AN AMPLENESS CRITERION FOR LINE BUNDLES ON ABELIAN VARIETIES

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Abstract. Let $A$ be an abelian variety defined over an algebraically closed field. We first show that a line bundle $L$ on $A$ is ample if its restriction to every curve in $A$ is ample. Using it we give a sufficient condition for a vector bundle on $A$ to be ample.

1. Introduction

Let $X$ be a projective variety over an algebraically closed field, and let $L$ be a line bundle on $X$. The Nakai-Moishezon criterion says that $L$ is ample if and only if $L^{\dim(Y)} \cdot Y > 0$ for every positive-dimensional subvariety $Y$ of $X$. In general, it is not sufficient to check this condition only for curves in $X$. Mumford gave an example of a non-ample line bundle on a surface which intersects every curve positively; see [Har, Example 10.6] or [La1, Example 1.5.2].

However, in some cases it turns out that to check ampleness of $L$ it suffices to verify $L \cdot C > 0$ for all curves $C \subset X$. In [HMP], this statement is proved for toric varieties; in fact, in [HMP] it is proved that a vector bundle $E$ on a toric variety $X$ is ample if the restriction of $E$ to the invariant rational curves on $X$ is ample. We recall that there are only finitely many invariant rational curves on $X$. For a flag variety $X$ over a curve defined over $\overline{F}_p$, a line bundle on $X$ is ample if its restriction to each curve is ample [BMP].

In this short note, we prove that a line bundle $L$ on an abelian variety $A$ is ample if the restriction of $L$ to every curve on $A$ is ample. Using it we give a similar sufficient condition for ampleness of vector bundles on $A$.

2. The ampleness criterion

Let $k$ be an algebraically closed field. Our first result is the following.

Theorem 2.1. Let $A$ be an abelian variety defined over $k$. Let $L$ be a line bundle over $A$ with the following property: for every pair $(C, f)$, where $C$ is an irreducible smooth projective curve defined over $k$, and $f : C \rightarrow A$ is a nonconstant morphism, the inequality

$$\text{degree}(f^*L) > 0$$

holds. Then $L$ is ample.
Proof. Take a line bundle $L$ on $A$. Let 
$$
\alpha : A \times A \to A, \ (x, y) \mapsto x + y
$$
be the addition map. Consider the family of line bundles 
$$(\alpha^* L) \otimes p_1^* L^* \to A \times A \xrightarrow{p_2} A,$$
where $p_1$ and $p_2$ are the projections of $A \times A$ to the first and second factor respectively. Let 
$$
\varphi_L : A \to A^\vee = \text{Pic}^0(A)
$$
be the classifying morphism for this family. This $\varphi_L$ is a group homomorphism. Let 
$$K(L) \subset A$$
be the (unique) maximal connected subgroup of the reduced kernel $\ker(\varphi_L)_{\text{red}}$.

If $L \in A^\vee = \text{Pic}^0(A)$, then $\varphi_L$ is the constant morphism $x \mapsto 0$ [MFK p. 120] (see after Definition 6.2), [GN p. 11, Lemma 2.1.6]. Using this it follows that if $L'$ is numerically equivalent to $L$, then $\varphi_L = \varphi_L'$, which in turn implies that 
$$K(L) = K(L'). \quad (2.2)$$

It is known that $L$ is ample if the following two conditions hold:

1. the line bundle $L$ is effective, and  
2. $K(L) = 0$.

(See [Mum1 p. 288, § 1], [GN p. 13, Theorem 2.2.2].)

We will use the following lemma:

**Lemma 2.2.** The line bundle $L$ in Theorem 2.1 is ample if $K(L) = 0$.

**Proof of Lemma 2.2.** Since $L$ is nef, it follows that $L$ is numerically equivalent to a $\mathbb{Q}$–effective $\mathbb{Q}$–Cartier divisor on $A$ (see [Mo p. 811, Proposition 3.1]). So $L^n$ is numerically equivalent to an effective divisor $D$ on $A$, for some positive integer $n$. Note that 
$$\varphi_{L^n} = n \cdot \varphi_L. \quad (2.3)$$

Assume that $K(L) = 0$. Consequently, from (2.3) and (2.2) it follows that $K(D) = 0$. Since $D$ is also effective, using the above mentioned criterion for ampleness it follows that $D$ is ample. This implies that $L$ is ample. \qed

Continuing with the proof of Theorem 2.1 in view of Lemma 2.2 it suffices to show that $\dim K(L) = 0$. Assume that 
$$\dim K(L) \geq 1.$$ 

The restriction of $L$ to the sub-abelian variety $K(L) \subset A$ will be denoted by $L_0$. For any closed point $x \in A$, define 
$$\alpha_x : A \to A, \ y \mapsto x + y. \quad (2.4)$$

For any closed point $x \in K(L)$, let $\tilde{\alpha}_x : K(L) \to K(L)$ be the restriction of $\alpha_x$ in (2.4) to $K(L)$. 

For any \( x \in K(L) \), we have \( \alpha_x^*L = L \); hence we have
\[
\hat{\alpha}_x^*L_0 = (\alpha_x^*L)|_{K(L)} = L|_{K(L)} = L_0.
\]
This implies that the line bundle \( L_0 \) on \( K(L) \) is numerically trivial \([\text{Mum2}, \text{p. 74, Definition}] \) and \([\text{Mum2}, \text{p. 86}] \). Consequently, for any pair \((C, f)\), where \( C \) is an irreducible smooth projective curve defined over \( k \), and \( f : C \to K(L) \subset A \) is a nonconstant morphism, we have
\[
\deg(f^*L) = 0.
\]
Since this contradicts \((2.1)\), we conclude that \( \dim K(L) = 0 \). Hence \( L \) is ample by Lemma 2.2.

\[\square\]

### 2.1. Ample vector bundles on \( A \)

Let \( E \) be a vector bundle of rank \( r \) on \( A \) satisfying the following two conditions:

1. The line bundle \( \det E := \bigwedge^r E \) has the property that for every pair \((C, f)\), where \( C \) is an irreducible smooth projective curve defined over \( k \), and \( f : C \to A \) is a nonconstant morphism, the inequality \( \deg(\det E) > 0 \) holds.
2. For every closed point \( x \in A \), there is a line bundle \( L(x) \) on \( A \) such that
\[
\alpha_x^*E = E \otimes L(x),
\]
where \( \alpha_x \) is the morphism in \((2.4)\).

**Proposition 2.3.** The above vector bundle \( E \) is ample.

**Proof.** Since \( E \) satisfies the condition in \((2.5)\), a theorem of Mukai says that there is an isogeny
\[
f : B \to A
\]
such that the vector bundle \( f^*E \) admits a filtration of subbundles
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E
\]
for which \( \operatorname{rank}(E_i) = i \), and the line bundle \( E_i/E_{i-1} \) is numerically equivalent to \( E_1 \) for every \( 1 \leq i \leq r \) \([\text{Muk}, \text{p. 260, Theorem 5.8}] \) (see also \([\text{MN}, \text{p. 2}] \)).

Consequently, the line bundle \( \det E = \bigwedge^r E = \bigotimes_{i=1}^r (E_i/E_{i-1}) \) is numerically equivalent to the line bundle \( E_1^{\geq r} \). From Theorem 2.1 we know that \( \det E \) is ample. This implies that \( E_1^{\geq r} \) is ample. Hence \( E_1 \) is ample. So \( E_i/E_{i-1} \) is ample for every \( 1 \leq i \leq r \). Consequently, from \((2.6)\) it follows that \( E \) is ample \([\text{La2}, \text{p. 13, Proposition 6.1.13}] \). \[\square\]

**Remark 2.4.** Let \( X \) be a projective variety over an algebraically closed field \( k \). A divisor \( D \) on \( X \) is said to be **big** if there is an ample divisor \( H \) on \( X \) such that \( mD - H \) is linearly equivalent to an effective divisor for some positive integer \( m \). A \( \mathbb{Q} \)-divisor \( D \) is **pseudo-effective** if \( D + B \) is big for any big \( \mathbb{Q} \)-divisor \( B \). Similarly one can define the notion of pseudo-effective \( \mathbb{R} \)-divisors. In the Néron-Severi space \( N^1(X)_{\mathbb{R}} \), the pseudo-effective \( \mathbb{R} \)-divisors form a cone which is the closure of the cone of effective \( \mathbb{R} \)-divisors.

If \( \dim(X) = 2 \) and the pseudo-effective cone of \( X \) is equal to the effective cone, then a line bundle \( L \) on \( X \) is ample if and only if \( L \cdot C > 0 \) for every curve \( C \) on \( X \). But, in
general, the pseudo-effective cone of a projective variety is not equal to the effective cone; see the example of Mumford described in [Har, Example 10.6] or [La1, Example 1.5.2].

If \( k \) is an algebraic closure of a finite field, Moriwaki showed that every pseudo-effective divisor (over \( \mathbb{Q} \) or \( \mathbb{R} \)) is effective when \( X \) is a projective bundle over a curve or when \( X \) is an abelian variety (see [Mo, p. 802, Theorem 0.4] and [Mo, p. 802, Proposition 0.5]). As our next example shows, this statement is false for abelian varieties over \( \mathbb{C} \).

**Example 2.5.** Let \( X \) be an elliptic curve defined over \( \mathbb{C} \). Let \( x \in X \) be a point of infinite order. Let \( D := 0 - x \), where 0 is the identity element of \( X \). Then \( D \) is a divisor of degree 0 and it is pseudo-effective. However, no multiple of \( D \) is effective, since \( x \) has is infinite order.

We end with the following question.

**Question 2.6.** Let \( A \) be an abelian variety over an algebraically closed field. Let \( E \) be a vector bundle on \( A \) such that the restriction \( E|_C \) is ample for every curve \( C \) on \( A \). Then is \( E \) ample?

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