Grand Unification in Non-Commutative Geometry

A. H. Chamseddine\textsuperscript{1} * G. Felder\textsuperscript{2} and J. Fröhlich\textsuperscript{3}

\textsuperscript{1} Theoretische Physik, Universität Zürich, CH 8001 Zürich Switzerland
\textsuperscript{2} Department of Mathematics, ETH, CH 8092 Zürich Switzerland
\textsuperscript{3} Theoretische Physik, ETH, CH 8093 Zürich Switzerland

Abstract

The formalism of non-commutative geometry of A. Connes is used to construct models in particle physics. The physical space-time is taken to be a product of a continuous four-manifold by a discrete set of points. The treatment of Connes is modified in such a way that the basic algebra is defined over the space of matrices, and the breaking mechanism is planted in the Dirac operator. This mechanism is then applied to three examples. In the first example the discrete space consists of two points, and the two algebras are taken respectively to be those of $2 \times 2$ and $1 \times 1$ matrices. With the Dirac operator containing the vacuum breaking $SU(2) \times U(1)$ to $U(1)$, the model is shown to correspond to the standard model. In the second example the discrete space has three points, two of the algebras are identical and consist of $5 \times 5$ complex matrices, and the third algebra consists of functions. With an appropriate Dirac operator this model is almost identical to the minimal $SU(5)$ model of Georgi and Glashow. The third and final example is the left-right symmetric model $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$.

* Supported by the Swiss National Foundation (SNF)
1. Introduction

At present energies the standard model of electroweak interactions has passed all experimental tests. One of the essential ingredients of this model is the Higgs field. The presence of the Higgs field is required to break the gauge symmetry spontaneously. From the four-dimensional point of view, there is no apparent geometrical reason for the Higgs field. Although there are some possible candidates, e.g. a Kaluza-Klein theory or a compactified string model, there are no compelling models yet. A new picture was put forward recently by Connes [1-2], where the experimental validity of the standard model was taken as an indication for a non-commutative picture of space-time. Space-time is taken to be a product of a continuous Euclidean manifold $M_4$ by a discrete ”two-point” space. The fibers in the two copies of space are taken to be $U(1)$ and $SU(2)$ respectively. The vector potential defined in this space will have as components $U(1)$ and $SU(2)$ gauge fields along the continuous directions and the scalar Higgs field along the discrete directions. Therefore, non-commutative geometry offers a geometrical picture for the unification of the gauge and Higgs field. The advantage of this approach over the Kaluza-Klein approach is that there is no truncation of any physical modes, while, in the latter, an infinite number of massive modes is truncated. In this formalism it was shown by Connes and Lott [3-4], and elaborated upon in great detail by Kastler [5], on how to construct the standard model. Inclusion of the $SU(3)$ strong interaction proved to be more difficult and was only achieved recently [3-5]. Other constructions were proposed by different authors, such as Coquereux et al., Dubois-Violette et al. and Balakrishna et al. [6], but they lack a compelling geometrical structure and will not be followed here.

It is usually expected that the standard model [7] will be replaced by a different theory at higher energies which one hopes to be more unified. In particular, the grand unified theories (GUTs) seem to provide (in their supersymmetric forms) acceptable models. The problem that will be addressed in this paper is to find a way to build GUTs models and other possible models at energies higher than the weak scale, within the non-commutative picture.

The strategy adopted in references [1-5] is only appropriate in the case of a product symmetry such as $SU(2) \times U(1)$. If one follows this strategy without modification, many difficulties will be encountered and no phenomenologically successful grand-unified model can be built. This strategy also excludes a single gauge group. To explore other possibilities, we note that a typical GUT involves at least two scales:
the grand unification scale, where the three coupling constants of $SU(3)$, $SU(2)$ and $U(1)$ coincide, and the electroweak scale. In addition, there could be intermediate scales. By choosing space-time to be a product of a continuous four-dimensional Riemannian manifold by a discrete set of points we immediately see that the simplest possibility for the choice of a Dirac operator including more than one scale is to take the discrete space to consist of three points. This is the situation we shall be mostly interested in, although the extension to a discrete space of $N$ points is straightforward. By generalizing the algebra of functions to be given by a direct sum of algebras of matrix-valued functions and by planting the symmetry breaking mechanism in the Dirac operator, it will turn out to be possible to construct unified models.

The plan of this paper is as follows: In section 2, we modify the prescription of Connes [1-2] in such a way that the discrete space consists of three points generalizable to $N$ points. We introduce the idea of planting the symmetry breaking in the Dirac operator and prove that this does not break gauge invariance. In section 3, and as a warm up, we apply this prescription to construct the standard model of the electroweak interactions. In section 4, we construct the $SU(5)$ model and obtain the minimal model (apart from an extra Higgs singlet) of Georgi and Glashow [8]. In section 5, we construct the model $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ of Pati and Mohapatra [9] by taking the discrete space to consist of four points. Section 6, contains our conclusions.

2. A new prescription for model building

Consider a model of non-commutative geometry consisting of the triple $(\mathcal{A}, h, D)$, where $h$ is a Hilbert space, $\mathcal{A}$ is an involutive algebra of operators on $h$, and $D$ is an unbounded self-adjoint operator on $h$. An example important for our construction is the following one: Let $X$ be a compact Riemannian spin-manifold, $\mathcal{A}_1$ the algebra of functions on $X$, and $(h_1, D_1, \Gamma_1)$ the Dirac-K cycle with $h_1 \equiv L^2(X, \sqrt{g} d^4x)$ on $\mathcal{A}_1$. Let $(\mathcal{A}_2, h_2, D_2)$ be given by $\mathcal{A}_2 = M_n(C) \oplus M_p(C) \oplus M_q(C)$, where $M_n(C)$ is the set of all $n \times n$ matrices and $h_2 = h_{2,1} \oplus h_{2,2} \oplus h_{2,3}$ where $h_{2,1}$, $h_{2,2}$ and $h_{2,3}$ are the Hilbert spaces $C^n$, $C^p$ and $C^q$, respectively. Then $\mathcal{A}$ and $D$ are taken to be

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

$$D = D_1 \otimes 1 + \Gamma_1 \otimes D_2 \quad (2.1)$$

To every $f \in \mathcal{A}$ we associate a triplet $(f_1, f_2, f_3)$ of matrix-valued functions on $X$, where $f_1$, $f_2$, and $f_3$ are $n \times n$, $p \times p$, and $q \times q$ matrices, respectively. The decomposition of $h_2$ corresponds to the decomposition $h = h_1 \oplus h_2 \oplus h_3$ for which the action of $f$ is block-diagonal

$$f \rightarrow \text{diag}(f_1, f_2, f_3). \quad (2.2)$$
In this decomposition, the operator $D$ becomes

$$D = \frac{n}{p} \begin{pmatrix} \partial \otimes 1 & \gamma_5 \otimes M_{12} & \gamma_5 \otimes M_{13} \\ \gamma_5 \otimes M_{21} & \partial \otimes 1 & \gamma_5 \otimes M_{23} \\ \gamma_5 \otimes M_{31} & \gamma_5 \otimes M_{32} & \partial \otimes 1 \end{pmatrix} \quad (2.3)$$

where $M_{mn}^* = M_{nm}$ and $m, n = 1, 2, 3, m \neq n$. The gamma matrices we use satisfy:

$\gamma_a^* = -\gamma_a$, $\{\gamma_a, \gamma_b\} = -2\delta_{ab}$, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, $\gamma_5^* = \gamma_5$, and $g_{ab} = -\delta_{ab}$ is the Euclidean metric.

An important difference between our approach and the prescription given by Connes et al. is that they choose all the matrices $M_{mn}$ to be of the same size, i.e. $n = p = q$, and proportional to the identity matrix. In our approach they can be general matrices and do not commute with elements of $A$. The novel idea that we will advance is that the matrices $M_{mn}$ of the model determine the tree level vacuum-expectation values of Higgs fields and the desired symmetry breaking scheme. This modification allows us, first, to simplify the construction of the standard model and then go beyond this model to grand unification models.

Let $E$ be a vector bundle characterized by the vector space $E$ of its sections. We shall consider the example where $E = A$. Let $\rho$ be a self-adjoint element in the space, $\Omega^1(A)$, of one forms

$$\rho = \sum_i a^i db^i, \quad (d1 = 0) \quad (2.4)$$

where $\Omega^*(A) = \oplus^{\infty}_{n=0} \Omega^n(A)$ is the universal differential algebra, with $\Omega^0(A) = A$; See [1]. (The space $\Omega^n(A)$ plays the role of $n$-forms in non-commutative geometry.)

An involutive representation of $\Omega^*(A)$ is provided by the map $\pi : \Omega^*(A) \rightarrow B(h)$ defined by

$$\pi(a_0 da_1...da_n) = a_0 [D, a_1] [D, a_2]...[D, a_n] \quad (2.5)$$

where $B(h)$ is the algebra of bounded operators on $h$. The image of the one-form $\rho$ is

$$\pi(\rho) = \sum_i a^i [D, b^i], \quad (2.6)$$

where the elements $a^i$ and $b^i$ are represented by

$$a^i \rightarrow \text{diag}(a^i_1, a^i_2, a^i_3)$$

$$b^i \rightarrow \text{diag}(b^i_1, b^i_2, b^i_3) \quad (2.7)$$

interpreted as bounded operators on $h$. The product $a^i [D, b^i]$ is defined in terms of standard multiplication. Using the expression of eq.(2.3) for $D$, the commutator
$[D, b]$ can be easily evaluated and is given by

$$[D, b] = \begin{pmatrix}
\partial b_1 & \gamma_5 \otimes (M_{12}b_2 - b_1M_{12}) & \gamma_5 \otimes (M_{13}b_3 - b_1M_{13}) \\
\gamma_5 \otimes (M_{21}b_1 - b_2M_{21}) & \partial b_2 & \gamma_5 \otimes (M_{23}b_3 - b_2M_{23}) \\
\gamma_5 \otimes (M_{31}b_1 - b_3M_{31}) & \gamma_5 \otimes (M_{32}b_2 - b_3M_{32}) & \partial b_3
\end{pmatrix}$$

(2.8)

Inserting eq.(2.8) in eq.(2.6), we obtain

$$\pi(\rho) = \begin{pmatrix}
A_1 & \gamma_5 \otimes \phi_{12} & \gamma_5 \otimes \phi_{13} \\
\gamma_5 \otimes \phi_{21} & A_2 & \gamma_5 \otimes \phi_{23} \\
\gamma_5 \otimes \phi_{31} & \gamma_5 \otimes \phi_{32} & A_3
\end{pmatrix}$$

(2.9)

where the new variables $A$ and $\phi$ are functions of the $a$’s and the $b$’s given by

$$A_m = \sum_i a_m^i \partial b_m^i, \quad m = 1, 2, 3,$$

$$\phi_{mn} = \sum_i a_m^i (M_{mn}b_n^i - b_m^iM_{mn}), \quad m \neq n,$$

(2.10)

and satisfy $A_m^* = A_m$ and $\phi_{mn}^* = \phi_{nm}$.

The two-form $d\rho$ is:

$$d\rho = \sum_i da_i^i db_i^i$$

(2.11)

and its image under the involutive representation $\pi$ is given by

$$\pi(d\rho) = \sum_i [D, a^i][D, b^i]$$

(2.12)

At this point we can address the question of gauge invariance. If one wishes for the action of a spinor field

$$< \Psi, (D + \pi(\rho))\Psi >$$

(2.13)

to be invariant under the transformation $\Psi \rightarrow g^* \Psi = g\Psi$, where $g \in \mathcal{A}$ satisfies $g \in U(\mathcal{A}) = \{g \in \mathcal{A}| g^*g = 1\}$ is unitary, then $\rho$ must transform inhomogeneously according to

$$g\rho = g\rho g^* + gdg^*$$

(2.14)

This is consistent with the definition of $\rho$ in eq.(2.4):

$$g\rho = \sum_i (ga_i^i)d(b_i^i g^*) - g((\sum_i a_i^i b_i^i) - 1)dg^*$$

(2.15)

where the second term could be included in the first term by enlarging the set of the $a_i^i$’s and $b_i^i$’s. It is possible to define gauge transformations explicitly on the elements $a_i$ and $b_i$:

$$a_i \rightarrow g a_i$$

$$b_i \rightarrow g b_i = b_i^*$$

(2.16)
provided one imposes the constraint
\[ \sum_i a^i b^i = 1 \] (2.17)

This is no loss in generality, as the field \( \sum_i a^i b^i \) is independent. We shall use the constraint (2.17) and the transformations (2.16) when convenient. Similarly, the transformation of \( dp \) could be easily derived to be
\[ dp \rightarrow g(dp) = dg\rho g^* + gd\rho g^* + gd\rho g^* - g\rho dg^* \] (2.18)

Working in the representation \( \pi \), we see from eq.(2.15) that
\[ \pi(g\rho) = g\pi(\rho)g^* + g[D, g^*] \] (2.19)

and this can be written in the form
\[ \pi(g\rho) = \sum_i g a^i [D, g b^i] \] (2.20)

As expected, the Dirac operator is not acted up on by the gauge transformations (i.e. \( gD \equiv D \)). The curvature \( \theta \), defined by
\[ \theta = d\rho + \rho^2 \] (2.21)

is easily seen to be covariant under the gauge transformations
\[ \theta \rightarrow g\theta = g\theta g^* \] (2.22)

To see how gauge transformations act on the components of \( \pi(\rho) \), we first give the representation of \( g \):
\[ g \rightarrow \text{diag}(g_1, g_2, g_3) \] (2.23)

where \( g_1, g_2 \) and \( g_3 \) are \( n \times n, p \times p \) and \( q \times q \) unitary matrix-valued functions respectively. A simple computation, using the commutator \([D, g]\) in eq.(2.8), gives the component form of eq.(2.19):
\[ g A_m = g_m A_m g_m^* + g_m\partial g_m^* \], \( m = 1, 2, 3 \)
\[ g(\phi_{mn} + M_{mn}) = g_m(\phi_{mn} + M_{mn})g_n^* \], \( m \neq n \) (2.24)

In this form it becomes manifest that the \( A_m \) are the usual gauge fields, while the combinations \( \phi_{mn} + M_{mn} \) are scalar fields transforming covariantly under the mixed gauge transformations \( g_m \) and \( g_n \). (We use that \( gM_{mn} = M_{mn} \) in \( D \).) The fields \( \phi_{mn} \)
are the physical fields, and the $M_{mn}$ are the vacuum expectation values of the Higgs fields. In other words, the Higgs potential will turn out to have its minimum when $\phi_{mn} = 0$, indicating that the scalar fields appearing are the fluctuations around the vacuum state, and that we are in the spontaneously broken phase. To pass to the symmetric phase, we must reexpress all the scalar fields in the combination $\phi_{mn} + M_{mn}$.

It was noted by Connes and Lott [4] that the representation $\pi$ is ambiguous, a fact that will explain the appearance of auxiliary fields. This can be seen from the fact that if $\pi(\rho)$ is set to zero $\pi(d\rho)$ is not necessarily zero, and the correct space of forms to work on is $\frac{\Omega^*(A)}{\ker \pi + d\ker \pi}$, where $\ker \pi$ is the kernel of the map $\pi$. Thus the auxiliary fields can be either quotiented out or eliminated through their equations of motion as they are non-dynamical. We choose to keep the auxiliary fields explicitly in our calculations (rather than modding them out) since the step of identifying which fields are genuinely independent is complicated and model-dependent. However, Proposition 4 in [4] shows that, for the Yang-Mills functional, the two procedures are equivalent.

Next we proceed to compute $\pi(d\rho)$ which is a lengthy calculation. The elements of this matrix are functions of the $a_i'$s and the $b_i'$s and must be reexpressed in terms of the fields $A_m$, $\phi_{mn}$ and possibly new independent fields. We first consider

$$
\pi(d\rho)_{11} = \sum_i \partial a^i_1 \partial b^i_1 + \sum_i (M_{12}a^i_2 - a^i_1 M_{12})(M_{21}b^i_1 - b^i_2 M_{21})
+ \sum_i (M_{13}a^i_3 - a^i_1 M_{13})(M_{31}b^i_1 - b^i_3 M_{31})
= \partial A_1 + M_{12}\phi_{21} + \phi_{12}M_{21} + M_{13}\phi_{31} + \phi_{13}M_{31} - X_{11}
$$

(2.25)

where the auxiliary field $X_{11}$ is given by

$$
X_{11} = \sum_i a^i_1 (\partial^2 b^i_1 + [M_{12}M_{21} + M_{13}M_{31}, b^i_1]).
$$

(2.26)

Before continuing our calculations, we would like to point out the following problem and the necessary modifications needed to remedy it. The part

$$
\sum_i a^i_1 \partial^2 b^i_1
$$

of the auxiliary field $X_{11}$ is an $n \times n$ matrix whose elements are arbitrary functions. Thus the terms of the scalar Higgs potential could be absorbed in it. This, of course, would be undesirable for any model (since all the scalar fields would remain massless at the classical level). What saves the potential from disappearing altogether is to include the information about the mixing between the three generations of quarks
and leptons in the Dirac operator. This mixing is related to the fermionic mass matrix. Therefore the Dirac operator used in eq.(2.3) should be modified to

\[
D = \begin{pmatrix}
\partial \otimes I \otimes I & \gamma_5 \otimes M_{12} \otimes K_{12} & \gamma_5 \otimes M_{13} \otimes K_{13} \\
\gamma_5 \otimes M_{21} \otimes K_{21} & \partial \otimes I \otimes I & \gamma_5 \otimes M_{23} \otimes K_{23} \\
\gamma_5 \otimes M_{31} \otimes K_{31} & \gamma_5 \otimes M_{32} \otimes K_{32} & \partial \otimes I \otimes I
\end{pmatrix}
\]  

(2.27)

where \( K_{mn} = K_{nm}^* \). The matrix \( K \) commutes with the \( a^i \) and \( b^i \). This modification implies that \( \pi(\rho) \) is obtained by substituting

\[
\phi_{mn} \rightarrow \phi_{mn} \otimes K_{mn}
\]

(2.28)

and \( \pi(\rho)_{11} \) given in eq.(2.25), now becomes *

\[
\pi(\rho)_{11} = \partial A_1 + (|K_{12}|^2(M_{12}\phi_{21} + \phi_{12}M_{21}) + |K_{13}|^2(M_{13}\phi_{31} + \phi_{13}M_{31})) - X_{11}
\]

(2.28)

where the new field \( X_{11} \) is given by

\[
X_{11} = \sum_i a_i^i (\partial^2 b_i^i + [K_{12}|^2 M_{12} M_{21} + |K_{13}|^2 M_{13} M_{31}, b_i^i])
\]

(2.29)

where \( |K_{ij}|^2 = K_{ij}^* K_{ij} \). The other elements of \( \pi(\rho) \) can be found easily and expressed in the compact and generalizable form

\[
\pi(\rho)_{mm} = \partial A_m + \sum_{n \neq m} |K_{mn}|^2 (M_{mn}\phi_{nm} + \phi_{mn} M_{nm}) - X_{mm}
\]

(2.30)

where the \( X_{mm} \) fields are defined by

\[
X_{mm} = \sum_i a_i^i (\partial^2 b_i^i + \sum_{n \neq m} |K_{mn}|^2 M_{mn} M_{nm}, b_m^i).
\]

(2.31)

The non-diagonal element \( \pi(\rho)_{12} \) is given by

\[
\pi(\rho)_{12} = \gamma_5 K_{12} (-\sum_i \partial a_i^i (M_{12} b_i^i - b_i^i M_{12}) + K_{13} K_{32} \sum_i (M_{12} a_i^j - a_i^j M_{12}) \partial b_i^j) \\
+ \sum_i (M_{13} a_i^j - a_i^j M_{13})(M_{32} b_i^j - b_i^j M_{32}).
\]

(2.32)

and can be rewritten in terms of the fields \( A_m \) and \( \phi_{mn} \) and a new field \( X_{12} \)

\[
\pi(\rho)_{12} = -\gamma_5 K_{12} (\partial \phi_{12} + A_1 M_{12} - M_{12} A_2) + K_{13} K_{32} (M_{13} \phi_{32} + \phi_{13} M_{32} - X_{12})
\]

(2.33)

* We omit the tensor product signs to simplify notation. Thus, e.g. \( K_{mn} \) means \( 1 \otimes 1 \otimes K_{mn} \) and \( M_{mn} \) means \( 1 \otimes M_{mn} \otimes 1 \).
where the new field $X_{12}$ is given by

$$X_{12} = \sum_i a_i^1 (M_{13} M_{32} b_i^1 - b_i^1 M_{13} M_{32}) \quad (2.34)$$

Similarly the other non-diagonal elements may be written in a compact and generalizable form:

$$\pi(d\rho)_{mn} = -\gamma_5 K_{mn} (\partial \phi_{mn} + A_m M_{mn} - M_{mn} A_n)$$

$$+ \sum_{p \neq m,n} K_{mp} K_{pn} (M_{mp} \phi_{pn} + \phi_{mp} M_{pn}) - X_{mn}, \quad m \neq n. \quad (2.35)$$

where the fields $X_{mn}$ are defined by

$$X_{mn} = \sum_i a_i^m \sum_{p \neq m,n} K_{mp} K_{pn} (M_{mp} b_n^i - b_m^i M_{mp} M_{pn}), \quad m \neq n, \quad (2.36)$$

The elements $\pi(d\rho)_{mn}$ are self adjoint,

$$\pi(d\rho)^{*}_{mn} = \pi(d\rho)_{nm} \quad (2.37)$$

Collecting all these results, the representation of the curvature $\pi(\theta)$ can be written in terms of components. First, the diagonal elements are given

$$\pi(\theta)_{mm} = \frac{1}{2} \gamma_{\mu\nu} F^m_{\mu\nu} + \left( \sum_{p \neq m} (|K_{mp}|^2 |\phi_{mp} + M_{mp}|^2 - Y_m) - X'_{mm} \right) - X'_{mm} \quad m = 1, 2, 3 \quad (2.38)$$

where we have defined

$$X'_{mm} = \sum_i a_i^m \partial^2 b_i^m + (\partial^\mu A_m^\mu + A^m_{\mu} A_\mu^m)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_m^\mu, A_m^\nu] \quad (2.39)$$

$$Y_m = \sum_{p \neq m} \sum_i a_i^m |K_{mp}|^2 |M_{mp}|^2 b_i^m$$

The non-diagonal elements of $\pi(\theta)$ are given by $(m \neq n)$:

$$\pi(\theta)_{mn} = -\gamma_5 K_{mn} (\partial \phi_{mn} + A_m (\phi_{mn} + M_{mn}) - (\phi_{mn} + M_{mn}) A_n) - X_{mn}$$

$$+ \sum_{p \neq m,n} K_{mp} K_{pn} ((\phi_{mp} + M_{mp})(\phi_{pn} + M_{pn}) - M_{mp} M_{pn}) \quad (2.40)$$

where we have used the notation $|\phi_{mp}|^2 = \phi_{mp} \phi_{pm}$. The curvature is self-adjoint:

$$\pi(\theta)^{*}_{mn} = \pi(\theta)_{nm}.$$
The fields $Y_m$ and $X_{mn}$ are not all independent. Depending on the structure of the mass matrices $M_{mn}$, some of them could be expressed in terms of $\phi_{mn}$. If it so happens that all the $X$-fields are independent then after eliminating all the auxiliary fields, the scalar potential will disappear. This does not happen if the mass matrices are chosen in such a way as to correspond to a possible vacuum with symmetry breaking. In the examples that we consider here, the potential will survive.

The Yang-Mills action is given by the positive-definite expression

$$I = \frac{1}{8} Tr_\omega(\theta^2 |D|^{-4}) \quad (2.41)$$

where $Tr_\omega$ is the Dixmier trace. It is defined by

$$Tr_\omega(|T|) = \lim_{\omega \to 0} \frac{1}{\log N} \sum_{i=0}^{N} \mu_i(T) \quad (2.42)$$

where $T$ is a compact operator, and $\mu_i$ are the eigenvalues of $|T|$. This trace effectively picks out the coefficient of the logarithmic divergences. For the Dirac operators we shall consider the Dixmier trace to be equivalently replaced with a heat kernel expression, using the identity

$$|D|^{-4} = \int_0^\infty d\epsilon \epsilon e^{-\epsilon|D|^2}. \quad (2.43)$$

and the expansion

$$\text{tr}(f e^{-|D|^2}) = \int d^4 x \sqrt{g} f(x) \left( \frac{a_0}{\epsilon^2} + \frac{a_1}{\epsilon} + \ldots \right) \quad (2.44)$$

where $a_0 = 1$, $g$ is the metric, and $a_1 = R$ is the curvature scalar. This can be used to show that the action (2.41) is equal to

$$I = \frac{1}{8} \int d^4 x \sqrt{g} \text{Tr}(\text{tr}(\pi^2(\theta))) \quad (2.45)$$

where $\text{tr}$ is taken over the Clifford algebra, and $\text{Tr}$ is taken over the matrix structure. Using eqs (2.38) and (2.45) the action takes the familiar form (in Euclidean space):
where we have used the notation $|D_\mu \phi|^2 \equiv D_\mu \phi D_\nu \phi \eta^{\mu\nu}$, and when we analytically continue to Minkowski space by the change $x_4 \to it$ the action changes by $I_E \to -I_M$. This action contains the Yang-Mills action for the gauge fields $A_{\mu m}$, kinetic energies for the scalar fields $\phi_{mn}, m \neq n$, and a potential for the scalar fields. In the last step, the independent fields from the set $X'_{mn}, X_{mn}$, and $Y_m$ must be eliminated. The result depends on the particular choices of $M_{mn}$ and is model-dependent. If the potential survives, it is positive definite being a sum of squares and it is minimized for $\phi_{mn} = 0$. Now we are ready to apply this construction to model building.

3. The $SU(2) \times U(1)$ standard model

To clarify the general formalism developed in the last section, we consider the simple example where the Riemannian manifold is extended by two points. In all the formulas of the last section we now set

$$a_3^i = b_3^i = 0, \quad M_{13} = M_{23} = 0 \quad (3.1)$$

We take the elements $a_1 \epsilon M_2(A)$, and $a_2 \epsilon M_1(A)$ to be $2 \times 2$ and $1 \times 1$ matrices, respectively. The matrix $M_{12}$ is then a $2 \times 1$ matrix and will be chosen to be

$$M_{12} \equiv \mu S = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2)$$

The choice of $M_{12}$ dictates the breaking mechanism. With these choices $\pi(\rho)$ takes the form

$$\pi(\rho) = \begin{pmatrix} (A_1)^{I}_I & H^J \\ H^I & A_2 \end{pmatrix} \quad (3.3)$$

where $H^I$ is a $2 \times 1$ doublet. We shall also impose the graded tracelessness condition, $\text{Tr}(\Gamma_1 \pi(\rho)) = 0$, where $\Gamma_1$ is the grading matrix $\Gamma_1 = \text{diag}(1, -1)$. This implies

$$\text{Tr} A_1 = A_2 \quad (3.4)$$

In terms of the elements $a^i$ and $b^i$ the Higgs field $H$ takes the form

$$H = \mu \sum_i a_1^i (S b_2^i - b_1^i S) \quad (3.5)$$

while the fields $X_{mn}$ and $Y_m$ are given by

$$X_{12} = 0 = X_{21}$$

$$Y_1 = \mu^2 \sum_i a_1^i T b_1^i \quad (3.6)$$

$$Y_2 = \mu^2$$
where $T$ is the matrix \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). This implies that the only auxiliary fields are $X'_{11}$, $X'_{22}$, and $Y_1$, and these should be eliminated. This step can be immediately taken and results in the disappearance of the terms involving these fields. The final action then takes the form (in Minkowski space):

\[
I = \frac{1}{4} \left( (F^1_{\mu\nu})^I_J (F^1_{\mu'\nu'})^I_J + (F^2_{\mu\nu})^I_J (F^2_{\mu'\nu'})^I_J \right) \\
+ \frac{1}{2} \text{Tr} K K^* \left| \partial_\mu (H^I + H^0_I) + (A_{\mu 1})^I_J (H^J + H^0_J) - (H^I + H^0_I) A_{\mu 2} \right|^2 \\
- \frac{1}{2} \left( \text{Tr}(K K^*)^2 - (\text{Tr} K K^*)^2 \right) \left( (H^I + H^0_I) (H^I + H^0_I) - \mu^2 \right)^2
\] (3.7)

where we have normalized the trace such that $\text{Tr} 1 = 1$. Note that, for this normalization of the trace, $\text{Tr}(K K^*)^2 - (\text{Tr} K K^*)^2$ is positive (non-negative), by the Schwarz inequality for $\text{Tr}$. Thus the coupling constant of the quartic term in the Higgs potential is non-negative which guarantees stability of the theory at tree level. We also have that $\text{Tr}(K K^*)^2 - (\text{Tr} K K^*)^2 \leq (n-1)(\text{Tr} K K^*)^2$, where $n$ is the number of rows and columns of $K$. Therefore, the order of magnitude of the bare quartic Higgs coupling constant is the same as the order of magnitude of the square of the bare gauge coupling constant. The potential is minimised when $H^I = 0$. The fields are already expanded around the vacuum state, and the minimum corresponds to the broken phase. To display gauge invariance explicitly, the action could be easily expressed in terms of the shifted field $H + H_0$.

The gauge fields are in the familiar form of the standard model [7], but in the broken phase. By writing

\[
(A_1)^I_J = i \begin{pmatrix} A^0 + A^3 \\ A^1 + i A^2 \\ A^0 - A^3 \end{pmatrix}
\]

\[
A_2 = 2i A^0
\]

one finds that $A_1 - i A_2 = W_\mu$ and $A^0 + A^3 = Z_\mu$ are the $W$ and $Z$ gauge fields. The leptons fit naturally in this scheme, and can be included by introducing the spinors $L$ subject to the chirality condition

\[
(\gamma_5 \otimes \Gamma_1) L = L
\] (3.9)

where this condition will only be imposed after we have performed the Wick rotation from Euclidean to Minkowski space. The spinors $L$ then take the form

\[
L = \begin{pmatrix} \chi_L \\ e^-_R \end{pmatrix}
\] (3.10)
where the left-handed electron and neutrino are in the first copy and form a doublet of $SU(2)$: $l_L = \begin{pmatrix} \nu_e \\ -e^- \end{pmatrix}_L$, while the right-handed electron is in the second copy and is a singlet as can be deduced from the form of the elements $a^i$ and $b^i$. The leptonic action is then given by

$$I_l = < L, (D + \pi(\rho)) L > = \int d^4 x \overline{L} (D + \pi(\rho)) L$$

(3.11a)

In terms of components this becomes

$$I_l = \int d^4 x \left[ \overline{l}_L (D + \pi(A)) l_L + \overline{e}_R (\partial + A) e_R ight. \\
\left. + \overline{l}_L (H + H_0) e_R K + \overline{e}_R (H^* + H^*_0) l_L K^* \right]$$

(3.11b)

Thus, as required, the electron becomes massive, while the neutrino remains massless.

Introducing $SU(3)$ and the quarks is more complicated in this approach. The reason is that $SU(3)$ is not broken, and no Higgs fields are necessary. It can be introduced in an essentially commutative way. The solution adopted in [3-4] was to introduce a bimodule. One must introduce, in addition, a new algebra $B$ which must be taken to be $M_1(C) \oplus M_3(C)$. The mass matrices in the Dirac operator along these directions are taken to be zero, forcing the vanishing of the Higgs fields along the same directions. Because the hypercharge assignments of the quarks are delicate, the different $U(1)$ factors must be related. This is achieved with the following assignments: On the algebra $A$ we must set $\text{Tr} A_1 = 0$, $A_2 = -Y$ and on the algebra $B$ we must set $B_1 = -Y = -\text{Tr} B_2$. This prescription guarantees the correct hypercharge assignments for the quarks and leptons. The following point is in order. Although the relation between the different $U(1)$ factors could be obtained from the mathematical condition of the unimodularity of the algebras considered, it is clear that this condition is not natural, especially since the main motivation behind the non-commutative picture is to explain the geometric origin of the Higgs fields, and of the phenomena of symmetry breaking. Introducing $SU(3)$ in a commutative way and decoupling it from the rest is not very convincing. However we shall proceed in our construction for illustration and to show that it is perfectly possible to obtain the standard model using this method.

The quarks are taken to be in the representation

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} q_L \\ d_R \end{pmatrix}$$

(3.12)
subject to the chirality condition $\gamma_5 \otimes \Gamma_1(Q) = Q$, and $q_L = \left( \begin{array}{c} u_L \\ d_L \end{array} \right)$ is a left-handed $SU(2)$ doublet. Unfortunately, an action similar to that of the leptons will leave the up quarks massless. To avoid this one must also introduce the "dual" quark representation

$$\tilde{Q} = \left( \frac{1}{\sqrt{2}} \begin{array}{c} \tilde{q}_L \\ u_R \end{array} \right)$$ (3.13)

where $\tilde{q} = \left( \begin{array}{c} d_L \\ -u_L \end{array} \right) = i\tau_2 q$ is also a doublet of $SU(2)$. By taking the matrix $K$ to be

$$K = \text{diag}(h^e_{\alpha\beta}, h^d_{\alpha\beta}, h^u_{\alpha\beta})$$ (3.14)

where the $h_{\alpha\beta}$'s are matrices in generation space. By defining the spinor

$$\psi^\alpha = \left( \begin{array}{c} L^\alpha \\ Q^\alpha \\ \tilde{Q}^\alpha \end{array} \right)$$ (3.15)

where $\alpha = 1, 2, 3$ refer to the three families, the full fermionic action can then be written as

$$I_f = \langle \psi, (D + \pi(\rho))\psi \rangle = \int d^4x \overline{\psi}(D + \pi(\rho))\psi$$ (3.16)

and when this action is expanded in terms of components, it gives exactly the fermionic action of the standard model. This shows that the standard model can be obtained within the non-commutative setting. But as mentioned earlier, the $SU(2) \times U(1)$ sector fits more naturally into this formalism than the $SU(3)$ sector.

4. The $SU(5)$ unified theory

The way the strong interactions were introduced in the standard model suggests that a unified picture is more desirable from the geometrical point of view. This was also one of the reasons why model builders constructed unified theories. Another reason is that it appears to be natural to assume that, at higher energies, the standard model is replaced by a more unified picture. The simplest example of such a scheme is the $SU(5)$ gauge theory [8], which is the lowest rank group containing $SU(3) \times SU(2) \times U(1)$ as a subgroup. The $SU(5)$ theory is spontaneously broken at two scales. At the grand unification scale $M$, $SU(5)$ is broken to $SU(3) \times SU(2) \times U(1)$ and this, in turn, is broken at the weak scale $\mu$ to $U(1)_{em}$. The role of scales in non-commutative geometry is to measure the distance between the different copies.
of the space. Thus to reproduce the $SU(5)$ theory we need to take the space to be the product of the Reimannian manifold times three points. In the minimal $SU(5)$ theory the first stage of breaking is achieved through the use of the adjoint Higgs representation $24$ and the second stage through the fundamental $5$ representation. The vacuum expectation value of the adjoint Higgs is taken to be

$$\Sigma_0 = M \text{diag} (2, 2, 2, -3, -3) \quad (4.1)$$

which breaks the symmetry from $SU(5)$ to $SU(3) \times SU(2) \times U(1)$. This is further broken to $U(1)_{em}$ when the fundamental Higgs acquires the vacuum expectation value

$$H_0 = \mu \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv \mu S \quad (4.2)$$

If the Riemannian manifold is extended by three points, the simplest possibility to obtain two scales and not three, as well as a Higgs field belonging to the adjoint representation and not to a product representation, is to identify two copies. In other words we must have a permutation symmetry under the exchange $1 \leftrightarrow 2$. Therefore we must identify

$$a_1^i = a_2^i, \quad b_1^i = b_2^i \quad (4.3)$$

The operators in the first and second copies must be taken to be $5 \times 5$ matrices. To obtain the fundamental Higgs fields, the third copy must correspond to $1 \times 1$ matrices, and, to avoid an extra $U(1)$ factor we take these elements to be real. Therefore we must consider the algebra

$$A = M_5(C) \oplus M_5(C) \oplus M_1(R) \quad (4.4)$$

With these choices the vector potential $\pi(\rho)$ becomes

$$\pi(\rho) = \begin{pmatrix} A & \Sigma & H \\ \Sigma & A & H \\ H^* & H^* & 0 \end{pmatrix} \quad (4.5)$$

where $A_1 = A_2 = A = \gamma^\mu (A_\mu)^I_I$ is a self-adjoint $5 \times 5$ gauge vector, $\Sigma^I_I$ is a self-adjoint $5 \times 5$ scalar field, and $H^I$ is a complex scalar field. The reason that $\Sigma$ is self-adjoint lies in the permutation symmetry, as $\Sigma_{21} = \Sigma_{12} = \Sigma_{12}^*$. This is also the reason why the $H'$s in the first and second row are equal. The vector $A_3$ vanishes, because the
self-adjointness condition implies that \( A_{\mu 3} = -A^*_{\mu 3} \), but as \( A_{\mu 3} = \sum_i a^i_3 \partial_\mu b^i_3 \) is real, it must vanish. The tracelessness condition

\[
\text{Tr}(\Gamma_1 \pi(\rho)) = 0 \quad (4.6)
\]

where \( \Gamma_1 = \text{diag}(1, 1, -1) \) implies

\[
\text{Tr} A = 0 \quad (4.7)
\]

reducing \( U(5) \) to \( SU(5) \). In our method, the symmetry breaking pattern is specified by choosing the mass matrices \( M_{mn} \) in the Dirac operator to correspond to the desired vacuum. We shall then take

\[
M_{12} = M_{21} = \Sigma_0 \\
M_{13} = M_{23} = H_0 \quad (4.8)
\]

Writing the fields explicitly we find (see eq. (2.27)):

\[
A^I_j = \sum_i (a^i_1 b^i_1)^I_j, \quad \sum_i a^i_1 b^i_1 = 1 \\
\phi_{12} \equiv \Sigma^I_j = \sum_i (a^i_1 [\Sigma_0, b^i_1])^I_j \\
\phi_{13} \equiv H^I = \mu \sum_i (a^i_1 (S b^i_3 - b^i_1 S))^I
\]

(4.9)

To determine the potential by eliminating the auxiliary fields, we must determine whether the new functions \( X_{mn}, m \neq n \), and \( Y_m \) are independent. First we find

\[
X_{12} = \mu^2 \sum_i a^i_1 [S S^*, b^i_1] \quad (4.10)
\]

which is equal to \( X_{21} \). Clearly this is a new field which cannot be expressed in terms of the \( \Sigma \) and the \( H \) and thus must be eliminated. Next we calculate

\[
X_{13} = \sum_i a^i_1 (M_{13} M_{32} b^i_3 - b^i_1 M_{13} M_{32}) \\
= -3 M H^I \quad (4.11)
\]

which is equal to \( X_{23} \), and where we have used the property that \( \Sigma_0 H_0 = -3 M H_0 \). Obviously these fields are not auxiliary and contribute to the potential. The other fields \( X_{31}, X_{32} \), are the conjugate of \( X_{13} \). The \( Y \)'s are more subtle. First, we have that

\[
Y_1 = \sum_i a^i_1 (|K_{12}|^2 \Sigma^2_0 + |K_{13}|^2 H_0 H^* b^i_1) \quad (4.12)
\]
which appears to be independent. However, because of the property of Σ

\[ \Sigma_0^2 = \frac{1}{5} \text{Tr} \Sigma_0^2 - M \Sigma_0 \]  

(4.13)

and eq.(4.10), and the definition of Σ: Σ = \( \sum_i a_i^\dagger \Sigma_0 b_i^\dagger \), we can rewrite it as

\[ Y_1 = |K_{12}|^2 (-M \Sigma + \Sigma_0^2) + |K_{13}|^2 (H_0 H_0^* + X_{12}) \]  

(4.14)

It thus must be kept, as it will contribute when \( X_{12} \) is eliminated. Finally \( Y_2 = Y_1 \) and \( Y_3 = 2\mu^2 |K_{31}|^2 \). Collecting all these results, the action takes the form

\[
I = \text{Tr} \left( \frac{1}{2} F_{\mu \nu} F^{\mu \nu} + |K_{12}|^2 \left( \partial_\mu (\Sigma + \Sigma_0) + [A_\mu, \Sigma + \Sigma_0] \right)_J \right|^2 + |K_{13}|^2 \left| \left( \left( \partial_\mu + A_\mu \right)(H + H_0) \right)_J \right|^2 - V(H, \Sigma) \]

(4.15)

where the potential \( V(H, \Sigma) \) is the scalar potential and is given by

\[
V = \text{Tr} \left( \left( |K_{12}|^2 (\Sigma + \Sigma_0)^2 + |K_{13}|^2 (H + H_0)(H + H_0)^* - Y_1 \right) - X_{11}' \right|^2 \\
+ |K_{13}| \left( |H + H_0|^2 - H_0 H_0^* - X_{12} I_J \right|^2 \\
+ |K_{12} K_{23}| \left( |(\Sigma + \Sigma_0 + 3M)(H + H_0) I_J \right|^2 \\
+ 2 |K_{31}| \left( (H + H_0)^* (H + H_0) - \mu^2 \right) - X_{33}' \right|^2 \]

(4.16)

We now eliminate the auxiliary field \( X_{12} \) from the first two terms of the potential. This is then followed by eliminating \( X_{11}' \) and \( X_{33}' \) to get the manifestly gauge invariant potential

\[
V(\Sigma, H) = \left( \text{Tr}|K_{12}|^4 - (\text{Tr}|K_{12}|^2)^2 \right) \left( (\Sigma + \Sigma_0)^2 + M(\Sigma + \Sigma_0) - \frac{1}{5} \text{Tr} \Sigma_0^2 \right)_J \right|^2 \\
+ \text{Tr}|K_{12} K_{23}|^2 \left( |(\Sigma + \Sigma_0 + 3M)(H + H_0) I_J \right|^2 \\
+ 2 \left( \text{Tr}|K_{31}|^4 - (\text{Tr}|K_{31}|^2)^2 \right) \left( H + H_0 \right)^* (H + H_0) - \mu^2 \right|^2 \]

(4.17)

Clearly the potential is positive-definite, and the minimum occurs when \( \Sigma = 0 = H \). Also, in order not to loose the Σ potential, the matrix \( K_{12} \) should be different from the identity matrix. So the picture we have is that of a space-time consisting of three copies where two of the copies are identical and separated by a distance of order \( M^{-1} \). These in turn are separated from the third copy by a distance of order \( \mu^{-1} \).

At the next stage, we introduce the fermions into the picture. As is known, the fermions fit neatly as left-handed chiral spinors, in 5 + 10 representations of \( SU(5) \).
denoted by $\psi_I + \psi^{IJ}$ [8]. The fermionic action contains, besides the kinetic energies interacting with the $SU(5)$ gauge fields, the Yukawa couplings

$$I_y = \int d^4x \left( f_{\alpha\beta} \bar{\psi}_{I\alpha}^c H^*_J \psi^c_{\beta J} + f'_{\alpha\beta} \epsilon_{IKLM} \bar{\psi}_{\alpha I}^c \psi^K_L H^M + h.c \right)$$

where $f_{\alpha\beta}$ and $f'_{\alpha\beta}$ are matrices in the family space, $\alpha, \beta = 1, 2, 3$, and $\psi^c = C\bar{\psi}$, is the charge conjugate spinor having the same chirality as $\psi$, ($C$ being the charge conjugation matrix).

In the present formulation of non-commutative geometry, the full fermionic action including both the kinetic terms and the Yukawa couplings must be obtained from an expression of the form $<\Psi, (D + \pi(\rho))\Psi>$, where $\Psi$ is some appropriate representation for the spinors. We wish to incorporate the $(\overline{5} + 10)_L$ into one spinor where we shall use the equivalence $\overline{5}_L = 5_R$. We define the spinor $\Psi$ which transforms as

$$\Psi \rightarrow g\Psi = g \otimes g\Psi$$

under the antisymmetric tensor product representation of the group $U(A)$ of unitary elements of the algebra $A$ (acting on $h \wedge h$). A useful representation for this spinor is $\Psi_{AB}$, where the indices $A$ and $B$ take the values $A = I_1, I_2, 1$ along the directions of the three spaces. It must then satisfy

$$\Psi_{AB} = -\Psi_{BA}.$$  

This together with the permutation symmetry $1 \leftrightarrow 2$ implies that $\Psi_{AB}$ has the following components:

$$\Psi_{I_1 J_1} = \Psi_{I_2 J_2} = \frac{1}{\sqrt{6}} \psi_{IJ}$$

$$\Psi_{I_1 J_2} = \Psi_{I_2 J_1} = 0$$

$$\Psi_{I_1 1} = -\Psi_{1 I_1} = \Psi_{J_2 1} = \frac{1}{\sqrt{2}} \psi_I$$

By further imposing the chirality condition $(\gamma_5 \Gamma_1 \otimes \Gamma_1)\Psi = \Psi$, which can be written in the form

$$\gamma_5 (\Gamma_1)^A_A' (\Gamma_1)^B_B' \Psi_{A'B'} = \Psi_{AB}$$

one finds that $\psi_{IJ}$ is left-handed and $\psi_I$ is right-handed:

$$\psi_{IJ} = \psi_{IJ} (L) \quad \psi_I = \psi_I (R)$$
To put it differently, the fermions fit neatly in one spinor in a representation transforming under the antisymmetric product of $U(A)$. The fermionic action is then

$$I_{1f} = <\Psi, (D + \pi(\rho) \otimes 1 + 1 \otimes \pi(\rho))\Psi>$$

$$= \int d^4x \overline{\Psi}_{AB}(D\Psi_{AB} + \pi(\rho)_A^C\Psi_{CB} + \pi(\rho)_B^C\Psi_{AC})$$

$$= \int d^4x \left( \frac{1}{3}\overline{\psi}_{IJ} (L)(\partial\psi_{IJ} (L) + A^K_I \psi_{KJ} (L) + A^K_J \psi_{IK} (L)) + \overline{\psi}_I (R)(\partial + A)_I^J \psi_J (R) + \frac{1}{\sqrt{3}}(K_{13}\overline{\psi}_J (R)(H + H_0)_J^I\psi_{IJ} (L) + h.c)\right)$$

(4.24)

and where the particle assignments are taken to be

$$\psi_{IJ} (L) = \begin{pmatrix}
0 & u^c_3 & -u^c_2 & u_1 & d_1 \\
-u^c_3 & 0 & u^c_1 & u_2 & d_2 \\
u^c_2 & -u^c_1 & 0 & u_3 & d_3 \\
u_1 & -u_2 & -u_3 & 0 & e^+ \\
d_1 & -d_2 & -d_3 & -e^+ & 0
\end{pmatrix}_L$$

(4.25)

for the 10 representation, and

$$\psi^c_I = \begin{pmatrix}
d_1 \\
d_2 \\
d_3 \\
e^c \\
\nu^c
\end{pmatrix}_R$$

(4.26)

for the 5 representation. It is clear that this interaction provides masses to the leptons and down quarks but not to the up quarks. (The neutrinos will be always massless since they have no right-handed partners). This situation is identical to the one we faced for the Yukawa couplings of the quarks in the standard model. In terms of the $SU(5)$ couplings such Yukawa couplings come from the second term in eq(4.18) and require the introduction of the epsilon tensor of $SU(5)$. We then introduce the spinor in the completely antisymmetric tensor product $U(A)$, (acting on $h \wedge h \wedge h$). It can be represented by a spinor $\chi^{ABC}$ completely antisymmetric in $A, B, C$. Because of the permutation symmetry $1 \leftrightarrow 2$ the non-vanishing components are $\chi^{L_1 J_1 K_1} = \chi^{L_2 J_2 K_2}$ and $\chi^{L_1 J_1} = \chi^{L_2 J_2}$. The chirality condition on $\chi$ is

$$\left(\gamma_5 \Gamma_1 \otimes \Gamma_1 \otimes \Gamma_1\right)\chi = \chi.$$  (4.27)

This implies that

$$\chi^{IJL} = \chi^{IJK}(L) \quad \chi^{IJ} = \chi^{IJ}(R)$$  (4.28)
However, since we do not wish to introduce more particles, these spinors will be related to $\Psi_{AB}$ by the identification

$$\chi^{IJK}_{(L)} = \frac{1}{6\sqrt{2}} \epsilon^{IJKMN} \psi_{MN}^{(L)}$$

$$\chi^{IJ}_{(R)} = \frac{1}{\sqrt{6}} C \psi_{IJ}^{(L)}$$

The action corresponding to the $\chi$-spinor is then

$$I_{2f} = \langle \chi, (D + \pi(\rho) \otimes 1 \otimes 1 + 1 \otimes \pi(\rho) \otimes 1 + 1 \otimes 1 \otimes \pi(\rho)) \chi \rangle = \int d^4x \chi^{ABC} (D\chi^{ABC} + 3\pi(\rho)A^{EBC})$$

The component form of this action then takes the form

$$I_{2f} = \int d^4x \left( \frac{2}{3} \psi_{IJ}^{(L)} (D\psi_{IJ}^{(L)} + 2(A)_I^K \psi_{KJ}^{(L)}) + \frac{1}{6\sqrt{3}} (K_{31}^{IJKMN} \psi_{IJ}^{(L)} (H_K^* + H_0^*) \psi_{MN} + \text{h.c.}) \right)$$

Notice that we have defined the spinors in such a way that their kinetic energies are properly normalized. Finally, in order to give different masses to the up and down quarks, we must introduce the spinor

$$\lambda = \left( \begin{array}{c} \Psi \\ \chi \end{array} \right)$$

and take the matrix $K_{13}$ to be of the form

$$K = \text{diag}(f_{\alpha\beta}, f'_{\alpha\beta})$$

where $f_{\alpha\beta}$ and $f'_{\alpha\beta}$ are matrices in generation space. In this way the fermionic action could be written compactly in the form

$$\langle \lambda, (D + \pi(\rho)) \lambda \rangle$$

Note that the fermionic mass matrix is proportional to $K_{13}$, while $K_{12}$ is necessary for the survival of the $\Sigma$ self couplings in the potential.

To summarize, the present picture looks very attractive: The discrete structure of space becomes apparent, first, at the weak scale, and the Higgs field $H$ is associated with the mediation between the third copy and the two identical copies which, at that scale, would appear to coincide. As we climb up in energy, probing the smaller distance scale, we encounter the Higgs field $\Sigma$ associated with the mediation between...
the two identical copies. The fermions fit into one representation (and its conjugate) and their action takes a very simple form. Of course, from the phenomenological point of view, the gauge group $SU(5)$ has a serious drawback connected with proton decay. The rate predicted by this model is ruled out experimentally, and only in more complicated models this problem is avoided. The analysis of such models, however, is beyond the scope of this paper. The construction of a phenomenologically successful model will be left to the future. Here we content ourselves with the construction of some prototype models, in order to master and illustrate the new techniques advanced here.

5. **Left-right $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ symmetric model**

Another class of attractive models which are of phenomenological interest is the left-right symmetric models. The simplest one of which is the $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ theory [9]. The non-commutative geometry setting is perfectly appropriate for product groups. The Higgs fields used in the breaking are usually taken to be a $(2,2)$ and $(3,1)+(1,3)$ with respect to $SU(2)_L \times SU(2)_R$. It is also possible to replace the $(3,1)+(1,3)$ by doublets $(2,1)+(1,2)$. But this choice is less preferred for phenomenological reasons [9]. As we have learned from the $SU(5)$ theory in order to get an adjoint Higgs representation, two copies must be identified, i.e. interchanged by permutation symmetry. Thus, for each $SU(2)$, the Riemannian manifold should be extended by two points. The total extension is by four points. One can immediately see that the algebra must be taken to be

$$A_2 = M_2(C) \oplus M_2(C) \oplus M_2(C) \oplus M_2(C) \quad (5.1)$$

The elements $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$, are $2 \times 2$ matrices. We must require the permutation symmetries $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. Then

$$a_1 = a_2 \quad b_1 = b_2 \quad a_3 = a_4 \quad b_3 = b_4 \quad (5.2)$$

The vector potential $\pi(\rho)$ takes the form

$$\pi(\rho) = \begin{pmatrix} A_1 & \Delta_1 & \Phi & \Phi \\ \Delta_1 & A_1 & \Phi & \Phi \\ \Phi^* & \Phi^* & A_2 & \Delta_2 \\ \Phi^* & \Phi^* & \Delta_2 & A_2 \end{pmatrix} \quad (5.3)$$

where $A_1$ and $A_2$ are $U(2)_L$ and $U(2)_R$ gauge fields, $\Delta_1$ and $\Delta_2$ are triplets in the adjoint representations of the respective groups, and $\Phi$ is $(2,2)$ with respect to the product groups.
The mass matrices entering the Dirac operator are taken to be

\[ M_{12} = M_{21} = \begin{pmatrix} 0 & 0 \\ v_1 & 0 \end{pmatrix} \equiv v_1 S \]

\[ M_{13} = M_{14} = M_{23} = M_{24} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \] \hspace{1cm} (5.4)

\[ M_{34} = M_{43} = \begin{pmatrix} 0 & 0 \\ v_2 & 0 \end{pmatrix} \]

where \( u_1, v_1, v_2 \) are taken to be real, and \( u_2 \) is taken to be complex, with the phase of \( u_2 \) related to CP violation. To reduce the gauge group from \( U(2)_L \times U(2)_R \) to \( SU(2)_L \times SU(2)_R \times U(1)_{B-L} \) we impose the tracelessness condition

\[ \text{Tr}(\Gamma_1 \pi(\rho)) = 0 \] \hspace{1cm} (5.5)

where \( \Gamma_1 = \text{diag}(1, 1, -1, -1) \). The scalar fields in \( \pi(\rho) \) are given in terms of the \( a^i \) and \( b^i \) by

\[ \Delta_1 = v_1 \sum_i a_1^i S b_1^i \]

\[ \Delta_2 = v_2 \sum_i a_3^i S b_3^i \] \hspace{1cm} (5.6)

\[ \Phi = \sum_i a_1^i (M_{13} b_3^i - b_1^i M_{13}) \]

These expressions are important in determining which of the auxiliary fields are independent. The \( X \)- and \( Y \)- fields are given by

\[ X_{12} = 2(|u_2|^2 - |u_1|^2) \sum_i a_1^i [T, b_1^i] \]

\[ X_{13} = u_1 \sum_i a_1^i (v_1 S b_3^i - b_1^i v_2 S^*) \]

\[ X_{34} = (|u_2|^2 - |u_1|^2) \sum_i a_3^i [T, b_3^i] \] \hspace{1cm} (5.7)

\[ Y_1 = \text{Tr} K K^* \left( 2|u_1|^2 + (2|u_2|^2 - 2|u_1|^2 + |v_1|^2) \sum_i a_1^i T b_1^i \right) \]

\[ Y_3 = \left( 2|u_1|^2 + (2|u_2|^2 - 2|u_1|^2 + |u_1|^2) \sum_i a_3^i T b_3^i \right) \]

where the other functions could be determined from the above using the permutation symmetry, and the matrix \( T \) is given by: \( T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). For simplicity we have assumed that \( K_{12} = K_{34} = K_{13} = K \).
From the form of the action in eq. (2.46) it is clear that the field $X_{13}$ is auxiliary and eliminating it will remove the whole term that appears in $\theta_{13}$. The remaining potential is given by

$$V = V_1(\Phi, \Delta_1) + V_2(\Phi, \Delta_2)$$

where the two parts are

$$V_1(\Phi, \Delta_1) = 2 \left( \text{Tr}(KK^*)^2 - (\text{Tr}KK^*)^2 \right) \left| 2|\Phi + M_{13}|^2 + |\Delta_1 + M_{12}|^2 \right|^2$$

$$- 2\alpha Z_1 - |M_{12}|^2 - 2|M_{13}|^2 \right|^2$$

$$+ 8\text{Tr}(KK^*)^2 \left| \Phi + M_{13} \right|^2 - |M_{13}|^2 - \beta Z_1 \right|^2$$

(5.9a)

where $\alpha = 2|u_2|^2 - 2|u_1|^2 + |v_1|^2$ and $\beta = |u_2|^2 - |u_1|^2$ and $Z_1 = \sum_i a_i^i T^i$ is an auxiliary field. The second part $V_2$ has a similar structure and can be obtained by the substitutions

$$V_2(\Phi, \Delta_2) = V_1(\Phi \to \Phi^*, \Delta_1 \to \Delta_2, v_1 \to v_2, Z_1 \to Z_2)$$

(5.9b)

where $Z_2 = \sum_i a_i^3 T^i$. Elimination of $Z_1$ and $Z_2$ will yield a potential of the desired form.

The leptons have the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_1' \\ \psi_2 \\ \psi_2' \end{pmatrix}$$

(5.10)

where $\psi_1$ and $\psi_2$ are doublets under the two $SU(2)$ groups. After imposing the chirality condition

$$(\gamma_5 \otimes \Gamma_1) \Psi = \Psi$$

(5.11)

one gets

$$\psi_1 = \psi_1 \ (L) \quad \psi_2 = \psi_2 \ (R)$$

(5.12)

By writing $\psi = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ one finds that the usual leptons (with neutrinos acquiring Majorana masses) emerge. The required coupling is

$$< \Psi, (D + \pi(\rho)) \Psi > = \int d^4x \Psi(D + \pi(\rho)) \Psi$$

(5.13)

However, in order to make the right fermions heavy, one must introduce the conjugate fermions

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_1 \\ \chi_2 \\ \chi_2 \end{pmatrix}$$

(5.14)
required to also satisfy the chirality condition. We shall make the identifications, \( \chi_1 = i\tau_2 \psi_1 \) and \( \chi_2 = i\tau_2 \psi_2 \). The other term needed in the action is

\[
<\chi, (D + \pi(\rho))\chi> = \int d^4x \overline{\chi} (D + \pi(\rho)) \chi
\]  

(5.15)

and provides, in addition to the kinetic terms which appear again, the coupling of the conjugate Higgs. The quarks can be introduced in a similar manner. However, the correct coupling to \( U(1) \) can only be achieved after introducing \( SU(3) \). This can be done in a way identical to that in the standard model, where the \( U(3) \times U(1) \) group is coupled through the bimodule structure with both \( U(1) \) and \( \text{Tr} U(3) \) related to the \( U(1)_{B-L} \) in order to provide the correct hypercharge assignments for the quarks. The phenomenological details of this model will be treated elsewhere.

6. Summary and conclusion

We have achieved the main objective set for this paper: The construction of a formalism using the framework of non-commutative geometry of Alain Connes [1-2]. We have modified one particular point in that we choose the basic algebra to be a direct sum of matrix algebras. This simplifies the computations and makes it possible to consider large groups, while in the original setting this becomes a very complicated task, since all the elements in the curvature have to be computed one by one. Another improvement is choosing the vacuum state of the potential to appear in the Dirac operator. Such a choice might appear to break gauge invariance. But this is not so, as the Dirac operator does not transform under gauge transformations, and the actions constructed are shown to be gauge invariant. This is also seen in detail in the component form of the actions. We have derived the formulas for the Yang-Mills action corresponding to a continuous space-time multiplied by three points, but have written the formulas in a way applicable to a space extended by \( N \) points. We have studied three examples in great detail, and showed, in a step by step calculation, how to extract the bosonic action by eliminating the auxiliary fields, and how to introduce fermions in a realistic way. Of course, this formalism does not pretend to solve the fundamental problem of explaining the fermionic mass matrices, and thus does not reduce the number of parameters associated with the fermion masses. It however specifies the Higgs sector and reduces the number of possible terms at the tree level, since the potential takes a very specific form. At the classical level, the cosmological constant always naturally comes out to be zero. All this provides a very good motivation to investigate some of the problems arising in this formalism. The first question that one may ask is on how restrictive this new formalism is. Obviously it is somewhat restrictive, but not to the point that only few models remain. One common feature is that the models that one can construct favor the minimal Higgs
representations. Indeed, if in the $SU(5)$ example one wanted to introduce the Higgs representation $45$ one finds that this can only be done by taking it as an external scalar field not associated to any vector. Of course, this would be self-defeating and cannot be considered to be natural. In this respect, an $SO(10)$ model which is acceptable phenomenologically is not easy to construct, since it would require complicated Higgs’s such as $120$ or $16_s$.

The second question that require further study is the question of whether space-time supersymmetry can be embedded in non-commutative geometry. This question does not appear to have an obvious answer. The difficulty is that the basic building block in non-commutative geometry is the Dirac operator, while in supersymmetry, what appears to be more fundamental is the supersymmetric covariant derivative, $D_\alpha$, which satisfies

$$\{D_\alpha, D^\beta\} = (\partial)^{\beta}_{\alpha}$$

It is like a square root of the Dirac operator. It would be extremely interesting if the second question could be answered in the affirmative and will have positive consequences for constructing models which are acceptable phenomenologically.

Most fundamental, however, is the question of quantization of theories in non-commutative geometry. At present we only have information about the classical action, and any quantum effects can only be dealt with by starting from the action extracted in the classical limit. The non-commutative geometry setting advocated in this paper (if preserved after quantization ) imposes certain constraints on the counterterms admissible in the renormalization of the quantum theories. It is likely that these constraints yield relations between the square of gauge coupling constants and certain quartic Higgs coupling constants. We hope to report on some of these questions in future projects.

**Acknowledgments**

We would like to thank D. Wyler for very useful discussions.
References

[1] A. Connes, *Publ. Math. IHES* 62 44 (1983).

[2] A. Connes, in *the interface of mathematics and particle physics*, Clarendon press, Oxford 1990, Eds D. Quillen, G. Segal and S. Tsou.

[3] A. Connes and J. Lott, *Nucl.Phys.B Proc.Suppl.* 18B 29 (1990), North-Holland, Amsterdam.

[4] A. Connes and J. Lott, *to appear in Proceedings of 1991 Summer Cargese conference*.

[5] D. Kastler, Marseille preprints.

[6] R. Coquereaux, G. Esposito-Farése, G. Vaillant, *Nucl. Phys.* B353 689 (1991);
M. Dubois-Violette, R. Kerner, J. Madore, *J. Math. Phys.* 31 (1990) 316;
B. Balakrishna, F. Gürsey and K. C. Wali, *Phys. Lett.* 254B (1991) 430.

[7] S. Glashow *Nucl. Phys.* 22,579 (1961);
A. Salam and J. Ward *Phys. Lett* 13,168 (1964);
S. Weinberg, *Phys. Rev. Lett.* 19, 1264 (1967);
A. Salam, in *Elementary Particle Theory* (editor N. Svartholm), Almquist and Forlag, Stockholm.

[8] H. Georgi and S. Glashow, *Phys. Rev. Lett* 32 438 (1976);
For a review of unified theories see the book by G. Ross *Grand Unified Theories*, Frontiers in Physics Series, vol 60, Benjamin Publishing.

[9] R. Mohapatra and J. Pati, *Phys. Rev.* D11 566 (1975);
R. Mohapatra and G. Senjanovich *Phys. Rev.* D21165 (1981);
For a review of left-right symmetric models see the book by R. Mohapatra *Unification and Supersymmetry* Springer-Verlag, Berlin.