$, so

$$E_{t}^{2,2} \lesssim \int_{0}^{t} ds \int_{0}^{s} d\bar{s} \sum_{k \neq 0, k \neq 0, k_{124} = k} \theta(2^{-q}k)^2 |k_1|^{-3+\eta'} |k_2|^{-3+\eta'} |k_4|^2 |t-s|^{-1+\frac{\eta''}{2}} |s-\bar{s}|^{-1+\frac{\eta''}{2}}$$

$$\lesssim t^\delta \sum_{k \neq 0, k \neq 0, k_{124} = k} \theta(2^{-q}k)^2 |k_1|^{-1+\eta''} |k_2|^{-3+\eta'} |k_4|^2 \max(|k_1|)^{-1''} \lesssim t^\delta 2^{q(1+\eta'')}

that we decompose as in the previous term whether $|k_1| \geq |k_4|$ or $|k_4| \geq |k_1|$.

**Terms in the fifth chaos.** For all suitable variables, we have

$$\mathbb{E} \left[ \hat{X}_t^y (k_1) \hat{X}_t^y (k_2) \hat{X}_t^y (k_3) \hat{X}_t^y (k_4) \hat{X}_t^y (k_5) \right] = 12 \prod_{i=1}^{5} \frac{f(\varepsilon k_i)}{|k_i|^2} \delta_{k_i} = -k_i e^{-|s-\pi|(|k_1|^2+|k_2|^2+|k_3|^2)}$$

$$+ 72 \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_i} = -k_i \delta_{k_2} = -k_2 \delta_{k_3} = -k_3 \delta_{k_4} = -k_4 \delta_{k_5} = -k_5 \delta_{k} = -k \frac{e^{-|s-\pi|(|k_1|^2+|k_2|^2)-|t-s||k_3|^2-|t-\pi||k_4|^2}}{e^{-|s-\pi|(|k_1|^2+|k_2|^2)+|t-s||k_3|^2-|t-\pi||k_4|^2} +}$$

$$+ 36 \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_i} = -k_i \delta_{k_2} = -k_2 \delta_{k_3} = -k_3 \delta_{k_4} = -k_4 \delta_{k_5} = -k_5 \delta_{k} = -k \frac{e^{-|s-\pi|(|k_1|^2-|t-s||k_3|^2)-|t-\pi||k_4|^2}}{e^{-|s-\pi|(|k_1|^2+|k_2|^2)+|t-s||k_3|^2-|t-\pi||k_4|^2}}$$

Then

$$\mathbb{E} \left[ |\Delta_q I_{t}^1|^2 \right] = E_{t}^{1,1} + E_{t}^{1,2} + E_{t}^{1,3}$$

with

$$E_{t}^{1,1} = 12 \int_{[0,t]^2} ds d\bar{s} \theta(2^{-q}k)^2 \sum_{k \neq 0, k \neq 0, k_{12345} = k} \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2} e^{-|k_{12345}|^2 |s-\bar{s}|} e^{-|k_1|^2+|k_2|^2+|k_3|^2}|s-\bar{s}|$$

$$E_{t}^{1,2} = 72 \int_{[0,t]^2} ds d\bar{s} \sum_{k \neq 0, k \neq 0, k_{12345} = k} \theta(2^{-q}k)^2 \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2}$$

$$\times e^{-|k_{12345}|^2+|k_3|^2} |t-s| e^{-|k_{12345}|^2+|k_4|^2} |t-\pi| e^{-|s-\pi|(|k_1|^2+|k_2|^2)}$$

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and

\[ E_{t}^{1,3} = 36 \int_{0}^{t} ds \int_{0}^{t} d\tau \sum_{k \neq 0, k \neq 0, k_{12345} = k} \theta(2^{-q}k)^{2} \prod_{i=1}^{5} \frac{f(\varepsilon k_{i})^{2}}{|k_{i}|^{2}} \]

\[ \times e^{-((k_{123})^{2} + (k_{2})^{2} + (k_{3})^{2})|t-s|} e^{-(k_{145})^{2} + (k_{5})^{2} + (k_{4})^{2})|t-\tau|} e^{-|s-\tau| |k_{1}|^{2}} \]

id est

\[ E_{t}^{1,3} = e^{-((k_{123})^{2} + (k_{2})^{2} + (k_{3})^{2})|t-s|} e^{-(k_{145})^{2} + (k_{5})^{2} + (k_{4})^{2})|t-\tau|} e^{-|s-\tau| |k_{1}|^{2}} \]

**Estimation of** \( E_{t}^{1,1} \). Let us rewrite it in a form better to handle

\[ E_{t}^{1,1} = 12 \int_{[0,t]^{2}} ds d\tau \sum_{k \neq 0, k_{1} + k_{2} + l = k, l_{1} + l_{2} + l_{3} = l} \theta(2^{-q}k)^{2} \prod_{i=1}^{2} \frac{f(\varepsilon k_{i})^{2}}{|k_{i}|^{2}} \prod_{i=1}^{3} \frac{f(\varepsilon l_{i})^{2}}{|l_{i}|^{2}} e^{-|l|^{2}|t-s|} e^{-(k_{1}^{2} + l_{2}^{2} + l_{3}^{2})|s-\tau|} \]

Thanks to the symmetries of this term, we can always assume that \(|k_{1}| = \max(|k_{i}|)\) and \(l_{1} = \max(|l_{i}|)\).

For \( l = 0 \), we have

\[ \int_{[0,t]^{2}} ds d\tau \sum_{k \neq 0, k_{1} + k_{2} = k, l_{1} + l_{2} + l_{3} = 0} \theta(2^{-q}k)^{2} \prod_{i=1}^{2} \frac{f(\varepsilon k_{i})^{2}}{|k_{i}|^{2}} \prod_{i=1}^{3} \frac{f(\varepsilon l_{i})^{2}}{|l_{i}|^{2}} e^{-|l|^{2}|t-s|} e^{-(k_{1}^{2} + l_{2}^{2} + l_{3}^{2})|s-\tau|} \]

\[ \leq 2^{q(2+\eta)} t. \]

Let us assume that \(|l| = \max(|l|, |k_{1}|)\); as we have the following estimate \(|l_{1}|^{-1} \leq |l|^{-1} \), the following bound holds

\[ \int_{[0,t]^{2}} ds d\tau \sum_{k \neq 0, k_{1} + k_{2} = 0, l_{2} + l_{3} = 0} \theta(2^{-q}k)^{2} |k|^{-1+\eta} \sum_{k_{2} \neq 0} |k_{2}|^{-3+\eta} \sum_{l_{2} \neq 0, l_{3} \neq 0} |l_{2}|^{-4+\eta} |l_{3}|^{-4+\eta} |s-\tau|^{-1+\eta} \]

\[ \leq 2^{q(2+\eta)} t \]

The case in which \(|k_{1}| = \max(|l|, |k_{1}|)\) is quite similar, and the conclusion holds for \( E_{t}^{1,1} \).

**Estimation of** \( E_{t}^{1,2} \). This term is symmetric in \( k_{1}, k_{2} \) and in \( k_{3}, k_{4} \). Hence, we can assume that \(|k_{1}| \geq |k_{2}|\) and \(|k_{3}| \geq |k_{4}|\) First let us assume that \(|k_{5}| = \max\{|k_{i}|\}\). Then

\[ E_{t}^{1,2} \leq \sum_{k_{12345} = k} \theta(2^{-q}k)^{2} \int_{0}^{t} ds \int_{0}^{s} d\tau (|t-s||s-\tau|)^{-1+\eta} \]

\[ \times |k_{1}|^{-4+2\eta} |k_{2}|^{-2} |k_{3}|^{-4+2\eta} |k_{4}|^{-2} |k_{5}|^{-1+\eta} |k_{1}|^{-(1-\eta)} \]

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\begin{align*}
&\lesssim t^n \sum_k \theta(2^{-q}k)^2 |k|^{-(1-\nu')} \sum_{k_{12345} = k} |k_1|^{-7/2+2\eta} |k_2|^{-3-\nu'/2} |k_3|^{-7/2+2\eta} |k_4|^{-3-\nu'/2} \\
&\lesssim t^n 2^{(2+\nu')q}
\end{align*}
for \( \eta \) small enough.

Then assume that \( \max\{|k_i|\} = |k_1| \)

\[ E_t^{1.2} \lesssim t^\delta \sum_{k \in k_{12345}} \theta(2^{-q}k)^2 |k_1|^{-4+2\eta}|k_2|^{-2}|k_3|^{-3+\nu'|} |k_4|^{-3+\nu'} |k_5|^{-2} \int_0^t ds \int_0^s d\varpi |t - s|^{-1+\nu'} |s - \varpi|^{-1+\eta} \]

\[ \lesssim t^\nu \sum_{k \in k_{12345}} \theta(2^{-q}k)^2 |k_1|^{-1+\nu''}|k_2|^{-3-\nu''}|k_3|^{-7/2+(2\eta+\nu''+\nu')/2} |k_4|^{-7/2+(2\eta+\nu''+\nu')/2} |k_5|^{-3-\nu''} \]

\[ \lesssim t^\delta 2^{(2+\nu')q} \]

For \( \max\{|k_i|\} = |k_3| \)

\[ \lesssim t^\delta \sum_{k \in k_{12345}} \theta(2^{-q}k)^2 |k_1|^{-4+\eta}|k_2|^{-2}|k_3|^{-4+\nu'|} |k_4|^{-2} |k_5|^{-2} \]

\[ \lesssim t^\delta \sum_{k \in k_{12345}} \theta(2^{-q}k)^2 |k_1|^{-3+\nu'+1/4}|k_2|^{-3+1/4} |k_1|^{-1+\nu'|} |k_4|^{-3+1/4} |k_5|^{-3+1/4} \lesssim t^\delta 2^{(2+\nu')q} \]

hence there exists \( \delta > 0 \) and \( \nu > 0 \) such that

\[ E_t^{1.2} \lesssim t^\delta 2^{(2+\nu')q}. \]

**Estimation of** \( E_t^{1.3} \). Let us deal with this last term. Here the symmetries are in \( k_2, k_3 \) and \( k_4, k_5 \). Then we can suppose that \( |k_2| \geq |k_3| \geq |k_4| \geq |k_5| \). Furthermore, the role of \( k_2, k_3 \) and \( k_4, k_5 \) are symmetrical, then we can assume that \( |k_1| \geq |k_4| \)

\[ E_t^{1.3} = \int_{[0,t]^2} ds d\varpi \sum_{k \in k_{12345}} \theta(2^{-q}k)^2 \sum_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \prod_{j \neq i} |k_j|^2 \]

\[ \times e^{-(|k_{12345}|^2 + |k_2|^2 + |k_3|^2)|t - s|} e^{-(|k_{12345}|^2 + |k_4|^2 + |k_4|^2)|t - \varpi|} e^{-|s - \varpi| |k_1|^2} \]

If \( |k_1| = \max\{|k_i|\} \) then

\[ \lesssim \int_{[0,t]^2} ds d\varpi (|t - s||t - \varpi|)^{1+\eta} \sum_{k \in k_{12345}} \theta(2^{-q}k)^2 |k|^{-1+\eta} |k_2||k_3||k_3||k_4|^{-7/4 + 3\eta/4} \]

\[ \lesssim t^\delta 2^{(2+\nu')q} t^\eta \]

If \( |k_2| = \max\{|k_i|\} \) then

\[ \lesssim 2 \int_0^t \int_0^s ds d\varpi (|t - s||s - \varpi|)^{1+\eta} \sum_{k \in k_{12345}} \theta(2^{-q}k)^2 |k|^{-1+\eta} |k_1||k_3||k_3||k_4|^{-7/4 + 3\eta/4} \]

\[ \lesssim t^\delta 2^{(2+\nu')q} \]
A A commutation lemma

We give the proof of the Lemma 2.5. This proof is from Gubinelli, Imkeller and Perkowski, and can be found in the first version of [7] and also in [14] Lemmas 5.3.20 and 5.5.7. In fact we give a stronger result, and apply it with \( \varphi(k) = \exp(-|k|^2/2) \).

Lemma A.1. Let \( \alpha < 1 \) and \( \beta \in \mathbb{R} \). Let \( \varphi \in \mathcal{S} \), \( \psi \in C^\alpha \), and \( v \in C^\beta \). Then for every \( \varepsilon > 0 \) and every \( \delta \geq -1 \) we have

\[
\|\varphi(\varepsilon D)\pi_{<}(u,v) - \pi_{<}(u,\varphi(\varepsilon D)v)\|_{\alpha+\beta+\delta} \lesssim \varepsilon^{-\delta} \|u\|_{\alpha} \|v\|_{\beta},
\]

where \( \varphi(D)u = \mathcal{F}^{-1}(\varphi \mathcal{F}u) \).

Proof. We define for \( j \geq -1 \),

\[
S_{j-1}u = \sum_{i=-1}^{j-2} \Delta_i u
\]

and every term of this series has a Fourier transform with support in an annulus of the form \( 2^j A \). Lemma 2.69 in [1] implies that it suffices to control the \( L^\infty \) norm of each term. Let \( \psi \in \mathcal{D} \) with support in an annulus be such that \( \psi \equiv 1 \) on \( A \). We have

\[
\varphi(\varepsilon D)(S_{j-1}u \Delta_j v) - S_{j-1}u \Delta_j \varphi(\varepsilon D)v = \left( (\psi(2^{-j} \cdot) \varphi(\varepsilon \cdot))(D)(S_{j-1}u \Delta_j v) - S_{j-1}u(\psi(2^{-j} \cdot) \varphi(\varepsilon \cdot))(\mathcal{D}\Delta_j v)ight)
\]

where \( [(\psi(2^{-j} \cdot) \varphi(\varepsilon \cdot))(D), S_{j-1}u] \Delta_j v \) denotes the commutator. In the proof of Lemma 2.97 in [1], it is shown that writing the Fourier multiplier as a convolution operator and applying a first order Taylor expansion and then Young’s inequality yields

\[
\|[(\psi(2^{-j} \cdot) \varphi(\varepsilon \cdot))(D), S_{j-1}u] \Delta_j v\|_{L^\infty} \lesssim \sum_{\eta \in \mathbb{N}^d: |\eta| = 1} \left\| \mathcal{F}^{-1}(\psi(2^{-j} \cdot) \varphi(\varepsilon \cdot)) \right\|_{L^1} \left\| \partial^\eta S_{j-1}u \right\|_{L^\infty} \left\| \Delta_j v \right\|_{L^\infty}.
\]

(25)

Now \( \mathcal{F}^{-1}(f(2^{-j} \cdot)g(\varepsilon \cdot)) = 2^{j+1} \mathcal{F}^{-1}(fg(\varepsilon 2^j \cdot))(2^{-j} \cdot) \) for every \( f, g \), and thus we have for every multi-index \( \eta \) of order one

\[
\left\| \mathcal{F}^{-1}(\psi(2^{-j} \cdot) \varphi(\varepsilon \cdot)) \right\|_{L^1} \leq 2^{-j} \left\| \mathcal{F}^{-1}((\partial^\eta \psi)(2^{-j} \cdot) \varphi(\varepsilon \cdot)) \right\|_{L^1} + \varepsilon \left\| \mathcal{F}^{-1}(\psi(2^{-j} \cdot) \partial^\eta \varphi(\varepsilon \cdot)) \right\|_{L^1}
\]

\[
= 2^{-j} \left\| \mathcal{F}^{-1}((\partial^\eta \psi)(\varepsilon 2^j \cdot)) \right\|_{L^1} + \varepsilon \left\| \mathcal{F}^{-1}(\psi(\partial^\eta \varphi(\varepsilon 2^j \cdot))) \right\|_{L^1}
\]

\[
\lesssim 2^{-j} \left\| (1 + |\cdot|)^{2d} \mathcal{F}^{-1}((\partial^\eta \psi)(\varepsilon 2^j \cdot)) \right\|_{L^\infty} + \varepsilon \left\| (1 + |\cdot|)^{2d} \mathcal{F}^{-1}(\psi(\partial^\eta \varphi(\varepsilon 2^j \cdot))) \right\|_{L^\infty}
\]

\[
= 2^{-j} \left\| \mathcal{F}^{-1}((1 - \Delta)^d((\partial^\eta \psi)(\varepsilon 2^j \cdot))) \right\|_{L^\infty} + \varepsilon \left\| \mathcal{F}^{-1}((1 - \Delta)^d(\psi(\partial^\eta \varphi(\varepsilon 2^j \cdot))) \right\|_{L^\infty}
\]

\[
\lesssim 2^{-j} \left\| (1 - \Delta)^d((\partial^\eta \psi)(\varepsilon 2^j \cdot)) \right\|_{L^\infty} + \varepsilon \left\| (1 - \Delta)^d(\psi(\partial^\eta \varphi(\varepsilon 2^j \cdot))) \right\|_{L^\infty}.
\]

(26)
where the last step follows because $\psi$ has compact support. For $j$ satisfying $\varepsilon 2^j \geq 1$ we obtain
\[
\| x^n \mathcal{F}^{-1} (\varphi (\varepsilon \cdot) \psi (2^{-j} \cdot)) \|_{L^1} \lesssim (\varepsilon + 2^{-j}) (\varepsilon 2^j)^{2d} \sum_{|\eta| \leq 2d+1} \| \partial^n \varphi (\varepsilon 2^j \cdot) \|_{L^\infty (\text{supp} (\psi))},
\] (27)
where we used that $\psi$ and all its partial derivatives are bounded, and where $L^\infty (\text{supp} (\psi))$ means that the supremum is taken over the values of $\partial^n \varphi (\varepsilon 2^j \cdot)$ restricted to $\text{supp} (\psi)$. Now $\varphi$ is a Schwartz function, and therefore it decays faster than any polynomial. Hence, there exists a ball $B_\delta$ such that for all $x \notin B_\delta$ and all $|\eta| \leq 2d + 1$ we have
\[
\| \partial^n \varphi (x) \| \lesssim |x|^{-2d-1-\delta}.
\] (28)
Let $j_0 \in \mathbb{N}$ be minimal such that $2^{j_0} \varepsilon A \cap B_\delta = \emptyset$ and $\varepsilon 2^{j_0} \geq 1$. Then the combination of (25), (27), and (28) shows for all $j \geq j_0$ that
\[
\| (\varphi (2^{-j} \cdot) \varphi (\varepsilon \cdot)) (D), S_{j-1} u \Delta_j v \|_{L^\infty} \lesssim (\varepsilon + 2^{-j}) (\varepsilon 2^j)^{2d} \sum_{|\eta| \leq 2d+1} \| (\partial^n \varphi) (\varepsilon 2^j \cdot) \|_{L^\infty (\text{supp} (\psi))} (2^{j(1-\alpha)}) \| u \|_{2^{-j} \beta} \| v \|_\beta
\lesssim (\varepsilon + 2^{-j}) (\varepsilon 2^j)^{2d} (\varepsilon 2^j)^{-2d-1-\delta} 2^{j(1-\alpha-\beta)} \| u \|_{2^{-j} \beta} \| v \|_\beta
\lesssim (1 + (\varepsilon 2^j)^{-1}) \varepsilon^{-\delta} 2^{-j(\alpha+\beta+\delta)} \| u \|_{\alpha} \| v \|_\beta.
\]
Here we used that $\alpha < 1$ in order to obtain $\| \partial^n S_{j-1} u \|_{L^\infty} \lesssim 2^{j(1-\alpha)} \| u \|_{L^\infty}$. Since $\varepsilon 2^j \geq 1$, we have shown the desired estimate for $j \geq j_0$. On the other side Lemma 2.97 in [1] implies for every $j \geq -1$ that
\[
\| (\varphi (\varepsilon D), S_{j-1} u \Delta_j v \|_{L^\infty} \lesssim \varepsilon \max_{\eta \in \mathbb{N}^d, |\eta| = 1} \| \partial^n S_{j-1} u \|_{L^\infty} \| \Delta_j v \|_{L^\infty} \lesssim \varepsilon 2^{j(1-\alpha-\beta)} \| u \|_{\alpha} \| v \|_\beta.
\]
Hence, we obtain for $j < j_0$, i.e. for $j$ satisfying $2^j \varepsilon \lesssim 1$, that
\[
\| (\varphi (\varepsilon D), S_{j-1} u \Delta_j v \|_{L^\infty} \lesssim (\varepsilon 2^j)^{1+\delta} \varepsilon^{-\delta} 2^{-j(\alpha+\beta+\delta)} \| u \|_{\alpha} \| v \|_\beta \lesssim \varepsilon^{-\delta} 2^{-j(\alpha+\beta+\delta)} \| u \|_{\alpha} \| v \|_\beta,
\]
where we used that $\delta \geq -1$. This completes the proof. \qed

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