Index formulas and charge deficiencies on the Landau levels

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Abstract

The notion of charge deficiency is studied from the view of \( K \)-theory of operator algebras and is applied to the Landau levels in \( \mathbb{R}^{2n} \). We calculate the charge deficiencies at the higher Landau levels in \( \mathbb{R}^{2n} \) by means of an Atiyah-Singer type index theorem.

1 Introduction

The paper is a study of the charge deficiencies at the Landau levels in \( \mathbb{R}^{2n} \). The Landau levels are the eigenspaces of the Landau Hamiltonian which is the energy operator for a quantum particle moving in \( \mathbb{R}^{2n} \) under the influence of a constant magnetic field of full rank.

In [1], the notion of charge deficiency was introduced as a measure of how much a flux tube changes a fermionic system in \( \mathbb{R}^2 \). The setting of [1] is a quantum system where the Fermi energy is in a gap and the question is what happens when the system is taken through a cycle. Letting \( P \) denote the projection onto the state space and \( U \) the unitary transformation representing the cycle, the projection \( Q \) onto the new state space after it had been taken through a cycle can be expressed as \( Q = UPU^* \). The relative index \( \text{ind} (Q, P) \) is defined as an infinite dimensional analogue of \( \dim Q - \dim P \) and is well defined whenever \( Q - P \) is a compact operator. The condition that \( Q - P \) is compact is equivalent to that \( [P, U] \) is compact. In the setting of [1] the relative index represents the change in the number of fermions that \( U \) produces. In [1] the following formula was proven:

\[
\text{ind} (Q, P) = \text{ind} (PUU^*).
\]

For sufficiently nice systems in \( \mathbb{R}^2 \) one can choose the particular unitary given by multiplication by the bounded function \( U := z/|z| \). The condition on the system that is needed is that \( P \) commutes with \( U \) up to a compact operator. The charge deficiency of a projection \( P \) in the sense of [1] is then defined using \( U \) as

\[
c(P) := \text{ind} (PUU^*).
\]

The viewpoint we will have in this paper is that the charge deficiency is a \( K \)-homology class. This viewpoint lies in line with the view on D-brane charges in string theory, see more in [4], [10]. In the case studied in [1] the charge
we let Bergman projection choosing up to a compact operator with a expressed as of the sphere $S$.

The charge deficiency $K$ homology class $[u]$ is Fredholm if and only if $[u] \in K_1(C(T))$. Thus the charge deficiency is the image of $[P]$ under the isomorphism

$$K^1(C(T)) = K K_1(C(T), \mathbb{C}) \cong \text{Hom}(K_1(C(T)), K_0(\mathbb{C})) \cong \mathbb{Z},$$

where the first isomorphism is the natural mapping coming from the Universal Coefficient Theorem for $KK$-theory and the second isomorphism comes from choosing $[u]$ as a generator for $K_1(C(T))$. So a better picture is that the $K$-homology class $[P] \in K^1(C(T))$ is the charge deficiency of $P$.

The system we will consider in this paper consists of a particle moving in $\mathbb{R}^{2n}$ under the influence of a constant magnetic field $B$ of full rank. If we choose a linear vector potential $A$ satisfying $dA = B$ the Hamiltonian of this system is given by

$$H_A := (-i \nabla - A)^2,$$

This Landau Hamiltonian should be viewed as a densely defined operator in the Hilbert space $L^2(\mathbb{R}^{2n})$. Taking $\mathcal{D}(H_A) = C_c^\infty(\mathbb{R}^{2n})$, the operator $H_A$ becomes essentially self-adjoint, see more in [9]. Due to the identification $\mathbb{R}^{2n} = \mathbb{C}^n$ we will use the complex structure and we will assume that $B = \frac{1}{2} \sum dz_j \wedge d\overline{z}_j$.

The Landau Hamiltonian has a discrete spectrum with eigenvalues $\Lambda_\ell = 2\ell + n$ for $\ell \in \mathbb{N}$ and the eigenspaces $\mathcal{L}^\ell$ are infinite dimensional. Let

$$P_\ell : L^2(\mathbb{R}^{2n}) \to \mathcal{L}^\ell$$

denote the orthogonal projection to the $\ell$:th eigenspace. Our point of view on the charge deficiencies for the Landau levels is that they are $K$-homology classes of the sphere $S^{2n-1}$. For a bounded continuous function $a : \mathbb{R}^{2n} \to M_N(\mathbb{C})$ we define the continuous function $a_\ell \in C(S^{2n-1})$ as

$$a_\ell(v) := a(rv).$$

We let $A_N$ be the subalgebra of $C_0(\mathbb{R}^{2n}) \otimes M_N(\mathbb{C})$ such that $a_\ell$ converges uniformly in $v$ to a continuous function $a_\theta$ on $S^{2n-1}$. The mapping $a \mapsto a_\theta$ defines a $\ast$-homomorphism $A_N \to C(S^{2n-1}) \otimes M_N(\mathbb{C})$. The projection $P_\ell$ commutes up to a compact operator with $a \in A_N$ (see below in Theorem 5.2) and

$$P_\ell a|_{\mathcal{L}^\ell \otimes \mathbb{C}^N} : \mathcal{L}^\ell \otimes \mathbb{C}^N \to \mathcal{L}^\ell \otimes \mathbb{C}^N$$

is Fredholm if and only if $a_\theta$ is invertible (see Proposition 5.6). Now we may present the main theorem of this paper:

**Theorem 1.** If $a_\theta$ is smooth and invertible, the index of $P_\ell a|_{\mathcal{L}^\ell \otimes \mathbb{C}^N}$ can be expressed as

$$\text{ind}(P_\ell a|_{\mathcal{L}^\ell \otimes \mathbb{C}^N}) = \frac{-(\ell + n - 1)!}{\ell!(2n - 1)!(2\pi i)^n} \int_{S^{2n-1}} \text{tr}((a_\theta^{-1}d a_\theta)^{2n-1}).$$

The charge deficiency $[P_\ell] \in K^1(C(S^{2n-1}))$ may be expressed in terms of the Bergman projection $P_B$ on the unit ball in $\mathbb{C}^n$ as

$$[P_\ell] = \frac{(\ell + n - 1)!}{\ell!(n-1)!} [P_B].$$

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2 The particular Landau levels

The spectral theory of the Landau Hamiltonian is well known and we will review it briefly. See more in [13]. We will let \( \varphi := \frac{iz}{\ell} \) and assume that the magnetic field \( B \) is of the form \( B = i\partial\bar{\partial}\varphi \). Here \( \partial \) is the complex linear part of the exterior differential \( d \). Define the annihilation operators as

\[
q_j := 2\frac{\partial}{\partial \bar{z}_j} + z_j \quad \text{for} \quad j = 1, \ldots, n.
\]

The adjoints are given by the creation operators \( q_j^* := -2\frac{\partial}{\partial z_j} + \bar{z}_j \). The annihilation and creation operators satisfy the following formulas:

\[
[q_j, q_i] = [q_j^*, q_i^*] = 0, \quad [q_i, q_j^*] = 2\delta_{ij} \quad \text{and} \quad H_A = \sum_{j=1}^n q_j^* q_j + n = \sum_{j=1}^n q_j q_j^* - n.
\]

Here we view \( H_A \) as a densely defined operator in \( L^2(\mathbb{C}^n) \). Thus the lowest eigenvalue is \( n \) with corresponding eigenspace \( L_0 = e^{-\varphi}F(\mathbb{C}^n) \) where \( F(\mathbb{C}^n) := L^2(\mathbb{C}^n, e^{-2\varphi}) \cap \mathcal{O}(\mathbb{C}^n) \) denotes the Fock space. Here \( \mathcal{O}(\mathbb{C}^n) \) denotes the space of holomorphic functions in \( \mathbb{C}^n \). In one complex dimension there is only one creation operator \( q^* \) and the eigenspaces are given by \( L_k = (q^*)^kL_0 \). Using multi-index notation, for \( \kappa = (k_1, \ldots, k_n) \in \mathbb{N}^n \) we define \( q_{\kappa} := q_1^{k_1} \cdots q_n^{k_n} \) and

\[
L_{\kappa} := q_{\kappa}^*L_0 = L_{k_1} \otimes L_{k_2} \otimes \cdots \otimes L_{k_n}.
\]

We will call this space for the particular Landau level of height \( \kappa \). Using that \( q_j \) and \( q_j^* \) define a representation of the Heisenberg algebra in \( n \) dimension we obtain the eigenvalues of \( H_A \) as \( \Lambda_{\kappa} = 2\ell + n \) with the corresponding eigenspaces

\[
L^\ell := \bigoplus_{|\kappa| = \ell} L_{\kappa} = \bigoplus_{|\kappa| = \ell} L_{k_1} \otimes L_{k_2} \otimes \cdots \otimes L_{k_n}.
\]

The \( \ell \)-th eigenspace \( L^\ell \) is called the Landau level of height \( \ell \). Since the Hamiltonian commutes with the representation of \( SU(n) \) on \( \mathbb{C}^n \), its eigenspaces are \( SU(n) \)-invariant. Also the orthogonal projections \( P_{\ell} : L^2(\mathbb{C}^n) \rightarrow L^\ell \) are invariant under the \( SU(n) \)-action.

Recall that the vacuum subspace \( L_0 \subseteq L^2(\mathbb{C}^n) \) has a reproducing kernel induced by the reproducing kernel on the Fock space. The reproducing kernel of \( F(\mathbb{C}^n) \) is given by \( K(z, w) = e^{2\varphi} \). So the reproducing kernel of \( L_0 \) is given by

\[
K_0(z, w) := e^{\frac{1}{4}(w \cdot z - |z|^2 - |w|^2)}.
\]

This expression for the reproducing kernel implies that the orthogonal projection \( P_0 : L^2(\mathbb{C}^n) \rightarrow L_0 \) is given by

\[
P_0 f(z) = \int_{\mathbb{C}^n} f(w)\overline{K_0(z, w)}dV.
\]

By [12] the orthogonal projection \( P_k : L^2(\mathbb{C}^n) \rightarrow L_k \) onto the particular Landau level of height \( k \) is also an integral operator with kernel

\[
K_k(z, w) = e^{\frac{1}{4}(w \cdot z - |z|^2 - |w|^2)} \prod_{j=1}^n L_{k_j} \left( \frac{1}{2} |z_j - w_j|^2 \right).
\]

Here \( L_k \) is the Laguerre polynomial of order \( k \). Notice that the projections \( P_k \) are not \( SU(n) \)-invariant in general.
3 Toeplitz operators on the Landau levels

We want to study topological properties of the particular Landau levels using Toeplitz operators. The symbols will be taken from a suitable subalgebra of $C_b(\mathbb{C}^n)$, the bounded functions on $\mathbb{C}^n$. The standard notation $\mathcal{B}(\mathcal{H})$ will be used for the bounded operators on a separable Hilbert space $\mathcal{H}$ and the compact operators will be denoted by $\mathcal{K}(\mathcal{H})$. We will let $\pi : C_b(\mathbb{C}^n) \to \mathcal{B}(L^2(\mathbb{C}^n))$ denote the representation given by pointwise multiplication. This is clearly an $SU(n)$-equivariant mapping. Define the linear map $T_h : C_b(\mathbb{C}^n) \to \mathcal{B}(\mathcal{L}_h)$ by $T_h(a) := P_h\pi(a)|_{\mathcal{L}_h}$.

**Lemma 3.1.** If $a \in C_0(\mathbb{C}^n)$ then $T_h(a) \in \mathcal{K}(\mathcal{L}_h)$ for all $h \in \mathbb{N}^n$.

The proof of this lemma is analogous to the proof for the same statement for Toeplitz operators on a pseudoconvex domain from [14].

**Proof.** It is sufficient to prove the claim for $a \in C_e(\mathbb{C}^n)$, since $T_h$ is continuous and $C_e(\mathbb{C}^n) \subseteq C_0(\mathbb{C}^n)$ is dense. Define the compact set $K := \text{supp}(a)$. Let $R : L_2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ denote the operator given by multiplication by $\chi_K$, the characteristic function of $K$. We have $T_h(a) = P_h\pi(a)R$ so the Lemma holds if $R$ is compact. That $R$ is compact follows from Cauchy estimates of holomorphic functions on a compact set. □

Define the $SU(n)$-invariant $C^*$-subalgebra $A \subseteq C_b(\mathbb{C}^n)$ as consisting of functions $a$ such that $a(rv)$ converges uniformly in $v$ as $r \to \infty$ to a continuous function $a_0 : S^{2n-1} \to \mathbb{C}$ when $r \to \infty$. Thus we obtain a surjective $SU(n)$-equivariant $*$-homomorphism $\pi_0 : A \to C(S^{2n-1})$ given by $\pi_0(a)(v) := \lim_{r \to \infty} a(rv)$.

The mapping $\pi_0$ satisfies $\ker \pi_0 = C_0(\mathbb{C}^n)$. We will henceforth consider $T_h$ as a mapping from $A$ to $\mathcal{B}(\mathcal{L}_h)$.

If we let $B_n$ denote the open unit ball in $\mathbb{C}^n$, another view on $A$ is as the image of the $SU(n)$-equivariant $*$-monomorphism $C(\overline{B_n}) \to C_b(B_n) \cong C_b(\mathbb{C}^n)$ where the last isomorphism comes from an $SU(n)$-equivariant homeomorphism $B_n \cong \mathbb{C}^n$.

**Theorem 3.2.** The projection $P_h$ satisfies $[P_h, \pi_0(a)] \in \mathcal{K}(L^2(\mathbb{C}^n))$ for all $a \in A$. Therefore the $*$-linear mapping $T_h : A \to \mathcal{B}(\mathcal{L}_h)$ satisfies $T_h(ab) - T_h(a)T_h(b) \in \mathcal{K}(\mathcal{L}_h)$.

The proof is based on a similar result from [2] where the Fock space was used to define a Toeplitz quantization of a certain subalgebra of $L^\infty(\mathbb{C}^n)$. The case of the Fock space is more or less the same as the case $h = 0$ for Landau quantization. To prove the Theorem we need a lemma similar to part (iv) of Theorem 5 of [2]. Using the isomorphism $A \cong C(\overline{B_n})$ we define the dense subalgebra $A_1 \subseteq A$ as the inverse image of the Lipschitz continuous functions in $C(\overline{B_n})$.

**Lemma 3.3.** For $a \in A_1$ then for any $\varepsilon > 0$ we may write $a = g_\varepsilon + h_\varepsilon$ where $h_\varepsilon \in C_0(\mathbb{C}^n)$ and $g_\varepsilon \in A$ satisfies $|g_\varepsilon(z) - g_\varepsilon(w)| \leq \varepsilon|z - w| \quad \forall z, w \in \mathbb{C}^n$. (2)
Proof. Let $C$ denote the Lipschitz constant of $\pi_\partial(a)$. Take an $\varepsilon > 0$ and let $\chi_\varepsilon$ be a Lipschitz continuous $SU(n)$-invariant cutoff such that $\chi_\varepsilon(z) = 0$ for $|z| \leq R$ and $\chi_\varepsilon(z) = 0$ for $|z| \geq 2R$ where $R = R(\varepsilon, C)$ is to be defined later. To shorten notation, define $a_\partial := \pi_\partial(a)$. Let

$$g_\varepsilon(z) := \chi_\varepsilon(z) \cdot a_\partial(z/|z|)$$

and $h_\varepsilon := a - g_\varepsilon$. Clearly $h_\varepsilon \in C_0(\mathbb{C}^n)$ and $g_\varepsilon \in A$ so what remains to be proven is that $R$ can be chosen in such a way that $g_\varepsilon$ satisfies equation (2).

We have elementary estimates

$$\left| \frac{z}{|z|} - \frac{w}{|w|} \right| \leq \frac{|z - w|}{|z|} + \frac{w}{|z|} - \frac{w}{|w|} \leq 2 \frac{|z - w|}{|w|}.$$ 

Thus for $z, w \neq 0$ the function $a_\partial$ satisfies

$$\left| a_\partial \left( \frac{z}{|z|} \right) - a_\partial \left( \frac{w}{|w|} \right) \right| \leq 2C \frac{|z - w|}{|w|}.$$ 

The function $\chi_\varepsilon$ has Lipschitz coefficient $1/R$ so if we take $R > 2C/\varepsilon$ then $g_\varepsilon$ satisfies equation (2). 

Let $C(L^2(\mathbb{C}^n)) := B(L^2(\mathbb{C}^n))/\mathcal{K}(L^2(\mathbb{C}^n))$ denote the Calkin algebra and $q$ the quotient mapping.

**Proof of Theorem 3.2** Since Lipschitz continuous functions are dense in $A$ we may assume that $a \in A_1$, so by Lemma 3.3 we can for any $\varepsilon > 0$ write $a = g_\varepsilon + h_\varepsilon$. In this case we have for $f \in L^2(\mathbb{C}^n)$

$$[P_k, \pi(g_\varepsilon)]f(z) = \int (g_\varepsilon(z) - g_\varepsilon(w))K_k(z, w)f(w)dw.$$ 

Define the operator

$$Bf(z) := \int |z - w|K_k(z, w)f(w)dw.$$ 

By equation (1) we have that for some $C$ the integral kernel of $B$ is bounded by

$$|z - w||K_k(z, w)| \leq C|z - w|^{|z| + 1}e^{-\frac{1}{8}|z - w|^2}.$$ 

Therefore the kernel of $B$ is dominated by the kernel of a bounded convolution operator and $\|B\| < \infty$. The estimate (2) for $g_\varepsilon$ implies that

$$\|[P_k, \pi(g_\varepsilon)]\| \leq \varepsilon \|B\|.$$ 

Using that $[P_k, \pi(g_\varepsilon)] = [P_k, \pi(a)]$ modulo compact operators, by Lemma 3.1 we have the inequality

$$\|q([P_k, a])\|_C(L^2(\mathbb{C}^n)) \leq \varepsilon \|B\| \quad \forall \varepsilon > 0.$$ 

Therefore $q([P_k, a]) = 0$ and $[P_k, a]$ is compact.
Theorem 3.2 implies that the mapping \( \tilde{\beta}_k := q \circ T_k : A \to C(\mathcal{L}_k) \) is a well defined \( * \)-homomorphism. Define the \( C^* \)-algebra
\[
\tilde{\mathcal{T}}_k := \{ a \oplus x \in A \oplus \mathcal{B}(\mathcal{L}_k) : \tilde{\beta}_k(a) = q(x) \}.
\]
This \( C^* \)-algebra contains \( \mathcal{K} \) as an ideal via the embedding \( k \mapsto 0 \oplus k \) and we obtain a short exact sequence
\[
0 \to \mathcal{K} \to \tilde{\mathcal{T}}_k \to A \to 0.
\]

**Lemma 3.4.** Let \( (\mathfrak{k}_p)_{p=1}^N \subseteq \mathbb{N}^n \) be a finite collection of distinct \( n \)-tuples of integers. Then the mapping
\[
A \ni a \mapsto q \left( \sum_{p=1}^N P_{\mathfrak{k}_p} \right) \pi(a) \left( \sum_{p=1}^N P_{\mathfrak{k}_p} \right) \in C(\oplus_{p=1}^N \mathcal{L}_{\mathfrak{k}_p})
\]
coincides with the mapping
\[
A \ni a \mapsto \oplus_{p=1}^N \tilde{\beta}_{\mathfrak{k}_p}(a) \in C(\oplus_{p=1}^N \mathcal{L}_{\mathfrak{k}_p}).
\]

**Proof.** The Lemma follows if we show that \( P_k \pi(a) P_{k'} \in \mathcal{K}(L^2(\mathbb{C}^n)) \) for \( k \neq k' \). But Theorem 3.2 implies that \( P_k \pi(a)(1 - P_k) \in \mathcal{K}(L^2(\mathbb{C}^n)) \). So the Lemma follows from
\[
P_k \pi(a) P_{k'} = P_k \pi(a)(1 - P_k) P_{k'}.
\]

In particular we can look at the collection of all \( \mathfrak{k} \)'s such that \( |\mathfrak{k}| = \ell \). We will define the \( SU(n) \)-equivariant mapping \( \tilde{\beta}_\ell : A \to C(\mathcal{L}_\ell) \) as
\[
a \mapsto \oplus_{|\mathfrak{k}|=\ell} \tilde{\beta}_\mathfrak{k}(a).
\]
Just as for the particular Landau levels we define
\[
\tilde{T}_\ell := \{ a \oplus x \in A \oplus \mathcal{B}(\mathcal{L}_\ell) : \tilde{\beta}_\ell(a) = q(x) \}.
\]
The projection map \( \tilde{T}_\ell \to A \) given by \( a \oplus x \mapsto a \) defines an \( SU(n) \)-equivariant extension
\[
0 \to \mathcal{K} \to \tilde{T}_\ell \to A \to 0.
\]

**Lemma 3.5.** The kernel of \( \tilde{\beta}_\ell \) is \( C_0(\mathbb{C}^n) \).

**Proof.** Lemma 3.1 implies that \( C_0(\mathbb{C}^n) \subseteq \ker \tilde{\beta}_\ell \). To prove the reverse inclusion we observe that the mapping \( \tilde{\beta}_\ell \) is a unital \( SU(n) \)-equivariant \( * \)-homomorphism. Since \( \tilde{\beta}_\ell \) is equivariant, the ideal \( \ker \tilde{\beta}_\ell \subseteq A \) is \( SU(n) \)-invariant. The inclusion \( C_0(\mathbb{C}^n) \subseteq \ker \tilde{\beta}_\ell \) implies that there is an equivariant surjection \( C(S^{2n-1}) \to A/\ker \tilde{\beta}_\ell \) which must be an isomorphism since \( C(S^{2n-1}) \) is \( SU(n) \)-simple and \( \tilde{\beta}_\ell \) is unital. It follows that \( \ker \tilde{\beta}_\ell = C_0(\mathbb{C}^n) \).

It is interesting that although the statement of Lemma 3.5 sounds algebraic, it is really the analytic statement that \( T_\ell(a) \) is compact if and only if \( a \) vanishes at infinity. And this is proven with algebraic methods!

**Proposition 3.6.** If \( u \in A \oplus M_N \), the operator \( T_\ell(u) \) is Fredholm if and only if \( \pi_\ell(u) \) is invertible.

**Proof.** By Atkinson’s Theorem \( T_\ell(u) \) is Fredholm if and only if \( \tilde{\beta}_\ell(u) \) is invertible. Lemma 3.4 implies that \( \ker \pi_\ell = \ker \tilde{\beta}_\ell \) so \( \tilde{\beta}_\ell(u) \) is invertible if and only if \( \pi_\ell(u) \) is invertible.
4 Pulling symbols back from $S^{2n-1}$

To put the Toeplitz operators on a Landau level in a suitable homological picture, we must pass from $A$ to $C(S^{2n-1})$. This is a consequence of the circumstance that $A$ is homotopy equivalent to $C$, so $A$ does not contain any relevant topological information. With Lemma 3.5 in mind we define the Toeplitz algebra $\mathcal{T}_k$ for $C(S^{2n-1})$ as if $\beta_k$ were injective. So let $\lambda : C(S^{2n-1}) \to B(L^2(\mathbb{C}^n))$ denote the $*$-representation defined by

$$\lambda(a)f(z) = a\left(\frac{z}{|z|}\right)f(z).$$

(4)

Take $\chi_0 \in C^\infty(\mathbb{R})$ to be a smooth function such that $\chi_0(x) = 0$ for $|x| \leq 1$ and $1 - \chi_0 \in C^\infty(\mathbb{R})$. We define the cut-off $\chi(z) := \chi_0(|z|)$ and the operator

$$\bar{P}_k := P_k \chi.$$  

(5)

For the operator $\bar{P}_k$, $q(\bar{P}_k)$ is a projection by Lemma 3.1. We let $\mathcal{T}_k$ be the $C^*$-algebra generated by $\bar{P}_k \lambda(C(S^{2n-1})) \bar{P}_k^*$.  

**Theorem 4.1.** For any $k, k' \in \mathbb{N}^n$ there exist a unitary

$$Q_{k,k'} : \mathcal{L}_{k'} \to \mathcal{L}_k$$

such that $\text{Ad}(Q_{k,k'}) : \mathcal{T}_k \to \mathcal{T}_{k'}$ is an isomorphism satisfying

$$q(\bar{P}_{k'} \lambda(a) \bar{P}_k^*) = q \circ \text{Ad}(Q_{k,k'})(\bar{P}_k \lambda(a) \bar{P}_k^*).$$

(6)

Furthermore, for any $k \in \mathbb{N}^n$, the representation of $\mathcal{T}_k$ on $\mathcal{L}_k$ given by the inclusion $\mathcal{T}_k \subseteq B(\mathcal{L}_k)$ is irreducible and has the cyclic vector $\xi_k$ defined by

$$\xi_k(z) := q_k^*(e^{-|z|^2/4}).$$

Up to normalization the cyclic vectors satisfy

$$Q_{k,k'}\xi_{k'} = \xi_k.$$  

**Proof.** Let us start with observing that for any $a, b \in C(S^{2n-1})$ we have

$$\bar{P}_k \lambda(ab) \bar{P}_k^* - \bar{P}_k \lambda(a) \bar{P}_k^* P_k \lambda(b) \bar{P}_k^* \in K.$$

So if $\mathcal{T}_k$ acts irreducibly on $\mathcal{L}_k$, then $K \subseteq \mathcal{T}_k$.

First we will construct a cyclic vector for the $\mathcal{T}_k$-action on $\mathcal{L}_k$ and use the cyclic vector in $\mathcal{L}_0$ to show that $\mathcal{T}_0$ acts irreducibly on $\mathcal{L}_0$. Then we will show that for $k$ such that $\mathcal{T}_k$ acts irreducibly on $\mathcal{L}_k$ and $1 \leq j \leq n$ there is an isomorphism $\mathcal{T}_k \cong \mathcal{T}_{k+e_j}$ induced by a unitary intertwining the $\mathcal{T}_k$-action on $\mathcal{L}_k$ with the $\mathcal{T}_{k+e_j}$-action on $\mathcal{L}_{k+e_j}$.

Consider the elements $\xi_{m,k} \in \mathcal{L}_k$ for $m \in \mathbb{N}^n$ defined by

$$\xi_{m,k}(z) := q_k^*(z^m e^{-|z|^2/4}).$$

The elements $\xi_{m,k}$ form an orthogonal basis for $\mathcal{L}_k$. As in the statement of the theorem, we define $\xi_k := \xi_{0,k}$. For $a \in C(S^{2n-1})$ we have

$$\langle \xi_{m,k}, \bar{P}_k a \bar{P}_k^* \xi_k \rangle = \langle \xi_{m,k}, \chi^2 a \xi_k \rangle =
\int_{\mathbb{C}^n} q_k^*(z^m e^{-|z|^2/4}) q_k^*(e^{-|z|^2/4}) \chi^2(z) a \left(\frac{z}{|z|}\right) dV = \int_{S^{2n-1}} p_m(z) a(z) dS,$$
for some polynomials \(p_m\) of degree at most \(2|k| + |m|\). It follows that \(\mathcal{T}_k \xi_k\) span \(\mathcal{L}_k\) and therefore \(\mathcal{T}_k \xi_k = \mathcal{L}_k\). Thus \(\xi_k\) is a cyclic vector for the \(\mathcal{T}_k\)-action.

By standard theory \(\mathcal{T}_0\) acts irreducibly on \(\mathcal{L}_0\) if and only if there are no non-zero \(\xi'_0, \xi''_0 \in \mathcal{L}_0\) such that \(\xi_0 = \xi'_0 + \xi''_0\) and \(\mathcal{T}_0 \xi'_0 \perp \mathcal{T}_0 \xi''_0\). Assume that for some \(\xi'_0 \in \mathcal{L}_0\) we have \(\mathcal{T}_0 \xi'_0 \perp \mathcal{T}_0 (\xi_0 - \xi'_0)\). The orthogonality condition implies that \(\langle P_0 a P_0^* (\xi_0 - \xi'_0), \xi''_0 \rangle = 0\) for all \(a \in C(S^{2n-1})\) and \(P_0\) is self-adjoint so this relation is equivalent to \(\langle \chi^2 a \xi_0, \xi''_0 \rangle = \langle \chi^2 a \xi'_0, \xi''_0 \rangle\) for all \(a \in C(S^{2n-1})\). There exist a holomorphic function \(f_0\) such that \(\xi''_0 (z) = f_0(z) e^{-|z|^2/4}\) and the equation \(\langle \chi^2 a \xi_0, \xi''_0 \rangle = \langle \chi^2 a \xi'_0, \xi''_0 \rangle\) implies

\[
\int_{C^n} T_0(z) e^{-|z|^2/4} \chi^2(z) a \left( \frac{z}{|z|} \right) dV = \int_{C^n} |f_0(z)|^2 e^{-|z|^2/4} \chi^2(z) a \left( \frac{z}{|z|} \right) dV.
\]

Hence \(f_0\) must be real, and since it is holomorphic it must be constant. Thus \(\xi'_0\) is in the linear span of \(\xi_0\) and \(\xi_0\) defines a pure state. Since the \(\mathcal{T}_0\)-action on \(\mathcal{L}_0\) has a pure state, it is irreducible.

Assume that \(\mathcal{T}_k\) acts irreducibly on \(\mathcal{L}_k\). Consider the polar decomposition of the unbounded operator \(q_j\) on \(L^2(\mathbb{C}^n)\), that is \(q_j^* = E_j Q_j\) where \(Q_j\) is a coisometry and \(E_j\) is a strictly positive unbounded operator. Clearly \(E_j\) is diagonal on the energy levels and

\[
E_j = \bigoplus_{k' \in \mathbb{C}^n} \sqrt{k_j} P_{k'}.
\]

We define the \(*\)-homomorphism \(\rho_j : \mathcal{T}_{k+e_j} \to B(\mathcal{L}_k)\) by \(\rho_j(T) := Q_j^* T Q_j |_{\mathcal{L}_k}\). Since \(Q_j\) is a coisometry this is clearly a \(*\)-homomorphism. It follows from the fact that \(q_j^* : \mathcal{L}_k \to \mathcal{L}_{k+e_j}\) is an isomorphism, that \(Q_j^* : \mathcal{L}_k \to \mathcal{L}_{k+e_j}\) is unitary, so \(\rho_j\) is unital. If \(a \in C^\infty(S^{2n-1})\) then for some non-zero constant \(c\) we have

\[
\rho_j (P_{k+e_j} A(a) \tilde{P}^*_{k+e_j}) = c q_j P_{k+e_j} A(a) \tilde{P}^*_{k+e_j} |_{\mathcal{L}_k} =
\]

\[
= c P_k \left[ \frac{\partial}{\partial \xi_j}, \chi^2 A(a) \right] P_{k+e_j} q_j^* |_{\mathcal{L}_k} + \tilde{P}_k A(a) \tilde{P}^*_{k} \in \mathcal{T}_k,
\]

because Theorem 5.2 implies \(P_k b P_{k+e_j} \in \mathcal{K}(L^2(\mathbb{C}^n))\) for \(b \in A\) and by the induction assumption \(K \subseteq \mathcal{T}_k\). So we obtain a \(*\)-monomorphism \(\rho_j : \mathcal{T}_{k+e_j} \to \mathcal{T}_k\). However, we have cyclic vectors \(\xi_k\) and \(\xi_{k+e_j}\) for \(\mathcal{T}_k\) respectively \(\mathcal{T}_{k+e_j}\). For these vectors, \(Q_j \xi_k\) is a multiple of \(\xi_{k+e_j}\), so

\[
\mathcal{L}_{k+e_j} = \mathcal{T}_{k+e_j} \xi_{k+e_j} \xrightarrow{Q_j^*} \mathcal{T}_k \xi_k.
\]

Therefore \(\rho_j\) is surjective and an isomorphism. We conclude that \(\mathcal{T}_k\) is independent of \(k\) and the representations on \(\mathcal{L}_k\) are irreducible since \(\xi_0\) is pure and the \(\mathcal{T}_k\)-actions are all equivalent.

In [5] a weaker, but more explicit, statement was proven in complex dimension 1. Lemma 9.2 of [5] gives an explicit expression of \(Q_{k,0}^* T_k(a) Q_{k,0}\) if \(a \in A\) is smooth as

\[
Q_{k,0}^* T_k(a) Q_{k,0} = T_0(D_k(a)),
\]

where \(D_k := \text{id} + \sum_{j=1}^{k} d_j \Delta_j\), for some explicit constants \(d_j, k\) and \(\Delta\) is the Laplacian on \(\mathbb{C}\).
For $i = 1, \ldots, n$ we let $z_i : S^{2n-1} \to \mathbb{C}$ denote the coordinate functions of the embedding $S^{2n-1} \subseteq \mathbb{C}^n$. Clearly $z_i \in C(S^{2n-1})$.

**Corollary 4.2.** The operators $P_k \lambda(z_i) P_k^*$ together with $K$ generate $T_k$ as a $C^*$-algebra.

**Proof.** Let $U$ denote the $C^*$-algebra generated by $P_k \lambda(z_i) P_k$ and $K$. The $C^*$-algebra $T_k$ is constructed as the $C^*$-algebra generated by the linear space $P_k \lambda(C(S^{2n-1})) P_k$ because $P_k \lambda(a) P_k - P_k \lambda(a) P_k^* \in K$. So it is sufficient to prove $P_k \lambda(C(S^{2n-1})) P_k \subseteq U$. Given a function $a \in C(S^{2n-1})$ the Stone-Weierstrass theorem implies that there is a sequence of polynomials $R_j = R_j(z, \bar{z})$ such that $R_j \to a$ in $C(S^{2n-1})$. The functions $R_j$ are polynomials so it follows that

$$P_k \lambda(R_j) P_k - R_j(P_k \lambda(z) P_k, P_k \lambda(z^*) P_k) \in K$$

and $P_k \lambda(R_j) P_k \in U$. Finally $\|P_k \lambda(R_j) P_k - P_k \lambda(a) P_k \|_{B(\mathcal{L}_k)} \leq \|R_j - a\|_{C(S^{2n-1})}$ which implies $P_k \lambda(a) P_k \in U$. □

**Corollary 4.3.** The mapping $\beta_k : C(S^{2n-1}) \to \mathcal{C}(\mathcal{L}_k)$ induced from $\tilde{\beta}_k$ is injective, so if $u \in A \otimes M_N$ the operator $T_k(u)$ is Fredholm if and only if $\pi_\sigma(u)$ is invertible.

**Proof.** Due to equation (9) in Theorem 4.1 and Lemma 4.5 the Corollary follows from Lemma 4.6. The proof of the second statement of the Corollary is proven in the same fashion as Proposition 4.6.

From the fact that the mapping $\beta_k$ is injective it follows that the symbol mapping $\tilde{P}_k \lambda(a) \tilde{P}_k^* \mapsto a$ gives a well defined surjection $\sigma_k : T_k \to C(S^{2n-1})$. Clearly the kernel of $\sigma_k$ is non-zero and $\ker \sigma_k \subseteq K$, so by Theorem 4.1 $\ker \sigma_k = K$. Therefore we can construct the exact sequence

$$0 \to K \to T_k \xrightarrow{\sigma_k} C(S^{2n-1}) \to 0. \tag{7}$$

A completely positive splitting of the symbol mapping $\sigma_k : T_k \to C(S^{2n-1})$ is given by $a \mapsto \tilde{P}_k \lambda(a) \tilde{P}_k^*$. The exact sequence (7) defines an extension class $[T_k] \in Ext(C(S^{2n-1}))$. To read more about $Ext$, $K$-theory and $K$-homology we refer the reader to the references. Since $C(S^{2n-1})$ is a nuclear $C^*$-algebra there is an isomorphism $Ext(C(S^{2n-1})) \cong K^1(C(S^{2n-1}))$ and we can describe the $K$-homology class of $[T_k]$ explicitly by a Fredholm module as follows; we let $\lambda : C(S^{2n-1}) \to B(L^2(\mathbb{C}^n))$ be as in equation (11) and define the operator

$$F_k = \frac{(1 + \tilde{P}_k)}{2}$$

where $\tilde{P}_k$ is as in equation (5). Clearly, $(L^2(\mathbb{C}^n), \lambda, F_k)$ defines a Fredholm module which represents the image of $[T_k]$ in $K^1(C(S^{2n-1}))$.

**Corollary 4.4.** The class $[T_k] \in Ext(C(S^{2n-1}))$ is independent of $k$.

**Proof.** The extension $T_k$ is equivalent to $T_k'$ since it follows from equation (6) that the following diagram with exact rows commute

$$
\begin{array}{ccc}
0 & \longrightarrow & K & \longrightarrow & T_k & \longrightarrow & C(S^{2n-1}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & T_k' & \longrightarrow & C(S^{2n-1}) & \longrightarrow & 0
\end{array}
$$

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The Bergman space on the unit ball $B_n \subseteq \mathbb{C}^n$ is defined as $A^2(B_n) := L^2(B_n) \cap O(B_n)$, that is; holomorphic functions on $B_n$ which are square integrable. The Bergman space is a closed subspace of $L^2(B_n)$ and we will denote the orthogonal projection $L^2(B_n) \to A^2(B_n)$ by $P_B$.

The Bergman projection defines a $K$-homology class $[P_B] \in K^1(C(S^{2n-1}))$ in the same fashion as for the Landau projections. That is, for $a \in C(B_n)$ the operator $[P_B,a] \in B(L^2(B_n))$ is compact. The reason that we can use $P_B$ to define a $K$-homology class for $S^{2n-1}$ instead of $T^n$ is analogously to above that $P_B[a|_{A^2(B_n)}]$ is compact if and only if $a \in C_0(B_n)$, see more in [14]. Thus $P_B[a|_{A^2(B_n)}]$ is Fredholm if and only if $a|_{S^{2n-1}}$ is invertible.

Furthermore $[P_B,a]$ is compact. So $[P_B]$ is a well defined $K$-homology class in $K^1(C(S^{2n-1}))$. By [3] the following index formula holds for the Toeplitz operator $P_B[a|_{A^2(B_n)}]$ if the symbol $a_0 := a|_{S^{2n-1}}$ is smooth:

$$\text{ind}(P_B[a|_{A^2(B_n)}]) = \frac{-(n-1)!}{(2n-1)!(2\pi i)^n} \int_{S^{2n-1}} \text{tr}((a_0^{-1}da_0)^{2n-1}).$$  \hspace{1cm} (8)

This formula was also proven in [8] by an elegant use of Atiyah-Singer's index theorem.

We will by $\mathcal{T}^n$ denote the $C^*$-algebra generated by $P_B C(B_n) P_B$ in $B(A^2(B_n))$. The $K$-homology class $[P_B] \in K^1(C(S^{2n-1}))$ can be represented by the extension class $[\mathcal{T}^n] \in \text{Ext}(C(S^{2n-1}))$ defined by means of the short exact sequence

$$0 \to K \to \mathcal{T}^n \xrightarrow{\varepsilon^n} C(S^{2n-1}) \to 0.$$ \hspace{1cm} (9)

5 The special cases $\mathbb{C}$ and $\mathbb{C}^2$

In this chapter we will study the special cases of complex dimension 1 and 2. Dimension 1 has been studied previously in [1] and provides a simpler picture than in higher dimensions. In the 1-dimensional case we have that $K_1(C(T)) \cong \mathbb{Z}$ and we can take the coordinate function $z : T \to \mathbb{C}$ to be a generator. So when we want to determine the class $[T_k]$ we only need to calculate the index of $P_k \lambda(z) P_k$ where $\lambda$ is as in equation (4). We recall the following Proposition from [1]:

**Proposition 5.1** (Proposition 7.3 from [1]). For any $k \in \mathbb{N}$ we have that

$$\text{ind}(P_k \lambda(z) P_k) = -1.$$
Theorem 5.2. For $n = 1$ there is an isomorphism $\mathcal{T}_k \cong \mathcal{T}^1$ making $[\mathcal{T}_k] = [\mathcal{T}^1] \in K^1(C(\mathbb{T}))$.

Proof. By Proposition 7.3 of [1]

$$[\mathcal{T}_k][u] = \text{ind}(P_k \lambda(u) P_k) = -\text{wind}(u) = [\mathcal{T}^1][u] \quad (10)$$

for an invertible function $u \in C(\mathbb{T})$. Here wind $(u)$ denotes the winding number of $u$ which is defined for smooth $u$ as

$$\text{wind}(u) := \frac{1}{2\pi i} \int_{\mathbb{T}} u^{-1} du$$

and defines an isomorphism $K_1(C(\mathbb{T})) \to \mathbb{Z}$. By the Universal Coefficient Theorem for $KK$-theory (see Theorem 4.2 of [1]) the mapping

$$K^1(C(\mathbb{T})) \to \text{Hom}(K_1(C(\mathbb{T})), \mathbb{Z})$$

is an isomorphism so equation (10) implies that $[\mathcal{T}_k] = [\mathcal{T}^1]$. By Theorem 13 of [6], the short exact sequence $0 \to \mathcal{K} \to \mathcal{T}_k \to C^*(\mathbb{T}) \to 0$ is characterized by an isometry $\mathcal{V}$ such that $\mathcal{V} u^* - 1$ is compact and $\mathcal{T}_k$ is generated by $\mathcal{V}$. Then $z \mapsto \mathcal{V}$ defines a splitting and the symbol mapping $\mathcal{T}_k \to C^*(\mathbb{T})$ is just $\mathcal{V} \mapsto z$. By equation (10), $1 - \mathcal{V} u^*$ is a rank one projection, so the theorem follows.

Also in dimension 2 we can find a generator for the odd $K$-theory. As generator for $K_1(C(S^3)) \cong \mathbb{Z}$ we can take the diffeomorphism $u : S^3 \to SU(2)$ defined as

$$u(z_1, z_2) := \begin{pmatrix} z_1 \\ -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}.$$ 

Proposition 5.3. The extension class $[\mathcal{T}_2]$ generate $K^1(C(S^3))$ and $[u]$ generate $K_1(C(S^3))$.

Proof. Recalling that $P_B$ denotes the Bergman projection we will start by calculating the index of the Toeplitz operator $P_B u P_B : A^2(B_2) \otimes \mathbb{C}^2 \to A^2(B_2) \otimes \mathbb{C}^2$. Using the index theorem by Boutet de Monvel ([3]) reviewed above in equation (3), the following index formula holds for smooth $u$:

$$\text{ind}(P_B u P_B) = \frac{1}{3!(2\pi i)^2} \int_{S^3} \text{tr}((u^* du)^3). \quad (11)$$

A straightforward calculation gives that

$$\text{tr}((u^* du)^3) = 3(z_1 dz_1 - \bar{z}_1 dz_1) \wedge dz_2 \wedge d\bar{z}_2 + 3(z_2 d\bar{z}_2 - \bar{z}_2 d\bar{z}_2) \wedge dz_1 \wedge d\bar{z}_1.$$

Invoking Stokes Theorem on equation (11) gives that

$$\frac{1}{3!(2\pi i)^2} \int_{S^3} \text{tr}((u^* du)^3) = \frac{1}{48 \cdot \text{vol}(B_2)} \int_{B_2} d\text{tr}((u^* du)^3) =$$

$$= \frac{1}{4 \cdot \text{vol}(B_2)} \int_{B_2} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = -\frac{1}{\text{vol}(B_2)} \int_{B_2} dV = 1.$$

This equation shows that

$$[\mathcal{T}_2][u] = \text{ind}(P_B u P_B) = -1. \quad (12)$$
Consider the split-exact sequence $0 \rightarrow C_0(\mathbb{R}^3) \rightarrow C(S^3) \rightarrow \mathbb{C} \rightarrow 0$ where the mapping $C(S^3) \rightarrow \mathbb{C}$ is point evaluation. Since the sequence splits, and $K_1(\mathbb{C}) = K^1(\mathbb{C}) = 0$ the embedding $C_0(\mathbb{R}^3) \rightarrow C(S^3)$ induces isomorphisms $K_1(C(S^3)) \cong K_1(C_0(\mathbb{R}^3)) = \mathbb{Z}$ and $K^1(C(S^3)) \cong K^1(C_0(\mathbb{R}^3)) = \mathbb{Z}$. So the Kasparov product $K_1(C(S^3)) \times K^1(C(S^3)) \rightarrow \mathbb{Z}$ is just a pairing $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$, and since $[T^2].[u] = -1$ it follows that $[T^2]$ generates $K^1(C(S^3))$ and $[u]$ generates $K_1(C(S^3))$. 

**Theorem 5.4.** For any $k \in \mathbb{N}^2$ we have

$$\text{ind}(P_k\lambda(u)P_k) = -1.$$  

Therefore $[T^2] = [T_k]$. 

**Proof.** If equation (13) holds, $[T^2] = [T_k]$ follows directly from equation (12) using the Universal Coefficient Theorem for $KK$-theory (see Theorem 4.2 of [11]). This is a consequence of the fact that the natural mapping

$$K^1(C(S^3)) \rightarrow K_1(C(S^3)), \mathbb{Z}$$

is an isomorphism. The injectivity of this map implies that if $[T^2].[u] = [T_k].[u]$ for a generator $[u]$, then $[T^2] = [T_k]$.  

To prove equation (13) we take $k = 0$, since Corollary 4.3 implies that the integer $\text{ind}(P_0\lambda(u)P_0)$ is independent of $k$. We claim that $P_0\lambda(u)P_0$ is an injective operator and the cokernel of $P_0\lambda(u)P_0$ is spanned by the $\mathbb{C}^2$-valued function $z \mapsto e^{-|z|^2/4} \oplus 0$. This statement will prove the theorem.  

To prove that $P_0\lambda(u)P_0$ is injective, assume $f \in \ker(P_0\lambda(u)P_0)$. Define the functions

$$\xi^m(z) := z^m e^{-|z|^2/4}$$

for $m \in \mathbb{N}^2$. The functions $\xi^m$ form an orthogonal basis for $L_0$ by Theorem 1.63 of [7]. Expand the function $f$ in an $L^2$-convergent series

$$f = \sum_{m \in \mathbb{N}^2} c_m \xi^m,$$

where $c_m = c_m^{(1)} \oplus c_m^{(2)} \in \mathbb{C}^2$. Since $f \in \ker(P_0\lambda(u)P_0)$ we have the following orthogonality condition

$$0 = \langle \xi^{m'} \oplus 0, \lambda(u)f \rangle = \sum_m \int_{\mathbb{C}^2} \left( c_m^{(1)} \frac{z^{m'}}{|z|} + c_m^{(2)} \frac{z^{m'} z^{m+e_1}}{|z|} \right) e^{|z|^2/2} dV =$$

$$= \sum_m t_{mn} \int_{S^3} \left( c_m^{(1)} e^{m'} z^{m+e_1} + c_m^{(2)} e^{m'} z^{m+e_1} \right) dS,$$

for some coefficients $t_{mn}$, for a detailed calculation of $t_{mn}$, see below in Proposition 6.1. Using that the functions $\xi^m$ are orthogonal we obtain that there exist $C_m > 0$ such that

$$c_m^{(1)} = -C_m c_m^{(2)}.$$  

(14)

On the other hand, we have

$$0 = \langle 0 \oplus \xi^{m'}, \lambda(u)f \rangle = \sum_m \int_{\mathbb{C}^2} \left( -c_m^{(1)} \frac{z^{m'+e_2} z^m}{|z|} + c_m^{(2)} \frac{z^{m'+e_1} z^m}{|z|} \right) e^{|z|^2/2} dV =$$

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\[
= \sum_{m} t_{m,m'} \int_{S^3} \left( -c_{m}^{(1)} z^{m+e_2} \bar{z}^{m} + c_{m}^{(2)} z^{m+e_1} \bar{z}^{m} \right) dS.
\]

Again using orthogonality of the functions \( \xi_{m} \) we obtain that there is a \( C'_{m} > 0 \) such that
\[
\xi_{m}^{(1)} = C'_{m} \xi_{m+e_1}^{(2)}, \tag{15}
\]
Equation (14) implies \( \xi_{m}^{(1)} = 0 \) for \( m_2 = 0 \). For \( m_2 > 0 \) equation (14) implies
\[
\xi_{m}^{(1)} = -C_{m+e_1} \xi_{m-e_2+e_1}^{(2)}.
\]
Then equation (15) for \( m - e_2 \) gives
\[
\xi_{m}^{(1)} \left( 1 + \frac{C_{m+e_1}}{C'_{m-e_2}} \right) = 0.
\]
So \( \xi_{m}^{(1)} = 0 \) for all \( m \). Equation (14) implies \( \xi_{m}^{(2)} = 0 \) for all \( m \). Thus \( f = 0 \) and \( \ker(P_0 \lambda(u) P_0) = 0 \).

The second statement, that the cokernel of \( P_0 \lambda(u) P_0 \) is spanned by the \( \mathbb{C}^2 \)-valued function
\[
z \mapsto e^{-|z|^2/4} \oplus 0,
\]
is proven analogously. There is a natural isomorphism
\[
coker P_0 \lambda(u) P_0 \cong (im P_0 \lambda(u) P_0)^\perp = \ker P_0 \lambda(u^*) P_0.
\]
Analogously to the reasoning above, for \( g \in \ker P_0 \lambda(u^*) P_0 \) we expand the function \( g \) in an \( L^2 \)-convergent series
\[
g = \sum_{m \in \mathbb{N}^2} d_{m} \xi_{m}^{m},
\]
where \( d_{m} = d_{m}^{(1)} \oplus d_{m}^{(2)} \in \mathbb{C}^2 \). After taking scalar product by \( \xi_{m} \)'s, for some \( D_{m}, D'_{m} > 0 \) we obtain the following conditions on the coefficients:
\[
d_{m}^{(1)} = D_{m} d_{m-e_2}^{(2)} \quad \text{and} \quad d_{m}^{(1)} + e_1 = -D'_{m} d_{m-e_2}^{(2)}.
\]
The second of these equations implies \( d_{m}^{(1)} = 0 \) for \( m_1 = 0 \) and \( m_2 > 0 \). Also, the first of these equations implies \( d_{m}^{(1)} = 0 \) for \( m_2 = 0 \) and \( m_1 > 0 \). For \( m_1, m_2 > 0 \), putting in \( m - e_2 \) in the first equation, gives
\[
d_{m}^{(1)} = D_{m-e_2} d_{m-e_2}^{(2)}.
\]
Finally, combining this relation with the second equation for \( m - e_2 \) we obtain
\[
d_{m}^{(1)} \left( 1 + \frac{D_{m-e_2}}{D'_{m-e_2}} \right) = 0 \quad \text{for} \quad m_1, m_2 > 0.
\]
Therefore \( d_{m}^{(1)} = 0 \) for all \( m \neq 0 \). The equations in (16) imply \( d_{m}^{(2)} = 0 \) for all \( m \). However, the function \( z \mapsto e^{-|z|^2/4} \oplus 0 \), corresponding to \( d_{0}^{(1)} = 1 \), is in the space \( \ker(P_0 \lambda(u^*) P_0) \) which completes the proof. \( \square \)
6 The index formula on the particular Landau levels

In this section we will prove an index formula for the particular Landau levels. On $S^{2n-1}$ we have the complex coordinates $z_1, \ldots, z_n$ and we denote by $Z_1, \ldots, Z_n$ the image of these coordinate functions under the representation $\lambda$ which was defined in equation (4). So $Z_i$ is the operator on $L^2(\mathbb{C}^n)$ given by multiplication by the almost everywhere defined function $z \mapsto \frac{z_i}{|z|}$. Consider the polar decompositions
\[ P_iZ_iP_i = V_{i,0}S_{i,0}, \]
where $V_{i,0}$ are partial isometries and $S_{i,0} > 0$. An orthonormal basis for $\mathcal{L}_0$ is given by
\[ \eta_{mn}(z) := \frac{z^m e^{-|z|^2/4}}{\sqrt{\pi^m 2^{2m}|m|!}}, \]
see more in [7].

Proposition 6.1. The operator $V_{i,0}$ is an isometry described by the equation
\[ V_{i,0}\eta_m = \eta_{m+1}, \]
and the operator $S_{i,0}$ is diagonal in the basis $\eta_{mn}$ with eigenvalues given by
\[ \lambda_{mn}^\theta = \Gamma\left(\frac{|m| + n + \frac{1}{2}}{n} \right) \frac{\sqrt{|m| + 1}}{(|m| + n)!}. \tag{18} \]

Proof. For $m, m' \in \mathbb{N}$ we have
\[ \langle \eta_{m'}, Z_i \eta_m \rangle = \int_{C^n} \frac{1}{\sqrt{\pi^n 2^{2m+|m'|+2|m|m'|/2}}} \frac{z^{m'+|m'|}}{|z|^2} e^{-|z|^2/2} dV = \]
\[ = \frac{1}{\sqrt{\pi^n 2^{2m+|m'|+2|m|m'|/2}}} \int_0^{\infty} \int_{S^{m+|m'|+n-1}} \frac{z^m e^{-r^2/2}}{r^{m+|m'|+n-1}} e^{-r^2/2} dr dS = \]
\[ = \frac{\Gamma(|m| + n + \frac{1}{2})}{2\pi^{|m|}! \sqrt{(m_j + 1)}} \int_{S^{m+|m'|}} z^m e^{-r^2/2} dS = \]
\[ = \delta_{m', m} \Gamma(|m| + n + \frac{1}{2}) \frac{\sqrt{|m| + 1}}{(|m| + n)!}. \]

It follows that $V_{i,0}\eta_m = \eta_{m+1}$ and $S_{i,0}\eta_m = \lambda_{mn}^\theta \eta_m$, where $\lambda_{mn}^\theta$ is as in equation (18).

On the other hand, we can, just as on $\mathcal{L}_0$, let $\tilde{Z}_1, \ldots, \tilde{Z}_n \in B(L^2(B_n))$ be the operators on $L^2(B_n)$ defined by the multiplication by the almost everywhere defined function $z \mapsto \frac{z}{|z|}$. Consider the polar decompositions
\[ P_B \tilde{Z}_i P_B = V_{i,B}S_{i,B}, \]
where again $V_{i,B}$ are partial isometries and $S_{i,B} > 0$. An orthonormal basis for $A^2(B_n)$ is given by
\[ \rho_m(z) := \frac{1}{|m|!} \frac{z^m}{\sqrt{(n+|m|)!}}. \]

Similar to the lowest Landau level, the partial isometries $V_{i,B}$ are just shifts in this basis:

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Proposition 6.2. The operator $V_{i,B}$ is an isometry described by the equation

$$V_{i,B} \mu_{m} = \mu_{m + e_i}$$

and the operator $S_{i,B}$ is diagonal in the basis $\mu_n$ with eigenvalues given by

$$\lambda_{i,m}^\mu = \frac{\sqrt{m_i + 1}}{\sqrt{n + |m| + 1}}.$$  \hspace{1cm} (19)

Proof. The proof is the analogous to that of Proposition 6.1. For $m, m' \in \mathbb{N}$ we have

$$\langle \mu_m, \tilde{Z}_i \mu_m \rangle = \int_{B_n} \pi^{-n} \frac{(n + |m|)(n + |m'|)!}{m!m'!} \frac{z^{m + e_i}}{|z|^x} dV = \pi^{-n} \frac{(n + |m|)(n + |m'|)!}{m!m'!} \int_0^{1} \frac{r^{n + |m'| + 2n - 1}}{2^{n} n^{m + 1}} \int_{S_2^{n-1}} z^{m'} z^{m + e_i} dS = \delta_{m',m + e_i} \frac{(n + |m|)!}{(2|m| + 2n)!} \frac{\sqrt{n + |m| + 1}}{m_i + 1} \int_{S_2^{n-1}} z^{m'} z^{m + e_i} dS = \delta_{m',m + e_i} \frac{\sqrt{m_i + 1}}{\sqrt{n + |m| + 1}}.$$  \hspace{1cm}

It follows that $V_{i,B} \mu_m = \mu_{m + e_i}$ and $S_{i,B} \mu_m = \lambda_{i,m}^\mu \mu_m$ where the eigenvalues $\lambda_{i,m}^\mu$ are given in equation (19).

Lemma 6.3. If $a$ is a real number then

$$\frac{\Gamma(x + a)}{\Gamma(x)} = x^a + O(x^{-1+a}) \text{ as } x \to +\infty.$$  \hspace{1cm}

Proof. By Stirling’s formula

$$\ln \Gamma(x) = \left( x - \frac{1}{2} \right) \ln x - x + \frac{\ln 2\pi}{2} + O(x^{-1}).$$  \hspace{1cm}

After Taylor expanding $\ln \Gamma(x + a)$ around $a = 0$ we obtain that

$$\ln \Gamma(x + a) - \ln \Gamma(x) = a \ln x + O(x^{-1}).$$  \hspace{1cm}

Lemma 6.4. With the unitary $U : A^2(B_n) \to L_0$ defined by $\mu_m \mapsto \eta_m$, the operators $S_{i,0}$ and $S_{i,B}$ satisfy

$$U^* S_{i,0} U - S_{i,B} \in \mathcal{K}.$$  \hspace{1cm}

Proof. The operators $U^* S_{i,0} U$ and $S_{i,B}$ are both diagonal in the basis $\mu_n$. So it is sufficient to prove that $|\lambda_{i,n}^\mu - \lambda_{i,n}^\eta| \to 0$. The proof of this statement is based on the estimate from Lemma 6.3. When $|m| \to \infty$, Lemma 6.3 implies

$$|\lambda_{i,n}^\mu - \lambda_{i,n}^\eta| = \left| \frac{\Gamma(|m| + n + 1/2)}{\Gamma(|m| + n)} \frac{\sqrt{m_i + 1}}{\sqrt{|m| + n - 1}} \right| = \frac{\sqrt{m_i + 1}}{\Gamma(|m| + n + 1)} \left| \frac{\Gamma(|m| + n + 1 - 1/2)}{\Gamma(|m| + n + 1)} - (|m| + n - 1)^{-1/2} \right| = O(|m|^{-1}).$$  \hspace{1cm}

Therefore we have that $U^* S_{i,0} U - S_{i,B} \in L^{n+}(A^2(B_n))$, the $n$:th Dixmier ideal. In particular $U^* S_{i,0} U - S_{i,B}$ is compact.  \hspace{1cm}

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Theorem 6.5. The unitary $U$ induces an isomorphism $\text{Ad}(U) : T_0 \rightarrow T^n$ such that

\[ \sigma^n \circ \text{Ad}(U) = \sigma_0. \]

where $\sigma^n$ and $\sigma_0$ are the symbol mappings.

Proof. Lemma 6.4 and the Propositions 6.1 and 6.2 imply

\[ U^* (P_B Z_i P_B) U = P_B \tilde{Z}_i P_B + K_i, \] (20)

for some compact operators $K_i$. Since $T^n$ contains the compact operators, $U^* (P_B Z_i P_B) U \in T^n$. Corollary 4.2 therefore implies $U^* T_0 U \subseteq T^n$. Theorem 4.1 states that $T_0$ acts irreducibly on $L_0$, so $U^* T_0 U$ acts irreducibly on $A_2(B_n)$. Therefore $K \subseteq U^* T_0 U$ and $P_B \tilde{Z}_i P_B \in U^* T_0 U$. The operators $P_B \tilde{Z}_i P_B$ together with $K$ generate $T^n$ so $U^* T_0 U \supseteq T^n$. The relation $\sigma^n \circ \text{Ad}(U) = \sigma_0$ holds since by equation (20) it holds on the generators of $C(S^{2n-1})$.

Corollary 6.6. Let $[T^n] \in \text{Ext}(C(S^{2n-1}))$ denote the Toeplitz quantization of the Bergman space defined in equation (9) and $[T_k] \in \text{Ext}(C(S^{2n-1}))$ the Toeplitz quantization of the particular Landau level of height $k$ defined in equation (7). Then

\[ [T^n] = [T_k]. \]

So for $u \in A \otimes M_N$ such that $u_\partial := \pi_\partial(u)$ is invertible and smooth

\[ \text{ind} (P_\partial u_{|L_\partial \otimes C_\infty}) = \frac{-(n-1)!}{(2n-1)!(2\pi i)^n} \int_{S^{2n-1}} \text{tr}((u_\partial^{-1} du_\partial)^{2n-1}). \] (21)

Proof. By Corollary 4.4 the class $[T_k]$ is independent of $k$, so take $k = 0$. In this case Theorem 6.5 implies that the unitary $U$ makes the following diagram commutative:

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & T_0 & \rightarrow & C(S^{2n-1}) & \rightarrow & 0 \\
\downarrow \text{Ad}(U) & & \downarrow \text{Ad}(U) & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & K & \rightarrow & T^n & \rightarrow & C(S^{2n-1}) & \rightarrow & 0
\end{array}
\]

Therefore $[T^n] = [T_0] = [T_k]$ and the index formula (21) follows from [8].

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