COMPLETENESS OF SETS OF SHIFTS IN INVARIANT BANACH SPACES OF TEMPERED DISTRIBUTIONS VIA TAUBERIAN THEOREMS

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Abstract. The main result of this paper is a far reaching generalization of the completeness result given by V. Katsnelson in a recent paper (28). Making use of Tauberian Theorems for Beurling algebras as found in the book of H. Reiter (31) as well as the theory of invariant Banach spaces of tempered distributions we demonstrate that instead of using all dilates of a given function with non-vanishing integral (this case is treated in 22) it is enough to use one single function $g$ such as the Gaussian and its translates in order to generate a dense subspace of the given invariant space. The key condition now is the non-vanishing of the Fourier transform $\hat{g}(y)$, for any $y \in \mathbb{R}^d$.

1. Introduction

This paper can be seen as an alternative approach to the question of completeness of sets of translates of a given test function for a large variety of Banach space of tempered distributions. The motivation for the current paper is the wish to demonstrate that the very specific results describing the density of the linear span of the set of all translates and dilates of the Gauss function as given in 28 can be generalized into several directions.

In the companion paper 22 we have shown that only the non-vanishing integral, i.e. without loss of generality the assumption $\int_{\mathbb{R}^d} g(x)dx = 1$ for the generating Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ is sufficient in order to guarantee the density for quite a few Banach spaces with double module structure. As it turned out, the setting of the paper 22 appeared to be most appropriate, which is quite similar to the setting of so-called “standard spaces” as used in papers on compactness (14) or double module structures (3). As it is clear that there is only a chance for such a statement if $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of $(B, \| \cdot \|_B)$, so we will make this minimality assumption throughout this paper.

Observing that the Gauss function has another important property, namely a nowhere vanishing Fourier transform, Tauberian Theorems come to mind (see 37, 38, 32, 29, 16), which show that this “Tauberian condition” implies that density of translates in $(L^1(\mathbb{R}^d), \| \cdot \|_1)$ resp. in Beurling algebras $(L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})$. It is this branch of analysis which we want to pursue here further by demonstrating that this density can be moved from Beurling algebras to dense Banach ideals in such Beurling algebras and using the double module structure to the same family of Banach spaces of tempered distributions which has already served well in 22.

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The main advantage of the approach given here is the fact that it does not make use of dilations, so we do not have to generate Dirac sequences by compressing the given building block \( g \in \mathcal{S}(\mathbb{R}^d) \) (with just \( \hat{g}(0) \neq 0 \)). Thus at least the formulation of the main result of the current paper can easily be transferred to the setting of LCA (locally compact Abelian) groups without significant changes.

One may also argue that the restrictions to polynomially moderate weight functions and hence to stay within the world of \( \mathcal{S}'(\mathbb{R}^d) \), the tempered distributions, is somewhat restrictive, and that one should look for such results also in the context of ultra-distributions. In this way much more sensitive norms are used for the space of test functions (so in a way the kind of approximation that has to be achieved is much more challenging), but also much bigger spaces have to be treated, where many more and less regular elements have to be approximated in the sense of the given norm.

Overall we have chosen to present our results in the context of tempered distributions over \( \mathbb{G} = \mathbb{R}^d \), in order to make the key steps more clear. The generalization (of basically the same principle ideas) in the context of ultra-distributions over LCA groups thus will be left to a future publication.

In order to keep the paper readable we have avoided this outmost level of generality, mentioning once more that this is still a very far-reaching extension of the results in [28].

2. Translation and modulation invariant spaces

Throughout this paper we will work with the following standard assumptions, similar to the setting chosen in [8] or [22]:

**Definition 1.** A Banach space \((B, \| \cdot \|_B)\) is called a minimal tempered standard space (abbreviated as minimal tempered TMIB, or equivalently a minimal TMIB in the sense of [7], [22]) if the following conditions are valid:

1. We assume the following chain of continuous embeddings:
   \[
   \mathcal{S}(\mathbb{R}^d) \hookrightarrow (B, \| \cdot \|_B) \hookrightarrow \mathcal{S}'(\mathbb{R}^d); 
   \]

2. \( \mathcal{S}(\mathbb{R}^d) \) is dense in \((B, \| \cdot \|_B)\) (minimality);
3. \((B, \| \cdot \|_B)\) is translation invariant, and for some \( s_1 \in \mathbb{N} \) and \( C_1 > 0 \) one has
   \[
   \| T_x f \|_B \leq C_1 (x)^{s_1} \| f \|_B \quad \forall x \in \mathbb{R}^d, f \in B. 
   \]
4. \((B, \| \cdot \|_B)\) is modulation invariant, and for some \( s_2 \in \mathbb{N} \) and \( C_2 > 0 \) one has
   \[
   \| M_y f \|_B \leq C_2 (y)^{s_2} \| f \|_B \quad \forall y \in \mathbb{R}^d, f \in B. 
   \]

Here we use the Japanese bracket symbol \( \langle x \rangle \) respectively \( \langle y \rangle \) for the function \( v_s(z) = (1 + |z|^2)^{s/2}, z \in \mathbb{R}^d \), which is also known as Beurling weight of polynomial type, because it satisfies for \( s \geq 0 \) the submultiplicativity property

\[
\langle x + y \rangle \leq \langle x \rangle \langle y \rangle, \quad x, y \in \mathbb{R}^d. 
\]

For any such (polynomial) weight \( v_s \) the corresponding weighted \( L^1 \)-space \( L^1_{v_s}(\mathbb{R}^d) \) is a Banach algebra with respect to convolution, continuously embedded into \((L^1(\mathbb{R}^d), \| \cdot \|_1)\) (since \( v_s(x) \geq 1 \)) and having bounded approximate units (obtained by \( L^1 \)-norm preserving compression, so-called Dirac sequences).

Whenever we use generic, submultiplicative Beurling weights (not necessarily of polynomial type) we will make use of the standard notation \( w \). The corresponding weighted
for any bounded, approximate unit \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w(\mathbb{R}^d)})\) are Banach algebras with respect to convolution (we use the symbol \(f * g\), called Beurling algebras. The natural norm to be used is (cf. [31])

\[
\|f\|_{L^1_w(\mathbb{R}^d)} = \|f\|_{1,w} := \|f \cdot w\|_{L^1}, \quad f \in L^1_w(\mathbb{R}^d).
\]

For the formulation of our arguments the terminology of the theory of Banach modules will be convenient (see [31]):

**Definition 2.** A Banach space \((B, \| \cdot \|_B)\) is called a Banach module over a Banach algebra \((A, \| \cdot \|_A)\) if there is a (usually natural) embedding of \((A, \| \cdot \|_A)\) into the operator algebra over \((B, \| \cdot \|_B)\), written as \((a, b) \mapsto a \bullet b\), such that this bilinear (and associative) mapping satisfies

\[
\|a \bullet b\|_B \leq \|a\|_A \|b\|_B, \quad a \in A, b \in B. \quad (5)
\]

If, in addition, \((B, \| \cdot \|_B) \hookrightarrow (A, \| \cdot \|_A)\) and the module operation is just the internal multiplication of the algebra \((A, \| \cdot \|_A)\), we call \((B, \| \cdot \|_B)\) a Banach ideal.

A Banach module is called essential if \(A \bullet B\) generates a dense subspace of \((B, \| \cdot \|_B)\).

For our applications the Banach algebras \((A, \| \cdot \|_A)\) have bounded approximate units, i.e. there are bounded sequences or nets \((e_\alpha)_{\alpha \in I}\) of elements in \((A, \| \cdot \|_A)\) such that

\[
\lim_{\alpha \to \infty} e_\alpha \bullet a = a, \quad a \in A.
\]

In such a case the Cohen-Hewitt factorization Theorem can be applied and gives

\[
B = A \bullet B = \{a \bullet b \mid a \in A, b \in B\}.
\]

We only consider Beurling algebras as Banach convolution algebras (in this case we write \(f * g\) for the abstract multiplication \(f \bullet g\) acting on a TMIB via convolution (we will speak of Banach convolution modules) or Banach algebras inside of \((C_0(\mathbb{R}^d), \| \cdot \|_1)\) acting on \((B, \| \cdot \|_B)\) via pointwise multiplication (we write \(f \cdot g\), or simply \(fg\)).

Essential Banach ideals in the Banach convolution algebra \((L^1(\mathbb{R}^d), \| \cdot \|_1)\) are exactly the Segal algebras in the sense of H. Reiter ([31]) (also called normed ideals in [5]), see [10]. Further references to abstract Segal algebras are given in papers of J.T. Burnham [4], see also [11].

The density of \(S(\mathbb{R}^d)\) in \((B, \| \cdot \|_B)\) and the conditions (2) and (3) imply a double module structure for such Banach spaces:

**Proposition 1.** Any minimal tempered TMIB \((B, \| \cdot \|_B)\) has a double module structure:

1. \((B, \| \cdot \|_B)\) is an essential Banach convolution module over the Beurling algebra \(L^1_{v_1} : \quad \|g * f\|_B \leq \|g\|_{L^1_{v_1}} \|f\|_B, \quad g \in L^1_{v_1}, f \in B, \quad (6)\)

and for any bounded, approximate unit \((e_\alpha)_{\alpha \in I}\) in \((L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})\) one has:

\[
\lim_{\alpha \to \infty} \|e_\alpha \ast f - f\|_B = 0, \quad \forall f \in B. \quad (7)
\]

2. \((B, \| \cdot \|_B)\) is an essential Banach module with respect to pointwise multiplication over the Fourier-Beurling algebra \(A_{v_2} := FL^1_{v_2}\), with the corresponding norm estimate. Correspondingly bounded approximate units for pointwise multiplication act accordingly.

3. The Wiener amalgam space \((W(A_{v_2}, L^1_{v_1})), \| \cdot \|_{W(A_{v_2}, L^1_{v_1})})\) with local component \(A_{v_2}\) and global component \(L^1_{v_1}\) is continuously and densely embedded into \((B, \| \cdot \|_B)\).
The statement in this proposition only summarize results which may be considered as folklore. Integrated action appears in a similar form in [3], [14] or [8], for example. For the last (minimality) statement see [13], or more explicitly [15, Proposition 6]. Also as folklore. Integrated action appears in a similar form in [3], [14] or [8], for example.

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that for some (resp. any) BUPU (bounded partition of unit, see Definition 3) in continuous functions on \( \mathbb{R} \), i.e.

**Lemma 1.** Assume that \((B, \| \cdot \|_B)\) is translation invariant Banach space of tempered distributions with \(\|T_x f\|_B \leq w(x) \|f\|_B\), for all \(f \in B, x \in \mathbb{R}^d\), for some submultiplicative function \(w\) on \(\mathbb{R}^d\), and \((L_w^1(\mathbb{R}^d), \| \cdot \|_{L_w^1})\) the corresponding Beurling algebra.

Then \((B_{1,w}, \| \cdot \|_{B_{1,w}})\), the vector space \(B_{1,w} = L_w^1 \cap B\) is a dense, essential Banach ideal inside the Banach convolution algebra \((L_w^1(\mathbb{R}^d), \| \cdot \|_{L_w^1})\) with respect to the norm

\[
\|f\|_{B_{1,w}} = \|f\|_{L_w^1} + \|f\|_B, \quad f \in B_{1,w}.
\]

This means \(L_w^1 \ast B_{1,w} \subset B_{1,w}\) is a Banach space with respect to the norm given by (9) and

\[
\|g \ast f\|_{B_{1,w}} \leq \|g\|_{L_w^1} \|f\|_{B_{1,w}}, \quad g \in L_w^1, f \in B_{1,w},
\]

and that for any bounded, approximate unit \((e_\alpha)_\alpha \subset \ell_1 (L_w^1(\mathbb{R}^d), \| \cdot \|_{L_w^1})\):

\[
\lim_{\alpha \to \infty} \|e_\alpha \ast f - f\|_{B_{1,w}} = 0, \quad \forall f \in B_{1,w},
\]

and consequently (by the Cohen-Hewitt Factorization Theorem):

\[
L_w^1(\mathbb{R}^d) \ast B_{1,w} = B_{1,w}.
\]

If \(w\) is a Beurling weight of polynomial growth, then \(S(\mathbb{R}^d)\) is dense in \(B_{1,w}(\mathbb{R}^d)\), hence \(D(\mathbb{R}^d) = S \cap C_c(\mathbb{R}^d)\) is a dense subspace of \((B_{1,w}, \| \cdot \|_{B_{1,w}})\).

**Proof.** The facts collected in the above lemma are essentially based on more detailed considerations published in [11]. The first step is to verify that \(B_{1,w}\) is a dense subspace of \(L_w^1(\mathbb{R}^d)\). For this purpose it is enough to note that compactly supported elements are dense in any Beurling algebra, i.e. for given \(\varepsilon > 0\) one finds \(\varphi \in C_c(\mathbb{R}^d)\) with

\[
\|f - \varphi\|_{L_w^1} < \varepsilon/4.
\]

By smoothing \(\varphi \in C_c(\mathbb{R}^d)\) with a sufficiently small supported test function \(\psi \in D(\mathbb{R}^d)\) one has

\[
\|\varphi - \ast \|_{L_w^1} < \varepsilon/4.
\]

Observing that \(\psi \ast \varphi \in D(\mathbb{R}^d) \subset S(\mathbb{R}^d) \subset B\) we note that \(\psi \ast \varphi \in B_{1,w}\) and satisfies

\[
\|f - \ast \|_{L_w^1} < \varepsilon/2.
\]

Since both \((L_w^1(\mathbb{R}^d), \| \cdot \|_{L_w^1})\) and \((B, \| \cdot \|_B)\) are translation invariant and have continuous translation the same is true for \((B_{1,w}, \| \cdot \|_{B_{1,w}})\).
The remaining consequences are just stated for easier reference but are standard results. The last step is a consequence of the Cohen-Hewitt factorization theorem (\cite{27}, Chap.32).

Next let us recall the important Beurling-Domar condition, which ensures the existence of band-limited elements (i.e. functions with a compactly supported Fourier transform) in general Beurling algebras. According to \cite{31} (and referring to the important paper by Y. Domar, \cite{9}) a Beurling weight $w$ satisfies the Beurling-Domar non-quasianalyticity condition if one has

$$\text{(BD)} \sum_{n \geq 1} \frac{w(nx)}{n^2} < \infty \quad \forall x \in \mathbb{R}^d. \tag{16}$$

It is an easy exercise to check that any polynomial weight satisfies the (BD)-condition. One of the advantages of increasing, radial symmetric weights is the fact that one can create easily bounded approximate units simply by applying to an arbitrary $g \in \mathcal{S}(\mathbb{R}^d) \subset L^1_{w_1}$ with $\hat{g}(0) = 1$ the $L^1$-norm preserving compression operator

$$\text{St}_\rho g(x) = \rho^{-d} g(x/\rho), \rho \to 0.$$

The next result is just a reminder concerning Fourier-Beurling algebras, i.e. pointwise Banach algebras obtained as $(\mathcal{F}L^1_w(\mathbb{R}^d), \| \cdot \|_{\mathcal{F}L^1_w})$, with the norm $\| \hat{f} \|_{\mathcal{F}L^1_w} = \| f \|_{L^1_w}$. The statements of the following lemma are restatements of facts found in \cite{31}.

**Lemma 2.**

i) If the weight function $w$ satisfies the (BD)-condition the Beurling algebra $(L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})$ has bounded approximate units consisting of band-limited elements.

ii) The Banach algebra $(\mathcal{F}L^1_w(\mathbb{R}^d), \| \cdot \|_{\mathcal{F}L^1_w})$ is a Wiener algebra in the sense of H. Reiter (\cite{31}), meaning that it is a regular Banach algebra of continuous functions, which allows (among others) to separate compact sets from open neighborhoods (see Chap.2.4 of \cite{31} resp. \cite{22}). In particular, the compactly supported elements are dense.

iii) For the case of weight functions of polynomial growth one has in addition: $\mathcal{S}(\mathbb{R}^d) \hookrightarrow (L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})$ and for any $g \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = 1$ one has

$$\| \text{St}_\rho g * f - f \|_{L^1_w(\mathbb{R}^d)} \to 0, \quad \text{for } \rho \to 0.$$

**Lemma 3.** For any minimal tempered TMIB $(\mathcal{B}, \| \cdot \|_\mathcal{B})$ the band-limited elements form a dense subspace of $(\mathcal{B}, \| \cdot \|_\mathcal{B})$. The same is true for $(\mathcal{B}_{1,w}, \| \cdot \|_{\mathcal{B}_{1,w}})$, which is in fact also a minimal tempered TMIB itself.

**Proof.** Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^1_w(\mathbb{R}^d)$ for the (polynomial) weights appearing in Proposition \ref{prop21} the (BD)-condition is clearly satisfied. Hence there are (even bounded) approximate units in $(L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})$ which are band-limited (i.e. compactly supported on the Fourier transform side), we write $(\epsilon_\alpha)_{\alpha \in I}$.

Since $\epsilon_\alpha * f$ approximates $f \in \mathcal{B}$ according to \ref{prop21} and

$$\text{supp}(\mathcal{F}(\epsilon_\alpha * f)) \subset \text{supp}(\hat{\epsilon}_\alpha) \quad \text{we find that the band-limited elements are dense in } (\mathcal{B}, \| \cdot \|_\mathcal{B}).$$

For the proof of the main result we will make use of the following key observations already explained in detail in \cite{22}. It makes use of so-called BUPUs, i.e. (bounded) uniform partitions of unity of size $\Psi = \delta > 0$, i.e. of countable collections of continuous functions $\psi_i, i \in I$, with $0 \leq \psi_i(x), \text{supp}(\psi_i) \subset B_{\delta}(x_i)$ and $\sum_{i \in I} \psi_i(x) \equiv 1$.\hfill $\square$
For this reason we will introduce a simple tool, the so-called BUPUs, the “bounded uniform partitions of unity”. For simplicity we only consider the regular case, i.e. BUPUs which are obtained as translates of a single function:

**Definition 3.** A (countable) family of translates $\Psi = (T_\lambda \psi)_{\lambda \in \Lambda}$, where $\psi$ is a compactly supported function (i.e. $\psi \in C_c(\mathbb{R}^d)$), and $\Lambda = A(\mathbb{Z}^d)$ a lattice in $\mathbb{R}^d$ (for some non-singular $d \times d$-matrix $A$) is called a regular BUPU (or more precisely a $\Lambda$-regular BUPU, or a $\Lambda$-invariant BUPU) if

$$\sum_{\lambda \in \Lambda} \psi(x - \lambda) \equiv 1. \quad (17)$$

We will write diam($\Psi$) $\leq \gamma$ if supp($\psi$) $\subset B_\gamma(0)$ for some $\gamma > 0$.

We will be interested in the case of “finer and finer” BUPUs, i.e. the case diam($\Psi$) $\to 0$. Sometimes we will simply write $|\Psi|$ instead of diam($\Psi$). The use of fine partitions of unity is also well established in the context of usual distribution theory, see e.g. [36].

The following results are required in the sequel, we refer [22] for details of the proof. The method can even be used to introduce convolution of measures (see [19]).

**Theorem 1.** Given two functions $g, f$ in some Beurling algebra $(L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any BUPU $\Psi = (\psi_i)_{i \in I}$ with $|\Psi| \leq \delta$ one has:

$$\|g * f - g * D_\Psi f\|_{L^1_w} < \varepsilon,$$

where the discrete (bounded) measure $D_\Psi f$ is given by

$$D_\Psi f = \sum_{i \in I} \int_{\mathbb{R}^d} f(x) \psi_i(x) \delta_{x_i} \quad (19)$$

and for some $C_1 > 0$ one has $\sum_{i \in I} |\mu(\psi_i)| w(x_i) \leq C_1 \|f\|_{L^1_w}, \quad f \in L^1_w(\mathbb{R}^d)$.

**Theorem 2.** Any minimal tempered TMIB $(B, \| \cdot \|_B)$ is an essential Banach module over some Beurling algebra $(L^1_w(\mathbb{R}^d), \| \cdot \|_{L^1_w})$. Moreover, one has for any $k \in L^1_w(\mathbb{R}^d)$:

$$\|g * k - g * D_\Psi k\|_B \to 0 \quad \text{for } |\Psi| \to 0, \quad g \in B. \quad (20)$$

3. The Completeness Result

**Theorem 3.** For any minimal tempered TMIB $(B, \| \cdot \|_B)$ any $g \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{g}(y) \neq 0$ for all $y \in \mathbb{R}^d$, the set of translates of $g$, i.e. $T(g) = \{ T_x g \mid x \in \mathbb{R}^d \}$ spans a dense subspace of $(B, \| \cdot \|_B)$.

**Proof.** The proof requires a couple of steps.

(1) First of all one has to check that $T(g)$ is a subset of $B$. In fact, we have $g \in \mathcal{S}(\mathbb{R}^d) \subset B_{1,w} \subset B$, and all the involved spaces are translation invariant;

(2) Given $f \in (B, \| \cdot \|_B)$ and $\varepsilon > 0$ we first approximate $f$ by some band-limited function $h \in B_{1,w}$ according to Lemma 3 i.e. we start with the estimate

$$\|f - h\|_B < \varepsilon. \quad (21)$$
(3) For such a band-limited \(h \in B_{1,w}(\mathbb{R}^d) \subset L^1_w(\mathbb{R}^d)\) with \(\text{spec}(h) := \text{supp}(\hat{h}) = Q\) (some compact set) we can find, according to Wiener’s inversion theorem for Beurling algebras (see [31], Chap.1.6.5., p.16, or a variant of Lemma 1.4.2, p.13 of [32]) some \(g_1 \in L^1_w(\mathbb{R}^d)\) with \(\hat{g}_1(y) = 1/\hat{g}(y)\) for all \(y \in Q\). As a consequence we can write

\[
h = (g \ast g_1) * h = g \ast (g_1 * h)
\]

where \(g \in \mathcal{S}(\mathbb{R}^d) \subset L^1_w(\mathbb{R}^d)\) and \(g_1 \ast h \in L^1_{w_1}(\mathbb{R}^d) \ast B_{1,w}(\mathbb{R}^d) \subset B_{1,w}(\mathbb{R}^d) \subset B\), in view of Lemma [1].

(4) In the next step we approximate \(g_1 \ast h \in B_{1,w}(\mathbb{R}^d) \subset (L^1_w(\mathbb{R}^d), \parallel \cdot \parallel_{L^1_w})\) by some compactly supported function \(k \in C_c(\mathbb{R}^d)\), i.e. we can have

\[
\|g_1 \ast h - k\|_{L^1_w} < \varepsilon/4\|g\|_B.
\]

As a consequence we obtain, recalling the identity (22) and (6):

\[
\|h - k \ast g\|_B = \|(g_1 \ast h - k) \ast g\|_B \leq \|g_1 \ast h - k\|_{L^1_w} \|g\|_B < \varepsilon/4.
\]

(5) By assumption we have \(g \in \mathcal{S}(\mathbb{R}^d) \subset B\), and thus for \(\varepsilon > 0\) we can find \(k \in B_{1,w} \subset L^1_w(G)\) Theorem [2] such that approximate the convolution product in (18), implying that for any sufficiently fine BUPU \(\Psi\) one has:

\[
\|g \ast k - g \ast D_\Psi k\|_B < \varepsilon/4.
\]

(6) Note that, due to the compact support condition imposed on \(k\) the discrete measure \(D_\Psi k\) obtained from it is a finite, discrete measure, hence \(\phi \in T(g)\), since only finitely many coefficients (integrals of the form \(\langle k, \psi_i \rangle\)) are non-zero:

\[
\phi := g \ast D_\Psi k = \sum_{i \in I} \int_{\mathbb{R}^d} k(x) \psi_i(x) dx T_x g.
\]

(7) By combining the estimates (21), (24), (25) we come up with the final estimate:

\[
\|f - \phi\|_B \leq \|f - h\|_B + \|h - g \ast k\|_B + \|g \ast k - g \ast D_\Psi k\|_B < \varepsilon.
\]

Remark 1. The proof of the main theorem clearly shows the crucial ideas behind the argument: Given the double module structure we can approximate a given element \(f \in (B, \parallel \cdot \parallel_B)\) or in \((L^1_w(\mathbb{R}^d), \parallel \cdot \parallel_{L^1_w})\) either by a compactly supported function or by a band-limited element. The band-limited functions, in turn, can be factorized over the given Schwartz function \(g\), thanks to Wiener’s Inversion Theorem (applied in the context of the relevant Beurling algebras) and the non-vanishing condition on \(\hat{g}\). Finally, the well known general idea that convolutions can be approximated by linear combinations of shifts (convolution by \(k\) is approximated by a convolution with some finite, discrete measure) is applied. Of course all these estimates require to use appropriate intermediate spaces and corresponding norm estimates.

Remark 2. Without loss of generality we may assume (by multiplying \(\hat{g}_1\) above by some plateau-like function in \(\mathcal{F}L^1_{w_1}(\mathbb{R}^d)\) with compact support) that the function \(\hat{g}_1\) has compact support (the existence of such plateau functions is granted by [1]).
4. Applications

Although we have indicated that there is room for further generalization we have to point out that the list of examples to which the above result applies is endless. The first author has tried to collect the construction principles widely (or occasionally) used in Fourier Analysis in a systematic way. It appears to be harder to identify space which are generally important, containing $S(\mathbb{R}^d)$ as a dense subspace, but not satisfying the assumptions formulated for this paper.

In order to mention at least some of the most important cases which are playing a major role in the literature let us mention:

1. Weighted $L^p$-spaces, for $1 \leq p < \infty$ with respect to polynomial weights (see [12], [24]);
2. Wiener amalgam spaces of the form $W(L^p, \ell^1_v)$, for $1 \leq p, q < \infty$ (see [20]), but also weighted Wiener amalgam spaces, as treated in [25];
3. The classical Besov-Triebel-Lizorkin spaces $(B^s_{p,q}(\mathbb{R}^d), \| \cdot \|_{B^s_{p,q}})$ resp. $(F^s_{p,q}(\mathbb{R}^d), \| \cdot \|_{F^s_{p,q}})$, for $1 \leq p, q < \infty$ (see [30] or the books of H. Triebel [34, 35]);
4. Modulation spaces in general (see [17]), but specifically the classical modulation spaces $(M^s_{p,q}(\mathbb{R}^d), \| \cdot \|_{M^s_{p,q}})$, also for the case $1 \leq p, q < \infty$. See [2] and [6];
5. More general decomposition spaces, as discussed in the literature, based on the approach given in [21];
6. Tauberian Theorems have been derived directly in the context of functions of bounded means in [16], where they have been a crucial step in order to extend Wiener’s Third Tauberian Theorem to the full range $1 < p \leq \infty$, and for $\mathbb{R}^d$ instead of just $p = 2$ and $d = 1$ (see [37, 38]);
7. The atomic space $H(q, p, \alpha)$ appearing in [20] as well as many other atomic spaces, including the exotic case discussed in [23], are minimal tempered TMIBs;
8. General construction principles for a further variety of function spaces as well as a long list of references are provided in [18].

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