Choreography solutions of the $n$-body problem on $S^2$

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Abstract

We try to prove the existence of choreography solutions for the $n$—body problem on $S^2$. For the three-body problem, we show the existence of the 8-shape orbit on $S^2$.

Key words: celestial mechanics, curved $n$-body problem, periodic solutions, choreographies.

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1 Introduction

The curved $n$—body problem is a generalization of the Newtonian gravitational problem. It has been studied for several authors, for example in [3, 4, 5, 6, 8, 9, 10, 11, 16, 22]. Particularly the interesting history of this problem can be found on [3, 4]. Here we consider the positive curvature case, i.e. particles moving on the unit sphere, $S^2$.

The motion of the $n$ particles with masses $m_i > 0$ and positions $q_i \in S^2$, $i = 1, \ldots, n$, is described by the following system

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i} - m_i (\dot{q}_i \cdot \dot{q}_i)q_i, \quad i = 1, \ldots, n,$$

(1)

where $U$ is the force function which generalizes the Newtonian one, and it is given by

$$U = \sum_{i<j} m_i m_j \cot(d(q_i, q_j)).$$

(2)

On classical $n$-body problems, Chenciner and Montgomery proved the existence of the eight-shape choreography for the three body problem in 2000 [1], which was described numerically by Moore in 1993 [14] and Simó in 2000 [18]. In the recent years Montenelli and Gushterov computed numerically the analogue solutions in the positive curved space [13].

The goal of this work is, based on the work of Zhang and co-authors [19, 20, 21], to prove the existence of periodic solutions for the $n$—body problem on $S^2$.

Motivated by Simó [18] for planar $N$—body problems, in this paper we seek for periodic solutions of (1) moving on the same orbit, i.e., setting the period as $T = 1$, we look for solutions such that
\[ q_i(t) = Q(t + k_i), \quad i = 1, \ldots, n, \]  

(3)

with \( 0 = k_1 < \cdots < k_n < 1 \), and for some periodic function \( Q : [0,1] \to S^2 \).

Since our problem is on the curved space, the solution is much more complicated than Euclidean space.

We define the following set

\[ D = \{ q = (q_1, \ldots, q_n) \in (S^2)^n \mid q_i \text{ is absolutely continuous and } q_i(t) \neq q_j(t), \]  

for \( 1 \leq i \neq j \leq n \}. \]  

(4)

The Lagrangian action associated to system (1) on \( D \) is

\[ f(q) = \int_0^1 \left( \frac{1}{2} \sum_{i=1}^n m_i |\ddot{q}_i(t)|^2 + U(q(t)) \right) dt. \]  

(5)

We are interested in showing the existence of new choreography solutions of (1). In other words, we will not only show that the Lagrangian action functional reaches its minimum in \( D \), but in a subset where the \( n \) particles follow the same orbit.

There are some works where circular choreography solutions have been found, see for instance [8, 10]. In order to find new families of choreographies we will introduce the following sets

\[ E_1 = \{ q = (q_1, \ldots, q_n) \in D \mid q_1(t) = q_n(t + 1/n), \quad q_i(t) = q_{i-1}(t + 1/n), \]  

\[ i = 2, \ldots, n \}, \]  

\[ E_2 = \{ q = (q_1, \ldots, q_n) \in D \mid q_1(t + 1/2) = diag\{1, -1, 1\}q_1(t) \}, \]  

\[ E_3 = \{ q = (q_1, \ldots, q_n) \in D \mid q_1(-t) = diag\{-1, -1, 1\}q_1(t) \}. \]  

It is not difficult to see that \( q_1(0) = (0, 0, 1) = q_1(1/2) \) for \( q \in E_2 \cap E_3 \). Hence circular orbits mentioned above do not belong to \( E_2 \cap E_3 \). The set of choreographies are orbits on

\[ H = \{ q = (q_1, \ldots, q_n) \in D \mid q_1 \in E_1 \cap E_2 \cap E_3 \}. \]  

Let \( B = diag\{1, -1, 1\} \) and \( C = diag\{-1, -1, 1\} \). We now define the following actions \( \Phi_1, \Phi_2 \) and \( \Phi_3 \) on \( D \)

\[ \Phi_1(q(t)) = (q_n(t + 1/n), q_1(t + 1/n), \ldots, q_{n-1}(t + 1/n)), \]  

\[ \Phi_2(q(t)) = (Bq_1(t + 1/2), Bq_2(t + 1/2), Bq_3(t + 1/2)), \]  

\[ \Phi_3(q(t)) = (Cq_1(-t), Cq_2(-t), Cq_3(-t)). \]

Then the fixed point of \( \Phi_i \) is \( E_i, \) \( i = 1, 2, 3. \)

We refer to Palais’ principle of symmetric criticality [15], in order to conclude that the critical points of \( f \) restricted to \( H \) are critical points of \( f \) on \( D \) as well.

We state our main theorem as follows

**Theorem 1.** Consider the \( n \)-body problem on \( S^2 \). The Lagrange action functional (5) reaches its minimum on \( H \). This minimum is a periodic non-collision solution of the equations of motion (1).
We first show that any critical point of \((6)\) on \(D\) satisfies \((1)\).

**Proposition 2.** A critical point of the Lagrange action functional on \(D\) is a solution of the equations of motion.

**Proof.** Let \(q_0 = (q_{01}, \ldots, q_{0n})\) be a critical point of the Lagrange action functional on \(D\).

For a given \(q\), a displacement \(\delta f\) is given by (the Gateaux derivative)

\[
\delta f = \frac{d}{d\varepsilon} \int_0^1 \left( \frac{1}{2} \sum_{i=1}^n m_i |\dot{q}_i| \right) d\varepsilon \bigg|_{\varepsilon=0},
\]

restricted to any \(p = (p_1, \ldots, p_n)\) such that \(|q_i(t) + \varepsilon p_i(t)|^2 = 1\), for every \(\varepsilon \to 0\), and \(i = 1, \ldots, n\). Let \(g_i\) be the function defined as \(g_i(q_i) = |q_i(t)|^2 - 1\) (the constraint \(g_i(q_i) = 0\) maintains the particle \(q_i\) on the sphere \(S^2\)). At a given time, for displacements of the constraint equation, the following should be held

\[
\delta g_i = \frac{d}{d\varepsilon} \left( |q_i(t) + \varepsilon p_i(t)|^2 - 1 \right) \bigg|_{\varepsilon=0} = 0, \quad i = 1, \ldots, n.
\]

Integrating both sides with respect to time we have

\[
\delta h_i = \int_0^1 \frac{d}{d\varepsilon} \left( |q_i(t) + \varepsilon p_i(t)|^2 - 1 \right) \bigg|_{\varepsilon=0} dt = 0, \quad i = 1, \ldots, n.
\]

From Hamilton principle we have

\[
0 = \delta f + \sum_{i=1}^n \lambda_i \delta h_i
\]

\[
= \frac{d}{d\varepsilon} \int_0^1 \left( \frac{1}{2} \sum_{i=1}^n m_i |\dot{q}_i|^2 + U(q(t) + \varepsilon p(t)) + \sum_{i=1}^n \lambda_i g_i \bigg|_{q_i=q_0} \right) dt \bigg|_{\varepsilon=0}
\]

where each \(\lambda_i\) is the Lagrange multiplier corresponding to the body \(i\), it will be computed later in the proof.

Then we have

\[
0 = \frac{d}{d\varepsilon} \int_0^1 \left( \frac{1}{2} \sum_{i=1}^n m_i |\dot{q}_i|^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n m_j m_j \text{cot}(d(q_i + \varepsilon p_i, q_j + \varepsilon p_j)) 
+ \lambda_i (|q_i(t) + \varepsilon p_i(t)|^2 - 1) \bigg|_{q_i=q_0} \bigg|_{\varepsilon=0} dt \right)
\]

\[
= \sum_{i=1}^n \left[ \int_0^1 \frac{d}{d\varepsilon} \left( \frac{1}{2} m_i |\dot{q}_i|^2 + \frac{m_i m_j}{\sqrt{(q_i + \varepsilon p_i) \cdot (q_i + \varepsilon p_i)} \sqrt{(q_j + \varepsilon p_j) \cdot (q_j + \varepsilon p_j)}} \right) \right. 
\]

\[
= \frac{1}{2} \sum_{j=1, j \neq i}^n \frac{(q_i + \varepsilon p_i) \cdot (q_j + \varepsilon p_j)}{\sqrt{(q_i + \varepsilon p_i) \cdot (q_i + \varepsilon p_i)} \sqrt{(q_j + \varepsilon p_j) \cdot (q_j + \varepsilon p_j)}} \left( 1 - \left( \frac{(q_i + \varepsilon p_i) \cdot (q_i + \varepsilon p_j)}{\sqrt{(q_i + \varepsilon p_i) \cdot (q_i + \varepsilon p_i)} \sqrt{(q_j + \varepsilon p_j) \cdot (q_j + \varepsilon p_j)}} \right)^2 \right)^{1/2}
\]

\[
+ \lambda_i (|q_i(t) + \varepsilon p_i(t)|^2 - 1) \bigg|_{q_i=q_0} \bigg|_{\varepsilon=0} dt \right].
\]
After considering $\varepsilon \to 0$, and $q_i \cdot q_i = 1$ we have

$$0 = \sum_{i=1}^{n} \left[ \int_{0}^{1} \left( m_i \dddot{q}_i \cdot \dot{p}_i + \frac{1}{2} \sum_{j=1, j \neq i}^{n} m_i m_j \frac{[q_i \cdot p_j + q_j \cdot p_i] - (q_i \cdot q_j)[q_j \cdot p_j + q_i \cdot p_i]}{(1 - (q_i \cdot q_j)^2)^{3/2}} \right. \\
+ 2\lambda_i (q_i \cdot p_i) \bigg|_{q_i=q_{0i}} \right) dt \right]$$

$$= \sum_{i=1}^{n} \left[ \int_{0}^{1} \left( m_i \dddot{q}_i \cdot \dot{p}_i + \sum_{j=1, j \neq i}^{n} m_i m_j \frac{[q_j \cdot p_i] - (q_i \cdot q_j)[q_i \cdot p_i]}{(1 - (q_i \cdot q_j)^2)^{3/2}} + 2\lambda_i (q_i \cdot p_i) \bigg|_{q_i=q_{0i}} \right) dt \right]$$

$$= \sum_{i=1}^{n} \left[ \int_{0}^{1} \left( m_i \dddot{q}_i \cdot \dot{p}_i + \sum_{j=1, j \neq i}^{n} m_i m_j \left( \frac{q_j - (q_i \cdot q_j)q_i}{(1 - (q_i \cdot q_j)^2)^{3/2}} \right) \cdot p_i + 2\lambda_i (q_i \cdot p_i) \bigg|_{q_i=q_{0i}} \right) dt \right].$$

Integrating the first term and considering that the variations vanish at the the end points

$$0 = \sum_{i=1}^{n} \left[ m_i \dddot{q}_i \cdot p_i \bigg|_{0}^{1} + \int_{0}^{1} \left( -m_i \dddot{q}_i \cdot p_i + \sum_{j=1, j \neq i}^{n} m_i m_j \left( \frac{q_j - (q_i \cdot q_j)q_i}{(1 - (q_i \cdot q_j)^2)^{3/2}} \right) \cdot p_i \right. \\
+ 2\lambda_i (q_i \cdot p_i) \bigg|_{q_i=q_{0i}} \right) dt \right]$$

$$= \sum_{i=1}^{n} \left[ \int_{0}^{1} \left( -m_i \dddot{q}_i \cdot p_i + \sum_{j=1, j \neq i}^{n} m_i m_j \left( \frac{q_j - (q_i \cdot q_j)q_i}{(1 - (q_i \cdot q_j)^2)^{3/2}} \right) \cdot p_i \right. \\
+ 2\lambda_i (q_i \cdot p_i) \bigg|_{q_i=q_{0i}} \right) dt \right]$$

$$= \sum_{i=1}^{n} \left[ \int_{0}^{1} \left( -m_i \dddot{q}_i + \sum_{j=1, j \neq i}^{n} m_i m_j \left( \frac{q_j - (q_i \cdot q_j)q_i}{(1 - (q_i \cdot q_j)^2)^{3/2}} \right) + \lambda_i q_i \right) \cdot p_i \bigg|_{q_i=q_{0i}} dt \right]$$

$$= \sum_{i=1}^{n} \left[ \int_{0}^{1} \left( -m_i \dddot{q}_i + \frac{\partial U}{\partial q_i} + 2\lambda_i q_i \right) \cdot p_i \bigg|_{q_i=q_{0i}} dt \right].$$

Since this must hold for any $p = (p_1, \ldots, p_n)$ in the interval $(0, 1)$, it follows that the critical point should satisfy

$$-m_i \dddot{q}_i + \frac{\partial U}{\partial q_i} - 2\lambda_i q_i \bigg|_{q_i=q_{0i}} = 0, \quad i = 1, \ldots, n,$$

(7)

where the multiplier $\lambda_i$ can be computed multiplying the last expression by $q_i$

$$-m_i \dddot{q}_i \cdot q_i + \frac{\partial U}{\partial q_i} \cdot q_i - 2\lambda_i q_i \cdot q_i \bigg|_{q_i=q_{0i}} = 0, \quad i = 1, \ldots, n.$$

Using the fact that the potential is a homogeneous function of degree zero, and that the expression $\dddot{q}_i \cdot q_i = -\dddot{q}_i \cdot \dddot{q}_i$ holds we have
\[ \lambda_i = \frac{m_i \dot{q}_i \cdot \dot{q}_i}{2}. \]

Substituting this expression into (7), we have

\[ -m_i \ddot{q}_i + \frac{\partial U}{\partial q_i} - m_i (\dot{q}_i \cdot \dot{q}_i) q_i \Bigg|_{q_i = q_0_i} = 0, \quad i = 1, \ldots, n. \]

Hence, any critical point \( q_0 \) of the Lagrangian action satisfies the equation of motion.

\[ \square \]

2 Proof of Theorem 1

Now we prove that the action functional reaches its minimum on \( D \). The proof of the theorem will be a consequence of the following result,

**Proposition 3.** [17] A weakly lower semicontinuous from below functional \( F(u) \), in a reflexive Banach space \( U \) is bounded from below on any bounded weakly closed set \( M \subset \text{Dom}F \) and attains its minimum on \( M \) at a point of \( M \).

Our task now is to prove that the functional (5) is weakly lower semicontinuous from below and that \( D \cup \partial D \) is weakly closed.

**Proposition 4.** \( f(q) \) is weakly lower semicontinuous from below on \( D \cup \partial D \)

**Proof.** Recall that \( f \) is called weakly lower semicontinuous from below if for any \( q^n \in D \cup \partial D \) such that \( q^n \to q \) weakly, the following inequality holds

\[ \liminf_{n \to \infty} f(q^n) \geq f(q). \]

If \( q \in D \), then there exists \( N \) such that for \( n > N \), \( q^n \in D \). The functions \( q^n_i \) are continuous and converges to \( q_i \) uniformly.

This implies that \( U(q^n_i) \to U(q_i) \) for \( t \in [0, 1] \).

By Fatou’s lemma we have

\[ \liminf_{n \to \infty} f(q^n) \geq \int_0^1 \frac{1}{2} \sum_1^3 |\dot{q}_i(t)|^2 + \int_0^1 \liminf_{n \to \infty} \left( \sum_{i<j} \cot d(q^n_i, q^n_j) \right) dt = f(q). \]

Now let us suppose that \( q^n_i \in \partial D \) and \( q^n_i \to q_i \) weakly.

There exist \( t_0 \in [0, 1] \) such that \( q^n_{i_0}(t_0) = q^n_{j_0}(t_0) \) for \( i_0 \neq j_0 \). Consider the set \( C = \{ t \in [0, 1] \mid \text{there exist } i_0 \neq j_0 \text{ with } q_{i_0}(t) = q_{j_0}(t) \} \).

Consider the Lebesgue measure, \( \mu(C) \), of \( C \). Firstly, let us suppose that \( \mu(C) = 0 \).

Since \( q^n \) converges to \( q \) uniformly, then the following holds almost everywhere,

\[ \cot d(q^n_i(t), q^n_j(t)) \to \cot d(q_i(t), q_n(t)). \]

This implies, by Fatou’s lemma

\[ \int_0^1 \cot(d(q_i(t_0), q_n(t_0))) = \int_0^1 \liminf_n \cot(d(q^n_i(t_0), q^n_j(t_0))) \]

\[ \leq \liminf_n \int_0^1 \cot(d(q^n_i(t_0), q^n_j(t_0))). \]
Hence \( f(q) \leq \lim \inf_n f(q^n) \). Secondly, if \( \mu(C) > 0 \), then

\[
\int_0^1 \cot(d(q_i(t), q_j(t))) = +\infty.
\]

Additionally we have,

\[
\cot(d(q^n_i(t), q^n_j(t))) \to \cot(d(q_i(t), q_j(t))),
\]

uniformly. This implies that

\[
\int_0^1 \cot(d(q^n_i(t), q^n_j(t))) \to +\infty.
\]

It follows that

\[
f(q) \leq \lim \inf_n f(q^n).
\]

\[\square\]

**Proposition 5.** \( D \cup \partial D \) is a weakly closed subset of \((W^{1,2}(\mathbb{R}/\mathbb{Z}, S^2))^3 := \{(q_1, q_2, q_3) \in (S^2)^3 | q_i \in L^2, \dot{q}_i \in L^2, q_i(t+1) = q_i(t), i = 1, 2, 3\}\)

**Proof.** Since \( q^n \to q \) weakly, then \( q^n \to q \) uniformly, then \( q \in D \cup \partial D \). Hence \( D \cup \partial D \) is a weakly closed subset of \((W^{1,2}(\mathbb{R}/\mathbb{Z}, S^2))^3\).

\[\square\]

### 3 Choreography solution for the three-problem on \( S^2 \)

In order to show a choreography solution for the three-body problem on \( S^2 \), we will firstly estimate the lower bound of the Lagrangian action for a binary collision generalized solution. We will consider masses equal to 1.

**Proposition 6.** Consider three bodies on \( S^2 \). Let \( q \in T^*(S^2)^3 \) be a periodic binary collision generalized solution, then the Lagrangian action satisfies \( f(q) \geq \frac{3}{2}(12\pi)^{2/3} - 3 \).

The following lemma will be useful to proof Proposition 6.

**Lemma 7.** Consider \( q_i \) and \( q_j \) on \( S^2 \) satisfying equations of motion \eqref{eq:eom}, then

\[
\frac{1}{r_{ij}} - 1 < \cot(d(q_i, q_j)) < \frac{1}{r_{ij}},
\]

where \( r_{ij} \) is the Euclidean distance between \( q_i \) and \( q_j \).

**Proof.** For this proof we will consider the origin of the system at the north pole of the unit sphere, i.e., at \( R = (0,0,1) \). The equations of motion takes the form

\[
\ddot{q}_i = \sum_{i=1, j \neq i}^n \frac{q_j - \left(1 - \frac{r^{2}_{ij}}{2}\right) q_i + \frac{r^{2}_{ij} R}{2}}{r^{2}_{ij} \left(1 - \frac{r^{2}_{ij}}{3}\right)} - (\dot{q}_i \cdot \dot{q}_i) (q_i + R).
\]

\(\square\)
The potential energy in $S^2$ is given by

$$U = \sum_{i<j} \cot(d(q_i, q_j)) = \sum_{i<j} \frac{1 - \frac{r_{ij}^2}{4}}{r_{ij} \left( 1 - \frac{r_{ij}^2}{4} \right)^{1/2}},$$

(10)

for more details about the equations of motion and potential energy written in this coordinates, please see [4].

Consider $n = 2$, then

$$\cot(d(q_i, q_j)) = \frac{1 - \frac{r_{ij}^2}{2}}{r_{ij} \left( 1 - \frac{r_{ij}^2}{4} \right)^{1/2}} > \frac{1 - \frac{r_{ij}^2}{2}}{r_{ij} \left( 1 - \frac{r_{ij}^2}{4} \right)^{1/2}} = \left( \frac{1 - \frac{r_{ij}^2}{2}}{r_{ij} \left( 1 - \frac{r_{ij}^2}{4} \right)^{1/2}} \right) > \frac{1}{r_{ij}} - 1. \quad (11)$$

On the other hand, we have

$$\cot(d(q_i, q_j)) = \frac{1 - \frac{r_{ij}^2}{2}}{r_{ij} \left( 1 - \frac{r_{ij}^2}{4} \right)^{1/2}} < \frac{1 - \frac{r_{ij}^2}{4}}{r_{ij} \left( 1 - \frac{r_{ij}^2}{4} \right)^{1/2}} = \left( \frac{1 - \frac{r_{ij}^2}{4}}{r_{ij} \left( 1 - \frac{r_{ij}^2}{4} \right)^{1/2}} \right) < \frac{1}{r_{ij}}. \quad (12)$$

Hence we conclude the proof of the lemma.

Proof. Consider three point particles $q_1, q_2, q_3 \in S^2$ with masses $m_1 = m_2 = m_3 = 1$ satisfying the equations of motion (11), and suppose that the particles $q_1$ and $q_2$ collide, without loss of generality, at the north pole.

The Lagrangian action is given by

$$f(q) = \int_0^1 \left( \frac{1}{2} \sum_{i=1}^3 |\dot{q}_i|^2 + \sum_{1 \leq i < j \leq 3} \cot(d(q_i, q_j)) \right) dt,$$

where the constrains $|q_i|^2 = 1$ and $q_i \cdot \dot{q}_i = 0$, $i = 1, 2, 3$, hold.

Notice that (19) (20)

$$\sum_{1 \leq i < j \leq 3} |\dot{q}_i - \dot{q}_j|^2 + \left| \sum_{i=1}^3 \dot{q}_i \right|^2 = \sum_{i=1}^3 |\dot{q}_i|^2.$$

We have
\[
\begin{align*}
    f(q) &= \int_0^1 \left( \frac{3}{2} |\dot{q}_k|^2 + \sum_{1 \leq i < j \leq 3} \cot(d(q_i, q_j)) \right) dt \\
    &\geq \int_0^1 \left( \sum_{1 \leq i < j \leq 3} \frac{1}{6} |\dot{q}_i - \dot{q}_j|^2 + \sum_{1 \leq i < j \leq 3} \cot(d(q_i, q_j)) \right) dt \\
    &\geq \int_0^1 \left( \sum_{1 \leq i < j \leq 3} \frac{1}{6} |\dot{q}_i - \dot{q}_j|^2 + \sum_{1 \leq i < j \leq 3} \frac{1}{r_{ij}} - 3 \right) dt \quad \text{(by Lemma 7)} \\
    &= \frac{1}{3} \int_0^1 \left( \sum_{1 \leq i < j \leq 3} \frac{1}{2} |\dot{q}_i - \dot{q}_j|^2 + \sum_{1 \leq i < j \leq 3} \frac{3}{r_{ij}} \right) dt \\
    &\geq \frac{1}{3} \int_0^1 \left( \sum_{1 \leq i < j \leq 3} \frac{1}{2} |\dot{q}_i - \dot{q}_j|^2 + \sum_{1 \leq i < j \leq 3} \frac{3}{r_{ij}} \right) dt - 3. 
\end{align*}
\]

If \( q_1(t_0) = q_2(t_0) \), then \( q_1(t_0 + 1/2) = q_2(t_0 + 1/2) \). Then using some estimates of \cite{19, 20} we have

\[
    \frac{1}{3} \int_0^1 \left( \frac{1}{2} |\dot{q}_1 - \dot{q}_2|^2 + \frac{3}{r_{12}} \right) dt = \frac{2}{3} \int_0^{1/2} \left( \frac{1}{2} |\dot{q}_1 - \dot{q}_2|^2 + \frac{3}{r_{12}} \right) dt \\
    = \frac{(12\pi)^{2/3}}{2}.
\]

Since \( q_1(t) = q_3(t + 1/3) = q_2(t + 2/3) \), then

\[
    f(q) \geq \frac{3(12\pi)^{2/3}}{2} - 3. \tag{14}
\]

**Proposition 8.** \( f^{-1}((0, \frac{3}{2}(12\pi)^{2/3} - 3)) \neq \emptyset \)

*Proof.* Consider the test loop

\[
    q_1(t) = (x(t), y(t), z(t)), \quad q_2(t) = q_1(t + 1/3), \quad q_3 = q_1(t + 2/3), \tag{15}
\]

where

\[
    \begin{align*}
        x(t) &= 0.15 \sin(4\pi t), \\
        y(t) &= 0.2275 \sin(2\pi t), \\
        z(t) &= \sqrt{1 - x^2(t) - y^2(t)}. 
    \end{align*}
\]

In \cite{19} the authors show that if \( \sin(2\pi t) = \sin(2\pi (t + \frac{i-1}{3})) \), then \( \sin(4\pi t) \neq \sin(4\pi (t + \frac{i-1}{3})) \), for \( t \in (0, 1), \ i = 2, 3 \). Hence \( q_i(t) \neq q_j(t), \ i \neq j \).

With the expressions \cite{15}, we have \( f(q) \approx 13.76572 < \frac{3}{2}(12\pi)^{2/3} - 3 \approx 13.8647 \).

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