ON THE IDEAL GENERATED BY ALL SQUAREFREE
MONOMIALS OF A GIVEN DEGREE

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Abstract. An explicit construction is given of a minimal free resolution of the ideal generated by all squarefree monomials of a given degree. The construction relies upon and exhibits the natural action of the symmetric group on the syzygy modules. The resolution is obtained over an arbitrary coefficient ring; in particular, it is characteristic free. Two applications are given: an equivariant resolution of De Concini-Procesi rings indexed by hook partitions, and a resolution of FI-modules.

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1. Introduction

Let $A$ be an associative, commutative ring with unit. Let $R_n$ denote $A[x_1, \ldots, x_n]$, the polynomial ring in $n$ variables with coefficients in $A$.

Definition 1.1. Let $I_{d,n}$ be the ideal of $R_n$ generated by all squarefree monomials of degree $d$ in $x_1, \ldots, x_n$.

The symmetric group $S_n$ acts on $R_n$ by permuting the variables. This action is $A$-linear, and is compatible with grading and multiplication in $R_n$. 

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Observe that the ideal $I_{d,n}$ is stable under the action of $\mathfrak{S}_n$. Therefore $\mathfrak{S}_n$ acts on the minimal free resolutions of $I_{d,n}$. The main goal of this paper is to construct an $\mathfrak{S}_n$-equivariant minimal free resolution of $I_{d,n}$ that describes this action explicitly. Our main theorem (Theorem 4.11) can be stated as follows.

**Theorem.** Let $n$, $d$, and $i$ be integers. For $1 \leq d \leq n$ and $0 \leq i \leq n - d$, we define the $A[\mathfrak{S}_n]$-module

$$U_{i}^{d,n} := \text{Ind}_{\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}}^{\mathfrak{S}_n} (S^{(d,1^i)} \otimes_A S^{(n-d-i)}),$$

For all other values of $n$, $d$, and $i$, we set $U_{i}^{d,n} := 0$. The ideal $I_{d,n}$ admits an $\mathfrak{S}_n$-equivariant minimal free resolution $F_{d,n}^\bullet$ with

$$F_{i}^{d,n} := U_{i}^{d,n} \otimes_A R_n(-d - i).$$

Our construction has some desirable features:

- the differentials are described explicitly (see Definition 4.3);
- the resolution is realized over an arbitrary coefficient ring, in particular it is characteristic free;
- the representation theory leads to an easy combinatorial interpretation of Betti numbers in terms of Young tableaux (Corollary 4.12).

The ideals $I_{d,n}$ appear in the study of De Concini-Procesi rings. Introduced in [De Concini and Procesi, 1981], De Concini-Procesi rings are quotients of a polynomial ring that give a presentation for the cohomology ring of Springer fibers (in type A). These rings also appear in the study of nilpotent orbits [Kraft, 1981]. In general, the Betti numbers and minimal free resolutions of De Concini-Procesi are not known. In the case of De Concini-Procesi rings indexed by hook partitions, the Betti numbers were described in [Biagioli et al., 2007]. The proof, which holds over a field, recognizes $I_{d,n}$ as a monomial ideal with linear quotients to compute its Betti numbers, and then uses an iterated mapping cone to obtain the Betti numbers of a certain De Concini-Procesi ring. Using the same mapping cone procedure, we give an application of our main result and describe the modules in an $\mathfrak{S}_n$-equivariant minimal free resolution of a De Concini-Procesi ring indexed by a hook partition (see Theorem 5.7). We believe that an approach using equivariant resolutions could facilitate the task of finding Betti numbers for other De Concini-Procesi rings as well.

The resolutions $F_{d,n}^\bullet$ of our main theorem have another interesting property. Namely, for a fixed $d$, there are natural $\mathfrak{S}_n$-equivariant maps of complexes $F_{d,n}^\bullet \to F_{d,n+1}^\bullet$. These maps allow us to assemble the complexes $F_{d,n}^\bullet$ to obtain a resolution of FI-modules in the sense of [Church et al., 2015] (see Theorem 5.14).

We also note that $I_{d,n}$ is the defining ideal of a star configuration of $n$ hyperplanes in projective $(n-1)$-space (see [Geramita et al., 2013]). The Betti numbers of these ideals were computed (over a field) in [Park and Shin, 2015, Corollary 3.5]. Despite being a special case of star configurations, it was
shown in [Geramita et al., 2015] that the ideals \( I_{d,n} \) actually govern the theory of arbitrary star configurations and their symbolic powers. For this reason we believe it would be particularly interesting to extend the main result of this paper to describe \( \mathfrak{S}_n \)-equivariant minimal free resolutions of the symbolic powers of \( I_{d,n} \).

I am grateful to J. Weyman for introducing me to De Concini-Procesi rings, which ultimately lead to this project. I am particularly indebted to A. Hoefel and D. Wehlau for many useful discussions in the early stages of this work. Special thanks go to H. Abe for providing a reference that shortened the proof of the main theorem. While working on this project, I was partially supported by an NSERC grant.

2. BETTI NUMBERS

As explained in the introduction, the Betti numbers of \( I_{d,n} \) were previously computed, over a field, by several authors using various techniques. Here we offer a different approach that works over more general coefficient rings.

**Theorem 2.1.** Let \( n \) and \( d \) be integers with \( 1 \leq d \leq n \). The Betti numbers of \( I_{d,n} \) are

\[
\beta_{i,j}(I_{d,n}) = \begin{cases} \binom{n}{d+i}(d+i-1), & \text{if } 0 \leq i \leq n - d \text{ and } j = d + i, \\ 0, & \text{otherwise.} \end{cases}
\]

**Proof.** The proof is by double induction on \( n \) and \( d \).

For \( n = d = 1 \), we have \( I_{1,1} = (x_1) \), for which the result clearly holds.

For \( n > 1, d = 1 \), the ideal \( I_{1,n} = (x_1, \ldots, x_n) \) is minimally resolved by a Koszul complex, whose Betti numbers match those in the statement.

Consider the case \( n > 1, d > 1 \). There is a short exact sequence

\[
(1) \quad 0 \to I_{d,n-1} \to I_{d-1,n-1} \to I_{d-1,n-1}/I_{d,n-1} \to 0
\]

of \( R_{n-1} \)-modules, the second arrow being an inclusion. Let \( F_{d,n-1}^d \) and \( F_{d-1,n-1}^d \) be minimal free resolutions of the left and middle modules in (1). The inclusion \( I_{d,n-1} \to I_{d-1,n-1} \) extends to a map of complexes \( F_{d,n-1}^d \to F_{d-1,n-1}^d \), whose mapping cone \( C^d \) is a free resolution of \( I_{d-1,n-1}/I_{d,n-1} \).

By the inductive hypothesis, \( C_i = F_{i-1}^{d,n-1} \oplus F_i^{d-1,n-1} \) is generated in degree \( d + i - 1 \). It follows that \( C^d \) is linear and hence minimal.

The squarefree monomials generating \( I_{d,n} \) are of two kinds: the ones that are not divisible by \( x_n \), and the ones that are. The former coincide with the generators of \( I_{d,n-1} \), the latter are the generators of \( I_{d-1,n-1} \) multiplied by \( x_n \). This leads to the short exact sequence

\[
(2) \quad 0 \to (I_{d-1,n-1}/I_{d,n-1}) \otimes_{R_{n-1}} R_n(-1) \xrightarrow{\cdot x_n} (R_{n-1}/I_{d,n-1}) \otimes_{R_{n-1}} R_n \to R_n/I_{d,n} \to 0
\]

of \( R_n \)-modules, where the second arrow is multiplication by \( x_n \).
Since $R_n$ is a free $R_{n-1}$-module, it is flat. Thus $F^{d,n-1}_{n-1} \otimes_{R_{n-1}} R_n$ is an exact complex. In fact, if we augment it with the obvious map $F^{d,n-1}_{0} \otimes_{R_{n-1}} R_n \to R_n$, we obtain a minimal free resolution of the middle term in (2). Similarly, $C_{\bullet} \otimes_{R_{n-1}} R_n(-1)$ is a minimal free resolution of the left term. The second map of sequence (2) extends to a map of complexes between the two resolutions just described. The mapping cone $D_{\bullet}$ of this map is a free resolution of $R_n/I_{d,n}$. Note $D_0 = R_n$ and, for $i > 0$, we have

$$D_i = (C_{i-1} \otimes_{R_{n-1}} R_n(-1)) \oplus (F^{d,n-1}_{i-1} \otimes_{R_{n-1}} R_n),$$

so $D_i$ is generated in degree $d + i - 1$. Therefore $D_{\bullet}$ is minimal.

If $F^{d,n}_{\bullet}$ is a minimal free resolution of $I_{d,n}$, then $F^{d,n}_{i} \cong D_{i+1}$ is generated in degree $d + i$. Moreover, equation (3) gives

$$F^{d,n}_i \cong ((F^{d,n-1}_{i-1} \oplus F^{d,n-1}_{i-1}) \otimes_{R_{n-1}} R_n(-1)) \oplus (F^{d,n-1}_{i} \otimes_{R_{n-1}} R_n).$$

We deduce that

$$\beta_{i,d+i}(I_{d,n}) = \binom{n-1}{d+i-1} \binom{i}{i-1} + \binom{n-1}{d+i-1} \binom{d+i-2}{i} + \binom{n-1}{d+i} \binom{d+i-2}{i} =$$

$$= \binom{n-1}{d+i-1} \binom{i}{i-1} + \binom{n-1}{d+i} \binom{d+i-1}{i} =$$

$$= \binom{n}{d+i-1} \binom{i}{i-1}$$

as desired. Clearly all other Betti numbers are zero. \hfill \Box

**Example 2.2** ($n = 4, d = 2$). The nonzero Betti numbers of $I_{2,4}$ are

$$\beta_{0,2}(I_{2,4}) = 6, \quad \beta_{1,3}(I_{2,4}) = 8, \quad \beta_{2,4}(I_{2,4}) = 3.$$ 

3. **Representations**

Specht modules are representations of the symmetric group. Their construction and properties are discussed in detail in [James, 1978]. We briefly recall a presentation for Specht modules following [Fulton, 1997, §7.4].

Let $A[\mathfrak{S}_n]$ be the group algebra of the symmetric group $\mathfrak{S}_n$ over $A$. Let $\lambda$ be a partition of $n$. A (Young) tableau of shape $\lambda$ is a filling of the Young diagram associated to $\lambda$ (we use the English notation for diagrams, cf. [Macdonald, 1995, p. 2]). The Specht module $S^\lambda$ is the $A[\mathfrak{S}_n]$-module generated by the equivalence classes $[T]$ of tableaux of shape $\lambda$ with entries in $[n] = \{1, \ldots, n\}$ modulo the following relations.

**Alternating columns:** $\sigma[T] = \text{sgn}(\sigma)[T]$ for all $\sigma \in \mathfrak{S}_n$ preserving the columns of $T$ ($\text{sgn}(\sigma)$ denotes the sign of $\sigma$).

**Shuffling relations:** $[T] = \sum [T']$, where the sum is over all $T'$ obtained from $T$ by exchanging the top $k$ elements of the $(j+1)$-st column of $T$ with $k$ elements in the $j$-th column of $T$, preserving the vertical order of each set of $k$ elements.

We are primarily interested in hook partitions, i.e., partitions of the form $(d, 1^i)$. We illustrate the relations above with an example using a hook partition.
Example 3.1. Consider the hook partition \((3, 1, 1)\) of 5. Since columns are alternating in \(S^{(3,1,1)}\), we have
\[
\begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 \\ 4 \end{pmatrix} = -\begin{pmatrix} 3 & 1 & 5 \\ 2 & 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 5 \\ 1 & 2 \\ 4 \end{pmatrix}.
\]
Using the shuffling relation involving the first two columns, we get
\[
\begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 \\ 4 \end{pmatrix},
\]
whereas the one involving the second and third columns gives
\[
\begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 3 & 4 \end{pmatrix}.
\]

Let us recall some facts from the theory of Specht modules.

• For all \(\lambda\), \(S^\lambda\) is a free \(A\)-module. Its rank can be computed with the hook length formula (cf. [James, 1978, Theorem 20.1]). In particular, in the case of a hook, we have
\[
\begin{align*}
\text{rank}(S^{(d,1,i)}) &= \frac{(d+i)!}{(d+i)(d-i)!} = \binom{d+i-1}{i}.
\end{align*}
\]

• A standard tableau is one whose rows are strictly increasing from left to right, and whose columns are strictly increasing from top to bottom. If \(\lambda\) is a partition of \(n\), then the equivalence classes of standard tableaux of shape \(\lambda\) with entries in \([n]\) form an \(A\)-basis of \(S^\lambda\).

We will denote by \(\text{SYT}(\lambda, [n])\) the set of standard tableaux of shape \(\lambda\) with entries in \([n]\). Note that \(\text{SYT}(\lambda, [n])\) is defined even when \(\lambda\) is a partition of some non negative integer \(m \neq n\).

Definition 3.2. Let \(n, d, \) and \(i\) be integers. For \(1 \leq d \leq n\) and \(0 \leq i \leq n-d\), we define an \(A[\mathfrak{S}_n]\)-module by setting
\[
U^{d,n}_i := \text{Ind}_{\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}}^{\mathfrak{S}_n} (S^{(d,1,i)} \otimes_A S^{(n-d-i)}),
\]
where the right hand side of the assignment is the \(A[\mathfrak{S}_n]\)-module induced by the \(A[\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}]\)-module \(S^{(d,1,i)} \otimes_A S^{(n-d-i)}\). The group \(\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}\) is realized as the subgroup of permutations of \(\mathfrak{S}_n\) that preserve the subsets \(\{1, \ldots, d+i\}\) and \(\{d+i+1, \ldots, n\}\). If any of the inequalities involving \(n\), \(d\), and \(i\) are violated, we set \(U^{d,n}_i := 0\).

The modules \(U^{d,n}_i\) will play a central role in the rest of this paper. We record some properties of the modules \(U^{d,n}_i\).

Proposition 3.3. Let \(n, d, \) and \(i\) be integers with \(1 \leq d \leq n\), and \(0 \leq i \leq n-d\).

(a) The module \(U^{d,n}_i\) is a free \(A\)-module and
\[
\text{rank}(U^{d,n}_i) = \binom{n}{d+i} \binom{d+i-1}{i}.
\]
(b) The module $U_{d,n}^{i}$ is isomorphic to the $A[\mathfrak{S}_n]$-module generated by the equivalence classes of tableaux of shape $(d,1^i)$ with entries in $[n]$ modulo alternating columns and shuffling relations. The equivalence classes of standard tableaux form an $A$-basis of $U_{d,n}^{i}$.

(c) The module $U_{d,n}^{i}$ is a principal $A[\mathfrak{S}_n]$-module generated by the equivalence class of any tableau of shape $(d,1^i)$ with entries in $[n]$.

Proof. (a) By definition of induced module, we have

$$U_{d,n}^{i} \cong \bigoplus \sigma(S^{(d,1^i)} \otimes_A S^{(n-d-i)}),$$

where the direct sum is over a set of representatives for the cosets of $\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}$ in $\mathfrak{S}_n$. Each summand is a free $A$-module, therefore so is $U_{d,n}^{i}$. Note that $\text{rank}(S^{(n-d-i)}) = 1$, hence

$$\text{rank}(U_{d,n}^{i}) = \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}|} = \frac{(d+i)!}{(d+i-1)!} = \frac{n}{(d+i)!} \binom{n}{d+i} \binom{d+i}{i}.$$

(b) Consider the module $S^{(d,1^i)} \otimes_A S^{(n-d-i)}$. The factor $S^{(n-d-i)}$ has rank one, hence it does not play a role in what follows. The factor $S^{(d,1^i)}$ is generated by the equivalence classes of tableaux of shape $(d,1^i)$ with entries in $[d+i]$ modulo alternating columns and shuffling relations. Moreover, $S^{(d,1^i)}$ has a basis given by the equivalence classes of standard tableaux.

For each coset of $\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}$ in $\mathfrak{S}_n$ there is a unique representative $\sigma$ which is an increasing function on $\{1,\ldots,d+i\}$ and on $\{d+i+1,\ldots,n\}$. Denote by $C$ the collection of all coset representatives with this property.

Given a representative $\sigma \in C$ and a tableau $T$ of shape $(d,1^i)$ with entries in $[d+i]$, we can apply $\sigma$ to all entries of $T$ to obtain a tableau $\sigma T$ of shape $(d,1^i)$ with entries in $\{\sigma(1),\ldots,\sigma(d+i)\}$. Moreover, if $T$ is standard, then $\sigma T$ is again standard because $\sigma$ is increasing on $\{1,\ldots,d+i\}$.

Therefore we can identify the direct summand $\sigma(S^{(d,1^i)} \otimes_A S^{(n-d-i)})$ in (6) with the $A$-module generated by equivalence classes of tableaux of shape $(d,1^i)$ with entries in $\{\sigma(1),\ldots,\sigma(d+i)\}$ modulo alternating columns and shuffling relations. In particular, the equivalence classes of standard tableaux form an $A$-basis. As $\sigma$ runs over $C$, we get the desired result.

(c) Fix a tableau $T$ of shape $(d,1^i)$ with entries in $[n]$. Given any other tableau $T'$ of shape $(d,1^i)$ with entries in $[n]$, there is a permutation $\sigma \in \mathfrak{S}_n$ such that $T' = \sigma T$. Thus we have $[T'] = \sigma[T]$ in $U_{d,n}^{i}$. Now the statement follows from part (b).

□

Remark 3.4. If $A$ is a field, then the Littlewood-Richardson rule (see [James, 1978, §16]) implies that the module $U_{d,n}^{i}$ admits a filtration whose associated graded object is $\bigoplus S^\lambda$, where the direct sum is over all partitions $\lambda = (\lambda_1,\lambda_2,\ldots)$ of $n$ such that $\lambda_1 \geq d$, $\lambda_2 \leq d$, and $\lambda_i = 1$ for all $i$ with $3 \leq i \leq n + 2 - \lambda_1 - \lambda_2$. In particular, note that this is a multiplicity-free decomposition in terms of Specht modules. The same description can also
be obtained using the simpler Pieri rule (cf. [Fulton, 1997, §2.2]). If \( A \) is a field of characteristic zero, then the same rules give a decomposition of \( U_i^{d,n} \) as a direct sum of simple \( A[\mathfrak{S}_n] \)-modules.

**Remark 3.5.** The restriction of \( U_i^{d,n} \) to \( \mathfrak{S}_{n-1} \) has the following interesting property:

\[
\text{Res}_{\mathfrak{S}_{n-1}}(U_i^{d,n}) \cong U_{i-1}^{d,n-1} \oplus U_i^{d-1,n-1} \oplus U_i^{d,n-1}.
\]

This is easy to prove using Proposition 3.3 (b). Notice also that this property matches the isomorphism in equation (4).

### 4. Equivariant resolution

In the section, we describe an \( \mathfrak{S}_n \)-equivariant minimal free resolution of \( I_{d,n} \). The free modules and the differentials are constructed using the representations \( U_i^{d,n} \) introduced in §3. As the definitions are very explicit, this allows for a direct proof of exactness, which is the content of our main theorem. We conclude this section with different interpretations of the Betti numbers of \( I_{d,n} \).

#### 4.1. Modules

**Definition 4.1.** Let \( n, d, \) and \( i \) be integers, with \( n > 0 \). We define an \( R_n[\mathfrak{S}_n] \)-module by setting

\[
F_i^{d,n} := U_i^{d,n} \otimes_A R_n(-d - i).
\]

We write \( p[T] \) for the simple tensor \( [T] \otimes p \in F_i^{d,n} \), where \( p \in R_n \) and \( T \) is a tableau of shape \((d, 1^i)\) with entries in \([n]\).

**Proposition 4.2.** Let \( n, d, \) and \( i \) be integers, with \( 1 \leq d \leq n \) and \( 0 \leq i \leq n - d \).

(a) The module \( F_i^{d,n} \) is a free \( R_n \)-module and

\[
\text{rank}(F_i^{d,n}) = \binom{n}{d+i} \binom{d+i-1}{i}.
\]

(b) The module \( F_i^{d,n} \) is isomorphic to the \( R_n[\mathfrak{S}_n] \)-module generated by the equivalence classes of tableaux of shape \((d, 1^i)\) with entries in \([n]\) modulo alternating columns and shuffling relations. The equivalence classes of standard tableaux form an \( R_n \)-basis of \( F_i^{d,n} \).

(c) The module \( F_i^{d,n} \) is a principal \( R_n[\mathfrak{S}_n] \)-module generated by the equivalence class of any tableau of shape \((d, 1^i)\) with entries in \([n]\).

**Proof.** Everything follows from the corresponding statements in Proposition 3.3. \( \square \)
4.2. **Differentials.** Consider the tableau

\[ T = \begin{array}{cccc}
    a_0 & b_1 & \ldots & b_{d-1} \\
    a_1 \\
    \vdots \\
    a_i \\
\end{array} \]

of shape \((d, 1^i)\) with entries in \([n]\). For \(0 \leq j \leq i\), define \(T \setminus a_j\) as the tableau of shape \((d, 1^{i-1})\) with entries in \([n]\) obtained from \(T\) by removing the box containing \(a_j\) and sliding upward the boxes below it. For clarity, we have

\[ T \setminus a_j = \begin{array}{cccc}
    a_0 & b_1 & \ldots & b_{d-1} \\
    a_1 \\
    \vdots \\
    a_{j-1} \\
    a_{j+1} \\
    \vdots \\
    a_i \\
\end{array} \]

respectively when \(1 \leq j \leq i\) and when \(j = 0\).

**Definition 4.3.** Let \(n, d,\) and \(i\) be integers, with \(1 \leq d \leq n\) and \(1 \leq i \leq n - d\). Fix a tableau \(T\) of shape \((d, 1^i)\) with entries \(a_0, \ldots, a_i, b_1, \ldots, b_{d-1}\) in \([n]\) as above. Since \(F_{i}^{d,n}\) is a principal \(R_n[S_n]\)-module generated by \([T]\), we define a map \(\partial^{d,n}_i: F_i^{d,n} \to F_{i-1}^{d,n}\) of \(R_n[S_n]\)-modules by setting

\[ \partial^{d,n}_i([T]) := \sum_{j=0}^{i} (-1)^{i-j} x_{a_j} [T \setminus a_j] \]

and extending by \(R_n[S_n]\)-linearity.

Note that the same formula for \(\partial^{d,n}_i\) holds for any other tableau \(T'\) of shape \((d, 1^i)\) with entries in \([n]\).

**Example 4.4** \((n = 4, d = 2, i = 2)\). We have, for example,

\[ \partial^{2,4}_2 \left( \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \right) = x_1 \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} - x_3 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}. \]
Expressing this in terms of standard tableaux, we get
\[ \partial_{2,4}^2 \begin{bmatrix} 2 & 4 \\ 3 & 4 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = x_1 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} - x_3 \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = x_1 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - x_1 \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} - x_3 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = x_1 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - x_1 \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} - x_3 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.
\]

In fact, the matrix of \( \partial_{2,4}^2 \) in the bases of standard tableaux is
\[
\begin{bmatrix}
2 & 3 \\
3 & 2 \\
1 & 2 \\
2 & 1 \\
4 & 3 \\
4 & 2 \\
3 & 2 \\
3 & 1
\end{bmatrix} 
\begin{bmatrix}
x_1 & x_1 & 0 \\
0 & -x_2 & 0 \\
x_3 & 0 & 0 \\
x_1 & 0 & x_1 \\
0 & 0 & -x_2 \\
0 & 0 & x_3 \\
x_4 & 0 & 0 \\
0 & x_4 & 0
\end{bmatrix} 
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
3 & 2 \\
1 & 3
\end{bmatrix}.
\]

**Proposition 4.5.** Let \( n \) and \( d \) be integers with \( 1 \leq d \leq n \). The sequence of maps \( \partial_{i}^{d,n} \) forms a minimal complex of free \( R_n \)-modules.

**Proof.** The computation showing that \( \partial_{i}^{d,n} \partial_{i+1}^{d,n} = 0 \) is essentially the same as the one for differentials in a Koszul complex. Minimality is obvious from the definition of \( \partial_{i}^{d,n} \).

**Definition 4.6.** Let \( n \) and \( d \) be integers with \( 1 \leq d \leq n \). We denote by \( F_{\bullet}^{d,n} \) the complex defined by the sequence of maps \( \partial_{i}^{d,n} \).

**Example 4.7** \((n = 4, d = 2)\). As \( R_4 \)-modules, we have \( F_{0}^{2,4} \cong R_4(-2)^6 \), \( F_{1}^{2,4} \cong R_4(-3)^8 \), and \( F_{2}^{2,4} \cong R_4(-4)^3 \). The complex \( F_{\bullet}^{2,4} \) looks as follows.

\[ F_{0}^{2,4} = \begin{bmatrix}
-x_2 & -x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & -x_1 & x_3 & 0 & -x_1 & 0 & 0 \\
x_4 & 0 & 0 & 0 & 0 & x_1 & 2 & 0 \\
0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\
0 & x_4 & 0 & 0 & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 \\
0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ F_{1}^{2,4} = \begin{bmatrix}
x_1 & 0 & -x_1 & 0 & 0 & 0 \\
-x_3 & 0 & 0 & -x_1 & x_1 & 0 \\
0 & 0 & -x_2 & 0 & 0 & -x_2 \\
0 & 0 & 0 & x_3 & 0 & 0 \\
x_4 & 0 & 0 & 0 & 0 & 0 \\
x_4 & 0 & 0 & 0 & 0 & 0 \\
0 & x_4 & 0 & 0 & 0 & 0 \\
0 & 0 & x_4 & 0 & 0 & 0
\end{bmatrix} \]

\[ F_{2}^{2,4} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

**Remark 4.8.** By Theorem 2.1, the complex \( F_{\bullet}^{d,n} \) has the same Betti numbers as a minimal free resolution of \( I_{d,n} \).
4.3. Exactness.

**Lemma 4.9.** Let $\varphi_{d,n}^i$ denote $(\partial_{d,n}^i)_{d+i} : (F_{d,n}^i)_{d+i} \to (F_{d-1,n}^i)_{d+i}$, i.e., the restriction of $\partial_{d,n}^i$ to degree $d + i$. Then $\varphi_{d,n}^i$ admits a left inverse.

Before proving the lemma, we illustrate $\varphi_{d,n}^i$ with an example.

**Example 4.10** ($n = 3, d = 2, i = 1$). We write an explicit matrix for $\varphi_{2,3}^1$. Note that the domain of $\varphi_{2,3}^1$ is isomorphic to $U_{2,3}^1$. On the other hand, the codomain of $\varphi_{2,3}^1$ is isomorphic to $U_{2,3}^0 \otimes A(R^1)$. For the domain, we choose the $A$-basis given by $\{T\}$, where $T \in \text{SYT}((2, 1), [3])$. For the codomain, we choose the $A$-basis given by elements $x_j[T']$, where $x_j$ is a variable in $R_n$ and $T' \in \text{SYT}((2), [3])$.

$$
\begin{bmatrix}
  x_1 & 2 & 3 \\
  x_2 & 2 & 3 \\
  x_3 & 2 & 3 \\
  x_1 & 1 & 3 \\
  x_2 & 1 & 3 \\
  x_3 & 1 & 3 \\
  x_1 & 1 & 2 \\
  x_2 & 1 & 2 \\
  x_3 & 1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
  -1 & -1 \\
   0 &   0 \\
   0 &   0 \\
   0 &   0 \\
   1 &   0 \\
   0 &   0 \\
   0 &   1 \\
   0 &   0 \\
   0 &   1 \\
\end{bmatrix}
\begin{bmatrix}
  [1 & 2] \\
  [3] \\
\end{bmatrix}
\begin{bmatrix}
  [1 & 3] \\
  [2] \\
\end{bmatrix}
$$

Note that the submatrix on the fifth and ninth row from the top is a permutation matrix, hence invertible over $A$. Therefore $\varphi_{2,3}^1$ admits a left inverse.

**Proof.** Observe that $(F_{d,n}^i)_{d+i} \cong U_{d,n}^i$. By Proposition 3.3, $U_{d,n}^i$ is a free $A$-module with a basis $B$ consisting of equivalence classes $[T]$ for $T \in \text{SYT}((d, 1^i), [n])$. Similarly, $(F_{d-1,n}^i)_{d+i} \cong U_{d-1,n}^i \otimes_A (R_n)_1$ is a free $A$-module with a basis $C$ consisting of elements $x_j[T']$, where $x_j$ is a variable in $R_n$ and $T' \in \text{SYT}((d, 1^{i-1}), [n])$. Let $M$ be the matrix of $\varphi_{d,n}^i$ relative to the bases $B$ and $C$. We will show that $M$ admits a left inverse by selecting some rows giving a submatrix of $M$ which is invertible over $A$.

Define the set

$$
C' := \{ x_{a_i}[T \setminus a_i] \in C : T \in \text{SYT}((d, 1^i), [n]) \}.
$$

The elements $C'$ are obtained from tableaux on $(d, 1^i)$ by removing the bottom box of the first column and multiplying by the variable with index the entry of the removed box. The cardinalities of $B$, $C$, and $C'$ are related by
the formula $|B| = |C'| < |C|$. In particular, this shows that $M$ always has more rows than columns. Let $N$ be the square submatrix of $M$ on the rows corresponding to elements of $C'$.

Given $[T] \in B$, we have

$$\partial^{d,n}_i([T]) = \sum_{j=0}^{i} (-1)^{i-j} x_{a_j} [T \setminus a_j].$$

For $1 \leq j \leq i$, the elements $x_{a_j} [T \setminus a_j]$ are in $C$. The only one belonging to $C'$ is $x_{a_i} [T \setminus a_i]$, which appears with coefficient 1. If $T \setminus a_0$ is standard, then $x_{a_0} [T \setminus a_0]$ is in $C$ but not in $C'$. If $T \setminus a_0$ is not standard, then $x_{a_0} [T \setminus a_0]$ can be expanded into an $A$-linear combination of basis elements $x_{a_0} [T']$ for certain tableaux $T' \in \text{SYT}((d,1^{i-1}),[n])$. Note that the entries of any such $T'$ form a subset of $\{a_1, \ldots, a_i, b_1, \ldots, b_{d-1}\}$; in particular, all entries of $T'$ are bigger than $a_0$. This implies that $x_{a_0} [T']$ is not in $C'$. We deduce that the column of $N$ corresponding to $[T]$ has a single nonzero entry, and this entry is is equal to 1. Therefore $N$ is a permutation matrix, hence it is invertible over $A$.

**Theorem 4.11.** Let $n$ and $d$ be integers with $1 \leq d \leq n$. The complex $F^d_{\bullet,n}$ is an $\mathfrak{S}_n$-equivariant minimal free resolution of the $R_n$-module $I_{d,n}$.

**Proof.** The module $F^d_{0,n}$ has an $R_n$-basis $[T]$ with $T \in \text{SYT}((d),[n])$. Define a map of $R_n$-modules $\partial^d_{0,n} : F^d_{0,n} \rightarrow I_{d,n}$ by sending the equivalence class of the tableau

$$\begin{bmatrix} b_1 & \cdots & b_d \end{bmatrix}$$

to $x_{b_1} \cdots x_{b_d} \in I_{d,n}$. The map $\partial^d_{0,n}$ is clearly $\mathfrak{S}_n$-equivariant. Moreover, $\partial^d_{0,n}$ is surjective because $I_{d,n}$ is generated by the squarefree monomials of degree $d$ in $R_n$.

Next we show that $\partial^d_{0,n} \partial^d_{1,n} = 0$. Consider the tableau

$$T = \begin{bmatrix} a_0 & b_1 & \cdots & b_{d-1} \\ a_1 & \end{bmatrix}$$

of shape $(d,1)$ with entries in $[n]$. We have

$$\partial^d_{0,n} \partial^d_{1,n}([T]) = \partial^d_{0,n}(-x_{a_0} [T \setminus a_0] + x_{a_1} [T \setminus a_1]) = -x_{a_0} x_{b_1} \cdots x_{b_{d-1}} + x_{a_1} x_{a_0} x_{b_1} \cdots x_{b_{d-1}} = 0.$$ 

Therefore $F^d_{\bullet,n}$ can be extended, via the map $\partial^d_{0,n}$, to an $\mathfrak{S}_n$-equivariant complex of $R_n$-modules $0 \leftarrow I_{d,n} \leftarrow F^d_{\bullet,n}$, which is exact at $I_{d,n}$. If we can show this complex is exact everywhere else, then the theorem will follow. We will prove the complex is exact at $F^d_{i,n}$ proceeding by induction on $i$.

For the base case, let $i = 0$. We need to show $\ker(\partial^d_{0,n}) = \text{im}(\partial^d_{1,n})$. Consider $\varphi^d_{1,n}$, the restriction of $\partial^d_{1,n}$ to degree $d+1$. By Lemma 4.9, $\varphi^d_{1,n}$ admits
a left inverse, which is a map of \(A\)-modules from \((F_{d,n}^{0})_{d+1}\) to \((F_{d,n}^{1})_{d+1}\). Denote by \(\psi_{1}^{d,n}: (\ker(\partial_{0}^{d,n}))_{d+1} \rightarrow (F_{d,n}^{1})_{d+1}\) the \(A\)-module map obtained by restricting the left inverse of \(\varphi_{1}^{d,n}\) to the degree \(d+1\) component of \(\ker(\partial_{0}^{d,n})\) (see the diagram below).

\[
\begin{array}{ccc}
(F_{0}^{d,n})_{d+1} & \xrightarrow{\varphi_{1}^{d,n}} & (F_{1}^{d,n})_{d+1} \\
\downarrow & & \downarrow \\
(\ker(\partial_{0}^{d,n}))_{d+1} & \xrightarrow{\psi_{1}^{d,n}} & (F_{1}^{d,n})_{d+1}
\end{array}
\]

Since \(\text{im}(\partial_{1}^{d,n}) \subseteq \ker(\partial_{0}^{d,n})\), we have \(\text{im}(\varphi_{1}^{d,n}) \subseteq (\ker(\partial_{0}^{d,n}))_{d+1}\). Then we can consider the composition \(\psi_{1}^{d,n} \varphi_{1}^{d,n}\), which is, by construction, the identity map of \((F_{d,n}^{1})_{d+1}\). It follows that \(\psi_{1}^{d,n}\) is surjective. Since \(\partial_{0}^{d,n}\) surjects onto \(I_{d,n}\), Remark 4.8 implies that \(\ker(\partial_{0}^{d,n})\) is generated in degree \(d+1\) and that \((\ker(\partial_{0}^{d,n}))_{d+1}\) is a free \(A\)-module of rank \(n^{d+1}d_{1}\). At the same time, \((F_{d,n}^{1})_{d+1}\) is also free of rank \(n^{d+1}d_{1}\). Since \(\psi_{1}^{d,n}\) is a surjection between free modules of the same rank, it is an isomorphism (see [Atiyah and Macdonald, 1969, Chapter 3, Exercise 15]). Now \(\varphi_{1}^{d,n}\) is the right inverse of \(\psi_{1}^{d,n}\), hence it gives an isomorphism of \(A\)-modules between \((F_{d,n}^{1})_{d+1}\) and \((\ker(\partial_{0}^{d,n}))_{d+1}\). We deduce that \((\ker(\partial_{0}^{d,n}))_{d+1} = \text{im}(\varphi_{1}^{d,n}) = (\text{im}(\partial_{1}^{d,n}))_{d+1}\). Given that \(\ker(\partial_{0}^{d,n})\) is generated in degree \(d+1\), we conclude that \(\ker(\partial_{0}^{d,n}) = \text{im}(\partial_{1}^{d,n})\) as desired.

For the inductive step, let \(i > 0\). The proof is essentially the same as for the base step. We need to show \(\ker(\partial_{i}^{d,n}) = \text{im}(\partial_{i+1}^{d,n})\). The map \(\varphi_{i+1}^{d,n}\), the restriction of \(\partial_{i+1}^{d,n}\) to degree \(d+i+1\), admits a left inverse by Lemma 4.9. Denote by \(\psi_{i+1}^{d,n}: (\ker(\partial_{i}^{d,n}))_{d+i+1} \rightarrow (F_{i+1}^{d,n})_{d+i+1}\) the \(A\)-module map obtained by restricting the left inverse of \(\varphi_{i+1}^{d,n}\) to the degree \(d+i+1\) component of \(\ker(\partial_{i}^{d,n})\) (see the diagram below).

\[
\begin{array}{ccc}
(F_{i}^{d,n})_{d+i+1} & \xrightarrow{\varphi_{i+1}^{d,n}} & (F_{i+1}^{d,n})_{d+i+1} \\
\downarrow & & \downarrow \\
(\ker(\partial_{i}^{d,n}))_{d+i+1} & \xrightarrow{\psi_{i+1}^{d,n}} & (F_{i+1}^{d,n})_{d+i+1}
\end{array}
\]

Since \(\text{im}(\varphi_{i+1}^{d,n}) \subseteq (\ker(\partial_{i}^{d,n}))_{d+i+1}\), the composition \(\psi_{i+1}^{d,n} \varphi_{i+1}^{d,n}\) is the identity map of \((F_{d,n}^{i+1})_{d+i+1}\). It follows that \(\psi_{i+1}^{d,n}\) is surjective. By induction, \(\text{im}(\partial_{i}^{d,n}) = \ker(\partial_{i-1}^{d,n})\). Hence Remark 4.8 implies that \(\ker(\partial_{i}^{d,n})\) is generated in degree \(d+i+1\) and that \((\ker(\partial_{i}^{d,n}))_{d+i+1}\) is a free \(A\)-module of rank.
\( \binom{n}{d+i+1} \binom{d+i}{i+1} \). Note that \((F_{i+1}^d, n/d+i+1)\) is also free of rank \( \binom{n}{d+i+1} \binom{d+i}{i+1} \). Being a surjection between free modules of the same rank, \( \varphi_{i+1}^{d,n} \) is an isomorphism. Thus \( \varphi_{i+1}^{d,n} \) gives an isomorphism between \((F_{i+1}^d, n/d+i+1)\) and \((\ker(\partial_{i+1}^d, n))_{d+i+1}\). It follows that \( \ker(\partial_{i+1}^d, n) = \text{im}(\varphi_{i+1}^{d,n}) = (\text{im}(\partial_{i+1}^d, n))_{d+i+1} \). Given that \( \ker(\partial_{i+1}^d, n) \) is generated in degree \( d + i + 1 \), we conclude that \( \ker(\partial_{i+1}^d, n) = \text{im}(\partial_{i+1}^d, n) \) as desired. \( \square \)

The following corollary gives a representation theoretic description of the syzygy modules, and a combinatorial interpretation of the Betti numbers of \( I_{d,n} \). The proof follows directly from Theorem 4.11.

**Corollary 4.12.** Let \( n \) and \( d \) be integers with \( 1 \leq d \leq n \).

(a) There are isomorphisms of \( A[S_n] \)-modules

\[
\text{Tor}^{R_n}_{i}(I_{d,n}, A) \cong \begin{cases} U_{i}^{d,n}, & \text{if } j = d + i, \\ 0, & \text{otherwise.} \end{cases}
\]

(b) The Betti numbers of \( I_{d,n} \) are given by the formula

\[
\beta_{i,j}(I_{d,n}) = \begin{cases} |\text{SYT}((d, 1^i), [n])|, & \text{if } j = d + i, \\ 0, & \text{otherwise.} \end{cases}
\]

5. Applications

**5.1. De Concini-Procesi rings.** Let \( n \) be a fixed positive integer. For each partition \( \mu \) of \( n \), there is a De Concini-Procesi ideal \( I_{\mu} \subset R_n = A[x_1, \ldots, x_n] \), and a corresponding De Concini-Procesi ring \( R_n/I_{\mu} \). Historically, these ideals and quotients have been studied in the case when the coefficient ring \( A \) is a field, although they can be defined more generally. The original definition is in [De Concini and Procesi, 1981], along with an explicit set of generators. Over the years, other generating sets have been described that are simpler and/or smaller than the original one; see for example [Biagioli et al., 2008, Tanisaki, 1982]. Since we are only interested in the case of hook partitions, we describe the De Concini-Procesi ideal in this case following [Biagioli et al., 2007, Proposition 3.4].

**Proposition 5.1.** Let \( \mu = (n - d + 1, 1^{d-1}) \), with \( 1 \leq d \leq n \). Then we have

\[
I_{\mu} = (e_1, \ldots, e_{d-1}) + I_{d,n},
\]

where \( e_i \) is the \( i \)-th symmetric polynomial.

In Proposition 5.1, we recognize our monomial ideal \( I_{d,n} \). The elementary symmetric polynomials making up the rest of the ideal have another useful property first presented in [Biagioli et al., 2007, Proposition 4.3].

**Proposition 5.2.** Let \( n \) and \( d \) be integers with \( 1 \leq d \leq n \). The residue classes of \( e_1, \ldots, e_{d-1} \) in \( R_n/I_{d,n} \) form a regular sequence.
Using this property, R. Biagioli, S. Faridi, and M. Rosas observed that a free resolution of $R_n/I_\mu$ can be obtained from one of $R_n/I_{d,n}$ via iterated mapping cones. Moreover, Proposition 5.2, guarantees that the mapping cones are minimal. The details of the procedure are discussed in [Biagioli et al., 2007].

This mapping cone construction extends immediately to the equivariant case. In fact, we have an $S_n$-equivariant resolution of $I_{d,n}$ from Theorem 4.11 and the mapping cones are induced from multiplication by an elementary symmetric polynomial, which is an $S_n$-invariant. We illustrate with an example.

**Example 5.3** ($n = 4, d = 2$). Based on Proposition 5.1, we have $I_{(3,1)} = (e_1) + I_{2,4}$. From Proposition 5.2, we know that $e_1$ is regular on $R_4/I_{2,4}$. Therefore we have a short exact sequence

$$0 \rightarrow R_4/I_{2,4} \xrightarrow{e_1} R_4/I_{2,4} \rightarrow R_4/I_{(3,1)} \rightarrow 0$$

where the second map is multiplication by $e_1$.

From Example 4.7, we have the following resolution of $R_4/I_{2,4}$.

$$R_4 \xleftarrow{\partial_0^{2,4}} F_0^{2,4} \xrightarrow{\partial_1^{2,4}} F_1^{2,4} \xleftarrow{\partial_0^{2,4}} F_2^{2,4} \xrightarrow{0}$$

The map $\partial_0^{2,4}$ was defined in the proof of Theorem 4.11. Multiplication by $e_1$ in the short exact sequence above extends to a map between two copies of the resolution of $R_4/I_{2,4}$. The mapping cone of this map of complexes looks as follows.

$$R_4 \xleftarrow{[\partial_0^{2,4} \quad e_1]} F_0^{2,4} \oplus R_4(-1) \xleftarrow{\begin{bmatrix} \partial_1^{2,4} & e_1 \\ 0 & -\partial_0^{2,4} \end{bmatrix}} F_1^{2,4} \oplus F_0^{2,4}(-1) \xrightarrow{\begin{bmatrix} \partial_2^{2,4} & e_1 \\ 0 & -\partial_2^{2,4} \end{bmatrix}} F_2^{2,4} \oplus F_1^{2,4}(-1) \xleftarrow{\begin{bmatrix} e_1 \\ -\partial_2^{2,4} \end{bmatrix}} F_2^{2,4}(-1) \xrightarrow{0}$$

By the general theory, this complex is an $S_n$-equivariant minimal free resolution of $R_4/I_{(3,1)}$.

A complete description of the differentials in the iterated mapping cone complex resolving $I_\mu$ would be notationally cumbersome. Instead we focus on a description of the modules. Following the example of Biagioli, Faridi, and Rosas, we will use a bigraded Poincaré series. We start by recalling some terminology.

The Grothendieck ring of $A[S_n]$ is the free abelian group generated by isomorphism classes $[P]$ of finitely generated projective $A[S_n]$-modules modulo the relations $[P'] - [P] + [P''] = 0$ for every short exact sequence

$$0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0.$$ Addition is defined by $[P_1] + [P_2] := [P_1 \oplus P_2]$, while multiplication is defined by $[P_1][P_2] := [P_1 \otimes P_2]$. Note that the class $[S^{(n)}]$ of the rank one trivial
A[S_n]-module is the identity element for multiplication; thus we will simply write 1 for [S(n)].

**Remark 5.4.** The A[S_n]-module \(U_i^{d,n}\) is projective. To see this, consider the following diagram of A[S_n]-modules with exact rows.

\[
\begin{array}{ccc}
U_i^{d,n} & \overset{\rho}{\rightarrow} & M \\
\downarrow{\varphi} & & \downarrow{\psi}
\end{array}
\]

Let \(T\) be a tableau of shape \((d,1^i)\) with entries in \([n]\), so that \([T]\) generates \(U_i^{d,n}\). Since \(\psi\) is surjective, there exists \(m \in M\) such that \(\psi(m) = \varphi([T])\).

The A-submodule of \(U_i^{d,n}\) generated by \([T]\) is free. Therefore we can define \(\rho: U_i^{d,n} \rightarrow M\) by setting \(\rho([T]) := m\), and extending by A[S_n]-linearity. This guarantees that \(\rho\) is a map of A[S_n]-modules. Moreover, the equality \(\rho([T]) = m\) implies that \(\varphi = \psi \rho\).

Since \(U_i^{d,n}\) is a finitely generated projective A[S_n]-module, we can consider its class \([U_i^{d,n}]\) in the Grothendieck ring of A[S_n].

**Definition 5.5.** Let \(M\) be a finitely generated graded \(R_n[S_n]\)-module. Suppose that, for all integers \(i,j\), Tor\(^R\(_n\_\_\)i\(\_\_\_\)M,A\_)\(_j\) is a finitely generated projective \(A[S_n]\)-module. The \(S_n\)-equivariant bigraded Poincaré series of \(M\) is the power series

\[
P_{M}^{S_n}(q,t) := \sum_{i,j} [\text{Tor}_i^R(M,A)]_{ij} q^i t^j
\]

in the variables \(q, t\) with coefficients in the Grothendieck ring of A[S_n].

This equivariant Poincaré series is simply a compact device to keep track of the representations appearing in an equivariant resolution.

**Example 5.6.** Comparing with Example 5.3, we have the equality

\[
P_{R_{4d}/I_{2,4}}^{S_n}(q,t) = 1 + [U_0^{2,4}] q^2 t^2 + [U_1^{2,4}] q^2 t^3 + [U_2^{2,4}] q^3 t^4,
\]

and also

\[
P_{R_{4d}/I_{3,1}}^{S_n}(q,t) = 1 + [U_0^{2,4}] q^2 t^2 + qt + ([U_1^{2,4}] + [U_0^{2,4}]) q^3 t^3 + ([U_2^{2,4}] + [U_1^{2,4}]) q^4 t^4 =
\]

\[
= (1 + qt)(1 + [U_0^{2,4}] q^2 t^2 + [U_1^{2,4}] q^2 t^3 + [U_2^{2,4}] q^3 t^4).
\]

Finally, we are in a position to state the main result for De Concini-Procesi rings indexed by hook partitions. This theorem is an immediate consequence of Theorem 4.11 and the discussion of this section.
Theorem 5.7. Let \( \mu = (n - d + 1, 1^{d-1}) \), with \( 1 \leq d \leq n \). Then we have
\[
P^S_{R_\mu/I_\mu}(q, t) = \prod_{k=1}^{d-1} (1 + qt^k) \left( 1 + \sum_{i=0}^{n-d} [U_i^{d,n}] q^{i+1} t^{d+i} \right).
\]

5.2. A resolution of FI-modules. The framework of FI-modules, developed by T. Church, J.S. Ellenberg, and B. Farb, allows us to assemble all the complexes \( F_{d,n}^d \), for a fixed \( d \), into a single comprehensive structure. For all details on FI-modules, we refer the reader to [Church et al., 2015]. Our base ring will be \( A \), and it should replace any instance of \( k \) occurring in the reference mentioned. We do not appeal to any deep result about FI-modules, as we are merely interested in the language they provide. Explicitly, we will use the notions of FI-module (Definition 1.1), morphism of FI-modules (Definition 2.1.1), graded FI-module (Remark 2.1.5), and exactness in the category of FI-modules (which follows from Remark 2.1.2).

Remark 5.8. The notation in [Church et al., 2015] uses a capital letter, say \( V \), for an FI-module, and \( V_n \) for the object that \( V \) associates to the set \([n]\). To distinguish FI-modules from other entities, we will use a capital calligraphic letter, say \( \mathcal{V} \), for an FI-module. Moreover, our FI-modules will carry subscripts for homological dimension and graded components. To avoid using another subscript, we will write \( \mathcal{V}(n) \) for the object that \( \mathcal{V} \) associates to the set \([n]\).

Definition 5.9. For a positive integer \( d \), define \( \mathcal{I}_d \) to be the graded FI-module that associates:
- to the set \([n]\) the graded \( A \)-module \( I_{d,n} \);
- to an injection \( \varepsilon : [n] \to [m] \) the map \( \mathcal{I}_d(\varepsilon) : I_{d,n} \to I_{d,m} \) of graded \( A \)-modules defined by \( p(x_1, \ldots, x_n) \mapsto p(x_{\varepsilon(1)}, \ldots, x_{\varepsilon(n)}) \).

Let \( T \) be a tableau of shape \((d, 1^i)\) with entries in \([n]\). If \( \varepsilon : [n] \to [m] \) is an injection, then denote by \( \varepsilon(T) \) the tableau of shape \((d, 1^i)\) with entries in \([m]\) obtained by replacing each entry \( i \) of \( T \) by \( \varepsilon(i) \).

Definition 5.10. For integers \( d \) and \( i \), with \( d > 0 \) and \( i \geq 0 \), define \( \mathcal{F}_i^d \) to be the FI-module that associates:
- to the set \([n]\) the graded \( A \)-module \( F_{i,n}^{d,n} \);
- to an injection \( \varepsilon : [n] \to [m] \) the map \( \mathcal{F}_i^d(\varepsilon) : F_{i,n}^{d,n} \to F_{i,m}^{d,m} \) of graded \( A \)-modules given by \( p(x_1, \ldots, x_n)[T] \mapsto p(x_{\varepsilon(1)}, \ldots, x_{\varepsilon(n)})[\varepsilon(T)] \), where \( T \) is a tableau of shape \((d, 1^i)\) with entries in \([n]\).

Proposition 5.11. Let \( d \) and \( i \) be positive integers. For all integers \( m, n \), with \( m \geq n \geq 0 \), and all injections \( \varepsilon : [n] \to [m] \), the diagrams of graded \( A \)-modules and maps
are commutative.

Proof. The proof of commutativity of the two diagrams is similar. We write a proof for the diagram on the right.

If \( F_{i,n}^d = 0 \), then the diagram is obviously commutative. Hence we assume that \( F_{i,n}^d \neq 0 \). It follows that all other modules appearing in the diagram are also nonzero.

The group \( S_m \) acts on \( F_{i-1}^d \) and \( F_i^d \), while \( S_n \) acts on \( F_{i-1}^{d,n} \) and \( F_i^{d,n} \). By construction, the map \( \partial_i^{d,m} \) is \( S_m \)-equivariant and \( \partial_i^{d,n} \) is \( S_n \)-equivariant.

Fix an injection \( \varepsilon : [n] \to [m] \). The choice of \( \varepsilon \) gives rise to a natural identification of \( S_n \) with a subgroup of \( S_m \). With this convention, all maps in the diagram are \( S_n \)-equivariant.

Let \( T \) be a tableau of shape \((d, 1^i)\) with entries in \([n]\). By Proposition 4.2, \( F_{i,n}^d \) is generated, as an \( R_n[S_n] \)-module, by \([T]\). Therefore it is enough to prove commutativity of the diagram holds for \([T]\). On one hand we have

\[
\mathcal{F}_{i-1}^{d} (\varepsilon) \partial_i^{d,n} ([T]) = \mathcal{F}_{i-1}^{d} (\varepsilon) \left( \sum_{j=0}^{i} (-1)^{i-j} x_{a_j} [T \setminus a_j] \right) = \\
= \sum_{j=0}^{i} (-1)^{i-j} x_{\varepsilon(a_j)} [\varepsilon (T \setminus a_j)] = \\
= \sum_{j=0}^{i} (-1)^{i-j} x_{\varepsilon(a_j)} [\varepsilon (T) \setminus \varepsilon(a_j)].
\]

On the other hand

\[
\partial_i^{d,m} \mathcal{F}_{i}^{d} (\varepsilon) (\varepsilon(T)) = \partial_i^{d,m} (\varepsilon(T)) = \sum_{j=0}^{i} (-1)^{i-j} x_{\varepsilon(a_j)} [\varepsilon (T) \setminus \varepsilon(a_j)].
\]

This shows that \( \mathcal{F}_{i-1}^{d} (\varepsilon) \partial_i^{d,n} ([T]) = \partial_i^{d,m} \mathcal{F}_{i}^{d} (\varepsilon) (\varepsilon(T)) \). Therefore the diagram commutes.

As a consequence of Proposition 5.11, we are allowed to make the following definitions.

**Definition 5.12.** For a positive integer \( d \), let \( \partial_0^d : \mathcal{F}_0^d \to \mathcal{I}_d \) be the morphism of graded FI-modules defined by setting \( \partial_0^d ([n]) := \partial_0^{d,n} \) for all integers \( n \).
Definition 5.13. For positive integers $d$ and $i$, let $\partial^d_i : F^d_i \to F^d_{i-1}$ be the morphism of graded FI-modules defined by setting $\partial^d_i([n]) := \partial^d_i(n)$ for all integers $n$.

Theorem 5.14. For each positive integer $d$, there is a resolution of graded FI-modules

$$0 \leftarrow I_d \leftarrow I_d \leftarrow I_3 \leftarrow I_2 \leftarrow I_1 \leftarrow I_0 \leftarrow F^d_0 \leftarrow F^d_1 \leftarrow F^d_2 \leftarrow \ldots \leftarrow F^d_{i-1} \leftarrow F^d_i \leftarrow \ldots$$

Proof. The sequence of graded FI-modules and morphisms in the statement of the theorem is functorial on the category of finite sets with injections. Thus, given a nonnegative integer $n$, the sequence applies to the set $[n]$ to produce a complex of $R_n$-modules, namely the complex $F^d_{\bullet,n}$. By definition, the sequence in the statement of the theorem is exact if and only if $F^d_{\bullet,n}$ is exact for all $n \in \mathbb{N}$. The latter is true by virtue of Theorem 4.11. □

Remark 5.15. All graded FI-modules appearing in this section are of finite type in the sense of [Church et al., 2015, Definition 4.2.1].

Remark 5.16. One can write an analogue of Theorem 5.14 for De Concini-Procesi rings indexed by partitions $\mu = (n-d+1,1^{d-1})$.

References

[Atiyah and Macdonald, 1969] Atiyah, M. F. and Macdonald, I. G. (1969). *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.

[Biagioli et al., 2007] Biagioli, R., Faridi, S., and Rosas, M. (2007). Resolutions of De Concini-Procesi ideals of hooks. *Comm. Algebra*, 35(12):3875–3891.

[Biagioli et al., 2008] Biagioli, R., Faridi, S., and Rosas, M. (2008). The defining ideals of conjugacy classes of nilpotent matrices and a conjecture of Weyman. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn117, 33.

[Church et al., 2015] Church, T., Ellenberg, J. S., and Farb, B. (2015). FI-modules and stability for representations of symmetric groups. *Duke Math. J.*, 164(9):1833–1910.

[De Concini and Procesi, 1981] De Concini, C. and Procesi, C. (1981). Symmetric functions, conjugacy classes and the flag variety. *Invent. Math.*, 64(2):203–219.

[Fulton, 1997] Fulton, W. (1997). *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge. With applications to representation theory and geometry.

[Geramita et al., 2013] Geramita, A. V., Harbourne, B., and Migliore, J. (2013). Star configurations in $\mathbb{P}^n$. *J. Algebra*, 376:279–299.

[Geramita et al., 2015] Geramita, A. V., Harbourne, B., Migliore, J., and Nagel, U. (2015). Matroid configurations and symbolic powers of their ideals. arXiv:1507.00380.

[James, 1978] James, G. D. (1978). *The representation theory of the symmetric groups*, volume 682 of *Lecture Notes in Mathematics*. Springer, Berlin.

[Kraft, 1981] Kraft, H. (1981). Conjugacy classes and Weyl group representations. In *Young tableaux and Schur functors in algebra and geometry (Toruń, 1980)*, volume 87 of *Astérisque*, pages 191–205. Soc. Math. France, Paris.

[Macdonald, 1995] Macdonald, I. G. (1995). *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition. With contributions by A. Zelevinsky, Oxford Science Publications.
[Park and Shin, 2015] Park, J. P. and Shin, Y.-S. (2015). The minimal free graded resolution of a star-configuration in $\mathbb{P}^n$. *J. Pure Appl. Algebra*, 219(6):2124–2133.

[Tanisaki, 1982] Tanisaki, T. (1982). Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups. *Tohoku Math. J. (2)*, 34(4):575–585.

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