Robust control design for home pension service mobile robots with passive and servo constraints

Yating Zhao¹,², Xiaolong Chen³ and Han Zhao³
¹College of Economics and Management, Hefei Normal University, Hefei, Anhui Province, People’s Republic of China
²School of Management, Hefei University of Technology, Hefei, Anhui Province, People’s Republic of China
³School of Mechanical Engineering, Hefei University of Technology, Hefei, Anhui Province, People’s Republic of China

Abstract
This paper presents a novel robust control design for a class of home pension service mobile robots (HPSMRs) with non-holonomic passive constraints, based on the Udwadia-Kalaba theory and Udwadia control. The approach has two portions: dynamics modeling and robust control design. The Udwadia-Kalaba theory is employed to deal with the non-holonomic passive constraints. The frame of the Udwadia control is employed to design the robust control to tracking the servo constraints. The designed approach is easy to implement because the analytical solution of the control force can explicitly be obtained even if the non-holonomic passive constraints exist. The uniform boundedness and uniform ultimate boundedness are demonstrated by the theoretical analysis. The effectiveness of the proposed approach is verified through the numerical simulation by a HPSMR.

Keywords
Home pension service robots, passive constraints, servo constraints, Udwadia-Kalaba theory, Udwadia control, dynamics modeling, robust control

Corresponding author:
Xiaolong Chen, School of Mechanical Engineering, Hefei University of Technology, No.193 Tunxi Road, Hefei, Anhui Province 230009, People’s Republic of China.
Email: c.xlong@qq.com

Creative Commons Non Commercial CC BY-NC: This article is distributed under the terms of the Creative Commons Attribution-NonCommercial 4.0 License (https://creativecommons.org/licenses/by-nc/4.0/) which permits non-commercial use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
Introduction

Data from the World Health Organization shows that the number of elderly people around the world is steadily increasing and it is expected to reach 2 billion elderly population by 2050. In order to improve the quality of life of elderly people and their ability of independent living, robotic solutions centered around human-centered and home-oriented have been greatly developed in the past decade. Therefore, home pension service mobile robots (HPSMRs) have been attended extensively. The HPSMRs can be regarded as a special kind of mobile robots serving human beings, which can carry out home service work for older adults.

In general, constraints acting on mechanical systems have many classification methods. If we focus on the way in which the constraint force arises, it can be divided into two categories: the passive one and servo one. The passive one shows more about the inherent characteristics of the system itself and the relationship between the systems and environment. Thus, the corresponding constraint force is generated by the system structure or environment. The servo one reflects more about the control requirements that the system needs to meet. Thus, the corresponding constraint force is provided by the servo control. For controlling mobile robots, some excellent methods have been reported in the literature. However, most of the methods mentioned above can not be applied in a straightforward manner to mechanical systems with non-holonomic passive constraints, or whose analysis and computation do not entirely take into account the complex nonlinear effects of the dynamical model of mobile robots.

As other mobile robots, the HPSMRs also include the non-holonomic passive constraints in their moving process. For example, a HPSMR with a non-holonomic passive constraint in a home is shown in Figure 1. When the HPSMR moves along a desired trajectory, it cannot move axially. Therefore, there exists a non-holonomic passive constraint on the axial direction of the HPSMR. The non-holonomic passive constraint contains a differential element which cannot be transformed into geometric constraint by integration, which makes HPSMRs’ dynamics modeling and control design difficult by the Lagrange multiplier method.

Udwadia and Kalaba devoted to research the motion of the constrained mechanical systems and they developed a novel theory to deal with the motion in 1992, which is praised as the Udwadia-Kalaba theory at present. In the Udwadia-Kalaba theory, the constraint is studied in the passive category that can be ideal, non-ideal, holonomic, and non-holonomic. In dynamics modeling, the theory presents an explicit, general equations of motion for passive constrained, discrete dynamical systems in terms of the generalized coordinates that describe their configurations. In 2003, when the constraint was related to the servo category, the Udwadia control arose at the historic moment, which is a novel tracking control method for non-linear mechanical systems from a new perspective in which the prescribed tracking trajectories are viewed as the servo constraints that are imposed on these systems. The applications of the Udwadia control was extensively reported in recent years. However, Udwadia control does not consider any uncertainty
that is inevitable in the actual mechanical systems. Motivated by these reasons, a robust control is designed based on the frame of the Udwadia control in this paper.

This paper is focused on the dynamics modeling and robust trajectory tracking control for HPSMRs. From the perspective of inverse dynamics, the main contributions of this paper are threefold. First, the Udwadia-Kalaba theory is employed to address the dynamics modeling when the non-holonomic passive constraints are involved. Second, taking the system uncertainty into consideration, a novel robust control is designed based on the frame of the Udwadia control to drive the systems to tracking the pre-specified servo constraints approximately. Third, the uniform boundedness and uniform ultimate boundedness of the controlled system are verified.

The remainder of this paper is outlined as follows. In Section 2, a class of HPSMRs is described, including the formulations of dynamic equation, passive constraints and servo constraints. In Section 3, a robust control is designed for HPSMRs. In Section 4, the effectiveness of the proposed approach for the HPSMRs is shown by a numerical experiment in Matlab, followed by the conclusion in Section 5.

**System description**

The motion equation of a class of constrained HPSMRs can be obtained by relation
\[ H(\phi(t), \chi(t), t)\ddot{t}(t) = F(\phi(t), \dot{\phi}(t), \chi(t), t) + Q(\phi(t), \dot{\phi}(t), \chi(t), t) + \tau(t) \]  (1)

Here, \( t \in R \) represents time, \( \phi \in R^n \) represents the position, \( \dot{\phi} \in R^n \) represents the velocity, \( \ddot{\phi} \in R^n \) represents the acceleration, \( \chi \in \sum \subset R^p \) represents the (possible fast time varying) uncertain parameter, and the set \( \sum \subset R^p \) which stands for the possible bound of \( \chi \), is compact and prescribed. Moreover, \( H(\cdot) \in R^{n \times n} \) is the inertia matrix. \( F(\cdot) \in R^n \) is the force imposed on the system whose constraints are released. \( Q(\cdot) \in R^n \) is the force imposed on the system whose constraints are released. \( \tau(\cdot) \in R^n \) is the servo constraint force, which constructively utilizes the Lagrange aspect, provided by the actuator which drives the system while the constraints evolves as time.

For each \((\phi, t) \in R^n \times R, \sum \subset R^p\), there is \( H(\cdot) > 0 \), and the functions \( H(\cdot), F(\cdot) \) and \( Q(\cdot) \) are of appropriate dimensions. For the passive constraint force \( Q \), we shall assume that the system is subjected to \( h \) holonomic passive constraints of the form

\[ \varphi_i(\phi, t) = 0 \quad i = 1, 2, \cdots, h \]  (2)

and \( m_p - h \) non-holonomic passive constraints of the form

\[ \varphi_i(\phi, \dot{\phi}, t) = 0 \quad i = h, h + 1, \cdots, m_p \]  (3)

Next, combine first derivative of holonomic passive constraints with respect to \( t \) with non-holonomic passive constraints, then put it in the matrix form

\[ A_p(\phi, t)\dot{\phi} = c_p(\phi, t) \]  (4)

where \( A_p \in R^{m_p \times n} \) and \( c_p \in R^{m_p \times 1} \), which is the first-order passive constraint form.

Differentiating holonomic passive constraints twice and non-holonomic passive constraints twice with respect to \( t \), put them in matrix form

\[ A_p(\phi, t)\ddot{\phi} = b_p(\phi, \dot{\phi}, t) \]  (5)

where \( b_p \in R^{m_p \times 1} \), which is the second-order passive constraint form.

**Remark 1.** The passive constraints (4) and (5) have to meet strictly in the whole process of the systems’ moving regardless of the initial condition and the uncertainty.

Now, the passive constraint force \( Q \) can be obtained based on following theorem:

**Theorem 1.** (Udwadia-Kalaba theory\textsuperscript{25}): Consider mechanical systems and passive constraints. The passive constraint force can be given as
\[
Q(\phi, \dot{\phi}, \chi, t) = H^1(\phi, \chi, t) (A_p(\phi, t) H^{-1}(\phi, \chi, t))^+ [b_p(\phi, \dot{\phi}, t) - A_p(\phi, t) H^{-1}(\phi, \chi, t) F(\phi, \dot{\phi}, \chi, t)]
\]

Here, "+" stands for the Moore-Penrose generalized inverse.\(^{37}\)

Remark 2. There is no elimination or transformation of coordinates when the
holonomic passive constraints are presented in the Udwadia-Kalaba theory,
which leads to a simple and new fundamental view of Lagrangian mechanics.
The coordinates describing the unconstrained systems are same as those describ-
ing the constrained systems, which is responsible for the simplicity of the expli-
cit equations and the fundamental insights about the nature of the constrained
motion. By the Udwadia-Kalaba theory, the explicit and general motion equa-
tions for a constrained HPSMR system can be established in following three
steps. Firstly, in terms of the generalized coordinates, the equation of the
unconstrained dynamical system of the HPSMR can be acquired by using
Lagrangian or Newtonian mechanics. Second, the passive constraint equations
are formed, which maybe include the holonomic and non-holonomic. In the
end, the passive constraint forces are imposed on the HPSMR system, namely,
the passive constraint forces resulted from the presence of passive constraints
are added to the unconstrained HPSMR system’s force.

By similar reasoning, we can get first-order and second-order servo constraints
and put them in the matrix form as follows
\[
A_s(\phi, t) \dot{\phi} = c_s(\phi, t) \tag{7}
\]
where \(A_s \in \mathbb{R}^{m_s \times n}\) and \(c_s \in \mathbb{R}^{m_s \times 1}\), which is the first-order servo constraint form.

\[
A_s(\phi, t) \ddot{\phi} = b_s(\phi, \dot{\phi}, t) \tag{8}
\]
where \(b_s \in \mathbb{R}^{m_s \times 1}\), which is the second-order servo constraint form.

Remark 3. The constraints in engineering can be classified into categories,
namely passive ones and servo ones, according to the constraint forces provided
by different subjects. In the passive constraint problem, the focus is to let the
environment include the structure of the mechanical system to generate the
required constraint forces. In the servo constraint problem, which is a control
problem in reality, the focus is to let a mechanical system, equipped with servo
controls follow a set of constraints by generating the required constraint forces
through servo controls. The passive constraints (4) and (5) will be met sponta-
neously by the natural environment, but regarding the servo constraints (7) and
(8), an appropriate servo constraint force shall be offered to make this type con-
straints satisfied. The servo constraints (7) and (8) are not like the passive con-
straints (4) and (5) that have to be strictly met. The servo constraints only need
to be approximately followed, because the strict control requirement is not
needed for people in real life. Moreover, with the uncertainty in presence and no restrictions on the initial condition, it is only reasonable to expect approximate constraint following. Without considering the model uncertainties and initial conditions, Udwadia\textsuperscript{30} creatively designed a servo constraint force by analytical form. In Udwadia control, the holonomic and non-holonomic servo constraints will be treated equivalently, thus it has a great potential in addressing the stability or trajectory following control problem because these issues can be converted to control problems.

\textit{Remark 4.} A constrained HPSMR could be described by (1), which is a second-order motion equation. A HPSMR is equipped with passive constraint force follow a set of passive constraints (4) and (5), and the passive constraint force can be obtained by using Theorem 1, which is a modeling problem. Then, the HPSMR constrained by passive constraints follows a set of servo constraints (7) and (8), which becomes a control problem in reality. The servo constraint problem can now be stated as follows: Determine the servo control \( \tau \) such that the resulting controlled system approximately observes the servo constraints (7) and (8), which can be the pre-given trajectory and set-point.

\section*{Robust control design}

Before designing the control \( \tau \), it is need to consider the uncertainties existing in the HPSMRs. So supposing the matrices/vectors \( H, F \) and \( Q \) in (1) can be decomposed as follows\textsuperscript{6,38}:

\begin{align*}
H(\phi, \chi, t) &= \tilde{H}(\phi, t) + \Delta H(\phi, \chi, t) \tag{9} \\
F(\phi, \dot{\phi}, \chi, t) &= \tilde{F}(\phi, \dot{\phi}, t) + \Delta F(\phi, \dot{\phi}, \chi, t) \tag{10} \\
Q(\phi, \dot{\phi}, \chi, t) &= \tilde{Q}(\phi, \dot{\phi}, t) + \Delta Q(\phi, \dot{\phi}, \chi, t) \tag{11}
\end{align*}

Here, \( \tilde{H}, \tilde{F} \) and \( \tilde{Q} \) represent the “nominal” portions, while \( \Delta H, \Delta F \) and \( \Delta Q \) are the corresponding uncertain ones. Let \( \Delta D(\phi, \chi, t) := H^{-1}(\phi, \chi, t) - \tilde{H}^{-1}(\phi, t), \Delta D(\phi, t) := H^{-1}(\phi, t), \) and \( E(\phi, \chi, t) := \tilde{H}(\phi, t)H^{-1}(\phi, \chi, t) - I. \) Thus, we notice that

\begin{align*}
\Delta D(\phi, \chi, t) = D(\phi, t)E(\phi, \chi, t).
\end{align*}

\textbf{Assumption 1.} The functions \( \tilde{H}(\cdot), \Delta H(\cdot), \tilde{F}(\cdot), \Delta F(\cdot), \tilde{Q}(\cdot) \) and \( \Delta Q(\cdot) \) are all continuous. For each \( (\phi, t) \in R^n \times R \), there is rank \( A_2(\phi, t) \geq 1 \), and \( A_2(\phi, t)A_2^T(\phi, t) \) is invertible.

\textbf{Assumption 2.} There exists \( \rho_E(\cdot) : R^n \times R \rightarrow (-1, \infty) \) such that for all \( (\phi, t) \in R^n \times R \)

\begin{align*}
\frac{1}{2} \min_{\chi \in C} \lambda_0(E(\phi, \chi, t) + E^T(\phi, \chi, t)) \geq \rho_E \tag{12}
\end{align*}
Remark 5. If there is no uncertainty in $H$ (i.e. $\dot{H} = H$, $E = 0$), one can choose $\rho_E = 0$ to meet the assumption. By continuity, there is a unidirectional threshold for the allowable uncertainty in $E$. For the current setting, it is less restrictive than the standard assumption in this area that $\max _{x \in \Sigma} \| E(\phi, \chi, t) \| < 1$.

Assumption 3. For given $P \in \mathbb{R}^{m \times m}$, $P > 0$, let

$$
\psi(\phi, t) := PA_s(\phi, t)D(\phi, t)D(\phi, t)A_s^T(\phi, t)P
$$

There exists a scalar constant $\Lambda > 0$ such that

$$
\inf _{(\phi, t) \in \mathbb{R}^n \times \mathbb{R}} \lambda_m(\psi(\phi, t)) \geq \Lambda
$$

Remark 6. Under Assumption 1, $\psi(\phi, t)$ is always positive definite. Thus the Assumption 3 is to assure $\lambda_m(\psi(\phi, t))$ positively bounded from below.

Now consider the approximate servo constraint following problem. That is, it is possible that $A_s \phi \neq c_s$ or $A_s \phi \neq b_s$ (let $\beta(\phi, \phi, t) = A_s(\phi, t)\phi - c_s(\phi, t)$, hence $\beta \neq 0$, where $\beta$ represents the error between actual trajectory and the desired trajectory. The aim is to control $\beta \to 0$ so that $(A_s \phi - c_s) \to 0$ when $t \to \infty$). This maybe due to system uncertainty. In addition, the system may not start with the constraint manifold in the beginning (i.e. $\beta \neq 0$ as $t = t_0$). Consider the following robust control design

$$
\tau(t) = p_1(\phi, \dot{\phi}, t) + p_2(\phi, \dot{\phi}, t) + p_3(\phi, \dot{\phi}, t)
$$

with

$$
p_1(\phi, \dot{\phi}, t) := \dot{H}(\phi, t)(A_s(\phi, t)\dot{H}^{-1}(\phi, t))^+ [b(\phi, \dot{\phi}, t)
$$

$$
- A_s(\phi, t)\dot{H}^{-1}(\phi, t)(\bar{F}(\phi, \dot{\phi}, t) + \bar{Q}(\phi, \dot{\phi}, t))]
$$

$$
p_2(\phi, \dot{\phi}, t) := -\kappa\dot{H}^{-1}(\phi, t)A_s^T(\phi, t)P(\phi, \dot{\phi}, t)
$$

$$
p_3(\phi, \dot{\phi}, t) := -\gamma(\phi, \dot{\phi}, t)\mu(\phi, \dot{\phi}, t)\rho(\phi, \dot{\phi}, t)
$$

where

$$
\gamma(\phi, \dot{\phi}, t) = \begin{cases} 
\frac{(1 + \rho_E)^{-1}}{\| \tilde{\mu}(\phi, \dot{\phi}, t) \|} & \text{if } \| \mu(\phi, \dot{\phi}, t) \| > \varepsilon \\
\frac{(1 + \rho_E)^{-1}}{\| \tilde{\mu}(\phi, \dot{\phi}, t) \|^2} & \text{if } \| \mu(\phi, \dot{\phi}, t) \| \leq \varepsilon 
\end{cases}
$$

$$
\mu(\phi, \dot{\phi}, t) = \eta(\phi, \dot{\phi}, t)\rho(\phi, \dot{\phi}, t)
$$
Based on (23), we have

$$\eta(\phi, \dot{\phi}, t) = \hat{\mu}(\phi, \dot{\phi}, t)$$ \hspace{1cm} (21)$$

$$\hat{\mu}(\phi, \dot{\phi}, t) = H^{-1}(\phi, t)A_s^T(\phi, t)P$$ \hspace{1cm} (22)

$\varepsilon, \kappa \in R, \varepsilon, \kappa > 0$. The function $\rho(\phi, \dot{\phi}, t) : R^n \times R^n \times R \rightarrow R$ is chosen such that

$$\rho(\phi, \dot{\phi}, t) \equiv \max_{\lambda \in \Sigma} \| PA_s \Delta \dot{D}(F + Q + p_1 + p_2) + PA_s \Delta F + \Delta Q \|$$ \hspace{1cm} (23)

**Remark 7.** The proposed robust control is designed as three portions: $p_1$, $p_2$, and $p_3$. $p_1$ is Udawida control. $p_2$ is developed to suppress initial condition deviation. $p_3$ is developed to compensate uncertainty.

**Theorem 2.** (In Chen: Consider the mechanical system of (1), and suppose that Assumption 1 to 3 are met. The robust control ((15)) renders the following performance:

1) Uniform boundedness: For any $r > 0$, there is a $d(r) < \infty$, such that if $\| \beta(\phi(t_0), \dot{\phi}(t_0), t_0) \| \leq r$, then $\| \beta(\phi(t), \dot{\phi}(t), t) \| \leq d(r)$ for all $t \geq t_0$.

2) Uniform ultimate boundedness: For any $r > 0$ with $\| \beta(\phi(t_0), \dot{\phi}(t_0), t_0) \| \leq r$, there exists a $d > 0$, such that $\| \beta(\phi(t), \dot{\phi}(t), t) \| \leq d$ for any $d > d$ as $t \geq t_0 + T(d, r)$, where $T(d, r) < \infty$. Furthermore, $\tilde{d} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

**Proof.** A Lyapunov function candidate is chosen as:

$$V(\beta) = \beta^TP\beta$$ \hspace{1cm} (24)

Now, we prove the stability of the HPSMR systems with the proposed control. For a given uncertainty $\chi(\cdot)$, the derivative of $V$ along a trajectory is evaluated as (henceforth, for simplicity, arguments of functions are sometimes omitted when no confusions are likely to arise):

$$\dot{V} = 2\beta^TP\dot{\beta}$$

$$= 2\beta^TP(A_s\ddot{\phi} - b_s)$$

$$= 2\beta^TP[A_sH^{-1}(F + Q) + A_sH^{-1}\tau - b_s]$$

$$= 2\beta^TP[A_sH^{-1}(F + Q) + A_sH^{-1}(p_1 + p_2 + p_3) - b_s]$$

$$= 2\beta^TP[A_sD(\bar{F} + \bar{Q}) - b_s + A_sD(\Delta F + \Delta Q) + A_s\Delta D(F + Q + p_1 + p_2) + A_sDp_1 + A_sDp_2 + A_s(D + \Delta D)p_3]$$ \hspace{1cm} (25)

By (16), we have

$$2\beta^TP[A_sD(\bar{F} + \bar{Q}) + A_sDp_1 - b_s] = 0$$ \hspace{1cm} (26)

Based on (23), we have...
$$2\beta^T P[A,D(\Delta F + \Delta Q) + A,\Delta D(F + Q + p_1 + p_2)]$$
\[\leq 2 \| \beta \| \| PA_sD(\Delta F + \Delta Q) + PA_s\Delta D(F + Q + p_1 + p_2) \| \tag{27}\]
\[\leq 2 \| \beta \| \rho \]

By (17), we get
\[2\beta^T PA_sDp_2 = 2\beta^T PA_sD(-\kappa H^{-1} A_s^T P\beta) = -2\kappa \eta^T \eta = -2\kappa \| \eta \|^2 \tag{28}\]

By $\Delta D = DE, H^{-1} = D$, and (18), we have
\[
2\beta^T PA_s(D + \Delta D)p_3 \\
= 2\beta^T PA_s(D + \Delta D)(-\gamma \mu \rho) \\
= 2(\mu A_sP\beta \rho)^T (I + E)(-\gamma \mu) \\
= 2\mu^T (I + E)(-\gamma \mu) \\
= -2\gamma \mu^T \mu - 2\gamma \mu^T E\mu \\
= -2\gamma \mu^T \mu - 2\gamma \frac{1}{2} (E + E^T)\mu \\
\leq -2\gamma \| \mu \|^2 - 2\gamma \frac{1}{2} \lambda_m(E + E^T) \| \mu \|^2 \\
\leq -2\gamma (1 + \rho \varepsilon) \| \mu \|^2
\]

As $\| \mu \| > \varepsilon$, by (19), we have
\[ -2\gamma (1 + \rho \varepsilon) \| \mu \|^2 = -2 \frac{(1 + \rho \varepsilon)^{-1}}{\| \mu \|^2} \| \mu \|^2 = -2 \| \beta \| \rho \tag{30}\]

As $\| \mu \| \leq \varepsilon$, by (19), we have
\[ -2\gamma (1 + \rho \varepsilon) \| \mu \|^2 = -2 \frac{(1 + \rho \varepsilon)^{-1}}{\| \mu \|^2} \| \mu \|^2 = -2 \frac{\| \beta \|^2 \rho^2}{\varepsilon} \tag{31}\]

With (26) to (31), we have, for all $\| \mu \| > \varepsilon$,
\[\dot{V} = 2\beta^T P\dot{\beta} \leq -2\kappa \| \eta \|^2 + 2 \| \beta \| \rho - 2 \| \beta \| \rho = -2\kappa \| \eta \|^2 \tag{32}\]

and for all $\| \mu \| \leq \varepsilon$,
\[\dot{V} = 2\beta^T P\dot{\beta} \leq -2\kappa \| \eta \|^2 + 2 \| \beta \| \rho - 2\frac{\| \beta \|^2 \rho^2}{\varepsilon} \leq -2\kappa \| \eta \|^2 + \frac{\varepsilon}{2} \tag{33}\]

Finally, we have
\[\dot{V} \leq -2\kappa \| \eta \|^2 + \frac{\varepsilon}{2} \tag{34}\]

By Rayleigh’s principle and Assumption 3
||\eta||^2 = \eta^T \eta = \beta^T P A_s D D A^T_s P \beta \geq \lambda_m (P A_s D D A^T_s P) \ || \beta||^2 \geq \underline{\lambda} \ || \beta||^2 \quad (35)

Therefore,

\dot{V} \leq -2\kappa \underline{\lambda} \ || \beta||^2 + \frac{\varepsilon}{2} \quad (36)

Upon invoking the standard arguments as in Chen\textsuperscript{39} and Khalil,\textsuperscript{40} we conclude uniform boundedness with

\begin{align*}
d(r) = \begin{cases} 
\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} R} & \text{if } r \leq R \\
\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} r} & \text{if } r > R
\end{cases}
\end{align*} \quad (37)

\begin{align*}
R = \sqrt{\frac{\varepsilon}{4 \kappa \underline{\lambda}}}
\end{align*} \quad (38)

Furthermore, uniform ultimate boundedness also follows with

\begin{align*}
\bar{d} \geq \bar{d} = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} R
\end{align*} \quad (39)

\begin{align*}
T(\bar{d}, r) = \begin{cases} 
0, & \text{if } r \leq \bar{d} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \\
\frac{\lambda_{\max}(P) r^2 - \lambda_{\min}(P) \bar{d}^2}{\lambda_{\max}(P)}, & \text{otherwise.}
\end{cases}
\end{align*} \quad (40)

The uniform ultimate boundedness ball size \( \bar{d} \) can be made arbitrary small by a suitable choice of \( \varepsilon \).

**Numerical experiment**

To verify the proposed approach above mentioned, a HPSMR with a non-holonomic constraint shown in Figure 2 is carried out. Its two rear wheels are the driving wheels. The configuration of the HPSMR can be represented by the triplet \( \phi = [x \ y \ \theta]^T \), where \( x \) and \( y \) are the coordinates of mass center on \( X - Y \) plane, and \( \theta \) is the orientation angle of the HPSMR. The symbol definition of the variables and parameters of the HPSMR is listed in Table 1 in detail.

Based on the Newtonian mechanics, the unconstrained HPSMR’s motion equation can be obtained easily as
Then, considering the passive constraint, servo constraints and uncertainty, the system can be described as

$$H(\phi, \chi, t)\ddot{\phi} = F(\phi, \dot{\phi}, \chi, t) + Q(\phi, \dot{\phi}, \chi, t) + \tau(t)$$

(42)

where
The non-holonomic passive constraint of the HPSMR is given as follows

$$\dot{y} \cos (\theta) - \dot{x} \sin (\theta) = 0$$

(43)

Differentiating (43) with respect to $t$, we have

$$\ddot{y} \cos (\theta) - \dot{y} \dot{\theta} \sin (\theta) - \ddot{x} \sin (\theta) - \dot{x} \dot{\theta} \cos (\theta) = 0$$

(44)

in the matrix form as

$$A_p(\phi, \dot{\phi}, t)\ddot{\theta} = b_p(\phi, \dot{\phi}, t)$$

(45)

where

$$A_p = \begin{pmatrix} -\sin (\theta) & \cos (\theta) & 0 \end{pmatrix},$$

$$b_p = \begin{pmatrix} \dot{x} \cos (\theta) + \ddot{x} \dot{\theta} \sin (\theta) \end{pmatrix}.$$ .

By using Theorem 1, we can get the passive constraint force of the HPSMR as

$$Q(\phi, \dot{\phi}, x, t) = \dot{H}(\phi, x, t)(A_p(\phi, t)H^{-1}(\phi, x, t))^+ \times [b_p(\phi, \dot{\phi}, t) - A_p(\phi, t)H^{-1}(\phi, x, t)F(\phi, \dot{\phi}, x, t)]$$

(46)

To perform the approximate trajectory tracking control of the HPSMR based on the designed robust control, we assume the pre-given servo constraints as follows

$$\begin{cases} x = 20 \sin (t) \\ y = 10 \cos (t) \end{cases}$$

(47)

Differentiating (47) once with respect to $t$ yields

$$\begin{cases} \dot{x} = 20 \cos (t) \\ \dot{y} = -10 \sin (t) \end{cases}$$

(48)

and differentiating (47) twice with respect to $t$ yields

$$\begin{cases} \ddot{x} = -20 \sin (t) \\ \ddot{y} = -10 \cos (t) \end{cases}$$

(49)

Equations (48) and (49) can be rewritten in the form of (7) and (8), then we can get

$$A_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, c_s = \begin{pmatrix} 20 \cos (t) \\ -10 \sin (t) \end{pmatrix}, b_s = \begin{pmatrix} -20 \cos (t) \\ -10 \sin (t) \end{pmatrix}.$$
By theorem 2, the analytical solution of servo control force based on the servo constraints (47) and (49) can be given as

\[
\tau = p_1 + p_2 + p_3 = \begin{pmatrix}
F_r \cos(\theta) + F_l \cos(\theta) \\
F_r \sin(\theta) + F_l \sin(\theta) \\
F_r l - F_l l
\end{pmatrix} = \begin{pmatrix}
\frac{\mu_r \cos(\theta)}{r} & \frac{\mu_1 \cos(\theta)}{r} \\
\frac{\mu_2 \sin(\theta)}{r} & \frac{\mu_3 \sin(\theta)}{r} \\
\frac{\mu_4 l}{r} & -\frac{\mu_5 l}{r}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{\cos(\theta)}{r} & \frac{\cos(\theta)}{r} \\
\frac{\sin(\theta)}{r} & \frac{\sin(\theta)}{r} \\
\frac{l}{r} & -\frac{l}{r}
\end{pmatrix} \begin{pmatrix}
\mu_r \\
\mu_l
\end{pmatrix} = : B \mu
\]

Finally, the control input can be given as

\[
\mu(t) = B^+ \tau(t) = B^+ (p_1 + p_2 + p_3)
\]

The solution of the numerical simulation can be obtained through ode15i algorithm in MATLAB. We consider the mass is uncertain parameter (hence, \(m = \bar{m} + \Delta m, \Delta m\) is unknown). Figures 3 to 9 show the simulation results by choosing \(\bar{m} = 10\text{Kg}, \quad \Delta m = \sin(0.5t)\text{kg}, \quad J = 0.1\text{Kg} \cdot \text{m}^2, \quad l = 0.5\text{m}, \quad r = 0.1\text{m}, \quad P = I_{2 \times 2}, \quad \kappa = 1, \quad \rho = 1, \quad \varepsilon = 0.01, \quad \mu_E = -0.9\). The initial values of the generalized coordinate are chosen \(\phi = [x \ y \ \theta]^T = [-3 \ 5 \ 0]^T\) and \(\dot{\phi} = [x \ y \ \theta]^T = [0 \ 0 \ 0]^T\). Figure 3 and 4 show the motion of the mass center of the HPSMR on \(x\) and \(y\) directions, respectively, under the servo constraints. At the same time, the actual trajectory are compared with the desired trajectory in Figure 5 \(X - Y\) plane, which denotes that the proposed approach is effective. The trajectory tracking errors are shown in Figures 6 and 7. Figures 8 and 9 show the driving torques of the left motor and right, respectively.

Figure 3. The motion of the mass center of the HPSMR on \(x\) direction.
Conclusion

A novel approach is proposed based on the frame of Udwadia’s and Kalaba’s methods for HPSMRs. This approach has two hierarchies: dynamics modeling and robust control design. The hierarchy of dynamics modeling is based on the Udwadia-Kalaba theory to address the problem of the non-holonomic constraint existing in the HPSMRs. The hierarchy of robust control design is based on the frame of the Udwadia control to solve the problem of HPSMRs’ approximate trajectory tracking control with uncertainty. The robust control guarantees the uniform boundedness and uniform ultimate boundedness of the controlled HPSMRs, which is analyzed by using Lyapunov method. The effectiveness of the proposed control is shown by a numerical experiment, and which shows that the errors are
Figure 6. The error of HPSMR in the $x$ direction.

Figure 7. The error of HPSMR in the $y$ direction.

Figure 8. The driving torque $\mu_i$ curve generated by the left motor.
stable at the least in the order $10^{-3}$, so the trajectory tracking control is approximately realized.

**Declaration of conflicting interests**

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work was supported by the University Synergy Innovation Program of Anhui Province (GXXT-2019-031)

**ORCID iD**

Xiaolong Chen https://orcid.org/0000-0003-1954-0069

**References**

1. World Health Organization. 10 facts on ageing and health, https://www.who.int/features/factfiles/ageing/en/, 2017.
2. Zachiotis GA, Andrikopoulos G, Gornez R, et al. A survey on the application trends of home service robotics. In: 2018 IEEE international conference on Robotics and Biomimetics (ROBIO), Kuala Lumpur, Malaysia, December 2018, pp.1999–2006. New York: IEEE.
3. Do HM, Pham M, Sheng W, et al. Rish: a robot-integrated smart home for elderly care. Rob Auton Syst 2018; 101: 74–92.
4. Chen HZ, Tian GH and Liu GL. A selective attention guided initiative semantic cognition algorithm for service robot. Int J Autom Comput 2018; 15(5): 559–569.
5. Ramoly N, Bouzeghoub A and Finance B. A framework for service robots in smart home: an efficient solution for domestic healthcare. IRBM 2018; 39(6): 413–420.
6. Chen YH. Constraint-following servo control design for mechanical systems. *J Vib Control* 2009; 15(3): 369–389.
7. Matraji I, Al-Durra A, Haryono A, et al. Trajectory tracking control of skid-steered mobile robot based on adaptive second order sliding mode control. *Control Eng Pract* 2018; 72: 167–176.
8. Zhai Jy and Song Zb. Adaptive sliding mode trajectory tracking control for wheeled mobile robots. *Int J Control* 2019; 92(10): 2255–2262.
9. Mohamed M and Abbas M. Design a fuzzy pid controller for trajectory tracking of mobile robot. *J Eng Technol* 2018; 36(1A).
10. Abbas MY and Mohamed MJ. Design interval type-2 fuzzy like (pid) controller for trajectory tracking of mobile robot. *Iraqi J Comp Commun Control Sys Eng* 2019; 19(3): 1–15.
11. Xu JX, Guo ZQ and Lee TH. Design and implementation of a takagi–sugeno-type fuzzy logic controller on a two-wheeled mobile robot. *IEEE Trans Ind Electron* 2012; 60(12): 5717–5728.
12. Staicu S. Dynamics equations of a mobile robot provided with caster wheel. *Nonlinear Dyn* 2009; 58(1–2): 237.
13. Alakshendra V and Chiddarwar SS. Adaptive robust control of mecanum-wheeled mobile robot with uncertainties. *Nonlinear Dyn* 2017; 87(4): 2147–2169.
14. Fu J, Tian F, Chai T, et al. Motion tracking control design for a class of nonholonomic mobile robot systems. *IEEE Trans Syst Man Cybern Syst* 2018; 50: 2150–2156.
15. Zhang X, Peng Z, Yang S, et al. Distributed fixed-time consensus-based formation tracking for multiple nonholonomic wheeled mobile robots under directed topology. *Int J Control* 2019; 1–10.
16. Fateh MM and Arab A. Robust control of a wheeled mobile robot by voltage control strategy. *Nonlinear Dyn* 2015; 79(1): 335–348.
17. Nascimento TP, Dorea CE and Goncalves LMG. Nonholonomic mobile robots’ trajectory tracking model predictive control: a survey. *Robotica* 2018; 36(5): 676–696.
18. Siciliano B, Sciavicco L, Villani L, et al. *Robotics: modelling, planning and control*. London: Springer Science and Business Media, 2010.
19. Nascimento TP, Basso GF, Dorea C, et al. Perception-driven motion control based on stochastic nonlinear model predictive controllers. *IEEE/ASME Trans Mechatron* 2019; 24: 1751–1762.
20. Ray JR. Nonholonomic constraints. *Am J Phys* 1966; 34: 406–408.
21. Sun H, Zhao H, Zhen S, et al. Application of the udwadia kalaba approach to tracking control of mobile robots. *Nonlinear Dyn* 2015; 83(1–2): 1–12.
22. Pappalardo CM and Guida D. On the dynamics and control of underactuated nonholonomic mechanical systems and applications to mobile robots. *Arch Appl Mech* 2019; 89: 669–698.
23. Kozlowski K. Robot motion and control. *J Intell Robot Syst* 2019; 93: 617–619.
24. Pappalardo CM and Guida D. Forward and inverse dynamics of a unicycle-like mobile robot. *Machines* 2019; 7(1): 5.
25. Udwadia FE and Kalaba RE. A new perspective on constrained motion. *Proc R Soc Lond A Math Phys Sci* 1992; 439: 407–410.
26. Zhao H, Zhen S and Chen YH. Dynamic modeling and simulation of multi-body systems using the udwadiakalaba theory. *Chin J Mech Eng* 2013; 26: 839–850.
27. Zhang B, Zhen S, Zhao H, et al. A novel study on kepler’s law and inverse square law of gravitation. *Eur J Phys* 2015; 36: 035018.
28. Yin H, Chen YH and Yu D. Vehicle motion control under equality and inequality constraints: a diffeomorphism approach. *Nonlinear Dyn* 2019; 95(1): 175–194.

29. Zhao XM, Chen YH, Zhao H, et al. Udwadia–kalaba equation for constrained mechanical systems: Formulation and applications. *Chin J Mech Eng* 2018; 31: 1–14.

30. Udwadia FE. A new perspective on the tracking control of nonlinear structural and mechanical systems. *Proc R Soc Lond A Math Phys Sci* 2003; 459: 1783–1800.

31. Kang H, Shunqiang S, Shengchao Z, et al. Dynamic analysis and tracking trajectory control of a crane. *Proc IMechE, Part E: J Process Mechanical Engineering* 2017; 231(5): 1045–1052.

32. Liu X, Zhen S, Huang K, et al. A systematic approach for designing analytical dynamics and servo control of constrained mechanical systems. *IEEE/CAA J Autom Sin* 2015; 2(4): 382–393.

33. Zhao H, Li C, Huang K, et al. Trajectory tracking control of parallel manipulator based on udwadia-kalaba approach. *Math Probl Eng* 2017; 2017.

34. Yin H, Chen YH and Yu D. Controlling an underactuated two-wheeled mobile robot: a constraint-following approach. *J Dyn Syst Meas Control* 2019; 141(7): 071002.

35. Li C, Zhao H, Zhen S, et al. Udwadia–kalaba theory for the control of bulldozer link lever. *Adv Mech Eng* 2018; 10(6): 1687814018779728.

36. Chen X, Zhao H, Zhen S, et al. Adaptive robust control for a lower limbs rehabilitation robot running under passive training mode. *IEEE/CAA J Autom Sin* 2019; 6(2): 493–502.

37. Goult RJ. *Applied linear algebra*. Chichester: John Wiley and Sons Inc, 1978.

38. Chen X, Zhao H, Sun H, et al. A novel adaptive robust control approach for underactuated mobile robot. *J Franklin Inst* 2019; 356(5): 2474–2490.

39. Chen YH. On the deterministic performance of uncertain dynamical systems. *Int J Control* 1986; 43(5): 1557–1579.

40. Khalil HK. *Nonlinear systems*. 3rd ed. Upper Saddle River, NJ: Prentice-Hall Inc, 2002.

41. Daniel JW and Noble B. *Applied linear algebra*. Englewood Cliffs, NJ: Prentice-Hall, 1977.

**Author biographies**

**Yating Zhao** was born in 1984. She received the M.B.A. from Hannam University, South Korea, in 2011. She is currently a Ph.D. candidate at Hefei University of Technology, China. Her research interests include marketing management, business administration, and pension service robot.

**Xiaolong Chen** was born in 1994. He received the B.E. from Hefei University of Technology, China, in 2016. He is currently a Ph.D. candidate at Hefei University of Technology, China, and a visiting scholar at National University of Singapore, Singapore. His research interests include robust control, adaptive robust control, and fuzzy optimization.
Han Zhao was born in 1957. He received the Ph.D. degree from Aalborg University, Denmark, in 1990. He is currently a professor at Hefei University of Technology, China. His research interests include mechanical transmission, magnetic machine, vehicles, digital design and manufacturing, information system, dynamics, and control.