SHALLOW-WATER AND DEEP-WATER LIMITS OF THE GENERALISED FINITE-DEPTH-FLUID EQUATION

GUOPENG LI

Abstract. We study convergence problems for the finite-depth-fluid equation (FDF) on $\mathbb{R}$ and $\mathbb{T}$ from a microscopic point of view. The FDF equation can be naturally viewed as an interpolation between the Korteweg-de Vries equation (KdV) and Benjamin-Ono equation (BO). In this paper, we prove that the solutions of FDF converge to those of KdV in the shallow-water limit (as the fluid depth goes to zero), and to those of BO in the deep-water limit (as the fluid depth reaches infinity). Our convergence results also apply to the FDF with an analytic nonlinearity $f(u)$. In the case of FDF $f(u) = u^2$, we improve the previous results by Abdelouhab-Bona-Felland-Saut (1989); and to the best of our knowledge, the quartic case $f(u) = u^4$ is the first time studied here; in the case where $f(u) = u^k$ for $k \geq 5$, our results improve those of Han-Wang (2008). Moreover, we provide the first convergence results on $\mathbb{T}$, while all previous studies focused on $\mathbb{R}$.

Contents

1. Introduction 2
   1.1. Finite-depth-fluid equation 2
   1.2. Limit equations and generalised finite-depth-fluid equations 5
   1.3. Main results 10
2. Preliminaries 14
   2.1. Notations 14
   2.2. Dispersion relation 15
   2.3. Function spaces and their basic properties 18
3. Linear and short-time Strichartz estimates 20
4. Energy estimates 27
   4.1. Preliminary technical estimates 27
   4.2. A priori bounds on solutions 31
   4.3. A priori bounds on the difference of solutions 44
5. On the construction of solution 58
6. On limiting behaviour of $g$FDF 63
   6.1. Deep-water limit on $\mathbb{T}$ 63
   6.2. Shallow-water limit on $\mathbb{T}$ 65
Appendix A. On the discussion of limiting behaviour on $\mathbb{R}$ 68
   A.1. Short-time Strichartz estimates on $\mathbb{R}$ 68
References 71

2020 Mathematics Subject Classification. 35A01, 35A02, 35Q53, 35Q35.
Key words and phrases. finite-depth-fluid equation, intermediate long-wave equation, Benjamin-Ono, KdV, limit behaviours, energy method, well-posedness.
1. Introduction

In this paper, we concern with the rigorous mathematical study of the limiting behaviour of the finite-depth-fluid equation (FDF), which is an equation that has captured much interest in the fields of mathematics and physics. The FDF equation is often known as the intermediate long-wave equation, and it is a one-way propagation asymptotic model for internal waves travel in an appropriate fluid regime. For example, one could view this as waves travelling in a stratified fluid of finite depth (the fluid system is separated into upper and lower regions by an interface).

The FDF equation is an important physical model because it forms a natural connection between the model equations of the shallow and deep water regions and also formulates wave propagation in the intermediate region. Different versions of studies have been presented in the literature to explain its features of behaviour and its application in modelling real-world situations. For example, FDF has played a crucial role in the context of long internal gravity waves [43, 46, 75, 78]. It is also viewed as the crucial model for wave propagation in localised regions of waveguides, such as ocean pycnocline, atmospheric fronts or regions of temperature inversion [51, 52]. Furthermore, in modelling nonlinear waves propagating on two opposite edges of a quantum Hall system, we refer to [8]. To read about further applications of FDF, we shall refer the interested readers to [18, 54, 57, 71, 74]. Alongside physical applications and its convergence features, the FDF displays other interesting features, such as the N-soliton solutions, Hamiltonian structure, complete integrability, etc. See for example [4, 17, 33, 45, 50, 19].

1.1. Finite-depth-fluid equation. The derivation of the FDF was initially done by Joseph [32] and Kubota-Ko-Dobbs [46], see also [2, 56] in the periodic boundary conditions. The FDF on the real line $\mathbb{R}$ and on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is given by:

$$\partial_t u - \left(1 + \frac{1}{\delta}\right)G_\delta(\partial_x^2 u) + \partial_x(u^2) = 0 \quad (t, x) \in \mathbb{R}_+ \times \mathcal{M}. \quad (1.1)$$

Here, $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ and $u$ is a real-valued function. The operator

$$G_\delta = -\coth(\delta \partial_x) - \frac{1}{\delta} \partial_x^{-1}$$

characterises the phase speed and it is understood as the Fourier multiplier defined by,

$$\hat{G}_\delta f(\xi) := -i\left\{ \coth(\delta \xi) - \frac{1}{\delta \xi} \right\} \hat{f}(\xi) \quad \text{for } \xi \in \hat{\mathcal{M}},$$

where $f \in \mathcal{D}'(\mathcal{M})$. Here, $\coth(x)$ denotes the usual hyperbolic cotangent function:

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}, \quad x \in \mathbb{R} \setminus \{0\}.$$

The frequency domain is defined by

$$\hat{\mathcal{M}} = \begin{cases} \mathbb{R} & \text{if } \mathcal{M} = \mathbb{R}, \\ \mathbb{Z} & \text{if } \mathcal{M} = \mathbb{T}. \end{cases}$$
Furthermore, the parameter $\delta > 0$ represents the fluid depth. The dispersion term of (1.1) comes from a specific choice of the kernel $K(x - y)$ of the following Whitham equation:

$$
\partial_t u + \partial_x \int_{-\infty}^{\infty} K(x - y) u(t, y) \, dy + \partial_x (u^2) = 0.
$$

(1.2)

The equation (1.2) was developed to understand the breaking of nonlinear dispersive water waves, see for example [84]. Indeed, in the very first work of Joseph [32], he observed that FDF is a particular form of the Whitham equation (1.2), by using the dispersion relation derived in [73]. In particular, such a choice of kernel typically arises in a two-layer fluid of finite depth, and FDF describes the wave phenomena at the intermediate depth of a two-layer system. More interestingly, FDF forms the natural connection between the waves on shallow water surfaces and the waves on infinitely deep fluids.

This connection leads to one of the most important properties of the FDF: It has the following Korteweg-de Vries equation (KdV) the limiting formulation in the shallow-water region:

$$
\partial_t u + \partial_x^3 u + \partial_x (u^2) = 0.
$$

(1.3)

The following Benjamin-Ono equation (BO) as its limiting formulation in the deep-water region:

$$
\partial_t u - \mathcal{H} (\partial_x^2 u) + \partial_x (u^2) = 0,
$$

(1.4)

where $\mathcal{H}$ in (1.4) is the spatial Hilbert transform defined by

$$
\widehat{\mathcal{H}f}(\xi) := -i \text{sgn}(\xi) \hat{f}(\xi) \quad \text{for} \ \xi \in \mathcal{M}.
$$

All of these equations: FDF (1.1), KdV (1.3), and BO (1.4) arise in the propagation of long internal waves in a stratified fluid. Moreover, by making a suitable choice of the Kernel in (1.2), we can yield the desired forms of FDF, KdV, or BO, respectively (i.e. (1.1), (1.4), and (1.3) are a special form of (1.2)). See for example [84, Section 6].

Our goal is to address the following question: the convergence of FDF to its limiting forms in the shallow-water and deep-water regions. We recall that the depth parameter of (1.1) characterises the dispersion phenomenon of the internal waves between the shallow-water and deep-water regions. By shallow (resp. deep) region, we are referring to those water regions whose total fluid depth is much smaller (resp. larger) than the wavelength. The differencing depth parameter $\delta$ leads to the physical observation that the internal waves (characterised by $\delta$) act as an intermediary between waves in shallow water surfaces and those in infinitely deep water. As an important consequence, formally we see FDF reduce to KdV and BO, it was also shown that the exact stationary wave solution of FDF converges to those solutions of KdV and BO, respectively [3, 32, 56]. For example, by following the discussion in [3, Chapter 3] that the FDF soliton has the following form:

$$
u_{\text{FDF}} = \frac{(1 + \delta^{-1})k_1 \sin(\delta k_1)}{\cos(\delta k_1) + \cosh(k_1 x - t[1 + \delta^{-1}]\{k_1 \delta^{-1} \cot(\delta k_1)\]) + k_2},
$$

where $k_1, k_2$ are arbitrary parameters. Then, it converges to the KdV soliton:

$$
u_{\text{KdV}} := \frac{k_1^2}{2} \operatorname{sech}^2 \left( \frac{1}{2} k_1 x - \frac{1}{3} k_1^2 t + k_2 \right),
$$

as $\delta \to 0$. Here, a restriction, $0 < k_1 \delta < \pi$ is required, and it is doable as $\delta$ can be taken arbitrarily small. On the other hand, in the case of limit $\delta \to \infty$, we observe that there is
no proper limit unless we send \( k_1 \to 0 \). Then, by taking \( \delta k_1 = \pi - \frac{k_1}{C} \) with a positive real constant \( C \) yields the BO rational soliton:

\[
\text{u}_{\text{BO}} := \frac{2C}{1 - C^2(x - Ct)^2}.
\]

This convergence of solitary solutions, together with the formal convergence at the equation level, give evidence that one can regard the properties of KdV and BO as the limiting properties of FDF, as the depth parameter \( \delta \to 0 \) and \( \delta \to \infty \), respectively.

Indeed, from the physical viewpoint, such convergence properties has attracted much attention around the 1980s \[3, 32, 45, 49, 50, 75, 76\]. Physicists have tried to understand the connection of waves propagation at the different levels of the two-layer model mainly in two ways: (i) taking a substitution of the limiting forms of the dispersion terms to show the convergence process; (ii) comparing those similar features of wave evolutions in the shallow-water, the water of intermediate-depth, and infinitely deep water. Although, the justifications of convergence in the above literature is rather formal from the mathematical viewpoint, the reduction process inspired our later analysis. In Section 1.2, we provide detailed discussions on the derivation of limiting formulations in shallow and deep water regions.

In order to rigorously study the question of convergence of solutions to FDF, we need to overcome the following mathematical difficulties: (i) First, construct analytical solutions to the FDF; (ii) the justification of the existence of the limiting objects. In this paper, the discussion is based on the deterministic theory. We show convergence of the solutions (individual ones) on a microscopic scale. Alternatively, we can answer this question from the probabilistic viewpoint, see \[53\] where they consider the initial data sampled from some statistical ensemble and then study the convergence properties of the resulting solutions. The FDF equation offers both physical and mathematical interested models. We acknowledge that one of the central motivations for studying the convergence properties of FDF is that physicists have observed that FDF plays the intermediary role between KdV and BO. In the rest of the present paper, we focus on the mathematical study of the convergence problem for FDF in shallow and deep water regions.

Before we go into the derivation of the limiting forms of (1.1), we make the following important comments: (i) First of all, FDF as formulated in (1.1) was introduced for considerations that obscure the difference between the scaling that leads to KdV and BO, see \[3\ p. 211\] and \[1\ p. 384\]. In particular, the constant \((1 + \frac{1}{\delta})\) of (1.1) was introduced to capture both limiting behaviours when \( \delta \) is small and large. For \( \delta \gtrsim 1 \) large, from (1.1) we obtain the formulation

\[
\partial_t u - G_\delta(\partial_x^2 u) + \partial_x(u^2) = 0,
\]

which we may often see in the physics studies for example in \[49, 75, 76\]. Moreover, such formulation as a mathematical model captures only the internal wave propagation in the deep-water region, see discussion in next section. As for the shallow-water behaviour \((\delta \ll 1 \) small), equation (1.1) gives the formulation

\[
\partial_t u - \frac{1}{\delta} G_\delta(\partial_x^2 u) + \partial_x(u^2) = 0,
\]

which is the correct model that captures the internal wave propagation in the shallow water region. Although the equation (1.1) leads to two formulations when the fluid depth is small and large, these two formulations are both correct models for the internal wave propagation
of the finite-depth fluid in an intermediate state. Here, we use different models to capture wave evolution in shallow and deep-water regions.

(ii) It is natural to see in our mathematical study that FDF has two formulations in the shallow and deep water regions separately. The physical FDF model was introduced to the two-layer system, for example, [46, 32], and it is a two-depth-parameter equation. In particular, the dispersive effects are characterised by the depths of the upper and lower layers. It was also suggested that the physical FDF model with two parameters degenerates into one parameter, see [46]. However, in our mathematical model, our equation has no physical parameters, which causes the lack of information on the ratio of the interface depths and wavelength. As a result, in the mathematical model, we need to separate into two cases according to the fluid depths. Therefore, introducing the constant \((1 + \frac{1}{\delta})\) as in (1.1) is efficient in capturing the wave evolution in shallow and deep water regions. Alternatively, introducing a rescaled amplitude, see (1.10) in the later discussion, can also resolve such issues of missing information on capturing the different water evolution.

In what follows, we give a careful analysis (at the equation level) of the correct FDF formulation, which converges to KdV in the shallow-water limit and BO in the deep-water limit. Moreover, we look into the generalisation of FDF, in the sense that one can replace the quadratic nonlinearity with some other generalised nonlinear terms.

1.2. Limit equations and generalised finite-depth-fluid equations. The correct mathematical FDF model in showing the shallow-water limit was discussed in [1, 3], where [1] applied a necessary rescaling to the FDF (1.5) and [3] introduced the factor \((1 + \frac{1}{\delta})\) as we used in (1.1). Both [1, 3] pointed out that in showing the shallow-water and deep-water limits, we should consider the corresponded mathematical formulation, which characterises the correct dispersive phenomenon.

In what follows first, we give the formal discussion on the limiting formulations of (1.5) and (1.9). The approach we follow is introduced in [1] and we also explain that these two formulations are reversible via a scaled amplitude.

- **Deep-water limit:**

We start with the formal derivations of BO limit in the deep-water region \((\delta \geq 1)\). The internal wave phenomena are characterised by the FDF of the following form

\[
\partial_t u - G_\delta(\partial_x^2 u) + \partial_x(u^2) = 0.
\]

(1.5)

We can restate the equation (1.5) as a perturbation of the Benjamin-Ono equation (BO), which we recall in the following

\[
\partial_t u - \mathcal{H}(\partial_x^2 u) + \partial_x(u^2) = 0.
\]

(1.6)

In particular, we make the following adjustment to (1.5), by adding and subtracting the BO dispersion term \(\mathcal{H}(\partial_x^2 u)\)

\[
\partial_t u - \mathcal{H}(\partial_x^2 u) + \partial_x(u^2) + K_\delta(\partial_x u) = 0,
\]

(1.7)

where \(K_\delta\) is defined as a Fourier multiplier with symbol \(q_\delta\):

\[
\widehat{K_\delta u}(\xi) = q_\delta(\xi)\widehat{u}(\xi) = \{|\xi| - \xi \coth(\delta \xi) + \frac{1}{\delta}\}\widehat{u}(\xi).
\]

(1.8)
One sees that for all $\xi \in \mathbb{R}$ (see for example Lemma 2.3),
\[ 0 \leq q_\delta(\xi) \leq \frac{2}{\delta}. \]
As $\delta$ tends to $\infty$, it is clear that $K_\delta(u_x)$ tends to 0. As a result, for all $\xi \in \mathbb{R}$, we can deduce that
\[ \lim_{\delta \to \infty} \left( \coth(\delta \xi) - \frac{1}{\delta \xi} \right) = \text{sgn}(\xi), \]
which together with (1.7) leads to the formal convergence of FDF (1.5) to BO (1.6).

- **Shallow-water limit:**

Here, to distinguish the formulations in capturing the solution behaviours in the deep-water and the shallow-water regions, we use $v(t, x)$ as the solution to the FDF formulation (1.9). In the shallow-water limit, the convergence of the solutions is considered for the following reformulated equation
\[ \partial_t v - \frac{3}{\delta} G_\delta(\partial_x^2 v) + \partial_x(v^2) = 0. \]
(1.9)
The reason we say (1.9) is the (reformulated) FDF is because (1.9) can be seen as a rescaled formulation of (1.5). In particular, by using some suitable scaled amplitude such as
\[ v(t, x) = 3\delta^{-1} u(3\delta^{-1} t, x). \]
(1.10)
One can readily recover (1.5) from substituting (1.10) into (1.9). Here, it is worth noticing that the rescaling (1.10) is harmless for fixed $\delta > 0$. In the sense that one can switch between (1.5) and (1.9) for any fixed $\delta$. We recall that the power series of $\coth(z)$ is such that
\[ \coth(z) = \frac{1}{z} + \frac{z^3}{3} - \frac{z^5}{45} + 2\frac{z^7}{45} + O(z^7), \]
(1.11)
where $0 < |z| < \pi$. Then, by using (1.11), one can consider the operator $G_\delta$ as a formal power series in $\delta$, which is valid since $\delta$ can be taken arbitrarily small. Therefore, we obtain the following expression:
\[ G_\delta = \coth(\delta \partial_x) - \frac{1}{\delta} \partial_x^{-1} = \frac{e^{\delta \partial_x} + e^{-\delta \partial_x}}{e^{\delta \partial_x} - e^{-\delta \partial_x}} - \frac{1}{\delta} \partial_x^{-1} = \frac{\delta \partial_x}{3} + \frac{\delta^3 \partial_x^5}{45} + O(\delta^5). \]
(1.12)
Next, by replacing (1.12) into the dispersion term of (1.9) and sending $\delta \to 0$, we obtain
\[ \lim_{\delta \to 0} 3\frac{\delta}{\partial_x} G_\delta(\partial_x^2 v) = \lim_{\delta \to 0} \left( \partial_x^3 + \delta^2 \frac{\partial_x^4}{15} + O(\delta^4) \right) v = \partial_x^3 v. \]
Hence, the (scaled) FDF (1.9) reduces to the following KdV:
\[ \partial_t v + \partial_x^3 v + \partial_x(v^2) = 0 \]
(1.13)
Furthermore, the power series expansion of $G_\delta$ (1.12) shows that equation (1.5) does not give a correct limiting form as $\delta \to 0$. In particular, let us substitute (1.12) into (1.5) and neglecting $O(\delta^3)$ terms. The resulting limit equation of (1.5) as $\delta \to 0$, is of the following form:
\[ \partial_t u + \frac{\delta}{3} \partial_x^3 u + \partial_x(u^2) = 0, \]
(1.14)
Moreover, we see from (1.14) that it is improper to send $\delta \to 0$. This limit yield the following inviscid Burgers’ equation:
\[ \partial_t u + \partial_x(u^2) = 0. \]
(1.15)
Therefore, without a suitable scaling on $\text{(1.5)}$ may be misleading, since the dispersion term of $\text{(1.14)}$ disappears along with $\delta \to 0$. Also, $\text{(1.14)}$ and $\text{(1.15)}$ suggest that a suitable rescaling of $\text{(1.5)}$ on dispersion term lead us to see the meaningful limit in the shallow-water region. As a conclusion, it is crucial that we only study the solutions to $\text{(1.9)}$ that tend to those of KdV $\text{(1.13)}$, as $\delta \to 0$.

**Generalised finite-depth-fluid equation:**

Previously, we see the equations with the quadratic nonlinearity, which causes the steepening behaviour of one-side travelling waves. Solitary wave solutions have a significant physical meaning, especially for the KdV and BO models, see [6, 7, 70, 46]. It is well-known that when the dispersive effects exactly balance the nonlinear effects, we obtain solitary waves (the shape-preserving feature of the nonlinear waves). For the mathematical reason, in the following, we introduce the FDF equation associated with stronger nonlinear steepening effects or more generalised nonlinearities. The Cauchy problem of the generalised finite-depth-fluid equation (gFDF) on $\mathbb{R}$ and $\mathbb{T}$ is given by:

$$
\begin{aligned}
\begin{cases}
\partial_t u - \left(1 + \frac{1}{2}\delta \right) G_\delta (\partial_x^2 u) + \partial_x (f(u)) = 0, \\
u|_{t=0} = u_0,
\end{cases}
\end{aligned}
$$

(1.16)

Here, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a real analytic function with an infinite radius of convergence. In particular, this covers all the power-type nonlinearities. See Remark 1.2 for more details. One observation from the shallow and deep water limits is that, the derivations of the limiting equations do not get affected by changing the nonlinearity, see for example [27, 28]. To study the convergence of the gFDF $\text{(1.16)}$, we also have similar limiting formulations in shallow-water and deep-water regions. In particular, the gFDF of the deep-water formulation

$$
\partial_t u - G_\delta (\partial_x^2 u) + \partial_x (f(u)) = 0
$$

(1.17)

converges to the generalised BO (gBO)

$$
\partial_t u - H (\partial_x^2 u) + \partial_x (f(u)) = 0,
$$

(1.18)

as $\delta \to \infty$. The (scaled) gFDF of the shallow-water formulation

$$
\partial_t v - \frac{3}{\delta} G_\delta (\partial_x^2 v) + \partial_x (f(v)) = 0
$$

(1.19)

converges to the the generalised KdV (gKdV)

$$
\partial_t v + \partial_x^2 v + \partial_x (f(v)) = 0,
$$

(1.20)

as $\delta \to 0$.

In the following, we go over the previous convergence results on the FDF and gFDF with power-type nonlinearities (i.e. $f(u) = u^k$ for $k \geq 3$).

**The FDF convergence:**

The question of convergence of FDF indeed arises naturally from the physical side, see [32, 46, 56], which presented the convergence of solitary-wave solutions to the FDF and numerically investigated the convergence of FDF. At the level of the equations, we have seen that KdV and BO can be regarded as the limiting forms of FDF.

However, the above information is insufficient to deduce the limiting behaviour of analytic solutions. In particular, without a priori information on the solutions to FDF, such as their the existence or how they evolve along with the fluid depth, we cannot pursue the
question of convergence. To answer such a question, the first mathematical study was done by Abdelouhab-Bona-Felland-Saut [1] (1989). They studied the convergence of FDF solutions in $C(\mathbb{R}^+; H^s(\mathbb{M}))$ by using PDE techniques: (i) deep-water limit for $s > \frac{2}{3}$; (ii) shallow-water limit for $s \geq 2$. See [50].

Their method did not exploit any dispersive nature of the equation, hence the, high regularity assumptions were needed for these two convergence results. Secondly, we notice that the regularities are different between the shallow and deep water limits. This is due to the $\delta$-independent estimates of solutions to the FDF (1.5) ([1, Theorem 6.1]) does not directly apply to the formulation (1.9). Although equations (1.9) and (1.5) are invariant under a suitable scaling, the $\delta$-uniformity breaks down under this scaling. We refer to [1, Lemma 8.2.2] for the details.

Furthermore, let us mention the interesting work [75] where it was shown that when $\delta \to \infty$, the inverse scattering transform scheme of FDF reduces to that of BO. This work provides additional evidence for the convergence properties of FDF to BO, although their mathematical interests rather focused on inverse scattering transform schemes.

• The modified FDF convergence:
In the case $f(u) = u^3$ of (1.16), the equation is known as the modified FDF equation (mFDF). The convergence of the solutions to mFDF was (not for the mKdV-limit) studied by Guo-Wang [27], where they showed that solutions of mFDF converges to those of mBO in $C(\mathbb{R}^+; H^s(\mathbb{R}))$ for $s \geq \frac{1}{2}$, as $\delta \to \infty$. Similar issues occur in the shallow-water region (for $\delta$ small), meaning that the $\delta$-independent estimates cannot be obtained in the same regularity space. Hence, the authors of [27] did not include the convergence of the solutions to the (scaled) mFDF in the shallow-water surface.

We notice that in [27], they first applied the modern techniques from dispersive PDEs. In particular, they applied the local smoothing and Strichartz estimates. This allowed them to study the convergence of the mFDF in rougher $L^2$-based Sobolev spaces.

• The gFDF convergence:
In the case of gFDF (1.16) where $f(u) = u^k$ for $k \geq 5$, this convergence problem was studied by Han-Wang [28], where they considered the Besov-type spaces on the real line. We shall refer the interested readers to [28].

In the following, we summarise the convergence results for FDF, mFDF, and gFDF. Moreover, for comparison reasons, we include our main deterministic results from Theorems 1.8 and 1.9.
Table 1. Summary table

| Nonlinearity | Space | Limit $\delta \to 0$ | Limit $\delta \to \infty$ |
|--------------|-------|-----------------------|---------------------------|
| $u^2$        | $u \in H^s(M)$ | $s \geq 2$: [1]. | $s > \frac{3}{2}$: [1]. |
|              |       | $s > \frac{1}{2}$ on $\mathbb{R}$: Thm 1.8 | |
|              |       | $s \geq \frac{2}{3}$ on $\mathbb{T}$: Thm 1.8 | |
| $u^3$        | $u \in H^s(M)$ | Thm 1.8 | $s \geq \frac{1}{2}$ on $\mathbb{R}$: [27]. |
|              |       |                        | Thm 1.9 |
| $u^4$        | $u \in H^s(M)$ | Thm 1.8 | Thm 1.9 |
| $u^{k+1}$ ($k \geq 4$) | $u \in \dot{B}^s_{2,1}(\mathbb{R})$ | $s_k = \frac{7}{2} - \frac{2}{k}$: [28]. | $s_k = \frac{3}{2} - \frac{1}{k}$: [28]. |
|              | $u \in H^s(M)$ | Thm 1.8 | Thm 1.9 |

We make a few comments on Table 1:

(i) One of the main goals of this paper is to treat this convergence question on the torus $\mathbb{T}$. We note that the arguments of [27, 28] work only in the real line setting. The method we use to study the convergence of gFDF works equally well on both $\mathbb{R}$ and $\mathbb{T}$. See Remark 1.11 for further discussion.

(ii) In [28], the authors considered gFDF (1.16) with $f(u) = u^{k+1}$ for $k \geq 4$. The convergence results of the quartic gFDF (i.e. $f(u) = u^4$) are missing there. Moreover, the function space in establishing the convergence of solutions in [28] was Besov spaces, we do not go into further details. In this paper, we focus on the $L^2$-based Sobolev spaces, and we give the first convergence results of the quartic gFDF. Moreover, for the mFDF and gFDF we give the first convergence results on $\mathbb{T}$.

(iii) It is worth emphasising that in [1], the KdV-limit required a higher regularity assumption (unknown for mKdV-limit in [27]). Our main convergence results require less regularity without relying on the complete integrability.

Remark 1.1. It is interesting to note from [1.8] that the same behaviour occurs in both the short-wave limit (high frequency) and the infinite-depth limit (arbitrary large $\delta$). This indicates that in the water of finite depth, the waves with relatively short wavelengths compared to the total depth, they behave similarly to long waves of infinitely deep water. On the other hand, long waves, along with the fluid interface in finite-depth water, behave similarly to those in shallow water surface.

1In [28], the Besov spaces were for the initial data, their convergence limit was taken in some other spaces, we only discuss their results here for the completeness of the table.
Moreover, numerical simulations have also suggested that solutions behave in a continuous manner with respect to the fluid depth, and the two limiting solutions are bridged continuously by the FDF solution, see in [46, Fig. 4-6].

**Remark 1.2.** The hypothesis on $f$ ensures that $f$ is of $C^\infty$ class. In particular,

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad \forall x \in \mathbb{R}.$$  

It is clear that any polynomial function is of this form. Moreover, the exponential functions as $e^x, \sin(x), \cos(x)$, and their products or compositions are also in this class.

1.3. **Main results.** The main goal of the deterministic theory is to study the convergence behaviour for the analytic solutions to FDF in low regularity Sobolev spaces $^{2}$ The method of constructing the solutions to FDF works for the generalised nonlinearities. Therefore, we focus on studying the convergence of gFDF (1.16), as the results easily cover FDF by letting $f(u) = u^2$. We emphasise our deterministic convergence problem in the following:

**Convergence problem.** Suppose the initial data is fixed and belongs to some Sobolev space $H^s(\mathcal{M})$. Then, the solution to the gFDF (1.16) converges, uniformly on bounded time intervals, to the solution to the gKdV (1.20) or to the gBO (1.18) with the same initial data, as $\delta$ approaches 0 or $\infty$, respectively. Recall here, $\delta$ denotes the fluid depth.

The dispersive effects $G_\delta$ vary along with the depth of the medium (as in Section 1.2), we separate our analysis into: (i) shallow-water theory; (ii) deep-water theory. In what follows, we first construct the solutions to the equations (1.19) and (1.17).

**• Solution construction:**

**Theorem 1.3 (Shallow-water theory).** Let $\delta_0 > 0$ to be fixed. The following two statements hold for any $\delta \in (0, \delta_0)$.

(i) (real line case). Let $s > \frac{1}{2}$. Then, the (scaled) gFDF (1.19) is locally well-posed in $H^s(\mathbb{R})$.

(ii) (torus case). Let $s \geq \frac{3}{4}$. Then, the (scaled) gFDF (1.19) is locally well-posed in $H^s(\mathbb{T})$.

Moreover, on the both real line and the torus the local existence time $T = T(\|v_0\|_{H^s(\mathcal{M})}, \delta_0) > 0$ is independent of $\delta$.

On the other hand, for the deep-water evolution, we have:

**Theorem 1.4 (Deep-water theory).** Let $s \geq \frac{3}{4}$, fix $\delta_0 > 0$, and $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. Then, for any $\delta \in [\delta_0, \infty)$ the gFDF (1.17) is locally well-posed in $H^s(\mathcal{M})$. Moreover, the local existence time $T = T(\|v_0\|_{H^s(\mathcal{M})}, \delta_0) > 0$ is independent of $\delta$.

**Remark 1.5.** The $\delta$-independence of Theorems 1.3 and 1.4 are crucial in our later convergence part, which needs to be considered separately for large $\delta$ and small $\delta$. The different formations capture the different evolution in the shallow-water and deep-water regions, which causes the different regularity assumptions in the different fluid regions.

---

$^2$ The convergence results in [1] were obtained in $H^s$ for some $s > \frac{3}{2}$. In our study, our goal is to go (at least) blow $s < 1$. 

G. Li

10
Remark 1.6. (i) Our method in showing Theorems 1.3 and 1.4 is closely related to [64, 63]. However, it requires us to carefully examine the dependence on the varying parameter $\delta$ for all the relevant estimates.

(ii) The Cauchy problem of gKdV (1.20) and gBO (1.18) have been studied extensively. Especially, for the most important case where we have the quadratic nonlinearity, there have been significant developments in the past decades, for instance, Kenig-Ponce-Vega [36, 37], Colliander-Keel-Staffilani-Takaoka-Tao [16], Abdelouhab-Bona-Felland-Saut [1], Koch-Tzvetkov [44], Tao [81], Molinet-Vento [64]. We mention the most recent breakthrough by Killip-Visan [42] and Gérard-Kappeler-Topalov [25] that explored the complete integrability of the equation. For the cubic nonlinearity, it has also been well-studied. For example, we refer to Bourgain [10], Nakanishi-Takaoka-Tsutsumi [68], and Molinet-Pilod-Vento [67]. For other related results and the case of higher-order nonlinearity, we refer to [11, 14, 5, 15, 16, 20, 31, 22, 23, 24, 29, 26, 39, 58, 30, 34, 38, 48, 40, 41, 47, 62, 61, 66, 59, 65, 63, 72, 83, 85].

Remark 1.7. Our solutions are understood as distributional solutions. In particular, for any test function $\phi \in C^\infty_c((-T, T) \times M)$, the following holds

$$\int_0^\infty \int_M \left( (\phi_t + (1 + \delta^{-1})G_\delta \phi_{xx})u + \phi_x f(u) \right) dx dt + \int_M \phi(0, \cdot)u_0 dx = 0. \tag{1.21}$$

Then, we note that for $u \in L^\infty((0, T); H^s(M))$ with $s > \frac{1}{2}$, $f(u)$ is well-defined and belongs to $L^\infty((0, T); H^s(M))$. Moreover, $\|u_0\|_{H^s} \leq \|u\|_{L^\infty_t H^s}$. Finally, we notice that this also ensures that $u$ satisfies the following Duhamel formulation:

$$u(t) = S(t)u_0 - i \int_0^t S(t-t')[\partial_x(f(u))] (t') dt',$n

where $S(t) := e^{-t(1+\delta^{-1})G_\delta \partial^2_x}$ denotes the linear propagator. See for example [63] Definition 1 and Remark 1.4 and [72] Definition 1.1 and Remark 1.2 for a similar discussion.

• Main convergence results:

We first study the limiting behaviour of solutions that evolve in the shallow-water region. Namely, we show that the solution to the (scaled) gFDF (1.19) converges to the solution to the gKdV (1.20), as $\delta \to 0$. We define the solution map to the (scaled) gFDF (1.19) by

$$S_{T,\delta}(v_0) = v_\delta,$n

where $S_{T,\delta}(v_0) := S_T(v_\delta) = v_\delta$.

Here, we notice that the solution map (1.22) can be used to define the solution maps of (1.20), (1.17), and (1.18) in a similar manner. Moreover, we recall that our solutions are actually characterised by $\delta$. Our goal in the shallow-water region is to show that for a given data $v_\delta$ of (1.19), by letting $\delta \to 0$, and then we have $v_\delta \to v$, where $v$ is a solution of (1.13). We write $v_\delta$ to emphasise that we pass the limit on $\delta$, and it also makes the distinction between the solution of gKdV (1.20) and the solution of gFDF (1.19). The first convergence result for solutions living in the shallow-water region is as follows.

Theorem 1.8 (Shallow-water limit). Let $S_{T,\delta}$ be given as in (1.22), and $S_{T,K}$ be the solution map for gKdV (1.20). Assume $v_0 \in H^s(M)$ such that

$$S_{T,\delta}(v_0) - S_{T,K}(v_0) = v_\delta - v.$n

Then
Then, the following two statements hold:

(i) (limit on the real line). Let \( s > \frac{1}{2} \). Then, for any \( T \) obtained from Theorem 1.3(i) we have
\[
\lim_{\delta \to 0} \| v_\delta - v \|_{C([0,T];H^s(\mathbb{R}))} = 0.
\]

(ii) (limit on the torus). Let \( s \geq \frac{3}{4} \). Then, for any \( T \) obtained from Theorem 1.3(ii) we have
\[
\lim_{\delta \to 0} \| v_\delta - v \|_{C([0,T];H^s(\mathbb{T}))} = 0.
\]

Next, we study the limiting behaviour of solutions living in the deep-water region. To distinguish between the shallow-water limit and the deep-water limit, we define analogously the solution map of gFDF (1.17) to be
\[
D_{T,\delta}(u_0) = u_\delta.
\]
The convergence result for solutions living in the deep-water region is as follows.

**Theorem 1.9** (Deep-water limit). Let \( s \geq \frac{3}{4} \), \( D_{T,\delta} \) be the solution map for gFDF (1.17), and \( S_{T,H} \) be the solution map for gBO (1.18). Assume \( u_0 \in H^s(M) \) such that
\[
D_{T,\delta}(u_0) - S_{T,H}(u_0) = u_\delta - u.
\]
Then, for any \( T \) obtained from Theorem 1.4, we have
\[
\lim_{\delta \to \infty} \| u_\delta - u \|_{C([0,T];H^s(M))} = 0.
\]

First of all, from Table 1 that our convergence results improve most of the previous works in [1, 28, 27] Secondly, our work also provides the convergence results on the torus \( \mathbb{T} \). Especially, the Theorems 1.8 and 1.9 improve the FDF convergence results of [1] for Sobolev index \( s < 1 \).

The last result in our deterministic part is to see how we can extend our convergence to hold globally-in-time. We first notice that, for any fixed \( \delta \), the Hamiltonian of (1.16) is of the following form:
\[
H(u) = \frac{1}{2} \int_M u(1 + \delta^{-1})g_\delta \partial_\alpha u \, dx + \int_M F(u) \, dx,
\]
where the potential energy part, \( F(x) \) is defined by \( F(x) := \int_0^x f(y) dy \).

**Corollary 1.10** (Global convergence). Let \( s \geq 1 \). Consider any of the following cases:

Case 1: Let \( \delta_0 > 0 \) to be fixed and \( \delta \geq \delta_0 \). The function \( F \) defined in (1.23) satisfying one of the following conditions: (i) there exists \( C > 0 \) such that \( |F(x)| \leq C(1 + |x|^{p+1}) \) for some \( 0 < p < 5 \); (ii) there exists \( B > 0 \) such that \( F(x) \leq B \), \( \forall x \in \mathbb{R} \).

Case 2: For any \( \delta > 0 \), and let \( f(u) = u^2 \).

Let Case 1 and Case 2 are satisfied, as \( \delta \to 0 \). Then, the convergence of solutions can be extended globally-in-time. Let Case 2 be satisfied, as \( \delta \to \infty \). Then, the convergence of solutions can be extended globally-in-time.

This corollary is a direct consequence of the GWP results. Namely, we show that the solutions to gFDF converge to those solutions to gKdV (as \( \delta \to 0 \)), and to gBO (as \( \delta \to \infty \)). Then, we know for Case 1 of Corollary 1.10 that the global-in-time solution of gKdV exists according to [63]. Therefore, we can upgrade local convergence to hold globally-in-time. On the other hand, we are not able to improve the results [27] in the case of cubic nonlinearity with \( \delta \to \infty \). However, we are able to establish convergence results as \( \delta \to 0 \), which is beyond the scope of the work [27].
the other hand, in the FDF case (quadratic nonlinearity), there are infinite many conservation laws. In particular, we can use the $H^1$-type of conserved quantity (see in Remark 1.12) to control the solution.

**Remark 1.11.** (i) Our approach applies well to both $\mathbb{R}$ and $\mathbb{T}$. As a comparison, the arguments in [27, 28] strongly rely on the local smoothing property, which is not available on the torus. Moreover, our argument handles not only the power-type of nonlinearity but also trigonometric and exponential nonlinearities.

(ii) In terms of writing, the main analysis is written on $\mathbb{T}$. For the real line case, we shall only provide the crucial improved Strichartz estimates. See Lemmas A.1 and A.3 for details. Then, by following the same type of argument (with a simpler decomposition) we can establish the energy estimates on $\mathbb{R}$, see Remark 4.11. Therefore, the convergence on $\mathbb{R}$ follows the same type of argument.

**Remark 1.12.** The condition of Case 1 and Case 2 in Corollary 1.10 allow us to show the GWP of gFDF for $s \geq 1$. This would rely on several conserved quantities associated with the equation (1.16), such as the $L^2$-conservation: $M(u) = \int_M u^2 \, dx$, and the Hamiltonian. For any fixed $\delta$ the Hamiltonian defined in (1.23) is at the $H^{2\frac{1}{2}}$-level. However, we can easily see the kinetic energy part of (1.23) is bounded uniformly in $\delta$, for $\delta$ small, by the kinetic energy part of gKdV. Moreover, the Hamiltonian associated with equation (1.17) (for large $\delta$) enjoys $H^{2\frac{1}{2}}$-level of conservation law uniformly in $\delta$, for large $\delta$. Due to the kinetic energy part of the Hamiltonian associated with equation (1.17) is bounded uniformly in $\delta$ by the kinetic energy part of gBO.

Hence, gathering these conservation laws with the above LWP results, let us assume $s \geq 1$, and then we are able to extend the solution constructed in Theorem 1.3 for any $T > 0$. This is analogous to [63, Theorems 1.2 & 1.3], so we omit details here and refer interested readers to [63].

We see from the Theorem 1.4 that the regularity threshold is $s > \frac{1}{2}$. However, according to the Hamiltonian associated with equation (1.17), we can only obtain the $H^{2\frac{1}{2}}$-level of conservation law in the deep-water region (for $\delta$ large). Therefore, the argument proposed in [63] (in proving GWP) cannot be applied to extend the Theorem 1.4 for any $T > 0$. However, if we restrict $f(u) = u^2$, then it is well-known that FDF enjoys infinite many conservation laws; see for example [50, 76]. In particular, we can use the following $H^1$-type quantity (see p. 368 of [1]):

$$E_2(u) := \int_M \left( \frac{1}{4} u^4 + \frac{3}{2} u^2 T_\delta(u_x) + \frac{1}{2} u_x^2 + \frac{3}{2} \{ T_\delta(u_x) \}^2 + \frac{1}{\delta} \left\{ \frac{3}{2} u^3 + \frac{9}{2} u T_\delta(u_x) \right\} + \frac{3}{2\delta^2} u^2 \right) \, dx,$$

where $\mathcal{T}_\delta u(\xi) = -i \coth(2\pi \delta \xi) \hat{f}(\xi)$. Therefore, this $H^1$-invariant quantity extends Theorem 1.4 globally-in-time.

**Remark 1.13.** In [1], the $\delta$-independent estimates for solutions to the equation (1.5) were obtained automatically from [1, Theorem 6.1]. For the convergence part, with the $\delta$-independent estimates they easily showed the Cauchy property of the family of solutions $\{ u^\delta \}_{\delta \geq 1}$ to (1.5). On the other hand, the statement [1, Theorem 6.1] does not directly apply to the formulation (1.9). Although we know that (1.9) and (1.5) are invariant under some suitable scaling, the $\delta$-uniformity breaks down under this scaling. As a result, $\delta$-independent a priori estimate for
solution $v_0$ to (1.9) was established independently. In particularly, see [1, Lemma 8.2.2] and it was done through the infinitely many conservation laws of the FDF, and we emphasise that an extra regularity assumption has been made on this lemma, which holds for $v_0 \in H^s(\mathbb{R})$ with $s \geq 2$.

In fact, for the gFDF case of [28], the uniform (in $\delta$) estimates were also established separately for small and large $\delta$. In the gKdV-limit, they also needed a high regularity assumption to handle the derivatives after applying the integration by parts. Moreover, the shallow-water limit was not covered in [27]. However, they conjectured that should still be true due to the results in [1].

Our deterministic approach is relatively straightforward (in general, this methodology applies to the gFDF (1.16) too): we analyse the dispersive effects of internal waves generated at the different depths, and then we identify regions for which dispersion phenomena become most likely to be KdV or BO-like. It turns out that it is possible to isolate dispersive effects that encode the leading order dynamics of the solution to FDF to have features of the KdV (or BO) dynamics. In particular, from the discussions of the shallow-water and deep-water limits, we see there are two FDF formulations. One captures the shallow-water evolution and another one captures the deep water evolution. This separation procedure is crucial in studying the convergence of solutions to the FDF, which is solvable by looking at the different formulations, and also possesses a family of explicit solutions that is $\delta$-dependent in a very regular way. We then search for solutions to FDF, which are also $\delta$-dependent but uniformly bounded in $\delta$. It appears to be possible to deduce the existence of such solutions to FDF itself using energy and compactness methods as well as dispersive PDEs techniques, since the $\delta$-dependent solutions to the model equation are stable in a very precise sense, see Propositions 4.9 and 4.12. In the end, we show that after passing $\delta$ to the limits, those solutions satisfy the limit equations regarded as KdV and BO.

2. Preliminaries

In this section, we start by introducing notations. Then, we will explore the behaviours of the dispersion terms to the equations (1.19) and (1.17). Lastly, we will introduce the function spaces we will use throughout the paper and their well-known properties.

2.1. Notations. For $a,b > 0$, we use $a \lesssim b$ to mean that there exists $C > 0$ such that $a \leq Cb$. By $a \sim b$, we mean that $a \lesssim b$ and $b \lesssim a$. Moreover, we denote $a \ll b$ if the estimate $b \lesssim a$ does not hold. For two non negative numbers $a,b$, we denote $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$. We also write $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

Given a function $u(t,x)$ on $\mathbb{R} \times \mathcal{M}$, we use $\hat{u}$ and $\mathcal{F}(u)$ to denote the space Fourier transform of $u$ given by

$$\hat{u}(k) = \int_{\mathcal{M}} e^{-ikx} u(t,x) \, dx \quad \text{for} \quad k \in \mathcal{M}.$$ 

In our writing, if there is no specific explanation to clarify the frequency variables. We use $n \in \mathbb{Z}$ to denote the frequency variable on the torus, and $\xi \in \mathbb{R}$ denotes the frequency variable on the real line. In the rest of the paper, we shall only restrict our writing on the torus ($n \in \mathbb{Z}$), except for Section A, where we give some estimates to support our argument on the real line.
For any $s \in \mathbb{R}$, we define $D^s f$ by its Fourier transform:

$$\hat{D^s f}(n) = |n|^s \hat{f}(n).$$

We fix a smooth cutoff function $\chi$ such that $\chi \in C_0^\infty(\mathbb{R})$ satisfy

$$0 \leq \chi \leq 1, \quad \chi([-1,1]) = 1, \quad \text{and} \quad \text{supp } \chi \subset [-2,2].$$

We set $\phi(n) := \chi(n) - \chi(2n)$. For any $l \in \mathbb{N}$, we define

$$\phi_2(n) := \phi(2^{-l} n), \quad \psi_2(\tau, n) := \phi_2(\tau - p_\delta(n)),$$

where $-ip_\delta(n)$ is the Fourier symbol of $G_\delta \partial_x^2$ or $\frac{3}{2\pi n} G_\delta \partial_x^2$. See (2.1) and (2.2). By convention, we also denote

$$\phi_0(n) = \chi(2n) \quad \text{and} \quad \psi_0(\tau, n) = \chi(2(\tau - p_\delta(n))).$$

Any summations of overcapitalised variables such as $K, L, M$ or $N$ are presumed to be dyadic. We work with non-homogeneous dyadic decompositions. In particular, these variables ranges over numbers of the form $\{2^k; k \in \mathbb{N}_0\}$. We call those numbers nonhomogeneous dyadic numbers. It is worth pointing out that

$$\text{supp}(\phi_N) \subset \{ \frac{N}{2} \leq |n| \leq 2N \} \quad \text{for } N \geq 1,$$

and for $N = 0$, we have

$$\text{supp}(\phi_0) \subset \{|n| \leq 1\}.$$

Finally, we define the Littlewood–Paley multipliers $P_N$ and $Q_L$ by

$$\hat{P_N}u = \phi_N \hat{u} \quad \text{and} \quad \hat{Q_L}u = \psi_L \hat{u}.$$

We also set

$$P_{\geq N} := \sum_{K \geq N} P_K, \quad P_{\leq N} := \sum_{K \leq N} P_K$$

as well as

$$Q_{\geq N} := \sum_{K \geq N} Q_K, \quad Q_{\leq N} := \sum_{K \leq N} Q_K.$$

Moreover, for $N \geq 0$ to be a nonhomogeneous dyadic numbers, we have the Littlewood-Paley decomposition:

$$u = \sum_{N} P_N u,$$

For simplicity, sometimes we also use $u_N = P_N u$ when there is no confusion.

2.2. Dispersion relation. In this subsection, we go over the properties of dispersion relation associated to the equation (1.19) and (1.17). First we recall the definition of the operator

$$G_\delta = - \coth(\delta \partial_x) - \frac{1}{\delta} \partial_x^{-1}$$

and on the frequency side, it is defined as

$$\hat{G_\delta} f(n) = -i \left\{ \coth(\delta n) - \frac{1}{\delta n} \right\} \hat{f}(n)$$

Let us fix $\delta_0 > 0$. For the deep-water ($\delta \geq \delta_0$) formulation (1.17), where the linear dispersion is $G_\delta \partial_x^2$. On the other hand, for the shallow-water ($\delta < \delta_0$) formulation (1.19), where the
linear dispersion is $\frac{3}{\delta}G_\delta \partial_x^2$. Hence, we denote the unitary group associated to the linear term of (1.17) and (1.19) by

$$S_{\delta_1}(t) = e^{-\frac{3}{\delta}G_\delta \partial_x^2} \quad \text{and} \quad S_{\delta_2}(t) = e^{-tG_\delta \partial_x^2},$$

separately. Moreover, in shallow-water region, $\frac{3}{\delta}G_\delta \partial_x^2$ is defined by

$$-ip_{\delta_1}(n) := 3n \coth(\delta n) - \frac{1}{\delta}.$$ \hspace{1cm} (2.1)

Also, in the deep-water region, $G_\delta \partial_x^2$ is defined by

$$-ip_{\delta_2}(n) := n \coth(\delta n) - \frac{1}{\delta}.$$ \hspace{1cm} (2.2)

The following lemma give an expression of the coth($z$) by using the expansion form.

**Lemma 2.1.** For all $n \in \mathbb{R}$ and $\delta > 0$, we have the following expansion,

$$n \coth(\delta n) = \frac{1}{\delta} + \frac{1}{3} \delta n^2 + \frac{1}{3} n^2 h(n, \delta),$$ \hspace{1cm} (2.3)

where the remainder $h(n, \delta)$ satisfies the following conditions:

(i) For any finite $N \in \mathbb{N}$, we have

$$\max_{|n| \leq N} |h(n, \delta)| \lesssim N \delta^3.$$  

(ii) There is some absolute constant $C_0$ such that for any $n \in \mathbb{Z}$,

$$|h(n, \delta)| \leq C_0 \delta.$$ \hspace{1cm} (2.4)

**Proof.** The proof was in [1, Lemma 8.2.1]. We keep it here for the reader’s convenience. First of all, we recall the Mittag-Leffler expansion [13] of coth($z$) such that

$$z \coth(z) = 1 + \sum_{k=1}^{\infty} \frac{2z^2}{z^2 + (k\pi)^2}$$

which is valid for $z \in \mathbb{C}$, $z \neq ik\pi$, and for any integer $k$. Therefore, for any $n \in \mathbb{R}$ and $\delta > 0$, we compute

$$n \coth(\delta n) = \frac{1}{\delta} + 2n^2 \delta \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2 + \delta^2 n^2}$$

$$= \frac{1}{\delta} + 2n^2 \delta \left[ \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} + \sum_{k=1}^{\infty} \left( \frac{1}{k^2\pi^2 + \delta^2 n^2} - \frac{1}{k^2\pi^2} \right) \right]$$

$$= \frac{1}{\delta} + \frac{n^2 \delta}{3} - n^2 \sum_{k=1}^{\infty} \frac{2\delta^3 n^2}{k^2\pi^2 (k^2\pi^2 + \delta^2 n^2)}.$$ 

That is, the remainder term $h(n, \delta)$ is explicitly given by

$$h(n, \delta) = \sum_{k=1}^{\infty} \frac{2\delta^3 n^2}{k^2\pi^2 (k^2\pi^2 + \delta^2 n^2)}.$$ 

For $|n| \leq N$, we have

$$|h(n, \delta)| \lesssim N \delta^3 \sum_{k=1}^{\infty} \frac{1}{k^4} \lesssim N \delta^3.$$
And for all \( n \in \mathbb{Z} \), we have
\[
\frac{|h(n, \delta)|}{\delta} \lesssim \sum_{k=1}^{\infty} \frac{1}{k^2} \lesssim 1.
\]
This concludes our proof. \( \square \)

**Remark 2.2.** Recall the definition of \( p_{\delta_1} \) from (2.1), we deduce from (2.3) that
\[
-p_{\delta_1} = n^3 + n^3 \frac{h(n, \delta)}{\delta}.
\]
Here, we notice from above definition that \( h(\delta, n) \leq 0 \), and from (2.4) so that \( h(n, \delta) \delta^{-1} \) is bounded by some absolute constant \( C \), uniformly for all \( n \in \mathbb{R} \) and \( \delta > 0 \). Moreover, for fixed each \( n \), \( h(n, \delta) \delta^{-1} = O(\delta^2) \) as \( \delta \to 0 \).

**Lemma 2.3.** For all \( n \in \mathbb{R} \) and every \( \delta > 0 \). Then,
\[
-\frac{1}{\delta} + |n| \leq n \coth(\delta n) \leq \frac{1}{\delta} + |n|.
\] (2.5)

**Proof.** See [1, Lemma 4.1]. Alternatively, it is enough to show the following
\[
|n \coth(\delta n) - |n|| \leq \delta^{-1}
\]
That is equivalent to show
\[
|\delta n \coth(\delta n) - \delta|n|| = |\delta n[\coth(\delta n) - \text{sgn}(\delta n)]| \leq 1.
\]
Now, it suffices to show for any \( x \in \mathbb{R} \) such that
\[
\sup_{x \in \mathbb{R}} |x[\coth(x) - \text{sgn}(x)]| \leq 1.
\]
It is trivial when \( x = 0 \). Therefore, we show the case when \( x \neq 0 \). Since for both \( \coth(x) \) and \( \text{sgn}(x) \) are odd function, it is enough to show when \( x > 0 \). Hence, we use the definition of \( \coth(x) \) yields
\[
x[\coth(x) - \text{sgn}(x)] = x \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{e^x - e^{-x}}{e^x - e^{-x}} \right) = \frac{2xe^{-x}}{e^x - e^{-x}} = \frac{2xe^{-x}}{e^{2x} - 1} \leq 1.
\]
\( \square \)

A direct computation gives the following corollary.

**Corollary 2.4.** For all \( n \in \mathbb{R} \) and fix \( \delta_0 > 0 \) such that the following statements hold:

(i) For all \( \delta \in (0, \delta_0) \), the shallow-water symbol satisfies
\[
|p_{\delta_1}(n)| \leq C|n|^3; \tag{2.6}
\]

(ii) For all \( \delta \in [\delta_0, \infty) \), the deep-water symbol satisfies
\[
|p_{\delta_2}(n)| \leq C|n| + 2\pi|n|^2. \tag{2.7}
\]

Here, constant \( C \) is independent of \( \delta \).

**Remark 2.5.** In [27, Lemma 3.1] showed that for all \( n \in \mathbb{R} \) the dispersion \( G_0 \partial_x^2 \) is uniform bounded for all \( \delta \geq 1 \), while this uniform bound is no longer true when \( \delta \leq 1 \). Therefore, we can see that the advantage of the scaling transformation in the shallow-water region is that we have a uniform control of \( \frac{2}{\delta} G_0 \) in \( \delta \), as \( \delta \to 0 \).
Proof. From (2.5) we have for all \( n \in \mathbb{R} \),
\[
0 \leq |n| - 2\pi n \coth(\delta n) + \frac{1}{\delta} \leq \frac{2}{\delta}
\]
Therefore, let \( \delta \geq \delta_0 \) for some positive \( \delta_0 \) such that we have
\[
\left| \frac{p_{\delta_2}(n)}{n} - |n| \right| \lesssim 1
\]
uniformly in \( \delta \). From Lemma 2.1 and Remark 2.2 we have
\[
|p_{\delta_1}(n)| = \left| \frac{3n}{\delta} \left( n \coth(\delta n) - \frac{1}{\delta} \right) \right| \leq |n|^3 + \left| \frac{n^3 h(n, \delta)}{\delta} \right| \leq C|n|^3.
\]
\[ \square \]

This corollary is in fact crucial in our later \( L^4 \)-Strichartz estimates, see Lemmas 3.1 and 3.2. In particular, it gives a uniform control on the dispersion symbol of the shallow-water and deep-water regions.

2.3. Function spaces and their basic properties. We finish this section by introducing the function spaces and their properties. Firstly, we slightly modify the classical Sobolev spaces in the following way: Let \( s \geq 0, \) and let \( \{\omega_N\} \) to be a dyadic sequence. Then, we define \( H^s_{\omega}(\mathbb{T}) \) associated with the norm
\[
\|u\|_{H^s_{\omega}} := \left( \sum_N |N|^2(1 \lor N)^{2s} \|P_Nu\|_{L^2_{\omega}}^2 \right)^{1/2}.
\]
One notices that when we choose \( \omega_N \equiv 1 \), then we recover our usual \( L^2 \)-based Sobolev space \( (i.e., H^s(\mathbb{T}) = H^s(\mathbb{T})) \). Moreover, we remark that this modification here is useful in showing the continuity of solution with respect to initial data. In particular, we will use the frequency envelope method, which was introduced in, for instance, [44, 81].

If \( B_x \) is one of spaces defined above, for \( 1 \leq p \leq \infty \) and \( T > 0 \), we define the space-time spaces, with short-hand notation, \( L^p_t B_x := L^p(\mathbb{R}; B_x) \) and \( L^p_T B_x := L^p([0, T]; B_x) \) equipped with the norms (with obvious modifications for \( p = \infty \))
\[
\|u\|_{L^p_t B_x} = \left( \int_0^T \|u(t, \cdot)\|^p_{B_x} \, dt \right)^{1/p} \quad \text{and} \quad \|u\|_{L^p_T B_x} = \left( \int_0^T \|u(t, \cdot)\|^p_{B_x} \, dt \right)^{1/p},
\]
respectively. For \( s, b \in \mathbb{R} \), we introduce the \( X^{s, b} \)-spaces associated to the operator \( G_\delta \) (or \( \frac{3}{2\pi \delta} G_\delta \)) endowed with the norm
\[
\|u\|_{X^{s, b}} = \left( \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \langle n \rangle^{2s} \langle \tau - p_\delta(n) \rangle^{2b} |\tilde{u}(\tau, n)|^2 \, d\tau \right)^{1/2}.
\]
We also use a slightly stronger space \( X^{s, b}_{\omega} \) with the norm
\[
\|u\|_{X^{s, b}_{\omega}} := \left( \sum_N \omega_N^2(1 \lor N)^{2s} \|P_Nu\|_{X^{s, b}_{\omega}}^2 \right)^{1/2}.
\]
We define the function spaces \( Z^s \) (resp. \( Z^s_{\omega} \)) for \( s \in \mathbb{R} \), as
\[
Z^s := L^\infty_t H^s \cap X^{s-1,1} \quad \text{(resp.} Z^s_{\omega} := L^\infty_t H^s_{\omega} \cap X^{s-1,1}_{\omega}),
\]
endowed with the following norm
\[ \| u \|_{Z^s} = \| u \|_{L^\infty T H^s} + \| u \|_{X^{s-1,1}} \] (resp. \( \| u \|_{Z^s} = \| u \|_{L^\infty T H^s} + \| u \|_{X^{s-1,1}} \)). \tag{2.8}

Let us also define the restriction in time versions of these spaces. Let \( T > 0 \) be a positive time and \( B \) be a normed space of space-time functions. The restriction space \( B_T \) is defined to the space of functions \( u : (0, T) \times \mathbb{T} \to \mathbb{R} \) or \( \mathbb{C} \) satisfying
\[ \| u \|_{B_T} := \inf \{ \| \tilde{u} \|_B \mid \tilde{u} : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \text{ or } \mathbb{C}, \ \tilde{u} = u \text{ on } (0, T) \times \mathbb{T} \} < \infty. \]

Finally, let us introduce a bounded linear operator \( \rho_T \) such that
\[ \rho_T : X^{s-1,1}_{\omega,T} \cap L^\infty_T H^s_\omega \to Z^s_\omega, \]
the bound is independent of \( s \) and \( T \). The existence of this operator ensures that \( Z^s_{\omega,T} = L^\infty_T H^s_\omega \cap X^{s-1,1}_{\omega,T} \). We follow the definition in \([55]\). Let \( i = 1, 2 \). Then, we define \( \rho_T \) as
\[ \rho_T(u)(t) := S_{\delta_i}(t) \chi(t) S_{\delta_i}(-\mu_T(t)) u(\mu_T(t)), \tag{2.9} \]
where \( \mu_T \) is the continuous piecewise affine function defined by
\[ \mu_T(t) = \begin{cases} 0 & \text{for } t \notin (0, 2T), \\ t & \text{for } t \in [0, T], \\ 2T - t & \text{for } t \in [T, 2T]. \end{cases} \]

**Lemma 2.6.** Let \( \kappa \geq 1 \), and suppose that the dyadic sequence \( \{ \omega_N \} \) of positive numbers satisfies \( \omega_N \leq \omega_{2N} \leq \kappa \omega_N \) for \( N \geq 1 \). Let \( 0 < T \leq 1 \) and \( s \in \mathbb{R} \). Then,
\[ \rho_T : X^{s-1,1}_{\omega,T} \cap L^\infty_T H^s_\omega \to Z^s_\omega \]
\[ u \mapsto \rho_T(u) \]
is a bounded linear operator. In particular,
\[ \| \rho_T(u) \|_{L^\infty_T H^s_\omega} + \| \rho_T(u) \|_{X^{s-1,1}} \lesssim \| u \|_{L^\infty_T H^s_\omega} + \| u \|_{X^{s-1,1}} \], \tag{2.10} \]
for all \( u \in X^{s-1,1}_{T,\omega} \cap L^\infty_T H^s_\omega \). Moreover, it holds that
\[ \| \rho_T(u) \|_{L^\infty_T H^s_\omega} \lesssim \| u \|_{L^\infty_T H^s_\omega} \] \tag{2.11}
for all \( u \in L^\infty_T H^s_\omega \). Here, the implicit constants in \((2.10)\) and \((2.11)\) can be chosen independent of \( 0 < T \leq 1, \delta, \) and \( s \in \mathbb{R} \).

**Proof.** See in \([67]\) Lemma 2.4] for \( \omega_N \equiv 1 \) but it is obvious that the result does not depend on \( \omega_N \). See also \([63]\).

In the rest of this section, we collect some fundamental estimates and well-known estimates are adapted for our setting \( H^s_\omega(\mathbb{T}) \) and \( f(u) \). The following lemma follows from \([82]\) in particular the proof of \((2.12)\) is identical to that of in \([82]\) Lemma A.8]. See also \([63]\) Lemma 2.2].

**Lemma 2.7.** Let \( s > 0, \kappa \geq 1, \) and suppose that the dyadic sequence \( \{ \omega_N \} \) of positive numbers satisfies \( \omega_N \leq \omega_{2N} \leq \kappa \omega_N \) for \( N \geq 1 \). Then, we have the estimate
\[ \| uv \|_{H^s_\omega} \lesssim \| u \|_{H^s_\omega} \| v \|_{L^\infty} + \| u \|_{L^\infty} \| v \|_{H^s_\omega}. \tag{2.12} \]
In particular, for any fixed real entire function $f$ with $f(0) = 0$, there exists a real entire function $G = G[f]$ that is increasing non-negative on $\mathbb{R}_+$ such that
\[
\|f(u)\|_{H^s} \lesssim G(\|u\|_{L^\infty})\|u\|_{H^s}.
\] (2.13)

**Lemma 2.8.** Let $s_1 + s_2 \geq 0$, $s_1 \wedge s_2 \geq s_3$, and $s_3 < s_1 + s_2 - \frac{1}{2}$. Then,
\[
\|uv\|_{H^{s_3}} \lesssim \|u\|_{H^{s_1}}\|v\|_{H^{s_2}}.
\] (2.14)

In particular, let $u, v \in H^s(\mathbb{T})$ for $s > \frac{1}{2}$. Then, there exists a increasing and non-negative real entire function $G = G[f]$ on $\mathbb{R}_+$ such that
\[
\|f(u) - f(v)\|_{H^{s-1}} \leq G(\|v\|_{H^s} + \|v\|_{H^s})\|u - v\|_{H^{s-1}}.
\] (2.15)

**Proof.** See [63, Lemma 2.3] and (2.14) can be found in [26, Lemma 3.4]. \qed

The next commutator-type estimate will be frequently used in proving the essential energy estimate and the estimate for the difference between two solutions. It can be seen as a variant of the integration by parts. The proof follows from [Lemma 2.4, [63], see also [Lemma 3.3, [44].

**Lemma 2.9.** Let $N \in 2^{N_0}$. Then, there exists a positive constant $C$ such that for every $u, v \in L^2(\mathbb{T})$, and $\partial_x w \in L^\infty(\mathbb{T})$, the following holds
\[
\left| \int_\mathbb{T} \Pi(u, v)wdx \right| \leq C\|u\|_{L^2_x}\|v\|_{L^2_x}\|\partial_x w\|_{L^\infty_x}.
\]

Here, we define
\[
\Pi(u, v) := v\partial_x P_N^2u + u\partial_x P_N^2v.
\] (2.16)

### 3. Linear and short-time Strichartz estimates

In this section, we establish improved Strichartz estimates which play an important role in our later energy estimates. In order to obtain the $\delta$-independent nonlinear estimate, it is important to check all the linear estimates are $\delta$-independent. Since the linear operator is varying with the depth parameter $\delta$, we first have to restate the standard Strichartz estimate and make sure it is uniform in (small or large) $\delta$. Here, we will use the Corollary 2.4 so that we can deduce the uniform bounds on phase functions.

**Definition 1.** For $\alpha = 1$ or 2, we define
\[
\beta(\alpha) := \frac{1}{4(\alpha + 1)}, \quad b(\alpha) := \beta(\alpha) + \frac{1}{4}\left( = \frac{\alpha + 2}{4(\alpha + 1)} \right).
\] (3.1)

Then, we denote that $\alpha = 1$, $\beta = \frac{1}{8}$ and $b = \frac{3}{8}$ correspond to the deep-water case ($\delta > \delta_0$), whereas $\alpha = 2$, $\beta = \frac{1}{12}$ and $b = \frac{5}{12}$ correspond to the shallow-water case ($\delta < \delta_0$). In the later analysis, we will repeatedly see these numbers.

The first lemma below is the $L^4$-Strichartz estimate in the $X^{a,b}_\tau$-space, which is originally introduced in [9, 10]. We will adopt it into our dispersion relations.
Lemma 3.1. Let $\delta \in (0, \infty)$, for any $u \in X^{0,b(\alpha)}$, where $b(\alpha)$ is defined in (3.1). Then, there exists some $\delta_0 > 0$ such that the following two estimates hold:

(i) For any $\delta \in (0, \delta_0)$, we have
$$\|u\|_{L^4_{t,x}} \leq C\|u\|_{X^0_{\frac{1}{3}}},$$
(3.2)

(ii) For any $\delta \in [\delta_0, \infty)$, we have
$$\|u\|_{L^4_{t,x}} \leq C\|u\|_{X^0_{\frac{2}{3}}}. $$
(3.3)

Here, the constant $C$ is independent of $\delta$.

Proof. The proof immediately follows from the next lemma (see the Appendix in [58] for similar considerations). We briefly explain it here for its completeness.

• For the inequality (3.2):
We will thus work in the function space $X^{0,b}$ endowed with the norm
$$\|u\|_{X^{0,b}} = \left( \sum_{\tau \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} |\tau - p_{\delta_1}(n)|^{2b} |\widehat{u}(\tau, n)|^2 \right)^\frac{1}{2} \right).$$

Let $u \in X^{0,b}$ for $b \geq \frac{1}{3}$. We introduce a Littlewood-Paley decomposition of $u$:
$$u = \sum_{M, \text{dyadic}} u_M,$$

where $\text{supp } \widehat{u}_M \subset \{(\tau, n) \in \mathbb{Z}^2 | \langle \tau - p_{\delta_1}(n) \rangle \sim M \}$. Then, we have
$$\sum_{M, \text{dyadic}} M^{2b} \|u_M\|_{L^2_{t,x}}^2 \sim \|u\|_{X^{0,b}}^2.$$

We also have
$$\|u\|_{L^4_{t,x}}^2 = \|uu\|_{L^2_{t,x}} \leq \sum_{M_1, M_2, \text{dyadic}} \|u_{M_1} u_{M_2}\|_{L^2_{t,x}} \leq \sum_{k=0}^{\infty} \sum_{M, \text{dyadic}} \|u_M u_{2^k M}\|_{L^2_{t,x}}.$$

Therefore, the rest of the proofs are readily followed from Lemma 3.2.

• For the inequality (3.3):
We use similar steps, but this time we work in the function space $X^{0,b}$ endowed with the norm
$$\|u\|_{X^{0,b}} = \left( \sum_{\tau \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} |\tau - p_{\delta_2}(n)|^{2b} |\widehat{u}(\tau, n)|^2 \right)^\frac{1}{2} \right).$$

By Plancherel theorem,
$$\|u_{M_1} u_{M_2}\|_{L^2_{t,x}} = \|\widehat{u}_{M_1} *_{\tau,n} \widehat{u}_{M_2}\|_{L^2_{t,x}}$$

it suffices to show the following lemma.

Lemma 3.2. Let $u, v \in L^2(\mathbb{R}; \ell^2(\mathbb{Z}))$ be any real-valued functions and $N_1, N_2 \in 2^\mathbb{N}$ any dyadic numbers. Then, there exists $\delta_0 > 0$ and $C > 0$ independent of $\delta$ such that the following two statements hold:

(i) For any $\delta \in (0, \delta_0)$ we have
$$\|\psi_{N_1} u *_{\tau,n} (\psi_{N_2} v)\|_{L^2_{t,x}} \leq C(N_1 \wedge N_2)\frac{1}{2} (N_1 \lor N_2)\frac{1}{2} \|\psi_{N_1} u\|_{X^{0,\frac{1}{2}}} \|\psi_{N_2} v\|_{X^{0,\frac{1}{2}}};$$
(ii) For $\delta \in [\delta_0, \infty)$ we have

$$\| (\psi_{N_1} u \ast_{\tau, n} \psi_{N_2} v) \|_{L^2 L^2_n} \leq C(N_1 \wedge N_2) \frac{1}{\delta} (N_1 \vee N_2)^\frac{1}{2} \| \psi_{N_1} u \|_{L^2 L^2_n} \| \psi_{N_2} v \|_{L^2 L^2_n}.$$ 

Here, $\psi_{N_i}$ is the projection on the modulation function.

The following counting lemma plays an important role in proving Lemma 3.2. It follows from [77, Lemma 2].

**Lemma 3.3.** Let $I$ and $J$ be two intervals on the real line and $g \in C^1(J; \mathbb{R})$. Then,

$$\# \{ x \in J \cap \mathbb{Z}; g(x) \in I \} \leq \frac{|I|}{\inf_{x \in J} |g'(x)|} + 1.$$ 

In what follows we prove Lemma 3.2. The argument closely related to [63, Lemma 3.2], see also [79, Lemma 3.1]. The main ingredient is that we obtained the uniform bounds on the phase functions from Corollary 2.4.

**Proof of Lemma 3.2.** Following [9], we may assume that

$$\sup_n \text{supp}(\psi_{N_1} u), \sup_n \text{supp}(\psi_{N_2} v) \subset \mathbb{N}$$

since $u$ and $v$ are real-valued. But using the relation

$$\tau = \tau_1 + \tau_2$$

and

$$n = n_1 + n_2,$$

the Cauchy-Schwarz inequality in $(\tau_1, n_1)$ gives

$$\| (\psi_{N_1} u \ast_{\tau, n} \psi_{N_2} v) \|_{L^2 L^2_n}^2 = \sum_{n=0}^{\infty} \int_{\tau} \left| \sum_{n_1 \geq n_1} \int_{\tau_1} \psi_{N_1}(\tau_1, n_1) u(\tau_1, n_1) \psi_{N_2}(\tau - \tau_1, n - n_1) v(\tau - \tau_1, n - n_1) d\tau_1 \right|^2 d\tau \lesssim \sup_{(\tau, n) \in \mathbb{R} \times \mathbb{N}} A(\tau, n) \| \psi_{N_1} u \|_{L^2 L^2_n} \| \psi_{N_2} v \|_{L^2 L^2_n},$$

where

$$A(\tau, n) = \# \{ (\tau, n_1) \in \mathbb{R} \times \mathbb{N} | (\tau_1, n_1) \in \text{supp}(\psi_{N_1} u) \text{ and } (\tau_2, n_2) \in \text{supp}(\psi_{N_2} v) \}$$

$$\lesssim \# \{ (\tau_1, n_1) \in \mathbb{R} \times \mathbb{N} | n - n_1 \geq 0, \langle \tau_1 - p_{\delta_1}(n_1) \rangle \sim N_1,$$

and

$$\langle \tau_1 - p_{\delta_1}(n_1) \rangle \sim N_2 \}$$

$$\lesssim (N_1 \wedge N_2) \# B(\tau, n)$$

with the following (let $L := N_1 \vee N_2$)

$$B(\tau, n) = \{ n_1 \geq 0 | n - n_1 \geq 0, \langle \tau - p_{\delta_1}(n_1) \rangle \sim N_1 \} \lesssim N_1 \vee N_2$$

$$= \{ n_1 \geq 0 | n - n_1 \geq 0, \langle \tau - p_{\delta_1}(n_1) \rangle \sim L \}$$

For different depth regions we need to consider the different phase functions. Therefore, let $\delta_0 > 0$ be some absolute constant, we split into the case when $\delta \in (0, \delta_0)$ and the case $\delta \in [\delta_0, \infty)$. Let us recall the definition of $p_{\delta_1}(n)$ and $p_{\delta_2}(n)$:

$$p_{\delta_1}(n) = n^3 + n^3 \frac{h(n, \delta)}{\delta}, \quad p_{\delta_2} = n \left( n \coth(\delta n) - \frac{1}{\delta} \right).$$
Moreover, $p_{\delta_1}(n), p_{\delta_2}(n) \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ are real-valued odd-functions, and there exists some $n_0 > 0$ such that for all $n \geq n_0$, the following hold
\begin{align}
 p'_{\delta_1}(n) &\sim n^2 \quad \text{and} \quad p''_{\delta_1}(n) \sim n; \quad \tag{3.4}
 p'_{\delta_2}(n) &\sim n \quad \text{and} \quad p''_{\delta_2}(n) \sim 1.
\end{align}

**Case I:** shallow-water region, when $\delta \in (0, \delta_0)$.

In this case, let $u, v$ be solutions to (1.19). Then, we consider the phase function defined in (2.1) and we recall it from (2.3) so that can be expressed to the following
\[ p_{\delta_1}(n) = \frac{3n}{\delta} \left( n \coth(\delta n) - \frac{1}{\delta} \right) = n^3 + n^2 \frac{h(n, \delta)}{\delta}. \]

Moreover, it follows from Lemma 2.1 and estimate (2.6), we can deduce that
\[ |p_{\delta_1}(n)| \lesssim |n|^3, \]
here the implicit constant is independent of $\delta$.

- **Low-frequency contribution:**

For the low-frequency contribution, let
\[ n \leq L^{\frac{1}{3}} + 2n_0 + 2, \]
where $n_0$ is fixed from the discussion of (3.4). Moreover, we have $0 \leq n_1 \leq n$ and $L \geq 1$. Thus, let $(\tau, n)$ to be fixed. Then, we obtain
\[ \#B(\tau, n) \leq L^{\frac{1}{3}} + 2n_0 + 2 \leq C(n_0)L^{\frac{1}{3}}, \]
for some constant depends only on the choice of $n_0$.

- **High-frequency contribution:**

On the other hand, for the high-frequency contribution, we consider
\[ n \geq L^{\frac{1}{3}} + 2n_0 + 2. \]

Let us define the following function,
\[ g(n_1) := \tau - p_{\delta_1}(n_1) - p_{\delta_1}(n - n_1). \]

Let $(\tau, n)$ to be fixed. Then, we obtain
\[ \#B(\tau, n) \lesssim \#\{n_1 \geq 0 \mid n \geq 2n_1 \quad \text{and} \quad |g(n_1)| \lesssim L\}
\lesssim \left[ n_0 \right] + 1 + \#\{n_1 \geq 0 \mid n \geq 2n_1, n_1 \geq n_0, \quad \text{and} \quad |g(n_1)| \lesssim L\}
\lesssim \#\{n_1 \in [n_0, \frac{3}{2}] \mid |n - 2n_1|^3 \geq L, \quad \text{and} \quad |g(n_1)| \lesssim L\} + [n_0] + L^{\frac{1}{3}}, \]
where $[n_0]$ is an integral part of $n_0$, and the implicit constant is independent of $\delta$. The first inequality, we used symmetry. Moreover, we notice that since $(n - L^{\frac{1}{3}}) \geq n_0$ by the assumption, which guarantees the set on the right-hand side is non-empty. We observe the following (allow us to apply Lemma 3.3)
\[ |g'(n_1)| = |p'_{\delta_1}(n_1) - p'_{\delta_1}(n - n_1)|
\lesssim \left| \int_{n_1}^{n-n_1} p''_{\delta_1}(\theta) d\theta \right| \sim (n - n_1)^2 - n_1^2 \geq (n - 2n_1)^2 \geq L^{\frac{1}{3}}. \]
For any $\delta$ in (2.2) is independent of $\delta$ in (2.2). Therefore, from Lemma 3.3, we have
\[ \#\{n_1 \in [n_0, \frac{n}{2}] | n - 2n_1| \geq L, \text{ and } |g(n_1)| \lesssim L \} \lesssim L^{\frac{1}{2}}. \] (3.5)

**Case II:** deep-water region, when $\delta \in [\delta_0, \infty)$.

In this case, we take $u, v$ to be solutions to (1.17). Then, we recall the phase function defined in (2.2) is
\[ p_{\delta_2} = n\left(n \coth(\delta n) - \frac{1}{\delta}\right) \]
Moreover, from Lemma 2.3 and (2.7), we can deduce that
\[ |p_{\delta_2}| \lesssim |n|^2 \] (3.6)
for some constant $C$ is independent of $\delta$.

By repeating the same argument as in case I, we consider the low and high-frequency contributions, separately.

- **Low-frequency contribution:**

For the low-frequency contribution, we assume that $n \leq C(n_0) L^{\frac{1}{2}}$. Let $(\tau, n)$ to be fixed. Then, the same argument shows
\[ \#B(\tau, n) \leq C(n_0) L^{\frac{1}{2}}. \]

- **High-frequency contribution:**

For the high-frequency contribution, we consider $n \geq C(n_0) L^{\frac{1}{2}}$, and we define
\[ f(n_1) := \tau - p_{\delta_2}(n_1) - p_{\delta_2}(n - n_1). \]
Let us fix $(\tau, n)$. Then, we yield the following
\[ \#B(\tau, n) \lesssim \#\{n_1 \geq \tau: n \geq 2n_1 \text{ and } |f(n_1)| \leq L \} \leq [n_0] + 1 + \#\{n_1 \geq \tau: n \geq 2n_1, n_1 \geq n_0, \text{ and } |f(n_1)| \leq L \} \lesssim \#\{n_1 \in [n_0, \frac{n}{2}] | n - 2n_1| \geq L, \text{ and } |f(n_1)| \leq L \} + [n_0] + L^{\frac{1}{2}}. \]

Now, we check the condition in Lemma 3.3,
\[ |g'(n_1)| = |p'_{\delta_2}(n_1) - p'_{\delta_2}(n - n_1)| = \left| \int_{n_1}^{n_1 - n_1} p''_{\delta_2}(\theta)d\theta \right| \sim (n_1) - n_1 \geq n - 2n_1 \geq L^{\frac{1}{2}} \]
Therefore, from Lemma 3.3, we have
\[ \#\{n_1 \in [n_0, \frac{n}{2}] | n - 2n_1| \geq L, \text{ and } |g(n_1)| \leq L \} \lesssim L^{\frac{1}{2}}. \]
This completes the proof. \[ \square \]

Lemma 3.1 enables us to establish the following Strichartz estimate:

**Lemma 3.4.** Let $T > 0$ and any $u \in L^2(\mathbb{T})$. Then, there exists $\delta_0 > 0$ and $C > 0$ is independent of $\delta$ such that the following two statements hold:

(i) For any $\delta \in (0, \delta_0)$, we have
\[ \|S_{\delta_1}(t)u\|_{L^4_{t,x}} \leq CT^{\frac{1}{2} - \frac{1}{2}} \|u\|_{L^2}. \]
(ii) For any \( \delta \in [\delta_0, \infty) \), we have
\[
\|S_{\delta}(t)u\|_{L^4_{t,x}} \leq CT^{\frac{1}{2} - \frac{3}{2}} \|u\|_{L^2_x}.
\]

**Proof.** See Lemma 2.1 in [62]. \( \Box \)

We are now ready to prove our improved Strichartz estimates for solutions to gFDF. We point out that it is crucial to state estimates in \( \ell^4(\mathbb{N}) \)-norm since we are not allowed to lose any derivatives in order to reach \( s = s(\alpha) \) for \( \alpha = 1 \) or 2. For that purpose, the way to choose \( c_{j,N} \) plays an important role in the proof below. This type of argument can be found, for example, in [69, Lemma 2.4].

**Proposition 3.5.** Let \( s \geq s_0 > \frac{1}{2} \), \( T \in (0, 1) \), \( \kappa \geq 1 \), and suppose that the dyadic sequence \( \{\omega_N\} \) of positive numbers satisfies
\[
\omega_N \leq \omega_{2N} \leq \kappa \omega_N \quad \text{for} \quad N \geq 1.
\]
Moreover, there exists \( \delta_0 > 0 \) such that the following statements hold.

(i) For \( 0 < \delta < \delta_0 \), let \( v \in C([0, T]; H^s_x(\mathbb{T})) \) satisfy (1.19) with \( v_0 \in H^s_x(\mathbb{T}) \) on \([0, T]\). Then,
\[
\left( \sum_N \omega_N^4 \|D_x^{-\frac{1}{8}} P_N v\|_{L^4_{t,x}}^4 \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{4}} G(\|v\|_{L^\infty_{t,x}}) \|v\|_{L^\infty_T H^s_x}.
\]
(3.7)
\[
\left( \sum_N \|D_x^{\frac{3}{4}} P_N v\|_{L^4_{t,x}}^3 \right)^{\frac{1}{3}} \lesssim T^{\frac{1}{4}} G(\|v\|_{L^\infty_{t,x}}) \|v\|_{L^\infty_T H^s_x}.\]
(3.8)

(ii) For \( \delta_0 \leq \delta < \infty \), let \( u \in C([0, T]; H^s_x(\mathbb{T})) \) satisfy (1.17) with \( u_0 \in H^s_x(\mathbb{T}) \) on \([0, T]\). Then,
\[
\left( \sum_N \omega_N^4 \|D_x^{\frac{3}{8}} P_N u\|_{L^4_{t,x}}^4 \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{4}} G(\|u\|_{L^\infty_{t,x}}) \|u\|_{L^\infty_T H^s_x}.
\]
(3.9)
\[
\left( \sum_N \|D_x^{\frac{5}{4}} P_N u\|_{L^4_{t,x}}^2 \right)^{\frac{1}{2}} \lesssim T^{\frac{1}{4}} G(\|u\|_{L^\infty_{t,x}}) \|u\|_{L^\infty_T H^s_x}.\]
(3.10)

Here, \( G = G[f] \) is an entire function that is increasing and non negative on \( \mathbb{R}_+ \), and the implicit constant is independent of \( \delta \).

**Proof.** It suffices to consider the case \( N \gg 1 \). We divide the interval in small intervals of length \( \sim N^{-1} \). In other words, we define \( \{I_{j,N}\}_{j \in J_N} \) so that
\[
\bigcup_{j \in J_N} I_{j,N} = [0, T], \quad |I_{j,N}| \lesssim N^{-1}, \quad \text{and} \quad |J_N| \lesssim N.
\]
By the hypothesis, we see that \( \|D_x^s P_N u(t)\|_{L^2_x} \in C([0, T]) \). For \( j \in J_N \), we choose \( c_{j,N} \in I_{j,N} \) at which \( \|D_x^s P_N u(t)\|_{L^2_x} \) attains its minimum on \( I_{j,N} \). For the We see from the Duhamel formulation that
\[
P_N u(t) = e^{-(t-c_{j,N})\partial_x^2} P_N u(c_{j,N}) + \int_{c_{j,N}}^t e^{-(t-t')\partial_x^2} P_N \partial_x (f(u))(t') dt'
\]
for \( t \in I_{j,N} \) and \( i = 1 \) or 2.
We first consider the linear contribution in the following. Let $\alpha = 1$ or $2$. Recall the definition of $\beta(\alpha)$ and $b(\alpha)$ from (3.1). Then, Lemma 3.4 and the Bernstein inequality show that for $i = 1$ or 2, we have

$$
\left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \| D_x^{s-\beta(\alpha)} e^{-(t-c_j,N)\partial_x} D_x^{s-\beta(\alpha)} P_N u(c_j,N) \|_{L^4(I_{j,N};L^2_x)}^4 \right)^{1/4}
\lesssim \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 |I_{j,N}| 2^{-4b(\alpha)} N^{-4\beta(\alpha)} \| D_x^{s} P_N u(c_j,N) \|_{L^2_x}^4 \right)^{1/4}
\lesssim \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 |I_{j,N}| \| D_x^{s} P_N u(c_j,N) \|_{L^2_x} \right)^{1/4}
\lesssim \left( \sum_{N \gg 1} \int_0^T \omega_N^4 \| D_x^{s} P_N u(t) \|_{L^2_x}^4 dt \right)^{1/4}
\lesssim T^{1/4} \| u \|_{L^{\infty}_{t,x} H^s_x}.
$$

In the last inequality, we used the fact $\ell^2(\mathbb{N}) \hookrightarrow \ell^4(\mathbb{N})$, and the implicit constant is independent of $\delta$.

Next, we estimate the contribution of the Duhamel term. To simplify the expressions we set $\tilde{f} = f - f(0)$. From Lemma 3.4, the Hölder inequality in time, and (2.13), we have that for $i = 1$ or 2

$$
\left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \left\| \int_{c_j,N}^t e^{-(t-t')\partial_x} D_x^{s-\beta(\alpha)} P_N \partial_x (\tilde{f}(u))(t') dt' \right\|_{L^4(I_{j,N};L^2_x)}^4 \right)^{1/4}
\lesssim \left( \sum_{N \gg 1} \sum_{j} \left( \omega_N^4 |I_{j,N}|^{1/2-b(\alpha)} N^{1-\beta(\alpha)} \int_{I_{j,N}} \| D_x^{s} P_N \tilde{f}(t') \|_{L^2_x} dt' \right)^4 \right)^{1/4}
\lesssim \left( \sum_{N \gg 1} \sum_{j} \left( \omega_N^4 |I_{j,N}|^{-3/4} \int_{I_{j,N}} \| D_x^{s} P_N \tilde{f}(t') \|_{L^2_x} dt' \right)^4 \right)^{1/4}
\lesssim \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \int_{I_{j,N}} \| D_x^{s} P_N \tilde{f}(t') \|_{L^2_x}^4 dt' \right)^{1/4}
\lesssim \left( \int_0^T \sum_{N \gg 1} \omega_N^4 \| D_x^{s} P_N \tilde{f}(t') \|_{L^2_x}^4 dt' \right)^{1/4}
\lesssim T^{1/4} \| \tilde{f}(u) \|_{L^{\infty}_{t,x} H^s_x} \lesssim T^{1/4} G(\| u \|_{L^{\infty}_{t,x}}) \| u \|_{L^{\infty}_{t,x} H^s_x}.
$$

This finishes the proofs of (3.7) and (3.9).

Next, we prove (3.8) and (3.10). For the spatial variable, we use the following estimate from the Bernstein inequality

$$
\| P_N u \|_{L^\infty} \lesssim N^{1/4} \| P_N u \|_{L^4}.
$$

Then, together with the Hölder inequality in time give the following

$$
\| D_x^{1/3} P_N u \|_{L^3(I_{j,N};L^\infty_x)} \lesssim |I_{j,N}|^{1/12} \| D_x^{7/12} P_N u \|_{L^4(I_{j,N};L^2_x)} \lesssim \| D_x^{1/2} P_N u \|_{L^{1}(I_{j,N};L^2_x)}.
$$
where recall $|I_{j,N}| \sim N^{-1}$ and $\#J \sim N$ from our above convention. This implies that

$$\left( \sum_N \| D^{1/3}_x P_N u \|_{L^3_x L^\infty_x}^3 \right)^{1/3} \lesssim \left( \sum_N \sum_j \| D^{1/2}_x P_N u \|_{L^4(I_{j,N};L^4_x)}^3 \right)^{1/3}.$$  

We can obtain (3.10) by the same way as (3.7) with $\omega_N = 1$. \hfill \Box

The following corollary estimates the difference between the two solutions.

**Corollary 3.6.** Let $s > \frac{1}{4}$, $T \in (0,1)$, and there exists some $\delta_0 > 0$ such that the following two estimates hold:

(i) For $\delta_0 \leq \delta < \infty$, let $u^{(1)}, u^{(2)} \in C([0,T];H_0^s(\mathbb{T}))$ satisfy (1.17) with $u_0^{(1)}, u_0^{(2)} \in H^s(\mathbb{T})$ on $[0,T]$. Then, for $w := u^{(1)} - u^{(2)}$

$$\left( \sum_N \{ (1 \vee N)^{s-\frac{9}{14}} \| P_N w \|_{L^4_x L^6_x} \}^4 \right)^{\frac{1}{4}} \lesssim T^\frac{1}{2} G(u^{(1)}, u^{(2)}) \| w \|_{H^s_x L^8_T}^{\frac{7}{5}};$$

$$\left( \sum_N \{ (1 \vee N)^{-\frac{9}{14}} \| P_N w \|_{L^4_x L^6_x} \}^3 \right)^{\frac{1}{3}} \lesssim T^\frac{1}{2} G(u^{(1)}, u^{(2)}) \| w \|_{L^8_T H^{s-\frac{1}{2}}_x}^{\frac{7}{5}}.$$  

(ii) For $0 < \delta < \delta_0$, let $v^{(1)}, v^{(2)} \in C([0,T];H^s(\mathbb{T}))$ satisfy (1.19) with $v_0^{(1)}, v_0^{(2)} \in H^s(\mathbb{T})$ on $[0,T]$. Then, for $w := v^{(1)} - v^{(2)}$

$$\left( \sum_N \{ (1 \vee N)^{s-\frac{11}{12}} \| P_N w \|_{L^4_x L^6_x} \}^4 \right)^{\frac{1}{4}} \lesssim T^\frac{1}{2} G(v^{(1)}, v^{(2)}) \| w \|_{H^s_x L^8_T}^{\frac{7}{5}};$$

$$\left( \sum_N \{ (1 \vee N)^{-\frac{11}{12}} \| P_N w \|_{L^4_x L^6_x} \}^3 \right)^{\frac{1}{3}} \lesssim T^\frac{1}{2} G(v^{(1)}, v^{(2)}) \| w \|_{L^8_T H^{s-\frac{1}{2}}_x}^{\frac{7}{5}}.$$  

Here, the implicit constants are independent of $\delta$, and has the following form

$$G(u,v) := G(\| u \|_{L^8_T H^s_x} + \| v \|_{L^8_T H^s_x})$$

such that $G = G[f]$ is an increasing and non-negative entire function on $\mathbb{R}_+$. \hfill \Box

**Proof.** The proof is the same as that of Proposition 3.5, but with using (2.15) instead of (2.13). \hfill \Box

### 4. Energy estimates

**4.1. Preliminary technical estimates.** Before we get our main $\delta$-independent a priori estimates, there are a few useful estimates we shall state in this subsection.

Let us denote by $1_T$ the characteristic function of the interval $(0,T)$. It has been pointed out in [64] that $1_T$ does not commute with $Q_L$. Therefore, we follow the approach in [64], we further decompose $1_T$ as

$$1_T = 1_{1_T,R}^{\text{low}} + 1_{1_T,R}^{\text{high}},$$

with $F_\tau(1_{1_T,R}^{\text{low}})(\tau) = \chi(\frac{\tau}{R}) F_\tau(1_T)(\tau)$, for some $R > 0$ to be fixed later. This further decomposition avoids the difficulty that $1_T$ does not commute with $Q_L$.

The following lemmas give some useful properties regarding the time cut-off and projection $Q_L$. The proof can be found in [67].
Lemma 4.1 (Lemma 3.5 in [67]). Let $L > 0$ be a inhomogeneous dyadic number, $1 \leq p \leq \infty$, and $s \in \mathbb{R}$. Then, the operator $Q_{\leq L}$ is bounded in $L^p_t H^s_x$ uniformly in $L$. In particular,

$$\|Q_{\leq L}u\|_{L^p_t H^s_x} \lesssim \|u\|_{L^p_t H^s_x},$$  \hspace{1cm} (4.1)

for all $u \in L^p_t H^s_x$ and the implicit constant appearing in (4.1) does not depend on $L$.

Lemma 4.2 (Lemma 3.6 in [67]). For any $R > 0$ and $T > 0$, it holds

$$\|1^\text{high}_{T,R}\|_{L^1} \lesssim T \land R^{-1},$$  \hspace{1cm} (4.2)

and

$$\|1^\text{high}_{T,R}\|_{L^\infty} + \|1^\text{low}_{T,R}\|_{L^\infty} \lesssim 1.$$  \hspace{1cm} (4.3)

Lemma 4.3 (Lemma 3.7 in [67]). Assume that $T > 0$, $R > 0$, and $L \gg R$. Then, it holds

$$\|Q_L(1^\text{low}_{T,R}u)\|_{L^2_t x} \lesssim \|Q_{\sim L}u\|_{L^2_t x},$$  \hspace{1cm} (4.4)

for all $u \in L^2(\mathbb{R}_t \times \mathbb{T}_x)$.

In what follows, we study the properties on the resonance function associated to (1.17) and (1.19). In particular, we see that when the nonlinear interaction is non-resonant, we gain of derivative from so-called “multilinear dispersion”. Namely, the resonance function has a uniform lower bound and from Bourgain’s Fourier restriction norm method, the modulation function provides derivative gain to balance the derivative loss on the nonlinearity. First of all we define the following notation.

Definition 2. Let $i = 1$ or $2$, and $j \in \mathbb{N}$. We define $\Omega^i_j(n_1, \ldots, n_{j+1}) : \mathbb{Z}^{j+1} \rightarrow \mathbb{R}$ as

$$\Omega^i_j(n_1, \ldots, n_{j+1}) := \sum_{k=1}^{j+1} p^i_k(n_k)$$

for $(n_1, \ldots, n_{j+1}) \in \mathbb{Z}^{j+1}$, where $p^i_k$ satisfies (2.1) and (2.2).

The resonance function $(\Omega_{k+1})$ can be resonant (i.e., very small) with respect to higher-order interaction such as the polynomial-type nonlinearity (i.e. $\partial_x(u^{k+1})$). In what follows, we clarify that in the non-resonant interaction situation in which we can recover the derivative loss of the nonlinearity by using a priori estimates in $X^{s,b}$-spaces of the solution to (1.19) or (1.17) proved in Lemma 4.8.

Lemma 4.4. Let $i = 1$ or $2$, $k \geq 1$, and $(n_1, \ldots, n_{k+2}) \in \mathbb{Z}^{k+2}$ satisfy \(\sum_{j=1}^{k+2} n_j = 0\). Assume that

$$\begin{cases} |n_1| \sim |n_2| \gtrsim |n_3|, & \text{if } k = 1 \\
1_1 |n_1| \sim |n_2| \gtrsim |n_3| \gg k \max_{j \geq 4} |n_j|, & \text{if } k \geq 2.\end{cases}$$

Then, there exists some $n_0 > 0$ such that for $|n_1| \gg \left( \max_{n \in [0,n_0]} |p^i_k(n)| \right)^{\frac{1}{\alpha(i)}}$,

$$|\Omega^i_{k+1}(n_1, \ldots, n_{k+2})| \gtrsim |n_3||n_1|^\alpha(i).$$

Here, $\alpha(1) = 2$, $\alpha(2) = 1$, and the implicit constant does not dependent on $\delta$. 
Proof. For simplicity we denote \( \alpha(i) = \alpha \) in the following proof. Also, we recall from (3.4) that \( p_{\delta_i}(n), p_{\delta_i}(n) \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}) \) are real-valued odd-functions, and there exists some \( n_0 > 0 \) such that for all \( n \geq n_0 \), the following hold
\[
p'_\delta(n) \sim n^\alpha \quad \text{and} \quad p''_\delta(n) \sim n^{\alpha-1}.
\]

First, we consider the case \( k \geq 2 \). We separate different cases:

**Case 1:** \( |n_2| \gg |n_3| \).

Firstly, by our hypothesis, it follows that \( |n_2| \gg n_0 \). The mean value theorem (MVT) implies that there exists \( k \in \mathbb{R} \) such that \( |k| \sim |n_2| \) and that
\[
|p_{\delta_i}(n_2 + \cdots + n_{k+2}) - p_{\delta_i}(n_2)| = |n_3 + \cdots + n_{k+2}|p'_\delta(k)| \sim |n_3||n_1|^{\alpha}.
\]

Here, we used \(|n_1| \sim |n_2|\) and \(|n_3| \gg k \max_{j \geq 4} |n_j|\).

Next, we consider \(|n_j| \leq n_0\) for \( j \geq 4 \), then by MVT again
\[
|p_{\delta_i}(n_j)| \leq |n_j| \max_{n \in [0,n_0]} |p'_\delta(n)| \ll \frac{|n_3||n_1|^{\alpha}}{k},
\]

where we have used \( p_{\delta_i}(0) = 0 \). On the other hand, if \(|n_j| \geq n_0\) for \( j \geq 4 \), then MTV twice, we have
\[
|p_{\delta_i}(n_j)| \leq |p_{\delta_i}(n_j) - p_{\delta_i}(n_0)| + |p_{\delta_i}(n_0) - p_{\delta_i}(0)| 
\lesssim |n_0| \max_{n \in [0,n_0]} |p'_\delta(n)| + |n_j|^{\alpha+1} \ll \frac{|n_3||n_1|^{\alpha}}{k}.
\]

Similarly, we can get \( |p_{\delta_i}(n_3)| \ll |n_3||n_1|^{\alpha} \). Gathering these estimates leads to
\[
|\Omega_{k+1}^i| \gtrsim |n_3||n_1|^{\alpha}.
\]

**Case 2:** \( |n_2| \sim |n_3| \).

Then we have \(|n_3| \gg n_0 \). By impossible interactions, \( n_1, n_2 \) and \( n_3 \) do not have the same sign. By the symmetry and \(|n_1| \sim |n_2| \sim |n_3|\), it suffices to consider the case \( n_2, n_3 > 0 \). We notice that
\[
-\Omega_{k+1}^i = \int_{n_0}^{n_2} \{p'_{\delta_i}(\theta + n_3 + \cdots + n_{k+2}) - p'_{\delta_i}(\theta)\} \, d\theta
\]
\[
+ \{p_{\delta_i}(n_0 + n_3 + \cdots + n_{k+2}) - p_{\delta_i}(n_3)\}
\]
\[
- p_{\delta_i}(n_0) - \sum_{j=4}^{k+2} p_{\delta_i}(n_j).
\]

By the MVT, we have
\[
|p_{\delta_i}(n_0 + n_3 + \cdots + n_{k+2}) - p_{\delta_i}(n_3)| \leq (|n_0| + k \max_{j \geq 4} |n_j|)|n_3|^{\alpha} \ll |n_3||n_1|^{\alpha}.
\]

Moreover, we see the following
\[
p'_{\delta_i}(\theta + n_3 + \cdots + n_{k+2}) - p'_{\delta_i}(\theta) = \int_0^{n_3 + \cdots + n_{k+2}} p''_{\delta_i}(\theta + \mu) \, d\mu.
\]
For some $\theta \geq n_0$, and $p''_\theta$ does not change sign since $|p''_\theta(\theta)| \sim |\theta|^{\alpha}$ and $p''_\theta$ is continuous outside 0. Therefore, for $\theta \in [n_0, n_2]$, we get

$$\int_{0}^{n_3+\cdots+n_{k+2}} p''_\theta(\theta + \mu) d\mu \sim \int_{0}^{n_3+\cdots+n_{k+2}} (\theta + \mu)^{\alpha-1} d\mu \sim n_3^\alpha,$$

since $n_3 \gg n_j$ for $j \geq 4$. Gathering these estimates, we obtain

$$|\Omega^i_{k+1}| \gtrsim |n_3||n_1|^\alpha.$$

For the case $k = 1$, we can argue exactly as above.

\[\Box\]

**Lemma 4.5.** Let $i = 1$ or $2$, $k \geq 2$, and $(n_1, \ldots, n_{k+2}) \in \mathbb{Z}^{k+2}$ satisfy $\sum_{j=1}^{k+2} n_j = 0$. Assume that

$$\left\{ \begin{array}{l}
|n_1| \sim |n_2| \gg |n_3| \gtrsim |n_4|, \quad \text{if} \quad k = 2; \\
|n_1| \sim |n_2| \gg |n_3| \gtrsim |n_4| \quad \text{with} \quad |n_3 + n_4| \gg k \max_{j \geq 5} |n_j| \quad \text{if} \quad k \geq 3.
\end{array} \right.$$ 

Then, there exists some $n_0 > 0$ such that for $|n_1| \gg (\max_{n \in [0, n_0]} |p'_\theta(n)|)^{1\over|n_1|}$

$$|\Omega^i_{k+1}(n_1, \ldots, n_{k+2})| \gtrsim |n_3 + n_4||n_1|^{\alpha(i)},$$

where $\alpha(1) = 2, \alpha(2) = 1$ as above, and the implicit constant does not dependent on $\delta$.

**Proof.** For simplicity we denote $\alpha(i) = \alpha$ in the following proof. Also, there exists some $n_0 > 0$ such that for all $n \geq n_0$, the following hold

$$|p'_\theta(n)| \sim n^\alpha \quad \text{and} \quad |p''_\theta(n)| \sim n^{\alpha-1}.$$

First we consider the case $k \geq 3$. We separate different cases:

**Case 1:** $|n_3| \gg |n_4|$.

In this situation, we have

$$|n_3| \gg k \max_{j \geq 5} |n_j| \quad \text{and} \quad |n_3| \sim |n_3 + n_4|.$$

Therefore, we can argue exactly as in Case 1 of Lemma 4.4 and obtain

$$|\Omega^i_{k+1}| \gtrsim |n_3||n_1|^\alpha \sim |n_3 + n_4||n_1|^\alpha.$$

**Case 2:** $|n_3| \sim |n_4|$.

Let us first consider $n_3n_4 \geq 0$. Then, it holds that $|n_3 + n_4| = |n_3| + |n_4|$. Moreover, we have

$$|n_3|, |n_4| \gg k \max_{j \geq 5} |n_j|.$$

Therefore, we argue by the same way as in Case 1 of Lemma 4.4 and obtain

$$|\Omega^i_{k+1}| \gtrsim |n_3||n_1|^\alpha \sim |n_3 + n_4||n_1|^\alpha.$$

In the case when $n_3n_4 < 0$. By using MVT, there exist $k_1, k_2 \in \mathbb{R}$ such that

$$\left\{ \begin{array}{l}
|k_1| \sim |n_1| \sim |n_2|, \\
|n_4| \lesssim |k_2| \lesssim |n_3|.
\end{array} \right.$$
We notice that \( n_1 + n_2 = -(n_3 + n_4 + \cdots + n_{k+2}) \), and we can see the following computation

\[
-\Omega^i_{k+1} = -(n_1 + n_2)p'_\delta(k_1) - (n_3 + n_4)p'_\delta(k_2) - \sum_{j=5}^{k+2} p_\delta(n_j)
\]

\[
= (n_3 + n_4 + \cdots + n_{k+2})p'_\delta(k_1) - (n_3 + n_4)p'_\delta(k_2) - \sum_{j=5}^{k+2} p_\delta(n_j)
\]

where we used the property of being an odd-function such that \( p_\delta(n) = -p_\delta(-n) \) for any \( n \in \mathbb{R} \). We also notice that \( |p'_\delta(k_2)| \ll |n_1|^{\alpha} \). Indeed, recall that \( |p'_\delta(k_2)| \sim |n_1|^{\alpha} \), and we have

\[
|p'_\delta(k_2)| \leq \max_{n \in [0, n_0]} |p'_\delta(n)| \ll |n_1|^{\alpha} \quad \text{if} \quad |k_2| \leq n_0;
\]

and

\[
|p'_\delta(k_2)| \sim |n_2|^{\alpha} \ll |n_3|^{\alpha} \ll |n_1|^{\alpha} \quad \text{if} \quad |k_2| \geq n_0.
\]

As in Case 1 of Lemma \( 4.4 \), we also have \( k \max_{j \geq 5} |p_\delta(n_j)| \ll |n_3 + n_4||n_1|^{\alpha} \). From these estimates, we obtain \( |\Omega^i_{k+1}| \ll |n_3 + n_4||n_1|^{\alpha} \).

For the case \( k = 2 \), we can argue exactly as above.

\[\square\]

**Lemma 4.6.** Let \( \kappa > 1 \), suppose the dyadic sequence \( \{\omega_N\} \) of positive numbers satisfies

\[
\omega_N \leq \omega_{2N} \leq \kappa \omega_N \quad \text{for} \quad N \geq 1,
\]

and \( \omega_N \to \infty \) as \( N \to \infty \). Then, for any \( 1 < \kappa' < \kappa \), there exists a dyadic sequence \( \{\bar{\omega}_N\} \) such that

\[
\bar{\omega}_N \leq \omega_N, \quad \bar{\omega}_N \leq \bar{\omega}_{2N} \leq \kappa' \bar{\omega}_N \quad \text{for} \quad N \geq 1
\]

and \( \bar{\omega}_N \to \infty \) as \( N \to \infty \).

**Proof.** See [63, Lemma 4.6].

\[\square\]

**Remark 4.7.** For the given dyadic sequence \( \{\omega_N\} \) of positive numbers, Lemma \( 4.6 \) allows us to assume \( \kappa \leq 2 \), by defining a new dyadic sequence. In the proof of Proposition \( 4.9 \), we use this fact assumption. Let \( N, M \) be dyadic numbers and take \( \ell \geq 2 \) such that \( \ell N \gtrsim M \gtrsim 1 \). By using the condition \( \omega_{2N} \leq \kappa \omega_N \), the following holds

\[
\frac{\omega_M}{\omega_N} \lesssim \kappa^{\log_2 \ell} \lesssim \ell
\]

which is uniformly in \( \kappa \).

### 4.2. A priori bounds on solutions

In this subsection, we will present the crucial energy estimates. We first notice that our solutions \( u_\delta \) and \( v_\delta \) to \( (1.19) \) and \( (1.17) \) are characterised by \( \delta \), but in this construction of solutions part, we do not need to worry about the parameter \( \delta \). The main goal in this step is to keep tracking all the estimates that are uniformly in \( \delta \) such that we can pass the limit to the solution of \( (1.19) \) and \( (1.17) \), separately. Hence, for the simplicity reason, we write \( u_\delta = u \) and \( v_\delta = v \) in this section and we keep in mind this notation.
Let $s \in \mathbb{R}$. We recall the function space $Z^s$ (resp. $Z^s_\omega$) from (2.8) that $Z^s := L^\infty_t H^s \cap X^{s-1,1}_t$ (resp. $Z^s_\omega := L^\infty_t H^s_\omega \cap X^{s-1,1}_\omega$), endowed with the natural norm
\[ \|u\|_{Z^s} = \|u\|_{L^\infty_t H^s} + \|u\|_{X^{s-1,1}_t} \quad \text{(resp. } \|u\|_{Z^s_\omega} = \|u\|_{L^\infty_t H^s_\omega} + \|u\|_{X^{s-1,1}_\omega} \text{).} \]

Let $s > \frac{1}{2}$ and $(u_0, v_0) \in (H^s_\omega(\mathbb{T}))^2$. We construct a solution to (1.19) and (1.17) in $Z^s_\omega$, whereas the difference of two solutions generated from initial data belonging to $H^s(\mathbb{T})$ are estimated in $Z^{s-1}_\omega$. The following lemma holds for (1.19) and (1.17). For simplicity, the following lemma stated in terms of the solution $u$ to (1.17). It is true for solutions of (1.19), which we replace $u$ by the solution $v$ to (1.19).

**Lemma 4.8.** Let $1 \leq \kappa \leq 2$ and suppose that the dyadic sequence $\{\omega_N\}$ of positive numbers satisfies
\[ \omega_N \leq \omega_{2N} \leq \kappa \omega_N \quad \text{for} \quad N \geq 1. \]

Consider $T \in (0,1)$, $s > \frac{1}{2}$, and $u \in L^\infty([0,T]; H^s_\omega(\mathbb{T}))$ be a solution to (1.19). Then, for $u \in Z^s_\omega$ and it holds
\[ \|u\|_{Z^s_\omega} \lesssim \|u\|_{L^\infty_T H^s_\omega} + G(\|u\|_{L^\infty_T H^s_\omega}) \|u\|_{L^\infty_T H^s_\omega}. \] (4.5)

Moreover, let $u^{(j)} \in L^\infty([0,T]; H^s(\mathbb{T})$ be solutions to (1.19) associated with initial data $v_0^{(j)} \in H^s(\mathbb{T}$, for $j = 1, 2$. Then, the following holds
\[ \|u^{(1)} - u^{(2)}\|_{Z^s_\omega} \lesssim \|u^{(1)} - u^{(2)}\|_{L^\infty_T H^{s-1}_\omega} + G(\|u^{(1)}\|_{L^\infty_T H^s_\omega} + \|u^{(2)}\|_{L^\infty_T H^s_\omega}) \|u^{(1)} - u^{(2)}\|_{L^\infty_T H^{s-1}_\omega}. \] (4.6)

Here, the implicit constants are independent of $\delta$.

**Proof.** By following the extension Lemma 2.6 in order to prove (4.5) we only have to estimate $u \in X^{s-1,1}_\omega$. Moreover, Remark 1.7 implies $u$ satisfies the Duhamel formula of (1.19) and
\[ \|u_0\|_{H^s_\omega} \leq \|u\|_{L^\infty_T H^s_\omega} \]
for any $\theta \leq s$. Hence, standard linear estimates in $X^{s,b}$-spaces and (2.13) lead to
\[ \|u\|_{X^{s-1,1}_\omega} \lesssim \|u_0\|_{H^s_\omega} + \|\partial_x(f(u))\|_{X^{s-1,1}_\omega} \lesssim \|u_0\|_{H^s_\omega} + \|f(u) - f(0)\|_{L^1_T H^s_\omega} \lesssim \|u\|_{L^\infty_T H^{s-1}_\omega} + G(\|u\|_{L^\infty_T H^s_\omega}) \|u\|_{L^\infty_T H^s_\omega}. \]

Similarly, by using (2.15), we get
\[ \|u^{(1)} - u^{(2)}\|_{X^{s-2,1}_\omega} \lesssim \|u^{(1)}_0 - u^{(2)}_0\|_{H^{s-1}_\omega} + \|f(u^{(1)}) - f(u^{(2)})\|_{L^1_T H^{s-1}_\omega} \lesssim \|u^{(1)} - u^{(2)}\|_{L^\infty_T H^{s-1}_\omega} + G(\|u^{(1)}\|_{L^\infty_T H^s_\omega} + \|u^{(2)}\|_{L^\infty_T H^s_\omega}) \|u^{(1)} - u^{(2)}\|_{L^\infty_T H^{s-1}_\omega} \]
which completes the proof of (4.6).

Moreover, for the solutions of (1.17), we note that the above computations did not use any dispersion. Therefore, we repeat the exact same argument, we get the claim for solution $v$ of (1.17).

\[ \square \]

\[ ^4 \text{See also [61] Remark 1.3 for the discussion.} \]
In the following we establish an energy estimate for solutions to the gFDF (1.19) and (1.17). This argument is very close in spirit to the improved energy method by Molinet-Vento [64]. See also [63, Proposition 4.8].

Let \( u \in C(\mathbb{R}; H^\infty(\mathbb{T})) \) be a smooth solution to (1.19). Then, by the Fundamental Theorem of Calculus, we have

\[
\|u(t)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 = -2 \int_{\mathbb{T}} \partial_x(f(u))u \, dx. \tag{4.7}
\]

We turn to study the nonlinear interactions on the RHS of (4.7). The computation (4.7) holds exactly for smooth solution \( v \in C(\mathbb{R}; H^\infty(\mathbb{T})) \) to (1.17).

**Proposition 4.9** (\( \delta \)-independent A priori estimate). Let \( \kappa \geq 1 \) and suppose that the dyadic sequence \( \{\omega_N\} \) of positive numbers satisfies

\[
\omega_N \leq \omega_{2N} \leq \kappa \omega_N \quad \text{for} \quad N \geq 1.
\]

Consider \( \alpha = 1 \) or 2 and

\[
2 \geq s \geq s(\alpha) := \frac{1}{2} + 2\beta(\alpha) = \frac{1}{2} + \frac{1}{2(\alpha + 1)}.
\]

Let \( T \in (0, 1) \), \( v \in L^\infty([0, T]; H^s(\mathbb{T})) \) be a solution to (1.19) associated with an initial data \( v_0 \in H^s(\mathbb{T}) \), and \( u \in L^\infty([0, T]; H^s(\mathbb{T})) \) be a solution to (1.17) associated with an initial data \( u_0 \in H^s(\mathbb{T}) \). Consider \( \delta \in (0, \infty) \). Then, there exists \( \delta_0 > 0 \) and an increasing and non negative entire function \( G = G(\cdot) \) on \( \mathbb{R}_+ \) such that the following two estimates hold.

(i) For any \( \delta \in (0, \delta_0) \) and let \( s \geq \frac{3}{4} \). Then, we have

\[
\|v\|_{L^\infty_t X^s H^s_x} \leq \|v_0\|_{H^s_x}^2 + T^{\frac{1}{2}} G(\|v\|_{L^\infty_t Z^s_x}) \|v\|_{L^\infty_t Z^s_{x,t}} \|v\|_{L^\infty_t H^s_x}; \tag{4.8}
\]

(ii) For any \( \delta \in (\delta_0, \infty) \) and let \( s \geq \frac{3}{4} \). Then, we have

\[
\|u\|_{L^\infty_t X^s H^s_x} \leq \|u_0\|_{H^s_x}^2 + T^{\frac{1}{2}} G(\|u\|_{L^\infty_t Z^s_x}) \|u\|_{L^\infty_t Z^s_{x,t}} \|u\|_{L^\infty_t H^s_x}. \tag{4.9}
\]

Here, the estimates are uniformly in \( \delta \).

**Remark 4.10.** The restriction on the regularity comes from the resonant interactions of the nonlinearity. In particular, see subcase 2.1 of \( A_1 \) contribution such that

\[
N \sim N_1 \sim N_2 \sim N_3 \sim kN_4.
\]

In this situation, the nonlinear effects happen to be resonant interactions. To recover the derivative loss on the nonlinearity, we share the derivative with 4 terms with high frequencies in \( L^4_{t,x} \). Then, we apply the Proposition 3.5 (short-time Strichartz estimates) to control them in the desired norms. In particular, we see from the definition (3.1) and Proposition 3.5 that by controlling 4 terms (with high frequencies) in \( L^4_{t,x} \), the recovery of the derivatives loss leads to the following regularity restriction:

\[
4(s - \beta(\alpha)) \geq 2s + 1 \quad \iff \quad s \geq \frac{1}{2} + 2\beta(\alpha) \quad \iff \quad s \geq s(\alpha).
\]

**Remark 4.11.** This proof is valid for the real line situation. Indeed, as for the following set-up step, we only used the integration by parts (IBP). The main strategy of our crucial nonlinear estimates is to use symmetry arguments. In practice, we distribute the lost derivative to several functions, and then to recover it by applying either improved Strichartz estimates.
or $X^{s,b}$-type estimates. To decide which way of recovering the loss of derivative depends on whether the nonlinear interactions are resonant. Therefore, in the case of $\mathbb{R}$, we shall replace the improved Strichartz estimates with the real line setting accordingly.

**Proof of 4.9.** First of all, the proofs for (4.8) and (4.9) are almost identical. For simplicity, we present our proof by using solution notation $u$. We notice that according to Lemma 4.8 it holds $u \in Z_{s,T}^{2}$. Moreover, by taking the Littlewood-Paley projection onto the solution $u$, it is clear that $P_{N} \in C([0, T]; H^{\infty}(\mathbb{T}))$ with $P_{N} \partial_{x} u \in L^{\infty}([0, T]; H^{\infty}(\mathbb{T}))$. Therefore, (4.7) applies, in particular, taking the $L^{2}$-scalar product of the resulting equation with $P_{N} u$, multiplying by $\omega_{N}^{2}(N)^{2s}$ and integrating over $[0, t]$ with $0 < t < T$, we yield

$$\omega_{N}^{2}(N)^{2s} \|P_{N} u(t)\|_{L^{2}}^{2} = \omega_{N}^{2}(N)^{2s} \|u_{0}\|_{L^{2}}^{2} - 2\omega_{N}^{2}(N)^{2s} \int_{0}^{t} \int_{\mathbb{T}} \partial_{x} P_{N}(f(u)) P_{N} u \, dx \, dt'.$$

We use integration by parts, apply Bernstein inequalities, and sum over in $N$, we obtain

$$\|u(t)\|_{H^{s}}^{2} = \sum_{N} \omega_{N}^{2}(N)^{2s} \left( \|P_{N} u_{0}\|_{L^{2}}^{2} - 2 \int_{0}^{t} \int_{\mathbb{T}} P_{N} \partial_{x}(f(u)) P_{N} u \, dx \, dt' \right) \leq \|u_{0}\|_{H^{s}}^{2} + 2 \sum_{N} \omega_{N}^{2}(N)^{2s} \left| \int_{0}^{t} \int_{\mathbb{T}} P_{N}(f(u) - f(0)) P_{N} \partial_{x} u \, dx \, dt' \right| \leq \|u_{0}\|_{H^{s}}^{2} + 2 \sum_{N \geq 1} \omega_{N}^{2}(N)^{2s} \left| \int_{0}^{t} \int_{\mathbb{T}} (f(u) - f(0)) P_{N}^{2} \partial_{x} u \, dx \, dt' \right|,$$

where we recall that $f(0) = 0$ and hence $\partial_{x}(f(u)) = \partial_{x}(f(u) - f(0))$, also we note that $P_{0} \partial_{x} u = 0$. Let us rewrite the difference of the nonlinearity in the following

$$f(u) - f(0) = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} u^{k}.$$

Then, for any fixed $N \in 2^{\mathbb{N}}$ we have the following

$$\int_{0}^{t} \int_{\mathbb{T}} (f(u) - f(0)) P_{N}^{2} \partial_{x} u \, dx \, dt' = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} \int_{0}^{t} \int_{\mathbb{T}} u^{k} P_{N}^{2} \partial_{x} u \, dx \, dt'. \quad (4.11)$$

Indeed, in order to interchange the summation and the integration in (4.11), we need to show the following boundedness. For each fixed $N$,

$$\sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_{0}^{t} \int_{\mathbb{T}} |u^{k} P_{N}^{2} \partial_{x} u| \, dx \, dt' \lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_{0}^{t} \|u^{k}\|_{L^{2}_{x}}^{2} \|u\|_{L^{2}_{x}} \, dt' \lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_{0}^{t} \|u\|_{L^{\infty}_{x}}^{k-1} \|u\|_{L^{2}_{x}}^{2} \, dt' \lesssim TNG(\|u\|_{L^{\infty}_{T,x}}) \|u\|_{L^{\infty}_{T,x}}^{2} \lesssim N^{2}.$$
Hence, the equation (4.11) follows from Fubini-Lebesgue’s theorem. Moreover, by (4.11) together with Fubini-Tonelli’s theorem, we obtain

$$\sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T (f(u) - f(0)) P_N^2 \partial_x u \, dx \, dt' \right|$$

$$= \sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T u^k P_N^2 \partial_x u \, dx \, dt' \right|$$

$$\leq \sum_{N \geq 1} \sum_{k \geq 1} \omega_N^2 N^{2s} \left| \frac{f(k)(0)}{k!} \right| \left| \int_0^t \int_T u^k P_N^2 \partial_x u \, dx \, dt' \right| =: \sum_{k \geq 1} \frac{|f(k)(0)|}{k!} I_k^k$$

where $I_k^k$ is defined by

$$I_k^k := \sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T u^k P_N^2 \partial_x u \, dx \, dt' \right|.$$

We notice here, it is easy to check that $I_1^1 = 0$ by IBP. Therefore, we shall prove that for any $k \geq 1$, the following holds

$$I_{k+1}^k \leq C^k T^{1/4} G(C_0) C_1^k \left( \| u \|_{X^{s-1,1}_{T,T}} + \| u \|_{L^\infty_T H^s_x} \right) \| u \|_{L^\infty_T H^s_x}. \quad (4.13)$$

Here, we define

$$C_0 := \| u \|_{Z^s_T}, \quad C_1 := \| u \|_{L^\infty_T x},$$

where recall $s(\alpha)$:

$$\begin{cases} s(\alpha) = s(2) = \frac{2}{3}, \text{ when } 0 < \delta < 1; \\ s(\alpha) = s(1) = \frac{3}{4}, \text{ when } 1 \leq \delta < \infty. \end{cases}$$

Since $\sum_{k \geq 1} \frac{|f(k+1)(0)|}{(k+1)!} C_0^k C_1^k < \infty$, it is clearly that (4.13) leads to (4.8) and (4.9) by taking (4.10) and (4.12) into account. Moreover, the bound on (4.13) is independent of $\delta$.

In the following, we fix $k \geq 1$. For simplicity, for any positive numbers $a$ and $b$, the notation $a \lesssim_k b$ means there exists a positive constant $C > 0$ independent of $k$ such that

$$a \leq C^k b. \quad (4.14)$$

Remark that $a \leq k^m b$ for $m \in \mathbb{N}$ can be expressed by $a \lesssim_k b$ too since an elementary calculation shows $k^m \leq m! c^k$ for $m \in \mathbb{N}$.

**Low frequency contribution.**

The contribution of the sum over $N \lesssim 1$ in $I_{k+1}^k$ is easily estimated by

$$\sum_{N \lesssim 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T u^{k+1} P_N^2 \partial_x u \, dx \, dt' \right| \leq T \sum_{N \lesssim 1} \| u \|_{L^\infty_T L^2} \| u \|_{L^\infty_T L^2} \| P_N^2 u \|_{L^\infty_T L^2} \lesssim_k T C_1^k \| u \|_{L^\infty_T H^s_x}^2.$$

**High frequency contribution.**

\footnote{Here, $e$ is Napier’s constant.}
It thus remains to bound the contribution of the sum over $N \gg 1$ in $I_{k+1}$. First, we define the following symbols,

\[ A(n_1, \ldots, n_{k+2}) := \sum_{j=1}^{k+2} \phi_N^2(n_j)n_j, \]
\[ A_1(n_1, n_2) := \phi_N^2(n_1)n_1 + \phi_N^2(n_2)n_2, \]
\[ A_2(n_4, \ldots, n_{k+2}) := \sum_{j=4}^{k+2} \phi_N^2(n_j)n_j. \]

Here, $\phi_N$ is defined in Section 2.1. It is clear that

\[ A(n_1, \ldots, n_{k+2}) = A_1(n_1, n_2) + \phi_N^2(n_3)n_3 + A_2(n_4, \ldots, n_{k+2}). \]

Moreover, we see from the symmetry that

\[ \int_{\mathbb{T}} u^{k+1} P_N^2 \partial_x u dx = \frac{i}{k+2} \sum_{n_1+\cdots+n_{k+2}=0} A(n_1, \ldots, n_{k+2}) \prod_{j=1}^{k+2} \hat{u}(n_j) \]
\[ = \frac{i}{k+2} \sum_{N_1, \ldots, N_{k+2}} \sum_{n_1+\cdots+n_{k+2}=0} A(n_1, \ldots, n_{k+2}) \prod_{j=1}^{k+2} \phi_{N_j}(n_j) \hat{u}(n_j). \tag{4.15} \]

By symmetry, we can assume that

\[
\begin{cases}
N_1 \geq N_2 \geq N_3, & \text{if } k = 1; \\
N_1 \geq N_2 \geq N_3 \geq N_4, & \text{if } k = 2; \\
N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j, & \text{if } k \geq 3.
\end{cases}
\]

We notice that the cost of this choice is a constant factor less than $(k + 2)^4$. It is also worth seeing that the frequency projection operator $P_N$ ensures that there is no contribution \[ \text{if } \{\supp(P_N) \cap \supp(P_{N_1})\} = \emptyset. \]

**Case 1: $A_2$ contribution.**

We start with the $A_2$ contribution, and note that we must have $k \geq 2$ since otherwise $A_2 = 0$. Also it suffices to consider the contribution of $(\phi_N(n_4))^2 n_4$, since the contributions of $(\phi_N(n_j))^2 n_j$ for $j \geq 5$, are simpler. Note that $N_4 \sim N$ in this case (0 otherwise, due to the observation before). First of all, by the Bernstein inequality, we have for $s_0 > \frac{1}{2}$

\[ \sum_K \|P_K u\|_{L^\infty_{T,x}} \lesssim \sum_K (1 \vee K^{1/2-s_0}) \|u\|_{L^\infty_{T} H^{s_0}_x} \lesssim \|u\|_{L^\infty_{T} H^{s_0}_x} \lesssim C_0. \tag{4.16} \]
This together with Hölder’s, Young’s convolution inequalities (see (4.17)), and Proposition 3.5, we have

\[
\sum_{N_0} \sum_{N_1, \ldots, N_k} \omega_{\alpha}^2 N^{2\alpha} \left| \int_0^T \int_\Omega \left( \frac{1}{\rho} \frac{\partial x}{\partial n} \rho P_N u \right) \prod_{j=1, j \neq 4}^{k+2} P_{N_j} u \, dx dt \right|
\]

\[
\lesssim_k \|u\|_{L^2_t H^0_x} \sum_{N_1, N_2, N_3, N_4} \omega_{N_4}^2 N^{4s+1} \prod_{j=1}^4 \|P_{N_j} u\|_{L^4_t \Omega}^4
\]

\[
\lesssim_k \left( \sum_{N_4} \|D^{1/2+\beta} P_{N_4} u\|_{L^4_t \Omega}^4 \right)^{1/4}
\]

\[
\times \left( \sum_{N_4} \left( \sum_{N_3 \geq N_4} \left( \frac{N_4}{N_3} \right)^{1/2+\beta} \|D^{1/2+\beta} P_{N_4} u\|_{L^4_t \Omega}^4 \right)^{1/4} \right)
\]

\[
\times \left( \sum_{N_4} \left( \sum_{K \geq N_4} \left( \frac{N_4}{K} \right)^{2(s-\beta)} \omega_K^2 \|D^{s-\beta} P_{K} u\|_{L^4_t \Omega}^2 \right)^{1/2} \right)
\]

\[
\lesssim_k T^{1/2} G(C_0) \|u\|_{L^2_t H^s_x}^2.
\]

To be precise in the application of Young’s convolution inequalities, let us denote \(2^k = K \lesssim M = 2^m\) for \(k < m\), and \(K, M\) to be the dyadic numbers. Then, we can see the following

\[
\left( \sum_{K} \left( \sum_{M \geq K} \left( \frac{K}{M} \right)^{\theta} \|D_x^\theta P_M u\|_{L^4_t \Omega}^4 \right)^{1/4} \right) \approx \left( \sum_{k} \left( \sum_{m \geq k} (2^k-m)^\theta \|D_x^\theta P_{2^m} u\|_{L^4_t \Omega}^4 \right)^{1/4} \right)
\]

\[
\approx \left( \sum_{k} \left\{ f \ast g(k) \right\}^4 \right)^{1/4} \lesssim \|f\|_{\ell^1} \|g\|_{\ell^4}, \quad (4.17)
\]

where \(f \ast g(k) = \sum_m f(k - m)g(m)\) and \(f, g\) are defined by the following

\[
g(m) := \|D_x^\theta P_{2^m} u\|_{L^4_t \Omega}, \quad f(k) := 2^k 1_{k < 0}(k).
\]

We note that Young’s inequality will be used frequently in our later analyses.

**Case 2: **A1 contribution.

Let us consider the A1 contribution. We notice that the frequency projector in A1 ensures that either \(N_1 \sim N\) or \(N_2 \sim N\) and thus in any case \(N \gtrsim N_3\). Moreover we can also assume that \(N_3 \geq 1\) since otherwise the contribution of A1 cancelled by integration by parts.

We divide the contribution A1 into three cases:

(i) \(N_2 \lesssim N_3 \lesssim kN_4\),

(ii) \(N_3 \gg kN_4\) or \(k = 1\),

(iii) \(N_2 \gg N_3\).
Moreover, let us define the following notation

\[ J_t := \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left| \int_0^t \int_T \Pi(P_{N_j} u, P_{N_j} u) \prod_{j=3}^{k+2} P_{N_j} u dx dt \right|, \]

where \( \Pi(f, g) \) is defined by (2.16). Note that \( N \gg 1 \) ensures that \( N_1 \gg 1 \).

**Subcase 2.1:** \( N_2 \lesssim N_3 \lesssim k N_4 \).

Since \( N \lesssim N_1 \lesssim k N_2 \lesssim k N_3 \lesssim k^2 N_4 \). Then, by Hölder’s, Bernstein’s, Young’s inequality (4.17), and Proposition 3.5 show that

\[ J_t \lesssim k^{2(2s+1)} \sum_{N_1, \ldots, N_{k+2}} \omega_{N_1}^2 N_4^{2s+1} \prod_{j=1}^4 \| P_{N_j} u \|_{L^4_T, x} \prod_{j=5}^{k+2} \| P_{N_j} u \|_{L^\infty_T, x} \]

\[ \lesssim_k \sum_{N_1, N_2, N_3, N_4, N_5 \geq N} \omega_{N_1} \omega_{N_2} \left( \frac{N_4}{N_1} \right)^{s-\beta(\alpha)} \left( \frac{N_4}{N_2} \right)^{s-\beta(\alpha)} \left( \frac{N_4}{N_3} \right)^{\beta(\alpha)+1/2} \]

\[ \times \prod_{j=1}^2 \| D_x^{s-\beta(\alpha)} P_{N_j} u \|_{L^4_T, x} \prod_{l=3}^4 \| D_x^{\beta(\alpha)+1/2} P_{N_l} u \|_{L^4_T, x} \]

\[ \lesssim_k \left( \sum_K \omega_K^4 \| D_x^{s-\beta(\alpha)} P_K u \|^4_{L^4_T, x} \right)^{1/2} \left( \sum_K \| D_x^{\beta(\alpha)+1/2} P_K u \|^4_{L^4_T, x} \right)^{1/2} \]

\[ \lesssim_k TG(C_0) \| u \|^6_{L^\infty_T H^s_x}, \]

where we used that \( s \geq 2 \) so that \( k^{2(2s+1)} \leq k^{10} \). Also Remark 4.7 \( \kappa \leq 2 \) and \( N_1 \lesssim k N_2 \) such that we have \( \omega_{N_1} / \omega_{N_2} \lesssim k \). Moreover, it is not difficult to see that the last inequality holds when

\[ s \geq s(\alpha) = \frac{1}{2} + 2\beta(\alpha) \]

for \( \alpha = 1 \) or 2, where we recall \( \beta(2) = \frac{1}{12} \) and \( \beta(1) = \frac{1}{8} \).

**Subcase 2.2:** \( N_3 \gg k N_4 \) or \( k = 1 \).

By impossible frequency interactions, the largest two frequencies must be comparable. We must have \( N \sim N_1 \sim N_2 \). We take the extensions \( \tilde{u} = \rho_T(u) \) of \( u \) defined in (2.9). For simplicity, we define the following functional:

\[ J^{(2)}_{\infty}(u_1, \ldots, u_{k+2}) := \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left| \int_0^t \int_T \Pi(u_1, u_2) \prod_{j=3}^{k+2} u_j dx dt \right|. \quad (4.18) \]

By setting \( R = N_1^{1/3} N_3^{4/3} \), and then we split \( J_t \) into

\[ J_t \leq J^{(2)}_{\infty}(P_{N_1}^{\text{high}} \tilde{u}, P_{N_2} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u}) \]

\[ + J^{(2)}_{\infty}(P_{N_1}^{\text{low}} \tilde{u}, P_{N_2} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u}) \]

\[ + J^{(2)}_{\infty}(P_{N_1} \tilde{u}, P_{N_2}^{\text{low}} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u}) \]

\[ =: J^{(2)}_{\infty, 1} + J^{(2)}_{\infty, 2} + J^{(2)}_{\infty, 3}. \]
In particular, we see the following expression

\[
J_i \leq \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left\| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} 1_{t, R}^{\text{high}} \ddot{u}, P_{N_2} 1_{t, R}^{\text{low}} \ddot{u}) \prod_{j=3}^{k+2} P_{N_j} \ddot{u} \, dx \, dt \right\|
\]

\[+ \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left\| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} 1_{t, R}^{\text{low}} \ddot{u}, P_{N_2} 1_{t, R}^{\text{high}} \ddot{u}) \prod_{j=3}^{k+2} P_{N_j} \ddot{u} \, dx \, dt \right\|
\]

\[+ \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left\| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} 1_{t, R}^{\text{low}} \ddot{u}, P_{N_2} 1_{t, R}^{\text{low}} \ddot{u}) \prod_{j=3}^{k+2} P_{N_j} \ddot{u} \, dx \, dt \right\| =: \sum_{n=1}^{3} J_{\infty, n}^{(2)}.
\]

For \( J_{\infty, 1}^{(2)} \), recall that \( N \sim N_1 \sim N_2 \). We see from (4.2) that

\[
\| 1_{t, R}^{\text{high}} \|_{L^1} \lesssim T^{1/4} N_1^{-1/4} N_3^{-1},
\]

which gives

\[
J_{\infty, 1}^{(2)} = \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left\| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} 1_{t, R}^{\text{high}} \ddot{u}, P_{N_2} 1_{t, R}^{\text{low}} \ddot{u}) \prod_{j=3}^{k+2} P_{N_j} \ddot{u} \, dx \, dt \right\|
\]

\[\lesssim \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \| 1_{t, R}^{\text{high}} \|_{L^1} \| P_{N_1} \ddot{u} \|_{L^\infty L^2} \| P_{N_2} \ddot{u} \|_{L^\infty L^2} \prod_{j=3}^{k+2} \| P_{N_j} \ddot{u} \|_{L^\infty L^2} \| \ddot{u} \|_{L^\infty H^3_x}^2
\]

(4.19)

\[
\lesssim k T^{1/4} \| \ddot{u} \|_{L^\infty H^{s_0}_x} \| \ddot{u} \|_{L^\infty H^s_x} \sum_{N_1} N_1^{-1/4} \lesssim k T^{1/4} C_k^\alpha \| u \|_{L^\infty T^x H^\alpha_x},
\]

for some \( s_0 > \frac{1}{2} \). In the last inequality, we used (2.11).

By using (4.3), \( J_{\infty, 2}^{(2)} \) can be estimated by the same bound as (4.19).

For \( J_{\infty, 3}^{(2)} \), we see from Lemma 4.4 that \( |Q_{k+1}^{1} | \gtrsim N_3 N_1^\alpha \gg R \). Then, by defining \( L := N_3 N_1^\alpha \), we further decompose \( J_{\infty, 3}^{(2)} \) into the following

\[
J_{\infty, 3}^{(2)} = \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left\| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} 1_{t, R}^{\text{low}} \ddot{u}, P_{N_2} 1_{t, R}^{\text{low}} \ddot{u}) \prod_{j=3}^{k+2} P_{N_j} \ddot{u} \, dx \, dt \right\|
\]

\[\leq J_{\infty}^{(2)} (P_{N_1} Q_{\geq L}(1_{t, R}^{\text{low}} \ddot{u}), P_{N_2} 1_{t, R}^{\text{low}} \ddot{u}, \cdots, P_{N_{k+2}} \ddot{u})
\]

\[+ J_{\infty}^{(2)} (P_{N_1} Q_{\ll L}(1_{t, R}^{\text{low}} \ddot{u}), P_{N_2} Q_{\geq L}(1_{t, R}^{\text{low}} \ddot{u}), P_{N_3} \ddot{u}, \cdots, P_{N_{k+2}} \ddot{u})
\]

\[+ J_{\infty}^{(2)} (P_{N_1} Q_{\ll L}(1_{t, R}^{\text{low}} \ddot{u}), P_{N_2} Q_{\ll L}(1_{t, R}^{\text{low}} \ddot{u}), P_{N_3} Q_{\ll L} \ddot{u}, \cdots, P_{N_{k+2}} \ddot{u})
\]

\[+ \cdots + J_{\infty}^{(2)} (P_{N_1} Q_{\ll L}(1_{t, R}^{\text{low}} \ddot{u}), P_{N_2} Q_{\ll L}(1_{t, R}^{\text{low}} \ddot{u}), P_{N_3} Q_{\ll L} \ddot{u}, \cdots, P_{N_{k+2}} Q_{\geq L} \ddot{u})
\]

\[=: J_{\infty, 3, 1}^{(2)} + \cdots + J_{\infty, 3, k+2}^{(2)}.
\]
In particular, we have

\[
J_{\infty,3}^{(2)} \leq \sum_{N_1 \gg 1} \sum_{N_1, \ldots, N_3} \omega_N^2 N_3^{2s} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} Q \geq N_3 N_1 \left(1_{t,R} \tilde{u}\right), P_{N_2} 1_{t,R} \tilde{u}) \prod_{j=3}^{k+2} P_{N_j} \tilde{u} dx dt' \right|
\]

\[
+ \sum_{N_1 \gg 1} \sum_{N_1, \ldots, N_3} \omega_N^2 N_3^{2s} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} Q \leq N_3 N_1 \left(1_{t,R} \tilde{u}\right), P_{N_2} Q \geq N_3 N_1 \left(1_{t,R} \tilde{u}\right)) \prod_{j=3}^{k+2} P_{N_j} \tilde{u} dx dt' \right|
\]

\[
+ \sum_{n=3}^{N} \sum_{N_1, \ldots, N_3} \omega_N^2 N_3^{2s} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_1} Q \leq N_3 N_1 \left(1_{t,R} \tilde{u}\right), P_{N_2} Q \leq N_3 N_1 \left(1_{t,R} \tilde{u}\right)) \prod_{j=3}^{k+2} P_{N_j} \tilde{u} dx dt' \right|
\]

\[
\times \left( \prod_{j=3}^{k} P_{N_j} Q \leq N_3 N_1 \tilde{u} \right) \prod_{j=n+1}^{k+2} P_{N_j} \tilde{u} dx dt' = \sum_{j=1}^{k+2} J_{\infty,3,j}^{(2)}
\]

It is worth noting that \( R \ll N_3 N_1^\alpha \) since \( N_1 \gg 1 \). We also see from (4.2) and (4.3) that \( \|1_{t,R} \tilde{u}\|_{L_t^2} \leq R^{-1/2} \), and we have

\[
\|P_{N_2} 1_{t,R} \tilde{u}\|_{L_t^2} \leq \|P_{N_2} 1_{t,R} \tilde{u}\|_{L_t^2} + \|P_{N_2} \|_{L_t^2} \leq \|P_{N_2} 1_{t,R} \tilde{u}\|_{L_t^2} + T^{1/4} R^{-1/4} \|P_{N_2} \tilde{u}\|_{L_t^2 L_x^6},
\]

Thus, for \( J_{\infty,3,1}^{(2)} \), we use Lemma 2.9, Lemma 2.6, Hölder’s inequality, (4.4) and (4.20) imply that

\[
J_{\infty,3,1}^{(2)} \lesssim \sum_{N_1, \ldots, N_3} \omega_N^2 N_3^{2s} \|P_{N_1} Q \geq L \left(1_{t,R} \tilde{u}\right)\|_{L_t^2 L_x^6} \|P_{N_2} Q \geq L \left(1_{t,R} \tilde{u}\right)\|_{L_t^2 L_x^6} \prod_{j=3}^{k+2} \|P_{N_j} \tilde{u}\|_{L_t^\infty L_x^6}
\]

\[
\lesssim k \|\tilde{u}\|_{L_t^\infty H_x^{s_0}} \sum_{N_1 \geq 1} \omega_N^2 N_1^{2s-\alpha} \|P_{N_1} \tilde{u}\|_{X_0^1} \|P_{N_1} 1_{t,R} \tilde{u}\|_{L_t^2}
\]

\[
+ T^{1/4} \|\tilde{u}\|_{L_t^\infty H_x^{s_0}} \sum_{N_1 \geq N_3} \omega_N^2 N_1^{2s-13/12} N_3^{-1/3} \|P_{N_1} \tilde{u}\|_{X_0^1} \|P_{N_1} \tilde{u}\|_{L_t^2 L_x^6} \|P_{N_4} \tilde{u}\|_{L_t^\infty L_x^6}
\]

\[
\lesssim T^{1/4} \|\tilde{u}\|_{L_t^\infty H_x^{s_0}} \|\tilde{u}\|_{L_t^\infty H_x^{s_0}} \|\tilde{u}\|_{X_0^{s_0-1}} \lesssim k T^{1/4} \|u\|_{L_t^\infty H_x^{s_0}} \|u\|_{Z_0^{s_0-1}}.
\]

In the above computation of \( J_{\infty,3,1}^{(2)} \), from line-3 to line-4 we used \( N_1^{\alpha} \leq N_1^{-1} \) since \( \alpha = 1 \) or 2.

We can evaluate the contribution \( J_{\infty,3,2}^{(2)} \) by the same way with (4.20).

Next, we consider the contribution \( J_{\infty,3,3}^{(2)} \). By using Lemma 2.6, Lemma 2.9, Hölder’s inequality and (4.1) yield

\[
J_{\infty,3,3}^{(2)} \lesssim \sum_{N_1, \ldots, \ldots, N_3} \omega_N^2 N_1^{2s} N_3 \|P_{N_1} Q \leq L \left(1_{t,R} \tilde{u}\right)\|_{L_t^2 L_x^6} \|P_{N_2} Q \leq L \left(1_{t,R} \tilde{u}\right)\|_{L_t^\infty L_x^6}
\]

\[
\times \|P_{N_3} Q \leq L \tilde{u}\|_{L_t^\infty L_x^6} \prod_{j=4}^{k+2} \|P_{N_j} \tilde{u}\|_{L_t^\infty L_x^6}
\]

\[
\lesssim k T^{1/2} \|\tilde{u}\|_{L_t^\infty H_x^{s_0}} \sum_{N_1 \geq N_3 \geq 1} \omega_N^2 N_1^{2s-\alpha} \|P_{N_1} \tilde{u}\|_{L_t^\infty L_x^6} \|D_x^{1/2} P_{N_3} \tilde{u}\|_{X_0^1}
\]

\[
\lesssim T^{1/2} \|\tilde{u}\|_{L_t^\infty H_x^{s_0}} \sum_{N_1 \geq N_3 \geq 1} N_1^{-\beta(\alpha)} N_3^{-\beta(\alpha)} \omega_N^2 N_1^{2s} \|P_{N_1} \tilde{u}\|_{L_t^\infty L_x^6} \|P_{N_3} \tilde{u}\|_{X_0^{s_0-1,1}}.
\]
we can share the lost derivative between three functions, where the last inequality we used
\(\alpha \in \mathbb{N} \setminus \mathbb{P} \),

Subcase 2.3: \(N_1 \sim N_2 \gg N_3\).

In this case, we need to compare the size \(|n_3 + n_4|\) and \(k|n_5|\). By symmetry we can assume \(|n_5| \geq |n_j|\), where \(n_j\) is the \(j\)-th largest frequency. Therefore, we consider the following two cases:

\(|n_3 + n_4| \gg k|n_5|\) \quad \text{and} \quad |n_3 + n_4| \lesssim k|n_5|.

If \(|n_3 + n_4| \gg k|n_5|\), we have a suitable non-resonance relation (see Lemma 4.5). Otherwise we can share the lost derivative between three functions, \(P_{N_j}u\), for \(j = 3, 4, 5\), see in (4.21).

Hence, let us split \(J_t\) into the following two terms:

\[
J_t \leq \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N_1^{2s} \int_0^t \int_{\mathbb{T}} \Pi(P_{N_1}u, P_{N_2}u) P_{\leq kN_5} \left( \prod_{j=3}^{k+2} P_{N_j}u \right) dxdt' + \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \sum_{M \leq N_3} \omega_N^2 N_1^{2s} \int_0^t \int_{\mathbb{T}} \Pi(P_{N_1}u, P_{N_2}u) P_{M} \left( \prod_{j=3}^{k+2} P_{N_j}u \right) dxdt'
=: I_t^{(3)} + J_t^{(3)}.
\]

One observation from Lemma 4.5\(^7\) that if \(k = 2\). Then, the term \(I_t^{(3)}\) does not appear. The above estimate holds that \(J_t \leq J_t^{(3)}\) with the summation \(\sum_{M \leq N_3}\) instead of \(\sum_{kN_5 \leq M \leq N_3}\).

Note also that the contribution in respectively \(I_t^{(3)}\) and \(J_t^{(3)}\) of respectively \(N_5 = 0\) and \(N_5 = M = 0\) does vanish by integration by parts. Therefore we can always assume that \(N_5 \geq 1\) in \(I_t^{(3)}\) and that \(M \geq 1\) in \(J_t^{(3)}\). For \(I_t^{(3)}\), since either \(N_1 \sim N\) or \(N_2 \sim N\), we observe that by the Young inequality and the assumption that \(\kappa \leq 2\)

\[
\sum_{N_1 \geq N_2, kN_2 \geq N_1} \left( \omega_{N_1}^2 N_1^{2s} + \omega_{N_2}^2 N_2^{2s} \right) \|P_{N_1}u\|_{L_x^2} \|P_{N_2}u\|_{L_x^2} \lesssim k^s \sum_{kN_2 \geq N_1} \omega_{N_1}^s \omega_{N_2} \left( \frac{N_1}{kN_2} \right)^s \|D_x^s P_{N_1}u\|_{L_x^2} \|D_x^s P_{N_2}u\|_{L_x^2}
+ \sum_{N_2 \leq N_1} \omega_{N_2}^s \omega_{N_1} \omega_{N_2} \left( \frac{N_2}{N_1} \right)^s \|D_x^s P_{N_1}u\|_{L_x^2} \|D_x^s P_{N_2}u\|_{L_x^2} \lesssim k^3 \|u\|_{H_x^s}^2.
\]

\(^7\)In this case if \(k = 2\), there is a suitable non-resonance.
By using above observation together with Lemma 2.9, Hölder’s, Young’s inequality (4.17), and Proposition 3.9 show

\[
J^{(3)}_t \lesssim \sum_{N \geq 2} \sum_{k \geq 2} (\omega_{N_1}^2 + \omega_{N_2}^2) k N \\
\times \int_0^t \|P_{N_j} u\|_{L^2} \|P_{N_j} u\|_{L^2} \prod_{j=3}^{k+2} \|P_{N_j} u\|_{L^\infty} dt' \tag{4.21}
\]

\[
\lesssim k \|u\|_{L_t^\infty H_x^\alpha}^2 \sum_{N \geq 2} \prod_{j=3}^{k+2} \left( \frac{N}{N_j} \right)^{1/3} \|D_x^{1/3} P_{N_j} u\|_{L_t^4 L_x^\infty}^2 \\
\lesssim k \|u\|_{L_t^\infty H_x^\alpha}^2 \sum_{N \geq 2} \|D_x^{1/3} P_{N_j} u\|_{L_t^4 L_x^\infty}^2 \lesssim k T^{5/8} G(C_0) \|u\|_{L_t^\infty H_x^\alpha}^2,
\]

where we used that \( \alpha \in \{1, 2\} \), and simple algebra shows that \( s(\alpha) = 1/2 + 2\beta(\alpha) \geq 7/12 + \beta(\alpha) \).

For \( J^{(3)}_t \), we take the extensions \( \tilde{u} = \rho_T(u) \) of \( u \) defined in (2.9). Note that we have \( N_1 \sim N_2 \sim N \). We further decompose \( J^{(3)}_t \) as in Case 2. In the same spirit as (4.18), we define the functional for the sake of notation:

\[
J^{(3)}(u_1, \ldots, u_{k+2}) := \sum_{N \geq 1} \sum_{k \geq 2} \sum_{k_{N_j} \leq M \leq N_3} \omega^2 N^{2s} \int \int \Pi(u_1, u_2) P_M \left( \prod_{j=3}^{k+2} u_j \right) dx dt.
\]

We let \( R = N_1^{1/3} M^{4/3} \) and yeild\[
J^{(3)}_t \leq J^{(3)}(P_{N_1} 1_{t_R}^\text{high} \tilde{u}, P_{N_2} 1_{t_R}^\text{low} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u}) \\
+ J^{(3)}(P_{N_1} 1_{t_R}^\text{low} \tilde{u}, P_{N_2} 1_{t_R}^\text{high} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u}) \\
+ J^{(3)}(P_{N_1} 1_{t_R}^\text{low} \tilde{u}, P_{N_2} 1_{t_R}^\text{low} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u}) \\
=: J^{(3)}_{t,1} + J^{(3)}_{t,2} + J^{(3)}_{t,3}.
\]

More precisely, we would have

\[
J^{(3)}_t \leq \sum_{N \geq 1} \sum_{N_1 \leq M \leq N_3} \omega^2 N^{2s} \int \int \Pi(1_{t_R}^\text{high} \tilde{u}, 1_{t_R}^\text{low} \tilde{u}) P_M \left( \prod_{j=3}^{k+2} P_{N_j} \tilde{u} \right) dx dt' \\
+ \sum_{N \geq 1} \sum_{N_1 \leq M \leq N_3} \omega^2 N^{2s} \int \int \Pi(1_{t_R}^\text{low} \tilde{u}, 1_{t_R}^\text{high} \tilde{u}) P_M \left( \prod_{j=3}^{k+2} P_{N_j} \tilde{u} \right) dx dt' \\
+ \sum_{N \geq 1} \sum_{N_1 \leq M \leq N_3} \omega^2 N^{2s} \int \int \Pi(1_{t_R}^\text{low} \tilde{u}, 1_{t_R}^\text{low} \tilde{u}) P_M \left( \prod_{j=3}^{k+2} P_{N_j} \tilde{u} \right) dx dt' \\
=: J^{(3)}_{t,1} + J^{(3)}_{t,2} + J^{(3)}_{t,3}.
\]

Obviously, let \( M, N_3 \) to be dyadic numbers. Then, for any \( \varepsilon > 0 \), we have

\[
\sum_{M \leq N_3} 1 \lesssim N_3^\varepsilon,
\]

\(^8\text{For instance, } \varepsilon = \beta(\alpha)/2 \land (s_0/2 - 1/4).\)
where the implicit constant does not depend on $\varepsilon$. By (4.22), (2.11) and the same argument as that of $J_{\infty,1}^{(2)}$, we obtain

$$J_{\infty,2}^{(3)} + J_{\infty,3}^{(3)} \lesssim_k T^{1/4} \|\bar{u}\|_{L_t^{\infty}H_x^0}^k\|\bar{u}\|_{L_t^{\infty}H_x^2}^2 \lesssim_k T^{1/4} \|u\|_{L_t^{\infty}H_x^2}^2.$$

For $J_{\infty,3}^{(3)}$, Lemma 4.5 implies that

$$|\Omega_{n+1}| \gtrsim MN_1^\alpha \sim |n_3 + n_4|N_1^\alpha.$$

So, defining $L := MN_1^\alpha$, we have

$$J_{\infty,3}^{(3)} \leq J_{\infty,3}^{(3)}(P_{N_1}Q_{\geq L}(1_{t,R}^\text{low} \bar{u}), P_{N_2}1_{t,R}^\text{low} \bar{u}, \cdots, P_{N_{k+2}}^\text{low} \bar{u})$$

$$+ J_{\infty,3}^{(3)}(P_{N_1}Q_{< L}(1_{t,R}^\text{low} \bar{u}), P_{N_2}Q_{\geq L}(1_{t,R}^\text{low} \bar{u}), P_{N_3} \bar{u}, \cdots, P_{N_{k+2}}^\text{low} \bar{u})$$

$$+ J_{\infty,3}^{(3)}(P_{N_1}Q_{< L}(1_{t,R}^\text{low} \bar{u}), P_{N_2}Q_{< L}(1_{t,R}^\text{low} \bar{u}), P_{N_3}Q_{L} \bar{u}, \cdots, P_{N_{k+2}}^\text{low} \bar{u})$$

$$+ \cdots + J_{\infty,3}^{(3)}(P_{N_1}Q_{< L}(1_{t,R}^\text{low} \bar{u}), P_{N_2}Q_{< L}(1_{t,R}^\text{low} \bar{u}), P_{N_3}Q_{L} \bar{u}, \cdots, P_{N_{k+2}}Q_{L} \bar{u})$$

$$=: J_{\infty,3,1}^{(3)} + \cdots + J_{\infty,3,k+2}^{(3)}.$$

In particular,

$$J_{\infty,3}^{(3)} \leq \sum \omega_N^2 N^{2s} \left| \int T \int \Pi(P_{N_1}Q_{\geq MN_1^\alpha}(1_{t,R}^\text{low} \bar{u}), P_{N_2}1_{t,R}^\text{low} \bar{u})P_M\left( \prod_{j=3}^{k+2} P_{N_j} \bar{u} \right) dx dt' \right|$$

$$+ \sum \omega_N^2 N^{2s} \left| \int T \int \Pi(P_{N_1}Q_{< MN_1^\alpha}(1_{t,R}^\text{low} \bar{u}), P_{N_2}Q_{\geq MN_1^\alpha}(1_{t,R}^\text{low} \bar{u}))P_M\left( \prod_{j=3}^{k+2} P_{N_j} \bar{u} \right) dx dt' \right|$$

$$+ \sum_{l=3}^{k+2} \sum \omega_N^2 N^{2s} \left| \int T \int \Pi(P_{N_1}Q_{< MN_1^\alpha}(1_{t,R}^\text{low} \bar{u}), P_{N_2}Q_{< MN_1^\alpha}(1_{t,R}^\text{low} \bar{u}))$$

$$\times P_M\left( \prod_{j=3}^{l-1} P_{N_j} Q_{\geq MN_1^\alpha} \bar{u} \right) P_{N_l} Q_{\geq MN_1^\alpha} \bar{u} \prod_{j=l+1}^{k+2} P_{N_j} \bar{u} \right) dx dt' \right| =: \sum J_{\infty,3,j}^{(3)},$$

where $\sum$ denotes $\sum_{N_1 \cdots N_{k+2} \sim N} \sum_{N_i \in [N_1, N_2, \ldots, N_{k+2}]} \sum_{k_5} \lesssim N_3$ for simplicity.

It is worth noting that $R \ll L = MN_1^\alpha$ since $N_1 \gg 1$. For $J_{\infty,3,1}^{(3)}$, we use the argument of $J_{\infty,3,1}^{(2)}$. Lemmas 2.9 and 2.6, the H"older inequality, (4.4), (4.20) and (4.22) show

$$J_{\infty,3,1}^{(3)} = \sum \omega_N^2 N^{2s} \left| \int T \int \Pi(P_{N_1}Q_{\geq MN_1^\alpha}(1_{t,R}^\text{low} \bar{u}), P_{N_2}(1_{t,R}^\text{low} \bar{u}))P_M\left( \prod_{j=3}^{k+2} P_{N_j} \bar{u} \right) dx dt' \right|$$

$$\lesssim \sum \sum \omega_N^2 N^{2s} M \|P_{N_1}Q_{L}(1_{t,R}^\text{low} \bar{u})\|_{L_t^{\infty}L_x^2} \|P_{N_2}1_{t,R}^\text{low} \bar{u}\|_{L_t^{\infty}L_x^2} \sum \|P_{N_j} \bar{u}\|_{L_t^{\infty}L_x^2}$$

$$\lesssim_k \sum \sum \omega_N^2 N^{2s-\alpha} \|P_{N_1} \bar{u}\|_{X^{1,0}} \|P_{N_2}1_{t,R}^\text{low} \bar{u}\|_{L_t^{\infty}L_x^2} \|P_{N_j} \bar{u}\|_{L_t^{\infty}L_x^2}$$

$$+ T^{1/4} \sum \sum \omega_N^2 N^{2s-\alpha-1/12} M^{-1/3} \|P_{N_1} \bar{u}\|_{X^{0,1}} \|P_{N_j} \bar{u}\|_{L_t^{\infty}L_x^2}.$$
For any 

with the initial data

\( \delta > 0 \), we can estimate the contribution \( J^{(3)}_{\infty,3,2} \) by the same way. For the contribution \( J^{(3)}_{\infty,3,3} \), similarly to \( J^{(2)}_{\infty,3,3} \), we see from (4.1), (4.22) and Lemma 2.6 that

\[
J^{(3)}_{\infty,3,3} = \sum \omega^2_N N^2 \int_\mathbb{R} \int_\mathbb{T} \Pi(P_{N_1} Q_{\leq MN^4} (1_{I,R} \tilde{u}), P_{N_2} Q_{\leq MN^4} (1_{I,R} \tilde{u}))
\times PM\left(P_{N_3} Q_{\geq MN^4} \tilde{u} \prod_{j=1}^{k+2} P_{N_j} \tilde{u}\right) dxdt'
\leq k T^{1/2} \| \tilde{u} \|_{L^\infty_{s} H^{s_0}}^k \sum_{N_1 \geq N_3 \geq 1} \omega^2_N N^2 \| \tilde{u} \|_{X^{s_0,- \alpha}}^{s_0} \| \tilde{u} \|_{L^\infty_{s} H^{s_0}}^2 \| D_x^{1/2} P_{N_3} \tilde{u} \|_{X^{0,1}}.
\]

This completes the proof.\( \Box \)

In a similar manner, we can estimate the contributions of \( J^{(3)}_{\infty,3,j} \) for \( j = 4, \ldots, k + 2 \) such that

\[
J^{(3)}_{\infty,3,j} \leq k T^{1/2} \| u \|_{L^\infty_T H^{s_0}_x}^2.
\]

Case 3: \( \phi_N^2(n_3) n_3 \) contribution.

Finally, we consider the contribution of \( \phi_N^2(n_3) n_3 \). We may assume that \( N_3 \gg k N_4 \). Otherwise, the proof is the same as the contribution of \( A_2 \). When \( N_3 \gg k N_4 \), we can obtain the desired estimate as in Case 2. This completes the proof.\( \Box \)

4.3. A priori bounds on the difference of solutions. In the following, let us establish the energy estimate at the regularity level \( s - 1 \) on the difference between the two solutions. This is because the symmetrisation argument as we used in the proof of Proposition 4.9 is weaker when we deal with the difference between two solutions. Moreover, we do not use the frequency envelope, we always argue with the usual Sobolev space \( H^s(\mathbb{T}) \). This sort of argument can be also seen in [64, 63].

We recall that \( \alpha = 1 \) or \( 2 \) and \( 2 \geq s \geq s(\alpha) := 1/2 + 2\beta(\alpha) \). In particular, in the shallow-water region, we restrict the regularity \( s \in [2/3, 2] \); in the deep-water region, we restrict the regularity \( s \in [3/4, 2] \).

Proposition 4.12. Consider \( \alpha = 1 \) or \( 2 \) and

\[
2 \geq s \geq s(\alpha) := \frac{1}{2} + 2\beta(\alpha) = \frac{1}{2} + \frac{1}{2(\alpha + 1)}.
\]

Let \( \delta_0 > 0 \). Then, the following two statements hold:

(i) For any \( \delta \in (\delta_0, \infty) \), let \( s \geq \frac{3}{4} \). Let \( (u^{(1)}, u^{(2)}) \in (Z^s_T)^2 \) be two solutions of (1.17) associated with the initial data \( (u^{(1)}_0, u^{(2)}_0) \in (H^s(\mathbb{T}))^2 \). Then, we have

\[
\| u^{(1)} - u^{(2)} \|_{L^\infty_T H^{s-1}_x}^2 \leq \| u^{(1)}_0 - u^{(2)}_0 \|_{H^{s-1}}^2 + T^{1/2} G(\| u^{(1)} \|_{Z^s_T} + \| u^{(2)} \|_{Z^s_T}) \times \| u^{(1)} - u^{(2)} \|_{Z^{s-1}_T} \| u^{(1)} - u^{(2)} \|_{L^\infty_T H^{s-1}_x}.
\]

(4.23)
(ii) For any \( \delta \in (0, \delta_0) \), let \( s \geq \frac{2}{3} \). Let \((v^{(1)}, v^{(2)}) \in (Z^2_T)^2\) be two solutions of (1.19) associated with the initial data \((v_0^{(1)}, v_0^{(2)}) \in (H^s(\mathbb{T}))^2\). Then, we have
\[
\|v^{(1)} - v^{(2)}\|_{L^\infty_T H^{s-1}_x} \leq \|v_0^{(1)} - v_0^{(2)}\|_{H^{s-1}_x} + T^{\frac{1}{2}} G(\|v^{(1)}\|_{Z^2_T} + \|v^{(2)}\|_{Z^2_T}) \\
\times \|v^{(1)} - v^{(2)}\|_{Z^2_T} \|v^{(1)} - v^{(2)}\|_{L^\infty_T H^{s-1}_x},
\]
where \( G = G[f] \) is an increasing and non-negative entire function on \( \mathbb{R}_+ \). Moreover, the estimate is independent of \( \delta \).

**Remark 4.13.** This proof also applies to the real-line situation, since the idea would be almost the same as in the proof of Proposition 4.9. One should notice here that the equation for the difference of two solutions enjoys fewer symmetries, this difference will be estimated in a space with lower regularity than that of the solution itself. Then, later in proving the local well-posedness part, we will use the frequency envelope approach to recover the continuity result with respect to initial data.

**Proof.** The proof of (4.23) and (4.24) share the same argument. For simplicity, let us denote the two solutions are \((u, v) \in (Z^2_T)^2\) associated with the initial data \((u_0, v_0) \in (H^s(\mathbb{T}))^2\). The difference \( w = u - v \) satisfies
\[
\partial_t w - \mathcal{G}_\delta \partial_x^2 w = -\partial_x (f(u) - f(v)),
\]
for \( i = 1 \) or 2 and recall that the dispersion terms are \( \mathcal{G}_\delta_1 := \frac{3}{2\pi^2} \mathcal{G}_\delta, \mathcal{G}_\delta_2 := \mathcal{G}_\delta \). By rewriting \( f(u) - f(v) \) as the following
\[
f(u) - f(v) = \sum_{k \geq 1} f^{(k)}(0) \frac{(u^k - v^k)}{k!} = \sum_{k \geq 1} f^{(k)}(0) \frac{k!}{(k-1)!} \sum_{i=0}^{k-1} u^i v^{k-1-i}.
\]
We proceed as in the proof of Proposition 4.9, we see from (4.25) that for \( t \in [0, T] \) and we obtain
\[
\|w(t)\|_{H^{s-1}_x}^2 \leq \|u_0 - v_0\|_{H^{s-1}_x}^2 + 2 \sum_{k \geq 1} |f^{(k)}(0)| \max_{i \in \{0, \ldots, k-1\}} I_{k,i}^t,
\]
where \( I_{k,i}^t \) is defined by
\[
I_{k,i}^t := \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} u^i v^{k-1-i} w P_N^2 \partial_x w \, dx \, dt' \right|.
\]
It is clear that \( I_{1,i}^t = 0 \) from IBP. Therefore we are reduced to estimating the contribution of
\[
I_{k+1}^t = \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} z^k w P_N^2 \partial_x w \, dx \, dt' \right|
\]
where \( z^k \) stands for \( u^i v^{k-i} \) for some \( i \in \{0, \ldots, k\} \). We set \( C_0 := \|u\|_{Z^2_T} + \|v\|_{Z^2_T} \), and we claim that for any \( k \geq 1 \) the following bound holds
\[
I_{k+1}^t \leq C_k^k T^{1/4} G(C_0) C_0^k \|w\|_{Z^2_T} \|w\|_{L^\infty_T H^{s-1}_x}.
\]
To this end, we conclude this leads to (4.23) and (4.24) by following the same sort of discussion after (4.13).
Now, let us proceed with the proof of our claim \(4.29\). In the sequel we fix \(k \geq 1\) and we estimate \(I_k^j\). We also use the notation \(a \lesssim b\) defined in \(4.14\).

- **Low frequency contribution.**

The contribution of the sum over \(1 \leq N \lesssim 1\) in \(4.28\) is easily estimated. For the situation \(1 - s < 0\), we notice that \(L^2(\mathbb{T}) \subset H^{0-}(\mathbb{T})\). Then, for \(1 - s > 0\) we apply \(2.12\), and obtain the following

\[
\sum_{N \leq 1} (1 \lor N)^{2(s-1)} \left| \int_{0}^{t} \int_{\mathbb{T}} z^k w P_N^2 \partial_x w dx dt \right|
\]

\[
\lesssim T \sum_{N \leq 1} \|w\|_{L^\infty_T H^{1-s}_x} \|z^k P_N^2 \partial_x w\|_{L^\infty_T L^1_{x,s}}
\]

\[
\lesssim T \sum_{N \leq 1} \|w\|_{L^\infty_T H^{1-s}_x} \left\{ \|z^k\|_{L_T^\infty L^1_{x,s}} \|P_N^2 \partial_x w\|_{L^\infty_T L^1_{x,s}} + \|z^k\|_{L^\infty_T L^1_{x,s}} \|P_N^2 \partial_x w\|_{L^\infty_T L^1_{x,s}} \right\}
\]

\[
\lesssim T \|w\|_{L^\infty_T H^{1-s}_x} \sum_{N \leq 1} \left\{ \|z^k\|_{L^\infty_T H^{1-s}_x} N^{3/2} \|P_N^2 w\|_{L^\infty_T L^1_{x,s}} + \|z^k\|_{L^\infty_T H^{1-s}_x} N^{1-s} \|P_N^2 w\|_{L^\infty_T L^1_{x,s}} \right\}
\]

\[
\lesssim k T \|w\|^2_{L^\infty_T H^{1-s}_x}
\]

Our assumption is that \(s > \frac{1}{2}\). Then, we have the Sobolev inequality such that \(H^{1/2}(\mathbb{T}) \subset L^\infty(\mathbb{T})\). In the last inequality, we used \(0 < 1 - s < \frac{1}{2}\) and iteratively applied \(2.12\) to obtain

\[
\|z^k\|_{H^{1-s}} \lesssim \|z\|_{H^s}^k.
\]

- **High frequency contribution.**

Therefore, in what follows, we can assume that \(N \gg 1\). A similar argument to \(4.15\) we see from the symmetry that

\[
\left| \int_{\mathbb{T}} (w P_N^2 \partial_x w) z^k \partial_x \right| \lesssim \sum_{n_1, \ldots, n_{k+2} = 0} A(n_1, n_2) w_1 w_2 \prod_{j=3}^{k+2} \bar{z}_j(n_j)
\]

\[
\lesssim \sum_{N_j, n_j} A(n_1, n_2) \prod_{\ell=1}^{2} \phi_{N_j}(n_\ell) \hat{w}_\ell(n_\ell) \prod_{j=3}^{k+2} \phi_{N_j}(n_j) \bar{z}_j(n_j),
\]

where

\[
A(n_1, n_2) = \phi_N(n_1) n_1 + \phi_N(n_2) n_2, \quad \text{and} \quad z_j \in \{u, v\}
\]

for \(j \in \{3, \ldots, k+2\}\). Therefore, we see the following expression immediately

\[
\sum_{N \gg 1} N^{2(s-1)} \left| \int_{0}^{t} \int_{\mathbb{T}} z^k w P_N^2 \partial_x w dx dt \right|
\]

\[
\leq \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{0}^{t} \prod_{j=1}^{k+2} P_{N_j} w \right| \prod_{j=3}^{k+2} P_{N_j} z_j dx dt
\]

\[
=: J_t,
\]
where $\Pi(f, g)$ is defined by (2.16). By symmetry, we may assume that

\[
\begin{aligned}
N_1 &\geq N_2; \\
N_3 &\geq N_4, \text{ for } k = 2; \\
N_3 &\geq N_4 \geq N_5 = \max_{j \geq 5} N_j, \text{ for } k \geq 3.
\end{aligned}
\]

Note again that the cost of this choice is a constant factor less than $(k+2)^4$. It is also worth noticing that the frequency projectors in $\Pi(\cdot, \cdot)$ ensure that

\[N_1 \sim N \quad \text{or} \quad N_2 \sim N,
\]
in particular, $N_1 \gtrsim N$. Moreover, we notice that we can assume that $N_3 \geq 1$, since one application of IBP shows that the contribution of $N_3 = 0$ vanish. Finally, we note that we can also assume that $N_2 \geq 1$. Due to the case when $N_2 = 0$, we must have $kN_3 \gtrsim N_1$, and it easy to see the following computation

\[
J_t = \sum_{N \geq 1} \cdots \cdots N^{2(s-1)} \left| \int_0^t \cdots \cdots \left| \Pi(P_{N_1} f, P_{N_2} f) \right| \right|
\]

Here, we have used that $P_0 \partial_x w = 0$. In the last inequality, we use the following observation

\[|P_0 w(x)| = |F_x^{-1} \{ \phi_0(n) \hat{w}(n) \}(x)| = |\hat{w}(0)| \leq \| \langle n \rangle^{s-2} \hat{w}(n) \|_\ell^2
\]

for $n \in \mathbb{Z}$. In proving the claim (4.29). We shall divide $J_t$ into following three cases:

1. $N_1 \lesssim kN_4 (k \geq 2)$;
2. $N_1 \gg kN_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_3$);
3. $N_1 \gg kN_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).

**Case 1:** $N_1 \lesssim kN_4 (k \geq 2)$.

Following from the Hölder’s inequality, Young’s convolution inequalities (same manner as in (4.17), (4.16), Proposition 3.5 and Corollary 3.6 imply that

\[
J_t \geq \sum_{N \geq 1} \cdots \cdots N^{2(s-1)} \left| \int_0^t \cdots \cdots \left| \Pi(P_{N_1} f, P_{N_2} f) \right| \right|\]

for $n \in \mathbb{Z}$. In proving the claim (4.29). We shall divide $J_t$ into following three cases:

1. $N_1 \lesssim kN_4 (k \geq 2)$;
2. $N_1 \gg kN_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_3$);
3. $N_1 \gg kN_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).

**Case 1:** $N_1 \lesssim kN_4 (k \geq 2)$.

Following from the Hölder’s inequality, Young’s convolution inequalities (same manner as in (4.17), (4.16), Proposition 3.5 and Corollary 3.6 imply that

\[
J_t = \sum_{N \geq 1} \cdots \cdots N^{2(s-1)} \left| \int_0^t \cdots \cdots \left| \Pi(P_{N_1} f, P_{N_2} f) \right| \right|
\]

for $n \in \mathbb{Z}$. In proving the claim (4.29). We shall divide $J_t$ into following three cases:

1. $N_1 \lesssim kN_4 (k \geq 2)$;
2. $N_1 \gg kN_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_3$);
3. $N_1 \gg kN_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).

**Case 1:** $N_1 \lesssim kN_4 (k \geq 2)$.

Following from the Hölder’s inequality, Young’s convolution inequalities (same manner as in (4.17), (4.16), Proposition 3.5 and Corollary 3.6 imply that

\[
J_t = \sum_{N \geq 1} \cdots \cdots N^{2(s-1)} \left| \int_0^t \cdots \cdots \left| \Pi(P_{N_1} f, P_{N_2} f) \right| \right|
\]

for $n \in \mathbb{Z}$. In proving the claim (4.29). We shall divide $J_t$ into following three cases:

1. $N_1 \lesssim kN_4 (k \geq 2)$;
2. $N_1 \gg kN_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_3$);
3. $N_1 \gg kN_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).

**Case 1:** $N_1 \lesssim kN_4 (k \geq 2)$.

Following from the Hölder’s inequality, Young’s convolution inequalities (same manner as in (4.17), (4.16), Proposition 3.5 and Corollary 3.6 imply that
\[ \lesssim_k TG(C_0)\|w\|_{L^\infty_t H_x^{s(\alpha)-1}}^2, \]

where it is not difficult to see that \( s(\alpha) = \frac{1}{2} + 2\beta(\alpha) \), where \( \alpha \in \{1, 2\} \), and \( \beta(2) = \frac{1}{12} \), \( \beta(1) = \frac{1}{8} \). Moreover, we notice the following

\[ 2s - \beta(\alpha) - 1/2 > 0 \quad \text{and} \quad s - \beta(\alpha) > 1/2 - \beta(\alpha) > 0. \]

**Case 2:** \( N_1 \gg k N_4 \) and \( N_2 \gtrsim N_3 \) (or \( k = 1 \) and \( N_2 \gtrsim N_3 \)).

Firstly, let us recall that \( N_1 \sim N \) or \( N_2 \sim N \), which implies that \( N \sim N_1 \sim N_2 \gtrsim N_3 \).

Therefore, in this case, the contribution of \( J_t \) can be estimated in the same way as the \( A_1 \) contribution in Proposition 4.9. In particular, we shall replace

\[ N_2^s, P_{N_1} u, P_{N_2} u, P_{N_j} u, \quad \text{for} \quad j = 3, \ldots, k + 2 \]

by the following

\[ N_2^{s-2}, P_{N_1} w, P_{N_2} w, P_{N_j} z_j, \quad \text{for} \quad j = 3, \ldots, k + 2, \]

respectively. Then, as the result, we can obtain

\[ J_t = \sum_{N \gtrsim 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_0^T \int_{\mathbb{T}} \Pi(P_{N_1} w, P_{N_2} w) \prod_{j=3}^{k+2} P_{N_j} z_j dx dt' \right| \]

\[ \lesssim C T^{1/4} (\|w\|_{X^{s-2,1}} + \|w\|_{L^\infty_t H_x^{s-1}}) \|w\|_{L^\infty_t H_x^{s-1}} \]

for \( s \gtrsim s(\alpha) \).

**Case 3:** \( N_1 \gg k N_4 \) and \( N_2 \ll N_3 \) (or \( k = 1 \) and \( N_2 \ll N_3 \)).

Observe that in this case \( N_1 \sim N_3 \sim N \gg N_2 \vee N_4 \). We further divide the contribution of \( J_t \) into the following three subcases:

1. \( N_2 \gg k N_4 \) or \( k = 1 \),
2. \( k N_4 \gtrsim N_2 \gtrsim N_5 \) (or \( k = 2 \) and \( N_4 \gtrsim N_2 \)),
3. \( N_2 \ll N_5 \) (\( k \geq 3 \)).

**Subcase 3.1:** \( N_2 \gg k N_4 \) or \( k = 1 \).

Recall that we have \( N_2 \geq 1 \) and thus this subcase contains the case \( N_4 = 0 \). We take the extensions

\[
\begin{cases}
\hat{w} = \rho_T(w) \quad \text{of} \quad w \\
\hat{z}_j = \rho_T(z_j) \quad \text{of} \quad z_j
\end{cases}
\]

for \( j \geq 3 \), see definition [2.9]. For simplicity, we shall use the following notation:

\[ J_{\infty}^{(3.1)}(u_1, \ldots, u_{k+2}) := \sum_{N \gtrsim 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(u_1, u_2) \prod_{j=3}^{k+2} u_j dx dt' \right|. \]
By setting \( R = N_1^{1/3} N_2^{4/3} \), we divide \( J_t \) as
\[
J_t \leq J_{(3.1)}^{(3.1)}(P_{N_1}^{high} t, R \tilde{w}, P_{N_2} \tilde{w}, P_{N_3} \tilde{z}_3, P_{N_4} \tilde{z}_4, \ldots, P_{N_{k+2}} \tilde{z}_{k+2})
\]
\[
+ J_{(3.1)}^{(3.1)}(P_{N_1}^{low} t, R \tilde{w}, P_{N_2} \tilde{w}, P_{N_3}^{high} t, R \tilde{z}_3, P_{N_4} \tilde{z}_4, \ldots, P_{N_{k+2}} \tilde{z}_{k+2})
\]
\[
+ J_{(3.1)}^{(3.1)}(P_{N_1}^{low} \tilde{w}, P_{N_2} \tilde{w}, P_{N_3}^{low} t, R \tilde{z}_3, P_{N_4} \tilde{z}_4, \ldots, P_{N_{k+2}} \tilde{z}_{k+2})
\]
\[=: J_{(3.1)}^{(3.1)} + J_{(3.1)}^{(3.1)} + J_{(3.1)}^{(3.1)}. \]

In more details, the above can be expressed as the following
\[
J_t \leq \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{R} \int_{T} \Pi(P_{N_1}^{high} t, R \tilde{w}, P_{N_2} \tilde{w}) \prod_{j=3}^{k+2} P_{N_j} \tilde{z}_j \ dx \ dt \right|
\]
\[+ \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{R} \int_{T} \Pi(P_{N_1}^{low} t, R \tilde{w}, P_{N_2} \tilde{w}) \prod_{j=3}^{k+2} P_{N_j} \tilde{z}_j \ dx \ dt \right|
\]
\[+ \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{R} \int_{T} \Pi(P_{N_1}^{low} \tilde{w}, P_{N_2} \tilde{w}) \prod_{j=3}^{k+2} P_{N_j} \tilde{z}_j \ dx \ dt \right|
\]
\[=: \sum_{n=1}^{3} J_{(3.1)}^{(3.1)}. \]

For \( J_{(3.1)}^{(3.1)} \), we see from \((4.2)\) that
\[
\|P_{N_1}^{high} t, R \tilde{w}\|_{L^1} \lesssim T^{1/4} N_1^{-1/4} N_2^{-1},
\]
which gives
\[
J_{(3.1)}^{(3.1)} = \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{R} \int_{T} \Pi(P_{N_1}^{high} t, R \tilde{w}, P_{N_2} \tilde{w}) \prod_{j=3}^{k+2} P_{N_j} \tilde{z}_j \ dx \ dt \right|
\]
\[\lesssim \sum_{N_1, \ldots, N_{k+2}} N_1^{2s-1} \left| \prod_{j=3}^{k+2} P_{N_j} \tilde{z}_j \right| \lesssim_k T^{1/4} C_0^{k-1} \sum_{N_1, N_2} N_1^{2s-5/4} N_2^{-1} \left| \prod_{j=3}^{k+2} P_{N_j} \tilde{z}_j \right| \lesssim_k T^{1/4} \left\| \tilde{w} \right\|_{L^\infty_t H^s_x}^2.
\]

Here, in the first inequality, we used
\[
N_2^{s-1} \leq N_1^{s-1} \quad \text{since} \quad s \geq s(\alpha) > 1/2.
\]

In the last inequality, we also used \((2.11)\).

In a similar manner, we can also estimate \( J_{(3.1)}^{(3.1)} \) by the same bound like that of \( J_{(3.1)}^{(3.1)} \).
\[
J_{(3.1)}^{(3.1)} = \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{R} \int_{T} \Pi(P_{N_1}^{low} t, R \tilde{w}, P_{N_2} \tilde{w}) \prod_{j=3}^{k+2} P_{N_j} \tilde{z}_j \ dx \ dt \right|
\]
\[\lesssim_k T^{1/4} C_0 \left\| \tilde{w} \right\|_{L^\infty_t H^s_x}^2 \lesssim_k T^{1/4} \left\| \tilde{w} \right\|_{L^\infty_t H^s_x}^2.
\]
where we have also used (4.3).

For \( J_{∞,3}^{(3.1)} \), we note that from Lemma 4.4, we have \(|Ω_{k+1}^n| \gtrsim N_2 N_1^α =: L\). This enables us to decompose \( J_{∞,3}^{(3.1)} \) as the following

\[
J_{∞,3}^{(3.1)} \leq J_{∞,3}^{(3.1)}(P_{N_1} Q_{≥L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} \tilde{w}, P_{N_3} 1_{t,R}^Low \, \tilde{z}_3, \ldots, P_{N_{k+2}} \tilde{z}_{k+2}) \\
+ J_{∞,3}^{(3.1)}(P_{N_1} Q_{≤L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} \tilde{w}, P_{N_3} Q_{≥L}(1_{t,R}^Low \, \tilde{z}_3), \ldots, P_{N_{k+2}} \tilde{z}_{k+2}) \\
+ J_{∞,3}^{(3.1)}(P_{N_1} Q_{≤L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} Q_{≥L \tilde{w}}, P_{N_3} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_3), \ldots, P_{N_{k+2}} \tilde{z}_{k+2}) + \ldots \\
+ J_{∞,3}^{(3.1)}(P_{N_1} Q_{≤L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} Q_{≤L \tilde{w}}, P_{N_3} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_3), P_{N_{k+2}} Q_{≥L \tilde{z}_{k+2}}) =: J_{∞,3,1}^{(3.1)} + \ldots + J_{∞,3,k+2}^{(3.1)},
\]

where \( J_{∞,3,n}^{(3.1)} \) for \( 4 \leq n \leq k + 2 \) corresponds to the term in which \( Q_{≥L} \) lands on \( P_{N_n} \, \tilde{z}_n \). More precisely, we have the following expression:

\[
J_{∞,3}^{(3.1)} \leq \sum N_2^{2(s-1)} \int_R \int_T \left| \Pi(P_{N_1} Q_{≥L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} \tilde{w}) P_{N_3} 1_{t,R}^Low \, \tilde{z}_3 \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \, dx dt' \right| \\
+ \sum N_2^{2(s-1)} \int_R \int_T \left| \Pi(P_{N_1} Q_{≤L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} \tilde{w}) P_{N_3} Q_{≥L}(1_{t,R}^Low \, \tilde{z}_3) \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \, dx dt' \right| \\
+ \sum N_2^{2(s-1)} \int_R \int_T \left| \Pi(P_{N_1} Q_{≤L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} Q_{≥L \tilde{w}}) P_{N_3} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_3) \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \, dx dt' \right| \\
\times P_{N_3} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_3) \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \, dx dt' \\
+ \sum_{n=4}^{k+2} \sum N_2^{2(s-1)} \int_R \int_T \left| \Pi(P_{N_1} Q_{≤L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} Q_{≤L \tilde{w}}) P_{N_3} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_3) \prod_{j=4}^{n-1} P_{N_j} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_j) \prod_{j=n+1}^{k+2} P_{N_j} \tilde{z}_j \, dx dt' \right| \\
\times P_{N_3} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_3) \prod_{j=4}^{n-1} P_{N_j} Q_{≤L}(1_{t,R}^Low \, \tilde{z}_j) \prod_{j=n+1}^{k+2} P_{N_j} \tilde{z}_j \, dx dt' \\
=: \sum_{j=1}^{k+2} J_{∞,3,j}^{(3.1)},
\]

where \( \sum \) denotes \( \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \) for simplicity. It is worth noting that for \( α \in \{1, 2\} \)

\[
R = N_1^{1/3} N_2^{4/3} \ll N_2 N_1^α = L.
\]

Then, by the Hölder’s inequality, (4.4), (4.20) and Lemma 2.6 imply that

\[
J_{∞,3,1}^{(3.1)} = \sum N_2^{2(s-1)} \int_R \int_T \left| \Pi(P_{N_1} Q_{≥L}(1_{t,R}^Low \, \tilde{w}), P_{N_2} \tilde{w}) P_{N_3} 1_{t,R}^Low \, \tilde{z}_3 \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \, dx dt' \right| \\
\lesssim \sum_{N_1, \ldots, N_{k+2}} N_2^{2s-1} \|P_{N_1} Q_{≥L}(1_{t,R}^Low \, \tilde{w})\|_{L_{t,x}^∞} \|P_{N_2} \tilde{w}\|_{L_{t,x}^∞} \|P_{N_3} 1_{t,R}^Low \, \tilde{z}_3\|_{L_{t,x}^∞} \prod_{j=4}^{k+2} \|P_{N_j} \tilde{z}_j\|_{L_{t,x}^∞}
\]
\[
\lesssim k C_0^{k-1} \sum_{N_1, N_2} N_1^{2s-1-\alpha} \| P_{N_1} \bar{w} \|_{X^{\alpha, 1}} \| P_{N_2} \bar{w} \|_{L_t^\infty H_x^{s-1/2}} \| P_{N_1} 1_{L,R} \bar{z}_3 \|_{L_t^2}
\]
\[
\lesssim k \sum_{N_1, N_2} N_1^{2s-1-\alpha} \| P_{N_1} \bar{w} \|_{X^{\alpha, 1}} \| P_{N_2} \bar{w} \|_{L_t^\infty H_x^{s-1/2}} \| P_{N_1} 1 \bar{z}_3 \|_{L_t^2}
\] + \[T^{1/4} \sum_{N_1, N_2} N_1^{2s-13/12-\alpha} N_2^{-1/3} \| P_{N_1} \bar{w} \|_{X^{\alpha, 1}} \| P_{N_2} \bar{w} \|_{L_t^\infty H_x^{s-1/2}} \| P_{N_1} \bar{z}_3 \|_{L_t^\infty L_x^2}
\]
\[
\lesssim k T^{1/4} \| \bar{w} \|_{L_t^\infty H_x^{s-1}} \| \bar{w} \|_{X^{s-1, 1}} \lesssim k T^{1/4} \| w \|_{Z_T^{s-1}} \| w \|_{L_T^\infty H_x^{s-1}}.
\]

Similar computation and (4.1) show
\[
J_{3,2}^{(3,1)} = \sum_{N_1, N_2} N_1^{2s-1} \left| \int_{\mathbb{R}} \int_{T} \Pi (P_{N_1} Q \ll N_2 N_1^\alpha (1_{L,R} \bar{w}), P_{N_2} \bar{w}) P_{N_3} Q \gg N_2 N_1^\alpha (1_{L,R} \bar{z}_3) \prod_{j=4}^{k+2} P_{N_j} \bar{z}_j \ dx \ dt \right|
\]
\[
\lesssim k T^{1/4} C_0^{k-1} \| \bar{w} \|_{L_t^\infty H_x^{s-1}} \| \bar{z}_3 \|_{X^{s-1, 1}} \lesssim k T^{1/4} \| w \|_{L_T^\infty H_x^{s-1}}.
\]

Next, we consider the contribution of \( J_{3,3}^{(3,1)} \). By using Lemma 2.6 the Hölder’s inequality, and (4.1) show
\[
J_{3,3}^{(3,1)} = \sum_{N_1, N_2} N_1^{2s-1} \left| \int_{\mathbb{R}} \int_{T} \Pi (P_{N_1} Q \ll N_2 N_1^\alpha (1_{L,R} \bar{w}), P_{N_2} Q \gg N_2 N_1^\alpha \bar{w}) \right|
\]
\[
\times P_{N_3} Q \ll N_2 N_1^\alpha (1_{L,R} \bar{z}_3) \prod_{j=4}^{k+2} P_{N_j} \bar{z}_j \ dx \ dt \right|
\]
\[
\lesssim \sum_{N_1, \ldots, N_{k+2}} N_1^{2s-1} \| P_{N_1} 1_{L,R} \bar{w} \|_{L_t^2} \| P_{N_2} Q \gg L \bar{w} \|_{L_t^2 L_x^\infty} \| P_{N_3} \bar{z} \|_{L_t^\infty L_x^2} \prod_{j=4}^{k+2} \| P_{N_j} \bar{z}_j \|_{L_t^\infty L_x^2}
\]
\[
\lesssim k T^{1/2} C_0^{k-1} \sum_{N_1, N_2} N_1^{2s-1-\alpha} N_2^{-1} \| P_{N_1} \bar{w} \|_{L_t^\infty L_x^2} \| D_x^{3/2} P_{N_2} \bar{w} \|_{X^{\alpha, 1}} \| P_{N_3} \bar{z} \|_{L_t^\infty L_x^2}
\]
\[
\lesssim k T^{1/2} \| \bar{w} \|_{X^{s-2, 1}} \| \bar{w} \|_{L_t^\infty H_x^{s-1}} \sum_{N_1, N_2} N_1^{-\alpha} N_2^{-1} \| P_{N_2} \bar{w} \|_{X^{1/2, 1}} \lesssim T^{1/2} \| w \|_{Z_T^{s-1}} \| w \|_{L_T^\infty H_x^{s-1}}
\]

since \( s \geq s(\alpha) > \frac{1}{2} \).

Similarly, we obtain the bound on \( J_{\infty,3,4}^{(3,1)} \), i.e. when the case
\[
\begin{cases}
Q \gg L \text{ hits } P_{N_2} \bar{z}_3; \\
Q \ll L \text{ hit } P_{N_1} \bar{w}, P_{N_2} \bar{w}, \text{ and } P_{N_3} \bar{z}_3.
\end{cases}
\]

Therefore, we have
\[
J_{\infty,3,4}^{(3,1)} \lesssim k C_0^{k-2} \sum_{N_1, N_2, N_4} N_1^{2s-1} \| P_{N_1} 1_{L,R} \bar{w} \|_{L_t^2} \| P_{N_2} \bar{w} \|_{L_t^\infty} \| P_{N_3} \bar{z}_3 \|_{L_t^\infty L_x^2} \| P_{N_4} Q \gg L \bar{z}_4 \|_{L_t^2 L_x^\infty}
\]
\[
\lesssim k T^{1/2} \sum_{N_1, N_2, N_4} N_1^{2s-\alpha-1} N_2^{-1} \| P_{N_1} \bar{w} \|_{L_t^\infty L_x^2} \| P_{N_2} \bar{w} \|_{L_t^\infty H_x^{s-1/2}} \| P_{N_3} \bar{z}_3 \|_{L_t^\infty L_x^2} \| P_{N_4} \bar{z}_4 \|_{X^{1/2, 1}} \lesssim T^{1/2} C_0^{k-2} \| \bar{w} \|_{L_t^\infty H_x^{s-1}} \| \bar{w} \|_{L_t^\infty L_x^2} \| \bar{z}_3 \|_{L_t^\infty H_x^s} \sum_{N_1, N_2} N_1^{-\alpha} \| P_{N_4} \bar{z}_4 \|_{X^{1/2, 1}}
\]
We conclude this subcase by noticing from the above argument that \( J_{\infty,3,j}^{(3,1)} \) for \( 5 \leq j \leq k+2 \) can be estimated by the same bound as that of \( J_{\infty,3,4}^{(3,1)} \). The same argument yields for \( 5 \leq j \leq k+2 \),

\[
J_{\infty,3,j}^{(3,1)} \lesssim_T k^{1/2} C_0^{k-1} \|w\|_{L_T^\infty H_x^{s-1}}^2 \lesssim_T k^{1/2} C_0^k \|w\|_{L_T^\infty H_x^{s-1}}^2.
\]

**Subcase 3.2:** \( N_5 \lesssim N_2 \lesssim kN_4 \) (or \( k = 2 \) and \( N_4 \gtrsim N_2 \)).

Note that the cases \( N_2 = 0 \) or \( N_4 = 0 \) have been already treated so that we can assume that \( N_2 \geq 1 \) and \( N_4 \geq 1 \). Let us recall the following relation:

\[
N \sim N_1 \sim N_3 \gg N_2 \vee N_4.
\]

It suffices to consider the case \( N_5 \lesssim N_2 \lesssim N_4 \) since \( k \geq 1 \). First, we observe the following, which separates the case into when the derivative hits the low-frequency part \( P_{N_2}w \) and when the derivative hits the high frequency part \( P_{N_1}w \).

\[
J_t \leq \sum_{N \gg 1} \sum_{N_1,\ldots,N_{k+2}} N^{2(s-1)} \int_0^t \int_T \partial_x P_N^2 P_{N_1}w P_{N_2}w \prod_{j=3}^{k+2} P_{N_j}z_j \, dx dt'
\]

\[
+ \sum_{N \gg 1} \sum_{N_1,\ldots,N_{k+2}} N^{2(s-1)} \int_0^t \int_T P_{N_1}w \partial_x P_N^2 P_{N_1}w \prod_{j=3}^{k+2} P_{N_j}z_j \, dx dt'
\]

\[
=: I_t^{(3,2)} + J_t^{(3,2)}.
\]

Furthermore, for the term \( I_t^{(3,2)} \), we further divide it as in Case 3 in Proposition 4.9. We will compare the size of \(|n_2 + n_4|\) and \( k|n_5|\), and we arrive

\[
I_t^{(3,2)} \leq \sum_{N \gg 1} \sum_{N_1,\ldots,N_{k+2}} N^{2(s-1)} \int_0^t \int_T \partial_x P_N^2 P_{N_1}w P_{N_2}z_3 P_{\leq kN_5 \vee 1} \left( P_{N_2}w \prod_{j=4}^{k+2} P_{N_j}z_j \right) \, dx dt'
\]

\[
+ \sum_{kN_5 \vee 1 \leq M \leq N_4} N^{2(s-1)} \int_0^t \int_T \partial_x P_N^2 P_{N_1}w P_{N_2}z_3 P_M \left( P_{N_2}w \prod_{j=4}^{k+2} P_{N_j}z_j \right) \, dx dt'
\]

\[
=: I_{t,1}^{(3,2)} + I_{t,2}^{(3,2)}.
\]

where \( \sum \) denotes \( \sum_{N \gg 1} \sum_{N_1,\ldots,N_{k+2}} \) for the simplicity. Let us remark that if \( k = 2 \), then the term \( I_{t,1}^{(3,2)} \) does not appear, and it holds that \( I_t^{(3,2)} \leq I_{t,2}^{(3,2)} \) with the summation \( \sum_{M \leq N_4} \) instead of \( \sum_{kN_5 \vee 1 \leq M \leq N_4} \).
To estimate $I_{t,1}^{(3,2)}$, we use Hölder’s, Bernstein’s, Minkowski’s inequalities, Proposition 3.5 and Corollary 3.6 to obtain the following

$$I_{t,1}^{(3,2)} \lesssim \sum_{N_1, \ldots, N_{k+2}} N_1^{2s-1} (k^{3/4} N_5^{3/4} \vee 1) \int_0^t \left\| P_{N_1} w \right\| L^2_t \left\| P_{N_5} z_3 \right\| L^2_t \left\| P_{N_2} w \right\| L^4_t$$

$$\times \left\| P_{N_4} z_4 \right\| L^4_t \left\| P_{N_5} z_5 \right\| L^4_t \prod_{j=6}^{k+2} \left\| P_{N_j} z_j \right\| L^\infty_t \, dt'$$

$$\lesssim_k C_0^{k-3} \left\| w \right\| L^{5/12}_T H^s T \left\| w \right\| \sum_{N_4 \geq N_5} \left( \frac{N_2}{N_4} \right)^{5/12} \left( \frac{N_5 \vee 1}{N_4} \right)^{1/6}$$

$$\times \left\| D_x^{-5/12} P_{N_2} w \right\| L^4_T L^2_x \left\| D_x^{7/12} P_{N_4} z_4 \right\| L^4_T L^2_x \left\| D_x^{7/12} P_{N_5} z_5 \right\| L^4_T L^2_x$$

$$\lesssim_k TG(C_0) \left\| w \right\| L^{5/12}_T H^{s(\alpha)-1}_T \left\| w \right\| L^{5/12}_T H^{s(\alpha)} T \prod_{i=4}^5 \left\| z_i \right\|^2 L^{5/12}_T H^{s(\alpha)} T \lesssim_k TG(C_0) \left\| w \right\|^2 L^{5/12}_T H^{s-1}_T$$

since for $\alpha \in \{1, 2\}$, we have $s(\alpha) \geq 7/12 + \beta(\alpha)$.

In order to estimate $I_{t,2}^{(3,2)}$, we shall further decompose it. Again, we take the extensions

$$\begin{cases} \bar{w} = \rho_T(w) & \text{of } w \\ \bar{z}_j = \rho_T(z_j) & \text{of } z_j \end{cases}$$

for $j \geq 3$, see definition (2.9). As in Subcase 3.1, we shall use the following notation:

$$I_{\infty,2}(u_1, \ldots, u_{k+2})$$

$$:= \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \sum_{kN_5 \geq 1} N^{2(s-1)} \left\| \int_0^t \int_T \left( \partial_x P_N^2 u_1 \right) u_3 P_M \left( \prod_{j=4}^{k+2} u_j \right) \, dx \, dt' \right\|.$$

Putting $R = N_1^{1/3} M^{4/3}$, we see that

$$I_{t,2}^{(3,2)} \leq I_{\infty,2}^{(3,2)} (P_{N_1}^{t,R} \bar{w}, P_{N_2} \bar{w}, P_{N_3} \bar{z}_3, P_{N_4} \bar{z}_4, \ldots, P_{N_{k+2}} \bar{z}_{k+2})$$

$$+ I_{t,2}^{(3,2)} (P_{N_1}^{t,R} \bar{w}, P_{N_2} \bar{w}, P_{N_3}^{t,R} \bar{z}_3, P_{N_4} \bar{z}_4, \ldots, P_{N_{k+2}} \bar{z}_{k+2})$$

$$+ I_{t,2}^{(3,2)} (P_{N_1}^{t,R} \bar{w}, P_{N_2} \bar{w}, P_{N_3}^{t,R} \bar{z}_3, P_{N_4} \bar{z}_4, \ldots, P_{N_{k+2}} \bar{z}_{k+2})$$

$$=: I_{\infty,2,1}^{(3,2)} + I_{\infty,2,2}^{(3,2)} + I_{\infty,2,3}^{(3,2)}.$$

In particular, the above inequality can be expressed as

$$I_{t,2}^{(3,2)} \leq \sum N^{2(s-1)} \left\| \int_{\mathbb{R}} \int_T \partial_x P_N^2 P_{N_1}^{t,R} \bar{w} P_{N_3} \bar{z}_3 P_M \left( \prod_{j=4}^{k+2} P_{N_j} \bar{z}_j \right) \, dx \, dt' \right\|$$

$$+ \sum N^{2(s-1)} \left\| \int_{\mathbb{R}} \int_T \partial_x P_N^2 P_{N_1}^{t,R} \bar{w} P_{N_3}^{t,R} \bar{z}_3 P_M \left( \prod_{j=4}^{k+2} P_{N_j} \bar{z}_j \right) \, dx \, dt' \right\|$$

$$+ \sum N^{2(s-1)} \left\| \int_{\mathbb{R}} \int_T \partial_x P_N^2 P_{N_1}^{t,R} \bar{w} P_{N_3}^{t,R} \bar{z}_3 P_M \left( \prod_{j=4}^{k+2} P_{N_j} \bar{z}_j \right) \, dx \, dt' \right\|$$

$$=: I_{\infty,2,1}^{(3,2)} + I_{\infty,2,2}^{(3,2)} + I_{\infty,2,3}^{(3,2)}.$$. 
where $\sum$ denotes $\sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \sum_{kN \leq M \leq N_4}$ for simplicity. For $I_{\infty,2,1}^{(3,2)}$, we see from (4.2) that
\[
\|1_{t,R}^{hi} \|_{L^1} \lesssim T^{1/4} N_1^{-1/4} M^{-1}.
\]
Then, we have
\[
I_{\infty,2,1}^{(3,2)} \lesssim \sum_{N_1, \ldots, N_{k+2}} \sum_{kN \leq M \leq N_4} N_1^{2s-1} \|1_{t,R}^{hi} \|_{L^1} \||P_{N_1} \hat{u}_l||P_{N_3} \hat{z}_3||P_{N_4} \hat{z}_4||L^\infty L^2_t\|
\times \left\| P_M \left( P_{N_2} \hat{u} \prod_{j=4}^{k+2} P_{N_j} \hat{z}_j \right) \right\|_{L^\infty t, \infty}
\lesssim T^{1/4} \sum_{N_1, \ldots, N_{k+2}} \sum_{kN \leq M \leq N_4} N_1^{2s-5/4} \|P_{N_1} \hat{u}_l||P_{N_3} \hat{z}_3||P_{N_4} \hat{z}_4||L^\infty L^2_t\|
\times \left\| P_{N_2} \hat{u} \prod_{j=4}^{k+2} P_{N_j} \hat{z}_j \right\|_{L^\infty t, \infty H^{-1/2}_x}.
\]
Note that (2.14) leads to
\[
\sum_{N_5, \ldots, N_{k+2}} \left\| P_{N_2} \hat{u} \prod_{j=4}^{k+2} P_{N_j} \hat{z}_j \right\|_{H^{-1/2}_x} \lesssim_k C_0^{k-2} \|P_{N_4} \hat{z}_4||H^{s_0}||P_{N_2} \hat{u}_l||H^{-1/2}_x}
\]
with $\frac{1}{2} < s_0 < s$. This together with (4.22) and (2.11) shows that
\[
I_{\infty,2,1}^{(3,2)} \lesssim_k T^{1/4} \left\| \hat{u}_l \right\|_{L^\infty H^{-1/2}_x} \left\| \hat{u}_l \right\|_{L^\infty H^{-1/2}_x} \left\| \hat{z}_3 \right\|_{L^\infty H^s_x} \left\| \hat{z}_4 \right\|_{L^\infty H^s_x} \sum_{N_1} N_1^{-1/4}
\lesssim_k T^{1/4} \left\| \hat{u}_l \right\|^2_{L^\infty H^{-1/2}_x}.
\]
Similarly, $I_{\infty,2,2}^{(3,2)}$ can be estimated by the same bound as above.
\[
I_{\infty,2,2}^{(3,2)} \lesssim_k T^{1/4} \left\| \hat{u}_l \right\|^2_{L^\infty H^{-1/2}_x}.
\]
For $I_{\infty,2,3}^{(3,2)}$, we see from Lemma 4.5 that we have a non-resonant function for $i \in \{1, 2\}$ such that
\[
|\Omega_i^{k+1} | \gtrsim MN_1 \sim |n_2 + n_4| N_1^0.
\]
Therefore, by setting $L := MN_1^0$, we further decompose as
\[
I_{\infty,2,3}^{(3,2)} \leq I_{\infty,2}^{(3,2)} (P_{N_1} Q_{\leq L}(1_{\ell/R}^{l_{hi}} \hat{u}), P_{N_2} \hat{u}, P_{N_3} 1_{\ell/R}^{l_{hi}} \hat{z}_3, \ldots, P_{N_{k+2}} 1_{\ell/R}^{l_{hi}} \hat{z}_{k+2})
+ I_{\infty,2}^{(3,2)} (P_{N_1} Q_{\leq L}(1_{\ell/R}^{l_{hi}} \hat{u}), P_{N_2} \hat{u}, P_{N_3} Q_{\leq L}(1_{\ell/R}^{l_{hi}} \hat{z}_3), \ldots, P_{N_{k+2}} 1_{\ell/R}^{l_{hi}} \hat{z}_{k+2})
+ I_{\infty,2}^{(3,2)} (P_{N_1} Q_{\leq L}(1_{\ell/R}^{l_{hi}} \hat{u}), P_{N_2} Q_{\leq L}(1_{\ell/R}^{l_{hi}} \hat{z}_3), \ldots, P_{N_{k+2}} 1_{\ell/R}^{l_{hi}} \hat{z}_{k+2})
+ I_{\infty,2}^{(3,2)} (P_{N_1} Q_{\leq L}(1_{\ell/R}^{l_{hi}} \hat{u}), P_{N_2} Q_{\leq L}(1_{\ell/R}^{l_{hi}} \hat{z}_3), \ldots, P_{N_{k+2}} 1_{\ell/R}^{l_{hi}} \hat{z}_{k+2})
=: I_{\infty,2,3,1}^{(3,2)} + \ldots + I_{\infty,2,3,k+2}^{(3,2)}.
\]
where \( I_{\infty, 2, 3}^{(3)} \), for \( 4 \leq n \leq k + 2 \), corresponds to the term in which \( Q_{\geq L} \) lands on \( P_{N_n} \). We see a more explicit explanation blow:

\[
I_{\infty, 2, 3}^{(3)} \leq \sum N^{2(s - 1)} \left| \int \int_T \partial_x P_N^2 P_{N_1} Q_{\geq MN_1} (1_{t, R} \tilde{w}) P_{N_3} 1_{t, R} \tilde{z}_3 \right|^2 \left( P_{N_2} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) dxdt \bigg| \\
+ \sum N^{2(s - 1)} \left| \int \int_T \partial_x P_N^2 P_{N_1} Q_{\leq MN_1} (1_{t, R} \tilde{w}) P_{N_3} Q_{\geq MN_1} (1_{t, R} \tilde{z}_3) \right|^2 \left( P_{N_2} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) dxdt \bigg| \\
+ \sum N^{2(s - 1)} \left| \int \int_T \partial_x P_N^2 P_{N_1} Q_{\leq MN_1} (1_{t, R} \tilde{w}) P_{N_3} Q_{\leq MN_1} (1_{t, R} \tilde{z}_3) \right|^2 \left( P_{N_2} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) dxdt \bigg| \\
\times P_M \left( P_{N_2} Q_{\geq MN_1} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) dxdt \bigg| \\
+ \sum_{n=4}^{k+2} \sum N^{2(s - 1)} \left| \int \int_T \partial_x P_N^2 P_{N_1} Q_{\leq MN_1} (1_{t, R} \tilde{w}) P_{N_3} Q_{\leq MN_1} (1_{t, R} \tilde{z}_3) \right|^2 \left( P_{N_2} \prod_{j=4}^{n-1} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) dxdt \bigg| \\
\times P_M \left( P_{N_2} Q_{\leq MN_1} \tilde{w} \prod_{j=4}^{n-1} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) dxdt \bigg| \\
= \sum_{j=1}^{k+2} I_{\infty, 2, 3, j}^{(3)}
\]

where \( \sum \) denotes \( \sum N_{\geq 1} \sum_{N_1, \ldots, N_{k+1}} \sum_{1 \ll N_5 \ll M \leq N_4} \) for simplicity.

The contribution of \( I_{\infty, 2, 3, 1}^{(3)} \) is estimated, thanks to Lemma 2.6 (4.1), (4.4), (4.20), and (4.31), by

\[
I_{\infty, 2, 3, 1}^{(3)} \leq \sum_{N_1, \ldots, N_{k+2}} \sum_{1 \ll N_5 \leq M \leq N_4} N^{2s - 1} \left\| P_{N_1} \tilde{w} \right\|_{X^{0,1}} \left\| P_{N_3} 1_{t, R} \tilde{z}_3 \right\|_{L^2 t,x} \\
\times \left\| P_M \left( P_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) \right\|_{L^\infty t,x} \\
\leq \sum_{N_1, \ldots, N_{k+2}} \sum_{M \leq N_4} N^{2s - 2} \left\| P_{N_1} \tilde{w} \right\|_{X^{0,1}} \left\| P_{N_3} 1_{t, R} \tilde{z}_3 \right\|_{L^2 t,x} \left\| P_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right\|_{L^\infty H^{s-1}} \\
\leq k \| \tilde{w} \|_{L^\infty t H^{s-1}} \sum_{N_1 \gg N_4 M \gg N_4} (N^{2s - 2} \left\| P_{N_1} \tilde{w} \right\|_{X^{0,1}} \left\| P_{N_3} 1_{t, R} \tilde{z}_3 \right\|_{L^2 t,x} \left\| P_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right\|_{L^\infty H^{s-1}} \\
+ T^{1/4} N_1^{2s - 25/12} M^{-1/3} \left\| P_{N_1} \tilde{w} \right\|_{X^{0,1}} \left\| P_{N_3} \tilde{z}_3 \right\|_{L^\infty t L^2} \left\| P_{N_2} \tilde{z}_4 \right\|_{L^\infty H^0} \right\|_{L^\infty t H^{s-1}} \\
\leq k T^{1/4} \| \tilde{w} \|_{L^\infty t H^{s-1}} \| \tilde{w} \|_{X^{-2,1}} \| \tilde{z}_3 \|_{L^\infty H^s} \| \tilde{z}_4 \|_{L^\infty H^s} \| \tilde{z}_4 \|_{L^\infty H^s} \leq k T^{1/4} \| w \|_{Z_t^{s-1}} \| w \|_{L^\infty t H^{s-1}}
\]

with \( 1/2 < s_0 < s \). Here, we used \( N_1^{-\alpha} \leq N_1^{-1} \) since \( \alpha \{1, 2\} \).

Similarly, we can estimate \( I_{\infty, 2, 3, 3}^{(3)} \) by

\[
I_{\infty, 2, 3, 3}^{(3)} \leq k T^{1/4} C_0^k \| \tilde{w} \|_{L^\infty t H^{s-1}}^2 \leq k T^{1/4} \| w \|_{L^\infty t H^{s-1}}^2.
\]
Next, for $I^{(3,2)}_{\infty,2,3,3}$, thanks to Lemma \[2,6\] \(4.3\), \(4.22\) \(4.11\) and \(4.31\), we have

\[
I^{(3,2)}_{\infty,2,3,3} \lesssim \sum_{N_1,\ldots,N_{k+2}} N_1^{2^{s-1}} \left\| P_{N_1}Q \ll L_t^{\text{low}} u \right\|_{L_t^\infty L_x^3} \left\| P_{N_3}Q \ll L_t^{\text{low}} \tilde{z}_3 \right\|_{L_t^2} \left\| P_M \left\{ (P_{N_2} Q \gg L \tilde{u}) \right\} \prod_{j=4}^{k+2} P_{N_j} \tilde{z} \right\|_{L_t^2 L_x^\infty}
\]

\[
\lesssim T^{1/2} \sum_{N_1,\ldots,N_{k+2}} \sum_{M \leq N_4} N_1^{2^{s-1}} M \left\| P_{N_1} \tilde{u} \right\|_{L_t^\infty L_x^2} \left\| P_{N_3} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} 
\times \left\| (P_{N_2} Q \gg L \tilde{u}) \prod_{j=4}^{k+2} P_{N_j} \tilde{z} \right\|_{L_t^2 H_x^{-1/2}}
\]

\[
\lesssim_k T^{1/2} \left\| \tilde{u} \right\|_{X^{s-1,1}} \sum_{N_1 \geq N_4} \sum_{M \leq N_4} N_1^{2^{s-2}N_3^{3/2-s}} \left\| P_{N_1} \tilde{u} \right\|_{L_t^\infty L_x^2} \left\| P_{N_3} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} \left\| P_{N_4} \tilde{z}_4 \right\|_{L_t^\infty H_x^{s-1}}
\]

with $\frac{1}{2} < s_0 < s$. Here, we used $N_1^{-\alpha} \leq N^{-1}$. In the first inequality, we have inserted $MM^{-1}$, which guarantees that we can apply \(4.31\). The second inequality we notice

\[
\left\| P_{N_3}Q \ll L_t^{\text{low}} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} \leq \left\| P_{N_3}Q \ll L_t^{\text{high}} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} + \left\| P_{N_3}Q \ll L_t^{\text{low}} \tilde{z}_3 \right\|_{L_t^\infty L_x^2}
\leq T^{1/4} R^{-1/4} \left\| P_{N_3}Q \ll L \tilde{z}_3 \right\|_{L_t^\infty L_x^2} + \left\| L_t \right\|_{L_t^2} \left\| P_{N_3}Q \ll L \tilde{z}_3 \right\|_{L_t^\infty L_x^2}
\leq T^{1/2} N^{-1/4} R^{-1/4} \left\| P_{N_3} \tilde{z}_3 \right\|_{L_t^1 L_x^3} + T^{1/2} \left\| P_{N_3} \tilde{z}_3 \right\|_{L_t^1 L_x^3}
\leq T^{1/2} \left\| P_{N_3} \tilde{z}_3 \right\|_{L_t^\infty L_x^2},
\]

where $R = N^{1/3} M^{4/3}$. Moreover, in this case we have $N_3 \sim N_1 \gg N_2$, and we notice the following observation

\[
N_4^\varepsilon N_1^{-2} N_3^{3/2-s} \lesssim N_1^{-1/2 + N_2^{-s}},
\]

which allows us to sum over all the dyadic pieces of $N_j$.

By a similar argument, we also have

\[
I^{(3,2)}_{\infty,2,3,4} \lesssim T^{1/2} \sum_{N_1,\ldots,N_{k+2}} \sum_{M \leq N_4} N_1^{2^{s-1}} M \left\| P_{N_1} \tilde{u} \right\|_{L_t^\infty L_x^2} \left\| P_{N_3} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} 
\times \left\| (P_{N_2} Q \gg M N_1^{\alpha} \tilde{u}) (P_{N_4} Q \gg M N_1^{\alpha} \tilde{z}_4) \prod_{j=5}^{k+2} P_{N_j} \tilde{z}_j \right\|_{L_t^2 H_x^{-1/2}}
\]

\[
\lesssim_k T^{1/2} \left\| \tilde{u} \right\|_{L_t^\infty H_x^{-1/2}} \sum_{N_1 \geq N_4} \sum_{N_1 \leq N_4} N_1^{2^{s-2}} \left\| P_{N_1} \tilde{u} \right\|_{L_t^\infty L_x^2} \left\| P_{\sim N_1} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} \left\| P_{N_4} \tilde{z}_4 \right\|_{X^{s,0,1}}
\]

\[
\lesssim_k T^{1/2} \left\| \tilde{u} \right\|_{L_t^\infty H_x^{-1/2}} \sum_{N_1 \geq N_4} \sum_{N_1 \leq N_4} N_1^{2^{s-2}} \left\| P_{N_1} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} \left\| P_{\sim N_1} \tilde{z}_4 \right\|_{L_t^\infty L_x^2}
\]

\[
\lesssim T^{1/2} \left\| \tilde{z}_3 \right\|_{L_t^\infty H_x^{-1/2}} \left\| \tilde{z}_4 \right\|_{X^{s,0,1,1}} \left\| \tilde{u} \right\|_{L_t^\infty H_x^{-1/2}} \lesssim_k T^{1/2} \left\| w \right\|_{L_t^\infty H_x^{-1}},
\]

Here, for $\alpha \{1, 2\}$, we chose $s_0 \in (1/2, s(\alpha))$, and $N_1^{-\alpha} \leq N_1^{-1}$.

Similarly, we can estimate $I^{(3,2)}_{\infty,2,3,n}$ for $5 \leq n \leq k + 2$ by the same bound as that of $I^{(3,2)}_{\infty,2,3,4}$,

\[
\sum_{n=5}^{k+2} I^{(3,2)}_{\infty,2,3,n} \lesssim_k T^{1/2} C_k \left\| w \right\|_{L_t^\infty H_x^{-1}} \lesssim_k T^{1/2} \left\| w \right\|_{L_t^\infty H_x^{-1}}.
\]
which completes the estimate of the contribution of \( I_{t}^{(3,2)} \).

On the other hand, the contribution of \( J_{t}^{(3,2)} \) can be controlled by \( I_{t}^{(3,2)} \) since \( s > 1/2 \) and \( N_{2}^{2s-1} \leq N_{1}^{2s-1} \). Indeed, we see the following expression

\[
J_{t}^{(3,2)} \leq \sum_{N \gg 1} N^{2(s-1)} \left| \int_{0}^{t} \int_{T} P_{N_{1}} w \sum_{j=4}^{k+2} P_{N_{j}} z \right| \quad \left| \int_{0}^{t} \int_{T} P_{N_{1}} w \sum_{j=4}^{k+2} P_{N_{j}} z \right|
\]

Observe that \( J_{t,1}^{(3,2)} \) is bounded by the right-hand side of the first inequality in (4.30).

**Subcase 3.3:** \( N_{2} \ll N_{5} \ (k \geq 3) \).

Recall that we can assume \( N_{2} \geq 1 \) and note that we can also assume that \( N_{5} \geq 1 \) since \( N_{5} = 0 \) is included in Subcase 3.2: \( N_{2} \geq N_{5} \). Also, recall that we assumed symmetry

\[
N_{1} \geq N_{2} \quad \text{and} \quad N_{3} \geq N_{4} \geq N_{5}.
\]

As well as in this case, \( N \sim N_{1} \sim N_{3} \). As in Subcase 3.2, we evaluate \( I_{t}^{(3,2)} \) and \( J_{t}^{(3,2)} \). Define \( L_{k} \) for \( k \geq 3 \) so that

\[
L_{3} := N_{2} \quad \text{and} \quad L_{k} := N_{2} \vee kN_{6} \quad \text{for} \quad k \geq 4.
\]

Therefore, we compare the size of \( |n_{5} + n_{4}| \) and \( |n_{2}| \vee k|n_{6}| \) to see a good resonance function as we have seen before. Moreover, we decompose

\[
I_{t}^{(3,2)} \leq \sum_{L_{k} \ll M \leq N_{4}} N^{2(s-1)} \left| \int_{0}^{t} \int_{T} \partial_{x} P_{N}^{2} P_{N_{1}} w P_{N_{3}} z \right| \quad \left| \int_{0}^{t} \int_{T} \partial_{x} P_{N}^{2} P_{N_{1}} w P_{N_{3}} z \right|
\]

where \( \sum \) denotes \( \sum_{N \gg 1} \sum_{N_{1}, \ldots, N_{k+2}} \) for simplicity. In what follows, we may assume that \( k \geq 4 \) since the case \( k = 3 \) can be treated by the same way as Subcase 3.2. For \( I_{t,1}^{(3,3)} \), Hölder’s, Bernstein’s and Minkowski’s inequalities, Proposition 3.5 and Corollary 3.6 give
We present the proof of Theorem 1.3 in terms of the solution $u$ which is analogue to [44]. The argument in general is applicable to both Theorems 1.3 and 4.12 plays a key role in this step. In the second part, we will show the existence of solutions 4.9 and 4.12 are $\delta$-independent. Moreover, we notice that the crucial energy estimates we obtained in Propositions 4.9 and 4.12 are $\delta$-independent. More precisely, our local well-posedness results are uniformly in $\delta$, i.e., the local existence time is $\delta$-independent.

Here, we will use the so-called energy method. First of all, we will show the uniqueness of the solution by considering the difference between the two solutions. Then, Proposition 4.12 plays a key role in this step. In the second part, we will show the existence of solutions to our desire equations. This is done by considering a smooth approximation and using our a priori estimates Proposition 4.9. In the final step, we will complete our proof by showing the continuous dependence on the initial data. This is done by using the frequency envelope, which is analogue to [44]. The argument in general is applicable to both Theorems 1.3 and 1.4. We present the proof of Theorem 1.3 in terms of the solution $u$ to gFDF (1.17) in the following. For the proof of Theorem 1.4 we just replace $u$ by $v$ and restrict the regularity assumption to $s \geq \frac{3}{4}$.
5.0.1. **Uniqueness.** Let \( s \geq \frac{2}{3} \), \( u_0 \in H^s(\mathbb{T}) \), and \( T \in (0,1) \). Consider \((u^{(1)}, u^{(2)}) \in (L^\infty([0,T]; H^s(\mathbb{T})))^2 \) be two solutions to the \eqref{GFDF} that are generated from \( u_0 \). Then, Lemma 4.8 implies that \((u^{(1)}, u^{(2)}) \in (Z^s_T)^2 \). Moreover, set the difference \( w := u^{(1)} - u^{(2)} \), we have
\[
\|w\|_{Z^s_T} \lesssim \|w\|_{L^\infty_T H^s_x} + G(\|u^{(1)}\|_{L^\infty_T H^s_x} + \|u^{(2)}\|_{L^\infty_T H^s_x}) w_{L^\infty_T H^s_x}.
\]
Proposition 4.12 and \( w(0) = 0 \), we obtain
\[
\|w\|^2_{L^\infty_T H^s_x} \leq T^{1/4} G(\|u^{(1)}\|_{Z^s_T} + \|u^{(2)}\|_{Z^s_T}) \|w\|_{Z^s_T} \|w\|_{L^\infty_T H^s_x},
\]
ensure that
\[
u^{(1)}(0) \equiv u^{(2)}(0) \quad \text{on} \quad [0,T_0]
\]
with \( 0 < T_0 \leq T \). We notice here the equivalency only depends on \( \|u^{(1)}\|_{Z^s_T} \) and \( \|u^{(2)}\|_{Z^s_T} \). Therefore, we conclude \( u^{(1)}(T_0) = u^{(2)}(T_0) \) and we can iterate the same argument on \([T_0,T]\). Finally, this proves that \( u^{(1)} \equiv u^{(2)} \) on \([0,T]\) after a finite number of iteration.

5.0.2. **Local existence.** First, we recall the well known results (see [1]) that the Cauchy problem \( \eqref{GFDF} \) is locally well-posed in \( H^s(\mathbb{T}) \) for \( s > \frac{3}{2} \) with a minimal time of existence \( T = T(\|u_0\|_{H^s_x}^2) \). Indeed, for any \( s \in \mathbb{R} \) and any smooth test function \( \phi \), the \( H^s \)-scalar product \( \langle G, \phi \rangle_{H^s} \) vanishes. The classical energy method (which consists of the parabolic regularisation, Kato–Ponce commutator estimate, and Bona–Smith argument [12]) imply the local well-posedness in \( C([0,T]; H^s(\mathbb{T})) \) for \( s > \frac{3}{2} \).

- **Existence of smooth approximation**

Let \( u \in C([0,T_0]; H^\infty(\mathbb{T})) \) be a smooth solution to \( \eqref{GFDF} \) emanating from a smooth initial data \( u_0 \in H^\infty(\mathbb{T}) \) with \( \|u_0\|_{H^s_x} < \infty \). Let \( w_\infty \equiv 1 \). Then, Lemma 4.8 gives the following
\[
\|u\|_{Z^s_T} \lesssim \|u\|_{L^\infty_T H^s_x} + G(\|u\|_{L^\infty_T H^s_x}) \|u\|_{L^\infty_T H^s_x}.
\]
Then, together with Proposition 4.9, there exists an increasing, non-negative, entire function \( G \) on \( \mathbb{R}_+ \) such that
\[
\|u\|^2_{L^\infty_T H^s_x} \leq \|u_0\|^2_{H^s_x} + T^{1/4} G(\|u\|_{L^\infty_T H^s_x}) \|u\|^2_{L^\infty_T H^s_x}.
\]
for any \( 0 < T \leq \min(1,T_0) \) and any \( \frac{2}{3} \leq s \leq 2 \).

Let \( T_* \geq T_0 \) denote the maximal time of existence of \( u(t) \in H^\infty(\mathbb{T}) \). We recall from [1] that the well-posedness result \( u(t) \in H^{s}(\mathbb{T}) \) for all \( 0 < T < \min(T_0, T_1) \), we have
\[
\|u\|^2_{L^\infty_T H^s_x} \leq 2\|u_0\|^2_{H^s_x} < \infty.
\]
For any \( \frac{3}{4} \leq s \leq 2 \), \eqref{5.1} leads to the following
\[
\|u\|^2_{L^\infty_T H^s_x} \leq 2\|u_0\|^2_{H^s_x} < \infty.
\]
Therefore, it ensures that \( T_1 < T_0 \), i.e., \( u \) does exist \([0,T_1]\) with \( u \in C([0,T_1]; H^\infty(\mathbb{T})) \). Moreover, without loss of generality, we take \( T_1 = 1 \) and \( T_* \geq 1 \).

- **Compactness**
We are now in the right place to perform compactness argument. Let
\[ u_0 \in H^s(\mathbb{T}) \quad \text{with} \quad 2 \geq s \geq \frac{3}{4}, \]
Consider that the smooth sequence \( \{u_{0,n}\}_n \) which converges to \( u_0 \) in \( H^s(\mathbb{T}) \). We deduce from previous steps that the sequence of solutions \( \{u_n\}_{n \geq 1} \) generated by \( \{u_{0,n}\}_{n \geq 1} \) is included in \( C([0,T];H^\infty(\mathbb{T})) \) and satisfies
\[
\| u_n \|_{L^\infty_T H^s_x}^2 \leq 2 \| u_0 \|_{H^s}^2 < \infty, \tag{5.2}
\]
where \( 0 < T < 1 \). This proves that the sequence \( \{u_n\}_{n \geq 1} \) is bounded in \( L^\infty([0,T];H^s(\mathbb{T})) \) and thus \( f(u_n) \) is bounded in \( L^\infty([0,T];H^s(\mathbb{T})) \).

The boundedness of \( \{u_n\}_{n \geq 1} \) together with the Banach-Alaoglu Theorem imply that \( \{u_n\}_{n \geq 1} \) converges in the weak* topology of \( L^\infty([0,T];H^s(\mathbb{T})) \) to some limit, say \( u \).

Moreover, in the following, we show the Cauchy property of \( \{u_n\}_{n \geq 1} \) in \( L^\infty([0,T];H^{s-1}(\mathbb{T})) \). Let \( (u_{n_1}, u_{n_2}) \in (L^\infty([0,T];H^s(\mathbb{T})))^2 \) to be the solutions generated by \( (u_{0,n_1}, u_{0,n_2}) \), where \( 1 \leq n_1 \leq n_2 \). For simplicity, let us set
\[ w = u_{n_1} - u_{n_2} \quad \text{with} \quad w(0) = u_{0,n_1} - u_{0,n_2}. \]

We first note from Proposition 4.12 that
\[
\| w \|_{L^\infty_T H^{s-1}_x}^2 \leq \| w(0) \|_{H^{s-1}} + T^{1/4}G(\| u_{n_1} \|_{L^\infty_T H^s_x} + \| u_{n_2} \|_{L^\infty_T H^s_x}) \| w \|_{L^\infty_T H^{s-1}_x} \| w \|_{L^\infty_T H^s_x}. \]

Lemma 4.8 second part (4.6) provide the following estimate
\[
\| w \|_{L^\infty_T H^{s-1}_x} \leq \| w(0) \|_{H^{s-1}} + T^{1/4}G(\| u_{n_1} \|_{L^\infty_T H^s_x} + \| u_{n_2} \|_{L^\infty_T H^s_x}) \| w \|_{L^\infty_T H^{s-1}_x}. \]

Let \( j = 1, 2 \). From (5.2) and Lemma 4.8 with \( w_N = 1 \) give the following
\[
\| u_{n_j} \|_{L^\infty_T H^s_x} \leq G(\| u_{n_j} \|_{L^\infty_T H^s_x}) \| u_{n_j} \|_{L^\infty_T H^s_x} < C(\| u_0 \|_{H^s}), \]
where the constant only depends on \( \| u_0 \|_{H^s} \). Thus, together with (5.2) we can conclude that there exist constants \( C(\| u_{0,n_1} \|_{H^s}, \| u_{0,n_2} \|_{H^s}) \) and \( C(\| u_{0,n_1} \|_{H^s}, \| u_{0,n_2} \|_{H^s}) \) depend only on \( \| u_{0,n_1} \|_{H^s}, \| u_{0,n_2} \|_{H^s} \) such that the following holds
\[
\| w \|_{L^\infty_T H^{s-1}_x}^2 \leq \| w(0) \|_{H^{s-1}} + T^{1/4}C(\| w \|_{L^\infty_T H^s_x}) \| w \|_{L^\infty_T H^{s-1}_x} \| w \|_{L^\infty_T H^s_x} \leq \| w(0) \|_{H^{s-1}}. \]

Since \( \{u_{0,n}\}_{n \geq 1} \) is a convergent sequence in \( H^s(\mathbb{T}) \) by assumption, therefore it is convergent in \( H^{s-1}(\mathbb{T}) \). For any \( \varepsilon > 0 \), there exist \( n_* \) large enough such that for all \( n_1 \geq n_* \), we have
\[
\| w \|_{L^\infty_T H^{s-1}_x} < \varepsilon. \]

Namely, we showed that
\[ \{u_n\}_{n \geq 1} \quad \text{is a Cauchy sequence in} \quad C([0,T];H^{s-1}(\mathbb{T})). \]

Thus, the weak* convergence of \( \{u_n\}_{n \geq 1} \) in \( L^\infty([0,T];H^s(\mathbb{T})) \) and the Cauchy property of \( \{u_n\}_{n \geq 1} \) in \( C([0,T];H^{s-1}(\mathbb{T})) \), we can deduce that \( \{u_n\}_{n \geq 1} \) converges to some function
\[ u \in C([0,T];H^s(\mathbb{T})) \cap L^\infty([0,T];H^s(\mathbb{T})). \]

\footnote{For example, we may take \( u_{0,n} = P_{\leq n} \ast u_0 \), it is clear that the sequence \( \{u_{0,n}\}_{n \geq 1} \) belongs to \( H^\infty(\mathbb{T}) \) and converges to \( u_0 \) in \( H^s(\mathbb{T}) \).}
for any $s' < s$. Thus, in view of (1.21), we can pass to the limit on the nonlinear term $f(u_n)$ that

$$\partial_x f(u_n) \overset{P'}{\to} \partial_x f(u),$$

i.e., convergence in the distributional sense. Moreover, the linear part converges in the distributional sense, which we can easily pass to the limit.

- **Strong continuity in $H^s(\mathbb{T})$**

Let $u$ be a solution of (1.17) generated by the initial data $u_0 \in H^s(\mathbb{T})$. Due to the time translation invariance, reversibility in time (invariance by the change of variables $(t, x) \mapsto (-t, -x)$ of (1.17)) and the uniqueness of the solutions, it suffices to check that the continuity of

$$u(t) \in H^s(\mathbb{T}) \quad \text{at} \quad t = 0.$$

Since the limiting object we obtained is

$$u \in C([0, T]; H^{s'}(\mathbb{T})) \cap L^\infty((0, T); H^s(\mathbb{T}))$$

for some $s' < s$. By the following theorem.

**Theorem 5.1** ([80]). Let $V$ and $Y$ be Banach spaces, $V$ reflexive, $V$ a dense subset of $Y$ and the inclusion map of $V$ into $Y$ continuous. Then,

$$C_w([0, T]; Y) \cap L^\infty([0, T]; V) = C_w([0, T]; V).$$

$C_w([0, T]; V)$ denotes the the subspace of $L^\infty([0, T]; V)$ consisting of those functions which are a.e. equal to weakly continuous functions with values in $V$.

It yields that $u(t)$ is is weakly continuous (w.r.t time) in $H^s(\mathbb{T})$ on $[0, T]$. Namely,

$$u \in C_w([0, T]; H^s(\mathbb{T})),
$$

and the following holds

$$\|u_0\|_{H^s} \leq \liminf_{t \to 0} \|u(t)\|_{H^s}. \quad (5.3)$$

On the other hand, we see from (4.9) with $\omega_N \equiv 1$ that

$$\limsup_{t \to 0} \|u(t)\|_{H^s} \leq \|u_0\|_{H^s} \quad (5.4)$$

From (5.3) and (5.4) we conclude that

$$\lim_{t \to 0} \|u(t)\|_{H^s} = \|u_0\|_{H^s}. \quad (5.5)$$

Finally, weak continuity of $u(t)$ in $H^s(\mathbb{T})$ on $[0, T]$ and continuous of norms (5.5) imply:

$$[0, T] \ni t \mapsto u(t) \in H^s(\mathbb{T})$$

is continuous. This finishes the proof that $u \in C([0, T]; H^s(\mathbb{T}))$. 
5.0.3. Continuity with respect to initial data. In this subsection, we show the continuity of the flow map. The proof of continuous dependence on the data in the following is different from the classical Bona–Smith approximation argument (see [12]). This approach is known as the frequency envelope argument, which relies on the following lemma.

Lemma 5.2 ([44]). Suppose that the sequence of spatial functions \( g_n \to g \) in \( H^s(\mathbb{T}) \). Then there exists a sequence \( \{\omega_M\} \) for \( M = 2^{N_0} \) of positive dyadic numbers which satisfies

\[
2^s \omega_M \leq \omega_{2M} \leq 2^{s+1} \omega_M \quad \text{and} \quad \frac{\omega_M}{M^s} \to \infty
\]

such that

\[
\sup_n \sum_M \omega_n^2 \|P_M g_n\|_{L^2}^2 < \infty.
\]

Remark 5.3. In the application of Lemma 5.2, the choice of \( \omega_N \) depends only on the initial data and the approximation sequence of the initial condition.

We make use of the frequency envelope \( \omega_N \). Let \( \{u_n\}_{n \geq 1} \) be a sequence of solutions in \( C([0,T];H^s(\mathbb{T})) \) with

\[
u_{0,n} \to u_0 \quad \text{in} \quad H^s(\mathbb{T}).
\]

We want to prove that the emanating solution \( u_n \) tends to \( u \) in \( C([0,T];H^s(\mathbb{T})) \). Let \( P_{\leq K} u = \sum_{N \leq K} P_N u \). Then, by the triangle inequality, we have the following

\[
\|u_n - u\|_{L_T^\infty H_x^s} \leq \|u_n - P_{\leq K} u_n\|_{L_T^\infty H_x^s} + \|P_{\leq K} (u_n - u)\|_{L_T^\infty H_x^s} + \|P_{\leq K} u - u\|_{L_T^\infty H_x^s}. \tag{5.6}
\]

First, for any fixed \( K \), a direct consequence of Proposition 4.12 gives us a Lipschitz bound of the difference, which tells us that there exists \( n_0 \) such that for \( n \geq n_0 \), the following holds:

\[
\|P_{\leq K} (u_n - u)\|_{L_T^\infty H_x^s} \leq (2K) \|P_{\leq K} (u_n - u)\|_{L_T^\infty H_x^{s-1}} \lesssim K \|u_{0,n} - u_0\|_{H^{s-1}} < \varepsilon. \tag{5.7}
\]

Secondly, for \( K \) large enough, we have

\[
\|P_{\leq K} u - u\|_{L_T^\infty H_x^s} < \varepsilon. \tag{5.8}
\]

Finally, by Lemma 5.2, there exist a dyadic sequence \( \{\omega_N\} \) of positive numbers satisfies \( \omega_N \leq \omega_{2^n} \leq \kappa \omega_N \) for \( n \geq 1 \) (by Remark 4.7, we may assume that \( 1 < \kappa \leq 2 \) and this is equivalent to Lemma 5.2) such that

\[
\|u_0\|_{H_x^s} < \infty, \quad \sup_{n \geq 1} \|u_{0,n}\|_{H_x^s} < \infty, \quad \text{and} \quad \omega_N \to \infty.
\]

For \( K > 0 \) large enough, by applying Proposition 4.9, we have the following

\[
\|P_{\leq K} u_n - u_n\|_{L_T^\infty H_x^s} = \sum_{N > K} \|P_N u_n\|_{L_T^\infty H_x^s} \leq \sup_n \sum_{N > K} \omega_n \|P_N u_n\|_{L_T^\infty H_x^s} \leq \sum_{N > K} \omega_n \|P_N u_n\|_{L_T^\infty H_x^s} \leq \omega_n \|u_n\|_{L_T^\infty H_x^s} \lesssim_K \omega_n \|u_0, n\|_{H_x^s} < \varepsilon. \tag{5.9}
\]

Therefore, we plug (5.7), (5.8), and (5.9) into (5.6), we obtain the continuity of the flow map. The proof of Theorem 1.4 is thus completed in the case \( M = \mathbb{T} \) and \( s \geq \frac{3}{4} \). The proof of Theorem 1.3 on \( \mathbb{T} \) and \( s \geq \frac{3}{4} \) can be done by following the exactly same argument from above three steps.

\footnote{take \( \omega_M = \omega_N N^s \) in view of Lemma 5.2}
6. On limiting behaviour of gFDF

In this section, we will explore the limit behaviour of gFDF. First, we note that solutions to gFDF are $\delta$-dependent. To emphasise that we take the limit on $\delta$, in this section we denote $u_\delta$ and $v_\delta$ to be the solutions. As we have discussed in the introduction, in order to capture the correct shallow-water and deep-water evolutions, we need to consider the different formulations (1.19) and (1.17), separately. The main idea of showing two limit cases is similar. We first establish the Cauchy sequences for the families solutions $\{v_\delta\}_{\delta>0}$ and $\{u_\delta\}_{\delta \geq 1}$ in $C_T H^s$. We know that in the complete metric space there exists some limit $v$ and $u$, correspondingly. Then, we verify the limiting objects are indeed the solutions to the limiting equations, as $\delta \to 0$ and $\delta \to \infty$.

6.1. Deep-water limit on $\mathbb{T}$. We first prove Theorem [1.9] which express the limit behaviour of the gFDF (1.17), as $\delta \to \infty$. We are considering waves in the deep-water region. Without loss of generality, we may assume $\delta_0 = 1$. According to the discussion of linear and nonlinear estimates, we have the uniform estimates in $\delta$, for any $\delta \geq 1$.

Let any $\delta, \gamma \geq 0$, for simplicity we denote $u_\gamma$ and $u_\delta$ are two solutions to gFDF (1.17). Then, the difference $w = u_\gamma - u_\delta$ solves the following equation:

$$
\begin{cases}
  \partial_t w + \mathcal{H}(\partial_x^2 u_\gamma) + K_\delta(\partial_x w) = (K_\gamma - K_\delta)\partial_x u_\gamma + \partial_x (f(u_\gamma) - f(u_\delta)) \\
  w(0) = 0.
\end{cases}
$$

(6.1)

Here, $K_\delta$ is defined in (1.8) and it is convenient to use notation $T_{\delta,\gamma}(u) = K_\gamma(u) - K_\delta(u)$

The goal here is to show that $\{u_\delta\}_{\delta \geq 1}$ is Cauchy in $C([0, T]; H^s(\mathbb{T}))$ for $s \geq \frac{3}{4}$.

**Lemma 6.1.** Let $s \geq \frac{3}{4}$, and any $\delta \geq 1$. Then, the one-parameter family $\{u_\delta\}_{\delta \geq 1}$ is Cauchy in $C([0, T]; H^s(\mathbb{T}))$ as $\delta \to \infty$, where $T$ is the local existence time.

Before we prove Lemma 6.1, we need the following estimate. It is a consequence of Proposition 4.12. Moreover, we find the explicit upper bound in terms of the depth parameters.

**Lemma 6.2.** There exists $C = C(T, \|u_0\|_{H^s(\mathbb{T})}) > 0$ independent of $\delta, \gamma \geq 1$ such that

$$
\sup_{0 \leq t \leq T} \|w(t)\|_{H^{s-1}(\mathbb{T})}^2 \leq C(\frac{1}{\delta^2} + \frac{1}{\gamma})^2.
$$

(6.2)

**Proof.** We notice that $s - 1$ could be negative, therefore we need to check the $L^2$-case. We multiply (6.1) by $w$ and use integration by parts, we obtain:

$$
\frac{d}{dt} \|w(t)\|_{L^2}^2 \leq \int_{\mathbb{T}} T_{\delta,\gamma}(\partial_x u_\gamma) w \, dx + \int_{\mathbb{T}} \partial_x (f(u_\gamma) - f(u_\delta)) \, w \, dx.
$$

(6.3)

By Hölder’s inequality, it follows easily from the definition (1.8) and Lemma 2.3 that

$$
\left| \int_{\mathbb{T}} T_{\delta,\gamma}(\partial_x u_\gamma) \, w \, dx \right| \leq \frac{1}{2}(\frac{1}{\delta^2} + \frac{1}{\gamma})^2 \|u_\gamma\|_{H^1}^2 + \|w(t)\|_{L^2}^2.
$$

Next, we integrate the second term of the RHS of (6.3), we get

$$
\int_0^t \int_{\mathbb{T}} \partial_x (f(u_\gamma) - f(u_\delta)) \, w \, dx \, dt'.
$$
Let us recall the proof of Proposition 4.12 in particular, (4.25)–(4.29). Since here \( u, v \) are two solutions of the same initial data, we can apply Proposition 4.12 to (6.3) and yield
\[
\|w(t)\|_{L^\infty_T L^2_x} \lesssim T(\frac{1}{\gamma} + \frac{1}{\gamma})^2 \|u\|_{L^\infty_T H^2_x}^2 + T \|w(t)\|_{L^\infty_T L^2_x}^2 + T \mathcal{I}G(\|u_\gamma\|_{Z^2_t} + \|u_\delta\|_{Z^2_t})\|w\|_{Z^2_t}\|w\|_{L^\infty_T L^2_x}.
\] (6.4)
Moreover, we recall that LWP result implies that the family \{\( u_\delta \)\}_{\delta \geq 1} is bounded in \( C([0,T];H^s(\mathbb{T})) \) for any \( \frac{3}{4} \leq s \leq 2 \). In particular, there exists a \( \delta \)-independent constant \( M \) such that
\[
\|u_\delta\|_{L^\infty_T H^s_x} \leq \|u_0\|_{H^s} < M.
\]
We can further deduce the following by using Lemma 4.8
\[
\|w(t)\|_{L^\infty_T L^2_x} \lesssim T(\frac{1}{\gamma} + \frac{1}{\gamma})^2 M + T \|w(t)\|_{L^\infty_T L^2_x}^2 + T \mathcal{I}\|w\|_{L^\infty_T L^2_x}.
\] (6.5)
Hence, by Gronwall’s lemma, we get
\[
\sup_{0 \leq t \leq T} \|w(t)\|_{L^2_x}^2 \leq MT(\frac{1}{\gamma} + \frac{1}{\gamma})^2 e^{MT} = C(\frac{1}{\gamma} + \frac{1}{\gamma})^2.
\] (6.6)
Next, we multiply the equation (6.1) with \( D^{s-1} \), then take the \( L^2_x \)-scalar product of the resulting equation with \( D^{s-1}w \), we have the following
\[
\frac{d}{dt} \|D^{s-1}w(t)\|_{L^2_x}^2 \lesssim \left| \int_T (D^{s-1}\{T_{\delta,\gamma}(\partial_x u_\gamma)\}D^{s-1}w)dx \right| + \left| \int_T (D^{s-1}\{\partial_x (f(u_\gamma) - f(u_\delta))\}D^{s-1}w)dx \right|
\]
\[
\lesssim \|D^{s-1}\{T_{\delta,\gamma}(\partial_x u_\gamma)\}\|_{L^2_x}\|D^{s-1}w\|_{L^2_x} + \left| \int_T (D^{s-1}\{\partial_x (f(u_\gamma) - f(u_\delta))\}D^{s-1}w)dx \right|
\]
Lemma 2.3 implies the following
\[
\|D^{s-1}\{T_{\delta,\gamma}(\partial_x u_\gamma)\}\|_{L^2_x}\|D^{s-1}w\|_{L^2_x} \leq \frac{1}{2}\|T_{\delta,\gamma} u_\gamma\|_{H^s}^2 + \frac{1}{2}\|w\|_{H^{s-1}}^2
\]
\[
\leq C(\frac{1}{\gamma} + \frac{1}{\gamma})^2 + \frac{1}{2}\|w\|_{H^{s-1}}^2.
\]
By using the same discussion as we obtained for (6.4), we can estimate the following
\[
\left| \int_0^T \int_T (D^{s-1}\{\partial_x (f(u_\gamma) - f(u_\delta))\})D^{s-1}w\, dx\, dt \right| \leq CT^\frac{1}{\gamma}\|w\|_{L^\infty_T H^{s-1}_x}^2,
\]
where Proposition 4.12 was applied. Finally, we follow the same argument as in the \( L^2 \) case (6.5), there exists some \( \theta > 0 \)
\[
\|w(t)\|_{L^\infty_T H^{s-1}_x} \leq T^\theta \|w\|_{L^\infty_T H^{s-1}_x} + CT(\frac{1}{\gamma} + \frac{1}{\gamma})^2
\]
Again, from Gronwall’s lemma, we have
\[
\sup_{0 \leq t \leq T} \|w(t)\|_{H^{s-1}_x} \leq MT(\frac{1}{\gamma} + \frac{1}{\gamma})^2 e^{MT} = C(\frac{1}{\gamma} + \frac{1}{\gamma})^2.
\] (6.7)
Hence, we conclude (6.2) from (6.6) and (6.7).

Proof of Lemma 6.7 By the triangle inequality, we first write the following:
\[
\|u_\delta - u_\gamma\|_{C_T H^s_x} \lesssim \|u_\delta - P_{\leq K} u_\delta\|_{C_T H^s_x} + \|P_{\leq K} u_\delta - P_{\leq K} u_\gamma\|_{C_T H^s_x}
\]
\[
+ \|P_{\leq K} u_\gamma - u_\gamma\|_{C_T H^s_x}
\]
We first fix $K$ large enough. Let $\eta > 0$ and the same argument we have seen in (5.9), we can deduce the following:

$$\left\|u_\delta - P_{\leq K}u_\delta\right\|_{C_T H^2_\delta} < \frac{\eta}{3}, \quad \left\|P_{\leq K}u_\gamma - u_\gamma\right\|_{C_T H^2_\delta} < \frac{\eta}{3}.$$  

Notice that Lemma 6.2 implies that for all $\delta, \gamma$ such that $1 \leq \delta \leq \gamma$, there exists a constant $C = C(T, \|u_0\|_{H^s})$ independent of $\delta$ and $\gamma$ such that for any $K$,

$$\left\|P_{\leq K}(u_\delta) - P_{\leq K}(u_\gamma)\right\|_{C_T H^2_\delta} \leq 2K \left\|u_\delta - u_\gamma\right\|_{C_T H^{s-1}_\delta} < \frac{CK}{\delta^2}$$

provided that same initial data such that $w(0) = 0$. Now, we choose $K = \delta^2$, so that as $\delta \to \infty$

$$\left\|u_\delta - u_\gamma\right\|_{C_T H^2_\delta} < \eta.$$

\[\square\]

To complete the proof of Theorem 1.9, we first note that from Lemma 6.1, for $s \geq \frac{3}{4}$, there exists $u \in C([0, T]; H^s(\mathbb{T}))$ such that $u_\delta$ converges to $u$ in $C([0, T]; H^s(\mathbb{T}))$, as $\delta \to \infty$. We now show $u$ is indeed a solution to gBO. We notice for any $\delta \geq 1$, the following bound is true

$$\left\|K_\delta(\partial_x u_\delta)\right\|_{H^{s-1}_\delta} \leq \frac{\lambda}{3} \left\|u_\delta\right\|_{H^s} \leq \frac{c}{\delta},$$

(6.8)

for some absolute constant $c$. Then, it is readily seen that $u$ is actually the solution of the gBO corresponding to the initial data $u_0$. Indeed, we have seen from (1.7) that we can write gFDF as perturbed gBO:

$$\partial_t u_\delta + H(\partial_x^2 u_\delta) + \partial_x(f(u_\delta)) + K_\delta(\partial_x u_\delta) = 0.$$  

We have the almost everywhere convergence of the linear part:

$$\partial_t u_\delta + H\partial_x^2 u_\delta \xrightarrow{\text{P}} \partial_t u + H\partial_x^2 u,$$

i.e. convergent in the distributional sense, as $\delta \to \infty$. Also, by the analyticity of $f(\cdot)$, we have $f(u_\delta)$ converges to $f(u)$ in $C([0, T]; H^s(\mathbb{T}))$ as $\delta \to \infty$. Moreover, by (6.8), $K_\delta(\partial_x u_\delta)$ vanishes as $\delta \to \infty$. Therefore, it is enough to conclude our claim.

### 6.2. Shallow-water limit on $T$.

We now express the limit behaviour of (scaled) gFDF as $\delta \to 0$. Without loss of generality, we can assume that the depth characteristic parameter satisfies $\delta \leq 1$.

Similar as in the previous section, we first let $\delta, \gamma > 0$ and we denote $v_\delta$ and $v_\gamma$ are two solutions to (scaled) gFDF (1.19). Then, the difference $w = v_\delta - v_\gamma$ solves the following equation:

$$\left\{\begin{array}{l}
\partial_t w + H_\delta(\partial_x w) = (H_\gamma - H_\delta)\partial_x v_\gamma - \partial_x(f(v_\delta) - f(v_\gamma)) \\
w(0) = 0.
\end{array}\right. \quad (6.9)$$

Here, $H_\delta$ is defined by the Fourier multiplier:

$$- H_\delta := n^2 \frac{h(n, \delta)}{\delta} \quad (6.10)$$

and $h(n, \delta)$ is defined in Lemma 2.1 similar definition apply to $H_\gamma$. Moreover, for convenience we denote

$$L_{\delta, \gamma} = H_\gamma - H_\delta.$$
We now show \( \{v_\delta\}_{\delta > 0} \) is Cauchy \( C([0,T]; H^s(\mathbb{T})) \) for \( s \geq \frac{2}{3} \).

**Lemma 6.3.** Let \( s \geq \frac{2}{3} \) and local existence time \( T > 0 \). Then, the one-parameter family \( \{v_\delta\}_{\delta > 0} \) is Cauchy in \( C([0,T]; H^s(\mathbb{T})) \) as \( \delta \to 0 \).

**Proof.** We start with \( s = 0 \). Multiply (6.9) by \( w \) and integrate over \( \mathbb{T} \). Then, we follow the same argument as we see for (6.3), by using Proposition 4.12 and Lemma 4.8, we reach the relation
\[
\|w(t)\|_{L^2_T L^2_x}^2 \lesssim T^{1/4}\|w(t)\|_{L^2_T L^2_x}(\|v_\delta\|_{L^2_T L^2_x} + \|v_\gamma\|_{L^2_T L^2_x} + 1) + T\|L_{\delta,\gamma}(\partial_x v_\delta)\|_{L^2_T L^2_x}^2.
\] (6.11)

In order to conclude (6.11), we need the following observations. From (6.10), we have
\[
\|L_{\delta,\gamma}(\partial_x v_\delta)\|_{L^2}^2 \leq C \sum_{n \in \mathbb{N}} n^6 \left( \frac{h^2(n,\delta)}{\delta^2} + \frac{h^2(n,\gamma)}{\gamma^2} \right) |\hat{v}_\delta(n)|^2.
\]

Next, from LWP result we know that for \( s > \frac{2}{3} \) and solution \( u = v_\delta \) to the equation (1.17) satisfies the \( \delta \)-independent bound
\[
\|v_\delta\|_{L^\infty_T H^s_x} \leq C
\]
where \( T \) is the local existence time we obtained from before and the constant \( C = C(T, \|v_0\|_{H^s}) \) is independent of \( \delta \in (0, \delta_0) \). Then, together with Lebesgue’s dominated convergence theorem and Remark 2.2, we obtain that
\[
\|L_{\delta,\gamma}(\partial_x v_\delta)\|_{L^2}^2 \leq c(\delta).
\] (6.12)

Here, we have used that \( 0 < \gamma < \delta \) and \( c(\delta) \to 0 \) as \( \delta \to 0 \). Then, by Gronwall’s lemma applied to (6.11) gives
\[
\|w(t)\|_{L^2_T L^2_x}^2 \leq c_1(\delta),
\]
where \( c_1(\delta) \to 0 \) as \( \delta \to 0 \) in light of (6.12).

By a similar argument for \( s = 0 \) and using Proposition 4.12 we have for any fixed \( K \) so that
\[
\|P_{\leq K} w(t)\|_{L^\infty_T H^s_x} \leq 2K \|P_{\leq K} w(t)\|_{L^\infty_T H^{s-1}_x} \lesssim Kc_2(\delta).
\]
Moreover, from triangle inequality we can conclude for \( s \geq \frac{2}{3} \), the following holds
\[
\|w(t)\|_{L^\infty_T H^s}^2 = \|v_\delta - v_\gamma\|_{L^\infty_T H^s}^2 \leq \|v_\delta - P_{\leq K}(v_\delta)\|_{C_T H^s_x} + \|P_{\leq K}(v_\delta) - P_{\leq K}(v_\gamma)\|_{C_T H^s_x} + \|P_{\leq K}(v_\gamma) - v_\gamma\|_{C_T H^s_x} \leq KC(\delta),
\]
where we choose \( K = C(\delta)^{-\frac{1}{2}} \) so that \( KC(\delta) \to 0 \) as \( \delta \to 0 \). Hence, it implies \( \{v_\delta\}_{\delta > 0} \) is Cauchy in \( C([0,T]; H^s(\mathbb{T})) \).

□

We proceed with the proof of Theorem 1.9. The following proposition is the final result regarding the convergence of gFDF solutions to those of gKdV.

**Proposition 6.4.** Let \( s \geq \frac{2}{3} \), \( v_0 \in H^s(\mathbb{T}) \), and \( v_\delta \) denote the solution of (1.19) with initial data \( u_0 \). Then, for any \( 0 < T < 1 \), \( v_\delta \to v \) in \( C([0,T]; H^s(\mathbb{T})) \) as \( \delta \to 0 \), where \( v \) is the solution of the gKdV (1.20) with initial data \( u_0 \).
Proof. Let \( v_{0,\varepsilon} \) be the specific smooth approximation \(^{11}\) of \( v_0 \) as defined in Section 5. Let \( v_{\delta,\varepsilon} \) denote the solution of (1.19) with initial data \( v_{0,\varepsilon} \) and let \( v_\varepsilon \) denote the solution of the Cauchy problem for the gKdV (1.20). Then, Lemma 6.3 implies that
\[
\lim_{\delta \to 0} \| v_{\delta,\varepsilon} - v_\varepsilon \|_{C_TH^s} = 0. 
\] (6.13)
Indeed, first of all \( \{ v_{\delta,\varepsilon} \}_{\delta > 0} \) is Cauchy in \( C([0,T];H^s(\mathbb{T})) \), which follows directly from Lemma 6.3. Therefore, \( \{ v_{\delta,\varepsilon} \}_{\delta > 0} \) converges to some function
\[
v_\varepsilon' \in C([0,T];H^s(\mathbb{T})) \quad \text{as} \quad \delta \to 0.
\]
Moreover, one application of Lemma 2.1 so that
\[
h(n,\delta)/\delta \to 0 \quad \text{as} \quad \delta \to 0,
\]
uniformly for \( n \) in any bounded set. Therefore, we can deduce that \( v_\varepsilon' \) is a solution to the Cauchy problem (1.20). Thus, by the uniqueness that \( v_\varepsilon' = v_\varepsilon \) and hence we have that solution \( v_{\delta,\varepsilon} \) converges to solution \( v_\varepsilon \) in \( C([0,T];H^s(\mathbb{T})) \), as \( \delta \to 0 \).

Finally, we show the convergence of the solutions of the gFDF as scaled in (1.19), to an associated solution of the gKdV (1.20).

Let us fix \( v_0 \in H^s(\mathbb{T}) \) for \( s \geq \frac{3}{2} \), and let \( v_{0,\varepsilon} \) be the smooth approximations to \( v_0 \) defined in Section 5. Let \( v_\varepsilon \) denote the solution of the gKdV (1.20) with initial data \( v_{0,\varepsilon} \) and let \( v \) denote the solution with the initial data \( v_0 \). The theory developed in \[63\] implies that for any \( s \geq \frac{3}{2} \) and \( 0 < T < 1 \),
\[
v_\varepsilon \to v \quad \text{in} \quad C([0,T];H^s(\mathbb{T})),
\]
as \( \varepsilon \to 0 \). Let \( v_{\delta,\varepsilon} \) denote the solution of the gFDF (1.19) with initial data \( v_{0,\varepsilon} \) and \( v_\delta \) denote the solution with initial data \( v_0 \). It is also known from the theory developed in Section 5 that implies
\[
v_{\delta,\varepsilon} \to v_\delta \quad \text{in} \quad C([0,T];H^s(\mathbb{T})),
\]
as \( \varepsilon \to 0 \). Moreover, the \( \delta \)-independent bounds on the solution \( v_\delta \) such that \( \| v_\delta \|_{C_TH^s_\varepsilon} \leq C \), which applies to solutions of (rescaled) gFDF (1.19). We can trace through the proofs that \( v_{\delta,\varepsilon} \to v_\delta \), uniformly for \( \delta \in (0,1] \), say. In particular, we see the following discussion.

Introduce the families of functions \( \{ v_{\delta,\varepsilon} \}_\varepsilon \) and \( \{ v_\varepsilon \}_\varepsilon \). Then, by the triangle
\[
\| v_\delta - v \|_{C_TH^s_\varepsilon} \leq \| v_\delta - v_{\delta,\varepsilon} \|_{C_TH^s_\varepsilon} + \| v_{\delta,\varepsilon} - v_\varepsilon \|_{C_TH^s_\varepsilon} + \| v_\varepsilon - v \|_{C_TH^s_\varepsilon}
\]
Let \( \eta > 0 \) be given. We can \( \varepsilon > 0 \) sufficient small such that
\[
\| v_{\delta,\varepsilon} - v_\varepsilon \|_{C_TH^s_\varepsilon} + \| v_\varepsilon - v \|_{C_TH^s_\varepsilon} \leq \eta
\]
for all \( \delta \in (0,1] \). Such a choice is possible because of the following uniform convergence:
\[
v_{\delta,\varepsilon} \to v_\delta \quad \text{as} \quad \varepsilon \to 0.
\]
Let \( \varepsilon > 0 \) now is fixed. Then, by (6.13) \( \{ v_{\delta,\varepsilon} \}_{\delta > 0} \) is Cauchy in \( \delta \), we have
\[
\lim_{\delta \to 0} \| v_\delta - v \|_{C_TH^s_\varepsilon} \leq \eta + \lim_{\delta \to 0} \| v_{\delta,\varepsilon} - v_\varepsilon \|_{C_TH^s_\varepsilon} = \eta
\]
As \( \eta > 0 \) was arbitrary, we can take it to be sufficiently small. Therefore, we obtain
\[
v_\delta \to v \quad \text{in} \quad C([0,T];H^s(\mathbb{T}))
\]
as \( \delta \to 0 \), which is what we desired for.

\(^{11}\)We can use the standard mollifier so that \( v_{\delta,\varepsilon} := \rho_\varepsilon \ast u_\delta \).
Appendix A. On the discussion of limiting behaviour on $\mathbb{R}$

In the following section, we will briefly discuss the well-posedness results and limiting behaviour for the gFDF on $\mathbb{R}$. The crucial part is to obtain a improved Strichartz estimates on smooth solutions to the gFDF of the forms (1.19) and (1.17), in the shallow-water and deep-water regions, separately.

A.1. Short-time Strichartz estimates on $\mathbb{R}$. In this subsection, we seek to prove a refined Strichartz estimate for solutions to the linearized equation with a general source term. For simplicity, we may assume $\delta_0 = 1$ in the rest of the section. The proof we present here is just a slight modification of the arguments already shown in [44, 66, 67].

Before getting into the details, let us recall the classical smoothing effect derived in [35]. Moreover, we make the following useful observations so that we can roughly see the situation in the shallow-water region would be better.

\[ \|e^{-ig_0\partial_x^2}((g_0\partial_x^2)u)^{\frac{1}{2}}u\|_{L^1_tL^\infty_x} \lesssim \|u_0\|_{L^2}. \]  

(A.1)

Here, the implicit constant is independent of $\delta$.

(i) In the shallow-water region $(0 < \delta < 1)$, we consider the dispersion symbol:

\[ G_\delta_1(\partial_x^2 v) = \frac{3}{2\pi^2} G_\delta(\partial_x^2 v), \]

and moreover in the view of (2.6), we have $G_\delta_1\partial_x^{n''} \lesssim D_x$.

(ii) In the deep-water region $(1 \leq \delta < \infty)$, we consider the dispersion symbol:

\[ G_\delta_2(\partial_x^2 u) = G_\delta(\partial_x^2 u) \]

and similarly in the view of (2.7), we have $G_\delta_2\partial_x^{n''} \lesssim 1$.

We notice that in part (ii), we have no smoothing in the sense that the above equation (A.1) has no derivative gain.

Let us state our refined Strichartz estimate for the shall-water surface case first. The following lemma is almost the same idea from [72, Lemma 2.9].

Lemma A.1. Let $T > 0$, $\delta \in (0,1)$, and consider $\kappa \geq 0$ to be a fixed parameter. Let $v(t,x)$ to be any solution defined on $[0,T]$ to the following linear equation

\[ \partial_t v + \frac{3}{2\pi^2} G_\delta(\partial_x^2 v) = F. \]  

(A.2)

Then, there exists $\kappa_1, \kappa_2 \in \left(\frac{1}{4}, \frac{1}{2}\right)$ such that, for any $\theta > 0$, the following inequality holds

\[ \|v\|_{L^2_tL^\infty_x} \lesssim T^{\kappa_1} \|D_x^{-\frac{1}{2}}D_x^{n+\theta} v\|_{L^\infty_t L^2_x} + T^{\kappa_2} \|D_x^{-\frac{1}{2}}D_x^{-\frac{3n}{4}+\theta} F\|_{L^\infty_t L^2_x}. \]  

(A.3)

Here, the implicit constant is independent of $\delta$.

Proof. Let $v(t,x)$ be a solution to equation (A.2) defined on $[0,T]$. We use a nonhomogeneous Littlewood-Paley decomposition for the solution, that is, we write $v = \sum_N v_N$, where $v_N = P_N v$, and $N$ is a nonhomogeneous dyadic number. For simplicity, in the following, we shall also use the notation $v_N$ for $P_N v$. At this point, it is important to notice that, on the one hand, from Minkowski inequality, we know that
Then, for all \( \theta > 0 \). While on the other hand, by using the low-frequency projector \( P_{\leq 1} \), from Hölder and Bernstein inequalities we see that
\[
\|P_{\leq 1}v\|_{L_t^2_L_x^\infty} \lesssim T^{1/2}\|P_{\leq 1}v\|_{L_t^{\infty}_L_x^2}.
\]
Then, from the inequalities above we infer that it is enough to show that, for any \( \kappa > 0 \) and any \( N > 1 \) dyadic number, the following holds
\[
\|v_N\|_{L_t^2L_x^\infty} \lesssim T^\kappa \|D_x^{1/3}D_x^2v_N\|_{L_t^\infty L_x^2} + T^{\kappa'2} \|D_x^{-1/4}D_x^{-3/4}F_N\|_{L_t^2L_x^2}.
\] (A.4)
Now, in order to prove (A.4), we chop the time-interval \([0,T]\) into several pieces of length \( T^{\kappa'}N^{-\kappa} \), where \( \kappa' > 0 \) stands for a small number that shall be fixed later. In particular, we have the following:
\[
[0,T] = \bigcup_{j \in J} I_j, \quad \text{where } I_j := [a_j, b_j], \quad |I_j| \sim T^{\kappa'}N^{-\kappa}, \quad \#J \sim T^{-1-\kappa'}N^\kappa.
\]
On the other hand, notice that \( v_N(t) \) solves the integral equation
\[
v_N(t) = e^{-(t-a_j)}\frac{1}{2\pi}\mathcal{G}_s\delta^2\phi(v_N(a_j)) + \int_{a_j}^{t} e^{-(t-t')}\frac{1}{2\pi}\mathcal{G}_s\delta^2\phi F_N(t')dt',
\]
for all \( t \in I_j \). Therefore, by using the classical Strichartz estimate (A.1), as well as Hölder in time, and Bernstein inequalities, we obtain
\[
\|v_N\|_{L_t^2L_x^\infty} = \left( \sum_j \|v_N\|_{L_t^2L_x^\infty}^2 \right)^{1/2} \leq \left( T^\kappa N^{-\delta} \right)^{1/4} \left( \sum_j \|v_N\|_{L_t^2L_x^\infty}^2 \right)^{1/2} \leq \left( T^{\kappa'}N^{-\kappa} \right)^{1/4} \left( \sum_j \|D_x^{-1/4}v_N(a_j)\|_{L_x^2}^2 \right)^{1/2} + \left( T^{\kappa'}N^{-\kappa} \right)^{1/4} \left( \sum_j \left\| \int_{a_j}^{t} e^{-(t-t')}\frac{1}{2\pi}\mathcal{G}_s\delta^2\phi F_N(t')dt' \right\|_{L_t^2}^2 \right)^{1/2} \leq \left( T^{\kappa'}N^{-\kappa} \right)^{1/4} \left( T^{-1-\kappa'}N^{\kappa} \right)^{1/2} \|D_x^{-1/4}v_N\|_{L_t^\infty L_x^2} + \left( T^{\kappa'}N^{-\kappa} \right)^{1/4} \left( \sum_j \left( T^{\kappa'}N^{-\kappa} \int_{I_j} \|D_x^{-1/4}F_N\|_{L_x^2}^2 dt \right)^{1/2} \right) \leq T^{1/2-\kappa'/4} \|D_x^{-1/4}D_x^{1/4}v_N\|_{L_t^\infty L_x^2} + T^{3\kappa'/4} \|D_x^{-1/4}D_x^{-3/4}F_N\|_{L_t^2L_x^2},
\]
what concludes the proof of (A.3) by choosing, for example, \( \kappa' = \frac{1}{2} \). \( \square \)

We now deal with the equation (1.17), and we shall recall the standard Strichartz estimates from [27, Lemma 3.5].

**Lemma A.2.** Let \( 1 \leq \delta < \infty \), \( (q,r) \) is admissible such that \( 2 \leq q, r \leq \infty \), and \( \frac{2}{q} = \frac{1}{2} - \frac{1}{r} \). Then, for all \( \phi \in L^2(\mathbb{R}) \), we have
\[
\|e^{-i\mathcal{G}_s\delta^2}\phi\|_{L_t^2L_x^2} \lesssim \|\phi\|_{L^2}.
\] (A.5)
Here, the implicit constant is independent of $\delta$.

**Lemma A.3.** Let $T > 0$, $\delta \in [1, \infty)$, and $u(t, x)$ to be any solution defined on $[0, T]$ to the following linear equation

$$\partial_t u + G_\delta \partial_x^2 u = F.$$  

Moreover, suppose that $(q, r) \in \mathbb{R}^2$ is admissible and $2 \leq p \leq q$. Then, for any $\theta > 0$, the following inequality holds

$$\|u\|_{L^p_t L^r_x} \lesssim \|D^\frac{1}{q} v\|_{L^p_t L^2_x} + \|D^\frac{1}{q} F\|_{L^p_t L^2_x}.$$  

Here, the implicit constant is independent of $\delta$.

**Proof.** This proof is essentially the same as Lemma A.1. We briefly go over it again.

It is enough to show that, for any any $N > 1$ dyadic number, the following holds

$$\|u_N\|_{L^p_t L^r_x} \lesssim N^\frac{1}{q} \|u_N\|_{L^p_t L^2_x} + N^\frac{1}{q} \|F_N\|_{L^p_t L^2_x}.$$  

For each $N$, it is enough to have

$$[0, T] = \cup_{j \in J_N} I_{j, N}, \quad \text{where} \quad I_{j, N} := [a_j, b_j], \quad |I_{j, N}| \sim N^{-1}, \quad \text{and} \quad \#J \sim N.$$  

On the other hand, notice that $u_N(t)$ solves the integral equation

$$u_N(t) = e^{-(t-a_j)G_\delta \partial_x^2} u_N(a_j) + \int_{a_j}^t e^{-(t-t')G_\delta \partial_x^2} F_N(t') dt',$$  

for all $t \in I_{j, N}$. Then, from the triangle inequality we have

$$\|u_N\|_{L^p_t L^r_x} \lesssim \left( \sum_j \|e^{-(t-a_j)G_\delta \partial_x^2} u_N(a_j)\|_{L^p (I_{j, N}; L^r_x)}^p \right)^{\frac{1}{p}}$$

$$+ \left( \sum_j \left\| \int_{a_j}^t e^{-(t-t')G_\delta \partial_x^2} F_N(t') dt' \right\|_{L^p (I_{j, N}; L^r_x)}^p \right)^{\frac{1}{p}}$$

$$:= I + II.$$  

Now, the linear term we use Hölder in time and (A.5),

$$I \leq \left( \sum_j |I_{j, N}|^{1-\frac{p}{q}} \|e^{-(t-a_j)G_\delta \partial_x^2} u_N(a_j)\|_{L^p (I_{j, N}; L^r_x)}^p \right)^{\frac{1}{p}}$$

$$\lesssim \left( \sum_j |I_{j, N}|^{1-\frac{p}{q}} \|u_N(a_j)\|_{L^p_x}^p \right)^{\frac{1}{p}} \sim N^\frac{1}{q} \left( \sum_j \|u_N(a_j)\|_{L^2_x}^p \right)^{\frac{1}{p}}$$

$$\lesssim N^\frac{1}{q} \left( \sum_j \int_{I_{j, N}} \|u_N(t')\|_{L^2_x}^p dt' \right)^{\frac{1}{p}} = N^\frac{1}{q} \|u_N\|_{L^p_t L^2_x}.$$
Next, for the Duhamel term we use Hölder in time, Minkowski, and (A.5),

\[
\Pi \leq \left( \sum_j |I_{j,N}|^{-\frac{2}{q}} \left\| \int_{t_j}^{t} e^{-(t-t')\delta^2} F_N(t') dt' \right\|_{L^q(I_{j,N};L^2)} \right)^{\frac{1}{p}} \\
\leq \left( \sum_j |I_{j,N}|^{-\frac{2}{q}} \left\{ \int_{I_{j,N}} \left\| e^{-(t-t')\delta^2} F_N(t') \right\|_{L^q(I_{j,N};L^2)} dt' \right\}^{p} \right)^{\frac{1}{p}} \\
\lesssim \left( \sum_j |I_{j,N}|^{-\frac{2}{q}} \left\{ \int_{I_{j,N}} \left\| F_N(t') \right\|_{L^2} dt' \right\}^p \right)^{\frac{1}{p}} \sim \left( \sum_j |I_{j,N}|^{-\frac{2}{q}} \left\| F_N(t') \right\|_{L^1(I_{j,N};L^2)}^p \right)^{\frac{1}{p}} \\
\lesssim \left( \sum_j |I_{j,N}|^{-\frac{2}{q}+\frac{p-1}{p}} \left\| F_N(t') \right\|_{L^p(I_{j,N};L^2)}^p \right)^{\frac{1}{p}} \\
\sim \mathcal{N}^{\frac{1}{q}-1} \left( \sum_j \left\| F_N(t') \right\|_{L^p(I_{j,N};L^2)}^p \right)^{\frac{1}{p}} = \mathcal{N}^{\frac{1}{q}-1} \left\| F_N(t') \right\|_{L^p L^2}. 
\]

Following a similar argument as in Proposition 4.9, we can obtain δ-independent nonlinear estimates. In the case of the shallow-water region, one can share the derivative with two functions. On the other hand, in the case of an infinitely deep-water situation, we see that if we apply Lemma A.1 directly, there is no smoothing effect regarding the equation (A.1). However, we can use the alternative classical Strichartz estimates on the real line (A.5) and obtain the Lemma A.3, which allows us to share the derivative with more than two functions.

Together with energy estimates and the estimates for the difference between two solutions, we can construct our solution on \( \mathbb{R} \) similarly. Moreover, we can conclude our convergence results on \( \mathbb{R} \) by running the same convergent argument.

Acknowledgments. The author would like to thank his advisors Tadahiro Oh and Yuzhao Wang, for proposing this problem and for their continuing support. The author would also like to thank Tomoyuki Tanaka for many useful discussions. G.L. were supported by The Maxwell Institute Graduate School in Analysis and its Applications, a Centre for Doctoral Training funded by the UK Engineering and Physical Sciences Research Council (Grant EP/L016508/01), the Scottish Funding Council, Heriot-Watt University and the University of Edinburgh.

References

[1] L. Abdelouhab, J. L. Bona, M. Felland, J.-C. Saut, Nonlocal models for nonlinear, dispersive waves, Phys. D 40 (1989), no. 3, 360–392.
[2] M. J. Ablowitz, A. S. Fokas, J. Satsuma, H. Segur, On the periodic intermediate long wave equation, J. Phys. A 15 (1982), no. 3, 781–786.
[3] M. J. Ablowitz, H. Segur, Solitons and the inverse scattering transform, SIAM Studies in Applied Mathematics, 4. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981. x+425 pp.
[4] J. P. Albert, J. F. Toland, On the exact solutions of the intermediate long-wave equation, Differential Integral Equations 7 (1994), no. 3-4, 601–612.
[5] A. Babin, A. Ilyin, E. Titi, On the regularization mechanism for the periodic Korteweg-de Vries equation, Comm. Pure Appl. Math. 64 (2011), no. 5, 591–648.
72 T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech. 29 (1967), 559–592.

[6] D. J. Benney, *Long nonlinear waves in fluid flows*, J. Math. and Phys. 45 (1966), 52–63.

[7] B.K. Berntson, E. Langmann, J. Lenells, *Non-chiral Intermediate Long Wave equation and interedge effects in narrow quantum Hall systems*, Phys. Rev. B 102 (2020), no.15, 155308-155322.

[8] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. 3 (1993), no. 2, 107–156.

[9] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation*, Geom. Funct. Anal. 3 (1993), no. 3, 209–262.

[10] J. Bourgain, *Periodic Korteweg de Vries equation with measures as initial data*, Selecta Math. (N.S.) 3 (1997), no. 2, 115–159.

[11] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation*, Geom. Funct. Anal. 3 (1993), no. 3, 209–262.

[12] J. Bourgain, *Periodic Korteweg de Vries equation with measures as initial data*, Selecta Math. (N.S.) 3 (1997), no. 2, 115–159.

[13] J. L. Bona, R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A 278 (1975), no. 1287, 555–601.

[14] J. Bromwich, *An Introduction to the Theory of Infinite Series*, Second edition, (1962) Macmillan.

[15] N. Burq, F. Planchon, *The Benjamin-Ono equation in energy space. Phase space analysis of partial differential equations*, 55–62, Progr. Nonlinear Differential Equations Appl., 69, Birkhäuser Boston, Boston, MA, 2006.

[16] M. Christ, J. Colliander, T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. 125 (2003), no. 6, 1235–1293.

[17] H. H. Chen, Y. C. Lee, *Internal-wave solitons of fluids with finite depth*, Phys. Rev. Lett. 43 (1979), no. 4, 264–266.

[18] D.R. Christie, K. Muirhead, A. Hales, *On solitary waves in the atmosphere*, J. Atmos. Sci. 35 (1978), no. 5, 805–825.

[19] W. Craig, P. Gueymer, H. Kalisch, *Hamiltonian long-wave expansions for free surfaces and interfaces*, Comm. Pure Appl. Math. 58 (2005), no.12, 1587–1641.

[20] G. Fonseca, F. Linares, G. Ponce, *Global well-posedness for the modified Korteweg-de Vries equation*, Comm. Partial Differential Equations 24 (1999), no. 3-4, 683–705.

[21] P. Gérard, T. Kappeler, P. Topalov, *Sharp well-posedness results of the Benjamin-Ono equation in H^s(T,R) and qualitative properties of its solution*, to appear in Acta. Math.

[22] A. Grünrock, *A bilinear Airy-estimate with application to gKdV-3*, Differential Integral Equations 18 (2005), no. 12, 1333–1339.

[23] Z. Guo, *Local well-posedness and a priori bounds for the modified Benjamin-Ono equation*, Adv. Differential Equations 16 (2011), no. 11-12, 1087–1137.

[24] Z. Guo, *Local well-posedness for dispersion generalized Benjamin-Ono equations in Sobolev spaces*, J. Differential Equations 252 (2012), no. 3, 2053–2084.

[25] P. Gérard, T. Kappeler, P. Topalov, *Sharp well-posedness results of the Benjamin-Ono equation in H^s(T,R) and qualitative properties of its solution*, preprint (2020), arXiv:2004.0485, to appear in Acta. Math.

[26] Z. Guo, B. Wang, *Global well-posedness and limit behavior for the modified finite-depth-fluid equation*, arXiv:0809.2318 [math.AP].

[27] Z. Guo, Y. Lin, L. Molinet, *Well-posedness in energy space for the periodic modified Benjamin-Ono equation*, J. Differential Equations 256 (2014), no. 8, 2778–2806.

[28] M. Ionescu, C.E. Kenig, *Global well-posedness of the Benjamin-Ono equation in low-regularity spaces*, J. Amer. Math. Soc. 20 (2007), no.3, 753–798.

[29] M.Ifrim, D. Tataru, *Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation*, Ann. Sci. Éc. Norm. Supér. (4) 52 (2019), no. 2, 297–335.

[30] R. Iório Jr., *On the Cauchy problem for the Benjamin-Ono equation*, Comm. Partial Differential Equations 11 (1986), no. 10, 1031–1081.

[31] R.I. Joseph, *Solitary waves in a finite depth fluid*, J. Phys. A 10 (1977), no. 12, 225–227.

[32] R. I. Joseph, R. Egri, *Multi-soliton solutions in a finite depth fluid*, J. Phys. A 11(1978), no.5 97–102.

[33] C.E. Kenig, K.D. Koenig, *On the local well-posedness of the Benjamin-Ono and modified equations*, Math. Res. Lett. 10 (2003), no. 5-6, 879–896.
[35] C. E. Kenig, G. Ponce, L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J. 40 (1991), no. 1, 33–69.

[36] C. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620.

[37] C. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996), no. 2, 573–603.

[38] C. Kenig, G. Ponce, L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Math. J. 106 (2001), no. 3, 617–633.

[39] C.E. Kenig, H. Takaoka, Global wellposedness of the modified Benjamin-Ono equation with initial data in $H^{1/2}$, Int. Math. Res. Not. 2006, Art. ID 95702, 44 pp.

[40] N. Kishimoto, Unconditional uniqueness for the periodic modified Benjamin-Ono equation by normal form approach, Int. Math. Res. Not. 2021.

[41] N. Kishimoto, Unconditional uniqueness for the periodic Benjamin-Ono equation by normal form approach, J. Hyperbolic Differ. Equ. 18 (2021), no. 4, 931–984.

[42] R. Killip, M. Visan, KdV is well-posed in $H^{-1}$, Ann. of Math. (2) 190 (2019), no. 1, 249–305.

[43] C.G. Koop, G. Butler, An investigation of internal solitary waves in a two-fluid system, J. Fluid Mech. 112 (1981) 225–251.

[44] H. Koch, N. Tzvetkov, On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$, Int. Math. Res. Not. 2003, no. 26, 1449–1464.

[45] B. Kupershmidt, Involutivity of conservation laws for a fluid of finite depth and Benjamin-Ono equations, Libertas Math. 1 (1981) 125–132.

[46] T. Kubota, D.R.S. Ko, L.D. Dobbs, Weakly-Nonlinear, Long Internal Gravity Waves in Stratified Fluids of Finite Depth, J. Hydronautics 12 (1978), no.4, 157–165.

[47] S. Kwon, T. Oh, H. Yoon, Normal form approach to unconditional well-posedness of nonlinear dispersive PDEs on the real line, Ann. Fac. Sci. Toulouse Math. (6) 29 (2020), no. 3, 649–720.

[48] K. Kim, R. Schippa, Low regularity well-posedness for generalized Benjamin-Ono equations on the circle, J. Hyperbolic Differ. Equ. 18 (2021), no. 4, 931–984.

[49] Y. Kodama, J. Satsuma, M.J. Ablowitz, Nonlinear intermediate long-wave equation: analysis and method of solution, Phys. Rev. Lett. 46 (1981), no.11, 687–690.

[50] D.R. Lebedev, A.O. Radul, Generalized internal long waves equations: construction, Hamiltonian structure, and conservation laws, Comm. Math. Phys. 91 (1983), no. 4, 543–555.

[51] V. D. Lipovskiy, On the nonlinear theory of internal waves in a fluid of finite depth, Bull. USSR Acad. Sci. Atmos. Oceanic Phys. 21 (1986), 665.

[52] A.K. Liu, J.R. Holbrook, J.R. Apel, Nonlinear internal wave evolution in the Sulu Sea, J. Phys. Oceanography 15 (1985), no.12, 1613–1624.

[53] G. Li, T. Oh, G. Zheng, On the shallow-water and deep-water limits of the finite-depth-fluid equation from a statistical point of view, preprint.

[54] S.A. Maslowe, G. Redekopp, Long nonlinear waves in stratified shear flows, J. Fluid Mech. 101 (1980), no.2, 321–348.

[55] N. Masmoudi, K. Nakanishi, Energy convergence for singular limits of Zakharov type systems, Invent. Math. 172 (2008), no. 3, 535–583.

[56] T. Miloh, On periodic and solitary wavelike solutions of the intermediate long-wave equation, J. Fluid Mech. 211 (1990), 617–627.

[57] T. Miloh, A theory of dead water phenomena, Proceedings of the 17th Symposium on Naval Hydrodynamics, Hague, 1988, 1988: 127–142.

[58] L. Molinet, Global well-posedness in the energy space for the Benjamin-Ono equation on the circle, Math. Ann. 337 (2007), no. 2, 353–383.

[59] L. Molinet, A note on ill posedness for the KdV equation, Differential Integral Equations 24 (2011), no. 7-8, 759–765.

[60] L. Molinet, F. Ribaud, Well-posedness in $H^1$ for generalized Benjamin-Ono equations on the circle, Discrete Contin. Dyn. Syst. 23 (2009), no. 4, 1295–1311.
[63] L. Molinet, T. Tanaka, Unconditional well-posedness for some nonlinear periodic one-dimensional dispersive equations, J. Funct. Anal. 283 (2022), no. 1, Paper No. 109490, 45 pp.
[64] L. Molinet, S. Vento, Improvement of the energy method for strongly nonresonant dispersive equations and applications, Anal. PDE 8 (2015), no. 6, 1455–1495.
[65] L. Molinet, D. Pilod, S. Vento, Unconditional uniqueness for the modified Korteweg-de Vries equation on the line, Rev. Mat. Iberoam. 34 (2018), no. 4, 1563–1608.
[66] L. Molinet, D. Pilod, S. Vento, On well-posedness for some dispersive perturbations of Burgers’ equation, Ann. Inst. H. Poincaré C Anal. Non Linéaire 35 (2018), no. 7, 1719–1756.
[67] L. Molinet, D. Pilod, S. Vento, On unconditional well-posedness for the periodic modified Korteweg-de Vries equation, J. Math. Soc. Japan 71 (2019), no. 1, 147–201.
[68] K. Nakanishi, H. Takaoka, Y. Tsutsumi, Local well-posedness in low regularity of the mKdV equation with periodic boundary condition, Discrete Contin. Dyn. Syst. 28 (2010), no. 4, 1635–1654.
[69] M. Nakamura, T. Wada, Global existence and uniqueness of solutions to the Maxwell-Schrödinger equations, Comm. Math. Phys. 276 (2007), no. 2, 315–339.
[70] H. Ono, Algebraic solitary waves in stratified fluids, J. Phys. Soc. Japan 39 (1975), no. 4, 1082–1091
[71] A. R. Osborne, T. L. Burch, Internal solitons in the Andaman Sea, Science 208 (1980), 451–460.
[72] J. Palacios, Local well-posedness for the gKdV equation on the background of a bounded function, arXiv:2104.15126 [math.AP].
[73] O.M. Phillips, The dynamics of the upper ocean, Cambridge University Press (1966).
[74] N.N. Romanovna, Long nonlinear waves in layers of drastic wind velocity changes, Bull. USSR Acad. Sci. Atmos. Oceanic Phys. 20 (1984), no. 6, 296.
[75] P.M. Santini, M.J. Ablowitz, A.S. Fokas, On the limit from the intermediate long wave equation to the Benjamin-Ono equation, J. Math. Phys. 25 (1984), no. 4, 892–899.
[76] J. Satsuma, M.J. Ablowitz, Y. Kodama, On an internal wave equation describing a stratified fluid with finite depth, Phys. Lett. A 73 (1979), no. 4, 283–286.
[77] J.-C. Saut, N. Tzvetkov, On periodic KP-I type equations, Comm. Math. Phys. 221 (2001), no. 3, 451–476.
[78] H. Segur, J.L. Hammack, Soliton models of long internal waves, J. Fluid Mech. 118 (1982), 285–304.
[79] R. Schippa, Local and global well-posedness for dispersion generalized Benjamin-Ono equations on the circle, Nonlinear Anal. 196 (2020), 111777, 38 pp.
[80] W. A. Strauss, On continuity of functions with values in various Banach spaces, Pacific J. Math. 19 (1966), 543–551.
[81] T. Tao, Global well-posedness of the Benjamin-Ono equation in $H^1 (\mathbb{R})$, J. Hyperbolic Differ. Equ. 1 (2004), no. 1, 27–49.
[82] T. Tao, Nonlinear dispersive equations. Local and global analysis, CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.
[83] H. Takaoka, Y. Tsutsumi, Well-posedness of the Cauchy problem for the modified KdV equation with periodic boundary condition, Int. Math. Res. Not. 2004, no. 56, 3009–3040.
[84] G. B. Whitham, Variational methods and applications to water waves, Proc. R. Soc. Lond. A 299 (1967), no. 1456, 6–25.
[85] Y. Zhou, Uniqueness of weak solution of the KdV equation, Internat. Math. Res. Notices 1997, no. 6, 271–283.

Guopeng Li, School of Mathematics and Maxwell Institute for Mathematical Sciences, The University of Edinburgh, James Clerk Maxwell Building, The King’s Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

Email address: guopeng.li@ed.ac.uk