Optimal stopping of marked point processes and reflected backward stochastic differential equations

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Abstract

We define a class of reflected backward stochastic differential equation (RBSDE) driven by a marked point process (MPP) and a Brownian motion, where the solution is constrained to stay above a given càdlàg process. The MPP is only required to be non-explosive and to have totally inaccessible jumps. Under suitable assumptions on the coefficients we obtain existence and uniqueness of the solution, using the Snell envelope theory. We use the equation to represent the value function of an optimal stopping problem, and we characterize the optimal strategy.

Keywords: reflected backward stochastic differential equations, optimal stopping, marked point processes.

1 Introduction

Nonlinear backward stochastic differential equations (BSDE) driven by a Brownian motion were first introduced by Pardoux and Peng in the seminal paper [30]. Later, BSDE have found applications in several fields of mathematics, such as stochastic control, mathematical finance, nonlinear PDEs (see for instance [15, 31, 11]). As the driving noise, the Brownian motion has been replaced by more general classes of martingales; the first example is perhaps [14], see [37] for a very general situation.

In particular, occurrence of marked point processes in the equation has been considered since long. In [36, 3], related to optimal control and PDEs respectively, an independent Poisson random measure is added to the driving Wiener noise. Motivated by several applications to stochastic optimal control and financial modelling, more general marked point processes were considered in the

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BSDE. Examples can be found in [4, 9] for $L^2$ solutions, [10] for the $L^1$ case and [8] for the $L^p$ case.

In connection with optimal stopping and obstacle problems, in [16] a reflected BSDE is introduced, where the solution is forced to stay above a certain continuous barrier process. This class of BSDE finds applications in various problems in finance and stochastic games theory. A number of generalizations has followed, both with variations on the nature of the barrier process and the type of noise. In the Brownian case, in [21] the author solves the problem when the obstacle is just càdlàg in $L^2$, the authors allow the obstacle to be only $L^2$. On the other hand, in [22] the authors solve the problem when a Poisson noise is added, and the barrier is càdlàg with inaccessible jump times. This is later generalized in [23] where the barrier can have partially accessible jumps too. Other specific results are [18] where a BSDE with two generators is solved in a Wiener framework and [34] in a Lévy framework; the papers [33] and [17] where the noise is a Teugels Martingale associated to a one-dimensional Lévy process. The paper [12] that considers a marked point process with compensator admitting a bounded density with respect to the Lebesgue measure.

Finally, very general barriers beyond the càdlàg case were recently considered in [20, 19].

It is the aim of the present work to address the case when the obstacle to be a càdlàg process and, in addition to the Wiener process, a very general marked point process occurs in the equation. The only assumptions we make is that it is non-explosive and has totally inaccessible jumps. This is equivalent to the requirement that the compensator of the counting process of the jumps has continuous trajectories. However, we do not require absolute continuity with respect to the Lebesgue measure. To our knowledge, only in [1, 2], in [29], and in [7, 6] even more general cases have been addressed, but without reflection.

The equation has the form

$$Y_t = \xi + \int_t^T f_s(Y_s, U_s) dA_s + \int_t^T g_s(Y_s, Z_s) ds$$

$$- \int_t^T \int_E U_s(e) q(\,dt\,de) - \int_t^T Z_s dW_s + K_T - K_t$$

$$Y_t \geq h_t.$$

(1)

Here $W$ is a Brownian motion and $q$, independent from $W$, is a compensated integer random measure corresponding to some marked point process $(T_n, \xi_n)_{n \geq 1}$: see [2, 25, 28] as general references on the subject. The data are the final condition $\xi$ and the generators $f$ and $g$. $A$ is a continuous stochastic increasing process related to the point process. The $Y$ part of the solution is constrained to stay above a given barrier process $h$, and the $K$ term is there to assure this condition holds. This equation is then used to solve a non-Markovian optimal stopping problem, where the running gain, stopping reward and final reward are the data used in the BSDE. Under additional assumptions on the barrier process, an optimal stopping time is characterized.
This work generalizes the results previously obtained by allowing a more
general structure in the jump component. This introduces some technical
difficulties and some assumptions. For instance, we work in “weighted $L^2$ spaces”,
with a weight of the form $e^{\beta A_t}$, and the data must satisfy this integrability
conditions. Direct use of standard tools, like the Gronwall lemma, becomes difficult
in our case, so we have to resort to direct estimates. Since there is no general
comparison theorem for BSDE with so general marked point process, we do
not use a penalization method, but rather a combination of the Snell envelope
theory and contraction theorem.

The paper is organized as follows: in section 2 we first recall some results
on marked point processes and describe the setting and the problem we want to
solve. In section 3 we prove the existence and uniqueness of a Reflected BSDE
driven by a marked point process and a Wiener process when the generators
do not depend on the solution of the BSDE. This is solved in some $L^2$ space,
appropriate for the Brownian motion. When the (given) generator and the
other data are adapted only to the filtration generated by the point process,
the solution can be found in a larger space. We then link these equations to an
optimal stopping problem. Lastly in section 4 we solve the BSDE in the general
case with the help of a contraction argument. Here we use the $L^2$ framework for
both the case with only marked point process or with both driving processes.

2 Preliminaries, assumptions, formulation of the
problems

2.1 Some reminders on point processes

We start by recalling some notions about marked point processes and then
defining the objectives of this paper. For a comprehensive treatment of marked
point processes, we refer the reader to [25], [3] or [28]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a
complete probability space and let $E$ be a Borel space, i.e. a topological space
homeomorphic to a Borel subset of a compact metric space (sometimes called
Lusin space; we recall that every separable complete metric space is Borel). We
call $E$ the mark space and we denote by $\mathcal{E}$ its Borel $\sigma$-algebra.

Definition 2.1. A marked point process (MPP) is a sequence of random variables
$(T_n, \xi_n)_{n \geq 0}$ with values in $[0, +\infty] \times E$ such that $\mathbb{P}$-a.s.

- $T_0 = 0$.
- $T_n \leq T_{n+1} \forall n \geq 0$.
- $T_n < \infty \Rightarrow T_n < T_{n+1} \forall n \geq 0$.

We will always assume the marked point process in the paper to be non-
explosive, that is $T_n \to +\infty$ $\mathbb{P}$-a.s. To each marked point process we associate
a random discrete measure $p$ on $((0, +\infty) \times E, \mathcal{B}((0, +\infty) \otimes \mathcal{E}))$:

$$p(\omega, D) = \sum_{n \geq 1} 1_{(T_n(\omega), \xi_n(\omega)) \in D}.$$
We refer to $p$ also as marked point process. For each $C \in \mathcal{E}$, define the counting process $N_t(C) = p((0, t] \times C)$ that counts how many jumps have occurred to $C$ up to time $t$. Denote $N_t = N_t(E)$. They are right continuous increasing process starting from zero. Each point process generates a filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ as follows: define for $t \geq 0$

$$\mathcal{G}_t = \sigma(N_s(C) : s \in [0, t], C \in \mathcal{E})$$

and set $\mathcal{G}_t = \sigma(\mathcal{G}_t^0, \mathcal{N})$, where $\mathcal{N}$ is the family of $\mathbb{P}$-null sets of $\mathcal{F}$. $\mathbb{G}$ is a right-continuous filtration that satisfies the usual hypotheses. Denote by $\mathcal{P}^\mathbb{G}$ the $\sigma$-algebra of $\mathcal{G}$-predictable processes.

For each marked point process there exists a unique predictable random measure $\nu$, called compensator, such that for all non-negative $\mathcal{P}^\mathbb{G} \otimes \mathcal{E}$-measurable process $C$ it holds that

$$E \left[ \int_0^{+\infty} \int_E C_t(e)p(dtde) \right] = E \left[ \int_0^{+\infty} \int_E C_t(e)\nu(dtde) \right].$$

Similarly, there exists a unique right continuous increasing process with $A_0 = 0$, the dual predictable projection of $N$, such that for all non-negative predictable processes $D$

$$E \left[ \int_0^{+\infty} D_t dN_t \right] = E \left[ \int_0^{+\infty} D_t dA_t \right].$$

It is known that there exists a function $\phi$ on $\Omega \times [0, +\infty) \times \mathcal{E}$ such that we have the disintegration $\nu(\omega, dtde) = \phi_t(\omega, de)dA_t(\omega)$. Moreover the following properties hold:

- for every $\omega \in \Omega$, $t \in [0, +\infty)$, $C \mapsto \phi_t(\omega, C)$ is a probability on $(\mathcal{E}, \mathcal{E})$.
- for every $C \in \mathcal{E}$, the process $\phi_t(C)$ is predictable.

We will assume in the following that all marked point processes in this paper have a compensator of this form.

From now on, fix a terminal time $T > 0$. Next we need to define integrals with respect to point processes.

**Definition 2.2.** Let $C$ be a $\mathcal{P}^\mathbb{G} \otimes \mathcal{E}$-measurable process such that

$$E \left[ \int_0^T \int_E |C_t(e)|\phi_t(de)dA_t \right] < \infty.$$ 

Then we can define the integral

$$\int_0^T \int_E C_t(e)q(dtde) = \int_0^T \int_E C_t(e)p(dtde) - \int_0^T \int_E C_t(e)\phi_t(de)dA_t$$

as difference of ordinary integrals with respect to $p$ and $\phi dA$.

**Remark 2.1.** In the paper we adopt the convention that $\int_a^b$ denotes an integral on $(a, b]$ if $b < \infty$, or on $(a, b)$ if $b = \infty$. 


Remark 2.2. Since \( p \) is a discrete random measure, the integral with respect to \( p \) is a sum:

\[
\int_0^t \int_E C_s(e)p(dsde) = \sum_{T_n \leq t} C_{T_n}(\xi_n)
\]

Given a process \( C \) as above, the integral defines a process \( \int_0^t \int_E C_s(e)q(dsde) \) that, by the definition of compensator, is a martingale.

2.2 Probabilistic setting

In this paper we will assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space and \( p(dtde) \) a marked point process on a Borel space \((E, \mathcal{E})\) as before, whose compensator is \( \phi_t(dx)dA_t \). In addition we assume we are given an independent Wiener process \( W \) in \( \mathbb{R}^d \). Let \( \mathcal{G} = (\mathcal{G}_t)_{t \geq 0} \) (resp. \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \)) be the completed filtration generated by \( p \) (resp. \( p \) and \( W \)), which satisfies the usual conditions. Let \( T_t \) be the set of \( \mathcal{F} \)-stopping times greater than \( t \). Denote by \( \mathcal{P} \) (resp. \( \text{Prog} \)) be the predictable (resp. progressive) \( \sigma \)-algebra relative to \( \mathcal{F} \). For \( \beta > 0 \), we introduce the following spaces of equivalence classes we will be using in the following

- \( L^{r,\beta}(A) \) (resp. \( L^{r,\beta}(A,\mathcal{G}) \)) is the space of all \( \mathcal{F} \)-progressive (resp. \( \mathcal{G} \)-progressive) processes \( X \) such that
  \[
  \|X\|_{L^{r,\beta}(A)} = \mathbb{E}\left[\int_0^T e^{\beta A_s}|X_s|^r dA_s\right] < \infty.
  \]

- \( L^{r,\beta}(p) \) (resp. \( L^{r,\beta}(p,\mathcal{G}) \)) is the space of all \( \mathcal{F} \)-predictable (resp. \( \mathcal{G} \)-predictable) processes \( U \) such that
  \[
  \|U\|_{L^{r,\beta}(p)} = \mathbb{E}\left[\int_0^T \int_E e^{\beta A_s}|U_s(e)|^r \phi_s(de)dA_s\right] < \infty.
  \]

- \( L^{r,\beta}(W,\mathbb{R}^d) \) (resp. \( L^{r,\beta}(W,\mathbb{R}^d,\mathcal{G}) \)) is the space of \( \mathcal{F} \)-progressive (resp. \( \mathcal{G} \)-progressive) processes \( Z \) in \( \mathbb{R}^d \) such that
  \[
  \|Z\|_{L^{r,\beta}(W)} = \mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^r ds\right] < \infty
  \]

- \( \mathcal{I}^2 \) (resp. \( \mathcal{I}^2(\mathcal{G}) \)) is the space of all càdlàg increasing \( \mathcal{F} \)-predictable (resp. \( \mathcal{G} \)-predictable) processes \( K \) such that \( \mathbb{E}[K^2_T] < \infty \).

One last tool we will need in the following is the martingale representation theorem: if \( M \) is a càdlàg square integrable \( \mathcal{F} \)-martingale on \([0,T]\), then there
exist two processes $U$ and $Z$ such that

$$
\mathbb{E} \left[ \int_0^T \int_E |U_t(e)|\phi_t(\omega,dy)dA_t \right] + \mathbb{E} \left[ \int_0^T |Z_t|^2 dt \right] < \infty
$$

$$
M_t = M_0 + \int_0^t \int_E U_s(e)\phi_s(\omega,dy)dA_s + \int_0^t Z_s dW_s.
$$

2.3 Assumptions and formulation of the problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$, $(E, \mathcal{E})$, $p(dt dx)$, $W$, $\mathcal{F}$ be as before. We will consider the following reflected BSDE.

$$
\begin{cases}
Y_t = \xi + \int_t^T f(s, Y_s, U_s)dA_s + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T \int_E U_s(y)q(dsdy) \\
- \int_t^T Z_s(y)dB_s + K_T - K_t, \quad \forall t \in [0, T] \text{ a.s.} \\
Y_t \geq h_t, \quad \forall t \in [0, T] \text{ a.s.} \\
\int_0^T (Y_s - h_s)dK_s^c = 0 \text{ and } \Delta K_t \leq (h_t - Y_t)^+1_{\{Y_t - h_t \leq 0\}} \forall t \in [0, T] \text{ a.s.,}
\end{cases}
$$

(2)

A solution is a quadruple $(Y, U, Z, K)$ that lies in $(L^{2,\beta}(A) \cap L^{2,\beta}(W)) \times L^{2,\beta}(p) \times L^{2,\beta}(W) \times \mathcal{L}^2$, with $Y$ càdlàg, that satisfies (2). The condition on the last line in (2) is called the Skorohod condition, or the minimal push condition. It can be expressed in an alternative way: see Remark 2.4 below.

Let us now state the general assumptions that will be used throughout the paper. Additional specific assumptions will be presented in section 4. The first one is an assumption on the compensator $A$ of the counting process $N$ relative to $p$.

**Assumption (A):** The process $A$ is continuous.

**Assumption (B):**

i) The final condition $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_T$-measurable and

$$
\mathbb{E} \left[ e^{\beta_A \xi^2} \right] < \infty.
$$

ii) For every $\omega \in \Omega$, $t \in [0, T]$, $r \in \mathbb{R}$ a mapping

$$
f(\omega, t, r, \cdot) : L^2(E, \mathcal{E}, \phi_t(\omega,dy)) \to \mathbb{R}
$$

is given and satisfies the following:

a) for every $U \in L^{2,\beta}(p)$ the mapping

$$
(\omega, t, r) \mapsto f(\omega, t, r, U_t(\omega, \cdot))
$$

is $\text{Prog} \otimes \mathcal{B}(\mathbb{R})$-measurable, where $\text{Prog}$ denotes the progressive $\sigma$-algebra.
b) There exist $L_f \geq 0$, $L_U \geq 0$ such that for every $\omega \in \Omega$, $t \in [0,T]$, $y, y' \in \mathbb{R}$, $u, u' \in L^2(E, \mathcal{E}, \phi_t(\omega, dy))$ we have

$$|f(\omega, t, y, u(\cdot)) - f(\omega, t, y', u'(\cdot))| \leq L_f |y - y'| + L_U \left( \int_E |u(e) - u'(e)|^2 \phi_t(\omega, de) \right)^{1/2}$$

c) we have

$$E \left[ \int_0^T e^{\beta A_s} |f(s, 0, 0)|^2 dA_s \right] < \infty.$$

iii) The mapping $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is given

a) $g$ is $\text{Proc} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$ measurable.

b) There exist $L_g \geq 0$, $L_Z \geq 0$ such that for every $\omega \in \Omega$, $t \in [0,T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$

$$|g(\omega, t, y, z) - g(\omega, t, y', z')| \leq L_g |y - y'| + L_Z |z - z'|$$

c) we have

$$E \left[ \int_0^T e^{\beta A_s} |g(s, 0, 0)|^2 ds \right] < \infty.$$

iv) $h$ is a càdlàg $\mathcal{F}$-adapted process such that $h_T \leq \xi$. There exists a $\delta > 0$ such that

$$E[ \sup_{t \in [0,T]} e^{(\beta + \delta) A_t} h_t^2 ]$$

Remark 2.3. We recall that Assumption [A] is equivalent to the fact that the jumps of the point process are totally inaccessible (relative to $\mathcal{F}$): see [24] Corollary 5.28. We will often use the following consequence: since $K$ is required to be predictable, its jumps (that are all non-negative) are disjoint from the jumps of $p$; so at any jump time of $K$ we also have a jump of $Y$ with the same size, but of opposite sign, in symbols we have a.s.

$$\Delta K_t \mathbb{1}_{\{\Delta K_t > 0\}} = (-\Delta Y_t)^+ \mathbb{1}_{\{\Delta K_t > 0\}}, \quad t > 0. \quad (3)$$

Remark 2.4. The Skorohod condition on the last line in (2) tells us that the process $K$ grows only when the solution is about to touch the barrier. We claim that it is in fact equivalent to

$$\int_0^T (Y_{s-} - h_{s-}) dK_s = 0, \quad a.s. \quad (4)$$

To check the equivalence, note first that

$$\int_0^T (Y_{s-} - h_{s-}) dK_s = \int_0^T (Y_s - h_s) dK_s^c + \sum_{0 < s \leq T} (Y_{s-} - h_{s-}) \Delta K_s, \quad a.s.$$
If the Skorohod condition in (2) holds then both terms in the right-hand side are zero, since jumps of $K$ can only happen when $Y_t^- = h_t^-$. Conversely, assume that (3) holds. Then clearly $\int_0^T (Y_t^- - h_t^-)dK_t^c = 0$ and so $\int_0^T (Y_t^- - h_t^-)dK_t^c = 0$. Also, $\sum_{0<s\leq T}(Y_t^- - h_t^-)\Delta K_s = 0$, so $\{t: \Delta K_t > 0\} \subset \{t: Y_t^- = h_t^-\}$ and, recalling (3), we have a.s.

$$\Delta K_t = \Delta K_t \mathbb{1}_{\{\Delta K_t > 0\}} = (-\Delta Y_t)^+ \mathbb{1}_{\{\Delta K_t > 0\}} \leq (-\Delta Y_t)^+ \mathbb{1}_{\{Y_t^- = h_t^-\}} = (Y_t^- - Y_t)^+ \mathbb{1}_{\{Y_t^- = h_t^-\}}.$$

Remark 2.5. In the simpler case when there is no Brownian component the reflected BSDE (2) becomes

$$\begin{cases}
Y_t = \xi + \int_t^T f(s,Y_s,U_s)\,dA_s - \int_t^T \int F_s(y)q(dy)\,ds + K_T - K_t & \forall t \in [0,T] \text{ a.s.} \\
Y \text{ càdlàg and } Y \in L^{2,\beta}(A,\mathbb{G}), & U \in L^{2,\beta}(p,\mathbb{G}), & K \in \mathcal{I}^2(\mathbb{G}) \\
Y_t \geq h_t & \forall t \in [0,T] \text{ a.s.} \\
\int_0^T (Y_s - h_s)\,dK_t^c = 0 & \text{and } \Delta K_t \leq (h_t^- - Y_t)^+ \mathbb{1}_{\{Y_t^- = h_t^-\}} \forall t \in [0,T] \text{ a.s.}
\end{cases}$$

(5)

Here we only assume we are given the space $(\Omega,\mathcal{F},\mathbb{P})$ and the marked point process $p$. The assumptions we need are the same as in [A] and [B] provided we set $g = 0$ and $\mathbb{G} = \mathbb{F}$.

3 Reflected BSDE with given generators and optimal stopping problem

In this section we first study the reflected BSDE in the case when the generators $g$ and $f$ do not depend on $(Y,Z,U)$ but are a given processes that satisfy

Assumption (B'): $f$ and $g$ are $\mathbb{F}$-progressive processes such that

$$\mathbb{E} \left[ \int_0^T e^{\beta A_t}|f_s|^2\,dA_s + \int_0^T e^{\beta A_t}|g_s|^2\,ds \right] < \infty.$$  

(6)

Equation (2) reduces to

$$\begin{cases}
Y_t = \xi + \int_t^T f_s\,dA_s + \int_t^T g_s\,ds - \int_t^T \int F_s(y)q(dy)\,ds + Z_s\,dW_s + K_T - K_t \\
Y \in L^{2,\beta}(A) \cap L^{2,\beta}(W), & U \in L^{2,\beta}(p), & Z \in L^{2,\beta}(W), & K \in \mathcal{I}^2 \\
Y_t \geq h_t & \forall t \in [0,T] \text{ a.s.} \\
\int_0^T (Y_s - h_s)\,dK_t^c = 0 & \text{and } \Delta K_t \leq (h_t^- - Y_t)^+ \mathbb{1}_{\{Y_t^- = h_t^-\}} \forall t \in [0,T] \text{ a.s.}
\end{cases}$$

(7)

In this case, the solution $Y$ to the equation is also the value function of an optimal stopping problem, as we will see later. First we define the càdlàg process $\eta_t$ as

$$\eta_t = \int_0^{t \wedge T} f_s\,dA_s + \int_0^{t \wedge T} g_s\,ds + h_t \mathbb{1}_{\{t < T\}} + \xi \mathbb{1}_{\{t \geq T\}}$$

(8)
Remark 3.1. In the following we will often use this kind of inequalities:

\[
\left( \int_0^t f_s dA_s \right)^2 = \left( \int_0^t e^{-\beta A_s/2} e^{\beta A_s/2} |f_s| dA_s \right)^2 \leq \int_0^t e^{-\beta A_s} dA_s \int_0^t e^{\beta A_s} f_s^2 dA_s
\]

\[
= \frac{1 - e^{\beta A_t}}{\beta} \int_0^t e^{\beta A_s} f_s^2 dA_s \leq \frac{1}{\beta} \int_0^t e^{\beta A_s} f_s^2 dA_s \tag{9}
\]

Lemma 3.1: Under assumptions (B)-(i)(iv) and (B') \( \eta \) is of class \([D]\) and

\[
E \left[ \sup_{0 \leq t \leq T} |\eta|^2 \right] < \infty
\]

Proof. Fix a stopping time \( \tau \). Clearly

\[
|\eta|^2 \leq 4 \left( \int_0^T |f_s| dA_s \right)^2 + 4 \left( \int_0^T |g_s| ds \right)^2 + 4 |h_\tau|^2 \mathbb{1}_{\{\tau < T\}} + 4 |\xi|^2
\]

\[
\leq \frac{4}{\beta} \int_0^T e^{\beta A_s} f_s^2 dA_s + 4T \int_0^T e^{\beta A_s} |g_s|^2 ds + 4 \sup_{t \in [0, T]} e^{\beta A_t} |h_\tau|^2 + 4e^{\beta A_T} |\xi|^2,
\]

and since the right-hand side has finite expectation we obtain the class \([D]\) property. Likewise, by taking the supremum over all \( t \in [0, T] \), and expectation after that, we obtain the second property. \( \square \)

Now, using the Snell envelope theory, we show that there exists a solution to the equation above. Appendix \( A \) lists the properties that we will need in the following.

Proposition 3.1: Let assumptions \([A], (B)-(i)(iv)\) and \((B')\) hold for some \( \beta > 0 \), then there exists a unique solution to (7).

Proof. The uniqueness property is stated and proved separately in Proposition 3.2 below. Existence is proved in several steps.

Step 1. We start by defining \( Y_t \), for all \( t \geq 0 \), as the optimal value of the stopping problem:

\[
Y_t = \esssup_{\tau \in T_t} \E \left[ \int_t^{\tau \land T} f_s dA_s + \int_t^{\tau \land T} g_s ds + h_\tau \mathbb{1}_{\{\tau < T\}} + \xi \mathbb{1}_{\{\tau \geq T\}} \left| F_t \right. \right]. \tag{11}
\]

From (10) it follows that \( Y_t \) is integrable for all \( t \) and \( Y_t = \xi \) for \( t \geq T \). We have the following a priori estimate on \( Y \), that we will prove later.

Lemma 3.2: Assume \([B](i)(iv)\) and \([B']\) above on \( \xi, f, h, \xi \). Then

\[
\E \left[ \sup_{t \in [0, T]} e^{\beta A_t} Y_t^2 \right] < \infty. \tag{12}
\]
It follows that
\[ Y_t + \int_0^{t \wedge T} f_s dA_s + \int_0^{t \wedge T} g_s ds = \text{ess sup} \left[ \eta_t \mid F_t \right] \]
so \( Y_t + \int_0^{t \wedge T} f_s dA_s + \int_0^{t \wedge T} g_s ds \) is the Snell envelope of \( \eta \), that is the smallest supermartingale such that \( Y_t + \int_0^{t \wedge T} f_s dA_s + \int_0^{t \wedge T} g_s ds \geq \eta_t \). Since \( \eta \) is càdlàg, its Snell envelope \( R(\eta) \), and hence \( Y \), have a càdlàg modification. We refer to the appendix for a review of the properties of the Snell envelope that we will use. Also, from now on all supermartingales that we consider in this proof are assumed to be càdlàg. Also, since \( \eta \) will use. Also, from now on all supermartingales that we consider in this proof are assumed to be càdlàg. Also, since \( \eta \) satisfies \( \mathbb{E}[\eta_T^2] < \infty \), so that \( M \) is a square integrable martingale. Furthermore, \( K \) can be decomposed into \( K^c + K^d \), the continuous and discontinuous part, and we have that \( \Delta K_t = \Delta K_t \mathbb{1}_{\{R(\eta)_{t^-} = \eta_{t^-}\}} \) (see A.1.iii)). However it is immediate to see that \( R(\eta)_{t^-} = \eta_{t^-} \) if and only if \( Y_{t^-} = h_{t^-} \mathbb{1}_{\{t \leq T\}} + \xi \mathbb{1}_{\{t > T\}} \) and it follows that
\[ \Delta K_t = \Delta K_t \mathbb{1}_{\{Y_{t^-} = h_{t^-}\}}, \quad t \in [0, T]. \]
By the martingale representation theorem, there exists some \( U \) and \( Z \) such that
\[ \mathbb{E} \left[ \int_0^T \int_E |U_t(e)| \phi_t(dde) dA_t \right] + \mathbb{E} \left[ \int_0^T |Z_t|^2 dt \right] < \infty \]
(15)
\[ M_t = M_0 + \int_0^t \int_E U_s(e) q(dsde) + \int_0^t Z_s dW_s. \]
(16)
Choosing \( \tau = t \) in (11) we see that a.s. \( Y_t \geq h_t \) for all \( t < T \) and \( Y_T = \xi \), so \( Y_t \geq h_t \) for all \( t \leq T \) a.s. Plugging (15) in (13) we conclude that the first equality in (12) is verified.

**Step 2.** In this step we prove that the Skorohod conditions in (7) hold. From (9) it follows that \( \Delta K_t \leq (-\Delta Y_t)^+ \) and, taking into account (11), we obtain
\[ \Delta K_t \leq (-\Delta Y_t)^+ \mathbb{1}_{\{Y_{t^-} = h_{t^-}\}} = (Y_{t^-} - Y_t)^+ \mathbb{1}_{\{Y_{t^-} = h_{t^-}\}}, \]
(17)
that gives us the second condition. Consider now \( \tilde{Y}_t = Y_t + \int_0^t f_s dA_s + \int_0^t g_s ds + \Delta K^d_t = M_t - K^c_t \) and \( \tilde{\eta}_t = \eta_t + K^d_t \). We claim that \( \tilde{Y}_t \) is the Snell envelope of \( \tilde{\eta}_t \). Indeed, it is a supermartingale that dominates \( \tilde{\eta}_t \). Let \( Q_t \) be another supermartingale that dominates \( \tilde{\eta}_t \). Then \( Q_t - K^d_t \) is still a supermartingale, and dominates \( \eta_t \). Then, since \( Y_t + \int_0^t f_s dA_s + \int_0^t g_s ds = R(\eta)_t \), \( Q_t \geq \tilde{Y}_t \). Then \( \tilde{Y}_t \) is the smallest supermartingale that dominates \( \tilde{\eta}_t \), and thus its Snell
envelope. Next, \( Y_t + \int_0^t f_s dA_s + K^c_t = M_t - K^c_t \) is regular (we recall that a process \( X_t \) is regular if \( X_t - \mathbb{P}X_t \), where \( pX_t \) denotes the predictable projection, see also [A.1.ii]) all uniformly integrable càdlàg martingales are regular). Then, the stopping time defined as

\[
D_t = \inf \{ s \geq t : M_s \neq R(\tilde{\eta}) \} = \inf \{ s \geq t : K^c_s > K^c_t \}
\]

is the largest optimal stopping time, and it satisfies:

\[
\tilde{Y}_{D_t} = \tilde{\eta}_{D_t}
\]

\( \tilde{Y}_{s \wedge D_t} \) is a \( \mathbb{F} \)-martingale

See [A.1.ii]). Define then

\[
D_t = \inf \{ s \geq t : \tilde{Y}_s \leq \tilde{\eta}_s \}
\]

Since \( \tilde{Y}_{D_t} = \tilde{\eta}_{D_t} \) we have \( D_t \leq D_t^* \), and it follows that

\[
0 = \int_t^{D_t} (\tilde{Y}_s - \tilde{\eta}_s) dK^c_s = \int_t^{D_t} (Y_s - h_s) dK^c_s,
\]

which implies \( K^c_{D_t} = K^c_t \) for arbitrary \( t \), and hence \( \int_0^T (Y_s - h_s) dK^c_s = 0 \), that together with [14] gives us the Skorohod conditions.

Step 3. We conclude the proof showing that the processes are in the right spaces. We have already noticed that \( \mathbb{E}[K^c_T] < \infty \). Next we define the sequence of stopping times:

\[
S_n = \inf \left\{ t \in [0, T] : \int_0^t e^{\beta s} |Y_s|^2 dA_s + \int_0^t \int_E e^{\beta A_s} |U_s(e)|^2 \phi_s(de) dA_s + \int_0^t e^{\beta A_s} |Z_s|^2 ds > n \right\},
\]

and consider the “Ito Formula” applied to \( e^{\beta(A_s + t)} Y^2_t \) between 0 and \( S_n \). We have

\[
e^{\beta(A_s + S_n)} Y^2_{S_n} = Y^2_0 + \beta \int_0^{S_n} e^{\beta(A_s + s)} Y^2_s dA_s + \beta \int_0^{S_n} e^{\beta(A_s + s)} Y^2_s ds
\]

\[
+ 2 \int_0^{S_n} \int_E e^{\beta(A_s + s)} Y_s U_s(e) q(dsde) + 2 \int_0^{S_n} e^{\beta(A_s + s)} Y_s Z_s dW_s
\]

\[
- 2 \int_0^{S_n} e^{\beta(A_s + s)} Y_s f_s dA_s - 2 \int_0^{S_n} e^{\beta(A_s + s)} Y_s g_s ds
\]

\[
- 2 \int_0^{S_n} e^{\beta(A_s + s)} Y_s dK_s + \int_0^{S_n} e^{\beta(A_s + s)} Z_s dK_s + \sum_{0 < s \leq S_n} e^{\beta(A_s + s)} \Delta K^2_s + \int_0^T \int_E e^{\beta(A_s + s)} U^2_s(e) p(dsde)
\]
Now we use the fact that
\[
\int_0^t \int_E U_s(e)p(dsde) = \int_0^t \int_E U_s(e)\phi_s(de)dA_s + \int_0^t \int_E U_s(e)q(dsde),
\]
and, by Remark 2.4,
\[
\int_0^t e^{\beta(A_s)} Y_s^2 dK_s = \int_0^t e^{\beta(A_s)} (Y_s^2 - h_s) dK_s + \int_0^t e^{\beta(A_s)} h_s dK_s.
\]
Neglecting the positive terms \(Y_0^2\) and \(\sum_{0<s \leq S_n} e^{\beta(A_s)} \Delta K_s^2\) the previous equation becomes
\[
e^{\beta(A_s + S_n)} Y_{S_n}^2 \geq \beta \int_0^{S_n} e^{\beta(A_s + S_n)} Y_s^2 dA_s + \beta \int_0^{S_n} e^{\beta(A_s + S_n)} Y_s^2 ds + 2 \int_0^{S_n} e^{\beta(A_s + S_n)} Y_s U_s(e)q(dsde) + 2 \int_0^{S_n} e^{\beta(A_s + S_n)} Y_s Z_s dW_s - 2 \int_0^{S_n} e^{\beta(A_s + S_n)} Y_s f_s dA_s - 2 \int_0^{S_n} e^{\beta(A_s + S_n)} Y_s g_s ds - 2 \int_0^{S_n} e^{\beta(A_s + S_n)} h_s ds + \int_0^{S_n} e^{\beta(A_s + S_n)} U_s^2(e)\phi_s(de)dA_s + \int_0^{S_n} e^{\beta(A_s + S_n)} Z_s^2 ds,
\]
By the definition of \(S_n\) and remembering that \(Y\) satisfies (12), and using Burkholder-Davis-Gundy inequality we have that
\[
\int_0^{t \wedge S_n} e^{\beta(A_s + S_n)} Y_s Z_s dW_s
\]
is a martingale. Indeed we have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^{t \wedge S_n} e^{\beta(A_s + S_n)} Y_s Z_s dW_s \right] \leq \mathbb{E} \left[ \left( \int_0^{S_n} e^{2\beta(A_s + S_n)} Y_s^2 Z_s^2 ds \right)^{1/2} \right]
\]
\[
\leq e^{\beta T} \mathbb{E} \left[ \sup_{t} e^{\beta A_t / 2} |Y_t| \left( \int_0^{S_n} e^{\beta A_s} Z_s^2 ds \right)^{1/2} \right]
\]
\[
\leq n^{1/2} e^{\beta T} \mathbb{E} \left[ \sup_{t} e^{\beta A_t Y_t^2} \right] < \infty.
\]
Similarly, since
\[
\mathbb{E} \left[ \int_0^t \int_E e^{\beta(A_s + S_n)} Y_s - U_s(e)\phi_s(de)dA_s \right] \leq \mathbb{E} \left[ \int_0^t e^{\beta(A_s + S_n)} Y_s^2 dA_s \right] + \mathbb{E} \left[ \int_0^t \int_E e^{\beta(A_s + S_n)} U_s^2(e)\phi_s(de)dA_s \right] \leq 2n < \infty,
\]
we obtain that \( \int_0^{t \wedge S_n} \int_E e^{\beta(A_\tau + s)} Y_s U_s(e) q(dsde) \) is a martingale. Reordering terms and taking expectation we obtain

\[
\beta E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} Y_s^2 dA_s \right] + \beta E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} Y_s^2 ds \right] + \frac{\beta}{2} E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} Y_s^2 dA_s \right] \\
\leq E \left[ \int_0^{S_n} e^{\beta(A_s + S_n)} Y_s^2 ds \right] + 2 \beta E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} Y_s f_s dA_s \right] \\
+ 2 \beta E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} Y_s g_s dA_s \right] + 2 \beta E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} h_s dK_s \right]
\]

\[
\leq E \left[ \sup_{\tau} e^{\beta(A_\tau + \tau)} Y_\tau^2 \right] + \frac{\beta}{2} E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} Y_s^2 dA_s \right] + \frac{\beta}{2} E \left[ \int_0^{S_n} e^{\beta(A_\tau + s)} Y_s^2 ds \right] \\
+ \frac{1}{\beta} E \left[ \int_0^{T} e^{\beta(A_\tau + s)} f_s^2 dA_s \right] + \frac{2}{\beta} E \left[ \int_0^{T} e^{\beta(A_\tau + s)} g_s^2 ds \right] \\
+ \gamma E \left[ \sup_{\tau} e^{(\beta+\delta)(A_\tau + \tau)} h_\tau^2 \right] + \frac{1}{\gamma} E \left[ \left( \int_0^{S_n} e^{(\beta-\delta)\frac{A_\tau + s}{2}} dK_s \right)^2 \right],
\]

where \( \gamma > 0 \) is a constant whose value will be chosen sufficiently large afterwards.

We only need to estimate the last term with the integral in \( dK \). In order to do that we apply Ito’s formula to \( e^{(\beta-\delta)\frac{A_\tau + s}{2}} Y_s \) between 0 and a stopping time \( \tau \), obtaining the following relation

\[
\left( \int_0^{T} e^{(\beta-\delta)\frac{A_\tau + s}{2}} dK_s \right)^2 = \left( Y_0 - e^{(\beta-\delta)\frac{A_\tau + s}{2}} Y_s \right) + \frac{\beta - \delta}{2} \int_0^{T} e^{(\beta-\delta)\frac{A_\tau + s}{2}} Y_s dA_s \\
+ \beta - \delta \int_0^{T} e^{(\beta-\delta)\frac{A_\tau + s}{2}} Y_s ds - \int_0^{T} e^{(\beta-\delta)\frac{A_\tau + s}{2}} f_s dA_s \\
- \int_0^{T} e^{(\beta-\delta)\frac{A_\tau + s}{2}} g_s ds + \int_0^{T} \int e^{(\beta-\delta)\frac{A_\tau + s}{2}} U_s(e) q(dsde) \\
+ \int_0^{T} e^{(\beta-\delta)\frac{A_\tau + s}{2}} Z_s dW_s \right)^2
\]

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Notice that the following holds:

\[
\left( \int_0^T e^{(\beta - \delta) \frac{\Delta s + \epsilon}{2}} Y_s \, dA_s \right)^2 \leq \int_0^T e^{-\delta(A_s + \epsilon)} \, dA_s \int_0^T e^{\beta(A_s + \epsilon)} Y_s^2 \, dA_s \\
\leq \frac{1}{\delta} \int_0^T e^{\beta(A_s + \epsilon)} Y_s^2 \, dA_s
\]

\[
\left( \int_0^T e^{(\beta - \delta) \frac{\Delta s + \epsilon}{2}} Y_s \, ds \right)^2 \leq \int_0^T e^{-\delta A_s} e^{-\delta s} ds \int_0^T e^{\beta(A_s + \epsilon)} Y_s^2 \, ds \\
\leq \frac{1}{\delta} \int_0^T e^{\beta(A_s + \epsilon)} Y_s^2 \, ds
\]

and similarly

\[
\left( \int_0^T e^{(\beta - \delta) \frac{\Delta s + \epsilon}{2}} f_s \, dA_s \right)^2 \leq \frac{1}{\delta} \int_0^T e^{\beta(A_s + \epsilon)} f_s^2 \, dA_s \\
\left( \int_0^T e^{(\beta - \delta) \frac{\Delta s + \epsilon}{2}} g_s \, ds \right)^2 \leq \frac{1}{\delta} \int_0^T e^{\beta(A_s + \epsilon)} g_s^2 \, ds
\]

We note that for a \( P \otimes E \) measurable process \( H \) we have

\[
\mathbb{E} \left[ \left( \int_0^t \int_E H_s(e) q(dsde) \right)^2 \right] \leq \mathbb{E} \left[ \int_0^t \int_E H_s^2(e) \phi_s(\phi_s(de) \, dA_s \right] .
\]

This can be checked for instance by applying the Ito formula to compute \( N_t^2 \) where \( N_t = \int_0^t \int_E H_s(y) q(dsdy) \) and taking expectation after appropriate localization. Now by taking expectation and using Ito Isometry we obtain the following bound for \( \left( \int_0^T e^{(\beta - \delta) \frac{\Delta s + \epsilon}{2}} dK_s \right)^2 \):

\[
\mathbb{E} \left[ \left( \int_0^T e^{(\beta - \delta) \frac{\Delta s + \epsilon}{2}} dK_s \right)^2 \right] \leq 16 \mathbb{E} \left[ \sup_t e^{\beta(A+t)} Y_t^2 \right] + \frac{8}{\delta} \mathbb{E} \left[ \int_0^T e^{\beta(A_s + \epsilon)} g_s^2 \, ds \right] \\
+ 2 \frac{(\beta - \delta)^2}{\delta} \mathbb{E} \left[ \int_0^T e^{\beta(A_s + \epsilon)} Y_s^2 \, ds \right] + 2 \frac{(\beta - \delta)^2}{\delta} \mathbb{E} \left[ \int_0^T e^{\beta(A_s + \epsilon)} Y_s^2 \, dA_s \right] \\
+ \frac{8}{\delta} \mathbb{E} \left[ \int_0^T e^{\beta(A_s + \epsilon)} f_s^2 \, dA_s \right] + 8 \mathbb{E} \left[ \int_0^T e^{\beta(A_s + \epsilon)} Z_s^2 \, ds \right] \\
+ 8 \mathbb{E} \left[ \int_0^T \int_E e^{\beta(A_s + \epsilon)} U_s^2(e) \phi_s(de) \, dA_s \right] .
\]

By plugging this last estimate into (20), by choosing \( \alpha, \gamma \) such that

\[
\gamma > \max \left( 8, 4 \frac{(\beta - \delta)^2}{\beta \gamma} \right)
\]
we obtain
\[
E \left[ \int_0^{S_n} e^{\beta(A_s + s)} Y_s^2 dA_s \right] + E \left[ \int_0^{S_n} e^{\beta(A_s + s)} Y_s^2 ds \right]
\]
\[
+ E \left[ \int_0^{S_n} \int_E e^{\beta(s)} \phi_\xi(ds) dA_s \right] + E \left[ \int_0^{S_n} e^{\beta(A_s + s)} Y_s^2 ds \right]
\]
\[
\leq C \left( E \left[ \sup_t e^{\beta A_t} Y_t^2 \right] + 2 \left( \frac{1}{\beta} + \frac{1}{\delta \gamma} \right) E \left[ \int_0^T e^{\beta A_s} f_s^2 dA_s \right] \right)
\]
\[
+ E \left[ \int_0^T e^{\beta A_s} g_s^2 ds \right] + \gamma E \left[ \sup_t e^{(\beta + \delta) A_t} h_t^2 \right] \right),
\]
for some constant $C$ independent of $n$. Now let $S = \lim_n S_n$ and by the last estimate, considering how $S_n$ are defined, we have $S = T$. This implies that $Y \in L^{2,\beta}(A) \cap L^{2,\beta}(W)$, $Z \in L^{2,\beta}(W)$ and $U \in L^{2,\beta}(p)$.

**Proof of lemma 3.2.** By the definition of $Y$ we have

\[
e^{\beta A_t/2} Y_t \leq E \left[ e^{\beta A_T/2} \xi \right] + e^{\beta A_t/2} \int_t^T |f_s| dA_s
\]
\[
+ e^{\beta A_t/2} \int_t^T |g_s| ds + \sup_{0 \leq s \leq T} e^{\beta A_s/2} |h_s| \right|_{\mathcal{F}_t}
\]

Proceeding as in Remark 3.1 we have

\[
\int_t^T |f_s| dA_s \leq \frac{e^{-\beta A_t/2}}{\beta^{1/2}} \left( \int_t^T e^{\beta A_s} |f_s|^2 dA_s \right)^{1/2}
\]

and it follows that

\[
e^{\beta A_t/2} Y_t \leq E \left[ e^{\beta A_T/2} \xi \right] + \frac{1}{\beta^{1/2}} \left( \int_0^T e^{\beta A_s} |f_s|^2 dA_s \right)^{1/2}
\]
\[
+ \int_0^T e^{\beta A_s/2} |g_s| ds + \sup_{0 \leq s \leq T} e^{\beta A_s/2} |h_s| \right|_{\mathcal{F}_t} =: S_t
\]

Under assumption (B) (i)(iv) and (B'), $S$ is a square integrable martingale.

Then by Doob’s martingale inequality $E \left[ \sup_{0 \leq t \leq T} e^{\beta A_t} Y_t \right] \leq C E \left[ S_T^2 \right] < \infty$.

**Remark 3.2.** Contrary to the diffusive (or diffusive and Poisson) case, the fact that $E \left[ \sup_{t \in [0,T]} e^{\beta A_t} Y_t^2 \right] < \infty$ does not imply that $Y \in L^{2,\beta}(A)$. For this to happen we would need additional conditions on $A$, for example $E[A_T^2] < \infty$. 

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Next we prove uniqueness.

**Proposition 3.2:** Let assumptions (A), (B)(i)(iv) and (B') hold for some $\beta > 0$, then the solution to (7) is unique.

**Proof.** Let $(Y', U', Z', K')$ and $(Y'', U'', Z'', K'')$ be two solutions. Define

\[ \bar{Y} = Y' - Y'' \quad \bar{U} = U' - U'' \quad \bar{Z} = Z' - Z'' \quad \bar{K} = K' - K'', \]

then $(\bar{Y}, \bar{U}, \bar{Z}, \bar{K})$ satisfies

\[ \bar{Y}_t = -\int_t^T \int_E \bar{U}(e)q(dsde) - \int_t^T \bar{Z}_s dW_s + \bar{K}_T - \bar{K}_t. \quad (21) \]

We compute $d(e^{\beta(A_{s+t})}\bar{Y}_t^2)$ by the Itô formula and we obtain

\[ -\bar{Y}_t^2 = \beta \int_0^T e^{\beta(A_s+\bar{Y}_s^2)A_s} + \beta \int_0^T e^{\beta(A_s+\bar{Y}_s^2)ds} - 2 \int_0^T \bar{Y}_s d\bar{K}_s + \int_0^T e^{\beta(A_s+\bar{Y}_s^2)Z_s}dW_s + \int_0^T e^{\beta(A_s+\bar{Y}_s^2)Z_s^2}ds + \sum_{0 < s \leq T} e^{\beta(A_s+\bar{Y}_s^2)(\Delta \bar{Y}_s)^2} \quad (22) \]

The last term can be divided in totally inaccessible jumps (from the martingale in $q(dsde)$) and predictable jumps, from the $K$ process, thus:

\[ \sum_{0 < s \leq T} e^{\beta(A_s+\bar{Y}_s^2)(\Delta \bar{Y}_s)^2} \geq \sum_{0 < T_n \leq T} e^{\beta(A_s+\bar{Y}_s^2)U^2_{T_n}(\xi_n)} = \int_0^T \int_E U^2_s(e)p(dsde) \]

\[ = \int_0^T \int_E U^2_s(e)q(dsde) + \int_0^T \int_E U^2_s(e)\phi_s(de)dA_s \]

Proceeding as in (18) and (19) we prove that the stochastic integrals with respect to $W$ and $q$ are martingales. By neglecting $Y_0^2$ and taking expectation in (22), we obtain

\[ \beta E \left[ \int_0^T e^{\beta(A_s+\bar{Y}_s^2)}A_s \right] + \beta E \left[ \int_0^T e^{\beta(A_s+\bar{Y}_s^2)}ds \right] \]

\[ + E \left[ \int_0^T \int_E e^{\beta(A_s+\bar{Y}_s^2)U^2_y\phi_s(dy)dA_s} \right] + E \left[ \int_0^T e^{\beta(A_s+\bar{Y}_s^2)Z_s^2}ds \right] \leq 2E \left[ \int_0^T e^{\beta(A_s+\bar{Y}_s^2)}\bar{Y}_s d\bar{K}_s \right]. \]

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Now, taking into account Remark 2.4 we have
\[
\int_0^T \bar{Y}_s - \bar{d} \bar{K}_s \, ds = \int_0^T (Y'_s - h_s -) \, dK'_s + \int_0^T (Y''_s - h_s -) \, dK''_s + \int_0^T (Y''_s - h_s -) \, dK''_s
\]
\[\leq 0,\]
and thus
\[
\beta ||\bar{Y}||^2_{L^2(A)} + \beta ||\bar{Y}||^2_{L^2(W)} + ||\bar{U}||^2_{L^2(\rho)} + ||\bar{Z}||^2_{L^2(W)} \leq 0,
\]
which gives the uniqueness of \( Y, U \) and \( Z \). From (21) we obtain
\[
\bar{K}_T = \bar{K}_t \quad \forall t \in [0, T].
\]
Then \( \bar{K}_T = 0 \) since \( \bar{K}_0 = 0 \) and consequently \( \bar{K}_t = 0 \) for all \( t \).

Consider now the optimal stopping problem with running gains \( f, g \), early stopping reward \( h \) and non stopping reward \( \xi \). This means we are interested in the quantity
\[
v(t) = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E} \left[ \int_t^\tau f_s \, dA_s + \int_0^\tau g_s \, ds + h_{\tau \wedge T} + \xi_{\gamma \wedge T} \big| \mathcal{F}_t \right].
\]
Notice that we have two running gains, \( f \) integrated with respect to the process \( A \), and \( g \) integrated with respect to Lebesgue measure in time. This could be used for example if we want to describe two different time dynamics, one depending on the speed of the point process.

It is possible to show that the solution to the RBSDE solves the optimal stopping problem and it is possible to identify an \( \epsilon \)-optimal stopping time. Under additional assumptions, it is possible to find an optimal stopping time. For this we need a definition, given in [27] for admissible families over stopping times, that we adapt to our simpler case:

**Definition 3.1.** We say that a process \( \phi \) is left (resp. right) upper semi-continuous over stopping times in expectation (USCE) if for all \( \theta \in \mathcal{T}_0 \), \( \mathbb{E}[\phi_\theta] < \infty \) and for all sequences of stopping times \( (\theta_n) \) such that \( \theta_n \uparrow \theta \) (resp. \( \theta_n \downarrow \theta \)) it holds that
\[
\mathbb{E}[\phi_\theta] \geq \limsup_{n \to \infty} \mathbb{E}[\phi_{\theta_n}].
\]

**Remark 3.3.** If \( \phi \) is a left upper semi continuous progressive process, then \( \phi \) is left upper semi continuous along stopping times. If also \( \mathbb{E}[\sup_t |\phi_t|] \) holds, then it is left USCE. Indeed we have
\[
\limsup_{n \to \infty} \mathbb{E}[\phi_{\theta_n}] \leq \mathbb{E}\left[\limsup_{n \to \infty} \phi_{\theta_n}\right] \leq \mathbb{E}[\phi_\theta],
\]
by using Reverse Fatou’s lemma with \( \sup_t |\phi_t| \) as dominant.
Proposition 3.3: Let assumptions (A), (B)-(i)(iv) and (B′) hold. Then we have:

1. The solution to the RBSDE (7) is a solution to the optimal stopping problem
\[ Y_t = \operatorname{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E} \left[ \int_t^\tau f_s dA_s + \int_0^\tau g_s ds + h_\tau \mathbb{1}_{\{\tau < T\}} + \xi \mathbb{1}_{\{\tau \geq T\}} \middle| \mathcal{F}_t \right]. \]

2. For all \( \epsilon > 0 \), define \( D^\epsilon_t \) as
\[ D^\epsilon_t = \inf \{ s \geq t : Y_s \leq h_s + \epsilon \} \wedge T. \]
Then \( D^\epsilon_t \) is an \( \epsilon \)-optimal stopping time in the sense that
\[ Y_t \leq \operatorname{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E} \left[ \int_t^{D^\epsilon_t} f_s dA_s + \int_0^{D^\epsilon_t} g_s ds + h_{D^\epsilon_t} \mathbb{1}_{\{D^\epsilon_t < T\}} + \xi \mathbb{1}_{\{D^\epsilon_t \geq T\}} \middle| \mathcal{F}_t \right] + \epsilon. \]

3. If in addition \( h_t \mathbb{1}_{\{t < T\}} + \xi \mathbb{1}_{\{t \geq T\}} \) is left USCE, then
\[ \tau^*_t = \inf \{ s \geq t : Y_s \leq h_s \} \wedge T. \]
is optimal and is the smallest of all optimal stopping times.

Remark 3.4. The condition on the third point may seem unusual, but it is satisfied for example if \( h_t \) is left upper semi continuous on \([0,T]\) and \( h_T < \xi \).

Proof. Let \( \tau \in \mathcal{T}_t \) and consider the first equation (7) between \( t \) and \( \tau \):
\[ Y_t = Y_\tau + \int_t^\tau f_s dA_s + \int_t^\tau g_s ds - \int_t^\tau \int_E Z_s(y) q(dsdy) + K_\tau - K_t. \]
By taking conditioning at \( \mathcal{F}_t \) we have
\[ Y_t = \mathbb{E} \left[ Y_\tau + \int_t^\tau f_s dA_s + \int_t^\tau g_s ds + K_\tau - K_t \middle| \mathcal{F}_t \right] \]
\[ \geq \mathbb{E} \left[ h_\tau \mathbb{1}_{\{\tau < T\}} + \xi \mathbb{1}_{\{\tau \geq T\}} + \int_t^\tau f_s dA_s + \int_t^\tau g_s ds \middle| \mathcal{F}_t \right], \]
since the integral on \( q \) is a martingale, \( K \) is increasing and \( Y_t \geq h_\tau \mathbb{1}_{\{t < T\}} + \xi \mathbb{1}_{\{t = T\}} \). To prove the reverse inequality, consider \( \epsilon > 0 \) and the corresponding \( D^\epsilon_t \). It holds that \( Y_{D^\epsilon_t} \leq h_{D^\epsilon_t} + \epsilon \) on \( \{D^\epsilon_t < T\} \). And on \( \{D^\epsilon_t = T\} \) we have
\[ Y_u > h_u + \epsilon \text{ for all } t \leq u < T. \]
Then, between \( t \) and \( D^\epsilon_t \), \( Y_{s-} > h_{s-} \) and thus
\[ \int_t^{D^\epsilon_t} (Y_{s-} - h_{s-}) dK_s = 0 \Rightarrow K_{D^\epsilon_t} = K_t. \]
Considering all this in (23) we have
\[ Y_t = \mathbb{E} \left[ Y_{D_t^c} + \int_t^{D_t^c} f_s dA_s + \int_t^{D_t^c} g_s ds \bigg| \mathcal{F}_t \right] \]

\[ \leq \mathbb{E} \left[ h_{D_t^c} 1_{\{D_t^c < T\}} + \xi 1_{\{D_t^c = T\}} + \int_t^{D_t^c} f_s dA_s + \int_t^{D_t^c} g_s ds \bigg| \mathcal{F}_t \right] + \epsilon. \quad (25) \]

This together with (24) proves points one and two. For the third point, notice that \( D_t^c \) are non increasing in \( \epsilon \) and that \( D_t^c \leq \tau^* \). Thus \( D_t^c \rightarrow D_t^0 \leq \tau^* \) when \( \epsilon \rightarrow 0 \). Now since \( h_t 1_{\{t < T\}} + \xi 1_{\{t = T\}} \) is left USCE and the integral part is too, we have from (25)

\[ \mathbb{E}[Y_t] \leq \limsup_{\epsilon \rightarrow 0} \mathbb{E} \left[ h_{D_t^c} 1_{\{D_t^c < T\}} + \xi 1_{\{D_t^c = T\}} + \int_t^{D_t^c} f_s dA_s + \int_t^{D_t^c} g_s ds \bigg| \mathcal{F}_t \right] \]

\[ \leq \mathbb{E} \left[ h_{D_t^0} 1_{\{D_t^0 < T\}} + \xi 1_{\{D_t^0 = T\}} + \int_t^{D_t^0} f_s dA_s + \int_t^{D_t^0} g_s ds \right]. \]

Thus we have

\[ \mathbb{E} [Y_t] = \mathbb{E} \left[ h_{D_t^0} 1_{\{D_t^0 < T\}} + \xi 1_{\{D_t^0 = T\}} + \int_t^{D_t^0} f_s dA_s + \int_t^{D_t^0} g_s ds \right], \]

so \( D_t^0 \) is optimal (see A.1.iii)). We only need to prove that \( D_t^0 = \tau^* \). We already know that \( D_t^0 \leq \tau^* \). On the other hand, since \( D_t^0 \) is optimal it holds that \( Y_{D_t^0} = \eta_{D_t^0} \), and thus by the definition of \( \tau^* \), \( \tau^* \leq D_t^0 \). This also proves that \( \tau^* \) is the smallest optimal stopping time.

A further interesting property holds when the reward is left USCE:

**Proposition 3.4:** Under assumptions (B)(i)(iv) and (B') if \( h_t 1_{\{t < T\}} + \xi 1_{\{t \geq T\}} \) is also left USCE, then \( K \) in the solution of (7) is continuous.

**Proof.** The proof is given in [27] in the case were the reward is a positive progressive process \( \phi \) of class [D]. We can adapt to our case by using the transformation

\[ I = \inf_t \eta_t \quad N_t = \mathbb{E} [I | \mathcal{F}_t] \quad \tilde{\eta}_t = \eta_t - N_t. \]

We have that \( \tilde{\eta}_t \) is USCE, as \( \mathbb{E}[\tilde{\eta}_t] = \mathbb{E}[\eta_t] - \mathbb{E}[I] \). Indeed let \( \theta_n \uparrow \theta \), then

\[ \limsup_{n \rightarrow \infty} \mathbb{E}[\tilde{\eta}_{\theta_n}] \leq \mathbb{E}[\tilde{\eta}_\theta]. \]

Then if \( R(\eta) \) denotes the Snell envelope of \( \eta \), it holds that \( R(\tilde{\eta}) = R(\eta) - N_t \). The Doob-Meyer decomposition for the càdlàg supermartingale \( R(\tilde{\eta}) \) holds:

\[ R(\tilde{\eta})_t = \tilde{M}_t - \tilde{K}_t \]

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With \( \bar{K} \) continuous thanks to Proposition B.10 in [27]. Then \( Y_t + \int_0^t f_s dA_s = R(\eta) = R(\bar{\eta}) + N_t = \bar{M}_t + N_t - \bar{K}_t \), but since the decomposition is unique, \( \int_0^t \int_E Z_s(y)q(dsdy) = M_t = \bar{M} + N_t \) and \( K_t = \bar{K}_t \). Thus the term \( K \) is continuous.

If we are interested only in \( \bar{M} \), and we have a filtration generated only by a MPP and \( g \equiv 0 \), the proofs above are still applicable. In this case, there is no particular reason to use a \( L^2 \) space, since the martingale representation theorem for marked point processes works in \( L^1 \) (see [25]). We thus obtain the following:

**Proposition 3.5:** Let assumption \([\text{A}]\) hold. Let \( \xi \) be a \( \mathcal{G}_T \)-measurable random variable. Let \( f, h \) be \( \mathcal{G} \)-progressive processes. Assume that

\[
\mathbb{E} \left[ |\xi| + \int_0^T |f_s| dA_s + \sup_{t \in [0, T]} |h_t| \right] < \infty.
\]

Then there exists a unique solution to the system

\[
\begin{cases}
Y_t = \xi + \int_0^T f_s dA_s - \int_0^T \int_E U_s(y)q(dsdy) + K_T - K_t \\
y_t \geq h_t \quad \forall t \in [0, T] \text{ a.s.} \\
\int_0^T (Y_s - h_s) dK_s = 0 \text{ and } \Delta K_s \leq (h_s - Y_s)^+ \mathbb{1}_{\{Y_s = h_s\}}.
\end{cases}
\]

where \( Y \) is a càdlàg \( \mathcal{G} \)-adapted process such that \( \mathbb{E}[|Y_t|] < \infty \) for all \( t \), \( K \) is a \( \mathcal{G} \)-predictable càdlàg increasing process with \( K_0 = 0 \) and \( \mathbb{E}[K_T] < \infty \) and \( U \) is a \( \mathcal{P}(\mathcal{G}) \otimes \mathcal{E} \)-measurable process such that \( \mathbb{E}\left[ \int_0^T \int_E |U_s(e)| \phi_s(de) dA_s \right] < \infty \).

**Proof.** Existence of a solution is obtained as in 3.4. The process \( \eta \) satisfies then the weaker condition \( \mathbb{E}[\sup_t |\eta_t|] < \infty \), but this is enough to apply the Snell’s envelope results (see appendix A in particular (30)). Integrability is straightforward. Now let \((Y', U', K')\) and \((Y'', U'', K'')\) be two solutions, their difference satisfies

\[
Y'_t - Y''_t = Y'_0 - Y''_0 + \int_0^t \left( U'_s(e) - U''_s(e) \right) q(dsde) - (K'_t - K''_t).
\]

Uniqueness of the component \( Y \) comes from the fact that if \((Y, U, K)\) satisfies the equation, the càdlàg process \( Y \) satisfies

\[
Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \left( \int_t^{\tau \wedge T} f_s dA_s + h_\tau \mathbb{1}_{\{\tau < T\}} + \xi \mathbb{1}_{\{\tau \geq T\}} \right| \mathcal{G}_t \),
\]

which can be shown as in proposition 3.3 adapted to the this case with less integrability. Relation (27) becomes

\[
\int_0^t \int_E U'_s(e)q(dsde) - K'_t = \int_0^t \int_E U''_s(e)q(dsde) - K''_t.
\]

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Since the predictable jumps of \( K \) and the totally inaccessible jumps of the integrals with respect to \( q \) are disjoint, we have that \( U'_{T_n}(\xi_n) = U''_{T_n}(\xi_n) \) for all \( n \). Then
\[
\int_0^T \int_E |U'_s(e) - U''_s(e)| \phi_s(de) dA_s = \int_0^T \int_E |U'_s(e) - U''_s(e)| p(dsde) = \sum_{n \geq 1} |U'_{T_n}(\xi_n) - U''_{T_n}(\xi_n)| = 0,
\]
and thus \( U'_s(e) = U''_s(e) \phi_s(de) dA_s d\mathbb{P}\) a.e. Then \( K'_t = K''_t \) a.s. and uniqueness is proven. \( \square \)

We have then a result for optimal stopping analogous to proposition 3.3:

**Proposition 3.6:** Assume that the conditions of proposition 3.5 hold. Then

1. The solution to the RBSDE (26) is a solution to the optimal stopping problem
   \[
   Y_t = \text{ess sup}_{\tau \in T_t} \mathbb{E} \left[ \int_t^\tau f_s dA_s + \int_0^\tau g_s ds + h_\tau \mathbbm{1}_{\{\tau < T\}} + \xi \mathbbm{1}_{\{\tau \geq T\}} \bigg| \mathcal{F}_t \right].
   \]
2. For all \( \epsilon > 0 \), define \( D_\epsilon^t \) as
   \[
   D_\epsilon^t = \inf \{ s \geq t : Y_s \leq h_s + \epsilon \} \wedge T.
   \]
   Then \( D_\epsilon^t \) is an \( \epsilon \)-optimal stopping time in the sense that
   \[
   Y_t \leq \text{ess sup}_{\tau \in T_t} \mathbb{E} \left[ \int_t^{D_\epsilon^t} f_s dA_s + \int_0^{D_\epsilon^t} g_s ds + h_{D_\epsilon^t} \mathbbm{1}_{\{D_\epsilon^t < T\}} + \xi \mathbbm{1}_{\{D_\epsilon^t \geq T\}} \bigg| \mathcal{F}_t \right] + \epsilon.
   \]
3. If in addition \( h_\tau \mathbbm{1}_{\{\tau < T\}} + \xi \mathbbm{1}_{\{\tau \geq T\}} \) is left USCE, then
   \[
   \tau^*_t = \inf \{ s \geq t : Y_s \leq h_s \} \wedge T.
   \]
   is optimal and is the smallest of all optimal stopping times. Moreover, the process \( K \) is continuous.

4 Reflected BSDE

We now turn to the case where the generators depend on the solution, that is equation (2). Denote by \( \lambda \) the Lebesgue measure on \([0, T]\), and introduce now \( L^{2,\beta}(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), (A(\omega, dt) + \lambda(dt)) \), the space of all \( \mathbb{F}\)-progressive processes such that
\[
\|Y\|_{L^{2,\beta}(A+\lambda)}^2 = \mathbb{E} \left[ \int_0^T e^{\beta A_s} Y_s^2 (dA_s + ds) \right] < \infty.
\]
For brevity we denote as \( L^{2,\beta}(A+\lambda) \) in the following. It is a Hilbert space equipped with the norm above. It is clear that a process is in \( L^{2,\beta}(A+\lambda) \) if and only if lies in \( Y \in L^{2,\beta}(A) \cap L^{2,\beta}(W) \).
Theorem 4.1: Let assumptions \( [A] \) and \( [B] \) hold for some \( \beta > L_f^2 + 2L_f \). Then there exists a unique solution to (2).

Proof. We can use a contraction theorem on

\[
L^\beta = L^{2,\beta}(\Omega \times [0,T], F \otimes B([0,T]), (A(\omega, dt) + \lambda(dt))P(dw)) \times L^{2,\beta}(p) \times L^{2,\beta}(W).
\]

We construct a mapping \( \Gamma \) that to each \( (P, Q, R) \in L^{2,\beta}(A + \lambda) \times L^{2,\beta}(p) \times L^{2,\beta}(W) \) associates \( Y, U, Z \) solution to equation (7) when the generators are given by \( f_t(P_t, Q_t) \) and \( g_t(P_t, R_t) \). Such map is well defined: indeed if we fix \( (P, Q, R) \in L^{2,\beta}(A + \lambda) \times L^{2,\beta}(p) \times L^{2,\beta}(W) \), thanks to assumption \( [B] \), the generators are known process that satisfy assumption \( [B] \) and proposition 3.2 and \( \beta > L_f^2 + 2L_f \) give us the existence and uniqueness of \( Y, U, Z \in L^{2,\beta}(A + \lambda) \times L^{2,\beta}(p) \times L^{2,\beta}(W) \). Notice that thanks to the Lipschitz conditions on \( g \) and \( f \), if we take two triplets \( (P', Q', R') \equiv (P'', Q'', R'') \in L^{2,\beta}(A + \lambda) \times L^{2,\beta}(p) \times L^{2,\beta}(W) \), then

\[
f_s(Y''', U''', Z''') \equiv f_s(Y', U', Z') \text{ in } L^{2,\beta}(A) \text{ and } g_s(Y', Z') = g_s(Y'', Z'') \text{ in } L^{2,\beta}(W).
\]

Consider now \( (P'', Q'', R'') \) in \( L^{2,\beta} \), and consider their images through \( \Gamma \), \( (Y', U', Z') = \Gamma(P', Q', R') \) and \( (Y'', U'', Z'') = \Gamma(P'', Q'', R'') \). Denote \( \bar{Y} = Y' - Y'' \), \( \bar{P} = P' - P'' \) and so on. Denote also \( \bar{f}_t = f_t(P'_t, Q'_t) - f_t(P''_t, Q''_t) \) and similarly denote \( \bar{g} \). \( (\bar{Y}, \bar{U}, \bar{Z}, \bar{K}) \) satisfies

\[
\bar{Y} = \int_0^T \bar{f}_s dA_s + \int_0^T \bar{g}_s ds - \int_0^T \int_E \bar{U}_s(c) q(dsdc) - \int_0^T \bar{Z}_s dW_s + \bar{K}_T - \bar{K}_t.
\]

We now apply Ito’s Lemma to \( e^{\lambda A_s} e^{\gamma s} \bar{Y}_s^2 \) obtaining, after taking expectation,

\[
\begin{aligned}
\beta \mathbb{E} \left[ \int_0^T e^{\lambda A_s} e^{\gamma s} \bar{Y}_s^2 dA_s \right] + \gamma \mathbb{E} \left[ \int_0^T e^{\lambda A_s} e^{\gamma s} \bar{Y}_s^2 ds \right] + \mathbb{E} \left[ \int_0^T e^{\lambda A_s} e^{\gamma s} \bar{Z}_s^2 dW_s \right] \\
+ \mathbb{E} \left[ \int_0^T \int_E e^{\lambda A_s} e^{\gamma s} \bar{U}_s^2 \phi_s(c) dA_s \right] \leq 2 \mathbb{E} \left[ \int_0^T e^{\lambda A_s} e^{\gamma s} \bar{f}_s^2 dA_s \right] \\
+ 2 \mathbb{E} \left[ \int_0^T e^{\lambda A_s} e^{\gamma s} \bar{g}_s^2 ds \right] + 2 \mathbb{E} \left[ \int_0^T \bar{Y}_s dK_s \right].
\end{aligned}
\]

As in the proof of proposition 3.2 we have that

\[
\int_0^T \bar{Y}_s dK_s \leq 0.
\]

Denote by \( || \cdot ||_{\beta, A} \) the norm (equivalent to \( || \cdot ||_{L^{2,\beta}(A)} \))

\[
\left( \mathbb{E} \left[ \int_0^T e^{\lambda A_s} e^{\gamma s} \bar{Y}_s^2 dA_s \right] \right)^{1/2},
\]

and similarly denote the norms \( || \cdot ||_{\beta, P} \) and \( || \cdot ||_{\beta, W} \). Using the Lipschitz properties of \( f \) and \( g \) this gives
\[ \beta \| \bar{Y} \|^2_{\beta, \gamma, A} + \gamma \| \bar{Y} \|^2_{\beta, \gamma, W} + \| \bar{U} \|^2_{\beta, \gamma, p} + \| \bar{Z} \|^2_{\beta, \gamma, W} \leq \]
\[ \leq 2L_f \mathbb{E}\left[ \int_0^T e^{\beta A_t} e^{\gamma_s} |\bar{Y}_t| \| \bar{P}_s \| dA_s \right] + 2L_p \mathbb{E}\left[ \int_0^T e^{\beta A_t} e^{\gamma_s} |\bar{Y}_s| \left( \int_E |\bar{Q}_s| \right)^{1/2} dA_s \right] \]
\[ + 2L_g \mathbb{E}\left[ \int_0^T e^{\beta A_t} e^{\gamma_s} |\bar{Y}_s| \| \bar{P}_s \| ds \right] + 2L_W \mathbb{E}\left[ \int_0^T e^{\beta A_t} e^{\gamma_s} |\bar{Z}_s| \| \bar{R}_s \| ds \right]. \]

Using the inequality \( 2ab \leq a^2 + b^2 / \alpha \) for \( a, b \geq 0 \) we obtain:

\[ \beta \| \bar{Y} \|^2_{\beta, \gamma, A} + \gamma \| \bar{Y} \|^2_{\beta, \gamma, W} + \| \bar{U} \|^2_{\beta, \gamma, p} + \| \bar{Z} \|^2_{\beta, \gamma, W} \leq \]
\[ \leq \frac{L_f}{\sqrt{\alpha_p}} \| \bar{Y} \|^2_{\beta, \gamma, A} + \frac{L_f^2}{\alpha} \| \bar{Y} \|^2_{\beta, \gamma, A} + \frac{L_p^2}{\alpha} \| \bar{Y} \|^2_{\beta, \gamma, A} + \alpha \| \bar{Q} \|^2_{\beta, \gamma, p} \]
\[ + \frac{L_g}{\sqrt{\alpha}} \| \bar{Y} \|^2_{\beta, \gamma, W} + L_g \sqrt{\alpha} \| \bar{P} \|^2_{\beta, \gamma, W} + \frac{L_g^2}{\alpha} \| \bar{Y} \|^2_{\beta, \gamma, A} + \alpha \| \bar{R} \|^2_{\beta, \gamma, A}. \]

Rewriting we obtain the following relation:

\[ \| \bar{U} \|^2_{\beta, \gamma, p} + \| \bar{Z} \|^2_{\beta, \gamma, W} \geq \left( \beta - \frac{L_p^2}{\alpha} - \frac{L_f^2}{\alpha} \right) \| \bar{Y} \|^2_{\beta, \gamma, A} \]
\[ + \left( \gamma - \frac{L_g^2}{\alpha} - \frac{L_f}{\sqrt{\alpha}} \right) \| \bar{Y} \|^2_{\beta, \gamma, W} \]
\[ \leq L_f \sqrt{\alpha} \| \bar{P} \|^2_{\beta, \gamma, A} + \alpha \| \bar{Q} \|^2_{\beta, \gamma, p} + L_g \sqrt{\alpha} \| \bar{P} \|^2_{\beta, \gamma, W} + \alpha \| \bar{R} \|^2_{\beta, \gamma, A}. \quad (28) \]

Since \( \beta > L_p^2 + 2L_f \), it is possible to choose \( \alpha \in (0, 1) \) such that

\[ \beta > \frac{L_p^2}{\alpha} + \frac{2L_f}{\sqrt{\alpha}}. \]

and for that \( \alpha \), choose \( \gamma \) such that \( \gamma > \frac{L_g^2}{\alpha} + 2L_g / \sqrt{\alpha} \). The relation (28) rewrites as

\[ \frac{L_f}{\sqrt{\alpha}} \| \bar{Y} \|^2_{\beta, \gamma, A} + \frac{L_g}{\sqrt{\alpha}} \| \bar{Y} \|^2_{\beta, \gamma, W} + \| \bar{U} \|^2_{\beta, \gamma, p} + \| \bar{Z} \|^2_{\beta, \gamma, W} \leq \]
\[ \leq L_f \sqrt{\alpha} \| \bar{P} \|^2_{\beta, \gamma, A} + \alpha \| \bar{Q} \|^2_{\beta, \gamma, p} + L_g \sqrt{\alpha} \| \bar{P} \|^2_{\beta, \gamma, W} + \alpha \| \bar{R} \|^2_{\beta, \gamma, A} \]
\[ = \alpha \left( \frac{L_f}{\sqrt{\alpha}} \| \bar{P} \|^2_{\beta, \gamma, A} + \| \bar{Q} \|^2_{\beta, \gamma, p} + \frac{L_g}{\sqrt{\alpha}} \| \bar{P} \|^2_{\beta, \gamma, W} + \| \bar{R} \|^2_{\beta, \gamma, A} \right). \quad (29) \]

Now

\[ \frac{L_f}{\sqrt{\alpha}} \| \bar{P} \|^2_{\beta, \gamma, A} + \frac{L_g}{\sqrt{\alpha}} \| \bar{P} \|^2_{\beta, \gamma, W} = \mathbb{E}\left[ \int_0^T e^{\beta A_t} e^{\gamma_s} \bar{P}_s^2 \left( \frac{L_f}{\sqrt{\alpha}} dA_s + \frac{L_g}{\sqrt{\alpha}} ds \right) \right]. \]
is a norm equivalent to \( \| \bar{P} \|_{L^2,\beta(A+\lambda)} \). We have thus that \( \Gamma \) is a contraction on \( L \) for the equivalent norm

\[
\| (Y, U, Z) \|_{L,\beta,\gamma}^2 = \frac{L_f}{\sqrt{\alpha}} \| Y \|_{\beta,\gamma,A}^2 + \frac{L_g}{\sqrt{\alpha}} \| Y \|_{\beta,\gamma,W}^2 + \| \bar{U} \|_{\beta,\gamma,p}^2 + \| \bar{Z} \|_{\beta,\gamma,W}^2.
\]

Since the space is complete, the contraction theorem assures us the existence of a unique triplet \((Y, U, Z)\) in \( L^\beta \) such that \((Y, U, Z) = \Gamma(Y, U, Z)\), and \((Y, U, Z, K)\) is the solution to (2), where \( K \) is the one associated to \((Y, Z, U)\) by the map \( \Gamma \).

Since we know

\[
\Gamma
\]

This last result generalizes the case of Brownian and Poisson noise, allowing for a more general structure in the jump part.

If we are interested only on a BSDE driven by a marked point process, the proof above still applies when the filtration \( G \) is generated only by \( p \) and the data are adapted to it. Then we have the counterpart of theorem 4.1

**Theorem 4.2:** Let assumptions [A] and (B)(i,ii,iv) hold for some \( \beta > L^2_p + 2L_f \), but with the data adapted to the filtration \( G \). Then the system (6) admits a unique solution in \( L^{2,\beta}(A) \times L^{2,\beta}(p) \times I^2 \).

**Proof.** This is proven exactly as the case with also a Brownian motion. First, we show as in 3.1 the solution lies in \( L^{2,\beta}(A) \times L^{2,\beta}(p) \times I^2 \) and, using Itô’s formula, that it is unique. Next we build a contraction on this space, and obtain existence and uniqueness when the generator depends on \((Y, U)\).

**Remark 4.1.** A similar result does not hold in general in \( L^1 \). Counter examples are given in [10], where additional hypotheses are then added to obtain an existence and uniqueness result. We also refer to [8] where the case \( L^p \) is analysed.

\[ A \quad \text{Some remarks on the Snell envelope theory} \]

The Snell envelope theory has been treated in various works. [13] considers the case for a positive process without any restrictions on the filtration, obtaining general results. For a bit less general results, but still enough for our work, [26] develops the theory for non-negative càdlàg processes, while [33] treats the case where the process is càdlàg and left continuous over stopping times, and satisfies the condition

\[
\mathbb{E} \left[ \sup_t |\eta_t| \right] < \infty. \tag{30}
\]

The recent work [27] treats the subject in the framework of family of random variables indexed by stopping times, using quite general assumptions. In the following, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) be a filtration satisfying the usual conditions. Let \( \eta \) be a càdlàg process. Several properties
that hold for positive processes can be shown under the condition $\text{(30)}$, as we will see in proposition A.1. We recall the following definition:

**Definition A.1.** An optional process $R$ of class $[D]$ is said to be regular if $R_t = p R_t$ for any $t < T$, where $p X$ indicates the predictable projection.

**Proposition A.1:** Let $\eta$ be a càdlàg process satisfying (30). Define

$$R_t = \text{ess sup}_{\tau \in T} E[\eta_\tau | \mathcal{F}_t]$$

(31)

It holds that

i) $R_t$ is the Snell envelope of $\eta_t$. This means it is the smallest càdlàg super-martingale that dominates $\eta$, i.e. $R_t \geq \eta_t$ for all $t$ $\mathbb{P}$-a.s.

ii) A stopping time $\tau^*$ is optimal in (31) (i.e. $R_t = E[\eta_{\tau^*} | \mathcal{F}_t]$) if and only if one of the following conditions hold

- $R_{\tau^*} = \eta_{\tau^*}$ and $R_{s \wedge \tau^*}$ is a $\mathbb{F}$-martingale
- $E[R_t] = E[\eta^*_t]$  

iii) $R_t$ is of class $[D]$, hence it admits decomposition

$$R_t = M_t - K_t,$$

where $M$ is a martingale, $K$ a predictable increasing process with $K_0 = 0$. $K$ can be decomposed as $K = K^c_t + K^d_t$, where $K^c$ indicates the continuous part and $K^d$ the discontinuous part. Moreover we have, a.s.

$$\{t : \Delta K_t > 0\} \subset \{t : R_t = \eta_t\}$$

or equivalently, $\Delta K_t = \Delta K_t \mathbb{1}_{(R_\tau)_t = \eta_t}$, $t \geq 0$.

iv) If the process $R_t$ is regular in the sense that $R_t = p R_t$, where $p R$ indicates the predictable projection, defining the stopping time

$$D_t^* = \inf\{s \geq t : R_s \neq M_s\},$$

then $D_t^*$ is an optimal stopping time and it is in fact the largest optimal stopping time.

**Proof.** Define

$$I = \inf_{t \in [0, T]} \eta_t$$

and since $\eta_t - I \geq 0$ for all $t$, we have $\eta_t - N_t \geq 0$ for all $t$. $N_t$ is a uniformly integrable martingale thanks to (30). Consider $\tilde{\eta} = \eta_t - N_t \geq 0$ and $\tilde{R}_t = R_t - N_t$. Notice that then

$$\tilde{R}_t = R_t - N_t = \text{ess sup}_{\tau \in T} E[\eta_\tau | \mathcal{F}_t] = \text{ess sup}_{\tau \in T} E[\tilde{\eta}_\tau | \mathcal{F}_t],$$

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i.e. \( \tilde{R} \) is the Snell envelope of the positive process \( \tilde{\eta} \). \( R \) inherits all the properties from \( \tilde{R} \). Let us see why the fourth property holds, as the rest are obtained similarly. If \( R_t \) is regular, so is \( \tilde{R}_t \) because we are adding a uniformly integrable martingale, which is regular (all uniformly quasi-left-continuous integrable càdlàg martingales are regular, see [24] Def 5.49). The result then holds by [13] pag 140.

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