The mildly nonlinear imprint of structure on the CMB

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Abstract

I outline some nonperturbative relativistic effects that arise from gravitational corrections to the Boltzmann equations. These may be important for the study of CMB temperature anisotropies, particularly their interpretation. These terms are not included in the canonical treatment as they arise from the exact equations. Here a weakly nonlinear investigation of these effects is defined and investigated with an emphasis on a Rees-Sciama sourced effect – the imprint of structure evolution on the CMB. It is shown that gravitational nonlinearity in the weakly nonlinear extension of almost-FLRW temperature anisotropies leads to cancellation on small-scales when threading in the Newtonian frame. In the general frame this cancellation does not occur. In the context of a flat almost-FLRW CDM model we provide a heuristic argument for a nonperturbative small scale correction, due to the Rees-Sciama effect, of not more than $\Delta T/T \sim 10^{-6} - 10^{-5}$ near $\ell \sim 100 - 300$. The effect of mild gravitational nonlinearity could be more sophisticated than previously expected.

1 Introduction

The 1+3 Lagrangian threading formalism gives generic equations for general CMB temperature anisotropies; from these it has been found, using qualitative arguments, that nonlinear terms dominate on small-scales [22]. In a previous article [2] I discussed some of the implications with specific emphasis on the scattering correction; here I deal exclusively with the scalar gravitational corrections. The key issue that is dealt with here is that such corrections are not included in the canonical treatment as they arise from the exact equations; specifically the exact multipole divergence equations [22]. The key advantage of using the 1+3 Lagrangian approach for these types of calculations is that it is not plagued by the necessity of constructing higher order gauge invariant variables when finding nonlinear corrections to the perturbative part of the theory.

This paper is constructed with the following framework in mind.

Next, in the introduction, I discuss why such effects could be important. I then summarize the general attitude towards nonlinear effects. This is followed by some general comments. The main text then follows with section 2 which gives a brief introduction to the 1+3 covariant and gauge invariant formulation as reduced to weakly nonlinear form. Here I give an explicit equation defining the weakly nonlinear theory in manifestly covariant and gauge invariant form.

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1The linear FRW treatment using the Bardeen perturbation theory
The core of the paper is a set of calculations using the canonical almost-FLRW results as ansatz seed solutions, so, a brief exposition of the canonical almost-FLRW temperature anisotropy results, found using [7, 6], is given in section 3. This is followed, in Section 3.1, by a recovery of the standard derivation of the Rees-Sciama effect; but here in the 1+3 covariant and gauge invariant formalism. This will be needed in the last section where we will compare the nonlinear (in the Boltzmann equations) corrections arising from the Rees-Sciama effect, to the Rees-Sciama correction itself. With the almost-FLRW theory in hand we will then be ready for main section, Section 4.

In the main section I calculate the gravitational corrections induced via the weakly nonlinear corrections using the almost-FLRW theories formulation of the Rees-Sciama (RS) effect over an Einstein-de Sitter background (EdS). Section 4 is divided into four subsections. The first introduces the gravitational corrections in terms of the 1+3 mode functions defined in [4], and explicitly constructs the Fourier coefficients for the situation of aligned wavenumbers – this simplifies the mode-mode coupling considerably. In this section the correction is written in terms of a conformal time integration from last scattering until now. It is constructed in a manifestly covariant and gauge invariant form for scalar perturbations. In section 4.1, I calculate the effect in the Newtonian frame (see [22, 6]) – the effect is shown to be suppressed. The third subsection, section 4.2, has the calculation in the total frame (see [22, 3]). The effect is shown to be non-trivial in the total frame. The fourth subsection, section 4.3, then attempts to approximate the resulting angular correlation function for the total frame effect. This is done by reducing it to a form that is similar to the Rees-Sciama calculation given in section 3.1. This then allows me to compare the nonlinear correction to the Rees-Sciama one. It all ends with a sparse conclusion and the references.

As promised, the back-of-the-envelope argument is now given.

From an order of magnitude argument using the COBE-Copernican limits [22] in the exact anisotropy equations or Eq. (5) one can approximate the weakly nonlinear correction by finding the order of magnitude of the induced temperature anisotropy multipole, \( \tau_A^\ell \): \( \mathcal{O}(\tau_A^\ell) \sim \ell \mathcal{O}((\sigma_{ab}/\Theta)\Pi_A^\ell) \). Here we have ignored the acceleration by assuming CDM domination, and have used the usual assumption that \( |\dot{\tau}_A^\ell| < \Theta |\tau_A^\ell| \); these are consistent with the almost-EGS approach [31]. Three results follow.

First, using \( \tau_A^\ell \sim 10^{-5} \), for almost-FLRW sources (using \( \langle \sigma_{ab}/\Theta \rangle \sim 10^{-4} \) [32] and that the lower bound on \( \Pi_A^\ell \) is that on \( \tau_A^\ell \) we find that almost-FLRW sourced nonlinear corrections would only dominate the anisotropies at \( \ell \sim \mathcal{O}((\sigma_{ab}/\Theta)^{-1}) \sim 10^4 \), long since diffusion damping and system noise have dominated the signal.

Second, if there are nonlinear sources of shear (due to structure evolution or formation), even at only \( 10^{-3} \), one can expect an effect near \( \ell \sim 1000 \).

An important point here is then that we have entirely ignored feedback of the anisotropy source into the correction terms, there could be an accumulation effect.

Third, we attempt to take the feedback into account by modeling the situation as \( N \) repeated sources along the line of sight, an accumulated effect: \( \mathcal{O}(\Pi_A^\ell) \sim \sqrt{N} \mathcal{O}(\tau_A^\ell) \). One then finds that \( \ell \sim (1/\sqrt{N}) \times 10^4 \).

Although this line of argument leads to an unreasonably low value of \( \ell \), even for a moderate number of iterations (or sources along line of sight), one gets a feeling for the situation – the universe needs to be very special for there to be no nonlinear effect. We have excluded the effect of dissipation via mode-mode coupling, which would smooth the effect and could also push \( \ell \) up to higher values, as will any inherent cancellation effect – as found in the Newtonian frame. However, in general, it seems that one should still expect an observable effect. It is for this reason that I

\(^2\) As in previous papers almost-FLRW refers to 1+3 covariant and gauge invariant linearizations while linear-FRW refers to the 3+1 linear perturbation theory of Bardeen [1].

\(^3\) The 1+3 covariant and gauge invariant equivalent of the Newtonian Gauge
investigate the gravitational effects in more depth in this paper.

Perhaps the point is best seen as two fold: first, nonlinear sources of shear (or $\Phi_H$) would result in a moderately amplified correction apparently not accounted for in the canonical treatment because of its, by construction, linearity, second, nonlinear feedback (or, if that does not do the trick, accumulated sources) could have an even more noticeable effect.

The general attitude towards the importance of nonlinearity in both observational cosmology and CMB studies can be understood in the context of popular lore surrounding the Rees-Sciama effect (see for example [23, 24]).

The two points that are usually made with regards to the Rees-Sciama effect are: first, that numerical simulations (where nonlinear effects dominate the linear effects only above $\ell > 5000$ or that the effects are a problem near to a sensitivity of $10^{-7}$) corroborate the scaling arguments and second, that the scaling arguments indicate that the effects are ignorable: that the effect suffers cancellation as $(k\delta \eta)^{-1}$ for $\delta \eta$ being the time-scale of change in the potential [16]. The idea is that the Poisson equation relates potentials to densities as $(k\eta)^{-4}$ while the volume from the mode coupling introduces terms like $k^3$ meaning that the effect scales as $k^2P(k)$ – which is ignorable, particularly when compared with the dominant Vishniac corrections (which cannot be ignored).

The novelty of using the Rees-Sciama effect as the source of nonlinear gravitational effects is that the peak in the nonlinear correction seems to be near to the peak in the radiation transfer function, $\ell \sim k(\eta_0 - \eta_*)$, somewhere near $\ell = 100 - 300$ [30]. The additional $\ell$ scaling from the nonlinear correction then introduces an additional $k^2$ scaling in the angular correlation function. It is for this reason that the scalar gravitational correction will be investigated in substantially more depth using the Rees-Sciama corrections as the source term. It does not seem unreasonable to suspect that it could dominate the acoustic effect.

The additional linear scaling in $\ell$ ([22] and Eq. (5)) may cause the scaling argument to fail, nonlinear effects could become important well before $\ell > 5000$. It is the latter point that needs to be properly understood, especially given that this scaling does not seem to arise in the perturbative analysis; it appears as a nonperturbative feature of the exact small scale treatment.

In addition, many established treatments carry out the analysis in the linear gauges which set the shear to zero: such as the longitudinal, Newtonian or conformal Newtonian choices [27, 23], which have no nonlinear corrections of the form discussed here – this will be shown here in detail.

A second order EdS extension of Pine-Carrol [27] was carried out by Mollerach and Matarrese [26] using the Poisson gauge, which gives the appropriate Newtonian extension at higher order; it requires that $D^a\tilde{\sigma}_{ab} \approx 0$, somewhat less restrictive than $\tilde{\sigma}_{ab} \approx 0$, it is however a perturbative analysis and so will not uncover the corrections discussed here. The main problem with a strict perturbation theory approach is the need to maintain gauge invariance at each order [3, 29]. In addition the strict perturbative approach is a computational tour de force, even in the case of investigating “mildly nonlinear effects”.

The points that are being made here are: (1) that when calculating the influence of nonlinearity on temperature anisotropies using a Newtonian gauge choice within the linear theory context one should view the results of such calculations with suspicion. This is because: (i) the gauge is inconsistent at higher order [33], and (ii), suppresses the mode-mode coupling artificially – the cancellation should not be a surprise. (2) When using a higher order perturbation theory, one should not expect to find these effects either; the effects discussed here are not of a perturbative nature.

Although these arguments may seem esoteric and somewhat obscure, the point at hand is understanding the stability and validity of the use of the almost-FLRW models on observationally interesting scales. Can the observations be explained using non-almost-FLRW corrections? are non-almost-FLRW corrections something worth worrying about? From the perspective put
forward here: the resolution of this is not obvious given the nature of the construction of the canonical linear-FRW programme. The situation may become even more complex if one has to worry about a gravitational wave (GW) background (or other large scale effects). Perhaps another way of phrasing the question is: why should the time derivatives and spatial gradients of the dynamical and kinematical quantities be small? when realistic cosmological models are apparently stable to inhomogeneity, particularly given the subtle nature of realistic models which have coupled inhomogeneity and anisotropy.

I now define the weakly nonlinear theory.

2 Weakly nonlinear almost-FLRW

The almost-FLRW model is expressed in terms of the 1+3 covariant and gauge invariant perturbation theory. The temperature anisotropies are defined in terms of the covariant total direction brightness temperature and the bolometric average in the almost-FLRW universe: $T(x, e) = T(x)(1 + \tau(x, e))$. The almost-FLRW temperature anisotropies for generic-$\ell$ (in particular $\ell > 2$) were the focus of [6] while the implication of the observational constraints on the dipole, quadrupole and octupole were the focus of [22] – where the theoretical foundations of the use of the postdecoupling CDM dominated almost-FLRW models were established, and hence the plausibility of the use of almost-FLRW models on large scales within the context of relativistic cosmology.

The notation used here follows [4, 22] as in the previous papers; in summary the temperature anisotropies $\tau(x, e)$ has been expanded in terms of a multipole expansion, $\tau = \sum_\ell \tau_\ell A_\ell$ forming a Projected Symmetric Trace-Free (PSTF) basis [4]. The equations are all expressed in terms of the 1+3 threading variables derived with respect to a preferred $u^a$-frame [4, 3]:

$$\nabla_b u_a = \frac{1}{3}h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c - A_a\dot{u}_b,$$

defining the expansion: $D^a u_a = \Theta$, the shear: $D_{\langle a} u_{b\rangle} = \sigma_{ab}$, the vorticity vector: $\omega^a = -\frac{1}{2}\epsilon_{abc}D^b u^c$ and the acceleration: $A_a = \dot{u}_a = \omega^b \nabla_b u_a$. The dot is indicative of comoving time with respect to the $u^a$-frame, while a prime, $'$, will be used to denote conformal time. The spatially projected covariant derivative is denoted: $h^{ab}\nabla_b = D^a$.

Here $u^au_a = -1$, and the direction vectors $e^a$ are defined such that $e^ae_a = +1$ and $e^au_a = 0$: the momentum is the given by $p^a = Eu^a + \lambda e^a$ such that $E^2 = m^2 + \lambda^2$. We refer the reader to references [12, 4, 22, 3] for further details on the formalism – though care should be taken between the differences in the formalism in [3] and [12, 4, 22], I will be following the later.

The Weakly nonlinear equations for the brightness temperature anisotropies are defined by:

$$\dot{\tau} + e^a D_a \tau \approx \mathcal{B} + C[\tau] + \left\{ (\delta \dot{\tau})_{OV} + (\delta \dot{\tau})_{NL} + \delta \dot{C}_{NL} \right\}.$$

Here we have used the exact formulation found in [22] in the small scale (high-$\ell$) limit and FLRW linearization discussed in [3] to construct the weakly nonlinear theory. The respective terms used in equation (1) are, the linear order coupling between the anisotropies and the field equations, the linear order Thompson scattering correction, the Ostriker-Vishniac (OV) correction and the dominant nonlinear (NL) corrections:

$$\mathcal{B}(x, e) \approx -\frac{1}{3}D^a \tau_a + (D_a \ln T + A_a)e^a + \sigma_{ab}e^ae^b,$$

$$C[\tau](x, e) \approx -\sigma_{a} n_{\nu} (f_1 e^a v_a^B + f_2 e^a \epsilon_{ab} \tau_{ab} - \tau),$$

$$\delta \dot{\tau}_{OV} \approx -\sigma_{a} n_{p} \rho^b D_a \rho v_b,$$

$$\delta \dot{\tau}_{NL} \approx -\ell O_{A_\ell} \left[ \frac{1}{4}\sigma_{bc} \tau^{bc} A_\ell + \sigma^{(a_\ell a_{\ell - 1}} \tau^{A_{\ell - 2})} \right].$$
\[-(A^{(a_\ell - A_\ell - 1)} - \frac{1}{2} A_b^b A_\ell^b A_\ell^c) + \omega^b \epsilon_{b c} (A_\ell^{a_\ell - 1})\],

(5)

\[\delta C_{NL} \approx +\sigma T n_h O A_\ell \left[ \tau (A_\ell - 1) v_B^B + \frac{1}{2} \tau A_\ell v_a^B \right].\]

(6)

As was shown in [22], the dominant nonlinear contribution to the temperature anisotropies at high-\(\ell\) will arise from term (Eq. 5) above; if gravitational nonlinearity between decoupling and now significantly alters the small scale angular correlations the dominant contribution will arise from this term. By considering this term in isolation, that is, outside of the context of the exact equations and within the context of corrections to the almost-FLRW theory I am trying to put bounds on the possible effects that this term may have on the small scale angular correlation functions and hence on the acoustic peak.

One notices that for almost constant coupling one may expect to find an effect that increases linearly in \(\ell\). Furthermore, these equations are valid for the so-called scalar, vector and tensor perturbation split – at least near to linear order where such a split is meaningful in the almost-FLRW theory.

The scattering term includes two weighting factors which are used to deal with the anisotropic scattering correction and the polarization correction; this has been discussed in detail elsewhere [13, 14, 19, 6].

The third and fifth terms are not dealt with any further in this paper, that is the Ostriker-Vishniac (OV) correction and an additional nonlinear scattering correction. The OV contribution represents additional variations in the baryon streaming velocities, induced by gradients in the matter energy flux that generate flows perpendicular to the line of sight \(e^a\).

A last comment on other outstanding effects is with regards to the Sunyaev-Zel’dovich (SZ) effect, in particular the kinetic-SZ effect, this arises through \(e_a v_B^a\) in re-ionized re-scattering. The thermal-SZ effect arises from including a electron-baryon pressure term. The kinetic effect is accumulative, and is due to along-line-of-sight streaming velocity differences – this tends to cancel in much the same manner as the baryon infall effect does during slow decoupling.

In order to maintain our focus here: we are primarily interested in bettering the understanding of the gravitational nonlinear effects on the CMB temperature anisotropies (Eq. 5). This is to attain two immediate goals: first, to demonstrate how these effects would arise as a correction to the standard gauge invariant treatment following the canonical treatment, the work of Hu et al [13, 14] in the formalism of Bardeen [1] and Wilson [35, 34], and second, to demonstrate how to implicitly calculate such corrections analytically by finding the corrections to the mode coefficients of the temperature anisotropies.

I will first investigate the scalar effect in the conformal Newtonian frame using weak-coupling [15] and high-\(\ell\) approximation techniques (see latter sections and [22]). These calculations represent new effects that do not arise in the canonical treatment. As pointed out before, a caveat here is that in the Newtonian frame any extension to second order must be treated with extreme care. Such a threading is inconsistent beyond linear order [3] – we are relying on weak coupling and sufficiently small peculiar velocity corrections in order to argue validity and hence consistency of our treatment.

4The energy flux in the Newtonian frame with respect to the total frame is : \(D_\alpha q_\alpha \approx D_\alpha q_\alpha - (D_\alpha \rho + D_\alpha p) v_a\). In the situation with dominate pressure free dust one then has an additional scattering contribution as :

\[\delta \tau_{OV} (n_\ell) \approx - \int_{n_\ell}^{n_\ell_1} (\kappa' e^{-\kappa}) \left\{ D^\alpha \rho_\alpha v_a \right\} d\eta.\]

Here, the divergence of the energy flux will vanish along the line of sight by cancellation due to falling in and out of potential wells on small scales – however the tranverse components of the quantities in the brackets will not vanish generically.
Ideally such a calculation should be carried out in a manifestly gauge invariant manner (using the generic Lagrangian threading in the total frame) or in frames that are known to be consistent at higher order; such as the constant expansion frame or CDM frame\textsuperscript{[3]}: we do this for a nonlinear Rees-Sciama imprint on the temperature anisotropies. With such frame choices one then has the additional freedom of moving away from weak nonlinearity. We now begin the explicit calculation of the weakly nonlinear correction (with no nonlinear feedback beyond the effect itself) using the weak coupling approximation\textsuperscript{[4]}.

\section{The almost-FLRW anisotropy sources}
The calculation in this section aims to, first, provide the background so as to, second, reproduce the results of Martinez-Gonzalez et al\textsuperscript{[24]} and Seljak\textsuperscript{[30]}.

I use the temperature anisotropy results for almost-FLRW models as found in\textsuperscript{[24]}\textsuperscript{[30]}. The key worry in this calculation is the possibility of cancellation between the primary source feedback and the secondary source feedback. The almost-FLRW primary (P) source\textsuperscript{[6, 13]} (using matter domination to get $\Phi_A \approx -\Phi_H$, $\eta_0$ is the conformal time now and $\eta^*$ is that near last scattering) is:

$$\frac{\beta\tau^P_\ell(k, \eta_0)}{(2\ell + 1)} \approx [\delta T - \Phi_H](k, \eta^*) j_\ell(k(\eta_0 - \eta^*))$$

(7)

The almost-FLRW secondary (S) source\textsuperscript{[1]}\textsuperscript{[13]} is:

$$\frac{\beta\tau^S_\ell(k, \eta_0)}{(2\ell + 1)} \approx -2\int_{\eta^*}^{\eta_0} d\eta' e^{-\kappa}\Phi_H(k, \eta') j_\ell(k(\eta_0 - \eta'))$$.

(8)

Using weak-coupling (W) and not specifying the end time ($\eta_0 \rightarrow \eta$) the latter becomes

$$\frac{\beta\tau^W_\ell(k, \eta_0)}{(2\ell + 1)} \approx -2\frac{(2\ell + 1)}{\beta_\ell} \sqrt{\frac{\pi}{2\ell k}} \Phi_H(k, \eta_0)e^{-\kappa}$$, and $\eta_\ell \approx \eta - (2\ell + 1)/2k$.

(9)

To summarize the power spectrum definitions\textsuperscript{[3]}, recall that: $P(k) = |\Delta(k, \eta_0)|^2$, for $\Delta = D(\eta)\Delta(k, \eta_0)$ and $D(\eta) \approx (\eta^2/\Omega_0^2)$, in addition\textsuperscript{[4]} one has

$$\frac{2}{3}H_0^2\Omega_0\Delta(\eta, k) \approx -k^2(a\Phi_A(k, 0))$$

(10)

I can then write the primary and secondary sources as (cf.\textsuperscript{[15]}):

$$\tau^P_\ell(k, \eta) \approx \frac{(2\ell + 1)}{\beta_\ell} \left[\delta T_0 - \frac{3}{2}H_0^2\Omega_0 k^{-2} \left(\frac{D_\ell}{a_\ell}\right) \Delta(k, \eta_0)\right] j_\ell(k(\eta - \eta_\ell))$$

(11)

$$\tau^W_\ell(k, \eta) \approx -\frac{(2\ell + 1)}{\beta_\ell} 3H_0^2\Omega_0 \sqrt{\frac{\pi}{2\ell k^3}} e^{-\kappa} \left(\frac{D(\eta)}{a(\eta)}\right)' \Delta(k, \eta_0)$$.  

(12)

I assume the standard Einstein-de-Sitter (EdS) results: $D(\eta) = a(t)/a_0$ from $a/a_0 = \eta^2$ and that $P(k, t) = P(k, 0)D(t)$ along with the useful relation ($\Omega_0 D_\alpha/a_\alpha)^2 \approx \Omega_0^{1.54}$\textsuperscript{[16]}. The intention here\footnote{Note that the definition using $\delta_k$ differs from that using $\Delta$ by $k^3$:}

$$\Delta^2 \propto \frac{d^2(\delta_k)^2}{d\ln k} \propto k^3 P(k) \text{ for } P(k) \equiv \langle |\delta_k\delta_k|^2 \rangle$$.

I will be using the dimensionless form; the variance per ln $k$.

$$\frac{1}{3}D_\alpha \ln \rho_\alpha \approx \rho_\alpha^{-1} D^3 E_{\alpha} \iff D_\alpha \ln \rho_\alpha \approx +\frac{1}{3} \Delta(k, \eta) Q_\alpha \text{ and } E_{\alpha} \approx \frac{1}{3} D(aD_\beta)(\Phi_A(x) - \Phi_H(x)) \approx D(aD_\beta)\Phi_A(x) \Rightarrow \frac{2}{3}H_0^2\Omega_0 \Delta(k, \eta) \approx -k^2\Phi_A(k, 0)$. 

(12)
is to use the EdS growth factors with a nonlinear correction to $\Delta(k)$. As pointed out before, we are modeling the effect of small-scale nonlinearity in the matter density so as to recover the Rees-Sciama like corrections for the exact theory. In addition one could include damping as $\kappa'/k$ to include cancellation and diffusion (see [13]).

It seems reasonable to assume that the dominate contribution will arise from the ISW terms via a Rees-Sciama correction; this will dominate the early-ISW, Sachs-Wolfe and acoustic sourced primary effects; assuming that there was little or no nonlinearity near last scattering. So I will only consider the secondary sourced effect.

3.1 The Rees-Sciama effect (RS)

In order to understand the nature of the nonperturbative corrections we need to understand the Rees-Sciama effect. Here, as promised, I reproduce the well know result using the results from the previous section. From the almost-FLRW temperature anisotropies, using that $\Phi_H(k,\eta)^\prime \approx D'(k,\eta)\Phi_H(k,\eta_0)^\prime$ and following [30], but using the covariant and gauge invariant notation (Eqs. 9 and 8):

$$\tau^{\text{RS}}_\ell(k,\eta_0) \approx -2(2\ell + 1)\beta_\ell \sqrt{\frac{\pi}{2}} e^{-\kappa} D'(k,\eta_\ell)\Phi_H(k,\eta_0), \quad \text{for } k\eta_0 \geq \ell.$$  

(13)

This term is zero for $k\eta_0 < \ell$. Now using the definition of the angular correlation function [7]:

$$C^{\text{RS}}_\ell = \frac{2}{\pi} \frac{\beta^2_\ell}{(2\ell + 1)^2} \int_0^{\infty} k^2 dk |\tau^{\text{RS}}_\ell(k,\eta_0)|^2,$$

(14)

We find the angular correlation function and the Rees-Sciama effect in terms of the power spectrum of the potential:

$$C^{\text{RS}}_\ell \approx \frac{4}{\ell} (4\pi)^2 \int_0^{\infty} k^2 dk (D'(k,\eta_\ell))^2 P_{\Phi_H}(k) \frac{1}{k^2}.$$  

(15)

The power spectrum of the time changing potential is defined as

$$P_{\Phi'_H}(k,\eta) = (D'(k,\eta))^2 P_{\Phi_H}(k)$$

[30] to find that (here we are using $\eta_\ell = \eta_0 - \ell/k$ such that $k = \ell/(\eta_0 - \eta)$ for all $k\eta_0 \geq \ell$):

Rees-Sciama: $$C^{\text{RS}}_\ell \approx \frac{4}{\ell} (4\pi)^2 \int_0^{\infty} dk P_{\Phi'_H}(k,\eta_0 - \ell/k).$$

(17)

I can rescale the integration variable from $k$ to $\eta$, $dk \approx [\ell/(\eta_0 - \eta)^2][d\eta]$, and change the integration limits (the integral is non-vanishing for $k \geq \ell/\eta_0$) to write the angular correlation function in terms of the area distance $r \approx (\eta_0 - \eta)$:

$$C^{\text{RS}}_\ell \approx 4(4\pi)^2 \int_0^{\eta_0} \frac{d\eta}{r^2} P_{\Phi'_H}(\ell/r,\eta).$$

(18)

This recovers the result of [30]. From [30] we have that

$$P_{\Phi'_H} = \frac{9}{4} \frac{H_0^2}{k^4} (a')^2 P_{(2)}(k).$$

(19)

\footnote{This is nonlinear for the CDM dominate flat almost-FLRW models – where $\Phi_H(k,\eta)^\prime \approx \Phi_H(k,0)^\prime \approx 0$. To understand the notation used here. The Rees-Sciama correction arises from (i) the term $\Phi_H(k,\eta) = D(k,\eta)\Phi_H(k,\eta_0)$, while the (ii) generic ISW effect arises from $\Phi_H(k,0)^\prime$ which is written as $(D(\eta)\Phi_H(k,0)/a(\eta))^\prime$.}
This then gives the standard Rees-Sciama effect. The nonlinear power spectrum is approximated from \( P_0(k) \propto k^{-4} P_2(k) \). The relationship between the second order power spectrum and the linear one can be found from:

\[
P_2(k) \propto \int_0^\infty q^2dqP(q)P(|k-q|)D_2^2(k,|k-q|)
\]

where \( D_2 \) is the second order growth factor, and \( P(k) = A(k/k_0) + k^4 e^{-k\alpha} \) for \( k_0 = h/30 \) Mpc, \( a = 8h^{-1} \) Mpc such that for \( k < k_0 \) one has \( P(k) \propto k^4 \) and for \( a^{-1} > k > k_0 \) one finds \( P(k) \propto 1/k \).

In this latter region one is lead to find: \( P_2(k) \propto k^3 P(k) \). With the additional cancellation as \((k\delta\eta)^{-1}\) one finds the usual result that the source scales as \( k^2 P(k) \) in this region of interest – as pointed out in the introduction. Here one then finds \( P_2^\phi(k) \propto k^{-2} P(k) \) as the power spectrum of the time rate of change of the potential in the region \( a^{-1} > k > k_0 \), leading to the usual scaling argument.

We are now ready to investigate the new effects in more detail.

4 Weakly nonlinear gravitational corrections

I use the usual scalar mode decomposition but now include a mode-mode coupling between the wavenumber (assuming that the directions are all aligned along the line of sight)\(^8\).

The nonlinear corrections multipole coefficient and the temperature anisotropy multipole coefficients can be put into mode coefficient form:

\[
\delta_{NL}^a = \sum_{k_0} (\delta\eta)^k Q^a_k, \quad \text{and} \quad \tau^a = \sum_{k} \tau^k Q^a_k.
\]

Using the Fourier convolution theorem, with a little algebra and integrating over the solid angle, these, along with the mode form of the scalar potential \[^\Phi\] using the PSTF relation \( e_{\ell+1} O^{a}_{\ell+1} \) \[^\Phi\] and an identity for the delta function, \( \delta(2k) = \frac{1}{2} \delta(k) \), will allow us to write Eq. \[^\Phi\] as a mode coefficient for the correction in terms of those for the temperature anisotropy and scalar potential.

Now there are four different cases; using in addition that \( \ell \gg 1 \) and that \( e_{\ell} O^{a}_{\ell} = (\ell+1)/(2\ell+1) O^{a}_{\ell} \):

\[
A_{a \ell A_{\ell-1}} \sim (4\pi) Q_{A_{\ell}}(x^a, k^a) \int_0^\infty \frac{k^2dk}{(2\pi)^3} A(k) \tau_{\ell-1}(|k^* - k|),
\]

\[
A_{a \ell a_{\ell+1}} \sim (4\pi) \frac{\ell + 1}{2\ell + 1} Q_{A_{\ell}}(x^a, k^a) \int_0^\infty \frac{k^2dk}{(2\pi)^3} A(k) \tau_{\ell+1}(|k^* - k|),
\]

\[
\sigma_{a \ell+1 a_{\ell-2}} \sim (4\pi) Q_{A_{\ell}}(x^a, k^a) \int_0^\infty \frac{k^2dk}{(2\pi)^3} \sigma(k) \tau_{\ell-2}(|k^* - k|),
\]

\[
\sigma_{a_{\ell+1} b_{\ell-2}} \sim (4\pi) Q_{A_{\ell}}(x^a, k^a) \frac{(\ell + 1)(\ell + 2)}{(2\ell + 1)(2\ell + 3)} \int_0^\infty \frac{k^2dk}{(2\pi)^3} \sigma(k) \tau_{\ell+2}(|k^* - k|).
\]

This approximation scheme only holds for high-\( \ell \), this will then give

\[
A_{a \ell A_{\ell-1}} \sim (4\pi) Q_{A_{\ell}}(x^a, k^a) \int_{k_0}^\infty \frac{k^2dk}{(2\pi)^3} A(k) \tau_{\ell-1}(|k^* - k|),
\]

\(^8\)We do not need to worry about the direction couplings then (which is of course essential for the Vishniac effect and the nonlinear scattering correction which would generate a nonlinear Vishniac effect). We therefore do not have terms such as: \( \frac{1}{2} \sum_k (A(k^*)M(k^* - k^*) + A(k^* - k^*)M(k^*)). \)
\[ A^a \tau_{ab} A^b \sim -\left(\frac{4\pi}{9}\right) \frac{1}{2} Q_{A_\ell}(x^a, k^a) \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} A(k) \tau_{\ell+1}(|k^* - k|), \]

\[ \sigma_{(a_\ell A_{\ell-1}) A_{\ell-2}} \sim (4\pi) Q_{A_\ell}(x^a, k^a) \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} \sigma(k) \tau_{\ell-2}(|k^* - k|), \]

\[ \sigma^{ab \tau_{ab} A_\ell} \sim (4\pi) Q_{A_\ell}(x^a, k^a) \frac{1}{4} \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} \sigma(k) \tau_{\ell+2}(|k^* - k|). \]

I drop the vorticity (we are assuming matter domination for which it becomes natural to treat vorticity effects as very small). I then have the weakly nonlinear correction on small scales, Eq. (3), in mode form:

\[ \delta \tau_{NL}^\ell \approx -\left(\frac{4\pi}{9}\right) \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} \left[ \sigma(k, t) \tau_{\ell-2}(|k^* - k|, t) + \frac{1}{16} \sigma(k, t) \tau_{\ell+2}(|k^* - k|, t) \right. \]

\[ \left. - A(k, t) \tau_{\ell-1}(|k^* - k|, t) + \frac{1}{4} A(k, t) \tau_{\ell+1}(|k^* - k|, t) \right]. \]

I am interested in finding the effect on scales \( k > k_0 \). The above equations are valid for any \( u^a \)-frame as I have not yet frame-fixed the theory. On dropping the mode functions I find the following equation:

\[ \frac{d}{dt} (\delta \tau)_{NL}^\ell(k^*, t) \approx -\left(\frac{4\pi}{9}\right) \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} \left[ \sigma(k, t) \tau_{\ell-2}(|k^* - k|, t) + \frac{1}{16} \sigma(k, t) \tau_{\ell+2}(|k^* - k|, t) \right. \]

\[ \left. - A(k, t) \tau_{\ell-1}(|k^* - k|, t) + \frac{1}{4} A(k, t) \tau_{\ell+1}(|k^* - k|, t) \right]. \]

Finally, we can write this out in terms of the conformal time through: \( dt = ad\eta \), and write the effect today in terms of a timelike integration (under the assumption that the effects are secondary – as opposed to primary):

\[ \delta \tau_{NL}^\ell(k^*, \eta_0) \approx -\left(\frac{4\pi}{9}\right) \int_{\eta_0}^{\eta_0} a(\eta) d\eta \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} \left[ \sigma(k, \eta) \tau_{\ell-2}(|k^* - k|, \eta) \right. \]

\[ \left. + \frac{1}{16} \sigma(k, \eta) \tau_{\ell+2}(|k^* - k|, \eta) - A(k, \eta) \tau_{\ell-1}(|k^* - k|, \eta) + \frac{1}{4} A(k, \eta) \tau_{\ell+1}(|k^* - k|, \eta) \right]. \]

The question is: "How does this compare to the linear primary and secondary sources from the canonical treatment?"

Towards answering this two distinct effects need to be highlighted: the first, is the result of mode-mode coupling between the linear primary and linear secondary anisotropies (Eq. (2)) and the linear shear and linear acceleration sources; this will most likely lead to a small smoothing effect at high-\( \ell \). Second, the effect of local nonlinear matter dynamics coupling through into the anisotropies. It is this latter effect that I will emphasize here, because it promises to be dominant.

### 4.1 The Newtonian-frame correction

Once again starting from Eq. (3), the high-\( \ell \) correction assumes the form

\[ (\delta \tau)^A_\ell \approx -\ell \left( \frac{1}{2} \sigma_{bc} \tau^{bc} A_\ell + \sigma^{(a_\ell A_{\ell-1}) A_{\ell-2}} - A^{(a_\ell A_{\ell-1})} + \frac{1}{2} A_{bc} \tau^{bc} \right) \]

\[ + \omega_{bc} \epsilon^{(a_\ell A_{\ell-1} c)} - \omega_{bc} \tau^{(a_\ell A_{\ell-1} c)} \right). \]

\[ (33) \]
4.1.1 Expanding out the correction

At time variable, into an integral equation and now work in the conformal Newtonian frame using the conformal time parameter \( \eta \) integrated up to some arbitrary time parameter. Here I have used that \( A(k) = -(|k^a|/a)\Phi_A(k, \eta) \) for \( \tilde{A}_a \approx D_0 \Phi_A \).

4.1.2 Primary Sourced correction

I change from comoving time, \( t \), to conformal time, \( \eta \), in Eq. (34). Invert the resulting equation into an integral equation and now work in the conformal Newtonian frame using the conformal time variable, \( dt = ad\eta \). The integral inversion will only work for weak nonlinearity as we have excluded the general feedback:

\[
(\delta \tau)^{NL}_\ell(k^*, \eta) \approx -(4\pi)\ell \int_{\eta_0}^{\eta_0} \int_{k_0}^{\infty} \frac{k^2dk'}{(2\pi)^2} \frac{k'}{a} \Phi_A(k', \eta) \left[ \tau_{\ell-1}(k - k', \eta) - \frac{1}{4} \tau_{\ell+1}(k - k', \eta) \right].
\]

It becomes convenient to rewrite this in terms of the following sources:

\[
T_\ell(k^*, \eta) = T^p_\ell(k^*, \eta) + T^s_\ell(k^*, \eta) = \frac{\beta_\ell}{(2\ell + 1)} \left[ \tau_{\ell-1}(k^*, \eta) - \frac{1}{4} \tau_{\ell+1}(k^*, \eta) \right],
\]

where \( S \) and \( P \) denote the secondary and primary induced corrections.

Essentially the argument in the following two sections is that the time integration over \( T_\ell(k, \eta) \) cancels when using weak coupling on small scales, that is for high-\( \ell \).

4.1.2 Primary Sourced correction

Using the almost-FLRW integral solution for the temperature anisotropies in the mode coefficient formulation of the nonlinear correction (Eq. 35) and using (36):

\[
\frac{\beta_\ell (\delta \tau)^{NL}_\ell(\eta_0, k^*)}{(2\ell + 1)} \approx +2\ell \int_{\eta_0}^{\eta_0} d\eta \int_{k_0}^{\infty} \frac{k^2dk}{(2\pi)^2} k \Phi_A(k, \eta) T^p_\ell(k^*, \eta),
\]

\[
T^p_\ell(k^*, \eta) \approx [\delta T + \Phi_A](k^* - k, \eta_*) \left[ \frac{(2\ell - 1)\beta_{\ell-1}(|k^* - k|)}{\beta_{\ell-1}} - \frac{1}{4} \frac{(2\ell + 3)\beta_{\ell+1}(|k^* - k|)}{\beta_{\ell+1}} \right].
\]

Notationally, what is important to realize is that the Bessel functions have conformal time arguments as \( \eta_0 - \eta \) inside the time integration. The co-efficients inside the square brackets are evaluated at \( \eta_* \) while the ones outside at \( \eta \). The best way to think of this is that the integral solution is integrated up to some arbitrary time parameter \( \eta \), this is then convolved as in the integral above and integrated from \( \eta_* \) until now, \( \eta_0 \).

Now, I will again use the weak-coupling approximation, arguing that the variations in the CMB temperature anisotropies are significantly more rapid than those in the potentials. This will allow me to reduce the potential term and remove the conformal time integration down the worldline:

\[
\frac{\beta_\ell}{(2\ell + 1)} \int_{\eta_*}^{\eta_0} d\eta k \Phi_A(k, \eta) T^p_\ell(k^*, \eta) \approx + \sqrt{\frac{\pi}{2 |k^* - k|}} \left[ \delta T_0 + \Phi_A \right](|k^* - k|, \eta_*)
\]

\[
\times \left[ \Phi_A(k, \eta_{\ell-1}^*) \frac{(2\ell - 1)\beta_{\ell-1}}{\beta_{\ell-1}(|k^* - k|)} - \frac{1}{4} \Phi_A(k, \eta_{\ell+1}^*) \frac{(2\ell + 3)\beta_{\ell+1}}{\beta_{\ell+1}(|k^* - k|)} \right].
\]

From \( 4 \), where \( \tilde{\omega}^a = n^a \approx u^a + v^a \) such that \( D_{(a} n_{b)} = \tilde{\sigma}_{ab} = 0 \), which if dominated by pressure free dust leads one to expect \( \omega^a \approx 0 \).
We should expect some cancellations to occur between the \( k^* \) and \(-k^*\) terms too. I use that 
\[
\beta_{\ell}/\beta_{\ell-1} = \frac{\ell}{(2\ell - 1)} \quad \text{and} \quad \beta_{\ell}/\beta_{\ell+1} = \frac{(2\ell + 1)\ell}{(\ell + 1)^2},
\]
This allows the following reduction:
\[
\frac{(2\ell + 3)\beta_{\ell}}{\beta_{\ell+1}\sqrt{\ell + 1}} = \frac{(2\ell + 3)(2\ell + 1)}{(\ell + 1)^{3/2}}, \quad \text{and} \quad \frac{(2\ell - 1)\beta_{\ell}}{\beta_{\ell-1}\sqrt{\ell - 1}} = \frac{\ell}{\sqrt{\ell - 1}}.
\]

I now take the high-\( \ell \) approximation (large-\( k \)), again:
\[
\ell(2\ell + 3)/(\ell + 1)^{3/2} \approx 2\sqrt{\ell}, \quad \text{and} \quad \ell^2/(2\ell + 1)\sqrt{\ell - 1} \approx \frac{1}{2}\sqrt{\ell},
\]
to find that by putting Eq. (40) into Eq. (39) and using Eq. (41), that upon collecting terms to
cancel factors of \( 1/4 \), we recover:
\[
\frac{\beta_{\ell}}{(2\ell + 1)} \int_{\eta_0}^{\eta_0} dk \Phi_A(k, \eta) T_{\ell}^\mu(k^*, \eta) \approx \frac{1}{2} \sqrt{\frac{\pi}{2\ell}} \frac{k}{|k^*-k|}
\]
\[
\times [\delta T_0 + \Phi_A] \left( \Phi_A(k, \eta_{\ell-1}^{k^*-k}) - \Phi_A(k, \eta_{\ell+1}^{k^*-k}) \right).
\]
Putting this back together with Eqs. (37 and 42) I find (the add itional \( \ell \) in front of (37) leads to
the \( \sqrt{\ell} \) scaling below and the factor 2 cancels):
\[
\frac{\beta_{\ell}}{(2\ell + 1)} (\delta T_0)^{NL-P}(\eta_0, k^*) \approx 0.
\]

First, for sufficiently high-\( \ell \) (that is small scales) we could argue that the peaks of the Bessel
functions are sufficiently close together (again) such that \( \eta_{\ell} \approx \eta_{\ell \pm 1} \); one immediately notices that
all the relevant terms will cancel out in Eq. (43), we then find that the primary sourced corrections
vanishes
\[
\frac{\beta_{\ell}}{(2\ell + 1)} (\delta T_0)^{NL-P}(\eta_0, k^*) \approx 0.
\]
So it would appear that in the weakly nonlinear case there is no effect. The cancellation has nothing
to do with the damping factor which we can safely ignore.

I have shown that the primary sourced effect cancels (this I have labeled NL-P). I now show that
the secondary sourced effect cancels too. This is more important as this is how the Rees-Sciama
effect would arise in the Newtonian frame.

4.1.3 Secondary source correction

From Eq. (35) the term of interest is
\[
T_{\ell}^\mu(k^*, \eta) = \frac{\beta_{\ell}}{(2\ell + 1)} \left[ \tau_{\ell-1}^{k^*} - \frac{1}{4} \tau_{\ell+1}^{k^*} \right]
\]

Now from the ISW source of anisotropy I have Eqs. (12 and 9)
\[
\tau_{ISW}^{\eta_0} \approx \left[ \frac{-(2\ell + 1)}{\beta_{\ell}} \sqrt{\frac{\pi}{2\ell}} \left( \frac{\Delta(k^*, \eta_0)}{k^* a^3} \right) \right] S_{ISW}(\eta_{\ell}^{k^*}),
\]
\[
S_{ISW}(\eta) \approx 3H_0^2\Omega_0 \frac{D'}{D} - \frac{a'}{a}.
\]
Putting Eq. (47) into Eq. (45) I find on using that $\beta_\ell/\beta_{\ell-1} \approx \frac{1}{2}$ and that $\beta_\ell/\beta_{\ell+1} \approx 2$ (for $\ell \gg 1$):

$$T_\ell^s(k^*, \eta) \approx -\frac{2\ell}{(2\beta_\ell)} \left( \frac{\beta_\ell}{2\ell} \right) \sqrt{\frac{\pi}{2\ell}} \left( \frac{\Delta(k^*, \eta_0)}{k^3} \right) S_{\text{ISW}}(\eta^s_\ell)$$

$$+ \frac{1}{4} \left( \frac{2\ell}{(\beta_{\ell+1})} \right) \sqrt{\frac{\pi}{2\ell}} \left( \frac{\Delta(k^*, \eta_0)}{k^3} \right) S_{\text{ISW}}(\eta^s_{\ell+1})$$

$$\approx \frac{1}{2} \frac{\Delta(k^*, \eta_0)}{k^3} \sqrt{\frac{\pi}{2\ell}} \left( S_{\text{ISW}}(\eta^s_k) - S_{\text{ISW}}(\eta^s_{k-1}) \right).$$

(48)

Now, using that $\eta_{\ell-1} \approx \eta_{\ell+1} \approx \eta \approx \eta - \ell/k$, I then find that

$$T^s_\ell(k^*, \eta) \approx 0, \quad \Rightarrow (\delta \tau)^{NL}_\ell \approx 0.$$  

(50)

I have arrived at advertised result; that acceleration sourced nonlinearity is at best small and in the limit vanishes.

This means that in the Newtonian frame there may be mild nonlinearity (in the sense of second order perturbation theory as pertaining to the dynamics of the matter) but there are no weakly nonlinear effects due to the acceleration potential in the Newtonian frame – as arising from nonperturbative small-scale effects in the radiation dynamics, which are the dominant gravitational effects on small scales [22].

The important point is that this provides some indications that the effects of weak nonlinearity are suppressed in the Newtonian frame; in some sense the Newtonian frame calculation is stable to weakly nonlinear contamination making it an ideal first approximation when trying to include additional nonperturbative source terms.

In the Newtonian frame one needs only include the linear Rees-Sciama effect (Eq. 17) which is known to be small [30]. In the CDM-frame (or the total frame) one would need to be more careful with the nonlinear Rees-Sciama corrections (NLRS).

### 4.2 Total-frame correction

In the CDM dominated almost-EdS universe we have that $A_d \approx 0$. One then only needs to consider the nonlinear corrections due to the shear (given that the vorticity is small) – this is a nonlinear shear sourced correction.

Here I consider the nonlinear corrections (Eq. 33) due to the shear in the generic $u^a$-frame (Eq. 32 in its manifestly covariant and gauge invariant form):

$$\langle \delta \tau \rangle^{NL}_\ell(k^*, \eta_0) \approx 4\pi \ell \int_{\eta_*}^{\eta_0} d\eta' \int_{k_0}^{\infty} k^2 dk \sigma(k, \eta) \left[ \tau_{\ell-2}(|k^* - k|) + \frac{1}{16} \tau_{\ell+2}(|k^* - k|) \right].$$  

(51)

This will not generically vanish on small-scales; however in the case of sufficient matter domination I would expect the effect to be small. In the context of an EdS background the ISW sourced effect will very small, because the effect of $(\Phi_H(k, \eta)/a)^\prime \approx (D/a)(D/D - a'/a)\Phi_H(k, 0)$ is vanishingly small – which sources the ISW terms in the temperature anisotropy. So I need to consider the nonlinear-ISW effect – the Rees-Sciama effect.

The coupling between the primary sourced effect and the linear-FLRW shear could be expected to, at best, lead to a smoothing of the acoustic peaks; this still needs to be properly shown. If we interpret the shear as sourced by nonlinear dynamics (which leads to the Rees-Sciama effect – the imprint of nonlinear dynamics on the temperature anisotropies) I can then isolate three possible effects:
1. The **primary correction** about the flat CDM background: due to the coupling between the almost-FLRW shear (a term like \(a'(a')\Phi_A(k,0)\)) to the primary sources of anisotropies - the acoustic and Sachs-Wolfe effects. This may be important on intermediate scales.

2. The **primary nonlinear correction** about a flat CDM background: due to the coupling between the primary sources of CMB temperature anisotropies and nonlinear shear, induced by nonlinear small scale matter dynamics. This would be important on small scales.

3. The **secondary nonlinear correction** (or nonlinear Rees-Sciama effect) about the flat CDM background: due to the coupling between the nonlinear-ISW effect (Rees-Sciama effect) in the CMB temperature anisotropies and the nonlinear corrections to the shear (a term like \(D'(a')\Phi_A(k,0)\)). This effect is expected to be the dominant small scale correction – and is due to small scale nonlinearity in the matter dynamics generating radiation anisotropies which couple back to the nonlinear shear.

I focus on this latter possibility, with the intent of getting a feel for the form of the angular correlation function due to this effect. The interest in the secondary correction, a Nonlinear Rees-Sciama effect, lies in the realization that the peak in the Rees-Sciama effect is in the range of \(\ell = 200 - 300\); where one expects to find the peak in the angular correlation function (apparently due to the projection of Doppler and acoustic oscillations \([18]\)). I now provide a simple calculation demonstrating this contribution. Once again this calculation should be viewed as the departure point towards more sophisticated treatments.

### 4.2.1 The almost-FLRW shear source

The shear in the energy-frame (which coincides with the total-frame in the CDM dominated case considered here), is given from \([5]\) as:

\[
\frac{1}{2} (\rho + p)\sigma_{ab} \approx \left(D_{(a}D_{b)}\Phi_H\right) + 3H D_{(a}D_{b)}\Phi_H - HD_{(a}D_{b)}(\Phi_H + \Phi_A). \tag{52}
\]

In addition, for sufficient matter domination \([6]\) \((A_0 \approx 0\) and hence \(\Phi_H \approx -\Phi_A\)):

\[
\rho(\eta)\sigma_{ab} \approx -2(a^3D_{(a}D_{b)}\Phi_H). \tag{53}
\]

Now \(a^2\rho(\eta_0) \approx 3K\Omega_{0m}/\Omega_{0m} - 1 \approx 3a^2H_0^2\Omega_0\) (where \(K\) is the FRW curvature) and \(\rho(\eta) = a^{-3}\rho(\eta_0)\), which for scalar perturbations \([5], [6]\) gives the mode coefficient for the shear\(^\text{16}\) using a flat CDM dominated model:

\[
\frac{3 H_0^2\Omega_0}{2 a^3} \sigma(k, \eta)Q_{ab} \approx \left(-\frac{k^2}{a^2}\Phi_H Q_{ab}\right) + 3H \left(-\frac{k^2}{a^2}\Phi_H Q_{ab}\right) \approx -\frac{k^2}{a^2} H\Phi_H(k,0)Q_{ab}. \tag{54}
\]

I then find, on using that \(\Phi_H(k,\eta) = D(\eta,k)\Phi_H(k,\eta_0)\) and \(\Phi_H(k,\eta_0) = \Phi_H(k,0)\), here for the dust case, and along with a change to conformal time \((dt = ad\eta)\), to find

\[
\frac{3 H_0^2\Omega_0}{2 a^3} \sigma(k, \eta) \approx -\frac{k^2}{a^2} D'/a\Phi_H(k,\eta_0). \tag{55}
\]

I have in mind \(\Phi_H \gg H\Phi_H\) for \(\Phi_H(k,\eta) = D(\eta)\Phi_H(k,0)\); the small scale nonlinear situation, where in the linear case \(D = a\). So I am then able to use the following result:

\[
\frac{3}{2} H_0^2\Omega_0\sigma(k, \eta) \approx -\frac{k^2}{a^2} D'/a\Phi_H(k,0). \tag{56}
\]

\(^{16}\text{Using } D_{a}\Phi_H = \frac{\partial}{\partial a}\Phi_H(k,\eta)Q_{a} \text{ and } D_{(a}D_{b)}\Phi_H = -\frac{k^2}{a^2}\Phi_H(k,\eta)Q_{ab}.\)
Equation (54) gives the almost-FLRW shear in terms of the scalar potentials $\Phi_A(k, 0) \approx -\Phi_H(k, 0)$. However, more importantly, it also shows how the nonlinear dynamics will affect the shear in the 1+3 covariant and gauge invariant approach, via the growth factor $D(k, \eta)$. The nonlinear source of $\Phi_H + H\Phi_H$ will give a mildly nonlinear shear effect.

An important point here is that we have used the shear in its total-frame formulation (that natural to the Lagrangian threading), while the CMB temperature anisotropies have been found in the Newtonian frame (that given by an Eulerian threading in the exact theory). To understand why we can use these two together we recall (i) that $E$ in the Newtonian frame (that given by an Eulerian threading in the exact theory), while the CMB temperature anisotropies have been found in [6], (ii) $\tau(\eta, k)$ is invariant to linear order for $\ell > 1$ (we already have the source terms for the CMB temperature perturbation, $\delta T(k, \eta)$, and the dipole, $\tau_1(\eta, k)$, from [8] in the Newtonian frame). We can use the scalar sourced anisotropies as found in the Newtonian frame [8] – we use the shear in the total frame along with the evolution equations for the other various dynamic quantities [7].

Notice that from the div-$E$ equation, $k^2\Phi_H(k, 0) \approx +\frac{3}{2}(D/a)H_0^2\Omega_0\Delta(k, \eta_0)$ [8]. Now I have that $\Delta(\eta, k) \approx a\delta + a^2\delta_2$, where in an EdS spacetime only the second-order part contributes to the time changing potential, hence to the growth factor as $D \sim D_+(\eta) \sim a^2(\eta)$. So I then find:

$$\sigma(k, \eta)^{\text{NL}} \approx -\frac{D'(\eta)}{a(\eta)}\Delta_{\text{NL}}(k, \eta_0),$$  

which is for sufficient matter domination on small scales.

In this section I have derived the form of the shear source term, I will use this result in the next section: $\sigma_{\text{NL}} \sim \sigma$.

### 4.2.2 Nonlinear Rees-Sciama effect

The nonlinear Rees-Sciama effect will take the form

$$\frac{\beta_\ell (\delta \tau)^{\text{NLRS}}}{(2\ell + 1)}(k^*, \eta_0) \approx \frac{\beta_\ell}{(2\ell + 1)(4\pi \ell)} \int_{\eta_0}^{\eta_\ell} d\eta \int_0^\infty \frac{k^2 dk}{(2\pi)^3} S^{\text{NLRS}}_\ell,$$

$$S^{\text{NLRS}}_\ell = \sigma(k, \eta) \left[ \tau^{\text{RS}}_{\ell-2}(|k^* - k|, \eta) + \frac{1}{16} \tau^{\text{RS}}_{\ell+2}(|k^* - k|, \eta) \right],$$

with the linear Rees-Sciama source terms

$$\sigma(k, \eta) \approx -\frac{D'}{a}\Delta(k, \eta_0) \approx -\frac{2}{3} \frac{1}{H_0^2\Omega_0} k^2 \Phi_H(k, 0) \frac{D'}{a},$$

$$\tau^{\text{RS}}_{\ell}(k, \eta) \approx -\frac{2(2\ell + 1)}{\beta_\ell} \sqrt{\frac{\pi}{2\ell k}} e^{-k} D'(\eta) \Phi_H(k, 0), \forall \eta \geq \ell,$$

where $\eta_\ell \approx \eta - (\ell + \frac{1}{2})/k$, as before. Now substituting Eq. (59) and Eq. (60) into Eq. (58) I am able to find:

$$\frac{\beta_\ell (\delta \tau)^{\text{NLRS}}}{(2\ell + 1)}(k^*, \eta_0) \approx \frac{8\pi}{3H_0^2\Omega_0} \sqrt{\frac{\pi \ell}{2}} (e^{-k})$$

$$\times \int_0^\infty \frac{k^2 dk}{(2\pi)^3} \left[ \frac{k^2}{|k^* - k|^3} \Phi_H(k) \Phi_H(|k^* - k|) \right] I_\ell(|k^* - k|),$$

$$I_\ell(|k^* - k|) \approx \int_{\eta_\ell}^{\eta_0} d\eta \frac{D'(\eta)}{a(\eta)} D'(\eta - \frac{k}{|k^* - k|}).$$

---

The Einstein field equations are solved in the total frame. See [8] for the scalar theory, the tensor part is given in [12]. The scalar theory is solved in terms of the scalar potentials, $\Phi_A(k, \eta)$ and $\Phi_H(k, \eta)$, and a peculiar velocity, $v(k, \eta)$. These can then be substituted into the Newtonian frame solution.
where I have used the high-\(\ell\) assumption, which gives the ratios of the \(\beta_{\ell}\) coefficients

\[
\frac{\beta_{\ell}}{\beta_{\ell-2}} \sim \frac{1}{4}, \quad \text{and,} \quad \frac{\beta_{\ell}}{\beta_{\ell+2}} \sim 4, \quad \forall \ell \gg 1, \tag{63}
\]

Using that \(\eta_{\ell-2} \sim \eta_{\ell+2}\), I can show that:

\[
I_{\ell}(|k^* - k|) \sim 16 \left[ \frac{\ell}{2} - \frac{1}{2} \frac{\ell}{k^*} + \frac{1}{2} \frac{\ell^2}{k^* k} \right] \sim 16 \left[ \frac{\ell}{2} - \frac{1}{2} \frac{\ell}{|k^*-k|} \right]. \tag{64}
\]

I also need to use that the second order growth parameter is \(D(a) \sim a^2(\eta)\), and that for \(\ell \sim k\), the higher order terms will drop off faster under the \(k\)-space integration.

### 4.3 Approximating the angular correlation function

In order to find the angular correlation function I first find the mean-square of the correction to the CMB temperature anisotropy.

I use that\[\footnote{This arises from \(\delta(k^* - k^\alpha) = \frac{1}{k^*} \delta(k - k') \delta(e^\alpha - e'^\alpha)\). Recall that \(\langle \Phi(k, 0) \Phi(k', 0) \rangle = (2\pi)^3 P_s(k) \delta(k^* - k^\alpha)\).} \]

\[
\langle \Phi_H(k^* - k) \Phi_H(k) \Phi_H(k'' - k') \Phi_H(k') \rangle = (2\pi)^6 P_{\Phi_H}(k^* - k) P_{\Phi_H}(k)
\]

\[
\times \left[ \frac{\delta(k^* - k'')}{k^2} \frac{\delta(k - k')}{k^2} + \frac{\delta(k^* - k'')}{k^2} \frac{\delta(k - k')}{(k^* - k)^2} \right], \tag{65}
\]

to then find the mean-square of the correction to the CMB temperature anisotropy due to the nonlinear Rees-Sciama coupling between the local matter nonlinearities and the CMB temperature anisotropies.

It is worthwhile pointing out that the basis of the argument used to construct the maximum mean square contribution is that it will be near the peak in the radiation transfer function: \(k \sim |k^* - k| \sim \ell(\eta_0 - \eta)\)\[\footnote{To many it may seem as though this is being written in by hand – this is not the case. One expects the Rees-Sciama effect to peak between, \(\ell = 100 - 300\)\[\footnote{[30]}, while the peak in the radiation transfer function is also in this region in the standard CDM model\[\footnote{[5]}].\}

This, along with \((D')^2 P_s \approx P_{\Phi_s'}\), shows

\[
\frac{\beta_{\ell}^2}{(2\ell + 1)^2} |\delta t_{\ell}^{NLRS}(k^*, \eta_0)|^2 = \left(4\pi\right)^2 \left[ \frac{8\pi}{3H_0^2\Omega_0} e^{-\kappa} \sqrt{\frac{\pi\ell}{2}} \right]^2
\]

\[
\times \int_{k_0}^{\infty} k^2 dk P_{\Phi_H}(k, \eta_0) P_{\Phi_H}(k^* - k, \eta_0) \frac{k^4}{k^2 + |k^* - k|^2}. \tag{66}
\]

The correction to the angular correlation function due to the nonlinear Rees-Sciama terms Eq. (64) is constructed from the definition of the angular correlation function\[\footnote{[3]}\]:

\[
C_{\ell}^{NLRS} \approx \frac{2}{\pi} \frac{\beta_{\ell}^2}{(2\ell + 1)^2} \int_0^{\infty} k^2 dk P_{\Phi_H}(k, \eta_0) P_{\Phi_H}(k^* - k, \eta_0) \left( \frac{k^2 \delta(k^* - k^\alpha)}{(\ell\eta_0)^2} \right). \tag{67}
\]

I then find that the approximate maximum correction to the angular correlation function is:

\[
C_{\ell}^{NLRS} \sim (16\pi^2) \left[ \frac{8\pi}{3H_0^2\Omega_0} e^{-\kappa} \right]^2 \ell \int_0^{\infty} dk^* \int_{k_0}^{\infty} k^2 dk P_{\Phi_H}(k, \eta_0) P_{\Phi_H}(k^* - k, \eta_0) \left( k^2 \delta(k^* - k^\alpha) \right) \tag{68}
\]

\[
\times \left( \frac{(k^* - k^\alpha)}{(\ell\eta_0)^2} \right).
\]
I have approximated the time integral by using the assumption that the effect is most prevalent near $\ell \sim k(\eta_0 - \eta)$ (again). We use that $P_{\Phi_H}(k, \eta) \approx D'(k, \eta)P_{\Phi_H}(k)$. I then have the following result:

$$C_{\ell, NLRS}^{NLRS} \sim (16\pi^2) \left[ \frac{8\pi}{3H_0^2\Omega_0} e^{-\kappa} \right]^2 \ell \int k^2 dk P_{\Phi_H}'(k, \eta_0) P_{\Phi_H}'(k, \eta_0 - \ell/k).$$  (69)

Now I argue that one can approximate the result by using $\ell \sim k\eta_0$ in the first power spectrum and using that $P_{\Phi_H}'(k) \propto k^{-1}P(k)$ (which excludes the cancellation of $(k\delta\eta)^{-1}$) to give on small scales $P_{\Phi_H}' \propto k^{-2}$. Then it can be conveniently shown that:

$$C_{\ell, NLRS}^{NLRS} \sim \alpha_0(32\pi^2) e^{-2\kappa_*\ell} \int_{k_0}^\infty dk P_{\Phi_H}'(k, \eta_0 - \ell/k).$$  (70)

Here $\alpha_0 = 2A [8\pi/3H_0^2\Omega_0]^2$. We once again use that $k \approx \ell(\eta_0 - \eta)$ and that $r \sim (\eta_0 - \eta)$ to then find that

$$C_{\ell, NLRS}^{NLRS} \sim \ell^2(32\pi^2) \int_0^{\eta_0} d\eta \int_0^{\eta_0} P_{\Phi_H}'(r, \eta).$$  (71)

This can be readily compared to the Rees-Sciama calculation (where we ignore the damping):

**Nonlinear Rees – Sciama**: $C_{\ell, NLRS}^{NLRS} \sim \ell^2 C_{\ell, RS}^{RS}$.  (72)

This is of course a gross approximation. However, I think that it does go some distance to demonstrate that if $C_{\ell, RS}^{RS}$ dominates the angular correlation functions near $\ell \sim 5000$, one can expect the nonlinear Rees-Sciama effect (Eq. 3) to dominate on scales much larger than this.

1. If $C_{\ell, RS}^{RS}$ gives $\Delta T_{RS}/T \sim 10^{-7} - 10^{-6}$ between $\ell \sim 100 - 300$ (where the Rees-Sciama effect peaks [31]), one can naively expect, from (72) to find that $\Delta T_{NLRS}/T \sim 10^{-5} - 10^{-4}$ between $\ell \sim 100 - 300$.

2. If I include damping and cancellation as $\dot{\kappa}/k$ I then find :

$$C_{\ell, NLRS}^{NLRS} \sim \ell C_{\ell, RS}^{RS}.$$  (73)

This then gives $\Delta T_{NLRS}/T \sim 10^{-6} - 10^{-5}$ between $\ell \sim 100 - 300$. The latter is probably more realistic. The keypoint is that it is does not seem to be a negligible effect.

5 **Conclusions**

Although my results on nonlinearity are by no means conclusive, I have evidence that the Newtonian threading suppresses nonperturbative nonlinearity on small scales, Eq. (44 - 50). This along with the well know result that the Newtonian threading is inconsistent beyond linear order give an indication that such treatments are generically inadequate for the study of relativistic cosmology unless the a priori assumption that the universe is close to almost-FLRW on all observationally relevant scales is made. Although the self-consistency of this assumption has be shown [31, 32] – such an approach is not generic.

If this assumption is relaxed one discovers that there could be additional small-scale effects that have been excluded, by definition, from the canonical treatments of CMB temperature anisotropies Eq. (3) and Eq. (6). One such small-scale effect, the Rees-Sciama effect, when included in the
frame work of the nonperturbative small-scale corrections to the radiation dynamics, leads one to the conclusion that there could be contributions of the same order of magnitude near the peak in the radiation transfer function as the anisotropies themselves due a coupling of the radiation via gravity to the nonlinear matter dynamics, Eq. (72). This conclusion is not consistent with the canonical treatment - from which it is excluded by construction.

I have provided some evidence demonstrating that gravitational nonlinearity may become problematic before the standard $\ell = 5000$ limit in the case of no feedback between the CMB temperature anisotropies – the scaling is different from the canonical treatment as an additional $k$ scaling is introduced from the exact treatment. It can be expected, when the feedback is included, that the effect can become more problematic if there is no significant smoothing effect, Eq. (73).

One weakness in the above approach is that the almost-FLRW anisotropies for high – $\ell$ are found in the Newtonian frame, then boosted to the total frame. As $\tau_{\ell}$ is invariant under small frame boost the solutions are the same as those found in the Newtonian frame. The relationship between the shear and the acceleration potentials, the potentials and the matter perturbations and the peculiar velocities and the matter perturbations are all found in the total frame. Such tricks are only consistent for small relative velocities – at best this treatment is valid for mild nonlinearity. The other obvious weakness lies in the applicability of the string of approximations required to reduce the angular correlation function to that which takes on a form similar to the usual Rees-Sciama effect – which is the trick that allows the comparison between the Rees-Sciama effect and the nonlinear (in the Boltzmann equations) Rees-Sciama effect discussed here.

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