CLASSIFICATION OF QUASI-TRIGONOMETRIC SOLUTIONS OF THE CLASSICAL YANG–BAXTER EQUATION

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ABSTRACT. It was proved by Montaner and Zelmanov that up to classical twisting Lie bialgebra structures on \( g[u] \) fall into four classes. Here \( g \) is a simple complex finite-dimensional Lie algebra. It turns out that classical twists within one of these four classes are in a one-to-one correspondence with the so-called quasi-trigonometric solutions of the classical Yang-Baxter equation. In this paper we give a complete list of the quasi-trigonometric solutions in terms of sub-diagrams of the certain Dynkin diagrams related to \( g \). We also explain how to quantize the corresponding Lie bialgebra structures.

1. INTRODUCTION

The present paper constitutes a step towards the classification of quantum groups. We describe an algorithm for the quantization of all Lie bialgebra structures on the polynomial Lie algebra \( P = g[u] \), where \( g \) is a simple complex finite-dimensional Lie algebra.

Lie bialgebra structures on \( P \), up to so-called classical twisting, have been classified by F. Montaner and E. Zelmanov in [9]. We recall that given a Lie co-bracket \( \delta \) on \( P \), a classical twist is an element \( s \in P \wedge P \) such that

\[
\text{CYB}(s) + \text{Alt}(\delta \otimes \text{id})(s) = 0,
\]

where CYB is the l.h.s. of the classical Yang-Baxter equation.

We also note that a classical twist does not change the classical double \( D_\delta(P) \) associated to a given Lie bialgebra structure \( \delta \). If \( \delta^s \) is the twisting co-bracket via \( s \), then the Lie bialgebras \( (P, \delta) \) and \( (P, \delta^s) \) are in the same class, i.e. there exists a Lie algebra isomorphism between \( D_\delta(P) \) and \( D_{\delta^s}(P) \), preserving the canonical forms and compatible with the canonical embeddings of \( P \) into the doubles.

According to the results of Montaner and Zelmanov, there are four Lie bialgebra structures on \( P \) up to classical twisting. Let us present them:

Case 1. Consider \( \delta_1 = 0 \). Consequently, \( D_1(P) = P + \varepsilon P^* \), where \( \varepsilon^2 = 0 \). The symmetric nondegenerate invariant form \( Q \) is given by the canonical pairing between \( P \) and \( \varepsilon P^* \).

Lie bialgebra structures which fall in this class are the elements \( s \in P \wedge P \) satisfying \( \text{CYB}(s) = 0 \). Such elements are in a one-to-one correspondence with finite-dimensional quasi-Frobenius Lie subalgebras of \( P \).

Case 2. Let us consider the co-bracket \( \delta_2 \) given by

\[
\delta_2(p(u)) = [r_2(u, v), p(u) \otimes 1 + 1 \otimes p(v)],
\]

where \( r_2(u, v) = \Omega/(u - v) \). Here \( \Omega \) denotes the quadratic Casimir element on \( g \).

It was proved in [10] that the associated classical double is \( D_2(P) = g((u^{-1})) \), together with the canonical invariant form

\[
Q(f(u), g(u)) = \text{Res}_{u=0} K(f, g),
\]

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where $K$ denotes the Killing form of the Lie algebra $\mathfrak{g}((u^{-1}))$ over $\mathbb{C}((u^{-1}))$.

Moreover, the Lie bialgebra structures which are obtained by twisting $\delta_3$ are in a one-to-one correspondence with so-called rational $r$-matrices of the CYBE, according to [11].

**Case 3.** In this case, let us consider the Lie bialgebra structure given by
\begin{equation}
\delta_3(p(u)) = [r_3(u), p(u) \otimes 1 + 1 \otimes p(v)],
\end{equation}
with $r_3(u, v) = v\Omega/(u - v) + \sum e_\alpha \otimes f_\alpha + \frac{1}{2}\Omega_0$, where $e_\alpha, f_\alpha$ are root vectors of $\mathfrak{g}$ and $\Omega_0$ is the Cartan part of $\Omega$.

It was proved in [6] that the associated classical double is $D_3(P) = \mathfrak{g}((u^{-1})) \times \mathfrak{g}$, together with the invariant nondegenerate form $Q$ defined by
\begin{equation}
Q((f(u), a), (g(u), b)) = K(f(u), g(u))_0 - K(a, b),
\end{equation}
where the index zero means that one takes the free term in the series expansion. According to [6], there is a one-to-one correspondence between Lie bialgebra structures which are obtained by twisting $\delta_3$ and so-called quasi-trigonometric solutions of the CYBE.

**Case 4.** We consider the co-bracket on $P$ given by
\begin{equation}
\delta_4(p(u)) = [r_4(u, v), p(u) \otimes 1 + 1 \otimes p(v)],
\end{equation}
with $r_4(u, v) = uv\Omega/(v - u)$.

It was shown in [13] that the classical double associated to the Lie bialgebra structure $\delta_4$ is $D_4(P) = \mathfrak{g}((u^{-1})) \times (\mathfrak{g} \otimes \mathbb{C}[\varepsilon])$, where $\varepsilon^2 = 0$. The form $Q$ is described as follows: if $f(u) = \sum a_k u^k$ and $g(u) = \sum b_k u^k$, then
\[Q(f(u) + A_0 + A_1 \varepsilon, g(u) + B_0 + B_1 \varepsilon) = \text{Res}_{u=0} u^{-2} K(f, g) - K(A_0, B_1) - K(A_1, B_0).
\]

Lie bialgebra structures which are in the same class as $\delta_4$ are in a one-to-one correspondence with quasi-rational $r$-matrices, as it was proved in [13].

Regarding the quantization of these Lie bialgebra structures on $P$, the following conjecture stated in [6] and proved by G. Halbout in [4] plays a crucial role.

**Theorem 1.1.** Any classical twist can be extended to a quantum twist, i.e., if $(L, \delta)$ is any Lie bialgebra, $s$ is a classical twist, and $(A, \Delta, \varepsilon)$ is a quantization of $(L, \delta)$, there exists $F \in A \otimes A$ such that

1. $F = 1 + O(h)$ and $F - F^{21} = hs + O(h^2)$,
2. $(\Delta \otimes \text{id})(F)F^{12} - (\text{id} \otimes \Delta)(F)F^{23} = 0$,
3. $(\varepsilon \otimes \text{id})(F) = (\text{id} \otimes \varepsilon)(F) = 1$.

Moreover gauge equivalence classes of quantum twists for $A$ are in bijection with gauge equivalence classes of $h$-dependent classical twists $s_h = hs_1 + O(h^2)$ for $L$.

Let us suppose that we have a Lie bialgebra structure $\delta$ on $P$. Then $\delta$ is obtained by twisting one of the four structures $\delta_i$ from Cases 1–4. This above theorem implies that in order to find a quantization for $(P, \delta)$, it is sufficient to determine the quantization of $\delta_i$ and then find the quantum twist whose classical limit is $s$. Let us note that the quantization of $(P, \delta_5)$ is well-known. The corresponding quantum algebra was introduced by V. Tolstoy in [14] and it is denoted by $U_q(\mathfrak{g}[u])$.

The quasi-trigonomometric solutions of the CYBE were studied in [7], where it was proved that they fall into classes, which are in a one-to-one correspondence with vertices of the extended Dynkin diagram of $\mathfrak{g}$. Let us consider corresponding roots, namely simple roots $\alpha_1, \alpha_2, \cdots, \alpha_r$ and $\alpha_0 = -\alpha_{\max}$. In [7] quasi-trigonometric solutions corresponding to the simple roots which have coefficient one in the decomposition of the maximal root were classified. It was also proved there that quasi-trigonometric solutions corresponding to $\alpha_0$ are in a one-to-one correspondence with constant solutions of the modified CYBE classified in [11] and the polynomial part of these solutions is constant. The aim of our paper
2. Lie bialgebra structures associated with quasi-trigonometric solutions

**Definition 2.1.** A solution $X$ of the CYBE is called quasi-trigonometric if it is of the form $X(u, v) = v(\Omega / (u - v) + p(u, v)$, where $p$ is a polynomial with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$.

The class of quasi-trigonometric solutions is closed under gauge transformations. We first need to introduce the following notation: Let $R$ be a commutative ring and let $L$ be a Lie algebra over $R$. Let us denote by $\text{Aut}_R(L)$ the group of automorphisms of $L$ over $R$. In other words we consider such automorphisms of $L$, which satisfy the condition $f(rl) = rf(l)$, where $r \in R$, $l \in L$.

At this point we note that there exists a natural embedding

$$\text{Aut}_{\mathbb{C}[u]}(\mathfrak{g}[u]) \hookrightarrow \text{Aut}_{\mathbb{C}(u-1)}(\mathfrak{g}((u^{-1}))),$$

defined by the formula

$$\sigma(u^{-k}x) = u^{-k}\sigma(x),$$

for any $\sigma \in \text{Aut}_{\mathbb{C}[u]}(\mathfrak{g}[u])$ and $x \in \mathfrak{g}[u]$.

Now if $X$ is a quasi-trigonometric solution and $\sigma(u) \in \text{Aut}_{\mathbb{C}[u]}(\mathfrak{g}[u])$, one can check that the function $Y(u, v) := (\sigma(u) \otimes \sigma(v))(X(u, v))$ is again a quasi-trigonometric solution. $X$ and $Y$ are said to be gauge equivalent.

**Theorem 2.2.** There exists a natural one-to-one correspondence between quasi-trigonometric solutions of the CYBE for $\mathfrak{g}$ and linear subspaces $W$ of ${\mathfrak{g}}((u^{-1})) \times \mathfrak{g}$ which satisfy the following properties:

1. $W$ is a Lie subalgebra of $\mathfrak{g}((u^{-1})) \times \mathfrak{g}$ and $W \supseteq u^{-N}\mathfrak{g}[[u^{-1}]]$ for some positive integer $N$.
2. $W \oplus \mathfrak{g}[u] = \mathfrak{g}((u^{-1})) \times \mathfrak{g}$.
3. $W$ is a Lagrangian subspace of $\mathfrak{g}((u^{-1})) \times \mathfrak{g}$ with respect to the invariant bilinear form $Q$ given by (1.5).

Let $\sigma(u) \in \text{Aut}_{\mathbb{C}[u]}(\mathfrak{g}[u])$. Let $\sigma'(u) = \sigma(u) \oplus \sigma(0)$ be the induced automorphism of $\mathfrak{g}((u^{-1})) \times \mathfrak{g}$.

**Definition 2.3.** We will say that $W_1$ and $W_2$ are gauge equivalent if there exists $\sigma(u) \in \text{Aut}_{\mathbb{C}[u]}(\mathfrak{g}[u])$ such that $W_1 = \sigma(u)W_2$.

It was checked in [6] that two quasi-trigonometric solutions are gauge equivalent if and only if the corresponding subalgebras are gauge equivalent.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with the corresponding set of roots $R$ and a choice of simple roots $\Gamma$. Denote by $\mathfrak{g}_\alpha$ the root space corresponding to a root $\alpha$. Let $\mathfrak{h}(\mathbb{R})$ be the set of all $h \in \mathfrak{h}$ such that $\alpha(h) \in \mathbb{R}$ for all $\alpha \in R$. Consider the valuation on $\mathbb{C}((u^{-1}))$ defined by $v(\sum_{k \geq n} a_k u^{-k}) = n$. For any root $\alpha$ and any $h \in \mathfrak{h}(\mathbb{R})$, set $M_\alpha(h) = \{ f \in \mathbb{C}((u^{-1})) : v(f) \geq \alpha(h) \}$. Consider

$$\Omega_h := \mathfrak{h}[[u^{-1}]] \oplus (\oplus_{\alpha \in R} M_\alpha(h) \otimes \mathfrak{g}_\alpha).$$

As it was shown in [7], any maximal order $W$ which corresponds to a quasi-trigonometric solution of the CYBE, can be embedded (up to some gauge equivalence) into $\Omega_h \times \mathfrak{g}$. Moreover $h$ may be taken as a vertex of the standard simplex $\Delta_{st} = \{ h \in \mathfrak{h}(\mathbb{R}) : \alpha(h) \geq 0 \ \text{for all} \ \alpha \in \Gamma \ \text{and} \ \alpha_{\text{max}} \leq 1 \}$. Vertices of the above simplex correspond to vertices of the extended Dynkin diagram of $\mathfrak{g}$, the correspondence being given by the following rule:

$0 \leftrightarrow \alpha_{\text{max}}$

$h_i \leftrightarrow \alpha_i,$

where $\alpha_i(h_j) = \delta_{ij}/k_j$ and $k_j$ are given by the relation $\sum k_j \alpha_j = \alpha_{\text{max}}$. We will write $\Omega_\alpha$ instead of $\Omega_h$ if $\alpha$ is the root which corresponds to the vertex $h$.

By straightforward computations, one can check the following two results:
Lemma 2.4. Let \( R \) be the set of all roots and \( \alpha \) an arbitrary simple root. Let \( k \) be the coefficient of \( \alpha \) in the decomposition of \( \alpha_{\text{max}} \).

For each \( r, -k \leq r \leq k \), let \( R_r \) denote the set of all roots which contain \( \alpha \) with coefficient \( r \). Let \( g_0 = h \oplus \sum_{\beta \in R_0} g_{\beta} \) and \( g_r = \sum_{\beta \in R_r} g_{\beta} \). Then

\[
O_\alpha = \sum_{r=1}^{k} u^{-1} g_r + \sum_{r=-k}^{0} O g_r + u O g_{-k},
\]

where \( O := \mathbb{C}[u^{-1}] \).

Lemma 2.5. Let \( \alpha \) be a simple root and \( k \) its coefficient in the decomposition of \( \alpha_{\text{max}} \). Let \( \Delta_\alpha \) denote the set of all pairs \((a, b)\), \( a \in g_0 + g_{-k}, b \in g_0 + g_{-1} + ... + g_{-k}, a = a_0 + a_{-k}, b = b_0 + b_{-1} + ... + b_{-k} \) and \( a_0 = b_0 \). Then

(i) The orthogonal complement of \( O_\alpha \times g \) with respect to \( Q \) is given by

\[
(O_\alpha \times g)^\perp = \sum_{r=-k}^{1} O g_r + \sum_{r=0}^{k-1} u^{-1} O g_r + u^{-2} O g_k.
\]

(ii) There exists an isomorphism \( \sigma \)

\[
\frac{O_\alpha \times g}{(O_\alpha \times g)^\perp} \cong (g_k \oplus g_0 \oplus g_{-k}) \times g
\]

given by

\[
\sigma((f, a) + (O_\alpha \times g)^\perp) = (a_0 + b_0 + c_0, a),
\]

where the element \( f \in O_\alpha \) is decomposed according to Lemma 2.4

\[
f = u^{-1}(a_0 + a_1 u^{-1} + ...) + (b_0 + b_1 u^{-1} + ...) + u(c_0 + c_1 u^{-1} + ...) + ...
\]

\( a_i \in g_k, b_i \in g_0, c_i \in g_{-k} \) and \( a \in g \).

(iii) \( (O_\alpha \times g) \cap g[u] \) is sent via the isomorphism \( \sigma \) to \( \Delta_\alpha \).

Let us make an important remark. The Lie subalgebra \( g_k \oplus g_0 \oplus g_{-k} \) of \( g \) coincides with the semisimple Lie algebra whose Dynkin diagram is obtained from the extended Dynkin diagram of \( g \) by crossing out \( \alpha \). Let us denote this subalgebra by \( L_\alpha \). The Lie algebra \( L_\alpha \times g \) is endowed with the following invariant bilinear form:

\[
Q'(a, b, c, d) = K(a, c) - K(b, d),
\]

for any \( a, c \in L_\alpha \) and \( b, d \in g \).

On the other hand, \( g_0 + g_{-k} \) is the parabolic subalgebra \( P_{\alpha_{\text{max}}}^+ \) of \( L_\alpha \) which corresponds to \( -\alpha_{\text{max}} \).

The Lie subalgebra \( g_0 + g_{-1} + ... + g_{-k} \) is the parabolic subalgebra \( P_{\alpha}^- \) of \( g \) which corresponds to the root \( \alpha \) and contains the negative Borel subalgebra. Let us also note that \( g_0 \) is precisely the reductive part of \( P_{\alpha}^- \) and of \( P_{\alpha_{\text{max}}}^+ \). We can conclude that the set \( \Delta_\alpha \) consists of all pairs \((a, b) \in P_{\alpha_{\text{max}}}^+ \times P_{\alpha}^- \) whose reductive parts are equal.

Theorem 2.6. Let \( \alpha \) be a simple root. There is a one-to-one correspondence between Lagrangian subalgebras \( W \) of \( g((u^{-1})) \times g \) which are contained in \( O_\alpha \times g \) and transversal to \( g[u] \), and Lagrangian subalgebras \( \mathfrak{l} \) of \( L_\alpha \times g \) transversal to \( \Delta_\alpha \) (with respect to the bilinear form \( Q' \)).

Proof. Since \( W \) is a subspace of \( O_\alpha \times g \), let \( \mathfrak{l} \) be its image in \( L_\alpha \times g \). Because \( W \) is transversal to \( g[u] \), one can check that \( \mathfrak{l} \) is transversal to the image of \((O_\alpha \times g) \cap g[u]\) in \( L_\alpha \times g \), which is exactly \( \Delta_\alpha \). The fact that \( W \) is Lagrangian implies that \( \mathfrak{l} \) is also Lagrangian.

Conversely, if \( \mathfrak{l} \) is a Lagrangian subalgebra of \( L_\alpha \times g \) transversal to \( \Delta_\alpha \), then its preimage \( W \) in \( O_\alpha \times g \) is transversal to \( g[u] \) and Lagrangian as well.

\( \Box \)
The Lagrangian subalgebras I of $L_\alpha \times g$ which are transversal to $\Delta_\alpha$, can be determined using results of P. Delorme [2] on the classification of Manin triples. We are interested in determining Manin triples of the form $(Q', \Delta_\alpha, I)$.

Let us recall Delorme's construction of so-called generalized Belavin-Drinfeld data. Let $r$ be a finite-dimensional complex, reductive, Lie algebra and $B$ a symmetric, invariant, nondegenerate bilinear form on $r$. The goal in [2] is to classify all Manin triples of $r$ up to conjugacy under the action on $r$ of the simply connected Lie group $R$ whose Lie algebra is $r$.

One denotes by $r_+$ and $r_-$ respectively the sum of the simple ideals of $r$ for which the restriction of $B$ is equal to a positive (negative) multiple of the Killing form. Then the derived ideal of $r$ is the sum of $r_+$ and $r_-$.

Let $j_0$ be a Cartan subalgebra of $r$, $b_0$ a Borel subalgebra containing $j_0$ and $b'_0$ be its opposite. Choose $b_0 \cap r_+$ as Borel subalgebra of $r_+$ and $b'_0 \cap r_-$ as Borel subalgebra of $r_-$. Denote by $\Sigma_+$ (resp., $\Sigma_-$) the set of simple roots of $r_+$ (resp., $r_-$) with respect to above Borel subalgebras. Let $\Sigma = \Sigma_+ \cup \Sigma_-$ and denote by $\mathcal{W} = (H_\alpha, X_\alpha, Y_\alpha)_{\alpha \in \Sigma_+}$ a Weyl system of generators of $[r, r]$.

**Definition 2.7** (Delorme, [2]). One calls $(A, A', i_\alpha, i_\alpha')$ generalized Belavin-Drinfeld data with respect to $B$ when the following five conditions are satisfied:

1. $A$ is a bijection from a subset $\Gamma_+$ of $\Sigma_+$ on a subset $\Gamma_-$ of $\Sigma_-$ such that
   
   
   \[ B(H_\alpha, H_\beta) = -B(H_\alpha, H_\beta), \alpha, \beta \in \Gamma_+. \]

2. $A'$ is a bijection from a subset $\Gamma'_+$ of $\Sigma_+$ on a subset $\Gamma'_-$ of $\Sigma_-$ such that
   
   \[ B(H_{A'\alpha}, H_{A'\beta}) = -B(H_\alpha, H_\beta), \alpha, \beta \in \Gamma'_+. \]

3. If $C = A^{-1}A'$ is the map defined on $\text{dom}(C) = \{ \alpha \in \Gamma'_+ : A'\alpha \in \Gamma_- \}$ by $C\alpha = A^{-1}A'\alpha$, then $C$ satisfies:
   
   For all $\alpha \in \text{dom}(C)$, there exists a positive integer $n$ such that $\alpha, \ldots, C^{n-1}\alpha \in \text{dom}(C)$ and $C^n\alpha \notin \text{dom}(C)$.

4. $i_\alpha$ (resp., $i'_\alpha$) is a complex vector subspace of $j_0$, included and Lagrangian in the orthogonal $a$ (resp., $a'$) to the subspace generated by $H_\alpha$, $\alpha \in \Gamma_+ \cup \Gamma_-$ (resp., $\Gamma'_+ \cup \Gamma'_-$).

5. If $f$ is the subspace of $j_0$ generated by the family $H_\alpha + H_{A\alpha}$, $\alpha \in \Gamma_+$, and $f'$ is defined similarly, then
   
   \[ (f \oplus i_\alpha) \cap (f' \oplus i'_\alpha) = 0. \]

Let $R_+$ be the set of roots of $j_0$ in $r$ which are linear combinations of elements of $\Gamma_+$. One defines similarly $R_-$, $R'_+$ and $R'_-$. The bijections $A$ and $A'$ can then be extended by linearity to bijections from $R_+$ to $R_-$ (resp., $R'_+$ to $R'_-$). If $A$ satisfies condition (1), then there exists a unique isomorphism $\tau$ between the subalgebra $m_+$ of $r$ spanned by $X_\alpha, H_\alpha$ and $Y_\alpha$, $\alpha \in \Gamma_+$, and the subalgebra $m_-$ spanned by $X_\alpha, H_\alpha$ and $Y_\alpha$, $\alpha \in \Gamma_-$, such that $\tau(H_\alpha) = H_{A\alpha}$, $\tau(X_\alpha) = X_{A\alpha}$, $\tau(Y_\alpha) = Y_{A\alpha}$ for all $\alpha \in \Gamma_+$. If $A'$ satisfies (2), then one defines similarly an isomorphism $\tau'$ between $m'_+$ and $m'_-$.

**Theorem 2.8** (Delorme, [2]). (i) Let $\mathcal{BD} = (A, A', i_\alpha, i'_\alpha)$ be generalized Belavin-Drinfeld data, with respect to $B$. Let $n$ be the sum of the root spaces relative to roots $\alpha$ of $j_0$ in $b_0$, which are not in $R_+ \cup R_-$. Let $i := f \oplus i_\alpha \oplus n$, where $f := \{ X + \tau(X) : X \in m_+ \}$.

Let $n'$ be the sum of the root spaces relative to roots $\alpha$ of $j_0$ in $b'_0$, which are not in $R'_+ \cup R'_-$. Let $i' := f' \oplus i'_\alpha \oplus n'$, where $f' := \{ X + \tau'(X) : X \in m'_+ \}$.

Then $(B, i, i')$ is a Manin triple.

(ii) Every Manin triple is conjugate by an element of $R$ to a unique Manin triple of this type.

Let us consider the particular case $r = L_\alpha \times g$. We set $\Sigma_+ := (\Gamma_{\text{ext}} \setminus \{ \alpha \}) \times \{ 0 \}$, $\Sigma_- := \{ 0 \} \times \Gamma$ and $\Sigma := \Sigma_+ \cup \Sigma_-$. Denote by $(X_\gamma, Y_\gamma, H_\gamma)_{\gamma \in \Gamma}$ a Weyl system of generators for $g$ with respect to the root system $\Gamma$. Denote $-\alpha_{\text{max}}$ by $\alpha_0$. Let $H_{\alpha_0}$ be the coroot of $\alpha_0$. We choose $X_{\alpha_0} \in g_{\alpha_0}$, $Y_{\alpha_0} \in g_{-\alpha_0}$ such that $[X_{\alpha_0}, Y_{\alpha_0}] = H_{\alpha_0}$. 

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A Weyl system of generators in $L_\alpha \times \mathfrak{g}$ (with respect to the root system $\Sigma$) is the following: $X_{(\beta,0)} = (X_\beta,0)$, $H_{(\beta,0)} = (H_\beta,0)$, $Y_{(\beta,0)} = (Y_\beta,0)$, for any $\beta \in \Gamma^{\text{ext}} \setminus \{\alpha\}$, and $X_{(0,\gamma)} = (0,X_\gamma)$, $H_{(0,\gamma)} = (0,H_\gamma)$, $Y_{(0,\gamma)} = (0,Y_\gamma)$, for any $\gamma \in \Gamma$.

By applying the general result of Delorme, one can deduce the description of the Manin triples of the form $(Q', \Delta_\alpha, I)$.

First, let us denote by $i$ the embedding

$$\Gamma \setminus \{\alpha\} \hookrightarrow \Gamma^{\text{ext}} \setminus \{\alpha\},$$

Recall that $\mathfrak{g}_0$ denotes the reductive part of $P^-_\alpha$ and has the Dynkin diagram $\Gamma \setminus \{\alpha\}$. Then $i$ induces an inclusion

$$\mathfrak{g}_0 \hookrightarrow L_\alpha.$$

We will also denote this embedding by $i$.

**Corollary 2.9.** Let $S := \Gamma \setminus \{\alpha\}$ and $\zeta_S := \{h \in \mathfrak{h} : \beta(h) = 0, \forall \beta \in S\}$. For any Manin triple $(Q', \Delta_\alpha, I)$, there exists a unique generalized Belavin-Drinfeld data $BD = (A, A', i_\alpha, i_\alpha')$ where $A : i(S) \times \{0\} \to \{0\} \times S$, $A(i(\gamma),0) = (0,\gamma)$ and $i_\alpha = \text{diag}(\zeta_S)$, such that $(Q', \Delta_\alpha, I)$ is conjugate to the Manin triple $T_{BD} = (Q', i, i')$. Moreover, up to a conjugation which preserves $\Delta_\alpha$, one has $i = i'$.

**Proof.** Let us suppose that $(Q', \Delta_\alpha, I)$ is a Manin triple. Then there exists a unique generalized Belavin-Drinfeld data $BD = (A, A', i_\alpha, i_\alpha')$ such that the corresponding $T_{BD} = (Q', i, i')$ is conjugate to $(Q', \Delta_\alpha, I)$. Since $i$ and $\Delta_\alpha$ are conjugate and $\Delta_\alpha$ is “under” the parabolic subalgebra $P^-_\alpha$, it follows that $i$ is also “under” this parabolic and thus $a = \zeta_S \times \zeta_S$. According to [2], p. 136, the map $A$ should be an isometry between $i(S) \times \{0\}$ and $\{0\} \times S$. Let us write $A(i(\gamma),0) = (0,\tilde{\alpha}(\gamma))$, where $\tilde{\alpha} : S \to S$ is an isometry which will be determined below.

Let $m$ be the image of $\mathfrak{g}_0$ in $L_\alpha$ via the embedding $i$. Then $m$ is spanned by $X_{i(\beta)}$, $H_{i(\beta)}$, $Y_{i(\beta)}$ for all $\beta \in S$.

According to Theorem 2.8, the Lagrangian subspace $\mathfrak{k}$ contains $\{\{X, \tau(X) \} : X \in m\}$, where $\tau$ satisfies the following conditions: $\tau(X_{i(\beta)}) = X_{\tilde{\alpha}(\beta)}$, $\tau(H_{i(\beta)}) = H_{\tilde{\alpha}(\beta)}$, $\tau(Y_{i(\beta)}) = Y_{\tilde{\alpha}(\beta)}$ for all $\beta \in S$.

We obtain $\tau_i(X_\beta) = X_{\tilde{\alpha}(\beta)}$, $\tau_i(Y_\beta) = Y_{\tilde{\alpha}(\beta)}$, $\tau_i(H_\beta) = H_{\tilde{\alpha}(\beta)}$ for all $\beta \in S$.

Since $i$ and $\Delta_\alpha$ are conjugate, we must have $i_\alpha = \text{diag}(\zeta_S)$. Moreover $\tau_i$ has to be an inner automorphism of $\mathfrak{g}_0$. It follows that the isometry $\tilde{\alpha} : S \to S$ which corresponds to this inner automorphism must be the identity. Thus $\tilde{\alpha} = id$ and this ends the proof.

We will consider triples of the form $(\Gamma'_1, \Gamma'_2, \hat{A}')$, where $\Gamma'_1 \subseteq \Gamma^{\text{ext}} \setminus \{\alpha\}$, $\Gamma'_2 \subseteq \Gamma$ and $\hat{A}'$ is an isometry between $\Gamma'_1$ and $\Gamma'_2$.

**Definition 2.10.** We say that a triple $(\Gamma'_1, \Gamma'_2, \hat{A}')$ is of type I if $\alpha \notin \Gamma'_2$ and $(\Gamma'_1, i(\Gamma'_2), i\hat{A}')$ is an admissible triple in the sense of [1]. The triple $(\Gamma'_1, \Gamma'_2, \hat{A}')$ is of type II if $\alpha \in \Gamma'_2$ and $A'(\beta) = \alpha$, for some $\beta \in \Gamma'_1$ and $(\Gamma'_1 \setminus \{\beta\}, i(\Gamma'_2 \setminus \{\alpha\}), i\hat{A}')$ is an admissible triple in the sense of [1].

Using the definition of generalized Belavin-Drinfeld data, one can easily check the following:

**Lemma 2.11.** Let $A : i(S) \times \{0\} \to \{0\} \times S$, $A(i(\gamma),0) = (0,\gamma)$ and $i_\alpha = \text{diag}(\zeta_S)$. A quadruple $(A, A', i_\alpha, i_\alpha')$ is generalized Belavin-Drinfeld data if and only if the pair $(A', i_\alpha')$ satisfies the following conditions:

1. $A' : \Gamma'_1 \times \{0\} \to \{0\} \times \Gamma'_2$ is given by $A'(\gamma,0) = (0,\hat{A}'(\gamma))$ and $(\Gamma'_1, \Gamma'_2, \hat{A}')$ is of type I or II from above.

2. Let $I'_1$ be the subspace of $\mathfrak{h} \times \mathfrak{h}$ spanned by pairs $(H_{(\gamma)}, H_\gamma)$ for all $\gamma \in S$ and $I'$ be the subspace of $\mathfrak{h} \times \mathfrak{h}$ spanned by pairs $(H_{\beta}, H_{A'(\beta)})$ for all $\beta \in \Gamma'_1$. Let $i_\alpha'$ be a Lagrangian subspace of $\mathfrak{a}' := \{(h_1, h_2) \in \mathfrak{h} \times \mathfrak{h} : \beta(h_1) = 0, \gamma(h_2) = 0, \forall \beta \in \Gamma'_1, \forall \gamma \in \Gamma'_2\}$. Then

\[\text{span}(I') \cap (I' \oplus i_\alpha') = 0.\]

**Remark 2.12.** One can always find $i_\alpha'$ which is a Lagrangian subspace of $\mathfrak{a}'$ and satisfies condition (2.6). This is a consequence of [2] Remark 2, p. 142. A proof of this elementary fact can also be found in [12], Lemma 5.2.
Summing up the previous results we conclude the following:

**Theorem 2.13.** Let $\alpha$ be a simple root. Suppose that $l$ is a Lagrangian subalgebra of $L_\alpha \times g$ transversal to $\Delta_\alpha$. Then, up to a conjugation which preserves $\Delta_\alpha$, one has $l = l'$, where $l'$ is constructed from a pair formed by a triple $(\Gamma'_1, \Gamma'_2, \Lambda')$ of type I or II and a Lagrangian subspace $i_{\alpha'}$ of $\alpha'$ such that (2.6) is satisfied.

**Remark 2.14.** The above algorithm covers also the case when the simple root $\alpha$ has coefficient $k = 1$ in the decomposition of $\alpha_{\text{max}}$, which was considered in [7]. In this form we do not need to use the Cartan involution which appeared there.

**Remark 2.15.** In [7] we constructed two examples of quasi-trigonometric solutions for $g = sl(n + 1)$ related to the Cremmer-Gervais $r$-matrix. Let us apply the present algorithm for the simple root $\alpha_n$. Let $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ and $a_0 = -a_{\text{max}}$.

We consider the triple $(\Gamma'_1, \Gamma'_2, \Lambda')$, where $\Gamma'_1 = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-2}\}$, $\Gamma'_2 = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\Lambda'(\alpha_j)$ is $\alpha_{j+1}$, for $j = 0, \ldots, n - 2$. This is a triple of type I, cf. Definition 2.14.

Another example is given by the following triple: $\Gamma'_1 = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$, $\Gamma'_2 = \{\alpha_1, \ldots, \alpha_n\}$ and $\Lambda'(\alpha_j) = \alpha_{j+1}$, for $j = 1, \ldots, n - 1$. This is a triple of type II, cf. Definition 2.10.

The above triples together with the corresponding subspaces $i_{\alpha'}$ (which we do not compute here) induce two Lagrangian subalgebras of Cremmer-Gervais type.

Theorem 2.13 can also be applied to roots which have coefficient greater than one. As an example, let us classify solutions in $g = o(5)$.

**Corollary 2.16.** Let $\alpha_1, \alpha_2$ be the simple roots in $o(5)$ and $a_0 = -2\alpha_1 - \alpha_2$. Up to gauge equivalence, there exist two quasi-trigonometric solutions with non-trivial polynomial part.

**Proof.** The root $\alpha_2$ has coefficient $k = 1$ in the decomposition of the maximal root. The only possible choice for a triple $(\Gamma'_1, \Gamma'_2, \Lambda')$ with $\Gamma'_1 \subseteq \{\alpha_0, \alpha_1\}, \Gamma'_2 \subseteq \{\alpha_1, \alpha_2\}$ to be of type I or II is $\Gamma'_1 = \{\alpha_0, \alpha_1\}$, $\Gamma'_2 = \{\alpha_1, \alpha_2\}$. One can check that $i_{\alpha'}$ is a 1-dimensional space spanned by the following pair $(\text{diag}(-2, 1, 0, -1, 2), \text{diag}(0, \sqrt{5}, 0, -\sqrt{5}, 0))$. The Lagrangian subalgebra $i_2$ constructed from this triple is transversal to $\Delta_{\alpha_1}$ in $g \times g$.

The root $\alpha_1$ has coefficient $k = 2$ in the decomposition of the maximal root. The only possible choice for a triple $(\Gamma'_1, \Gamma'_2, \Lambda')$ with $\Gamma'_1 \subseteq \{\alpha_0, \alpha_2\}$, $\Gamma'_2 \subseteq \{\alpha_1, \alpha_2\}$ is again $\Gamma'_1 = \{\alpha_0, \alpha_1\}$, $\Gamma'_2 = \{\alpha_2\}$, $\Lambda'(\alpha_0) = \alpha_2$ and $i_{\alpha'}$ is as in the previous case. The Lagrangian subalgebra $i'_1$ constructed from this triple is transversal to $\Delta_{\alpha_1}$ in $L_{\alpha_1} \times g$.

\[ \square \]

### 3. Quantization of Quasi-Trigonometric Solutions

Let us consider the Lie bialgebra structure on $g[u]$ given by the simplest quasi-trigonometric solution:

\[ (3.1) \]

$$
\delta(p(u)) = [r(u, v), p(u) \otimes 1 + 1 \otimes p(v)],
$$

with $r(u, v) = v\Omega/(u - v) + \sum_\alpha e_\alpha \otimes f_\alpha + \frac{1}{2}\Omega_0$, where $e_\alpha, f_\alpha$ are root vectors of $g$ and $\Omega_0$ is the Cartan part of $\Omega$. Here $\delta$ and $r$ are exactly the same as $\delta_3$ and $r_3$ from the Introduction.

Quasi-trigonometric solutions correspond to Lie bialgebra structures on $g[u]$ which are obtained by twisting $\delta$. The quantization of the Lie bialgebra $(g[u], \delta)$ is the quantum algebra $U_q(g[u])$ introduced by V. Tolstoy in [14]. We will recall its construction with a slight modification.

Let $g$ be a finite-dimensional complex simple Lie algebra of rank $l$ with a standard Cartan matrix $A = (a_{ij})_{i,j=1}^l$, with a system of simple roots $\Gamma = \{\alpha_1, \ldots, \alpha_l\}$, and with a Chevalley basis $h_\alpha, e_\pm \alpha, (i = 1, 2, \ldots, l)$. Let $\theta$ be the maximal (positive) root of $g$. The corresponding non-twisted affine algebra with zero central charge $\tilde{g}$ is generated by $g$ and the additional affine elements $e_{-\delta - \theta} := u^{-1}e_\theta$.

The Lie algebra $g[u]$ is generated by $g$, the positive root vector $e_{\delta - \theta}$ and the Cartan element $h_{\delta - \theta} = [e_{\delta - \theta}, e_{-\delta + \theta}]$. The standard defining relations of the universal enveloping algebra $U(g[u])$ are
given by the formulas:

\[
\begin{align*}
(h_{\alpha_i}, h_{\alpha_j}) &= 0, \\
(h_{\alpha_i}, e_{\pm \alpha_j}) &= \pm (\alpha_i, \alpha_j)e_{\pm \alpha_j}, \\
[e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{ij}h_{\alpha_i}, \\
(ade_{\pm \alpha_i})_{n_{ij}}e_{\pm \alpha_j} &= 0 \quad \text{for } i \neq j, \ n_{ij} := 1 - a_{ij}, \\
(h_{\alpha_i}, e_{\delta - \theta}) &= -(\alpha_i, \theta)e'_{\delta - \theta}, \\
[e_{-\alpha_i}, e_{\delta - \theta}] &= 0, \\
(ade_{\alpha_i})_{n_{i\theta}}e_{\delta - \theta} &= 0 \quad \text{for } n_{i\theta} = 1 + 2(\alpha_i, \theta)/(\alpha_i, \alpha_i), \\
[[e_{\alpha_i}, e_{\delta - \theta}], e_{\delta - \theta}] &= 0 \quad \text{for } \mathfrak{g} \neq \mathfrak{sl}_2 \text{ and } (\alpha_i, \theta) \neq 0, \\
[[e_{\alpha_i}, e_{\delta - \alpha}], e_{\delta - \alpha}] &= 0 \quad \text{for } \mathfrak{g} = \mathfrak{sl}_2 \ (\theta = \alpha).
\end{align*}
\]

The quantum algebra $U_q(\mathfrak{g}[u])$ is a $q$-deformation of $U(\mathfrak{g}[u])$. The Chevalley generators for $U_q(\mathfrak{g}[u])$ are $k_{\alpha_i}^{\pm 1} := q^{\pm h_{\alpha_i}}, e_{\pm \alpha_i}$, $(i = 1, 2, \ldots, l)$, $e_{\delta - \theta}$ and $k_{\delta - \theta} := q^{h_{\delta - \theta}}$. Then the defining relations of $U_q(\mathfrak{g}[u])$ are the following:

\[
\begin{align*}
k_{\alpha_i}^{\pm 1}k_{\alpha_j}^{\pm 1} &= k_{\alpha_j}^{\pm 1}k_{\alpha_i}^{\pm 1}, \\
k_{\alpha_i}k_{\alpha_i}^{-1} &= k_{\alpha_i}^{-1}k_{\alpha_i} = 1, \\
k_{\alpha_i}e_{\pm \alpha_j}k_{\alpha_i}^{-1} &= q^{\pm (\alpha_i, \alpha_j)}e_{\pm \alpha_j}, \\
[e_{\alpha_i}, e_{-\alpha_j}] &= \frac{k_{\alpha_i} - k_{\alpha_j}^{-1}}{q - q^{-1}}, \\
(ade_{\pm \alpha_i})_{n_{ij}}e_{\pm \alpha_j} &= 0 \quad \text{for } i \neq j, \ n_{ij} := 1 - a_{ij}, \\
k_{\alpha_i}e_{\delta - \theta}k_{\alpha_i}^{-1} &= q^{-(\alpha_i, \theta)}e_{\delta - \theta}, \\
[e_{-\alpha_i}, e_{\delta - \theta}] &= 0, \\
(ade_{\alpha_i})_{n_{i\theta}}e_{\delta - \theta} &= 0 \quad \text{for } n_{i\theta} = 1 + 2(\alpha_i, \theta)/(\alpha_i, \alpha_i), \\
[[e_{\alpha_i}, e_{\delta - \theta}], e_{\delta - \theta}] &= 0 \quad \text{for } \mathfrak{g} \neq \mathfrak{sl}_2 \text{ and } (\alpha_i, \theta) \neq 0, \\
[[e_{\alpha_i}, e_{\delta - \alpha}], e_{\delta - \alpha}] &= 0 \quad \text{for } \mathfrak{g} = \mathfrak{sl}_2,
\end{align*}
\]

where $(ad_q e_{\beta})e_{\gamma}$ is the q-commutator:

\[
(\text{ad}_q e_{\beta})e_{\gamma} := [e_{\beta}, e_{\gamma}]_q := e_{\beta}e_{\gamma} - q^{(\beta, \gamma)}e_{\gamma}e_{\beta}.
\]

The comultiplication $\Delta_q$, the antipode $S_q$, and the co-unit $\varepsilon_q$ of $U_q(\mathfrak{g}[u])$ are given by

\[
\begin{align*}
\Delta_q(k_{\alpha_i}^{\pm 1}) &= k_{\alpha_i}^{\pm 1} \otimes k_{\alpha_i}^{\pm 1}, \\
\Delta_q(e_{-\alpha_i}) &= e_{-\alpha_i} \otimes k_{\alpha_i} + 1 \otimes e_{-\alpha_i}, \\
\Delta_q(e_{\alpha_i}) &= e_{\alpha_i} \otimes 1 + k_{\alpha_i}^{-1} \otimes e_{\alpha_i}, \\
\Delta_q(e_{\delta - \theta}) &= e_{\delta - \theta} \otimes 1 + k_{\delta - \theta}^{-1} \otimes e_{\delta - \theta},
\end{align*}
\]
\[ S_q(k_{\pm i}^{\pm 1}) = k_{\pm i}^{\pm 1}, \]
\[ S_q(e_{-\alpha_i}) = -e_{-\alpha_i}k_{\alpha_i}^{-1}, \]
\[ S_q(e_{\alpha_i}) = -k_{\alpha_i}e_{\alpha_i}, \]
\[ S_q(\varepsilon_{\delta - \theta}) = -k_{\delta - \theta}^{-1}e_{\delta - \theta}, \]
\[ \varepsilon_q(e_{\pm \alpha_i}) = \varepsilon_q(\varepsilon_{\delta - \theta}) = 0, \quad \varepsilon_q(k_{\alpha_i}^{\pm 1}) = 1. \]

We see that
\[ k_{\delta - \theta} = k_{\alpha_1}^{-n_1}k_{\alpha_2}^{-n_2} \cdots k_{\alpha_l}^{-n_l} \]
if \( \theta = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_l\alpha_l. \)

Let us consider now \( U_\hbar(\mathfrak{g}[u]) \), which is an algebra over \( \mathbb{C}[\hbar] \) defined by the previous relations in which we set \( q = \exp(\hbar) \).

**Theorem 3.1.** The classical limit of \( U_\hbar(\mathfrak{g}[u]) \) is \( (\mathfrak{g}[u], \delta) \).

**Proof.** Since the image of \( e_{\delta - \theta} \) in \( U_\hbar(\mathfrak{g}[u])/\hbar U_\hbar(\mathfrak{g}[u]) \) is \( ue_{\delta} \), we can identify the classical limit of \( U_\hbar(\mathfrak{g}[u]) \) with \( U(\mathfrak{g}[u]) \).

It remains to prove that for any \( a \in \mathfrak{g}[u] \) and its preimage \( \tilde{a} \in U_\hbar(\mathfrak{g}[u]) \), we have
\[ h^{-1}(\Delta(\tilde{a}) - \Delta^\alpha(\tilde{a})) \mod h = \delta(a). \]

It is clear that \( U_\hbar(\mathfrak{g}[u]) \) is a Hopf subalgebra of \( U_\hbar(\tilde{g}) \) with zero central charge. It follows from [3] that \( \Delta^\alpha(\tilde{a}) = R\Delta(\tilde{a})R^{-1} \), where \( R = 1 \otimes 1 + h\mathfrak{r} + \cdots \) is the universal \( R \)-matrix for \( U_\hbar(\mathfrak{g}) \) (see also [8] and [9]).

Writing \( \tilde{a} = a + hs \), we see that
\[ h^{-1}(\Delta(\tilde{a}) - R\Delta(a)R^{-1}) \mod h = [r, a \otimes 1 + 1 \otimes a] = \delta(a). \]

This ends the proof of the theorem. \( \square \)

The Lie bialgebra structures induced by quasi-trigonometric solutions are obtained by twisting \( \delta \) via a classical twist. According to Theorem 2.13 any such twist is given by a triple \( (\Gamma'_1, \Gamma'_2, \tilde{A}') \) of type I or II together with a Lagrangian subspace \( i_{\mathfrak{r}'} \) of \( \mathfrak{r}' \) such that (2.6) is satisfied.

On the other hand, Theorem 2.11 states that classical twists can be extended to quantum twists.

We conclude that any data of the form \( (\Gamma'_1, \Gamma'_2, \tilde{A}', i_{\mathfrak{r}'}) \) provides a twisted comultiplication and antipode in the quantum algebra \( U_\hbar(\mathfrak{g}[u]) \). In case \( \mathfrak{g} = \mathfrak{sl}(n) \), some exact formulas were obtained in [7].

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