Algebra and calculus for Tsallis thermostatistics

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Abstract
We construct generalized additions and multiplications, forming fields, and division algebras inspired by the Tsallis thermo-statistics. We also construct derivations and integrations in this spirit. These operations do not reduce to the naively expected ones, when the deformation parameter approaches zero.

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1. Introduction

Non-extensive thermo-statistics has attracted a lot of attention in recent years. This is due, in part, to the contributions of C. Tsallis and his collaborators who have been advocating the use of the non-extensive entropy

\[ S_q = k_B \left( 1 - \sum_{i \in W} p_i^q \right)^{1 \over q - 1} \]  

associated with a probability distribution \( p_i, \ i \in W \), where \( W \) is the set of cells in which one divides the phase space of a system in a coarse grained description, and \( k_B \) is the Boltzmann constant. Applications of this form of entropy have been found in very diverse areas ranging from Dynamical Systems theory and Physics to Medicine, Linguistics and Social Sciences [2]. Other forms of entropy have also been postulated and advocated over the years, (see [2] and references therein for some definitions) claiming to provide generalizations of the Boltzmann-Gibbs entropy to a variety of systems. In all these cases someone recovers the Boltzmann-Gibbs entropy in the limit of deformation disappearing \( q \to 1 \). Someone can notice that the Tsallis entropy (1) for two probabilistically independent systems \( A \) and \( B \), obeys

\[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \]  

as opposed to the usual addition \( S_1(A + B) = S_1(A) + S_1(B) \) that the Boltzmann-Gibbs entropy follows. The name “non-extensive entropy” is a result of (2). The generalized additivity (2) forces us to review the conventional definition of additivity [3] and to construct a generalized algebraic and analytic framework which will express such ideas more naturally [4]-[12]. The present work follows the spirit of [7], [8], [11] and modifies as well as extends the work of [12] and [13].

A brief summary of the contents of this paper is as follows: In Section 2 we introduce the \( \bigcirc \) operations of generalized addition and multiplication, and based on them, we construct a vector space and a division algebra. In Section 3, we introduce the \( \bigtriangleup \) operations and construct
similar algebraic structures as in Section 2. In Section 4, we construct generalized derivatives and integrals corresponding to these operations. In Section 5, we make some general comments and point to some topics for future research.

2. The $k$ algebraic operations

There is, a priori, an infinity of ways in which we can define a generalized addition and a generalized multiplication. We narrow down our choices by the requirement that the sought after operations should reflect, as much as possible, the algebraic properties of Tsallis’ entropy. Let $k \in \mathbb{R}$ indicate the non-extensive parameter [7], [8], [10], [11]. Other authors use $1 - q$, $q - 1$ and $\alpha$ instead of $k$ [1]-[6], [9], [12], [13].

We start by using (2.1) of [11] with the $k$-deformed logarithm $\ln_k(x)$

$$x_{\{k\}} = \ln_k(x) := \frac{x^k - 1}{k}$$

and consequently (2.2) of [11] with the $k$-deformed exponential $e_k(x)$

$$x^{(k)} = e_k(x) := (1 + kx)^{\frac{1}{k}}$$

With this identification, the requirements (2.3) and (2.4) of [11] are satisfied if we define for $x, y \in \mathbb{R}$ the generalized addition

$$x \kay y = (x^k + y^k - 1)^{\frac{1}{k}}$$

We see that the $\kay$ addition reduces to the usual multiplication in $\mathbb{R}$ as $k \to 0$, namely

$$\lim_{k \to 0} x \kay y = xy$$

We can verify that $\kay$ is commutative, associative and has 1 as neutral element. The opposite of $x$, denoted by $\kay x$, is given by

$$\kay x = (2 - x^k)^{\frac{1}{k}}$$
Subtraction is defined as \( x \ominus y = x \oplus (\ominus y) \) and (5),(6) give
\[
x \ominus y = (x^k - y^k + 1)^{\frac{1}{k}}
\] (7)

The generalized multiplication is defined by
\[
x \otimes y = \left\{ \frac{(xy)^k - x^k - y^k + (k + 1)}{k} \right\}^{\frac{1}{k}}
\] (8)

We notice that
\[
\lim_{k \to 0} x \otimes y = e^{(\ln x)(\ln y)}
\]

We see that \( \otimes \) is commutative, associative and has identity element \((k + 1)^{\frac{1}{k}}\). In addition, we observe that the distributivity property holds between \( \oplus \) and \( \otimes \), namely
\[
x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)
\] (9)

Therefore, the structure \( \mathcal{R}_1 = (\mathbb{R}, \oplus, \otimes) \) is a commutative ring with identity [14]. We can prove, by induction, that by \( \oplus \)-adding \( n \in \mathbb{Z}_+ \) times \( x \),
\[
x \oplus \cdots \oplus x = \left\{ nx^k - (n - 1) \right\}^{\frac{1}{k}}
\] (10)

By using (10) we check that \( \mathcal{R}_1 \) has zero characteristic. Motivated by (10), we define the multiplication \( \ominus \), for \( n \in \mathbb{R} \)
\[
n \ominus x = \left\{ nx^k - (n - 1) \right\}^{\frac{1}{k}}
\] (11)

We can prove, by induction, that by \( \otimes \)-multiplying \( x \) by itself \( n \in \mathbb{Z}_+ \) times, one gets
\[
x \otimes \cdots \otimes x = \left\{ \frac{(x^k - 1)^n + k^{n-1}}{k^{n-1}} \right\}^{\frac{1}{k}}
\] (12)

which can be used to prove that \( \mathcal{R}_1 \) has no non-trivial nilpotent elements. The inverse element of \( x \in \mathbb{R} \setminus \{1\} \) denoted by \( \ominus x \) is
\[
\ominus x = \left\{ 1 + \frac{k^2}{x^k - 1} \right\}^{\frac{1}{k}}
\] (13)
It is natural to define [11]-[13] the division as \( x \odot y = x \odot (\odot y) \) and find, by combining (8) and (13),

\[
x \odot y = \left\{ \frac{k^x - 1}{y^k - 1} + 1 \right\}^{\frac{1}{k}}
\]

(14)

So \( \mathcal{R}_1 \) is actually a field [14]. It can also be checked that (2.6) and (2.7) of [11] become in this case

\[
e_k(x) \odot e_k(y) = e_k(x + y)
\]

(15)

\[
e_k(x) \odot e_k(y) = e_k(xy)
\]

(16)

respectively. To proceed, we use (11) to form an algebra \( \mathcal{A} \) over \((\mathbb{R}, +, \cdot)\). As sets \( \mathcal{A} = \mathcal{R}_1 \).

Let \( r, s \in \mathbb{R} \) and \( x, y \in \mathcal{R}_1 \). We readily check that

\[
r \odot (x \oplus y) = (r \odot x) \oplus (r \odot y)
\]

(17)

\[
(r + s) \odot x = (r \odot x) \oplus (s \odot x)
\]

(18)

\[
r \odot (s \odot x) = (rs) \odot x
\]

(19)

\[
1 \odot x = x
\]

(20)

We see therefore that the structure \( \mathcal{V}_1 = (\mathbb{R}, \odot, \oplus) \) is a vector space over \((\mathbb{R}, +, \cdot)\). In addition, we find

\[
r \odot (x \odot y) = (r \odot x) \odot y = x \odot (r \odot y)
\]

(21)

which means that \( \mathcal{V}_1 \) becomes a commutative algebra \( \mathcal{A} \) over \( \mathbb{R} \), and since \( \mathcal{R}_1 \) is a field, \( \mathcal{A} \) is actually a commutative division algebra. It is obvious that

\[
\dim_{\mathbb{R}} \mathcal{A} = 1
\]

(22)

Let \( \mathbb{R}[x] \) denote the ring of polynomials of one variable, with coefficients in \( \mathbb{R} \). All elements of \( \mathcal{A} \) are algebraic over \( \mathbb{R} \), namely they are roots of some polynomial in \( \mathbb{R}[x] \). Therefore \( \mathcal{A} \) is an algebraic algebra over \( \mathbb{R} \). Then a theorem of Frobenius [14] and (22) imply that
\( \mathcal{A} \) is isomorphic to \( \mathbb{R} \) or to \( \mathbb{C} \) as division algebras. This formally justifies and extends the assertion of [11] concerning the isomorphism of \( \mathcal{A} \) and \( \mathbb{R} \).

3. The \( k \) Algebraic Operations

Instead of (3),(4) one could have initially made the “reverse” identifications in (2.1),(2.2) of [11], namely

\[
x_k = e_k(x)
\]

and consequently

\[
x_k = \ln_k(x)
\]

With this identification, the requirements (2.3) and (2.4) of [11] are satisfied if we define the following generalized addition

\[
x \boxplus y = \frac{(1 + kx)^{\frac{1}{k}} + (1 + ky)^{\frac{1}{k}}}{k} - 1
\]

We notice that

\[
\lim_{k \to 0} x \boxplus y = (e^x + e^y) \ln(e^x + e^y)
\]

We easily check that \( \boxplus \) is commutative, associative and has neutral element \( -\frac{1}{k} \). As in the Section 3, the opposite of \( x \), denoted by \( \boxminus x \), is given by

\[
\boxminus x = \frac{(-1)^k(1 + kx) - 1}{k}
\]

We can stop at this point to comment on the apparently indiscriminate use of powers and logarithms throughout this work. We have tacitly assumed that any real number can be raised to any real power, or equivalently, that we can calculate the logarithm of any non-zero real number. This is clearly possible, if we consider the inclusion \( j : \mathbb{R} \hookrightarrow \mathbb{C} \). Then the logarithm \( \ln z \) of \( z = |z|e^{i\theta} \in \mathbb{C}\{0\} \), and the complex power \( (w \in \mathbb{C}) \) are the (multi-valued) functions

\[
\ln z = \log |z| + i\theta \quad \quad z^w = e^{w \ln z}
\]
where \( \log |z| \) stands for the usual logarithm of the positive function of the modulus \( |z| \).

We choose, arbitrarily, one branch of the logarithm \( \ln \) and we work with it everywhere [15].

Considering the reals as a subset of the complex numbers is necessary, if we want to avoid complications arising from the possible lack of closure of the operations in the sets of interest. A drawback of this approach is that such an inclusion \( j \) may obscure the direct physical interpretation of some of the functions of physical interest.

Subtraction is defined by \( x \kappa y = x \bigoplus (\kappa y) \) and (25),(26) give

\[
x \kappa y = \frac{(1 + kx)^{\frac{1}{k}} + (-1)^k(1 + ky)^{\frac{1}{k}}}{k} - 1
\]

(27)

The generalized multiplication is defined by

\[
x \bowtie y = x + y + kxy
\]

(28)

We notice that in the limit \( k \to 0 \), this operation reduces to the usual addition in \( \mathbb{R} \), namely

\[
\lim_{k \to 0} x \kappa y = x + y
\]

We can check that \( \bowtie \) is commutative, associative and has identity element \( 0 \). In addition, we observe that the distributivity property holds between \( \bigoplus \) and \( \bowtie \) namely

\[
x \kappa (y \bigoplus z) = (x \kappa y) \bigoplus (x \bowtie z)
\]

(29)

Therefore the structure \( \mathcal{R}_2 = (\mathbb{R}, \bigoplus, \bowtie) \) is a commutative ring with identity. We can prove, by induction, that by \( \bigoplus \)-adding \( n \in \mathbb{Z}_+ \) times \( x \),

\[
x \bigoplus \cdots \bigoplus x = \frac{n^k(1 + kx) - 1}{k}
\]

(30)

By using (30) we check that \( \mathcal{R}_2 \) has zero characteristic. Motivated by (30), we define the multiplication \( \kappa \), for \( n \in \mathbb{R} \)

\[
n \kappa x = \frac{n^k(1 + kx) - 1}{k}
\]

(31)
We can also prove, by induction, that $\# - $multiplying $x$ by itself $n \in \mathbb{Z}_+$ times gives

$$x \# \cdots \# x = \frac{1}{k} \{(1 + kx)^n - 1\}$$

(32)

which can be used to prove that $\mathcal{R}_2$ has no non-trivial nilpotent elements. The inverse element of $x \in \mathbb{R} \setminus \{-\frac{1}{k}\}$, denoted by $\# x$, is

$$\# x = \frac{-x}{1 + kx}$$

(33)

The division, as usual [11]-[13], is defined by $x \# y = x \# (\# y)$ and we find, by combining (28) and (33),

$$x \# y = \frac{x - y}{1 + ky}$$

(34)

So $\mathcal{R}_2$ is actually a field. We can also check that (2.6) and (2.7) of [11] become in this case

$$\ln_k(x) \# \ln_k(y) = \ln_k(x + y)$$

(35)

$$\ln_k(x) \# \ln_k(y) = \ln_k(xy)$$

(36)

respectively. We proceed as in the construction of $\mathcal{A}$ in the previous paragraph. Using (31), we form an algebra $\mathcal{B}$ over $(\mathbb{R}, +, \cdot)$. Let $r, s \in \mathbb{R}$ and $x, y \in \mathcal{R}_2$. The following relations, analogues of (17) - (20), are satisfied

$$r \# (x \# y) = (r \# x) \# (r \# y)$$

(37)

$$(r + s) \# x = (r \# x) \# (s \# x)$$

(38)

$$r \# (s \# x) = (rs) \# x$$

(39)

$$1 \# x = x$$

(40)

Therefore, the structure $\mathcal{V}_2 = (\mathbb{R}, \#, \#)$ is a vector space over $(\mathbb{R}, +, \cdot)$. The analogue of (21)

$$r \# (x \# y) = (r \# x) \# y = x \# (r \# y)$$

(41)
is also satisfied. By the same arguments as for $\mathcal{A}$, one can prove that $\mathcal{B}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ as division algebras.

It is worth mentioning that the matrix operations [16] over $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined in the same way as these over $\mathbb{R}$ if we replace $+$ and $\cdot$ with $\oplus$ and $\otimes$ or with $\boxplus$ and $\boxtimes$. Since there is no obvious and compelling reason to radically change the definition of the Lie bracket between two matrices over $\mathcal{R}_1$ or $\mathcal{R}_2$, we define it by, respectively,

$$[A, B]_\oplus = \{A^k \otimes B\} \oplus \{B^k \otimes A\} \quad [A, B]_\boxplus = \{A^k \boxtimes B\} \boxplus \{B^k \boxtimes A\}$$

4. The $\oplus$ and $\boxplus$ differentials and integrals

As in the case of algebraic operations, there is a lot of freedom in defining the $k$-deformed derivative [11], [13]. We work, in most of this Section, with functions $f, g : \mathcal{R}_1 \to \mathcal{R}_1$ which will be differentiable as many times as needed, as we usually do in Physics. Unlike previous works [11]-[13], we believe that a more “natural” definition of the derivative $D_\oplus$ operator for such functions is

$$D_\oplus f(x) = \lim_{y \to x} \{f(y)^k \ominus f(x)^k\} \ominus \{y^k \ominus x\} \quad (42)$$

By using (7) and (14) we find

$$D_\oplus f(x) = \left\{1 + \frac{1}{x^{k-1}} \frac{d}{dx}[f(x)]^k \right\}^\frac{1}{k} \quad (43)$$

We see that (43) gives

$$\lim_{k \to 0} D_\oplus f(x) = e^{\frac{d}{dx} \ln f(x)}$$

which is clearly not equal to $\frac{df}{dx}$. If $r \in \mathbb{R}$, we readily verify that

$$D_\oplus \{f(x)^k \oplus g(x)\} = \{D_\oplus f(x)^k\} \oplus \{D_\oplus g(x)\} \quad (44)$$

$$D_\oplus \{r^k \otimes f(x)\} = r^k \{D_\oplus f(x)\} \quad (45)$$
Therefore $D_\otimes$ is a linear operator with respect to $\oplus, \otimes$. We can also verify that Leibniz’s rule holds

$$D_\otimes \{ f(x) \otimes g(x) \} = \{ D_\otimes f(x) \} \otimes g(x) \oplus f(x) \otimes \{ D_\otimes g(x) \} \quad (46)$$

Therefore $D_\otimes$ is a derivation on the space of functions $f : \mathcal{R}_1 \rightarrow \mathcal{R}_1$. It is also interesting to notice that although

$$D_\otimes \{ r \otimes f(x) \} = r \otimes \{ D_\otimes f(x) \} \quad (47)$$

Leibniz’s rule for $\otimes$ does not hold, namely

$$D_\otimes \{ f(x) \otimes g(x) \} \neq \{ D_\otimes f(x) \} \otimes g(x) \oplus f(x) \otimes \{ D_\otimes g(x) \} \quad (48)$$

It is not difficult to find the reason for the failure of this identity: $D_\otimes$ is defined through $\otimes$ which is defined through $\otimes$, and the two “circle” multiplications $\otimes, \otimes$ do not obey obvious, “nice” identities with each other. We notice that $D_\otimes r = 1, \forall r \in \mathcal{R}_1$ with 1 being the neutral element with respect to $\oplus$ and $D_\otimes (x^k) = (k + 1)^\frac{1}{k}$ where the right-hand-side is the identity with respect to $\otimes$.

Having defined the derivative operator $D_\otimes$, the next step is to find an expression for the deformed exponential on $\mathcal{R}_1$. Following [11], we demand the deformed exponential $\tilde{e}(x)$ to be an eigenfunction of $D_\otimes$, namely to satisfy $D_\otimes \tilde{e}(x) = \tilde{e}(x)$. The general solution of this differential equation is parametrized by $c \in \mathbb{R}$ and is given by

$$\tilde{e}_c(x) = \left\{ 1 + ce^\frac{x}{k} \right\}^\frac{1}{k} \quad (49)$$

The deformed logarithm $\tilde{\ln}_c(x)$, i.e. the inverse of $\tilde{e}_c(x)$ with respect to composition, is

$$\tilde{\ln}_c(x) = \left\{ k \ln \left( \frac{x^k - 1}{c} \right) \right\}^\frac{1}{k} \quad (50)$$

In the special case in which $c = k$, (49), (50) can be re-expressed in terms of (3), (4) as

$$\tilde{e}_k(x) = e_k \left( e^{\frac{x}{k}} \right) \quad (51)$$
\[ \ln_k(x) = k \ln(\ln_k(x)) \] (52)

One can continue and define generalized hyperbolic and trigonometric functions on \( R_1 \) in terms of \( \tilde{e}_k(x) \) and verify similar relations between them [4]-[13]. Since this path is straightforward, we continue by studying the integral operator. In the sequel will need the differential

\[ d_{\boxplus}x := \lim_{y \to x} y^k \ominus x = (1 + k x^{k-1} dx)^{\frac{1}{k}} \] (53)

The integral operator \( \int_{\boxplus} \) is operationally defined as the inverse with respect to composition of the differential operator \( D_{\boxplus} \), namely, by demanding that

\[ D_{\boxplus} \int_{\boxplus} f(x) \otimes d_{\boxplus}x = f(x) \] (54)

We can check that (54) is satisfied, if we define

\[ \int_{\boxplus} f(x) \otimes d_{\boxplus}x := \left\{ 1 + \int (f(x))^k - 1 \right\} x^{k-1} dx \] (55)

We immediately see that

\[ \lim_{k \to 0} \int_{\boxplus} f(x) \otimes d_{\boxplus}x = e^{\int \ln(f(x)) dx} \]

We can also verify that this integral operation is linear with respect to \( \boxplus \) and \( \otimes \), namely

\[ \int_{\boxplus} \left\{ f(x) \boxplus g(x) \right\} \otimes d_{\boxplus}x = \left\{ \int_{\boxplus} f(x) \otimes d_{\boxplus}x \right\} \boxplus \left\{ \int_{\boxplus} g(x) \otimes d_{\boxplus}x \right\} \] (56)

and

\[ \int_{\boxplus} \left\{ r \otimes f(x) \right\} \otimes d_{\boxplus}x = r \otimes \left\{ \int_{\boxplus} f(x) \otimes d_{\boxplus}x \right\} \] (57)

We can briefly turn our attention to the corresponding constructions for the \( \boxslash \) operations. The \( D_{\boxslash} \) derivative is defined by analogy to (42) as

\[ D_{\boxslash} f(x) = \lim_{y \to x} \{ f(y) \boxslash f(x) \} \boxslash \{ y \boxslash x \} \] (58)

A straightforward computation shows that (58) gives

\[ D_{\boxslash} f(x) = \begin{cases} \frac{f(x) - x}{1 + k x}, & \text{if } k \neq 2m + 1, \ m \in \mathbb{Z} \\ \frac{1}{k} \left( \frac{df(x)}{dx} \right)^k \left( \frac{1 + k x}{1 + k f(x)} \right)^{k-1} - 1, & \text{if } k = 2m + 1, \ m \in \mathbb{Z} \end{cases} \] (59)
We notice here that for any \( k \in \mathbb{R} \setminus \{2m + 1, m \in \mathbb{Z}\} \), \( D_k \) actually reduces to a difference rather than an expression involving the derivative \( \frac{df(x)}{dx} \). The case where \( k \) is an odd integer is the exception and has very little practical physical impact. Indeed, one cannot determine the value of \( k \) with infinite precision for a system. When \( k \) is an odd integer we can always perturb it by adding an arbitrarily small decimal part to it. We expect that the physical predictions arising from the appropriate thermodynamic expressions should not be too sensitive to such perturbations of \( k \). Accordingly, the entropy for such a small perturbation of \( k \) should not change dramatically. This requirement is a special case of the Lesche stability criterion [17],[18] which is obeyed by the Tsallis’ entropy [2] as well as other forms of entropy [19],[20].

Therefore, the case with \( k \) being an odd integer effectively never occurs in practice, so we do not have to elaborate on a mathematical structure describing it. For \( k \in \mathbb{R} \setminus \{\text{Odd integer}\} \), we mention for completeness that the “integral” corresponding to \( D_k \) is formally given by

\[
\int f(x)^k d_k x = x + (1 + kx)f(x)
\]

(60)

We see that the \( k \) calculus operations do not lead to anything particularly new and that is why we do not pursue in this work any further properties they may satisfy.

5. Discussion and conclusions

We defined the deformed operations \( \otimes, \circ, \circlearrowleft, \circlearrowright, \boxtimes, \boxdiv, \boxdot \). We have chosen not to modify the usual composition of functions, since no such need arises, and operations like matrix multiplication depend crucially on its definition.

In previous works, either the distributivity property did not hold, or the resulting structures were monoids and rigs at best. With the operations that we defined, the corresponding sets
become fields $\mathcal{R}_1$, $\mathcal{R}_2$, and subsequently division algebras $\mathcal{A}$, $\mathcal{B}$ which turn out to be isomorphic to $\mathbb{R}$, $\mathbb{C}$. We constructed a generalized derivative $D_{\mathcal{B}}$ and integral $\int_{\mathcal{B}}$ on $\mathcal{R}_1$ and found analogous, but not very interesting, structures on $\mathcal{R}_2$. Most of these structures do not reduce to the “usual” ones on $\mathbb{R}$ as $k \rightarrow 0$. The unusual identification and the unexpected un-deformed limits of $k$ and $\mathcal{B}$ can be considered as being, partly, responsible for the fact that the structures developed in this paper were previously missed [12],[13]. The characterization as “deformed” should be understood to refer to the way the algebraic and calculus operations were generated, rather than to their actual $k \rightarrow 0$ limit. Although the above structures are mathematically attractive, their physical relevance in quantifying ideas of non-extensive thermo-statistics is not clear at this point.

We have performed a very heuristic, operational, treatment of the concepts of limit, derivative and integral in $\mathcal{R}_1$ and $\mathcal{R}_2$. A more careful approach may probably be warranted in the future, but our treatment seems to be adequate for present purposes. Someone can extend our results by constructing more elaborate algebraic structures, like bi-algebras and Hopf algebras or can follow the steps of homological algebra [21] and proceed by constructing differential complexes [22] etc. Apart from some potential mathematical significance, it is not clear to us what the use of such structures would be in non-extensive thermo-statistics, or in any other context, that is why we did not pursue their further development in this work.

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References

1. C. Tsallis, *J. Stat. Phys.* 52, 479 (1988).
2. *Nonextensive entropy; Interdisciplinary Applications*, C. Tsallis, M.Gell-Mann, Eds., Oxford Univ. Press (2004).
3. C. Tsallis, arXiv:cond-mat/0409631
4. E.P. Borges, *J. Phys. A* 31, 5281 (1998).
5. C. Tsallis, *Braz. J. Phys.* 29, 1 (1999).
6. E.K. Lenzi, E.P. Borges, R.S. Mendes, *J. Phys. A* 32, 8551 (1999).
7. G. Kaniadakis, *Physica A* 296, 405 (2001).
8. G. Kaniadakis, A.M. Scarfone, *Physica A* 305, 69 (2002).
9. T. Yamano, *Physica A* 305, 486 (2002).
10. J. Naudts, *Physica A* 316, 323 (2002).
11. G. Kaniadakis, *Phys. Rev. E* 66, 056125 (2002).
12. L. Nivanen, A. Le Méhauté, Q.A. Wang, *Rep. Math. Phys.* 52, 437 (2003).
13. E.P. Borges, *Physica A* 340, 95 (2004).
14. T.W. Hungerford, *Algebra*, Springer (1980).
15. A.I. Markushevich, *Theory of Functions of a Complex Variable*, 2nd Ed., Chelsea (1977).
16. W. Greub, *Linear Algebra*, 4th Ed., Springer (1981).
17. B. Lesche, *J. Stat. Phys* 27, 419 (1982).
18. S. Abe, *Phys. Rev. E* 224, 046134 (2002).
19. G. Kaniadakis, A.M. Scarfone, *Physica A* 340, 102 (2004).
20. S. Abe, G. Kaniadakis, A. M. Scarfone, arXiv:cond-mat/0401290
21. P.J. Hilton, U. Stammbach, *A course in Homological Algebra*, 2nd Ed., Springer (1997).
22. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer (1995).