Some Variational Principles for the Metric Mean Dimension of a Semigroup Action

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Abstract
In this manuscript, we show that the metric mean dimension of a semigroup action satisfies three variational principles: (a) in our first result, we consider the local entropy function for a free semigroup action and show that the metric mean dimension satisfies a variational principle in terms of such function; (b) the second one is about a definition of Katok’s entropy for a free semigroup action introduced in Carvalho et al. (Ergod, 42, 65–85, 2022); (c) in our third result, based on the definition of Shapira’s entropy, introduced in Shapira (Israel J Math, 158, 225–247, 2007) for a single dynamic, we extend the definition of Shapira’s entropy for a semigroup action. We also obtain a formula which relates the Shapira’s entropy of a free semigroup action and the Shapira’s entropy of the induced skew product; (d) in our fourth result, we obtain a variational principle involving the metric mean dimension and the Shapira’s entropy of a free semigroup action; (e) in the last two theorems, we extend the definition of metric mean dimension and the topological entropy when we have a finitely generated semigroup inspired in the definition of topological entropy introduced in Ghys et al. (Acta Math, 160, 105–142, 1988). In this context, we obtain a partial variational principle for the metric mean dimension. Our results are inspired in the ones obtained by (Lindenstrauss et al. 2019) and Velozo and Velozo (2017) and Gutman and Śpiewak (Stud Math, 261, 345–360, 2021) and Shi (IEEE Trans Inf Theory, 68, 4282–4288, 2022).

Keywords Metric mean dimension · Variational principle · Semigroup action · Free semigroup action

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1 Introduction

The aim of this note is to explore the notion of metric mean dimension for a compactly generated semigroup action and for a compactly generated free semigroup action. The notion of metric mean dimension for a dynamical system $f: (X, d) \to (X, d)$, denoted by $\text{mdim}_M(X, \phi, d)$, was introduced in [12] and may be related to the problem of whether or not a given dynamical system can be embedded in the shift space $((0, 1)^{\mathbb{N}}, \sigma)$, where $k$ depends on the metric mean dimension of the system. It refines the topological entropy for systems with infinite entropy, which, in the case of a manifold of dimension greater than one, form a residual subset of the set consisting of homeomorphisms defined on the manifold (see [22]). In fact, every system with finite topological entropy has metric mean dimension equal to zero. The metric mean dimension depends on the metric $d$, therefore it is not a topological invariant. However, for a metrizable topological space $X$, $\text{mdim}_M(X, \phi) = \inf_{d'} \text{mdim}_M(X, \phi, d')$ is invariant under topological conjugacy, where the infimum is taken over all the metrics on $X$ which induce the topology on $X$. On the other hand, as showed in [13, 17], and in [21], the metric mean dimension is strongly related with the ergodic behaviour of the system, since it satisfies several variational principles.

In [6], the authors considered the compact metric space $(Y^\mathbb{N}, D)$ and $(X, d)$, where $(Y, d_Y)$ is a compact metric space and $D$ is the product metric induced by $d_Y$. In this setting, they introduced the notion of metric mean dimension for a free semigroup action and proved that for a certain class of random walks, the ones induced by homogeneous probability measures on $Y$, it is possible to obtain a kind of Bufetov’s formula (see [2] for Bufetov’s formula for the topological entropy of a free semigroup action).

Inspired by the works of Lindenstrauss and Tsukamoto [13], Velozo and Velozo [21] and, specially, by Shi [17] and Gutman and Śpiewak [18], where several variational principles involving the metric mean dimension and some quantities related with the metric entropy of invariant measures for a single dynamics were obtained, in this note we consider finitely generated semigroups and compactly generated free semigroups of continuous maps acting on a compact metric and prove that the metric mean dimension satisfies several variational principles for these two settings: (a) in our first result, we consider the local entropy function for a free semigroup action and show that the metric mean dimension satisfies a variational principle in terms of such function; (b) the second one is about a definition of Katok’s entropy for a free semigroup action introduced in [6]; (c) in our third result, based on the definition of Shapira’s entropy, introduced in [16] for a single dynamics, we extend the definition of Shapira’s entropy for a semigroup action. We also obtain a formula which relates the Shapira’s entropy of a free semigroup action and the Shapira’s entropy of the induced skew product; (d) in our fourth result, we obtain a variational principle involving the metric mean dimension and the Shapira’s entropy of a free semigroup action; (e) in the last two theorems, we extend the definition of metric mean dimension and the topological entropy when we have a finitely generated semigroup inspired in the definition of topological entropy introduced in [11]. In this context, we obtain a partial variational principle for the metric mean dimension. Such variational principle is inspired on the results of [1].

This paper is organized as follows. In Section 2, we present the main definitions and the main results. In Section 3, we recall some results and definitions about box dimension, homogeneous measures and $G$-homogeneous measures. In Section 4, we prove the main theorems.
2 Definitions and Main Results

We start recalling the main concepts we use and describing the systems we will work with.

2.1 Metric Mean Dimension of a Map

Let \((X, d)\) be a compact metric space. Given a continuous map \(f: X \to X\) and a non-negative integer \(n\), define the dynamical metric \(d_n: X \times X \to [0, \infty)\) by

\[
d_n(x, z) = \max\{d(x, z), d(f(x), f(z)), \ldots, d(f^{n-1}(x), f^{n-1}(z))\}
\]

which generates the same topology as \(d\). Having fixed \(\varepsilon > 0\), we say that a set \(E \subset X\) is \((n, \varepsilon)\)-separated by \(f\) if \(d_n(x, z) > \varepsilon\) for every \(x, z \in E\) with \(x \neq z\). In the particular case of \(n = 1\), we will call such a set \(\varepsilon\)-separated. Denote by \(s(f, n, \varepsilon)\) the maximal cardinality of all \((n, \varepsilon)\)-separated subsets of \(X\) by \(f\). Due to the compactness of \(X\), the number \(s(f, n, \varepsilon)\) is finite for every \(n \in \mathbb{N}\) and \(\varepsilon > 0\). We say that \(R \subset X\) is a \((n, \varepsilon)\)-spanning set if for any \(x \in X\) there exists \(z \in R\) such that \(d_n(x, z) < \varepsilon\). When \(n = 1\), we say that the set is \(\varepsilon\)-spanning. Let \(b(n, \varepsilon)\) be the minimum cardinality of the \((n, \varepsilon)\)-spanning subsets of \(X\).

**Definition 2.1** The lower metric mean dimension of \(f\) with respect to the fixed metric \(d\) is given by

\[
\overline{\text{mdim}}_M \left( X, f, d \right) = \liminf_{\varepsilon \to 0^+} \frac{h(f, \varepsilon)}{|\log \varepsilon|}
\]

where

\[
h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s(f, n, \varepsilon).
\]

Similarly, the upper metric mean dimension of \(f\) with respect to \(d\) is the limit

\[
\underline{\text{mdim}}_M \left( X, f, d \right) = \limsup_{\varepsilon \to 0^+} \frac{h(f, \varepsilon)}{|\log \varepsilon|}.
\]

Clearly, \(\overline{\text{mdim}}_M \left( X, f, d \right) = \underline{\text{mdim}}_M \left( X, f, d \right) = 0\) whenever the topological entropy of \(f\), given by \(h_{\text{top}}(f) = \lim_{\varepsilon \to 0^+} h(f, \varepsilon)\), is finite.

2.2 Compactly Generated Semigroup Action of Continuous Maps

Let \((X, d)\) and \((Y, d_Y)\) be compact metric spaces and \((g_y)_{y \in Y}\) be a family of continuous maps \(g_y: X \to X\). We denote by \(G\) the free semigroup having the set \(G_1 = \{g_y: y \in Y\}\) as generator, where the semigroup operation \(\circ\) is the composition of maps. Let \(\mathbb{S}\) be the induced free semigroup action

\[
\mathbb{S}: G \times X \to X \quad (g, x) \mapsto g(x)
\]

which is said to be compactly generated by \(Y\), and denote by \(T_G\) the associated skew product given by

\[
T_G: Y^\mathbb{N} \times X \to Y^\mathbb{N} \times X \quad (\omega, x) \mapsto \left( \sigma(\omega), g_{\omega_1}(x) \right),
\]
where \( \omega = (\omega_1, \omega_2, \ldots) \) is an element of the full unilateral space of sequences \( Y^N \) and \( \sigma \) denotes the shift map acting on the compact metric space \( (Y^N, D) \), where

\[
D(\omega, \theta) := \sum_{n=1}^{\infty} \frac{1}{2^n} d_Y(\omega_n, \theta_n).
\]

(2.2)

It will be a standing assumption that \( T_G \) is a continuous map. If for every \( n \in \mathbb{N} \) and \( \omega = (\omega_1, \omega_2, \ldots) \in Y^N \) we write

\[
f^n_{\omega} = g_{\omega_n} \cdots g_{\omega_1},
\]

then

\[
T^n_G(\omega, x) = \left( \sigma^n(\omega), f^n_{\omega}(x) \right).
\]

Consider the set \( G^*_1 = G_1 \setminus \{id\} \) and, for each \( n \in \mathbb{N} \), let \( G^*_n \) denote the space of concatenations of \( n \) elements in \( G^*_1 \). Similarly, define \( G = \bigcup_{n \in \mathbb{N}} G_n \), where \( G_0 = \{id\} \) and \( g \in G_n \) if and only if \( g = g_{\omega_n} \cdots g_{\omega_2} g_{\omega_1} \), with \( g_{\omega_j} \in G_1 \) (for notational simplicity’s sake we will use \( g_j g_i \) instead of the composition \( g_j \circ g_i \)). In what follows, we will assume that the generator set \( G_1 \) is minimal, meaning that no function \( g_y \in G_1 \), for \( y \in Y \), can be expressed as a composition of the remaining generators. To summon an element \( g \) of \( G_n^* \), we will write \( |g| = n \) instead of \( g \in G_n^* \). Each element \( g \) of \( G_n \) may be seen as a word which originates from the concatenation of \( n \) elements in \( G_1 \). Yet, different concatenations may generate the same element in \( G \). Nevertheless, in the computations to be done, we shall consider different concatenations instead of the elements in \( G \) they create.

### 2.3 Random Walks

A random walk \( \mathbb{P} \) on \( Y^N \) is a Borel probability measure in this space of sequences which is invariant by the shift map \( \sigma \). For instance, we may consider a finite subset \( F = \{p_1, \ldots, p_k\} \) of \( Y \), a probability vector \( (a_1, \ldots, a_k) \) (that is, a selection of positive real numbers \( a_i \) such that \( \sum_{i=1}^{k} a_i = 1 \)), the probability measure \( \nu = \sum_{i=1}^{k} a_i \delta_{p_i} \) on \( F \) and the Borel product measure \( \nu^N = \nu_N \) on \( Y^N \). Such a \( \nu^N \) will be called a Bernoulli measure, which is said to be symmetric if \( a_i = \frac{1}{k} \) for every \( i \in \{1, \ldots, k\} \), in which case we denote it by \( \nu_k \). If \( Y \) is a Lie group, a natural symmetric random walk is given by \( \nu_N \) where \( \nu \) is the Haar measure. We denote by \( \mathcal{P}(Y^N) \) the space of Borel probability measures on \( Y^N \) and by \( \mathcal{P}_B(Y^N) \) its subset of Bernoulli elements. It will be clear later on that the role of each random walk is to point out a particular complex feature of the dynamics, here defined in terms of either the topological entropy (definition in Section 2.4) or the metric mean dimension (definition in Section 2.6).

### 2.4 Topological Entropy of an Action

Given \( \varepsilon > 0 \) and \( g := g_{\omega_n} \cdots g_{\omega_2} g_{\omega_1} \in G_n \), the \( n \)-th dynamical ball \( B_n(x, g, \varepsilon) \) is the set

\[
B_n(x, g, \varepsilon) := \{ z \in X : d(g_j(z), g_j(x)) \leq \varepsilon, \ \forall \ 0 \leq j \leq n \}
\]

where, for every \( 0 \leq j \leq n \), the notation \( g_j \) stands for the concatenation \( g_{\omega_j} \cdots g_{\omega_2} g_{\omega_1} \) in \( G_j \), and \( g_0 = id \). Observe that this is a classical ball with respect to the dynamical metric \( d_\varepsilon \), defined by

\[
d_\varepsilon(x, z) := \max_{0 \leq j \leq n} d(g_j(x), g_j(z)).
\]

(2.3)
Notice also that both the dynamical ball and the dynamical metric depend on the underlying concatenation of generators $g_{\omega_n} \cdots g_{\omega_1}$ and not on the semigroup element $g$, since the latter may have distinct representations.

Given $g = g_{\omega_n} \cdots g_{\omega_1} \in G_n$, we say that a set $K \subset X$ is $(g, n, \varepsilon)$–separated if $d_g(x, z) > \varepsilon$ for any two distinct elements $x, z \in K$. The largest cardinality of any $(g, n, \varepsilon)$–separated subset on $X$ is denoted by $s(g, n, \varepsilon)$ (or, equivalently, $s(g_{\omega_n} \cdots g_{\omega_1}, n, \varepsilon)$). A set $K \subset X$ is said to be $(g, n, \varepsilon)$–spanning if for every $x \in X$ there is $k \in K$ such that $d_g(x, k) \leq \varepsilon$. The smallest cardinality of any $(g, n, \varepsilon)$–spanning subset on $X$ is denoted by $b(g, n, \varepsilon)$ (or $b(g_{\omega_n} \cdots g_{\omega_1}, n, \varepsilon)$).

**Definition 2.2** The topological entropy of the semigroup action $\mathbb{S}$ with respect to a fixed set of generators $G_1$ and a random walk $\mathbb{P}$ in $Y^\mathbb{N}$ is given by

$$h_{\text{top}}(\mathbb{S}, \mathbb{P}) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^\mathbb{N}} s(g_{\omega_n} \cdots g_{\omega_1}, n, \varepsilon) \, d\mathbb{P}(\omega)$$

where $\omega = \omega_1 \omega_2 \cdots \omega_n \cdots$. The topological entropy of the semigroup action $\mathbb{S}$ is then defined by

$$h_{\text{top}}(\mathbb{S}) = \sup_{\mathbb{P}} h_{\text{top}}(\mathbb{S}, \mathbb{P}).$$

We observe that the semigroup may have multiple generating sets, and the dynamical or ergodic properties (as the topological entropy) depend on the chosen generator set. More information regarding these concepts in the case of finitely generated free semigroup actions may be read in [3–5].

### 2.5 Entropy Function

Let $(X, d)$ be a compact metric space. For each $\varepsilon > 0$ and $x \in X$, define

$$h_d(x, \varepsilon) = \inf \{ B(K, \mathbb{S}, \varepsilon) : K \text{ is a compact neighbourhood of } x \},$$

where

$$B(K, \mathbb{S}, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \int_{\Sigma^+_n} b(K, g_{\omega_n} \cdots g_{\omega_1}, \varepsilon) \, d\mathbb{P}(\omega) \right),$$

and $b(K, g_{\omega_n} \cdots g_{\omega_1}, \varepsilon)$ denotes the minimum cardinality of a $(g_{\omega_n} \cdots g_{\omega_1}, \varepsilon)$–spanning set of $K$. As $h_d(x, \varepsilon)$ increases as $\varepsilon$ decreases to zero, it is well defined in the following

$$h_d(x) = \lim_{\varepsilon \to 0^+} h_d(x, \varepsilon)$$

(2.4)

and it is less or equal to $h_{\text{top}}(X, \mathbb{S})$. In fact, it depends only on the topology of $X$ and we can denote it by $h_{\text{top}}(x)$.

**Definition 2.3** Let $\mathbb{S} : G \times X \to X$ be a continuous finitely generated free semigroup action. The function $h_{\text{top}} : X \to [0, h_{\text{top}}(X, \mathbb{S})]$, $x \mapsto h_{\text{top}}(x)$ is called the entropy function of $\mathbb{S}$.

Since $B(K, \mathbb{S}, \varepsilon) \leq S(K, \mathbb{S}, \varepsilon) \leq B(K, \mathbb{S}, \varepsilon/2)$, we have

$$h_{\text{top}}(x) = \lim_{\varepsilon \to 0} \inf \{ S_d(K, \mathbb{S}, \varepsilon) : K \text{ is a compact neighbourhood of } x \}. $$
By [15, Theorem C] we have that
\[
\sup_{x \in X} \lim_{\varepsilon \to 0^+} h_d(x, \varepsilon) = h_{\text{top}}(\mathcal{S}, \mathbb{P}).
\]

2.6 Metric Mean Dimension of a Semigroup Action

Let \((X, d)\) be a compact metric space and \(\mathcal{S}\) be the free semigroup action induced on \((X, d)\) by a family of continuous maps \((g_y : X \to X)_{y \in Y}\). The following definition for the semigroup setting was introduced in [7].

Definition 2.4 The upper and lower metric mean dimensions of the free semigroup action \(\mathcal{S}\) on \((X, d)\) with respect to a fixed set of generators \(G_1\) and a random walk \(\mathbb{P}\) in \(Y^\mathbb{N}\) are given respectively by
\[
\overline{\text{mdim}}_M \left( X, \mathcal{S}, d, \mathbb{P} \right) = \limsup_{\varepsilon \to 0^+} \frac{h(X, \mathcal{S}, \mathbb{P}, \varepsilon)}{-\log \varepsilon},
\]
\[
\underline{\text{mdim}}_M \left( X, \mathcal{S}, d, \mathbb{P} \right) = \liminf_{\varepsilon \to 0^+} \frac{h(X, \mathcal{S}, \mathbb{P}, \varepsilon)}{-\log \varepsilon},
\]
where
\[
h(X, \mathcal{S}, \mathbb{P}, \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log \int_{Y^\mathbb{N}} s_{\mathbb{P}}(g_{\omega_1} \ldots g_{\omega_n}, n, \varepsilon) \ d\mathbb{P}(\omega). \tag{2.5}
\]

Our first result shows that the metric mean dimension of a semigroup action may be computed in terms of the entropy function.

Theorem A Let \((X, d)\) be a compact metric space and \(\mathcal{S}\) be the free semigroup action induced on \((X, d)\) by a family of continuous maps \((g_y : X \to X)_{y \in Y}\). Then,
\[
\overline{\text{mdim}}_M \left( X, \mathcal{S}, d, \mathbb{P} \right) = \limsup_{\varepsilon \to 0^+} \sup_{x \in X} h_d(x, \varepsilon),
\]
for every \(\mathbb{P} \in \mathcal{M}(Y^\mathbb{N})\).

2.7 Katok’s Entropy

In [5], the authors considered an extension of the Katok’s entropy when the dynamical system under consideration is a free semigroup action.

Definition 2.5 Given probability measure \(\mathbb{P}\) on \(Y^\mathbb{N}\) and a Borel probability measure \(\nu \in \mathcal{M}(X)\) (\(\mathcal{M}(X)\) is the set of probability measures on \(X\)), \(\delta \in (0, 1)\) and \(\varepsilon > 0\), define
\[
h^K_\nu \left( \mathcal{S}, \varepsilon, \delta \right) = \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^\mathbb{N}} s_{\nu}(g_{\omega_1} \ldots g_{\omega_n}, n, \varepsilon, \delta) \ d\mathbb{P}(\omega) \tag{2.6}
\]
where \(\omega = \omega_1 \omega_2 \ldots \omega_n \ldots\),
\[
s_{\nu}(g_{\omega_1} \ldots g_{\omega_n}, n, \varepsilon, \delta) = \inf_{E \subseteq X : \nu(E) > 1-\delta} \{ s(g_{\omega_1} \ldots g_{\omega_n}, n, \varepsilon, E) \}
\]
and \(s(g_{\omega_1} \ldots g_{\omega_n}, n, \varepsilon, E)\) denotes the maximal cardinality of the \((g_{\omega_1} \ldots g_{\omega_n}, n, \varepsilon)\)-separated subsets of \(E\).
The entropy of the semigroup action $\mathbb{S}$ with respect to $\nu$ and $\mathbb{P}$ is defined by
\begin{equation}
    h^K_{\nu}(\mathbb{S}, \mathbb{P}) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^N} s_\nu(g_{\omega_0} \ldots g_{\omega_1}, n, \varepsilon, \delta) \, d\mathbb{P}(\omega) \tag{2.7}
\end{equation}

Observe that the previous limit is well defined due to the monotonicity of the function $(\varepsilon, \delta) \mapsto \frac{1}{n} \log \int_{Y^N} s_\nu(g_{\omega_0} \ldots g_{\omega_1}, n, \varepsilon, \delta) \, d\mathbb{P}(\omega)$ on the unknowns $\varepsilon$ and $\delta$. Moreover, if the set of generators is $G_1 = \{Id, f\}$, we recover the notion proposed by Katok for a single dynamics $f$.

In [17], the author proved that for a compact metric space $(X, d)$ and a continuous map $f : X \to X$ holds the following variational principle for the metric mean dimension
\begin{equation}
    \overline{\text{mdim}}_M \left( X, f, d, \mathbb{P} \right) = \lim_{\varepsilon \to 0^+} \sup_{\nu \in \mathcal{E}_f(X)} \frac{h^K_{\nu}(X, f, \varepsilon, \delta)}{-\log \varepsilon}, \quad \text{for every } \delta \in (0, 1),
\end{equation}

where $\mathcal{E}_f(X)$ denotes the set of ergodic probability measures on $X$. In the case where the dynamical systems is given by a free semigroup action, the last may be extended as:

**Theorem B** Let $(X, d)$ be a compact metric space and $\mathbb{S}$ be the free semigroup action induced on $(X, d)$ by a family of continuous maps $(g_y : X \to X)_{y \in Y}$. Then,
\begin{equation}
    \overline{\text{mdim}}_M \left( X, \mathbb{S}, d, \mathbb{P} \right) \geq \lim_{\delta \to 0} \limsup_{\varepsilon \to 0^+} \frac{\sup_{\nu \in \mathcal{M}(X)} h^K_{\nu}(\mathbb{S}, \mathbb{P}, \varepsilon, \delta)}{-\log \varepsilon},
\end{equation}

for every $\mathbb{P} \in \mathcal{M}(Y^N)$. If $\mathbb{P} = \gamma^N$, with $\gamma$ an homogeneous probability measure on $Y$ and $\text{supp}(\gamma) = Y$, then
\begin{equation}
    \overline{\text{mdim}}_M \left( X, \mathbb{S}, d, \mathbb{P} \right) = \lim_{\delta \to 0} \limsup_{\varepsilon \to 0^+} \frac{\sup_{\nu \in \mathcal{M}(X)} h^K_{\nu}(\mathbb{S}, \mathbb{P}, \varepsilon, \delta)}{-\log \varepsilon}.
\end{equation}

### 2.8 Entropy of an Open Cover for a Free Semigroup Action

Consider $\mathbb{P} \in \mathcal{M}(Y^N)$. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a finite open cover of $X$. For each $\omega \in Y^N$ and $n \in \mathbb{N}$ define
\begin{equation}
    \mathcal{U}(\omega, n) = \left\{ U_{i_0} \cap (f_{\omega}^{-1})^{-1}(U_{i_1}) \cap \cdots \cap (f_{\omega}^{-1})^{-1}(U_{i_{n-1}}) : U_{i_j} \in \mathcal{U} \right\}.
\end{equation}

Let $N(\mathcal{U}, w, n)$ be the minimal cardinality of a subcover of $\mathcal{U}(w, n)$. Finally, define
\begin{equation}
    h_{\text{top}}(\mathcal{U}, \mathbb{S}, \mathbb{P}) = \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^N} N(\mathcal{U}(\omega, n)) \, d\mathbb{P}(\omega).
\end{equation}

As a consequence of [20, Theorem 2.4] we have that
\begin{equation}
    h_{\text{top}}(Y^N \times X, \mathbb{S}, \mathbb{P}) = \sup_{\text{id}} h_{\text{top}}(\mathcal{U}, \mathbb{S}, \mathbb{P}),
\end{equation}

where the open covers under consideration in the above supremum are those which are finite and with finite topological entropy.
2.9 Shapira’s Entropy of a Semigroup Action

For \( v \in \mathcal{M}(X) \), for \( \delta \in (0, 1) \) let \( N_v(\mathcal{U}, w, n, \delta) \) the minimal cardinality of a subcover of \( \mathcal{U}(w, n) \), up to a set of \( \nu \)-measure less than \( \delta > 0 \). Define

\[
\hat{h}^S_v(\mathcal{U}, \mathcal{S}, \mathbb{P}) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \int_{Y^\mathbb{N}} N_v(\mathcal{U}, \omega, n, \delta) \, d\mathbb{P}(\omega).
\] (2.8)

We call \( h^S_v(\mathcal{U}, \mathcal{S}, \mathbb{P}) \) the metric entropy of the cover \( \mathcal{U} \) with respect to \( v \). As

\[
N_v(\mathcal{U}, \omega, n, \delta) \leq N(\mathcal{U}, \omega, n) \text{ for every } \delta \in (0, 1),
\]

we have that \( h^S_v(\mathcal{U}, \mathcal{S}, \mathbb{P}) \leq h_{\text{top}}(\mathcal{U}, \mathcal{S}, \mathbb{P}) \). It is important to mention that when \( G_1 = \{id, f\} \), our definition coincides with the classical one given in [16].

Before we state our theorem, we need to introduce some notations. Associated to an open cover \( \mathcal{U} \) of \( X \), let \( \mathcal{U} = \{[i] \times V : i = 1, \ldots, p \text{ and } V \in \mathcal{U}\} \) and, for \( \nu \in \mathcal{M}(X) \), denote \( \Pi(\sigma, \nu)_{\text{erg}} \) the set of \( T_G \)-invariant and ergodic probability measures so that the marginal in \( \Sigma^+_\nu \) is \( \sigma \)-invariant and \( \nu \) is the marginal in \( X \).

Now, we present an example where we have \( \Pi(\sigma, \nu)_{\text{erg}} = \emptyset \).

**Example 2.6** Let \( X = S^1 \) be the circle, and consider \( f_i : S^1 \to S^1 \), where \( i = 1, 2 \), given by \( f_1(x) = 2x \text{ (mod 1)} \) and \( f_2(x) = 3x \text{ (mod 1)} \). We denote by \( G \) the free semigroup generated by \( G_1 = \{id, f_1, f_2\} \) and by \( \mathbb{S} \) the induced free semigroup action on \( S^1 \). If \( \text{Leb}_{S^1} \) is the Lebesgue measure on \( S^1 \) and \( \mathbb{P} \) is any \( \sigma \)-ergodic probability measure, then \( \mu = \mathbb{P} \times \text{Leb}_{S^1} \in \Pi(\sigma, \text{Leb}_{S^1})_{\text{erg}} \) (see [5] for more details).

If we consider a finite set \( G_1 = \{id, f_1, \ldots, f_p\} \) of continuous maps acting on a compact metric space \( X \), so that the maps in \( G_1 \) admit a common ergodic probability measure \( \nu \in \mathcal{M}(X) \) then \( \eta_{\nu} \times \nu \in \Pi(\sigma, \nu)_{\text{erg}} \).

**Theorem C** Let \( (X, d) \) be a compact metric space and \( \mathbb{S} \) be the free semigroup action induced on \( (X, d) \) by a finite family of continuous maps \((g_i : X \to X)_{i=1}^p \). Under the above conditions we have that

(a) \( h_{\text{top}}(\mathcal{U}, \mathcal{S}, \eta_p) = h_{\text{top}}(\mathcal{U}, T_G) - \log p \); 
(b) \( h_{\text{top}}(\mathcal{U}, \mathcal{S}, \eta_p) = \sup \left\{ h^S_v(\mathcal{U}, \mathcal{S}, \eta_p) : v \in \mathcal{M}(X) \text{ and } \Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset \right\} \),

where \( \eta_p = \left( \frac{1}{p}, \ldots, \frac{1}{p} \right)^\mathbb{N} \).

As a direct consequence of Theorem C and [5], we have the following.

**Corollary 1** Let \( (X, d) \) be a compact metric space and \( \mathbb{S} \) be the free semigroup action induced on \( (X, d) \) by a finite family of continuous maps \((g_i : X \to X)_{i=1}^p \). Then,

\[
h_{\text{top}}(\mathbb{S}, \eta_p) = \sup_{\mathcal{U}} \sup_{\left\{ v \in \mathcal{M}(X) \text{ and } \Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset \right\}} h^S_v(\mathcal{U}, \mathcal{S}, \eta_p)) = h_{\text{top}}(T_G) - \log p,
\]

where \( \eta_p = \left( \frac{1}{p}, \ldots, \frac{1}{p} \right)^\mathbb{N} \).

In [17], it was proved that, for a compact metric space \( (X, d) \) and a continuous map \( f : X \to X \),
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\[
\text{mdim}_M \left( X, f, d \right) = \limsup_{\varepsilon \to 0^+} \sup_{\nu \in \mathcal{E}(X)} \inf_{\text{diam}(U) \leq \varepsilon} h_{\nu}^S(U, f)
\]

In the next theorem, we extend such result for compactly generated free semigroup actions.

**Theorem D** Let \((X, d)\) be a compact metric space and \(S\) be the free semigroup action induced on \((X, d)\) by a family of continuous maps \(\{g_y : X \to X\}_{y \in Y}\). If \(\mathbb{P} = \gamma^\mathbb{N}, \gamma \in \mathcal{M}(Y)\) is homogeneous and \(\text{supp}(\gamma) = Y\), then

\[
\text{mdim}_M \left( X, S, d, \mathbb{P} \right) = \limsup_{\varepsilon \to 0^+} \sup_{\nu \in \mathcal{M}(X) : \Pi(\sigma, \nu) \neq \emptyset} \inf_{\text{diam}(U) \leq \varepsilon} h_{\nu}^S(S, U) - \log \varepsilon.
\]

### 2.10 Ghys-Langevin-Walczack Entropy

Ghys, Langevin and Walczak proposed in [11] the following definition of topological entropy of a semigroup action given by a finitely generated semigroup \(G\). A subset \(E\) of a compact metric space \((X, d_X)\) is \((n, \varepsilon)\)-separated points by elements of \(G\) if for any \(x \neq y\) in \(E\) there exists \(0 \leq j \leq n\) and \(g \in G_j\) such that \(d(g(x), g(y)) > \varepsilon\). The topological entropy of the semigroup action \(S\), induced by a semigroup \(G\) generated by a finite set \(G_1\) of continuous maps, is given by

\[
h_{GLW}(S) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \varepsilon)
\]

where \(s(n, \varepsilon)\) is the largest cardinality of \((n, \varepsilon)\)-separated points by elements of \(G_n\). Observe that, since \(X\) is compact, \(s(n, \varepsilon)\) is finite for every \(n \in \mathbb{N}\) and \(\varepsilon > 0\). Moreover, the map

\[
\varepsilon > 0 \quad \mapsto \quad h_{GLW}(S, \varepsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \varepsilon)
\]

is monotonic, so \(h_{GLW}(S)\) is well defined (though it depends on the set \(G_1\) of generators). This is a purely topological notion, independent of any previously fixed random walk on the semigroup. Observe also that

\[
\sup_{g \in G_1} \text{htop}(g) \leq h_{GLW}(S)
\]

but this inequality may be strict (cf. [11]).

**Remark 2.7** We observe that \(h_{GLW}(S)\) could be made in terms of \((n, \varepsilon)\)-spanning sets or \((n, \varepsilon)\)-covers. More precisely, a subset \(F\) of a compact metric space \((X, d_X)\) is \((n, \varepsilon)\)-spanning if for any \(x \in X\) there exists \(y \in F\) so that for any \(0 \leq j \leq n\) and any \(g \in G_j\) we have that \(d(g(x), g(y)) < \varepsilon\). We denote by \(b(n, \varepsilon)\) the minimal cardinality of a \((n, \varepsilon)\)-spanning set.

For \(n \in \mathbb{N}\) let

\[
B_n^G(x, \varepsilon) = \{ y \in X : d(g(x), g(y)) < \varepsilon \text{ for all } g \in G_j, \ 0 \leq j \leq n \},
\]

called the \(nth\)-dynamical ball of center \(x\) and radius \(\varepsilon\), which is an open subset of \(X\) (see [1, 11] for more details). We denote by \(\text{cov}(n, \varepsilon)\) the minimal number of dynamical balls necessary to cover \(X\). It is possible to show that

\[
\text{cov}(n, 2\varepsilon) \leq b(n, \varepsilon) \leq s(n, \varepsilon),
\]
and it implies that \( h_{GLW}(S) \) may be computed in terms of \((n, \varepsilon)\)-spanning sets or \((n, \varepsilon)\)-covers.

An important remark here is that the previous definition of entropy could be made for compactly generated semigroups.

### 2.10.1 Metric Mean Dimension in the GLW Setting

As a natural extension of the metric mean dimension for a single dynamics, we can consider the upper GLW-metric mean dimension as

\[
\overline{\text{mdim}}_{M}^{GLW} \left( X, S, d \right) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(S, \varepsilon)}{- \log \varepsilon}.
\tag{2.10}
\]

As a direct consequence of the above definition, we have that for any \( \mathbb{P} \in \mathcal{M}(\mathbb{N}) \),

\[
\overline{\text{mdim}}_{M} \left( X, S, \mathbb{P}, d \right) \leq \overline{\text{mdim}}_{M}^{GLW} \left( X, S, d \right),
\]

and in the case where the generating set consists of a single dynamics the two definitions coincide with the classical one.

### 2.10.2 Local Measure Entropy and Measure Metric Mean Dimension

For any \( \nu \in \mathcal{M}(X) \) the quantity

\[
h_{\nu}^{G}(x) = \lim_{\varepsilon \to 0} h_{\nu}^{G}(x, \varepsilon),
\]

where

\[
h_{\nu}^{G}(x, \varepsilon) = \limsup_{n \to \infty} - \frac{1}{n} \log \nu(B_{n}^{G}(x, \varepsilon)),
\]

is called the local upper \( \nu \)-measure entropy at the point \( x \). If one takes lim inf with respect to \( n \) in the above definition, we call the local lower \( \nu \)-measure entropy at the point \( x \), denoted by \( h_{\nu, G}(x) \). These quantities were defined and explored in [1], where the author proved that in the case of \( \nu \) being a \( G \)-homogeneous measure \( h_{\nu}^{G}(x) = h_{GLW}(S) \), for all \( x \in X \) (see Section 3 for the definition of \( G \)-homogeneous measure).

In order to have a concept related to the metric mean dimension, we define the upper local measure metric mean dimension as

\[
\overline{\text{mdim}}_{\nu}(x, d) = \limsup_{\varepsilon \to 0} \frac{h_{\nu}^{G}(x, \varepsilon)}{- \log \varepsilon}.
\tag{2.11}
\]

If one takes lim inf in \( \varepsilon \), we have the lower local upper measure metric mean dimension, denoted by \( \underline{\text{mdim}}_{\nu}(x, d) \).

If, instead of \( h_{\nu}^{G}(x, \varepsilon) \), we consider \( h_{\nu, G}(x, \varepsilon) \), we have the upper local lower measure metric mean dimension and lower local lower measure metric mean dimension, denoted by \( \overline{\text{mdim}}'_{\nu}(x, d) \) and \( \underline{\text{mdim}}'_{\nu}(x, d) \), respectively.

**Remark 2.8** All the above definitions could be made in terms of dynamical balls.

In the case where the ambient space \( X \) is an oriented manifold, it admits a volume form \( dV \) which induces a natural volume measure \( \nu_v \) on the Borel sets defined as

\[
\nu_v(A) = \int_{A} dV.
\]
The next theorem gives a kind of partial variational principle for the metric mean dimension of the action in terms of the volume measure, in the case where $G$ is a group of homeomorphisms.

**Theorem E** Let $(G, G_1)$ be a finitely generated group of homeomorphisms of a compact closed and oriented manifold $(X, d)$. Let $s \in (0, \infty)$ and $\nu_v$, the natural volume form on $X$. If

$$\overline{\mathrm{mdim}}_{\nu_v}(x, d) \leq s \text{ for all } x \in X \text{ then } \overline{\mathrm{mdim}}_M^{GLW}(X, \mathcal{S}, d) \leq s.$$ 

Our last theorem shows that, in the case where the group action admits a strongly $G$-homogeneous measure $\nu$ we have an equality between the local measure metric mean dimension of $\nu$ and the metric mean dimension of the group action (see Section 3 for the definition of strongly $G$-homogeneous measure).

**Theorem F** Let $(X, d)$ be a compact metric space and $\mathcal{S}$ be the semigroup action induced on $(X, d)$ by a finite family of continuous maps $(g_i : X \to X)_{i=1}^p$.

(a) If $\nu \in \mathcal{M}(X)$ is strongly $G$-homogeneous, then

$$\overline{\mathrm{mdim}}_M^{GLW}(X, \mathcal{S}, d) = \limsup_{\varepsilon \to 0} \frac{h_G^\nu(x, \varepsilon)}{-\log \varepsilon}. $$

(b) Let $\nu$ be a Borel measure on $X$ and $s \in (0, \infty)$. If

$$\inf_{x \in X} \overline{\mathrm{mdim}}_{\nu_v}(x, d) \geq s \text{ then } \overline{\mathrm{mdim}}_M^{GLW}(X, \mathcal{S}, d) \geq s.$$ 

### 3 Some Facts About Homogeneous Measures and $G$-Homogeneous Measures

In order to obtain a text as self-contained as possible, in this section, we recall the definitions of upper box dimension, homogeneous measure and $G$-homogeneous measure.

#### 3.1 Upper Box Dimension

Let $(Y, d_Y)$ be a compact metric space.

**Definition 3.1** The upper box dimension of $(Y, d_Y)$ is given by

$$\overline{\dim}_B Y = \limsup_{\varepsilon \to 0^+} \frac{\log N(\varepsilon)}{|\log \varepsilon|},$$ (3.1)

where $N(\varepsilon)$ stands for the maximal cardinality of an $\varepsilon$–separated set in $(Y, d_Y)$.

Consider now a Borel probability measure $\nu$ on $Y$.

**Definition 3.2** The upper box dimension of $\nu$ is given by

$$\overline{\dim}_B \nu = \lim_{\delta \to 0^+} \inf \left\{ \overline{\dim}_B Z : Z \subset Y \text{ and } \nu(Z) \geq 1 - \delta \right\}.$$
It is worth mentioning that, although the upper box dimension of a set \( Z \) coincides with the upper box dimension of its closure, the upper box dimension of a probability measure is intended to estimate the size of subsets rather than the entire support of the measure (that is, the smallest closed subset with full measure). Indeed, it may happen that \( \dim_B \nu < \dim_B (\text{supp } \nu) \) (cf. Example 7.1 in [19]). We refer the reader to [10, 19] for excellent accounts on dimension theory.

### 3.2 Homogeneous Measures

Let \( \nu \) be a Borel probability measure on the compact metric space \((Y, d_Y)\). A balanced measure should give the same probability to any two balls with the same radius, but this is in general a too strong demanding. Instead, we weaken the request in the following way.

**Definition 3.3** We say that \( \nu \) is homogeneous if there exists \( L > 0 \) such that
\[
\nu(B(y_1, 2\epsilon)) \leq L \nu(B(y_2, \epsilon)) \quad \forall \, y_1, y_2 \in \text{supp } \nu \quad \forall \, \epsilon > 0.
\] (3.2)

For instance, the Lebesgue measure on \([0, 1]\), atomic measures and probability measures absolutely continuous with respect to the latter ones, with densities bounded away from zero and infinity, are examples of homogeneous probability measures. We denote by \( \mathcal{H}_Y \) the set of such homogeneous Borel probability measures on \( Y \).

By definition, every homogeneous measure satisfies
\[
\nu(B(y, 2\epsilon)) \leq L \nu(B(y, \epsilon)) \quad \forall \, y \in \text{supp } \nu \quad \forall \, \epsilon > 0
\] (3.3)
and, as \( \nu(B(y_1, \epsilon)) \leq \nu(B(y_1, 2\epsilon)) \),
\[
\nu(B(y_1, \epsilon)) \leq L \nu(B(y_2, \epsilon)) \quad \forall \, y_1, y_2 \in \text{supp } \nu \quad \forall \, \epsilon > 0.
\] (3.4)

A measure \( \nu \) satisfying Eq. (3.3) is said to be a *doubling measure*. Although the two concepts Eqs. (3.3) and (3.4) are unrelated in general, if \( Y \) is a subset of an Euclidean space \( \mathbb{R}^k \) then any probability \( \nu \) satisfying Eq. (3.4) is a doubling measure. Indeed, as there is a constant \( C_k \) such that \( \text{Leb}(B(y, r)) = C_k r^k \) for every \( y \in Y \) and every \( r > 0 \), any ball \( B(y, 2\epsilon) \) can be covered by at most \( 2^k \) balls of radius \( \epsilon \); we now apply Eq. (3.2). For a discussion on conditions on \( Y \) which ensure the existence of homogeneous measures and further relations between homogeneity and the doubling property, we refer the reader to [1, Section 4] and references therein.

### 3.3 G-Homogeneous Measures

For a compactly generated semigroup by a continuous family \((g_y : X \to X)_{y \in Y}\) acting on a metric space, we say that a Borel measure \( \nu \in \mathcal{M}(X) \) is **G-homogeneous** if

(a) \( \nu(K) < \infty \), for any compact set \( K \subset X \);
(b) there exists \( K_0 \subset X \) such that \( \nu(K_0) > 0 \);
(c) for any \( \epsilon > 0 \) there exist \( \delta(\epsilon) > 0 \) and \( c > 0 \) such that
\[
\nu(B_n^G(x, \delta(\epsilon))) \leq c \cdot \nu(B_n^G(y, \epsilon))
\]
holds for any \( n \in \mathbb{N} \) and all \( x, y \in X \). In the case where \( \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} = A > 0 \), we say that \( \nu \) is strongly **G-homogeneous**.

As examples of spaces which admit a strongly G-homogeneous measure, we have the following:
1. The canonical volume form $dV$ on a closed, compact and oriented Riemannian manifold $X$ determines a strongly $G$-homogeneous measure $\nu$ if $G$ is a finitely generated group of isometries.

2. If $X$ is a locally compact topological group, $\mu$ is a right invariant measure and $G$ is a finitely generated group by $G_1 = \{id_X, T_1, T^{-1}_1, T_2, T^{-1}_2, \ldots, T_p, T^{-1}_p\}$, a finite and symmetric set of homeomorphisms, then $\mu$ is strongly $G$-homogeneous (see [1, Proposition 4.6]).

4 Proofs

In this section, we prove our main results.

4.1 Proof of Theorem A

It is clear from the definition of the entropy function that $h_d(x, \varepsilon) \leq h(X, S, P, \varepsilon)$, for all $x \in X$, and it implies that

$$\text{mdim}_M \left( X, S, d, P \right) \geq \limsup_{\varepsilon \to 0^+} \sup_{x \in X} \frac{h_d(x, \varepsilon)}{-\log \varepsilon}.$$ 

To prove the converse inequality, we start noticing that, for a fixed $\varepsilon > 0$, if $X = \bigcup_{i=1}^k F_i$, finite union of closed sets, then $B(X, \varepsilon, P) \leq \max_i B(F_i, \varepsilon, P)$. Then, cover $X$ by closed balls of radius 1, say $B_1 = \{B_1^1, \ldots, B_{1}^{1_1}\}$ such cover. Let $B_{1_j}$ be the closed ball in the given cover where the maximum occurs. Now, cover $B_1$ by a finite family of closed balls of radius at most $\frac{1}{2}$ denoted by $B_2 = \{B_2^1, \ldots, B_2^{1_2}\}$. Again, there exists $B_2^j \in B_2$ for which $B(X, S, \varepsilon, P) \leq B(B_2^j, S, \varepsilon, P)$. Following by induction, for each $k \in \mathbb{N}$, there exists a closed ball of radius at most $\frac{1}{k}$ so that $B(X, S, \varepsilon, P) \leq B(B_k, S, \varepsilon, P)$. Moreover, by the previous construction, we have a sequence of nested closed balls $\{B_k\}_{k \in \mathbb{N}}$ whose diameter goes to zero. So, there exists $\bar{x} = \cap_{k \in \mathbb{N}} B_k$ and for any closed neighbourhood $F$ of $\bar{x}$ we have $B_k \subset F$, for $k \in \mathbb{N}$ large enough. So,

$$B(F, S, \varepsilon, P) \geq B(B_k, S, \varepsilon, P) \geq B(X, S, \varepsilon, P),$$

which implies, by the definition of $h_d(\bar{x}, \varepsilon)$, $h_d(\bar{x}, \varepsilon) \geq B(X, S, \varepsilon, P)$. Hence,

$$\limsup_{\varepsilon \to 0^+} \sup_{x \in X} \frac{h_d(x, \varepsilon)}{-\log \varepsilon} \geq \text{mdim}_M \left( X, S, d, P \right)$$

and it finishes the proof.

4.2 Proof of Theorem B

First we notice that for any $\nu \in \mathcal{M}(X)$ and $\delta > 0$, $\nu(X) > 1 - \delta$ and so, for every $\varepsilon > 0$, $n \in \mathbb{N}$ and $\omega \in Y^n$

$$s(g_{o_n} \ldots g_{o_1}, n, \varepsilon) \geq s(\nu, g_{o_n} \ldots g_{o_1}, n, \varepsilon, \delta).$$

It implies that, for any $\nu \in \mathcal{M}(X)$

$$h(X, S, \varepsilon, \delta) \geq h^K(\nu, X, S, P, \varepsilon, \delta).$$
Hence,

$$\text{mdim}_M \left( X, \mathcal{S}, d, \mathbb{P} \right) \geq \lim_{\delta \to 0} \lim_{\varepsilon \to 0^+} \sup_{\nu \in \mathcal{M}(X)} \frac{h^K_\nu (\mathbb{S}, \mathbb{P}, \varepsilon, \delta)}{- \log \varepsilon}. \quad (4.1)$$

For the second part, we notice that Definition 2.5 could be made in terms of spanning sets. More precisely, given \( \varepsilon > 0 \), a positive integer \( n \) and \( g = g_{\omega_1} \ldots g_{\omega_1} \), we say that a subset \( A \) of \( E \subset X \) is a \((g_{\omega_1} \ldots g_{\omega_1}, n, \varepsilon, E)\)-spanning set if for any \( x \in E \) there exists \( y \in A \) so that \( d(g(x, y) < \varepsilon) \). By the compactness of \( X \), given \( \varepsilon, n \) and \( g \) as before, there exists a finite \((g, n, \varepsilon, E)\)-spanning set. Given \( \delta > 0 \), if we set

$$b_\nu (g_{\omega_1} \ldots g_{\omega_1}, n, \varepsilon, \delta) = \inf_{\{ E \subseteq X : \nu(E) > 1 - \delta \}} \frac{\nu(E)}{n \log \nu(E)}.$$

it is not difficult to see that

$$h^K_\nu (\mathbb{S}, \mathbb{P}) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \int_{Y} b_\nu (g_{\omega_1} \ldots g_{\omega_1}, n, \varepsilon, \delta) \, d\mathbb{P}(\omega).$$

Now, let \( \mathbb{P} = \gamma^\mathbb{N} \) with \( \gamma \in \mathcal{H}_Y \) and consider \( \nu \in \mathcal{M}(X) \). Fix \( \varepsilon > 0 \) and consider a positive integer \( k = k(\varepsilon) \geq 1 \) so that \( \sum_{i \geq k} \text{diam}(\gamma_i) < \frac{\varepsilon}{2} \). Choose a maximal \( \frac{\varepsilon}{4} \)-separated set \( E \subset Z = \text{supp}(\gamma) \), whose cardinality is denoted by \( N_Z(\varepsilon) \). By the definition of upper box dimension,

$$\lim_{\varepsilon \to 0} \frac{N_Z(\varepsilon)}{- \log \varepsilon} = \text{dim}_B(Z).$$

For each \( n \in \mathbb{N} \) and each point \( (p_1, \ldots, p_n, \varepsilon) \) consider the cylinder

$$C_{i_1 \ldots i_n} = \left\{ \omega \in Y^\mathbb{N} : \omega_i \in B\left( p_i, \frac{\varepsilon}{4} \right), \text{ for } i = 1, \ldots, n \right\}. \quad (4.2)$$

Note that the collection of cylinders defined above covers \( Z^\mathbb{N} \) and has diameter less than \( \varepsilon \). It follows that

$$\int_{Y^\mathbb{N}} b_\nu (g_{\omega_1} \ldots g_{\omega_1}, n, \varepsilon, \delta) \, d\mathbb{P}(\omega) \geq \sum_{i \in (i_1 \ldots i_{n+1})} \min_{\omega \in C_{i_1 \ldots i_n}(Z^\mathbb{N})} b_\nu (g_{\omega_1} \ldots g_{\omega_1}, n, \varepsilon, \delta) \times \min \mathbb{P}(C_{i_1} \cap Z^\mathbb{N}). \quad (4.3)$$

Now, we notice that the image of \( b_\nu (g_{\cdot}, n, \varepsilon, \delta) : C_{\cdot} \to \mathbb{Z}_+ \) has a minimum in \( \mathbb{Z}_+ \) and such minimum is attained by some \( \omega(\cdot) \in C_{\cdot} \). So, this together with Eq. (4.3) and the fact that \( \mathbb{P} \) is a product measure gives

$$\int_{Y^\mathbb{N}} b_\nu (g_{\omega_1} \ldots g_{\omega_1}, n, \varepsilon, \delta) \, d\mathbb{P}(\omega) \geq \left[ \sum_{i \in (i_1, i_2, \ldots, i_{n+1})} \min_{\omega \in C_{i_1} \cap Z^\mathbb{N}} b_\nu (g_{\omega_1} \ldots g_{\omega_1}, n, \varepsilon, \delta) \right] \times \min \mathbb{P}(C_{i_1} \cap Z^\mathbb{N})$$

$$\geq \sum_{i} b_\nu (g_{\omega(\cdot)}, n, \varepsilon, \delta) \times \min_{i} \prod_{j=0}^{n+k-1} \gamma(B(p_{ij}, \frac{\varepsilon}{4}) \cap Z).$$

Claim 1. For \( \mu \in \prod(\sigma, \nu)_{\text{erg}} \), we have that

$$\sum_{i} b_\nu (g_{\omega(\cdot)}, n, \varepsilon, \delta) \geq b_\mu (T_G |_{Z^\mathbb{N} \times X}, n, \varepsilon, \delta).$$
In fact, if \( \{x_1^{(i)}, \ldots, x_{b(g_{ω(i)}, n, ε)}^{(i)}\} \) is a \((g_{ω(i)}, n, ε)\)-spanning set for a subset \( Z \subset Z \), satisfying \( ν(Z) \geq 1 - δ \), with smallest cardinality, then

\[
\bigcup_i \left\{ \left( \omega^{(i)}, x_1^{(i)} \right), \ldots, \left( \omega^{(i)}, x_{b(g_{ω(i)}, n, ε)}^{(i)} \right) \right\}
\]

is a \((TG, n, ε)\)-spanning set for \( Y^N \times Z \) and \( µ(Y^N \times Z) = ν(Z) = 1 - δ \).

**Claim 2.** There exists \( L > 0 \), given by the homogeneity of \( γ \), such that

\[
\min_i \left( \prod_{j=0}^{n+k-1} γ(B(p_{ij}, \frac{ε}{4})) \right) \geq \left( \frac{1}{L^2} \right)^{n+k} \left( \frac{1}{NZ(ε)} \right)^{n+k}.
\]

In fact, due to the homogeneity of \( γ \), which implies that, for every \( q \in \text{supp} γ \), any \( p_{ij} \) and all \( i \),

\[
γ(B(p_{ij}, ε)) \geq \frac{1}{L} γ(B(q, ε)) \quad \forall \epsilon > 0
\]

and the fact that, as \( \bigcup_{e \in E} B(e, \frac{ε}{4}) = Z \),

\[
1 = γ\left( \bigcup_{e \in E} B\left(e, \frac{ε}{4}\right) \right) \leq \sum_{e \in E} γ\left(B\left(e, \frac{ε}{4}\right)\right) \leq NZ(ε) L γ\left(B\left(q, \frac{ε}{4}\right)\right)
\]

thus

\[
γ\left(B(q, \frac{ε}{4})\right) \geq \frac{1}{L} \frac{1}{NZ(ε)}.
\]

By Claim 1 and Claim 2, we obtain

\[
\int_{Y^N} b_ν(g_{ω_0} \ldots g_{ω_1}, n, ε, δ) dP(ο) \geq b_µ(TG | Z^N \times X, n, ε, δ) \left( \frac{1}{L^2} \right)^{n+k} \left( \frac{1}{NZ(ε)} \right)^{n+k}
\]

which implies

\[
h^K_ν(X, S, P, ε, δ) \geq \sup_{μ ∈ Π(σ, ν)_{erg}} h^K_µ(Y^N \times X, TG, ε, δ) - \log NZ(ε).
\]

Since \( Z = \text{supp}(γ) = Y \), by [17, Theorem 4.2] and [5, Theorem A],

\[
\lim \limsup_{δ → 0} \sup_{ε → 0^+} \frac{h^K_ν(S, P, ε, δ)}{− \log ε} \geq \lim \limsup_{δ → 0} \sup_{ε → 0^+} \frac{h^K_µ(TG, ε, δ)}{− \log ε} - \overline{\dim}_B Y = \overline{\dim}_M \left( Y^N \times X, TG, D \times d \right) − \overline{\dim}_B Y = \overline{\dim}_M \left( X, S, d, P \right).
\]

By Eq. (4.1), we have the desired equality and conclude the proof.

**Remark 4.1** We notice that the hypotheses \( \text{supp}(γ) = Y \) could be replaced by assuming that the set \( \text{supp}(γ) \times X \) is \( TG \)-invariant. To get the same conclusion as in Theorem B, we only need to combine the previous proof and [17, Theorem 4.2] and [5, Proposition 4.2].
4.3 Proof of Theorem C

Take $i_0, \ldots, i_{n-1} \in \{1, \ldots, p\}$, $U_{j_0}, \ldots, U_{j_{n-1}} \in \mathcal{U}$ and consider

$$
\left( [i_0] \times U_{j_0} \right) \cap \left( T_G^{-1}([i_1] \times U_{j_1}) \cap \cdots \cap T_G^{-n}([i_{n-1}] \times U_{j_{n-1}}) \right)
$$

where $\omega$ belongs to the cylinder set $[i_0 \ldots i_{n-1}]$. If we denote by $\mathcal{U}(\omega, n) = \{ V_{j_0} \cap \cdots \cap (f_{\omega}^{-n})^{-1}(U_{j_{n-1}}) : V_{j_t} \in \mathcal{U} \}$ the open cover of $X$ induced by $\omega$, we have that $N(\mathcal{U}(\omega, n))$ coincides with the minimum number of open sets of $\tilde{\mathcal{U}}(n)$ necessary to cover $[i_0 \ldots i_{n-1}] \times X$. So,

$$
h_{\text{top}}(\mathcal{U}, \mathcal{S}, \eta_p) + \log p = \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{p^n} \sum_{g \in G_n} N(\mathcal{U}, g, n) \right) + \log p
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \log N(\tilde{\mathcal{U}}, T_G, n)
$$

$$
= h_{\text{top}}(\tilde{\mathcal{U}}, T_G).
$$

It proves item (i).

For the second item, take $\delta \in (0, 1)$ and $\nu \in \mathcal{M}(X)$ so that $\Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset$. For $\mu \in \Pi(\sigma, \nu)_{\text{erg}}$, we have that

$$
\sum_{g \in G_n} N_\nu(\mathcal{U}, g, n, \delta) = N_\mu(\mathcal{U}, T_G, n, \delta).
$$

The equality comes from the fact that if

$$
\sum_{g \in G_n} N_\nu(\mathcal{U}, g, n, \delta) > N_\mu(\mathcal{U}, T_G, n, \delta),
$$

there exists a cylinder $[i_0 \ldots i_{n-1}]$ so that $[i_0 \ldots i_{n-1}] \times X$ is covered by at most $N_\nu(\mathcal{U}, g, n, \delta) - 1$ open sets, where $w = i_0 \ldots i_{n-1}$. As $(\pi_X)_*(\mu) = \nu$, it contradicts the minimality of $N_\nu(\mathcal{U}, g, n, \delta)$. So,

$$
\sup_{\{\nu \in \mathcal{M}(X) \text{ and } \Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset\}} h^S_\nu(\mathcal{U}, \mathcal{S})
$$

$$
= \sup_{\{\nu \in \mathcal{M}(X) \text{ and } \Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset\}} \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{p^n} \sum_{g \in G_n} N_\nu(\mathcal{U}, g, n) \right)
$$

$$
= \sup_{\mu \in \mathcal{E}(\Sigma_p^+ \times X)} \lim_{n \to \infty} \frac{1}{n} \log N_\mu(\tilde{\mathcal{U}}, T_G, n) - \log p
$$

$$
= h_{\text{top}}(\tilde{\mathcal{U}}, T_G) - \log p
$$

$$
= h_{\text{top}}(\mathcal{U}, \mathcal{S}, \eta_p),
$$

which concludes the proof of the second item.
4.4 Proof of Theorem D

For a fixed $\epsilon > 0$, let $U_0$ be an open cover of $X$ with $\text{diam}(U_0) \leq \epsilon$ and $\text{Leb}(U_0) \geq \frac{\epsilon}{8}$ (the existence of such open cover is guaranteed by [18, Lemma 3.4]). If $U$ is an open cover of $X$ with diameter less than $\frac{\epsilon}{8}$, $\omega \in Y^N$, as $\text{Leb}(U_0) \geq \text{diam}(U)$, $U(\omega, n)$ refines $U_0(\omega, n)$. Thus, once

$$N_v(S, U, g_{\omega_0}, \ldots, g_{\omega_1}, n, \delta) \geq s_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \geq b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta),$$

for all $\omega \in Y^N$ and $n \in \mathbb{N}$, we have that

$$\int_{Y^N} N_v(S, U, g_{\omega_0}, \ldots, g_{\omega_1}, n, \delta) \, d\mathbb{P}(\omega) \geq \int_{Y^N} N_v(S, U_0, g_{\omega_0}, \ldots, g_{\omega_1}, n, \delta) \, d\mathbb{P} \geq \int_{Y^N} b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \, d\mathbb{P}$$

where $Z = \text{supp}(\gamma)$ and $C_z$ denotes the cylinder set defined in Eq. (4.2). Now, we notice that the image of $b_v(S, \cdot, n, \epsilon, \delta) : C_z \to \mathbb{Z}_+$ has a minimum in $\mathbb{Z}_+$ and such minimum is attained by some $\omega(i) \in C_z$. So, adapting Claim 1 and Claim 2 obtained in Theorem B together with Eq. (4.4), we have that

$$\int_{Y^N} N_v(S, U, g_{\omega_0}, \ldots, g_{\omega_1}, n, \delta) \, d\mathbb{P}(\omega) \geq \int_{Y^N} b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \, d\mathbb{P} \geq \sum_{i=(i_1, \ldots, i_{i+k})} \min_{\omega \in C_z} b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \times \min_{\omega \in C_z} \mathbb{P}(C_z \cap Z^N),$$

where $Z = \text{supp}(\gamma)$ and $C_z$ denotes the cylinder set defined in Eq. (4.2). Now, we notice that the image of $b_v(S, \cdot, n, \epsilon, \delta) : C_z \to \mathbb{Z}_+$ has a minimum in $\mathbb{Z}_+$ and such minimum is attained by some $\omega(i) \in C_z$. So, adapting Claim 1 and Claim 2 obtained in Theorem B together with Eq. (4.4), we have that

$$\int_{Y^N} N_v(S, U, g_{\omega_0}, \ldots, g_{\omega_1}, n, \delta) \, d\mathbb{P}(\omega) \geq \int_{Y^N} b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \, d\mathbb{P} \geq \sum_{i=(i_1, \ldots, i_{i+k})} \min_{\omega \in C_z} b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \times \min_{\omega \in C_z} \mathbb{P}(C_z \cap Z^N),$$

where $Z = \text{supp}(\gamma)$ and $C_z$ denotes the cylinder set defined in Eq. (4.2). Now, we notice that the image of $b_v(S, \cdot, n, \epsilon, \delta) : C_z \to \mathbb{Z}_+$ has a minimum in $\mathbb{Z}_+$ and such minimum is attained by some $\omega(i) \in C_z$. So, adapting Claim 1 and Claim 2 obtained in Theorem B together with Eq. (4.4), we have that

$$\int_{Y^N} N_v(S, U, g_{\omega_0}, \ldots, g_{\omega_1}, n, \delta) \, d\mathbb{P}(\omega) \geq \int_{Y^N} b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \, d\mathbb{P} \geq \sum_{i=(i_1, \ldots, i_{i+k})} \min_{\omega \in C_z} b_v(S, g_{\omega_0}, \ldots, g_{\omega_1}, n, \epsilon, \delta) \times \min_{\omega \in C_z} \mathbb{P}(C_z \cap Z^N),$$

where by $g_{\omega(i)}$ we mean $g_{\omega_{i_1}} \cdots g_{\omega_{i_{i+k}}}$ if $\omega(i)_{[1, n]} = \omega(i_1) \cdots \omega(i_{i+k})$ and $\mu \in \Pi(\sigma, \nu)\text{erg}$, $V_0$ is an open cover with $\text{Leb}(V_0) \leq \epsilon$ and $L > 0$ is specified by the homogeneity of $\gamma$ and does not depend on neither $\epsilon$ nor $n$. Then, we notice that, since $Z = \text{supp}(\gamma) = Y$, if

$$h^S_v(S, \epsilon, \mathbb{P}) := \inf_{\text{diam}(U) \leq \epsilon} h^S_v(S, U, \mathbb{P}) \text{ and } h^S(S, \epsilon, \mathbb{P}) := \sup_{\{v \in M: \Pi(\sigma, \nu)\text{erg} \neq \emptyset\}} h^S_v(S, \epsilon, \mathbb{P}),$$
by Eq. (4.5) and [17, Lemma 2.3],

\[ h^S(S, \varepsilon, \mathbb{P}) = \sup_{\nu \in \mathcal{M} : \Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset} h^S(S, \varepsilon, \mathbb{P}) \geq \sup_{\mu \in \mathcal{E}(T_G)} h^S(T_G, V_0) - \log N_Y(\varepsilon) = h_{\text{top}}(T_G, V_0) - \log N_Y(\varepsilon) \geq h(T_G, 3\varepsilon) - \log N_Z(\varepsilon). \]

Therefore, by [17, Theorem 4.2] and [5, Theorem A],

\[ \limsup_{\varepsilon \to 0^-} \frac{h^S(S, \varepsilon, \mathbb{P}) - \log \varepsilon}{\varepsilon} \geq \frac{\overline{\dim}_M \left( Y_N \times X, T_G, D \times d \right)}{\overline{\dim} \left( Y_N \times X, T_G, D \times d \right) - \overline{\dim}_B Y} \]

\[ = \frac{\overline{\dim}_M (X, S, d, \mathbb{P})}{\overline{\dim}_M (X, S, d, \mathbb{P})}. \quad (4.6) \]

For the converse inequality, we observe that we fix \( \varepsilon > 0 \) and consider \( U_0 \) so that \( \text{diam}(U_0) \leq \varepsilon \) and \( \text{Leb}(U_0) \geq \varepsilon^4 \). We also notice that

\[ \int_{Y_N} N_v(S, U_0, g_{\omega_n} \ldots g_{\omega_1}, n, \delta) d\mathbb{P}(\omega) \leq \sum_{i = (i_1 \ldots i_{n+k})} \max_{\omega \in C_i \cap Z_N} s_v(S, \text{Leb}(U_0), g_{\omega_n} \ldots g_{\omega_1}, n, \delta) \times \mathbb{P}(C_i) \]

As the image of \( s_v(S, \cdot, n, \text{Leb}(U_0)) : C_i \to \mathbb{Z}_+ \) is contained in \( [0, s(T_G, n, \text{Leb}(U_0))] \), it has a maximum in \( \mathbb{Z}_+ \) and such maximum is attained by some \( \omega^{(i)} \in C_i \). So, using the fact that \( \gamma \) is homogeneous and the inequality

\[ \sum_{i = (i_1 \ldots i_{n+k})} s_v(S, g_{\omega_n} \ldots g_{\omega_1}, n, \text{Leb}(U_0), \delta) \leq s_{\mu}(T_G, n, \text{Leb}(U_0)) \]

we obtain

\[ \int_{Y_N} N_v(S, U_0, g_{\omega_n} \ldots g_{\omega_1}, n, \delta) d\mathbb{P}(\omega) \leq s_{\mu}(T_G, n, \text{Leb}(U_0)) \left( \frac{1}{N_Z(\text{Leb}(U_0))} \right)^{n+K} \leq N_{\mu}(T_G, V_0, n, \delta) \left( \frac{1}{N_Z(\text{Leb}(U_0))} \right)^{n+K}. \]
where \( \mathcal{V}_0 \) is a finite collection of open sets which covers \( Y^\mathbb{N} \times X \) up to a set of \( \mu \)-measure less than \( \delta \) and \( \frac{3\varepsilon}{4} \leq 3\text{Leb}(\mathcal{U}_0) = \text{diam}(\mathcal{V}_0) \leq 3\varepsilon \) and \( \text{Leb}(\mathcal{V}_0) \geq \frac{3\varepsilon}{16} \). So,

\[
h^S(\mathbb{S}, \varepsilon, \mathbb{P}) = \sup_{\{v \in \mathcal{M}(X) : \Pi(\sigma, v)_{\text{erg}} \neq 0\}} \inf_{\text{diam}(\mathcal{U}) \leq \varepsilon} h^S_\nu(\mathcal{S}, \mathcal{U}, \mathbb{P})
\]

\[
\leq \sup_{\{v \in \mathcal{M}(X) : \Pi(\sigma, v)_{\text{erg}} \neq 0\}} h^S_\nu(\mathcal{S}, \mathcal{U}_0, \mathbb{P})
\]

\[
= \sup_{\{v \in \mathcal{M}(X) : \Pi(\sigma, v)_{\text{erg}} \neq 0\}} \lim_{n \to \infty} \frac{1}{n} \log \int_{Y^\mathbb{N}} N_v(\mathcal{S}, \mathcal{U}_0, g_{\omega_0} \ldots g_{\omega_1}, n, \delta) \, d\mathbb{P}(\omega)
\]

\[
\leq \sup_{\{v \in \mathcal{M}(X) : \Pi(\sigma, v)_{\text{erg}} \neq 0\}} \lim_{n \to \infty} \frac{1}{n} \log N_\mu(T_G, \mathcal{V}_0, n, \delta) - \log N_Y(\text{Leb}(\mathcal{U}_0))
\]

\[
= h_{\text{top}}(T_G, \mathcal{V}_0) - \log N_Y \left( \frac{\varepsilon}{4} \right)
\]

\[
\leq \lim_{n \to \infty} \frac{1}{n} \log s(T_G, n, \text{Leb}(\mathcal{V}_0)) - \log N_Y \left( \frac{\varepsilon}{4} \right)
\]

\[
\leq \lim_{n \to \infty} \frac{1}{n} \log s(T_G, n, \frac{3\varepsilon}{16}) - \log N_Y \left( \frac{\varepsilon}{4} \right).
\]

Hence,

\[
\lim\sup_{\varepsilon \to 0} \frac{h^S(\mathbb{S}, \varepsilon, \mathbb{P})}{-\log \varepsilon} \leq \lim\sup_{\varepsilon \to 0} \left[ \frac{h(T_G, \frac{3\varepsilon}{16})}{-\log \varepsilon} - \frac{\log N_Y \left( \frac{\varepsilon}{4} \right)}{-\log \varepsilon} \right]
\]

\[
= \overline{\text{mdim}}_M \left( Y^\mathbb{N} \times X, T_G, D \times d \right) - \overline{\text{dim}}_B(Y)
\]

\[
= \overline{\text{mdim}}_M \left( X, \mathbb{S}, d, \mathbb{P} \right). \quad (4.7)
\]

By Eqs. (4.6) and (4.7), we obtain the result.

**Example 4.2** Let \( X = (\mathbb{S}^1)^\mathbb{N} \) with the metric \( D \) defined in (Eq. (2.2)). For \( \alpha \in [0, 1] \), consider the map

\[
f_1 : \omega \in X \mapsto (e^{2\pi i \alpha} \omega_1, e^{2\pi i \alpha} \omega_2, \ldots)
\]

and \( f_2 = \sigma \) be the shift map on \( X \). Let \( G \) be the free semigroup generated by \( G_1 = \{1d_X, f_1, f_2\} \) and let \( \mathbb{P} = \left( \frac{1}{2}, \frac{1}{2} \right)^\mathbb{N} \). As \( f_1 \) is an isometry and commutes with \( f_2 \), we have that \( s(g_\omega, n, \varepsilon) = s(f_2, m, \varepsilon) \), where \( m \) denotes the number of times that \( f_2 \) appears in \( \omega_{[1, \ldots, n]} \), for every \( \omega \in \{1, 2\}^\mathbb{N} \).

Consider \( \varepsilon > 0 \) and define \( \ell = \lfloor \log(4/\varepsilon) \rfloor \). In this case, we have that \( \sum_{n>\ell} 2^{-n} \leq \varepsilon/2 \).

We consider an open cover of \([0, 1]\) given by

\[
I_k = \left( \frac{(k - 1)\varepsilon}{12}, \frac{(k + 1)\varepsilon}{12} \right), \quad 0 \leq k \leq \left\lfloor \frac{12}{\varepsilon} \right\rfloor.
\]

Each \( I_k \) has length less than \( \varepsilon/6 \). Let \( n \geq 1 \). Now, consider the following open cover of \([0, 1]\) given by

\[
\{x : x_1 \in I_{k_1}, x_2 \in I_{k_2}, \ldots, x_{n+\ell} \in I_{k_{n+\ell}}\}, \quad \text{for } 0 \leq k_1, k_2, \ldots, k_{n+\ell} \leq \left\lfloor \frac{12}{\varepsilon} \right\rfloor.
\]

Each open set has diameter less than \( \varepsilon \) with respect to the dynamical distance \( D_\varepsilon \) induced by \( f_2 \). Again, as \( f_1 \) is an isometry and commutes with \( f_2 \), we obtain that, for \( \omega \in \{1, 2\}^\mathbb{N} \) and \( m \) defined above,
\[ b(g_{\omega_n} \cdots g_{\omega_1}, n, \varepsilon) = b(f_2, m, \varepsilon) \leq \left( 1 + \left\lfloor \frac{12}{\varepsilon} \right\rfloor \right)^{m+2[\log(4/e)]+1} \]

So,

\[ h(X, S, P, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{2^n} \sum_{m=1}^{n-1} b(f_2, m, \varepsilon) \right) \]

\[ \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{2^n} \sum_{m=1}^{n-1} \frac{n!}{(n-m)!m!} \left( 1 + \left\lfloor \frac{12}{\varepsilon} \right\rfloor \right)^{m+2[\log(4/e)]+1} \right) \]

\[ = \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{2^n} \left( 1 + \left\lfloor \frac{12}{\varepsilon} \right\rfloor \right) \left( 1 - \left( 1 + \left\lfloor \frac{12}{\varepsilon} \right\rfloor \right)^{n+2[\log(4/e)]} \right) \right) \]

\[ = \log \left( 1 + \left\lfloor \frac{12}{\varepsilon} \right\rfloor \right) - \log 2, \]

which implies \( \overline{\text{mdim}}(X, S, D, P) \leq 1. \)

Now, we prove the reverse inequality. For \( k \geq 1 \), consider the set \( P_k = \{p_1, p_2, \ldots, p_k\} \), where \( p_i = \frac{2i-1}{2^k} \). Define \( \lambda_k = \frac{1}{k} \sum_{i=1}^{k} \delta_{p_i} \) on \([0, 1]\) and let \( \mu_k = \lambda_k^{\otimes n} \), the product probability measure on \([0, 1]^n\).

Define

\[ C_{i_1, \ldots, i_m} = \{x \in [0, 1]^n : x_1 = p_{i_1}, \ldots, x_n = p_{i_m}\}. \]

**Claim.** Let \( r < \frac{1}{4k} \), \( q \in [0, 1]^n \) and \( g \in G_n \) so that \( f_2 \) appears in \( g \). Then, there is a unique set \( C_{i_1, \ldots, i_m} \), with \( 1 \leq m \leq n \), such that

\[ \text{supp}(\mu_k) \cap B_n(q, g, r) \subset C_{i_1, \ldots, i_m}. \]

**Proof of the Claim.** Let \( y \in \text{supp}(\mu_k) \cap B_n(q, g, r) \). Assume \( g = g_{\omega_n} \cdots g_{\omega_1} \) and let \( t_1 \leq t_2 \leq \cdots \leq t_m \) be the positions in the finite word \( g \) for which \( g_{\omega_j} = f_2 \). By definition

\[ d(y_j, q_j) \leq D(g_{\omega_{i_1}}, g_{\omega_{i_1}}(y), g_{\omega_{i_1}} \cdots g_{\omega_1}(q)) \leq D_g(y, q) < r < \frac{1}{2k}, \]

for all \( j \in \{1, \ldots, m\} \). Since \( y \in \text{supp}(\mu_k) \) we have that \( y_j \in P_k \), for \( j \in \{1, \ldots, m\} \). By the choice of \( r \) and Eq. (4.8), we have that for a fixed \( j \in \{1, \ldots, m\} \) there is a unique \( p_{ij} \in P_k \) so that \( y_j = p_{ij} \). It guarantees the uniqueness of \( C_{i_1, \ldots, i_m} \) and proves the claim.

Now, for \( r \) and \( g \in G_n \) satisfying the condition of the previous claim, we have that

\[ \mu_k(B_n(q, g, r)) \leq \mu_k(C_{i_1, \ldots, i_m}) = \frac{1}{km}, \]

for every \( q \in [0, 1]^n \). Then, if \( A \) satisfies \( \mu_k(A) > 1 - \delta \) and \( A \subset \bigcup_{i=1}^{L} B_n(q^{(i)}, g, r) \) then

\[ 1 - \delta < \mu_k(A) \leq \sum_{i=1}^{L} \mu_k(B_n(q^{(i)}, g, r)) \leq \frac{L}{km}. \]
and it implies that $N_{\mu_k}(g, r, \delta) \geq (1 - \delta) k^n$, for every $\delta \in (0, 1)$. So,

$$h_{\mu_k}(S, P, \frac{1}{2k}, \delta) = \lim_{n \to \infty} \sup \frac{1}{n} \log \left( \frac{1}{2^n} \sum_{|g|=n} N_{\mu_k}(g, r, \delta) \right) \geq \lim_{n \to \infty} \sup \frac{1}{n} \log \left( \frac{1}{2^n} \sum_{m=0}^{n} \frac{n!}{n-m)!m!} (1 - \delta) k^n \right) \geq \lim_{n \to \infty} \sup \frac{1}{n} \log \left( \frac{1}{2^n} \sum_{m=0}^{n} (1 - \delta) k^n \right) = \log k - \log 2.$$

Finally, we obtain

$$\overline{mdim}_M \left( X, S, D, P \right) = \lim_{\delta \to 0} \limsup_{r \to 0^+} \frac{\sup_{\nu \in \mathcal{M}(X)} h_\mu(S, P, r, \delta)}{-\log r} \geq \lim_{\delta \to 0} \limsup_{k \to \infty} \frac{h_{\mu_k}(S, P, \frac{1}{2k}, \delta)}{\log 2k} = \lim_{\delta \to 0} \lim_{k \to \infty} \frac{\log k - \log 2}{\log 2k} = 1,$$

and then $\overline{mdim}_M \left( X, S, D, P \right) = 1$.

### 4.5 Proof of Theorem E

For the proof of Theorem E, we need to recall the definition of metric mean dimension of a semigroup action on non-compact subsets. It is the content of the next subsubsection.

#### 4.5.1 The Metric Mean Dimension for Non-compact Subset

We now present the notion of metric mean dimension on non-compact sets introduced, in the case where the dynamics under consideration is given by a continuous map on a compact metric space, in [8]. The definition presented here is a slight adaptation of the one given in [8] combined with the notions of lower and upper Carathéodory capacities introduced in [1].

Given a set $Z \subset X$, let us consider

$$m(Z, s, N, \varepsilon) = \inf_{\Gamma} \left\{ \sum_{i \in I} \exp(-sn_i) \right\},$$

where the infimum is taken over all covers $\Gamma = \{B_{n_i}(x_i, \varepsilon)\}_{i \in I}$ of $Z$ with $n_i \geq N$. We also consider

$$m(Z, s, \varepsilon) = \lim_{N \to \infty} m(Z, s, N, \varepsilon).$$

One can show (see for instance [19]) that there exists a certain number $s_0 \in [0, +\infty)$ such that $m(Z, s, \varepsilon) = 0$ for every $s > s_0$ and $m(Z, s, \varepsilon) = +\infty$ for every $s < s_0$. In particular, we may consider
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\[ h_{GLW} \left( Z, S, \varepsilon \right) = \inf \{ s : m(Z, s, \varepsilon) = 0 \} = \sup \{ s : m(Z, s, \varepsilon) = +\infty \}. \]

The upper metric mean dimension of \( f \) on \( Z \) is then defined as the following limit

\[ \overline{\text{mdim}}_M \left( Z, S, d \right) = \limsup_{\varepsilon \to 0} \frac{h_{GLW} \left( Z, S, \varepsilon \right)}{| \log \varepsilon |}. \tag{4.9} \]

In the case where \( Z = X \), one can check that the two definitions of metric mean dimension given above actually coincide.

Another important tool to prove the theorem is the following.

**Lemma 4.1** (Vitali Covering Lemma (see [14])) Let \( X \) be a boundedly compact metric space and \( \mathcal{B} \) a family of closed balls in \( X \) such that

\[ \sup \{ \text{diam}(B) : B \in \mathcal{B} \} < \infty. \]

Then, there is a finite or countable sequence \( B_i \in \mathcal{B} \) of disjoint balls such that

\[ \bigcup_{B \in \mathcal{B}} B \subset \bigcup_i 5 \cdot B_i, \]

where \( 5 \cdot B_i \) denotes the closed ball of same center as \( B_i \) and radius five times the radius of \( B_i \).

Let \( \nu_v \) be the natural volume measure on \( X \) and assume that \( \overline{\text{mdim}}_{\nu_v} (x, d) \leq s \), for all \( x \in X \). Fix \( \delta > 0 \) and let

\[ X_k = \left\{ x \in X : \limsup_{n \to \infty} -\frac{1}{n} \log \nu_v(B^n_{n}(x, \varepsilon)) < (s + \delta/2) \text{ for all } \varepsilon \in (0, \frac{1}{k}) \right\}. \]

By hypothesis, \( X = \bigcup_{k \in \mathbb{N}} X_k \). For \( \varepsilon \in (0, \frac{1}{5k}] \) and \( x \in X_k \), there exists \( n(x) \in \mathbb{N} \) and an increasing sequence \( \{n_\ell(x)\}_{\ell \in \mathbb{N}} \) which converges to infinite and is contained in \( \mathbb{N} \) so that for any \( n_\ell(x) \geq n(x) \) we have

\[ \nu_v(B^n_{n}(x, \varepsilon)) \geq e^{-\left(s+\delta\right)n_\ell(x)} - \log \varepsilon. \]

Since \( X \) is a compact Riemannian manifold, it has bounded geometry (see [9] for more details on manifolds of bounded geometry). It implies that each function \( f_m : X_k \to \mathbb{R} \) given by \( f_m(x) := \nu_v(B^n_{n}(x, \varepsilon)) \) is continuous and so

\[ N_0 := \sup \{ n(x) : x \in X_k \} < \infty. \]

By Vitali Covering Lemma, for any \( N \geq N_0 \), it is possible to choose from the cover \( \mathcal{B}_N := \{B^n_{n}(x, \varepsilon) : x \in X_k \text{ and } n_\ell(x) \geq N \} \) of \( X_k \) a subset \( F_N \subset X_k \) and a family \( \mathcal{D}_N := \{B^n_{n}(x, \varepsilon) : x \in F_N \} \) of disjoint balls for which we have

\[ X_k \subset \overline{X_k} \subset \bigcup_{x \in F_N} B^n_{n}(x, 5\varepsilon) \subset \bigcup_{x \in F_N} B^n_{n}(x, 6\varepsilon), \]

and

\[ \nu_v(B^n_{n}(x, \varepsilon)) \geq e^{-\left(s+\delta\right)n_\ell(x)} - \log \varepsilon \text{ for all } x \in F_N. \]

So, as the family \( \mathcal{D}_N \) is given by disjoint balls,

\[ m(X_k, (s + \delta) \cdot - \log \varepsilon, N, 6\varepsilon) \leq \sum_{x \in F_N} e^{-\left(s+\delta\right)n_\ell(x)} - \log \varepsilon \leq \sum_{x \in F_N} \nu_v(B^n_{n}(x, \varepsilon)) \leq 1, \]
and consequently
\[ m(X_k, (s + \delta) \cdot -\log \epsilon, 6\epsilon) = \limsup_{N \to \infty} m(X_k, (s + \delta) \cdot -\log \epsilon, N, \epsilon) \leq 1, \]

which in turn implies that \( h_{GLW}(X_k, \mathbb{S}, 6\epsilon) \leq (s + \delta) \cdot -\log \epsilon \) and then
\[ \sup_{k \in \mathbb{N}} \text{mdim}^{GLW}_M(X_k, \mathbb{S}, d) = \limsup_{\epsilon \to 0} \frac{h_{GLW}(X_k, \mathbb{S}, 6\epsilon)}{-\log \epsilon} \leq s + \delta. \]

So,
\[ \text{mdim}^{GLW}_M(X, \mathbb{S}, d) = \sup_{k \in \mathbb{N}} \text{mdim}^{GLW}_M(X_k, \mathbb{S}, d) \leq s + \delta. \]

As \( \delta \geq 0 \) may be considered arbitrarily small, we have
\[ \text{mdim}^{GLW}_M(X, \mathbb{S}, d) \leq s \]
and it finishes the proof.

### 4.6 Proof of Theorem F

The following lemma is an important tool in the proof.

**Lemma 4.2** Let \( \nu \in \mathcal{M}(X) \) be a strongly \( G \)-homogeneous probability measure. Then,
\[ \text{mdim}_\nu(x, d) = \text{mdim}_\nu(y, d), \text{ for all } x, y \in X. \]

**Proof** For \( \epsilon > 0 \), by the \( G \)-homogeneity, there exist \( \delta(\epsilon) > 0 \) and \( c > 0 \) so that
\[ \nu(B_n^G(x, \delta(\epsilon))) \leq c \cdot \nu(B_n^G(y, \epsilon)) \]
and it implies
\[ h^G_\nu(x, \delta(\epsilon)) = \limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n^G(x, \delta(\epsilon))) \geq \limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n^G(y, \epsilon)) = h^G_\nu(y, \epsilon). \]

So
\[ \limsup_{\epsilon \to 0} \frac{h^G_\nu(x, \delta(\epsilon)) \log \delta(\epsilon)}{-\log \delta(\epsilon)} \geq \limsup_{\epsilon \to 0} \frac{h^G_\nu(y, \epsilon)}{-\log \epsilon}, \text{ for all } x, y \in X, \]
which gives \( \text{mdim}_\nu(x, d) \geq \text{mdim}_\nu(y, d) \). By switching the roles of \( x \) and \( y \) in the previous computations, one obtains the converse inequality and finishes the proof. \( \square \)

As a consequence of Lemma 4.2, we obtain that makes sense to define the measure metric mean dimension of a semigroup action with respect to a strongly \( G \)-homogeneous measure as the following:
\[ \text{mdim}_\nu(X, \mathbb{S}, d) = \limsup_{\epsilon \to 0} \frac{h^G_\nu(x, \epsilon)}{-\log \epsilon}, \text{ for any } x \in X, \]
since the limsup considered is constant in \( X \).

**Proposition 4.1** Let \( G \) be a compactly generated semigroup and \( \nu \) be a strongly \( G \)-homogeneous probability measure on a compact metric space \( (X, d) \). Then,
\[ \text{mdim}_\nu(X, \mathbb{S}, d) = \text{mdim}^{GLW}_M(X, \mathbb{S}, d). \]
Proof Fix $\varepsilon > 0$ and take $E$ a maximal $(n, \varepsilon)$-separated set in $X$. Then, by the maximality property of $E$, $B^G_n(x, \varepsilon/2) \cap B^G_n(y, \varepsilon/2) = \emptyset$ for any $x, y \in E$. By the strongly $G$-homogeneity, there exist $0 < \delta(\varepsilon) < \frac{\varepsilon}{2}$ and $c > 0$ so that $v(B^G_n(y, \delta(\varepsilon))) \leq c \cdot v(B^G_n(x, \varepsilon/2))$, for all $x, y \in X$. In particular, for a fixed $x \in E$

$$v(X) \geq \sum_{y \in E} v\left(B^G_n(y, \varepsilon/2)\right) \geq s(n, \varepsilon) \cdot v(B^G_n(x, \varepsilon/2)) \cdot \frac{1}{c}.$$ 

It follows that

$$\limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon) \leq \limsup_{n \to \infty} -\frac{1}{n} \log v(B^G_n(x, \delta(\varepsilon))).$$

Now, using again the strongly $G$-homogeneity

$$\limsup_{\varepsilon \to 0} \frac{h_{GLW}(S, \varepsilon)}{-\log \varepsilon} \leq \limsup_{\varepsilon \to 0} \frac{h^G_v(\delta(\varepsilon)) \log \delta(\varepsilon)}{-\log \delta(\varepsilon) \log \varepsilon} = \frac{c}{c} \cdot \nu(Y) \geq \nu(X) > 0$$

and so, by the strong $G$-homogeneity, we have

$$\liminf_{n \to \infty} \frac{-\log v(B^G_n(x, \varepsilon))}{-\log \varepsilon} > (s - \delta/2) \text{ for all } x, y \in X.$$

It guarantees that

$$c \cdot b(n, \delta(\varepsilon)) \cdot v\left(B^G_n(y, \varepsilon)\right) \geq v(X) > 0$$

and so, by the strong $G$-homogeneity, we have

$$\limsup_{\varepsilon \to 0} \frac{h_{GLW}(S, \varepsilon)}{-\log \varepsilon} \leq \limsup_{\varepsilon \to 0} \frac{h^G_v(\delta(\varepsilon)) \log \delta(\varepsilon)}{-\log \delta(\varepsilon) \log \varepsilon} = \frac{c}{c} \cdot \nu(Y) \geq \nu(X) > 0$$

and it ends the proof.

Let us proceed to the proof of Theorem F. For part (a), we notice that it is a consequence of Proposition 4.1. For part (b) let $\nu$ be a Borel measure on $X$ so that $\overline{\text{mdim}}_M^{\text{GLW}}(X, S, d) \geq s$, for all $x \in X$. Fix $\delta > 0$ and let

$$X_k = \left\{ x \in X : \liminf_{n \to \infty} -\frac{1}{n} \log v(B^G_n(x, \varepsilon)) > (s - \delta/2) \text{ for all } \varepsilon \in (0, 1/k) \right\}.$$

By hypothesis, $X = \bigcup_{k \in \mathbb{N}} X_k$. It follows that $0 < v(X) \leq \sum_k v(X_k)$, which guarantees the existence of some $k_0 \in \mathbb{N}$ for which we have $v(X_{k_0}) > 0$. Again, we can write $X_{k_0} = \bigcup_{N \in \mathbb{N}} X_{k_0,N}$ where

$$X_{k_0,N} = \left\{ x \in X_{k_0} : \frac{-\log v(B^G_n(x, \varepsilon))}{n \log \varepsilon} > (s - \delta/2) \text{ for all } n \geq N \right\}.$$

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In such case, there exists $N_0 \in \mathbb{N}$ for which $\nu(X_{k_0},N_0) > 0$. In particular,

$$\nu(B^G_N(x, \varepsilon)) \leq e^{-(s-\delta)(-\log \varepsilon)}, \text{ for all } x \in X_{k_0}, \varepsilon \in (0, \frac{1}{k_0}) \text{ and } n \geq N_0.$$  

Now, for each integer $N \geq N_0$, consider the open cover of $X_{k_0}$ given by $\mathcal{B}_N = \{B^G_N(x, \varepsilon) : x \in X_{k_0} \}$. In such case, we have that for a subcover $\mathcal{C}$ of $\mathcal{B}_N$

$$\inf_{\mathcal{C}} \#(\mathcal{C}) e^{-(s-\delta)(-\log \varepsilon)} = \inf_{\mathcal{C}} \sum_{B^G_N(x, \varepsilon) \in \mathcal{C}} e^{-(s-\delta)(-\log \varepsilon)} \geq \nu(X_{k_0},N_0).$$

As $\text{cov}(X, N, \varepsilon) \geq \text{cov}(X_{k_0},N_0, N, \varepsilon)$, for all $N$ and $\varepsilon > 0$, we have

$$\text{cov}(X, N, \varepsilon) e^{-(s-\delta)(-\log \varepsilon)} \geq \nu(X_{k_0},N_0),$$

and it implies that

$$\limsup_{N \to \infty} \frac{1}{N} \log \text{cov}(X, N, \varepsilon) e^{-(s-\delta)(-\log \varepsilon)} \geq 0$$

and so,

$$h_{GLW}(X, S, \varepsilon) \geq (s-\delta) \cdot (-\log \varepsilon).$$

Hence,

$$\overline{\text{mdim}}_{GLW}(X, S, d) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(X, S, \varepsilon)}{-\log \varepsilon} \geq s - \delta.$$  

As the inequality was obtained for an arbitrary $\delta$, we conclude that

$$\overline{\text{mdim}}_{GLW}(X, S, d) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(X, S, \varepsilon)}{-\log \varepsilon} \geq s,$$

as part (b) states.

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References

1. Biš A. An analogue of the variational principle for group and pseudogroup actions. Ann Inst Fourier. 2013;63(3):839–863.
2. Bufetov A. Topological entropy of free semigroup actions and skew-product transformations. J Dynam Contr Syst. 1999;5:137–143.
3. Carvalho M, Rodrigues F, Varandas P. Semigroups actions of expanding maps. J Stat Phys. 2017;116(1):114–136.
4. Carvalho M, Rodrigues F, Varandas P. Quantitative recurrence for free semigroup actions. Nonlinearity. 2018;31(3):864–886.
5. Carvalho M, Rodrigues F, Varandas P. A variational principle for free semigroup actions. Adv Math. 2018;334:450–487.
6. Carvalho M, Rodrigues F, Varandas P. A variational principle for the metric mean dimension of free semigroup actions. Ergod Th Dynam Sys. 2022;42:65–85.
7. Carvalho M, Rodrigues F, Varandas P. Generic homeomorphisms have full metric mean dimension. Ergod Th Dynam Sys. 2022;42:42–64.
8. Cheng D, Liand Z, Selmi B. Upper metric mean dimensions with potential on subsets. Nonlinearity. 2021;34:852–867.
9. Eldering J. Manifolds of bounded geometry. Normally hyperbolic invariant manifolds. Atlantis series in dynamical systems, vol 2. Atlantis Press, Paris.
10. Falkoner K. Fractal geometry: mathematical foundations and applications. Wiley. 1990.
11. Ghys E, Langevin R, Walczak P. Entropie géométrique des feuilletages. Acta Math. 1988;160(1-2):105–142.
12. Lindenstrauss E, Weiss B. Mean topological dimension. Israel J Math. 2000;115:1–24.
13. Lindenstrauss E, Tsukamoto M. Double variational principle for mean dimension. Geom Funct Anal. 2019;29:1048–1109.
14. Mattila P. Geometry of sets and measures in euclidean spaces. Cambridge studies in advanced mathematics, vol 44. Cambridge University Press, Cambridge, Fractals and rectifiability. 1995.
15. Rodrigues F, Jacobus T, Silva M. Entropy points and applications for free semigroup actions. J Stat Phys, vol 186(6). 2022.
16. Shapira U. Measure theoretical entropy of covers. Israel J Math. 2007;158(1):225–247.
17. Shi R. On variational principle for the metric mean dimension. IEEE Trans Inf Theory. 2022;68(7):4282–4288.
18. Gutman Y, Spiewak A. Around the variational principle for metric mean dimension. Stud Math. 2021;261:345–360.
19. Pesin Y. Dimension theory in dynamical systems: contemporary views and applications. Lectures in mathematics, Chicago Press. 1997.
20. Tang J, Li B, Cheng W-C. Some properties on topological entropy of free semigroup action. Dynamical Syst. 2018;33(1):54–71.
21. Velozo A, Velozo R. Rate distortion theory, metric mean dimension and measure theoretic entropy. arXiv:1707.05762 [math.DS][preprint]. 2017.
22. Yano K. A remark on the topological entropy of homeomorphisms. Inventiones Math. 1980;59(3):215–220.

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