NON-NESTING ACTIONS OF POLISH GROUPS ON PRETREES

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Abstract

We study non-nesting actions of groups on \(\mathbb{R}\)-trees. We prove some fixed point theorems for such actions under the assumption that a group is Polish and has a comeagre conjugacy class.

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1. Introduction

Non-nesting actions by homeomorphisms on \(\mathbb{R}\)-trees frequently arise in geometric group theory (when actions on spaces more general than trees are considered). Explicitly they were introduced in \cite{13}. We concentrate on the question when a Polish group \(G\) has fixed points under non-nesting actions on \(\mathbb{R}\)-trees. In Section 3 we prove the following theorem:

Let a Polish group \(G\) have a non-nesting action on an \(\mathbb{R}\)-tree \(T_0\) without \(G\)-fixed points in \(T_0\). Let \(X \subseteq G\) be a comeagre set. Then the following statements hold.

If every element of \(X\) fixes a point, then every element of \(G\) fixes a point.
If \(G\) fixes an end and \(X\) is a conjugacy class, then every element of \(G\) fixes a point.

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These results are related to the paper of D. Macpherson and S. Thomas [14] where they study actions of Polish groups on simplicial trees. Moreover in Section 3 we generalize the main result of [14] that if a Polish group has a comeagre conjugacy class then every element of the group fixes a point under any action on a $\mathbb{Z}$-tree without inversions. Our generalization concerns a wide class of non-nesting actions on pretrees covering the case of isometric actions on $\mathbb{R}$-trees. We apply this to actions of the group $Sym(\omega)$.

On the other hand we are not able to extend the theorem of Macpherson and Thomas to non-nesting actions on $\mathbb{R}$-trees in general. The main difficulty is that in this case we lose several basic properties of isometric actions (for example based on the material of Section 1 of [7]). To remedy the situation we have made some general investigation of non-nesting actions on $\mathbb{R}$-trees. This material is contained in Sections 1 and 2.

In these sections we apply some axiomatic approach and as a result we really study a more general class of actions. First of all it turns out that the most appropriate language for actions on trees by homeomorphisms is that of the betweenness relation $B(x; y, z)$; the corresponding structures are called pretrees. The author has introduced in [10] classical actions on pretrees and has noticed there that $\mathbb{R}$-trees with isometric $G$-actions are classical. Moreover in Section 2.2 below we show that non-nesting actions by homeomorphisms on $\mathbb{R}$-trees are classical too. The most interesting thing is that generalizing the theorem of Macpherson and Thomas we use methods which work for classical actions on median pretrees in general. In particular we apply some new statement about products of loxodromic elements (Proposition 1.5 below) which can be considered as a metric-free version of sections 1.5 - 1.11 from [7].

It is also worth noting that in our considerations we really use some algebraic property of comeagre classes (Condition (1) of Proposition 5.1); thus the results can be formulated in elementary terms.

Trying to generalize the theorem of Macpherson and Thomas to non-nesting actions on $\mathbb{R}$-trees we cannot eliminate the case when the comeagre conjugacy class consists of loxodromic elements. This case is investigated in Section 4 where we prove the most complicated result of the paper. It roughly says that the presence
of a comeagre conjugacy class of loxodromic elements implies that orbits of any end-stabilizer are much smaller that the corresponding orbits of the group. It is also based on Proposition 1.5.

Several preliminary versions of this paper have appeared since 1999. A very close material is contained in Section 8 of the recent paper 15 of Rosendal, were the theorem of Macpherson and Thomas is extended to Λ-trees. Papers 12 and 16 also study Polish groups with comeagre conjugacy classes and their actions on metric spaces. It is worth noting that some related problems (for example of embedding of generalized trees into R-trees and Λ-trees) have been studied before (see 3, 6, 9 and 13). Our motivation is partially based on these investigations.

1. Median pretrees

In this section we develop our basic tools which we will later apply to the main theorems of the paper. We start with very general results held for group actions on median pretrees.

The following definitions are taken from 10. Basically they appear in 3. The definition of a pretree is related to the definition of a B-relation given in 1.

**Definition 1.1.** A ternary structure \((T, B)\) is a pretree if the following axioms are satisfied:

* \((\forall x, y, z)(\neg B(y; x, x) \land \neg(B(y; x, z) \land B(z; x, y)))\);
* \((\forall x, y, z)(B(y; x, z) \leftrightarrow B(y; z, x))\);
* \((\forall x, y, z, w)(B(z; x, y) \land z \neq w \rightarrow (B(z; x, w) \lor B(z; y, w)))\);

Define \([t, t'] = \{x \in T : B(x; t, t') \lor x = t \lor x = t'\}\) the (closed) interval (segment) with endpoints \(t, t'\). We say that \([t, t'] (and (t, t'], (t, t']) under the natural definition) is an interval too. A nonempty subset \(S \subseteq T\) is an arc, if \(S\) is full (that is \((\forall x, y \in S)[x, y] \subseteq S\) and linear (for all distinct \(x, y, z \in S\) we have \(B(y; x, z) \lor B(z; x, y) \lor B(x; y, z)\)).

A pretree is complete if every arc is an interval, not necessarily closed.

A point \(x \in T\) is terminal, if \((\forall y, z \in T)\neg B(x; y, z)\). The pretree \(T\) can be naturally decomposed \(T = T_0 \cup P\), where \(P\) is the set of all terminal points.
The pretree \((T, B)\) is median if for any \(x, y, z \in T\) there is an element \(c \in [x, y] \cap [y, z] \cap [z, x]\). In this case \(c\) is unique and is called the median of \(x, y, z\); we will write \(c = m(x, y, z)\).

**Assumption 1.2.** From now on we consider only median pretrees.

It is clear that every simplicial or real tree can be considered as a complete median pretree by adding ends as terminal points and taking the reduct to the natural betweenness relation.

The following notion will be applied below several times (for subpretrees of median pretrees). It has not been formulated before. We say that a pretree is quasi-median if for any triple \(t, q, r\), if the interval \([t, q)\) is not closed and is contained in \([t, r]\) then \(q \in [t, r]\). To see that a median pretree is quasimedian let \(c = m(t, q, r)\). If \(c = q\) then \(q \in [t, r]\). If \(c \neq q\), then since \([t, q)\) is not closed and \(c \in [t, q)\), the interval \((c, q) = [t, q) \setminus [t, r]\) is not empty, contradicting the assumptions.

Let \(x \in T_0\). A maximal arc of the form \(L_x = \bigcup [x, t_\gamma]\) where all \(t_\gamma\) are not terminal, is called a half-line. Half-lines \(L_x\) and \(L_y\) are equivalent if there is a half-line \(L_z \subseteq L_x \cap L_y\). An end \(e\) of \(T\) is an equivalence class of half-lines. Define a partial order \(\prec_e\) by \(x \prec_e y\) if the half-line \(L_x\) representing \(e\) contains \(y\).

It is clear that an arc of the form \([x, p)\), with \(x, p \in P\), is a half-line. Since \(p\) is a terminal point, any pair of half-lines \([x, p)\) and \([y, p)\) with \(x, y \in T_0\), have a common point from \(T_0\) (which is the corresponding median). This shows that the set of all half-lines \([t, p), t \in T_0\) forms an end (the end corresponding to \(p \in P\)).

A maximal arc of the form \(\bigcup [t_\gamma, t'_\gamma]\) where \(t_\gamma, t'_\gamma\) are not terminal, is called a line. It is worth noting that the ends of a line of a complete pretree \(T\) are presented by a pair of terminal points of \(T\). The following lemma is conceivably known (see [18], [4] and Section 2 of [3]) and is based on existence of medians. A complete proof of the lemma (in a slightly more general situation) is given in [10].

**Lemma 1.3.** 1. The intersection of a finitely many (distinct) segments or half-lines with a common extremity \(t \in T_0\) is a segment having \(t\) as an extremity.

2. If \(t, q, r \in T_0\) satisfy \([t, q) \cap [q, r] = \{q\}\), then \([t, q] = [t, q) \cup [q, r]\). If the interval \([t, q)\) is not closed and \([t, q) = [t, r]\), then \(q = r\).
3. Let $T$ be complete and $A$ and $B$ be two full subsets of $T_0$ whose intersection consists of at most one point. Then there exists a segment $[t, q]$ which is contained in every full set which has a non-empty intersection with both $A$ and $B$, and, moreover, $A \cup \{t\}$ and $B \cup \{q\}$ are full. If for some $\epsilon, \tau \in \{0, 1\}$, $t$ and $q$ satisfy $t \in^{\epsilon} A \land q \in^{\tau} B$ (where $\in^0$ denotes $\notin$), then the condition $t \in^{\epsilon} A \land q \in^{\tau} B$ determines the segment uniquely.

4. Under the circumstances of the previous statement if $A$ is a segment, a line or a half-line, then $t$ as above can be found in $A$. Moreover in the case when $A$ and $B$ are lines these statements hold without the assumption that $T$ is complete.

Let $T$ and $A, B \subseteq T_0$ with $A \cap B = \emptyset$, satisfy the assumptions of Lemma 1.3 (3 or 4). Then there exists a segment $[t, q]$ which is contained in every full set which has a non-empty intersection with both $A$ and $B$, and, moreover, $A \cup \{t\}$ and $B \cup \{q\}$ are full. Then we call the interval $[t, q] \setminus (A \cup B)$ the bridge between $A$ and $B$. It can happen that the bridge is an open (or empty) interval. In this case we define the bridge by its extremities $t$ and $q$ as $(t, q)$.

Let $G$ be a group acting on a median pretree $T$ by automorphisms of the structure $(T, B)$. It is clear that the set $P$ of terminal points is $G$-invariant. Let $g \in G$. The set of $g$-fixed points is denoted by $T^g$. The element $g \in G$ is loxodromic, if $T^g_0 = \emptyset$, $|g| = \infty$ and there exists a unique $g$-invariant line in $T_0$ such that $g$ preserves the natural orders on the line. It is called the axis (characteristic line) of $g$. In the case of an isometric action on an $\mathbb{R}$-tree a loxodromic element is hyperbolic. The proof of the following lemma is standard (by arguments from [18], Section 3.1) and can be found in [10].

**Lemma 1.4.** Let $G$ act on $T$ and $g \in G$. If $g$ is loxodromic, then

(a) for any $p \in T_0$ the segment $[p, g(p)]$ meets the characteristic line $L_g$ and $[p, g(p)] \cap L_g = [q, g(q)]$ for some $q \in L_g$.

(b) $x \in L_g$ if and only if $x$ is the median of $x, g^{-1}(x), g(x)$.

The following proposition can be considered as a metric-free version of statements 1.5 - 1.11 of [7]. It is very convenient in applications and will be one of the basic tools of the paper.
Proposition 1.5. Let $G$ act on a median pretree $T$. Let $h_1, h_2, h_3 \in G$ be loxodromic and $h_2 \cdot h_1 = h_3$. Then one of the following cases holds:

1. There are $d \in L_{h_1}$, $b \in L_{h_2}$ and $c \in L_{h_3}$ such that one of the segments $[d, h_1(d)]$, $[b, h_2(b)]$ and $[c, h_3(c)]$ properly contains the others;

2. There is a non-linear triple $d, b, c$ with $m(d, b, c) \in L_{h_1} \cap L_{h_2} \cap L_{h_3}$ such that $c \in L_{h_2} \cap L_{h_3}$, $b \in L_{h_1} \cap L_{h_2}$ and $h_1(d) = b$, $h_2(b) = c$, $h_3(d) = c$. In this case the element $h_4 = h_2^{-1} h_1$ has no fixed points and the set $\bigcup \{(h_4(d), h_4^{j+1}(d)) : j \in \mathbb{Z}\}$ is an arc such that its segment $[d, h_4(d)]$ properly contains $[d, h_1(d)]$ and $[b, h_2^{-1}(b)]$.

Proof. Consider the case when $|L_{h_1} \cap L_{h_2}| \leq 1$. Let a segment $[a, b]$ satisfy Lemma 1.3 (3,4) with respect to $L_{h_1}$ and $L_{h_2}$ (thus $a \in L_{h_1}$ and $b \in L_{h_2}$). Then applying Lemma 1.3 (2) three times,

$$[a, h_2(h_1(a))] = [a, b] \cup [b, h_2(b)] \cup [h_2(b), h_2(a)] \cup [h_2(a), h_2(h_1(a))].$$

Moreover, by Lemma 1.4 the segment $[a, h_2(h_1(a))]$ meets $L_{h_3}$. Similarly the segment $[h_1^{-1}(b), h_2(b)]$ meets $L_{h_3}$ and

$$[h_1^{-1}(b), h_2(b)] = [h_1^{-1}(b), h_1^{-1}(a)] \cup [h_1^{-1}(a), a] \cup [a, b] \cup [b, h_2(b)].$$

Since $[h_1^{-1}(b), h_2(h_1(a))] = [h_1^{-1}(b), h_1^{-1}(a)] \cup [h_1^{-1}(a), a] \cup [a, b] \cup [b, h_2(b)] \cup [h_2(b), h_2(a)] \cup [h_2(a), h_2(h_1(a))]$, the line $L_{h_3}$ must meet $[a, b] \cup [b, h_2(b)]$.

If $[b, h_2(b)] \cap L_{h_3} \neq \emptyset$, then $[h_1^{-1}(h_2^{-1}(b)), h_1^{-1}(b)] \cap L_{h_3} \neq \emptyset$ and the elements between those intersections lie in $L_{h_3}$. Thus $h_1^{-1}(b), h_1^{-1}(a), a, b \in L_{h_3}$. This implies $h_2(h_1(a)) \in L_{h_3}$ and $h_2(b) \in L_{h_3}$.

If $[a, b] \cap L_{h_3} \neq \emptyset$, then $[h_2(h_1(a)), h_2(h_1(b))] \cap L_{h_3} \neq \emptyset$ and the elements between those intersections lie in $L_{h_3}$. Thus $b, h_2(b) \in L_{h_3}$. This implies $h_1^{-1}(b) \in L_{h_3}$ and $h_1^{-1}(a), a \in L_{h_3}$.

Let $d := h_1^{-1}(a)$ and $c := h_1^{-1}(b)$. Then $[b, h_2(b)]$ and $[d, h_1(d)]$ are proper subsegments of $[c, h_3(c)] = [h_1^{-1}(b), h_2(b)]$.

If $|L_{h_1} \cap L_{h_2}| \geq 1$, let $e_1, e_1'$ be the ends of (half-lines of) $L_{h_1}$ and $e_2, e_2'$ be the ends of $L_{h_2}$. For ease of notation we extend the betweenness relation of $T_0$.
to \( T_0 \cup \{ e_1, e_1', e_2, e_2' \} \) in the obvious way (so that \( L_{h_1} = [e_1, e_1'] \cap T_0 \) and \( L_{h_2} = [e_2, e_2'] \cap T_0 \)).

If the lines \( L_{h_1} \) and \( L_{h_2} \) represent the same end \( e \in \{ e_1, e_1' \} \cap \{ e_2, e_2' \} \), then \( L_{h_3} \) represents \( e \). Thus there exists \( d \in L_{h_1} \cap L_{h_2} \cap L_{h_3} \) with

\[
  h_1(d), h_1^{-1}(d), h_2(d), h_2^{-1}(d), h_3(d), h_3^{-1}(d) \in L_{h_1} \cap L_{h_2} \cap L_{h_3}.
\]

Now the lemma is obvious for \( b := h_1(d) \) and \( c = d \).

If \( \{ e_1, e_1' \} \cap \{ e_2, e_2' \} = \emptyset \), find \( a', a'' \in L_{h_1} \cap L_{h_2} \) such that \([a', a''] = L_{h_1} \cap L_{h_2}\) (notice that the points \( a' \) and \( a'' \) can be found as medians of appropriate triples from \( \{ e_1, e_1', e_2, e_2' \} \)). Assume that there are \( q_1 \in \{ e_1, e_1' \} \) and \( q_2 \in \{ e_2, e_2' \} \), such that \( a' = m(q_2, a'', q_1) \) and the following condition holds: \((h_1^{-1}(a') \in [a', q_1]) \) or \((h_2(a') \in [a', q_2]) \). Then \( a' \in [h_2(a'), h_1^{-1}(a')] \). When the conjunction \((h_1^{-1}(a') \in [a', q_1]) \land (h_2(a') \in [a', q_2]) \) does not hold we put \( b = a := a' \) and apply the argument of the case \( |L_{h_1} \cap L_{h_2}| \leq 1 \) (with some simplifications) to an appropriate pair of the segments \([h_1^{-1}(h_2^{-1}(a')), a'], [h_1^{-1}(b), h_2(b)] \) and \([h_1^{-1}(b), h_2(b)], [a, h_2(h_1(a))] \). This argument shows that the segments \([h_1^{-1}(a'), a'] \) and \([a', h_2(a')] \) lie on \( L_{h_3} \) as proper subsegments of \([c, h_3(c)] \) where \( c = h_1^{-1}(a') \).

The case of the conjunction \((h_1^{-1}(a') \in [a', q_1]) \land (h_2(a') \in [a', q_2]) \) is divided into two subcases. The first one appears if \( h_1(a') \in [a', a''] \) or \( h_2^{-1}(a') \in [a', a''] \). Then applying arguments as above we obtain that \( a' \in L_{h_3} \) and either \([a', h_3(a') \cup [a', h_1(a')] \subseteq [h_1(a'), h_2(h_1(a'))] \) or \([h_1^{-1}(h_2^{-1}(a'), a') \cup [h_2^{-1}(a'), a'] \subseteq [h_1^{-1}(h_2^{-1}(a'), h_2^{-1}(a')] \).

In the case of the conjunctions \((h_1^{-1}(a') \in [a', q_1]) \land (h_2(a') \in [a', q_2]) \) and \( h_1(a') \not\in [a', a''] \land h_2^{-1}(a') \not\in [a', a''] \) let \( b = a'' \), \( d = h_1^{-1}(a'') \) and \( c = h_2(a'') \).

It is easy to see that the second condition of the statement of the proposition is satisfied (where \( a' = m(d, b, c) \)). It is straightforward that for all \( j \in \mathbb{Z} \),

\[
[h_4^{-1}(d), h_4^{+1}(d)] \cap [h_4^{-1}(d), h_4^{+1}(d)] = \{h_4^j(d)\} ; \text{ thus } \bigcup([h_4^{-1}(d), h_4^{+1}(d)] : j \in \mathbb{Z}\} \text{ is an arc. To see that } h_4 \text{ does not fix any point assume that } h_4^{-1}(h_4(v)) = v. \text{ Then the segment } [v, h_1(v)] \text{ meets both } L_{h_1} \text{ and } L_{h_2} \text{ and thus } [a', a'']. \text{ Let } [v, h_1(v)] \cap L_{h_1} = [w, h_1(w)]. \text{ The assumption } h_1(a') \not\in [a', a''] \text{ implies that one of the extremities } w \text{ or } h_1(w) \text{ is outside of } [a', a'']. \text{ If } w \not\in [a', a''] \text{ the assumption } h_2^{-1}(a') \not\in [a', a''] \text{ now implies that } [w, a'] \cup [a', a''] \text{ is a non-trivial subsegment of } [v, h_2^{-1}(h_1(v))], \text{ a contradiction with } h_2^{-1}(h_1(v)) = v. \text{ If } h_1(w) \not\in [a', a''] \text{ then }
then similarly \([a'', h_2^{-1}(a'')] \cup [h_2^{-1}(a''), h_2^{-1}(h_1(u))]\) is a non-trivial subsegment of 
\([v, h_2^{-1}(h_1(v))]\), a contradiction.

Now assume that \(\{e_1, e_1'\} \cap \{e_2, e_2'\} = \emptyset\), but none of the cases \((h_1^{-1}(a') \in [a', q_1])\)
and \((h_2(a') \in [a', q_2])\) holds where \([a', a''] = L_{h_1} \cap L_{h_2}\) and \(q_1\) and \(q_2\) are as above.

Now we have that \(h_1(a') \in [a', q_1]\) and \(h_2^{-1}(a') \in [a', q_2]\). We may assume that \(q_1 = e_1\) and \(q_2 = e_2\). Since \(\{e_1, e_1'\} \cap \{e_2, e_2'\} = \emptyset\), we have \(a'' = m(e_2', a', e_1')\), \(h_1(a'') \in [a'', e_1']\) and \(h_2^{-1}(a'') \in [a'', e_2']\) (otherwise the arguments of the paragraphs above work for \(a''\) instead of \(a'\)). Then \(h_1^{-1}(a'), h_2(a') \in [a', a'']\) and 
\(h_1^{-1}(a''), h_2(a'') \in [a', a'']\) (because \(h_1\) and \(h_2\) preserve the natural orderings of 
their axises). If \(h_1^{-1}(a') \in L_{h_3}\) then taking \(d = c = h_1^{-1}(a')\) and \(b = a'\) we see that 
one of the segments \([d, h_1(d)]\) or \([b, h_2(b)]\) properly contains the remaining ones 
(including \([c, h_3(c)]\)). The same argument works or the case \(h_1^{-1}(a'') \in L_{h_3}\).

Let \(c' = h_1^{-1}(a')\) and \(c'' = h_1^{-1}(a'')\). We know that \([c', h_3(c')] \cup [c'', h_3(c'')] \subseteq [a', a'']\). Since \([c', h_3(c')]\) and \([c'', h_3(c'')]\) meet \(L_{h_3}\), in the case \([c', h_3(c')] \cap [c'', h_3(c'')] = \emptyset\) we have \(c', c'' \in L_{h_3}\) and by the previous paragraph, this is enough for the proposition. Assume \([c', h_3(c')] \cap [c'', h_3(c'')] \neq \emptyset\). Since \(h_1\) and \(h_2\) preserve the natural orders of their axises, \(c' \in [c', h_3(c')] \cap [c'', h_3(c'')]\) or \(c'' \in [c', h_3(c')] \cap [c'', h_3(c'')]\).

We may assume that \(c' \in [c', h_3(c')] \cap [c'', h_3(c'')]\) (then \(h_3(c'') \subseteq [c', h_3(c')]\)). Since \([c', h_3(c')]\) and \([c'', h_3(c'')]\) meet \(L_{h_3}\), there is \(u \in L_{h_3} \cap [c', h_3(c')]\). Then 
\(h_2^{-1}(u) \in [e_2', a'']\) and \((h_1^{-1}(h_2^{-1}(u)), c'') \cap [c'', a'] = \{c''\}\) (because \([h_1^{-1}(h_2^{-1}(u)), c''] = h_1^{-1}([h_2^{-1}(u), a''])\)). This implies \(c', c'' \in [h_3^{-1}(u), u]\). By \(h_3^{-1}(u), u \in L_{h_3}\) we have \(c', c'' \in L_{h_3}\), which is enough for the proposition. \(\square\)

2. Classical actions on pretrees

This section contains preliminary results on non-nesting classical actions.

2.1. Classical actions and non-nesting actions. In the following definition we collect usual properties of typical actions (for example isometric ones).

**Definition 2.1.** The action of \(G\) on a complete median pretree \(T (= T_0 \cup P)\) is classical if the following conditions hold:

(C0) If \(x, y \in T_0^g := T^g \cap T_0\), then \([x, y] \subseteq T_0^g\).

(C1) If \(x \notin T_0^g \neq \emptyset\), then \([x, g(x)] \cap T_0^g = 1\);

(C2) If \(T_0^g = \emptyset\), then \(g\) is loxodromic;
(C3) If $g_1, g_2, g_3 \in G$ are not loxodromic and $g_1 = g_2 \cdot g_3$, then $T^g_0 \cap T^{g_2}_0 \cap T^{g_3}_0 \neq \emptyset$;

(C4) If $g$ is loxodromic, $L$ is the axis of $g$ and $g = h \cdot h'$ with $T^h_0 \neq \emptyset \neq T^{h'}_0$, then $|T^h_0 \cap L| = 1 = |T^{h'}_0 \cap L|$.

It is worth noting here that we do not really need condition (C1) in this paper. We include it into the definition of classical actions because non-nesting actions satisfy it (see below). On the other hand any pretree satisfies a weaker form of (C1): if $x \notin T^g_0 \neq \emptyset$, then $|[x, g(x)] \cap T^g_0| \leq 1$. Indeed, if $u, u' \in T^g_0 \cap [x, g(x)]$ and $u \in [x, u']$, then $[x, g(x)] = [x, u'] \cup [u', g(x)]$ and $u = g(u) \in g([x, u']) = [g(x), u']$, a contradiction.

Let $\Lambda$ be linearly ordered abelian group. A $\Lambda$-metric space $(X, d)$ is called a $\Lambda$-tree if:

1. $X$ is geodesically linear: for any $a, b \in X$ there exists a unique metric morphism $\alpha : [0, d(a, b)] \to X$ such that $\alpha(0) = a$ and $\alpha(d(a, b)) = b$ (then $[a, b] := \alpha([0, d(a, b)])$).
2. $\forall x, y, z \exists! w([x, y] \cap [x, z] = [x, w])$.
3. $\forall x, y, z([x, y] \cap [y, z] = \{y\} \to [x, y] \cup [y, z] = [x, z])$.

For $\mathbb{R}$-trees and isometric actions on them Lemmas 1.3 and 1.4 are known [18] (moreover in Lemma 1.3 for closed $A$ and $B$ we have $t \in A$ and $q \in B$). It is also known that every action of a group on an $\mathbb{R}$-tree by isometries induces a classical action on the pretree extended by all ends with respect to the natural betweenness relation. The proof can be extracted from [18] (a more convenient reference is [8], where Lemma 1.2 corresponds to property (C4) of Definition 2.1). It is worth noting that $T^g_0$ is a closed set if $g$ is an isometry of an $\mathbb{R}$-tree $T_0$.

The following axiom (non-nesting) defines classical actions quite close to isometric ones.

$$\forall g \in G \forall t, t' \in T_0 \land ([g(t), g(t')] \subset [t, t']).$$

The axiom of non-nesting has the following immediate consequences. (The following lemma appears in a different form in [4].)

**Lemma 2.2.** Let a group $G$ have a classical non-nesting action on a pretree $T$ and let $g \in G$ be loxodromic. Then under an appropriate choice of $+\infty$ on $L_g$, the element $g$ is strictly increasing on $L_g$. 

Proof. Let $a, b \in L_g$ and $g(a) < a < b \leq g(b)$. Then $g^{-1}$ maps $[g(a), g(b)]$ properly into itself. The remaining cases are similar. □

Under assumptions of the lemma let $L$ be a line of $T$ and $G_{\{L\}}$ (and $G_L$) be the stabilizer (pointwise stabilizer) of $L$ in $G$. Assume that $G_{\{L\}}$ does not have elements reversing the terminal points of $L$ (this happens when it does not have subgroups of index 2). Then non-nesting implies that given an ordering of $L$ corresponding to the betweenness relation of $T$, the following relation makes $G_{\{L\}}/G_L$ a linearly ordered group: $g < g'$ if and only if $\exists t \in L(g(t) < g'(t))$.

Now assume that $H$ is a subgroup of the stabilizer $G_{\{L\}}$. By completeness $L$ consists of $H$-invariant intervals. If $[a, b)$ is such an interval then by non-nesting, $h(a) \notin [a, b)$, where $h \in H$. We see that $H$ acts trivially on $L$ or there are no proper $H$-invariant arcs in $L$.

This argument also shows that for a loxodromic $g$ the axis $L_g$ does not have proper $g$-invariant subintervals. In the case when $L_g$ is homeomorphic with $\mathbb{R}$ we immediately have that up to topological conjugacy $g$ can be viewed as a translation by a real number. On the other hand we also have that for every $h \in G_{\{L_g\}}$ there exists $n \in \omega$ such that $h < g^n$.

2.2. Non-nesting actions on $\mathbb{R}$-trees and their end stabilizers. Let $G$ be an infinite group acting on an $\mathbb{R}$-tree $T_0$ by homeomorphisms $^1$. By $T$ we denote $T_0$ together with the set of ends. As above the action is called non-nesting if no $g \in G$ maps an arc properly into itself. In this case if $T_0^g$ is not empty then $T_0^g$ is closed. If $g$ does not fix any point, then by Theorem 3(2) from [13] there exists a geodesic $\mathbb{R}$-line $L_g \subseteq T_0$ such that $g$ acts on $L_g$ by an order preserving transformation, which is a translation up to topological conjugacy. The following proposition develops these observations.

**Proposition 2.3.** Let $G$ be a group. Every non-nesting action of $G$ on an $\mathbb{R}$-tree $T_0$ induces a classical action on the corresponding pretree.

**Proof.** Since an $\mathbb{R}$-tree is complete and median we must verify (C0)-(C4) of Definition[2,1]. Condition (C0) is clear. Conditions (C1) and (C2) are proved in Theorem 3(1,2) of [13].

$^1$preserving the betweenness relation
To see (C3) let each of $h_1, h_2, h_3 \in G$ fix points in $T_0$ and $h_2 \cdot h_1 = h_3$. We want to show that $T_0^{h_1} \cap T_0^{h_2} \cap T_0^{h_3} \neq \emptyset$.

Take $t \in T_0^{h_1}$ with minimal distance $d(t, T_0^{h_2})$ (with minimal distance from $T_0^{h_2}$). Then the segment $[t, h_2(t)]$ meets $T_0^{h_2}$ at precisely one point, say $c$. Since $h_3(t) = h_3(h_2(t))$, $[t, h_2(t)]$ meets $T_0^{h_3}$ at precisely one point too, say $d$. If these points are not the same then one of the maps $h_2^{-1}h_3$ or $h_3^{-1}h_2$ maps an arc $[t, d]$ or $[t, c]$ into itself (the first case corresponds to the situation when $c$ is between $t$ and $d$). So, there is a point fixed by $h_2$ and $h_3$. It must be fixed by $h_1$.

The proof of (C4) is related to Proposition 1.2 from [8]. Let $g = h'h$ be loxodromic, but $h, h' \in G$ fix some points in $T_0$. Let $a \in T_0^h$, $b \in T_0^{h'}$ be chosen with minimal $d(a, b)$ (the existence of such $a$ and $b$ follows from the fact that $T_0^h$ and $T_0^{h'}$ are closed). Let us prove that $L_g \cap T_0^h = \{a\}$ and $L_g \cap T_0^{h'} = \{b\}$.

The segment $[a, g(a)] = [a, h'(a)]$ meets $L_g$ and contains $b$ (by (C1) and Lemma 2.2). Assume $[a, g(a)] = [a, q] \cup [q, g(q)] \cup [g(q), g(a)]$ where $[q, g(q)] = L_g \cap [a, g(a)]$. If $b \notin L_g$, for example $b \in [a, q]$, then $g(h')^{-1}([b, g(a)]) = [g(b), g(a)]$ is properly contained in $[b, g(a)]$ (because $[b, g(a)] = [b, q] \cup [q, g(q)] \cup [g(q), g(b)] \cup [g(b), g(a)]$). The case when $b$ is between $g(a)$ and an element from $L_g$ is similar. Thus $b \in L_g$ and $h^{-1}(b) = g^{-1}(b) \in L_g$. Then the segment $[g^{-1}(b), b]$ belongs to $L_g$. Thus $a \in L_g$. If there exists $a' \in T_0^h \setminus \{a\}$ belonging to $L_g$, then $[a', g(a')] \subset L_g \cap T_0^h$. Thus $[g^{-1}(b), b] \cap [a', a] = \{a\}$ and we see that $a', b, g^{-1}(b)$ belong to $L_g$ but are not linear, a contradiction.

The proof that $L_g \cap T_0^{h'} = \{b\}$ is similar. □

In a sense Lemma 2.2 describes elements stabilizing a line under the assumption of non-nesting. We now concentrate on end stabilizers of non-nesting actions on $\mathbb{R}$-trees. We will see that some kind of Lemma 2.2 holds in this case. Let a group $G$ have a non-nesting action on an $\mathbb{R}$-tree $T$ without $G$-fixed points in $T_0$. Assume that there is a loxodromic $g \in G$. Let $a_0 \in L_g$ and $e$ be the end represented by $(-\infty, a_0]$ ($-\infty$ is chosen so that $g$ is increasing). Consider the stabilizer $G_e$. Let $G_{ae}$ be the subset of $G_e$ of all elements fixing some points in $T_0$. Note that any $h \in G_e$ defines a map $(-\infty, a) \rightarrow (-\infty, b)$ for some $a, b \leq a_0$. Thus $G_{ae}$ be the subgroup of elements fixing pointwise $(-\infty, a)$ for some $a \leq a_0$ (by non-nesting).
We also see that it is normal in \( G_e \). In the following lemma we consider the group \( G_e/G(e) \).

**Lemma 2.4.** Let a group \( G \) have a non-nesting action on an \( \mathbb{R} \)-tree \( T \) without \( G \)-fixed points in \( T_0 \). Let \( g \in G \) be loxodromic and \( e \) be the \((-\infty)\)-end of \( L_g \). Then the group \( G_e/G(e) \) is embeddable into \( (\mathbb{R},+) \) as a linearly ordered group under the ordering: \( gG(e) \prec g'G(e) \iff \exists t \in T_0 (g'(t) <_e g(t)) \).

**Proof.** By Lemma 2.2 any \( h \in G_e \setminus G(e) \) defines a map \((-\infty,a] \to (-\infty,b] \) with some \( a,b \in (-\infty,a_0] \), which is strictly monotonic on \((-\infty,a] \). Now notice that under the induced ordering \( \prec \) the group \( G_e/G(e) \) is a linearly ordered group. Indeed, linearity follows from Lemma 2.2. If \( g_1, g_2 \in G_e \) satisfy \( g_1(a) < g_2(a) \) for some \( a \leq a_0 \), then by non-nesting for all \( a' \leq a \), \( g_1(a') < g_2(a') \). We see that for every \( g' \in G_e \) there exists \( b \leq a \) such that \( g_1 \cdot g'(x) \leq g_2 \cdot g'(x) \) for all \( x \leq b \). On the other hand if \( g_1, g_2 \in G_e \) satisfy \( g_1G(e) \preceq g_2G(e) \) then obviously \( g' \cdot g_1G(e) \preceq g' \cdot g_2G(e) \) for all \( g' \in G_e \). This shows that \( G_e/G(e) \) is a linearly ordered group.

Since the elements of \( G_e \) act by translations up to topological conjugacy, \( G_e/G(e) \) is Archimedean. By Hölder’s theorem it is a subgroup of \( (\mathbb{R},+) \). \( \square \)

We now define an action \( *_g \) of \( G_e \) on \( L_g \). Let \( h \in G_e \) and \( c \in L_g \). Find a natural number \( n_0 \) such that \( g^{-n_0}(c) \) is greater with respect to \( <_e \) than any of the elements \( a_0, h(a_0), h^{-1}(a_0) \). Now let \( h *_g c = g^{n_0}h g^{-n_0}(c) \). By the choice of \( n_0 \) we see \( h *_g c \in L_g \).

It is worth noting that for every \( n \geq n_0 \), \( h *_g c = g^n h g^{-n}(c) \). This follows from the fact that the element \( h^{-1} g^n h g^{n_0-n} \) belongs to \( G(e) \) (as \( G_e/G(e) \) is a subgroup of \( (\mathbb{R},+) \)) and then (by non-nesting) the transformations \( h g^{n_0-n} \) and \( g^{n_0-n}h \) are equal at \( g^{-n_0}(c) \). We now see:

\[
g^n h g^{-n}(c) = g^{n_0}h(h^{-1} g^n h g^{n_0-n}(g^{-n_0}(c))) = g^{n_0}h g^{-n_0}(c).
\]

Now it is easy to see that \( *_g \) is an action and the elements of \( G(e) \) act on \( L_g \) trivially.

Let \( L_g^{a_0} = G_e a_0 \cap L_g \) where \( G_e a_0 \) is the orbit of \( a_0 \). Then there exists a surjection \( \nu_{a_0} : G_e \rightarrow L_g^{a_0} \) defined by \( \nu_{a_0}(h) = h *_g a_0 \) (with respect to the action defined
above). It is easy to see that the map \( \nu_{a_0} \) is surjective. Moreover, for any \( h, h' \in G_e \), 
\[ \nu_{a_0}(h \cdot h') = h \ast g \nu_{a_0}(h'). \]

**Lemma 2.5.** The map \( \nu_{a_0} \) defines an order-preserving bijection from \( G_e/G(e) \) onto \( L_{g}^{a_0} \) under the order induced by \( L_{g} \).

**Proof.** Notice that if \( \nu_{a_0}(h_1) = \nu_{a_0}(h_2) \) then \( h_1 h_2^{-1} \) fixes some \((-\infty, a]\), \( a \leq a_0 \), pointwise. Indeed, let \( n \) and \( a = g^{-n}(a_0) \) be chosen so that \( a, h_1(a), h_2(a), h_1^{-1}(a), h_2^{-1}(a) \in L_{g} \cap L_{h_1} \cap L_{h_2} \) (for \( h_i \in G(e) \) we replace \( L_{h_i} \) by \( T_{h_i}^{a_0} \)). Then 
\[ g^n h_1 g^{-n}(a_0) = \nu_{a_0}(h_1) = \nu_{a_0}(h_2) = g^n h_2 g^{-n}(a_0) \]

and we see that \( h_1(a) = h_2(a) \). Now the claim follows from non-nesting.

Applying non-nesting again we obtain that the preimage of \( a_0 \) (with respect to \( \nu_{a_0} \)) equals the subgroup \( G(e) \) of elements fixing pointwise \((-\infty, a]\) for some \( a \leq a_0 \).

The proof of Lemma 2.4 shows that the condition \( h_1 \prec h_2 \) means the existence of \( a \in L_{g} \) with \( h_1(a') < h_2(a') \) for all \( a' < a \). This obviously implies \( \nu_{a_0}(h_1) < \nu_{a_0}(h_2) \). We see that \( (L_{g}^{a_0}, <) \) can be identified with the group \( (G_e/G(e), <) \).

**Lemma 2.6.** If the ordering of \( L_{g}^{a_0} \) is not dense, then \( L_{g}^{a_0} \) is a cyclic group with respect to the structure of \( G_e/G(e) \).

**Proof.** Notice that if there is an interval \((a, b)\), \( a, b \in L_{g}^{a_0} \), which does not have elements from \( L_{g}^{a_0} \), then every \( c \in L_{g}^{a_0} \) has a successor from \( L_{g}^{a_0} \). This follows from the fact that \([a, g(a)] \) and \([c, g(c)] \) can be taken onto \([g^{-1}(a_0), a_0]\) by an element from \( G_e \).

### 3. Polish groups with comeagre conjugacy classes

Dugald Macpherson and Simon Thomas have proved in [14] that if a Polish group has a comeagre conjugacy class then every element of the group fixes a point under any action on a \( Z \)-tree without inversions. In this section we generalize that result to the situation which covers the case of isometric actions. Our method is different and is based on some algebraic property of comeagre classes (Condition (1) of Proposition 3.1). As a result the theorem can be formulated in elementary terms not involving Polish groups. We apply this to actions of the group \( \text{Sym}(\omega) \). In the second part of the section we study actions of groups with comeagre conjugacy classes and invariant ends.
3.1. Fixed points. In the case of an isometric action if \( h_2 \cdot h_1 = h_3 \) and \( h_1, h_2, h_3 \) are loxodromic, then Proposition 1.5 implies that \( h_1, h_2, h_3 \) and \( h_2^{-1} h_1 \) cannot belong to the same conjugacy class. Indeed, if for example \([b, h_2(b)]\) is a proper subsegment of \([c, h_3(c)]\), where \( b \in L_{h_2}, c \in L_{h_3} \), then for no \( c' \in L_{h_3} \), the segment \([c', h_3(c')]\) can be mapped to \([b, h_2(b)]\) by a map induced by some \( g \in G \).

This observation motivates the following proposition.

**Proposition 3.1.** Let a group \( G \) have a classical action on a pretree \( T \). Let \( X \subset G \) satisfy the following conditions.

1. For every sequence \( g_1, ..., g_m \in G \) there exist \( h_0, h_1, ..., h_m \in X \) such that for every \( 1 \leq i \leq m \), \( g_i = h_0 h_i \).

2. If \( T_0^h = \emptyset \) for some \( h \in X \), then all \( h \in X \) are loxodromic and there are no \( h_1, h_2 \in X \) and \( c_1 \in L_{h_1}, c_2 \in L_{h_2}, \) such that \([c_1, h_1(c_1)]\) properly contains \([c_2, h_2(c_2)]\).

Then for any \( g \in G \), \( T_0^g \neq \emptyset \).

**Proof.** If all \( h \in X \) are loxodromic, find \( h_1, h_2, h_3, h_4 \in X \) such that \( h_1 \cdot h_3 = h_2 \) and \( h_1 h_4 = h_2^{-1} \) (by (1)). Then \( h_4^{-1} = h_2 h_1 \), \( h_3^{-1} = h_2^{-1} h_1 \) and by Proposition 1.5 there are \( i, j \in \{1, 2, 3, 4\}, i \neq j \), and \( c \in L_{h_i} \) and \( d \in L_{h_j} \) such that the segment \([c, h_i(c)]\) properly contains \([d, h_j(d)]\). This refutes condition (2).

We now have that for all \( h \in X \), \( T_0^h \neq \emptyset \). Let \( g \in G \) have no fixed points in \( T_0 \) and \( P^g = \{p_1, p_2\} \). Let \( g = h' \cdot h \), \( h, h' \in X \). Find \( a \) and \( b \) satisfying the properties that \( \{a\} \cup T_0^h \) and \( \{b\} \cup T_0^{h'} \) are full and for all \( a' \in T_0^h \) and \( b' \in T_0^{h'}, \) \( \{a, b\} \subset [a', b'] \) (Lemma 1.5(3)); note that formally it may happen that \( a \notin T_0^g \) or \( b \notin T_0^{h'} \). By (C4) of Definition 2.1 there are \( a_0 \in T_0^h \cap [p_1, p_2] \) and \( b_0 \in T_0^{h'} \cap [p_1, p_2] \). This implies that \( a, b \in [p_1, p_2] \).

By (1) there exist \( h_0, h_1, h_2, h_3 \in X \) such that \( g = h_0 h_3, h = h_0 h_1, h' = h_0 h_2 \). By (C3) of Definition 2.1 there are \( a_1 \in T_0^{h_0} \cap T_0^h \) and \( b_1 \in T_0^{h_0} \cap T_0^{h'} \). Then by (C0), \( a, b \in T_0^{h_0} \). Applying (C0) again we see \( m(a, a_0, a_1) \in T_0^{h_0} \cap T_0^h \cap [p_1, p_2] \) and \( m(b, b_0, b_1) \in T_0^{h_0} \cap T_0^{h'} \cap [p_1, p_2] \). On the other hand, by (C4) applied to \( h_0 \) and \( h_3 \), the intersection \( T_0^{h_0} \cap [p_1, p_2] \) is a singleton; then \( m(a, a_0, a_1) = m(b, b_0, b_1) \) and \( T_0^h \cap T_0^{h'} \neq \emptyset \) contradicting the assumption that \( g \) does not have fixed points in \( T_0 \).

\( \Box \)
It is worth noting that condition (1) of Proposition 3.1 can be weakened assuming that \( m = 3 \).

A Polish group is a topological group whose topology is Polish (a Polish space is a separable completely metrizable topological space). A subset is comeagre if it contains an intersection of a countable family of dense open sets.

**Proposition 3.2.** Let \( G \) be a Polish group with a classical action on a pretree \( T = T_0 \cup P \). Let \( X \subseteq G \) be comeagre.

1. Then \( X \) satisfies condition (1) of Proposition 3.1. If \( X \) is a conjugacy class then \( X = X^{-1} \).
2. Let \( X \) consist of elements fixing points in \( T_0 \). Then every \( g \in G \) has a fixed point in \( T_0 \).

**Proof.** (1) Let \( g_1, \ldots, g_m \in G \). Since the set \( X \) is comeagre in \( G \), all \( g_iX^{-1} \) are comeagre and have a common element \( h_0 \). Now find \( h_1, \ldots, h_m \in X \) such that for any \( 1 \leq i \leq m \), \( g_i = h_0h_i \). If \( X \) is a conjugacy class then \( X = g^G \), where \( g \in X \cap X^{-1} \). Thus \( X = X^{-1} \).

(2) By (1) we can apply the proof of Proposition 3.1 \( \square \)

We now see that Proposition 3.1 generalizes the result of Macpherson and Thomas mentioned above. Indeed, let a Polish group \( G \) have a comeagre conjugacy class \( X \). By Proposition 3.2 the group \( G \) satisfies condition (1) of Proposition 3.1. If \( G \) has an isometric action on an \( \mathbb{R} \)-tree then condition (2) of Proposition 3.1 is obvious. Thus every element of \( G \) fixes a point.

**3.2. Permutation groups.** Here we give an application of Proposition 3.2. Let \( A(\mathbb{Q}) \) be the group of order-preserving permutations of the rationals. Then \( A(\mathbb{Q}) \) can be considered as a subgroup of \( Sym(\omega) \). The following theorem shows that classical actions of the symmetric group are determined by \( A(\mathbb{Q}) \).

**Theorem 3.3.** Let \( Sym(\mathbb{Q}) \) have a classical action on a pretree \( T = T_0 \cup P \). If for the corresponding action of \( A(\mathbb{Q}) \) every element fixes a point from \( T_0 \), then so does every element of \( Sym(\mathbb{Q}) \).
Proof. We define an expansion of the structure $(\mathbb{Q}, <)$ by relations $^2 \left( P_n : n \in \omega \setminus \{0\} \right)$. The expansion satisfies the following properties:

$$\forall x_1, \ldots, x_n(P_n(x_1, \ldots, x_n) \rightarrow x_1 < \ldots < x_n);$$

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_m((\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_m\} \neq 0) \land P_n(x_1, \ldots, x_n) \land P_m(y_1, \ldots, y_m) \rightarrow$$

$$\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_m\};$$

$$\forall n, y_1, y_2 \exists x_1, \ldots, x_n(y_1 < y_2 \rightarrow P_n(x_1, \ldots, x_n) \land y_1 < x_1 < \ldots < x_n < y_2);$$

$$\forall x(\exists n, x_1, \ldots, x_n)(P_n(x_1, \ldots, x_n) \land (x \in \{x_1, \ldots, x_n\})).$$

Such an expansion can be easily obtained using the fact that the rationals form a countable dense linear ordering without ends (then having an expansion where all conditions till the last one are satisfied, put the elements for which the last condition does not hold, into $P_1$).

By back-and-forth $^3$ we now find an increasing $f \in A(\mathbb{Q})$ such that:
- each orbit of $f$ is cofinal in $\mathbb{Q}$;
- the union of all $f$-orbits included in $P_1$ is dense in $\mathbb{Q}$ and

$$\forall n, x_1, \ldots, x_n(P_n(x_1, \ldots, x_n) \rightarrow (\exists k \in \mathbb{Z})(\text{assuming that } B = \mathbb{Z} \setminus A$$

and $A = \{i : k < i\}$ or $A = \{i : i \leq k\}$ we have

$$\left( (\forall i \in A)P_n(f^i(x_1), \ldots, f^i(x_n)) \lor (\forall i \in A) \bigwedge_{j \leq n} P_1(f^i(x_j)) \right) \land$$

$$\left( (\forall i \in B)P_n(f^i(x_1), \ldots, f^i(x_n)) \lor (\forall i \in B) \bigwedge_{j \leq n} P_1(f^i(x_j)) \right);$$

$$\forall n, y_1, y_2 \exists x_1, \ldots, x_n, z_1, \ldots, z_n(y_1 < y_2 \rightarrow (y_1 < x_1 < \ldots < x_n < z_1 < \ldots < z_n < y_2 \land$$

$$\forall i \leq 0)(P_n(f^i(x_1), \ldots, f^i(x_n)) \land \bigwedge_{j \leq n} P_1(f^i(z_j))) \land$$

$$\forall i > 0)(P_n(f^i(z_1), \ldots, f^i(z_n)) \land \bigwedge_{j \leq n} P_1(f^i(x_j)))).$$

Let $h$ be the permutation of $\mathbb{Q}$ defined by: $P_n(x_1, \ldots, x_n) \rightarrow h(x_1) = x_2 \land \ldots \land h(x_n) = x_1$. It is easily seen that the permutation $g = fh^{-1}f^{-1}h$ has infinitely many cycles of each length.

$^2$denoting cycles of length $n$

$^3$see [5]: Ex.1.3.15 for the definition and Theorem 1.4.2 for an illustration
Define a permutation $h'$ as follows. For each $n > 1$ and $(x_1, ..., x_n) \in P_n$ with $\bigwedge P_1(f(x_j))$ replace every $h$-cycle $f^{-i}(x_j), j \leq n$, with odd $i > 0$ by $n$ single cycles and create $n$-element cycles $f^i(x_1) \to ... \to f^i(x_n) \to f^i(x_1)$ for all $i > 0$.

If $\bigwedge P_1(f^{-1}(x_j)), j \leq n$, holds for $(x_1, ..., x_n) \in P_n$ (by the definition of $f$ this is incompatible with the situation of the previous paragraph), then create $n$-element cycles $f^{-i}(x_1) \to ... \to f^{-i}(x_n) \to f^{-i}(x_1)$ for even $i > 0$ and remove all $h$-cycles $f^i(x_j), j \leq n$, for $i \geq 0$. We now see that $h'f^{-2}(h')^{-1}(x) = f^{-1}h^{-1}f^{-1}h(x)$.

As a result the permutation $g$ is the product of $f$ and $h^{-1}f^{-1}h$, where $f \in A(Q)$, and the permutation $h^{-1}f^{-1}h$ is the product of $f$ and $h'f^{-2}(h')^{-1} (= f^{-1}h^{-1}f^{-1}h)$.

If $Sym(Q)$ has a classical action on a pretree $T$, then by the assumption each element of a conjugacy class meeting $A(Q)$ fixes a point of $T_0$. Now by $(C3)$, $f$ and $h^{-1}f^{-1}h$ have a common fixed point in $T_0$, which is a fixed point of $g$, so $g$ is not loxodromic.

The permutation $g$ represents the comeagre conjugacy class in $Sym(\omega)$, [19]. Now Proposition 3.2 (assuming that $X$ is the comeagre conjugacy class) works in our case. □

3.3. Comeagre conjugacy classes and invariant ends. In the following proposition we consider a situation which appears in the case when a Polish group acts with an invariant end.

**Proposition 3.4.** Let a group $G$ have a non-nesting action on an $\mathbb{R}$-tree $T_0$ with an invariant end. Let $X \subset G$ be a conjugacy class with $X^{-1} = X$ and the following condition:

For any $g_1, g_2, g_3 \in G$ there exist $h_0, h_1, h_2, h_3 \in X$ such that for any $1 \leq i \leq 3$, $g_i = h_0h_i$.

Then for any $g \in G$, $T_0^g \neq \emptyset$.

**Proof.** If $T_0^h \neq \emptyset$ for some $h \in X$, then all $h \in X$ fix some points. This case can be considered as in the proof of Proposition 3.2.

If $T_0^h = \emptyset$ for some $h \in X$, then all $h \in X$ are loxodromic. We want to show that this case is impossible. Let a half-line $[t_0, \infty)$ represent the invariant end.
Find $h_0, h_1, h_2 \in X$ with $h_0 \cdot h_1 = h_2$ and $a \in [t_0, \infty)$ belonging to the axes of these elements. Replacing appropriate $h_i$ by $h_i^{-1}$ if necessary (and moving elements from one side to another), we may assume that all $h_i$ are increasing on $[a, \infty)$. Let $h_0(a) \leq h_1(a)$ (the case $h_0(a) > h_1(a)$ is similar) and let $g \in G$ satisfy $gh_0g^{-1} = h_2$.

Since $g$ fixes the same end with $h_0, h_1, h_2$, it must be loxodromic (otherwise $h_0$ and $h_2$ eventually coincide on $[t_0, \infty)$ and $h_1$ is not loxodromic).

Assume that $L_g$ contains $[a, \infty)$. Find $b_1, b_2 \in [a, \infty)$ such that $g(b_1) = b_2$. Assume $b_1 \leq b_2$. Let $m$ be the minimal number such that $h_0^m(b_1) > b_2$ (notice $1 \leq m$). Non-nesting and the condition $h_0(a) \leq h_1(a)$ imply $h_0^m(b_1) \leq h_1(b_2)$. Then $h_0^m g^{-1}(b_2) = h_0^m(b_1) > b_2$ and

$$h_0^m g^{-1}(h_2(b_2)) = h_0^{m+1}(b_1) = h_0 h_0(h_0^{m-1}(b_1)) \leq h_0 h_1(b_2) = h_2(b_2).$$

We now see that the element $h_0^m g^{-1}$ maps $[b_2, h_2(b_2)]$ properly into itself. This is a contradiction.

If $b_2 < b_1$ let $m$ be the minimal number such that $h_0^{-m}(b_1) \leq b_2$ (notice $1 \leq m$). Non-nesting and the condition $h_0(a) \leq h_1(a)$ imply $h_0^{-m+1}(b_1) \leq h_1(b_2)$. Then $h_0^{-m+1} g^{-1}(b_2) = h_0^{-m+1}(b_1) > b_2$ and

$$h_0^{-m+1} g^{-1}(h_2(b_2)) = h_0^{-m+2}(b_1) = h_0(h_0^{-m+1}(b_1)) \leq h_0 h_1(b_2) = h_2(b_2).$$

We now see that the element $h_0^{-m+1} g^{-1}$ maps $[b_2, h_2(b_2)]$ properly into itself. This is a contradiction. □

We now conclude the material above by the following theorem.

**Theorem 3.5.** Let a Polish group $G$ have a non-nesting action on an $\mathbb{R}$-tree $T_0$ without $G$-fixed points in $T_0$. Let $X \subseteq G$ be a comeagre set. Then the following statements hold.

If every element of $X$ fixes a point, then every element of $G$ fixes a point.

If $G$ fixes an end and $X$ is a conjugacy class, then every element of $G$ fixes a point.

**Proof.** We already know that the assumptions imply that the action is classical (Proposition 2.8). Now the first claim of the theorem follows from Proposition 3.2(2). By Proposition 3.2(1) and Proposition 3.4 we have that if a Polish group
$G$ has a comeagre conjugacy class then every element of $G$ fixes a point under any non-nesting action on an $\mathbb{R}$-tree with an invariant end. This is the second claim of the theorem. □

It is known that $\text{Sym}(\omega)$ and $A(\mathbb{Q})$ have comeagre conjugacy classes \cite{19}.

4. Comeagre conjugacy classes and end stabilizers

To formulate the main result of the section we need the following definition. For a subset $A$ of a median pretree $T$ define the closure $c(A)$ as the minimal subpretree of $T$ with the property that the $T$-median of any triple from $c(A)$ belongs to $c(A)$.

It is clear that in the case when a group $G$ acts on a pretree $T$ and $A$ is $G$-invariant, the pretree $c(A)$ is $G$-invariant too.

The following theorem roughly says that the presence of a comeagre loxodromic conjugacy class implies that the $G_e$-orbits are much smaller than the corresponding $G$-orbits.

**Theorem 4.1.** Let a group $G$ have a non-nesting action on an $\mathbb{R}$-tree $T_0$. Let $X \subset G$ be a conjugacy class of loxodromic elements satisfying the following condition.

For every triple $g_1, g_2, g_3 \in G$ there exist $h_0, h_1, h_2, h_3 \in X$ such that $g_i = h_0 h_i$ for all $1 \leq i \leq 3$.

Then for any $g \in X$, an end $e$ represented by $L_g$ and a point $a_0 \in L_g$ the ordering $L_g^{a_0} = G_e a_0 \cap L_g$ is not dense in $c(G a_0) \cap L_g$. In particular this conclusion holds if $X$ is a comeagre conjugacy class of loxodromic elements.

To illustrate some aspects of the formulation let $g \in G$ be loxodromic and $a_0 \in L_g$ be as in the theorem. It is clear that the set $L_g^{a_0} = G_e a_0 \cap L_g$ is cofinal (in both directions) in the line $L_g$. On the other hand it may happen that the ordering $c(G a_0) \cap L_g$ (induced by a natural ordering of $L_g$) is dense but not dense in $L_g$.

**Example.** Consider $\mathbb{R}$ as $\mathbb{Z} \times \{a, b\} \times (0, 1]$, where the elements of $(2k, 2k + 1]$ are denoted by triples $(k, a, r)$, $r \in (0, 1]$ and the elements of $(2k + 1, 2k + 2]$ are denoted by triples $(k, b, r)$, $r \in (0, 1]$, $k \in \mathbb{Z}$. Let $G = \mathbb{Q}$ act on $\mathbb{R}$ as follows. If $q + k + r = k' + r'$, with $k' \in \mathbb{Z}$ and $r' \in (0, 1]$, then put $(k, a, r) + q = (k', a, r')$ and $(k, b, r) + q = (k', b, r')$. It is easy to see that the action of $\mathbb{Q}$ obtained on $\mathbb{R}$
is non-nesting. On the other hand the interval \((0, 1]\) (consisting of all \((0, a, r)\) with \(r \in (0, 1]\)) does not contain any element of the orbit of \(0 = (-1, b, 1)\). The fact that the orbit of 0 is a dense ordering follows from density of \(\mathbb{Q}\). Its closure coincides with the orbit. □

We now describe one of our tools. A binary relation \(r\) (a partial ordering, where \(r(a, b) \lor (a = b)\) is interpreted as \(a \leq b\)) on a pretree \(T\) is called a flow (\[3\], pp. 23 - 25) if it satisfies the following axioms:

\[
\neg (r(x, y) \land r(y, x)) , \ B(z; x, y) \rightarrow r(x, z) \lor r(y, z) \text{ and } \\
(r(x, y) \land z \neq y) \rightarrow (B(y; x, z) \lor r(z, y)).
\]

The material of the next paragraph is based on pp. 26 - 28 of \[3\]. It would be helpful for the reader (but not necessary) to recall some formulations given there.

We say that a flow \(r\) is induced by an endless directed arc \((C, <)\) if \((x, y) \in r \iff \exists z \in C \forall w > z B(y; x, w)\) (see \[3\], p. 26). Then it is easy to see that for any arc \(J\) if \(J\) does not have maximal elements with respect to \(r\), then the formula \(r(x, y) \lor r(y, x) \lor x = y\) defines an equivalence relation on \(J\) with at most two classes. We say that \(r\) lies on \(J\) if \(J\) contains a maximal element of \(r\) in \(T\) or \(J\) does not have \(J\)-maximal elements with respect to \(r\) and the equivalence relation \(r(x, y) \lor r(y, x) \lor x = y\) defines a non-trivial cut \(J = J^- \cup J^+\), such that for any \(a \in J^+, b \in J^-\) there is no \(c \in T\) with \(r(a, c) \land r(b, c)\) (so it may happen that \(C \cap J\) is cofinal with \(C\)). It is easy to see that for any line \(L\) there is a natural function from the set of all flows of \(T\) induced by endless directed arcs and lying on \(L\) onto the set of all Dedekind cuts on \(L\): the Dedekind cut corresponding to a flow \(r\) is determined by a maximal element or the equivalence relation \(r(x, y) \lor r(y, x) \lor x = y\). The following lemma shows that when \(T\) is median and dense, this correspondence is bijective for flows without maximal elements.

**Lemma 4.2.** Assume that \(T\) is a median pretree and \(r\) is a flow induced by an endless directed arc. Let \(C, D\) be arcs of \(T_0\) which are linearly ordered with respect to \(r\) and do not have upper \(r\)-bounds in \(T_0\). Then \(C\) and \(D\) are cofinal or for any \(c \in C\) and \(d \in D\), \([c, d] = \{a : a \in C \land r(c, a) \lor a \in D \land r(d, a)\}\). Each of these arcs defines the flow \(r\) as in the definition above.

\[\text{by Lemma 3.8 of } [3] \text{ for any endless directed arc } (C, <) \text{ this formula defines a flow}\]
Proof. To see the first statement we start with the case when $r$ is induced by $C$. Let $c$ and $d$ be as in the formulation. If $C$ and $D$ are not cofinal then $C \cap D = \emptyset$ (apply the fact that if $t \in C \cap D$ and $D \models t < t'$, then $t'$ must belong to $C$). By the definition of $r$, if $D \models d < d'$, then $d'$ belongs to some $[d, c']$ with $C \models c < c'$. If $d' \not\in [d, c]$, then $d'$ belongs to $[c^*, c']$, where $c^*$ is the median of $c, c', d$, a contradiction with $C \cap D = \emptyset$. It is now easy to see that the set $[c, d] \cap D$ consists of all $d' \in D$ with $r(d, d')$.

This implies that there is no $a \in [c, d] \setminus (C \cup D)$ (otherwise $a$ is an upper $r$-bound for $D$) and there is no $c' \in C$ with $c' \not\in [c, d]$ and $c < c'$ (otherwise $c'$ is an upper $r$-bound for $D$). Now a straightforward argument gives the formula for $[c, d]$ as above.

In the case when $r$ is induced by some ordering $A$ and $A$ is cofinal with $C$ or $D$, the argument above works again. If $A$ is not cofinal with these orderings then for every $a \in A$ we have $[c, a] = \{a' : a' \in C \land r(c, a') \lor a' \in A \land r(a, a')\}$ and $[a, d] = \{a' : a' \in D \land r(d, a') \lor a' \in A \land r(a, a')\}$. Then the median of $a, c, d$ must belong to one of the intervals $C, D$ or $A$. The case $m(a, c, d) \in A$ is impossible, because the elements of $A$ greater than $m(a, c, d)$ cannot belong both to $[c, a]$ and $[d, a]$. When $m(a, c, d) \in C \cup D$, the arcs $C$ and $D$ are cofinal.

To show that the flow $r$ is induced by any of its linear orderings without upper $r$-bounds in $T_0$ we apply similar arguments as above. Indeed, if $A$ and $C$ are linear orderings without upper $r$-bounds in $T_0$, $r$ is induced by $A$ and $C$ is not cofinal with $A$, then any $D$ as above is cofinal with $A$ or $C$ (as $T$ is median). If $D$ is cofinal with $A$, then $D$ induces $r$. If $D$ is cofinal with $C$, then for any $d \in D$ and $a \in A$, $[a, d] = \{a' : a' \in D \land r(d, a') \lor a' \in A \land r(a, a')\}$. Let $t \in T_0$ and $t^* = m(t, a, d)$ for some $d \in D$ and $a \in A$. Now it is straightforward that for any $t' \in T_0$ the condition $r(t, t')$ is equivalent to $t' \in (t, t^*) \lor t' \in (t^*, d) \cap A \lor t' \in (t^*, a) \cap D$. Using this formula it is easy to verify that $r$ is induced by $D$ (notice that $A$ and $D$ are symmetric in this condition). ☐

By Lemma 4.2 we see that when $r$ is a flow of a dense median pretree $T$ defined by an endless directed arc and $r$ lies on a line $L$ but does not have a maximal element lying on $L$, then each of the half-lines on $L$ defined by the corresponding equivalence relation is an endless directed arc inducing $r$. 

\begin{itemize}
\item \textbf{Proof.} To see the first statement we start with the case when $r$ is induced by $C$. Let $c$ and $d$ be as in the formulation. If $C$ and $D$ are not cofinal then $C \cap D = \emptyset$ (apply the fact that if $t \in C \cap D$ and $D \models t < t'$, then $t'$ must belong to $C$). By the definition of $r$, if $D \models d < d'$, then $d'$ belongs to some $[d, c']$ with $C \models c < c'$. If $d' \not\in [d, c]$, then $d'$ belongs to $[c^*, c']$, where $c^*$ is the median of $c, c', d$, a contradiction with $C \cap D = \emptyset$. It is now easy to see that the set $[c, d] \cap D$ consists of all $d' \in D$ with $r(d, d')$.

This implies that there is no $a \in [c, d] \setminus (C \cup D)$ (otherwise $a$ is an upper $r$-bound for $D$) and there is no $c' \in C$ with $c' \not\in [c, d]$ and $c < c'$ (otherwise $c'$ is an upper $r$-bound for $D$). Now a straightforward argument gives the formula for $[c, d]$ as above.

In the case when $r$ is induced by some ordering $A$ and $A$ is cofinal with $C$ or $D$, the argument above works again. If $A$ is not cofinal with these orderings then for every $a \in A$ we have $[c, a] = \{a' : a' \in C \land r(c, a') \lor a' \in A \land r(a, a')\}$ and $[a, d] = \{a' : a' \in D \land r(d, a') \lor a' \in A \land r(a, a')\}$. Then the median of $a, c, d$ must belong to one of the intervals $C, D$ or $A$. The case $m(a, c, d) \in A$ is impossible, because the elements of $A$ greater than $m(a, c, d)$ cannot belong both to $[c, a]$ and $[d, a]$. When $m(a, c, d) \in C \cup D$, the arcs $C$ and $D$ are cofinal.

To show that the flow $r$ is induced by any of its linear orderings without upper $r$-bounds in $T_0$ we apply similar arguments as above. Indeed, if $A$ and $C$ are linear orderings without upper $r$-bounds in $T_0$, $r$ is induced by $A$ and $C$ is not cofinal with $A$, then any $D$ as above is cofinal with $A$ or $C$ (as $T$ is median). If $D$ is cofinal with $A$, then $D$ induces $r$. If $D$ is cofinal with $C$, then for any $d \in D$ and $a \in A$, $[a, d] = \{a' : a' \in D \land r(d, a') \lor a' \in A \land r(a, a')\}$. Let $t \in T_0$ and $t^* = m(t, a, d)$ for some $d \in D$ and $a \in A$. Now it is straightforward that for any $t' \in T_0$ the condition $r(t, t')$ is equivalent to $t' \in (t, t^*) \lor t' \in (t^*, d) \cap A \lor t' \in (t^*, a) \cap D$. Using this formula it is easy to verify that $r$ is induced by $D$ (notice that $A$ and $D$ are symmetric in this condition). ☐

By Lemma 4.2 we see that when $r$ is a flow of a dense median pretree $T$ defined by an endless directed arc and $r$ lies on a line $L$ but does not have a maximal element lying on $L$, then each of the half-lines on $L$ defined by the corresponding equivalence relation is an endless directed arc inducing $r$. 

\end{itemize}
Proof of Theorem 4.1. By Theorem 3.5 we may assume that $T_0$ is not a line. If the theorem is not true there are $g \in X$, a point $a_0 \in L_g$ and an end $e$ represented by $L_g$ such that the ordering $L_g^{a_0} = G(a_0 \cap L_g)$ is dense in the line $c(Ga_0) \cap L_g$ of the pretree $c(Ga_0)$. Note that $c(Ga_0) \cap L_g$ is cofinal in $L_g$ and this implies the same statement for any line $L_h$ with $h \in X$.

It is also worth noting that the subspace $\bigcup \{L_h : h \in G\}$ is a full subtree of $T_0$. Indeed, for any $h, h' \in X$ there exist $h_0, h_1, h_2 \in X$ such that $h = h_0 h_1$ and $h' = h_0 h_2$. By Proposition 1.5 there is an arc in $L_0 \cup L_{h_1} \cup L_{h_2}$ joining $L_h$ and $L_{h'}$. We may assume that $T_0 = \bigcup \{L_h : h \in X\}$.

Notice that if $a \in L_{h'} \cap L_{h''}$ defines an $L_{h'}$-half-line without other common elements with $L_{h''}$, then $a$ is the median of three non-linear elements from $Ga_0$ and thus belongs to $c(Ga_0)$. In particular in the situation above the arc joining $L_h$ and $L_{h''}$ consists of at most three intervals with extremities from $c(Ga_0)$.

We want to embed the $G$-pretree $c(Ga_0)$ into some special $\mathbb{R}$-tree with an isometric action of $G$. We start with the case when $L_g^{a_0}$ is not a dense ordering. Here we apply the results of Section 2.2 and consider $L_g^{a_0}$ as the ordered group $G_e/G(e)$. In this case all $h^{-1}(L_g^{a_0})$ are discrete. Since $T_0 = \bigcup \{L_h : h \in X\}$ we see that $c(Ga_0)$ is discrete. Thus $c(Ga_0)$ can be considered as a simplicial tree with an isometric action of $G$. Now the pretree $c(Ga_0)$ can be naturally embedded into an $\mathbb{R}$-tree with an isometric action of $G$. We have obtained a classical action of $G$ on a tree satisfying the conditions of Proposition 3.1 (the second one holds because the action is isometric). By Proposition 4.1 the elements of $X$ are not loxodromic, a contradiction.

From now on we consider the case when $L_g^{a_0}$ is dense. Denote $c(Ga_0)$ by $T'$. If elements $a_1$ and $a_2$ belong to $T'$, say $a_1 \in L_h$ and $a_2 \in L_{h'}$, then as above we find $h_0, h_1, h_2 \in X$ such that there is a $T_0$-arc in $L_{h_0} \cup L_{h_1} \cup L_{h_2}$ joining $L_h$ and $L_{h'}$. Since $c(Ga_0)$ is median, the intervals of the corresponding lines have extremities belonging to $c(Ga_0)$. This implies that the $T'$-interval $[a_1, a_2]$ is decomposed in $T'$ into at most five intervals from the corresponding lines. We now see that all intervals of $T'$ are dense. Let $T^*$ be the set consisting of $T'$ and all flows of $T'$ which are induced by endless directed arcs and which do not have maximal elements. We

\[\text{as any simplicial tree}\]
will show below that $T^*$ can be presented as an $\mathbb{R}$-tree where the action of $G$ is isometric. We start with some helpful observation.

Claim 1. Every endless directed arc $I$ from $T'$ which does not define an end of $T_0$, is cofinal with an endless directed arc from some $L_h \cap T'$, $h \in X$.

Indeed, let $a \in I$. Since $T_0$ together with the set of ends forms a complete tree, there is $c \in T_0$ such that $I$ is cofinal with $[a,c) \cap c(Ga_0)$. If $a \in L_{h'}$ and $c \in L_h$, then as above we find $h_0,h_1,h_2 \in X$ such that there is an arc in $L_{h_0} \cup L_{h_1} \cup L_{h_2}$ joining $L_h$ and $L_{h'}$. We see that $[a,c)$ is decomposed into at most five intervals from the corresponding lines. The last interval (which is of the form $[a',c)$ with $a' \in c(Ga_0)$) is cofinal with $I$.

We now extend the betweenness relation to $T^*$ as in [p.30]: for points $x,y$ and a flow $r$ we say $B(y;x,r)$ if $(x,y) \in r$. Then we can define $B(r;x,y)$ as the case when there is no point $z$ with $B(z;x,r) \wedge B(z;y,r)$. For flows $r,r'$ and a point $x$ we say $B(r;x,r')$ if for any point $y \neq x$, $(y,x) \in r \vee (y,x) \in r'$. We also put $B(r;x,r')$ if $B(r;x,y)$ for some $y \in T'$ with $(x,y) \in r'$. If $p,q,r$ are flows then $B(p;q,r)$ means that there is no point $z$ with $B(z;p,q) \wedge B(z;p,r)$.

This definition implies that for a non-linear triple $u,v,w \in T^*$ the median $m(u,v,w)$ can be presented as $m(x,y,z)$ for some non-linear $x,y,z \in T'$; thus $m(u,v,w) \in T'$.

By Claim 1 we know that all flows of $T^* \setminus T'$ are induced by endless directed arcs of lines $L_h \cap T'$, $h \in X$.

It is also worth noting that the definition of the betweenness relation on $T^*$ implies that a flow $r \in T^* \setminus T'$ lies on a line $L' \subset T'$ if and only if $r$ forms a linear triple with any two points of $L'$.

As a flow $r \in T^* \setminus T'$ is induced by any of its linear orderings without upper bounds (Lemma 4.2), if $r$ belongs to a $T^*$-line $L^*$, then $r$ is defined by a class of the corresponding Dedekind cut of $L' = T' \cap L^*$. Moreover any cut $L' = C^- \cup C^+$, which does not define an element of $T'$, defines a flow from $T^*$. Indeed suppose for a contradiction that $r$ is the flow defined by $C^-$ and $c$ is a maximal element of $r$. Then for any $c' \in C^-$ and $c'' \in C^+$, the interval $(c',c)$ (in $T'$) consists of elements of $L'$ $r$-greater than $c'$ (by axioms of flows and maximality of $c$) and is contained in $[c',c'']$ (see Lemma 4.2). Since $T'$ is quasimedian, $c \in [c',c'']$, contradicting the
assumption that the cut does not define an element of the line. As a result we obtain that the $T^*$-line $L^*$ is the Dedekind completion of $L'$ and can be identified with $\mathbb{R}$.

Since $T'$ is dense, $T'$ is dense in $T^*$.

The action of $G$ on $T'$ uniquely defines an action on $T^*$. We want to show that this action is a non-nesting action on an $\mathbb{R}$-tree. Let us start with the following claim.

Claim 2. If $g' \in G$ is not loxodromic in $T_0$, then $T'$ contains a point fixed by $g'$ or a segment which is inversed by $g'$.

Indeed, if $g'$ has a fixed point $a \in T_0$ such that for some $h_1, h_2, h_3 \in X$ the point $a$ defines three pairwise disjoint half-lines (without the extremity, half-lines of the form $(a, \infty)$) on the corresponding $L_{h_1}$, $L_{h_2}$ and $L_{h_3}$, then $a$ is the median of three elements of $Ga_0$ and thus belongs to $c(Ga_0)$. In particular if $g'$ fixes pointwise a segment or a half-line of some $L_{h} \neq g'(L_{h})$, then $g'$ has a fixed point in $c(Ga_0)$.

Using non-nesting we see that in the remaining case we must consider the situation when $a$ with $g'(a) = a$ belongs to some line $L_{h}$, $h \in X$, and to some segment $[b, g'(b)]$ with $b \in Ga_0$ and no point of $[b, a]$ is fixed by $g'$. Since $a$ is not a median of three non-linear points in $T_0$, we may assume that $b \in L_{h}$. By the same reason the interval $(a, g'g'(b))$ meets $([a, b] \cup (a, g'(b))]$. Our assumptions (together with non-nesting) imply that there is $c \in (a, b) \cap (a, g'(b)]$ and this $c$ can be found in $c(Ga_0)$. By non-nesting, $g'g'(c) = c$. Thus $[c, g'(c)]$ is inversed by $g'$.

Claim 3. An element $g' \in G$ fixes a point in $T_0$ if and only if it fixes a point in $T^*$.

If $g'$ does not fix a point in $T_0$, then there is a line $L \subseteq T_0$ which is the axis of $g'$. Since every segment of $T_0$ can be presented as the union of at most five segments from lines $L_{h}$, $h \in X$, with common extremities belonging to $c(Ga_0)$, the set $L \cap c(Ga_0)$ is not empty and thus is cofinal with $L$. By the definition of $T^*$ and the corresponding betweenness relation, $L \cap Ga_0$ is cofinal with some $T^*$-line $L^*$. It is clear, that $L^*$ is the axis of $g'$ in $T^*$. Thus $g'$ is loxodromic in $T^*$ (by Lemma 1.4).

To see the converse we apply Claim 2. By this claim we only have to consider the case when $g'$ fixes a point in $T_0$, does not fix any point of $T'$ and inverses a
As we already know, this contradicts the assumption that the action of $c^*T$ is non-nesting. This finishes the proof of the claim.

Claim 4. The action of the group $G$ on $T^*$ is non-nesting.

Indeed, if $g' \in G$ is loxodromic in $T_0$, then it is loxodromic in $T^*$. Lemma 1.3 implies that if $g'$ maps an segment of $T^*$ properly into itself, then such a segment can be chosen in the axis $L^*_g$ of $g'$. The latter is impossible, because $g'$ has a non-nesting action on $L^*_g \cap T'$ with a cofinal orbit and $L^*_g \cap T'$ is dense in $L^*$. If $g'$ fixes a point $a \in T^*$ and maps a segment $[b, b']$ properly into itself then $a$ and $[b, b']$ can be chosen so that $a \in [b, b']$. This can be shown by straightforward arguments depending on the place where the median of $a$ and the extremities of the segment lie.

If it happens that $a = b'$, then as $T'$ is dense in $[a, b]$, we can arrange that $b \in T'$. Since the action of $G$ on $T'$ is non-nesting we see that $a \notin T'$. Using Claims 2 and 3 find a segment $[c, g'(c)]$ with $(g')^2(c) = c \in c(Ga_0)$ and $a \in [c, g'(c)]$. Replacing $c$ by $m(a, b, c)$ or $m(a, b, g'(c))$ if necessary, we may assume that $c \in [a, b]$. Then $g'$ maps $[g'(c), b]$ properly into itself, contradicting non-nesting on $T'$.

Consider the case when neither $b$ nor $b'$ are fixed by $g'$. If $g'(b) \in [a, b]$ or $g'(b') \in [a, b']$ then apply the argument of the previous paragraph. Assume $g'(b) \in [a, b']$. Then $g'(b') \in [a, b]$ and $(g')^2$ maps the segment $[a, b']$ properly into itself. As we already know this contradicts the assumption that the action of $G$ on $T'$ is non-nesting. This finishes the proof of the claim.

We now consider the action $*_g$ of $G_e$ on $L_g \subseteq T_0$ (as in Section 2.2), where $g$ is as in the formulation of the theorem. Since every $g' \in G_e$ acts on $L_g$ as a translation (up to topological equivalence with $\mathbb{R}$) so it does on the corresponding axis $L^*_g$ from $T^*$ (by non-nesting). It is clear that all elements of $G(e)$ fix $L^*_g$ pointwise.

As we already know, the line of $T^*$ containing a copy $t^{-1}(L^*_{g_0})$ can be identified with the Dedekind completion of the $T'$-line $t^{-1}(c(Ga_0) \cap L_g)$. Since $L^*_{g_0}$ is dense in $c(Ga_0) \cap L_g$, this completion coincides with the Dedekind completion of $t^{-1}(L^*_{g_0})$. We will denote this line by $t^{-1}(L^*_{g_0})^{D}$. It is clear that $t^{-1}(L^*_{g_0})^{D}$ is the axis of $t^{-1}gt$ in $T^*$. 
The line $L_g^* = (L_g^{a_0})^D$ can be identified with $\mathbb{R}$ so that the group $L_g^{a_0}$ (under the structure of the Archimedean group $G_e/G_{(e)}$ defined in Section 2.2) acts on $(L_g^{a_0})^D$ by translations defined by real numbers. We assume that $a_0$ corresponds to 0. Let $d$ denote the metric obtained on $(L_g^{a_0})^D$ in this way.

Now for any $h \in G$ consider $h^{-1}(L_g^{a_0})^D$ under the metric $d^h$ induced by $d$ and the map $h^{-1}$ (so that $h^{-1}$ is an isometry).

Claim 5. The metrics $d'$ and $d^h$ agree on every common segment of the corresponding lines.\footnote{\textbf{6}If the segment contains more than one element, then it contains infinitely many elements both from $h^{-1}(L_g^{a_0})$ and $t^{-1}(L_g^{a_0})$}. In particular, if $(h^{-1}(L_g^{a_0}))^D = (t^{-1}(L_g^{a_0}))^D$, then $d' = d^h$.

Suppose $x_1, x_2 \in t^{-1}(L_g^{a_0})^D \cap h^{-1}(L_g^{a_0})^D$, $x_1 < x_2$ and $d'(x_1, x_2) < d^h(x_1, x_2)$ (the case $d^h(x_1, x_2) < d'(x_1, x_2)$ is similar). Since $t^{-1}(L_g^{a_0})$ is dense in $t^{-1}(L_g^{a_0})^D$, we can find $x'_1 \leq x''_1 < x'_2 \leq x''_2 < x'_2 \in t^{-1}(L_g^{a_0})^D \cap h^{-1}(L_g^{a_0})^D$ with $d'(x'_1, x'_2) < d^h(x'_1, x'_2)$ where $x'_1, x'_2 \in t^{-1}(L_g^{a_0})$ and $x''_1, x''_2 \in h^{-1}(L_g^{a_0})$. Then the inequality $t(x'_2) - t(x'_1) < h(x''_2) - h(x''_1)$ holds in the group $L_g^{a_0}$. Applying a translation from $L_g^{a_0}$ we can take $h(x''_1)$ to $t(x'_1)$; then $h(x''_2)$ must go to an element of $L_g^{a_0}$ greater that $t(x'_2)$, a contradiction with non-nesting.

Extend the metric $d$ to the space $T^*$ as follows. Since $T^*$ consists of $\bigcup\{h^{-1}(c(Ga_0)) \cap L_g) : h \in G\}$ and flows are defined by arcs from lines of the form $h^{-1}(c(Ga_0) \cap L_g)$, any $a \in T^*$ belongs to some $T^*$-line of the form $h^{-1}(L_g^{a_0})^D$. If $a \in h^{-1}(L_g^{a_0})^D$ and $b \in (h')^{-1}(L_g^{a_0})^D$ then find $h_0, h_1, h_2 \in X$ such that $h = h_0h_1$ and $h' = h_0h_2$. There is a bridge $[a', c_1] \cup [c_1, c_2] \cup [c_2, b']$ where $a' \in h^{-1}(L_g^{a_0})^D, b' \in (h')^{-1}(L_g^{a_0})^D$ and every segment above belongs to an appropriate line $h_{i}^{-1}(L_g^{a_0})^D$, $h_0^{-1}(L_g^{a_0})^D$ or $h_2^{-1}(L_g^{a_0})^D$ (it can happen that $a' = c_1$ or $c_1 = c_2$ or $c_2 = b'$). Now define the distance between $a$ and $b$ as the sum of distances in the sequence $a, a', c_1, c_2, b', b$ where each distance is taken from the corresponding axis.

By Claim 5 to prove that $d$ is invariant under the $G$-action it suffices to show that if $h$ maps $h_1^{-1}(L_g^{a_0})^D$ onto $h_2^{-1}(L_g^{a_0})^D$, then $h$ maps the metric $d^{h_1}$ onto $d^{h_2}$. The latter condition can be verified as follows. Since $h' = h_2hh_{1}^{-1}$ preserves the line $L_g$, $h'(L_g^{a_0}) = L_g^{a_0}$. This shows that $h^{-1}$ maps $h_2^{-1}(L_g^{a_0})$ onto $h_{1}^{-1}(L_g^{a_0})$. By Claim 5 the metrics $d^{h_1}$ and $d^{h_2h}$ coincide on $h_{1}^{-1}(L_g^{a_0})^D$. The latter one is the image of $d^{h_2}$ under $h^{-1}$ (by the definition).
The definition of $d$ implies that $T^*$ satisfies the definition of an $\mathbb{R}$-tree (see Section 2.1). Moreover every line $h^{-1}(L^a_g)^D$ becomes an $\mathbb{R}$-line where $g^h$ acts by a translation.

We have obtained a classical action on a tree satisfying the conditions of Proposition 3.1. By Proposition 3.1, the elements of $X$ are not loxodromic, a contradiction. □

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