Efficient Estimation of Graph Trussness

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Abstract

A \(k\)-truss is an edge-induced subgraph \(H\) such that each of its edges belongs to at least \(k - 2\) triangles of \(H\). This notion has been introduced around ten years ago in social network analysis and security, as a form of cohesive subgraph that is rich of triangles and less stringent than the clique.

The trussness of a graph is the maximum \(k\) such that a \(k\)-truss exists.

The problem of computing \(k\)-trusses has been largely investigated from the practical and engineering point of view. On the other hand, the theoretical side of the problem has received much less attention, despite presenting interesting challenges. The existing methods share a common design, based on iteratively removing the edge with smallest support, where the support of an edge is the number of triangles containing it.

The aim of this paper is studying algorithmic aspects of graph trussness. While it is possible to show that the time complexity of computing exactly the graph trussness and that of counting/listing all triangles is inherently the same, we provide efficient algorithms for estimating its value, under suitable conditions, with significantly lower complexity than the exact approach. In particular, we provide a \((1 \pm \epsilon)\)-approximation algorithm that is asymptotically faster than the exact approach, on graphs which contain \(\omega(m \text{ polylog}(n))\) triangles, and has the same running time on graphs that do not. For the latter case, we also show that it is impossible to obtain an approximation algorithm with faster running time than the one of the exact approach when the number of triangles is \(O(m)\), unless well known conjectures on triangle-freeness and Boolean matrix multiplication are false.

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1 Introduction

Consider an undirected graph \(G = (V(G), E(G))\) with \(n = |V(G)|\) nodes and \(m = |E(G)|\) edges, where \(N_G(v)\) represents the neighborhood of a node \(v\).\(^1\) Triangles in \(G\) are popular patterns studied in social networks to identify cohesive subgraphs. Recall that a triangle is a set of three pairwise-connected nodes \(u, v, z\) (hence, edges \(\{u, v\}, \{v, z\}, \{z, u\}\) belong to that triangle). For each edge \(e = \{u, v\}\) in \(G\), its support \(\text{sup}_G(e) = |N_G(u) \cap N_G(v)|\) is the number of triangles to which \(e\) belongs.

Among the cohesive subgraphs based on triangles, \(k\)-trusses have quickly spread in security and social sciences, as they enforce the presence of many triangles \(^{\text{III}}\). For an integer \(k \geq 0\),

\(^1\) We assume wlog that \(G\) does not contain isolated nodes, thus its size is \(O(n + m) = O(m)\).
Efficient Estimation of Graph Trussness

Figure 1 Left: a graph containing (dashed lines, starting from the outside) a 2-core (orange), 1-truss (green) and 4-clique (blue); a 4-clique is also a 2-truss—according to the 0-based definition—, so the graph trussness is $t_G = 2$. Note that a $k$-truss is a subgraph of a $(k + 1)$-core, and that a $k$-clique is a $(k - 2)$-truss. Right: a graph (complete bipartite plus an extra node $v_x$) with trussness $t_G = 1$, arboricity $\alpha_G = \Theta(n)$, and $T_G = \Theta(n^2)$ triangles.

we define the $k$-truss of $G$ as the maximal (edge-induced) subgraph $H$ of $G$ such that each edge $e$ of $H$ belongs to at least $k$ triangles of $H$. Specifically, $H = (V(H), E(H))$ where $E(H) \subseteq E(G)$, and $V(H) = \{x \in V(G) \mid \{x, y\} \in E(H)\}$, such that $\sup_{H(e)} \geq k$ for every $e \in E(H)$. Visual examples are given in Figure 1 (left), comparing $k$-trusses with the known notions of $k$-cores (i.e., each node in the subgraph has degree at least $k$) and $k$-cliques (i.e., the $k$ nodes in the subgraph are pairwise connected).

Over the years, the notion of $k$-truss has become popular in community detection, and is gaining momentum for purposes other than security [11, 40, 24, 9, 36, 39, 26, 43, 15], providing a remarkable benchmark (along with triangle counting and listing) to test new ideas, such as in the MIT/Amazon/IEEE GraphChallenge [26, 30, 36, 31, 14]. It has also been considered under different names, such as $k$-dense subgraph [33], triangle $k$-core [45], $k$-community [38], and $k$-brace [37].

Many existing highly-engineered solutions [30, 24, 30] are sophisticated implementations of the same basic algorithmic idea presented in [11], and sometimes referred to as peeling: the $k$-truss of $G$ is obtained by recursively deleting edges with support smaller than $k$ in the residual graph.

As a matter of fact, these algorithms compute a truss decomposition, namely, for each edge $e$ in $G$, they find its trussness $t_G(e)$ which is the largest $k$ such that a $k$-truss contains $e$.

Problem studied In this paper we study the problem of approximating the trussness $t_G$ of $G$, defined as the maximum $k$ such that there exists a $k$-truss in $G$. Note that $G$ is triangle-free if and only if $t_G = 0$. As $t_G = \max_{e \in E(G)} t_G(e)$, we can clearly compute exactly the trussness of $G$ from its truss decomposition. However, we observe in Appendix A that (1) not only the truss decomposition, and thus $t_G$, can be computed in $O(m \alpha_G)$ time and $O(m)$ space, where the arboricity $\alpha_G$ is the minimum number of forests into which the edges

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2 The original definition in [11] is 2-based, i.e., assumes that $k \geq 2$ and requires that $\sup_{H(e)} \geq k - 2$, so a $k$-clique is a $k$-truss. However, it is more convenient algorithmically, as we will see, to adopt our equivalent 0-based definition.

3 In a similar fashion to $k$-cores, obtained by recursively removing nodes with less than $k$ neighbors.
of $G$ can be partitioned but that (2) a conditional lower bound exists, suggesting that $O(m \alpha_G)$ time is difficult to improve for an exact computation of the value of $t_G$.

Because of (1) and (2), it is an interesting algorithmic question to see if the trussness $t_G$ can be approximated in less than $O(m \alpha_G)$ time without resorting to the exact truss decomposition.

We say that $\tilde{t}_G$ is an $r$-approximation for $t_G$, where $r > 1$ is a constant, if the relation $\frac{t_G}{r} \leq \tilde{t}_G \leq rt_G$ holds. For the special and interesting case $r = 1 + \epsilon$ with $0 < \epsilon < 1$, this implies the condition $(1 - \epsilon)t_G \leq \tilde{t}_G \leq (1 + \epsilon)t_G$, and we call $\tilde{t}_G$ a $(1 \pm \epsilon)$-approximation.

Given graph $G$, we want to compute a $(1 \pm \epsilon)$-approximation $\tilde{t}_G$ faster than computing $t_G$.

**Results** Let $T_G$ be the number of triangles in $G$. We show that we can improve the $O(m \alpha_G)$-time bound when $T_G = \omega(m \text{polylog}(n))$ (for all polylog$(n)$). This follows from our general result that, for any $0 < \epsilon < 1$, a $(1 \pm \epsilon)$-approximation of the trussness $t_G$ can be obtained, with high probability, in expected time $O\left(\epsilon^{-3} \min\left\{m \log m \over t_G + 1\right\} m \alpha_G \log(t_G + 2)\right)$ and space $O(\epsilon^{-2} m \log m)$ (Theorem 1).

Looking at the above bound, we also observe that in case $T_G$ is smaller, e.g. $T_G = \Theta(m)$, our approximation algorithm substantially has the same time cost $O(m \alpha_G)$ as the exact computation of $t_G$. However, we prove a matching conditional lower bound for this case, stating that there is no combinatorial algorithm which approximates $t_G$ within a multiplicative factor or an additive term taking $O(m \alpha_G)$ time, unless there exists a truly subcubic “combinatorial” algorithm for Boolean Matrix Multiplication (BMM) [42]. Indeed we prove that such an algorithm would improve the complexity of BMM and triangle-freeness, breaking the well known cubic lower bound (see [42]) as well as the 3-SUM conjecture [27].

Under the same conditions, i.e., the non-existence of a truly subcubic combinatorial algorithm for BMM, we also show that no combinatorial algorithm that provides an approximation by either a multiplicative factor or an additive term, may run in time $O(m(t_G + 1))$.

If, on the other hand, we move away from the scope of combinatorial algorithms and use matrix multiplication, where $\omega$ denotes, as in [42], the smallest real number such that an $(n \times n)$-matrix multiplication can be computed in $O(n^\omega)$, we show that a $(2 + \epsilon)$-approximation of the trussness $t_G$ can be computed in either $O(\epsilon^{-1} n^\omega \log 2^n / \epsilon)$ or $O(\epsilon^{-1} n^\omega \frac{2n}{1+1/\epsilon} + t_G)$ time (Theorem 10).

The exact computations of $t_G$ and $T_G$ are thus related since they share the same upper and conditional lower bounds. However, for the approximate computation, the trussness exhibits a hybrid behavior: when $T_G$ is polylogarithmically close to the number of edges in $G$, or smaller, approximation is as hard as listing the triangles in $G$; when $T_G$ is sufficiently large, instead, it can be much faster, as is the case for triangle counting. In particular, for $T_G = \Theta(m)$ and constant $\epsilon$, finding a $(1 \pm \epsilon)$-approximation for the trussness with a combinatorial algorithm is provably harder (unless BMM is truly subcubic) than finding a $(1 \pm \epsilon)$-approximation for triangle counting: the former can-

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4 A bound different but equivalent to $O(m \alpha_G)$ for the truss decomposition is also shown in [7].
5 In turn this implies that $(1 - \epsilon)(t_G + 2) \leq \tilde{t}_G + 2 \leq (1 + \epsilon)(t_G + 2)$. As the trussness originally defined in [13] is $t_G^* = t_G + 2$, we observe that $\tilde{t}_G + 2$ is a $(1 \pm \epsilon)$-approximation for $t_G^*$ as well.
6 We use $T_G + 1$ and $t_G + 2$ in place of $T_G$ and $t_G$ simply to preserve coherence of the bound when $T_G = 0$ or $t_G \leq 1$.
7 A definition of “combinatorial” algorithm, and a more detailed discussion of this conditional lower bound, which also includes the problems of (i) detecting if a graph is triangle-free and (ii) listing up to $n^{3-\delta}$ triangles in a graph for constant $\delta > 0$, are given in Section 5. We remark for the moment that, as a rule-of-the-thumb, an algorithm is typically combinatorial if it does not use fast matrix multiplication.
not be done in significantly less $O(m\alpha_G)$ time (Theorem 17), whereas the latter takes $O\left(\left(\frac{n}{T_G} + \min\{m, \frac{m^{2/3}}{T_G^{1/3}} + 1\}\right) \cdot \text{poly}(\log n, \frac{1}{\epsilon})\right) = O\left(n^{2/3} + \sqrt{m} \log(n)\right) = \tilde{o}(m)$ expected time using the result in [16].

We also provide further satellite results on triangle enumeration, trussness and truss decomposition in the rest of the paper.

Related work. The seminal paper by Cohen [11] presents the first algorithm to compute a $k$-truss and a variety of highly optimized implementations have tuned that algorithm: sequential for massive networks [10] in $O(m^{3/2})$ time, improved in [32], parallel [36, 26], distributed [9], using data-centric models [39], or parallel matrix multiplication on GPUs [5, 22]. Some of these were awarded in the 2017 and 2018 editions of the GraphChallenge [30].

Furthermore, [24] considered querying $k$-trusses under edge deletion, and [3] proposed an index computable in $O(m^{3/2})$ time, and maintainable on dynamic graphs, that permits to retrieve the $k$-truss containing a vertex $v$ in output-sensitive time.

A notion of trussness has also been given for probabilistic graphs [25], and uncertain graphs [16]. The $k$-truss notion has been also extended with additional constraint of co-location, i.e., requiring connectivity and bounded diameter within the $k$-truss [8].

From the algorithmic perspective, the problem has received much less attention. Recently, [7] studied novel combinatorial properties on $k$-trusses and proposed an exact algorithm for computing the truss-decomposition of a graph running with time $O(m\bar{\delta}(G))$, where $\bar{\delta}(G)$ is the average degeneracy of $G$. Moreover, the same paper provided an exact randomized algorithm which takes a parameter $K_{max}$ as input, and with high probability computes the $k$-truss-decomposition for all $k \leq K_{max}$, using matrix multiplication.

In this paper, we study approximation algorithms and conditional lower bounds for computing trussness, using several ideas: we introduce gadgets of controlled size to suitably amplify the trussness while preserving other parameters; we study several properties of a related hypergraph which equivalently represents the triangles; we provide a simple technique for triangle sampling based on wedge sampling [35, 34, 18], to name a few.

## Approximating the Trussness

We describe how to obtain a $(1 \pm \epsilon)$-approximation of the trussness $t_G$ w.h.p. using a combinatorial algorithm. Let $T_G$ be the number of triangles in $G$, observing that $T_G = O(m\alpha_G)$.

**Theorem 1.** The trussness of a graph $G$ can be $(1 \pm \epsilon)$-approximated w.h.p., for any $0 < \epsilon < 1$, in expected time $O\left(\epsilon^{-3} \min\left\{\frac{m}{T_G + 1}, 1\right\} m \alpha_G \log(t_G + 2)\right)$ and space $O(\epsilon^{-2} m \log m)$.

In the rest of section, we give an algorithm which meets the requirements of Theorem 1 whose structure is divided into main blocks according to the following roadmap.

1. In Section 2.1, we firstly recall the concepts of degeneracy and truss order; in order to approximate the trussness, and introduce a notion of approximate truss order.
2. In Section 2.2, we show how to link the truss order of $G$ to its trussness: we add gadgets to $G$, whose trussness is known, and deduce the trussness of the edges of $G$ by looking at their position in the truss order with respect to the edges of the gadget. When the truss order is not exact, understanding the trussness of $G$ is more complex. Nonetheless, we show that a suitable approximate order can also be used to approximate $t_G$. 
3. In Section 2.3, we focus on efficiently computing an approximate truss order of $G$. In particular, in Section 2.3.1, we introduce a transformation which maps the graph $G$ into a hypergraph $G^\triangle$, whose degeneracy turns out to be exactly the trussness of the original graph. We build a sample of $G^\triangle$, called $G_p$ obtained by sampling triangles uniformly at random from $G$ with probability $O(p)$ for a suitable $p$. Finally, in Section 2.3.2, we show that w.h.p. we can get an approximate degeneracy ordering of $G^\triangle$ using $G_p$, which also induces an approximate truss order.

2.1 Degeneracy order and truss order

Before getting into the details of the approximation algorithm, we need to introduce some concepts that will help us to achieve our goal.

The degeneracy of a graph $G$ is a well-known sparsity measure \[^{(13, 14, 11, 20)}\], and is indicated here as $d_G$. It is defined as the largest integer $k$ such that there exists a $k$-core in $G$. It is known that $d_G \leq 2 \cdot \alpha_G$ \[^{(7)}\] and thus $d_G = O(\sqrt{m})$. We will use the degeneracy order of the nodes in a graph, which can be defined as the order obtained by repeatedly removing the node of minimum (residual) degree from $G$. It takes $O(m)$ time to obtain a degeneracy ordering, and the $d_G$ is equal to the maximum among the residual degrees (called forward degrees in the following).

We introduce similar notions for $k$-trusses. A truss order of $G$ is an ordering $\langle e_1, \ldots, e_m \rangle$ of its edges obtained by repeatedly removing the edge of smallest (residual) support in $G$, breaking ties arbitrarily. For an edge $e_i$ in the order, we call forward triangles the triangles that $e_i$ forms using two other edges chosen from the residual graph (i.e., induced by $e_i, \ldots, e_m$), and forward support their number $t_G(e_i)$. We observe that $t_G = \max_{1 \leq i \leq m} t_G(e_i)$ by definition. In the truss order, the edges of $E(G)$ are numbered consistently with their trussness, i.e., $t_G(e_i) < t_G(e_j)$ implies $i < j$.

In the following, for a given order of the edges $\langle e_1, \ldots, e_m \rangle$, let $G_{\geq e_i}$ be the graph induced by the edges $e_i, \ldots, e_m$ and $\min\text{-sup}(G_{\geq e_i}) = \min_{j \geq i} \text{sup}_{G_{\geq e_i}}(e_j)$ be the minimum support of an edge in $G_{\geq e_i}$. Note that a truss order guarantees that $t_G(e_i) = \text{sup}_{G_{\geq e_i}}(e_i) = \min\text{-sup}(G_{\geq e_i}) \leq t_G$.

We introduce a definition of approximate truss order, inspired by the approximate degeneracy order introduced in \[^{(20)}\], that can be computed faster than $t_G$ under certain conditions. For $\epsilon > 0$, we say that $\langle e_1, \ldots, e_m \rangle$ is a $(1 + \epsilon)$-approximate truss order of $G$ if, for every edge $e_i$, the number of forward triangles to which it belongs is upper bounded as

\[ \sup_{G_{\geq e_i}}(e_i) \leq \max \left\{ \frac{T_G}{m}, (1 + \epsilon) \min\text{-sup}(G_{\geq e_i}) \right\} \] (1)

where $T_G$ is the number of triangles in $G$. At this point it is relevant to mention how $T_G$ relates to $t_G$ by an extension of Nash-Williams’ result \[^{(1)}\] to trussness, proven in \[^{(12)}\]:

**Theorem 2** (From \[^{(12)}\]). Given an undirected graph $G$ with trussness $t_G$, let $T_S$ be the number of triangles and $m_S$ be the number of edges in any subgraph $S$ of $G$. Then $\max_{S \subseteq G} \frac{T_S}{m_S} \leq t_G \leq 3 \max_{S \subseteq G} \frac{T_S}{m_S}$.

Indeed, this result implies $t_G \geq T_G/m$, meaning that in formula (1) each edge in the $(1 + \epsilon)$-approximate truss order has at most $t_G \cdot (1 + \epsilon)$ forward triangles.

\[^8\] This is proven by the fact that $G$ can be partitioned in $\alpha_G$ forests, so it has $\leq \alpha_G(n - 1)$ edges and an average degree $\leq 2\alpha_G$. As any subgraph of $G$ has at most the same arboricity, any subgraph has a vertex of degree at most $2\alpha_G$, so $d_G \leq 2\alpha_G$.

\[^9\] Whenever needed, we define $G_{\geq v_i}$ as the graph induced by nodes $v_j, \ldots, v_n$ in the order $\langle v_1, \ldots, v_n \rangle$. 
Gadget $G^{×q}$ Given a graph $G$ we use the gadget $G^{×q} = (V(G^{×q}), E(G^{×q}))$, known as balanced blow-up [20] and defined as follows: Let $G$ be called $G^1$ and, for each $i = 2, \ldots, q$, let $G^i = (V(G^i), E(G^i))$ be an exact copy of $G$. For each $v \in V(G)$, we call $v^i$ the corresponding node in $V(G^i)$. Thus, $V(G^{×q}) = V(G^1) \cup \cdots \cup V(G^q)$ contains $nq$ nodes. For each edge $\{u, v\} \in E(G)$, connect all pairs of nodes $u^i, v^j$ with $i, j \in \{1, \ldots, q\}$. Thus $E(G^{×q}) = E(G^1) \cup \cdots \cup E(G^q) \cup \{\{u^i, v^j\} \mid \{u, v\} \in E(G) \text{ and } i \neq j\}$ contains $O(mq^2)$ edges.

The main purpose of $G^{×q}$ is amplifying the trussness of $G$ by a factor $q$, introducing a limited amount of new nodes, edges and triangles. More formally, we can prove that:

Lemma 3. Given an undirected graph $G$ with $n$ nodes, $m$ edges, trussness $t_G$, and an integer $q > 1$, we can obtain a graph $G^{×q}$ with $|V(G^{×q})| = \Theta(nq)$ vertices, $|E(G^{×q})| = \Theta(mq^2)$ edges, and trussness $t_{G^{×q}} = qt_G$ (and $q^3T_G$ triangles). Given access to $G$, it is possible to implicitly navigate in $G^{×q}$ without materializing it.

Proof. Let us recall the definition of $G^{×q} = (V(G^{×q}), E(G^{×q}))$, given $G$: Let $G$ be called $G^1$ and, for each $i = 2, \ldots, q$, let $G^i = (V(G^i), E(G^i))$ be an exact copy of $G$. For each $v \in V(G)$, we call $v^i$ the corresponding node in $V(G^i)$. Thus, $V(G^{×q}) = V(G^1) \cup \cdots \cup V(G^q)$ contains $nq$ nodes. For each edge $\{u, v\} \in E(G)$, connect all pairs of nodes $u^i, v^j$ with $i, j \in \{1, \ldots, q\}$. Thus $E(G^{×q}) = E(G^1) \cup \cdots \cup E(G^q) \cup \{\{u^i, v^j\} \mid \{u, v\} \in E(G) \text{ and } i \neq j\}$ contains $O(mq^2)$ edges. For any $i, j$, we call $\{u^i, v^j\}$ the mirror edges of $\{u, v\}$.

We now show that $G^{×q}$ has trussness $t_{G^{×q}} = qt_G$. Consider any triplet of distinct nodes $a^h, b^i, c^j \in G^{×q}$, where $h, i, j \in \{1, \ldots, q\}$. They form a triangle in $G^{×q}$ if and only if $a, b, c$ form a triangle in $G$, so we have $q$ choices per edge, and thus the number of triangles is increased by a factor of $q^3$.

Furthermore, consider any edge $\{a, b\} \in E(G)$, recalling that it belongs to $\text{sup}_G(\{a, b\})$ triangles in $G$, and let $abc$ one such triangle. Looking at edge $\{a^h, b^i\} \in E(G^{×q})$, we observe that it belongs to the triangles $a^h b^i c^j, \ldots, a^h b^i c^q$. In other words, $\text{sup}_{G^{×q}}(\{a, b\}) = q \text{sup}_G(\{a, b\})$. Note this property is preserved for any subgraph $H$ of $G$, that is, $\text{sup}_{H^{×q}}(\{a, b\}) = q \text{sup}_H(\{a, b\})$, where $H^{×q}$ is the subgraph of $G^{×q}$ made up of the mirror edges of those in $H$.

Since $G$ has trussness $t_G$, let $H$ be a $k$-truss in $G$ with $k = t_G$. Since each edge $\{a, b\} \in E(H) \subseteq E(G)$ has support $\text{sup}_H(\{a, b\}) \geq t_G$, each mirror edge $\{a^h, b^i\}$ has support $\text{sup}_{H^{×q}}(\{a, b\}) \geq qt_G$, and thus $H^{×q}$ is a $k$-truss in $G^{×q}$ with $k \geq qt_G$. Note that the edge(s) of smallest support in $H$ (i.e., with support $t_G$) will have support exactly $qt_G$ in $H^{×q}$ and thus its trussness is exactly $qt_G$.

On the other hand, suppose by contradiction that $G^{×q}$ has a $k$-truss $J$, where $k > qt_G$: thus the edge of smallest support in $J$ has support $> qt_G$ in $J$. Define $J^* = H^{×q}$, where $E(H) = \{\{a, b\} \mid \{a^h, b^i\} \in E(J)\} \subseteq E(G)$. For each triangle $a^h b^i c^j \in J^*$, there is now at least one triangle $a^h b^i x$ in $J$ by definition of $J^*$. This means that each edge $\{a^h, b^i\}$ in $J^*$ has support at least equal to the support of some edge $\{a^h, b^i\}$ in $J$ by construction, thus $J^*$ is also a $k$-truss $J$, where $k > qt_G$. This is a contradiction as it implies $H$ is a $k$-truss where $k > t_G$, and thus $G$ has trussness $> t_G$.

It is worth noting that we can navigate in $G^{×q}$ without materializing it. We discuss five navigation operations needed in this paper, but it can be extended to more.

- Iteration through all the nodes or edges: $O(nq)$ and $O(mq^2)$ time.
- Adjacency check: same cost as in $G$, as $a^h$ and $b^i$ are neighbors iff $a$ and $b$ are neighbors in $G$.
- Adjacency list: $O(q\delta(a^h))$ time for any node $a^h$, as its neighbors can be iterated as there are $q$ copies of the neighborhood of $a$ in $G$.
Algorithm 1 Approximating the graph trussness

Input: graph $G_m$ and $0 < \epsilon < 1$

Output: w.h.p. $(1 + \epsilon)$-approximation $t_{G_m}$ of the trussness $t_{G_m}$

1. $G \leftarrow$ disjoint union of $G_m^{6}$ and a triangle
2. $\epsilon' \leftarrow \frac{\epsilon}{9}$, $x \leftarrow 1$, $t_G \leftarrow 1$
3. $(e_1, \ldots, e_{|E(G(x))|}) \leftarrow (1 + \epsilon')$-approximate truss order of $G(x)$ // w.h.p. by Lemma 6
4. if the first spurious edge $e_s$ appears before the last edge $e_g$ of $G$, then
   a. $t_G \leftarrow x$
   b. $x \leftarrow [(1 + \epsilon)x]$
   c. goto step 4
5. if $t_G < 2$ then return 0 // exact value of $t_{G_m}$
6. if a single $r \in \{G(1 + \epsilon)^{-1}, (t_G + 1)(1 + 3\epsilon')\}$ is a multiple of 6, then return $\frac{2}{3}$ // exact value
t7. return $\frac{2r}{3}$ // $(1 + \epsilon)$-approximation $t_{G_m}$

= The i-th neighbor: same cost as in $G$, as there are q copies of the neighborhood.

Node degree: same cost as in $G$, as there are q copies of the neighborhood.

Our algorithms using the above operations can run on $G^{xq}$ without computing it, if given access to $G$.

Gadget $G(x)$ We introduce another gadget, whose size and trussness are suitably bounded.

Lemma 4. Given a graph $G$ with $n$ nodes and $m$ edges, and an integer $x \geq 0$ with $x = O(\sqrt{m})$, let $G(x)$ be the graph obtained from $G$ by adding $\lceil m/(x+2) \rceil$ disjoint $(x+2)$-cliques, called spurious cliques (whose edges are also called spurious). Then $|V(G(x))| = O(m)$, $|E(G(x))| = O(m)$, and $\frac{T_G}{|E(G(x))|} \leq \min\{\frac{x^2}{2}, \frac{x}{3}, \max\{t_G, \frac{x}{3}\}\}$.

Proof. Let $\ell = \lceil m/(x+2) \rceil$. We observe that $G(x)$ has $T' = T_G + \ell(x+2)$ triangles and $m' = m + \ell(x+2)$ edges. Moreover, $m' \geq 2m$ and $m' \geq (2\ell - 1)(x+2)$. Also note that $m' = \Theta(m)$, since $m \geq (\ell - 1)(x+2)$ and $x = O(\sqrt{m})$. As for $|V(G(x))|$, this is $n + (x+2)\ell = O(m)$. We have $\frac{T_G}{m} = \frac{T_G + \ell(x+2)}{m + \ell(x+2)} \leq \frac{T_G}{m} + \frac{\ell(x+2)}{(2\ell - 1)(x+2)} = \frac{T_G}{m} + \frac{\ell x}{m(2\ell - 1)} \leq \frac{T_G}{m} + \frac{2}{3}$ (as $\ell \geq 1$). Moreover, as $\frac{a + \frac{a}{b}}{b} \leq \max\{\frac{a}{b}, \frac{a}{d}\}$ when $a,b,c,d$ are all positive, and since $\frac{T_G}{m} \leq t_G$, we also have that $\frac{T'}{m} = \frac{T_G + \ell(x+2)}{m + \ell(x+2)} \leq \max\{\frac{T_G}{m}, \frac{\ell(x+2)}{m(x+2)}\} \leq \max\{t_G, \frac{x}{3}\}$. From this we have $\frac{T'}{m} \leq \min\{\frac{T_G}{m} + \frac{x}{3}, \max\{t_G, \frac{x}{3}\}\}$.

2.2 Approximation algorithm

Suppose that we have a $(1 + \epsilon)$-approximate truss order. We show that using the gadgets described in Section 2.1 we obtain a $(1 + \epsilon)$-approximation of $t_G$. The basic idea is to expand the input graph using suitably $G^x$ and then obtain $G(x)$; after that, the approximate order on $G(x)$ where $x$ increases each time by a factor $(1 + \epsilon)$ is employed to spot the spurious cliques as “markers” to guess $t_G$. We give the pseudocode in Algorithm 1 whose rationale is explained in Lemma 6. The algorithm relies on the crucial test performed at line 4 that uses spurious edges as markers, according to what claimed in Lemma 5.

Lemma 5. Given a graph $G$ and an integer $x \geq 0$, let $(e_1, \ldots, e_{|E(G(x))|})$ be a $(1 + \epsilon)$-approximate truss order of $G(x)$. Let $e_g$ be the last edge of $G$ appearing in the order, and $e_s$ the first spurious edge appearing in the order. Then:
Efficient Estimation of Graph Trussness

1. if \( x < (1 + \epsilon)^{-1}t_G \) then \( e_g \) appears after \( e_s \)
2. if \( x > (1 + \epsilon) t_G \) then \( e_g \) appears before \( e_s \)

Proof. By (1) for \( G(x) \), each edge \( e_i \) has \( \text{sup} G(x)_{\geq e_i} (e_i) \leq \max \{ t_G, (1 + \epsilon) \text{min-sup}(G(x)_{\geq e_i}) \} \), where \( T' \) the number of triangles in \( G(x) \) and \( m' \) is the number of edges. From Lemma 4 we also have \( T' / m' \leq \min \{ (2 \epsilon^2 + 1), (1 + \epsilon) \text{sup}(G(x)_{\geq e_i}) \} \).

We now analyze the two cases in the statement.

1. Case \( x < (1 + \epsilon)^{-1}t_G \): First observe that

\[
\max \left\{ \frac{t_G}{2} + \frac{x}{3}, (1 + \epsilon) x \right\} < t_G
\]

by applying \( x < t_G \) to the first argument of max, and \( x(1 + \epsilon) < t_G \) to the second argument. Consider now an arbitrary edge \( e_i \), recalling that

\[
\text{sup} G(x)_{\geq e_i} (e_i) \leq \max \left\{ \frac{t_G}{2} + \frac{x}{3}, (1 + \epsilon) \text{min-sup}(G(x)_{\geq e_i}) \right\}
\]

holds by definition of approximate truss order, where \( T' / m' \leq \frac{t_G}{2} + \frac{x}{3} \) from Lemma 4.

If \( G(x)_{\geq e_i} \) contains a spurious edge, as the latter has support at most \( x \), we have \( \text{min-sup}(G(x)_{\geq e_i}) \leq x \) in (3) as each spurious clique is isolated in \( G(x) \). Hence we can upper bound (3) using this and (2), obtaining

\[
\text{sup} G(x)_{\geq e_i} (e_i) \leq \max \left\{ \frac{t_G}{2} + \frac{x}{3}, (1 + \epsilon) x \right\} < t_G
\]

In other words, \( e_i \) cannot be the first edge in the order of a \( t_G \)-truss because its forward triangles would be at least \( t_G \). Consequently, all the spurious edges (including \( e_s \)) occur before the first edge of a \( t_G \)-truss, and thus before \( e_g \).

2. Case \( x > (1 + \epsilon) t_G \): To prove this case we use a similar analysis to the one above, observing that \( x > \max \{ \text{max} \{ t_G, \frac{x}{3} \}, (1 + \epsilon) t_G \} \), since \( x > t_G \).

We use the fact \( T' / m' \leq \max \{ t_G, \frac{x}{3} \} \) from Lemma 4. Hence, in the approximate truss order

\[
\text{sup} G(x)_{\geq e_i} (e_i) \leq \max \left\{ \frac{t_G}{3}, (1 + \epsilon) \text{min-sup}(G(x)_{\geq e_i}) \right\}
\]

If \( G(x)_{\geq e_i} \) contains an edge from \( G \), we have \( \text{min-sup}(G(x)_{\geq e_i}) \leq t_G \) (by definition of trussness as the minimum support for an edge in any subgraph of \( G \) cannot be larger than \( t_G \)). Hence we can upper bound (5) as

\[
\text{sup} G(x)_{\geq e_i} (e_i) \leq \max \left\{ \frac{t_G}{3}, (1 + \epsilon) t_G \right\} < x
\]

It follows that \( e_i \) cannot be the first spurious edge in the ordering, which would have \( x \) forward triangles as each spurious clique in \( G(x) \) is isolated. Thus \( e_s \) must occur in the order after \( e_g \), which proves the statement for this case. (To get a full picture, all the spurious edges are at the end of this order.)

Lemma 6. Consider any undirected graph \( G \) with \( m \) edges, \( T_G \) triangles, arboricity \( \alpha_G \), and trussness \( t_G \). For any \( 0 < \epsilon < 1 \), suppose that a \( (1 + \epsilon) \)-approximate truss order can be found in expected \( f(\epsilon, m, T_G, \alpha_G) = O \left( \epsilon^{-2} \min \left\{ \frac{m \log m}{t_G + 1}, 1 \right\} m \alpha_G \right) \) time and with probability \( \Omega(1 - m^{-2}) \), using \( s(\epsilon, m) = O(\epsilon^{-2} m \log m) \) space. Then, Algorithm 4 provides w.h.p. a \( (1 + \epsilon) \)-approximation for the trussness \( t_G \) in expected \( O \left( \epsilon^{-1} \log(t_G + 2) \cdot f(\epsilon, m, T_G, \alpha_G) \right) \) time, using \( O(m + s(\epsilon, m)) \) space.
Proof. We replace the input graph $G_{in}$ with $G$, the graph obtained by uniting $G \times 6$ and a $K_3$ (line 1): in this way, we guarantee that $t_G \geq 1$ and that we can take $\epsilon' = \frac{\epsilon}{6}$. It is not difficult to see at this point that a $(1 \pm \epsilon)$-approximation of the trussness of the original input graph follows by dividing by 6 (line 2) the $(1 \pm \epsilon')$-approximation of the trussness of the new $G$, as $G \times 6$ amplifies trussness by 6 (Lemma 5), unless $t_{G_{in}}$ is 0, in which case the algorithm will output the correct result (that is clearly also a $(1 \pm \epsilon)$-approximation).

The algorithm performs some initialization steps (Line 2), then outputs the approximation of $G_{in}$ based on the value of $t$, which is computed in Lines 3–4 and bounded as follows:

1. We prove that $\tilde{t}_G \leq (1 + \epsilon')t_G \leq (1 + \epsilon)t_G$.
   
   As by assumption $t_G \geq 1$ (because of the added $K_3$), in the beginning $\tilde{t}_G = 1 < (1 + \epsilon')t_G$.
   
   Furthermore, if in a step we have $x > (1 + \epsilon')t_G$, then by Lemma 5 the first spurious edge $e_s$ appears after the last edge $e_g$ of $G$, meaning that $\tilde{t}_G$ will not be updated, so whenever we update $\tilde{t}_G$ we maintain that $\tilde{t}_G \leq (1 + \epsilon')t_G$.

2. We prove that $t_G \geq (1 + 3\epsilon')^{-1}t_G - 1 \geq (1 - \epsilon)t_G - 1$.

   Note that $\tilde{t}_G$ is the largest value of $x$ for which, in the $(1 + \epsilon')$-approximate truss order of $G(x)$, the condition at Line 4 is true.

   By Lemma 5, if $x < (1 + \epsilon')^{-1}t_G$ then the condition at Line 4 is true. Hence, let $x_F$ be first value of $x$ for which the condition is false (i.e., $e_s$ appears after $e_g$), and let $x_T$ be the value of $x$ soon before $x_F$. It must be $x_F \geq \frac{t_G}{1+\epsilon'}$ and $t_G = x_T$. By looking at step 1a we have $x_F = [x_T(1 + \epsilon')] = [t_G(1 + \epsilon')]$. Thus $[t_G(1 + \epsilon')] \geq \frac{t_G}{1+\epsilon'}$, meaning that $t_G(1 + \epsilon') + 1 \geq \frac{t_G}{1+\epsilon'}$, and $t_G \geq \frac{t_G}{(1+\epsilon')^2} - \frac{1}{1+\epsilon'} \geq \frac{t_G}{(1+\epsilon')^2} - 1 \geq \frac{t_G}{(1+\epsilon')^2} - 1 \geq t_G(1 - 3\epsilon') - 1 \geq t_G(1 - \epsilon) - 1$ \(
\)

   We observe that our $(1 \pm \epsilon)$-approximation still has an off-by-1 in point 2 above: we get $t_G \geq (1 - \epsilon)t_G - 1$ instead of $\tilde{t}_G \geq (1 - \epsilon)t_G$. The latter can be obtained with some extra math, requiring also the check at line 6 as discussed in the two cases below.

   - Case $t_G \geq \frac{1}{1+\epsilon'}$: Since $3\epsilon't_G \geq 1$, we get $t_G \geq (1 + 3\epsilon')^{-1}t_G - 1 \geq t_G(1 - 3\epsilon') - 1 \geq t_G(1 - 3\epsilon') - 3\epsilon't_G = t_G(1 - 6\epsilon')$, proving the statement as $\epsilon = 6\epsilon'$.

   - Case $t_G < \frac{1}{1+\epsilon'}$: Recall that $t_G \geq (1 + \epsilon')^{-1}\tilde{t}_G$ by point 2 and that $t_G \leq (\tilde{t}_G + 1)(1 + 3\epsilon')$ by point 2. We want to prove that $(\tilde{t}_G + 1)(1 + 3\epsilon') - (1 + \epsilon')^{-1}\tilde{t}_G \leq 4$ holds, thus ensuring that the algorithm will output the correct value of $t_G$ as it is the only multiple of 6 in the given range (check done at line 6). We have $(\tilde{t}_G + 1)(1 + 3\epsilon') - (1 + \epsilon')^{-1}\tilde{t}_G \leq \tilde{t}_G \left(1 + 3\epsilon' - \frac{1}{1+\epsilon'}\right) + 1 + 3\epsilon' \leq \tilde{t}_G \left(\frac{1}{1+\epsilon'} + 3\epsilon'\right) + 2 \leq t_G \left(\epsilon' + 3\epsilon'(1 + \epsilon')\right) + 2 \leq t_G(4\epsilon' + 3\epsilon'^2) + 2 \leq 5\epsilon't_G + 2 \leq 4$, where we used the bounds on $\epsilon'$ which imply $3\epsilon'^2 \leq \epsilon'$ and the upper bound $t_G(1 + \epsilon')$ on $t_G$.

   As for the correctness, we compute the approximate truss order $O\left(\frac{\log(t_G + 2)}{\log(1 + \epsilon')}\right) = O(\epsilon^{-1}\log(t_G + 2))$ times. Our algorithm still succeeds w.h.p. as the latter probability is $\Omega(1 - m^{-2})$ and we can generously bound $\epsilon^{-1}\log(t_G + 2)$ as $O(n \log n)$.

   As for the complexity, at each iteration we pay $O(f(\epsilon, m, T_G, \alpha_G))$. As the number of iterations is $O(\epsilon^{-1}\log(t_G + 2))$, and we use $O(m)$ extra space, the statement follows.

In order to conclude the proof of Theorem 4 we need to show in Section 2.3 how to compute a $(1 + \epsilon)$-approximate truss order in expected $O(\epsilon^{-2} \min \left\{ \frac{m \log m}{7\epsilon^2 + 1}, m \alpha_G \right\})$ time and with probability $\Omega(1 - m^{-2})$, using $s(\epsilon, m) = O(\epsilon^{-2} m \log m)$ space.

\footnote{Note that $1 + 3\epsilon' > (1 + \epsilon')^2$ and $\frac{1}{1+\epsilon'} > 1 - 3\epsilon'$}
2.3 Computing a \((1 + \epsilon)\)-approximate truss order

Given a graph \(G\), we denote as \(G^{\sim}\) the triangle hypergraph. This hypergraph, originally introduced in [7], is a 3-uniform hypergraph whose nodes are the edges of \(G\), and whose hyperedges are the triplets of edges which form a triangle in \(G\). More formally, \(V(G^{\sim}) = \{v_{e_i} : e_i\) is an edge of \(G\}\), and for each triplet \(e_1, e_2, e_3 \in E(G)\) which forms a triangle, there is a hyperedge \((v_{e_1}, v_{e_2}, v_{e_3}) \in E(G^{\sim})\). Since each pair of edges defines a unique triangle (if any), each pair of hyperedges in \(G^{\sim}\) can overlap by at most one vertex, and the degree of \(v_{e_i}\) in \(G^{\sim}\) is equal to the support of \(e_i\) in \(G\). Thus, \(G^{\sim}\) has \(|E(G)| = m\) vertices and \(T_G\) hyperedges. Moreover, building \(G^{\sim}\) takes \(O(md_G)\) time.

We will use \(G^{\sim}\) for computing the approximate truss order, since there is correspondence between the trussness of \(G\) and the degeneracy of \(G^{\sim}\). For each node \(u\) of a hypergraph \(H\) its degree, indicated as \(\delta_H(u)\), is the number of hyperedges in \(H\) containing \(u\). The degeneracy of the hypergraph \(G^{\sim}\), denoted as \(d_{G^{\sim}}\), is the natural extension of degeneracy for graphs: it is the maximum \(d\) such that there is an induced subgraph \(H'\) of \(H\) in which, for each vertex \(u\), \(\delta_{H'}(u) \geq d\).

Given the correspondence between degree in \(G^{\sim}\) and support in \(G\), as noted in [7], it is straightforward to observe that a \(k\)-core in \(G^{\sim}\) (a subgraph where all nodes have degree \(k\) or more) corresponds to a \(k\)-truss in \(G\) (a subgraph where all edges have support \(k\) or more). Hence we obtain:

> Remark 7. \(d_{G^{\sim}} = t_G\)

Our method for computing an approximate truss order consists in efficiently computing a sample of \(G^{\sim}\), as explained in Section 2.3.1, then using that sample to compute the approximate truss order, as shown in Section 2.3.2

2.3.1 Sampling

In the following, we show how to sample a subhypergraph \(G^{\sim}_p\) of \(G^{\sim}\) such that each hyperedge of \(G^{\sim}\) is in \(G^{\sim}_p\) independently with probability \(p \geq \zeta \frac{m \log m}{(md_G + 1) \epsilon^2}\), where \(\zeta\) is a suitable constant such that Lemma [12] holds, in less than \(O(m a_G)\).

Note that we must do this without access to \(G^{\sim}\) or even the value of \(p\): A simple procedure to obtain \(G^{\sim}_p\) would scan the hyperedges of \(G^{\sim}\), i.e. the probability space, retaining them with probability \(p\). This requires knowledge of \(p\) and scanning all the triangles of \(G\), with \(O(m a_G)\) time cost. As we cannot afford this cost, we use an alternative equivalent strategy for generating \(G^{\sim}_p\).

Sampling uniformly at random hyperedges of \(G^{\sim}\) corresponds to uniformly sampling triangles of \(G\): there is a 1-to-1 correspondence between triangles in \(G\) and edges in \(G^{\sim}\), thus our probability space is the set of triangles of \(G\), each one corresponding to a different hyperedge of \(G^{\sim}\). As obtaining the \(i\)th triangle of \(G\) is not easy, we use a different strategy to sample them, based on wedges. In particular, we observe that if the nodes are ordered, each triangle corresponds to a unique forward wedge (i.e., a wedge in which the middle node is the smallest of the three). For this reason, we sample forward wedges, which can be handled more easily, rather than triangles, and we accept them, adding the corresponding hyperedge of \(G^{\sim}\) to \(G^{\sim}_p\) only if they are closed.

More precisely, let \(W\) be the number of forward edges in \(G\). Our sampling procedure works as follows.

1. Start setting \(p = \zeta \frac{m \log m}{W}\).
2. Extract each forward wedge with probability \(p\).
3. For each forward wedge \( w \) sampled, if \( w \) is closed, i.e. it corresponds to a triangle, add to \( G_p^\circ \) the hyperedge of \( G^\circ \) which corresponds to \( w \).

4. If the number of hyperedges of \( G_p^\circ \) is not at least \( \frac{3}{2} \Delta \frac{m \log m}{\epsilon^2} \), double \( p \). If \( p \geq 1 \) run the exact algorithm, else clear \( G_p^\circ \) and go to step 2.

First of all, \( W \) can be computed in \( O(m) \) time, using a degeneracy ordering of the graph: Let \( \delta(u) \) be the forward degrees of a node \( u \) in \( G \), i.e. the number of edges going forward in the order from \( u \). As each distinct pair of forward neighbors of \( u \) makes with \( u \) a distinct forward wedge, we have \( W = \sum_{u \in V(G)} \delta(u)(\delta(u) - 1)/2 \).

In step 2 in order avoid paying \( O(W) \) time to choose which forward edges should be kept, we use Method 9 in [19] (as it is often done to generate \( k \geq \) geometrically distributed variables with parameter \( p \), where \( k \) is the number of sampled wedges. To generate a geometric random variable we use the method in [6] whose expected cost is \( O(1 + \log(1/p)/w) \), where \( w = \Omega(\log n) \), and in our case is constant as \( 1/p = O(\frac{1}{n}) \) and \( \epsilon = 1/n \).\(^{11}\)

We prove that step 2 and 3 sample triangles independently at random with probability \( p \).

\[\text{Lemma 8.} \quad \text{Step 2 and 3 of our sampling process to obtain } G_p^\circ \text{ are equivalent to scanning each hyperedge of } G^\circ \text{ and accepting it with probability } p.\]

\[\text{Proof.} \quad \text{Step 3 adds the hyperedge corresponding to a sampled wedge } w \text{ only if } w \text{ is closed. Scanning all the pairs } p \text{ and sampling each of them with probability } p \text{ implies that also each closed wedge } w \text{ is sampled with probability } p. \text{ Since in } G^\circ \text{ each hyperedge is involved in just one triangle and hence it corresponds to a unique forward wedge } w, \text{ we obtain that a hyperedge of } G^\circ \text{ is also added to } G_p^\circ \text{ with probability } p. \quad \blacksquare\]

The sampling procedure repeats step 2 and step 3, performing doubling on \( p \), and ends up sampling triangles with probability at least \( \zeta \frac{m \log m}{(T_G + 1)\epsilon^2} \) w.h.p. as shown next.

\[\text{Lemma 9.} \quad \text{Our sampling process of } G_p^\circ \text{ samples edges of } G^\circ \text{ independently and uniformly at random with probability } \zeta \frac{m \log m}{(T_G + 1)\epsilon^2} \leq p \leq 4\zeta \frac{m \log m}{(T_G + 1)\epsilon^2}, \text{ w.h.p..}\]

\[\text{Proof.} \quad \text{From Lemma 8 we know that } G_p^\circ \text{ is an uniform and independent sample of } G^\circ \text{ with probability } p. \text{ Thus, we only have to prove that when the algorithm terminates we have that } p \text{ is bounded as in the statement.}\]

To prove the lower bound on \( p \), we show that if \( p \leq \zeta \frac{m \log m}{(T_G + 1)\epsilon^2} \), then the number of edges in \( G_p^\circ \) is less than \( \frac{3}{2} \Delta \frac{m \log m}{\epsilon^2} \) w.h.p.. We use a well-known application of the Chernoff bounds on the tail of a binomial distribution, i.e. that for a binomial variable \( X \sim B(T_G, p) \) (where \( X \) represents the number of edges in \( G_p^\circ \)) we have \( \mathbb{P} \{ X \geq (1 + \delta)T_Gp \} \leq \exp \left( -\frac{\delta^2}{2} T_G p \right) \).

Setting \( \delta = \frac{1}{2} \frac{3 \Delta \frac{m \log m}{(T_G + 1)\epsilon^2}}{2 \zeta \frac{m \log m}{(T_G + 1)\epsilon^2} - 1} \), which trivial calculations show that is at least \( \frac{1}{2} \frac{m \log m}{(T_G + 1)\epsilon^2} \) if we assume \( p \leq \frac{1}{2} \zeta \frac{m \log m}{(T_G + 1)\epsilon^2} \), we obtain

\[\mathbb{P} \{ X \geq \frac{3}{2} \Delta \frac{m \log m}{\epsilon^2} \} \leq \exp \left( -\frac{1}{12} \Delta^2 \frac{m^2 \log^2 m}{(T_G + 1)^2} \right) \leq \exp \left( -\frac{1}{12} \epsilon^2 \frac{m^2 \log^2 m}{\epsilon^2} \right)\]

\(^{11}\)As \( T_G \leq n, \epsilon < 1/n \) does not give any more information than setting \( \epsilon = 1/n.\)
which is exponentially small in $m$ and $\epsilon^{-1}$.

We can obtain a similar upper bound on $P\{X \leq \frac{3}{2} \zeta \frac{m \log m}{2 \epsilon^2} \}$ when $p \geq 2\zeta \frac{m \log m}{2 \epsilon^2}$ using the Chernoff bound on the lower tail of the distribution of $X$. Thus, w.h.p. we have that the stopping condition holds if $p \geq 2\zeta \frac{m \log m}{(T_G+1)\epsilon^2}$ and does not hold if $p < \zeta \frac{m \log m}{(T_G+1)\epsilon^2}$. Since we double $p$ at each step, $p$ is twice the value $p'$ it had in the last step where the stopping condition did not hold. This means that, w.h.p., $p' < 2\zeta \frac{m \log m}{(T_G+1)\epsilon^2}$ and thus $p < 4\zeta \frac{m \log m}{(T_G+1)\epsilon^2}$. On the other hand, the stopping condition does not hold when $p < \zeta \frac{m \log m}{(T_G+1)\epsilon^2}$, w.h.p., thus w.h.p. $p$ must be at least $\zeta \frac{m \log m}{(T_G+1)\epsilon^2}$, proving the statement.

Which means the size of $G^\alpha_p$ is at most $4\zeta \frac{m \log m}{\epsilon^2}$ in expectation. Finally, the complexity of the process is as follows (we recall $d_G \leq 2 \cdot \alpha_G$):

**Lemma 10.** Sampling $G^\alpha_p$ costs $O\left(\min\left\{\frac{m \log m}{(T_G+1)\epsilon^2}, 1\right\} md_G\right)$ time in expectation.

**Proof.** Since the expected cost of extracting a geometrically distributed random variable is constant, the expected running time of the sampling in step 2 is the expected number of sampled wedges, i.e. $O(pW)$. Since at each iteration $p$ doubles and $p$ w.h.p. is at most $4\zeta \frac{m \log m}{(T_G+1)\epsilon^2}$, the last iteration dominates the running time. If $p \geq 1$ the same happens with the exact algorithm. Hence, we obtain the claimed running time, noticing that $W = O(md_G)$ if the graph is in degeneracy order. Note that $\min\left\{\frac{m \log m}{(T_G+1)\epsilon^2}, 1\right\} md_G \geq m$, meaning that the cost of computing a degeneracy order is dominated by the running time of the rest of the algorithm.

### 2.3.2 Approximate truss order

By Remark 8 the degeneracy of $G^\alpha$ corresponds to the trussness of $G$. In this section we aim to compute an approximate truss order of $G$ by computing an approximate degeneracy ordering of $G^\alpha$. This can be done using $G^\alpha_p$, and extending some of the techniques in [21] to hypergraphs.

Given a hypergraph $H$ and a subset of its nodes $U$, we indicate with $H[U]$ the subhypergraph induced by the nodes in $U$, i.e. the hypergraph having as nodes $U$ and as hyperedges the ones of $H$ subset of $U$.

**Lemma 11.** Let $H$ be a hypergraph with $n$ nodes and $m$ hyperedges, and $H_p$ be the hypergraph obtained from $H$ by retaining each hyperedge independently with probability $p = \zeta \frac{n \log n}{T_G m}$ for any constant $\zeta > 54$. Moreover, let $U$ be a subset of the nodes of $H$ and $v$ be the smallest-degree node in $H_p[U]$. Then, with probability at least $1 - n^{1 - \frac{1}{T_G}}$, the following holds: $\delta_{H[U]}(v) \leq \max\left\{\frac{m}{n}, (1 + \epsilon)\delta_{\min}(H[U])\right\}$.

**Proof.** Let $\zeta = 27\gamma$, so that $\gamma \geq 2$. We prove that, for any two nodes $x$ and $y$ with $\delta_{H[U]}(x) \geq \max\left\{\frac{m}{n}, (1 + \epsilon)\delta_{H[U]}(y)\right\}$, we have $\mathbb{P}\left\{\delta_{H_p[U]}(x) < \delta_{H_p[U]}(y)\right\} \leq \frac{1}{\gamma^2}$. This implies that the thesis holds w.h.p., as the probability that the thesis fails is bounded by the probability that at least one node $x$ in $U$ has a degree lower than the one of $v$ in $H_p[U]$ while satisfying $\delta_{H[U]}(x) \geq \max\left\{\frac{m}{n}, (1 + \epsilon)\delta_{H[U]}(v)\right\}$. This probability is bounded by $n \cdot \frac{1}{n^{1 - \frac{1}{T_G}}} = \frac{1}{n^{1 - \frac{1}{T_G}}}$, which means that the statement holds.

Note that we only need to consider the case when $\delta_{H[U]}(y) \geq \frac{1}{1 + \epsilon} \frac{m}{n}$. Indeed, when $\delta_{H[U]}(y) < \frac{1}{1 + \epsilon} \frac{m}{n}$, we have $\max\left\{\frac{m}{n}, (1 + \epsilon)\delta_{H[U]}(y)\right\} = \frac{m}{n}$. Moreover, if we replace the nodes $y$ such that $\delta_{H[U]}(y) < \frac{1}{1 + \epsilon} \frac{m}{n}$ with nodes of degree equal to $\frac{1}{1 + \epsilon} \frac{m}{n}$, the set of possible values of $x$ does not change, and for a fixed $x$ the probability $\mathbb{P}\left\{\delta_{H_p[U]}(x) \leq \delta_{H_p[U]}(y)\right\}$ increases when
\(\delta_{H_p[u]}(y)\) increases, meaning that bounding this probability in the case \(\delta_{H_p[u]}(y) \geq \frac{1}{1+c} \frac{m}{n}\) implies a bound for the case \(\delta_{H_p[u]}(y) < \frac{1}{1+c} \frac{m}{n}\).

Thus, as \(\delta_{H_p[u]}(x) \geq \max\left\{ \frac{m}{n}, (1+c)\delta_{H_p[u]}(y) \right\}\), we can assume that \(\delta_{H_p[u]}(y) \geq \frac{1}{1+c} \frac{m}{n}\), \(\delta_{H_p[u]}(x) \geq (1+c)\delta_{H_p[u]}(y)\), defining \(c = \frac{1}{1+c}\), so that \((1-c)\delta_{H_p[u]}(x) \geq (1+c)\delta_{H_p[u]}(y)\), we have that the event \(E_1 = (\delta_{H_p[u]}(x) \leq \delta_{H_p[u]}(y))\) implies the event \(E_2 = (\delta_{H_p[u]}(x) \leq (1-c)p\delta_{H_p[u]}(x) \vee \delta_{H_p[u]}(y) \geq (1+c)p\delta_{H_p[u]}(y))\), as \(\neg E_2\) implies \(\neg E_1\) because of the choice of \(c\). Hence, we have:

\[
p \{ \delta_{H_p[u]}(x) \leq \delta_{H_p[u]}(y) \} \leq \exp \left( -\frac{c^2}{2} p \delta_{H_p[u]}(x) \right) \leq 2 \exp \left( -\gamma \log n \right)
\]

where the second inequality follows from the fact that the degree distribution of a vertex in \(H_p\) is binomial, as well as the Chernoff bounds, and the third one from the fact that \(c^2 p \delta_{H_p[u]}(y) \geq c^2 p \frac{m}{n} \geq 3\gamma \log n\) because of our hypothesis on \(p\), and similarly for \(c^2 p \delta_{H_p[u]}(x)\).

In the following, we obtain the main property of our approximate truss order.

**Lemma 12.** Given a graph \(G\) with \(n\) nodes and \(m\) edges, let \(v_{c_1}, \ldots, v_{c_m}\), be the order of the edges of \(E(G)\) corresponding to a degeneracy ordering the nodes of \(G^\prec\), for \(p = \frac{\zeta \log m}{m^{1/2}c^2}\), for any constant \(\zeta > 81\). Then the following holds on \(G\) with probability at least \(1 - m^{2-\frac{\gamma}{\zeta}}\):

the forward triangles of \(e_i\) in \(G\) are at most \(\max\left\{ \frac{T_m}{m}, (1+\varepsilon)s \right\}\), where \(s\) is the smallest support of an edge in \(G_{\geq c_i}\).

**Proof.** By construction (see Section 2.3.1), we have that each edge of \(G^\prec\) appears in \(G_p^\prec\) with probability \(\zeta \frac{\log m}{m^{1/2}c^2}\), independently from the others. Since a degeneracy ordering is computed by iteratively removing the lowest-degree vertex from \(G_p^\prec\), applying \(m\) times Lemma 11 we get that \(v_{c_1}, \ldots, v_{c_m}\) is, with probability at least \(1 - m^{2-\frac{\gamma}{\zeta}}\), an approximate degeneracy ordering of \(G^\prec\).

This means that the forward neighbors of each \(v_{c_i}\) are bounded by \(\max\left\{ \frac{E(G^\prec_v)}{V(G^\prec_v)}, (1+\varepsilon)\delta_{\min} \right\}\) where \(\delta_{\min}\) is the smallest degree of a node in \(G_{\geq c_i}\). By construction of \(G^\prec\), the corresponding order \(e_{1}, \ldots, e_{m}\) on the edges of \(G\) has the property that the forward triangles of each \(e_i\) are bounded by \(\max\left\{ \frac{E(G^\prec_v)}{V(G^\prec_v)}, (1+\varepsilon)s \right\}\) where \(s\) is the smallest support of an edge in \(G_{\geq c_i}\). Thus, the statement follows.

As a result, the cost of the getting a \((1+\varepsilon)\)-approximate truss order follows. We can finally prove the following statement (corresponding to that of Theorem 1).

**Lemma 13.** There is an algorithm that computes w.h.p. a \((1+\varepsilon)\)-approximate truss order of \(G\) in expected time \(O\left( \min\left\{ \frac{m \log m}{(\zeta+1)c^2}, 1 \right\} m d_G \right)\) and space \(O(\varepsilon^{-2} m \log m)\).

**Proof.** By Lemma 10, \(G_p^\prec\) can be obtained with cost \(O\left( \min\left\{ \frac{m \log m}{(\zeta+1)c^2}, 1 \right\} m d_G \right)\) and has \(O(\varepsilon^{-2} m \log m)\) hyperedges. As the cost of computing a degeneracy ordering is bounded by the number of hyperedges, the statement follows, since if we end up running the exact algorithm

\[ \text{To guarantee a failure probability that is } o(m^{-2}), \text{ as required by Lemma 6, we need } \zeta > 108. \]
we have a total running time of $O(md_G)$, and if not then $\frac{m \log m}{(G+1)^2} md_G \geq \epsilon^{-2} m \log m$, i.e. the cost of sampling $G_p$ dominates over the cost of computing its degeneracy order.

\section{Conditional Lower Bounds}

The trivial algorithm for counting triangles in a graph takes $O(n^3)$ time: determining whether this bound is inherent or it can be improved to a truly subcubic one (e.g., $O(n^{2.9})$) by a “combinatorial” algorithm has been a long standing question, even referred to as a “holy grail” of graph algorithms [42].

What is a “combinatorial” algorithm is hard to define formally, but the term aims at designating practical approaches which have reasonable constant factors [42, 44, 2]. As a rule-of-the-thumb, combinatorial algorithms are not based on fast matrix multiplication, which has better asymptotic complexity (i.e., $O(n^\omega)$ for some $\omega < 2.373$ to multiply two $n \times n$ matrices) but very large hidden constant factors due to a high number of subproblems generated in the matrix multiplication that makes it not practically efficient.

This conditional lower bound for combinatorial algorithms is further strengthened in [42], which shows its equivalence to other important problems, such as Boolean Matrix Multiplication (BMM):

\begin{itemize}
  \item \textbf{Theorem 14} (from [42]). \textit{The following problems either all have truly subcubic combinatorial algorithms, or none of them do:}
    \begin{itemize}
      \item Boolean matrix multiplication (BMM).
      \item Detecting if a graph has a triangle.
      \item Listing up to $n^{3-\delta}$ triangles in a graph for constant $\delta > 0$.
      \item Verifying the correctness of a matrix product over the Boolean semiring.
    \end{itemize}

    Conditional lower bounds hold also for the (multiplicative or additive) approximation of trussness if we want to design a combinatorial algorithm that takes $O(m(t_G + 1))$ time in the worst case as shown by the following Lemma, whose proof is based on $G^{\times q}$ introduced in Lemma 6.

\begin{itemize}
  \item \textbf{Lemma 15.} \textit{Given any undirected graph $G$ with $n$ nodes, $m$ edges, and trussness $t_G$, there is no combinatorial algorithm that, for any $\epsilon > 0$, provides an approximation for the value of $t_G$ by either a multiplicative factor of $1 + \epsilon$ or an additive term of $O(n^{2/3-\epsilon})$, taking $O(m(t_G + 1))$ time in the worst case, unless BMM is truly subcubic.}
\end{itemize}

\begin{proof}
  The arguments for the multiplicative factor of $1 + \epsilon$ are analogous to those for Lemma 22 if $t_G = 0$ then its approximate value is zero and thus we can test if $G$ is triangle-free. Hence we focus on the additive term of $O(n^{2/3-\epsilon})$. To this aim, we need the gadget described in Lemma 6 to amplify the (unknown) trussness $t_G$ by any integer factor $q > 1$. Suppose that such an approximation algorithm $A$ exists for any $G'$ in $O(m(t_G + 1))$ time, and let it run on $G' = G^{\times q}$ where $q = 3w$ and $G$ is the graph that we want to check for triangle-freeness. If so, its trussness is $t_G = 0$, and thus $A$ on $G^{\times q}$ returns an approximate value $t_{G^{\times q}} \leq t_{G^{\times q}} + q(t_G + w) = t_G + w = w$. Otherwise, $G$’s trussness is $t_G \geq 1$, and $A$ on $G^{\times q}$ returns an approximate value $t_{G^{\times q}} \geq t_{G^{\times q}} - w \geq q - w > w$. In other words, checking if the output of $A$ is smaller or equal to $w$, we can tell whether $G$ is triangle-free. The time complexity of $A$ when $G$ is triangle-free is $O(|E(G^{\times q})| (G^{\times q} + 1)) = O(|E(G^{\times q})|) = O(mq^2) = O(mw^2) = O(n^2(n^{2/3-\epsilon})^2) = O(n^{3-2\epsilon})$ which improves BMM by Theorem 14 as $\epsilon > 0$: indeed we can stop $A$ if it runs longer, and declare that $G$ is triangle-free if $A$ outputs an approximation of value $\leq w$.
\end{proof}
Lemma 16. Suppose that there exists a (combinatorial) algorithm to approximate the trussness of any graph $G$ containing $\Omega(m)$ triangle in $O(m^6n^k)$ time, within either a multiplicative factor $c$ or an additive term $\frac{c^2}{2}$, for $c \geq 1$. Then there is a (combinatorial) algorithm to recognize whether a graph $G$ is triangle free in $O(m^{1+\epsilon}n^{2+\epsilon}c^{4h+2k})$ time.

Proof. Let $T_G = \Omega(m)$ be the number of triangles in $G$, recalling that $T_G = 0$ if and only if the trussness is $t_G = 0$. Given $G$ and its parameters:

1. Consider the gadget $G^{(c^2+2)}$ introduced in Lemma 3, which has trussness $(c^2 + 2)t_G$ and $O(c^6T_G)$ triangles. (Recall that we can navigate implicitly $G^{(c^2+2)}$).
2. Let $K$ be a graph consisting of a complete bipartite graph $K_{\sqrt{cn}, \sqrt{cn}}$, plus one node $v_x$ connected to all the others. $K$ has $O(c^2\sqrt{m}) = O(c^2n)$ nodes and $\Theta(c^3m)$ edges, and can be accessed implicitly similarly to $G^{(c^2+2)}$. Furthermore, we can observe that it contains $\Theta(c^3m)$ triangles, and its trussness is $t_K = 1$, since every edge not adjacent to $v_x$ belongs to exactly one triangle.

Now we take a new graph $G'$, which is simply the disjoint union of $G^{(c^2+2)}$ and $K$. We observe that $G'$ has $\Theta(cn + c^2\sqrt{m}) = O(c^2n)$ nodes, $\Theta(c^3m)$ edges, and $\Theta(c^6T_G) + \Theta(4^2m) = O(c^4m)$ triangles.

Furthermore, if $t_G = 0$, then the trussness of $G'$ is $t_{G'} = 1$ because of $K$; otherwise, $G'$ has trussness $t_{G'} = (c^2 + 2)t \geq c^2 + 2$.

Consider $A$ be an algorithm which can approximate the trussness of $G'$ either within a multiplicative factor of $c$ or within an additive factor of $\frac{c^2}{2}$, in $O(m^6n^k)$ time, provided that the graph has $\Omega(m)$ triangles. We observe that the assumption on the number of triangles is satisfied by $G'$. Thus, we can apply $A$ to $G'$ to get the approximate value $\tilde{t}_{G'}$, say within a multiplicative factor of $c$. If $t_G = 0$ then $\tilde{t}_{G'} \leq ct_{G'} = c$; else $\tilde{t}_{G'} \geq \frac{t_{G'}}{c} \geq \frac{c^2 + 2}{c} > c$. When the approximation is within an additive term $\frac{c^2}{2}$, we have the following. If $t_G = 0$ then $\tilde{t}_{G'} \leq t_{G'} + \frac{c^2}{2} = 1 + \frac{c^2}{2}$; else $\tilde{t}_{G'} \geq \tilde{t}_{G'} - \frac{c^2}{2} \geq (c^2 + 2) - \frac{c^2}{2} > 1 + \frac{c^2}{2}$. Hence we can decide whether $G$ is triangle-free in $O(|V(G')|^2|E(G')|^{1/2}) = O((c^4m)^{1/2}(c^2n)^{1/2})) = O(m^{2+\epsilon}n^{2+\epsilon}c^{4h+2k})$ time.

Theorem 17. Consider any undirected graph $G$ with $n$ nodes, $m$ edges, arboricity $\alpha_G$, and trussness $t_G$. Even if its number of triangles is $\Omega(m)$ then there is no combinatorial algorithm that, for any given $c \geq 1$, provides an approximation of $t_G$ within a multiplicative factor of $c$ or an additive term of $c^2/2$, taking $\tilde{o}(m\alpha_G)$ time in the worst case, unless BMM is truly subcubic.

Proof. Follows from Theorem 13 and the more general statement of Lemma 16, recalling that $m\alpha_G = \Theta(n^3)$ in the worst case.

Theorem 17 says that we need more than order $m$ triangles in $G$ to hope to be significantly faster the $O(m\alpha_G)$ time bound for the exact computation.

4 Approximating the Trussness (with Matrix Multiplication)

For the sake of completeness, it is interesting to see what can be done if matrix multiplication is allowed. To do so we use Lemma 18 which enables us to approximate the trussness using triangle counting algorithms.

Lemma 18. Given an undirected graph $G$ with $n$ nodes, $m$ edges, and trussness $t_G$, suppose that the support for every edge of $G$ can be computed in $f(n, m) = \Omega(m)$ time and $S(n, m) = \Omega(m)$ space. For any $\epsilon > 0$, a $(3 + \epsilon)$-approximation of $t_G$ can be computed in $O\left(\frac{\log m}{\epsilon^2} \frac{f(n, m)}{\epsilon^2} \left(\frac{3}{\epsilon^2}\right)^{h} \right) = O(f(n, m))$ time and $O(S(n, m))$ space.
Efficient Estimation of Graph Trussness

Proof. We follow the scheme in [20], and fix $c = 3 + \epsilon$. Let $A$ be the algorithm that computes the support for every edge of $G$. Let $G^1 = G$. For $i = 1, 2, \ldots$, run $A$ on $G^i$ and delete the edges with support less than or equal to $c \frac{t_{G^i}}{m_i}$, where $T_i$ is the number of triangles in $G^i$ and $m_i$ is its number of edges. We obtain $G^{i+1}$ in this way. Continue the iteration on $i := i + 1$ until the current graph has no more edges. As the sum of the supports is three times the number of triangles, we have that $m_{i+1} \leq \frac{3T_i}{cT_i/m_i} = \frac{3m_i}{c}$. In other words, at most a fraction $\frac{3}{c} = \frac{3}{3+\epsilon} < 1$ of the previous edges survive at each iteration.

The returned approximate value $t_G$ is $\max_{i \geq 1} T_i/m_i$, for the values $T_i$ and $m_i$ seen in the whole process. Since all the edges have been removed, there exists a value of $i$ such that $t_G \leq c \frac{T_i}{m_i}$, as otherwise no edge would have been removed from the $t_G$-truss of $G$. Hence, $c t_G \geq t_G$.

Applying Theorem 2, we also have $t_G \geq T_i/m_i$ for each $i$, thus $t_G \geq t_G$. Hence, $t_G$ is a $(3 + \epsilon)$-approximation.

Using the previous lemma, we can finally prove Theorem 19.

Theorem 19. Given an undirected graph $G$ with $n$ nodes, $m$ edges, and any $\epsilon > 0$, a $(2 + \epsilon)$-approximation of the trussness $t_G$ can be computed in time $O(c^{-1} m 1+\frac{n}{m})$ or $O(c^{-1} m 1+\frac{n}{m})$, where $O(n^w)$ is any upper bound for the $(n \times n)$-matrix multiplication cost.

Proof. We employ fast matrix multiplication for computing the support of each edge, with a cost of $O(\min(n,m)^{2w})$ or $O(m 1+\frac{n}{m})$ time [28], to the adjacency matrix of $G$. Considering cell $(a, b)$, its value is given by the scalar product of the adjacency vectors of nodes $a$ and $b$. Hence, if $\{a, b\} \in E(G)$, cell $(a, b)$ contains $\sup_G\{\{a, b\}\}$. From this follows that we can compute the support of all edges in $f(n,m) = O(n^w)$ or $f(n,m) = O(m 1+\frac{n}{m})$ time by following the approach of [1].

We obtain the required running times by replacing the cost of support counting in the formulas provided by Lemma 15, observing that in the case of the cost of $O(\min(n,m)^{2w})$ we can only consider the first $O(\log \frac{m}{n})$ terms of the cost, as after that we obtain a geometric series which is bounded by the first term. In the second case, the geometric series starts from the first term, thus removing all logarithmic factors.

We observe that it is possible to obtain a fast and practical algorithm from what described in the proof of Lemma 15, by replacing the matrix multiplication to compute the support of each edge, with a practical method to list triangles.

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We use the following relations:

$$\sum_{\{u,v\} \in E(G)} \min\{\delta(u), \delta(v)\} \leq 2m \alpha_G, \quad \alpha_G = O(\sqrt{m}), \quad \text{and} \quad \alpha_G \geq \frac{t_G + 1}{2}$$

See [10] for the first two. For the third one, since the degree of each node \(u\) in a \(k\)-truss is \(\delta(u) \geq k + 1\), we have that an inclusion-maximal \(k\)-truss for \(k = t_G\) has at least \(n'(t_G + 1)/2\) edges, where \(n'\) is the number of its nodes. As each forest cover less than \(n'\) of the edges in the \(k\)-truss, it yields \(\alpha_G \geq \frac{t_G + 1}{2}\).

**Lemma 20.** Given an undirected graph \(G\) with \(m\) edges and arboricity \(\alpha_G\), its triangles can be combinatorially counted or listed and its trussness can be computed in \(O(m \alpha_G)\) time and \(O(m)\) space.

**Proof.** Let \(\delta(v) = |N_G(v)|\) denote the degree of a node \(v\). Consider the algorithm that scans all the edges in \(E(G)\) and, for each edge \(e = \{u, v\}\), it computes \(\text{SUP}_G(e) = |N_G(u) \cap N_G(v)|\) in \(O(\min\{\delta(u), \delta(v)\})\) time by taking the node of minimum degree between \(e\)’s endpoints, say \(u\), and checking if its neighbors (in \(N_G(u)\)) are adjacent to the other endpoint, say \(v\). This gives both triangle listing and counting, where the latter quantity is obtained as \(T_G = \sum_{e \in E(G)} \text{SUP}_G(e)\). Total time is \(O(m + \sum_{\{u,v\} \in E(G)} \min\{\delta(u), \delta(v)\})\) which is proved in [10] to be \(O(m \alpha_G)\).

As for the trussness \(t_G\), an extra postprocessing step is needed for the algorithm. After computing the support of each edge, as mentioned above, it sorts the edges \(e \in E(G)\) in non-decreasing order with respect to \(\text{SUP}_G(e)\) in \(O(m)\) time. It then keeps the edges in buckets corresponding to their support, so that any two edges \(e\) and \(e'\) are in the same bucket iff \(\text{SUP}_G(e) = \text{SUP}_G(e')\). These buckets can be built in \(O(m)\) time, and they can be easily managed dynamically, thus taking the edge of smallest support and changing the bucket of an edge (whose supports has to change), in \(O(1)\) time. As long as the buckets are nonempty, the algorithm removes an edge \(e = \{u, v\}\) from the nonempty bucket with the smallest associated edge support, and decreases by 1 the support of the edges in

APPENDIX

## A. Trussness and truss decomposition

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\[
\{\{u, z\}, \{v, z\} \mid z \in N_G(u) \cap N_G(v)\} \text{ (while updating their buckets). As previously observed, this can be done by looking at the smallest-degree endpoint of } e, \text{ in } O(\min\{\delta(u), \delta(v)\}) \text{ time. It returns as } t_G \text{ the maximum among the supports of the extracted edges. Since each edge is removed once, the total cost is } O(m + \sum_{\{u,v\} \in E(G)} \min\{\delta(u), \delta(v)\}) = O(m \alpha_G) \text{ time and } O(m) \text{ space.} \\

We remark that a seemingly equivalent result is obtained with different techniques in [7] using the “average degeneracy” of the graph in, where the average degeneracy is 
\[
\frac{1}{|E(G)|} \sum_{\{u,v\} \in E(G)} \min\{\delta(u), \delta(v)\}.
\]

\textbf{Lemma 21.} \textit{There is an algorithm to compute the truss order of an undirected graph } G \textit{ with } n \textit{ nodes and } m \textit{ edges in } O(f(m, n)) \textit{ time iff there is an algorithm to compute the truss decomposition in } O(f(m, n)) \textit{ time, for a polynomial } f(m, n) = \Omega(m) \textit{.}

\textbf{Proof.} \textit{As the other implication is easy, let us discuss how to obtain the truss decomposition from the truss order. First, let us consider the gadget } G_x, \textit{ that is a graph with } 2x \textit{ nodes: } G_x \textit{ is build by adding } x \textit{ nodes to a clique } K_x, \textit{ where the } i^{\text{th}} \textit{ added node (} 1 \leq i \leq x \textit{) is connected arbitrarily to } i \textit{ nodes in } K_x. \textit{ It can be easily noted how for each value } j \in \{0, \ldots, x\} \textit{ there is at least one edge in } G_x \textit{ with trussness } j. \textit{ Now, compute the degeneracy } d_G \textit{ of } G, \textit{ which can be done in } O(m) \textit{ time, and recall that } t_G \leq d_G = O(\sqrt{m}). \textit{ Then, let } G' \textit{ be the disjoint union } G^{x^2} \cup G_{d_G}: \textit{ as the number of nodes and edges } G' \textit{ is } O(n) \textit{ and } O(m) \textit{ respectively, we can compute its truss order in } O(f(m, n)) \textit{ time. Recalling that the trussness of each edge of } G^{x^2} \textit{ is even, and } G_{d_G} \textit{ generates edges with all values of trussness up to } d_G \textit{ (which are known beforehand), each edge in } G^{x^2} \textit{ will appear in the truss order of } G' \textit{ between two consecutive edges of } G_{d_G}, \textit{ of which at least one has even trussness, which will be the exact trussness of the edge. After that, we divide the value by 2 to obtain the exact trussness of each edge in } E(G). \textbf{\textit{\blacksquare}}} 

\textbf{Lemma 21} \textit{implies that we cannot find a more efficient algorithm for the trussness } t_G \textit{ using the truss order, as the latter costs as much as computing } t_G \textit{ (see Lemmas 20 and 22).}

Since } m \alpha_G = \Theta(n^3) \textit{ in the worst case, Theorem 14 implies that improving the worst-case cost in Lemma 20 to significantly less than } O(m \alpha_G) \textit{ time using combinatorial algorithms is quite hard. However, since trussness } t_G \textit{ is also a parameter for complexity analysis, one could hope to replace } \alpha_G \textit{ with } (t_G + 1) \textit{ in Lemma 20 thus getting } O(m(t_G + 1)) \textit{ time. Not even this is possible, as } t_G = 0 \textit{ would mean that we can test if } G \textit{ is triangle-free. In summary, we immediately derive the following conditional lower bound.}

\textbf{\textit{\blacksquare}}} 

\textbf{Lemma 22.} \textit{Given any undirected graph } G \textit{ with } m \textit{ edges, arboricity } \alpha_G \textit{ and trussness } t_G, \textit{ triangle counting/listing and graph trussness cannot be computed by combinatorial algorithms in either } \tilde{O}(m \alpha_G) \textit{ time or } O(m(t_G + 1)) \textit{ time in the worst case, unless BMM is truly subcubic.}

\textbf{By Lemma 20 and Lemma 22} \textit{it makes sense to investigate the problem of approximating the graph trussness } t_G.