Research Article

New Fifth-Kind Chebyshev Collocation Scheme for First-Order Hyperbolic and Second-Order Convection-Diffusion Partial Differential Equations

Mahmoud M. Mokhtar 1 and M. H. El Dewaik 2

1 Department of Basic Science, Faculty of Engineering, Modern University for Technology and Information (MTI), El Mokattam, Egypt
2 Department of Basic Science, The British University in Egypt, El-Shorouk, Cairo, Egypt

Correspondence should be addressed to Mahmoud M. Mokhtar; drmmmokhtar@gmail.com

Received 14 July 2022; Revised 11 August 2022; Accepted 22 August 2022; Published 4 October 2022

1. Introduction

The first-order partial differential equations (PDEs) modelled various real-life and physical problems. Hyperbolic PDEs characterize the time-dependent physical systems and may be worn to model many phenomena including wave and advection transportation of the material. Advection equations form a special category of conservative hyperbolic first-order PDEs which deliver a given property at a fixed rate through a method. In advection equations, the space and time derivatives of the conserved quantity \( u(x, t) \) are proportional to each other, and the interested readers are referred to [1–4], for further implementations on hyperbolic PDE.

Spectral methods play a very important role in numerical analysis, especially in the field of numerical solution of ordinary, partial, and differential equations. The key advantage of spectral methods is that differential problems can be quickly transformed into solutions of linear or nonlinear algebraic equations. For more information on spectral methods, please see [5–13].

Chebyshev polynomials are very important in many mathematical divisions, especially in numerical analysis. The key idea of Chebyshev polynomials is that they form the foundation for the extension of the differential and integral equation solution. Four well-known Chebyshev polynomial groups are used in the literature. The author in [14] presented the extended Sturm–Liouville differential problem in the fascinating PhD thesis of Masjed-Jamei, and he introduced a basic category of orthogonal-symmetric polynomials. That class has four criteria. Some basic properties are also included, such as a seminal differential equation of order two and a generic relationship containing a three-term recursive relation. For more essential formulae about these polynomials, the reader is referred to the work of [14]. The advantage of this class is that two new types of Chebyshev polynomials, fifth and sixth, are inherited. Such polynomials were used only once in literature by seminal work in the numerical solution of fractional differential equations, and Abd-Elhameed and Youssri [15–20] first used the fifth-type Chebyshev polynomials to handle ODEs and PDEs.
1.1. Mathematical Preliminaries. This section presents the properties of the basic class of symmetric orthogonal polynomials (BCSOP) that formed in [14]. The key concept to develop this class of polynomial is focused on the use of an extended Sturm–Liouville differential problem. More precisely, in [14], the author assumed that $y = \phi_i(z)$ is a sequence of symmetric functions that satisfies the following differential equation of second order:

$$A_1(z)\phi_i''(z) + A_2(z)\phi_i'(z) + \left(\mu_i A_3(z) + A_4(z) + \frac{1 - (-1)^i}{2}A_5(z)\right)\phi_i(z) = 0,$$  \hspace{1cm} (1)$$

where $A_i(z)$, $1 \leq i \leq 5$ are independent functions, and $\{\mu_i\}$ are constants. In [14], it has been shown that $A_i(z)$, $i = 1, 3, 4, 5$ is even, and $A_2(z)$ is odd. You may obtain the desired symmetric category of orthogonal polynomials if $A_i(z), 1 \leq i \leq 5$, and $\{\mu_i\}$ are chosen as follows:

$$A_1(z) = z^2 \left( rz^2 + s \right),$$
$$A_2(z) = z \left( mz^2 + n \right),$$
$$A_3(z) = z^2,$$$A_4(z) = 0,$$
$$A_5(z) = -n, \mu_i = -i((i - 1)r + m),$$

The solution of (3) is the generalized polynomials $G_i^{m,n,r,s}(z)$ which have the explicit form:

$$G_i^{m,n,r,s}(z) = \sum_{k=0}^i \left( \binom{i}{k} \left( \binom{i}{2} \prod_{\ell=0}^{i-k-1} \frac{1}{\Gamma_{i,j,m,n,r,s}} \right) \right) \frac{i}{2} - k, \hspace{1cm} (4)$$

where

$$\Gamma_{i,j,m,n,r,s} = \frac{2j + (-1)^{i+1} + 2}{{(-1)^{i+1} + 2j + 2[2i/2]}r + m}. \hspace{1cm} (5)$$

Moreover, the author introduced orthogonal symmetric polynomials in [14], denoted by $\overline{G}_i^{m,n,r,s}(z)$ defined as

$$\overline{G}_i^{m,n,r,s}(z) = \left( \prod_{j=0}^{i-1} \Gamma_{i,j,m,n,r,s} \right) G_i^{m,n,r,s}(z). \hspace{1cm} (6)$$

The polynomials $\overline{G}_i^{m,n,r,s}(z)$ satisfy the following recurrence relation:

$$\overline{G}_{i+1}^{m,n,r,s}(t) = z\overline{G}_i^{m,n,r,s}(z) + A_{i,m,n,r,s}^{m,n,r,s}(z), \hspace{1cm} (7)$$

with the initials:

$$\overline{G}_0^{m,n,r,s}(z) = 1, \overline{G}_1^{m,n,r,s}(z) = z. \hspace{1cm} (8)$$

And

$$A_{i,m,n,r,s}^{m,n,r,s}(z) = \frac{1}{2}(m - 2r)n(1 - (-1)^i)(2r + m - 3r). \hspace{1cm} (9)$$

Many properties of $\overline{G}_i^{m,n,r,s}(t)$ may be found in [14]. There are many specific categories of important orthogonal polynomials of $\overline{G}_i^{m,n,r,s}(z)$ of the four different types of Chebyshev polynomials could be formed through the expressions:

$$T_i(z) = \overline{G}_i^{1,0,-1,1}(z),$$
$$U_i(z) = \overline{G}_i^{1,0,-1,1}(z),$$
$$V_i(z) = 2^{1/2} \overline{G}_i^{1,0,-1,1}(\sqrt{\frac{1 + z}{2}}),$$
$$W_i(z) = 2^{1/2} \overline{G}_i^{1,0,-1,1}(\sqrt{\frac{1 - z}{2}}). \hspace{1cm} (10)$$

and $T_i(z), U_i(z), V_i(z), W_i(z)$ are the first, second, third, and fourth kinds of Chebyshev polynomials. All these polynomials can be obtained as specific special cases of $\overline{G}_i^{m,n,r,s}(z)$. The two types of orthogonal polynomials in [14],
especially, Chebyshev’s fifth- and sixth-kind polynomials, may also be defined, respectively, as
\[
X_i(z) = C_{i-3,2,-1,1}^i(z),
\]
\[
Y_i(z) = C_{i-5,2,-1,1}^i(z).
\]

(11)

We focus our study on the Chebyshev fifth kind and their shifted polynomials. The property of orthogonality of \(X_i(z)\) is
\[
\int_{-1}^{1} \frac{z^2 X_i(z) X_j(z)}{\sqrt{1 - z^2}} dz = \begin{cases} 
\frac{\pi}{2} & \text{if } i = j, \\
0 & \text{otherwise}, 
\end{cases}
\]
\[
\int_{-1}^{1} \frac{z^2 Y_i(z) Y_j(z)}{\sqrt{1 - z^2}} dz = \begin{cases} 
\frac{\pi}{2} & \text{if } i = j, \\
0 & \text{otherwise}, 
\end{cases}
\]
\[
\int_{-1}^{1} \frac{z^2 X_i(z) Y_j(z)}{\sqrt{1 - z^2}} dz = 0.
\]

(12)

where \(A_{i,m,a,r,s}\) is defined in (9).

Alternatively, the orthogonality formula above is written as
\[
\int_{-1}^{1} \frac{z^2 X_i(z) X_j(z)}{\sqrt{1 - z^2}} dz = \begin{cases} 
h_i, & \text{if } i = j, \\
0, & \text{if } i \neq j. 
\end{cases}
\]
\[
\int_{-1}^{1} \frac{z^2 Y_i(z) Y_j(z)}{\sqrt{1 - z^2}} dz = \begin{cases} 
h_i, & \text{if } i = j, \\
0, & \text{if } i \neq j. 
\end{cases}
\]
\[
\int_{-1}^{1} \frac{z^2 X_i(z) Y_j(z)}{\sqrt{1 - z^2}} dz = 0.
\]

(13)

And

\[
h_i = \begin{cases} 
\frac{\pi}{2^{2i+1}}, & i \text{ even}, \\
\frac{\pi(i + 2)}{2^{2i+1}}, & i \text{ odd.} 
\end{cases}
\]

(14)

It is more reasonable to normalize fifth-type Chebyshev polynomials. For this specific purpose, we set
\[
X_i(z) = \frac{1}{\sqrt{h_i}} X_i(z).
\]

(15)

Accordingly, \(X_i(z)\) are orthonormal on \(I = [-1, 1]:(16)
\[
\int_{-1}^{1} \frac{z^2 X_i(z) X_j(z)}{\sqrt{1 - z^2}} dz = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j. 
\end{cases}
\]

1.2. Fifth Type of Shifted Orthonormal Chebyshev Polynomials (SSOCP). The SSOCP \(C_i(z)\) can be defined on \(I^* = [0, 1]\) by
\[
C_i(z) = \frac{1}{\sqrt{h_i}} (2z - 1).
\]

(17)

Also, \(h_i\) is defined in (14).

From (17), it is easy to note that \(C_i(z), i \geq 0\) are orthonormal on \(I^*\). Directly, we have
\[
\int_{0}^{1} w^* (z) C_i(z) C_j(z) dz = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{otherwise.} 
\end{cases}
\]
\[
\int_{0}^{1} w^* (z) C_i(z) C_j(z) dx = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{otherwise.} 
\end{cases}
\]
\[
\int_{0}^{1} w^* (z) C_i(z) C_j(z) dx = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{otherwise.} 
\end{cases}
\]

And \(w^* (z) = (1 - 2z^2)/\sqrt{z^2 - z^2}. The following results are needed in the sequel.

**Theorem 1.** The polynomials \(C_i(z)\) in (9) are connected with \(T_i^* (z)\) by the following formula:
\[
C_i(z) = \sum_{j=0}^{i} g_{i,j} T_j^* (z),
\]

(19)

where

\[
g_{i,j} = 2^{\frac{i-j}{2}} (-1)^j \frac{j}{\pi} \delta_{i,j}.
\]

(20)

And

\[
\delta_{i,j} = \begin{cases} 
\frac{1}{2}, & \text{if } j = 0, \\
1, & \text{if } j > 0.
\end{cases}
\]

(21)

**Proof.** See [15]

□

**Theorem 2.** The polynomials \(C_i(z)\) in (9) are connected with \(T_i^* (z)\) by two formulæ as follows:
\[
C_{2i}(z) = 2\sqrt{\pi} \sum_{r=0}^{i} (-1)^{i+r} \delta_{i,r} T_{2r}^* (z),
\]

(22)

where \(\delta_{i,r}\) is defined in (12).

And
\[
C_{2i+1}(z) = \frac{2\sqrt{2}}{\sqrt{\pi (2i + 1)}} \sum_{r=0}^{i} (-1)^{i+r} (2r + 1) T_{2r+1}^* (z).
\]

(23)

The next corollary shows the above-intended purpose.

**Corollary 1.** Chebyshev polynomials of the fifth type have the following trigonometric representations:
\[
X_{2i}(\cos \phi) = \frac{\sqrt{2}}{\pi} \frac{\cos (1 + 2i)}{\cos \phi},
\]

(24)

And
\[
X_{2i+1}(\cos \phi) = \frac{2}{\sqrt{\pi (4 + i^2 + 3)}} \frac{(3 + 2i) \cos ((2 + 2i) \phi) \cos \phi - \cos ((3 + 2i) \phi)}{\cos^2 \phi}.
\]

(25)
Proof. See [15]

The following connection theorem is needed in the sequel. The following two theorems are important.

**Theorem 3.** The analytical form $C_i(z)$ is specifically given as

$$C_i(z) = \sum_{r=0}^{i} q_{r,i} z^r,$$  \hspace{1cm} (26)

where

$$q_{r,i} = 2^{2r+\frac{3}{2}} \sqrt{\phi(2r)!} \left\{ \begin{array}{ll}
\frac{i}{2} \sum_{j=\lfloor r/2 \rfloor}^{\lfloor i/2 \rfloor} (-1)^{j-r} j \delta_j (2j+r-1)! (2j-r)!, & i \text{ even}, \\
\frac{1}{\sqrt{i(i+2)}} \sum_{j=\lfloor i/2 \rfloor}^{i-1} (-1)^{j-r} (2j+1)^2 (2j+r)! (2j-r-1)! & i \text{ odd}.
\end{array} \right.$$  \hspace{1cm} (27)

**Theorem 4.** The reflection relation (17) of the analytical relation may be stated as

$$z^m = \sum_{\ell=0}^{m} q_{m,\ell} C_\ell(z),$$  \hspace{1cm} (28)

where

$$q_{m,\ell} = \sqrt{\pi} 2^{-2m} \frac{1}{2} (2m)! \left\{ \begin{array}{ll}
\frac{2(m^2 + m + 1 + \ell)^2}{(m-\ell)! (\ell + m + 2)!} & \ell \text{ even}, \\
\frac{\ell}{2+\ell} (m-2+\ell)! (m+2+\ell)! + \frac{\sqrt{2\ell+1}}{(m+\ell)! (m-\ell)!} & \ell \text{ odd}.
\end{array} \right.$$  \hspace{1cm} (29)

Proof. See [15]

The derivative formula is also needed.

**Corollary 2.** These two identities hold for all nonnegative integer $q$:

$$D^q C_i(z) |_{t=1} = \alpha_{i,q},$$  \hspace{1cm} (30)

$$D^q C_i(z) |_{t=0} = (-1)^{i+q} \alpha_{i,q},$$

And

$$\alpha_{i,q} = \sum_{j=0}^{i} \bar{\alpha}_{i,j,q},$$  \hspace{1cm} (31)

where

$$\bar{\alpha}_{i,j,q} = 2 \sqrt{2} (-1)^{j-1} \Gamma(q+j) (-q+j+1) \Gamma(j+1/2) \left( \begin{array}{c}
\delta_{ij}, \\
\frac{j}{\sqrt{2+i}}, \\
0
\end{array} \right) \left\{ \begin{array}{ll}
i, j \text{ even}, \\
i, j \text{ odd}, \\
otherwise.
\right.$$  \hspace{1cm} (32)
where \((j)_q\) is the pochhammer symbol.

Proof. See [15] □

**Theorem 5.** (See [20]). Let \(j \geq 1\), the following relation holds

\[
DC_j(\xi) = \sum_{k=0}^{j-1} M_{k,j} C_k(\xi).
\]

And \(M_{k,j}\) are given by

\[
M_{k,j} = 2^{2-j+k}
\]

\[
\begin{cases}
(j+1)(k+1)/(k+2), & j \text{ even } k \text{ odd}, \\
(j^2 + 2j + 2k + 2)/(j), & j \text{ odd } j-k+1/2 \text{ even}, \\
(j^2 + 2j - 2k - 2)/(j), & j \text{ odd } j-k-1/2 \text{ even}, \\
0, & \text{otherwise}.
\end{cases}
\]

**Theorem 6.** (See [20]). For every \(j \geq 2\), the following relation holds

\[
D^2C_j(\xi) = \sum_{k=0}^{j-2} M_{k,j} C_k(\xi),
\]

where

\[
M_{k,j} = \sqrt{\pi}2^{2-j}j!(k+1)\left(\frac{1}{2}\left\lfloor\frac{j-3}{2}\right\rfloor + 3\right)(j+k)\left(\frac{j-k}{2} - 1\right)\Gamma\left(\frac{1}{2}(-j+k+7) + \left\lfloor\frac{j-3}{2}\right\rfloor\right)\Gamma\left(\frac{1}{2}(-j+k+2) + \left\lfloor\frac{j}{2}\right\rfloor\right)
\]

\[
\times\text{$_4F_3$}\left(-\frac{j-k}{2},\frac{j-k+1}{2},\frac{1}{2}\left\lfloor\frac{j+1}{2}\right\rfloor - 1,\frac{j+1}{2}\right|1\right).
\]

\[
D_t v = \xi_1 D_x v + \xi_2 v + \delta, x \in [0,1], t \in [0,1],
\]

subject to the initial condition

\[
v(x, 0) = k_0(x), x \in [0,1].
\]

And the boundary condition

\[
v(0, t) = k_1(t), t \in [0,1],
\]

2. **Implementation of the Method**

The aim of this section is to obtain two numerical algorithms for the solution of the first-order hyperbolic differential equation and the second-order convection-diffusion equation.

2.1. **First-Order Hyperbolic Equation.** Consider the following first-order hyperbolic partial differential equation

\[
v_t(x, t) = k_1 v(x, t), x \in [0,1], t \in [0,1],
\]

subject to the initial condition

\[
v(x, 0) = k_0(x), x \in [0,1].
\]

And the boundary condition

\[
v(0, t) = k_1(t), t \in [0,1],
\]
where $\xi_1, \xi_2$ are two constants. We expand the exact solution $v$, the derivatives $D_t v$ and $D_x v$ by the fifth-type Chebyshev expansion as

$$v(x, t) \approx v_N(x, t) = \sum_{i,j=0}^{N} a_{ij} C_i(x) C_j(t),$$

$$\frac{\partial v_N(x, t)}{\partial t} = \sum_{i,j=0}^{N} a_{ij} C_i(x) \frac{\partial C_j(t)}{\partial t},$$

$$\frac{\partial v_N(x, t)}{\partial x} = \sum_{i,j=0}^{N} a_{ij} \frac{\partial C_i(x)}{\partial x} C_j(t),$$

$$R_N(x, t) = D_t v_N - \xi_1 D_x v_N - \xi_2 v_N - \delta.$$

We apply the typical tau method and make use of the boundary conditions to get a system of $(N + 1) \times (N + 1)$ algebraic equations in the required double-shifted fifth-kind Chebyshev coefficients $a_{ij}, i, j = 0, 1, \ldots, N$ that can be solved using any standard iteration technique, such as the iteration method of Newton. It is therefore possible to evaluate the semianalytic solution $v_N(x, t)$.

2.2: Second-Order Convection-Diffusion Equation. Consider the following second-order convection-diffusion equation:

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} (x, t) \in (0, 1) \times (0, 1),$$

subject to the initial condition

$$u(x, 0) = f (x), x \in (0, 1).$$

And the boundary conditions

$$u(0, t) = g_0(t), u(1, t) = g_1(t), t \in (0, 1),$$

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} (x, t) \in (0, 1) \times (0, 1),$$

subject to the initial condition

$$u(x, 0) = f (x), x \in (0, 1).$$

And the boundary conditions

$$u(0, t) = g_0(t), u(1, t) = g_1(t), t \in (0, 1),$$

where $\alpha, \beta$ are two constants. We expand the exact solution in terms of the shifted fifth-kind Chebyshev polynomials and derivatives of $u$ based on the above-mentioned derivatives theorems as

$$u(x, t) = u_N(x, t) = \sum_{i,j=0}^{N} a_{ij} C_i(x) C_j(t),$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{i,j=0}^{N} a_{ij} M_{K,i} C_i(x) C_j(t),$$

$$R(x, t) = \sum_{i,j=0}^{N} a_{ij} M_{K,i} C_i(x) C_j(t)
+ \sum_{i,j=0}^{N} \delta_{ij} M_{K,i} C_i(x) C_j(t)
- \sum_{i,j=0}^{N} a_{ij} M_{K,i} C_i(x) C_j(t).$$

$$\sum_{i,j=0}^{N} a_{ij} C_i(x) C_j(0) = f (x),$$

$$\sum_{i,j=0}^{N} a_{ij} C_i(0) C_j(0) = g_0(t),$$

$$\sum_{i,j=0}^{N} a_{ij} C_i(1) C_j(1) = g_1(t).$$

Applying the inner product and using the orthogonality relation, we get

$$\sum_{i,j=0}^{N} \sum_{k=0}^{N} a_{ij} M_{K,i} \delta_{i,j} h_k h_j = \sum_{i,j=0}^{N} \sum_{k=0}^{N} \delta_{i,j} h_k h_j h_i = 0,$$

$$\sum_{i,j=0}^{N} a_{ij} \delta_{i,j} h_i C_j(0) = f_m,$$

$$\sum_{i,j=0}^{N} a_{ij} \delta_{i,j} h_i C_j(0) = g_{m,0},$$

$$\sum_{i,j=0}^{N} a_{ij} C_i(1) \delta_{i,j} h_i = g_{m,1}.$$
The obtained system of algebraic equations is solved by the use of Gaussian elimination to get the unknown expansion coefficients and, hence, the numerical solution.

3. Discussion of Convergence and Error Analysis

The following lemma is needed.

**Lemma 1.** The fifth-kind Chebyshev polynomials \( C_x(t) \) satisfy the following inequality:

\[
|C_x(t)| < (2 + \ell) \sqrt{\frac{2}{\pi}} \forall t \in (0, t),
\]

(49)

**Theorem 7.** If \( f(t) \in L_{\infty}^T \) (the set of all square Lebesgue integrable functions), \( |f^{(3)}(t)| \leq L \), and its expansion is

\[
f(t) = \sum_{\ell=0}^{\infty} a_{\ell}C_x(t).
\]

(50)

The series in (50) uniformly converges to \( f(t) \). Also, we have

\[
|a_{\ell}| < \sqrt{\frac{2\pi}{2\ell}}, \text{for all } \ell > 3.
\]

(51)

**Theorem 8.** Let \( u(t) \) satisfies the conditions of Theorem 7, \( u_{N+1}(t), u_N(t) \) are two approximate solutions of \( u(t) \), and we define \( \tau_N(t) = u_{N+1}(t) - u_N(t) \), and then, we get the following estimation of the error:

\[
\|\tau_N(t)\|_{L^2} = O\left(\frac{1}{N^3}\right),
\]

(52)

where \( \|\tau_N(t)\|_{L^2} \) means the \( L^2 \)-norm of \( \tau_N(t) \).

**Theorem 9.** Let \( f(t) \) ascertain conditions of Lemma 1, let \( e_N(t) = \sum_{\ell=0}^{\infty} a_{\ell}C_x(t) \) be the truncation error, and then, \( e_N(t) \) satisfy the following estimate

\[
|e_N(t)| < \frac{3L}{N^2}
\]

(53)

**Theorem 10.** Assume that \( v(x,t) \) is separable \( C^3 \) function with bounded third-order derivatives, and then, the coefficients in (31) satisfy:

\[
|a_{ij}| < \Omega \frac{\ell}{j^3} \forall i, j > 3,
\]

(54)

where \( \Omega \) is a generic positive constant.

**Proof.** The proof is a direct consequence of Theorem 6.

**Theorem 11.** If \( v(x,t) \) satisfies the conditions of Theorem 10 and \( v_N(x,t) \) is the approximation of \( v(x,t) \), then we get the following estimation of the error:

\[
\|v - v_N\|_{L^2} \leq O(N^{-3}),
\]

(55)

where \( \|\cdot\|_{L^2} \) denotes the \( L^2 \)-norm.

**Proof.** The proof is a direct consequence of Theorem 6.

Using the orthogonality of \( C_x(t)C_y(t) \) and with the aid of Theorem 9 and applying Parseval’s identity, we get the desired result.

4. Numerical Results

We offer some numerical tests in this section to illustrate the precision, efficacy, and the wide applicability of the proposed system. We compare our method with Laguerre-Gauss-Radau scheme [21] which shows that our method is very
efficient. Therefore, we affirm that the proposed technique is more suitable for solving problems of this type. The following tables depict the absolute errors.

\[ E(x, t) = |y_{N,M}(x, t) - y(x, t)|, \]  

(56)

where \( y_{N,M}(x, t) \) is the numerical, and \( y(x, t) \) is the exact solution at the node \((x, t)\). The point-wise errors are evaluated by

\[ E = \max_{(x,t) \in [0,1] \times [0,1]} E(x,t). \]  

(57)

**Example 1** (see [21]). Let us start with the following hyperbolic PDE:

\[ D_t y = D_x y + y + s, \quad x \in [0,1], t \in [0,1], \]  

(58)

with the initials

\[ y(0, t) = e^{-\sqrt{t}t}, \quad y(x, 0) = e^{-x}, \quad x \in [0,1], t \in [0,1], \]  

(59)

where

\[ s(x, t) = -\sqrt{2}e^{-\sqrt{t}t - x}. \]  

(60)

The smooth solution is given by

\[ y = e^{-t\sqrt{t} t + x}. \]  

(61)

In Tables 1–3, we compare between our method for the case \( N = 16, t = 0.1, 0.5, 1 \). These are obtained by the generalized collocation method Laguerre–Gauss–Radau [21]. In
Example 2 (see [21]). And, then the following hyperbolic PDE:

\[
\frac{\partial y(x,t)}{\partial t} = \frac{\partial y(x,t)}{\partial x} + \gamma (x,t) = -\left( \sin(t)e^{-t-x} - \cos(t)e^{-t-x} \right), \quad x \in [0,1], t \in [0,1],
\]

with the initials,

\[
\theta_1 = \theta_2 = \frac{1}{2}, \quad \theta_1 = \theta_2 = 0, \quad \theta_1 = \theta_2 = \frac{1}{2}, \quad \theta_2 = \theta_1 = 0.
\]

Table 5: Absolute errors for Example 2 when \( N = 16. \)

| \( (x,t) \) | \( \theta_1 = \theta_2 = \frac{1}{2} \) | \( \theta_1 = \theta_2 = 0, \theta_1 = \theta_2 = 0 \) | \( \theta_1 = \theta_2 = \frac{1}{2}, \theta_2 = \theta_1 = 0 \) | Current method |
|-------------|---------------------------------|---------------------------------|---------------------------------|----------------|
| (0.1, 0.1)  | 9.61 \times 10^{-7}             | 4.22 \times 10^{-7}            | 1.95 \times 10^{-7}            | 1.98 \times 10^{-19} |
| (0.2, 0.2)  | 6.37 \times 10^{-7}             | 5.92 \times 10^{-7}            | 7.08 \times 10^{-7}            | 2.35 \times 10^{-19} |
| (0.3, 0.3)  | 9.63 \times 10^{-7}             | 6.47 \times 10^{-7}            | 1.67 \times 10^{-7}            | 1.17 \times 10^{-19} |
| (0.4, 0.4)  | 1.44 \times 10^{-6}             | 8.90 \times 10^{-7}            | 7.52 \times 10^{-7}            | 6.03 \times 10^{-20} |
| (0.5, 0.5)  | 2.04 \times 10^{-6}             | 9.36 \times 10^{-7}            | 2.47 \times 10^{-7}            | 7.97 \times 10^{-17} |
| (0.6, 0.6)  | 1.28 \times 10^{-8}             | 4.81 \times 10^{-8}            | 1.35 \times 10^{-7}            | 6.08 \times 10^{-14} |
| (0.7, 0.7)  | 2.23 \times 10^{-6}             | 1.23 \times 10^{-6}            | 8.48 \times 10^{-7}            | 2.73 \times 10^{-12} |
| (0.8, 0.8)  | 7.97 \times 10^{-7}             | 4.56 \times 10^{-7}            | 1.16 \times 10^{-7}            | 3.94 \times 10^{-11} |
| (0.9, 0.9)  | 3.33 \times 10^{-6}             | 1.55 \times 10^{-6}            | 6.54 \times 10^{-7}            | 2.51 \times 10^{-10} |

Figure 3: The solution \( y(x,t) \) for \( N = 16 \) in Example 2.

Figure 4: The error in Example 2.
In this example, we consider

\[ y(0,t) = \sin(t)e^{-t}, t \in [0,1], y(x,0) = 0, x \in [0,1]. \]  \hspace{1cm} (63)

The exact smooth solution is

\[ y(x,t) = e^{-t-x} \sin(t). \] \hspace{1cm} (64)

Table 5 presents the results developed by our method when \( N = 16 \) and the method in [21]; in Figure 3, we illustrate the solution \( y(x,t) \), while in Figure 4, we depict the error when \( N = 16 \).

**Example 3** (see [22]). In this example, we consider

\[ f(x) = \exp \left( \frac{(x+0.5)^2}{0.00125} \right). \]

\[ g_0(0,t) = \frac{0.25}{\sqrt{0.000625 + 0.02t}} \exp \left( -\frac{(0.5-t)^2}{0.00125 + 0.04t} \right). \]

\[ g_1(1,t) = \frac{0.25}{\sqrt{0.000625 + 0.02t}} \exp \left( -\frac{(1.5-t)^2}{0.00125 + 0.04t} \right). \]

\[ \alpha = 0.01, \beta = 1.0, \] \hspace{1cm} (65)

with the exact smooth solution

\[ u(x,t) = \frac{0.25}{\sqrt{0.000625 + 0.02t}} \exp \left( -\frac{(x+0.5-t)^2}{0.00125 + 0.04t} \right). \] \hspace{1cm} (66)

In Table 6, we list the maximum pointwise error for different values of \( N = 16 \). In Figure 5, we depict the maximum absolute error when \( N = 16 \).

| \( N \)  | 2    | 4    | 6    | 8    | 10   | 12   | 14   | 16   |
|--------|------|------|------|------|------|------|------|------|
| MAE    | 4.10 \cdot 10^{-1} | 2.54 \cdot 10^{-2} | 4.26 \cdot 10^{-3} | 5.28 \cdot 10^{-5} | 5.67 \cdot 10^{-6} | 5.63 \cdot 10^{-8} | 2.67 \cdot 10^{-10} | 2.69 \cdot 10^{-13} |

**5. Conclusions**

A precise numerical technique for solving the hyperbolic partial differential equations is being constructed and applied in the present work. The fifth-type approach of Chebyshev spectral tau was used to simplify the solution of hyperbolic partial differential equations to a set of algebraic equations, which can be more conveniently solved. The numerical findings showed our method to be extremely effective and reliable.

**Data Availability**

No data were associated with this research.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to this work.

**References**

[1] A. N. Kochubei, *Fractional-hyperbolic Equations and Systems. Cauchy Problem*, pp. 197–222, Fractional Differential Equations, 2019.

[2] E. H. Doha, R. M. Hafez, and Y. H. Youssri, “Shifted Jacobi spectral-galerkin method for solving hyperbolic partial differential equations,” Computers & Mathematics with Applications, vol. 78, no. 3, pp. 889–904, 2019.

[3] J. Auriol and F. Di Meglio, “An explicit mapping from linear first order hyperbolic pdes to difference systems,” Systems & Control Letters, vol. 123, pp. 144–150, 2019.

[4] S. Cheng, L. I. Xin-zhu, Y. Yong, and Y. A. N. G. Zai-lin, “Overlapping grid method for solving the partial differential equations using the sbp-sat technique,” in *Proceedings of the 2019 Symposium on Piezoelectricty, Acoustic Waves and Device Applications (SPAWDA)*, pp. 1–6, IEEE, Harbin, China, April 2019.

[5] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Courier Corporation, North Chelmsford, 2001.

[6] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods*, Springer, Berlin, 2006.

[7] J. Shen, T. Tang, and Li-L. Wang, “Spectral methods: algorithms, analysis and applications,” *Springer Science & Business Media*, vol. 41, 2011.

[8] A. S. Mohamed and M. M. Mokhtar, “Spectral tau-Jacobi algorithm for space fractional advection–dispersion problem,” *Applications and Applied Mathematics: International Journal*, vol. 14, no. 1, pp. 548–561, 2019.

[9] N. H. Sweilam, A. M. Nagy, I. K. Youssef, and M. M. Mokhtar, “New spectral second kind Chebyshev wavelets scheme for solving systems of integro-differential equations,” *International Journal of Algorithms, Computing and Mathematics*, vol. 3, no. 2, pp. 333–345, 2017.
[10] N. H. Sweilam, A. M. Nagy, and M. M. Mokhtar, "On the numerical treatment of a coupled nonlinear system of fractional differential equations," *Journal of Computational and Theoretical Nanoscience*, vol. 14, no. 2, pp. 1184–1189, 2017.

[11] M. Izadi, Ş. Yüzbaşı, and W. Adel, "A new Chebyshev matrix method to solve linear and nonlinear fractional delay differential equations with error analysis," *The Mathematical Scientist*, 2022.

[12] M. Izadi, H. M. Srivastava, and W. Adel, "An effective approximation algorithm for second-order singular functional differential equations," *Axioms*, vol. 11, no. 3, p. 133, 2022.

[13] M. Izadi, Ş. Yüzbaşı, and W. Adel, "Accurate and efficient matrix techniques for solving the fractional Lotka–Volterra population model," *Physica A: Statistical Mechanics and Its Applications*, vol. 600, Article ID 127558, 2022.

[14] M. Masjed-Jamei, *Some New Classes of Orthogonal Polynomials and Special Functions: A Symmetric Generalization of Sturm-Liouville Problems and its Consequences*, PhD thesis, Germany, 2006.

[15] W. M. Abd-Elhameed and Y. H. Youssri, "Fifth-kind orthonormal Chebyshev polynomial solutions for fractional differential equations," *Computational and Applied Mathematics*, vol. 37, no. 3, pp. 2897–2921, 2018.

[16] W. M. Abd-Elhameed and Y. H. Youssri, "Sixth-kind Chebyshev spectral approach for solving fractional differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 20, no. 2, pp. 191–203, 2019.

[17] A. G. Atta, W. M. Abd-Elhameed, G. M. Moatimid, and Y. H. Youssri, "A fast galerkin approach for solving the fractional Rayleigh–Stokes problem via sixth-kind Chebyshev polynomials," *Mathematics*, vol. 10, no. 11, p. 1843, 2022.

[18] A. G. Atta, W. M. Abd-Elhameed, and Y. H. Youssri, "Shifted fifth-kind Chebyshev polynomials Galerkin-based procedure for treating fractional diffusion-wave equation," *International Journal of Modern Physics C*, vol. 33, no. 8, Article ID 2250102, 2022.

[19] A. G. Atta, W. M. Abd-Elhameed, G. M. Moatimid, and Y. H. Youssri, "Advanced shifted sixth-kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem," *Mathematical Sciences*, pp. 1–15, 2022.

[20] W. M. Abd-Elhameed and Y. H. Youssri, "New formulas of the high-order derivatives of fifth-kind Chebyshev polynomials: spectral solution of the convection–diffusion equation," *Numerical Methods for Partial Differential Equations*, 2021.

[21] A. H. Bhrawy, R. M. Hafez, E. O. Alzahrani, B. Dumitru, and A. A. Alzahrani, "Generalized laguerregauss-radau scheme for first order hyperbolic equations on semi-infinite domains," vol. 16, no. 4, pp. 490–498, 2015.

[22] M. Dehghan, "Weighted finite difference techniques for the one-dimensional advection–diffusion equation," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 307–319, 2004.