REGULARIZATION OF $p$-ADIC STRING AMPLITUDES, AND MULTIVARIATE LOCAL ZETA FUNCTIONS

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Abstract. We prove that the $p$-adic Koba-Nielsen type string amplitudes are bona fide integrals. We attach to these amplitudes Igusa-type integrals depending on several complex parameters and show that these integrals admit meromorphic continuations as rational functions. Then we use these functions to regularize the Koba-Nielsen amplitudes. As far as we know, there is no a similar result for the Archimedean Koba-Nielsen amplitudes. We also discuss the existence of divergencies and the connections with multivariate Igusa’s local zeta functions.

1. Introduction

This article aims to discuss some connections between $p$-adic string amplitudes and $p$-adic local zeta functions (also called Igusa’s local zeta functions). In the 80s, Volovich posed the conjecture that the space-time has a non-Archimedean structure at the level of the Planck scale and initiated the $p$-adic string theory [45], see also [44, Chapter 6], [46]. Volovich noted that the integral expression for the Veneziano amplitude of the open bosonic string can be generalized to a $p$-adic integral and to an adelic integral giving rise to non-Archimedean Veneziano amplitudes. Then Freund and Witten established (formally) that the ordinary Veneziano and Virasoro-Shapiro four-particle scattering amplitudes can be factored in terms of an infinite product of non-Archimedean string amplitudes [19], see also [3]. In $p$-adic string theory, justly as in the case of usual string theory, the problem of computing the scattering amplitudes in the perturbative theory is formulated as follows. Consider a certain number $N$ of vertex operators $V_1(x_1), V_2(x_2), \ldots, V_N(x_N)$, which are inserted at arbitrary ordered points $x_1, x_2, \ldots, x_N$ in the world-sheet manifold $\Sigma_{g, N}$. This manifold is a Riemann surface of genus $g$ and $N$ ordered marked points. The computation of these amplitudes implies to perform formal integrations over the moduli space of Riemann surfaces $\mathcal{M}_{g, N}$. Tree amplitudes correspond to the moduli space of $N$ ordered points on $\Sigma_{0,N}$. In this article we will workout the tree level amplitudes.

As a consequence of the ensuing interest on $p$-adic models of quantum field theory, which is motivated by the fact that these models are exactly solvable, there is a large list of $p$-adic type Feynman and string amplitudes that are related with local zeta functions of Igusa-type, and it is interesting to mention that it seems that the mathematical community working on local zeta functions is not aware of

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this fact, see e.g. [3]-[5], [8], [10]-[11], [17]-[19], [20], [31]-[33], [35], [39]-[40], and the references therein.

The connections between Feynman amplitudes and local zeta functions are very old and deep. Let us mention that the works of Speer [41] and Bollini, Giambiagi and González Domínguez [9] on regularization of Feynman amplitudes in quantum field theory are based on the analytic continuation of distributions attached to complex powers of polynomial functions in the sense of Gel’fand and Shilov [20], see also [4]-[5], [8], [35], among others. There are several types of local zeta functions, for instance $p$-adic, Archimedean, topological and motivic, among others, see e.g. [13]-[14], [20], [27]-[28], and the references therein. In the Archimedean setting, the local zeta functions were introduced in the 50s by Gel’fand and Shilov. The main motivation was that the meromorphic continuation of Archimedean local zeta functions implies the existence of fundamental solutions (i.e. Green functions) for differential operators with constant coefficients. This fact was establıshed, independently, by Atiyah [2] and Bernstein [6]. It is important to mention here, that in the $p$-adic framework, the existence of fundamental solutions for pseudodifferential operators is also a consequence of the fact that the Igusa local zeta functions admit a meromorphic continuations, see [51, Chapter 5], [30, Chapter 10]. This analogy turns out to be very important in the rigorous construction of quantum scalar fields in the $p$-adic setting, see [36] and the references therein.

In the 60s, Weil studied local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula [47]. In the 70s, Igusa developed a uniform theory for local zeta functions in characteristic zero [27]-[28]. In the $p$-adic setting, the local zeta functions are connected with the number of solutions of polynomial congruences mod $p^m$ and with exponential sums mod $p^m$. Recently Denef and Loeser introduced the motivic zeta functions which constitute a vast generalization of $p$-adic local zeta functions [14]-[15].

Take $N \geq 4$ and $s_{ij} \in \mathbb{C}$ satisfying $s_{ij} = s_{ji}$ for $1 \leq i < j \leq N - 1$. In this article we study the following multivariate Igusa-type zeta function:

$$Z^{(N)}(s) = \int_{\mathbb{Q}_p^{N-3} \setminus \Lambda} \prod_{i=2}^{N-2} |x_i|_{p}^{s_{1i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{p}^{s_{ij}} \prod_{i=2}^{N-2} dx_i,$$

where $s = (s_{ij}) \in \mathbb{C}^D$, here $D$ denotes the total number of possible subsets $\{i, j\}$, $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, and

$$\Lambda := \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; \prod_{i=2}^{N-2} x_i (1 - x_i) \prod_{2 \leq i < j \leq N-2} (x_i - x_j) = 0 \right\}.$$

We study integrals of type (1.1) by using the theory of local zeta functions, in this framework, it is not convenient, neither necessary, to assume some algebraic dependency between the variables $s_{ij}$. We call this type of integrals $p$-adic open string $N$-point zeta functions because they appeared in connection with the $p$-adic open string $N$-tachyon tree amplitudes, see e.g. [10]-[11], [17]-[19], [20], [31], [40], and the references therein. In all the published literature about $p$-adic string amplitudes, integrals of type (1.1) have been used without considering the convergence properties of them, i.e. the problem of the regularization of $p$-adic open string $N$-tachyon amplitudes has not been considered before. In the light of the theory of local zeta functions,
functions, the possible convergence of integrals of type (1.1) is a new and remarkable aspect. Theorem 1 is the main result of this article, establishes that the $p$-adic open string $N$-point zeta function is a holomorphic function in a certain domain of $\mathbb{C}^D$ and that it admits an analytic continuation to $\mathbb{C}^D$ (denoted as $Z(N)(s)$) as a rational function in the variables $p^{-s_{ij}}, i, j \in \{1, \ldots, N-1\}$. In addition, if $s = (s_{ij}) \in \mathbb{R}^D$, with $s_{ij} \geq 0$ for $i, j \in \{1, \ldots, N-1\}$, then the integral in (1.1) diverges to $+\infty$.

The typical approach to establish the existence of a meromorphic continuation for an integral of Igusa-type (which is holomorphic in a certain domain) is via Hironaka’s resolution of singularities theorem, see e.g. [28, Chapters 3, 5, 8]. Roughly speaking Hironaka’s resolution theorem provides a finite sequence of changes of variables (blow-ups) that allows to express an Igusa-type integral as a linear combination of integrals involving monomials, for this type of integrals the existence of an analytic continuation is easy to show. If the analyticity of the initial Igusa-type integral is unknown, then the above described approach cannot be used. In addition, nowadays, Hironaka’s resolution theorem is only valid in characteristic zero. Here, we use an approach inspired in the calculations presented in [10] and in the Igusa $p$-adic stationary phase formula, see [28, Theorem 10.2.1], [49]-[50]. This approach works on non-Archimedean local fields of arbitrary characteristic, for instance in $\mathbb{F}_p((t))$, the field of formal Laurent series over a finite field $\mathbb{F}_p$. In addition, our approach provides an algorithm for computing the $p$-adic open string $N$-point zeta functions.

Take $\phi(x_2, \ldots, x_{N-2})$ a locally constant function with compact support, then

$$Z^N(\phi)(s) = \int_{\mathbb{Q}^{N-3}_p \setminus \Lambda} \phi(x_2, \ldots, x_{N-2}) \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i,$$

for $\text{Re}(s_{ij}) > 0$ for any $ij$, is a multivariate Igusa local zeta function. In characteristic zero, a general theory for this type of local zeta functions was elaborated by Loeser in [34]. In particular, these local zeta functions admit analytic continuations as rational functions of the variables $p^{-s_{ij}}$. If we take $\phi$ to be the characteristic function of $B_r^{-N-3}$, the ball centered at the origin with radius $p^r$, the dominated convergence theorem and Theorem 1 imply that $\lim_{r \to \infty} Z_{B_r^{-N-3}}(s) = Z(N)(s)$ for any $s$ in the natural domain of $Z(N)(s)$.

In [10], Brekke, Freund, Olson and Witten work out the $N$-point amplitudes in explicit form and investigate how these can be obtained from an effective Lagrangian. The $p$-adic open string $N$-point tree amplitudes are defined as

$$A^{(N)}(k) = \int_{\mathbb{Q}^{N-3}_p} \prod_{i=2}^{N-2} |x_i|_p^{k_{1i}} |1 - x_i|_p^{k_{N-1i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{k_{ij}} \prod_{i=2}^{N-2} dx_i,$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}^{N-3}_p$, $k = (k_1, \ldots, k_N)$, $k_i = (k_{0i}, \ldots, k_{25i}), i = 1, \ldots, N, N \geq 4$, is the momentum vector of the $i$-th tachyon...
(with Minkowski product \( k_i, k_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \ldots + k_{25,i}k_{25,j} \)) obeying
\[
\sum_{i=1}^{N} k_i = 0, \quad k_i^2 = 2 \quad \text{for} \quad i = 1, \ldots, N.
\]

A central problem is to know whether or not integrals of type (1.2) converge for some values \( k_i, k_j \in \mathbb{C} \). Our Theorem 1 allows us to solve this problem. We take the \( p \)-adic open string \( N \)-point tree integrals \( Z^{(N)}(s) \) as regularizations of the amplitudes \( A^{(N)}(k) \). More precisely, we define
\[
A^{(N)}(k) = Z^{(N)}(s) \mid_{s_{ij} = k_i k_j} \text{with} \quad i \in \{1, \ldots, N-1\}, \quad j \in T \text{or} \quad i, j \in T,
\]
where \( T = \{2, \ldots, N-2\} \). By Theorem 1 \( A^{(N)}(k) \) are well-defined rational functions of the variables \( p^{-k_i k_j} \), \( i, j \in \{1, \ldots, N-1\} \), which agree with integrals (1.2) when they converge. This definition allows us to recover all the calculations made in [10] and other similar publications. At this point, it is relevant to mention that there is no similar result for the Archimedean string amplitudes at the three level, as Witten pointed out in [48, p. 4]. We notice that the string amplitudes \( A^{(N)}(k) \) are limits of local zeta functions when they are considered as distributions. By a slight abuse of notation, this means that
\[
A^{(N)}(k) = \lim_{r \to \infty} Z^{(N)}_{B^{N-3}}(k),
\]
for \( k \) in the natural domain of \( Z^{(N)}(s) \). Another important problem is to determine the existence of (in the sense of quantum field theory) ultraviolet and infrared divergences for \( A^{(N)}(k) \). If we use the Euclidean product instead of the Minkowski product to define \( s_{ij} = k_i k_j \), then \( A^{(N)}(k) \) has infrared divergences \( A^{(N)}(0) = +\infty \) and ultraviolet divergences \( A^{(N)}(k) = +\infty \) for \( k_i k_j > 0 \). The determination of the ultraviolet and infrared divergences, in the sense of quantum field theory, in the signature \(-++\ldots+\) for \( A^{(N)}(k) \) is an open problem. This problem requires the determination of the geometry of the natural domain of function \( Z^{(N)}(s) \). This type of problems has been not studied in the case of multivariate local zeta functions.

Lerner and Missarov studied a class of \( p \)-adic integrals that includes certain type of Feynman integrals and Koba-Nielsen amplitudes. They showed, see [31, Theorem 2], that this type of integrals can be computed recursively by using hierarchies, but they did not investigate the convergence, or more generally the holomorphy, of the Koba-Nielsen amplitudes, which is a delicate matter. On the other hand, the problem of regularizing string amplitudes has been recently considered by Witten in [48], by using an analog of ‘the \( i\epsilon \) method’ for regularizing Feynman integrals. Our approach to the regularization of \( p \)-adic string amplitudes is close to the technique of analytic regularization in quantum field theory, see e.g. [41], [29, Chapter 8] and references therein.

In a more general framework, we point out that the string amplitudes at the tree level and in general the Feynman amplitudes are ‘essentially’ local zeta functions (in the sense of Gel’fand, Sato, Weil, Bernstein, Tate, Igusa, Denef and Loeser, among others), and thus, they are algebraic-geometric objects that can be studied over several ground fields, for instance \( \mathbb{R}, \mathbb{C}, \mathbb{Q}_p \), \( \mathbb{C}(t) \). Over each these fields these objects have similar mathematical properties. As a consequence of our results and the theory of motivic Igusa zeta functions due to Denef and Loeser [13]-[15], a natural step is the construction of motivic string amplitudes (motivic in the sense
of motivic integration) which specialized to the $p$-adic string amplitudes. In this framework the limit $p \to 1$ of Igusa’s local zeta function makes mathematical sense.

There is empirical evidence that $p$-adic strings are related to the ordinary strings in the $p \to 1$ limit, see e.g. [21, 23, 38], and the references therein. Denef and Loeser established that the limit $p \to 1$ of an Igusa’s local zeta function gives rise to an object, called topological zeta function [16]. By using Denef-Loës-...
$S_0^1 = \mathbb{Z}_p^*$ (the group of units of $\mathbb{Z}_p$), but $(\mathbb{Z}_p^*)^n \subseteq S_0^1$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p^n, || \cdot ||_p)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$; see e.g. [16, Section 1.3], or [1] Section 1.8. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^n, || \cdot ||_p)$ is a locally compact topological space.

**Remark 1.** There is a natural map, called the reduction mod $p$ and denoted as $\rmod{\cdot}$, from $\mathbb{Z}_p$ onto $\mathbb{F}_p$, the finite field with $p$ elements. More precisely, if $x = \sum_{j=0}^{\infty} x_j p^j \in \mathbb{Z}_p$, then $\rmod{x} = \overline{x}_0 \in \mathbb{F}_p = \{ \overline{0}, \ldots, \overline{p-1} \}$. If $a = (a_1, \ldots, a_n) \in \mathbb{Z}_p^n$, then $\rmod{a} = (\overline{a}_1, \ldots, \overline{a}_n)$.

### 2.2. Integration on $\mathbb{Q}_p^n$.

Since $(\mathbb{Q}_p^+, +)$ is a locally compact topological group, there exists a Borel measure $dx$, called the Haar measure of $(\mathbb{Q}_p^+, +)$, unique up to multiplication by a positive constant, such that $\int_U dx > 0$ for every non-empty Borel open set $U \subset \mathbb{Q}_p$, and satisfying $\int_{E+x} dx = \int_E dx$ for every Borel set $E \subset \mathbb{Q}_p$, see e.g. [23, Chapter XI]. If we normalize this measure by the condition $\int_{\mathbb{Q}_p^n} dx = 1$, then $dx$ is unique. From now on we denote by $dx$ the normalized Haar measure of $(\mathbb{Q}_p^+, +)$ and by $d^n x$ the product measure on $(\mathbb{Q}_p^n, +)$.

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is said to be locally constant if for every $x \in \mathbb{Q}_p^n$ there exists an open compact subset $U, x \in U$, such that $\varphi(x) = \varphi(u)$ for all $u \in U$. Any locally constant function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ can be expressed as a linear combination of characteristic functions of the form $\varphi(x) = \sum_{k=1}^{\infty} c_k 1_{U_k}(x)$, where $c_k \in \mathbb{C}$ and $1_{U_k}(x)$ is the characteristic function of $U_k$, an open compact subset of $\mathbb{Q}_p^n$, for every $k$. If $\varphi$ has compact support, then $\varphi(x) = \sum_{k=1}^{L} c_k 1_{U_k}(x)$ and in this case

$$\int_{\mathbb{Q}_p^n} \varphi(x) d^n x = c_1 \int_{U_1} d^n x + \ldots + c_L \int_{U_L} d^n x.$$

A locally constant function with compact support is called a Bruhat-Schwartz function. These functions form a $\mathbb{C}$-vector space denoted as $D(\mathbb{Q}_p^n)$. By using the fact that $D(\mathbb{Q}_p^n)$ is a dense subspace of $C_c(\mathbb{Q}_p^n)$, the $\mathbb{C}$-space of continuous functions on $\mathbb{Q}_p^n$ with compact support, with the topology of the uniform convergence, the functional $\varphi \mapsto \int_{\mathbb{Q}_p^n} \varphi(x) d^n x, \varphi \in D(\mathbb{Q}_p^n)$ has a unique continuous extension to $C_c(\mathbb{Q}_p^n)$, as an unbounded linear functional. For integrating more general functions, say locally integrable functions, the following notion of improper integral will be used.

**Definition 1.** A function $\varphi \in L^1_{loc}$ is said to be integrable in $\mathbb{Q}_p^n$ if

$$\lim_{m \to +\infty} \int_{B_{n,m}} \varphi(x) d^n x = \lim_{m \to +\infty} \sum_{j=-\infty}^{m} \int_{Q_j} \varphi(x) d^n x$$

exists. If the limit exists, it is denoted as $\int_{\mathbb{Q}_p^n} \varphi(x) d^n x$, and we say that the (improper) integral exists.
2.3. Analytic change of variables. A function $h : U \to \mathbb{Q}_p$ is said to be analytic on an open subset $U \subset \mathbb{Q}_p^n$, if for every $b \in U$ there exists an open subset $\bar{U} \subset U$, with $b \in \bar{U}$, and a convergent power series $\sum_i a_i (x - b)^i$ for $x \in \bar{U}$, such that $h(x) = \sum_{i \in \mathbb{N}^n} a_i (x - b)^i$ for $x \in U$, with $x^i = x_1^{i_1} \cdots x_n^{i_n}$, $i = (i_1, \ldots, i_n)$. In this case, $\frac{\partial h}{\partial x_i}(x) = \sum_{i \in \mathbb{N}^n} a_i \frac{\partial}{\partial x_i} (x - b)^i$ is a convergent power series. Let $U$, $V$ be open subsets of $\mathbb{Q}_p^n$. A mapping $h : U \to V$, $h = (h_1, \ldots, h_n)$ is called analytic if each $h_i$ is analytic.

Let $\varphi : V \to \mathbb{C}$ be a continuous function with compact support, and let $h : U \to V$ be an analytic mapping. Then

$$\int_V \varphi(y) d^n y = \int_\mathbb{C} \varphi(h(x)) |\text{Jac}(h(x))|_p d^n x,$$

where $\text{Jac}(h(z)) := \det \left[ \frac{\partial h_i}{\partial x_j}(z) \right]_{1 \leq i, j \leq n}$, see e.g. [12, Section 10.1.2].

2.4. The multivariate Igusa zeta functions. Let $f_i(x) \in \mathbb{Q}_p [x_1, \ldots, x_n]$ be non-constant polynomials for $i = 1, \ldots, l$, and let $\Phi$ be a Bruhat-Schwartz function. The multivariate local zeta function attached to $(f_1, \ldots, f_l, \Phi)$ (also called Igusa local zeta function) is defined by the integral

$$Z_{\Phi}(s_1, \ldots, s_l; f_1, \ldots, f_l) = \int_{\mathbb{Q}_p^n \setminus \bigcup_{i=1}^l f_i^{-1}(0)} \Phi(x) \prod_{i=1}^l |f_i(x)|_p^{s_i} d^n x,$$

for $(s_1, \ldots, s_l) \in \mathbb{C}^n$ with $\text{Re}(s_i) > 0$, $i = 1, \ldots, l$. This integral defines a holomorphic function of $(s_1, \ldots, s_l)$ in the half-space $\text{Re}(s_i) > 0$, $i = 1, \ldots, l$. In the case $l = 1$, this assertion corresponds to Lemma 5.3.1 in [28]. For the general case, we recall that a continuous complex-valued function defined in an open set $A \subseteq \mathbb{C}^n$, which is holomorphic in each variable separately, is holomorphic in $A$. The multivariate local zeta functions admit analytic continuations to the whole $\mathbb{C}^n$ as rational functions of the variables $p^{-s_i}$, $i = 1, \ldots, l$, see [34]. The Igusa local zeta functions are related with the number of solutions of polynomial congruences mod $p^m$ and with exponential sums mod $p^m$. There are many intriguing conjectures relating the poles of local zeta functions with the topology of complex singularities, see e.g. [13, 28].

We want to highlight that the convergence of the local zeta functions depends crucially on the fact that $\Phi$ has compact support. Consider the following integral:

$$I(s) = \int_{\mathbb{Q}_p} |x|^s dx, \ s \in \mathbb{C}.$$

Assume that $I(s_0)$ exists for some $s_0 \in \mathbb{R}$, then necessarily the integrals

$$I_0(s_0) = \int_{\mathbb{Z}_p} |x|_p^{s_0} dx \ \text{and} \ I_1(s_0) = \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |x|_p^{s_0} dx$$
exist. The first integral is well-known, \( I_0(s_0) = \frac{1-p^{-1}}{p^{s_0}} \) for \( s_0 > -1 \). For the second integral, we use that \( |x|_p^{s_0} \) is locally integrable, and thus

\[
I_1(s_0) = \sum_{j=1}^{\infty} \int_{\mathbb{Z}_p^N} |x|_p^{s_0} \, dx = \sum_{j=1}^{\infty} p^{j+s_0} \int_{\mathbb{Z}_p^N} dx = (1-p^{-1}) \sum_{j=1}^{\infty} p^{j(1+s_0)} < \infty
\]

if and only if \( s_0 < -1 \). Then, integral \( I(s) \) does not exist for any \( s \in \mathbb{R} \) and consequently \( I(s) \) does not exist for any complex value \( s \).

For an in-depth discussion on local zeta functions the reader may consult \([13],[27],[28]\) and the references therein.

3. \textit{p-adic String Zeta Functions}

We fix an integer \( N \geq 4 \). To each set \( \{i,j\} \) with \( i,j \in \{1,\ldots,N-1\} \) we attach a complex variable \( s_{ij} \). We assume that the variables \( s_{ij} \) are algebraically independent. We set \( T := \{2,\ldots,N-2\} \), then \( D = 2(N-3) + \frac{(N-3)(N-4)}{2} = \frac{N(N-3)}{2} \), and \( \mathbb{C}^D \) is

\[
\left\{ \begin{array}{ll}
\{ s_{ij} \in \mathbb{C}; i \in \{1,N-1\}, j \in T \} & \text{if } N = 4 \\
\{ s_{ij} \in \mathbb{C}; i \in \{1,N-1\}, j \in T \text{ or } i,j \in T \text{ with } i < j \} & \text{if } N \geq 5.
\end{array} \right.
\]

We set \( s = (s_{ij}) \in \mathbb{C}^D, \ x = (x_2,\ldots,x_{N-2}) \in \mathbb{Q}_p^{N-3} \), and

\[
F(s,x;N) = \prod_{i=2}^{N-2} |x_i|_p^{s_{ij}} |1-x_i|_p^{s_{ij}(N-1)-s_{ij}} \prod_{2 \leq i < j \leq N-2} |x_i-x_j|_p^{s_{ij}}.
\]

**Definition 2.** The \textit{p-adic open string N-point zeta function} is defined as

\[
Z^{(N)}(s) := \int_{\mathbb{Q}_p^{N-3} \setminus \Lambda} F(s,x;N) \prod_{i=2}^{N-2} dx_i
\]

for \( s = (s_{ij}) \in \mathbb{C}^D \), where \( \Lambda \) denotes the divisor

\[
\left\{ (x_2,\ldots,x_{N-2}) \in \mathbb{Q}_p^{N-3}; \prod_{i=2}^{N-2} x_i (1-x_i) \prod_{2 \leq i < j \leq N-2} (x_i-x_j) = 0 \right\},
\]

and \( \prod_{i=2}^{N-2} dx_i \) is the normalized Haar measure of \( \mathbb{Q}_p^{N-3} \).

**Remark 2.** We notice that the domain of integration in \((3.1)\) is taken to be \( \mathbb{Q}_p^{N-3} \setminus \Lambda \) in order to use \( a^s = e^{s \ln a} \), with \( a > 0 \) and \( s \in \mathbb{C} \), as the definition of the complex power function. The convergence of integral \((3.1)\), as well as its holomorphy, will be discussed later on.

We define for \( I \subseteq T \), the sector attached to \( I \) as

\[
\text{Sect}(I) = \left\{ (x_2,\ldots,x_{N-2}) \in \mathbb{Q}_p^{N-3}; |x_i|_p \leq 1 \iff i \in I \right\}
\]

and

\[
Z^{(N)}(s;I) = \int_{\text{Sect}(I)} F(s,x;N) \prod_{i=2}^{N-2} dx_i.
\]
Hence

\[ Z^{(N)}(s) = \sum_{I \subseteq T} Z^{(N)}(s; I). \]

**Notation 1.** (i) The cardinality of a finite set \( A \) will be denoted as \(|A|\).

(ii) We will use the symbol \( \bigcup \) to denote the union of disjoint sets.

(iii) Given a non-empty subset \( I \) of \( \{2, \ldots, N - 2\} \) and \( B \) a non-empty subset of \( \mathbb{Q}_p \), we set

\[ B^{|I|} = \{(x_i)_{i \in I}; x_i \in B\}. \]

(iv) By convention, we define \( \prod_{i \in \emptyset} := 1 \), \( \sum_{i \in \emptyset} := 0 \), and if \( J = \emptyset \), then \( \int_{J^{|I|}} := 1 \).

(v) The indices \( i, j \) will run over subsets of \( T \), if we do not specify any subset, we will assume that is \( T \).

**Lemma 1.** With the above notation the following formulas hold: (i) \( F(s, x; N) \big|_{\text{Sect}(I)} = F_0(s, x; N) F_1(s, x; N) \), where

\[ F_0(s, x; N) := \prod_{i \in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \]

and

\[ F_1(s, x; N) := \prod_{i \in T \setminus I} |x_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{2 \leq j \leq N-2} s_{ij}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}}. \]

(ii) If \( \Re(s_{1i}) + \Re(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2} s_{ij} \Re(s_{ij}) + 1 < 0 \) for \( i \in T \setminus I \), and \( \Re(s_{ij}) > -1 \) for \( i, j \in T \setminus I \), then

\[ \int_{(\mathbb{Q}_p \setminus \mathbb{Z}_p)^{|T\setminus I|}} F_1(s, x; N) \prod_{i \in T \setminus I} dx_i \]

\[ = p^{M(s)} \int_{\mathbb{Z}_p^{|T\setminus I|}} \prod_{2 \leq i < j \leq N-2} |y_i - y_j|_p^{s_{ij}} \prod_{i \in T \setminus I} dy_i, \]

where \( M(s) = |T \setminus I| + \sum_{i \in T \setminus I} (s_{1i} + s_{(N-1)i}) + \sum_{2 \leq i < j \leq N-2} s_{ij} + \sum_{i \in T \setminus I, j \in T \setminus I} s_{ij} \).

(iii) If \( \Re(s_{1i}) + \Re(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2} s_{ij} \Re(s_{ij}) + 1 < 0 \) for \( i \in T \setminus I \), \( \Re(s_{ij}) > -1 \) for \( i, j \in T \setminus I \), \( \Re(s_{1i}) > -1 \) for \( i \in I \) and \( \Re(s_{(N-1)i}) > -1 \) for...
\[ i \in I, \text{ then} \]

\[
Z^{(N)}(s; I) = p^M(s) \left\{ \int_{\mathbb{C}} F_0(s, x; N) \prod_{i \in I} dx_i \right\} \times \left\{ \int_{\mathbb{C}^{|I|}} \prod_{2 \leq i < j \leq N-2} \frac{|x_i - x_j|^s_{|p}}{i \in I, j \in I} \prod_{2 \leq i \leq N-2} \prod_{i \in I, j \in I} |x_i|^s_{|p} \right\}
\]

\[
= p^M(s) Z_0^{(N)}(s; I) Z_1^{(N)}(s; T \setminus I).
\]

**Remark 3.** Later on we will show that the integrals in the right-hand side in the formulas given in (ii) and (iii) are convergent and holomorphic functions on a certain subset of \( \mathbb{C}^D \) for all \( I \subseteq T \).

**Proof.** (i) Notice that \( F(s, x; N)|_{\text{Sec}(I)} \) equals

\[
\prod_{i \in I} |x_i|^{s_{ij}} |1 - x_i|^{s_{(N-1)i}} \prod_{i \in I} |x_i|^{s_{ij} + s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^s_{|p} \times
\]

\[
\prod_{2 \leq i < j \leq N-2} |x_i - x_j|^s_{|p} \prod_{2 \leq i < j \leq N-2} |x_i|^s_{|p} \prod_{2 \leq i < j \leq N-2} |x_j|^s_{|p}.
\]

Now, by using that \( s_{ij} = s_{ji} \),

\[
\prod_{2 \leq i < j \leq N-2} |x_i|^{s_{ij}} \prod_{i \in I} |x_i|^{s_{ij}} = \prod_{2 \leq i < j \leq N-2} |x_i|^s_{|p} \prod_{i \in I} |x_i|^s_{|p} \prod_{2 \leq i < j \leq N-2} \frac{\sum_{2 \leq j \leq N-2} s_{ij}}{\sum_{j \neq i, i \in I} s_{ij}}.
\]

The announced formula follows from (3.2)–(3.3).

(ii) For \( |T \setminus I| \geq 1 \), we set

\[
J(s; T \setminus I) := \int_{(Q_p \setminus \mathbb{Z}_p)^{|T \setminus I|}} F_1(s, x; N) \prod_{i \in T \setminus I} dx_i,
\]

and for \( l \in \mathbb{N} \setminus \{0\} \),

\[
(Q_p \setminus \mathbb{Z}_p)^{|T \setminus I|} := \left\{ (x_i)_{i \in T \setminus I} \in (Q_p \setminus \mathbb{Z}_p)^{|T \setminus I|}; -l \leq \text{ord}(x_i) \leq -1 \text{ for } i \in T \setminus I \right\},
\]

\[
(p\mathbb{Z}_p)^{|T \setminus I|} := \left\{ (x_i)_{i \in T \setminus I} \in (p\mathbb{Z}_p)^{|T \setminus I|}; 1 \leq \text{ord}(x_i) \leq l \text{ for } i \in T \setminus I \right\},
\]

and

\[
J_{-l}(s; T \setminus I) := \int_{(Q_p \setminus \mathbb{Z}_p)^{|T \setminus I|}} F_1(s, x; N) \prod_{i \in T \setminus I} dx_i.
\]
Notice that \((\mathbb{Q}_p \setminus \mathbb{Z}_p)^{[T \setminus I]}\), \((p\mathbb{Z}_p)^{[T \setminus I]}\) are compact sets and that
\[
(\mathbb{Q}_p \setminus \mathbb{Z}_p)^{[T \setminus I]} \rightarrow (p\mathbb{Z}_p)^{[T \setminus I]},
\]
with \(\sigma (x_i) = \frac{1}{y_i}\) is an analytic change of variables satisfying \(\prod_{i \in T \setminus I} dx_i = \prod_{i \in T \setminus I} dy_i\), then by using this change of variables and the fact that
\[
\prod_{i \in T \setminus I} |y_{i}|_p^{s_{i1} + s_{i(N-1)} + \sum_{2 \leq j \leq N-2} s_{ij}} \prod_{i \neq j \in T} \prod_{i \neq j \in T \setminus I} |y_{i}|_p^{s_{ij}} = \prod_{i \in T \setminus I} |y_{i}|_p^{s_{i1} + s_{i(N-1)} + \sum_{2 \leq j \leq N-2} s_{ij}},
\]
we have
\[
J_{-l} (s; T \setminus I) = \int_{(p\mathbb{Z}_p)^{[T \setminus I]}} \prod_{i \in T \setminus I} |y_{i}|_p^{s_{i1} + s_{i(N-1)} + \sum_{2 \leq j \leq N-2} s_{ij}} \prod_{i \in T \setminus I} dy_i.
\]
in Lemmas 3 and 4, respectively). The proof of the propositions relies on a recursive expression for the $Z(s; J)$s in terms of auxiliary functions $L_0(s; I)$, $L_1(s; I, J)$, and $L_2(s; I, J, K)$. Lemmas 5 and 6 demonstrate the form of the respective analytic continuations of $L_1(s; I, J)$ and $L_2(s; I, J, K)$; Lemma 2 computes $L_0(s; I)$ in terms of $L_1(s; I, J)$. (A degeneration formula, relating $L_1(s; I, J)$ to $L_2(s; I, J, K)$, appears as Remark 7). Lemmas 4 and 5 address the computation of other auxiliary functions, $Z(s_1, s_2, s_3)$ and $M_1(s; J)$; Lemma 3 is a preliminary result regarding indexing.

A natural question is to know if the auxiliary integrals introduced to compute $Z(s)$ have a natural interpretation in the context of $p$-adic string theory. In our opinion, these auxiliary integrals are merely organizational devices, since they are defined over particular regions in the moduli space.

### 3.2. Some $p$-adic Integrals

We compute some $p$-adic integrals needed for calculating $Z_0(s; I)$ and $Z_1(s; I)$.

Let $J$ be a subset of $T$ with $|J| \geq 2$. We define

\begin{equation}
L_0 \left( (s_{ij})_{2 \leq i < j \leq N-2}; J \right) := L_0(s; J) = \int \prod_{i, j \in J} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i
\end{equation}

for $\text{Re}(s_{ij}) > 0$ for any $ij$, and

\begin{equation}
L_1 \left( (s_{ij})_{2 \leq i < j \leq N-2}; J, K \right) := L_1(s; J, K) = \int \prod_{i, j \in J \cup K} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i
\end{equation}

where $K \subseteq T_J := \{(i, j) \in T \times T; 2 \leq i < j \leq N - 2, i, j \in J\}$ and $\text{Re}(s_{ij}) > 0$ for any $ij$. Notice that if $|J| = 1$, then $L_0(s; J) = 1 - p^{-1}$ and $K = \emptyset$ which implies $L_1(s; J, K) = 1$. A precise definition of integrals $L_0(s; J)$ requires to integrate on

\[
\left( \mathbb{Z}_p^\times \right)^{|J|} \setminus \left\{ x \in \left( \mathbb{Z}_p^\times \right)^{|J|}; \prod_{2 \leq i < j \leq N-2} (x_i - x_j) = 0 \right\}.
\]

A similar consideration is required for $L_1(s; J, K)$. However, for the sake of simplicity we use definitions (3.6)-(3.7). We will use this simplified notation later on for similar integrals. The integrals $L_0(s; J)$, $L_1(s; J, K)$ are $p$-adic multivariate local zeta function, these functions were studied by Loeser in [34]. In particular, it is known that these functions have an analytic continuation to $\mathbb{C}^D$ as rational functions in the variables $p^{-s_{ij}}$ and that they are holomorphic functions on $\text{Re}(s_{ij}) > 0$ for any $ij$.

**Remark 4.** Let $J$ be subset of $T$, with $|J| \geq 2$. Set

\[
T_J = \{(i, j) \in T \times T; 2 \leq i < j \leq N - 2, i, j \in J\}
\]

as before. For $\overline{\alpha} = (\overline{\alpha})_{i \in J} \in (\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Delta}(J)$, with

\[
\overline{\Delta}(J) := \left\{ \overline{\alpha} \in (\mathbb{F}_p^\times)^{|J|}; \overline{\alpha}_i \neq \overline{\alpha}_j \text{ for } i \neq j, \text{ with } i, j \in J \right\},
\]

we set

\[
K(\overline{\alpha}) := \{(i, j) \in T_J; \overline{\alpha}_i = \overline{\alpha}_j\}.
\]
Now, we introduce on \((\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Delta}(J)\), the following equivalence relation:
\[
\overline{a} \sim \overline{b} \iff K(\overline{a}) = K(\overline{b}).
\]
We denote by \(\overline{A}(\overline{a}) = \{\overline{b} \in (\mathbb{F}_p^\times)^{|J|} \setminus \Delta(J) : \overline{a} \sim \overline{b}\}\), the equivalence class defined by \(\overline{a} \in (\mathbb{F}_p^\times)^{|J|} \setminus \Delta(J)\). For instance, if \(\overline{a} = \overline{1} = (\overline{1})_{i \in J}\), then \(\overline{A}(\overline{1}) = \bigcup_{\overline{b} \in \mathbb{F}_p^\times} \{\overline{b}(\overline{1})_{i \in J}\}\).
By taking a unique representative in each equivalence class, we obtain \(\mathcal{R}(J) \subset (\mathbb{F}_p^\times)^{|J|} \setminus \Delta(J)\) such that
\[
(\mathbb{F}_p^\times)^{|J|} = \bigsqcup_{\overline{a} \in \mathcal{R}(J)} \overline{A}(\overline{a}) \bigcap \Delta(J).
\]

Given a subset \(K \subseteq T_J\) with \(K = \{(i_1, j_1), \ldots, (i_m, j_m)\}\), we define \(K_{\text{list}} = \{i_1, j_1, \ldots, i_m, j_m\} \subset J\). We will use the notation \(K_{\text{list}}(\overline{a})\) to mean \(K(\overline{a})\), for \(\overline{a} \in (\mathbb{F}_p^\times)^{|J|}\). Notice that \(K(\overline{a}) \subset K_{\text{list}}(\overline{a}) \times K_{\text{list}}(\overline{a}), |K_{\text{list}}(\overline{a})| \geq 2\) for any \(\overline{a} \in (\mathbb{F}_p^\times)^{|J|} \setminus \Delta(J)\) and that \(K_{\text{list}}(\overline{1}) = J\).

**Lemma 2.** If \(|J| \geq 2\), then, with the notation of Remark 4, the following formula holds:
\[
L_0(s; J) = \sum_{\overline{a} \in \mathcal{R}(J)} |\overline{A}(\overline{a})| p^{-|J| - \sum_{(i,j) \in K(\overline{a})} s_{ij}} L_1(s; K_{\text{list}}(\overline{a}), K(\overline{a})) + |\overline{\Delta}(J)| p^{-|J|}
\]
for \(\Re(s_{ij}) > 0\) for all \(i, j \in J\).

**Proof.** For \(\overline{a} \in (\mathbb{F}_p^\times)^{|J|} \setminus \Delta(J)\), set \(A(\overline{a}) := \{b + px : \overline{b} \in A(\overline{a})\}\), and for \(\Delta(J), \Delta(J) := \{a + px : \overline{a} \in \Delta(J)\}\). Now
\[
L_0(s; J) = \sum_{\overline{a} \in (\mathbb{F}_p^\times)^{|J|} \setminus A(\overline{a})_{|J|}} \int \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i
\]

\[
= \sum_{\overline{a} \in \mathcal{R}(J)} \sum_{\overline{b} \in A(\overline{a})_{|J|}} \int \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i + \sum_{\overline{a} \in \Delta(J) \setminus A(\overline{a})_{|J|}} \int \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i
\]

\[
= \sum_{\overline{a} \in \mathcal{R}(J)} |\overline{A}(\overline{a})| p^{-|J| - \sum_{(i,j) \in K(\overline{a})} s_{ij}} \int \prod_{(i,j) \in K(\overline{a})} |x_i - x_j|_p^{s_{ij}} \prod_{i \in K_{\text{list}}(\overline{a})} dx_i + |\overline{\Delta}(J)| p^{-|J|}.
\]

**Lemma 3.** We use all the notation introduced in Remark 4. Given \(\overline{a} = (\overline{a}_i)_{i \in J} \in (\mathbb{F}_p^\times)^{|J|} \setminus \Delta(J)\) and \((i, j) \in K(\overline{a})\), we set
\[
K((i, j), \overline{a}) := \left\{ (\overline{i}, \overline{j}) \in K(\overline{a}) : \overline{a}_i = \overline{a}_j \right\}
\]
and use \(K_{\text{list}}((i, j), \overline{a}) := K((i, j), \overline{a})_{\text{list}}\). Then the following assertions hold:

(i) \(K((i, j), \overline{a}) = T_{K_{\text{list}}((i, j), \overline{a})} = \{r,s : 2 \leq r < s \leq N-2, r,s \in K_{\text{list}}((i, j), \overline{a})\}\);
(ii) the subsets \( K((i, j), \mathbf{a}) \) form a partition of \( K(\mathbf{a}) \), i.e. there exists a finite set \( R(\mathbf{a}) \) of elements \( (i, j) \in K(\mathbf{a}) \), such that \( K(\mathbf{a}) = \bigcup_{(i, j) \in R(\mathbf{a})} K((i, j), \mathbf{a}) \).

Proof. (i) By definition \( K((i, j), \mathbf{a}) \subseteq T_{K_{\text{list}}}((i, j), \mathbf{a}) \). Conversely, let \( (\tilde{i}_m, \tilde{j}_m) \in T_{K_{\text{list}}}((i, j), \mathbf{a}) \), then there exists \( \tilde{j}_m \in K_{\text{list}}((i, j), \mathbf{a}) \) such that \( (\tilde{i}_m, \tilde{j}_m) \in K((i, j), \mathbf{a}) \) or \( (\tilde{j}_m, \tilde{i}_m) \in K((i, j), \mathbf{a}) \). In any case, either \( (\tilde{i}_m, \tilde{j}_m) \) or \( (\tilde{j}_m, \tilde{i}_m) \) belongs to \( K(\mathbf{a}) \) and \( \tilde{i}_i = \tilde{a}_m = \tilde{a}_j \). Similarly, there exists \( \tilde{i}_i \in K_{\text{list}}((i, j), \mathbf{a}) \) such that either \( (\tilde{i}_i, \tilde{j}_i) \) or \( (\tilde{j}_i, \tilde{i}_i) \) belongs to \( K((i, j), \mathbf{a}) \) and \( \tilde{i}_i = \tilde{a}_m = \tilde{a}_j \). Therefore \( \tilde{a}_m = \tilde{a}_j \). i.e. \( (\tilde{i}_m, \tilde{j}_m) \in K(\mathbf{a}) \), furthermore \( (\tilde{i}_m, \tilde{j}_m) \in K((i, j), \mathbf{a}) \). Hence \( K((i, j), \mathbf{a}) = T_{K_{\text{list}}}((i, j), \mathbf{a}) \).

(ii) Let \( (i_m, j_m) \in K((i, j), \mathbf{a}) \cap K((\tilde{i}, \tilde{j}), \mathbf{a}) \), then \( \tilde{a}_i = \tilde{a}_m = \tilde{a}_j \) and \( (\tilde{i}, \tilde{j}) \in K((i, j), \mathbf{a}) \), and consequently \( K((\tilde{i}, \tilde{j}), \mathbf{a}) \subseteq K((i, j), \mathbf{a}) \). Similarly, one verifies that \( K((i, j), \mathbf{a}) \subseteq K((\tilde{i}, \tilde{j}), \mathbf{a}) \).

Remark 5. As a consequence of Lemmas 3 and 4, we have

\[
L_1 (s; K_{\text{list}}(\mathbf{a}), K(\mathbf{a})) = \prod_{(i, j) \in R(\mathbf{a})} L_1 (s; K_{\text{list}}((i, j), \mathbf{a}), T_{K_{\text{list}}}((i, j), \mathbf{a})).
\]

Example 1. Take \( p \geq 3 \), \( \mathbf{a} = (\mathbb{T}, \mathbb{T}, \mathbb{T}, \mathbb{T}) \in \mathbb{F}_p \), and \( J = \{2, 3, 4, 5, 6\} \). Hence \( T_J = \{2, 3\}, (2, 4) = (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\} \), and by Lemma 3

\[
K(\mathbf{a}) = \{(2, 4), (3, 5), (3, 6), (5, 6)\} = K((2, 4), \mathbf{a}) \cup K((3, 5), \mathbf{a}),
\]

where \( K((2, 4), \mathbf{a}) = \{(2, 4)\} \), \( K((3, 5), \mathbf{a}) = \{(3, 5), (3, 6), (5, 6)\} \). Thus \( K_{\text{list}}((2, 4), \mathbf{a}) = \{2, 4\} \) and \( K_{\text{list}}((3, 5), \mathbf{a}) = \{3, 5\} \).

With this notation, \( L_1 (s; K_{\text{list}}(\mathbf{a}), K(\mathbf{a})) \) equals

\[
\left\{ \begin{array}{l}
\int_{\mathbb{Z}_p^2} |x_2 - x_4| |x^3 - x_5| |x_3 - x_6| |x_5 - x_6| \ dx_2 dx_3 dx_4 dx_5 dx_6 \\
= \left\{ \begin{array}{l}
\int_{\mathbb{Z}_p^2} |x_2 - x_4| |x_2| |x_3 - x_5| |x_3 - x_6| |x_5 - x_6| \ dx_2 dx_3 dx_4 dx_5 dx_6 \\
= L_1 (s; K_{\text{list}}((2, 4), \mathbf{a}), T_{K_{\text{list}}(2, 4), \mathbf{a}}), L_1 (s; K_{\text{list}}((3, 5), \mathbf{a}), T_{K_{\text{list}}(3, 5), \mathbf{a}}).
\end{array} \right.
\end{array}
\]

Lemma 4. Set \( F(s_1, s_2, s_3, x, y) := |x| |y| |x - y| \), \( s_1, s_2, s_3 \in \mathbb{C}, \) and

\[
Z(s_1, s_2, s_3) := \int_{\mathbb{Z}_p^2} F(s_1, s_2, s_3, x, y) \ dx dy \text{ for } \Re(s_i) > 0, i = 1, 2, 3.
\]

Then \( Z(s_1, s_2, s_3) \) is a holomorphic function on

\[
\{(s_1, s_2, s_3) \in \mathbb{C}^3; \Re(s_i) > -1 \text{ for } i = 1, 2, 3, \text{ and } \Re(s_1) + \Re(s_2) + \Re(s_3) > -2\}.
\]
In addition,

\[ Z(s_1, s_2, s_3) := \frac{Q(p^{-s_1}, p^{-s_2}, p^{-s_3})}{(1 - p^{-2-s_1-s_2-s_3}) \prod_{i=1}^{3} (1 - p^{-1-s_i})} \]

where \( Q(p^{-s_1}, p^{-s_2}, p^{-s_3}) \) denotes a polynomial with rational coefficients in the variables \( p^{-s_1}, p^{-s_2}, p^{-s_3} \).

**Remark 6.** If \( s_1 = s_2 = 0 \), then the denominator of \( Z(s_1, s_2, s_3) \) is \( 1 - p^{-1-s_3} \).

**Proof.** By using that \( Z_p^2 = (pZ_p)^2 \sqcup S_0^2 \) with \( S_0^2 = pZ_p \times Z_p^\times \sqcup Z_p^\times \times pZ_p \sqcup Z_p^\times \times Z_p^\times \), and then by changing variables, we get

\[ Z(s_1, s_2, s_3) = \int_{Z_p^\times \times Z_p^\times} F(s_1, s_2, s_3, x, y) dxdy =: \frac{Z_0(s_1, s_2, s_3)}{1 - p^{-2-s_1-s_2-s_3}}. \]

On the other hand,

\[ Z_0(s_1, s_2, s_3) = \int_{pZ_p \times Z_p^\times} F(s_1, s_2, s_3, x, y) dxdy \]

\[ + \int_{Z_p^\times \times pZ_p} F(s_1, s_2, s_3, x, y) dxdy + \int_{Z_p^\times \times Z_p^\times} F(s_1, s_2, s_3, x, y) dxdy \]

\[ =: Z_{0,1}(s_1, s_2, s_3) + Z_{0,2}(s_1, s_2, s_3) + Z_{0,3}(s_1, s_2, s_3). \]

First, we compute \( Z_{0,1}(s_1, s_2, s_3) \). By a change of variables, we get

\[ Z_{0,1}(s_1, s_2, s_3) = p^{-1-s_1}(1 - p^{-1}) \int_{Z_p^\times} |x|_{p}^{s_1} dx = \frac{(1 - p^{-1})^2 p^{-1-s_1}}{1 - p^{-1-s_1}} \]

for \( \text{Re}(s_1) > -1 \). By a similar computation we obtain

\[ Z_{0,2}(s_1, s_2, s_3) = \frac{(1 - p^{-1})^2 p^{-1-s_2}}{1 - p^{-1-s_2}} \text{ for } \text{Re}(s_2) > -1. \]

In order to compute

\[ Z_{0,3}(s_1, s_2, s_3) = \int_{Z_p^\times \times Z_p^\times} |x - y|_{p}^{s_3} dxdy, \]

we use that \( (Z_p^\times)^2 = \sqcup_{\mathfrak{m}_0, \mathfrak{m}_1 \in \mathbb{Z}_p} a_0 + pZ_p \times a_1 + pZ_p \), where \( \mathbb{F}_p = \{1, 2, ..., p-1\} \) as sets, to get

\[ Z_{0,3}(s_1, s_2, s_3) = \sum_{\mathfrak{m}_0, \mathfrak{m}_1 \in \mathbb{F}_p} \int_{a_0 + pZ_p \times a_1 + pZ_p} |x - y|_{p}^{s_3} dxdy \]

\[ = p^{-2} \sum_{\mathfrak{m}_0, \mathfrak{m}_1 \in \mathbb{F}_p} \int_{Z_p \times Z_p} |a_0 + px - a_1 - py|_{p}^{s_3} dxdy + p^{-2} \sum_{\mathfrak{m}_0, \mathfrak{m}_1 \in \mathbb{F}_p} \int_{Z_p \times Z_p} |x - y|_{p}^{s_3} dxdy \]

\[ = p^{-2}(p - 1)(p - 2) + p^{-2-s_3}(p - 1) \frac{1 - p^{-1}}{1 - p^{-1-s_3}}. \]

□
Lemma 5. Let \( I \) be a subset of \( T \) satisfying \(|I| \geq 2\). Then \( L_1 (s; I, T) \) admits an analytic continuation as a rational function of the form

\[
L_1 (s; I, T) = \frac{Q_I \left( \{ p^{-s_{ij}} \}_{i,j \in I} \right)}{\prod_{J \in \mathcal{F}(I)} \left( 1 - p \left[ (|J| - 1 + \sum_{2 \leq i < j \leq N - 2} s_{ij}) \right] e_{ij} \right) \prod_{ij \in S_I} (1 - p^{-1 - s_{ij}} e_{ij})}
\]

where \( Q_I \left( \{ p^{-s_{ij}} \}_{i,j \in I} \right) \) is a polynomial with rational coefficients in the variables \( \{ p^{-s_{ij}} \}_{i,j \in I} \), \( \mathcal{F}(I) \) is a family of subsets of \( I \), with \( I \in \mathcal{F}(I) \), \( S_I \) is a non-empty subset of \( \{ 2 \leq i < j \leq N - 2, i, j \in I \} \), and the \( e_{ij}, e_{ij}'s \) are positive integers.

Proof. By using the partition \( \mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|} \), where \( (p\mathbb{Z}_p)^{|I|} = \{(x_i)_{i \in I} : x_i \in \mathbb{Z}_p \} \), \( (p\mathbb{Z}_p)^{|I|} = \{(x_i)_{i \in I} \in \mathbb{Z}_p^{|I|} : \max_{i \in I} \{|x_i|_p \} = 1 \} \). By a change of variables, we get

\[
L_1 (s; I, T) = \frac{\int \prod_{2 \leq i < j \leq N - 2} (x_i - x_j)^{s_{ij}} \prod_{i \in I} dx_i}{\left( 1 - p \right) \prod_{i,j \in I} (1 - s_{ij})} = : \frac{B_0 (s; I)}{1 - p}.
\]

For every non-empty subset \( J \subseteq I \), we define

\[
S_J^{|I|} = \{(x_i)_{i \in I} \in (p\mathbb{Z}_p)^{|I|} : |x_i|_p = 1 \Rightarrow i \in J \},
\]

then \( S_0^{|I|} = \bigcup_{J \subseteq I, J \neq \emptyset} S_J^{|I|} \) and \( A_0 (s; I) = \sum_{J \subseteq I, J \neq \emptyset} A_{0,J} (s) \) where

\[
B_{0,J} (s) := \int_{S_J^{|I|}} \prod_{2 \leq i < j \leq N - 2} (x_i - x_j)^{s_{ij}} \prod_{i \in I} dx_i,
\]

for this reason

\[
L_1 (s; I, T) = \frac{B_{0,1} (s) + \sum_{J \subseteq I, J \neq \emptyset} B_{0,J} (s)}{1 - p}.
\]

On the other hand,

\[
|x_i - x_j|_p^{s_{ij}} |_{S_J^{|I|}} = \begin{cases} |x_i - x_j|_p^{s_{ij}} & \text{if } i, j \in J \\ |x_i - x_j|_p^{s_{ij}} & \text{if } i, j \in I \setminus J \\ 1 & \text{if } i \in J, j \in I \setminus J \\ 1 & \text{if } j \in J, i \in I \setminus J. \end{cases}
\]

Then

\[
B_{0,1} (s) = L_0 (s; I),
\]
and if \( J \subseteq I \),

\[
B_{0,I}(s) = \left\{ \int_{(p^2p)^{I \setminus J}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^{s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} L_0(s;J)
\]

(3.11)

Therefore, from (3.9)-(3.11), \( L_1(s;I,T_I) \) equals

\[
\frac{L_0(s;I) + \sum_{J \subseteq I, J \neq I} p^{1-|I|} \cdot \sum_{ij \in I} L_1(s;J, T_{I \setminus J}) L_0(s;J)}{p^{1-|I|} \cdot \sum_{i<j} s_{ij}}.
\]

(3.12)

Now, by Lemma 2 and the fact that \( \overline{A}(I) = \bigcup_{b \in p} \{(b)\}_{b \neq I} \), \( K_{\text{list}}(I) = I \), see Remark 4

\[
L_0(s;I) = \sum_{\pi \in \mathcal{R}(I) \setminus \{I\}} |\overline{A}(\pi)| p^{1-|I|} \cdot \sum_{ij \in I} L_1(s;K_{\text{list}}(\pi), K(\pi)) + (p-1) p^{1-|I|} \cdot \sum_{ij \in I} L_1(s;I, T_I)
\]

(3.13)

with \( |K_{\text{list}}(\pi)| \geq 2 \), hence from (3.14)-(3.15),

\[
\sum_{\pi \in \mathcal{R}(I) \setminus \{I\}} d_{\pi}(s) L_1(s;K_{\text{list}}(\pi), K(\pi)) + \sum_{J \subseteq I \setminus \varnothing} c_J(s) L_1(s;J, T_{I \setminus J}) L_0(s;J) + |\overline{A}(I)| p^{-|I|}.
\]

This formula and Lemmas 2-4 give a recursive algorithm for computing integrals \( L_1(s;I,T_I) \), from which we get 5.8.

From Lemmas 2-4, we obtain the following result:

**Corollary 1.** If \( |I| \geq 2 \), then

\[
L_0(s;I) = \frac{R_{I, \binom{p^{-s_{ij}}}{i,j \in I}}}{\prod_{J \in \mathcal{G}(I)} \left( 1 - p^{1-1+\sum_{ij \in J} s_{ij}} \right)^{f_J}} \prod_{i,j \in I} \left( 1 - p^{-1-s_{ij}} f_{ij} \right)
\]

where \( R_{I, \binom{p^{-s_{ij}}}{i,j \in I}} \) is a polynomial with rational coefficients in the variables \( \{p^{-s_{ij}}\}_{i,j \in I} \), \( \mathcal{G}(I) \) is a family of non-empty subsets of \( I \), with \( I \in \mathcal{G}(I) \), \( G_I \) is a non-empty subset of \( \{2 \leq i < j \leq N-2, i, j \in I\} \), and the \( f_J, f_{ij} \)'s are positive integers.
Given $I \subseteq T$, with $|I| \geq 2$, and $K \subseteq I$, with $|K| \geq 1$, and $M \subseteq T_I$, with $|M| \geq 1$, we define
\[
L_2(s; I, K, M) = \int \prod_{i \in K} |x_i|^{s_{ti}} \prod_{(i, j) \in M} |x_i - x_j|^{s_{ij}} \prod_{i \in I} dx_i
\]
for $\text{Re}(s_{ij}) > 0$ for any $ij$. If $|M| = 0$, then
\[
L_2(s; I, K, M) = \int \prod_{i \in K} |x_i|^{s_{ti}} \prod_{i \in I} dx_i.
\]

**Lemma 6.** Let $t \in \{1, N - 1\}$. Then
\[
L_2(s; I, K, T_t) = \int \prod_{i \in K} |x_i|^{s_{ti}} \prod_{2 \leq i, j \leq N - 2, i, j \in I} |x_i - x_j|^{s_{ij}} \prod_{i \in I} dx_i
\]
admits an analytic continuation as a rational function of the form
\[
(3.14) \quad L_2(s; I, K, T_t) = \frac{Q_{I, K} \left( \{p^{-s_{ij}}\}_{i, j \in I}, \{p^{-s_{ii}}\}_{i \in \{1, N - 1\}, j \in I} \right)}{R_0(s; I, K) R_1(s; I, K) R_2(s; I, K)},
\]
where
\[
R_0(s; I, K) = \prod_{J \in \mathcal{G}_2(I)} \left( 1 - p^{-|J|-\sum_{i < j, i \in J} N-2 s_{ij}} \right)^{f_J},
\]
\[
R_1(s; I, K) = \prod_{i \in U_K} \left( 1 - p^{-s_{ii}} \right)^{h_i},
\]
\[
R_2(s; I, K) = \prod_{(J, R) \in \mathcal{G}_2(I \times I)} \left( 1 - p^{-|J|-\sum_{i \in R} k_{ii} - \sum_{2 \leq i, j \leq N - 2, i, j \in J} s_{ij}} \right),
\]
where $Q_{I, K} \left( \{p^{-s_{ij}}\}_{i, j \in I}, \{p^{-s_{ii}}\}_{i \in \{1, N - 1\}, j \in I} \right)$ denotes a polynomial with rational coefficients in the variables $\{p^{-s_{ii}}\}_{i \in I}, \{p^{-s_{ij}}\}_{i \in \{1, N - 1\}, j \in I}$, $\mathcal{G}_1(I)$ is a non-empty family of subsets of $I$, with $I \in \mathcal{G}_1(I)$, $\mathcal{G}_2(I \times I)$ is a non-empty family of subsets $J \times R$ of $I \times I$, with $R \subseteq J$ and $(I, K) \in \mathcal{G}_2(I \times I)$, $U_K$ is a non-empty subset of $K$, $S_I$ is a non-empty subset of $\{2 \leq i < j \leq N - 2, i, j \in I\}$, and the $f_J$’s, $g_{ij}$’s, and the $h_i$’s are positive integers.

**Remark 7.** The integral $L_2(s; I, K, M)$ is also a multivariate $p$-adic local zeta function. If $|I| \geq 2$ and $|K| = 0$, then $L_2(s; I, K, M) = L_1(s; I, M)$.

**Proof.** We use the partition $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$ as in the proof of Lemma 3 and a change of variables, to get
\[
L_2(s; I, K, T_t) = \int \prod_{i \in K} |x_i|^{s_{ti}} \prod_{2 \leq i, j \leq N - 2, i, j \in I} |x_i - x_j|^{s_{ij}} \prod_{i \in I} dx_i
\]
\[
= \frac{B_0(s; I, K, T_t)}{1 - p^{-|J|-\sum_{i \in R} k_{ii} - \sum_{2 \leq i, j \leq N - 2, i, j \in J} s_{ij}}}.
\]
We now use the partition \( S_0^{[I]} = \bigsqcup_{J \subseteq I, J \neq \emptyset} S_J^{[I]} \) to obtain
\[
B_0 (s; I, K, T_I) = \sum_{J \subseteq I, J \neq \emptyset} B_{0, J} (s),
\]
where
\[
B_{0, J} (s) := \int \prod_{S_J^{[I]} \ni i \in K} |x_i|_p^{s_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{|s_{ij}|} \prod_{i \in I} dx_i.
\]
Consequently
\[
L_2 (s; I, K, T_I) = \frac{B_0 (s) + \sum_{J \subseteq I, J \neq \emptyset} B_{0, J} (s)}{1 - p^{-|I| - \sum_{J \subseteq I} \sum_{2 \leq i < j \leq N-2, i, j \in J} s_{ij}}).
\]
On the other hand, \( |x_i - x_j|_p^{s_{ij}} |_{S_J^{[I]}} \) is given in (5.11) and
\[
\prod_{i \in K} |x_i|_p^{s_i} |_{S_J^{[I]}} = \prod_{i \in K} |x_i|_p^{s_i} |_{(p^{\infty})|K \setminus J|}.
\]
Then \( B_{0, I} (s) = L_0 (s; I) \), and if \( J \not\subseteq I \), \( B_{0, J} (s) \) equals
\[
\left\{ \int \prod_{(p^{\infty})|K \setminus J| \ni i \in K \setminus J} |x_i|_p^{s_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} L_0 (s; J) = p^{-|I| - \sum_{J \subseteq K, J \neq \emptyset} \sum_{2 \leq i < j \leq N-2, i, j \in J} s_{ij}} \times
\]
\[
\left\{ \int \prod_{J \subseteq K, J \neq \emptyset \setminus J} |x_i|_p^{s_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} L_0 (s; J) = p^{-|I| - \sum_{i \in K \setminus J} s_i - \sum_{2 \leq i < j \leq N-2, i, j \in I \setminus J} s_{ij}} L_2 (s; I \setminus J, K \setminus J, T_{I \setminus J}) L_0 (s; J).
\]
Hence \( (1 - p^{-|I| - \sum_{i \in K} s_i - \sum_{2 \leq i < j \leq N-2, i, j \in I} s_{ij}}) L_2 (s; I, K, T_I) \) equals
\[
(3.15) \quad L_0 (s; I) + \sum_{J \subseteq I, J \neq \emptyset} p^{-|I| - \sum_{J \subseteq K, J \neq \emptyset} \sum_{2 \leq i < j \leq N-2, i, j \in J} s_{ij}} L_2 (s; I \setminus J, K \setminus J, T_{I \setminus J}) L_0 (s; J).
\]
By using that \( |I \setminus J| < |I| \) if \( J \not\subseteq I \), \( J \neq \emptyset \), and that integrals \( L_0 (s; I) \), \( L_0 (s; J) \) can be computed effectively, see Corollary 1 formula (3.15) gives a recursive algorithm for computing \( L_2 (s; I, K, T_I) \), by using it, we obtain (3.14). Notice the integrals of type \( L_2 (s; I, K, T_I) \), with \( |I| = 1 \) and \( K = \{i\} \) contribute with terms of the form \( \frac{1-p^{-1}}{1-p^{-|I|-s_i}} \).

**Lemma 7.** Given \( J \) a non-empty subset of \( T \), with \( |J| \geq 2 \), we define
\[
M_1 (s; J) = \int \prod_{(\mathbb{Z}/p) \setminus J \ni i \in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i
\]
for \( \text{Re}(s_{(N-1)} i) > 0, \ i \in J, \) and \( \text{Re}(s_{ij}) > 0, \) for \( i, j \in J. \) Then, \( M_1(s; J) \) admits an analytic continuation as a rational function of the form

\[
M_1(s; J) = \frac{Q_J \left( \{p^{-s_{ij}}\}_{i,j \in J}, \{p^{-s_{(N-1)} i}\}_{i \in J} \right)}{\prod_{i=1}^{J} U_i(s; J)},
\]

where \n
\[
U_1(s; J) = \prod_{M \in F_1(J)} \left( 1 - p^{-|M|-1 + \sum_{2 \leq l < j \leq N-2} s_{ij}} \right)^{e_M} \prod_{i,j \in J} (1 - p^{-1-s_{ij}})^{f_{ij}},
\]

\[
U_2(s; J) = \prod_{(M,S) \in F_2(J)} \left( 1 - p^{-|M|-\sum_{i \in S} s_{(N-1)i} - \sum_{2 \leq l < j \leq N-2, i,j \in M} s_{ij}} \right)^{g_{(M,S)}},
\]

and

\[
U_3(s; J) = \prod_{i \in S^{(2)}} (1 - p^{-1-s_{(N-1)i}})^{h_i},
\]

where \( F_1(J) \) is a non-empty family of subsets of \( J, \) with \( J \in F_1(J), \) \( F_2(J) \) is a non-empty family of subsets of \( \mathbb{N} \times S \subseteq J \times J, \) with \( S \subseteq M, S^{(1)} \) and \( S^{(2)} \) are non-empty subsets of \( T, \) and the \( e_M \)'s, \( f_{ij} \)'s, \( g_{(M,S)} \)'s and the \( h_i \)'s are positive integers.

Remark 8. If \( |J| = 1, \) then \( M_1(s; J) = p^{-1} \left( \frac{1-p^{-1}}{1-p^{-s_{(N-1)i}}} + p - 2 \right). \)

Proof. To compute \( M_1(s; J), \) we proceed as follows. We set

\[
T_J = \{(i,j) \in T \times T; 2 \leq i < j \leq N-2, i, j \in J\}
\]

as before, and for \( \overrightarrow{a} = (\overrightarrow{a}_i)_{i \in J} \in (\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Pi}(J), \) with

\[
\overline{\Pi}(J) := \left\{ \overrightarrow{a} \in (\mathbb{F}_p^\times)^{|J|}; \overrightarrow{a}_i \neq \overrightarrow{a}_j \text{ if } i \neq j, \text{ for } i, j \in J \text{ and } \overrightarrow{a}_s \neq \overrightarrow{a}_i \text{ for any } s \in J \right\},
\]

we define

\[
K(\overrightarrow{a}) = \{(i,j) \in T_J; \overrightarrow{a}_i = \overrightarrow{a}_j\}, \quad K^{(1)}(\overrightarrow{a}) = \{(i,j) \in T_J; \overrightarrow{a}_i = \overrightarrow{a}_j = 1\},
\]

and

\[
K^{(2)}(\overrightarrow{a}) = \{i \in J; \overrightarrow{a}_i = 1 \text{ and } \overrightarrow{a}_i \neq \overrightarrow{a}_s \text{ for any } (i,s) \in T_J\}.
\]

Notice that \( K^{(1)}(\overrightarrow{a}) \subseteq K(\overrightarrow{a}) \text{ and } K^{(2)}(\overrightarrow{a}) \cap \text{K_{list}}(\overrightarrow{a}) = \emptyset. \) Now, we introduce on \( (\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Pi}(J), \) the following equivalence relation:

\[
\overrightarrow{a} \sim \overrightarrow{b} \Leftrightarrow K(\overrightarrow{a}) = K(\overrightarrow{b}) \text{ and } K^{(1)}(\overrightarrow{a}) = K^{(1)}(\overrightarrow{b}) \text{ and } K^{(2)}(\overrightarrow{a}) = K^{(2)}(\overrightarrow{b}).
\]

We denote by \( \overline{A}(\overrightarrow{a}) = \{\overrightarrow{b} \in (\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Pi}(J); \overrightarrow{a} \sim \overrightarrow{b}\}, \) the equivalence class defined by \( \overrightarrow{a} \in (\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Pi}(J). \) By taking a unique representative in each equivalence class, we obtain \( \mathcal{R}(J) \subset (\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Pi}(J) \) such that

\[
(F_p^\times)^{|J|} = \bigsqcup_{\overrightarrow{a} \in \mathcal{R}(J)} \overline{A}(\overrightarrow{a}) \bigsqcup \overline{\Pi}(J).
\]
Given a subset $K \subseteq T_j$ with $K = \{(i_1, j_1), \ldots, (i_m, j_m)\}$, we define $K_{\text{list}} = \{i_1, j_1, \ldots, i_m, j_m\} \subseteq J$ as before. With this notation, $M_1(s; J)$ equals

$$
(3.18) \quad \sum_{\bar{a} \in \mathcal{R}(J)} \sum_{b \in (p\mathbb{Z}_p)^{|J|}} \int \prod_{i \in J} [1 - x_i]^{s_{i-1}} \prod_{2 \leq i < j \leq N-2} [x_i - x_j]^{s_{ij}} \prod_{i \in J} dx_i
$$

$$
+ \sum_{\bar{a} \in \mathcal{P}(J)} \sum_{b \in (p\mathbb{Z}_p)^{|J|}} \int \prod_{i \in J} [1 - x_i]^{s_{i-1}} \prod_{2 \leq i < j \leq N-2} [x_i - x_j]^{s_{ij}} \prod_{i \in J} dx_i
$$

$$
:= M_1(s; J) + M_2(s; J).
$$

We now use that for each $\bar{a} \in (\mathbb{F}_p^\times)^{|J|} \setminus \mathcal{P}(J),$ $T_j = K(\bar{a}) \cup \{(i, j) \in T_j; \bar{a}_i \neq \bar{a}_j\}$

and

$$
J = K^{(1)}_{\text{list}}(\bar{a}) \cup K^{(2)}(\bar{a}) \cup \{i \in J; \bar{a}_i \neq \bar{a}_j\},
$$

to obtain

$$
\prod_{i \in J} [1 - x_i]^{s_{i-1}} = \prod_{i \in K^{(1)}_{\text{list}}(\bar{a})} [1 - x_i]^{s_{i-1}} \prod_{i \in K^{(2)}(\bar{a})} [1 - x_i]^{s_{i-1}}
$$

on $b + (p\mathbb{Z}_p)^{|J|}$, and

$$
\prod_{2 \leq i < j \leq N-2} [x_i - x_j]^{s_{ij}} = \prod_{(i, j) \in K(\bar{a})} [x_i - x_j]^{s_{ij}}
$$

on $b + (p\mathbb{Z}_p)^{|J|}$. With $J(\bar{a}) := K^{(2)}(\bar{a}) \cup K_{\text{list}}(\bar{a})$, we have

$$
(3.19) \quad M_1(s; J) = \sum_{\bar{a} \in \mathcal{R}(J)} \left[ -\left| J \right| - \sum_{i \in K^{(1)}_{\text{list}}(\bar{a}) \cup K^{(2)}(\bar{a})} s_{i-1} - \sum_{(i, j) \in K(\bar{a})} s_{ij} \right] \times
$$

$$
\int \prod_{i \in K^{(1)}_{\text{list}}(\bar{a}) \cup K^{(2)}(\bar{a})} [x_i]^{s_{i-1}} \prod_{(i, j) \in K(\bar{a})} [x_i - x_j]^{s_{ij}} \prod_{i \in J(\bar{a})} dx_i
$$

$$
= \sum_{\bar{a} \in \mathcal{R}(J)} \left[ \left| T_j(\bar{a}) \right| \prod_{i \in K^{(1)}_{\text{list}}(\bar{a}) \cup K^{(2)}(\bar{a})} [x_i]^{s_{i-1}} \prod_{i \in J(\bar{a})} dx_i \right] L_2 \left( s; K_{\text{list}}(\bar{a}), K^{(1)}_{\text{list}}(\bar{a}), K(\bar{a}) \right).
$$

Now, by using the partition of $K(\bar{a})$ given in Lemma 3, we obtain

$$
(3.20) \quad L_2 \left( s; K_{\text{list}}(\bar{a}), K^{(1)}_{\text{list}}(\bar{a}), K(\bar{a}) \right) = L_2 \left( s; K^{(1)}_{\text{list}}(\bar{a}), K^{(1)}_{\text{list}}(\bar{a}), T_{K^{(1)}_{i, j}(\bar{a})} \right)
$$

$$
\times \prod_{(i, j) \in K(\bar{a}) \setminus K^{(1)}(\bar{a})} L_1 \left( s; K_{\text{list}}((i, j), \bar{a}), T_{K_{i, j}(i, j, \bar{a})} \right)
$$

with the convention that $L_2(s, \emptyset, \emptyset, \emptyset) := 1$. Finally,

$$
(3.21) \quad M_1(s; J) = \sum_{\bar{a} \in \mathcal{P}(J)} \int \prod_{i \in J} dx_i = p^{-|J|} |\mathcal{P}(J)|.
$$
Hence, formula (3.10) follows from (3.18)-(3.21) by using Lemma 6 and Remark 7.

3.3. Computation of $Z_1(s; I)$.

**Proposition 1.** Let $I$ be a non-empty subset of $T$. Then, the integral

$$Z_1(s; I) = \begin{cases} \displaystyle \int_{2^{|I|}} \frac{1}{\prod_{i \in I} |x_i|^{|s_i|}} \prod_{1 \leq i < j \leq N-2} \prod_{x_{ij} \in I} dx_i & \text{if } |I| \geq 2, \\ \int_{2^{|I|}} \frac{1}{\prod_{i \in I} |x_i|^{|s_i|} (N-1)^{s_i} + \Sigma_{2 \leq j \leq N-2, j \neq i} s_j} \prod_{1 \leq i < j \leq N} dx_i & \text{if } |I| = 1 \end{cases}$$

converges on the set

$$\{(s_{ij}) \in \mathbb{C}^D; \Re(s_{ij}) > -1 \text{ for } 2 \leq i < j \leq N-2, i,j \in I\} \cap \\{(s_{ij}) \in \mathbb{C}^D; 1 + \Re(s_{1i} + s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \Re(s_{ij}) < 0 \text{ for } i \in I\},$$

which is an open and connected subset of $\mathbb{C}^D$. In addition, $Z_1(s; I)$ admits an analytic continuation to $\mathbb{C}^D$ as a rational function of the form

$$Z_1(s; I) = \frac{Q_{I,1}(\{p^{-s_{ij}}; i,j \in \{1, \ldots, N-1\}\})}{S_1(s; I) S_2(s; I) S_3(s; I) S_4(s; I)},$$

where $Q_{I,1}(\{p^{-s_{ij}}; i,j \in \{1, \ldots, N-1\}\})$ denotes a polynomial with rational coefficients in the variables $p^{-s_{ij}}$, $i,j \in \{1, \ldots, N-1\}$,

$$S_1(s; I) = \prod_{J \in \mathcal{H}_1(I)} \left( 1 - p^{\sum_{i \in J}(s_{1i} + s_{(N-1)i}) + \Sigma_{2 \leq j \leq N-2} s_{ij} + \Sigma_{2 \leq i < j \leq N-2} s_{ij}} \right),$$

where $\mathcal{H}_1(I)$ is a family of non-empty subsets of $I$, with $I \in \mathcal{H}_1(I)$,

$$S_2(s; I) := \prod_{J \in \mathcal{H}_2(J)} \prod_{K \in \mathcal{H}_2(J)} \left( 1 - p^{\sum_{i \in K}(s_{1i} + s_{(N-1)i}) + \Sigma_{1 \leq i < j \leq N-2} s_{ij} + \Sigma_{1 \leq \alpha < j, \beta \in K} s_{j\alpha} s_{j\beta}} \right)^{e_K},$$

where $\mathcal{H}_2(J)$ is a family of non-empty subsets of $J$, with $J \in \mathcal{H}_2(J)$, and the $e_K$'s are positive integers,

$$S_3(s; I) := \prod_{J \subseteq I} \prod_{i \in J} \left( 1 - p^{-s_{ij}} \right),$$

where $G^{(0)}_J$ is a non-empty subset $\{2 \leq i < j \leq N-2, i,j \in J\}$,

$$S_4(s; I) := \prod_{i \in G^{(1)}_I} \left( 1 - p^{1+s_{1i} + s_{(N-1)i} + \Sigma_{2 \leq j \leq N-2, j \neq i} s_{ij}} \right),$$

where $G^{(1)}_I$ is a non-empty subset $\{2 \leq i < j \leq N-2, i,j \in I\}$. 
Proof. By using the partition $Z_p^{|I|} = (pZ_p)^{|I|} \cup S_0^{|I|}$ as in the proof of Lemma 3.10 and a change of variables, we get

$$Z_1 (s; I) = \frac{\prod_{i \in I} |x_i - x_j|_p^{s_{ij}}}{\prod_{i \in I} |x_i|_p^{2s_{i1} + s_{i(N-1)i} + 2s_{i2j} + s_{i2j} - s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij}} \prod_{i \in I} dx_i}$$

$$C_0 (s) := \frac{C_0 (s)}{1 - p} =: \frac{|I| + \sum_{i \in I} (s_{i1} + s_{i(N-1)i}) + \sum_{2 \leq i < j \leq N} - 2 s_{ij} + \sum_{2 \leq i < j \leq N} - 2 s_{ij}}{1 - p}$$

We now use the partition $S_0^{|I|} = \cup_{J \subseteq I, j \in J} S_j^{|I|}$ to obtain

$$C_0 (s) = \sum_{j \subseteq I, j \neq \emptyset} C_{0,j} (s),$$

where

$$C_{0,j} (s) := \frac{\prod_{i \in J} |x_i - x_j|_p^{s_{ij}}}{\prod_{i \in J} |x_i|_p^{2s_{i1} + s_{i(N-1)i} + 2s_{i2j} + s_{i2j} - s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij}} \prod_{i \in J} dx_i},$$

and consequently,

$$Z_1 (s; I) = \frac{C_{0,I} (s) + \sum_{j \subseteq I, j \neq \emptyset} C_{0,j} (s)}{1 - p}$$

On the other hand, by using Lemma 3.10, we have $C_{0,I} (s) = L_0 (s; I)$, and if $J \subseteq I$,

$$C_{0,j} (s) = \left\{ \begin{array}{l} \int_{(pZ_p)^{|I\setminus J|}} \left\{ \prod_{i \in J} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} |x_i|_p^{2s_{i1} + s_{i(N-1)i} + 2s_{i2j} + s_{i2j} - s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} L_0 (s; J) \\
\end{array} \right. \times$$

$$= p \left\{ \int_{Z_p^{|I\setminus J|}} \left\{ \prod_{i \in I \setminus J} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} |x_i|_p^{2s_{i1} + s_{i(N-1)i} + 2s_{i2j} + s_{i2j} - s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij} + \Sigma_{2 \leq i < j \leq N} - 2 s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} L_0 (s; J) \\
\right. \times$$

$$Z_1 (s; I \setminus J) L_0 (s; J).$$
Notice that in (3.23), 
\[ Z_i \] 
for (3.24) 
\[ Z(J) \] 
where \( I \) all \( G \subset \) (3.23).

Therefore

\[ (3.23) \quad Z_1 (s; I) = \frac{L_0 (s; I) + \sum_{J \subseteq I, J \neq \emptyset} p^{M(s, J)} Z_1 (s; I \setminus J) L_0 (s; J)}{1 - p} \]

where

\[ M(s, J) := |I \setminus J| + \sum_{i \in I \setminus J} (s_{i1} + s_{(N-1)i}) + \sum_{2 \leq i < j \leq N-2} s_{ij} \]

\[ + \sum_{2 \leq i < j \leq N-2 \atop i \in T \setminus (I \setminus J), j \in I \setminus J} s_{ij}. \]

Notice that in (3.23), \( Z_1 (s; I \setminus J) \) may occur with \(|I \setminus J| = 1\), say \( I \setminus J = \{i\} \), in this case \( Z_1 (s; I) \) becomes

\[ (3.24) \quad \int_{2p} 1 \left| x_i \right|^{2+s_{i1}+s_{(N-1)i}+\sum_{2 \leq i < j \leq N-2, i \neq j} s_{ij}} dx_i = \frac{1 - p^{-1}}{1 - p^{1+s_{i1}+s_{(N-1)i}+\sum_{2 \leq i < j \leq N-2, i \neq j} s_{ij}}} \]

for \( \text{Re}(s_{i1}) + \text{Re}(s_{(N-1)i}) + \sum_{2 \leq i < j \leq N-2, i \neq j} \text{Re}(s_{ij}) < -1 \).

Finally, formula (3.24) gives a recursive algorithm for computing \( Z_1 (s; I) \), since \( I \setminus J \nsubseteq I \subseteq T \) and \( L_0 (s; I), \ L_0 (s; J) \) can be effectively computed, see Corollary [1] by using this algorithm and [3.22], we obtain [3.24].

\[
\begin{align*}
\text{Remark 9.} & \quad \text{Given positive integers } N_i, \ i \in I \subseteq T, \ v, \ \text{and complex numbers } s_i \ \text{for } i \in I, \ \text{we notice that the function} \ \frac{1 - p^{1+s_{i1}+s_{(N-1)i}+\sum_{2 \leq i < j \leq N-2, i \neq j} s_{ij}}}{1 - p^{-1}} \ \text{gives rise to a holomorphic function of} \ s_i \ \text{on the half-plane} \ \sum_{i \in I} N_i \text{Re}(s_i) + v > 0. \ \text{As a consequence of Proposition [1] there exist families} \ \mathcal{F}_1, \ \mathcal{F}_2 \ \text{of non-empty subsets of} \ T, \ \text{and a non-empty subset} \ \mathcal{G} \ \text{of} \ \{ij; 2 \leq i < j \leq N-2, i, j \in T\}, \ \text{such that all the integrals} \ Z_1 (s; I) \ \text{for all} \ I \subseteq T \ \text{are holomorphic functions of} \ s \ \text{on the solution set of the conditions:}
\end{align*}
\]

\[
\begin{align*}
(C1) \quad |J| + \sum_{i \in J} \left( \text{Re} (s_{i1}) + \text{Re} (s_{(N-1)i}) \right) + \sum_{2 \leq i < j \leq N-2} \text{Re}(s_{ij}) & < 0 \ \text{for} \ J \in \mathcal{F}_1; \\
& + \sum_{2 \leq i < j \leq N-2} \text{Re}(s_{ij}) < 0 \ \text{for} \ J \in \mathcal{F}_1;
\end{align*}
\]

\[
\begin{align*}
(C2) \quad |K| - 1 + \sum_{2 \leq i < j \leq N-2} \text{Re}(s_{ij}) & > 0 \ \text{for} \ K \in \mathcal{F}_2;
\end{align*}
\]

\[
\begin{align*}
(C3) \quad 1 + \text{Re}(s_{ij}) & > 0 \ \text{for} \ ij \in \mathcal{G} \subseteq \{ij; 2 \leq i < j \leq N-2\}.
\end{align*}
\]

Notice that the condition

\[
1 + \text{Re}(s_{i1}) + \text{Re}(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \text{Re}(s_{ij}) < 0
\]
is included in Condition C1 taking |J| = 1. This fact follows from the following identities:

\[
\sum_{2 \leq i \leq j \leq N-2} s_{ij} + \sum_{i \in J, j \in T} s_{ij} = \sum_{2 \leq i \leq j \leq N-2} s_{ij} + \sum_{i \in T, j \in J} s_{ij} - \sum_{i \leq j \leq N-2} s_{ij} =
\]

\[
\sum_{2 \leq i \leq j \leq N-2} s_{ij} + \sum_{i \in J, j \in T \setminus J} s_{ij} = \sum_{2 \leq i \leq j \leq N-2} s_{ij} + \sum_{i \in J, j \in T \setminus J} s_{ij} + \sum_{j \in T \setminus J, i \in T} s_{ij} =
\]

\[
\sum_{2 \leq i \leq j \leq N-2} s_{ij} + \sum_{i \in J, j \in T \setminus J} s_{ij} + \sum_{i \in J, j \in T \setminus J} s_{ij} = \sum_{2 \leq i \leq j \leq N-2} s_{ij} + \sum_{j \neq i, i \in J, j \in T \setminus J} s_{ij}.
\]

Finally, by taking \( J = \{i\} \), the last formula becomes \( \sum_{2 \leq j \leq N-2} s_{ij} \).

Denote by \( D_{1,1} \) the natural domain of definition of \( Z_1(s; I) \), i.e. \( D_{1,1} \) is an open and connected subset of \( \mathbb{C}^D \) in which \( Z_1(s; I) \) is holomorphic and there no exists a larger domain where this property holds.

**Lemma 8.** Take \( I \) to be a non-empty subset of \( T \) and set \( H_{1,1}(\mathbb{C}) \) to be the solution set in \( \mathbb{C}^D \) of the following conditions:

\[
(3.25) \quad 1 + \Re(s_{1i}) + \Re(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \Re(s_{ij}) < 0, \text{ for } i \in I.
\]

Then \( D_{1,1} \) is contained in \( H_{1,1}(\mathbb{C}) \).

**Proof.** Denote by \( H_{1,1}(\mathbb{R}) \) the solution set of (3.25) in \( \mathbb{R}^D \). Set \( \Re(D_{1,1}) = \{ \Re(s_{ij}) \in \mathbb{R}^D; (s_{ij}) \in D_{1,1} \} \). With this notation, it is sufficient to show that \( \Re(D_{1,1}) \subset H_{1,1}(\mathbb{R}) \). In order to do this, we show that the integral \( Z_1(\tilde{s}; I) < +\infty \) for any \( \tilde{s} \in \mathbb{R}^D \setminus H_{1,1}(\mathbb{R}) \). We prove this last assertion by contradiction. Assume that \( Z_1(\tilde{s}; I) < +\infty \) for \( \tilde{s} = (\tilde{s}_{ij}) \in \mathbb{R}^D \) with \( \tilde{s}_{ij} \geq 0 \) for \( 2 \leq i < j \leq N-2 \), \( i, j \in I \) and that \( \tilde{s} \notin H_{1,1}(\mathbb{R}) \). This last condition implies that at least a condition of the form

\[
(3.26) \quad 1 + \tilde{s}_{1i_0} + \tilde{s}_{(N-1)i_0} + \sum_{2 \leq j \leq N-2, j \neq i_0} \tilde{s}_{ij} \geq 0
\]

for some \( i_0 \in I \), holds. Then, from \( Z_1(\tilde{s}; I) < +\infty \), we have

\[
I(\tilde{s}; A) := \int_A \left( \prod_{2 \leq i < j \leq N-2} \frac{|x_i - x_j|_p^{\tilde{s}_{ij}}}{\left( \prod_{i \in I} |x_i|_p \right)^{2+\tilde{s}_{1i} + \tilde{s}_{(N-1)i} + \sum_{2 \leq j \leq N-2, j \neq i} \tilde{s}_{ij}}} \right) \prod_{i \in I} dx_i < +\infty
\]

for any \( A \subset \mathbb{Z}_p^{|I|} \). Take

\[
A_0 = \left\{ (x_i)_{i \in I} \in \mathbb{Z}_p^{|I|}; |x_{i_0}|_p < 1 \text{ and } |x_i|_p = 1 \text{ for } i \in I \setminus \{i_0\} \right\}.
\]

Then, by (3.26) and some \( \epsilon \geq 0 \),

\[
I(\tilde{s}; A_0) = \int_{A_0} \left( \prod_{2 \leq i < j \leq N-2} \frac{|x_i - x_j|_p^{\tilde{s}_{ij}}}{|x_{i_0}|_p^{1+\epsilon}} \right) \prod_{i \in I} dx_i = +\infty.
\]
Therefore, if $Z_1(\mathbf{s}; I) < +\infty$, necessarily $\mathbf{s} \in H_{1,1}(\mathbb{R})$.

**Corollary 2.** If $\mathbf{s} = (s_{ij}) \in \mathbb{R}^D$, with $s_{ij} \geq 0$ for $i, j \in \{1, \ldots, N - 1\}$, then the integral $Z_1(\mathbf{s}; I)$ diverges to $+\infty$, for any non-empty subset $I$ of $T$.

### 3.4. Computation of $Z_0(\mathbf{s}; I)$

**Proposition 2.** Let $I$ be a subset of $T$ satisfying $|I| \geq 2$. Then, the integral

$$Z_0(\mathbf{s}; I) = \int_{\mathcal{Z}_p(I)} \prod_{i \in I} |x_i|^{s_{ii}} |1 - x_i|^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^{s_{ij}} \prod_{i \in I} dx_i$$

gives rise to a holomorphic function on $H_{1,0} := \{(s_{ij}) \in \mathbb{C}^D; \Re(s_{ij}) > 0 \text{ for } i, j \in I \} \cap \{(s_{ij}) \in \mathbb{C}^D; \Re(s_{ii}) > 0 \text{ for } i \in I \} \cap \{(s_{ij}) \in \mathbb{C}^D; \Re(s_{(N-1)i}) > 0 \text{ for } i \in I \},$

which is an open and connected subset of $\mathbb{C}^D$. Furthermore $Z_0(\mathbf{s}; I)$ has an analytic continuation as a rational function of the form

$$Z_0(\mathbf{s}; I) = \frac{Q_{1,0}(\{p^{-s_{ii}}, p^{-(N-1)i}, p^{-s_{ij}}; i, j \in T\})}{\prod_{i=0}^{2} R_i(\mathbf{s}; I) \prod_{i=1}^{3} U_i(\mathbf{s}; I)}$$

where $Q_{1,0}(\{p^{-s_{ii}}, p^{-(N-1)i}, p^{-s_{ij}}; i, j \in T\}$ is a polynomial in the variables $p^{-s_{ii}}$, $p^{-(N-1)i}$, $p^{-s_{ij}}$ for $i, j \in T$, $U_i(\mathbf{s}; I)$, $i = 1, 2, 3$ are as in Lemma 4

$$R_1(\mathbf{s}; I, I) = (1 - p^{-1-s_{ii}})^{h_1},$$

$$R_2(\mathbf{s}; I, K) = \prod_{(J,R) \in \mathcal{G}_2(I \times I)} \left(1 - p^{-|J|-\sum_{i \in R} s_{ii} - \sum_{2 \leq i < j \leq N-2} s_{ij}}\right)^{l_{(J,R)}},$$

$R_0(\mathbf{s}; I), \mathcal{G}_2(I \times I)$ are as in Lemma 4 and the $l_{(J,R)}$'s are positive integers.

**Proof.** By using that $\mathcal{Z}_p(I) = (p\mathcal{Z}_p)[I] \cup S_0[1]$ we have

$$Z_0(\mathbf{s}; I) = V_1(\mathbf{s}; I) + V_2(\mathbf{s}; I),$$

where

$$V_1(\mathbf{s}; I) := \int_{(p\mathcal{Z}_p)[I]} \prod_{i \in I} |x_i|^{s_{ii}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^{s_{ij}} \prod_{i \in I} dx_i,$$

$$V_2(\mathbf{s}; I) := \int_{S_0[1]} \prod_{i \in I} |x_i|^{s_{ii}} |1 - x_i|^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^{s_{ij}} \prod_{i \in I} dx_i.$$

Now, by changing variables and using Lemma 4 with $t = 1$, $V_1(\mathbf{s}; I)$ equals

$$p^{-|I|-\sum_{i \in I} s_{ii} - \sum_{2 \leq i < j \leq N-2} s_{ij}} \int_{(p\mathcal{Z}_p)[I]} \prod_{i \in I} |x_i|^{s_{ii}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^{s_{ij}} \prod_{i \in I} dx_i$$

$$= p^{-|I|-\sum_{i \in I} s_{ii} - \sum_{2 \leq i < j \leq N-2} s_{ij}} L_2(\mathbf{s}; I, I, T_1).$$
To compute $V_2(s; I)$, we use the partition $S_0^{[I]} = \bigcup_{J \subseteq I, J \neq \emptyset} S_J^{[I]}$, with $S_J^{[I]} = \{(x_i)_{i \in I} \in \mathbb{Z}_p^{|I|}; |x_i|_p = 1 \Leftrightarrow i \in J\}$, then $V_2(s; I)$ equals

\begin{equation}
\sum_{J \subseteq I, J \neq \emptyset} \int \prod_{i \in I} |x_i|_p^{s_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx
= \sum_{J \subseteq I, J \neq \emptyset} M_J(s),
\end{equation}

where

\[
M_J(s) = \int \prod_{S_J^{[I]}} |x_i|_p^{s_i} \prod_{i \in I} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx
\]

\[
\prod_{2 \leq i < j \leq N-2} \prod_{i \in I} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx = \int \prod_{(\mathbb{Z}_p^2)_{i < j}} |x_i|_p^{s_i} \prod_{i \in I} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i
\]

\[
\times \int \prod_{(\mathbb{Z}_p^2)_{i < j}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i := H_0(s; I \setminus J) M_1(s; J).
\]

We notice that if $J = I$, then, by convention, $H_0(s; I \setminus J) = 1$. Now suppose that $J \subsetneq I$. From Lemma 6 with $t = 1$, we have

\begin{equation}
H_0(s; I \setminus J) = p^{-|I \setminus J| - \sum_{i \in I \setminus J} s_i i - \sum_{2 \leq i < j \leq N-2} s_{ij}}\prod_{i \in I \setminus J} L_2(s; I \setminus J, I \setminus J, T_{i \setminus J}).
\end{equation}

The announced result follows from formulas (3.27)-(3.30), and $M_1(s; J)$ by using Lemmas 5 and Remark 8. \qed

**Remark 10.** As a consequence of Proposition 9 all the integrals $Z_0(s; I)$ for all $I \subseteq T$ are holomorphic functions of $s$ on the solution set in $\mathbb{C}^D$ of the following conditions:

\begin{enumerate}[(C4)]
\item $|J| + \sum_{i \in S} \text{Re}(s_i) + \sum_{2 \leq i < j \leq N-2, i, j \in J} \text{Re}(s_{ij}) > 0$ for $J \times S \in \mathfrak{F}_3$,
\item with $S \subseteq J$, $t \in \{1, N-1\}$, and $\mathfrak{F}_3$ a family of non-empty subsets of $I \times I$;
\item $|K| - 1 + \sum_{2 \leq i < j \leq N-2} \text{Re}(s_{ij}) > 0$ for $K \in \mathfrak{F}_4$,
\end{enumerate}

where $\mathfrak{F}_4$ is a family of non-empty subsets of $I$;

\begin{enumerate}[(C6)]
\item $1 + \text{Re}(s_{ij}) > 0$ for $i, j \in G_T$,
\end{enumerate}

where $G_T$ is a non-empty subset of $\{2 \leq i < j \leq N-2, i, j \in J\}$ with $(N-1)i$, $1i \in G_T$.

**Remark 11.** If $s = (0)_{i,j}$ for $i, j \in \{1, \ldots, N-1\}$, then $Z_0(0; I) = 1$, for any non-empty subset $I$ of $T$.

**Definition 3.** Denote by $H(\mathbb{R})$, respectively by $H(\mathbb{C})$, the solution set of conditions C1-C6 in $\mathbb{R}^D$, respectively in $\mathbb{C}^D$. 

3.5. Main Theorem.

Lemma 9. Consider the following conditions:

\( (C1') \quad |J| + \sum_{i \in J} (\Re (s_{1i}) + \Re (s_{(N-1)i})) + \sum_{2 \leq i < j \leq N-2} \Re (s_{ij}) \)

\[ + \sum_{2 \leq i < j \leq N-2} \Re (s_{ij}) < 0 \text{ for } J \subseteq T, \ |J| \geq 1; \]

\( (C2') \quad |J| - 1 + \sum_{2 \leq i < j \leq N-2} \Re (s_{ij}) > 0 \text{ for } J \subseteq T, \ |J| \geq 2; \]

\( (C3') \quad |J| + \sum_{i \in J} \Re (s_{1i}) + \sum_{2 \leq i < j \leq N-2, i,j \in J} \Re (s_{ij}) > 0 \)

for \( t \in \{1, N-1\}, \ J \times S \subseteq T \times T \) with \( |J| \geq 2 \) or \( |S| \geq 1, S \subseteq J; \)

\( (C4') \quad 1 + \Re (s_{ij}) > 0 \text{ for } ij \in \{(i,j) ; 1 \leq i < j \leq N-1\}. \)

Denote by \( H_0(\mathbb{R}) \), respectively by \( H_0(\mathbb{C}) \), the solution set of conditions \( C1' - C4' \) in \( \mathbb{R}^D \), respectively in \( \mathbb{C}^D \). Then \( H_0(\mathbb{R}) \) is convex and bounded set with non-empty interior, and \( H_0(\mathbb{C}) \) contains an open and connected subset of \( \mathbb{C}^D \). Furthermore, \( H_0(\mathbb{R}) \subseteq H(\mathbb{R}) \) and \( H_0(\mathbb{C}) \subseteq H(\mathbb{C}). \)

Proof. We first notice that for all \( N \geq 4 \), the solution set \( H_0(\mathbb{R}) \) is an open convex set because it is a finite intersection of open half-spaces.

Claim \( H_0(\mathbb{R}) \) is a non-empty bounded subset. We consider the case \( N \geq 5 \) in which \( |T| \geq 2 \). Set \( N_1 = \frac{(N-4)(N-3)}{2} \). We define, for \( i,j \in \{2, ..., N-2\} \), the following conditions:

\( (C1'') \quad -\frac{2}{3N_1} < \Re (s_{ij}) < 0, \)

\( (C2'') \quad -\frac{2}{3} < \Re (s_{1i}) < -\frac{1}{2}, \)

\( (C3'') \quad -\frac{2}{3} < \Re (s_{(N-1)i}) < -\frac{1}{2}. \)

We notice that the solution set of conditions \( C1'' - C3'' \) is a non-empty open and connected subset in \( \mathbb{R}^D \). We now verify that the conditions \( C1'' - C2'' \) imply conditions \( C1' - C4' \). First, consider \( J \subseteq T \) such that \( |J| = 1 \). We can assume that \( J = \{i_0\} \) for some \( i_0 \in T \). By conditions \( C1'' - C3'' \), we have

\( (3.31) \quad 1 + \Re (s_{1i_0}) + \Re (s_{(N-1)i_0}) < 1 - 1/2 - 1/2 = 0, \)

\( (3.32) \quad \sum_{2 \leq i < j \leq N-2} \Re (s_{ij}) + \sum_{2 \leq i < j \leq N-2, i \in T\setminus J} \Re (s_{ij}) < 0, \)

thus, \( C1' \) follows from \( (3.31) \) and \( (3.32) \). Conditions \( C2', C3' \) and \( C4' \) follow directly from \( C1'' - C3'' \).
We now consider $J \subseteq T$ such that $|J| \geq 2$. Condition $C1'$ is obtained with a similar calculation to (3.31) and (3.32). Now, by condition $C1''$, we get

$$|J| - 1 + \sum_{2 \leq i < j \leq N-2, i, j \in J} \text{Re}(s_{ij}) > |J| - 1 - \frac{2}{3} > |J| - \frac{5}{3} > 0,$$

which implies $C2'$. We now verify Condition $C3'$. Let $t \in \{1, N-1\}$, by using conditions $C2''$ and $C3''$,

$$|J| + \sum_{i \in S} \text{Re}(s_{ti}) + \sum_{2 \leq i < j \leq N-2, i, j \in J} \text{Re}(s_{ij})$$

$$> |J| - \frac{2}{3}|S| - \frac{2}{3}|s_{(i,j)}| 0, \text{ if } |S| \geq 2,$

by using $\frac{2}{3}|S| - \frac{2}{3} > |S|$ and $|J| \geq |S|$, then $|J| - \frac{2}{3}|S| - \frac{2}{3} \geq |J| - |S| \geq 0$.

Finally, conditions $C4'$ follows from conditions $C1''-C3''$. Therefore, $H_0(\mathbb{R})$ is a convex and bounded set with non-empty interior, and $H_0(\mathbb{C})$ contains an open and connected subset of $\mathbb{C}^D$. Finally, since conditions $C1'-C4'$ imply conditions $C1-C6$, we conclude that $H_0(\mathbb{R}) \subset H(\mathbb{R})$ and that $H_0(\mathbb{C}) \subset H(\mathbb{C})$.

In the case $N = 4$, $|T| = 1$, the verification of the claim is straightforward. \qed

**Theorem 1.** (1) The p-adic open string $N$-point zeta function, $Z^{(N)}(s)$, gives rise to a holomorphic function on $H(\mathbb{C})$, which contains an open and connected subset of $\mathbb{C}^D$. Furthermore, $Z^{(N)}(s)$ admits an analytic continuation to $\mathbb{C}^D$, denoted also as $Z^{(N)}(s)$, as a rational function in the variables $p^{-s_{ij}}, i, j \in \{1, \ldots, N-1\}$. The real parts of the poles of $Z^{(N)}(s)$ belong to a finite union of hyperplanes, the equations of these hyperplanes have the form $C1-C6$ with the symbols $'<'$,$'>'$ replaced by $'=$. (2) If $s = (s_{ij}) \in \mathbb{C}^D$, with $\text{Re}(s_{ij}) \geq 0$ for $i, j \in \{1, \ldots, N-1\}$, then the integral $Z^{(N)}(s)$ diverges to $+\infty$.

**Proof.** (1) We recall that

$$Z^{(N)}(s) = \sum_{I \subseteq T} Z^{(N)}(s; I) = \sum_{I \subseteq T} p^{M(s)} Z_0^{(N)}(s; I) Z_1^{(N)}(s; T \setminus I),$$

see Remark 3.1. Now, by Propositions 1,2 and Lemma 9 for any $I \subseteq T$, $Z_0^{(N)}(s; I)$ and $Z_1^{(N)}(s; T \setminus I)$ are holomorphic functions of $s \in H_0(\mathbb{C})$, which is an open and connected subset, and consequently the analytic continuations of the integrals $Z_0^{(N)}(s; I)$ and $Z_1^{(N)}(s; T \setminus I)$ and formula (3.33) give rise to an analytic continuation of $Z^{(N)}(s)$ with the announced properties.

(2) It follows from formula (3.33) by Corollary 2 and Remark 11. \qed

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