ON DIFFERENCE SETS WITH SMALL $\lambda$

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Dedicated to K.T. Arasu on the occasion of his 65th birthday.

Abstract. In a 1989 paper [1], Arasu used an observation about multipliers to show that no $(352, 27, 2)$ difference set exists in any abelian group. The proof is quite short and required no computer assistance. We show that it may be applied to a wide range of parameters $(v, k, \lambda)$, particularly for small values of $\lambda$. With it a computer search was able to show that the Prime Power Conjecture is true up to order $2 \cdot 10^{10}$, extend Hughes and Dickey’s computations for $\lambda = 2$ and $k \leq 5000$ up to $10^{10}$, and show nonexistence for many other parameters.

1. Introduction

A $(v, k, \lambda)$-difference set $D$ in a group $G$ of order $v$ is a set $\{d_1, d_2, \ldots, d_k\}$ of elements from $G$ such that every nonzero element of $G$ has exactly $\lambda$ representations as $d_i - d_j$. The order of $D$ is $n = k - \lambda$.

A (numerical) multiplier is an integer $m$ for which multiplication of each $d_i$ by $m$ produces a shift of the original difference set: $mD = D + g$ for some $g \in G$. The set of multipliers form a group $M$, and it is well-known that some translate of $D$ is fixed by $M$. This implies that a shift of $D$ can be written as a union of orbits of $G$ under $M$.

The First Multiplier Theorem states that any prime $p > \lambda$ which divides $n$ and not $v$ must be a multiplier of $D$. The Multiplier Conjecture is that the $p > \lambda$ condition is not needed. This is still open, but there have been many strengthenings of the First Multiplier Theorem; see [8] for recent results.

Many difference set parameters can be dealt with by finding a group of multipliers $M$ and looking at the resulting orbits. For instance, it may be that no union of orbits has size $k$, or the set of orbits may be small enough that all possibilities may be checked with a short search. Lander, in [10], gives many such examples.

Arasu [1] showed that no abelian biplanes (difference sets with $\lambda = 2$) of order 25 exist. Our main tool will be a generalization of his argument, which we restate here.
Theorem 1. No $(352, 27, 2)$ difference set exists in any abelian group $G$.

Proof. Any such difference set has 5 as a multiplier. Take $p = 11$, and $H$ a group of order 32 so that $G = \mathbb{Z}_{11} \times H$. Then $5^8 \equiv 1 \pmod{32}$, and so fixes $H$. The orbits of $\langle 5^8 \rangle$ in $\mathbb{Z}_{11}$ are $\{0\}$, $\{1, 3, 4, 5, 9\}$, and $\{2, 6, 7, 8, 10\}$. The orbits in $G$ are just these orbits with a fixed element $h \in H$.

A difference set $D$ made up of these orbits will have a certain number $a$ of $5$-orbits $\langle (1, h) \rangle$ and $\langle (2, h) \rangle$, and $b = 27 - 5a$ 1-orbits. There are $b(b-1)$ differences of the singleton orbits, each of which is of the form $(0, h)$ with $h \neq 0$. There are 31 such elements, and each must occur exactly twice as a difference of elements of $D$, and so $b(b-1) \leq 31 \cdot 2 = 62$.

This means that we must have $b < 9$, and so $a \geq 4$. But the 20 differences from elements in one 5-orbit are all of the form $(x, 0)$, $x \neq 0$. There are 10 such elements, and in fact each of them occurs exactly twice in the differences of one 5-orbit. Since we have multiple 5-orbits, these elements will occur as differences too many times. □

One nice feature of this argument is that it takes care of all abelian groups $G$ of order 352 at once. Other arguments ([2], [10]) only handle specific groups.

2. Extending the Method

It is clear that Arasu’s method can be applied to other parameter sets. In this section we give a generalization of Theorem 1.

Lemma 2. Let $G = \mathbb{Z}_p \times H$, where $H$ is abelian and gcd$(p, |H|) = 1$. Let $m$ be a multiplier of a $(v, k, \lambda)$ difference set, and $s$ be the smallest positive integer for which $m^s \equiv 1 \pmod{\exp(H)}$. Then the orbits of $G$ under $\langle m^s \rangle$ are of the form $(\mathcal{O}, h)$, for fixed $h \in H$. There are exactly $|H|$ orbits $(0, h)$ of size 1, and the remaining orbits all have the same size $o = \ord_p(m^s)$.

Proof. The proof of this is the same as for Theorem 1. The group of multipliers generated by $m^s$ will fix all $h \in H$ Because $p$ is prime, all the nonzero orbits of $\mathbb{Z}_p$ under this group will have the same size, some divisor of $p - 1$. □

Now for any $(v, k, \lambda)$, if we can find a prime $p|v$ and multiplier $m$ for which $m^s$ has a reasonably large order mod $p$, we can look at differences of the 1-orbits and $o$-orbits and try to get a contradiction: if there are $a$ orbits of size $o$, and $b$ 1-orbits, then we have:
Theorem 3. Let $G = \mathbb{Z}_p \times H$, where $H$ is abelian and $\gcd(p, |H|) = 1$. Let $m$ be a multiplier of a $(v, k, \lambda)$ difference set, and $s$ be the smallest positive integer for which $m^s \equiv 1 \pmod{\exp(H)}$, and $o = \text{ord}_p(m^s)$. If there is no solution in positive integers $a$ and $b$ to:

$$k = ao + b$$

$$b(b - 1) \leq \lambda(|H| - 1)$$

$$a \cdot o(o - 1) \leq \lambda(p - 1)$$

then no $(v, k, \lambda)$ difference set exists in $G$.

This method will be most useful when $\lambda$ is small, since each element can only occur $\lambda$ times as a difference, so whatever the choice of orbits either elements of the form $(x, 0)$ or $(0, h)$ are likely to occur too many times. Still, when $n$ and $v$ have large prime factors ($n$ so that we have a known multiplier, and $v$ so that we have a suitable $p$ to use in Theorem 3), it can still often be applied.

When Theorem 3 fails, if $G$ is cyclic we will sometimes use the theorem of Xiang and Chen [12]:

Theorem 4. Let $D$ be a $(v, k, \lambda)$ difference set in a cyclic group $G$ with multiplier group $M$. Except for the $(21, 5, 1)$ difference set, $|M| \leq k$.

This theorem may be extended to contracted multipliers as well (see Section VI.5 of [4] for information about difference lists and contracted multipliers).

Theorem 5. Let $D$ be a $(v, k, \lambda)$ difference set in a cyclic group $G$, and $H$ be the subgroup of $G$ of order $h$ and index $u$. Then with the same exception, the group $M$ of $G/H$-multipliers has order $|M| \leq k$.

Proof. The proof is exactly the same as the proof of Theorem 4 in [12], replacing multipliers with contracted multipliers. $M$ is isomorphic to a subgroup of $\text{Gal} \mathbb{Q}(\zeta_u)/\mathbb{Q}$, where $\zeta_u$ is a primitive $u$th root of unity. Let

$$S = \overline{D} = \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_k}\}$$

be the $(u, k, h, \lambda)$ difference list over $G/H$ obtained by sending the elements of $D$ to their image in $G/H$. By Theorem 5.14 of [4], we may assume that $S$ is fixed by $M$. Let $\chi$ be a generator of the character group of $G/H$, $K = \mathbb{Q}(\chi(S), \chi^2(S), \ldots, \chi^{u-1}(S))$, and $\alpha_t$ be the field automorphism sending $\zeta_u \mapsto \zeta_{u}^t$. As in [12], we may show that $\text{Gal} \mathbb{Q}(\zeta_u)/K = M$. If $t \in M$ it fixes $S$, so $\alpha_t$ fixes $\chi(S)$. If $\alpha_t$ fixes $\chi^i(S)$ for $i = 1, 2, \ldots, u - 1$, then by Fourier inversion $t$ fixes $S$, and so is in $M$. 
Now let
\[ f(X) = \prod_{i=1}^{k} \left( X - \chi(d_i) \right). \]

The coefficients of \( f(X) \) are elementary symmetric polynomials in the \( \chi(d_i) \), which are fixed by \( \alpha_t \) for any \( t \in M \), so \( f(X) \in K[X] \).

By Theorem 1 of Cohen [5], if \( D \) is not the \((21,5,1)\) difference set, then at least one of the \( d_i \) is relatively prime to \( v \), and so \( \chi(d_i) \) is a primitive \( u \)th root of unity. It is also a root of \( f(X) \), and so
\[ |M| = [\mathbb{Q}(\zeta_u) : K] \leq \deg f(X) = k. \]

\[ \square \]

3. The Prime Power Conjecture

A \((v,k,1)\) difference set is called a planar abelian difference set. These exist if \( n = k - 1 \) is a prime power, and the Prime Power Conjecture (PPC) is that these are the only ones. In [6] it was shown that the PPC is true for all groups for orders up to \( 2 \cdot 10^6 \), and in [3] for cyclic groups for orders up to \( 2 \cdot 10^9 \). Peluse [11] recently showed that the PPC is asymptotically true; the number of orders up to \( N \) for which planar abelian difference sets exist is \( O(N/\log N) \), the same as the number of prime powers.

In these papers non-prime power orders were eliminated by a series of tests; see [6] for details. The initial tests only depended on the prime factors of \( n \) and \( v \), and were very fast. Tables 1 and 2 in [6] gave lists of \((v,k,1)\) planar abelian difference set parameters which could not be eliminated with these tests. To show they did not exist, Proposition 5.11 of Lander [10] was used:

**Theorem 6.** If \( t_1, t_2, t_3, t_4 \) are numerical multipliers of a \((v,k,1)\) difference set in \( G \), and
\[ t_1 - t_2 \equiv t_3 - t_4 \pmod{\exp(G)}, \]
then \( \exp(G) \) divides \( \text{lcm}(t_1 - t_2, t_1 - t_3) \).

For each case a large number of multipliers were generated, until either a prime known not to be an extraneous multiplier was discovered, or two pairs of multipliers with the same difference modulo \( \exp(G) \) were found, so that Theorem 6 could be applied. These calculations required a substantial amount of computation time and memory.

With Theorem 3 the hard cases from [6] can be eliminated quickly. To illustrate the power of the theorem, Table 4 gives parameters used in Theorem 3 to eliminate some of the parameters in the tables in [6];
with the value of $o$ in the last column, it is easy to check that there are no positive integers $a$ and $b$ solving equations (1), (2) and (3).

Using Arasu's method allows the computations to be redone in a different manner. In addition, it requires far less work for the hard cases, so it was possible to take the computations further. Replicating the search up to $2 \cdot 10^6$ took under a minute on a workstation. A longer run using the fast tests from [6] and Theorem 3 eliminated every order up to $2 \cdot 10^{10}$ except for the ones given in Table 2, which were then eliminated using Theorem 6. Note that the first two values of $k$ were missing from the tables in [6].

Unlike the fast tests in [6], for which the number passing was roughly linear in the bound on $n$, Theorem 3 gets more effective for larger orders, since it becomes increasingly likely that $v$ will have a large prime factor $p$ for which some prime divisor of $n$ has large order mod $p$. All values of $k$ between $7.7 \cdot 10^9$ and $2 \cdot 10^{10}$ were eliminated, and

| $k$  | $p$    | $|H|$ | $m^*$ | $\text{ord}_p(m^*)$ |
|------|--------|------|-------|----------------------|
| 2436 | 5931661| 1    | $5^4$ | 435                  |
| 24452| 199291951| 3   | 499$^1$| 6175                |
| 45152| 22651   | 90003| $27^789$| 25                  |
| 56408| 24781   | 128397| 4339$^{63}$| 295                |
| 58724| 450601  | 7653 | 8389$^{75}$| 751                |
| 2444 | 109    | 54777| $7^{465}$| 9                   |
| 3234 | 4759   | 2197 | 61$^{507}$| 61                  |
| 72012| 35911  | 144403| 673$^{245}$| 513                |
| 73482| 149113 | 36211| 373$^9$ | 2071               |

Table 1. Small $(v, k, 1)$ parameters from Tables 1 and 2 of [6] eliminated by Theorem 3

| $k$  | $n$    | $v$                   |
|------|--------|-----------------------|
| 1096386| 5 \cdot 219277 | 79 \cdot 109 \cdot 1951 \cdot 71551 |
| 1320794| 373 \cdot 3541  | 3 \cdot 11551 \cdot 50341831 |
| 2378196| 5 \cdot 475639 | 211 \cdot 631 \cdot 3319 \cdot 12799 |
| 20846324| 61 \cdot 341743 | 3 \cdot 88951 \cdot 1628496601 |
| 40027524| 107 \cdot 374089 | 7 \cdot 13 \cdot 3541 \cdot 54163 \cdot 91801 |
| 2830957656| 5 \cdot 566191531 | 109^2 \cdot 1171 \cdot 1231 \cdot 1951 \cdot 239851 |
| 7700562788| 9817 \cdot 784411 | 3 \cdot 61^2 \cdot 1831 \cdot 1703287^2 |

Table 2. $(v, k, 1)$ parameters up to $k = 2 \cdot 10^{10}$ not eliminated by Theorem 3
a heuristic argument suggests that the number of cases up to order $n$ passing Theorem 3 will be at most $O(\log n)$.

4. Biplanes

Theorem 1 was also shown by Hughes in [9]. Computations by Hughes and Dickey reported in that paper showed that no abelian $(v, k, 2)$ difference sets exist with order less than 5000, except for the known cases $k = 3, 4, 5, 6$ and 9. They give few details about their method; it is possible that their method was something similar to that of Arasu.

A run up to order $10^{10}$ eliminated all but 24 parameters. Most of the rest were dealt with using Theorems 4.19 and 4.38 of Lander [10]. Table 3 gives the remaining open cases.

Theorem 5 was an important tool for eliminating open cases in this and the next table. Biplanes of order a power of 4, such as $(525826, 1026, 2)$, pass Theorem 8 and have no known multipliers, so the standard methods are no help. However, in each case up to order $2^{30}$ we have that $G$ is cyclic, 2 is a $G/H$ multiplier for $H$ the group of order 2 by the Contracted Multiplier Theorem (Corollary 5.13 of [4]), and the order $ord_{v/2}(2)$ is larger than $k$, showing that those biplanes do not exist.

5. General Parameters

Theorem 8 may be applied for larger $\lambda$; while more parameters will slip through because of a lack of known multipliers or Equations (2) and (3) being less restrictive, many may still be eliminated. A run was done for difference sets with $\lambda = 3$ up to order $10^{10}$. There were 269 parameters that passed Theorem 8 but most were then eliminated with Theorems 4 and 5, the Lander tests, and the Mann test ([4], Theorem VI.6.2). Table 4 shows the six remaining cases.
The author has set up the La Jolla Difference Set Repository [7], an online database containing existence results for parameters up to $v = 10^6$, as well as a large number of known difference sets. There are 1.44 million parameters that pass basic counting and the BRC theorem, of which about 180,000 were open. Applying Theorems 3 and 5 resolved over 50,000 of them.

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