Interplay of Delay and multiplexing: Impact on Cluster Synchronization

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Communication delays and multiplexing are ubiquitous features of real-world networked systems. We here introduce a simple model where these two features are simultaneously present, and report the rich phenomenology which is actually due to their interplay on cluster synchronization. A delay in one layer has non trivial impacts on the collective dynamics of the other layers, enhancing or suppressing synchronization. At the same time, multiplexing may also enhance cluster synchronization of delayed layers. We elucidate several nontrivial (and anti-intuitive) scenarios, which are of interest and potential application in various real-world systems, where the introduction of a delay may render synchronization of a layer robust against changes in the properties of the other layers.

Keywords: Synchronization, Coupled maps, Chaos, Networks

Synchronization plays a vital role in proper functioning of the brain, social, ecological and many other complex real-world systems. These real-world systems possess multiple relations among its units (nodes) and thus can be represented by a multiplex network. Furthermore, communication delays naturally arise in real-world systems, and they may affect either a single layer of all the layers of the network. We introduce a simple model where two features are simultaneously present, and elucidate several phenomena of cluster synchronization due to their interplay. Interestingly, cluster synchronization of a layer in a multiplex network can be tuned by the delay value in the other layers. Furthermore, by introducing delay, cluster synchronization in one layer can be made robust against changes in the delay value as well as network architecture of the other layer. These results are important for multiplex systems with more than one type of vital resources, as for instance water supply and power plants of a city, where controlling synchronizability is a task of fundamental importance.

Synchronization is one of the most common process through which real-world systems (such as chirping crickets, or flashing fireflies1) act synergically, and it is a crucial feature which facilitates the collective functioning of man-made systems, like power grids or communication systems2. In particular, cluster synchronization (CS) refers to the case where only few components of the system form clusters of synchronized units2.

On the other side, multiple interactions co-exist in general among the same set of constituents of a system, which led to a spurt of modeling activities under the so-called multiplex network framework, where nodes are copied in different layers featuring different shared relationships. An example is a transport network, where cities can form different layers depending on whether they are connected by rail, bus or air3,5. Furthermore, layers are usually strongly inter-dependent (for instance, a strike of the bus service may result in...
overloading the rail and air traffic routes), and major consequences of such dependencies are known to be, for instance, explosive synchronization, diffusion dynamics, emergence of chimera, epidemic spreading, and intra- and inter-layer synchronization. CS, in particular, of sparse networks, gets enhanced upon multiplexing with other sparse networks, whereas it is suppressed upon multiplexing with dense networks.

In addition, communication delays naturally arise in real world systems, and they may affect either a single layer or all the layers of the network. Global and breathing synchronization are phenomena emerging due to delays in the inter-layer coupling, but still, the impact of delay on CS is not fully investigated. In this Letter, we focus on the simultaneous presence of delay and multiplexing, and examine the impact of their interplay on CS. We will show that delayed interactions in a layer may lead to an enhancement or suppression in CS of other layers. Our findings will be exemplified for multiplex networks having different combinations of 1-d lattices, scale-free (SF) and Erdös-Rényi (ER) random graphs. The used 1-d lattices have circular boundary conditions with each node having exactly k nearest neighbors. The scale-free networks are constructed using the Barabási-Albert algorithm by starting with \( k \) nodes and adding one node with \( k \) connections at each step. The Erdös-Rényi networks are constructed by connecting all pairs of the nodes with a probability \( p \).

Our starting point is a multiplex network (with \( N \) nodes in each layer, \( l \) layers, and with a number of connections \( N_c \)). The system is described by a time dependent vector state \( X_i \equiv \{ x_i(t), i = 1, 2, \ldots, l \times N \} \), which evolves following a model of coupled maps:

\[
x_i(t+1) = (1 - \varepsilon) f(x_i(t)) + \frac{\varepsilon}{k_i} \sum_{j=1}^{N_l} A_{ij} g(x_j(t - \tau_{ij})),
\]

where \( A \) is a supra-adjacency matrix: if only two layers are present in the network, \( A = \left( \frac{A^1_{ij}}{A^2_{ij}} \right) \), where \( A^1 \) (\( A^2 \)) is the adjacency matrix of layer 1 (2). In Eq. (1), \( \varepsilon \) is the overall coupling strength, and \( k_i = \sum_{j=1}^{N_l} A_{ij} \) is the degree of the \( i \)th node. The functions \( f \) and \( g \) are both taken to be the logistic map, so that \( f(x) = g(x) = \mu x (1 - x) \), with \( \mu = 4 \). Coupled logistic map, due to its simplicity yet ability to display complex dynamical behavior, have been widely investigated to understand various complex phenomena manifested by a diverse range of real-world systems. The delay \( \tau_{ij} \) is the communication time that information takes to travel from unit \( i \) to unit \( j \). We here consider a symmetric (\( \tau_{ij} = \tau_{ji} \)) and homogeneous (\( \tau_{ij}^1 = \tau_1 \) and \( \tau_{ij}^2 = \tau_2 \) \( \forall i, j \)) case, with \( \tau \) denoting the delay matrix given by \( \tau = \left( \begin{array}{cc} \tau_1^1 & 0 \\ 0 & \tau_2 \end{array} \right) \) (see Fig. 1 for a sketch of the system under study). Notice that we do not consider the delay in the inter-layer couplings, and only intra-layer communication delay is accounted for.

We focus on phase synchronization, which is here defined as follows: let \( n_i \) and \( n_j \) denote the number of times at which the variables \( x_i(t) \) and \( x_j(t) \) (\( t = 1, 2, \ldots, T \)) exhibit
local minima (or maxima) during a long time interval $T$, and let $n_{ij}$ denote the number of times these local minima (maxima) match with each other (i.e. occur at the same time). Then, a phase distance between two nodes $i$ and $j$ can be defined as $d_{ij} = 1 - 2n_{ij} / (n_i + n_j)$. Two nodes $i$ and $j$ are phase synchronized if $d_{ij} = 0$, and all phase synchronized nodes form a cluster of size $N_{\text{clus}}$. Note that the definition of $d_{ij}$ does not involve the network architecture and is measured between all the pairs of the nodes irrespective of if there exists a connection between a pair of nodes or not. However, phase distance between a pair of the nodes may change depending upon the network architecture, coupling value and the delay value. CS is said to be enhanced (reduced) if the the fraction of nodes participating in the cluster formation, $f_{\text{clus}} = N_{\text{clus}}/N$, increases (decreases). We define robustness in terms of the phenomenon of enhancement or suppression in CS of a layer upon multiplexing with another layer against change in the network parameter of the layer it is multiplexed with. When the case under consideration is a biplex (a network formed by only two layers), delays can be limited to one of the two layers, or they can affect the interactions in both layers. The first step is then focusing on the case in which only one layer is made of delayed interactions. Still, two relevant issues arise: (i) how the delay in a layer affects the dynamics of that same layer, and (ii) what is the impact of a delay acting in one layer on the dynamics of the other (un-delayed) layer.

We evolve Eq. 1 starting from a set of random initial conditions and study the phase synchronized clusters after an initial transient. We find that delayed interactions in a layer may significantly affect the CS of the other layer. The most intriguing case is possibly that of a biplex with regular architecture in both layers, which exhibit very poor CS.
FIG. 3. (Color online) Node vs. node diagram exhibiting enhancement in CS of the first layer as a delay is introduced in the second layer. Circles (◦) represent nodes. Node numbers are reorganized such that those belonging to the same cluster come consecutively and form a square block. Thus square blocks indicate the phase synchronized clusters, while circles along the diagonal indicate the nodes which are not synchronized with any other node in the network. For all the graphs second layer of the multiplex network is represented by the 1-d lattice, whereas the first layer is represented by (a) & (b) 1-d lattice, (c) & (d) ER network, (e) & (f) SF networks. Other network parameters are \( N_1 = N_2 = 100, \langle k_1 \rangle = \langle k_2 \rangle = 4 \) and \( \varepsilon = 0.75 \).

For instance, a 1-d lattice with \( \langle k_1 \rangle = 4 \) exhibits no CS (at both intermediate and strong couplings) when multiplexed with another un-delayed 1-d lattice with the same number of connections (see Figs. 2(a) and 3(a)), whereas it displays about 98% of the nodes taking part in the cluster formation (see Fig. 2(a)) upon multiplexing with a delayed 1-d lattice (for \( \tau_2 = 1 \)). A further increase in the delay, however, suppresses such an enhancement, and once again no CS is exhibited for \( \tau_2 > 4 \) at both intermediate and strong couplings (Fig. 3(b)).

A second interesting case is that of a regular network (again a 1-d lattice) multiplexed with a random one (an ER or an SF). Multiplexing with un-delayed random networks has
been shown, indeed, to enhance CS of a regular network. Here, however, the presence of a delay has an important effect: the multiplexing may lead to a further enhancement (or to a suppression) of CS, depending upon the parity of the delay value. Namely, odd (even) values of the delay enhance (suppress) CS. For instance, CS is enhanced in a 1-d lattice with \( \langle k_1 \rangle = 4 \) upon multiplexing with a un-delayed ER network of \( \langle k_2 \rangle = 4 \), it gets further enhanced for \( \tau_2 = 1, 3 \), whereas it is suppressed for \( \tau_2 = 2, 4 \) (see Fig. 2(b)). Similarly, ER and SF networks (when multiplexed with a 1-d lattice) exhibit an enhancement in CS for \( \tau_2 = 1, 3 \), and suppression for \( \tau_2 = 2, 4 \) (Figs. 2(c), (d)). Figs. 3 reveal that the observed enhancement is significantly more prominent for the 1-d lattice and SF network cases, as compared with the ER network case.

In order to grab the essential reason behind the observed CS enhancement, we notice first that a delay in a layer already enhances CS of that same layer (as also reported for isolated networks). Then, due to multiplexing, such an effect is cooperatively turned into an enhancement of CS in the other, un-delayed, layer. For instance, Fig. 4(a) shows that the introduction of a delay in an ER network with \( \langle k_2 \rangle = 4 \) conveys more than 60% of the nodes to participate in the cluster formation for \( 0.32 \lesssim \varepsilon \lesssim 0.96 \) (0.5 \( \lesssim \varepsilon \lesssim 0.96 \)) when \( \tau_2 = 1 \) (\( \tau_2 = 4 \)). In the same Figure, it is seen that the consequence is that the other un-delayed layer (another ER network with \( \langle k_1 \rangle = 4 \)), starts exhibiting a good CS in the same coupling regimes.

Such a cooperation mechanism between the delay and the multiplexing is controlled and regulated by the degree-degree correlations of the replica nodes, calculated using the Pearson correlation coefficient \( r \). The Pearson correlation coefficient \( r \) is given by \( r = \frac{\sum_{jk} e_{jk}(q_k - q_k^0)}{\sigma_q^2} \), where \( e_{jk} \) (\( q_k \)) is the fraction of nodes of degree \( j \) (the fraction of all nodes) connected with nodes of degree \( k \), so that \( \sum_{jk} e_{jk} = 1 \), and \( \sum_j e_{jk} = q_k \).

When the delayed layer is denser than the un-delayed one \( \langle k_2 \rangle > \langle k_1 \rangle \), the enhancement in CS of the un-delayed layer is less prominent than the case of both layers having comparable average degrees \( \langle k_1 \rangle \approx \langle k_2 \rangle \). Interestingly, in the extreme regime (i.e., when \( \langle k_2 \rangle \gg \langle k_1 \rangle \)), CS of the un-delayed layer turns out to be significantly suppressed. As an example, Fig. 5(a) shows that CS of an un-delayed 1-d lattice with \( \langle k_1 \rangle = 4 \) is maximum upon its multiplexing with another delayed \( \tau_2 = 1 \) 1-d lattice of \( \langle k_2 \rangle = 4 \), whereas it is completely suppressed when \( \langle k_2 \rangle \gtrsim 8 \). A similar scenario emerges for a pair of ER layers (one delayed, and one un-delayed) representing multiplex network, as depicted by Fig. 5(b).

The conclusion of this first part of the study is that, in sharp contrast to the case of multiplex networks made of 1-d lattices, the delayed evolution of a layer seems to lead (not to lead) to an enhancement in CS of the other un-delayed layer when degree-degree
correlations in the replica nodes are neutral (highly disassortative), irrespectively of the topology of each layer.

Another relevant scenario is illustrated in Fig. 6. First, in the absence of delay, an isolated 1-d lattice ($\langle k \rangle = 4$) does not exhibit CS for intermediate and strong couplings (see open circles in Fig. 6(a)). Multiplexing it with a un-delayed globally connected network does not lead to enhancement in CS as well, even when $f_{clus}^{1} \approx 1$ for the latter network (Fig. 6(b)). When a delay is considered ($\tau = 1$) in the initial lattice, CS instead increases: in the range $0.2 \lesssim \varepsilon \lesssim 0.6$ more than 75% of the nodes is participating in the cluster formation, while CS is almost negligible for $\varepsilon > 0.62$ (see the full circles of Fig. 6(a)). Finally, upon multiplexing, the delayed lattice starts exhibiting an excellent CS in the range $0.14 \lesssim \varepsilon \lesssim 0.81$ (with more than 80% of clustered nodes), and even for $\varepsilon > 0.82$ $f_{clus}^{1}$ remains at around 0.3 (see Fig. 6(c)). This demonstrates a clear enhancement of CS for regular delayed networks, upon multiplexing them with the globally connected networks. Thus, enhancement in CS is the combined impact of multiplexing and delayed coupling. Similarly, our results indicate that delayed ER networks exhibit strong enhancements in CS upon multiplexing them with regular, ER, and SF networks.

A third scenario is the case in which both layers are affected by communication delays. The main result is that CS of a delayed layer (say layer 1) remains almost unaffected by changes in the delay value of the second layer. For instance, a 1-d lattice with coupling delay $\tau_1 = 1$ exhibits an excellent CS upon multiplexing with another 1-d lattice (or with a SF network) irrespective of the delay value in the other layer [see triangles in Figs. 4(b) and (c)]. In contrast to the un-delayed case (Fig. 2), a change in the delay value in one layer does not lead to drastic changes on CS of another delayed layer.

So far, we focused on the impact of delay in one layer on CS of another (delayed or un-delayed) layer. Next, we show the effect of changing the connection density of a layer on CS of the other delayed layers. Higher connection densities are known to suppress CS in sparse layers. Interestingly, however, the nodes’ interactions of a sparse layer are affected by a delay (or if the nodes of both layers have delayed interactions), CS always gets enhanced upon multiplexing, even in the case of maximum disassortativity in the replica nodes (i.e. for $r = -1$, see curves with squares in Figs. 5(a) and (b) for the 1-d lattice and the ER network of $\langle k_1 \rangle = 4$). Notice that the level of CS exhibited by the delayed layer of a multiplex network is much higher than that of the corresponding isolated network. At higher delay values, instead, CS is suppressed. Therefore, degree-degree disassortativity in the mirror (replica) nodes does not have effects on CS of the sparse networks if delays affect both layers. Therefore, in contrast to CS of the un-delayed layer, which is controlled by the degree-degree correlations ($r$) of the replica nodes, presence of the communication delay makes CS of the delayed layer robust against the change in $r$ value.
FIG. 7. (Color online) $f_{clus}^1$ vs. $\varepsilon$ in a three-layers multiplex network. (a) an ER layer multiplexed with two other ER layers and (b) a 1-d lattice multiplexed with two other 1-d lattices. In both panels $(k_1) = (k_2) = (k_3) = 4$ and $\tau_3 = 1$ (circles), 2 (triangles), and 3 (squares). (c) a 1-d lattice with $(k_1) = 4$ multiplexed with other two 1-d lattices with $(k_2) = (k_3) = 20$ for $\tau_1 = \tau_2 = 0, \tau_3 = 1$ (circles), $\tau_1 = 1, \tau_2 = \tau_3 = 0$ (triangles) and for $\tau_1 = \tau_3 = 1, \tau_2 = 0$ (squares). In all layers, $N = 100$. Data refer to averages over 100 random realizations of the initial conditions.

Finally, in order to demonstrate the generality of our observation, we present results for the case of a three-layers multiplex network, whose supra-adjacency matrix can be written as

$$A = \begin{pmatrix} A^1 & I & I \\ I & A^2 & I \\ I & I & A^3 \end{pmatrix},$$

(2)

where $A^1$, $A^2$ and $A^3$ are the adjacency matrices corresponding to layers 1, 2 and 3, respectively. In particular, we consider the case in which one of the three layers has delayed interactions (say $\tau_3 \neq 0$) whereas the other two layers are un-delayed ($\tau_1 = \tau_2 = 0$). The results are reported in Fig. 4. There, a delayed layer consisting of an ER network (a 1-d lattice) is multiplexed with other two un-delayed layers of ER networks (of 1-d lattices). Figs. 7(a) and (b) show that CS is strongly enhanced for $\tau_3 = 1, 3$, while it is suppressed for $\tau_3 = 2, 4$. Once again, therefore, the system’s behavior is crucially different for odd and even values of the delay time. A very interesting result is shown in Fig. 7(c): if the delay affects a denser layer, the sparser (and un-delayed) layer do not show enhancements in CS. However, the delayed sparser layer exhibits better CS, irrespective of the delay values in the denser layers.

Furthermore, following two phenomena remain unaffected with an increase in the size of the network; (a) CS of an un-delayed layer gets affected with a change in the delay value of the other layer and the degree-degree correlation ($r$) of the mirror nodes. (b) In contrast to the un-delayed evolution, CS of a delayed layer is neither governed by the degree-degree correlation of the mirror nodes nor is controlled by the parameters (such as delay and network architecture) of the other layer. CS of the delayed layer is predominately governed by its own delay value. For example, Fig. 5(a) displays an enhancement in CS of a delayed sparse 1-d lattice of $N_1 = N_2 = 500$ upon multiplexing. This phenomenon is irrespective of $r$ of the mirror nodes as also observed for smaller size networks ($N_1 = N_2 = 100$) (Fig. 5). Furthermore, the fraction of nodes in the largest cluster ($f_L$) exhibits a similar behavior as revealed by $f_{clus}$. In real-world systems, not all the nodes may be similar, and there may be parameter mismatch in the local dynamical evolution. To present generality of the phenomena, we present results for the coupled non-identical nodes. We consider the coupled non-identical logistic maps with the parameter $\mu$ uniformly distributed between the range $0.38 \lesssim \mu \lesssim 4$. We find that non-identical nodes exhibits the similar behavior as observed for the identical nodes. Fig. 5 depicts robustness of the enhancement in CS of a delayed layer due to the multiplexing against changes in the connection density of the other layer of a biplex network.
Further, we study CS of a real-world social network constructed using the celebrity data. The network is created considering actors as the nodes and interaction between a pair of the nodes is defined based on if they have co-acted in a movie. The multiplex network is constructed by dividing the nodes into different layers depending on their genre. A pair of actors i and j are connected in a layer if they have worked in a movie of the corresponding genre. Note that not every actor has worked in movies of every genre. Thus the number of nodes may differ across the genres. However, since we have considered multiplex network with each node in a layer having its counterpart in the other layers, we consider only those nodes which are common in the movie actor network consisting of two layers. Fig. 8 demonstrates change in $f^1_{\text{clus}}$ of an un-delayed layer with respect to the change in the delay value in the other delayed ($\tau_2$) for a layer of a multiplex network consisting of layers denoting comedy and crime genre for the 2008-2012 data set. Whereas, the delayed layer exhibits robustness in the enhancement in CS against changes in the delay value of the other layer (Fig. 8).

In summary, we have investigated the interplay of multiplexing and communication delays on cluster synchronization in multiplex networks, and elucidated several nontrivial (and anti-intuitive) results. Namely, we have first shown that delayed interactions in a layer enhance or suppress CS of the other layers, depending on the parity of the delay value. Second, and in contrast to the un-delayed case, we gave evidence that CS of a regular network gets always enhanced upon multiplexing it with a delayed network, which may display arbitrary architecture and delay values. As a third result, we have shown that enhancement in CS is maximum if the mirror (replica) nodes have the same degree in both layers, while a mismatch in the degree-degree correlation of the mirror nodes leads to the suppression in CS of a un-delayed sparse network. All these results suggest that two cases have to be distinguished: a un-delayed layer coupled with delayed layers, and a delayed layer multiplexing with other (delayed or un-delayed) layers. In the former case, the enhancement of CS is susceptible to the delay value and the connection density of the other layers, in the latter CS is instead always enhanced. Therefore, the overall conclusion is that CS of a layer in a multiplex network can be properly tuned by the delay value in the other layers. Applications of this result are actually relevant, as earlier works have shown that various properties, particularly, the anti-correlation of mirror nodes make a network vulnerable to the random failure. Our results suggest, instead, that the synchronization in one layer
can be made robust against changes in the delay value (as well as network architecture) of the other layer. These results are therefore of value for multiplex systems with more than one type of vital resources, as for instance water supply and power plants of a city,21,22 where controlling synchronizability is a task of fundamental importance.

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The Editor

The first Referee has already recommended the manuscript for publication in the present form. The second Referee recommends the publication after the manuscript incorporate few changes. In the following we consider all the suggestions made by the Referee as well as make the amendments in the manuscript in the lines of these suggestions. We submit the revised version of the manuscript.

Sincerely yours,
Authors

Reply to the second Referee:

We thank the Referee for carefully reviewing our manuscript. Her (his) comments are indeed very helping in improving the manuscript. In the following, we address all the points one by one.

1 Referee’s Comment 1: Figure 4: Caption for part (a) You have captioned \( f^1_{clus} \) as corresponding to circles in the figure when in fact the symbols are squares. Please fix.
   
   Our Reply: Modifications are done in the revised manuscript.

2 Referee’s Comment 2: Page 5: In your definition of Pearson correlation coefficient you claim \( \sum_j e_j = q_k \). I believe it should be \( \sum_j e_{jk} = q_k \).
   
   Our Reply: We thank the Referee for pointing it out and make the corrections.

3 Referee’s Comment 3: Figure 6: Your y-axis labeling of \( f^1_{clus}, f^2_{clus} \) is only true for Figure 6(b) and 6(c). In Figure 6(a) the label should be simply \( f_{clus} \) since there is no multiplexing in this experiment.
   
   Our Reply: Concurring with the Referee we modify caption of Fig. 6(a) in the revised manuscript.

4 Referee’s Comment 4: Page 7: You analysis of Figure 7(c) is still incorrect despite having been brought to your attention in previous decision letters.
   From the text: ”if the delay affects a denser layer, the other two sparser (and un-delayed) layers do not show enhancements in CS.” From the caption: ”(c) a 1-d lattice with \( \langle k_1 \rangle = 4 \) multiplexed with other two 1-d lattices with \( \langle k_2 \rangle = \langle k_3 \rangle = 20...”
   
   These are inconsistent, the caption claims there are two dense layers and one sparser layer. I do not know whether the body of the text or the caption is correct. You must address this.
   
   Our Reply: We apologize for the mistake in the text. Fig. 7(c) is plotted for a multiplex network consists of two denser and one sparser layer. Motive behind presenting this figure is to demonstrate that a delayed sparser layer exhibits a better CS irrespective of the delay value of the denser layer. In the revised manuscript we make changes in text describing Fig. 7(c) and hope that now there is no ambiguity.
List of major changes in the manuscript:

1. Revision of captions of Fig. 4(a).

2. Modification of the labeling of Fig. 6(a). and Fig. 4(a).

3. Modification in the Fig. 8

4. Correction of the typo in the definition of the Pearson’s correlation coefficient.

5. Modification in the first paragraph of page 5 describing Fig. 7(c).

6. We add one new reference [Ref. 21] in the revised manuscript.