Duality in linearized gravity

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We show that duality transformations of linearized gravity in four dimensions, i.e., rotations of the linearized Riemann tensor and its dual into each other, can be extended to the dynamical fields of the theory so as to be symmetries of the action and not just symmetries of the equations of motion. Our approach relies on the introduction of two “superpotentials”, one for the spatial components of the spin-2 field and the other for their canonically conjugate momenta. These superpotentials are two-index, symmetric tensors. They can be taken to be the basic dynamical fields and appear locally in the action. They are simply rotated into each other under duality. In terms of the superpotentials, the canonical generator of duality rotations is found to have a Chern-Simons like structure, as in the Maxwell case.

I. INTRODUCTION

Duality is a fascinating symmetry which keeps appearing in many contexts. Originally developed for electromagnetism, where duality invariance of the Maxwell equations leads to the introduction of magnetic sources and the quantization of electric charges [1], duality has been at the origin of many remarkable developments in Yang-Mills theory [2]. The generalization of duality to extended objects and p-form gauge fields was carried out in [3, 4]. More recently, duality has revolutionized the understanding of string theory by providing non-perturbative insight. These latter developments indicate that duality should play a central role in gravitation theory as well (for a recent review, see [5]).

The idea that duality should be important in gravitation theory has in fact a long history that goes back to the recognition that the mixed space-time components $g_{i0}$ of the metric are analogous, in the stationary context, to the spatial components of the vector potential of Maxwell theory (“gravitomagnetism”). This enables one to interpret, for instance, the Taub-NUT solution as the metric of an object that carries both electric-type and magnetic-type gravitational masses. More generally, duality appears among the “hidden symmetries” of dimensionally-reduced gravitational theories [6], a subject that has received recently a new impetus [7, 8].

Understanding gravitational duality in the general, Killing vector free situation, remains, however, a challenge which is still unsolved to this date. In this article, we investigate gravitational duality in the linearized limit but without assuming the existence of spacetime symmetries so that usual dimensional reduction techniques are not available. Our main result is the proof that the action of standard linearized four-dimensional Einstein gravity is duality invariant. This implies that duality defines a conserved Noether charge, which acts as the generator of duality rotations through the Poisson bracket (classically) or the commutator (quantum-mechanically). The existence of a Noether generator would not hold if duality was a mere equations-of-motion symmetry as it is sometimes assumed. Furthermore, duality-invariance is clearly independent (at the linearized level) from dimensional reduction since it holds for arbitrary field configurations.

Our approach relies on the introduction of two local “superpotentials”, which are two-index, symmetric tensors. In terms of these superpotentials, the linearized Einstein action takes a form similar to that of the Maxwell action. One can then follow the approach of [9], where duality-invariance of the Maxwell action was established. The Noether charge exhibits an interesting Chern-Simons like structure when expressed in terms of the superpotentials.

A. $SO(2)$ duality-symmetry of spin-2 free field equations in four spacetime dimensions

The linearized Riemann tensor $R_{\lambda\mu\rho\sigma} = -R_{\mu\lambda\rho\sigma} = -R_{\lambda\mu\sigma\rho}$ fulfills the following identities,

$$R_{\lambda[\mu\rho\sigma]} = 0,$$

$$R_{\lambda\mu\rho\sigma\alpha} = 0.$$  (1.1)  (1.2)
It follows from (I.1) that $R_\lambda^\mu_\rho_\sigma$ is symmetric for the exchange of the pairs $(\lambda, \mu)$ and $(\rho, \sigma)$, $R_\lambda^\mu_\rho_\sigma = R_\rho^\sigma_\lambda_\mu$. Moreover, the identities (I.1) and (I.2) imply the familiar fact that there exists a symmetric tensor gauge field $h_\mu_\nu = h_\nu_\mu$ of which $R_\lambda^\mu_\rho_\sigma$ is the curvature,

$$ R_\lambda^\mu_\rho_\sigma = \partial_\lambda h_\mu_\nu^{[\rho, \sigma]} $$

In the absence of sources, the linearized Einstein equations take the form

$$ R_\mu_\nu = 0, $$

where $R_\mu_\nu$ is the linearized Ricci tensor. It follows that the dual $S_\lambda^\mu_\rho_\sigma = -S_\rho_\sigma^\lambda_\mu$ of the curvature, defined by

$$ S_\lambda^\mu_\rho_\sigma = \frac{1}{2} \epsilon_{\lambda\alpha_\beta_\gamma} R_{\gamma_\alpha_\beta_\rho_\sigma}, $$

enjoys also the properties

$$ S_\lambda^{[\mu_\rho_\sigma]} = 0, $$

$$ S_{\lambda_\mu_\rho_\sigma_\alpha} = 0 $$

(implying the existence of a dual potential) and

$$ S_\mu_\nu = 0. $$

Our conventions are as follows: the Minkowskian metric is $\eta_{\mu_\nu} = \text{diag}(-1,1,1,1)$ while $\epsilon^{0123} = 1 = -\epsilon_{0123}$. Indices are lowered, raised and contracted with the Minkowskian metric, e.g., $R_\mu_\nu = R_{\alpha_\mu_\beta_\nu} \eta^{\alpha_\beta}$ and $S_\mu_\nu = S_{\alpha_\mu_\beta_\nu} \eta^{\alpha_\beta}$. Square brackets $[\ ]$ denote antisymmetrization with strength one, e.g., $F_{[\lambda_\mu]} = (1/2)(F_{\lambda_\mu} - F_{\mu_\lambda})$.

Comparing (I.1), (I.2) and (I.4) with (I.6), (I.7) and (I.8) shows that the equations of the vacuum linearized Einstein theory are invariant under duality transformations, in which the curvature and its dual are rotated into each other,

$$ R'_\lambda^\mu_\rho_\sigma = \cos \alpha R_{\lambda^\mu_\rho_\sigma} + \sin \alpha S_{\lambda^\mu_\rho_\sigma}, $$

$$ S'_\lambda^\mu_\rho_\sigma = -\sin \alpha R_{\lambda^\mu_\rho_\sigma} + \cos \alpha S_{\lambda^\mu_\rho_\sigma} $$

[Actually, the equations remain invariant if we replace the rotation in (I.9), (I.10) by a general invertible matrix. However, as we will see in what follows, only rotations will leave the action invariant, just as for electromagnetism.]

It is useful to rewrite the duality transformations in terms of the electric and magnetic components of the Weyl tensor (which coincides on-shell with the Riemann tensor). One defines

$$ E_{mn} = R_{0mn0}, \quad B_{mn} = -\frac{1}{2} \epsilon_{npq} R_{0mn}^{pq} $$

The electric and magnetic tensors $E_{mn}$ and $B_{mn}$ are both traceless and symmetric on-shell. Thus, they have 5 independent components each, corresponding to the 10 independent components of the Weyl tensor. It is easy to verify that the transformations (I.9) and (I.10) are equivalent to

$$ E'_{mn} = \cos \alpha E_{mn} + \sin \alpha B_{mn}, $$

$$ B'_{mn} = -\sin \alpha E_{mn} + \cos \alpha B_{mn} $$

when the equations of motion hold.

**B. Is duality a symmetry of the action?**

The question investigated in this paper is: do the duality rotations (I.9) and (I.10) define symmetries of the action - and thus of the theory?

In [9], a similar question was asked for the Maxwell theory. It was shown that duality rotations of the field strength $F_{\mu_\nu}$ into its dual $* F_{\mu_\nu}$ do define symmetries of the Maxwell action. This might seem surprising at first sight since the Maxwell Lagrangian $\sim (E^2 - B^2)$ is not invariant under the (Euclidean) rotations $E' = \cos \alpha E + \sin \alpha B$, $B' = -\sin \alpha E + \cos \alpha B$. As explained in [9], this computation (i.e., evaluating the variation of the Lagrangian under the duality rotations of the curvatures just written) is meaningless, however, because the dynamical variables in the
manifest duality invariance is achieved at the cost of manifest Lorentz invariance.

analogous to the double potential formulation of electromagnetism of Refs [9, 10]. In terms of these superpotentials, potential components under duality and it turns out that these are such that the variations of the curvature take a different form off-shell than the ones written above. When the correct transformation rules are used, one finds that the Maxwell action is invariant under duality rotations [9].

We show in this paper that the same property holds for linearized gravity, described by the Pauli-Fierz action

\[ S[h_{\mu\nu}] = -\frac{1}{4} \int d^4x \left( \partial^\rho h^{\mu\nu} \partial_\rho h_{\mu\nu} - 2 \partial_\mu h^{\mu\nu} \partial_\rho h^\rho_\nu + 2 \partial^\mu h^\rho_\rho \partial^\nu h_{\mu\nu} - \partial^\mu h^\rho_\rho \partial_\mu h^\sigma_\sigma \right). \]  

(I.14)

By “lifting” the transformations (I.9) and (I.10) to the fields \( h_{\mu\nu} \), we are able to prove that the action is duality-invariant. We also compute the corresponding conserved charge.

The proof of duality-invariance rests on the introduction of two spatial superpotentials, leading to a formulation analogous to the double potential formulation of electromagnetism of Refs [9, 10]. In terms of these superpotentials, manifest duality invariance is achieved at the cost of manifest Lorentz invariance.

II. SUPERPOTENTIALS

A. Hamiltonian form of the action

As in [9], we work in the Hamiltonian formalism. Any symmetry of the Hamiltonian action is a symmetry of the original second order action when the momenta (which can be viewed as auxiliary fields) are eliminated through their own equations of motion (see concluding section below).

When written in Hamiltonian form, the Pauli-Fierz action (I.14) becomes

\[ S[h_{mn}, \pi^{mn}, n, n_m] = \int dt \left[ \int d^3x \pi^{mn} \dot{h}_{mn} - H - \int d^3x \left( nH + n_m H^m \right) \right] \]  

(II.1)

where \( \pi^{mn} \) are the conjugate momenta to the spatial components \( h_{mn} \) of the spin-2 field, while \( n \) and \( n_m \) are respectively the linearized lapse and (minus 2 times) the linearized shift. The Hamiltonian \( H \) reads

\[ H = \int d^3x \left[ \pi^{mn} \pi_{mn} - \frac{1}{2} \pi^2 + \frac{1}{4} \partial^m h^{mn} \partial_m h_{mn} - \frac{1}{2} \partial_m h^{mn} \partial_r h^r_m + \frac{1}{2} \partial^m h \partial^n h_{mn} - \frac{1}{4} \partial^m h \partial_m h \right] \]  

(II.2)

where \( h \equiv h^m_m \) is the trace of the spatial \( h_{mn} \) and \( \pi \equiv \pi^m_m \) is the trace of \( \pi^{mn} \). The constraints, obtained by varying the action with respect to the lapse and the shift, are \( \mathcal{H} = 0 \) and \( \mathcal{H}^m = 0 \) with

\[ \mathcal{H} = \partial^m \partial^n h_{mn} - \Delta h \]  

(II.3)

\[ \mathcal{H}^m = \pi^{mn}, n \]  

(II.4)

where \( \Delta \equiv \partial^m \partial_m \) is the spatial Laplacian.

B. Solution of the momentum constraint - First Superpotential

In order to exhibit the duality symmetry, we solve the constraints. This can be achieved while maintaining locality of the action principle by introducing “superpotentials”.

The general solution of the constraint \( \mathcal{H}^m = 0 \) is (see Appendix)

\[ \pi^{mn} = \partial_\rho \partial_r U^{mqr} \]  

(II.5)

where the tensor \( U^{mqr} \) has the antisymmetry properties \( U^{mpnr} = -U^{mpnr} = U^{nrmp} = -U^{mprn} \). By dualizing \( U^{mqr} \) in terms of a symmetric tensor \( P_{qs} = P_{sq} \),

\[ U^{mqr} = \epsilon^{mpq} \epsilon^{rs} P_{qs}, \]  

(II.6)

this expression can be rewritten

\[ \pi^{mn} = \epsilon^{mpq} \epsilon^{nrs} \partial_\rho \partial_r P_{qs} \]  

\[ = \delta^{mn} (\Delta P - \partial^r \partial^s P_{rs}) + \partial^m \partial^r P^r_m + \partial^n \partial^r P^m_r - \partial^m \partial^n P - \Delta P^{mn} \]  

(II.7)

(II.8)
where $P$ is the trace of $P_{mn}$. We shall call the symmetric tensor $P_{mn}$, which is equivalent to $U^{mnpq}$, “superpotential” (for the momenta).

Given $\pi^{mn}$, the superpotential $P_{mn}$ is determined up to

$$P_{mn} \rightarrow P_{mn} + \partial_m \xi_n + \partial_n \xi_m.$$  

A quick way to see this is to observe that $\pi^{mn}$ may be viewed as the Einstein tensor of $P_{mn}$ regarded as a metric. In three dimensions, the Einstein tensor completely determines the Riemann tensor, and hence $\pi^{mn}$ determines $P_{mn}$ up to a gauge transformation $\partial_m \xi_n + \partial_n \xi_m$. A particular solution is

$$P_{mn} = -\Delta^{-1} \pi_{mn} + \delta_{mn} \Delta^{-1} \pi$$  

(II.9)

One easily verifies that (II.9), when inserted in (II.8), reproduces $\pi^{mn}$ satisfying the constraint $\mathcal{H}^m = 0$.

One may use $P_{mn}$ instead of $\pi^{mn}$ as fundamental field in the action principle. When this is done, the momentum constraint and its Lagrange multiplier $n_{mn}$ drop out from the action principle because the constraint is identically satisfied. Although the expression of $\pi^{mn}$ is local in terms of $P_{mn}$ (which is what matters for the locality of the action expressed in terms of $P_{mn}$), the inverse transformation is non-local.

The momenta $\pi^{mn}$ are not gauge invariant but transform as $\pi^{mn} \rightarrow \pi^{mn} - \partial^m \partial^n \xi + \delta^{mn} \Delta \xi$ under the transformation generated by the Hamiltonian constraint. This transformation is simply generated by a conformal change of the superpotential $P_{mn}$, $\delta P_{mn} = \delta_{mn} \xi$. Hence, the total ambiguity in $P_{mn}$ is

$$P_{mn} \rightarrow P_{mn} + \partial_m \xi_n + \partial_n \xi_m + \delta_{mn} \xi$$  

(II.10)

The transformations (II.10) appear as gauge transformations of the formulation in which $P_{mn}$ is regarded as the fundamental field.

C. Solution of the Hamiltonian constraint - Second Superpotential

Similarly, one can also solve the “Hamiltonian constraint” $\mathcal{H} = 0$ in terms of a symmetric superpotential $\Phi_{mn} = \Phi_{nm}$ and a vector $u_m$ as

$$h_{mn} = \epsilon_{mrs} \partial^r \Phi^s \n + \epsilon_{nrs} \partial^r \Phi^s \m + \partial_m u_n + \partial_n u_m$$  

(II.11)

(see Appendix).

Given $h_{mn}$ up to a gauge transformation, there is some ambiguity in the superpotential $\Phi_{mn}$, which reads exactly as in (II.10),

$$\Phi_{mn} \rightarrow \Phi_{mn} + \partial_m \eta_n + \partial_n \eta_m + \delta_{mn} \eta$$  

(II.12)

This is also shown in the Appendix.

One can express $\Phi_{mn}$ non-locally in terms of the metric. A particular solution is

$$\Phi_{mn} = -\frac{1}{4} \Delta^{-1} (\epsilon_{mrs} \partial^r h^s \n + \epsilon_{nrs} \partial^r h^s \m)$$  

(II.13)

with

$$u_m = \frac{1}{4} \Delta^{-1} (3 \partial^r h_{prm} - \partial_m h) .$$  

(II.14)

We leave it to the reader to verify that this expression leads back to an $h_{mn}$ obeying the constraint $\mathcal{H} = 0$ when inserted in (II.11). When $\Phi_{mn}$ is used as fundamental field instead of $h_{mn}$, there is no constraint left and (II.12) appears as a gauge transformation.

Note that the first order constraint $\mathcal{H}^m = 0$ yields an expression for the momenta that involves two derivatives of the superpotential $P_{mn}$, while the second order constraint $\mathcal{H} = 0$ yields an expression for $h_{mn}$ that involves only one derivative of $\Phi_{mn}$ and $u_m$. Accordingly, the Hamiltonian is a homogeneous polynomial quadratic in the second derivatives of both superpotentials.
III. DUALITY TRANSFORMATIONS IN TERMS OF SUPERPOTTENTIALS

It turns out that the duality rotations are simply $SO(2)$ rotations of the superpotentials into each other,

\[ P_{mn}^{\prime} = \cos \alpha P^{mn} + \sin \alpha \Phi^{mn}, \]
\[ \Phi_{mn}^{\prime} = -\sin \alpha P^{mn} + \cos \alpha \Phi^{mn} \]  

(III.1)

(III.2)

To verify this assertion, we check that (III.1) and (III.2) imply (I.12) and (I.13) on-shell. To this end, we observe that on-shell, $\mathcal{E}_{mn} = (3) R_{mn}$. Substituting the expression for $h_{mn}$ in terms of the superpotential $\Phi_{mn}$, one gets

\[ E_{mn} = -\varepsilon_{mpq} \partial_n \Phi^k \Phi^q_k - \varepsilon_{npq} \partial_m \Phi^p \Phi^q_p + \varepsilon_{mpq} \partial_k \Phi^q_k \Delta \Phi^q_n + \varepsilon_{npq} \partial_k \Delta \Phi^q_m \]  

(III.3)

Similarly, one finds that

\[ B_{mn} = -\varepsilon_{mpq} \partial^q \left( \pi^q_n - \frac{1}{2} \delta^q_n \right) - \varepsilon_{npq} \partial^q \left( \pi^q_m - \frac{1}{2} \delta^q_m \right) \]

when the equations of motion hold, which easily yields

\[ B_{mn} = -\varepsilon_{mpq} \partial_n \Phi^k \Phi^q_k - \varepsilon_{npq} \partial_m \Phi^p \Phi^q_p + \varepsilon_{mpq} \partial_k \Phi^q_k \Delta \Phi^q_n + \varepsilon_{npq} \partial_k \Delta \Phi^q_m \]  

(III.4)

This expression for the magnetic components of the Weyl tensor is the same as the expression (III.3) for the electric components of the Weyl tensor, with $P_{mn}$ replacing $\Phi_{mn}$. Since these expressions are linear in $P_{mn}$ and $\Phi_{mn}$, rotations of $P_{mn}$ and $\Phi_{mn}$ into each other induce indeed the electromagnetic duality rotations.

IV. DUALITY INVARIANCE OF THE ACTION

A. Duality invariance of the Hamiltonian

We now insert the above expressions (II.8) and (II.11) in the Hamiltonian. Tiedous but straightforward computations give for the kinetic energy density (up to total derivatives that are being dropped)

\[ \pi^i \pi_{ij} - \frac{1}{2} \pi^2 = \Delta P_{ij} \Delta P_{ij} + \frac{1}{2} (\partial^k \partial^m P_{km})^2 + \partial^k \partial^m P_{km} \Delta P - 2 \partial_m \partial_i P_{ij} \partial^m \partial^k P_{kj} - \frac{1}{2} (\Delta P)^2 \]  

(IV.1)

Similarly, the potential energy density becomes (up to total derivatives)

\[ \frac{1}{4} \partial^r h^{mn} \partial_r h_{mn} - \frac{1}{2} \partial_m h^{mn} \partial_n h_{mn} + \frac{1}{2} \partial^m h \partial^m h_{mn} - \frac{1}{4} \partial^m h \partial_m h = \Delta \Phi_{ij} \Delta \Phi_{ij} + \frac{1}{2} (\partial^k \partial^m \Phi_{km})^2 + \partial^k \partial^m \Phi_{km} \Delta \Phi - 2 \partial_m \partial_i \Phi_{ij} \partial^m \partial^k \Phi_{kj} - \frac{1}{2} (\Delta \Phi)^2 \]  

(IV.2)

[This computation is simplified once it is recalled that the Hamiltonian is gauge-invariant. One may thus set $u_m = 0$ in the expression (II.11) when evaluating $H$.]

Because the kinetic and potential energies take exactly the same form in terms of their respective superpotentials, one sees that the Hamiltonian is invariant under $SO(2)$-rotations in the plane of $P_{mn}$ and $\Phi_{mn}$, i.e., the Hamiltonian is duality invariant.

Note that although not manifestly so, the Hamiltonian is positive. This can be be seen for instance by Fourier-transforming $P_{mn}$ and $\Phi_{mn}$. Assuming a single Fourier mode propagating in the third spatial direction, one gets for the energy density in $k$-space

\[ (k^3)^4 \left[ 2(P^{11} - P^{22})^2 + 8(P^{12})^2 + 2(\Phi^{11} - \Phi^{22})^2 + 8(\Phi^{12})^2 \right] \]  

(IV.3)

B. Duality invariance of the kinetic term

The invariance of the kinetic term $\pi h$ can also be checked easily. Injecting the expressions (II.8) and (II.11) into $\pi_{mn} h_{mn}$, one gets

\[ \int dt \, d^3x \, \pi_{mn} \dot{h}_{mn} = 2 \int dt \, d^3x \, \varepsilon^{mrs} (\partial^p \partial_r P_{ps} - \Delta \partial_x P_{qs}) \dot{\Phi}^q_m \]  

(IV.4)

Because this expression changes sign (up to a total derivative) under the exchange of $P_{mn}$ with $\Phi_{mn}$, it is invariant under the rotations (III.1) and (III.2) (up to a total derivative). This ends the proof of the duality-invariance of the free massless spin-2 theory in four dimensions.
C. $SO(2)$-vector notations

By introducing $SO(2)$-vector notations and adding a total derivative to make the kinetic term strictly antisymmetric under the exchange of the superpotentials, one may rewrite the free spin-$2$ action – with the superpotentials as basic dynamical fields – as

$$S[Z_{a}^{mn}] = \int dt \left[ \int d^{3}x \epsilon^{ab} \epsilon^{mrs} (\partial^{p} \partial^{q} Z_{aps} - \Delta \partial_{r} Z_{aqs}) Z_{b}^{q}_{\, m} - H \right]$$  \hspace{1cm} (IV.5)

with

$$(Z_{a}^{mn}) = (P_{mn}, \Phi_{mn}), \quad a, b = 1, 2$$  \hspace{1cm} (IV.6)

and

$$H = \int d^{3}x \delta^{ab} \left( \Delta Z_{aij} \Delta Z_{b}^{ij} + \frac{1}{2} \partial^{k} \partial^{m} Z_{akm} \partial^{q} \partial^{m} Z_{bqn} + \partial^{k} \partial^{m} Z_{akm} \Delta Z_{b} - 2 \partial_{m} \partial_{i} Z_{a}^{ij} \partial^{m} Z_{bq} - \frac{1}{2} \Delta Z_{a} \Delta Z_{b} \right)$$  \hspace{1cm} (IV.7)

with $Z_{a} \equiv Z_{a}^{mn}$.

This expression is manifestly duality invariant because the tensors $\epsilon^{ab}$ and $\delta^{ab}$ are $SO(2)$-invariant. It should be compared with the analogous expression for the Maxwell action considered in [10]. We also see that linear transformations that leave the action invariant must preserve both $\epsilon^{ab}$ and $\delta^{ab}$ and thus necessarily belong to $SO(2)$.

V. DUALITY GENERATOR

The conserved charge that generates infinitesimal duality rotations is found from the Noether theorem to be

$$Q = \frac{1}{2} \int d^{3}x \epsilon^{mrs} \left[ (\partial^{p} \partial^{q} P_{ps} - \Delta \partial_{r} P_{qs}) P_{mq}^{p} - (\partial^{p} \partial^{q} \Phi_{ps} - \Delta \partial_{r} \Phi_{qs}) \Phi_{mq}^{p} \right]$$  \hspace{1cm} (V.1)

It is invariant under the respective gauge transformations (II.10) and (II.12) of $P_{mn}$ and $\Phi_{mn}$.

One may rewrite the conserved charge more suggestively by introducing the curvatures and spin connections of $P_{mn}$ and $\Phi_{mn}$. These are defined by

$$R(P)_{pqrs} = \partial_{[q} P_{p][r,s]} \hspace{1cm} (V.2)$$
$$R(\Phi)_{pqrs} = \partial_{[q} \Phi_{p][r,s]} \hspace{1cm} (V.3)$$

Upon integration by parts, (V.1) becomes

$$Q = \frac{1}{2} \int d^{3}x \epsilon^{mrs} (R(P)_{pqrs} \omega(P)_{pq}^{m} - R(\Phi)_{pqrs} \omega(\Phi)_{pq}^{m})$$  \hspace{1cm} (V.4)

In terms of the curvature two-forms $R(P)_{pq} = \frac{1}{2} R(P)_{pqrs} \, dx^{r} \wedge dx^{s}$, $R(\Phi)_{pq} = \frac{1}{4} R(\Phi)_{pqrs} \, dx^{r} \wedge dx^{s}$ and the connection one-forms $\omega(P)_{pq} = \omega(P)_{pq}^{m} \, dx^{m}$, $\omega(\Phi)_{pq} = \omega(\Phi)_{pq}^{m} \, dx^{m}$, one can rewrite this expression as

$$Q = \int \left[ R(P)_{pq} \wedge \omega(P)_{pq} - R(\Phi)_{pq} \wedge \omega(\Phi)_{pq} \right]$$  \hspace{1cm} (V.5)

exhibiting a Chern-Simons structure analogous to that found in the Maxwell case.

Under the gauge transformations $P_{pq} \rightarrow P_{pq} + \partial_{m} \xi_{n} + \partial_{n} \xi_{m}$, the curvature $R(P)_{pqrs}$ is invariant, while $\omega(P)_{pq}^{m}$ transforms as the gradient of the rotation parameter $\partial^{p} \xi^{q} - \partial^{q} \xi^{p}$,

$$\omega(P)_{pq}^{m} \rightarrow \omega(P)_{pq}^{m} + \partial_{m} (\partial^{p} \xi^{q} - \partial^{q} \xi^{p}).$$

This is the transformation property of the spin connection in linearized gravity when the local rotation gauge freedom is fixed by the gauge condition that the triads should be symmetric. Hence, the name “spin connection” for $\omega(P)_{pq}^{m}$. The same properties hold for $R(\Phi)_{pqrs}$ and $\omega(\Phi)_{pq}^{m}$ under the gauge transformations $\Phi_{mn} \rightarrow \Phi_{mn} + \partial_{m} \eta_{n} + \partial_{n} \eta_{m}$. 
VI. CONCLUSION AND SUMMARY

In this paper, we have shown that duality is a symmetry not only of the equations of motion of the free spin-2 theory but also of the Pauli-Fierz action itself. Hence, duality is a symmetry of (linearized) gravity in the standard sense. This was achieved by introducing symmetric superpotentials. In terms of these superpotentials, the action takes a form very similar to that of electromagnetism (compare with “conformal gravity” analyzed in [11]). We have also computed the canonical generator of duality rotations and found the same Chern-Simons structure as in the spin-1 case. The theory is invariant under the gauge transformations (II.10) and (II.12) of the superpotentials, which take the form of independent linearized diffeomorphisms and conformal transformations. As in electromagnetism, the price paid for achieving manifest duality-invariance of the action is the loss of manifest Lorentz-invariance.

We have explicitly written the duality transformation rules in terms of the electric and magnetic superpotentials and verified duality-invariance only for the reduced action where the constraints have been eliminated, but this is sufficient to establish invariance of the original Pauli-Fierz action itself. This is in fact a standard general result, but for the sake of completeness, we repeat the reasoning here. The argument proceeds in two steps:

1. First, one proves duality invariance of the unreduced Hamiltonian action (II.1). From the transformation properties of the superpotentials, one can infer the transformation properties of $h_{mn}$ and $\pi^{mn}$ using $\delta P_{mn} = -\Phi_{mn}$, (II.11), (II.9), (II.8) and (II.13). One gets (assuming $\delta u_m = 0$ for simplicity, which is legitimate)

$$\delta h_{mn} = -\varepsilon_{mrs} \partial^r (\Delta^{-1} \pi^s_n) - \varepsilon_{mrs} \partial^r (\Delta^{-1} \pi^s_m)$$

$$\delta \pi^{mn} = \frac{1}{4} \varepsilon^{mrs} (\partial^s \partial^m \Delta^{-1} h^p_r - \partial^p \partial^r \Delta^{-1} h^m_s)$$

$$+ \frac{1}{4} \varepsilon^{mrs} (\partial^p \partial^r \Delta^{-1} h^s_n - \partial^s \partial^n \Delta^{-1} h^r_m)$$

These transformation rules hold on the constraint surface and we choose to extend them off the constraint surface using the same expressions. From what we have shown, the unreduced Hamiltonian action (II.1) is invariant under these transformations when the constraints hold, or, what is the same, its variation under duality is a combination of the constraints. One may thus adjust the variations of the Lagrange multipliers $a$ and $n_m$, which are free so far, so as to cancel the constraint terms that appear. This is always possible since the Lagrange multipliers multiply the constraints. The computation is cumbersome and will not be reproduced here.

2. Second, one eliminates the momenta $\pi^{mn}$ using their own equations of motion, i.e., through

$$\pi^{mn} = \frac{1}{2} (\dot{h}^{mn} - \delta^{mn} \dot{h})$$

The standard general theorems on auxiliary fields guarantee that the resulting action, which is the just the Pauli-Fierz action (I.14), is invariant under the transformations in which the momenta are replaced by their on-shell values (VI.3). This gives in particular, the following duality transformation rules for the spatial components of the metric

$$\delta h_{mn} = -\frac{1}{2} \left[ \varepsilon_{mrs} \partial^r (\Delta^{-1} \dot{h}^s_n) + \varepsilon_{mrs} \partial^r (\Delta^{-1} \dot{h}^s_m) \right]$$

Note that this expression is non-local in space but local in time.

Although we have not carried it explicitly, we expect the discussion of the duality properties of higher spins gauge fields actions[12] to proceed similarly. Much more challenging would the understanding of how the results can be extended to the full, non linear Einstein theory (in the same Killing vector free context considered here). The inclusion of dynamical sources of both electric and magnetic types in the general context is also an intricate question.

It is well known that the (full) Einstein action dimensionally reduced to 3 spacetime dimensions exhibits a “hidden” $SL(2, \mathbb{R})$ symmetry, of which a $SO(2)$ subgroup acts linearly in the small field limit and is the duality group considered above [13]. We have shown that this subgroup is already a symmetry of the Einstein action prior to dimensional reduction, at least in the linearized theory. The independence of the existence of duality (as a symmetry of the action) on dimensional reduction appears to be very suggestive.

Acknowledgments

The work of MH is partially supported by IISN - Belgium (convention 4.4505.86), by the “Interuniversity Attraction Poles Programme – Belgian Science Policy ” and by the European Commission RTN programme HPRN-CT-00131,
in which he is associated to K. U. Leuven. Institutional support to the Centro de Estudios Científicos (CECS) from Empresas CMPC is gratefully acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

APPENDIX A: PROOF OF EQUATIONS (II.5) (II.11) AND (II.12)

We supply in this appendix the proofs of results used - but left unproved - in the main text. These results have a cohomological content in that they may be viewed as special cases of the "Poincaré lemma" for a first-order differential operator $D$ fulfilling $D^3 = 0$ and generalizing to tensors characterized by two-column Young tableaux, the familiar exterior derivative operator $d$ ($d^2 = 0$) appropriate to antisymmetric tensors (characterized by one-column Young tableaux) [14, 15]. In that spirit, the Bianchi identity (I.2) can for instance be written as $DR = 0$ while (I.3) becomes $R = D^2h$ [14]. We shall, however, not use the general results established in [14, 15] to derive (II.5) (II.11) and (II.12) but instead, we shall follow a more explicit approach, which is in fact rather direct in the present case.

1. Proof of Eq. (II.5)

We first show that (II.5) is the general solution of $\pi^{mn}, n = 0$. The standard Poincaré lemma for closed 2-forms, with $m$ viewed as a spectator index, yields

$$\pi^{mn} = \partial_k M^{mnk}$$

with $M^{mnk} = -M^{mkn}$. The symmetry of $\pi^{mn}$ for the exchange of $m$ with $n$ implies then $\partial_k (M^{mnk} - M^{mkn}) = 0$, from which one infers, using again the standard Poincaré lemma (with $m$ and $n$ regarding as an antisymmetric pair of spectator indices) that

$$M^{mnk} - M^{mkn} = \partial_s A^{mnks}$$

with $A^{mnks} = -A^{nmks} = -A^{smkn}$. This leads to Eq. (II.5),

$$\pi^{mn} = \partial_k \partial_s U^{mkns}$$

with

$$U^{mkns} \equiv -\frac{1}{2} (A^{mkns} + A^{nmks}) = -U^{kmsn} = -U^{nsmk}.$$

2. Proof of Eq. (II.11)

We now prove that the general solution of

$$\partial^m \partial^n h_{mn} - \Delta h = 0 \quad (A.1)$$

is

$$h_{mn} = \epsilon_{mrs} \partial^r \Phi^s + \epsilon_{nrs} \partial^r \Phi^s + \partial_m v_n + \partial_n v_m \quad (A.2)$$

where the superpotential $\Phi^{mn}$ is symmetric, $\Phi^{mn} = \Phi^{nm}$.

To that end, we first note that that one can write $h_{mn}$ as

$$h_{mn} = j_{mn} + \partial_m v_n + \partial_n v_m$$

where $j_{mn}$ is symmetric and traceless. The tensor $h_{mn}$ is a solution of (A.1) if and only if $j_{mn}$ is a solution of

$$\partial^m \partial^n j_{mn} = 0 \quad (A.3)$$

This equation can be written as $\partial^m (\partial^n j_{mn}) = 0$ from which we get, using the standard Poincaré lemma, $\partial^n j_{mn} = \epsilon_{mnr} \partial^r M^q$ for some $M^q$ or equivalently

$$\partial^n (j_{mn} - \epsilon_{mnr} M^q) = 0.$$
Using again the standard Poincaré lemma with $m$ a spectator index yields

\[ j_{mn} - \epsilon_{mnq} M^q = 2 \epsilon_{nrs} \partial^r \Psi^s_m \]

for some $\Psi^s_m$. Taking the symmetric part gives

\[ j_{mn} = \epsilon_{mrs} \partial^r \Psi^s_n + \epsilon_{nrs} \partial^r \Psi^s_m. \quad (A.4) \]

For arbitrary $j_{mn}$, $\Psi^{rs}$ would not be symmetric. However, $j_{mn}$ is not arbitrary but is traceless. This implies further conditions on $\Psi^{rs}$. From $j^m_m = 0$, one gets $\frac{\partial}{\partial y} \Psi_{[m]} = 0$ from which it follows that $\psi_{[m]} = \partial_q w_m - \partial_m w_q$ for some $w_m$. The antisymmetric part of $\Psi^{rs}$ contributes therefore to Eq. (A.4) a term of the form

\[ \partial_n (-\epsilon_{mrs} \partial^r w^s) + \partial_m (-\epsilon_{nrs} \partial^r w^s) \]

which can be absorbed in the vector $v_m$. When this is done, one finds that $h_{mn}$ takes indeed the form (A.2), as announced.

3. Proof of Equ. (II.12)

Given $h_{mn}$, what is the ambiguity in $(\Phi_{mn}, u_m)$? To answer this question, we must analyze the homogeneous equation

\[ 0 = \epsilon_{mrs} \partial^r \Phi^s_n + \epsilon_{nrs} \partial^r \Phi^s_m + \partial_m u_n + \partial_n u_m. \quad (A.5) \]

Taking the trace of that equation yields $\partial^m u_m = 0$, i.e., $u_m = \epsilon_{mpq} \partial^p \chi^q$ for some $\chi^q$. This yields

\[ 0 = \epsilon_{mrs} \partial^r \Psi^s_n + \epsilon_{nrs} \partial^r \Psi^s_m \]

for

\[ \Psi_{mn} = \Phi_{mn} + \partial_m \chi_n + \partial_n \chi_m. \quad (A.7) \]

Taking the divergence of (A.6) with respect to $m$ gives $\epsilon_{npq} \partial^p (\partial^m \Psi^q_m) = 0$, i.e. $\partial^m \Psi^q_m = \partial^q C$ for some $C$. Eq. (A.6) becomes then

\[ \partial^i (\Psi^j_n + \delta^i_n B) - \partial^i (\Psi^i_n + \delta^i_n B) = 0 \]

with $B = (1/2)(C + \Psi)$, from which one infers

\[ \Psi_{in} = \partial_i k_n - \delta_{in} B \]

for some $k_n$. The symmetry of $\Psi_{in}$ implies then $k_n = \partial_n D$ for some $D$ and thus we get finally $\Psi_{mn} = \partial_m \partial_n D - \delta_{mn} B$. Combined with (A.7), this shows that the ambiguity in $\Phi_{mn}$ is indeed of the form

\[ \Phi_{mn} \rightarrow \Phi_{mn} + \partial_m \eta_n + \partial_n \eta_m + \delta_{mn} \eta \]

as in (II.12). The ambiguity in $u_m$ is $u_m \rightarrow u_m - \epsilon_{mrs} \partial^r \eta^s$. If one gives $h_{mn}$ only up to a gauge transformation, one finds that $\Phi_{mn}$ is still given up to (A.8) while $u_m$ can be shifted at will, $u_m \rightarrow u_m + \xi_m$.

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