On the generalized Apostol-type Frobenius-Genocchi polynomials

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Abstract. The main object of this work is to introduce a new class of the generalized Apostol-type Frobenius-Genocchi polynomials and is to investigate some properties and relations of them. We derive implicit summation formulae and symmetric identities by applying the generating functions. In addition a relation in between Array-type polynomials, Apostol-Bernoulli polynomials and generalized Apostol-type Frobenius-Genocchi polynomials is also given.

1. Introduction

Let \( a \in \mathbb{Z}, \lambda \in \mathbb{C}, a, b, c \in \mathbb{R}^+, a \neq b \) and \( x \in \mathbb{R} \). The generalized Apostol-Bernoulli, Euler and Genocchi polynomials with the parameters are given by means of the following generating function as follows (see [1-15]):

\[
\left( \frac{t}{\lambda b^t - a^t} \right)^\alpha c^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}, \quad |t \ln \frac{b}{a}| < 2\pi,
\]

\[
\left( \frac{2}{\lambda b^t + a^t} \right)^\alpha c^\alpha = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}, \quad |t \ln \frac{b}{a}| < \pi,
\]

and

\[
\left( \frac{2t}{\lambda b^t + a^t} \right)^\alpha c^\alpha = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}, \quad |t \ln \frac{b}{a}| < \pi.
\]

Obviously, we have

\[ B_n^{(\alpha)}(x; \lambda; 1, c, e) = B_n(x; \lambda), \quad E_n^{(\alpha)}(x; \lambda; 1, c, c) = E_n(x; \lambda), \quad \text{and} \quad G_n^{(\alpha)}(x; \lambda; 1, c, c) = G_n(x; \lambda). \]

Recently, Kurt et al. [1] and Simsek [11, 12] introduced the Apostol type Frobenius-Euler polynomials as follows.

Let \( a \in \mathbb{Z}, \lambda \in \mathbb{C}, a, b, c \in \mathbb{R}^+, a \neq b, x \in \mathbb{R} \). The generalized Apostol type Frobenius-Euler polynomials are defined by means of the following generating function:

\[
\left( \frac{a^t - u}{\lambda b^t - u} \right)^\alpha c^\alpha = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; u, a, b, c, \lambda) \frac{t^n}{n!}.
\]

2010 Mathematics Subject Classification. 11B68, 05A10, 05A15, 33C45, 26B99

Keywords. (Frobenius-Genocchi polynomials, Apostol-type Genocchi polynomials)

Received: 19 June 2018; Accepted: 28 September 2019

Communicated by Dragan S. Djordjević

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For $x = 0$ and $\alpha = 1$ in (1.4), we get
\[
\frac{a^t - u}{\lambda b^t - u} = \sum_{n=0}^{\infty} H_n(u, a, b; \lambda) \frac{t^n}{n!},
\]
(1.5)
where $H_n(u, a, b; \lambda)$ denotes the generalized Apostol type Frobenius-Euler numbers (see [9], [12], [14]).

On setting $a = 1$, $b = e$, $\lambda = 1$ in (1.4), the result reduces to
\[
\left(\frac{1 - u}{e^t - u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; u) \frac{t^n}{n!},
\]
(1.6)
where $H_n^{(\alpha)}(x; u)$ is called classical Frobenius-Euler polynomial of order $\alpha$ (see [1], [12], [15]).

Observe that
\[
H_n^{(1)}(x; u) = H_n(x; u),
\]
which denotes the Frobenius-Euler polynomials and
\[
H_n^{(\alpha)}(0; u) = H_n^{(\alpha)}(u),
\]
which denotes the Frobenius-Euler numbers of order $\alpha$.

Recently, Yaşar and Özarslan [15] introduced the Frobenius-Genocchi polynomials by means of the following generating function:
\[
\frac{(1 - \lambda)t}{e^t - \lambda} e^{xt} = \sum_{n=0}^{\infty} G_n(x; \lambda) \frac{t^n}{n!},
\]
(1.7)
and the generalized array type polynomials are defined by Simsek [11, p.6, Eq. (3.1)] as:
\[
\sum_{n=0}^{\infty} S_n^{(\alpha)}(x; a, b; \lambda) \frac{t^n}{n!} = \frac{(\lambda b^t - a^t)^{\nu}}{\nu!}.
\]
(1.10)
Kurt and Simsek [1] introduced the polynomial $Y_n(x; \lambda; a)$, which is given by the following generating function:
\[
\frac{1}{\lambda a^t - 1} a^{xt} = \sum_{n=0}^{\infty} Y_n(x; \lambda; a) \frac{t^n}{n!} \quad (a \geq 1).
\]
(1.11)
For $x = 0$ in (1.11), we get
\[
Y_n(0; \lambda; a) = Y_n(\lambda; a), (see[15]).
\]
(1.12)
Again if we set $x = 0$ and $a = 1$, in (1.11), we obtain
\[
Y_n(\lambda; 1) = \frac{t}{\lambda - 1}.
\]
(1.13)
Remark 2.1. Theorem 2.1

The following recurrence relation holds true:

\[ G_n^{(a)}(x; u, a, b, c; \lambda) = \sum_{r=0}^{\infty} \binom{n}{r} \lambda^r \sum_{m=0}^{\infty} G_m(u; a, b, c; \lambda) \frac{t^m}{m!}, \]

(2.1)

where \((a \in \mathbb{Z}, \lambda \in \mathbb{C}, a, b, c \in \mathbb{R}^+, a \neq b, x \in \mathbb{R}).

Remark 2.2. If we set \(a = 1, b = c = e, u = -1, (2.1)\) immediately reduces to the Apostol-type Genocchi polynomials (see [4], [12], [15]).

\[ \left( \frac{e^t - u}{e^{bt} - u} \right)^a e^x = \sum_{n=0}^{\infty} G_n(x; \lambda, |t| < \pi). \]

(2.2)

We have the following properties of (2.1), which are stated in terms of theorems as:

Theorem 2.1. The following recurrence relation holds true:

\[ (2u - 1) \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) G_r(x; u, a, b, c; \lambda) G_{n-r}(y; 1 - u; a, b, c; \lambda) \]

\[ = n(n-1) G_{n-1}(x+y; a, b, c; \lambda) + nu G_{n-1}(x+y; 1-u, a, b, c; \lambda) \]

\[ + \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (\ln a)^{n-r} G_r(x+y; a, b, c; \lambda) \]

\[ - \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (\ln a)^{n-r} G_r(x+y; 1-u, a, b, c; \lambda). \]

(2.4)

Proof. In order to prove (2.4), we set

\[ (2u - 1) \left( \frac{e^t - u}{e^{bt} - u} \right)^a e^x \left( \frac{e^t - (1-u)}{e^{bt} - (1-u)} \right)^c e^y. \]
Employing the result (2.2), equation (2.5) reduces as

\[
(2u - 1) \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left( \frac{n}{r} \right) G_r(x; u, a, b, c; \lambda) \frac{t^r}{r!} \sum_{n=0}^{\infty} G_n(y; 1 - u; a, b, c; \lambda) \frac{t^n}{n!} = (a' - (1 - u)t)
\]

Using [13, p. 100, Eq. 2], we get

\[
(2u - 1) \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left( \frac{n}{r} \right) G_r(x; u, a, b, c; \lambda) G_{n-r}(y; 1 - u; a, b, c; \lambda) \frac{t^r}{r!} \]

\[
= (a' - (1 - u)t) \sum_{r=0}^{\infty} G_r(x + y; u, a, b, c; \lambda) \frac{t^r}{r!} - (a' - u)t
\]

\[
\times \sum_{r=0}^{\infty} G_r(x + y; 1 - u; a, b, c; \lambda) \frac{t^r}{r!}.
\]

(2.7)

On comparing the coefficients of \(t^n\) in both sides, we arrive at the desired result (2.4).

**Theorem 2.2.** The following relation holds true:

\[
(G_{n+1}(x; u, a, b, b; \lambda) - \ln(b) \sum_{k=0}^{n+1} \left( \frac{n+1}{k} \right) Y_{n+1-k}\left(1; \frac{1}{u}; a\right) \frac{t^k}{k!})
\]

\[
= \ln(a)^{n+1} \sum_{k=0}^{n+1} \left( \frac{n+1}{k} \right) Y_{n+1-k}\left(1; \frac{1}{u}; a\right) G_k(x; u, a, b, b; \lambda)
\]

\[
+ \ln(b) \sum_{k=0}^{n+2} \left( \frac{n+2}{k} \right) Y_{n+2-k}\left(1; \frac{1}{u}; b\right) G_k^{(2)}(x; u, a, b, b; \lambda).
\]

(2.9)
Proof. In order to prove (2.9), we set $c = b$ and $\alpha = 1$ in equation (2.1) and then taking its derivative with respect to $t$, we have

$$
\sum_{n=0}^{\infty} g_{n+1}(x; u; a, b, c; \lambda) \frac{t^n}{n!} = \left[ \frac{(\lambda b^2 - u)(a^2 - u) + t a^2 \ln(a) - (a^2 - u) t \lambda b^2 \ln(b)}{(\lambda b^2 - u)^2} \right] b^{x t} \tag{2.10}
$$

On arranging the above equation and making use of (1.11) and (2.1), we get

$$
\sum_{n=0}^{\infty} g_{n+1}(x; u; a, b, c; \lambda) \frac{t^n}{n!} = \frac{1}{\lambda b^2 - u} \sum_{n=0}^{\infty} g_{n}(x; a, b, c; \lambda) \frac{t^n}{n!} + \frac{1}{\lambda b^2 - u} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda b^{x t} \ln(b) \tag{2.11}
$$

Making use of Lemma [13, p.100, Eq.2], above equation reduces as

$$
\sum_{n=0}^{\infty} g_{n+1}(x; u; a, b, c; \lambda) \frac{t^n}{n!} = \frac{1}{\lambda b^2 - u} \sum_{n=0}^{\infty} g_{n}(x; a, b, c; \lambda) \frac{t^n}{n!} + \frac{1}{\lambda b^2 - u} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda b^{x t} \ln(b) \tag{2.12}
$$

On equating the coefficients of $t^n$ in both sides of the above equation, we arrive at the required result (2.9). □

Theorem 2.3. The following relationship holds true

$$
g_n^{(-m)}(u; a, b, c; \lambda) = \sum_{k=0}^{n} g_k^{(-m)}(-x; u; a, b, c; \lambda) g_{(n-k)}^{(a-m)}(x; u; a, b, c; \lambda), \tag{2.13}
$$

Proof. In order to prove (2.13), replacing $x$ by $-x$ and $a$ by $-a$ in (2.1), to get

$$
\sum_{n=0}^{\infty} g_{n}^{(-a)}(-x; u; a, b, c; \lambda) \frac{t^n}{n!} = \left( \frac{(a^2 - u)t}{\lambda b^{2} - u} \right)^{(-a)} c^{-xt}. \tag{2.14}
$$

Making use of the above equation, we can write

$$
\sum_{k=0}^{\infty} g_k^{(-a)}(-x; u; a, b, c; \lambda) \frac{t^k}{k!} \sum_{n=0}^{\infty} g_n^{(a-m)}(x; u; a, b, c; \lambda) \frac{t^n}{n!} = \left( \frac{(a^2 - u)t}{\lambda b^{2} - u} \right)^{m}. \tag{2.15}
$$
Using Lemma [13, p.100, Eq.2], and comparing the coefficients of \( t^n \) from the resulting equation, we acquire the result (2.13). \( \square \)

**Theorem 2.4.** The following relationships hold true:

\[
\mathcal{G}_n^{(a)}(x; u, a, b, c; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{n-k}^{(a)}(y; u, a, b, c; \lambda) (x \ln c)^{(n-k)}.
\] (2.17)

\[
\mathcal{G}_n^{(a+b)}(x + y; u, a, b, c; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{n-k}^{(a)}(y; u, a, b, c; \lambda) \mathcal{G}_{k}^{(b)}(y; u, a, b, c; \lambda).
\] (2.18)

\[
((x + y) \ln c)^n = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k}^{(a)}(y; u, a, b, c; \lambda) \mathcal{G}_{n-k}^{(a)}(y; u, a, b, c; \lambda).
\] (2.19)

\[
\mathcal{G}_n^{(-a)}(2x; u, a, b, c; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k}^{(-a)}(y; u, a, b, c; \lambda) \mathcal{H}_{n-k}^{(-a)}(x; u, a, b, c; \lambda).
\] (2.20)

**Proof.** By using (1.4) and (2.1), we can easily find the results (2.17)-(2.20). We omit the proof. \( \square \)

3. **Implicit Summation Formulae Involving Generalized Apostol-type Frobenius-Genocchi Polynomials**

Here in this section, we provide some implicit formulae for generalized Apostol-type Frobenius-Genocchi polynomials.

**Theorem 3.1.** The following implicit formula for the generalized Apostol-type Frobenius-Genocchi polynomials holds true:

\[
\mathcal{G}_{k+l}^{(a)}(z; u, a, b, c; \lambda) = \sum_{n,m=0}^{k,l} \binom{l}{m} \binom{k}{n} (\ln c)^{(m+n)}(z-x)^{m+n} \mathcal{G}_{k-n-l}^{(a)}(y; u, a, b, c; \lambda).
\] (3.1)

**Proof.** Replacing \( t \) by \( (t + w) \) in (2.1) and rewriting equation (2.1) as

\[
\left(\frac{e^{t(1+w)} - 1}{t(1+w) - u}\right)^a = \sum_{k,l=0}^{\infty} \mathcal{G}_{k+l}^{(a)}(x; u, a, b, c; \lambda) \frac{t^k u^l}{k! \cdot l!}.
\] (3.2)

Replacing \( x \) by \( z \), and equating the obtained equation with the above equation, we have

\[
c^{((z-x)(1+w))} \sum_{k,l=0}^{\infty} \mathcal{G}_{k+l}^{(a)}(x; u, a, b, c; \lambda) \frac{t^k u^l}{k! \cdot l!} = \sum_{k,l=0}^{\infty} \mathcal{G}_{k+l}^{(a)}(z; u, a, b, c; \lambda) \frac{t^k u^l}{k! \cdot l!}.
\] (3.3)

Expanding the exponent part in the above equation, we have

\[
\sum_{N=0}^{\infty} \frac{[(z-x)(t + w)]^N}{N!} \sum_{k,l=0}^{\infty} \mathcal{G}_{k+l}^{(a)}(x; u, a, b, c; \lambda) \frac{t^k u^l}{k! \cdot l!}.
\] (3.4)
resulting equation, we obtain the required result (3.1).

The following relation holds true:

**Theorem 3.2.**

For

\[ G_{k+l}(z; a, b, c; \lambda) = \sum_{j=0}^{\infty} G_{k+j}(z; u; a, b, c; \lambda) \frac{t^j}{k! \frac{l!}{l!}}. \]

Proof. Using the result [13, p.52(2)], we have

\[ \sum_{N=0}^{\infty} f(N) \frac{(x + y)^N}{N!} = \sum_{m,n=0}^{\infty} f(m + n) \frac{x^m y^n}{m! n!} \]

in the left-hand side, we get

\[ \sum_{n,m=0}^{\infty} \frac{(\ln c)^{n+m}(z - x)^{n+m} x^n y^m}{n! m!} \sum_{k=0}^{\infty} G_{k+j}(z; u; a, b, c; \lambda) \frac{t^j}{k! \frac{l!}{l!}} = \sum_{k=0}^{\infty} G_{k+l}(z; u; a, b, c; \lambda) \frac{t^j}{k! \frac{l!}{l!}}. \]

Replacing \( k \) by \( k - n \) and \( l \) by \( l - m \) in the above equation and equating the coefficients of \( t^k \) and \( t^j \) from the resulting equation, we obtain the required result (3.1).

**Corollary 3.1.** For \( l = 0 \) in Theorem 3.1, we have the following relation

\[ G_{k+n}(z; u; a, b, c; \lambda) = \sum_{n=0}^{k} \binom{k}{n} \ln(c)^{n-k} G_{n}(z; u; a, b, c; \lambda). \]

**Theorem 3.3.** The following relation holds true:

\[ G_{n}^{(a+1)}(z; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(u; a, b, c; \lambda) G_{m}^{(a)}(z; u; a, b; \lambda). \]

Proof. Replacing \( x \) by \( x + 1 \) in equation (2.1), we get

\[ \left( \frac{(a' - u) t}{\lambda b' - u} \right)^{x+1} c^{x+1} = \sum_{n=0}^{\infty} G_{n}^{(a)}(x + 1; u; a, b, c; \lambda) \frac{t^n}{n!}. \]

On replacing \( k \) by \( k - n \) and equating the coefficients of \( t^n \) in the resulting equation, we obtain the desired result (3.7).
Proof. Let
\[ G_n(t; a, b, c; \lambda) = \sum_{n=0}^{\infty} G_n^{(a)}(x; u, a, b, c; \lambda) \frac{t^n}{n!}. \] (3.11)

On setting \( n \) by \( n - m \) in the above equation and equating the coefficients of \( t^n \), we obtain the required result. \( \square \)

**Theorem 3.4.** The following implicit summation formula holds true:
\[ \sum_{m=0}^{n} (-1)^m \binom{n}{m} (\ln(ab))^m (\alpha)^m G_{n-m}^{(a)}(x; u, a, b, c; \lambda) = (-1)^n G_n^{(a)}(x; u, a, b, c; \lambda), \] (3.12)

**Proof.** First, we replace \( t \) by \( -t \) in (2.1) and then we subtract the obtained equation with (2.1), we get
\[ \left( \frac{\alpha^t - u t}{\lambda b^t - u} \right)^n [c^{x^t} - (ab)^{x^t}(-1)^t c^{-x^t}] = \sum_{m=0}^{\infty} \left[ 1 - (-1)^m \right] G_n^{(a)}(x; u, a, b, c; \lambda) \frac{t^n}{n!}. \] (3.13)

By using (2.1) and Lemma [13, p.100, Eq.2] we get
\[ \sum_{n=0}^{\infty} G_n^{(a)}(x; u, a, b, c; \lambda) \frac{t^n}{n!} = (-1)^n \sum_{m=0}^{n} \binom{n}{m} (\ln(ab))^m (\alpha)^m G_{n-m}^{(a)}(x; u, a, b, c; \lambda) \frac{t^n}{(n-m)!} \]
\[ = \sum_{n=0}^{\infty} \left[ 1 - (-1)^m \right] G_n^{(a)}(x; u, a, b, c; \lambda) \frac{t^n}{n!}. \] (3.14)

On equating the coefficients of \( t^n \) in the both sides of above equation, yield result (3.12). \( \square \)

## 4. Symmetric identities for the generalized Apostol-type Frobenius-Genocchi polynomials

In this section, we establish symmetric identities for the generalized Apostol-type Frobenius-Genocchi polynomials by applying the generating function (2.1). The results extends some known identities of Khan et al. [3-5] and Pathan and Khan [8-10].

**Theorem 4.1.** The following identity holds true:
\[ \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} G_{n-k}^{(a)}(bx, by; u, A, B, c; \lambda) G_k^{(a)}(ax, ay; u, A, B, c; \lambda) \]
\[ = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} G_{n-k}^{(a)}(ax, ay; u, A, B, c; \lambda) G_k^{(a)}(bx, by; u, A, B, c; \lambda). \] (4.1)

**Proof.** Let
\[ H(t) = \left[ \left( \frac{A^{at} - ut}{\lambda B^{at} - u} \right) \left( \frac{A^{bt} - ut}{\lambda B^{bt} - u} \right) \right]^n e^{a(t(x+y)t}. \] (4.2)

The above expression is symmetric in \( a \) and \( b \), we can write \( H(t) \) into two ways as:
\[ H(t) = \sum_{n=0}^{\infty} G_n^{(a)}(bx, by; u, A, B, c; \lambda) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} G_k^{(a)}(ax, ay; u, A, B, c; \lambda) \frac{(bt)^k}{k!} \]
\[ H(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} G_{n-k}^{(a)}(bx, by; u, A, B, c; \lambda) G_k^{(a)}(ax, ay; u, A, B, c; \lambda) \frac{t^n}{n!}. \] (4.3)
On the other hand, we have

\[
H(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} t^{\alpha} b^{n-k} G_{n-k}^{(\alpha)}(ax; u; A, B, c; \lambda) G_k^{(\alpha)}(bx; u; A, B, c; \lambda) \frac{t^n}{n!}.
\]  (4.4)

On equating the coefficients of \(t^n\) from equations (4.3) and (4.4), we arrive at the desired result. \(\square\)

**Corollary 4.1.** For \(\alpha = 1\), Theorem 4.1 reduces to:

\[
\sum_{k=0}^{n} \binom{n}{k} t^{\alpha} b^{n-k} G_{n-k}^{(\alpha)}(ax; u; A, B, c; \lambda) G_k^{(\alpha)}(bx; u; A, B, c; \lambda)
= \sum_{k=0}^{n} \binom{n}{k} t^{\alpha} b^{n-k} G_{n-k}^{(\alpha)}(ax; u; A, B, c; \lambda) G_k^{(\alpha)}(bx; u; A, B, c; \lambda).
\]  (4.5)

**Theorem 4.2.** The following identity holds true:

\[
\sum_{k=0}^{n} \binom{n}{k} t^{\alpha} b^{n-k} G_{n-k}^{(\alpha)}(ax; u; A, B, c; \lambda) G_k^{(\alpha)}(bx; u; A, B, c; \lambda)
= \sum_{k=0}^{n} \binom{n}{k} t^{\alpha} b^{n-k} G_{n-k}^{(\alpha)}(ax; u; A, B, c; \lambda) G_k^{(\alpha)}(bx; u; A, B, c; \lambda)
\times G_k^{(\alpha)}(by; u; A, B, c; \lambda).
\]  (4.6)

**Proof.** Consider

\[
I(t) = \left[ \frac{(A^a - u)at}{\lambda B^a - u} \right] \left[ \frac{(A^a - u)bt}{\lambda B^a - u} \right] \frac{1}{(\lambda c^a + 1)(\lambda c^b + 1)} c^{\alpha(x+y)t}.
\]  (4.7)

\[
I(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} t^{\alpha} b^{n-k} G_{n-k}^{(\alpha)}(ax; u; A, B, c; \lambda) G_k^{(\alpha)}(bx; u; A, B, c; \lambda)
\times G_k^{(\alpha)}(by; u; A, B, c; \lambda) \frac{t^n}{n!}.
\]  (4.8)

On the other hand, we have

\[
I(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} t^{\alpha} b^{n-k} G_{n-k}^{(\alpha)}(ax; u; A, B, c; \lambda) G_k^{(\alpha)}(bx; u; A, B, c; \lambda)
\times G_k^{(\alpha)}(by; u; A, B, c; \lambda) \frac{t^n}{n!}.
\]  (4.9)

On equating the coefficients of \(t^n\) from last two equations (4.8) and (4.9), we acquire at the desired result (4.6). \(\square\)
5. Relation between $\lambda$-type Stirling polynomials, Apostol-Bernoulli polynomial and generalized Apostol-type Frobenius-Genocchi polynomial

This section deals with some relationships in between Array-type polynomials, Apostol-Bernoulli polynomial and generalized Apostol-type Frobenius-Genocchi polynomial.

**Theorem 5.1.** The following relationship holds true:

$$G_n^{(v)}(x; u, a, b; \lambda) = \frac{(v)!}{(-n)_{2v}} \sum_{k=0}^{n} \binom{n}{k} S_k(x; 1, b; \frac{\lambda}{u}) \beta^{(v)}_{n-k} \left( \frac{1}{u}; a \right). \quad (5.1)$$

**Proof.** On replacing $c$ by $b$ and $\alpha$ by $-v$ in equation (2.1), we get

$$\sum_{n=0}^{\infty} G_n^{(-v)}(x; u, a, b; \lambda) \frac{t^n}{n!} = \frac{(a^t - u)^{(-v)}}{\lambda b^t - u} b^v. \quad (5.2)$$

Using equations (1.10) and (1.11), the above equation reduces to

$$\sum_{n=0}^{\infty} G_n^{(-v)}(x; u, a, b; \lambda) \frac{t^n}{n!} = v! \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{n}{k} S_k(x; 1, b; \frac{\lambda}{u}) \beta^{(v)}_{n-k} \left( \frac{1}{u}; a \right) \frac{t^n}{n!}. \quad (5.3)$$

Replacing $m$ by $m - k$ in the above equation, we get

$$\sum_{n=0}^{\infty} G_n^{(-v)}(x; u, a, b; \lambda) \frac{t^{n+2v}}{n!} = v! \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{n}{k} S_k(x; 1, b; \frac{\lambda}{u}) \beta^{(v)}_{n-k} \left( \frac{1}{u}; a \right) \frac{t^n}{n!}. \quad (5.4)$$

On equating the coefficients of $t^n$, we arrive at the required result. \hfill \Box

**Theorem 5.2.** The following relationship holds true:

$$G_n^{(-v)}(x; u, a, b; \lambda) = \frac{(v)!}{(-n)_{2v}} \sum_{k=0}^{n} \binom{n}{k} S_{2k}(x; 1, b; \frac{\lambda}{u}) \beta^{(v)}_{n-k} \left( \frac{1}{u}; 1, a, b \right). \quad (5.6)$$

**Proof.** Making replacement of $c$ with $b$ and $\alpha$ with $-v$ in equation (2.1), we get

$$\sum_{n=0}^{\infty} G_n^{(-v)}(x; u, a, b; \lambda) \frac{t^n}{n!} = \frac{(a^t - u)^{(-v)}}{\lambda b^t - u} b^v. \quad (5.7)$$

Using equations (1.10) and (1.11), the above equation reduces to

$$\sum_{n=0}^{\infty} G_n^{(-v)}(x; u, a, b; \lambda) \frac{t^{n+2v}}{n!} = v! \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{n}{k} S_{2k}(x; 1, b; \frac{\lambda}{u}) \beta^{(v)}_{n-k} \left( \frac{1}{u}; 1, a, b \right) \frac{t^n}{n!}. \quad (5.8)$$
Using Lemma [13, p.100, Eq.2], we get

\[
\sum_{n=0}^{\infty} \mathcal{G}_n^{(\nu)}(x; u, a, b, \lambda) \frac{t^{n+2\nu}}{n!} = \nu! \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} S(k, \nu, 1, b, \lambda) \times B_{n-k}^{(\nu)} \left( x, \frac{1}{u}, 1, a, b \right) \frac{t^n}{n!}.
\]

(5.10)

On equating the coefficients of \( t^n \), we arrive at the required result. □

**Acknowledgment**

The authors are thankful to Integral University, Lucknow, India for acknowledging the paper with the manuscript number “IU/R&D/2018-MCN-000399”.

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