ON A GENERALIZATION OF JACOBI SUMS

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Abstract. We prove an estimate for multi-variable multiplicative character sums over affine subspaces of $\mathbb{A}^n_k$, which generalize the well known estimates for both classical Jacobi sums and one-variable polynomial multiplicative character sums.

1. Introduction

Let $k = \mathbb{F}_q$ be a finite field, with $q = p^a$ a prime power. Given $n$ non-trivial multiplicative characters $\chi_1, \ldots, \chi_n : \mathbb{F}_q^\times \to \mathbb{C}^\times$, the classical Jacobi sum is given by [BEW98, 10.1]

$$J(\chi_1, \ldots, \chi_n) := \sum_{x_1 + \cdots + x_n = 1} \chi_1(x_1) \cdots \chi_n(x_n).$$

More generally, given a hyperplane $L \subseteq \mathbb{A}^n_k$ defined by the equation $a_1x_1 + \cdots + a_nx_n = b$ with $a_1, \ldots, a_n, b \neq 0$, the sum

$$S(L; \chi_1, \ldots, \chi_n) := \sum_{x \in L} \chi_1(x_1) \cdots \chi_n(x_n)$$

(1)

can be rewritten, via the change of variables $y_i = a_ix_i/b$, as

$$\chi_1 \cdots \chi_n(b \overline{x}_1(a_1) \cdots \overline{x}_n(a_n)) J(\chi_1, \ldots, \chi_n).$$

It is well known [BEW98] Theorem 10.3.1] that the Jacobi sum can be expressed in terms of Gauss sums:

$$J(\chi_1, \ldots, \chi_n) = \begin{cases} G(\chi_1) \cdots G(\chi_n)/G(\chi_1 \cdots \chi_n) & \text{if } \chi_1 \cdots \chi_n \neq 1 \\ -G(\chi_1) \cdots G(\chi_n)/q & \text{if } \chi_1 \cdots \chi_n = 1 \end{cases}$$

where $G(\chi) := \sum_{x \in \mathbb{F}_q^\times} \chi(x)\psi(x)$ is the Gauss sum associated to $\chi$, $\psi : k \to \mathbb{C}^\times$ being an additive character. In particular, since $|G(\chi)| = \sqrt{q}$, we get

$$|S(L; \chi_1, \ldots, \chi_n)| = \begin{cases} q^{(n-1)/2} & \text{if } \chi_1 \cdots \chi_n \neq 1 \\ q^{(n-2)/2} & \text{if } \chi_1 \cdots \chi_n = 1 \end{cases}$$

Inspired by a question from Ivan Meir, in this article we study, more generally, sums of the form (1) where $L \subseteq \mathbb{A}^n_k$ is a sufficiently general affine subspace of arbitrary dimension. Before stating the main results, let us give a suitable definition of “sufficiently general”. For any $i = 1, \ldots, n$ let $H_i \subseteq \mathbb{A}^n_k$ be the hyperplane defined by the equation $x_i = 0$ and, for every subset $I \subseteq \{1, \ldots, n\}$, denote by $H_I$ the intersection $\cap_{i \in I} H_i$. By convention, we set $\dim(\emptyset) = -1$.

Definition 1. Let $L \subseteq \mathbb{A}^n_k$ be an affine subspace of dimension $d$. 
(1) We say that \( L \) is in \textbf{general position} (with respect to the coordinate hyperplanes) if, for every \( I \subseteq \{1, \ldots, n\} \) with \( |I| \leq d+1 \), we have \( \dim(L \cap H_I) = d - |I| \).

(2) We say that \( L \) is in \textbf{general position among its translates} if, for every \( I \subseteq \{1, \ldots, n\} \) with \( |I| \leq d+1 \), we have \( \dim(L \cap H_I) \leq d - |I| \) (so either \( \dim(L \cap H_I) = d - |I| \) or \( L \cap H_I = \emptyset \)).

Here is some justification for the chosen terminology. Consider the (affine) Grassmannian variety \( G(n, d) \) parameterizing all \( d \)-dimensional affine subspaces of \( \mathbb{A}_d^n \). The functions \( f_I : G(n, d) \to \mathbb{Z} \) given by \( f_I(L) = \dim(L \cap H_I) \) define a stratification of \( G(n, d) \) by locally closed subsets, such that all functions \( f_I \) are constant in each stratum. Then the subspaces in general position are those that belong to the dense open stratum. Similarly, \( L \) is in general position among its translates if it is in the dense open stratum of the restriction of the stratification to the closed subset of \( G(n, d) \) consisting of all translates of \( L \).

If \( A \cdot x = b \) is a system of linear equations defining \( L \), where \( A \) is an \( m \times n \) matrix of maximal rank (with \( m = n - d \)), then \( L \) is in general position if and only if all \( m \times m \) minors of the augmented matrix \( (A \mid b) \) are non-zero, and \( L \) is in general position among its translates if and only if \( b \) is not in any proper subspace of \( k^m \) generated by a subset of the columns of \( A \). In particular, for a fixed \( A \), there is a dense open set of \( b \in k^m \) such that the affine subspace defined by \( A \cdot x = b \) is in general position among its translates.

For example, if \( d = 0 \) (i.e. \( L = \{ P \} \) is a single point) then \( L \) is in general position if and only if it is in general position among its translates, if and only if all coordinates of \( P \) are non-zero. If \( d = 1 \) (i.e. \( L \) is a line) then \( L \) is in general position among its translates if and only if all points of \( L \) have at most one coordinate equal to 0, and it is in general position if, additionally, it is not parallel to any of the coordinate hyperplanes. If \( L \) is a hyperplane with equation \( a_1 x_1 + \cdots + a_n x_n = b \) then \( L \) is in general position among its translates if and only if \( b \neq 0 \), and it is in general position if, additionally, all \( a_i \neq 0 \).

For any \( L \subseteq \mathbb{A}_d^n \), we define the sum

\[
S(L; x_1, \ldots, x_n) := \sum_{x \in L(k)} \chi_1(x_1) \cdots \chi_n(x_n)
\]

and, more generally, for every finite extension \( k_r \) of \( k \) of degree \( r \),

\[
S_r(L; x_1, \ldots, x_n) := \sum_{x \in L(k_r)} \chi_1(N_{k_r/k}(x_1)) \cdots \chi_n(N_{k_r/k}(x_n))
\]

and we construct the associated \( L \)-function

\[
L(L; x_1, \ldots, x_n) := \exp \left( \sum_{r=1}^{\infty} \frac{T^r}{r} S_r(L; x_1, \ldots, x_n) \right).
\]

The main result of this article is the following

\textbf{Theorem 1.} Suppose that \( L \) is in general position among its translates. Then \( L(L; x_1, \ldots, x_n)^{(−1)^d} \) is a polynomial, whose degree is given by

\[
D_L := (-1)^d + \sum_{j=1}^{d} (-1)^{d+j} a_j
\]

where \( a_j \) is the number of subsets \( I \subseteq \{1, \ldots, n\} \) with \( |I| = j \) such that \( L \cap H_I \neq \emptyset \), and all whose reciprocal roots are algebraic integers of absolute value \( q^{i/2} \) for some \( i \leq d \).

If \( L \) is in general position, we can give a more precise result:
Theorem 2. Suppose that \( L \) is in general position. Then \( L(L; \chi_1, \ldots, \chi_n)^{(−1)^d} \) is a polynomial of degree \( \binom{n-1}{d-1} \), all whose reciprocal roots are algebraic integers. If \( \chi_1 \cdots \chi_n \) is non-trivial, then all of its reciprocal roots have absolute value \( q^{d/2} \). If \( \chi_1 \cdots \chi_n \) is trivial, then \( \binom{n-2}{d-2} \) of its reciprocal roots have absolute value \( q^{1/2} \) and \( \binom{d-2}{d-1} \) of them have absolute value \( q^{(d-1)/2} \).

This matches the known cases where \( d = n - 1 \) (classical Jacobi sums) or \( d = 1 \) (one-variable polynomial sums, cf. \[Kat02\] §1 for the case where all \( \chi_i \) are equal). Since the sums \( S(L; \chi_1, \ldots, \chi_n) \) can be written (up to sign) as the sum of the reciprocal roots of \( L(L; \chi_1, \ldots, \chi_n)^{(−1)^d} \), as the main consequence of the theorems we get an estimate for them:

Corollary 1. Suppose that \( L \) is in general position among its translates. Then we have an estimate

\[
|S(L; \chi_1, \ldots, \chi_n)| \leq D_L \cdot q^{d/2}.
\]

If \( L \) is in general position, we have

\[
|S(L; \chi_1, \ldots, \chi_n)| \leq \binom{n-1}{d} \cdot q^{d/2}
\]

if \( \chi_1 \cdots \chi_n \) is non-trivial, and

\[
|S(L; \chi_1, \ldots, \chi_n)| \leq \binom{n-2}{d} \cdot q^{d/2} + \frac{n-2}{d-1} \cdot q^{(d-1)/2}
\]

if \( \chi_1 \cdots \chi_n \) is trivial.

We also have the following variant, where \( L \) is given by a parametrization:

Corollary 2. Let \( L_1, \ldots, L_n : \mathbb{A}^d_k \to \mathbb{A}^1_k \) be affine linear forms, with \( L_i(t) = a_i t_1 + \cdots + a_i t_d + b_i \), and let \( V_i \subseteq \mathbb{A}^d_k \) be the hyperplane defined by \( L_i(t) = 0 \). Suppose that the affine map \( \mathbb{A}^d_k \to \mathbb{A}^1_k \) defined by the \( L_i \) is injective (that is, that the matrix \( (a_{ij}) \) has rank \( d \)), and that for every \( I \subseteq \{1, \ldots, n\} \) with \( |I| \leq d + 1 \) we have \( \dim(\bigcap_{i \in I} V_i) \leq d - |I| \). Then we have an estimate

\[
\left| \sum_{t \in \mathbb{G}^d_k} \chi_1(L_1(t)) \cdots \chi_n(L_n(t)) \right| \leq D_L \cdot q^{d/2}
\]

where \( D_L := (-1)^d + \sum_{j=1}^d (-1)^{d+j} a_j \) and \( a_j \) is the number of subsets \( I \subseteq \{1, \ldots, n\} \) with \( |I| = j \) such that \( \bigcap_{i \in I} V_i \neq \emptyset \).

Note that, even in the case where all \( \chi_i \) are equal to the same character \( \chi \), the subvariety of \( \mathbb{A}^d_k \) defined by \( L_1(t) \cdots L_n(t) = 0 \) is highly singular, so the estimates in \[Kat02\] and \[RL05\] for multi-variable multiplicative character sums do not give good bounds for these sums.

2. Proof of Theorem 1

In this section we will prove Theorem 1. We keep the same notation as in the previous section, in particular, we fix a finite field \( k = \mathbb{F}_q \) of characteristic \( p \), and \( n \) non-trivial multiplicative characters \( \chi_1, \ldots, \chi_n : \mathbb{F}_q^\times \to \mathbb{C}^\times \). Let \( A \subseteq \mathbb{A}^n_k \) be an affine subspace of dimension \( d \), defined as the solution set of the linear system

\[
A \cdot x = b
\]

where \( A \) is an \( m \times n \) matrix of maximal rank \( m := n - d \).

Fix a prime \( \ell \neq p \). We will work on the category of \( \ell \)-adic constructible sheaves on \( \mathbb{A}^n_k \) and, more generally, its derived category \( D^b_{\text{c}}(\mathbb{A}^n_k, \mathbb{Q}_\ell) \). For every \( i = 1, \ldots, n \) there is a rank 1, smooth sheaf \( \mathcal{L}_{\chi_i} \) on \( \mathbb{G}_{m,k} \) (the Kummer sheaf) whose Frobenius
trace at a point \( t \in \mathbb{G}_m(k, r) = k_r^{\times} \) is \( \chi_i(N_{k, k}(t)) \) [Del77a]. We will also denote by \( \mathcal{L}_{\chi_i} \), its extension by zero to \( k_1^{\times} \). Consider the product \( \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n} \) on \( k_1^{\times} \), then the sum of the traces of the Frobenius action on its stalks at the points of \( L(k) \) is precisely \( S_r(L, \chi_1, \ldots, \chi_n) \), so \( L(L, \chi_1, \ldots, \chi_n) \) is the \( L \)-function of the sheaf \( \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n} \) on \( L \). By the Grothendieck-Lefschetz trace formula we get

\[
L(L; \chi_1, \ldots, \chi_n) = \prod_{i=0}^{2d} \det(1 - T \cdot F|\mathcal{H}_{\chi_i}(L \otimes \bar{k}, \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}))(1)^{-1}
\]

where \( L \otimes \bar{k} \subseteq k_1^{\times} \) is obtained from \( L \) by extension of scalars, so Theorem [1] then a consequence of the following result, since \( \mathcal{L}_{\chi_i} \otimes \cdots \otimes \mathcal{L}_{\chi_n} \) is pure of weight 0 (so \( \mathcal{H}_{\chi_i}(L, \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}) \) is mixed of weights \( \leq i \) by [Del80]).

**Theorem 3.** Suppose that \( L \) is in general position among its translates. Then \( \mathcal{H}_{\chi_i}^j(L \otimes \bar{k}, \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}) = 0 \) for \( i \neq d \), and \( \mathcal{H}_{\chi_i}^d(L \otimes \bar{k}, \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}) \) has dimension \( D_L \).

Let \( K = \mathbb{R}A0(\mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}) \in D_r^b(k_m, \mathbb{Q}_l) \), where \( A \) is viewed as a linear map \( k_m \rightarrow k_m \). Then \( \mathcal{H}_{\chi_i}^j(L \otimes \bar{k}, \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}) \) is the stalk of \( \mathcal{H}^j(K) \) at \( b \). Recall that an object \( P \) in \( D_r^b(k_m, \mathbb{Q}_l) \) is called **perverse** if the dimension of the support of the cohomology sheaves \( \mathcal{H}^j(P) \) and \( \mathcal{H}^j(DP) \) is \( \leq -i \) for every \( i \in \mathbb{Z} \), where \( DP \) is the Verdier dual of \( P \) [BBDS2].

**Proposition 1.** The shifted object \( K[n] \in D_r^b(k_m, \mathbb{Q}_l) \) is perverse.

**Proof.** Let \( a_1, \ldots, a_n \in k_m \) be the columns of \( A \), and let \( \mathcal{L}_{\chi_i}(a_i) \) be the pull-back of the Kummer sheaf \( \mathcal{L}_{\chi_i} \) via the linear form \( k_m \rightarrow k_1 \) given by \( t \mapsto a_i \cdot t \) (which we will also denote by \( a_i \)). We define

\[
P := \mathcal{L}_{\chi_i}(a_1) \otimes \cdots \otimes \mathcal{L}_{\chi_n}(a_n)[m] \in D_r^b(k_m, \mathbb{Q}_l).
\]

This is a perverse object: we have \( P = j \cdot j^* P \), where \( j : U \hookrightarrow k_m \) is the inclusion of the dense open complement of the hyperplanes \( \{ A_i \cdot t = 0; \ i = 1, \ldots, n \} \). Since \( \mathcal{L}_{\chi_i}(a_1) \otimes \cdots \otimes \mathcal{L}_{\chi_n}(a_n) \) is smooth on \( U \), \( j^* P \) is perverse on \( U \) by [BBDS2] 4.0], and so is \( P \) by [BBDS2] Corollaire 4.1.3], since \( j \) is quasi-finite and affine.

We will show that \( K[n] \) is geometrically isomorphic to the Fourier transform \[Lau87\] of \( P \). Since the Fourier transform preserves perversity \[Lau87\] 1.3.2.3], this will prove the proposition. Equivalently, since the Fourier transform is (geometrically) involutive, we will show that the Fourier transform of \( K[n] \) is geometrically isomorphic to \( P \).

Fix an additive character \( \psi : \mathbb{k} \rightarrow \mathbb{C} \), with respect to which we will take the Fourier Transform of \( K[n] \). This is defined as

\[
\mathcal{F}T_\psi(K[n]) = \mathcal{R}\pi_2(\mu^* \mathcal{L}_\psi \otimes \pi_1^* K[n])[m]
\]

where \( \pi_1, \pi_2 : k_m \times k_m \rightarrow k_m \) are the projections, \( \mu : k_m \times k_m \rightarrow k_1 \) is the multiplication map \( \mu(t, u) = t \cdot u = \sum_{i=1}^m t_i u_i \) and \( \mathcal{L}_\psi \) is the Artin-Schreier smooth sheaf on \( k_1 \) associated to \( \psi \) [Del77a]. We have

\[
\mathcal{F}T_\psi(K[n])[-n - m] = \mathcal{R}\pi_2(\mu^* \mathcal{L}_\psi \otimes \pi_1^* \mathcal{R}A_0(\mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n})) \cong
\]

\[
\cong \mathcal{R}\pi_2(\mu^* \mathcal{L}_\psi \otimes R(A, I_d) \pi_1^* (\mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n})) \cong
\]

\[
\cong \mathcal{R}\pi_2(R(A, I_d) (\mu^* \mathcal{L}_\psi \otimes \pi_1^* (\mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}))) \cong
\]

\[
\cong \mathcal{R}\pi_2(\mu^* \mathcal{L}_\psi \otimes \pi_1^* (\mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n}))
\]

by proper base change and the projection formula, where \( \pi_1 : k_m \times k_m \rightarrow k_m \) and \( \pi_2 : k_m \times k_m \rightarrow k_m \) are the projections and \( \mu : k_m \times k_m \rightarrow k_1 \) is given by \( (x, u) \mapsto (Ax) \cdot u = \sum_{i=1}^m x_i (a_i \cdot u) \). In particular, \( \mu^* \mathcal{L}_\psi = \bigotimes_{i=1}^m \tilde{\mu}^i \mathcal{L}_\psi \), where...
\[ \tilde{\mu}_i = \sigma_i(\varpi, Id) : A^n_k \times A^m_k \rightarrow A^n_k \times A^m_n \rightarrow A^n_k, \quad \varpi_i : A^n_k \rightarrow A^n_k \text{ being the } i\text{-th projection and } \sigma_i : A^n_k \times A^m_k \rightarrow A^n_k \text{ being given by } (x, u) \mapsto x(a_i, u). \]

We then have

\[ \mathcal{F}^\psi(K(n))[-n - m] \cong R\pi_2(\mu^*\mathcal{L}_\psi \otimes \cdots \otimes \tilde{\mu}_n^*\mathcal{L}_\psi \otimes \tilde{\pi}_1^*(\mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n})) \]

\[ \cong R\pi_2((\varpi_1, Id)^*\sigma_1^*\mathcal{L}_\psi \otimes \cdots \otimes (\varpi_n, Id)^*\sigma_n^*\mathcal{L}_\psi \otimes (\varpi_1, Id)^*\tilde{\pi}_1^*(\mathcal{L}_{\chi_1}) \otimes \cdots \otimes (\varpi_n, Id)^*\tilde{\pi}_n^*(\mathcal{L}_{\chi_n})) \]

where \( \tilde{\pi}_1 : A^n_k \times A^m_n \rightarrow A^n_k \), \( \tilde{\pi}_2 : A^n_k \times A^m_n \rightarrow A^n_k \) are the projections. By the Künneth formula, this is

\[ R\tilde{\pi}_2(\sigma_1^*\mathcal{L}_\psi \otimes \tilde{\pi}_1^*\mathcal{L}_{\chi_1}) \otimes \cdots \otimes R\tilde{\pi}_2(\sigma_n^*\mathcal{L}_\psi \otimes \tilde{\pi}_n^*\mathcal{L}_{\chi_n}) \]

so it only remains to show that

\[ R\tilde{\pi}_2(\sigma_i^*\mathcal{L}_\psi \otimes \tilde{\pi}_1^*\mathcal{L}_{\chi_1}) \cong \mathcal{L}_{\chi_i}(a_i)[-1] = a_i^*\mathcal{L}_{\chi_i}[-1] \]

given geometrically for every \( i = 1, \ldots, n \). If \( \tilde{\mu} : A^n_k \times A^n_k \rightarrow A^n_k \) is the multiplication map, we can write

\[ R\tilde{\pi}_2(\sigma_1^*\mathcal{L}_\psi \otimes \tilde{\pi}_1^*\mathcal{L}_{\chi_1}) = R\tilde{\pi}_2((Id, a_i)^*\tilde{\mu}^*\mathcal{L}_\psi \otimes \tilde{\pi}_1^*\mathcal{L}_{\chi_1}) \cong \]

\[ \cong R\tilde{\pi}_2((Id, a_i)^*(\tilde{\mu}^*\mathcal{L}_\psi \otimes \tilde{\pi}_1^*\mathcal{L}_{\chi_1})) \cong a_i^*R\pi_2(\tilde{\mu}^*\mathcal{L}_\psi \otimes \tilde{\pi}_1^*\mathcal{L}_{\chi_1}) = a_i^*\mathcal{F}^\psi(\mathcal{L}_{\chi_i})[-1] \]

again by proper base change, where \( \pi_1, \pi_2 \) are now the projections \( A^n_k \times A^n_k \rightarrow A^n_k \).

Then (2) follows from the fact [Lan87, 1.4.3.1] that the Fourier transform of \( \mathcal{L}_{\chi_i} \) is geometrically isomorphic to \( \mathcal{L}_{\chi_i} \) for any non-trivial multiplicative character \( \chi_i \). \( \square \)

Next, we will show that, if \( L \) (defined by \( A \cdot x = b \)) is in general position among its translates, then the cohomology sheaves of \( K \) are smooth at \( b \in A^m_k \).

**Proposition 2.** Let \( b \in k^m \) such that the affine subspace \( L \subseteq A^n_k \) defined by \( A \cdot x = b \) is in general position among its translates. Then the cohomology sheaf \( \mathcal{H}(K) \) is smooth at \( b \) for every \( i \in \mathbb{Z} \).

**Proof.** Let \( \overline{X} \subseteq \mathbb{P}^n_k \times A^m_k \) be the projective closure of the graph \( X = \Gamma_A \) of the map \( A : A^n_k \rightarrow A^n_k \). Then \( K = R\pi_2_\mathcal{F} \), where \( \mathcal{F} \) is the extension by zero to \( \overline{X} \) of the sheaf \( \mathcal{L}_{\chi_1(x_1)} \otimes \cdots \otimes \mathcal{L}_{\chi_n(x_n)} \) on \( X \). We claim that \( \pi_2 : \overline{X} \rightarrow A^m_k \) is locally acyclic for \( \mathcal{F} \) at every point of its fibre at \( b \) [Del77b 2.12]. By [Del77b A.2.2], this implies that all cohomology sheaves of \( K \) are smooth at \( b \).

Following [Del80 3.7.3] we will show that, over some neighborhood \( U \) of \( b \) in \( A^m_k \), locally for the étale topology on \( \overline{X} \), the pair \( (\overline{X}, \mathcal{F}) \) is \( U \)-isomorphic to the product of \( U \) and a scheme endowed with a \( \mathcal{G} \)-sheaf \( \mathcal{G} \) (in other words, that \( \mathcal{F} \) is étale-locally isomorphic to a sheaf independent of \( b \)). We consider different cases depending on the position of the point \( x \in \pi_2^{-1}(b) \) (viewed in \( \mathbb{P}^n_k \)).

a) Suppose that \( x \in A^m_k \) has all its coordinates different from zero. Then \( \mathcal{F} \) is smooth in a neighborhood of \( x \), and in fact can be trivialized by taking a finite étale map, so the assertion is clear in this case.

b) Suppose that \( x \in A^m_k \) has some coordinates equal to zero. Without loss of generality, assume that \( x_1 = \cdots = x_e = 0 \) and \( x_i \neq 0 \) for \( e + 1 \leq i \leq n \). Then \( \mathcal{L}_{\chi_{e+1}(x_{e+1})} \otimes \cdots \otimes \mathcal{L}_{\chi_n(x_n)} \) is smooth at \( x \), so \( \mathcal{F} \) is étale-locally isomorphic to \( \mathcal{L}_{\chi_1(x_1)} \otimes \cdots \otimes \mathcal{L}_{\chi_e(x_e)} \).

Since \( L \) is in general position among its translates, \( L \cap H_{1, \ldots, c} \) has dimension \( d - e \). In particular, \( t_1 = x_1, \ldots, t_e = x_e \) is part of a system of parameters of \( L \) at \( x \). Solving the system \( A \cdot x = z \) for a generic \( z \) we get a parametrization

\[ \phi_z : (t_1, \ldots, t_d) \mapsto (t_1, \ldots, t_e, \alpha_{e+1}(z) + \sum_{i=1}^d c_{e+1,i}t_i, \ldots, \alpha_n(z) + \sum_{i=1}^d c_{n,i}t_i) \]

of \( \pi_2^{-1}(z) \) for \( z \) in a neighborhood \( U \) of \( b \). The pull-back of \( \mathcal{L}_{\chi_1(x_1)} \otimes \cdots \otimes \mathcal{L}_{\chi_e(x_e)} \) to \( A^m_k \times U \) via the map \((t, z) \mapsto (\phi_z(t), t) \) is the \( \mathcal{G} \) we are looking for.
c) Suppose that $\mathbf{x} = (0 : x_1 : \ldots : x_n) \in \mathbb{P}_k^n \setminus A^n_k$. Without loss of generality, assume that $x_1 = \ldots = x_e = 0$ and $x_i \neq 0$ for $e + 1 \leq i \leq n$ for some $0 \leq e \leq n - 1$. Dehomogenizing with respect to $x_n$ and letting $y_i = x_i/x_n$ for $0 \leq i \leq n - 1$ be the new affine variables, we see that

$$L_{x_1}(x_1, y_0) \otimes \cdots \otimes L_{x_n}(x_n, y_0) \cong L_{x_1}(y_1) \otimes \cdots \otimes L_{x_{n-1}}(y_{n-1}) \otimes L_{x_n}(y_n)$$

which is étale-locally isomorphic to $L_{x_1}(y_1) \otimes \cdots \otimes L_{x_{n-1}}(y_{n-1}) \otimes L_{x_n}(y_n)$.

Note that the intersection of $\mathcal{T}$ (the projective closure of $L$) with the hyperplane at infinity $H_\infty = \{y_0 = 0\}$, of dimension $d - 1$, does not depend on $b$, only on the map $A$. With respect to the new coordinates, $\mathcal{T}\setminus \{x_0 = 0\}$ is the set of solutions of the system of equations

$$(-b | A') \mathbf{y} = -a_n$$

where $A'$ is the matrix $A$ with its last column $a_n$ removed.

Suppose that $\dim (\mathcal{T} \cap H_\infty \cap \mathcal{T}_{(1, \ldots, e)}) = d - 1 - e$. Then $\{y_0, y_1, \ldots, y_e\}$ is part of a system of parameters of $\mathcal{T}$ at $\mathbf{x}$, and we conclude as in (b) (note that $\chi_1 \cdots \chi_n$ might be trivial, but (b) is proved without any non-triviality condition on the $\chi_i$'s).

Finally, suppose that $\dim (\mathcal{T} \cap H_\infty \cap \mathcal{T}_{(1, \ldots, e)}) > d - 1 - e$. Solving the system $(-\mathbf{z} | A') \mathbf{y} = -a_n$ via Gaussian elimination, we get a parametrization $\phi_2(t) = (\phi_{2,0}(t), \phi_1(t), \ldots, \phi_{n-1}(t))$ of $\mathbb{P}_{e-1}^1(\mathbf{z})$ for $\mathbf{z}$ in a neighborhood of $\mathbf{b}$, where $\phi_1, \ldots, \phi_{n-1}$ do not depend on $\mathbf{z}$. Furthermore, for $\mathbf{z} \neq (0, \ldots, 0)$ (which is the case in a neighborhood of $\mathbf{b}$, since an affine subspace which is in general position among its translates can never contain the origin), $\phi_{2,0}(t)$ has the form $\phi_0(0)/\alpha(\mathbf{z})$ where $\phi_0(0)$ does not depend on $\mathbf{z}$ and $\alpha(\mathbf{z}) = \alpha_1 z_1 + \cdots + \alpha_m z_m$ is linear on $\mathbf{z}$ and $\alpha(\mathbf{b}) \neq 0$.

The pull-back of $L_{x_1}(y_1) \otimes \cdots \otimes L_{x_{n-1}}(y_{n-1}) \otimes L_{x_n}(y_n)$ to $A_x \times A_{\mathbb{C}}$ via $\phi(0, \mathbf{z}) = (\phi_2(t), \mathbf{z})$ is

$$L_{x_1}(\phi_1(t)) \otimes \cdots \otimes L_{x_n}(\phi_n(t)) \otimes L_{x_1}(\chi_\alpha(\mathbf{z})) \otimes L_{x_1}(\cdots)$$

where the last factor is smooth in a neighborhood $U$ of $\mathbf{b}$. It is then étale-locally isomorphic to $L_{x_1}(\phi_1(t)) \otimes \cdots \otimes L_{x_n}(\phi_n(t)) \otimes L_{x_1}(\chi_\alpha(\mathbf{z}))$ over $U$, which is independent of $\mathbf{z}$.

Since $K[\mathbf{n}]$ is perverse (so all its cohomology sheaves except for $H_\infty^m(K)$ are supported on proper closed subsets), proposition 2 implies that $H_\infty^m(K) = 0$ for $i \neq n - m = d$, which proves the first part of Theorem 3. The dimension of $H_\infty^d(L \otimes \mathbb{k}, L_{x_1} \boxtimes \cdots \boxtimes L_{x_n})$ is then $(-1)^d$ times its Euler characteristic $\chi_c(L \otimes \mathbb{k}, L_{x_1} \boxtimes \cdots \boxtimes L_{x_n})$.

Let $H := \bigcup_{i=1}^n H_i \subseteq A^n_\mathbb{k}$ be the union of the coordinate hyperplanes. We will show that, for any affine subvariety $L \subseteq A^n_\mathbb{k}$, the Euler characteristic $\chi_c(L \otimes \mathbb{k}, L_{x_1} \boxtimes \cdots \boxtimes L_{x_n})$ is just the Euler characteristic of $U := L \setminus H$. We proceed by induction on $n$: for $n = 1$ or, more generally, whenever $d = 0$ or $d = n$, the statement is clear (in the latter case, $\chi_c(A^n_\mathbb{k}, L_{x_1} \boxtimes \cdots \boxtimes L_{x_n}) \cong \chi_c(A_1^1, L_{x_1}) \cdots \chi_c(A_1^1, L_{x_n}) = 0 = \chi_c(G_{m,k}, \mathbb{P}_k)$).

In the general case, assume $d < n$ and pick some $i \in \{1, \ldots, n\}$ such that the projection $\pi_i$ of $L$ on the hyperplane $H_i$ is injective. Without loss of generality, assume $i = n$. Let $\pi = \pi_n : A^n_\mathbb{k} \to A^1_\mathbb{k}$ be the projection, then

$$R\pi_!L_!(L_{x_1} \boxtimes \cdots \boxtimes L_{x_n}) \cong L_{x_1} \boxtimes R\pi_!L_!(L_{x_1} \boxtimes \cdots \boxtimes L_{x_{n-1}} \boxtimes \mathbb{Q}_l)$$

by the projection formula. Since $L_{x_n}$ is smooth of rank 1 on $G_m$ and tamely ramified at both 0 and $\infty$, the Grothendieck-Ogg-Shafarevich formula implies that $\chi_c(A_1^1, K \boxtimes L_{x_n}) = \chi_c(G_{m,k}, K)$ for any object $K \in D^b(A_1^1, \mathbb{Q}_l)$. In particular,

$$\chi_c(L \otimes \mathbb{k}, L_{x_1} \boxtimes \cdots \boxtimes L_{x_n}) = \chi_c(A_1^1, R\pi_!L_!(L_{x_1} \boxtimes \cdots \boxtimes L_{x_{n-1}} \boxtimes \mathbb{Q}_l)) = \chi_c(L(H_n) \otimes \mathbb{k}, L_{x_1} \boxtimes \cdots \boxtimes L_{x_{n-1}} \boxtimes \mathbb{Q}_l)$$
\[ \chi_c(L \otimes \bar{k}, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_n}) = \chi_c((L \cap H_n) \otimes \bar{k}, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_{n-1}}) = \]
\[ = \chi_c(\pi(L) \otimes \bar{k}, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_{n-1}}) - \chi_c((L \cap H_n) \otimes \bar{k}, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_{n-1}}) = \]
\[ = \chi_c(L) - \chi_c((L \cap H_n) \cup H_i) = \]
\[ = \chi_c(L) - \chi_c((L \cap H_n) \cup H_i) = \chi_c(L - \bigcup_{i=1}^{n-1} H_i) \]
by induction hypothesis.

More precisely, we have

\[ \chi_c(L \otimes \bar{k}, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_n}) = \chi_c(L \setminus H) = 1 - \chi_c(L \cap \bigcup_{i=1}^{n-1} H_i) = \]
\[ = 1 + \left( \sum_{j=1}^{n} (-1)^j \sum_{i \subseteq \{1, \ldots, n\} \mid |i| = j} \chi_c(L \cap H_i) \right). \]

For every \( I \subseteq \{1, \ldots, n\} \), \( L \cap H_I \) is either empty (which is always the case for \( |I| > d \) since \( L \) is in general position among its translates) or it is an affine subvariety of \( \mathbb{A}^n_k \), which has Euler characteristic 1. Therefore we get

\[ \chi_c(L \otimes \bar{k}, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_n}) = 1 + \sum_{j=1}^{d} (-1)^j a_j \]
where \( a_j \) is the number of subsets \( I \subseteq \{1, \ldots, n\} \) with \( j \) elements such that \( L \cap H_I \neq \emptyset \). In particular, if \( L \) is in general position, \( a_j = \binom{n}{j} \) for every \( j = 1, \ldots, d \), so

\[ \chi_c(L \otimes \bar{k}, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_n}) = 1 + \sum_{j=1}^{d} (-1)^j \binom{n}{j} = \]
\[ = 1 + \sum_{j=1}^{d} (-1)^j \left( \binom{n-1}{j-1} + \binom{n-1}{j} \right) = (-1)^d \binom{n-1}{d}. \]
This completes the proof of Theorem 1 (and of the first statement of Theorem 2).

3. Proof of Theorem 2

In this section we will compute the absolute values of the reciprocal roots of the \( L \) function \( L(L; \chi_1, \ldots, \chi_n) \) in the case where \( L \) is in general position, proving Theorem 2.

We proceed by induction on \( n \), the case \( n = 1 \) (or, in general, \( d = 0 \) or \( d = n \)) being trivial. Assume \( n > 1 \) and \( 0 < d < n \). Suppose first that there is some \( i \in \{1, \ldots, n\} \) such that the product of the \( \chi_j \) for \( j \neq i \) is non-trivial (this is always the case except when all \( \chi_i \) are equal to the same character \( \chi \) with \( \chi^{n-1} = 1 \)). Without loss of generality, we may assume that \( i = n \). Consider the projection \( \pi = \pi_n : L \subseteq \mathbb{A}^n_k \rightarrow \mathbb{A}^n_k \). Since \( L \) is in general position, \( \pi(L) = \mathbb{A}_k^n \). By the projection formula, we have

\[ \text{Res} \pi_L(\mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_n}) = \mathcal{L}_{x_n} \otimes \text{Res} \pi_L(\mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_{n-1}} \boxtimes \mathcal{Q}_L). \]

Let \( K := \text{Res} \pi_L(\mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_{n-1}} \boxtimes \mathcal{Q}_L) \in D_c^b(\mathbb{A}^n_k, \mathcal{Q}_L) \). Then \( \text{Res} \pi_L(A_k^1, K) = \text{Res} \pi_L(\mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_{n-1}} \boxtimes \mathcal{Q}_L) = \text{Res} \pi_L(\mathcal{Q}_L, \mathcal{L}_{x_1} \boxtimes \cdots \boxtimes \mathcal{L}_{x_{n-1}}) \) where \( \mathcal{Q}_L : A_k^n \rightarrow A_k^{n-1} \) is the projection onto the first \( n-1 \) coordinates. The last equality holds because \( \mathcal{Q}_L : L \rightarrow \mathcal{Q}_L(L) \) is an isomorphism (otherwise \( L \) would be parallel to the coordinate axis \( x_n \), contradicting the general position condition).
Note that \( \varpi(L) \subseteq A_{k}^{n-1} \) is also in general position: for every \( I \subseteq \{1, \ldots, n-1\} \) with \( |I| \leq d+1 \), if we denote by \( \hat{H}_I \) the intersection of the coordinate hyperplanes \( H_i \subseteq A_{k}^{n-1} \) for \( i \in I \), then \( \dim(\varpi(L) \cap \hat{H}_I) = \dim(L \cap \varpi^{-1}(\varpi(L) \cap \hat{H}_I)) = \dim(L \cap \varpi^{-1}(\varpi(L) \cap \hat{H}_I)) \). Therefore, by theorem, \( H^i_c(A_{k}^{1}, K) = H^i_c(\varpi(L), L_{\chi_1} \boxtimes \cdots \boxtimes L_{\chi_{n-1}}) \) vanishes for \( i \neq d \), and \( H^2_c(A_{k}^{1}, K) \) has dimension \( \binom{n-2}{d-2} \) and is pure of weight \( d \) by induction hypothesis.

Next, we check that for all but finitely many \( t \in k^\infty \) the fibre \( \pi^{-1}(t) \in A_{k}^{n-1} \) is in general position. Fix \( I \subseteq \{1, \ldots, n-1\} \) with \( |I| \leq d \). Then \( \pi^{-1}(t) \cap \hat{H}_I = L \cap \hat{H}_I \cap \{x_n = t\} \). Since \( L \) is in general position, \( \dim(L \cap \hat{H}_I) = d - |I| \). So, either \( \dim(L \cap \hat{H}_I \cap \{x_n = t\}) = d - 1 - |I| \), or \( L \cap \hat{H}_I \subseteq \{x_n = t\} \). The latter is not possible if \( |I| < d \), because then \( L \cap \hat{H}_I \cup \{x_n = t\} \) would be empty, contradicting the general position hypothesis. And, if \( |I| = d \), it can only happen for at most \( \binom{n-1}{d-1} \) values of \( t \). Furthermore, the fibre at \( t = 0 \) (and, more generally, the intersection of \( L \) with any subset of the coordinate hyperplanes) is in general position (with respect to the remaining coordinate hyperplanes).

By induction hypothesis and Proposition \( \[ \] \) we conclude that there is a finite subset \( Z \subseteq A_{k}^{1} \) not containing \( 0 \) such that on \( U := A_{k}^{1} \setminus Z \) all cohomology sheaves of \( K \) other than \( H^{d-1}(K) \) vanish, and \( H^{d-1}(K) \) is smooth of rank \( \binom{n-2}{d-1} \) and pure of weight \( d-1 \). Furthermore, by the previous section \( K \) is the restriction to a line of a perverse sheaf on \( A_{k}^{1} \) (namely, \( RA'(\mathcal{L}_{1\chi_1} \boxtimes \cdots \boxtimes \mathcal{L}_{1\chi_{n-1}}) \)) where \( A' \) is the matrix \( A \) with its last column deleted), which implies that \( H^i_c(K) = 0 \) for \( i < d-1 \) and \( H^{d-1}(K) \) has no punctual sections [Kat03, Propositions 7 and 9]. Let \( F := H^{d-1}(K) \). From the truncation distinguished triangle

\[
F[1-d] \to K \to \tau_{\geq d}K \to .
\]

we get

\[
R\Gamma_c(A_{k}^{1}, F)[1-\ell] \to R\Gamma_c(A_{k}^{1}, K) \to R\Gamma_c(A_{k}^{1}, \tau_{\geq d}K) \cong \bigoplus_{z \in Z} \tau_{\geq d}K_z
\]

since \( \tau_{\geq d}K \) is supported on \( Z \). This gives an exact sequence

\[
0 \to H^1_c(A_{k}^{1}, F) \to H^i_c(A_{k}^{1}, K) \to \bigoplus_{z \in Z} H^d(K)_z \to H^2_c(A_{k}^{1}, F) \to 0
\]

and

\[
\bigoplus_{z \in Z} H^i_c(K)_z \cong H^{i+2-d}_c(A_{k}^{1}, F) = 0
\]

for \( i > d \). Also, \( H^2_c(A_{k}^{1}, F) = H^2_c(U, F) \) is pure of weight \( d+1 \) since \( F \) is smooth and pure of weight \( d-1 \) on \( U \), so the map \( \bigoplus_{z \in Z} H^d(K)_z \to H^2_c(A_{k}^{1}, F) \) must be trivial as \( H^d(K)_z \) has weights \( \leq d \). Therefore \( H^2_c(A_{k}^{1}, F) = 0 \) and the exact sequence reduces to

\[
0 \to H^1_c(A_{k}^{1}, F) \to H^d_c(A_{k}^{1}, K) \to \bigoplus_{z \in Z} H^d(K)_z \to 0.
\]

Since \( H^d_c(A_{k}^{1}, K) \) is pure of weight \( d \), so are \( H^1_c(A_{k}^{1}, F) \) and \( \bigoplus_{z \in Z} H^d(K)_z \). If we denote by \( j : U \hookrightarrow P^1_{k} \) the inclusion, we have an exact sequence

\[
0 \to F \to j_*j^*F \to (j_*j^*F)/F \to 0
\]

since \( F \) has no punctual sections, which gives rise to an exact sequence (where we identify \( F \) and its extension by zero to \( P^1_{k} \))

\[
0 \to H^0(Z \cup \{\infty\}, (j_*j^*F)/F) \to H^1_c(A_{k}^{1}, F) \to H^1(P^1_{k}, j_*j^*F) \to 0
\]

and, since \( H^0(Z \cup \{\infty\}, (j_*j^*F)/F) \) has weights \( \leq d-1 \), we conclude that \( (j_*j^*F)/F \) is pure of weight \( d \) and \( F \cong j_*j^*F \).
We now tensor the distinguished triangle
\[ \mathcal{F}[1 - d] \to K \to \tau_{\geq d}K = \bigoplus_{z \in Z} \mathcal{H}^d(K)_z[-d] \to . \]
with \( \mathcal{L}_\chi \) to obtain a triangle
\[ \mathcal{L}_\chi \otimes \mathcal{F}[1 - d] \to \mathcal{L}_\chi \otimes K \to \bigoplus_{z \in Z} \mathcal{L}_\chi \otimes \mathcal{H}^d(K)_z[-d] \to . \]

We know that \( H^0_c(A^1_k, \mathcal{L}_\chi \otimes K) = H^0_c(L \otimes k, \mathcal{L}_\chi, \boxtimes \cdots \boxtimes \mathcal{L}_\chi) = 0 \) for \( i \neq d \), and \( H^0_c(A^1_k, \bigoplus_{z \in Z} \mathcal{L}_\chi \otimes \mathcal{H}^d(K)_z[-d]) \) also vanishes for \( i \neq d \) since \( \bigoplus_{z \in Z} \mathcal{L}_\chi \otimes \mathcal{H}^d(K)_z \) is punctual. We have therefore an exact sequence
\[ 0 \to H^0_c(A^1_k, \mathcal{L}_\chi \otimes \mathcal{F}) \to H^0_c(L \otimes k, \mathcal{L}_\chi, \boxtimes \cdots \boxtimes \mathcal{L}_\chi) \to \bigoplus_{z \in Z} \mathcal{L}_\chi \otimes \mathcal{H}^d(K)_z \to \]
\[ \to H^2_c(A^1_k, \mathcal{L}_\chi \otimes \mathcal{F}) \to 0 \]
and the same argument as above shows that \( H^2_c(A^1_k, \mathcal{L}_\chi \otimes \mathcal{F}) = 0 \). Since \( \bigoplus_{z \in Z} \mathcal{L}_\chi \otimes \mathcal{H}^d(K)_z = \mathcal{L}_\chi \otimes \bigoplus_{z \in Z} \mathcal{H}^d(K)_z \) is pure of weight \( d \), we conclude that the weight \( < d \) subspace of \( H^d(L \otimes k, \mathcal{L}_\chi, \boxtimes \cdots \boxtimes \mathcal{L}_\chi) \) is isomorphic to that of \( H^2_c(A^1_k, \mathcal{L}_\chi \otimes \mathcal{F}) \).

Let \( j : \mathbb{G}_{m,k} \hookrightarrow \mathbb{P}^1_k \) be the inclusion. Since \( \mathcal{F} \) is isomorphic to \( j_*(-)F \) and \( \mathcal{L}_\chi \) is smooth on \( \mathbb{G}_{m,k} \), \( j_*j^*(\mathcal{L}_\chi \otimes \mathcal{F}) \cong \mathcal{L}_\chi \otimes j_*(\mathcal{F}) \cong \mathcal{L}_\chi \otimes \mathcal{F} \) on \( \mathbb{G}_{m,k} \), and we get an exact sequence
\[ 0 \to \mathcal{L}_\chi \otimes \mathcal{F} \to j_*(\mathcal{L}_\chi \otimes \mathcal{F}) \to i_{0*}j_!i^*_\infty j_*\mathcal{L}_\chi \otimes \mathcal{F} \to 0 \]
and, taking cohomology,
\[ 0 \to j_*(\mathcal{L}_\chi \otimes \mathcal{F})_0 \otimes j_!(\mathcal{L}_\chi \otimes \mathcal{F})_{\infty} \to H^1_c(A^1_k, \mathcal{L}_\chi \otimes \mathcal{F}) \to H^1_c(\mathbb{P}^1_k, j_*j^*(\mathcal{L}_\chi \otimes \mathcal{F})) \to 0 . \]

But \( \mathcal{L}_\chi \otimes \mathcal{F} \) is totally ramified at 0 (since \( \mathcal{F} \) is unramified and \( \mathcal{L}_\chi \) is totally ramified), so \( j_*(\mathcal{L}_\chi \otimes \mathcal{F})_0 \to 0 \). On the other hand, \( H^1_c(\mathbb{P}^1_k, j_*j^*(\mathcal{L}_\chi \otimes \mathcal{F})) \) is pure of weight \( d \) by Theorem 2. We conclude that the weight \( < d \) subspace of \( H^1_c(A^1_k, \mathcal{L}_\chi \otimes \mathcal{F}) \) is isomorphic to \( j_*(\mathcal{L}_\chi \otimes \mathcal{F})_{\infty} \).

We claim that, as a representation of the inertia group \( I_\infty \) of \( \mathbb{P}^1_k \) at infinity, \( \mathcal{L}_\chi \otimes \mathcal{F} \) is isomorphic to a direct sum of \( \binom{n+1}{d} \) copies of \( \mathcal{L}_{\chi_1} \otimes \cdots \otimes \mathcal{L}_{\chi_n} \).

Equivalently, \( \mathcal{L}_{(\chi_1, \ldots, \chi_{n-1})} \otimes \mathcal{F} \) is unramified at infinity. We have
\[ \mathcal{L}_{(\chi_1, \ldots, \chi_{n-1})} \otimes \mathcal{F} = \mathcal{L}_{(\chi_1, \ldots, \chi_{n-2})} \otimes R^{d-1} \pi \mathcal{L}_{(\chi_1, x_1)} \otimes \cdots \otimes \mathcal{L}_{\chi_{n-1}(x_{n-1})} \cong \]
\[ \cong R^{d-1} \pi \mathcal{L}_{(\chi_1, x_1)} \otimes \cdots \otimes \mathcal{L}_{\chi_{n-1}(x_{n-1})} \cong \]
\[ \cong R^{d-1} \pi \mathcal{L}_{(\chi_1, x_1, x_2)} \otimes \cdots \otimes \mathcal{L}_{\chi_{n-1}(x_{n-1})} \]
where \( M \subset A^0_k \) is \( \sigma^{-1}(L) \), \( \sigma : A^0_n \to A^0_n \) is the map \( (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1 x_1, \ldots, x_{n-1} x_n, x_n) \) (which is an isomorphism away from the coordinate hyperplanes) and \( \pi_M \) is the restriction of \( \pi \) to \( M \). If \( A \cdot x = b \) is a system of independent linear equations that define \( L, M \) is defined by \( (a_0 x_1 + \cdots + a_{n-1} x_{n-1} + a_n)x_n = b \), where \( a_0, \ldots, a_n \) are the columns of the matrix \( A \). If \( \iota : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k} \) denotes the inversion map, we have then
\[ \iota^*(\mathcal{L}_{(\chi_1, \ldots, \chi_{n-1})} \otimes \mathcal{F}) \cong R^{d-1} \pi \mathcal{L}_{(\chi_1, x_1)} \otimes \cdots \otimes \mathcal{L}_{\chi_{n-1}(x_{n-1})} \]
where \( N \subset A^0_k \) is defined by \( a_0 x_1 + \cdots + a_{n-1} x_{n-1} + a_n = b x_n \) or \( A' \cdot x = b x_n - a_n \), where \( A' \) is the matrix \( A \) with the last column deleted. That is, \( \iota^*(\mathcal{L}_{(\chi_1, \ldots, \chi_{n-1})} \otimes \mathcal{F}) \)

is isomorphic to the restriction via the map \( \mathbb{G}_{m,k} \to A^0_k \) given by \( \lambda \mapsto \lambda b - a_n \) of the sheaf \( R^{d-1} A' \mathcal{L}_{(\chi_1, \cdots, \chi_{n-1})} \) on \( A^0_k \). But this sheaf is smooth at \(-a_n\), as seen in the previous section, since the affine variety defined by \( A' \cdot x = -a_n \) is in general position (all size \((n-r)\) minors of the augmented matrix are minors of the matrix \( A \), and therefore non-zero since \( L \) itself is in general position).
So $\eta^*(\mathcal{L}_{(x_1, \ldots, x_{n-1})^{-1}} \otimes F)$ extends to a sheaf on $\mathbb{A}^1_k$ which is smooth at 0 (and purely of weight $d - 1$ by induction hypothesis), so it is unramified there. Hence $\mathcal{L}_{(x_1, \ldots, x_{n-1})^{-1}} \otimes F$ is unramified at infinity.

Thus, if $\chi_1, \ldots, \chi_n$ is non-trivial, $\mathcal{L}_{\chi_n} \otimes F \cong \mathcal{L}_{\chi_1, \ldots, \chi_n} \otimes (\mathcal{L}_{(x_1, \ldots, x_{n-1})^{-1}} \otimes F)$ is totally ramified at infinity, so $j_* (\mathcal{L}_{\chi_n} \otimes F)_{\infty} = 0$ and $H^1_{\chi_n}(\mathbb{A}^1_k, \mathcal{L}_{\chi_n} \otimes F)$ is pure of weight $d$. On the other hand, if $\chi_1, \ldots, \chi_n$ is trivial, $\mathcal{L}_{\chi_n} \otimes F$ is unramified at infinity, so $j_* (\mathcal{L}_{\chi_n} \otimes F)_{\infty}$ has dimension rank$(\mathcal{F}) = \binom{n-2}{d-1}$ and, as seen above, is pure of weight $d - 1$. We conclude that $H^1_{\chi_n}(\mathbb{A}^1_k, \mathcal{L}_{\chi_n} \otimes F)$ has $(n-2)/d$ Frobenius eigenvalues of weight $d - 1$ and $\binom{n-1}{d-1} - \binom{n-2}{d-1} = \binom{n-2}{d}$ of weight $d$.

It remains to prove the case where all $\chi_i$ are equal to a fixed non-trivial $\chi$ such that $\chi^{n-1} = 1$, in which case we need to show that $H^1_{\chi}(\mathbb{A}^1_k, K \otimes \mathcal{L}_\chi)$ is pure of weight $d$. By the previous case, we know that $H^1_{\chi}(\mathbb{A}^1_k, K \otimes \mathcal{L}_\chi) \cong H^1_{\chi}(L \otimes k, \mathcal{L}_\chi) \oplus \cdots \oplus \mathcal{L}_{\chi_{n-1}} \boxtimes \mathcal{L}_\eta)$ is pure of weight $d$ for every non-trivial character $\eta$ of $k^\times$ other than $\chi$.

Like in the previous case, the distinguished triangle

$$F[1-d] \to K \to \tau_{\geq d}K \to .$$

gives rise to an exact sequence

$$0 \to H^1_{\chi}(\mathbb{A}^1_k, F) \to H^1_{\chi}(\mathbb{A}^1_k, K) \to \bigoplus_{z \in Z} \mathcal{H}^d(K)_z \to H^2_{\chi}(\mathbb{A}^1_k, F) \to 0$$

and, similarly,

$$0 \to H^1_{\chi}(\mathbb{A}^1_k, F \otimes \mathcal{L}_\eta) \to H^1_{\chi}(\mathbb{A}^1_k, K \otimes \mathcal{L}_\eta) \to \bigoplus_{z \in Z} \mathcal{H}^d(K \otimes \mathcal{L}_\eta)_z \to 0$$

for any non-trivial character $\eta$, since $H^1_{\chi}(\mathbb{A}^1_k, F \otimes \mathcal{L}_\eta) = 0$ because $F$ is unramified at 0. In particular, $H^1_{\chi}(\mathbb{A}^1_k, F \otimes \mathcal{L}_\eta)$ and $\bigoplus_{z \in Z} \mathcal{H}^d(K \otimes \mathcal{L}_\eta)_z$ are pure of weight $d$ for $\eta \neq \chi$, and therefore so is $\bigoplus_{z \in Z} \mathcal{H}^d(K \otimes \mathcal{L}_\eta)_z = \mathcal{L}_{\chi/\eta} \otimes (\bigoplus_{z \in Z} \mathcal{H}^d(K \otimes \mathcal{L}_\eta)_z)$.

So $H^1_{\chi}(\mathbb{A}^1_k, K \otimes \mathcal{L}_\chi)$ is pure of weight $d$ if and only if $H^1_{\chi}(\mathbb{A}^1_k, F \otimes \mathcal{L}_\chi)$ is.

Next, we consider the exact sequence

$$0 \to j_* j^* F \to F \to i_* i^* F \to 0$$

where $j : U \hookrightarrow \mathbb{A}^1_k$ and $i : Z \hookrightarrow \mathbb{A}^1_k$ are the inclusions, which gives rise to a cohomology exact sequence

$$0 \to \bigoplus_{z \in Z} (F \otimes \mathcal{L}_\eta)_z \to H^1_{\chi}(U, F \otimes \mathcal{L}_\eta) \to H^1_{\chi}(\mathbb{A}^1_k, F \otimes \mathcal{L}_\eta) \to 0$$

for any non-trivial character $\eta$, since $F$ has no punctual sections. Since the weights of $\bigoplus_{z \in Z} (F \otimes \mathcal{L}_\eta)_z$ clearly do not depend on $\eta$ (because $\mathcal{L}_\eta$ is always pure of weight 0), the result is then a consequence of the following lemma, which we can apply in this case given that $F \cong \mathcal{L}_{(x_1, \ldots, x_{n-1})^{-1}} \otimes F$ is unramified at infinity as shown in the proof of the previous case.

**Lemma 1.** Let $F$ be a smooth $\mathbb{Q}_l$-sheaf on some non-empty open set $U \subseteq \mathbb{A}^1_k$, mixed of some weights and unramified at 0 and $\infty$. Then the dimension and the weights of $H^1_{\chi}(U, F \otimes \mathcal{L}_\eta)$, for $\eta$ a non-trivial multiplicative character of $k^\times$, are independent of $\eta$.

**Proof.** Since $\mathcal{L}_\eta$ vanishes at 0, we can and will assume that 0 $\notin U$. Since $F$ is smooth an unramified at 0 and $\infty$, $F \otimes \mathcal{L}_\eta$ is totally ramified at both points, so $H^1_{\chi}(U, F \otimes \mathcal{L}_\eta) = 0$ and the Euler characteristic of $F \otimes \mathcal{L}_\eta$ is $-\dim H^1_{\chi}(U, F \otimes \mathcal{L}_\eta)$. By the Grothendieck-Ogg-Shafarevic formula, this Euler characteristic is $(\# Z) \cdot \text{rk}(F \otimes \mathcal{L}_\eta) + \sum_{z \in Z \cup \{0, \infty\}} \text{swan}_z(F \otimes \mathcal{L}_\eta)$ where $Z := \mu_{m,k} \setminus U$, which is clearly
independent of \( \eta \) (tensoring with a rank one tame representation does not change the Swan conductor).

For any exact sequence \( 0 \to G \to F \to H \to 0 \) of smooth sheaves we get a cohomology exact sequence \( 0 \to H^1_c(U, G \otimes \mathcal{L}_\eta) \to H^1_c(U, F \otimes \mathcal{L}_\eta) \to H^1_c(U, H \otimes \mathcal{L}_\eta) \to 0 \) (since both \( G \otimes \mathcal{L}_\eta \) and \( H \otimes \mathcal{L}_\eta \) are also totally ramified at 0 and \( \infty \)) so we may assume that \( F \) is pure of some weight \( w \).

Let \( j : U \hookrightarrow \mathbb{P}^1_k \) be the inclusion. Note that, since \( F \otimes \mathcal{L}_\eta \) is totally ramified at 0 and \( \infty \), \( j_! F \) vanishes at both points. If we denote by \( i : Z \hookrightarrow \mathbb{P}^1_k \) the inclusion, the exact sequence of sheaves
\[
0 \to j_!(F \otimes \mathcal{L}_\eta) \to j_* (F \otimes \mathcal{L}_\eta) \to i_* i^* j_!(F \otimes \mathcal{L}_\eta) \to 0
\]
induces an exact sequence
\[
0 \to H^0_c(Z, i^* j_!(F \otimes \mathcal{L}_\eta)) \to H^1_c(U, F \otimes \mathcal{L}_\eta) \to H^1_c(\mathbb{P}^1, j_!(F \otimes \mathcal{L}_\eta)) \to 0
\]
where the last non-zero term is pure of weight \( w + 1 \) by [Del80, Théorème 2]. So it remains to show that the weights of \( H^0_c(Z, i^* j_!(F \otimes \mathcal{L}_\eta)) = \oplus_{z \in Z} j_!(F \otimes \mathcal{L}_\eta)_z \) are independent of \( \eta \). But \( \mathcal{L}_\eta \) is smooth at every \( z \in Z \), so \( j_!(F \otimes \mathcal{L}_\eta)_z \cong \mathcal{L}_\eta_z \otimes (j_! F)_z \) has the same weights as \((j_! F)_z\), which are independent of \( \eta \). \( \square \)

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