The 6-strand braid group is $\text{CAT}(0)$

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Abstract. We show that braid groups with at most 6 strands are $\text{CAT}(0)$ using the close connection between these groups, the associated non-crossing partition complexes and the embeddability of their diagonal links into spherical buildings of type A. Furthermore, we prove that the orthoscheme complex of any bounded graded modular complemented lattice is $\text{CAT}(0)$, giving a partial answer to a conjecture of Brady and McCammond.

1 Introduction

Our paper addresses the open question whether braid groups are $\text{CAT}(0)$ and gives a positive answer for the ones with at most 6 strands.

In [8] Charney stated the more general question about the curvature of Artin groups, and suggested that all of them should be $\text{CAT}(0)$. Several partial answers to this question are known. For example Brady and McCammond showed in [5] that the $n$-strand braid groups are $\text{CAT}(0)$ if $n = 4$ or 5. However, their proof for $n = 5$ relies heavily on a computer program. They also conjectured that the same statement should hold for arbitrary $n$ [5, Conjecture 8.4]. The same authors studied new presentations for certain three-generator Artin groups [6] and showed that the associated presentation 2-complex admits a metric of non-positive curvature.

Charney and Davis [10] introduced the Salvetti complex, a piecewise Euclidean cube complex, associated to an Artin group. They show that its universal cover, on which the Artin group acts geometrically, is $\text{CAT}(0)$ if and only if the Artin group is right-angled (each $m_{ij} = 2$ or $\infty$).

Brady [4] studied a class of Artin groups with three generators and constructed certain complexes using the associated Coxeter groups. He showed that these complexes carry a piecewise Euclidean metric of non-positive curvature and have as fundamental group the Artin groups under consideration. A generalisation of this was proved by Bell [1].

Explicit examples of Artin groups with two-dimensional Eilenberg-McLane spaces which act geometrically on 3-dimensional $\text{CAT}(0)$ complexes (but not so on 2-dimensional ones) were constructed by Brady and Crisp [3].

This paper exploits the close relationship between braid groups, non-crossing partitions of a regular $n$-gon and the geometry of spherical buildings. Associated to the lattice non-crossing partitions, there is a metric polyhedral complex, its...
orthoscheme complex, whose geometry was studied in [5]. (See Section 2 for further details.) Brady and McCammond showed that the CAT(0) property for braid groups can be deduced from the fact that the orthoscheme complex of the non-crossing partition lattice $NCP_n$ is a CAT(0) space. This is done by inspecting the diagonal link of the orthoscheme complex of $NCP_n$ and proving that this diagonal link is CAT(1).

The diagonal link of the orthoscheme complex of the lattice $NCP_n$ can be embedded into a spherical building of type $A_{n-2}$. Our approach is based on investigating the relationship between the geometry of the diagonal link of $NCP_n$ and the ambient building. Following the criterion of Gromov (see [12]), made precise by Bowditch (see [2]), Charney and Davis (see [9]), we will show inductively that the diagonal link is locally CAT(1) and that it does not contain any locally geodesic loop of length smaller than $2\pi$. We follow this strategy in the proof of the following theorem, which is entirely geometric and does not rely on a computer program.

**Theorem 3.37.** For every $n \leq 6$ the diagonal link in the orthoscheme complex of non-crossing partitions $NCP_n$ is CAT(1).

As a consequence we obtain

**Corollary 3.38.** For every $n \leq 6$, the $n$-strand braid group is CAT(0).

We are thus giving a new proof of the theorem in case $n = 4$ or 5, and provide more evidence (with the newly covered case $n = 6$) towards [5, Conjecture 8.4]. Note that our proof at no point relies on computer-assisted calculations.

Brady and McCammond conjectured further that the orthoscheme complex of any bounded graded modular lattice is CAT(0) [5, Conjecture 6.10]. We are able to give a partial result towards the solution of this problem.

**Theorem 3.42.** The orthoscheme complex of any bounded graded modular complemented lattice is CAT(0).

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**Outline of the proof**

The main result contained in this paper is Theorem 3.37. To prove it we first embed the diagonal link of the orthoscheme complex of non-crossing partitions $NCP_n$ into a spherical building. Then we assume (for a contradiction) that the diagonal link contains an unshrinkable (and hence locally geodesic) short loop. The image of such a loop $l$ contains a positive finite number of points of special interest (called turning points) which characterise the positions at which the loop fails to be locally geodesic in the ambient space.

By inspection we show that there is a path $p$ between any turning point of $l$ and the point opposite to it in $l$, such that the two new loops we obtained by following half of $l$ and then $p$ are short and shrinkable. Then a result of Bowditch [2] concludes the argument.
2 Definitions

Definition 2.1 (Intervals of integers). We will use \([n, m]\) to denote the interval in \(\mathbb{Z}\) between \(n\) and \(m\) (with \(n \leq m\)), that is
\[
[n, m] = [n, m] \cap \mathbb{Z}.
\]

Definition 2.2 (Posets). A poset \(P\) is a set with a partial order. \(P\) is bounded if it has a minimum element 0 and a maximum element 1.

Definition 2.3 (Subposets). Let \(P \subseteq Q\) be two posets. We say that \(P\) is a subposet of \(Q\) if and only if the order on \(P\) is induced by the order on \(Q\).

Definition 2.4 (Rank). A bounded poset \(P\) has rank \(n - 1\) if and only if every chain is contained in a maximal chain with \(n\) elements. For \(x \leq y\) in \(P\), the interval between \(x\) and \(y\) is the subposet \(P_{xy} = \{z \in P : x \leq z \leq y\}\). If every interval in \(P\) has a rank, \(P\) is graded. Let \(x\) be an element in a bounded graded poset \(P\), then the rank of \(x\) is the rank of the interval \(P_{0x}\).

Definition 2.5 (Joins, meets, lattices). A poset \(P\) is called a lattice if and only if for every \(x, y \in P\):

- there exists a unique minimal element \(x \lor y\) of the set \[\{z \in P \mid x \leq z \text{ and } y \leq z\}\], called the join of \(x\) and \(y\), and
- there exists a unique maximal element \(x \land y\) of the set \[\{z \in P \mid x \geq z \text{ and } y \geq z\}\], called the meet of \(x\) and \(y\).

Definition 2.6 (Geometric realisation). Let \(P\) be a bounded lattice. The simplicial realisation \(|P|\) of \(P\) is the simplicial complex whose vertex set is \(P\), and whose \(k\)-simplices correspond to chains \(x_0 < x_1 < \ldots < x_k\) of length \(k\).

A metric space \(X\) is a geometric realisation of \(P\) if and only if it is homeomorphic to \(|P|\).

Note that we will often abuse notation by using \(P\) to denote both a poset, and some fixed geometric realisation of it.

Definition 2.7 (Diagonal link). Given a bounded lattice \(P\), we define the diagonal link of \(P\) to be the link
\[LK(e_{01}, P)\]
in the geometric realisation of \(P\) of the diagonal edge \(e_{01}\) connecting the minimum 0 to the maximum 1.

Definition 2.8 (Linear lattice). If \(V\) is an \((n-1)\)-dimensional vector space over a division algebra, we will denote by \(S(V)\) the rank \((n-1)\) lattice consisting of all vector subspaces of \(V\), with the order given by inclusion. We call \(S(V)\) the linear lattice of \(V\).
Lemma 2.11 \[(Duality)\]
order-reversing bijection \[x \mapsto \frac{1}{n} \pi_i \omega_i \] \[\in \mathbb{Z}/n\mathbb{Z}\]
deNote that the diagonal link of \(S(V)\) is the spherical building of \(PGL(V)\).

Definition 2.9 \[(Partition lattice)\]
Let \(U_n\) be the set of \(n\)th roots of unity inside the plane \(\mathbb{C}\). Let \(P_n\) denote the set of partitions of the set \(U_n\). An element of a partition is called a block. Note that \(P_n\) is a bounded graded lattice of rank \(n - 1\), where the order is given by: \(p \leq p'\) if and only if every block of \(p\) is contained in a block of \(p'\).

Definition 2.10 \[(Non-crossing partition lattice)\]
We define \(NCP_n\) to be the subposet of \(P_n\) consisting of non-crossing partitions of \(U_n\), i.e. partitions such that for every distinct blocks \(x, y\) of the partition, the convex hulls \(Hull(x)\) and \(Hull(y)\) in \(\mathbb{C}\) do not intersect. The poset \(NCP_n\) is a bounded graded lattice of rank \(n - 1\).

Lemma 2.11 \[(Duality)\]
For \(n \geq 2\), there is a duality on \(NCP_n\), i.e. an order-reversing bijection \(x \mapsto x^*\) from \(NCP_n\) to itself.

Proof. Denote by \(\{\omega_i\}_{i \in \mathbb{Z}/n\mathbb{Z}} = U_n\) the \(n\)th roots of unity. If \(x\) is a non-crossing partition of \(U_n\), then its dual \(x^*\) is the partition of the shifted set \(U_n^* = \{m_i = e^{\frac{2\pi}{n}\omega_i}\}_{i \in \mathbb{Z}/n\mathbb{Z}}\) defined by: \(m_i\) and \(m_j\) belong to the same block of \(x^*\) if and only if the geodesic segment \([m_i, m_j]\) in \(\mathbb{C}\) does not intersect the convex hull of any block of \(x\). Then \(x^*\) is a non-crossing partition of \(U_n^*\), with \(\text{rk}(x^*) = n - 1 - \text{rk}(x)\), and \((x^*)^T = x\). Now choose some identification between \(U_n^*\) and \(U_n\) (like multiplying by \(e^{\frac{2\pi}{n}\omega_i}\)) to get a map from \(NCP_n\) to itself. \(\square\)

Remark 2.12. Note that we will only use duality to reduce the number of cases that will need checking in the later stage of our proof. It is not used in an essential way.

Lemma 2.13 \[(NCPs are linearly embedded)\]
For every \(n \geq 2\), \(P_n\) and \(NCP_n\) are isomorphic to subposets of \(S(V)\), where \(V\) is an \((n - 1)\)-dimensional vector space.

Proof. Fix a field \(\mathbb{F}\), and let \(V = \{(y_i) \in \mathbb{F}^n \mid \sum_{i=1}^n y_i = 0\}\). Then \(V\) is an \((n - 1)\)-dimensional \(\mathbb{F}\)-vector space. Identify \(U_n\) with \([1, n]\). If \(x \in P_n\) let \(f(x) = \{(y_i) \in V \mid \forall \text{ block } Q \in x: \sum_{i \in Q} y_i = 0\}\). Then \(f\) is an injective rank-preserving poset map from \(P_n\) to \(S(V)\). It clearly restricts to \(NCP_n \subseteq P_n\). \(\square\)

3 Proof of the main theorem

3.1 Turning faces

Definition 3.1 \[(Turning points)\]
Let \(X\) be a path-connected subspace of a geodesic metric space \(B\), and endow \(X\) with the induced length metric. Suppose that \(l: D \to X\) is a local isometry, where \(D\) is a metric space. We say that a point \(t \in D\) is a turning point of \(l\) in \(B\) if and only if \(l \circ l\) fails to be a local isometry at \(t\), where \(l: X \to B\) is the inclusion map.
Definition 3.2 (Locally geodesic loops). Let $X$ be a metric space. We say that $l: S^1 \to X$ is a \textit{locally geodesic loop} in $X$ if and only if $l$ is a local isometry, where $S^1$ is given the length metric of the quotient of some closed interval $I$ of $\mathbb{R}$ by its endpoints. The \textit{length} of $l$ is defined to be the length of $I$.

We say that $l: I \to X$ is a \textit{locally geodesic path} in $X$ if and only if $l$ is a local isometry, where $I$ is a closed interval of $\mathbb{R}$. The \textit{length} of $l$ is defined to be the length of $I$.

Lemma 3.3. Let $X$ be a path-connected subset of a geodesic metric space $B$, and endow $X$ with the induced length metric. Suppose that $l: D \to X$ is a locally geodesic path or loop, with $D$ being respectively $I$ or $S^1$. Let $t \in D$. Suppose that there exists a subset $N \subseteq X$, such that $N$ contains the convex hull in $B$ of the image under $l$ of some neighbourhood of $t$ in $D$. Then $t$ is \textit{not} a turning point.

Proof. Suppose (for a contradiction) that $t$ is a turning point. We claim that then $l$ fails to be a local geodesic at $t$.

To prove the claim let us fix $\varepsilon > 0$. As $i \circ l$ fails to be a local geodesic at $t$, there exist $t_1, t_2 \in D$ with

$$d_B(l(t_1), l(t_2)) < d(t_2, t_1) < \varepsilon$$

and such that $l(t_j) \in N$ for $j \in \{1, 2\}$.

Since $B$ is a geodesic metric space, we can realise the distance between $l(t_1)$ and $l(t_2)$ with a geodesic segment $g$ in $B$. Since $N$ contains the endpoints of $g$, it contains the whole of $g$. Hence in particular $g$ lies in $X$, which (as was claimed) contradicts the fact that $l$ was a local geodesic.

Remark 3.4. We will often identify (isometrically) a neighbourhood of a point $t \in S^1$ with an interval in $\mathbb{R}$ containing $t$ in its interior. We will therefore feel free to write $[t - \varepsilon, t + \varepsilon]$ etc. (for a small $\varepsilon$) to denote a subset of $S^1$.

Definition 3.5 (Consecutive turning points). Suppose that we have a subset $T \subseteq S^1$. We will say that $t, t' \in T$ are \textit{consecutive} if and only if there is a path in $S^1$ with endpoints $t$ and $t'$ not containing any other point in $T$. A shortest such path will be denoted by $[t, t']$.

Remark 3.6. Note that $[t, t']$ defined above is unique provided that the cardinality of $T$ is at least 3.

Lemma 3.7. Let $X$ be a path-connected subset of a CAT(1) space $B$, and endow $X$ with the induced length metric. If $l: S^1 \to X$ is a locally geodesic loop in $X$ of length $L < 2\pi$, then the cardinality of the set of turning points $T$ of $l$ is greater than 2. Moreover, $l|[t, t']$ is a geodesic in $B$ for any pair of consecutive turning points $t, t'$.

Proof. Suppose that we can find three distinct points $t_1, t_2, t_3 \in S^1$ such that each pairwise distance is strictly bounded above by $\frac{1}{2}L < \pi$, and such that $T$ is contained in $[t_1, t_2] \cup \{t_3\}$, where $[t_i, t_j]$ denotes the shortest segment of $S^1$ with endpoints $t_i$ and $t_j$ not containing $t_k$ for

$$\{i, j, k\} = [1, 3].$$
We define the set \( r_k(F) \) and \( \text{corank}(F) \) of \( F \) has dimension \( n \). The metric \((\text{Rank and corank})\) Definition 3.9. \( \pi \) than \( \pi \) of length at most \(|T|\). Thus shown that \( \pi \) cannot be a local geodesic in \( B \). But then \( \pi \) cannot be a local geodesic in \( S^1\). Therefore \( \Delta' \) cannot be a great circle in \( S^2\). Suppose that \( t_3 \notin T \). Then the angle of \( \Delta \) at \( l(t_3) \) is equal to \( \pi \), and the same is true in the comparison triangle \( \Delta' \) (by the CAT(1) inequality). But then the triangle is degenerate, and hence so is \( \Delta \). In particular the geodesic from \( l(t_1) \) to \( l(t_2) \) goes via \( l(t_3) \). But this contradicts the assumption that the distance (in \( B \)) between \( l(t_1) \) and \( l(t_2) \) is smaller than \( \frac{1}{2}L \). So \( t_3 \in T \). We will now use this trick to prove our claims.

If \(|T| \leq 1\) then we immediately get a contradiction by taking either any three points in \( S^1\) satisfying the conditions above, or the turning point and two other points so that the triple satisfies the condition.

If \(|T| = 2\) and the two points are not antipodal in \( S^1\), then we can always (very easily indeed) find a third point so that the triple satisfies our condition. If the turning points are antipodal, then the two local geodesics given by \( l \), which connect the images of the turning points, coincide. This is because local geodesics of length smaller than \( \pi \) are geodesics in \( B \), and such geodesics in CAT(1) spaces are unique. But then \( l \) cannot be a local geodesic in \( X \). We have thus shown that \(|T| \geq 3\).

Now suppose we have two consecutive turning points, \( t_1 \) and \( t_2 \). If \([t_1, t_2]\) is of length at most \( \pi \), then \([l(t_1, t_2)]\) is a geodesic as before. If the length is larger than \( \pi \), then in particular it is larger than \( \frac{1}{2}L \), and so we can take the midpoint \( t_3 \in [t_1, t_2]\) and (applying the argument above) conclude that \( t_3 \in T \), which in turn contradicts the definition of \([t_1, t_2]\).

The following definition combines all the assumptions that will become standard in the rest of the paper.

**Definition 3.8.** We say that \( X \) is linearly embedded in \( B \) if and only if there exists an integer \( n \geq 3 \), a vector space \( V \) of dimension \((n - 1)\) over a division algebra, and a bounded graded lattice \( P \) of rank \((n - 1)\), such that

1. \( B \) is a geometric realisation of the diagonal link \( LK(e_{01}, S(V)) \) isometric to the spherical building of \( PGL(V) \) (equivalently, it is given the spherical orthoscheme metric), and

2. \( P \subseteq S(V) \) is a subposet, and

3. \( X \) is a geometric realisation of the diagonal link \( LK(e_{01}, P) \subseteq B \), and

4. \( X \) is given the length metric induced from \( B \).

The metric \( X \) is given is called the spherical orthoscheme metric. Note that \( X \) has dimension \( n - 3 \).

**Definition 3.9** (Rank and corank). Let \( F \) be a face of codimension \( m \) in \( X \). Then \( F \) is the span of vertices \( x_1, \ldots, x_{n-1-m} \) of ranks \( r_1, \ldots, r_{n-1-m} \). We define the set \( \text{rk}(F) = \{r_1, \ldots, r_{n-1-m}\} \) to be the rank of \( F \), and the set \( \text{crk}(F) = \llbracket 1, n-1 \rrbracket \setminus \text{rk}(F) \) to be the corank of \( F \).
Definition 3.10 (Turning face). If $t$ is a turning point of a locally geodesic loop $l$ in $B$, the smallest (with respect to inclusion) intersection $F$ of a chamber in $X$ containing $l([t, t - \varepsilon])$ and a chamber containing $l([t, t + \varepsilon])$ will be called the turning face of $t$.

Lemma 3.11. Let $X$ be linearly embedded in $B$, and let $l$ be a locally geodesic loop or path in $X$. Then the set $T$ of turning points of $l$ is finite.

Proof. Let $l : D \to X$, where $D = S^1$ or $D = I = [0, L]$ and $l(0) \neq l(L)$. If $D = I$, notice that for $\varepsilon > 0$ small enough $l([0, \varepsilon])$ is contained in a chamber in $X$, hence by Lemma 3.13 the point $0 \in I$ is not a turning point of $l$, and similarly $L$ isn’t either.

We claim that $T$ is a discrete subset of $D$. Suppose that $t \in T$ is a turning point. Let $F$ be the unique open face containing $l(t)$. Since $l$ is a local geodesic in $X$, there exists $\varepsilon > 0$ such that $l([t - \varepsilon, t + \varepsilon])$ is a geodesic in $X$ and is included in the star of $F$ in $X$, which is a neighbourhood of $l(t)$ in $X$.

Then $l(t - \varepsilon)$ belongs to a chamber $C$ in $X$ containing $F$, hence $l([t - \varepsilon, t])$ is the geodesic segment from $l(t - \varepsilon)$ to $l(t)$ in $C \subset X$. In particular it is also locally geodesic in $B$, so there is not turning point in $(t - \varepsilon, t)$, and similarly in $(t, t + \varepsilon)$. So $T$ is discrete.

Note that $T$ is closed – this follows directly from the fact that if $t \in D$ is not a turning point, then $l$ is a geodesic (in $B$) at some open neighbourhood of $t$ in $D$, and so in particular none of the points in this open neighbourhood are turning points themselves. Hence $T$ is closed, and therefore compact since $D$ is compact. We have thus shown that $T$ is compact and discrete, and so it is finite.

Lemma 3.12. Let $X$ be linearly embedded in $B$, and let $l$ be a locally geodesic loop in $X$. Then for every turning point of $l$ in $B$, its turning face has a corank which contains two consecutive integers.

Proof. Suppose that we have $t \in S^1$, a turning point of $l$, whose image $x$ under $l$ is contained in the turning face $F$ in $X$. As $l$ is a local geodesic and $X$ is a union of chambers, there exists $\varepsilon > 0$ such that $l([t - \varepsilon, t + \varepsilon]) \subseteq C^-$ and $l([t, t + \varepsilon]) \subseteq C^+$, where $C^-$ and $C^+$ are chambers of $X$. Choose $C^\pm$ such that $F = C^+ \cap C^-$. Assume that the corank of $F$ does not contain two consecutive integers. Then the sets of vertices of $C^-$ and $C^+$ differ at vertices of ranks

$$1 \leq r_1 < \ldots < r_k \leq n - 2,$$

with $\forall i \in [1, k - 1] : r_{i+1} - r_i \geq 2$. Then for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{\pm\}^k$, consider the chamber $C^\varepsilon$ spanned by $C^+ \cap C^-$ and, for every $1 \leq i \leq k$, by the vertex of rank $r_i$ in $C^\varepsilon$. Since all vertices of $C^\varepsilon$ belong to $X$, we know that $C^\varepsilon$ belongs to $X$.

Observe that the link of $F = C^+ \cap C^-$ in $B$ is a spherical building of type $A_k$, hence its apartments are $k - 1$-spheres. Let $N = \bigcup_{\varepsilon \in \{\pm\}^k} C^\varepsilon$. Remark that the image of $N$ in the link of $F$ in $B$ is precisely one of the apartments, and therefore it is convex. Since the link $\text{lk}(x, N)$ of $x$ in $N$ is isometric to the spherical join $\text{lk}(x, F) \ast \text{lk}(F, N)$, it is convex in the link $\text{lk}(x, B) \simeq \text{lk}(x, F) \ast \text{lk}(F, B)$ of $x$ in $B$. For $\varepsilon > 0$ small, the $\varepsilon$-ball around $x$ is isometric to the $\varepsilon$-ball around the cone point in the cone over the link of $x$, according to [7, Theorem 7.16]. Since $\text{lk}(x, N)$ is convex in $\text{lk}(x, B)$, we conclude that the $\varepsilon$-ball around $x$ in $N$
is convex in $B$. Since $N$ contains the image under $l$ of some neighbourhood of $t$ in $S^1$, using Lemma 3.5 we show that $t$ is not a turning point. □

**Definition 3.13 (Failing modularity).** Let $P$ be a subposet of a linear poset $S(V)$. We say that two points $x, y \in P$ fail modularity if and only if

$$x \lor y \notin P \text{ or } x \land y \notin P,$$

where by $\lor$ and $\land$ we mean the join and meet in $S(V)$ (i.e. unions and intersections of vector subspaces).

We say that a point $x$ fails modularity around a face $F$ if and only if $x$ is adjacent to $F$ and there exists $y$ adjacent to $F$ such that $x, y$ fail modularity.

**Lemma 3.14.** Let $B$ be a geometric realisation of the diagonal link

$$LK(e_{01}, S(V))$$

is isometric to the spherical building of $PGL(V)$, where $V$ is a $(n-1)$-dimensional vector space over a division algebra. Let $F, F'$ be two faces in $B$. Consider the minimal set $M$ of vertices of $B$ containing the vertices of $F$ and $F'$ which is stable under joins and meets (i.e. intersections and unions of vector subspaces). Then the full subcomplex spanned by $M$ is the simplicial convex hull of $F \cup F'$.

**Proof.** Let $H$ denote the simplicial convex hull of $F \cup F'$ which is by definition the intersection of all apartments in $B$ containing $F \cup F'$.

Let $A$ be an apartment in $B$ containing $F \cup F'$, and let $\{e_1, \ldots, e_{n-1}\}$ be a basis of $V$ corresponding to $A$. Then every element of $M$ is spanned by some elements of $\{e_1, \ldots, e_{n-1}\}$, so it belongs to the apartment $A$. Since this is true for every apartment containing $F \cup F'$ it holds for their intersections and hence we have proved that $M \subseteq H$.

Conversely, fix an apartment $A$ containing $M$, and assume there exists

$$v \in A \setminus M.$$

We will show that there exists another apartment containing $M$ but not $v$, thus showing that $v \notin H$. Let $\{e_1, \ldots, e_{n-1}\}$ be a basis of $V$ corresponding to $A$. Since $M$ is stable under joins and $v \notin M$, there exists $i \in [1, n-1]$ such that $e_i \notin v$, and

$$\forall m \in M, e_i \preceq m \implies m \notin v.$$

Consider $m_0 = \bigwedge\{m \in M \mid e_i \preceq m\} \in M$, where $m_0 = V$ if there is no $m \in M$ such that $e_i \leq m$. Since $e_i \leq m_0$, we know that $m_0 \notin v$, so there exists $j \in [1, n-1] \setminus \{i\}$ such that $e_j \leq m_0$ and $e_j \notin v$. Then the apartment corresponding to the basis $\{e_1, \ldots, e_{i-1}, e_i + e_j, e_{i+1}, \ldots, e_{n-1}\}$ contains $M$ but not $v$. So $v \notin H$, and we have proved that $H \subseteq M$. □

**Lemma 3.15.** Let $X$ be linearly embedded in $B$, and let $l: I \to X$ be a locally geodesic segment in $X$ with a turning point $t$ in $B$. Let $E^+$ (respectively $E^-$) be minimal faces in $X$ containing the image under $l$ of a right (respectively left) $\varepsilon$-neighbourhood of $t$ for some $\varepsilon > 0$. Then there exist vertices $x^+ \in E^+$ and $x^- \in E^-$ which fail modularity.
Proof. Suppose (for a contradiction) that no two vertices in $E^+ \cup E^-$ fail modularity. Consider $N$, the full subcomplex of $B$ spanned by the closure under taking joins and meets in $(S(V))$ of the set of all vertices of $E^- \cup E^+$. By our assumption $N \subseteq X$. Also, by Lemma 3.14, $N$ is simplicially convex, hence is metrically convex in $B$, and contains the image under $l$ of some neighbourhood of $t$ in $S^1$. Therefore using Lemma 3.3, we show that $t$ is not a turning point, and this concludes the proof.

3.2 Criteria for CAT(1)

In this section we recall some of Bowditch’s results about locally CAT(1) spaces (see [2]), and how Brady and McCammond use them to give a sufficient condition for braid groups to be CAT(0) (see [5]).

Definition 3.16 (Locally CAT(1)). A complete, locally compact, path-metric space $X$ is said to be locally CAT(1) if each point of $X$ has a CAT(1) neighbourhood.

Definition 3.17 (Shrinking and shrinkable loops). Let $X$ be a complete, locally compact path-metric space. A rectifiable loop $l$ in $X$ is said to be shrinkable to $l'$ if and only if $l'$ is another rectifiable loop in $X$, and there exists a homotopy between $l$ and $l'$ going through rectifiable loops of non-increasing lengths.

A rectifiable loop $l$ is shrinkable if and only if it is shrinkable to a constant loop.

The loop $l$ is said to be short if its length is smaller than $2\pi$.

Theorem 3.18 (Bowditch [2, Theorem 3.1.2]). Let $X$ be a locally CAT(1) space. Then $X$ is CAT(1) if and only if every short loop is shrinkable.

The following theorem will be an important tool in our argument.

Theorem 3.19 (Bowditch [2, Theorem 3.1.1]). Let $X$ be a locally CAT(1) space. Let $x, y \in X$, and consider three paths $\alpha_1, \alpha_2, \alpha_3 : [0,1] \rightarrow X$ joining $x$ to $y$. For all $i \in \{1,2,3\}$, consider the loop $\gamma_i = \alpha_{i+1} \circ \alpha_i$ based at $x$ (with indices modulo 3). Assume that for all $i \in \{1,2,3\}$ the loop $\gamma_i$ is short. Assume further that $\gamma_1$ and $\gamma_2$ are shrinkable. Then $\gamma_3$ is shrinkable.

Theorem 3.20 (Brady, McCammond [5, Theorem 5.10 and Lemma 5.8]). Assume that for all $3 \leq k \leq n$, the diagonal link in $NCP_k$ does not contain any unshrinkable short loop. Then $NCP_n$ is CAT(0).

Proposition 3.21 (Brady, McCammond [5, Proposition 8.3]). If $NCP_m$ is CAT(0) for all $m \leq n$, then the $n$-strand braid group is CAT(0).

3.3 Non-crossing partitions

In this section we will restrict our attention to $X$ linearly embedded in $B$, where $X$ is the diagonal link in $NCP_n$ for some $n \geq 3$, $B$ is the diagonal link of $|S(V)|$ and $V$ is a $(n-1)$-dimensional vector space (for instance, given by Lemma 2.13). Note that $B$ is a spherical building, hence it is a CAT(1) metric space, which is why (as Brady and McCammond remarked, see [5, Remark 8.5]) the spherical orthoscheme metric on $X$ is natural, and it is a good candidate to be CAT(1) for all $n \geq 3$. 

Remark that the geometric realisation of $NCP_n$ has dimension $n - 1$. Since the $n$-strand braid group has a free abelian subgroup of rank $n - 1$, the complex constructed by Brady and McCammond on which the $n$-strand braid group acts properly and cocompactly has the minimum dimension to be such a CAT(0) space.

**Remark 3.22.** Note that for $n = 3$, the diagonal link $X$ in $NCP_3$ is the disjoint union of 3 points, so it is CAT(1). And for $n = 4$, the diagonal link $X$ in $NCP_4$ is a subgraph of the incidence graph of the Fano plane, so it has combinatorial girth at least 6. Since each edge has length $\frac{\pi}{3}$, its girth is at least $6\frac{\pi}{3} = 2\pi$, so $X$ is CAT(1). A picture of the diagonal link of $NCP_4$ can be found in Figure 3.1.

![Figure 3.1: The diagonal link of $NCP_4$ is shown with solid lines. Dotted lines represent the missing two vertices of the Fano plane, one of which is a vertex of the diagonal link of the crossing partition complex.](image)

**Definition 3.23** (Non-crossing trees). A non-crossing forest of $U_n$ is a metric forest embedded in $\mathbb{C}$ with vertex set $U_n$, whose edges are geodesic segments in $\mathbb{C}$. when such a forest has only one connected component, we call it a non-crossing tree.

**Remark 3.24.** Note that every non-crossing forest corresponds to a vertex in $X$. The correspondence is obtained by saying that two points in $U_n$ lie in the same block if and only if they lie in a single connected component of the forest. In particular this gives a one-to-one correspondence between vertices of rank 1 in $X$ and non-crossing forests with only one edge.

This way we also can associate a subset of $X$ to a non-crossing tree by taking the span of all vertices associated to proper subforests of our non-crossing tree.

**Proposition 3.25.** Let $A$ be an apartment in $B$. Then $A$ is included in $X$ if and only if its $(n - 1)$ rank 1 vertices lie in $X$ and correspond to the edges of a non-crossing tree.

**Proof.** Suppose $A$ is an apartment lying in $X$. Each rank 1 vertex $v_i$ of $A$ corresponds to a basis vector $e_i$ of $V$, the vector space that is used to define $B$ as the diagonal link in $S(V)$. Each such vertex also corresponds to an edge $e_i$. 
as explained in the remark above. We claim that \( T \), the union of edges \( e_i \), is an embedded tree.

Let \( v_i \) and \( v_j \) be two distinct vertices of \( A \) of rank 1. Then their join in \( B \) has rank 2 (it is the plane \( \langle e_i, e_j \rangle \)), and lies in \( A \). But \( A \subseteq X \), and so the partition \( v_i \lor v_j \) has rank 2. Observe that if \( e_i \) intersects \( e_j \) away from \( U_n \), then the join \( v_i \lor v_j \) has to contain the convex hull of \( e_i \cup e_j \) as a block (since it is non-crossing), and therefore its rank is at least 3. Hence \( e_i \) can intersect \( e_j \) only at \( U_n \), and therefore \( T \) is embedded.

Now suppose that \( T \) contains a cycle. Without loss of generality let us suppose that the shortest cycle is given by the concatenation of edges \( e_1, \ldots, e_k \) for some \( k \). Then note that the joins in \( X \) satisfy

\[
\bigvee_{i=1}^{k} v_i = \bigvee_{i=1}^{k-1} v_i.
\]

But, as before, they are equal to the joins in \( B \) (since \( A \subseteq X \)). This yields the equality

\[
\langle e_1, \ldots, e_k \rangle = \langle e_1, \ldots, e_{k-1} \rangle,
\]

which contradicts the fact that the vectors \( e_i \) are linearly independent.

We have thus shown that \( T \) is an embedded forest. But \( T \) consists of \( n-1 \) edges, and so an Euler characteristic count yields that it has exactly one connected component. Therefore \( T \) is a tree as required.

Now suppose the vertices of rank 1 of an apartment \( A \) lie in \( X \) and form a non-crossing tree \( T \). The apartment is the span of the closure of the set of its rank 1 vertices under taking joins in \( B \). The fact that \( T \) is non-crossing tells us that the joins of these vertices taken in \( B \) or \( X \) coincide, and hence all vertices of \( A \) lie in \( X \). Therefore \( A \subseteq X \).

**Definition 3.26** (Universal points). A point \( x \in X \) is said to be *universal* if it belongs to a face in \( X \), all of whose vertices are partitions with exactly one block containing more than one vertex, and such that this block only contains consecutive elements of \( U_n \). Such a face is also called *universal*.

Note that every universal face is contained in a universal chamber.

**Example 3.27.** Consider for \( n = 6 \) the edge between the two partitions \( \{(1, 2, 3), \{4\}, \{5\}, \{6\}\} < \{(1, 2, 3, 4, 5), \{6\}\} \). Then any point on this edge is universal.

![Figure 3.2: All points on this edge in X are universal.](image)

**Lemma 3.28.** Let \( x \in X \) be a universal point, and let \( y \in X \) be non opposite to \( x \) in \( B \). Then the geodesic in \( B \) between \( x \) and \( y \) lies in \( X \). In other words, \( X \) is \( \pi \)-star-shaped at \( x \).
Proof. Let $C$ be a universal chamber with $x \in C$, and let $C'$ be a chamber containing $y$. We will construct an apartment $A \subseteq X$ containing both.

Let $\{x_1, \ldots, x_{n-1}\}$ denote the vertices of $C$, with indices corresponding to ranks. We are going to construct a total order on $U_n$. Note that, seen as partitions, $x_{i+1}$ is obtained from $x_i$ by expanding the unique block with multiple elements (which we will refer to as the big block of $x_i$) by a vertex adjacent to the block. We will call this vertex the new vertex of $x_{i+1}$. We take our order to be one in which a vertex $v \in U_n$ is larger than $u$ whenever there exists $i$ such that $v$ is new for $x_{i+1}$, and $u$ belongs to the big block of $x_i$ (we allow $i = n$ and set $x_n = 1$, the partition with one element). Note that there are precisely two such total orders, depending on how we order vertices in the big block of $x_1$. Note also that (crucially) given any vertex $v \in U_n$, we get a non-crossing partition $o_v$ with blocks
\[
\{w \in U_n | w \leq v\} \text{ and } \{w \in U_n | w > v\}.
\]

Now let $\{y_1, \ldots, y_{n-1}\}$ denote the vertices of $C'$, with indices corresponding to ranks. Note that, seen as partitions, $y_{i+1}$ is obtained from $y_i$ by combining two blocks into one. We are now going to inductively construct embedded forests with vertex set $U_n$, and edges given by geodesic segments.

We set $T_1$ to be the forest with vertex set $U_n$, and a single edge connecting the two vertices of the unique non-trivial block of $y_1$. Suppose we have already defined $T_i$. Then $T_{i+1}$ is obtained from $T_i$ by adding an edge connecting vertices $v$ and $w$ such that
\begin{itemize}
  \item $v$ and $w$ do not lie in a common block in $y_i$;
  \item $v$ and $w$ do lie in a common block in $y_{i+1}$;
  \item $v > w$;
  \item $v$ is the minimal vertex in its block in $y_i$; and
  \item the new forest $T_{i+1}$ is embedded.
\end{itemize}

To show that such a pair $v, w$ exists let us look at minimal vertices in the two blocks of $y$, that become one in $y_{i+1}$. We let $v$ be the larger of the two. Then we know that the block not containing $v$ contains at least one smaller vertex. Together with the fact that $o_v$ defined above is non-crossing, the existence of a suitable $w$ is guaranteed.

Now it is clear that $T = T_n$ (with $y_n = 1$, the full partition) is an embedded tree with vertex set $U_n$. It is also clear that the apartment $A$ defined by $T$ (using Proposition 3.25) contains $C'$.

Observe that every vertex (except the minimal one) is connected with an edge to a smaller vertex. This is due to the fact that every vertex except the minimal stops being the smallest vertex in its block for some $i$ (when we add $y_n = 1$ to our considerations). When it stops being minimal, it plays the role of $v$ above, and so is connected to a smaller vertex. From this we easily deduce that $x_i \in A$ for every $i$, and so that $C \subseteq A$.

Now both points $x$ and $y$ lie in a common apartment $A$, and the distance between them is smaller than $\pi$. Hence there exists a unique geodesic in $B$ between them, and it lies in $A$. But $A \subseteq X$, so this concludes the proof.
Lemma 3.29. Let $x \in X$ be a universal point, and let $l$ be a short loop in $X$ through $x$. Then $l$ is shrinkable in $X$. 

Proof. View $l$ as a path $l : [0, L] \to X$ from $x$ to $x$. The Arzelà–Ascoli theorem tells us that we can assume without loss of generality that $l$ (seen as a loop) cannot be shrunk to a shorter loop going through $x$. Then, since every point in $X$ has a neighbourhood isometric to a metric cone over the point, the loop $l$ is a locally geodesic path in $X$. We want to prove that $l$ is constant.

If $l$ is not locally geodesic in $B$, then it has a turning point in $(0, L)$. According to Lemma 3.11 the set of turning points of $l$ is finite. Consider a turning point closest to 0 or $L$; without loss of generality assume that $0 < t < \frac{L}{2} < \pi$ is a turning point such that there is not turning point in $(0, t)$. Then $l|_{[0, t]}$ is a locally geodesic segment in $B$ of length smaller than $\pi$, hence it is a geodesic segment in $B$.

Then for $\varepsilon > 0$ small, the geodesic segments $[x, l(t + \alpha \varepsilon)] \subset B$ for $\alpha \in (0, 1]$ lie in $X$ by Lemma 3.28 and are shorter than $l|_{[0, t + \alpha \varepsilon]}$. They also vary continuously with $\alpha$, since $B$ is CAT(1) (compare Figure 3.3). Therefore $l$ can be shrunk by replacing $l|_{[0, t + \alpha \varepsilon]}$ by $[x, l(t + \alpha \varepsilon)]$, which contradicts the assumption on $l$.

So $l$ is locally geodesic in $B$, and therefore $l|_{[0, \frac{L}{2}]}$ and $l^{-1}|_{[\frac{L}{2}, L]}$ are two locally geodesic paths in $B$ from $x$ to $l(\frac{L}{2})$ of length smaller than $\pi$. They must be equal, since $B$ is CAT(1), and hence $l$ is constant.

Figure 3.3: Illustration of curve shortening in the proof of Lemma 3.29.

Lemma 3.30. When $n = 5$, turning faces in $X$ are universal vertices.

Proof. When $n = 5$, by Lemma 3.12 we know that the corank of a turning face contains at least 2 consecutive integers. Since the rank of $X$ is equal to 3, we conclude that $F$ is a vertex of rank either 1 or 3. By Lemma 3.15 we know that $F$ has two neighbours which fail modularity, hence $F$ is necessarily (by inspection) a universal vertex.

Corollary 3.31. The non-crossing partition complex $NCP_5$ is CAT(1).

Proof. Assume there is an unshrinkable short loop in $X$, the diagonal link of $NCP_5$. Then it has a turning point by Lemma 3.7 which is a universal vertex by 3.30, so the loop can be shrunk by Lemma 3.29. Hence by Theorem 3.20 (and Remark 3.22), $NCP_5$ is CAT(1).

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Example 3.32 (Vertices failing modularity in $NCP_5$). Figure 3.4 pictures a turning vertex corresponding to the partition $\{\{1, 2, 3, 4\}, \{5\}\}$ in $NCP_5$, with a pair of neighbours having partitions $\{\{1, 2\}, \{3, 4\}, \{5\}\}$ and $\{\{2, 3\}, \{1, 4\}, \{5\}\}$ failing modularity.

The same turning vertex with crossing neighbours that fail modularity is illustrated in Figure 3.5 where we also show how these vertices fit into the diagonal link of the crossing partition complex.

Definition 3.33 (Dominant vertex). A vertex $v$ of a face $F$ of $X$ is called dominant if and only if every apartment $A$ in $B$ having $v \in A \subseteq X$, contains $F$.

The following lemma will be accompanied by figures illustrating the subcases, in which the turning face is an edge, and giving examples of adjacent vertices failing modularity.

Lemma 3.34. When $n = 6$, a non-universal turning face $F$ in $X$, up to symmetries of $U_6$, falls into one of the following cases.
1. a vertex $v$ of rank 1, or an edge of rank $(1, 2)$ or $(1, 4)$ with a dominant vertex $v$ of rank 1, where $v$ is the partition $\{2, 4\}, \{1\}, \{3\}, \{5\}, \{6\}$.

2. a vertex $v$ of rank 2, or an edge of rank $(1, 2)$ with a dominant vertex $v$ of rank 2, where $v$ is the partition $\{1, 2\}, \{3, 4\}, \{5\}, \{6\}$ or $\{1, 2\}, \{4, 5\}, \{3\}, \{6\}$.

3. a vertex $v$ of rank 3, or an edge of rank $(3, 4)$ with a dominant vertex $v$ of rank 3, where $v$ is the partition $\{1, 2, 3, 5\}, \{4\}, \{6\}$ or $\{1, 2, 4, 5\}, \{3\}, \{6\}$.

4. a vertex $v$ of rank 4, or an edge of rank $(3, 4)$ or $(1, 4)$ with a dominant vertex $v$ of rank 4, where $v$ is the partition $\{1, 2, 3, 4\}, \{5, 6\}$.

Proof. When $n = 6$, by Lemma 3.12 we know that the corank of a turning face contains at least 2 consecutive integers. Since the rank of $X$ is equal to 4, we conclude that $F$ is either a vertex or an edge. Using duality, we can restrict to the cases where $F$ is a vertex of rank 1 or 2, or an edge of rank $\{1, 2\}$ or $\{1, 4\}$.
{1,4}. In these cases it is not hard to see that the cases listed are the only ones which allow for a pair of adjacent vertices failing modularity, in view of Lemma 3.16.

When $F$ is a face in $X$ and $i \in U_n$, let $F_i \subseteq U_n$ denote the smallest subset of $U_n$ that appears as a block in a vertex of $F$ and that contains $i$ properly. If the set of such subsets is empty we set $F_i = U_n$.

**Lemma 3.35.** Let $C$ be a chamber in $X$, and let $i, j$ be consecutive elements of $U_n$. If $C_i$ contains $j$, then there exists an apartment in $X$ containing $C$, $v$ and $w$, where $v$ is the boundary edge having the single nontrivial block $\{i, j\}$ and $w$ is the universal vertex opposite to $v$ in $B$ given by the partition $w = \{\{i\}, [1,n] \setminus \{i\}\}$.

**Proof.** Write $v_1, \ldots, v_{n-2}$ for the vertices of $C$, with the indices corresponding to the ranks, and let $v_{n-1} = 1$ be the maximal element in NCP$_n$. Denote the edge $\{i, j\}$ by $e$.

Let $k$ be minimal such that $C_k$ is a block of $v_i$.

Any apartment in $X$ containing both $v$ and $w$ is represented by a non-crossing tree $T$ which contains the edge $e$ in $U_n$, and such that the subforest obtained by removing $e$ from the tree corresponds to $w$ (since $w$ is opposite $v$), using the correspondence from Remark 3.24. We will now construct such a tree $T$ by inductively picking edges $e_l = \{1, \ldots, n-1\} \in U_n$.

Take $e_1$ to be the edge corresponding to $v_1$. For each $2 \leq l \leq k-1$ choose $e_l$ to be an edge such that the edges $e_1, \ldots, e_l$ form a non-crossing forest corresponding to $v_l$ (again using Remark 3.24). This is possible since vertices of $C$ are non-crossing partitions. Choose $e_k = e$. Observe that the edges $e_1, \ldots, e_k$ still form a non-crossing forest, since the edge $e$ cannot cross any other edge, and the vertex $v_i$ was isolated in the forest formed by $e_1, \ldots, e_{k-1}$, and so no cycles appear.

Now we continue choosing edges $e_l$ for $k+1, \ldots, n-1$ as before, with the additional requirement that none of the edges $e_l$ with $l \geq k+1$ connects to $i$.

Choosing the remaining edges like this is possible since in each step two blocks of $v_l$ are joined to form a block of $v_{l+1}$, and the block containing $i$ always contains at least also the vertex $j$, hence if a block is joined to the one containing $i$ then we may do this using an edge emanating from $j$ (or some other vertex in this block different from $i$).

The resulting tree is by construction non-crossing. The apartment $A$ spanned by $T$ contains $v$ and $C$. Further, since $e_k$ is the only edge connected to $i$, the apartment $A$ does also contain the vertex $w$.

**Lemma 3.36.** When $n = 6$, let $F$ be a non-universal turning face in $X$, and let $C$ be any chamber in $X$. Then there exists a pair $v, w$ of universal vertices in $X$, which are opposite in $B$, such that $F, v, w$ are contained in an apartment in $X$, and $C, v, w$ are contained in a (possibly different) apartment in $X$.

**Proof.** According to Lemma 3.34 let $u \in F$ denote the universal vertex of $F$. Our strategy here is to find consecutive $i, j \in U_n$ such that $j \in C_i \cap u_j$. Then Lemma 3.35 will give a pair $v, w$ of universal vertices in $X$, which are opposite in $B$, such that there exists an apartment in $X$ containing $C, v, w$, and another apartment in $X$ containing $u, v, w$. Since $u$ is universal for the face $F$, this last apartment contains $F, v, w$. 

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The following table lists all possibilities for $u_i$ (up to duality), depending on $i$ and the dominant vertex $u$ of $F$ (listed as in Lemma 3.34).

| Dominant vertex $u$ | $i = 1$ | 2   | 3   | 4   | 5   | 6   |
|---------------------|---------|-----|-----|-----|-----|-----|
| $\{(2, 4), \{1\}, \{3\}, \{5\}, \{6\}\}$ | $U_6$   | $\{2, 4\}$ | $U_6$ | $\{2, 4\}$ | $U_6$ | $U_6$ |
| $\{(1, 2), \{3, 4\}, \{5\}, \{6\}\}$ | $\{1, 2\}$ | $\{1, 2\}$ | $\{3, 4\}$ | $\{3, 4\}$ | $U_6$ | $U_6$ |
| $\{(1, 2), \{4, 5\}, \{3\}, \{6\}\}$ | $\{1, 2\}$ | $\{1, 2\}$ | $U_6$ | $\{4, 5\}$ | $\{4, 5\}$ | $U_6$ |

Now let us consider $C_i$. If $5 \in C_6$ or $1 \in C_6$ then our table tells us that we are done. Suppose that neither of these two occurs. If $4 \in C_6$ then $4 \in C_5$ and again we are done. Similarly if $2 \in C_6$ then $2 \in C_1$.

We are left with the case $C_6 = \{3, 6\}$. Here $6 \in C_5$ or $4 \in C_5$, which deals with the first two possibilities for $u_5$. The third one requires the observation that if $4 \notin C_5$ then $5 \in C_4$.

Theorem 3.37. The non-crossing partition complex $NC_6$ is $CAT(1)$.

Proof. Assume there is an unshrinkable short loop $l : \mathbb{R}/LZ \to X$, where $X$ is the diagonal link of $NC_6$ and $L < 2\pi$. Then by Lemma 3.37 this loop has a turning point with image $x$ in $X$. Let us reparametrise $l$ so that $x = l(0)$. Consider $y = l(L/2)$. By Lemma 3.36 there exists a pair $v, w$ of universal vertices in $X$, which are opposite in $B$, such that both $\{x, v, w\}$ and $\{y, v, w\}$ lie in apartments in $X$. Hence we know that

$$d(x, v) + d(x, w) = d(y, v) + d(y, w) = \pi.$$ 

So at least one element of $\{v, w\}$, say $v$, satisfies

$$d(x, v) + d(v, y) \leq \pi.$$

Let $\alpha_1 = l|_{[0, L/2]}$ and $\alpha_2 = l|_{[L/2, L]}$ be the two subpaths of $l$ from $x$ to $y$. Let $\alpha_3 : [0, d(x, v) + d(v, y)] \to X$ denote the concatenation of the geodesic segments $[x, v]$ and $[v, y]$.

Consider the loop $\alpha_3^{-1} \circ \alpha_1$ – since it is short and passes through the universal vertex $v$, by Lemma 3.24 it can be shrunk. Similarly, the loop $\alpha_3^{-1} \circ \alpha_2$ can be shrunk. Now we can apply Theorem 3.19 which tells us that the loop $l = \alpha_2^{-1} \circ \alpha_1$ can be shrunk. Hence by Theorem 3.20 $NC_6$ is $CAT(1)$.

Now we apply Proposition 3.21 to conclude the following.

Corollary 3.38. For every $n \leq 6$, the $n$-strand braid group is $CAT(0)$.

3.4 The orthoscheme complex of a modular complemented lattice is $CAT(0)$

Independently of the rest, we prove that the orthoscheme complex of a bounded graded modular complemented lattice is $CAT(0)$, thus giving a partial result towards [5, Conjecture 6.10]. It appears to be the best result we can get using an embedding into a space we already know to be $CAT(1)$, namely a spherical building; the complemented condition is needed to allow us to apply Frink’s embedding theorem.
Definition 3.39 (Modular lattice). A lattice $P$ is said to be modular if
\[ \forall x, y, z \in P, \quad x \geq z \implies x \land (y \lor z) = (x \land y) \lor z. \]

Definition 3.40 (Complemented lattice). A bounded lattice $P$ is said to be complemented if
\[ \forall x \in P, \exists y \in P, \quad x \land y = 0 \text{ and } x \lor y = 1. \]

Theorem 3.41 (Frink’s embedding Theorem). Let $P$ be a bounded graded modular complemented lattice. Then $P$ is isomorphic to a direct product
\[ P = \prod_{i=1}^{r} P_i \]
of bounded graded modular complemented lattices, such that for all $i \in [1, r]$, the lattice $P_i$ can be embedded as a subposet of a linear lattice (over a division algebra) or of a non-Arguesian plane lattice, where the embedding preserves the meets and the joins.

Proof. According to [11, Theorem 279] and [11, Lemma 99], $P$ is isomorphic to a direct product $P = \prod_{i=1}^{r} P_i$ of simple bounded graded modular complemented lattices, where the fact that each $P_i$ is simple implies that $P_i$ cannot be embedded non-trivially as a subposet of a non-trivial product of lattices. According to [11, Corollary 439], each $P_i$ is then embedded as a subposet of a product of linear lattices (over a division algebra) or of non-Arguesian plane lattices, such that the joins and the meets are preserved. Since $P_i$ is simple, the product consists of only one non-trivial factor.

Theorem 3.42. The orthoscheme complex of a bounded graded modular complemented lattice is $\text{CAT}(0)$.

Proof. Let $P$ be a bounded graded modular complemented lattice, and let $|P|$ be its orthoscheme complex. By Theorem 3.41, write $P = \prod_{i=1}^{r} P_i$. Since the orthoscheme complex of $|P|$ is the Euclidean product of the orthoscheme complexes of the $P_i$’s, we only need to show that each $|P_i|$ is $\text{CAT}(0)$.

Fix $i \in [1, r]$. The lattice $P_i$ is embedded as a subposet of a linear lattice $S(V)$ (over a division algebra) or of a non-Arguesian plane lattice $L$, where the embedding preserves the meets and the joins.

In the first case, $P_i$ is linearly embedded in $S(V)$ in the sense of Definition 3.8. Since joins and meets coincide in $P_i$ and in $S(V)$, and as $P_i$ is modular, we deduce that $P_i$ has no pair of elements failing modularity in the sense of Definition 3.13. Let $X = LK(e_{01}, |P_i|) \subseteq LK(e_{01}, |S(V)|) = Y$ be their diagonal links. Since $Y$ is a spherical building, it is $\text{CAT}(1)$. Assume there is a short local geodesic loop $l$ in $X$. By Lemma 3.7, the loop $l$ has a turning point in $Y$. By Lemma 3.15, the image of that turning point has neighbours which fail modularity, which contradicts the assumption that $P_i$ is modular. Hence $l$ cannot exist, and so according to [5, Theorem 5.10] this implies that $|P_i|$ is $\text{CAT}(0)$.

In the second case, $|P_i|$ is a subgraph of the incidence graph $|L|$ of a non-Arguesian projective plane. This graph has combinatorial girth at least $6$, and since edges have length $\frac{\pi}{3}$, we conclude that $|P_i|$ has girth at least $2\pi$, hence it is $\text{CAT}(0)$.

\[ \square \]
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