Magnetostatic interaction between two Bubble Skyrmions

M. A. Castro¹, D. Mancilla-Almonacid¹, J. A. Valdivia², and S. Allende¹

¹Departamento de Física, CEDENNA,
Universidad de Santiago de Chile, USACH,
Av. Ecuador 3493, Santiago, Chile and
²Departamento de Física, Facultad de Ciencias,
Universidad de Chile, Casilla 653, 7800024, Santiago, Chile

Abstract

A detailed analytic and numerical analysis of the interaction between two bubble skyrmions has been carried out. Results from micromagnetic calculations show a strong dependence of the parameters of the skyrmion magnetic profile as a function of the magnetostatic interaction. The magnetic core and edge-width sizes of the skyrmion increase or decrease depending on the relative position between the skyrmions and the uniaxial perpendicular anisotropy. In particular, when a magnetic disk is over another, there is a transition from a Bloch-like skyrmion configuration to a Néel-like skyrmion configuration as the distance between the disks decreases. This transition is due to the magnetostatic interaction between them. Therefore, it is possible to stabilize a bubble skyrmion with a Néel configuration without the Dzyaloshinskii-Moriya interaction. Thus, these results can be used for the parameters control of the skyrmions in magnetic spintronic devices that need to use these configurations.
INTRODUCTION

During the last decade, a great deal of attention has been focused on the study of the magnetic skyrmions in magnetic structures because they have potential applications in magnetic storage devices of high density, spintronic devices, etc. [1-6]. For example, in nanostructures such as nanodisks it is possible to find different type of skyrmions, like Néel, Bloch or bubble configurations, among others. The Néel and Bloch skyrmion configurations can be obtained by introducing a Dzyaloshinskii-Moriya interaction due to the strong spin-orbit coupling between two materials [1, 2, 17]. Similarly, the bubble skyrmions can be stabilized through an uniaxial magnetic anisotropy perpendicular to the plane of the disk [8-12].

It is interesting to note that the magnetic particles possess a long-range magnetostatic field, which is present in the formation of a great variety of magnetic textures like vortices or skyrmions. Recently, arrays of bubble skyrmions in nanodisks with perpendicular anisotropy have been proposed for the implementation of spintronic devices [13-16]. In these systems, it should be emphasized that the interaction between the skyrmions through the magnetostatic field can be strong depending on their locations [2, 17]. In terms of analysis, the interaction between bubble skyrmions can be decomposed as the magnetostatic field interaction of cores and edges. This magnetostatic interaction could even influence their movements and may also affect their magnetic structures [17, 18] affecting the operation of the device. Therefore it becomes necessary to study in detail the interaction between two bubble skyrmions.

Hence, in this paper, we study the magnetostatic interaction between two magnetic dots that have a magnetic bubble skyrmion. They are stabilized by an effective anisotropy without the Dzyaloshinskii-Moriya interaction. Specifically, we focus our attention on the skyrmion core and edge that vary in size as a function of the magnetostatic interaction between these two magnetic dots. Based on micromagnetic calculations and micromagnetic simulations, we have carried out numerical calculations, in which we have observed a strong variation of the parameters of the skyrmion magnetic profile. The magnetic core and the edge-width sizes of the skyrmion increase or decrease depending on the relative position between the skyrmions and the uniaxial perpendicular anisotropy. In particular, it is possible to stabilize bubble skyrmions with a Néel-like skyrmion magnetic profile when a magnetic disk is over another, in the absence of the Dzyaloshinskii-Moriya interaction. This transition from the Bloch-like skyrmion configuration to the Néel-like skyrmion configuration is due to
the magnetostatic interaction between the magnetic disks. These results could be useful for the realization of future bubble skyrmion devices.

**THEORY**

We start with two dots that have a magnetic Co/Pt bubble skyrmion configuration. These dots are separated, center to center, by a horizontal distance $x$ and a vertical distance $z$, as shown in Fig. 1. The skyrmions are then allowed to interact through the magnetostatic interaction. Each magnetic dot has a radius $R$, a height $H$, and an effective magnetic uniaxial anisotropy perpendicular to the plane of the dot characterized by $K_u > 0$. The magnetic parameters for each dot are $M_s = 500$ kA/m and $A = 1.5 \times 10^{-11}$ J/m, so that the exchange length is equal to $L_{ex} = \sqrt{2A/\mu_0 M_s^2} \approx 9.8$ nm [19]. We approach the study of these systems with the micromagnetic theory by using analytical and numerical calculations, and micromagnetic simulations.

FIG. 1. Schematic representation of two bubble skyrmions separated, center to center, by a horizontal distance $x$ and a vertical distance $z$. They are coupled by the magnetostatic interaction.

The micromagnetic simulations are performed with the Object Oriented Micromagnetic Framework (OOMMF) code [20]. We consider that each dot has a thickness $H = 10$ nm and a radius $R = 300$ nm, a cubic mesh size of $2 \times 2 \times 2$ nm$^3$ and the Gilbert damping constant equal to 0.5. To obtain the minimum energy configuration, we consider different initial states of the magnetization such as vortex, in plane, out plane, and skyrmion configurations. To relax the system into the most stable configuration, we use the Runge Kutta fourth order method.
Horizontal separation between two magnetic disks with low anisotropy

In the first place, we consider two disks, with low magnetic anisotropy, separated by a horizontal distance \((x > R)\) and in the same plane \((z = 0)\). From the micromagnetic simulations, we propose a magnetic profile of the form of a Bloch-like skyrmion characterized by a magnetization that rotates in the plane and perpendicular to the radial direction, i.e., \(\vec{M}(\vec{r}) = M_sm_\phi(\rho) \hat{\phi} + M_sm_z(\rho) \hat{z}\), where \(M_s\) is the saturation magnetization of the dot, and \(m_\phi^2(\rho) + m_z^2(\rho) = 1\). Then, the analytical and numerical calculations are done by parameterizing the magnetization function for the bubble skyrmion in cylindrical coordinates by \([12, 21]\)

\[
m^B_1(\rho) = \begin{cases} 
(1 - \frac{\rho^2}{b^2})^4 & 0 < \rho \leq b \\
0 & b < \rho \leq R - c \\
-\left(1 - \frac{(R - \rho)^2}{c^2}\right)^4 & R - c < \rho \leq R
\end{cases}
\tag{1}
\]

where \(b\) and \(c\) are the sizes of the core and the edge-width of the magnetic bubble skyrmion, respectively. The abbreviation \(B_1\) in the superindex of \(m_z\) is used with the aim of referring to a Bloch-like skyrmion. Figure 2 illustrates the \(z\)-component of the magnetization of the magnetic profile for \(K_u = 143\) kJ/m\(^3\). The top row illustrates a comparison between the analytic magnetic profile given by Eq. (1) and the magnetic profile obtained by the micromagnetic simulation with OOMMF. The bottom row illustrates a top view of the magnetization obtained with OOMMF. Figures 2(a) and 2(d) considers an isolated magnetic dot. Figures 2(b) and 2(e) considers the magnetostatic interaction between two disks with a parallel-configuration of the magnetic bubbles with \(x = 610\) nm. Figures 2(c) and 2(f) considers the magnetostatic interaction between two disks with an anti parallel-configuration of the magnetic bubbles with \(x = 610\) nm. The analytical magnetic profiles shows a good agreement when the disk is isolated and also when there is a strong magnetostatic interaction between two of them. Therefore, these OOMMF simulations suggest that the cylindrical angular variation of the perturbations in the magnetic profile of these bubble skyrmions can be disregarded when they are interacting by the magnetostatic interaction, as suggested in Figs. Figs. 2(d), 2(e), and 2(f). Hence, for simplicity, in the analytical analysis below we will consider that their magnetic profiles do not depend on the polar angle. Therefore, we use the magnetic profile given by Eq. (1) when \(z = 0\) and \(K_u = 143\) kJ/m\(^3\).
FIG. 2. Comparison between the analytic magnetic profile given by Eq. (1) and the magnetic profile obtained by the micromagnetic simulation with OOMMF with $R = 300$, $H = 10$ nm, $K_u = 143$ kJ/m$^3$, and $z = 0$. Magnetic profile of (a) an isolated disk, (b) two parallel $B_2$ configuration with $x = 610$ nm, and (c) two antiparallel $B_1$ configuration with $x = 610$ nm. In addition, Figures 2(d), 2(e), and 2(f) show a top view of the minimum energy configuration obtained by OOMMF for the Figures 2(a), 2(b), and 2(c), respectively.

The total magnetic energy of two disks with the $B_1$ configuration, $E_{tot}^{B_1}$, is given by the sum of the exchange, magnetostatic, and anisotropy energies, whose forms are suggested by the micromagnetic theory[22]. The exchange energy for the bubble skyrmion with the $B_1$ configuration, $E_{ex}^{B_1}$, is given by [12]

$$E_{ex}^{B_1} = 2\pi HA \left[ \frac{1}{2} \mathcal{H}(8) - 4\mathcal{H}\left(\frac{-1}{8}\right) + \ln\left(\frac{R-c}{b}\right) + \int_{R-c}^{R} f(\rho)\rho d\rho \right], \quad (2)$$

where $A$ is the stiffness constant, $\mathcal{H}(\xi)$ is the harmonic number function of the complex variable $\xi$ [23] given by the Euler’s integral

$$\mathcal{H}(\xi) = \int_{0}^{1} \frac{1-w^{\xi}}{1-w} dw, \quad (3)$$

and the function $f(\rho)$ in Eq. (2) is

$$f(\rho) = \frac{64(\zeta(\rho)^2 - 1)^6(\zeta(\rho)^2 + 1)}{c^2(1 - (\zeta(\rho)^2 - 1)^8)} + \frac{(1 - (\zeta(\rho)^2 - 1)^8)}{\rho^2} \quad (4)$$
with \( \zeta(\rho) = (R - \rho)/c \). The magnetostatic contribution is given by the self-magnetostatic interaction for every dot defined by \( E_{m,\text{self}} \), and the magnetostatic interaction between the dots called \( E_{m,\text{int}} \). The self magnetostatic interaction is \( E_{m,\text{self}} = (\mu_0/2) \int \vec{M}(\vec{r}) \cdot \nabla U_{\text{self}}(\vec{r}) \, dv \), where \( U_{\text{self}}(\vec{r}) \) is the magnetostatic potential in a dot due to the same magnetic dot.\(^{[22]}\)

Observing that we do not have volumetric charges in the magnetic profile of Eq. (1) \( \nabla \cdot \vec{M}(\vec{r}) = 0 \), then \( E_{m,\text{self}} \) has the form \(^{[12]}\)

\[
E_{m,\text{self}} = \pi \mu_0 M_s^2 \int_0^\infty dq \left[ F_1(q, b) + F_2(q, c, R) \right]^2 \left( 1 - e^{-qH} \right)
\]

where \( F_1(q, b) \) and \( F_2(q, c, R) \) are:

\[
F_1(q, b) = \int_0^b \rho J_0(q\rho) m_z^B(\rho) \, d\rho = \frac{384 J_5(bq)}{b^3 q^5},
\]

\[
F_2(q, c, R) = \int_{R-c}^R \rho J_0(q\rho) m_z^B(\rho) \, d\rho.
\]

The magnetostatic interaction between the two dots is given by \( E_{m,\text{int}} = \mu_0 \int \vec{M}(\vec{r}) \cdot \nabla U_{\text{int}}(\vec{r}) \, dv \), where \( U_{\text{int}}(\vec{r}) \) is the magnetostatic potential in a dot due to the other magnetic dot.\(^{[22]}\) Then \( E_{m,\text{int}} \) with the magnetic profile given by Eq. (1) is equal to:

\[
E_{m,\text{int}}(x, z) = -\pi \mu_0 M_s^2 \sigma_A \sigma_B \int_0^\infty dq J_0(qx) \left[ F_1(q, b) + F_2(q, c, R) \right]^2 e^{-q(H+z)} g(q, H, z),
\]

where the letters \( A \) and \( B \) represent the dot \( A \) and the dot \( B \), respectively. \( \sigma_A \) and \( \sigma_B \) take values \( \pm 1 \), and their values define if the magnetization profile is given by Eq. (1) (value \( +1 \)) or minus the magnetization profile given by Eq. (1) (value \( -1 \)). In essence we can parametrize the skyrmions with two orientations, namely, up or down. When \( \sigma_A = \sigma_B \) the configuration is called parallel, and when \( \sigma_A \neq \sigma_B \) the configuration is called anti-parallel.

The function \( g(q, H, z) \) is:

\[
g(q, H, z) = \begin{cases} 
(1 - 2e^{qH} + e^{2qz}) & 0 \leq z < H \\
(e^{qH} - 1)^2 & z \geq H
\end{cases}
\]

The anisotropy contribution, \( E_{\text{ani}} \), is given by \( E_{\text{ani}} = -K_u \int m_z^2(\rho) \, dv \). Then, \( E_{\text{ani}} \), with the magnetic profile given by Eq. (1), is \(^{[12]}\):

\[
E_{\text{ani}}^{B_1} = -2\pi K_u H \left( \frac{b^2}{18} - \frac{c^2}{18} + \frac{32768}{109395} c R \right).
\]

Hence, the expression of the total energy of the system is equal to

\[
E_{\text{tot}}^{B_1} = 2E_{\text{ex}}^{B_1} + 2E_{m,\text{self}}^{B_1} + 2E_{\text{ani}}^{B_1} + E_{m,\text{int}}^{B_1}.
\]
This expression, Eq. (11), depends on the parameters $x$, $z$, $b$, and $c$. Therefore, to obtain the energy of the system, we need to minimize $E_{\text{tot}}^{B_1}$ as a function of the parameters $b$ and $c$; for a fixed $R$, $H$, $x$, $z$, and $K_u$.

**Horizontal separation between two magnetic disks with high anisotropy**

In this section we consider two disks, with high magnetic anisotropy, separated by a horizontal distance ($x > R$) and in the same plane ($z = 0$). From the micromagnetic simulations, we propose a magnetic profile of the form $\vec{M}(\vec{r}) = M_s m_{\phi}(\rho) \hat{\phi} + M_s m_z(\rho) \hat{z}$, where the magnetic profile for $m_z(\rho)$, is given by

$$m_z^{B_2}(\rho) = \tanh\left(\frac{\rho - \gamma}{\Delta}\right),$$

where $b = \gamma$ and $c = R - \gamma$ are the core and the edge-width of the magnetic bubble skyrmion, respectively. The abbreviation $B_2$ in the superindex of $m_z$ is used with the aim of referring to a Bloch-like skyrmion configuration. Figure 3 illustrates the $z$-component of the magnetization of the magnetic profile for $R = 300$ nm, $H = 10$ nm, and $K_u = 150$ kJ/m$^3$. The top row illustrates a comparison between the analytic magnetic profile given by Eq. (12) and the magnetic profile obtained by the micromagnetic simulation with OOMMF. The bottom row illustrates a top view of the magnetization obtained with OOMMF. Figures 3(a) and 3(d) considers an isolated magnetic bubble. Figures 3(b) and 3(e) consider the magnetostatic interaction between two disks with a parallel-configuration of the magnetic bubbles with $x = 610$ nm. Figures 3(c) and 3(f) consider the magnetostatic interaction between two disks with an anti parallel-configuration of the magnetic bubbles with $x = 610$ nm. The analytical magnetic profiles shows a good agreement both when the disk is isolated and also when there is a strong magnetostatic interaction between two of them. Therefore, these OOMMF simulations suggest that the cylindrical angular variation of the perturbations in the magnetic profile of these bubble skyrmion can be disregarded when they are interacting by the magnetostatic interaction, as suggested in Figs. 3(d), 3(e), and 3(f). Hence, for simplicity, in the analytical analysis below we will consider that their magnetic profiles do not depend on the polar angle. Therefore, we use the magnetic profile given by Eq. (12) when $z = 0$ and $K_u = 150$ kJ/m$^3$.

The total magnetic energy of the system with the $B_2$ configuration for the dots, $E_{\text{tot}}^{B_2}$, is
FIG. 3. Comparison between the analytic magnetic profile given by Eq. (12) and the magnetic profile obtained by the micromagnetic simulation with OOMMF with $R = 300$, $H = 10$ nm, $K_u = 150$ kJ/m$^3$, and $z = 0$. Magnetic profile of (a) an isolated disk, (b) two parallel $B_2$ configuration with $x = 610$ nm, and (c) two antiparallel $B_2$ configuration with $x = 610$ nm. In addition, Figures (d), (e), and (f) show a top view of the minimum energy configuration obtained by OOMMF for the Figures (a), (b), and (c), respectively.

given by the sum of the exchange, magnetostatic, and anisotropy energies, whose form are suggested by the micromagnetic theory\cite{22}. The exchange energy for the bubble skyrmion configuration with the $B_2$ configuration, $E_{ex}^{B_2}$, is given by

$$E_{ex}^{B_2} = 2\pi H A \int_0^R \frac{(\Delta^2 + \rho^2) \text{sech}^2 \left( \frac{\rho - q}{\Delta} \right) \rho d\rho.}$$ \hspace{1cm} (13)

The self-magnetostatic energy with the $B_2$ configuration is

$$E_{m,self}^{B_2} = \pi \mu_0 M_s^2 \int_0^\infty dq \left[ \int_0^R J_0(k\rho) m_{z}^{B_2}(\rho) \rho d\rho \right]^2 \left(1 - e^{-qH}\right).$$ \hspace{1cm} (14)

The magnetostatic interaction between the two dots by using the magnetic profile given by Eq. (12) is equal to:

$$E_{m,int}(x, z) = -\pi \mu_0 M_s^2 \sigma_A \sigma_B \int_0^\infty dq J_0(qx) \left[ \int_0^R J_0(q\rho) m_{z}^{B_2}(\rho) \rho d\rho \right]^2 e^{-q(H+z)}g(q, H, z),$$ \hspace{1cm} (15)
The anisotropy contribution with the magnetic profile given by Eq. (12), $E_{\text{ani}}^{B_2}$, is:

$$E_{\text{ani}}^{B_2} = -\pi HR^2K_u - 2\pi HK_u \left( \Delta^2 \ln \left[ \cosh \left( \frac{\gamma - R}{\Delta} \right) \right. \right.$$  
$$\left. \left. \left. \left. - \sech \left( \frac{\gamma}{\Delta} \right) \right) + \Delta R \tanh \left( \frac{\gamma - R}{\Delta} \right) \right] \right). \quad (16)$$

Therefore, the total energy expression of the system with the $B_2$ configuration for the dots is equal to

$$E_{\text{tot}}^{B_2} = 2E_{\text{ex}}^{B_2} + 2E_{\text{m, self}}^{B_2} + 2E_{\text{ani}}^{B_2} + E_{\text{m, int}}^{B_2}. \quad (17)$$

This expression, Eq. (17), depends on the parameters $x$, $z$, $\gamma$, and $\Delta$. Therefore, to obtain the energy of the system, we need to minimize $E_{\text{tot}}^{B_2}$ as a function of the parameters $\gamma$ and $\Delta$; for a fixed $R$, $H$, $x$, $z$, and $K_u$.

**Vertical separation between two magnetic disks**

In this section we consider two disks, separated by a vertical distance ($z > H$) and in the same axis ($x = 0$). Our micromagnetic simulations show that the magnetic configuration of each dot changes from a Bloch-like skyrmion configuration to a Néel-like skyrmion configuration as the distance between the dots decreases. In a Néel-like skyrmion the magnetization rotates in the plane parallel to the radial direction, then we propose a magnetic profile of the form $\vec{M}_A (\vec{r}) = -M_s m_\rho (\rho) \hat{\rho} + M_s m_z (\rho) \hat{z}$ and $\vec{M}_B (\vec{r}) = M_s m_\rho (\rho) \hat{\rho} + M_s m_z (\rho) \hat{z}$, for the dot A and for the dot B, respectively. We have that $m_\rho^2 (\rho) + m_z^2 (\rho) = 1$ and $m_z (\rho)$ is given by Eq. (12). Figure 4 illustrates the $z$-component of the magnetization of the magnetic profile for $z = 20$ nm and $K_u = 150$ kJ/m$^3$. Figure 4(a) illustrates a comparison between the analytic magnetic profile given and the magnetic profile obtained by the micromagnetic simulation with OOMMF. Figure 4(b) shows a top view of the magnetization of the two dots. Figure 4(c) shows a schematic representation of the front view of the magnetization of both dots, showing clearly how the rotating field lines can connect the rotating magnetization of the two dots.

The total magnetic energy of the system with the Néel-like skyrmion configuration for the dots, $E_{\text{tot}}^N$, is given by the sum of the exchange, magnetostatic, and anisotropy energies. The abbreviation $N$ is used with the aim of referring to a Néel-like skyrmion. The exchange energy for the two dots with the $N$ configuration, $E_{\text{ex}}^N$, is the same given by the $B_2$ configuration, i.e., $E_{\text{ex}}^N = E_{\text{ex}}^{B_2}$. 
FIG. 4. Magnetization of the minimum energy state for two dots, one over the other, with $R = 300$, $H = 10$ nm, $K_u = 150$ kJ/m$^3$, $x = 0$, and $z = 20$ nm. (a) Comparison between the analytic magnetic profile of the Néel-like skyrmion configuration and the magnetic profile obtained by the micromagnetic simulation with OOMMF. (b) Top view of the magnetization obtained by OOMMF for the two dots. (c) Schematic representation of the front view of the magnetization of the two dots.

The self-magnetostatic energy in this case, have superficial and volumetric contributions. Then, the self-magnetostatic energy is:

$$E_{\text{m, self}}^N = \pi \mu_0 M_s^2 \int_0^\infty dk \left( kH + e^{-kH} - 1 \right) \left( \int_0^R \rho J_1(k\rho)\rho d\rho \right)^2 + \pi \mu_0 M_s^2 \int_0^\infty dk \left( 1 - e^{-kH} \right) \left( \int_0^R \rho J_0(k\rho)\rho d\rho \right)^2$$

(18)

The magnetostatic interaction between the two dots is equal to:

$$E_{\text{m, int}}^N (x = 0, z) = -\pi \mu_0 M_s \int_0^\infty dk e^{-k(H+y)}(e^{kH} - 1)^2 \left( \int_0^R m_\rho(\rho)J_1(k\rho)\rho d\rho \right)^2$$

$$- \pi \mu_0 M_s \int_0^\infty dk e^{-k(H+y)}(e^{kH} - 1)^2 \left( \int_0^R m_z(\rho)J_0(k\rho)\rho d\rho \right)^2$$

$$- 2\pi \mu_0 M_s \int_0^\infty dk e^{-k(H+y)}(e^{kH} - 1)^2 \int_0^R m_\rho(\rho)J_0(k\rho)\rho d\rho \int_0^R m_\rho(\rho)J_1(k\rho)\rho d\rho$$

(19)

The anisotropy contribution of the $N$ bubble skyrmion configuration is equal to the $B_2$ bubble configuration, i.e., $E_{\text{ani}}^N = E_{\text{ani}}^{B_2}$. Therefore, the total energy expression of the system with the $N$ configurations for the two dots at $x = 0$ is equal to

$$E_{\text{tot}}^N = 2E_{\text{ex}}^N + 2E_{\text{m, self}}^N + 2E_{\text{ani}}^N + E_{\text{m, int}}^N.$$  

(20)
This expression, Eq. (20), depends on the parameters \( x, z, \gamma, \) and \( \Delta \). Therefore, to obtain the energy of the system, we need to minimize \( E_{tot}^N \) as a function of the parameters \( \gamma \) and \( \Delta \); for a fixed \( R, H, x, z, \) and \( K_u \).

**RESULTS AND DISCUSSION**

We start with two Co/Pt magnetic dots with geometrical parameters \( R = 300 \) nm and \( H = 10 \) nm. The center of the dot \( A \) is set up at the origin \( x = z = 0 \). With these parameters, we study two scenarios in the following subsections: the disks adjacent to each other is when the dot \( B \) is at \( x > 2R \) and \( z = 0 \) (dots in the same plane) and the disks vertically stacked corresponds to \( x = 0 \) and \( z > H \) (dots in the same axis \( z \)). In the following sections we study and discuss the magnetostatic interaction energy and also the dependence of the parameters of the skyrmion, the core size \( b \) and the end-width size \( c \), as a function of the distance between the dots.

**Horizontal separation between two magnetic disks with bubble skyrmion configurations, \( x > 2R \) and \( z = 0 \).**

In this section we study the magnetostatic interaction between two disks where the disks are one beside another, i.e., according to Fig. 1, \( z = 0 \) and the distance between their centers is \( x > 2R \). Figure 5 illustrates the magnetostatic interaction energy normalized by \( \mu_0 M_s^2 L_{ex}^3 \), \( E_{m, int} = E_{m, int} / (\mu_0 M_s^2 L_{ex}^3) \), as a function of \( x \) for \( K_u = 143 \) kJ/m\(^3\) and 150 kJ/m\(^3\). In this configuration, the magnetostatic interaction energy is negative when \( \sigma_A \neq \sigma_B \), so that the two skyrmions are oriented antiparallel. Similar results were obtained with magnetic vortices in disks, where the antiparallel alignment of the vortices have the lowest magnetostatic energy [24, 25]. The orientation of the skyrmions is energetically favorable because the magnetic field lines produced at the edge of one of the skyrmions, can naturally and immediately match the orientation of the magnetization and magnetic field produced at the closest edge of the other skyrmion, minimizing the energy associated with the magnetostatic interaction (similar explanation is observed in two coupled vortices, where the magnetic cores close the magnetic field lines [24, 25]). For this reason, we start to analyze the antiparallel configuration in this subsection, as it corresponds to the configuration of lowest energy.
It is interesting to observe the variation of the initial decay of the magnetostatic interaction energy as a function of the separation $x$ for different anisotropies. Although for $K_u = 150 \text{ kJ/m}^3$ the decay becomes faster than for $K_u = 143 \text{ kJ/m}^3$, the former is more stable at least for small distances (e.g. $x < 80L_{ex}$). However, this system is less energetically stable than the parallel configuration of the disks vertically stacked (see subsection), since in this case the magnetic field produced by the core and the far edge of one skyrmion have no simple way to couple to the magnetization of the other skyrmion. As a general trend, the interaction between the skyrmions grows stronger as more magnetic field lines produced at one skyrmion connect to the other skyrmion in the correct direction, as to increase the magnetostatic interaction energy.

To study the dependence of the core size, $b$, and the edge-width size, $c$, of the two skyrmions as a function of $x$, we will choose the two following regimes. The first one is when $K_u = 143 \text{ kJ/m}^3$ and we use the configuration $B_1$ [Eq. (1)]. The second one is when $K_u = 150 \text{ kJ/m}^3$ and we use the configuration $B_2$ [Eq. (12)]. Figure 6 illustrates the parameters of the skyrmions, $b$ and $c$, as a function of $x$. Figure 6(a) considers $K_u = 143 \text{ kJ/m}^3$. Figure 6(b) considers $K_u = 150 \text{ kJ/m}^3$. When the distance between the disks

![FIG. 5. Normalized magnetostatic interaction energy as a function of the horizontal separation $x$ between the two disks with the antiparallel bubble skyrmion configurations for different anisotropies. The symbols represent the numerical points at different anisotropies. The lines are obtained by fitting these numerical points. The dot (solid line) represents $K_u = 150 \text{ kJ/m}^3$ and the square (dashed line) represents $K_u = 143 \text{ kJ/m}^3$. Every disk has $h = 10 \text{ nm} \times L_{ex} \approx 1.02$ and $r = 300 \text{ nm} / L_{ex} \approx 30.67$.](image-url)
decreases, the edge-width size of the skyrmions increases while the size of the cores decreases. This behavior can be explained by the following argument: when the two magnetic disks (skyrmions \( A \) and \( B \)) are near each other, the magnetization directions for the edges of \( A \) and \( B \) are oriented antiparallel, allowing the magnetostatic field produce by one skyrmion to naturally close at the edge of the other skyrmion. This results in a decrease of the magnetostatic energy making the system to be more stable. When the skyrmions are close to each other, it is energetically favorable to have a relatively large edge \( c \). However, when the disks are moved away from each other, the magnetostatic interaction between them is reduced compared to the other energy terms, so that \( c \) begins to decrease until it reaches the value corresponding to an isolated skyrmion. In order to explain the behavior of the core of the skyrmion, we must first consider that the magnetic interaction between the core \( A \) and the edge-width \( B \) is stronger than the interaction between the core \( A \) and the core \( B \). This is because the magnetic volume due to the magnetization perpendicular to the plane of the disk from the cores is less than the magnetic volume due to the magnetization perpendicular to the plane of the disk from the edge-widths, and also because the distance between the core \( A \) and the core \( B \) is greater than the distance between the core \( A \) and the edge-width \( B \). If we focus on the magnetic interaction between the core of the skyrmion \( A \) and the edge-width of the skyrmion \( B \) when they are close together, we see that both magnetization directions are the same. Such parallel configuration is not favorable, so that it is energetically favorable to have a small magnetic volume, i.e., the size of the core, \( b \), should be small. For the purpose of increasing \( b \), the magnetostatic interaction should diminish, therefore, we have to increase \( x \). The opposite behavior occurs for the cores of the magnetic disk that have magnetic vortices, i.e., the core radius of the vortices decreases as the distance between them increases. This discussion for the vortices is analogous to what happens with the edges of the skyrmions, which correspond to the strongest interaction in this scenario, see Refs. [24, 25]. Analogously to the edge-width size, the core size of each skyrmion takes the value of an isolated disk when the separation distance is large enough to consider the magnetostatic interaction energy between the disks equal to zero.

From an application point of view, both antiparallel and parallel configurations are well worth investigating, because the core orientation of the skyrmion can be used to encode binary information and could be either up or down, depending on the information. For this reason, in addition to the previous study of the antiparallel configuration, the parallel con-
The core, $b$, and the edge-width, $c$, sizes of two disks with the antiparallel bubble skyrmion configurations as a function of $x$ at $h = 10\,\text{nm}$ $/L_{\text{ex}} \approx 1.02$ and $r = 300\,\text{nm}/L_{\text{ex}} \approx 30.67$. The anisotropy is (a) $K_u = 143\,\text{kJ/m}^3$ and (b) $K_u = 150\,\text{kJ/m}^3$.

The configuration of two skyrmions with a horizontal separation is investigated. Figure 7 illustrates the normalized magnetostatic interaction energy, as a function of $x$ for 143 kJ/m$^3$ and 150 kJ/m$^3$. In this configuration, the magnetostatic interaction energy is positive. For larger values of $K_u$ the decay becomes slower.

Figure 8 illustrates the parameters of the skyrmions, $b$ and $c$, as a function of $x$. Figure 8(a) corresponds to $K_u = 143\,\text{kJ/m}^3$ and Figure 8(b) corresponds to $K_u = 150\,\text{kJ/m}^3$. In both cases, when the distance between the disks decreases, the edge-width size of the skyrmions decreases while the core size increases. This can be explained through the core-edge and edge-edge interaction. The strongest interaction is between the edges and it is unfavorable, so the parameter $c$ decreases when $x$ decreases. On the other hand, the edge-core interaction is favorable, then the parameter $b$ increases when $x$ decreases.

Vertical separation between two magnetic disks with bubble skyrmion configurations, $x = 0$ and $z > H$.

In this section we study the magnetostatic interaction between two disks where one disk is over the other, i.e., $x = 0$ and $z > H$. Figure 9(a) and Figure 9(b) illustrate the normalized magnetostatic interaction energy as a function of $z$ for $K_u = 143\,\text{kJ/m}^3$ and $K_u = 150\,\text{kJ/m}^3$, respectively. In this scenario, we observe that the configuration of
FIG. 7. Normalized magnetostatic interaction energy as a function of the horizontal separation $x$ between the two disks with the parallel bubble skyrmion configurations for different anisotropies. The symbols represent the numerical points at different anisotropies. The lines are obtained by fitting these numerical points. The dot (solid line) represent $K_u = 143 \text{ kJ/m}^3$ and the square (dashed line) represents $K_u = 150 \text{ kJ/m}^3$. Every disk has $h = 10 \text{ nm}/L_{ex} \approx 1.02$ and $r = 300 \text{ nm}/L_{ex} \approx 30.67$.

FIG. 8. The sizes of the core, $b$, and the edge-width, $c$, of two disks with the parallel bubble skyrmion configurations as a function of $x$ for $h = 10 \text{ nm}/L_{ex} \approx 1.02$ and $r = 300 \text{ nm}/L_{ex} \approx 30.67$. The anisotropy is (a) $K_u = 143 \text{ kJ/m}^3$ and (b) $K_u = 150 \text{ kJ/m}^3$.

negative magnetostatic interaction energy occurs when the skyrmions are oriented parallel to each other, since both the directions of the core magnetization and also the directions of the edge-width magnetization of the skyrmions are the same. Analogous results have been
reported in stacked ferromagnetic disks with magnetic vortices, where the parallel alignment of the vortices have the lowest magnetostatic energy\cite{26, 27}. For this reason, we study the parallel configuration, $\sigma_A = \sigma_B$, as this is the configuration of lowest energy.

As mentioned before, this system is much more stable than the antiparallel configuration of the disks adjacent to each other, since a relatively large fraction of the magnetic field lines produced by the core and edge of one skyrmion can close at the other skyrmion increasing the magnetostatic interaction energy. Furthermore, it becomes interesting that the strength of the interaction decreases with $z$ much slower than in the disks adjacent to each other with $x$, and this is due to the fact that a large portion of the magnetic field lines produced at one skyrmion can naturally close on the other skyrmion. This fact is not true in the disks adjacent to each other, as a large portion of the magnetic field lines produced at one skyrmion cannot close at the other skyrmion, and must close on itself or infinity, not contributing as strongly to the total energy as in these disks vertically stacked.

To study the magnetic parameters $b$ and $c$ of the skyrmions, first we need to know the magnetic configuration of the dots when the vertical distance $z$ varies. Figure\cite{10} (a) shows the normalized total energy for the three magnetic configurations ($B_1$, $B_2$, and $N$) as a function
of \( z \) for \( K_u = 143 \text{ kJ/m}^3 \). We observe that the bubble skyrmion with the \( B_1 \) configuration is observed when the disk are isolated until the disks have a separation of \( z \approx 14.33 L_{ex} \). From \( z \approx 14.33 L_{ex} \) to \( z \approx 8.19 L_{ex} \), we observe that the disks have a \( B_2 \) configuration. For \( z \) lower than \( z = 8.19 L_{ex} \), we observe the \( N \) configuration for the disks. Figure 10(b) illustrates de normalized total energy for the configurations \( B_2 \) and \( N \) as a function of \( z \) for \( K_u = 150 \text{ kJ/m}^3 \). We observe that a distance \( z = 8.39 L_{ex} \), there is a transition from the \( B_2 \) configuration to the \( N \) configuration as \( z \) decreases. Then, for both anisotropies, we observe that the bubble skyrmion configuration with a Néel-like skyrmion configuration is stabilized by the magnetostatic interaction, without the Dzyaloshinskii-Moriya interaction, i.e., there is a transition from the Bloch-like to the Néel-like configuration.

![Graph](image)

**FIG. 10.** Normalized total energy as a function of the vertical distance \( z \) between the two disks for (a) \( K_u = 143 \text{ kJ/m}^3 \) and (b) \( K_u = 150 \text{ kJ/m}^3 \), at \( x = 0 \). The symbols represent the different configurations: \( B_1 \) (triangles), \( B_2 \) (squares), and \( N \) (dots).

Figure 11 shows the variation of the magnetic parameters \( b \) and \( c \) of the two skyrmions as a function of \( z \). We consider \( K_u = 143 \text{ kJ/m}^3 \) and \( K_u = 150 \text{ kJ/m}^3 \) for Fig. 11(a) and Fig. 11(b), respectively. We observe that the magnetostatic interaction energy decreases, as the disks are moved closer, until it reaches the lowest value at \( z = 10 \text{ nm} \). By decreasing the distance between the disks, the cores sizes of the skyrmions increase. However the edge-width sizes of the skyrmions decrease. To understand this behavior we observe the magnetic charges (related to the normal component of \( \vec{M} \)) on the surfaces of the magnetic disks. We will call \( B \) the upper disk and \( A \) the lower disk. Both disks have a parallel magnetic configuration. We consider that they are very close together so that \( z \approx H \). Disk \( A \) has a magnetic charge.
FIG. 11. The core, \( b \), and the edge-width, \( c \), sizes of two disks with the parallel bubble skyrmion configurations as a function of \( z \) at \( h = 10 \) nm / \( L_{\text{ex}} \approx 1.02 \) and \( r = 300 \) nm/\( L_{\text{ex}} \approx 30.67 \). (a) The anisotropy is \( K_u = 143 \) kJ/m\(^3\). (b) The anisotropy is \( K_u = 150 \) kJ/m\(^3\).

\(-q\) on the top surface of the edge-width while disk \( B \) has a magnetic charge \( q \) on the bottom surface of the edge-width. This configuration is stable because the interaction between the disks reduces the magnetic energy causing that the core sizes of the skyrmions increase. It should be noted that the magnetic volumes due to the magnetization perpendicular to the plane of the disks from the core are higher than the edge-width. Therefore, their interaction is stronger than the interaction of the edge-width volumes. For this reason the core sizes of the skyrmions increase when \( z \) decreases because opposite magnetic charges are attracted. The edge-width sizes decrease because for close distance in \( K_u = 143 \) kJ/m\(^3\) and \( K_u = 150 \) kJ/m\(^3\), the condition \( b + c = R \) occurs. Then if the magnetostatic interaction of the cores is stronger than the magnetostatic interaction of the edge-widths, the core size must to increase and the edge-width must to decrease as the distance between the dots decreases. In addition, from Fig. 11(a), we observe a discontinuity in both curves of the magnetic parameters \( b \) and \( c \). Figure 12 illustrates a comparison between the analytical profile of \( m_z \) and the micromagnetic simulations. Figs. 12 (a)-(b)-(c) are for \( K_u = 143 \) kJ/m\(^3\) and Figs. 12 (d)-(e)-(f) are for \( K_u = 150 \) kJ/m\(^3\). For a low anisotropy \( K_u = 143 \) kJ/m\(^3\) we observe that the magnetic profiles of \( m_z \) obtained by micromagnetic simulations, for the \( B_1 \) and the \( B_2 \) configurations, are different from the analytical profile that we consider. In fact, we observe that there is a transition of the magnetic profile from Eq. (1) to Eq. (12). Then, the discontinuity in Fig. 11(a) is given because we do not consider this transition in the
analytical calculation. In addition, we observe that for a high anisotropy $K_u = 150 \text{ kJ/m}^3$ the magnetic profiles of $m_z$ obtained by micromagnetic simulations are very similar to the analytical profile that we consider. Another point that is interesting to discuss is that in our analytical model, we do not consider a continuous transition from the Néel configuration (with $m_\phi = 0$ and $m_\rho \neq 0$) to the Bloch configuration (with $m_\phi \neq 0$ and $m_\rho = 0$). This also must have an effect related to the discontinuity observed in Fig. 11(a). Furthermore, we note that that there is no continuous manner to move from profile $B_1$ of Eq. (1) to profile $B_2$ of Eq. (1) keeping the same exact definition of $b$ and $c$. Therefore it is not that unexpected, within our analytical approximation, to observe discontinuous changes in $b$ and $c$ as we change profiles.

To finalize, we did not consider the case of antiparallel configuration when one disk is over the other. The reason is that we did not observe this configuration in the micromagnetic simulation performed by OOMMF when the disks have a strongly magnetostatic interaction.

![Graphs showing magnetic profiles](image)

**FIG. 12.** Component $z$ of the normalized magnetization at different vertical distance $z$ of two disks with the parallel bubble skyrmion configuration. The top row is for $K_u = 143 \text{ kJ/m}^3$ and the bottom row is for $K_u = 150 \text{ kJ/m}^3$. Each disk has $h = 10 \text{ nm}$, $r = 300 \text{ nm}$, and $L_{ex} \approx 1.02$ and $r = 300 \text{ nm}$, $L_{ex} \approx 30.67$. 
CONCLUSIONS

In summary, by means of an analytic model and numerical calculations, we have studied the dependence of the core and edge-width sizes for two magnetic disks, that have a bubble skyrmion configuration, that are interacting by the magnetostatic interaction. By using different ansatzs for the magnetic profile of a bubble skyrmion, it was possible to obtain an expression for the magnetostatic interaction energy between the two disks. We observed that the magnetic parameters that describe a skyrmion vary in different ways depending on the location of the disks. When the disks are separated by a horizontal distance, the configuration with minimum energy corresponds to the skyrmions that have an anti-parallel orientation. Results show that if the horizontal distance decreases, the core sizes of the skyrmions decrease and the edge-width sizes of the skyrmions increase. These results can be explained by the magnetic interactions between the magnetostatic fields created by the magnetizations of the cores and the edge-widths of the skyrmions. When one disk is over the other, the configuration with minimum energy corresponds to skyrmions that have a parallel orientation. As the vertical distance decreases, we observe that the bubble skyrmion configuration with a Néel-like skyrmion configuration is stabilized by the magnetostatic interaction, without the Dzyaloshinskii-Moriya interaction, i.e., there is a transition from the Bloch-like to the Néel-like configuration. Thus, these results can be used in the fabrication of future magnetic devices in which two or more bubble-type skyrmions are present.

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