Twisted determinants and bosonic open strings in an electromagnetic field.

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Abstract: The bosonization equivalence between the 2-dimensional Dirac and Laplacian operators can be used to derive new interesting identities involving Theta functions. We use these formulae to compute the multiloop partition function of the bosonic open string in presence of a constant electromagnetic field.

1 Twisted determinants

Among the basic building blocks of string amplitudes one finds the determinants of the Laplacian and of the Dirac operator. Since these operators are present in the 2D action describing the free string propagation, their determinants will appear in all perturbative amplitudes, starting from the simplest one: the vacuum energy. A detailed study of these determinants was performed in \cite{1} (and references therein). Moreover the equivalence between fermionic and the bosonic determinants, through the bosonization duality, can be used as a tool for finding or proving (in a “physicist’s way”) interesting mathematical identities among Theta functions.

Different approaches can be used in string computations. For instance one can use the Polyakov path integral and evaluate it directly on the appropriate Riemann surface (see \cite{2} for a classical review and \cite{3} and references therein for the recent applications of this technique to the two-loop case). Another powerful constructive way to derive the building blocks of string amplitudes is the sewing technique. This is a very old idea \cite{4} allowing to construct higher loop amplitudes from tree diagrams: pairs of external legs are sewn together taking the trace over all possible states with the insertion of a propagator that geometrically identifies the neighborhoods around two punctures. In this way one gets a particular parametrization of the $g$–loop Riemann surface known as Schottky uniformization, where all geometrical quantities are written in terms of products over the Schottky group (see \cite{5} and references therein). For instance, at 1–loop one gets the

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partition function of a chiral scalar field in the form \( \prod_{n=1}^{\infty} (1 - q^n)^3 \), where \( q \) is a complex parameter representing the modulus of the torus; it is related to period matrix entering in the corresponding Theta–function by \( q = \exp(2\pi i \tau) \), with \( \text{Im} \tau > 0 \iff |q| < 1 \).

The formulation in terms of Schottky products and the one with Theta–functions are complementary: the geometrical expressions in terms of Theta–functions make manifest the modular properties, while the Schottky uniformization makes manifest the unitarity properties of string amplitudes; in fact the Schottky multipliers \( q \) are basically the equivalent of the (exponential of the) Schwinger parameters in field theory, but they are not suitable to deal with modular transformations, which can be non–analytic in \( q \) (for instance, consider \( \tau \to -1/\tau \)).

Following the ideas of [6], we use the equivalence between bosonic and fermionic systems as a device allowing to recast Schottky products in terms of Theta–functions for any genus \( g \). By using the techniques of [7], we generalize previous analysis [6] in two directions. On one hand we consider fermions of spin \( (\lambda, 1 - \lambda) \) (the system \( b, c \)) and the dual bosonic field \( \phi \) with the background charge \( Q = 1 - 2\lambda \), instead of focusing just on the simplest case \( \lambda = 1/2 \). On the other hand, for general \( \lambda \) we also consider twisted periodicity conditions along the \( b \)–cycles\(^{1}\); the twisted boundary conditions along the \( b \)–cycles are enforced by the insertion together with the propagator of the operator \( e^{2\pi i j_0} \) on the fermionic side, where \( j_0 \) is the fermionic number, and of the operator \( e^{2\pi i \epsilon \phi p_0} \) on the bosonic side. Notice, however, that \( \epsilon_{\phi} = \epsilon - 1/2 \), where the additional factor of \( 1/2 \) is necessary in order to reproduce in the bosonic language the usual \((-1)^F\)–twist of the fermionic traces.

This twisting can be equivalently thought as the effect of a flat gauge connection along the \( b \)–cycles on a minimally coupled fermionic system. Thus the periodicity parameters \( (\epsilon_{\mu}) \) can be naturally thought as non–geometrical parameters of the Riemann surface.

The sewing technique for computing multiloop amplitudes is discussed in detail in [7]. The main tool for the sewing technique is a generalization [8], of the usual vertex operators, where also the emitted states are described by a whole Hilbert space

\[
\langle \mathcal{V}_I^v | = \langle 0, x_0 = 0 | : \exp \left\{ \int_0^1 dz \left( -\phi^v (1 - z) \partial_z \phi_I (z) \right) \right\} : , \tag{1}
\]

\[
\langle \mathcal{V}_I^{bc} | = \langle 0, q = -Q | : \exp \left\{ \int_0^1 dz \left( b^v (1 - z) c_I (z) - c^v (1 - z) b_I (z) \right) \right\} : .
\]

Here the coordinates with the superscript \( v \) describe a propagating (virtual) string, while those with a subscript \( I \) describe a generic emitted state in the Hilbert space \( \mathcal{H}_I \). The modes of the two types of fields (anti)–commute among them, since they refer to completely independent states. Let us stress that the bra–vector in Eq. (1) is the vacuum in the Hilbert space of the emitted string, so that \( \langle \mathcal{V} | \) is an operator in the \( v \)–Hilbert space of the propagating string. The vertex \( \langle \mathcal{V} | \) can be seen as the generator of all possible three string interactions which are obtained by saturating it with highest weight states.

Following [7] it is easy to construct the generator of the \( N \)-point Green functions on

\(^{1}\)As usual, we call \( b \)–cycles the loops in the worldsheet along the \( \tau \) direction and \( a \)–cycles the spatial loops along \( \sigma \).
where the $N$ Hilbert spaces labeled by $I$ describe the external fields and are all independent, while $\gamma_I$ are (arbitrary) projective transformations mapping the interaction point from $z = 1$ as in (1) to the arbitrary point $z_I$. The generalization $\langle V_{N;g}^{bc} \rangle$ of the vertex (2) at genus $g$ is built starting from $\langle V_{N;0}^{bc} \rangle$ with $N' = 2g + N$ and then identifying $g$ pairs of Hilbert spaces (labeled by the index $\mu = 1, \ldots, 2g$) by means of twisted propagators. For a genus $g$ surface, the Riemann–Roch theorem shows that the simplest non–trivial amplitude is the correlation function with $N_b = \lvert Q \rvert (g - 1)$ insertions of the $b$ field:

\[
Z_\epsilon^\lambda(z_1, \ldots, z_{N_b}) = \langle \prod_{I=1}^{N_b} b(z_I) \rangle_{(\epsilon, \lambda)} = \langle V_{N_b;g}^{bc} \rangle \prod_{I=1}^{N_b} \langle b_{-\lambda}^{(I)}(q = 0)I \rangle ,
\]

where the subscript $\epsilon$ reminds the non–trivial boundary conditions along the $b$–cycles. As it is shown in Ref. [5] one gets

\[
Z_\epsilon^\lambda(z_1, \ldots, z_{N_b}) \equiv \det(1 - H) \mathcal{F}^{(\lambda)} ,
\]

where

\[
\det(1 - H) = \prod_{\alpha} \prod_{n = \lambda}^{\infty} (1 - e^{-2\pi i \epsilon \cdot N_\alpha q_n^\alpha}) (1 - e^{2\pi i \epsilon \cdot N_\alpha q_n^\alpha}) ;
\]

$N_\alpha$ is a vector with $g$ integer entries; the $\mu^{th}$ entry counts how many times the Schottky generator $S_\mu$ enters in the element of the Schottky group $T_\alpha$, whose multiplier is $q_\alpha$ ($S_\mu$ contributes 1, while $(S_\mu)^{-1}$ contributes $-1$). The product $\prod_{\alpha}$ is over the primary classes of the Schottky group excluding the identity and counting $T_\alpha$ and its inverse only once. For the expression of $\mathcal{F}^{(\lambda)}$ in the general case, see [5]; in this talk we focus on the case $Q = -1$, where the conformal weight of $b(z)$ is $\lambda = 1$ and the one of $c(z)$ is zero; we consider generic values for the twist $\epsilon_\mu$.

In the hypothesis that at least one $\epsilon$ is non–trivial, for instance $\epsilon_g \neq 0$, one gets:

\[
\mathcal{F}^{(1)} = \det \begin{pmatrix} \zeta_1(z_1) & \ldots & \zeta_\mu(z_1) \\ \vdots & \ddots & \vdots \\ \zeta_1(z_{g-1}) & \ldots & \zeta_\mu(z_{g-1}) \\ e^{2\pi i \epsilon_1} - 1 & \ldots & e^{2\pi i \epsilon_\mu} - 1 \end{pmatrix} = (e^{2\pi i \epsilon_-} - 1) \det [\Omega_i(z_j)] ,
\]

where $i, j = 1, \ldots, g - 1$ and the expressions of the $\Omega_\mu(z)$ and $\zeta_\mu(z)$ are given in Ref. [5].

From these expressions it is possible to show that the both $\Omega_\mu(z)$ and $\zeta_\mu(z)$ are periodic along the $a$–cycles and get a phase $e^{2\pi i \epsilon_\nu}$ going around the cycle $b_\nu$. Moreover the $\Omega$‘s are completely regular on the Riemann surface, and thus give an explicit representation for the $g - 1$ twisted abelian differentials\footnote{These 1–forms are known as Prym differentials, see [5], where the unique $\Omega$ of the $g = 2$ case is characterized in terms of Theta functions.}. On the other hand, for $\epsilon_\mu \neq 0$ the $\zeta_\mu$’s are not analytic, but they reduce to the $g$ untwisted abelian differentials $\omega_\mu$ when $\epsilon \to 0$.
In an analogous way the sewing technique allows to derive the bosonic correlation functions corresponding to $Z^Q_\epsilon$ of Eq. (8). In the notation of [6] these correlators are

$$Z^Q_\epsilon(z_1, \ldots, z_{N_b}) = \prod_{I=1}^{N_b} :e^{-\phi(z_I)}: e^{\phi(z_I)} = \langle V^\phi_{N_b g} \prod_{I=1}^{N_b} (|q = -1 \rangle_I) \rangle,$$

where the bosonic system has background charge $Q = 1 - 2\lambda$.

By comparing [5] the bosonic and the fermionic results one gets for $\lambda = 1$ the identity

$$C^{(1)}_\epsilon \ F^{(1)} = B_\epsilon(z_I; \tau, \Delta, \omega),$$

where

$$B_\epsilon(z_I; \tau, \Delta, \omega) = \prod_{I=1}^{g-1} \sigma(z_I) \prod_{l<j} E(z_I, z_J) \theta \left[ \frac{0}{\epsilon} \right] \left( \Delta - \sum_{I=1}^{g-1} J(z_I) \right) \tau, \quad \epsilon = 1 + \sum_{I=1}^{g-1} \lambda_I \left( 1 - q_{\alpha}^n \right),$$

$$C^{(1)}_\epsilon = \prod_{n=1}^{\infty} (1 - q_{\alpha}^n) (1 - e^{2\pi i \epsilon N_\alpha} q_{\alpha}^n) (1 - e^{2\pi i \epsilon N_{\alpha} q_{\alpha}^n}).$$

The complex $g$-component vector $J(z)$ is given by the Jacobi map $J_\mu(z) = \frac{1}{2\pi i} \int_{z_0}^{z} \omega_\mu$ and the explicit expressions in terms of the Schottky parametrization of $\omega_\mu$, $\sigma(z_I)$, of the prime form $E(z_I, z_J)$ and of the Riemann constant $\Delta$ are given in Appendix A of [7].

One can easily, but not trivially, check that the two sides of the identity have the same periodicity properties; moreover, for $g = 2$, we have successfully compared the first four terms of the expansion in powers of $q_\mu$ of the Schottky series in the two sides of [8].

2 Modular properties

We now turn to the study of the modular properties of the infinite product (10) and, in particular, we are interested in the modular transformation $\tau \to (A \tau + B)(C \tau + D)^{-1}$, with $A = D = 0$, $B = -C = 1$. This map is highly non-analytic in the language of the Schottky parametrization and can be analyzed only by using the identity derived in the previous section. The strategy is clear: we first rewrite the product (10) in terms of geometrical objects, whose modular properties are known (see for instance [4], [5]), then perform explicitly the modular transformation we are interested in, and finally use again Eq. (8) to recast the result in the Schottky language. Of course the Schottky multipliers $k_\mu$ appearing in the final result are different from the original ones and are related to the $q_\mu$’s in a complicated non-analytic way. So, as a first step, we need to study the modular transformation of the “geometrical” expression $B_\epsilon$; by using the results collected in Appendix A of [6] one gets

$$B_\epsilon(z_I; \tau, \Delta, \omega) = \xi K^{g-1} \tilde{B}_\epsilon(z_I; \tilde{\tau}, \tilde{\Delta}, \tilde{\omega}) \sqrt{\det \tilde{\tau}} \ e^{i \pi i \tilde{\tau}^{-1} \tilde{\epsilon} + 2 \pi i \tilde{\tau}^{-1} \tilde{\Delta} - \sum_{I=1}^{g-1} \tilde{J}(z_I)},$$

where $K$ is an undetermined factor independent of the $z_I$ and $\xi$ is a phase due to the quantum Weyl anomaly in two dimension. Both of them will not be important for the final result. Finally the tilded quantities are related to the untilded ones by

$$\tilde{\tau} = -\tau^{-1}, \quad \tilde{\Delta} = \Delta \tilde{\tau}, \quad \tilde{\omega} = \omega \tilde{\tau}, \quad \tilde{\epsilon} = \epsilon \tilde{\tau}.$$

Finally qubits are related to the untilded ones by

$$\tilde{q} = -q^{-1}, \quad \tilde{\omega} = \omega \tilde{q}, \quad \tilde{\epsilon} = \epsilon \tilde{q}.$$
By using again the identity (8) for $\tilde{B}_\epsilon$, we can write
\[ C_\epsilon \mathcal{F}^{(1)}_\epsilon = \xi \ K^{g-1} \sqrt{\det \tilde{\tau}} \ e^{i \pi \epsilon \tilde{\tau} + 2i \pi \epsilon (\Delta - \sum_{l=1}^{g-1} J(z_l))} \tilde{C}_\epsilon \mathcal{F}^{(1)}_\epsilon, \]
where all quantities are expressed in terms of the multipliers $q_\mu$ on the l.h.s., and in terms of the $k_\mu$’s on the r.h.s.; moreover the rôle of the $a$ and the $b$–cycles is exchanged in the two sides ($\tilde{a}_\mu = -b_\mu$, $\tilde{b}_\mu = a_\mu$). By studying the $\epsilon \to 0$ limit of this relation, one can derive the multiloop generalization of the usual modular transformation for the 1–loop Dedekind $\eta$–function. In fact, at the first order in $\epsilon$, one has $\tilde{F}^{(1)}_\epsilon = \mathcal{F}^{(1)}_\epsilon \det \tilde{\tau}$, since these determinants become linear in the $\epsilon_\mu$’s or the $\tilde{\epsilon}_\mu$’s, the $\zeta_\mu$’s and the $\tilde{\zeta}_\mu$’s reduce to the usual differentials, and all the elements are connected by Eq. (12). So the determinants of the abelian differentials on the two sides of (13) cancel, and one obtains
\[ \prod_{\alpha}^{\infty} \prod_{n=1}^{\infty} (1 - q_\alpha^n)^3 = \xi \ K^{g-1} (\det \tilde{\tau})^{3/2} \prod_{\alpha}^{\infty} \prod_{n=1}^{\infty} (1 - k_\alpha^n)^3. \]

In order to avoid the appearance of undetermined factors $\xi$ and $K$ we will focus on the ratio
\[ D_{\epsilon}^q = \prod_{\alpha}^{\infty} \prod_{n=1}^{\infty} \frac{(1 - e^{-2i \epsilon N_a q_\alpha^n})(1 - e^{2i \epsilon N_a q_\alpha^n})}{(1 - q_\alpha^n)^2}. \]

The modular transformation of $D_{\epsilon}^q$ can be again derived from (13). The additional complication brought by the presence of a non-trivial $\epsilon$ is that the determinants in the two sides of (13) do not cancel any more. However we can simplify the l.h.s. of the relation by integrating each $z_I$ along the $a_I$ cycle, so that the matrix defined in (6) becomes diagonal and one simply gets $\mathcal{F}^{(1)} = e^{2i \pi \epsilon g} - 1$. On the r.h.s., the integration has to be taken along the $b_I$ cycles and is non-trivial since the $z_I$’s appear both in the determinant $\tilde{F}^{(1)}_\epsilon$ and in the exponent. In this way one obtains from Eqs. (13) and (14)
\[ (e^{2i \pi \epsilon g} - 1) D_{\epsilon}^q = D_{\epsilon}^k \ e^{i \pi \epsilon \tilde{\tau} + 2i \pi \epsilon \Delta} \frac{\det \rho}{\det \tilde{\tau}}. \]

where $\rho_{\nu \mu}$ is a $g \times g$ matrix
\[ \rho_{\nu \mu} = \frac{1}{2\pi i} \int_{w} dz \left[ \tilde{\zeta}_\mu(z) e^{-z \tilde{z} \omega} \right] \text{with } \nu \neq g, \text{ and } \rho_{g \mu} = e^{2i \epsilon (\tilde{\epsilon} \mu)_g} - 1; \]

the tilde on $\zeta$, $S_\nu$, $\omega$ and $\tau$ indicates that their expression is in terms of the $k_\mu$ multipliers, with twisting parameter for the $\tilde{z}$’s equal to $\tilde{\epsilon} = \epsilon \cdot \tilde{\tau}$. One can check that $\rho_{\nu \mu}$ does not depend on the variable $w$. Analogously, Eq. (13) is independent of the base-point $z_0$ of the Jacobi map, since the dependence on $z_0$ of $\Delta$ and of $\rho_{\nu \mu}$ compensate each other.

This result is quite interesting for two reasons. From the mathematical point of view, Eq. (17) suggests how to extend the relations presented in the previous section (in particular Eq. (8)) to the most general twists. In fact, Eq. (8) can be generalized by replacing $B_\epsilon$ in Eq. (9) with
\[ B_{a_\epsilon, \epsilon b} = \prod_{l=1}^{g} \sigma(z_l) \prod_{I < J} E(z_I, z_J) \theta \left[ \frac{\epsilon_a}{\epsilon_b} \right] \left( \Delta - \sum_{I=1}^{g-1} J(z_I) \right) \tau, \]
by replacing the twist $\epsilon$ in the product (10) with $\epsilon = \epsilon_b - \tau \cdot \epsilon_a$ and, finally, by modifying the determinant $F^{(1)}$ in the following way
\[ e^{i\pi (\epsilon \cdot \tau - 1 \cdot \epsilon_b)} \widehat{F}^{(1)}_{\epsilon_a, \epsilon_b}, \]
where $\widehat{F}^{(1)}_{\epsilon_a, \epsilon_b}$ is given by (6) with the $\zeta_{\mu}$’s replaced by
\[ \hat{\zeta}^{(1)}_{\epsilon_a, \epsilon_b}(z_I) = \zeta^{(1)}_{\epsilon}(z_I) \exp \left( \frac{-2\pi i}{g-1} \epsilon_a \cdot \Delta z_I \right). \]

One can check that both $B_{\epsilon_a, \epsilon_b}$ (18) and the $\hat{\zeta}^{(1)}_{\epsilon_a, \epsilon_b}$’s get a phase factor of $\exp (2\pi i \epsilon)$ (or $\exp (2\pi i \epsilon)$) when $z_I$ goes once around the cycle $a_\nu$ (or $b_\nu$). Therefore the $\hat{\zeta}^{(1)}_{\epsilon_a, \epsilon_b}$’s in (20) can be used as building blocks for constructing the abelian differentials with twists both along the $a$ and the $b$ cycles.

A physical application of the results just presented is to the multiloop partition function of a charged open bosonic string. In fact, by using the boundary state approach as explained in [10], it is possible to derive the interaction amplitude among many (bosonic) D-branes. In [10] one can find this amplitude in absence of external field (or for neutral strings). If the “last” D-brane is singled out and a non trivial gauge field strength $F_{12} = - F_{12} = f = \tan \pi \epsilon$ is switched on, the D-brane interaction is obviously modified. However it turns out that this modification is quite simple: the final result for the integrand representing the D-brane interaction is just the usual untwisted bosonic measure (see [10]) multiplied by the factor $\cos(\pi \epsilon) D_q [\hat{\tau}]^{-1}$, where the twist in $D_q$ is given by the $g$-vector $\vec{\epsilon} = (0, \ldots, 0, \epsilon)$. Notice that the factor of $1 / \cos(\pi \epsilon)$ is just a rewriting of the Born-Infeld contribution to the boundary state normalization (see for instance Sect. 3 of [11]). As in [10] this result is valid for the closed string channel expression, but thanks to the analysis of this section, it is now possible to rewrite the same result in the open string channel. From this point of view the interaction is just the partition function $P_\epsilon$ for a charged open bosonic string and becomes
\[ P_\epsilon = \frac{e^{i\pi \epsilon} - 1}{\cos(\pi \epsilon)} \left[ D^k_{\hat{\tau}} \right]^{-1} e^{-i\pi \epsilon \hat{\tau} \cdot \hat{\epsilon} - 2\pi \epsilon \Delta} \frac{\det \hat{\tau}}{\det \rho} P_0. \]

Notice that for $g = 1$, the modification of the usual partition function (i.e. the coefficient of $P_0$) is exactly equal to Eq. (22) of [12]. In order to make the comparison explicit one needs to change the magnetic field here introduced into an electric one, as considered in [12], ($\epsilon \rightarrow i\epsilon$) and also to change the conventions on the period matrix (here purely imaginary for open strings, $\hat{\tau} \rightarrow i\tau/2$). Then it is easy to see that the two results agree, by using $f = \tan \pi \epsilon$ and the explicit formula for the $g = 1$ Riemann class: $\Delta = -\hat{\tau}/2 + 1/2$.

We hope to give more details on both these developments in a subsequent publication.

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