BSDEs driven by $G$-Brownian motion under degenerate case and its application to the regularity of fully nonlinear PDEs

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Abstract. In this paper, we obtain the existence and uniqueness theorem for backward stochastic differential equation driven by $G$-Brownian motion ($G$-BSDE) under degenerate case. Moreover, we propose a new probabilistic method based on the representation theorem of $G$-expectation and weak convergence to obtain the regularity of fully nonlinear PDE associated to $G$-BSDE.

Key words. $G$-expectation; $G$-Brownian motion; Backward stochastic differential equation; Fully nonlinear PDE

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1 Introduction

Motivated by volatility uncertainty in finance, Peng [18–21] introduced the notions of $G$-expectation $\hat{E}[\cdot]$ and $G$-Brownian motion $B$ for each monotone and sublinear function $G : S_d \to \mathbb{R}$. The Ito’s calculus with respect to $G$-Brownian motion was constructed. Furthermore, he studied stochastic differential equation driven by $G$-Brownian motion ($G$-SDE) and a special type of backward stochastic differential equation (BSDE) containing only the solution $Y$, and then established the relevant theory. Denis et al. [2] (see also [10]) obtained that the $G$-expectation can be represented as an upper expectation over a family of weakly compact and non-dominated probability measures $\mathcal{P}$, and gave the characterizations of some spaces by inner capacity associated to $\mathcal{P}$. By quasi-surely stochastic analysis based on outer capacity, Denis and Martini [3] made a great contribution to study super-pricing of contingent claims under volatility uncertainty. The relationship between these two capacities has been clearly explained in notes and comments of Chapter 6 in [22].
Hu et al. [7] studied the following BSDE driven by $G$-Brownian motion ($G$-BSDE)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d(B)_s - \int_t^T Z_s dB_s - (K_T - K_t) \quad (1.1)$$

under the non-degenerate $G$, i.e., there exists a constant $\sigma^2 > 0$ such that

$$G(A) - G(B) \geq \frac{1}{2} \sigma^2 \text{tr}[A - B] \quad \text{for} \quad A \geq B.$$ 

They proved that the above $G$-BSDE has a unique solution $(Y, Z, K)$, where $K$ is a non-increasing $G$-martingale with $K_0 = 0$. Soner et al. [25] studied a new type of fully nonlinear BSDE, called 2BSDE, by different formulation and method, and obtained the deep result of the existence and uniqueness theorem for 2BSDE. For recent advances in these two directions, the reader may refer to [4–6, 12, 14–16, 23] and the references therein.

The key step to obtain the solution of $G$-BSDE (1.1) under non-degenerate $G$ is to use Krylov’s regularity estimate for fully nonlinear PDEs (see Appendix C.4 in [22]). But under degenerate $G$, we have to get round the difficulty that the regularity estimation condition (see Definition C.4.3 in [22]) is not satisfied. A natural idea is to construct a family of non-degenerate $G_\varepsilon$ with $\varepsilon \in (0, \varepsilon_0]$ such that $G_\varepsilon \uparrow G$ as $\varepsilon \downarrow 0$. The corresponding $G_\varepsilon$-expectation and the set of probability measures are denoted by $\hat{E}_\varepsilon[\cdot]$ and $\mathcal{P}_\varepsilon$, respectively. By the definition of $G$-expectation, we know that $\hat{E}_\varepsilon[X] \uparrow \hat{E}[X]$ for $X \in L_0^G(\Omega_T)$ and $\mathcal{P}$ is the closure of $\mathcal{P}_1 := \cup_{\varepsilon > 0} \mathcal{P}_\varepsilon$ under the topology of weak convergece. It is important to note that the quasi-surely stochastic analysis with respect to $\mathcal{P}$ (i.e. $\mathcal{P}$-q.s.) and $\mathcal{P}_1$ (i.e. $\mathcal{P}_1$-q.s.) are different (see [11]). Following the method proposed in the proof of Proposition A.1. in [22], we can get a process $Z$ in the $\mathcal{P}_1$-q.s. sense such that

$$\inf_{\eta \in M^0(0, T)} \hat{E}_\varepsilon \left[ \left( \int_0^T |Z_s - \eta_s|^2 d(B)_s \right)^{p/2} \right] = 0 \quad \text{for} \quad \varepsilon > 0, \ p > 1,$$

where the definition of $M^0(0, T)$ can be found in Section 3. At this point, there is a natural misconception that $Z \in H^{2,p}_G(0, T; (B))$ holds. But we notice Sion’s minimax theorem can not be used to obtain

$$\inf_{\eta \in M^0(0, T)} \sup_{\varepsilon \in (0, \varepsilon_0)} \hat{E}_\varepsilon \left[ \left( \int_0^T |Z_s - \eta_s|^2 d(B)_s \right)^{p/2} \right] = \sup_{\varepsilon \in (0, \varepsilon_0)} \inf_{\eta \in M^0(0, T)} \hat{E}_\varepsilon \left[ \left( \int_0^T |Z_s - \eta_s|^2 d(B)_s \right)^{p/2} \right],$$

because $(0, \varepsilon_0]$ is not compact. Therefore, whether $Z$ belongs to $H^{2,p}_G(0, T; (B))$ remains unsolved, even for $G$-martingale representation theorem, which is a special case of $G$-BSDE (1.1), i.e., $f = g = 0$. Thus, one purpose of this paper is to investigate the existence and uniqueness theorem for $G$-BSDE (1.1) under degenerate $G$.

It is well known that the theory of classical BSDEs provides a tool to study the regularity of quasilinear PDEs (see [17]). However, we all know that this classical tool is not suitable for the regularity of fully nonlinear PDEs, and up to our knowledge there is no result on this field. So, the other purpose of this paper is to establish the regularity of fully nonlinear PDEs by $G$-BSDEs.

In this paper, we introduce a quite different method to study the existence and uniqueness theorem for a type of well-posed $G$-BSDEs under degenerate $G$ (see [5.1]), which has two major contributions. The first one is to obtain the solution $(Y, Z, K)$ for $G$-BSDE under degenerate $G$ in the extended $\hat{G}$-expectation space,
which is essential to show that $K$ is a $G$-martingale in the key Lemma 3.8. The second one is to propose a new probabilistic method based on the representation theorem of $G$-expectation and weak convergence to obtain the uniform lower bound for $\partial^2_{xx}u_\varepsilon$ with $\varepsilon > 0$, where $u_\varepsilon$ is a solution to a fully nonlinear PDE associated to a $G_\varepsilon$-BSDE under non-degenerate $G_\varepsilon$ (see (3.17) and (3.31)). This uniform lower bound for $\partial^2_{xx}u_\varepsilon$ plays a key role in proving $Z \in H^2_{G,\varepsilon}(0, T; \langle B \rangle)$ in Lemma 3.8 and up to our knowledge, it is completely new in the literature because it does not depend on $\varepsilon$ as the bound by Krylov’s regularity estimate for fully nonlinear PDEs. Finally, we use the above probabilistic method to obtain the regularity of fully nonlinear PDE associated to $G$-BSDE under degenerate $G$.

The paper is organized as follows. In Section 2, we present some basic results of $G$-expectations. The existence and uniqueness theorem for $G$-BSDE under degenerate case is established in Section 3. In Section 4, we obtain the regularity of fully nonlinear PDE associated to $G$-BSDE under degenerate $G$.

2 Preliminaries

We recall some basic results of $G$-expectations. The readers may refer to Peng’s book [22] for more details.

Let $T > 0$ be given and let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $[0, T]$ with $\omega_0 = 0$. The canonical process $B_t(\omega) := \omega_t$, for $\omega \in \Omega_T$ and $t \in [0, T]$. For any fixed $t \leq T$, set

$$
\text{Lip}(\Omega_t) := \{ \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) : N \geq 1, t_1 < \cdots < t_N \leq t, \varphi \in C_{b,Lip}(\mathbb{R}^{d \times N}) \},
$$

where $C_{b,Lip}(\mathbb{R}^{d \times N})$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^{d \times N}$.

Let $G : \mathbb{S}_d \to \mathbb{R}$ be a given monotonic and sublinear function, where $\mathbb{S}_d$ denotes the set of $d \times d$ symmetric matrices. Then there exists a unique bounded, convex and closed set $\Sigma \subset \mathbb{S}_d^+$ such that

$$
G(A) = \frac{1}{2} \sup_{\gamma \in \Sigma} \text{tr}[A\gamma] \text{ for } A \in \mathbb{S}_d,
$$

where $\mathbb{S}_d^+$ denotes the set of $d \times d$ nonnegative matrices. If there exists a $\pi > 0$ such that $\gamma \geq \pi I_d$ for any $\gamma \in \Sigma$, $G$ is called non-degenerate. Otherwise, $G$ is called degenerate.

Peng [20, 21] constructed the $G$-expectation $\hat{E} : \text{Lip}(\Omega_T) \to \mathbb{R}$ and the conditional $G$-expectation $\hat{E}_t : \text{Lip}(\Omega_T) \to \text{Lip}(\Omega_t)$ as follows:

(i) For each $s_1 \leq s_2 \leq T$ and $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, define $\hat{E}^{\varphi}(B_{s_2} - B_{s_1}) = u(s_2 - s_1, 0)$, where $u$ is the viscosity solution (see [1]) of the following $G$-heat equation:

$$
\partial_t u - G(D^2_x u) = 0, \quad u(0, x) = \varphi(x).
$$

(ii) For each $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) \in \text{Lip}(\Omega_T)$, define

$$
\hat{E}_t[X] = \varphi_i(B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}) \text{ for } i = N - 1, \ldots, 1 \text{ and } \hat{E}[X] = \hat{E}[\varphi(B_{t_1})],
$$

where $\varphi_{N-1}(x_1, \ldots, x_{N-1}) := \hat{E}[\varphi(x_1, \ldots, x_{N-1}, B_{t_{N-1}} - B_{t_{N-1}})]$ for $(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{d \times (N-1)}$ and

$$
\varphi_i(x_1, \ldots, x_i) := \hat{E}[\varphi_{i+1}(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i})] \text{ for } i = N - 2, \ldots, 1.
$$
The space \((\Omega_T, \text{Lip}(\Omega_T), \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t \in [0,T]} )\) is a consistent sublinear expectation space, where \(\hat{\mathbb{E}}_0 = \hat{\mathbb{E}}\). The canonical process \((B_t)_{t \in [0,T]}\) is called the G-Brownian motion under \(\hat{\mathbb{E}}\).

For each \(t \in [0,T]\), denote by \(L^p_G(\Omega_t)\) the completion of \(\text{Lip}(\Omega_t)\) under the norm \(\|X\|_{L^p_G} := (\hat{\mathbb{E}}[|X|^p])^{1/p}\) for \(p \geq 1\). It is clear that \(\hat{\mathbb{E}}_t\) can be continuously extended to \(L^1_G(\Omega_T)\) under the norm \(\|\cdot\|_{L^1_G}\).

The following theorem is the representation theorem of G-expectation.

**Theorem 2.1** ([2, 10]) There exists a unique weakly compact and convex set of probability measures \(\mathcal{P}\) on \((\Omega_T, \mathcal{B}(\Omega_T))\) such that
\[
\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \quad \text{for all } X \in L^1_G(\Omega_T),
\]
where \(\mathcal{B}(\Omega_T) = \sigma(B_s: s \leq T)\).

For this \(\mathcal{P}\), define
\[
L^p_G(\Omega_t) := \left\{ X \in \mathcal{B}(\Omega_t) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \right\} \quad \text{for } p \geq 1.
\]
It is easy to check that \(L^p_G(\Omega_t) \subset L^p(\Omega_t)\). For each \(X \in L^1_G(\Omega_T)\),
\[
\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X]
\]
is still called the G-expectation.

The capacity associated to \(\mathcal{P}\) is defined by
\[
c(A) := \sup_{P \in \mathcal{P}} P(A) \quad \text{for } A \in \mathcal{B}(\Omega_T).
\]
A set \(A \in \mathcal{B}(\Omega_T)\) is polar if \(c(A) = 0\). A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables \(X\) and \(Y\) if \(X = Y\) q.s.

**Definition 2.2** A process \((M_t)_{t \leq T}\) is called a G-martingale if \(M_t \in L^1_G(\Omega_t)\) and \(\hat{\mathbb{E}}_s[M_t] = M_s\) for any \(0 \leq s \leq t \leq T\).

The following Doob’s inequality for G-martingale can be found in [24, 26]. The following proof is based on [9, 24].

**Theorem 2.3** Let \(1 \leq p < p'\) and \(\xi \in L^{p'}_G(\Omega_T)\). Then
\[
\left( \hat{\mathbb{E}}\left[ \sup_{t \leq T} (\hat{\mathbb{E}}_t[|\xi|^p])^{1/p} \right] \right)^{1/p} \leq \left( \hat{\mathbb{E}}\left[ \sup_{t \leq T} \hat{\mathbb{E}}_t[|\xi|^p] \right] \right)^{1/p} \leq C \left( \hat{\mathbb{E}}[|\xi|^{p'}] \right)^{1/p'},
\]
where
\[
C = \left( 1 + \frac{p}{p' - p} \right)^{1/p}.
\]

**Proof.** By the definition of \(L^{p'}_G(\Omega_T)\), we only need to prove the inequality for \(\xi \in \text{Lip}(\Omega_T)\). Define
\[
M_t = \hat{\mathbb{E}}_t[|\xi|] \quad \text{for } t \leq T.
\]
For each fixed \(\lambda > 0\) and integer \(n \geq 1\), define a stopping time
\[
\tau = \inf\{ t_i : M_{t_i} \geq \lambda, i = 0, \ldots, n \},
\]
where
\[
\hat{\mathbb{E}}_t = \hat{\mathbb{E}}_t[|\xi|] \quad \text{for } t \leq T.
\]
where \( t_i = iT/n \) and \( \inf \emptyset = \infty \). It is easy to check that
\[
\{ \tau = t_i \} \in \mathcal{B}(\Omega_t), \quad \{ \tau = \infty \} \in \mathcal{B}(\Omega_T) \quad \text{and} \quad \{ \tau = t_i \} \cap \{ \tau = t_j \} = \emptyset \quad \text{for} \ i \neq j.
\]

By Proposition 3.9 in [9], we have
\[
\mathbb{E} \left[ \sum_{i=0}^{n} |\xi| I_{\{\tau = t_i\}} + 0I_{\{\tau = \infty\}} \right] = \mathbb{E} \left[ \sum_{i=0}^{n} \mathbb{E}_i \{ |\xi| I_{\{\tau = t_i\}} + \mathbb{E}_T[0]I_{\{\tau = \infty\}} \} \right],
\]
which implies
\[
\mathbb{E} \left[ |\xi| I_{\{\tau \leq t_n\}} \right] = \mathbb{E} \left[ \sum_{i=0}^{n} M_t I_{\{\tau = t_i\}} \right] \geq \lambda \mathbb{E} \left[ I_{\{\tau \leq t_n\}} \right].
\]

Note that \( \{ \tau \leq t_n \} = \{ \sup_i M_t \geq \lambda \} \), then we have
\[
\lambda \mathbb{E} \left[ I_{\{\sup_i M_t \geq \lambda\}} \right] \leq \mathbb{E} \left[ |\xi| I_{\{\sup_i M_t \geq \lambda\}} \right] \leq \left( \mathbb{E} \left[ |\xi|^{p'} \right] \right)^{1/p'} \left( \mathbb{E} \left[ I_{\{\sup_i M_t \geq \lambda\}} \right] \right)^{1/q'}
\]
where \( 1/p' + 1/q' = 1 \). Thus,
\[
\mathbb{E} \left[ I_{\{\sup_i M_t \geq \lambda\}} \right] \leq \frac{1}{\lambda^p} \mathbb{E} \left[ |\xi|^{p'} \right] \quad \text{for each} \ \lambda > 0.
\]

For each fixed \( \lambda_0 > 0 \), we have
\[
\mathbb{E} \left[ \sup_i M_{t_i}^p \right] = \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ \sup_i M_{t_i}^p \right]
\]
\[
= \sup_{P \in \mathcal{P}} \int_0^{\lambda_0} P(\sup_i M_t \geq \lambda) \lambda^{p-1} d\lambda
\]
\[
\leq \int_0^{\lambda_0} p\lambda^{p-1} d\lambda + \int_{\lambda_0}^{\infty} p\lambda^{p-1} \mathbb{E}[|\xi|^{p'}] d\lambda
\]
\[
= (\lambda_0)^p + \frac{p\lambda_0^{p-p'}}{p'-p} \mathbb{E}[|\xi|^{p'}].
\]

Taking \( \lambda_0 = \left( \mathbb{E}[|\xi|^{p'}] \right)^{1/p'} \), we get
\[
\mathbb{E} \left[ \sup_i M_{t_i}^p \right] \leq \left( 1 + \frac{p}{p'-p} \right) \left( \mathbb{E}[|\xi|^{p'}] \right)^{p/p'}.
\]

Since \( |\xi| \in L_{ip}(\Omega_T) \), we have
\[
\sup_{t \leq T} M_{t_i}^p \uparrow \sup_{t \leq T} M_{t}^p.
\]

Then we obtain
\[
\mathbb{E} \left[ \sup_{t \leq T} \left( \mathbb{E}_t[|\xi|] \right)^p \right] \leq \left( 1 + \frac{p}{p'-p} \right) \left( \mathbb{E}[|\xi|^{p'}] \right)^{p/p'}.
\]

It is obvious that \( \left( \mathbb{E}_t[|\xi|] \right)^p \leq \mathbb{E}_t[|\xi|^p] \). Since inequality (2.3) holds for \( |\xi|^p \in L_{ip}(\Omega_T) \) and \( 1 < p'/p' \), we have
\[
\mathbb{E} \left[ \sup_{t \leq T} \mathbb{E}_t[|\xi|^p] \right] \leq \left( 1 + \frac{1}{p'/p'-1} \right) \left( \mathbb{E}[|\xi|^{p'}] \right)^{p/p'} = \left( 1 + \frac{p}{p'-p} \right) \left( \mathbb{E}[|\xi|^{p'}] \right)^{p/p'}.
\]

Thus we obtain (2.2) □
3 BSDEs driven by G-Brownian motion under degenerate case

Let $B_t = (B^1_t, \ldots, B^d_t)^T$ be a $d$-dimensional G-Brownian motion satisfying

$$G(A) = G'(A') + \frac{1}{2} \sum_{i=d'+1}^d \sigma_i^2 a_i^+,$$

where $d' < d$, $A' \in S_{d'}$, $a_i \in \mathbb{R}$ for $d' < i < d$,

$$A = \begin{pmatrix} A' & \cdots & \cdots & \cdots \\ \cdots & a_{d'+1} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & a_d \end{pmatrix} \in S_d,$$

$G' : S_{d'} \to \mathbb{R}$ is non-degenerate, $\sigma_i > 0$ for $i = d' + 1, \ldots, d$. By Corollary 3.5.8 in Peng [22], we know that

$$(B^i, B^j)_{t+s} - (B^i, B^j)_t \in s\Sigma$$ for any $t, s \geq 0,$

where $(B^i, B^j)$ is the mutual variation process of $B^i$ and $B^j$, and $\Sigma \subset S^+_d$ is the unique bounded, convex and closed set satisfying $[21]$. It follows from $[21]$ and $[3.2]$ that, for any $t, s \geq 0$,

$$cs \leq (B^i)_{t+s} - (B^i)_t \leq Cs$$ for $i \leq d', (B^i)_{t+s} - (B^i)_t \leq \sigma_i^2 s$$ for $d' < i \leq d$,

$$(B^i, B^j)_t = 0$$ for $i \leq d, d' < j \leq d, i \neq j,$

where $(B^i) = (B^i, B^i)$, $0 < c \leq C < \infty$. We consider the following type of G-BSDE under degenerate case:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z^*_s)ds + \sum_{i,j=1}^{d'} \int_t^T g_{ij}(s, Y_s, Z^*_s)dB^i_s d\nu(s),$$

$$= \sum_{i=d'+1}^d \int_t^T Z^*_i dB^i_s - \sum_{k=1}^d \int_t^T Z^*_k dB^k_s - (K_T - K_t),$$

where $Z^*_i = (Z^1_i, \ldots, Z^d_i)^T$, $f, g_{ij} : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{d'} \to \mathbb{R}, g_r : [0, T] \times \Omega \times \mathbb{R}^{d'} \times \mathbb{R} \to \mathbb{R}.$

The following spaces and norms are needed to define the solution of the above G-BSDE.

- $M^0(0, T) := \left\{ \eta_t = \sum_{k=0}^{N-1} \xi_k I_{(t_k, t_{k+1})}(t) : N \in \mathbb{N}, 0 = t_0 < \cdots < t_N = T, \xi_k \in L^p(\Omega_{t_k}) \right\};$
- $\|\eta\|_{M^p_{G}(0, T)} := \left( \mathbb{E} \left[ \left( \int_0^T |\eta|^p + dB^i_t \right)^{\frac{p}{p}} \right] \right)^{1/p};$
- $M^p_{G}(0, T) := \left\{ \text{the completion of } M^0(0, T) \text{ under the norm } \|\cdot\|_{M^p_{G}(0, T)} \right\}$ for $p, \bar{p} \geq 1;$
- $H^p_{G}(0, T; (B^i)) := \left\{ \text{the completion of } M^0(0, T) \text{ under the norm } \|\cdot\|_{H^p_{G}(0, T; (B^i))} \right\}$ for $p, \bar{p} \geq 1;$
- $M^p_G(0, T) := M^p_G(0, T), H^p_G(0, T; (B^i)) := H^p_G(0, T; (B^i));$
• $S^0(0, T) := \{h(t, B_{t_1}, \ldots, B_{t_N}) : N \in \mathbb{N}, 0 < t_1 < \cdots < t_N = T, h \in C_b L^p(\mathbb{R}^{N+1})\} ;$

• $\|\eta\|_{S^p_G(0, T)} := \left(\hat{E}\left[\sup_{t \leq T} |\eta|^p\right]\right)^{1/p} ;$

• $S^p_G(0, T) := \text{the completion of } S^0(0, T) \text{ under the norm } \| \cdot \|_{S^p_G(0, T)} \text{ for } p \geq 1.$

By (3.3), we know that

\[ c^{1/p}\|\eta\|_{M^p_G(0, T)} \leq \|\eta\|_{H^p_G(0, T; \langle B^i \rangle)} \leq c^{1/p}\|\eta\|_{M^p_G(0, T)} \text{ for } i \leq d' \]

and

\[ \|\eta\|_{H^p_G(0, T; \langle B^i \rangle)} \leq \bar{\sigma}^{2/p}_i\|\eta\|_{M^p_G(0, T)} \text{ for } d' < i \leq d. \]

Thus $M^p_G(0, T) = H^p_G(0, T; \langle B^i \rangle)$ for $i \leq d'$ and $M^p_G(0, T) \subset H^p_G(0, T; \langle B^i \rangle)$ for $d' < i \leq d.$

Throughout the paper, we use the following assumptions:

\textbf{(H1)} There exists a $p > 1$ such that $\xi \in L^p_G(\Omega_T),$ $f(\cdot, y, z'), g_{ij}(\cdot, y, z') \in M^p_G(0, T)$ and $g_i(\cdot, y, z', z) \in H^p_G(0, T; \langle B^i \rangle)$ for any $y, z \in \mathbb{R}, z' \in \mathbb{R}^{d'}, i, j \leq d', d' < l \leq d; \]

\textbf{(H2)} There exists a constant $L > 0$ such that, for any $(t, \omega) \in [0, T] \times \Omega_T,$ $(y, z', z), (\tilde{y}, \tilde{z}', \tilde{z}) \in \mathbb{R} \times \mathbb{R}^{d'} \times \mathbb{R},$

\[ |f(t, \omega, y, z') - f(t, \omega, \tilde{y}, \tilde{z}')| + \sum_{i=d'+1}^{d} |g_{ij}(t, \omega, y, z') - g_{ij}(t, \omega, \tilde{y}, \tilde{z}')| \]

\[ + \sum_{l=d'+1}^{d} |g_l(t, \omega, y, z', z) - g_l(t, \omega, \tilde{y}, \tilde{z}', \tilde{z})| \leq L(|y - \tilde{y}| + |z' - \tilde{z}'| + |z - \tilde{z}|). \]

Now we give the $L^p$-solution of G-BSDE (3.4) and will be given at the end of this section. In the following, the constant $C$ will change from line to line for simplicity.

**Definition 3.1** $(Y, Z^1, \ldots, Z^d, K)$ is called an $L^p$-solution of G-BSDE (3.4) if the following properties hold:

(i) $Y \in S^p_G(0, T), Z^i \in H^p_G(0, T; \langle B^i \rangle)$ for $i \leq d,$ $K$ is a non-increasing G-martingale with $K_0 = 0$ and $K_T \in L^p_G(\Omega_T);$

(ii)

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z'_s)ds + \sum_{i,j=1}^{d'} \int_t^T g_{ij}(s, Y_s, Z'_s)d(B^i)_s \]

\[ + \sum_{l=d'+1}^{d} \int_t^T g_l(s, Y_s, Z'_s, Z'^l_s)d(B^l)_s - \sum_{k=1}^{d-1} \int_t^T Z^k_s dB^k_s - (K_T - K_t), \]

where $Z'_s = (Z^1_s, \ldots, Z^d'_s)^T$ and $t \leq T.$

For simplicity of representation, we only give the proof for the following G-BSDE:

\[ Y_t = \xi + \int_t^T f(s, Y_s)ds + \int_t^T g(s, Y_s, Z_s)d(B)_s - \int_t^T Z_s dB_s - (K_T - K_t), \tag{3.5} \]

where $B$ is a 1-dimensional G-Brownian motion, $G(a) := \frac{1}{2\sigma^2}a^+ \text{ for } a \in \mathbb{R} \text{ with } \sigma > 0.$ The results still hold for G-BSDE (5.5), and will be given at the end of this section. In the following, the constant $C$ will change from line to line for simplicity.
3.1 Prior estimates of G-BSDEs

In this subsection, we give some useful prior estimates of G-BSDEs.

**Proposition 3.2** Suppose that \( \xi_i, f_i \) and \( g_i \) satisfy (H1) and (H2) for \( i = 1, 2 \). Let \((Y^i, Z^i, K^i)\) be the \( L^p \)-solution of G-BSDE corresponding to \( \xi_i, f_i \) and \( g_i \) for some \( p \in (1, \bar{p}) \). Then there exists a positive constant \( C \) depending on\( p, \tilde{\sigma}, L \) and \( T \) satisfying

\[
|\dot{Y}_t^i|^p \leq C_\tilde{E}_t \left[ |\xi_i|^p + \left( \int_t^T |\dot{f}_s| ds \right)^p + \left( \int_t^T |\dot{g}_s| d(B)_s \right)^p \right] , \tag{3.6}
\]

\[
|Y_t^i|^p \leq C_\tilde{E}_t \left[ |\xi_i|^p + \left( \int_t^T |f_i(s,0)| ds \right)^p + \left( \int_t^T |g_i(s,0,0)| d(B)_s \right)^p \right] \text{ for } i = 1, 2, \tag{3.7}
\]

\[
\tilde{E} \left[ \left( \int_0^T |Z_t^i|^2 d(B)^{+}_s \right)^{p/2} \right] \leq C \left\{ \tilde{E} \left[ \sup_{t \leq T} |\dot{Y}_t|^p \right] + (\tilde{\Lambda} + \tilde{\Lambda}^2)^{1/2} \left( \tilde{E} \left[ \sup_{t \leq T} |\dot{Y}_t|^p \right] \right)^{1/2} \right\} , \tag{3.9}
\]

where

\[
\tilde{\Lambda} = \tilde{E} \left[ \sup_{t \leq T} |\dot{Y}_t|^p \right] + \tilde{E} \left[ \left( \int_0^T |f_i(s,0)| ds \right)^p \right] + \tilde{E} \left[ \left( \int_0^T |g_i(s,0,0)| d(B)_s \right)^p \right] \text{ for } i = 1, 2,
\]

\( \dot{Y}_t = Y_t^1 - Y_t^2, \dot{\xi} = \xi_1 - \xi_2, \dot{f}_s = f_1(s,Y_s^2) - f_2(s,Y_s^2), \dot{g}_s = g_1(s,Y_s^2,Z_s^2) - g_2(s,Y_s^2,Z_s^2), \dot{Z}_t = Z_t^1 - Z_t^2. \)

**Proof.** The method is the same as that in [3]. For convenience of the reader, we sketch the proof.

For each given \( t < T \), consider the following SDE for \( r \in [t, T] \)

\[
X_r = \int_t^r (f_1(s,Y_s^2 - X_s) - f_2(s,Y_s^2)) ds + \int_t^r (g_1(s,Y_s^2 - X_s,Z_s^2) - g_2(s,Y_s^2,Z_s^2)) d(B)_s.
\]

Noting that \( \langle B \rangle_{t+s} - \langle B \rangle_t \leq \tilde{\sigma}^2 s \) for any \( t, s \geq 0 \), we obtain

\[
|X_r| \leq \int_t^r |\dot{f}_s| ds + \int_t^r |\dot{g}_s| d(B)_s + L(1 + \tilde{\sigma}^2) \int_t^r |X_s| ds \text{ for } r \in [t, T].
\]

By the Gronwall inequality, we have

\[
|X_T| \leq C \left( \int_t^T |\dot{f}_s| ds + \int_t^T |\dot{g}_s| d(B)_s \right), \tag{3.10}
\]

where \( C \) depends on \( \tilde{\sigma}, L \) and \( T \). For each \( \varepsilon > 0 \), noting that

\[
p(|x|^2 + \varepsilon)^{(p/2)-1} + p(p - 2)(|x|^2 + \varepsilon)^{(p/2)-2}|x|^2 \geq p((p - 1) \wedge 1)(|x|^2 + \varepsilon)^{(p/2)-1}
\]

for \( x \in \mathbb{R} \) and taking \( \lambda = pL(1 + \tilde{\sigma}^2) + pL^2\tilde{\sigma}^22^{-1}(p - 1)^{-1} \vee 1 \), we get by applying Itô’s formula to \( (|\dot{Y}_r + X_r|^2 + \varepsilon)^{p/2} e^{\lambda r} \) on \( [t, T] \) that

\[
(|\dot{Y}_t|^2 + \varepsilon)^{p/2} e^{\lambda T} + M_T - M_t \leq (|\dot{\xi} + X_T|^2 + \varepsilon)^{p/2} e^{\lambda T}, \tag{3.11}
\]
where $M_T - M_t = \int_t^T p(\|\hat{Y}_s + X_s\|^2 + \varepsilon)^{(p/2)-1}e^{\lambda s}[\hat{Y}_s + X_s] \hat{Z}_s dB_s + (\hat{Y}_s + X_s)^+ dK_1^s + (\hat{Y}_s + X_s)^- dK_2^s$. By Lemma 3.4 in [7], we know that $\hat{E}_t[M_T - M_t] = 0$. Taking $\hat{E}_t$ on both sides of (3.11) and letting $\varepsilon \downarrow 0$, we get (3.10). Taking $\xi_j = f_j = g_j = 0$ for $j \neq i$, we have $(Y^j_i, Z^j_i, K^j_i) = 0$. Thus we obtain (3.7) by (3.6).

Applying Itô’s formula to $|Y_t^i|^2$ on $[0, T]$, by the B-D-G inequality, we get

$$
\hat{E} \left[ \left( \int_0^T |Z_{t}^i|^2 d(B)_s \right)^{p/2} \right] \leq C \left\{ \Lambda_i + \left( \hat{E} \left[ \sup_{t \leq T} |Y_t^i|^p \right] \right)^{1/2} \left( \hat{E} \left[ |K_{T}^i|^p \right] \right)^{1/2} \right\},
$$

where $C$ depends on $p, \sigma, L$ and $T$. Then we deduce (3.8) by (3.12) and (3.13).

Applying Itô’s formula to $|\hat{Y}_t|^2$ on $[0, T]$, by the B-D-G inequality, we get

$$
\hat{E} \left[ \left( \int_0^T |\hat{Z}_{t}^i|^2 d(B)_s \right)^{p/2} \right] \leq C \left\{ \hat{E} \left[ \sup_{t \leq T} |\hat{Y}_t|^p \right] + (\tilde{\Lambda}_1 + \tilde{\Lambda}_2)^{1/2} \left( \hat{E} \left[ \sup_{t \leq T} |\hat{Y}_t|^p \right] \right)^{1/2} \right\},
$$

where $C$ depends on $p, \sigma, L$ and $T$, and

$$
\tilde{\Lambda}_i = \Lambda_i + \hat{E} \left[ \left( \int_0^T |Z_{t}^i|^2 d(B)_s \right)^{p/2} \right] + \hat{E} \left[ |K_{T}^i|^p \right] \text{ for } i = 1, 2.
$$

Thus we obtain (3.9) by (3.8) and (3.11). □

### 3.2 Solution in the extended $\tilde{G}$-expectation space

Following [7], the key point to obtain the solution of $G$-BSDE (3.5) is to study the following type of $G$-BSDE:

$$
Y_t = \varphi(B_T) + \int_t^T h(Y_s, Z_s) d(B)_s - \int_t^T Z_s dB_s - (K_T - K_t),
$$

where $\varphi \in C_0^\infty(\mathbb{R})$, $h \in C_0^\infty(\mathbb{R}^2)$.

In order to obtain the solution of $G$-BSDE (3.15), we introduce the extended $\tilde{G}$-expectation space. Set $\tilde{\Omega}_T = C_0([0, T]; \mathbb{R}^2)$ and the canonical process is denoted by $(B, \tilde{B})$. For each $a_{11}, a_{12}, a_{22} \in \mathbb{R}$, define

$$
\tilde{G} \left( \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \right) = G(a_{11}) + \frac{1}{2} a_{22} = \frac{1}{2} \sup_{\gamma \in \tilde{\Sigma}} \text{tr} \left( \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \right) \gamma,
$$

where

$$
\tilde{\Sigma} = \left\{ \begin{pmatrix} \sigma^2 & 0 \\ 0 & 1 \end{pmatrix} : \sigma \in [0, \sigma] \right\}.
$$
The \( \tilde{G} \)-expectation is denoted by \( \tilde{\mathbb{E}} \), and the related spaces are denoted by

\[
\text{Lip}(\tilde{\Omega}_t), L^p_G(\tilde{\Omega}_t), M^0(0, T), M^p_G(0, T), H^p_G(0, T; \{B\}), S^p_G(0, T).
\]

For each \( \mathbf{a} = (a_1, a_2)^T \in \mathbb{R}^2 \), by Proposition 3.1.5 in Peng [22], we know that \( B^a := a_1 B + a_2 \tilde{B} \) is a \( G_a \)-Brownian motion, where \( G_a(b) = \frac{1}{2}[(\bar{\sigma}^2 a_1^2 + |a_2|^2)b^+ - |a_2|^2 b^-] \) for \( b \in \mathbb{R} \). In particular, \( B \) is a \( G \)-Brownian motion and \( \tilde{B} \) is a classical Brownian motion. Thus \( \tilde{\mathbb{E}}|_{\text{Lip}(\tilde{\Omega}_T)} = \tilde{\mathbb{E}} \), which implies that the completion of \( M^0(0, T) \) (resp. \( S^0(0, T) \)) under the norm \( \| \cdot \|_{H^p_G(0, T; \{B\})} \) (resp. \( \| \cdot \|_{S^p_G(0, T)} \)) is \( H^p_G(0, T; \{B\}) \) (resp. \( S^p_G(0, T) \)). Similar to [3.2], we know that \( \langle B, \tilde{B} \rangle_t = 0 \) and \( \langle \tilde{B} \rangle_t = t \) in the \( \tilde{G} \)-expectation space.

**Lemma 3.3** Let \( \varphi \in C_0^\infty(\mathbb{R}) \) and \( h \in C_0^\infty(\mathbb{R}^2) \). Then, for each given \( p > 1 \), \( G \)-BSDE \((3.15)\) has a unique \( L^p \)-solution \((Y, Z, K)\) in the extended \( \tilde{G} \)-expectation space such that \( Y \in S^p_G(0, T), Z \in H^p_G(0, T; \{B\}) \) and \( K_T \in L^p_G(\tilde{\Omega}_T) \).

**Proof.** The uniqueness is due to \([3.6]\) and \([3.9]\) in Proposition 3.2. The proof of existence is divided into two parts.

Part 1. The purpose of this part is to find a solution \((Y, Z, K)\) in the extended \( \tilde{G} \)-expectation space such that \( Y \in S^p_G(0, T) \) and \( Z \in H^p_G(0, T; \{B\}) \).

For each fixed \( \varepsilon \in (0, \bar{\sigma}) \), define

\[
B^\varepsilon_t = B_t + \varepsilon \tilde{B}_t \quad \text{for} \quad t \in [0, T].
\]

Then \( (B^\varepsilon_t)_{t \in [0, T]} \) is the \( G_\varepsilon \)-Brownian motion under \( \tilde{\mathbb{E}} \), where

\[
G_\varepsilon(a) = \frac{1}{2}[(\bar{\sigma}^2 + \varepsilon^2) a^+ - \varepsilon^2 a^-] \quad \text{for} \quad a \in \mathbb{R}.
\]

Let \( u_\varepsilon \) be the viscosity solution of the following PDE

\[
\partial_t u + G_\varepsilon(\partial^2_{xx} u + 2h(u, \partial_x u)) = 0, \quad u(T, x) = \varphi(x).
\]

By Theorem 6.4.3 in Krylov [13] (see also Theorem C.4.4 in Peng [22]), there exists a constant \( \alpha \in (0, 1) \) satisfying

\[
\|u_\varepsilon\|_{C^{1+\alpha/2, 2+\alpha}(0, T-\delta) \times \mathbb{R}} < \infty \quad \text{for any} \quad \delta > 0.
\]

Applying Itô’s formula to \( u_\varepsilon(t, B^\varepsilon_t) \) on \([0, T-\delta]\), we obtain

\[
Y^\varepsilon_t = Y^\varepsilon_{T-\delta} + \int_t^{T-\delta} h(Y^\varepsilon_s, Z^\varepsilon_s) d\langle B^\varepsilon \rangle_s - \int_t^{T-\delta} Z^\varepsilon_s dB^\varepsilon_s - (K^\varepsilon_{T-\delta} - K^\varepsilon_t),
\]

where \( Y^\varepsilon_t = u_\varepsilon(t, B^\varepsilon_t), Z^\varepsilon_t = \partial_x u_\varepsilon(t, B^\varepsilon_t) \) and

\[
K^\varepsilon_t = \int_0^t \frac{1}{2} \left[ \partial^2_{xx} u_\varepsilon(s, B^\varepsilon_s) + 2h(Y^\varepsilon_s, Z^\varepsilon_s) \right] d\langle B^\varepsilon \rangle_s - \int_0^t G_\varepsilon(\partial^2_{xx} u_\varepsilon(s, B^\varepsilon_s) + 2h(Y^\varepsilon_s, Z^\varepsilon_s)) ds.
\]

By Lemma 4.2.1 in Peng [22], we obtain that \( K^\varepsilon \) is non-increasing and \( K^\varepsilon_t = \tilde{\mathbb{E}}_t[K^\varepsilon_{T-\delta}] \) for \( t \leq T-\delta \).

The same analysis as in the proof of inequality (4.3) in [7], we get that there exists a positive constant \( C \) depending on \( \varphi, h, \bar{\sigma} \) and \( T \) such that

\[
|u_\varepsilon(t_1, x_1) - u_\varepsilon(t_2, x_2)| \leq C(\sqrt{|t_1 - t_2| + |x_1 - x_2|}) \quad \text{for} \quad \varepsilon \in (0, \bar{\sigma}), t_1, t_2 \leq T, \quad x_1, x_2 \in \mathbb{R}.
\]
From this we can easily deduce that \( \hat{E}_t [ |Y_{T-\delta}^\varepsilon - \varphi (B_T^\varepsilon) |^2 ] \to 0 \) as \( \delta \downarrow 0 \) and
\[
|u_\varepsilon (t, x)| \leq |\varphi (x)| + C \sqrt{T}, \quad |\partial_x u_\varepsilon (t, x)| \leq C \text{ for } \varepsilon \in (0, \sigma), \quad t \leq T, \quad x \in \mathbb{R}.
\]
(3.20)

Taking \( \delta \downarrow 0 \) in (3.19), we obtain
\[
Y_t^\varepsilon = \varphi (B_T^\varepsilon) + \int_t^T h (Y_s^\varepsilon, Z_s^\varepsilon) d (B_s) + \int_t^T Z_s^\varepsilon d B_s - (K_T^\varepsilon - K_t^\varepsilon),
\]
(3.21)
where \( Y^\varepsilon \) and \( Z^\varepsilon \) are uniformly bounded for \( \varepsilon \in (0, \sigma) \) by (3.20).

For each given \( \varepsilon, \varepsilon' \in (0, \sigma) \), set
\[
\hat{Y}_t^{\varepsilon, \varepsilon'} = Y_t^{\varepsilon} - Y_t^{\varepsilon'}, \quad \hat{Z}_t^{\varepsilon, \varepsilon'} = Z_t^{\varepsilon} - Z_t^{\varepsilon'}, \quad \hat{K}_t^{\varepsilon, \varepsilon'} = K_t^{\varepsilon} - K_t^{\varepsilon'}, \quad \hat{\xi}^{\varepsilon, \varepsilon'} = \varphi (B_T^{\varepsilon'}) - \varphi (B_T^{\varepsilon}).
\]
Then, by (3.21) and \( \langle B_s \rangle_s = \langle B_s \rangle_s + \varepsilon^2 s \), we have
\[
\hat{Y}_t^{\varepsilon, \varepsilon'} = \hat{\xi}^{\varepsilon, \varepsilon'} + \int_t^T \hat{h}_s^{\varepsilon, \varepsilon'} d (B_s) + \int_t^T \hat{Z}_s^{\varepsilon, \varepsilon'} d B_s - \int_t^T \hat{Z}_s^{\varepsilon, \varepsilon'} d B_s - (\hat{K}_T^{\varepsilon, \varepsilon'} - \hat{K}_t^{\varepsilon, \varepsilon'}).
\]

Applying Itô's formula to \( |\hat{Y}_s^{\varepsilon, \varepsilon'}|^2 e^{\lambda s} \) on \([t, T]\) for some positive constant \( \lambda \), we obtain
\[
|\hat{Y}_t^{\varepsilon, \varepsilon'}|^2 e^{\lambda T} + \lambda \int_t^T e^{\lambda s} |\hat{Y}_s^{\varepsilon, \varepsilon'}|^2 d s + \int_t^T e^{\lambda s} |\hat{Z}_s^{\varepsilon, \varepsilon'}|^2 d (B_s) + M_T - M_t
\leq |\hat{\xi}^{\varepsilon, \varepsilon'}|^2 e^{\lambda T} + 2 \int_t^T e^{\lambda s} |\hat{Y}_s^{\varepsilon, \varepsilon'}| |\hat{h}_s^{\varepsilon, \varepsilon'}| d (B_s) + 2 \int_t^T e^{\lambda s} |\hat{Y}_s^{\varepsilon, \varepsilon'}||\hat{Z}_s^{\varepsilon, \varepsilon'}| d s,
\]
(3.22)
where
\[
M_T - M_t = 2 \int_t^T e^{\lambda s} |\hat{Z}_s^{\varepsilon, \varepsilon'}| d B_s + |\hat{Z}_s^{\varepsilon, \varepsilon'}| d B_s + \int_t^T e^{\lambda s} (|\hat{Y}_s^{\varepsilon, \varepsilon'}|^2 d K_s^{\varepsilon} + (\hat{Y}_s^{\varepsilon, \varepsilon'})^{-1} d K_s^{\varepsilon}).
\]
Since
\[
2|\hat{Y}_s^{\varepsilon, \varepsilon'}||\hat{h}_s^{\varepsilon, \varepsilon'}| \leq 2 L_1 |\hat{Y}_s^{\varepsilon, \varepsilon'}| (|\hat{Y}_s^{\varepsilon, \varepsilon'}|^2 + |\hat{Z}_s^{\varepsilon, \varepsilon'}|^2) \leq (|\hat{Y}_s^{\varepsilon, \varepsilon'}|^2 + |\hat{Z}_s^{\varepsilon, \varepsilon'}|^2),
\]
\[
2|\hat{Y}_s^{\varepsilon, \varepsilon'}||\hat{Z}_s^{\varepsilon, \varepsilon'}| \leq |\hat{Y}_s^{\varepsilon, \varepsilon'}|^2 + |\hat{Z}_s^{\varepsilon, \varepsilon'}|^2 \leq |\hat{Y}_s^{\varepsilon, \varepsilon'}|^2 + 2 L_2 (\varepsilon^4 + (\varepsilon')^4),
\]
where \( L_1 = \sup_{(y, z) \in \mathbb{R}^2} (|\partial_y h (y, z)| + |\partial_z h (y, z)|) \) and \( L_2 = \sup_{(y, z) \in \mathbb{R}^2} |h (y, z)| \), we get by taking \( \lambda = (|L_1| + 2 L_1) \sigma^2 + 1 \) in (3.22) that
\[
|\hat{Y}_t^{\varepsilon, \varepsilon'}|^2 e^{\lambda T} + M_T - M_t \leq |\hat{\xi}^{\varepsilon, \varepsilon'}|^2 e^{\lambda T} + 2 L_2^2 (\varepsilon^4 + (\varepsilon')^4) T e^{\lambda T}.
\]
(3.23)

By Lemma 3.4 in [1], we know that \( \hat{E}_t [ M_T - M_t ] = 0 \). Taking \( \hat{E}_t \) on both sides of (3.23), we obtain
\[
|\hat{Y}_t^{\varepsilon, \varepsilon'}|^2 \leq C \left( \hat{E}_t [ \hat{\xi}^{\varepsilon, \varepsilon'}|^2 ] + \varepsilon^4 + (\varepsilon')^4 \right)
\leq C \left( L_\varphi^2 \varepsilon - \varepsilon'^2 + \hat{E}_t [ |\hat{B}_T|^2 ] + \varepsilon^4 + (\varepsilon')^4 \right),
\]
where \( L_\varphi = \sup_{x \in \mathbb{R}} |\varphi' (x)| \) and \( C \) depends on \( \bar{\sigma}, h \) and \( T \). Thus, for each given \( p > 1 \), we obtain
\[
\hat{E}_t \left[ \sup_{t \leq T} |\hat{Y}_t^{\varepsilon, \varepsilon'}|^p \right] \leq C \left( |\varepsilon - \varepsilon'|^p + \varepsilon^2p + (\varepsilon')^2p \right) \to 0 \text{ as } \varepsilon, \varepsilon' \to 0,
\]
(3.24)
where $C$ depends on $p, \sigma, \varphi, h$ and $T$. Applying Itô’s formula to $|Y_{t,0}|^2$ on $[0, T]$, we get

$$
\int_0^T |\dot{Z}_{t,0} + 2Y_{t,0} |dB_t| \leq \int_0^T \left| \phi_t + \frac{\sigma h_t}{\sqrt{1 + \sigma^2}} \right|^2 dB_t + \int_0^T \left| \frac{\sigma h_t}{\sqrt{1 + \sigma^2}} \right|^2 dB_t,
$$

(3.25)

By (3.20), (3.21), (3.23) and (3.25), we obtain

$$
\hat{E} \left[ \int_0^T |\dot{Z}_{t,0} + 2Y_{t,0} |dB_t| \right] \leq C \left\{ \hat{E} \left[ \sup_{t \leq T} |Y_{t,0}|^2 \right] + \left( \hat{E} \left[ \sup_{t \leq T} |Y_{t,0}|^2 \right] \right)^{1/2} \right\} \rightarrow 0 \text{ as } \varepsilon, \varepsilon' \rightarrow 0,
$$

(3.26)

where $C$ depends on $\sigma, \varphi, h$ and $T$. Since $Z^\varepsilon$ is uniformly bounded for $\varepsilon \in (0, \sigma)$, we deduce from (3.26) that, for each given $p > 1$,

$$
\hat{E} \left[ \left( \int_0^T |\dot{Z}_{t,0} + 2Y_{t,0} |dB_t| \right)^{p/2} \right] \rightarrow 0 \text{ as } \varepsilon, \varepsilon' \rightarrow 0.
$$

(3.27)

Thus, for each given $p > 1$, there exist $Y \in S^p_G(0, T)$ and $Z \in H^{2,p}_G(0, T; \langle B \rangle)$ such that

$$
\hat{E} \left[ \sup_{t \leq T} |Y_t - Y_t|^p + \left( \int_0^T |Z_t - Z_t| dB_t \right)^{p/2} \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
$$

(3.28)

It follows from (3.21) and (3.28) that there exists a $K_T \in L^p_G(\Omega_T)$ such that $\hat{E} \left[ |K_T - K_T|^p \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$. Taking $\varepsilon \rightarrow 0$ in (3.21), we obtain

$$
Y_t = \varphi(B_T) + \int_t^T h(Y_s, Z_s) dB_s - \int_t^T Z_s dB_s - (K_T - K_t),
$$

(3.29)

where $K$ is non-increasing and $K_t = \hat{E} \left[ K_T \right]$ for $t \leq T$.

Part 2. The purpose of this part is to prove that $Y \in S^p_G(0, T)$ for each $p > 1$.

Noting that $Y_t^{\varepsilon} = u_{\varepsilon}(t, B_t^{\varepsilon})$ and (3.20), we have

$$
\hat{E} \left[ \sup_{t \leq T} |Y_t^{\varepsilon} - u_{\varepsilon}(t, B_t^{\varepsilon})|^p \right] \leq C \hat{E} \left[ \sup_{t \leq T} |B_t^{\varepsilon}|^p \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,
$$

which implies

$$
\hat{E} \left[ \sup_{t \leq T} |u_{\varepsilon}(t, B_t) - Y_t|^p \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
$$

(3.30)

Thus $Y \in S^p_G(0, T)$. \hfill \Box

### 3.3 Estimates of partial derivatives of $u_{\varepsilon}$

In order to show that $Z$ obtained in Lemma 3.3 belongs to $H^{2,p}_G(0, T; \langle B \rangle)$, we need to prove that $\partial_{xx}^2 u_{\varepsilon}$ is uniformly bounded from below for $\varepsilon \in (0, \sigma)$, where $u_{\varepsilon}$ is the solution of PDE (3.17).

For each fixed $\varepsilon \in (0, \sigma)$, $G_{\varepsilon}$ is defined in (3.19). Let $\hat{E}^{\varepsilon}$ be the $G_{\varepsilon}$-expectation on $(\Omega_T, Lip(\Omega_T))$. The canonical process $(B_t)_t \in [0, T]$ is the 1-dimensional $G_{\varepsilon}$-Brownian motion under $\hat{E}^{\varepsilon}$. For each given $(t, x) \in [0, T) \times \mathbb{R}$, denote

$$
B_{t,s}^{x} = x + B_s - B_t \text{ for } s \in [t, T].
$$
Similar to (3.21), applying Itô's formula to $u_{c}(s, B_{s}^{t,x})$ under $\hat{\mathbb{E}}$, we obtain that the following $G_{\varepsilon}$-BSDE

\[ Y_{s}^{t,x} = \varphi(B_{T}^{t,x}) + \int_{t}^{T} h(Y_{r}^{t,x}, Z_{r}^{t,x})d(B)_{r} - \int_{t}^{T} Z_{r}^{t,x}dB_{r} - (K_{T}^{t,x} - K_{s}^{t,x}) \]  

(3.31)

has a unique solution $(Y_{s}^{t,x}, Z_{s}^{t,x}, K_{s}^{t,x})_{s \in [t,T]}$ satisfying $Y_{s}^{t,x} = u_{c}(s, B_{s}^{t,x})$, $Z_{s}^{t,x} = \partial_{s}u_{c}(t, B_{s}^{t,x})$ and $K_{1}^{t,x} = 0$.

Let $\mathcal{P}$ be a weakly compact and convex set of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

\[ \hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_{P}[X] \text{ for all } X \in L_{\mathcal{B}_{c}}^{1}(\Omega). \]

For each given $(t, x) \in [0, T) \times \mathbb{R}$, denote

\[ \mathcal{P}_{t,x}^{\varepsilon} = \{ P \in \mathcal{P} : E_{P}[K_{T}^{t,x}] = 0 \}. \]

The following estimates for $G_{\varepsilon}$-BSDE (3.31) are useful.

**Proposition 3.4.** Suppose that $\varphi \in C_{0}^{\infty}(\mathbb{R})$ and $h \in C_{0}^{\infty}(\mathbb{R}^{2})$. For each $(t, x, \Delta) \in [0, T) \times \mathbb{R} \times \mathbb{R}$, let $(Y_{s}^{t,x}, Z_{s}^{t,x}, K_{s}^{t,x})_{s \in [t,T]}$ and $(Y_{s}^{t,x+\Delta}, Z_{s}^{t,x+\Delta}, K_{s}^{t,x+\Delta})_{s \in [t,T]}$ be two solutions of $G_{\varepsilon}$-BSDE (3.31). Then, for each given $p > 1$,

\[ \sup_{s \in [t,T]} |Y_{s}^{t,x+\Delta} - Y_{s}^{t,x}|^{p} \leq C|\Delta|^{p} \]  

(3.32)

\[ \hat{\mathbb{E}}^{\varepsilon} \left[ \sup_{s \in [t,T]} |Y_{s}^{t,x}|^{p} + \left( \int_{t}^{T} |Z_{r}^{t,x}|^{2} d(B)_{r} \right)^{p/2} + |K_{T}^{t,x}|^{p} \right] \leq C(1 + |x|^{p}) \]  

(3.33)

\[ E_{P} \left[ \left( \int_{t}^{T} |Z_{r}^{t,x+\Delta} - Z_{r}^{t,x}|^{2} d(B)_{r} \right)^{p/2} + |K_{T}^{t,x+\Delta}|^{p} \right] \leq C|\Delta|^{p} \text{ for } P \in \mathcal{P}_{t,x}^{\varepsilon}, \]  

(3.34)

\[ E_{P\Delta} \left[ \left( \int_{t}^{T} |Z_{r}^{t,x+\Delta} - Z_{r}^{t,x}|^{2} d(B)_{r} \right)^{p/2} + |K_{T}^{t,x}|^{p} \right] \leq C|\Delta|^{p} \text{ for } P\Delta \in \mathcal{P}_{t,x+\Delta}^{\varepsilon}, \]  

(3.35)

where the constant $C > 0$ depends on $p$, $\sigma$, $\varphi$, $h$ and $T$.

**Proof.** Similar to the proof of (3.3), (3.7) and (3.8), we obtain

\[ \sup_{s \in [t,T]} |Y_{s}^{t,x+\Delta} - Y_{s}^{t,x}|^{p} \leq C \sup_{s \in [t,T]} \hat{\mathbb{E}}^{\varepsilon} \left[ |\varphi(B_{T}^{t,x+\Delta}) - \varphi(B_{T}^{t,x})|^{p} \right] \leq C|\Delta|^{p} \]

and

\[ \hat{\mathbb{E}}^{\varepsilon} \left[ \sup_{s \in [t,T]} |Y_{s}^{t,x}|^{p} + \left( \int_{t}^{T} |Z_{r}^{t,x}|^{2} d(B)_{r} \right)^{p/2} + |K_{T}^{t,x}|^{p} \right] \leq C \left( 1 + \hat{\mathbb{E}}^{\varepsilon} \left[ |\varphi(B_{T}^{t,x})|^{p} \right] \right) \]

\[ \leq C \left( 1 + |x|^{p} + \hat{\mathbb{E}}^{\varepsilon} \left[ \sup_{s \in [t,T]} |B_{r} - B_{t}|^{p} \right] \right) \]

\[ \leq C(1 + |x|^{p}), \]
where the constant $C > 0$ depends on $p$, $\bar{\sigma}$, $\varphi$, $h$ and $T$.

Set $\hat{Y}_s^\Delta = Y_s^{t,x+\Delta} - Y_s^{t,x}$ and $\hat{Z}_s^\Delta = Z_s^{t,x+\Delta} - Z_s^{t,x}$ for $s \in [t, T]$. For each given $P \in \mathcal{P}_{t,x}$, we know that $K_t^{t,x} = 0$ $P$-a.s. by $E_P[K_T^{t,x}] = 0$. Applying Itô’s formula to $|\hat{Y}_s^\Delta|^2$ on $[t, T]$ under $P$, we obtain

$$|\hat{Y}_t^\Delta|^2 + \int_t^T |\hat{Z}_r^\Delta|^2 dB_r = |\hat{Y}_T^\Delta|^2 + 2 \int_t^T \hat{Y}_r^\Delta \hat{h}_r dB_r - 2 \int_t^T \hat{Y}_r^\Delta \hat{Z}_r^\Delta dB_r - \int_t^T \hat{Y}_r^\Delta dK_r^{t,x+\Delta},$$

(3.36)

where

$$|\hat{h}_r| = |h(Y_r^{t,x+\Delta}, Z_r^{t,x+\Delta}) - h(Y_r^{t,x}, Z_r^{t,x})| \leq \sup_{(y, z) \in \mathbb{R}^2} (|h'_y(y, z)| + |h'_z(y, z)|)(|\hat{Y}_r^\Delta| + |\hat{Z}_r^\Delta|).$$

(3.37)

Since $K_t^{t,x+\Delta}$ is non-increasing with $K_t^{t,x+\Delta} = 0$ and $d(B)_r \leq (\bar{a}^2 + \epsilon^2)dr \leq 2\bar{\sigma}^2dr$ under $P$, we deduce by (3.36) and (3.37) that

$$E_P \left[ \left( \int_t^T |\hat{Z}_r^\Delta|^2 dB_r \right)^{p/2} \right] \leq CE_P \left[ \sup_{r \in [t, T]} |\hat{Y}_r^\Delta|^p + \left( \sup_{r \in [t, T]} |\hat{Y}_r^\Delta|^p \right)^{p/2} \right],$$

(3.38)

where $C > 0$ depends on $p$, $\bar{\sigma}$, $h$ and $T$. Noting that

$$K_T^{t,x+\Delta} = \hat{Y}_T^\Delta - \hat{Y}_t^\Delta + \int_t^T \hat{h}_r dB_r - \int_t^T \hat{Z}_r^\Delta dB_r, \text{ $P$-a.s.,}$$

we get

$$E_P \left[ K_T^{t,x+\Delta} \right] \leq CE_P \left[ \sup_{r \in [t, T]} |\hat{Y}_r^\Delta|^p \right] + \left( \int_t^T |\hat{Z}_r^\Delta|^2 dB_r \right)^{p/2},$$

(3.39)

where $C > 0$ depends on $p$, $\bar{\sigma}$, $h$ and $T$. Thus we obtain by (3.38) and (3.39) that

$$E_P \left[ \left( \int_t^T |\hat{Z}_r^\Delta|^2 dB_r \right)^{p/2} + \left| K_T^{t,x+\Delta} \right|^p \right] \leq CE_P \left[ \sup_{r \in [t, T]} |\hat{Y}_r^\Delta|^p \right],$$

(3.40)

where $C > 0$ depends on $p$, $\bar{\sigma}$, $h$ and $T$. By 3.32 and 3.30, we obtain 3.34j. By the same method, we obtain 3.35. □

In the following theorem, we obtain the formula of $\partial_x u_\epsilon$ based on $u_\epsilon(t, x) = Y_t^{t,x}$.

**Theorem 3.5** Suppose that $\varphi \in C_0^\infty(\mathbb{R})$ and $h \in C_0^\infty(\mathbb{R}^2)$. Let $u_\epsilon$ be the solution of PDE (3.17). Then, for each $(t, x) \in [0, T) \times \mathbb{R}$, we have

$$\partial_x u_\epsilon(t, x) = E_P \left[ \Gamma_t^{t,x} \varphi(B_t^{t,x}) \right] \text{ for any } P \in \mathcal{P}_{t,x},$$

(3.41)

where $(\Gamma_s^{t,x})_{s \in [t, T]}$ is the solution of the following G-SDE:

$$d\Gamma_s^{t,x} = h'_y(Y_s^{t,x}, Z_s^{t,x})\Gamma_s^{t,x} dB_s + h'_z(Y_s^{t,x}, Z_s^{t,x})\Gamma_s^{t,x} dB_s, \Gamma_t^{t,x} = 1.$$  

(3.42)

**Proof.** For each $\Delta \in \mathbb{R}$, we use the notations $(\hat{Y}_s^\Delta)_{s \in [t, T]}$ and $(\hat{Z}_s^\Delta)_{s \in [t, T]}$ as in the proof of Proposition 3.3. Then, for any given $P \in \mathcal{P}_{t,x}$, we have

$$\hat{Y}_s^\Delta = \hat{Y}_t^\Delta + \int_t^s \hat{h}_r dB_r - \int_s^T \hat{Z}_r^\Delta dB_r - \int_s^T dK_r^{t,x+\Delta}, \text{ $P$-a.s.,}$$

(3.43)
where
\[
\dot{h}_r = h(Y^{t,x+ \Delta}_r, Z^{t,x+ \Delta}_r) - h(Y^{t,x}_r, Z^{t,x}_r) = h'(Y^{t,x}_r, Z^{t,x}_r)\dot{Y}^{\Delta}_r + h'_r(Y^{t,x}_r, Z^{t,x}_r)\dot{Z}^{\Delta}_r + I^{\Delta}_r.
\]
Since $h \in C^\infty_c(\mathbb{R}^2)$, we get $|I^{\Delta}_r| \leq C(|\dot{Y}^{\Delta}_r|^2 + |\dot{Z}^{\Delta}_r|^2)$, where $C > 0$ depends on $h$. Applying Itô’s formula to $\dot{Y}^{\Delta}_r \Gamma^{t,x}_r$ on $[t, T]$ under $P$, we obtain
\[
\dot{Y}^{\Delta}_r = \dot{Y}^{\Delta}_r \Gamma^{t,x}_r + \int_t^T \Gamma^{t,x}_r I^{\Delta}_r d\langle B \rangle_r - \int_t^T (\Gamma^{t,x}_r \dot{Z}^{\Delta}_r + h'_r(Y^{t,x}_r, Z^{t,x}_r)\Gamma^{t,x}_r \dot{Y}^{\Delta}_r) dB_r - \int_t^T \Gamma^{t,x}_r dK^{t,x+\Delta}_r. \tag{3.43}
\]
Noting that $\dot{Y}^{\Delta}_r = u_\varepsilon(t, x + \Delta) - u_\varepsilon(t, x)$, we get
\[
\frac{u_\varepsilon(t, x + \Delta) - u_\varepsilon(t, x)}{\Delta} = \frac{1}{\Delta} E_P \left[ \dot{Y}^{\Delta}_r \Gamma^{t,x}_r + \int_t^T \Gamma^{t,x}_r I^{\Delta}_r d\langle B \rangle_r - \int_t^T \Gamma^{t,x}_r dK^{t,x+\Delta}_r \right]. \tag{3.44}
\]
By \textbf{3.32}, \textbf{3.34}, $\varphi \in C^\infty_c(\mathbb{R})$ and $h \in C^\infty_c(\mathbb{R}^2)$, we can easily deduce that
\[
\lim_{\Delta \to 0} \frac{1}{\Delta} E_P \left[ \dot{Y}^{\Delta}_r \Gamma^{t,x}_r + \int_t^T \Gamma^{t,x}_r I^{\Delta}_r d\langle B \rangle_r \right] = E_P \left[ \Gamma^{t,x}_r \varphi'(B^{t,x}_r) \right]. \tag{3.45}
\]
Since $\Gamma^{t,x}_r > 0$, $dK^{t,x+\Delta}_r \leq 0$ and $\partial_x u_\varepsilon(t, x)$ exists, we obtain by \textbf{3.44} and \textbf{3.45} that
\[
E_P \left[ \Gamma^{t,x}_r \varphi'(B^{t,x}_r) \right] \leq \partial_x u_\varepsilon(t, x) = \partial_x u_\varepsilon(t, x) = \partial_x u_\varepsilon(t, x) \leq E_P \left[ \Gamma^{t,x}_r \varphi'(B^{t,x}_r) \right],
\]
which implies the desired result. \hfill \Box

Now we give the estimate for $\partial_{x}^{2} u_{\varepsilon}$. 

\textbf{Theorem 3.6} Suppose that $\varphi \in C^\infty_c(\mathbb{R})$ and $h \in C^\infty_c(\mathbb{R}^2)$. Let $u_\varepsilon$ be the solution of PDE (3.17). Then
\[
\partial_{x}^{2} u_{\varepsilon}(t, x) \geq -C \text{ for } (t, x) \in [0, T) \times \mathbb{R},
\]
where the constant $C > 0$ depends on $\tilde{\sigma}$, $\varphi$, $h$ and $T$.

\textbf{Proof.} For each $(t, x, \Delta) \in [0, T) \times \mathbb{R} \times \mathbb{R}$, we use the notations $(\dot{Y}^{\Delta}_s)_{s \in [t, T]}$ and $(\dot{Z}^{\Delta}_s)_{s \in [t, T]}$ as in the proof of Proposition 3.3. For any given $P \in \mathcal{P}_{t,x}$, we obtain by \textbf{3.43} that
\[
\dot{Y}^{\Delta}_r = E_P \left[ \dot{Y}^{\Delta}_r \Gamma^{t,x}_r + \int_t^T \Gamma^{t,x}_r I^{\Delta}_r d\langle B \rangle_r - \int_t^T \Gamma^{t,x}_r dK^{t,x+\Delta}_r \right].
\]
Since $\Gamma^{t,x}_r > 0$ and $dK^{t,x+\Delta}_r \leq 0$, we get
\[
\dot{Y}^{\Delta}_r \geq E_P \left[ \dot{Y}^{\Delta}_r \Gamma^{t,x}_r + \int_t^T \Gamma^{t,x}_r I^{\Delta}_r d\langle B \rangle_r \right].
\]
Noting that $|\dot{Y}^{\Delta}_r - \varphi'(B^{t,x}_r)\Delta| \leq C\Delta^2$ and $|I^{\Delta}_r| \leq C(|\dot{Y}^{\Delta}_r|^2 + |\dot{Z}^{\Delta}_r|^2)$, where $C > 0$ depends on $\varphi$ and $h$, we deduce by Proposition 3.3 that
\[
\dot{Y}^{\Delta}_r \geq E_P \left[ \dot{Y}^{\Delta}_r \Gamma^{t,x}_r + \int_t^T \Gamma^{t,x}_r I^{\Delta}_r d\langle B \rangle_r \right] \geq E_P \left[ \Gamma^{t,x}_r \varphi'(B^{t,x}_r) \right] - C\Delta^2, \tag{3.46}
\]
where $C > 0$ depends on $\sigma$, $\varphi$, $h$ and $T$. For any given $P^\Delta \in \mathcal{P}^\varepsilon_{t,x+\Delta}$, applying Itô’s formula to $\hat{Y}^\Delta_{t,x+\Delta}$ on $[t,T]$ under $P^\Delta$, we obtain

$$
\hat{Y}^\Delta_{t,x} = E_{P^\Delta} \left[ \hat{Y}^\Delta_{T,x+\Delta} + \int_t^T \Gamma^\Delta r_{t,x+\Delta} d(B) r + \int_t^T \Gamma^\Delta dK_{t,x} \right],
$$

where $\hat{I}^\Delta = h(Y^r_{t,x+\Delta}, Z^r_{t,x+\Delta}) - h(Y^r_{t,x}, Z^r_{t,x}) - h'_y(Y^r_{t,x+\Delta}, Z^r_{t,x+\Delta})\hat{Y}^\Delta - h'_z(Y^r_{t,x+\Delta}, Z^r_{t,x+\Delta})\hat{Z}^\Delta$. Since $\Gamma^\Delta r_{t,x+\Delta} > 0$ and $dK_{t,x} \leq 0$, we get

$$
\hat{Y}^\Delta_{t,x} \leq E_{P^\Delta} \left[ \hat{Y}^\Delta_{T,x+\Delta} + \int_t^T \Gamma^\Delta dK_{t,x} \right].
$$

Similar to the proof of (3.48), we have

$$
\hat{Y}^\Delta_{t,x} \leq E_{P^\Delta} \left[ \hat{Y}^\Delta_{T,x+\Delta} + \int_t^T \Gamma^\Delta dK_{t,x} \right] \leq E_{P^\Delta} \left[ \Gamma^\Delta r_{T,x+\Delta} \varphi'(B^r_{T,x+\Delta}) \right] + C\Delta^2,
$$

(3.47)

where $C > 0$ depends on $\sigma$, $\varphi$, $h$ and $T$. By Theorem 3.5, (3.46) and (3.47), we obtain

$$
\frac{\partial_x u^p(t,x+\Delta) - \partial_x u^p(t,x)}{\Delta} = \frac{1}{\Delta^2} \left( E_{P^\Delta} \left[ \Gamma^\Delta r_{T,x+\Delta} \varphi'(B^r_{T,x+\Delta}) \right] - E_P \left[ \Gamma^\Delta r_{T,x} \varphi'(B^r_{T,x}) \right] \right) \geq -C,
$$

which implies the desired result. $\square$

**Remark 3.7** The constant $C$ in the above theorem is independent of $\varepsilon \in (0,\bar{\epsilon})$.

### 3.4 Existence and uniqueness of $G$-BSDEs

We first give the following existence and uniqueness result of $G$-BSDE (3.15).

**Lemma 3.8** Let $\varphi \in C^\infty_0(\mathbb{R})$ and $h \in C^\infty_0(\mathbb{R}^2)$. Then, for each given $p > 1$, $G$-BSDE (3.15) has a unique $L^p$-solution $(Y,Z,K)$ in the $G$-expectation space.

**Proof.** The uniqueness is due to (3.6) and (3.9) in Proposition 3.2. In the following, we give the proof of existence.

By Lemma 3.3, for each given $p > 1$, $G$-BSDE (3.15) has a unique $L^p$-solution $(Y,Z,K)$ in the extended $\hat{G}$-expectation space, i.e.,

$$
Y_t = \varphi(B_T) + \int_t^T h(Y_s, Z_s) d(B)_s - \int_t^T Z_s dB_s - (K_T - K_t).
$$

(3.48)

Let $u_\varepsilon$ be the solution of PDE (3.17) for $\varepsilon \in (0,\bar{\epsilon})$. Applying Itô’s formula to $u_\varepsilon(t,B_t)$ under $\hat{G}$-expectation, we get by (3.20) that

$$
\hat{Y}_t^\varepsilon = \varphi(B_T) + \int_t^T h(\hat{Y}_s^\varepsilon, \hat{Z}_s^\varepsilon) d(B)_s - \int_t^T \frac{1}{2}\varepsilon^2 \left( \partial_{xx} u_\varepsilon(s,B_s) + 2h(\hat{Y}_s^\varepsilon, \hat{Z}_s^\varepsilon) \right) ds
$$

$$
- \int_t^T \hat{Z}_s dB_s + (L_T - L_t),
$$

(3.49)
Lemma 3.9

Since $0 \leq d(B)_t \leq \sigma^2 d_t$ under $\tilde{E}$, we deduce that $L^\varepsilon$ is non-increasing with $L^0_0 = 0$ under $\tilde{E}$.

In the proof of Lemma 3.3, we know that, for each given $p > 1$,

$$\tilde{E} \left[ \sup_{t \leq T} |\tilde{Y}^\varepsilon_t| + \left( \int_0^T |\partial_x u_x(t, B_t + \varepsilon \tilde{B}_t) - Z_t| d(B)_t \right)^{p/2} \right] \to 0 \text{ as } \varepsilon \to 0. \quad (3.50)$$

Thus $|Y| + |Z| \leq C$ by (3.20), where $C > 0$ depends on $\tilde{\sigma}, \phi, h$, and $T$. By (3.48), we get

$$\tilde{E} \left[ |K_T|^2 \right] \leq C \tilde{E} \left[ \sup_{t \leq T} |Y_t|^2 + 1 + \int_0^T |Z_t|^2 d(B)_t \right] \leq C,$$

where $C > 0$ depends on $\tilde{\sigma}, \phi, h$, and $T$. By Theorem 3.6, we know $\partial_x^2 u_x \geq -C$ for $\varepsilon \in (0, \tilde{\sigma})$, where $C > 0$ depends on $\tilde{\sigma}, \phi, h$, and $T$. Thus

$$\left( \partial_x^2 u_x(t, B_s + 2h(\tilde{Y}^\varepsilon_s, \tilde{Z}^\varepsilon_s)) \right) \leq C \text{ for } s \in [0, T] \text{ and } \varepsilon \in (0, \tilde{\sigma}),$$

where $C > 0$ depends on $\tilde{\sigma}, \phi, h$, and $T$. By (3.20) and (3.49), we have $|\tilde{Y}^\varepsilon| + |\tilde{Z}^\varepsilon| \leq C$ for $\varepsilon \in (0, \tilde{\sigma})$ and

$$\tilde{E} \left[ |L^\varepsilon|^2 \right] \leq C \tilde{E} \left[ \sup_{t \leq T} |\tilde{Y}_t|^2 + 1 + \int_0^T |\tilde{Z}_t|^2 d(B)_t \right] \leq C \text{ for } \varepsilon \in (0, \tilde{\sigma}),$$

where $C > 0$ depends on $\tilde{\sigma}, \phi, h$, and $T$.

Applying Itô’s formula to $|\tilde{Y}^\varepsilon_t - Y_t|^2$ on $[0, T]$, we obtain

$$\tilde{E} \left[ \int_0^T |\tilde{Z}^\varepsilon_t - Z_t|^2 d(B)_t \right] \leq C \tilde{E} \left[ \int_0^T |\tilde{Y}^\varepsilon_t - Y_t| dt + (|L^\varepsilon_t| + |K_T|) \sup_{t \leq T} |\tilde{Y}^\varepsilon_t - Y_t| \right]$$

$$\leq C \left( \tilde{E} \left[ \sup_{t \leq T} |\tilde{Y}^\varepsilon_t - Y_t|^2 \right] \right)^{1/2},$$

where $C > 0$ depends on $\tilde{\sigma}, \phi, h$, and $T$. By (3.50) and (3.49), we get

$$\lim_{\varepsilon \downarrow 0} \tilde{E} \left[ \left( \int_0^T |\tilde{Z}^\varepsilon_t - Z_t|^2 d(B)_t \right)^{p/2} \right] = 0 \text{ for each } p > 1.$$

Thus $Z \in H_{G}^{2,p}(0, T; (B))$, and then $K_T \in L_{G}^{p}(\Omega_T)$ by (3.48). □

Moreover, we extend the above result to the following two lemmas.

**Lemma 3.9** Let $t_1 \in [0, T)$, $\varphi \in C_{b,Lip}(\mathbb{R})$, $h_1 \in C_{0}^{\infty}(\mathbb{R})$ and $h_2 \in C_{0}^{\infty}(\mathbb{R}^2)$. Then, for each given $p > 1$, $G$-BSDE

$$Y_t = \varphi(B_T - B_{t_1}) + \int_t^T h_1(Y_s) ds + \int_t^T h_2(Y_s, Z_s) d(B)_s - \int_t^T Z_s dB_s - (K_T - K_t) \quad (3.51)$$
has a unique \( L^p \)-solution \((Y, Z, K)\) in the G-expectation space. Furthermore, \(Y_t = u(t, B_t - B_{t_1})\) for \(t \in [t_1, T]\), where \(u(t, x) = Y^{t,x}_t\) and \((Y^{t,x}_t)_{s \in [t, T]}\) satisfies the following G-BSDE:

\[
Y^{t,x}_s = \varphi(x + B_T - B_t) + \int_t^T h_1(Y^{t,x}_r)dr + \int_s^T h_2(Y^{t,x}_r, Z^{t,x}_r)dB_r - \int_s^T Z^{t,x}_r dB_r - (K^{t,x}_T - K^{t,x}_s).
\] (3.52)

**Proof.** The uniqueness is due to (3.6) and (3.9) in Proposition 3.2. For each given \(p \geq 1\), we can find a sequence \(\varphi_n \in C_0^\infty(\mathbb{R})\), \(n \geq 1\), such that \(\mathbb{E}[|\varphi_n(B_T - B_1) - \varphi(B_T - B_1)|^{p+1}] \to 0\) as \(n \to \infty\). Similar to the proof of Lemma 3.8, the following G-BSDE

\[
Y^n_t = \varphi_n(B_T - B_t) + \int_t^T h_1(Y^n_s)ds + \int_t^T h_2(Y^n_s, Z^n_s)dB_s - \int_t^T Z^n_s dB_s - (K^n_T - K^n_t)
\] (3.53)

has a unique \( L^p \)-solution \((Y^n, Z^n, K^n)\) in the G-expectation space. By (3.6), (3.9) in Proposition 3.2 and Theorem 2.3, we can easily deduce

\[
\lim_{n,m \to \infty} \mathbb{E} \left[ \sup_{t \leq T} |Y^n_t - Y^m_t|^p + \left( \int_0^T |Z^n_t - Z^m_t|^2 dB_t \right)^{p/2} + |K^n_T - K^m_T|^p \right] = 0.
\]

Thus there exist \(Y \in S^p_G(0, T), Z \in H^{2,p}_G(0, T; (B))\) and a non-increasing G-martingale \(K\) with \(K_0 = 0\) and \(K_T \in L^p_G(\Omega_T)\) such that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \leq T} |Y^n_t - Y_t|^p + \left( \int_0^T |Z^n_t - Z_t|^2 dB_t \right)^{p/2} + |K^n_T - K_T|^p \right] = 0.
\]

From this we can easily get

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \leq T} \left( \int_t^T |h_1(Y^n_s) - h_1(Y_s)|ds + \int_t^T |h_2(Y^n_s, Z^n_s) - h_2(Y_s, Z_s)|dB_s + \int_t^T |Z^n_s - Z_s|dB_s \right)^p \right] = 0.
\]

Thus \((Y, Z, K)\) satisfies G-BSDE 3.51 by taking \(n \to \infty\) in 3.53.

From the above proof, we know that G-BSDE 3.52 has a unique \( L^p \)-solution \((Y^{t,x}, Z^{t,x}, K^{t,x})\) and \(Y^{t,x} \in \mathbb{R}\). By (3.6) in Proposition 3.2, we obtain that

\[
|u(t, x) - u(t, x')| \leq C|x - x'| \text{ and } |Y_t - u(t, x)| \leq C|B_t - B_{t_1} - x|
\]

where the constant \(C > 0\) depends on \(\sigma, \varphi, h_1, h_2\) and \(T\). Thus we get \(Y_t = u(t, B_t - B_{t_1})\) for \(t \in [t_1, T]\). \(\square\)

**Lemma 3.10** Let \(\xi \in \text{Lip}(\Omega_T), f(t, y) = \sum_{i=1}^{N_1} f_i h_1^i(y)\) and \(g(t, y, z) = \sum_{j=1}^{N_2} g_j h_2^j(y, z)\) with \(f^i, g^j \in M^0(0, T), h_1^i \in C_0^\infty(\mathbb{R}), h_2^j \in C_0^\infty(\mathbb{R}^2), i \leq N_1, j \leq N_2\). Then G-BSDE (5.3) has a unique \( L^p \)-solution \((Y, Z, K)\) for each given \(p > 1\).

**Proof.** The uniqueness is due to (3.6) and (3.9) in Proposition 3.2. For the existence, we only prove the special case \(\xi = \varphi(B_t, B_T - B_1), f(t, y) = 0\) and \(g(t, y, z) = (I_{[0,1]}(t) + \psi(B_t)) I_{[t_1, T]}(t) h_2(y, z)\), the general case is similar.

By Lemma 3.9 G-BSDE

\[
Y^n_t = \varphi(x, B_T - B_t) + \int_t^T \psi(x)h_2(Y^n_s, Z^n_s)dB_s - \int_t^T Z^n_s dB_s - (K^n_T - K^n_t)
\] (3.54)
has a unique $L^p$-solution $(Y^x, Z^x, K^x)$ for each given $p > 1$. Furthermore, $Y^x_t = u(t, x, B_t - B_{t_1})$ for $t \in [t_1, T]$, where $u(t, x, x') = Y^{t,x,x'}_t$ and $(Y^{t,x,x'}_s)_{s \in [t, T]}$ satisfies the following G-BSDE:

$$Y^{t,x,x'}_s = \varphi(x, x' + BT - B_t) + \int_s^T \psi(x)h_2(Y^{r,t,x,x'}_r, Z^{r,t,x,x'}_r) d(B)_r - \int_s^T Z^{r,t,x,x'}_r dB_r - (K^{r,t,x,x'}_T - K^{r,t,x,x'}_s).$$

By (3.6) in Proposition 3.2, we obtain that, for $t \in [t_1, T]$, $x, x', \tilde{x}, \tilde{x}' \in \mathbb{R}$,

$$|u(t, x, x')| \leq C \text{ and } |u(t, x, x') - u(t, \tilde{x}, \tilde{x}')| \leq C(|x - \tilde{x}| + |x' - \tilde{x}'|)$$

(3.55)

where the constant $C > 0$ depends on $\sigma, \varphi, \psi, h_2$ and $T$.

For each positive integer $n$, by partion of unity theorem, we can find $l^n_i \in \mathcal{C}_b^\infty(\mathbb{R})$, $i = 1, \ldots, k_n$, such that

$$0 \leq l^n_i \leq 1 \text{ and } \lambda(\text{supp}(l^n_i)) \leq \frac{1}{n} \text{ for } i \leq k_n, I_{[-n,n]}(x) \leq \sum_{i=1}^{k_n} l^n_i(x) \leq 1,$$

where $\lambda(\cdot)$ is the Lebesgue measure. For $t \in [t_1, T]$, set

$$Y^n_t = \sum_{i=1}^{k_n} l^n_i(B_t)Y^n_t^{i}, \quad Z^n_t = \sum_{i=1}^{k_n} l^n_i(B_t)Z^n_t^{i}, \quad K^n_t = \sum_{i=1}^{k_n} l^n_i(B_t)K^n_t^{i},$$

where $l^n_i(x^n_i) > 0$. Then, by (3.54), we get that, for $t \in [t_1, T]$,

$$Y^n_t = Y^n_T + \int_t^T \sum_{i=1}^{k_n} l^n_i(B_t)\psi(x^n_i)h_2(Y^n_s, Z^n_s) d(B)_s - \int_t^T Z^n_s dB_s - (K^n_T - K^n_s).$$

(3.56)

It follows from (3.55) that

$$\left|Y^n_t - u(t, B_{t_1}, B_t - B_{t_1})\right| \leq \sum_{i=1}^{k_n} l^n_i(B_t_1)|u(t, x^n_i, B_t - B_{t_1}) - u(t, B_{t_1}, B_t - B_{t_1})| + \left(1 - \sum_{i=1}^{k_n} l^n_i(B_t_1)\right) |u(t, B_{t_1}, B_t - B_{t_1})|$$

$$\leq \frac{C}{n} + \frac{C}{n} |B_{t_1}|,$$

which implies

$$\lim_{n,m \to \infty} \mathbb{E} \left[ \sup_{t \in [t_1, T]} \left|Y^n_t - Y^m_t\right|^p \right] = 0. \quad (3.57)$$

Noting that $|\sum_{i=1}^{k_n} l^n_i(B_t_1)\psi(x^n_i)h_2(Y^n_s, Z^n_s)| \leq C$, where $C$ depends on $\psi$ and $h_2$, we obtain by (3.54) in Proposition 3.2 that

$$\lim_{n,m \to \infty} \mathbb{E} \left[ \left( \int_{t_1}^T \left|Z^n_t - Z^m_t\right|^2 d(B)_t \right)^{p/2} \right] = 0. \quad (3.58)$$

It is easy to verify that

$$\left|\sum_{i=1}^{k_n} l^n_i(B_t_1)\psi(x^n_i)h_2(Y^{x^n_i, Z^n_i}_s) - \psi(B_{t_1})h_2(Y^n_s, Z^n_s)\right|$$

$$\leq \frac{C}{n}(1 + |B_{t_1}|) + C \left|\sum_{i=1}^{k_n} l^n_i(B_t_1)\right| \left|h_2 \left(\sum_{j=1}^{k_n} l^n_j(B_t_1)Y^{x^n_j, Z^n_j}_s, \sum_{j=1}^{k_n} l^n_j(B_t_1)Z^{x^n_j, Z^n_j}_s\right) - h_2(Y^n_s, Z^n_s)\right|$$

$$\leq \frac{C}{n}(1 + |B_{t_1}|) + C \sum_{i,j=1}^{k_n} l^n_i(B_t_1)l^n_j(B_t_1) \left(\left|Y^{x^n_j, Z^n_j}_s - Y^n_s\right| + \left|Z^{x^n_j, Z^n_j}_s - Z^n_s\right|\right).$$
By (3.55), we know that \( |Y_t^{x^n} - Y_t^{x^n}| \leq C|x_t^n - x_j^n| \). Similar to the proof of (3.9) in Proposition 3.2, we deduce that, for each \( P \in \mathcal{P} \),

\[
E_P \left[ \left( \int_{t_1}^T \left| Z_t^{x^n} - Z_t^{x^n} \right|^2 d(B)_t \right)^{p/2} \mid \mathcal{B}(\Omega_t) \right] \leq C \left\{ E_P \left[ \sup_{t \in [t_1, T]} \left| Y_t^{x^n} - Y_t^{x^n} \right|^p \mid \mathcal{B}(\Omega_t) \right] + \left( E_P \left[ \sup_{t \in [t_1, T]} \left| Y_t^{x^n} - Y_t^{x^n} \right|^p \mid \mathcal{B}(\Omega_t) \right] \right)^{1/2} \right\} \leq C \left( |x_t^n - x_j^n|^p + |x_t^n - x_j^n|^p \right).
\]

Noting that \( l_t^n(B_{t_1})l_t^n(B_{t_1})|x_t^n - x_j^n| = 0 \) if \( |x_t^n - x_j^n| > \frac{2}{n} \), we obtain

\[
\lim_{n \to \infty} \sup_{P \in \mathcal{P}} E_P \left[ \left( \int_{t_1}^T \left| \sum_{i=1}^{k_1} l_t^n(B_{t_1}) \psi(x_t^n) b_2(Y_s^{x^n}, Z_s^{x^n}) - \psi(B_{t_1}) b_2(Y_s^{x^n}, Z_s^{x^n}) \right| d(B)_s \right)^{p} \right] = 0. \tag{3.59}
\]

By (3.55), (3.56), (3.58) and (3.59), we get \( \lim_{m, n \to \infty} \mathbb{E} \left[ \left| (K^n_T - K^n_T) - (K^n_T - K^n_T) \right|^p \right] = 0 \). Thus there exist \( Y \in S^m_{L^p_G(t_1, T)}, Z \in H^m_{L^p_G(t_1, T)}(B) \) and a non-increasing \( K \) with \( K_t = 0 \) and \( K_T \in L^p_G(\Omega_T) \) such that

\[
Y_t = \varphi(B_{t_1}, B_T - B_{t_1}) + \int_t^T \psi(B_{t_1}) b_2(Y_s, Z_s) d(B)_s - \int_t^T Z_s dB_s - (K_T - K_t) \quad \text{for } t \in [t_1, T].
\]

In the following, we prove that \( K \) is a \( G \)-martingale. For each positive integer \( n \), set

\[
\tilde{l}_t^n(x) = I_{[-\frac{n}{2}, n+\frac{1}{2}]}(x) \text{ for } i = 0, \ldots, 2n^2 - 1, \quad \tilde{l}_t^n(x) = I_{[-n, n]}(x)
\]

and

\[
\tilde{Y}_t^n = \sum_{i=0}^{2n^2} \tilde{l}_t^n(B_{t_1})Y_t^{-n+i}, \quad \tilde{Z}_t^n = \sum_{i=0}^{2n^2} \tilde{l}_t^n(B_{t_1})Z_t^{-n+i}, \quad \tilde{K}_t^n = \sum_{i=0}^{2n^2} \tilde{l}_t^n(B_{t_1})K_t^{-n+i}.
\]

Then, for \( t \in [t_1, T] \),

\[
\tilde{Y}_t^n = \tilde{Y}_T^n + \int_t^T \int_{i=0}^{2n^2} \tilde{l}_t^n(B_{t_1}) \psi \left( -n + \frac{i}{n} \right) b_2(Y_s^{n+i}, Z_s^{n+i}) d(B)_s - \int_t^T \tilde{Z}_s^n dB_s - (\tilde{K}_T^n - \tilde{K}_t^n).
\]

Similar to the above proof, we have \( \lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [t_1, T]} |\tilde{Y}_t^n - Y_t|^p \right] = 0 \), which implies

\[
\lim_{n \to \infty} \mathbb{E} \left[ \left| (\tilde{K}_T^n - \tilde{K}_T^n) - (K_T - K_t) \right|^p \right] = 0.
\]

By Proposition 2.5 in [1], we know that, for \( t \in [t_1, T] \), \( \mathbb{E}_t \left[ \tilde{K}_T^n - \tilde{K}_t^n \right] = 0 \) and

\[
\mathbb{E} \left[ \mathbb{E}_t[K_T - K_t] \right] = \mathbb{E} \left[ \mathbb{E}_t[K_T - K_t] - \mathbb{E}_t[\tilde{K}_T^n - \tilde{K}_t^n] \right] \leq \mathbb{E} \left[ (K_T - K_t) - (\tilde{K}_T^n - \tilde{K}_t^n) \right],
\]

which implies \( \mathbb{E}_t[K_T] = K_t \) by letting \( n \to \infty \). Thus we obtain an \( L^p \)-solution \((Y, Z, K)\) on \([t_1, T]\). Noting that \( Y_{t_1} = u(t_1, B_{t_1}, 0) \), we obtain the desired result by applying Lemma 3.9 to find an \( L^p \)-solution on \([0, t_1]\). \( \square \)

Now, we give the following existence and uniqueness result of \( G \)-BSDE (3.55).
Theorem 3.11 Suppose that $\xi$, $f$ and $g$ satisfy (H1) and (H2). Then G-BSDE (3.5) has a unique $L^p$-solution $(Y, Z, K)$ for each given $p \in (1, \bar{p})$.

Proof. The uniqueness is due to (3.3) and (3.4) in Proposition 3.2. For each positive integer $n$, by partition of unity theorem, we can find $h_i^n \in C_0^\infty(\mathbb{R})$, $h_i^n \in C_0^\infty(\mathbb{R})$, $i \leq k_n$, $j \leq \bar{k}_n$, such that

$$0 \leq h_i^n \leq 1 \quad \text{and} \quad \lambda(\text{supp}(h_i^n)) \leq \frac{1}{n} \quad \text{for} \quad i \leq k_n, \quad I_{[-n,n]}(y) \leq \sum_{i=1}^{k_n} h_i^n(y) \leq 1,$$

$$0 \leq \tilde{h}_j^n \leq 1 \quad \text{and} \quad \lambda(\text{supp}(\tilde{h}_j^n)) \leq \frac{1}{n} \quad \text{for} \quad j \leq \bar{k}_n, \quad I_{[-n,n] \times [-n,n]}(y, z) \leq \sum_{j=1}^{\bar{k}_n} \tilde{h}_j^n(y, z) \leq 1.$$

For each $N > 0$, set $\tilde{f}(t, y) = f(t, y) - f(t, 0, 0)$, $\tilde{g}(t, y, z) = g(t, y, z) - g(t, 0, 0)$,

$$\tilde{f}^N(t, y) = (\tilde{f}(t, y) \wedge N) \vee (-N), \quad \tilde{g}^N(t, y, z) = (\tilde{g}(t, y, z) \wedge N) \vee (-N),$$

$$f^N(t, y) = f(t, 0) + \tilde{f}^N(t, y), \quad g^N(t, y, z) = g(t, 0, 0) + \tilde{g}^N(t, y, z),$$

where $h_i^n(y_i^n) > 0$, $\tilde{h}_j^n(y_j^n, z_j^n) > 0$ for $i \leq k_n$, $j \leq \bar{k}_n$. By Proposition 3.2 and Lemma 3.10 we can easily deduce that G-BSDE

$$Y_t^{N,n} = \xi + \int_t^T f^N(s, Y_s^{N,n})ds + \int_t^T g^N(s, Y_s^{N,n}, Z_s^{N,n})dB_s - \int_t^T Z_s^{N,n}dB_s - (K_T^{N,n} - K_t^{N,n}) \quad (3.60)$$

has a unique $L^p$-solution $(Y^{N,n}, Z^{N,n}, K^{N,n})$ for each given $p \in (1, \bar{p})$. Noting that

$$f_n^N(s, Y_s^{N,n}) = f^N(s, Y_s^{N,n}) + \tilde{f}_n^N(s) \quad \text{and} \quad g_n^N(s, Y_s^{N,n}, Z_s^{N,n}) = g^N(s, Y_s^{N,n}, Z_s^{N,n}) + \tilde{g}_n^N(s),$$

where $|\tilde{f}_n^N(s)| = |f_n^N(s, Y_s^{N,n}) - f^N(s, Y_s^{N,n})| \leq \left( \frac{1}{n} + \frac{\xi}{N} \right)(|Y_s^{N,n}| \vee (2N))$, $|\tilde{g}_n^N(s)| \leq \left( \frac{1}{N} + \frac{\xi}{(2N)} \right)(|Y_s^{N,n}| + |Z_s^{N,n}|) \vee (2N)$. By Proposition 3.2 we can easily deduce that, for each given $p \in (1, \bar{p})$,

$$\lim_{n,m \to \infty} \hat{E} \left[ \sup_{t \in [0, T]} |Y_t^{N,n} - Y_t^{N,m}|^p + \left( \int_0^T |Z_t^{N,n} - Z_t^{N,m}|^2 dB_t \right)^{p/2} + |K_T^{N,n} - K_T^{N,m}|^p \right] = 0,$$

which implies that G-BSDE

$$Y_t^N = \xi + \int_t^T f^N(s, Y_s^N)ds + \int_t^T g^N(s, Y_s^N, Z_s^N)dB_s - \int_t^T Z_s^NdB_s - (K_T^N - K_t^N) \quad (3.61)$$

has a unique $L^p$-solution $(Y^N, Z^N, K^N)$ for each given $p \in (1, \bar{p})$. By (3.7), (3.8) in Proposition 3.2 and Theorem 2.3 we obtain that, for each $p \in (1, \bar{p})$,

$$\sup_{N > 0} \hat{E} \left[ \sup_{t \in [0, T]} |Y_t^{N}|^p + \left( \int_0^T |Z_t^{N}|^2 dB_t \right)^{p/2} + |K_T^{N}|^p \right] \leq C, \quad (3.62)$$

where the constant $C > 0$ depends on $p$, $\bar{p}$, $\bar{\sigma}$, $L$ and $T$. For each fixed $p \in (1, \bar{p})$, we have

$$|f^{N_1}(s, Y_s^{N_1}) - f^{N_2}(s, Y_s^{N_2})| \leq (N_1 \wedge N_2)^{-\delta} |f(s, Y_s^{N_1})|^{1+\delta} \leq L^{1+\delta}(N_1 \wedge N_2)^{-\delta} |Y_s^{N_1}|^{1+\delta},$$

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\[|g^{N_1}(s, Y_s^{N_1}, Z_s^{N_1}) - g^{N_2}(s, Y_s^{N_2}, Z_s^{N_2})| \leq L^{1+\delta}(N_1 \wedge N_2)^{-\delta}(|Y_s^{N_1}| + |Z_s^{N_2}|)^{1+\delta},\]

where \(\delta = [\frac{1}{2}(p - 1)] \wedge 1\). Thus, by (3.10), (3.12) in Proposition 3.2, 3.12 and Theorem 2.3, we get

\[
\lim_{N_1, N_2 \to \infty} \mathbb{P} \left[ \sup_{t \in [0,T]} |Y_t^{N_1} - Y_t^{N_2}|^p + \left( \int_0^T |Z_t^{N_1} - Z_t^{N_2}|^2 d(B)_t \right)^{p/2} + |K_t^{N_1} - K_t^{N_2}|^p \right] = 0,
\]

which implies the desired result by letting \(N \to \infty\) in (3.61). \(\square\)

The following example shows that \(f\) can not contain \(z\) in G-BSDE (3.5).

**Example 3.12** Let \(B\) be a 1-dimensional G-Brownian motion with \(G(a) := \frac{1}{2}a^2a^1\) for \(a \in \mathbb{R}\). For each \(n \geq 1\), we know that \(((n^{-1} + \langle B \rangle_s)^{-1/5})_{s \in [0,T]} \in H^{2,p}_{\mathbb{G}}(0, T; \langle B \rangle)\) for each \(p > 1\). Since

\[
\left| (n^{-1} + \langle B \rangle_s)^{-1/5} - (\langle B \rangle_s)^{-1/5} \right| \leq (\langle B \rangle_s)^{-2/5} \left| (n^{-1} + \langle B \rangle_s)^{1/5} - (\langle B \rangle_s)^{1/5} \right| \leq n^{-1/5}(\langle B \rangle_s)^{-2/5},
\]

we have

\[
\int_0^T \left| (n^{-1} + \langle B \rangle_s)^{-1/5} - (\langle B \rangle_s)^{-1/5} \right|^2 d(B)_s \leq n^{-2/5} \int_0^T (\langle B \rangle_s)^{-1/5} d(B)_s = 5n^{-2/5}(\langle B \rangle_T)^{1/5}.
\]

Thus \(((\langle B \rangle_s)^{-1/5})_{s \in [0,T]} \in H^{2,p}_{\mathbb{G}}(0, T; \langle B \rangle)\) for each \(p > 1\), which implies \(\int_0^T (\langle B \rangle_s)^{-1/5} d(B)_s \in L^p(\Omega_T)\) for each \(p > 1\). Consider the following linear G-BSDE:

\[
Y_t = \int_0^T (\langle B \rangle_s)^{-1/5} d(B)_s + \int_t^T Z_s dJ_t - \int_t^T Z_s dB_s - (K_T - K_t),
\]

(3.63)

we assert that, for each given \(p > 1\), the above G-BSDE has no \(L^p\)-solution \((Y, Z, K)\). Otherwise, there exists an \(L^p\)-solution \((Y, Z, K)\) for some \(p > 1\).

For each \(\varepsilon > 0\), we introduce the following \(\mathcal{G}\)-expectation \(\mathbb{E}^\varepsilon\). Set \(\tilde{\Omega}_T = C_0([0,T]; \mathbb{R}^2)\) and the canonical process is denoted by \((B, \tilde{B})\). For each \(A \in \mathbb{S}_2\), define

\[
\mathcal{G}^\varepsilon(A) = \frac{1}{2} \sup_{\sigma^2 \leq \varepsilon^2} \text{tr} \left[ A \left( \begin{array}{cc} \varepsilon^2 & 1 \\ 1 & \varepsilon^{-2} \end{array} \right) \right].
\]

By Proposition 3.1.5 in Peng [23], we know that \(\varepsilon \tilde{B}\) is a classical 1-dimensional standard Brownian motion under \(\mathbb{E}^\varepsilon\) and \(\mathbb{E}^\varepsilon_{L^p(\Omega_T)} \leq \mathbb{E}\). Thus G-BSDE (3.63) still holds under \(\mathbb{E}^\varepsilon\). Similar to (3.3), we know that \(\langle B, \tilde{B}\rangle_t = t\) and \(\langle \tilde{B}\rangle_t = \varepsilon^{-2}t\) under \(\mathbb{E}^\varepsilon\). Consider the following G-SDE under \(\mathbb{E}^\varepsilon\):

\[
dX_t = X_t dB_t, \quad X_0 = 1.
\]

The solution is \(X_t = \exp(\tilde{B}_t - 2^{-1}\varepsilon^{-2}t) > 0\). Applying Itô’s formula to \(X_t Y_t\) on \([0,T]\) under \(\mathbb{E}^\varepsilon\), we get

\[
X_T Y_T = Y_0 + \int_0^T X_t Z_t dB_t + \int_0^T X_t Y_t dB_t + \int_0^T X_t dK_t.
\]

Since \(\int_0^T X_t dK_t \leq 0\), we obtain

\[
Y_0 \geq \mathbb{E}^\varepsilon[X_T Y_T] = \mathbb{E}^\varepsilon \left[ X_T \int_0^T (\langle B \rangle_s)^{-1/5} dB_s \right] \quad \text{for each } \varepsilon > 0.
\]

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Applying Itô’s formula to \( X_t \int_0^T ((B)_s)^{-1/5} dB_s \) on \([0, T]\) under \( \tilde{\mathbb{E}}^\varepsilon \), we deduce

\[
Y_0 \geq \tilde{\mathbb{E}}^\varepsilon \left[ X_T \int_0^T ((B)_s)^{-1/5} dB_s \right] = \mathbb{E}^\varepsilon \left[ \int_0^T X_s((B)_s)^{-1/5} ds \right] \text{ for each } \varepsilon > 0.
\]

Let \( \mathbb{E}^\varepsilon \) be the linear \( \bar{G}^\varepsilon \)-expectation with

\[
\bar{G}^\varepsilon (A) = \frac{1}{2} \text{tr} \left[ A \begin{pmatrix} \varepsilon^2 & 1 \\ 1 & \varepsilon^{-2} \end{pmatrix} \right] \text{ for } A \in \mathbb{S}_2.
\]

Since \( \bar{G}^\varepsilon \leq \tilde{G}^\varepsilon \), we know that \( \mathbb{E}^\varepsilon \leq \tilde{\mathbb{E}}^\varepsilon \). By Proposition 3.15 in Peng [22], we know that \( \varepsilon^{-1} B \) and \( \varepsilon \tilde{B} \) are two classical 1-dimensional standard Brownian motion under \( \mathbb{E}^\varepsilon \). Then we get

\[
Y_0 \geq \mathbb{E}^\varepsilon \left[ X_T \int_0^T ((B)_s)^{-1/5} dB_s \right] = \frac{5}{4} T^{4/5} \varepsilon^{-2/5} \text{ for each } \varepsilon > 0,
\]

which contradicts to \( Y_0 \in \mathbb{R} \). Thus, for each given \( p > 1 \), G-BSDE (3.4) has no \( L^p \)-solution \((Y, Z, K)\).

Finally, we give the following existence and uniqueness result of G-BSDE

\[3.4\]

**Theorem 3.13** Suppose that \( \xi, f, g, h, i, j, l \leq d', d' < l \leq d \), satisfy (H1) and (H2). Then G-BSDE (3.4) has a unique \( L^p \)-solution \((Y, Z, K)\) for each given \( p \in (1, \hat{p}) \).

**Proof.** The proof of this theorem is similar to Theorem 3.11 we omit it. \( \Box \)

### 4 Application to the regularity of fully nonlinear PDEs

For simplicity of representation, we only consider 1-dimensional G-Brownian motion with \( G(u) = \frac{1}{2} u^2 a^+ \), the methods still hold for the \( d \)-dimensional G-Brownian motion with \( G(\cdot) \) given in (3.1). For each fixed \( t \in [0, T] \) and \( \xi \in \cap_{p \geq 2} L_p^G(\Omega_t) \), consider the following G-FBSDE:

\[
dX^{t, \xi}_s = b(s, X^{t, \xi}_s) ds + h(s, X^{t, \xi}_s) d(B)_s + \sigma(s, X^{t, \xi}_s) dB_s, \quad X^{t, \xi}_t = \xi, \quad s \in [t, T], \tag{4.1}
\]

\[
Y^{t, \xi}_s = \varphi(X^{t, \xi}_T) + \int_t^T f(r, X^{t, \xi}_r, Y^{t, \xi}_r, Z^{t, \xi}_r) dr + \int_t^T g(r, X^{t, \xi}_r, Y^{t, \xi}_r, Z^{t, \xi}_r) d(B)_r, \quad s \in [t, T], \tag{4.2}
\]

where \( b, h, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}, \varphi : \mathbb{R} \to \mathbb{R}, f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}, g : [0, T] \times \mathbb{R}^3 \to \mathbb{R} \) satisfy the following conditions:

**A1** \( b, h, \sigma, f, g \) are continuous in \((s, x, y, z)\).
There exist a constant $L_1 > 0$ and a positive integer $m$ such that for any $s \in [0, T]$, $x, x', y, y', z, z' \in \mathbb{R}$,
\[
|b(s, x) - b(s, x')| + |h(s, x) - h(s, x')| + |\sigma(s, x) - \sigma(s, x')| \leq L_1|x - x'|,
\]
\[
|\varphi(x) - \varphi(x')| \leq L_1(1 + |x|^m + |x'|^m)|x - x'|,
\]
\[
|f(s, x, y) - f(s, x', y')| + |g(s, x, y, z) - g(s, x', y', z')| \\
\leq L_1[(1 + |x|^m + |x'|^m)|x - x'| + |y - y'| + |z - z'|].
\]

Under the assumptions (A1) and (A2), for each $p \geq 2$, SDE (4.1) has a unique solution $(X_{s}^{t, \xi})_{s \in [t, T]} \in S_{p}^{p}(t, T)$ and G-BSDE (4.2) has a unique $L^p$-solution $(Y_{s}^{t, \xi}, Z_{s}^{t, \xi}, K_{s}^{t, \xi})_{s \in [t, T]}$ with $K_{s}^{t, \xi} = 0$. The following standard estimates of SDE can be found in Chapter 5 in Peng (22).

**Proposition 4.1** Suppose that (A1) and (A2) hold. Let $\xi, \xi' \in \cap_{p \geq 2} L_{p}^{p}(\Omega_{n})$ with $t < T$. Then, for each $p \geq 2$ and $\delta \in [0, T - t]$, we have
\[
\mathbb{E} \left[ |X_{t+\delta}^{t, \xi} - X_{t+\delta}^{t, \xi'}|^p \right] \leq C|\xi - \xi'|^p \quad \text{and} \quad \mathbb{E} \left[ |X_{t+\delta}^{t, \xi}|^p \right] \leq C(1 + |\xi|^p),
\]
where the constant $C > 0$ depends on $L_1$, $\bar{\sigma}$, $p$ and $T$.

Set $\xi = x \in \mathbb{R}$, define
\[
u(t, x) = Y_{t}^{t, x} \quad \text{for} \quad (t, x) \in [0, T] \times \mathbb{R}. \tag{4.3}
\]

Since $(B_{t+r} - B_{t})_{r \geq 0}$ is still a $G$-Brownian motion, we have $Y_{t}^{t, x} \in \mathbb{R}$.

**Proposition 4.2** Suppose that (A1) and (A2) hold. Then

(i) For each $(t, x) \in [0, T] \times \mathbb{R}$, we have $Y_{s}^{t, x} = u(s, X_{s}^{t, x})$ for $s \in [t, T]$.

(ii) $u(\cdot, \cdot)$ is the unique viscosity solution of the following fully nonlinear PDE:
\[
\begin{aligned}
&\partial_t u + G(\sigma^2(t, x)\partial_{xx}^2 u + 2h(t, x)\partial_x u + 2g(t, x, u, \sigma(t, x)\partial_x u)) \\
&+ b(t, x)\partial_x u + f(t, x, u) = 0,
\end{aligned}
\]
\[
u(T, x) = \varphi(x).
\]

**Proof.** The proof is the same as Theorems 4.4 and 4.5 in [8], we omit it. □

In the following, we discuss the regularity properties of $u(\cdot, \cdot)$. First, we study $\partial_x u(t, x)$. For each $(t, x) \in [0, T] \times \mathbb{R}$ and $\Delta \in [-1, 1]$, by Proposition 4.1 we have, for each $p \geq 2$,
\[
\sup_{s \in [t, T]} \mathbb{E} \left[ |X_{s}^{t, x+\Delta} - X_{s}^{t, x}|^p \right] \leq C|\Delta|^p \quad \text{and} \quad \sup_{s \in [t, T]} \mathbb{E} \left[ |X_{s}^{t, x}|^p \right] \leq C(1 + |x|^p), \tag{4.4}
\]
where $C > 0$ depends on $L_1$, $\bar{\sigma}$, $p$ and $T$. It follows from Proposition 3.2, Theorem 2.3 and (4.4) that, for each $p \geq 2$,
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y_{s}^{t, x+\Delta} - Y_{s}^{t, x}|^p \right] \leq C(1 + |x|^p)|\Delta|^p, \tag{4.5}
\]
where $C > 0$ depends on $L_1$, $\bar{\sigma}$, $p$ and $T$.

Let $\mathcal{P}$ be a weakly compact and convex set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$ satisfying

$$
\hat{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \text{ for } \xi \in L^1_{\mathcal{N}}(\Omega_T).
$$

For each $(t,x) \in [0,T) \times \mathbb{R}^n$, set

$$
\mathcal{P}_{t,x} = \{ P \in \mathcal{P} : E_P[K_T^{t,x}] = 0 \}.
$$

Similar to the proof of Proposition 3.4, we obtain that, for each $p \geq 2$, \[ E_P \left[ \left( \int_t^T \left| \int_t^T \mathbb{E} \left[ Z^{t,x+\Delta}_s - Z^{t,x}_s \right]^2 d(B_s) \right|^{p/2} \right)^2 \right] \leq C(1 + |x|^p) \Delta \] for $P \in \mathcal{P}_{t,x},$ \[ E_P \left[ \left( \int_t^T \left| \int_t^T \mathbb{E} \left[ Z^{t,x+\Delta}_s - Z^{t,x}_s \right]^2 d(B_s) \right| \right)^{p/2} \right] \leq C(1 + |x|^p) \Delta \] for $P \in \mathcal{P}_{t,x+\Delta}$. (4.6) \[ E_P \left[ \left( \int_t^T \left| \int_t^T \mathbb{E} \left[ Z^{t,x+\Delta}_s - Z^{t,x}_s \right]^2 d(B_s) \right| \right)^{p/2} \right] \leq C(1 + |x|^p) \Delta \] for $P \in \mathcal{P}_{t,x+\Delta}$. (4.7) In order to obtain $\partial_x u(t,x)$, we need the following assumption.

(A3) \[ b_x(s, x), h_x(s, x), \sigma_x(s, x), \varphi_x(s, x), f_x(s, x), g_x(s, x), g_x^2(s, x), g_x^2(s, x) \text{ are continuous in } (s, x, y, z). \]

Remark 4.3 Under the assumptions (A2) and (A3), we can easily deduce that, for any $s \in [0,T], x, y, z \in \mathbb{R}$,

\[ |b_x(s, x)| + |h_x(s, x)| + |\sigma_x(s, x)| \leq L_1, \quad |\varphi_x(s)| \leq L_1(1 + 2|x|^m), \quad |g_x(s, x, y, z)| \leq L_1, \]

\[ |f_x(s, x)| + |g_x(s, x, y, z)| \leq L_1(1 + 2|x|^m), \quad |g_x(s, x, y, z)| \leq L_1. \]

Lemma 4.4 Suppose that (A1)–(A3) hold. Then, for each $(t,x) \in [0,T) \times \mathbb{R}$ and $p \geq 2$, we have

$$
\lim_{\Delta \to 0} \sup_{s \in [t,T]} \hat{E}[\left| X^{t,x+\Delta}_s - X^{t,x}_s \right|^p] = 0,
$$

where $(\hat{X}^{t,x}_s)_{s \in [t,T]}$ is the solution of the following G-SDE:

$$
d\hat{X}^{t,x}_s = b_x(s, X^{t,x}_s) \hat{X}^{t,x}_s ds + h_x(s, X^{t,x}_s) \hat{X}^{t,x}_s d\langle B \rangle_s + \sigma_x(s, X^{t,x}_s) \hat{X}^{t,x}_s dB_s, \quad \hat{X}^{t,x}_t = 1. \tag{4.9}
$$

Proof. Set $\hat{X}^{t,x}_s = X^{t,x+\Delta}_s - X^{t,x}_s$, $\hat{X}^{t,x}_s = \hat{X}^{t,x}_s - \hat{X}^{t,x}_t$ for $s \in [t,T]$, we have

$$
d\hat{X}^{t,x}_s = (b_x(s) \hat{X}^{t,x}_s + \hat{b}(s)) ds + (h_x(s) \hat{X}^{t,x}_s + \hat{h}(s)) d\langle B \rangle_s + (\sigma_x(s) \hat{X}^{t,x}_s + \hat{\sigma}(s)) dB_s, \quad \hat{X}^{t,x}_t = 1,
$$

where $b_x(s) = b_x(s, X^{t,x}_s)$,

$$
\hat{b}(s) = b(s, X^{t,x+\Delta}_s) - b(s, X^{t,x}_s) - b_x(s, X^{t,x}_s) \hat{X}^{t,x}_s = X^{t,x}_s \int_0^1 \left[ b_x(s, X^{t,x}_s + \theta \hat{X}^{t,x}_s) - b_x(s, X^{t,x}_s) \right] d\theta.
$$

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similar for \(h_s'(s), \tilde{h}(s), \sigma_s'(s)\) and \(\tilde{\sigma}(s)\). By standard estimates of SDE, we get

\[
\sup_{s \in [t,T]} \mathbb{E}\left[|\tilde{X}_s^\Delta|^p\right] \leq C \mathbb{E}\left[\left(\int_t^T |\tilde{b}(s)| ds\right)^p + \left(\int_t^T |\tilde{h}(s)| d\langle B \rangle_s\right)^p + \left(\int_t^T |\tilde{\sigma}(s)|^2 d\langle B \rangle_s\right)^{p/2}\right] \leq C \int_t^T \mathbb{E}[|\tilde{b}(s)|^p + |\tilde{h}(s)|^p + |\tilde{\sigma}(s)|^p] ds,
\]

where \(C > 0\) depends on \(L_1, \tilde{\sigma}, p\) and \(T\). By (4.3) and Hölder’s inequality, we obtain

\[
\mathbb{E}[|\tilde{b}(s)|^p] \leq C |\Delta|^p \left(\mathbb{E}\left[\left(\int_0^s |b_s'(s, X_s^{t,x} + \theta \tilde{X}_s^\Delta}) - b_s'(s, X_s^{t,x})| d\theta\right)^{2p}\right]\right)^{1/2},
\]

where \(C > 0\) depends on \(L_1, \tilde{\sigma}, p\) and \(T\). For each \(N > 0\) and \(\varepsilon > 0\), define

\[
\omega_N(\varepsilon) = \sup\{|b_r'(r, x_1) - b_r'(r, x_2)| : r \in [0, T], |x_1| \leq N, |x_1 - x_2| \leq \varepsilon\}.
\]

Under assumption (A3), we know that \(\omega_N(\varepsilon) \to 0\) as \(\varepsilon \to 0\). Noting that

\[
|b_s'(s, X_s^{t,x} + \theta \tilde{X}_s^\Delta) - b_s'(s, X_s^{t,x})| \\
\leq |b_s'(s, X_s^{t,x} + \theta \tilde{X}_s^\Delta) - b_s'(s, X_s^{t,x})| I[|\tilde{X}_s^\Delta| \leq \varepsilon] + 2L_1 I[|\tilde{X}_s^\Delta| > \varepsilon] \\
\leq |b_s'(s, X_s^{t,x} + \theta \tilde{X}_s^\Delta) - b_s'(s, X_s^{t,x})| I[|\tilde{X}_s^\Delta| \leq N] + 2L_1 I[|\tilde{X}_s^\Delta| > N] + 2L_1 I[|\tilde{X}_s^\Delta| > \varepsilon]
\]

we obtain by (4.4), (4.10) and (4.12) that

\[
\mathbb{E}[|\tilde{b}(s)|^p] \leq C |\Delta|^p \left(\omega_N(\varepsilon)^p + \frac{1 + |x|^p}{N^p} + \frac{|\Delta|^p}{\varepsilon^p}\right),
\]

where \(C > 0\) depends on \(L_1, \tilde{\sigma}, p\) and \(T\). Thus

\[
\limsup_{\Delta \to 0} \frac{1}{|\Delta|^p} \int_t^T \mathbb{E}[|\tilde{b}(s)|^p] ds \leq C \left(\omega_N(\varepsilon)^p + \frac{1 + |x|^p}{N^p}\right),
\]

which implies \(\limsup_{\Delta \to 0} \frac{1}{|\Delta|^p} \int_t^T \mathbb{E}[|\tilde{b}(s)|^p] ds = 0\) by letting \(\varepsilon \to 0\) first and then \(N \to \infty\). Similarly, we can obtain

\[
\limsup_{\Delta \to 0} \frac{1}{|\Delta|^p} \int_t^T \mathbb{E}[|\tilde{b}(s)|^p + |\tilde{\sigma}(s)|^p] ds = 0,
\]

which implies the desired result.

**Theorem 4.5** Suppose that (A1)-(A3) hold. Then, for each \((t, x) \in [0, T) \times \mathbb{R}\), we have

\[
\partial_{x^+} u(t, x) = \sup_{p \in P_{t, x}} E_p \left[\phi'(X_t^{t,x}) \tilde{X}_t^{t,x} \Gamma_t^{t,x} + \int_t^T f_s'(s) \tilde{X}_s^{t,x} \Gamma_s^{t,x} d\langle B \rangle_s + \int_t^T g_s'(s) \tilde{X}_s^{t,x} \Gamma_s^{t,x} d\langle B \rangle_s\right],
\]

\[
\partial_{x^-} u(t, x) = \inf_{p \in P_{t, x}} E_p \left[\phi'(X_t^{t,x}) \tilde{X}_t^{t,x} \Gamma_t^{t,x} + \int_t^T f_s'(s) \tilde{X}_s^{t,x} \Gamma_s^{t,x} d\langle B \rangle_s + \int_t^T g_s'(s) \tilde{X}_s^{t,x} \Gamma_s^{t,x} d\langle B \rangle_s\right],
\]

where \((\tilde{X}_s^{t,x})_{s \in [t,T]}\) satisfies (4.9). \((\Gamma_s^{t,x})_{s \in [t,T]}\) satisfies the following G-SDE:

\[
d\Gamma_s^{t,x} = f_s'(s) \Gamma_s^{t,x} ds + g_s'(s) \Gamma_s^{t,x} d\langle B \rangle_s + \bar{g}_s'(s) \Gamma_s^{t,x} dB_s, \quad \Gamma_t^{t,x} = 1,
\]

similar for \(f_s'(s), g_s'(s), f'_s(s)\) and \(f''_s(s)\).
Proof. Set \( \hat{X}_s^{\Delta} = X_s^{t+s,\Delta} - X_s^{t,\Delta} \), \( \hat{Y}_s^{\Delta} = Y_s^{t+s,\Delta} - Y_s^{t,\Delta} \), \( \hat{Z}_s^{\Delta} = Z_s^{t+s,\Delta} - Z_s^{t,\Delta} \) for \( \Delta > 0 \) and \( s \in [t,T] \). For each \( P \in \mathcal{P}_{t,x} \), we have

\[
\hat{Y}_s^{\Delta} = \varphi'(X_T^s,\Delta)\hat{X}_T^s - \varphi(T) + \int_t^T f'_x(s)\hat{X}_s^{t,s}ds + \int_t^T g'_s(s)\hat{X}_s^{t,s}d\langle B \rangle_s \]

where \( \hat{Y}_s^{\Delta} \) converges weakly to \( \hat{Y}_s^{\Delta} \) and \( \hat{Z}_s^{\Delta} \), similar for \( \hat{Y}_s^{\Delta} \) and \( \hat{Z}_s^{\Delta} \). Applying Itô’s formula to \( \hat{Y}_s^{\Delta} \) on \([t,T]\) under \( P \), we obtain

\[
\Delta^{-1}\hat{Y}_s^{\Delta} = E_P \left[ \varphi'(X_T^s,\Delta)\hat{X}_T^s + \int_t^T f'_x(s)\hat{X}_s^{t,s}ds + \int_t^T g'_s(s)\hat{X}_s^{t,s}d\langle B \rangle_s \right] + \Delta^{-1}E_P \left[ \varphi(T)\Gamma_T^x + \int_t^T f(s)\Gamma_s^xds + \int_t^T g(s)\Gamma_s^x d\langle B \rangle_s - \int_t^T \Gamma_s^xdK_s^x \right].
\]

Noting that \( \varphi(T) = \varphi'(X_T^s,\Delta)\hat{X}_T^s - \varphi(T) + \int_t^T \varphi'(X_T^s,\Delta)\hat{X}_s^{t,s}ds + \int_t^T \varphi'(X_T^s,\Delta)\hat{X}_s^{t,s}d\langle B \rangle_s \), similar for \( \varphi(T) \) and \( g(s) \), by \( (4.4), (4.5), (4.6), (4.8) \) and using the method in \( (4.12) \), we get

\[
\lim_{\Delta \to 0} \Delta^{-1}E_P \left[ \varphi(T)\Gamma_T^x + \int_t^T f(s)\Gamma_s^xds + \int_t^T g(s)\Gamma_s^x d\langle B \rangle_s - \int_t^T \Gamma_s^xdK_s^x \right] = 0.
\]

Since \( \Delta > 0 \), \( \Gamma_{t,x}^{x,\Delta} \) and \( dK_{t,x}^{x,\Delta} \), we deduce by \( (1.10) \) and \( (1.17) \) that

\[
\lim_{\Delta \to 0} \Delta^{-1}E_P \left[ \varphi(T)\Gamma_T^x + \int_t^T f(s)\Gamma_s^xds + \int_t^T g(s)\Gamma_s^x d\langle B \rangle_s - \int_t^T \Gamma_s^xdK_s^x \right] = 0.
\]

For each \( P^\Delta \in \mathcal{P}_{t,x+\Delta} \) for \( \Delta > 0 \), similar to \( (1.16) \), we have

\[
\Delta^{-1}\hat{Y}_s^{\Delta} = E_{P^\Delta} \left[ \varphi'(X_T^s,\Delta)\hat{X}_T^s + \int_t^T f'_x(s)\hat{X}_s^{t,s}ds + \int_t^T g'_s(s)\hat{X}_s^{t,s}d\langle B \rangle_s \right] + \Delta^{-1}E_{P^\Delta} \left[ \varphi(T)\Gamma_T^x + \int_t^T f(s)\Gamma_s^xds + \int_t^T g(s)\Gamma_s^x d\langle B \rangle_s - \int_t^T \Gamma_s^xdK_s^x \right].
\]

Similar to \( (1.17) \), we get

\[
\lim_{\Delta \to 0} \Delta^{-1}E_{P^\Delta} \left[ \varphi(T)\Gamma_T^x + \int_t^T f(s)\Gamma_s^xds + \int_t^T g(s)\Gamma_s^x d\langle B \rangle_s - \int_t^T \Gamma_s^xdK_s^x \right] = 0.
\]

Since \( \mathcal{P} \) is weakly compact, for any sequence \( \Delta_j \to 0 \), we can find a subsequence \( \Delta_j \to 0 \) such that \( P_{\Delta_j} \) converges weakly to \( P^\ast \in \mathcal{P} \). By Proposition \( (3.2) \) and \( (4.13) \), we have \( \hat{E} \left[ |K_{T,x}^{x+\Delta} - K_{T,x}^{x}| \right] \to 0 \) as \( \Delta \to 0 \). Due to

\[
|E_{P^\ast}[K_{T,x}^{x}]| = |E_{P^\ast}[K_{T,x}^{x}] - E_{P_{\Delta_j}}[K_{T,x}^{x+\Delta_j}]| \leq |E_{P^\ast}[K_{T,x}^{x}] - E_{P_{\Delta_j}}[K_{T,x}^{x}]| + \hat{E} \left[ |K_{T,x}^{x+\Delta_j} - K_{T,x}^{x}| \right]
\]

and \( E_{P_{\Delta_j}}[K_{T,x}^{x}] \to E_{P^\ast}[K_{T,x}^{x}] \) as \( \Delta_j \to 0 \), we get \( E_{P^\ast}[K_{T,x}^{x}] = 0 \), which implies \( P^\ast \in \mathcal{P}_{t,x} \). Noting that \( \int_t^T \Gamma_s^x dK_s^x \leq 0 \) and

\[
\varphi'(X_T^s,\Delta)\hat{X}_T^s + \int_t^T f'_x(s)\hat{X}_s^{t,s}ds + \int_t^T g'_s(s)\hat{X}_s^{t,s}d\langle B \rangle_s \in L_0(T),
\]

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Lemma 4.6 Suppose that (A1)-(A4) hold. Then, for each $\bar{s} \in [t,T]$, we deduce by (4.19) and (4.20) that
\[
\limsup_{\Delta \downarrow 0} \frac{\hat{Y}^\Delta_{\bar{s}}}{\Delta} \leq \sup_{p \in \mathcal{P}_{t,x}} E_p \left[ \varphi^\prime(X^t,x_T)X^t,x_T \Gamma^{t,x}_T + \int_t^T f^\prime_x(s)X^t,x_s \Gamma^{t,x}_s ds + \int_t^T g^\prime_x(s)X^t,x_s \Gamma^{t,x}_s d(B)^s \right]. \tag{4.21}
\]
Thus we obtain (4.13) by (4.18) and (4.21). Similarly, we can get (4.14).

Proof. Theorem 4.7

Now we study $\partial_t u(t,x)$. For each $(t,x) \in (0,T) \times \mathbb{R}$ and $|\Delta| < t \wedge (T-t)$, noting that
\[
\sqrt{\frac{T-t}{T-t-\Delta}} \left( B_{t+x} - B_{t+x-\Delta} - B_{t+x} \right)_{s \in [t,T]}
\]
and $(B_{s-B_t})_{s \in [t,T]}$ have the same distribution, we obtain $u(t+\Delta,x) = \hat{Y}^t,x,\Delta$, where $(\hat{X}^t,x,\Delta, \hat{Y}^t,x,\Delta, \hat{Z}^t,x,\Delta, \hat{K}^t,x,\Delta)$ satisfies the following G-FBSDE:
\[
\hat{X}^t,x,\Delta = x + \int_t^{t+\Delta} b_x(r,\hat{X}^t,x_\Delta) dr + \int_t^{t+\Delta} \sigma_x(r,\hat{X}^t,x_\Delta) d\hat{B}_r,
\]
\[
\hat{Y}^t,x,\Delta = \varphi(\hat{X}^t,x_\Delta) + \int_t^{t+\Delta} g_x(r,\hat{X}^t,x_\Delta,\hat{Y}^t,x_\Delta,\hat{Z}^t,x_\Delta,\hat{K}^t,x_\Delta,\sqrt{T-t}) d\hat{B}_r - \int_t^{t+\Delta} \int_T^t \left( \int_s^t \hat{Z}^t,x_\Delta d\hat{B}_r \right) ds.
\]
In order to obtain $\partial_t u(t,x)$, we need the following assumption.

(A4) $b^\prime_x, b^\prime_x, \sigma^\prime_x, f^\prime_x, g^\prime_x$ are continuous in $(s,x,y,z)$, and there exist a constant $L_2 > 0$ and a positive integer $m_1$ such that for any $s \in [0,T)$, $x, y, z \in \mathbb{R}$,
\[
|b^\prime_x(s,x)| + |b^\prime_x(s,x)| + |\sigma^\prime_x(s,x)| + |f^\prime_x(s,x,y)| + |g^\prime_x(s,x,y,z)| \leq L_2(1 + |x|^{m_1} + |y|^{m_1} + |z|^{m_1}).
\]

Lemma 4.6 Suppose that (A1)-(A4) hold. Then, for each $(t,x) \in (0,T) \times \mathbb{R}$ and $p \geq 2$, we have
\[
\lim_{\Delta \to 0} \sup_{s \in [t,T]} \hat{E} \left[ \left| \frac{\hat{X}^t,x,\Delta - \hat{X}^t,x}{\Delta} \right|^p \right] = 0,
\]
where $(\hat{X}^t,x)_{s \in [t,T]}$ is the solution of the following G-SDE:
\[
\hat{X}^t,x = \int_t^s \left[ b_x(r,\hat{X}^t,x_s) \hat{X}^t,x_s + \frac{T-t}{T-t-\Delta} \int_t^s b_x(r,\hat{X}^t,x_s) \hat{X}^t,x_s \right] dr + \int_t^s \left[ \sigma_x(r,\hat{X}^t,x_s) \hat{X}^t,x_s + \frac{T-t}{T-t-\Delta} \sigma_x(r,\hat{X}^t,x_s) \hat{X}^t,x_s \right] d\hat{B}_r,
\]
\[
\hat{Y}^t,x = \varphi(\hat{X}^t,x_T) \hat{X}^t,x_T + \int_t^T \left( f^\prime_x(s) \hat{X}^t,x_s + \frac{T-t}{T-t-\Delta} \int_t^s f^\prime_x(r) \hat{X}^t,x_s \right) ds + \int_t^T g^\prime_x(s) \hat{X}^t,x_s \Gamma^{t,x}_s d\hat{B}_r \tag{4.22}
\]

Proof. The proof is similar to Lemma 4.3 we omit it. □

Theorem 4.7 Suppose that (A1)-(A4) hold. Then, for each $(t,x) \in (0,T) \times \mathbb{R}$, we have
\[
\partial_t u(t,x) = \sup_{p \in \mathcal{P}_{t,x}} E_p \left[ \varphi^\prime(X^t,x_T)X^t,x_T \Gamma^{t,x}_T + \int_t^T \left( f^\prime_x(s)X^t,x_s + \frac{T-t}{T-t-\Delta} \int_t^s f^\prime_x(r) \hat{X}^t,x_s \right) \Gamma^{t,x}_s ds + \int_t^T \left( g^\prime_x(s)Z^t,x_s + \frac{T-t}{T-t-\Delta} g^\prime_x(s) \hat{X}^t,x_s \right) \Gamma^{t,x}_s d\hat{B}_r \right],
\]

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\[ \partial_{-} u(t,x) = \inf_{P \in \mathcal{P}_{r,T}} E_P \left[ \varphi' (X^{1,x}_T) X^{1,x}_T \Gamma^{1,x}_T + \int_t^T \left( f_t'(s) X^{1,x}_s + \frac{T-s}{T-t} f_t'(s) - \frac{1}{T-t} f(s) \right) \Gamma^{1,x}_s ds \right] \]
\[ + \int_t^T \left( g'_t(s) Z^{1,x}_s \frac{T-s}{2(T-t)} + g'_t(s) X^{1,x}_s + \frac{T-s}{T-t} g'_t(s) - \frac{1}{T-t} g(s) \right) \Gamma^{1,x}_s d(B)_s, \]

where \((\hat{X}^{1,x}_s)_{s \in [t,T]}\) satisfies (4.23), \((\Gamma^{1,x}_s)_{s \in [t,T]}\) satisfies (4.13), \(f_t'(s) = f_t'(s, X^{1,x}_s, Y^{1,x}_s, Z^{1,x}_s)\), similar for \(f(s), f'_t(s), g(s), g'_t(s), g''_t(s)\) and \(g''_t(s)\).

**Proof.** The proof is similar to Theorem 4.5, we omit it. \( \square \)

The following theorem gives the condition for \(\partial_{x} u(t,x) = \partial_{x}^{-} u(t,x)\).

**Theorem 4.8** Suppose that (A1)-(A4) hold. If \(\sigma(t,x) \neq 0\) for some \((t,x) \in (0,T) \times \mathbb{R}\), then \(\partial_{x} u(t,x) = \partial_{x}^{-} u(t,x)\).

**Proof.** We first sketch the properties of \(u\), which is the same as in the proof of Theorem 4.5 in [8]. By Propositions 4.2 and 4.3, we can get that, for \(s \in [0,T], x_1, x_2 \in \mathbb{R}, p \geq 2,\)

\[ |u(s,x_1) - u(s,x_2)| \leq C(1 + |x_1|^m + |x_2|^m)|x_1 - x_2|, |u(s,x_1)| \leq C(1 + |x_1|^{m+1}), \]  

\[ \mathbb{E} \left[ \sup_{s \leq t \leq T} |Y^{x_{1},x_{1}}_r|^p + \left( \int_s^T |Z^{x_{1},x_{1}}_r|^2 d(B)_r \right)^{p/2} + |K^{x_{1},x_{1}}_T|^p \right] \leq C(1 + |x_1|^{(m+1)p}), \]

where \(C > 0\) depends on \(L_1, \sigma, p\) and \(T\). For each \(0 \leq t_1 < t_2 \leq T\) and \(x_1 \in \mathbb{R}\), by (i) of Proposition 1.2, we know

\[ u(t_1,x_1) = \mathbb{E} \left[ u(t_2,x_{t_2}^{x_1,x_1}) + \int_{t_1}^{t_2} f(r, X^{x_1,x_1}_r, Y^{x_1,x_1}_r) dr + \int_{t_1}^{t_2} g(r, X^{x_1,x_1}_r, Y^{x_1,x_1}_r, Z^{x_1,x_1}_r) d(B)_r \right]. \]  

It follows from (14), (123), (122), (125) and Hölder’s inequality, we obtain

\[ |u(t_1,x_1) - u(t_2,x_1)| \leq C(1 + |x_1|^{m+1}) \sqrt{t_2 - t_1}, \]

where \(C > 0\) depends on \(L_1, \sigma, p\) and \(T\).

We then take \(t_1 = t - \delta\) with \(\delta \in (0,t), t_2 = t\) and \(x_1 = x\) in (125). By Theorem 4.7, we know that

\[ \lim_{\delta \downarrow 0} \delta^{-1} (u(t - \delta, x) - u(t,x)) = -\partial_{x} u(t,x) \in \mathbb{R}. \]  

In the following, we will prove that

\[ \mathbb{E} \left[ u(t,x_{t}^{t-\delta,x}) - u(t,x + \sigma(t,x)(B_t - B_{t-\delta})) \right] \leq C\delta, \]

\[ \mathbb{E} \left[ \int_{t-\delta}^t |f(r, X^{t-\delta,x}, Y^{t-\delta,x})| dr + \int_{t-\delta}^t |g(r, X^{t-\delta,x}, Y^{t-\delta,x}, Z^{t-\delta,x})| d(B)_r \right] \leq C\delta, \]

\[ \lim_{\delta \downarrow 0} \delta^{-1} \mathbb{E} [u(t,x + \sigma(t,x)(B_t - B_{t-\delta})) - u(t,x)] = \infty \text{ if } \partial_{x} u(t,x) > \partial_{x}^{-} u(t,x), \]

where the constant \(C > 0\) depends on \(x, L_1, L_2, m, m_1, \sigma\) and \(T\). If (123), (129) and (130) hold, we can get \(\partial_{x} u(t,x) = \partial_{x}^{-} u(t,x)\) by (127).
Noting that
\[ \mathbb{E} \left[ \int_{t-\delta}^{t} |\sigma(r, X_{r}^{t-\delta,x}) - \sigma(t, x)|^2 d(B)_r \right] \leq C \int_{t-\delta}^{t} \mathbb{E}[|X_{r}^{t-\delta,x} - x|^2] dr + C\delta^3 \leq C\delta^2, \]
we get (4.28) by (4.23). By (i) of Proposition 3.2, we know \( Y_{r}^{t-\delta,x} = u(r, X_{r}^{t-\delta,x}) \). Then we get
\[ Y_{r}^{t-\delta,x} - u(t, x) = \left( u(t, X_{r}^{t-\delta,x}) - u(t, x) + \int_{t}^{r} g(r, X_{r}^{t-\delta,x}, u(r, X_{r}^{t-\delta,x}), Z_{r}^{t-\delta,x})dB_r \right. \]
\[ + \left. \int_{t}^{r} f(r, X_{r}^{t-\delta,x}, u(r, X_{r}^{t-\delta,x})) dr - \int_{s}^{r} Z_{r}^{t-\delta,x} dB_r - (K_{r}^{t-\delta,x} - K_{s}^{t-\delta,x}). \]

By (3.3) in Proposition 5.2, (4.23) and (4.26), we obtain
\[ \mathbb{E} \left[ \int_{t-\delta}^{t} |Z_{r}^{t-\delta,x}|^2 d(B)_r \right] \leq C\mathbb{E} \left[ \sup_{s \in [t-\delta, t]} \left| u(s, X_{s}^{t-\delta,x}) - u(t, x) \right|^2 \right] + C\delta^2 \]
\[ \leq C\mathbb{E} \left[ \sup_{s \in [t-\delta, t]} \left| u(s, X_{s}^{t-\delta,x}) - u(s, x) \right|^2 \right] + C\delta \]
\[ \leq C\delta. \]

Then we can easily get (4.29) by H"older's inequality.

Now we prove (4.30). Set \( \xi_{\delta} = \sigma(t, x)(B_{t} - B_{t-}) \), we have
\[ \frac{u(t, x + \xi_{\delta}) - u(t, x)}{\delta} = \frac{\left| u(t, x + \xi_{\delta}) - u(t, x) - \partial_{x,+}u(t, x)\xi_{\delta}|I_{\{\xi_{\delta} > 0\}} + \partial_{x,-}u(t, x)\xi_{\delta}^+ + \partial_{x,-}u(t, x)\xi_{\delta}^- \right|}{\delta} \]
\[ \leq C(1 + |\xi_{\delta}|^{m}) \frac{\left| \xi_{\delta} \right|}{\delta} I_{\{\xi_{\delta} > 0\}} + \frac{\gamma}{4} \delta \]
\[ \leq C(1 + |\xi_{\delta}|^{m}) \frac{\left| \xi_{\delta} \right|}{\delta} + \frac{\gamma}{4} \delta, \]

where the constant \( C > 0 \) depends on \( t, L_{1}, m, \sigma \) and \( T \). Similarly, we have
\[ \frac{\left| u(t, x + \xi_{\delta}) - u(t, x) - \partial_{x,-}u(t, x)\xi_{\delta}|I_{\{\xi_{\delta} < 0\}} \right|}{\delta} \leq C(1 + |\xi_{\delta}|^{m}) \frac{\left| \xi_{\delta} \right|}{\delta} + \frac{\gamma}{4} \delta. \]

Noting that \( \partial_{x,+}u(t, x)\xi_{\delta}^+ - \partial_{x,-}u(t, x)\xi_{\delta}^- = \frac{\gamma}{2} |\xi_{\delta}| + \frac{\delta}{2} \gamma + 2\partial_{x,-}u(t, x)\xi_{\delta} \) we get
\[ \frac{u(t, x + \xi_{\delta}) - u(t, x)}{\delta} \geq \frac{\gamma}{4} \delta + \frac{1}{2} \left( \gamma + 2\partial_{x,-}u(t, x) \right) \frac{\left| \xi_{\delta} \right|}{\delta} - 2C(1 + |\xi_{\delta}|^{m}) \frac{\left| \xi_{\delta} \right|}{\delta^2}. \]
Under the assumption (A5), it is easy to check that

\[ \Delta^{-1} |\partial_x u(t, x + \Delta) - \partial_x u(t, x)| \geq -C(1 + |x|^{2m}) \text{ for } \Delta \in (0, 1], \]

(4.31)

\[ \Delta^{-1} |\partial_x u(t, x + \Delta) - \partial_x u(t, x)| \geq -C(1 + |x|^{2m}) \text{ for } \Delta \in [-1, 0), \]

(4.32)

where the constant \( C > 0 \) depends on \( L_1, L_3, \bar{\sigma} \) and \( T \).

Proof. By the definition of \( \mathcal{P}_{t,x} \), it is easy to verify that \( \mathcal{P}_{t,x} \) is weakly compact. Then we can choose a \( P \in \mathcal{P}_{t,x} \) such that

\[
\partial_x u(t, x) = E_P \left[ \varphi' (X_T^{t,x}) \dot{X}_t^{t,x} \Gamma_t^{t,x} + \int_t^T f'_s(s) \dot{X}_s^{t,x} \Gamma_s^{t,x} ds + \int_t^T g'_s(s) \dot{X}_s^{t,x} \Gamma_s^{t,x} d(B)_s \right]
\]

in (4.19). Using the same notations as in the proof of Theorem 4.9 for \( \Delta \in (0, 1] \), we get by (4.16) that

\[
\check{Y}^{\Delta}_t \geq \Delta \partial_x u(t, x) + E_P \left[ \varphi(T) \Gamma_T^{t,x} + \int_t^T \check{f}(s) \Gamma_s^{t,x} ds + \int_t^T \check{g}(s) \Gamma_s^{t,x} d(B)_s \right].
\]

Under the assumption (A5), it is easy to check that

\[
|\varphi(T)| \leq C(1 + |X_T^{t,x}|^m) |\check{X}_T^{t,x} - \dot{X}_t^{t,x} \Delta| + C|\check{X}_T^{t,x}|^2, \quad |\check{f}(s)| \leq C(1 + |X_s^{t,x}|^m) |\check{X}_s^{t,x} - \dot{X}_t^{t,x} \Delta| + C(|\check{X}_s^{t,x}|^2 + |\check{Y}_s^{t,x}|^2),
\]

(4.33)

\[
|\check{g}(s)| \leq C(1 + |X_s^{t,x}|^m) |\check{X}_s^{t,x} - \dot{X}_t^{t,x} \Delta| + C(|\check{X}_s^{t,x}|^2 + |\check{Y}_s^{t,x}|^2 + |\check{Z}_s^{t,x}|^2),
\]

where \( C > 0 \) depends on \( L_1 \) and \( L_3 \). We can also get

\[
\sup_{s \in [t,T]} \hat{E} \left[ |\check{X}_s^{t,x} - \dot{X}_t^{t,x} \Delta|^p \right] \leq C \Delta^{2p}
\]

(4.34)

for \( p \geq 2 \) in the proof of Lemma 4.14 by \( |\hat{b}(s)| + |\hat{h}(s)| + |\hat{\sigma}(s)| \leq C|\check{X}_s^{t,x}|^2 \), where \( C > 0 \) depends on \( L_1, L_3, \bar{\sigma}, p \) and \( T \). It follows from (4.14), (4.15) and (4.16) that

\[
E_P \left[ \varphi(T) \Gamma_T^{t,x} + \int_t^T \check{f}(s) \Gamma_s^{t,x} ds + \int_t^T \check{g}(s) \Gamma_s^{t,x} d(B)_s \right] \leq C(1 + |x|^{2m}) \Delta^2,
\]

(4.35)

where \( C > 0 \) depends on \( L_1, L_3, \bar{\sigma} \) and \( T \). Thus

\[
\check{Y}_t^{\Delta} \geq \Delta \partial_x u(t, x) - C(1 + |x|^{2m}) \Delta^2.
\]

(4.36)
We can also choose a $P^\Delta \in \mathcal{P}_{t,x+\Delta}$ such that

$$\partial_{x-} u(t, x + \Delta) = E_{P^\Delta} \left[ f'(X^t_{t,x+\Delta}) X^t_{t,x+\Delta} \Delta + \int_t^T f'_x(s, X^t_s, Y^t_s, Z^t_s) \hat{X}^t_s \Delta \Gamma^t_s \Delta ds \right].$$

Applying Itô’s formula to $\hat{Y}^\Delta_t \Gamma^t_s \Delta$ on $[t, T]$ under $P^\Delta$, similar to (4.16) and the analysis of (4.33), we can get

$$\hat{Y}^\Delta_t \leq \Delta \partial_{x-} u(t, x + \Delta) + C(1 + |x|^{2m}) \Delta^2,$$

(4.35)

where $C > 0$ depends on $L_1, L_3, \bar{\sigma}$ and $T$. Then we obtain (4.31) by (4.34) and (4.35). Similarly, we can deduce (4.32). □

**Remark 4.10** We can get similar estimates under the assumption

$$|b''_{x,x}(s, x)| \leq L_4 (1 + |x|^{m_2})$$

for positive constant $L_4$ and positive integer $m_2$, similar for the second derivatives of $h$, $\sigma$, $f$ and $g$.

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