Fluctuation-induced phase separation in metric and topological models of collective motion

David Martin,1 Hugues Chaté,2,3 Cesare Nardini,2 Alexandre Solon,4 Julien Tailleur,1 and Frédéric Van Wijland1

1 Université de Paris, Laboratoire Matière et Systèmes Complexes (MSC), UMR 7057 CNRS, F-75205 Paris, France
2 Service de Physique de l’État Condensé, CNRS UMR 3680, CEA-Saclay, 91919 Gif-sur-Yvette, France
3 Computational Science Research Center, Beijing 100094, China
4 Sorbonne Université, CNRS, Laboratoire Physique Théorique de la Matière Condensée, 75005 Paris, France

(Dated: August 5, 2020)

We study the role of noise on the nature of the transition to collective motion in dry active matter. Starting from field theories that predict a continuous transition at the deterministic level, we show that fluctuations induce a density-dependent shift of the onset of order, which in turns changes the nature of the transition into a phase-separation scenario. Our results apply to a range of systems, including the topological models in which particles interact with a fixed number of nearest neighbors, which were believed so far to exhibit a continuous onset of order. Our analytical predictions are confirmed by numerical simulations of fluctuating hydrodynamics and microscopic models.

Within active matter studies, the transition to collective motion is a problem of both historical and paradigmatic value, which has led to a wealth of theoretical [1–5], numerical [6–8] and experimental works, both on biological [9–13] and synthetic [14–16] systems. Thanks to its simplicity, the setting of dry polar flocks, in which self-propelled particles stochastically and locally align their velocities, continues to inspire the research of many [5–7, 17–26].

In metric models, particles align with all their neighbors within a finite distance. At the microscopic level, the nature of the transition is now well established [7, 8], notably thanks to the introduction of the active Ising model (AIM) in which rotational symmetry of the Vicsek model (VM) is explicitly broken [5]. For both the AIM and the VM, the scenario is that of a phase-separation between a disordered gas/paramagnetic phase and a polarly ordered liquid/ferromagnetic phase, with a coexistence region whose properties depend on the symmetry of the model [27].

At the continuous level, these results have first been accounted for using deterministic hydrodynamic theories, which typically couple a density field \( \rho \) and an order parameter field \( \alpha \) [3, 5, 28–30]. The nature of the phase transition can indeed be understood by considering their dynamics in one spatial dimension, a minimal model of which is given by

\[
\partial_t \rho = D \partial_x \rho - v \partial_x m \\
\partial_t m = D \partial_x m - v \partial_x \rho - F(\rho, m).
\]

Here, \( m \) is akin to a magnetisation field in a spin system and \( F(\rho, m) = \alpha m + \gamma \frac{m^3}{3} \) is a Landau term that controls ferromagnetic alignment [5]. Equations (1) and (2) have been derived for the AIM [5]; many similar hydrodynamic models have been proposed or derived in one and two dimensions [2–5, 16, 28, 30]. All these models lead to the same conclusion: the first-order nature of the transition stems from the density-dependence of the linear term: \( \alpha = \alpha(\rho) \). To continue the ferromagnetic analogy, as detailed in [31], any density-dependent critical temperature such that \( \alpha' (\rho) \neq 0 \) leads to a phase-separation transition. The latter is characterized by two main features: homogeneous ordered profiles are linearly unstable close to the transition, when \( \alpha \lesssim 0 \), and this instability leads to the emergence of inhomogeneous propagating solutions [7, 29, 32]. Interestingly, a number of microscopic metric models have also been described using continuous descriptions in which, in some scaling limit, \( \alpha \) is independent of the density, thus being compatible with continuous scenarios [33, 34].

In a second class of models, referred to as topological [35] and motivated by animal-behavior studies, the interaction between two particles is not decided based on their relative distance. For instance, particles can align with their Voronoi neighbors [36] or with their \( k \) nearest neighbors [10]. Doubling the distance between all particles, and hence reducing the particle density, does not impact the aligning dynamics so that these models are expected to be less sensitive to density variations. The coarse-graining of a Voronoi-based model has indeed led to a density-independent critical temperature, hence predicting a continuous onset of order [37]. Existing numerics appear consistent with such a continuous scenario [36].

Finally, deterministic hydrodynamic equations only capture part of the large-scale physics and the role of fluctuations at the coarse-grained level must also be assessed. In the homogeneous ordered phase, this was done using the celebrated Toner-Tu equations to establish the existence of long-range orientational order in two dimensions [1, 2, 38]. For metric systems with density-dependent critical temperature, the phase-separation scenario is robust to the introduction of noise [4]. The latter, however, selects the nature of the propagating bands observed in the coexistence regime [27]. On the contrary, the impact of fluctuations on the continuous scenario, for both metric and topological models with density-independent critical temperature, remains to be explored.

In this Letter, we show how fluctuations generically turn continuous transitions to collective motion predicted by deterministic hydrodynamics into the standard phase-separation scenario. We first show that dressing the PDEs (1)–(2) with noise generically yields a renormalized density-dependent critical temperature, hence leading to phase separation. We note that the hydrodynamic theory of topological interactions...
and we ignore it here, although our approach can be extended to noise. The latter is expected to be subdominant at large scales.

As an alternative field theory of topological models, we propose a modification of the Landau term in (2) that preserves the topological nature of the interaction at the coarse-grained level. Surprisingly, this still leads to a fluctuation-induced first-order scenario. To probe whether this prediction is an artifact of our continuous theory or a generic feature of topological interactions, we consider a microscopic model in which particles align with their k nearest neighbors. Numerical simulations confirm a liquid-gas phase separation scenario. Finally, we show how measuring the dependency of the onset of order on the average density is a simple quantitative test that allows predicting the nature of the transition. All calculations below are based on the one-dimensional hydrodynamic theory (1)-(2) and its topological generalization. Our results can be extended to two dimensions and to other hydrodynamic models [40]. We complement our analytical approach by numerical simulations, mostly in 2D, which are all detailed in [31].

**Fluctuation-induced first-order transitions.** We first study the impact of fluctuations on the deterministic dynamics (1) and (2), which encompass metric models, but also the hydrodynamic theory proposed for Voronoi neighbors [37]. We consider α independent of ρ to study the fate of the continuous transition predicted by (1)-(2) in this case. To do so, we complement Eq. (2) with a noise term:

\[
\partial_t m = D \partial_{xx} m - v \partial_x \rho - F (\rho, m) + \sqrt{2 \sigma \rho \eta}, \quad (3)
\]

where \( \eta(x, t) \) is a zero-mean delta-correlated Gaussian white noise field. Note that, hereafter, \( \rho(x, t) \) and \( m(x, t) \) represent fluctuating fields. The order parameter \( m(x, t) \) represents the sum of the orientations of particles located around position \( x \). The noise acting on \( m(x, t) \) will thus be multiplicative; it describes the fluctuations of a sum over \( \infty \) \( \rho \) particles and we take it proportional to \( \sqrt{\rho(x, t)} \). We now construct the hydrodynamics of the average fields \( \rho_0(x, t) = \langle \rho(x, t) \rangle \) and \( m_0(x, t) = \langle m(x, t) \rangle \) to leading order in the noise strength \( \sigma \), where brackets represent average over noise realizations. In principle, we could also complement Eq. (1) with a conserved noise. The latter is expected to be subdominant at large scales and we ignore it here, although our approach can be extended to this case. Introducing \( \delta \rho = \rho - \rho_0 \) and \( \delta m = m - m_0 \), the dynamics of \( \rho_0 \) and \( m_0 \) can be approximated as

\[
\begin{align*}
\partial_t \rho_0 &= D \partial_{xx} \rho_0 - v \partial_x \rho_0 \\
\partial_t m_0 &= D \partial_{xx} m_0 - v \partial_x \rho_0 - F (\rho_0, m_0) \\
&\quad - \frac{\partial^2 F}{\partial m^2} (\delta m^2) - \frac{\partial^2 F}{\partial \rho^2} (\delta \rho^2) - \frac{\partial^2 F}{\partial m \partial \rho} (\delta m \delta \rho) 
\end{align*}
\]

(4)

To close Eqs. (4) and (5), we need to compute \( \langle \delta m^2 \rangle, \langle \delta \rho^2 \rangle, \langle \delta m \delta \rho \rangle \) as functions of \( \rho_0 \) and \( m_0 \). In the small noise limit, the fluctuations \( \delta \rho \) and \( \delta m \) are assumed to be small so that we compute these correlators at the linear, Gaussian fluctuations level [41–44]. The dynamics of \( \delta \rho(x, t), \delta m(x, t) \) then read

\[
\begin{align*}
\partial_t \delta \rho &= D \partial_{xx} \delta \rho - v \partial_x \delta m \\
\partial_t \delta m &= D \partial_{xx} \delta m - v \partial_x \delta \rho - \frac{\partial F}{\partial \rho} \delta \rho - \frac{\partial F}{\partial m} \delta m + \sqrt{2 \sigma \rho_0 \eta} 
\end{align*}
\]

(6)

(7)

Away from the deterministic transition \( \alpha = 0 \), this linear system of equations leads to bounded fluctuations of \( \delta \rho, \delta m \) around the homogeneous solutions of Eq. (1) and (2). It can be solved in Fourier space and the correlators appearing in (5) can be obtained explicitly as integrals over \( k \) space, e.g. \( \langle \delta m^2 \rangle = \int dk \langle \delta m_k \delta m_{-k} \rangle / (2\pi) \) [31]. The alignment terms in Eq. (5) can then, consistently with the approximation leading to a Landau form, be expanded as \( F (\rho_0, m_0) = \bar{\alpha} m_0 + \bar{\gamma} m_0^3 / \rho_0^2 \). Fluctuations have thus, to this order in \( \sigma \), dressed \( \alpha \) and \( \gamma \) into \( \bar{\alpha} \) and \( \bar{\gamma} \). For a given set of parameters, the integrals over \( k \) space can be computed numerically. It is, however, more enlightening to compute them explicitly in the high temperature phase, where \( \alpha > 0 \). The precise expression of \( \bar{\gamma} \) is irrelevant for our purpose, and is presented in [31].

The linear term is renormalized into

\[
\bar{\alpha} = \alpha + \frac{3 \gamma}{4 \rho_0 v} f \left( \frac{\alpha D}{v^2} \right) \text{ with } f(u) = \frac{\sqrt{2u} + \sqrt{1+u}}{2+u} 
\]

(8)

Importantly, \( \bar{\alpha} \) now depends explicitly on the density. To first order in \( \sigma \), fluctuations thus renormalize the continuous transition predicted by Eqs. (1) and (2) into the standard liquid-gas phase separation. Note that higher orders in \( \sigma \) have no reason to cancel the dependence of \( \bar{\alpha} \) on density and we thus expect our conclusions to hold non-perturbatively in \( \sigma \).

To confirm our predictions, we carried out simulations of the 2D generalization of the stochastic PDEs (1) and (2) [31]. As shown in Fig. 1, the continuous transition predicted by (1) and (2) is replaced by the standard liquid-gas framework [5, 45]. This is best illustrated by the emergence of a coexistence region in which ordered bands travel in a disordered background [3, 4, 7, 28].

**Field theory for topological interactions.** The study of the dynamics (1) and (3) thus showed that fluctuations genericize the transition to collective motion first order in metric models. This applies, in particular, to the hydrodynamic theory proposed for Voronoi-based interactions in [37]. We now propose an alternative hydrodynamic description which preserves the topological nature of the interactions at the coarse-grained level. To do so, we focus on models in which particles align with their k nearest neighbors. We introduce a coarse-grained field \( y(x) \) which measures the interaction range of a particle at \( x \):

\[
\int_{x-y(x)}^{x+y(x)} \rho(z) dz = k .
\]

(9)

Particles at position \( x \) then align with a ‘topological’ field \( \bar{m}(x, t) \) computed over their k nearest neighbours through

\[
\bar{m}(x) = \frac{1}{k} \int_{x-y(x)}^{x+y(x)} m(z) dz .
\]

(10)
Homogeneous solutions \( \rho_0, m_0 \) correspond to \( y(x) = k/(2\rho_0) \) and \( \bar{m} = m_0/\rho_0 \). The linear term in \( F_{\text{topo}} \) then reduces to \( 2G(1-\beta)m_0 \), leading to a density-independent transition at \( \beta_m = 1 \). Linear stability analysis of the homogeneous solutions then shows that disordered and ordered solutions are linearly stable for \( \beta < \beta_m \) and \( \beta > \beta_m \), respectively [31]. Our topological field theory thus predicts a continuous transition at the mean-field level.

Let us now assess the effect of dressing the dynamics of the order parameter with noise: we consider the stochastic dynamics (1) and (3), albeit with \( F \) replaced by \( F_{\text{topo}} \). Here also, \( m(x,t) \) is the sum of the orientations of particles located around position \( x \) and we consider a multiplicative noise proportional to \( \sqrt{\rho(x,t)} \). To construct the dynamics of the average fields \( \rho_0 \) and \( m_0 \) perturbatively in \( \sigma \), we first stress that Eq. (9) directly enslaves the field \( y(x) \) to \( \rho(x) \). There are thus, again, only two independent fields, \( \rho(x,t) \) and \( m(x,t) \), so that Eq. (7) is still valid, up to \( F \to F_{\text{topo}} \). The expression of \( F_{\text{topo}} \) being, however, more complicated than in the metric case, the algebra is correspondingly more involved. We detail in [31] the renormalization of the linear part of \( F_{\text{topo}} \), which controls the nature of the transition. To first order in the noise strength \( \sigma \), we find

\[
(\beta - 1)m_0 \to \left[ \beta - 1 - g(\beta, k, \Gamma, \nu, \Gamma_D) \right] m_0, \tag{12}
\]

with \( g(\beta, k, \Gamma, \nu, \Gamma_D) \) a positive function whose expression is provided as an integral in [31].

Importantly, the linear term in the aligning dynamics of \( m_0 \) has again become density-dependent, hence predicting a phase-separation scenario. This is confirmed by numerical simulations of (1) and (3) with \( F \to F_{\text{topo}} \), which again reveal the existence of inhomogeneous propagating bands [31]. Our results thus also predict a fluctuation-induced phase-separation scenario for topological models.

**Microscopic models with k-nearest-neighbors interactions.** To test the above predictions, we first consider an off-lattice active Ising model [5, 45] in which \( N \) particles move in an...
\[ L_x \times L_y \text{ domain with periodic boundary conditions. Each particle carries a spin } s_i = \pm 1 \text{ and evolves according to the Langevin dynamics} \]

\[
\dot{\vec{r}}_i = s_i v_0 \vec{u}_x + \sqrt{2D}\vec{\eta}_i, \quad (13)
\]

where \( v_0 \) sets the self-propulsion speed and \( D \) sets the strength of the Gaussian white noise \( \vec{\eta}_i \). Spins flip from \( s_i \) to \(-s_i\) at rate \( W(s_i) \) given by

\[
W(s_i) = \Gamma e^{-\beta s_i \bar{m}_i}, \quad \text{where} \quad \bar{m}_i = \frac{1}{k} \sum_{j \in N_i} s_j, \quad (14)
\]

where \( N_i \) is the set of the \( k \)-nearest neighbours of particle \( i \) and \( \bar{m}_i \) their average magnetization. Note that a mean-field treatment of the aligning dynamics (14) indeed leads to Eq. (11). In agreement with our predictions, the model exhibits a first order transition to collective motion, akin to a liquid-gas phase separation: the onset of order at \( \beta \gtrsim \beta_c(\bar{\rho}_0) \) occurs through the emergence of an ordered propagating band (Fig. 2). Unlike the mean-field critical temperature, the boundaries of the coexistence region show a clear dependence on the mean density \( \bar{\rho} \).

To probe the generality of our results, we then implemented a topological version of the Vicsek model [6, 7] in which the particles align with their \( k \) nearest neighbors. We considered \( N \) scalar spins moving in a \( L_x \times L_y \) domain with periodic boundary conditions. At every time step, the particles align with the average direction of their \( k \) nearest neighbors:

\[
\arg[\vec{u}_i] \rightarrow \arg\left[\frac{1}{k} \sum_{j \in N_i} \vec{u}_j\right] + \sigma \eta, \quad (15)
\]

where \( \eta \) is uniformly drawn in \([-\pi, \pi]\). The particles then move a distance \( v_0 \Delta t \) along their propulsion vector. Once again, propagating bands are observed close to the onset of collective motion (Fig. 3).

Overall, despite being usually considered resilient to density fluctuations, the topological interactions studied in this article thus lead to a phase-separation scenario. Our results suggest that the nature of the transition to collective motion can be simply assessed by estimating whether its threshold depends on density or not. We illustrate this by considering simple models in which the aligning dynamics is disconnected from spatial positions, hence ensuring that the transition remains continuous.

**Loyal and random aligning models.** We consider \( N \) scalar spins moving in a \( L_x \times L_y \) domain with periodic boundary conditions according to the Langevin equation (13) with two different aligning dynamics [46]. In the Random Alignment Model (RAM), the aligning dynamics is given by (14), with \( \bar{m}_i \) computed over \( k \) spins chosen at random at every time step. In the Loyal Alignment Model (LAM), on the contrary, alignment occurs with the same set of \( k \) neighbours throughout the simulations, irrespective of the particle positions. In our simulations, we chose \( k = 4 \) and assigned to each particle its nearest neighbours on an initial square lattice. Simulations of both systems lead to continuous transitions, without the emergence of inhomogeneous phases. Figure 4 shows that measurements of the global magnetization as the temperature is varied leads to behaviors which are hard to distinguish between LAM, RAM and our topological microscopic model. Repeating these measurements at different densities reveals a density-dependence of the onset of order in the latter case, but not in LAM & RAM. Measuring \( \beta_c \) as \( \bar{\rho} \) varies thus constitutes a simple and robust test of the nature of the transition.

**Conclusion.** We have shown that dressing hydrodynamic equations for polar flocks with noise generically leads to a density-dependent renormalization of the onset of order. Noise thus makes homogeneous ordered profiles linearly unstable close to the transition and leads to a phase-separation scenario. This surprisingly holds for metric-free topological interactions. We confirmed our field-theoretical computations by numerical simulations of microscopic models in which particles interact with their \( k \) nearest neighbors. Finally, we have argued that measuring the dependence of the onset on the global density allows assessing the nature of the transition. This is likely to be a more stringent test than measuring putative critical exponents, which are unlikely to distinguish weakly-first-order transitions from the liquid-gas sce-
nario. Finally, it would be interesting to reproduce our studies on related models in which the coupling between density and alignment is qualitatively altered, from Malthusian [26] to incompressible [47] or Lévy flocks [48].

Acknowledgment: JT, HC and DM acknowledge support from the ANR grant Bacttemrs. We thank John Toner for interesting discussions. CN acknowledges the support of an Aide Investissements dAvenir du LabEx PALM (ANR-10-LABX-0039-PALM).

[1] J. Toner and Y. Tu, Physical Review Letters 75, 4326 (1995).
[2] J. Toner, Y. Tu, and S. Ramaswamy, Annals of Physics 318, 170 (2005).
[3] E. Bertin, M. Droz, and G. Grégoire, Physical Review E 74, 022101 (2006).
[4] S. Mishra, A. Baskaran, and M. C. Marchetti, Physical Review E 81, 061916 (2010).
[5] A. Solon and J. Tailleur, Physical Review Letters 111, 078101 (2013).
[6] T. Vicsik, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Physical Review Letters 75, 1226 (1995).
[7] G. Grégoire and H. Chaté, Physical Review Letters 92, 025702 (2004).
[8] H. Chaté, F. Ginelli, G. Grégoire, and F. Raynaud, Phys. Rev. E 77, 046113 (2008).
[9] J. Buhl, D. J. Sumpter, I. D. Couzin, J. J. Hale, E. Despland, E. R. Miller, and S. J. Simpson, Science 312, 1402 (2006).
[10] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Moretti, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, Proceedings of the National Academy of Sciences 105, 1232 (2008), https://www.pnas.org/content/105/4/1232.full.pdf.
[11] V. Schaller, C. Weber, C. Semmrich, E. Frey, and A. R. Bausch, Nature 467, 73 (2010).
[12] A. Cavagna, D. Conti, C. Creato, L. Del Castillo, I. Giardina, T. S. Grigera, S. Melillo, L. Parisi, and M. Viale, Nature Physics 13, 914 (2017).
[13] N. Bain and D. Bartolo, Science 363, 46 (2019).
[14] V. Narayan, S. Ramaswamy, and N. Menon, Science 317, 105 (2007).
[15] J. Deseigne, O. Dauchot, and H. Chaté, Physical Review Letters 105, 098001 (2010).
[16] A. Bricard, J.-B. Caussin, N. Desreumaux, O. Dauchot, and D. Bartolo, Nature 503, 95 (2013).
[17] F. Ginelli, F. Peruani, M. Bär, and H. Chaté, Physical Review Letters 104, 184502 (2010).
[18] F. Peruani, T. Klauss, A. Deutsch, and A. Voss-Boehme, Physical Review Letters 106, 128101 (2011).
[19] V. Dossetti and F. J. Sevilla, Physical Review Letters 115, 058301 (2015).
[20] A. Morin, N. Desreumaux, J.-B. Caussin, and D. Bartolo, Nature Physics 13, 63 (2017).
[21] L. Chen, C. F. Lee, and J. Toner, New Journal of Physics 20, 113035 (2018).
[22] F. A. Lavergne, H. Wendeheenne, T. Bäuerle, and C. Bechinger, Science 364, 70 (2019).
[23] B. Mahault, F. Ginelli, and H. Chaté, Physical Review Letters 123, 218001 (2019).
[24] D. Geyer, D. Martin, J. Tailleur, and D. Bartolo, Physical Review X 9, 031043 (2019).
[25] H. J. Charlesworth and M. S. Turner, Proceedings of the National Academy of Sciences 116, 15362 (2019).
[26] L. Chen, C. F. Lee, and J. Toner, arXiv preprint arXiv:2001.01300 (2020).
[27] A. P. Solon, H. Chaté, and J. Tailleur, Physical Review Letters 114, 068101 (2015).
[28] E. Bertin, M. Droz, and G. Grégoire, Journal of Physics A: Mathematical and Theoretical 42, 445001 (2009).
[29] J.-B. Caussin, A. Solon, A. Peshkov, H. Chaté, T. Dauxois, J. Tailleur, V. Vitelli, and D. Bartolo, Physical Review Letters 112, 148102 (2014).
[30] T. Ihle, Journal of Statistical Mechanics: Theory and Experiment 2016, 083205 (2016).
[31] See Supplemental Material [url], which includes theoretical and numerical details, as well as Refs. XXX.