Challenging the paradigm of singularity excision in gravitational collapse

Luca Baiotti\textsuperscript{1}, Luciano Rezzolla\textsuperscript{1,2,3}
\textsuperscript{1}Max-Planck-Institut f"ur Gravitationsphysik, Albert-Einstein-Institut, 14476 Golm, Germany
\textsuperscript{2}SISSA, International School for Advanced Studies and INFN, Via Beirut 2, 34014 Trieste, Italy and
\textsuperscript{3}Department of Physics, Louisiana State University, Baton Rouge, LA 70803 USA

(Dated: March 24, 2022)

PACS numbers: 04.25.Dm, 04.30.Db, 04.70.Bw, 95.30.Lz, 97.60.Jd

A paradigm deeply rooted in modern numerical relativity calculations prescribes the removal of those regions of the computational domain where a physical singularity may develop. We here challenge this paradigm by performing three-dimensional simulations of the collapse of uniformly rotating stars to black holes without excision. We show that this choice, combined with suitable gauge conditions and the use of minute numerical dissipation, improves dramatically the long-term stability of the evolutions. In turn, this allows for the calculation of the waveforms well beyond what previously possible, providing information on the black-hole ringing and setting a new mark on the present knowledge of the gravitational-wave emission from the stellar collapse to a rotating black hole.

Numerical relativity simulations have recently recorded important breakthroughs, which have allowed for long-term stable and accurate calculations of curved and highly dynamical spacetimes, thus increasing their potential of providing physically relevant predictions for gravitational-wave astronomy. Behind this rapid progress are various novel approaches, some of which involve the dropping of assumptions or techniques that were considered to be important or simply necessary. Firstly, the dismissal of the $3+1$ ADM formulation of the field equations, which, after large-scale efforts\textsuperscript{1}, has shown not to have the stability properties necessary for long-term fully three-dimensional (3D) simulations. In lieu of the ADM equations, new formulations have been either reconsidered (as in the case of the conformal and traceless formulation of the ADM equations\textsuperscript{2}) or investigated for the first time in 3D simulations (as in the case of the harmonic formulation of the Einstein equations\textsuperscript{3}). Both approaches have been shown to provide long-term stability on timescales sufficiently large to evolve accurately a large class of spacetimes, including black holes\textsuperscript{4,5} and neutron stars\textsuperscript{6,7}. Secondly, the abandoning of the use of numerical grids with uniform spacing, in lieu of which several codes now use mesh-refinement techniques (either fixed\textsuperscript{8} or with adaptivity\textsuperscript{9}). This conceptually simple but technologically challenging improvement allows to concentrate computational resources where the truncation error needs to be the smallest, while saving them where they may not be needed. In turn, non-uniform grids have allowed to place the outer boundaries of the computational domain at very large distances, thus reducing the influence of inaccurate outer-boundary conditions and making it possible to have the wave-zone within the domain and to extract there the precious gravitational-wave signal\textsuperscript{10}. Thirdly, for some years now, successful long-term 3D evolutions of black-hole vacuum spacetimes have been possible only thanks to the use of excision techniques (see, e.g.,\textsuperscript{11} for a first example), that is by ignoring the spacetime regions inside black-hole horizons. These are causally disconnected from the outside and should have no effect on the rest of the evolution as long as a suitable treatment of the equations is made at the excision surface. Furthermore, recent simulations of the collapse of rotating neutron stars to Kerr black holes\textsuperscript{2,11} have shown the effectiveness of excision techniques also for spacetimes with matter, where they were applied separately to the field equations and to the hydrodynamical equations. In those simulations, in fact, the use of excision has extended considerably the lifetime of the simulations, allowing for an accurate investigation of the dynamics of the trapped surfaces formed during the collapse and for the extraction of the first gravitational waveforms from 3D collapse to rotating black holes\textsuperscript{2}.

Although the assumption that a region of spacetime that is causally disconnected can be ignored without this affecting the solution in the remaining portion of the spacetime is certainly true for signals and perturbations travelling at physical speeds, numerical signals, such as gauge waves or constraint violations, may travel at velocities larger than that of light and thus leave the physically disconnected region. Indeed, this is what is commonly observed when excising a topologically spherical surface in Cartesian coordinates within a conformal and traceless formulation of the Einstein equations and without the massive use of numerical dissipation as in ref.\textsuperscript{10}.

In this paper we challenge the paradigm of singularity excision and show that accurate numerical simulations can be carried out even in the absence of an excised region and independently of whether the spacetime is vacuum and the singularity modelled as a “puncture”\textsuperscript{4,5}. In addition, we show that the absence of an excised region improves dramatically the long-term stability, allowing for the calculation of the gravitational waveforms well beyond what previously possible and past the black-hole quasi-normal-mode (QNM) ringing.

The simplest and most impressive way of proving the improvements resulting from not excising the region inside the apparent horizon (AH) is by reconsidering the same initial data of ref.\textsuperscript{11}, which lead to gravitational collapse to rotating black holes. More specifically, we consider rotating relativistic stars calculated as equilibrium solutions of the Einstein equations, with a polytropic equation of state $p = K \rho^\gamma$, where $K$ and $\gamma$ are constants.
with $\Gamma = 2$ and with the polytropic constant initially set to
$K_{\alpha} = 100$ \cite{23}. To illustrate the collapse for the range of
neutron stars rotating either very slowly or near the mass-
shedding limit, we consider two representative models, indi-
cated as D1 and D4 in ref. \cite{11}, with different initial angular
momenta. The first one is a slowly rotating star of mass
$M = 1.67 M_\odot$, circumferential equatorial radius $R_e = 11.43$
km and $a \equiv J/M^2 = 0.21$, where $J$ is the angular momen-
tum; the second one has with $M = 1.86 M_\odot$, $R_e = 14.25$ km
and is rotating close to the mass-shedding limit with $a = 0.54$.

The numerical methods used for these evolutions are the
same as discussed in ref. \cite{6}. In particular, we use a con-
formal and traceless formulation of the ADM equations \cite{12},
which are solved with fourth-order finite-difference oper-
ators, while we evolve the hydrodynamical equations with
the Whisky code \cite{13}, implementing high-resolution shock-
capturing (HRSC) techniques with a variety of approximate
Riemann solvers and reconstruction algorithms and an over-
all second-order truncation error \cite{11}. The numerical grid
setup makes use of the same mesh refinement implemented
in the numerical infrastructure described in ref. \cite{8}. Besides
a fixed number of refinement levels which are present already
on the initial slice, new refined levels are added at predefined
positions during the evolution. More specifically, as the star
collapses, the innermost (most refined) grid-box is progress-
evously further refined with box-in-box grids, the final one hav-
ing a resolution of $\Delta x \simeq 0.02 M$; the outermost grid, instead,
which is not further refined, has a resolution of $\Delta x \simeq 1.5 M$,
sufficient to capture the details of the gravitational radiation.
In this way, our outer boundaries are placed at $\approx 165 M$; so
as to minimize the influence of our imperfect outer-boundary
conditions on the very small gravitational-wave signal. We
note that the “switching-on” of different levels of resolution
does have a small but appreciable effect on its dynamics, es-
sentially due to the necessary interpolation to the new refined
grids, which also changes the growth-time of the most unsta-
ble, exponentially growing mode. This effect is negligible in
the evolution of the matter but can influence the waveforms.
To provide a direct comparison with \cite{9}, we have here fol-
lowed the same approach.

Fig. 1 summarises the dynamics of the matter for model
D4 by showing the time evolution of the maximum of the
rest-mass density $\rho_{\text{max}}$ (continuous line) and of the total rest-
mass $M_*$ (dashed line) for model D4. Circles indicate the times at which the AH is
first found and thick lines the results obtained with excision.

(i.e. $t \gtrsim 300 M$) without appreciable loss of accuracy or sign
of instability. Indeed, as the matter collapses and concentrates
over a very few gridpoints (and ultimately on only one), the
high accuracy of the HRSC methods is able to conserve the
rest mass to very high precision with a loss of less than 0.2%
up to when the rest-mass density distribution is diffused as a
result of the poorly resolved gradients.

While very little extra is needed for the evolution of the
hydrodynamical quantities in the absence of an excision algo-
rithm, a stable evolution of the Einstein equations requires at
least two important ingredients. The first one is represented
by gauge conditions for the lapse function $\alpha$ with suitable
singularity-avoidance properties. Our experience has shown
that hyperbolic $K$-driver slicing conditions of the form
$\delta t \alpha = -f(\alpha) \alpha^2 (K - K_0)$, with $f > 0$ and $K_0$ being the trace
of the extrinsic curvature at $t = 0$, are essential to “freeze”
the evolution in those regions of the computational domain in-
side the AH, where the metric functions experience the growth
of very large gradients. In practice, we confirm that using
the generalized “1+log” slicing condition \cite{14} as obtained by
setting $f = 2/\alpha$ provides the desired singularity avoidance
and is computationally efficient. With a good choice for the
slicing condition, the results do not depend sensitively on the
gauge condition for the shift. We have found that the use of the
“Gamma-driver” shift conditions discussed in ref. \cite{11} is
sufficient to compensate for the “slice-stretching” induced by
the singularity-avoiding slicing.

The second important ingredient is the introduction of an
artificial dissipation of the Kreiss-Oliger type \cite{17} on the
right-hand-sides of the evolution equations for the spacetime
variables and the gauge quantities (No dissipation is intro-
duced for the hydrodynamical variables.). This is needed
mostly because all the field variables develop very steep gra-
In the region within the AH, the solution of the Einstein equations becomes increasingly inaccurate as the collapse proceeds. As in ref. [9], we have extracted the gravitational-wave information through an approach in which the spacetime is matched with the non-spherical perturbations of a Schwarzschild black hole described in terms of gauge-invariant odd \( Q_{\ell m}^{(o)} \) and even-parity \( \Psi_{\ell m}^{(e)} \) metric perturbations, where \( \ell, m \) are the indices of the angular decomposition. In Fig. 3 we report the lowest-order multipoles \( Q_{20}^{(o)} \equiv \lambda \Psi_{20}^{(e)} \), where \( \lambda \equiv \sqrt{2(\ell + 2)!/\ell!} \). The left panel, in particular, shows the signal detected by an observer at a distance of \( 42 M \) for the slowly rotating model D1; the right panel, instead, shows the same multipole extracted at \( 37 M \) for the rapidly rotating model D4. In both cases the thick lines refer to the evolutions carried out with the excision technique and terminate when the code crashed (cf. left panel of Fig. 1 in ref. [3]). This comparison shows that it is now possible to detect the complete gravitational-wave signal produced by the collapse of a relativistic star to a rotating black hole well beyond what was previously possible in either 2D [13] or 3D simulations [4]. The estimated error on the phase is \( \lesssim 1\% \) and \( \lesssim 5\% \) on the L2 norm of the amplitude.

Fig. 3 also highlights that the signal can be rather different both in amplitude and form in the two cases with the excep-

errors are considerably different in amplitude, the one inside the AH being much larger. Second, the sudden decrease in the long-dashed and continuous lines at the time of AH formation at \( t \approx 60 M \) is due to the fact that some gridpoints (those with the largest violations) are no longer included in the calculation of the \( L^2 \) norm. Third, the violation outside the AH grows only at a slow pace over the remaining evolution. Finally, shown with circles is the violation for a high-resolution simulation. The values are rescaled to second order before AH formation (when the dominant truncation error comes from hydrodynamics and the HRSC methods used are only second-order) and to third order afterwards (when the smooth accretion flow of the atmosphere boosts the accuracy of the hydrodynamics to third-order).

The loss of accuracy by showing the time evolution of the Hamiltonian-constraint violation is seen inside the AH. No convergence of the violation is seen inside the AH. Overall, Fig. 2 shows that the solution of the Einstein equations inside the AH is by and large incorrect, but these errors remain confined within the AH and do not contaminate the overall accuracy of the simulation. Shown for comparison in the inset is the violation computed when the singularity is excised, showing an exponential growth soon after the AH is found.

Note that the ability to perform these long-term evolutions cannot be related in any manner to the use of a “puncture” prescription for the physical singularity, as recently done in simulations of binary black holes [14,15]. However, it is possible that the stability provided here by the singularity-avoiding gauge used and made numerically more robust by the use of dissipation is closely related to the one discussed for punctures [18].

Besides a long-term stability and the possibility of following the collapse well beyond what was possible with the use of excision techniques, the most dramatic advantage produced by the approach suggested here can be appreciated in the study of the gravitational radiation produced during the collapse. As in ref. [9], we have extracted the gravitational-wave information through an approach in which the spacetime is matched with the non-spherical perturbations of a Schwarzschild black hole described in terms of gauge-invariant odd \( Q_{\ell m}^{(o)} \) and even-parity \( \Psi_{\ell m}^{(e)} \) metric perturbations, where \( \ell, m \) are the indices of the angular decomposition.
tion of the final parts, when the signal is dominated by the black-hole QNM ringing. This happens between $\sim 80 \, M$ and $\sim 120 \, M$, where the signal we extract matches very well with a perturbative one computed using the frequencies given in ref. [20] for the angular momenta and masses of our models. The very good agreement with the perturbative results is an important confirmation of the accuracy of our results, which are intrinsically plagued by the extreme weakness of the emitted gravitational radiation. Furthermore, the richness of details in the two waveforms opens the prospects that a careful characterization of the waveforms will provide important information on the properties of the black hole as well as on those of the collapsing matter [21].

A straightforward analysis of the now-complete gravitational-wave signal computed in these simulations allows to improve the estimates provided in ref. [6] for the detectability of the gravitational collapse of a uniformly rotating polytropic star at a distance of 10 kpc. More specifically, the energy efficiency in the emission of gravitational radiation is $E_{\text{D1}}/M = 3.3 \times 10^{-7}$ and $E_{\text{D4}}/M = 3.7 \times 10^{-6}$ respectively, with an overall accuracy of $\sim 10\%$. The resulting signal-to-noise ratios are then: $(S/N)_{\text{Virgo}} \simeq 0.27 - 2.1$, $(S/N)_{\text{D1-D4}} \simeq 1.2 - 11$, and $(S/N)_{\text{D4}} \simeq 3.3 - 28$ for detectors such as Virgo/LIGO, advanced LIGO or Dual [22].

In conclusion, we have presented the first 3D calculations of the gravitational collapse of uniformly rotating stars to black holes without excision. This choice, together with suitable gauge conditions and the use of minute numerical dissipation, improves dramatically the long-term stability of the evolutions, providing the most accurate waveforms of this process to date. While our approach represents a challenge to the paradigm of singularity excision, it does not necessarily imply that all excision techniques should be expected to lead to instabilities. Rather, it highlights that, for a conformal traceless formulation of the Einstein equations and in highly dynamical spacetimes, the excision of a spherical surface in Cartesian coordinates may be more of a problem than a solution.

We thank D. Pollney, B. Szelágyi and J. Thornburg for useful discussions. Support comes also through the SFB-TR7 of the German DFG and the Italian INFN “OG51”.

![FIG. 3: Lowest gauge-invariant multipole for a slowly rotating star (left panel) and a rapidly rotating one (right panel). Thick lines refer to evolutions carried out with excision and terminate at a code crash, while the circles indicate the time of the AH formation time.](image-url)