A NEW COMPLEX REFLECTION GROUP IN $PU(9,1)$ AND THE BARNES-WALL LATTICE

TATHAGATA BASAK

Abstract. We show that the projectivized complex reflection group $\Gamma$ of the unique $(1+i)$-modular Hermitian $\mathbb{Z}[i]$-module of signature $(9,1)$ is a new arithmetic reflection group in $PU(9,1)$. We find 32 complex reflections of order four generating $\Gamma$. The mirrors of these 32 reflections form the vertices of a sort of Coxeter-Dynkin diagram $D$ for $\Gamma$ that encode Coxeter-type generators and relations for $\Gamma$. The vertices of $D$ can be indexed by sixteen points and sixteen affine hyperplanes in $\mathbb{P}^2$. The edges of $D$ are determined by the finite geometry of these points and hyperplanes. The group of automorphisms of the diagram $D$ is $2^4 : (2^3 : L_3(2)) : 2$. This group transitively permutes the 32 mirrors of generating reflections and fixes an unique point $\tau$ in $CH^9$. These 32 mirrors are precisely the mirrors closest to $\tau$. These results are strikingly similar to the results satisfied by the complex hyperbolic reflection group at the center of Allcock’s monstrous proposal.

1. Introduction

Let $G = \mathbb{Z}[i]$ be the ring of Gaussian integers. Let $p = (1+i)$. We study the projectivized complex reflection group $\Gamma = \Gamma_1$ of the unique $p$-modular Hermitian $G$-lattice of signature $(9,1)$. In particular, we show that $\Gamma_1$ is arithmetic and we find nice generators and relations for $\Gamma_1$ (The notation $\Gamma_1$ is only used in the introduction. Afterwards we shall simply write $\Gamma$ instead of $\Gamma_1$). We mention three reasons for our interest in $\Gamma_1$:

- There are only few known examples of lattices in $PU(n,1)$ generated by complex reflections when $n > 3$ and these are all arithmetic. The two largest values of $n$ for which an example is known are 13 and 9. Sources for these examples are Deligne-Mostow [DM, M1, M2], Thurston [T] and Allcock [A1, A2]. The “largest” example found in [M2] and [T] are identical; it is an arithmetic lattice in $PU(9,1)$. This lattice is denoted by $\Gamma_5$ later in this introduction. A single example in dimension thirteen ($\Gamma_2$ in our notation) was found in [A2]. The group $\Gamma_1$ of this article clearly gives a new example in $PU(9,1)$.

- Allcock’s monstrous proposal conjecture states that the fundamental group of the ball quotient constructed from $\Gamma_2$ maps onto $(M \wr 2)$ where $M$ is the monster simple group. The arithmetic lattice $\Gamma_2 \subseteq PU(13,1)$ plays a central role in the monstrous proposal; see [A3, Ba2, AB1, AB2]. Our results for $\Gamma_1$ have striking similarity with results for $\Gamma_2$ obtained in [Ba2].

Date: April 12, 2018.

2010 Mathematics Subject Classification. Primary: 11H56, 20F55; Secondary: 20F05, 51M10.

Key words and phrases. complex reflection group, hyperbolic reflection group, arithmetic lattices, Barnes-Wall lattice.
\( \diamond \) The results for \( \Gamma_1 \) and \( \Gamma_2 \) (and a few other lattices in \( PU(n, 1) \)) tie into a general pattern of phenomenon that have analogy in the theory of Weyl groups; see Theorem 1.1. Understanding this analogy may be useful in finding more interesting examples and in studying complex reflection groups in general.

Before discussing \( \Gamma_1 \) in more detail, we want to describe this general pattern of phenomenon. Let \( K \) denote one of three real division algebras \( \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \). Let \( V \) be a \( K \)-module with a non-degenerate \( K \)-valued Hermitian form which is either positive definite or Lorentzian (i.e. of signature \( (n, 1) \)). Let \( G \) denote the real Lie group of isometries of \( V \). Let \( \Gamma \) be a discrete subgroup of \( G \) generated by real, complex or quaternionic reflections. Let \( X \) be the projective space over \( K \) if \( V \) is positive definite and let \( X \) be the hyperbolic space over \( K \) if \( V \) is Lorentzian. There is a natural \( G \)-invariant metric on the symmetric space \( X \). The discrete group \( \Gamma \) acts properly discontinuously on \( X \) by isometries. The fixed points of a reflection \( s \in \Gamma \) is a totally geodesic \( K \)-hypersurface in \( X \) called the mirror of \( s \). Nice generators and relations for \( \Gamma \) can often be found as follows: Choose a suitable point \( \tau \) in \( X \) that is a point of symmetry of the mirrors of \( \Gamma \) in an appropriate sense. Let \( S \) be the set of reflections in \( \Gamma \) whose mirrors are closest to \( \tau \). In many examples, the reflections in \( S \) generate \( \Gamma \) and these reflections satisfy nice Coxeter-type relations.

First, we give some well known examples:

\( \diamond \) Let \( \Gamma \) be a Weyl group of \( A-D-E \) type acting on the real vector space \( V \) by its natural reflection representation. Let \( \tau \) be the line in \( V \) containing the Weyl vector. Then the mirrors of the simple reflections are exactly the mirrors in \( P(V) \) closest to \( \tau \). So in this case \( S \) is just the set of simple mirrors. In analogy with this classical case, in all the examples described below, the reflections in \( S \) will be called simple reflections and the corresponding mirrors will be called simple mirrors.

\( \diamond \) Let \( \Gamma \) be an irreducible finite complex reflection group acting on \( V = \mathbb{C}^n \). For most \( \Gamma \) including the infinite family \( G(de, e, n) \), there exists a point \( \tau \) in \( P(V) \) such that the mirrors closest to \( \tau \) generate \( \Gamma \) (see [Ba4], section 3.10 and section 3.11, remark (5)). For the infinite family \( G(de, e, n) \) one can choose \( \tau \) such that \( S \) is the standard set of generators, given, for example in [BMR]. However, for some exceptional \( \Gamma \), one can choose a \( \tau \) that are analogous to a Weyl vector (in a certain sense, explained in [Ba4]) and that yield a set of generators \( S \) different from the standard ones given in [BMR].

\( \diamond \) The reflection group \( \Gamma \) of \( \Pi_{25,1} \) (the unique even self-dual \( \mathbb{Z} \)-lattice of signature \( (25, 1) \)) acts on the real hyperbolic space \( X = \mathbb{R}H^{25} \). Chooses \( \tau \) to be a “Leech cusp”, which means that \( \tau \) is a line containing a primitive norm zero vector \( \tau \), such that \( \tau^\perp / \tau \) is isomorphic to the Leech lattice. Here we are stretching our discussion a little bit since \( \tau \) is not really a point in \( \mathbb{R}H^{25} \) but a point on its boundary. The mirrors closest to \( \tau \) in horocyclic distance are parametrized by the vectors of the Leech lattice (modulo \( \pm 1 \)). So they are called Leech mirrors. The reflections in the Leech mirrors generate \( \Gamma \). These generating reflections obey Coxeter relations governed by the Leech lattice, leading to Conway’s observation: “the Leech lattice is the Dynkin diagram of the reflection group of \( \Pi_{25,1} \)” [C].

Counting the example studied in this article, we now have at least six examples of complex and quaternionic hyperbolic reflection groups, where similar results hold.
To state these results in an uniform manner, we need some notation. Let \( \mathcal{O} \), \( \mathcal{E} \) and \( \mathcal{H} \) denote the ring of Gaussian integers, the ring of Eisenstein integers and the quaternionic ring of Hurwitz integers respectively. Let \( \mathcal{O} \) be one of these three rings. Let \( l \) be a nonzero prime in \( \mathcal{O} \) of smallest possible norm. If \( \mathcal{O} = \mathcal{E} \) or \( \mathcal{O} = \mathcal{H} \), we may choose \( l = p = (1 + i) \). If \( \mathcal{O} = \mathcal{E} \), we may choose \( l = \sqrt{3} \). Let \( \mathcal{O}_{n,1} \) denote a \( l \)-modular Hermitian \( \mathcal{O} \)-lattice of signature \((n, 1)\), if such a lattice exists. In particular, we define \( \mathcal{O}_{1,1} = \mathcal{O}_{1,1} \oplus \mathcal{O}_{e_2} \) where \( e_1^2 = e_2^2 = 0 \) and \( \langle e_1 | e_2 \rangle = l \). The lattice \( \mathcal{O}_{1,1} \) is the unique \( l \)-modular Hermitian \( \mathcal{O} \)-lattice of signature \((1, 1)\) and we call it the hyperbolic cell. We shall consider the following six lattices:

\[
L_1 = \mathcal{O}_{3,1}, \quad L_2 = \mathcal{E}_{13,1}, \quad L_3 = \mathcal{H}_{7,1}, \quad L_4 = \mathcal{O}_{5,1}, \quad L_5 = \mathcal{E}_{9,1}, \quad L_6 = \mathcal{H}_{5,1}.
\]

In each case \( L_j \) is the unique \( l \)-modular Hermitian \( \mathcal{O} \)-lattice in its given signature.

Note that \( L_{3+j} \) is a sub-lattice of \( L_j \). Let \( R(L_j) \) be the (complex or quaternionic) reflection group of \( L_j \) and let \( \Gamma_j = \text{PR}(L_j) \) be the image of \( R(L_j) \) in \( PU(n, 1) \). The mirrors of \( \Gamma_j \) are determined by the orthogonal complements of vectors of minimal positive norm in \( L_j \). Since \( L_j \) is indefinite, there are infinitely many mirrors. If \( \mathcal{O} = \mathcal{E} \), then the reflection group \( \Gamma_j \) contains order 3 reflections around the mirrors. If \( \mathcal{O} = \mathcal{G} \) or \( \mathcal{H} \), then \( \Gamma_j \) contains order 4 and order 2 reflections around the mirrors. The projectivized reflection group \( \Gamma_j \) acts faithfully on the complex or quaternionic hyperbolic space \( X_j \) of appropriate dimension. The following results hold:

1.1. Theorem. (a) \( \Gamma_j \) has finite index in \( P\text{Aut}(L_j) \). So \( \Gamma_j \) is arithmetic.

(b) There is a point \( \tau_j \) in \( X_j \) such that the set of reflections \( S_j \) in the mirrors closest to \( \tau_j \) (i.e. the simple mirrors) generate \( \Gamma_j \). Further, \( P\text{Aut}(L_j) \) has a finite subgroup \( Q_j \) that acts transitively on the simple mirrors and \( \tau_j \) is the unique point of \( X_j \) fixed by \( Q_j \).

(c) The Coxeter relations between the simple reflections \( S_j \) are encoded by the edges of a diagram \( D_j \) which we call the Dynkin diagram of \( \Gamma_j \). The vertices of \( D_j \) correspond to the simple reflections. The diagram \( D_2 \) (resp. \( D_3 \)) is the incidence graph of \( PU(2, \mathbb{F}_3) \) (resp. \( PU(2, \mathbb{F}_2) \)). The diagram \( D_1 \) has 32 vertices and is defined by the incidence relations of 16 points and sixteen hyperplanes in \( \mathbb{F}_2^4 \). A precise description of \( D_1 \) is given later in this introduction. Finally, \( D_{3+j} \) is a maximal circuit in \( D_j \).

In each case, \( Q_j \) is roughly the automorphism group of the diagram \( D_j \). Theorem 1.1 is the summary of results from a few articles. For \( \Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6 \), part (a) is due to Allcock [A1, A2]. For \( \Gamma_4 \), theorem 1.1 is due to Goertz [Goe] (unpublished). The rest of the results are due to the author. In this paper, we prove Theorem 1.1 for \( \Gamma_1 \). Theorem 1.1 for \( \Gamma_2, \Gamma_3, \Gamma_5 \) follow from the results in [Ba2], [Ba3] and section 4.1 of [Ba1] respectively. The generators and relations for \( \Gamma_2 \) encoded in the diagram \( D_2 \) form the basis for Allcock’s monstrous proposal conjecture [A3]. One of our motivation for studying \( \Gamma_1 \) in detail is the close similarity between \( \Gamma_1 \) and \( \Gamma_2 \) and our interest in \( \Gamma_2 \) stemming from the monstrous proposal conjecture. The proofs of part (b), (c) for \( \Gamma_6 \) have not been written up. However the proofs for \( \Gamma_{3+j} \) are entirely similar to the proofs for \( \Gamma_j \) and easier. A detailed study of the references mentioned and this paper reveal many more similarities between these reflection groups.

\footnote{This means \( \mathcal{O}_{n,1} \) is a free (right) \( \mathcal{O} \)-module of rank \((n + 1)\) with a \( \mathcal{O} \)-valued Hermitian form \( \langle \cdot | \cdot \rangle : \mathcal{O}_{n,1} \times \mathcal{O}_{n,1} \to \mathcal{O} \) of signature \((n, 1)\), and \( l^{-1} \mathcal{O}_{n,1} \) is equal to the dual lattice of \( \mathcal{O}_{n,1} \).}
For the rest of the introduction, we shall focus on \( \Gamma_1 \) and give some more details. To maintain notational consistency with the references cited above, we shall drop the subscript and write \( L = L_1 \), \( \tau = \tau_1 \) and so on. We work over the ring \( \mathcal{G} = \mathbb{Z}[i] \).

Let \( p = (1 + i) \). Our objective is to study the complex reflection group of the unique \( p \)-modular \( \mathcal{G} \)-lattice \( L = \mathcal{G}_{9,1} \) of signature \((9, 1)\). One has

\[
L \simeq 4D_4^T \oplus \mathcal{G}_{1,1} \simeq \text{BW}_{16}^T \oplus \mathcal{G}_{1,1}.
\]

where \( D_4^T \) and \( \text{BW}_{16}^T \) are the Gaussian forms of the \( D_4 \) root lattice and the Barnes-Wall lattice respectively. The projective reflection group \( \Gamma \) is a discrete subgroup of \( PU(9, 1) \) and acts faithfully by isometries on the complex hyperbolic space

\[
\mathbb{B}(L) = \{ C v : v \in L \otimes_{\mathbb{G}} \mathbb{C}, \langle v | v \rangle < 0 \} \simeq \mathbb{C} H^9.
\]

By definition, \( R(L) \) is generated by \( i \)-reflections (order 4 complex reflections) in the norm 2 vectors of \( L \). By definition, \( \Gamma = PR(L) \) is the image of \( R(L) \) in \( PU(9, 1) \).

1.2. **Theorem.** (a) (See \[4.4\]) \( \Gamma \) has finite index in \( \text{PAut}(L) \). So \( \Gamma \) is arithmetic.

(b) (See \[5.11, 5.9\]) \( \Gamma \) is generated by thirteen \( i \)-reflections satisfying the Coxeter relations of the diagram \( X_{3333} \) shown in figure \[4\].

(c) (See \[5.2, 5.6, 5.5\]) The above generating set of thirteen \( i \)-reflections can be extended to a set of thirty-two \( i \)-reflections whose mirrors are equidistant from a point \( \tau \) in \( \mathbb{B}(L) \). These 32 mirrors are precisely the mirrors closest to \( \tau \). A subgroup \( Q \) of \( \text{PAut}(L) \) isomorphic to \((2^4 : (2^3 : L_3(2))) : 2 \) (in \[ATLAS\] notation) acts transitively on the 32 mirrors and \( \tau \) is the unique point in \( \mathbb{B}(L) \) fixed by \( Q \).

The configuration of the 32 mirrors closest to \( \tau \) has appealing symmetry related to the geometry of the finite vector space \( \mathbb{F}_2^3 \). To describe this, fix a point \( a \in \mathbb{F}_2^3 \).

For \( u \in \mathbb{F}_2^3 \), let \( t_u : \mathbb{F}_2^3 \to \mathbb{F}_2^3 \) be the translation \( t_u(v) = u + v \). Let \( K_0 \) be the set of hyperplanes in \( \mathbb{F}_2^3 \) that do not contain \( a \). Let \( K \) be the set of translates of the hyperplanes in \( K_0 \). So \( K \) consists of 8 homogeneous and 8 affine hyperplanes in \( \mathbb{F}_2^3 \).

Let \( D = \mathbb{F}_2^3 \cup K \). Note that each \( t_u \) permutes \( D \) and permutes \( K \) and thus defines a permutation of \( D \). The thirty-two \( i \)-reflections closest to \( \tau \) can be labeled by \( D \) such that the relations among these \( i \)-reflections are dictated by the configuration \( D \).

More precisely, let \( d, d' \in D \) and let \( R, R' \) be the corresponding \( i \)-reflections.

- If \( \{d, d'\} \) is an incident pair of point and hyperplane, then \( RR'R = R'R'R' \).
- This is denoted in the diagram \( D \) (fig. \[1\]) by a solid edge joining \( d \) to \( d' \).
- If \( d' = t_a(d) \), then \( RR'R = R'R'R' \). This is denoted in the diagram \( D \) by a dotted edge joining \( d \) to \( d' \).
- Otherwise, \( RR'R = R'R \).

We picture \( D \) as a graph with two kinds of edges. The subgroup \( 2^3 : L_3(2) \) in \( Q \) is the stabilizer of \( a \) in \( L_4(2) \), the \( 4^3 \) corresponds to the translation action of \( \mathbb{Z}_2^4 \) on itself, and the extra \( \mathbb{Z}_2/2 \) is a symmetry that interchanges the points in \( \mathbb{F}_2^4 \) and hyperplanes in \( K \) (see \[5.5\] for details). The group \( Q \) acts on the set \( D \) preserving both kind of edges. We may think of \( D \) as the Dynkin diagram for \( R(L) \) and \( Q \) as the group of diagram automorphisms.

The proof showing that \( \Gamma = PR(\mathcal{G}_{9,1}) \) is arithmetic is similar to the proof for \( \Gamma_2 = PR(E_{13,1}) \) in \[A1\] which in turn is adapted from an argument in \[C\]. The statements and proofs in this article often closely parallel those in \[A1, Bn2, Bn3\]. We shall refrain from mentioning them at every step, but a couple of remarks comparing \( \mathcal{G}_{9,1} \) and \( E_{13,1} \) are worthwhile. Below \( \text{Leech}^2 \) denotes the complex Leech lattice (studied in detail in \[W\]) scaled to have minimal norm 6.
Figure 1. The $X_{3333}$ diagram on the left. A shorthand drawing of the 32 node diagram $D$ on the right. In the 32 node diagram $1 \leq i < j \leq 4$. So the node $c_i$ (resp. $e_{ij}$) stands for four (resp. six) nodes. A solid (resp. dotted) edge between two vertices $x$ and $y$ indicates the relation $xyx = yxy$ (resp. $xyxy = yxyx$). No edge between $x$ and $y$ means $xy = yx$. The following shorthands are used: The edge between $a$ and $b_i$ means that $a$ and $b_i$ are connected for all $i$. A single (resp. triple) edge between $b_i$ and $c_i$ (resp. $g_i$) means that $b_i$ and $c_j$ (resp. $g_j$) are connected if $i = j$ (resp. $i \neq j$). Notice that there are two kinds of edges out of $e_{ij}$. The edge between $e_{ij}$ and $d_i$ (resp. $b_i$) means that $e_{ij}$ is connected to $d_k$ (resp. $b_k$) if $k \in \{i, j\}$ (resp. $k \notin \{i, j\}$). Finally, the dotted loop joining $e_{ij}$ to itself means that $e_{ij}$ is joined to $e_{kl}$ if $\{i, j\} \cap \{k, l\} = \emptyset$.

The description $G_{9,1} \simeq BW_{16} \oplus G_{1,1}$ is crucial in our proofs just like the description $E_{13,1} \simeq \text{Leech}^e \oplus E_{1,1}$ is crucial in the proofs in \cite{A2, Ba2}. This is because we use two key properties that are shared by the (real) Barnes-Wall and the (real) Leech lattice: they contain no norm 2 vectors and they are very dense, in fact the densest lattices known in the respective dimensions. The absence of norm 2 vector in $BW_{16}$ provides us with a “Barnes-Wall cusp” at the boundary of $\mathbb{B}(L)$ such that no mirror passes through it. These cusps play a key role in our arguments.

The results on $E_{13,1}$ use the fact that the covering radius of the Leech lattice is $\sqrt{2}$. In our results on $G_{9,1}$, the density of the Barnes-Wall lattice is used via lemma 6.11 of \cite{A1} which is a special case of results in \cite{Bo3}. This lemma gives a covering of the underlying real vector space of the Barnes-Wall lattice using balls of two sizes. It is curious to note that this lemma does not use the covering radius of the Barnes-Wall lattice. Rather, it uses the covering radius of the Leech lattice and an embedding of the Barnes-Wall lattice in the Leech lattice. Just like the results in \cite{A1, Ba2, AB1}, the results of this paper would all fail to hold if the covering radius of the Leech lattice was any bigger than $\sqrt{2}$. 
2. Preliminaries

2.1. Definition (Gaussian lattices). Let $\mathcal{G} = \mathbb{Z}[i]$ be the ring of Gaussian integers. Let $p = (1+i)$. Let $K$ be a free $\mathcal{G}$-module of finite rank with a $\mathbb{Q}[i]$-valued Hermitian form $\langle \cdot | \cdot \rangle : K \times K \rightarrow \mathbb{Q}[i]$. Hermitian forms are always assumed to be linear in second variable. We shall always identify $K$ inside the $\mathbb{Q}[i]$-vector space $K \otimes G \mathbb{Q}[i]$ and further, inside the complex vector space $K \otimes G \mathbb{C}$. The Hermitian form linearly extends to these vector spaces. For $v \in K \otimes G \mathbb{C}$, write $v^2 = \langle v | v \rangle$. We say that $v^2$ is the norm of $v$. A nonzero vector of norm zero is called a null vector.

Let $A \subseteq K \otimes G \mathbb{C}$ and $m \in \mathbb{R}$. It will be convenient to use the notation:

$$A(m) = \{a \in A : a^2 = m\} \quad \text{and} \quad A(\leq m) = \{a \in A : a^2 \leq m\}.$$ 

Let $A^\perp = \{v \in K \otimes G \mathbb{C} : \langle v | a \rangle = 0 \ \forall a \in A\}$. The radical of $K$ is defined as $\text{rad}(K) = K^\perp \cap K$. The Hermitian form is nonsingular if $\text{rad}(K) = 0$. If $\text{rad}(K) = 0$, then $K$ is called a $G$-lattice or a Gaussian lattice. If $\text{rad}(K) \neq 0$, then $K$ is called a singular $G$-lattice. Say that $K$ is integral if the Hermitian form takes values in $\mathcal{G}$. Say that $K$ is Lorentzian if it has signature $(n,1)$.

Let $K$ be a $G$-lattice. Define $K^\vee = \{v \in K \otimes G \mathbb{Q}[i] : \langle v | K \rangle \subseteq \mathcal{G}\}$. Then $K^\vee$ is a $G$-lattice called the dual lattice of $K$. Note that $K$ is integral if and only if $K \subseteq K^\vee$.

Let $l$ be a prime in $\mathcal{G}$. A $G$-lattice $K$ is called $l$-modular if $K^\vee \cong l^{-1} K$. Clearly if $K_1$ and $K_2$ are $l$-modular, then so is $K_1 \oplus K_2$. Let $K_2$ denote the underlying $\mathbb{Z}$-lattice of $K$. This means that $K_2$ is the underlying $\mathbb{Z}$-module of $K$ with the bilinear form $\text{Re}(\cdot | \cdot)$. Note that $K$ is an integral $G$-lattice if and only if $K_2$ is an integral $\mathbb{Z}$-lattice and $(K^\vee)_2 = (K_2)^\vee$.

2.2. The $D^G_4$ lattice: Let $D^G_{2n}$ be the sub-lattice of $G^n$ consisting of all $(x_1, \ldots, x_n)$ in $G^n$ such that $(x_1 + \cdots + x_n) \equiv 0 \ mod \ p$ with the standard positive definite Hermitian form

$$\langle x | x' \rangle = \bar{x}_1 x'_1 + \cdots + \bar{x}_n x'_n.$$ 

The underlying $\mathbb{Z}$-module of $D^G_{2n}$ with the inner product $\text{Re}(x | y)$ is the root lattice $D_{2n}$, where

$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : (x_1 + \cdots + x_n) \equiv 0 \ mod \ 2\}.$$ 

Note that $D^G_2$ is $p$-modular but $D^G_4$ is not $p$-modular for $n > 2$. A $G$-basis for $D^G_2$ is $v_1 = (1,1)$ and $v_2 = (0, \bar{p})$. The discriminant group of $D^G_4$ is $(D^G_2)^\vee/D^G_2 \cong (\mathbb{Z}/2)^2$. Coset representatives for $p^{-1}D^G_4/D^G_2$ can be chosen to be $\{(0,0), v_1/\bar{p}, v_2/\bar{p}, (v_1 - v_2)/\bar{p}\}$.

2.3. The Barnes-Wall lattice over Hurwitz quaternions and Gaussian integers: Express real quaternions in the form $(x + yj)$ where $x, y \in \mathbb{C}$. The ring $\mathcal{H}$ of Hurwitz integers consists of all $(x + yj)$ such that $(x, y) \in \mathbb{G}^2$ or $(x + \bar{p}, y + \bar{p}) \in \mathbb{G}^2$. Note that $\mathcal{G}$ is a subring of $\mathcal{H}$. The map $(x + yj) \mapsto (x, y)$ defines an isomorphism $\mathcal{H} \cong p^{-1} D_4$ as $\mathcal{G}$-modules. Define

$$\text{BW}^G_{16} = \{(x_1, \cdots, x_4) : x_j \in p^{-1}D^G_4, x_j \equiv x_k \ mod \ D^G_4 \ \forall j, k, \sum x_j \in pD^G_4\}.$$ 

Allcock [A1] describes a four dimensional Hurwitz lattice whose real form (appropriately scaled) is the Barnes-Wall lattice. It is immediate that $\text{BW}^G_{16}$, as defined above, is the Gaussian form of this Hurwitz Lattice with the norms scaled by $\sqrt{2}$. So the real form of $\text{BW}^G_{16}$ is the usual rank sixteen Barnes-Wall lattice of minimum
norm 4. For more information on the Barnes-Wall lattice see [CS, VS, NRS]. The sixteen dimensional Barnes-Wall lattice is part of a family of lattices studied widely in the coding theory literature: see [GP, NRS] and the references in there. An alternative quick definition of the \(G\)-lattice \(BW_{16}^G\) is the \(G\)-span of the rows of \((1,1)^\otimes 4\). It is straightforward to verify the equivalence of the two definitions.

2.4. A common over-lattice of \(4D_4\) and \(BW_{16}^G\): Let \(M_{16}^G\) be the rank 8 Gaussian lattice

\[
M_{16}^G = \{(x_1,x_2,x_3,x_4) : x_j \in p^{-1}D_4^G : x_j \equiv x_k \mod D_4^G \forall j, k\}.
\]

Then \(M_{16}^G\) is an integral Gaussian lattice of minimum norm 2 that contains both \(4D_4^G\) and \(BW_{16}^G\). It is easy to verify the following inclusions among lattices with the indices indicated next to the edges:

\[
\begin{array}{c}
(4D_4^G)^\vee \\
\downarrow 4 \\
(4D_4^G)^\vee \\
\downarrow 4 \\
\downarrow 4 \\
M_{16}^G \\
\downarrow 16 \\
4D_4^G \\
\downarrow 4 \\
BW_{16}^G
\end{array}
\]

In particular \(|(BW_{16}^G)^\vee /BW_{16}^G| = 2^8\). On the other hand, from the definition of \(BW_{16}^G\) one verifies that \((BW_{16}^G)^\vee \supseteq p^{-1}BW_{16}^G\). So

\[
2^8 = |(BW_{16}^G)^\vee /BW_{16}^G| \geq |p^{-1}BW_{16}^G/BW_{16}^G| = |p^{-1}G^8/G^8| = 2^8
\]

It follows that equality must hold everywhere and that \((BW_{16}^G)^\vee = p^{-1}BW_{16}^G\). In particular, all vectors in \(BW_{16}^G\) have even norm.

One crucial property of \(BW_{16}^G\) is that it does not have any norm 2 vector, so it has minimum norm 4. This can be seen quickly from our definition as follows: Note that \(p^{-1}D_4^G\) has minimum norm 1. Take \(x \in BW_{16}^G(2)\). Write \(x = (x_1,x_2,x_3,x_4)\) with each \(x_j \in p^{-1}D_4^G\). Then \(\sum_{j=1}^4 x_j^2 = 2\) implies that at least 2 of the \(x_j\)'s must be 0. Since the \(x_j\)'s are all congruent modulo \(D_4^G\), it follows that \(x_j \in D_4^G\) for all \(j\). Since \(D_4^G\) has minimum norm 4, it follows that there exists a \(k \in \{1,2,3,4\}\) such that \(x_j = 0\) for \(j \neq k\). But now \(x_k = \sum_{j} x_j \in pD_4^G\), so \(x^2 = x_k^2\) has norm at least 4, which is a contradiction.

The lemma below is taken from [A1] and is a special case of the results in [Bo3] where it was used to find interesting reflection groups in real hyperbolic space \(\mathbb{R}H^{17}\) (see theorem 3.1 of [Bo3] and the examples following theorem 3.3 on page 232). As mentioned in the introduction, the proof depends on the covering radius of the Leech lattice. We quote it below for convenience:

2.5. **Lemma** (same as lemma 6.11 of [A1]). The real vector space \(BW_{16}^G \otimes\mathbb{Z} \mathbb{R}\) is covered by the closed balls of radius \(\sqrt{2}\) around the vectors of \(BW_{16}^G\) together with closed balls of radius 1 around the vectors \(p^{-1}v\) with \(v \in BW_{16}^G\) and \(v^2 \equiv 2 \mod 4\).
2.6. Definition (roots, reflection groups). Let \( V \) be a complex vector space with a Hermitian form \( \langle \cdot | \cdot \rangle \). Let \( v \in V \) be a vector of nonzero norm. A complex reflection \( R \) in \( v \) is a linear automorphism of \( V \) of finite order that point-wise fixes \( v^\perp \). If \( R \) has order \( n \), then it follows that \( R(v) = \xi v \) where \( \xi \) is a primitive \( n \)-th root of unity. We shall write \( R = R_\xi^n \). One has

\[
R_\xi^n(x) = x - (1 - \xi) \frac{\langle v | x \rangle}{v^2} v.
\]

Let \( K \) be a \( \mathcal{G} \)-lattice. A root of \( K \) means a primitive vector \( v \) of \( K \) of positive norm such that \( R_\xi^n \in \text{Aut}(K) \) for some non-trivial root of unity \( \xi \). The reflection group of \( K \), denoted \( R(K) \), is the subgroup of \( \text{Aut}(K) \) generated by the reflections in the roots of \( K \). Write \( R_v = R_1^v \). Let \( s \) and \( t \) be two norm 2 vectors of \( K \). One verifies that \( R_s \) and \( R_t \) commutes (reps. braids) if \( \langle s | t \rangle = 0 \) (resp. \( |\langle s | t \rangle| = \sqrt{2} \)). Further, if \( |\langle s | t \rangle| = 2 \), then \( tR_sR_tR_s(t) = t \), and hence \( R_sR_tR_sR_t = R_tR_sR_tR_s \).

2.7. Reflection groups of \( p \)-modular lattices: Assume \( K \) is a \( p \)-modular \( \mathcal{G} \)-lattice. Then the minimal norm of \( K \) is at least 2. From the formula for complex reflection we find that \( K(2) \) is the set of roots of \( K \) and the reflections in \( R(K) \) are precisely the order 4 and order 2 reflections in these roots.

2.8. The reflection group of \( D^2_q \): The lattice \( D^2_q \) has six roots counted up to roots of unity. These are \((p, 0), (0, p), (1, 0), (0, 1), \) for \( r = 0, 1, 2, 3 \). If \( s \) and \( t \) are any two non-proportional and non-orthogonal roots of \( D^2_q \), then the finite complex reflection group \( R(D^2_q) \) is generated by the \( i \)-reflections \( R_s \) and \( R_t \). These two reflections obey the braiding relation \( R_sR_tR_sR_t = R_tR_sR_s \). The relations \( R_2^2 = R_4^4 = 1 \) and the braiding relation are sufficient to give a presentation of \( R(D^2_q) \). This group is called \( G_8 \) in the table of finite complex reflection groups given in \([BMR]\).

2.9. Complex hyperbolic space: Let \( V \) be a complex vector space of dimension \((n + 1)\) with the standard non-degenerate Hermitian form of signature \((n, 1)\). Let \( \mathbb{B}(V) \) be the set of one dimensional negative definite subspaces of \( V \). This is an open subset of the projective space \( \mathbb{P}(V) \). If \( L \) is a Lorentzian \( \mathcal{G} \)-lattice, we write \( \mathbb{B}(L) = \mathbb{B}(L \otimes_G \mathbb{C}) \).

The group of isometries \( V \) is \( U(n, 1) \) and \( \mathbb{B}(V) \) is a concrete model for the corresponding Hermitian symmetric space, sometimes called the complex hyperbolic space and denoted by \( CH^n \). Up to scaling, there is a unique \( PU(n, 1) \) invariant metric on \( CH^n \) called the Bergman metric. We shall only need some facts about the associated distance function.

A negative norm vector \( v \) in \( V \) determines a point \( C\mathbb{B}v \) in \( \mathbb{B}(V) \). A positive norm vector \( r \) in \( V \) determines a totally geodesic hyperplane \( \mathbb{B}(r^\perp) \) in \( \mathbb{B}(V) \). For simplicity we shall write \( v \) instead of \( C\mathbb{B}v \) and \( r^\perp \) instead of \( \mathbb{B}(r^\perp) \) when there is no chance of confusion.

Let \( u, v \) be two negative norm vectors in \( V \). The distance between the corresponding points in the complex hyperbolic space is

\[
d(u, v) = \cosh^{-1} \sqrt{\frac{|\langle u | v \rangle|^2}{u^*v^2}}.
\]

Let \( r \) be a negative norm vector in \( V \). Then

\[
d(r^\perp, v) = \sinh^{-1} \sqrt{\frac{|\langle r | v \rangle|^2}{-r^*v^2}}.
\]
Let \( r, s \) be two negative norm vectors in \( V \). If \( C_r + C_s \) is positive definite then the hyperplanes \( B(r^+) \) and \( B(s^+) \) meet in \( B(L) \). Otherwise, one has
\[
d(r^+, s^+) = \cosh^{-1} \sqrt{\frac{|(r,s)|^2}{r^*s}}.
\]
Our distance function differs from the ones in [Gol] by a factor of 2. This is not an issue because we only use these formulas only to compare distances. Let \( v \in V \) be a positive norm vector. Let \( \xi \) be a primitive \( k \)-th root of unity for some \( k > 1 \). Unless it is necessary, we shall not distinguish between \( R_k^\xi \) and its image in \( PU(n,1) \) and refer to either as a \( \xi \)-reflection in \( s \). This reflection is an isometry of \( B(V) \) that point-wise fixes the totally geodesic hyperplane \( v^+ \) (called the mirror of reflection) and acts as anti-clockwise rotation of angle \( 2\pi/k \) in the normal bundle to the mirror.

2.10. **Definition** (horocyclic distance). Let \( V \) be as in [2.8]. Let \( z \) be a null vector in \( V \). If \( v \) is a negative norm vector in \( V \), define
\[
d_z(v) = \frac{1}{2} \log(|\langle z|v\rangle|^2/(-v^2)).
\]
Note that \( d_z \) determines a function on the complex hyperbolic space \( B(V) \). We denote this function also by \( d_z \). The null vector \( z \) determines a point \( \mathbb{C}z \) in the boundary \( \partial B(V) \) of \( B(V) \). As before, we write \( z \) instead of \( \mathbb{C}z \) if there is no chance of confusion. We say that \( d_z(v) \) is the horocyclic distance between \( z \) and \( v \). This terminology is justified by the lemma below. We include a proof because we could not find a convenient reference.

2.11. **Lemma** (ideal triangle inequality). (a) Let \( x, y \) be negative norm vectors in \( V \). Then one has \( |d_z(x) - d_z(y)| \leq d(x, y) \). (b) The equality \( d_z(x) - d_z(y) = d(x, y) \) holds if and only if \( y \) lies on the geodesic ray joining \( x \) and \( z \).

**Proof.** (a) Let \( \alpha = \langle z|x \rangle \), \( \beta = \langle y|z \rangle \) and \( \gamma = \langle x|y \rangle \). By changing \( x, y \) by units if necessary, we may assume, without loss, that, \( |x|^2 = |y|^2 = -1 \). If \( z, x, y \) are linearly independent then their span has signature \((2,1)\), so \( \det(\text{gram}(z, x, y)) < 0 \), otherwise \( \det(\text{gram}(z, x, y)) = 0 \). So we have
\[
0 \geq \det(\text{gram}(z, x, y)) = \det\begin{pmatrix}
0 & \alpha & \beta \\
\bar{\alpha} & 1 & \gamma \\
\bar{\beta} & \bar{\gamma} & 1
\end{pmatrix} = |\alpha|^2 + |\beta|^2 + 2 \Re(\alpha\beta\gamma) \\
|\alpha|^2 + |\beta|^2 - 2|\alpha\beta\gamma|.
\]
It follows that
\[
\cosh d(x, y) = |\gamma| \geq \frac{1}{2} \left( |\alpha|^2 + |\beta|^2 \right) = \frac{1}{2} \left( e^{d_z(x)} - d_z(y) + e^{d_z(y)} - d_z(x) \right) \\
= \cosh d_z(x) - d_z(y).
\]
Since \( t \mapsto \cosh t \) is strictly increasing for \( t \in [0, \infty) \), part (a) follows.

(b) Suppose \( y \) lies on the geodesic ray joining \( z \) and \( x \). Then \( z, x, y \) are linearly dependent. So the calculation in part (a) show that \( d(x, y) = |d_z(x) - d_z(y)| \). Now, without loss, assume \( \langle z|x \rangle \) is a negative real number and \( |x|^2 = -1 \). If \( y \) is on the geodesic ray joining \( x \) and \( z \), then \( Cy = C(x + tz) \) for some \( t \geq 0 \). So
\[
e^{2d_z(y)} = \frac{|(y|z)|^2}{1 - 2t(x|z)} = \frac{|(z|x)|^2}{1 - 2t(x|z)} < |(x|z)|^2 = e^{2d_z(x)}.
\]
So \( d_z(x) > d_z(y) \) and hence \( d(x, y) = d_z(x) - d_z(y) \). The other implication follows from uniqueness of geodesic which is a consequence of negative curvature. We shall skip the details since we do not need this for our application. \( \square \)
2.12. Definition (Horoballs). Let $z$ be a null vector in $V$. Let $c$ be a positive real number. A subset $B \subseteq \mathbb{B}(V)$ of the form $B = \{v: d_z(v) < c\}$ is called an open horoball around $z$. Similarly define a closed horoballs. The boundary of a horoball around $z$ is called a horosphere around $z$. Pick $v \in \mathbb{B}(V) - B$. Let $p$ be the point where the geodesic ray joining $v$ and $z$ intersects $\partial B$. Then, one verifies that $p$ is the unique point of the closed horoball $\bar{B}$ that is closest to $v$, that is
\[ d(p, v) = d(B, v). \]
In other words, $p$ is the projection of $v$ on $B$. Lemma 2.11 implies that
\[ d_z(v) = c + d(B, v). \]
Let $\zeta \in \partial \mathbb{B}(V)$ be the point determined by $z$. We shall say that $v_1$ is closer to $\zeta$ than $v_2$ in horocyclic distance if and only if $d_z(v_1) < d_z(v_2)$. If we scale $z$, then both sides of the inequality gets multiplied by the same positive factor; so this notion does not depend on the choice of $z$. Another way to see this is to note that $v_1$ is closer to $\zeta$ than $v_2$ if and only if $v_1$ is closer to $B$ than $v_2$, where $B$ is any small horoball around $\zeta$ that misses $v_1$ and $v_2$.

3. Reflection groups of $p$-modular $G$-lattices: height reduction

In this section we prove some results about the reflection group of a general $p$-modular Lorentzian $G$-lattice $L$. A null vector $z \in L$ or the point of $\mathbb{B}(L)$ determined by $z$ is called a cusp (of $R(L)$). Our first goal is to prove some lemmas that are useful for finding sets of mirrors close to a cusp $z$ such the reflections in them generate $R(L)$. Formally, these results are of the following sort:

3.1. Lemma. Let $G$ be a group of isometries of a metric space $X$. Let $\mathcal{H}$ be $G$-stable collection subsets of $X$ and $A \subseteq X$ such that \( \{d(A, H) : H \in \mathcal{H}\} \) is a discrete subset of $[0, \infty)$. Let $d_0 \in [0, \infty)$. Assume that for all $H \in \mathcal{H}$ with $d(A, H) > d_0$, there exists $g \in G$ such that $d(A, gH) < d(A, H)$. Then \( \{H \in \mathcal{H} : d(A, H) \leq d_0\} \) meets every $G$-orbit in $\mathcal{H}$.

In this situation we say that $d(A, H)$ (or some suitable increasing function of it) is the height of $H$ (with respect to $A$). The proof is an obvious induction on height. We call these height reduction arguments. In our application, $G$ will be some subgroup of $R(L)$, $X = \mathbb{B}(L)$, and $A$ will be either a point in $\mathbb{B}(L)$ or a small horoball around some cusp of $L$. The collection $\mathcal{H}$ will be either the set of mirrors of $R(L)$ or a suitable collection of horoballs around the cusps of $L$.

Our second goal of this section is to introduce a discrete Heisenberg group $T$ sitting in the stabilizer of a cusp in $\text{Aut}(L)$ and show that the reflection group $R(L)$ contains a finite index subgroup of $T$.

3.2. Notation: For this section, let $\Lambda$ denote a $p$-modular positive definite $G$-lattice of rank $n$. Let $\Lambda(r \mod 4) = \{\lambda \in \Lambda : \lambda^2 \equiv r \mod 4\}$. Since the underlying $\mathbb{Z}$-lattice of $\Lambda$ is even, $\lambda \mapsto \frac{1}{2} \lambda^2 \mod 2$ is a homomorphism $\Lambda \to \mathbb{Z}/2$. The kernel of this homomorphism is $\Lambda(0 \mod 4)$ and the complement of the kernel is $\Lambda(2 \mod 4)$. If $a \in p\mathbb{G}$, then the homomorphism $\Lambda \to \mathbb{Z}/2$ factors through $\Lambda/a$. In other words, all the elements in a coset in $\Lambda/a$ either have norm $0 \mod 4$ or have norm $2 \mod 4$.

3.3. $p$-modular Gaussian Lorentzian lattice: Let $L = \Lambda \oplus G_{1,1}$. Since $G_{1,1}$ is $p$-modular, so is $L$. Note that all vectors of $L$ have even norm. Vectors of $L$ will be
written in the form \((\sigma; m, n)\) where \(\sigma \in \Lambda\) and \(m, n \in \mathcal{G}\). The Hermitian form on \(L\) is given by
\[
\langle (\sigma; m, n) | (\sigma'; m', n') \rangle = \langle \sigma | \sigma' \rangle + m\bar{m}' + n\bar{n}'.
\]
In particular,
\[
(\sigma; m, n)^2 = \sigma^2 + 2 \text{Re}(\bar{m}n).
\]
Let
\[
\rho = (0; 0, 1).
\]
This norm zero vector plays a special role throughout. Note that
\[
\langle \rho | (\sigma; m, n) \rangle = pm.
\]
If \(s \in L(N) - \rho^{-1}\), then one can write \(s\) in the form
\[
s = (\sigma; m_p \bar{m}^{-1}(\bar{N} - \sigma^2 / 2) + \nu),
\]
where \(\nu \in \text{Im}(\mathbb{C})\) is chosen so that the last coordinate of \(s\) lies in \(\mathcal{G}\). We define the *height* of a primitive lattice vector \(s\) (with respect to \(\rho\)) to be
\[
\text{ht}(s) = \begin{cases} \frac{|\langle s | \rho \rangle|^2}{|s|^2} & \text{if } s^2 \neq 0, \\ |\langle s | \rho \rangle|^2 & \text{if } s^2 = 0. \end{cases}
\]
Given \(s \in L(N)\) and \(s' \in L(N')\) written in the form \(\square\), we record an useful formula for their inner product which is verified by direct calculation:
\[
\text{Re}\langle \frac{s}{m} | \frac{s'}{m'} \rangle = \frac{N}{|m|^2} + \frac{N'}{|m'|^2} - \frac{1}{2}(\frac{\sigma}{m} - \frac{\sigma'}{m'})^2, \quad \text{Im}\langle \frac{s}{m} | \frac{s'}{m'} \rangle = \text{Im}\langle \frac{s}{m} | \frac{s'}{m'} \rangle + \frac{2\nu}{|m|^2} - \frac{2\nu'}{|m'|^2}.
\]

3.4. **The roots of \(L\) near the cusp \(\rho\):** The roots of \(L\) are the vectors of minimum norm 2. Let \(s \in L(2)\) be a root. As in \(\square\), we write
\[
s = (\sigma; m_p \bar{m}^{-1}(\frac{1}{2}(1 - \frac{\sigma^2}{2}) + \nu)).
\]
Note that \(\text{ht}(s) = |m|^2\). The roots having height 1, 2, 4, \cdots are called the roots in the first shell, second shell, third shell and so on. Mirror of a \(j\)-th first shell root is called a \(j\)-th shell mirror Among the mirrors that do not pass through \(\rho\), the first shell mirrors are the mirrors closest to the cusp \(\rho\) in horocyclic distance. The second shell mirrors are the next closest and so on. The lemma below explicitly describes the roots in the first two shell. One verifies easily that the first shell roots of \(L\) are of the form
\[
\text{i}^\nu(\sigma; 1, p(\frac{1}{2}(1 - \frac{\sigma^2}{2}) + \nu)) \quad \text{where } \sigma \in \Lambda, \nu \in \frac{1}{2}\mathbb{Z} \quad \text{and } \frac{2}{i} \nu \equiv (1 - \frac{\sigma^2}{2}) \mod 2.
\]
Writing \(\nu = ik - \frac{1}{2}(1 - \frac{\sigma^2}{2})\) we find that the first shell roots are of the form
\[
\text{i}^\nu(\sigma; 1, 1 - \frac{\sigma^2}{2} + ipk), \quad \text{where } \sigma \in \Lambda \quad \text{and } k \in \mathbb{Z}.
\]
One verifies easily that the second shell roots of \(L\) are of the form
\[
\text{i}^\nu(\sigma; \bar{\rho}, \frac{1}{2}(1 - \frac{\sigma^2}{2}) + \nu) \quad \text{where } \sigma \in \Lambda(2 \mod 4) \quad \text{and } \nu \in i\mathbb{Z}.
\]
It is useful to note that if \(s\) is a first or second shell root written as above and we change \(\nu\) to \(\nu' \in \nu + i\mathbb{Z}\), then we again get a root in the same shell.
Let \( l = (\lambda; h, \ast) \in L \). Let \( s = (\sigma; m, \ast) \) be a root of \( L \). Assume \( h, m \neq 0 \). The lemma below gives us condition for a reflection in \( s \) to decrease the height of \( l \). If \( l \) is a root, then this is equivalent to saying that a reflection in \( s \) brings the mirror \( \mathbb{B}(l^\perp) \) closer to \( \rho \). The condition is conveniently expressed in terms of the quantity

\[
y = y(s, l) = |m|^2 \langle \frac{\lambda}{h}, \frac{\lambda}{h} \rangle.
\]

3.5. Lemma. An order four reflection in \( s \) decreases the height of \( l \) if and only if \( y = y(s, l) \) belongs to \( B(1 + i, \sqrt{2}) \cup B(1 - i, \sqrt{2}) \), that is, the union of radius \( \sqrt{2} \) open discs in the complex plane centered at \((1 \pm i)\).

**Proof.** Let \( \xi = \pm i \) and let \( R \) denote the \( \xi \)-reflection in \( s \). We calculate

\[
\langle \rho | R(\frac{l}{h}) \rangle = \langle R^{-1}(\rho) | \frac{l}{h} \rangle = \langle \rho - \frac{1}{2}(1 - \xi)\bar{m}\bar{s}| \frac{l}{h} \rangle = p - \frac{1}{2}(1 - \xi)py.
\]

The reflection \( R \) brings \( l^\perp \) closer to \( \rho \) if and only if \( |\langle \rho | R(\frac{l}{h}) \rangle| > |\langle \rho | R(\frac{l}{h}) \rangle| \), that is,

\[
|p| > |p - \frac{1}{2}(1 - \xi)py|.
\]

Multiplying both sides by \(|p|^{-1}(1 - \xi)|\), the inequality becomes

\[
\sqrt{2} > |(1 - \xi) - y|
\]

which is equivalent to \( y \) lying in \( B(1 + i, \sqrt{2}) \cup B(1 - i, \sqrt{2}) \). \( \square \)

3.6. Lemma. Let \( l = (\lambda; h, p\bar{m}^{-1}((l^2 - \lambda^2)/4 + \nu)) \) be a root of \( L \).

(a) If there exists \( \sigma \in \Lambda \) with \((\sigma - \lambda/h)^2 \leq 2 \) and \(|h| > 1 \), then a first shell reflection decreases the height of \( l \).

(b) If there exists \( \sigma \in p^{-1}\Lambda(2 \text{ mod } 4) \) with \((\sigma - \lambda/h)^2 \leq 1 \) and \(|h| > \sqrt{2} \), then a second shell reflection decreases the height of \( l \).

**Proof.** In part (a) (resp. (b)) we show that we can choose a root \( s \) in the first (resp. second) shell such that \( y = y(s, l) \) belongs to the rectangle \((0, 2) \times [-i, i]\). The lemma then follows from \ref{1.3}, since this rectangle is a subset of \( B(1 + i, \sqrt{2}) \cup B(1 - i, \sqrt{2}) \).

(a) Choose \( \sigma \in \Lambda \) such that \((\sigma - \lambda/h)^2 \leq 2 \). Consider a first shell root written in the form \( s = (\sigma; 1, p(\frac{1}{2}(1 - \frac{1}{2}h^2) + \nu)) \) as in \ref{1.3} with \( \nu \) still to be determined. Using \( \ref{2} \), we compute

\[
\Re(y) = 1 + \frac{l^2}{|h|^2} - \left( \frac{\lambda}{h} - \frac{\lambda}{h} \right)^2 \quad \text{and} \quad \Im(y) = \Im(\sigma | \frac{\lambda}{h}) + \frac{2\nu}{|h|^2} - 2\nu. \quad \tag{3}
\]

Since \(|h| > 1 \) and \( l^2 = 2 \), The choice of \( \sigma \) ensures that \( \Re(y) \in (0, 2) \). From \ref{1.3} note that we are free to choose \( 2\nu/i \) either from \( 2\mathbb{Z} \) or from \( 2\mathbb{Z} + 1 \). So we can choose \( \nu \) to ensure that \( \Im(y) \in [-i, i] \).

(b) Choose \( \sigma \in \Lambda(2 \text{ mod } 4) \) such that \((\sigma/p - \lambda/h)^2 \leq 1 \). Consider a second shell root written in the form \( s = (\sigma; p, \frac{1}{2}(1 - \frac{1}{2}h^2) + \nu)) \) as in \ref{1.3} with \( \nu \) still to be determined. Using \( \ref{2} \), we compute

\[
\Re(y) = 1 + \frac{l^2}{|h|^2} - \left( \frac{\lambda}{h} - \frac{\lambda}{h} \right)^2 \quad \text{and} \quad \Im(y) = 2\Im(\sigma | \frac{\lambda}{h}) + \frac{4\nu}{|h|^2} - 2\nu. \quad \tag{4}
\]

Since \(|h| > \sqrt{2} \) and \( l^2 = 2 \), the choice of \( \sigma \) ensures that \( \Re(y) \in (0, 2) \). From \ref{1.3} Note that we are free to choose \( 2\nu/i \) from \( 2\mathbb{Z} \). So we can choose \( \nu \) to ensure that \( \Im(y) \in [-i, i] \). \( \square \)

3.7. Lemma. Let \( l = (\lambda; h, \ast) \) be a primitive null vector of \( L \) with \( h \neq 0 \). Assume that \( l \) is not orthogonal to any root of \( L \).
(a) If there exists $\sigma \in \Gamma$ with $(\sigma - \lambda/h)^2 \leq 2$, then a reflection in a first shell root decreases height of $h$.

(b) If there exists $\sigma \in p^{-1}\Gamma(2 \mod 4)$ with $(\sigma - \lambda/h)^2 \leq 1$, then a reflection in a second shell root decreases the height of $h$.

Proof. The proof is almost identical to the proof of lemma 3.4. The computation is only slightly different. Equations (3) and (4) still hold but now since $l^2 = 0$, we obtain $y(s,l) \in [0,1] \times [-i,i]$. This rectangle minus the origin is a subset of $B(1+i, \sqrt{2}) \cup B(1-i, \sqrt{2})$. Since $l$ is assumed to be not orthogonal to any root, we have $y(s,l) \neq 0$.

3.8. Definition (The Heisenberg group of translations). Let $T$ be the group of automorphisms of $L$ that fix $\rho$ and act trivially on $\rho^*/\rho$. One verifies that

$$T = \{T_{\lambda,z}: \lambda \in \Lambda, z \in i(\lambda^2/2 + 2\mathbb{Z})\} = \{T_{\lambda,i(\lambda^2/2) + 2ik}: \lambda \in \Lambda, k \in \mathbb{Z}\}$$

where $T_{\lambda,z} \in \text{Aut}(L)$ is defined by

$$T_{\lambda,z}(l; a, b) = (l + a\lambda; a, -\bar{\lambda}^{-1}\lambda|l| + a\bar{\lambda}^{-1}(z - \lambda^2/2) + b).$$

Note that for each $\lambda \in \Lambda$, the integer $z/i$ either runs over $2\mathbb{Z}$ or $2\mathbb{Z}+1$. We call $T$ the group of translations. One verifies that the translations form a discrete Heisenberg group whose multiplication is given by

$$T_{\lambda,z}T_{\lambda',z'} = T_{\lambda+\lambda', z+z' + \text{Im}(\lambda'|\lambda)}. \quad (5)$$

Note that the translations of the form $T_{0,z}$ are central, $T_{\lambda,z}^{-1} = T_{-\lambda,-z}$, and

$$T_{\lambda,z}T_{\lambda',z'}T_{\lambda,z}^{-1}T_{\lambda',z'}^{-1} = T_{0,2\text{Im}(\lambda'|\lambda)}. \quad (6)$$

3.9. Lemma. Let $R_1$ and $R_2$ be the $i$-reflections in the roots $r_1 = (0^n; 1,1)$ and $r_2 = (0^n; 1, i)$ respectively. Let $\lambda \in \Lambda$. Choose $z$ such that $T_{\lambda,z} \in T$. Let $G$ be the group generated by the reflections in $T_{\lambda,z}(r_1), T_{\lambda,z}(r_2), r_1, r_2$. Then $\overline{T_{\rho,\lambda,z}} \subseteq G$.

Proof. Let $\beta$ be the automorphism of $L$ that is identity on $\Lambda$ and acts on $\mathcal{G}_{0,1,1}$ as multiplication by $-i$. Then one verifies that $R_1R_2 = \beta T_{0,-4i}$. Since $T_{0,-4i}$ is a central translation, it follows that

$$(R_1R_2)T_{\lambda,z}(R_1R_2)^{-1} = \beta T_{\lambda,z} \beta^{-1} = T_{i\lambda,z}. \quad \Box$$

So $G$ contains $T_{\lambda,z}(R_1R_2)T_{\lambda,z}^{-1}(R_1R_2)^{-1} = \overline{T_{\rho,\lambda,z}}$.

We finish this section by showing that $R(L)$ contains many translations, specifically, a finite index subgroup of $T$.

3.10. Corollary. (a) Fix a $G$-basis $\lambda_1, \cdots, \lambda_n$ of $\Lambda$. For each $j = 1, \cdots, n$, fix $z_j \in \text{Im}(\mathbb{C})$ such that $T_{\lambda_j,z_j} \in T$. Let $G$ be the subgroup of $R(L)$ generated by reflections in the following set of roots:

$$\{r_k, T_{\lambda_j,z_j}(r_k), T_{\lambda_j,z_j}(r_k): k = 1, 2, j = 1, 2, \cdots, n\}.$$

Then $G$ contains a translation of the form $T_{\lambda,*}$ for each $\lambda \in p\Lambda$ and the central translations $T_{0,4im}$ for all $m \in \mathbb{Z}$.

(b) A full set of coset representatives for $(R(L) \cap T)\backslash T$ can be chosen from the finite set $T_* = \{T_{\sigma,2i}^{\sigma^2/2}, T_{\sigma,2i+1}^{\sigma^2/2}: \sigma \in (\Lambda/p)^{\times}\}$ where $(\Lambda/p)^{\times}$ is a full set of coset representatives for $\Lambda/p$. 

Proof: Lemma 3.9 implies that \( G \) contains a translation of the form \( T_{p\lambda}^* \) and a translation of the form \( T_{ip\lambda}^* \) for each \( i \). From equation (5), it follows that \( R(L) \) contains a translation of the form \( T_{\lambda}^* \) for all \( \lambda \in p\Lambda \). Since \( \Lambda \) is \( p \)-modular, choose \( \lambda, \lambda' \in p\Lambda \) with \( \langle \lambda' \mid \lambda \rangle = 2p \). Then equation (6) implies that \( G \) also contains \( T_{0,4i} \). So \( G \) contains the central translations of the form \( T_{0,4im} \) for all \( m \in \mathbb{Z} \).

(b) Let \( T = T_{\lambda^*} \in T \). Choose \( \sigma \in (\Lambda/p)^* \) such that \( \sigma - \lambda' = p\lambda \) for some \( \lambda \in \Lambda \). By part (a) we can choose a translation \( T_1 \) in \( R(L) \) having the form \( T_1 = T_{p\lambda}^* \). Then \( T_1T = T_{p\lambda^*,*} = T_{p\lambda + \lambda^*,*} = T_{\sigma,*} \). So \( T_1T = T_{\sigma, 2ik + (\alpha^2/2)} \) for some \( k \in \mathbb{Z} \). Choose \( m \in \mathbb{Z} \) such that \( k + 2m \in \{0, 1\} \). Let \( T' = T_{0,4m}T_1 \). Then \( T' \in R(L) \) and \( T'T = T_{\sigma, 2i(k + 2m) + i(\alpha^2/2)} \in T^* \).

4. REFLECTION GROUP OF THE \( p \)-MODULAR \( G \)-LATTICE OF SIGNATURE \((9,1)\)

4.1. Lemma. There is a unique \( p \)-modular Gaussian lattice of signature \((4n+1,1)\).

Lemma 4.1 quickly follows from the uniqueness of even self-dual \( \mathbb{Z} \)-lattice of signature \((8n + 2, 2)\); as in the proof of lemma 2.6 of [Ba2]. The only implication of lemma 4.1 that we need is the isomorphism \( 4D_4^G \oplus \mathcal{G}_{1,1} \simeq \text{BW}_{16}^G \oplus \mathcal{G}_{1,1} \). However, for our computational needs, we shall exhibit an explicit isomorphism \( 4D_4^G \oplus \mathcal{G}_{1,1} \simeq \text{BW}_{16}^G \oplus \mathcal{G}_{1,1} \) in [5,10]. So we omit the proof of lemma 4.1.

4.2. Notation: In the last section, \( \Lambda \) denoted any positive definite \( p \)-modular Gaussian lattice. From here on, unless otherwise stated, we specialize to the case

\[
\Lambda = \text{BW}_{16}^G \quad \text{and} \quad L = \mathcal{G}_{0,1} \simeq 4D_4^G \oplus \mathcal{G}_{1,1} \simeq \Lambda \oplus \mathcal{G}_{1,1}.
\]

Both descriptions of \( L \) are going to be useful for us. In this section, unless otherwise stated, we identify \( L = \Lambda \oplus \mathcal{G}_{1,1} \). In the next section we shall use the identification \( L = 4D_4^G \oplus \mathcal{G}_{1,1} \). Let \( \rho = (0; 0, 1) \in L \). Since \( \Lambda \) has no root, there are no mirrors through the cusp \( \rho \). As before, the mirrors closest to \( \rho \) are called the first shell mirrors; the mirrors that are next closest to \( \rho \) are called second shell mirrors, and so on. The corresponding roots are called first shell roots, second shell roots etc.

Our first goal is to show that the projective reflection group \( \Gamma = PR(L) \) is an arithmetic lattice in \( PU(9,1) \). The plan of the proof follows Theorem 4.1 of [A2].

4.3. Lemma. The reflection group \( R(L) \) acts transitively on primitive null vector of \( L \) (considered up to \( 4 \)-th roots of unity) that are not orthogonal to any roots.

Proof. Let \( z = (\zeta; h,*) \) be a primitive null vector of \( L \) that is not orthogonal to any root. Suppose \( h \neq 0 \). Lemma 2.3 implies that either there exists \( \sigma \in \Lambda \) such that \( (\sigma - \zeta/h)^2 \leq \sqrt{2} \) or there exists \( \sigma \in p^{-1}\Lambda(2 \text{ mod } 4) \) such that \( (\sigma - \zeta/h)^2 \leq 1 \). Lemma 3.7 then shows that the height of \( z \) can be reduced by a reflection in some root (lemma 3.7 only uses reflections in the first and second shell roots but this is irrelevant for this argument). By induction, it follows that a finite sequence of reflection can bring \( z \) to a null vector of the form \( z_1 = (\zeta_1; 0,*) \). Now, \( z_1^2 = 0 \) implies \( \zeta_1^2 = 0 \). Since \( z_1 \) is primitive, it follows that \( z_1 \) is an unit multiple of \( \rho \).

4.4. Theorem. \( |\Gamma \setminus \text{PAut}(L)| \leq 2^p|\text{Aut}(\Lambda)| \). In particular \( \Gamma \) is arithmetic.

Proof. Let \( \rho_1 = (0; 1, 0) \). Take \( g \in \text{Aut}(L) \). Then \( gp \) is a primitive null vector that is not orthogonal to any root. Lemma 4.3 implies that there exists \( g_1 \in R(L) \) such that \( i^{-m}g_1^{-1}gp = \rho \) for some \( m \in \mathbb{Z}/4 \). So \( i^{-m}g_1^{-1}g_1 \rho_1 \) is a null vector of the
form \((*)1,*)\). One verifies that the group of translations act transitively on the null vectors of the form \((*)1,*\)). So there exists a translation \(T \in \text{Aut}(L)\) such that \(T^{-1}i^{-m}g^{-1}_{1}g\) fixes \(\rho\) and \(g\). So \(T^{-1}i^{-m}g^{-1}_{1}g = \alpha \in \text{Aut}(\Lambda)\), a finite group. So \(g = g_{1}T\alpha i^{m}\). By lemma 3.10(b), there exists a \(g_{2} \in R(L)\) and \(t \in T_{\ast}\) such that such that \(T = g_{2}t\). So \(g = g_{1}T\alpha i^{m} = g_{1}g_{2}T\alpha i^{m} \in R(L)\alpha i^{m}\). It follows that a full set of coset representatives for \(\Gamma \setminus \text{PAut}(L)\) can be chosen from the finite set \(\{\alpha : t \in T_{\ast}, \alpha \in \text{Aut}(\Lambda)\}\).

The goal of the rest of the section is to find a finite set of generators for \(R(L)\).

4.5. Lemma. The \(i\)-reflections in the first and second shell roots generate \(R(L)\).

Proof: The argument is similar to the proof of lemma 4.3 except that one needs lemma 3.10 instead of lemma 3.7. From lemma 2.8, we know that closed balls of radius \(\sqrt{2}\) around vectors in \(\Lambda\) together with closed balls or radius 1 around the vectors in \(p^{-1}\Lambda(2 \text{ mod } 4)\) cover the underlying real vector space of \(\Lambda\). So the lemma follows from 3.6 parts (a) and (b) using induction on height of a root.

Lemma 4.5 gives us an infinite set of reflections that generate \(R(L)\). The next lemma shows that an explicit finite subset of these reflections are enough to generate \(R(L)\). To list the roots of these generating reflections, fix a \(G\)-basis \(\lambda_{1}, \cdots, \lambda_{8}\) of \(\Lambda\). For each \(j = 1, \cdots, 8\), fix a \(z_{j} \in \text{Im}(\mathbb{C})\) such that \(T_{\lambda_{j},z_{j}} \in T\). Recall that if \(m \in G\), then \((\Lambda/m)^{\sim}\) denotes a full set of coset representatives for \(\Lambda/m\). Define

\[
\begin{align*}
S_{0} & = \{r_{k}, T_{\lambda_{j},z_{j}}(r_{k}) : j = 1, \cdots, 8, k = 1, 2\}, \\
S_{1} & = \{(\sigma; 1, 1 - \frac{\sigma^{2}}{2} + ipk) : \sigma \in (\Lambda/p)^{\sim}, k = 0, 1\}, \\
S_{2} & = \{(\sigma; \bar{p}, \frac{1}{2}(1 - \frac{\sigma^{2}}{2}) + ik) : \sigma \in (\Lambda/2)^{\sim} \cap \Lambda(2 \text{ mod } 4), k = 0, 1, 2, 3\}.
\end{align*}
\]

From the discussion in 3.2, recall that all the vectors in a coset in \(\Lambda/2\) either have norm 0 mod 4 or have norm 2 mod 4. So \((\Lambda/2)^{\sim} \cap \Lambda(2 \text{ mod } 4)\) is a set of representatives of the cosets that consist of vectors of norm 2 mod 4.

4.6. Lemma. The \(i\)-reflections in the roots in \(S_{0} \cup S_{1} \cup S_{2}\) generate \(R(L)\).

Proof. Let \(G\) be the group generated by the reflections listed. Since \(G\) contains the reflections in the roots in \(S_{0}\), lemma 3.10(a) implies that \(G\) contains a translation of the form \(T_{p\lambda_{n}}\) for all \(\lambda \in \Lambda\) and the central translations of the form \(T_{0, \text{im}}\). For \(n \in \mathbb{Z}\). We make the following claim:

Claim: If \(s\) is a root of the form \((*)1,*)\) (resp. \((*)\bar{p},*)\)), then there exists a translation \(T \in G\) such that \(Ts\) belongs to \(S_{1}\) (resp. \(S_{2}\)).

Let \(s\) be a root of the form \(s = (\sigma; 1,*)\). Choose \(\sigma_{0} \in (\Lambda/p)^{\sim}\) such that \(\sigma_{0} - \sigma = p\lambda\) for some \(\lambda \in \Lambda\). Choose a translation of the form \(T_{p\lambda,s} \in G\) that takes \(s\) to a root of the form \(s' = (\sigma_{0}; 1,*)\). Next, one can choose a central translation in \(G\) that takes \(s'\) to a root in \(S_{1}\). This proves the claim for roots of the form \((*)1,*\)2

The argument for roots of the form \((*)\bar{p},*)\) is similar.

Now let \(R\) be an \(i\)-reflection in a root in the first or second shell. Then there exists a root \(s\) of the form \((*)1,*)\) or \((*)\bar{p},*)\) such that \(R = R_{s}\). Choose \(T \in G\) such that \(Ts \in S_{1} \cup S_{2}\). So \(R_{T}s \in G\). It follows that \(R_{s} = T^{-1}R_{T}s \in G\). Thus, \(G\) contains the \(i\)-reflections in all the roots in the first two shells. Lemma 4.5 completes the proof.

\[\text{2}\] acts simply transitively on the roots of the form \((*)1,*)\). So the argument here is essentially a repeat of the proof of lemma 3.10(b).
5. The thirty two mirrors closest to a point in $\mathbb{C}H^9$

The goal of this section is to find nice generators and relations for $R(L)$, analogous to the Coxeter generators and relations of Weyl groups. This is only a rough analogy; for example the generators are not a minimal set and we do not know if the our relations are sufficient to give a presentation of $R(L)$. However, the analogy is reinforced because similar phenomenon repeats for other complex hyperbolic reflection groups of interesting lattices, as illustrated by Theorem 1.1. In particular, everything in this section closely parallels the results of [Ba2] and [Ba3].

As for Coxeter groups, our generators and relations can be encoded in a diagram $D$, a sort of Coxeter-Dynkin diagram of $R(L)$. This diagram can be defined from the intersection pattern of a configuration of points and hyperplanes in the finite vector space $\mathbb{F}_2^4$. We shall define the lattice $L$ starting from $D$, rather like defining the root lattice from a Cartan matrix. We start by describing these points and hyperplanes and by working out the symmetries of this configuration.

5.1. A configuration of points and hyperplanes in $\mathbb{F}_2^4$: Let

$$g_i = (1,0,0,0), \cdots, g_4 = (0,0,0,1)$$

be the standard basis vectors of $\mathbb{F}_2^4$. The sixteen points of $\mathbb{F}_2^4$ are named $a$, $c_j$, $g_j$, $e_{ij}$, $z$ where

$$a = (1,1,1,1), \quad c_i = a + g_i, \quad e_{ij} = a + g_i + g_j, \quad z = (0,0,0,0)$$

and $1 \leq i < j \leq 4$. Next, we name sixteen hyperplanes in $\mathbb{F}_2^4$. Let

$$d_k = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}_2^4 : x_k = 0\} \quad \text{and} \quad f_k = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}_2^4 : x_k = \sum_j x_j\}.$$

So $K_0 = \{d_1, \cdots, d_4, f_1, \cdots, f_4\}$ is the set of homogeneous hyperplanes in $\mathbb{F}_2^4$ not containing $a$. Let

$$b_k = \mathbb{F}_2^4 - f_k \quad \text{and} \quad h_k = \mathbb{F}_2^4 - d_k.$$

Finally let $K = \{d_k, f_k, b_k, h_k : k = 1, 2, 3, 4\}$ be the set of translates of $K_0$ and let

$$D = \mathbb{F}_2^4 \cup K.$$

Let $Q_+$ be the subgroup of the group of affine transformations of $\mathbb{F}_2^4$ that preserve $D$. One verifies that $Q_+ \simeq 2^4 : (2^3 : L_3(2))$ where the $2^4$ comes from the translation action of $\mathbb{F}_2^4$ on itself and the $2^3 : L_3(2)$ is the stabilizer of $a$ in $L_4(2)$. It will be useful to note that the symmetric group $S_4$ acting by coordinate permutation on $\mathbb{F}_2^4$ fixes $a$. So this $S_4$ is a subgroup of $Q_+$.

The diagram for our reflection group $R(L)$ is shown in figure 11. It is a graph with vertex set $D$ and with two kinds of edges as shown in figure 11. Two vertices $u$ and $v$ are joined by a dotted edge if $v = t_a(u)$ where $t_a$ is the automorphism of $D$ induced by the translation $t_a : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ defined by $t_a(v) = a + v$. If we ignore the dotted edges, then $D$ should be thought of a directed bipartite graph with a directed edge from $v$ to $u$ if $v \in K$, $u \in \mathbb{F}_2^4$ and $u$ is incident on $v$. The automorphism group $Q_+$ acts on this graph $D$ preserving both kinds of edges.
5.2. Lemma. Let $L^\circ$ be the Gaussian lattice of rank 32 with a basis $\{s_v^\circ : v \in D\}$ indexed by $D$ and inner product defined by

$$\langle s_u^\circ | s_v^\circ \rangle = \begin{cases} 
2 & \text{if } u = v, \\
\rho & \text{if } u \in \mathbb{F}_2^d, v \in \mathbb{K} \text{ and } u \in v, \\
-\rho & \text{if } u \in \mathbb{K}, v \in \mathbb{F}_2^d \text{ and } v \in u, \\
-2 & \text{if } v = t_a(u), \\
0 & \text{otherwise.} 
\end{cases} \quad (7)$$

Then $L \simeq L^\circ / \text{Rad}(L^\circ)$.

Proof. Identify $L = 4D_4^2 \oplus \mathbb{G}_{1,1}$. We claim that there are 32 roots $\{s_v : v \in D\}$ in $L$ such that the inner products between them are governed by $D$ as in equation (7). In other words, there is an inner product preserving linear map $L^\circ \to L$ by that sends $s_v^\circ$ to $s_v$. Recall that there is an obvious $S_4$ action on $D$. The symmetric group $S_4$ also acts on $4D_4^2 \oplus \mathbb{G}_{1,1}$ by permuting the four copies of $D_4^2$. The map $s_v^\circ \to s_v$ is going to be $S_4$ equivariant. Define

$$s_a = [0, 0, 0, 0, 0, 0; -1, -1],$$

$$s_{c_1} = [-1, 1, 0, 1, 0, 0; 0, 0],$$

$$s_{c_{12}} = [-1, 1, 1, 0, 0, 0; i, 1],$$

$$s_{g_1} = [-1, 0, 1, 0, 1, 1; 0, 2],$$

$$s_{g_2} = [-1, 1, 1, 1, 1, 1; -1 + 2i, 1],$$

$$s_{f_1} = [0, 0, 0, p, 0, 0; ip, p],$$

$$s_{h_1} = [0, 0, 0, 0, 0, 0; -1, 0],$$

$$s_{d_1} = [-p, 0, 0, 0, 0, 0; 0, 0],$$

$$s_{h_1} = [p, 0, 0, 0, 0, 0; p - 3, p].$$

The other roots are obtained by using the $S_4$ symmetry (that is, by permuting the four copies of $D_4^2$). For example $s_{c_2} = [0, 0, -1, 1, 0, 0; 0, 0]$. The lemma is proved once one verifies that these roots have the required inner products. \qed

5.3. Remark. We want to mention how we came across the 32 roots $\{s_v : v \in D\}$. The complex reflection group of $D_4^2$ is generated by two braiding $i$-reflections. In other words, $R(D_4^2)$ has Dynkin diagram $A_2$ with each vertex having order 4. Write $L_* = 3D_4^2 \oplus \mathbb{G}_{1,1}$. Affinizing and hyperbolizing (in the sense of [CS], chapter 30) one gets a diagram $Y_{333}$ inside $R(L_*)$. The graph $Y_{333}$ is a maximal sub-tree in $\text{Inc}(P^2(\mathbb{F}_2))$, which means the incidence graph of finite projective plane $P^2(\mathbb{F}_2)$. One can extend the $Y_{333}$ diagram uniquely to a $\text{Inc}(P^2(\mathbb{F}_2))$ diagram in $R(L_*)$. The diagram automorphisms $L_{3}(2) : 2$ act on $\mathbb{B}(L_*)$ with a unique fixed point $\tau_*$ and the 14 mirrors corresponding to the vertices of $\text{Inc}(P^2(\mathbb{F}_2))$ are exactly the mirrors closest to $\tau$. We tried to prove the results similar to theorem [L] for the reflection group $R(L_*)$ but could not make the arguments work because the covering radius of the lattice $3D_4^2$ is not small enough. However, these arguments work for $L = 4D_4^2 \oplus \mathbb{G}_{1,1}$ because of the alternative description $L \simeq \mathbb{B}W_{16} \oplus \mathbb{G}_{1,1}$. In the root system of $L = 4D_4^2 \oplus \mathbb{G}_{1,1}$, we can naturally extend the $\text{Inc}(P^2(\mathbb{F}_2))$ diagram, first by using the obvious $S_4$ symmetry permuting the four copies of $D_4^2$ and then by looking for a regular graph. This leads to a set of 32 root diagram $D$ in $L$. 

5.4. Linear relations among the 32 roots: One could also prove lemma 5.2 by working out the 22 dimensional radical of $L^\circ$. It is useful for us to at least write down enough linear relations among the 32 roots $\{s_v: v \in D\}$, where enough means that the corresponding vectors of $\text{Rad}(L^\circ)$ span $\text{Rad}(L^\circ) \otimes \mathbb{C}$. If both $v, w \in \mathbb{F}_2^4$ or both $v, w \in K$, then we have the relation

$$ s_v + s_{t_a(v)} = s_w + s_{t_a(w)}. $$

We define

$$ p_\infty = s_v + s_{t_a(v)} \quad \text{if} \quad v \in \mathbb{F}_2^4, $$

and

$$ l_\infty = s_v + s_{t_a(v)} \quad \text{if} \quad v \in K. $$

Using (7), one verifies that $p_\infty$ and $l_\infty$ are primitive null vectors of $L$. Explicitly, one computes

$$ p_\infty = -(1, 1, 1, 1, 1, 1, 1, 1; 2i, 2) \quad \text{and} \quad l_\infty = (0, p, 0, p, 0, p, p; p - 3, p). $$

Further, for each $u \in \mathbb{F}_2^4$ and for each $w \in K$, we have the relations

$$ -2(1 + i)s_u + \sum_{v \in K: w \in v} s_v = 4l_\infty - (1 + i)p_\infty, $$

and

$$ -2(1 - i)s_w + \sum_{v \in \mathbb{F}_2^4: v \in K} s_v = 4p_\infty - (1 - i)l_\infty. $$

(9)

(10)

To verify the relation (8), one just checks, using (7), that both sides of (8) have the same inner product with each vector in $\{s_v: v \in D\}$. The relations in (9) and (10) can be verified similarly, or by direct computation.

5.5. The configuration of the 32 mirrors in complex hyperbolic space:

One verifies that the mirrors $\{s_v^+: v \in \mathbb{F}_2^4\}$ meet at the cusp determined by $p_\infty$ and the mirrors $\{s_v^-: v \in K\}$ meet at the cusp determined by $l_\infty$. The group $Q_+ = 2^4$ (of $L_3(2)$) transitively permutes these two sets of sixteen mirrors and fixes the $CH^1$ spanned by $p_\infty$ and $l_\infty$. The set of thirty two mirrors $\{s_v^\pm: v \in D\}$ has an extra symmetry, that is described below. Let $\sigma_K: K \to \mathbb{F}_2^4$ be the bijection

$$ \sigma_K\left((x_1, \cdots, x_4) \in \mathbb{F}_2^4: \sum_j u_j x_j = \epsilon\right) = (u_1, u_2, u_3, 1) + (u_4 + \epsilon)(1, 1, 1, 1). $$

Define $\sigma_D: D \to D$ by $\sigma_D|_{\mathbb{F}_2^4} = \sigma_K$ and $\sigma_D|_{\mathbb{F}_2^4} = \sigma_K^{-1}$. One verifies that the involution $\sigma_D$ preserves the incidence relations between points and hyperplanes and commutes with the action of the translation $t_a$. It follows that $\sigma_D$ is an

---

3If the definition of $\sigma_K$ seem too ad-hoc, the following discussion may help. Identify $\mathbb{F}_2^4$ and $K$ with two affine hyperplanes $\mathfrak{H} = \{x \in \mathbb{F}_2^4: x_5 = 1\}$ and $\mathfrak{H} = \{x \in \mathbb{F}_2^4: x_1 + x_2 + x_3 + x_4 = 1\}$ in $\mathbb{F}_2^4$ respectively via

$$ i_{\mathfrak{H}}: v \mapsto (\gamma) \quad \text{and} \quad i_{\mathfrak{H}}: \{x \in \mathbb{F}_2^4: u^T x = \epsilon\} \mapsto (\gamma). $$

Note that $v \in \mathbb{F}_2^4$ belongs to the hyperplane $\{x \in \mathbb{F}_2^4: u \cdot x = \epsilon\}$ if and only if $(\gamma)$ and $(\epsilon)$ are orthogonal with respect to the standard inner product of $\mathbb{F}_2^4$. We need to choose an appropriate $\sigma_\mathfrak{H} \in L_3(2)$ taking $\mathfrak{H}$ to $\mathfrak{H}$. For this, let $J_{m,n}$ denote the $m \times n$ matrix all whose entries are equal to 1. Let $H = (1 \ 0 \ 1)$. Let $J_n$ denote the $n \times n$ identity matrix. Define $\sigma_\mathfrak{H} = (J_{3,2} \ J_{1,2}) \in L_3(2)$. Verify that $\sigma_\mathfrak{H}^{-1}: \mathfrak{H} \to \mathfrak{H}$ and $\sigma_\mathfrak{H}: \mathfrak{H} \to \mathfrak{H}$ are mutually inverse bijections and $\sigma_\mathfrak{H}$ is an involution. Let $f \in \mathfrak{H}$ and $k \in \mathfrak{H}$. Since $\sigma_\mathfrak{H}$ is self adjoint, $\sigma_\mathfrak{H}^{-1}f$ is orthogonal to $\sigma_\mathfrak{H}k$ if and only if $k$ is orthogonal to $f$. Let $\sigma_\mathfrak{H} = i_{\mathfrak{H}} \circ \sigma_\mathfrak{H} \circ i_{\mathfrak{H}}$. Then $\sigma_K: K \to \mathbb{F}_2^4$ is a bijection such that $v \in h$ if and only if $\sigma(h) \in \sigma(v)$ for all $v \in \mathbb{F}_2^4$ and for all $h \in K$. In other words, $\sigma_K$ and $\sigma_K^{-1}$ interchanges
the points in \( F \) reversing involution of \( D \). In other words, the involution \( \sigma_L \) acts on \( D \) by preserving both kinds of edges and reversing the orientation on the solid edges. Define

\[
\sigma_{L^o} : L^o \to L^o \quad \text{by} \quad \sigma_{L^o}(s^o_v) = \begin{cases} s^o_{\tau(v)} & \text{if } v \in K, \\ -is^o_{\tau(v)} & \text{if } v \in \mathbb{F}_2. \end{cases}
\]

From definition of inner product on \( L^o \), it follows that \( \sigma_{L^o} \) is an inner product preserving isomorphism of \( L^o \) whose square is multiplication by \(-i\). So \( \sigma_{L^o} \) descends to define an automorphism \( \sigma = \sigma_L \) of \( L \) and an involution of \( \mathbb{B}(L) \), also denoted by \( \sigma \). This involution \( \sigma \) interchanges the sixteen mirrors meeting at \( p_\infty \) with the sixteen mirrors meeting at \( l_\infty \). One verifies that the group \( Q \) generated by \( \sigma \) and \( Q_+ \) permutes the 32 mirrors transitively and fixes a unique point in \( \mathbb{B}(L) \) which can be represented by the vector

\[
\tau = e^{-\pi i/4}\tau_\infty - p_\infty.
\]

The 32 mirrors are all equidistant from \( \tau \). Let \( d_0 \) be this distance. We compute

\[
d_0 = d(\tau, s^o_w) \approx 0.4090 \quad \text{for all } \; w \in D.
\]

As already mentioned, we are using the same notation for a vector, say \( \tau \) (resp. a hyperplane, say \( s^o_w \)) in \( L \) and the point (resp. hyperplane) in \( \mathbb{B}(L) \) it determines. Let \( B \) be a small horoball around \( p_\infty \) not containing \( \tau \) and let \( B' \) be the image of \( B \) under any automorphism of \( L \) taking \( p_\infty \) to \( l_\infty \). Then \( \tau \) is the point on the real geodesic joining \( p_\infty \) and \( l_\infty \) that is equidistant from \( B \) and \( B' \). We should think of \( \tau \) as the mid-point between \( p_\infty \) and \( l_\infty \).

5.6. **Theorem.** The 32 mirrors \( \{ s^o_w : w \in D \} \) are precisely the mirrors closest to \( \tau \). In particular, \( \tau \) does not lie on any mirror.

Let \( r \) be any root of \( L \) such that \( d(\tau, r^\perp) \leq d_0 \). The lemma below gives us conditions that allow us to restrict the possibilities for \( r \) to a finite set.

5.7. **Lemma.** (a) Let \( w \) be a root of \( L \). Then

\[
|p^{-1}(w|r)|^2 \leq 2 \cosh^2(d_0 + d(\tau, w^\perp)).
\]

(b) Let \( w \) be a primitive null vector of \( L \). From \( 2.10 \) recall that \( d_w(\tau) = 1/2 \log(|\langle w|\tau\rangle|^2/( -\tau^2)) \). One has

\[
|p^{-1}(w|r)|^2 \leq e^{2(d_0 + d_w(\tau))}. \]

In each case, note that \( p^{-1}(w|r) \in \mathcal{G} \). So \( p^{-1}(w|r) \) belongs to \( \{0, 1, 2, 4, 5, 8, 9, \ldots \} \). Thus, in each case, there are finitely many possibilities for \( \langle w|r\rangle \).

**Proof.** (a) If the hyperplanes \( w^\perp \) and \( r^\perp \) meet in \( \mathbb{B}(L) \cup \partial \mathbb{B}(L) \), then \( |p^{-1}(w|r)|^2 \leq 2 \), so the required inequality holds trivially. Otherwise, the triangle inequality implies

\[
d(r^\perp, w^\perp) \leq d(r^\perp, \tau) + d(\tau, w^\perp) \leq d_0 + d(\tau, w^\perp). \]

Part (a) follows using the formula given in \( 2.9 \) for distance between two hyperplanes.

(b) Let \( p_\tau \) be the projection of \( \tau \) on \( r^\perp \). Then \( d(\tau, p_\tau) = d(\tau, r^\perp) \leq d_0 \). Choose a small horoball \( B \) around \( w \) that does not meet \( \tau \) and \( r^\perp \). Let \( p_w \) be the point of \( r^\perp \) that is nearest \( B \). In other words \( p_w \) is the projection of \( w \) on \( r^\perp \). Then \( p_w \) is the points in \( \mathbb{F}_2^4 \) with the hyperplanes in \( \mathcal{K} \) preserving the incidence relations. For \( w \in \mathbb{F}_2^4 \), the translation \( t_w \) acts on \( \mathcal{K} \) and \( \mathcal{R} \) as the matrices \( t_w|\mathcal{K} = \begin{pmatrix} t_w & 0 \\ 0 & 1 \end{pmatrix} \) and \( t_w|\mathcal{R} = \begin{pmatrix} t_w & 0 \\ 0 & \tau \end{pmatrix} \). One verifies that \( (t_w|\mathcal{K}) = \mathcal{R} (t_w|\mathcal{R}) \) if and only if \( w = a \) or \( w = 0 \).
closer to \( w \) than \( p_r \), that is, \( d_w(p_w) \leq d_w(p_r) \). From the ideal triangle inequality, we have \[
\frac{1}{2} \log\left(\frac{|\langle w | p_w \rangle|^2}{p_w^2}\right) = d_w(p_w) \leq d_w(p_r) \leq d_w(\tau) + d(\tau, p_r) \leq d_w(\tau) + d_0.
\]
So \[
|\langle w | p_w \rangle|^2 / (p_w^2) \leq e^{2(d_0 + d_w(\tau))}.
\]
The projection \( p_w \) of \( w \) on \( r^\perp \) is represented by the intersection of \( Cw + Cr \) and \( r^\perp \). So \( p_w \) can be represented by the vector \( p_w = w - \langle r | w \rangle r / r^2 \). One computes that \( p_w^2 = -|\langle r | w \rangle|^2 / r^2 \) and \( |\langle w | p_w \rangle|^2 / (p_w^2) = |\langle r | w \rangle|^2 / r^2 \). Part (c) follows. \( \square \)

Lemma 5.7(a) implies, in particular, that \( |p^{-1}(r | s_v \rangle)^2 \leq 2 \cosh^2(2d_0) \approx 3.6642 \) for all \( v \in D \). So \[
p^{-1}(r | s_v \rangle) \in \mathcal{G}(\leq 2) \text{ for all } v \in D.
\]
(Recall that \( \mathcal{G}(\leq k) \) denotes the set of elements of \( \mathcal{G} \) of norm \( \leq k \)). To obtain further restrictions on \( r \), it will be convenient to use the basis \( v_1, \ldots, v_{10} \) for \( L \otimes \mathbb{C} \) given below:

\[
(v_1, v_2, \cdots, v_{10}) = (-s_4, s_4, -s_3, -s_2, s_2, -s_1, s_1, 0^8; 1, 0, l_\infty).
\]
The roots \( s_d, s_b \) were defined in the proof of lemma 5.2. The inner products between \( v_1, \ldots, v_{10} \) are described as follows: \( v_1, \ldots, v_8 \) are eight roots that form an orthogonal basis for a maximal positive definite subspace of \( L \otimes \mathbb{C} \) whose orthogonal complement has a basis consisting of the two null vectors \( v_9, v_{10} \). Finally \( \langle v_9 | v_{10} \rangle = 2 \). Write \( r \) as a linear combination of \( v_1, \ldots, v_{10} \) in the form

\[
r = p^{-1}(c_1v_1 + c_2v_2 + \cdots + c_{10}v_{10}). \tag{11}
\]
The lemma below gives enough conditions on \( c_1, \ldots, c_{10} \) to allow a computer enumeration of all possible \( (c_1, \ldots, c_{10}) \) and hence, of all possible \( r \).

**5.8. Lemma.** One has \( c_1 \cdots, c_9 \in \mathcal{G}(\leq 2) \) and \( c_{10} \in \mathcal{G}(\leq 9) \). One has \( c_1 + c_2 \equiv c_3 + c_4 \equiv c_5 + c_6 \equiv c_7 + c_8 \equiv c_{10} \mod p \) and \( |c_1|^2 + \cdots + |c_8|^2 = 2 - 2 \Re(c_9c_{10}) \in \{2, 4, 6, 8, 10\} \).

**Proof.** Taking inner product of \( r \) with \( v_1, \ldots, v_{10} \) we find,

\[
c_9 = p^{-1}(v_{10} | r \rangle), c_{10} = p^{-1}(v_9 | r \rangle), \text{ and } c_j = p^{-1}(v_j | r \rangle) \text{ for } j = 1, \ldots, 8.
\]
Since \( L \) is \( p \)-modular, \( c_1, \ldots, c_{10} \in \mathcal{G} \). Write \( r \) in the coordinate system \( 4D_4^* \oplus \mathcal{G}_{1,1}: \)

\[
r = (c_1, c_2 + c_{10}, c_3, c_4 + c_{10}, c_5, c_6 + c_{10}, c_7, c_8 + c_{10}; r_9, c_{10})
\]
where \[
r_9 = (c_9 + (p - 3)c_{10} - c_2 - c_4 - c_6 - c_8) / p. \tag{12}
\]
From the definition of \( D_4^* \), it follows that \( c_{2j-1} + c_{2j} + c_{10} \equiv 0 \mod p \) for \( j = 1, 2, 3, 4 \). This implies the congruences on the \( c_j \)’s.

The bounds on norms of \( c_j \)’s follow from lemma 5.7. As already noted, lemma 5.7(a) implies \( |p^{-1}(r | s_v \rangle) \in \mathcal{G}(\leq 2) \) for \( v \in D \). In particular, \( c_1, \ldots, c_8 \in \mathcal{G}(\leq 2) \).

Next, lemma 5.7(b) implies that \[
|p^{-1}(v_9 | r \rangle)|^2 \leq e^{2(d_0 + d_{v_9}(\tau))} \approx 9.3379,
\]
and

\[ |p^{-1}(v_9)|^2 \leq e^{2(d_0 + d_{16}(\tau))} \approx 3.2043. \]

This implies that \( c_{10} = \bar{p}^{-1}(v_9) \in \mathcal{G}(\leq 9) \) and \( c_9 = \bar{p}^{-1}(v_{10}) \in \mathcal{G}(\leq 2) \). Taking norm on both sides of equation (11) and rearranging, we obtain

\[ |c_1|^2 + \cdots + |c_8|^2 = 2 - 2 \text{Re}(c_9c_{10}). \]

We already know that there are a small number of possibilities for \( c_9 \) and \( c_{10} \). Enumerating these, we find that \( \text{Re}(c_9c_{10}) \in [-4, 4] \cap \mathbb{Z} \). Since \( |c_1|^2 + \cdots + |c_8|^2 \geq 0 \), it follows that \( \text{Re}(c_9c_{10}) \in [-4, 1] \cap \mathbb{Z} \). The lemma follows once we argue that \( \text{Re}(c_9c_{10}) \neq 1 \). If possible, suppose \( \text{Re}(c_9c_{10}) = 1 \). Then \( \sum_{j=1}^{8} |c_j|^2 = 0 \), so \( c_1 = \cdots = c_8 = 0 \). The congruences satisfied by \( c_j \)'s now implies \( c_{10} \equiv 0 \mod p \). Now the condition \( r_9 \in \mathcal{G} \) implies that \( c_9 \equiv 0 \mod p \). But this implies that \( \text{Re}(c_9c_{10}) \in 2\mathbb{Z} \) which is a contradiction. \( \square \)

**proof of theorem 5.9** We use a computer program to enumerate all possible tuples \((c_1, \cdots, c_{10})\) satisfying the conditions of lemma 5.8 and subject to the further restriction that \( c_9 \in \{0, 1, p\} \). We may assume \( c_9 \in \{0, 1, p\} \) since it is enough to enumerate the possible tuples \((c_1, \cdots, c_{10})\) up to units. Let \( r = (\sum_j c_j v_j)/p \). Then \( r \) is a root of \( L \) if and only if \( r_9 \in \mathcal{G} \) (see equation (12)). We run through the possibilities for \( r \) and list those for which \( r_9 \in \mathcal{G} \) and \( d(\tau, r^+) \leq d_0 \). This produces only the unit multiples of \( \{s_v : v \in D\} \). \( \square \)

**5.9. Theorem.** The \( i \)-reflections in the 32 roots \( \{s_v : v \in D\} \) generate \( R(L) \). These generators obey the Coxeter relations dictated by \( D \) as stated in the introduction.

**Proof.** Write \( S = \{s_v : v \in D\} \). Let \( G \) denote the subgroup of \( R(L) \) generated by the reflections in \( S \). Lemma 4.6 provides us a finite set of roots \( S_0 \cup S_1 \cup S_2 \) such that the \( i \)-reflections in them generate \( R(L) \). We take a root \( x_0 \in S_0 \cup S_1 \cup S_2 \) and try to find some \( s \in S \) and \( \xi \in \{i, i^2, i^3\} \) such that \( x_1 = R_{\xi}(x_0) \) is closer to \( \tau \) than \( x_0 \). We repeat this to obtain a sequence of roots \( x_0, x_1, x_2, \cdots \). If some \( x_j \) is an unit multiple of \( S \), then we say that height reduction (with respect to \( \tau \)) is successful for \( x_0 \) and in this case, we obtain \( R_{x_0} \in G \).

In a computer calculation, height reductions is successful for most of the 123426 roots in \( S_0 \cup S_1 \cup S_2 \). For 401 roots (all from \( S_2 \)) height reductions is not successful. In these cases, we end up with a root \( x_j \) whose distance from \( \tau \) cannot be decreased by any reflection in \( S \). Let \( S' \) be the set of these 401 roots. To deal with these cases, by little experimentation, we found a root \( y \in S_2 - S' \) such that height reduction is successful for all the root in \( \{R_y(x) : x \in S'\} \). This means that \( R_y \in G \) and further that for each \( x \in S' \), one has \( R_y R_x R_y^{-1} \in G \); hence \( R_x \in G \). This proves that the reflections in \( S \) generate \( R(L) \). The Coxeter relations between these generators are consequences of the inner products between the roots in \( S \), as given in (7). \( \square \)

**5.10. Remarks on computer calculations:** In the proof of theorem 5.9 we glossed over one step. The roots \( S_0 \cup S_1 \cup S_2 \) in lemma 4.6 are given in the coordinate system \( BW_{16}^0 \oplus \mathcal{G}_{1,1} \) while the 32 roots in \( S \) are given in the coordinate system \( 4D_4^2 \oplus \mathcal{G}_{1,1} \). So we need to find an explicit isomorphism from \( BW_{16}^0 \oplus \mathcal{G}_{1,1} \)
to $4D_4^2 \oplus G_{1,1}$. Our computation used the following 10 vectors in $BW_{16}^2 \oplus G_{1,1}$:

$$v_1 = [p, p, \bar{p}, p, p, \bar{p}, p, 2, -2]/2,$$
$$v_2 = [-\bar{p}, p, p, \bar{p}, p, \bar{p}, p, 2, -2]/2,$$
$$v_3 = [0, p, 0, 0, 0, 0, p, 1, -1],$$
$$v_4 = [1, i, 0, 0, 1, i, 0, 1, -1],$$
$$v_5 = [\bar{p}, p, p, \bar{p}, p, \bar{p}, p, 2, -2]/2,$$
$$v_6 = [-\bar{p}, p, p, \bar{p}, p, \bar{p}, p, 2, -2]/2,$$
$$v_7 = [p, p, p, p, p, p, 1, 1, -1],$$
$$v_8 = [-\bar{p}, p, p, \bar{p}, p, \bar{p}, p, 2, -2]/2,$$
$$v_9 = [4 + p, p, 2 + \bar{p}, 1 + 3i, 4 + \bar{p}, 4 - 6i, -4\bar{p}]/2,$$
$$v_{10} = [0, 0, 0, 0, 0, 1, i] - v_9.$$ 

One verifies that the inner products between these 10 vectors are the same as the 10 vectors $(d_1, c_1, d_2, c_2, d_3, c_3, d_4, c_4, (0^8;1,0), (0^8;0,1))$ that form a basis of $4D_4^2 \oplus H$. So sending $v_1, v_2, \cdots$ to $d_1, c_1, \cdots$ defines an isomorphism from $BW_{16}^2 \oplus G_{1,1}$ to $4D_4^2 \oplus G_{1,1}$. Finding the vectors $v_1, \cdots, v_{10}$ required considerable computation using a list of the 4320 short vectors of $BW_{16}^2$. We shall omit these computational details, since, for the purpose of proving theorem 5.9, these computations are irrelevant once the vectors $v_1, \cdots, v_{10}$ have been found. One has to simply verify that $v_1, v_2, \cdots$ has the same gram matrix as $d_1, c_1, \cdots$. The root $y \in 4D_4^2 \oplus G_{1,1}$ used in the proof of theorem 5.9 to perturb the 401 elements of $S'$ is

$$y = [1 + 2i, 3, 1 + i, 5 + i, 1 + 2i, 4 + i, 1 + 2i, 4 + i, 7, -6i].$$

All the computer calculations needed in this article were performed using the pari/gp calculator. The calculations are contained in the file bw2.gp, available on the website math.iastate.edu/~tathagat/codes. The calculations needed for theorem 5.9 only use exact arithmetic. We should note that a lot of calculations performed while trying height reduction on the 123426 roots become redundant after the fact. To aid verification of our proof, below, we sketch the computations one needs to perform. The 32 roots in $S$ are named $ss[1], \cdots, ss[32]$ in bw2.gp. The function $generate_S_all()$ generates the 123426 roots in $S \cup S_1 \cup S_2$. These are named $s_list[1], \cdots, s_list[123426]$.

The function $generate_path()$ uses $generate_S_all()$ and runs the height reduction algorithm on the 123426 roots. It outputs a large file bwpath that contains a list of vectors $bwpath[1], \cdots, bwpath[123426]$. This program takes about an hour and half to run on a laptop. All the other codes take at most a few minutes. Each $bwpath[1]$ is a string of integers $(n_1, \cdots, n_k)$ with each $n_j \in \{1, \cdots, 64\}$. Rename the file bwpath as bwpath.gp and read it into pari/gp. For each $j$, let

$$R_j = \begin{cases} R_{ss[n_j]} & \text{if } 1 \leq n_j \leq 32, \\ R_{ss[n_j-32]}^{-1} & \text{if } 33 \leq n_j \leq 64. \end{cases}$$

The code $verify_path_all()$ checks that for each applying $R_k R_{k-1} \cdots R_2 R_1$ to $s_list[1]$ produces an unit multiple of an element of $S$.

5.11. Theorem. The thirteen $i$-reflections in $s_a, s_b, s_c, s_d$ for $k = 1, 2, 3, 4$ generate $R(L)$. 
Theorem 5.11 follows quickly from Theorem 5.9. Before giving the proof, we recall a definition from [Ba4].

5.12. Definition. Let \( \{x_j : j \in \mathbb{Z}/k\} \) be the elements in a monoid and let \( m \) be a positive integer. Let \( C_m(x_0, \cdots, x_{k-1}) \) denote the positive homogeneous relation \( x_0 x_1 \cdots x_{m-1} = x_1 x_2 \cdots x_m \).

For example \( C_2(x,y) \) (resp. \( C_3(x,y) \)) denote the relations \( xy = yx \) (resp. \( xyx = yxy \)). Let \( A_{n-1} \) be the affine Dynkin diagram of type \( A_{n-1} \) with vertices labeled by \( \mathbb{Z}/n \). Let \( \{y_j : j \in \mathbb{Z}/n\} \) be the generators for the corresponding Artin group:

\[
y_{ij}y_{kj} = y_{kj}y_{ij} \quad \text{if} \quad j \text{ and } k \text{ are adjacent and } y_{ij}y_k = y_{kj}y_j \quad \text{otherwise.} \tag{13}
\]

If \( y_j^2 = 1 \), then we have a presentation of the Affine Weyl group of type \( A_n \). The relation \( C_{n-1}(y_1, \cdots, y_n) \) collapses this affine Weyl group to the spherical Weyl group or the symmetric group. Following [CS], we call this relation deflating the \( n \)-gon \( (y_1, \cdots, y_n) \). One can verify that in the presence of the braiding and commuting relations of (13), the deflation relation \( C_{n-1}(y_1, \cdots, y_n) \) is equivalent to \( C_{n-1}(y_{j+1}, \cdots, y_{j+n}) \) for any \( j \) (see [Ba4], lemma 4.3 (a)).

Proof of 5.11. Write \( r_v = R_{a_v} \). Let \( G \) be the subgroup of \( R(L) \) generated by the thirteen reflections \( r_a, r_b, r_c, r_k, r_d \). One verifies that \( (d_2, c_2, b_2, a, b_1, c_1, d_1, c_{12}) \) is an octagon in the graph \( D \) and the deflation relation

\[
C_7(r_{d_2}, r_{c_2}, r_{b_2}, r_a, r_{b_1}, r_{c_1}, r_d, r_{c_{12}})
\]

holds in \( R(L) \). This shows that \( r_{c_{12}} \in G \). By \( S_4 \) symmetry we obtain \( r_{e_{jk}} \in G \) for all \( j, k \). Next, one verifies that one has the octagons \( (d_1, c_1, b_1, c_{34}, b_2, c_2, d_2, z), (c_2, b_2, a, b_3, c_{24}, d_4, c_4, f_1), (c_4, b_4, c_{13}, d_3, z, d_2, c_2, h_1), (d_2, c_{23}, f_3, c_4, b_3, a, b_2, g_1) \) in \( D \) and deflation relation holds for each of these octagons. Applying \( S_4 \) symmetry it successively follows that \( r_2, r_{f_k}, r_{h_k}, r_{g_k} \in G \) as well. Now the theorem follows from [Ba4].

\[\square\]

References

[A1] D. Allcock, New complex- and quaternionic-hyperbolic reflection groups. Duke Math. J. 103 (2000) 303-333.

[A2] D. Allcock, The Leech lattice and complex hyperbolic reflections. Invent. Math. 140 (2000) 283–301.

[A3] D. Allcock, A monstrous proposal, in Groups and Symmetries, From neolithic Scots to John McKay, ed. J. Harnad. AMS and CRM, (2009).

[AB1] D. Allcock and T. Basak, Geometric generators for braid like groups, Geom. Topol. 20 no. 2 (2016) 747–778.

[AB2] D. Allcock and T. Basak, Generators for a complex hyperbolic braid group, to appear in Geom. Topol. (2018) arXiv: 1702.05707.

[Ba1] T. Basak, Complex reflection groups and Dynkin diagrams, Ph.D thesis, U.C. Berkeley (2006).

[Ba2] T. Basak, The complex Lorentzian Leech lattice and the bimonster, J. Alg. 309 (2007) 32–56.

[Ba3] T. Basak, Reflection group of the quaternionic Lorentzian Leech lattice, J. Alg. 309 (2007) 57–68.

[Ba4] T. Basak, On Coxeter diagrams of complex reflection groups, Trans. Amer. Math. Soc. 364 No. 9 (2012) 4909–4936.

[Bo1] R. E. Borcherds, The Leech lattice and other lattices, Ph.D. thesis, Cambridge (1985) arXiv:math/9911195

[Bo2] R. E. Borcherds, The Leech lattice, Proc. Royal Soc. London A398 (1985) 365-376.

[Bo3] R. E. Borcherds, Lattices like the Leech lattice, J. Alg. 130 219–234 (1990).
[BMR] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. reine angew. Math. 500 (1998) 127–190.

[ATLAS] J. H. Conway et. al., *Atlas of finite groups*, Oxford University Press, Eynsham, (1985).

[C] J. H. Conway, *The automorphism group of the 26-dimensional even unimodular Lorentzian lattice*, J. Alg. 80 No. 1 (1983) 159–163. (reprinted as chapter 27 of [CS]).

[CSI] J.H. Conway and C.S. Simons, *26 implies the Bimonster* J. Alg. 235 (2001) 805–814.

[CS] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups 3rd ed*. Springer-Verlag (1998).

[DM] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy groups*, Inst. Hautes Etudes. Sci. Publ. Math. 63 (1986) 5–90.

[Goe] J. Goertz, *Reflection group diagrams for a sequence of Gaussian Lorentzian lattices*, Ph.D Thesis, Iowa State University (2017).

[Gol] W. M. Goldman, *Complex Hyperbolic Geometry* Oxford mathematical monographs. Oxford University press, 1999.

[GP] E. Grigorescu and C. Peikert, *List-decoding Barnes-Wall lattices*, Computational complexity, 26 Issue 2 (2017) 365–392.

[M1] G. D. Mostow, *Generalized Picard lattices arising from half-integral conditions*, Inst. Hautes Etudes. Sci. Publ. Math. 63 (1986) 91–106.

[M2] G. D. Mostow, *On Discontinuous action of monodromy groups on the complex n-ball*, J. Am. Math. Soc. 1 No. 3 (1988) 555–586.

[NRS] G. Nebe, E.M. Rains, N.J.A. Sloane, *A simple construction for the Barnes-Wall lattices*, chapter 19 in Codes, Graphs and Systems, Chapter 19, Springer Science+Business Media (2002) 333–342.

[VS] R. Scharlau and B. Venkov, *The genus of the Barnes-Wall lattice*, Comment. Math. Helvetici 69 (1994) 322–333.

[T] W. Thurston, *Shapes of polyhedra and triangulations of the sphere*, Geometry and Topology Monographs, 1 (1998) 511–549.

[W] R. A. Wilson, *The complex Leech Lattice and maximal subgroups of the Suzuki group*, J. Alg. 84 (1983) 151–188.

Department of Mathematics, Iowa State University, Ames, IA 50011

E-mail address: tathagat@iastate.edu

URL: http://orion.math.iastate.edu/tathagat