REGULARITY OF PROJECTION OPERATORS ATTACHED TO WORM DOMAINS

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ABSTRACT. We construct a projection operator on an unbounded worm domain which maps subspaces of $W^s$ to themselves. The subspaces are determined by a Fourier decomposition of $W^s$ according to a rotational invariance of the worm domain.

INTRODUCTION

Our work is on the non-smooth unbounded worm domains
$$D_\beta = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re} \left( z_1 e^{-i \log z_2} \right) > 0, |\log z_2| < \beta - \pi/2 \} \quad \beta > \pi/2.$$ On a bounded version of the domains $D_\beta$, given by
$$\Omega_c = \{(z_1, z_2) : |z_1 + e^{i \log z_2}|^2 < 1, |\log z_2| < \beta - \pi/2 \},$$
C. Kiselman showed the failure of the Bergman projection to preserve $C^\infty(\Omega_c)$ [3]. The model domains, $D_\beta$, were important in [1], where the first author used them in his construction of a counterexample to regularity of the Bergman projection on a smoothly bounded pseudoconvex domain. In a detailed analysis of the Bergman kernel, Krantz and the third author, in [4], studied the $L^p$ mapping properties of the Bergman projection on $D_\beta$, obtaining the exact range of values of $p$ for which the mapping is bounded.

In this article we look at regularity in terms of Sobolev spaces. We denote by $W^s(D_\beta)$ the space of functions whose derivatives of order $\leq s$ are in $L^2(D_\beta)$. The first author’s counterexample on smooth domains relied on the fact (proved in the same paper) that the Bergman projection on the model domain, $D_\beta$, fails to map $W^s(D_\beta)$ to $W^s(D_\beta)$ for large enough $s$ [1]. More precisely, the failure to preserve Sobolev spaces was proved on subspaces (defined as $W^s_j(D_\beta)$ below). The question remained whether there exists another (oblique) projection operator which preserves the Sobolev spaces. We construct such an operator in the present article.

We now state our main result. From the rotational invariance of $D_\beta$ with respect to the rotations, $\rho_\theta(z) = (z_1, e^{i\theta} z_2)$, we can decompose the Bergman space $B(D_\beta) = L^2(D_\beta) \cap \mathcal{O}(D_\beta)$ by
$$B(D_\beta) = \bigoplus_{j \in \mathbb{Z}} B_j(D_\beta),$$
where $B_j(D_\beta)$ consists of functions $f \in B(D_\beta)$ satisfying $f \circ \rho_\theta \equiv e^{ij\theta} f$. The space $L^2(D_\beta)$ admits a similar decomposition into subspaces $L^2_j(D_\beta)$, and we can define $W^s_j(D_\beta) = L^2_j(D_\beta) \cap W^s(D_\beta)$.

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Our main theorem is grounded on adjustments to factors which imply the obstruction to regularity of the Bergman projection on worm domains. The Bergman kernel for each space, $B_j(D_\beta)$, is explicitly calculated and expressed as an integral in the form:

$$K_j(z, w) = \frac{1}{2\pi^2} \frac{(\xi - i^{j+1})\xi}{z_1^{\xi-1}w_1^{\xi-1}} \int_\mathbb{R} \frac{\sinh[(2\beta - \pi)(\xi - i^{j+1})]}{\sinh[2\pi\xi]} \frac{\sinh[\pi(\xi - i^{j+1})]}{\sinh[\pi(\xi - i^{j+1})]} \, d\xi.$$  

Using the residue calculus, one can compute an asymptotic expansion of the kernel (see [1]). The poles corresponding to non-integer multiples of $i$ of the kernel lead to non-integer powers of $z_1$ and $w_1$ which ultimately lead to the obstruction of regularity of the operator.

We construct a kernel which, when added to the Bergman kernel, eliminates all such poles, and in this way we successfully remove the obstruction to regularity of the Bergman projection on the model domains, $D_\beta$, and construct new projections which preserve the spaces $W^s_j(D_\beta)$:

**Main Theorem.** Let $\beta > \pi/2$, and $D_\beta$ be defined as above. For all $j \in \mathbb{Z}$ there exists a bounded linear projection

$$T_j : L^2(D_\beta) \to B_j(D_\beta)$$

which satisfies

$$T_j : W^s(D_\beta) \to W^s_j(D_\beta)$$

for every $s \geq 0$.

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1. **The Bergman projection on $D_\beta$**

Following [1], we introduce the domains

$$D'_\beta = \left\{(z_1, z_2) \in \mathbb{C}^2 : |\text{Im } z_1 - \log z_2\overline{z_2}| < \frac{\pi}{2}, |\log z_2\overline{z_2}| < \beta - \frac{\pi}{2}\right\}$$

to aid in our study of the Bergman kernels on $D_\beta$. $D'_\beta$ is related to $D_\beta$ via the biholomorphic mapping

(1.1) \hspace{1cm} \Psi : D'_\beta \to D_\beta \hspace{1cm} (z_1, z_2) \mapsto (e^{z_1}, z_2).

Let $K_{D_\beta}(z, w)$ be the Bergman kernel for $D_\beta$, and $K_j(z, w)$ the reproducing kernel for $B_j(D_\beta)$; we have the relation

$$K_{D_\beta}(z, w) = \sum_j K_j(z, w).$$

We calculate $K_j(z, w)$ using Fourier transforms as in [1].

Let $S_\beta$ denote the strip

$$S_\beta := \{z = x + iy \in \mathbb{C} : |y| < \beta\},$$

and let $\omega_j(y)$ be the continuous bounded function on the interval $I_\beta := \{y : |y| < \beta\}$, given by $\omega_j = \pi (e^{i(j+1)\pi} \chi_{\beta-\pi/2}) \ast \chi_{\pi/2}$, where for $a > 0$, $\chi_a := \chi(-a, a)$, the
characteristic function of the interval \((-a, a)\). We denote by \(\| \cdot \|_{\omega_j}\) the \(L^2(S_\beta)\)-norm weighted with the function \(\omega_j\):

\[
\| f \|_{\omega_j} := \left( \int_{S_\beta} |f(x, y)|^2 \omega_j(y) dxdy \right)^{1/2}.
\]

We further define the weighted Bergman spaces on the strip \(S_\beta\) by

\[B_{\omega_j} = \{ f \text{ holomorphic on } S_\beta : \| f \|_{\omega_j}^2 < \infty \}.
\]

For \(f \in B_{\omega_j}\),

\[
\hat{f}(\xi, y) = \int_\mathbb{R} f(x + iy)e^{-ix\xi} dx
\]
satisfies

\[(1.2) \quad \hat{f}(\xi, y) = e^{-y\xi} \hat{f}_0(\xi),
\]

where \(\hat{f}_0(\xi) = \hat{f}(\xi, 0)\).

Here and throughout we use the notation for complex variables

\[z_1 = x + iy, \quad w_1 = x' + iy'.\]

Define

\[k'_j(\xi, y, w_1) = \frac{1}{\hat{\omega}_j(-2i\xi)} e^{i(y - w_1)\xi},\]

where \(\hat{\omega}_j\) refers to the Fourier-Laplace transform of \(\omega_j\), and satisfies

\[(1.3) \quad \hat{\omega}_j(-2i\xi) = \frac{\pi \sinh \left[ (2\beta - \pi)(\xi - \frac{j+1}{2}) \right]}{\sinh \left( \frac{j+1}{2} \right) \xi}.
\]

We note that \(\hat{\omega}_j\) extends to an entire function. We claim that \(k'_j\) corresponds to the kernel for the orthogonal projection on \(D'_\beta\) according to the following lemma:

**Lemma 1.1.** Let \(K'_j(z, w)\) denote the reproducing kernel of the space \(B_j(D'_\beta)\). Then

\[K'_j(z, w) = \frac{1}{2\pi^2 z_1^2 w_1^2} \int_\mathbb{R} \frac{(\xi - \frac{j+1}{2})\xi}{\sinh \left[ (2\beta - \pi)(\xi - \frac{j+1}{2}) \right]} \sinh \pi\xi e^{i(z_1 - w_1)\xi} d\xi.
\]

**Proof.** Let \(\Gamma : B_1 \rightarrow B_2\) be a surjective isometry of two Bergman spaces. Let \(K_1(z, w)\) be the reproducing kernel of the space \(B_1\) and \(K_2(z, w)\) the kernel for \(B_2\). Then

\[(1.4) \quad K_2(z, w) = \Gamma \cdot K_1(z, w).
\]

We now apply (1.4) to the spaces \(B_1 = B_{\omega_j}\) and \(B_2 = B_j(D'_\beta)\). From \(\Gamma\)

\[K_j(z_1, w_1) = \frac{1}{2\pi} \int_\mathbb{R} k'_j(\xi, y, w_1)e^{ix\xi} d\xi
\]
is the reproducing kernel for \(B_{\omega_j}\), and

\[\Gamma : B_{\omega_j} \rightarrow B_j(D'_\beta)
\]

\[f(z_1) \mapsto z_1^j f(z_1)
\]
is the isometry between Bergman spaces. Thus, by (1.4)
\[ K_j'(z, w) = \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{R}} k_j(\xi, y, w_1)e^{iz\xi}d\xi \]
from which the lemma follows. \(\square\)

2. Improving the Bergman projection

Crucial to the proof in [1] of the failure of the Bergman projection to preserve \(W^s(D_\beta)\) is the existence of poles of \(k_j'(\xi, y, w_1)\) in the \(\xi\) variable whose imaginary part is a non-integer multiple of \(i\). We see from (1.3) that such poles of \(k_j'(\xi, y, w_1)\) are due to the zeros of \(\hat{\omega}_j(-2i\xi)\) at \((j + 1)/2 + ik\pi/(2\beta - \pi)\) for \(k\) a non-zero integer. In this section we deal with this obstruction by adding a correction term which eliminates such poles.

We assume initially that \(j = -1\). The goal in this section is to find a function, denoted by \(\hat{h}(\xi, y)\), defined in \(\mathbb{C} \times I_\beta\) such that \(\hat{h}(\xi, y)\), cancels the poles of the function
\[
(2.1) \frac{1}{\omega_{-1}(2i\xi)}e^{-\xi y}
\]
at \(\xi = ik\nu_\beta\), for \(k\) a non-zero integer, and \(\nu_\beta = \pi/(2\beta - \pi)\). The function \(\hat{h}(\xi, y)\) will have an inverse transform which is orthogonal to \(B_{\omega_{-1}}\) and satisfy certain \(L^2\) estimates which will be used in Section 3 to construct an integral operator.

To ease notation we set
\[
\tau_k(\xi) = (-1)^k e^{-2k^2\nu_\beta^2} \frac{\xi^2}{(2\beta - \pi)\pi \sinh(\pi\xi)} e^{-\xi^2} \quad k \in \mathbb{Z}.
\]
We define
\[
(2.2) \quad \hat{h}_k(\xi, y) = \frac{\tau_k(\xi)e^{(\xi - 2ik\nu_\beta)y}}{\xi - ik\nu_\beta}.
\]
We note that the pole of (2.1) at \(\xi = ik\nu_\beta\), for \(k\) a non-zero integer is the same as the pole of \(\hat{h}_k\). Our aim is to sum \(\hat{h}_k\) over \(k\) in order to produce a function which will be used to eliminate all such poles of (2.1). The following proposition shows that we can sum over \(k\).

To keep track of the poles, we introduce the set \(P\) of all poles:
\[
P := \{ik\nu_\beta : k \neq 0\} \cup \{ik : k \neq 0\}.
\]

Proposition 2.1. Let \(\hat{h}_k(\xi, y)\) be defined as above. The infinite sum
\[
\sum_{k \neq 0} \hat{h}_k(\xi, \cdot)
\]
converges in \(L^\infty(I_\beta)\) to a function \(\hat{h}(\xi, \cdot)\) uniformly in \(\xi\) on compact subsets of \(\mathbb{C} \setminus P\).

Let \(B_r = \bigcup B(ik\nu_\beta; r)\) denote the union of balls centered at elements of \(P\) for some fixed radius \(r > 0\). Let \(U\) be any neighborhood of \(P\) containing \(B_r\). Then on \(\mathbb{C} \setminus U \times I_\beta\)
\[
(2.3) \quad |\hat{h}(\xi, y)| \lesssim |\xi|^2 e^{-Re\xi^2} e^{(\beta - \pi)Re\xi},
\]
with the constant of inequality depending only on \(U\).
Proof:

\[ \sum_{k \neq 0} \hat{h}_k(\xi, y) = \sum_{k \neq 0} \frac{\tau_k(\xi)e^{(\xi-2ik\nu\beta)y}}{\xi - ik\nu\beta} \]

is a sum of terms of the form

\[ e^{\xi y} \sum_{k \neq 0} a_k(\xi) \]

where

\[ |a_k(\xi)| \lesssim \frac{1}{k} e^{-k^2\nu^2|\xi|^2} e^{-Re\xi^2} e^{-\pi|Re\xi|} \quad k \neq 0. \]

Inequality (2.4) is then a consequence of

\[ |\hat{h}(\xi, y)| = \left| e^{\xi y} \sum_{k} a_k(\xi) e^{-2ik\nu\beta y} \right| \]

\[ \lesssim e^{\beta Re\xi} \sum_{k} |a_k(\xi)| \]

\[ \lesssim |\xi|^2 e^{-Re\xi^2} e^{(\beta-\pi)|Re\xi|}. \]

We note for \( f \in B_{\omega^{-1}} \):

\[ \int_{\mathbb{R}} \int_{I_\beta} \hat{h}(\xi, y) \hat{f}(\xi, y) \omega_{-1}(y) dy d\xi = \int_{\mathbb{R}} \int_{I_\beta} \hat{h}(\xi, y) e^{-y\xi} \hat{f}_0(\xi) \omega_{-1}(y) dy d\xi \]

\[ = \int_{\mathbb{R}} \frac{\tau_k(\xi)}{\xi + ik\nu\beta} \hat{f}_0(\xi) \left[ \int_{I_\beta} e^{2ik\nu\beta y} \omega_{-1}(y) dy \right] d\xi \]

\[ = 0, \]

where we use the representation of \( f \) in (1.2) in the first step, and the fact that \( \int_{I_\beta} e^{2ik\nu\beta y} \omega_{-1}(y) dy = \hat{\omega}_{-1}(-2k\nu\beta) = 0 \) in the last.

We collect the essential properties, which follow directly from the above, of the kernel function \( h(x, y) \) in the following theorem:

**Theorem 2.2.** There exists \( h(x, y) \in L^2_{\omega_{-1}}(S_\beta) \) with the following properties:

(i) For each \( y \in I_\beta \), the poles of

\[ \hat{h}(\xi, y) + \frac{1}{\omega_{-1}(2i\xi)} e^{-\xi y} \]

with respect to \( \xi \) lie at only integer multiples of \( i \).

(ii) The kernel given by

\[ \mathcal{H}'(z, w) = \frac{1}{2\pi} \frac{1}{z_2 w_2} \int_{\mathbb{R}} \hat{h}(\xi, y) e^{i(x-w_1)\xi} d\xi \]

is orthogonal to the space \( B_{-1}(D'_\beta) \) in the sense that \( \mathcal{H}'(\cdot, w) \perp B_{-1}(D'_\beta) \).

(iii) Let \( U \) be any neighborhood of \( P \) containing \( B_r \) for some \( r > 0 \). Then on \( \mathbb{C} \setminus U \times I_\beta \)

\[ |\hat{h}(\xi, y)| \lesssim |\xi|^2 e^{-Re\xi^2} e^{(\beta-\pi)|Re\xi|}, \]

with the constant of inequality depending only on \( U \).
We also denote the horizontal strips
\[ S_t = \{ \mathbb{R} + it \} \]
for \( t \in \mathbb{R} \). From the Theorem 2.2 iii), we have in particular, on any given \( S_t \) such that \( S_t \cap P = \emptyset \), \( \hat{h}(\xi, y) \) satisfies the following estimates uniformly, i.e. with constant of inequality independent of \( \xi \):
\[
\int_{I_\beta} \left| \hat{h}(\xi, y) \right|^2 dy \lesssim |\xi|^4 e^{-2\text{Re}\xi^2} e^{2(\beta-\pi)|\text{Re}\xi|}.
\]

3. Mapping properties

We begin this section with some integral estimates for our constructed correction term. We let \( H'(z, w) \) be as in Theorem 2.2. Due to the \( \frac{1}{z} \) factor in \( H'(z, w) \), the operator determined by the kernel, \( H'(z, w) \), will have its action restricted to the \( L^2(D_\beta') \) component of a given function in \( L^2(D_\beta') \).

We use the equivalence between Bergman spaces given in Lemma 1.1 in the proof of the next proposition: for \( G \in B^{-1}_-(D_\beta') \), \( G \) is of the form \( G = g(z_1) \frac{1}{z^2} \), where \( g \in B_{\omega-1} \), and \( \| G \|_{B^{-1}_-(D_\beta')} = \| g \|_{B_{\omega-1}} \).

**Proposition 3.1.** Let \( \beta > \pi/2 \), and \( H' \) be the integral operator
\[
H'f(w) = \int_{D_\beta'} f(z) \overline{H'(z, w)} dV(z),
\]
where
\[
H'(z, w) = \frac{1}{2\pi^2 z_2 \overline{w}_2} \int_{\mathbb{R}} \hat{h}(\xi, y) e^{i(x-w_1)\xi} d\xi.
\]
Then
\[
H' : L^2(D_\beta') \rightarrow B^{-1}_-(D_\beta'),
\]
and
\[
\| H'f \|_{B^{-1}_-(D_\beta')} \lesssim \| f \|_{L^2(D_\beta')}.
\]

**Proof.** We write \( D_\beta' = \mathbb{R} \times d_\beta' \), where
\[
d_\beta' = \{ (y, z_2) \in \mathbb{R} \times \mathbb{C} : |y - \log z_2 \overline{z}_2| < \pi/2, |\log z_2 \overline{z}_2| < \beta - \pi/2 \}.
\]
Then,
\[
H'f(w) = \frac{1}{2\pi} \frac{1}{w_2} \int_{D_\beta'} \frac{1}{z_2} \int_{\mathbb{R}} \hat{h}(\xi, y) e^{-i(x-w_1)\xi} d\xi f(z) dV(z)
\]
\[
= \frac{1}{2\pi} \frac{1}{w_2} \int_{d_\beta'} \frac{1}{z_2} \int_{\mathbb{R}} \hat{h}(\xi, y) e^{-i(x-w_1)\xi} f(x, y, z_2) dx d\xi dy dV(z_2)
\]
\[
= \frac{1}{2\pi} \frac{1}{w_2} \int_{d_\beta'} \frac{1}{z_2} \int_{\mathbb{R}} \hat{h}(\xi, y) e^{iw_1\xi} \hat{f}(\xi, y, z_2) d\xi dy dV(z_2).
\]
Also, 
\[
\|H^f\|_{B_{-1}(D_\beta')}^2 = \frac{1}{4\pi^2} \left\| \int_{d_\beta} \frac{1}{2} \int_{z_2} \hat{h}(\xi, \eta)e^{i(\xi, \eta)\xi} \hat{f}(\xi, \eta, z_2)d\xi dV(z_2) \right\|_{B_{-1}}^2
\]
\[
= \frac{1}{4\pi^2} \int_{d_\beta} \int_{R} \frac{1}{2} \int_{R} \hat{h}(\xi, \eta)e^{-\eta'\xi}e^{i\eta')\xi} \hat{f}(\xi, \eta, z_2)d\xi dV(z_2) d\xi' d\omega_{-1}(\xi')d\eta'
\]
\[
= \frac{1}{4\pi^2} \int_{d_\beta} \int_{R} \int_{d_\beta} \frac{1}{2} \hat{h}(\zeta, \eta)e^{-\eta'\zeta} \hat{f}_{-1}(\zeta, \eta, z_2)d\eta dV(z_2) d\zeta d\omega_{-1}(\zeta')d\eta'
\]
\[
\lesssim \int_{d_\beta} \int_{R} \left( \int_{d_\beta} \hat{h}(\zeta, \eta)^2 d\omega_{-1}(\eta) \right) \times \left( \int_{d_\beta} \hat{f}_{-1}(\zeta, \eta, z_2)^2 d\eta dV(z_2) \right) e^{-2\eta'\zeta} d\zeta d\omega_{-1}(\zeta')d\eta'.
\]

In the third step above we use the fact that integrating over \(z_2\) kills off all terms of \(f\) in the decomposition of
\[
f(z) = \sum_j f_j(z), \quad f_j(z) \in L^2(D_\beta')
\]
except \(f_{-1}(z)\).

From Theorem 2.2 (iii) and (2.4) we have that 
\[
\int_{d_\beta} \hat{h}(\zeta, \eta)^2 \omega_{-1}(\eta) d\eta \lesssim \zeta^4 e^{-2Re\zeta^2} e^{2(\beta-\tau)|Re\zeta|}.
\]

We continue with our estimate of \(\|H^f\|_{B_{-1}(D_\beta')}^2\):
\[
\|H^f\|_{B_{-1}(D_\beta')}^2 \lesssim \int_{R} \int_{d_\beta} \hat{f}_{-1}(\zeta, \eta, z_2)^2 d\eta dV(z_2) \zeta^4 e^{-2\zeta^2} e^{2(\beta-\tau)|\zeta|} \omega_{-1}(-2i\zeta) d\zeta
\]
\[
\lesssim \|f_{-1}\|_{L^2(D_\beta')}^2,
\]
where the last estimate follows by the fact that the term \(\zeta^4 e^{-2\zeta^2} e^{2(\beta-\tau)|\zeta|} \omega_{-1}(-2i\zeta)\) is bounded with respect to \(\zeta\). \(\square\)

We recall the biholomorphic mapping \(\Psi : D_\beta' \to D_\beta\) from [11]. We define the kernel
\[
(3.1) \quad \mathcal{H}(z, w) = \frac{1}{z_1 \overline{w}_1} \mathcal{H}'(\Psi^{-1}z, \Psi^{-1}w).
\]

Let \(H\) be the integral operator
\[
Hf(w) = \int_{D_\beta} f(z) \overline{\mathcal{H}(z, w)} dV(z),
\]
where \(\mathcal{H}(z, w)\) is given by (3.1).

Then as a result of Proposition 3.1 we have the following

**Corollary 3.2.** We have that
\[
H : L^2(D_\beta) \to B_{-1}(D_\beta),
\]
and
\[ \|Hf\|_{B^{-1}(D_\beta)} \lesssim \|f\|_{L^2_1(D_\beta)}. \]

We now define the projection operator \( T_{-1} \) as
\[ T_{-1} = P_{-1} + H, \]
where \( P_{-1} : L^2(D_\beta) \to B^{-1}(D_\beta) \) is the orthogonal projection operator.

4. Properties of the projection \( T_{-1} \)

**Theorem 4.1.** Let \( \beta > \pi/2 \) and \( T_{-1} = P_{-1} + H \). Then
\[ T_{-1} : L^2(D_\beta) \to B^{-1}(D_\beta). \]
Furthermore, \( T_{-1} \) is a projection, and has the regularity property
\[ (4.1) \quad T_{-1} : W^k(D_\beta) \to W^{-k}_{-1}(D_\beta) \quad \forall k, \]
and
\[ \|T_{-1}f\|_{W^k_{-1}(D_\beta)} \lesssim \|f\|_{W^k(D_\beta)}. \]

**Proof.** The mapping from \( L^2(D_\beta) \) to \( B^{-1}(D_\beta) \) follows from the corresponding properties of \( P_{-1} \) and \( H \) (see Corollary 3.2).

That \( T_{-1} \) is a projection follows from \( P_{-1} \) being a projection and from the restriction of \( H \) to \( B^{-1}(D_\beta) \) being equivalently 0 (from Theorem 2.2(ii)):
\[ T_{-1}^2 = P_{-1}^2 + P_{-1}H + HP_{-1} + H^2 = P_{-1} + H = T_{-1}. \]

Since \( T_{-1}f \) is holomorphic, to prove \( 4.1 \) we estimate the \( L^2 \) norm of holomorphic derivatives of \( T_{-1}f \). Also, since \( T_{-1}f \) is of the form \( g(w_1)w_2^{-1} \) for \( g \in B_{\omega_{-1}} \), we only need to estimate the derivatives with respect to the first variable. To prove the theorem we thus show
\[ (4.2) \quad \left\| \frac{\partial^k}{\partial w_1^k} T_{-1}f \right\|_{L^2(D_\beta)} \lesssim \|f\|_{W^k(D_\beta)}. \]

The domain \( D_{\beta}' \) is related to \( D_\beta \) via the biholomorphic mapping \( \Psi \). We can then read off the kernels attached to the domain \( D_{\beta}' \) from the transformation formula applied to the corresponding kernels on \( D_{\beta}' \), as in (3.1). We have the relations
\[ K_{-1}(z, w) = \frac{1}{z_1 w_1} K'_{-1}(\Psi^{-1}z, \Psi^{-1}w) \]
\[ H(z, w) = \frac{1}{z_1 w_1} H'(\Psi^{-1}z, \Psi^{-1}w) \]
\[ T_{-1}(z, w) = \frac{1}{z_1 w_1} T'_{-1}(\Psi^{-1}z, \Psi^{-1}w), \]
where \( K_{-1}, H, T_{-1} \) (resp. \( K'_{-1}, H', T'_{-1} \)) are the kernels for, respectively, \( P_{-1}, H, T_{-1} \) (resp. \( P'_{-1}, H', T'_{-1} \)).

We relate \( \frac{\partial^k}{\partial w_1^k} T_{-1}f \) to \( D^k f \), where \( D^k f \) will be used to denote any \( k^{th} \) order tangential differential operator.
From above, we have
\[ T_{-1}f(w) = \int_{D_{\beta}} \overline{T_{-1}(z, w)} f(z) dV(z), \]
where
\[ \overline{T_{-1}(z, w)} = \frac{1}{2\pi i z_2 w_2} \int_{\mathbb{R}} \frac{1}{\omega_{-1}(2i\xi)} \frac{1}{z_1} w_1^{i\xi-1} \]
\[ + \hat{h}(\xi, (\log z_1 - \log z_1)/2i) z_1^{-i\xi/2-1} w_1^{i\xi/2} \]
\[ d\xi. \]
By virtue of the factor \( z_2^{-1} \) in \( T_{-1}(z, w) \), all action is isolated on \( f_{-1}(z) \). Thus,
\[ T_{-1}f(w) = \int_{D_{\beta}} \overline{T_{-1}(z, w)} f(z) dV(z) \]
\[ = \int_{D_{\beta}} \overline{T_{-1}(z, w)} f_{-1}(z) dV(z). \]
Furthermore,
\[ (4.3) \quad \frac{\partial^k}{\partial w_1^k} T_{-1}f = \int_{D_{\beta}} \frac{\partial^k}{\partial w_1^k} \overline{T_{-1}(z, w)} f_{-1}(z) dV(z), \]
and
\[ \frac{\partial^k}{\partial w_1^k} \overline{T_{-1}(z, w)} = \frac{1}{2\pi i z_2 w_2} \int_{\mathbb{R}} \frac{1}{\omega_{-1}(2i\xi)} \frac{1}{z_1} w_1^{i\xi-1} \]
\[ + \hat{h}(\xi, (\log z_1 - \log z_1)/2i) z_1^{-i\xi/2-1} w_1^{i\xi/2} \]
\[ d\xi. \]
Our strategy is roughly as follows: we use shifts of contours of integration to write the integrands of (4.3) using derivatives with respect to \( z_1 \); we make sure Fubini’s theorem applies with respect to the \( z \) and \( \xi \) integrals and then we take the \( z_1 \) derivatives outside the \( \xi \) integrals; finally we can then perform an integration by parts in the \( z_1 \) variable in (4.3).
When shifting the contour of integration, in order to verify that Fubini’s theorem applies, we work with the two cases, each of which determines a different direction of shift:

\( i \) \(|w_1| < |z_1|\)
ii \( |z_1| < |w_1|\).
To illustrate the cases, we consider integrals of the form
\[ \phi_1(w_1) = \int_U \int_{\mathbb{C}} \sigma_{w_1}(\xi, z_1, \overline{z_1}) f_{-1}(z) d\xi dV(z), \]
where \( \sigma_{w_1} \) will be either
\[ (i\xi - 1)(i\xi - 2) \cdots (i\xi - k) \cdot \frac{1}{\omega_{-1}(2i\xi)} \frac{1}{z_1 w_1^{k+1}} \]
\[ \cdot \left( \frac{w_1}{z_1} \right)^i \]
therefore deform the contour of integration in (4.4) to
the sides of the contour are null due to the exponential decay in
\( \xi \) be shown in both cases we are lead to the same expression.

\[ \text{We use } (4.5) \]

We see Fubini’s theorem applies in case \( t < 0 \) and in case \( t > 0 \) t functions of derivatives with respect to the variable, corresponding to the two cases. It will
be shown in both cases we are lead to the same expression.

We now proceed to the write an expression for the kernel \( \partial \) for \( \hat{h} \), we have

\[ |\phi_t(w_1)| \lesssim \int_U \left| \frac{1}{z_2} \frac{1}{w_1 w_1^k + 1} \right| \left( \frac{|z_1|}{|w_1|} \right)^{\beta-1} |f_{-1}(z)| dV(z). \]

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We now work with the contour of integration in (4.4) deformed to \( \mathbb{R} - ik \). We therefore deform the contour of integration in \( 1 \) to \( \mathbb{R} - ik \). The contribution of the sides of the contour are null due to the exponential decay in \( \xi \) of the integrand.

We now work with the contour of integration in \( 1 \) deformed to \( \mathbb{R} - ik \). We first consider

\[ (i\xi + k - 1)(i\xi + k - 2) \cdots (i\xi) \frac{1}{\omega_{-1}(2i(\xi - ik))} \zeta_1^{-i\xi - k - 1} w_1^{i\xi - 1}. \]

We use

\[ \frac{1}{\omega_{-1}(2i(\xi - ik))} = (-1)^k \frac{1}{\pi} \frac{(\xi - ik)^2}{\sinh((2\beta - \pi)(\xi - ik)) \sinh \pi \xi}, \]

hence

\[ \frac{(i\xi + k - 1)(i\xi + k - 2) \cdots (i\xi)}{\omega_{-1}(2i(\xi - ik))} \zeta_1^{-i\xi - k - 1} w_1^{i\xi - 1} \]

\[ = (-1)^k \frac{1}{\pi} \frac{(\xi - ik)(\xi - ik)}{\sinh((2\beta - \pi)(\xi - ik)) \sinh \pi \xi} \times \]

\[ (i\xi + k - 1)(i\xi + k - 2) \cdots (i\xi) \zeta_1^{-i\xi - k - 1} w_1^{i\xi - 1} \]

\[ = (-1)^k \frac{1}{\pi} \frac{(\xi - ik)(\xi - ik)}{\sinh((2\beta - \pi)(\xi - ik)) \sinh \pi \xi} \times \]

\[ (\xi - ik)(i\xi + k - 1) \cdots (i\xi + 1) \zeta_1^{-i\xi - k - 1} w_1^{i\xi - 1} \]

\[ = \frac{1}{\pi} \frac{(\xi - ik)(\xi - ik)}{\sinh((2\beta - \pi)(\xi - ik)) \sinh \pi \xi} \frac{\partial^k}{\partial \xi^k} \zeta_1^{-i\xi - k - 1} w_1^{i\xi - 1}. \]

Similarly, we work with

\[ (4.6) \frac{1}{2\pi} \frac{1}{z_2 w_2} \int_{\mathbb{R} - ik} (i\xi - 1)(i\xi - 2) \cdots (i\xi - k) \hat{h}(\xi, (\log z_1 - \log \pi_1)/2i) \times \]

\[ \zeta_1^{-i\xi - k - 1} w_1^{i\xi - k - 1} d\xi. \]

Let us write

\[ \hat{h}(\xi, (\log z_1 - \log \pi_1)/2i) = \frac{\xi}{\sinh(\pi \xi)} g(\xi, z_1), \]
and note that $g(\xi, z_1)$ has the property

$$\Lambda_t g(\xi, z_1) = 0,$$

where

$$\Lambda_t := \left( \frac{z_1}{\overline{z}_1} \right)^{1/2} \frac{\partial}{\partial z_1} + \left( \frac{\overline{z}_1}{z_1} \right)^{1/2} \frac{\partial}{\partial \overline{z}_1}.$$

The integral in (4.6) can thus be written according to

$$\frac{1}{2\pi} \frac{1}{\overline{\Xi}_{2W}} \int_{\mathbb{R}} \left[ (i\zeta + k - 1)(i\zeta + k - 2) \cdots (i\zeta) \hat{h}(\zeta - ik, (\log z_1 - \log \overline{z}_1)/2i) \times \right.$$

$$z_1^{(-i\zeta - k)/2} \overline{z}_1^{(-i\zeta - k)/2} w_1^{i\zeta - 1} \left. \right] d\zeta$$

$$= \frac{(-1)^k}{2\pi} \frac{1}{\overline{\Xi}_{2W}} \int_{\mathbb{R}} \left[ \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - ik, z_1) \times \right.$$

$$(i\zeta + k)(i\zeta + k - 1) \cdots (i\zeta + 1) z_1^{(-i\zeta - k)/2} \overline{z}_1^{(-i\zeta - k)/2} w_1^{i\zeta - 1} \left. \right] d\zeta$$

$$= \frac{1}{2\pi} \frac{1}{\overline{\Xi}_{2W}} \int_{\mathbb{R}} \left[ \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - ik, z_1) (\Lambda_t)^k z_1^{(-i\zeta - k)/2} \overline{z}_1^{(-i\zeta - k)/2} w_1^{i\zeta - 1} \right] d\zeta.$$

Therefore,

$$\int_{|w_1| < |z_1|} \frac{\partial^k}{\partial w_1^k} \hat{T}_{-1}(z, w) f(z) dV(z) = -\frac{1}{2\pi} \frac{1}{\overline{\Xi}_{2W}} \times$$

$$\int_{\mathbb{R}} \left[ \frac{1}{\pi} \int_{\mathbb{R}} \frac{(\zeta - ik)\zeta}{\sinh(2\beta - \pi)(\zeta - ik)} \sinh(\pi \zeta) \frac{\partial^k}{\partial\zeta^k} z_1^{(-i\zeta - k)/2} \overline{z}_1^{(-i\zeta - k)/2} w_1^{i\zeta - 1} d\zeta \right. +$$

$$\left. \int_{\mathbb{R}} \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - ik, z_1) (\Lambda_t)^k z_1^{(-i\zeta - k)/2} \overline{z}_1^{(-i\zeta - k)/2} w_1^{i\zeta - 1} d\zeta \right] f_{-1}(z) dV(z).$$

We remark that, as outlined above, the $\zeta$ and $z$ integrations can be switched (just consider the integral $\phi_k$ in (4.5)).

Case ii). We begin by writing (4.4) in the form:

$$\frac{\partial^k}{\partial w_1^k} \hat{T}_{-1}(z, w) = \frac{1}{2\pi} \frac{1}{\overline{\Xi}_{2W}} \times$$

$$\int_{\mathbb{R}} \left[ \frac{(-1)^k}{\pi} \frac{(\xi + ik)(\xi)}{\sinh \left( (2\beta - \pi)(\xi) \right)} \sinh(\pi \xi) \frac{\partial^k}{\partial \xi^k} z_1^{(-i\xi + (k - 1)/2)} w_1^{i\xi - 1 - k} + \right.$$
which is also obtained by deforming the contour of integration to $R + ik$ (using that the sides of the contour give no contributions in the same manner as that of case $i$) of the following integral

$$\int \frac{\partial^k}{\partial w_1^k} T_{-1}(z, w) f(z) dV(z) =$$

$$- \frac{1}{2\pi} \int_{[w_1]} \frac{1}{\pi} \frac{\partial^k}{\partial w_1^k} \left[ \frac{\zeta(\zeta - ik)}{\pi \sinh((2\beta - \pi)(\zeta - ik)) \sin \pi \zeta} \frac{\partial^k}{\partial z_1} e^{-i\zeta-1} w_1^{i\zeta-1} \right] d\zeta,$$

noting that the contribution from the poles at integer multiples of $i$ are cancelled due to the differential operators.

Combining the results in cases $i$ and $ii$), we have

$$\int_{[w_1]} \frac{\partial^k}{\partial w_1^k} T_{-1}(z, w) f(z) dV(z) =$$

$$- \frac{1}{2\pi} \int_{[w_1]} \frac{1}{\pi} \frac{\partial^k}{\partial w_1^k} \left[ \frac{\zeta(\zeta - ik)}{\pi \sinh((2\beta - \pi)(\zeta - ik)) \sin \pi \zeta} \frac{\partial^k}{\partial z_1} e^{-i\zeta-1} w_1^{i\zeta-1} d\zeta + \int_{\mathbb{R}} \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - ik, z_1)(\Lambda t)^{k-z_1} e^{-i\zeta/2-1} w_1^{i\zeta/2} d\zeta \right)f_{-1}(z) dV(z)$$

and

$$\int_{[w_1]} \frac{\partial^k}{\partial w_1^k} T_{-1}(z, w) f(z) dV(z) =$$

$$- \frac{1}{2\pi} \int_{[w_1]} \frac{1}{\pi} \frac{\partial^k}{\partial w_1^k} \left[ \frac{\zeta(\zeta - ik)}{\pi \sinh((2\beta - \pi)(\zeta - ik)) \sin \pi \zeta} \frac{\partial^k}{\partial z_1} e^{-i\zeta-1} w_1^{i\zeta-1} d\zeta + \int_{\mathbb{R}} \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - ik, z_1)(\Lambda t)^{k-z_1} e^{-i\zeta/2-1} w_1^{i\zeta/2} d\zeta \right)f_{-1}(z) dV(z).$$

We now use Fubini’s theorem in both case $i$ and $ii$) to take the derivatives outside of the $\zeta$ integrals, and then combine the results above. Before doing so, we note

$$\Lambda t = \left( \frac{z_1}{\bar{z}_1} \right)^{1/2} \frac{\partial}{\partial z_1} + \left( \frac{\bar{z}_1}{z_1} \right)^{1/2} \frac{\partial}{\partial \bar{z}_1}$$

$$= \partial_{r_1},$$
where } r_1 = |z_1| \text{, is a tangential differential operator. Furthermore,}
\[(\Lambda_1)^* = -\partial_{r_1}.\]

For fixed } z_2 \text{ we note } z_1 \text{ can be written with coordinates } r_1, d_1 \text{ representing the distance to the boundary } \Re z_1 e^{-i \log z_2 z^*_2} = 0, \text{ via}
\[z_1 = (r_1 + id_1) e^{i\alpha}\]
where } \alpha = \log |z_2|^2 - \pi/2. \text{ Then it follows that}
\[
\frac{\partial}{\partial z_1} = \alpha_1 \frac{\partial}{\partial z_1} + \alpha_2 \frac{\partial}{\partial r},
\]
where } \alpha_1(|z_2|) \text{ and } \alpha_2(|z_2|) \text{ are bounded away from 0 and depend smoothly on } |z_2|.

We recall that that } g(\xi, z_1) \text{ has the property } \Lambda_1 g(\xi, z_1) = 0, \text{ and so }
\[g(\zeta - ik, z_1)(\Lambda_1)^k z_1^{-i\kappa/2 - 1} \pi^{-i\kappa/2} = (\Lambda_1)^k [z_1^{-i\kappa/2 - 1} \pi^{-i\kappa/2} g(\zeta - ik, z_1)]
= (\partial_{r_1})^k [z_1^{-i\kappa/2 - 1} \pi^{-i\kappa/2} g(\zeta - ik, z_1)],\]

We thus have, after commuting the } z \text{ derivatives with the } \zeta \text{ integrals,
\[
\int_{D_\beta} \frac{\partial^k}{\partial w_1^k} T_{\alpha}(z, w) f(z) dV(z) =
- \frac{1}{2\pi} \int_{D_\beta \times 2w_2} \left[ \frac{1}{\pi} \frac{\partial^k}{\partial z_1^k} \int_R \frac{(\zeta - ik)\zeta}{\sinh(2\beta - \pi)(\zeta - ik)} \sinh \pi \zeta^{-i\kappa-1} w_1^{i\kappa-1} d\zeta \right]
\]
\[\left( \partial_{r_1} \right)^k \int_R \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - ik, z_1) z_1^{-i\kappa/2 - 1} \pi^{-i\kappa/2} w_1^{i\kappa-1} d\zeta \right] f_{-1}(z) dV(z).\]

Integrating by parts in the first integral on the right in (4.7) gives
\[
- \frac{1}{2\pi^2} \int_{D_\beta} \frac{1}{z_2 w_2} \times
\left[ \frac{\partial^k}{\partial z_1^k} \int_R \frac{(\zeta - ik)\zeta}{\sinh(2\beta - \pi)(\zeta - ik)} \sinh \pi \zeta^{-i\kappa-1} w_1^{i\kappa-1} d\zeta \right] f_{-1}(z) dV(z)
= - \frac{1}{2\pi^2} \int_{D_\beta} \frac{1}{z_2 w_2} \times
\left[ (\alpha_2 \partial_{r_1})^k \int_R \frac{(\zeta - ik)\zeta}{\sinh(2\beta - \pi)(\zeta - ik)} \sinh \pi \zeta^{-i\kappa-1} w_1^{i\kappa-1} d\zeta \right] f_{-1}(z) dV(z)
= (-1)^{k+1} \frac{1}{2\pi^2} \int_{D_\beta} \frac{1}{z_2 w_2} \times
\left[ \int_R \frac{(\zeta - ik)\zeta}{\sinh(2\beta - \pi)(\zeta - ik)} \sinh \pi \zeta^{-i\kappa-1} w_1^{i\kappa-1} d\zeta \right] (\alpha_2 \partial_{r_1})^k f_{-1}(z) dV(z).\]

Similarly, we perform an integration by parts in the second integral in (4.7).
And so the second integral in (4.7) gives
\[ -\frac{1}{2\pi} \int_{D_\beta} \frac{1}{z_2 w_2} (\partial_{r_1})^k \int_{\mathbb{R}} \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - i k, z_1) z_1^{-i\zeta/2-1} z_1^{i\zeta/2} w_1^{i\zeta-1} d\zeta f_{-1}(z) dV(z) \]
\[ = (-1)^{k+1} \frac{1}{2\pi} \int_{D_\beta} \frac{1}{z_2 w_2} \left[ \int_{\mathbb{R}} \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - i k, z_1) z_1^{-i\zeta/2-1} z_1^{i\zeta/2} w_1^{i\zeta-1} d\zeta \right] \times \]
\[ (\partial_{r_1})^k f_{-1}(z) dV(z). \]

(4.9)

To finish the proof we note that the proof of Proposition 3.1, with \( \hat{h}(\xi, y) e^{-ix\xi} \) replaced with
\[ \frac{(\xi - i k)\zeta}{\sinh(2\beta - \pi)(\xi - i k)} \sinh \pi \xi \]
may be followed to show that the operator from (4.8) with kernel
\[ \int_{D_\beta} \frac{1}{z_2 w_2} \int_{\mathbb{R}} \frac{(\zeta - i k)\zeta}{\sinh(2\beta - \pi)(\zeta - i k)} \sinh \pi \zeta \]
maps \( L^2(D_\beta) \) to \( L^2_{-1}(D_\beta) \). Similarly, the proof of Proposition 3.1 shows that the operator with kernel
\[ \frac{1}{2\pi} \frac{1}{z_2 w_2} \int_{\mathbb{R}} \frac{\zeta}{\sinh(\pi \zeta)} g(\zeta - i k, z_1) z_1^{-i\zeta/2-1} z_1^{i\zeta/2} w_1^{i\zeta-1} d\zeta \]
occuring in (4.9) maps \( L^2(D_\beta) \) to \( L^2_{-1}(D_\beta) \). Then, together (4.8) and (4.9) show
\[ \left\| \frac{\partial^k}{\partial w_k^1} T_{-1} f \right\|_{L^2(D_\beta)} \lesssim \| f_{-1} \|_{W^k(D_\beta)}. \]
The estimate in (4.2) is verified, completing the proof of the theorem. \( \square \)

5. The case \( j \neq -1 \)

Analogous to our work in the sections above, we can construct operators
\[ T_j : W^k(D_\beta) \to W^k_j(D_\beta) \quad \forall k, \]
for the cases \( j \neq -1 \) as follows.

We let \( Q_j \) be the projection from \( L^2_j(D_\beta) \) to \( L^2_{j-1}(D_\beta) \) given by
\[ Q_j f(z_1, z_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_1, e^{i\theta} z_2) e^{-ij\theta} d\theta. \]
Then the operator \( T_j \) is given by
\[ T_j f = w_2^{j+1} T_{-1}(z_2^{-j-1} Q_j f). \]
For each \( T_j \), due to properties of the operator \( T_{-1} \), we have a theorem similar to Theorem 4.1.

**Theorem 5.1.** Let \( \beta > \pi/2 \), and \( D_\beta \) be defined as above. For all \( j \in \mathbb{Z} \) there exists a bounded linear projection
\[ T_j : L^2(D_\beta) \to B_j(D_\beta) \]
which satisfies
\[ T_j : W^k(D_\beta) \to W^k_j(D_\beta) \quad \forall k, \]
and
\[ \|T_j f\|_{W^k_j(D_\beta)} \lesssim \|f\|_{W^k_j(D_\beta)}. \]

This proves the Main Theorem.

6. Remarks

We end with a few remarks. We first note that in our proof of Theorem 4.1, we worked with Sobolev spaces, \( W^k \) for integer \( k \). The general case for all \( s \geq 0 \) follows by interpolation.

Secondly, there are infinitely many projection operators which have the same regularity properties as our constructed projection in the Main Theorem. Other projections can be constructed for instance by changing the factor \( \tau_k(\xi) \) in Section 2 with the replacement of the term \( e^{-\xi^2} \) with another \( e^{-m\xi^2} \) for any positive integer \( m \). Then the rest of the arguments could be followed verbatim.

We lastly note that, while it would be ideal to obtain an operator which would map \( W^s \) to itself, without the restriction to the space \( W^s_j \), by summing the operators in Main Theorem\( ^6 \) over \( j \), the dependence of the norms in Theorem 5.1 on \( j \) prohibit the convergence of such a summation. Following the calculations of the proof of Proposition 3.1 leads to the estimates for the norms of \( T_j \):
\[ \|T_j\| \lesssim \frac{\sinh[(j + 1)(\beta - \pi/2)]}{j + 1}. \]

This exponential growth of the estimates thus prohibits us from using results such as the Cotlar–Stein almost orthogonality lemma to conclude any convergence of a sum over the operators \( T_j \).

References

[1] D. Barrett. Behavior of the Bergmann projection on the Diederich-Fornæss worm. Acta. Math., 168:1–10, 1992.
[2] K. Diederich and J. E. Fornæss. Pseudoconvex domains: an example with nontrivial Nebenhulle. Math. Ann., 225:275–292, 1977.
[3] C. Kiselman. A study of the Bergman projection in certain Hartogs domains. Proc. Symposia in Pure Math., 52:219–231, 1991.
[4] S. Krantz and M. Peloso. The Bergman kernel and projection on non-smooth worm domains. Houston J. Math., 34(3):873–950, 2008.