The electrogravity transformation and global monopoles in scalar-tensor gravity

Sukanta Bose * and Naresh Dadhich†

*Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind,
Pune 411007, India
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Abstract

The electrogravity transformation is defined by an interchange of the “active” and “passive” electric parts of the Riemann tensor. Such a transformation has been used to find new solutions that are “dual” to the Kerr family of black hole spacetimes in general relativity. In such a case, the dual solution is a similar black hole spacetime endowed with a global monopole charge. Here, we extend this formalism to obtain solutions dual to the static, spherically symmetric solutions of two different scalar-tensor gravity theories. In particular, we first study the duals of the charged black hole solutions of a four-dimensional low-energy effective action of heterotic string theory. Next, we study dual of the Xanthopoulos-Zannias solution in Brans-Dicke theory, which contains a naked singularity. We show that, analogous to general relativity, in these scalar-tensor gravity theories the dual solutions are similar to the original spacetimes, but with a global monopole charge.

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*Electronic address: sbose@iucaa.ernet.in
†Electronic address: nkd@iucaa.ernet.in

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I. INTRODUCTION

Ever since the work of Barriola and Vilenkin (BV) [1], it has been of interest to explore the existence of spacetimes with a global monopole charge in alternative theories of gravity. The motivation for considering gravity theories other than general relativity (GR), which was the domain of the BV paper, is that topological defects such as global monopoles are believed to have formed as an inevitable consequence of phase transitions that occurred in the early universe. There is also reason to believe that (GR) may not necessarily be the only viable theory of gravity. Therefore, it is interesting to ask if the global monopole spacetimes in viable alternative gravity theories are sufficiently different from the BV solution. On the other hand, it is also important to study if there are any common underlying features shared by these spacetimes arising as solutions in different gravity theories. Indeed this constitutes one of the main motivations for this paper. Here, we show that in some scalar-tensor theories of gravity, spacetimes with a global monopole charge, which by definition are static and spherically symmetric (SSS), arise from other SSS solutions by an electrogravity transformation [2] defined below.

It is well known that in general relativity, in analogy with Maxwell’s theory of electromagnetism, one can extract the “electric” and “magnetic” parts of the gravitational field from the Riemann curvature tensor [3–5]. These parts are defined relative to a timelike unit vector field and are represented by second rank tensors in three-dimensional space transverse to that field. There are two types of electric parts, “active” and “passive”. The former is obtained by projecting two of the indices of the Riemann tensor along the timelike unit vector, while the latter is obtained by a similar operation on the double-dual of the Riemann tensor (see section II below). Between them, the active and passive parts comprise of 6 independent components each since they are given by symmetric tensors. An analogous projection of the left dual of the Riemann tensor yields a traceless second rank tensor in three space dimensions, which is called the magnetic part. It has 8 independent components. Therefore, the two electric parts and the magnetic part completely determine the Riemann tensor.

In general relativity (GR), a field transformation analogous to the electromagnetic-duality transformation in Maxwell’s theory can be defined that keeps the Einstein-Hilbert action invariant. Such a transformation simultaneously maps the active electric part and the magnetic part into the magnetic part and minus the passive electric part, respectively. In fact, such a set of transformation equations implies the Einstein vacuum field-equations, with a vanishing cosmological constant [2,6]. The electromagnetic-duality transformation in Maxwell’s theory does not exhibit such a property since it involves only the fields, which are defined in terms of only the first derivative of the gauge potential. By contrast, the electric and magnetic parts of the gravitational field are obtained from projections of the Riemann tensor and its duals, and hence, contain second-order derivatives of the metric. These parts, therefore, can be appropriately combined to lay down the equations governing the dynamics of the gravitational field.

Another kind of transformation, termed as the electrogravity transformation (EGT), can be defined by interchanging the active part of the gravitational field with the passive one [7]. Such a transformation is a symmetry of the Einstein-Hilbert action: It maps this action into itself, provided one simultaneously transforms the gravitational constant as, $G \rightarrow -G$. 
Therefore, the vacuum field equations are left invariant under EGT [7]. In this sense it is like a duality transformation. The need to implement the change in sign of $G$ in the "dual" solutions can be understood as follows. The source term for the active electric part stems from the matter stress tensor and can be argued to represent matter energy, while the passive part is associated with the gravitational-field energy. For an attractive field, these two kinds of energy must bear opposite signs. Thus, $G$ has to change sign under the interchange of active and passive electric parts.

The field equations coupled to matter, however, are not invariant under EGT. By a common abuse of language, however, we will refer to the field equations transformed under the EGT as the "dual" equations, and their solutions as the "dual" solutions.

Importantly, EGT can be effectively used to obtain new solutions from known ones. In this context, note that EGT transforms the Ricci tensor into the Einstein tensor, and vice versa. This is because contraction over a pair of Riemann tensor indices yields the Ricci tensor while a similar contraction on its double dual yields the Einstein tensor. For all vacuum solutions, it is possible to introduce matter terms suitably in the field equations such that the modified field equations still admit the original vacuum solutions. However, since the field equations coupled to matter are not EGT invariant, the solutions to the dual equations will be, in general, different from the original vacuum solutions. This is what happens for the static, spherically symmetric family of black hole solutions in GR. In fact, it has been shown that in such a case, a typical dual solution represents a black hole endowed with a global monopole charge. Here, we extend this formalism to obtain solutions dual to the static, spherically symmetric solutions of Brans-Dicke theory as well as a four-dimensional (4D) low-energy effective action of heterotic string theory, namely, dilaton gravity coupled to a $U(1)$ gauge field. We show that analogous to general relativity, in these scalar-tensor gravity theories the dual solutions are similar to the original spacetimes, but with a global monopole charge, which, therefore, appears to be a rather generic feature of EGT. Our work also implies that the effect of a spontaneous symmetry breaking on the global monopole field is tantamount to performing an EGT on the active and passive electric components of the spacetime curvature.

The layout of the paper is as follows. In section II, we define the electrogravity transformation and show how the vacuum Einstein field equations remain unchanged under it. We briefly recapitulate how the solution dual to Schwarzschild can be obtained by implementing this transformation, in section III. The global monopole field configuration and the associated matter stress tensor are discussed in section IV. In section V, we discuss the charged black hole solutions of dilaton gravity and find its dual by effecting the EGT. In section VI, we obtain the dual of the Xanthopoulos-Zannias solution in Brans-Dicke theory. A few thoughts on these solutions and scope for future work are presented in section VII. We work with the metric signature ($-$, $+$, $+$, $+$) and employ geometrized units $G = 1 = c$.

## II. ELECTROGRAVITY DUALITY

The electric and magnetic parts of the gravitational field in general relativity are defined as follows. Consider a timelike unit vector field $u^a$, with $u^a u_a = -1$. Then the active and passive parts of the Riemann tensor relative to $u^a$ are
respectively. Above, $* R_{abcd}$ is the double-dual of the Riemann tensor given by:

$$* R_{abcd} = \frac{1}{4} \epsilon_{abef} \epsilon_{cdgh} R^{efgh},$$  \hspace{1cm} (2.2)$$

where $\epsilon_{abcd}$ is the canonical four-volume element of the spacetime. The magnetic part is the projection of left or right dual of the Riemann tensor and is given by

$$H_{ac} = -* R_{abcd} u^b u^d = H_{(ac)} - H_{[ac]},$$  \hspace{1cm} (2.3)$$

where we have used the left-dual,

$$* R_{abcd} = \frac{1}{2} \epsilon_{abef} R^{ef}_{cd}.$$  \hspace{1cm} (2.4)$$

Also, the symmetric and antisymmetric parts of $H_{ac}$ can be expressed as:

$$H_{(ac)} = -* C_{abcd} u^b u^d$$  and  $$H_{[ac]} = -\frac{1}{2} \epsilon_{abce} R^e_{d} u^b u^d,$$  \hspace{1cm} (2.5)$$

where $C_{abcd}$ is the Weyl tensor. Thus, the symmetric part is equal to the Weyl magnetic part, whereas the anti-symmetric part represents energy flux. Note that $E_{ab}$ and $\tilde{E}_{ab}$ are both symmetric while $H_{ac}$ is trace-free and they are all purely spacelike, i.e., $(E_{ab}, \tilde{E}_{ab}, H_{[ab]}) u^b = 0$.

The Ricci tensor can then be expressed in terms of the electric and magnetic parts as

$$R_a^b = E_a^b + \tilde{E}_a^b - (E + \tilde{E}) u_a u^b - \tilde{E} \delta_a^b - (\epsilon_{amn} H^{mn} u^b + \epsilon^{bmn} H_{mn} u_a)$$  \hspace{1cm} (2.6)$$

where $E = E_a^a$ and $\tilde{E} = \tilde{E}_a^a$.

The EGT is defined by an interchange of the active and passive parts of the electric field, and mapping the magnetic part to minus itself:

$$E_{ab} \longleftrightarrow \tilde{E}_{ab}, \quad H_{ab} \longrightarrow -H_{ab}.$$  \hspace{1cm} (2.7)$$

To see the effect of this transformation on vacuum solutions, note that the vacuum field equations, $R_{ab} = 0$, are in general equivalent to

$$E \text{ or } \tilde{E} = 0, \quad H_{[ab]} = 0 = E_{ab} + \tilde{E}_{ab}$$  \hspace{1cm} (2.8)$$

which are symmetric in $E_{ab}$ and $\tilde{E}_{ab}$. Thus the vacuum field equations (2.8) are invariant under EGT (2.7).

III. SCHWARZSCHILD DUAL

To set the notation and to aid the discussion of SSS solutions and their duals in scalar-tensor gravity, we briefly study how one arrives in GR at the dual of the Schwarzschild solution [4]. Birkhoff’s theorem implies that the Schwarzschild solution, characterized by its
mass, is the unique spherically symmetric solution to Einstein’s vacuum field equations. Any spherically symmetric metric can be cast in the form:

\[ ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + h^2d\omega^2, \] (3.1)

where \( \nu, \lambda, \) and \( h \) are functions of time, \( t, \) and the radial coordinate \( r, \) and \( d\omega^2 \) is the line element on a unit two-sphere. It is well known that under the above conditions we can choose a gauge where \( h = r. \) This is what we do first. A natural choice for the timelike vector \( u^a \) in this case is the timelike unit normal to \( t = \text{constant} \) hypersurfaces. Then, a subset of the conditions in Eqs. (2.8) that ensure a vacuum solution, namely, \( H_{[ab]} = 0 \) and \( E_\theta^\theta + \tilde{E}_\theta^\theta = 0 \) imply that \( \nu + \lambda = 0. \) A supplementary requirement of \( \vec{E} = 0 \) yields \( e^{-2\lambda} = (1 - 2M/r). \) This leads to the Schwarzschild solution. Here, it is important to realize that we were not required to impose the remaining condition in Eqs. (2.8), namely, \( E_r^r + \tilde{E}_r^r = 0, \) in order to obtain this solution. In fact, this equation is implied by the rest. We thus have a choice for introducing some matter distribution in the \( r \)-direction without affecting the Schwarzschild solution. We modify the vacuum field equations (2.8) to read as

\[ H_{[ab]} = 0 = \vec{E}, \quad E_{ab} + \tilde{E}_{ab} = kw_au_b \] (3.2)

where \( k \) is a scalar and \( w_a \) is a spacelike unit vector along the radial acceleration vector \( \dot{u}_a = u^b\nabla_b u_a. \) It is clear that the above equations once again admit the Schwarzschild solution as the unique spherically symmetric solution with \( k = 0. \)

We now perform the electrogravity transformation (2.7) on the above set of equations (3.2) to obtain:

\[ H_{[ab]} = 0 = \vec{E}, \quad E_{ab} + \tilde{E}_{ab} = kw_au_b. \] (3.3)

Its general solution is given by the metric (3.1) with

\[ e^{2\nu} = e^{-2\lambda} = \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right), \] (3.4)

which is the Barriola-Vilenkin solution [1] for a Schwarzschild particle with global monopole charge parameter, \( \sqrt{2k}. \) This solution is obtained as follows. The condition \( \nu + \lambda = 0 \) is implied by the equation \( E_\theta^\theta + \tilde{E}_\theta^\theta = 0. \) In addition to this, the condition \( E = 0 \) yields \( e^{2\nu} = (1 - 8\pi\eta^2 - 2M/r) \) and \( k = 4\pi\eta^2/r^2. \) This spacetime has non-zero stresses given by

\[ 8\pi T^t_t = 8\pi T^r_r = \frac{2k}{r^2}. \] (3.5)

Just like the Schwarzschild solution, the monopole solution (3.4) is also the unique solution of Eq. (3.3).

One may now ask if there are any common features that the dual solutions share. This can be easily found for spherically symmetric solutions with the metric (3.1). For such solutions, the condition \( E_\theta^\theta + \tilde{E}_\theta^\theta = 0 \) from Eq. (3.2) is tantamount to \( R^t_t = R^r_r \) in terms of the Ricci-tensor components. It implies that \( \nu + \lambda = 0. \) Together with \( \vec{E} = 0 \) this implies that \( R_\theta^\theta = 0 = R_\phi^\phi. \) The other components of \( R^a_b \) are zero owing to the remaining condition in Eq. (3.2), namely, \( H_{[ab]} = 0. \) These conditions on the components of the Ricci tensor can
be satisfied even by a matter stress tensor that does not necessarily represent vacuum. In fact, a particular example of $R_{a b}$ conforming to these requirements is

$$R^b_a = k \left( w^a w^b - u^a u^b \right). \quad \text{(3.6)}$$

The matter stress tensor associated with the above form of the Ricci tensor is

$$8\pi T^b_a = k \left( w^a w^b - u^a u^b - \delta^b_a \right), \quad \text{(3.7)}$$

where, in general, $k$ can be a function of $r$ and $t$.

Since the EGT interchanges $R^b_a$ with $G^b_a$, the dual spacetimes are solutions of Eq. (3.6), with $R^b_a$ replaced by $G^b_a$. Hence, they are solutions to a different matter distribution, given by

$$8\pi T^b_a = k \left( w^a w^b - u^a u^b \right). \quad \text{(3.8)}$$

In this case, the Einstein field equation implies that

$$R^b_a = k \left( w^a w^b - u^a u^b - \delta^b_a \right). \quad \text{(3.9)}$$

Consequently, the equations of motion are:

$$R^t_t = R^r_r = 0, \quad \text{(3.10a)}$$
$$R^t_r = 0, \quad \text{(3.10b)}$$
$$R^\theta_\theta = k. \quad \text{(3.10c)}$$

As before, Eq. (3.10a) implies $\nu + \lambda = 0$. However, for a non-vanishing $k$, we can expect $h$ to be different from $r$ here, unlike in the case of the Schwarzschild solution. Since

$$R^\theta_\theta = \left\{ 1 + e^{-2\lambda} \left[ r(\Lambda' - \nu') - 1 \right] \right\} + e^{-2\lambda} \left[ (hh' - r)(\Lambda' - \nu') - (hh'' + h'^2) + 1 \right], \quad \text{(3.11)}$$

one immediately notices that $k \propto 1/r^2$ will always act as a source for the above Ricci tensor component if $h = \text{const} \times r^2$, where the constant is different from 1. In fact, for SSS line-elements in general relativity this gives rise to a general prescription to obtain dual solutions by changing $h$ in the above manner [8].

**IV. GLOBAL MONOPOLE**

Global monopoles are stable topological defects. They are supposed to be produced when global symmetry is spontaneously broken in phase transitions in the early Universe [9]. A global monopole is described by an isoscalar triplet, $\psi^a$, with $a = 1, 2, 3$. The associated Lagrangian density is [1]:

$$L_m = \frac{1}{2} (\nabla \psi^a)^2 + \frac{\lambda}{4} (\psi^a \psi_a - \eta^2)^2. \quad \text{(4.1)}$$

Such a system has a global $O(3)$ symmetry and offers topologically non-trivial self-supporting solutions. The global monopole is obtained by implementing the ansatz that $\psi^a(r) =$
\[ \eta f(r)x^a/r, \]  
where \( x^a x^a = r^2 \). Here \( \eta \) is a constant whose value defines the energy scale of symmetry breaking.

For a given spacetime metric, the stress tensor associated with a global monopole can be inferred from the above Lagrangian density in a standard manner. Consider the SSS metric \([\text{3.I}]\) with \( \nu \) and \( \lambda \) independent of \( t \). Then \( x^a \) is interpreted as a “Cartesian” coordinate, and the field equation for \( \psi^a \) reduces to the following equation for \( f(r) \).

\[
e^{-2\lambda} f'' + \left[ 2e^{-2\lambda} \frac{r^2}{r^2} + e^{-2\nu} \left( e^{2(\nu-\lambda)} \right)^2 \right] f' - \frac{2f}{r^2} - \lambda \eta^2 f(f^2 - 1) = 0. \quad (4.2)
\]

The stress-tensor components of the monopole are:

\[
T^t_t = \frac{\eta^2 f'^2 e^{-2\lambda}}{2} + \frac{\eta^2 f^2}{r^2} + \frac{1}{4} \lambda \eta^4 (f^2 - 1)^2, \]

\[
T^r_r = -\frac{\eta^2 f'^2 e^{-2\lambda}}{2} + \frac{\eta^2 f^2}{r^2} + \frac{1}{4} \lambda \eta^4 (f^2 - 1)^2, \]

\[
T^\theta_\theta = T^\phi_\phi = \frac{\eta^2 f'^2 e^{-2\lambda}}{2} + \frac{1}{4} \lambda \eta^4 (f^2 - 1)^2. \quad (4.3)
\]

The monopole core is defined by values of \( r \) for which \( f(r) \approx 1 \). Outside and at large distances from the monopole core the stresses would approximate to \([\text{4.4}]\), which is precisely of the form given in Eq. \([\text{3.5}]\).

The dual solution to flat spacetime can also be obtained as follows. Note that flat spacetime is a solution to the following equations of motion:

\[
\tilde{E}_{ab} = 0 = H_{[ab]}, \quad E_{ab} = k w_a w_b, \quad (4.5)
\]

which are solved to give \( \nu = \lambda = 0 \). As before, the condition \( k = 0 \) is implied by the fact that such a solution corresponds to an isotropic spacetime. Its dual is the solution of the equation dual to \([\text{4.5}]\), which reads as

\[
E_{ab} = 0 = H_{[ab]}, \quad \tilde{E}_{ab} = k w_a w_b \quad (4.6)
\]

yielding the general solution,

\[
\nu' = \lambda' = 0 \implies e^{2\nu} = 1, \quad e^{2\lambda} = (1 - 2k)^{-1} = \text{constant}. \quad (4.7)
\]

The resulting spacetime is non-flat and represents a global monopole of zero mass. Note that such a spacetime is the same as the one described by Eq. \([\text{3.4}]\) in the limit of vanishing mass \( M \). This could as well be considered as a spacetime of constant relativistic potential. It can also be viewed as a “minimally” curved spacetime (see Refs. \([6,7]\)). The EGT thus generates topological defects in vacuum solutions of the Einstein field equations, which is a remarkable property of this transformation.

To summarize, the above procedure for obtaining solutions dual to any known spacetime solution would work as long as there occurs a free equation in the field equations \([\text{2.3}]\) that is not used in finding that solution. Note that this holds for all solutions in the family of charged Kerr black holes \([\text{11,12}]\) as well as for the NUT solution \([\text{13}]\). Then the dual set admits a solution similar to the original one, but with a topological defect, namely, global monopole charge.
V. 4D DILATON GRAVITY

In the spirit of the Barriola-Vilenkin solution (3.4), one may expect analogous solutions to exist even in some scalar-tensor theories of gravity. A particular class of candidates among these classical theories, which are posed as leading alternatives to general relativity, are the 4D low-energy effective theories derived from heterotic string theory. Here, we consider the specific case of 4D dilaton gravity action coupled to a $U(1)$ gauge field, which has charged black hole solutions [14,15] (see Refs. [16,17] for reviews):

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left\{ e^{-2\phi} [\bar{R} + 4(\bar{\nabla}\phi)^2 - 2\Lambda] - \bar{F}^2 \right\} ,$$

where $\bar{g}_{\mu\nu}$ is the string metric, $\bar{R}$ is the 4D Ricci scalar, $\Lambda$ is a cosmological constant and $\bar{F}_{\mu\nu}$ is the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$. Here, we shall consider the case where $\Lambda = 0$. The conformal transformation $g_{\mu\nu} = e^{-2\phi} \bar{g}_{\mu\nu}$ can be implemented to recast the above action in the “Einstein-Hilbert” form:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2(\nabla\phi)^2 - e^{-2\phi} F^2 \right] ,$$

where $g_{\mu\nu}$ is the Einstein-frame metric. The corresponding equations of motion are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 2\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}(\nabla\phi)^2 - 2e^{-2\phi} F_{\mu\lambda} F^\lambda_{\nu} + \frac{1}{2} g_{\mu\nu} e^{-2\phi} F^2 = 0 ,$$

$$2\nabla^2 \phi + e^{-2\phi} F^2 = 0 ,$$

$$\nabla^\nu (e^{-2\phi} F_{\mu\nu} = 0 .$$

Taking the trace of (5.3a) gives:

$$R = 2(\nabla\phi)^2 ,$$

which, together with (5.3a) and (5.3c), implies

$$R_{\mu\nu} = 2\nabla_\mu \phi \nabla_\nu \phi + 2e^{-2\phi} F_{\mu\lambda} F^\lambda_{\nu} + g_{\mu\nu} \nabla^2 \phi .$$

This equation will play a pivotal role below in our study of charged black hole solutions and their duals.

If a matter action, say, corresponding to the Lagrangian density (4.1) of a global monopole, were present in Eq. (5.2), then a matter stress tensor following from such a term would contribute to the right-hand side of the field equation (5.3a). Since it is not known how the dilaton couples to the monopole field $\psi^a$, one can obtain different theories incorporating $\psi^a$ based on the choice of this coupling. Global monopoles in 4D dilaton gravity have been studied for some choices of coupling and for both massive and massless dilaton in Ref. [18]. We will later consider solutions to action (5.2) coupled to $\psi^a$ by adding to it the matter action

$$S_m = -\frac{1}{16\pi} \int d^4x \sqrt{-g} L_m .$$

8
where $L_m$ is given in Eq. (4.1). Such a choice of coupling and $L_m$ is different from that considered in Ref. [18].

We begin by briefly recalling how the above equations of motion are solved to obtain the charged black hole solution. Consider the spherically symmetric static (SSS) metric (3.1), where $\nu$, $\lambda$, and $h$ are now functions of the radial coordinate $r$ only. For such a metric the only non-vanishing components of the Ricci tensor are the diagonal elements. For Eq. (5.5) to have an SSS solution, the following conditions on $R_{tt}$ and $R_{rr}$ must be obeyed:

$$R_{tt} = e^{2(\nu - \lambda)} \left\{ \nu'' + \nu'^2 - \lambda' \nu' + \frac{2\nu'}{r} \right\} - 2\nu' e^{2(\nu - \lambda)} \left[ \frac{1}{r} - \frac{h'}{h} \right] = e^{2\nu - 2\phi} \frac{Q^2}{h^4}, \quad (5.7a)$$

$$R_{rr} = \left\{ -\nu - \nu'^2 + \lambda' \nu' + \frac{2\lambda'}{r} \right\} - 2\lambda' \left[ \frac{1}{r} - \frac{h'}{h} \right] - \frac{2h''}{h} = \left[ 2\phi'^2 - e^{2\lambda - 2\phi} \frac{Q^2}{h^4} \right]. \quad (5.7b)$$

Similarly, the field equation for the component $R_{\theta\theta}$,

$$\left\{ 1 + e^{-2\lambda} \left[ r (\lambda' - \nu') - 1 \right] \right\} + e^{-2\lambda} \left[ (hh' - r)(\lambda' - \nu') - (hh'' + h'^2) + 1 \right] = e^{-2\phi} \frac{Q^2}{h^2}, \quad (5.8)$$

must also be satisfied. Spherical symmetry ensures that the $R_{\phi\phi}$ equation implies the same condition on $\nu$, $\lambda$, and $h$ as in Eq. (5.8). If $Q = 0$, then the right-hand side of the above equation vanishes. Therefore, the following expressions,

$$e^{2\nu} = e^{-2\lambda} = 1 - 2m/r, \quad \text{and} \quad h = r \quad (5.9)$$

constitute a solution to the above equations. This simply corresponds to the Schwarzschild metric, which indeed is a solution to the equations of motion (5.3) with $F_{\mu\nu} = 0$ and $\phi$ = constant.

Finding an SSS metric as a solution to (5.7) and (5.8) for $Q \neq 0$ is also straightforward. Note that $\nu$ and $\lambda$ given in Eq. (5.9) makes the braces on the left-hand sides of Eqs. (5.7) and (5.8) vanish. Thus, the problem reduces to finding an $h$ that makes the remaining term on the left-hand sides of Eqs. (5.7) and (5.8) equal to their right-hand sides, respectively. Such an $h$ exists and is given by

$$h^2 = r^2 \left( 1 - \frac{Q^2}{mr} \right). \quad (5.10)$$

This, therefore, constitutes the charged black hole solution of 4D dilatonic gravity. The corresponding fields are:

$$ds^2 = - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 \left( 1 - \frac{Q^2}{mr} \right) d\omega^2. \quad (5.11)$$

$$e^{-2\phi} = \left( 1 - \frac{Q^2}{mr} \right) = U(\phi), \quad (5.12)$$

$$F_{rt} = \frac{Q}{r^2}. \quad (5.13)$$
In obtaining the above solution from the equations of motion, we have assumed that \( \phi \to 0 \) as \( r \to \infty \).

The effect of the electrogravity-duality transformation on the field equations (5.3) is to modify them by the addition of the asymptotic form of the global-monopole stress tensor (3.5) on its right-hand side. This is completely analogous to what happens in the case of the Schwarzschild black hole in GR (see section III). It, therefore, follows that the following new choices for \( \nu \) and \( \lambda \) will solve such a set of equations:

\[
e^{2\nu} \to e^{2\tilde{\nu}} = 1 - 8\pi \eta^2 - \frac{2\tilde{m}}{r},
\]

\[
e^{2\lambda} \to e^{2\tilde{\lambda}} = 1 - 8\pi \eta^2 - \frac{2\tilde{m}}{r},
\]

(5.14)

where \( \tilde{m} \) is just an integration constant. In other words, for the choice in (5.14), the braces on the lhs of Eqs. (5.7) and (5.8) are exactly equal to the components of the global-monopole stress tensor. This suggests the possibility that there exists an \( h \to \tilde{h} \) and \( \phi \to \tilde{\phi} \), for which \( \tilde{\nu} \) and \( \tilde{\lambda} \) solve the field equations (5.7) and (5.8) modified by the presence of source terms arising from the global monopole stress tensor.

In fact, it turns out that such a choice for \( h \) is available. This can be understood by noting that \( \tilde{\nu} \) and \( \tilde{\lambda} \) can be cast in the same form as (5.9):

\[
e^{2\tilde{\nu}} = e^{-2\kappa} \left( 1 - \frac{2\tilde{m}}{r} \right) = e^{-2\tilde{\lambda}},
\]

(5.15)

where \( e^{-2\kappa} = (1 - 8\pi \eta^2) \) and \( \tilde{m} = e^{-2\kappa} m \). Using such scaling relations between tilded and untilded parameters, it is easy to see that

\[
\tilde{h}^2 = r^2 \left( 1 - \frac{Q^2}{\tilde{m}r} \right), \quad \text{and} \quad e^{-2\tilde{\phi}} = \left( 1 - \frac{Q^2}{\tilde{m}r} \right).
\]

(5.16)

Calling \( \tilde{m} = M \), we finally arrive at the metric of the spacetime dual to the charged dilatonic black holes:

\[
ds^2 = - \left( 1 - 8\pi \eta^2 - \frac{2M}{r} \right) dt^2 + \left( 1 - 8\pi \eta^2 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 \left( 1 - \frac{Q^2}{Mr} \right) d\omega^2.
\]

(5.17)

This is a solution to the modified equations of motion. The corresponding dilaton and \( U(1) \) field solutions are given by Eqs. (5.16) and (5.13), respectively. The resulting field configuration solves the equations of motion (3.3). In the limit \( Q = 0 \), one recovers the Barriola-Vilenkin spacetime. This is expected since in that limit the dilaton field acquires a constant value. Consequently, 4D dilaton gravity reduces to general relativity. Additionally, if \( M = 0 \), then the above metric describes a locally flat spacetime with a global monopole charge, which is the electrogravity dual of flat spacetime.

It, however, remains to be shown that the stress tensor (3.5) is indeed the asymptotic form of the global monopole stress tensor arising from (5.6) for the above spacetime metric. To see that this is indeed true, we cast the above metric as
\[ ds^2 = - \left(1 - \frac{4M^2}{Q^2 + \sqrt{1 + \frac{4M^2}{Q^2}}\rho^2}\right) dt^2 \]
\[ + \frac{1}{4} \left(1 + \frac{Q^2}{4M^2\rho^2}\right)^{-1} \left(1 - \frac{4M^2}{Q^2 + \sqrt{1 + \frac{4M^2}{Q^2}}\rho^2}\right)^{-1} d\rho^2 + \rho^2 d\omega^2 , \] (5.18)

where \( \rho^2 = r^2 - rQ^2/M \). This metric is of the same form as Eq. (3.1) with \( r \) replaced by \( \rho \) there. Using Eqs. (4.3) to compute the matter stress-tensor components gives

\[ T_t^t \approx T_\rho^\rho \approx \frac{\eta^2}{\rho^2} , \quad T_\theta^\theta = T_\phi^\phi \approx 0 , \] (5.19)

outside the monopole core. In the limit of large \( r \) these go over to the expected stress tensor components (4.4) of a global monopole. Moreover, it is straightforward to verify that the field equation for \( f(\rho) \), which is given by Eq. (4.2) with \( r \) replaced by \( \rho \), can be solved asymptotically with \( f(\rho) \approx 1 \) outside the core.

VI. BRANS-DICKE THEORY

Another alternative candidate for the theory of gravity is the Brans-Dicke theory (BD). Analogous to the dilaton field in string theory, BD also includes a scalar field as part of the spacetime geometry. The BD action is, however, different from those considered in the previous sections [19]. Vacuum BD can be conformally transformed to 4D Einstein gravity coupled to a massless scalar field, \( \varphi \). The corresponding field equations are:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa (T^\varphi_{\mu\nu} + T_{\mu\nu}) , \] (6.1)

where \( \kappa \) is a coupling constant, \( T^\varphi_{\mu\nu} \) is the stress tensor of the scalar field,

\[ T^\varphi_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \varphi \nabla_\lambda \varphi , \] (6.2)

and \( T_{\mu\nu} \) is the stress tensor of any matter distribution that may be present. Also, the scalar field obeys the equation,

\[ \nabla^\mu \nabla_\mu \varphi = 0 , \] (6.3)

Equations (6.1) and (6.2) can be combined to yield

\[ R_{\mu\nu} = \kappa (\nabla_\mu \varphi \nabla_\nu \varphi + T_{\mu\nu}) . \] (6.4)

Thus, Eqs. (6.4) and (6.3) comprise the field equations of this theory. Note that these equations can be obtained from the more general class of field equations, Eqs. (5.3) by setting \( F_{\mu\nu} = 0, \kappa = 2 \), and identifying the dilaton field with the scalar field in those equations.

The spherically symmetric neutral black hole solutions to the above equations are known. The no scalar-hair theorem [20,21] guarantees that they are just the Schwarzschild solution.
with a constant scalar field \[22,23\]. Also, the spherically symmetric charged black hole solution to the 4D Brans-Dicke-Maxwell theory still remains the Reissner-Nordstrom solution with a constant scalar field \[24\]. However, as was shown by Xanthopoulos and Zannias (XZ) \[23\], there is an interesting solution to the above equations with a varying scalar field and with a vanishing matter stress tensor, \(T_{\mu\nu}\). The corresponding fields are:

\[
\begin{align*}
\text{ds}^2 &= -dt^2 + dr^2 + (r^2 - r_0^2) d\omega^2 , \\
\phi(r) &= \frac{1}{\sqrt{2\kappa}} \ln \left\{ \frac{r - r_0}{r + r_0} \right\} ,
\end{align*}
\]

where \(r_0\) is a constant. The scalar curvature behaves as:

\[
R = \frac{2r_0}{(r^2 - r_0^2)} .
\]

Although not a black hole, nevertheless the above solution has a naked curvature singularity at \(r = r_0\). We call this the XZ solution. Below, we seek the dual of this solution.

Under EGT, the field equation (6.1) gets transformed to:

\[
G_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa (T_{\mu\nu}^\phi + T_{\mu\nu}) .
\]

It can be shown that the spacetime solution to the above equations is:

\[
\text{ds}^2 = -(1 - 8\pi\eta^2)dt^2 + (1 - 8\pi\eta^2)^{-1} dr^2 + (r^2 - r_0^2) d\omega^2 .
\]

The above form of the dual metric is expected since the original metric is spherically symmetric. The only non-vanishing component of the Einstein tensor for the above metric is

\[
G^r_r = \frac{2r_0^2}{(1 - 8\pi\eta^2)(r^2 - r_0^2)} .
\]

The corresponding scalar field solution is the same as that given by (6.6). This is because, under the duality transformation, the determinant of the XZ metric (6.5) remains invariant and \(g^{rr}\) continues to remain constant.

The matter stress tensor, \(T_{\mu\nu}\), for this solution is non-vanishing; in fact, it is the same as that for a global monopole (4.4), as is typical of solutions obtained by applying EGT on known SSS solutions. Indeed, in the limit \(r_0 \to 0\), the metric (6.9) clearly describes a space with a deficit solid angle. This limit corresponds to a constant scalar field and, therefore, yields a solution in GR. Such a solution was indeed observed in Ref. [1], and is associated with a global monopole.
VII. DISCUSSION

An important observation made in Refs. [6,7] was that as long as one of the Einstein field equations remains unused in obtaining a particular solution, one can obtain a dual solution, which is different from the original, by modifying that equation. The modification is to introduce the term on the right in Eqs. (3.2), where the spacelike unit vector $w_a$ is along the radial four-acceleration vector. By this prescription solutions dual to all isolated sources have been obtained. The next question is: How good are Eqs. (3.2) as “non-vacuum” field equations? The first part of the equation implies vanishing of energy density (i.e., $\tilde{E} = 0$) and of energy flux (i.e., $H_{[ab]} = 0$). This means that they cannot have a physically meaningful non-vacuum solution. In the case of spacetimes corresponding to non-localized sources, such as those associated with the Kasner, the Weyl, the Levi-Civita metrics, and the metric of a plane gravitational wave, it turns out that Eqs. (3.2) (with $kw_a w_b$ replaced by $k(g_{ab} + u_a u_b)$ for the homogeneous case) admits them as solutions, and so does the dual set of equations [25]. Thus, such solutions describing non-localized sources are electrogravity self-dual. Also, Eqs. (3.2) admit meaningful solutions only when they are vacuum spacetimes. Even if they admit any non-vacuum solution, it would correspond to a matter distribution with vanishing energy density and, hence, would be unphysical. Hence, although the electrogravity duality transformation can be effected to obtain dual spacetime solutions in the above manner, it is not guaranteed to correspond to physically acceptable matter distributions. For instance, there does not occur a physically reasonable dual solution to any spherically symmetric static perfect fluid spacetime other than the de Sitter solution.

The application of the electrogravity-dual transformation on perfect fluid spacetimes has been considered [26]. It maps the perfect-fluid stress-tensor components as $\rho \rightarrow (\rho + 3p)/2$ and $p \rightarrow (\rho - p)/2$, where $\rho$ and $p$ are the matter energy and pressure densities, respectively. Thus, the transformed stress tensor also describes a perfect fluid. In particular, the equations of state corresponding to radiation, $\rho = 3p$, and to the cosmological constant term, $\rho + p = 0$, remain invariant. In the former case, the solution is self-dual because the Ricci tensor is identical to the Einstein tensor. Whereas in the latter case we have $\Lambda \rightarrow -\Lambda$, which implies that the de Sitter and the anti-de Sitter spacetimes are dual to each other. Also, the dust distribution, where $p = 0$, is dual to the stiff fluid, where $\rho = p$.

Since electrogravity duality transformation only involves the Riemann tensor and hence is quite general, it should be applicable in other metric theories as well. Here we have seen its application on black hole solutions in the 4D low-energy effective heterotic string theory as well as on the Xanthopoulos-Zannias solution in Brans-Dicke theory. The procedure works exactly along the lines of GR and we obtain their dual solutions, which exhibit the presence of a global monopole charge. Thus we can make the general statement that by implementing the electrogravity duality transformation one can always generate a global monopole charge in a static spherically symmetric solution in GR, Brans-Dicke theory, and in 4D dilaton gravity. This is because as in GR, even in these scalar-tensor theories, the static spherically symmetric sector of their solution space is determined by only a subset of the corresponding field equations. In fact this is true even in lower dimensional Einstein-gravity, e.g., the theory corresponding to the 3D Einstein-Hilbert action involving a cosmological constant term. Hence, these theories, which have played an important role in this decade in the understanding of black hole physics and related quantum aspects, also hold the promise of
harboring yet unknown solutions with topological defects that may play an important role in alternative cosmological models. We are currently involved in studying such solutions [27].

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