Quasinormal Modes
and Black Hole Quantum Mechanics
in 2+1 Dimensions

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Abstract

We explore the relationship between classical quasinormal mode frequencies and black hole quantum mechanics in 2+1 dimensions. Following a suggestion of Hod, we identify the real part of the quasinormal frequencies with the fundamental quanta of black hole mass and angular momentum. We find that this identification leads to the correct quantum behavior of the asymptotic symmetry algebra, and thus of the dual conformal field theory. Finally, we suggest a further connection between quasinormal mode frequencies and the spectrum of a set of nearly degenerate ground states whose multiplicity may be responsible for the Bekenstein-Hawking entropy.

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1. Introduction

Since the earliest days of black hole thermodynamics, the belief has been widely held that black hole quantum mechanics would one day lead to deep insights into quantum gravity. The form of the Bekenstein-Hawking entropy strongly suggests that black holes have a discrete area spectrum \[1\], hinting in turn at the quantization of fundamental geometric quantities. This idea received new impetus from a suggestion of Hod \[3\], following an earlier qualitative observation of Bekenstein \[2\], that the classical “ringing” modes of a black hole—the quasinormal modes—might be related by way of the correspondence principle to the elementary quanta of mass and angular momentum. This suggestion received dramatic support from Dreyer’s discovery \[4\] that such a correspondence correctly fixes the Immirzi parameter \[5\], an undetermined prefactor in the area operator of loop quantum gravity.

Hod’s starting point was the observation—originally numerical, but later confirmed analytically by Motl \[6, 7\]—that in the large damping limit, the classical quasinormal modes of a Schwarzschild black hole of mass \(M\) have a real part \(\omega_{\text{QNM}}\) \(~\sim\~\ln \frac{3}{8\pi GM}\). (1.1) Since the horizon area of a Schwarzschild black hole is \(16\pi G^2 M^2\), an identification of \(\hbar \omega_{\text{QNM}}\) with the elementary quantum of mass \(\Delta M\) gives an area spacing of \(\Delta A = 4 \ln 3 \hbar G\), a spectrum of a form suggested earlier for rather different reasons \[8, 9, 10\]. Dreyer subsequently adapted this argument to loop quantum gravity. In that approach to quantization, areas are not equally spaced, but one can still define an “elementary” transition \(\Delta A\), corresponding to a change \(\Delta j = \pm 1\) of a state represented by a spin network (see \[11\]). Remarkably, Dreyer found that the identification \(\Delta M = \hbar \omega_{\text{QNM}}\) fixed the Immirzi parameter to be

\[
\gamma = \frac{\ln 3}{2\pi \sqrt{2}}. \tag{1.2}
\]

But the entropy of a black hole in loop quantum gravity is inversely proportional to \(\gamma\) \[12\], and (1.2) is precisely the right value to give the correct Bekenstein-Hawking entropy.

The value (1.2) is odd enough that it is hard to attribute its appearance to coincidence. But tests of the universality of this approach—looking, for example, at black holes in more than four dimensions, at de Sitter and anti-de Sitter space, and at charged and rotating black holes \[6, 7, 13, 14, 15, 16, 17, 18, 19\]—have so far been, at best, inconclusive. In particular, quasinormal mode boundary conditions depend not only on the geometry near the black hole horizon, but also on the asymptotic geometry. This can cause quasinormal modes to behave very differently in asymptotically flat and asymptotically anti-de Sitter spacetimes. On the other hand, one
might expect black hole energy levels to depend only on local physics near the horizon, and it is not obvious how to reconcile such locality with the quasinormal mode behavior. We therefore decided to consider the (2+1)-dimensional asymptotically anti-de Sitter black hole of Bañados, Teitelboim, and Zanelli (BTZ) [20] to look for connections between classical quasinormal modes and quantization.

The BTZ black hole differs in important ways from black holes in more than three spacetime dimensions (see [21] for a review). In particular, it seems likely that the horizon length, the lower dimensional analog of area, is not quantized [22]. On the other hand, a good deal is known about the quantum mechanics of the BTZ black hole, and the quasinormal modes are known exactly [23,24]. If Hod’s proposed correspondence between quasinormal modes and black hole quantization is correct, some version should appear in this setting as well.

We find that there is, indeed, a relation between quasinormal mode frequencies and black hole quantization in 2+1 dimensions, but also that it is quite different from the Schwarzschild case considered by Hod and Dreyer. As we explain below, the quantum mechanics of the BTZ black hole is characterized by a Virasoro algebra—that is, a two-dimensional conformal algebra—at infinity, which characterizes the dual conformal theory in the AdS/CFT correspondence [25,26]. Identifying the real part of the quasinormal frequencies with the fundamental quanta of black hole mass and angular momentum, we find that an elementary excitation corresponds exactly to a correctly quantized shift of the Virasoro generator \( L_0 \) or \( \bar{L}_0 \) in this algebra. We also argue, somewhat more speculatively, that the quasinormal frequencies may also be associated with the spectrum of a collection of nearly degenerate ground states whose multiplicity may be responsible for the BTZ black hole entropy.

2. Quasinormal Modes and Mass Spectrum

The BTZ black hole is a solution of the vacuum Einstein equations in three spacetime dimensions with a negative cosmological constant \( \Lambda = -1/l^2 \). A rotating black hole is parametrized by its ADM mass \( M \) and angular momentum \( J \). In terms of the location of the inner and outer horizons \( r_\pm \), we have

\[
M = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad J = \frac{r_+ r_-}{4Gl}.
\]  

(2.1)

The BTZ metric is asymptotically anti-de Sitter, and as Brown and Henneaux first showed [25], the diffeomorphisms that preserve the asymptotic structure generate a left- and right-moving Virasoro algebra, with [26,27]

\[
L_0 = \frac{1}{2} (Ml + J) + \frac{l}{16G}, \quad \bar{L}_0 = \frac{1}{2} (Ml - J) + \frac{l}{16G}.
\]  

(2.2)

This algebra can be recognized as a two-dimensional conformal algebra, and the BTZ black hole provides one of the earliest and most explicit examples of the AdS/CFT correspondence.
A detailed study of the quasinormal mode spectrum for the rotating BTZ black hole has been given in [23, 24], following earlier work in [28, 29, 30]. For a perturbation by a field with conformal weight \((h_L, h_R)\), the quasinormal modes take the general form

\[
R(r) \exp\{i(\text{Re} \omega)(t \pm \ell \phi)\} \exp\{i(\text{Im} \omega)t\} \tag{2.3}
\]

with frequencies that can be written naturally in terms of the left and right temperatures as

\[
\omega_L = \frac{k_L}{l} - 4\pi i T_L(n + h_L),
\]

\[
\omega_R = \frac{k_R}{l} - 4\pi i T_R(n + h_R), \tag{2.4}
\]

with a mode number \(n \in \mathbb{N}\) and \(k_L, k_R \in \mathbb{Z}\). The left and right temperatures \(T_{L,R} = (r_+ \mp r_-)/2\pi l^2\) are related to the Hawking temperature \(T_H\) by

\[
\frac{1}{T_L} + \frac{1}{T_R} = \frac{2}{T_H}. \tag{2.5}
\]

We can now apply Hod’s argument to the BTZ black hole. We first note that the real part of the quasinormal frequencies depends only on \(k_{L,R}\) and \(l\), and in particular is independent of the mode number \(n\). Thus, in contrast to the \((3+1)\)-dimensional case, there is no need to go to the large damping limit. Since by (2.3) the quasinormal modes carry angular momentum as well as energy, we require that

\[
\Delta M = \omega_L + \omega_R = \frac{k_L}{l} + \frac{k_R}{l},
\]

\[
\Delta \left(\frac{J}{l}\right) = \omega_L - \omega_R = \frac{k_L}{l} - \frac{k_R}{l}. \tag{2.6}
\]

Using (2.2), we see that we are led directly to a quantization of the Virasoro operators, with

\[
\Delta L_0 = k_L, \quad \Delta \bar{L}_0 = k_R. \tag{2.7}
\]

This result can be viewed both as corroboration of Hod’s correspondence principle and as a new hint about the quantization of the BTZ black hole. Strominger’s asymptotic symmetry analysis [26, 31] does not in itself tell us what conformal field theory describes the quantum black hole, and there is still a great deal of uncertainty about this issue [32]. There are, however, strong arguments that, at least for pure gravity, the relevant theory should be related to an \(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})\) WZW model [32], probably reduced to a Liouville theory [33]. Theories of this class may have “ground states” whose values of \((L_0, \bar{L}_0)\) do not differ by integers, but each such ground state lies at the base of a tower of states with integrally spaced \((L_0, \bar{L}_0)\). For the WZW model, for instance, these states are built up by acting on a ground state with currents \(J^a_n\). The quantization (2.7) is thus exactly what one would expect for the transitions between such states.
3. Ground States and Liouville Theory

We argued above that the BTZ quasinormal frequencies give the correct quantization for transitions among states built on a single ground state. It is well known, however, that in Liouville theory such states are not sufficient to account for the BTZ black hole entropy \[34\]. So it is interesting to ask whether quasinormal modes can tell us anything about possible degenerate or nearly degenerate ground states. Although the answer is more speculative, it is possible that they can.

We begin with a slight detour. A point particle in (2+1)-dimensional anti-de Sitter space can be described as a conical singularity, with deficit angle \(2\beta\) and “time jump” \(2\pi A\). Such a solution has an ADM mass lying between the anti-de Sitter value of \(-1/8G\) and the extremal BTZ black hole value of zero \[25\]. There is, however, another definition of the mass and angular momentum of a conical singularity \[35, 36\], coming from the fact that such a spacetime also solves the Einstein field equations with a delta function stress-energy tensor. In terms of these quantities, the deficit angle is \(2\beta = 8\pi Gm\) and the “time jump” is \(2\pi A = 8\pi Gj\). Since a point particle, like a black hole, is asymptotically anti-de Sitter, it is characterized by Virasoro charges \[25\]:

\[
L_0 = -\frac{l}{16G} \left[ 1 - 4G \left( m + \frac{j}{l} \right) \right]^2 + \frac{l}{16G},
\]

\[
\bar{L}_0 = -\frac{l}{16G} \left[ 1 - 4G \left( m - \frac{j}{l} \right) \right]^2 + \frac{l}{16G},
\]

(3.1)

where we have used the conventions of \[27\]. Applied to \(m\) and \(j\), Hod’s correspondence principle would give

\[
\Delta \left( m + \frac{j}{l} \right) = \frac{2k_L}{l}, \quad \Delta \left( m - \frac{j}{l} \right) = \frac{2k_R}{l},
\]

(3.2)

again relating left- and right-moving quasinormal modes to the holomorphic and antiholomorphic generators of the Virasoro algebra.

Equation (3.2) does not yet have any obvious relationship to the quantum mechanics of black holes. Remarkably, though, there are a set of states in Liouville theory with exactly the same range of conformal weights \[34\] as those of point particles. These states—which are picked out naturally by the quantum group structure of Liouville theory—lie in the “nonnormalizable sector” \[37\], the sector that is known to be needed to give the right counting of states for black hole entropy. While the primary states in this sector have negative norm, recent work indicates that they have descendant states of positive norm that decouple from the negative-norm states in much the same way that the null states decouple in minimal models \[38\]. These states thus provide a natural candidate for a set of nearly degenerate “ground states” on which to build the excited states of the preceding section.
The “decoupled” positive norm states in question have conformal weights

\[ L_0 = -\frac{1}{2\gamma^2}(1 - \alpha\gamma)^2 + \frac{1}{2\gamma^2}, \]

\[ \bar{L}_0 = -\frac{1}{2\gamma^2}(1 - \bar{\alpha}\gamma)^2 + \frac{1}{2\gamma^2}, \]  \hspace{1cm} (3.3)

where \( \gamma^2 = 8G/l \) up to small quantum corrections. Identifying (3.3) with (3.1), we see that the transitions (3.2) correspond to

\[ \Delta \left( \frac{\alpha}{\gamma} \right) = k_L, \quad \Delta \left( \frac{\bar{\alpha}}{\gamma} \right) = k_R. \]  \hspace{1cm} (3.4)

This condition picks out a certain natural set of quantum group representations, and calculations [38] indicate that these representations, combined with their towers of excited states, give exactly the right statistical mechanical counting of the BTZ black hole entropy.

4. Discussion

We began with the question of whether Hod’s suggested correspondence principle, relating classical quasinormal mode frequencies to black hole quantization, could be tested in three spacetime dimensions. In one sense, we have found that it could, and that the quantum mechanics of the BTZ black hole gives support to Hod’s proposal. One part of the test is fairly clear: quasinormal mode energies and angular momenta describe transitions among excited states that have a clear interpretation in terms of the Virasoro algebra of the dual conformal field theory. A second piece is more tentative, but potentially even more exciting: the quasinormal mode correspondence may offer us insight into the detailed construction of black hole states in Liouville theory.

On the other hand, our results are also rather puzzling. Although we have found a connection between quasinormal modes and quantum black holes in 2+1 dimensions, the connection is very different from the one that has been suggested for asymptotically flat black holes in 3+1 dimensions. We have not found a quantization of horizon size, and have no analog of the Immirzi parameter (1.2). Given the contrast between quasinormal modes in asymptotically flat and asymptotically AdS spaces, differences are perhaps to be expected, but it is then surprising that a strong connection to quantization survives. In short, while we find evidence for a correspondence of the sort suggested by Hod, the details of this correspondence are apparently considerably more complicated than has been appreciated. We may learn more by studying the large class of string theory black holes whose near-horizon geometry contains the BTZ solution. It will be interesting to see whether
the quasinormal mode structure of these higher-dimensional black holes is consistent with the quantization we have found in 2+1 dimensions.

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