Causality and Impulse Mode Freeness of an NDS with Descriptor Form Subsystems

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Abstract—A novel necessary and sufficient condition is derived in this paper for the causality/impulse mode freeness of a descriptor system. From this result, a matrix rank based necessary and sufficient condition is established for the causality/impulse mode freeness of a networked dynamic system (NDS) that is constituted from several descriptor subsystems, in which the associated matrix depends affinely on subsystem connections. A prominent property of this condition is that all the involved numerical computations are performed independently on each individual subsystem, which is quite attractive in reducing computation costs and improving numerical stability for large scale NDS analysis and synthesis. Situations have also been clarified in which NDS causality/impulse mode freeness is independent of subsystem connections.

Index Terms—causality, descriptor system, large scale system, networked dynamic system, singular system.

I. INTRODUCTION

In describing plant dynamics, descriptor systems, which are sometimes also called singular systems, have been extensively recognized as an appropriate model. Compared with the well adopted state space model, a descriptor system is believed to be more efficient in keeping structural information of plant dynamics, describing system evolutions with its natural variables, etc. These properties are quite important in analyzing influences of a system parameter on its performances, as well as in understanding responses of its natural variables to external stimulus, etc. [1], [3], [8], [12], [15]. Similar to the state space model, this model has also been attracting extensive research attentions for a long time, and has been frequently utilized in various fields. Some examples include economy, engineering, biology, etc.

When a descriptor model is adopted in system analysis and synthesis, various particular issues arise. Among them, an essential one is its causality when it is in a discrete time form, which requires that system states and outputs do not depend on its future inputs. A closely related issue in a continuous time descriptor model is that there do not exist impulse terms in the response of its states and outputs to external stimulus [3], [12]. Obviously, requirements on system future inputs are in general not reasonable. Similarly, existence of impulse modes in a system is usually not greatly appreciated also, as it may significantly deteriorate system performances, and even destroy a system. To avoid occurrence of these undesirable phenomena, significant efforts have been devoted from many researchers and various results have been obtained. For example, some matrix rank based necessary and sufficient conditions are derived in [2] for the causality of a discrete time descriptor system, as well as for the non-existence of impulse modes in a continuous time descriptor system, using the concept of restricted system equivalence. On the basis of graph theory, some necessary conditions, as well as some sufficient conditions, are obtained in [12] for verifying generic causality of a discrete time descriptor system. With the help of the Kronecker canonical form of a matrix pencil, some necessary and sufficient conditions are given in [2] for the causal observability of a continuous time descriptor system, which are also based on matrix ranks. And so on.

While one of the motivations for introducing a descriptor model is to investigate large scale interconnected systems [9], [8], all the above conditions are based on a lumped model, which may not be very suitable when the dimension of the system state vector is large, and/or the system is composed from a large number of subsystems, noting that rank verification for a high dimensional matrix is usually computationally very intensive and numerically quite sensitive [3], [4], [6].

To overcome these difficulties, a novel necessary and sufficient condition is derived in this paper for the causality of a discrete time descriptor system, which is also valid for the non-existence of an impulse mode in a continuous time descriptor system. From this result, a matrix rank based necessary and sufficient condition is established for the causality of a networked dynamic system (NDS) composed of several discrete time descriptor subsystems, as well as for the non-existence of impulse modes in an NDS constituted from several continuous time descriptor subsystems. In this condition, the associated matrix depends affinely on subsystem connections. A prominent property of this condition is that all the involved numerical computations are performed independently on each individual subsystem. This makes the condition efficient in reducing computation costs and improving numerical stability, scalable for large scale NDS analysis and synthesis, as well as helpful in subsystem parameter selections and NDS topology designs. From this condition, situations have also become clear in which the causality/impulse mode freeness of an NDS is completely and independently determined by its subsystem dynamics, which means that no matter how its subsystem connections are changed, the NDS remains causal/free from impulse modes.

The outline of the remaining of this paper is as follows. At first, in Section II, a descriptor form like model is given for subsystems of an NDS, together with some preliminary results. NDS causality is studied in Section III, together with its non-existence of impulse modes. Some concluding remarks are given in Section IV in which several further issues are discussed. Finally, an appendix is included to give proofs of some technical results.

The following notation and symbols are adopted. $\mathbb{R}^{m\times n}$
and $\mathbb{R}^n$ stand respectively for the set of $m \times n$ dimensional real matrices and the $n$ dimensional real Euclidean space. $(\cdot)^\dagger$ represents the Moore-Penrose inverse of a matrix, $\det(\cdot)$ the determinant of a square matrix, $\text{rank}(\cdot)$ the rank of a matrix, $\text{null}(\cdot)$ the (right) null space of a matrix, $\text{span}(\cdot)$ the space spanned by the columns of a matrix, while $^T$ the matrix whose columns form a base of the (right) null space of a matrix.

$\text{diag}\{X_{i,i}\}_{i=1}^n$ denotes a block diagonal matrix with its $i$-th diagonal block being $X_i$, while $\text{col}\{X_{i,i}\}_{i=1}^n$ the vector/matrix stacked by $X_{i,i}$ with its $i$-th row block/matrix/vector being $X_i$, $I_n$, $0_m$ and $0_{m \times n}$ represent respectively the $m$ dimensional identity matrix, the $m$ dimensional zero column vector and the $m \times n$ dimensional zero matrix. The subscript is usually omitted if it does not lead to confusions. The superscript $T$ is used to denote the transpose of a matrix/vector.

II. SYSTEM DESCRIPTION AND SOME PRELIMINARIES

In various actual engineering/biological/economic problems, a plant is usually constituted from several subsystems that may have distinctive dynamics. A plant with this characteristic is usually called NDS, and a promising method to describe its dynamics is to represent the dynamics of its subsystems with an ordinary model, divide their inputs/outputs into two different classes, which are respectively called external and internal inputs/outputs. With these classifications, subsystem interactions are expressed through transmitting an internal output of one subsystem to some other subsystems as one of their internal inputs.

When subsystem dynamics are linear time invariant (LTI) and subsystem connections are time invariant, this approach has been adopted in [13], [16], [14], [15] for investigating NDS regularity/controllability/observability, in which the dynamics of each subsystem is described respectively by a state vector, $X$ describe the dynamics of its subsystems, the following model is utilized in this paper to represent the dynamics of its subsystems.

$$\Sigma \{ z(t, i) | i = 1, \ldots, N \}$$

respectively for its regularity, controllability and observability. $\Phi$ of Equation (2) is also able to include parameters of a subsystem in the NDS $\Sigma$, provided that the system matrices of that subsystem depend on these parameters through a linear fractional transformation. Details can be found in [14], [15].

In system analysis and synthesis, a frequently adopted model is the following descriptor system.

$$E \delta(x(t)) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

in which $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$, $D \in \mathbb{R}^{q \times p}$ and $E \in \mathbb{R}^{n \times n}$ are some constant real matrices. When the matrix $E$ is not invertible, this model is sometimes also referred to as a singular system. Compared with the so-called state space model, this model is widely believed to be more natural and more convenient in expressing system constraints and keeping system structure information [11], [3], [3], [9].

A descriptor system is said to be regular if $\det(\lambda E - A) \neq 0$, which is a specific and important concept for descriptor systems. When a descriptor system is not regular, its outputs can not be uniquely determined by its inputs and initial states, even if they are consistent.

The following results are well known about a descriptor system [11], [2], [7].

**Lemma 1:** Assume that the descriptor system of Equation (4) is regular.

- Let $\delta(\cdot)$ represent the forward time shift operation. If $\text{rank} \begin{bmatrix} E & B \end{bmatrix} = n$, then the associated descriptor system is causal if and only if

$$\text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \text{rank}(E)$$

(4)

- Let $\delta(\cdot)$ represent the derivative of a function with respect to time. Then the associated descriptor system is free of impulse modes, if and only if the condition of Equation (4) is satisfied.

This lemma reveals that the conditions for the nonexistence of impulse modes in a continuous time descriptor system are actually part of the conditions for the causality of a discrete time descriptor system. This makes it possible to investigate this two problems together.

The following results are well known in linear algebra [4], [5].

**Lemma 2:** Let $A$ be an $m \times n$ dimensional real matrix and $y$ an $n$ dimensional real column vector. Then there always exist an $m \times m$ dimensional orthogonal matrix $U$ and an $n \times n$
dimensional orthogonal matrix \( V \), as well as positive numbers \( \sigma_i \mid i=1 \), such that
\[
A = U \begin{bmatrix}
\text{diag}(\sigma_i^{-1})_{i=1} & 0_{r \times (n-r)} \\
0_{(m-r) \times r} & 0_{(m-r) \times (n-r)}
\end{bmatrix} V^T
\]

(5)

Moreover, the Moore-Penrose inverse \( A^\dagger \) of this matrix can be expressed as
\[
A^\dagger = V \begin{bmatrix}
\text{diag}(\sigma_i^{-1})_{i=1} & 0_{r \times (n-r)} \\
0_{(n-r) \times r} & 0_{(n-r) \times (m-r)}
\end{bmatrix} U^T
\]

(6)

In addition, there exists a vector \( x \in \mathbb{R}^n \) satisfying \( Ax = y \) if and only if
\[
(I_m - AA^\dagger) y = 0
\]

(7)

When this condition is satisfied, all the vectors \( x \in \mathbb{R}^n \) satisfying \( Ax = y \) can be parametrized as
\[
x = A^\dagger y + (I_n - A^\dagger A) z
\]

(8)
in which \( z \) is an arbitrary \( n \) dimensional real column vector.

The expression of a matrix in the form of Equation (10) is equivalent to that the matrix \( \Sigma \) is well-posed. Satisfaction of this condition of Equation (4) is always satisfied by the NDS \( \Sigma \). Obviously, all the conditions of Lemma 1 are simultaneously satisfied when the matrix \( E \) is invertible. This means that when this matrix is regular, the associated descriptor system is always causal/impulse mode free, and the associated verification problem is trivial. Therefore, the following theorem only investigates the situation in which the matrix \( E \) does not have an inverse.

**Theorem 1**: Concerning the descriptor system of Equation (6), assume that the matrix \( E \) is rank deficient. Let
\[
E = U_E \begin{bmatrix}
\text{diag}(\sigma_E^{-1})_{i=1} & 0_{r \times (n-r)} \\
0_{(n-r) \times r} & 0_{(n-r) \times (n-r)}
\end{bmatrix} V_E^T
\]

be the SVD of the matrix \( E \). Partition the matrices \( U_E \) and \( V_E \) respectively as follows,
\[
U_E = [U_{E_1} \ U_{E_2}], \quad V_E = [V_{E_1} \ V_{E_2}]
\]

in which \( U_{E_1} \in \mathbb{R}^{n \times r}, \ U_{E_2} \in \mathbb{R}^{n \times (n-r)}, \ V_{E_1} \in \mathbb{R}^{n \times r} \) and \( V_{E_2} \in \mathbb{R}^{n \times (n-r)} \). Then
\[
\text{rank} \left( \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} \right) = n + \text{rank} (E)
\]

if and only if the matrix \( U_{E_2}^T \ A \ V_{E_2} \) is invertible.

The proof of this theorem is given in the appendix.

On the basis of this theorem, as well as properties of a linear fractional transformation, a computationally efficient condition is derived for the satisfaction of Equation 6 by the NDS \( \Sigma \).

**Theorem 2**: Let matrices \( U_{E_2} \) and \( V_{E_2} \) have the same definitions as those of Theorem 1 with \( n \) being replaced by \( n_x \). If the matrix \( U_{E_2}^T \ [A_{xx} \ V_{E_2} \ A_{sv}] \) is of FCR, then the condition of Equation 6 is always satisfied by the NDS \( \Sigma \), no matter how the subsystems are connected. Otherwise, partition the matrix \( \left( U_{E_2}^T \ [A_{xx} \ V_{E_2} \ A_{sv}] \right)^\dagger \)
\[
\left( U_{E_2}^T \ [A_{xx} \ V_{E_2} \ A_{sv}] \right)^\dagger = \begin{bmatrix} N_{xx} \\ N_{sv} \end{bmatrix}
\]

(11)
in which the matrix \( N_{xx} \) has \( n_x - r \) rows. Then the condition of Equation 6 is satisfied by the NDS \( \Sigma \), if and only if the matrix \( N_{sv} = \Phi (A_{xx} \ V_{E_2} \ N_{xx} + A_{sv} \ N_{sv}) \) is of FCR.

The proof of the above theorem is deferred to the appendix.

Clearly, matrix inversions of Equation 6 are no longer required in the above condition. This is quite attractive in large scale NDS analysis and synthesis. In addition, the matrix
$N_{xx} - \Phi (A_{xx}V_{E2}N_{xx} + A_{xx}N_{xx})$ depends affinely on the SCM $\Phi$, which is helpful in NDS parameter selections and topology designs, recalling that subsystem parameters can also be included in this matrix.

It is worthwhile to mention that a large scale NDS usually has a sparse structure \([11], [10], [16]\). This implies that the dimension of the matrix $N_{xx} - \Phi (A_{xx}V_{E2}N_{xx} + A_{xx}N_{xx})$ is usually significantly smaller than that of the state vector $x(i)$ of the NDS $\Sigma$. That is, compared with the dimension of the matrices in Equation \([4]\), which are respectively $2n_x \times 2n_x$ and $n_x \times n_x$, this matrix often has a much lower dimension. This is also very attractive from the computational viewpoint.

Note that the matrices $E$, $A_{xx}$, $A_{xx}$, $A_{xx}$ and $A_{xx}$ have a consistent block diagonal structure. This means that the SVD of the matrix $E$, as well as the computation of the matrices $(U_{E2}^T [A_{xx} V_{E2} A_{xx}])^{-1}$ and $A_{xx} V_{E2} N_{xx} + A_{xx} N_{xx}$, can be calculated with each individual subsystem separately. Moreover, the matrices $N_{xx}$ and $(A_{xx} V_{E2} N_{xx} + A_{xx} N_{xx})$ are also block diagonal. These characteristics are completely the same as those of the conditions established in \([13], [14], [15]\) for NDS controllability/observability. This means that the condition of Theorem \([3]\) shares the same computational advantages as those of Theorem \([2]\) in NDS analysis and synthesis.

From the consistent block diagonal structure of the associated matrices, it is clear that the condition of Theorem \([3]\) has the same computational advantages as those of Theorem \([2]\) in NDS analysis and synthesis. Note that the matrix col \(\{U_{E2}^T B_x, B_x\} \) is of FRR only when the matrix $U_{E2}^T B_x$ holds this property. From the consistent block diagonal structure of the matrices $E$ and $B_x$, as well as the proof of Theorem \([3]\) the latter is equivalent to that for each $i = 1, 2, \ldots, N$, rank \(\{E(i) B_x(i)\} = n_x\), in which $n_x(i)$ stands for the dimension of the state vector $x(t, i)$ of the $i$-th subsystem $\Sigma_i$. In addition, Theorem \([3]\) also makes it clear that this condition is not necessary for the whole NDS $\Sigma$ to meet the requirement rank \(\{E B\} = n_x\). Once again, this implies that even if there are some subsystems that are not causal, it is still possible to build a causal NDS $\Sigma$ though selecting appropriate subsystem connections.

For each $i = 1, 2, \ldots, N$, let $E(i) = U_E(i) \begin{bmatrix} \text{diag}\{E(i)\}^{-1} & 0_{r(i) \times r(i)} & 0_{r(i) \times r(i)} & 0_{r(i) \times r(i)} \end{bmatrix} V_E(i)$ be the SVD of the matrix $E(i)$. Moreover, divide the matrices $U_E(i)$ and $V_E(i)$ respectively as $U_E(i) = [U_{E1}(i) U_{E2}(i)]$, \(V_E(i) = [V_{E1}(i) V_{E2}(i)]\) in which $U_{E1}(i) \in \mathbb{R}^{n(i) \times r(i)}$, $U_{E2}(i) \in \mathbb{R}^{n(i) \times (n(i) - r(i))}$, $V_{E1}(i) \in \mathbb{R}^{n(i) \times r(i)}$ and $V_{E2}(i) \in \mathbb{R}^{n(i) \times (n(i) - r(i))}$. From Theorems \([2]\) and \([3]\) as well as the consistent block diagonal structure of the associated matrices, it can be straightforwardly shown that if for each $i \in \{1, 2, \ldots, N\}$, the matrix $U_{E2}(i) [A_{xx}(i) V_{E2}(i) A_{xx}(i)]$ is of FCR, then the NDS $\Sigma$ is always free from impulse modes, no matter how its subsystems are connected, provided that it is composed from continuous time descriptor subsystems. Similarly, when the NDS $\Sigma$ consists of only discrete time descriptor subsystems, then it is always causal, in other words, its causality does not depend on any particular value of its SCM, if in addition that the matrix col \(\{U_{E2}(i) B_x(i), B_x(i)\} \) is of FRR for every $i = 1, 2, \ldots, N$.

Note that the above conditions are imposed on each individual subsystem independently. This means that there exist some kinds of plants which always form a causal/impulse mode free NDS, no matter how these plants are connected to each other. These characteristics are quite important in some real world problems. For example, in a system consisting of several autonomous agents, connections among these agents may change according to working situations, and/or may be difficult to predict in practice \([11], [16]\).

IV. CONCLUDING REMARKS

Through analyzing the structure of the null space of the associated matrix, a novel rank based necessary and sufficient condition is derived in this paper for the causality/impulse mode freeness of a descriptor system. On the basis of this result, a matrix rank based necessary and sufficient condition is established for the causality/impulse mode freeness of a networked dynamic system which is composed from several descriptor subsystems. This condition successfully
Proof of Theorem 1:

Then \( \phi \) and \( \psi \) are vectors in the null space of the matrix \( E \). By Lemma 2 and Equation (a.2), we have that \( \psi = \text{col} \{ \psi_1, \psi_2, \ldots, \psi_k \} \) such that \( \psi \) satisfies Equation (a.7).

On the contrary, for any real vector \( \psi \) with a compatible dimension, define vector \( \phi \) as that of Equation (a.7). Then it can be proven through direct matrix multiplications that this vector always satisfies Equation (a.7).

Then it can be proven through direct matrix multiplications that this vector always satisfies Equation (a.7).

The above arguments show that a vector \( \phi \) belongs to the null space of the matrix \( [E, 0] \), if and only if there is a vector \( \psi \) satisfying Equation (a.7).

On the other hand, from the SVD of the matrix \( E \) and properties of orthogonal matrices, as well as Lemma 2 direct matrix manipulations show that

\[
I - EE^\dagger = U_{E2}U_{E2}^T, \quad I - E^\dagger E = V_{E2}V_{E2}^T
\]

In addition, the matrices \( E^\dagger \) and \( E^\perp \) can be respectively expressed as

\[
E^\dagger = V_{E1} \Sigma_E^{-1} U_{E1}^T, \quad E^\perp = V_{E2}
\]

in which \( \Sigma_E \) represents \( \text{diag} \{ \sigma_E( \tau_{\sum_1} ) \} \) for brevity.

Define matrices \( \Theta \) and \( \Xi \) respectively as

\[
\Theta = \begin{bmatrix}
V_{E2} & 0 \\
-V_{E2} \Sigma_E^{-1} U_{E1}^T A & V_{E2}
\end{bmatrix}, \\
\Xi = \begin{bmatrix}
[I - EE^\dagger] A E^\perp & 0 \\
0 & V_{E2}^T
\end{bmatrix}
\]

From the definition of SVD, we have that the matrix \( V_E = [V_{E1}, V_{E2}] \) is orthogonal, which means that its submatrix \( V_{E2} \) is of FCR. Hence, the matrix \( \Theta \) is also of FCR. More specifically, if there exist vectors \( \alpha_1 \) and \( \alpha_2 \), such that

\[
\Theta \Delta = \begin{bmatrix}
V_{E2} \alpha_1 \\
-V_{E2} \Sigma_E^{-1} U_{E1}^T A \alpha_1 + V_{E2} \alpha_2
\end{bmatrix} = 0
\]

Then it is certain that \( \alpha_1 = 0 \) as the matrix \( V_{E2} \) is of FCR. Hence \( -V_{E2} \Sigma_E^{-1} U_{E1}^T A \alpha_1 + V_{E2} \alpha_2 = 0 \), which further implies that \( \alpha_2 = 0 \). Therefore the matrix \( \Theta \) is of FCR as its null space consists only of the zero vector.

In addition to these, from Equations (a.8) and (a.9), straightforward algebraic operations prove that

\[
\Theta \Xi = \begin{bmatrix}
[I - EE^\dagger] A E^\perp & 0 \\
-E^\dagger A E^\perp & I - E^\dagger E
\end{bmatrix}
\]

From this equation and Equation (a.7), it can be directly claimed that for any element in the null space of the matrix \( [E, 0] \), say \( \phi \), there is a vector \( \psi \) satisfying

\[
\phi = \Theta \Xi \psi
\]

Let \( k \) be an arbitrary positive integer not smaller than \( d + 1 \), and \( \psi_1^{[1]}, \psi_2^{[2]}, \ldots, \psi_k^{[k]} \) be \( k \) different but arbitrary nonzero vectors with a dimension compatible with the matrix \( \Xi \). For each \( i = 1, 2, \ldots, k \), define a vector \( \phi_i^{[i]} \) as

\[
\phi_i^{[i]} = \Theta \Xi \psi_i^{[i]}
\]
Then the above arguments reveals that all these vectors $\phi^{[1]}$, $\phi^{[2]}$, …, and $\phi^{[k]}$ belong to the null space of the matrix
\[
\begin{bmatrix}
E & 0 \\
A & E
\end{bmatrix}
\]
Recall that $d$ is the dimension of the aforementioned null space. This means that the vectors $\phi^{[i]}_{i=1}^{k}$ can not be linearly independent of each other. These observations mean that the vector $\alpha = [a_1 a_2 \cdots a_k]^T$ is not a zero vector and
\[
\sum_{i=1}^{k} a_i \phi^{[i]} = 0
\]
(a.14)
Substitute Equation (a.13) into Equation (a.14), we have that
\[
\sum_{i=1}^{k} a_i \Theta \Xi \phi^{[i]} = \Theta \sum_{i=1}^{k} a_i \Xi \phi^{[i]} = 0
\]
(a.15)
Denote the matrix $[\psi^{[1]} \psi^{[2]} \cdots \psi^{[k]}]$ by $\Psi$. As the matrix $\Theta$ is proved to be FCR, the above equality means that
\[
\sum_{i=1}^{k} a_i \Xi \phi^{[i]} = \Xi \Psi \alpha = 0
\]
(a.16)
That is, the vectors $\Xi \phi^{[1]}$, $\Xi \phi^{[2]}$, … and $\Xi \phi^{[k]}$ are linearly dependent.

Assume now that $d = n - r$. Let $k$ equal to $n - r + d$, in which $d_*$ stands for the column number of the matrix $[(I - E E^\dagger) A E^\perp]^{\perp}$. Then the vectors $\psi^{[1]}$, $\psi^{[2]}$, …, and $\psi^{[k]}$ can be selected to satisfy
\[
\Psi = \begin{bmatrix}
I_{d_*} & 0 \\
0 & V_{E_2}
\end{bmatrix}
\]
in which $I_{d_*}$ denotes the $d_* \times d_*$ dimensional identity matrix. Then for each $n - r + d_*$ dimensional real vector $\alpha = [a_1 a_2 \cdots a_{n-r+d_2}]^T$, we have that
\[
\Xi \Psi \alpha = \begin{bmatrix} [(I - E E^\dagger) A E^\perp]^{\perp} \text{col} \{ a_i \}_{i=1}^{d_*} \\ \text{col} \{ a_i \}_{i=n-r+d+1}^{d}
\end{bmatrix}
\]
(a.17)
which is equal to the $2n - r$ dimensional zero vector if and only if $\text{col} \{ a_i \}_{i=n-r+d+1}^{d} = 0$ and
\[
[(I - E E^\dagger) A E^\perp]^{\perp} \text{col} \{ a_i \}_{i=1}^{d_*} = 0
\]
(a.18)
From Equation (a.16), it can be claimed that there is a nonzero vector $\alpha$ such that $\Xi \Psi \alpha = 0$ for these particular $\phi^{[i]}_{i=1}^{k}$. According to Equation (a.18), this is possible only when $\text{col} \{ a_i \}_{i=n-r+d+1}^{d} = 0$ is not a zero vector. Note that the columns of the matrix $[(I - E E^\dagger) A E^\perp]^{\perp}$ constitute a base of the right null space of the matrix $(I - E E^\dagger) A E^\perp$. These columns are linearly independent of each other. These observations mean that the matrix $[(I - E E^\dagger) A E^\perp]^{\perp}$ is in fact a zero vector, which further means that the matrix $(I - E E^\dagger) A E^\perp$ is of FCR.

From Equations (a.8) and (a.9), it is obvious that
\[
(I - E E^\dagger) A E^\perp = U_{E_2} U_{E_2}^T A V_{E_2}
\]
Hence, the matrix $(I - E E^\dagger) A E^\perp$ is of FCR only when the matrix $U_{E_2} A V_{E_2}$ is invertible.

On the other hand, assume that the matrix $U_{E_2} A V_{E_2}$ is invertible. As the matrix $U E_2 = [U_{E_1} U_{E_2}]$ is orthogonal, it is clear that its submatrix $U_{E_2}$ is of FCR. Hence, the matrix $(I - E E^\dagger) A E^\perp$ is also of FCR. More specifically, if this matrix is not of FCR, then according to Equation (a.19), it can be claimed that there exists a nonzero vector $\zeta$, such that
\[
U_{E_2} U_{E_2}^T A V_{E_2} \zeta = 0
\]
(a.20)
As the matrix $U_{E_2} U_{E_2}^T A V_{E_2}$ is invertible, it is obvious that $U_{E_2} U_{E_2}^T A V_{E_2} \zeta \neq 0$ for every nonzero vector $\zeta$. This observation, together with Equation (a.20), imply that the matrix $U_{E_2}$ is not of FCR, which is clearly a contradiction.

When the matrix $(I - E E^\dagger) A E^\perp$ is of FCR, we have that $[(I - E E^\dagger) A E^\perp]^{\perp} = 0$. Substitute this equality and Equation (a.8) into Equation (a.7), we have that a vector $\phi$ belongs to the null space of the matrix $[E 0 A E]$, if and only if there is a vector $\psi$, such that
\[
\phi = \begin{bmatrix} 0 \\ 0 \\ V_{E_2} V_{E_2}^T \end{bmatrix} \psi
\]
(a.21)
in which the zero matrices in general have different dimensions. That is, the null space of the matrix $[E 0 A E]$ is spanned by the columns of the matrix $[V_{E_2} V_{E_2}^T]$, which can be straightforwardly proved to have a dimension of $n - r$, noting that the matrix $V_{E_2}$ is of FCR.

This completes the proof.

\textbf{Proof of Theorem 2}

For brevity, define matrices $\Omega$ and $\Pi$ respectively as
\[
\Omega = U_{E_2}^T A_{xx} + A_{xv} (I - A_{xv})^{-1} A_{xx} V_{E_2}
\]
\[
\Pi = \begin{bmatrix} U_{E_2}^T A_{xx} V_{E_2} \\ U_{E_2}^T A_{xv} \\ -\Phi A_{xx} V_{E_2} \\ I - \Phi A_{xv}
\end{bmatrix}
\]
From Theorem 1 and Equation (10), we have that the condition of Equation (4) is satisfied by the NDS $\Sigma$, if and only if the matrix $\Omega$ is invertible. The latter is equivalent to that
\[
\Omega \alpha = 0
\]
(a.22)
if and only if $\alpha = 0$.

Let $\alpha$ be an arbitrary vector satisfying Equation (a.22). Define a vector $\beta$ as $\beta = (I - \Phi A_{xv})^{-1} \Phi A_{xx} V_{E_2} \alpha$. Noting that when the matrix $I - A_{xv} \Phi$ is invertible, the matrix $I - \Phi A_{xv}$ is also invertible. Moreover, $\Phi (I - A_{xv} \Phi)^{-1} = \Phi - (I - A_{xv} \Phi)^{-1}$. From these observations, it can be claimed that the vector $\text{col} \{ \alpha, \beta \}$ is a solution to the following equation,
\[
\Pi \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0
\]
(a.23)
Note that when $\alpha \neq 0$, the vector $\text{col} \{ \alpha, \beta \}$ is also not equal to zero. This means that if the matrix $\Omega$ is not invertible, then the matrix $\Pi$ is certainly not of FCR.

On the contrary, assume that the matrix $\Pi$ is not of FCR. Then there is a nonzero vector $\text{col} \{ \alpha, \beta \}$ satisfying Equation
Proof of Theorem 3:

\[ \beta = (I - \Phi A \xi \xi^T) \Phi A \xi \xi^T \alpha, \]
which implies that \( \alpha \neq 0 \). In addition, this nonzero vector \( \alpha \) also satisfies Equation (a.22). It can therefore be declared that if the matrix \( \Pi \) is not of FCR, then the matrix \( \Omega \) is certainly not invertible.

These arguments mean that the matrix \( \Omega \) is invertible, if and only if matrix II is of FCR. Obviously, if the matrix \( U_{E_2}^T [A_{xx} V_{E_2} \ A_{xv}] \) is of FCR, then the matrix II is also of FCR, no matter what value the SCM \( \Phi \) takes. This means that the condition of Equation (4) is always satisfied by the NDS \( \Sigma \), no matter how its subsystems are connected.

In the remaining of this proof, it is assumed that the matrix \( U_{E_2}^T [A_{xx} V_{E_2} \ A_{xv}] \) is not of FCR, which means that the matrix \( (U_{E_2}^T [A_{xx} V_{E_2} \ A_{xv}])^T \) is not a zero vector.

Assume that the matrix II is not of FCR. Then there exists a nonzero vector \( \xi \) such that \( \Pi \xi = 0 \). From the definition of the matrix II, it is obvious that \( U_{E_2}^T [A_{xx} V_{E_2} \ A_{xv}] \xi = 0 \). That is, this vector \( \xi \) must belong to the right null space of the matrix \( U_{E_2}^T [A_{xx} V_{E_2} \ A_{xv}] \), which means that there is a nonzero vector \( \xi \) satisfying

\[ \xi = \begin{bmatrix} N_{xx} \\ N_{xv} \end{bmatrix} \zeta \]  
(a.24)

We therefore have that

\[ \Pi \xi = \begin{bmatrix} 0 \\ N_{xv} - \Phi (A_{xx} V_{E_2} N_{xx} + A_{xv} N_{xv}) \zeta \end{bmatrix} \]  
(a.25)

Hence

\[ N_{xv} - \Phi (A_{xx} V_{E_2} N_{xx} + A_{xv} N_{xv}) \zeta = 0 \]  
(a.26)

As \( \xi \) is a nonzero vector, this means that the matrix \( N_{xv} - \Phi (A_{xx} V_{E_2} N_{xx} + A_{xv} N_{xv}) \) is not of FCR.

On the contrary, assume that the matrix \( N_{xv} - \Phi (A_{xx} V_{E_2} N_{xx} + A_{xv} N_{xv}) \) is not of FCR. Then there is a nonzero vector \( \zeta \) satisfying

\[ (N_{xv} - \Phi (A_{xx} V_{E_2} N_{xx} + A_{xv} N_{xv})) \zeta = 0 \]. Define a vector \( \xi \) as \( \xi = [N_{xx} T N_{xv}^T] \zeta \). Then \( \xi \neq 0 \) and \( \Pi \xi = 0 \). Hence, the matrix II is not of FCR also.

These observations imply that when the matrix \( U_{E_2}^T [A_{xx} V_{E_2} \ A_{xv}] \) is not of FCR, the matrix II is in FCR, if and only if the matrix \( N_{xv} - \Phi (A_{xx} V_{E_2} N_{xx} + A_{xv} N_{xv}) \) is.

The proof can now be completed by recalling that the condition of Equation (4) is satisfied by the NDS \( \Sigma \), if and only if the matrix II is of FCR.

Proof of Theorem 3

Obviously, the condition \( \text{rank} ([E \ B]) = n_x \) is equivalent to the matrix \( [E \ B] \) is of FRR.

Let \( \alpha \) be an arbitrary nonzero \( r_x - r \) dimensional row vector. According to the definition of the matrix \( U_{E_2} \), it is obvious that \( \alpha U_{E_2}^T \neq 0 \). On the other hand, direct matrix multiplications show that

\[ \alpha U_{E_2}^T [E \ B] = [0 \ \alpha U_{E_2}^T B] \]  
(a.27)

This implies that if the matrix \( [E \ B] \) is of FRR, then the matrix \( U_{E_2}^T B \) is also of FRR.

On the contrary, assume that the matrix \( U_{E_2}^T B \) is of FRR. Let \( \xi \) be an arbitrary nonzero row vector satisfying \( \xi E = 0 \).

From the SVD of the matrix \( E \), we have that

\[ \xi E = \xi U_{E_1} \text{diag} \{ \sigma \} U_{E_1}^T \]  
(a.28)

As the matrix \( V_{E_1} \) is of FCR and \( \sigma_{E_1,i} > 0 \) for each \( i = 1, 2, \ldots, r \), it is straightforward to prove that \( \xi E = 0 \) if and only if there is a vector \( \alpha \) such that \( \xi = \alpha U_{E_2}^T \). Moreover, \( \alpha \neq 0 \) whenever \( \xi \neq 0 \), noting that the matrix \( U_{E_2}^T \) is also of FCR. Hence

\[ \xi [E \ B] = [0 \ \alpha U_{E_2}^T B] \neq 0 \]  
(a.29)

This means that the matrix \( [E \ B] \) is also of FRR, noting that any nonzero vector \( \xi \) satisfying \( \xi E \neq 0 \) certainly satisfying \( \xi [E \ B] \neq 0 \).

It can therefore be declared that the matrix \( [E \ B] \) is of FRR, if and only if the matrix \( U_{E_2}^T B \) is.

Note that a matrix is of FRR if and only if its transpose is of FCR. On the other hand, recall from Equation (10) that \( B = B_x + A_{xx} \Phi (I - A_{xv} \Phi)^{-1} B_x \). Conclusions of this theorem can be established using similar arguments as those in the proof of Theorem 2 noting that the matrix \( (U_{E_2}^T B)^T \) has completely the same form as the matrix \( U_{E_2}^T A V_{E_2} \).

The details are omitted due to their close similarities.

This completes the proof. \( \diamond \)