INTEGRABILITY OF DIFFERENTIAL–DIFFERENCE EQUATIONS WITH DISCRETE KINKS

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Abstract. In this article we discuss a series of models introduced by Barashenkov, Oxtoby and Pelinovsky to describe some discrete approximations to the $\phi^4$ theory which preserve travelling kink solutions. We show, by applying the multiple scale test that they have some integrability properties as they pass the $A_1$ and $A_2$ conditions. However they are not integrable as they fail the $A_3$ conditions.

1. Introduction

In a recent article Barashenkov, Oxtoby and Pelinovsky [1] considered those exceptional discretizations of the $\phi^4$ theory in which the stationary kink can be centered anywhere between the lattice sites. The request that the kink be translational invariance generates three families of exceptional discretizations which provide new differential–difference equations which are not known to be integrable. The continuous $\phi^4$ model in itself is not completely integrable, i.e. it has not an infinite number of generalized symmetries, conservation laws and exact solutions, however it is an approximation of many C or S integrable nonlinear models like the Liouville equation or the sine–Gordon equation. So the analysis of their integrability is an interesting problem worthwhile to be investigated. To do so we apply the multiple scale analysis to the nonlinear discrete equations as this provides an integrability test and gives an integrability grading of the equations.

In Sections 2 and 3 we present a review of the models presented in [1] and of the results necessary to apply the multiple scale analysis to them. Then in Section 4 we present the results of the calculations and in the last Section we give some concluding remarks. The details of the calculations are left to an Appendix.

2. Exceptional models

$\phi^4$ theory has been one of the most important nonlinear models for the description of statistical mechanics and field theory systems [2,19]. The discrete analogs of the $\phi^4$ kinks are solutions of equations of the form

$$
\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + \frac{u_n}{2} + Q_n(u_{n-1}, u_n, u_{n+1}),
$$

where $u_n = u(hn, t)$, with $h$ a constant lattice spacing. The function $Q_n$ is chosen so as to give in the continuous limit, at first order in $h$, the $\phi^4$ potential $Q = -\frac{1}{2}u^3$. These kinks have been used to describe charge-density waves in polymers and metals [21], narrow domain walls in ferroelectrics [5], discommensurations in dielectric crystals [8], and topological excitations in hydrogen-bonded chains [3,10]. Physically, one of the most significant properties of domain walls and topological defects is their mobility [4]. Mathematically, discrete equations [1] are known to admit stationary kink solutions [3,18]; however whether traveling discrete kinks exist remains an open question [9,11,12,21].

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the existence of traveling discrete kinks becomes a nontrivial matter. Indeed, if the equation is non-au-thonomous, the discretization even breaks the translation invariance of \( [3] \) due to the presence of a Peierls-Nabarro barrier, an additional periodic potential induced by discreteness. Miraculously, there are several exceptional discretizations which, while breaking the translation invariance of the equation, allow for the existence of translationally invariant kinks, that is, kinks centered at an arbitrary point between the sites. One such discretization was discovered by Speight and Ward using a Bogomolny-type energy-minimality argument \([24, 25]\). In \([1]\) we have a systematic study of these exceptional cases based on the observation that the translational invariance of the kink implies the existence of an underlying one-dimensional map \( u_{n+1} = F(u_n) \).

A simple algorithm based on this observation gives three classes of exceptional discretizations:

\begin{align}
(2a) \quad Q_1 & \equiv \frac{1}{20} \left[ \mu_1 (u_{n+1} + u_{n-1})(u_{n+1}^2 + u_{n-1}^2) + 
\mu_2 (u_{n+1}^2 + u_{n-1}^2) + 
\mu_3 (u_{n+1}^2 + u_{n-1}^2) \right], \\
(2b) \quad Q_2 & \equiv \mu_1 \left[ u_{n+1}^3 + u_{n-1}^3 + 2 \mu_2 (u_{n+1} + u_{n-1}) \right] + 
2 \mu_1 (u_{n+1} + u_{n-1})^2 + 2 \mu_2 (u_{n+1} + u_{n-1}) \\
(2c) \quad Q_3 & \equiv \mu_1 \left[ \frac{1}{4} + \frac{h^2}{4} \right] \left( u_{n+1}^2 + u_{n-1}^2 \right) + 
\mu_2 \left( u_{n+1}^2 + u_{n-1}^2 \right) + 
\mu_3 \left( u_{n+1}^2 + u_{n-1}^2 \right).
\end{align}

where in the first two equations the coefficients \( \mu_1, \mu_2, \mu_3 \), as presented in \([1]\), are not independent.

3. Multiple scale expansion of lattice equations

For completeness we present here the basic ideas of the multiple scale expansion of lattice equations as presented in \([13]\) in the case of expansions in terms of analytic functions.

3.1. Expansion of real dispersive partial difference equations.

3.1.1. From shifts to derivatives. We consider a function \( u_n : \mathbb{Z} \to \mathbb{R} \) depending on a discrete index \( n \in \mathbb{Z} \) and suppose that:

(a) The dependence of \( u_n \) on \( n \) is realized through the slow variable \( n_1 = \varepsilon n \in \mathbb{R}, \varepsilon \in \mathbb{R}, \ 0 < \varepsilon \ll 1 \), i.e. \( u_n \equiv u(n_1) \);
(b) \( n_1 \) varies in a region of the integer axis such that \( u(n_1) \) is therein analytical;
(c) The radius of convergence of the Taylor series in \( n_1 \) is wide enough to include as inner points all the points involved in the discrete equation.

Under these hypotheses we can write the action of the shift operator \( T_n \), defined by \( T_n u_n = u_{n+1} = u(n_1 + \varepsilon) \) in the variable \( n_1 \), as

\begin{align}
(3) \quad T_n u(n_1) & = u(n_1) + \varepsilon u^{(1)}(n_1) + \varepsilon^2 u^{(2)}(n_1) + \ldots + \varepsilon^i u^{(i)}(n_1) + \ldots \\
& = \sum_{i=0}^{+\infty} \frac{\varepsilon^i}{i!} u^{(i)}(n_1),
\end{align}
where \( u^{(i)}(n_1) = \frac{d^i u(n_1)}{dn_1^i} = d_n u(n_1) \), being \( d_n \) the total derivative operator. From (3) it follows that we can write the following formal expansion for the shift operator \( T_n \):

\[
T_n = \sum_{i=0}^{+\infty} \frac{\varepsilon^i}{i!} d_n^i = e^{\varepsilon d_n}.
\]

If \( u_n \) depends simultaneously on the fast variable \( n \) and on the slow variable \( n_1 \) i.e. \( u_n = u(n, n_1) \), the action of the total shift operator \( T_n \) will give \( T_n u_n = u_n + 1 = u(n + 1, n_1 + \varepsilon) \). So we can split it in terms of partial shift operators \( T_n \) and \( T^{(\varepsilon)}_n \) which are defined respectively by \( T_n u(n, n_1) = u(n + 1, n_1) \) and \( T^{(\varepsilon)}_n u(n, n_1) = u(n, n_1 + \varepsilon) \), \( T_n = T_n T^{(\varepsilon)}_n \), where \( T^{(\varepsilon)}_n \) is also given by Eq. (4). The dependence of \( u_n \) on \( n \) can be easily extended to the case of one fast variable \( n \) and \( K \) slow variables \( n_j = \varepsilon_j n, \varepsilon_j \in \mathbb{R}, 1 \leq j \leq K \). The total shift operator \( T_n \) will now be written as: \( T_n = T_n \prod_{j=1}^{K} T^{(\varepsilon_j)}_n \).

Let us now consider a nonlinear partial difference equation

\[
F \left[ u_{\{n+1\}^{(\varepsilon+)}_{\varepsilon \in \mathbb{N}_{\varepsilon-}},\{m+j\}^{(\varepsilon+)}_{\varepsilon \in \mathbb{N}_{\varepsilon-}}} \right] = 0, \quad \mathcal{N}^{(\pm)}, \mathcal{M}^{(\pm)} \geq 0,
\]

for a function \( u_{n,m} : \mathbb{Z}^2 \to \mathbb{R} \) which depends on two integer indexes \( n \) and \( m \) which we will call respectively space and time variables. Eq. (5) contains shifts of \( m \) contained in the intervals \( (m - \mathcal{M}^{(-)}, m + \mathcal{M}^{(+)}), \) and \( n \)-shifts in the interval \( (n - \mathcal{N}^{(-)}, n + \mathcal{N}^{(+)}) \). Under the hypotheses (a, b, c) we can give a series representation of the shifted values of \( u_{n,m} \) around the point \( (n, m) \). Choosing

\[
\varepsilon_{n_1} = N_1 \varepsilon, \quad \varepsilon_{m_j} = M_j \varepsilon, \quad 1 \leq j \leq K, \quad (\varepsilon, N_1, M_j) \in \mathbb{R}
\]

we can write

\[
T_n = T_n T^{(\varepsilon_{n_1})}_n = T_n \sum_{j=0}^{+\infty} \varepsilon^j A^{(j)}_n, \quad A^{(j)}_n = \frac{N_1^j}{j!} \partial_{n_1}^j,
\]

\[
T_m = T_m \prod_{j=1}^{K} T^{(\varepsilon_{m_j})}_m = T_m \sum_{j=0}^{+\infty} \varepsilon^j A^{(j)}_m,
\]

\[
T_n T_m = T_n T_m T^{(\varepsilon_{n_1})}_n \prod_{j=1}^{K} T^{(\varepsilon_{m_j})}_m = T_n T_m \sum_{j=0}^{+\infty} \varepsilon^j A^{(j)}_{n,m},
\]

where the operators \( A^{(j)}_{n,m} \) are appropriate combinations of \( A^{(j)}_{m_k} = \frac{M_j^k}{j!} \partial_{m_k}^j \) and \( A^{(j)}_n \) (see [22] for more details). When we insert the explicit expressions (6) of the shift operators in term of the derivatives with respect to the slow variables into eq. (5) we get an \( \varepsilon \)-dependent PDE of infinite order. Then we assume for the function \( u = u(n, m, n_1, \{m_j\}_{j=1}^{K}, \varepsilon) \) a double expansion in harmonics and in the perturbative parameter \( \varepsilon \)

\[
u(n, m, n_1, \{m_j\}_{j=1}^{K}, \varepsilon) = \sum_{\gamma=1}^{+\infty} \sum_{\alpha=-\gamma}^{\gamma} \varepsilon^\gamma u^{(\alpha)}_\gamma \left( n_1, \{m_j\}_{j=1}^{K} \right) E^{\alpha}_{n,m},
\]

where the index \( \gamma \) is chosen \( \geq 1 \) in order to let any nonlinear part of eq. (5) to enter as a perturbation in the multiscale expansion.
3.1.2. From derivatives to shifts. Splitting (5) in the various powers of $\epsilon$ and in the different harmonics the multiple scale approach produces from a given partial difference equation partial differential equations for the amplitudes $u^{(\alpha)}$. Starting from the obtained partial differential equation we can write down a partial difference equation inverting the expression of the shift operator as
\[
\partial_{n_1} = \ln T_{n_1} = \ln \left(1 + h_{n_1} \Delta_{n_1}^{(+)}\right) = \sum_{i=1}^{+\infty} \frac{(-1)^{i-1} h_{n_1}^i}{i} \Delta_{n_1}^{(+i)},
\]
where $\Delta_{n_1}^{(+)} = (T_{n_1} - 1)/h_{n_1}$ is just the first forward difference operator with respect to the slow-variable $n_1$.

Only when we impose that the function $u_{n_1}$ is a slow–varying function of order $l$ in $n_1$, i.e. $\Delta_{n_1}^{l+1} u_{n_1} \approx 0$, the $\partial_{n_1}$ operator reduces to polynomials in $\Delta_{n_1}$ of order at most $l$. In [14], choosing $l = 2$ for the indexes $n_1$ and $m_1$ and $l = 1$ for $m_2$, it was shown that the integrable lattice potential $KdV$ equation [17] reduces to a completely discrete and local Nonlinear Schrödinger Equation (NLSE) which has been proved to be not integrable by singularity confinement and algebraic entropy [20, 27]. Consequently, if one passes from derivatives to shifts, one ends up in general with a nonlocal partial difference equation in the slow variables $n_\kappa$ and $m_\delta$.

3.2. The orders beyond the NLSE equation and the integrability conditions. The multiscale expansion of a difference equation (5) for analytic functions will give rise to a continuous PDE. So a multiple scale integrability test will require that an equation of the class of equations (1) is integrable if its multiscale expansion will go into the hierarchy of the NLSE. To be able to do so we need to consider here the orders beyond that at which one obtains for the harmonic $u^{(1)}$ the (integrable) NLSE. The first attempts to go beyond the NLSE order has been presented by Kodama and Mikhailov and by Santini, Degasperis and Manakov in [6, 13]. Starting from S integrable models (models integrable via a Scattering Transform), through a combination of an asymptotic functional analysis and spectral methods, one succeeds in removing all the secular terms from the reduced equations order by order. The results could be summarized as follows:

1. The number of slow-time variables required to \cdots for the amplitudes $u^{(\alpha)}$'s coincides with the number of nonvanishing coefficients $\omega_j(\kappa) = \frac{1}{j!} \frac{d \omega(k)}{d k}$;
2. The amplitude $u^{(1)}_1$ evolves at the slow-times $t_\sigma, \sigma \geq 3$ according to the $\sigma$–th equation of the NLS hierarchy;
3. The amplitudes of the higher perturbations of the first harmonic $u^{(1)}_j, j \geq 2$, taking into account some asymptotic boundary conditions, evolve at the slow-times $t_\sigma, \sigma \geq 2$ according to certain linear, nonhomogeneous equations.

Thus the cancellation at each stage of the perturbation process of all the secular terms is a sufficient condition to uniquely fix the evolution equations followed by every $u^{(1)}_j, j \geq 1$ for each slow-time $t_\sigma$. Point 2 implies that a hierarchy of integrable equations always provide compatible evolutions for a unique function $u$ depending on different times, i.e. the equations in its hierarchy are generalized symmetries of each other. In this way this procedure provides necessary and sufficient conditions to get secularity-free reduced equations [6].

Then, following Degasperis and Procesi [7] we state:

**Definition 3.1.** A nonlinear PDE is said to be integrable if it possesses a nontrivial Lax pair and consequently an infinity of generalized symmetries.

As a consequence of this Definition we have the following Proposition:
Proposition 3.1 If equation (5) is $S$ integrable, then under a multiscale expansion the functions $u_j^{(1)}$, $j \geq 1$ satisfy the equations

\begin{align}
\partial_{t_\sigma} u_1^{(1)} &= K_\sigma \left[ u_1^{(1)} \right], \\
M_\sigma u_j^{(1)} &= f_\sigma(j), \quad M_\sigma \doteq \partial_{t_\sigma} - K_\sigma' \left[ u_1^{(1)} \right],
\end{align}

\forall j, \ \sigma \geq 2, \text{ where } K_\sigma \left[ u_1^{(1)} \right] \text{ is the } \sigma\text{-th flow in the nonlinear Schrödinger hierarchy. All the other } u_j^{(\kappa)}, \kappa \geq 2 \text{ are expressed in terms of differential monomials of } u_\rho^{(1)}, \rho \leq j.

In (9b), $f_\sigma(j)$ is a nonhomogeneous nonlinear forcing term and $K_\sigma' [u] v$ is the Frechet derivative of the nonlinear term $K_\sigma [u]$ along the direction $v K_\sigma' [u] v \equiv \frac{d}{ds} K_\sigma [u + sv] |_{s=0}$, i.e. the linearization of $K_\sigma [u]$ along the direction $v$. Eqs. (9) are a necessary condition for integrability and represent a hierarchy of compatible evolutions for the same function $u_1^{(1)}$ at different slow-times. The compatibility of eqs. (9) is not always guaranteed but is subject to a sort of commutativity conditions among their r.h.s. terms $f_\sigma(j)$. Then it is easy to prove that the operators $M_\sigma$ defined in eq. (9) commute among themselves. Once we fix the index $j \geq 2$ in the set of eqs. (9), this commutativity condition implies the following compatibility conditions

\begin{align}
M_\sigma f_{\sigma'}(j) &= M_{\sigma'} f_\sigma(j), \quad \forall \sigma, \sigma' \geq 2,
\end{align}

where, as $f_\sigma(j)$ and $f_{\sigma'}(j)$ are functions of the different perturbations of the fundamental harmonic up to degree $j-1$, the time derivatives $\partial_{t_\sigma}, \partial_{t_{\sigma'}}$ of those harmonics appearing respectively in $M_\sigma$ and $M_{\sigma'}$ have to be eliminated using the evolution equations (9) up to the index $j-1$.

The commutativity conditions (10) turn out to be an integrability test.

Following [6] we conjecture that the relations (9) are a sufficient condition for the integrability or that the integrability is a necessary condition to have a multiscale expansion where eqs. (9) are satisfied. To construct the functions $f_\sigma(j)$ according to the Proposition 3.1 we define:

Definition 3.2. A differential monomial $\rho \left[ u_j^{(1)} \right]$, $j \geq 1$ in the functions $u_j^{(1)}$, its complex conjugate and its $\xi$-derivatives is a monomial of "gauge" 1 if it possesses the transformation property

\begin{align}
\rho \left[ \bar{u}_j^{(1)} \right] &= e^{i\theta} \rho \left[ u_j^{(1)} \right], \\
\bar{u}_j^{(1)} &= e^{i\theta} u_j^{(1)};
\end{align}

Definition 3.3. A finite dimensional vector space $P_n$, $n \geq 2$ is the set of all differential polynomials in the functions $u_j^{(1)}$, $j \geq 1$, their complex conjugates and their $\xi$-derivatives of order $n$ in $\varepsilon$ and gauge 1 when

\begin{align}
\text{order} \left( \partial_{\xi}^m u_j^{(1)} \right) &= \text{order} \left( \partial_{\xi}^m \bar{u}_j^{(1)} \right) = m + j, \quad m \geq 0;
\end{align}

Definition 3.4. $P_n(m)$, $m \geq 1$ and $n \geq 2$ is the subspace of $P_n$ whose elements are differential polynomials in the functions $u_j^{(1)}$ s, their complex conjugates and their $\xi$-derivatives of order $n$ in $\varepsilon$ and gauge 1 for $1 \leq j \leq m$.

From the definition (3.4) one can see that in general $K_\sigma \left[ u_1^{(1)} \right] \in \partial_{\xi}^2 u_1^{(1)} \cup P_{\sigma+1}(1)$ and that $f_\sigma(j) \in P_{j+\sigma}(j-1)$ where $j, \ \sigma \geq 2$. The basis monomials of the spaces $P_n(m)$ can be found, for example, in [22]

Proposition 3.2 If for each fixed $j \geq 2$ the equation (10) with $\sigma = 2$ and $\sigma' = 3$, namely $M_2 f_3(j) = M_3 f_2(j)$, is satisfied, then there exist unique differential polynomials $f_\sigma(j)$ for $\sigma \geq 4$ such that the flows $M_\sigma u_j^{(1)} = f_\sigma(j)$ commute for any $\sigma \geq 2$. 


Hence among the relations (10) only those with $\sigma = 2$ and $\sigma' = 2$ have to be tested.

**Proposition 3.3** The homogeneous equation $M_u u = 0$ has no solution $u$ in the vector space $P_m$, i.e. $\text{Ker}(M_u) \cap P_m = \emptyset$.

Consequently the multiscale expansion (9) is secularity-free. Finally we define the degree of integrability of a given equation:

**Definition 3.5.** If the relations (10) are satisfied up to the index $j$, $j \geq 2$, we say that our equation is asymptotically integrable of degree $j$ or $A_j$ integrable.

3.2.1. Integrability conditions for the NLSE hierarchy. We specify here the conditions for asymptotic integrability of order $k$ or $A_k$ integrability conditions. To simplify the notation, we will use for $u^{(j)}$ the concise form $u(j)$. Moreover, for convenience of the reader, we list the fluxes $K$ of the NLSE hierarchy for $u$ up to $\sigma = 4$:

\begin{align*}
(11a) & \quad K_1[u] \doteq Au_{\xi}, \\
(11b) & \quad K_2[u] \doteq -i\rho_1 \left[ u_{\xi\xi} + \frac{\rho_2}{\rho_1} |u|^2 u \right], \\
(11c) & \quad K_3[u] \doteq B \left[ u_{\xi\xi\xi} + \frac{3\rho_2}{\rho_1} |u|^2 u_{\xi} \right], \\
(11d) & \quad K_4[u] \doteq -iC \left\{ u_{\xi\xi\xi\xi} + \frac{\rho_2}{\rho_1} \left[ \frac{3\rho_2}{2\rho_1} |u|^4 u + 4 |u|^2 u_{\xi\xi} + 3 \bar{u}\xi_{\xi\xi} \xi + 2 |u| \xi^2 u + u^2 \bar{u}_{\xi\xi\xi} \right] \right\},
\end{align*}

where $\rho_1, \rho_2, A, B$ and $C$ are arbitrary complex constants.

The $A_1$ integrability condition is given by the reality of the coefficient $\rho_2$ of the nonlinear term in the NLSE.

The $A_2$ integrability conditions are obtained choosing $j = 2$ in the compatibility conditions (10) with $\sigma = 2$ and $\sigma' = 3$

\begin{equation}
M_2 f_3(j) = M_3 f_2(j).
\end{equation}

In this case we have that $f_2(2) \in P_4(1)$ and $f_3(2) \in P_5(1)$ with $\dim(P_4(1)) = 2$ and $\dim(P_5(1)) = 5$, so that $f_2(2)$ and $f_3(2)$ will be respectively identified by 2 and 5 complex constants.

\begin{align*}
(13a) & \quad f_2(2) \doteq au_\xi(1)|u(1)|^2 + b\bar{u}_\xi(1)u(1)^2, \\
(13b) & \quad f_3(2) \doteq \alpha|u(1)|^4 u(1) + \beta|u_\xi(1)|^2 u(1) + \gamma u_{\xi\xi}(1)^2 \bar{u}(1) + \delta \bar{u}_{\xi\xi}(1) u(1)^2 + \epsilon |u(1)|^2 u_{\xi\xi\xi}(1).
\end{align*}

In this way, if $\rho_2 \neq 0$, eliminating from eq. (12) the derivatives of $u(1)$ with respect to the slow-times $t_2$ and $t_3$ using the evolutions (9a) with $\sigma = 2$ and $\sigma' = 3$ and equating term by term, we obtain the $A_2$ integrability conditions

\begin{equation}
a = \bar{a}, \quad b = \bar{b}.
\end{equation}

So at this stage we have two conditions obtained requiring the reality of the coefficients $a$ and $b$. The expression of $\alpha, \beta, \alpha, \delta$ in terms of $a$ and $b$ are:

\begin{equation}
\alpha = \frac{3iB\rho_2 a}{4\rho_1^2}, \quad \beta = \frac{3iBb}{\rho_1}, \quad \gamma = \frac{3iBa}{2\rho_1}, \quad \delta = 0, \quad \epsilon = \gamma.
\end{equation}

The $A_3$ integrability conditions are derived in a similar way setting $j = 3$ in eq. (12). In this case we have that $f_2(3) \in P_5(2)$ and $f_3(3) \in P_6(2)$ with $\dim(P_5(2)) = 12$ and $\dim(P_6(2)) = 26,$
so that \( f_2(3) \) and \( f_3(3) \) will be respectively identified by 12 and 26 complex constants

\[
\begin{align*}
\text{(16a)} \quad f_2(3) &\doteq \tau_1 |u(1)|^4u(1) + \tau_2|u(1)|^2u(1) + \tau_3|u(1)|^2u_{\xi}(1) + \tau_4u_{\xi}(1)u(1)^2 + \tau_5u_{\xi}(1)^2\bar{u}(1) + \\
&+ \tau_6u(2)u(1)^2 + \tau_7u(2)u(1)^2 + \tau_8u(2)^2\bar{u}(1) + \tau_9u(2)^2u(1) + \tau_{10}u(2)u(1)\bar{u}(1) + \\
&+ \tau_{11}u(2)\bar{u}(1)u(1) + \tau_{12}\bar{u}(2)u(1)u(1),
\end{align*}
\]

\[
\begin{align*}
\text{(16b)} \quad f_3(3) &\doteq \gamma_1|u(1)|^4u(1) + \gamma_2|u(1)|^2u(1)^2u_{\xi}(1) + \gamma_3|u(1)|^2u_{\xi}(1) + \gamma_4u(1)^2u_{\xi}(1) + \\
&+ \gamma_5|u(1)|^2u(1) + \gamma_6u_{\xi}(1)u(1) + \gamma_7u_{\xi}(1)u_{\xi}(1)u(1) + \gamma_8u_{\xi}(1)u_{\xi}(1)\bar{u}(1) + \\
&+ \gamma_9|u(1)|^2u(2) + \gamma_{10}|u(1)|^2u(2) + \gamma_{11}u_{\xi}(1)u(2) + \gamma_{12}u_{\xi}(1)u(2)^2 + \\
&+ \gamma_{13}|u(1)|^2u(2) + \gamma_{14}|u(2)|^2u(2) + \gamma_{15}|u_{\xi}(1)|^2u_{\xi}(2) + \\
&+ \gamma_{16}|u(1)|^2u_{\xi}(2) + \gamma_{17}u(2)u_{\xi}(1)u(1) + \gamma_{18}u(2)u_{\xi}(1)u(1) + \gamma_{19}u(2)u_{\xi}(1)u(1) + \\
&+ \gamma_{21}u(2)u_{\xi}(2)u(1) + \gamma_{22}u(2)u_{\xi}(2)u(1) + \gamma_{23}u(2)u_{\xi}(2)u(1) + \gamma_{24}u(2)u_{\xi}(2)u(1) + \gamma_{25}u_{\xi}(2)u(1)u(1) + \\
&+ \gamma_{26}u_{\xi}(2)u(2)u(1).
\end{align*}
\]

Let us eliminate from eq. (12) with \( j = 3 \) the derivatives of \( u(1) \) with respect to the slow-times \( t_2 \) and \( t_3 \) using the evolutions (9a) respectively with \( \sigma = 2 \) and \( \sigma' = 3 \) and the same derivatives of \( u(2) \) using the evolutions (9b) with \( \sigma = 2 \) and \( \sigma' = 3 \). Let us equate the remaining terms term by term, if \( \rho_2 \neq 0 \), and indicating with \( R_i \) and \( I_i \) the real and imaginary parts of \( \tau_i, i = 1, \ldots, 12 \), we obtain the \( A_3 \) integrability conditions

\[
\begin{align*}
R_1 &= -\frac{aI_6}{4\rho_1}, \quad R_3 = \frac{(b-a)I_6}{2\rho_2} - \frac{aI_{12}}{2\rho_2}, \quad R_4 = \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{aI_{12}}{4\rho_2}, \\
R_5 &= \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{(b-a)I_{12}}{4\rho_2}, \quad R_6 = -\frac{aI_8}{\rho_2}, \quad R_7 = R_{12} + \frac{(a-b)I_8}{\rho_2}, \\
R_8 &= R_9 = 0, \quad R_{10} = R_{12}, \quad R_{11} = R_{12} + \frac{(a-2b)I_8}{\rho_2}, \\
I_4 &= \frac{(b+a)R_{12}}{4\rho_2} + \frac{\rho_1I_1}{\rho_2} + \frac{I_3 - 2I_5}{4} + \frac{[2b(a-b) + a^2]I_8}{4\rho_2^2}, \quad I_7 = 0,
\end{align*}
\]

(17)

For completeness we give the expressions of the \( \gamma_j, j = 1, \ldots, 26 \) as functions of the \( \tau_i, i = 1, \ldots, 12 \):

\[
\begin{align*}
\gamma_1 &= \frac{3B}{8\rho_1} \left[ -2bR_{12} - 8\rho_1 I_1 + 2(I_2 - 2I_3 - 2I_5)\rho_2 + i(b - 5a)I_6 + \frac{2a^2I_8}{\rho_2} - 3iaI_{12} \right], \\
\gamma_2 &= -\frac{3Ba}{4\rho_1} \left[ iI_6 + \frac{(a-2b)I_8}{\rho_2} + \tau_2 \right], \quad \gamma_3 = \frac{3iB\tau_3}{2\rho_1}, \quad \gamma_4 = 0, \quad \gamma_5 = \frac{3iB\tau_2}{2\rho_1}, \\
\gamma_6 &= \frac{3iB\tau_4}{\rho_1}, \quad \gamma_7 = \gamma_5, \quad \gamma_8 = \gamma_3 + \frac{3iB\tau_5}{\rho_1}, \quad \gamma_9 = -\frac{3B(\rho_2I_6 + 3aiI_8)}{4\rho_1^2}, \\
\gamma_{10} &= \frac{3iB\rho_2R_{12}}{2\rho_1^2}, \quad \gamma_{11} = 0, \quad \gamma_{12} = \frac{3iB\tau_9}{2\rho_1}, \quad \gamma_{13} = \frac{3iB\tau_{11}}{2\rho_1}, \quad \gamma_{14} = 0, \quad \gamma_{15} = \frac{3iB\tau_{12}}{2\rho_1}, \\
\gamma_{16} &= \frac{3iB\tau_6}{2\rho_1}, \quad \gamma_{17} = \gamma_{18} = 0, \quad \gamma_{19} = \frac{3iB\tau_{10}}{2\rho_1}, \quad \gamma_{20} = \gamma_{15}, \quad \gamma_{21} = \frac{3iB\tau_8}{\rho_1}, \\
\gamma_{22} &= \gamma_{12}, \quad \gamma_{23} = \gamma_{16} + \gamma_{19}, \quad \gamma_{24} = \gamma_{13}, \quad \gamma_{25} = \frac{3iB\tau_7}{\rho_1}, \quad \gamma_{26} = 0.
\end{align*}
\]

(18)

The conditions in the case of \( C \) integrable or linearizable equations are similar to those presented here and can be found in ref. [23].
4. Multiple scale Expansions of the exceptional models

Let us perform the multiscale analysis of the differential-difference models (1) for the real function $u_n(t)$ with the functions $Q_n$ given by (2). To do so we expand the variable $u_n$ according to the prescriptions of the previous section and split the obtained system in terms of the different harmonics and various powers of $\epsilon$. The equations at the various orders are presented in the Appendix A, here we just present the final results.

**Proposition 4.1** All models (2) pass the $A_1$ and $A_2$ integrability conditions however fail the $A_3$ one for any choice of the involved parameters $\mu_i$.

Moreover, even the particular choice done by Barashenkov, Oxtoby and Pelinovsky in the case of $Q_1$ and $Q_2$ is no more integrable than the others.

In Appendix A it is also shown that these nonlinear model can never be nontrivially linearized.

5. Concluding remarks

In this article we have shown, applying the multiple scale expansion integrability test, that the set of discrete $\phi^4$ models constructed by Barashenkov, Oxtoby and Pelinovsky are not integrable but have just some nontrivial degree of integrability as they pass the two lowest integrability conditions.

The models do not satisfy the symmetries of the continuous $\phi^4$ theory but just possess travelling kink solutions. It would be interesting to construct discrete models which preserve the symmetries of the continuous one. This can be done [16], however this implies that we will need to have a variable non uniform lattice. As the continuous model is not completely integrable, we do not expect such discrete symmetry preserving model to be completely integrable.

An interesting extension of the calculation done here consists in expressing the multiple scale expansion in term of normal forms. In such a way we would be able to use the results to give approximations to the solutions of the starting equations. Work on both these aspect is in progress.

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**Appendix A. Details of the calculations**

We present here in the following the equations one obtains by splitting (1), once we substitute the field $u_n$ by its multiple scale expansion (7), into the various harmonics and different orders of $\epsilon$. Except when explicitly stated, the equations are valid for any function $Q_n$ given in (2).

- **Order $\epsilon$ and $\alpha = 0$**: One obtains

  \[ u_1^{(0)} = 0; \]

- **Order $\epsilon$ and $\alpha = 1$**: If one requires that $u_1^{(1)} \neq 0$, one obtains the dispersion relation

  \[ \omega^2 = \frac{4 \sin^2 (\kappa h/2)}{h^2} - \frac{1}{2}. \]

- **Order $\epsilon^2$ and $\alpha = 0$**: We obtain

  \[ u_2^{(0)} = 0; \]
• Order $\varepsilon^2$ and $\alpha = 1$: Taking into account the dispersion relation (20), we have

\begin{equation}
\partial_{t_1} u_1^{(1)} - \alpha_1 \partial_n u_1^{(1)} = 0, \quad \alpha_1 = \frac{\sin(\kappa h)}{h^2 \omega},
\end{equation}

which tells us that $u_1^{(1)}$ has the form

\begin{equation}
u_1^{(1)} = g(\xi, t_j, j \geq 2), \quad \xi = h n_1 + \frac{\sin(\kappa h)}{h \omega} t_1,
\end{equation}

where $g$ is an arbitrary function of its arguments going to zero as $\xi \to \pm \infty$;

• Order $\varepsilon^2$ and $\alpha = 2$: Taking into account the dispersion relation (20), we have

\begin{equation}
u_2^{(2)} = 0.
\end{equation}

• Order $\varepsilon^3$ and $\alpha = 0$: Taking into account relation (19), we have

\begin{equation}
u_3^{(3)} = 0;
\end{equation}

• Order $\varepsilon^3$ and $\alpha = 1$: Taking into account the dispersion relation (20) and the eqs. (19, 22), we have

\begin{equation}
\begin{aligned}
\partial_{t_1} u_2^{(1)} - \alpha_1 \partial_n u_2^{(1)} &= -\partial_{t_2} u_1^{(1)} - i \rho_1 \partial_\xi^2 u_1^{(1)} - i \rho_2 |u_1^{(1)}|^2, \\
\rho_1 &= \frac{3 + (h^2 - 4) \cos(\kappa h) + \cos(2\kappa h)}{2h^2 \omega [h^2 - 4 + 4 \cos(\kappa h)]}, \\
\rho_2 &= \frac{-4 \mu_2 + (5 \mu_1 + 3 \mu_3) \cos(\kappa h) + 2 \mu_2 \cos(2\kappa h) + \mu_1 \cos(3\kappa h)}{20 \omega}, \\
\rho_3 &= \frac{-3 (4 + h^2) \mu_1 + 2 \mu_2 + 3 (1 - 2 \mu_2) \cos(\kappa h) + [3 (4 + h^2) \mu_1 + 4 \mu_2] \cos(2\kappa h)}{4 \omega},
\end{aligned}
\end{equation}

where (26d) is obtained in the case of the model $Q_1$, (26d) for $Q_2$ and (26c) for $Q_3$. As a consequence of eq. (22), the right hand side of eq. (26a) is secular. Hence we have to require that

\begin{equation}
\begin{aligned}
\partial_{t_1} u_2^{(1)} - \alpha_1 \partial_n u_2^{(1)} &= 0, \\
\partial_{t_2} u_1^{(1)} &= -i \rho_1 \partial_\xi^2 u_1^{(1)} - i \rho_2 |u_1^{(1)}|^2 \doteq K_2 [u_1^{(1)}].
\end{aligned}
\end{equation}

Eq. (27a) tells us that $u_2^{(1)}$ also depends on $\xi$ while (27b) is an integrable NLSE, as from the definitions (26c, 26d, 26e) we can see that $\rho_2$ is a real coefficient in all of the three cases. Hence all the models are A$_1$-integrable.
If we require that our models be $A_1$-linearisable \[23\], then we need $\rho_2 = 0$ for any $\kappa$. In this case we have respectively

\[(28a)\] $Q_1 : \quad \mu_1 = \mu_2 = \mu_3 = 0,$
\[(28b)\] $Q_2 : \quad \mu_1 = \mu_3 = 0,$
\[(28c)\] $Q_3 : \quad \text{no solution},$

i.e. in the first two models only the trivial linear cases are selected while the third model doesn’t admit an $A_1$-linearisable reduction.

- **Order $\varepsilon^3$ and $\alpha = 2$:** Taking into account the dispersion relation (20) and \[19, 24\], we have

\[(29)\] $u_3^{(2)} = 0;$

- **Order $\varepsilon^3$ and $\alpha = 3$:** Taking into account the dispersion relation (20), we obtain

\[(30a)\] $u_3^{(3)} = \alpha_2 u_1^{(1)};$
\[(30b)\] $\alpha_2 = \frac{h^2 [\mu_3 + 2 \mu_2 \cos (\kappa h) + 2 \mu_1 \cos (2 \kappa h)] \cos (\kappa h)}{20 [8 - 2 h^2 - 9 \cos (\kappa h) + \cos (3 \kappa h)]},$
\[(30c)\] $\alpha_2 = \frac{h^2 [\mu_3 + 2 \mu_1 \cos (\kappa h)] [\mu_1 - \mu_2 + (\mu_2 + \mu_3) \cos (\kappa h) + \mu_1 \cos (2 \kappa h)]}{8 - 2 h^2 - 9 \cos (\kappa h) + \cos (3 \kappa h)},$
\[(30d)\] $\alpha_2 = \frac{h^2 [(4 + h^2) \mu_1 + 2 \mu_2 + (1 - 2 \mu_2) \cos (\kappa h) + (4 + h^2) \mu_1 \cos (2 \kappa h)]}{4 [8 - 2 h^2 - 9 \cos (\kappa h) + \cos (3 \kappa h)]},$

where \[(30b)\] is obtained in the case of the model $Q_1$, \[(30c)\] for $Q_2$ and \[(30d)\] for $Q_3$.

- **Order $\varepsilon^4$ and $\alpha = 0$:** Taking into account \[19\] \[21\] \[24\], we get

\[(31)\] $u_4^{(0)} = 0;$

- **Order $\varepsilon^4$ and $\alpha = 1$:** Taking into account the dispersion relation (20) and eqs. \[19\] \[22\] \[27a\] \[27b\], we get

\[(32a)\] $\partial_\xi u_3^{(1)} - \alpha_1 \partial_{w_1} u_3^{(1)} = - \left( \partial_{\xi} u_1^{(1)} - K_3 \left[ u_1^{(1)} \right] \right) - \left( \partial_{\xi} u_2^{(1)} - K_2 \left[ u_1^{(1)} \right] u_2^{(1)} - f_2 (2) \right),$
\[(32b)\] $B = \frac{[6 - 8 h^2 + h^4 + 2 (h^2 - 4) \cos (\kappa h) + 2 \cos (2 \kappa h)] \sin (\kappa h)}{6 h^2 \omega [h^2 - 4 + 4 \cos (\kappa h)]^2},$

where $K_3 \left[ u_1^{(1)} \right]$ \[11c\] is the cmKdV flux, the second flux of the NLSE hierarchy, $K_2 \left[ u_1^{(1)} \right] u_2^{(1)}$ is the Frechet derivative along the direction $u_2^{(1)}$ of the NLSE flux $K_2 \left[ u_1^{(1)} \right]$ defined by relation \[27b\] and $f_2 (2)$ \[13a\] is a nonlinear forcing term depending on $u_1^{(1)}$ and defined by the coefficients $a$ and $b$. As a consequence of \[22\] \[27a\]
the right hand side of (32a) is secular, so that

\[(33a) \quad \partial_{t_1} u_3^{(1)} - \alpha_1 \partial_{n_1} u_3^{(1)} = 0,\]
\[(33b) \quad \partial_{t_2} u_2^{(1)} - K'_{2} \left[ u_1^{(1)} \right] u_2^{(1)} = - \left( \partial_{\xi} u_1^{(1)} - K_{3} \left[ u_1^{(1)} \right] \right) + f_{2}(2).\]

Eq. (33a) tells us that \(u_{3}^{(1)}\) depends on \(\xi\) too while in (33b), as a consequence of (27a),
the first term of the right hand side is secular, so that

\[(34a) \quad \partial_{t_2} u_2^{(1)} - K'_{2} \left[ u_1^{(1)} \right] u_2^{(1)} = f_{2}(2),\]
\[(34b) \quad \partial_{\xi} u_1^{(1)} - K_{3} \left[ u_1^{(1)} \right] = 0.\]

The coefficients \(a\) and \(b\) of the forcing term \(f_{2}(2)\) (13a) are given by

\[(35a) \quad a = \frac{-36 \left( t^2 - 4 \right) \mu_1 + 3 \left( 4 - 8h^2 + h^4 \right) \mu_2 - 5 \left( h^2 - 4 \right) \mu_3}{\Delta_{a}} + \frac{\left[ -153 + 32h^2 + 4h^4 \right] \mu_1 - 6 \left( h^2 - 4 \right) \mu_2 - 2 \left( 2 + 8h^2 - 4h^4 \right) \mu_3}{\Delta_{a}} \sin(2ch) + \frac{23 \left( h^2 - 4 \right) \mu_1 + 3 \left( 8h^2 - h^4 \right) \mu_2 - 3 \left( h^2 - 4 \right) \mu_3}{\Delta_{a}} \sin(3ch) + \frac{\left[ -42 + 8h^2 - h^4 \right] \mu_1 - 3 \left( h^2 - 4 \right) \mu_2 + 5 \mu_3}{\Delta_{a}} \sin(4ch) + \frac{3 \left( h^2 - 4 \right) \mu_1 \sin(5ch) + \mu_1 \sin(6ch)}{\Delta_{a}}.
\[(35b) \quad b = \frac{12 \mu_2 + \left( h^2 - 4 \right) \mu_3 + 2 \left[ 7\mu_1 + \left( h^2 - 4 \right) \mu_2 + 5 \mu_3 \right] \cos(ch)}{\Delta_{b}} + \frac{2 \left[ \left( h^2 - 4 \right) \mu_1 + 4 \mu_2 \right] \cos(2ch) + 6 \mu_1 \cos(3ch)}{\Delta_{b}} \sin(\chi),
\[(35c) \quad a = \frac{\left[ 50 - 48h^2 + 7h^4 \right] \mu_2 - 46 \left( h^2 - 4 \right) \mu_1 + 6 \left( 4 - 8h^2 + h^4 \right) \mu_3}{\Delta_{a}} + \frac{\left[ 70 - 21h^2 + 2h^4 \right] \mu_2 - 5 \left( h^2 - 4 \right) \mu_3}{\Delta_{a}} \sin(ch) + \frac{\left[ 1 + 10h^2 \right] \mu_2 + \left( -157 + 16h^2 - 2h^4 \right) \mu_1 + \mu_2}{\Delta_{a}}\sin(2ch) + \frac{\left[ -58 + 6h^2 + h^4 \right] \mu_2 + 12 \left( h^2 - 4 \right) \mu_1 + \left( -2 + 8h^2 + h^4 \right) \mu_3}{\Delta_{a}} \sin(3ch)\sin(\chi) + \frac{3 \left( h^2 + 4 \right) \mu_2 + \left( h^2 - 4 \right) \mu_3}{\Delta_{a}} \sin(3ch)\sin(\chi) + \frac{\left[ 46 - 11h^2 + h^4 \right] \mu_2 - 17 \left( h^2 - 4 \right) \mu_1 + 2 \left( 30 - 8h^2 + h^4 \right) \mu_3}{\Delta_{a}} \sin(4ch) + \frac{-3 \left( 13 - 3h^2 \right) \mu_2 + 3 \left( h^2 - 4 \right) \mu_1 + 2 \mu_2\mu_3}{\Delta_{a}} \sin(5ch) + \left( \mu_1 - \mu_2 \right) \mu_1 \sin(6ch)\sin(\chi),
\[(35d) \quad b = \frac{\left[ \left( h^2 - 4 \right) \mu_2 + 3 \left( h^2 - 7 \right) \mu_2 + 16 \mu_3 \right] \mu_1 + \left[ \left( h^2 - 8 \right) \mu_2 + \left( h^2 - 4 \right) \mu_3 \right]}{\Delta_{b}} \sin(ch) + \frac{\left[ 2 \left( 7 \mu_1 - 6 \mu_2 \right) \mu_1 + \left[ 2 \left( h^2 - 4 \right) \mu_1 + (13 - 2h^2) \mu_2 + 5 \mu_3 \right] \mu_3}{\Delta_{b}} \sin(2ch) + \frac{\left[ \left( h^2 - 4 \right) \mu_1 - \left( h^2 - 5 \right) \mu_2 \right] \mu_1 + 2 \left( 4 \mu_1 - 3 \mu_2 \right) \mu_3}{\Delta_{b}} \sin(3ch) + 3 \left( \mu_1 - \mu_2 \right) \mu_1 \sin(4ch)\sin(\chi),
\[(35e) \quad \Delta_{a} = 4\omega \left[ h^2 - 4 + 4 \cos(\chi) \right] \sin(2ch) + \frac{3 + \left( h^2 - 4 \right) \cos(ch) + \cos(2ch)}{\Delta_{a}}
\[(35f) \quad \Delta_{b} = 2\omega \left[ h^2 - 4 + 4 \cos(\chi) \right],
\[(35g) \quad \Delta_{b} = \omega \left[ h^2 - 4 + 4 \cos(\chi) \right].
\[
\begin{align*}
\Delta_a &= 8\omega \left[ k^2 - 4 + 4 \cos (\kappa h) \right] \left[ 3 + \left( k^2 - 4 \right) \cos (\kappa h) + \cos (2\kappa h) \right], \\
\Delta_b &= 4\omega \left[ k^2 - 4 + 4 \cos (\kappa h) \right],
\end{align*}
\]

where (35a, 35b) is obtained in the case of the model \( Q_1 \), (35c, 35d) for \( Q_2 \) and (35e, 35f) for \( Q_3 \). As one can see, in all the three cases the coefficients \( a \) and \( b \) are real. As a consequence all the three models are \( A_2 \)-integrable;

- **Order \( \varepsilon^4 \) and \( \alpha = 2 \)**: Taking into account the dispersion relation (20) and (19, 21, 24, 29), it results

\[
u_4^{(2)} = 0;
\]

- **Order \( \varepsilon^4 \) and \( \alpha = 3 \)**: Taking into account the dispersion relation (20) and (19, 22, 30), it results

\[
u_4^{(3)} = \left( \alpha_3 u_2^{(1)} + \alpha_4 \partial_x u_1^{(1)} \right) u_1^{(1)2},
\]

As \( u_4^{(3)} \) do not enter into the final result, the coefficients \( \alpha_3 \) and \( \alpha_4 \) are not explicitly written down here.

- **Order \( \varepsilon^4 \) and \( \alpha = 4 \)**: Taking into account the dispersion relation (20) and (24), it results

\[
u_4^{(4)} = 0.
\]

- **Order \( \varepsilon^5 \) and \( \alpha = 0 \)**: Taking into account (19, 21, 22, 24, 27a, 27b, 30, 33a, 34a, 34b), we get

\[
u_5^{(0)} = 0;
\]

- **Order \( \varepsilon^5 \) and \( \alpha = 1 \)**: Taking into account the dispersion relation (20) and (19, 21, 22, 24, 27a, 27b, 30, 33a, 34a, 34b), we get

\[
\begin{align*}
\partial_{t_4} u_4^{(1)} - \alpha_1 \partial_{t_1} u_4^{(1)} &= - \left( \partial_{t_2} u_3^{(1)} - K_2' u_1^{(1)} \right) u_3^{(1)} - f_2 (3) - \\
&\quad - \left( \partial_{t_3} u_2^{(1)} - K_3' u_1^{(1)} \right) u_2^{(1)} - f_3 (2) - \left( \partial_{t_4} u_1^{(1)} - K_4 u_1^{(1)} \right), \\
\end{align*}
\]

\[
C = \frac{7 \left( 5 - 8 k^2 + 4 h \right) + \left( h^2 - 4 \right) \left( 14 - 8 k^2 + 4 h \right) \cos (\kappa h) + \cos (2\kappa h)}{24 h^2 \omega \left[ h^2 - 4 + 4 \cos (\kappa h) \right]^3} + \frac{28 + 8 k^2 - 4 h \cos (2\kappa h) + 2 \left( h^2 - 4 \right) \cos (3\kappa h) + \cos (4\kappa h)}{24 h^2 \omega \left[ h^2 - 4 + 4 \cos (\kappa h) \right]^3}.
\]

In the above relations \( K_4 u_1^{(1)} \) is the third flux of the NLSE hierarchy, \( K_3' u_2^{(1)} \) is the Frechet derivative along the direction \( u_2^{(1)} \) of the cmKdV flux.
\( K_3 \left[ u_1^{(1)} \right] \) defined by relation \( (11c) \) and \( f_2 \left( 3 \right) \), \( f_3 \left( 2 \right) \) are the nonlinear forcing terms defined in \( (13b) \). As a consequence of \( (22) \), \( (27a) \) the right hand side of \( (40a) \) is secular, so that

\[
\begin{align*}
(41a) \quad & \partial_t u_4^{(1)} - \alpha_1 \partial_n u_4^{(1)} = 0, \\
(41b) \quad & \partial_t u_3^{(1)} - K_2' \left[ u_1^{(1)} \right] u_3^{(1)} = - \left( \partial_t u_2^{(1)} - K_3' \left[ u_1^{(1)} \right] u_2^{(1)} - f_3 \left( 2 \right) \right) - \left( \partial_t u_1^{(1)} - K_4 \left[ u_1^{(1)} \right] \right) + f_2 \left( 3 \right). 
\end{align*}
\]

The first relation tells us that \( u_4^{(1)} \) depends on \( \xi \) too while in \( (41b) \), as a consequence of \( (34a) \) and of

\[
\left( \partial_t - K_2' \left[ u_1^{(1)} \right] \right) f_3 \left( 2 \right) = \left( \partial_t - K_3' \left[ u_1^{(1)} \right] \right) f_2 \left( 2 \right),
\]

the first term on the right hand side is secular. Moreover also the second term on the right hand side is secular because, when we equal it to zero, we obtain a generalized symmetry the \( NLSE \), the third equation of the corresponding hierarchy. Hence

\[
\begin{align*}
(42a) \quad & \partial_t u_3^{(1)} - K_2' \left[ u_1^{(1)} \right] u_3^{(1)} = f_2 \left( 3 \right), \\
(42b) \quad & \partial_t u_2^{(1)} - K_3' \left[ u_1^{(1)} \right] u_2^{(1)} = - \left( \partial_t u_1^{(1)} - K_4 \left[ u_1^{(1)} \right] \right) + f_3 \left( 2 \right).
\end{align*}
\]

Finally the first term on the right hand side of \( (42b) \) is secular because, when we equal it to zero, we obtain a generalized symmetry the \( cmKdV \) equation as both equations belong to the same \( NLSE \) hierarchy. Hence

\[
\begin{align*}
(43a) \quad & \partial_t u_2^{(1)} - K_3' \left[ u_1^{(1)} \right] u_2^{(1)} = f_3 \left( 2 \right), \\
(43b) \quad & \partial_t u_1^{(1)} - K_4 \left[ u_1^{(1)} \right] = 0.
\end{align*}
\]

The real and imaginary parts \( (R_j, I_j) \) of the coefficients \( \tau_j \), \( j = 1, \ldots, 12 \) of the forcing term \( f_2(3) \) are given by

\[
\begin{align*}
(44) \quad & R_1 = R_2 = R_3 = R_4 = R_5 = R_8 = R_9 = 0, \quad R_{10} = R_{12}, \\
(45) \quad & I_6 = I_7 = I_{10} = I_{11} = I_{12} = 0, \quad I_9 = 2I_8,
\end{align*}
\]
independently from the model.

\[
I_1 = \frac{2\alpha_2 \mu_2 \rho_2^2 - 5 \left(2 \rho_2^2 \rho_2 + 3aB_\omega - 6C_\rho_2 \omega \right) \rho_2 + \alpha_2 \rho_2^2 \left[\mu_2 \left[3 \cos (2\kappa h) + \cos (4\kappa h)\right] + \frac{20 \kappa h^2 \alpha}{20 \kappa h^2 \rho_2}\right] + \left[\mu_3 + 2\mu_1 \cos (2\kappa h)\right] \left[2 \cos (2\kappa h) + \cos (3\kappa h)\right]}{20 \kappa h^2 \rho_2},
\]

\[
I_2 = \frac{2\mu_2 \rho_1 + 20 \left[(a + b) \alpha_1 \rho_1 + (3B_{\alpha_1} - 2\rho_1^2 + 2C_\omega) \rho_2 - 3B_\omega\right] + 6\mu_1 \cos (2\kappa h) - \mu_2 \cos (3\kappa h)}{20 \kappa h^2 \rho_2},
\]

\[
I_3 = \frac{2a_1 \rho_1 + 2 \left(3B_{\alpha_1} - 2\rho_1^2 + 4C_\omega\right) \rho_2 - 3B_\omega + \left[4\mu_1 + \mu_3 + 3\mu_2 \cos (2\kappa h) + \mu_3 \cos (2\kappa h)\right] \cos (2\kappa h) + \frac{2 \kappa h}{20 \kappa h^2 \rho_2}}{20 \kappa h^2 \rho_2},
\]

\[
I_4 = \frac{2 \left[20 \mu_1 + \mu_2\right] \rho_1 + 20 \left[\left[5 \mu_1 + \mu_2\right] \cos (2\kappa h) + \mu_3 \cos (3\kappa h)\right] \rho_1 + 20 \left[2 + \cos (2\kappa h)\right] \mu_2 + \mu_3 \cos (3\kappa h)}{40 \kappa h^2 \rho_2},
\]

\[
I_5 = \frac{2 \left(2 \alpha_1 \rho_1 - 3B_\omega\right) a - \mu_2 \rho_1 + 40 \left(3B_{\alpha_1} - \rho_1^2 + 3C_\omega\right) \rho_2 + 2 \left[\mu_2 + \mu_1 \cos (2\kappa h)\right] \rho_1 + 20 \left[2 + \cos (2\kappa h)\right] \mu_2 + \mu_1 \cos (2\kappa h)}{40 \kappa h^2 \rho_2},
\]

\[
R_6 = \frac{2a_1 \rho_1 + 6 \left[3B_{\alpha_1} - \rho_1^2 + 2C_\omega\right] \rho_2 + 3 \left[2 \mu_2 + \mu_1 \cos (2\kappa h)\right] \cos (2\kappa h)}{10 \kappa h^2 \rho_2},
\]

\[
R_7 = \frac{2a_1 \rho_1 + 6 \left[3B_{\alpha_1} - \rho_1^2 + 2C_\omega\right] \rho_2 - \mu_2 \rho_1 + 2 \left[\mu_2 + \mu_1 \cos (2\kappa h)\right] \mu_1 \rho_1 + \left[4 \mu_1 - \mu_2\right] \mu_1 \rho_1 + \frac{2 \kappa h}{20 \kappa h^2 \rho_2}}{10 \kappa h^2 \rho_2},
\]

\[
R_8 = \frac{\left(5 \mu_1 + 3 \mu_2\right) \cos (2\kappa h) + 2 \left[2 + \cos (2\kappa h)\right] \mu_2 + \mu_1 \cos (3\kappa h)}{20 \kappa h^2 \rho_2},
\]

\[
R_9 = \frac{\left(2a_1 \rho_1 - 3B_\omega\right) \rho_2 + \left[4 \mu_1 + \mu_3 + 3\mu_2 \cos (2\kappa h) + 2 \mu_1 \cos (2\kappa h)\right] \sin (2\kappa h)}{10 \kappa h^2 \rho_2},
\]

\[
R_{10} = \frac{2 \left(2a_1 \rho_1 - 3B_\omega\right) \rho_2 - \left[4 \mu_1 + \mu_3 + 3\mu_2 \cos (2\kappa h) + 2 \mu_1 \cos (2\kappa h)\right] \sin (2\kappa h) + \mu_2 \sin (2\kappa h)}{10 \kappa h^2 \rho_2},
\]

\[
R_{11} = \frac{2 \left(2a_1 \rho_1 - 3B_\omega\right) \rho_2 - \left[4 \mu_1 + \mu_3 + 3\mu_2 \cos (2\kappa h) + 2 \mu_1 \cos (2\kappa h)\right] \sin (2\kappa h) + \mu_2 \sin (2\kappa h)}{10 \kappa h^2 \rho_2},
\]

for the model $Q_1$

\[
I_1 = \frac{4 \left(3 \mu_1 + \mu_2 + 4 \mu_1 \mu_3 - \mu_2 \mu_3\right) \alpha_2 \rho_2^2 - \left(2 \rho_2^2 \rho_2 + 3aB_\omega - 6C_\rho_2 \omega\right) \rho_2 + \frac{4 \kappa h^2 \rho_2}{4 \kappa h^2 \rho_2}}{4 \kappa h^2 \rho_2},
\]

\[
I_2 = \frac{\left(a + b\right) \alpha_1 \rho_1 + \left(3aB_{\alpha_1} - 2\rho_1^2 + 2C_\omega\right) \rho_2 - 3aB_\omega + \left[2 \mu_2 + \mu_1 \cos (2\kappa h)\right] \cos (2\kappa h)}{20 \kappa h^2 \rho_2},
\]

\[
I_3 = \frac{2 \left(3aB_{\alpha_1} - 2\rho_1^2 + 4C_\omega\right) \rho_2 + 3aB_\omega + \left[2 \mu_2 + \mu_1 \cos (2\kappa h)\right] \cos (2\kappa h) + \left(3 \mu_1 + \mu_3\right) \mu_1 \rho_1}{20 \kappa h^2 \rho_2},
\]

\[
I_4 = \frac{2 \left[\alpha_1 + \left(2 \mu_2 + \mu_1\right) \mu_1 \right] \rho_1 + 2 \mu_2 \rho_1 \cos (2\kappa h) + \left[\left(7 \mu_1 - 3 \mu_2\right) \mu_1 + \left(2 \mu_2 + \mu_1\right) \mu_3\right] \cos (3\kappa h) + \left(5 \mu_1 - \mu_2\right) \mu_1 \cos (5\kappa h)}{2 \kappa h^2 \rho_2},
\]

\[
I_5 = \frac{6 \left(A_{\alpha_1} + C_\omega\right) \rho_2 - 2 \left[\left(5 \mu_1 - \mu_2\right) \mu_1 + \mu_1 \rho_1\right] \rho_1 + \left(2 \alpha_1 \rho_1 - 3B_\omega\right) a + \left(\mu_1 - \mu_2\right) \mu_1 \cos (2\kappa h) + \mu_1 \cos (3\kappa h)}{2 \kappa h^2 \rho_2},
\]

\[
R_6 = \frac{2 \left(3aB_{\alpha_1} - 3B_\omega\right) \rho_2 - \left(2 \left(3 \mu_1 - \mu_2\right) \mu_1 + \mu_2 + 3 \mu_3\right) \mu_1 \rho_1}{20 \kappa h^2 \rho_2},
\]

\[
R_7 = \frac{2 \left(3aB_{\alpha_1} - 3B_\omega\right) \rho_2 - \left(2 \left(3 \mu_1 - \mu_2\right) \mu_1 + \mu_2 + 3 \mu_3\right) \mu_1 \rho_1}{20 \kappa h^2 \rho_2},
\]

\[
R_{11} = \frac{2 \left(3aB_{\alpha_1} - 3B_\omega\right) \rho_2 - \left(2 \left(3 \mu_1 - \mu_2\right) \mu_1 + \mu_2 + 3 \mu_3\right) \mu_1 \rho_1}{20 \kappa h^2 \rho_2},
\]
for the model \(Q_2\) and

\[
I_1 = \left( (h^2 + 4) \mu_1 + 2 \mu_2 \right) \alpha_2 p_1^2 - \left( 2p_2^2 \rho_2 + 3aB_\omega - 6C_\omega \right) p_2 + \alpha_2 p_1^2 \left( 2 (1 - 2 \mu_2) \cos (\omega k) + \frac{3 (h^2 + 4) \mu_1 + 2 \rho \mu_2}{4p_2^2} \cos (2\omega k) \right),
\]

\[
I_2 = 2 \left( a + 2b \right) \alpha_1 p_1 + \left( 3B_\alpha - 2p_1^2 + 2C_\omega \right) p_2 - 3aB_\omega + \frac{(h^2 + 4) \mu_1 - 4p_1 p_2 - 2 (h^2 + 4) \mu_1 + 2 \mu_2}{2p_1} \cos (2\omega k),
\]

\[
I_3 = \frac{2a \alpha_1 p_1 + 2 (3B_\alpha - 2p_1^2 + 4C_\omega) p_2 - 3aB_\omega}{2p_1} + \frac{1 - 2 \mu_2 + 2 (h^2 + 4) \mu_1 + 2 \mu_2}{4p_1} \cos (\omega k),
\]

\[
I_5 = \frac{12 \left( B_\alpha + C_\omega \right) p_2 - 2 \left( h^2 + 4 \right) \mu_1 + 2 (\mu_2 + 2 \mu_2) \rho \mu_2 + 2 \left( h^2 + 4 \right) \mu_1 + 2 \mu_2 \cos (2\omega k)}{8p_1},
\]

\[
R_0 = \frac{2 \left( \alpha \alpha_1 - 3B_\omega \right) p_2 + (2 \mu_2 - 1) \sin (\omega k) - 2 \left( h^2 + 4 \right) \mu_1 + 2 \mu_2 \sin (2\omega k)}{2\omega},
\]

\[
R_2 = \frac{4\alpha \rho_2 + (1 - 2 \mu_2) \sin (\omega k) + 2 \left[ h^2 + 4 \right] \mu_1 + 2 \mu_2 \sin (2\omega k)}{2\omega},
\]

\[
I_7 = \frac{3 (h^2 + 4) \mu_1 + 2 \mu_2 + 3 (1 - 2 \mu_2) \cos (\omega k) + 2 \left( h^2 + 4 \right) \mu_1 + 4 \mu_2 \cos (2\omega k)}{4\omega},
\]

\[
R_{10} = \frac{2 \alpha \alpha_1 - 3B_\omega p_2 + \left( 1 - 2 \mu_2 \right) \sin (\omega k) + 2 \left[ h^2 + 4 \right] \mu_1 + 2 \mu_2 \sin (2\omega k)}{2\omega},
\]

\[
R_{11} = \frac{4\alpha \rho_2 + \left( h^2 + 4 \right) \mu_1 + 2 \mu_2}{2\omega},
\]

for the model \(Q_3\). As the coefficients \([16][17][18][19]\) do not satisfy the algebraic relations \([17]\), for any \(Q\) the \(A_3\) integrability is never satisfied.

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