Exact Black String Solutions in Three Dimensions

James H. Horne and Gary T. Horowitz

Department of Physics
University of California
Santa Barbara CA 93106-9530
jhh@cosmic.physics.ucsb.edu
gary@cosmic.physics.ucsb.edu

ABSTRACT: A family of exact conformal field theories is constructed which describe charged black strings in three dimensions. Unlike previous charged black hole or extended black hole solutions in string theory, the low energy spacetime metric has a regular inner horizon (in addition to the event horizon) and a timelike singularity. As the charge to mass ratio approaches unity, the event horizon remains but the singularity disappears.
1. Introduction

In a recent paper [1], it was shown that string theory has a rich variety of solutions describing extended objects surrounded by event horizons. In particular there are black string solutions in ten dimensions characterized by three parameters: the mass and axion charge per unit length, and the asymptotic value of the dilaton. These solutions were obtained by solving the low energy string equations of motion. Although this is sufficient to establish the existence of exact solutions with these qualitative features, it was not clear how to construct directly the conformal field theory with these properties.

Witten has recently shown [2] that a simple gauged WZW model [3,4] yields a two dimensional black hole. This raises the possibility of using similar constructions to find exact conformal field theories corresponding to higher dimensional black holes or extended black holes. (The conformal field theory associated with an extremal limit of the charged black fivebranes has recently been found [5].) In this paper we will show that a simple extension of Witten’s construction yields three dimensional charged black strings. These solutions are also characterized by three parameters: the mass $M$ and axion charge $Q$ per unit length, and a constant $k$ related to the asymptotic value of the derivative of the dilaton. The low energy metric, antisymmetric tensor, and dilaton take the form:

$$ds^2 = - \left(1 - \frac{M}{r}\right) dt^2 + \left(1 - \frac{Q^2}{Mr}\right) dx^2 + \left(1 - \frac{M}{r}\right)^{-1} \left(1 - \frac{Q^2}{Mr}\right)^{-1} k \frac{dr^2}{8r^2}$$

$$H_{rtx} = \frac{Q}{r^2}$$

$$\Phi = \ln r + \frac{1}{2} \ln \frac{k}{2}.$$  \hspace{1cm} (1)

Perhaps the most important feature of these solutions is that their global structure is qualitatively different from all previously discussed string solutions. The existing solutions describing black holes [2], charged black holes [6,7,8,9,10], and extended black holes [1] (away from their extremal values) are all qualitatively similar to the Schwarzschild solution of general relativity: The low energy spacetime metrics have an event horizon and a spacelike singularity inside. In contrast, when $0 < |Q| < M$, our black strings are similar to the Reissner–Nordström solution which describes charged black holes in general relativity. In addition to the event horizon, there is a second inner horizon and a timelike singularity. When $|Q| = M$, we will show that the black string solutions have the unusual property of possessing an event horizon, but no singularity! Finally, when $|Q| > M$, we will see that the spacetime has neither a horizon nor a curvature singularity. However there is in general a conical singularity which can be removed by identifying the $x$ coordinate with an appropriate period. This changes the spacetime at infinity from $\mathbb{R}^3$ to $\mathbb{R}^2 \times S^1$. In a certain limit, our solution reduces to the two dimensional Euclidean black hole solution with a time direction added.
Although the three dimensional black strings are most naturally described in terms of the string metric (the metric appearing in the sigma model), it is also of interest to consider the rescaled Einstein metric (with the standard Einstein-Hilbert action). We will see that the Einstein metric also describes a black string in an asymptotically flat spacetime. But it is not static! There is still a timelike symmetry outside the event horizon, but it resembles a boost at infinity rather than a time translation.

### 2. Derivation of Black String Solutions

We now describe the conformal field theory construction which yields the black strings. Since our target space is going to have Lorentz signature, we will use a Lorentz metric $ds^2 = 2d\sigma_+d\sigma_-$ on the world sheet $\Sigma$. If $g$ is an element of a group $G$, then the ungauged Wess–Zumino–Witten action can be written

$$L(g) = \frac{k}{4\pi} \int_{\Sigma} d^2\sigma \, \text{Tr}(g^{-1} \partial_+ gg^{-1} \partial_- g) - \frac{k}{12\pi} \int_{B} \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg),$$

(2)

where $B$ is a three manifold with boundary $\Sigma$.

We are interested in gauging a one dimensional subgroup $H$ of the symmetry group of eq. (2), with action $g \rightarrow hgh$. We can make this global symmetry local by introducing a gauge field $A_i$ which takes values in the Lie algebra of $H$. If $\epsilon$ is an infinitesimal gauge parameter, then the local axial symmetry is generated by

$$\delta g = \epsilon g + g\epsilon, \quad \delta A_i = -\partial_i \epsilon.$$

(3)

This local axial symmetry is a symmetry of the gauged WZW action

$$L(g, A) = L(g) + \frac{k}{2\pi} \int_{\Sigma} d^2\sigma \, \text{Tr}(A_+ \partial_- gg^{-1} + A_- g^{-1} \partial_+ g + A_+ A_- + A_+ g A_- g^{-1}).$$

(4)

Witten showed that if $G$ is $SL(2, \mathbb{R})$ and $H$ is the subgroup generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the gauged WZW action describes a black hole in two dimensions. We now generalize this construction by adding one free boson $x$ to the action, which is equivalent to letting $G = SL(2, \mathbb{R}) \times \mathbb{R}$. We then gauge the one dimensional subgroup generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ together with a translation of $x$. To be explicit, parameterize the group manifold of $SL(2, \mathbb{R})$ by

$$g_{SL(2,\mathbb{R})} = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \quad \text{with} \quad ab + uv = 1.$$  

(5)

*Since the two dimensional Einstein action is a topological invariant, this is not possible for the two dimensional black hole.*
This gives the ungauged action

\[ L(g) = -\frac{k}{4\pi} \int_{\Sigma} d^2\sigma \left( \partial_+ u \partial_- v + \partial_- u \partial_+ v + \partial_+ a \partial_- b + \partial_- a \partial_+ b \right) \]
\[ + \frac{k}{2\pi} \int_{\Sigma} d^2\sigma \log u(\partial_+ a \partial_- b - \partial_- a \partial_+ b) + \frac{1}{\pi} \int_{\Sigma} d^2\sigma \partial_+ x \partial_- x, \]  
(6)

We then gauge

\[ \delta a = 2\epsilon a, \delta b = -2\epsilon b, \delta u = \delta v = 0, \delta x = 2\epsilon c, \delta A_i = -\partial_t \epsilon. \]  
(7)

where \( c \) is an arbitrary constant*. The full action is now

\[ L(g, A) = L(g) + \frac{k}{2\pi} \int_{\Sigma} d^2\sigma A_+(b \partial_- a - a \partial_- b - u \partial_- v + v \partial_- u + \frac{4c}{k} \partial_- x) \]
\[ + A_-(b \partial_+ a - a \partial_+ b + u \partial_+ v - v \partial_+ u + \frac{4c}{k} \partial_+ x) \]
\[ + 4A_+ A_-(1 + \frac{2c^2}{k} - uv). \]  
(8)

As in ref. [2], we can now gauge fix by setting \( a = \pm b \), depending on the sign of \( 1 - uv \). After making this gauge choice and eliminating \( A \) the action becomes

\[ L = -\frac{k}{8\pi} \int_{\Sigma} d^2\sigma \frac{\lambda v^2 \partial_+ u \partial_- u + \lambda u^2 \partial_+ v \partial_- v + (2 - 2uv + 2\lambda - \lambda uv)(\partial_+ u \partial_- v + \partial_- u \partial_+ v)}{(1 - uv)(1 + \lambda - uv)} \]
\[ + \frac{1}{\pi} \int_{\Sigma} d^2\sigma \frac{1 - uv}{1 + \lambda - uv} \partial_+ x \partial_- x \]
\[ + \frac{1}{2\pi} \int_{\Sigma} d^2\sigma \frac{c}{1 + \lambda - uv} (v \partial_+ u \partial_- x - v \partial_- u \partial_+ x - u \partial_+ v \partial_- x + u \partial_- v \partial_+ x) \]  
(9)

where \( \lambda \equiv 2c^2/k \). This action can be greatly simplified by making the field redefinition

\[ u = e^{\sqrt{2t}/\sqrt{k(1+\lambda)}} \sqrt{\hat{r} - (1 + \lambda)}, \quad v = -e^{-\sqrt{2t}/\sqrt{k(1+\lambda)}} \sqrt{\hat{r} - (1 + \lambda)}, \]  
(10)

after which the action becomes

\[ L = \frac{1}{\pi} \int_{\Sigma} d^2\sigma \frac{k \partial_+ \hat{r} \partial_- \hat{r}}{8\hat{r}^2(1 - \lambda/\hat{r})(1 - (1 + \lambda)/\hat{r})} - \left( 1 - \frac{1 + \lambda}{\hat{r}} \right) \partial_+ t \partial_- t \]
\[ + \left( 1 - \frac{\lambda}{\hat{r}} \right) \partial_+ x \partial_- x + \sqrt{\frac{\lambda}{1 + \lambda}} \left( 1 - \frac{1 + \lambda}{\hat{r}} \right) (\partial_+ x \partial_- t - \partial_- x \partial_+ t). \]  
(11)

This describes a string propagating in a spacetime with metric

\[ ds^2 = -\left( 1 - \frac{1 + \lambda}{\hat{r}} \right) dt^2 + \left( 1 - \frac{\lambda}{\hat{r}} \right) dx^2 + \left( 1 - \frac{1 + \lambda}{\hat{r}} \right)^{-1} \left( 1 - \frac{\lambda}{\hat{r}} \right)^{-1} \frac{k \hat{r}^2}{8\hat{r}^2}, \]  
(12)

*A related construction with a compactified \( x \) has been used to obtain two dimensional charged black holes [10].
and an antisymmetric tensor field

\[ B_{tx} = \sqrt{\frac{\lambda}{1 + \lambda}} \left( 1 - \frac{1 + \lambda}{\hat{r}} \right). \] (13)

The exact central charge of this gauged WZW model is \( 3k/(k - 2) \). This is one larger than the two dimensional black hole since we have added an extra boson. Eqs. (12) and (13) yield the lowest order expressions for the metric and antisymmetric tensor field, but quantum (sigma model) corrections will introduce corrections. There is also a dilaton which arises from quantum effects. This is most easily obtained from the requirement that the fields must be an extremum of the low energy string action

\[ S = \int e^\Phi \left[ R + (\nabla \Phi)^2 - \frac{1}{12} H^2 + \frac{8}{k} \right] \] (14)

where the cosmological constant \( 8/k \) arises from the fact that the central charge is not equal to the spacetime dimension. It is straightforward to verify that the above metric and antisymmetric tensor field are indeed an extrema of this action if

\[ \Phi = \ln \hat{r} + a \] (15)

for an arbitrary constant \( a \).

The metric components (12) are ill behaved at \( \hat{r} = 0, \lambda, \) and \( 1 + \lambda \). We can test whether these are true singularities by looking at the scalar curvature, which is

\[ R = \frac{4(2\hat{r} + 4\lambda \hat{r} - 7\lambda - 7\lambda^2)}{k\hat{r}^2}. \] (16)

Thus \( \hat{r} = 0 \) is a curvature singularity. As suggested by eq. (16), the difficulties at \( \hat{r} = \lambda \) and \( \hat{r} = 1 + \lambda \) can be completely removed by an appropriate change of coordinates. In fact, the original \( u, v, x \) coordinates in eq. (9) are well behaved at \( uv = 0 \) which corresponds to \( \hat{r} = 1 + \lambda \). (The \( u, v, x \) coordinates are not well behaved at \( uv = 1 \) (\( \hat{r} = \lambda \)). This is a direct consequence of the fact that our gauge fixing breaks down there. Note that unlike the case of the two dimensional black hole, the spacetime is nonsingular where the gauge fixing breaks down.) We will see that \( \hat{r} = 1 + \lambda \) is an event horizon. The solution is clearly invariant under translations of both \( t \) and \( x \), and for large \( \hat{r} \) the metric is asymptotically flat. Thus the solution represents a straight, static, black string.

We now wish to reexpress the free parameters \( \lambda \) and \( a \) in terms of the physical mass per unit length and axion charge per unit length of the black string. First note that the overall scaling for the \( t \) and \( x \) coordinates is fixed by the condition that as \( \hat{r} \) goes to infinity, the metric components \( g_{tt} \) and \( g_{xx} \) approach unity. It is not possible to similarly fix the overall scaling of the coordinate \( \hat{r} \) since the metric asymptotically approaches \( k\hat{r}^2/8\hat{r}'^2 \).
It will be convenient to fix the scaling of \( \hat{r} \) so that the dilaton is exactly \( \Phi = \ln r + \frac{1}{2} \ln \frac{k}{2} \). In other words we set
\[
\hat{r} = re^{-a} \sqrt{\frac{k}{2}}
\] (17)
in eqs. (12), (13), and (15). This has the virtue that the metric now depends on two parameters which we will see are simply related to the physical mass and charge per unit length. The fact that the metric depends on both the mass and charge is of course the familiar situation with higher dimensional black holes and black strings.

The axion charge is computed as follows. In \( n \) dimensions, it follows from the action (14) that the \( n - 3 \) form \( K = \frac{1}{2} e^\Phi \ast H \) is curl free where \( \ast \) denotes the Hodge dual. The axion charge per unit length is the integral of this form over the \( n - 3 \) sphere at large transverse directions from the string\(^\dagger\). Since we are in three dimensions, \( K \) is just a function which must be constant by the field equations. The axion charge per unit length is simply the value of this constant. For the black string solutions we obtain
\[
Q = e^a \sqrt{\frac{2\lambda(1 + \lambda)}{k}}.
\] (18)

To calculate the mass per unit length of the string we follow the standard ADM procedure. For large \( r \) the black string solutions approach the asymptotic solution
\[
ds^2 = -dt^2 + dx^2 + d\rho^2
\]
\[
\Phi = \rho \sqrt{\frac{8}{k}}
\] (19)
where \( \rho = \sqrt{\frac{k}{8}} \ln(r \sqrt{k/2}) \) and \( H = 0 \). To calculate the mass, one extremizes the action (14) to obtain the metric field equation, linearizes this expression about the asymptotic solution (19), and integrates the time-time component of this equation over a constant time surface. Since the integrand is a total derivative, the result can be expressed as a surface integral at infinity. The antisymmetric tensor field appears quadratically in the metric field equation and vanishes in the background, so it does not explicitly appear in the formula for the mass. We therefore only need to keep track of the metric and dilaton. Their contributions to the field equation are
\[
e^\Phi \left[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi + g_{\mu\nu} \left( \nabla^2 \Phi + \frac{1}{2} (\nabla \Phi)^2 - \frac{4}{k} \right) \right].
\] (20)

We now linearize this expression. Since \( k \) appears in the background solution, it cannot be changed by the perturbation*. We have chosen our radial coordinate so that, in our

\(^\dagger\) This definition differs by a factor of \( \frac{1}{2} \) from the one used in ref. [1]

* It is as meaningless to compare the masses of two solutions with different \( k \), as it is to compare the masses of two Kaluza-Klein solutions with different compactifications.

6
solutions, \( \Phi \) depends only on \( k \). So to calculate their mass, we do not need to include a perturbation of \( \Phi \). We need only perturb the metric \( g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \). Integrating the time-time component of the linearized form of eq. (20) over a spacelike surface yields the following formula for the total mass:

\[
\mathcal{M}_{\text{tot}} = \frac{1}{2} \oint e^\Phi [-\partial_i \gamma_{00} + \partial^\mu \gamma_{\mu i} - \partial_i \gamma + \gamma_{ij} \partial^\mu \Phi] dS^i.
\]  

Using the specific form of \( \gamma_{\mu\nu} \) for our solutions (12), and the fact that \( x \) measures proper distance at infinity we obtain the mass per unit length

\[
M = \sqrt{\frac{2}{k}} (1 + \lambda) e^a
\]  

Combining the above results, we obtain our final expression for the black string solutions

\[
ds^2 = - \left( 1 - \frac{M}{r} \right) dt^2 + \left( 1 - \frac{Q^2}{M r} \right) dx^2 + \left( 1 - \frac{M}{r} \right)^{-1} \left( 1 - \frac{Q^2}{M r} \right)^{-1} k \frac{dr^2}{r^2}  
\]  

with antisymmetric field strength and dilaton

\[
H_{rtx} = \frac{Q}{r^2}, \\
\Phi = \ln r + \frac{1}{2} \ln \frac{k}{2}.
\]

When \( Q = 0 \), \( H \) vanishes and our black string solutions become a simple product of \( dx^2 \) and the two dimensional metric

\[
ds^2 = - \left( 1 - \frac{M}{r} \right) dt^2 + \left( 1 - \frac{M}{r} \right)^{-1} k \frac{dr^2}{r^2}.
\]

This is exactly Witten’s black hole solution [2]. It can be put into his form by the coordinate transformation \( r = M \cosh^2 \rho, t = \sqrt{\frac{2}{k}} \tau \). The form of the metric (25) shows clearly that the region beyond the singularity (\( r < 0 \)) has negative mass.

3. Global Structure

Before discussing the global properties of the metric (23), let us first review the charged black hole solution in four dimensional general relativity. This is described by the Reissner–Nordström metric

\[
ds^2 = - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2
\]  

(26)
with $|Q| \leq M$. The metric components are ill defined at $r = 0$ and $r = r_+ \equiv M \pm \sqrt{M^2 - Q^2}$. Only the first is a true curvature singularity. The surface $r = r_+$ is the event horizon. The surface $r = r_-$ is another null surface inside the black hole called the inner horizon. It has the following interpretation. Given initial data for some fields on a $t = \text{constant}$ surface, one can uniquely evolve only up to the inner horizon. After that, the evolution is affected by the boundary conditions at the singularity. Since $r = \text{constant}$ surfaces are timelike near the singularity, the singularity is also timelike, in contrast to the familiar Schwarzschild solution. This means that observers falling into the black hole are not required to hit the singularity, but can pass by it into another asymptotically flat region of spacetime. In fact, one can show that freely falling test particles never hit the singularity: The Reissner–Nordström solution is timelike geodesically complete. Since the metric is spherically symmetric, its global structure can be described by a two dimensional figure in which each point represents a sphere. (Equivalently, one can view the figure as representing the two dimensional spacetime obtained by holding $\theta$ and $\phi$ fixed.) It is convenient to rescale the metric so that infinity is at a finite distance, and have light rays travel along 45 degree lines. The result is called a Penrose diagram. For the $|Q| < M$ Reissner–Nordström solution, the Penrose diagram is shown in fig. 1. (The regions VII and VIII correspond to $r < 0$ or equivalently $M < 0$ and are usually not included. They describe asymptotically flat spacetimes with naked singularities.) The result is an infinite family of asymptotically flat regions joined by black holes. For more details see Hawking and Ellis [11]. When $|Q| = M$, the horizons coincide, and the global structure is described in fig. 2. Finally, when $|Q| > M$, the spacetime contains a naked singularity. The Penrose diagram is equivalent to region VII of fig. 1.

The inner horizon is believed to be unstable. It has been shown that generic time dependent perturbations blow up there [12]. When the dilaton is included as in string theory, even static spherically symmetric charged black holes do not have an inner horizon [6-10]. It is thus surprising that the charged black string does have an inner horizon, as we now discuss.

A. The black string with $0 < |Q| < M$

Consider the black string solutions (1). Near $r = M$, the metric is similar to the region near $r = r_+$ of the Reissner–Nordström solution. As in that case, $r = M$ is an event horizon. Near $r = Q^2/M$, the metric is similar to the region near $r = r_-$ and this is indeed an inner horizon. However there is one important difference. In Reissner–Nordström, the Killing vector which is timelike at infinity becomes spacelike between the two horizons and timelike again inside the inner horizon. In the black string solution, the Killing vector $\partial/\partial t$ which is a time translation at infinity becomes spacelike inside the event horizon and stays spacelike all the way to the singularity. The Killing vector $\partial/\partial x$ which translates
along the string at infinity becomes timelike inside the inner horizon. Equivalently, the
time coordinate in region I of fig. 1 is \( t \), the time coordinate in region II is \( r \), and the time
coordinate in region V is \( x \).† For this reason it is not possible to represent all aspects of the
causal structure by a two dimensional diagram. Nevertheless, most features are faithfully
indicated by fig. 1 with \( r_+ = M \) and \( r_- = Q^2/M \). Each point now represents a line in
spacetime. However it is the line in the \( x \) direction for \( r > Q^2/M \) and in the \( t \) direction for
\( r < Q^2/M \). Unlike Reissner–Nordström, the black string solution also contains the regions
labeled VII and VIII in fig. 1 which correspond to naked singularities. This is because
these regions correspond to \( r < 0 \) or \( uv < -(1 + \lambda) \) which is certainly part of the original
gauged WZW model.

We now consider geodesics in the black string solutions. This is particularly simple
due to the two conserved quantities associated with the two translational symmetries. Let \( \xi^\mu \) be tangent to an affinely parametrized geodesic, and let \( E = -\xi \cdot \partial/\partial t \), \( P = \xi \cdot \partial/\partial x \) denote the conserved quantities. Then geodesics satisfy

\[
\frac{k r^2}{8r^2} = E^2 - P^2 + \frac{1}{r} \left( P^2 M - \frac{E^2 Q^2}{M} \right) + \alpha (1 - M/r) (1 - Q^2/Mr) \tag{27}
\]

where the dot denotes derivative with respect to an affine parameter and \( \alpha \) is zero for
null geodesics and \(-1\) for timelike geodesics. In either case, if the right hand side is
positive for large \( r \), it stays positive for all \( r > Q^2/M \). This shows that timelike and
null geodesics which begin at large \( r \) cross both the event horizon at \( r = M \) and the inner
horizon at \( r = Q^2/M \). For timelike geodesics, the term \(-Q^2/r^2\) eventually dominates
causing \( r \) to achieve a minimum value. Thus timelike geodesics never reach the singularity:
The spacetime is timelike geodesically complete. Eq. (27) has a simple solution for null
geodesics. If \( P^2 < E^2 \), then

\[
r = \frac{(e^\lambda - P^2 M^2 + E^2 Q^2)^{1/2}}{4 e^\lambda M (E^2 - P^2)}, \tag{28}
\]

where \( \lambda \) is an affine parameter. If \( P^2 = E^2 \), then \( r = \lambda^2 \). Null geodesics can reach the
singularity, but only if \( P^2 M^2 - E^2 Q^2 > 0 \). Otherwise, the null geodesics reach a minimum
value of

\[
r_{\text{min}} = \frac{E^2 Q^2 - P^2 M^2}{M (E^2 - P^2)}. \tag{29}
\]

This is qualitatively the same behavior as geodesics in the Reissner–Nordström solution.

There appears to be a problem with geodesics having \( P = 0 \). These geodesics are
orthogonal to \( \partial/\partial x \) everywhere, but inside the inner horizon this vector becomes timelike
and cannot be orthogonal to any timelike or null geodesic. As we can see from eqs. (27)

† This shows that it is not possible to compactify the \( x \) direction and view this as a two
dimensional solution.
and (29), the resolution is that geodesics with $P = 0$ cross the point labeled $p$ in fig. 1 (recall this is really a line in spacetime). At $p$ the Killing vector $\partial/\partial x$ is not only null, but actually vanishes.

We have seen that the three dimensional black string is qualitatively very similar to the Reissner-Nordström solution. This analogy appears to extend to Hawking evaporation. We can define a Hawking temperature for the black string by analytically continuing $t = i\tau$ in eq. (1). The horizon $r = M$ is a regular point only if we identify $\tau$ with period $\pi M \sqrt{2k/(M^2 - Q^2)}$ which corresponds to a temperature

$$T = \frac{1}{\pi M} \sqrt{\frac{M^2 - Q^2}{2k}}. \quad (30)$$

Therefore, the temperature vanishes as $Q \rightarrow M$. (This is also true of Reissner-Nordström.) Thus, if the charge cannot be radiated away, the black strings would settle down to $|Q| = M$.

**B. The extremal limit: $|Q| = M$**

What does the extremal configuration look like? If one sets $|Q| = M$ in eq. (1) one obtains

$$ds^2 = (1 - M/r)(-dt^2 + dx^2) + (1 - M/r)^{-2}k\frac{dr^2}{8r^2}. \quad (31)$$

Notice that this extremal metric is not only static and translationally invariant, it is also boost invariant along the string. (Higher dimensional extended black holes are also boost invariant in the extremal limit [1].) At first sight, the global structure of the metric (31) appears to be analogous to the extreme Reissner-Nordström metric with a single horizon at $r = M$ and a singularity at $r = 0$. However this is misleading. The proper continuation across the horizon is not to let $r$ become less than $M$. To find the correct extension across the horizon we must introduce a new radial coordinate. Starting in the region outside the horizon, set

$$\tilde{r}^2 = r - M \quad (33)$$

Then the metric becomes

$$ds^2 = \frac{\tilde{r}^2}{\tilde{r}^2 + M}(-dt^2 + dx^2) + \frac{k\frac{dr^2}{2\tilde{r}^2}}{2\tilde{r}^2}. \quad (34)$$
One can easily verify that geodesics now cross the horizon \( \tilde{r} = 0 \) from positive to negative values of \( \tilde{r} \) so the metric (34) and not (31) describes the correct extension across the horizon. But the region \( \tilde{r} < 0 \) is identical to the region \( \tilde{r} > 0 \), and the metric (34) is nonsingular! Nevertheless, \( \tilde{r} = 0 \) is still an event horizon. Thus one has the unusual situation of a spacetime with an event horizon but no singularity*. The global structure is described in fig. 3. Observers in this spacetime who cross the event horizon are not able to return, but (fortunately for them) find themselves in another asymptotically flat region of spacetime which is identical to the one they started in.

What happens to the region \( V \) in fig. 1, near the singularity, as \( |Q| \rightarrow M \)? Setting \( \tilde{r}^2 = M - r \), the metric in this region becomes

\[
ds^2 = \frac{\tilde{r}^2}{M - \tilde{r}^2} (dt^2 - dx^2) + \frac{k d\tilde{r}^2}{2\tilde{r}^2}
\]

which is singular at \( \tilde{r} = \pm \sqrt{M} \) and has a horizon at \( \tilde{r} = 0 \). The causal structure looks like fig. 4.

There is in fact another way to take the extremal limit. To motivate it, let us return to the conformal field theory construction. One starts with \( SL(2, \mathbb{R}) \times \mathbb{R} \) and gauges a one dimensional subgroup of \( SL(2, \mathbb{R}) \) and a translation in \( \mathbb{R} \). The free parameter \( c \) determines the relative weights of the two gaugings. One limiting case \((c = 0)\), corresponds to not gauging \( \mathbb{R} \) at all. This yields the uncharged black string. The other limit \((c \rightarrow \infty)\) corresponds to not gauging the \( SL(2, \mathbb{R}) \). The result is just the group invariant metric on \( SL(2, \mathbb{R}) \) which has constant negative curvature and hence is three dimensional anti-de Sitter space. But this limit also corresponds to \( |Q| \rightarrow M \). Thus the ungauged \( SL(2, \mathbb{R}) \) WZW model can be viewed as describing the extremal black string. Notice that this solution has constant dilaton, and is not asymptotically flat. It can be obtained from the metric (1) by rescaling \( y = \frac{8(r-M)}{k(M^2-Q^2)} \), \( \hat{t} = (1 - Q^2/M^2)^{1/4} t \), \( \hat{x} = (1 - Q^2/M^2)^{1/4} x \) and then taking the limit \( |Q| \rightarrow M \). The resulting metric is

\[
ds^2 = \frac{k}{8} \left( -y dt^2 + y dx^2 + \frac{dy^2}{y^2} \right).
\]

To see that this is indeed anti-de Sitter spacetime (albeit in unusual coordinates) one can calculate the curvature and find \( R_{\mu\nu} = -\frac{1}{k} g_{\mu\nu} \). The limiting antisymmetric tensor is simply \( H = \epsilon \sqrt{8/k} \) where \( \epsilon \) is the volume form.

In ten dimensions, the extremal limit of the black string solutions [1] is of particular interest. It agrees precisely with the solution found by Dabholkar et al. [13] describing the

* The authors of ref. [10] have found a two dimensional solution with similar properties by modifying the construction described in their paper.

† This extremal limit is analogous to the one used in ref. [5] for black fivebranes.
fields outside of a fundamental macroscopic string. Dabholkar et al. also found the fields outside of a fundamental macroscopic string in any dimension. In three dimensions, their solution is

$$ds^2 = \frac{1}{\tilde{y}} (-d\tilde{t}^2 + d\tilde{x})^2 + d\tilde{y}^2$$

$$\Phi = \ln \tilde{y}$$

$$H = \frac{\epsilon}{\tilde{y}}.$$  

(37)

This solution, like the ten dimensional analog, has a singularity at \(\tilde{y} = 0\) and no horizon. It does not resemble the extremal limit of our black string. However, in order to compare the two solutions, we must take into account the fact that we have included a correction to the central charge proportional to \(1/k\) which was not included in ref. [13]. If one wants to view the three dimensional black string, by itself, as a solution to critical string theory, it is necessary to include this modification to the central charge. However if one wants to add an internal conformal field theory, then the central charge need not be modified. To compare the two solutions, we must take the limit as \(k \to \infty\). Since large \(k\) means that the metric is rescaled by a large factor, the limit \(k \to \infty\) is usually thought to yield a flat metric. However, if one starts with a singular solution, one can take \(k \to \infty\) staying close to the singularity and obtain a nontrivial limit. More precisely, set \(\tilde{y} = (k/8)^{1/2} r/M, \tilde{t} = (k/8)^{1/4} x, \tilde{x} = (k/8)^{1/4} t\) in eq. (31). Then in the limit \(k \to \infty\), the solution agrees exactly with eq. (37).

C. The solutions with \(|Q| > M\)

We now consider the solution when \(|Q| > M\). The metric (1) appears to change signature at \(r = Q^2/M\). But this is just another indication that an incorrect extension is being used. The correct extension can again be found by considering the motion of geodesics. It corresponds to setting \(\tilde{r}^2 = r - Q^2/M\). In terms of \(t, x, \tilde{r}\) the metric becomes

$$ds^2 = -\frac{Q^2 - M^2 + M\tilde{r}^2}{Q^2 + M\tilde{r}^2} dt^2 + \frac{M\tilde{r}^2}{Q^2 + M\tilde{r}^2} dx^2 + \frac{M k}{2(Q^2 - M^2 + M\tilde{r}^2)} d\tilde{r}^2.$$  

(38)

This metric, with \(0 \leq \tilde{r} < \infty\), is globally static. It has no horizons and no curvature singularity. It does have a conical singularity at \(\tilde{r} = 0\) which can be removed by identifying \(x\) with period \(\pi Q \sqrt{2k/(Q^2 - M^2)}\). The resulting spacetime is completely nonsingular. Notice that the identification changes the structure of the spacetime at infinity from \(R^3\) to \(R^2 \times S^1\). A spacelike surface \(t = \) constant now resembles an infinite cigar. This is reminiscent of the form of a Euclidean two dimensional black hole. In fact if we take the

* They chose to work with the Einstein metric and different scaling for the dilaton. We have reexpressed their solution in terms of the string metric and dilaton used throughout this paper.
limit $M \to 0, Q \to 0$ keeping $Q^2/M = m$ fixed, one finds that (38) reduces to exactly the product of $-dt^2$ and the two dimensional Euclidean black hole discussed in ref. [2].

This insight helps us to resolve another aspect of these solutions. The conformal field theory construction described in the previous section only yields the solutions (1) with $|Q| < M$. However, the fields (38) with $|Q| > M$ also solve the low energy string equations and its natural to ask what is the exact conformal field theory that they correspond to. The answer is a slight modification of the construction in sec. 2. One again starts with $SL(2,\mathbb{R}) \times \mathbb{R}$ but now puts a timelike metric on $\mathbb{R}$. One then gauges a translation of $\mathbb{R}$ together with the subgroup of $SL(2,\mathbb{R})$ generated by \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]
The result is exactly the solutions (38) with $|Q| > M$. In the limit that $\mathbb{R}$ is not gauged, one obtains the two dimensional Euclidean black hole cross $-dt^2$.

Investigations of the two dimensional Euclidean black hole have shown that this nonsingular space is dual to the Euclidean negative mass solution which has a curvature singularity [14,15,16,17,18,19]. In other words, these two different geometries are equivalent as conformal field theories. Although this sounds intriguing, the physical interpretation of this result remains unclear. The reason is that in the Euclidean context, the duality involves string winding modes in Euclidean time. In the Lorentzian context, both the original conformal field theory and its dual contain all six regions of the black hole, so the spacetime metric does not change at all!* In three dimensions there is an analog of this duality with a clear physical interpretation and a rather striking conclusion. By the usual two dimensional arguments, the spacetime

\begin{equation}
 ds^2 = -dt^2 + \left(1 - \frac{M}{r}\right)^{-1} \frac{k}{8r^2} dr^2 + \left(1 - \frac{M}{r}\right) d\theta^2
\end{equation}  \tag{39}  

(where the spatial metric – with $r \geq M$ – is the positive mass Euclidean black hole) is completely equivalent as a conformal field theory to the spacetime

\begin{equation}
 ds^2 = -dt^2 + \left(1 + \frac{M}{r}\right)^{-1} \frac{k}{8r^2} dr^2 + \left(1 + \frac{M}{r}\right) d\theta^2
\end{equation}  \tag{40}  

(where the spatial metric is now the negative mass Euclidean solution). The duality is now the familiar one involving winding modes in a spacelike direction. There is no need for further analytic continuation. This shows that the curvature singularity in eq. (40) is not seen by strings. String propagation in this spacetime is completely equivalent to string propagation in the nonsingular geometry (39). This duality can be extended to the general solution with $|Q| > M$, and will be discussed in detail elsewhere [20].

* We thank E. Verlinde and H. Verlinde for a discussion on this point.
4. Conclusions

We have been describing the black string solutions in terms of the metric which appears in the sigma model. This is the metric that the strings couple directly to and is the most natural one to use in string theory. However to compare with results in general relativity, it is sometimes useful to rescale this string metric by a power of the dilaton to obtain a metric with the standard Einstein action. Since conformal transformations do not change the causal structure, both the original string metric and the new Einstein metric will have the same horizons. But since the dilaton is growing linearly at infinity one might think that the Einstein metric will not be asymptotically flat. This is incorrect. In three dimensions, the Einstein metric $\tilde{g}_{\mu\nu}$ is related to the string metric by $\tilde{g}_{\mu\nu} = e^{2\Phi}g_{\mu\nu}$. Thus asymptotically, the Einstein metric approaches

$$\tilde{ds}^2 = r^2(-dt^2 + dx^2) + \frac{k}{8}dr^2.$$ (41)

One can check that the curvature of this metric vanishes at large $r$ like $r^{-2}$. In $n$ dimensions, the usual definition of asymptotic flatness requires that the curvature fall off like $r^{-(n-1)}$. So it is reasonable to view eq. (41) as an asymptotically flat three dimensional spacetime*. Thus the Einstein metric also describes a black string. But it is not static. The timelike symmetry at large $r$ does not approach a time translation but rather a boost. One can check that the Einstein metric is still singular at $r = 0$ (for $|Q| < M$).

The fact that the Einstein metric describes a nonstatic string resolves an apparent contradiction with an earlier result on the existence of black strings in low dimensions. It was shown in ref. [1] that if the strong energy condition is satisfied, there cannot exist static, cylindrically symmetric, black strings in less than five dimensions which are locally asymptotically flat in the transverse directions. This does not rule out three (or four) dimensional solutions in which the string metric describes static black strings, since the low energy field theory does not satisfy the energy condition. However, when expressed in terms of the Einstein metric, the low energy field theory does satisfy the energy condition. Since the causal structure is unchanged, and the asymptotic boundary conditions on the metric are unchanged, the only way these solutions can exist is if they are not static. It would appear that they cannot even “settle down” to a static configuration unless they evaporate completely by Hawking radiation. (Recall that the extremal case with zero temperature still has a horizon and hence cannot be static.)

To summarize, we have found that a simple gauged WZW model yields charged black string solutions in three dimensions. When $0 < |Q| < M$, the solutions are qualitatively

* In higher dimensions, if one starts with a flat string metric and a linear dilaton, and rescales to the Einstein metric, one again finds that the curvature falls off like $r^{-2}$. Thus higher dimensional analogs could not be considered asymptotically flat in the usual sense.
similar to the Reissner–Nordström solution. They have an event horizon, an inner horizon and a timelike singularity. When $|Q| = M$ the spacetime has an event horizon but no singularity. (Another way to take the extremal limit, which does not preserve the boundary conditions, yields anti-de Sitter spacetime.) When $|Q| > M$, both the horizon and the curvature singularity disappear. To avoid a conical singularity one must compactify one of the directions. A limiting case yields just the product of time and the two dimensional Euclidean black hole.

Since our black string solutions are three dimensional and have a linear dilaton at infinity, they presumably are not of direct physical interest. Their importance is twofold. First, they illustrate that a wide range of causal structures (including some having no analog in general relativity) can occur in string theory. Indeed, we find it surprising that a simple conformal field theory construction can result in such nontrivial spacetime structure. This encourages the hope that an exact conformal field theory describing higher dimensional black holes and black strings will soon be found. Second, like the two dimensional black hole, they provide an important test of whether gravitational collapse will lead to singularities in string theory. It has been shown that string theory does have exact solutions which are singular [21]. However the known singular solutions do not have event horizons and hence do not describe gravitational collapse.

It is still not clear whether the two dimensional black hole is singular in string theory. (Recall that although one has an exact description of the conformal field theory in terms of a gauged WZW model, the spacetime metric (25) is only the lowest order approximation to the geometry. Higher order corrections can be large near the singularity.) It has been argued [2] that even though the conformal field theory may be regular at $r = 0$, it does not make sense to consider signals propagating from $r > 0$ to $r < 0$ for two reasons. One is that, in the two dimensional black hole, the surface $r = 0$ is spacelike on one side and timelike on the other. Thus if signals can propagate across, there would appear to be a violation of causality. The other is that $r = 0$ appears to be unstable in that generic perturbations blow up there. For our three dimensional black strings (with $0 < |Q| < M$) the surface $r = 0$ is timelike on both sides, so no causality problems should arise. On the other hand, general arguments suggest that the inner horizon $r = Q^2/M$ is now unstable. So once again, it appears to be impossible to propagate signals from large positive $r$ to large negative $r$.

Acknowledgements

We wish to thank S. Giddings, N. Ishibashi, M. Li, A. Steif, A. Strominger, and especially D. Garfinkle for helpful discussions. J.H.H. would like to thank the Aspen Center for Physics, where this work was begun. This work was supported in part by NSF grant PHY-9008502.
References

1. G. Horowitz and A. Strominger, “Black Strings and $p$-Branes,” *Nucl. Phys.* B360, 197 (1991).

2. E. Witten, “On String Theory and Black Holes,” *Phys. Rev.* D44, 314 (1991).

3. K. Gawedzki and A. KUPIainen, *Phys. Lett.* B215, 119 (1988); *Nucl. Phys.* B320, 625 (1989).

4. I. Bars and D. Nemeschansky, “String Propagation in Backgrounds with Curved Space-time,” *Nucl. Phys.* B348, 89 (1991).

5. S. Giddings and A. Strominger, “Exact Black Fivebranes in Critical Superstring Theory,” UCSB preprint, UCSBTH-91-35, July 1991.

6. G. Gibbons, *Nucl. Phys.* B207, 337 (1982).

7. G. Gibbons and K. Maeda, *Nucl. Phys.* B298, 741 (1988).

8. B. Ivanov, “Black Holes and the Heterotic String,” ICTP preprint, IC/89/3, January 1989.

9. D. Garfinkle, G. Horowitz and A. Strominger, “Charged Black Holes in String Theory,” *Phys. Rev.* D43, 3140 (1991).

10. N. Ishibashi, M. Li and A. Steif, “Two Dimensional Charged Black Holes in String Theory,” UCSB preprint, UCSBTH-91-23, July 1991.

11. S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge) 1973.

12. S. Chandrasekhar and J. Hartle, *Proc. Roy. Soc. Lond.* A384, 301 (1982).

13. A. Dabholkar, G. Gibbons, J. Harvey and F. Ruiz, “Superstrings and Solitons,” *Nucl. Phys.* B340, 33 (1990).

14. A. Giveon, “Target Space Duality and Stringy Black Holes,” Berkeley preprint, LBL-30671, April 1991.

15. E. Kiritsis, “Duality in Gauged WZW Models,” Berkeley preprint, LBL-30747, May 1991.

16. A. Tseytlin, “Space-Time Duality, Dilaton, and String Cosmology,” to appear in *Proceedings of the First International Sakharov Conference on Physics*, May 1991.
17. R. Dijkgraaf, E. Verlinde and H. Verlinde, “String Propagation in a Black Hole Geometry,” Princeton preprint, PUPT-1252, May 1991.

18. E. Martinec and S. Shatasvili, “Black Hole Physics and Liouville Theory,” Enrico Fermi preprint, EFI-91-22, May 1991.

19. I. Bars, “Curved Space-Time Strings and Black Holes,” USC preprint, USC-91-HEP-B4, June 1991; “String Propagation on Black Holes,” USC preprint, USC-91-HEP-B3, May 1991.

20. J. Horne, G. Horowitz, and A. Steif, to appear.

21. G. Horowitz and A. Steif, *Phys. Rev. Lett.* 64, 260 (1990); *Phys. Rev.* D42, 1950 (1990); *Phys. Lett.* B258, 91 (1991).

Figure Captions

Figure 1: The global structure for both the Reissner–Nordström solution and for the black string when $0 < |Q| < M$. The jagged lines represent singularities, and $r = r_\pm$ represent horizons. For the Reissner–Nordström solution, $r = r_\pm \equiv M \pm \sqrt{M^2 - Q^2}$ and each point in the figure represents a sphere in spacetime. For the black string, $r_+ = M$ and $r_- = Q^2/M$. Each point in the figure represents a line in spacetime. Geodesics with $P = 0$ cross the point $p$ in the diagram. (The regions VII and VIII correspond to $r < 0$ and are usually not considered part of the Reissner–Nordström solution.)

Figure 2: The global structure for the extreme Reissner–Nordström solution, $|Q| = M$. The two horizons in fig. 1 have become a single horizon.

Figure 3: The global structure for the extreme black string solution, $|Q| = M$, keeping the region $r \to \infty$. The spacetime has an event horizon, but no singularities.

Figure 4: The global structure for the extreme black string solution, $|Q| = M$, keeping the region near the singularity.