Extending partial isometries of generalized metric spaces

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Abstract

We consider generalized metric spaces taking distances in an arbitrary ordered commutative monoid, and investigate when a class $K$ of finite generalized metric spaces satisfies the Hrushovski extension property: for any $A \in K$ there is some $B \in K$ such that $A$ is a subspace of $B$ and any partial isometry of $A$ extends to a total isometry of $B$. Our main result is the Hrushovski property for the class of finite generalized metric spaces over a semi-archimedean monoid $R$. When $R$ is also countable, this can be used to show that the isometry group of the Urysohn space over $R$ has ample generics. Finally, we prove the Hrushovski property for classes of integer distance metric spaces omitting triangles of uniformly bounded odd perimeter. As a corollary, given odd $n \geq 3$, we obtain ample generics for the automorphism group of the universal, existentially closed graph omitting cycles of odd length bounded by $n$.

1 Introduction and Summary of Results

A well-known result of Hrushovski [7] from 1992 says that, given a finite graph $\Gamma_1$, there is a finite graph $\Gamma_2$ such that $\Gamma_1$ is a subgraph of $\Gamma_2$ and any partial automorphism of $\Gamma_1$ extends to a total automorphism of $\Gamma_2$. Since then, several authors have investigated the occurrence of similar behavior in other classes of finite structures. For example, Herwig ([3],[4]) proves analogous results for the classes of finite $K_n$-free graphs; and Solecki [14] proves analogous results for the class of finite metric spaces.

In [6], Hodges, Hodkinson, Lascar, and Shelah use Hrushovski’s original result, along with an analysis of generic automorphisms, to prove that the automorphism group of the countable random graph has the small index property. In [8], Kechris and Rosendal generalize this result substantially by determining conditions on a Fraïssé class $\mathcal{K}$, which characterize when the Fraïssé limit $\mathcal{M}$ of $\mathcal{K}$ has “ample” generic automorphisms. They then show that this implies the small index property for the automorphism group of $\mathcal{M}$. Translated to a general Fraïssé class $\mathcal{K}$, Hrushovski’s extension property for partial automorphisms can make considerable progress toward showing that $\mathcal{K}$ satisfies the conditions in Kechris and Rosendal’s characterization.

In this paper, we consider classes of generalized metric spaces, which take distances in arbitrary ordered commutative monoids, called distance monoids (Definition 1.1). This is a very robust setting, which naturally encompasses many of the previous examples, and also provides a convenient framework for obtaining new results. In particular, we investigate natural algebraic properties of a distance monoid $\mathcal{R}$, which imply that the class of finite metric spaces over $\mathcal{R}$ has the Hrushovski property. As a starting point, we generalize Solecki’s work in [14] to show the Hrushovski property in the case that $\mathcal{R}$ is archimedean. This provides the base case for proving the same result when $\mathcal{R}$ is semi-archimedean (Definition 1.9), which is much less restrictive and includes, for example,
ultrametric spaces as a special case. Altogether we conclude that, when \( \mathcal{R} \) is countable and semi-archimedean, the isometry group of the Urysohn space over \( \mathcal{R} \) has ample generics. Finally, we extend our results to classes of generalized metric spaces omitting some fixed class of subspaces. This will yield the Hrushovski property for certain classes of metric spaces omitting triangles of odd perimeter. Using this, we obtain ample generics for the automorphism group of the universal, existentially closed graph omitting cycles of uniformly bounded odd length.

Some portions of the main proofs work with \( \mathcal{L} \)-structures, where \( \mathcal{L} \) is a first-order relational language. Therefore, we recall that a \textit{relational language} \( \mathcal{L} \) consists of a set \( \{ R_i : i \in I \} \) of symbols together with a sequence of positive integers \( \{ n_i : i \in I \} \). An \( \mathcal{L} \)-structure is a set \( M \) together with an \textit{interpretation} of the symbols in \( \mathcal{L} \), i.e., a subset \( R_i^M \subseteq M^{n_i} \) for each \( R_i \). We then write \( M \models R_i(a) \) if \( a \in R_i^M \). Two \( \mathcal{L} \)-structures \( M \) and \( N \) are \textit{isomorphic} if there is a bijection \( f : M \to N \) such that for all \( R_i \) and \( a \in M^{n_i} \), \( M \models R_i(a) \) if and only if \( N \models R_i(f(a)) \). By convention, we assume all classes of structures are closed under isomorphism.

A natural example of a relational structure is a graph, in which the language contains a single binary relation symbol interpreted as the edge relation. Our focus is on metric spaces, which we often consider as complete graphs with edges labeled by nonzero distances. In this way, metric spaces can be viewed as relational structures in a language containing a binary relation for each nonzero distance.

We begin by defining a reasonable generalization of the notion of a metric space, in which distances between points can come from arbitrary ordered additive structures.

\textbf{Definition 1.1.}

1. A structure \( \mathcal{R} = (R, \oplus, \leq, 0) \) is a \textbf{distance monoid} if

   (i) \( (R, \leq, 0) \) is a linear order with least element \( 0 \);

   (ii) \( (R, \oplus, 0) \) is a commutative monoid with identity \( 0 \);

   (iii) for all \( r, s, t, u \in R \), if \( r \leq s \) and \( t \leq u \) then \( r \oplus t \leq s \oplus u \).

2. Suppose \( \mathcal{R} = (R, \oplus, \leq, 0) \) is a distance monoid. Given a set \( A \) and a function \( d : A \times A \to R \), we call \( (A, d) \) an \textbf{\( \mathcal{R} \)-metric space} if

   (i) for all \( a, b \in A \), \( d(a, b) = 0 \) if and only if \( a = b \);

   (ii) for all \( a, b \in A \), \( d(a, b) = d(b, a) \);

   (iii) for all \( a, b, c \in A \), \( d(a, c) \leq d(a, b) \oplus d(b, c) \).

3. Given a distance monoid \( \mathcal{R} \), we let \( \mathcal{K}_R \) denote the class of finite \( \mathcal{R} \)-metric spaces.

If \( \mathcal{R} \) is a \textit{countable} distance monoid, then \( \mathcal{K}_R \) is a countable class modulo isometry. In this case, it straightforward to verify that \( \mathcal{K}_R \) is a Fraïssé class. Indeed, the only nontrivial property to verify is the amalgamation property, and this can be seen using the natural notion of free amalgamation of \( \mathcal{R} \)-metric spaces. In particular, given finite \( \mathcal{R} \)-metric spaces \( A = (A, d_A) \) and \( B = (B, d_B) \), with \( A \cap B \neq \emptyset \), we define the \textit{free amalgamation of \( A \) and \( B \)}, denoted \( A \oplus B \), to be the \( \mathcal{R} \)-metric space \( C = (C, d_C) \), where \( C = A \cup B \) and, for all \( x, y \in C \),

\[
d_C(x, y) = \begin{cases} 
  d_A(x, y) & \text{if } x, y \in A \\
  d_B(x, y) & \text{if } x, y \in B \\
  \min_{z \in A \cap B}(d_A(x, z) \oplus d_B(z, y)) & \text{if } x \in A \setminus B \text{ and } y \in B \setminus A.
\end{cases}
\]
Definition 1.2. Given a countable distance monoid $\mathcal{R}$, define the $\mathcal{R}$-Urysohn space, denoted $\mathcal{U}_\mathcal{R}$, to be the countable Fraïssé limit of $\mathcal{K}_\mathcal{R}$.

By standard results in classical Fraïssé theory, we have the following fact.

Fact 1.3. Suppose $\mathcal{R}$ is a countable distance monoid. Then $\mathcal{U}_\mathcal{R}$ is a countable, ultrahomogeneous $\mathcal{R}$-metric space, which is universal for the class $\mathcal{K}_\mathcal{R}$ of finite $\mathcal{R}$-metric spaces. Moreover, $\mathcal{U}_\mathcal{R}$ is the unique such $\mathcal{R}$-metric space (up to isometry).

Example 1.4.

1. Classical metric spaces are generalized metric spaces over the standard distance monoid $\mathbb{R}^\geq = (\mathbb{R}^\geq, +, \leq, 0)$. A countable example is obtained by restricting to the submonoid $\mathbb{Q}^\geq = (\mathbb{Q}^\geq, +, \leq, 0)$ of rational numbers, which yields the rational Urysohn space $\mathcal{U}_{\mathbb{Q}^\geq}$. The completion of $\mathcal{U}_{\mathbb{Q}^\geq}$ is known as the Urysohn space, and is the unique separable ultrahomogeneous metric space, which is universal for the class of separable metric spaces.

2. Ultrametric spaces are generalized metric spaces over the distance monoid $(\mathbb{R}^\geq, \max, \leq, 0)$. In general, if $(\mathbb{R}, \leq, 0)$ is a linear order with least element 0, then $\mathbb{R} = (\mathbb{R}, \max, \leq, 0)$ is a distance monoid and, when $\mathbb{R}$ is countable, we have the associated ultrametric Urysohn space $\mathcal{U}_\mathbb{R}$. Any distance monoid arising in this way is called ultrametric.

3. We will often focus on metric spaces taking only integer distances. Therefore, given an integer $n > 0$, we define the distance monoid $\mathcal{R}_n = (\{0, 1, \ldots, n\}, +_n, \leq, 0)$, where $+_n$ is addition truncated at $n$. We also define $\mathcal{N} = (\mathbb{N}, +, \leq, 0)$.

4. The previous example is a special case of the following. Fix a subset $S \subseteq \mathbb{R}^\geq$, which contains 0 and is closed under the binary operation $r +_S s = \sup\{x \in S : x \leq r + s\}$. When $+_S$ is associative, we have a distance monoid $\mathcal{S} = (S, +_S, \leq, 0)$. Associativity of $+_S$ is equivalent to what is known as the four-values condition for $S$, which essentially describes the ability to amalgamate metric spaces with distances in $S$. This condition was defined by Delhommé, Laflamme, Pouzet, and Sauer in [2], where the authors study metric spaces with restricted (real) distances. See also Appendix A of [12], in which Nguyen Van Thé classifies all subsets $S \subseteq \mathbb{N}$, which satisfy the four-values condition and have at most four nonzero elements.

In particular, part (3) of the previous example provides a convenient setting in which to work with the natural metrics on graphs.

Definition 1.5. Suppose $(\Gamma, E)$ is a graph.

1. If $(\Gamma, E)$ is connected, then we define the **path-metric**, $d : \Gamma \times \Gamma \rightarrow \mathbb{N}$, by setting $d(x, y)$ to be the number of edges in the shortest path from $x$ to $y$. In this case, $(\Gamma, d)$ is an $\mathcal{N}$-metric space. If $(\Gamma, d)$ has bounded diameter then it is an $\mathcal{R}_n$-metric space for large enough $n$.

2. Given $n > 0$, we define the **path-metric truncated at** $n$, $d : \Gamma \times \Gamma \rightarrow \{0, 1, \ldots, n\}$, by setting $d(x, y)$ to be the minimum of $n$ and the number of edges in the shortest path from $x$ to $y$ (if one exists). Then $(\Gamma, d)$ is an $\mathcal{R}_n$-metric space. When $n = 2$, we also refer to this metric as the **edge-metric** on $(\Gamma, E)$.

Our interest lies in questions around extending partial isometries of generalized metric spaces, and the resulting consequences for the isometry groups of universal generalized metric spaces (e.g. $\mathcal{U}_\mathcal{R}$ for a distance monoid $\mathcal{R}$). We begin this investigation with the basic definitions.
Definition 1.6. Fix a distance monoid $\mathcal{R}$.

1. Given an $\mathcal{R}$-metric space $A$, a partial isometry of $A$ is an isometry $\varphi : A_1 \rightarrow A_2$, where $A_1$ and $A_2$ are subspaces of $A$.

2. Suppose $K$ is a class of $\mathcal{R}$-metric spaces and $A \in K$. Then $A$ has the extension property in $K$ if there is some $B \in K$ such that $A$ is a subspace of $B$ and any partial isometry of $A$ extends to a total isometry of $B$.

3. A class $K$ of finite $\mathcal{R}$-metric spaces has the Hrushovski property if every element of $K$ has the extension property in $K$.

Note that the edge-metric on graphs associates $\mathcal{K}_{\mathcal{R}_2}$ with the class of finite graphs, and therefore associates $\mathcal{U}_{\mathcal{R}_2}$ with the countable random graph. Viewed this way, Hrushovski’s original result in [7] is that $\mathcal{K}_{\mathcal{R}_2}$ has the Hrushovski property. In [14], Solecki shows that $\mathcal{K}_{\mathcal{R}_{\geq 0}}$, the class of all finite metric spaces, also has the Hrushovski property. More precisely, it is shown that if $(G, +, 0)$ is a subgroup of $(\mathbb{R}, +, 0)$, then $\mathcal{K}_G$ has the Hrushovski property, where $\mathcal{G}$ is the distance monoid $(\mathbb{R}_{\geq 0}, +, \leq, 0)$. As it turns out, the only significant obstacle in extending Solecki’s result to arbitrary distance monoids comes from the fact that $\mathbb{R}_{\geq 0}$ is archimedean.

Definition 1.7. A distance monoid $\mathcal{R} = (\mathbb{R}, \oplus, \leq, 0)$ is archimedean if, for all $r, s \in \mathbb{R}_{> 0}$, there exists some integer $n > 0$ such that $s \leq nr$.

Our first result is the following theorem, which is proved in Section 2.

Theorem 1.8. If $\mathcal{R}$ is an archimedean distance monoid then $\mathcal{K}_\mathcal{R}$ has the Hrushovski property.

It is worth noting here that, unlike the situation with archimedean groups, it is not the case that any archimedean monoid is isomorphic to a submonoid of $\mathbb{R}_{\geq 0}$ (even when allowing for submonoids truncated at a maximal element).

The proof of Theorem 1.8 is a very straightforward generalization of Solecki’s work in [14], which relies on a deep theorem of Herwig and Lascar [5, Theorem 3.2]. The archimedean assumption is used exactly once in the proof (specifically, Claim 2.1), and is crucial in order to use Herwig and Lascar’s result in our more general setting. In particular, their result deals with extending partial automorphisms of finite $\mathcal{L}$-structures, which omit some finite class $\mathcal{F}$ of finite $\mathcal{L}$-structures, where $\mathcal{L}$ is a finite relational language. A key insight in Solecki’s work is the translation from metric spaces to $\mathcal{L}$-structures, with appropriate choice of $\mathcal{L}$ and $\mathcal{F}$. As is evident in the proof of Theorem 1.8, this translation can be directly generalized, but the assumption that $\mathcal{R}$ is archimedean is necessary in order to maintain the feature that $\mathcal{F}$ is finite.

On the other hand, many interesting examples of generalized metric spaces can be obtained using non-archimedean monoids. In particular, any ultrametric monoid $\mathcal{R} = (\mathbb{R}, \max, \leq, 0)$ is severely non-archimedean, as every element of $\mathcal{R}$ is the sole member of its archimedean class. Therefore, we define the following notion of a semi-archimedean distance monoid, which generalizes both archimedean and ultrametric.

Definition 1.9. A distance monoid $\mathcal{R} = (\mathbb{R}, \oplus, \leq, 0)$ is semi-archimedean if, for all $r, s \in \mathbb{R}_{> 0}$, if $nr < s$ for all $n > 0$ then $r \oplus s = s$.

In other words, a semi-archimedean distance monoid can have multiple nontrivial archimedean classes, but addition between elements of distinct archimedean classes is as trivial as possible. Our second main result is the following theorem, which is proved in Section 3.

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Theorem 1.10. If \( R \) is a semi-archimedean distance monoid, then \( K_R \) has the Hrushovski property.

The proof of this result proceeds by induction, with Theorem 1.8 providing the base case. The inductive step consists of an explicit method for extending partial isometries by hand. In some sense, the definition of a semi-archimedean distance monoid is formulated to capture the the largest class of distance monoids for which such an inductive argument will work. Therefore, the question of extending partial isometries of \( R \)-metric spaces, where \( R \) is a general distance monoid, remains open and would seem to demand a new method of proof.

Question 1.11. Let \( R \) be an arbitrary distance monoid. Does \( K_R \) have the Hrushovski property?

In addition to being a combinatorially interesting property of finite structures, the Hrushovski property can also have significant consequences for automorphism groups of countable structures. Recall that, if \( M \) is a countable structure, then \( G = \text{Aut}(M) \) is a separable topological group under the pointwise convergence topology. Quoting [8], we say \( G \) has ample generics if, for all \( n > 0 \), the conjugation action of \( G \) on \( G^n \) has a comeager orbit. If \( G \) has ample generics then any subgroup of index less than \( 2^\aleph_0 \) is open (and therefore has countable index). Using this, it then follows that any homomorphism from \( G \) to a separable topological group is continuous. See [8] for details.

In [8], Kechris and Rosendal characterize the existence of ample generics for \( \text{Aut}(M) \), in the case that \( M \) is the Fraïssé limit of a Fraïssé class \( \mathcal{K} \). This characterization is given in terms of the joint embedding property and weak amalgamation property, which are defined for the class \( K_{p,n}^{\mathcal{K}} \) consisting of tuples \((A, \varphi_1, \ldots, \varphi_n)\) where \( A \in \mathcal{K} \) and each \( \varphi_i \) is a partial automorphism of \( A \). For the class \( \mathcal{K}_{Q_{\geq 0}} \) of finite metric spaces with rational distances, it is shown in [8] Section 6.2 that the Hrushovski property for \( \mathcal{K}_{Q_{\geq 0}} \) implies the weak amalgamation property for \( K_{p,n}^{\mathcal{K}_{Q_{\geq 0}}} \). Together with joint embedding for \( K_{p,n}^{\mathcal{K}_{Q_{\geq 0}}} \) (which is also done in [8]), one obtains Solecki’s result that \( \text{Isom}(\mathcal{U}_{Q_{\geq 0}}) \) has ample generics. For any distance monoid \( R \), the demonstrations of joint embedding for \( K_{p,n}^{K_R} \), and weak amalgamation for \( K_{p,n}^{K_R} \) as a consequence of the Hrushovski property for \( K_R \), go through exactly in [8] for the case \( R = Q_{\geq 0} \). Altogether, we have the following conclusion.

Fact 1.12. Suppose \( R \) is a countable distance monoid and \( K_R \) has the Hrushovski property. Then \( \text{Isom}(\mathcal{U}_R) \) has ample generics.

Applying Theorem 1.10 we then have:

Corollary 1.13. If \( R \) is a countable semi-archimedean distance monoid, then \( \text{Isom}(\mathcal{U}_R) \) has ample generics.

We should mention that recent work of Malicki [11] shows ample generics for the isometry groups of Polish ultrametric Urysohn spaces, even in the case of uncountably many distances. On the other hand, the isometry group of the complete Urysohn space does not have ample generics (although Sabok [13] has shown automatic continuity still holds).

In the final section of this paper (Section 4) we extend our investigation of the Hrushovski property to classes of generalized metric spaces omitting some fixed class of subspaces. The motivation for this pursuit comes from the following well-known examples.

Example 1.14.

1. Given an integer \( n \geq 3 \), the class of finite \( K_n \)-free graphs is a Fraïssé class. We let \( \mathcal{H}_n \) denote the Fraïssé limit, which is often called a Henson graph.
2. Fix an odd integer \( n \geq 3 \). We consider the class \( K_n^c \) of finite graphs omitting cycles of odd length bounded by \( n \). By work of Komjáth, Mekler, and Pach ([9], [10]), there is a countable, existentially closed graph \( \Gamma_n^c \), which omits cycles of odd length bounded by \( n \) and is universal for \( K_n^c \). Using existential closure, it is not too difficult to see that \( \Gamma_n^c \) has diameter \( n_s := \frac{n+1}{2} \) with respect to the path-metric. Therefore, we consider \( \Gamma_n^c \) as an \( R_n^c \)-metric space, and we consider the elements of \( K_n^c \) as \( R_n^c \)-metric spaces under the path-metric truncated at \( n_s \).

With this interpretation, \( K_n^c \) is precisely the class of finite \( F_n^c \)-free \( R_n^c \)-metric spaces, where \( F_n^c \) is the class of triangles of odd perimeter bounded by \( n \) (i.e. 3-point metric spaces with integer distances, in which the sum of the three nonzero distances is odd and bounded by \( n \)).

The main theorem of Section 4 will imply the Hrushovski property for the classes of generalized metric spaces defined in the previous example. For \( K_n^c \)-free graphs, this result is due to Herwig ([9], [10]), and there is no substantive difference between viewing these structures as graphs versus \( R_2 \)-metric spaces. On the other hand, the Hrushovski property for metric spaces omitting triangles of odd perimeter has not been previously shown. Moreover, we will use this result to obtain ample generics for the automorphism group of the graph \( \Gamma_n^c \). This is interesting because \( \Gamma_n^c \) is not homogeneous as a graph and, as a class of graphs, \( K_n^c \) is not a Fraïssé class and does not have the Hrushovski property. However, considered as a class of \( R_n^c \)-metric spaces, \( K_n^c \) is a Fraïssé class. In particular, if \( A \) and \( B \) are \( F_n^c \)-free \( R_n^c \)-metric spaces, then the free amalgamation \( A \otimes B \) is still \( F_n^c \)-free. It is also a somewhat folkloric exercise to show that the Fraïssé limit of \( K_n^c \) (as a class of \( R_n^c \)-metric spaces) is isometric to \( \Gamma_n^c \) (under the path-metric), and so \( \Gamma_n^c \) is a metrically homogeneous graph. Some detail can be found in Cherlin’s catalog ([1]) of such graphs.

To obtain the results described above, we will prove a metric space analog of the previously mentioned theorem of Herwig and Lascar. In particular, we show that if \( R \) is an archimedean distance monoid, and \( F \) is a finite class of finite \( R \)-metric spaces satisfying certain conditions, then any finite \( F \)-free \( R \)-metric space, which has the extension property in the class of all \( F \)-free \( R \)-metric spaces, has the extension property in the class of finite \( F \)-free \( R \)-metric spaces. Provided that there is a universal, homogeneous object for the class of finite \( F \)-free \( R \)-metric spaces (e.g. a Fraïssé limit), this will be enough to conclude the Hrushovski property.

**Corollary 1.15.** Fix an odd integer \( n \geq 3 \).

\( a \) Let \( n_s = \frac{n+1}{2} \). Then the class \( K_n^c \) of \( R_n^c \)-metric spaces, which omit triangles of odd perimeter bounded by \( n \), has the Hrushovski property.

\( b \) Let \( \Gamma_n^c \) be the countable, universal, existentially closed graph omitting cycles of odd length bounded by \( n \). Then \( \text{Aut}(\Gamma_n^c) \) has ample generics.

Returning to the general picture, this is only partial progress. In particular, the restrictions in Section 4 placed on the omitted family \( F \) are quite strong. Moreover, we can still only work in the context of archimedean distance monoids.

**Question 1.16.** Let \( R \) be an arbitrary distance monoid and let \( F \) be a finite class of finite \( R \)-metric spaces. Suppose the class \( K \) of \( F \)-free \( R \)-metric spaces satisfies the amalgamation properties of a Fraïssé class (but perhaps is uncountable). Under what conditions can we conclude that \( K \) has the Hrushovski property?

## 2 Proof of Theorem 1.8: The Archimedean Case

Throughout the proof, we fix an archimedean distance monoid \( R \) and a finite \( R \)-metric space \( A = (A, d_A) \). We want to find a finite \( R \)-metric space \( B = (B, d_B) \) such that \( A \subseteq B \), \( d_B|_{A \times A} = d_A \),
and any partial isometry of $A$ extends to a total isometry of $B$. After replacing $\mathcal{R}$ by the submonoid of $\mathcal{R}$ generated by the distances in $A$, we may assume $\mathcal{R}$ is countable.

Following Solecki [14], we set $S = \{d_A(a,b) : a, b \in A, a \neq b\}$, and define the finite first-order language $\mathcal{L} = \{d_r(x,y) : r \in S\}$, where each $d_r(x,y)$ is a binary relation. Given an $\mathcal{R}$-metric space $X = (X,d_X)$, we interpret $X$ as an $\mathcal{L}$-structure, where $X \models d_r(x,y)$ if and only if $d_X(x,y) = r$.

Given $\mathcal{L}$-structures $M$ and $N$, a function $f : M \to N$ is a weak homomorphism if, for all $r \in S$ and $x, y \in M$, if $M \models d_r(x,y)$ then $N \models d_r(f(x),f(y))$. A weak embedding is an injective weak homomorphism. We say $M$ weakly embeds in $N$ if there is a weak embedding $f : M \to N$. Given a class $\mathcal{F}$ of $\mathcal{L}$-structures, and an $\mathcal{L}$-structure $M$, we say $M$ is $\mathcal{F}$-free if no element in $\mathcal{F}$ weakly embeds in $M$. We say $M$ is homomorphically $\mathcal{F}$-free if there is no weak homomorphism from an element of $\mathcal{F}$ to $M$.

Define $\Sigma = \{(r_0, \ldots, r_n) : n > 0, \ r_i \in S, \ r_0 > r_1 \oplus \ldots \oplus r_n\}$.

Claim 2.1. $\Sigma$ is finite.

Proof. Since $S$ is finite, it suffices to fix $r_0 \in S$ and find an integer $k > 0$ such that, if $r_1, \ldots, r_n \in S$ and $r_0 > r_1 \oplus \ldots \oplus r_n$, then $n < k$. So fix $r_0 \in S$ and let $s = \min S$. Since $\mathcal{R}$ is archimedean, there is some $k > 0$ such that $r_0 \leq ks$. Therefore, if $n \geq k$ and $r_1, \ldots, r_n \in S$, we have $r_0 \leq ns \leq r_1 \oplus \ldots \oplus r_n$, as desired. \[ \]

Fix $\sigma = (r_0, \ldots, r_n) \in \Sigma$. We define an $\mathcal{L}$-structure $P_\sigma = \{x_0, \ldots, x_n\}$ satisfying:

(i) $P_\sigma \models d_{r_0}(x_0, x_n) \land d_{r_0}(x_n, x_0)$,

(ii) for all $1 \leq i \leq n$, $P_\sigma \models d_{r_i}(x_{i-1}, x_i) \land d_{r_i}(x_i, x_{i-1})$.

Let $\mathcal{F} = \{P_\sigma : \sigma \in \Sigma\}$. Let $\mathcal{K}$ be the class of homomorphically $\mathcal{F}$-free $\mathcal{L}$-structures. By the triangle inequality, $\mathcal{K}$ contains any $\mathcal{R}$-metric space (interpreted as an $\mathcal{L}$-structure). Note that any partial isometry of an $\mathcal{R}$-metric space can be viewed as a partial $\mathcal{L}$-automorphism. On the other hand, by choice of $S$, any partial $\mathcal{L}$-automorphism of $A$ can be viewed as a partial isometry of $A$. Therefore, by ultrahomogeneity of $\mathcal{U}_\mathcal{R}$, we have that any partial $\mathcal{L}$-automorphism of $A$ extends to a total $\mathcal{L}$-automorphism of $\mathcal{U}_\mathcal{R} \in \mathcal{K}$. This observation allows us to apply a powerful theorem of Herwig and Lascar [5 Theorem 3.2], to conclude that there is a finite (homomorphically) $\mathcal{F}$-free $\mathcal{L}$-structure $C$ such that any partial $\mathcal{L}$-automorphism of $A$ extends to a total $\mathcal{L}$-automorphism of $C$. We now continue to follow Solecki’s strategy in [14] to obtain, from $C$, the desired $\mathcal{R}$-metric space $B$.

Define a graph relation $E$ on $C$ such that, given distinct $x, y \in C$,

$$E(x,y) \iff C \models d_r(x,y) \land d_r(y,x) \ \text{for some} \ r \in S.$$ 

Note that $(A, E)$ is a complete subgraph of $(C, E)$. Let $B \subseteq C$ be the connected component of $C$ containing $A$. Given distinct $x, y \in B$, let $\Delta(x,y)$ denote the set of finite sequences $(r_1, \ldots, r_n)$ of elements of $S$ such that there is a simple path $(x_0, \ldots, x_n)$ in $(B, E)$ with $x_0 = x, x_n = y$, and $B \models d_{r_i}(x_{i-1}, x_i) \land d_{r_i}(x_i, x_{i-1})$ for all $0 < i \leq n$. Then $\Delta(x,y)$ is nonempty since $(B, E)$ is connected. We define $d_B : B \times B \to R$ such that $d_B(x,x) = 0$ for all $x \in B$ and, given distinct $x, y \in B$,

$$d_B(x,y) = \min\{r_1 \oplus \ldots \oplus r_n : (r_1, \ldots, r_n) \in \Delta(x,y)\}.$$ 

It is easy to see that $d_B$ is an $\mathcal{R}$-metric on $B$. Note also that, since $B$ is $\mathcal{F}$-free (as an $\mathcal{L}$-substructure of $C$), it follows that $d_B|_{A \times A} = d_A$. To finish the proof, we fix a partial isometry $\varphi$ of $A$, and show that $\varphi$ extends to a total isometry of $B$. 

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By choice of \( C \), there is an \( \mathcal{L} \)-automorphism \( \hat{\varphi} \) of \( C \) extending \( \varphi \). Then \( \hat{\varphi} \) is a graph automorphism of \( (C,E) \), and therefore fixes \( B \) setwise. In particular, \( \varphi_* := \hat{\varphi}|_B \) is an \( \mathcal{L} \)-automorphism of \( B \) extending \( \varphi \). For any distinct \( x,y \in B \) and any finite sequence \((r_1,\ldots,r_n)\) from \( S \), we have \((r_1,\ldots,r_n) \in \Delta(x,y)\) if and only if \((r_1,\ldots,r_n) \in \Delta(\varphi_*(x),\varphi_*(y))\). It then follows that \( d_B(x,y) = d_B(\varphi_*(x),\varphi_*(y)) \) for all \( x,y \in B \), and so \( \varphi_* \) is an isometry of \( B \) extending \( \varphi \). This finishes the proof of Theorem [L.8]

3 Proof of Theorem [1.10]: The Semi-Archimedean Case

Throughout the proof, we fix a semi-archimedean distance monoid \( \mathcal{R} \), and a finite \( \mathcal{R} \)-metric space \( A = (A,d) \). We want to find a finite \( \mathcal{R} \)-metric space \( B \) such that \( A \subseteq B \) and any partial isometry of \( A \) extends to a total isometry of \( B \). Set \( S = \{d(a,b) : a,b \in A, a \neq b\} \subseteq R^{>0} \). We again replace \( \mathcal{R} \) with the submonoid of \( \mathcal{R} \) generated by \( S \). Then \( \mathcal{R} \) is countable with only finitely many archimedean classes.

We proceed by induction on the number \( n \) of nontrivial archimedean classes of \( \mathcal{R} \). If \( n = 1 \) then the result follows from Theorem [L.8]. For the induction hypothesis, suppose \( n > 1 \) and assume that, if \( \mathcal{R}' \) is a semi-archimedean distance monoid with \( n - 1 \) nontrivial archimedean classes, then \( K_{\mathcal{R}'} \) has the Hrushovski property.

Partition \( R = R_1 \cup R_2 \) so that \( R_2 \) is a single archimedean class and \( r < s \) for all \( r \in R_1 \) and \( s \in R_2 \). Then \( R_1 \) and \( R_2 \) are each closed under \( \oplus \), and so we may define the distance monoids \( \mathcal{R}_1 = (R_1,\oplus,\leq,0) \) and \( \mathcal{R}_2 = (R_2 \cup \{0\},\oplus,\leq,0) \). Note that \( \mathcal{R}_2 \) is archimedean, and \( \mathcal{R}_1 \) is semi-archimedean with \( n - 1 \) nontrivial archimedean classes. Let \( \sim \) denote the binary relation on \( A \) given by

\[
a \sim b \iff d(a,b) \in R_1.
\]

Then \( \sim \) is an equivalence relation by the triangle inequality and the fact that \( R_1 \) is an initial segment of \( R \) closed under \( \oplus \). Let \( A = A_1 \cup \ldots \cup A_m \) be the partition of \( A \) into \( \sim \)-classes. Note that, for all \( 1 \leq i \leq m \), \( (A_i,d) \) is an \( \mathcal{R}_1 \)-metric space.

**Claim 3.1.** Given \( 1 \leq i < j \leq m \), there is \( s_{i,j} \in R_2 \) such that \( d(a,b) = s_{i,j} \) for all \( a \in A_i \) and \( b \in A_j \).

**Proof.** Fix \( a,a' \in A_i \) and \( b,b' \in A_j \). Then \( d(a,b),d(a',b') \in R_2 \) and \( d(a,a'),d(b,b') \in R_1 \). Since \( \mathcal{R} \) is semi-archimedean, it follows that

\[
d(a,b) \leq d(a,a') \oplus d(a',b') \oplus d(b,b') = d(a',b') \leq d(a',a) \oplus d(a,b) \oplus d(b,b') = d(a,b).
\]

Therefore \( d(a,b) = d(a',b') \). \( \square \)

**Claim 3.2.** We may assume \( A_i \) and \( A_j \) are isometric for all \( i,j \leq m \).

**Proof.** If \( S \cap R_1 = \emptyset \) then each \( A_i \) is a single point and there is nothing to prove. Assume \( S \cap R_1 \neq \emptyset \). We will extend \( A \) to an \( \mathcal{R} \)-metric space \( A^* \) such that, if \( A_1^*,\ldots,A_m^* \) are the \( \sim \)-classes of \( A^* \), then \( A_i^* \) and \( A_j^* \) are isometric for all \( i,j \leq m \). Let \( s = \max(S \cap R_1) \). We define an \( \mathcal{R}_1 \)-metric \( d_0 \) on \( A \) such that, given \( a,b \in A \),

\[
d_0(a,b) = \begin{cases} d(a,b) & \text{if } a,b \in A_i \text{ for some } i, \\ s & \text{if } a \in A_i, b \in A_j \text{ for distinct } i,j. \end{cases}
\]

Note that, for all \( 1 \leq i \leq m \), \( (A_i,d_0) \) is a subspace of \( (A,d) \). Therefore, we may extend each \( (A_i,d) \) to an \( \mathcal{R}_1 \)-metric space \( (A_i^*,d) \) isometric to \( (A,d_0) \), with \( A_i^* \cap A_j^* = \emptyset \) for \( i \neq j \). Now set \( A^* = \ldots \)
Claim 3.3. Given isometry of \( A_i \) and \( A_j \) are isometric for all \( 1 \leq i, j \leq m \). Using Claim 3.1, we define an \( R_2 \)-metric on \([m] := \{1, \ldots, m\} \) such that, given \( 1 \leq i < j \leq m \), \( \rho(i, j) = \rho(j, i) = s_{i,j} \).

By the base case (Theorem 1.8), there is some \( m_1 \geq m \), and an \( R_2 \)-metric space extension \((\{m_1\}, \rho)\) of \((\{m\}, \rho)\) such that any partial isometry of \((\{m\}, \rho)\) extends to a total isometry of \((\{m_1\}, \rho)\).

Next, by induction, there is an \( R_1 \)-metric space \((B_1, d)\) such that \( A_1 \subseteq B_1 \) and any partial isometry of \( A_1 \) extends to a total isometry of \( B_1 \). We may assume \( B_1 \cap A_i = \emptyset \) for all \( 1 \leq i \leq m \). Given \( 1 < i \leq m_1 \), we let \( (B_i, d) \) be an isometric copy of \((B_1, d)\) such that \( B_i \cap B_j = \emptyset \) for all \( 1 \leq i < j \leq m \), and, if \( i \leq m \), \( A_i \subseteq B_i \). Given \( i, j \in [m_1] \), we let \( \theta_{i,j} \) denote a fixed isometry from \( B_i \) to \( B_j \).

Set \( B = B_1 \cup \ldots B_{m_1} \). Extend \( \rho \) (which is defined on each individual \( B_i \)) to an \( R \)-metric on all of \( B \) by setting \( \rho(a, b) = \rho(i, j) \) where \( a \in B_i \) and \( b \in B_j \) for some \( i \neq j \). To verify the triangle inequality, fix \( a, b, c \in B \). If \( a, b, c \) are all in the same \( B_i \), or each in distinct \( B_i \), then triangle inequality follows from the fact that \((B_i, d) \) and \((\{m_1\}, \rho)\) are \( R \)-metric spaces. So we may assume \( a, b \in B_i \) and \( c \in B_j \) for some \( i \neq j \). Then \( d(a, c) = d(b, c) \in R_2 \) and \( d(a, b) \in R_1 \), and so \( d(a, b) < d(a, c) = d(b, c) \), which means the triple \((d(a, b), d(b, c), d(a, c))\) satisfies all permutations of the triangle inequality.

To complete the proof of the theorem, we fix a partial isometry \( \varphi \) of \( A \), and show \( \varphi \) extends to a total isometry \( \varphi_* \) of \( B \). Let \( I = \{1 \leq i \leq m : \text{dom}(\varphi) \cap A_i \neq \emptyset\} \).

Claim 3.3. Given \( i \in I \), there is a unique \( i' \in [m] \) such that \( \varphi(A_i) \cap A_{i'} \neq \emptyset \).

Proof. First, since \( \text{dom}(\varphi) \cap A_i \neq \emptyset \) and \( \varphi \) is a partial isometry of \( A \), there is some \( i' \in [m] \) such that \( \varphi(A_i) \cap A_{i'} \neq \emptyset \). Suppose we have \( a, b \in A_i \cap \text{dom}(\varphi) \) such that \( \varphi(a) \in A_j \) and \( \varphi(b) \in A_k \) for some \( j \neq k \). Then \( d(\varphi(a), \varphi(b)) \in R_2 \) and \( d(a, b) \in R_1 \), which contradicts that \( \varphi \) is a partial isometry.

Claim 3.4. \( f \) is a partial isometry of \((\{m\}, \rho)\).

Proof. Using a similar argument as in the proof of Claim 3.3 one may show that \( f \) is injective. Fix distinct \( i, j \in I \). We want to show \( \rho(i, j) = \rho(f(i), f(j)) \). Fix \( a_i \in \text{dom}(\varphi) \cap A_i \) and \( a_j \in \text{dom}(\varphi) \cap A_j \).

We have \( \varphi(a_i) \in A_{f(i)} \) and \( \varphi(a_j) \in A_{f(j)} \), which means
\[
\rho(f(i), f(j)) = d(\varphi(a_i), \varphi(a_j)) = d(a_i, a_j) = s_{i,j} = \rho(i, j),
\]
as desired.

We now define the desired extension \( \varphi_* \) of \( \varphi \). First, given \( i \in I \), let \( \varphi_i = \varphi|_{A_i} \), and define \( \chi_i = \theta_{1,f(i)}^{-1} \circ \varphi_i \circ \theta_{1,i} \). Then \( \chi_i \) is a partial isometry of \( A_1 \), and so \( \chi_i \) extends to a total isometry \( \hat{\chi}_i \) of \( B_1 \). Set \( \hat{\varphi}_i = \theta_{1,f(i)}^{-1} \circ \hat{\chi}_i \circ \theta_{1,i}^{-1} \). Then \( \hat{\varphi}_i \) is a total isometry from \( B_i \) to \( B_{f(i)} \), and we claim that \( \hat{\varphi}_i \) extends \( \varphi_i \). Indeed, if \( a \in \text{dom}(\varphi_i) \) then \( \theta_{1,i}^{-1}(a) \in \text{dom}(\chi_i) \). Therefore, since \( \chi_i \circ \theta_{1,i}^{-1} = \theta_{1,f(i)}^{-1} \circ \varphi_i \) and \( \hat{\chi}_i \) extends \( \chi_i \), we have
\[
\hat{\varphi}_i(a) = \theta_{1,f(i)}(\hat{\chi}_i(\theta_{1,i}^{-1}(a))) = \theta_{1,f(i)}(\chi_i(\theta_{1,i}^{-1}(a))) = \theta_{1,f(i)}(\theta_{1,f(i)}^{-1}(\varphi_i(a))) = \varphi_i(a).
\]
Next, by Claim 3.4 we may extend $f$ to a total isometry $f_*$ of $([m_*], \rho)$. For $i \in [m_*]\setminus I$, let $\hat{\phi}_i = \theta_{i, f_*(i)}$. Altogether, for any $i \in [m_*]$, $\hat{\phi}_i$ is a total isometry from $B_i$ to $B_{f_*(i)}$. Set
\[
\phi_* = \bigcup_{i=1}^{m_*} \hat{\phi}_i.
\]
Since $f_*$ is a permutation of $[m_*]$ and $\{B_1, \ldots, B_{m_*}\}$ is a partition of $B$, it follows that $\phi_* : B \to B$ is a well-defined bijection. Since $\hat{\phi}_i$ extends $\phi_i$ for all $i \in I$, and $\phi = \bigcup_{i \in I} \hat{\phi}_i$, it follows that $\phi_*$ extends $\phi$. To verify $\phi_*$ is an isometry, fix $a, b \in B$. We may assume $a \in B_i$ and $b \in B_j$ for some $i \neq j$. Then $\phi_*(a) \in B_{f_*(i)}$, $\phi_*(b) \in B_{f_*(j)}$, and
\[
d(a, b) = \rho(i, j) = \rho(f_*(i), f_*(j)) = d(\phi_*(a), \phi_*(b)).
\]
This completes the proof of Theorem 1.10.

4 Metric Spaces with Omitted Subspaces

In this section, we refine the proof of Theorem 1.8 in order to obtain the Hrushovski property for certain classes of generalized metric spaces, which omit some fixed finite class of subspaces. Using the notation in the proof of Theorem 1.8, our strategy is to enlarge the class $\mathcal{F}$ of finite $\mathcal{L}$-structures, used in the application of the Herwig-Lascar theorem, in order to obtain the feature that the resulting $\mathcal{R}$-metric space $B = (B, d_B)$ omits some fixed class of finite $\mathcal{R}$-metric spaces. Since we have limited control of the $\mathcal{R}$-metric $d_B$, it will be necessary to also omit partially defined $\mathcal{R}$-metric spaces obtained, in a certain way, from the $\mathcal{R}$-metric spaces we wish to omit. The next definitions formulate these remarks precisely.

Definition 4.1. Fix a distance monoid $\mathcal{R} = (R, +, \leq, 0)$.

1. Given a set $X$ and a partial function $\delta : X \times X \to R$, we call $(X, \delta)$ a partial $\mathcal{R}$-semimetric space if it satisfies the following conditions:
   
   (i) for all $x \in X$, $(x, x) \in \text{dom}(\delta)$ and $\delta(x, x) = 0$;
   
   (ii) for all $x, y \in X$, if $(x, y) \in \text{dom}(\delta)$ then $(y, x) \in \text{dom}(\delta)$ and $\delta(x, y) = \delta(y, x)$.

2. Given a partial $\mathcal{R}$-semimetric space $(X, \delta)$, define $\text{Im}^+(\delta) := \text{Im}(\delta) \setminus \{0\} \subseteq R^>.$

3. Given an $\mathcal{R}$-metric space $A = (A, d)$ and a partial $\mathcal{R}$-semimetric space $X = (X, \delta)$, we say $X$ weakly embeds in $A$ if there is an injective function $f : X \to A$ such that, for all $(x, y) \in \text{dom}(\delta)$, $d(f(x), f(y)) = \delta(x, y)$.

   We continue to work with the same notions of weak homomorphism, weak embedding, $\mathcal{F}$-free, and homomorphically $\mathcal{F}$-free as established at the beginning of Section 2.

Definition 4.2. Fix a distance monoid $\mathcal{R} = (R, +, \leq, 0)$. Suppose $A = (A, d)$ is an $\mathcal{R}$-metric space and $(X, \delta)$ is a partial $\mathcal{R}$-semimetric space.

1. $(X, \delta)$ is an extension of $A$ if $A \subseteq X$ and $d(a, b) = \delta(a, b)$ for all $(a, b) \in \text{dom}(\delta) \cap (A \times A)$. In this case, we set $A_\delta := (A \times A) \setminus \text{dom}(\delta)$.

2. $(X, \delta)$ is a path extension of $A$ if it is an extension of $A$ and, for all $(x, y) \in A_\delta$ there is a sequence $\pi_{x, y} = (x_0, \ldots, x_n)$ of elements of $X$ such that the following conditions hold:
Proof. Since if, for all \( A \), many path extensions of \( A \)

(i) \( X = A \cup \bigcup_{(x,y) \in A_\delta} \pi_{x,y} \);  

(ii) for all \( (x,y) \in A_\delta \), if \( \pi_{x,y} = (x_0, \ldots, x_n) \) then \( x_0 = x, x_n = y, (x_i, x_{i+1}) \in \text{dom}(\delta) \) for all \( 0 \leq i < n \), and

\[
\delta(x_0, x_1) \oplus \delta(x_1, x_2) \oplus \ldots \oplus \delta(x_{n-1}, x_n) = d(x,y)
\]

(we call \( \pi_{x,y} \) a distance path for \( (x,y) \) in \( (X, \delta) \)).

3. If \( (X, \delta) \) is an extension of \( A \), and \( \text{Im}^+(\delta) \subseteq S \subseteq R^{>0} \), then we say \( (X, \delta) \) is an extension of \( A \) over \( S \).

In order to omit a certain \( \mathcal{R} \)-metric space \( A \) using the method of Theorem \[1\] it will be necessary to also omit all path extensions of \( A \). Therefore, we must determine conditions under which \( A \) has only finitely many such extensions.

Definition 4.3. Let \( \mathcal{R} = (R, \oplus, \leq, 0) \) be a distance monoid. An \( \mathcal{R} \)-metric space \((A, d)\) has dominated spectrum if, for all \( a, b \in A \) there is some \( r \in R \) such that \( d(a, b) < r \).

Lemma 4.4. Suppose \( \mathcal{R} \) is an archimedean distance monoid and \( S \subseteq R^{>0} \) is finite and nonempty. Let \( A = (A, d) \) be a finite \( \mathcal{R} \)-metric space with dominated spectrum. Then there are only finitely many path extensions of \( A \) over \( S \).

Proof. Since \( S \) is finite, it suffices to produce a uniform bound on the size of a path extension \((X, \delta)\) of \( A \) over \( S \). Since \( A \) is finite with dominated spectrum, we may fix some \( r \in R \) such that \( d(a, b) < r \) for all \( a, b \in A \). We also set \( s = \min S \). Since \( \mathcal{R} \) is archimedean, there is some \( k > 0 \) such that \( r \leq ks \). We show that if \((X, \delta)\) is a path extension of \( A \) over \( S \), then \( |X| < |A| + k|A|^2 \).

To see this, fix a path extension \((X, \delta)\) of \( A \) over \( S \). Then we may write \( X = A \cup \bigcup_{(x,y) \in A_\delta} \pi_{x,y} \), where \( \pi_{x,y} \) is a distance path for \( (x,y) \) in \((X, \delta)\). Since \( |A_\delta| \leq |A|^2 \), it suffices to show that, for all \((x,y) \in A_\delta\), if \( \pi_{x,y} = (x_0, \ldots, x_n) \), then \( n < k \). For this, note that

\[
ns \leq \delta(x_0, x_1) \oplus \ldots \oplus \delta(x_{n-1}, x_n) = d(x,y) < r \leq ks.
\]

If \( \mathcal{F} \) is a class of \( \mathcal{R} \)-metric spaces, then there is no guarantee that an \( \mathcal{F} \)-free \( \mathcal{R} \)-metric space will also omit path extensions of the spaces in \( \mathcal{F} \). Therefore, we must artificially impose this condition.

Definition 4.5. Let \( \mathcal{R} \) be a distance monoid and \( \mathcal{F} \) a class of \( \mathcal{R} \)-metric spaces. Fix \( S \subseteq R^{>0} \).

1. Define \( \mathcal{F}_S^* = \{(X, \delta) : (X, \delta) \text{ is a path extension of some } A \in \mathcal{F} \text{ over } S\} \).

2. \( \mathcal{F} \) is closed under path extensions over \( S \) if any \( \mathcal{F} \)-free \( \mathcal{R} \)-metric space is homomorphically \( \mathcal{F}_S^* \)-free.

We now prove the main theorem of this section, the statement of which mirrors the Herwig-Lascar theorem [1] Theorem 3.2.

Theorem 4.6. Fix an archimedean distance monoid \( \mathcal{R} \). Suppose \( \mathcal{F} \) is a finite class of finite \( \mathcal{R} \)-metric spaces with dominated spectrum, and \( A = (A, d_A) \) is a finite \( \mathcal{F} \)-free \( \mathcal{R} \)-metric space. Suppose \( S \subseteq R^{>0} \) is finite, with \( \text{Im}^+(d_A) \cup \bigcup_{(Y,dy) \in \mathcal{F}} \text{Im}^+(dy) \subseteq S \), and assume \( \mathcal{F} \) is closed under path extensions over \( S \). If \( A \) has the extension property in the class of \( \mathcal{F} \)-free \( \mathcal{R} \)-metric spaces, then \( A \) has the extension property in the class of finite \( \mathcal{F} \)-free \( \mathcal{R} \)-metric spaces.
Theorem. The proof closely follows that of Theorem \ref{t:1.8}. Let $L = \{d_r(x, y) : r \in S\}$, where $d_r(x, y)$ is a binary relation. Then any partial $R$-semimetric space $X = (X, \delta)$ can be interpreted as an $L$-structure by $X \models d_r(x, y)$ if and only if $(x, y) \in \text{dom}(\delta)$ and $\delta(x, y) = r$. Given an $L$-structure $M$, $x, y \in M$, and $r \in S$, we write $M \models d(x, y) = r$ if $M \models d_r(x, y) \wedge d_r(y, x)$.

Next, let $F_0 = \{P_\sigma : \sigma \in \Sigma\}$, where $\Sigma$ and $P_\sigma$ are defined as in the proof of Theorem \ref{t:1.8} (relative to our larger set $S$). Recall that $F_0$ is finite (see Claim \ref{cl:2.1}). By Lemma \ref{lem:4.4}, $F_S^*$ is also a finite class of finite $R$-semimetric spaces, which we consider as finite $L$-structures. Note that any $R$-metric space is a path extension of itself, and so, by assumption on $S$, we have $F \subseteq F_S^*$. Altogether, if we set $F^* = F_0 \cup F_S^*$, then $F^*$ is a finite class of finite $L$-structures, with $F \subseteq F^*$.

As observed in Section \ref{sec:2}, any $R$-metric space is homomorphically $F_0$-free, and so, by assumption on $F$, any $F$-free $R$-metric space is homomorphically $F^*$-free.

Suppose $A$ has the extension property in the class of $F$-free $R$-metric spaces, and so there is an $F$-free $R$-metric space $U$ such that $A$ is a substructure of $U$ and any partial isometry of $A$ extends to a total isometry of $U$. Then, as in the proof of Theorem \ref{t:1.8} (with $U$ replacing $U_R$), we have that any partial $L$-automorphism of $A$ extends to a total $L$-automorphism of $U$. By the Herwig-Lascar theorem, there is a finite (homomorphically) $F^*$-free $L$-structure $C$ such that $A$ is a substructure of $C$ and any partial $L$-automorphism of $A$ extends to a total $L$-automorphism of $C$.

We define $B \subseteq C$ and $d_B : B \times B \rightarrow R$ exactly as in the proof of Theorem \ref{t:1.8} By the same arguments, $(B, d_B)$ is an $R$-metric space extension of $(A, d_A)$ and any partial isometry of $A$ extends to a total isometry of $B$. It remains to show that $(B, d_B)$ is $F$-free. Note first that, since $B$ is $F_0$-free as an $L$-structure, it follows that for any distinct $a, b \in B$ there is at most one $r \in S$ such that $B \models d(a, b) = r$ and, in this case, we have $d_B(a, b) = r$.

Suppose, toward a contradiction, that there is $Y \subseteq B$, with $(Y, d_B) \in F$. Then, for any $x, y \in Y$, there is a sequence $\pi_{x, y} = (x_0, \ldots, x_n)$ of elements of $B$ such that:

(i) $x_0 = x$ and $x_n = y$;
(ii) for all $0 \leq i < n$, $d_B(x_i, x_{i+1}) \in S$ and $B \models d(x_i, x_{i+1}) = d_B(x_i, x_{i+1})$;
(iii) $d_B(x_0, x_1) \oplus d_B(x_1, x_2) \oplus \ldots \oplus d_B(x_{n-1}, x_n) = d_B(x, y)$.

Let $X = Y \cup \bigcup_{(x, y) \in Y \times Y} \pi_{x, y} \subseteq B$. Define a partial function $\delta : X \times X \rightarrow R$ as follows:

1. $\text{dom}(\delta) = \{(x_i, x_{i+1}) : 0 \leq i < n, (x_0, \ldots, x_n) = \pi_{x, y}, (x, y) \in Y \times Y\}$;
2. if $(x, y) \in Y \times Y$, $\pi_{x, y} = (x_0, \ldots, x_n)$, and $0 < i \leq n$, then $\delta(x_i, x_{i+1}) = \delta(x_{i+1}, x_i) = d_B(x_i, x_{i+1})$.

By construction, $(X, \delta)$ is a path extension of $Y$ over $S$. But this is a contradiction since, as an $L$-structure $B$ is $F_S^*$-free and $(X, \delta)$ weakly embeds in $B$.

We now apply this theorem to the structures in Example \ref{ex:1.14}. For Example \ref{ex:1.14}(1), fix $n \geq 3$ and note that, as an $R_2$-metric space, $K_n$ has dominated spectrum and has no proper path extensions (over $\{1, 2\}$). Moreover, any finite $K_n$-free graph has the extension property in the class of $K_n$-free graphs (witnessed by the Henson graph $H_n$). Therefore, the class of finite $K_n$-free graphs has the Hrushovski property. Of course, the direct proof of this fact from the Herwig-Lascar theorem is much more straightforward; and the translation to $R_2$-metric spaces is quite convoluted. Therefore, the interesting case is Example \ref{ex:1.14}(2).

Lemma 4.7. Fix $n \geq 3$ odd, and let $n_+ = \frac{n+1}{2}$. Let $F_n^*$ be the class of triangles of odd perimeter bounded by $n$. Let $S = \{1, \ldots, n_+\}$. Then $F_n^*$ is closed under path extensions over $S$. 

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Proof. Suppose \((X, \delta)\) is a path extension over \(S\) of some triangle in \(F^c_n\). If there is a weak homomorphism from \((X, \delta)\) to an \(R_{n_*}\)-metric space \((A, d)\), then there is a (possibly non-simple) cycle \((x_1, x_2, \ldots, x_m, x_1)\) in \(A\), with \(m \geq 3\), such that \(d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_0)\) is odd and bounded by \(n\). Therefore, it suffices to fix an \(F^c_n\)-free \(R_{n_*}\)-metric space \((A, d)\) and a sequence \((x_1, \ldots, x_m)\) of elements of \(A\), with \(m \geq 3\), and show that, if \(p := d(x_1, x_2) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_1)\) and \(p \leq n\), then \(p\) is even.

We proceed by induction on \(m\). If \(m = 3\) and \((x_1, x_2, x_3)\) is injective then the claim follows since \(A\) is \(F^c_n\)-free. Otherwise, after a cyclic rotation, we may assume \(x_1 = x_2\), and so \(p = 2d(x_2, x_3)\), which is even. Assume the result for \(m\) and fix a sequence \((x_1, \ldots, x_{m+1})\) of elements of \(A\). Set

\[
\begin{align*}
p &= d(x_1, x_2) + \ldots + d(x_m, x_{m+1}) + d(x_{m+1}, x_1) \\
q &= d(x_1, x_2) + \ldots + d(x_{m-1}, x_m) \\
r &= d(x_m, x_{m+1}) + d(x_{m+1}, x_1).
\end{align*}
\]

Assume \(p \leq n\). We want to show that \(p\) is even. Since \(d(x_m, x_1) \leq r\), we have \(q + d(x_m, x_1) \leq q + r = p \leq n\). By induction, \(q + d(x_m, x_1)\) is even, and so \(q\) and \(d(x_m, x_1)\) have the same parity. We also have \(d(x_m, x_1) \leq q\), and so \(d(x_1, x_m) + r \leq q + r = p \leq n\). By the base case, \(d(x_1, x_m) + r\) is even, and so \(d(x_m, x_1)\) and \(r\) have the same parity. Altogether, \(q\) and \(r\) have the same parity, and so \(p = q + r\) is even.

We can now prove our desired results. Recall that, given \(n \geq 3\) odd, \(\Gamma^c_n\) denotes the countable, universal and existentially closed graph omitting cycles of odd length bounded by \(n\). Equipped with the path metric, \(\Gamma^c_n\) is the Fraïssé limit of the class of finite \(F^c_n\)-free \(R_{n_*}\)-metric spaces.

Proof of Corollary 4.7. Part (a). We apply Theorem 4.6 with \(R = R_{n_*}\), \(F = F^c_n\), and \(S = \{1, \ldots, n_*\}\). Clearly, \(R_{n_*}\) is archimedean and \(F^c_n\) is a finite class of finite \(R_{n_*}\)-metric spaces. By Lemma 4.7, \(F^c_n\) is closed under path extensions over \(S\). Moreover, \(\Gamma^c_n\) witnesses that any finite \(F^c_n\)-free \(R_{n_*}\)-metric space has the extension property in the class of \(F^c_n\)-free \(R_{n_*}\)-metric spaces. Altogether, we only need to show that each triangle in \(F^c_n\) has dominated spectrum. For, this it suffices to show that \(n_*\) does not appear as a distance in any triangle in \(F^c_n\). Indeed, if \(\{a_1, a_2, a_3\}\) is a 3-point \(R_{n_*}\)-metric space and \(d(a_1, a_2) = n_*\), then \(d(a_2, a_3) + d(a_3, a_1) \geq n_*\), and so \(d(a_1, a_2) + d(a_2, a_3) + d(a_3, a_1) \geq 2n_* = n + 1\).

Part (b). Let \(d\) denote the path metric on \(\Gamma^c_n\). Then \(\text{Aut}(\Gamma^c_n) = \text{Isom}(\Gamma^c_n, d)\), and so we want to show \(\text{Isom}(\Gamma^c_n, d)\) has ample generics. Once again, we use the characterization in [8] discussed in Section 4. Since the standard amalgamation of \(R_{n_*}\)-metric spaces preserves the property of being \(F^c_n\)-free, the Hrushovski property for \(\mathcal{K}^c_n\) implies weak amalgamation for \((\mathcal{K}^c_n)^{p,m}\), for all \(m > 0\), using the same argument as for \(\mathcal{K}_{R_{n_*}}\). The joint embedding property for \((\mathcal{K}^c_n)^{p,m}\) is similarly straightforward.

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