Abstract—Non-data-aided (NDA) parameter estimation is considered for binary-phase-shift-keying transmission in an additive white Gaussian noise channel. Cramér-Rao lower bounds (CRLBs) for signal amplitude, noise variance, channel reliability constant and bit-error rate are derived and it is shown how these parameters relate to the signal-to-noise ratio (SNR). An alternative derivation of the iterative maximum likelihood (ML) SNR estimator is presented together with a novel, low complexity NDA SNR estimator. The performance of the proposed estimator is compared to previously suggested estimators and the CRLB. The results show that the proposed estimator performs close to the iterative ML estimator at significantly lower computational complexity.

I. INTRODUCTION

Most iterative decoders, e.g. turbo decoders [1], rely on knowledge of the signal-to-noise ratio (SNR) or the channel reliability constant [1–3]. The SNR is also required for other functionalities in the receiver. Many SNR estimators have been proposed, both data-aided (DA) — that require pilot symbols or feedback from the decoders [3–5], and non-data-aided (NDA) — that are only based on the received observables [2, 4, 6].

A comparison of both DA and NDA SNR estimators was performed in [4] and compared to the Cramér-Rao lower bound (CRLB) for DA estimators. The CRLB for NDA estimators was later derived in [7]. The NDA maximum likelihood (ML) estimator based on the expectation maximization (EM) algorithm was proposed in [6] and also compared to other NDA estimators. This NDA ML estimator was found iteratively, but unfortunately requiring processing of all observables for each iteration, making it computationally complex.

The contributions in this paper are as follows. To complement the NDA CRLB for SNR in [7], we derive the NDA CRLB for the signal amplitude, the noise variance, the channel reliability constant, and the bit-error rate (BER). It is also shown how to estimate the a priori probability of the transmitted symbols, in the case when they are not equally likely. Furthermore, we provide a more direct, alternative derivation of the NDA ML estimator and we propose a new, low complexity NDA SNR estimator. The performance of the new estimator is compared to previously suggested NDA estimators and found to be similar to that of the NDA ML estimator. This performance is achieved with significantly lower computational complexity than the ML estimator. Only binary-phase-shift-keying (BPSK) transmission is considered here, but generalization to M-PSK is straightforward [4, 7].

II. PROBLEM STATEMENT

Let $X \in \{-1, +1\}$ denote a binary random variable with equally likely symbols. Further, let $W$ represent a zero-mean Gaussian random variable with unit variance. Define a new random variable $Y$ according to

$$Y \triangleq \mu X + \sigma W,$$

(1)

with probability density function (PDF) expressed as [6, 7]

$$p_Y(y) = \frac{1}{\sqrt{2\pi}\sigma^2} \left( e^{-\frac{(y-\mu)^2}{2\sigma^2}} + e^{-\frac{(y+\mu)^2}{2\sigma^2}} \right) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{y^2}{2\sigma^2}} \cosh \left( \frac{\mu}{\sigma^2} \right).$$

(2)

Let $x$, $y$, and $w$ denote samples from $X$, $Y$, and $W$, respectively. $N$ independent samples of $Y$ is observed and collected in a column vector $y = [y_1, y_2, \ldots, y_N]^T$. If $\mu = \sqrt{E_s}$ and $\sigma^2 = N_0/2$, the model in (1) represents BPSK transmission in additive white Gaussian noise (AWGN)

$$y = \mu x + \sigma w,$$  

(3)

where $y$ is the matched filter output, $x = [x_1, x_2, \ldots, x_N]^T$ the transmitted data, and $w = [w_1, w_2, \ldots, w_N]^T$ white Gaussian noise. $E_s$ is the transmitted energy and $N_0/2$ is the double-sided noise power spectral density. Define the SNR as

$$\gamma \triangleq \frac{E_s}{N_0} = \frac{\mu^2}{2\sigma^2}. $$

(4)

Since all samples in $y$ are assumed independent, the logarithm of their joint PDF is given by [7]

$$\ln p_Y(y) = \ln \left( \prod_{n=1}^{N} p_Y(y_n) \right)$$

$$= -\frac{N}{2} \ln(2\pi\sigma^2) - \frac{N\mu^2}{2\sigma^2} - \sum_{n=1}^{N} \frac{y_n^2}{2\sigma^2} + \sum_{n=1}^{N} \ln \left( \cosh \left( \frac{\mu y_n}{\sigma^2} \right) \right).$$

The average BER can be expressed as

$$P(\gamma) \triangleq \Pr(Y < 0 | X = +1) = Q \left( \sqrt{2\gamma} \right),$$

(6)
where \( Q(\alpha) \triangleq 1/\sqrt{2\pi} \int_\alpha^\infty e^{-\beta^2/2} d\beta \) is the Gaussian Q-function. The average mutual information (MI) [8] between \( X \) and \( Y \) in (1) can be calculated as

\[
I(\gamma) = I(X; Y) = J(\sqrt{8\gamma}),
\]

where \( J \) is defined as [8]

\[
J(\alpha) \triangleq 1 - \frac{1}{\sqrt{2\pi}\alpha^2} \int_{-\infty}^{+\infty} \log_2(1 + e^{-\beta}) e^{-\beta^2/(2\alpha^2)} d\beta.
\]  

The corresponding instantaneous MI for a symbol at position \( n \) can be estimated by [9]

\[
P_n(\lambda) \triangleq \frac{1}{1 + e^{-\lambda y_n}}.
\]  

The instantaneous BER for a specific received symbol at position \( n \) can be estimated by [10]

\[
R_n(\gamma) \triangleq 1 - 2 \log_2(1 + e^{-\lambda y_n} / 1 + e^{-\lambda y_n}).
\]  

With no knowledge of the transmitted symbols the average BER in (11) or the average MI in (12) depend solely on the SNR, \( \gamma \). Also, in order to use an LLR-based decoder (basically all turbo-like decoders [1,8] or soft decoders), to estimate the channel reliability constant in (12), the channel reliability constant in (12) needs to be known. However, as we show in Section IV, the SNR and the channel reliability constant are related through the second moment of the observables. We therefore only need to estimate one of the two. Here, we have chosen to estimate the SNR, \( \gamma \).

### III. CRAMÉR-RAO LOWER BOUND

The CRLB, here denoted by \( \Gamma \), provides a lower bound on the variance of any unbiased estimator [11]. Let \( g(\mu, \sigma) \) represent an arbitrary function of the parameters \( \mu \) and \( \sigma \), and define \( \delta \triangleq g(\mu, \sigma) \). The normalized CRLB (NCRLB) for \( \delta \) can then be calculated as [11]

\[
\Gamma_\delta \triangleq \frac{1}{\delta^2} \left[ \partial^2 \ln p_Y(y) / \partial \mu \partial \sigma \right] J^{-1} \left[ \partial^2 \ln p_Y(y) / \partial \mu \partial \sigma \right]^T,
\]

where \( J \) is the Fisher information matrix [11], defined as

\[
J \triangleq \begin{bmatrix}
-\mathbb{E} \left\{ \partial^2 \ln p_Y(y) / \partial \mu^2 \right\} & -\mathbb{E} \left\{ \partial^2 \ln p_Y(y) / \partial \mu \partial \sigma \right\} \\
-\mathbb{E} \left\{ \partial^2 \ln p_Y(y) / \partial \sigma \partial \mu \right\} & -\mathbb{E} \left\{ \partial^2 \ln p_Y(y) / \partial \sigma^2 \right\}
\end{bmatrix}.
\]

Here \( \mathbb{E} \{ \cdot \} \) denotes the expectation over \( Y \). A similar Fisher information matrix as in (13) has been derived in [7] and the inverse can be expressed as

\[
J^{-1} = \frac{\sigma^2}{N} \left[ \begin{array}{cc}
\frac{1}{2 - 2f(\gamma) - 8\gamma f(\gamma)} & 2 - 8\gamma f(\gamma) \\
\sqrt{8\gamma} f(\gamma) & 1 - f(\gamma)
\end{array} \right],
\]

where \( f(\gamma) \) is a function of \( \gamma \) [7]

\[
f(\gamma) \triangleq \sqrt{\frac{4\gamma}{\pi}} \int_{-\infty}^{+\infty} e^{-\beta^2/(8\gamma)} d\beta.
\]  

The NCRLB for \( I(\gamma) \) can also be found in a similar way by replacing \( P(\gamma) \) in (11) with \( J(Y) \). Unfortunately, there is no simple form to express

\[
\frac{\partial J(Y)}{\partial \gamma}.
\]

The NCRLBs for DA estimation of \( \mu, \sigma, \gamma, \lambda \), and \( P(\gamma) \) are easily found by letting \( f(\gamma) = 0 \) in (14)–(21).

### IV. ESTIMATORS

Define moment \( k \) of \( Y \) as

\[
M_k \triangleq \mathbb{E} \{ Y^k \} \approx \frac{1}{N} \sum_{n=1}^{N} y_n^k,
\]

which can be approximated by its sample average [4]. The second moment of \( Y \) is

\[
M_2 = \mu^2 + \sigma^2.
\]  

Assume that an estimate of \( \gamma \) exists, denoted by \( \hat{\gamma} \). Combining (25) with (4) and (10) gives the following estimators for \( \mu, \sigma, \gamma, \lambda \), and \( P(\gamma) \)

\[
\hat{\gamma} = \sqrt{\frac{8\gamma + 16\gamma^2}{M_2}}.
\]  

The next sub-sections present different estimators for \( \gamma \) that can be used to estimate the above parameters.
A. Conventional Method

The absolute moment (AM) of $Y$ is defined as [2, 12]

$$A \triangleq E\{|Y|\} = \mu + \sigma \sqrt{\frac{2}{\pi}} e^{-\frac{\mu^2}{2\sigma^2}} - 2\mu Q\left(\frac{\mu}{\sigma}\right),$$  
(27)

and can also be approximated by the sample average [4]

$$A \approx \frac{1}{N} \sum_{n=1}^{N} |y_n|.$$  
(28)

For large $\mu$ or small $\sigma$, the AM will tend to $\mu$

$$\lim_{\sigma \to 0} A = \lim_{\mu \to \infty} A = \mu.$$  
(29)

In other words, for high values of $\gamma$, $\hat{\gamma}$ can be closely approximated by

$$\hat{\gamma} = \frac{A^2}{2(M_2 - A^2)} = \frac{\sigma^2}{2 \left(1 - \frac{\sigma^2}{M_2}\right)},$$  
(30)

using [4] and [23]. This estimator was first introduced in [13] and will here be referred to as the conventional method (CM) estimate.

B. Maximum Likelihood

The ML estimator maximizes the joint PDF in [5]. Taking the partial derivatives of $\ln p_Y(y)$ gives

$$\frac{\partial \ln p_Y(y)}{\partial \mu} = \frac{N}{\sigma^2} \left(\frac{1}{N} \sum_{n=1}^{N} y_n \tanh\left(\frac{\mu y_n}{\sigma^2}\right) - \mu\right),$$  
(31)

$$\frac{\partial \ln p_Y(y)}{\partial \sigma} = \frac{N}{\sigma^3} \left(\mu^2 - \sigma^2 + \frac{1}{N} \sum_{n=1}^{N} y_n^2 - \frac{2\mu}{N} \sum_{n=1}^{N} y_n \tanh\left(\frac{\mu y_n}{\sigma^2}\right)\right).$$  
(32)

Setting the derivatives in (31) and (32) to zero and solve for $\sigma^2$ gives

$$\sigma^2 = \frac{1}{N} \sum_{n=1}^{N} y_n^2 - \mu^2 = M_2 - \mu^2.$$  
(33)

Inserting (33) in (31) gives an expression depending only on $y$, $\mu$, and $M_2$, which can be solved iteratively by

$$\hat{\mu}_{k+1} = \frac{1}{N} \sum_{n=1}^{N} y_n \tanh\left(\frac{\hat{\mu}_k y_n}{M_2 - \hat{\mu}_k^2}\right),$$  
(34)

where $\hat{\mu}_k$ denotes the estimate of $\mu$ after $k$ iterations. The iteration in (34) is identical to the iteration in the EM algorithm presented in [6], but here derived in a different way. A good starting point for the iterative estimator is the CM estimate of $\mu$, $\mu_0 = A$. After $K$ iterations the SNR can be estimated by

$$\hat{\gamma} = \frac{\hat{\mu}_K^2}{2(M_2 - \hat{\mu}_K^2)},$$  
(35)

which will be referred to as the ML estimator.

C. Method of Moments

The approach of estimating a parameter based on the moments of the observables is known as the method of moments (MM) [11]. The fourth moment of $Y$ is [4]

$$M_4 \triangleq E\{Y^4\} = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.$$  
(36)

Combining (36) with (24) gives the MM estimator, e.g. [4],

$$\hat{\gamma} = \frac{\sqrt{6M_2^2 - 2M_4}}{4M_2 - 2\sqrt{6M_2^2 - 2M_4}}.$$  
(37)

where $M_2$ and $M_4$ are approximated by the sample average [22]. If $M_4 > 3M_2^2$, (37) is no longer real and $\hat{\gamma}$ is set to zero. This will be referred to as the MM estimator.

D. Absolute Moment

An estimator based on the second moment and the AM can be found by combining [23] with [27]. Unfortunately, there is no closed-form analytical solution for $\mu$ and $\sigma$ as for the MM estimator. However, dividing the square of $A$ by $M_2$ gives an expression that only depends on $\gamma$,

$$h(\gamma) \triangleq \frac{A^2}{M_2} = \frac{2\gamma}{2\gamma + 1} \left(1 + \frac{1}{\sqrt{\pi} \gamma} e^{-\gamma} - 2Q\left(\sqrt{2\gamma}\right) \right)^2.$$  
(38)

An estimator for $\gamma$ can therefore be stated as

$$\hat{\gamma} = h^{-1}\left(\frac{A^2}{M_2}\right).$$  
(39)

Since there is no closed-form solution to (39), alternative methods must be explored. In [12], a table-lookup for $h^{-1}(\alpha)$ is suggested. A different approach is to approximate $h^{-1}(\alpha)$ with a simple closed-form function, which was done in [2] as,

$$h^{-1}(\alpha) \approx \frac{1}{2} 10^{\left(-34.0516/\alpha^2 + 65.9548/\alpha - 23.6184\right)}/10.$$  
(40)

The estimator in (39), using the approximation in (40), is referred to as the second-order polynomial (P2) estimator.

From (39), it is easy to verify that

$$h(0) = \frac{2}{\pi} \approx 0.6366, \quad \text{and} \quad h(\infty) = 1.$$  
(41)

Therefore, we suggest the following approximation of $h(\alpha)$ and its inverse

$$h(\alpha) \approx 1 - \left(1 - \frac{2}{\pi}\right) (H_1 \alpha^{H_2} + 1)^{H_3},$$  
(42)

$$h^{-1}(\alpha) \approx \left(\frac{1 - \alpha^{1/H_2}}{H_1} - 1\right)^{1/H_2}.$$  
(43)

Numerical optimization, using the Nelder-Mead simplex method [14] to minimize the mean squared difference between (39) and (42) gives $H_1 = 0.6153$, $H_2 = 1.5296$, and $H_3 = -0.6575$. The estimator in (39), using the novel approximation in (43), with $\hat{\gamma} = 0$ whenever $A^2/M_2 \leq 2/\pi$, is a new approach we propose and is here referred to as the AM estimator.
Fig. 1 shows the analytical expression from (43) together with its indistinguishable approximation in (42). Since the CM estimator in (39) depends only on $A^2/M_2$ it is also shown in Fig. 1. It is clear that the function used by the CM estimator converges to the analytical one for large $\gamma$, but differs for small $\gamma$. The same figure also shows the approximation in (40). Since this polynomial approximation was only optimized between $-3$ to $3$ dB ($\gamma = 0.5–2$) [2] it differs from the analytical expression outside this region.

### E. Non-Equiprobable Symbols

Define $q$ to be the *a priori* probability of $X$ in (1)

$$ q \triangleq \Pr(X = +1). $$

The ML estimator is invariant to non-equiprobable symbols and gives the same results even if $q = 1$ [6]. It is straightforward to show that $A$, $M_2$, and $M_4$ are independent of $q$ [10]. Since the CM, the MM, the P2, and the AM estimators are based only on these quantities, they will give the same results independent of $q$.

However, when $q \neq 0.5$, the odd moments are non-zero,

$$ M_1 \triangleq \mathbb{E}\{Y\} = \mu(2q - 1) = \frac{1}{N} \sum_{n=1}^{N} y_n. $$

This means that the *a priori* probability $q$ can be estimated using $M_1$, by combining (45) with (3) and (25).

$$ \hat{q} = \frac{M_1}{2} \sqrt{\frac{1 + 2\hat{\gamma}}{2\hat{\gamma}M_2}} + \frac{1}{2}. $$

### V. Numerical Examples

The performance of the SNR estimators is evaluated based on their normalized mean squared error (NMSE)

$$ \frac{1}{L} \sum_{j=1}^{L} \frac{(\hat{\gamma}_j - \gamma)^2}{\gamma^2} $$

and their normalized bias (NB)

$$ \frac{1}{L} \sum_{j=1}^{L} \frac{\hat{\gamma}_j - \gamma}{\gamma}. $$

where $\hat{\gamma}_j$ is estimated based on $N$ samples. The number of trials was chosen to $L = 100,000$. The best estimator is an unbiased estimator with minimum NMSE [11].

Fig. 2 shows the NMSE and Fig. 3 shows the NB, both for $N = 64$ observables. The NDA and the DA NCRLB are also included as a reference, even though they are only bounds for unbiased estimators. All the estimators presented here are biased when $N = 64$, even for high $\gamma$ which is evident from Fig. 4. Different approaches to reduce the bias has been suggested, e.g., [4–6]. The CM estimator has a large NB (and therefore also a large NMSE) for low $\gamma$. For large $\gamma$ the CM estimator approaches the ML estimator, which was shown analytically in [15]. In fact, Figs. 2–3 show that all estimators, except the P2 estimator converges to the same constant NMSE and constant NB for high $\gamma$ (the NB is around 5% above the true $\gamma$). The P2 estimator only works well between -3 to 3 dB, the interval for which it was optimized. The MM estimator has the second highest NMSE for low $\gamma$. The ML estimator after $K = 10$ iterations has the lowest NMSE at -6 dB, but Fig. 5 shows that it at the same time has the second highest NB. Finally, the suggested AM estimator has almost identical performance (both in NMSE and NB) as the ML estimator for all $\gamma$, even though it has a computationally complexity that is less than the first ML iteration.

Figs. 4–5 show the NMSE and the NB for different $N$ at -2 dB. This corresponds to an $E_b/N_0$ around 1 dB for a half-rate code, e.g. the original turbo code [1]. At this low SNR, the CM estimator has bad performance. The NB saturates around 60% above the true value (not shown here), which gives the high NMSE in Fig. 4. The P2 estimator has a negative NB for large $N$ at this SNR. The ML estimator after 10 iterations and the AM estimator have a small positive NB (around 1%) for large $N$. The ML estimator and the MM estimator are the only two estimators that are unbiased for large $N$, but only the ML estimator approaches the NCRLB in Fig. 5 which makes it asymptotically optimal [11]. The second best estimator, after the ML estimator, over the whole range of $N$ is the suggested AM estimator.

### VI. Conclusions

In this paper we have derived the NDA NCRLB for the signal amplitude, the noise variance, the channel reliability constant, and the BER in an AWGN channel with BPSK modulated transmission. It was also shown that these parameters, as well as the *a priori* probability of the transmitted symbols and the instantaneous MI can all be estimated based on the SNR estimate. A novel SNR estimator with low computationally complexity was introduced and shown to be surpassed in performance only by the iterative ML estimator among previously suggested estimators. The proposed estimator performs close to the performance of the iterative ML estimator at significantly lower computationally complexity.
Fig. 2. Normalized MSE for $\hat{\gamma}$ when $N = 64$ samples are observed.

Fig. 3. Normalized bias for $\hat{\gamma}$ when $N = 64$ samples are observed.

Fig. 4. Normalized MSE for $\hat{\gamma}$ when $\gamma = -2$ dB.

Fig. 5. Normalized bias for $\hat{\gamma}$ when $\gamma = -2$ dB.

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