Semi–vector spaces
and units of measurement

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Abstract
This paper is aimed at introducing an algebraic model for physical scales and units of measurement. This goal is achieved by means of the concept of “positive space” and its rational powers. Positive spaces are 1–dimensional “semi–vector spaces” without the zero vector. A direct approach to this subject might be sufficient. On the other hand, a broader mathematical understanding requires the notions of sesqui and semi–tensor products between semi–vector spaces and vector spaces.

So, the paper is devoted to an original contribution to the algebraic theory of semi–vector spaces, to the algebraic analysis of positive spaces and, eventually, to the algebraic model of physical scales and units of measurement in terms of positive spaces.

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Introduction

Most objects in physics (like metric field, electromagnetic field, gauge fields, masses, charges, and so on) have physical dimensions, as their numerical values depend on the choice of units of measurement. Usually, these physical dimensions are dealt with in an “informal” way, without a precise mathematical setting.

In recent years, a geometric formulation of covariant classical and quantum mechanics includes also a formal mathematical setting of physical scales and units of measurement in an original algebraic way (see, for instance, [4, 7, 8, 9, 13, 14, 15]). Such a rigorous mathematical setting of scales plays an essential role in some aspects of the covariant theory, in particular in the classification of covariant operators (see, for instance, [7]).

This approach is based on the notion of “positive space” and its rational powers, as model for the spaces of scales and units of measurement. Then, tensor products between positive spaces and vector bundles arising from spacetime yield “scaled objects”, i.e. objects with physical dimension.

In the above papers, this mathematical scheme is sketched very briefly. So, a comprehensive mathematical foundation of this subject is required.

“Positive spaces” are 1–dimensional “semi–vector” spaces. In principle, we could develop a formalism for positive spaces directly, but a broader mathematical understanding of this subject suggests to insert the theory of positive spaces in the wider framework of semi–vector spaces. This concept is not new, and has been used in very different contexts: in [3] for the analysis of some properties of $\mathbb{Z}_2$-valued matrices, in [5] for problems of fuzzy analysis, in [10] for problems of measure theory, in [11, 12] for topological fixed point problems. Here, we introduce the sesqui and semi–tensor products between semi–vector spaces and vector space and prove several results as well. Such results are new and of independent interest in the algebraic theory of semi–vector spaces.

Thus, the goal of the paper is two folded: giving a contribution to the algebraic theory of semi–vector spaces, with special emphasis to tensor products, and proposing an algebraic model for physical scales and units of measurement.

Thus, the paper is organised as follows.

In section 1, we discuss semi–vector spaces over the semi–field $\mathbb{R}^+$, introduce the sesqui and semi–tensors products between semi–vector spaces and vector spaces.

In section 2, we restrict our attention to positive spaces, which are one-dimensional semi–vector spaces without zero vector, and introduce their rational powers.

In section 3, we discuss the algebraic model of physical scales and units of measurement. We start by assuming three positive spaces, $T$, $L$ and $M$ as representatives of the spaces of time, length and mass scales. Then, we describe all possible derived scales in terms of semi–tensor products of rational powers of $T$, $L$ and $M$. Next, we introduce the “scaled objects”, by considering the sesqui–tensor product of a positive space with a vector bundle arising from spacetime. We also sketch differential calculus with scaled objects. Several of the above results are related to the dimensional analysis; we briefly analyse the interplay between that theory and our algebraic setting.
1 Semi–vector spaces

1.1 Semi–vector spaces

We define the semi–vector spaces analogously to vector spaces, by substituting the field of real numbers with the semi–field of positive real numbers. Semi–vector spaces are similar to vector spaces in some respects; however, the lack of some standard properties yields subtle problems, which require a careful treatment.

Let \( \mathbb{R}^+ \subset \mathbb{R} \) and \( \mathbb{R}^- \subset \mathbb{R} \) be the subsets of positive and negative real numbers; moreover, let us set \( \mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\} \) and \( \mathbb{R}_0^- := \mathbb{R}^- \cup \{0\} \).

The set \( \mathbb{R}^+ \) is a semi–field \([6]\) with respect to the operations of addition and multiplication. This means that \( (\mathbb{R}^+, +) \) is a commutative semi–group, \( (\mathbb{R}^+, \cdot) \) is a commutative group and the distributive law holds. Note that, for each \( x, y, z \in \mathbb{R}^+ \), the “cancellation law” holds: if \( x + z = y + z \), then \( x = y \).

1.1 Definition. A semi–vector space (over \( \mathbb{R}^+ \)) is defined to be a set \( U \) equipped with the operations \( + : U \times U \to U \) and \( \cdot : \mathbb{R}^+ \times U \to U \), which fulfill the following properties, for each \( r, s \in \mathbb{R}^+ \), \( u, v, w \in U \),

\[
\begin{align*}
  u + (v + w) &= (u + v) + w, \\
  u + v &= v + u, \\
  (rs) u &= r (su), \\
  1 u &= u, \\
  r (u + v) &= ru + rv, \\
  (r + s) u &= ru + su. \quad \square
\end{align*}
\]

Despite the fact that some authors (see [5]) require that a semi–vector space has a zero vector, here we do not make such a general assumption. However, interesting properties arise for semi–vector spaces with a zero vector.

Let \( U \) be a semi–vector space.

An element \( 0 \in U \) is said to be (additively) neutral (or, equivalently, a zero vector) if, for each \( u \in U \), we have \( 0 + u = u \).

If two elements \( 0, 0' \in U \) are neutral, then they coincide; in fact, we have \( 0' = 0 + 0' = 0' + 0 = 0 \).

If \( 0 \in U \) is a neutral element, then, for each \( r \in \mathbb{R}^+ \), we have \( r 0 = 0 \); in fact, for each \( u \in U \), we have \( r 0 + u = r(0 + \frac{1}{r} u) = r(\frac{1}{r} u) = u \).

A semi–vector space equipped with a zero vector is said to be complete.

A complete semi–vector space \( U \) turns out to be also a semi–vector space over \( \mathbb{R}_0^+ \), by setting \( 0 u = 0 \), for each \( u \in U \). But we shall always refer to the scalar multiplication with respect to the semi–field \( \mathbb{R}^+ \) even for complete semi–vector spaces.

Let \( U \) be a complete semi–vector space. We say that \( u \in U \) is invertible if there exists a vector \( v \in U \) such that \( u + v = 0 \).

A non complete semi–vector space, or a complete semi–vector space with no invertible elements, is said to be simple.

We say that a semi–vector space \( U \) is regular if the “cancellation law” holds, i.e. if,
for each \( u, u', v \in U \), the equality \( u + v = u' + v \) implies \( u = u' \).

In regular complete semi–vector spaces further properties hold.

1.2 Note. Let \( U \) be regular and complete. If \( 0' \in U \) is an element such that \( 0' + u = u \) for a certain \( u \in U \), then \( 0' = 0' \); in fact, we have \( 0' + u = u = 0 + u \), hence, the cancellation law yields \( 0' = 0 \). Moreover, if \( u \in U \) is invertible, then its inverse is unique; in fact, \( u + v = 0 = u + v' \) implies \( v = v' \), in virtue of the cancellation law.

1.3 Note. Let \( V \) be a vector space and \( U \subset V \) a subset, which turns out to be a semi–vector space with respect to the sum and product inherited from \( V \). Then, \( U \) is regular.

1.4 Definition. Let us consider a simple semi–vector space \( U \).

A subset \( B \subset U \) is said to be a semi–basis if any non neutral \( u \in U \) can be written in a unique way as \( u = \sum_{i \in I_u} u^i b_i \), with \( u^i \in \mathbb{R}^+ \) and \( b_i \in B \), where the sum is extended to a finite family of indices \( I_u \), which is uniquely determined by \( u \).

Accordingly, we denote by \( B_u \) the finite subset \( B_u := \{b_i\}_{i \in I_u} \subset B \), which is uniquely determined by \( u \).

The semi–vector space \( U \) is said to be semi–free if it admits a semi–basis.

We can easily see that if \( U \) is a complete semi–free semi–vector space and \( B \) is a semi–basis, then \( 0 \notin B \).

1.5 Proposition. If \( U \) is semi–free, then it is regular.

Proof. Let \( B \) be a semi–basis. If \( u, v, w \in U \), then we have unique expressions of the type \( u = \sum_i u^i b_i \), \( v = \sum_j v^j b_j \), \( w = \sum_k w^k b_k \), with \( u^i, v^j, w^k \in \mathbb{R}^+ \) and \( b_i \in B_u, b_j \in B_v, b_k \in B_w \).

We have \( B_{u+w} = B_u \cup B_w \) and \( u+w = \sum (\hat{u}^k + \hat{w}^k) b_k \), where \( b_k \in B_{u+w} \), \( \hat{u}^k = u^k \), or \( \hat{w}^k = w^k \), if \( b_k \in B_u \), or \( b_k \in B_w \), respectively, and \( \hat{u}^k = 0 \), or \( \hat{w}^k = 0 \), if \( b_k \notin B_u \), or \( b_k \notin B_w \), respectively. Analogous considerations hold for \( v+w \).

Then, from the uniqueness of the expression of semi–vectors, if \( u + w = v + w \), then we have \( B_{u+w} = B_{v+w} \) and \( \sum_k (\hat{u}^k + \hat{w}^k) b_k = \sum_k (\hat{u}^k + \hat{w}^k) b_k \), which implies \( \hat{u}^k = \hat{v}^k \), i.e. \( u^k = v^k \). Hence, \( u = v \). QED

1.6 Proposition. Let \( U \) be a semi–free semi–vector space. Moreover, let \( B \) be a semi–basis and \( B \subset U \) a subset. Then, the following facts hold.

1) Let \( \bar{B} \) consist of non vanishing elements and suppose that there is a bijection \( \bar{B} \rightarrow B : \bar{b}_i \mapsto b_i \) and, for each \( \bar{b}_i \in \bar{B} \), we have \( B_{\bar{b}_i} = \{b_i\} \), so that we obtain the expression \( \bar{b}_i = S_i b_i \), with \( S_i \in \mathbb{R}^+ \). Then, \( B \) is a semi–basis and, for each \( b_i \in B \), we have \( \bar{B}_{b_i} = \{\bar{b}_i\} \), so that we obtain the expression \( b_i = S_i \bar{b}_i = (1/\bar{S}_i) \bar{b}_i \).

2) Suppose that \( B \) is a semi–basis. Then, there is a bijection \( \bar{B} \rightarrow B : \bar{b}_i \mapsto b_i \) and, for each \( \bar{b}_i \in \bar{B} \), we have \( B_{\bar{b}_i} = \{b_i\} \), so that we obtain the expression \( \bar{b}_i = S_i b_i \), with \( S_i \in \mathbb{R}^+ \); moreover, for each \( b_i \in B \), we have \( \bar{B}_{b_i} = \{\bar{b}_i\} \), so that we obtain the expression \( b_i = S_i \bar{b}_i = (1/\bar{S}_i) \bar{b}_i \).

Proof. 1) For each \( b_i \in B \), the expression \( \bar{b}_i = S_i b_i \) implies \( b_i = (1/\bar{S}_i) \bar{b}_i \). Then, for each \( 0 \neq v \in U \), we have \( v = \sum_{i \in I_v} v^i b_i = \sum_{j \in I_v} v^j (1/\bar{S}_j) \bar{b}_j \). Indeed, the 2nd expression is unique. In fact,
if \( v = \sum_{h \in I_v} \bar{v}^h \bar{b}_h \), then we have also \( v = \sum_{h \in \bar{I}_v} \bar{v}^h \bar{S}_h b_h \); hence, in virtue of the uniqueness of the expression with respect to the semi–basis \( B \), we obtain \( I_v = \bar{I}_v \) and \( \bar{v}^i \bar{S}_i = v^i \), i.e. \( \bar{v}^i = v^i \left(1/\bar{S}_i \right) \).

2) For each \( b_i \in B \) and \( \bar{b}_j \in \bar{B} \), we have unique expressions of the type

\[
b_i = \sum_{h \in I_{b_i}} \bar{S}_h \bar{b}_h \quad \text{and} \quad \bar{b}_j = \sum_{k \in I_{\bar{b}_j}} S_k b_k , \quad \text{with} \quad S_i, \bar{S}_j \in \mathbb{R}^+ ,
\]

Then, by substituting the 2nd expressions into the 1st one, we obtain

\[
b_i = \sum_{h \in I_{b_i}} \sum_{k \in I_{\bar{b}_h}} S_h^i \bar{S}_k b_k .
\]

On the other hand, by recalling the uniqueness of the decomposition of each element \( b_i \in B \) with respect to this semi–basis \( (b_i) \) and observing that all coefficients \( S_i^j \) and \( \bar{S}_j^h \) are positive, we deduce that the above equality holds if and only if the following conditions hold:

a) there is a bijection \( B \rightarrow \bar{B} : b_i \mapsto \bar{b}_i \),
b) \( B_{\bar{b}_i} = \{\bar{b}_i\} \), \( B_{b_i} = \{b_i\} \),
c) \( \bar{b}_i = \bar{S}_i b_i \) and \( b_i = S_i \bar{b}_i = \left(1/\bar{S}_i \right) \bar{b}_i \), with \( \bar{S}_i \in \mathbb{R}^+ \). QED

We stress that the transition law between semi–bases of semi–free semi–vector spaces given by the above Proposition 1.6 is essentially more restrictive than the transition law between bases of vector spaces.

1.7 Corollary. If \( U \) is semi–free, then all semi–bases have the same cardinality. □

1.8 Definition. If \( U \) is a semi–free semi–vector space, then we define the semi–dimension of \( U \) to be the cardinality of its semi–bases. We denote the semi–dimension of \( U \) by \( \text{s-dim} U \). □

In the following, we shall refer only to semi–free semi–vector spaces with finite semi–dimension in those formulas where we compute explicitly the semi–dimension.

1.2 Examples of semi–vector spaces

We have examples of quite standard semi–vector spaces and odd examples as well.

1.9 Example. A vector space (over \( \mathbb{R} \)) turns out to be a complete and regular semi–vector space. But, it is not simple and semi–free. □

A semi–vector space is said to be proper if it is not a vector space.

1.10 Example. If \( V \) is a vector space, then the subset \( U := V - \{0\} \subset V \) is not a semi–vector space, because the sum of an element and of its negative is not defined in \( U \). □

1.11 Note. Let \( V \) be a vector space and \( S \subset V \) a subset consisting of \( n \geq 1 \) vectors.
Then, we define the cone generated by \( S \) over \( \mathbb{R}^+ \) to be the subset \( \langle S \rangle_+ \subset V \) consisting of all linear combinations of the type \( r^1s_1 + \cdots + r^ms_m \), with \( m = 1, \ldots, n \) and \( s_1, \ldots, s_m \in S \), \( r^1, \ldots, r^m \in \mathbb{R}^+ \).

Analogously, we define the cone generated by \( S \) over \( \mathbb{R}^- \) to be the subset \( \langle S \rangle_- \subset V \) consisting of all linear combinations of the type \( r^1s_1 + \cdots + r^ms_m \), with \( m = 1, \ldots, n \) and \( s_1, \ldots, s_m \in S \), \( r^1, \ldots, r^m \in \mathbb{R}^- \).

Clearly, if \( S_- \subset V \) is the subset consisting of the the negatives of the set \( S \), then we have \( \langle S \rangle_- = \langle S \rangle_- \). \( \square \)

1.12 Note. Let \( V \) be a vector space and \( S \subset V \) a subset consisting of \( n \geq 1 \) vectors. The set \( \langle S \rangle_+ \) is convex in \( V \).

Proof. The set \( \langle S \rangle_+ \) is convex if and only if for any \( u, v \in \langle S \rangle_+ \) the vector \( tu + (1 - t)v \in \langle S \rangle_+ \) for all \( 0 \leq t \leq 1 \). On the other hand, by observing that \( u, v \in \langle S \rangle_+ \), we can say that the set \( \langle S \rangle_+ \) is convex if and only if for any \( u, v \in \langle S \rangle_+ \) the vector \( tu + (1 - t)v \in \langle S \rangle_+ \) for all \( 0 < t < 1 \).

But, according to the definition of \( \langle S \rangle_+ \), we have \( u = u^1s_1 + \cdots + u^ns_m, v = v^1s'_1 + \cdots + v^ks'_k \), for \( m, k = 1, \ldots, n \) and \( s_1, \ldots, s_m, s'_1, \ldots, s'_k \in S \), \( u^1, \ldots, u^n, v^1, \ldots, v^k \in \mathbb{R}^+ \). Then, \( tu + (1 - t)v = \sum_i((1-t)u^iv^i)s_i + \sum_i(1-t)u^is'_i + \sum_j(tu^j)s'_j \), where \( s_i \in \{s_1, \ldots, s_m\} \cap \{s'_1, \ldots, s'_k\} \), \( s'_j \notin \{s_1, \ldots, s_m\} \). So, for any \( 0 < t < 1 \), \( tu + (1 - t)v \) is a linear combination of vectors in \( \{s_1, \ldots, s_m\} \cup \{s'_1, \ldots, s'_k\} \subset S \) with positive coefficients, i.e. it is a vector in \( \langle S \rangle_+ \). QED

1.13 Example. Let \( V \) be a vector space and \( S \subset V \) a subset consisting of \( n \geq 1 \) linearly independent vectors.

Then, \( \langle S \rangle_+ \) turns out to be a non complete, simple, proper, regular and semi-free semi–vector space, with the semi–basis \( B = S \).

Moreover, any “transversal section” of the cone \( \langle S \rangle_+ \) is a polyhedron with \( n \) vertices; they are of the type \( p_i = r^is_i \), with \( r^i \in \mathbb{R}^+ \) and \( s_i \in S \). Additionally, the boundary of the cone \( \langle S \rangle_+ \) consists of the union of the \( n \) cones generated by the \( n \) subsets \( S_i := S - \{s_i\} \subset S \), with \( s_i \in S \).

In particular, \( \mathbb{R}^+ \) is a non complete, simple, proper, regular and semi-free semi–vector space with semi–dimension 1; its semi–bases are of the type \( B = (s) \), with \( s \in \mathbb{R}^+ \). Clearly, analogous considerations hold for \( \mathbb{R}^- \).

More generally, \( (\mathbb{R}^+)^n \) is a non complete, simple, proper and regular semi–vector space and \( (\mathbb{R}_0^+)^n \) is a complete, simple, proper, regular and semi-free semi–vector space with semi–dimension \( n \); its semi–bases are of the type

\[
B = \left((s^1, 0, \ldots, 0), \ldots, (0, \ldots, 0, s^n)\right) \subset (\mathbb{R}^+)^n.
\]

Moreover, we have

\[
\langle B \rangle_+ = (\mathbb{R}^+_0)^n - \{0\}.
\]

Clearly, analogous considerations hold for \( (\mathbb{R}^-)^n \). \( \square \)

1.14 Example. Let \( V \) be a vector space and \( S \subset V \) a subset consisting of \( n \geq 1 \) linearly independent vectors.

Then, \( \langle S \rangle_+ \) minus its boundary is a non complete, simple, proper and regular semi–vector space, but it is not semi–free because it does not admit any semi–basis. \( \square \)
1.15 Example. Let $V$ be a vector space of dimension $n$ and $S \subset V$ a subset consisting of $n$ linearly independent vectors. Moreover, let us consider a further non vanishing vector $v \in V - \langle S \rangle_+$ (thus, by hypothesis $v$ turns out to be a linear combination of $S$).

Then, $\langle \{v\} \cup S \rangle_+$ is not semi–free because the uniqueness of the positive decomposition of semi-vectors is no longer true.

For instance, a cone in $\mathbb{R}^3$, whose “transversal section” is a square or a disk, turns out to be a non complete, simple, proper and regular semi–vector space. □

1.16 Example. Let $S$ be a set and $U$ a semi–vector space; then, the set $\text{Map}(S, U)$ consisting of all maps $f : S \to U$ turns out to be a semi–vector space in a natural way.

Moreover, if $U$ is complete, then $\text{Map}(S, U)$ is complete. Furthermore, if $U$ is regular, then $\text{Map}(S, U)$ is regular.

Additionally, let $U$ be complete and semi–free with a semi–basis $B \subset U$. Define the subset

$$B' := \{ f(\sigma, b_i) \mid \sigma \in S, b_i \in B \} \subset \text{Map}(S, U),$$

where, for each $\sigma \in S$ and $b_i \in B$, we have defined the map

$$f(\sigma, b_i) : S \to U : s \mapsto \delta_s^\sigma b_i,$$

where $\delta_s^\sigma = 1$ if $s = \sigma$ and $\delta_s^\sigma = 0$ if $s \neq \sigma$.

Then, $\text{Map}(S, U)$ is semi–free with a semi–basis $B' := \{ f(\sigma, b_i) \}$. □

1.17 Example. Let $V$ be a vector space. Then, the set of positive definite symmetric bilinear forms $f : V \times V \to \mathbb{R}$ is a proper semi–vector space. □

Let us exhibit the following non standard example of semi–vector space.

1.18 Example. Let us consider two semi–vector spaces $U$ and $V$ and define the set $W := U \sqcup V$ equipped with an addition and a scalar multiplication as follows

$$u_1 + u_2, \lambda u \quad \text{as in} \quad U, \quad u + v = u = v + u, \quad v_1 + v_2, \lambda v \quad \text{as in} \quad V,$$

for each $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, $\lambda \in \mathbb{R}^+$. 

We can easily see that $W$ turns out to be a semi–vector space. In fact, we can verify the axioms step by step. For instance, in order to verify the associativity of the addition, it suffices to consider the following equalities, for each $u_1, u_2 \in U$, $v_1, v_2 \in V$,

$$(u_1 + v_1) + v_2 = u_1 + v_2 = u_1 = u_1 + (v_1 + v_2)$$

$$(u_1 + v_1) + u_2 = u_1 + u_2 = u_1 + (v_1 + u_2)$$

$$(u_1 + u_2) + v_1 = u_1 + u_2 = u_1 + (u_2 + v_1)$$

$$(v_1 + u_1) + u_2 = u_1 + u_2 = v_1 + (u_1 + u_2)$$

$$(v_1 + u_1) + v_2 = u_1 + v_2 = u_1 + u_1 = v_1 + (u_1 + v_2)$$

$$(v_1 + v_2) + u_1 = u_1 + v_1 + u_1 = v_1 + (v_2 + u_1).$$

We can verify the other axioms in a similar way.
1.19 Proposition. If $U$ is a non complete semi–vector space, then we can naturally extend $U$ to a complete semi–vector space $\hat{U} := U \sqcup \{0\}$ defined by a procedure as in the above Example 1.18.

1.20 Remark. With reference to the above Proposition 1.19, if the semi–vector space $U$ is already complete, then the original $0_U$ is no longer a neutral element of the new completed semi–vector space, because we have $0_U + 0 = 0_U$ and not $0_U + 0 = 0$.

1.21 Example. With reference to Example 1.13, the sets $\langle \hat{S} \rangle_+ := \langle S \rangle_+ \cup \{0\}$ and $\langle \hat{S} \rangle_- := \langle S \rangle_- \cup \{0\}$ are complete semi–free and proper semi–vector spaces with semi–dimension $n$.

Moreover, $\mathbb{R}_0^+ \equiv \mathbb{R}_0^+$ and $\mathbb{R}_0^- \equiv \mathbb{R}_0^-$ are complete semi–free and proper semi–vector spaces with semi–dimension 1.

We can easily define the concepts of semi–vector subspace and product of semi–vector spaces.

1.3 Semi–linear maps

The notion of semi–linear map is similar to that of linear map. However, the lack of some standard properties of semi–vector spaces yields subtle problems which require additional care.

1.22 Definition. A map $f : U \to V$ between semi–vector spaces is said to be semi–linear if, for each $u, v \in U$, $r \in \mathbb{R}^+$, we have $f(u+v) = f(u) + f(v)$ and $f(ru) = rf(u)$.

Of course, a linear map between vector spaces is also semi–linear.

If $U$ and $V$ are semi–vector spaces, then we obtain the semi–vector subspace

$$s\text{-Lin}(U, V) := \{ f : U \to V \mid f \text{ is semi–linear} \} \subset \text{Map}(U, V).$$

In particular, if $U$ is a semi–vector space, then its semi–dual is defined to be the semi–vector space

$$U^* := s\text{-Lin}(U, \mathbb{R}^+).$$

If $U$ and $V$ are vector spaces, then $s\text{-Lin}(U, V)$ turns out to be a semi–vector subspace $s\text{-Lin}(U, V) \subset \text{Lin}(U, V)$.

The composition of two semi–linear maps is semi–linear. Hence, semi–vector spaces and semi–linear maps constitute a category.

If $f : U \to V$ is a bijective semi–linear map, then the inverse map $f^{-1} : V \to U$ is also semi–linear.
Moreover, let \( V \) be any semi–vector space and \( \alpha \in \text{trans} \). Then, we have \( \alpha = \alpha \circ f \) is also semi–linear.

1.23 Proposition. Let \( U \) be a complete semi–vector space and \( V \) a regular and complete semi–vector space. If \( f : U \rightarrow V \) is a semi–linear map, then \( f(0_U) = 0_V \).

Proof. For each \( u \in U \), we have \( f(u) + 0_V = f(u) = f(u + 0_U) = f(u) + f(0_U) \). Hence, we obtain \( f(0_U) = 0_V \). QED

1.24 Proposition. Let \( U \) be complete semi–vector space and \( V \) be semi–vector space. If there exists a surjective semi–linear map \( f : U \rightarrow V \), then \( V \) is complete and \( f(0_U) = 0_V \).

Proof. In virtue of the surjectivity of \( f \), for any \( v \in V \) there exists \( u \in U \) such that \( f(u) = v \). Then, we have \( v = f(u) = f(u + 0_U) = f(u) + f(0_U) = v + f(0_U) \), which implies that \( f(0_U) = 0_V \) is a neutral vector in \( V \). QED

1.25 Proposition. Let \( U \) be a semi–free semi–vector space and \( B \subseteq U \) a semi–basis. Moreover, let \( V \) be any semi–vector space and \( S \subseteq V \) any subset. Then, there exists a unique semi–linear map \( f : U \rightarrow V \) such that \( f(b_i) = v_i \), with \( b_i \in B \) and \( v_i \in S \). □

1.26 Corollary. Let \( U \) be a semi–free semi–vector space of semi–dimension \( n \).
If \( U \) is non complete, then it is semi–linearly isomorphic to \((\mathbb{R}_0^+)^n - \{0\}\).
If \( U \) is complete, then it is semi–linearly isomorphic to \((\mathbb{R}_0^+)^n\).

Proof. Let \( B = \{b_1, \ldots, b_n\} \) be a semi–basis of \( U \). Then the map \( U \rightarrow (\mathbb{R}_0^+)^n \) characterised by \( b_i \mapsto (0, \ldots, 1, \ldots, 0) \), with 1 on the \( i \)-th position, defines the semi-linear isomorphism. QED

1.27 Lemma. Let \( U \) be a semi–free semi–vector space and \( V \) a vector space. Moreover, let \( u, \bar{u} \in U \). If, for any semi–linear map \( f : U \rightarrow V \) we have \( f(u) = f(\bar{u}) \), then \( u = \bar{u} \).

Proof. By considering a semi–basis \( B \), the equality \( f(u) = f(\bar{u}) \) implies \( \sum_i u^i f(b_i) = \sum_i \bar{u}^i f(b_i) \). Then, the arbitrariness of \( f \) implies \( u^i = \bar{u}^i \). QED

1.28 Proposition. Let \( U \) and \( U' \) be semi–free semi–vector spaces and let \( U' \) be complete; moreover, let \( B \subseteq U \) and \( B' \subseteq U' \) be semi–bases. Then, the semi–vector space \( \text{s-Lin}(U, U') \) turns out to be complete and semi–free and with a semi–basis \( S := \{f_{ij}\} \subseteq \text{s-Lin}(U, U') \) consisting of the semi–linear maps \( f_{ij} : U \rightarrow U' \) uniquely defined by the condition \( f_{ij} : b_h \mapsto \delta_{hi} b'_j \). □

1.4 Sesqui–tensor products

Next, we introduce the notion of tensor product between semi–vector spaces and vector spaces. This construction is quite similar to the analogous construction for vector spaces, but the lack of standard properties of semi–vector spaces requires an additional care. In particular, it is worth specifying explicitly the different roles of \( \mathbb{R} \) and \( \mathbb{R}^+ \).
Let $U$ be a semi–vector space and $V$ a vector space.

1.29 Definition. A (right) sesqui–tensor product between $V$ and $U$ is defined to be a vector space $V \hat{\otimes} U$ along with a map $\hat{\otimes} : V \times U \rightarrow V \hat{\otimes} U$, which is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor, and which fulfills the following universal property:

if $W$ is a vector space and $f : V \times U \rightarrow W$ a map which is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor, then there exists a unique linear map $\tilde{f} : V \hat{\otimes} U \rightarrow W$, such that $f = \tilde{f} \circ \hat{\otimes}$.

1.30 Theorem. The sesqui–tensor product exists, is unique up to a distinguished linear isomorphism and is linearly generated by the image of the map $\hat{\otimes} : V \times U \rightarrow V \hat{\otimes} U$.

Proof. The proof is analogous to that for the tensor product of vector spaces, with an additional care.

Existence. We consider the vector space $F$ consisting of all maps $\phi : V \times U \rightarrow \mathbb{R}$, which vanish everywhere except on a finite subset of $V \times U$. Clearly, the set $F$ becomes a vector space in a natural way. Accordingly, each $\phi \in F$ can be written as a formal sum of the type

$$\phi = \phi^{11}(v_1, u_1) + \cdots + \phi^{nm}(v_n, u_m),$$

where $\phi^{ij} \equiv \phi(v_i, u_j) \in \mathbb{R}$ are the (possibly) non vanishing values of $\phi$.

Next, we consider the subset $S \subset F$ consisting of elements of the type

$$(v + v', u) - (v, u) - (v', u), \quad (v, u + u') - (v, u) - (v, u'),$$
$$(sv, u) - s(v, u), \quad (v, ru) - r(v, u),$$

with $v, v' \in V$, $u, u' \in U$, $s \in \mathbb{R}$, $r \in \mathbb{R}^+$. Then, we consider the vector subspace $(S)_{\mathbb{R}} \subset F$ linearly generated by $S$ on $\mathbb{R}$. Eventually, we obtain the quotient vector space and the bilinear map

$$V \hat{\otimes} U \equiv F/(S)_{\mathbb{R}}$$

and $\hat{\otimes} := q \circ j : V \times U \rightarrow V \hat{\otimes} U$,

where $j : V \times U \rightarrow F$ and $q : F \rightarrow F/(S)_{\mathbb{R}}$ are the natural inclusion and the quotient projection.

Clearly, $V \hat{\otimes} U$ is linearly generated by the image of the map $\hat{\otimes}$.

Now, let us refer to the universal property (Definition 1.29). If the linear map $\tilde{f} : V \hat{\otimes} U \rightarrow W$ such that $f = \tilde{f} \circ \hat{\otimes}$ exists, then it is unique because $V \hat{\otimes} U$ is linearly generated by the image of the map $\hat{\otimes}$. Indeed, such a map exists. In fact, we can easily prove that the bilinear map $f : V \times U \rightarrow W$ yields naturally a linear map $f' : F \rightarrow W$, which passes to the quotient yielding the required linear map $\tilde{f} : V \hat{\otimes} U \rightarrow W$.

Uniqueness. The sesqui–tensor product is “unique” in the following sense. If $V \hat{\otimes} U$ and $V \hat{\otimes} U$ are sesqui–tensor products, then the universal properties of the two sesqui–tensor products yield the following commutative diagram

$$\begin{array}{ccc}
V \hat{\otimes} U & \xrightarrow{\sim} & V \hat{\otimes} U \\
\hat{\otimes} & \Downarrow & \hat{\otimes} \\
V \times U & \xleftarrow{\sim} & V \hat{\otimes} U
\end{array}$$

where $\tilde{\otimes} : V \hat{\otimes} U \rightarrow V \hat{\otimes} U$ and $\tilde{\otimes} : V \hat{\otimes} U \rightarrow V \hat{\otimes} U$ are mutually inverse linear isomorphisms. QED
1.31 Note. Clearly, for each \( v, v' \in V \), \( u, u' \in U \), \( s \in \mathbb{R} \), \( r \in \mathbb{R}^+ \), we have
\[
(v + v') \otimes u = v \otimes u + v' \otimes u, \quad v \otimes (u + u') = v \otimes u + v \otimes u',
\]
\[
(s \, v) \otimes u = s \, (v \otimes u), \quad v \otimes (ru) = r \, (v \otimes u).
\]
The following annihilation rules follow from the universal property:

1.32 Proposition. If \( u \in U \), \( 0_V \in V \) then we have \( 0_V \otimes u = 0 \in V \hat{\otimes} U \).

Proof. If \( f : V \hat{\otimes} U \to W \) is any linear map, then \( \phi := f \circ \hat{\otimes} : V \times U \to W \) is a linear map with respect to the 1st factor. Hence, we have \( \phi(0_V, u) = 0 \), which, in virtue of the universal property, implies \( f(0_V \otimes u) = 0_W \). Therefore, in virtue of the arbitrariness of \( f \), we obtain \( 0_V \otimes u = 0 \). QED

1.33 Proposition. Let \( U \) be complete. If \( 0_U \in U \), \( v \in V \), then we have \( v \otimes 0_U = 0 \in V \hat{\otimes} U \).

Proof. If \( f : V \hat{\otimes} U \to W \) is any linear map, then \( \phi := f \circ \hat{\otimes} : V \times U \to W \) is a semi–linear map with respect to the 2nd factor. Hence, we have \( \phi(v, 0_U) = 0_W \), which, in virtue of the universal property, implies \( f(v \otimes 0_U) = 0_W \). Therefore, in virtue of the arbitrariness of the linear map \( f \), we obtain \( v \otimes 0_U = 0 \). QED

1.34 Proposition. Let \( U \) be semi–free. Moreover, let \( B \) be a basis of \( V \) and \( C \) a semi–basis of \( U \). Then,
\[
B \hat{\otimes} C := \{ b_i \otimes c_j \mid b_i \in B, c_j \in C \}
\]
is a basis of \( V \hat{\otimes} U \). Thus, we have
\[
\dim(V \hat{\otimes} U) = (\dim V) \,(s\text{-dim } U).
\]

Proof. Clearly, \( B \hat{\otimes} C \) linearly generates \( V \hat{\otimes} U \).

Next, let us prove that the elements of \( B \hat{\otimes} C \) are linearly independent. For this purpose, let us observe that the universal property of the sesqui–linear tensor product yields a bijection \( f \mapsto \hat{f} \) between the maps \( f : V \times U \to \mathbb{R} \) which are linear with respect to the 1st factor and semi–linear with respect to the 2nd factor and the linear maps \( \hat{f} : V \hat{\otimes} U \to \mathbb{R} \), according to the rule \( \hat{f}(v \otimes u) = f(v, u) \), for each \( v \in V \), \( u \in U \). Now, let us consider an element \( t := \sum_{ij} t^{ij} b_i \otimes c_j \in V \hat{\otimes} U \). Indeed, for any \( \hat{f} \) as above, we have \( \hat{f}(\sum_{ij} t^{ij} b_i \otimes c_j) = \sum_{ij} t^{ij} f(b_i, c_j) \). Then, a property of semi–bases and bases implies that \( \sum_{ij} t^{ij} f(b_i, c_j) = 0 \), for all \( f \) as above, if and only if \( t^{ij} = 0 \). Hence, \( \hat{f}(\sum_{ij} t^{ij} b_i \otimes c_j) = 0 \), for all \( \hat{f} \) as above, if and only if \( t^{ij} = 0 \). Therefore, a property of bases implies that \( (b_i \otimes c_j) \) is a basis of the sesqui–tensor product. QED

1.35 Remark. If \( U \) and \( V \) are vector spaces, then we can consider their sesqui–tensor product and tensor product, by considering, respectively, one factor as a semi–vector spaces, or both factors as vector spaces. We stress that the above tensor products are different. For instance, if \( v \in V \) and \( u \in U \), then
\[
- (v \hat{\otimes} u) \neq v \hat{\otimes} (-u) \neq (-v) \hat{\otimes} u = -(v \hat{\otimes} u), \quad (-v) \hat{\otimes} (-u) \neq v \hat{\otimes} u,
\]
\[
- (v \otimes u) = v \otimes (-u) = (-v) \otimes u = -(v \otimes u), \quad (-v) \otimes (-u) = v \otimes u. \square
\]
1.36 Proposition. Let $V$ and $U$ be vector spaces. Then, we have

$$\dim(V \hat{\otimes} U) = 2 \dim V \dim U.$$ 

Indeed, the universal properties of the tensor products yield the natural surjective semi-linear map

$$\hat{\pi} : V \otimes U \to V \hat{\otimes} U : v \otimes u \mapsto v \hat{\otimes} u,$$

whose kernel is semi-linearly generated by the elements of the type

$$v \hat{\otimes} u + v \hat{\otimes} (-u) \in V \hat{\otimes} U,$$

with $v \in V$, $u \in U$. Thus, we have $\dim(\ker \hat{\pi}) = \dim V \dim U$. $\square$

We can introduce the left sesqui–tensor product $U \hat{\otimes} V$ analogously to the right sesqui–tensor product $V \hat{\otimes} U$. Clearly, we have a natural linear isomorphism $V \hat{\otimes} U \simeq U \hat{\otimes} V$, which is characterised by the map $v \hat{\otimes} u \mapsto u \hat{\otimes} v$.

1.5 Universal vectorialising space

The sesqui–tensor product of a semi–free semi–vector space with $\mathbb{R}$ yields a distinguished extension of the semi–vector space into a vector space.

Let us consider a semi–vector space $U$ and the distinguished vector space $\mathbb{R} \hat{\otimes} U$.

1.37 Lemma. Each element $t \in \mathbb{R} \hat{\otimes} U$ can be written as

$$t = a \hat{\otimes} u_+ + b (-\hat{\otimes}) u_- = a \hat{\otimes} u_+ - b \hat{\otimes} u_-,$$

where each of the the parameters $a$ and $b$ might assume the value 1 or 0 and $u_+, u_- \in U$.

Proof. Each element $t \in \mathbb{R} \hat{\otimes} U$ can be written as

$$t = \sum_{ij} t^{ij} (\lambda_i \hat{\otimes} u_j) = \sum_j (\sum_i t^{ij} \lambda_i) \hat{\otimes} u_j = \sum_j r^j \hat{\otimes} u_j,$$

with $t^{ij}, \lambda_i \in \mathbb{R}$, $u_j \in U$ and $r^j := \sum_i t^{ij} \lambda_i \in \mathbb{R}$.

Hence, we obtain the result by recalling that $0 \hat{\otimes} u = 0$ and by setting

$$u_+ := \sum_j r^j u_j, \quad \text{for} \quad r^j > 0 \quad \text{and} \quad u_- := \sum_j (-r^j) u_j, \quad \text{for} \quad r^j < 0. \quad \text{QED}$$

1.38 Proposition. The sesqui–tensor product $\mathbb{R} \hat{\otimes} U$ is the union of three semi–vector subspaces

$$\mathbb{R} \hat{\otimes} U = U_+ \cup \{0\} \cup U_-,$$

where

$$U_+ := \{1 \hat{\otimes} u \mid u \in U\}, \quad U_- := \{-1 \hat{\otimes} u \mid u \in U\}.$$
Thus, $\mathbb{R} \hat{\otimes} U$ is linearly generated by the subset consisting of elements of the type $1 \hat{\otimes} u$, with $u \in U$.

Moreover, we have the natural semi–linear map

$$i : U \to \mathbb{R} \hat{\otimes} U : u \mapsto 1 \hat{\otimes} u.$$ 

**1.39 Lemma.** Let $U$ be semi–free. Then, for each non neutral element $u \in U$, we have $1 \hat{\otimes} u \neq 0$ and $(-1) \hat{\otimes} u \neq 0$.

**Proof.** By the hypothesis that $U$ is semi–free, there exist a semi–linear map $\phi : U \to \mathbb{R}$, with positive values, excepts on the possible neutral element of $U$. This map yields the map $f : \mathbb{R} \times U \to \mathbb{R} : (r, v) \mapsto r \phi(v)$; indeed, the map $f$ is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor and fulfills the equalities $f(1, u) = \phi(u)$ and $f(-1, u) = -\phi(u)$. Then, in virtue of the universal property, we obtain the linear map $\tilde{f} : \mathbb{R} \hat{\otimes} U \to \mathbb{R}$. Thus, we have $\tilde{f}(1 \hat{\otimes} u) = \phi(u) \neq 0$ and $\tilde{f}((-1) \hat{\otimes} u) = -\phi(u) \neq 0$; hence $1 \hat{\otimes} u \neq 0$ and $(-1) \hat{\otimes} u \neq 0$. QED

**1.40 Lemma.** Let $U$ be semi–free. Then, for each non neutral elements $u, v \in U$, we have $1 \hat{\otimes} u \neq (-1) \hat{\otimes} v$.

**Proof.** By the hypothesis that $U$ is semi–free, there exists a semi–linear map $\phi : U \to \mathbb{R}$, with positive values, excepts on the possible neutral element of $U$. This map yields the map $f : \mathbb{R} \times U \to \mathbb{R} : (r, v) \mapsto r \phi(v)$; indeed, the map $f$ is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor. Then, in virtue of the universal property, we obtain the linear map $\tilde{f} : \mathbb{R} \hat{\otimes} U \to \mathbb{R}$. Indeed, we obtain $\tilde{f}(1 \hat{\otimes} u) = \phi(u) \in \mathbb{R}^+$ and $\tilde{f}((-1) \hat{\otimes} v) = -\phi(v) \in \mathbb{R}^-$. Hence, $1 \hat{\otimes} u \neq (-1) \hat{\otimes} v$, because there is a linear map which takes different values on these elements. QED

**1.41 Lemma.** Let $U$ be semi–free. Then, for each $u, \tilde{u} \in U$, $1 \hat{\otimes} u = 1 \hat{\otimes} \tilde{u}$ implies $u = \tilde{u}$.

**Proof.** Let us consider any semi–linear map $\phi : U \to \mathbb{R}$ and the induced map $f : \mathbb{R} \times U \to \mathbb{R} : (r, u) \mapsto r \phi(u)$, which is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor. Thus, we obtain the induced linear map $\tilde{f} : \mathbb{R} \hat{\otimes} U \to \mathbb{R} : r \hat{\otimes} u \mapsto r \phi(u)$.

Then, the equality $1 \hat{\otimes} u = 1 \hat{\otimes} \tilde{u}$ implies $\tilde{f}(1 \hat{\otimes} u) = \tilde{f}(1 \hat{\otimes} \tilde{u})$, hence $\phi(u) = \phi(\tilde{u})$. Therefore, the arbitrariness of $\phi$ implies $u = \tilde{u}$, in virtue of Lemma 1.27. QED

**1.42 Note.** Let $U$ be semi–free.

If $U$ is non complete, then $U_+ \cap U_- = \emptyset$.

If $U$ is complete, then $U_+ \cap U_- = \{0\}$. QED

**1.43 Proposition.** Let $U$ be semi–free. Then, the map $i$ is injective. Moreover, if $B$ is a semi–basis of $U$, then $(1 \hat{\otimes} b_i)$ is a basis of $\mathbb{R} \hat{\otimes} U$. Hence,

$$\dim(\mathbb{R} \hat{\otimes} U) = s \dim U.$$ QED

**1.44 Definition.** If $U$ is semi–free, then we say $\bar{U} := \mathbb{R} \hat{\otimes} U$ to be the **universal vector extension** of $U$. QED

In fact, this vector space allows us to transform any semi–linear map into a linear map. Moreover, this space is the smallest one with this property.
1.45 Proposition. Let $U$ be semi-free. Moreover, let $W$ be a vector space and $f : U \to W$ a semi-linear map. Then, the following facts hold.

1) There exists a unique linear map $\bar{f} : \bar{U} \to W$, such that $f = \bar{f} \circ \iota$. This map is given by

$$\bar{f}(1 \otimes u) = f(u), \quad \bar{f}((-1) \otimes u) = -f(u), \quad \forall u \in U.$$ 

2) If $U'$ is another vector space and $\iota' : U \to U'$ a semi-linear inclusion which fulfill the above property, then there exist a unique linear inclusion $\rho : U \hookrightarrow U'$, such that $\iota' = \rho \circ \iota$ and $\bar{f} = \bar{f}' \circ \rho$.

Proof. 1) Let us consider the map $\phi : \mathbb{R} \times U \to W : (r, u) \mapsto rf(u)$, which is linear with respect to the 1st factor and semi-linear with respect to the 2nd factor. Then, in virtue of the universal property of the sesqui–tensor product, there is a unique linear map $\bar{f} : \bar{U} \to W$, such that $\bar{f}(r, u) := rf(u)$. In particular, we have $\bar{f}(1 \otimes u) = f(u)$.

Moreover, if $\bar{f}' : \bar{U} \to W$ is another linear map such that $\bar{f}'(1 \otimes u) = f(u)$, then $\bar{f}'(1 \otimes u) = f(u) = \bar{f}(1 \otimes u)$ implies also $\bar{f}'((-1) \otimes u) = -f(u) = \bar{f}((-1) \otimes u)$. Hence, being $\mathbb{R} \otimes U$ semi-linearly generated by the elements of the type $1 \otimes u$ and $(-1) \otimes u$, we obtain $\bar{f}' = \bar{f}$.

2) The existence and uniqueness of the semi–linear map $\rho : U \to U'$ follow from the universal property of the sesqui–tensor product $\mathbb{R} \otimes U$, by considering the map $\mathbb{R} \times U \to U' : (r, u) \mapsto r \iota'(u)$. QED

For instance, the sesqui–tensor product $\bar{U}' := \mathbb{C} \otimes U$ is also a vector extension of $U$ and we have the distinguished real linear inclusion $\mathbb{R} \otimes U \hookrightarrow \mathbb{C} \otimes U$.

In an analogous way we can prove the following fact.

1.46 Proposition. Let $U$ and $V$ be semi-free semi–vector spaces. Moreover, let $W$ be a vector space and $f : U \times V \to W$ a semi–bilinear map. Then, the following facts hold:

1) There exists a unique bilinear map $\bar{f} : \bar{U} \times \bar{V} \to W$, such that $f = \bar{f} \circ (\iota_U \times \iota_V)$.

This map is given by

$$\bar{f}(1 \otimes u, 1 \otimes v) = f(u, v), \quad \bar{f}((-1) \otimes u, 1 \otimes v) = \bar{f}(1 \otimes u, (-1) \otimes v) = -f(u, v), \quad \bar{f}((-1) \otimes u, (-1) \otimes v) = f(u, v), \quad \forall (u, v) \in U \times V.$$ 

2) If $U' \times \bar{V}'$ is another vector space and $(\iota'_U \times \iota'_V) : U \times V \to U' \times \bar{V}'$ a semi–linear inclusion which fulfill the above property, then there exist a unique linear inclusion $\rho : U \times V \hookrightarrow U' \times \bar{V}'$, such that $(\iota'_U \times \iota'_V) = \rho \circ (\iota_U \times \iota_V)$ and $\bar{f} = \bar{f}' \circ \rho$. □

1.47 Note. From Proposition 1.43 and Example 1.13 it follows that all semi–free semi–vector spaces can be regarded as cones in a vector space. □

1.48 Remark. Let $V$ be a vector space. Then the map $\iota : V \to \mathbb{R} \otimes V$ is not a linear isomorphism; in fact, for each $v \in V$, we have

$$\iota(-v) = 1 \otimes (-v) \neq -(1 \otimes v) = -\iota(v).$$
Indeed, if $B = (b_1, \ldots, b_n)$ is a basis of $V$, then
\[
\left(1 \otimes b_1, \ldots, 1 \otimes b_n, (-1) \otimes b_1, \ldots, (-1) \otimes b_n\right) \subset \mathbb{R} \otimes V
\]
is a basis of $\mathbb{R} \otimes V$.

Hence, we have
\[
\dim(\mathbb{R} \otimes V) = 2 \dim V.
\]
Thus, we have
\[
\bar{V} := \mathbb{R} \otimes V \neq V \simeq \mathbb{R} \otimes V.
\]
On the other hand, we have a surjective linear map
\[
\bar{V} \to V : r \otimes v \mapsto rv,
\]
whose kernel is linearly generated by the elements of the type
\[
1 \otimes v + 1 \otimes (-v), \quad \text{with} \quad v \in V. \quad \square
\]

1.49 Proposition. Let $U$ and $V$ be semi–free semi–vector spaces. Then, we have the natural injective linear map
\[
\overline{s-Lin}(U, V) \to \text{Lin}(\bar{U}, \bar{V}) : r \otimes f \mapsto \phi, \quad \text{where} \quad \phi : \bar{U} \to \bar{V} : s \otimes u \mapsto r(s \otimes f(u)).
\]
Indeed, we have
\[
\dim(\overline{s-Lin}(U, V)) = s-dim(U) \quad s-dim(V),
\]
\[
\dim(\text{Lin}(\bar{U}, \bar{V})) = s-dim(U) \quad s-dim(V). \quad \square
\]

1.50 Corollary. Let $U$ be a semi–free semi–vector space. Then, we have a natural injective linear map
\[
\bar{U}^* \to \bar{U}^*. \quad \square
\]

1.6 Semi–tensor products

Now, we are in a position to introduce the notion of tensor product between semi–free semi–vector spaces.

The procedure is similar to that for vector spaces, but we need to pass through the universal vector extension, in order to overcome the lack of standard properties of semi–vector spaces.

Let $U$ and $V$ be semi–free semi–vector spaces.

1.51 Definition. A *semi–tensor product* between $U$ and $V$ is defined to be a semi–vector space $U \hat{\otimes} V$ along with a semi–bilinear map $\hat{\otimes} : U \times V \to U \hat{\otimes} V$, which fulfills the following universal property:

if $W$ is a semi–vector space and $f : U \times V \to W$ a semi–bilinear map, then there exists a unique semi–linear map $\tilde{f} : U \hat{\otimes} V \to W$, such that $f = \tilde{f} \circ \cdot$. \quad \square
1.52 Theorem. The semi–tensor product exists and is unique up to a distinguished semi–linear isomorphism.

Proof. The uniqueness can be proved by a standard procedure as in Theorem 1.30. Then, we have to prove the existence of the semi–tensor product.

For this purpose, we consider the subset \( U \hat{\otimes} V \subset \bar{U} \otimes \bar{V} \) consisting of the semi–linear combinations of elements of the type \((1 \otimes u) \otimes v\), with \(u \in U\) and \(v \in V\), and the map \( \hat{\otimes} : U \times V \rightarrow U \hat{\otimes} V : (u, v) \mapsto (1 \otimes u) \otimes v\).

We can easily see that \( U \hat{\otimes} V \) is a semi–vector space and \( \hat{\otimes} : U \times V \rightarrow U \hat{\otimes} V \) a semi–bilinear map.

Next, we prove that the above objects fulfill the required universal property. Clearly, if the map \( \tilde{f} : U \hat{\otimes} V \rightarrow W \) of the universal property exists, then it is unique because \( U \hat{\otimes} V \) is semi–linearly generated by the image of the map \( \hat{\otimes} \).

Moreover, this map is well defined by the equality \( \tilde{f}((1 \otimes u) \otimes v) = f(u, v) \), according to the following commutative diagram

\[
\begin{array}{ccc}
U \times V & \xrightarrow{i} & \bar{U} \times \bar{V} \\
\hat{\otimes} & \downarrow{f} & \downarrow{\bar{f}} \\
W & \xrightarrow{j} & \bar{W} \\
\end{array}
\]

where the maps \( i, \iota, j \) are natural inclusions and where the map \((\bar{\circ} f) \circ j\) uniquely factorises through a map \( f \). We can easily see that the map \( f \) is semi–linear and that \( \tilde{f} \circ \hat{\otimes} = f \). QED

In an analogous way, we can construct the semi–tensor product via the left sesqui–tensor product (instead of via the right sesqui–tensor product). We can also easily prove that the two constructions are naturally isomorphic.

1.53 Proposition. The semi–tensor product is a semi–free semi–vector space. Moreover, if \( B \) and \( C \) are semi–bases of \( U \) and \( V \), respectively, then

\[ B \hat{\otimes} C := \{ b_i \otimes c_j | b_i \in B, c_j \in C \} \]

is a semi–basis of \( U \hat{\otimes} V \). Moreover, we have

\[ s \text{–dim}(U \hat{\otimes} V) = s \text{–dim } U \cdot s \text{–dim } V. \]

1.54 Note. We have the natural semi–linear isomorphisms

\[ \mathbb{R}^+ \hat{\otimes} U \rightarrow U : r \otimes u \mapsto ru \quad \text{and} \quad U \hat{\otimes} \mathbb{R}^+ \rightarrow U : u \otimes r \mapsto ru. \]

1.55 Note. We have a distinguished semi–linear isomorphism

\[ \overline{U \otimes V} \simeq \bar{U} \otimes \bar{V}. \]

1.56 Proposition. We have the natural semi–linear inclusion

\[ U^* \hat{\otimes} V \hookrightarrow \text{Lin}(U, V), \]
characterised by
\[ \alpha \otimes v : U \to V : u \mapsto \alpha(u)v, \quad \forall \alpha \in U^*, v \in V. \]

Moreover, if the semi–dimensions of \( U \) and \( V \) are finite, then the above inclusion is a semi–linear isomorphism.

**Proof.** It follows easily from the universal property in the standard way. QED

We obtain the contravariant “semi–tensor algebra” of a semi–free semi–vector space in a way analogous to that of vector spaces.

Let \( m \) be a positive integer. If \( U_1, \ldots, U_m \) are semi–free semi–vector spaces, then we can easily define the semi–tensor product \( U_1 \hat{\otimes} \ldots \hat{\otimes} U_m \) and prove its basic properties along the same lines as for vector spaces. In particular, if \( U = U_1 = \cdots = U_m \), then we set \( \hat{\otimes}^m U := U_1 \hat{\otimes} \ldots \hat{\otimes} U_m \) and \( \hat{\otimes}^0 U := \mathbb{R}^+ \). Moreover, the semi–direct sum \( \bigoplus_{m \in \mathbb{N}} U^m \) turns out to be a “semi–algebra” over \( \mathbb{R}^+ \).

## 2 Positive spaces

The positive spaces constitute a distinguished elementary type of semi–vector spaces, which allows us to introduce further notions, such as rational maps and rational powers.

We use these spaces for achieving an algebraic model of scales and units of measurement. In fact, this is one of the main goals of this paper.

### 2.1 The notion of positive space

We start by introducing the positive spaces and their basic properties.

**2.1 Definition.** A **positive space** is defined to be a non–complete semi–free semi–vector space \( U \) of dimension 1.

**2.2 Note.** In other words, a positive space is a semi–vector space generated over \( \mathbb{R}^+ \) by a non vanishing element.

Thus, each positive space \( U \) is semi–linearly isomorphic to \( \mathbb{R}^+ \). More precisely, each semi–linear isomorphism \( U \to \mathbb{R}^+ \) is of the type \( U \to \mathbb{R}^+ : u \mapsto u/b \), where \( b \in U \) and \( r := u/b \in \mathbb{R}^+ \) is the unique positive number such that \( rb = u \).

Clearly, for each \( b \in U \), the subset \( (b) \subseteq U \) turns out to be a semi–basis.

If \( U \) and \( V \) are positive spaces, then each semi–linear map \( f : U \to V \) is an isomorphism. In fact, we can easily see that \( f \) is surjective and injective.

Positive spaces and semi–linear maps constitute a category.

**2.3 Note.** Let \( U \) be a positive space. The scalar multiplication \( s : \mathbb{R}^+ \times U \to U \) turns out to be a free and transitive action of the group \((\mathbb{R}^+,\cdot)\) on the set \( U \).
2.4 Note. The semi–tensor product of positive spaces is a positive space.

In particular, if \( U \) is a positive space, then \( \hat{\otimes}^n U \) is a positive space. Moreover, each element \( t \in \hat{\otimes}^n U \) is decomposable; even more, it can be uniquely written as \( t = u \hat{\otimes} \ldots \hat{\otimes} u \), with \( u \in U \). □

For positive spaces we shall often adopt a notation similar to the standard notation used for numbers. Namely, if \( U \) and \( U' \) are positive spaces, we shall often write \( u u' \equiv u \hat{\otimes} u' \in U \hat{\otimes} U' \), for each \( u \in U \) and \( u' \in U' \).

Moreover, if \( U \) is a positive space and \( u \in U \), then the unique element \( 1/u \in U^* \), such that \( (1/u, u) = 1 \), is called the inverse of \( u \) (not to be confused with the additive inverse). Clearly, for each \( u \in U \) and \( r \in \mathbb{R}^+ \), we have \( \frac{1}{ru} = \frac{1}{r} \frac{1}{u} \). Moreover, \( (1/u) \) is just the dual semi–basis of \( (u) \).

2.2 Rational maps between positive spaces

Next, we discuss the notion of \( q \)–rational maps between positive spaces.

Let us consider two positive space \( U \) and \( V \) and a rational number \( q \in \mathbb{Q} \).

2.5 Definition. A map \( f : U \rightarrow V \) is said to be \( q \)–rational (or, rational of degree \( q \)) if, for each \( u \in U \) and \( r \in \mathbb{R}^+ \), we have \( f(r u) = r^q f(u) \). □

We denote by \( \text{Rat}^q(U, V) \subset \text{Map}(U, V) \) the subspace of \( q \)–rational maps between the positive spaces \( U \) and \( V \).

2.6 Proposition. If \( u \in U \) and \( v \in V \), then there exists a unique \( q \)–rational map \( f : U \rightarrow V \), such that \( f(u) = v \). □

The composition of two rational maps is a rational map, whose degree is the product of the degrees. Hence, positive spaces and rational maps constitute a category.

2.7 Note. Let \( q' \) be another rational number. If \( f : U \rightarrow \mathbb{R}^+ \) is a \( q \)–rational map, then the map \( f^{q'} : U \rightarrow \mathbb{R}^+ : u \mapsto (f(u))^{q'} \) is \( \left( qq' \right) \)–rational. □

2.8 Proposition. The subspace \( \text{Rat}^q(U, V) \subset \text{Map}(U, V) \) turns out to be a semi–vector subspace and a positive space.

Proof. The 1st statement is trivial. Moreover, \( \text{Rat}^q(U, V) \) is a positive space because, for any given \( u \in U \), the map \( \text{Rat}^q(U, V) \rightarrow V : f \mapsto f(u) \) is a semi–linear isomorphism. QED

2.9 Corollary. A \( q \)–rational map \( f : U \rightarrow V \) is a bijection if and only if \( q \neq 0 \); in this case the inverse map is \( (1/q) \)–rational. □

2.10 Example. We have the following distinguished cases.
2.3 Rational powers of a positive space

a) The 0–rational maps \( f : U \to V \) are just the constant maps. Hence, we have the natural semi–linear isomorphism
\[
\text{Rat}^0(U, V) \simeq V : f \mapsto f(u),
\]
which turns out to be independent of the choice of \( u \in U \). In particular, we have \( \text{Rat}^0(U, \mathbb{R}^+) \simeq \mathbb{R}^+ \).

b) The 1–rational maps \( f : U \to V \) are just the semi–linear maps. Hence, we can write
\[
\text{Rat}^1(U, V) = \text{s-Lin}(U, V).
\]
In particular, we have \( \text{Rat}^1(U, \mathbb{R}^+) = \text{s-Lin}(U, \mathbb{R}^+) = U^* \).

c) The \((-1)–\)rational maps \( f : U \to V \) can be identified with the semi–linear maps \( \bar{f} : U^* \to V \), through the natural semi–linear isomorphism
\[
\text{Rat}^{-1}(U, V) \to \text{s-Lin}(U^*, V) : f \mapsto \bar{f},
\]
where \( \bar{f} : U^* \to V \) is the unique semi–linear map such that \( \bar{f}(1/u) = f(u) \), with reference to a chosen element \( u \in U \). Indeed, this isomorphism turns out to be independent of the choice of \( u \in U \).

In particular, the map
\[
\text{inv} : U \to U^* : u \mapsto 1/u,
\]
which associates with each element \( u \in U \) its dual form \( 1/u \in U^* \), is a \((-1)–\)rational map.

Indeed, \( \text{inv} \in \text{Rat}^{-1}(U, U^*) \) is the distinguished element which corresponds to the element \( \text{id}_{U^*} \in \text{s-Lin}(U^*, U^*) \), through the isomorphism \( \text{Rat}^{-1}(U, U) \simeq \text{s-Lin}(U^*, U) \).

We have also the map
\[
\text{inv} : U^* \to U^{**} \simeq U.
\]

2.3 Rational powers of a positive space

Eventually, we introduce the rational powers of a positive space.

The basic idea is quite simple and could be achieved in an elementary way, by referring to a base and showing that the result is independent of this choice.

However, a full understanding of this concept is more subtle than it might appear at first insight and suggests a more sophisticated formal approach.

Let us consider a positive space \( U \) and a rational number \( q \in \mathbb{Q} \).

2.11 Lemma. The map
\[
\pi^q : U \to \text{Rat}^q(U^*, \mathbb{R}^+) : u \mapsto u^q,
\]
where \( u^q \in \text{Rat}^q(U^*, \mathbb{R}^+) \) is the unique element such that \( u^q(1/u) = 1 \), turns out to be \( q–\)rational.
Proof. In fact, we have $1 = u^q(1/u)$ and $1 = (ru)^q(1/(ru)) = (ru)^q(1/r \cdot 1/u) = (1/r)^q (ru)^q(1/u)$.
Hence, we obtain $u^q(1/u) = (1/r)^q (ru)^q(1/u)$, which yields $(ru)^q = r^4 u^q$. QED

2.12 Definition. The $q$–power of $\mathbb{U}$ is defined to be the pair $(\mathbb{U}^q, \pi^q)$ defined by

$$\mathbb{U}^q := \text{Rat}^q(\mathbb{U}^*, \mathbb{R}^+) \quad \text{and} \quad \pi^q : \mathbb{U} \rightarrow \mathbb{U}^q : u \mapsto u^q,$$

where $u^q : \mathbb{U}^* \rightarrow \mathbb{R}^+$ is the unique $q$–rational map such that $u^q(1/u) = 1$. □

We can reinterpret the above notion in a natural way in terms of semi–tensor powers as follows.

2.13 Note. Clearly, for each $r \in \mathbb{R}^+$, the following diagram commutes

$$\begin{array}{ccc}
\mathbb{U} & \longrightarrow & \mathbb{U}^q \\
\downarrow{s_r} & & \downarrow{s_{rq}} \\
\mathbb{U} & \longrightarrow & \mathbb{U}^q,
\end{array}$$

where $s_r$ and $s_{rq}$ denote the scalar multiplications by $r$ and $r^q$.

Thus, the rational power of positive spaces emulates the rational power of positive numbers, according to the above commutative diagram. □

2.14 Note. We have the following distinguished cases.
1) If $q = 0$, then we have a natural semi–linear isomorphism

$$\mathbb{U}^0 := \text{Rat}^0(\mathbb{U}^*, \mathbb{R}^+) \simeq \mathbb{R}^+,$$

and $\pi^0$ turns out to be the constant map with value 1.

2) Let $q \equiv n$ be a positive integer.

Then, we have the natural mutually inverse semi–linear isomorphisms

$$\begin{align*}
\otimes^n \mathbb{U} & \rightarrow \text{Rat}^n(\mathbb{U}^*, \mathbb{R}^+) : u \otimes \ldots \otimes u \mapsto f_u, \\
\text{Rat}^n(\mathbb{U}^*, \mathbb{R}^+) & \rightarrow \otimes^n \mathbb{U} : f \mapsto u_f \otimes \ldots \otimes u_f,
\end{align*}$$

where $f_u : \mathbb{U}^* \rightarrow \mathbb{R}^+ : \omega \mapsto \omega(u) \ldots \omega(u)$ and where $u_f := 1/\omega_f \in \mathbb{U}$ being $\omega_f \in \mathbb{U}^*$ the unique element such that $f(\omega_f) = 1$.

Moreover, according to the above isomorphisms, the map $\pi^n$ is given by

$$\pi^n : \mathbb{U} \rightarrow \text{Rat}^n(\mathbb{U}^*, \mathbb{R}^+) : u \mapsto f_u.$$

In particular, in the case $q \equiv n = 1$ we have the natural semi–linear isomorphism

$$\mathbb{U}^1 := \text{Rat}^1(\mathbb{U}^*, \mathbb{R}^+) = \text{s-Lin}(\mathbb{U}^*, \mathbb{R}^+) := \mathbb{U}^{**} \simeq \mathbb{U}.$$

3) Let $q \equiv 1/n$ be the inverse of a positive integer $n$.
2.3 Rational powers of a positive space

Then, we have the natural mutually inverse semi-linear isomorphisms

\[ \hat{\otimes}^n \operatorname{Rat}^{1/n}(\mathbb{U}^*, \mathbb{R}^+) \to s-\operatorname{Lin}(\mathbb{U}^*, \mathbb{R}^+) := \mathbb{U}^{**} \simeq \mathbb{U} : f \hat{\otimes} \ldots \hat{\otimes} f \mapsto f^n,\]
\[ \mathbb{U} \simeq \mathbb{U}^{**} := s-\operatorname{Lin}(\mathbb{U}^*, \mathbb{R}^+) \to \hat{\otimes}^n \operatorname{Rat}^{1/n}(\mathbb{U}^*, \mathbb{R}^+) : f \mapsto f^{1/n} \hat{\otimes} \ldots \hat{\otimes} f^{1/n},\]

where \( f^n : \mathbb{U}^* \to \mathbb{R}^+ : \omega \mapsto f(\omega) \ldots f(\omega) \) and \( f^{1/n} : \mathbb{U}^* \to \mathbb{R}^+ : \omega \mapsto (f(\omega))^{1/n} \).

Moreover, according to the above isomorphisms, the map \( \pi^{1/n} \) is given by

\[ \pi^{1/n} : \mathbb{U} \simeq \mathbb{U}^{**} \to \mathbb{U}^{1/n} := \operatorname{Rat}^{1/n}(\mathbb{U}^*, \mathbb{R}^+) : f \mapsto f^{1/n}.\]

4) Let \( q \equiv -n \) be a negative integer.

Then, we have the natural mutually inverse semi-linear isomorphisms

\[ \hat{\otimes}^n \mathbb{U}^* \to \operatorname{Rat}^{-n}(\mathbb{U}^*, \mathbb{R}^+) : \omega \hat{\otimes} \ldots \hat{\otimes} \omega \mapsto f_\omega,\]
\[ \operatorname{Rat}^{-n}(\mathbb{U}^*, \mathbb{R}^+) \to \hat{\otimes}^n \mathbb{U}^* : f \mapsto \omega_f \hat{\otimes} \ldots \hat{\otimes} \omega_f,\]

where \( f_\omega : \mathbb{U}^* \to \mathbb{R}^+ : \alpha \mapsto \omega(1/\alpha) \ldots \omega(1/\alpha) \) and \( \omega_f \in \mathbb{U}^* \) is the unique element such that \( f(\omega_f) = 1 \).

Moreover, according to the above isomorphisms, the map \( \pi^{-n} \) is given by

\[ \pi^{-n} : \mathbb{U} \to \operatorname{Rat}^{-n}(\mathbb{U}^*, \mathbb{R}^+) : u \mapsto f_{1/u}.\]

In particular, in the case \( q = -1 \), we have the natural semi-linear isomorphism

\[ \mathbb{U}^{-1} := \operatorname{Rat}^{-1}(\mathbb{U}^*, \mathbb{R}^+) \simeq s-\operatorname{Lin}(\mathbb{U}^{**}, \mathbb{R}^+) \simeq s-\operatorname{Lin}(\mathbb{U}, \mathbb{R}^+) := \mathbb{U}^*. \]

Next, we analyse the natural behaviour of the exponents of rational powers. Indeed, this behaviour is just what we expect and is analogous to that of powers of positive real numbers. We leave to the reader the easy proofs of the following Propositions.

2.15 Proposition. Let \( p \) and \( q \) be rational numbers. Then, we obtain the natural semi-bilinear map

\[ b : \operatorname{Rat}^p(\mathbb{U}^*, \mathbb{R}^+) \times \operatorname{Rat}^q(\mathbb{U}^*, \mathbb{R}^+) \to \operatorname{Rat}^{p+q}(\mathbb{U}^*, \mathbb{R}^+) : (f, g) \mapsto fg,\]

which yields the unique semi-linear isomorphism \( \tilde{b} : \mathbb{U}^p \hat{\otimes} \mathbb{U}^q \to \mathbb{U}^{p+q} \), such that \( \pi^{p+q} = \tilde{b} \circ (\pi^p \hat{\otimes} \pi^q) \), in virtue of the universal property of the semi-tensor product. ❑

2.16 Proposition. If \( p \) and \( q \) are rational numbers, then we have the natural semi-linear isomorphism

\[ c : (\mathbb{U}^p)^q := \operatorname{Rat}^q \left( \operatorname{Rat}^p(\mathbb{U}^*, \mathbb{R}^+), \mathbb{R}^+ \right) \to \mathbb{U}^{p+q} := \operatorname{Rat}^{p+q}(\mathbb{U}^*, \mathbb{R}^+) : f \mapsto gf,\]

where \( gf : \mathbb{U}^* \to \mathbb{R}^+ : 1/u \mapsto f(1/u^p) \). Moreover, we have \( \pi^{pq} = c \circ (\pi^q \circ \pi^p) \). ❑
2.17 Corollary. If \( q \) is a rational number, then
\[
(U^q)^* \simeq (U^*)^q.
\]
If \( p < q \) are two positive integers, then
\[
\hat{\otimes} q U \hat{\otimes} (\hat{\otimes} p U^*) \simeq U^q \hat{\otimes} U^{-p} = U^{q-p} \simeq \hat{\otimes} q-p U
\]
\[
\hat{\otimes} p U \hat{\otimes} (\hat{\otimes} q U^*) \simeq U^p \hat{\otimes} U^{-q} = U^{p-q} \simeq \hat{\otimes} p-q U^*.
\]

3 Algebraic model of physical scales

Next, we discuss the physical model of scales and units of measurement through positive spaces. Thus, we introduce the fundamental scale spaces and related notions, including scale dimension, scale basis and coupling scales. Finally, we show how unit spaces can be employed in physical theories by the language of differential geometry.

The formalism discussed in this section has been widely used in several papers dealing with physical theories (see, for instance, [4, 7, 8, 9, 13, 14, 15]). In the present paper we analyse the mathematical foundations of this formalism for the first time. We also discuss the interplay of our theory with dimensional analysis.

3.1 Units and scales

We introduce the fundamental scale spaces and related notions.

In this paper, we shall be concerned just with scales derived from time, length and mass scales via rational powers. Of course, the treatment could be extended to other types of systems in an analogous way.

We assume the following positive spaces as basic spaces of scales:
(1) the space \( \mathbb{T} \) of time scales,
(2) the space \( \mathbb{L} \) of length scales,
(3) the space \( \mathbb{M} \) of mass scales.

The elements of the above spaces are called basic scales. More precisely,
(1) each element \( u_0 \in \mathbb{T} \) is said to be a time scale,
(2) each element \( l \in \mathbb{L} \) is said to be a length scale,
(3) each element \( m \in \mathbb{M} \) is said to be a mass scale.

For each time scale \( u_0 \in \mathbb{T} \), we denote its dual by \( u^0 := 1/u_0 \in \mathbb{T}^* \).

3.1 Definition. A scale space is defined to be a positive space of the type

\[
\mathbb{S} \equiv \mathbb{S}[d_1, d_2, d_3] := \mathbb{T}^{d_1} \hat{\otimes} \mathbb{L}^{d_2} \hat{\otimes} \mathbb{M}^{d_2}, \quad \text{where} \quad d_i \in \mathbb{Q}.
\]

A scale is defined to be an element \( k \in \mathbb{S} \).
3.1 Units and scales

A scale \( k \in \mathbb{S} \), regarded as a semi–basis of the scale space \( \mathbb{S} \), is called a unit of measurement.

For each scale space \( \mathbb{S} = \mathbb{T}^d_1 \hat{\otimes} \mathbb{L}^d_2 \hat{\otimes} \mathbb{M}^d_3 \) and for each scale \( k \in \mathbb{S} \), we set

\[
|\mathbb{S}| \equiv (|\mathbb{S}|_1, |\mathbb{S}|_2, |\mathbb{S}|_3) := (d_1, d_2, d_3)
\]

\[
|k| \equiv (|k|_1, |k|_2, |k|_3) := (d_1, d_2, d_3).
\]

The above 3-plet \((d_1, d_2, d_3)\) of rational numbers is called the scale dimension of \( \mathbb{S} \) and of \( k \).

For instance, we have \( \mathbb{S}[d_1, d_2, 0] := \mathbb{T}^d_1 \hat{\otimes} \mathbb{L}^d_2 \), \( \mathbb{S}[d_1, 0, 0] := \mathbb{T}^d_1 \), \( \mathbb{S}[0, 0, 0] := \mathbb{R}^+ \), and the other similar examples.

We stress that the scale dimension should not be confused with the semi–dimension: indeed all these semi–vector spaces have semi–dimension 1.

3.2 Note. The scale dimension \(|k|\) of a scale \( k \) determines the corresponding scale space \( \mathbb{S} \). In other words, for two scales \( k \) and \( k' \), we have \(|k| = |k'|\) if and only if the two scales belong to the same scale space. If this is the case, then \( k = rk' \), where \( r = k/k' \). Hence, the scale dimension \(|k|\) of a scale \( k \) determines \( k \) up to a positive real factor.

The map \( k \mapsto |k| \) fulfills the following properties, for each \( k \in \mathbb{S} \), \( k' \in \mathbb{S}' \), \( r \in \mathbb{R}^+ \) and \( q \in \mathbb{Q} \),

\[
|rk| = |k|, \quad |1/k| = -|k|, \quad |k \hat{\otimes} k'| = |k| + |k'|, \quad |k|^q = q|k|.
\]

3.3 Definition. A 3–plet of scales \((e_1, e_2, e_3)\) is said to be a scale basis if each scale \( k \) can be written in a unique way as

\[
k = r (e_1)^{c_1} \hat{\otimes} (e_2)^{c_2} \hat{\otimes} (e_3)^{c_3}, \quad \text{with} \quad r \in \mathbb{R}^+, \ c_i \in \mathbb{Q}.\]

3.4 Proposition. A 3–plet of scales \((e_1, e_2, e_3)\) is a scale basis if and only if

\[
\text{det}(|e_j|_i) \neq 0.
\]

Moreover, let \((e_1, e_2, e_3)\) be a scale basis and \( k \) a scale. Then, the 3–plet of rational exponents \((c_1, c_2, c_3)\) is the unique solution of the linear rational system

\[
|k|_i = \sum_j |e_j|_i c_j.
\]

Proof. Let us consider a 3–plet of scales \((e_1, e_2, e_3)\) and a scale \( k \). Then,

\[
k = r (e_1)^{c_1} \hat{\otimes} (e_2)^{c_2} \hat{\otimes} (e_3)^{c_3} \quad \Leftrightarrow \quad |k|_i = \sum_j |e_j|_i c_j.
\]

Hence, the above left hand side expression holds and is unique if and only if \( \text{det}(|e_j|_i) \neq 0 \). QED
3.5 Example. Clearly, each 3–plet of the type
\[(u_0, l, m) \in T \otimes L \otimes M\]
is a scale basis. More generally, each 3–plet of the type
\[(u_0^{d_1}, l^{d_2}, m^{d_3}) \in T^{d_1} \otimes L^{d_2} \otimes M^{d_3}, \quad \text{with} \quad d_i \in \mathbb{Q} - \{0\},\]
is a scale basis. □

Of course, we can also consider variable scales. Indeed, given a manifold \(M\), we define a scale of \(M\) to be a map of the type \(k : M \to \mathbb{S}\).

3.2 Scaled objects

In geometric models of physical theories one is often concerned with vector bundle valued maps which have physical dimensions. Our theory of positive spaces allows us to keep into account this fact in a formal algebraic way. In fact, we consider maps with values in vector bundles tensorialised with positive spaces. The positive factors can be treated as numerical constants, with respect to differential operators.

Let \(U\) be a positive space. We observe that \(U\) has a natural structure of 1-dimensional manifold. Moreover, it is easy to prove that the tangent space \(TU\) is naturally isomorphic to a cartesian product. More precisely
\[TU \simeq U \times \overline{U}.\]

Now, let us consider a scale space \(S\), two vector bundles \(p : F \to B\) and \(q : G \to B\) and a manifold \(M\).

We can easily define the sesqui–tensor product bundle \((U \hat{\otimes} F) \to B\). We can regard this vector bundle as the sesqui–tensor product over \(B\) of the trivial semi–vector bundle \(\overline{U} := (B \times U) \to B\) and the vector bundle \(F \to B\).

3.6 Definition. The bundle \((\mathbb{S} \hat{\otimes} F) \to B\) and its sections \(s : B \to \mathbb{S} \hat{\otimes} F\) are said to be scaled. Moreover, the bundle \(\mathbb{S} \hat{\otimes} F\), its sections \(s \in \text{sec}(B, \mathbb{S} \hat{\otimes} F)\), and the linear differential operator
\[\phi : \text{sec}(B, G) \to \text{sec}(B, \mathbb{S} \hat{\otimes} F)\]
are said to have scale dimension
\[|\mathbb{S} \hat{\otimes} F| = |s| = |\phi| = |\mathbb{S}| . \Box\]

3.7 Note. Let \(D : \text{sec}(B, F) \to \text{sec}(B, G)\) be a linear differential operator. Then, we obtain the linear differential operator (defined by the same symbol)
\[D : \text{sec}(B, \mathbb{S} \hat{\otimes} F) \to \text{sec}(B, \mathbb{S} \hat{\otimes} G) : s \mapsto Ds := u \hat{\otimes} D(\alpha, 1/u),\]
where \( u \in \mathbb{S} \) and \( \langle \alpha, 1/u \rangle \in \text{sec}(B, F) \). Of course, this definition does not depend on the choice of \( u \). \( \Box \)

The above construction applies, for instance, to the cases when \( D \) is the exterior differential, a Lie derivative, a covariant derivative, and so on.

**3.8 Example.** If \( \alpha \in \text{sec}(M, \mathbb{S} \otimes \Lambda^r T^* M) \) is a scaled form, then we obtain the “scaled exterior differential”

\[
d \alpha := u \hat{\otimes} d \alpha' \in \text{sec}(M, \mathbb{S} \otimes \Lambda^{r+1} T^* M),
\]

where \( u \in \mathbb{S} \) and \( \alpha' := \langle \alpha, 1/u \rangle \in \text{sec}(M, \Lambda^r T^* M) \). \( \Box \)

**3.9 Example.** If \( t \in \text{sec}(M, \otimes TM) \) is a form and \( X \in \text{sec}(M, \mathbb{S} \otimes TM) \) a scaled vector field. Then, we obtain the “scaled Lie derivative”

\[
L_X t := u \hat{\otimes} L_X t' \in \text{sec}(M, \mathbb{S} \hat{\otimes} (\otimes' TM)),
\]

where \( u \in \mathbb{S} \) and \( X' \) is the vector field \( X' := \langle X, 1/u \rangle \in \text{sec}(M, TM) \). \( \Box \)

**3.10 Example.** If \( c \) is a linear connection of the vector bundle \( F \to B \), \( X \in \text{sec}(B, TB) \) a vector field and \( s \in \text{sec}(B, \mathbb{S} \otimes F) \) a section. Then, we obtain the “scaled covariant derivative”

\[
\nabla_X s := u \hat{\otimes} \nabla_X s' \in \text{sec}(B, \mathbb{S} \hat{\otimes} V F),
\]

where \( u \in \mathbb{U} \) and \( s' \) is the section \( s' := \langle s, 1/u \rangle \in \text{sec}(B, F) \).

We can re–interpret the above result in the following way.

The trivial linear connection of \( (B \times \mathbb{U}) \to B \) and the linear connection \( c \) of \( F \to B \) yield a linear connection \( c' \) of \( (\mathbb{U} \hat{\otimes} F) \to B \), which has the same symbols of \( c \). The scaled covariant derivative can be regarded as the covariant derivative with respect to the above product connection.

By abuse of language, we shall denote by \( c \) also the product connection \( c' \). \( \Box \)

### 3.3 Distinguished scales

In this section, we discuss the algebraic model of distinguished scales occurring in physics.

Let us consider a vector bundle \( F \to B \). Suppose that in a physical theory we meet two scaled sections with different scale factors

\[
s : M \to \mathbb{S} \hat{\otimes} F \quad \text{and} \quad s' : M \to \mathbb{S}' \hat{\otimes} F.
\]

Then, we can compare the two scales and write \( s = k \hat{\otimes} s' \), provided we avail of a scale factor \( k : B \to \mathbb{S} \hat{\otimes} \mathbb{S}'^* \), whose scale dimension is \( |k| = |s| - |s'|\). We call such a factor a coupling scale (or, according to the traditional use, a coupling constant).
Some coupling scales, such as, for instance, the speed of the light, the Planck constant, the gravitational constant and the positron charge have a fixed value, without reference to specific systems. For this reason, we shall call these coupling scales \textit{universal}.

Other types of coupling scales, such as, for instance, masses and charges, arise, case by case, and are associated with different particles.

For instance, we have the following universal coupling scales:

1) the \textit{speed of the light} $c \in T^{-1} \otimes L$,
2) the \textit{Planck constant} $\hbar \in T^{-1} \otimes L^2 \otimes M$,
3) the \textit{gravitational constant} $g \in T^{-2} \otimes L^3 \otimes M^{-1}$,
4) the \textit{positron charge} $e \in L^{3/2} \otimes M^{1/2}$.

Besides the above universal coupling scales, there are the following scales which depend on the choice of a particle:

1) a \textit{mass} $m \in M$,
2) a \textit{charge} $q \in T^{-1} \otimes L^{3/2} \otimes M^{1/2}$. We stress that a charge is a scale tensorialised with real numbers; hence, a charge might be positive, vanishing, or negative.

\subsection*{3.11 Note.} The following 3–plets are scale bases (for \(q \neq 0\)):

1) \((e_1, e_2, e_3) := (m, q, \hbar)\),
2) \((e_1, e_2, e_3) := (m, \hbar, g)\),
3) \((e_1, e_2, e_3) := (q, \hbar, g)\).

Conversely, the following 3–plet is not a scale basis (for \(q \neq 0\)):

4) \((e_1, e_2, e_3) := (m, q, g)\).

In fact, we have the following values of determinants in the above cases, respectively:

1) $\det(|e_j|_i) = -1/2$,
2) $\det(|e_j|_i) = 1$,
3) $\det(|e_j|_i) = 1$,
4) $\det(|e_j|_i) = 0$.

Note that $|g| = |q^2/m^2|$.

It may be algebraically correct, but not physically reasonable to express certain scales by means of some of the above scale bases. For instance, it may not be physically reasonable to express the gravitational coupling scale $g$ through the scale basis $(m, q, \hbar)$, because $g$ is a universal coupling scale, while $m$ and $q$ depend on the choice of a specific particle.

\subsection*{3.4 Interplay with dimensional analysis}

The \textit{dimensional analysis} (here, we use \cite{2} as a reference) is the branch of mathematical physics which studies the properties of physical models which depend on units of measurement.

Many of the foundational ideas of dimensional analysis become very natural facts in our algebraic theory of the units of measurement. Below, we list some of these facts.
A class of systems of units [2, p. 14] is, in our language, the choice of basic spaces of scales.

The dimension of a physical quantity [2, p. 16] is what we call the scale dimension. The dimension function of a physical quantity is the expression of the physical quantity with respect to a given scale basis. Such an expression is always a rational function (provided that the quantity depends only on the chosen basic spaces of scales).

It can be proved [2, p. 17] that the dimension function is always a power-law monomial. This justifies our algebraic setting: we obtain a rigorous formulation of these powers via tensor products and semi–linear duality. On the other hand, polynomials or power series would require additional constructions which seem not to be justified in view of the above property of the dimension function. Indeed, when in physics formulae containing power series occur they always involve real numbers, i.e. unscaled quantities (usually called “pure numbers”), obtained as ratio of two scales belonging to the same positive space. Often one of the two scales plays the role of a variable and the other one is regarded as a fixed distinguished scale.

The independence of dimensions for some quantities [2, p. 20] is just the property of those quantities of being a scale basis.

Any function that defines some relationship between quantities is homogeneous; this is a proposition from [2, p. 24]. It is a natural consequence of our setting that functions between scale spaces are rational. This property leads to the Π-theorem of dimensional analysis [2].

Summarising, after realising that physical quantities transform with a power-law monomial, it is natural to implement scales in physical models as rational tensor powers, and maps between them as rational maps.

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