POLYNOMIAL REPRESENTATIVES OF
SCHUBERT CLASSES IN $QH^*(G/B)$

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Abstract. We show how the quantum Chevalley formula for $G/B$, as stated by Peterson and proved rigorously by Fulton and Woodward, combined with ideas of Fomin, S. Gelfand and Postnikov, leads to a formula which describes polynomial representatives of the Schubert cohomology classes in the canonical presentation of $QH^*(G/B)$ in terms of generators and relations. We generalize in this way results of [FGP].

§1 Introduction

A theorem of Borel [B] describes the cohomology$^1$ ring of the generalized complex flag manifold $G/B$ as the “co-invariant algebra” of the Weyl group of $G$, which is essentially a quotient of a certain polynomial ring. The Schubert cohomology classes (i.e. Poincaré duals of Schubert varieties) are a basis of $H^*(G/B)$. In order to determine the structure constants of the cup-multiplication on $H^*(G/B)$ with respect to this basis, we need to describe the Schubert cohomology classes in Borel’s presentation. According to Bernstein, I. M. Gelfand and S. I. Gelfand [BGG], we obtain polynomial representatives of Schubert classes in Borel’s ring by starting with a representative of the top cohomology and then applying successively divided difference operators associated to the simple roots of $G$. More details concerning the Bernstein-Gelfand-Gelfand construction can be found in section 2 of our paper.

When dealing with the (small) quantum cohomology ring $QH^*(G/B)$ we face a similar situation. There exists a canonical presentation of that ring, again as a quotient of a polynomial algebra, where the variables are the same as in the classical case, plus the “quantum variables” $q_1, \ldots, q_l$. As about the ideal of relations, it is generated by the “quantum deformations” of the relations from Borel’s presentation of $H^*(G/B)$ (for more details, see section 3). The Schubert classes are a basis of $QH^*(G/B)$ as a $\mathbb{R}[q_1, \ldots, q_l]$-module. A natural aim (see the next paragraph) is to describe them in the previous

$^1$The coefficient ring for cohomology will always be $\mathbb{R}$. 
presentation of $QH^*(G/B)$. Our main result gives a method for obtaining such polynomial representatives. It can be described briefly as follows: we start with an arbitrary polynomial representing the Schubert class $\sigma_w$ in Borel’s description (e.g. by using the B-G-G construction); this is transformed into a polynomial representing $\sigma_w$ in the canonical description of $QH^*(G/B)$ after successive applications of divided difference operators, multiplications by $q_j$’s and integer numbers and additions. The precise formula is stated in Theorem 3.6 (also see Lemma 3.4 and relation (5) in order to understand the notations). For $G = SL(n, \mathbb{C})$, the same result was proved by Fomin, S. Gelfand and Postnikov [FGP].

The main ingredient of our proof is a result of D. Peterson [P] (we call it the “quantum Chevalley formula”, since Chevalley obtained a similar result for the cup-multiplication on $H^*(G/B)$) which describes the quantum multiplication by degree 2 Schubert classes.

Finally, a few words should be said about the importance of our result. The standard presentation of $QH^*(G/B)$ mentioned above is explicitly determined by Kim [K] (see also [M]). Our description could be relevant for finding the structure constants of the quantum multiplication with respect to the basis consisting of Schubert classes, which would lead immediately to the Gromov-Witten invariants of $G/B$. The efficiency of this strategy depends very much on the input: we dispose of the choice of polynomial representatives of Schubert classes in Borel’s ring and this has to be made judiciously (see again [FGP], as well as Billey and Haiman [BH] and Fomin and Kirillov [FK]).

§2 The Bernstein-Gelfand-Gelfand construction

The main object of study of this paper is the generalized complex flag manifold $G/B$, where $G$ is a connected, simply connected, semisimple, complex Lie group and $B \subset G$ a Borel subgroup. Let $t$ be the Lie algebra of a maximal torus of a compact real form of $G$ and $\Phi \subset t^*$ the corresponding set of roots. The negative of the Killing form restricted to $t$ gives an inner product $\langle , \rangle$. To any root $\alpha$ corresponds the coroot

$$\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

which is an element of $t$, by using the identification of $t$ and $t^*$ induced by $\langle , \rangle$. If $\{\alpha_1, \ldots, \alpha_l\}$ is a system of simple roots then $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ is a system of simple coroots. Consider $\{\lambda_1, \ldots, \lambda_l\} \subset t^*$ the corresponding system of fundamental weights, which are defined by $\lambda_i(\alpha_j^\vee) = \delta_{ij}$. To any positive root $\alpha$ we assign the reflection $s_\alpha$ of $(t, \langle , \rangle)$ about the hyperplane $\ker \alpha$. The Weyl group $W$ is generated by all reflections $s_\alpha, \alpha \in \Phi^+$: it is actually generated by a smaller set, namely by the simple reflections $s_1 = s_{\alpha_1}, \ldots, s_l = s_{\alpha_l}$. To any $w \in W$ corresponds a length, $l(w)$, which is the smallest number of factors in a decomposition of $w$ as a product of simple reflections.

There are two different ways to describe $H^*(G/B)$: On the one hand, we can take $B^- \subset G$ the Borel subgroup opposite to $B$ and assign to each $w \in W$ the Schubert variety
\( \tilde{C}_w = B_{w}, \) which has real codimension \( 2l(w) \); its Poincaré dual \( \sigma_w \) is an element of \( H^{2l(w)}(G/B) \); the set \( \sigma_w, w \in W \) is a basis of \( H^*(G/B) \). On the other hand, let us consider the symmetric algebra \( S(t^*) \), which consists of polynomial functions on \( t \). A theorem of Borel says that the ring homomorphism \( S(t^*) \to H^*(G/B) \) induced by \( \lambda_i \mapsto \sigma_{s_i}, 1 \leq i \leq l \), is surjective; moreover it induces the ring isomorphism

\[
H^*(G/B) \simeq \mathbb{R}[\lambda]/I_W,
\]

where \( I_W \) is the ideal of \( S(t^*) = \mathbb{R}[\lambda_1, \ldots, \lambda_l] = \mathbb{R}[\lambda] \) generated by the \( W \)-invariant polynomials of strictly positive degree.

One is looking for a Giambelli type formula, which connects these two descriptions by assigning to each Schubert cycle \( \sigma_w \) a polynomial representative in the quotient ring \( \mathbb{R}[\lambda]/I_W \). We are going to sketch the construction of such polynomials, as performed by Bernstein, I. M. Gelfand and S. I. Gelfand in [BGG]. It relies on the following facts:

- \( H^*(G/B) \) and \( \mathbb{R}[\lambda]/I_W \) are generated as rings by \( \sigma_{s_i} \), respectively \( \lambda_i, 1 \leq i \leq l \),
- we have a formula of Chevalley which gives the matrix of the cup multiplication by \( \sigma_{s_i} \) on \( H^*(G/B) \) with respect to the basis \( \{\sigma_w : w \in W\} \),
- there is another, “very similar”, formula, which involves the divided difference operators \( \Delta_w, w \in W \) (see below) on the polynomial ring \( \mathbb{R}[\lambda] \).

The following result was proved by Chevalley [Ch] (see also Fulton and Woodward [FW]).

**Lemma 2.1.** (Chevalley’s formula). For any \( 1 \leq i \leq l \) and any \( w \in W \) we have

\[
\sigma_{s_i} \sigma_w = \sum_{\alpha \in \Phi^+, l(ws_\alpha) = l(w) + 1} \lambda_i(\alpha^\vee) \sigma_{ws_\alpha}.
\]

To each positive root \( \alpha \) we assign the **divided difference operator** \( \Delta_\alpha \) on the ring \( \mathbb{R}[\lambda] \) (the latter being just the symmetric ring \( S(t^*) \), it admits a natural action of the Weyl group \( W \)):

\[
\Delta_\alpha(f) = \frac{f - s_\alpha f}{\alpha}
\]

If \( w \) is an arbitrary element of \( W \), take \( w = s_{i_1} \ldots s_{i_k} \) a reduced expression and then set

\[
\Delta_w = \Delta_{\alpha_{i_1}} \circ \cdots \circ \Delta_{\alpha_{i_k}}.
\]

One can show (see for instance [Hi]) that the definition does not depend on the choice of the reduced expression. The operators obtained in this way have the following property:

\[
\Delta_w \circ \Delta_{w'} = \begin{cases} 
\Delta_{ww'}, & \text{if } l(ww') = l(w) + l(w') \\
0, & \text{otherwise.}
\end{cases}
\]

The importance of those operators for our present context is revealed by the similarity of the following formula with Lemma 2.1:
Lemma 2.2. (Hiller [Hi]) If $\lambda_i^*$ denotes the operator of multiplication by $\lambda_i$ on $\mathbb{R}[\lambda]$, then for any $w \in W$ we have

$$\Delta_w \lambda_i^* - w \lambda_i^* w^{-1} \Delta_w = \sum_{\beta \in \Phi^+, l(ws) = l(w) - 1} \lambda_i(\beta^\vee) \Delta_{ws}. $$

Let $w_0$ be the longest element of $W$. The polynomial

$$c_{w_0} := \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha$$

is homogeneous, of degree $l(w_0)$ and has the property that $\Delta_{w_0} c_{w_0} = 1$. But $l(w_0)$ is at the same time the complex dimension of $G/B$, and it can be easily shown that the class of $c_{w_0}$ in $\mathbb{R}[\lambda]/I_W$ generates the top cohomology of $G/B$. To any $w \in W$ we assign $c_w := \Delta_{w^{-1}w_0} c_{w_0}$ which is a homogeneous polynomial of degree $l(w)$ satisfying

$$\Delta_v c_w = \begin{cases} c_{wv^{-1}}, & \text{if } l(wv^{-1}) = l(w) - l(v) \\ 0, & \text{otherwise} \end{cases}$$

for any $v \in W$ (see (2)). In particular, if $l(v) = l(w)$, then $\Delta_v (c_w) = \delta_{vw}$. Since $\Delta_w$ leaves $I_W$ invariant, it induces an operator on $\mathbb{R}[\lambda]/I_W$ which also satisfies $\Delta_v ([c_w]) = \delta_{vw}$, provided that $l(v) = l(w)$. Because $\dim \mathbb{R}[\lambda]/I_W = |W|$, it follows that the classes $[c_w]$, $w \in W$, are a basis of $\mathbb{R}[\lambda]/I_W$. We can easily determine any of the coefficients $a_v$ from

$$\lambda_i [c_w] = \sum_{l(v) = l(w) + 1} a_v [c_v],$$

by applying $\Delta_v$ on both sides and using Lemma 2.2. It follows that

$$(3) \quad \lambda_i [c_w] = \sum_{\alpha \in \Phi^+, l(ws) = l(w) + 1} \lambda_i (\alpha^\vee) [c_{ws}],$$

From $\Delta_{s_i} (\lambda_j) = \delta_{ij}$, $1 \leq i, j \leq l$, we deduce that $c_{s_i} = \lambda_i$. We just have to compare (3) with Lemma 2.1 to conclude:

Theorem 2.3. (Bernstein, I. M. Gelfand and S. I. Gelfand [BGG]) Let $[c_{w_0}]$ be the image of $\sigma_{w_0}$ by the identification $H^*(G/B) = \mathbb{R}[\lambda]/I_W$ indicated above. Then the map $\sigma_w \mapsto [c_w] := \Delta_{w^{-1}w_0} [c_{w_0}]$ is a ring isomorphism.

The polynomial $c_w = \Delta_{w^{-1}w_0} c_{w_0}$ being a representative of the Schubert cycle $\sigma_w$ in $\mathbb{R}[\lambda]/I_W$, is a solution of the classical (i.e. non-quantum) Giambelli problem for $G/B$. 

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§3 Quantization map

Additively, the quantum cohomology \( QH^*(G/B) \) of \( G/B \) is just \( H^*(G/B) \otimes \mathbb{R}[q_1, \ldots, q_l] \), where \( l \) is the rank of \( G \) and \( q_1, \ldots, q_l \) are some variables. The multiplication \( \circ \) is uniquely determined by \( \mathbb{R}[q] \)-linearity and the general formula

\[
\sigma_u \circ \sigma_v = \sum_{d=(d_1, \ldots, d_l) \geq 0} q^d \sum_{w \in W} \langle \sigma_u | \sigma_v | \sigma_{w_0} \rangle d \sigma_w,
\]

\( u, v \in W \), where \( q^d \) denotes \( q_1^{d_1} \ldots q_l^{d_l} \). The coefficient \( \langle \sigma_u | \sigma_v | \sigma_{w_0} \rangle d \) is the Gromov-Witten invariant, which counts the number of holomorphic curves \( \varphi : \mathbb{C}P^1 \to G/B \) such that \( \varphi_*([\mathbb{C}P^1]) = d \) in \( H_2(G/B) \) and \( \varphi(0) \), \( \varphi(1) \) and \( \varphi(\infty) \) are in general translates of the Schubert varieties dual to \( \sigma_u \), \( \sigma_v \), respectively \( \sigma_{w_0} \). It turns out that this number can be nonzero and finite only if \( l(u) + l(v) = l(w) + 2 \sum_{i=1}^l d_i \); if it is infinity, we set \( \langle \sigma_u | \sigma_v | \sigma_{w_0} \rangle d = 0 \). The ring \( (QH^*(G/B), \circ) \) is commutative and associative (for more details about quantum cohomology we refer the reader to Fulton and Pandharipande [FP]).

One can show that the quantum cohomology ring of \( G/B \) is generated by \( H^2(G/B) \otimes \mathbb{R}[q_1, \ldots, q_l] \), i.e. by \( q_1, \ldots, q_l, \lambda_1, \ldots, \lambda_l \). To determine the ideal of relations, we only have to take any of the fundamental \( W \)-invariant polynomials \( u_i, 1 \leq i \leq l \) as generators of the ideal \( I_W \) of relations in \( H^*(G/B) \) — and find its “quantum deformation” \( R_i \). The latter is a polynomial in \( \mathbb{R}[q, \lambda] \), uniquely determined by:

(a) the relation \( R_i(q_1, \ldots, q_l, \sigma_{s_i} \circ, \ldots, \sigma_{s_i} \circ) = 0 \) holds in \( QH^*(G/B) \),

(b) the component of \( R_i \) free of \( q \) is \( u_i \).

If \( I_W^a \) denotes the ideal of \( \mathbb{R}[q, \lambda] \) generated by \( R_1, \ldots, R_l \), then we have the ring isomorphism

\[ QH^*(G/B) \simeq \mathbb{R}[q, \lambda]/I_W^a. \]

The challenge is now to solve the “quantum Giambelli problem”: via the isomorphism (4), find a polynomial representative in \( \mathbb{R}[q, \lambda]/I_W^a \) for each Schubert class \( \sigma_w, w \in W \). We can actually use Theorem 2.3 in order to rephrase the problem as follows: Describe (the image of \([c_w]\) via) the map

\[
\mathbb{R}[q, \lambda]/(I_W \otimes \mathbb{R}[q]) = \mathbb{R}[\lambda]/I_W \otimes \mathbb{R}[q] \xrightarrow{\sim} H^*(G/B) \otimes \mathbb{R}[q] = QH^*(G/B) \xrightarrow{\cong} \mathbb{R}[q, \lambda]/I_W^a.
\]

Note that the latter is an isomorphism of \( \mathbb{R}[q] \)-modules, but not of algebras; following [FGP], we call it the quantization map. So the main goal of our paper is to give a presentation of the quantization map. For \( G = SL(n, \mathbb{C}) \), the problem has been solved by Fomin, Gelfand and Postnikov [FGP]. We are going to extend their result to an arbitrary semisimple Lie group \( G \).
As in the non-quantum case, we will essentially rely on the Chevalley formula, this time in its quantum version: the formula was obtained by D. Peterson in [P] (for more details, see section 10 of Fulton and Woodward [FW]). If $\alpha^\vee$ is a positive coroot, we consider its height

$$|\alpha^\vee| = m_1 + \ldots + m_l,$$

where the positive integers $m_1, \ldots, m_l$ are given by $\alpha^\vee = m_1\alpha_1^\vee + \ldots + m_l\alpha_l^\vee$. We also put

$$q^\alpha = q_1^{m_1} \ldots q_l^{m_l}.$$

**Theorem 3.1.** (Quantum Chevalley Formula; Peterson [P], Fulton and Woodward [FW])

In $(QH^*(G/B), \circ)$ one has

$$\sigma_{s_i} \circ \sigma_w = \sigma_{s_i} \sigma_w + \sum_{l(ws_\alpha) = l(w) - 2|\alpha^\vee| + 1} \lambda_i(\alpha^\vee)q^\alpha \sigma_{ws_\alpha}.$$

The following inequality can be found in Peterson’s notes [P], as well as in Brenti, Fomin and Postnikov [BFP]. For the sake of completeness, we will give our own proof of it.

**Lemma 3.2.** For any positive root $\alpha$ we have $l(s_\alpha) \leq 2|\alpha^\vee| - 1$.

**Proof.** We prove the lemma by induction on $l(s_\alpha)$. If $l(s_\alpha) = 1$, then $\alpha$, as well as $\alpha^\vee$, is simple, so $|\alpha^\vee| = 1$. Let now $\alpha$ be a positive, non-simple root. There exists a simple root $\beta$ such that $\alpha(\beta^\vee) > 0$ (otherwise we would be led to $\alpha(\alpha^\vee) \leq 0$). Consequently, $\beta(\alpha^\vee)$ is a strictly positive number, too, hence

$$s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$$

must be a negative root. Also

$$s_\beta s_\alpha(\beta) = (\alpha(\beta^\vee)\beta(\alpha^\vee) - 1)\beta - \beta(\alpha^\vee)\alpha$$

is a negative root. By Lemma 3.3, chapter 1 of [Hi], we have $l(s_\beta s_\alpha s_\beta) = l(s_\alpha) - 2$. Because

$$s_\beta(\alpha)^\vee = s_\beta(\alpha^\vee) = \alpha^\vee - \beta(\alpha^\vee)\beta^\vee,$$

we have $|s_\beta(\alpha)^\vee| = |\alpha^\vee| - \beta(\alpha^\vee)$. By the induction hypothesis we conclude:

$$l(s_\alpha) = l(s_\beta s_\alpha s_\beta) + 2 \leq 2|s_\beta(\alpha)^\vee| - 1 + 2 = 2|\alpha^\vee| - 1 + 2(1 - \beta(\alpha^\vee)) \leq 2|\alpha^\vee| - 1.$$
Denote by $\tilde{\Phi}^+$ the set of all positive roots $\alpha$ with the property $l(s_\alpha) = 2|\alpha^\vee| - 1$. The following operators

\[ \Lambda_i = \lambda_i + \sum_{\alpha \in \tilde{\Phi}^+} \lambda_i(\alpha^\vee)q^{\alpha^\vee} \Delta_{s_\alpha} \]

on $\mathbb{R}[q, \lambda]$, $1 \leq i \leq l$ have been considered by Peterson in [P]. His key observation is that we have

\[ \Lambda_i[c_w] = \lambda_i[c_w] + \sum_{l(ws_\alpha)=l(w)-2|\alpha^\vee|+1} \lambda_i(\alpha^\vee)q^{\alpha^\vee}[c_{ws_\alpha}], \]

the right hand side being, by the quantum Chevalley formula, just $\lambda_i \circ [c_w]$. In order to justify (6), we only have to say that if $w \in W$ and $\alpha$ is a positive root with $l(ws_\alpha) = l(w) - 2|\alpha^\vee| + 1$, then, by Lemma 3.2, $\alpha$ must be in $\tilde{\Phi}^+$.

From the associativity of the quantum product $\circ$ it follows that any two $\Lambda_i$ and $\Lambda_j$ commute as operators on $(\mathbb{R}[\lambda]/I_W) \otimes \mathbb{R}[q]$. In fact the following stronger result (also stated by Peterson in [P]) holds:

**Lemma 3.3.** The operators $\Lambda_1, \ldots, \Lambda_l$ on $\mathbb{R}[q, \lambda]$ commute.

**Proof.** Put $w = s_\alpha$ in Lemma 2.2 and obtain:

\[ \Delta_{s_\alpha}\lambda^*_i = (\lambda^*_i - \lambda_i(\alpha^\vee)\alpha^*)\Delta_{s_\alpha} + \sum_{\gamma \in \tilde{\Phi}^+, l(s_\alpha s_\gamma)=l(s_\alpha)-1} \lambda_i(\gamma^\vee)\Delta_{s_\alpha s_\gamma}. \]

It follows

\[ \Lambda_j \Lambda_i = (\lambda_j \lambda_i)^* + \sum_{\alpha \in \tilde{\Phi}^+} \lambda_j(\alpha^\vee)q^{\alpha^\vee} \lambda^*_i \Delta_{s_\alpha} \]
\[ + \sum_{\alpha \in \tilde{\Phi}^+} \lambda_j(\alpha^\vee)q^{\alpha^\vee} \lambda_i^* \Delta_{s_\alpha} - \sum_{\alpha \in \tilde{\Phi}^+} \lambda_j(\alpha^\vee)\lambda_i(\alpha^\vee)q^{\alpha^\vee} \alpha^* \Delta_{s_\alpha} \]
\[ + \sum_{\alpha \in \tilde{\Phi}^+, \gamma \in \tilde{\Phi}^+, l(s_\alpha s_\gamma)=l(s_\alpha)-1} \lambda_j(\alpha^\vee)\lambda_i(\gamma^\vee)q^{\alpha^\vee} \Delta_{s_\alpha s_\gamma} \]
\[ + \sum_{\alpha, \beta \in \tilde{\Phi}^+, l(s_\alpha s_\beta)=l(s_\alpha)+l(s_\beta)} \lambda_j(\alpha^\vee)\lambda_i(\beta^\vee)q^{\alpha^\vee+\beta^\vee} \Delta_{s_\alpha s_\beta}. \]

Denote by $\Sigma_{ij}$ the sum of the last two sums: the rest is obviously invariant by interchanging $i \leftrightarrow j$. 

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Let us return to the Bernstein-Gelfand-Gelfand construction described in the first section: Fix $c_{\omega_0} \in \mathbb{R}[\lambda]$ such that $[c_{\omega_0}] = \sigma_{\omega_0}$ and then set $c_w = \Delta_{\omega^{-1} \omega} c_{\omega_0}$, $w \in W$; their classes modulo $I_W$ are a basis of $\mathbb{R}[\lambda]/I_W$. As we said earlier, from the associativity of the quantum product we deduce that $\Lambda_j \Lambda_i [c_w]$ is symmetric in $i$ and $j$, for any $w \in W$. In particular, $\Sigma_{ij} [c_{\omega_0}]$ is symmetric in $i$ and $j$. Because $l(w_0 v) = l(w_0) - l(v)$ for any $v \in W$, we have

$$\Sigma_{ij} [c_{\omega_0}] = \sum_{\alpha \in \tilde{\Phi}^+, l(s_{\alpha}s_\gamma) = l(s_\alpha) - 1} \lambda_j (\alpha^\vee) \lambda_i (\gamma^\vee) q^{\alpha^\vee} [c_{\omega_0} s_\gamma s_\alpha] + \sum_{\alpha, \beta \in \tilde{\Phi}^+, l(s_\alpha s_\beta) = l(s_\alpha) + l(s_\beta)} \lambda_j (\alpha^\vee) \lambda_i (\beta^\vee) q^{\alpha^\vee + \beta^\vee} [c_{\omega_0} s_\beta s_\alpha].$$

The latter reproduces exactly the expression of $\Sigma_{ij}$ itself: $\{[c_w] : w \in W\}$ (actually $\{[c_{\omega_0} w^{-1}] : w \in W\}$) are linearly independent, exactly like the operators $\{\Delta_w : w \in W\}$. So $\Sigma_{ij}$ is symmetric in $i$ and $j$ and the lemma is proved. \(\blacksquare\)

The next result is a generalization of Lemma 5.3 of [FGP].

**Lemma 3.4.** The map $\psi : \mathbb{R}[q, \lambda] \to \mathbb{R}[q, \lambda]$ given by

$$f \mapsto f(\Lambda_1, \ldots, \Lambda_l)(1)$$

is an $\mathbb{R}[q]$-linear isomorphism. If $f \in \mathbb{R}[q, \lambda]$ has degree $d$ with respect to $\lambda_1, \ldots, \lambda_l$, then we can express $\psi^{-1}(f)$ as follows

$$\psi^{-1}(f) = \frac{I - (I - \psi)^d}{\psi}(f) = (\binom{d}{1}) f - \binom{d}{2} \psi(f) + \ldots + (-1)^{d-2} \binom{d}{d-1} \psi^{d-2}(f) + (-1)^{d-1} \psi^{d-1}(f) + (-1)^d f,$$

where $\binom{d}{1}, \ldots, \binom{d}{d-1}$ are the binomial coefficients.

**Proof.** The degrees of elements of $\mathbb{R}[q, \lambda]$ we are going to refer to here are taken only with respect to $\lambda_1, \ldots, \lambda_l$. First, $\psi$ is injective, because if $g \in \mathbb{R}[q, \lambda]$ has the property that $g(\Lambda_1, \ldots, \Lambda_l)(1) = 0$, then obviously $g$ must be 0. In order to prove both surjectivity and the formula for $\psi^{-1}$, we notice that the operator $I - \psi$ lowers the degree of a polynomial by at least one, so if $f$ is a polynomial of degree $d$, then $(I - \psi)^d(f) = 0$. \(\blacksquare\)

The next result is a direct consequence of the quantum Chevalley formula.
Proposition 3.5. For any of the generators $R_1, \ldots, R_l$ of the ideal $I_W^q$, $\psi(R_i)$ is an $\mathbb{R}[q]$-linear combination of elements of $I_W$, the free term with respect to $q_1, \ldots, q_l$ being $u_i$. Hence $\psi(I_W^q) = I_W \otimes \mathbb{R}[q]$ and $\psi$ gives rise to a bijection

$$\psi : \mathbb{R}[q, \lambda]/I_W^q \rightarrow \mathbb{R}[q, \lambda]/(I_W \otimes \mathbb{R}[q]).$$

Proof. We just have to use the fact that

$$\lambda_{i_1} \circ \ldots \circ \lambda_{i_k} = \Lambda_{i_1} \ldots \Lambda_{i_k}(1) \mod I_W \otimes \mathbb{R}[q]$$

so that

$$\psi(R_i) \mod I_W \otimes \mathbb{R}[q] = R_i(q_1, \ldots, q_l, \Lambda_1, \ldots, \Lambda_l)(1) \mod I_W \otimes \mathbb{R}[q]$$

$$= R_i(q_1, \ldots, q_l, \lambda_1 \circ, \ldots, \lambda_l \circ)$$

$$= 0.$$

Our polynomial representatives of Schubert classes in $QH^*(G/B)$ are described by the following theorem, which is the central result of the paper. The proof is governed by the same ideas that have been used in the non-quantum case (see section 2).

Theorem 3.6. The quantization map $\mathbb{R}[q, \lambda]/(I_W \otimes \mathbb{R}[q]) \rightarrow \mathbb{R}[q, \lambda]/I_W^q$ is just $\psi^{-1}$. More precisely, if $w \in W$ has length $l(w) = l$, then the class of $c_w$ in $\mathbb{R}[q, \lambda]/(I_W \otimes \mathbb{R}[q])$ is mapped to the class of

$$\frac{I - (I - \psi)^l}{\psi}(c_w) = \binom{l}{1} c_w - \binom{l}{2} \psi(c_w) + \ldots + (-1)^{l-2} \binom{l}{l-1} \psi^{l-2}(c_w) + (-1)^{l-1} \psi^{l-1}(c_w)$$

in $\mathbb{R}[q, \lambda]/I_W^q$, where $\psi$ has been defined in Lemma 3.4.

Proof. For any polynomial $f \in \mathbb{R}[q, \lambda]$, we denote by $[f], [f]_q$ its classes modulo $I_W \otimes \mathbb{R}[q]$, respectively modulo $I_W^q$. By the definition of $\psi$, the polynomial $\hat{c}_w := \psi^{-1}(c_w)$ is determined by

$$\hat{c}_w(\Lambda_1, \ldots, \Lambda_l)(1) = c_w.$$

We take into account (6), where $\Lambda_i[c_w]$ is the same as

$$[\Lambda_i(c_w)] = [\Lambda_i(\hat{c}_w(\Lambda_1, \ldots, \Lambda_l)(1))] = \psi([\Lambda_i \hat{c}_w]_q).$$

In view of Theorem 5.5 of [FGP], we could actually expect to have $\psi(R_i) = u_i$. 

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Because $[c_v] = \psi([\hat{c}_v]_q)$ for any $v \in W$ and the map $\psi$ is bijective, it follows that in $\mathbb{R}[q,\lambda]/I_W^q$ we have

\begin{equation}
\lambda_i[\hat{c}_w]_q = \sum_{l(ws_\alpha) = l(w)+1} \lambda_i(\alpha^\vee)[\hat{c}_{ws_\alpha}]_q + \sum_{l(ws_\alpha) = l(w)-2|\alpha^\vee|+1} \lambda_i(\alpha^\vee)q^{\alpha^\vee}[\hat{c}_{ws_\alpha}]_q.
\end{equation}

As $\mathbb{R}[q]$-algebras, both $QH^*(G/B)$ and $\mathbb{R}[q,\lambda]/I_W^q$ are generated by their degree 2 elements; this is why their structure is uniquely determined by the bases $\{\sigma_w : w \in W\}$, respectively $\{[\hat{c}_w] : w \in W\}$ and the matrices of multiplication by $\sigma_{s_i}$, respectively $\lambda_i$, $1 \leq i \leq l$. Since $\hat{c}_{s_i} = \lambda_i$, $1 \leq i \leq l$, it follows from Theorem 3.1 and relation (8) that the map

$$QH^*(G/B) \to \mathbb{R}[q,\lambda]/I_W^q$$

given by $\sigma_w \mapsto \hat{c}_w$, $w \in W$

is an isomorphism of algebras and the proof is finished. ■

**Example.** We will illustrate our main result by giving concrete solutions to the quantum Giambelli problem for $G/B$, where $G$ is simple of type $B_2$. This is the first interesting case, different from $A_n$ and for which $\tilde{\Phi}^+ \neq \Phi^+$. We will use the following presentation of the root system: if $x_1, x_2$ are an orthogonal coordinate system of the plane and $e_1, e_2$ the unit direction vectors of the coordinate axes, then

- the simple roots are $\alpha_1 := x_1$ and $\alpha_2 := x_2 - x_1$.
- the positive roots are $\alpha_1, \alpha_2, \alpha_3 := \alpha_1 + \alpha_2 = x_2$ and $\alpha_4 := 2\alpha_1 + \alpha_2 = x_1 + x_2$.
- the positive coroots are $\alpha_1^\vee = 2e_1$, $\alpha_2^\vee = e_2 - e_1$, $\alpha_3^\vee = 2e_2 = \alpha_1^\vee + 2\alpha_2^\vee$ and $\alpha_4^\vee = e_1 + e_2 = \alpha_1^\vee + \alpha_2^\vee$.
- the fundamental weights $\lambda_1, \lambda_2$ are determined by

\begin{align*}
x_1 &= 2\lambda_1 - \lambda_2 \\
x_2 &= \lambda_2
\end{align*}

- the simple reflections are $s_1 : (x_1, x_2) \mapsto (-x_1, x_2)$ and $s_2 : (x_1, x_2) \mapsto (x_2, x_1)$. The generators of $I_W$ are obviously $x_1^2 + x_2^2$ and $x_1^2x_2^2$.
- following [FK], we can obtain polynomial representatives of Schubert classes in $\mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2, x_1^2x_2^2)$ as indicated in the following table:
\[
\begin{array}{c|c}
 w & c_w \\
\hline
 w_0 = s_1s_2s_1s_2 & (x_1 - x_2)^3(x_1 + x_2)/16 \\
 s_2s_1s_2 & -x_2(x_1 - x_2)(x_1 + x_2)/4 \\
 s_1s_2s_1 & -(x_1 - x_2)^2(x_1 + x_2)/8 \\
 s_2s_1 & (x_1 + x_2)^2/4 \\
 s_1s_2 & -(x_1 - x_2)(x_1 + x_2)/4 \\
 s_2 & x_2 \\
 s_1 & (x_1 + x_2)/2 \\
\end{array}
\]

Note that we have started the B-G-G algorithm with \( c_{w_0} \) which differs from \( \alpha_1\alpha_2\alpha_3\alpha_4/8 \) by a multiple of \( x_1^2 + x_2^2 \).

Theorem 2.6 will allow us to describe the quantization map without knowing anything about the ideal \( I_W^q \) of quantum relations. But for the sake of completeness we will also obtain the two generators of \( I_W^q \), by using the theorem of Kim as presented in our paper \([M]\).

We have to consider the Hamiltonian system which consists of the standard 4-dimensional symplectic manifold \( (\mathbb{R}^4, dr_1 \wedge ds_1 + dr_2 \wedge ds_2) \) with the Hamiltonian function

\[
E(r, s) = \sum_{i,j=1}^2 \langle \alpha_i^\vee, \alpha_j^\vee \rangle r_i r_j + \sum_{i=1}^2 e^{-2s_i} = (2r_1 - r_2)^2 + r_2^2 + e^{-2s_1} + e^{-2s_2}.
\]

The first integrals of motion of the system are \( E \) and — by inspection — the function

\[
F(r, s) = (2r_1 - r_2)^2r_2^2 + r_2^2e^{-2s_1} - (2r_1 - r_2)r_2e^{-2s_2} + 2e^{-2s_1}e^{-2s_2} + \frac{1}{4}(e^{-2s_2})^2.
\]

By the main result of \([M]\), the quantum relations are obtained from \( E \), respectively \( F \), by the formal replacements:

\[
2r_1 - r_2 \mapsto x_1, r_2 \mapsto x_2 \\
e^{-2s_1} \mapsto -\langle \alpha_1^\vee, \alpha_1^\vee \rangle q_1 = -4q_1, e^{-2s_2} \mapsto -\langle \alpha_2^\vee, \alpha_2^\vee \rangle q_2 = -2q_2.
\]

In conclusion, \( I_W^q \) is the ideal of \( \mathbb{R}[q_1, q_2, x_1, x_2] \) generated by

\[
x_1^2 + x_2^2 - 4q_1 - 2q_2 = 0 \quad \text{and} \quad x_1^2x_2^2 - 4q_1x_2^2 + 2q_2x_1x_2 + 16q_1q_2 + q_2^2.
\]

Now, we will determine explicitly the image of each Schubert class \( \sigma_w, w \in W \) via the isomorphism

\[
QH^*(G/B) \simeq \mathbb{R}[q_1, q_2, x_1, x_2]/I_W^q.
\]
The place of the operators $\Lambda_1, \Lambda_2$ is taken by $X_1, X_2$ where

$$X_i = x_i + x_i(\alpha_i^\vee)q_1\Delta_{s_1} + x_i(\alpha_i^\vee)q_2\Delta_{s_2} + x_i(\alpha_i^\vee)q_1q_2\Delta_{s_1}\Delta_{s_2}\Delta_{s_1}, \quad i = 1, 2.$$  

More precisely, we have

$$X_1 = x_1 + 2q_1\Delta_{s_1} - q_2\Delta_{s_2} + q_1q_2\Delta_{s_1}\Delta_{s_2}\Delta_{s_1}$$

and

$$X_2 = x_2 + q_2\Delta_{s_2} + q_1q_2\Delta_{s_1}\Delta_{s_2}\Delta_{s_1}.$$  

Rather than using the formula for $\psi^{-1}$ given by (7), it seems more convenient to determine $\hat{c}_w := \psi^{-1}(c_w) \in \mathbb{R}[q_1, q_2, x_1, x_2]$ by the definition of $\psi$, i.e. from the condition

$$\hat{c}_w(X_1, X_2)(1) = c_w(x_1, x_2).$$

We will explain the details just for the case $w = w_0$, which is the most illustrative one. The polynomial we are looking for has the form $\hat{c}_{w_0} = c_{w_0} + q_1a_1 + q_2a_2 + b_1q_1^2 + b_2q_2^2 + b_3q_1q_2$, where $a_1, a_2$ are homogeneous polynomials of degree 2 in $x_1, x_2$ and $b_1, b_2, b_3$ are constant. The condition that determines $a_1, a_2, b_1, b_2, b_3$ is

$$c_{w_0}(X_1, X_2)(1) = c_{w_0}(1) + q_1a_1(1) + q_2a_2(1) + b_1q_1^2 + b_2q_2^2 + b_3q_1q_2$$

(9)

The first step is to compute $c_{w_0}(X_1, X_2)(1)$ and determine $a_1$ and $a_2$. Using

$$\Delta_{s_1}(f) = \frac{f(x_1, x_2) - f(-x_1, x_2)}{x_1} \quad \text{and} \quad \Delta_{s_2}(f) = \frac{f(x_1, x_2) - f(x_2, x_1)}{x_2 - x_1},$$

$f \in \mathbb{R}[x_1, x_2]$ we obtain

$$c_{w_0}(X_1, X_2)(1) = c_{w_0}(x_1, x_2) + \frac{1}{8}q_1(3x_1^2 - 4x_1x_2 + x_2^2) + \frac{1}{4}q_2(x_1^2 - x_2^2) + q_1^2 + q_1q_2.$$  

Since the coefficients of $q_1$, respectively $q_2$ in the left hand side of (9) must vanish, we deduce:

$$a_1 = -\frac{1}{8}(3x_1^2 - 4x_1x_2 + x_2^2), \quad a_2 = -\frac{1}{4}(x_1^2 - x_2^2).$$

The second step is to compute $a_1(X_1, X_2)(1)$ and $a_2(X_1, X_2)(1)$ and determine $b_1, b_2$ and $b_3$. We take into account that

$$X_1 - X_2 = x_1 - x_2 + 2q_1\Delta_{s_1} - 2q_2\Delta_{s_2}.$$  

and find

\[
a_1(X_1, X_2)(1) = -\frac{1}{8}(X_1 - X_2)(3x_1 - x_2) = a_1(x_1, x_2) - \frac{3}{2}q_1 - q_2
\]

\[
a_2(X_1, X_2)(1) = -\frac{1}{4}(X_1 - X_2)(x_1 + x_2) = a_2(x_1, x_2) - q_1.
\]

Coming back to (9), we deduce

\[
b_1 = \frac{1}{2}, \quad b_2 = 0, \quad b_3 = 1,
\]

hence

\[
\hat{c}_{w_0} = c_{w_0} - \frac{1}{8}q_1(3x_1^2 - 4x_1x_2 + x_2^2) - \frac{1}{4}q_2(x_1^2 - x_2^2) + \frac{1}{2}q^2 + q_1q_2.
\]

The other \(\hat{c}_w, w \in W\), can be obtained by similar computations. They are described in the following table:

| \(w\) | \(\hat{c}_w - c_w\) |
|---|---|
| \(s_2s_1s_2\) | \(q_1x_2\) |
| \(s_1s_2s_1\) | \(\frac{1}{2}(x_1 - x_2)q_1 + \frac{1}{2}(x_1 + x_2)q_2\) |
| \(s_2s_1\) | \(-q_1\) |
| \(s_1s_2\) | \(q_1\) |
| \(s_2\) | 0 |
| \(s_1\) | 0 |

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