I. INTRODUCTION.

Since the pioneering work of Dirac (1), who proposed, motivated by the occurrence of large numbers in Universe, a theory with a time variable gravitational coupling constant $G$, cosmological models with variable $G$ and nonvanishing and variable cosmological term, $\Lambda$, have been intensively investigated in the physical literature (see for example [2]-[14]).

In modern cosmological theories, the cosmological constant remains a focal point of interest (see [15]-[18] for reviews of the problem). A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant. Some of the recent discussions on the cosmological constant “problem” and on cosmology with a time-varying cosmological constant point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”. However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying $\Lambda$ can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation. Recent observations strongly favour a significant and a positive value of $\Lambda$ with magnitude $\Lambda (Gh/c^3) \approx 10^{-123}$. These observations suggest on accelerating expansion of the universe, $q < 0$.

Our current understanding of the physical universe is anchored on the analysis of expanding, isotropic and homogeneous models with a cosmological constant, and linear perturbations thereof. This model successfully accounts for the late time universe, as is evidenced by the observation of large scale cosmic microwave background observations. Parameter determination from the analysis of CMB fluctuations appears to confirm this picture. However, further analyses seem to suggest some inconsistency. In particular, its appears that the universe could have a preferred direction. Followup analyses of various sets of WMAP data sets, with different techniques seem to lead to the same conclusion. It is still unclear whether or not the directional preference is intrinsic to the underlying model, and what implications this has on our understanding of cosmology. For this reason Bianchi models are important in the study of anisotropies.

Recently, the cosmological implications of a variable speed of light during the early evolution of the Universe have been considered. Varying speed of light (VSL) models proposed by Moffat (19) and Albrecht and Magueijo (20), in which light was travelling faster in the early periods of the existence of the Universe, might solve the same problems as inflation. Hence they could become a valuable alternative explanation of the dynamics and evolution of our Universe and provide an explanation for the problem of the variation of the physical “constants”. Einstein’s field equations (EFE) for FRW spacetime in the VSL theory have been solved by Barrow (21) and (22) for anisotropic models, who also obtained the rate of variation of the speed of light required to solve the flatness and cosmological constant problems (see J. Magueijo (23) for a review of these theories).

This model is formulated under the strong assumption that a $c$ variable (where $c$ stands for the speed of light) does not introduce any corrections into the curvature tensor, furthermore, such formulation does not verify the covariance and the Lorentz invariance as well as the resulting field equations do not verify the Bianchi identities either (see Bassett et al (24)).

Nevertheless, some authors (T. Harko and M. K. Mak (24), P.P. Avelino and C.J.A.P. Martins (26) and H. Shojae et al (27) have proposed a new generalization of General Relativity which also allows arbitrary changes in the speed of light, $c$, and the gravitational constant, $G$, but in such a way that variations in the speed of light introduces corrections to the curvature tensor in the Einstein equations in the cosmological frame. This new formulation is both covariant and Lorentz invariant and as we will show the resulting field equations (FE) verify the Bianchi identities.

The purpose of this paper is to study a perfect fluid Bianchi II model with time varying constants through the self-similarity (SS) hypothesis (see foe example [34]-[42]). The study of SS models is quite important since a large class of orthogonal spatially homogeneous models are asymptotically self-similar at the initial singularity.
and are approximated by exact perfect fluid or vacuum self-similar power law models. Exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution. This last point is of particular importance in relating Bianchi models to the real Universe. At the same time, self-similar solutions can describe the behaviour of Bianchi models at late times i.e. as $t \to \infty$.

We would like to emphasize that in this work we are more interested in mathematical respects (exact solutions) than in studying strong physical consequences. Nevertheless we consider some observational data in order to rule out some of the obtained solutions.

Therefore the paper is organized as follows. In the second section, we begin by defining the metric and calculating its Killing vectors as well as their algebra. We also calculate the four velocity and the quantities derived from it, i.e. the expansion, Hubble parameter etc... In section three, we review some basic ideas about self-similarity. Once we have clarified some concepts then we calculate the homothetic vector field as well as the constrains for the scale factors. In section fourth we review the classical solution i.e. a perfect fluid model with “constant” constants. We show that the obtained solution belongs to the LRS Bianchi type II by calculating the fourth Killing vector field. We also calculate all the curvature invariants showing that the obtained solution is singular. These results will be valid for the rest of the studied models. We end this section showing that there is not solution for the vacuum model. In section five, we study a perfect fluid model with variable $G$ and $\Lambda$. We compare the solution with the obtained one in the above section showing that we have been able to relax the constrains obtained for the classical model. We rule out some of the obtained solution taking into account the current observations which suggest us that $\Lambda_0 > 0$ and $q < 0$. In section six, we study a perfect fluid model with variable speed of light (VSL). We outline the field equations taking into account the effects of a c-var into the curvature tensor. This point is essential in this approach as we have showed in a previous paper [28]. We calculate the homothetic vector field for the new metric and we show the new constrains for the scale factors. If we assume a particular solution for $c(t)$ then we show that it may be a decreasing time function. Only taking into account observational restriction we are able to rule out some of the obtained solutions as in the above section. We end summarizing the conclusions in the last section.

II. THE METRIC.

Throughout the paper $M$ will denote the usual smooth (connected, Hausdorff, 4-dimensional) spacetime manifold with smooth Lorentz metric $g$ of signature $(-,+,+,+)$ (see for example [28]). Thus $M$ is paracompact. A comma, semi-colon and the symbol $\mathcal{L}$ denote the usual partial, covariant and Lie derivative, respectively, the covariant derivative being with respect to the Levi-Civita connection on $M$ derived from $g$. The associated Ricci and stress-energy tensors will be denoted in component form by $R_{ij}(=\mathcal{R}^a_{jcd})$ and $T_{ij}$ respectively.

A Bianchi II space-time is a spatially homogeneous space-time which admits an abelian group of isometries $G_3$, acting on spacelike hypersurfaces, generated by the spacelike KVs,

$$\xi_1 = K \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = -Ky \partial_x + \partial_z, \quad (1)$$

and their algebra is:

$$ [\xi_i, \xi_j] = C^k_{ij} \xi_k, \quad C^1_{23} = 1. $$

In synchronous co-ordinates the metric is:

$$ds^2 = -c^2 dt^2 + a^2(t) dx^2 + \left( b^2(t) + K^2 z^2 a^2(t) \right) dy^2 + 2K a^2(t) dy dz + d^2(t) dz^2 \quad (2)$$

where the metric functions $a(t), b(t), d(t)$ are functions of the time co-ordinate only and $K \in \mathbb{R}$. The introduction of this constant is essential since if we set $K = 1$, as it is the usual way, then there is not SS solution for the outlined field equations. In this paper we are interested only in Bianchi II space-times, hence all metric functions are assumed to be different and the dimension of the group of isometries acting on the spacelike hypersurfaces is three.

Once we have defined the metric and we know which are its killing vectors, then we calculate the four velocity. It must verify, $\mathcal{L}_u u_i = 0$, so we may define the four velocity as follows:

$$u^i = \left( \frac{1}{c}, 0, 0, 0 \right),$$

in such a way that it is verified, $g(u^i, u^j) = -1$.

From the definition of the 4-velocity we find that:

$$\theta = u^i_{,i} = \frac{1}{c} \left( \frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d} \right) = \frac{1}{c} H,$$

$$H = H_3 + H_2 + H_3,$$

$$q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1,$$

$$\sigma^2 = \frac{1}{3c^2} \left( H^2_1 + H^2_2 + H^2_3 - H_1 H_2 - H_1 H_3 - H_2 H_3 \right).$$

III. SELF-SIMILARITY.

In general relativity, the term self-similarity can be used in two ways. One is for the properties of spacetimes, the other is for the properties of matter fields.
These are not equivalent in general. The self-similarity in general relativity was defined for the first time by Cahill and Taub (see [34], and for general reviews [37, 10]). Self-similarity is defined by the existence of a homothetic vector $V$ in the spacetime, which satisfies

$$\mathcal{L}_V g_{ij} = 2\alpha g_{ij},$$

where $g_{ij}$ is the metric tensor, $\mathcal{L}_V$ denotes Lie differentiation along $V$ and $\alpha$ is a constant. This is a special type of conformal Killing vectors. This self-similarity is called homothety. If $\alpha \neq 0$, then it can be set to be unity by a constant rescaling of $V$. If $\alpha = 0$, i.e. $\mathcal{L}_V g_{ij} = 0$, then $V$ is a Killing vector.

Homothety is a purely geometric property of spacetime so that the physical quantity does not necessarily exhibit self-similarity such as $\mathcal{L}_V Z = dZ$, where $d$ is a constant and $Z$ is, for example, the pressure, the energy density and so on. From equation (4) it follows that $\mathcal{L}_V R^i_{jkl} = 0$, and hence $\mathcal{L}_V R_{ij} = 0$ and $\mathcal{L}_V G_{ij} = 0$. A vector field $V$ that satisfies the above equations is called a curvature collineation, a Ricci collineation and a matter collineation, respectively. It is noted that such equations do not necessarily mean that $V$ is a homothetic vector.

We consider the Einstein equations $G_{ij} = 8\pi GT_{ij}$, where $T_{ij}$ is the energy-momentum tensor. If the spacetime is homothetic, the energy-momentum tensor of the matter fields must satisfy $\mathcal{L}_V T_{ij} = 0$. For a perfect fluid case, the energy-momentum tensor takes the form $T_{ij} = (p + \rho)u_i u_j + pg_{ij}$, where $p$ and $\rho$ are the pressure and the energy density, respectively. Then, equations (4) result in

$$\mathcal{L}_V u^i = -\alpha u^i, \quad \mathcal{L}_V \rho = -2\alpha \rho, \quad \mathcal{L}_V p = -2\alpha p. \quad (5)$$

As shown above, for a perfect fluid, the self-similarity of the spacetime and that of the physical quantity coincide. However, this fact does not necessarily hold for more general matter fields. Thus the self-similar variables can be determined from dimensional considerations in the case of homothety. Therefore, we can conclude homothety as the general relativistic analogue of complete similarity.

From the constraints 3, we can show that if we consider the barotropic equation of state, i.e., $p = f(\rho)$, then the equation of state must have the form $p = \omega \rho$, where $\omega$ is a constant. In this paper, we would like to show how taking into account this class of hypothesis one is able to find exact solutions to the field equations within the framework of the time varying constants.

From equation (4), we find the following homothetic vector field for the BI metric (2):

$$V = t \partial_t + \left(1 - t \frac{a'}{a} \right) x \partial_x + \left(1 - t \frac{b'}{b} \right) y \partial_y + \left(1 - t \frac{d'}{d} \right) z \partial_z,$$

with the following constrains for the scale factors:

$$a(t) = a_0 t^{a_1}, \quad b(t) = b_0 t^{a_2}, \quad d(t) = d_0 t^{a_3} \quad (7)$$

with $a_1, a_2, a_3 \in \mathbb{R}$, in such a way, that the constants $a_i$ must verify the following restriction:

$$a_2 + a_3 - a_1 = 1. \quad (8)$$

Therefore, the resulting homothetic vector field is:

$$V = t \partial_t + (1 - a_1) x \partial_x + (1 - a_2) y \partial_y + (1 - a_3) z \partial_z. \quad (9)$$

which is the same as the found one in the Bianchi I model (see for example [32]).

**IV. THE PERFECT FLUID MODEL. THE CLASSICAL SOLUTION.**

Taking into account the field equations (FE)

$$R_{ij} - \frac{1}{2} Rg_{ij} = \frac{8\pi G}{c^2} T_{ij} - \Lambda g_{ij}, \quad (10)$$

where, in this section, we shall consider that $\Lambda$ vanish, and the energy-momentum tensor, $T_{ij}$, is defined as follows:

$$T_{ij} = (p + \rho)u_i u_j + pg_{ij}. \quad (11)$$

with the usual equation of state $p = \omega \rho$, $\omega \in \mathbb{R}$, as we have discussed above.

The resulting FE for the metric (2) with a perfect fluid matter model (11) are:

$$\begin{align*}
\frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d} - \frac{K^2 a'^2}{4 b^2 d^2} &= \frac{8\pi G}{c^2} \rho, \\
\frac{b''}{b} + \frac{d''}{d} - \frac{3K^2 a'^2}{4 b^2 d^2} &= -\frac{8\pi G}{c^2} \omega \rho, \\
\frac{d''}{d} + \frac{a''}{a} + \frac{b''}{b} - \frac{K^2 a'^2}{4 b^2 d^2} &= \frac{8\pi G}{c^2} \omega \rho, \\
\rho' + \rho (1 + \omega) \left(\frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d}\right) &= 0.
\end{align*} \quad (12)$$

Now, if we take into account the obtained SS restrictions for the scale factors i.e.

$$a(t) = a_0 t^{a_1}, \quad b(t) = b_0 t^{a_2}, \quad d(t) = d_0 t^{a_3},$$

where $a_1, a_2, a_3 \in \mathbb{R}$, must satisfy $a_2 + a_3 - a_1 = 1$, we get from eq. (10):

$$\rho = \rho_0 t^{-\gamma}, \quad (17)$$

where $\gamma = \alpha (1 + \omega) = (a_1 + a_2 + a_3) (1 + \omega)$. Therefore the system to solve is the following one:

$$\begin{align*}
a_2(a_2 - 1) + a_3(a_3 - 1) + a_2a_3 - \frac{3K^2}{4} &= -A\omega, \\
a_3(a_3 - 1) + a_1a_3 + a_1(a_1 - 1) + \frac{K^2}{4} &= -A\omega, \\
a_2(a_2 - 1) + a_1a_2 + a_1(a_1 - 1) + \frac{K^2}{4} &= -A\omega, \\
a_1 + a_2 + a_3 &= \frac{2}{1 + \omega}.
\end{align*} \quad (18)$$
where we have set the constant $A$ as follows

$$A = a_1 a_2 + a_1 a_3 + a_3 a_2 - \frac{K^2}{4},$$

and $a_2 + a_3 - a_1 = 1$.

The obtained solution is the following one:

$$a_1 = \frac{1 - \omega}{2(\omega + 1)}, \quad a_2 = a_3 = \frac{\omega + 3}{4(\omega + 1)},$$

therefore the constrain $a_2 + a_3 - a_1 = 1$ collapses to $2a_2 - a_1 = 1$, and

$$K^2 = \frac{1 + 2\omega - 3\omega^2}{4(\omega + 1)^2}, \quad \iff \quad \omega \in \left(\frac{-1}{3}, 1\right),$$

so $a_2 \in \left(\frac{1}{3}, 1\right), \quad a_1 \in (0, 1), \quad K^2 \in (0, 1/4)$, and $A = 2a_1 a_2 + a_2 - \frac{K^2}{4} = \frac{(5-\omega)}{4(\omega+1)^2}$. Taking into account all these results the homotetic vector field collapses to

$$V = t\partial_t + (1 - a_1) x\partial_x + (1 - a_2) (y\partial_y + z\partial_z),$$

this is the already solution obtained by Hsu et al. This solution has been already obtained by Collins.

**Remark 1** As is it observed, we may find the relation $a_2 + a_3 - a_1 = 1$ from dimensional considerations, only looking at the FE. Note that the quantity

$$\frac{a_2^2 c^2}{b^2 d^2} \approx \frac{a_0^2 t^{2}}{b_0^2 t^{2} + a_0^2 t^{2} + a_3^2 t^{2}} \approx t^{2 a_1 - 2 a_2 - 2 a_3} \approx t^{-2},$$

after substitution, it must have dimensions of $T^{-2}$ as the rest of the factors. Therefore we find the relation $a_2 + a_3 - a_1 = 1$.

Therefore we have obtained a LRS BII model, note that $d = b$, and hence the metric collapses to this one:

$$ds^2 = -c^2 dt^2 + a^2(t) dx^2 + (b^2(t) + K^2 z^2 a^2(t)) dy^2 + 2 Ka^2(t) z dx dy + b^2(t) dz^2, \quad (22)$$

which admits the three Killing vector fields $[1]$ and this new one

$$\xi_4 = -K \left( \frac{z^2}{2} - \frac{y^2}{2} \right) \partial_x + z \partial_y - y \partial_z. \quad (23)$$

Hence their algebra is:

|  \xi_1 |  \xi_2 |  \xi_3 |  \xi_4 |
|------|------|------|------|
|  \xi_1 |  0 |  0 |  0 |  0 |
|  \xi_2 |  0 |  0 |  \xi_1 | -\xi_4 |
|  \xi_3 |  0 | -\xi_1 |  0 |  \xi_2 |
|  \xi_4 |  0 |  \xi_3 | -\xi_2 |  0 |

i.e. $C_{13} = 1$, $C_{34} = -1$, and $C_{24} = 1$.

With the obtained results we can see that

$$H = \frac{4a_2 - 1}{t} = \left( \frac{2}{(\omega + 1)} \right) \frac{1}{t},$$

$$q = \frac{4a_2}{4a_2 - 1} = \frac{3 + \omega}{2} < 0, \quad \forall \omega \in (-1/3, 1),$$

while the shear behaves as:

$$\sigma^2 = \frac{(a_2 - 1)^2}{3c^2 t^2} = \left( \frac{3\omega + 1}{4(\omega + 1)} \right)^2 \frac{1}{3c^2 t^2}.$$

As it is observed, these quantities fit perfectly with the current observations by High-Z Supernova Team and Supernova Cosmological Project see for example: Garnavich et al. 1998; Perlmutter et al., 1997, 1998; Riess et al., 1998; Schmidt et al., 1998.

**A. Curvature behaviour.**

With all these results, we find the following behaviour for the curvature invariants (see for example [45]-[48]).

Ricci Scalar, $I_0$, yields

$$I_0 = \frac{2}{c^2 t^2} \left( 11a_2^2 - 10a_2 + 2 - \frac{K^2 c^2}{4} \right), \quad (24)$$

while Kretschmann scalar, $I_1 := R_{ijkl} R^{ijkl}$, yields:

$$I_1 = \frac{1}{4c^4 t^4} \left( 432a_2^4 - 960a_2^3 + 896a_2^2 - 384a_2 + 64 + K^2 c^2 \left( 11K^2 c^2 - 40a_2^2 + 80a_2 - 48 \right) \right).$$

The full contraction of the Ricci tensor, $I_2 := R_{ij} R^{ij}$, is:

$$I_2 = \frac{1}{4c^4 t^4} \left( 528a_2^4 - 1024a_2^3 + 768a_2^2 - 256a_2 + 32 + K^2 c^2 \left( 3K^2 c^2 - 16a_2 + 8 \right) \right),$$

this means that the model is singular.

The non-zero components of the Weyl tensor. The following components of the Weyl tensor run to $\pm \infty$ when $t \to 0$,

$$C_{ttxx} \approx C_{ttxy} \approx t^{4(a_2-1)}, \quad C_{tttx} \approx t^{2(a_2-1)},$$

and

$$C_{tyty} \approx t^{2(a_2-1)} + z^2 t^{4(a_2-1)},$$

these others depend on the value of $a_2$,

$$C_{txyz} \approx C_{tyxz} \approx C_{tzyz} \approx C_{txzy} \approx t^{4a_2-3},$$

$$C_{xxzz} \approx C_{xxzy} \approx C_{xzyx} \approx t^{2(3a_2-2)},$$

$$C_{yzyz} \approx t^{2(2a_2-1)} + z^2 t^{2(3a_2-2)}.$$
i.e. they may run to zero as well as to $\pm \infty$ when $t \to 0$, depending on the value of $a_2$.

The Weyl scalar, $I_3 = C^{abcd}C_{abcd} = I_1 - 2I_2 + \frac{1}{3}I_0^2$, (this definition is only valid when $n = 4$)

$$I_3 = \frac{4}{3c^4t^4} \left[ \left((a_2 - 1) \left(-2(a_2 - 1) - 3Kc\right) + K^2c^2\right) \times \left((a_2 - 1) \left(-2(a_2 - 1) + 3Kc\right) + K^2c^2\right) \right].$$

The electric part scalar $I_4 = E_iE^i$, (see W.C. Lim et al [49])

$$I_4 = \frac{1}{6c^4t^4} \left(-2(a_2 - 1)^2 + K^2c^2\right)^2,$$

while the magnetic part scalar $I_5 = H_{ij}H^{ij}$, yields

$$I_5 = \frac{3}{2c^2t^4}(a_2 - 1)^2 K^2.$$

Therefore the Weyl parameter $W^2$ (see W.C. Lim et al [49])

$$W^2 = \frac{1}{6} \left(E_{ij}E^{ij} + H_{ij}H^{ij}\right),$$

therefore

$$W^2 = \frac{1}{36c^4t^4} \left((a_2 - 1)^2 + K^2c^2\right)^2 \left((2a_2 - 2)^2 + K^2c^2\right)^2,$$

note that the value of $W^2$ is really small.

The gravitational entropy (see [10]–[17])

$$P^2 = \frac{I_3}{I_2} = \frac{I_2 - 2I_2 + \frac{1}{3}I_0^2}{I_2} = \frac{I_2}{I_2} + \frac{1}{3}I_0^2 - 2 = \text{const.},$$

since the spacetime is SS (see [50] and [43] for a discussion).

**B. Vacuum model.**

In this section we shall show that there is no self-similar solution for the vacuum model. In this case, after substitution, the resulting FE are:

$$a_1a_2 + a_1a_3 + a_3a_2 - \frac{K^2}{4} = 0, \quad (25)$$

$$a_2(a_2 - 1) + a_3(a_3 - 1) + a_2a_3 - \frac{3K^2}{4} = 0, \quad (26)$$

$$a_3(a_2 - 1) + a_1a_3 + a_1(a_1 - 1) + \frac{K^2}{4} = 0, \quad (27)$$

$$a_2(a_2 - 1) + a_1a_2 + a_1(a_1 - 1) + \frac{K^2}{4} = 0, \quad (28)$$

so, by solving this system, we only obtain the following unphysical solutions:

$$a_1 = a_2 = a_3 = K = 0,$$

$$a_1 = 1, \quad a_2 = a_3 = K = 0,$$

$$a_1 = \frac{1}{3}, \quad a_2 = a_3 = \frac{1}{3}, \quad K^2 = -\frac{4}{9},$$

$$a_1 = \frac{1}{3}, \quad a_2 = a_3 = \frac{1}{3}, \quad K = 0.$$

Therefore, we conclude that there is no physical solution for the vacuum model. Note that when we set $K = 0$, the metric is reduced to Bianchi I one. In the same way, it is also verified the relation $a_2 + a_3 - a_1 = 1$.

**V. Perfect fluid model with G and Lambda variable.**

The resulting FE for the metric (22) with a perfect fluid matter model (11) and with $G$ and $\Lambda$ time-varying are:

$$\rho' + \rho(1 + \omega) \left(\frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d}\right) = 0, \quad (33)$$

$$\Lambda' = -\frac{8\pi G}{c^2}G^2\rho. \quad (34)$$

Now, we shall take into account the obtained SS restrictions for the scale factors i.e.

$$a(t) = a_0t^{\alpha_1}, \quad b(t) = b_0t^{\alpha_2}, \quad d(t) = d_0t^{\alpha_3},$$

where $a_1, a_2, a_3 \in \mathbb{R}$, in such a way that they must verify $a_2 + a_3 - a_1 = 1$.

From eq. (33) we get

$$\rho = \rho_0t^{-\gamma}, \quad (35)$$

where $\gamma = (\omega + 1)\alpha$ and $\alpha = (a_1 + a_2 + a_3)$.

From eq. (29) we obtain:

$$\Lambda = \frac{1}{c^2} \left[At^{-2} - \frac{8\pi G}{c^2}\rho_0t^{-(\omega + 1)\alpha}\right] \quad (36)$$

where $A = a_1a_2 + a_1a_3 + a_2a_3 - \frac{K^2}{4}$.

Now, taking into account eq. (34) and eq. (36) algebra brings us to obtain

$$G = G_0t^{\gamma - 2}, \quad (37)$$

where $G_0 = \frac{\pi c^2A}{8\pi \rho_0(\omega + 1)\alpha}$.

While the cosmological “constant” behaves as:

$$\Lambda = \frac{A}{c^2} \left(1 - \frac{2}{(\omega + 1)\alpha}\right) t^{-2} = \Lambda_0 t^{-2}. \quad (38)$$
With all these result we find that the system to solve is the following one:

\[
a_2(a_2 - 1) + a_3(a_3 - 1) + a_2a_3 - \frac{3K^2}{4} = A\left(\frac{\alpha - 2}{\alpha}\right), \quad (39)
\]

\[
a_3(a_3 - 1) + a_1a_3 + a_1(a_1 - 1) + \frac{K^2}{4} = A\left(\frac{\alpha - 2}{\alpha}\right), \quad (40)
\]

\[
a_2(a_2 - 1) + a_1a_2 + a_1(a_1 - 1) + \frac{K^2}{4} = A\left(\frac{\alpha - 2}{\alpha}\right), \quad (41)
\]

whose solution is:

\[
a_1 = 2a_2 - 1, \quad a_2 = a_3, \quad K^2 = -4a_2^2 + 6a_2 - 2, \quad (42)
\]

therefore this solution has only sense if \(a_2 \in (\frac{1}{4}, 1)\). Note that for these values, \(a_1 > 0, a_1 = 2a_2 - 1 \in (0, 1)\). This result is valid for all equation of state. We would like to point out that this solution is more general that the obtained one in the perfect fluid case with “constant” constants, since it is valid for \(\omega \in (-1, \infty)\), although we are only interested in \(\omega \in (-1, a], a \geq 1\). It is important to emphasize that \(\omega \neq -1\).

With this solution we may see that all the quantities depend on two variables, \(a_2\) and \(\omega\). Newton gravitational constant behaves as follows:

\[
G = \frac{c^2A}{4\pi \rho_0 (\omega + 1) \alpha} t^{\tau - 2} = G_0 t^2, \quad G_0 > 0, \quad (43)
\]

with \(g = (\omega + 1)(a_2 - 1) - 2 \in (-2, 4)\), and \(A = 5a_2^2 - 2a_2 - \frac{K^2}{4} = 6a_2^2 - \frac{5}{2}a_2 + \frac{1}{2} \in (\frac{1}{4}, 3)\). So depending of the different combinations between \(a_2\) and \(\omega\), \(G\) may be a growing or a decreasing time function.

With regard to the behaviour of the cosmological constant, as we can see, it is a decreasing time function, as it is expected. Only rest to know which is the sign of \(\Lambda_0\). We may express its behaviour in the following tables:

| \(\omega\) | \(a_2\) | \(\Lambda_0\) |
|----------|------|--------|
| \((-1, 1]\) | \(\frac{1}{2}\) | \(\leq 0\) |
| 1 | \(\frac{1}{2}\) | 0 |
| > 1 | \(\frac{1}{2}\) | > 0 |

| \(\omega\) | \(a_2\) | \(\Lambda_0\) |
|----------|------|--------|
| \((-1, -1/2]\) | \(-1/2\) | \(-1\) |
| > -1/2 | \(-1\) | > 0 |

this means, for example, that when we fix \(a_2 \notin \frac{1}{2}\) i.e. that \(a_2\) tends to \(1/2\), for all value of \(\omega\) between -1 and 1, \(\Lambda_0 < 0, \Lambda_0 = 0\), if \(\omega = 1\) and only \(\Lambda_0 > 0\) when \(\omega > 1\). In the say way we may say that if we fix \(a_2 \not\in \frac{1}{2}\), then \(\forall \omega \in (-1, -1/2]\), we find that \(\Lambda_0 < 0, \Lambda_0 = 0\), if \(\omega = -1/2\) and only it is found \(\Lambda_0 > 0\) when \(\omega > -1/2\). So may only say that the sign of \(\Lambda_0\) depends on the parameters \(a_2\) and \(\omega\). If we take into account the current observations which suggest that \(\Lambda_0 > 0\), then we may rule out the values which make \(\Lambda_0 < 0\).

As it is observed the behaviour of \(G\) and the sign of \(\Lambda_0\), are related. If \(G\) is growing then \(\Lambda_0 > 0\), if \(G = const.\) then \(\Lambda_0 = 0\), i.e. the model collapses to the standard one studied above, and if \(G\) is decreasing then \(\Lambda_0 < 0\). Therefore if we consider the recent observations which suggest us that \(\Lambda_0 > 0\), we must rule out the other cases, concluding that \(G\) is a growing time function and \(\Lambda\) is a positive decreasing function.

The energy-density behaves as

\[
\rho = \rho_0 t^{-\gamma}, \quad \text{with} \quad \gamma \in (0, 6) \quad (44)
\]

so it is always a time decreasing function if \(\omega > -1\), and to end, we find the following behaviour for the Hubble, deceleration and shear parameters:

\[
H = \frac{4a_2 - 1}{t}, \quad q = -\frac{4a_2}{4a_2 - 1} < 0,
\]

\[
\sigma^2 = \frac{(a_2 - 1)^2}{3c^2t^2} \rightarrow 0. \quad (45)
\]

As in the above model, the metric of this one collapses to a LRSBII type \((22)\).

VI. PERFECT FLUID MODEL WITH VSL.

We start by defining the new metric as follows:

\[
ds^2 = -c(t)^2 dt^2 + a^2(t) (dx^2 + (b^2(t) + K^2 z^2 a^2(t)) dy^2 + 2Ka(t) z dx dy + d^2(t) dz^2, \quad (46)
\]

note that we have simply replaced \(c\) by \(c(t)\), so we shall consider the four velocity:

\[
u = \left(\frac{1}{c(t)}, 0, 0, 0\right), \quad / \quad u_i u^i = -1. \quad (47)
\]

The time derivatives of \(G, c\) and \(\Lambda\) are related by the Bianchi identities

\[
\left(R_{ij} - \frac{1}{2} R g_{ij}\right)^\beta = \left(\frac{8\pi G}{c^4} T_{ij} - \Lambda g_{ij}\right)^\beta, \quad (48)
\]

in our case we obtain

\[
\rho' + \rho (1 + \omega) H + \frac{\Lambda' c^4}{8\pi G} + \rho \left(\frac{G'}{G} - 4\frac{c'}{c}\right) = 0, \quad (49)
\]

where \(H = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\), but if we take into account the condition, \(T_{ij}^{\beta} = 0\), it is obtained the following set of equations:

\[
\rho' + \rho (1 + \omega) H = 0, \quad (50)
\]

\[
\frac{\Lambda' c^4}{8\pi G} + \frac{G'}{G} - 4\frac{c'}{c} = 0. \quad (51)
\]

Therefore, the resulting FE are:
Note that we have taken into account the effects of a $c$-var into the curvature tensor (see [28] for a discussion about the issue).

In this model, the homothetic vector field is:

$$X = \left( \int \frac{cdt}{c(t)} \right) \partial_t + \left( 1 - \frac{\int cdt}{c(t)} \right) x \partial_x + \left( 1 - \frac{\int cdt}{c(t)} \right) y \partial_y + \left( 1 - \frac{\int cdt}{c(t)} \right) z \partial_z,$$ 

with the following restrictions

$$a = a_0 \left( \int \frac{cdt}{c(t)} \right)^{\alpha_1}, \ b = b_0 \left( \int \frac{cdt}{c(t)} \right)^{\alpha_2}, \ d = d_0 \left( \int \frac{cdt}{c(t)} \right)^{\alpha_3},$$

with $(\alpha_i)_{i=1}^3 \in \mathbb{R}$, in such a way that they must satisfy the relation $\alpha_2 + \alpha_3 - \alpha_1 = 1$. Since we expect to get a growing scale factors, we shall consider that $(\int cdt)$ must be a positive growing time function. Note that $c$ only needs to be integrable but it may be decreasing as we shall show below.

**Remark 2** As it is observed, if $c = \text{const.}$, we regain the usual homothetic vector field. i.e.

$$X = t \partial_t + \left( 1 - tH_1 \right) x \partial_x + \left( 1 - tH_2 \right) y \partial_y + \left( 1 - tH_3 \right) z \partial_z,$$

while the scale factors behave as

$$a = a_0 \left( t \right)^{\alpha_1}, \ b = b_0 \left( t \right)^{\alpha_2}, \ d = d_0 \left( t \right)^{\alpha_3},$$

with $\alpha_2 + \alpha_3 - \alpha_1 = 1$, as in the above studied cases.

By defining

$$H_i = \alpha_i \frac{c}{\int cdt} \quad \Rightarrow \quad H = \alpha \frac{c}{\int cdt},$$

where $\alpha = \sum_{i=1}^3 \alpha_i$, we find, from eq. [54], the behavior of the energy density i.e.

$$\rho = \rho_0 \left( \int cdt \right)^{-\gamma},$$

where $\gamma = (1 + \omega)\alpha$.

In the same way it is easily calculated the shear

$$\sigma^2 = \frac{1}{3c^2} \left( H_1^2 + H_2^2 + H_3^2 - H_1H_2 - H_1H_3 - H_2H_3 \right),$$

i.e.

$$\sigma^2 = \frac{2}{3} \left( \sum_{i,j} \alpha_i^2 - \sum_{i \neq j} \alpha_i \alpha_j \right) \left( \int \frac{cdt}{c(t)} \right)^{-2}.$$ 

As it is observed all the quantities depend on $\int \frac{c(t)dt}{c(t)}$.

Now only rest to calculate $G$ and $\Lambda$.

From eqs. [52] and [63] we get:

$$A \left( \frac{c}{\int c} \right)^2 = \frac{8\pi G}{c^2} \rho_0 \left( \int c \right)^{-\gamma} + \Lambda c^2,$$

where we have written, for simplicity, $\int c$ instead of $\int cdt$, and we have set $A = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 - K^2$, therefore

$$\Lambda' = -\frac{2Ac}{\left( \int c \right)^3} - \frac{8\pi \rho_0 G}{c^4} \left[ \frac{G'}{G} - \frac{4\epsilon}{c} - \frac{\gamma \epsilon}{\int c} \right].$$

Now, taking into account eq. (57), we get that

$$\Lambda' = -\frac{2Ac}{\left( \int c \right)^3} + \frac{8\pi \rho_0 G}{c^4} \left[ \frac{G'}{G} - \frac{4\epsilon}{c} - \frac{\gamma \epsilon}{\int c} \right] +$$

$$+ \frac{G'}{G} - \frac{4\epsilon}{c} = 0,$$

and hence we obtain

$$G = \frac{A}{4\pi \rho_0 \gamma} c^4 \left( \int c \right)^{-2}\gamma^2,$$

and in this way we find that the cosmological constant behaves as

$$\Lambda = A \left( 1 - \frac{2}{\gamma} \right) \left( \int c \right)^{-2}.$$
As we can see, from eqs. (68) and (69), we have that are verified the following relationships

\[ \frac{G \rho}{c^4} \approx \left( \int c \right)^{-2}, \quad \Lambda \left( \int c \right)^2 = \text{const.} \quad (70) \]

Now, we will try to find the value of the constants \((a_i)_{i=1}^3\). Taking into account the field eqs. (52-55) we find that, obviously eq. (52) vanish, but from eqs. (53, 55) we get

\[ a_2 (a_2 - 1) + a_3 (a_3 - 1) + a_2 a_3 - \frac{3K^2}{4} = A \left( \frac{\alpha - 2}{\alpha} \right), \quad (71) \]

\[ a_3 (a_3 - 1) + a_1 a_3 + a_1 (a_1 - 1) + \frac{K^2}{4} = A \left( \frac{\alpha - 2}{\alpha} \right), \quad (72) \]

\[ a_2 (a_2 - 1) + a_1 a_2 + a_1 (a_1 - 1) + \frac{K^2}{4} = A \left( \frac{\alpha - 2}{\alpha} \right), \quad (73) \]

finding therefore the same solution as in the above model, i.e.

\[ a_1 = 2a_2 - 1, \quad a_2 = a_2, \quad K^2 = -4a_2^2 + 6a_2 - 2, \quad (74) \]

this solution has only sense if \(a_2 \in (\frac{1}{2}, 1)\), note that for these values, \(a_1 > 0\).

We would like to emphasize that to out line the system (71,72) it is essential to take into account the effects of a c-var into the curvature tensor. If we do not consider such effects then the system depends on \(\int c\) and therefore it is not algebraic (see [28] for a detailed discussion).

Therefore the behaviour of the main quantities is the following one. The energy density behaves as

\[ \rho = \rho_0 \left( \int c(t) dt \right)^{-\gamma}, \quad (75) \]

where \(\gamma = (\omega + 1) \alpha\), and \(\alpha = 2a_2 + a_1\), while Hubble and the deceleration parameters behave as

\[ H = \alpha \frac{c}{\int c dt}, \quad q = \frac{1}{\alpha} \left( 1 - \frac{c'}{c} \frac{\int c}{c} \right) - 1, \quad (76) \]

and the shear yields

\[ \sigma^2 = \frac{1}{3} (a_2 - 1)^2 \left( \int c dt \right)^{-2}. \]

On the other hand, \(G\) behaves as, \(G = G_0 \omega^4 \left( \int c dt \right)^{-2}\), where \(G_0 > 0\). Hence, its behaviour will depend on the different values of \(\gamma\). If \(\gamma \in (0, 2)\) we find that \(G\) is a decreasing time function. If \(\gamma = 2\), \(G = \text{const.}\) and all the model collapses to the standard one studied in the above sections (\(G = \text{const.}, \omega = \text{const.}\) and \(\Lambda = 0\)). While if \(\gamma \in (2, 6)\) then \(G\) is a growing time function.

The cosmological constant behaves as

\[ \Lambda = A \left( 1 - \frac{2}{\gamma} \right) \left( \int c dt \right)^{-2} = \Lambda_0 \left( \int c dt \right)^{-2}, \quad (76) \]

where the sign of \(\Lambda_0\) depends on the different values of \(\gamma\);

\[ \Lambda_0 = \begin{cases} < 0 & \text{if } \gamma \in (0, 2) \quad (77) \end{cases} \]

\[ = 0 & \text{if } \gamma = 2 \quad (78) \]

\[ > 0 & \text{if } \gamma \in (2, 6) \quad (79) \]

Since our model is formally self-similar, then (77-79) have shown, that all the quantities must follow a power law, so, we may assume that for example, \(c\) takes the following form: \(c(t) = c_0 t^\epsilon\), with \(\epsilon \in \mathbb{R}\). Hence, the scale factors behaves as:

\[ a = a_0 t^\alpha (\epsilon + 1), \quad b = b_0 t^\beta (\epsilon + 1) = d, \quad (77) \]

in such a way that they behave as growing time function if \(\epsilon \in (-1, 0)\), \(a_i > 0\). This means that \(c\) may be a decreasing time function. The homothetic vector field collapses to this one

\[ V = \frac{t}{\epsilon + 1} \partial_t + (1 - a_2) (2x \partial_x + y \partial_y + z \partial_z) \quad (78) \]

The Hubble parameter behaves as: \(H = (4a_2 - 1)(\epsilon + 1)t^{-1} = \dot{a}t^{-1}\), so we find again the restriction for \(\epsilon, \epsilon \in (-1, \infty)\). In the same way, we find from eq. \(\Lambda = \Lambda_0 \left( \int c \right)^{-2}\), that, since \(\Lambda\) must be a decreasing time function, this is only possible if \(\epsilon \in (-1, \infty)\), the special case, \(\epsilon = -1\), is forbidden, note that \(\int c dt = \frac{\rho_0}{\epsilon + 1} t^{\epsilon + 1} > 0, \forall \epsilon \in (-1, \infty)\).

So, we find that

\[ \rho \approx t^{-\gamma(\epsilon + 1)}, \quad G \approx t^{\gamma(\epsilon + 1) + 2 \epsilon - 2}, \quad \Lambda \approx t^{-2(\epsilon + 1)}, \quad (79) \]

with \(\epsilon \in (-1, \infty)\), and \(\gamma = (\omega + 1) \alpha = (\omega + 1)(4a_2 - 1)\). Therefore the energy density is always a decreasing time function if \(\omega \in (-1, 1)\) and \(\epsilon \in (-1, \infty)\). As above, the behaviour of \(G\) depends on \((a_2, \omega, \epsilon)\), hence it may be a growing function as well as a decreasing one. With regard to the cosmological constant, the most important thing is to know its sign. Recalculating all the computations we find that \(\Lambda_0 = \tilde{A} c_0^2 \left( 1 - \frac{2}{\gamma} \right)\), where \(\tilde{\gamma} = \gamma (\epsilon + 1) = (\omega + 1)(4a_2 - 1)(\epsilon + 1)\), in such a way that this sign is affected by the perturbing parameter \((\epsilon + 1)\). We have set \(\tilde{A} = (5a_2^2 - 2a_2) (\epsilon + 1)^2 - \frac{K^2}{\omega^4}\), remember that, \(a_1 = 2a_2 - 1\).

To clarify this results we may fix \(\omega = 1, a_2 = 2/3\) and \(\epsilon = \pm 1/2\), finding

| \(\epsilon\) | \(\rho\) | \(G\) | \(\Lambda_0\) |
|---|---|---|---|
| -1/2 | \(t^{-5/3}\) | \(t^{-4/3}\) | < 0 |
| 1/2 | \(t^{-5}\) | \(t^4\) | > 0 |
Hence, if we take into account the current observation, then we must rule out those values which make $\Lambda_0 < 0$. Therefore we conclude that $c$ and $G$ are growing time functions while $\Lambda$ is a positive time decreasing time function.

With regard to the deceleration parameter we find that it has the following behavior:

$$q = -1 + \frac{1}{\tilde{\alpha}},$$

(80)

where $\tilde{\alpha} = (4a_2 - 1) (\epsilon + 1)$, indicating as that it is quite unlikely that $\epsilon \to -1$. We find that $q < 0$, if $\tilde{\alpha} > 1$.

VII. CONCLUSIONS.

We have studied several perfect fluid Bianchi II models under the self-similarity hypothesis. We have started our study reviewing the classical solution and showing that it belongs to the LRS BII kind by calculating the fourth Killing vector. The self-similar solution brings us to get a power law solution for the scale factors i.e. they behave as $a = a_0 t^{a_1}, b = b_0 t^{a_2} = d$, in such a way the the constants $a_0$ must satisfy the relation $2a_2 - a_1 = 1$. The only drawback is that this solution is only valid for an equation of state $\omega \in (-1/3, 1)$. We have also shown that this solution fits perfectly with the current observations for $H$ and $q$. With regard to its curvature behaviour we may conclude that the model is singular. The Weyl parameter is quite small and the gravitational entropy is constant as it is expected for a self-similar space-time. This happens because the definition of gravitational entropy does not work in this kind of space-times. We have finished our study of the classical model showing that there is not solution for the vacuum model.

With regard to the perfect fluid model with $G$ and $\Lambda$ variable we have arrived to the conclusion that its solution is quite similar to the obtained one for the classical model. But we have been able to enlarge the set of values for the equation of state, in this case is $\omega \in (-1, \infty)$. As we have mentioned above the solution is similar to the classical model so it also belongs to the LRS BII kind. With regard to the behaviour of $G$ and $\Lambda$ we have showed that $G$ is a growing time function while $\Lambda$ is a positive decreasing time function. We have ruled out the rest of the solution taking into account the recent observation which suggest us that $\Lambda_0 > 0$ and $q < 0$.

In the VSL model we have arrived to similar conclusions as in the above model i.e. the solution is valid for $\omega \in (-1, \infty)$. We have started outlining the FE but taking into account the effects of a $c-$var into the curvature tensor. In this approach is essential to take into account such effects, in other way, it is impossible to get, after substitution, and algebraic system of equations. We have also calculated the homothetic vector field for the new metric as well as the constrains for the scale factors. In this case all the quantities depend on $\int c(t)dt$. By solving the associated algebraic system we have obtained the same solution as in the above studied case. Hence this solution belongs to the LRS BII type. In a generic way we have discussed the behaviour of the main quantities ruling out solutions like $\Lambda_0 \leq 0$. If we assume a power law for $c(t)$, $c = c' \times t$, then we arrive to the conclusion that it may be a decreasing time function. The only restriction is $\int c(t)dt$ must be a growing time function, hence $c$ must be integrable but it is allowed that it may be decreasing. But if we consider the possibility of a $c$ decreasing then $\Lambda_0 < 0$. Therefore we have concluded that $G$ and $c$ are growing time functions while $\Lambda$ is a positive decreasing time function.

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