TWO-PARAMETRIC ERROR ESTIMATES IN HOMOGENIZATION OF SECOND ORDER ELLIPTIC SYSTEMS IN $\mathbb{R}^d$ INCLUDING LOWER ORDER TERMS

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Abstract. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider a selfadjoint operator $B_\varepsilon$, $0 < \varepsilon \leq 1$, given by the differential expression $b(D)^* g(x/\varepsilon) b(D) + \sum_{j=1}^d (a_j(x/\varepsilon) D_j + D_j a_j(x/\varepsilon)^*) + Q(x/\varepsilon)$, where $b(D) = \sum_{l=1}^d b_l D_l$ is the first order differential operator, and $g, a_j, Q$ are matrix-valued functions in $\mathbb{R}^d$ periodic with respect to some lattice $\Gamma$. It is assumed that $g$ is bounded and positive definite, while $a_j$ and $Q$ are, in general, unbounded. We study the generalized resolvent $(B_\varepsilon - \xi Q_0(x/\varepsilon))^{-1}$, where $Q_0$ is a $\Gamma$-periodic, bounded and positive definite matrix-valued function, and $\xi$ is a complex-valued parameter. Approximations for the generalized resolvent in the $(L_2 \rightarrow L_2)$- and $(L_2 \rightarrow H^1)$-norms with two-parametric error estimates (with respect to the parameters $\varepsilon$ and $\xi$) are obtained.

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INTRODUCTION

The paper concerns homogenization theory of periodic differential operators (DO’s). A broad literature is devoted to homogenization problems. First, we mention the books [BeLPap, BaPan, ZhKO].

0.1. Statement of the problem. We study matrix elliptic DO’s $B_\varepsilon$, $0 < \varepsilon \leq 1$, acting in the space $L^2(\mathbb{R}^d; \mathbb{C}^n)$. Let $\Gamma$ be a lattice in $\mathbb{R}^d$, and let $\Omega$ be the cell of $\Gamma$. For $\Gamma$-periodic functions in $\mathbb{R}^d$, we use the notation $\psi_\varepsilon(x) := \psi(\varepsilon^{-1}x)$ and $\overline{\psi} := |\Omega|^{-1} \int_\Omega \psi(x) \, dx$.

The principal part $A_\varepsilon$ of the operator $B_\varepsilon$ is given in a factorized form

$$A_\varepsilon = b(D)^* g_\varepsilon(x) b(D),$$

(0.1)

where $b(D)$ is a matrix homogeneous first order DO and $g(x)$ is a bounded and positive definite $\Gamma$-periodic matrix-valued function in $\mathbb{R}^d$. (The precise assumptions on $b(D)$ and $g(x)$ are given below in Subsection 1.3.) The simplest example of the operator $A_\varepsilon$ is the acoustics operator $-\text{div} g_\varepsilon(x) \nabla$; the operator of elasticity theory also can be represented in the required form. The homogenization problem for the operator $A_\varepsilon$ was studied in a series of papers [BSu1, BSu2, BSu3] and also in [Su5, Su7] in detail. In the present paper we consider a more general selfadjoint DO $B_\varepsilon$ including lower order terms:

$$B_\varepsilon = b(D)^* g_\varepsilon(x) b(D) + \sum_{j=1}^d \left( a_j^\varepsilon(x) D_j + D_j a_j^\varepsilon(x)* \right) + Q_\varepsilon(x).$$

(0.2)

Here $a_j(x)$ are $\Gamma$-periodic matrix-valued functions, in general, unbounded. In general, the coefficient $Q(x)$ is a distribution generated by a periodic matrix-valued measure. (The precise assumptions on the coefficients are given below in Subsections 1.4, 1.5.) The precise definition of the operator $B_\varepsilon$ is given in terms of the corresponding quadratic form.

The coefficients of the differential expression (0.2) oscillate rapidly as $\varepsilon \to 0$. A typical homogenization problem for the operator $B_\varepsilon$ is to approximate the resolvent $(B_\varepsilon - \zeta I)^{-1}$ or the generalized resolvent $(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1}$ for small $\varepsilon$. Here $Q_0(x)$ is a positive definite and bounded $\Gamma$-periodic matrix-valued function.

0.2. A survey of the results on the operator error estimates. The homogenization problem for the operator $A_\varepsilon$ was studied in a series of papers [BSu1, BSu2, BSu3] by M. Sh. Birman and T. A. Suslina. In [BSu1], it was proved that

$$\| (A_\varepsilon + I)^{-1} - (A_0^0 + I)^{-1} \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C \varepsilon.$$

(0.3)
Here $A^0 = b(D)^* g^0 b(D)$ is the effective operator with a constant effective matrix $g^0$. The definition of the effective matrix (see Subsection [subsection1.8] below) is well known in homogenization theory. Next, in [BSu3] approximation of the resolvent $(A_\varepsilon + I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d; \mathbb{C}^n)$ was obtained:

$$\| (A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon) \|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon. \quad (0.4)$$

Here $K(\varepsilon)$ is a corrector. The corrector has zero order with respect to $\varepsilon$, but it involves rapidly oscillating factors. Therefore, $\| K(\varepsilon) \|_{L_2 \rightarrow H^1} = O(\varepsilon^{-1})$. Estimates of the form (0.3) and (0.4) are called operator error estimates; they are order-sharp, and the constants in estimates are controlled explicitly in terms of the problem data. The method of [BSu1, BSu2, BSu3] is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory.

Later the method of [BSu1, BSu2, BSu3] was developed by T. A. Suslina [Su1, Su2, Su6] for the operator (0.2). In [Su2], the following analogs of estimates (0.3) and (0.4) were obtained:

$$\| (B_\varepsilon + \lambda Q_\varepsilon - B^0 + \lambda Q_0)^{-1} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon, \quad (0.5)$$

$$\| (B_\varepsilon + \lambda Q_\varepsilon - B^0 + \lambda Q_0)^{-1} - \varepsilon K(\varepsilon) \|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon. \quad (0.6)$$

Here the real-valued parameter $\lambda$ is such that the operator $B_\varepsilon + \lambda Q_\varepsilon$ is positive definite; $B^0$ is the corresponding effective operator with constant coefficients. Estimates (0.5) and (0.6) are order-sharp, and the constants in estimates are controlled explicitly in terms of the problem data and $|\lambda|$ (however, in the cited papers the optimal dependence of the constants in estimates on the spectral parameter $\lambda$ was not searched out).

A different approach to operator error estimates in homogenization theory was suggested by V. V. Zhikov. In [Zh1, Zh2] and [ZhPas], estimates of the form (0.3) and (0.4) for the acoustic operator and the operator of elasticity theory were obtained. The method was based on analysis of the first order approximation to the solution and introduction of an additional parameter. Besides the problems in $\mathbb{R}^d$, in [Zh1, Zh2, ZhPas] homogenization problems for elliptic equations in a bounded domain $O \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were studied.

Also, operator error estimates for the Dirichlet and Neumann problems for a second order elliptic equation in a bounded domain (without lower order terms) have been studied by different methods in the papers [Gr1, Gr2], [KeLiSi], [PSu], [Su3, Su4, Su5, Su7]; see a detailed survey in the introduction to [Su4, Su7].

In the presence of the lower order terms, homogenization problem for the operator of the form (0.2) in $\mathbb{R}^d$ was studied in the paper [Bo] by D. I. Borisov. In [Bo], expression for the effective operator $B^0$ was found and error estimates of the form (0.5) and (0.6) were obtained. Moreover, it was assumed that the coefficients of the operator depend not only on the rapid
variable, but also on the slow variable. However, in [Bo] it was assumed that the coefficients are sufficiently smooth.

Up to now, we have discussed the results about approximation of the resolvent at a fixed regular point. Approximations of the operator \((A_\varepsilon - \zeta I)^{-1}\) with error estimates depending on \(\varepsilon\) and \(\zeta = |\zeta e^{i\phi}| \in \mathbb{C} \setminus \mathbb{R}_+\) were obtained in the recent papers [Su5, Su7]. It was proved that

\[
\| (A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\varepsilon)|\zeta|^{-1/2} \varepsilon, \quad (0.7)
\]

\[
\| (A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon K(\varepsilon; \zeta) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C(\varepsilon)(1 + |\zeta|^{-1/2})\varepsilon. \quad (0.8)
\]

The dependence of the constants in estimates (0.7) and (0.8) on the angle \(\phi\) was traced. Estimates (0.7) and (0.8) are two-parametric (with respect to \(\varepsilon\) and \(|\zeta|\)); they are uniform with respect to \(\phi\) in any sector \(\phi \in [\phi_0, 2\pi - \phi_0]\) with arbitrarily small \(\phi_0 > 0\). Also, in [Su5, Su7] the operators \(A_{D,\varepsilon}\) and \(A_{N,\varepsilon}\) given by the expression (0.1) in a bounded domain with the Dirichlet or Neumann boundary conditions were studied. Approximations of the resolvents of these operators with two-parametric error estimates were found.

Investigation of the two-parametric estimates of the form (0.7) and (0.8) was stimulated by the study of homogenization for the parabolic problems. This study is based on the following representation for the operator exponential:

\[
e^{-A_{D,\varepsilon}t} = -(2\pi i)^{-1} \int_\gamma e^{-\zeta t} (A_{D,\varepsilon} - \zeta I)^{-1} d\zeta, \quad \gamma = D, N,
\]

where \(\gamma \subset \mathbb{C}\) is a contour on the complex plane enclosing the spectrum of the operator \(A_{D,\varepsilon}\) (in the positive direction). See details in [MSu1, MSu2].

0.3. Main results. Before we formulate the results, it is convenient to turn to the nonnegative operator \(B_\varepsilon = B_\varepsilon + cQ_0^\varepsilon\), choosing an appropriate constant \(c\). Then the corresponding effective operator is \(B^0 = B^0 + cQ_0^0\). Our goal is to approximate the generalized resolvent \((B_\varepsilon - \zeta Q_0^\varepsilon)^{-1}\) with error estimates depending on \(\varepsilon\) and the spectral parameter \(\zeta\).

Our main results are the following estimates:

\[
\| (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (B^0 - \zeta Q_0^0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\varepsilon)|\zeta|^{-1/2}, \quad (0.9)
\]

\[
\| (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (B^0 - \zeta Q_0^0)^{-1} - \varepsilon K(\varepsilon; \zeta) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C(\varepsilon), \quad (0.10)
\]

for \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| \geq 1, 0 < \varepsilon \leq 1\). The dependence of the constants in estimates on the angle \(\phi = \arg \zeta\) is traced. The two-parametric estimates (0.9) and (0.10) are uniform with respect to \(\phi\) in any domain \(\{\zeta = |\zeta e^{i\phi}| \in \mathbb{C} : |\zeta| \geq 1, \phi_0 \leq \phi \leq 2\pi - \phi_0\}\) with arbitrarily small \(\phi_0 > 0\).

In the general case, the corrector in (0.10) contains a smoothing operator. We distinguish the cases where a simpler corrector can be used.

Also, we find approximation in the \((L_2 \to L_2)\)-norm for the operator \(g^\varepsilon b(D)(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1}\) corresponding to the flux.
Estimates (0.9) and (0.10) are generalizations of estimates (0.7) and (0.8) from [Su5, Su7] for the case of the operator $B_\varepsilon$ including the lower order terms. However, there is a difference: estimates (0.7) and (0.8) are valid for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, while estimates (0.9) and (0.10) are proved under the additional assumption that $|\zeta| \geq 1$. This is related to the presence of the lower order terms.

For completeness, we find approximation of the operator $(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1}$ in a wider domain of the parameter $\zeta$; the corresponding error estimates have a different behavior with respect to $\zeta$. (See Section 8 below.)

0.4. Method. The method is based on application of the known results at a fixed point, the scaling transformation, and appropriate identities for generalized resolvents. We consider the auxiliary operator family $B_\varepsilon(\vartheta)$ depending on the additional parameter $0 < \vartheta \leq 1$. The operator $B_\varepsilon(\vartheta)$ is given by $B_\varepsilon(\vartheta) = b(D)^* g b(D) + \vartheta \sum_{j=1}^d \left( a_j^2 D_j + D_j (a_j^*) \right) + \vartheta^2 (Q^\varepsilon + c Q_0^\varepsilon)$.

Estimates (0.5) and (0.6) at a fixed point $\lambda$ are valid for $(B_\varepsilon(\vartheta) + \lambda Q_0^\varepsilon)^{-1}$ with the common constants for all $0 < \vartheta \leq 1$.

Let us discuss the proof of estimate (0.9). Using (0.5) for $B_\varepsilon(\vartheta)$, by the scaling transformation, we obtain
\[
\|(B(\varepsilon; \vartheta) + \lambda \varepsilon^2 Q_0)^{-1} - (B^0(\varepsilon; \vartheta) + \lambda \varepsilon^2 Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C \varepsilon^{-1}, \quad 0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1}.
\]
(For $1 < \varepsilon \leq \vartheta^{-1}$ we use rather rough estimates.) Here
\[
B(\varepsilon; \vartheta) = b(D)^* g b(D) + \vartheta \sum_{j=1}^d (a_j(x) D_j + D_j a_j(x)^*) + \vartheta^2 \varepsilon^2 Q(x) + \vartheta^2 \varepsilon^2 c Q_0(x),
\]
and the operator $B^0(\varepsilon; \vartheta)$ is obtained from the effective operator $B^0$ in a similar way. By an appropriate identity for the generalized resolvents, we carry estimate (0.11) over to the point $\tilde{\zeta} = e^{i\phi}$, $\phi \in (0, 2\pi)$:
\[
\|(B(\varepsilon; \vartheta) - \tilde{\zeta} \varepsilon^2 Q_0)^{-1} - (B^0(\varepsilon; \vartheta) - \tilde{\zeta} \varepsilon^2 Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\phi) \varepsilon^{-1}.
\]
In this inequality, we put $\varepsilon = \varepsilon|\zeta|^{1/2}$ and $\vartheta = |\zeta|^{-1/2}$, where $0 < \varepsilon \leq 1$. The restriction $|\zeta| \geq 1$ ensures that $0 < \vartheta \leq 1$. By the inverse scaling transformation, renaming $\varepsilon =: \varepsilon$, we arrive at (0.9). Estimate (0.10) is checked by the same method.

The trick based on the scaling transformation and the appropriate resolvent identities was applied before in [Su5, Su7]. In the present paper, we use one more trick, namely, introduction of the additional parameter $\vartheta$; this is related to the presence of the lower order terms.

The authors plan to apply the results of the present paper to homogenization of elliptic systems in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$; a separate paper [MSu3] is in preparation.
0.5. **Plan of the paper.** The paper consists of nine sections. In Section 1, the class of operators is introduced, the effective operator is described, and the results of the paper [Su2] are formulated. Section 2 contains the auxiliary material. In Section 3, from (0.6) we deduce a similar estimate, but with a different smoothing operator in the corrector (we turn to the Steklov smoothing, since it is more convenient for further investigation of the problems in a bounded domain). Main results are formulated in Section 4. The proof of estimate (0.9) is given in Section 5. Section 6 contains the proof of estimate (0.10) and the proof of approximation for the ,,flux" 
\[ g^\varepsilon_b(D) - 1 \]
In Section 7, we distinguish the cases where the smoothing operator in the corrector can be removed, and discuss some special cases. Approximations of the generalized resolvent for a wider domain of \( \zeta \) are obtained in Section 8. In Section 9, some applications of the general results are discussed.

The scalar elliptic operator of the form
\[ B_\varepsilon = (D - A^\varepsilon(x))^*g^\varepsilon(D - A^\varepsilon(x)) + \varepsilon^{-1}v^\varepsilon(x) + V^\varepsilon(x) \]
is considered; it can be treated as the periodic Schrödinger operator with rapidly oscillating metric \( g^\varepsilon \), magnetic potential \( A^\varepsilon \), and electric potential \( \varepsilon^{-1}v^\varepsilon + V^\varepsilon \) containing the singular first term. Also, the periodic Schrödinger operator involving a strongly singular potential \( \varepsilon^{-2}v^\varepsilon \) is studied.

0.6. **Notation.** Let \( H \) and \( H^\ast \) be complex separable Hilbert spaces. The symbols \( (\cdot, \cdot)_H \) and \( \| \cdot \|_H \) stand for the inner product and the norm in \( H \); the symbol \( \| \cdot \|_{H \to H^\ast} \) denotes the norm of a continuous linear operator acting from \( H \) to \( H^\ast \).

The symbols \( (\cdot, \cdot) \) and \( | \cdot | \) denote the inner product and the norm in \( \mathbb{C}^n \); \( 1_n \) is the identity \( (n \times n) \)-matrix. For \( z \in \mathbb{C} \), we write \( z^* \) for the complex conjugate number. (This nonstandard notation is employed because \( \overline{f} \) denotes the mean value of a periodic function \( g(x) \).) If \( a \) is an \( (m \times n) \)-matrix, then \( |a| \) denotes the norm of the corresponding operator from \( \mathbb{C}^n \) to \( \mathbb{C}^m \). We use the notation \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), \( iD_j = \partial_j = \partial/\partial x_j \), \( j = 1, \ldots, d \), \( D = -i\nabla = (D_1, \ldots, D_d) \). The \( L_p \)-classes of \( \mathbb{C}^n \)-valued functions in a domain \( O \subset \mathbb{R}^d \) are denoted by \( L_p(O; \mathbb{C}^n) \), \( 1 \leq p \leq \infty \). The Sobolev classes of \( \mathbb{C}^n \)-valued functions in a domain \( O \) are denoted by \( H^s(O; \mathbb{C}^n) \). If \( n = 1 \), we write simply \( L_p(O) \), \( H^s(O) \), but sometimes we use such abbreviated notation also for spaces of vector-valued or matrix-valued functions.

We denote \( \mathbb{R}_+ = [0, \infty) \). Various constants in estimates are denoted by the symbols \( c, \varepsilon, C, \mathcal{C} \) (possibly, with indices and marks).

1. **The class of operators. Approximation of the generalized resolvent \( (B_\varepsilon + \lambda_0Q_0^\varepsilon)^{-1} \)**

In this section, we introduce the class of operators under consideration, describe the effective operator, and formulate the results of the paper [Su2].
1.1. **Lattices in** $\mathbb{R}^d$. Let $\Gamma \subset \mathbb{R}^d$ be a lattice generated by a basis $a_1, \ldots, a_d$:

$$\Gamma = \left\{ a \in \mathbb{R}^d : a = \sum_{j=1}^{d} \nu_j a_j, \nu_j \in \mathbb{Z} \right\}.$$

Let $\Omega$ be the elementary cell of the lattice $\Gamma$:

$$\Omega = \left\{ x \in \mathbb{R}^d : x = \sum_{j=1}^{d} \tau_j a_j, -\frac{1}{2} < \tau_j < \frac{1}{2} \right\}.$$

We denote $|\Omega| = \text{meas } \Omega$ and $2r_1 = \text{diam } \Omega$.

The basis $b_1, \ldots, b_d \in \mathbb{R}^d$ dual to the basis $a_1, \ldots, a_d$ is defined by the relations $\langle b_j, a_i \rangle = 2\pi \delta_{ji}$, where $\delta_{ji}$ is the Kronecker delta. The lattice

$$\tilde{\Gamma} = \left\{ b \in \mathbb{R}^d : b = \sum_{j=1}^{d} \mu_j b_j, \mu_j \in \mathbb{Z} \right\}$$

generated by the dual basis is called the lattice dual to $\Gamma$. We denote by $\tilde{\Omega}$ the central Brillouin zone of the lattice $\tilde{\Gamma}$:

$$\tilde{\Omega} = \left\{ k \in \mathbb{R}^d : |k| < |k - b|, \ 0 \neq b \in \tilde{\Gamma} \right\}.$$

Note that $\tilde{\Omega}$ is a fundamental domain of $\tilde{\Gamma}$. Let $r_0$ be the radius of the ball inscribed in $\text{clos } \tilde{\Omega}$, i.e., $2r_0 = \min_{0 \neq b \in \tilde{\Gamma}} |b|$.

For $\Gamma$-periodic measurable matrix-valued functions, we systematically use the following notation:

$$f^\varepsilon(x) := f(x/\varepsilon), \ \varepsilon > 0;$$

$$\tilde{f} := |\Omega|^{-1} \int_{\Omega} f(x) dx, \ \tilde{f} := \left( |\Omega|^{-1} \int_{\Omega} f(x)^{-1} dx \right)^{-1}.$$

Here, in the definition of $\tilde{f}$ it is assumed that $f \in L_{1,\text{loc}}(\mathbb{R}^d)$, and in the definition of $\tilde{f}$ is is assumed that the matrix $f$ is square and nondegenerate, and $f^{-1} \in L_{1,\text{loc}}(\mathbb{R}^d)$. Let $[f^\varepsilon]$ denote the operator of multiplication by the matrix-valued function $f^\varepsilon(x)$.

By $H^1(\Omega)$ we denote the subspace of all functions in $H^1(\Omega)$ whose $\Gamma$-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\text{loc}}(\mathbb{R}^d)$.

1.2. **The smoothing operator** $\Pi_\varepsilon$. Let $\Pi_\varepsilon^{(k)}$, $\varepsilon > 0$, be the pseudodifferential operator with the symbol $\chi_{\tilde{\Omega}/\varepsilon}(\xi)$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^k)$ (where $k \in \mathbb{N}$), i.e.,

$$([\Pi_\varepsilon^{(k)}] u)(x) = (2\pi)^{-d/2} \int_{\tilde{\Omega}/\varepsilon} e^{i(x,\xi)} \tilde{u}(\xi) d\xi, \ u \in L_2(\mathbb{R}^d; \mathbb{C}^k). \quad (1.1)$$
Here \(\hat{u}\) is the Fourier-image of the function \(u\). Obviously, for \(u \in H^s(\mathbb{R}^d; \mathbb{C}^k)\) we have \(\Pi_{(k)}^{(k)} a D^m u = D^m \Pi_{(k)}^{(k)} u\) for any multiindex \(\alpha\) such that \(|\alpha| \leq s\). In what follows, we drop the index \(k\) in the notation \(\Pi_{(k)}\) and write simply \(\Pi_{(k)}\).

Below we need the following properties of the operator \(\Pi_{(k)}\) proved in [PSu, Proposition 1.4] and [BSu3, Subsection 10.2].

**Proposition 1.1.** For any function \(u \in H^1(\mathbb{R}^d; \mathbb{C}^k)\) and \(\varepsilon > 0\) we have
\[
\|\Pi_{(k)} u - u\|_{L^2(\mathbb{R}^d)} \leq \varepsilon r_0^{-1}\|Du\|_{L^2(\mathbb{R}^d)}.
\]

**Proposition 1.2.** Let \(f\) be a \(\Gamma\)-periodic function in \(\mathbb{R}^d\) such that \(f \in L_2(\Omega)\). Then the operator \([f_{(k)}^{(k)}] \Pi_{(k)}\) is continuous in \(L_2(\mathbb{R}^d; \mathbb{C}^k)\), and
\[
\|[f_{(k)}^{(k)}] \Pi_{(k)}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2}\|f\|_{L_2(\Omega)}, \quad \varepsilon > 0.
\]

1.3. **The operator \(A\).** In \(L_2(\mathbb{R}^d; \mathbb{C}^n)\), we consider the operator \(A\) given formally by the differential expression
\[
A = b(D)^* g(x) b(D), \quad x \in \mathbb{R}^d.
\]
Here \(g\) is a \(\Gamma\)-periodic Hermitian matrix-valued function of size \(m \times m\), in general, with complex entries. It is assumed that
\[
g(x) > 0, \quad g, g^{-1} \in L_\infty(\mathbb{R}^d).
\]
Next, \(b(D)\) is the first order DO with constant coefficients given by
\[
b(D) = \sum_{l=1}^{d} b_l D_l.
\]
Here \(b_l, l = 1, \ldots, d,\) are constant \((m \times n)\)-matrices, in general, with complex entries. Let \(b(\xi) = \sum_{l=1}^{d} b_l \xi_l\) be the symbol of the operator \(b(D)\). We assume that \(m \geq n\) and
\[
\text{rank } b(\xi) = n, \quad 0 \neq \xi \in \mathbb{R}^d.
\]
This condition is equivalent to the existence of positive constants \(\alpha_0\) and \(\alpha_1\) such that
\[
\alpha_0 1_n \leq b(\theta)^* b(\theta) \leq \alpha_1 1_n, \quad \theta \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty.
\]
From (1.4) it follows that
\[
|b_l| \leq \alpha_1^{1/2}, \quad l = 1, \ldots, d.
\]

The precise definition is the following: \(A\) is the selfadjoint operator in \(L_2(\mathbb{R}^d; \mathbb{C}^n)\) generated by the quadratic form
\[
a[u, u] = \int_{\mathbb{R}^d} \langle g(x) b(D) u(x), b(D) u(x) \rangle dx, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
The form \(a\) is closed and nonnegative because of the estimates
\[
\alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \int_{\mathbb{R}^d} |Du(x)|^2 dx \leq a[u, u] \leq \alpha_1 \|g\|_{L_\infty} \int_{\mathbb{R}^d} |Du(x)|^2 dx, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n),
\]
that follow from (1.2) and (1.4).

1.4. The operators $\mathcal{Y}$ and $\mathcal{Y}_2$. We consider the closed operator $\mathcal{Y}$ acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^{dn})$ and defined by

$$\mathcal{Y} u = D u = \text{col} \{D_1 u, \ldots, D_d u\}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

The lower estimate (1.6) means that

$$\|\mathcal{Y} u\|^2_{L_2(\mathbb{R}^d)} \leq c_1^2 |a[u, u]|, \quad c_1 = \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \quad (1.7)$$

Let $a_j(x)$, $j = 1, \ldots, d$, be $\Gamma$-periodic $(n \times n)$-matrix-valued functions in $\mathbb{R}^d$ (in general, with complex entries) such that

$$a_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \geq 2; \quad j = 1, \ldots, d. \quad (1.8)$$

We consider the operator $\mathcal{Y}_2 : L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^{dn})$ that acts as multiplication by the $(dn \times n)$-matrix-valued function consisting of the matrices $a_j(x)^*$, $j = 1, \ldots, d$. In other words,

$$\mathcal{Y}_2 u(x) = \text{col} \{a_1(x)^* u(x), \ldots, a_d(x)^* u(x)\}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Using the Hölder inequality and the Sobolev embedding theorem, it is easy to check (see [Su2, (5.11)–(5.14)]) that for any $\nu > 0$ there exist constants $C_j(\nu) > 0$ such that

$$\|a_j^* u\|^2_{L_2(\mathbb{R}^d)} \leq \nu \|D u\|^2_{L_2(\mathbb{R}^d)} + C_j(\nu)\|u\|^2_{L_2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad j = 1, \ldots, d.$$ 

Summing over $j$ and taking the lower estimate (1.6) into account, we see that for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$\|\mathcal{Y}_2 u\|^2_{L_2(\mathbb{R}^d)} \leq \nu |a[u, u]| + C(\nu)\|u\|^2_{L_2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \quad (1.9)$$

For $\nu$ fixed, the constant $C(\nu)$ depends only on $d$, $\rho$, $\alpha_0$, $\|g^{-1}\|_{L_\infty}$, the norms $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \ldots, d$, and the parameters of the lattice $\Gamma$.

1.5. The form $q$. Suppose that $d\mu(x)$ is a $\Gamma$-periodic Borel $\sigma$-finite measure in $\mathbb{R}^d$ with values in the class of Hermitian $(n \times n)$-matrices. In other words, $d\mu(x) = \{d\mu_{jl}(x)\}$, $j, l = 1, \ldots, n$, where $d\mu_{jl}(x)$ is a complex-valued $\Gamma$-periodic measure in $\mathbb{R}^d$, and $d\mu_{jl} = d\mu_{lj}^*$. Suppose that the measure $d\mu$ is such that the function $|u(x)|^2$ is integrable with respect to each measure $d\mu_{jl}$ for any $u \in H^1(\mathbb{R}^d)$.

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the quadratic form

$$q[u, u] = \int_{\mathbb{R}^d} \langle d\mu(x) u, u \rangle = \sum_{j, l = 1}^n \int_{\mathbb{R}^d} u_j(x) u_j(x)^* d\mu_{jl}(x), \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \quad (1.10)$$

The measure $d\mu$ is subject to the following condition.
Condition 1.3. There exist constants $c_0 \geq 0$, $\bar{c}_2 \geq 0$, $c_3 \geq 0$, and $0 \leq \bar{c} < \alpha_0 \lVert g^{-1} \rVert_{L_\infty}^{-1}$ such that

$$-\bar{c} \lVert \text{D} u \rVert_{L_2(\Omega)}^2 - c_0 \lVert u \rVert_{L_2(\Omega)}^2 \leq \int_{\Omega} \langle d\mu(x)u, u \rangle \leq \bar{c}_2 \lVert \text{D} u \rVert_{L_2(\Omega)}^2 + c_3 \lVert u \rVert_{L_2(\Omega)}^2,$$

$$u \in H^1(\Omega; \mathbb{C}^n). \quad (1.11)$$

For $u \in H^1(\mathbb{R}^d; \mathbb{C}^n)$, writing inequalities of the form $(1.11)$ over the shifted cells $\Omega + a, a \in \Gamma$, and summing up, we obtain similar inequalities in $\mathbb{R}^d$. Together with $(1.6)$, this yields

$$-(1 - \kappa) a[u, u] - c_0 \lVert u \rVert_{L_2(\mathbb{R}^d)}^2 \leq q[u, u] \leq c_2 a[u, u] + c_3 \lVert u \rVert_{L_2(\mathbb{R}^d)}^2,$$

$$u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad (1.12)$$

where

$$c_2 = \bar{c}_2 \alpha_0^{-1} \lVert g^{-1} \rVert_{L_\infty}, \quad \kappa = 1 - \bar{c}_0^{-1} \lVert g^{-1} \rVert_{L_\infty}, \quad 0 < \kappa \leq 1. \quad (1.13)$$

Examples of forms $(1.10)$ are given in [Su2, Subsection 5.5]. Here we give only the main example (see [Su2, Example 5.3]).

Example 1.4. Suppose that the measure $d\mu$ is absolutely continuous with respect to Lebesgue measure: $d\mu(x) = Q(x) \, dx$, where $Q(x)$ is a $\Gamma$-periodic Hermitian $(n \times n)$-matrix-valued function in $x \in \mathbb{R}^d$ such that

$$Q \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2. \quad (1.14)$$

Then $q[u, u] = (Qu, u)_{L_2(\mathbb{R}^d)}$, $u \in H^1(\mathbb{R}^d; \mathbb{C}^n)$, and for any $\nu > 0$ there exists a constant $C_Q(\nu) > 0$ such that

$$\int_{\Omega} |\langle Q(x)u, u \rangle| \, dx \leq \nu \int_{\Omega} |\text{D} u|^2 \, dx + C_Q(\nu) \int_{\Omega} |u|^2 \, dx, \quad u \in H^1(\Omega; \mathbb{C}^n).$$

If $\nu$ is fixed, the constant $C_Q(\nu)$ is controlled in terms of $d, s, \lVert Q \rVert_{L_s(\Omega)}$, and the parameters of the lattice $\Gamma$. Putting $\nu = 2^{-1} \alpha_0^{-1} \lVert g^{-1} \rVert_{L_\infty}^{-1}$, we see that Condition $(1.3)$ is satisfied with $\bar{c} = \nu, c_0 = C_Q(\nu), \bar{c}_2 = 1$, and $c_3 = C_Q(1)$. Then, by $(1.13)$, $c_2 = \alpha_0^{-1} \lVert g^{-1} \rVert_{L_\infty}$ and $\kappa = 1/2$.

1.6. The operator $B(\varepsilon)$. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the family of operators $B(\varepsilon), 0 < \varepsilon \leq 1$, formally given by the differential expression

$$B(\varepsilon) = A + \varepsilon (\mathcal{Y}_2^* \mathcal{Y} + \mathcal{Y}^* \mathcal{Y}_2) + \varepsilon^2 Q$$

$$= b(D)^* g(x) b(D) + \varepsilon \sum_{j=1}^d (a_j(x) D_j + D_j a_j(x)^*) + \varepsilon^2 Q(x),$$

where $Q(x)$ can be interpreted as the generalized matrix-valued potential generated by the measure $d\mu(x)$. Precisely, $B(\varepsilon)$ is the selfadjoint operator generated by the quadratic form

$$b(\varepsilon)[u, u] = a[u, u] + 2\varepsilon \text{Re} \langle \mathcal{Y} u, \mathcal{Y}_2 u \rangle_{L_2(\mathbb{R}^d)} + \varepsilon^2 q[u, u], \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \quad (1.15)$$
Let us check that the form (1.15) is closed and lower semibounded. By (1.7) and (1.9),
\[
2\varepsilon |\Re (Y_u, Y_2 u)_{L_2(\mathbb{R}^d)}| \leq \frac{K}{2} a[u, u] + c_4 \varepsilon^2 \|u\|_{L_2(\mathbb{R}^d)}^2,
\]
\(u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad c_4 = 4\kappa^{-1}c_1^2 C(\nu_0) \) for \(\nu_0 = \kappa^2(16c_1^2)^{-1} \).

Now, the lower estimate (1.12), relation (1.16), and the restriction \(0 < \varepsilon \leq 1\) imply the lower estimate for the form (1.15):
\[
b(\varepsilon)[u, u] \geq \frac{K}{2} a[u, u] - (c_0 + c_4)\varepsilon^2 \|u\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
Together with the lower estimate (1.6) this yields
\[
b(\varepsilon)[u, u] \geq c_* \|Du\|_{L_2(\mathbb{R}^d)}^2 - (c_0 + c_4)\varepsilon^2 \|u\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1; \quad c_* := \frac{\kappa}{2} \alpha_0 \|g^{-1}\|_{L_1}^{-1}.
\]
Combining (1.7), (1.9) with \(\nu = 1\), and the upper estimate (1.12), we arrive at
\[
b(\varepsilon)[u, u] \leq (2 + c_1^2 + c_2) a[u, u] + (C(1) + c_3)\varepsilon^2 \|u\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
Together with the upper estimate (1.6) this leads to
\[
b(\varepsilon)[u, u] \leq C_* \|Du\|_{L_2(\mathbb{R}^d)}^2 + (C(1) + c_3)\varepsilon^2 \|u\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1; \quad C_* := (2 + c_1^2 + c_2)\alpha_1 \|g\|_{L_\infty}.
\]

1.7. The operator \(B_\varepsilon\). Let \(T_\varepsilon, \varepsilon > 0\), be the unitary scaling transformation in \(L_2(\mathbb{R}^d; \mathbb{C}^n)\) defined by
\[
(T_\varepsilon u)(x) = \varepsilon^{d/2} u(\varepsilon x), \quad u \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad \varepsilon > 0.
\]
Let \(A_\varepsilon\) be the selfadjoint operator in \(L_2(\mathbb{R}^d; \mathbb{C}^n)\) corresponding to the quadratic form
\[
a_\varepsilon[u, u] := \varepsilon^{-2} a[T_\varepsilon u, T_\varepsilon u] = (g^\varepsilon b(D)u, b(D)u)_{L_2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
Formally, we have \(A_\varepsilon = b(D)^* g^\varepsilon(x) b(D)\).

We introduce the operator \(Y_{2,\varepsilon}\) defined by
\[
(Y_{2,\varepsilon} u)(x) = \text{col} \{a_1^\varepsilon(x)^* u(x), \ldots, a_n^\varepsilon(x)^* u(x)\}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
Let \(d\mu\) be a measure defined in Subsection 1.3. We define the measure \(d\mu^\varepsilon\) as follows. For any Borel set \(\Delta \subset \mathbb{R}^d\), we consider the set \(\varepsilon^{-1}\Delta := \{x \in \mathbb{R}^d : \varepsilon x \in \Delta\}\) and put \(\mu^\varepsilon(\Delta) := \varepsilon^d \mu(\varepsilon^{-1}\Delta)\). Consider the quadratic form \(q_\varepsilon\) defined by
\[
q_\varepsilon[u, u] = \int_{\mathbb{R}^d} \langle d\mu^\varepsilon(x) u, u \rangle, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
We put $B_\varepsilon := \varepsilon^{-2}T_\varepsilon^* B(\varepsilon) T_\varepsilon$, $0 < \varepsilon \leq 1$. In other words, $B_\varepsilon$ is the selfadjoint operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ generated by the quadratic form

$$b_\varepsilon[u, u] := \varepsilon^{-2} b(\varepsilon)[T_\varepsilon u, T_\varepsilon u]$$

$$= a_\varepsilon[u, u] + 2\text{Re} (\mathcal{V} u, \mathcal{V}_\varepsilon u)_{L_2(\mathbb{R}^d)} + q_\varepsilon[u, u], \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

(1.20)

Relations (1.20), (1.17), and (1.18) imply that

$$b_\varepsilon[u, u] \geq c_* \|Du\|_{L_2(\mathbb{R}^d)}^2 - (c_0 + c_4) \|u\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

(1.21)

$$b_\varepsilon[u, u] \leq C_* \|Du\|_{L_2(\mathbb{R}^d)}^2 + (C(1) + c_3) \|u\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

for $0 < \varepsilon \leq 1$. Thus, the form $b_\varepsilon$ is closed and lower semibounded. Formally, we can write

$$B_\varepsilon = b(D)^* g^\varepsilon(x)b(D) + \sum_{j=1}^d \left( a_j^\varepsilon(x) D_j + D_j a_j^\varepsilon(x)^* \right) + Q^\varepsilon(x).$$

(1.22)

Here $Q^\varepsilon$ can be interpreted as the generalized matrix potential generated by the measure $d\mu^\varepsilon$. We see that the coefficients of the operator $B_\varepsilon$ oscillate rapidly for small $\varepsilon$.

1.8. The effective matrix. The effective operator for $A_\varepsilon = b(D)^* g^\varepsilon(x)b(D)$ is given by $A^0 = b(D)^* g^0 b(D)$. Here $g^0$ is a constant positive $(m \times m)$-matrix called the effective matrix. The matrix $g^0$ is defined in terms of the solution of the auxiliary problem on $\Omega$. Suppose that a $\Gamma$-periodic $(n \times m)$-matrix-valued function $\Lambda(x)$ is the solution of the problem

$$b(D)^* g(x)(b(D)\Lambda(x) + 1_m) = 0, \quad \int_{\Omega} \Lambda(x) \, dx = 0. \quad (1.23)$$

(The equation is understood in the weak sense.) Then the effective matrix is given by

$$g^0 = |\Omega|^{-1} \int_{\Omega} \bar{g}(x) \, dx,$$

(1.24)

where

$$\bar{g}(x) := g(x)(b(D)\Lambda(x) + 1_m).$$

(1.25)

It is easily seen that $g^0$ is positive definite.

We need the following estimates for $\Lambda$ proved in [BSu2], (6.28) and Subsection 7.3:

$$\|\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{1/2} M_1, \quad M_1 := m^{1/2}(2r_0)^{-1} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}, \quad (1.26)$$

$$\|D\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{1/2} M_2, \quad M_2 := m^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}. \quad (1.27)$$

The effective matrix has the following properties (see [BSu1], Chapter 3, Theorem 1.5)].
Proposition 1.5. Let $g^0$ be the effective matrix (1.24). Then
\[ g \leq g^0 \leq \overline{g}. \] (1.28)
If $m = n$, then $g^0 = g$.

Estimates (1.28) are known in homogenization theory as the Voigt-Reuss bracketing. From (1.28) it follows that
\[ |g^0| \leq \|g\|_{L_{\infty}}, \quad |(g^0)^{-1}| \leq \|g^{-1}\|_{L_{\infty}}. \]

Now we distinguish the cases where one of the inequalities in (1.28) becomes an identity (see [BSu1, Chapter 3, Propositions 1.6 and 1.7]).

Proposition 1.6. The identity $g^0 = \overline{g}$ is equivalent to the relations
\[ b(D)^*g_k(x) = 0, \quad k = 1, \ldots, m, \] (1.29)
where $g_k(x)$, $k = 1, \ldots, m$, are the columns of the matrix $g(x)$.

Proposition 1.7. The identity $g^0 = g$ is equivalent to the representations
\[ l_k(x) = l_k^0 + b(D)w_k, \quad l_k^0 \in \mathbb{C}^m, \quad w_k \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad k = 1, \ldots, m, \] (1.30)
where $l_k(x)$, $k = 1, \ldots, m$, are the columns of the matrix $g(x)^{-1}$.

1.9. The effective operator $B^0$. The effective operator for $B_\varepsilon$ was introduced in [Su2] (see also [Bo]).

Suppose that a $\Gamma$-periodic $(n \times n)$-matrix-valued function $\tilde{\Lambda}(x)$ is the (weak) solution of the problem
\[ b(D)^*g(x)b(D)\tilde{\Lambda}(x) + \sum_{j=1}^{d} D_j a_j(x)^* = 0, \quad \int_{\Omega} \tilde{\Lambda}(x) \, dx = 0. \] (1.31)

Define constant matrices $V$ and $W$ as follows:
\[ V = |\Omega|^{-1} \int_{\Omega} (b(D)\Lambda(x))^* g(x)(b(D)\tilde{\Lambda}(x)) \, dx, \] (1.32)
\[ W = |\Omega|^{-1} \int_{\Omega} (b(D)\tilde{\Lambda}(x))^* g(x)(b(D)\tilde{\Lambda}(x)) \, dx. \] (1.33)

The effective operator for the operator (1.22) is given by
\[ B^0 = b(D)^*g^0b(D) - b(D)^*V - V^*b(D) + \sum_{j=1}^{d} (a_j + a_j^*) D_j - W + \tilde{Q}. \] (1.34)

The operator $B^0$ is a second order elliptic operator with constant coefficients.

According to [Su6 (5.7)], if $\lambda > c_0 + c_4$, then the symbol $L_\lambda(\xi)$ of the operator $B^0 + \lambda I$ satisfies
\[ L_\lambda(\xi) \geq c_\lambda(|\xi|^2 + 1)1_n, \quad \xi \in \mathbb{R}^d, \]
where $c_\lambda = \min\{c_\varepsilon; \lambda - c_0 - c_4\}$. Putting $\lambda_\varepsilon := c_0 + c_4 + c_\varepsilon$, we obtain the following estimate for the symbol $L_{\lambda}(\xi)$ of the operator $B^0 + \lambda_\varepsilon I$:
\[ L_{\lambda}(\xi) \geq c_\varepsilon(|\xi|^2 + 1)1_n, \quad \xi \in \mathbb{R}^d. \]
Consequently, the quadratic form $b^0$ of the operator $B^0$ satisfies
\[ b^0[u, u] + \lambda_* \|u\|_{L^2(\mathbb{R}^d)}^2 \geq c_* \left( \|Du\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}^2 \right), \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \]

Since $\lambda_* = c_0 + c_4 + c_5$, we deduce
\[ b^0[u, u] \geq c_* \|Du\|_{L^2(\mathbb{R}^d)}^2 - (c_0 + c_4)\|u\|_{L^2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \] (1.35)

Below we need the following estimates for $\tilde{\Lambda}$ proved in [Su2] (7.52) and (7.51):
\[ \|\tilde{\Lambda}\|_{L^2(\Omega)} \leq (2r_0)^{-1}C\alpha n^{1/2}d_0^{-1}\|g^{-1}\|_{L^\infty}, \] (1.36)
\[ \|D\tilde{\Lambda}\|_{L^2(\Omega)} \leq C\alpha n^{1/2}d_0^{-1}\|g^{-1}\|_{L^\infty}, \] (1.37)

where $C_a^2 = \sum_{j=1}^d \int_\Omega |a_j(x)|^2 \, dx$.

1.10. The generalized resolvent. Let $Q_0(x)$ be a $\Gamma$-periodic $(n \times n)$-matrix-valued function such that
\[ Q_0(x) > 0, \quad Q_0, Q_0^{-1} \in L_\infty(\mathbb{R}^d). \]

We study the generalized resolvent of the operator $B_\varepsilon$, i.e., the operator $(B_\varepsilon - zQ_0^{-1})^{-1}$. We rely on the results of [Su2], where approximations for this resolvent at a fixed real point $z$ were obtained. Before we formulate the results, it is convenient to turn from the operator $B_\varepsilon$ to the nonnegative operator
\[ B_\varepsilon := B_\varepsilon + c_5 Q_0^\varepsilon, \] (1.38)
putting $c_5 := (c_0 + c_4)\|Q_0^{-1}\|_{L^\infty}$. From (1.24) it follows that $B_\varepsilon \geq 0$.

Observe that the operator $B_\varepsilon$ can be considered as the operator of the form (1.22) with the initial coefficients $g^\varepsilon$, $\alpha^\varepsilon$, and the „new” matrix-valued potential $\tilde{Q}^\varepsilon = Q^\varepsilon + c_5 Q_0$. The corresponding form $\int_\Omega \langle d\mu(x)u, u \rangle + c_5 \int_\Omega \langle Q_0(x)u, u \rangle \, dx$ satisfies Condition 1.3 with $c_0 = 0$ in place of $c_0$, the constant $c_3 = c_3 + c_5\|Q_0\|_{L^\infty}$ in place of $c_3$, and the initial $\tilde{c}$ and $\tilde{c}_2$.

Let us fix $\lambda_0$ as follows:
\[ \lambda_0 = 2\|Q_0^{-1}\|_{L^\infty}c_4, \] (1.39)
cf. [Su2] (5.27). The operator $B_\varepsilon + \lambda_0 Q_0^\varepsilon$ is positive definite, whence the generalized resolvent $(B_\varepsilon + \lambda_0 Q_0^\varepsilon)^{-1}$ is a bounded operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$.

Remark 1.8. In [Su2], the generalized resolvent $(B_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ was studied for $\lambda \geq \lambda_0$, while in [Su6] it was studied under the weaker assumption that $\lambda > 0$. For our goals, it suffices to formulate the results for $\lambda$ fixed; for convenience of referring to [Su2], we use condition (1.39).

For convenience of further references, the set of parameters
\[ d, m, n, p; \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|a_j\|_{L^p(\Omega)}, j = 1, \ldots, d, \]
\[ \tilde{c}, c_0, \tilde{c}_2, c_3 \text{ from Condition 1.3: } \|Q_0\|_{L^\infty}, \|Q_0^{-1}\|_{L^\infty}; \] (1.40)
the parameters of the lattice $\Gamma$
is called the „initial data”. Note that the constants \( c_1, C(1), \kappa, c_2, \) and \( c_4 \) are determined by the initial data.

The effective operator for the operator (1.38) is given by
\[
B^0 = B^0 + c_5 \mathcal{Q}_0.
\] (1.41)

Relation (1.35) implies the lower estimate for the quadratic form \( b^0 \) of the operator \( B^0 \):
\[
b^0[\mathbf{u}, \mathbf{u}] \geq c_4 \| \mathbf{D}\mathbf{u} \|^2_{L^2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n). \] (1.42)

This is equivalent to the following estimate for the symbol \( L_0(\boldsymbol{\xi}) \) of the operator \( B^0 \):
\[
L_0(\boldsymbol{\xi}) \geq c_4 |\boldsymbol{\xi}|^2 \mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d.
\] (1.43)

The operator \( B^0 + \lambda_0 \mathcal{Q}_0 \) is the second order DO with constant coefficients with the symbol
\[
L(\boldsymbol{\xi}) = L_0(\boldsymbol{\xi}) + \lambda_0 \mathcal{Q}_0
\]
\[
= b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) - b(\boldsymbol{\xi})^* V - V^* b(\boldsymbol{\xi}) + \sum_{j=1}^{d} (a_j + a_j^*) \mathbf{\xi}_j + \mathcal{Q} - W + (c_5 + \lambda_0) \mathcal{Q}_0.
\]

From (1.43) and (1.39) it follows that
\[
L(\boldsymbol{\xi}) \geq \tilde{c}_4(|\boldsymbol{\xi}|^2 + 1) \mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d; \quad \tilde{c}_4 = \min\{c_4; 2c_4\}. \] (1.44)

1.11. **Approximation of the operator** \( (B_{\varepsilon} + \lambda_0 \mathcal{Q}_0)^{-1} \). Applying Theorem 9.2 from \( \text{[Su2]} \) to the operator (1.38), we obtain the following result.

**Theorem 1.9 (Su2).** Suppose that the assumptions of Subsections 1.3–1.10 are satisfied. Let \( \lambda_0 \) be given by (1.39). Then for \( 0 < \varepsilon \leq 1 \) we have
\[
\|(B_{\varepsilon} + \lambda_0 \mathcal{Q}_0)^{-1} - (B^0 + \lambda_0 \mathcal{Q}_0)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_1 \varepsilon.
\]

The constant \( C_1 \) is controlled in terms of the initial data (1.40).

In order to approximate the operator \( (B_{\varepsilon} + \lambda_0 \mathcal{Q}_0)^{-1} \) in the \( (L^2 \to H^1) \)-operator norm, we introduce a corrector
\[
K(\varepsilon) = \left( [\Lambda^\varepsilon] b(\mathbf{D}) + [\Lambda^\varepsilon] \right) \Pi_{\varepsilon} (B^0 + \lambda_0 \mathcal{Q}_0)^{-1}.
\] (1.45)

Here \( \Pi_{\varepsilon} \) is given by (1.1). The corrector (1.45) is a continuous mapping of \( L^2(\mathbb{R}^d; \mathbb{C}^n) \) to \( H^1(\mathbb{R}^d; \mathbb{C}^n) \). This can be easily checked by using Proposition 1.2 and the relations \( \Lambda, \tilde{\Lambda} \in H^1(\Omega) \). Herewith, \( \|\varepsilon K(\varepsilon)\|_{L^2 \to H^1} = O(1) \).

The following result was obtained in \( \text{[Su2], Theorem 9.7} \).

**Theorem 1.10 (Su2).** Suppose that the assumptions of Theorem 1.9 are satisfied. Let \( K(\varepsilon) \) be the operator given by (1.45). Then for \( 0 < \varepsilon \leq 1 \) we have
\[
\|(B_{\varepsilon} + \lambda_0 \mathcal{Q}_0)^{-1} - (B^0 + \lambda_0 \mathcal{Q}_0)^{-1} - \varepsilon K(\varepsilon)\|_{L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_2 \varepsilon.
\]

The constant \( C_2 \) is controlled in terms of the initial data (1.40).
Remark 1.11. Below in Sections 5 and 6 we will apply Theorems 1.9 and 1.10 (precisely, instead of Theorem 1.10 we will use its analog, Theorem 5.3) in order to approximate the generalized resolvent \((B_\varepsilon(\vartheta) + \lambda_0 Q_0^\varepsilon)^{-1}\) of the operator family \(B_\varepsilon(\vartheta)\) depending on the auxiliary parameter \(\vartheta \in (0, 1]\). The operator \(B_\varepsilon(\vartheta)\) has the same principal part as \(B_\varepsilon\) and the lower order coefficients \(\vartheta a_j^\varepsilon\) and \(\vartheta^2 Q^\varepsilon\) (see Subsection 5.2). Note that, for all \(\vartheta \in (0, 1]\), the form \(\vartheta^2 \left(\int_\Omega \langle dp(x)u, u\rangle + c_5 \int_\Omega \langle Q_0(x)u, u\rangle \, dx\right)\) satisfies Condition 1.3 with the same constants: \(\tilde{c}_0 = 0\) in the role of \(c_0\), \(\tilde{c}_3 = c_3 + c_5 \|Q_0\|_{L^\infty}\) in the role of \(c_3\), and the previous \(\tilde{c}\) and \(\tilde{c}_2\). Next, the constant \(\nu_0 = \kappa^2(16\pi^2)^{-1}\) does not depend on \(\vartheta\). Multiplying (1.9) with \(\nu = \nu_0\) by \(\vartheta^2\), we see that the inequality of the form (1.9) (in the case of the coefficients \(\vartheta a_j\)) is valid with the constant \(C(\nu_0)\) for all \(\vartheta \in (0, 1]\). Hence, the constant \(c_4\) (see (1.16)), and then also \(\lambda_0\) (see 1.39) can be taken independently of \(\vartheta\). Note also that the norms of the coefficients \(\vartheta a_j\) in \(L^p(\Omega)\) are majorized by the norms \(\|a_j\|_{L^p(\Omega)}\) for all \(\vartheta \in (0, 1]\). In [Sh2], the dependence of the constants \(C_1\) and \(C_2\) on the initial data was searched out (in particular, these constants increase, as the norms \(\|a_j\|_{L^p(\Omega)}\) increase). What was said allows us to choose the constants \(C_1\) and \(C_2\) in approximations for the operator \((B_\varepsilon(\vartheta) + \lambda_0 Q_0^\varepsilon)^{-1}\) to be independent of the parameter \(\vartheta\).

2. Auxiliary statements

2.1. Properties of the matrix-valued function \(\Lambda\). We need the following result proved in [PSu] Lemma 2.3:

Lemma 2.1. Suppose that \(\Lambda\) is the \(\Gamma\)-periodic solution of problem (1.28). Then for any \(u \in C_c^\infty(\mathbb{R}^d)\) and \(\varepsilon > 0\) we have

\[
\int_{\mathbb{R}^d} |(D\Lambda)(x)|^2 |u(x)|^2 \, dx \leq \beta_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2 \varepsilon^2 \int_{\mathbb{R}^d} |\Lambda(x)|^2 |D u(x)|^2 \, dx.
\]

The constants \(\beta_1\) and \(\beta_2\) depend only on \(m, d, \alpha_0, \alpha_1, \|g\|_{L^\infty},\) and \(\|g^{-1}\|_{L^\infty}\).

From Lemma 2.1 and the density of \(C_c^\infty(\mathbb{R}^d)\) in \(H^1(\mathbb{R}^d)\) we deduce the following statement.

Corollary 2.2. Suppose that \(\Lambda\) is the \(\Gamma\)-periodic solution of problem (1.28). Assume also that \(\Lambda \in L^\infty\). Then for any \(u \in H^1(\mathbb{R}^d)\) and \(\varepsilon > 0\) we have

\[
\int_{\mathbb{R}^d} |(D\Lambda)(x)|^2 |u(x)|^2 \, dx \leq \beta_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2 \varepsilon^2 \|\Lambda\|^2_{L^\infty} \int_{\mathbb{R}^d} |D u(x)|^2 \, dx.
\]

2.2. Properties of the matrix-valued function \(\tilde{\Lambda}\). The proof of the following statement is similar to the proof of Lemma 2.1 from [PSu].

Lemma 2.3. Suppose that \(\tilde{\Lambda}\) is the \(\Gamma\)-periodic solution of problem (1.31). Then for any \(u \in C_c^\infty(\mathbb{R}^d)\) we have

\[
\int_{\mathbb{R}^d} |D\tilde{\Lambda}(x)|^2 |u(x)|^2 \, dx \leq \tilde{\beta}_1 \|u\|^2_{H^1(\mathbb{R}^d)} + \tilde{\beta}_2 \int_{\mathbb{R}^d} |\tilde{\Lambda}(x)|^2 |D u(x)|^2 \, dx.
\]
The constants $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are given below in (2.12) and depend only on $n$, $d$, $\alpha_0$, $\alpha_1$, $\rho$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the norms $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \ldots, d$, and the parameters of the lattice $\Gamma$.

Proof. Let $w_k(x)$, $k = 1, \ldots, n$, be the columns of the matrix $\bar{A}(x)$. By (1.31), for any function $\eta \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ such that $\eta(x) = 0$ for $|x| > R$ (with some $R > 0$) we have

$$\int_{\mathbb{R}^d} \left( \langle g(x)b(D)w_k(x), b(D)\eta(x) \rangle + \sum_{j=1}^{d} \langle a_j(x)^*e_k, D_j\eta(x) \rangle \right) \, dx = 0. \tag{2.2}$$

Here $e_k$, $k = 1, \ldots, n$, is the standard orthonormal basis in $\mathbb{C}^n$.

Let $u \in C_0^\infty(\mathbb{R}^d)$. We put $\eta(x) := w_k(x)|u(x)|^2$. Then, by (1.3),

$$b(D)\eta(x) = (b(D)w_k(x))|u(x)|^2 + \sum_{l=1}^{d} b_lw_k(x) D_l|u(x)|^2, \quad \tag{2.3}$$
$$D_j\eta(x) = (D_jw_k(x))|u(x)|^2 + w_k(x)D_j|u(x)|^2. \quad \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.2), we obtain

$$J := \int_{\mathbb{R}^d} \langle g(x)b(D)w_k(x), b(D)w_k(x) \rangle |u(x)|^2 \, dx$$
$$= -\sum_{j=1}^{d} \int_{\mathbb{R}^d} \langle a_j(x)^*e_k, D_jw_k(x) \rangle |u(x)|^2 \, dx$$
$$+ \sum_{l=1}^{d} \int_{\mathbb{R}^d} \langle g(x)b(D)w_k(x), b_lw_k(x) \rangle (u^*D_lu + uD_lu^*) \, dx$$
$$+ \sum_{j=1}^{d} \int_{\mathbb{R}^d} \langle a_j(x)^*e_k, w_k(x) \rangle (u^*D_ju + uD_ju^*) \, dx. \tag{2.5}$$

Denote the consecutive summands in the right-hand side of (2.5) by $J_1$, $J_2$, and $J_3$. We have

$$|J_1| \leq 4\alpha_0^{-1}\|g^{-1}\|_{L_\infty} \sum_{j=1}^{d} \int_{\mathbb{R}^d} |a_j(x)^*e_k|^2 |u(x)|^2 \, dx$$
$$+ \frac{1}{4} \left( 4\alpha_0^{-1}\|g^{-1}\|_{L_\infty} \right)^{-1} \int_{\mathbb{R}^d} |Dw_k(x)|^2 |u(x)|^2 \, dx.$$

Using condition (1.3) on the coefficients $a_j$, we see that

$$\int_{\mathbb{R}^d} |a_j(x)^*e_k|^2 |u(x)|^2 \, dx \leq \int_{\mathbb{R}^d} |a_j(x)|^2 |u(x)|^2 \, dx$$
$$\leq C_{\Omega, \rho}^2 \|a_j\|^2_{L_\rho(\Omega)} \|u\|^2_{H^1(\mathbb{R}^d)}, \quad \tag{2.6}$$
According to (1.3),

\[ |J_1| \leq 4\alpha_0^{-1} g^{-1} \| L_\infty C_{\Omega, \rho}^2 \sum_{j=1}^{d} \| a_j \| _{L_\rho(\Omega)}^2 \| u \| _{H^1(\mathbb{R}^d)}^2 + \frac{1}{4} \left( 4\alpha_0^{-1} g^{-1} \| L_\infty \right)^{-1} \int_{\mathbb{R}^d} |Dw_k(x)|^2 |u(x)|^2 \, dx. \]  

(2.7)

Combining (2.5) and (2.7)–(2.9), we arrive at

\[ \text{Taking (2.6) into account, we obtain} \]

\[ |J_2| \leq 2 \int_{\mathbb{R}^d} |g(x)^{1/2} b(D)w_k(x)||u(x)| \left( \sum_{l=1}^{d} |g(x)^{1/2} b_l w_k(x)||D_lu(x)| \right) \, dx \]

\[ \leq \frac{1}{2} J + 2\alpha_1 \| g \| _{L_\infty} \int_{\mathbb{R}^d} |w_k(x)|^2 |Du(x)|^2 \, dx. \]  

(2.8)

Finally, we estimate \( J_3 \):

\[ |J_3| \leq 2 \sum_{j=1}^{d} \int_{\mathbb{R}^d} |a_j(x)||w_k(x)||u(x)||Du(x) \, dx \]

\[ \leq \sum_{j=1}^{d} \int_{\mathbb{R}^d} |a_j(x)||u(x)||^2 \, dx + \int_{\mathbb{R}^d} |w_k(x)||Du(x)||^2 \, dx. \]

Taking (2.6) into account, we obtain

\[ |J_3| \leq C_{\Omega, \rho}^2 \sum_{j=1}^{d} \| a_j \| _{L_\rho(\Omega)}^2 \| u \| _{H^1(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} |w_k(x)||Du(x)||^2 \, dx. \]  

(2.9)

Combining (2.5) and (2.7)–(2.9), we arrive at

\[ J \leq 2 \left( 4\alpha_0^{-1} g^{-1} \| L_\infty \right) + 1 \left( C_{\Omega, \rho}^2 \sum_{j=1}^{d} \| a_j \| _{L_\rho(\Omega)}^2 \| u \| _{H^1(\mathbb{R}^d)}^2 \right. \]

\[ + \frac{1}{2} \left( 4\alpha_0^{-1} g^{-1} \| L_\infty \right)^{-1} \int_{\mathbb{R}^d} |Dw_k(x)|^2 |u(x)|^2 \, dx \]

\[ + 2 \left( 2\alpha_1 \| g \| _{L_\infty} + 1 \right) \int_{\mathbb{R}^d} |w_k(x)||Du(x)||^2 \, dx. \]  

(2.10)

Now we deduce the required estimate from (2.10). By (1.4),

\[ \int_{\mathbb{R}^d} |D(w_ku)(x)||^2 \, dx \leq \alpha_0^{-1} \int_{\mathbb{R}^d} |b(D)(w_ku)(x)||^2 \, dx. \]

According to (1.3), \( b(D) = \sum_{l=1}^{d} b_l D_l \), whence

\[ b(D)(w_ku) = (b(D)w_k)u + \sum_{l=1}^{d} b_l w_k D_l u. \]
Using (1.5) and the expression for $J$ (see (2.5)), we have
\[
\int_{\mathbb{R}^d} |D(w_k u)(x)|^2 \, dx \leq 2\alpha_0^{-1} \int_{\mathbb{R}^d} |b(D)w_k(x)|^2 |u(x)|^2 \, dx \\
+ 2\alpha_0^{-1}\alpha_1 d \int_{\mathbb{R}^d} |w_k(x)|^2 |D u(x)|^2 \, dx
\]
\[
\leq 2\alpha_0^{-1} \|g^{-1}\|_{L_{\infty}} J + 2\alpha_0^{-1}\alpha_1 d \int_{\mathbb{R}^d} |w_k(x)|^2 |D u(x)|^2 \, dx.
\]
Obviously,
\[
\int_{\mathbb{R}^d} |Dw_k(x)|^2 |u(x)|^2 \, dx \leq 2 \int_{\mathbb{R}^d} |D(w_k u)(x)|^2 \, dx \\
+ 2 \int_{\mathbb{R}^d} |w_k(x)|^2 |D u(x)|^2 \, dx.
\]
Combining this with (2.10) and (2.11), we obtain
\[
\int_{\mathbb{R}^d} |Dw_k(x)|^2 |u(x)|^2 \, dx
\]
\[
\leq 16\alpha_0^{-1} \|g^{-1}\|_{L_{\infty}} (4\alpha_0^{-1} \|g^{-1}\|_{L_{\infty}} + 1) C_{\Omega,\rho}^2 \sum_{j=1}^{d} \|a_j\|_{L_{\rho}(\Omega)}^2 \|a\|_{H_1(\mathbb{R}^d)}^2 \\
+ (16\alpha_0^{-1} \|g^{-1}\|_{L_{\infty}} (2d\alpha_1 \|g\|_{L_{\infty}} + 1) + 8\alpha_0^{-1}\alpha_1 d + 4) \\
\times \int_{\mathbb{R}^d} |w_k(x)|^2 |D u(x)|^2 \, dx.
\]
Summing up over $k$, we arrive at (2.1) with
\[
\tilde{\beta}_1 = 16\alpha_0^{-1} \|g^{-1}\|_{L_{\infty}} (4\alpha_0^{-1} \|g^{-1}\|_{L_{\infty}} + 1) C_{\Omega,\rho}^2 \sum_{j=1}^{d} \|a_j\|_{L_{\rho}(\Omega)}^2, \quad (2.12)
\]
\[
\tilde{\beta}_2 = 16\alpha_0^{-1} \|g^{-1}\|_{L_{\infty}} (2d\alpha_1 \|g\|_{L_{\infty}} + 1) + 8\alpha_0^{-1}\alpha_1 d + 4.
\]
\]
By the scaling transformation, Lemma 2.3 implies the following result.

**Lemma 2.4.** Under the assumptions of Lemma 2.3, for $0 < \varepsilon \leq 1$ we have
\[
\int_{\mathbb{R}^d} |(D\bar{\Lambda}^\varepsilon(x))|^2 |u(x)|^2 \, dx \leq \tilde{\beta}_1 \|u\|_{H_1(\mathbb{R}^d)}^2 + \tilde{\beta}_2 \varepsilon^2 \int_{\mathbb{R}^d} |\bar{\Lambda}^\varepsilon(x)|^2 |D u(x)|^2 \, dx.
\]

Below in Section 7, we will need the following simple statement.

**Lemma 2.5.** Let $f(x)$ be a $\Gamma$-periodic function in $\mathbb{R}^d$ such that
\[
f \in L_p(\Omega), \quad p = 2 \text{ for } d = 1, \quad p > 2 \text{ for } d = 2, \quad p = d \text{ for } d \geq 3. \quad (2.13)
\]
Then for $0 < \varepsilon \leq 1$ the operator $[f^\varepsilon]$ is a continuous mapping of $H_1(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$, and
\[
\|[f^\varepsilon]\|_{H_1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|f\|_{L_p(\Omega)} C_{\Omega}^{(p)},
\]

\]
where \( C^{(p)}_\Omega \) is the norm of the embedding \( H^1(\Omega) \hookrightarrow L_{2(p/2)'}(\Omega) \). Here \((p/2)' = \infty\) for \( d = 1\), and \((p/2)' = p(p-2)^{-1}\) for \( d \geq 2\).

Proof. Let \( d \geq 2\), and let \( u \in H^1(\mathbb{R}^d)\). Substituting \( x = \varepsilon y\), \( u(x) = v(y)\), and using the Hölder inequality and the Sobolev embedding theorem, for \( 0 < \varepsilon \leq 1 \) we obtain

\[
\int_{\mathbb{R}^d} |f'(x)|^2 |u(x)|^2 \, dx = \varepsilon^d \int_{\mathbb{R}^d} |f(y)|^2 |v(y)|^2 \, dy
\]

\[
= \varepsilon^d \sum_{a \in \Gamma} \int_{\Omega + a} |f(y)|^2 |v(y)|^2 \, dy
\]

\[
\leq \varepsilon^d \|f\|^2_{L_p(\Omega)} \sum_{a \in \Gamma} \left( \int_{\Omega + a} |v(y)|^{2(p/2)'} \, dy \right)^{1/(p/2)'}
\]

\[
\leq \varepsilon^d \|f\|^2_{L_p(\Omega)} (C^{(p)}_\Omega)^2 \|v\|^2_{H^1(\mathbb{R}^d)} \leq \|f\|^2_{L_p(\Omega)} (C^{(p)}_\Omega)^2 \|u\|^2_{H^1(\mathbb{R}^d)}.
\]

Here \((p/2)^{-1} + ((p/2)')^{-1} = 1\). For the case where \( d = 1 \) the proof is similar (the necessary changes are obvious).

\( \square \)

Lemma 2.4 and Lemma 2.5 directly imply the following corollary.

Corollary 2.6. Suppose that \( \tilde{\Lambda} \) is the \( \Gamma \)-periodic solution of problem (1.31). Suppose also that \( \Lambda \) satisfies condition of the form (2.13). Then for any \( u \in H^2(\mathbb{R}^d) \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\int_{\mathbb{R}^d} |(D\tilde{\Lambda})^\varepsilon(x)|^2 |u(x)|^2 \, dx \leq \beta_1 \|u\|^2_{H^1(\mathbb{R}^d)} + \beta_2 \varepsilon^2 \|\tilde{\Lambda}\|^2_{L_p(\Omega)} (C^{(p)}_\Omega)^2 \|Du\|^2_{H^1(\mathbb{R}^d)}.
\]

2.3. Lemma about \( Q_0^\varepsilon - Q_0 \). The following lemma will be needed for the proof of the main results.

Lemma 2.7. Suppose that \( Q_0(x) \) is a \( \Gamma \)-periodic \((n \times n)\)-matrix-valued function such that \( Q_0 \in L_\infty \). Then for \( \varepsilon > 0 \) the operator \( Q_0^\varepsilon - \overline{Q_0} \) is a continuous mapping by the function \( Q_0^\varepsilon - \overline{Q_0} \) is a continuous mapping of \( H^1(\mathbb{R}^d; \mathbb{C}^n) \) to \( H^{-1}(\mathbb{R}^d; \mathbb{C}^n) \), and

\[
\|(Q_0^\varepsilon - \overline{Q_0})\|_{H^1(\mathbb{R}^d) \rightarrow H^{-1}(\mathbb{R}^d)} \leq C_{Q_0} \varepsilon, \quad \varepsilon > 0.
\]

The constant \( C_{Q_0} \) is controlled in terms of \( d \), \( \|Q_0\|_{L_\infty} \), and the parameters of the lattice \( \Gamma \).

Proof. Since \( Q_0 - \overline{Q_0} \in L_\infty \) and

\[
\int_{\Omega} (Q_0(x) - \overline{Q_0}) \, dx = 0,
\]

we have the following representation

\[
Q_0^\varepsilon(x) - \overline{Q_0} = -\varepsilon \sum_{j=1}^d D_j h_j^\varepsilon(x),
\]
where \( h_j, j = 1, \ldots, d, \) are \( \Gamma \)-periodic \((n \times n)\)-matrix-valued functions such that \( h_j \in L_\infty \).

Indeed, let \( \Phi(x) \) be the \( \Gamma \)-periodic solution of the problem
\[
\triangle \Phi(x) = Q_0(x) - Q_0, \quad \int_\Omega \Phi(x) \, dx = 0.
\]

By (2.15), the solvability condition is satisfied. Since \( Q_0 - Q_0 \in L_\infty \), then \( \Phi \in W_p^2(\Omega) \) for any \( 1 \leq p < \infty \), and
\[
\|\Phi\|_{W_p^2(\Omega)} \leq c_1(p)\|Q_0 - Q_0\|_{L_p(\Omega)} \leq \tilde{c}_1(p)\|Q_0\|_{L_\infty}.
\] (2.17)
The constants \( c_1(p) \) and \( \tilde{c}_1(p) \) depend only on \( p \) and the parameters of the lattice \( \Gamma \). This follows from the Marcinkiewicz theorem about the Fourier multipliers [Ma]. We put \( h_j(x) = D_j\Phi(x) \). Then
\[
Q_0(x) - Q_0 = -\sum_{j=1}^d D_j h_j(x),
\]
and \( h_j \in W_p^1(\Omega) \) for any \( 1 \leq p < \infty \). Let \( p = d + 1 \). Then, by the embedding theorem, \( h_j \in L_\infty \). Combining this with (2.17), we obtain
\[
\|h_j\|_{L_\infty} \leq c_2\|h_j\|_{W_p^1(\Omega)} \leq c_2\|\Phi\|_{W_p^2(\Omega)} \leq c_2\tilde{c}_1(d + 1)\|Q_0\|_{L_\infty}.
\]
(Here \( c_2 \) is the norm of the corresponding embedding.)

Let \( F \in H^1(\mathbb{R}^d; \mathbb{C}^n) \). By (2.16),
\[
\|(Q_0^\varepsilon - Q_0^\varepsilon)F\|_{H^{-1}(\mathbb{R}^d)} = \sup_{0 \neq v \in H^1(\mathbb{R}^d; \mathbb{C}^n)} \left| \sum_{j=1}^d \left( (D_j h_j^\varepsilon)F, v \right)_{L_2(\mathbb{R}^d)} \right| / \|v\|_{H^1(\mathbb{R}^d)} \leq \varepsilon \sup_{0 \neq v \in H^1(\mathbb{R}^d; \mathbb{C}^n)} \sum_{j=1}^d \left| \left( (D_j h_j^\varepsilon)F, v \right)_{L_2(\mathbb{R}^d)} \right| / \|v\|_{H^1(\mathbb{R}^d)}. \] (2.18)

Using the identity \( (D_j h_j^\varepsilon)F = D_j(h_j^\varepsilon F) - h_j^\varepsilon D_j F \) and integrating by parts, we obtain
\[
\left( (D_j h_j^\varepsilon)F, v \right)_{L_2(\mathbb{R}^d)} = (h_j^\varepsilon F, D_j v)_{L_2(\mathbb{R}^d)} - (h_j^\varepsilon D_j F, v)_{L_2(\mathbb{R}^d)}, \quad j = 1, \ldots, d.
\]

Together with (2.18) this yields
\[
\|(Q_0^\varepsilon - Q_0^\varepsilon)F\|_{H^{-1}(\mathbb{R}^d)} \leq \varepsilon \sup_{0 \neq v \in H^1(\mathbb{R}^d; \mathbb{C}^n)} \sum_{j=1}^d \left| \left( (h_j^\varepsilon F, D_j v \right)_{L_2(\mathbb{R}^d)} \right| / \|v\|_{H^1(\mathbb{R}^d)} + \varepsilon \sup_{0 \neq v \in H^1(\mathbb{R}^d; \mathbb{C}^n)} \sum_{j=1}^d \left| \left( (h_j^\varepsilon D_j F, v \right)_{L_2(\mathbb{R}^d)} \right| / \|v\|_{H^1(\mathbb{R}^d)}. \] (2.19)
Obviously,
\[
\sum_{j=1}^{d} \left| \left( h_{j}^{\varepsilon}F, D_{j}v \right)_{L_{2}(\mathbb{R}^{d})} \right| \leq C_{h} \| F \|_{L_{2}(\mathbb{R}^{d})} \| Dv \|_{L_{2}(\mathbb{R}^{d})}, \tag{2.20}
\]
\[
\sum_{j=1}^{d} \left| \left( h_{j}^{\varepsilon}D_{j}F, v \right)_{L_{2}(\mathbb{R}^{d})} \right| \leq C_{h} \| DF \|_{L_{2}(\mathbb{R}^{d})} \| v \|_{L_{2}(\mathbb{R}^{d})}, \tag{2.21}
\]
where \( C_{h}^{2} \) := \( \operatorname{ess} \sup_{x \in \mathbb{R}^{d}} \sum_{j=1}^{d} |h_{j}(x)|^{2} \). Note that \( C_{h} \leq C_{3} \| Q_{0} \|_{L_{\infty}}, \) where the constant \( C_{3} \) depends only on \( d \) and the parameters of the lattice \( \Gamma \).

Relations (2.19)–(2.21) imply (2.14) with the constant \( C_{Q_{0}} = 2C_{h} \). \( \square \)

3. The Steklov smoothing. Another approximation of the generalized resolvent \( (B_{\varepsilon} + \lambda_{0}Q_{0}^{\varepsilon})^{-1} \)

3.1. The Steklov smoothing operator. The operator \( S_{\varepsilon}^{(k)}, \varepsilon > 0, \) acting in \( L_{2}(\mathbb{R}^{d}; C^{k}) \) (where \( k \in \mathbb{N} \)) and defined by
\[
(S_{\varepsilon}^{(k)}u)(x) = |\Omega|^{-1} \int_{\Omega} u(x - \varepsilon z) \, dz, \quad u \in L_{2}(\mathbb{R}^{d}; C^{k}), \tag{3.1}
\]
is called the Steklov smoothing operator. We will omit the index \( k \) in the notation and write simply \( S_{\varepsilon} \). Obviously, \( S_{\varepsilon} D^{\alpha}u = D^{\alpha} S_{\varepsilon}u \) for \( u \in H^{s}(\mathbb{R}^{d}; C^{k}) \) and any multiindex \( \alpha \) such that \( |\alpha| \leq s \).

We need the following properties of the operator \( S_{\varepsilon} \) (see [ZhPas] Lemmas 1.1 and 1.2 or [PSu] Propositions 3.1 and 3.2).

**Proposition 3.1.** For any \( u \in H^{1}(\mathbb{R}^{d}; C^{k}) \) and \( \varepsilon > 0 \) we have
\[
\| S_{\varepsilon}u - u \|_{L_{2}(\mathbb{R}^{d})} \leq \varepsilon r_{1} \| Du \|_{L_{2}(\mathbb{R}^{d})}, \tag{3.2}
\]
where \( 2r_{1} = \operatorname{diam} \Omega \).

**Proposition 3.2.** Let \( f \) be a \( \Gamma \)-periodic function in \( \mathbb{R}^{d} \) such that \( f \in L_{2}(\Omega) \). Then the operator \( [f^{\varepsilon}] S_{\varepsilon} \) is continuous in \( L_{2}(\mathbb{R}^{d}) \), and
\[
\| [f^{\varepsilon}] S_{\varepsilon} \|_{L_{2}(\mathbb{R}^{d})} \leq |\Omega|^{-1/2} \| f \|_{L_{2}(\Omega)}, \quad \varepsilon > 0.
\]

3.2. Another approximation of the operator \( (B_{\varepsilon} + \lambda_{0}Q_{0}^{\varepsilon})^{-1} \). We put
\[
(\tilde{K}(\varepsilon) = \left( [\Lambda^{\varepsilon}]b(D) + [\Lambda^{\varepsilon}] \right) S_{\varepsilon}, \quad \varepsilon > 0, \tag{3.3}
\]
The operator \( \tilde{K}(\varepsilon) \) is a continuous mapping of \( L_{2}(\mathbb{R}^{d}; C^{n}) \) to \( H^{1}(\mathbb{R}^{d}; C^{n}) \).

Along with Theorem 1.10 the following result is true.

**Theorem 3.3.** Suppose that the assumptions of Theorem 1.9 are satisfied. Let \( \tilde{K}(\varepsilon) \) be defined by (3.2). Then for \( 0 < \varepsilon \leq 1 \) we have
\[
\| (B_{\varepsilon} + \lambda_{0}Q_{0}^{\varepsilon})^{-1} - (B^{0} + \lambda_{0}Q_{0})^{-1} - \varepsilon \tilde{K}(\varepsilon) \|_{L_{2}(\mathbb{R}^{d}) \rightarrow H^{1}(\mathbb{R}^{d})} \leq C_{3} \varepsilon.
\]
The constant \( C_{3} \) depends only on the initial data (1.41).
Remark 3.4. Theorems 1.10 and 3.3 show that for homogenization problem in \( \mathbb{R}^d \) different smoothing operators can be involved in the corrector (both \( \Pi_\varepsilon \) and \( S_\varepsilon \) are suitable). However, for homogenization problems in a bounded domain (see, e.g., \[\text{ZhPas}^{-3/4}, \text{PSu}, \text{Su}^{3,4,7}\]), it is more convenient to use the Steklov smoothing. Since we are aimed on application of the results of the present paper to study homogenization problems in a bounded domain \[\text{MSu}^{3,4,7}\], we have turned to the Steklov smoothing.

Remark 3.5. Under the assumptions of Remark 1.11 the constant \( C_3 \) can be taken independent of the parameter \( \vartheta \in (0,1] \).

3.3. Proof of Theorem 3.3. We deduce Theorem 3.3 from Theorem 1.10

Lemma 3.6. For any \( u \in H^1(\mathbb{R}^d) \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\int_{\mathbb{R}^d} |(D\tilde{\Lambda})^\varepsilon(x)|^2 |(\Pi_\varepsilon - S_\varepsilon)u|^2 \, dx \leq \tilde{\beta}_1 \| (\Pi_\varepsilon - S_\varepsilon)u \|_{H^1(\mathbb{R}^d)}^2 + \tilde{\beta}_2 \varepsilon^2 \int_{\mathbb{R}^d} |\tilde{\Lambda}^\varepsilon(x)|^2 |(\Pi_\varepsilon - S_\varepsilon)Du|^2 \, dx.
\]

(3.3)

Proof. By Propositions 1.4, 3.2 and the relation \( \tilde{\Lambda} \in \tilde{H}^1(\mathbb{R}^d) \), all the terms in (3.3) are continuous functionals of \( u \) in the \( H^1(\mathbb{R}^d) \)-norm. Therefore, it suffices to check (3.3) for \( u \in C_0^\infty(\mathbb{R}^d) \).

We fix a function \( F \in C_0^\infty(\mathbb{R}^d \setminus 0) \) such that \( 0 \leq F(t) \leq 1 \), \( F(t) = 1 \) for \( 0 \leq t \leq 1 \), and \( F(t) = 0 \) for \( t \geq 2 \). Next, define the function \( F_R(x) := F(|x|/R) \) in \( \mathbb{R}^d \) depending on the parameter \( R > 0 \). If \( u \in C_0^\infty(\mathbb{R}^d) \), then \( F_R(\Pi_\varepsilon - S_\varepsilon)u \in C_0^\infty(\mathbb{R}^d) \). Hence, by Lemma 2.4

\[
\int_{\mathbb{R}^d} |(D\tilde{\Lambda})^\varepsilon|^2 |F_R(\Pi_\varepsilon - S_\varepsilon)u|^2 \, dx \leq \tilde{\beta}_1 \| F_R(\Pi_\varepsilon - S_\varepsilon)u \|_{L^2(\mathbb{R}^d)}^2
\]

\[+ \tilde{\beta}_1 \sum_{j=1}^d \int_{\mathbb{R}^d} |(\partial_j F_R)(\Pi_\varepsilon - S_\varepsilon)u + F_R(\Pi_\varepsilon - S_\varepsilon)\partial_j u|^2 \, dx\]

\[+ \tilde{\beta}_2 \varepsilon^2 \sum_{j=1}^d \int_{\mathbb{R}^d} |\tilde{\Lambda}^\varepsilon|^2 |(\partial_j F_R)(\Pi_\varepsilon - S_\varepsilon)u + F_R(\Pi_\varepsilon - S_\varepsilon)\partial_j u|^2 \, dx\]

Using the estimate \( \max |\partial_j F_R| \leq c/R \) and passing to the limit as \( R \to \infty \), we deduce (3.3) with the help of the Lebesgue theorem. \( \square \)

Now we are ready to prove Theorem 3.3. Obviously,

\[
\varepsilon \| K(\varepsilon) - \tilde{K}(\varepsilon) \|_{L^2 \to H^1} \leq \varepsilon \| |\Lambda^\varepsilon| (\Pi_\varepsilon - S_\varepsilon)b(D)(B^0 + \lambda_0 \overline{Q_0})^{-1} \|_{L^2 \to H^1} + \varepsilon \| |\tilde{\Lambda}^\varepsilon| (\Pi_\varepsilon - S_\varepsilon)(B^0 + \lambda_0 \overline{Q_0})^{-1} \|_{L^2 \to H^1}.
\]

(3.4)

For the first term in the right-hand side of (3.4), we have:

\[
\varepsilon \| |\Lambda^\varepsilon| (\Pi_\varepsilon - S_\varepsilon)b(D)(B^0 + \lambda_0 \overline{Q_0})^{-1} \|_{L^2 \to H^1} \leq \varepsilon \| |\Lambda^\varepsilon| (\Pi_\varepsilon - S_\varepsilon)b(D) \|_{H^2 \to H^1} \| (B^0 + \lambda_0 \overline{Q_0})^{-1} \|_{L^2 \to H^2}.
\]

(3.5)
By \textbf{(1.44)},
\[
\| (B^0 + \lambda_0 \mathcal{Q}_0)^{-1} \|_{L_2(\mathbb{R}^d) \rightarrow H^2(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d} (|\xi|^2 + 1) |L(\xi)|^{-1} \leq \varepsilon_*^{-1}. \tag{3.6}
\]
According to \textbf{[PSu] Lemma 3.5],
\[
\varepsilon \|[\Lambda^{\varepsilon}](\Pi - S_\varepsilon) \tilde{b}(D)\|_{H^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_{\Lambda} \varepsilon, \tag{3.7}
\]
where the constant $C_\Lambda$ depends only on $m, d, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \alpha_0, \alpha_1$, and the parameters of the lattice $\Gamma$. Combining \textbf{(3.5)}, \textbf{(3.7)}, we arrive at
\[
\varepsilon \|[\Lambda^{\varepsilon}](\Pi - S_\varepsilon) \tilde{b}(D)(B^0 + \lambda_0 \mathcal{Q}_0)^{-1} \|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq \varepsilon_c^{-1} C_{\Lambda} \varepsilon. \tag{3.8}
\]
Now we estimate the second term in the right-hand side of \textbf{(3.4)}. By \textbf{(3.6)},
\[
\varepsilon \|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon)(B^0 + \lambda_0 \mathcal{Q}_0)^{-1} \|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq \varepsilon \varepsilon_*^{-1} \|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon)\|_{H^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)}. \tag{3.9}
\]
Let $\Phi \in H^2(\mathbb{R}^d; C_0)$. Then
\[
\varepsilon \|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \Phi\|_{H^1(\mathbb{R}^d)} \leq \varepsilon \|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \Phi\|_{L_2(\mathbb{R}^d)} + \|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \tilde{b}(D) \Phi\|_{L_2(\mathbb{R}^d)}. \tag{3.10}
\]
Using Propositions \textbf{1.2} \textbf{3.2} and taking \textbf{(1.36)} into account, we obtain
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2}(2r_0)^{-1}C_a \varepsilon n^{1/2}\alpha_0^{-1} \leq: \tilde{M}_1, \varepsilon > 0, \tag{3.11}
\]
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \Phi\|_{L_2(\mathbb{R}^d)} \leq \tilde{M}_1 \|\Phi\|_{L_2(\mathbb{R}^d)}. \tag{3.12}
\]
By \textbf{(3.11)} and \textbf{(3.12)},
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \Phi\|_{L_2(\mathbb{R}^d)} \leq 2\tilde{M}_1 \|\Phi\|_{L_2(\mathbb{R}^d)}, \tag{3.13}
\]
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \tilde{b}(D) \Phi\|_{L_2(\mathbb{R}^d)} \leq 2\tilde{M}_1 \|\tilde{b}(D) \Phi\|_{L_2(\mathbb{R}^d)}. \tag{3.14}
\]
For $0 < \varepsilon \leq 1$, the second term in the right-hand side of \textbf{(3.10)} is estimated with the help of Lemma \textbf{3.6}
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \Phi\|_{L_2(\mathbb{R}^d)}^2 \leq \beta_1 \|\tilde{\Lambda}^{\varepsilon} \Phi\|_{H^1(\mathbb{R}^d)^2}^2
\]
\[
+ \beta_2 \varepsilon^2 \int_{\mathbb{R}^d} |\tilde{\Lambda}^{\varepsilon}|^2 |(\Pi - S_\varepsilon) \tilde{b}(D) \Phi|^2 dx. \tag{3.15}
\]
Using Propositions \textbf{1.1} and \textbf{3.1} we obtain
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \Phi\|_{H^1(\mathbb{R}^d)} \leq \varepsilon (r_0^{-1} + r_1) \|\Phi\|_{H^2(\mathbb{R}^d)}. \tag{3.16}
\]
Together with \textbf{(3.14)} and \textbf{(3.15)} this implies
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon) \Phi\|_{L_2(\mathbb{R}^d)} \leq \varepsilon \left( \beta_1 (r_0^{-1} + r_1)^2 + 4\beta_2 \tilde{M}_1^2 \right)^{1/2} \|\Phi\|_{H^2(\mathbb{R}^d)}. \tag{3.16}
\]
Combining \textbf{(3.10)}, \textbf{(3.13)}, \textbf{(3.14)}, and \textbf{(3.16)}, we conclude that
\[
\|[\tilde{\Lambda}^{\varepsilon}](\Pi - S_\varepsilon)\|_{H^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_{\Lambda} \varepsilon, \tag{3.17}
\]
where \( C_\Lambda = 4\tilde{M}_1 + (\tilde{\beta}_1(r_0^{-1} + r_1)^2 + 4\tilde{\beta}_2\tilde{M}_1^2)^{1/2} \). Relations (3.9) and (3.17) yield
\[
\varepsilon \|\hat{\Lambda}^\varepsilon(\Pi_\varepsilon - S_\varepsilon)(B^0 + \lambda_0\bar{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \tilde{c}_*C_\Lambda^\varepsilon. \tag{3.18}
\]
Finally, applying Theorem 4.10 and estimates (3.4), (3.8), (3.18), we complete the proof.

\[\Box\]

4. Main results

4.1. Formulation of the results. In the present subsection, we formulate the main results of the paper.

**Theorem 4.1.** Suppose that the assumptions of Subsections 1.3–1.10 are satisfied. Let \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \), \( \zeta = |\zeta|e^{i\phi}, \phi \in (0, 2\pi) \), and let \( |\zeta| \geq 1 \). We put
\[
c(\phi) = \begin{cases} |\sin \phi|^{-1} , & \phi \in (0, \pi/2) \cup (3\pi/2, 2\pi); \\ 1 , & \phi \in [\pi/2, 3\pi/2]. \end{cases} \tag{4.1}
\]
Then for \( 0 < \varepsilon \leq 1 \) we have
\[
\|(B_\varepsilon - \zeta\bar{Q}_0^\varepsilon)^{-1} - (B^0 - \zeta\bar{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_4|\phi|^2|\zeta|^{-1/2}. \tag{4.2}
\]
The constant \( C_4 \) depends only on the initial data (1.40).

To formulate the result about approximation of the operator \((B_\varepsilon - \zeta\bar{Q}_0^\varepsilon)^{-1}\) in the \((L_2 \to H^1)\)-operator norm, we introduce a corrector
\[
K(\varepsilon; \zeta) = \left([\Lambda^\varepsilon]b(D) + [\hat{\Lambda}^\varepsilon]\right)S_\varepsilon(B^0 - \zeta\bar{Q}_0)^{-1}. \tag{4.3}
\]
Here \( S_\varepsilon \) is the Steklov smoothing operator given by (3.1). The operator (4.3) is a continuous mapping of \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) to \( H^1(\mathbb{R}^d; \mathbb{C}^n) \). This follows from Proposition 3.2 and the relations \( \Lambda, \hat{\Lambda} \in \tilde{H}^1(\Omega) \).

**Theorem 4.2.** Suppose that the assumptions of Theorem 4.1 are satisfied. Let \( K(\varepsilon; \zeta) \) be the operator given by (4.3). Then for \( 0 < \varepsilon \leq 1 \) and \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \), \( |\zeta| \geq 1 \), we have
\[
\|(B_\varepsilon - \zeta\bar{Q}_0^\varepsilon)^{-1} - (B^0 - \zeta\bar{Q}_0)^{-1} - \varepsilon K(\varepsilon; \zeta)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_5|\phi|^2\varepsilon|\zeta|^{-1/2}, \tag{4.4}
\]
\[
\|D\left((B_\varepsilon - \zeta\bar{Q}_0^\varepsilon)^{-1} - (B^0 - \zeta\bar{Q}_0)^{-1} - \varepsilon K(\varepsilon; \zeta)\right)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_6|\phi|^2\varepsilon. \tag{4.5}
\]
The constants \( C_5 \) and \( C_6 \) are controlled in terms of the problem data (1.40).

Theorem 4.2 directly implies the following corollary.

**Corollary 4.3.** Under the assumptions of Theorem 4.2, we have
\[
\|(B_\varepsilon - \zeta\bar{Q}_0^\varepsilon)^{-1} - (B^0 - \zeta\bar{Q}_0)^{-1} - \varepsilon K(\varepsilon; \zeta)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq (C_5 + C_6)|\phi|^2\varepsilon, \quad 0 < \varepsilon \leq 1.
\]
Using Theorem 4.2, we obtain the result about approximation of the operator \( g^\varepsilon b(D)(B_\varepsilon - \zeta Q_0^{-1}) \) (corresponding to the "multiflux").

**Theorem 4.4.** Suppose that the assumptions of Theorem 4.1 are satisfied. Let \( \tilde{g}(x) \) be the matrix-valued function defined by (1.25). Denote

\[
G(\varepsilon; \zeta) := \tilde{g}^* S_2 b(D)(B_0^{-1} - \zeta Q_0) + g^\varepsilon (b(D)\tilde{A})^\varepsilon S_2 (B_0^{-1} - \zeta Q_0)^{-1}. \tag{4.6}
\]

Then for \( 0 < \varepsilon \leq 1 \) and \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \ |\zeta| \geq 1 \), we have

\[
\|g^\varepsilon b(D)(B_\varepsilon - \zeta Q_0^{-1}) - G(\varepsilon; \zeta)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_7 c(\phi)^2 \varepsilon. \tag{4.7}
\]

The constant \( C_7 \) depends only on the initial data (1.40).

4.2. **Discussion.** Homogenization of the resolvent of the operator \( A_\varepsilon \) in dependence of the spectral parameter was studied in [Su7]. The two-parametric error estimates that we obtain (Theorems 4.1, 4.2, and 4.4) have the same behavior as the estimates from [Su7] Theorems 2.2, 2.4, and 2.6.

However, there is a difference between our results and the results of [Su7]. Approximations of the resolvent \( (A_\varepsilon - \zeta I)^{-1} \) with error estimates of the form (4.2), (4.4), (4.5), and (4.7) were proved for any \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \). For the operator \( B_\varepsilon \) the situation is different: in Theorems 4.1, 4.2, and 4.4 it is assumed in addition that \( |\zeta| \geq 1 \). This restriction is related to the presence of the lower order terms in \( B_\varepsilon \). Below in Section 5, we extend the domain of admissible values of \( \zeta \), but the order of estimates (with respect to \( \zeta \)) will be less precise.

5. **Proof of Theorem 4.1**

5.1. **The operator** \( B(\varepsilon; \vartheta) \). The proof of Theorems 4.1 and 4.2 is based on application of Theorems 4.1 and 5.1 to the auxiliary family of operators depending on the additional parameter \( 0 < \vartheta \leq 1 \). In Subsections 5.1–5.3, we introduce the necessary objects.

Suppose that the assumptions of Subsections 1.3–1.5 are satisfied. Let \( Q_0 \) be the matrix-valued function introduced in Subsection 1.10 and let \( c_5 = (c_0 + c_4)\|Q_0^{-1}\|_{L^\infty} \). In \( L_2(\mathbb{R}^d; \mathbb{C}^n) \), consider the quadratic form

\[
b(\varepsilon; \vartheta)[u, u] = a[u, u] + 2\vartheta \varepsilon \text{Re} (\mathcal{Y} u, \mathcal{Y}_2 u)_{L_2(\mathbb{R}^d)} + \vartheta^2 c_5 (Q_0 u, u)_{L_2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \tag{5.1}
\]

Here \( \varepsilon > 0 \) and \( 0 < \vartheta \leq 1 \). We assume that \( 0 < \varepsilon \vartheta \leq 1 \).

Note that, by (1.15) and (5.1), we have \( b(\varepsilon; \vartheta)[u, u] = b(\varepsilon \vartheta)[u, u] + c_5 \varepsilon^2 \vartheta^2 (Q_0 u, u)_{L_2} \) for any \( u \in H^1(\mathbb{R}^d; \mathbb{C}^n) \). Combining this with (1.17), (1.18) and the formula for \( c_5 \), we see that the form \( b(\varepsilon; \vartheta) \) is closed and nonnegative, and

\[
c_5 \|Du\|^2_{L_2(\mathbb{R}^d)} \leq b(\varepsilon; \vartheta)[u, u] \leq C_s \|Du\|^2_{L_2(\mathbb{R}^d)} \tag{5.2}
\]

\[
+ (C(1 + c_3 + c_5 \|Q_0\|_\infty \varepsilon^2) u^2)^2_{L_2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq \vartheta^{-1}.
\]
By $B(\varepsilon; \vartheta)$ we denote the selfadjoint operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ generated by this form. Formally, we have

$$B(\varepsilon; \vartheta) = b(D)^* g(x) b(D) + \varepsilon \sum_{j=1}^{d} (a_j(x) D_j + D_j a_j(x)^*) + \varepsilon^2 Q(x) + \varepsilon^2 c_5 Q_0(x).$$

(5.3)

5.2. The operator $B_\varepsilon(\vartheta)$. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the quadratic form

$$b_\varepsilon(\vartheta)[u,u] = a_\varepsilon[u,u] + 2 \varepsilon^2 \text{Re}(\langle \mathcal{V} u, \mathcal{V}_2 \varepsilon u \rangle_{L_2(\mathbb{R}^d)}) + \varepsilon^2 q_\varepsilon[u,u] + \varepsilon^2 c_5(Q_0 u, u)_{L_2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

where $0 < \varepsilon \leq 1$ and $0 < \varepsilon \leq \vartheta^{-1}$. We have

$$b_\varepsilon(\vartheta)[u,u] = \varepsilon^{-2} b(\varepsilon; \vartheta)[T_\varepsilon u, T_\varepsilon u], \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

(5.4)

(5.5)

where $T_\varepsilon$ is the scaling transformation given by (1.19). Relations (5.2) and (5.5) imply that the form (5.4) is closed and nonnegative, and satisfies the estimates

$$c_\varepsilon \|Du\|_{L_2(\mathbb{R}^d)}^2 \leq b_\varepsilon(\vartheta)[u,u] \leq c_0 \|u\|_{H^1(\mathbb{R}^d)}^2,$$

(5.6)

$$u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq \vartheta^{-1},$$

where $c_0 = \max\{C_s; C(1) + c_3 + c_5\|Q_0\|_{L_\infty}\}$. Let $B_\varepsilon(\vartheta)$ be the selfadjoint operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ corresponding to the form (5.5).

We will apply Theorem 1.9 to approximate the generalized resolvent $(B_\varepsilon(\vartheta) + \lambda_0 Q_0)^{-1}$, where $\lambda_0$ is given by (1.39) (see Remark 1.11).

5.3. The operators $B^0(\vartheta)$ and $B^0(\varepsilon; \vartheta)$. Obviously (see (1.23)–(1.25), (1.31)–(1.34), and (1.41)), the effective operator for $B_\varepsilon(\vartheta)$ takes the form

$$B^0(\vartheta) = A^0 + \vartheta \left( -b(D)^* V - V^* b(D) + \sum_{j=1}^{d} (a_j + a_j^*) D_j \right) + \vartheta^2 (Q + c_5 Q_0 - W).$$

According to (1.42), the quadratic form $b^0(\vartheta)$ of the operator $B^0(\vartheta)$ satisfies

$b^0(\vartheta)[u,u] \geq c_\varepsilon \|Du\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).$

(5.7)

This is equivalent to the following estimate for the symbol $L_0(\xi; \vartheta)$ of the operator $B^0(\vartheta)$:

$$L_0(\xi; \vartheta) \geq c_\varepsilon |\xi|^2 \mathbf{1}_n, \quad \xi \in \mathbb{R}^d, \quad 0 < \vartheta \leq 1.$$  

(5.8)

Hence, by (1.39), the symbol

$$L(\xi; \vartheta) = b(\xi)^* g^0 b(\xi) - \vartheta b(\xi)^* V - \vartheta V^* b(\xi)$$

$$+ \vartheta \sum_{j=1}^{d} (a_j + a_j^*) \xi_j + \vartheta^2 (Q + c_5 Q_0) - \vartheta^2 W + \lambda_0 Q_0$$

is the symbol of $B^0(\vartheta)$. 

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of the operator \( B^0(\vartheta) + \lambda_0 \overline{Q}_0 \) satisfies
\[
L(\xi; \vartheta) \geq \tilde{c}_s (|\xi|^2 + 1) \mathbf{1}_n, \quad \xi \in \mathbb{R}^d, \quad 0 < \vartheta \leq 1, \quad \tilde{c}_s = \min\{c_s; 2c_4\}. \tag{5.9}
\]
Note that
\[
B^0(\vartheta) = \varepsilon^{-2} T^*_\varepsilon B^0(\varepsilon; \vartheta) T_\varepsilon,
\]
where \( B^0(\varepsilon; \vartheta) \) is the selfadjoint operator in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) given by
\[
B^0(\varepsilon; \vartheta) = A^0 + \varepsilon \partial \left( -b(D)^* V - V^* b(D) + \sum_{j=1}^{d} (a_j + a_j^*) D_j \right) \tag{5.10}
\]
\[+ \varepsilon^2 \vartheta^2 (Q - W + c_5 \overline{Q}_0).\]

5.4. The operators \( \overline{B}_\varepsilon(\vartheta) \) and \( \overline{B}^0(\vartheta) \). We factorize the matrices \( Q_0(x) \) and \( \overline{Q}_0 \) as follows:
\[
Q_0(x) = (f(x)^*)^{-1} f(x)^{-1}, \tag{5.11}
\]
\[
\overline{Q}_0 = f_0^{-2}. \tag{5.12}
\]
Let \( \overline{B}_\varepsilon(\vartheta) \) be the selfadjoint operator in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) generated by the quadratic form
\[
\overline{b}_\varepsilon(\vartheta)[u, u] := b_\varepsilon(\vartheta)[f^* u, f^* u] \tag{5.13}
\]
derived on the domain
\[
\text{Dom} \overline{b}_\varepsilon(\vartheta) = \{ u \in L_2(\mathbb{R}^d; \mathbb{C}^n) : f^* u \in H^1(\mathbb{R}^d; \mathbb{C}^n) \}. \tag{5.14}
\]
Here \( 0 < \vartheta \leq 1 \) and \( 0 < \varepsilon \leq \vartheta^{-1} \). Since the operator \( B_\varepsilon(\vartheta) \) is nonnegative, the operator \( \overline{B}_\varepsilon(\vartheta) \) is also nonnegative.

Let \( \overline{B}^0(\vartheta) = f_0 B^0(\vartheta) f_0 \). Note that
\[
(B_\varepsilon(\vartheta) - \zeta \overline{Q}_0)^{-1} = f^*(\overline{B}_\varepsilon(\vartheta) - \zeta I)^{-1}(f^*)^*, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \tag{5.15}
\]
\[
(B^0(\vartheta) - \zeta \overline{Q}_0)^{-1} = f_0 (\overline{B}^0(\vartheta) - \zeta I)^{-1} f_0, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}_+. \tag{5.16}
\]

5.5. Proof of Theorem 4.1. Applying Theorem 1.9 to the operator \( B_\varepsilon(\vartheta) \) and taking Remark 1.11 into account, we obtain
\[
\|(B_\varepsilon(\vartheta) + \lambda_0 \overline{Q}_0)^{-1} - (B^0(\vartheta) + \lambda_0 \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \lambda_1 \varepsilon, \quad 0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq 1. \tag{5.17}
\]
Now we carry this estimate over to all \( 0 < \varepsilon \leq \vartheta^{-1} \). By (5.14),
\[
\|(B_\varepsilon(\vartheta) + \lambda_0 \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \|f\|_{L_\infty}^2 \|(\overline{B}_\varepsilon(\vartheta) + \lambda_0 I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \lambda_0^{-1} \|f\|^2_{L_\infty}. \tag{5.18}
\]
Similarly, (5.15) and (5.12) imply that
\[
\|(B^0(\vartheta) + \lambda_0 \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \lambda_0^{-1} \|f\|^2_{L_\infty}. \tag{5.19}
\]
From (5.17) and (5.18) it follows that the left-hand side of (5.16) does not exceed $2\lambda_0^{-1}\|f\|_{L_\infty}^2 \varepsilon$ for $1 < \varepsilon \leq \vartheta^{-1}$. Together with (5.16) this implies

$$
\|(B_\varepsilon(\vartheta) + \lambda_0 Q_0^{-1} - (B^0(\vartheta) + \lambda_0 Q_0^{-1})L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \leq \tilde{C}_1 \varepsilon, \\
0 < \vartheta \leq 1, 0 < \varepsilon \leq \vartheta^{-1},
$$

(5.19)

with $\tilde{C}_1 = \max\{C_1; 2\lambda_0^{-1}\|f\|_{L_\infty}^2\}$.

Now we obtain an analog of estimate (5.19) for $(B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}$, where $\tilde{\zeta} = e^{i\phi}$ with $\phi \in (0, 2\pi)$. We rely on the identity

$$
(B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1} - (B^0(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}
= (B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}(B_\varepsilon(\vartheta) + \lambda_0 Q_0^{-1})
\times ((B_\varepsilon(\vartheta) + \lambda_0 Q_0^{-1})^{-1} - (B^0(\vartheta) + \lambda_0 Q_0^{-1})^{-1})
\times (B^0(\vartheta) + \lambda_0 Q_0^{-1})(B^0(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}
+ (\lambda_0 + \tilde{\zeta})(B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}(Q_0 - \bar{Q}_0)(B^0(\vartheta) - \tilde{\zeta} Q_0^{-1}).
$$

(5.20)

By (5.14), we have

$$
\|(B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1})(B_\varepsilon(\vartheta) + \lambda_0 Q_0^{-1})\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}
= \|f \tilde{\zeta}(\mathbf{\tilde{B}}_\varepsilon(\vartheta) - \tilde{\zeta} I)^{-1}(\mathbf{\tilde{B}}_\varepsilon(\vartheta) + \lambda_0 I)(f \tilde{\zeta})^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}
\leq \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \|\tilde{B}_\varepsilon(\vartheta) - \tilde{\zeta} I^{-1}(\tilde{B}_\varepsilon(\vartheta) + \lambda_0 I)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}.
$$

(5.21)

Thus, if $\tilde{B}_\varepsilon(\vartheta) \geq 0$, then

$$
\|(\mathbf{\tilde{B}}_\varepsilon(\vartheta) - \tilde{\zeta} I)^{-1}(\mathbf{\tilde{B}}_\varepsilon(\vartheta) + \lambda_0 I)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \sup_{\nu \geq 0} \frac{\nu + \lambda_0}{\nu + \tilde{\zeta}}
\leq \sup_{\nu \geq 0} \frac{\nu + \lambda_0}{\nu + 1} \frac{\nu + 1}{\nu - \tilde{\zeta}} \leq 2(1 + \lambda_0)c(\phi).
$$

(5.22)

Here $c(\phi)$ is given by (4.11). Relations (5.21) and (5.22) imply that

$$
\|(B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1})(B_\varepsilon(\vartheta) + \lambda_0 Q_0^{-1})\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}
\leq 2(1 + \lambda_0)\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} c(\phi).
$$

(5.23)

Similarly,

$$
\|(B^0(\vartheta) + \lambda_0 Q_0^{-1})(B^0(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}
\leq 2(1 + \lambda_0)\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} c(\phi).
$$

(5.24)

Let us estimate the norm of the second term in the right-hand side of (5.20):

$$
\|\tilde{\zeta} + \lambda_0)(B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1})(Q_0 - \bar{Q}_0)(B^0(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}
\leq (1 + \lambda_0)(\|B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^{-1}\|_{H^{-1}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}
\times \|Q_0 - \bar{Q}_0\|_{H^1(\mathbb{R}^d) \to H^{-1}(\mathbb{R}^d)} \|(B^0(\vartheta) - \tilde{\zeta} Q_0^{-1})^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)}.
$$

(5.25)
By the duality argument, we have
\[
\|(B_\varepsilon (\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = \|(B_\varepsilon (\vartheta) - \tilde{\zeta}^* Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)}.
\] (5.26)

By (5.14),
\[
\|(B_\varepsilon (\vartheta) - \tilde{\zeta}^* Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \|f\|_{L_\infty}^2 \|(\tilde{B}_\varepsilon (\vartheta) - \tilde{\zeta}^* I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\
\leq \|f\|_{L_\infty}^2 c(\phi).
\] (5.27)

Next, the lower estimate (5.6), (5.13), and (5.14) imply that
\[
\|D(B_\varepsilon (\vartheta) - \tilde{\zeta}^* Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\
\leq c_1^{-1/2} \|B_\varepsilon (\vartheta)^{1/2} (B_\varepsilon (\vartheta) - \tilde{\zeta}^* Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\
= c_1^{-1/2} \|\tilde{B}_\varepsilon (\vartheta)^{1/2} (\tilde{B}_\varepsilon (\vartheta) - \tilde{\zeta}^* I)^{-1} (f^\varepsilon)^*\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\
\leq c_1^{-1/2} \|f\|_{L_\infty} \sup_{\nu \geq 0} \left| \frac{\nu^{1/2}}{\nu - \tilde{\zeta}^*} \right| \leq c_1^{-1/2} \|f\|_{L_\infty} c(\phi).
\] (5.28)

Combining this with (5.26) and (5.27), we see that
\[
\|(B_\varepsilon (\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1}\|_{H^{-1}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_1 c(\phi),
\] (5.29)

where \(C_1 := \|f\|_{L_\infty}^2 + c_1^{-1/2} \|f\|_{L_\infty}^2\).

By analogy with (5.27) and (5.28), we estimate the \((L_2 \rightarrow H^1)\)-norm of the operator \((B^0(\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1}\), using (5.7), (5.12), and (5.15). This yields
\[
\|(B^0(\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_2 c(\phi).
\] (5.30)

Relations (2.14), (5.25), (5.29), and (5.30) imply that
\[
\|(\tilde{\zeta} + \lambda_0)(B_\varepsilon (\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1} (Q_0^\varepsilon - Q_0^\varepsilon)(B^0(\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_2 \varepsilon c(\phi)^2,
\]
where \(C_2 = (1 + \lambda_0)C_{Q_0} \mathcal{C}_1 \mathcal{C}_2^2\). Combining this with (5.19), (5.20), (5.23), and (5.24), we obtain
\[
\|(B_\varepsilon (\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1} - (B^0(\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_4 \varepsilon c(\phi)^2,
\] (5.31)

with \(C_4 = 4(1 + \lambda_0)^2 \|f\|_{L_\infty}^2 \|f^{-1}\|_{L_\infty}^2 \tilde{C}_1 + \mathcal{C}_2\).

By the scaling transformation, from (5.31) we deduce
\[
\|(B(\varepsilon; \vartheta) - \tilde{\zeta}^2 Q_0)^{-1} - (B^0(\varepsilon; \vartheta) - \tilde{\zeta}^2 Q_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_4 \varepsilon c(\phi)^2 \varepsilon^{-1}.
\] (5.32)

In this estimate, we substitute \(\tilde{\varepsilon}|\zeta|^{1/2}\) in place of \(\varepsilon\), assuming that \(0 < \tilde{\varepsilon} \leq 1\), \(\zeta = |\zeta| e^{i\phi} \in \mathbb{C} \setminus \mathbb{R}_+, \phi \in (0, 2\pi), |\zeta| \geq 1\), and put \(\vartheta = |\zeta|^{-1/2}\). Then the
conditions \(0 < \vartheta \leq 1\) and \(0 < \varepsilon \vartheta \leq 1\) are satisfied. We have (see (5.3) and (5.10))
\[
B(\varepsilon|\zeta|^{1/2}; |\zeta|^{-1/2}) = B(\varepsilon; 1), \\
B^0(\varepsilon|\zeta|^{1/2}; |\zeta|^{-1/2}) = B^0(\varepsilon; 1).
\]
Therefore, (5.32) implies that
\[
\| (B(\varepsilon; 1) - \zeta \overline{\varepsilon^2 Q_0})^{-1} - (B^0(\varepsilon; 1) - \zeta \overline{\varepsilon^2 Q_0})^{-1} \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} 
\leq C_4 c(\phi) \varepsilon^{-1} |\zeta|^{-1/2}, \quad 0 < \varepsilon \leq 1, \quad \zeta = |\zeta| e^{i\phi}, \quad |\zeta| \geq 1, \quad 0 < \phi < 2\pi.
\]
Renaming \(\varepsilon := \varepsilon\) and applying the inverse scaling transformation, we arrive at (4.2). This completes the proof of Theorem 4.1.

6. Proof of Theorems 4.2 and 4.4

6.1. The operator \(K(\varepsilon; \vartheta)\). As above, assume that \(0 < \vartheta \leq 1\) and \(0 < \varepsilon \leq \vartheta^{-1}\). Consider the generalized resolvent \((B_0(\vartheta) + \lambda_0 Q_0)^{-1}\). Taking into account the form of the problems for \(\Lambda\) and \(\widetilde{\Lambda}\) (see (1.23) and (1.31)), we see that the analog of the corrector (3.2) for the generalized resolvent \((B_0(\vartheta) + \lambda_0 Q_0)^{-1}\) takes the form
\[
K(\varepsilon; \vartheta) = \left( [\Lambda^c] b(D) + \vartheta [\Lambda^c] \right) S_\varepsilon (B^0(\vartheta) + \lambda_0 \overline{Q_0})^{-1}.
\]
(6.1)
The operator \(K(\varepsilon; \vartheta)\) is a continuous mapping of \(L^2(\mathbb{R}^d; \mathbb{C}^n)\) to \(H^1(\mathbb{R}^d; \mathbb{C}^n)\). This follows from the next lemma.

Lemma 6.1. Let \(K(\varepsilon; \vartheta)\) be the operator given by (6.1). Then for \(0 < \vartheta \leq 1\) and \(\varepsilon > 0\) the operator \(K(\varepsilon; \vartheta)\) is a continuous mapping of \(L^2(\mathbb{R}^d; \mathbb{C}^n)\) to \(H^1(\mathbb{R}^d; \mathbb{C}^n)\), and
\[
\|K(\varepsilon; \vartheta)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C^{(1)}_K, \\
\|\varepsilon D K(\varepsilon; \vartheta)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C^{(2)}_K \varepsilon + C^{(3)}_K.
\]
(6.2) (6.3)
The constants \(C^{(1)}_K\), \(C^{(2)}_K\), and \(C^{(3)}_K\) depend only on the initial data (1.40).

Proof. First, we estimate the \((L^2 \to L^2)\)-norm of the corrector:
\[
\|K(\varepsilon; \vartheta)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \| [\Lambda^c] S_\varepsilon \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \|b(D)(B^0(\vartheta) + \lambda_0 \overline{Q_0})^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} 
\]  
\[ + \| [\Lambda^c] S_\varepsilon \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \|(B^0(\vartheta) + \lambda_0 \overline{Q_0})^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}.\]
(6.4)
Proposition 3.2 and (1.26) imply that
\[
\| [\Lambda^c] S_\varepsilon \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq M_1.
\]
(6.5)
According to (1.4),
\[ \|b(D)(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \alpha_1^{1/2}\|D(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}. \]  
(6.6)

Since the symbol of the operator \( B^0(\vartheta) + \lambda_0Q_0 \) satisfies (5.9), we have
\[ \|D(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \tilde{c}_*^{-1} \sup_{\xi \in \mathbb{R}^d} \frac{|\xi|}{|\xi|^2 + 1} \leq (2\tilde{c}_*)^{-1}. \]  
(6.7)

Relations (3.12), (5.18), and (6.4)–(6.7) imply (6.2) with the constant \( C^{(1)}_K = \alpha_1^{1/2}(2\tilde{c}_*)^{-1}M_1 + \lambda_0^{-1}\tilde{M}_1\|f\|^2_{L^\infty}. \)

Now we check (6.3). Obviously,
\[ \varepsilon D_jK(\varepsilon; \vartheta) = [(D\Lambda)^\varepsilon]S_\varepsilon b(D)(B^0(\vartheta) + \lambda_0Q_0)^{-1} \]
\[ + \varepsilon[\Lambda^\varepsilon]S_\varepsilon b(D)D_j(B^0(\vartheta) + \lambda_0Q_0)^{-1} \]
\[ + \varepsilon[(D\Lambda)^\varepsilon]S_\varepsilon B^0(\vartheta) + \lambda_0Q_0)^{-1} \]
\[ + \varepsilon\theta[(D\Lambda)^\varepsilon]S_\varepsilon D_j(B^0(\vartheta) + \lambda_0Q_0)^{-1}. \]

Hence,
\[ \|\varepsilon DK(\varepsilon; \vartheta)\|^2_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \]
\[ \leq 4\|[(D\Lambda)^\varepsilon]S_\varepsilon b(D)(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|^2_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \]
\[ + 4\|\varepsilon[\Lambda^\varepsilon]S_\varepsilon b(D)D_j(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|^2_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \]
\[ + 4\|[(D\Lambda)^\varepsilon]S_\varepsilon B^0(\vartheta) + \lambda_0Q_0)^{-1}\|^2_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \]
\[ + 4\|\varepsilon\theta[(D\Lambda)^\varepsilon]S_\varepsilon D_j(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|^2_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}. \]  
(6.8)

Applying Proposition 3.2 and (1.27), (1.37), we obtain
\[ \|[(D\Lambda)^\varepsilon]S_\varepsilon\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq M_2, \]  
(6.9)
\[ \|[(D\Lambda)^\varepsilon]S_\varepsilon\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq |\Omega|^{-1/2}C_\alpha n^{-1/2}a_0^{-1}\|g^{-1}\|_{L^\infty} =: \tilde{M}_2. \]  
(6.10)

By (1.4) and (5.8),
\[ \|b(D)D(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \alpha_1^{1/2}c_*^{-1}. \]  
(6.11)

Combining (3.12), (5.18), and (6.5)–(6.11), we arrive at (6.3) with
\[ C^{(2)}_K = (4\alpha_1M_1^2c_*^{-2} + \tilde{M}_1^2\tilde{c}_*^{-2})^{1/2}, \]
\[ C^{(3)}_K = (\alpha_1M_2^2\tilde{c}_*^{-2} + 4\lambda_0^{-2}\|f\|^4_{L^\infty}\tilde{M}_2^2)^{1/2}. \]
6.2. Proof of Theorem 4.2. Applying Theorem 3.3 to $B_\varepsilon(\vartheta)$ and taking Remark 3.5 into account, we obtain

$$
\|(B_\varepsilon(\vartheta) + \lambda_0Q_0)^{-1} - (B^0(\vartheta) + \lambda_0Q_0)^{-1} - \varepsilon K(\varepsilon; \vartheta, \vartheta)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_3 \varepsilon,
$$

$$
0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq 1.
$$

(6.12)

Now we carry this estimate over to all $0 < \varepsilon \leq \vartheta^{-1}$. For $1 < \varepsilon \leq \vartheta^{-1}$ we use rather rough estimates. By analogy with (5.27) and (5.28), we see that

$$
\|(B_\varepsilon(\vartheta) + \lambda_0Q_0)^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_3,
$$

(6.13)

where $C_3 = \lambda_0^{-1} \|f\|_{L_\infty}^2 + \frac{1}{2} \lambda_0^{-1/2} c_*^{-1/2} \|f\|_{L_\infty}$.

Similarly, from (5.7) and (5.12) it follows that

$$
\|(B^0(\vartheta) + \lambda_0Q_0)^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_3.
$$

(6.14)

For $1 < \varepsilon \leq \vartheta^{-1}$ we use Lemma 6.1 and (6.13), (6.14), while for $0 < \varepsilon \leq 1$ we apply (6.12). Then

$$
\|(B_\varepsilon(\vartheta) + \lambda_0Q_0)^{-1} - (B^0(\vartheta) + \lambda_0Q_0)^{-1} - \varepsilon K(\varepsilon; \vartheta, \vartheta)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \hat{C}_3 \varepsilon,
$$

$$
0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1},
$$

(6.15)

where $\hat{C}_3 = \max\{C_3; 2C_3 + C^{(1)}_K + C^{(2)}_K + C^{(3)}_K\}$.

We put

$$
K(\varepsilon; \vartheta; \zeta) := \left( [A^\varepsilon] b(D) + \vartheta [\tilde{A}^\varepsilon] \right) S_\varepsilon (B^0(\vartheta) - \zeta Q_0)^{-1}, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}_+.
$$

(6.16)

Note that $K(\varepsilon; \vartheta; -\lambda_0) = K(\varepsilon; \vartheta)$.

We prove an analog of estimate (6.15) for the operator $(B_\varepsilon(\vartheta) - \tilde{Q}_0)^{-1}$, where $\tilde{\zeta} = e^{i\phi}$ with $\phi \in (0, 2\pi)$, with the help of the identity

$$
(B_\varepsilon(\vartheta) - \tilde{Q}_0)^{-1} - (B^0(\vartheta) - \zeta Q_0)^{-1} - \varepsilon K(\varepsilon; \vartheta; \tilde{\zeta})
$$

$$
= (B_\varepsilon(\vartheta) - \tilde{Q}_0)^{-1}(B_\varepsilon(\vartheta) + \lambda_0Q_0)
$$

$$
\times \left((B_\varepsilon(\vartheta) + \lambda_0Q_0)^{-1} - (B^0(\vartheta) + \lambda_0Q_0)^{-1} - \varepsilon K(\varepsilon; \vartheta; -\lambda_0)\right)
$$

$$
\times (B^0(\vartheta) + \lambda_0Q_0)(B^0(\vartheta) - \zeta Q_0)^{-1}
$$

$$
+ \varepsilon(\lambda_0 + \tilde{\zeta})(B_\varepsilon(\vartheta) - \tilde{Q}_0)^{-1}Q_0K(\varepsilon; \vartheta; \tilde{\zeta})
$$

$$
+ (\lambda_0 + \tilde{\zeta})(B_\varepsilon(\vartheta) - \tilde{Q}_0)^{-1}(Q_0 - Q_0)(B^0(\vartheta) - \zeta Q_0)^{-1}.
$$

(6.17)

Denote the consecutive summands in the right-hand side by $J_l(\varepsilon; \vartheta; \tilde{\zeta})$, $l = 1, 2, 3$. First, we estimate the $(L_2 \to L_2)$-norm of each summand. By (5.11), (6.14), and (6.10),

$$
\|Q_0K(\varepsilon; \vartheta; \tilde{\zeta})\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|f^{-1}\|_{L_\infty}^2 \|K(\varepsilon; \vartheta)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}
$$

$$
\times \|(B^0(\vartheta) + \lambda_0Q_0)(B^0(\vartheta) - \zeta Q_0)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}.
$$

Together with (5.24) and (6.2) this yields
\[ \|Q^0_\varepsilon K(\varepsilon; \vartheta; \hat{C})\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq 2(1 + \lambda_0)C^{(1)}_K \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}^3 c(\phi). \]  
(6.18)

Relations (5.27) and (6.18) imply the following estimate for the operator \( \mathcal{J}_2(\varepsilon; \vartheta; \hat{\zeta}) \):
\[ \|\mathcal{J}_2(\varepsilon; \vartheta; \hat{\zeta})\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq 2\varepsilon(\lambda_0 + 1)^2 \|f\|_{L_\infty}^3 \|f^{-1}\|_{L_\infty}^3 c(\phi)^2. \]

To estimate the \((L_2 \rightarrow L_2)\)-norm of the operator \( \mathcal{J}_1(\varepsilon; \vartheta; \hat{\zeta}) \), we use (5.23), (5.24), and (6.15); to estimate the term \( \mathcal{J}_3(\varepsilon; \vartheta; \hat{\zeta}) \), we apply (2.14), (5.25), (5.29), and (5.30). Then we arrive at
\[ \|(B_\varepsilon(\vartheta) - \hat{\zeta}Q^0_\varepsilon)^{-1} - (B^0(\vartheta) - \hat{\zeta}Q_0)^{-1} - \varepsilon K(\varepsilon; \vartheta; \hat{\zeta})\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_5 \varepsilon c(\phi)^2, \quad 0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1}, \quad \hat{\zeta} = e^{i\phi}, \quad 0 < \phi < 2\pi. \]
(6.19)
Here
\[ C_5 = 4(1 + \lambda_0)^2 \|f\|_{L_\infty}^2 \|f^{-1}\|_{L_\infty}^2 \|f\|_{L_\infty}^3 \|f^{-1}\|_{L_\infty}^3 c(\phi)^2 + (1 + \lambda_0)C_0 C^2_\varepsilon. \]

Now, we apply the operator \( B_\varepsilon(\vartheta)^{1/2} \) to both sides of (6.17):
\[ B_\varepsilon(\vartheta)^{1/2} \left( (B_\varepsilon(\vartheta) - \hat{\zeta}Q^0_\varepsilon)^{-1} - (B^0(\vartheta) - \hat{\zeta}Q_0)^{-1} - \varepsilon K(\varepsilon; \vartheta; \hat{\zeta}) \right) = B_\varepsilon(\vartheta)^{1/2} \mathcal{J}_1(\varepsilon; \vartheta; \hat{\zeta}) + B_\varepsilon(\vartheta)^{1/2} \mathcal{J}_2(\varepsilon; \vartheta; \hat{\zeta}) + B_\varepsilon(\vartheta)^{1/2} \mathcal{J}_3(\varepsilon; \vartheta; \hat{\zeta}), \]
(6.20)
and estimate the \((L_2 \rightarrow L_2)\)-norm of each term on the right.

By (5.13), for any \( \mathbf{w} \in H^1(\mathbb{R}^d; \mathbb{C}^n) \) we have
\[ \|B_\varepsilon(\vartheta)^{1/2} \mathbf{w}\|^2_{L_2(\mathbb{R}^d)} = B_\varepsilon(\vartheta)[\mathbf{w}, \mathbf{w}] \]
\[ = \overline{b}_\varepsilon(\vartheta)|f^\varepsilon|^{-1} \mathbf{w}, f^\varepsilon|^{-1} \mathbf{w}| = \|\overline{B}_\varepsilon(\vartheta)^{1/2}(f^\varepsilon|^{-1} \mathbf{w}|^2_{L_2(\mathbb{R}^d)}. \]
(6.21)

Next, from (5.14) it follows that
\[ (f^\varepsilon)^*(B_\varepsilon(\vartheta) + \lambda_0 Q_0^0) = (\overline{B}_\varepsilon(\vartheta) + \lambda_0 I)(f^\varepsilon)^{-1}. \]
(6.22)

Using (5.14), (5.22), (6.21), and (6.22), we obtain
\[ \|B_\varepsilon(\vartheta)^{1/2} (B_\varepsilon(\vartheta) - \hat{\zeta}Q^0_\varepsilon)^{-1}(B_\varepsilon(\vartheta) + \lambda_0 Q_0^0)\|_{H^1(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \|(\overline{B}_\varepsilon(\vartheta) - \hat{\zeta}I)^{-1}(\overline{B}_\varepsilon(\vartheta) + \lambda_0 I)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \times \|\overline{B}_\varepsilon(\vartheta)^{1/2}(f^\varepsilon)^{-1}\|_{H^1(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \]
\[ \leq 2(\lambda_0 + 1)c(\phi)\|B_\varepsilon(\vartheta)^{1/2}\|_{H^1(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}. \]
(6.23)
Combining the upper estimate (5.0), (5.24), (6.15), and (6.23), we deduce the following estimate for the first term in the right-hand side of (6.20):
\[ \|B_\varepsilon(\vartheta)^{1/2} \mathcal{J}_1(\varepsilon; \vartheta; \hat{\zeta})\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_4 \varepsilon c(\phi)^2, \quad 0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1}, \quad \hat{\zeta} = e^{i\phi}, \quad 0 < \phi < 2\pi, \]
(6.24)
where \( C_4 := 4(1 + \lambda_0)^2 c^2_0 \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}. \)
Next, 
\[
\|B_\varepsilon(\vartheta)^{1/2} f_2(\varepsilon; \vartheta; \hat{\zeta})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq \varepsilon (\lambda_0 + 1) \|B_\varepsilon^*(\vartheta)^{1/2} (B_\varepsilon^*(\vartheta) - \hat{\zeta} Q_0^\varepsilon)^{-1}\|_{L^2 \to L^2} \|Q_0^\varepsilon K(\varepsilon; \vartheta; \hat{\zeta})\|_{L^2 \to L^2}. 
\] (6.25)

By (5.14) and (6.21),
\[
\|B_\varepsilon(\vartheta)^{1/2} (B_\varepsilon(\vartheta) - \hat{\zeta} Q_0^\varepsilon)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
= \|\tilde{B}_\varepsilon(\vartheta)^{1/2} (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1} (f^\varepsilon)^*\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}. 
\] (6.26)

Obviously,
\[
\|\tilde{B}_\varepsilon(\vartheta)^{1/2} (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \sup_{\nu \geq 0} \frac{\nu^{1/2}}{|\nu - \zeta|} \leq c(\phi). 
\] (6.27)

Relations (6.26) and (6.27) imply that
\[
\|B_\varepsilon(\vartheta)^{1/2} (B_\varepsilon(\vartheta) - \hat{\zeta} Q_0^\varepsilon)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \|f\|_{L^\infty} c(\phi). 
\] (6.28)

From (6.18), (6.25), and (6.28) we deduce the following estimate for the second term in the right-hand side of (6.20):
\[
\|B_\varepsilon(\vartheta)^{1/2} f_2(\varepsilon; \vartheta; \hat{\zeta})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \mathfrak{c}_5 \varepsilon c(\phi)^2, \\
0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1}, \quad \hat{\zeta} = e^{i\phi}, \quad 0 < \phi < 2\pi, 
\] (6.29)

where \(\mathfrak{c}_5 := 2(1 + \lambda_0)^2 C_K^{(1)} \|f\|_{L^\infty}^2 \|f^{-1}\|_{L^\infty}^3\).

Now we proceed to the third term in the right-hand side of (6.20). Obviously,
\[
\|B_\varepsilon(\vartheta)^{1/2} f_3(\varepsilon; \vartheta; \hat{\zeta})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq (\lambda_0 + 1) \|B_\varepsilon(\vartheta)^{1/2} (B_\varepsilon^*(\vartheta) - \hat{\zeta} Q_0^\varepsilon)^{-1}\|_{H^{-1}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\times \|\|Q_0^\varepsilon - Q_0^\varepsilon\|_{H^1(\mathbb{R}^d) \to H^{-1}(\mathbb{R}^d)} \| (B_\varepsilon^*(\vartheta) - \hat{\zeta} Q_0^\varepsilon)^{-1}\|_{L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)}. 
\] (6.30)

Using (5.14), (6.21), and the duality arguments, we have
\[
\|B_\varepsilon(\vartheta)^{1/2} (B_\varepsilon(\vartheta) - \hat{\zeta} Q_0^\varepsilon)^{-1}\|_{H^{-1}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
= \|\tilde{B}_\varepsilon(\vartheta)^{1/2} (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1} (f^\varepsilon)^*\|_{H^{-1}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
= \|f^\varepsilon (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1} (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1}\|_{L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\
= \|f^\varepsilon (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1}\|_{L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)}. 
\] (6.31)

The lower estimate (5.6) and (5.13) imply that
\[
\|D[f^\varepsilon] (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq c^\varepsilon^{-1/2} \|B_\varepsilon(\vartheta)^{1/2} f^\varepsilon (\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
= c^\varepsilon^{-1/2} \|\tilde{B}_\varepsilon(\vartheta)(\tilde{B}_\varepsilon(\vartheta) - \hat{\zeta} I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}. 
\] (6.32)
Obviously,
\[ \| \tilde{B}_\varepsilon(\vartheta)(\tilde{B}_\varepsilon(\vartheta) - \tilde{\zeta}^* I)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \sup_{\nu \geq 0} \frac{\nu}{|\nu - \tilde{\zeta}|} \leq c(\phi). \]  
(6.33)

From (6.27) at the point \( \tilde{\zeta}^* \) and (6.31)–(6.33) it follows that
\[ \| B_\varepsilon(\vartheta)^{1/2} (B_\varepsilon(\vartheta) - \tilde{\zeta}^* Q_0^\varepsilon)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq (\| f \|_{L_\infty} + c_\varepsilon^{-1/2}) c(\phi). \]  
(6.34)

Inequalities (2.14), (5.30), (6.30), and (6.31) imply that
\[ \| B_\varepsilon(\vartheta)^{1/2} J_3(\vartheta; \tilde{\zeta}) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_6 \varepsilon c(\phi)^2, \]
\[ 0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1}, \quad \tilde{\zeta} = e^{i\phi}, \quad 0 < \phi < 2\pi, \]  
(6.35)

where \( C_6 := (1 + \lambda_0)(\| f \|_{L_\infty} + c_\varepsilon^{-1/2}) C_{Q_0} C_1. \)

Finally, relations (6.20), (6.21), (6.29), and (6.35) lead to the estimate
\[ \| B_\varepsilon(\vartheta)^{1/2} (B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1} - (B_0^0(\vartheta) - \overline{\tilde{\zeta} Q_0})^{-1} - \varepsilon K(\varepsilon; \vartheta; \tilde{\zeta}) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \]
\[ \leq \tilde{C}_6 \varepsilon c(\phi)^2, \quad 0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1}, \quad \tilde{\zeta} = e^{i\phi}, \quad 0 < \phi < 2\pi, \]  
(6.36)

with \( \tilde{C}_6 = C_4 + C_5 + C_6. \)

By the lower estimate (5.6), from (6.36) it follows that
\[ \| D((B_\varepsilon(\vartheta) - \tilde{\zeta} Q_0^\varepsilon)^{-1} - (B_0^0(\vartheta) - \overline{\tilde{\zeta} Q_0})^{-1} - \varepsilon K(\varepsilon; \vartheta; \tilde{\zeta})) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \]
\[ \leq C_6 \varepsilon c(\phi)^2, \quad 0 < \vartheta \leq 1, \quad 0 < \varepsilon \leq \vartheta^{-1}, \quad \tilde{\zeta} = e^{i\phi}, \quad 0 < \phi < 2\pi, \]  
(6.37)

where \( C_6 = c_\varepsilon^{-1/2} \tilde{C}_6. \)

Applying the scaling transformation, from (6.19) and (6.37) we deduce
\[ \|(B(\varepsilon; \vartheta) - \tilde{\zeta}^2 Q_0)^{-1} - (B_0^0(\varepsilon; \vartheta) - \tilde{\zeta}^2 Q_0)^{-1} - \tilde{K}(\varepsilon; \vartheta; \tilde{\zeta}) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \]
\[ \leq C_5 \varepsilon^{-1} c(\phi)^2, \]  
(6.38)

\[ \| D((B(\varepsilon; \vartheta) - \tilde{\zeta}^2 Q_0)^{-1} - (B_0(\varepsilon; \vartheta) - \tilde{\zeta}^2 Q_0)^{-1} - \tilde{K}(\varepsilon; \vartheta; \tilde{\zeta})) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \]
\[ \leq C_6 c(\phi)^2, \]  
(6.39)

where
\[ \tilde{K}(\varepsilon; \vartheta; \tilde{\zeta}) := \left( [\Lambda] b(D) + \varepsilon \vartheta [\Lambda] \right) S_1(B_0^0(\varepsilon; \vartheta) - \tilde{\zeta}^2 Q_0)^{-1}. \]

As in the proof of Theorem 4.1 in (6.38) and (6.39) we substitute \( \tilde{\varepsilon}|\zeta|^{1/2} \) in place of \( \varepsilon, \) assuming that \( 0 < \tilde{\varepsilon} \leq 1, \ \tilde{\zeta} = |\zeta| e^{i\phi}, \ |\zeta| \geq 1, \) and put \( \vartheta = |\zeta|^{-1/2}. \)

Next, we rename \( \tilde{\varepsilon} := \varepsilon \) and apply the inverse scaling transformation, taking into account that
\[ \varepsilon K(\varepsilon; \zeta) = \varepsilon^2 T_{\varepsilon^*}^* \tilde{K}(\varepsilon|\zeta|^{1/2}; |\zeta|^{-1/2}; \tilde{\zeta}) T_{\varepsilon}, \quad \zeta = |\zeta| e^{i\phi}. \]
Here $T_\varepsilon$ is given by (1.19), and the operator $K(\varepsilon; \zeta)$ is given by (4.3). This leads to the required estimates (4.4), (4.5), and completes the proof of Theorem 4.2.

\[ T_\varepsilon \]

6.3. Proof of Theorem 4.4. We deduce the statement of Theorem 4.4 from Theorem 4.2. From (4.5) and (1.4) it follows that

\[ \|g^\varepsilon b(D)(B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2} \]

By (1.3),

\[ \varepsilon g^\varepsilon b(D) \left( [\Lambda] b(D) + [\tilde{\Lambda}] \right) S_\varepsilon (B_0 - \zeta Q_0^{-1}) \]

\[ = g^\varepsilon (b(D)\Lambda) \varepsilon S_\varepsilon b(D)(B_0 - \zeta Q_0^{-1}) + g^\varepsilon (b(D)\tilde{\Lambda}) \varepsilon S_\varepsilon (B_0 - \zeta Q_0^{-1}) \]

\[ + \varepsilon \sum_{l=1}^d g^\varepsilon b_l \left( [\Lambda] S_l b(D) D_l + [\tilde{\Lambda}] S_l D_l \right) (B_0 - \zeta Q_0^{-1}). \]

Relations (1.4), (1.5), (3.12), and (6.5) imply the following estimate for the third term in the right-hand side of (6.41):

\[ \left\| \varepsilon \sum_{l=1}^d g^\varepsilon b_l \left( [\Lambda] S_l b(D) D_l + [\tilde{\Lambda}] S_l D_l \right) (B_0 - \zeta Q_0^{-1}) \right\|_{L^2 \to L^2} \]

\[ \leq \varepsilon \|g\|_{L^\infty} \alpha_{1/2} M_1 \sum_{l=1}^d \|b(D) D_l (B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2} \]

\[ + \varepsilon \|g\|_{L^\infty} \alpha_{1/2} M_1 \sum_{l=1}^d \|D_l (B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2} \]

\[ \leq \varepsilon \|g\|_{L^\infty} \alpha_1 d^{1/2} M_1 \|D^2 (B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2} \]

\[ + \varepsilon \|g\|_{L^\infty} \alpha_1 d^{1/2} M_1 \|D (B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2}. \]

From (1.43) it follows that

\[ \|D^2 (B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2} \leq c_\varepsilon^{-1} \|B^0 (B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2}. \]

Using (5.12) and the relation between the operators $B^0 = B^0(1)$ and $\tilde{B}^0 = \tilde{B}^0(1)$ (see Subsection 5.4), we obtain

\[ \|B^0 (B_0 - \zeta Q_0^{-1})\|_{L^2 \to L^2} = \|f_0^{-1} \tilde{B}^0 (\tilde{B}^0 - \zeta I)^{-1} f_0\|_{L^2 \to L^2} \]

\[ \leq \|f\|_{L^\infty} \|f^{-1}\|_{L^\infty} \sup_{\nu \geq 0} \|\nu - \zeta\| \leq \|f\|_{L^\infty} \|f^{-1}\|_{L^\infty} c(\phi). \]

\[ \square \]
Together with (6.43) this implies
\[ \| D^2 (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2 \to L_2} \leq c_s^{-1} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} c(\phi). \] (6.45)

Similarly, using (1.42), we see that
\[ \| D (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2 \to L_2} \leq c_s^{-1/2} \| (B^0)^{1/2} (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2 \to L_2} \]
\[ = c_s^{-1/2} \| (B^0)^{1/2} (B^0 - \zeta I)^{-1} f_0 \|_{L_2 \to L_2} \]
\[ \leq c_s^{-1/2} \| f \|_{L_\infty} \sup_{\nu \geq 0} \| \nu^{1/2} \|_{L_\infty} \leq c_s^{-1/2} \| f \|_{L_\infty} c(\phi) |\zeta|^{-1/2}. \] (6.46)

From (6.42), (6.45), (6.46), and the restriction |\zeta| \geq 1 it follows that
\[ \left\| \varepsilon \sum_{l=1}^{d} g^l b_l \left( [\Lambda^\varepsilon] S_{\varepsilon} b(D) D_l + [\overline{\Lambda}^\varepsilon] S_{\varepsilon} D_l \right) (B^0 - \zeta \overline{Q}_0)^{-1} \right\|_{L_2 \to L_2} \leq C_7 \varepsilon c(\phi), \] (6.47)

where
\[ C_7 = \| g \|_{L_\infty} d^{1/2} \| f \|_{L_\infty} \left( \alpha_1 M_1 c_s^{-1} \| f^{-1} \|_{L_\infty} + \alpha_1^{1/2} \overline{M}_1 c_s^{-1/2} \right). \]

By Proposition 3.1 and relations (1.4), (6.45), we have
\[ \| g^l b(D) (I - S_{\varepsilon}) (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \]
\[ \leq \| g \|_{L_\infty} \varepsilon r_1 \| D b(D) (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \]
\[ \leq \varepsilon r_1 \alpha_1^{1/2} \| g \|_{L_\infty} \| D^2 (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_8 \varepsilon c(\phi), \] (6.48)

where \( C_8 := r_1 \alpha_1^{1/2} \| g \|_{L_\infty} c_s^{-1} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty}. \)

Finally, relations (1.25), (6.40), (6.41), (6.47), and (6.48) imply the required estimate (4.7) with \( C_7 = \| g \|_{L_\infty} \alpha_1^{1/2} C_6 + C_7 + C_8. \) This completes the proof of Theorem 4.4.

\[ \] \[ \]

7. Removal of the smoothing operator.

Special cases

7.1. Removal of \( S_{\varepsilon} \) in the corrector. It turns out that the smoothing operator \( S_{\varepsilon} \) in the corrector can be removed under some additional assumptions on the matrix-valued functions \( \Lambda(x) \) and \( \overline{\Lambda}(x). \)

Condition 7.1. Suppose that the \( \Gamma \)-periodic solution \( \Lambda(x) \) of problem (1.23) is bounded, i. e., \( \Lambda \in L_\infty(\mathbb{R}^d). \)

Some cases where Condition 7.1 is satisfied were distinguished in [BSu3, Lemma 8.7].

Proposition 7.2 ([BSu3]). Suppose that at least one of the following assumptions is satisfied:

1°) \( d \leq 2; \)
$2^o)$ $d \geq 1$, and the operator $\mathcal{A}_\varepsilon$ is of the form \( \mathcal{A}_\varepsilon = D^*g^\varepsilon(x)D \), where $g(x)$ is symmetric matrix with real entries;

$3^o)$ the dimension $d$ is arbitrary, and $g^0 = g$, i.e., relations (1.30) are satisfied.

Then Condition 7.1 is fulfilled.

In order to remove $S_\varepsilon$ in the term of the corrector containing $\tilde{\Lambda}^\varepsilon$, it suffices to impose the following condition.

**Condition 7.3.** Suppose that the $\Gamma$-periodic solution $\tilde{\Lambda}(x)$ of problem (1.31) is such that

\[ \tilde{\Lambda} \in L^p(\Omega), \quad p = 2 \text{ for } d = 1, \quad p > 2 \text{ for } d = 2, \quad p = d \text{ for } d \geq 3. \]

The following result was obtained in [Su2, Proposition 8.11].

**Proposition 7.4 ([Su2]).** Suppose that at least one of the following assumptions is satisfied:

$1^o)$ $d \leq 4$;

$2^o)$ the dimension $d$ is arbitrary and the operator $\mathcal{A}_\varepsilon$ is of the form $D^*g^\varepsilon(x)D$, where $g(x)$ is symmetric matrix with real entries.

Then Condition 7.3 is fulfilled.

**Remark 7.5.** If $\mathcal{A}_\varepsilon = D^*g^\varepsilon(x)D$, where $g(x)$ is symmetric matrix with real entries, from Theorem 13.1 of [LaU, Chapter III] it follows that the norm $\|\Lambda\|_{L^\infty}$ does not exceed a number depending only on $d$, $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, and $\Omega$, while the norm $\|\tilde{\Lambda}\|_{L^\infty}$ is controlled in terms of $d$, $\rho$, $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, $\|a_j\|_{L^\rho(\Omega)}$, $j = 1, \ldots, d$, and $\Omega$. Herewith, Conditions 7.1 and 7.3 are satisfied.

In this subsection, our goal is to prove the following theorem.

**Theorem 7.6.** Suppose that the assumptions of Theorem 4.1 are satisfied.

$1^o)$. Suppose that Condition 7.1 is satisfied. Then for $0 < \varepsilon \leq 1$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, we have

\[
\|(B_\varepsilon - \zeta Q_0)\varepsilon^{-1} - (I + \varepsilon[\Lambda^\varepsilon]b(D) + \varepsilon[\tilde{\Lambda}^\varepsilon]S_\varepsilon)(B^0 - \zeta \overline{Q_0})\varepsilon^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq C_8^\varepsilon c(\phi)^2 \varepsilon |\zeta|^{-1/2},
\]

\[
\|D((B_\varepsilon - \zeta Q_0)\varepsilon^{-1} - (I + \varepsilon[\Lambda^\varepsilon]b(D) + \varepsilon[\tilde{\Lambda}^\varepsilon]S_\varepsilon)(B^0 - \zeta \overline{Q_0})\varepsilon^{-1})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq C_9^\varepsilon c(\phi)^2 \varepsilon.
\]

The constants $C_8^\varepsilon$ and $C_9^\varepsilon$ depend only on the initial data (1.40) and the norm $\|\Lambda\|_{L^\infty}$. 

2°. Suppose that Condition (7.3) is satisfied. Then for $0 < \varepsilon \leq 1$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| > 1$, we have

$$
\| (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (I + \varepsilon \Lambda^\varepsilon)b(D)S_\varepsilon + \varepsilon \Lambda^\varepsilon ) (B^0 - \zeta Q_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} 
\leq C_8^\varepsilon \epsilon c(\phi)^2 \epsilon |\zeta|^{-1/2},
$$

$$
\| D \left( (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (I + \varepsilon \Lambda^\varepsilon)b(D)S_\varepsilon + \varepsilon \Lambda^\varepsilon ) (B^0 - \zeta Q_0)^{-1} \right) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} 
\leq C_9^\varepsilon \epsilon c(\phi)^2 \epsilon.
$$

The constants $C_8^\varepsilon$ and $C_9^\varepsilon$ are controlled in terms of the initial data (1.40), $p$, and the norm $\| \Lambda \|_{L_p(\Omega)}$.

3°. Suppose that Conditions (7.1) and (7.3) are satisfied. Then for $0 < \varepsilon \leq 1$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| > 1$, we have

$$
\| (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (I + \varepsilon \Lambda^\varepsilon)b(D) + \varepsilon \Lambda^\varepsilon ) (B^0 - \zeta Q_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} 
\leq C_8 \epsilon c(\phi)^2 \epsilon |\zeta|^{-1/2},
$$

$$
\| D \left( (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (I + \varepsilon \Lambda^\varepsilon)b(D) + \varepsilon \Lambda^\varepsilon ) (B^0 - \zeta Q_0)^{-1} \right) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} 
\leq C_9 \epsilon c(\phi)^2 \epsilon.
$$

(7.1)

The constants $C_8$ and $C_9$ depend only on the initial data (1.40), $p$, and the norms $\| \Lambda \|_{L_\infty}$, $\| \Lambda \|_{L_p(\Omega)}$.

From Corollary 2.2, Lemma 2.5, and Corollary 2.6 it follows that the operators under the norm sign in the estimates of Theorem 7.6 are bounded (under the corresponding assumptions). The statements of Theorem 7.6 follow from (7.1), (7.3), and Lemmas 7.7, 7.8 proved below.

**Lemma 7.7.** Suppose that the assumptions of Theorem 4.1 and Condition (7.1) are satisfied. Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| > 1$, and $0 < \varepsilon \leq 1$ we have

$$
\varepsilon \| \Lambda^\varepsilon \|_{L_\infty} \varepsilon c(\phi) \| (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_\Lambda^{(1)} \varepsilon c(\phi) |\zeta|^{-1/2},
$$

$$
\varepsilon \| D \Lambda^\varepsilon \|_{L_\infty} \varepsilon c(\phi) \| (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_\Lambda^{(2)} \varepsilon c(\phi).
$$

(7.2)

(7.3)

The constants $C_\Lambda^{(1)}$ and $C_\Lambda^{(2)}$ depend only on the initial data (1.40) and $\| \Lambda \|_{L_\infty}$.

**Proof.** To check (7.2), we apply (1.3) and (6.46):

$$
\| \Lambda^\varepsilon \|_{L_\infty} \| D (B_\varepsilon - \zeta Q_0) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_\Lambda^{(1)} \varepsilon c(\phi) |\zeta|^{-1/2},
$$

where $C_\Lambda^{(1)} = 2 \alpha_1^{1/2} \alpha_2^{1/2} \varepsilon \epsilon c(\phi) |\zeta|^{-1/2}$.
Now let us check \((7.3)\). We have
\[
\varepsilon \partial_j [\Lambda^\varepsilon] b(\mathbf{D})(S_{\varepsilon} - I)(B^0 - \xi_0)^{-1} = [(\partial_j \Lambda)^\varepsilon](S_{\varepsilon} - I)b(\mathbf{D})(B^0 - \xi_0)^{-1} + \varepsilon [\Lambda^\varepsilon](S_{\varepsilon} - I)b(\mathbf{D})\partial_j (B^0 - \xi_0)^{-1}.
\]
Hence,
\[
\varepsilon^2 \|\mathbf{D}[\Lambda^\varepsilon] b(\mathbf{D})(S_{\varepsilon} - I)(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2 \\
\leq 2\|[(\mathbf{D}\Lambda)^\varepsilon](S_{\varepsilon} - I)b(\mathbf{D})(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2 \\
+ 2\varepsilon^2 \|\Lambda\|_{L^\infty}^2 \sum_{j=1}^d \|(S_{\varepsilon} - I)b(\mathbf{D})\partial_j (B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2.
\]
Combining this with Corollary \((2.2)\) we obtain
\[
\varepsilon^2 \|\mathbf{D}[\Lambda^\varepsilon] b(\mathbf{D})(S_{\varepsilon} - I)(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2 \\
\leq 2\beta \|((S_{\varepsilon} - I)b(\mathbf{D})(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2 \\
+ 2\varepsilon^2 \|\Lambda\|_{L^\infty}^2 (\beta_2 + 1) \sum_{j=1}^d \|(S_{\varepsilon} - I)b(\mathbf{D})\partial_j (B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2.
\]
Applying Proposition \((3.1)\) to estimate the first term on the right and using \((1.4)\), we arrive at
\[
\varepsilon^2 \|\mathbf{D}[\Lambda^\varepsilon] b(\mathbf{D})(S_{\varepsilon} - I)(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2 \\
\leq \varepsilon^2 \alpha_1 (2\beta_2 \beta_1 + 8(\beta_2 + 1)\|\Lambda\|_{L^\infty}^2) \|\mathbf{D}^2(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}^2 \|f\|_{L^\infty} \|f^{-1}\|_{L^\infty}.
\]

**Lemma 7.8.** Suppose that the assumptions of Theorem \((4.1)\) and Condition \((7.3)\) are satisfied. Then for \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+\), \(|\zeta| \geq 1\), and \(0 < \varepsilon \leq 1\) we have
\[
\varepsilon \|(\tilde{\Lambda}^\varepsilon)(S_{\varepsilon} - I)(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \\
\leq \varepsilon \lambda_{\xi}(1 + 2\beta_2 \beta_1 + 8(\beta_2 + 1)\|\tilde{\Lambda}\|_{L^\infty}) \|f\|_{L^\infty} \|f^{-1}\|_{L^\infty}.
\]

The constants \(\lambda_{\xi}^{(1)}(\tilde{\Lambda})\) and \(\lambda_{\xi}^{(2)}(\tilde{\Lambda})\) depend only on the initial data \((1.4)\), \(p\), and \(\|\tilde{\Lambda}\|_{L^p(\Omega)}\).

**Proof.** By Lemma \((2.3)\) and Condition \((7.3)\) we have
\[
\|(\tilde{\Lambda}^\varepsilon)(S_{\varepsilon} - I)(B^0 - \xi_0)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \\
\leq 2\lambda_{\xi}(\tilde{\Lambda}) \|\tilde{\Lambda}\|_{L^p(\Omega)} \|f\|_{L^\infty} \|f^{-1}\|_{L^\infty}.
\]

Using the relation between the operators $B^0 = B^0(1)$ and $\tilde{B}^0 = \tilde{B}^0(1)$ (see Subsection 5.4), and also (5.12), we obtain
\[
\| (B^0 - \zeta \Omega_0)^{-1} \|_{L^2(\mathbb{R}^d)} \| L^2(\mathbb{R}^d) \leq \| f_0 \|^2 \| (\tilde{B}^0 - \zeta I)^{-1} \|_{L^2(\mathbb{R}^d)} \leq \| f \|^2 L^\infty c(\phi) |\zeta|^{-1}.
\] (7.8)

Relations (6.46), (7.7), and (7.8) imply (7.5) with $c^{(1)}_A = 2C^{(p)}_1 \| \tilde{\Lambda} \|_{L^p(\Omega)} (\| f \|^2 L^\infty + c_*^{-1/2} \| f \| L^\infty )$.

Now we prove (7.6). Similarly to (7.4),
\[
\varepsilon \partial_j [\tilde{\Lambda}^e](S_\varepsilon - I)(B^0 - \zeta \Omega_0)^{-1} = [\partial_j \tilde{\Lambda}^e](S_\varepsilon - I)(B^0 - \zeta \Omega_0)^{-1} + \varepsilon [\tilde{\Lambda}^e](S_\varepsilon - I)\partial_j (B^0 - \zeta \Omega_0)^{-1}.
\]
Together with Corollary 2.6 and Lemma 2.5 this implies
\[
\varepsilon^2 \| (D[\tilde{\Lambda}^e])(S_\varepsilon - I)(B^0 - \zeta \Omega_0)^{-1} \|_{L^2(\mathbb{R}^d)} \leq 2\tilde{\beta}_1 \| (S_\varepsilon - I)(B^0 - \zeta \Omega_0)^{-1} \|^2_{L^2(\mathbb{R}^d)} + 2(\tilde{\beta}_2 + 1)\varepsilon^2 \| \tilde{\Lambda} \|^2_{L^p(\Omega)} (C^{(p)}_1)^2 \| (S_\varepsilon - I)D(B^0 - \zeta \Omega_0)^{-1} \|^2_{L^2(\mathbb{R}^d)}.
\]

Applying Proposition 3.1 to estimate the first term on the right, we arrive at
\[
\varepsilon \| (D[\tilde{\Lambda}^e])(S_\varepsilon - I)(B^0 - \zeta \Omega_0)^{-1} \|_{L^2(\mathbb{R}^d)} \leq \varepsilon \left( 2\tilde{\beta}_1 r_1^2 + 8(\tilde{\beta}_2 + 1) (C^{(p)}_1)^2 \| \tilde{\Lambda} \|^2_{L^p(\Omega)} \right)^{1/2} \| D(B^0 - \zeta \Omega_0)^{-1} \|_{L^2(\mathbb{R}^d)}.
\]

Combining this with (6.45) and (6.46), we obtain (7.6) with $c^{(2)}_A = \left( 2\tilde{\beta}_1 r_1^2 + 8(\tilde{\beta}_2 + 1) (C^{(p)}_1)^2 \| \tilde{\Lambda} \|^2_{L^p(\Omega)} \right)^{1/2} \times (c_*^{-1} \| f \| L^\infty \| f^{-1} \| L^\infty + c_*^{-1/2} \| f \| L^\infty )$.

7.2. Removal of $S_\varepsilon$ in approximation of the flux.

**Theorem 7.9.** Suppose that the assumptions of Theorem 4.1 are satisfied. Let $\tilde{g}(x)$ be given by (1.25).

1°. Suppose that Condition 7.1 is satisfied. Denote
\[
G_1(\varepsilon; \zeta) := g^\varepsilon b(D)(B^0 - \zeta \Omega_0)^{-1} + g^\varepsilon (b(D)\tilde{\Lambda})^\varepsilon S_\varepsilon (B^0 - \zeta \Omega_0)^{-1}.
\] (7.9)

Then for $0 < \varepsilon \leq 1$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, we have
\[
\| g^\varepsilon b(D)(B^0 - \zeta \Omega_0)^{-1} - G_1(\varepsilon; \zeta) \|_{L^2(\mathbb{R}^d)} \leq C_1^\prime c(\phi)^2 \varepsilon.
\]

The constant $C_1^\prime$ is controlled in terms of the problem data (1.40) and $\| \Lambda \|_{L^\infty}$.
2°. Suppose that Condition 7.3 is satisfied. Denote
\[ G_2(\varepsilon; \zeta) := \tilde{g} \tilde{S}(D)(B^0 - \zeta Q_0)^{-1} + g^\varepsilon(b(D)\tilde{\Lambda})^\varepsilon(B^0 - \zeta Q_0)^{-1}. \] (7.10)
Then for \( 0 < \varepsilon \leq 1 \) and \( \zeta \in C \setminus \mathbb{R}_+, \ |\zeta| \geq 1 \), we have
\[ \|g^\varepsilon(b(D)(B_\varepsilon - \zeta Q_0)^{-1} - G_2(\varepsilon; \zeta)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C'_{10}c(\phi)^2\varepsilon. \]
The constant \( C'_{10} \) depends only on the initial data (1.40), \( p \), and \( \|\tilde{\Lambda}\|_{L_p(\Omega)} \).

3°. Suppose that Conditions 7.1 and 7.3 are satisfied. Denote
\[ G_3(\varepsilon; \zeta) := \tilde{g} \tilde{S}(D)(B^0 - \zeta Q_0)^{-1} + g^\varepsilon(b(D)\tilde{\Lambda})^\varepsilon(B^0 - \zeta Q_0)^{-1}. \] (7.11)
Then for \( 0 < \varepsilon \leq 1 \) and \( \zeta \in C \setminus \mathbb{R}_+, \ |\zeta| \geq 1 \), we have
\[ \|g^\varepsilon(b(D)(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - G_3(\varepsilon; \zeta)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C_{10}c(\phi)^2\varepsilon. \] (7.12)
The constant \( C_{10} \) depends only on the initial data (1.40), \( p \), and the norms \( \|\tilde{\Lambda}\|_{L_\infty}, \ \|\tilde{\Lambda}\|_{L_p(\Omega)} \).

Proof. The statement of Theorem 7.9 is deduced from Theorem 7.6. The proof is similar to that of Theorem 4.1 (see Subsection 6.3). To be concrete, let us prove statement 3°. By analogy with (6.40), from (7.11) it follows that
\[ \|g^\varepsilon(b(D)(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} \]
\[ - g^\varepsilon(b(D) \left( I + \varepsilon \left( [\Lambda^\varepsilon]b(D) + [\tilde{\Lambda}^\varepsilon] \right) \right)(B^0 - \zeta Q_0)^{-1}\|_{L_2(\mathbb{R}^d)} \]
\[ \leq \|g\|_{L_\infty}C_1^{1/2}C_9c(\phi)^2\varepsilon. \] (7.13)

Next, similarly to (6.41),
\[ \varepsilon g^\varepsilon(b(D) \left( [\Lambda^\varepsilon]b(D) + [\tilde{\Lambda}^\varepsilon] \right) (B^0 - \zeta Q_0)^{-1} \]
\[ = g^\varepsilon(b(D)\Lambda)^\varepsilon b(D)(B^0 - \zeta Q_0)^{-1} + g^\varepsilon(b(D)\tilde{\Lambda})^\varepsilon(B^0 - \zeta Q_0)^{-1} \]
\[ + \varepsilon \sum_{l=1}^d g^\varepsilon b_l \left( [\Lambda^\varepsilon]b(D)D_l + [\tilde{\Lambda}^\varepsilon]D_l \right) (B^0 - \zeta Q_0)^{-1}. \] (7.14)

The difference with the proof of Theorem 4.1 is related to estimation of the third term in the right-hand side of (7.13). Condition 7.4 and relations (1.4), (1.5) imply that
\[ \varepsilon \sum_{j=1}^d \|g^\varepsilon b_l[\Lambda^\varepsilon](b(D)D_l(B^0 - \zeta Q_0)^{-1}\|_{L_2(\mathbb{R}^d)} \]
\[ \leq \varepsilon C_1 d^{1/2}\|g\|_{L_\infty}\|\Lambda\|_{L_\infty}\|D^2(B^0 - \zeta Q_0)^{-1}\|_{L_2(\mathbb{R}^d)} \]
(7.15)

Using Condition 7.3, (1.5), and Lemma 2.5, we obtain
\[ \varepsilon \sum_{j=1}^d \|g^\varepsilon b_l[\tilde{\Lambda}^\varepsilon]D_l(B^0 - \zeta Q_0)^{-1}\|_{L_2(\mathbb{R}^d)} \]
\[ \leq \varepsilon C_1^{1/2}d^{1/2}C_{\Omega}(p)\|g\|_{L_\infty}\|\tilde{\Lambda}\|_{L_p(\Omega)}\|D(B^0 - \zeta Q_0)^{-1}\|_{L_2(\mathbb{R}^d)} \]
(7.16)
From (6.45), (6.46), (7.15), and (7.16) it follows that the \((L_2 \rightarrow L_2)\)-norm of the third term in the right-hand side of (7.14) does not exceed \(\tilde{C}_{10}c(\phi)\varepsilon\), where
\[
\tilde{C}_{10} = \alpha_1 d^{1/2} \|g\|_{L_\infty} \|\Lambda\|_{L_\infty} c_\varepsilon^{-1} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}
+ \alpha_1^{1/2} d^{1/2} C_\Omega \|g\|_{L_\infty} \|\Lambda\|_{L_\nu(\Omega)} \left( c_\varepsilon^{-1} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} + c_\varepsilon^{-1/2} \|f\|_{L_\infty} \right) .
\]
Together with (7.13) and (7.14) this implies (7.12) with \(C_{10} = \|g\|_{L_\infty} \alpha_1^{1/2} C_0 + \tilde{C}_{10}\).

Statements 1° and 2° are proved in a similar fashion; to check 2°, in addition we have to take (6.48) into account. \(\square\)

7.3. **The case of the zero corrector.** Suppose that \(g^0 = \overline{g}\), which is equivalent to (1.29). Then the \(\Gamma\)-periodic solution of problem (1.23) is equal to zero: \(\Lambda(x) = 0\). Suppose also that
\[
\sum_{j=1}^{d} D_j a_j(x)^* = 0 . \tag{7.17}
\]
Then the \(\Gamma\)-periodic solution of problem (1.31) is equal to zero: \(\tilde{\Lambda}(x) = 0\).

**Proposition 7.10.** Suppose that the assumptions of Theorem 4.11 are satisfied. Suppose also that relations (1.29) and (7.17) are satisfied. Then for \(0 < \varepsilon \ll 1\) and \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| \gg 1\), we have
\[
\|D \left( (B_\varepsilon - \zeta Q_0^5)^{-1} - (B^0 - \zeta Q_0^{-1}) \right) \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_0 c(\phi)^2 \varepsilon.
\]

7.4. **The special case.** Suppose that \(g^0 = g\), which is equivalent to (1.30). Then, by Proposition 7.2 (3°), Condition 7.4 is satisfied. According to [BSu2, Remark 3.5], in this case the matrix-valued function (1.25) is constant and coincides with \(g^0\), i.e., \(\overline{g}(x) = g^0 = g\). Thus, \(\overline{g}b(D)(B^0 - \zeta Q_0^{-1}) = g^0 b(D)(B^0 - \zeta Q_0^{-1})\).

In addition, suppose that relation (7.17) is satisfied. Then \(\tilde{\Lambda}(x) = 0\), and Theorem 7.9 (3°) implies the following result.

**Proposition 7.11.** Suppose that the assumptions of Theorem 4.11 are satisfied. Suppose also that relations (1.30) and (7.17) are satisfied. Then for \(0 < \varepsilon \ll 1\) and \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| \gg 1\), we have
\[
\|g^\varepsilon b(D)(B_\varepsilon - \zeta Q_0^5)^{-1} - g^0 b(D)(B^0 - \zeta Q_0^{-1})\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_{10} c(\phi)^2 \varepsilon.
\]

8. **Another approximation of the generalized resolvent** \((B_\varepsilon - \zeta Q_0^5)^{-1}\)

8.1. **The result in the general case.** In Theorems of Sections 4 and 7, it was assumed that \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+\) and \(|\zeta| \gg 1\). In the present subsection, we obtain the results in a wider domain of \(\zeta\).
The constants $C_{11}$ are given in Subsection 5.3. Let $c_0 \geq 0$ be a common lower bound of the operators $\tilde{B}_\epsilon = (f^\epsilon)^*B_\epsilon f^\epsilon$ and $\tilde{B}^0 = f_0B^0f_0$. We put $\zeta - c_0 = |\zeta - c_0|e^{i\varphi}$, $\psi \in (0, 2\pi)$, and denote
\[
\varrho(\zeta) = \begin{cases}
(c(\psi))^2|\zeta - c_0|^{-2}, & |\zeta - c_0| < 1, \\
(c(\psi))^2, & |\zeta - c_0| \geq 1.
\end{cases}
\] (8.1)

Then for $0 < \epsilon \leq 1$ we have
\[
\begin{align*}
\|B_\epsilon - \zeta\bar{Q}_0^{-1} - (B^0 - \zeta\bar{Q}_0^{-1})\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} & \leq C_{11}\varrho(\zeta)\epsilon, \\
\|B_\epsilon - \zeta\bar{Q}_0^{-1} - (B^0 - \zeta\bar{Q}_0^{-1} - \epsilon\mathcal{K}(\epsilon; \zeta))\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} & \leq C_{12}\varrho(\zeta)\epsilon,
\end{align*}
\] (8.2) (8.3)

\[
\|\mathbf{D}(B_\epsilon - \zeta\bar{Q}_0^{-1} - (B^0 - \zeta\bar{Q}_0^{-1} - \epsilon\mathcal{K}(\epsilon; \zeta)))\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}
\leq (C_{13} + |\zeta + 1|^{1/2}C_{14})\varrho(\zeta)\epsilon.
\] (8.4)

Let $G(\epsilon; \zeta)$ be the operator defined by (4.6). Then for $0 < \epsilon \leq 1$ we have
\[
\|G^2b(D)(B_\epsilon - \zeta\bar{Q}_0^{-1} - G(\epsilon; \zeta))\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq (C_{15} + |\zeta + 1|^{1/2}C_{16})\varrho(\zeta)\epsilon.
\] (8.5)

The constants $C_{11}, C_{12}, C_{13}, C_{14}, C_{15}$, and $C_{16}$ are controlled in terms of the initial data (3.40) and $c_0$.

**Corollary 8.2.** Under the assumptions of Theorem 8.1 we have
\[
\begin{align*}
\|B_\epsilon - \zeta\bar{Q}_0^{-1} - (B^0 - \zeta\bar{Q}_0^{-1} - \epsilon\mathcal{K}(\epsilon; \zeta))\|_{L^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} & \leq (C_{12} + C_{13} + |\zeta + 1|^{1/2}C_{14})\varrho(\zeta)\epsilon, \\
0 < \epsilon & \leq 1.
\end{align*}
\] (8.6)

**Remark 8.3.** 1) We do not control the lower edges of the spectra of the operators $\tilde{B}_\epsilon \geq 0$ and $\tilde{B}^0 \geq 0$ explicitly. However, we can always take $c_0 = 0$. In this case, we have $\psi = \phi$, and for $|\zeta| = |\zeta - c_0| \geq 1$ estimates of Theorem 8.1 are more than the results of Theorems 4.1, 4.2, and 4.3. 2) For large $|\zeta|$ it is more convenient to apply Theorems 4.1, 4.2, and 4.4 while for bounded values of $|\zeta|$ Theorem 8.1 may be preferable.

**Proof.** To check (8.2), we apply Theorem 4.1 for $\zeta = -1$. According to (4.2),
\[
\|B_\epsilon + Q_0^{-1} - (B^0 + \bar{Q}_0^{-1})\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq C_4\epsilon, \quad 0 < \epsilon \leq 1.
\]

Similarly to (5.21)–(5.24), using the analog of identity (5.20) for $\vartheta = 1$ (with $\tilde{\lambda}_0$ replaced by $\lambda_0$ and $\lambda_0$ replaced by $\zeta$ and 1, respectively), we obtain
\[
\begin{align*}
\|B_\epsilon - \zeta\bar{Q}_0^{-1} - (B^0 - \zeta\bar{Q}_0^{-1})\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} & \leq C_4\epsilon\|f\|_{L^\infty}^2\|f^{-1}\|_{L^\infty}^2\|\sup_{\nu > c_0} (\nu + 1)^2 \nu^{-\zeta}\| f\|_{L^\infty}^2
+ |\zeta||B_\epsilon - \zeta\bar{Q}_0^{-1}\|_{H^{-1}(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}\|\bar{Q}_0^{-1} - \bar{Q}_0\|_{H^{-1}(\mathbb{R}^d) \rightarrow H^{-1}(\mathbb{R}^d)}\|B^0 - \bar{Q}_0^{-1}\|_{L^2(\mathbb{R}^d) \rightarrow H^{-1}(\mathbb{R}^d)}.
\end{align*}
\] (8.7)
A calculation shows that
\[
\sup_{\nu \geq \nu_0} \frac{(\nu + 1)^2}{|\nu - \zeta|^2} \leq (c_0 + 2)^2 \varrho(\zeta), \quad \zeta \in \mathbb{C} \setminus [c_0, \infty).
\] (8.8)

Next, by the duality argument,
\[
\|(B_\varepsilon - \zeta \mathcal{Q}_0^\varepsilon)^{-1}\|_{H^{-1} \rightarrow L^2} = \|(B_\varepsilon - \zeta^* \mathcal{Q}_0^\varepsilon)^{-1}\|_{L^2 \rightarrow H^1}.
\] (8.9)

By (5.14), we have
\[
\|(B_\varepsilon - \zeta^* \mathcal{Q}_0^\varepsilon)^{-1}\|_{L^2 \rightarrow L^2} \leq \|f\|_{L^\infty}^2 \sup_{\nu \geq \nu_0} \frac{1}{|\nu - \zeta^*|} = \|f\|_{L^\infty}^2 c(\psi)|\zeta - c_0|^{-1}.
\] (8.10)

Note that
\[
|\zeta + 1|^{1/2} \leq (2 + c_0)^{1/2} \text{ for } |\zeta - c_0| < 1, \quad (8.11)
\]
\[
|\zeta + 1|^{1/2}|\zeta - c_0|^{-1} \leq (2 + c_0)^{1/2} \text{ for } |\zeta - c_0| \geq 1.
\] (8.12)

Therefore,
\[
|\zeta + 1|^{1/2}\|(B_\varepsilon - \zeta^* \mathcal{Q}_0^\varepsilon)^{-1}\|_{L^2 \rightarrow L^2} \leq \|f\|_{L^\infty}^2 (2 + c_0)^{1/2} \varrho(\zeta)^{1/2}.
\] (8.13)

By analogy with (5.28), we obtain
\[
\|\mathbf{D}(B_\varepsilon - \zeta^* \mathcal{Q}_0^\varepsilon)^{-1}\|_{L^2 \rightarrow L^2} \leq c_*^{-1/2} \|f\|_{L^\infty} \sup_{\nu \geq \nu_0} \frac{\nu^{1/2}}{|\nu - \zeta^*|}.
\] (8.14)

A calculation shows that
\[
\sup_{\nu \geq \nu_0} \frac{\nu}{|\nu - \zeta^*|^2} \leq \begin{cases} (c_0 + 1)c(\psi)^2|\zeta - c_0|^{-2}, & |\zeta - c_0| < 1, \\ (c_0 + 1)c(\psi)^2|\zeta - c_0|^{-1}, & |\zeta - c_0| \geq 1. \end{cases}
\] (8.15)

Using (8.11) and the estimate \(|\zeta + 1||\zeta - c_0|^{-1} \leq 2 + c_0\) for \(|\zeta - c_0| \geq 1\), we see that
\[
|\zeta + 1| \sup_{\nu \geq \nu_0} \frac{\nu}{|\nu - \zeta^*|^2} \leq (c_0 + 2)(c_0 + 1) \varrho(\zeta).
\] (8.16)

By (8.14) and (8.16),
\[
|\zeta + 1|^{1/2}\|\mathbf{D}(B_\varepsilon - \zeta^* \mathcal{Q}_0^\varepsilon)^{-1}\|_{L^2 \rightarrow L^2} \leq c_*^{-1/2} \|f\|_{L^\infty} (c_0 + 2)^{1/2} \|c(\psi)^2\|^{1/2} \varrho(\zeta)^{1/2}.
\] (8.17)

Relations (8.9), (8.13), and (8.17) imply that
\[
|\zeta + 1|^{1/2}\|(B_\varepsilon - \zeta \mathcal{Q}_0)^{-1}\|_{H^{-1} \rightarrow L^2} \leq \mathcal{C}_0 \varrho(\zeta)^{1/2},
\] (8.18)

where
\[
\mathcal{C}_0 = \|f\|_{L^\infty}^2 (2 + c_0)^{1/2} + c_*^{-1/2} \|f\|_{L^\infty} (c_0 + 2)^{1/2} (c_0 + 1)^{1/2}.
\]

Similarly to (8.10) and (8.14), using (1.42) and (5.12), we obtain
\[
\|(B^0 - \zeta \mathcal{Q}_0)^{-1}\|_{L^2 \rightarrow L^2} \leq \|f\|_{L^\infty}^2 c(\psi)|\zeta - c_0|^{-1},
\] (8.19)

\[
\|\mathbf{D}(B^0 - \zeta \mathcal{Q}_0)^{-1}\|_{L^2 \rightarrow L^2} \leq c_*^{-1/2} \|f\|_{L^\infty} \sup_{\nu \geq \nu_0} \frac{\nu^{1/2}}{|\nu - \zeta^*|},
\] (8.20)
Together with (8.11), (8.12), and (8.16) this yields
\[ |\zeta + 1|^{1/2} \| (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2 \to H^1} \leq C_9 \rho(\zeta)^{1/2}. \] (8.21)

Combining (2.14), (8.17), (8.18), and (8.21), we arrive at (8.2) with
\[ C_1 = C_4 (c_\gamma + 2)^2 \| f \|^2 + \| f^{-1} \|^2 + C_Q c_0^2. \]

To prove (8.3), we rely on the already proved estimate (8.2):
\[ \| (B_\varepsilon - \zeta \overline{Q}_0)^{-1} - (B^0 - \zeta \overline{Q}_0)^{-1} - \varepsilon K(\varepsilon; \zeta) \|_{L_2 \to L_2} \leq C_1 \rho(\zeta) \varepsilon + \varepsilon \| K(\varepsilon; \zeta) \|_{L_2 \to L_2}. \] (8.22)

By (1.4), (3.12), and (6.5), the operator (4.3) satisfies
\[ \| K(\varepsilon; \zeta) \|_{L_2 \to L_2} \leq M_1 \alpha_1^{1/2} \| D(B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2 \to L_2} + \hat{M}_1 \| (B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2 \to L_2}. \] (8.23)

Taking (8.15) and (8.20) into account, we have
\[ \| D(B^0 - \zeta \overline{Q}_0)^{-1} \|_{L_2 \to L_2} \leq \kappa^{-1/2}(c_\gamma + 1)^{1/2} \| f \|_{L_\infty} \rho(\zeta)^{1/2}. \] (8.24)

By (8.19), (8.23), and (8.24),
\[ \| K(\varepsilon; \zeta) \|_{L_2 \to L_2} \leq C_{10} \rho(\zeta)^{1/2}, \] (8.25)

where \( C_{10} = M_1 \alpha_1^{1/2} c_\gamma^{-1/2}(c_\gamma + 1)^{1/2} \| f \|_{L_\infty} + \hat{M}_1 \| f \|_{L_\infty}^2 \). Combining (8.22) and (8.25), and noting that \( \rho(\zeta)^{1/2} \leq \rho(\zeta) \), we arrive at estimate (8.3) with
\[ C_2 = C_1 + C_{10}. \]

In order to prove (8.4), we write down the analog of (6.17) for \( \vartheta = 1 \) (at the points \( \zeta \) and 1) and apply the operator \( B_\varepsilon^{1/2} \) to both sides of the corresponding identity:
\[ B_\varepsilon^{1/2}((B_\varepsilon - \zeta \overline{Q}_0)^{-1} - (B^0 - \zeta \overline{Q}_0)^{-1} - \varepsilon K(\varepsilon; \zeta)) = B_\varepsilon^{1/2}(B_\varepsilon - \zeta \overline{Q}_0)^{-1}(B_\varepsilon + Q_0)^{-1}((B_\varepsilon + Q_0)^{-1} - (B^0 + \overline{Q}_0)^{-1} - \varepsilon K(\varepsilon; -1)) \times (B^0 + \overline{Q}_0)(B^0 - \zeta \overline{Q}_0)^{-1} + \varepsilon(1 + \zeta)B_\varepsilon^{1/2}(B_\varepsilon - \zeta \overline{Q}_0)^{-1}Q_0 \hat{K}(\varepsilon; \zeta) + (1 + \zeta)B_\varepsilon^{1/2}(B_\varepsilon - \zeta \overline{Q}_0)^{-1}(Q_0 - \overline{Q}_0)(B^0 - \zeta \overline{Q}_0)^{-1}. \] (8.26)

Denote the consecutive summands in the right-hand side by \( \mathcal{I}_j(\varepsilon; \zeta) \), \( j = 1, 2, 3. \)

Using the analogous of (6.1), (6.21), and (6.22) with \( \vartheta = 1 \), we see that
\[ \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta \overline{Q}_0)^{-1}(B_\varepsilon + Q_0)w \|_{L_2(\mathbb{R}^d)} = \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta I)^{-1}(B_\varepsilon + I)(f^\varepsilon)^{-1}w \|_{L_2(\mathbb{R}^d)} \] (8.27)
According to (5.13) and (5.14) (with ∥I∥1, the analog of (6.44), and (8.8)). Now we estimate the second term in the right-hand side of (8.26):

\[ \| I_2(\varepsilon; \zeta)\|_{L_2 \to L_2} \leq \varepsilon |\zeta + 1| \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta Q_0^{\flat})^{-1}\|_{L_2 \to L_2} f^{-1} \| f \|_{L_\infty} \| K(\varepsilon; \zeta)\|_{L_2 \to L_2}. \]

According to (8.13) and (8.14) (with \( \vartheta = 1 \)), we have

\[ \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta Q_0^{\flat})^{-1}\|_{L_2 \to L_2} \leq \| f \|_{L_\infty} \sup_{\nu \geq c_9} \frac{\nu^{1/2}}{\nu - \zeta}. \]

By (8.16) this implies that

\[ |\zeta + 1|^{1/2} \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta Q_0^{\flat})^{-1}\|_{L_2 \to L_2} \leq (c_9 + 2)^{1/2}(c_9 + 1)^{1/2} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} (\alpha_1^{1/2} M_1 + \tilde{M}_1) \| \zeta \|_{L_\infty} \| K(\varepsilon; \zeta)\|_{L_2 \to L_2}. \]

From (8.21) and (8.23) it follows that

\[ |\zeta + 1|^{1/2} \| K(\varepsilon; \zeta)\|_{L_2 \to L_2} \leq (\alpha_1^{1/2} M_1 + \tilde{M}_1) \| \zeta \|_{L_\infty} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} \| K(\varepsilon; \zeta)\|_{L_2 \to L_2}. \]

Relations (8.29)–(8.31) imply that

\[ \| I_2(\varepsilon; \zeta)\|_{L_2 \to L_2} \leq \mathcal{C}_{12} \rho(\zeta) \varepsilon, \]

where \( \mathcal{C}_{12} = (c_9 + 2)^{1/2}(c_9 + 1)^{1/2} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} (\alpha_1^{1/2} M_1 + \tilde{M}_1) \| \zeta \|_{L_\infty} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} \| K(\varepsilon; \zeta)\|_{L_2 \to L_2} \| \zeta \|_{L_\infty} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} \| K(\varepsilon; \zeta)\|_{L_2 \to L_2}. \]

For the third term in the right-hand side of (8.26), we have

\[ \| I_3(\varepsilon; \zeta)\|_{L_2 \to L_2} \leq |\zeta + 1| \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta Q_0^{\flat})^{-1}\|_{H^{-1} \to L_2} \| Q_0^{-1} - Q_0^{\flat}\|_{H^{1} \to H^{-1}} \| (B_\varepsilon - \zeta Q_0^{\flat})^{-1}\|_{L_2 \to H^1}. \]

By the duality argument (cf. (8.31)),

\[ \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta Q_0^{\flat})^{-1}\|_{H^{-1} \to L_2} = \| f \|_{L_\infty} \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta^* I)^{-1}\|_{L_2 \to H^1}. \]

Similarly to (8.32), using (8.8), we obtain

\[ \| \mathcal{D}(f) B_\varepsilon^{1/2}(B_\varepsilon - \zeta^* I)^{-1}\|_{L_2 \to L_2} \leq (c_9 + 1)^{1/2} \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta^* I)^{-1}\|_{L_2 \to L_2} \leq (c_9 + 2) \rho(\zeta)^{1/2}. \]

Next, taking (8.15) into account, we have

\[ \| f \varepsilon B_\varepsilon^{1/2}(B_\varepsilon - \zeta^* I)^{-1}\|_{L_2 \to L_2} \leq \| f \|_{L_\infty} (c_9 + 1)^{1/2} \rho(\zeta)^{1/2}. \]

From (8.34)–(8.36) it follows that

\[ \| B_\varepsilon^{1/2}(B_\varepsilon - \zeta Q_0^{\flat})^{-1}\|_{H^{-1} \to L_2} \leq \mathcal{C}_{13} \rho(\zeta)^{1/2}, \]
where $C_{13} = \|f\|_{L_\infty} (c_3 + 1)^{1/2} + c_3^{-1/2} (c_3 + 2)$. Relations (2.14), (8.21), (8.33), and (8.37) yield

$$\|I_3(\varepsilon; \zeta)\|_{L_2 \to L_2} \leq C_{Q_0} C_9 C_{13} |\zeta| + 1^{1/2} \rho(\zeta) \varepsilon. \quad (8.38)$$

Combining (8.26), (8.28), (8.32), (8.38), and the lower estimate (5.6) (with $\vartheta = 1$), we arrive at (8.41) with $C_{13} = c_3^{-1/2} (C_{11} + C_{12})$, $C_{14} = c_3^{-1/2} C_{Q_0} C_9 C_{13}$. It remains to check (8.5). From (8.4) and (1.4) it follows that

$$\|g^\varepsilon b(D)(I + \varepsilon [\Lambda^\varepsilon] S_\varepsilon b(D) + \varepsilon [\tilde{\Lambda}^\varepsilon] S_\varepsilon) (B^0 - \zeta Q_0) - 1\|_{L_2 \to L_2} \leq \alpha_1^{1/2} \|g\|_{L_\infty} (C_{13} + |\zeta| + 1^{1/2} C_{14}) \rho(\zeta) \varepsilon. \quad (8.39)$$

By analogy with (6.41), (6.42), and (6.48), we obtain

$$\|g^\varepsilon b(D)(I + \varepsilon [\Lambda^\varepsilon] S_\varepsilon b(D) + \varepsilon [\tilde{\Lambda}^\varepsilon] S_\varepsilon) (B^0 - \zeta Q_0) - 1 - G(\varepsilon; \zeta)\|_{L_2 \to L_2} \leq \varepsilon \|g\|_{L_\infty} \alpha_1^{1/2} ((\alpha_1 d)^{1/2} M_1 + r_1) \|D^2 (B^0 - \overline{\zeta Q_0})^{-1}\|_{L_2 \to L_2} \quad (8.40)$$

Similarly to (6.43) and (6.44), taking (8.8) into account, we have

$$\|D^2 (B^0 - \overline{\zeta Q_0})^{-1}\|_{L_2 \to L_2} \leq c_3^{-1} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \sup_{\nu > c_3} |\nu - \zeta| \quad (8.41)$$

Together with (8.24) and (8.40) this implies

$$\|g^\varepsilon b(D)(I + \varepsilon [\Lambda^\varepsilon] S_\varepsilon b(D) + \varepsilon [\tilde{\Lambda}^\varepsilon] S_\varepsilon) (B^0 - \zeta Q_0) - 1 - G(\varepsilon; \zeta)\|_{L_2 \to L_2} \leq C_{14} \varepsilon \rho(\zeta)^{1/2}, \quad (8.42)$$

where $C_{14} = \|g\|_{L_\infty} \|f\|_{L_\infty} \alpha_1^{1/2} ((\alpha_1 d)^{1/2} M_1 + r_1) c_3^{-1} \|f^{-1}\|_{L_\infty} (c_3 + 2) + d^{1/2} M_1 c_3^{-1/2} (c_3 + 1)^{1/2}$. Now, relations (8.39) and (8.42) imply estimate (8.5) with $C_{15} = \alpha_1^{1/2} \|g\|_{L_\infty} C_{13}$ and $C_{16} = \alpha_1^{1/2} \|g\|_{L_\infty} C_{14}$. \hfill \Box

Note that, if $Q_0$ is a constant matrix, then $Q_0 = \overline{Q_0}$ and the third summand in the right-hand side of (8.26) is equal to zero: $I_3(\varepsilon; \zeta) = 0$. Therefore, from the proof of Theorem 8.1 we derive the following.

**Remark 8.4.** If the assumptions of Theorem 8.1 are satisfied and $Q_0$ is a constant matrix, then estimates (8.41)–(8.6) are true with $C_{14} = C_{16} = 0$. It means that in estimates (8.41)–(8.6) there are no terms containing $|\zeta| + 1^{1/2}$.

8.2. Removal of $S_\varepsilon$. Now we distinguish the cases where the smoothing operator $S_\varepsilon$ in the corrector can be removed.

**Theorem 8.5.** Suppose that the assumptions of Theorem 8.1 are satisfied.
1°. Suppose that Condition (7.1) is satisfied. Let \( G_1(\varepsilon; \zeta) \) be the operator defined by (7.9). Then for \( 0 < \varepsilon \leq 1 \) we have

\[
\| (B_\varepsilon - \zeta Q_0^{-1} - (I + \varepsilon [\Lambda^\varepsilon] b(D) + \varepsilon [\Lambda^\varepsilon] S_\varepsilon) (B^0 - \zeta Q_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{17}' \varrho(\zeta) \varepsilon,
\]

\[
\| \mathbf{D} \left( (B_\varepsilon - \zeta Q_0^{-1} - (I + \varepsilon [\Lambda^\varepsilon] b(D) + \varepsilon [\Lambda^\varepsilon] S_\varepsilon) (B^0 - \zeta Q_0)^{-1} \right) \|_{L_4(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq (C_{18}' + |\zeta + 1|^{1/2} C_{19}') \varrho(\zeta) \varepsilon,
\]

\[
\| g^\varepsilon b(D)(B_\varepsilon - \zeta Q_0^{-1} - G_1(\varepsilon; \zeta)) \|_{L_2(\mathbb{R}^d) \to L_4(\mathbb{R}^d)} \leq (C_{20}' + |\zeta + 1|^{1/2} C_{21}') \varrho(\zeta) \varepsilon.
\]

The constants \( C_{17}', C_{18}', C_{19}', C_{20}', \) and \( C_{21}' \) are controlled in terms of the problem data (1.40), \( c_0, p, \) and \( \| \Lambda \|_{L_\infty} \).

2°. Suppose that Condition (7.3) is satisfied. Let \( G_2(\varepsilon; \zeta) \) be the operator given by (7.10). Then for \( 0 < \varepsilon \leq 1 \) we have

\[
\| (B_\varepsilon - \zeta Q_0^{-1} - (I + \varepsilon [\Lambda^\varepsilon] b(D) S_\varepsilon + \varepsilon [\Lambda^\varepsilon]) (B^0 - \zeta Q_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{17}'' \varrho(\zeta) \varepsilon,
\]

\[
\| \mathbf{D} \left( (B_\varepsilon - \zeta Q_0^{-1} - (I + \varepsilon [\Lambda^\varepsilon] b(D) S_\varepsilon + \varepsilon [\Lambda^\varepsilon]) (B^0 - \zeta Q_0)^{-1} \right) \|_{L_4(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq (C_{18}'' + |\zeta + 1|^{1/2} C_{19}'') \varrho(\zeta) \varepsilon,
\]

\[
\| g^\varepsilon b(D)(B_\varepsilon - \zeta Q_0^{-1} - G_2(\varepsilon; \zeta)) \|_{L_2(\mathbb{R}^d) \to L_4(\mathbb{R}^d)} \leq (C_{20}'' + |\zeta + 1|^{1/2} C_{21}'') \varrho(\zeta) \varepsilon.
\]

The constants \( C_{17}'', C_{18}'', C_{19}'', C_{20}'', \) and \( C_{21}'' \) depend only on the initial data (1.40), \( c_0, p, \) and \( \| \Lambda \|_{L_p(\Omega)} \).

3°. Suppose that Conditions (7.1) and (7.3) are satisfied. Let \( G_3(\varepsilon; \zeta) \) be the operator given by (7.11). Then for \( 0 < \varepsilon \leq 1 \) we have

\[
\| (B_\varepsilon - \zeta Q_0^{-1} - (I + \varepsilon [\Lambda^\varepsilon] b(D) + \varepsilon [\Lambda^\varepsilon]) (B^0 - \zeta Q_0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{17}''' \varrho(\zeta) \varepsilon,
\]

\[
\| \mathbf{D} \left( (B_\varepsilon - \zeta Q_0^{-1} - (I + \varepsilon [\Lambda^\varepsilon] b(D) + \varepsilon [\Lambda^\varepsilon]) (B^0 - \zeta Q_0)^{-1} \right) \|_{L_4(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq (C_{18}''' + |\zeta + 1|^{1/2} C_{19}''') \varrho(\zeta) \varepsilon,
\]

\[
\| g^\varepsilon b(D)(B_\varepsilon - \zeta Q_0^{-1} - G_3(\varepsilon; \zeta)) \|_{L_2(\mathbb{R}^d) \to L_4(\mathbb{R}^d)} \leq (C_{20}''' + |\zeta + 1|^{1/2} C_{21}''') \varrho(\zeta) \varepsilon.
\]

The constants \( C_{17}, C_{18}, C_{19}, C_{20}, \) and \( C_{21} \) are controlled in terms of the initial data (1.40), \( c_0, p, \) and \( \| \Lambda \|_{L_\infty}, \) and \( \| \Lambda \|_{L_p(\Omega)} \).

**Remark 8.6.** If the assumptions of Theorem (8.3) are satisfied and \( Q_0 \) is a constant matrix, then estimates of Theorem (8.5) are valid with \( C_{19}'' = C_{21}' = C_{19}'' = C_{21}'' = C_{19} = C_{21} = 0.\)

**Proof.** Approximations with corrector for the operator \( (B_\varepsilon - \zeta Q_0^{-1}) \) are deduced from Theorem (8.1) by analogy with the proof of Theorem (7.6). The difference is that, instead of (6.45), (6.46), and (7.8) we use (8.41), (8.24), and (8.19), respectively.
The statements concerning the flux are derived from approximations for the operator \((B_\varepsilon - \Omega Q_0^{\varepsilon})^{-1}\), by analogy with the proof of Theorem 7.9. The difference is that, instead of (6.45) and (6.46) we apply (8.41) and (8.24), respectively. 

**8.3. Special cases.** Similarly to Proposition 7.10 with the help of Theorem 8.1 we distinguish the case where the corrector is equal to zero.

**Proposition 8.7.** Suppose that the assumptions of Theorem 8.1 are satisfied. Suppose also that relations (1.29) and (7.17) are satisfied. Then for \(0 < \varepsilon \leq 1\) we have

\[
\|D((B_\varepsilon - \Omega Q_0^{\varepsilon})^{-1} - (B_0 - \Omega Q_0)^{-1})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq (C_{13} + |\zeta + 1|^{1/2}C_{14})\rho(\zeta)\varepsilon.
\]

Similarly to Proposition 7.11 we deduce the following statement from Theorem 8.5(3).

**Proposition 8.8.** Suppose that the assumptions of Theorem 8.1 are satisfied. Suppose also that relations (1.30) and (7.17) are satisfied. Then for \(0 < \varepsilon \leq 1\) we have

\[
\|g^\varepsilon b(D)((B_\varepsilon - \Omega Q_0^{\varepsilon})^{-1} - (B_0 - \Omega Q_0)^{-1})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq (C_{20} + |\zeta + 1|^{1/2}C_{21})\rho(\zeta)\varepsilon.
\]

**9. Application of the general results**

In this section, we consider examples that were studied before in [Su2] and [Su6].

**9.1. The scalar elliptic operator.** We consider the case where \(n = 1, m = d\), \(b(D) = D\), and \(g(x)\) is a \(\Gamma\)-periodic symmetric \((d \times d)\)-matrix-valued function with real entries such that \(g(x) > 0\) and \(g, g^{-1} \in L^\infty\). Then, obviously, \(\alpha_0 = \alpha_1 = 1\) (see (1.4)). The operator \(A_\varepsilon\) takes the form \(A_\varepsilon = -\text{div} g^\varepsilon(x)\nabla\).

Next, let \(A(x) = \text{col}\{A_1(x), \ldots, A_d(x)\}\), where \(A_j(x), j = 1, \ldots, d\), are \(\Gamma\)-periodic real-valued functions such that

\[
A_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \geq 2; \quad j = 1, \ldots, d. \tag{9.1}
\]

Let \(v(x)\) and \(V(x)\) be real-valued \(\Gamma\)-periodic functions such that

\[
v, V \in L_s(\Omega), \quad \int_\Omega v(x) \, dx = 0, \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2. \tag{9.2}
\]

In \(L^2(\mathbb{R}^d)\), we consider the operator \(B_\varepsilon\) given formally by the differential expression

\[
B_\varepsilon = (D - A_\varepsilon(x))^* g^\varepsilon(x)(D - A_\varepsilon(x)) + \varepsilon^{-1} v^\varepsilon(x) + \nu^\varepsilon(x). \tag{9.3}
\]
Precisely, $B_\varepsilon$ is the operator generated by the quadratic form

$$\mathfrak{b}_\varepsilon[u, u] = \int_{\mathbb{R}^d} \left( \langle g^\varepsilon(D - \mathbf{A}^\varepsilon)u, (D - \mathbf{A}^\varepsilon)u \rangle + \langle \varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon \rangle |u|^2 \right) \, dx,$$

$u \in H^1(\mathbb{R}^d)$.

The operator \eqref{9.3} can be treated as the periodic Schrödinger operator with the metric $g^\varepsilon$, the magnetic potential $\mathbf{A}^\varepsilon$, and the electric potential $\varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon$ containing the singular summand $\varepsilon^{-1} v^\varepsilon$.

It is easy to check (see \cite{Su2, Subsection 13.1}) that the operator \eqref{9.3} can be written in the required form \eqref{1.22}:

$$B_\varepsilon = D^* g^\varepsilon(x) D + \sum_{j=1}^{d} (a^j_\varepsilon(x) D_j + D_j(a^j_\varepsilon(x))^*) + Q^\varepsilon(x).$$

Here the real-valued function $Q(x)$ is given by

$$Q(x) = \mathcal{V}(x) + \langle g(x) \mathbf{A}(x), \mathbf{A}(x) \rangle.$$ \hfill (9.4)

The complex-valued functions $a_j(x)$ are given by

$$a_j(x) = -\eta_j(x) + i\gamma_j(x), \quad j = 1, \ldots, d,$$ \hfill (9.5)

where $\eta_j(x)$ are the components of the vector-valued function $\eta(x) = g(x)\mathbf{A}(x)$, and $\gamma_j(x) = -\partial_j \Phi(x)$. Here $\Phi(x)$ is the $\Gamma$-periodic solution of the problem $\triangle \Phi(x) = v(x), \int_\Omega \Phi(x) \, dx = 0$. Then

$$v(x) = -\sum_{j=1}^{d} \partial_j \gamma_j(x).$$ \hfill (9.6)

It is easy to check that the functions \eqref{9.5} satisfy condition \eqref{1.8} with suitable $\rho'$ depending on $\rho$ and $s$; the norms $\|a_j\|_{L_{\rho'}(\Omega)}$ are controlled in terms of $\|g\|_{L_{\infty}}$, $\|\mathbf{A}\|_{L_{\rho}(\Omega)}$, $\|v\|_{L_{s}(\Omega)}$, and the parameters of the lattice $\Gamma$. (See \cite{Su2, Subsection 13.1} for details). The function \eqref{9.4} satisfies condition \eqref{1.14} with suitable $s' = \min\{s; \rho'/2\}$. Thus, now the form $q$ is as in Example 1.4.

Suppose that $Q_0(x)$ is a positive definite and bounded $\Gamma$-periodic function. According to \eqref{1.38}, we consider the nonnegative operator $B_\varepsilon := B_\varepsilon + c_5 Q_0^{-1}$. Here $c_5 = (c_0 + c_4)\|Q_0^{-1}\|_{L_{\infty}}$, and the constants $c_0$ and $c_4$ correspond to the operator \eqref{9.3}. We are interested in the behavior of the operator $(B_\varepsilon - \zeta Q_0^{-1})^{-1}$, where $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. In the case under consideration, the initial data (see \eqref{1.40}) reduces to the following set

$$d, \rho, s; \|g\|_{L_{\infty}}, \|g^{-1}\|_{L_{\infty}}, \|\mathbf{A}\|_{L_{\rho}(\Omega)}, \|v\|_{L_{s}(\Omega)}, \|\mathcal{V}\|_{L_{s}(\Omega)}, \|Q_0\|_{L_{\infty}}, \|Q_0^{-1}\|_{L_{\infty}}; \text{the parameters of the lattice } \Gamma.$$ \hfill (9.7)

Let us find the effective operator. In the case under consideration, the $\Gamma$-periodic solution of problem \eqref{1.23} is the row

$$\Lambda(x) = i\Psi(x), \quad \Psi(x) = (\psi_1(x), \ldots, \psi_d(x)),$$
where \( \psi_j \in \tilde{H}^1(\Omega) \) is the solution of the problem

\[
\text{div } g(x) (\nabla \psi_j(x) + e_j) = 0, \quad \int_{\Omega} \psi_j(x) \, dx = 0.
\]

Here \( e_j, j = 1, \ldots, d, \) is the standard orthonormal basis in \( \mathbb{R}^d. \) Clearly, the functions \( \psi_j(x) \) are real-valued, and \( \Lambda(x) \) has purely imaginary entries. By \( (1.25), \) \( g(x) \) is the \( (d \times d) \)-matrix-valued function with the columns \( g(x)(\nabla \psi_j(x) + e_j), j = 1, \ldots, d. \) According to \( (1.24), \) the effective matrix is given by \( g^0 = [\Omega]^{-1} \int_{\Omega} g(x) \, dx. \) Clearly, the matrices \( \tilde{g}(x) \) and \( g^0 \) have real entries.

By \( (9.5) \) and \( (9.6), \) the periodic solution of problem \( (1.31) \) can be represented as \( \tilde{\Lambda}(x) = \tilde{\Lambda}_1(x) + i\tilde{\Lambda}_2(x), \) where the real-valued \( \Gamma \)-periodic functions \( \tilde{\Lambda}_1(x) \) and \( \tilde{\Lambda}_2(x) \) are the solutions of the following problems:

\[
- \text{div } g(x) \nabla \tilde{\Lambda}_1(x) + v(x) = 0, \quad \int_{\Omega} \tilde{\Lambda}_1(x) \, dx = 0,
\]

\[
- \text{div } g(x) \nabla \tilde{\Lambda}_2(x) + \text{div } g(x) A(x) = 0, \quad \int_{\Omega} \tilde{\Lambda}_2(x) \, dx = 0.
\]

The column \( V \) (see \( (1.32) \)) can be written as \( V = V_1 + i V_2, \) where the columns \( V_1 \) and \( V_2 \) with real entries are given by

\[
V_1 = |\Omega|^{-1} \int_{\Omega} (\nabla \Phi(x))^t g(x) \nabla \tilde{\Lambda}_2(x) \, dx,
\]

\[
V_2 = -|\Omega|^{-1} \int_{\Omega} (\nabla \Phi(x))^t g(x) \nabla \tilde{\Lambda}_1(x) \, dx.
\]

According to \( (1.33), \) the constant \( W \) is given by

\[
W = |\Omega|^{-1} \int_{\Omega} \left( \langle g(x) \nabla \tilde{\Lambda}_1(x), \nabla \tilde{\Lambda}_1(x) \rangle + \langle g(x) \nabla \tilde{\Lambda}_2(x), \nabla \tilde{\Lambda}_2(x) \rangle \right) \, dx.
\]

The effective operator for \( B_\varepsilon \) takes the form

\[
B_\varepsilon^0 u = -\text{div } g^0 \nabla u + 2i\langle \nabla u, V_1 + \eta \rangle + (-W + \overline{Q} + c_5 \overline{Q_0}) u, \quad u \in H^2(\mathbb{R}^d).
\]

In other words,

\[
B^0 = (\mathbf{D} - \mathbf{A}^0)^* g^0(\mathbf{D} - \mathbf{A}^0) + V^0 + c_5 \overline{Q_0},
\]

where

\[
\mathbf{A}^0 = (g^0)^{-1}(V_1 + \overline{gA}), \quad V^0 = \overline{\langle gA, A \rangle} - \langle g^0 \mathbf{A}, \mathbf{A} \rangle - W.
\]

According to Remark \( 7.5, \) in the case under consideration Conditions \( 7.3, \) and \( 7.3 \) are satisfied, and the norms \( \| \Lambda \|_{L_\infty} \) and \( \| \tilde{\Lambda} \|_{L_\infty} \) are controlled in terms of the initial data \( (9.7). \) Therefore, it is possible to use the simpler corrector given by

\[
K^0(\varepsilon; \zeta) := ([\Lambda^\varepsilon]\mathbf{D} + [\tilde{\Lambda}^\varepsilon])(B^0 - \zeta \overline{Q_0})^{-1} = ([\Psi^\varepsilon]\nabla + [\tilde{\Lambda}^\varepsilon])(B^0 - \zeta \overline{Q_0})^{-1}.
\]
The operator (7.11) takes the form

\[ G_3(\varepsilon; \zeta) = \tilde{g}^\varepsilon D(B^0 - \zeta \bar{Q}_0)^{-1} + g^\varepsilon (D\bar{\Lambda})^\varepsilon (B^0 - \zeta \bar{Q}_0)^{-1}. \]

Applying Theorems 4.1, 7.6(3°) and 9.9(3°), we deduce the following result.

**Proposition 9.1.** Let \( B_\varepsilon \) be the operator (9.3) whose coefficients satisfy the assumptions of Subsection 9.1. Let \( Q_0(x) \) be a \( \Gamma \)-periodic positive definite and bounded function, and let \( c_5 = (c_0 + c_4)\|Q_0^{-1}\|_{L_\infty} \), where the constants \( c_0 \) and \( c_4 \) correspond to the operator (9.3). Let \( B_\varepsilon = B_\varepsilon + c_5Q_0^\varepsilon \), and let \( B^0 \) be the effective operator (9.10), whose coefficients are defined by (9.8), (9.9), and (9.11). Let \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \), \( \zeta = |\zeta|e^{i\phi}, 0 < \phi < 2\pi \), and \( |\zeta| \geq 1 \). Suppose that \( K^0(\varepsilon; \zeta) \) is defined by (9.12). Then for \( 0 < \varepsilon \leq 1 \) we have

\[
\|(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (B^0 - \zeta \bar{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_4 \varepsilon c(\phi)^2 |\zeta|^{-1/2},
\]

\[
\|(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (B^0 - \zeta \bar{Q}_0)^{-1} - \varepsilon K^0(\varepsilon; \zeta)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq (C_8 + C_9)c(\phi)^2 \varepsilon,
\]

\[
\|g^\varepsilon \nabla (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - \left( \tilde{g}^\varepsilon \nabla + g^\varepsilon (\nabla \bar{\Lambda})^\varepsilon \right) (B^0 - \zeta \bar{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{10} c(\phi)^2 \varepsilon.
\]

Here \( c(\phi) \) is given by (4.1). The constants \( C_4, C_8, C_9, \) and \( C_{10} \) depend only on the initial data (9.7).

In order to obtain „another“ approximation of the operator \((B_\varepsilon - \zeta Q_0^\varepsilon)^{-1}\), we use Theorem 8.1 (for the principal term of approximation) and Theorem 8.5(3°).

**Proposition 9.2.** Suppose that the assumptions of Proposition 9.1 are satisfied. Denote \( f(x) = Q_0(x)^{-1/2} \) and \( f_0 = (\bar{Q}_0)^{-1/2} \). Let \( \zeta \in \mathbb{C} \setminus [c_0, \infty) \), where \( c_0 > 0 \) is the common lower bound of the operators \( \bar{B}_\varepsilon := f^\varepsilon B_\varepsilon f^\varepsilon \) and \( \bar{B}^0 := f_0 B^0 f_0 \). Let \( \varrho(\zeta) \) be given by (8.1). Then for \( 0 < \varepsilon \leq 1 \) we have

\[
\|(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (B^0 - \zeta \bar{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{11} \varrho(\zeta) \varepsilon,
\]

\[
\|(B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - (B^0 - \zeta \bar{Q}_0)^{-1} - \varepsilon K^0(\varepsilon; \zeta)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq (C_{17} + C_{18} + |\zeta + 1|^{1/2} C_{19}) \varrho(\zeta) \varepsilon,
\]

\[
\|g^\varepsilon \nabla (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} - \left( \tilde{g}^\varepsilon \nabla + g^\varepsilon (\nabla \bar{\Lambda})^\varepsilon \right) (B^0 - \zeta \bar{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq (C_{20} + |\zeta + 1|^{1/2} C_{21}) \varrho(\zeta) \varepsilon.
\]

The constants \( C_{11}, C_{17}, C_{18}, C_{19}, C_{20}, \) and \( C_{21} \) depend only on the initial data (9.7) and \( c_0 \). In the case where \( Q_0 \) is constant, estimates (9.16) and (9.17) are true with \( C_{19} = C_{21} = 0 \).
9.2. The periodic Schrödinger operator. In $L_2(\mathbb{R}^d)$, we consider the operator $\tilde{A} = D^*\tilde{g}(x)D + \tilde{v}(x)$, where $\tilde{g}(x)$ is a $\Gamma$-periodic symmetric $(d \times d)$-matrix-valued function with real entries such that $\tilde{g}(x) > 0$, $\tilde{g}$, $\tilde{g}^{-1} \in L_\infty(\mathbb{R}^d)$; and $\tilde{v}(x)$ is a real-valued $\Gamma$-periodic function such that

$$\tilde{v} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2.$$  

As usual, the precise definition of the operator $\tilde{A}$ is given in terms of the quadratic form

$$\tilde{a}[u, u] = \int_{\mathbb{R}^d} (\langle \tilde{g}(x)Du, Du \rangle + \tilde{v}(x)|u|^2) \, dx, \quad u \in H^1(\mathbb{R}^d). \quad (9.18)$$  

Adding an appropriate constant to $\tilde{v}(x)$, we may assume that the bottom of the spectrum of the operator $\tilde{A}$ is the point zero. Under this condition, the operator $\tilde{A}$ admits a convenient factorization (see, e. g., [BSu1, Chapter 6, Subsection 1.1]). To describe this factorization, we consider the equation

$$D^*\tilde{g}(x)D\omega(x) + \tilde{v}(x)\omega(x) = 0. \quad (9.19)$$  

This equation has a $\Gamma$-periodic solution $\omega \in \tilde{H}^1(\Omega)$ defined up to a constant factor. This factor can be fixed so that $\omega(x) > 0$ and

$$\int_\Omega \omega^2(x) \, dx = |\Omega|. \quad (9.20)$$  

Moreover, this solution is positive definite and bounded: $0 < \omega_0 \leq \omega(x) \leq \omega_1 < \infty$, and the norms $||\omega||_{L_\infty}, ||\omega^{-1}||_{L_\infty}$ are controlled in terms of $||\tilde{g}||_{L_\infty}$, $||\tilde{g}^{-1}||_{L_\infty}$, and $||\tilde{v}||_{L_s(\Omega)}$. The function $\omega$ is a multiplier in $H^1(\mathbb{R}^d)$, as well as in $\tilde{H}^1(\Omega)$. After the substitution $u = \omega z$, the form (9.18) turns into

$$\tilde{a}[u, u] = \int_{\mathbb{R}^d} \omega^2(x)\langle \tilde{g}(x)Dz, Dz \rangle \, dx, \quad u = \omega z, \quad z \in H^1(\mathbb{R}^d).$$  

Thus, the operator $\tilde{A}$ admits the following factorization:

$$\tilde{A} = \omega^{-1}D^*gD\omega^{-1}, \quad g = \omega^2\tilde{g}. \quad (9.21)$$  

Now we consider the operator

$$\tilde{A}_\varepsilon = (\omega^\varepsilon)^{-1}D^*g^\varepsilon D(\omega^\varepsilon)^{-1}. \quad (9.22)$$  

In the initial terms, (9.22) can be written as

$$\tilde{A}_\varepsilon = D^*\tilde{g}^\varepsilon D + \varepsilon^{-2}\tilde{v}^\varepsilon. \quad (9.23)$$  

We emphasize that the expression (9.23) involves the large factor $\varepsilon^{-2}$ standing at the rapidly oscillating potential $\tilde{v}^\varepsilon$. The operator $\tilde{A}_\varepsilon$ can be viewed as the Schrödinger operator with the rapidly oscillating metric $\tilde{g}^\varepsilon$ and the strongly singular potential $\varepsilon^{-2}\tilde{v}^\varepsilon$.

Next, as above, we assume that $A = \text{col} \{A_1(x), \ldots, A_d(x)\}$, where $A_j(x)$, $j = 1, \ldots, d$, are $\Gamma$-periodic real-valued functions satisfying (9.1). Let $\tilde{v}(x)$
and \( \mathcal{V}(x) \) be \( \Gamma \)-periodic real-valued functions such that
\[
\hat{\nu}, \hat{\mathcal{V}} \in L_4(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2,
\]
\[
\int_{\Omega} \hat{\nu}(x) \omega^2(x) \, dx = 0. \tag{9.24}
\]
Consider the operator \( \mathcal{B}_\varepsilon \) given formally by the expression
\[
\mathcal{B}_\varepsilon = (\mathbf{D} - \mathbf{A}_\varepsilon)^* \hat{g}_\varepsilon (\mathbf{D} - \mathbf{A}_\varepsilon) + \varepsilon^{-2} \hat{\nu}_\varepsilon + \varepsilon^{-1} \hat{\mathcal{V}}_\varepsilon + \hat{\mathcal{V}}^\varepsilon. \tag{9.25}
\]
(The precise definition is given in terms of the corresponding quadratic form.) The operator \( \mathcal{B}_\varepsilon \) can be treated as the Schrödinger operator with the metric \( \hat{g}_\varepsilon \), the magnetic potential \( \mathbf{A}_\varepsilon \), and the electric potential \( \varepsilon^{-2} \hat{\nu}_\varepsilon + \varepsilon^{-1} \hat{\mathcal{V}}_\varepsilon + \hat{\mathcal{V}}^\varepsilon \) containing the singular summands \( \varepsilon^{-2} \hat{\nu}_\varepsilon \) and \( \varepsilon^{-1} \hat{\mathcal{V}}_\varepsilon \).

We put
\[
v(x) := \hat{\nu}(x) \omega^2(x), \quad \mathcal{V}(x) := \hat{\mathcal{V}}(x) \omega^2(x). \tag{9.26}
\]
Using (9.22) and (9.23), we see that \( \mathcal{B}_\varepsilon = (\omega^\varepsilon)^{-1} \mathcal{B}_\varepsilon(\omega^\varepsilon)^{-1}, \) where the operator \( \mathcal{B}_\varepsilon \) is given by (9.3) with \( g \) as in (9.21), and \( v, \mathcal{V} \) as in (9.26). From (9.24) and the properties of \( \omega \) it follows that the coefficients \( v \) and \( \mathcal{V} \) satisfy (9.2).

Let \( \hat{Q}_0(x) \) be a \( \Gamma \)-periodic real-valued function; we assume that \( \hat{Q}_0(x) \) is positive definite and bounded. We put \( Q_0(x) := \hat{Q}_0(x) \omega^2(x) \). Denote \( c_5 = (c_0 + c_4) \| Q_0^{-1} \|_{L_\infty} \), where the constants \( c_0 \) and \( c_4 \) correspond to the operator \( \mathcal{B}_\varepsilon \) described above. The operators \( \mathcal{B}_\varepsilon := \mathcal{B}_\varepsilon + c_5 \hat{Q}_0^\varepsilon \) and \( \mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon + c_5 Q_0^\varepsilon \) satisfy the following relation: \( \mathcal{B}_\varepsilon = (\omega^\varepsilon)^{-1} \mathcal{B}_\varepsilon(\omega^\varepsilon)^{-1}. \) Obviously,
\[
(\mathcal{B}_\varepsilon - \zeta \hat{Q}_0^\varepsilon)^{-1} = \omega^\varepsilon (\mathcal{B}_\varepsilon - \zeta Q_0^{-1})^{-1}. \tag{9.27}
\]
Now the initial data reduces to the following set
\[
d, \rho, s; \| \hat{g} \|_{L_\infty}, \| \hat{g}^{-1} \|_{L_\infty}, \| \hat{\mathbf{A}} \|_{L_\infty(\Omega)}, \| \hat{\nu} \|_{L_4(\Omega)}, \| \hat{\mathcal{V}} \|_{L_4(\Omega)}, \| \hat{\mathcal{V}}^\varepsilon \|_{L_4(\Omega)}, \| \hat{Q}_0 \|_{L_\infty}, \| \hat{Q}_0^{-1} \|_{L_\infty}; \text{the parameters of the lattice } \Gamma. \tag{9.28}
\]
Using (9.27) and Proposition 9.1 we obtain the following result.

**Proposition 9.3.** Let \( \mathcal{B}_\varepsilon \) be the operator (9.25) whose coefficients \( \hat{g}_\varepsilon, \hat{\nu}_\varepsilon, \) and \( \hat{\mathcal{V}}^\varepsilon \) satisfy the assumptions of Subsection 9.2. Let \( \omega(x) \) be the \( \Gamma \)-periodic positive solution of equation (9.19) satisfying condition (9.20). Let \( \mathcal{B}_\varepsilon \) be the operator (9.3) with the coefficients \( g_\varepsilon = \hat{g}_\varepsilon(\omega^\varepsilon)^2, \mathbf{A}_\varepsilon, \nu_\varepsilon = \hat{\nu}_\varepsilon(\omega^\varepsilon)^2, \) and \( \mathcal{V}_\varepsilon = \hat{\mathcal{V}}^\varepsilon(\omega^\varepsilon)^2. \) Let \( \hat{Q}_0 \) be a \( \Gamma \)-periodic bounded and positive definite function, and let \( Q_0 := \hat{Q}_0 \omega^2 \). Denote \( c_5 = (c_0 + c_4) \| Q_0^{-1} \|_{L_\infty}, \) where the constants \( c_0 \) and \( c_4 \) correspond to the operator \( \mathcal{B}_\varepsilon \). Let \( \mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon + c_5 Q_0^\varepsilon \) and \( \mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon + c_5 Q_0^\varepsilon. \) Let \( B^0 \) be the effective operator for \( B_\varepsilon \) defined by (9.10). Let \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \zeta = |\zeta|e^{i\phi}, \phi \in (0, 2\pi), \) and \( |\zeta| \geq 1. \) Let \( K^0(\varepsilon; \zeta) \) be the
corrector (9.12) for the operator $B_\varepsilon$. Then for $0 < \varepsilon \leq 1$ we have

$$
\|(\tilde{B}_\varepsilon - \zeta \tilde{Q}_0^\varepsilon)^{-1} - \omega^\varepsilon (B^0 - \zeta \tilde{Q}_0^0)^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C_4 \|\omega\|_{L_\infty}^2 c(\phi)^2 \varepsilon |\zeta|^{-1/2},
$$

(9.29)

$$
\|\omega^\varepsilon)^{-1}(\tilde{B}_\varepsilon - \zeta \tilde{Q}_0^\varepsilon)^{-1} - (B^0 - \zeta \tilde{Q}_0^0)^{-1}\omega^\varepsilon - \varepsilon K^0(\varepsilon; \zeta) \omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\
\leq (C_8 + C_9) \|\omega\|_{L_\infty} c(\phi)^2 \varepsilon,
$$

(9.30)

$$
\|g^\varepsilon \nabla (\omega^\varepsilon)^{-1}(\tilde{B}_\varepsilon - \zeta \tilde{Q}_0^\varepsilon)^{-1} - \left(g^\varepsilon \nabla + g^\varepsilon (\nabla \tilde{A})^\varepsilon\right) (B^0 - \zeta \tilde{Q}_0^0)^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\
\leq C_{10} \|\omega\|_{L_\infty} c(\phi)^2 \varepsilon.
$$

(9.31)

Here $c(\phi)$ is given by (4.1). The constants $C_4 \|\omega\|_{L_\infty}^2$, $(C_8 + C_9) \|\omega\|_{L_\infty}$, and $C_{10} \|\omega\|_{L_\infty}$ depend only on the initial data (9.28).

Proof. Multiplying the operators under the norm sign in (9.13) by $\omega^\varepsilon$ from both sides and using (9.27), we arrive at (9.29).

By (9.27), $(\omega^\varepsilon)^{-1}(\tilde{B}_\varepsilon - \zeta \tilde{Q}_0^\varepsilon)^{-1} = (B^0 - \zeta \tilde{Q}_0^0)^{-1}\omega^\varepsilon$. Multiplying the operators under the norm sign in (9.14) by $\omega^\varepsilon$ from the right, we obtain (9.30). Similarly, (9.15) implies (9.31).

In order to obtain approximation of the operator $(\tilde{B}_\varepsilon - \zeta \tilde{Q}_0^\varepsilon)^{-1}$ in a wider domain of the parameter $\zeta$, we use Proposition 9.2. By analogy with the proof of Proposition 9.3 it is easy to check the following statement.

**Proposition 9.4.** Suppose that the assumptions of Proposition 9.3 are satisfied. Denote $f(x) = Q_0(x)^{-1/2}$ and $f_0 = (Q_0)^{-1/2}$. Let $\zeta \in \mathbb{C} \setminus [c_0, \infty)$, where $c_0 \geq 0$ is a common lower bound of the operators $B_\varepsilon := f^\varepsilon B_\varepsilon f^\varepsilon$ and $B^0 := f_0 B_0 f_0^n$. Then for $0 < \varepsilon \leq 1$ we have

$$
\|(\tilde{B}_\varepsilon - \zeta \tilde{Q}_0^\varepsilon)^{-1} - \omega^\varepsilon (B^0 - \zeta \tilde{Q}_0^0)^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C_{11} \|\omega\|_{L_\infty}^2 \varrho(\zeta) \varepsilon,
$$

(9.32)

$$
\|\omega^\varepsilon)^{-1}(\tilde{B}_\varepsilon - \zeta \tilde{Q}_0^\varepsilon)^{-1} - (B^0 - \zeta \tilde{Q}_0^0)^{-1}\omega^\varepsilon - \varepsilon K^0(\varepsilon; \zeta) \omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\
\leq (C_{17} + C_{18} + |\zeta + 1^{1/2}C_{19}|) \|\omega\|_{L_\infty} \varrho(\zeta) \varepsilon,
$$

(9.33)

Here $\varrho(\zeta)$ is given by (8.1). The constants $C_{11} \|\omega\|_{L_\infty}^2$, $(C_{17} + C_{18}) \|\omega\|_{L_\infty}$, $C_{19} \|\omega\|_{L_\infty}$, $C_{20} \varrho(\zeta) \|\omega\|_{L_\infty}$, and $C_{21} \|\omega\|_{L_\infty}$ are controlled in terms of the initial data (9.28) and $c_0$. In the case where $Q_0$ is constant, estimates (9.32) and (9.33) are valid with $C_{19} = C_{21} = 0$. 
Remark 9.5. Propositions 9.3 and 9.4 demonstrate that for the operator (9.25) the nature of the results is different from the results for the operator (9.3). Because of the presence of the strongly singular potential $\varepsilon^{2}\tilde{v}_{\varepsilon}$, the generalized resolvent $(\tilde{B}_{\varepsilon} - \zeta \tilde{Q}_{0})^{-1}$ has no limit in the $L_{2}(\mathbb{R}^{d})$-operator norm; it is approximated by the generalized resolvent $(B^{0} - \zeta Q_{0})^{-1}$ sandwiched between the rapidly oscillating factors $\omega_{\varepsilon}$.

9.3. The two-dimensional Pauli operator. Note that it is also possible to apply the general results to the two-dimensional periodic Pauli operator with a singular magnetic potential perturbed by a singular electric potential. This operator was considered in [Su2, §14] by using a convenient factorization for the Pauli operator. The nature of the results for this operator is similar to Propositions 9.3 and 9.4. We will not dwell on this.

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