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The neutrino mixing angle $\theta_{13}$ in an $S_3$ flavour symmetric model

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**Abstract.** In this work, we discuss neutrino masses and mixings in the framework of a minimal $S_3$ symmetric extension of the Standard Model. In this model, the mass matrices of all fermions take the same generic form with two texture zeroes. The mass matrices of the neutrinos and charged leptons are re-parameterized in terms of their eigenvalues, then the neutrino mixing matrix, $V_{PMNS}$, is computed and exact, explicit analytical expressions for the neutrino mixing angles as functions of the masses of neutrinos and charged leptons are obtained in excellent agreement with the latest experimental data.

1. Introduction

The observation of flavour oscillations of solar, atmospheric, reactor, and accelerator neutrinos established that they have non-vanishing masses and mix among themselves, much like the quarks do [1]. In these observations and experiments, the differences of the squared neutrino masses, as well as the neutrino mixing angles are measured. These discoveries brought out very forcibly the need of extending the Standard Model (SM) in order to accomodate in the theory the new data on neutrino physics in a consistent way that would allow for a unified and systematic treatment of the observed hierarchy of masses and mixings of all fermions. At the same time, the number of free parameters in the extended form of the SM had to be drastically reduced in order to give predictive power to the theory. These two seemingly contradictory demands are met by a flavour symmetry under which the families transform in a non-trivial fashion. The observed pattern of neutrino mixing and, in particular, the non vanishing and sizable value of the reactor mixing angle strongly suggests a flavour permutational symmetry $S_3$.

The results of a combined analysis of all available neutrino oscillation data, including the recent results from long-baseline $\nu_\mu \rightarrow \nu_e$ searches at the Tokai to Kamioka (T2K) [2] and Double CHOOZ experiments [3], as well as, the Main Injector Neutrino Oscillation Search (MINOS) experiment, give the following values for the differences of the squared neutrino masses and the mixing angles in the lepton mixing matrix, $U_{PMNS}$, at $1\sigma$ confidence level [4]:

$$\Delta m^2_{21} = 7.59^{+0.20}_{-0.18} \times 10^{-5} \text{ eV}^2, \quad \Delta m^2_{31} = \begin{cases} -2.46^{+0.08}_{-0.09} \times 10^{-3} \text{ eV}^2, \\ +2.50^{+0.09}_{-0.16} \times 10^{-3} \text{ eV}^2. \end{cases}$$

(1)
the upper (lower) row corresponds to inverted (normal) neutrino mass hierarchy, see also Gonzalez Garcia et al [5] and J. F. W. Valle et al [6]. Recently, the Daya Bay [7] and RENO experiments [8] found the following values for the reactor neutrino mixing angle: 
\[ \sin^2 \theta_{13}^R = 0.092 \pm 0.016 \text{ (stat)} \pm 0.005 \text{ (syst)} \] which is equivalent to \( \theta_{13}^R \simeq 8.8^\circ \pm 0.8^\circ \) at 5.2 \( \sigma \) level, and 
\[ \sin^2 \theta_{13}^L = 0.113 \pm 0.013 \text{ (stat)} \pm 0.019 \text{ (syst)} \] which is equivalent to \( \theta_{13}^L \simeq 9.8^\circ \) at 4.9 \( \sigma \), respectively.

In the last ten years, important theoretical advances have been made in the understanding of the mechanism for fermion mass generation and flavour mixing [9, 10, 11]. A phenomenologically and theoretically meaningful approach for reducing the number of free parameters in the Standard Model and its extensions is the imposition of flavour symmetries and/or texture zeros [9, 10]. In this approach, certain texture zeros may be obtained from a flavour symmetry [12, 13]. For recent review of flavour symmetry models see [11]. In the case of the Minimal \( S_3 \)-Invariant Extension of the Standard Model [14, 15, 16, 17, 18, 19, 20], the concept of flavour and generations is extended to the Higgs sector in such a way that all the matter fields – Higgs, quarks, and lepton fields, including the right-handed neutrino fields– have three species and transform under the flavour symmetry group as the three dimensional representation \( 1 \oplus 2 \) of the permutational group \( S_3 \). A model with more than one Higgs \( SU(2) \) doublet has tree level flavour changing neutral currents whose exchange may give rise to lepton flavour violating processes and may also contribute to the anomalous magnetic moment of the muon [14, 17]. An effective test of the phenomenological success of the model is obtained by verifying that all flavour changing neutral current processes and the magnetic anomaly of the muon, computed in the \( S_3 \)-Invariant extended form of the Standard Model, agree with the experimental values [17, 18].

2. The Minimal \( S_3 \)-invariant Extension of the Standard Model

In the Standard Model analogous fermions in different generations have identical couplings to all gauge bosons of the strong, weak and electromagnetic interactions. Prior to the introduction of the Higgs boson and mass terms, the Lagrangian is chiral and invariant with respect to permutations of the left and right fermionic fields, that is, it is invariant under the action of the group of permutations acting on the flavour indices of the matter fields [14, 17].

The six possible permutations of three objects \( (f_1, f_2, f_3) \) are elements of the permutational group \( S_3 \). This is the discrete, non-Abelian group with the smallest number of elements. The three-dimensional real representation is not an irreducible representation of \( S_3 \). It can be decomposed into the direct sum of a doublet \( f_D \) and a singlet \( f_s \), where

\[ f_s = \frac{1}{\sqrt{3}} (f_1 + f_2 + f_3), \quad f_D^T = \left( \frac{1}{\sqrt{2}} (f_1 - f_2), \frac{1}{\sqrt{6}} (f_1 + f_2 - 2f_3) \right), \] (3)

The direct product of two doublets \( p_D^T = (p_{D1}, p_{D2}) \) and \( q_D^T = (q_{D1}, q_{D2}) \) may be decomposed into the direct sum of two singlets \( r_s \) and \( r_s' \), and one doublet \( r_D^T \) where

\[ r_s = p_{D1}q_{D1} + p_{D2}q_{D2}; \quad r_s' = p_{D1}q_{D2} - p_{D2}q_{D1}, \] (4)

\[ r_D^T = (r_{D1}, r_{D2}) = (p_{D1}q_{D2} + p_{D2}q_{D1}, p_{D1}q_{D1} - p_{D2}q_{D2}). \] (5)

The antisymmetric singlet \( r_s' \) is not invariant under \( S_3 \) [14, 17].
Since the Standard Model has only one Higgs $SU(2)_L$ doublet, which can only be an $S_3$ singlet, it can only give mass to the quark or charged leptons in the $S_3$ singlet representation, one in each family, without breaking the $S_3$ symmetry \cite{13, 21}.

Hence, in order to impose $S_3$ as a fundamental symmetry, unbroken at the Fermi scale, we are led to extend the Higgs sector of the theory. The quark, lepton and Higgs fields are

\[ Q^T = (u_L, d_L), \quad u_R, \quad d_R, \quad L^T = (\nu_L, e_L), \quad e_R, \quad \nu_R \quad \text{and} \quad H, \]  

(6)

in an obvious notation. All of these fields have three species, and we assume that each one forms a reducible representation $1_3 \otimes 2$. The doublets carry capital indices $I$ and $J$, which run from 1 to 2, and the singlets are denoted by

\[ q_3, \ u_{3R}, \ d_{3R}, \ L_3, \ e_{3R}, \ \nu_{3R}, \ \text{and} \ H_S. \]  

(7)

Note that the subscript 3 denotes the singlet representation and not the third generation. The most general renormalizable Yukawa interactions of this model are given by \cite{14, 17}

\[ \mathcal{L}_Y = \mathcal{L}_{Y_D} + \mathcal{L}_{Y_U} + \mathcal{L}_{Y_E} + \mathcal{L}_\nu, \]  

(8)

where

\[ \mathcal{L}_{Y_D} = -Y_U^i \bar{Q}_I H_S d_{JR} - Y_D^i \bar{Q}_3 H_S d_{3R} - Y_{2D}^d [\bar{Q}_I \kappa_{IJ} H_{1d_{JR}} + \bar{Q}_I \eta_{IJ} H_{2d_{JR}} ] - Y_3^d \bar{Q}_I H_{1d_{3R}} + \text{h.c.}, \]  

(9)

\[ \mathcal{L}_{Y_U} = -Y_U^i \bar{Q}_I (i \sigma_2) H_S^* u_{JR} - Y_D^i \bar{Q}_3 (i \sigma_2) H_S^* u_{3R} - Y_{2U}^u [\bar{Q}_I \kappa_{IJ} H_{1u_{JR}} + \bar{Q}_I \eta_{IJ} (i \sigma_2) H_{2u_{JR}} ] - Y_3^u \bar{Q}_I (i \sigma_2) H_{1u_{3R}} + \text{h.c.}, \]  

(10)

\[ \mathcal{L}_{Y_E} = -Y_T^i \bar{T}_I H_S e_{eJR} - Y_3^e \bar{T}_3 H_S e_{3R} - Y_{2E}^\nu [\bar{T}_I \kappa_{IJ} H_{1\nu_{eJR}} + \bar{T}_I \eta_{IJ} (i \sigma_2) H_{2\nu_{eJR}} ] - Y_3^\nu \bar{T}_I (i \sigma_2) H_{1\nu_{3R}} + \text{h.c.}, \]  

(11)

\[ \mathcal{L}_\nu = -Y_1^\nu \bar{\nu}_I (i \sigma_2) H_S^* \nu_{JR} - Y_3^\nu \bar{\nu}_3 (i \sigma_2) H_S^* \nu_{3R} - Y_{2\nu}^\nu [\bar{\nu}_I \kappa_{IJ} (i \sigma_2) H_{1\nu_{JR}} + \bar{\nu}_I \eta_{IJ} (i \sigma_2) H_{2\nu_{JR}} ] - Y_3^\nu \bar{\nu}_I (i \sigma_2) H_{1\nu_{3R}} + \text{h.c.}, \]  

(12)

and

\[ \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(13)

Furthermore, we add to the Lagrangian the Majorana mass terms for the right-handed neutrinos

\[ \mathcal{L}_M = -M_1 \nu_R^T C \nu_R - M_2 \nu_{2R}^T C \nu_{2R} - M_3 \nu_{3R}^T C \nu_{3R}, \]  

(14)

where $M_i$ with $i = 1, 2, 3$ are the right-handed neutrino masses.

Due to the presence of three Higgs fields, the Higgs potential $V_H(H_S, H_D)$ is more complicated than that of the Standard Model \cite{15, 22, 23}. This potential was first analyzed by Pakvasa and Sugawara \cite{24} who found that in addition to the $S_3$ symmetry, it has a permutational symmetry $S'_2$: $H_1 \leftrightarrow H_2$, which is not a subgroup of the flavour group $S_3$. In this communication, we will assume that the vacuum respects the accidental $S'_2$ symmetry of the Higgs potential and that $\langle H_1 \rangle = (H_2)$. With these assumptions, the Yukawa interactions, eqs. (9)-(12) yield mass matrices, for all fermions in the theory, of the general form \cite{14}

\[ M = \begin{pmatrix} \mu_1 + \mu_2 & \mu_2 & \mu_5 \\ \mu_2 & \mu_1 - \mu_2 & \mu_5 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}. \]  

(15)
Table 1. \(Z_2\) assignment in the leptonic sector.

The left-handed Majorana neutrinos \(\nu_L\) naturally acquire their small masses through the see-saw mechanism type I of the form

\[
M_\nu = M_{\nu D} M_R^{-1} (M_{\nu D})^T,
\]

where \(M_{\nu D}\) and \(M_R\) denote the Dirac and right handed Majorana neutrino mass matrices, respectively.

In principle, all entries in the mass matrices can be complex since there is no restriction coming from the flavour symmetry \(S_3\). The mass matrices are diagonalized by bi-unitary transformations as

\[
U^\dagger_{\text{fl}} M_i U_R = \text{diag}(m_1, m_2, m_3), \quad U^T_\nu M_\nu U_\nu = \text{diag}(m_1, m_2, m_3),
\]

where \(i = d, u, e\). The entries in the diagonal matrices may be complex, so the physical masses are their absolute values.

The quark and lepton flavor mixing matrices, \(V_{PMNS}\) and \(V_{CKM}\), arise from the mismatch between diagonalization of the mass matrices of \(u\) and \(d\) type quarks and the diagonalization of the mass matrices of charged leptons and left-handed neutrinos, respectively,

\[
V_{CKM} = U^\dagger_{uL} U_{dL}, \quad V_{PMNS} = U^\dagger_{eL} U_\nu K,
\]

where \(K\) is the diagonal matrix of the Majorana phase factors. Therefore, in order to obtain the unitary matrices appearing in eq. (18) and get predictions for the flavor mixing angles and CP violating phases, we should specify the mass matrices.

3. The mass matrices in the leptonic sector and \(Z_2\) symmetry

A further reduction of the number of parameters in the leptonic sector may be achieved by means of an Abelian \(Z_2\) symmetry. A possible set of charge assignments of \(Z_2\), compatible with the experimental data on masses and mixings in the leptonic sector is given in Table 1.

These \(Z_2\) assignments forbid the following Yukawa couplings, \(Y^e_i = Y^d_i = Y^\nu_i = 0\). Therefore, the corresponding entries in the mass matrices vanish, \(i.e., \mu^e_1 = \mu^e_3 = 0\) and \(\mu^d_1 = \mu^d_5 = 0\).

3.1. The mass matrix of the charged leptons

The mass matrix of the charged leptons takes the form

\[
M_e = m_\tau \begin{pmatrix} \tilde{\mu}_2 & \tilde{\mu}_5 \\ \tilde{\mu}_2 & \tilde{\mu}_5 \\ \mu_4 & \mu_4 & 0 \end{pmatrix}.
\]

The unitary matrix \(U_{eL}\) that enters in the definition of the mixing matrix, \(V_{PMNS}\), is calculated from \(U^\dagger_{eL} M_e M^\dagger_e U_{eL} = \text{diag}(m^e_2, m^e_5, m^e_5)\), where \(m_e, m_\mu\) and \(m_\tau\) are the masses of the charged leptons, and

\[
\frac{M_e M^\dagger_e}{m^2_\tau} = \begin{pmatrix} 2|\tilde{\mu}_2|^2 + |\tilde{\mu}_5|^2 & |\tilde{\mu}_5|^2 & 2|\tilde{\mu}_2||\tilde{\mu}_4|e^{-i\delta_e} \\ |\tilde{\mu}_5|^2 & 2|\tilde{\mu}_2|^2 + |\tilde{\mu}_5|^2 & 0 \\ 2|\tilde{\mu}_2||\tilde{\mu}_4|e^{i\delta_e} & 0 & 2|\tilde{\mu}_4|^2 \end{pmatrix}.
\]
Notice that this matrix has only one non-ignorable phase factor [16].

Once \(M_eM^\dagger_e\) has been reparametrized in terms of the charged lepton masses, it is straightforward to compute \(M_e\) and \(U_{eL}\) also as functions of the charged lepton masses [16]. The resulting expression for \(M_e\), written to order \((m_\mu m_e/m_\mu^2)^2\) and \(x^4 = (m_e/m_\mu)^4\), is

\[
M_e \approx m_\tau \begin{pmatrix}
\frac{1}{\sqrt{2}} \frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}} \frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}} \frac{1}{1+x^2} \\
\frac{1}{\sqrt{2}} \frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & -\frac{1}{\sqrt{2}} \frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}} \frac{1}{1+x^2} \\
\frac{\tilde{m}_e (1+x^2)}{\sqrt{1+x^2}} e^{i\delta_e} & \frac{\tilde{m}_e (1+x^2)}{\sqrt{1+x^2}} e^{i\delta_e} & 0
\end{pmatrix},
\]

(21)

This approximation is numerically exact up to order \(10^{-9}\) in units of the \(\tau\) mass. Notice that this matrix has no free parameters other than the Dirac phase \(\delta_e\).

The unitary matrix \(U_{eL}\) that diagonalizes \(M_eM^\dagger_e\) and enters in the definition of the neutrino mixing matrix \(V_{PMNS}\) may be written as

\[
U_{eL} = P_e O_{eL},
\]

(22)

where \(P_e = \text{diag}(1, 1, e^{i\delta_e})\) and the orthogonal matrix \(O_{eL}\) written to the same order of magnitude as \(M_e\), is

\[
O_{eL} \approx \begin{pmatrix}
\frac{1}{\sqrt{2}} x & \frac{(1+2\tilde{m}_\mu^2+4x^2+\tilde{m}_\mu^2+2\tilde{m}_\mu^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^2+\tilde{m}_\mu^2+12x^2}} & -\frac{1}{\sqrt{2}} \frac{(1+2\tilde{m}_\mu^2+\tilde{m}_\mu^2-2\tilde{m}_\mu^2)}{\sqrt{1+4\tilde{m}_\mu^2+6x^2+6\tilde{m}_\mu^2-4\tilde{m}_\mu^2-5\tilde{m}_\mu^2}} \\
\frac{1}{\sqrt{2}} x & \frac{(1+2\tilde{m}_\mu^2+4x^2-\tilde{m}_\mu^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^2+\tilde{m}_\mu^2+12x^2}} & \frac{1}{\sqrt{2}} \frac{(1+2\tilde{m}_\mu^2+\tilde{m}_\mu^2)}{\sqrt{1+4\tilde{m}_\mu^2+6x^2+6\tilde{m}_\mu^2-4\tilde{m}_\mu^2-5\tilde{m}_\mu^2}} \\
\frac{-x}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^2+\tilde{m}_\mu^2+12x^2}} & \frac{(1-x^2-\tilde{m}_\mu^2+2\tilde{m}_\mu^2)}{\sqrt{1+4\tilde{m}_\mu^2+6x^2+6\tilde{m}_\mu^2-4\tilde{m}_\mu^2-5\tilde{m}_\mu^2}} & \frac{-x}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^2+\tilde{m}_\mu^2+12x^2}} \\
\frac{-x}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^2+\tilde{m}_\mu^2+12x^2}} & \frac{(1-x^2-\tilde{m}_\mu^2-2\tilde{m}_\mu^2)}{\sqrt{1+4\tilde{m}_\mu^2+6x^2+6\tilde{m}_\mu^2-4\tilde{m}_\mu^2-5\tilde{m}_\mu^2}} & \frac{x}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^2+\tilde{m}_\mu^2+12x^2}}
\end{pmatrix},
\]

(23)

where, as before, \(\tilde{m}_\mu = m_\mu/m_\tau, \tilde{m}_e = m_e/m_\tau\) and \(x = m_e/m_\mu\).

3.2. The mass matrix of the neutrinos

In the minimal \(S_3\)-invariant extension of Standard Model, the Yukawa interactions and the \(S_3 \times Z_2\) flavour symmetry yield a mass matrix for the Dirac neutrinos of the form

\[
M_{\nu_D} = \begin{pmatrix}
\mu_D^\nu & \mu_D^\nu & 0 \\
\mu_D^\nu & -\mu_D^\nu & 0 \\
\mu_D^\nu & \mu_D^\nu & \mu_D^\nu
\end{pmatrix},
\]

(24)

In principle, all entries in the mass matrix \(M_{\nu_D}\) can be complex, since there is no restriction coming from the \(S_3 \times Z_2\) flavour symmetry.

The masses of the left-handed Majorana neutrinos, \(M_\nu\), are generated by the see-saw mechanism type I, \(M_\nu = M_R M_R^{-1} (M_{\nu_D})^T\), where \(M_R\) is the mass matrix of the right-handed neutrinos, which we take to be real and diagonal but non-degenerate \(M_R = \text{diag}(M_1, M_2, M_3)\). In previous works it was assumed that the first two right-handed neutrino masses are equal, in
which case the resulting reactor mixing angle is different from zero but very small in comparison with the recent experimental data [14, 17]. Hence, in this work we will assumed that \( M_1 \neq M_2 \).

Then, the mass matrix \( M_\nu \) takes the form

\[
M_\nu = \begin{pmatrix}
\frac{2(\mu_2^e)^2}{M^2} & \frac{2\lambda(\mu_2^e)^2}{M^2} & \frac{2\mu_2^e\mu_3^e\lambda}{M^2} \\
\frac{2\lambda(\mu_3^e)^2}{M^2} & \frac{2(\mu_3^e)^2}{M^2} & 2\mu_3^e\mu_4^e\lambda \\
\frac{2\mu_2^e\mu_3^e\lambda}{M^2} & \frac{2\mu_3^e\mu_4^e\lambda}{M^2} & \frac{2(\mu_4^e)^2}{M^2} + \left(\mu_3^e\right)^2
\end{pmatrix}
\]  

(25)

where

\[
\lambda = \frac{1}{2} \left( \frac{M_2 - M_1}{M_1 + M_2} \right) \quad \text{and} \quad \overline{M} = 2 \frac{M_1 M_2}{M_2 + M_1}
\]  

(26)

When the first two right-handed neutrino masses are equal, the parameter \( \lambda \) vanishes and we recover the expression for \( M_\nu \) given in Kubo et al [14]. Then, in this model the magnitude of the reactor mixing angle is sensitive to the difference of the values of the first and second masses of the right-handed neutrinos, while the solar and atmospheric mixing angles are almost insensitive to the value of this difference of masses.

As we have assumed the right-handed neutrino mass matrix \( \tilde{M}_R \) to be real, the complex symmetric neutrino mass matrix \( M_\nu \) has only three independent phase factors that come from the parameters \( \mu_2, \mu_3 \) and \( \mu_4 \). Here, to simplify the analysis we will consider the case when \( \arg \{\mu_2^e\} = \arg \{\mu_3^e\} \) or \( 2 \arg \{\mu_4^e\} = \arg \left\{ \frac{2(\mu_4^e)^2}{\overline{M}} + \left(\mu_3^e\right)^2 \right\} \). The general case, with three independent phase factors will be considered in detail elsewhere.

In the case considered here, the phase factors may be taken out of \( M_\nu \) as

\[
M_\nu = Q \overline{M}_\nu Q
\]  

(27)

where

\[
Q = \begin{pmatrix}
e^{i\phi_2} & 0 & 0 \\
0 & e^{i\phi_2} & 0 \\
0 & 0 & e^{i\phi_4}
\end{pmatrix} \quad \text{and} \quad \overline{M}_\nu = \begin{pmatrix}
a & d & b \\
d & a & c \\
b & c & a
\end{pmatrix} \quad (28)
\]

with

\[
\phi_2 = \arg \{\mu_2^e\} , \quad \phi_4 = \arg \{\mu_4^e\}
\]  

(29)

\[
a = \frac{2|\mu_2^e|^2}{|M|} , \quad b = \frac{2|\mu_3^e||\mu_2^e|}{|M|} , \quad c = \frac{2|\mu_4^e|^2}{|M|} + \frac{|\mu_3^e|^2}{M_3} , \quad d = \frac{2|\lambda||\mu_3^e|^2}{|M|} , \quad e = \frac{2|\mu_2^e||\mu_4^e||\lambda|}{|M|}
\]  

(30)

The real symmetric matrix \( \overline{M}_\nu \) may be brought to diagonal form by means of a similarity transformation with an orthogonal matrix \( O_\nu \) as \( \overline{M}_\nu = O_\nu \text{diag} \{m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\} O_\nu^T \), the columns in \( O_\nu \) are the normalized eigenvectors of \( \overline{M}_\nu \).

In order to compute \( O_\nu \) we notice that the diagonalization of \( \overline{M}_\nu \) is equivalent to the diagonalization of a mass matrix \( \tilde{M} \) with two texture zeroes. First define a new mass matrix
\( \overline{M}_\nu' \) obtained from \( \overline{M}_\nu \) by a \( \frac{\pi}{4} \) rotation

\[
U_{\frac{\pi}{4}} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0
\end{pmatrix}, \tag{31}
\]

through the similarity transformation \( \overline{M}_\nu' = U_{\frac{\pi}{4}}^\dagger \overline{M}_\nu U_{\frac{\pi}{4}} \). Then, the matrix \( \overline{M}_\nu' \) can be written in the following form:

\[
\overline{M}_\nu' = \mu_0 I_{3\times3} + \tilde{M} \tag{32}
\]

where \( I_{3\times3} \) is the identity matrix,

\[
\mu_0 = a - d = \frac{2 |\mu_0|^2}{|\bar{M}|} (1 - |\lambda|) \quad \text{and} \quad \tilde{M} = \begin{pmatrix} 0 & A & 0 \\ A & B & C \\ 0 & C & D \end{pmatrix} \tag{33}
\]

with \( A = \frac{b - \mu_0}{\sqrt{2}}, \ B = c + d - a, \ C = \frac{b + \mu_0}{\sqrt{2}} \) and \( D = 2d \). As mentioned before the diagonalization of \( \overline{M}_\nu \) is reduced to the diagonalization of the real symmetric matrix \( \tilde{M} \), which is a matrix with two texture zeroes of class I [12]. In the literature, these similarity transformations are also known as weak basis transformations, since they leave invariant the gauge currents [25].

As in the case of the charged leptons, the matrices \( M_\nu \) and \( U_\nu \) can be reparametrized in terms of the neutrino masses. For this we use the information that we already have about the diagonalization of a matrix with two texture zeroes of class I [12, 13, 21]. Then, the mass matrix \( M_\nu \) for a normal [inverted] hierarchy in the mass spectrum takes the form

\[
M_{\nu}^{N(I)} = \begin{pmatrix}
\mu_0 + d & d & \frac{1}{\sqrt{2}} \left( C^{N(I)} + A^{N(I)} \right) \\
d & \mu_0 + d & \frac{1}{\sqrt{2}} \left( C^{N(I)} - A^{N(I)} \right) \\
\frac{1}{\sqrt{2}} \left( C^{N(I)} + A^{N(I)} \right) & \frac{1}{\sqrt{2}} \left( C^{N(I)} - A^{N(I)} \right) & m_{\nu_1} + m_{\nu_2} + m_{\nu_3} - 2 (\mu_0 + d)
\end{pmatrix} \tag{34}
\]

where

\[
A^{N(I)} = \sqrt{\left( m_{\nu_2} - \mu_0 \right) \left( m_{\nu_3} - \mu_0 - m_{\nu_1} \right) \left( \mu_0 - m_{\nu_3} \right) / 2d}, \tag{35}
\]

\[
C^{N(I)} = \sqrt{\left( 2d + \mu_0 - m_{\nu_1} \right) \left( 2d + \mu_0 - m_{\nu_2} \right) \left( m_{\nu_2} - m_{\nu_1} - 2d \right) / 2d}. \tag{36}
\]

The values allowed for the parameters \( \mu_0 \) and \( 2d + \mu_0 \) are in the following ranges:

\[
m_{\nu_2} > \mu_0 > m_{\nu_3} \quad \text{and} \quad m_{\nu_3} > 2d + \mu_0 > m_{\nu_2} \tag{37}
\]

Now, the unitary matrix \( U_\nu \) takes the following form:

\[
U_\nu = P_{\tilde{M}} U_{\frac{\pi}{4}} O^{N(I)} \tag{38}
\]
where $U_\pm$ is given in eq.(31), $P_\nu = \text{diag} \{1, 1, e^{i\delta_\nu}\}$ with $\delta_\nu = \phi_2 - \phi_4$, and $\mathbb{D}^{N[I]}$ is the matrix that diagonalizes $\mathbf{M}$ whose elements are

\[
O^{N[I]}_{11} = \frac{\sqrt{\Delta}}{D^{N[I]}_1}, \quad O^{N[I]}_{12} = \sqrt{\frac{\Delta}{D^{N[I]}_2}}, \quad O^{N[I]}_{13} = \frac{\sqrt{\Delta}}{D^{N[I]}_3}, \quad O^{N[I]}_{21} = \sqrt{\frac{\Delta}{D^{N[I]}_2}}, \quad O^{N[I]}_{22} = \sqrt{\frac{\Delta}{D^{N[I]}_3}}, \quad O^{N[I]}_{23} = \frac{\sqrt{\Delta}}{D^{N[I]}_3}, \quad O^{N[I]}_{31} = \frac{\sqrt{\Delta}}{D^{N[I]}_3}, \quad O^{N[I]}_{32} = \sqrt{\frac{\Delta}{D^{N[I]}_3}}, \quad O^{N[I]}_{33} = \frac{\sqrt{\Delta}}{D^{N[I]}_3},
\]

where

\[
D^{N[I]}_1 = 2d (m_{\nu_2} - m_{\nu_\tau}) (m_{\nu_3[1]} - m_{\nu_\tau[1]}), \quad D^{N[I]}_2 = 2d (m_{\nu_2} - m_{\nu_\tau}) (m_{\nu_3[2]} - m_{\nu_\tau[3]}),
\]

\[
D^{N[I]}_3 = 2d (m_{\nu_\tau[1]} - m_{\nu_\tau[3]}) (m_{\nu_3[2]} - m_{\nu_\tau[3]}), \quad f_1 = (2d + m_\nu - m_{\nu_\tau}),
\]

\[
f_2^{N[I]} = [-1] (2d + m_\nu - m_{\nu_\tau}), \quad f_3^{N[I]} = [-1] (m_{\nu_3} - m_\nu - 2d)
\]

The superscripts $N$ and $I$ denote the normal and inverted hierarchies respectively.

It is well known [21] that in the hadronic sector the masses of quarks may be obtained from a matrix with two texture zeroes which successfully reproduces the strong mass hierarchy of up and down type quarks. Also, the numerical values of the quark mixing angles determined in this framework are in good agreement with the experimental data [13]. Additionally, in a unified treatment in which the mass matrices of all fermions have a similar form with two texture zeroes of class I and a normal hierarchy, the numerical values obtained for masses and mixing of the neutrinos are in very good agreement with all available experimental data [13, 10]. Therefore, it is to be expected that the mixing matrix $V_{\text{PMNS}}$ that is obtained from the mass matrices $M_\mu$ and $M_\tau$, eqs. (21)-(25) will also reproduce the current experimental data on masses and mixings in the leptonic sector of the theory.

4. The neutrino mixing matrix

The neutrino mixing matrix $V_{\text{PMNS}}$, is the product $U_L^\dagger U_\nu K$, where $K$ is the diagonal matrix of the Majorana phase factors, defined by $K = \text{diag}(1, e^{i\alpha_\tau}, e^{i\beta})$. Now, we obtain the theoretical expression of the elements of the lepton mixing matrix, $V_{\text{PMNS}}$. This expression has the following form:

\[
V_{\text{PMNS}}^{\text{th}} = \left(\begin{array}{ccc}
V_{\text{th}}^{\mu_1} & V_{\text{th}}^{\mu_2} & V_{\text{th}}^{\mu_3} \\
V_{\text{th}}^{\tau_1} & V_{\text{th}}^{\tau_2} & V_{\text{th}}^{\tau_3} \\
\end{array}\right)
\]
where
\[ V_{e1}^{th} = \frac{\bar{m}_e}{m_\mu} O_{12}^{N[\tau]} - O_{21}^{N[\tau]} e^{i \delta_1}, \quad V_{e2}^{th} = \frac{\bar{m}_e}{m_\mu} O_{13}^{N[\tau]} - O_{22}^{N[\tau]} e^{i \delta_2}, \quad V_{\tau_1}^{th} = O_{31}^{N[\tau]}, \]
\[ V_{e3}^{th} = \frac{\bar{m}_e}{m_\mu} O_{11}^{N[\tau]} - O_{23}^{N[\tau]} e^{i \delta_1}, \quad V_{\mu_1} = -O_{11}^{N[\tau]} - \frac{\bar{m}_\mu}{m_\mu} O_{21}^{N[\tau]} e^{i \delta_1}, \quad V_{\nu_1}^{th} = O_{31}^{N[\tau]}, \]
\[ V_{\mu_2}^{th} = -O_{12}^{N[\tau]} - \frac{\bar{m}_\mu}{m_\mu} O_{22}^{N[\tau]} e^{i \delta_2}, \quad V_{\nu_1}^{th} = -O_{13}^{N[\tau]} - \frac{\bar{m}_\mu}{m_\mu} O_{23}^{N[\tau]} e^{i \delta_2}, \quad V_{\tau_3}^{th} = O_{33}^{N[\tau]} \]
with \( \delta_1 = \delta_\nu - \delta_\tau \) and the elements \( O_{ij}^{N[\tau]} \) (\( i, j = 1, 2, 3 \)) in the eqs. (42) are given in the eq. (39).

4.1. The Mixing Angles

The theoretical expression for the lepton mixing angles as functions of the lepton mass ratios and one phase are:

\[ \sin^2 \theta_{12}^{th} = \frac{\left( \frac{\bar{m}_e}{m_\mu} \right)^2 \left( O_{12}^{N[\tau]} \right)^2 + \left( O_{22}^{N[\tau]} \right)^2 - 2 \frac{\bar{m}_e}{m_\mu} O_{12}^{N[\tau]} O_{22}^{N[\tau]} \cos \delta_1}{1 - \left( \frac{\bar{m}_e}{m_\mu} \right)^2 \left( O_{13}^{N[\tau]} \right)^2 - \left( O_{23}^{N[\tau]} \right)^2 + 2 \frac{\bar{m}_e}{m_\mu} O_{13}^{N[\tau]} O_{23}^{N[\tau]} \cos \delta_1} \]
\[ \sin^2 \theta_{23}^{th} = \frac{\left( \frac{\bar{m}_e}{m_\mu} \right)^2 \left( O_{13}^{N[\tau]} \right)^2 + \left( O_{23}^{N[\tau]} \right)^2 + 2 \frac{\bar{m}_e}{m_\mu} O_{13}^{N[\tau]} O_{23}^{N[\tau]} \cos \delta_1}{1 - \left( \frac{\bar{m}_e}{m_\mu} \right)^2 \left( O_{13}^{N[\tau]} \right)^2 - \left( O_{23}^{N[\tau]} \right)^2 + 2 \frac{\bar{m}_e}{m_\mu} O_{13}^{N[\tau]} O_{23}^{N[\tau]} \cos \delta_1} \]
\[ \sin^2 \theta_{13}^{th} = \frac{\left( \frac{\bar{m}_e}{m_\mu} \right)^2 \left( O_{13}^{N[\tau]} \right)^2 + \left( O_{23}^{N[\tau]} \right)^2 - 2 \frac{\bar{m}_e}{m_\mu} O_{13}^{N[\tau]} O_{23}^{N[\tau]} \cos \delta_1}{1 - \left( \frac{\bar{m}_e}{m_\mu} \right)^2 \left( O_{13}^{N[\tau]} \right)^2 - \left( O_{23}^{N[\tau]} \right)^2 + 2 \frac{\bar{m}_e}{m_\mu} O_{13}^{N[\tau]} O_{23}^{N[\tau]} \cos \delta_1} \]

where, as before, \( \bar{m}_\mu = m_\mu/m_\tau \) and \( \bar{m}_\tau = m_\tau/m_\tau \).

In a first, preliminary analysis for the mixing angle \( \theta_{13}^{th} \) and for an inverted neutrino mass hierarchy \( (m_\nu_2 > m_\nu_1 > m_\nu_3) \) the eq. (45) takes the form:

\[ \sin^2 \theta_{13}^{th} \approx \frac{(\mu_0 + 2d - m_\nu_3)(\mu_0 - m_\nu_3)}{(m_\nu_2 - m_\nu_3)(m_\nu_3 - m_\nu_3)} \]

Now, with the following values for neutrino masses \( m_\nu_2 = 0.056 \, eV \), \( m_\nu_1 = 0.053 \, eV \) and \( m_\nu_3 = 0.048 \, eV \), and the parameters values \( \delta_1 = \pi/2, \mu_0 = 0.049 \, eV \) and \( d = 8 \times 10^{-5} \, eV \) we get \( \sin^2 \theta_{13}^{th} \approx 0.029 \rightarrow \theta_{13}^{th} \approx 9.8^\circ \).

5. Conclusions

In the minimal \( S_3 \)-invariant extension of the Standard Model [14] a well defined structure of the Yukawa couplings is obtained, which permits the calculations of mass and mixing matrices for quark and leptons in a unified way. A further reduction of redundant parameters is achieved in the leptonic sector by introducing a \( Z_2 \) symmetry. The flavour symmetry group \( S_3 \times Z_2 \) relates the neutrino mass spectrum and mixings. This allowed us to compute the neutrino mixing matrix, \( V_{PMNS} \), explicitly in terms of the masses of the charged leptons and neutrinos and one phase \( \delta_1 \). In this model, the mass matrices of all fermions are brought to the same generic form with two texture zeroes by means of a similarity transformation. Then, we computed the neutrino mixing matrix \( V_{PMNS} \) which has three CP-violating phases, namely, one Dirac phase \( \delta_1 = \delta_\nu - \delta_\tau \) and two Majorana phases \( \alpha \) ans \( \beta \) which are functions of the charged leptons and neutrino masses. The neutrino mixing angles are given as functions of the charged leptons and neutrino masses. In this way, the numerical value of the reactor mixing angle, \( \theta_{13} \), is related to the numerical values of the neutrinos masses, which in this analysis have an inverted mass hierarchy with the values |\( m_\nu_3 \) = 0.056 eV, |\( m_\nu_1 \) = 0.053 eV and |\( m_\nu_3 \) = 0.048 eV. The numerical value obtained for \( \theta_{13} \) is \( \theta_{13}^{th} \approx 9.8^\circ \) in agreement with the latest analysis of the experimental data on neutrino oscillations and mixings [2, 3].
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