Scattering of two photons from two distant qubits: Exact solution

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We consider the inelastic scattering of two photons from two qubits separated by an arbitrary distance $R$ and coupled to a one-dimensional transmission line. We present an exact, analytical solution to the problem, and use it to explore a particular configuration of qubits which is transparent to single-photon scattering, thus highlighting non-Markovian effects of inelastic two-photon scattering: Strong two-photon interference and momentum dependent photon (anti-)bunching. This latter effect can be seen as an inelastic generalization of the Hong–Ou–Mandel effect.

Introduction. — Efficient manipulation of states of photons is a necessary ingredient for the development of quantum networks [1], which have a great potential for applications in quantum information and communications. Scattering of microwave photons in one-dimensional transmission lines coupled to superconducting qubits [2] is one of the natural tools to achieve this goal in the framework of the waveguide QED. Recent experiments [3] on the scattering of microwave photons have demonstrated the fundamental possibility to achieve a long-range photon-mediated interaction between qubits. The reciprocal effect – emergence of an effective interaction between photons – manifests itself in the inelastic component of the resonance fluorescence power spectrum [4, 5].

The scattering problem of an arbitrary multiphoton state from an arbitrary local scatterer is known to have an explicit analytic solution [2, 6]. However, for the purposes of quantum networking one requires more than one qubit, ideally, a distributed array of qubits. The single-photon – elastic – scattering from several qubits is theoretically studied either by the application of the transfer matrix method [7, 8] or by a direct solution of the Schr¨odinger equation [9]. Most studies (see, e.g., [10]) of the inelastic scattering of two or more photons use the seminal approach of Lehembre [11] which is based on the Markov approximation. It breaks down for a rather large qubit separation, $R ≳ \hbar v_g/Γ$, where $Γ$ is a qubit’s relaxation rate and $v_g$ is the group velocity of photons. An infinite continuum array of quantum scatterers can be used to model a nonlinear medium, and this inelastic scattering problem has been solved in the adiabatic approximation leading to the nonlinear Schrödinger equation [12]. An earlier take on the inelastic scattering from several scatterers used the elastic approximation [13].

The Lippmann–Schwinger equation for the two-photon scattering from two distant qubits has recently been solved numerically and used to study the regime $R \sim \hbar v_g/Γ$, where non-Markovian effects start to appear [14].

In this Letter we present an exact analytic solution to the two-photon, two-qubit, scattering problem. By virtue of our analytical approach we can study arbitrary system parameters and photon energies, analyze limiting cases, and specify relaxation rates. We apply our method to the scattering of two counter-propagating photons from a configuration of distance $(R \gg \hbar v_g/Γ)$ qubits, which is fully transparent to single-photon scattering. Therefore, this configuration allows us to focus solely on inelastic two-photon scattering in the non-Markovian regime.

We find that the probability density for the momentum exchange of the photons exhibits a strong interference pattern, and that its envelope is peaked at different positions for photons scattering to the same and opposite directions. This is essentially an inelastic counterpart to the Hong–Ou–Mandel effect – destructive interference of photons scattering in opposite directions [15]. We also find that the two-photon correlation function exhibits photon (anti)bunching at times which are integer multiples of $R/v_g$. Model. — Our model consists of two qubits separated by a distance $R$ coupled to photons with a linear dispersion. Coupling to the individual qubits is energy independent, but the coupling to the combined scatterer becomes effectively energy dependent. This leads to non-Markovian effects, going beyond the approaches of Refs. [2, 6] dealing only with local scatterers. Introducing the operators for right-, $a_{1k} = a_{R,k}$, and left-moving photons, $a_{2k} = a_{L,-k}$, our Hamiltonian reads $H = H_0 + v$, where the free part (in units where $\hbar = v = 1$)

$$H_0 = H_{0p} + H_{0q} = \sum_{\alpha=1,2} \int dk k a^\dagger_{\alpha k} a_{\alpha k} + \frac{\Omega_2}{2} \sigma^z(1)$$

consists of the photon $H_{0p}$ and qubit $H_{0q}$ parts, and the interaction is given by $v = \sum_{\alpha=1,2} \int dk v_{\alpha k} a^\dagger_{\alpha k} a^\dagger_{\alpha k} + h.c. \equiv v_s a^\dagger_{\alpha k} a^\dagger_{\alpha k} + h.c.$, where $s = \{\alpha k\}$ is a collective index combining the channel index $\alpha$ and mode $k$, and summation over repeated indices is implied. The bare single-photon vertex

$$v_s v_{\alpha k} = \sum_{\beta=1,2} g_{\beta\sigma} e^{-i\alpha c_\beta (kR+\varphi)/2},$$

describes the dipole interaction in the rotating wave approximation (RWA) between the one-dimensional field and the two qubits placed at positions $x_1 = +R/2$ and $x_2 = -R/2$ with coupling constants $g_{1,2} \equiv \sqrt{\Gamma_{1,2}/\pi}$. 

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Here $\varphi = k_0 R$, $k_0$ is the spectrum linearization point, and $c_{\alpha(\beta)} = (-1)^{\alpha(\beta)+1}$ [17].

Scattering formalism.—The scattering matrix [16], $S = 1 - 2\pi iT(E = E_{\text{in}})\delta(E_i - E_f)$, is generally defined in terms of the so-called $T$-matrix

$$T(E) = \frac{1}{E - H + \imath\eta} v = \sum_{n=1}^{\infty} \left( \frac{1}{E - H_0 + \imath\eta} v \right)^n,$$

which is projected on-shell $E = E_i = E_f$; here $E_{i/f}$ is the energy of the initial/final state of the entire system. The infinitesimal parameter $\eta > 0$ regulates the adiabatic switching on/off of the coupling between the field and the qubits. The energy conservation in the absence of losses to external degrees of freedom (which can be made small in experiments [5]) is enforced in the scattering matrix by the delta-function.

Given the initial state $|\Psi_i\rangle$, the scattering matrix prescribes which final state $|\Psi_f\rangle = S|\Psi_i\rangle$ is established in the stationary limit $t \to \infty$. To find $|\Psi_f\rangle$, it suffices to evaluate the $T$-matrix, which has the following properties [6].

First, nonzero matrix elements of the $T$-matrix belong to the subset of the qubit states with zero relaxation rate (dark states). For the model (1),(2), like for many models based on the RWA, this state is unique and corresponds to the initial deexcitation of all qubits.

Second, the $T$-matrix can be represented as a sum $T(E) = \sum_{N=1}^{\infty} T^{(N)}(E)$ of the normal-ordered contributions $T^{(N)}(E) = T_{s_1'...s_N'}(s_1...s_N)(a_{s_1'}^\dagger a_{s_1}^\dagger a_{s_N}...a_{s_1})$, each acting in the corresponding $N$-photon Hilbert space.

Third, the rightmost (leftmost) vertex in (3) does not have to be additionally normal-ordered. In fact, if we start from (want to end up with) a state of two deexcited qubits, then in the first (last) step we have to excite (deexcite) one of the qubits. This corresponds to the appearance of only $v_{s_1'} a_{s_1}$ ($v_{s_1} a^{\dagger}_{s_1}$) in the rightmost (leftmost) position. This means that the representation of each component $T^{(N)}_{s_1'...s_N'}(s_1...s_N)(E)$ (going from left to right) always begins with a bare vertex $v_{s_1'}$ and ends with a bare vertex $v_{s_1}$. In particular, for $N = 1, 2$ we have $T^{(1)}(E) = v_{s_1'} M(k_1) v_{s_1}$ and $T^{(2)}_{s_1'...s_2'}(s_1...s_2)(E) = v_{s_1'} M(k_1) W_{s_2's_2'}(E) M(k_1) v_{s_1}$, where $M(E) = (1 - P_{++} - P_{--}) G(E)(1 - P_{++} - P_{--})$ is the projection of the qubit Green’s function $G(E) = [E - H_0 - \Sigma(E)]^{-1}$ onto the one-excitation subspace, with $P_{cc'} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{(c,c')} \begin{pmatrix} 1 & 1 \end{pmatrix}$, $c, c' = \pm; \Sigma(E)$ being the qubit self-energy; and $W_{s_2's_2'}(E)$ the effective two-photon vertex. We have also taken into account the on-shell projection $E = k_1 + k_2 = k_1 + k_2$, which fixes the arguments of the Green’s functions in $T^{(2)}$, and implicitly assumed the projection $T^{(1,2)} \rightarrow P_{--} T^{(1,2)} P_{--}$.

Summarizing these observations we find the following representation for the one- and two-photon scattering matrices

$$S^{(1)} = \begin{cases} \delta_{s's} - 2\pi i v_{s'} M(k) v_s \delta_{k k} \end{cases} a_{s'}^\dagger a_s,$$

$$S^{(2)} = \begin{cases} \frac{1}{2} \delta_{s's_1} \delta_{s's_2} - 2\pi i v_{s_1} M(k_1) \left[ \delta_{k_1 k_1} \delta_{s_2 s_2} + \\
+ W_{s_2's_2'}(E) M(k_1) \delta_{k_1 + k_2 - k_1 + k_2} \right] v_{s_2'}^\dagger a_{s_1}^\dagger a_{s_2} a_{s_1}, \end{cases}$$

Exact solution.—The two yet unknown objects in (4) and (5) are $M(E)$ and $W_{s_2's_2'}(E)$. It turns out that in our model they can be found in an explicit analytic form.

Suppose we have excited one of the qubits by the rightmost bare vertex and thereby created one excitation in the system. The next options are either to excite the second qubit (1 → 2) or to deexcite the already excited one (1 → 0). An irreducible scattering process corresponds to the alternation of the singly excited and the deexcited states 1 → 0 → 1 → ... → 1. This chain of transitions may have an arbitrary length. Resumming all corresponding diagrams (see Fig. 1), we obtain the irreducible contribution $W^{(i)}_{s_2's_2'}$ to $W_{s_2's_2'}(E)$. It obeys the following integral equation

$$W^{(i)}_{s_2's_2'}(E) = w^{(i)}_{s_2's_2'}(E) + w^{(i)}_{s_2's_2'}(E) M(E - k_1) W^{(i)}_{s_2's_2'}(E),$$

where the two-photon vertex $w^{(i)}_{s_2's_2'}(E) = v_{s_2'} P_{++} - k_{++} v_{s_2}$ describes the elementary process $1 → 0 → 1$. Note that the self-energy $P_{--} \Sigma(E) P_{--} = -i\eta$ of the deexcited state remains infinitesimally small, which reflects that this is the only dark state in the model.

The other, reducible contribution $W^{(r)}_{s_2's_2'}(E)$ to $W_{s_2's_2'}(E)$ is obtained if we insert into the sequence of zeros and ones the process $1 → 2 → 1$. Resumming all fragments of chains $1 → ... → 2$ up to the first occurring doubly excited state, we get the effective single-photon vertex $\tilde{V}^{(i)}_{s_2'}(E)$ creating one excitation. Resumming all fragments $2 → ... → 1$ from the last occurring doubly excited state, we get the effective single-photon vertex $V^{(r)}_{s_2'}(E)$ annihilating one excitation. Resumming all fragments between the first and the last occurring doubly excited states $2 → ... → 2$, we get the Green’s function $G_{++}(E) = P_{++} G(E) P_{++}$ of the doubly excited state. As a result

$$W^{(r)}_{s_2's_2'} = V^{(r)}_{s_2'} G_{++} V^{(i)}_{s_2'}(E).$$

The effective vertices $\tilde{V}^{(i)}_{s_2'}(E)$ and $V^{(r)}_{s_2'}(E)$ appearing in this expression are obtained by dressing the bare single-photon vertices $v_{s_2'}$ and $v_{s_2}$ with the vertex corrections

$$\tilde{V}^{(i)}_{s_2'}(E) = v_{s_2'}^f + v_{s_2'}^f M(E - k_1) W^{(i)}_{s_2's_2'}(E),$$

$$V^{(r)}_{s_2'}(E) = v_{s_2'} + W^{(i)}_{s_2's_2'}(E) M(E - k_1) v_{s_2},$$

which are again expressed via the irreducible two-photon vertex $W^{(i)}$. Note that $\tilde{V}^{(i)}_{s_2'}(E) \neq [V_{s_2'}(E)]^\dagger$.

The qubit self-energy is given by

$$\Sigma(E) = v_{s_2'}^f \left[ \delta_{s's} + M(E - k) W^{(i)}_{s_2's_2'}(E) \right] M(E - k) v_{s_2}.$$
where the part containing $W^{(i)}$ contributes only to $\Sigma_{++} = P_{++}\Sigma(E)P_{++}$; see the discussion of the reducible processes above. In turn, the first term in (10) contributes both to the one-excitation sector $\Sigma_M(E) \equiv (1 - P_{++} - P_{--})\Sigma_E(1 - P_{++} - P_{--}) = -2i[\Gamma_1 P_{+-} + \Gamma_2 P_{--} + \sqrt{\gamma_1} \Sigma_{b} e^{i\phi_1 + i(\sigma_1^{(1)} \sigma_2^{(2)} + \sigma_1^{(1)} \sigma_2^{(2)})}],$ and to $\Sigma_{++}$ with the constant term $-2i[\Gamma_1 + \Gamma_2] P_{+-}.$

We see that the number of classes of diagrams contributing to $T^{(2)}(E)$ is finite, and that a full resummation within each class can be performed. This is a direct consequence of the finiteness of the Hilbert space of the qubits and the number of photons, and of the conservation of the number of excitations in RWA. The closed set of equations necessary for the complete solution of the two-photon scattering problem is summarized in Fig. 1.

Equation (6) plays a central role in the problem, since the knowledge of $W^{(i)}$ allows us to find all the other objects involved. It admits an explicit analytic solution. In particular, decomposing $W^{(i)}_{a'a}(E) = w^{(i)}_{a'a}(E) + \bar{W}^{(i)}_{a'a}(E)$ into a sum of the Markovian $w^{(i)}_{a'a}(E)$ and the non-Markovian $\bar{W}^{(i)}_{a'a}(E)$ contributions (see the corresponding discussion below), we establish that $\bar{W}^{(i)}_{a'a}(E) = \int_0^R dx \int_0^R dx'e^{-ikx - ik'x'} \sum_{\delta',\delta} F^{\delta\delta'}_{\alpha\alpha}(x', x; E)|\delta'(\delta)|,$ where $\delta', \delta = +, -, ++$ label the states in the one-excitation sub-space. The continuous functions $F^{\delta\delta'}_{\alpha\alpha}(x', x; E)$ are defined for $0 \leq x', x \leq R$, and may have jumps of the derivatives (up to the third order) along the lines $x = x'$ and $x + x' = R$. Away from these lines, $F^{\delta\delta'}_{\alpha\alpha}(x', x; E)$ is given by a sum of terms $\propto e^{ip_j(E)x' + ip_l(E)x}$ over $j, l = 1 \ldots 4$. The parameters $p_j = p_j(E)$ include the renormalized, energy-dependent transition frequencies $\sim \Re p_j(E)$ and relaxation rates $\sim |\Im p_j(E)|$, and are given by

$$p_{1,3} = \sqrt{\lambda^2 + b^2 + 2\lambda b - \nu^2}, \quad p_{2,4} = -p_{1,3}. \tag{11}$$

Here $\lambda = (\frac{1}{2}(E - \tilde{\Omega}_1 - \tilde{\Omega}_2), \nu = 4\Gamma_1 \Omega_2 e^{2\nu + 2\nu_2} E^{-\Omega_1 - \Omega_2}$, $\tilde{\Omega}_1, \tilde{\Omega}_2 = \Omega_{1,2} - 2i\Gamma_{1,2},$ and $b = \frac{1}{2}(\tilde{\Omega}_1 - \tilde{\Omega}_2)$ is a measure of the asymmetry between the qubits.

The details of this solution including explicit pre-exponential factors in $\bar{W}^{(i)}_{a'a}$ as well as explicit expressions for $\bar{V}_{a'a}$, $\Sigma_{++}$, and $\bar{W}^{(i)}_{a'a}$, are presented in the Supplemental Material [17].

**Single-photon scattering.**—This result is independent of $W^{(i)}$ and follows straightforwardly from $\Sigma_M(E)$ quoted above. We recover the familiar expressions [8] for the transmission $S_{11}(k) = S_{22}(k) = \frac{2}{1 - e^{i(k - \bar{k}) E + 2s(k - \bar{k})}}$ and reflection $S_{12/21}(k) = e^{i(k - \bar{k}) E + 2s(k - \bar{k})}$ amplitudes (here $\bar{k} = k + k_0$), expressed via the amplitudes $t_{k}^{(2,1)} = k - \Omega_{1,2}, r_{k}^{(1,2)} = k^{(1,2)} - 1$ of the individual qubits. One can directly check the unitarity of the single-photon scattering matrix $S_{a'a}(k) = \delta_{a'a}.$

**Two-photon scattering.**—In order to discuss effects of the emerging effective interaction between photons, we explicitly separate the elastic $S_{a'a}^{(2)}$ and the inelastic $S_{a'a}^{(inel)}$ contributions of the two-photon scattering matrix (5). To this end, we define the delta-part $w^{(d)}_{a'a} = \frac{1}{2}[w^{(i)}_{a'a} - w^{(d)}_{a'a}]$ and the principal part $w^{(p)}_{a'a} = \frac{1}{2}[w^{(i)}_{a'a} + w^{(d)}_{a'a}]$ of $w^{(i)}_{a'a}.$ Then, the elastic term $S_{a'a}^{(el)} = \frac{1}{2}S_{a'a}^{(i)}S_{a'a}^{(i)}$ is obtained by combining the terms independent of $W$ with $w^{(d)}$ contained in $W$, and the inelastic term receives the remaining contribution from $W - w^{(d)} \equiv w^{(i)} + \bar{W}$ (the superscript (i) of the scattering matrix (5) [17]. Using the set of equations depicted in Fig. 1 one can prove the unitarity of the scattering matrix (5) [17].

**Markov approximation and non-Markovian effects.**—If the qubits are situated sufficiently close to each other, $\Gamma_{1,2} R < 1$, the approach of Lehmann [11] becomes valid. This approach has recently been applied [10] for the theoretical description of an actual experiment [3]. It has also been recognized that Lehmann’s approximation coincides with the Markov approximation in the master equation treatment of the problem [10, 14].

In the present framework, Markovian and non-Markovian contributions can be readily distinguished from each other. Noticing that $p_j \sim \Gamma_{1,2}$ and $\bar{W}^{(i)} \sim e^{-\Gamma_{1,2} R} - \sim \Re((\Gamma_{1,2} R)^2)$ for small $R$, we
can neglect in this regime all contributions containing $W^{(i)}$. Thus, Markov approximation implies $W^{(i)} \to w^{(i)}$, $V_{s'} \to v_{s'}$, $V_{s}^{\dagger} \to v_{s}^{\dagger} M(E - k) v_{s}$, and $W_{s,s'}^{(r)} \to v_{s'} E - \Omega_{s} - \Omega_{s'} v_{s}$. Inspecting the diagrams surviving after this replacement, we establish that Markov approximation is equivalent to the non-crossing approximation (NCA) which neglects all vertex corrections.

Non-Markovian effects are neither present in the single-photon scattering nor in the elastic part of the two-photon scattering (because of their independence of $W^{(i)}$). They also vanish, if either of the couplings $g_{1,2}$ becomes zero. Thus, non-Markovian effects are only present in the inelastic part of the two-photon scattering, and require at least two qubits. For this reason, we conclude that they originate from the interference of the two-photon states.

**Results.**— With our exact solution we can study the two-photon scattering in the whole parameter space. We consider the scattering of two counter-propagating photons at equal momenta $k_0$ from two qubits: One qubit is detuned from resonance to a lower frequency, $-\Omega$, whereas the other qubit is detuned to a higher frequency, $\Omega$ [see Fig. 2(a)]. The coupling strengths are given by $\Gamma_{1,2} = \Omega/2$, and $k_0 R = n \cdot 2\pi$, $n \in \mathbb{Z}$. This corresponds to full transparency for single-photon scattering, as can be seen by using the equations for $S^{(1)}$ presented above.

For practical purposes, it is convenient to choose the linearization point $k_0 = E/2$ for given $E$, and to define the energy differences $\Delta = k - (-k) = 2k$ and $\Delta' = k' - (-k') = 2k'$ of the initial and final states of photons, respectively. We also note that we can reproduce the results of Ref. [14] in the regime studied there.

The probability density for the scattering in the same or opposite directions is shown in Fig. 2(b). Both probability densities exhibit an interference pattern characteristic of a sinc-function: The envelope has a width of $\Gamma$ and it is filled with spikes broadened by $\sim 1/R$. The appearance of this interference pattern is only possible if $\Gamma R > 1$; it is essentially a non-Markovian effect. In the ultimate limit $R \to \infty$, the spikes become delta-peaked and dense, but the envelope vanishes, removing all inelastic effects. For finite $R$, the probability density to scatter in opposite directions dominates for $\Delta'/2 < \Omega$ whereas for $\Delta'/2 > \Omega$ photons tend to scatter in the same direction.

As the second qubit is gradually decoupled from the photons, the interference pattern weakens and ultimately vanishes. This can be seen in Fig. 2(c). Simultaneously, the overall probability for the photons to scatter in the same direction decreases.

The second order correlation function, $g_{a_1 a_2}^{(2)}(t) = \langle \Psi_f | a_{a_1}^\dagger(x) a_{a_2}^\dagger(x + t) a_{a_2}(x + t) a_{a_1}(x) | \Psi_f \rangle$, is an experimentally accessible quantity. It is shown in Fig. 3. For $t$ being an even multiple of $R/v_g$ we find strong bunching of photons scattering to the same direction, whereas for odd multiples photons exhibit antibunching when scattering to opposite directions. This can be understood as photons making multiple round-trips between the qubits, and the antibunching as an inelastic counterpart of the Hong–Ou–Mandel effect.

**Summary.**— We have presented an exact analytical solution to the problem of two-photon scattering from two distant qubits based on a full resummation of diagrams. We expect that the problem of $N$ photons scattering from two qubits can be solved exactly in terms of the two-photon vertex $W$, since simultaneous absorption of three or more photons is not possible (i.e., no emergence of irreducible three- or more-photon vertices). We also conjecture that the problem of $M$ photons scattering from $M$ qubits is solved exactly by the $2(M - 1)$-crossing
approximation, by generalizing our approach. We have also explored the inelastic effects in the scattering from a particular two-qubit configuration, and found prominent signatures of two-photon interference, reminiscent of the Hong–Ou–Mandel effect.

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[1] H. J. Kimble, Nature (London) 453, 1023 (2008).
[2] J.-T. Shen and S. Fan, Phys. Rev. Lett. 95, 213001 (2005); J.-T. Shen and S. Fan, Phys. Rev. A 76, 062709 (2007).
[3] A. F. van Loo, A. Fedorov, K. Lalumiere, B. C. Sanders, A. Blais, and A. Wallraff, Science 342, 1494 (2013).
[4] O. Astafiev, A. M. Zagoskin, A. A. Abdumalikov, Y. A. Pashkin, T. Yamamoto, K. Inomata, Y. Nakamura, and J. S. Tsai, Science 327, 840 (2010).
[5] I.-C. Hoi, C. M. Wilson, G. Johansson, T. Palomaki, B. Peropadre, and P. Delsing, Phys. Rev. Lett. 107, 073601 (2011).
[6] M. Pletyukhov and V. Gritsev, New. J. Phys. 14, 095028 (2012).
[7] J. M. Bendickson, J. P. Dowling, and M. Scalora, Phys. Rev. E 53, 4107 (1996).
[8] T. S. Tsoi and C. K. Law, Phys. Rev. A 78, 063832 (2008).
[9] C. Gonzalez-Ballestero, F. J. Garcia-Vidal, and E. Moreno, New J. Phys. 15, 073015 (2013).
[10] K. Lalumiere, B. C. Sanders, A. F. van Loo, A. Fedorov, A. Wallraff, and A. Blais, Phys. Rev. A 88 043806 (2013).
[11] R. H. Lehmberg, Phys. Rev. 2, 883 (1970); ibid 2, 889 (1970).
[12] M. Hafezi, D. E. Chang, V. Gritsev, E. Demler, and M. D. Lukin, Phys. Rev. A 85, 013822 (2012).
[13] R. Konik and A. LeClair, Phys. Rev. B 58, 1872 (1998).
[14] H. Zheng and H. U. Baranger, Phys. Rev. Lett. 110, 113601 (2013); Y.-L. L. Fang, H. Zheng, and H. U. Baranger, EPJ Quantum Technology 1, 3 (2014).
[15] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. 59, 2044 (1987).
[16] J. R. Taylor, Scattering Theory: The Quantum Theory of Nonrelativistic Collisions (John Wiley, 1972).
[17] See the Supplemental Material.
Scattering of two photons from two distant qubits: Exact solution
Supplemental Material

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I. DERIVATION OF THE HAMILTONIAN

We consider the following model

\[
H = \int dk \left\{ \left[ \omega_0 + (k - k_0) \right] a_{R,k-k_0}^\dagger a_{R,k-k_0} + \left[ \omega_0 - (k + k_0) \right] a_{L,k+k_0}^\dagger a_{L,k+k_0} \right\} + \frac{\omega_1}{2} \sigma_z^{(1)} + \frac{\omega_2}{2} \sigma_z^{(2)} + v, \tag{S1}\]

where \(k_0, \omega_0\) and \((-k_0, \omega_0)\) are the linearization points of the right-moving (R) and left-moving (L) branches of the photon spectrum. For convenience, we count the states of photons starting from these points. The qubits’ transition frequencies \(\omega_{1,2}\) are commensurate with \(\omega_0\).

Performing the gauge transformation \(U^\dagger HU + i(\partial_t U^\dagger)U\)

\[
U = \exp \left[ -i \omega_0 t \left( a_{R,k-k_0}^\dagger a_{R,k-k_0} + a_{L,k+k_0}^\dagger a_{L,k+k_0} + \frac{\sigma_z^{(1)}}{2} + \frac{\sigma_z^{(2)}}{2} \right) \right], \tag{S3}\]

we eliminate the large energy scales \(\omega_{1,2}\) and \(\omega_0\), and obtain the low-energy Hamiltonian

\[
H = \int dk \left\{ (k - k_0) a_{R,k-k_0}^\dagger a_{R,k-k_0} - (k + k_0) a_{L,k+k_0}^\dagger a_{L,k+k_0} \right\} + \frac{\Omega_1}{2} \sigma_z^{(1)} + \frac{\Omega_2}{2} \sigma_z^{(2)} + v, \tag{S4}\]

where \(\Omega_{1,2} = \omega_{1,2} - \omega_0\). We note that the interaction \(v\) remains intact under the transformation (S3), since \([v, U] = 0\).

Next, we shift the integration variable \(k \rightarrow k + k_0 (k \rightarrow k - k_0)\) for the right- (left-) movers, and define the two modes \(a_{1k} = a_{Rk}\) and \(a_{2k} = a_{L,-k}\). Then, assuming an infinite range of integration, we obtain the Hamiltonian

\[
H = \sum_{\alpha=1,2} \int dk k a_{\alpha k}^\dagger a_{\alpha k} + \sum_{\beta=1,2} \frac{\Omega_\beta}{2} \sigma_z^{(\beta)} + v, \tag{S5}\]

\[
\begin{align*}
v &= \sum_{\alpha=1,2} \sum_{\beta=1,2} g_{\beta} \int dk \left\{ \sigma_z^{(\beta)} e^{-ic_{\alpha} c_{\beta} (kR + \varphi)/2} a_{\alpha k}^\dagger + \text{h.c.} \right\}, \tag{S6}\end{align*}\]

where \(c_1 = 1, c_2 = -1\), and the dependence on \(k_0\) appears only through the phase \(\varphi = k_0R\).

II. DETAILS OF THE SOLUTION

A. Parameterization of the effective two-photon vertex \(W_s^{(i)}\)

Introducing the bare vertex

\[
v_s = v_{ak} = \sum_{\beta=1,2} g_{\beta} \sigma_z^{(\beta)} e^{-ic_\alpha c_\beta (kR + \varphi)/2}, \tag{S7}\]

where \(s = \{\alpha, k\}\) is the multi-index including the channel index \(\alpha\), we represent the interaction term (S6) as

\[
v = v_s a_s^\dagger + v_s^\dagger a_s. \tag{S8}\]

From here on we assume the convolution (the summation and the integration) over repeated indices, unless explicitly stated otherwise.
We also introduce another set of field operators

\[ a_r \equiv a_k^{(\beta)} = \sum_\alpha a_{\alpha k} e^{ic_\alpha c_\beta (kR+\phi)/2} = t_{rs} a_s, \quad t_{rs} = e^{ic_\alpha c_\beta (kR+\phi)/2}, \]  

(S9)

where \( r = \{ \beta, k \} \) is the multi-index including the qubit’s index \( \beta \). Note \( a_r \) and \( a^\dagger_r \) do not obey the standard bosonic commutation relations, since the matrix \( t_{rs} \) is not unitary. In the basis (S9) the interaction (S6) reads

\[ v = v_r a^\dagger_r + v^\dagger_r a_r, \]  

(S10)

where \( v^\dagger_r = g_\beta a^\dagger_+ = \sqrt{\frac{\Gamma_s}{\pi}} a^\dagger_+ \), and \( v_r = t_{rs} v^\dagger_r \). Analogously, we have the two representations for the effective one- and two-photon vertices

\[ V = V_s(E) a^\dagger_+ + V_+^\dagger_+ (E) a_s = V_r(E) a^\dagger_+ + V^\dagger_r (E) a_r, \]  

(S11)

\[ W = W_s(E) a^\dagger_+ a_s = W_r(E) a^\dagger_+ a_r, \]  

(S12)

with the following transformation laws

\[ V_s(E) = V_r(E) t^\ast_{rs}, \]  

(S13)

\[ V^\dagger_s(E) = V^\dagger_r(E) t_{rs}, \]  

(S14)

\[ W_s(E) = W_r(E) t^\ast_{rs} t_{rs}. \]  

(S15)

In particular, using the last one we establish

\[ w^{(i)}_{\ast s}(E) = \frac{P_{- \ast}}{E - k - k' + i\eta} v^\dagger_{s'} = \sqrt{\frac{\Gamma_\beta}{\pi}} \frac{\sigma^\dagger_+ \sigma^\dagger_+}{E - k - k' + i\eta} t^\ast_{s's'} t_{rs} = w^{(i)}_r (E) t^\ast_{s's'} t_{rs}, \]  

(S16)

\[ w^{(i)}_{r'r'}(E) = \frac{\sqrt{\frac{\Gamma_\beta}{\pi}} \sigma^\dagger_+ \sigma^\dagger_+}{E - k - k' + i\eta}, \]  

(S17)

We rewrite the equation for the irreducible two-photon vertex \( W^{(i)}_{s's} = w^{(i)}_{s's} + w^{(i)}_{s's} M(E - k_1) W^{(i)}_{s's} = w^{(i)}_{s's} + W^{(i)}_{s's} M(E - k_1) W^{(i)}_{s's} \) in the new basis (S9)

\[ W^{(i)}_{r'r'}(E) = w^{(i)}_{r'r'}(E) + w^{(i)}_{r'r'}(E) t_{r_1 s_1} M(E - k_1) t^\ast_{r_1 s_1} \delta_{k_1 k_1} w^{(i)}_{r'r'}(E) \]  

(S18)

\[ = w^{(i)}_{r'r'}(E) + W^{(i)}_{r'r'}(E) t_{r_1 s_1} M(E - k_1) t^\ast_{r_1 s_1} \delta_{k_1 k_1} w^{(i)}_{r'r'}(E). \]  

(S19)

Here

\[ M(E) = \frac{(E - \tilde{\Omega}_2) P_{-} + (E - \tilde{\Omega}_1) P_{-} - 2i \sqrt{\Gamma_1 \Gamma_2} e^{iER+i\phi} (\sigma^\dagger_+ \sigma^\dagger_+ (2) + \sigma^\dagger_+ \sigma^\dagger_+ (2))}{(E - \Omega_1)(E - \Omega_2) + 4 \Gamma_1 \Gamma_2 e^{2iER+i\phi}} \]  

(S20)

is the projection of the qubits’ Green’s function onto the one-excitation subspace, and \( \tilde{\Omega}_{1,2} = \Omega_{1,2} - 2i\Gamma_{1,2} \).

Next, we note that all multiplicands in (S18) and (S19) depending on the integration variable \( k_1 (k'_1) \) are analytic in the lower complex half-plane of this variable, which is inferred from (S16) and (S20). Then, in order to get a finite contribution from the integration over \( k_1 \), it is necessary to prohibit the closure of the integration contour in this half-plane. Considering the factor

\[ t_{r_1 s_1} t^\ast_{r_1 s_1} = e^{i(\beta_1 - \beta'_1)(kR+\phi)/2} + e^{-i(\beta_1 - \beta'_1)(kR+\phi)/2}, \]  

(S21)

we see that this is only possible if \( c_{\beta_1} = -c_{\beta'_1} \), or \( \beta_1 = \tilde{\beta}_1 \), where \( \tilde{\beta} = (\beta + 1) \mod 2 \). In this case we acquire the factor \( e^{i(k_1 R + i\phi)} \), under the integral over \( k_1 \) in (S18) and (S19). This is, however, insufficient for the off-diagonal part of \( M(E - k_1) \), which contains itself the factor \( e^{i(E - k_1) R + i\phi} \), canceling the exponential dependence on \( k_1 \) in the numerator of the integrand and thus annihilating the off-diagonal contribution. As for the diagonal part of \( M(E - k_1) \), we notice that

\[ \int dk_1 h(k_1) e^{ik_1 R + i\phi} \left[ \frac{E - k_1 - \tilde{\Omega}_{2,1}}{(E - k_1 - \tilde{\Omega}_1)(E - k_1 - \tilde{\Omega}_2) + 4 \Gamma_1 \Gamma_2 e^{2i(E - k_1) R + 2i\phi} - \frac{1}{E - k_1 - \tilde{\Omega}_{1,2}}} \right] = 0 \]  

(S22)
for an arbitrary function $h(k_1)$, which is analytic in the lower half-plane and does not grow there faster than $e^{ik_1 R}$. Since in practice we always have $h \sim e^0$, we can therefore replace in (S18) and (S19)

$$
M(E - k_1) \rightarrow \tilde{M}_{k_1} = e^{ik_1 R + i\varphi} \left[ \frac{P_+}{E - k_1 - \Omega_1} + \frac{P_-}{E - k_1 - \Omega_2} \right] \equiv \tilde{M}_{k_1}^+ P_+ + \tilde{M}_{k_1}^- P_-. 
$$

(S33)

Thus, introducing $\tilde{r}_1 = \{\tilde{\beta}_1, k_1\}$ we cast (S18) and (S19) to

$$
\mathcal{W}^i_{r' r} = w^{(i)}_{r' r} + w^{(i)}_{r' r} \tilde{M}_{k_1} w^{(i)}_{r' r} = w^{(i)}_{r' r} + \mathcal{W}^i_{r' r} \tilde{M}_{k_1} w^{(i)}_{r' r}. 
$$

(S24)

Let us now define the kernels

$$
\tau^{(r-r)}_{r' r} = w^{(i)}_{r' r} \tilde{M}_{k_1} w^{(i)}_{r' r}, \quad \tau^{(r-r)}_{r' r} = \tilde{M}_{k_1} w^{(i)}_{r' r} \tilde{M}_{k_1} w^{(i)}_{r' r},
$$

(S25)

which are related to each other by $\tilde{M}_{k_1} \tau^{(r-r)}_{r' r} = \tau^{(r-r)}_{r' r} \tilde{M}_{k_1}$ (no integration over $k, k'$); and represent

$$
\mathcal{W}^i_{r' r} = \mathcal{W}^i_{r' r} + \mathcal{W}^i_{r' r} + \mathcal{W}^i_{r' r},
$$

(S27)

where $\mathcal{W}^a_{r' r}$ are the sums (over $n = 1, 2, \ldots$) of every $(4n + a)$-th term, $a = 0, 1, 2, 3$, in the series resulting from the iteration of the equation (S24). The objects $\mathcal{W}^a_{r' r}$ obey the following equations

$$
\mathcal{W}^0_{r' r} = \tau^{(r-r)}_{r' r},
$$

(S28)

$$
\mathcal{W}^1_{r' r} = \mathcal{W}^i_{r' r} + \mathcal{W}^i_{r' r},
$$

(S29)

$$
\mathcal{W}^2_{r' r} = \mathcal{W}^i_{r' r} \tilde{M}_{k_1} \mathcal{W}^i_{r' r} = \mathcal{W}^i_{r' r} \tilde{M}_{k_1} w^{(i)}_{r' r},
$$

(S30)

$$
\mathcal{W}^3_{r' r} = \mathcal{W}^i_{r' r} \tilde{M}_{k_1} \mathcal{W}^2_{r' r} = \mathcal{W}^i_{r' r} \tilde{M}_{k_1} w^{(i)}_{r' r},
$$

(S31)

We also note the alternative representation of Eqs. (S28) and (S29)

$$
\mathcal{W}^0_{r' r} = w^{(i)}_{r' r} \tilde{M}_{k_1} w^{(i)}_{r' r}, \quad \mathcal{W}^3_{r' r} = w^{(i)}_{r' r} + \tau^{(r-r)}_{r' r} \mathcal{W}^i_{r' r},
$$

(S32)

The crucial observations are that $\tau^{(r-r)}_{r' r}, \tau^{(r-r)}_{r' r} \sim \delta^{r_2}_{r_1}$ in the photon basis, and that they both are diagonal and spanned by $P_+$ and $P_-$ in the qubits’ basis. This allows us to parameterize

$$
\tau^{(r-r)}_{r' r} = \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^0_{r' r} + \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^1_{r' r} + \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^2_{r' r} + \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^3_{r' r},
$$

(S33)

and consequently

$$
\mathcal{W}^0_{r' r} = \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^0_{r' r} + \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^1_{r' r} + \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^2_{r' r} + \tau^{(i)}_{r' r} \delta^{r_2}_{r_1} \mathcal{W}^3_{r' r},
$$

(S34)

Thus, there are sixteen components $\mathcal{W}^a_{r' r}$ parameterizing the two-photon vertex $\mathcal{W}^a_{r' r}$. However, they are not independent but are related to each other by the operations of transposition $T$ (i.e., $k' \rightarrow k k'$) and exchange of the qubits’ parameters $I$ (i.e., $\Omega_1 \leftrightarrow \Omega_2$) as follows

$$
T_{k_1} \rightarrow T_{k_1}, \quad S_{k_1} \rightarrow S_{k_1}, \quad \hat{S}_{k_1} \rightarrow \hat{S}_{k_1}, \quad \hat{T}_{k_1} \rightarrow \hat{T}_{k_1},
$$

(S39)

$$
T_{k_1} \rightarrow T_{k_1}, \quad S_{k_1} \rightarrow S_{k_1}, \quad \hat{S}_{k_1} \rightarrow \hat{S}_{k_1}, \quad \hat{T}_{k_1} \rightarrow \hat{T}_{k_1},
$$

(S40)

$$
T_{k_1} \rightarrow T_{k_1}, \quad S_{k_1} \rightarrow S_{k_1}, \quad \hat{S}_{k_1} \rightarrow \hat{S}_{k_1}, \quad \hat{T}_{k_1} \rightarrow \hat{T}_{k_1},
$$

(S41)

$$
T_{k_1} \rightarrow T_{k_1}, \quad S_{k_1} \rightarrow S_{k_1}, \quad \hat{S}_{k_1} \rightarrow \hat{S}_{k_1}, \quad \hat{T}_{k_1} \rightarrow \hat{T}_{k_1},
$$

(S42)
To establish an explicit form of all these components, it is sufficient to solve an equation for one of them, e.g., $T_{k'k}^{(1)2+}$. Inserting the parameterization (S36) into (S33) we obtain the equation

$$T_{k'k}^{(1)2+} = \frac{\Gamma_1}{\pi} \frac{1}{E - k' - k + i\eta} + \tau_{k'k}^{(1)2+},$$  
(S43)

which is decoupled from the rest components. Here

$$\tau_{k'k}^{(1)2+} = \frac{\Gamma_1^2 \Gamma_2^2}{\pi^4} \frac{1}{E - k' - k_1 + i\eta} \tilde{M}_{k_1}^+ \frac{1}{E - k_1 - k_2 + i\eta} \tilde{M}_{k_2}^- \frac{1}{E - k_2 - k_3 + i\eta} \tilde{M}_{k_3}^- \frac{1}{E - k_3 - k + i\eta} \tilde{M}_k^+$$  
(S44)

(note that there is no integration over $k$).

The equation (S43) is an inhomogeneous Fredholm integral equation. It is the core equation in the paper, since it cannot be further reduced to a simpler form. Nevertheless, it allows for an explicit analytic solution, see the next section.

The other components $T_{k'k}^{(a)2+}$ are obtained from $T_{k'k}^{(1)2+}$ by means of the convolutions

$$T_{k'k}^{(2)2+} = \frac{\sqrt{\Gamma_1 \Gamma_2}}{\pi} \frac{1}{E - k' - k + i\eta} \tilde{M}_{k_1}^+ T_{k'k}^{(1)2+},$$  
(S45)

$$T_{k'k}^{(3)2+} = \frac{\Gamma_2}{\pi} \frac{1}{E - k' - k_1 + i\eta} \tilde{M}_{k_1}^- T_{k'k}^{(2)2+},$$  
(S46)

$$T_{k'k}^{(2)1+} = \frac{\sqrt{\Gamma_1 \Gamma_2}}{\pi} \frac{1}{E - k' - k_1 + i\eta} \tilde{M}_{k_1}^- T_{k'k}^{(3)2+},$$  
(S47)

following from (S30),(S31), and (S32), respectively. Additionally we quote the relations following from (S29) and (S31), which are necessary for computing $T_{k'k}^{(1)1+}$ and $T_{k'k}^{(3)1+}$,

$$T_{k'k}^{(1)1+} = \frac{\sqrt{\Gamma_1 \Gamma_2}}{\pi} \frac{1}{E - k' - k + i\eta} + \frac{\Gamma_1}{\pi} \frac{1}{E - k' - k_1 + i\eta} \tilde{M}_{k_1}^+ T_{k'k}^{(1)1+},$$  
(S48)

$$T_{k'k}^{(3)1+} = \frac{\sqrt{\Gamma_1 \Gamma_2}}{\pi} \frac{1}{E - k' - k_1 + i\eta} \tilde{M}_{k_1}^- T_{k'k}^{(3)1+}.$$  
(S49)

The components $T_{k'k}^{(1)1+}, T_{k'k}^{(2)1+}$ appearing here as well as all the other remaining ones follow from the symmetry relations (S39)-(S42).

### B. Solution of the integral equation (S43)

To solve (S43), we make the ansatz

$$T_{k'k}^{(1)2+} = \frac{\Gamma_1}{\pi} \frac{1}{E - k' - k + i\eta} - \frac{i\Gamma_1}{\pi} \int_0^R dx \int_0^R dx' e^{i(E/2-k')x'} F(x', x) e^{i(E/2-k)x},$$  
(S50)

where $F(x', x) = F(x, x')$ is a continuous function in the square $[0, R] \times [0, R]$. Inserting it into (S43), we obtain the following integral equation for $F(x', x)$ in the coordinate representation

$$F(x', x) = f^{(4)}(x', x) + \int_0^R dx_1 f^{(4)}(x_1, x) F(x_1, x),$$  
(S51)

where

$$f^{(4)}(x', x) = \Theta(x') \Theta(x) \int_0^R dx_1 \int_0^R dx_2 \int_0^R dx_3 f^+(x' + x_1) f^-(x_1 + x_2) f^- (x_2 + x_3) f^+(x_3 + x)$$  
(S52)

and

$$f^{\pm}(x) = 2\Gamma_{1,2} \Theta(R - x) e^{iER/2+i\varphi} e^{i(E/2-\Omega_1,\lambda)(R-x)} = 2\Gamma_{1,2} \Theta(R - x) e^{iER/2+i\varphi} e^{i(\lambda\mp b)(R-x)}. $$  
(S53)

Here we have introduced $\lambda = \frac{1}{2}(E - \tilde{\Omega}_1 - \tilde{\Omega}_2)$ and $b = \frac{1}{2}(\tilde{\Omega}_1 - \tilde{\Omega}_2)$. 
Let us now define the differential operators

\[ l_\pm \left( \frac{d}{dx} \right) = -\frac{e^{-iER/2-i\varphi}}{2\Gamma_{1,2}} \left[ \frac{d}{dx} + i(\lambda \mp b) \right], \tag{S54} \]

such that

\[ l_+ \left( \frac{d}{dx} \right)^2 f^+(x) = \delta(R-x), \tag{S55} \]
\[ l_- \left( \frac{d}{dx} \right)^2 f^-(x) = \delta(R-x). \tag{S56} \]

This leads us to the equality

\[ l_+ \left( -\frac{d}{dx} \right)^2 l_- \left( -\frac{d}{dx} \right)^2 l_- \left( -\frac{d}{dx} \right) \left( \frac{d}{dx} \right) f^{(4)}(x', x) = \delta(x' - x), \tag{S57} \]

which helps us to convert the integral equation (S43) into the differential one

\[ \left[ l_+ \left( -\frac{d}{dx} \right)^2 l_- \left( -\frac{d}{dx} \right)^2 l_- \left( -\frac{d}{dx} \right) \right] F(x', x) = \delta(x' - x), \tag{S58} \]

or

\[ \left[ \frac{d^4}{dx'^4} + 2(\lambda^2 + b^2)\frac{d^2}{dx'^2} + (\lambda^2 - b^2)^2 + 4\lambda^2 \nu^2 \right] F(x', x) = -4\lambda^2 \nu^2 \delta(x' - x), \tag{S59} \]

where \( \nu = \frac{4i\Gamma_1\Gamma_2 e^{E_R/2+i\varphi}}{E - \Omega_1 - \Omega_2} \). It is accompanied with the following boundary conditions

\[ F(R, x) = 0, \tag{S60} \]
\[ l_+ \left( \frac{d}{dx} \right) F(x', x) \bigg|_{x' = 0} = 0 \quad \quad \Rightarrow \quad \quad \frac{dF}{dx'}(0, x) = -i(\lambda - b)F(0, x), \tag{S61} \]
\[ l_- \left( -\frac{d}{dx} \right) l_+ \left( -\frac{d}{dx} \right) F(x', x) \bigg|_{x' = R} = 0 \quad \quad \Rightarrow \quad \quad \frac{d^2F}{dx'^2}(R, x) = 2ib \frac{dF}{dx'}(R, x), \tag{S62} \]
\[ l_- \left( -\frac{d}{dx} \right) l_- \left( -\frac{d}{dx} \right) l_- \left( -\frac{d}{dx} \right) F(x', x) \bigg|_{x' = 0, x = 0} = f^+(R^-) \quad \Rightarrow \quad \frac{d^3F}{dx'^3}(0^+, 0) = -i(\lambda - b) \frac{d^2F}{dx'^2}(0^+, 0) - 4\lambda^2 \nu^2. \tag{S63} \]

We solve (S59) by means of the ansatz

\[ F(x', x) = \Theta(x' - x) \sum_j C_{ji} e^{ip_j x' + ip_j x} + \Theta(x - x') \sum_j C_{ji} e^{ip_j x' + ip_j x'}, \tag{S64} \]

where \( p_{1,3} = \sqrt{\lambda^2 + b^2 \pm 2\lambda \sqrt{\nu^2 + b^2}}, p_{2,4} = -p_{1,3}, \) are the roots of the characteristic equation

\[ p^4 - 2(\lambda^2 + b^2)p^2 + (\lambda^2 - b^2)^2 + 4\lambda^2 \nu^2 = 0. \tag{S65} \]

Inserting (S64) into (S59) we establish after a lengthy calculation the following form of the coefficients \( C_{ji} \):

\[ C_{11} = Z \left( \frac{p_1 + b - \lambda}{|p_1|} \right)^2 - \frac{i\lambda \nu^2}{2p_1 \sqrt{b^2 - \nu^2}} \frac{e^{-ip_1 R(p_1 + b - \lambda)}}{|p_1|}, \tag{S66} \]
\[ C_{22} = Z \left( \frac{p_1 + \lambda - b}{|p_1|} \right)^2 + \frac{i\lambda \nu^2}{2p_1 \sqrt{b^2 - \nu^2}} \frac{e^{ip_1 R(p_1 + \lambda - b)}}{|p_1|}, \tag{S67} \]
\[ C_{33} = Z \left( \frac{p_3 + b - \lambda}{|p_3|} \right)^2 + \frac{i\lambda \nu^2}{2p_3 \sqrt{b^2 - \nu^2}} \frac{e^{-ip_3 R(p_3 + b - \lambda)}}{|p_3|}, \tag{S68} \]
\[ C_{44} = Z \left( \frac{p_3 + \lambda - b}{|p_3|} \right)^2 - \frac{i\lambda \nu^2}{2p_3 \sqrt{b^2 - \nu^2}} \frac{e^{ip_3 R(p_3 + \lambda - b)}}{|p_3|}. \tag{S69} \]
\[ C_{12} = Z \frac{(p_1 + b - \lambda)(p_1 + \lambda - b)}{[p_1]^2} - \frac{i\lambda \nu^2}{2p_1 \sqrt{b^2 - \nu^2}} e^{-ip_1 R (p_1 + \lambda - b)}, \] 
(S70)

\[ C_{21} = Z \frac{(p_1 + b - \lambda)(p_1 + \lambda - b)}{[p_1]^2} + \frac{i\lambda \nu^2}{2p_1 \sqrt{b^2 - \nu^2}} e^{ip_1 R (p_1 + b - \lambda)}, \] 
(S71)

\[ C_{34} = Z \frac{(p_3 + b - \lambda)(p_3 + \lambda - b)}{[p_3]^2} + \frac{i\lambda \nu^2}{2p_3 \sqrt{b^2 - \nu^2}} e^{-ip_3 R (p_3 + \lambda - b)}, \] 
(S72)

\[ C_{43} = Z \frac{(p_3 + b - \lambda)(p_3 + \lambda - b)}{[p_3]^2} - \frac{i\lambda \nu^2}{2p_3 \sqrt{b^2 - \nu^2}} e^{ip_3 R (p_3 + \lambda - b)}, \] 
(S73)

\[ C_{13} = C_{31} = -Z \frac{(p_1 + b - \lambda)(p_3 + b - \lambda)}{[p_1][p_3]}, \] 
(S74)

\[ C_{14} = C_{41} = -Z \frac{(p_1 + b - \lambda)(p_3 + \lambda - b)}{[p_1][p_3]}, \] 
(S75)

\[ C_{23} = C_{32} = -Z \frac{(p_1 + \lambda - b)(p_3 + b - \lambda)}{[p_1][p_3]}, \] 
(S76)

\[ C_{24} = C_{42} = -Z \frac{(p_1 + \lambda - b)(p_3 + \lambda - b)}{[p_1][p_3]}, \] 
(S77)

where

\[ Z = -\frac{2i\lambda \nu^2 b}{\sqrt{b^2 - \nu^2}} \frac{1}{p_1^2 - 2bp_1 \frac{[p_1]}{p_1} - p_3^2 + 2bp_3 \frac{[p_3]}{p_3}}, \] 
(S78)

and

\[
[p] = e^{ipR} (p + b - \lambda) + e^{-ipR} (p + \lambda - b),
\]
(S79)

\[
\{p\} = e^{ipR} (p + b - \lambda) - e^{-ipR} (p + \lambda - b).
\]
(S80)

We note that the matrix \(C_{jl}\) is not symmetric, and the jumps of the derivatives of \(F(x', x)\) at \(x' = x\) are encoded in finite \(\Delta C_{12} = C_{12} - C_{21} \neq 0\) and \(\Delta C_{34} = C_{34} - C_{43} \neq 0\).

Performing the convolutions in Eqs. (S45)-(S47), we obtain

\[
\mathcal{T}_{k'k}^{(2)2+} = e^{-3iER/2-3i\varphi} \frac{2\pi}{4\Gamma_2 \sqrt{\Gamma_1 \Gamma_2}} \int_0^R dx \int_0^R dx' e^{i(E/2-k')x'} e^{i(E/2-k)x} 
\]

\[
\times \left[ \Theta(R - x' - x) \sum_{jl} C_{jl}[(p_j - b)^2 - \lambda^2](p_j + \lambda + b)e^{ip_j (R-x') + ip_j x} + \Theta(x + x' - R) \sum_{jl} C_{jl}[(p_l - b)^2 - \lambda^2](p_l + \lambda + b)e^{ip_l x' + ip_l x} \right], \] 
(S81)

\[
\mathcal{T}_{k'k}^{(3)2+} = -i e^{-iER-2i\varphi} \frac{4\pi}{4\Gamma_2 \sqrt{\Gamma_1 \Gamma_2}} \int_0^R dx \int_0^R dx' e^{i(E/2-k')x'} e^{i(E/2-k)x} 
\]

\[
\times \left[ \Theta(x' - x) \sum_{jl} C_{jl}[(p_j - b)^2 - \lambda^2]e^{ip_j x' + ip_j x} + \Theta(x - x') \sum_{jl} C_{jl}[(p_l - b)^2 - \lambda^2]e^{ip_l x + ip_l x'} \right], \] 
(S82)
\[ T_{k'k}^{2+} = e^{-iER/2-i\phi} \left\{ \int_0^R dx \int_0^R dx' e^{i(E/2-k')x'} e^{i(E/2-k)x} \right. \]
\[ \times \left. \left[ \Theta(R-x') \sum_{jl} C_{jl}(p_j + \lambda - b)e^{ip_j(R-x') + ip_lx} + \Theta(x + x' - R) \sum_{jl} C_{jl}(p_l + \lambda - b)e^{ip_l(R-x') + ip_jx} \right] \right\} . \]

(S83)

From the relations (S48),(S49) we establish

\[ T_{k'k}^{(1)+} = \sqrt{\frac{\Gamma_1 \Gamma_2}{\pi}} \frac{1}{E-k'-k+\imath\eta} + i\sqrt{\frac{\Gamma_1 \Gamma_2}{\pi}} \int_0^R dx \int_0^R dx' e^{i(E/2-k')x'} e^{i(E/2-k)x} \]
\[ \times \left[ \Theta(x-x') \sum_{jl} C_{jl}(p_j + \lambda - b)e^{ip_j(R-x') + ip_l(R-x)} + \Theta(x' - x) \sum_{jl} C_{jl}(p_j + \lambda - b)e^{ip_j(R-x') + ip_l(R-x')} \right] \]

(S84)

and

\[ T_{k'k}^{(3)+} = \frac{ie^{-iER-2i\phi}}{4\pi \Gamma_1} \int_0^R dx \int_0^R dx' e^{i(E/2-k')x'} e^{i(E/2-k)x} \]
\[ \times \left[ \Theta(x-x') \sum_{jl} C_{jl}(p_j + \lambda - b)(p_l + \lambda - b)e^{ip_j(R-x') + ip_l(R-x)} + \Theta(x' - x) \sum_{jl} C_{jl}(p_j + \lambda - b)(p_l + \lambda - b)e^{ip_j(R-x') + ip_l(R-x')} \right] . \]

(S85)

C. Explicit form of the effective one-photon vertices \( V_{v', v} \), the self-energy \( \Sigma_{++} \), and the reducible two-photon vertex \( W_{s's}^{(v)} \)

The equations \( V_{v'}(E) = v_{v'} + W_{s's}^{(1)}(E)M(E - k_1)v_{s_1} \) and \( \tilde{V}_s^{(1)}(E) = v_{s_1} M(E - k_1)W_{s's}^{(1)}(E) \) in the basis (S9) read

\[ V_{v'}(E) = v_{v'} + W_{s's}^{(1)}(E)M_{k_1}v_{s_1} = \sqrt{\frac{\Gamma_1}{\pi}} \sigma_-(1) \delta_{\beta'1} + \sqrt{\frac{\Gamma_2}{\pi}} \sigma_-(2) \delta_{\beta'2} \]
\[ + \int dk \left[ \left( \sqrt{\frac{\Gamma_1}{\pi}} T_{k'k}^{(1)+} \tilde{M}_{-k} + \sqrt{\frac{\Gamma_2}{\pi}} T_{k'k}^{(2)+} \tilde{M}_{-k} \right) \sigma_-(1) P_+ + \left( \sqrt{\frac{\Gamma_1}{\pi}} T_{k'k}^{(1)+} \tilde{M}_{k} + \sqrt{\frac{\Gamma_2}{\pi}} T_{k'k}^{(2)+} \tilde{M}_{k} \right) P_+ \sigma_-(2) \right] \delta_{\beta'1} \]
\[ + \int dk \left[ \left( \sqrt{\frac{\Gamma_1}{\pi}} T_{k'k}^{(1)+} \tilde{M}_{-k} + \sqrt{\frac{\Gamma_2}{\pi}} T_{k'k}^{(2)+} \tilde{M}_{-k} \right) P_+ \sigma_-(1) + \left( \sqrt{\frac{\Gamma_1}{\pi}} T_{k'k}^{(1)+} \tilde{M}_{k} + \sqrt{\frac{\Gamma_2}{\pi}} T_{k'k}^{(2)+} \tilde{M}_{k} \right) \sigma_+(1) P_+ \right] \delta_{\beta'2} \]
\[ = \left[ \sqrt{\frac{\Gamma_1}{\pi}} \sigma_- P_+ + f_{k'}^2 \sigma_- P_+ + f_{k'}^2 \sigma_+(1) P_+ \right] \delta_{\beta'1} + \left[ \sqrt{\frac{\Gamma_2}{\pi}} P_+ \sigma_-(2) + f_{k'} \sigma_+ P_+ \right] \delta_{\beta'2} \]

(S86)
and

\[ \tilde{V}^\dagger_r (E) = \tilde{V}^\dagger_s (E) = v^\dagger_r M_k \tilde{W}_{rr} (E) = \sqrt{\frac{\Gamma_1}{\pi}} \sigma_+ (1) \delta_{\beta_1} + \sqrt{\frac{\Gamma_2}{\pi}} \sigma_+ (2) \delta_{\beta_2} \]

(S87)

\[ + \int dk^r \left[ \left( \sqrt{\frac{\Gamma_1}{\pi}} M_{k r} T_{k r}^{-2} \right)^{\sigma_+ (1)} + \sqrt{\frac{\Gamma_2}{\pi}} M_{k r} T_{k r}^{-1} \right]^\dagger \left( \left( \sqrt{\frac{\Gamma_1}{\pi}} M_{k r} T_{k r}^{-2} \right)^{\sigma_+ (2)} + \sqrt{\frac{\Gamma_2}{\pi}} M_{k r} T_{k r}^{-1} \right)^\dagger \right] \delta_{\beta_2} \]

\[ + \int dk^r \left[ \left( \sqrt{\frac{\Gamma_1}{\pi}} M_{k r} T_{k r}^{1} \right)^{\sigma_+ (1)} + \sqrt{\frac{\Gamma_2}{\pi}} M_{k r} T_{k r}^{1} \right]^\dagger \left( \left( \sqrt{\frac{\Gamma_1}{\pi}} M_{k r} T_{k r}^{1} \right)^{\sigma_+ (2)} + \sqrt{\frac{\Gamma_2}{\pi}} M_{k r} T_{k r}^{1} \right)^\dagger \right] \delta_{\beta_2} \]

= \left[ \sqrt{\frac{\Gamma_1}{\pi}} \sigma_+ (1) f_k^{-1} + \sqrt{\frac{\Gamma_2}{\pi}} \sigma_+ (2) f_k^{-2} \right] \delta_{\beta_2} \]

where

\[ f_k^{-1} = \frac{i}{\sqrt{\pi \Gamma_1}} \int dk^r T_{k r}^{-2} \frac{2 \sqrt{\pi \Gamma_2}}{i \pi} \sum p_j \left[ 1 - \frac{2i \lambda \nu}{p_j (p_j + b)^2 - \lambda^2} \right] e^{ip_j R C_{jl}} e^{i(E/2 - k + p_i)R - 1}, \]

(S88)

\[ f_k^{-2} = \frac{i}{\sqrt{\pi \Gamma_1}} \int dk^r T_{k r}^{-1} \frac{2 \sqrt{\pi \Gamma_2}}{i \pi} \sum p_j \left[ 1 - \frac{2i \lambda \nu}{p_j (p_j + b)^2 - \lambda^2} \right] e^{ip_j R C_{jl}} e^{i(E/2 - k + p_i)R - 1}, \]

(S89)

and \( f_k^{1/2} = I[f_k^{2}], f_k^{-2} = I[f_k^{1/2}] \). We explicitly see that \( \tilde{V}^\dagger_r (E) \neq [V_r (E)]^\dagger \), since the functions \( f_k^{1/2, \pm} \neq (f_k^{2, \pm})^* \) are not real-valued.

The self-energy of the doubly excited state reads

\[ \Sigma_{++} = -2i(\Gamma_1 + \Gamma_2)P_{++} + (\tilde{V}^\dagger_r (E) - v^\dagger_r)M(E - k) v_s \]

\[ = -2i(\Gamma_1 + \Gamma_2)P_{++} + (\tilde{V}^\dagger_r (E) - v^\dagger_r)M(E - k) v_r \equiv -2i(\Gamma_1 + \Gamma_2 + \sigma_{++})P_{++}, \]

(S90)

where

\[ \sigma_{++} = \sqrt{\frac{\Gamma_1}{\pi}} \int dk^r f_k^{-1} M_{kr} + \sqrt{\frac{\Gamma_2}{\pi}} \int dk^r f_k^{-2} M_{kr}^* \]

\[ = \frac{8i \lambda^2 \nu}{p_j^2 - 2b p_j \frac{p_1}{p_1} - p_3^2 + 2b p_3 \frac{p_3}{p_3}} \left[ (\nu + ib) \left( \frac{\sin p_1 R}{p_1} - \sin p_3 R \frac{p_3}{p_3} \right) + i \sqrt{b^2 - \nu^2} \left( \frac{\sin p_1 R}{p_1} + \frac{\sin p_3 R}{p_3} \right) \right] \]

\[ + (b \to -b). \]

(S91)

Finally, we present the components of the reducible two-photon vertex \( W^{(r)} \) in the basis (S9)

\[ g_{++}^{-1} W^{(r)} = f_k^{1+} f_k^{1+} \delta_{\beta_1} \delta_{\beta_2} P_{++} + f_k^{2+} f_k^{2+} \delta_{\beta_1} \delta_{\beta_2} P_{++} + f_k^{1-} f_k^{1-} \delta_{\beta_1} \delta_{\beta_2} P_{--} + f_k^{2-} f_k^{2-} \delta_{\beta_1} \delta_{\beta_2} P_{--}, \]

(S92)

\[ + f_k^{1-} f_k^{1+} \delta_{\beta_1} (\sigma_{++}^{(1)} + \sigma_{--}^{(2)} + \sigma_{+1}^{(2)} + \sigma_{-1}^{(1)}) + f_k^{2-} f_k^{2+} \delta_{\beta_1} (\sigma_{++}^{(1)} + \sigma_{--}^{(2)} + \sigma_{+1}^{(2)} + \sigma_{-1}^{(1)}) \]

(S93)

\[ + f_k^{1-} f_k^{1+} \delta_{\beta_2} (\sigma_{++}^{(1)} + \sigma_{--}^{(2)} + \sigma_{+1}^{(2)} + \sigma_{-1}^{(1)}) + f_k^{2-} f_k^{2+} \delta_{\beta_2} (\sigma_{++}^{(1)} + \sigma_{--}^{(2)} + \sigma_{+1}^{(2)} + \sigma_{-1}^{(1)}) \]

(S94)

\[ + f_k^{1-} f_k^{1+} \delta_{\beta_2} (\sigma_{++}^{(1)} + \sigma_{--}^{(2)} + \sigma_{+1}^{(2)} + \sigma_{-1}^{(1)}) + f_k^{2-} f_k^{2+} \delta_{\beta_2} (\sigma_{++}^{(1)} + \sigma_{--}^{(2)} + \sigma_{+1}^{(2)} + \sigma_{-1}^{(1)}) \]

(S95)

where \( g_{++}^{-1} = E - \tilde{\Omega}_1 - \tilde{\Omega}_2 - \sigma_{++} \).
III. SCATTERING MATRIX AND THE UNITARITY CONDITION

The two-photon scattering matrix in the second-quantized representation and its hermitian conjugate read

\[
S^{(2)} = P_\sigma \left\{ \frac{1}{2} \delta_{s_1 s_2} \delta_{s_1 s_2} - 2\pi i \, v_{s_1} M(k_1') \left[ \delta_{k_1' k_1} \delta_{s_1 s_2} + W_{s_2 s_2} M(k_1) \delta_{k_1' k_1, k_1 + k_2} \right] v_{s_1}^\dagger \right\} P_\sigma a_{s_1}^\dagger a_{s_2} a_{s_1},
\]

(S96)

\[
S^{(2)*} = P_\sigma \left\{ \frac{1}{2} \delta_{s_1 s_2} \delta_{s_1 s_2} + 2\pi i \, v_{s_1} M(k_1') \left[ \delta_{k_1' k_1} \delta_{s_1 s_2} + W_{s_2 s_2}^\dagger M(k_1') \delta_{k_1' k_1, k_1 + k_2} \right] v_{s_1}^\dagger \right\} P_\sigma -a_{s_1}^\dagger a_{s_2}^\dagger a_{s_1} a_{s_2},
\]

(S97)

The unitarity condition \(S^{(2)*} S^{(2)} = 1\) implies the following optical theorem for the two-photon \(T\)-matrix

\[
0 = P_\sigma v_{s_1}^\dagger W_{s_2 s_2} M(k_1) \left[ \delta_{k_1' k_1} \delta_{s_1 s_2} + W_{s_2 s_2} M(k_1) \delta_{k_1' k_1, k_1 + k_2} \right] - M(k_1') \left[ \delta_{k_1' k_1} \delta_{s_1 s_2} + W_{s_2 s_2}^\dagger M(k_1) \delta_{k_1' k_1, k_1 + k_2} \right] v_{s_1} - v_{s_2} M(k_1) \left[ \delta_{k_1' k_1} \delta_{s_2 s_2} + W_{s_1 s_2} M(k_1) \delta_{k_1' k_1, k_1 + k_2} \right] v_{s_1} - v_{s_2} M(k_2) \left[ \delta_{k_1' k_1} \delta_{s_2 s_2} + W_{s_1 s_2} M(k_1) \delta_{k_1' k_1, k_1 + k_2} \right] v_{s_1} - v_{s_2},
\]

(S98)

where the convolutions should be performed over all indices. Below we prove that the term in the curly brackets in (S98) vanishes identically, which is sufficient for the validity of (S98), i.e.

\[
0 = M(k_1) \delta_{k_1' k_1} \delta_{s_1 s_2} - M(k_1') \delta_{k_1' k_1} \delta_{s_1 s_2} + 2\pi i M(k_1) \left[ \delta_{k_1' k_1} v_{s_1}^\dagger P_\sigma v_{s_1} \right] M(k_1) \delta_{k_1' k_1} \delta_{s_1 s_2},
\]

(S99)

where the convolutions should be performed only over the indices \(s_1'\) and \(s_2'\). Expanding (S99), we obtain

\[
0 = M(k_1) \delta_{k_1' k_1} \delta_{s_1 s_2} - M(k_1') \delta_{k_1' k_1} \delta_{s_1 s_2} + 2\pi i M(k_1) \left[ \delta_{k_1' k_1} v_{s_1}^\dagger P_\sigma v_{s_1} \right] M(k_1) \delta_{k_1' k_1} \delta_{s_1 s_2} + 2\pi i M(k_1) \left[ \delta_{k_1' k_1} v_{s_1}^\dagger P_\sigma v_{s_1} \right] M(k_1) \delta_{k_1' k_1} \delta_{s_1 s_2}
\]

Next, we notice the identities

\[
2\pi i M(k_1) \left[ \delta_{k_1' k_1} v_{s_1}^\dagger P_\sigma v_{s_1} \right] M(k_1) = M(k_1) \left[ \Sigma^\dagger_{M}(k_1) - \Sigma_{M}(k_1) \right] M(k_1)
\]

(S100)

and

\[
-2\pi i \delta_{E,k_1' k_1} v_{s_1}^\dagger P_\sigma v_{s_1'} \left\{ \delta_{E,k_1' k_1} v_{s_1}^\dagger P_\sigma v_{s_1'} \right\} = w_{s_1 s_1'}^{(i)} - w_{s_1 s_1'}^{(i)\dagger},
\]

(S101)
With their help we simplify (S100)
\[
0 \equiv M(V^{(k')}_{\pm}) \left\{ \overline{W}(i)_{s_2}^{(i)} + W^{(r)}_{s_2} - w^{(i)}_{s_2} M(E - k'_2) W_{s_2} - \overline{W}_{s_2} + W^{(r)}_{s_2} M'(E - k'_2) w^{(i)}_{s_2} \\
+ W^{(r)}_{s_2} M'(E - k'_2) \left( \overline{W}(i)_{s_2}^{(i)} + W^{(r)}_{s_2} - w^{(i)}_{s_2} M(E - k'_2) W_{s_2} \right) \right\}.
\] (S103)

After accounting the defining equation of the irreducible component \( \overline{W}^{(i)}_{s_2} = w^{(i)}_{s_2} M(E - k'_2) W_{s_2} \), it remains to prove
\[
0 \equiv W^{(r)}_{s_2} - w^{(i)}_{s_2} M(E - k'_2) W_{s_2} - W^{(r)}_{s_2} + W^{(r)}_{s_2} M'(E - k'_2) w^{(i)}_{s_2} \]
\[
+ W^{(r)}_{s_2} M'(E - k'_2) \left( W^{(r)}_{s_2} - w^{(i)}_{s_2} M(E - k'_2) W_{s_2} \right) \]
\[
- \left( W^{(r)}_{s_2} - W^{(r)}_{s_2} M'(E - k'_2) w^{(i)}_{s_2} \right) M(E - k'_2) W_{s_2}.
\] (S104)

Using the representation \( W^{(r)}_{s_2} = V_{s_2} G_{++} \overline{V}_{s_2} \) and the equation
\[
V_{s_2} - w^{(i)}_{s_2} M(E - k'_2) V_{s_2} = v_{s_2},
\] (S105)

following from the inversion of the equation defining \( V_{s_2} \), we conclude
\[
W^{(r)}_{s_2} - w^{(i)}_{s_2} M(E - k'_2) W^{(r)}_{s_2} = \left( V_{s_2} - w^{(i)}_{s_2} M(E - k'_2) V_{s_2} \right) G_{++} \overline{V}_{s_2} = v_{s_2} G_{++} \overline{V}_{s_2}.
\] (S106)

Then, (S104) acquires the form
\[
0 \equiv \left( v^{(i)}_{s_2} + v^{(i)}_{s_2} M(E - k'_2) W_{s_2} \right) G_{++} \overline{V}_{s_2} + V_{s_2} G_{++} \left( v^{(i)}_{s_2} + v^{(i)}_{s_2} M(E - k'_2) W_{s_2} \right) \]
\[
= \left( \overline{V}_{s_2} + v^{(i)}_{s_2} M(E - k'_2) W_{s_2} \right) G_{++} \overline{V}_{s_2} + V_{s_2} G_{++} \left( \overline{V}_{s_2} + v^{(i)}_{s_2} M(E - k'_2) W_{s_2} \right) \]
\[
= \overline{V}_{s_2} G_{++} \left( G_{++}^{-1} + v^{(i)}_{s_2} M(E - k'_2) v_{s_2} - v^{(i)}_{s_2} M(E - k'_2) V_{s_2} \right) G_{++} \overline{V}_{s_2}.
\] (S107)

Finally, we notice that \( \Sigma_{++} = P_{++} v^{(i)}_{s_2} M(E - k'_2) V_{s_2} P_{++} \), and the equality (S107) indeed holds, since \( G_{++}^{-1} - G_{++}^{-1} + \Sigma_{++}^{-1} - \Sigma_{++} \equiv 0 \). Thus, the two-photon scattering matrix (S96) is unitary.

### IV. VALIDITY OF THE MARKOV APPROXIMATION

Validity of the Markov approximation can be estimated by comparing results for the second order correlation function, shown in Fig. 1. It is evident that already for \( \Gamma R = 1 \) there are significant deviations between the exact and the Markovian results. Markov approximation tends to overestimate the amplitude of the oscillations, and it also misses the sharp feature (see inset) appearing at \( t/R = 1 \).
FIG. 1: Second order correlation function for two photons scattered to the same, or opposite directions for $\Gamma_1 = \Gamma_2 = \Gamma$, $\Omega_1 = -\Omega_2 = 2\Gamma$, and $\Gamma_R = 1$. Initial state consists of two counter-propagating photons at equal momenta. Left: exact solution. Right: Markovian approximation.