Revisiting Atiyah-Hitchin manifold in the generalized Legendre transform

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Abstract

We revisit construction of the Atiyah-Hitchin manifold in the generalized Legendre transform approach. This is originally studied by Ivanov and Roček and is subsequently investigated more by Ionas, in the latter of which the explicit forms of the Kähler potential and the Kähler metric are calculated. There is a difference between the former and the latter. In the generalized Legendre transform approach, a Kähler potential is constructed from the contour integration of one function with holomorphic coordinates. The choice of the contour in the latter is different from the former’s one, whose difference may yield a discrepancy in the Kähler potential and eventually in the Kähler metric. We show that the former only gives the real Kähler potential, which is consistent with its definition, while the latter yields the complex one. We derive the Kähler potential and the metric for the Atiyah-Hitchin manifold in terms of holomorphic coordinates for the contour considered by Ivanov and Roček for the first time.

Keywords

Hyperkähler manifolds; Generalized Legendre transform; Kähler potential; Kähler metric

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1 Introduction

So far several constructions of hyperkähler metrics have been proposed in extended supersymmetric nonlinear sigma models. One of ways is hyperkähler quotient framework, where the nonlinear sigma models are described by dynamical hypermultiplets and an auxiliary gauge supermultiplet (vectormultiplet). In this framework, the cotangent bundle over $\mathbb{C}P^n$ model [1, 2, 3] and its generalization such as the cotangent bundle over Grassmannian [4] have been constructed. Another interesting approach is to use the projective superspace formalism [5], which is an $\mathcal{N} = 2$ off-shell superfield formulation. Based on this formulation, constructions of the cotangent bundles over $\mathbb{C}P^1$ [6] and $\mathbb{C}P^n$ [7] and the other hermitian symmetric spaces [8, 9, 10, 11] have been worked out.

The other novel construction of hyperkähler metric is based on the generalized Legendre transform approach [4, 12, 13, 14]. This approach relates the Kähler potentials of certain hyperkähler manifolds to a linear space. It has also connection to the theory of twistor spaces of hyperkähler manifolds [12]. A merit of the generalized Legendre transform approach is that the complex structure is manifest. Namely geometrical quantities such as the Kähler metric are described by holomorphic coordinates. In this approach, a Kähler potential is constructed from the contour integration of one function with holomorphic coordinates (see [3] and [9]). This integration is called the $F$-function. The $F$-function of the several hyperkähler metrics such as Eguchi-Hanson family of self-dual instantons [15, 16], Taub-NUT family of self-dual instantons [16] and the metric due to Calabi [17] have been constructed [12].
After the above studies, the Atiyah-Hitchin manifold [18], which is a metric on the centered moduli space of two Bogomol’nyi-Prasad-Sommerfield SU(2) monopoles, has been also constructed [19]. In [19], the F-function giving the Atiyah-Hitchin manifold is proposed and corresponding Kähler 2-form is derived, which precisely coincides with one of the Atiyah-Hitchin manifold. On the other hand, the explicit form of the Kähler potential and the metric are not derived there because the calculation becomes complicated and messy. A different form of the F-function of the Atiyah-Hitchin manifold has been also derived through the twistor space in [20]. In this paper too, the Kähler potential and the metric have not been calculated. The generalization of the Atiyah-Hitchin manifold to the k-monopole case [21] and the asymptotically locally flat hyperkahler manifold called $D_k$ type [22] have been discussed. The $k = 2$ case for the former and $D_0$ case for the latter corresponds to the Atiyah-Hitchin manifold. In [21], the Kähler potential and the metric have not been derived explicitly while in [22] the metric has been evaluated but it is written by integral form. The first explicit calculation of the Kähler potential and the metric for the Atiyah-Hitchin manifold in the generalized Legendre transform approach has been worked out in [23, 24]. In this paper too, the Kähler potential and the metric have not been calculated. The generalization of the Atiyah-Hitchin manifold to the k-monopole case [21] and the asymptotically locally flat hyperkahler manifold called $D_k$ type [22] have been discussed. The $k = 2$ case for the former and $D_0$ case for the latter corresponds to the Atiyah-Hitchin manifold. In [21], the Kähler potential and the metric have not been derived explicitly while in [22] the metric has been evaluated but it is written by integral form. The first explicit calculation of the Kähler potential and the metric for the Atiyah-Hitchin manifold in the generalized Legendre transform approach has been worked out in [23, 24]. In the calculation, the original holomorphic coordinates in the F-function are kept and the metric is derived in terms of them. Deriving the metric where the complex structure is manifestly kept is important, because it would be useful to investigate geometrical properties. However, we stress that several points in [23, 24] should be reconsidered. First of all, the contour of the integration in the F-function chosen in [23] is different from one of [19], but it can be shown that this choice does not yield a real Kähler potential but a complex one. The former is consistent with the definition of the Kähler potential. In fact, it is possible to show that the choice in [19] gives a real Kähler potential. Thus, the calculation deriving the Kähler potential and the Kähler metric in [23] should be discussed again with the choice of the contour in [19]. In addition, the detailed derivation of the Kähler potential and the Kähler metric and the proof of the necessary formulas for the calculation should be also included. In the derivation in [23, 24], the elliptic integrals, their differentiation, and their related formulas are heavily used, they are partially explained in [23] and are not sufficient. Therefore, they should be explained in a comprehensive way. Moreover, some formulas for the derivation and their proof should be incorporated and placed properly. Indeed, the formulas necessary in [23] are given in [24] and the proof of the formulas is not fully provided. We emphasize that it is difficult to derive the Kähler manifold and the metric for the Atiyah-Hitchin manifold for the contour in [19] simply by referring [23, 24] and that a detailed explicit calculation to derive them is necessary.

In this paper, we restudy the Atiyah-Hitchin manifold in the generalized Legendre transform approach. We employ the contour of the integration in the F-function in [19] and derive the Kähler potential and the Kähler metric with holomorphic coordinates for that choice of the contour for the first time. We show that the choice of the contour in [19] gives a real Kähler potential. We provide all the necessary steps to derive the Kähler potential and the metric, starting from the F-function (see [20]) which defines the Atiyah-Hitchin manifold. The formulas related to elliptic integrals, elliptic functions and the other formulas are given in the main body and Appendices. We find that the resultant Kähler potential and the metric for the Atiyah-Hitchin manifold (which are given in [125] and [143]-[146]) slightly different from ones in [24]. The coefficients of them are different. This stems from the difference of the choice of the contour.

The same contour in [23] is also chosen in other literatures [25, 26]. In [21], it seems that the same contour as in [23] is chosen. The contours in [20, 22] cannot be directly compared with one in [23] since the forms of the F-functions are different from one in [23].
This paper is organized as follows. In Sec. 2, we briefly review the generalized Legendre transform approach. In Sec. 3, we give the $F$-function defining the Atiyah-Hitchin manifold and perform the integration in the $F$-function by using the theory of Weierstrass elliptic function. Finally, we derive the Kähler potential and the Kähler metric by means of the elliptic integrals. Sec. 4 is devoted to conclusion. In Appendix A, we summarize about Weierstrass $\wp$-function, $\zeta$-function and $\sigma$-functions. In Appendix B, the proofs of relations used in the calculation of the Kähler potential are given. In Appendix C, differential formulas in Weierstrass normal form which is necessary to derive the Kähler metric are explained.

2 The generalized Legendre transform

We briefly review of the generalized Legendre transform construction of hyperkähler manifold [14]. We start with a polynomial

$$\eta^{(2j)} = \frac{\bar{z}}{\zeta^j} + \frac{\bar{v}}{\zeta^{j-1}} + \frac{\bar{t}}{\zeta^{j-2}} + \cdots + x + (-)^j (-t\zeta^{-j-2} - v\zeta^{-j-1} + z\zeta^j),$$

where $z, t, \cdots, x$ are holomorphic coordinates and $\zeta$ is the coordinate of the Riemann sphere $\mathbb{C}P^1 = S^2$. This polynomial is called an $O(2j)$-multiplet. Eq. (1) should obey the reality condition

$$\eta^{(2j)}(-1/\zeta) = \overline{\eta^{(2j)}(\zeta)}.$$  

The Kähler potential for a hyperkähler manifold is constructed from a function with $\eta^{(2j)}$:

$$F = \oint_C \frac{d\zeta}{\zeta} G(\eta^{(2j)}),$$

where $G$ is an arbitrary holomorphic (possibly single or multi-valued) function and the contour $C$ is chosen such that the result of the integration is real. We call (3) the $F$-function. The $F$-function satisfies the following set of second order differential equations

$$F_{zz} = -F_{v\bar{v}} = F_{tt} = \cdots = (-)^j F_{xx},$$

$$F_{z\bar{v}} = -F_{\bar{v}t} = \cdots,$$

$$F_{zt} = F_{vv} \quad \text{etc},$$

$$F_{zv} = F_{vz} \quad \text{etc},$$

where

$$F_{zz} = \frac{\partial F^2}{\partial z\partial \bar{z}}, \quad \text{etc.}$$

The Kähler potential can be constructed from the $F$-function by performing a two dimensional Legendre transform with respect to $v$ and $\bar{v}$

$$K(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}, t, \bar{t}, \cdots, x) - uv - \bar{u}\bar{v},$$

together with the extremizing conditions

$$\frac{\partial F}{\partial v} = u,$$

$$\frac{\partial F}{\partial t} = \cdots = \frac{\partial F}{\partial x} = 0.$$
These equations tell us that \(v, \bar{v}, t, \bar{t}, \ldots, x\) are implicit functions of \(z, \bar{z}, u, \bar{u}\). Considering that fact, differentiating (10) and (11) with respect to \(z\) gives

\[
F_{zb} + \frac{\partial a}{\partial z} F_{ab} = 0, \tag{12}
\]

where \(a, b\) run over \(v, \bar{v}, t, \bar{t}, \ldots, x\) and summation over repeated indices is assumed. Eq. (12) yields

\[
\frac{\partial a}{\partial z} = -F_{ab} F_{bz}, \tag{13}
\]

where we have used \(F_{zb} = F_{bz}\) and \(F_{ab}\) is the inverse matrix of \(F_{ab}\). On the other hand, differentiating (11) with respect to \(u\), we have

\[
\frac{\partial a}{\partial u} = F_{av}. \tag{14}
\]

Eqs. (13) and (14) are used to derive the Kähler metric in terms of derivatives of \(F\) with respect to the holomorphic coordinates. Taking the derivatives of (9) with respect to \(z\) and \(u\), we obtain

\[
\frac{\partial K}{\partial z} = \frac{\partial F}{\partial z}, \tag{15}
\]

\[
\frac{\partial K}{\partial u} = -v. \tag{16}
\]

Further taking the derivatives of (15) and (16) and using (13) and (14), we have the Kähler metric as

\[
K_{z\bar{z}} = F_{z\bar{z}} - F_{za} F^{ab} F_{b\bar{z}}, \tag{17}
\]

\[
K_{z\bar{u}} = F_{z\bar{u}} F^{av}, \tag{18}
\]

\[
K_{u\bar{z}} = F^{va} F_{a\bar{z}}, \tag{19}
\]

\[
K_{u\bar{u}} = -F^{v\bar{v}}. \tag{20}
\]

3 The Atiyah-Hitchin metric

In this section we give the \(F\)-function for the Atiyah-Hitchin manifold and derive the Kähler potential and the Kähler metric. For the Atiyah-Hitchin manifold, the polynomial in (3) is an \(O(4)\)-multiplet. In derivation of the Kähler potential and the Kähler metric, we heavily use the elliptic integrals, the elliptic functions and their relations. The formulas are written in the main body and the necessary proofs of the relations are given in Appendices.

3.1 The function \(F\) for the Atiyah-Hitchin manifold

Let us consider an \(O(4)\)-multiplet \(\eta^{(4)} = \eta^{(4)}(\zeta)\) expressed in a Majorana normal form:

\[
\eta^{(4)} = \frac{\bar{\zeta}}{\zeta^2} + \frac{\bar{\eta}}{\zeta} + x - v\zeta + z\zeta^2. \tag{21}
\]

It is convenient to rewrite this form in terms of its roots and a scale factor. Since it obeys the reality condition \([2]\), the four roots of \(\eta^{(4)}\) are invariant under the antipodal map \(\zeta \mapsto -1/\bar{\zeta}\). Hence [21] is expressed as

\[
\eta^{(4)} = \frac{\rho}{\zeta^2} \frac{(\zeta - \alpha)(\bar{\alpha}\zeta + 1)(\zeta - \beta)(\bar{\beta}\zeta + 1)}{(1 + |\alpha|^2)(1 + |\beta|^2)}. \tag{22}
\]
The relations between $z, v, x$ and the roots are obtained by comparing (21) with (22) as

\begin{align}
  z &= \frac{\rho \bar{\alpha} \bar{\beta}}{(1 + |\alpha|^2)(1 + |\beta|^2)}, \\
  v &= -\frac{\rho (\bar{\alpha} + \bar{\beta} - |\alpha|^2 \bar{\beta} - \bar{\alpha} |\beta|^2)}{(1 + |\alpha|^2)(1 + |\beta|^2)}, \\
  x &= \frac{\rho (-\bar{\alpha} \beta - \alpha \bar{\beta} + (1 - |\alpha|^2)(1 - |\beta|^2))}{(1 + |\alpha|^2)(1 + |\beta|^2)}. 
\end{align}

(23)

(24)

(25)

Since $x$ is real, so is the scale factor $\rho$ by (25). Without loss of generalities we may assume that the scale factor $\rho$ is positive.

Following to [19] and [23], the $F$-function, $F = F(z, \bar{z}, v, \bar{v}, x)$, of the Atiyah-Hitchin manifold is given by

\begin{equation}
  F = F_2 + F_1 = -\frac{1}{2\pi i h} \oint_{\Gamma_0} \frac{d\zeta}{\zeta} \eta^{(4)} + \oint_{\Gamma} \frac{d\zeta}{\zeta} \sqrt{\eta^{(4)}},
\end{equation}

(26)

where $h$ is a constant coupling scale, $\Gamma_0$ is an integration contour encircling the origin of $\zeta$-plane in a counterclockwise direction, and $\Gamma = \Gamma_m \cup \Gamma'_m$ is a double contour that winds once around two branch-cuts between $\alpha$ and $-1/\bar{\beta}$, and $\beta$ and $-1/\bar{\alpha}$ (see Fig. 1). As we will see later, the contour $\Gamma_m$ corresponds to a meridian on a torus associated with $\eta^{(4)}$ (see [13] for the definition of the torus). Then, we find that $\Gamma'_m$ is homologous to $\Gamma_m$ on the torus.

Figure 1: Contour $\Gamma$

Note that the choice of the double contour is firstly proposed in [19] while only $\Gamma_m$ is chosen as $\Gamma$, which we call a single contour, in [23] as well as the other literatures such as [25, 26]. In the case of the single contour, it can be shown that the function $F_1$ is not real and eventually the Kähler potential is not real. However, the double contour makes $F_1$ and the Kähler potential real-valued. We shall give the proof for completeness. The function $F_1 = F_1(z, \bar{z}, v, \bar{v}, x)$ satisfies

\begin{equation}
  F_1(\lambda z, \lambda \bar{z}, \lambda v, \lambda \bar{v}, \lambda x) = \lambda^{1/2} F_1(z, \bar{z}, v, \bar{v}, x),
\end{equation}

(27)

\*\*We follow the sign in [19] while in [23] the overall sign is different.\*\*
for any $\lambda \in \mathbb{R}$, so that, by applying Euler’s homogeneous function theorem to $F_1$, we obtain

$$F_1 = 2z \frac{\partial F_1}{\partial z} + 2\bar{z} \frac{\partial F_1}{\partial \bar{z}} + 2v \frac{\partial F_1}{\partial v} + 2\bar{v} \frac{\partial F_1}{\partial \bar{v}} + 2x \frac{\partial F_1}{\partial x}.$$  \hfill (28)

If we put, for $n \in \mathbb{Z}$,

$$I_n = \int_{\Gamma} \zeta^n \frac{d\zeta}{2\zeta \sqrt{\eta(\zeta)}},$$  \hfill (29)

then the partial derivatives in (28) are rewritten as follows:

$$\frac{\partial F_1}{\partial z} = I_2, \quad \frac{\partial F_1}{\partial \bar{z}} = I_{-2}, \quad \frac{\partial F_1}{\partial v} = -I_1, \quad \frac{\partial F_1}{\partial \bar{v}} = I_{-1}, \quad \frac{\partial F_1}{\partial x} = I_0.$$  \hfill (30)

Thanks to the choice of $\Gamma$ we can prove the following relation $^1$:

$$I_{-n} = (-1)^n I_n.$$  \hfill (31)

Its proof is given in Appendix B.1. It follows from (31) that $I_0$ is real-valued, and that (28) is rewritten as

$$F_1 = 2xI_0 - 2(vI_1 + v\bar{I}_1) + 2(zI_2 + z\bar{I}_2).$$  \hfill (32)

Therefore we have shown that $F_1$ is real-valued. As will be shown in (33), since $F_2$ is also real-valued, from (9), the Kähler potential is real-valued.

In order to derive the Kähler potential from (26) by the generalized Legendre transformation, we need to perform the integrals in (26). Explicitly they are $F_2$, and $I_n(n = 0, 1, 2)$ in $F_1$. $F_2$ can be evaluated by means of a straightforward application of Cauchy’s integral formula. Then, we get

$$F_2 = -\frac{x}{h}.$$  \hfill (33)

The evaluation of $I_n$ in $F_1$ needs several steps. First of all, let us rewrite $I_n$ in terms of only the single contour $\Gamma_m$. When deforming $\Gamma_m$ to $\Gamma_m$, we need to pick up the residues of the integrand of $I_n$ (see Fig. 2).

---

$^1$A similar relation for the single contour is given in [23], where there is an additional term in the right-hand side in (31). In our case, such a term does not exist due to the choice of the double contour.
Since this integrand has two simple poles (except $n = 0$ case, see (40)); one at $\zeta = 0$ and the other at $\zeta = \infty$, we obtain

$$I_n = \left( \oint_{\Gamma_m} + \oint_{\Gamma_m'} \right) \frac{\zeta^n}{2\sqrt{\eta^{(4)}}} = 2 \oint_{\Gamma_m} \frac{\zeta^n}{2\sqrt{\eta^{(4)}}} + 2\pi i \left\{ R(0, n) + R(\infty, n) \right\}, \quad (34)$$

where $R(\zeta, n)$ denotes the residue for the integrand of $I_n$ at $\zeta \in \{0, \infty\}$. In order to evaluate the right-hand side of (34), we shall obtain the residues $R(\zeta, n)$. The integrand of $I_n$ can be expanded around $\zeta = 0$ as follows:

$$\frac{1}{2\sqrt{\eta^{(4)}}} = \left\{ 1 - \frac{v}{2z} \zeta + \left( \frac{3v^2}{8z^2} - \frac{x}{2z} \right) \zeta^2 + \cdots \right\}, \quad (35)$$

from which we get

$$R(0, n) = 0. \quad (36)$$

The residue at $\zeta = \infty$ can be computed by changing the variable $\zeta = 1/\zeta'$. If we denote by $\Gamma'$ which is the image of $\Gamma$ mapped by $\zeta = 1/\zeta'$ on the $\zeta'$-plane, then $I_n$ can be rewritten as

$$I_n = \oint_{\Gamma'} (\zeta'^{-n}) \frac{d\zeta'}{2\sqrt{\eta^{(4)}}(\zeta')} \quad (37)$$

Since the expansion of its integrand around $\zeta' = 0$ is given by

$$\frac{1}{2\sqrt{\eta^{(4)}}(\zeta')} = \left\{ 0 - \frac{1}{2\sqrt{z}} \right\} \left\{ 1 + \frac{v}{2z} \zeta' + \left( \frac{3v^2}{8z^2} - \frac{x}{2z} \right) \zeta'^2 + \cdots \right\}, \quad (38)$$

we obtain

$$R(\infty, n) = \begin{cases} 0 & (n = 0), \\ -\frac{1}{2\sqrt{z}} & (n = 1), \\ -\frac{1}{2\sqrt{z}} \cdot \frac{v}{2z} & (n = 2). \end{cases} \quad (39)$$

Substituting (36) and (39) into (34), we obtain

$$I_n = \begin{cases} 2 \oint_{\Gamma_m} \frac{d\zeta}{2\sqrt{\eta^{(4)}}} & (n = 0), \\ 2 \oint_{\Gamma_m} \frac{\zeta d\zeta}{2\sqrt{\eta^{(4)}}} + 2\pi i \left( -\frac{1}{2\sqrt{z}} \right) & (n = 1), \\ 2 \oint_{\Gamma_m} \zeta^2 d\zeta \quad 2\pi i \left( -\frac{1}{2\sqrt{z}} \cdot \frac{v}{2z} \right) & (n = 2). \end{cases} \quad (40)$$

From the above argument, $F_1$ is rewritten as

$$F_1 = 2x I_0 - 2(v I_1 - z I_2 + c.c.)$$

$$= 4 \left\{ x I_0(\Gamma_m) - \left( v I_1(\Gamma_m) - z I_2(\Gamma_m) - \frac{\pi i}{4} \cdot \frac{v}{\sqrt{z}} + c.c. \right) \right\}, \quad (41)$$

where we put

$$I_n(\Gamma_m) = \oint_{\Gamma_m} \frac{\zeta^n d\zeta}{2\sqrt{\eta^{(4)}}} \quad (n = 0, 1, 2). \quad (42)$$

In the following subsections, we will evaluate $I_n(\Gamma_m)$.

**In [23], the residue contribution does not exist since the single contour for the function $F$ is considered.**
3.2 Calculation of $I_0(\Gamma_m)$

We first consider a $(\zeta, \eta)$-plane curve defined by

$$C: \quad \eta^2 = 4\zeta^2 \eta^{(4)}(\zeta).$$

(43)

It is verified that the projectivization of $C$ is isomorphic to a torus, so that this elliptic curve defines a globally defined holomorphic 1-form $\varpi$ as follows:

$$\varpi = \frac{d\zeta}{\eta} = \frac{d\zeta}{2\zeta \sqrt{\eta^{(4)}}}.$$  

(44)

This is the integrand in $I_0(\Gamma_m)$. Such a form is uniquely determined up to a constant multiple. We call this form the abel form of $C$. The two periods of $\varpi$ are described by its integral over canonical cycles. Namely, one period is equal to

$$2\omega = \int_{\Gamma_m} \varpi,$$

(45)

and the other is

$$2\omega' = \oint_{\Gamma_l} \varpi,$$

(46)

where $\Gamma_l$ is a contour that winds once around the roots $-1/\bar{\beta}$ and $-1/\bar{\alpha}$ (see Fig. 3), which corresponds to a longitude on the torus.

![Figure 3: Contour $\Gamma_l$](image)

It can be shown that the periods are calculated as

$$\omega = \frac{1}{\sqrt{\rho}} K(k), \quad \omega' = \frac{i}{\sqrt{\rho}} K(k'),$$

(47)

where $K(k)$ denotes the complete elliptic integral of the first kind with modulus

$$k = \frac{|1 + \bar{\alpha}\beta|}{\sqrt{(1 + |\alpha|^2)(1 + |\beta|^2)}},$$

(48)
and \( k' = \sqrt{1 - k^2} \) denotes the complementary modulus. We give the details of the derivation of (47). We make use of the following birational transformation:

\[
\begin{align*}
\nu &= [\zeta, -1/\bar{\alpha}, \alpha, \beta] = (\zeta - \alpha)(1 + \bar{\alpha}\beta)/(\zeta - \beta)(1 + |\alpha|^2), \\
\mu &= \eta \frac{\partial \nu}{\partial \zeta}.
\end{align*}
\] (49)

Then the curve \( C \) in (43) is expressed as a Riemann normal form (cf. [27, p. 64]):

\[
\mu^2 = 4 \rho \nu (\nu - 1)(\nu - k^2),
\] (50)

and the four roots \( \alpha, -1/\bar{\beta}, -1/\bar{\alpha}, \beta \) correspond to 0, \( k^2 \), 1, \( \infty \), respectively. In addition, the abel form \( \varpi \) is expressed by

\[
\varpi = \frac{d\nu}{\mu} = \frac{d\nu}{2\sqrt{\rho \nu (\nu - 1)(\nu - k^2)}}.
\] (51)

Hence we obtain

\[
\omega = \int_0^{k^2} \frac{d\nu}{2\sqrt{\rho \nu (\nu - 1)(\nu - k^2)}} = \frac{1}{\sqrt{\rho}} \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}} = \frac{1}{\sqrt{\rho}} K(k).
\] (52)

Here, we have changed the variable from \( \nu \) to \( t \) by \( \nu = k^2t^2 \) in the second equality. A similar calculation shows the second equality in (47). Indeed, we observe

\[
\omega' = \int_1^{k^2} \frac{d\nu}{2\sqrt{\rho \nu (\nu - 1)(\nu - k^2)}} = \frac{i}{\sqrt{\rho}} \int_0^1 \frac{dx}{\sqrt{(k^2x^2 + k^2)(1 - x^2)}}.
\] (53)

Here, in the second equality, we have changed the variable from \( \nu \) to \( x \) by

\[
x = \sqrt{\frac{\nu - k^2}{k^2}}.
\] (54)

Using

\[
\frac{1}{\sqrt{(k^2x^2 + k^2)(1 - x^2)}} = \frac{1}{\sqrt{k^2}} \cdot \frac{1}{\sqrt{(1 + \kappa^2x^2)(1 - x^2)}}, \quad \kappa := \frac{k'}{k},
\] (55)

and making substitution

\[
x = \frac{t}{\sqrt{1 + \kappa^2(1 - t^2)}},
\] (56)

we have

\[
\omega' = \frac{i}{\sqrt{\rho}} \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - (\kappa / \sqrt{1 + \kappa^2})^2t^2)}} = \frac{i}{\sqrt{\rho}} K(k').
\] (57)

Thus, we have verified (47).

The period (45) is actually \( I_0(\Gamma_m) \), so we have

\[
I_0(\Gamma_m) = \int_{\Gamma_m} \varpi = 2\omega.
\] (58)
3.3 Calculation of $I_1(\Gamma_m)$

In order to evaluate $I_1(\Gamma_m)$ and $I_2(\Gamma_m)$, we heavily use the theory of Weierstrass elliptic functions (see (96) and (108) for our results). Our conventions for the elliptic functions are those of [27]. We start with rewriting the elliptic curve $C$ in (43) by means of the following transformation:

\[
\begin{align*}
\nu &= \frac{X}{\rho} + \frac{1 + k^2}{3}, \\
\mu &= \frac{Y}{\rho}.
\end{align*}
\]  

(59)

By using this transformation, we see that the three roots $1, k^2, 0$ of the curve correspond to:

\[ e_1 = -\frac{\rho}{3}(k^2 - 2), \quad e_2 = \frac{\rho}{3}(2k^2 - 1), \quad e_3 = -\frac{\rho}{3}(k^2 + 1), \]

(60)

respectively. Then, $0 < k^2 < 1$ implies $e_3 < e_2 < e_1$. Clearly, we get:

\[ e_1 - e_3 = \rho, \quad \frac{e_2 - e_3}{e_1 - e_3} = k^2. \]

(61)

Hence, the curve $C$ expressed in (50) becomes a Weierstrass normal form:

\[ Y^2 = 4(X - e_1)(X - e_2)(X - e_3) = 4X^3 - g_2X - g_3. \]

(62)

The relation between roots and coefficients are given by:

\[ e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \]

(63)

This obeys:

\[ g_2 = \frac{4}{3}\rho^2(1 - k^2 + k^4), \quad g_3 = \frac{4}{27}\rho^3(k^2 - 2)(2k^2 - 1)(k^2 + 1). \]

(64)

Furthermore, the discriminant $\Delta$ of $C$ is not equal to zero because:

\[ \Delta = g_2^3 - 27g_3^2 = 16\rho^6k^4k^4 \neq 0. \]

(65)

The abel form $\varpi$ is now expressed by:

\[ \varpi = \frac{dX}{Y} = \frac{dX}{\sqrt{4X^3 - g_2X - g_3}}. \]

(66)

Here, we give a summary of the relations among Majorana normal form, Riemann normal form and Weierstrass normal form as in Table 1.

| Normal form | Elliptic curve | Roots | Abel form |
|-------------|----------------|-------|-----------|
| Majorana    | $\eta^2 = \zeta^2\eta^{(4)}(\zeta)$ | $\alpha, -\frac{1}{\beta}, -\frac{1}{\alpha}, \beta$ | $\frac{d\zeta}{\eta}$ |
| Riemann     | $\mu^2 = \rho\nu(\nu - 1)(\nu - k^2)$ | $0, k^2, 1, \infty$ | $\frac{d\nu}{\mu}$ |
| Weierstrass | $Y^2 = 4X^3 - g_2X - g_3$ | $e_3, e_2, e_1, \infty$ | $\frac{dX}{Y}$ |
We shall rewrite $\mathcal{I}_n(\Gamma_m)$ by using the Weierstrass normal form. Combined (49) and (59), we have a single transformation
\[
\frac{(\zeta - \alpha)(1 + \bar{\alpha} \beta)}{(\zeta - \beta)(1 + |\alpha|^2)} = \frac{X - e_3}{e_1 - e_3}.
\]
(67)
We denote by $X_\zeta$ the image of $\zeta$ through this birational map. Under this convention we have
\[
X_0 = e_3 + (e_1 - e_3) \frac{\alpha}{\beta} \frac{1 + \bar{\alpha} \beta}{1 + |\alpha|^2} = e_3 + \rho \cdot \frac{\alpha}{\beta} \frac{1 + \bar{\alpha} \beta}{1 + |\alpha|^2},
\]
(68)
\[
X_\infty = e_3 + \rho \cdot \frac{1 + \bar{\alpha} \beta}{1 + |\alpha|^2}.
\]
(69)
Then, (67) is rewritten as
\[
\zeta = \beta \frac{X - X_0}{X - X_\infty}.
\]
(70)
The contours $\Gamma_m$ and $\Gamma_l$ on $\zeta$-plane are mapped to ones on $X$-plane via (70), which we write the same symbols, namely, $\Gamma_m$ (resp. $\Gamma_l$) winds once around the branch-cut between the roots $e_3$ and $e_2$ (resp. the roots $e_2$ and $e_1$) on $X$-plane. Therefore, $\mathcal{I}_n(\Gamma_m)$ has the following expression:
\[
\mathcal{I}_n(\Gamma_m) = \oint_{\Gamma_m} \left( \beta \frac{X - X_0}{X - X_\infty} \right)^n \frac{dX}{Y}.
\]
(71)
Clearly, (58) is rewritten as
\[
\mathcal{I}_0(\Gamma_m) = \oint_{\Gamma_m} \frac{dX}{Y} = 2\omega.
\]
(72)
In the following, we set
\[
\omega_1 = \omega, \quad \omega_2 = \omega + \omega', \quad \omega_3 = \omega'.
\]
(73)
For the evaluation of (71), we need to use Weierstrass $\wp$-function, $\zeta$-function and $\sigma$-function. For reader’s convenience, we briefly review the basics of them in Appendix A.

To evaluate (71), we first define $u_\zeta \in \mathbb{C}/\Lambda$ for $\zeta \in \mathbb{C} \cup \{\infty\}$, by the following equation
\[
X_\zeta = \wp(u_\zeta),
\]
(74)
and we set $Y_\zeta = \wp'(u_\zeta)$. With the use of $X_\zeta$ and $Y_\zeta$, it can be shown that the following equalities hold:
\[
\frac{Y_\infty}{X_0 - X_\infty} = 2\beta \sqrt{\zeta}, \quad \frac{Y_0}{X_\infty - X_0} = \frac{2\sqrt{\zeta}}{\beta},
\]
(75)
which are frequently used below. Their proofs are given in Appendix B.2.

It is convenient to divide $u_\zeta$ into the real part and the imaginary part with respect to the antiholomorphic involution $\zeta \mapsto -1/\zeta$ on $\mathbb{C} \cup \{\infty\}$, that is,
\[
u_\zeta^\pm = u_\zeta \pm u_{-1/\zeta},
\]
(76)
so that we have
\[
u_\infty^\pm = u_\infty \pm u_0.
\]
(77)
We write $(x_\pm, y_\pm)$ as the $(X, Y)$-coordinates of the point corresponding to $u_\zeta^\pm$ via the abel map $\psi$, (A.5). Thanks to (75), we can prove that the following relation holds:
\[
x_\pm = \frac{x \pm 6|z|}{3}.
\]
(78)
Its proof is given in Appendix B.3. The coordinate \( y_\pm \) is calculated by substituting \((u, v, w) = (u_\infty, \pm u_0, -u_\infty)\) into (A.8). Then we have

\[
\det \begin{pmatrix}
1 & X_\infty & Y_\infty \\
1 & X_0 & \pm Y_0 \\
1 & x_\pm & -y_\pm
\end{pmatrix} = 0,
\]

that is,

\[
y_\pm = (x_\pm - X_0) \frac{Y_\infty}{X_0 - X_\infty} \pm (x_\pm - X_\infty) \frac{Y_0}{X_\infty - X_0}.
\]

By using (23), (24), (25), (68), (69), (75) and (78), it can be shown that (80) is rewritten as

\[
y_+ = iv_+(x_+ - x_-), \quad y_- = v_-(x_- - x_+),
\]

where we put

\[
v_+ = \text{Im} \frac{v}{\sqrt{z}}, \quad v_- = \text{Re} \frac{v}{\sqrt{z}}.
\]

Now let us evaluate \( I_1(\Gamma_m) \). First, we observe

\[
I_1(\Gamma_m) = \beta \int_{\Gamma_m} \frac{dX}{Y} + \int_{\Gamma_m} \frac{Y_\infty - X_0}{Y_\infty - X_\infty} dX.
\]

The first term in (83) can be easily calculated by using (72). In the second term, we need to evaluate the integral

\[
\pi(X_\zeta) \equiv -\int_{\Gamma_m} \frac{Y_\zeta}{X - X_\zeta} dX, \quad \zeta \in \mathbb{C} \cup \{\infty\},
\]

with \( \zeta = \infty \). We calculate it for arbitrary \( \zeta \). Using the abel map (A.5) and the following formula (cf. [27, p. 41]):

\[
\frac{\varphi'(v)}{\varphi(u) - \varphi(v)} = -\zeta(u + v) + \zeta(u - v) + 2\zeta(v),
\]

we obtain

\[
\pi(X_\zeta) = -2 \int_{\omega_3}^{\omega_2} \frac{Y_\zeta}{X - X_\zeta} dX
\]

\[
= -2 \int_{\omega_3}^{\omega_2} \frac{\varphi'(u_\zeta)}{\varphi(u) - \varphi(u_\zeta)} du
\]

\[
= -2 \int_{\omega_3}^{\omega_2} \{ -\zeta(u + u_\zeta) + \zeta(u - u_\zeta) + 2\zeta(u_\zeta) \} du
\]

\[
= -2 \int_{\omega_3}^{\omega_2} \{ -\zeta(u + u_\zeta) + \zeta(u - u_\zeta) \} du - 4\omega_1 \zeta(u_\zeta).
\]

Here, in the last equality we have used \( \omega_1 = \omega_2 - \omega_3 \). By using the definition of \( \sigma \)-function given in (A.16), we have

\[
\pi(X_\zeta) = 2 \log \frac{\sigma(\omega_2 + u_\zeta)}{\sigma(\omega_2 - u_\zeta)} - 2 \log \frac{\sigma(\omega_3 + u_\zeta)}{\sigma(\omega_3 - u_\zeta)} - 4\omega_1 \zeta(u_\zeta).
\]

The first and second terms are evaluated by means of the monodromy property of \( \sigma \)-function for \( j = 2, 3 \), (A.18)

\[
\sigma(\omega_j + u_\zeta) = e^{2\eta_j u_\zeta} \sigma(\omega_j - u_\zeta),
\]

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so that the following relation holds:
\[
2 \log \frac{\sigma(\omega_j + u_\zeta)}{\sigma(\omega_j - u_\zeta)} = 4\eta_j u_\zeta \pmod{2\pi i\mathbb{Z}}, \tag{89}
\]
where \(\eta_j\) is the quasi-half period defined in \((A.11)\) and \((A.12)\). Substituting \((89)\) into \((87)\), we obtain
\[
\pi(X_\zeta) = 4 \det \begin{pmatrix} u_\zeta & \omega_1 \\ \zeta(u_\zeta) & \zeta(\omega_1) \end{pmatrix} \pmod{2\pi i\mathbb{Z}}, \tag{90}
\]
where we have used \((A.12)\).

Next we evaluate \((84)\) in the case for \(\zeta = \infty\). We find that \(\zeta(u_\infty)\) is rewritten as
\[
\zeta(u_\infty) = 2 \left\{ \zeta(u_\infty^+) + \zeta(u_\infty^-) - \frac{Y_\infty}{X_\infty - X_0} \right\}. \tag{91}
\]
Indeed, it follows from the formula \((85)\) for \(u = u_\infty, v = u_0\) that
\[
\frac{Y_\infty}{X_\infty - X_0} = \frac{\varphi'(u_\infty)}{\varphi(u_\infty) - \varphi(u_0)} = \zeta(u_\infty^+) + \zeta(u_\infty^-) - 2\zeta(u_\infty), \tag{92}
\]
where we have used \((77)\). Hence, we get \((91)\). By using \((91)\), we have
\[
\det \begin{pmatrix} u_\infty & \omega_1 \\ \zeta(u_\infty) & \zeta(\omega_1) \end{pmatrix} = \frac{1}{2} \left\{ \det \begin{pmatrix} u_\infty^+ & \omega_1 \\ \zeta(u_\infty^+) & \zeta(\omega_1) \end{pmatrix} + \det \begin{pmatrix} u_\infty^- & \omega_1 \\ \zeta(u_\infty^-) & \zeta(\omega_1) \end{pmatrix} + \omega_1 \frac{Y_\infty}{X_\infty - X_0} \right\}, \tag{93}
\]
from which we obtain
\[
\pi(X_\infty) = \frac{1}{2} \left\{ \pi(x_+) + \pi(x_-) \right\} + 2\omega_1 \frac{Y_\infty}{X_\infty - X_0} \pmod{\pi i\mathbb{Z}}. \tag{94}
\]
Thus, there exists \(a \in \mathbb{Z}\) such that
\[
\pi(X_\infty) = \frac{1}{2} \left\{ \pi(x_+) + \pi(x_-) \right\} + 2\omega_1 \frac{Y_\infty}{X_\infty - X_0} + a\pi i. \tag{95}
\]
As will seen later, the exact value of the integer \(a\) does not matter in our derivation of the Kähler potential for Atiyah-Hitchin manifold (see \((125)\) for the expression of the Kähler potential).

Substituting \((72)\) and \((95)\) into \((83)\), we obtain
\[
I_1(\Gamma_m) = 2\beta\omega_1 - \beta \frac{X_\infty - X_0}{Y_\infty} \left[ \frac{1}{2} \left\{ \pi(x_+) + \pi(x_-) \right\} + 2\omega_1 \frac{Y_\infty}{X_\infty - X_0} + a\pi i \right] \\
= \frac{1}{4\sqrt{2}} \left\{ \pi(x_+) + \pi(x_-) + 2a\pi i \right\}. \tag{96}
\]
Here, in the last equality we have used the first equality in \((75)\).
3.4 Calculation of $\mathcal{I}_2(\Gamma_m)$

We evaluate (71) for $n = 2$ case, that is,

$$\mathcal{I}_2(\Gamma_m) = \oint_{\Gamma_m} \left( \frac{\beta}{X - X_0} \right)^2 \frac{dX}{Y}. \quad (97)$$

We first observe

$$\left( \frac{X - X_0}{X - X_\infty} \right)^2 = 1 + \frac{2(X_\infty - X_0)}{Y_\infty} \cdot \frac{Y_\infty}{X - X_\infty} + \left( \frac{X_\infty - X_0}{Y_\infty} \right)^2 \left( \frac{Y_\infty}{X - X_\infty} \right)^2, \quad (98)$$

so that we will evaluate (97) by integrating each term in the right-hand side of (98). We can get the integrals of the first term and the second term by means of (72) and (95), respectively. The integral of the second term is calculated as follows:

$$\oint_{\Gamma_m} \left\{ \frac{2(X_\infty - X_0)}{Y_\infty} \cdot \frac{Y_\infty}{X - X_\infty} \right\} \frac{dX}{Y} = \frac{1}{2\beta\sqrt{z}} \left\{ \pi(x_+) + \pi(x_-) + 2a\pi i \right\} - 4\omega_1. \quad (99)$$

In order to perform the integration of the third term, we observe

$$\left( \frac{Y_\infty}{X - X_\infty} \right)^2 = 2(X - X_\infty) - \frac{12X_\infty^2 - g_2}{2Y_\infty} \cdot \frac{Y_\infty}{X - X_\infty} - Y \frac{d}{dX} \left( \frac{Y}{X - X_\infty} \right). \quad (100)$$

Here we use an integral expression of the quasi-period $\eta_1$ of $\zeta$-function (see (A.13)):

$$2\eta_1 = -2 \int_{e_3}^{e_2} X \frac{dX}{Y} = -\oint_{\Gamma_m} X \frac{dX}{Y}. \quad (101)$$

By using this, calculating the integral of the first term in (100) yields:

$$\oint_{\Gamma_m} 2(X - X_\infty) \frac{dX}{Y} = -4\eta_1 - 4\omega_1 \left( \frac{x}{3} - \beta v + 2\beta^2 z \right). \quad (102)$$

In the equality in (102), we have used (101) and

$$X_\infty = \frac{x}{3} - \beta v + 2\beta^2 z. \quad (103)$$

Using the following relation

$$\frac{12X_\infty^2 - g_2}{4(X_0 - X_\infty)} = \beta v - 4\beta^2 z, \quad (104)$$

we can calculate the integral of the second term in (100):

$$\oint_{\Gamma_m} \frac{12X_\infty^2 - g_2}{2Y_\infty} \cdot \frac{Y_\infty}{X - X_\infty} \frac{dX}{Y} = -\frac{1}{2} \left( \frac{v}{\sqrt{z}} - 4\beta\sqrt{z} \right) \left\{ \pi(x_+) + \pi(x_-) + 2a\pi i \right\} + 4\omega_1(\beta v - 4\beta^2 z). \quad (105)$$

The integral of the third term in (100) can be evaluated as

$$\oint_{\Gamma_m} Y \frac{d}{dX} \left( \frac{Y}{X - X_\infty} \right) \frac{dX}{Y} = \oint_{\Gamma_m} d \left( \frac{Y}{X - X_\infty} \right) = 0. \quad (106)$$
From (102), (105) and (106), we obtain
\[
\int_{\Gamma_m} \left( \frac{Y_\infty}{X - X_\infty} \right)^2 \frac{dX}{Y} = -4\eta_1 - 4\omega_1 \left( \frac{x}{3} - 2\beta^2 z \right) + \frac{1}{2} \left( \frac{v}{\sqrt{z}} - 4\beta \sqrt{z} \right) \{ \pi(x_+) + \pi(x_-) + 2a\pi i \}. \tag{107}
\]
Thus, by using (99) and (107), we conclude
\[
\mathcal{I}_2(\Gamma_m) = \beta^2 \left[ 2\omega_1 + \int_{\Gamma_m} \left( \frac{2(Y_\infty - X_0)}{Y_\infty} \cdot \frac{Y_\infty}{X - X_\infty} \right) \frac{dX}{Y} + \left( \frac{X_\infty - X_0}{Y_\infty} \right)^2 \int_{\Gamma_m} \left( \frac{Y_\infty}{X - X_\infty} \right)^2 \frac{dX}{Y} \right] = -\frac{1}{z} \left[ \eta_1 + \omega_1 \cdot \frac{x}{3} - \frac{1}{8\sqrt{z}} \{ \pi(x_+) + \pi(x_-) + 2a\pi i \} \right]. \tag{108}
\]

### 3.5 The function $F$ in terms of elliptic integrals

We return to our calculation of $F_1$. We will calculate the second term in (41) by using (96) and (108). We get
\[
v\mathcal{I}_1(\Gamma_m) - z\mathcal{I}_2(\Gamma_m) - \frac{\pi i}{4} \frac{v}{\sqrt{z}} = \eta_1 + \omega_1 \cdot \frac{x}{3} + \frac{v}{8\sqrt{z}} \{ \pi(x_+) + \pi(x_-) \} + \frac{\pi i}{4} (a - 1) \cdot \frac{v}{\sqrt{z}}. \tag{109}
\]
Thanks to (81), we find that the integrands of
\[
\pi(x_+) = -2 \int_{\epsilon_3}^{\epsilon_2} \frac{y_+}{X - x_+} \frac{dX}{Y}, \quad \pi(x_-) = -2 \int_{\epsilon_3}^{\epsilon_2} \frac{y_-}{X - x_-} \frac{dX}{Y}, \tag{110}
\]
are pure imaginary and real, respectively, so that we have
\[
\pi(x_+) \in i\mathbb{R}, \quad \pi(x_-) \in \mathbb{R}. \tag{111}
\]
This obeys
\[
\frac{v}{\sqrt{z}} \left\{ \pi(x_+) + \pi(x_-) \right\} + \text{c.c.} = 2 \left\{ iv_+ \pi(x_+) + v_- \pi(x_-) \right\}, \tag{112}
\]
where $v_\pm$ are defined in (82). Hence we obtain
\[
v\mathcal{I}_1(\Gamma_m) - z\mathcal{I}_2(\Gamma_m) - \frac{\pi i}{4} \frac{v}{\sqrt{z}} + \text{c.c.} = 2\eta_1 + \omega_1 \cdot \frac{2x}{3} + \frac{1}{4} \left\{ iv_+ \pi(x_+) + v_- \pi(x_-) \right\} - \frac{\pi}{2} (a - 1)v_+. \tag{113}
\]
Here, we have used that $\eta_1$ is real as noted in (A.14) below. By using (113), $F_1$ expressed in (41) is rewritten as
\[
F_1 = -8\eta_1 + 8(x_+ + x_-)\omega_1 - \left\{ iv_+ \pi(x_+) + v_- \pi(x_-) \right\} + 2\pi (a - 1)v_+. \tag{114}
\]
Here, we have used $x = 6(x_+ + x_-)$. Thus, the final result for $F = F_2 + F_1$ is
\[
F = -8\eta_1 + \left( \frac{3}{2\hbar} \right) (x_+ + x_-) - \left\{ iv_+ \pi(x_+) + \pi(x_-) \right\} + 2\pi (a - 1)v_+. \tag{115}
\]
3.6 Deriving the Kähler potential by the generalized Legendre transformation

The Kähler potential $K = K(z, \bar{z}, u, \bar{u})$ for the Atiyah-Hitchin manifold has the following expression by the generalized Legendre transformation:

$$K(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}, x) - (uv + \bar{u}\bar{v}) \quad (116)$$

with the conditions (10) and (11), the latter of which, in our setting, reduces to

$$\frac{\partial F}{\partial x} = 0 \quad (117)$$

Then, $K$ satisfies hyperkähler Monge-Ampère equation (cf. [28, (4.5)]):

$$\det \begin{pmatrix} K_{zz} & K_{z\bar{u}} \\ K_{u\bar{z}} & K_{u\bar{u}} \end{pmatrix} = 1 \quad (118)$$

Let us consider the condition (10). By (33) we get

$$\frac{\partial F}{\partial v} = \frac{\partial}{\partial v} \left( -\frac{x}{\hbar} \right) = 0 \quad (119)$$

From the third equality in (30) we find

$$\frac{\partial F_1}{\partial v} = -\frac{1}{2} \cdot \frac{1}{\sqrt{z}} \left\{ \pi(x_+) + \pi(x_-) \right\} - \frac{\pi i}{\sqrt{z}} (a - 1) \quad (120)$$

where we have used (96). Hence we have

$$u = -\frac{1}{2} \cdot \frac{1}{\sqrt{z}} \left\{ \pi(x_+) + \pi(x_-) \right\} - \frac{(a - 1)i}{\sqrt{z}} \quad (121)$$

By using this expression we get

$$uv + \bar{u}\bar{v} = -\{iv_+ \pi(x_+) + v_- \pi(x_-)\} + 2(a - 1)v_+ \quad (122)$$

where we have used (112). In a similar manner, the condition (117) is rewritten as

$$\frac{1}{\hbar} = 4\omega_1 \quad (123)$$

Indeed, from the last equality in (30) we have

$$\frac{\partial F_1}{\partial x} = I_0 = 4\omega_1, \quad \frac{\partial F_2}{\partial x} = -\frac{1}{\hbar} \quad (124)$$

Thus, by substituting (115), (122) and (123) into (116) we obtain the final expression of the Kähler potential

$$K = -8\eta_1 + 2(x_+ + x_-)\omega_1 \quad (125)$$

This result is consistent with one in [23], where differences are coefficients and an overall sign. The former stems from the fact that we choose the double contour while the latter is a due to a difference of the overall sign of the $F$-function. It should be stressed that this Kähler potential is real-valued, which is consistent with the definition of the Kähler potential. This is the consequence of the choice of the double contour. The differences are also reflected to the Kähler metric as will be shown below.
3.7 Deriving the Kähler metric

Let us introduce holomorphic coordinates \( Z, U \) defined by

\[
Z = 2\sqrt{z}, \quad U = u\sqrt{z}.
\]  

(126)

We remark that this coordinate change preserves hyperkähler Monge-Ampère equation \([118]\), namely,

\[
\det \begin{pmatrix}
K_{ZZ} & K_{ZU} \\
K_{UZ} & K_{UU}
\end{pmatrix} = 1.
\]  

(127)

In this subsection, we will derive the components \( K_{ZZ}, K_{UZ}, K_{ZU} \) and \( K_{UU} \) of the metric with respect to the coordinates \((Z, U)\). Note that in this subsection, we derive the components by differentiating the Kähler potential \( K \) not using the formula \((17)-(20)\), where the function \( F \) is directly differentiated. The latter also yields the same results given below though we do not show it in this paper.

We start with evaluating \( d\eta_1 \) and \( dx_\pm \) by means of \( dZ, d\bar{Z}, dU \) and \( d\bar{U} \). It follows from \((121)\) that \( U \) and \( \bar{U} \) have the following expressions:

\[
U = -\frac{1}{2}\{\pi(x_+) + \pi(x_-)\} - \pi i(a-1), \quad \bar{U} = -\frac{1}{2}\{-\pi(x_+) + \pi(x_-)\} + \pi i(a-1),
\]  

(128)

where \( a \) is the integer given by \((95)\). This obeys

\[
dU = -\frac{1}{2}\{d\pi(x_+) + d\pi(x_-)\}, \quad d\bar{U} = -\frac{1}{2}\{-d\pi(x_+) + d\pi(x_-)\}.
\]  

(129)

Here \( d\pi(x_\pm) \) can be expressed in terms of \( dx_\pm, dg_2 \) and \( dg_3 \). Then, \( dg_2 \) and \( dg_3 \) can be converted to \( d\omega_1 \) and \( d\eta_1 \). The detailed calculations to obtain \( d\pi(x_\pm) \) are summarized in Appendices C.1 and C.2. The result is

\[
d\pi(x_\pm) = \frac{4(x_\pm\omega_1 + \eta_1)}{y_\pm} dx_\pm + \frac{8(x_\pm^2 - V\eta_1)}{y_\pm} d\omega_1 - \frac{8(x_\pm + V\omega_1)}{y_\pm} d\eta_1,
\]  

(130)

where

\[
V = \frac{-3g_3\omega_1 + 2g_2\eta_1}{12\eta_1^2 - g_2\omega_1^2}.
\]  

(131)

Using \((123)\), we have

\[
d\pi(x_\pm) = 4A_\pm dx_\pm - 8B_\pm d\eta_1,
\]  

(132)

where we put

\[
A_\pm = \frac{x_\pm\omega_1 + \eta_1}{y_\pm}, \quad B_\pm = \frac{x_\pm + V\omega_1}{y_\pm}.
\]  

(133)

Substituting this into \((129)\) and making \( dU - d\bar{U} \) and \( dU + d\bar{U} \), we obtain

\[
dU - d\bar{U} = -4A_+ dx_+ + 8B_+ d\eta_1,
\]  

(134)

\[
dU + d\bar{U} = -4A_- dx_- + 8B_- d\eta_1.
\]  

(135)

By the definition, we get \( |Z|^2 = x_+ - x_- \). This yields

\[
\bar{Z}dZ + Zd\bar{Z} = dx_+ - dx_-.
\]  

(136)
By using this, we obtain

\[ A_-(dU - d\bar{U}) - A_+(dU + d\bar{U}) = -4A_+A_- (\bar{Z}dZ + Zd\bar{Z}) + 8(A_+B_+ - A_-B_-)d\eta_1. \]  
(137)

This yields

\[ d\eta_1 = \frac{A_-(dU - d\bar{U}) - A_+(dU + d\bar{U}) + 4A_+A_- (\bar{Z}dZ + Zd\bar{Z})}{8(A_+B_+ - A_-B_-)}. \]  
(138)

Furthermore, we have

\[ (-B_+ + B_-)dU - (B_+ + B_-)d\bar{U} = 4(A_+B_+ - A_-B_-)dx_+ - 4A_-B_+(\bar{Z}dZ + Zd\bar{Z}), \]  
(139)
equivalently,

\[ dx_+ = \frac{(-B_+ + B_-)dU - (B_+ + B_-)d\bar{U} + 4A_+B_+(\bar{Z}dZ + Zd\bar{Z})}{4(A_+B_+ - A_-B_-)}. \]  
(140)

A similar calculation shows

\[ dx_- = \frac{(-B_+ + B_-)dU - (B_+ + B_-)d\bar{U} + 4A_-B_+(\bar{Z}dZ + Zd\bar{Z})}{4(A_+B_+ - A_-B_-)}. \]  
(141)

Now we are ready to evaluate the components \( K_{\bar{Z}Z}, K_{U\bar{Z}}, K_{ZU} \) and \( K_{U\bar{U}} \). Introducing the notation

\[ Q = (\eta_1 + e_1\omega_1)(\eta_1 + e_2\omega_1)(\eta_1 + e_3\omega_1) = \eta_1^3 - \frac{g_2}{4}\omega_1^2\eta_1 + \frac{g_3}{4}\omega_1^3, \]  
(142)

we will have

\[ K_{\bar{Z}Z} = -\frac{2}{Q|Z|^2}K_1, \]  
(143)

\[ K_{U\bar{Z}} = \frac{v_-K_{3+} + iv_+K_{3-}}{2Q\bar{Z}}, \]  
(144)

\[ K_{ZU} = \frac{v_-K_{3+} - iv_+K_{3-}}{2QZ}, \]  
(145)

\[ K_{U\bar{U}} = -\frac{1}{2Q|Z|^2}K_2, \]  
(146)

where

\[ K_2 = \left( \frac{g_2}{4} - 3\xi_+\xi_- \right)\eta_1^2 - \left\{ \frac{3g_3}{2} + (\xi_+ + \xi_-)\frac{g_2}{2} \right\} \omega_1\eta_1 \]
\[ + \left\{ \frac{g_2^2}{16} + 3(\xi_+ + \xi_-)\frac{g_3}{4} + \xi_+\xi_-\frac{g_2}{4} \right\} \omega_1^3, \]  
(147)

\[ K_{3\pm} = \eta_1^3 + 3\xi_+\omega_1\eta_1^2 + \frac{g_2}{4}\omega_1^2\eta_1 - \left( \frac{g_3}{2} + \xi_+\frac{g_2}{4} \right) \omega_1^3, \]  
(148)

\[ K_4 = \eta_1^4 + 2(\xi_+ + \xi_-)\omega_1\eta_1^3 + \left( \frac{g_2^2}{4} + 3\xi_+\xi_- \right) \omega_1^2\eta_1^2 - \frac{g_3}{2}\omega_1^3\eta_1 \]
\[ - \left\{ (\xi_+ + \xi_-)\frac{g_3}{4} + \xi_+\xi_-\frac{g_2}{4} \right\} \omega_1^4. \]  
(149)
We start with calculating $K_Z$ and $K_{\bar{Z}}$ in terms of (125). Then, we have

$$K_Z = 2\bar{Z}\frac{-2A_+A_- + (A_-B_+ + A_+B_-)\omega_1}{A_-B_+ - A_+B_-}$$

$$= -\frac{2}{Z}\{2\eta_1 + (x_+ + x_-)\omega_1\} . \tag{150}$$

Here we have used

$$A_-B_+ - A_+B_- = |Z|^2\frac{\eta_1 - V\omega_1}{y_+y_-} , \tag{151}$$

$$-2A_+A_- + (A_-B_+ + A_+B_-)\omega_1 = -\frac{\eta_1 - V\omega_1^2}{y_+y_-}\{2\eta_1 + (x_+ + x_-)\omega_1\} . \tag{152}$$

A similar calculation shows

$$K_{\bar{Z}} = -\frac{2}{Z}\{2\eta_1 + (x_+ + x_-)\omega_1\} . \tag{153}$$

Hence we obtain

$$K_{ZZ} = -\partial_{\bar{Z}}K_Z$$

$$= -\frac{2(A_+A_-) + 2(A_-B_+ + A_+B_-)\omega_1}{A_-B_+ - A_+B_-}$$

$$= -\frac{2}{12\mathcal{K}_4} \frac{12\mathcal{Q}}{|Z|^2(\eta_1 - V\omega_1^2)}(12\eta_1^2 - \omega_1^2g_2)$$

$$= -\frac{2}{Q|Z|^2}\mathcal{K}_4 , \tag{154}$$

where, in the last equality, we have used

$$\eta_1 - V\omega_1^2 = \frac{12\mathcal{Q}}{12\eta_1^2 - \omega_1^2g_2} . \tag{155}$$

A similar calculation shows (144) and (145). Substituting (143)–(145) into (127), we find that $K_{U\bar{U}}$ is rewritten as

$$K_{U\bar{U}} = \frac{1}{K_{ZZ}}(1 + K_{Z\bar{U}}K_{U\bar{Z}}) = -\frac{1}{2Q|Z|^2}\mathcal{K}_2 . \tag{156}$$

The metric obtained has a similar form with the one in [23]. A difference is just a coefficient of the metric. Such a difference stems from the choice of contour: We have taken the double contour while in [23] the single contour is chosen. The double contour yields a different coefficient for the first three terms of the $F$-function (115) from one of the single contour case and produces the residue contribution. The latter does not appear in the single contour case. However, the residue contribution does not affect the metric since it cancels through the generalized Legendre transformation (116). One can also understand this when deriving the metric by using (17)-(20) with (115), where the $F$-function is directly differentiated. Since the residue is linear in $v$ and holomorphic in $z$ and vanishes by derivative of $F$ with respect to them, it does not contribute to the metric.
4 Conclusion

We have restudied the construction of the Atiyah-Hitchin manifold in the generalized Legendre transform approach. The $F$-functions for the Atiyah-Hitchin manifold are given in [19] and [23], but the contours are different, which might yield a discrepancy of the Kähler potential and the metric. We have shown that the choice of the double contour of $F$-function in [19] actually yields the real Kähler potential which is consistent with a definition of a Kähler potential. We have calculated the Kähler potential and the Kähler metric in terms of holomorphic coordinates by the choice in [19] for the first time. They had been only evaluated in the single contour in [23]. We have shown all the detailed steps in their derivation, which were missing in [23, 24]. In the derivation of the Kähler potential, the calculations of the integrations $I_n(\Gamma_m)$ ($n = 0, 1, 2$) in the $F$-function are necessary. We have performed them by using the theory of Weierstrass elliptic function. The necessary formulas related to the Weierstrass elliptic functions and the other relations have been explained in a comprehensive way. In the calculation of the Kähler metric, the main result one has to use is the differentiation of the Weierstrass elliptic integrals, which has been also given in detail in our paper. We have shown that the resultant Kähler potential and metric are slightly different from the ones in [23]. A difference is a coefficient of the Kähler potential and the metric. This stems from the choice of contour. Now the Kähler potential and the metric have been obtained in terms of the holomorphic coordinates and so one of the complex structures is manifest. Such a construction has a potential application to fields in geometry and physics, which may be found elsewhere in the future.

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Appendix

A Review of Weierstrass $\wp$-function, $\zeta$-function and $\sigma$-functions

We first review Weierstrass $\wp$-functions. Let $\Lambda$ denote the orthogonal lattice in $\mathbb{C}$ defined by $\Lambda = \mathbb{Z} \cdot 2\omega \oplus \mathbb{Z} \cdot 2\omega'$ where $\omega$ and $\omega'$ are in [17]. The half-periods $\omega$ and $\omega'$ satisfy $\text{Im}(\omega'/\omega) > 0$. The Weierstrass $\wp$-function $\wp(u) = \wp(u, \Lambda)$ is defined by

$$
\wp(u) = \frac{1}{u^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left\{ \frac{1}{(u - \lambda)^2} - \frac{1}{\lambda^2} \right\}, \quad u \in \mathbb{C}. \quad (A.1)
$$

This function is even and has the double periodicity, that is,

$$
\wp(u + 2\omega) = \wp(u), \quad \wp(u + 2\omega') = \wp(u), \quad u \in \mathbb{C}. \quad (A.2)
$$
We also have the following differential equation:

\[(\varphi'(u))^2 = 4\varphi(u)^3 - g_2\varphi(u) - g_3. \quad (A.3)\]

We denote by \(C^*\) the projectivization of \(C\), i.e.,

\[C^* = \{[x_0, x_1, x_2] \in \mathbb{CP}^2 \mid x_0 x_2^2 = 4x_1^3 - g_2 x_0 x_1 - g_3 x_0^3\}. \quad (A.4)\]

Thanks to (A.2), the \(\varphi\)-function induces a function on the torus \(\mathbb{C}/\Lambda\), which we write the same symbol \(\varphi(u)\). Then, we obtain a map \(\psi : \mathbb{C}/\Lambda \to \mathbb{CP}^2\) defined by

\[\psi(u) = [1, \varphi(u), \varphi'(u)] = [1, X, Y], \quad u \in \mathbb{C}/\Lambda. \quad (A.5)\]

From Abel’s theorem, \(\psi\) gives an isomorphism between \(\mathbb{C}/\Lambda\) and \(C^*\). We call this map the abel map of \(C^*\). Then the differential \(du\) on \(\mathbb{C}/\Lambda\) is mapped to the abel form \(\varpi = dX/Y\).

The inverse map of \(\psi\) is given by

\[u = \psi^{-1}(p) = \int_{\infty}^{p} \frac{dX}{Y}, \quad p \in C, \quad (A.6)\]

which is defined modulo \(\Lambda\). Here, \(\infty\) corresponds to the point \([0, 0, 1]\) of \(C^*\). We set \(\omega_1 = \omega, \omega_2 = \omega + \omega'\) and \(\omega_3 = \omega'\) as in (73). It is verified that \(e_i = \varphi(\omega_i)\) holds for \(i = 1, 2, 3\). Indeed, from

\[\omega_1 = \int_{\infty}^{e_1} \frac{dX}{Y}, \quad \omega_3 = \int_{\infty}^{e_3} \frac{dX}{Y} \quad (mod \ \Lambda), \quad (A.7)\]

we get \(e_1 = \varphi(\omega_1), e_3 = \varphi(\omega_3)\). We also get \(e_2 = \varphi(\omega_2)\) by means of the following formula (cf. [31, p. 332]):

\[
\det \begin{pmatrix} 1 & \varphi(u) & \varphi'(u) \\ 1 & \varphi(v) & \varphi'(v) \\ 1 & \varphi(w) & \varphi'(w) \end{pmatrix} = 0, \quad u + v + w = 0. \quad (A.8)
\]

Indeed, by applying \((u, v, w) = (\omega_1, -\omega_2, \omega_3)\) to this formula we have

\[
\det \begin{pmatrix} 1 & e_1 & 0 \\ 1 & \varphi(\omega_2) & -\varphi'(\omega_2) \\ 1 & e_3 & 0 \end{pmatrix} = 0, \quad (A.9)
\]

that is, \(\varphi'(\omega_2) = 0\). Here, we have used that \(\varphi'(u)\) is odd. This means that \(\varphi(\omega_2)\) is a root of \(4X^3 - g_2X - g_3 = 0\). In addition, \(\varphi(\omega_2)\) is different from \(e_1\) and \(e_3\). Thus we have \(e_2 = \varphi(\omega_2)\).

Next we review Weierstrass \(\zeta\)-function. The \(\zeta\)-function is defined by the following formula (cf. [27, p. 35]):

\[\zeta(u) = -\int_{0}^{u} \varphi(u)du, \quad (A.10)\]

so that \(\zeta'(u) = -\varphi(u)\) holds. Then the \(\zeta\)-function is odd. It is also verified that this function has the following double periodicity (cf. [27, p. 35]): If we put \(\eta_1 = \zeta(\omega_1)\) and \(\eta_3 = \zeta(\omega_3)\), then

\[\zeta(u + 2\omega_1) = \zeta(u) + 2\eta_1, \quad \zeta(u + 2\omega_3) = \zeta(u) + 2\eta_3, \quad u \in \mathbb{C}. \quad (A.11)\]
It follows from $\omega_2 = \omega_1 + \omega_3$ that $\eta_2 = \zeta(\omega_2)$ satisfies the following relation:

$$\eta_2 = \eta_1 + \eta_3. \quad (A.12)$$

Here, $\eta(i = 1, 2, 3)$ is called a quasi-half period, whose integral expression is obtained from (A.10) with the abel map (A.5) and the inverse map (A.6). For instance, $\eta_1$ is expressed as

$$\eta_1 = \zeta(\omega_1) = -\int_{\omega_3}^{\omega_2} \varphi(u) du = -\int_{\omega_3}^{\omega_2} X d\frac{X}{Y}. \quad (A.13)$$

It can be verified that $\eta_1$ (the proof is given in Appendix B in [29]):

$$\eta_1 = \frac{-1}{\sqrt{e_1 - e_3}} \{e_1 K(k) - (e_1 - e_3)E(k)\}, \quad (A.14)$$

where $E(k)$ denotes the complete elliptic integral of the second kind with modulus $k$. From this expression, we find that $\eta_1$ is in $\mathbb{R}$.

Lastly we review Weierstrass $\sigma$-function, $\sigma(u) = \sigma(u, \Lambda)$, This is defined by the following equality (cf. [27, p. 37]):

$$\log \frac{\sigma(u)}{u} = \int_0^u \left\{ \zeta(u) - \frac{1}{u} \right\} du, \quad (A.15)$$

so that

$$\zeta(u) = \frac{d}{du} \log \sigma(u). \quad (A.16)$$

It is shown that the $\sigma$-function is odd. Weierstrass $\sigma$-function satisfies the monodromy property, which is derived as follows: By the double periodicity (A.11) for the $\zeta$-function, we have

$$\frac{\sigma'(u + 2\omega_j)}{\sigma(u + 2\omega_j)} = \zeta(u + 2\omega_j) = \zeta(u) + 2\eta_j = \frac{\sigma'(u)}{\sigma(u)} + 2\eta_j, \quad j = 1, 3. \quad (A.17)$$

This implies that

$$\sigma(u + 2\omega_j) = -e^{2\eta_j(u + \omega_j)} \sigma(u), \quad j = 1, 3. \quad (A.18)$$

Following to [27] (5), p. 36 it is shown that

$$\eta_1\omega_3 - \eta_3\omega_1 = \frac{\pi i}{2}. \quad (A.19)$$

By using this, (A.18) yields the same relation for $j = 2$.

B Proofs of relations

B.1 Proof of (31)

We start to define the function as

$$f_n(\zeta) = \zeta^n \frac{1}{2\sqrt[4]{\eta(\zeta)}(\zeta)}. \quad (B.1)$$

Then we have

$$I_n = \oint f_n(\zeta) d\zeta = \oint_{\Gamma_m} f_n(\zeta) d\zeta + \oint_{\Gamma_m'} f_n(\zeta) d\zeta. \quad (B.2)$$
Set \( g_n(\bar{\zeta}) = \frac{f_n(\zeta)}{n} \) and \( \zeta' = -1/\zeta \). The first term of the right-hand side of (B.2) is rewritten as

\[
\frac{1}{\Gamma_m} \int f_n(\zeta) d\zeta = 2 \int_{-1/\alpha}^{\beta} g_n(\bar{\zeta}) \frac{d\bar{\zeta}}{\bar{\zeta}^2} = 2 \int_{-1/\alpha}^{\beta} g_n \left( -\frac{1}{\zeta} \right) \frac{d\bar{\zeta}}{\bar{\zeta}^2}.
\]

(B.3)

By using the reality condition (2),

\[
\eta^{(4)}(-1/\zeta') = \eta^{(4)}(\bar{\zeta}')
\]

(B.4)

we have

\[
g_n \left( -\frac{1}{\zeta} \right) = (-1)^{-n+1} \bar{\zeta}^{-n+2} \frac{1}{2\zeta' \sqrt{\eta^{(4)}(\zeta')}} = (-1)^{-n+1} f_{-n+2}(\bar{\zeta}').
\]

(B.5)

Substituting (B.5) into (B.3), we obtain

\[
\frac{1}{\Gamma_m} \int f_n(\zeta) d\zeta = 2 \int_{-1/\alpha}^{\beta} (-1)^{-n+1} f_{-n+2}(\bar{\zeta}') \frac{d\bar{\zeta}}{\bar{\zeta}^2} = (-1)^n \int_{\Gamma_m} f_{-n}(\zeta) d\zeta.
\]

(B.6)

In a similar manner, we find

\[
\frac{1}{\Gamma_m} \int f_n(\zeta) d\zeta = (-1)^n \int_{\Gamma_m} f_{-n}(\zeta) d\zeta.
\]

(B.7)

Hence, by substituting (B.6) and (B.7) into (B.2) we have (31). This completes the proof.

B.2 Proof of (75)

By using (68) and (69), we have

\[
\left( \frac{Y_\infty}{X_0 - X_\infty} \right)^2 = \frac{4 \bar{X}_\infty^2 - g_2 X_\infty - g_3}{(X_0 - X_\infty)^2} = 4\beta^2 \frac{\rho \bar{\alpha} \bar{\beta}}{(1 + |\alpha|^2)(1 + |\beta|^2) = 4\beta^2 z}.
\]

(B.8)

Here, in the last equality we have used (23). By the choice of the branch cut of the square root \( \sqrt{4 \bar{X}_\infty^2 - g_2 X_\infty - g_3} \), we have

\[
\frac{Y_\infty}{X_0 - X_\infty} = 2\beta \sqrt{z},
\]

(B.9)

from which the first equality in (75) holds. A similar argument shows the second equality in (75).

B.3 Proof of (78)

We start with recalling the following formula:

\[
\varphi(u + v) = -(\varphi(u) + \varphi(v)) + \frac{1}{4} \left( \frac{\varphi'(u) - \varphi'(v)}{\varphi(u) - \varphi(v)} \right)^2, \quad u, v \in \mathbb{C}.
\]

(B.10)

Substituting \( u = u_\infty \) and \( v = \pm u_0 \) into this formula, we obtain

\[
\varphi(u_\infty^\pm) = -(\varphi(u_\infty) + \varphi(u_0)) + \frac{1}{4} \left( \frac{\varphi'(u_\infty) \mp \varphi'(u_0)}{\varphi(u_\infty) - \varphi(u_0)} \right)^2
\]

\[
= -(X_\infty + X_0) + \frac{1}{4} \left( \frac{Y_\infty}{X_0 - X_\infty} \pm \frac{Y_0}{X_\infty - X_0} \right)^2.
\]

(B.11)
By using (68) and (69) it is verified that the first term in (B.11) is rewritten as
\[-(X_\infty + X_0) = \frac{x}{3} - \left(\beta^2 z + \frac{\bar{z}}{\beta^2}\right).\]  
(B.12)

From (75) the second term in (B.11) is rewritten as
\[\frac{1}{4} \left(\frac{Y_\infty}{X_0 - X_\infty} \pm \frac{Y_0}{X_\infty - X_0}\right)^2 = \beta^2 z + \frac{\bar{z}}{\beta^2} \pm 2|z|.\]  
(B.13)

Combined (B.12) and (B.13), we have
\[x_\pm = \varphi(u^\pm_\infty) = \frac{x \pm 6|z|}{3},\]  
(B.14)

from which (78) holds.

C Differential formulas in Weierstrass normal form

This appendix provides some formulas, which are used to calculate the metric of Atiyah-Hitchin manifold in Subsection 3.7. In our derivation of the formulas, we make use of differential formulas for complete elliptic integrals (see [27, 30] for basic references).

C.1 Evaluations of $d\omega$ and $d\eta$

The purpose of this subsection is to evaluate the differentials for $\omega$ and $\eta$. We start with the curve of the Weierstrass normal form, $C$, in (62). The half-period and the quasi-period of $C$ are written by means of integral forms, i.e.,

\[\omega = \omega(g_2, g_3) = \int_{e_3}^{e_2} \frac{dX}{Y},\]  
(C.1)

\[\eta = \eta(g_2, g_3) = -\int_{e_3}^{e_2} X \frac{dX}{Y}.\]  
(C.2)

We note that $\omega$ and $\eta$ are equal to $\omega_1$ in (A.7) above and $\eta_1$ in (A.11), respectively.

We first derive the Jacobian in
\[
\begin{pmatrix}
  dg_2 \\
  dg_3
\end{pmatrix} = J \begin{pmatrix}
  d\rho \\
  dk^2
\end{pmatrix}.
\]  
(C.3)

This can be easily calculated by using (64):
\[
J = \frac{D(g_2, g_3)}{D(\rho, k^2)} = \begin{pmatrix}
\frac{8\rho^3}{3}(1-k^2+k^4) & \frac{4\rho^2}{3}(-1+2k^2) \\
\frac{4\rho^2}{9}(k^2-2)(2k^2-1)(k^2+1) & \frac{4\rho^3}{9}(-1-2k^2+2k^4)
\end{pmatrix}.
\]  
(C.4)

With the use of (65) we obtain
\[
det(J) = -\frac{\Delta}{3k^2k^2\rho^2}.\]  
(C.5)

The inverse matrix $J^{-1}$ is given by
\[
J^{-1} = -\frac{3k^2k^2\rho^2}{\Delta} \begin{pmatrix}
\frac{4\rho^3}{9}(-1-2k^2+2k^4) & \frac{4\rho^2}{3}(-1+2k^2) \\
\frac{4\rho^2}{9}(k^2-2)(2k^2-1)(k^2+1) & \frac{8\rho}{3}(1-k^2+k^4)
\end{pmatrix}.
\]  
(C.6)
Next, we recall the differentiation formulas for the complete elliptic integrals of the first kind $K(k)$ and the second kind $E(k)$ (cf. [30]):
\[
\frac{dK(k)}{dk^2} = \frac{E(k) - k^2 K(k)}{2k^2 k'^2}, \quad \frac{dE(k)}{dk^2} = \frac{E(k) - K(k)}{2k^2}.
\] (C.7)

We are ready to derive our differentiation formulas for $\omega$ and $\eta$. We first evaluate $d\omega$. We observe (see Subsection 3.2)
\[
\omega = \frac{1}{\sqrt{\rho}} K(k).
\] (C.8)

By differentiating the both sides of (C.8), we get
\[
d\omega = -\frac{1}{2\sqrt{\rho}} K(k) d\rho + \frac{1}{2\sqrt{\rho}} \frac{E(k) - k^2 K(k)}{k^2 k'^2} d\rho,
\] (C.9)
where we have used the first equality in (C.7). Combined (C.6) and (C.9) we can obtain an explicit description of the partial derivatives \(\frac{\partial \omega}{\partial g_2}\) and \(\frac{\partial \omega}{\partial g_3}\). Indeed, we have
\[
\frac{\partial \omega}{\partial g_2} = \frac{\partial \omega}{\partial \rho} \frac{\partial \rho}{\partial g_2} + \frac{\partial \omega}{\partial k^2} \frac{\partial k^2}{\partial g_2} = -\frac{g_2^2 \omega + 18g_3 \eta}{4 \Delta},
\] (C.10)
By a similar calculation, we get
\[
\frac{\partial \omega}{\partial g_3} = \frac{3(3g_3 \omega - 2g_2 \eta)}{2 \Delta},
\] (C.11)
so that (C.9) is rewritten as
\[
d\omega = -\frac{g_2^2 \omega + 18g_3 \eta}{4 \Delta} d\rho_2 + \frac{3(3g_3 \omega - 2g_2 \eta)}{2 \Delta} d\rho_3.
\] (C.12)

Next, let us evaluate $d\eta$. From (A.14), we have
\[
\eta = \frac{\sqrt{\rho}}{3} \left\{ (k^2 - 2)K(k) + 3E(k) \right\},
\] (C.13)
whose differentiation yields the following expression:
\[
d\eta = -\frac{g_2(3g_3 \omega - 2g_2 \eta)}{8 \Delta} d\rho_2 - \frac{g_2^2 \omega + 18g_3 \eta}{4 \Delta} d\rho_3.
\] (C.14)

### C.2 Evaluation of $d\pi$

We introduce the following function $\pi = \pi(x, g_2, g_3)$ defined by
\[
\pi = \pi(x, g_2, g_3) = -\int_{\Gamma_m} \frac{y}{X - x} dX,
\] (C.15)
where $y^2 = 4x^3 - g_2 x - g_3$ holds. This integral coincides with that in (84). In order to derive the differential of $\pi$, we need to calculate $\partial \pi / \partial x$, $\partial \pi / \partial g_2$ and $\partial \pi / \partial g_3$. To this end, we make use of the results in [29]. In this paper, the following integrals was calculated:
\[
I_1^{(1)} = I_1^{(1)}(g_2, g_3) = \int_{\Gamma_m} \frac{dX}{Y},
\] (C.16)
\[
I_2^{(1)} = I_2^{(1)}(g_2, g_3) = \int_{\Gamma_m} X \frac{dX}{Y},
\] (C.17)
\[
I_3^{(1)} = I_3^{(1)}(x) = I_3^{(1)}(x, g_2, g_3) = \int_{\Gamma_m} \frac{1}{X - x} dX.
\] (C.18)
where \( \Gamma_m \) is an integration contour encircling \( e_3 \) and \( e_2 \) of \( X \)-plane. Then our \( \omega (= \omega_1) \), \( \eta (= \eta_1) \) and \( \pi \) are related with the above expression as
\[
\omega = \frac{1}{2} I_1^{(1)} \quad \eta = -\frac{1}{2} I_2^{(1)} \quad \pi = -y I_3^{(1)} \quad (C.19)
\]

From \([29, (B.16)]\) the relation between \( I_1^{(1)}, I_2^{(1)} \) and \( I_3^{(1)} \) is given by
\[
y^2 \frac{\partial I_3^{(1)}}{\partial x} = -2x I_1^{(1)} + 2I_2^{(1)} - \frac{1}{2} (12x^2 - g_2) I_3^{(1)}. \quad (C.20)
\]

From this we have \( \partial \pi / \partial x \) as
\[
\frac{\partial \pi}{\partial x} = \frac{4x\omega + 4\eta}{y}. \quad (C.21)
\]

We next evaluate \( \partial \pi / \partial g_2 \). We observe
\[
\frac{\partial I_3^{(1)}}{\partial g_2} = \frac{1}{8} \int_{\Gamma_m} X \frac{dX}{(X-x)(X-e_1)(X-e_2)(X-e_3) Y}. \quad (C.22)
\]

If we put
\[
a = \frac{x}{(x-e_1)(x-e_2)(x-e_3)} = \frac{4x}{y^2}, \quad (C.23)
\]
\[
b = \frac{e_1}{(x-e_1)(e_1-e_2)(e_3-e_1)}, \quad (C.24)
\]
\[
c = \frac{e_2}{(x-e_2)(e_1-e_2)(e_2-e_3)}, \quad (C.25)
\]
\[
d = \frac{e_3}{(x-e_3)(e_2-e_3)(e_3-e_1)}, \quad (C.26)
\]

then we get
\[
\frac{X}{(X-x)(X-e_1)(X-e_2)(X-e_3)} = \frac{a}{X-x} + \frac{b}{X-e_1} + \frac{c}{X-e_2} + \frac{d}{X-e_3}, \quad (C.27)
\]
from which we obtain
\[
\frac{\partial I_3^{(1)}}{\partial g_2} = \frac{1}{8} \left\{ a I_3^{(1)}(x) + b I_3^{(1)}(e_1) + c I_3^{(1)}(e_2) + d I_3^{(1)}(e_3) \right\}. \quad (C.28)
\]

We remark that the integrals \( I_3^{(1)}(e_1), I_3^{(1)}(e_2) \) and \( I_3^{(1)}(e_3) \) have the following expression (cf. \([29, (B.11–13)]\)):
\[
I_3^{(1)}(e_1) = \frac{-2}{(e_1-e_3)^{3/2}} \frac{E(k)}{1-k^2}, \quad (C.29)
\]
\[
I_3^{(1)}(e_2) = \frac{2}{(e_1-e_3)^{3/2}} \frac{1}{k^2} \left( \frac{E(k)}{1-k^2} - K(k) \right), \quad (C.30)
\]
\[
I_3^{(1)}(e_3) = \frac{2}{(e_1-e_3)^{3/2}} \frac{1}{k^2} (K(k) - E(k)). \quad (C.31)
\]

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Thus we obtain
\[
\frac{\partial \pi}{\partial g_2} = - \frac{\partial y}{\partial g_2} \frac{f_1^{(1)}}{y} - \frac{\partial f_3^{(1)}}{\partial g_2} = - \frac{1}{4y(e_1 - e_3)^{3/2}} y^2 \left\{ -c + d \frac{k^2}{k^2 - 1} K(k) + \left( -\frac{b}{1 - k^2} + \frac{c}{1 - k^2} \right) \frac{1}{k^2} - \frac{d}{k^2} \right\} E(k) \right\}.
\]
(C.32)

Here, we have used
\[
\frac{x}{2y} - \frac{ay}{8} = \frac{x}{2y} - \frac{4x y}{y^2} = 0.
\]
(C.33)

Furthermore, a direct calculation shows
\[
y^2 \left\{ -\frac{c + d}{k^2} K(k) + \left( -\frac{b}{1 - k^2} + \frac{c}{1 - k^2} \right) \frac{1}{k^2} - \frac{d}{k^2} \right\} E(k) = - \frac{1}{4k^4 k^4 \rho^4 \sqrt{\rho}} \left\{ (g_2 x + 3g_3)(3g_3 \omega - 2\eta g_2) + 2(-g_2^2 \omega + 18g_3 \eta) x^2 \right\}.
\]
(C.34)

Hence we obtain
\[
\frac{\partial \pi}{\partial g_2} = \frac{(g_2 x + 3g_3)(3g_3 \omega - 2\eta g_2) + 2(-g_2^2 \omega + 18g_3 \eta) x^2}{y \Delta}.
\]
(C.35)

In a similar manner, \(\partial \pi/\partial g_3\) is obtained as
\[
\frac{\partial \pi}{\partial g_3} = \frac{2}{y \Delta} \left\{ (6x^2 - g_2)(3g_3 \omega - 2g_2 \eta) - g_2^2 \omega + 18g_3 \eta) x \right\}.
\]
(C.36)

It follows from (C.21), (C.35) and (C.36) that the differential of \(\pi\) has the following expression:
\[
d\pi = \frac{4x \omega + 4\eta}{y} \frac{d x}{d x} + \frac{(g_2 x + 3g_3)(3g_3 \omega - 2\eta g_2) + 2(-g_2^2 \omega + 18g_3 \eta) x^2}{y \Delta} \frac{d g_2}{d x} + 2 \left\{ (6x^2 - g_2)(3g_3 \omega - 2g_2 \eta) - g_2^2 \omega + 18g_3 \eta) x \right\} \frac{d g_3}{d x}.
\]
(C.37)

It is convenient to rewrite \(d g_2\) and \(d g_3\) by means of \(d \omega\) and \(d \eta\). Calculating the Jacobian as
\[
J' = \frac{D(\omega, \eta)}{D(g_2, g_3)} = \begin{pmatrix}
\frac{-g_2^2 \omega + 18g_3 \eta}{4 \Delta} & \frac{3(3g_3 \omega - 2g_2 \eta)}{8 \Delta} \\
\frac{-g_2(3g_3 \omega - 2g_2 \eta)}{8 \Delta} & \frac{2(g_2(3g_3 \omega - 2g_2 \eta) - 4(-g_2^2 \omega + 18g_3 \eta))}{4 \Delta}
\end{pmatrix},
\]
(C.38)

we obtain
\[
\det(J') = \frac{12 \eta^2 - g_2^2 \omega^2}{16 \Delta}.
\]
(C.39)

In what follows, we assume that \(J'\) is non-singular, equivalently, \(12 \eta^2 - \omega^2 g_2 \neq 0\). The inverse matrix \(J'^{-1}\) is given by
\[
J'^{-1} = \frac{1}{12 \eta^2 - g_2^2 \omega^2} \begin{pmatrix}
4(g_2^2 \omega - 18g_3 \eta) & -24(3g_3 \omega - 2g_2 \eta) \\
2g_2(3g_3 \omega - 2g_2 \eta) & 4(-g_2^2 \omega + 18g_3 \eta)
\end{pmatrix}.
\]
(C.40)

From the above argument we obtain the following expression of \(d \pi\):
\[
d\pi = \frac{\partial \pi}{\partial x} dx + \frac{\partial \pi}{\partial g_2} \frac{d g_2}{d x} \frac{d x}{d x} + \frac{\partial \pi}{\partial g_3} \frac{d g_3}{d x} \frac{d x}{d x} + \left( \frac{\partial \pi}{\partial g_2} \frac{d g_2}{d \eta} + \frac{\partial \pi}{\partial g_3} \frac{d g_3}{d \eta} \right) d \eta
\]
\[
= \frac{4x \omega + 4\eta}{y} dx + \frac{8(x^2 - V \eta)}{y} d x - \frac{8(x + V \omega)}{y} d \eta,
\]
(C.41)

where \(V\) is given in (131).
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