Local and global well-posedness for the 2D Zakharov-Kuznetsov-Burgers equation in low regularity Sobolev space

Hiroyuki Hirayama
Organization for Promotion of Tenure Track, University of Miyazaki,
Miyazaki, 889-2192, Japan
E-mail address, h.hirayama@cc.miyazaki-u.ac.jp

Abstract
In the present paper, we consider the Cauchy problem of the 2D Zakharov-Kuznetsov-Burgers (ZKB) equation, which has the dissipative term $-\partial_x^2 u$. This is known that the 2D Zakharov-Kuznetsov equation is well-posed in $H^s(\mathbb{R}^2)$ for $s > 1/2$, and the 2D nonlinear parabolic equation with quadratic derivative nonlinearity is well-posed in $H^s(\mathbb{R}^2)$ for $s \geq 0$. By using the Fourier restriction norm with dissipative effect, we prove the well-posedness for ZKB equation in $H^s(\mathbb{R}^2)$ for $s > -1/2$.

Key Words and Phrases. Zakharov-Kuznetsov equation, Burgers equation, well-posedness, Cauchy problem, Fourier restriction norm.

2010 Mathematics Subject Classification. 35Q53.

1 Introduction
We consider the Cauchy problem of the 2D Zakharov-Kuznetsov-Burgers (ZKB) equation:

\[
\begin{cases}
\partial_t u + \partial_x (\partial_x^2 + \partial_y^2) u - \partial_x^2 u = \partial_x (u^2), & t > 0, \ (x, y) \in \mathbb{R}^2, \\
u(0, x, y) = u_0(x, y), & (x, y) \in \mathbb{R}^2,
\end{cases}
\]

(1.1)

where the unknown function $u$ is $\mathbb{R}$-valued. This equation is two dimensional model of the Korteweg-de Vries-Burgers (KdVB) equation

\[
\partial_t u + \partial_x^3 u - \partial_x^2 u = \partial_x (u^2), \ t > 0, \ x \in \mathbb{R},
\]

(1.2)

and appears in the dust-ion-acoustic-waves in dusty-plasmas (See, [22], [24]). We can see that (1.1) has both dissipative term and dispersive term. The aim of this paper is to prove the well-posedness of (1.1) in the Sobolev space $H^s(\mathbb{R}^2)$.

First, we introduce some known results for related problems for 1D case. In [11], Kenig, Ponce, and Vega proved that the Korteweg-de Vries (KdV) equation

\[
\partial_t u + \partial_x^3 u = \partial_x (u^2), \ t > 0, \ x \in \mathbb{R},
\]

is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/4$. Colliander, Keel, Stafillani, Takaoka, and Tao ([11]) extended the local result to globally in time. For the
critical case, Kishimoto ([13]) and Guo ([10]) obtained the global well-posedness of KdV equation in $H^{-\frac{3}{4}}(\mathbb{R})$. While, it is proved that the flow map of KdV equation is not uniformly continuous for $s < -\frac{3}{4}$ by Kenig, Ponce, and Vega in [12] (for $\mathbb{C}$-valued KdV) and Christ, Colliander, and Tao in [5] (for $\mathbb{R}$-valued KdV). Therefore, $s = -\frac{3}{4}$ is optimal regularity to obtain the well-posedness of KdV equation by using the iteration argument. For the Burgers equation

$$\partial_t u - \partial_x^2 u = \partial_x (u^2), \quad t > 0, \; x \in \mathbb{R},$$

Dix ([7]) proved the local well-posedness in $H^s(\mathbb{R})$ for $s > -\frac{1}{2}$ and nonuniqueness of solution for $s < -\frac{1}{2}$. For the critical case, Bekiranov ([2]) obtained the local well-posedness of the Burgers equation in $H^{-\frac{1}{2}}(\mathbb{R})$. These results say that $-\frac{1}{2}$ is optimal regularity to obtain the well-posedness of the Burgers equation.

In [20], Molinet and Ribaud considered the KdV-Burgers equation

$$\partial_t u + \partial_x^3 u - \partial_x^2 u = \partial_x (u^2), \quad t > 0, \; x \in \mathbb{R}$$

and obtained the global well-posedness in $H^s(\mathbb{R})$ for $s > -1$. For the critical case, Molinet and Vento ([21]) proved the global well-posedness of the KdV-Burgers equation in $H^{-\frac{1}{2}}(\mathbb{R})$. They also proved that the flow map is discontinuous for $s < -1$. We note that the regularity $s = -1$ is lower than both $-3/4$ and $-1/2$. It means that both the dispersive term and the dissipative term are essentially effective for well-posedness.

Next, we introduce some known results for related problems for 2D case. Grünrock and Herr ([9]), and Molinet and Pilod ([19]) proved that the 2D Zakharov-Kuznetsov equation

$$\partial_t u + \partial_x (\partial_x^2 + \partial_y^2) u = \partial_x (u^2), \quad t > 0, \; (x, y) \in \mathbb{R}^2$$

is locally well-posed in $H^s(\mathbb{R}^2)$ for $s > 1/2$. Especially, Grünrock and Herr used the linear transform

$$v(t, x, y) = u \left( t, \frac{4 + \sqrt{3}}{2} (x + y), \frac{4 \sqrt{3}}{2} (x - y) \right),$$

and rewrote (1.3) to the symmetric form

$$\partial_t v + (\partial_x^2 + \partial_y^2) v = 4 \cdot \frac{\sqrt{3}}{2} (\partial_x + \partial_y) (v^2), \quad t > 0, \; (x, y) \in \mathbb{R}^2.$$  

Such transform is introduced by Artzi, Koch, and Saut in [1]. We note that the well-posedness of (1.3) in $H^s(\mathbb{R}^2)$ is equivalent to the well-posedness of (1.4) in $H^s(\mathbb{R}^2)$. This transform is not essentially needed to obtain the well-posedness (Actually, Molinet and Pilod did not used such transform), but the symmetry helps us to find the structure of the equation and to write some parts of proof simply. Well-posedness of (1.3) for $s \leq 1/2$ is still open. But, Kinoshita gave the author the comment that there is a counter example for the $C^2$-well-posedness of (1.4) in $H^s(\mathbb{R}^2)$ for $s < -1/4$. His counter example is given as

$$\tilde{u}_0(\xi, \eta) := N^{-s+\frac{1}{2}} (\chi_A(\xi, \eta) + \chi_B(\xi, \eta)).$$
The initial-boundary problem of ZKB equation is studied by Larkin ([15], [16]). There is no results for the well-posedness of (1.1) as far as we know. But the also considered the high dimensional cases and obtained more general results.

ξ, η and the leading term of Φ( the well-posedness of this equation with η the dimensional dissipative KdV type equation

\[ \|\langle \xi, \eta \rangle \|_{H^s} \sim 1 \] and

\[ \sup_{0 < t < T} \left\| \int_0^t e^{-i(t-t')(\partial_x^2 + \partial_y^2)} \partial_x + \partial_y \left( (e^{-i(t' \partial_x^2 + \partial_y^2)} u_0)^2 \right) dt' \right\|_{H^s} \gtrsim N^{-s-\delta}. \]

While for the nonlinear parabolic equation

\[ \partial_t u - \Delta u = P(D)F(u), \quad t > 0, \ (x, y) \in \mathbb{R}^d, \]

Ribaud ([23]) obtained some well-posedness results. His results contain that the well-posedness of the 2D nonlinear parabolic equation

\[ \partial_t u - (\partial_x^2 + \partial_y^2) u = \partial_x (u^2), \quad t > 0, \ (x, y) \in \mathbb{R}^2 \] (1.5)

in \( H^s(\mathbb{R}^2) \) for \( s > 0 \) and nonuniqueness of solution for \( s < 0 \). Therefore, our interest is the well-posedness of (1.1) in \( H^s(\mathbb{R}^2) \) for lower \( s \) than both \(-1/4\) and 0.

Here, we introduce the results for 2D dispersive-dissipative models. The KP-Burgers equation

\[ \partial_x \left( \partial_t u + \partial_x^2 u - \partial_x^2 u - \partial_x (u^2) \right) + \epsilon \partial_y^2 u = 0, \quad t > 0, \ (x, y) \in \mathbb{R}^2, \quad \epsilon \in \{-1, 1\}, \]

is also two dimensional model of KdV-Burgers equation. We call KP-Burgers equation “KP-I-Burgers equation” if \( \epsilon = -1 \), and “KP-II-Burgers equation” if \( \epsilon = 1 \). The well-posedness of KP-Burgers equation is obtained in \( H^{s,0}(\mathbb{R}^2) \) for \( s > -1/2 \) by Kojok in [14] (for \( \epsilon = 1 \)) and Mohamad in [18] (for \( \epsilon = -1 \)). Where \( H^{s,0}(\mathbb{R}^2) \) is anisotropic Sobolev space defined by the norm \( \|f\|_{H^{s,0}} = \|\xi^s \hat{f}(\xi, \eta)\|_{L_{x,y}^{s,0}} \). Carvajal, Esfahani, and Panthee ([4]) considered the two dimensional dissipative KdV type equation

\[ \partial_t u + \partial_x^2 u + L_{x,y} u + \partial_x (u^2) = 0, \quad t > 0, \ (x, y) \in \mathbb{R}^2, \]

where the operator \( L_{x,y} \) is defined by

\[ \mathcal{F}_{xy}[L_{x,y}f](\xi, \eta) = -\Phi(\xi, \eta) \hat{f}(\xi, \eta) \]

and the leading term of \( \Phi(\xi, \eta) = -(|\xi|^{p_1} + |\eta|^{p_2}) \) with \( p_1, p_2 > 0 \). They obtained the well-posedness of this equation with \( p_2 > 1 \) in \( H^{s,0}(\mathbb{R}^2) \) for \( s > -3/4 \). They also considered the high dimensional cases and obtained more general results.

There is no results for the well-posedness of (1.1) as far as we know. But the initial-boundary problem of ZKB equation is studied by Larkin ([15], [16]).
Now, we give the main results in this paper. To begin with, we rewrite (1.1) to the symmetric form based on (9). We put
\[ v(t,x,y) = 4u(16t, 2(x+y), 2\sqrt{3}^{-1}(x-y)). \]
Then, (1.1) can be rewritten
\[
\begin{aligned}
\partial_t v + (\partial_x^2 + \partial_y^2)v - (\partial_x + \partial_y)^2 v &= (\partial_x + \partial_y)^2 v, \\
v(0, x, y) &= v_0(x, y) := 4u_0(2(x+y), 2\sqrt{3}^{-1}(x-y)).
\end{aligned}
\]
We note that the well-posedness of (1.1) in \( H^s(\mathbb{R}^2) \) is equivalent to the well-posedness of (1.6) in \( H^s(\mathbb{R}^2) \). Therefore, we consider (1.6) instead of (1.1).

**Theorem 1.1.** Let \( s > -\frac{1}{2} \). Then (1.6) is locally well-posed in \( H^s(\mathbb{R}^2) \). (Therefore (1.1) is also locally well-posed in \( H^s(\mathbb{R}^2) \).) More precisely, for any \( v_0 \in H^s(\mathbb{R}^2) \), there exist \( T > 0 \), and an unique solution \( v \in X_T^{s,\frac{1}{2}} \) \( \hookrightarrow C([0, T]; H^s(\mathbb{R}^2)) \) (See, Definition 2.1) to (1.6) in \([0, T] \). Furthermore, the data-to-solution map is Lipschitz continuous from \( H^s(\mathbb{R}^2) \) to \( C([0, T]; H^s(\mathbb{R}^2)) \).

**Theorem 1.2.** Let \( s > -\frac{1}{2} \). For any \( v_0 \in \tilde{H}^s(\mathbb{R}^2) \), the solution \( v \) obtained in Theorem 1.1 can be extended globally in time and \( v \) belongs to \( C((0, \infty); \tilde{H}^s(\mathbb{R}^2)) \), where \( \tilde{H}^s(\mathbb{R}^2) \) is the completion of the Schwartz class \( S(\mathbb{R}^2) \) with the norm \( \|f\|_{\tilde{H}^s} = \|\xi + \eta\|^s \hat{f}(\xi, \eta)\|_{L^2_{\xi \eta}} \), and \( \tilde{H}^s(\mathbb{R}^2) = \bigcap_{s \in \mathbb{R}} \tilde{H}^s(\mathbb{R}^2) \).

**Remark 1.3.** (i) Although (1.1) does not have the dissipative term with respect to \( y \), the well-posedness of (1.1) is obtained in isotropic Sobolev space \( H^s(\mathbb{R}^2) \) for lower regularity than both (1.3) and (1.5).

(ii) Theorem 1.1 says that (1.6) is globally well-posed in \( H^{s,0}(\mathbb{R}^2) \) for \( s > -\frac{1}{2} \).

To obtain Theorem 1.1 we have to treat the dissipative term carefully, because the symbol \( (\xi + \eta)^2 \) is vanished on the line \( \{ (\xi, -\xi) \mid \xi \in \mathbb{R} \} \). But the nonlinear term is also vanished on the same line. It helps us to obtain the key bilinear estimate (Proposition 3.1). We will use the iteration argument with the Fourier restriction norm to obtain the local well-posedness. While, the global well-posedness will be proved by using the smoothing effect from the dissipative term and non-increasing of \( L^2 \)-norm of the solution.

**Notation.** We denote the spatial Fourier transform by \( \hat{\cdot} \) or \( \mathcal{F}_{xy} \), the Fourier transform in time by \( \mathcal{F}_t \), and the Fourier transform in all variables by \( \hat{\cdot} \) or \( \mathcal{F} \).

The operator \( U(t) = e^{-it(\partial_x^2 + \partial_y^2)} \) and \( W(t) = e^{i|t|(\partial_x + \partial_y)^2} e^{-it(\partial_x^2 + \partial_y^2)} \) on \( H^s(\mathbb{R}^2) \) is given as a Fourier multiplier
\[
\mathcal{F}_{xy}[U(t)f](\xi, \eta) = e^{i(t(\xi^2 + \eta^2))} \hat{f}(\xi), \quad \mathcal{F}_{xy}[W(t)f](\xi, \eta) = e^{-i(t(\xi + \eta)^2)} e^{i(t(\xi^2 + \eta^2))} \hat{f}(\xi).
\]
\( U(t) \) and \( W(t) \) give a solution to
\[
\partial_t u + (\partial_x^2 + \partial_y^2)u = 0
\]
and
\[
\partial_t u + (\partial_x^2 + \partial_y^2)u - \text{sgn}(t)(\partial_x + \partial_y)^2 u = 0
\]
respectively. We note that \( \mathcal{F}[U(-\cdot)F(\cdot)](\tau, \xi, \eta) = \hat{F}(\tau + \xi^3 + \eta^3, \xi, \eta) \).
We will use \( A \lesssim B \) to denote an estimate of the form \( A \leq CB \) for some constant \( C \) and write \( A \approx B \) to mean \( A \approx B \) and \( A \approx A \). We will use the convention that capital letters denote dyadic numbers, e.g. \( N = 2^n \) for \( n \in \mathbb{Z} \) and for a dyadic summation we write \( \sum_{N \in \mathbb{Z}, 2^n \geq N} a_N := \sum_{n \in \mathbb{Z}, 2^n \geq N} a_{2^n} \), \( \sum_{N \in \mathbb{Z}, 2^n \leq N} a_N := \sum_{n \in \mathbb{Z}, 2^n \leq N} a_{2^n} \) for brevity. Let \( \chi \in C_0^\infty((-2,2)) \) be an even, non-negative function such that \( \chi(t) = 1 \) for \( |t| \leq 1 \). We define \( \varphi(t) := \chi(t) - \chi(2t) \) and \( \varphi_N(t) := \varphi(N^{-1}t) \). Then, \( \sum_N \varphi_N(t) = 1 \) whenever \( t \neq 0 \). We define the projections

\[
\begin{align*}
\widetilde{P}_N u(\xi, \eta) &:= \varphi_N(\langle \xi, \eta \rangle) \hat{u}(\xi, \eta), \\
\widetilde{Q}_L u(\tau, \xi, \eta) &:= \varphi_L(\tau - \xi^3 - \eta^3) \hat{u}(\tau, \xi, \eta),
\end{align*}
\]

where \( \varphi_{N,M}(\xi, \eta) := \varphi_N(\langle \xi, \eta \rangle) \varphi_M(\xi + \eta) \).

The rest of this paper is planned as follows. In Section 2, we will give the definition of the solution space, and prove the linear estimates. In Section 3, we will prove the bilinear estimate which is main part of this paper. In Section 4, we will give the proof of the well-posedness (Theorems 1.1 and 1.2).

## 2 Function space and linear estimate

In this section, we define the function space, and prove the estimate for linear solution and Duhamel term. First, we consider the standard Fourier restriction norm \( \|u\|_{X_{s,b}} \) for \( s,b \) defined by

\[
\|u\|_{X_{s,b}} = \|\langle \xi, \eta \rangle^s \langle \xi + \eta \rangle^2 + i(\tau - \xi^3 - \eta^3)\hat{u}(\tau, \xi, \eta)\|_{L_{t\xi}^2}.
\]

Such Fourier restriction norm was introduced by J. Bourgain \([3]\) for the non-linear Schrödinger equation and the KdV equation. Let \( \psi \in C_0^\infty(\mathbb{R}) \) denotes a cut-off function such that supp \( \psi \subset [-2,2] \), \( \psi = 1 \) on \([-1,1]\). We note that, the estimate

\[
\|\psi(t)W(t)u_0\|_{X_{s,b}} \lesssim \|\langle \xi, \eta \rangle^s \langle \xi + \eta \rangle^b \hat{u}_0(\xi, \eta)\|_{L_{t\xi}^2}
\]

holds. Therefore, if \( b \leq 1/2 \), then \( \psi W(\cdot)u_0 \in X_{s,b} \) for \( u_0 \in H^s \). But the embedding \( X_{s,b} \hookrightarrow C(\mathbb{R}; H^s(\mathbb{R}^2)) \) does not hold for \( b \leq 1/2 \). Therefore, we use the Besov type Fourier restriction norm defined as follows.

**Definition 2.1.** Let \( s \in \mathbb{R}, b \in \mathbb{R} \).

(i) We define the function space \( X_{s,b}^{1,1} \) as the completion of the Schwartz class \( \mathcal{S}(\mathbb{R} \times \mathbb{R}^2_{\xi\eta}) \) with the norm

\[
\|u\|_{X_{s,b}^{1,1}} = \left\{ \sum_{N \in 2^\mathbb{Z}} \sum_{M \in 2^\mathbb{Z}} \left( \sum_{L \in 2^\mathbb{Z}} (N)^s (M^2 + L^4) \|P_{N,M}Q_L u\|_{L_{t\xi}^2} \right) \right\}^{\frac{1}{2}}.
\]

(ii) For \( T > 0 \), we define the time localized space \( X_{s,b}^{1,1}(T) \) as

\[
X_{s,b}^{1,1}(T) = \{ u|_{[0,T]} | u \in X_{s,b}^{1,1} \}
\]

with the norm

\[
\|u\|_{X_{s,b}^{1,1}(T)} = \inf \{ \|v\|_{X_{s,b}^{1,1}} | v \in X_{s,b}^{1,1}, v|_{[0,T]} = u|_{[0,T]} \}.
\]
Remark 2.2. (i) The embedding $X_T^{s, \frac{1}{2}, 1} \hookrightarrow C([0, T]; H^s(\mathbb{R}^2))$ holds.
(ii) The size of $|\xi + \eta|$, which comes from the symbol of the dissipative term of \( L_0 \), is not decided by the size of $|(\xi, \eta)|$. Therefore, to use the dissipative effect strictly, we focus on not only $|(\xi, \eta)| \sim N$, but also $|\xi + \eta| \sim M$. This is a different point from 1D case.
(iii) We can assume $\sum_{M \in \mathbb{Z}} = \sum_{M \leq N}$ since $|\xi + \eta| \lesssim |(\xi, \eta)|$ holds.

We choose $X_T^{s, \frac{1}{2}, 1}$ as the solution space. Now, we define the operator $K$ and $L$ by

\[
K \psi(t)(\xi, \eta) := \int_{\mathbb{R}^2} e^{it\tau} - e^{-|t||\xi + \eta|^2} F(\psi(t)(\cdot, \xi, \eta))d\tau
\]

\[
L \psi(t) := U(t) \int_{\mathbb{R}^2} e^{ix\xi} e^{iy\eta} K \psi(t)(\xi, \eta) d\xi d\eta = U(t) \mathcal{F}^{-1}_{\mathbb{Z}^2}[K \psi(t)].
\]

Then, we note that

\[
L \psi(t) = \int_0^t W(t - t') F(t') dt'
\]

holds for $t \geq 0$ and the integral form of \( L_0 \) on $[0, \infty)$ is given by

\[
v(t) = W(t)v_0 + \int_0^t W(t - t')(\partial_x + \partial_y)(v(t')^2) dt'
\]

(2.1)

Proposition 2.3. Let $s \in \mathbb{R}$. There exists $C_1 > 0$, such that for any $u_0 \in H^s(\mathbb{R}^2)$, we have

\[
\| \psi(t)W(t)u_0 \|_{X_T^{s, \frac{1}{2}, 1}} \leq C_1 \| u_0 \|_{H^s}.
\]

Proof. Since

\[
\left( \sum_N \sum_M (N)^{2s} \| P_{N,M} u_0 \|^2_{L^2_x} \right)^{\frac{1}{2}} \sim \| u_0 \|_{H^s},
\]

holds, it suffice to prove

\[
\sum_L (M^2 + L)^{\frac{1}{2}} \| P_{N,M} Q_L(\psi(t)W(t)u_0) \|_{L^2_{xy}} \lesssim \| P_{N,M} u_0 \|_{L^2_{xy}}
\]

for each $N, M \in \mathbb{Z}$. By using Plancherel’s theorem, we have

\[
\| P_{N,M} Q_L(\psi(t)W(t)u_0) \|_{L^2_{xy}} \sim \| \varphi_N \varphi_L \varphi(t) e^{-|t|(|\xi + \eta|^2)} u_0(\xi, \eta) \|_{L^2_{xy}}
\]

\[
\lesssim \| P_{N,M} u_0 \|_{L^2_{xy} \varphi} \| \varphi_N(\xi + \eta) \varphi_L(\tau) \mathcal{F}[\psi(t)e^{-i|\xi + \eta|^2}] \|_{L^2_{\xi \tau}}
\]

\[
= \| P_{N,M} u_0 \|_{L^2_{xy} \varphi} \| \varphi_M(\xi) \varphi_L(\tau) \mathcal{F}[\psi(t)e^{-i|\xi|^2}] \|_{L^2_{\xi \tau}},
\]

where $\phi_M = \varphi_M + \varphi_M + \varphi_M$ and we used $\varphi_M = \varphi_M \varphi_M$. Therefore, it suffice to prove

\[
\sum_L (M^2 + L)^{\frac{1}{2}} \| \varphi_M(\xi) \varphi(t) e^{-|t| |\xi|^2} \|_{L^2_{\xi}} \lesssim 1.
\]

(2.2)

It is obtained in the proof of Proposition 4.1 in [21].
**Proposition 2.4.** Let $s \in \mathbb{R}$. There exists $C_2 > 0$, such that for any $F \in X^{s-\frac{3}{2},1}$, we have

$$
\|\psi(t)\mathcal{L}F(t)\|_{X^{s-\frac{3}{2},1}} \leq C_2 \|F\|_{X^{s-\frac{3}{2},1}}
$$

**Proof.** We use the argument in the proof of Lemma 4.1 in [21]. Since

$$
\|P_{N,M}Q_L(\psi(t)\mathcal{L}F(t))\|_{L^2_{\xi\eta}} \sim \|\varphi_{N,M}(\xi,\eta)\varphi_L(\tau)F_1[\psi\mathcal{K}F](\tau,\xi,\eta)\|_{L^2_{\xi\eta}},
$$

it suffice to show that

$$
\sum_{L}(M^2 + L)^{\frac{7}{2}}\|\varphi_{N,M}(\xi,\eta)\varphi_L(\tau)F_1[\psi\mathcal{K}F](\tau,\xi,\eta)\|_{L^2_{\xi\eta}}
\lesssim \sum_{L}(M^2 + L)^{-\frac{7}{2}}\|\varphi_{N,M}(\xi,\eta)\varphi_L(\tau)F_1[\psi\mathcal{K}F](\tau,\xi,\eta)\|_{L^2_{\xi\eta}}
$$

(2.3)

We put $w(t) = U(-t)F(t)$ and split $\psi\mathcal{K}F$ into $K_1 + K_2 + K_3 - K_4$, where

$$
K_1(t, \xi, \eta) = \psi(t) \int_{|\tau| \leq 1} \frac{e^{it\tau} - 1}{(\xi + \eta)^2 + i\tau} \tilde{w}(\tau, \xi, \eta)d\tau,
$$

$$
K_2(t, \xi, \eta) = \psi(t) \int_{|\tau| \leq 1} \frac{1 - e^{-i|\xi + \eta|^2}}{(\xi + \eta)^2 + i\tau} \tilde{w}(\tau, \xi, \eta)d\tau,
$$

$$
K_3(t, \xi, \eta) = \psi(t) \int_{|\tau| \geq 1} \frac{e^{it\tau}}{(\xi + \eta)^2 + i\tau} \tilde{w}(\tau, \xi, \eta)d\tau,
$$

$$
K_4(t, \xi, \eta) = \psi(t) \int_{|\tau| \geq 1} \frac{e^{-i|\xi + \eta|^2}}{(\xi + \eta)^2 + i\tau} \tilde{w}(\tau, \xi, \eta)d\tau.
$$

Furthermore, we put $w_{N,M} = P_{N,M}w$. We note that $\tilde{w}_{N,M}(\tau, \xi, \eta) = \phi_M(\xi + \eta)\tilde{w}_{N,M}(\tau, \xi, \eta)$ since $\varphi_M = \varphi_M\phi_M$.

**Estimate for $K_1$**

By using the Taylor expansion, we have

$$
\|\varphi_{N,M}(\xi,\eta)\varphi_L(\tau)F_1[K_1(\tau,\xi,\eta)]\|_{L^2_{\xi\eta}}
\lesssim \sum_{n=1}^{\infty} \frac{1}{n!} \left\| \left( \int_{|\tau| \leq 1} \frac{|\tau|^n|\tilde{w}_{N,M}(\tau,\xi,\eta)|}{(\xi + \eta)^2 + |\tau|} d\tau \right) \|\varphi_L(\tau)F_1[t^n\psi(t)](\tau)\|_{L^2_{\xi\eta}} \right\|_{L^2_{\xi\eta}}.
$$

By the Cauchy-Schwartz inequality, we obtain

$$
\int_{|\tau| \leq 1} \frac{|\tau|^n|\tilde{w}_{N,M}(\tau,\xi,\eta)|}{(\xi + \eta)^2 + |\tau|} d\tau
\lesssim \left( \int_{|\tau| \leq 1} \frac{|\tau|^2((\xi + \eta)^2 + |\tau|)^2}{(\xi + \eta)^2 + |\tau|} \phi_M(\xi + \eta)^2 d\tau \right)^{\frac{1}{2}} \left( \int_{|\tau| \leq 1} \frac{|\tilde{w}_{N,M}(\tau,\xi,\eta)|^2}{((\xi + \eta)^2 + |\tau|)^2} d\tau \right)^{\frac{1}{2}}
\lesssim (M)^{-1} \sum_{L} (M^2 + L)^{-\frac{1}{2}} \|\varphi_L(\tau)\tilde{w}_{N,M}(\tau,\xi,\eta)\|_{L^2_{\xi\eta}}
$$
for n ≥ 1. Therefore, we get
\[
\sum_L (M^2 + L)^{\frac{1}{2}} \| \varphi_{N,M}(\xi,\eta)\varphi_L(\tau)F_t[K_1](\tau,\xi,\eta) \|_{L^2_{\xi \eta}} \leq \sum_{n=1}^{\infty} \frac{1}{n!} \| |t|^n \psi \|_{B_{2,1}^{\frac{1}{2}}} \sum_L (M^2 + L)^{-\frac{1}{2}} \| \varphi_L(\tau)\tilde{w}_{N,M}(\tau,\xi,\eta) \|_{L^2_{\xi \eta}} \leq \sum_L (M^2 + L)^{-\frac{1}{2}} \| \varphi_L(\tau)\tilde{w}_{N,M}(\tau,\xi,\eta) \|_{L^2_{\xi \eta}}
\]

since \( (M^2 + L)^{\frac{1}{2}} (M)^{-1} \lesssim (L)^{\frac{1}{2}} \).

**Estimate for \( K_2 \)**

By Plancherel’s theorem, we have
\[
\| \varphi_{N,M}(\xi,\eta)\varphi_L(\tau)F_t[K_2](\tau,\xi,\eta) \|_{L^2_{\xi \eta}} \lesssim \left\| \left( \int_{|\tau| \leq 1} \frac{|\tilde{w}_{N,M}(\tau,\xi,\eta)|}{(\xi + \eta)^2 + |\tau|^2} \right) \| \phi_M(\xi + \eta)\varphi_L(\tau)F_t[\psi(t)(1 - e^{-|t|^2(\xi + \eta)^2})](\tau) \|_{L^2_{\xi \eta}} \right\|_{L^2_{\xi \eta}}.
\]

By the Cauchy-Schwartz inequality, we obtain
\[
\int_{|\tau| \leq 1} \frac{|\tilde{w}_{N,M}(\tau,\xi,\eta)|}{(\xi + \eta)^2 + |\tau|^2} \| \phi_M(\xi + \eta)\varphi_L(\tau)F_t[\psi(t)(1 - e^{-|t|^2(\xi + \eta)^2})](\tau) \|_{L^2_{\xi \eta}} \lesssim M^{-2}(M) \sum_L (M^2 + L)^{-\frac{1}{2}} \| \varphi_L(\tau)\tilde{w}_{N,M}(\tau,\xi,\eta) \|_{L^2_{\xi \eta}}
\]

Therefore if \( M \geq 1 \), then we get
\[
\sum_L (M^2 + L)^{\frac{1}{2}} \| \varphi_{N,M}(\xi,\eta)\varphi_L(\tau)F_t[K_2](\tau,\xi,\eta) \|_{L^2_{\xi \eta}} \lesssim \sum_L (M^2 + L)^{-\frac{1}{2}} \| \varphi_L(\tau)\tilde{w}_{N,M}(\tau,\xi,\eta) \|_{L^2_{\xi \eta}}
\]

by \( 2.2 \) and
\[
\sum_L (M^2 + L)^{\frac{1}{2}} \| \varphi_L(\tau)F_t[\psi](\tau) \|_{L^2_{\xi \eta}} \lesssim M \| \psi \|_{B_{2,1}^{\frac{1}{2}}} \lesssim M.
\]

While if \( M \leq 1 \), then by using the Taylor expansion, we have
\[
\| \phi_M(\xi + \eta)\varphi_L(\tau)F_t[\psi(t)(1 - e^{-|t|^2(\xi + \eta)^2})](\tau) \|_{L^2_{\xi \eta}} \lesssim \sum_{n=1}^{\infty} \frac{(\xi + \eta)^{2n}}{n!} \phi_M(\xi + \eta)\varphi_L(\tau)F_t[\psi(t)|t|^n](\tau) \|_{L^2_{\xi \eta}} \lesssim M^2 \sum_{n=1}^{\infty} \frac{1}{n!} \| \varphi_L(\tau)F_t[\psi(t)|t|^n](\tau) \|_{L^2_{\xi \eta}}.
\]
Therefore, we get
\[
\sum_{L} (M^2 + L)^{\frac{1}{2}} \| \varphi_{N,M}(\tau, \xi, \eta) \varphi_{L}(\tau) \mathcal{F}_{l}[K_2](\tau, \xi, \eta) \|_{L^2_{\xi,\eta}} \\
\lesssim \sum_{n=1}^{\infty} \frac{1}{n!} \| |\tau|^n \psi | \|_{B^{\frac{1}{2}, 1}_{2,1}} \langle M^2 + L \rangle^{\frac{1}{2}} \| \varphi_{L}(\tau) \tilde{w}_{N,M}(\tau, \xi, \eta) \|_{L^2_{\xi,\eta}} \\
\lesssim \sum_{L} (M^2 + L)^{-\frac{1}{2}} \| \varphi_{L}(\tau) \tilde{w}_{N,M}(\tau, \xi, \eta) \|_{L^2_{\xi,\eta}}.
\]

Estimate for $K_3$

We put $g_{N,M}(t) = \mathcal{F}_n^{-1}[\mathbf{1}_{|\tau| \geq 1}((\xi + \eta)^2 + i\tau)^{-1} \tilde{w}_{N,M}(\tau, \xi, \eta)](t)$. Then, we have
\[
| \varphi_{N,M}(\xi, \eta) \varphi_{L}(\tau) \mathcal{F}_{l}[K_3](\tau) | \sim | \varphi_{L}(\tau) (\mathcal{F}_{l}[\psi] * \mathcal{F}_{l}[g_{N,M}(\tau)]) | \\
\lesssim \sum_{L_1} \sum_{L_2} | \varphi_{L_1}(\tau) (\varphi_{L_2} \mathcal{F}_{l}[\psi]) *_{\tau} (\varphi_{L_2} \mathcal{F}_{l}[g_{N,M}]) |.
\]

(i) Summation for $L_1 \ll L$ (then, $L_2 \sim L$)

By the Young inequality, we have
\[
| \varphi_{L_1}(\tau) (\varphi_{L_2} \mathcal{F}_{l}[\psi]) *_{\tau} (\varphi_{L_2} \mathcal{F}_{l}[g_{N,M}]) | \\
\lesssim | \varphi_{L_1}(\tau) \mathcal{F}_{l}[\psi](\tau) \|_{L^1} \| \varphi_{L_2}(\tau) \mathcal{F}_{l}[g_{N,M}(\tau)] \|_{L^2} \\
\lesssim | \varphi_{L_1}(\tau) \mathcal{F}_{l}[\psi](\tau) \|_{L^1} (M^2 + L_2)^{\frac{1}{2}} \| \varphi_{L_2}(\tau) \tilde{w}_{N,M}(\tau, \xi, \eta) \|_{L^2_{\xi,\eta}}.
\]

Therefore, we obtain
\[
\sum_{L_1} \langle M^2 + L \rangle^{\frac{1}{2}} \sum_{L_1 \ll L, L_2 \sim L} \| \varphi_{L_1}(\tau) (\varphi_{L_2} \mathcal{F}_{l}[\psi]) *_{\tau} (\varphi_{L_2} \mathcal{F}_{l}[g_{N,M}]) \|_{L^2_{\xi,\eta}} \\
\lesssim \left( \sum_{L_1} \| \varphi_{L_1}(\tau) \mathcal{F}_{l}[\psi](\tau) \|_{L^1} \right) \left( \sum_{L_2} \| (M^2 + L_2)^{\frac{1}{2}} | \varphi_{L_2}(\tau) \tilde{w}_{N,M}(\tau, \xi, \eta) \|_{L^2_{\xi,\eta}} \right) \\
\lesssim \sum_{L_2} (M^2 + L_2)^{-\frac{1}{2}} \| \varphi_{L_2}(\tau) \tilde{w}_{N,M}(\tau, \xi, \eta) \|_{L^2_{\xi,\eta}}.
\]

since
\[
\sum_{L_1} \| \varphi_{L_1}(\tau) \mathcal{F}_{l}[\psi](\tau) \|_{L^1} \lesssim \sum_{L_1} L_1^{\frac{1}{2}} \| \varphi_{L_1}(\tau) \mathcal{F}_{l}[\psi](\tau) \|_{L^2} \lesssim \| \psi \|_{B_{2,1}^{\frac{1}{2}}} \lesssim 1.
\]

(ii) Summation for $L \lesssim M^2, L_1 \gtrsim L$.

By the H"older inequality and the Young inequality, we have
\[
| \varphi_{L}(\tau) (\varphi_{L_1} \mathcal{F}_{l}[\psi]) *_{\tau} (\varphi_{L_2} \mathcal{F}_{l}[g_{N,M}]) | \\
\lesssim \| \varphi_{L} \|_{L^2} \| \varphi_{L_1}(\tau) \mathcal{F}_{l}[\psi](\tau) \|_{L^2} \| \varphi_{L_2}(\tau) \mathcal{F}_{l}[g_{N,M}(\tau)] \|_{L^2} \\
\lesssim L_1^{\frac{1}{2}} \| \varphi_{L_1}(\tau) \mathcal{F}_{l}[\psi](\tau) \|_{L^2} (M^2 + L_2)^{-\frac{1}{2}} \| \varphi_{L_2}(\tau) \tilde{w}_{N,M}(\tau, \xi, \eta) \|_{L^2_{\xi,\eta}}.
\]
\[
\sum_{L \geq M^2} \langle M^2 + L \rangle^{\frac{1}{2}} \sum_{L \geq L_1} \sum_{L_2} \| \varphi_L(\tau)(\varphi_L \mathcal{F}_i \psi) \rangle \ast \tau (\varphi_L \mathcal{F}_i \hat{g}_{N,M})(\tau)\|_{L^2_{\xi \eta r}} \\
\lesssim \langle M \rangle \left( \sum_{L_1} L_1^{-\frac{1}{2}} \| \varphi_{L_1}(\mathcal{F}_i \psi) \rangle \|_{L^2_\xi} \right) \left( \sum_{L_2} \langle M^2 + L_2 \rangle^{-\frac{1}{2}} \| \varphi_{L_2}(\tau) \hat{w}_{N,M}(\tau, \xi, \eta)\|_{L^2_{\xi \eta r}} \right) \\
\lesssim \sum_{L_2} \langle M^2 + L_2 \rangle^{-\frac{1}{2}} \| \varphi_{L_2}(\tau) \hat{w}_{N,M}(\tau, \xi, \eta)\|_{L^2_{\xi \eta r}} 
\] 

since \( \langle M \rangle \lesssim \langle M^2 + L_2 \rangle^{\frac{1}{2}} \) and
\[
\sum_{L_1} L_1^{\frac{1}{2}} \| \varphi_{L_1}(\mathcal{F}_i \psi) \rangle \|_{L^2_\xi} \lesssim \| \psi \|_{B^\frac{1}{2}_\infty} \lesssim 1.
\]

(iii) Summation for \( L_1 \gtrsim L \gtrsim M^2 \). By the Young inequality and the Cauchy-Schwartz inequality, we have
\[
\| \varphi_L(\tau)(\varphi_L \mathcal{F}_i \psi) \rangle \ast \tau (\varphi_L \mathcal{F}_i \hat{g}_{N,M})(\tau)\|_{L^2_\xi} \\
\lesssim \| \varphi_{L_1}(\mathcal{F}_i \psi) \rangle \|_{L^2_\xi} \| \varphi_{L_2}(\tau) \mathcal{F}_i \hat{g}_{N,M}(\tau)\|_{L^2_\xi} \\
\lesssim \| \varphi_{L_1}(\tau) \mathcal{F}_i \psi(\tau)\|_{L^2_\xi} \langle M^2 + L_2 \rangle^{-\frac{1}{2}} \| \varphi_{L_2}(\tau) \hat{w}_{N,M}(\tau, \xi, \eta)\|_{L^2_{\xi \eta r}}.
\]
Therefore, we obtain
\[
\sum_{L \gtrsim M^2} \langle M^2 + L \rangle^{\frac{1}{2}} \sum_{L \geq L_1} \sum_{L_2} \| \varphi_L(\tau)(\varphi_L \mathcal{F}_i \psi) \rangle \ast \tau (\varphi_L \mathcal{F}_i \hat{g}_{N,M})(\tau)\|_{L^2_{\xi \eta r}} \\
\lesssim \left( \sum_{L_1} L_1^{-\frac{1}{2}} \| \varphi_{L_1}(\mathcal{F}_i \psi) \rangle \|_{L^2_\xi} \right) \left( \sum_{L_2} \langle M^2 + L_2 \rangle^{-\frac{1}{2}} \| \varphi_{L_2}(\tau) \hat{w}_{N,M}(\tau, \xi, \eta)\|_{L^2_{\xi \eta r}} \right) \\
\lesssim \sum_{L_2} \langle M^2 + L_2 \rangle^{-\frac{1}{2}} \| \varphi_{L_2}(\tau) \hat{w}_{N,M}(\tau, \xi, \eta)\|_{L^2_{\xi \eta r}} 
\] 

since
\[
\sum_{L_2} \langle L_1 \rangle^{\frac{1}{2}} \| \varphi_{L_1}(\mathcal{F}_i \psi) \rangle \|_{L^2_\xi} \lesssim \| \psi \|_{B^\frac{1}{2}_\infty} \lesssim 1.
\]

Estimate for \( K_4 \)

By Plancherel’s theorem, we have
\[
\| \varphi_{N,M}(\xi, \eta) \varphi_L(\tau) \mathcal{F}_i[K_4](\tau, \xi, \eta)\|_{L^2_{\xi \eta r}} \\
\lesssim \left\| \left( \int_{|\tau| \geq 1} \frac{1}{(|\xi + \eta|^2 + |\tau|)} \| \phi_{M}(\xi + \eta) \varphi_L(\tau) \mathcal{F}_i[\psi(t)] e^{-|t|(|\xi + \eta|^2)}(\tau)\|_{L^2_\xi} \right) \right\|_{L^2_{\xi \eta}}.
\]

By the Cauchy-Schwartz inequality, we obtain
\[
\int_{|\tau| \geq 1} \frac{1}{(|\xi + \eta|^2 + |\tau|)} d\tau \lesssim \sum_{L} \langle M^2 + L \rangle^{-1} \| \varphi_L(\tau) \hat{w}_{N,M}(\tau, \xi, \eta)\|_{L^1} \\
\lesssim \sum_{L} \langle M^2 + L \rangle^{-\frac{1}{2}} \| \varphi_L(\tau) \hat{w}_{N,M}(\tau, \xi, \eta)\|_{L^2}.
\]
Therefore, by (2.2), we get
\[
\sum_L (M^2 + L) \frac{1}{2} \| \varphi_{N,M}(\xi,\eta) \varphi_L(\tau) \mathcal{F}_x[K_4](\tau,\xi,\eta) \|_{L_2^\tau}^2 \leq \sum_L (M^2 + L) \frac{1}{2} \| \varphi_L(\tau) \tilde{w}_{N,M}(\tau,\xi,\eta) \|_{L_2^\tau}^2.
\]
\[
\square
\]

3 Bilinear estimate

In this section, we prove the estimate for nonlinear terms as follows.

**Proposition 3.1.** Let \( s \geq s_0 > -\frac{1}{2} \). There exist \( 0 < \delta \ll 1 \) and \( C_3 > 0 \), such that for any \( u, v \in X^s \), we have
\[
\|(\partial_x + \partial_y)(uv)\|_{X^{s-\frac{1}{2}}} \leq C_3\|u\|_{X^s} \|\frac{1}{\sqrt{1 + \langle \xi^2 \rangle}} 1_{\delta} \|v\|_{X^s}.
\]

To prove Proposition 3.1, we first give some Strichartz estimates.

**Proposition 3.2.** Let \( (p, q) \in \mathbb{R}^2 \) satisfy \( p \geq 3 \) and \( \frac{1}{p} + \frac{2}{q} = 1 \). For any \( u_0 \in L^2(\mathbb{R}^2) \), we have
\[
\|U(t)u_0\|_{L_t^p L_x^q} \lesssim \|u_0\|_{L_x^2}.
\]

Proposition 3.2 is obtained by using the variable transform \((x, y) \mapsto (4^{-\frac{2}{3}}(x + \sqrt{3}y), 4^{-\frac{2}{3}}(x - \sqrt{3}y))\) in Proposition 2.4 in [17].

**Proposition 3.3.** For any \( u_0 \in L^2(\mathbb{R}^2) \), we have
\[
\|D_x^s D_y^s U(t)u_0\|_{L_t^p L_x^q} \lesssim \|u_0\|_{L_x^2},
\]
where \( D_x^s = \mathcal{F}_x^{-1}[|\xi|^s \mathcal{F}_{xy}] \), \( D_y^s = \mathcal{F}_x^{-1}[|\eta|^s \mathcal{F}_{xy}] \) for \( s \in \mathbb{R} \).

Proposition 3.3 is obtained by applying \( \Omega(\xi, \eta) = \xi^3 + \eta^3 \) in Corollary 3.4 in [19].

By using the same argument as in Lemma 2.3 in [8], we obtain the following estimates from Proposition 3.2 and Proposition 3.3.

**Corollary 3.4.** Let \( (p, q) \in \mathbb{R}^2 \) satisfy \( p \geq 3 \) and \( \frac{1}{p} + \frac{2}{q} = 1 \). For \( N, L \in 2\mathbb{Z} \), we have
\[
\|P_N P_L u\|_{L_t^p L_x^q} \lesssim L^{\frac{1}{2}} \|P_N P_L u\|_{L_t^p L_x^q}.
\]

Furthermore, if \( \mathcal{F}_{xy}[P_N u] \) is supported in \( \{\xi, \eta\} \sim |\xi| \sim |\eta|\}, \) then we have
\[
\|P_N P_L u\|_{L_t^p L_x^q} \lesssim N^{-\frac{1}{3}} L^{\frac{1}{2}} \|P_N P_L u\|_{L_t^p L_x^q}.
\]

To get a positive power of \( M \), we give the following estimates.

**Corollary 3.5.** Let \( 0 < \delta \ll 1 \), \( 0 < \epsilon < 1 - \delta \). For \( N, M, L \in 2\mathbb{Z} \), we have
\[
\|P_{N,M} P_L u\|_{L_t^p L_x^q} \lesssim (NM)^{\frac{1}{2}} L^{\frac{1}{2}(1-\delta)} \|P_{N,M} P_L u\|_{L_t^p L_x^q}.
\]

Furthermore, if \( \mathcal{F}_{xy}[P_N u] \) is supported in \( \{\xi, \eta\} \sim |\xi| \sim |\eta|\}, \) then we have
\[
\|P_{N,M} P_L u\|_{L_t^p L_x^q} \lesssim (NM)^{\frac{1}{2}} N^{-\frac{1}{3}(1-\delta-\epsilon)} L^{\frac{1}{2}(1-\delta)} \|P_N P_L u\|_{L_t^p L_x^q}.
\]
Proof. By \( 3.1 \) with \( p = q = 5 \), we have the \( L^5 \)-Strichartz estimate
\[
\| P_{N,M} Q_L u \|_{L^2_t L^6_x} \lesssim L^{7/2} \| P_{N,M} Q_L u \|_{L^{16}_x} .
\]
By the interpolation between (3.6) and (3.7), we obtain (3.3).

Next, we give the bilinear Strichartz estimates.

Proposition 3.6. Let \( R^{(j)}_K \) \((j = 1, 2)\) denote the bilinear operator defined by
\[
R^{(j)}_K(P_{N_1} Q_L u_1, P_{N_2} Q_L u_2) = \int \varphi_K(\xi_1 - (\xi - \xi_1)^2)\hat{u}_1(\xi, \eta)\hat{u}_2(\xi - \xi_1, \eta - \eta_1) d\xi d\eta,
\]
\[
R^{(2)}_K(P_{N_1} Q_L u_1, P_{N_2} Q_L u_2) = \int \varphi_K(\eta_2 - (\eta - \eta_1)^2)\hat{u}_1(\xi_1, \eta_1)\hat{u}_2(\xi - \xi_1, \eta - \eta_1) d\xi d\eta.
\]
For \( N_1, N_2, L_1, L_2, K \in 2\mathbb{Z} \) with \( N_1 \geq N_2 \), and \( j \in \{1, 2\} \), we have
\[
\| R^{(j)}_K(P_{N_1} Q_L u_1, P_{N_2} Q_L u_2) \|_{L^2_t L^6_x} \lesssim K^{-\frac{3}{4}} N_1^\frac{3}{2} L_1^\frac{1}{2} L_2^\frac{1}{2} \| P_{N_1} Q_L u_1 \|_{L^2_t L^6_x} \| P_{N_2} Q_L u_2 \|_{L^2_t L^6_x} .
\]

We only prove for \( j = 1 \) because the case \( j = 2 \) can be proved by the same way. We put \( f_i = F[P_{N_i} Q_L u_i] \), \( \zeta_i = (\xi_i, \eta_i) \) \((i = 1, 2)\). By the duality argument, it suffice to show that
\[
\left| \int_{\Omega} f_1(\tau_1, \eta_1) f_2(\tau_2, \eta_2) f(\tau_1 + \tau_2, \xi_1 + \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim K^{-\frac{3}{4}} N_1^\frac{3}{2} L_1^\frac{1}{2} L_2^\frac{1}{2} \| f_1 \|_{L^2_{\xi_1 \eta_1}} \| f_2 \|_{L^2_{\xi_2 \eta_2}} \| f \|_{L^6_{\xi_2 \eta_2}}
\]
for any \( f \in L^2(\mathbb{R} \times \mathbb{R}^2) \), where
\[
\Omega = \{(\tau_1, \tau_2, \xi_1, \xi_2) \mid |\xi_i \sim N_i, |\tau_i - \xi_i^3 - \eta_i^3| \sim L_i \(i = 1, 2\), |\xi_i - \xi_2^3| \sim K\}.\]
By the Cauchy-Schwartz inequality, we have
\[
\left| \int_{\Omega} f_1(\tau_1, \zeta_1) f_2(\tau_2, \zeta_2) f(\tau_1 + \tau_2, \zeta_1 + \zeta_2) d\tau_1 d\tau_2 d\zeta_1 d\zeta_2 \right| \lesssim \|f_1\|_{L^2_{r_0}} \|f_2\|_{L^2_{r_0}} \left( \int_{\Omega} |f(\tau_1 + \tau_2, \zeta_1 + \zeta_2)|^2 d\tau_1 d\tau_2 d\zeta_1 d\zeta_2 \right)^{\frac{1}{2}}. \tag{3.10}
\]

By applying the variable transform \((\tau_1, \tau_2) \mapsto (\theta_1, \theta_2)\) and \((\zeta_1, \zeta_2) \mapsto (\mu, w, z, \nu)\) as
\[
\theta_i = \tau_i - \xi_i^3 - \eta_i^3 \quad (i = 1, 2),
\mu = \theta_1 + \theta_2 + \xi_1^3 + \eta_1^3 + \eta_2^3, \quad w = \xi_1 + \xi_2, \quad z = \eta_1 + \eta_2, \quad \nu = \eta_2,
\]
we have
\[
\int_{\Omega} |f(\tau_1 + \tau_2, \zeta_1 + \zeta_2)|^2 d\tau_1 d\tau_2 d\zeta_1 d\zeta_2 \lesssim \int_{[\theta_1 \sim \xi_1^3 - \eta_1^3 \sim N_2]} \left( \int_{[\nu \sim N_2]} |f(\mu, w, z)|^2 1_{[|\xi_1^3 - \eta_2^3| \sim K]}(\xi_1, \xi_2) J(\xi_1, \xi_2)^{-1} d\mu dw dz dv \right) d\theta_1 d\theta_2,
\]
where
\[
J(\xi_1, \xi_2) = \left| \det \frac{\partial (\mu, w, z, \nu)}{\partial (\xi_1, \eta_1, \xi_2, \eta_2)} \right| = 3|\xi_1^2 - \xi_2^2|.
\]
Therefore, we obtain
\[
\int_{\Omega} |f(\tau_1 + \tau_2, \zeta_1 + \zeta_2)|^2 d\tau_1 d\tau_2 d\zeta_1 d\zeta_2 \lesssim K^{-1} N_2 L_2 \|f\|_{L^2_{\tau_0}}. \tag{3.11}
\]
As a result, we get (3.9) from (3.10) and (3.11). \(\square\)

**Remark 3.7.** In particular, if \(N_1 \gg N_2\), then we have
\[
\|P_{N_1} Q_L u_1 \cdot P_{N_2} Q_L u_2\|_{L^2_{x,y}} \lesssim \|P_{N_1} Q_L u_1\|_{L^2_{x,y}} \|P_{N_2} Q_L u_2\|_{L^2_{x,y}}, \tag{3.12}
\]
since the equality
\[
P_{N_1} Q_L u_1 \cdot P_{N_2} Q_L u_2 = R_K^{(j)}(P_{N_1} Q_L u_1, P_{N_2} Q_L u_2)
\]
with \(K \sim N_1^2\) holds for \(j = 1\) or 2.

**Corollary 3.8.** Let \(0 < \delta \ll 1\), \(0 < \epsilon \ll 1 - \delta\). For \(N_1\), \(N_2\), \(M_1\), \(M_2\), \(L_1\), \(L_2 \in 2^\mathbb{Z}\) with \(N_1 \gg N_2\), we have
\[
\|P_{N_1, M_1} Q_L u_1 \cdot P_{N_2, M_2} Q_L u_2\|_{L^2_{x,y}} \lesssim J_{\delta, \epsilon} (L_1 L_2)^{\frac{1}{2} - \frac{1}{2} - \frac{1}{2}} \|P_{N_1, M_1} Q_L u_1\|_{L^2_{x,y}} \|P_{N_2, M_2} Q_L u_2\|_{L^2_{x,y}} \tag{3.13}
\]
where
\[
J_{\delta, \epsilon} = J_{\delta, \epsilon}(N_1, M_1, N_2, M_2) = (N_1 M_1 N_2 M_2)^{\frac{1}{2}} (N_1^{-1} N_2^2)^{1 - \delta - \epsilon}.
\]

13
Proof. By the Hölder inequality and \(3.7\), we have

\[
\|P_{N_1,M_1}Q_{L_1}u_1 \cdot P_{N_2,M_2}Q_{L_2}u_2\|_{L_{t,x,y}^2} \lesssim \|P_{N_1,M_1}Q_{L_1}u_1\|_{L_{t,x,y}^2}^\frac{1}{2} \|P_{N_2,M_2}Q_{L_2}u_2\|_{L_{t,x,y}^2}^\frac{1}{2} \lesssim (N_1M_1L_1N_2M_2L_2)^\frac{1}{2} \|P_{N_1,M_1}Q_{L_1}u_1\|_{L_{t,x,y}^2} \|P_{N_2,M_2}Q_{L_2}u_2\|_{L_{t,x,y}^2}.
\]

By the interpolation between this estimate and \(3.12\), we obtain

\[
\|P_{N_1,M_1}Q_{L_1}u_1 \cdot P_{N_2,M_2}Q_{L_2}u_2\|_{L_{t,x,y}^2} \lesssim \int_{\mathcal{E}_N} (L_1L_2)^\frac{1}{2} \frac{|vM|^2}{v} \|P_{N_1,M_1}Q_{L_1}u_1\|_{L_{t,x,y}^2} \|P_{N_2,M_2}Q_{L_2}u_2\|_{L_{t,x,y}^2}. \tag{3.14}
\]

While, by the Cauchy-Schwartz inequality, we have

\[
\|P_{N_1,M_1}Q_{L_1}u_1 \cdot P_{N_2,M_2}Q_{L_2}u_2\|_{L_{t,x,y}^2} \lesssim \|P_{N_1,M_1}Q_{L_1}u_1\|_{L_{t,x,y}^2} \|P_{N_2,M_2}Q_{L_2}u_2\|_{L_{t,x,y}^2}. \tag{3.15}
\]

By the interpolation between \(3.14\) and \(3.15\), we obtain \(3.13\). \(\square\)

Here, we prove Proposition \(3.1\).

Proof of Proposition \(3.1\) By using the embedding \(l^1 \hookrightarrow l^2\) for the summation \(\sum \sum \sum M\), and the duality argument, we have

\[
\|(\partial_x + \partial_y)(uv))\|_{X_{s-\frac{1}{2},1}} \lesssim \sum_{N,M,L} \sum_{N_1,M_1,L_1} \left( \sum_{N_2,M_2,L_2} \frac{(N)^sM}{(M^2 + L)^{\frac{3}{2}}} \times \sup_{\|u\|_{L^2} = 1} \left| \int P_{N_1,M_1}Q_{L_1}u \cdot P_{N_2,M_2}Q_{L_2}v \cdot P_{N,M}Q_Lwdtdxdy \right| \right).
\]

We put

\[
\begin{align*}
N_1, M_1, L_1 &= P_{N_1,M_1}Q_{L_1}u, \quad N_2, M_2, L_2 = P_{N_2,M_2}Q_{L_2}v, \quad W_{N,M,L} = P_{N,M}Q_Lw, \\
N_1, M_1, L_1 &= \langle N_1 \rangle^s(M^2 + L_1)^{\frac{1}{2} + b}u_{N_1,M_1,L_1}, \quad g_{N_2,M_2,L_2} = \langle N_2 \rangle^s(M_2^2 + L_2)^{\frac{1}{2} + b}v_{N_2,M_2,L_2},
\end{align*}
\]

for \(0 < \delta \ll 1\) and

\[
I = \int u_{N_1,M_1,L_1} \cdot v_{N_2,M_2,L_2} \cdot w_{N,M,L}dt \, dx \, dy.
\]

We note that \(L_{t,x,y}^1\|u_{N_1,M_1,L_1}\|_{L_{t,x,y}^2} \lesssim \langle N_1 \rangle^{-s}\|f_{N_1,M_1,L_1}\|_{L_{t,x,y}^2}\) and \(L_{t,x,y}^2\|v_{N_2,M_2,L_2}\|_{L_{t,x,y}^2} \lesssim \langle N_2 \rangle^{-s}\|g_{N_2,M_2,L_2}\|_{L_{t,x,y}^2}\) hold for \(b \leq \frac{1 - \delta}{2}\) since \(L_i \lesssim \langle M_i^2 + L_i \rangle\) \((i = 1, 2)\).

By the symmetry, we can assume \(N_1 \gtrsim N_2\). We first consider the case \(1 \gtrsim N_1 \gtrsim N_2\). We note that

\[
\|P_{N,M}Q_Lu\|_{L_{t,x,y}^2} \lesssim L_{t,x,y}^{\frac{b}{2}}\|P_{N,M}Q_Lu\|_{L_{t,x,y}^2}. \tag{3.16}
\]
holds by the interpolation between (3.3) and a trivial equality \( \|P_{N,M,Q,L}u\|_{L^2_{x,y}} = L^0\|P_{N,M,Q,L}u\|_{L^2_{x,y}} \). By the H"{o}lder inequality, (3.3), and (3.10), we have

\[
I \lesssim \|u_{N_1,M_1,L_1}\|_{L^2_{x,y}} \|v_{N_2,M_2,L_2}\|_{L^2_{x,y}} \|w_{N,M,L}\|_{L^2_{x,y}} \lesssim (N_1 M_1 N_2 M_2)^{\frac{5}{6}} L^{\frac{5}{6}} \|f_{N_1,M_1,L_1}\|_{L^2_{x,y}} \|g_{N_2,M_2,L_2}\|_{L^2_{x,y}} \|w_{N,M,L}\|_{L^2_{x,y}}
\]

since \( \langle N_i^s \rangle \sim 1 \) (\( i = 1, 2 \)) for any \( s \in \mathbb{R} \). Therefore, we obtain

\[
\sum_{N \leq 1} \sum_{M \leq N} \sum_{L} \frac{\langle N \rangle^s M}{(M^2 + L)^{\frac{7}{6}}} \sup_{\|u\|_{L^2_x}} I \lesssim (N_1 M_1 N_2 M_2)^{\frac{5}{6}} \|f_{N_1,M_1,L_1}\|_{L^2_{x,y}} \|g_{N_2,M_2,L_2}\|_{L^2_{x,y}}
\]

since

\[
\sum_{L} \frac{L_{\frac{5}{6}}^{\frac{5}{6}}}{(M^2 + L)^{\frac{7}{6}}} \lesssim \sum_{M \leq (M + L)^{\frac{5}{6}}} \frac{L}{(M^2 + L)^{\frac{7}{6}}} \lesssim (M)^{\frac{5}{6}} \lesssim 1
\]

and

\[
\sum_{N \leq 1} \sum_{M \leq N} \sum_{N \leq 1} \sum_{M \leq N} \sum_{L \leq (M + L)^{\frac{5}{6}}} \langle N \rangle^s M \lesssim \sum_{N \leq 1} \sum_{M \leq N} \sum_{N \leq 1} \sum_{M \leq N} \sum_{L \leq (M + L)^{\frac{5}{6}}} \langle N \rangle^s M \lesssim \sum_{N \leq 1} \sum_{M \leq N}
\]

for any \( s \in \mathbb{R} \). By using (3.17) and the Cauchy-Schwartz inequality for the summations \( \sum_{N_1,M_1 \leq 1} \) and \( \sum_{N_2,M_2 \leq 1} \), we have

\[
\sum_{N_1,M_1 \leq 1} \sum_{L_1} \sum_{N_2,M_2 \leq 1} \sum_{L_2} \left( \sum_{N,M,L} \frac{\langle N \rangle^s M}{(M^2 + L)^{\frac{7}{6}}} \sup_{\|u\|_{L^2_x}} I \right) \lesssim \|u\|_{X^s_{1,\frac{1}{6} \frac{1}{6},\frac{1}{6}}} \|v\|_{X^s_{1,\frac{1}{6} \frac{1}{6},\frac{1}{6}}}
\]

for any \( s \in \mathbb{R} \).

Next, we consider the case \( N_1 \geq N_2, N_1 \geq 1 \). It suffice to show that

\[
\sum_{N,M,L} \frac{\langle N \rangle^s M}{(M^2 + L)^{\frac{7}{6}}} \sup_{\|u\|_{L^2_x}} I \lesssim N_1^{-\epsilon} (M_1 M_2)^{\frac{5}{6}} \|f_{N_1,M_1,L_1}\|_{L^2_{x,y}} \|g_{N_2,M_2,L_2}\|_{L^2_{x,y}}
\]

for small \( \epsilon > 0 \). Indeed, (3.18) and the Cauchy-Schwartz inequality for the summations \( \sum_{N_1,M_1} \) and \( \sum_{N_2,M_2} \) imply

\[
\sum_{N_1,M_1,L_1} \sum_{N_2,M_2,L_2} \left( \sum_{N,M,L} \frac{\langle N \rangle^s M}{(M^2 + L)^{\frac{7}{6}}} \sup_{\|u\|_{L^2_x}} I \right) \lesssim \left( \sum_{N_1 \geq 1} \sum_{M_1 \geq N_1} \sum_{N_2 \leq N_1} \sum_{M_2 \leq N_1} N_1^{-2\epsilon} (M_1 M_2)^{\frac{5}{6}} \right)^{\frac{1}{2}} \times \left( \sum_{N_1} \sum_{M_1} \sum_{L_1} \|f_{N_1,M_1,L_1}\|_{L^2_{x,y}} \right)^{\frac{1}{2}} \left( \sum_{N_2} \sum_{M_2} \sum_{L_2} \|g_{N_2,M_2,L_2}\|_{L^2_{x,y}} \right)^{\frac{1}{2}} \lesssim \|u\|_{X^s_{1,\frac{1}{6} \frac{1}{6},\frac{1}{6}}} \|v\|_{X^s_{1,\frac{1}{6} \frac{1}{6},\frac{1}{6}}}
\]

15
Now, we prove (3.18).

Case 1: \( N_1 \sim N_2 \gg N, N_1 \geq 1 \).

We note that \( M \lesssim \max\{M_1, M_2\} \) since \( \xi + \eta = (\xi_1 + \eta_1) + (\xi - \xi_1 + \eta - \eta_1) \).

By the symmetry, we can assume \( M \lesssim M_1 \). By the Hölder inequality, we have

\[
I \lesssim \|u_{N_1, M_1, L_1}\|_{L^2} \|v_{N_2, M_2, L_2, \cdot \cdot \cdot, W_{N,M,L}}\|_{L^2}
\]

Furthermore, we have

\[
\|u_{N_1, M_1, L_1}\|_{L^2} \lesssim N_1^{-s} \left( \frac{N_1}{M_1 + L_1} \right)^{\frac{s}{2}} \|f_{N_1, M_1, L_1}\|_{L^2}
\]

by (3.16), and we have

\[
\|v_{N_2, M_2, L_2, \cdot \cdot \cdot, W_{N,M,L}}\|_{L^2} \lesssim \sum_{N < N_1} \left( \frac{N}{M^2 + L} \right)^{\frac{s}{2}} \|N_1^{-s} N_2^{-s - 1 + \delta} \|f_{N_1, M_1, L_1}\|_{L^2} \|g_{N_2, M_2, L_2}\|_{L^2}
\]

by (3.18) and \( M \lesssim M_1 \). Therefore, if we choose \( \epsilon > 0 \) as \( \epsilon = \frac{1}{10} \delta \), we obtain

\[
\sum_{N < N_1} \sum_{M \leq M_1} \sum_{L} \frac{\langle N \rangle^s M}{\langle M^2 + L \rangle^{\frac{s}{2}}} \sup_{\|w\|_{L^2} = 1} I
\]

\[
\lesssim (M_1 M_2)^\frac{s}{2} N_1^{-s} N_2^{-s - 1 + \delta} \|f_{N_1, M_1, L_1}\|_{L^2} \|g_{N_2, M_2, L_2}\|_{L^2}
\]

\[
\times \left( \sum_{N < N_1} \langle N \rangle^s N_1^{-\frac{s}{2}} \sum_{M \leq M_1} \frac{M}{\langle M^2 + L \rangle^{\frac{s}{2}}} \sum_{L} \frac{L^\frac{1-s}{2}}{\langle M^2 + L \rangle^{\frac{s}{2}}} \right)
\]

\[
\lesssim N_1^{-s} (M_1 M_2)^{\frac{s}{2}} N_1^{-\frac{s}{2} + \delta + 2\epsilon} \|f_{N_1, M_1, L_1}\|_{L^2} \|g_{N_2, M_2, L_2}\|_{L^2}
\]

for \( s \geq -\frac{1}{2} + \frac{\delta}{2} \) since

\[
\sum_{M \leq M_1} \frac{M}{\langle M^2 + L \rangle^{\frac{s}{2}}} \sum_{L} \frac{L^\frac{1-s}{2}}{\langle M^2 + L \rangle^{\frac{s}{2}}} \lesssim \sum_{M \leq M_1} \frac{M^{1-\delta - \frac{s}{2}}}{\langle M^2 \rangle^{\frac{s}{2}}} < 1.
\]

As a result, we get (3.18) for \( s > -\frac{1}{2} \) if we choose \( \delta > 0 \) as \( 0 < \delta < \frac{1}{10} (s + \frac{1}{2}) \).

Case 2: \( N \sim N_1 \gg N_2, N_1 \geq 1 \).

By the Hölder inequality, we have

\[
I \lesssim \|u_{N_1, M_1, L_1, \cdot \cdot \cdot, W_{N,M,L}}\|_{L^2} \|w_{N,M,L}\|_{L^2}
\]

Furthermore, we have

\[
\|w_{N,M,L}\|_{L^2} \lesssim L^\frac{s}{2} \|w_{N,M,L}\|_{L^2}
\]
by (3.16), and we have
\[
\|u_{N_1,M_1,L_1} \cdot v_{N_2,M_2,L_2}\|_L^{\frac{\gamma}{\gamma_0}}_{xy} \\
\lesssim J_{0,0}(N_1, M_1, N_2, M_2)(L_1 L_2)^{\frac{1}{4} - \frac{\gamma}{4}} \|u_{N_1,M_1,L_1}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \|v_{N_2,M_2,L_2}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}}
\]
\[
\lesssim (M_1 M_2)^{\frac{r}{6}} N_1^{-s - \frac{1}{2} + \frac{\delta}{4} + \epsilon} \langle N^{-s} N_2^{1 + \frac{\delta}{4} - \frac{\gamma}{4}} \|f_{N_1,M_1,L_1}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \|g_{N_2,M_2,L_2}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}}
\]
by (3.13). Therefore, if \(s \leq \frac{1 - \delta}{2} - \frac{\gamma}{4}\), we obtain
\[
\sum_{N \sim N_1} \sum_{M \leq N} \sum_L \frac{(N)^s M}{(M^2 + L)^{\frac{\gamma}{2}}} \sup_{\|u\|_{L^2}} I
\]
\[
\lesssim (M_1 M_2)^{\frac{r}{6}} N_1^{-s - \frac{1}{2} + \frac{\delta}{4} + \epsilon} \|f_{N_1,M_1,L_1}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \|g_{N_2,M_2,L_2}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \left( \sum_{M \leq N_1} M \sum_L \frac{L^{\frac{\delta}{4}}}{(M^2 + L)^{\frac{\gamma}{2}}} \right)
\]
\[
\lesssim N_1^{-\epsilon} (M_1 M_2)^{\frac{r}{6}} N_1^{-s - \frac{1}{2} + \frac{\delta}{4} + 2\epsilon} \|f_{N_1,M_1,L_1}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \|g_{N_2,M_2,L_2}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}},
\]
since
\[
\sum_L \frac{L^{\frac{\delta}{4}}}{(M^2 + L)^{\frac{\gamma}{2}}} \lesssim N^{-\frac{1}{2} - \frac{\delta}{4}}.
\]
As a result, we get (3.13) for \(\frac{1}{4} > s > -\frac{1}{2}\) if we choose \(\delta > 0\) and \(\epsilon > 0\) as
\[0 < \epsilon < \frac{1}{4} \left( s + \frac{1}{2} \right), \ 0 < \delta < \min \left\{ \frac{1}{12} \left( s + \frac{1}{2} - 2\epsilon \right), 2 \left( \frac{1}{2} - s - \frac{\gamma}{4} \right) \right\}.
\]
While if \(s \geq \frac{1}{2}\), then we have
\[
I \lesssim (M_1 M_2)^{\frac{r}{6}} N_1^{-s - \frac{1}{2} + \frac{\delta}{4} + \epsilon} \|f_{N_1,M_1,L_1}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \|g_{N_2,M_2,L_2}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \|u_{N,M,L}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}}
\]
by the same argument with using \(\langle N_2 \rangle^{-\epsilon} \lesssim 1\). Therefore, we obtain
\[
\sum_{N \sim N_1} \sum_{M \leq N} \sum_L \frac{(N)^s M}{(M^2 + L)^{\frac{\gamma}{2}}} \sup_{\|u\|_{L^2}} I
\]
\[
\lesssim N_1^{-\epsilon} (M_1 M_2)^{\frac{r}{6}} N_1^{-s - \frac{1}{2} + \frac{\delta}{4} + 2\epsilon} \|f_{N_1,M_1,L_1}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}} \|g_{N_2,M_2,L_2}\|_{L_{xy}^{\frac{\gamma}{\gamma_0}}},
\]
which implies (3.13) since \(-\frac{\gamma}{4} + \frac{1}{4} - \frac{\delta}{4} + 2\epsilon < 0\).

Case 3: \(N \sim N_1 \sim N_2 \geq 1\)

We can assume \(M \lesssim M_1\) such as Case 1. We split \(v_{N_2,M_2,L_2}\) and \(w_{N,M,L}\) into
\[
v_{N_2,M_2,L_2} = \sum_{i=1}^3 R_i v_{N_2,M_2,L_2}, \quad w_{N,M,L} = \sum_{j=1}^3 R_j w_{N,M,L}.
\]
We put
\[
I_{i,j} = \left| \int u_{N_1,M_1,L_1} \cdot R_i v_{N_2,M_2,L_2} \cdot R_j w_{N,M,L} \, dt \, dx \, dy \right|
\]
where \(R_i\) (\(i = 1, 2, 3\)) are projections given by
\[
F_{xy}[R_i f] = 1_{\{|x| \gg \eta\}} \hat{f}, \quad F_{xy}[R_2 f] = 1_{\{|x| \sim \eta\}} \hat{f}, \quad F_{xy}[R_3 f] = 1_{\{|x| \ll \eta\}} \hat{f}.
\]
We note that \( \mathcal{F}_{xy}[w_{N,M,L}] \) is supported in at least one of \( \{ (\xi, \eta) \mid |\xi| \sim N \} \) or \( \{ (\xi, \eta) \mid |\eta| \sim N \} \). By the symmetry, we can assume \( \text{supp} \mathcal{F}_{xy}[w_{N,M,L}] \subset \{ (\xi, \eta) \mid |\xi| \sim N \} \). Then, it suffices to show the estimate for \( I_{i,j} \) with \( i = 1, 2, 3, j = 1, 2 \).

**Estimate for \( I_{1,1} \)**

In this case, we note that \( N \sim N_1 \sim N_2 \sim M \sim M_1 \sim M_2 \) and

\[
|\xi_1 \xi_2 + \eta_1 \eta_2| \sim |\xi_1 \xi_2| \sim N_1^3
\]

for \((\xi_1, \eta_1) \in \text{supp} \mathcal{F}_{xy}[u_{N_1,M_1,L_1}], (\xi_2, \eta_2) \in \text{supp} \mathcal{F}_{xy}[v_{N_2,M_2,L_2}]\) with \( \xi_1 + \xi_2 = \xi, \eta_1 + \eta_2 = \eta \). It implies

\[
\max \{L_1, L_2, L \} \gtrsim N_1^3
\]

since

\[
(\tau_1 - \xi_1^3 - \eta_1^3) + (\tau_2 - \xi_2^3 - \eta_2^3) - (\tau - \xi_3 - \eta_3^3) = 3|\xi_1 \xi_2 + \eta_1 \eta_2|.
\]

holds for \((\tau_1, \xi_1, \eta_1) (i = 1, 2) \) with \((\tau, \xi, \eta) = (\tau_1 + \tau_2, \xi_1 + \xi_2, \eta_1 + \eta_2)\).

(i) For the case \( L \gtrsim N_1^3 \)

By the Hölder inequality, (3.3), and (3.16), we have

\[
I \lesssim \|u_{N_1,M_1,L_1}\|_{L^1_{txy}} \|v_{N_2,M_2,L_2}\|_{L^1_{txy}} \|w_{N,M,L}\|_{L^\infty_{txy}}
\]

\[
\lesssim (N_1M_1N_2M_2)^\frac{5}{18} (L_1L_2)^{\frac{5}{18}} \|u_{N_1,M_1,L_1}\|_{L^2_{txy}} \|v_{N_2,M_2,L_2}\|_{L^2_{txy}} \|w_{N,M,L}\|_{L^2_{txy}}
\]

\[
\sim N_1^{-\epsilon} (M_1M_2) \frac{\tau}{2^s + \delta} \|f_{N_1,M_1,L_1}\|_{L^2_{txy}} \|g_{N_2,M_2,L_2}\|_{L^2_{txy}} \|w_{N,M,L}\|_{L^2_{txy}}.
\]

Therefore, we obtain

\[
\sum_{N \sim N_1} \sum_{M \gtrsim N} \sum_{L \gtrsim N_1^3} \frac{(N)^s M}{(M^2 + L)^\frac{s}{2}} \sup_{\|w\|_{L^2=1}} I
\]

\[
\lesssim N_1^{-\epsilon} (M_1M_2) \frac{\tau}{2^s + \delta} \|f_{N_1,M_1,L_1}\|_{L^2_{txy}} \|g_{N_2,M_2,L_2}\|_{L^2_{txy}} \left( \sum_{M \lesssim N_1} M \sum_{L \gtrsim N_1^3} \frac{L^{\frac{s}{2}}}{(M^2 + L)^\frac{s}{2}} \right)
\]

\[
\lesssim N_1^{-\epsilon} (M_1M_2) \frac{\tau}{2^s + \delta} \|f_{N_1,M_1,L_1}\|_{L^2_{txy}} \|g_{N_2,M_2,L_2}\|_{L^2_{txy}} \left( \sum_{L \gtrsim N_1^3} \frac{L^{\frac{s}{2}}}{(M^2 + L)^\frac{s}{2}} \right)
\]

\[
\lesssim N_1^{-\epsilon} (M_1M_2) \frac{\tau}{2^s + \delta} \|f_{N_1,M_1,L_1}\|_{L^2_{txy}} \|g_{N_2,M_2,L_2}\|_{L^2_{txy}} \lesssim N_1^{-\epsilon} (M_1M_2) \frac{\tau}{2^s + \delta} \|f_{N_1,M_1,L_1}\|_{L^2_{txy}} \|g_{N_2,M_2,L_2}\|_{L^2_{txy}}.
\]

As a result, we get (3.18) for \( s > -\frac{1}{3} \) if we choose \( \delta > 0 \) and \( \epsilon > 0 \) as

\[
0 < \epsilon < \frac{1}{2} (s + \frac{1}{2}), \quad 0 < \delta < \frac{1}{2} (s + \frac{1}{2} - \frac{1}{2} \epsilon).
\]

(ii) For the case \( L_1 \gtrsim N_1^3 \)

By the Hölder inequality, (3.16), and (3.3), we have

\[
I \lesssim \|u_{N_1,M_1,L_1}\|_{L^1_{txy}} \|v_{N_2,M_2,L_2}\|_{L^1_{txy}} \|w_{N,M,L}\|_{L^\infty_{txy}}
\]

\[
\lesssim N_1^{-\epsilon} (M_2M_1M) \frac{\tau}{2^s + \delta} \|f_{N_1,M_1,L_1}\|_{L^2_{txy}} \|g_{N_2,M_2,L_2}\|_{L^2_{txy}} \|w_{N,M,L}\|_{L^2_{txy}}
\]

\[
\lesssim N_1^{-\epsilon} (M_1M_2) \frac{\tau}{2^s + \delta} \|f_{N_1,M_1,L_1}\|_{L^2_{txy}} \|g_{N_2,M_2,L_2}\|_{L^2_{txy}} \|w_{N,M,L}\|_{L^2_{txy}}
\]

18
By the Hölder inequality, (3.16), (3.4), and argument as in Case 1 since $L_s > \frac{1}{5}$ holds. While, by the Cauchy-Schwartz inequality, we have $\sum_{M \leq N} M \sum_{L} \frac{(N^2)^M M}{(M^2 + L)^{\frac{1}{2}}} \langle f_{N_1, M_1, L_1} \parallel g_{N_2, M_2, L_2} \parallel L_{ty} \rangle$. Therefore, we get (3.18) for $N \gtrsim 1$.

Estimate for $I_{2,2}$

In this case, we have

$$|\xi_2| \sim |\eta_2| \sim N_2, |\xi| \sim |\eta| \sim N.$$ 

By the Hölder inequality, (3.16), (3.4), and $M \lesssim M_1$, we have

$$I \lesssim \|u_{N_1, M_1, L_1}\|_{L_{ty}^{(s-\frac{1}{2})}} \|v_{N_2, M_2, L_2}\|_{L_{ty}^{s+\frac{1}{4}+\frac{\epsilon}{2}}} \|w_{N, M, L}\|_{L_{ty}^{\frac{1}{2}}} \lesssim \frac{(M_2 M_1 N M)^{\frac{1}{2}}}{(M_1 + M_2)^{\frac{1}{2}}} \langle f_{N_1, M_1, L_1} \parallel g_{N_2, M_2, L_2} \parallel L_{ty} \rangle \|w_{N, M, L}\|_{L_{ty}^{\frac{1}{2}}}.$$ 

Therefore, we get (3.13) for $s > -\frac{1}{2}$ by the same argument as in Case 1.

Estimate for $I_{1,2}$

In this case, we have $|\eta_2^2 - \eta_1^2| \sim |\eta|^2 \sim N^2$.

Therefore, we obtain

$$\|R_1 v_{N_2, M_2, L_2} \cdot R_2 w_{N, M, L}\|_{L_{txy}^{\frac{1}{2}}} \lesssim N^{-\frac{1}{2}} L_N^{\frac{1}{2}} L_T^{\frac{1}{2}} \|R_1 v_{N_2, M_2, L_2}\|_{L_{txy}^{\frac{1}{2}}} \|R_2 w_{N, M, L}\|_{L_{txy}^{\frac{1}{2}}}$$

by (3.18) since

$$R_1 v_{N_2, M_2, L_2} \cdot R_2 w_{N, M, L} = R_K^{(2)}(R_1 v_{N_2, M_2, L_2} \cdot R_2 w_{N, M, L})$$

with $K \sim N$ holds. While, by the Cauchy-Schwartz inequality, we have

$$\|R_1 u_{N_1, M_1, L_1} \cdot R_1 v_{N_2, M_2, L_2}\|_{L_{txy}^{\frac{1}{2}}} \lesssim \|R_1 u_{N_2, M_2, L_2}\|_{L_{txy}^{\frac{1}{2}}} \|R_2 w_{N, M, L}\|_{L_{txy}^{\frac{1}{2}}}.$$ 

Therefore, we obtain the bilinear Strichartz estimate such as (3.13) for the product $R_1 v_{N_2, M_2, L_2} \cdot R_2 w_{N, M, L}$, and we get (3.13) for $s > -\frac{1}{2}$ by the same argument as in Case 1 since $M \lesssim M_1$. The estimates for $I_{2,1}, I_{3,1}$, and $I_{3,2}$ are obtained by the same way.
Remark 3.9. We can also obtain the bilinear estimate
\[ \| (\partial_x + \partial_y) (uv) \|_{X^{s, -\frac{1}{2}}} \leq \frac{C_3}{2} \left( \| u \|_{X^{s, \frac{1}{2}}}, \| v \|_{X^{s, \frac{1}{2}}} + \| u \|_{X^{s, 0}}, \| v \|_{X^{s, \frac{1}{2}}} \right) \]
for \( s \geq s_0 > -\frac{1}{2} \) by using
\[ \langle \xi \rangle^s \lesssim \langle \xi \rangle^{s_0} \left( \langle \xi_0 \rangle^{s-s_0} + \langle \xi - \xi_0 \rangle^{s-s_0} \right). \]

4 Proof of the well-posedness

In this section, we prove Theorem 1.1 and 1.2. For \( T > 0 \) and \( v_0 \in H^s(\mathbb{R}^2) \), we define the map \( \Phi_{T, v_0} \) as
\[ \Phi_{T, v_0}(v)(t) := \psi(t) \left( W(t) v_0 + \int_0^t W(t-t') (\partial_x + \partial_y) v(t')^2 dt' \right), \]
where \( \psi \) is cut-off function defined in Section 2, and \( \psi(t) = \psi \left( \frac{t}{T} \right) \). For \( R > 0 \) and Banach space \( X \), we define \( B_R(X) := \{ u \in X \mid \| u \|_X \leq R \} \). To obtain the well-posedness of (1.0) in \( H^s(\mathbb{R}^2) \), we prove that \( \Phi_{T, v_0} \) is a contraction map on closed subset of \( X^{s, \frac{3}{2}-1} \).

Lemma 4.1. Let \( 0 < T \leq 1, 0 < \delta \leq 1 \). There exist \( C_4 > 0 \) and \( \mu = \mu(\delta) > 0 \), such that for any \( u \in X^{s, \frac{3}{2}-1} \), we have
\[ \| \psi_T u \|_{X^{s, \frac{3}{2}-1}} \leq C_4 T^\mu \| u \|_{X^{s, \frac{3}{2}-1}}. \]

The proof of Lemma 4.1 is almost same as the proof of Lemma 2.5 and 3.1 in [8].

Proof of Theorem 1.1 Let \( s \geq s_0 > -\frac{1}{2} \) and \( v_0 \in H^s(\mathbb{R}^2) \) are given, and \( T \in (0, 1], R > 0 \) will be chosen later. We define the function space \( Z^s \) as
\[ Z^s := \{ v \in X^{s, \frac{3}{2}-1} \mid \| v \|_{Z^s} := \| v \|_{X^{s, \frac{3}{2}-1}} + \alpha \| v \|_{X^{s, \frac{1}{2}}}, \alpha < \infty \}, \]
where \( \alpha = \| v_0 \|_{H^\infty}/\| v_0 \|_{H^s} \). For \( v, v_1, v_2 \in B_R(Z^s) \), we have
\[ \| \Phi_{T, v_0}(v) \|_{Z^s} \leq C_1 (1 + \alpha) \| v_0 \|_{H^\infty} + C_2 C_3 C_4^2 T^{2\mu} \| v \|_{Z^s}^2 \]
\[ \leq C_1 (1 + \alpha) \| v_0 \|_{H^\infty} + C_2 C_3 C_4^2 T^{2\mu} R^2 \]
and
\[ \| \Phi_{T, v_0}(v_1) - \Phi_{T, v_0}(v_2) \|_{Z^s} \leq C_2 C_3 C_4^2 T^{2\mu} \| v_1 + v_2 \|_{Z^s} \| v_1 - v_2 \|_{Z}, \]
\[ \leq C_2 C_3 C_4^2 T^{2\mu} R \| v_1 - v_2 \|_{Z^s} \]
by Proposition 2.3, 2.4, 3.1, Remark 3.9 and Lemma 4.1. Therefore, if we choose \( T, R \) as
\[ R = 2C_1(1 + \alpha) \| v_0 \|_{H^\infty}, \quad 0 < T^{2\mu} < \left( 4C_1 C_2 C_3 C_4^2 (1 + \alpha) \| v_0 \|_{H^\infty} \right)^{-1}, \]
then \( \Phi_{T, v_0} \) is contraction map on \( B_R(Z^s) \). We note that \( T = T(\| v_0 \|_{H^\infty}) \). By Banach’s fixed point theorem, there exists a solution \( v \in X^{s, \frac{3}{2}-1} \) to \( v(t) = \Phi_{T, v_0}(v)(t) \) and \( v|_{[0, T]} \in X^{s, \frac{3}{2}-1} \) satisfies (2.1) on \([0, T]\). The Lipschitz continuous dependence on initial data is obtained by the similar argument as above. The uniqueness is obtained by the same argument as in Section 4.2 of [20].
Next, to prove the global well-posedness of \((1.6)\) in \(\hat{H}^s(\mathbb{R}^2)\), we define the function space \(\tilde{X}^{s,b,1}\) as the completion of the Schwartz class \(S(\mathbb{R} \times \mathbb{R}^2_x, y)\) with the norm

\[
\|u\|_{\tilde{X}^{s,b,1}} = \left\{ \sum_{N \in \mathbb{Z}^2} \sum_{M \in \mathbb{Z}^2} \left( \sum_{L \in \mathbb{Z}^2} (M)^s(M^2 + L)^b \|P_{N,M} Q_L u\|_{L^2_x} \right)^2 \right\}^{\frac{1}{2}}.
\]

We also define \(\tilde{X}_T^{s,b,1}\) as the time localized space of \(\tilde{X}^{s,b,1}\).

**Remark 4.2.** We can see that \(\tilde{X}_T^{s,\frac{1}{2},1} \hookrightarrow L^2((0,T); \hat{H}^{s+1}(\mathbb{R}^2))\) since \((M)^s(M^2 + L)^{\frac{1}{2}} \leq (M)^s(M^2 + L)^{\frac{1}{2}}\) and \(l_1^2 \leq l_2^2\) hold.

**Proposition 4.3.** Let \(s \in \mathbb{R}\). There exists \(C_1 > 0\), such that for any \(u_0 \in \hat{H}^s(\mathbb{R}^2)\), we have

\[
\|\psi(t)W(t)u_0\|_{\tilde{X}_T^{s,\frac{1}{2},1}} \leq C_1 \|u_0\|_{\hat{H}^s}.
\]

**Proposition 4.4.** Let \(s \in \mathbb{R}\). There exists \(C_2 > 0\), such that for any \(F \in \tilde{X}^{s-\frac{1}{2},1}\), we have

\[
\|\phi(t)L F(t)\|_{\tilde{X}_T^{s,\frac{1}{2},1}} \leq C_2 \|F\|_{\tilde{X}_T^{s-\frac{1}{2},1}}.
\]

The proof of Proposition 4.3 and 4.4 are same as the proof of Proposition 2.3 and 2.4.

**Proposition 4.5.** Let \(s > -\frac{1}{2}\). There exist \(0 < \delta \ll 1\) and \(C_3 > 0\), such that for any \(u, v \in \tilde{X}^{s-\frac{1}{2},1}\), we have

\[
\| (\partial_x + \partial_y)(uv)\|_{\tilde{X}_T^{s-\frac{1}{2},1}} \leq C_3 \|u\|_{\tilde{X}_T^{s-\frac{1}{2},1}} \|v\|_{\tilde{X}_T^{s-\frac{1}{2},1}}.
\]

The proof of Proposition 4.5 is similar to the proof of Proposition 3.1. We will give the proof at the last part of this section.

**Proof of Theorem 1.2.** Let \(s \geq s_0 > -\frac{1}{2}\) are given. By Proposition 4.3, 4.4, and using the same argument as in the proof of Theorem 1.1, we obtain the solution \(v \in \tilde{X}_T^{s,\frac{1}{2},1}\) to (1.6) on \([0, T]\) with \(T = T(\|v_0\|_{\hat{H}^{s_0}})\). Let \(T' \in (0, T)\) be fixed. Since \(\tilde{X}_T^{s,\frac{1}{2},1} \hookrightarrow L^2((0,T); \hat{H}^{s+1}(\mathbb{R}^2))\) holds, there exists \(t_0 \in (0, T')\) such that \(v(t_0) \in \hat{H}^{s+1}(\mathbb{R}^2)\). Therefore, by choosing \(v(t_0)\) as the initial data and using the uniqueness of the solution, we obtain \(v(t_0 + \cdot) \in \tilde{X}_T^{s+1,\frac{1}{2},1}\). In particular, we have \(v(T') \in \hat{H}^{s+1}(\mathbb{R}^2)\). By repeating this argument, we get \(v(T') \in \hat{H}^{\infty}(\mathbb{R}^2)\). Since we can choose \(T' > 0\) arbitrary small, \(v\) belongs to \(C((0,T]; \hat{H}^{\infty}(\mathbb{R}^2))\). This arrows us to take the \(L^2\)-scalar product of (1.6) with \(v\), and we have

\[
\frac{d}{dt}\|v(t)\|_{L^2_x}^2 = (\partial_t v(t), v(t))_{L^2_x} = -\| (\partial_x + \partial_y) v(t) \|_{L^2_x}^2 \leq 0
\]

for any \(t \in (0,T)\). Therefore, \(\|v(t)\|_{L^2_x}\) is non-increasing, and we can extend the solution \(v\) globally in time. \(\square\)
Remark 4.6. We note that the embedding $X^s_T \rightarrow L^2([0,T]; H^{s+1}(\mathbb{R}^2))$ does not hold. Therefore, we cannot use the above argument for initial data $v_0 \in H^s(\mathbb{R}^2)$.

Finally, we give the proof of Proposition 4.5

Proof of Proposition 4.5. We put

$$ u_{N_1,M_1,L_1} = P_{N_1,M_1} Q_{L_1} u, \quad v_{N_2,M_2,L_2} = P_{N_2,M_2} Q_{L_2} v, \quad w_{N,M,L} = P_{N,M} Q_{L} w, $$

$$ f_{N_1,M_1,L_1} = (M_1)^s (M_2^2 + L_1) \frac{1-s}{s} u_{N_1,M_1,L_1}, \quad g_{N_2,M_2,L_2} = (M_2)^s (M_2^2 + L_2) \frac{1-s}{s} v_{N_2,M_2,L_2} $$

for $0 < \delta \ll 1$ and

$$ I = \left| \int u_{N_1,M_1,L_1} \cdot v_{N_2,M_2,L_2} \cdot w_{N,M,L} dt dx dy \right|. $$

We use $L^2_\mu ||u_{N_1,M_1,L_1}||_{L^2_{xy}}^2 \lesssim (M_1)^{-s} \|f_{N_1,M_1,L_1}\|_{L^2_{xy}}^2$ and $L^2_\mu ||v_{N_2,M_2,L_2}||_{L^2_{xy}}^2 \lesssim (M_2)^{-s} \|g_{N_2,M_2,L_2}\|_{L^2_{xy}}^2$ instead of $L^2_\mu ||u_{N_1,M_1,L_1}||_{L^2_{xy}}^2 \lesssim (N_1)^{-s} \|f_{N_1,M_1,L_1}\|_{L^2_{xy}}^2$ and $L^2_\mu ||v_{N_2,M_2,L_2}||_{L^2_{xy}}^2 \lesssim (N_2)^{-s} \|g_{N_2,M_2,L_2}\|_{L^2_{xy}}^2$ in the proof of Proposition 3.1.

By the same argument as in the proof of Proposition 3.1, we have

$$ \sum_{N_1, M_1 \leq 1} \sum_{L_1} \sum_{N_2, M_2 \leq 1} \sum_{L_2} \left( \sum_{N,M,L} \frac{(M)^s M}{(M^2 + L)^\frac{\epsilon}{2}} \sup_{\|w\|_{L^2} = 1} I \right) \lesssim ||u||_X, ||v||_X, ||w||_X, $$

for any $s \in \mathbb{R}$ and it suffice to show that

$$ \sum_{N,M,L} \frac{(M)^s M}{(M^2 + L)^\frac{\epsilon}{2}} \sup_{\|w\|_{L^2} = 1} I \lesssim N_1^{-\epsilon} (M_1 M_2)^{\frac{\epsilon}{2}} \|f_{N_1,M_1,L_1}\|_{L^2_{xy}} \|g_{N_2,M_2,L_2}\|_{L^2_{xy}} $$

for $N_1 \geq N_2$, $N_1 \geq 1$, and small $\epsilon > 0$.

Case 1': $N_1 \sim N_2 \gg N$

We only have to modify little in the proof of Proposition 3.1. Case 1. Since it hold that

$$ (M_1)^{-s} \sum_{M \leq M_1} \frac{(M)^s M}{(M^2 + L_1)^{\frac{\epsilon}{2}}} \sum_{L} \frac{L^{1-s}}{(M^2 + L)^{\frac{\epsilon}{2}}} \lesssim \sum_{M \leq M_1} \frac{M^{s+1-\epsilon} \frac{\epsilon}{2}}{(M_1)^{s+1-\epsilon} \frac{\epsilon}{2}} \lesssim 1 $$

for $\epsilon = \frac{10}{3} \delta$, $s > -1 + \frac{8}{3} \delta$, and

$$ (M_2)^{-s} \lesssim N_1^{-s} $$

for $s < 0$, we get (4.18) for $-\frac{2}{3} < s < 0$ by the same way as in the proof of Proposition 3.1. Case 1.

Case 2': $N \sim N_1 \gg N_2$
If $M \geq M_1$, then we have
\[ \langle M \rangle^s (M_1)^{-s} (M_2)^{-s} \lesssim \langle M_2 \rangle^{-s} \lesssim N_1^{-s} \]
for $s < 0$. Therefore, we get (3.18) for $-\frac{1}{2} < s < 0$ by the same way as in the proof of Proposition 3.1 Case 2.

While, if $M \leq M_1$, then we have
\[ J_{\delta,\epsilon}(N, M, N_2, M_2) \lesssim J_{\delta,\epsilon}(N_1, M_1, N_2, M_2). \]
Therefore, by estimating
\[ I \lesssim \|u_{N_1, M_1, L_1}\|_{L^{2}_{\text{txy}}} \|v_{N_2, M_2, L_2} \cdot w_{N, M, L}\|_{L^{2}_{\text{txy}}} \]
instead of
\[ I \lesssim \|u_{N_1, M_1, L_1} \cdot v_{N_2, M_2, L_2}\|_{L^{2}_{\text{txy}}} \|w_{N, M, L}\|_{L^{2}_{\text{txy}}} \]
in the proof of Proposition 3.1 Case 2, we get (3.18) for $-\frac{1}{2} < s < 0$ by the same modification such as Case 1’

Case 3’: $N \sim N_1 \sim N_2 \geq 1$
If $\text{supp} \mathcal{F}_{x,y}[w_{N,M,L}] \subset \{(\xi, \eta) \mid \|\xi\| \gg \|\eta\| \text{ or } \|\xi\| \ll \|\eta\|\}$, then $M \sim N$ holds. Therefore, we have
\[ \langle M \rangle^s (M_1)^{-s} (M_2)^{-s} \lesssim \langle N \rangle^s (N_1)^{-s} (N_2)^{-s} \lesssim N_1^{-s} \]
for $s < 0$ and get (3.18) for $-\frac{1}{2} < s < 0$ by the same way as in the proof of Proposition 3.1 Case 3.
We assume $\text{supp} \mathcal{F}_{x,y}[w_{N,M,L}] \subset \{(\xi, \eta) \mid \|\xi\| \sim \|\eta\|\}$. It suffice to show the estimate for $I_{1,2}$ and $I_{2,2}$, which are defined in Proposition 3.1 Case 3. By the same modification such as in Case 1’, we can obtain (3.18) for $-\frac{1}{2} < s < 0$. \[\square\]

Acknowledgements
This work is financially supported by JSPS KAKENHI Grant Number 17K14220 and Program to Disseminate Tenure Tracking System from the Ministry of Education, Culture, Sports, Science and Technology. The author would like to his appreciation to Shinya Kinoshita (Nagoya university) for his useful comments and discussions.

References
[1] M. B.-Artzi, H. Koch, and J.-C. Saut, Dispersion estimates for third order equations in two dimensions, Commun. Partial Differ. Equ 28 (2003), 1943–1974.
[2] D. Bekiranov, The initial value problem for the generalized Burgers’ equation, Differ. Integral Equ. 9 (1996), 1253–1265.
[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I, II., Geom. Funct. Anal. 3 (1993), no. 3, 107–156, 209–262.
[4] X. Carvajal, A. Esfahani, and M. Panthee, *Well-posedness results and dissipative limit of high dimensional KdV-type equations*, Bull Bras Math Soc, New Series 48 (2017), 505–550.

[5] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. 125 (2003), 1235–1293.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on R and T*, J. Amer. Math. Soc. 16 (2003), 705–749.

[7] D. B. Dix, *Nonuniqueness and uniqueness in the initial-value problem for Burgers’ equation*, SIAM J. Math. Anal. 27 (1996), 708–724.

[8] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. 151 (1997), 384–436.

[9] A. Grünrock, and S. Herr, *The Fourier restriction norm method for the Zakharov-Kuznetsov equation*, Discrete. Contin. Dyn. Syst. 34 (2014), 2061–2068.

[10] Z. Guo, *Global well-posedness of Korteweg-de Vries equation in H^−3/4(R)*, J. Math. Pure. Appl. 91 (2009), 583–597.

[11] C. Kenig, G. Ponce, and G. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. 9 (1996), 573–603.

[12] C. Kenig, G. Ponce, and G. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke. Math. J. 106 (2001), 617–633.

[13] N. Kishimoto, *Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity*, Differ. Integral Equ. 22 (2009), 447–464.

[14] B. Kojok, *Sharpe well-posedness for Kadomtsev-Petviashvili-Burgers (KP-BII) equation in R^2*, J. Diff. Eqns 242 (2007), 211–247.

[15] N. A. Larkin, *2D Zakharov-Kuznetsov-Burgers equations on a strip*, [arXiv:1404.4638v1 [math.AP]].

[16] N. A. Larkin, *2D Zakharov-Kuznetsov-Burgers equations with variable dissipation on a strip*, Electron. J. Differential Equations, 2015 (2015), 1–20.

[17] F. Linares, and A. Pastor, *Well-posedness for the two-dimensional modified Zakharov-Kuznetsov equation*, SIAM J. Math. Anal. 41 (2009), 1323–1339.

[18] D. Mohamad, *On the well-posedness for Kadomtsev-Petviashvili-Burgers I equation*, J. Diff. Eqns 253 (2012), 1584–1603.

[19] L. Molinet, and D. Pilod, *Bilinear Strichartz estimates for the Zakharov-Kuznetsov equation and applications*, Ann. I. H. Poincaré - AN. 32 (2015), 347–371.

[20] L. Molinet, and F. Ribaud, *On the low regularity of the Korteweg-de Vries-Burgers equation*, Int. Math. Res. Not. 37 (2002), 1979–2005.
[21] L. Molinet, and S. Vento, *Sharpe ill-posedness and well-posedness results for the KdV-Burgers equation: the real line case*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 2011, 10 (3), pp.531-560. (hal-00436652v2)

[22] W. M. Moslem, and R. Sabry, *Zakharov-Kuznetsov-Burgers equation for dust ion acoustic waves*, Chaos, Solitons and Fractals 36 (2008), 628–634.

[23] F. Ribaud, *Cauchy problem for semilinear parabolic equations with initial data in $H^s_p(\mathbb{R}^n)$ spaces*, Rev. Mat. Iberoamericana 14 (1998), no. 1, 1–46.

[24] H. L. Zhen, B. Tian, H. Zhong, W. R. Sun, and M. Li, *Dynamics of the Zakharov-Kuznetsov-Burgers equations in dusty plasmas*, Physics of Plasmas 20 (2013), doi: http://dx.doi.org/10.1063/1.4818508