An integrated approach to test for missingness not at random

A. A. ADEDIRAN\textsuperscript{1} J. NOONAN\textsuperscript{2,*} R. MITRA\textsuperscript{2,**} S. BIEDERMANN\textsuperscript{3}

\textsuperscript{1}a.a.adediran@soton.ac.uk, School of Mathematical Sciences, University of Southampton, U.K.
\textsuperscript{2}noonanj1@cardiff.ac.uk, mitrar5@cardiff.ac.uk\textsuperscript{*}, School of Mathematics, Cardiff University, U.K
\textsuperscript{3}stefanie.biedermann@open.ac.uk, School of Mathematics and Statistics, The Open University, U.K.

Abstract

Missing data can lead to inefficiencies and biases in analyses, in particular when data are missing not at random (MNAR). It is thus vital to understand and correctly identify the missing data mechanism. Recovering missing values through a follow up sample allows researchers to conduct hypothesis tests for MNAR, which are not possible when using only the original incomplete data. Our results shed new light on the properties of one such test, and provide an integrated framework for hypothesis testing and designing follow up samples in an efficient cost-effective way.

Keywords: Missing data; Missing not at random; Optimal design; Selection model.

1 Introduction

Missing data mechanisms [1] model the process generating the missing values and are fundamental in formulating appropriate analysis methods for the incomplete data. MNAR is the most challenging mechanism to handle. Its presence introduces biases in the analysis, and is an untestable assumption based on the original incomplete sample [2]. As effects of MNAR should only be corrected if we have good confidence in its presence and functional form, it is important to first design a test that permits this. This is the area this article focuses on.

To construct a test for MNAR, we assume a proportion of the missing records can be followed up. Subsequently, a regression model can be formulated to test for MNAR based on the originally observed sample together with the newly recovered values [3]. The choice of regression model to use depends on whether the Pattern Mixture or Selection Model Framework (SMF) is adopted. In this article we choose the latter as it more closely corresponds with [1]’s definition of mechanisms and allows intuitive interpretation of the results. Here, a binary response regression can test for MNAR (typically through a logistic link) where the outcome (missing indicator) is modelled as a function of the covariates and the response variable (which now comprises a mix of observed and recovered values). The test is formally defined in Section 2 but, intuitively, detecting MNAR is equivalent to testing whether the regression coefficient on the response is 0 or not.

A natural question on which missing records to follow up then arises. The choice of follow up sample could affect the power of this test, and also, crucially, the possibly non-representative sample of observed plus recovered values could alter the distribution properties of the data and thus affect estimation of model parameters, as seen in other settings involving binary response models fit to subsamples [4–6]. In these papers, to compute the optimal weights used within the subsampling procedure, all values must be observed. Therefore an immediate application of the results in [4–6] is not clear in the presence of missing data.

We provide an expression for the missing data mechanism in the augmented data, and relate it to the original missing data mechanism we wish to classify. This enables us to conclude the maximum likelihood estimator (MLE) for the regression coefficient of interest, using the augmented data, will still be (asymptotically) unbiased, and thus the null hypothesis that this coefficient is 0 in the original missing data mechanism can still reliably be tested using the augmented data.
Following up a large number of missing records may be costly, so to optimise the practicality of our approach, we develop a strategy for selecting the follow up sample in such a way that the test will achieve higher power than when using a random follow up sample. Exploiting the relationship between $T$- and $D_1$-optimality [7], we propose an algorithm that can be used to construct follow up samples, or recovery designs, that minimise the variance of the MLE.

2 Testing for MNAR

2.1 Problem formulation

To motivate our methodology, consider a setting involving a univariate response, $Y$, and $p$ covariates, $X_1, \ldots, X_p$, where

$$Y|(X_1 = x_1, \ldots, X_p = x_p) \sim N(\beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p, \sigma_y^2),$$

(1)

with $\beta = (\beta_0, \ldots, \beta_p)^T \in \mathbb{R}^p$ and $X_j$ having arbitrary density function $f_j(x)$, $j = 1, \ldots, p$. For the construction of designs in Section 4, we will assume $\beta$ and $\sigma_y^2$ are known. Whilst this is restrictive, the aim of this paper is to demonstrate a proof of concept.

We assume fully observed covariates $X_j$, $j = 1, \ldots, p$, and that missing values only occur within the response $Y$. Let $M$ denote an indicator random variable that equals one if $Y$ is missing and zero if observed. The conditional distribution of $M$, $\Pr(M = 1|X_1 = x_1, \ldots, X_p = x_p, Y = y)$, determines the missing data mechanism [1]. Under MAR, we have

$$\Pr(M = 1|X_1 = x_1, \ldots, X_p = x_p, Y = y) = \Pr(M = 1|X_1 = x_1, \ldots, X_p = x_p);$$

i.e. $M$ is conditionally independent of $Y$ given the covariates. Under MNAR, this is not present. To infer the missing data mechanism we assume a logistic regression model of the form

$$\text{logit} \Pr(M_i = 1) = \alpha_0 + \alpha_1 x_{1,i} + \ldots + \alpha_p x_{p,i} + \alpha_{p+1} y_i,$$

(2)

although in principle any binary response regression model could be used. Determining the presence and type of MNAR thus involves determining the value of $\alpha_{p+1}$. However, this parameter is inestimable based on the original sample as $Y_i$ is only observed for $M_i = 0$.

In order to address this we formulate a two-stage experiment to collect the data. At stage one, let $Y_1, \ldots, Y_n$, $X_{j,1}, \ldots, X_{j,n}$ ($j = 1, \ldots, p$) and $M_1, \ldots, M_n$ denote samples of size $n$ from the model in (1). Within the sample of $Y_1, \ldots, Y_n$, suppose $n_{\text{obs}} < n$ of $Y_i$ are observed, meaning $n_{\text{miss}} := n - n_{\text{obs}}$ observations of $Y_i$ are missing. Without loss of generality, we assume the first $n_{\text{obs}}$ of $Y_1, \ldots, Y_n$ are observed, meaning $Y_{n_{\text{obs}}+1}, \ldots, Y_n$ are all initially missing. At stage 2, assume resources permit follow up of a number of experimental units with missing responses to obtain (recovr) their responses, e.g. through home visits to patients in a clinical trial or follow-up telephone calls in a survey. We denote the number of recovered responses by $n^*$, where $n^* \leq n_{\text{miss}}$. As the number of recovered responses is a proportion of the missing observations, we can express $n^* = \lfloor c \cdot n_{\text{miss}} \rfloor$ with $0 < c \leq 1$. We assume the choice of which responses to recover is in the practitioner’s control, and thus this gives rise to the concept of a recovery design.

Definition. A recovery design $R = R(n^*)$ is a subset of size $n^*$ from $M := \{n_{\text{obs}}+1, n_{\text{obs}}+2, \ldots, n\}$.

The follow-up design $R$ instructs the practitioner what missing values to recover as follows. Specifically, a set of $n^*$ observations among the $n_{\text{miss}}$ missing observations to be recovered. To facilitate notation, assume these units are relabelled so that the recovery design $R = \{k_1, k_2, \ldots, k_{n^*}\}$. This results in the recovery of the previously missing response vector $Y_R := (Y_{k_1}, \ldots, Y_{k_{n^*}})$. By combining this with the observed responses $Y_O := (Y_1, Y_2, \ldots, Y_{n_{\text{obs}}})$, we obtain the augmented responses $Y_A := (Y_O, Y_R)^T$. The analogous vectors of augmented data for the covariates and the indicator random variables are: $X_{j,A} := (X_{j,O}, X_{j,R})^T$ with $X_{j,R} := (X_{j,k_1}, \ldots, X_{j,k_{n^*}})$ and $X_{j,O} := (X_{j,1}, X_{j,2}, \ldots, X_{j,n_{\text{obs}}})$; $M_A := (M_O, M_R)^T$ with $M_O = (0, \ldots, 0)$ and $M_R = (1, \ldots, 1)$.
With the above formulation we are now able to make inferences about the missing data mechanism. In particular we can address the following two objectives: 1) Improve the power of a test to detect the presence of MNAR if present, and 2) Obtain unbiased estimates of parameters characterising the missing data mechanism.

A key complication is that, even if the missing data mechanism can be well modelled by \( p \) based on the complete sample, the observed plus recovered sample may not be representative of the wider population. Thus the original model for the missing data mechanism may no longer be plausible, both in terms of its functional form as well as its parameter values.

The problem of estimating parameters in logistic regression models based on a subset of the data has been noted in \([4–6]\). However, they assume a subset is being taken from a fully observed population which is a less challenging problem than the one considered here, where we are constructing the subsample from an unobserved group of observations.

A key finding is as follows. If the missing mechanism is of the following form

\[
\Pr(M=1|X_1 = x_1, \ldots, X_p = x_p, Y = y) = \frac{\exp(\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_p x_p + \alpha_{p+1} y)}{1 + \exp(\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_p x_p + \alpha_{p+1} y)},
\]

where \( \alpha_{p+1} = 0 \) implies MAR, then the augmented data using a sample recovered randomly within a specified \( p \)-dimensional region of the covariate space maintains this \textit{expit} form, but with potentially different parameter values. This has two main benefits. Firstly, better estimates of the parameters in the missing data mechanism pertaining to the augmented data can be obtained. Secondly, and related to point one, propriety of tests for MNAR in this setting can be established and subsequently benefit from being optimised with respect to power.

### 2.2 A test for MNAR with logistic regression

For \( Y^*_i \in Y_A \), let \( X^*_{j,i} \in X_{i,A} \) be the corresponding value in the \( j^{th} \) covariate and let \( M^*_i \in M_A \) be the corresponding indicator value. A test for MNAR utilising the Selection Model framework (SMF) fits the following model to the augmented data,

\[
\logit\Pr(M^*_i = 1) = \alpha_0 + \alpha_1 X^*_{1,i} + \cdots + \alpha_p X^*_{p,i} + \alpha_{p+1} Y^*_i.
\]

Under the null hypothesis of MAR, we have \( \alpha_{p+1} = 0 \), which can be tested using a likelihood ratio test within the logistic regression framework; otherwise we conclude MNAR. Of course, a key assumption of this test for MNAR is that the missing mechanism has an \textit{expit} form like \( 3 \). If this is not the case then any inferences can only be made approximately.

### 3 Main Results

In this section, we formulate the theoretical foundations of the paper. One of the key messages from this section is that the augmented data used for determining whether MNAR is present represents a sample from a mixture distribution. This arises from a weighted combination of the distribution of the observed data and the distribution of the missing data. A careful construction of the marginal distribution allows us to mathematically formulate the missing data mechanism for the augmented data. This is formalised in the next section.

#### 3.1 A mixture distribution for the augmented data

Let \( C_A \subseteq \mathbb{R}^p \) be a \( p \)-dimensional region constructed by the cartesian product of intervals in \( \mathbb{R} \) of positive length. For example, we could have \( C_A = \prod_{j=1}^p [a_j, b_j] \) with \( a_j < b_j \) and \( a_j, b_j \in \mathbb{R} \); in this example \( C_A \) becomes a \( p \)-dimensional cuboid if \( a_j \) and \( b_j \) are finite. However, we could also permit more general sets such as e.g. \( C_A = \bigcup_{j=1}^p [a_j, b_j] \cup [e_j, f_j] \) with \( e_j < f_j \) and \( e_j, f_j \in \mathbb{R} \). For a random vector \( X = (X_1, \ldots, X_p) \in \mathbb{R}^p \), define the intersection of events:

\[
\mathcal{M}_O := \{ M = 0 \} \cap \{ X \in C_A \} \quad \text{and} \quad \mathcal{M}_R := \{ M = 1 \} \cap \{ X \in C_A \}.
\]
We assume \( C_A \) is chosen such that \( \Pr(M_O) > 0 \). We could also assume \( C_A \) is chosen such that
\[
\Pr(M_R) = c \cdot \Pr(M = 1). \tag{5}
\]
These requirements ensure enough observations (missing and observed) are present within the region where the augmented data lie, as well as ensuring the recovered sample comprises a fixed proportion, \( c \), of the missing values. However, when recovery designs involve random selection (without replacement) amongst the missing values, \( C_A \) can be relaxed to permit \( C_A \) such that
\[
\Pr(M_R) \geq c \cdot \Pr(M = 1). \tag{6}
\]
If we set \( C_A = \mathbb{R}^p \) such that \( \Pr(M_R) = \Pr(M = 1) \) then \( R \) would be a random sample of size \( n^* \) from all the missing responses. We enforce all observed data lying within \( C_A \) will be included in the augmented sample. With this framework the following results can be derived.

**Lemma 1.** The indicator variable for the augmented data, which we denote by \( M_A \), satisfies
\[
M_A = \begin{cases}
1 & \text{with probability } \frac{c \cdot \Pr(M = 1)}{c \cdot \Pr(M = 1) + \Pr(M_O)} \\
0 & \text{with probability } \frac{\Pr(M_O)}{c \cdot \Pr(M = 1) + \Pr(M_O)}
\end{cases}.
\]
The proof of Lemma 1 can be found in the supplement.

Accordingly, define the random variables and random vectors
\[
Y_O := Y \mid M_O; \quad Y_R := Y \mid M_R; \quad X_O := X \mid M_O; \quad X_R := X \mid M_R.
\]

**Lemma 2.** The augmented response/covariates are realisations from random variable/vector:
\[
Y_A := (1 - M_A)Y_O + M_AY_R \tag{7}
\]
\[
X_A := (1 - M_A)X_O + M_AX_R. \tag{8}
\]

These expressions arise by construction, being a combination of data with observed outcomes and data with recovered outcomes. Averaging over the distribution of \( M_A \), Lemma 2 can equivalently express the distributions of \( Y_A \) and \( X_A \) as mixtures satisfying:
\[
\mu_{Y_A}(\cdot) := \Pr(M_A = 0) \cdot \mu_{Y_O}(\cdot) + \Pr(M_A = 1) \cdot \mu_{Y_R}(\cdot)
\]
\[
\mu_{X_A}(\cdot) := \Pr(M_A = 0) \cdot \mu_{X_O}(\cdot) + \Pr(M_A = 1) \cdot \mu_{X_R}(\cdot),
\]
where, \( \mu_{Y_A}(\cdot) \), and \( \mu_{X_A}(\cdot) \), are probability measures for the random variable/vector, \( Y_A \) and \( X_A \) respectively, and similarly define probability measures \( \mu_{Y_O}(\cdot) \), \( \mu_{X_O}(\cdot) \), \( \mu_{Y_R}(\cdot) \), \( \mu_{X_R}(\cdot) \),

Using the above, and problem formulation, we state the key theorem and corollary below.

**Theorem 1.** For \( 0 < c \leq 1 \), provided \( C_A \) satisfies \( 5 \) or \( 6 \), then for \( x := (x_1, \ldots, x_p) \) the missing data mechanism in the augmented data has the following form. For \( x \in C_A \)
\[
\Pr(M_A = 1 \mid X_A = x, Y_A = y) = \frac{c^*\Pr(M = 1 \mid X = x, Y = y)}{c^*\Pr(M = 1 \mid X = x, Y = y) + \Pr(M = 0 \mid X = x, Y = y)},
\]
where \( c^* = c / \Pr(X \in C_A \mid M = 1) \). Otherwise zero.

The proof is in the supplement.

**Corollary 1.** If the original missing data mechanism is of the form
\[
\Pr(M = 1 \mid Y = y, X = x) = \frac{\exp(\alpha_0 + \alpha_1x_1 + \cdots + \alpha_p x_p + \alpha_{p+1}y)}{1 + \exp(\alpha_0 + \alpha_1x_1 + \cdots + \alpha_p x_p + \alpha_{p+1}y)},
\]
where \( (\alpha_0, \ldots, \alpha_{p+1})^T \in \mathbb{R}^{p+2} \), then the missing mechanism in the augmented data has the form
\[
\Pr(M_A = 1 \mid Y_A = y, X_A = x) = \frac{\exp(\alpha_0 + \log(c^*) + \alpha_1x_1 + \cdots + \alpha_p x_p + \alpha_{p+1}y)}{1 + \exp(\alpha_0 + \log(c^*) + \alpha_1x_1 + \cdots + \alpha_p x_p + \alpha_{p+1}y)}. \tag{9}
\]
While Corollary 4 assumes a MNAR mechanism it also holds under MAR as \( \alpha_{p+1} \) can equal 0. Importantly, if we consider our augmented sample within \( C_A \), then only the intercept in the mechanism’s linear predictor changes. In particular, the coefficient on \( y \), \( \alpha_{p+1} \), in the mechanism based on the augmented data is the same as its counterpart based on the full sample, permitting MNAR’s presence to be reliably inferred using the augmented sample.

4 Designing the region \( C_A \)

In this section, we consider how to choose the region \( C_A \), and thus the recovery design, to optimise the power of the SMF test [4]. Note, under regularity conditions, the MLE satisfies

\[
(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, I^{-1}(\alpha)) \quad \text{as} \quad n_A \to \infty,
\]

where \( n_A \) is the number of observations in the augmented data and \( I(\alpha) \) is the observed Fisher Information matrix of \( \alpha \). A design that leads to higher power for the SMF test [4] among MLEs is one that minimises the variance of \( \hat{\alpha} \) [10]. We avoid inverting \( I(\alpha) \) by appealing to Cramer's rule to obtain

\[
I^{-1}(\alpha)[p+1,p+1] = \frac{\det(I(\alpha)[1:p,1:p])}{\det(I(\alpha))},
\]

where \( I(\alpha)[1:p,1:p] \) is the \( p \times p \) submatrix of \( I(\alpha) \) comprising its first \( p \) rows and columns.

\( I(\alpha) \) satisfies

\[
I(\alpha) = -\mathbb{E}[H(l(\alpha))] \quad \text{where} \quad H \quad \text{is the Hessian and} \quad l \quad \text{is the log-likelihood function of the augmented data.}
\]

If we solely consider missing mechanisms of the form [6], properties of this mechanism (continuity, differentiability, and being bounded on \((0,1)\)) mean we can take the expectation within the Hessian. Exploiting iid observations and Wald's identity, one can show

\[
I(\alpha) = -\mathbb{E}[H(l(\alpha))] = -\mathbb{E}[\mathbb{E}[n_A] \cdot \mathbb{E} \log \left( \frac{\exp(M_{A\alpha} + \log(c^*) + \alpha_1 X_{1,A} + \cdots + \alpha_p X_{p,A} + \alpha_{p+1} Y_A)}{1 + \exp(M_{A\alpha} + \log(c^*) + \alpha_1 X_{1,A} + \cdots + \alpha_p X_{p,A} + \alpha_{p+1} Y_A)} \right)].
\]

To compute the expectation in [12], one can exploit the forms of [7] and [8]. After conditioning on \( M_A = 1 \) and \( M_A = 0 \), one needs to know the joint distributions of \((X_R, Y_R)\) and \((X_O, Y_O)\) respectively to compute the required expectation.

**Lemma 3.** For \( x \in C_A \):

\[
\begin{align*}
\Pr(X_R \in dx, Y_R \in dy) &= \frac{\Pr(M = 1 \mid X = x, Y = y) \Pr(Y \in dy \mid X = x) \Pr(X \in dx)}{\Pr(M_R)} \\
\Pr(X_O \in dx, Y_O \in dy) &= \frac{\Pr(M = 0 \mid X = x, Y = y) \Pr(Y \in dy \mid X = x) \Pr(X \in dx)}{\Pr(M_O)},
\end{align*}
\]

otherwise zero. Here, \( dx = (dx_1, \ldots, dx_p) \) is a vector of infinitesimals and the relation \( \Pr(Y \in dy \mid X = x) \) can be obtained from [4].

The proof of Lemma 3 is provided in the supplement. From Lemma 3 it is clear we need to have knowledge of the initial missing mechanism \( \Pr(M = 1 \mid X = x, Y = y) \), the regression relation \( \Pr(Y \in dy \mid X = x) \) and the distribution of \( X \) to compute the Fisher Information matrix. We now propose an algorithm for choosing our recovery design. For simplicity, we only consider \( C_A \) of the form of a \( p \)-dimensional cuboid.

**Algorithm 1.** For a given \( 0 < c \leq 1 \), select the \( p \)-dimensional cuboid \( C_A \) such that the r.h.s of [11] is minimised subject to the constraint \( \Pr(M_R) \geq c \cdot \Pr(M = 1) \) and \( \Pr(M_O) > 0 \). The recovery design \( R \) should consist of a random sample of \( n^* \) points within \( C_A \).

In practice, any implementation of Algorithm 1 is likely to lead to locally optimal solutions. Furthermore, in a finite regime there will likely be scenarios where the region \( C_A \) includes slightly fewer points than \( n^* \).

In this case, we recommend a uniform enlargement until \( n^* \) points lie within \( C_A \). We now investigate the performance of recovery designs found by Algorithm 1.
5 Simulation study for MSE and power

Example 1. We first consider $p = 1$ (single covariate). We generate $n = 1000$ points as follows,

$$ Y | (X = x) \sim N(2 - 2x, 4), $$

with $X \sim N(0,16)$. MNAR missingness is introduced into the model using:

$$ P(M = 1 | Y = y, X = x) = \exp(-2 + 0.4x_1 - 0.15y)/(1 + \exp(-2 + 0.4x - 0.15y)). $$

We apply test (4), with $H_0 : \alpha_2 = 0$, over 10000 replications. Figure 1 presents the mean squared error (MSE) of the MLE, $\hat{\alpha}_2$, as well as the power of test (4), for different recovery proportions, $c$, under designs constructed from Algorithm 1 (red dashed). We also present equivalent results from randomly recovered observations, i.e., $C_A = \mathbb{R}$ (solid black). We see considerable gains over random sampling, in both MSE and power, when using designs found by Algorithm 1. E.g., when $c = 0.3$ the power of test (4) is $\approx 60\%$ for our method while only $\approx 45\%$ for random sampling.

Figure 1: MSE (L) and Power (R) for different recovery proportions and designs; $p = 1$.

Example 2. We also consider $p = 2$, specifically we generate $n = 1000$ points by,

$$ Y | (X_1 = x_1, X_2 = x_2) \sim N(2 - 2x_1 + 2x_2, 4), $$

with $X_1 \sim N(0,16)$ and $X_2 \sim N(2, 4)$. MNAR missingness is introduced into the model using:

$$ P(M = 1 | Y = y, X_1 = x_1, X_2 = x_2) = \frac{\exp(-2 + 0.4x_1 + 0.2x_2 - 0.15y)}{1 + \exp(-2 + 0.4x_1 + 0.2x_2 - 0.15y)}. $$

(13)

Figure 2: MSE (L) and Power (R) for different recovery proportions and designs, now with $p = 2$.

Figure 2 presents equivalent summaries to Figure 1 and confirms our conclusions from Example 1. The green rectangles in Figures 3 and 4 illustrate regions $C_A$ found by Algorithm 1 for $c = 0.2$ and 0.9 respectively. Red/blue points depict $(x_1, x_2)$ values with observed/missing $Y$.
6 Conclusion

This research provides two key take-away messages. Firstly, our mathematically expressions for the missing data mechanism based on the subsample of observed plus recovered data permits principled inferences to be made in this setting. Notably, for the commonly used expit model, irrespective of the recovery design used, we determine that there is a shift in the intercept of the model by $\ln(c^*)$ while the other coefficients in the linear predictor, and indeed the form of the model, remain unchanged. Consequently, the SMF test (4) with $H_0: \alpha_{p+1} = 0$ based on the augmented data will reliably test for the presence of MNAR in the original data, giving analysts confidence in their results. Secondly, using experimental design methods to construct the recovery sample allows inferences to be optimised. In particular, constructing recovery designs that minimise the variance of the MLE for $\alpha_{p+1}$ ($D_1$-optimality) increases efficiency and thereby improves the power of the SMF test (4). We note the equivalence to $T$-optimality [7] here, and the potential to considerably outperform random recovery sampling merit their use.

Overall, this research provides a unified approach for detecting MNAR missingness in an incomplete data set in practice, combining a design strategy with a corresponding reliable test for MNAR. A well-chosen recovery design can achieve higher power to detect MNAR compared with random recovery sampling (for fixed recovery proportions). Similarly, if the power is fixed in advance, an efficient recovery design can result in a smaller proportion of missing responses needing to be followed up to achieve this power, reducing costs. Note that designs constructed from Algorithm 1 are locally optimal since $I(\alpha)$ depends on values of $\alpha$.

In future work, robustness of our designs, as well as strategies for constructing robust designs, should be investigated.

Acknowledgements

We gratefully acknowledge EPSRC grant EP/V00641X/1 and the Commonwealth Scholarship Commission that supported this research.

References

[1] D. Rubin. Inference and missing data. *Biometrika*, 63(3):581–592, 1976.

[2] R. Little and D. Rubin. Bayes and multiple imputation. *Statistical Analysis with Missing Data*, pages 200–220, 2002.

[3] J. Carpenter and M. Kenward. *Multiple Imputation and its Application*. John Wiley & Sons, 2012.

[4] H. Wang, R. Zhu, and P. Ma. Optimal subsampling for large sample logistic regression. *Journal of the American Statistical Association*, 113(522):829–844, 2018.
Appendix

Proof of Lemma 1.
We begin the proof by firstly assuming the condition in (5) holds. For this case, we have \( M_A = M|\{M_O \cup M_R\} \).
Recognising that events \( \{M = 0\} \) and \( \{M = 1\} \) are mutually exclusive, we can deduce by definition events \( M_O \) and \( M_R \) are also mutually exclusive. Thus,

\[
Pr(M_A = 1) = \frac{Pr(M = 1|\{M_O \cup M_R\})}{Pr(M_O \cup M_R)} = \frac{Pr(M = 1 \cap \{M_O \cup M_R\})}{Pr(M = 1 \cap \{M_O \cup M_R\}) + Pr(M = 0 \cap \{M_O \cup M_R\})} \frac{Pr(M = 1 \cap \{M_O \cup M_R\})}{Pr(M = 1 \cap \{M_O \cup M_R\}) + Pr(M = 0 \cap \{M_O \cup M_R\})} \\
= \frac{Pr(M = 1 \cap \{M_O \cup M_R\})}{Pr(M = 1 \cap \{M_O \cup M_R\}) + Pr(M = 0 \cap \{M_O \cup M_R\})} \frac{c \cdot Pr(M = 1)}{Pr(M = 1) + Pr(M_O)}.
\]

If instead of condition (5), we assume that condition (6) holds then we need to modify the claim \( M_A = M|\{M_O \cup M_R\} \).
Let \( U \) be a uniform random variable on \([0, 1]\) and define the event \( B = \{U \leq c \cdot Pr(M = 1)/Pr(M_R)\} \). Then \( M_A = M|\{M_O \cup \{M_R \cap B\}\} \). The result then follows from almost identical calculations to the above.

Proof of Lemma 3.
It follows for \( x \in C_A \):

\[
Pr(X_R \in dx, Y_R \in dy) = Pr(X \in dx, Y \in dy|M_R) = \frac{Pr(X \in dx, Y \in dy,M_R)}{Pr(M_R)} = \frac{Pr(X \in dx, Y \in dy,M = 1)}{Pr(M_R)}.
\]

The result then follows by applying rules of conditional probability in the numerator. The same approach can be used to derive the second result of the lemma for \( Pr(X_O \in dx, Y_O \in dy) \).

Proof of Theorem 1

Auxiliary lemma
The following auxiliary lemma will be used to prove Theorem 1.

Lemma 4.

\[
Pr(M_A = 1|Y_A = y, X_A = x) = Pr(M_A = 1) \times \frac{Pr(X_R \in dx, Y_R \in dy)}{Pr(X_A \in dx, Y_A \in dy)}.
\]
Proof of Theorem 1

Combining the results of Lemma 3 with Lemma 4 provides

\[ \Pr(M_A = 1|X_A = x, Y_A = y) = \frac{\Pr(M_A = 1)\Pr(M = 1|X = x, Y = y)\Pr(X \in dx, Y \in dy)}{A + B}, \]

where

\[ A = \Pr(M_A = 1)\Pr(M = 1|X = x, Y = y)\Pr(X \in dx, Y \in dy) \]
\[ B = \frac{\Pr(M_R)}{\Pr(M_O)}\Pr(M = 0)\Pr(M = 0|X = x, Y = y)\Pr(X \in dx, Y \in dy). \]

By cancelling the common term \( \Pr(X \in dx, Y \in dy) \) present in the numerator and denominator, and performing further rearrangement, we obtain:

\[ \Pr(M_A = 1|X_A = x, Y_A = y) = \frac{\Pr(M = 1|X \in dx, Y = y)\Pr(M = 0, X \in C_A)}{C + D}, \]

where

\[ C = \Pr(M = 1|X = x, Y = y)\Pr(M_O) \]
\[ D = \frac{\Pr(M = 0)}{\Pr(M_A = 1)} \cdot \Pr(M_R) \cdot \Pr(M = 0|X = x, Y = y). \]

From Lemma 1, one notices the following relation:

\[ \frac{\Pr(M_A = 0)}{\Pr(M_A = 1)} = \frac{\Pr(M_O)}{c \cdot \Pr(M = 1)}. \]

By then cancelling the common term \( \Pr(M_O) \), and dividing the numerator and denominator by \( \Pr(M_R) \), we obtain:

\[ \Pr(M_A = 1|X_A = x, Y_A = y) = \frac{c^*\Pr(M = 1|X = x, Y = y)}{c^*\Pr(M = 1|X = x, Y = y) + \Pr(M = 0|X = x, Y = y)} \]

where,

\[ c^* = \frac{c \cdot \Pr(M = 1)}{\Pr(M_R)}. \]

By then substituting the original expit model (3) for \( \Pr(M = 1|X = x, Y = y) \) and \( \Pr(M = 0|X = x, Y = y) \), we arrive at the result of Corollary 1 by expressing \( c^* = \exp(\log(c^*)) \).