THE HAMILTONIAN APPROACH AND PHASE SPACE PATH INTEGRATION
FOR NONLINEAR SIGMA MODELS WITH AND WITHOUT FERMIONS

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ABSTRACT

Instead of imposing the Schrödinger equation to obtain the configuration space propagator \( \langle x, t_2 | y, t_1 \rangle \) for a quantum mechanical nonlinear sigma model, we directly evaluate the phase space propagator \( \langle x | \exp \left( -\frac{i}{\hbar} \hat{H} \Delta t \right) | p \rangle \) by expanding the exponent and pulling all operators \( \hat{p} \) to the right and \( \hat{x} \) to the left. Contrary to the widespread belief that it is sufficient to keep only terms linear in \( \Delta t \) in the expansion if one is only interested in the final result through order \( \Delta t \), we find that all terms in the expansion must be retained. We solve the combinatorical problem of summing the infinite series in closed form through order \( \Delta t \). Our results straightforwardly generalize to higher orders in \( \Delta t \). We then include fermions for which we use coherent states in phase space. For supersymmetric \( N=1 \) and \( N=2 \) quantum mechanics, we find that if the super Van Vleck determinant replaces the original Van Vleck determinant the propagator factorizes into a classical part, this super determinant and the extra scalar curvature term which was first found by DeWitt for the purely bosonic case by imposing the Schrödinger equation. Applying our results to anomalies in \( n \)-dimensional quantum field theories, we note that the operator ordering in the corresponding quantum mechanical Hamiltonians is fixed in these cases. We present a formula for the path integral action, which corresponds one to one to any given covariant or noncovariant \( \hat{H} \). We then evaluate these path integrals through two loop order, and reobtain the same propagators in all cases.

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1. Introduction.

In this article we settle a long standing problem, namely the direct computation of the phase space matrix element \( \langle x | \exp \left( -\frac{i}{\hbar} \hat{H} \Delta t \right) | p \rangle \) for quantum mechanical systems with a kinetic term of the form \( a(\hat{x})\hat{p}^2 + b(\hat{x})\hat{p} + c(\hat{x}) \). The standard example, on which we concentrate, is the nonlinear sigma model describing a point particle coupled to an external gravitational field. Another example is the Hamiltonian for a point particle minimally coupled to an electromagnetic field. We shall also consider the inclusion of fermions, in particular supersymmetric quantum mechanics. Although our results generalize straightforwardly to quantum field theory, we shall restrict our discussion to quantum mechanical systems.

Through the work of Alvarez-Gaumé and Witten on chiral and gravitational anomalies \([1,2]\), and later the work of Bastianelli and van Nieuwenhuizen on trace (Weyl) anomalies \([3,4]\), it is known that anomalies of an \( n \)-dimensional quantum field theory can in general be written as configuration space path integrals for nonlinear sigma models. However, the precise relation between the quantum mechanical Hamiltonian, which corresponds to the regulator in the quantum field theory and whose operator ordering is unambiguously fixed by this quantum field theory \([5]\), and the corresponding action \( S_{\text{conf}} \) to be used in the configuration space integral, is unclear. For chiral anomalies, the precise form of this action is not needed, as due to their topological nature the naive classical action \( S_{\text{cl}} \) is sufficient. However, for trace anomalies the extra terms by which \( S_{\text{conf}} \) differs from \( S_{\text{cl}} \) are crucial. We therefore decided to settle the relation between \( S_{\text{conf}} \) and \( S_{\text{cl}} \) in general.

Before going on, let us eliminate a possible source of confusion. We are aware of an enormous amount of literature on configuration space path integrals, in which geometric operators like the Laplace-Beltrami operator and the Lichnerowicz operator play a role. In these articles these operators are selected because of their nice mathematical properties. In the present article we do not begin by selecting such nice operators on mathematical grounds; rather, on physical grounds we begin with certain Hamiltonians \( \hat{H} \) whose operator orderings are fixed a priori by corresponding quantum field theories, and our aim is to find the corresponding \( S_{\text{conf}} \) which produce exactly the same propagator as \( \hat{H} \). When it will so happen that these \( S_{\text{conf}} \) do not have nice mathematical properties, so be it. For us, the relevant question is: which \( S_{\text{conf}} \) corresponds to which \( \hat{H} \)? Of course, this question only makes sense if we also give a precise definition of how to evaluate the path integrals. We shall only consider perturbative expansions, and the precise rules are given in \([1,3,4]\). They will be reviewed in the appendix. (For the evaluation of anomalies, perturbation theory gives exact results).

In the configuration space path integrals considered in \([1,2,3,4]\), the internal symmetry sector or the fermionic sector were still treated by an operator formalism. We shall treat the bosonic and fermionic sectors on equal footing, starting with an operator formalism and deducing from it the corresponding path integral formalism. Our final path integrals will be genuine path integrals, without sectors where one still uses operators. To achieve this, we need to evaluate the matrix element \( \langle x | \exp \left( -\frac{i}{\hbar} \hat{H} \Delta t \right) | p \rangle \) for bosons or \( \langle x, \bar{\eta} | \exp \left( -\frac{i}{\hbar} \hat{H} \Delta t \right) | p, \xi \rangle \) for fermions where \( \langle \bar{\eta} \rangle \) and \( | \xi \rangle \) are fermionic coherent states.
At first sight the task seems appalling (which may account for the absence of this result in the literature) because expanding the exponent one finds momentum operators $\hat{p}$ in all possible places between position operators $\hat{x}$ which appear in the arbitrary metric $g_{ij}(\hat{x})$. Pulling all $\hat{p}$ to the right and keeping track of all possible commutators of $\hat{p}$ and $\hat{x}$ seems a hopelessly complicated combinatorial problem. It seems universally believed that it is sufficient to expand $\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)$ into $1 - \frac{i}{\hbar}\hat{H}\Delta t$, evaluate the matrix element $\langle x|\hat{H}|p\rangle \equiv h(x,p)\langle x|p\rangle$ or $\langle z^*|\hat{H}|z\rangle \equiv h(z^*,z)\langle z^*|z\rangle$ and reexponentiate to obtain $\exp\left(-\frac{i}{\hbar}h(x,p)\Delta t\right)\langle x|p\rangle$ or $\exp\left(-\frac{i}{\hbar}h(z^*,z)\Delta t\right)\langle z^*|z\rangle$ (here $|z\rangle$ denote bosonic coherent states)[2,5]. This we have found to be incorrect: one must keep all terms in the expansion of the exponent, but for each term one needs to take into account only a finite number of commutators between $\hat{p}$ and $\hat{x}$ if one is only interested in the result to a given order in $\Delta t$. This makes the problem tractable after all. Including fermions, one obtains in addition to $\hat{x}$ and $\hat{p}$ also operators $\hat{\psi}$ and $\hat{\psi}^\dagger$. Again, expanding the exponent, all terms contribute but only a finite number of anticommutators of $\hat{\psi}$ and $\hat{\psi}^\dagger$ need be kept in each term, and the transition element (also called propagator) $\langle x,\bar{\eta}|\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|p,\xi\rangle$ is obtained to any given order in $\Delta t$.

To avoid confusion, let us define what we mean by ‘to a given order in $\Delta t$’. We say that the propagator is given up to order $l$ in $\Delta t$ if it can be written as a product of some function $f(x,y;\Delta t)$ and a factor $[1 + O(\Delta t)^{l+1/2}]$. The function $f$ itself will in general be singular in $\Delta t$.

The (incorrect) neglecting of terms of order $(\Delta t)^2$ in nonlinear sigma models should not be confused with the (correct) neglecting of terms of order $(\Delta t)^2$ in the approach of Feynman and others for Hamiltonians of the form $\hat{T}(\hat{p}) + V(\hat{x})$, where the Trotter formula[6] actually proves the correctness of this procedure. Nonlinear sigma models are simply not in the class of models to which the Trotter formula can be applied.

It is often claimed that after integrating out the momenta, one can use the resulting propagators $\langle x_j|\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|x_{j-1}\rangle$ to order $\Delta t$ to obtain the configuration space path integrals by taking products of these building blocks and integrating over the intermediate points $x_1 \ldots x_{N-1}$. However, there is a problem: usually one wants to use local actions $S_{\text{conf}}$, and the matrix elements $\langle x_j|\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|x_{j-1}\rangle$ cannot in general be written as the exponent of a local action. For example, for the nonlinear sigma model we consider below, one finds terms $\Delta t R(x)$ and $R_{ij}(x)(x-y)^i(x-y)^j$, and whereas the former can be written to order $\Delta t$ as $\int dt R(x)$, the latter term cannot be written in such a way. In fact, the nonlocal $\langle x_j|\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|x_{j-1}\rangle$ satisfy the product property of path integrals, and thus trivially produce in a path integral the propagator $\langle x|\exp\left(-\frac{i}{\hbar}\hat{H}T\right)|y\rangle$. The problem is rather to find a local action $S_{\text{conf}}$, which also produces in a path integral this propagator. From the example of the point particle in curved space it is known that $S_{\text{conf}}$ contains local terms of higher order in $\hbar$, but in general the problem of constructing $S_{\text{conf}}$ from a given Hamiltonian (with given operator ordering) seems unsolved. DeWitt has recently made great progress in this direction by establishing a formal (unregularized) connection between $\hat{H}$ and $S_{\text{conf}}$ using the concept of time-ordering[7]. Using his results and the two and three-loop results of [4] on trace anomalies, we shall give the exact local action $S_{\text{conf}}$ which corresponds to any, covariant or noncovariant, Hamiltonian $\hat{H}$ with at most two $\hat{p}$ operators. We have not been able to give a rigorous proof of this result, but we have checked our result through two-loop order, and give
general arguments which hold at higher loop order.

Actually, Feynman already suspected that for nonlinear sigma models integrating out the momenta would not lead to path integrals with the classical action in the exponent\[8\]. In the literature an indirect method has been used to obtain the propagator \( \langle x, t_2 | y, t_1 \rangle \) in configuration space. Namely, one requires that it satisfies the Schrödinger equation
\[
H_x \langle x, t_2 | y, t_1 \rangle = i\hbar \left( \frac{\partial}{\partial t_2} \right) \langle x, t_2 | y, t_1 \rangle,
\]
where \( H_x \) is the Hamiltonian in the \( x \)-representation, \( \langle x | \hat{H} | x' \rangle = H_x \delta(x-x') \). In this way, DeWitt found that for small \( t_2 - t_1 \), this propagator does not approach Feynman’s result, but an extra term proportional to the scalar curvature is present\[9\]. We shall be able to trace its origin by following the direct calculation step by step. Bastianelli and one of us\[4\] found the extension of DeWitt’s result to fermions. They took a very general ansatz for the action \( S_{\text{conf}} \) in the configuration space path integral, and after evaluating this configuration space path integral by a loop expansion, they were able to determine the propagator by again imposing the Schrödinger equation.

The reader may start wondering at this moment why we use at all the direct but cumbersome method of evaluating
\[
\int dp \langle x | \exp \left( -i \frac{\Delta t}{\hbar} \hat{H} \right) | p \rangle \langle p | y \rangle,
\]
rather than just following the heat kernel approach. We have two reasons. First of all, since Feynman, most physicists derive path integrals as we do, and it is desirable to understand this approach in detail. Secondly, it will help us in determining the action \( S_{\text{conf}} \). In fact, in flat spacetime, one obtains directly \( S_{\text{conf}} \) after integrating out the momenta which are introduced by \( \langle x | \exp \left( -i \frac{\Delta t}{\hbar} \hat{H} \right) | p \rangle \); only in curved spacetime complications set in.

There are two distinct results one is interested in: the propagator and its trace. Let us first consider the case of flat spacetime. Then both results are obtained from a formula which is obtained in the spirit of Feynman by inserting complete sets of \( x \) and \( p \) states or coherent states, but in the former case the propagator can only be written as a phase space path integral provided one adds boundary terms depending on a complex classical trajectory, while for the trace one directly obtains the path integral without the extra boundary terms and without the complex classical trajectory. These boundary terms can be found in textbooks\[5\], whereas the results for the trace formula were used by Alvarez-Gaumé in \[2\]. In section 3 we explain the difference between both approaches. In curved spacetime, as we already mentioned, even for small \( \Delta t \) the propagator may contain terms which cannot be written as local actions, and in these cases the straightforward Hamiltonian approach does not yield the action \( S_{\text{conf}} \). As we already stated, we have a formula for the correct \( S_{\text{conf}} \) in general, and it contains the same kind of boundary conditions as found from the Hamiltonian approach in flat spacetime.

Since it is sometimes claimed that phase space path integrals are less well-defined than configuration space path integrals, and that path integrals for fermions cannot be as rigorously defined as for bosons, and because we were also ourselves confused for a long time, we decided to give a detailed account of the resolution of these problems, starting from scratch in section 2. In particular, we want to stress here that path integrals for fermions are phase space path integrals, and perfectly well defined in terms of coherent states. (The Grassmann integration makes them actually more convergent than their bosonic counter parts). This is very well explained in some textbooks\[5\],

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and we have nothing new to add, but for completeness we also include these aspects in section 5. In reference [5] the Bargmann-Wigner formalism of holomorphic functions is used to define inner products, Hilbert spaces, and only afterwards one tackles quantum mechanics. This we found an unnecessary detour; one can directly define coherent states in quantum mechanics, and use the inner product for these coherent states which follows straightforwardly from the (anti)commutation relations of the annihilation and creation operators.

The fermionic path integrals need Grassmann numbers \( \bar{\eta}^a \) and \( \eta^a \) \((a = 1 \ldots n)\) which are independent. However, the Hamiltonian which corresponds to the regulator \( \bar{D} D \) of spin-\( \frac{1}{2} \) quantum field theory, contains only one kind of fermionic operators, \( \psi^a_1 \). (One identifies the Dirac matrices \( \gamma^a \) with \( \sqrt{2} \psi^a_1(t) \)). This \( \psi^a_1 \) satisfies the equal-time anticommutation relations \( \{ \psi^a_1, \psi^b_1 \} = \delta^{ab} \), but in order to obtain creation and annihilation operators, one needs operators \( \psi^a\dagger \) and \( \psi^a \) satisfying \( \{ \psi^a, \psi^a\dagger \} = \delta^{ab} \) and \( \{ \psi^a, \psi^b \} = \{ \psi^a\dagger, \psi^b\dagger \} = 0 \). It is suggested in the literature that one may double the number of fermion species in the Hamiltonian, and replace the spin connection \( \omega_{iab} \psi^a_1 \psi^b_1 \) by \( \omega_{iab} \psi^a_1 \psi^b_1 \) with \( \alpha = 1, 2 \). Afterwards one is supposed to divide the final results by \( 2^{n/2} \) in order to undo this doubling. This practise is well-known for one-loop problems in quantum field theory[10], but it is incorrect to use it in this context because the one-loop anomalies of a quantum field theory correspond to multiloop corrections of the corresponding quantum mechanical model[3,4].

We resolve this problem by adding a second set of free fermions.

We begin in section 2 with a discussion of phase space path integrals based on \( x \) and \( p \) eigenstates. In section 3 we discuss the bosonic phase space path integrals based on coherent states. Section 4 contains the heart of this paper: a direct evaluation of the propagator through order \( \Delta t \) of bosonic nonlinear sigma models. We use here \( x \) and \( p \) eigenstates. The result leads us to the complex classical trajectories, and the trace formula. In section 5 we study the \( N=2 \) supersymmetric nonlinear sigma model and discuss the relation between operator ordering of the Hamiltonian and the supersymmetry charge on the one hand, and the various supersymmetric or non-supersymmetric actions on the other hand. In section 6 we repeat the analysis of section 4 including fermions and using fermionic coherent states. In both cases, the resulting propagator can be written as a product of three factors: the classical action, the (super) Van Vleck determinant, and a term which involves only the scalar curvature and which is related to the trace anomaly in two dimensions. In section 7 we discuss the \( N=1 \) supersymmetric nonlinear sigma model, in particular the resolution of the problem of fermion doubling alluded to above. Again the propagator can be written as the product of the super Van Vleck determinant, the exponent of the classical action, and a term involving the scalar curvature which determines the trace anomaly of in this case a spin-\( \frac{1}{2} \) field. Section 8 contains the conclusions. In particular, we give here, without proof, the one to one correspondence between \( \hat{H} \) and \( S_{\text{conf}} \) for any, covariant or noncovariant, \( \hat{H} \). In the appendix we perform the 2-loop calculations based on configuration space path integrals, using covariant and noncovariant actions \( S_{\text{conf}} \).
2. Phase space path integrals in general.

Path integrals were (almost) introduced into quantum mechanics by Dirac in 1933[11]. In those days, quantum mechanics was obtained from classical mechanics by replacing the canonical brackets of the Hamiltonian formalism by corresponding quantum commutators. Dirac set out to find a formulation which was based on the Lagrangian in order to be able to retain manifest Lorentz invariance at the quantum level. His starting point were the following relations of classical mechanics between coordinates, conjugate momenta and action

\[
\frac{\partial}{\partial x} S_{cl}(x,y,t) = p(t) \quad ; \quad \frac{\partial}{\partial y} S_{cl}(x,y,t) = -p(0) \tag{2.1}
\]

where \(S_{cl}(x,y,t)\) is the classical action for a point particle as a function of its initial coordinate \(y\) and its final coordinate \(x\). These relations follow immediately from the Euler-Lagrange variational equations for \(\int_0^t dt' L(x(t'), \dot{x}(t'))\) at fixed \(t\). Dirac found a corresponding operator equation in quantum mechanics as follows. Introducing “moving frames” \(^\dagger\) \(|x,t\rangle = \exp \left( -i \hat{H} t \right) |x\rangle\), where \(|x\rangle\) are the eigenstates of the position operator \(\hat{x} = \hat{x}(t=0)\) at time \(t = 0\), and \(|x,t\rangle\) are the eigenstates of the position operator \(\hat{x}(t) = \exp \left( -i \hat{H} t \right) \hat{x} \exp \left( i \hat{H} t \right)\) at time \(t\), he noted that the inner product between these complete sets of states, called “transformation function” by him, satisfies a relation similar to (2.1). Namely, defining a function \(U(x,y,t)\) by

\[
\langle x,t | y,t=0 \rangle = \exp \left( \frac{i}{\hbar} U(x,y,t) \right) \tag{2.2}
\]

one has

\[
\langle x,t | \hat{p}(t) | y,t=0 \rangle = \int dx' \langle x,t | \hat{p}(t) | x',t \rangle \langle x',t | y,t=0 \rangle = -i\hbar \frac{\partial}{\partial x} \langle x,t | y,t=0 \rangle = \left( \frac{\partial}{\partial x} U(x,y,t) \right) \langle x,t | y,t=0 \rangle \tag{2.3}
\]

where \(\hat{p}(t)\) is the momentum operator at time \(t\). A similar relation holds for \(\hat{p}(0)\). Dirac then introduced an operator \(\hat{U}(\hat{x}(t), \hat{x}(0))\) whose matrix elements are the overlap functions in (2.2)

\[
\langle x,t | \hat{U}(\hat{x}(t), \hat{x}(0)) | y,t=0 \rangle = U(x,y,t) \tag{2.4}
\]

He called such operators “well-ordered”, meaning that all \(\hat{x}(t)\) should be put on the left and all \(\hat{x}(0)\) should be put on the right, where they can be replaced by their eigenvalues \(x\) and \(y\), respectively.

\(^\dagger\)These moving frames project a ket \(|\psi\rangle\) onto the corresponding Schrödinger wave function \(\psi(x,t) = \langle x,t | \psi\rangle\), and in this \(x\) representation, \(\hat{\rho}(t)\) is represented on \(\psi(x,t)\) as \(p_x = -i\hbar \frac{\partial}{\partial x}\), as follows from \(\langle x,t | \hat{\rho}(t) | x',t \rangle = -i\hbar \frac{\partial}{\partial x} \delta(x-x')\). The overlap function in (2.2) gives then the probability amplitude for the point particle to be at position \(x\) at time \(t\), if it is initially at time \(t = 0\) at position \(y\). In the nonlinear sigma models we consider we shall use as inner product \(\sqrt{g(x)} \langle x|x'\rangle = \delta(x-x')\) where we define \(\delta(x-x')\) by \(\int dx' f(x') \delta(x-x') = f(x)\). There is then an ambiguity in the representation of \(\hat{\rho}(t)\), namely \(p_x = -i\hbar g^{\alpha \beta} \frac{\partial}{\partial x} g^{-\alpha} \) with \(g = \det g_{ij}(x)\), which we fix by requiring that \(\hat{\rho}(t)\) be hermitian with respect to this inner product. The result is \(\alpha = -1/4\).
Combining (2.3) and (2.4), Dirac obtained operator equations corresponding to (2.1)

\[ \dot{p}(t) = \frac{\partial}{\partial \dot{x}(t)} \hat{U}(\dot{x}(t), \dot{x}(0)) \quad ; \quad \dot{\hat{x}}(t) = -\frac{\partial}{\partial \dot{x}(0)} \hat{U}(\dot{x}(t), \dot{x}(0)) \]  

(2.5)

He then noted that according to the completeness postulate of quantum mechanics one has

\[ \langle x, t | y, t = 0 \rangle = \int dx_1 \ldots dx_{N-1} \langle x, t | x_{N-1}, t_{N-1} \rangle \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \ldots \langle x_1, t_1 | y, t = 0 \rangle \]  

(2.6)

and stated that for fixed \( t_j \) and small \( \hbar \) only stationary points of \( U \) contribute.

The relation (2.5) suggests that the operator \( \hat{U} \) is the kind of quantum mechanical extension of the classical action Dirac was looking for. In fact, he went further and carefully stated that the propagator \( \langle x_{j+1}, t_{j+1} | x_j, t_j \rangle \) “corresponds to” the classical action \( S_{\text{cl}}(x_{j+1}, x_j; t_{j+1} - t_j) \) for small \( t_{j+1} - t_j \). He did not say that they are equal but suggested that they were proportional, in which case he obtained an equivalence principle for small \( \hbar \), provided the stationary trajectories of \( U \) coincide with the classical trajectories of \( S_{\text{cl}} \).

In this article we consider quantum mechanical systems with a kinetic term of the form \( a(x)\dot{p}^2 + b(x)\dot{p} + c(x) \), where \( \dot{x} \) and \( \dot{p} \) have \( n \) components (\( \dot{x}^i \) and \( \dot{p}_i \) with \( i = 1 \ldots n \)), also called nonlinear sigma models. The models we shall consider are point particles, or point spinors coupled to an external gravitational field. An example of the kind of Hamiltonians we study is given by

\[ \hat{H} = \frac{1}{2} g^{-1/4}(\dot{x}) \hat{\pi}_i g^{1/2}(\dot{x}) g^{ij}(\dot{x}) \hat{\pi}_j g^{-1/4}(\dot{x}) - \frac{1}{8} R_{abcd}(\omega(\dot{x}))(\hat{\psi}_a^\dagger \hat{\psi}_b)(\hat{\psi}_b^\dagger \hat{\psi}_a) \]  

(2.7)

where \( g(x) = \det g_{ij}(x) \), \( \alpha \) and \( \beta \) run from 1 to 2, and \( a,b \) and \( i,j \) run from 1 to \( n \). The Riemann tensor is defined in (5.4), while

\[ \hat{\pi}_i = \hat{p}_i + \frac{1}{2\hbar} \omega_{iab}(\dot{x}) \hat{\psi}_a^\dagger \hat{\psi}_b ; \quad \{ \hat{\psi}_a^\dagger, \hat{\psi}_b^\dagger \} = \hbar \delta^{ab} \delta_{\alpha\beta} ; \quad [\hat{\pi}_i, \dot{x}^j] = -i\hbar \delta_i^j \]  

(2.8)

This is the Hamiltonian for \( N=2 \) supersymmetric quantum mechanics. For the \( N=1 \) case \( \hat{\psi}_1^\dagger = \hat{\psi}_1^\dagger = \frac{1}{\sqrt{2}} \hat{\psi}_1 \), and \( \hat{\pi}_i \) can be identified with the covariant derivative \( \partial_i + \frac{i}{2} \omega_{iab} \gamma^a \gamma^b \) if one identifies \( \hat{\psi}_1 \) with \( \sqrt{\hbar} \gamma^a \). In fact, for the \( N=1 \) case, and also for the purely bosonic case, the particular ordering of all operators in \( \hat{H} \) (including the position operator \( \dot{x} \) which we shall write from now on for notational simplicity without hat), is dictated by the \( n \)-dimensional quantum field theory from which they come. We now explain this point; however, readers not interested in quantum field theory may skip the following paragraph and simply restrict their attention to the particular orderings of \( \hat{H} \) which we consider.  

There is a general construction[12] for any quantum field theory which yields the regulator \( \hat{R} \) one must use in order to obtain consistent anomalies. If the Jacobian for an infinitesimal transformation of a classical symmetry is given by \( 1 + J \), the anomaly is given by

\[ A = \lim_{M^2 \to \infty} \text{Tr} \hat{J} \exp \left( \frac{\hat{R}}{M^2} \right) \]  

(2.9)
For a scalar field in $n$ dimensions this construction yields $\hat{R}_s = g^{-1/4} \partial_i g^{1/2} g^{ij} \partial_j g^{-1/4}$ for the consistent regulator which preserves general coordinate invariance. On the other hand, the consistent regulator which preserves Weyl (local scale) invariance but, as a consequence, breaks Einstein (general coordinate) invariance, is given by $\partial_i g^{1/2} g^{ij} \partial_j$. We call these regulators consistent because they lead to anomalies which satisfy the consistency relations. The corresponding quantum mechanical Hamiltonians are obtained by replacing $\partial_i$ by $i\hbar \hat{p}_i$. The matrix elements of $\hat{p}_i$ in the $x$-representation will not be needed (they are given in the footnote above), since all we will use are the commutation relations and the fact that $\hat{p}_i | p \rangle \equiv p | p \rangle$ on eigenfunctions $| p \rangle$. However, for completeness we mention that in the $x$-representation with inner product $(f_1, f_2) = \int dx \sqrt{g} f_1^* f_2$ one must replace $\hat{p}_i$ by $\hbar^{-1/4} \partial_i g^{1/4}$. Then the quantum mechanical operator corresponding to the regulator becomes $g^{-1/2} \partial_i g^{1/2} g^{ij} \partial_j$, which is indeed hermitian with the inner product with $\sqrt{g}$.

For spin-$\frac{1}{2}$ fields $\hat{R}_{fer} = g^{1/4} \not{\partial} \not{D} g^{-1/4}$ which can be written as $\hat{R}_{fer} = \hat{R}_{fer}^0 + \frac{1}{4} R$ where $\hat{R}_{fer}^0$ is the first term on the right hand side of (2.7) and $R$ is the scalar curvature. The Dirac matrices $\gamma^a$ satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and are represented in the corresponding quantum mechanical model by operators $\sqrt{2} \hat{\psi}^a_i$ satisfying $\{\hat{\psi}^a_i, \hat{\psi}^b_j\} = \eta^{ab}$. In this case there is thus no ordering ambiguity in the Hamiltonian. For other applications[2] one may need an $N = 2$ action with $(1,1)$ supersymmetry in which case $\alpha$ and $\beta$ range from 1 to 2. In this case we shall fix the operator orderings in another way, because there is no quantum field theory for which the regulator corresponds to the $N=2$ Hamiltonian. The result is that the last term in the Hamiltonian in (2.7) is then the same (up to a sign) as the 4-fermion term in the classical action. Even though there are no ordering ambiguities in the Hamiltonian which corresponds to $\hat{R}_s$ or $\hat{R}_{fer}$, it remains to be seen whether the corresponding action we will produce in the configuration space path integrals is the invariant classical action.

We shall obtain the anomalies of the corresponding $n$-dimensional quantum field theory from the propagator $\langle x, t | y, t=0 \rangle$ by multiplying with the Jacobian $J(x)$, putting $x = y$, integrating over $x$ and taking the limit $t$ tending to zero, retaining only the finite part. ($t^{-1}$ plays the role of the square of the regulator mass, $M^2$). As a result the classical terms (of order $\hbar^{-1}$) in the propagator as well as the one-loop ($\hbar$-independent) terms will not contribute, and only the two-loop terms of order $\hbar$ will contribute for small $\hbar$ in $n = 2$ dimensions. This is not in disagreement with Dirac’s results, since he only made a statement about the propagator for $\hbar \to 0$ at fixed $t$.

Dirac did not write down an equation relating the propagator $\langle x, t | y, t=0 \rangle$ for small $t$ to the classical action $S_{cl}(x, y; t)$ because he probably did not know the normalization factor between them. Before him, it was already known to some mathematicians (for example Wiener in 1923[13]) that one could define functionals of functions $x(t)$ by time discretization and taking ratios of two such functionals. For example, for the functional $F[V[x]] = \int d\mu \exp \left( \int_0^t V(x)dt \right)$ with $V(x)$ a local smooth function of $x(t)$ such as $x^2(t)$, and $\int d\mu = \int Dx \exp \left( -\frac{1}{2}m \int_0^t \dot{x}^2(t)dt \right)$, the following limit exists and defines the symbol $\int Dx$

$$F[V[x]] = \lim_{N \to \infty} \frac{F[V, N]}{F[0, N]}$$

(2.10)
Here $F[V, N]$ contains the time discretization

$$F[V, N] = \int dx_1 \ldots dx_N \exp \left( \sum_{j=0}^{N-1} \left[ -\frac{1}{2} m (x_{j+1} - x_j)^2 \epsilon^{-1} + V(x_j) \right] \right)$$

(2.11)

with $N \epsilon = t$ and $x_0 = x(t=0) = y$ is fixed. The path integral in the denominator can be viewed as a measure for the path integration. In the Hamiltonian approach, to which we now turn, one derives the measure from first principles. This has the advantage that one need not worry whether the measure in (2.10) is the only reasonable choice of measure.

Path integrals as such, and their normalization, were found by Feynman in 1948[8], who went back to Dirac’s work, but inserted not only the $N - 1$ complete sets of intermediate coordinate eigenstates, but also $N$ complete sets of momentum eigenstates. He considered quantum mechanical systems with a Hamiltonian of the form $\hat{H} = T(\hat{p}) + V(\hat{x})$, and using the relation

$$\exp \left( -\frac{i}{\hbar} H \Delta t \right) = \exp \left( -\frac{i}{\hbar} T \Delta t \right) \exp \left( -\frac{i}{\hbar} V \Delta t \right) + \mathcal{O}(\Delta t)^2$$

(2.12)

he obtained phase space path integrals for a point particle with $n$ components by substituting the following expression into (2.6)

$$\langle x_{j+1}, t_{j+1} | x_j, t_j \rangle = \sum_{p_{j+1}} \langle x_{j+1} | \exp \left( -\frac{i}{\hbar} T \Delta t \right) | p_{j+1} \rangle \langle p_{j+1} | \exp \left( -\frac{i}{\hbar} V \Delta t \right) | x_j \rangle$$

$$= \int \frac{d^n p_{j+1}}{(2\pi \hbar)^n} \exp \left( \frac{i}{\hbar} (x_{j+1} - x_j)|p_{j+1}\rangle \right) \exp \left( -\frac{i}{\hbar} T(p_{j+1}) \Delta t \right) \exp \left( -\frac{i}{\hbar} V(x_j) \Delta t \right)$$

(2.13)

where $\Delta t = t_{j+1} - t_j$. For $T(p) = \frac{1}{2m} p^2$, he could integrate over $p$ and obtain the configuration space path integrals of Dirac, but with their normalization

$$\langle x_{j+1}, t_{j+1} | x_j, t_j \rangle = \left( \frac{m}{2\pi i \hbar \Delta t} \right)^{n/2} \exp \left( \frac{i}{\hbar} S(x_{j+1}, x_j; t_{j+1} - t_j) \right) + \mathcal{O}(\Delta t)^2$$

(2.14)

Here $S = T - V$, but in Euclidean space one may start with $\exp \left( -\frac{i}{\hbar} H \Delta t \right)$ in (2.12) and then one finds $\exp \left( -\frac{i}{\hbar} S_E \right)$ in (2.14) with $S_E = T + V$.

In the actual loop by loop calculations we perform in the appendix, we do not use (2.6) and (2.14), but rather go over to so-called mode-variables in the continuum theory $a_k$. One can carefully follow the steps leading from the discrete-time variables to the variables $a_k$, and then take the limit $N \to \infty$. The result is that one obtains a nontrivial measure $d\mu = (2\pi \hbar)^{-n/2} \prod \left( \frac{\pi k^2}{2 \hbar} \right)^{1/2} da_k$. Using this measure, propagators and vertices follow, and loops can be calculated. For a detailed derivation of this measure see [3,4,14].

Since Feynman, path integrals have taken an enormous flight. There are two approaches: the phase space approach and the configuration space approach. In the phase space approach there always comes a moment where $\langle x_j \rangle \exp \left( -\frac{i}{\hbar} H \Delta t \right) | p_j \rangle$ has to be evaluated. It is our contention that the evaluation of this basic matrix element is always done incorrectly for nonlinear sigma models,
even though for models in flat space the usual procedure gives the correct result. Namely, without exception (to our knowledge), it is always assumed that the following is a correct approximation if one is interested in a result which is correct up to order $\Delta t$

$$\langle x_j | \exp \left( -\frac{i}{\hbar} \hat{H} \Delta t \right) | p_j \rangle = \langle x_j | 1 - \frac{i}{\hbar} \hat{H} \Delta t | p_j \rangle$$  \hspace{1cm} (2.15)$$

We have evaluated this matrix element exactly for nonlinear sigma models, and found, as conjectured by Feynman, that one cannot neglect terms of order $(\Delta t)^k$ in the expansion of the exponent for any $k$. Most of these $\Delta t$ will be re-exponentiated and appear as $-\frac{i}{2m \pi \epsilon} \hbar^2 \Delta t$ in the exponent; they yield the normalization factor in (2.14) after integrating over $p$. However, some terms remain, proportional to the same exponent times a polynomial in $p$. One should really keep all terms in the expansion of the exponent, move all $\hat{p}$ to the right and all $\hat{x}$ to the left, and each time a $\hat{p}$ passes a $\hat{x}$ it produces a factor $-i\hbar$. In terms of rescaled variables $q = p \sqrt{\epsilon/\hbar}$ where $\epsilon = \Delta t$ one then obtains expressions of the form

$$\langle x | \exp \left( -\frac{i}{\hbar} \hat{H} \Delta t \right) | y \rangle = (2\pi \hbar)^{-n} g^{-1/4}(x) g^{-1/4}(y) \left( \frac{\hbar}{\epsilon} \right)^{n/2} \int d^n q \exp \left( \frac{i}{(\epsilon \hbar)^{1/2}} q_j (x^j - y^j) \right) \exp \left( -\frac{i}{2m} q_i g^{ij}(x) q_j \right) \left[ 1 + \sqrt{\epsilon} \hbar A(q, x) + \epsilon \hbar B(q, x) + \ldots \right]$$  \hspace{1cm} (2.16)$$

where $A$ contains one or three factors of $q$ and contains the terms due to one $p, x$ commutator, while $B$ contains zero, two, four or six $q$ momenta and contains the terms due to two $p, x$ commutators. Integration over $q$ replaces $q$ by $m(x - y) / \sqrt{\epsilon} \hbar$, and since $x - y$ is of order $\sqrt{\epsilon}$, we can stop the expansion in (2.16) at $B$ for a two-dimensional quantum field theory. The factors $g^{-1/4}(x)$ and $g^{-1/4}(y)$ come from the inner product

$$\langle x | p \rangle = (2\pi \hbar)^{-n/2} g^{-1/4}(x) \exp \left( \frac{i}{\hbar} p_i x^i \right)$$  \hspace{1cm} (2.17)$$

which is a direct consequence of the orthogonality relations $\sqrt{g(x)} \langle x | x' \rangle = \delta(x - x')$ and $\langle p | p' \rangle = \delta(p - p')$. Putting $A = B = 0$ amounts to the approximation in (2.15), but is is clear from the way we have displayed the further terms in (2.16) that after integration over $q$, the terms with $A$ and $B$ will contribute corrections to the propagator of order $\sqrt{\epsilon} \hbar$ and $\epsilon \hbar$. This last term will give the trace anomaly for $n = 2$. For a quantum field theory in $n_0$ dimensional space (or spacetime), one must determine the expression within square brackets in (2.16) to order $(\epsilon \hbar)^{n_0/2}$, which is cumbersome. In the configuration space approach\[1,2,3,4\] which uses the Schrödinger equation, one must compute “graphs” on the worldline with $n_0$ loops, which is also cumbersome. Only for chiral anomalies where the Grassmann integration over the fermionic zero modes brings in factors of $\epsilon \hbar$ one need only expand the configuration space path integral to second order in quantum fluctuations\[1,2\]. There are then already enough factors of $\hbar$ to ensure that it makes no difference whether one takes the classical action or another action which corresponds to a different operator ordering of the Hamiltonian.
3. Path integrals with coherent states.

In this section we want to discuss phase space path integrals based on coherent states. These are the path integrals one uses for fermions (because for fermions $\psi$ and $\psi^\dagger$ are more like $a$ and $a^\dagger$ then like $p$ and $x$). There are some misconceptions about such path integrals, having to do with the observation that a path connecting a point in phase space to another point in phase space cannot in general be a classical path. We begin by quoting Schulman[15] who in his very readable textbook on path integrals already anticipated the extra terms in (2.16): “... No one (to my knowledge) has made a serious investigation of the neglected terms $\epsilon(z_j - z_{j-1})$ [he refers here to the approximation in (2.15) and the extra terms in (2.16)]. My own guess is that they can contribute and that this contribution will be related to the operator ordering problems in quantum mechanics ...”. We agree with his observation on operator ordering: if the exponent of the Hamiltonian of the nonlinear sigma model would have been well-ordered in Dirac’s sense, no extra terms would have appeared in (2.16), but physics (the $n$-dimensional quantum field theory) has given us another ordering. However, Schulman considers at this point coherent states $|z,t\rangle$ (see below), rather than the eigenstates $|x,t\rangle$ and $|p,t\rangle$ we considered so far, and claims that the extra terms are produced because the curves $z(t)$ are more singular than the Brownian motion paths that enter the usual path integrals (for which $(\Delta x)^2 \sim \Delta t$). We do not agree with him on this point; already in (2.16) which does not use coherent states one encounters the extra terms. And he goes on[15]: “... A well known feature (see Section 31) of the phase space path integral is that paths in phase space are discontinuous so that a term $\epsilon \Delta p$ need not go to zero any faster than $\epsilon$ ...”. Indeed, if $p(t)$ is the time derivative of $x(t)$, and the paths $x(t)$ are continuous but not differentiable, then the paths $p(t)$ will be discontinuous. However, looking up this Section 31, one discovers that he considers the following path integrals

$$\langle x,t|y,t=0 \rangle = \lim_{N \to \infty} \int \frac{dp_Ndx_{N-1}\ldots dx_1dp_1}{(2\pi\hbar)^N} \exp \left( \frac{i}{\hbar} \sum_{j=0}^{N-1} [p_{j+1}(x_{j+1} - x_j) - H(x_j,p_j)] \right)$$

(3.1)

He then states[15]: “... we interpret the formula (3.1) as a sum over paths [correct]. The integration variables suggest that one is adding trajectories in phase space [incorrect in our opinion]... However, ... the fact that a single point in phase space determines the classical path, causes some difficulty [indeed]. Here are two interpretations ...”. And he goes on to consider either nonclassical paths in phase space for which $x_j$ is continuous but $p_j$ jumps at the end of each time segment, or classical paths for which $p(t_j + \frac{1}{2}\Delta t) = p_j$ and $x(t_j + \Delta t) = x_{j+1}$ with $t_{j+1} - t_j = \Delta t$. We claim that in (3.1) one is considering ordinary paths in configuration space, with beginpoint $x_0 = y$ and endpoint $x_N = x$. For coherent states things are more subtle as we now discuss.

We define coherent states for bosonic point particles by

$$|z\rangle = e^{a^\dagger z}|0\rangle \quad ; \quad \langle z^*\rangle = \langle 0|e^{z^*a} \quad ; \quad [a, a^\dagger] = 1$$

(3.2)

where $z^* = (z)^*$ is the complex conjugate of $z$, and $z = (x + ip)/\sqrt{2\hbar}$ while $z^* = (x - ip)/\sqrt{2\hbar}$. (This corresponds to a harmonic oscillator with $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2 = \hbar(a^\dagger a + \frac{1}{2})$. For $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$,}
one must use $\hat{x}\sqrt{m\omega}$ and $\hat{p}/\sqrt{m\omega}$ in these definitions). The vacuum is annihilated by $a$, and one has the following inner product and decomposition of unity

$$\langle w^*|z \rangle = e^{w^*z} ; \quad 1 = \int \frac{dz^*}{2\pi i} |z\rangle e^{-z^*z} \langle z^*|$$  \hspace{1cm} (3.3)

The symbol $\frac{dz^*}{2\pi i}$ always denotes $\frac{dx dp}{2\pi i}$, and we stress that $z$ and $z^*$ are not independent complex variables. The propagator $\langle w^*, t=T|z, t=0 \rangle \equiv \langle w^*| \exp \left(-\frac{i}{\hbar}\hat{H}T \right)|z \rangle$ is then equal to

$$\langle w^*, t=T|z, t=0 \rangle = \int \prod_{j=1}^{N-1} \frac{d\zeta_j}{2\pi i} \langle w^*| \exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right)|\zeta_{N-1} \rangle e^{-\zeta_N z}$$  \hspace{1cm} (3.4)

$$\langle \zeta_{N-1} | \exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right) | \zeta_{N-2} \rangle \ldots e^{-\zeta_1 z} \langle \zeta_1^* | \exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right) | z \rangle$$

with $\Delta t = T/N$. Note that this is an exact result for any $N$. In section 4 we shall compute for small $\Delta t$ the matrix elements of $\exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right)$. Anticipating this calculation, let us define a function $h(z_{j+1}^*, z_j)$ by

$$\langle \zeta_{j+1}^* | \exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right) | \zeta_j \rangle = \exp \left(-\frac{i}{\hbar}h(z_{j+1}^*, z_j)\Delta t \right) \langle \zeta_{j+1}^* | \zeta_j \rangle$$  \hspace{1cm} (3.5)

Then the integrand in the propagator can be written as

$$\exp \left( w^*z_{N-1} - z_{N-1} z_{N-1}^* + z_{N-2}^* z_{N-2} \ldots + z_1^* z - \frac{i}{\hbar} \sum_{j=0}^{N-1} h(z_{j+1}^*, z_j)\Delta t \right)$$  \hspace{1cm} (3.6)

where $z_0 = z$ and $z_N^* = w^*$. This is the formula referred to in the introduction, from which the propagator and its trace are obtained.

For the trace anomaly one should take the trace of the propagator and evaluate

$$\text{Tr} \exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right) = \int \frac{dw^* dz}{2\pi i} e^{-w^*z} \langle w^*, \Delta t|z, t=0 \rangle$$  \hspace{1cm} (3.7)

where $w^*$ must be equated to $(z)^*$. The integration in (3.7) is then again an ordinary Gaussian integral. To prove (3.7), one may either use (3.3) or expand the coherent states into harmonic oscillator eigenfunctions $|n\rangle = (n!)^{-1/2}(a^d)^n|0\rangle$. The result reads then indeed $\sum_n \langle n| \exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right)|n \rangle$. The extra term $e^{-w^*z}$ in (3.7) completes the exponent of (3.6) to an action provided we identify $z = z_0$ with $z_N$, hence $z = z_0 = z_N$

$$\exp \left( \int dt \left[ -z^* \dot{z} - \frac{i}{\hbar}h(z^*, z) \right] \right)$$  \hspace{1cm} (3.8)

This path integral has then periodic boundary conditions. In terms of $x$ and $p$ the kinetic term reduces to the familiar $-\frac{1}{\hbar^2}p^2$.

For the computation of other anomalies one must evaluate the trace of $\langle w^*| \hat{J} \exp \left(-\frac{i}{\hbar}\hat{H}\Delta t \right)|z \rangle$, where $\hat{J}$ is the quantum mechanical operator which corresponds to the Jacobian of an infinitesimal symmetry transformation of the quantum field theory. Depending on
the problem, the presence of $\dot{J}$ can be translated into a change in boundary conditions (periodic, antiperiodic, etc.[2]).

For other problems than anomalies, one may need the propagator itself. In order to still be able to write (3.6) as a path integral, we no longer have the extra exponent due to the trace operation but rather we must add and subtract a suitable term by hand. In principle, any term will do, but for practical calculations it is very useful to decompose $z(t)$ and $z^*(t)$ into a classical part which satisfies the equation of motion, and a quantum part over which one still integrates. Terms linear in the quantum fields then cancel from the action, and the corrections begin with the harmonic (quadratic) approximation, and then higher terms. To achieve this one considers two independent classical trajectories $z_{cl}(t)$ and $w_{cl}^*(t)$, uniquely specified by $z_{cl}(0) = z$ and $w_{cl}^*(t) = w^*$. Of course, the value of $z_{cl}(T)$ at $t = T$ is then not $w$, nor is $w_{cl}^*(t=0)$ equal to $(z)^*$. Rather, to avoid confusion, we keep denoting them by $z_{cl}(T)$ and $w_{cl}^*(0)$. We then complete (3.6) by adding and subtracting a term $w^*z_{cl}(T)$. This yields as path integral

$$e^{w^*z_{cl}(T)} \int \frac{Dz^*Dz}{2\pi \hbar} \exp \left( \int dt \left[-z^*\dot{z} - \frac{i}{\hbar}h(z^*, z) \right] \right)$$

(3.9)

One can also treat the beginpoint and endpoint more symmetrically, by adding one-half the above term, and another term given by $\frac{1}{2}w_{cl}^*(0)z$. In this case one finds the path integral[5]

$$e^{\frac{1}{2}[w^*z_{cl}(T)+w_{cl}^*(0)z]} \int \frac{Dz^*Dz}{2\pi \hbar} \exp \left( \int dt \left[-\frac{1}{2}(z^*\dot{z} - \dot{z}^*z) - \frac{i}{\hbar}h(z^*, z) \right] \right)$$

(3.10)

The boundary conditions are now: $z^*(T) = w^*$, $z^*(0) = w_{cl}^*(0)$, $z(T) = z_{cl}(T), z(0) = z$. In the action one is then to replace $z(t)$ by $z_{cl}(t) + z_{qu}(t)$ and similarly $z^*(t)$ by $w_{cl}^*(t) + z_{qu}^*(t)$. The quantum fields vanish at the boundaries

$$z_{qu}(0) = z_{qu}(T) = z_{qu}^*(0) = z_{qu}^*(T) = 0$$

(3.11)

and we still have $z = (x + ip)/\sqrt{2\hbar}, z^* = (x - ip)/\sqrt{2\hbar}$, with real $x$ and $p$. The measure $\frac{Dz^*Dz}{2\pi \hbar}$ defines an ordinary Gaussian integral. One could define a classical $p$ and $x$ by

$$w_{cl}^*(t) = (x_{cl}(t) - ip_{cl}(t))/\sqrt{2\hbar} ; \quad z_{cl}(t) = (x_{cl}(t) + ip_{cl}(t))/\sqrt{2\hbar}$$

(3.12)

In that case, $x_{cl}(t) = (w_{cl}^*(t) + z_{cl}(t))\sqrt{\hbar}/2$ and in particular $x_{cl}(t=0), x_{cl}(t=T)$ are complex (because $w_{cl}^*(t=0) \neq z^*$). Similarly for $p_{cl}(t)$. There is no problem with classical paths connecting two points in phase space because we have here two classical paths which do not connect $w^*$ and $z$, but one of which starts at $z$ and ends wherever the dynamics may take it, while the other starts at $w^*$ and ends somewhere else. No discontinuity or jumps of paths need be considered.

To evaluate (3.10) for the harmonic oscillator is trivial [5]. Terms linear in quantum fields cancel whereas the terms quadratic in quantum fields are independent of the classical fields and yield an overall constant, so that the classical factor in front of the integral in (3.10) contains the whole result up to a constant. This constant can be fixed by considering the limit $T \to 0$, or by doing the path integral over $z$ and $z^*$ explicitly. For more complicated systems, one can
evaluate the path integral by expanding the action in terms of classical and quantum fields, but then \( z_{\text{qu}}(t) \) is not an independent complex variable, but rather expressed in terms of \( (z_{\text{qu}}(t))^* \) by \( z_{\text{qu}}^*(t) = (z_{\text{cl}}(t))^* - w_{\text{cl}}^*(t) + (z_{\text{qu}}(t))^* \). We can define \( x_{\text{qu}} \) and \( p_{\text{qu}} \) by \( z_{\text{qu}} = (x_{\text{qu}} + ip_{\text{qu}})/\sqrt{2\hbar} \) and \( z_{\text{qu}}^* = (x_{\text{qu}} - ip_{\text{qu}})/\sqrt{2\hbar} \), but then also \( x_{\text{qu}}(t) \) and \( p_{\text{qu}}(t) \) are complex. Using

\[
x_{\text{qu}}(t) = x(t) + \frac{1}{2} (z_{\text{cl}}(t) + w_{\text{cl}}^*(t)) \quad ; \quad p_{\text{qu}}(t) = p(t) + \frac{1}{2i} (z_{\text{cl}}(t) - w_{\text{cl}}^*(t))
\] (3.13)

it is clear that the Jacobian for the transformation of the integration variables \( x(t) \) and \( p(t) \) to \( x_{\text{qu}}(t) \) and \( p_{\text{qu}}(t) \) is unity, so that the measure becomes

\[
Dx_{\text{qu}}Dp_{\text{qu}} ; \quad -\infty \leq x_{\text{qu}} \leq \infty, \quad -\infty \leq p_{\text{qu}} \leq \infty
\] (3.15)

For fermions we consider operators \( \hat{\psi}^\dagger \) and \( \hat{\psi} \) satisfying \( \{ \hat{\psi}, \hat{\psi}^\dagger \} = 1 \), and \textit{independent} Grassmann variables \( \bar{\eta} \) and \( \eta \). This will force us to double the number of fermionic operators, from \( \hat{\psi}_1^a \) to \( \hat{\psi}_a^\dagger \) as in (2.7). The operators \( \hat{\psi}^a \) and \( \hat{\psi}^a\dagger \) are defined by \( (\hat{\psi}_1^a + i\hat{\psi}_2^a)/\sqrt{2} \) and \( (\hat{\psi}_1^a - i\hat{\psi}_2^a)/\sqrt{2} \), respectively. We continue in this section with the case \( n = 1 \), omitting the superscript \( a \). The coherent states are now defined by

\[
|\eta\rangle = e^{\hat{\psi}^\dagger \eta}|0\rangle ; \quad \langle \bar{\eta}| = \langle 0|e^{\bar{\eta} \hat{\psi}}
\] (3.16)

satisfying \( \hat{\psi}|\eta\rangle = \eta|\eta\rangle \) and \( \langle \bar{\eta}| \hat{\psi}^\dagger = \langle \bar{\eta}|\bar{\eta} \). The inner product and decomposition of unity read formally the same as in the bosonic case

\[
\langle \bar{\eta}|\xi\rangle = e^{\bar{\eta}\xi} ; \quad 1 = \int d\bar{\eta}d\xi \ |\xi\rangle e^{-\bar{\eta}\xi} \langle \bar{\eta}|
\] (3.17)

but now the ordering of \( d\bar{\eta} \) and \( d\xi \) as well as the ordering of \( \bar{\eta} \) and \( \xi \) in \( \bar{\eta}\xi \) is important, while the factor \( (2\pi i)^{-1} \) is absent from the measure because we now have Grassmann integration according to which \( \int d\xi = \int d\bar{\eta} \bar{\eta} = 1 \). The results in (3.17) follow directly by expanding the exponent; one finds \( 1 = |0\rangle\langle 0| + |1\rangle\langle 1| \) with \( \hat{\psi}^\dagger|0\rangle = |1\rangle \), which is evidently correct. We can repeat the same steps as before, leading to (3.6). The propagator is then given by

\[
e^{\bar{\eta}\xi cl(T)} \int D\bar{\eta}D\xi \exp \left( \int_0^T dt \left[ -\bar{\eta}\dot{\xi} - \frac{i}{\hbar} h(\bar{\eta},\xi) \right] \right)
\] (3.18)
or, in the symmetric case, by

\[ e^{\frac{i}{\hbar}[\bar{\eta}\xi(T)+\bar{\eta}(0)]} \int D\bar{\eta} D\xi \exp \left( \int_0^T dt \left[ -\frac{1}{2}(\bar{\eta}\dot{\xi} - \dot{\bar{\eta}}\xi) - \frac{i}{\hbar}h(\bar{\eta},\xi) \right] \right) \quad (3.19) \]

Again we obtain the well known extra boundary terms in these path integrals with fermionic coherent states[5]. On the other hand, the trace formula is given by

\[ \text{Tr} \hat{A} = \langle 0|\hat{A}|0 \rangle + \langle 1|\hat{A}|1 \rangle = \int d\xi d\bar{\eta} e^{i\bar{\eta}\xi} \langle \bar{\eta}|\hat{A}|\xi \rangle \quad (3.20) \]

where now \( \xi \) and \( \bar{\eta} \) are independent variables. The proof follows directly from Grassmann integration and shows that \( d\xi \) stands to the left of \( d\bar{\eta} \). The trace of the propagator becomes then

\[ \int d\xi d\bar{\eta} \prod_{j=1}^{N-1} d\bar{\eta}_j d\xi_j \exp \left( \bar{\eta}\xi + \bar{\eta}_j \xi_{N-1} - \bar{\eta}_{N-1} \xi_{N-1} \ldots - \bar{\eta}_1 \xi_1 + \bar{\eta}_1 \xi_1 - \frac{\bar{\eta}}{\hbar} \sum_{j=0}^{N-1} h(\bar{\eta}_{j+1},\xi_j)\Delta t \right) \quad (3.21) \]

This can be interpreted as a phase space path integral with \( \int dt (-\bar{\eta}\dot{\xi} - \frac{i}{\hbar}h) \), provided one interprets \( \xi(t=0) \equiv \xi_0 = -\xi(t=T) \). By putting the term \( \bar{\eta}\xi \) in the exponent to the far right, one again finds a path integral, now with \( \int dt (\bar{\eta}\xi - \frac{i}{\hbar}h) \), provided \( \bar{\eta}(t=0) \equiv -\bar{\eta} = -\bar{\eta}(t=T) \). Taking the symmetric case, we recover (3.8), but now with antiperiodic boundary conditions for \( \xi \) and \( \bar{\eta} \). For other anomalies with \( \hat{J} \) not equal to unity, the boundary conditions on \( \xi \) and \( \bar{\eta} \) may be different. For example, for chiral anomalies one obtains periodic boundary conditions for \( \xi \) and \( \bar{\eta} \) [2].

4. Transition amplitude for the bosonic case.

We will consider the matrix element

\[ \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | p \rangle \quad (4.1) \]

with

\[ \hat{H} = \frac{1}{2}g^{-1/4} \hat{p}_i g^{1/2} g^{ij} \hat{p}_j g^{-1/4} - \frac{1}{2} \xi h^2 R \quad (4.2) \]

We are working from now on in Euclidean space; to obtain the results for Minkowski spacetime one should replace \( \epsilon \) by \( ie \). For \( \xi = \frac{n-2}{2n-1} \), \( \hat{H} \) is the regulator for a classically conformal invariant scalar quantum field theory in \( n \) dimensions[3]. Expanding the exponent in (4.1), we define

\[ \langle x | \left( \hat{H} \right)^k | p \rangle = \sum_{l=0}^{2k} A_l^k(x) p^l \langle x | p \rangle \quad (4.3) \]

where \( A_l^k(x) \) is a \( c \)-number function and \( p^l \) denotes a homogeneous polynomial of order \( l \) in the momenta.
In order to compute the transition amplitude \( \langle x | \exp \left( -\frac{i}{\hbar} \hat{H} \right) | y \rangle \) to order \( \epsilon \) compared to the leading terms, it will turn out that we only need terms on the right hand side of (4.3) with \( l = 2k \), \( 2k - 1 \), and \( 2k - 2 \). We find, defining \( p^2 = g^{ij}(x)p_i p_j \),

\[
A_{2k}^k(x)p^{2k} = \left( \frac{1}{\epsilon p^2} \right)^k
\]

(4.4)

Since this is the term containing the maximal number of \( \hat{p} \)'s, it can be easily computed because all \( \hat{p} \) operators are just replaced by the corresponding \( c \)-numbers when acting on \( |p\rangle \).

The next term is

\[
A_{2k-1}^k(x)p^{2k-1} = -i\hbar k \left( \frac{1}{2} p^2 \right)^{k-1} \left( \partial_i g^{ij} \right) p_j \\
- i\hbar \left( \frac{1}{2} p^2 \right)^{k-2} \frac{1}{2} g^{ij} \left( \partial_j g^{kl} \right) p_j p_k p_l
\]

(4.5)

In this expression one of the \( \hat{p} \)'s acts as a derivative, whereas the other \( 2k - 1 \) are replaced by the corresponding \( c \)-numbers. The first term in (4.5) comes about when the derivative acts within the same factor \( \hat{H} \) in which it appears, and is multiplied by \( k \) since there are \( k \) factors of \( \hat{H} \). The second term arises if this derivative acts on a different factor of \( \hat{H} \). For this to occur there are \( \binom{k}{2} \) possible combinations, and taking into account that there are two \( \hat{p} \)'s in each factor of \( \hat{H} \) we get an extra factor 2. Notice that in both cases the terms involving a derivative acting on \( g \) cancel.

The last term we have to calculate is

\[
A_{2k-2}^k(x)p^{2k-2} = \\
\hbar^2 k \left( \frac{1}{2} p^2 \right)^{k-1} \left[ \frac{1}{32} g^{ij} \left( \partial_i \log g \right) \left( \partial_j \log g \right) + \frac{1}{8} g^{ij} \left( \partial_i \partial_j \log g \right) + \frac{1}{8} \left( \partial_i g^{ij} \right) \left( \partial_j \log g \right) \right] \\
- \hbar^2 \left( \frac{k}{2} \right) \left( \frac{1}{2} p^2 \right)^{k-2} \left[ \frac{1}{2} g^{ij} \left( \partial_i \partial_k g^{kl} \right) + \frac{1}{4} \left( \partial_i g^{ij} \right) \left( \partial_k g^{kl} \right) \right. \\
\left. + \frac{1}{4} \left( \partial_i g^{ik} \right) \left( \partial_k g^{il} \right) + \frac{1}{4} g^{ik} \left( \partial_i \partial_k g^{il} \right) \right] p_j p_l \\
- \hbar^2 \left( \frac{k}{3} \right) \left( \frac{1}{2} p^2 \right)^{k-3} \left[ \frac{1}{2} g^{ik} g^{jl} \left( \partial_i \partial_j g^{mn} \right) + \frac{3}{4} g^{im} \left( \partial_i g^{kl} \right) \left( \partial_j g^{jn} \right) \right. \\
\left. + \frac{1}{4} g^{il} \left( \partial_j g^{ik} \right) \left( \partial_i g^{mn} \right) + \frac{1}{4} g^{ij} \left( \partial_i g^{kl} \right) \left( \partial_j g^{mn} \right) \right] p_k p_l p_m p_n \\
- \hbar^2 \left( \frac{k}{4} \right) \left( \frac{1}{2} p^2 \right)^{k-4} \left[ \frac{3}{4} g^{ij} g^{mn} \left( \partial_i g^{kl} \right) \left( \partial_n g^{pq} \right) \right] p_j p_k p_l p_m p_n p_p p_q - \frac{1}{2} \xi \hbar^2 k \left( \frac{1}{2} p^2 \right)^{k-1} R
\]

(4.6)

In this expression two of the \( \hat{p} \)'s act as derivatives, except for the last term which is already of order \( p^{2k-2} \).

The first set of terms appears when both derivatives act within the same factor \( \hat{H} \); again there are \( k \) terms of this kind.

The next set of terms arises when only two of the factors \( \hat{H} \) play a rôle. There are four possibilities: (i) one \( \hat{p} \) from the left factor acts on the right factor, while another \( \hat{p} \) from the right
factor acts within the right factor, (ii) the first $\hat{p}$ acts within the first $\hat{H}$, while the second $\hat{p}$ acts within the second $\hat{H}$, (iii) both $\hat{p}$’s come from the left $\hat{H}$, but one of them acts inside the left $\hat{H}$ while the other acts on the right $\hat{H}$, and (iv) both $\hat{p}$’s from the left $\hat{H}$ act on the right $\hat{H}$. In all cases it is easy to see that again the derivatives on $g$ cancel.

The following set of terms comes from combinations using three factors $\hat{H}$, hence its overall factor $\left(\frac{\epsilon}{\hbar}\right)^k$. There are again four cases: (i) a $\hat{p}$ from the first $\hat{H}$ and a $\hat{p}$ from the second $\hat{H}$ hit the third $\hat{H}$, (ii) one $\hat{p}$ acts inside the factor $\hat{H}$ in which it appears whereas a $\hat{p}$ from another $\hat{H}$ hits the remaining $\hat{H}$ (there are 3 terms of this kind), (iii) a $\hat{p}$ from the first $\hat{H}$ hits the second $\hat{H}$, and a $\hat{p}$ from the second $\hat{H}$ hits the third $\hat{H}$, and (iv) of the two $\hat{p}$’s from the first $\hat{H}$ one acts on the second, and one on the third $\hat{H}$.

Finally, the term with $\left(\frac{\epsilon}{\hbar}\right)^k$ involves four factors $\hat{H}$, such that one $\hat{p}$ from one $\hat{H}$ hits another $\hat{H}$, and the other $\hat{p}$ from one of the remaining factors $\hat{H}$ hits the last $\hat{H}$.

The reason further terms do not contribute can be most easily seen if we rescale $q = \sqrt{\pi} p$. Then the transition amplitude becomes

$$
\langle x | \exp \left( - \frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = \int d^n p \langle x | \exp \left( - \frac{\epsilon}{\hbar} \hat{H} \right) | p \rangle \langle p | y \rangle
$$

$$
= g^{-1/4}(x)g^{-1/4}(y)(2\pi\hbar)^{-n/2} \left( \frac{\hbar}{\epsilon} \right)^{n/2} \int d^n q \exp \left( i \frac{q_i(x - y)^i}{\sqrt{\epsilon\hbar}} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{\epsilon}{\hbar} \right)^k \sum_{l=0}^{2k} A_l^k(x) q^l \left( \frac{\epsilon}{\hbar} \right)^{-1/2}
$$

(4.7)

where we have used (2.17). So only the $A_{2k-1}^k$ and $A_{2k-2}^k$ terms contribute through order $\epsilon$ compared to the leading term $A_{2k}^k$.

The sum over $k$ in (4.7) can be performed, leading to

$$
\langle x | \exp \left( - \frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = g^{-1/4}(x)g^{-1/4}(y) (4\pi^2 \hbar \epsilon)^{-n/2} \int d^n q \exp \left( - \frac{1}{8} g^{ij} q_i q_j + i \frac{q_i(x - y)^i}{\sqrt{\epsilon\hbar}} \right)
$$

$$
\left[ 1 + i \sqrt{\epsilon\hbar} \left\{ \frac{1}{2} \left( \partial_{ij} g^{ijkl} \right) q_j - \frac{1}{4} g^{ij} \left( \partial_{ik} g^{kl} \right) q_j q_k q_l \right\} + \epsilon \hbar \left\{ \frac{1}{2} \xi R - \frac{1}{32} g^{ij} \left( \partial_{ij} \log g \right) \left( \partial_{ij} \log g \right) - \frac{1}{8} g^{ij} \left( \partial_{ij} \partial_{ij} \log g \right) - \frac{1}{8} \left( \partial_{ij} g^{ij} \right) \left( \partial_{ij} \log g \right) \right. \right.
$$

$$
\left. - \frac{1}{4} g^{ij} \left( \partial_{ij} g^{kl} \right) + \frac{1}{8} \left( \partial_{ij} g^{kl} \right) \left( \partial_{ij} g^{kl} \right) + \frac{1}{8} \left( \partial_{ij} g^{kl} \right) \left( \partial_{ij} g^{kl} \right) + \frac{1}{8} \left( \partial_{ij} g^{kl} \right) \left( \partial_{ij} g^{kl} \right) \right] q_j q_l + O(\epsilon^{3/2}) \right]
$$

(4.8)

We can now complete the square in the exponent and integrate out the momenta $q_i$, since the integral becomes just a sum of Gaussian integrals which can easily be evaluated. The problem is
now to factorize the result such that it is manifestly a scalar both in $x$ and in $y$ (a ‘bi-scalar’) under general coordinate transformations. We expect, of course, to find at least the classical action integrated along a geodesic (see (4.11)). In the expansion of this functional around $x(0) = x$, we recognize many of the terms in (4.8). However, there are terms left over. They combine into $R$ or $R_{ij}$, while expansion of $g(y)$ yields terms with $\partial \log g$ or derivatives thereof. With this in mind, we write the result in a factorized form, where in one factor we put all terms which possibly can come from expanding some power of $g(y)$, while into another factor we put the expanded action and curvature terms. It is quite nontrivial, and an excellent check on the results obtained so far, that this is at all possible. The resulting expression is

$$
\langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = (2\pi \hbar \epsilon)^{-n/2} g^{-1/4}(x) g^{-1/4}(y) \left[ g^{1/2}(x) + g^{1/4}(y-x)^i \left( \partial_i g^{1/4}(x) + \frac{1}{2} g^{1/4}(x)(y-x)^j \left( \partial_j \partial_j g^{1/4}(x) \right) \right) \right] \exp \left( -\frac{1}{2\hbar} g_{ij}(x)(y-x)^i(y-x)^j \right) \left[ 1 - \frac{1}{4\hbar} \left( \partial_k g_{ij}(x) \right) (y-x)^i(y-x)^j(y-x)^k + \frac{1}{2} \left( \frac{1}{4\hbar} \left( \partial_k g_{ij}(x) \right) (y-x)^i(y-x)^j(y-x)^k \right)^2 \right. \\
- \frac{1}{12\hbar} \left( \partial_k \partial_i g_{ij}(x) - \frac{1}{2} g_{mn}(x) \Gamma^m_{ij}(x) \Gamma^n_{kl}(x) \right) (y-x)^i(y-x)^j(y-x)^k(y-x)^l + \left( \frac{1}{2} \epsilon - \frac{1}{12} \right) \hbar R(x) - \frac{1}{12} R_{ij}(x)(y-x)^i(y-x)^j + \mathcal{O}(\epsilon^{3/2}) \right] (4.9)
$$

where the Ricci tensor is defined by $R_{ij} = R^k_{\ ijk}$ and $R_{ijkl}$ is given in (5.4). The terms within the first pair of square brackets are, through order $\epsilon$, equal to $g^{1/4}(x)g^{1/4}(y)$ and cancel the factors $g^{-1/4}(x)g^{-1/4}(y)$ in front of the whole expression. Note that from the term in the exponent in (4.9) it follows that the difference $(y-x)$ is of order $\sqrt{\epsilon}$, thus one indeed finds this expansion through order $\epsilon$. Similarly the terms within the second pair of square brackets are of order $\epsilon$ or less. The terms with $\partial_k g_{ij}$ and its square are the first two terms in the expansion of an exponent. This suggests to exponentiate all terms, yielding

$$
\langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = (2\pi \hbar \epsilon)^{-n/2} \exp \left\{ -\frac{\epsilon}{\hbar} \left[ \frac{1}{2} g_{ij}(x) + \frac{1}{4} \partial_k g_{ij}(x)(y-x)^k \right. \right. \\
+ \frac{1}{12} \left( \partial_k \partial_i g_{ij}(x) - \frac{1}{2} g_{mn}(x) \Gamma^m_{ij}(x) \Gamma^n_{kl}(x) \right) (y-x)^k(y-x)^l \left( \frac{y-x)^i(y-x)^j}{\epsilon} \right) \left( \frac{y-x)^i(y-x)^j}{\epsilon} \right) \left. \right. \\
+ \hbar \left( \frac{1}{2} \epsilon - \frac{1}{12} \right) R(x) - \frac{1}{12} R_{ij}(x)(y-x)^i(y-x)^j + \mathcal{O}(\epsilon^{3/2}) \right\} (4.10)
$$

All terms in the exponent except the last two just correspond to an expansion around $x$ of the classical action, which is equal to the integral along the geodesic joining $x$ and $y$ of the invariant
line element (cf. [9])

\[ S_{\text{cl}}(x, y; \epsilon) = \int_{t=-\epsilon}^{t=0} dt \frac{1}{2} g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t) \]

\[ = \frac{1}{\epsilon} \left[ \frac{1}{2} g_{ij}(x)(y-x)^i (y-x)^j + \frac{1}{4} \partial_k g_{ij}(x)(y-x)^i (y-x)^j (y-x)^k 
+ \frac{1}{12} \left( \partial_k \partial_l g_{ij}(x) - \frac{1}{2} g_{mn}(x) \Gamma^m_{ij}(x) \Gamma^n_{kl}(x) \right)(y-x)^i (y-x)^j (y-x)^k (y-x)^l \right] + \frac{1}{\epsilon} \mathcal{O}(y-x)^5 \]

(4.11)

when boundary conditions \( x(-\epsilon) = y, \ x(0) = x \) are imposed.

Our final result for the transition amplitude can thus be written as

\[
\langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = (2\pi \hbar \epsilon)^{-n/2} \exp \left( -\frac{1}{\hbar} S_{\text{cl}} \right) \left[ \frac{1}{2} \epsilon \hbar \left( \frac{1}{2} \epsilon - \frac{1}{12} \right) \left( R(x) + R(y) \right) \right.
- \frac{1}{24} \left( R_{ij}(x) + R_{ij}(y) \right) (x-y)^i (x-y)^j + \mathcal{O}(\epsilon^{3/2}) \right]
\]

(4.12)

This shows that (4.10) is, to the order we are expanding, symmetric under the exchange of \( x \) and \( y \). One may check that it satisfies the composition rule

\[
\int dz \sqrt{g(z)} \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | z \rangle \langle z | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = \langle x | \exp \left( -\frac{2\epsilon}{\hbar} \hat{H} \right) | y \rangle
\]

(4.13)

A quick way to check this is to use normal coordinates, in which case one only needs to retain the leading term in the classical action (since \( \partial_i g_{jk} = \partial_i \partial_j g_{kl} = 0 \)), while \( g_{ij}(z) = g_{ij}(x) + \frac{1}{3} R_{kijl}(x)(z-x)^k(z-x)^l \) through order \( \epsilon \). Taking the opposite point of view, we can impose (4.13) as a consistency condition on the amplitude. This fixes the higher order terms in the expansion of (4.12), with exception of the terms proportional to the scalar curvature, which are related to the trace anomaly. One should expect this to be the case since we can always add such a term to the Lagrangian and interpret it as an external potential, which should not be fixed by the requirement (4.13).

The transition element can of course also be evaluated using path integral methods. One should be careful however in choosing the correct action in the configuration space path integral, since our Hamiltonian (4.2) is not Weyl-ordered[16] or time-ordered[7]. The corresponding Weyl-ordered or time-ordered Hamiltonian is (see e.g. [17,7]), dropping a noncovariant term proportional to the product of two Christoffel symbols for reasons given in the conclusions,

\[ \hat{H}_{\text{Weyl}} = \hat{H} - \frac{1}{8} \hbar^2 R \]

(4.14)

Since it is \( \hat{H}_{\text{Weyl}} \) that corresponds to the classical action in the path integral, the (Euclidean) action that corresponds to \( \hat{H} \) in (4.2) is \( S_{\text{conf}} = S_{\text{cl}} + \frac{1}{8} \hbar^2 \int dt R \), in agreement with the results in [3].

We have obtained the result for \( \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle \) from a direct computation; the extra \( R \) and \( R_{ij} \) terms are due to ordering the \( \hat{p} \) operators in all terms of the expansion of \( \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) \). The terms with \( R_{ij} \) can be expressed in terms of the classical action as follows. By using

\[ D^{1/2}(x, y; \epsilon) = e^{-\epsilon/2} g^{1/4}(x) g^{1/4}(y) \left[ 1 - \frac{1}{12} R_{ij}(y-x)^i (y-x)^j \right] \]

(4.15)
where \( D(x, y; \epsilon) = \det D_{ij}(x, y; \epsilon) \) is the Van Vleck determinant \([18,19]\)
\[
D_{ij}(x, y; \epsilon) \equiv -\frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} S_{\text{cl}}(x, y; \epsilon) \tag{4.16}
\]
we can rewrite (4.12) as
\[
\langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = (2\pi \hbar)^{-n/2} \tilde{D}^{1/2} \exp \left( -\frac{1}{\hbar} S_{\text{cl}} \right) \left[ 1 + \frac{1}{2} \epsilon \hbar \left( \frac{1}{12} - \frac{1}{2} \right) (R(x) + R(y)) \right] \tag{4.17}
\]
where \( \tilde{D} = g^{-1/2}(x)D(x, y; \epsilon)g^{-1/2}(y) \) and we have neglected terms of higher order in \( \epsilon \). Note that \( \tilde{D} \) as well as \( S_{\text{cl}}(x, y; \epsilon) \) transform as biscalars under general coordinate transformations (\( D \) itself is a bi-density \([20]\)), hence also the infinitesimal transition amplitude \( \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle \) does not depend on the coordinates chosen. Obviously this result is not affected by the terms proportional to the scalar curvature.

The Van Vleck determinant gives the harmonic approximation of the quantum terms in the path integrals\([15]\). For example, the Feynman factor \((\frac{m}{\hbar})^n \)^{1/2} in (2.14) is a Van Vleck determinant. Since \( \frac{\partial}{\partial y^i} S_{\text{cl}}(x, y; \epsilon) = -p_j \) with \( p_j \) the momentum conjugate to \( y^j \), \( D \) is the Jacobian for the transition of the phase space measure \( dz_2 dp_1 dx_1 \) to the configuration space measure \( dx_2 dy dx_1 \) where \( y \) divides the interval \( (x_1, x_2) \) into two smaller intervals of size \( \frac{1}{2} \Delta t \). To the order we working in, \( \tilde{D} \) for the half-interval is equivalent to \( \tilde{D}^{1/2} \) over the whole interval and reproduces the \( R_{ij} \) terms. One might think that one could explain the \( R_{ij} \) terms in the propagator by taking as \( \exp(S_{\text{conf}}) \) the expression \( \tilde{D}^{1/2} \exp(S_{\text{cl}}) \). However, this is incorrect because the path integral for this \( S_{\text{conf}} \) produces its own \( R_{ij} \) terms, on top of those contained in \( \tilde{D} \), as was already noted in \([3]\).

We conclude that the final result can be written as the product of three factors: one term which is only related to the trace anomaly, the exponent of the classical action, and the square root of the Van Vleck determinant.

As one would expect, the transition amplitude is the Green function associated with the Schrödinger equation (diffusion equation for Euclidean time)
\[
\left[ \hat{H}(y) + \hbar \frac{\partial}{\partial \epsilon} \right] \theta(\epsilon) \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = \hbar g^{-1/2}(x) \delta(\epsilon) \delta^n(x - y) \tag{4.18}
\]
where
\[
\hat{H}(y) = -\frac{1}{2} \hbar^2 g^{-1/2}(y) \frac{\partial}{\partial y^i} g^{1/2}(y) g^{ij}(y) \frac{\partial}{\partial y^j} - \frac{1}{2} \xi \hbar^2 R(y) \tag{4.19}
\]
To prove (4.18) we begin with the identity
\[
-\hbar \frac{\partial}{\partial \epsilon} \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = \int dz \sqrt{g(z)} \langle x | \hat{H} | z \rangle \langle z | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) y \rangle = \hat{H}(x) \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) y \rangle \tag{4.20}
\]
If we would expand the propagator around \( y \), and act on it with \( \hat{H}(x) \) as in (4.20), we obtain the same result as acting with \( \hat{H}(y) \) on our propagator (which is expanded around \( x \)), because the right hand side of (4.18) is symmetric in \( x \) and \( y \). Using the footnote in section 2 and (4.18) we may check (4.10).

We can now simply compute the anomaly in \( n = 2 \) by taking the trace in (4.12) and singling out the \( \epsilon \) independent part. The result is\([3,4]\)
\[
\mathcal{A}_{2, \text{scalar}} = -\frac{\hbar}{24\pi} R \tag{4.21}
\]
5. The classical and quantum Hamiltonians for the $N=2$ supersymmetric nonlinear sigma model.

As an example of a case where the operator orderings in the Hamiltonian are unambiguously fixed by requiring that the corresponding configuration space path integral has the original classical action in its exponent, consider $N=2$ supersymmetric quantum mechanics. The classical action in Minkowski time is given by

$$L = \frac{1}{2} g_{ij}(\phi) \dot{\phi}^i \dot{\phi}^j + \frac{i}{2} g_{ij}(\phi) \chi^i \chi_j \partial_t \chi^j + \frac{1}{8} R_{ijkl}(\Gamma)(\chi^i \chi^j)(\chi^k \chi^l)$$

$$D_t \chi^i = \partial_t \chi^i + \dot{\phi}^j \Gamma_{i}^{jkl} \chi^l ; \quad \alpha, \beta = 1, 2$$

and is invariant under the following supersymmetry transformations

$$\delta \phi^i = i \left( \epsilon_1 \chi^1_i - \epsilon_2 \chi^2_i \right) = \epsilon^T \sigma_2 \chi^i$$

$$\delta \chi^i = -i \dot{\phi}^j \sigma_2 \epsilon - \Gamma^i_{jk} \delta \phi^j \chi^k$$

This action can either be obtained by dimensional reduction of the $N=(1,1)$ supersymmetric nonlinear $\sigma$ model from $1+1$ dimensions to one time dimension\cite{1}, or directly by requiring invariance under $(1,1)$ supersymmetry transformations with parameter $\epsilon_\alpha$. One finds either way the classical supersymmetry transformations, and the classical supersymmetry charge. The latter is given by

$$Q^{(cl)}_{\alpha} = g_{ij}(\phi) \chi^i_{\alpha} \dot{\phi}^j$$

To define the corresponding quantum generator, it is helpful to introduce fermions with tangent-space indices, because their brackets are field-independent. Therefore define $\psi^a_\alpha = e^a_i(\phi) \chi^i_{\alpha}$ where $e^a_i e^b_j \delta_{ab} = g_{ij}$. Then the action in Minkowski time becomes

$$L = \frac{1}{2} g_{ij}(\phi) \dot{\phi}^i \dot{\phi}^j + \frac{i}{2} \psi^a_\alpha \partial_t \psi^a_\alpha + \frac{1}{8} R_{abcd}(\omega)(\psi^a_\alpha \psi^b_\beta)(\psi^c_\beta \psi^d_\alpha)$$

$$D_t \psi^a_\alpha = \partial_t \psi^a_\alpha + \dot{\phi}^j \omega^a_{j b} \psi^b_\alpha$$

$$R_{ijab}(\omega) = \partial_i \omega_{j ab} + \omega_{ia} \omega_{j cb} - (i \leftrightarrow j) = R_{ijkl}(\Gamma) e^a_i e^b_j$$

$$\partial_i e^a_i + \Gamma^a_{ik} e^k_a + \omega_{ia} e^i_b = 0$$

The supersymmetry charge becomes

$$Q^{(cl)}_{\alpha} = e_{ia}(\phi) \psi^a_\alpha \dot{\phi}^i$$

So far, this is standard supergravity.

To quantize this system, we first observe that there are second class constraints

$$C^a_\alpha = p(\psi^a_\alpha) + \frac{i}{2} \psi^a_\alpha$$

where

$$p(\psi^a_\alpha) = \partial_t \psi^a_\alpha + \dot{\phi}^j \omega^a_{j b} \psi^b_\alpha$$

and

$$\psi^a_\alpha = e^a_i(\phi) \chi^i_{\alpha}$$
The Poisson brackets are given by \( \{ p(\psi^a_\alpha), \psi^b_\beta \}_P = -\delta^{ab}\delta_{\alpha\beta} \), where \( p(\psi^a_\alpha) = \frac{\partial}{\partial \psi^a_\alpha} \mathcal{L} \) (the minus sign in the Poisson bracket is not a matter of convention but follows from requiring the Heisenberg field equations to hold). Using \( \{ C^a_\alpha, C^b_\beta \}_P = -i\delta^{ab}\delta_{\alpha\beta} \), we find the Dirac brackets

\[
\begin{align*}
\{ p(\psi^a_\alpha), \psi^b_\beta \}_D &= \left\{ p(\psi^a_\alpha), \psi^b_\beta \right\}_P - \left\{ p(\psi^a_\alpha), C^c_\gamma \right\}_P i \left\{ C^c_\gamma, \psi^b_\beta \right\}_P \\
&= -\frac{1}{2}\delta^{ab}\delta_{\alpha\beta}
\end{align*}
\]  

which is as usual half the Poisson brackets. Thus the quantum anticommutator reads (we set \( \hbar = 1 \) in this section)

\[
\{ \psi^a_\alpha, \psi^b_\beta \} = \delta^{ab}\delta_{\alpha\beta}
\]  

Furthermore, the conjugate momentum of \( \phi^i \) is given by

\[
p_i = g_{ij}\dot{\phi}^j + \frac{i}{2}\omega^{ij}_{\alpha\beta}\psi^a_\alpha\psi^b_\beta \; ; \; \; \; \; [p_i, \phi^j] = -i\delta^j_i
\]  

We now choose as quantum supersymmetry charge

\[
Q^{(\text{qu})}_\alpha = e^{\alpha}_a(\phi)\psi^a_\alpha g^{1/4}\pi_i g^{-1/4} \; ; \; \; \; \pi_i \equiv p_i - \frac{i}{2}\omega^{ij}_{\alpha\beta}\psi^a_\alpha\psi^b_\beta
\]

We choose this particular ordering because then \( Q^{(\text{qu})}_\alpha \) is hermitian, and hence also \( H^{(\text{qu})} \) will be hermitian. To prove that this \( Q^{(\text{qu})}_\alpha \) is hermitian, one may show that after hermitian conjugation one gets three extra terms which cancel due to the “vielbein postulate” that the total covariant derivative of the vielbein (with Cristoffel and spin connection) cancels. We evaluate the quantum anticommutator \( \{ Q^{(\text{qu})}_\alpha, Q^{(\text{qu})}_\beta \} \) and show that the result is equal to \( \delta_{\alpha\beta}H^{(\text{qu})} \) with \( H^{(\text{qu})} \) by definition the quantum Hamiltonian. Classically, the Hamiltonian is given by

\[
\frac{1}{2}g^{ij}\pi_i\pi_j - \frac{1}{8}R_{abcd}(\omega)\left( \psi^a_\alpha\psi^b_\beta \right)\left( \psi^c_\gamma\psi^d_\delta \right),
\]

but quantum mechanically there are many possibilities for the Hamiltonian, because \( \{ \psi^a_\alpha, \psi^b_\beta \}X^{\alpha\beta}_{ab} \) vanishes classically for any \( X \), but equals \( \delta^{ab}\delta_{\alpha\beta}X^{\alpha\beta}_{ab} \) at the quantum level. We define the Hamiltonian as the square of the supersymmetry charge because, as we shall discuss in section 6, when this \( H^{(\text{qu})} \) is used in \( \langle x, \bar{\eta} \rangle \exp \left( -\epsilon H \right) |y, \chi \rangle \) it gives the same propagator as the path integral in which the original supersymmetric action appears. Hence, we have chosen what might be called supersymmetric ordering for \( Q^{(\text{qu})}_\alpha \) and \( H^{(\text{qu})} \). Also, with this ordering prescription, \( Q^{(\text{qu})}_\alpha \) and \( H^{(\text{qu})} \) transform as scalars under general coordinate transformations applied to operators (cf. [7]).

The direct evaluation of \( \{ Q^{(\text{qu})}_\alpha, Q^{(\text{qu})}_\beta \} \) is tedious, but straightforward. Using

\[
[\pi_i, \psi^a_\alpha] = i\omega^{ab}_i\psi^b_\alpha \; ; \; \; \; \; [\pi_i, e^i_a] = -i(\partial_i e^i_a)
\]

one finds, keeping all operators in the order they come,

\[
\{ Q^{(\text{qu})}_\alpha, Q^{(\text{qu})}_\beta \} = g^{1/4}\left[ \delta_{\alpha\beta}\delta^{ab}\pi_i\pi_j - \psi^a_\alpha\psi^b_\beta \pi_i - \psi^a_\alpha\psi^b_\beta \right]|g^{-1/4}
\]

(We used the vielbein postulate \( \partial_j e^i_a = -\Gamma^{k}_{ij}a^k_{a} - \omega^{b}_{ja}e^i_b \) and found that all \( \Gamma \) terms cancel). The last term vanishes classically, but at the quantum level it is nonvanishing; in fact, it just covariantizes the \( \pi^2 \) term to the complete d’Alembertian for a scalar particle! Using

\[
[\pi_i, \pi_j] = -\frac{1}{2}R_{ijab}\psi^a_\alpha\psi^b_\beta
\]
and the cyclic identity
\[
\psi^a_2 \left( \bar{\psi}^b_1 \psi^c_1 \psi^d_1 + \bar{\psi}^c_1 \psi^d_1 \psi^b_1 + \psi^d_1 \psi^b_1 \bar{\psi}^c_1 \right) R_{abcd} = 0
\]
\[
= 3 \psi^a_2 \left( \bar{\psi}^b_1 \psi^c_1 \psi^d_1 - \delta^{bc} \psi^d_1 \right) R_{abcd} = 3 \psi^a_2 \psi^b_1 \psi^c_1 \psi^d_1 R_{abcd}
\]
we see that also the four-fermion term is proportional to \( \delta_{\alpha\beta} \). Our final result reads
\[
\left\{ Q^{(qu)}_\alpha, Q^{(qu)}_\beta \right\} = 2 \delta_{\alpha\beta} H^{(qu)}
\]
\[
H^{(qu)} = \frac{1}{2} \delta^{ab} g^{1/4} \left( e^i_a \pi^i_b \pi^j_c + e^i_a \omega_{ib} e^j_c \pi^j_i \right) g^{-1/4} - \frac{1}{8} R_{abcd}(\omega) \left( \bar{\psi}^a_\alpha \psi^a_\beta \right) \left( \bar{\psi}^c_\beta \psi^c_\alpha \right)
\]
(5.15)

Thus, imposing the quantum algebra \( \left\{ Q^{(qu)}_\alpha, Q^{(qu)}_\beta \right\} = \delta_{\alpha\beta} H^{(qu)} \) with given \( Q^{(qu)}_\alpha \) has fixed the operator ordering in \( H^{(qu)} \) as given in (5.15). Note that this quantum Hamiltonian is already Weyl-ordered\(^\dagger\): the term with \( \pi^2 \) yields upon Weyl-ordering again a term \( \frac{1}{8} R \), see (4.14), whereas the \( R \psi^4 \) term contributes \(-\frac{1}{8} R \), cancelling the bosonic contribution. To show this latter result, write the Weyl-ordered operator corresponding to \( \frac{1}{8} R_{abcd} \bar{\psi}^a_\alpha \psi^a_\beta \bar{\psi}^c_\beta \psi^c_\alpha \) as
\[
\frac{1}{2} R_{abcd} \left[ \theta(t_a, t_b) \theta(t_b, t_c) \theta(t_c, t_d) \bar{\psi}^a_\alpha \psi^a_\beta \bar{\psi}^c_\beta \psi^c_\alpha - \theta(t_a, t_b) \theta(t_b, t_d) \theta(t_c, t_d) \bar{\psi}^a_\alpha \psi^a_\beta \bar{\psi}^d_\alpha \psi^d_\beta + 22 \text{ more} \right]
\]
(5.16)
where \( t_a \) denotes the time associated with \( \psi^a \), and the limit \( t_a, t_b, t_c, t_d \to t \) is understood. We do not have to include the curvature term in the Weyl ordering because it commutes with all the fermions. To compute the difference between (5.15) and (5.16), we decompose unity in a sum of products of \( \theta \)-functions, and write the operator in (5.15) as follows
\[
\frac{1}{2} R_{abcd} \bar{\psi}^a_\alpha \psi^a_\beta \bar{\psi}^c_\beta \psi^c_\alpha = \frac{1}{2} R_{abcd} \bar{\psi}^a_\alpha \psi^a_\beta \bar{\psi}^c_\beta \psi^c_\alpha \left[ \theta(t_a, t_b) \theta(t_b, t_c) \theta(t_c, t_d) + \theta(t_a, t_b) \theta(t_b, t_d) \theta(t_c, t_d) + 22 \text{ more} \right]
\]
(5.17)
Now subtract (5.17) from (5.16), using the anticommutation relations (5.8). When we make use of identities such as
\[
\theta(t_a, t_d) \theta(t_c, t_b) = \theta(t_a, t_d) \theta(t_d, t_c) \theta(t_c, t_b) + \theta(t_a, t_c) \theta(t_c, t_d) \theta(t_d, t_b) + \theta(t_a, t_b) \theta(t_b, t_d) \theta(t_d, t_c) + \theta(t_c, t_a) \theta(t_a, t_b) \theta(t_b, t_d) + \theta(t_c, t_b) \theta(t_b, t_d) \theta(t_d, t_a) \theta(t_a, t_c)
\]
(5.18)
then, in the limit that all times coincide, we find that the difference equals \(-\frac{1}{8} R \). So the two extra terms due to Weyl ordering the two terms in \( H^{(qu)} \) in (5.15) cancel, and indeed the corresponding configuration space path integral which follows from this particular \( \tilde{H}^{(qu)} \) has the classical supersymmetric action in its exponent.

\(^\dagger\)We obtain this covariant result if we use a Weyl ordering in which we consider \( \pi \) as an independent operator, and not \( p \). Note that DeWitt in [7] at this point switches notation, and denotes \( g_i \tilde{x}^j \) by \( p_i \) (his equation 6.7.26). Weyl ordering (which he discusses when \( p \) and \( \pi \) coincide) in terms of his new \( p \) and \( x \) then indeed agrees with our observation that (5.15) is already Weyl-ordered. However, from a canonical point of view, \( \pi \) is not an independent operator. We discuss this point further in the conclusions.
6. Transition amplitude for $N=2$ supersymmetric sigma model.

We will now compute the matrix element

$$
\langle x, \bar{\eta} | \exp \left( -\frac{\xi}{\hbar} \hat{H} \right) | p, \xi \rangle
$$

(6.1)

where

$$
\hat{H} = \frac{1}{2} g^{-1/4} \pi_i g^{1/2} g^{ij} \pi_j g^{-1/4} - \frac{1}{8} \hbar^2 R_{abcd} \langle \psi^a \psi^b \rangle \langle \psi^c \psi^d \rangle
$$

(6.2)

As shown in (5.15), this is the quantum Hamiltonian for the $N=2$ supersymmetric nonlinear sigma model. The states $|\chi\rangle$ and $\langle \bar{\eta}|$ are eigenstates of the operators $\psi^a$ and $\bar{\psi}^b$ respectively, $\psi^a |\chi\rangle = \chi^a |\chi\rangle$ and $\langle \bar{\eta}| \bar{\psi}^b = (\bar{\eta}) \bar{\psi}^b$. They are coherent states, $|\chi\rangle = \exp (\bar{\psi}^a \chi^a) |0\rangle$ and $\langle \bar{\eta}| = \langle 0 | \exp (\bar{\eta}^a \psi^a)$, and satisfy the completeness relation $1 = \int (d\xi^1 d\xi^2 \ldots d\xi^n) \exp (-\bar{\xi}^a \xi^a) |\chi\rangle \langle \xi|$, and inner product $\langle \bar{\eta}| \xi\rangle = \exp (\bar{\eta} \xi)$.

Compared to the $D_i$ appearing in the regulator for the trace anomaly of a spin-$\frac{1}{2}$ field ($\hat{R}_{f e v} = g^{-1/4} \hat{P} \hat{\Phi} g^{1/4}$), we have twice as many fermionic degrees of freedom ($\alpha = 1, 2$). The case with half as many fermionic degrees of freedom (with $\alpha = 1$ in (2.8)), does correspond to a quantum field theory for spin-$\frac{1}{2}$ fields and will be discussed in section 7. We define

$$
\psi^a = \frac{1}{\sqrt{2}} (\psi_1^a + i \psi_2^a) ; \quad \bar{\psi}^a = \frac{1}{\sqrt{2}} (\psi_1^a - i \psi_2^a)
$$

(6.3)

Analogously to the bosonic case we expand the exponent in (6.1) and define

$$
\langle x, \bar{\eta} | (\hat{H})^k | p, \xi \rangle \equiv \sum_{l=0}^{2k} B_{l}^k (x, \bar{\eta}, \xi) p^l \langle x, \bar{\eta} | p, \xi \rangle
$$

(6.4)

Again we will only need the terms with $l = 2k, 2k - 1, \text{and} 2k - 2$. It will be convenient to express the Hamiltonian as $\hat{H} = \hat{\alpha} + \hat{\beta} + \hat{\gamma}$, with

$$
\hat{\alpha} = \frac{1}{2} g^{-1/4} \tilde{p}_i g^{1/2} g^{ij} \tilde{p}_j g^{-1/4} = \hat{H}_{\text{bos}}
$$

$$
\hat{\beta} = -i \hbar g^{ij} \omega_{iab} \bar{\psi}^a \psi^b g^{1/4} \tilde{p}_j g^{-1/4}
$$

$$
\hat{\gamma} = -\frac{1}{2} \hbar^2 g^{-1/2} \partial_i (g^{1/2} g^{ij} \omega_{jab}) \bar{\psi}^a \psi^b - \frac{1}{2} \hbar^2 g^{ij} \omega_{iab} \omega_{jab} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d - \frac{1}{2} \hbar^2 R_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d
$$

(6.5)

The leading term is the same as in the bosonic case

$$
B_{2k}^k (x, \bar{\eta}, \xi) p^{2k} = A_{2k}^k (x) p^{2k} = \left( \frac{1}{2} p^2 \right)^k
$$

(6.6)

The next term equals

$$
B_{2k-1}^k (x, \bar{\eta}, \xi) p^{2k-1} = A_{2k-1}^k (x) p^{2k-1} - i \hbar \left( \frac{1}{2} p^2 \right)^{k-1} g^{ij} \omega_{iab} \bar{\psi}^a \xi^b \psi^d
$$

(6.7)
The first term comes from $k$ factors $\hat{\alpha}$, and is identical to what the purely bosonic case yields, whereas the second term arises when we combine $k - 1$ factors $\hat{\alpha}$ with one factor $\hat{\beta}$, and replace all operators by their corresponding c-number or Grassmann number values. The last term we have to compute is

$$B_{2k-2}^k(x, \bar{\eta}, \xi)p^{2k-2} = A_{2k-2}^k(x)p^{2k-2} - i\hbar A_{2k-3}^{k-1}(x)p^{2k-3}g^{ij}\omega_{iab}\bar{\eta}^a\xi^b p_j - \frac{1}{2}k(k - 1)\hbar^2 (\frac{1}{2}p^2)^{k-2}g^{ij}\partial_i(g^{kl}\omega_{kab})\bar{\eta}^a\xi^bp_jp_l$$

$$+ \frac{1}{4}k\hbar^2 (\frac{1}{2}p^2)^{k-1}g^{ij}(\partial_i\log g)\omega_{iab}\bar{\eta}^a\xi^b - \frac{1}{4}k(k - 1)\hbar^2 (\frac{1}{2}p^2)^{k-2}g^{ij}(\partial_ig^{kl})\omega_{kab}\bar{\eta}^a\xi^bp_kp_l$$

$$- \frac{1}{2}k(k - 1)\hbar^2 (\frac{1}{2}p^2)^{k-2}g^{ij}\omega_{iab}g^{kl}\omega_{kcd}(\bar{\eta}^a\xi^d\delta^{bc} - \bar{\eta}^a\bar{\xi}^c\xi^d)p_jp_l$$

(6.8)

There are four possible combinations that give a contribution: (i) $k$ factors $\hat{\alpha}$, yielding the same expression as in the bosonic case, (ii) $k - 1$ factors $\hat{\alpha}$ and one factor $\hat{\beta}$; now one $\hat{p}$, either from one of the $\hat{\alpha}$’s or from $\hat{\beta}$ acts as a derivative, again either on one of the $\hat{\alpha}$’s or on $\hat{\beta}$, yielding the next four terms, (iii) $k - 1$ factors $\hat{\alpha}$ and one factor $\hat{\gamma}$ and replacing all $\hat{p}$’s by their c-number values, and (iv) $k - 2$ factors $\hat{\alpha}$ and two factors $\hat{\beta}$, which gives rise to the last term. Notice that to evaluate the last two terms we first needed to normal order the fermionic operators which produced extra terms proportional to $\delta^{ab}$ according to (6.3). In detail, we write $\tilde{\psi}^a\psi_b\tilde{\psi}^c\psi_d$ as $\tilde{\psi}^a\psi_d\delta^{bc} - \tilde{\psi}^a\tilde{\psi}^c\psi^b\psi^d$. We can now proceed in the same way as in the bosonic case, by doing the sum over $k$, extracting a factor $\exp(-\frac{1}{2\epsilon}p^2)$, and performing the Gaussian integral over the momenta. In the last term of (6.8), the integral over $p_jp_l$ contributes a term with $g_{ji}$ which cancels the complete $\omega^2$ term in the line above it, plus a term with $(y - x)_j(y - x)_i$. In this result, the contribution proportional to $\omega^2\bar{\eta}^2\xi^2$ is the square of the second term on the right hand side of (6.7) and is removed by exponentiating. The two-fermion part of the last term in (6.8) survives and is also exponentiated. We then compute the transition amplitude

$$\langle x, \bar{\eta} | \exp\left(-\frac{\epsilon}{\hbar}\hat{H}\right) | y, \chi\rangle = \int d^n\bar{p}d^nx\xi^a\xi^b e^{-\xi\bar{\xi}}\langle x, \bar{\eta} | \exp\left(-\frac{\epsilon}{\hbar}\hat{H}\right) | p, \xi\rangle \langle p, \bar{\xi} | y, \chi\rangle$$

(6.9)

Using the inner product

$$\langle x, \bar{\eta} | p, \xi\rangle = \langle x | p\rangle \langle \bar{\eta} | \xi\rangle = (2\pi\hbar)^{-n/2}g^{-1/4}(x)e^{\frac{x}{2}\bar{\eta}}e^{\xi}$$

(6.10)

we find for the amplitude

$$\langle x, \bar{\eta} | \exp\left(-\frac{\epsilon}{\hbar}\hat{H}\right) | y, \chi\rangle = (2\pi\epsilon\hbar)^{-n/2}\exp\left(-\frac{1}{\hbar}S_B - \frac{1}{\hbar}\bar{S}_F\right)$$

$$\left[1 - \frac{1}{12}\epsilon\hbar R(x) - \frac{1}{12}R_{ij}(y - x)_i(y - x)_j + \frac{1}{2}\epsilon\hbar R_{ab}(x)\bar{\eta}^a\chi^b\right]$$

(6.11)
where

\[ S_B = \frac{1}{2\epsilon} g_{ij}(x)(y-x)^i(y-x)^j + \frac{1}{4\epsilon} \partial_k g_{ij}(x)(y-x)^i(y-x)^j(y-x)^k + \frac{1}{12\epsilon} \left( \partial_k \partial_l g_{ij}(x) - \frac{1}{2} g_{mn}(x) \Gamma^m_{ij} \Gamma^n_{kl}(x) \right) (y-x)^i(y-x)^j(y-x)^k(y-x)^l \]  

(6.12)

is the expansion through order \( \epsilon \) of the length of the geodesic joining \( x \) and \( y \) (cf. (4.11)), and

\[ \tilde{S}_F = -\hbar \delta_{ab} \bar{\eta}^a \chi^b - \hbar (y(x) \omega_{ab}(x) \bar{\eta}^a \chi^b - \frac{1}{2} \hbar (y(x) \tilde{\chi}^a \omega_{ab}(x) + \omega_{iab}(x) \tilde{\chi}^i \omega^c(x) \omega_{jcb}(x) \bar{\eta}^a \chi^b \]  

- \frac{1}{2} \epsilon \hbar^2 R_{abcd}(x) \bar{\eta}^a \chi^b \bar{\eta}^c \chi^d \]

(6.13)

\[ S_F = S_f - \hbar \delta_{ab} \bar{\eta}^a \psi^b(0) \]  

\[ S_f = \hbar \int_{-\epsilon}^0 dt \left( \delta_{ab} \bar{\eta}^a \psi^b + \bar{x}^i \omega_{iab} \bar{\eta}^a \psi^b - \frac{1}{2} \hbar R_{abcd} \bar{\eta}^a \psi^b \bar{\eta}^c \psi^d \right) \]  

(6.14)

If we add an extra boundary term \( \hbar \delta_{ab} \bar{\psi}_{c_i}(0) \psi^b(0) = \hbar \delta_{ab} \bar{\eta}^a \psi^b(0) \) to (6.13), then the sum of the first term in (6.13) together with this extra term becomes the leading term in the fermionic action.

\[ \tilde{S}_F = S_F - \hbar \delta_{ab} \bar{\eta}^a \psi^b(0) ; \quad S_F = \hbar \int_{-\epsilon}^0 dt \left( \delta_{ab} \bar{\eta}^a \psi^b + \bar{x}^i \omega_{iab} \bar{\eta}^a \psi^b - \frac{1}{2} \hbar R_{abcd} \bar{\eta}^a \psi^b \bar{\eta}^c \psi^d \right) \]

We can easily check that the expansion through order \( \epsilon \) of (6.14) indeed equals the expression in (6.13) when the equations of motion are imposed. These read (of course we also need the bosonic part of the action to find the full equations of motion)

\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k - R^i_{jabc} \dot{x}^j \bar{\eta}^a \psi^b + \frac{1}{2} \hbar g^{ij} \left( \partial_j R_{abc} \right) \bar{\eta}^a \psi^b \bar{\eta}^c \psi^d = 0 \]

(6.15)

\[ \dot{\psi}^a + \dot{x}^i \omega_{iab} \psi^b - R_{abc} \psi^b \bar{\eta}^c \psi^d = 0 \]

\[ \dot{\bar{\psi}}^a + \dot{x}^i \omega_{iab} \bar{\psi}^b - R_{abc} \bar{\eta}^c \psi^d = 0 \]

We can now expand the Lagrangian in a Taylor series around its value at \( t = 0 \), and then do the trivial time integrations. This yields

\[ S = \epsilon L(0) - \frac{1}{2} \epsilon^2 \hat{L}(0) + \ldots \]  

(6.16)

We thus expand all fields in the Lagrangian around their values at \( t = 0 \), making use of the equations of motion (6.15). The expansions up to the order \( \epsilon \) we need are given by

\[ \dot{x}^i(0) = \frac{(x-y)^i}{\epsilon} + \frac{1}{2}(x-y)^j R^i_{jabc} \bar{\eta}^a \chi^b \]

\[ \psi^a(0) = \bar{\eta}^a \]

(6.17)

\[ \psi^a(0) = \chi^a - (x-y)^i \omega^i_{ab} \chi^b \]

\[ \dot{\psi}^a(0) = \frac{\psi^a(0) - \chi^a}{\epsilon} - \frac{1}{2} \frac{(x-y)^i (x-y)^j}{\epsilon} \left( \partial_j \omega_{ab}^i - \omega^i_{ac} \omega^c_{jb} \right) \chi^b \]

Inserting these expansions into \( \tilde{S}_F \) in (6.14) yields the expression (6.13).

Again, it is easy to check that the composition rule holds

\[ \langle x, \bar{\eta} \rangle \exp \left( -\frac{2\epsilon}{\hbar} \hat{H} \right) |y, \chi \rangle = \int d^3 z d^3 \xi d^3 \xi \sqrt{g(z)} e^{-\xi t} \langle x, \bar{\eta} \rangle \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) |z, \xi \rangle \langle z, \xi \rangle \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) |y, \chi \rangle \]

(6.18)
Imposing this rule as a consistency condition fixes the term proportional to the Ricci curvature times $(y - x)^i(y - x)^j$ in the expansion of (6.11), whereas it leaves arbitrary all curvature terms which have a factor $\epsilon$ in front.

Also for this $N = 2$ supersymmetric case one can derive the resulting transition element from a path integral. As we explained in section 5, the Hamiltonian to be used in the path integral is the supersymmetric one, leading to the following configuration space path integral

$$S_{\text{conf}} = \int dt \left( \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \delta_{ab} \bar{\psi}^a \psi^b + \dot{x}^i \omega_{iab} \bar{\psi}^a \psi^b - \frac{1}{2} R_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d \right) - \delta_{ab} \bar{\eta}^a \psi^b(0)$$

(6.19)

where we have now rescaled the fermions by a factor $\sqrt{\hbar}$, in order to obtain the usual anticommutation relation $\{\psi^a, \psi^b\} = \hbar g^{ab}$. The last term in (6.19) is the extra term one always needs when writing down path integrals for coherent states as is discussed in [5]. We indeed find that it must be included to obtain the correct propagator. The actual computation of this path integral is worked out in the appendix.

We will now show that the final result for the transition amplitude can again be written as the product of three factors: a term containing only the scalar curvature which is related to the trace anomaly, the exponent of the classical action, and the square root of in this case the supersymmetric generalisation of the Van Vleck determinant. The latter is defined by [7,21]

$$D_S = \text{sdet} D_A \quad ; \quad D_A \equiv - \frac{\partial}{\partial \Phi^A} \left( S_B + S_F \right) \frac{\partial}{\partial \Phi^B}$$

(6.20)

where $\Phi^A = (x^i, \bar{\eta}^a)$ and $\Phi^B = (y^j, \chi^b)$, and for $S_B$ and $S_F$ we substitute the expressions (6.12) and (6.13), with the fermions rescaled by a factor $\sqrt{\hbar}$. To evaluate $D_S$ write

$$D_{AB} = \begin{pmatrix} A_{ij} & B_{ib} \\ C_{aj} & D_{ab} \end{pmatrix}$$

(6.21)

We find, expanding in normal coordinates around $x$ to simplify the expressions,

$$A_{ij} = \frac{1}{\epsilon} g_{ij}(x) + \partial_i \omega_{j ab}(x) \bar{\eta}^a \chi^b$$

$$- \frac{1}{2} \left( \partial_j \omega_{j ab}(x) + \partial_j \omega_{iab}(x) + \omega_{ia} \omega_{j cb}(x) + \omega_{ja} \omega_{ibc}(x) \right) \bar{\eta}^a \chi^b$$

$$B_{ib} = - \omega_{iab}(x) \bar{\eta}^a$$

$$C_{aj} = \omega_{j ab}(x) \chi^b$$

$$D_{ab} = \delta_{ab} + (y - x)^i \omega_{iab}(x) + \frac{1}{2} (y - x)^i (y - x)^j \left( \partial_i \omega_{j ab}(x) + \omega_{ia} \omega_{j cb}(x) \right)$$

$$+ \epsilon \left( R_{abcd}(x) - R_{adcb}(x) \right) \bar{\eta}^c \chi^d$$

(6.22)

We do not need terms of order $\epsilon$ in $B$ and $C$, since $D_S = \det A \det^{-1} \left( D - CA^{-1} B \right)$ and $A^{-1}$ is already of order $\epsilon$. Writing $A_{ij} = \frac{1}{\epsilon} g_{ij}(\delta^k_j + \epsilon a^k_j)$ and $D_{ab} = (\delta_{ab} + d_{ab})$, we can write the expansion of the super Van Vleck determinant as

$$D_S^{1/2} = (\epsilon)^{-n/2} g^{1/2}(x) \left[ 1 + \frac{1}{2} \epsilon \text{tra} + \frac{1}{2} \epsilon \text{tr}CB - \frac{1}{2} \epsilon \text{trd} + \frac{1}{4} \epsilon \text{trd}^2 \right]$$

(6.23)
Multiplying by $g^{-1/4}(x)g^{-1/4}(y)$ to transform $D_S^{1/2}$ into a bi-scalar, we obtain

$$\tilde{D}_S^{1/2} = g^{-1/4}(x)D_S^{1/2}g^{-1/4}(y) = (e)^{-n/2}\left[1 - \frac{1}{12} R_{ij}(x)(y - x)^i(y - x)^j + \frac{1}{2} \epsilon_{abc} \chi^a \chi^b\right]$$ (6.24)

So indeed we can write

$$\langle x, \bar{q} | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y, \chi \rangle = (2\pi\hbar)^{-n/2} \tilde{D}_S^{1/2} \exp \left( -\frac{1}{\hbar} (S_B + \tilde{S}_F) \right) \left[1 - \frac{1}{12} \epsilon \hbar R\right]$$ (6.25)

similarly to (4.17). All terms involving the Ricci curvature in (6.11) are thus completely accounted for in the super Van Vleck determinant! This would not have been the case if we would have used the ordinary determinant. Note that it is $\tilde{S}_F$ and not $S_F$ which appears in the exponent, for reasons explained in (3.18). As we already mentioned, also in the path integral we shall need $\tilde{S}_F$.

7. The $N=1$ supersymmetric sigma model.

For the case of $N=(0,1)$ supersymmetry we start again with the supersymmetric sigma model as defined in (5.4), but now truncate the theory by putting in the action $\psi_1^a \psi_2^b = \frac{1}{\sqrt{2}} \psi_1^a$. This then requires $\epsilon_1 + \epsilon_2 = 0$, see (5.2). Canonical quantization then yields $\{\psi_1^a, \psi_1^b\} = \delta^{ab}$, and $p_i = g_{ij} \bar{\phi}^j + \frac{i}{2} \omega_{iab} \psi_1^a \psi_1^b$. The truncation of $H^{(q)}$ would yield again (5.15) but now with $\psi_1^a \bar{\psi}_1^b \psi_1^c \bar{\psi}_1^d$ instead of $\left(\psi_1^a \bar{\psi}_1^b \right) \left(\bar{\psi}_1^c \psi_1^d\right)$. Using the cyclic identity of the Riemann tensor, one obtains

$$H_{N=2,\text{truncated}}^{(q)} = \frac{1}{2} g^{-1/4} \pi_1 \gamma^{1/2} g^{ij} \pi_j g^{-1/4} - \frac{1}{16} \hbar^2 R$$ (7.1)

However, this is not the Hamiltonian we are interested in for two reasons: (i) we want a Hamiltonian which corresponds to the regulator of a spin-$\frac{1}{2}$ field in quantum field theory. The latter is given by $-\frac{1}{8} \hbar^2 g^{1/4} \hat{D} \hat{D} g^{-1/4}$ which differs from the truncated Hamiltonian because it has a term $-\frac{1}{16} \hbar^2 R$ instead of $-\frac{1}{16} \hbar^2 R$, and (ii), we want the $N=1$ supersymmetric quantum Hamiltonian, given by the square of the $N=1$ supersymmetry charge. Truncating the $N=2$ supersymmetry charge in (5.10) to the $N=1$ case, one obtains the operator $Q_{N=1}^{(q)} = g^{1/4} e_a^i (\partial \bar{\phi}) \psi_i^a \pi_i g^{-1/4}$. Since this operator corresponds to the Dirac operator $g^{1/4} \hat{D} g^{-1/4}$ whose square contains a term $\frac{1}{4} R$ we clearly obtain the Hamiltonian with $-\frac{1}{8} \hbar^2 R$.

The truncated Hamiltonian is thus no longer supersymmetric. The reason is that the field $\psi_1 - \psi_2$ can consistently be set to zero at the classical level, as its classical supersymmetry variation vanishes, but at the quantum level the anticommutator of this field with itself is nonvanishing. So instead of the Hamiltonian in (7.1) we should use the Hamiltonian

$$\hat{H} = \frac{1}{2} g^{-1/4} \pi_1 \gamma^{1/2} g^{ij} \pi_j g^{-1/4} - \frac{1}{8} \hbar^2 R$$

$$\pi_i = p_i - \frac{i \hbar}{2} \omega_{iab} \psi_1^a \psi_1^b$$ (7.2)
We can now repeat the computation of the transition element of section 6. Note that in contrast with (6.2) we have only one species of fermions present, namely $\psi_a$. In order to compute the transition element (and subsequently if needed its trace), we need two operators $\psi^a$ and $\bar{\psi}^a$ as in (3.16). Hence we enlarge our Hilbert space by introducing a second set of free fermions, $\psi^a_{II}$, satisfying $\{\psi^a_{II}, \bar{\psi}^b_{II}\} = \delta^{ab}$ and $\{\psi^a_{II}, \psi^b_{II}\} = 0$. We then again define $\psi^a = \frac{1}{\sqrt{2}}(\psi^a + i\psi^a_{II})$ and $\bar{\psi}^a = \frac{1}{\sqrt{2}}(\psi^a_{II} - i\bar{\psi}^a_{II})$. We rewrite the operators $\psi^a$ appearing in the Hamiltonian (7.2) in terms of the operators $\psi^a$ and $\bar{\psi}^a$ as $\psi^a = \frac{1}{\sqrt{2}}(\psi^a + \bar{\psi}^a)$, and proceed analogously to the computation in section 6 to obtain the transition element. Compared to this section the only difference is that we replace the operators $1/2\omega_{ia}\bar{\psi}^a\psi^b = \omega_{ia}\bar{\psi}^a\psi^b$ of section 6 by $1/2\omega_{ia}\psi^a\bar{\psi}^b = 1/2\omega_{ia}(\bar{\psi}^a + \psi^a)(\bar{\psi}^b + \psi^b)$, and $R_{abcd}\psi^a\bar{\psi}^b\psi^c\bar{\psi}^d$ by $R$. The net result is that (i) all $\bar{\eta}^a\xi^b$ terms in (6.7) and (6.8) are replaced by $1/4(\bar{\eta}^a + \xi^a)(\bar{\eta}^b + \xi^b)$, (ii) in (6.8) the coefficient $(\bar{\eta}^a\xi^b\bar{\eta}^c\xi^d + \delta^{bc}\bar{\eta}^a\xi^d)$ of the $\omega^2$ term is replaced by $1/16[(\bar{\eta}^a + \xi^a)(\bar{\eta}^b + \xi^b)(\bar{\eta}^c + \xi^c)(\bar{\eta}^d + \xi^d) + 2\delta^{bc}\delta^{ad}]$ (we have used that the expressions are antisymmetric under $a \leftrightarrow b$ or $c \leftrightarrow d$, and symmetric under $ab \leftrightarrow cd$), and (iii) all contributions from the $R\psi^4$ term in the Hamiltonian disappear and are replaced by $1/8\hbar R$. One finds

$$
\langle x, \bar{\eta} | \exp \left( -\frac{c}{\hbar} H \right) | y, \chi \rangle = (2\pi\hbar)^{-n/2} \exp \left( -\frac{1}{\hbar} (S_B + \bar{S}_F) \right) \left[ 1 + \frac{1}{24} \hbar R(x) \right.
- \frac{1}{12} R_{ij}(x)(y - x)^i(y - x)^j + \frac{1}{16}(y - x)^i(y - x)^j \omega_{ia}^b(x) \omega_{ja}^a(x) \left. \right]\times
\left(7.3\right)
$$

where

$$
\frac{1}{\hbar} \bar{S}_F = -\delta_{ab}\bar{\eta}^a\chi^b - \frac{1}{4}(y - x)^i \omega_{ia}(x)(\bar{\eta}^a + \chi^a)(\bar{\eta}^b + \chi^b)
- \frac{1}{8}(y - x)^i(y - x)^j \partial_{ij}\omega_{ab}(x)(\bar{\eta}^a + \chi^a)(\bar{\eta}^b + \chi^b)
\left(7.4\right)
$$

and $S_B$ is defined in (6.12).

The various terms in (7.3) and (7.4) can be understood as follows. The $\bar{\eta}\chi$ term comes from the inner product of coherent states in (6.10). The rest of the terms in (7.4) can be obtained from (6.13) by making the replacement $\bar{\eta}^a\chi^b \rightarrow \frac{1}{4}(\bar{\eta}^a + \chi^a)(\bar{\eta}^b + \chi^b)$. Since this substitution yields a result antisymmetric in $a \leftrightarrow b$, the $(x - y)^2\omega^2\bar{\eta}\chi$ term can not have a counterpart in (7.4). However, there is a $(x - y)^2\omega^2$ term in (7.3), and this term is the counterpart of the $(x - y)^2\omega^2\bar{\eta}\chi$ in (6.13) as one may trace by recalling that $(\bar{\eta}^a\xi^b\bar{\eta}^c\xi^d + \delta^{bc}\bar{\eta}^a\xi^d)$ was replaced by $1/16[(\bar{\eta}^a + \xi^a)(\bar{\eta}^b + \xi^b)(\bar{\eta}^c + \xi^c)(\bar{\eta}^d + \xi^d) + 2\delta^{bc}\delta^{ad}]$. The first part of this term is the square of the $(x - y)^2\omega^2$ term in $\bar{S}_F$, whereas the second part explains the presence of the $(x - y)^2\omega^2$ term in (7.3). Finally, the $-1/12\hbar R$ from the bosonic case in (4.10) combines with the $1/8\hbar R$ from (7.2) into the term $1/2\hbar R$.

We will now show that the expression for the propagator can again be written as the product of the super Van Vleck determinant, the exponent of the classical action, and a term involving the scalar curvature which, as shown at the end of this section, determines the trace anomaly of a
spins-1/2 field. Defining the super Van Vleck determinant as in (6.20) and (6.21), we find from (7.4)

\[ A_{ij} = \frac{1}{\epsilon} g_{ij}(x) + \frac{1}{8} \left( \partial_i \omega_{jab}(x) - \partial_j \omega_{iab}(x) \right) (\tilde{\eta}^a + \chi^a)(\tilde{\eta}^b + \chi^b) \]

\[ B_{ib} = -\frac{1}{2} \omega_{iab}(x)(\tilde{\eta}^a + \chi^a) \]

\[ C_{aj} = \frac{1}{2} \omega_{jab}(x)(\tilde{\eta}^b + \chi^b) \]

\[ D_{ab} = \delta_{ab} + \frac{1}{2} (y - x)^i \omega_{iab}(x) + \frac{1}{4} (y - x)^i (y - x)^j \partial_i \omega_{jab}(x) \]

which yields, using again (6.23),

\[ \tilde{D}^{1/2}_S = (\epsilon)^{-n/2} \left[ 1 - \frac{1}{12} R_{ij}(x)(y - x)^i (y - x)^j + \frac{1}{16} (y - x)^i (y - x)^j \omega_{ia}^b(x) \omega_{ja}^a(x) \right] \]

So we can indeed write

\[ \langle x, \tilde{\eta}| \exp \left( -\frac{\epsilon}{\hbar} \tilde{H} \right) |y, \chi \rangle = (2\pi\hbar)^{-n/2} \tilde{D}^{1/2}_S \exp \left[ -\frac{1}{\hbar} (S_B + \tilde{S}_F) \right] \left[ 1 + \frac{1}{24}\epsilon\hbar R \right] \]

similarly to (4.17) and (6.25).

Again, the same result for the propagator can also be obtained from a configuration space path integral with a suitable action. As explained in the appendix, also in this case we have to introduce an extra set of free fermions, \( \psi^a_2 \), in order to make the path integral well-defined. The action one should use in the configuration space path integral is (we again rescaled the fermions by a factor \( \sqrt{\hbar} \))

\[ S_{\text{conf}} = \int dt \left( \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + \frac{1}{2} \delta_{ab} \psi^a_1 \dot{\psi}^b_1 + \frac{1}{2} \delta_{ab} \psi^a_2 \dot{\psi}^b_2 + \frac{1}{2} \dot{x}^i \omega_{iab} \psi^a_1 \psi^b_1 \right) - \delta_{ab} \tilde{\eta}^a \psi^b_1(0) \]

where compared to the naive action corresponding to the Hamiltonian in (7.2) the \( \frac{1}{8} \hbar^2 R \) term is cancelled because of Weyl ordering of the bosonic term (in this N=1 case there is no \( R \psi^4 \) term which would yield another \( -\frac{1}{8} R \) term in the action, upsetting the cancellation of \( R \) terms in (7.8)). We have introduced an extra set of fermions that do not couple to any of the other fields, this way making certain that we do not alter the dynamics. In order to keep local Lorentz invariance we require that the fermions \( \psi^a_2 \) are inert under local Lorentz transformations.

We can now easily compute the trace anomaly for a spin-1/2 field by taking the trace of the transition element (7.3) and singling out the \( \epsilon \)-independent part. Since by introducing the free fermions \( \psi^a_2 \) (\( a = 1 \ldots n \)), we have (for even \( n \)) 2\( n/2 \) states in the \( \psi_1 \) sector and 2\( n/2 \) states in the \( \psi_2 \) sector (combining the \( n \) \( \psi^a_1 \) into half as many pairs of creation and absorption operators, and similarly for \( \psi^a_2 \)). Hence, we must divide the trace over \( \psi_1 \) and \( \psi_2 \) by a factor 2\( n/2 \), since we really should only take the trace in the \( \psi_1 \) sector. The trace anomaly for a spin-1/2 field in \( n \) dimensions is therefore given by (cf. [4], equation (2.9))

\[ \mathcal{A}_{n}^{\text{spin-1/2}} = -\frac{\hbar}{2^{n/2}} \lim_{\epsilon \to 0} \int d^3 \chi d^3 \bar{\eta} e^{\bar{\eta} \chi} \langle x, \bar{\eta}| \exp \left( -\frac{\epsilon}{\hbar} \tilde{H} \right) |x, \chi \rangle ; \quad a = 1 \ldots n \]

For \( n = 2 \) the trace anomaly becomes \( \frac{\hbar}{2^2} R \), which is indeed the result for a Dirac fermion; for the anomaly of a Majorana fermion we have to divide this expression by two.
8. Conclusions.

We have considered the evaluation of the propagator in quantum mechanics for a point particle in curved space, and for a point particle with its fermionic extension in curved space. In the latter case we considered both $N=1$ and $N=2$ supersymmetric quantum mechanics. We first used the Hamiltonian approach and obtained the propagators through order $\Delta t = \epsilon$. In this approach there are no ambiguities at all, but the calculations are somewhat tedious. Afterwards we considered the problem of finding an action for a path integral, together with a prescription for evaluating this path integral, which reproduces the propagator. This problem has been studied at great length in the literature, but in our opinion no complete solution has yet been found. We now summarize the contribution of this paper to these problems.

In the Hamiltonian approach, the propagator is defined by $\langle x | \exp \left( -\frac{i}{\hbar} H \right) | y \rangle$ where we assume $H$ to be an operator with a given, a priori fixed, ordering of the operators $\hat{x}^i$ and $\hat{p}_j$ but without an explicit time dependence. We evaluated this matrix element, following Feynman, by inserting a complete set of states $\int d^n p \langle p | \langle p |$ expanding the exponent, and moving in each term all $\hat{x}$ to the left and all $\hat{p}$ to the right, where we replace them by their eigenvalues $x$ and $p$, and finally reexponentiating the result. It seems widely believed that for small $\epsilon$ it is sufficient to only retain the term linear in $H$ in the expansion, to define $\langle x | \hat{H} | p \rangle = h(x,p)\langle x | p \rangle$, and to replace $\langle x | \left( 1 - \frac{i}{\hbar} H \right) | p \rangle$ by $\exp \left( -\frac{i}{\hbar} h(x,p) \right) \langle x | p \rangle$. If $\hat{H}(\hat{x},\hat{p})$ is of the form $T(\hat{p}) + V(\hat{x})$, this procedure is correct, and proven by the Trotter formula (see e.g. [15]). However, for more general systems, such as a particle in curved space with Hamiltonian

$$\hat{H} = \frac{1}{2} g^{-1/4}(\hat{x}) \hat{p}_i g^{ij}(\hat{x}) g^{1/2}(\hat{x}) \hat{p}_j g^{-1/4}(\hat{x})$$

(8.1)

where $\int d^n x \sqrt{g(x)} | x \rangle | x \rangle = 1$, or a point particle in flat space coupled to electromagnetism with Hamiltonian

$$\hat{H} = \frac{1}{2} \left( \hat{p}_i - \frac{e}{c} A_i(\hat{x}) \right) \left( \hat{p}^i - \frac{e}{c} A^i(\hat{x}) \right)$$

(8.2)

where $\int d^n x | x \rangle | x \rangle = 1$, this linear approximation is incorrect. In both cases, the commutator of $\hat{p}_i$ with a function $f(\hat{x})$ is given by $\frac{\hbar}{i} \frac{df}{d\hat{x}} (\hat{x})$ (we never need to use the fact that for hermiticity $\hat{p}_i$ must be represented in curved space by $\frac{\hbar}{i} g^{-1/4}(\hat{x}) \partial_i g^{1/4}(\hat{x})$ if one uses an inner product defined by $\int d^n x \sqrt{g(x)} | x \rangle | x \rangle = 1$, see section 2). The reason that terms linear in $\epsilon$ in the expansion of $\exp \left( -\frac{i}{\hbar} H \right)$ is that the $\hat{p}$-integration uses a Gaussian integrand $\exp \left( -\epsilon \hat{p}^2 \right)$, so that under the integral $\hat{p}$ is of order $\epsilon^{-1/2}$. Hence each commutator $[\hat{x},\hat{p}]$ which removes an operator $\hat{p}$, contributes a factor $\epsilon^{1/2}$ to the final answer. Consequently, to obtain the terms of order $\epsilon^{m/2}$ in the propagator for (8.1) it is sufficient and necessary to retain all terms with at most $m$ commutators in the evaluation of the matrix element $\langle x | \sum \frac{1}{k!} (-\frac{\hbar}{i})^k \hat{H}^k | p \rangle$. In particular, terms proportional to $\epsilon$ (which are needed to check that the propagator satisfies the Schrödinger equation to lowest order in $\epsilon$) come from terms containing none, one or two commutators in the evaluation of $\langle x | \hat{H}^k | p \rangle$ for any $k$! Forgetting these commutators leads to an incorrect result.

We have been able to resum this infinite series both for the bosonic and for the supersymmetric cases. It should be stressed that all terms in $\langle x | \hat{H}^k | p \rangle$ are well defined, and no ambiguities exist
in the evaluation of the propagator in the Hamiltonian approach. Only the commutation relations between \( \hat{p} \) and \( \hat{x} \) were needed, and no regularization of products of operators at the same time was used. Our final answer for the bosonic point particle in curved space agrees with the literature, proving that it was indeed necessary to keep terms beyond the linear approximation. The result through order \( \epsilon \) is a product of (i) the classical action for a solution of the classical equation of motion for \( x^i(t) \), describing a particle moving from \( y \) to \( x \) in time \( \epsilon \), (ii) the square root of the Van Vleck determinant, which is anyhow needed to make the propagator a general coordinate scalar both in \( x \) and in \( y \) (a bi-scalar [20]), and (iii) the trace anomaly. Namely

\[
\langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle = (2\pi \hbar)^{-n/2} \exp \left( -\frac{1}{\hbar} S_{\text{cl}}(x, y; \epsilon) \right) \epsilon^{-n/2} \left( 1 - \frac{1}{12} R_{ij}(x)(x - y)^i(x - y)^j \right) \exp \left( -\frac{1}{12} \hbar R(x) \epsilon \right)
\]

(8.3)

We would like to remark that the extension of this result to include coupling to an electromagnetic field can easily be deduced from its supersymmetric extensions we considered in section 6 and 7. One simply identifies \( \frac{\epsilon}{\hbar} A_i(x) \) with \( \frac{\hbar}{2} \omega_{ij} \psi^a_i \psi^b_j \) in section 6, or with \( \frac{\hbar}{2} \omega_{ij} \psi^a_i \psi^b_j \) in section 7, and retains only the terms which are not coming from either anticommuting the fermions or from the inner product of the fermionic coherent states. This yields again the expression (8.3), where now \( S_{\text{cl}} \) is the expansion through order \( \epsilon \) of the classical action of a point particle in curved space coupled to electromagnetism.

Although in (8.3) the term \( R(x) \epsilon \) can be rewritten as an integral of a local functional of \( x(t) \), \( R(x) \epsilon = \int dt R(x_{\text{cl}}(t)) \) to linear order in \( \epsilon \), the term with \( R_{ij}(x)(x - y)^i(x - y)^j \) can not be obtained by expanding the integral of a local functional. For this reason, one cannot simply string a set of matrix elements \( \langle x_i | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | x_{i+1} \rangle \) together and, by integrating over the intermediate points \( x_1, \ldots x_{N-1} \), obtain a path integral. The path integrals we are interested in have, by definition, a local action, and the problem one is faced with is to find a local functional \( S_{\text{conf}}[x(t)] \) which produces the propagator \( \langle x | \exp \left( -\frac{\epsilon}{\hbar} \hat{H} \right) | y \rangle \) when inserted into a path integral of the form \( \int D x(t) \exp \left( -\frac{1}{\hbar} S_{\text{conf}}[x(t)] \right) \). The paths all run from \( x(t=-\epsilon) = y \) to \( x(t=0) = x \). Of course, one should also specify how to evaluate the path integral, and this we do by “mode-expansion”[1,3,4], according to which \( x(t) \) is expanded in terms of a complete set: \( x(t) = \sum a_n \phi_n(t) \). The obvious and most convenient choice for the complete set of \( \phi_n(t) \) are the eigenfunctions of the kinetic operator, because in that case the kinetic terms become a diagonal quadratic expression in terms of the \( a_n \). The problem is therefore to find a local Lagrangian \( L_{\text{conf}}[x(t)] \), which can be split into a kinetic part \( L_{\text{conf}}^0 \) and the rest, denoted by \( L_{\text{conf}}^{\text{int}} \), which will provide the vertices. As shown in [4] the correct trace anomaly, in \( d = 2 \) due to the terms proportional to \( \epsilon \), and in \( d = 4 \) due to the terms proportional to \( \epsilon^2 \), is obtained by using the covariant action

\[
L_{\text{conf}} = \frac{1}{2} g_{ij} x^i x^j + \frac{\hbar^2}{8} R
\]

(8.4)

provided one uses normal coordinates. In this paper for supersymmetric systems, and in [3] for a scalar particle, it was found that the complete propagator through order \( \epsilon \) is produced by the covariant Lagrangians we have discussed, again provided one uses normal coordinates.
It is not sufficient only to know how to calculate in normal coordinates, because some Hamiltonians one encounters are not generally covariant. We have in mind the consistent anomalies in quantum field theories which maintain local scale (Weyl) invariance and consequently break general coordinate (Einstein) invariance. For these systems, the regulator can be constructed using the algorithm of reference [12], and, as expected, the corresponding Hamiltonians are not generally covariant. Thus, one must know the action $S_{\text{conf}}$ and the rules to evaluate path integrals in arbitrary coordinates. Of course, for generally covariant Hamiltonians, the propagator is a coordinate scalar, hence one may evaluate it in normal coordinates to obtain the correct answer.

For general coordinates, the form of $L_{\text{conf}}$ is unknown. In the literature, some attempts have been made to deduce $H_{\text{conf}}$ (related to $L_{\text{conf}}$ by a Legendre transformation, both are functions, not operators) from $\hat{H}$ by a functional differential equation. In particular, DeWitt has pursued this problem in the latest version of his book on supermanifolds[7], but he runs into the well-known problem how to define products of operators at the same point (i.e., at the same time). He proposes to use “time-ordering”, according to which one first defines all operators at different times, then considers all permutations of these operators, and finally takes the limit that all times become equal. Following this procedure for the bosonic case, one obtains not only the $\frac{h^2}{8} R$ term mentioned above, but also an extra noncovariant term

$$H_{\text{conf}} = \frac{1}{2} g^{ij} p_i p_j - \frac{1}{8} h^2 R - \frac{1}{8} h^2 \Gamma^i_{jk} \Gamma^j_{il} g^{kl}$$

(8.5)

Concluding that time-ordering breaks general coordinate covariance, he chooses a pseudo-Euclidean frame in which $\Gamma^i_{jk}(x(t=0)) = 0$ and computes (for the supersymmetric case) some two loop diagrams and finds that they are in agreement with the result for the propagator (which he obtained long ago by heat kernel methods).

We believe that this is not the whole story. First of all, the presence of a $\Gamma^2$ term at all $t$ will show up in calculations using normal coordinates at order $\epsilon^2$ (where 3-loop diagrams contribute). Although these contributions do look covariant in normal coordinates, one obtains an incorrect result for the $d = 4$ trace anomaly. Hence the $\frac{h^2}{8} \Gamma^2$ term without further modifications should not be present in $L_{\text{conf}}$ when one uses normal coordinates. Of course, once the correct action $S_{\text{conf}}$ is known in normal coordinates, one can find the corresponding action in general coordinates by making a change of integration variables. The Jacobian will contribute new vertices, and although we have not worked out the path integrals for general metrics, we have a remark. The background decompositions $x^i(t) = x^i_0(t) + x^i_{\text{qu}}(t)$ seems too linear to us for such a nonabelian theory as gravity. Let us recall that in superspace Yang-Mills theory, the Yang-Mills field is contained in $\exp(V)$, and a background field approach replaces this by $\exp(V_{\text{back}}) \exp(V_{\text{qu}})$. This is clearly an infinitely nonlinear split. Analogously, we expect also in gravity a nonlinear split of the form $x^i(t) = x^i_0(t) + x^i_{\text{qu}}(t) + (x^i_{\text{qu}}(t))^2 \text{terms} + \ldots$. We do not claim to have exhaustively studied this problem, and intend to return to it in the future. However, we have made some introductory calculations which we now describe.

In our explicit computation of the path integral to order $\epsilon$ (2 loops), we took $L_0 = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j$ where the $x$ in $g_{ij}(x)$ is the endpoint of the interval. Decomposing the action into the sum of this
kinetic term and vertices we computed loops. We then proceeded to evaluate all diagrams up to 2 loops in an arbitrary coordinate frame

\[ S = \frac{1}{\epsilon} \int_{-1}^{0} d\tau \frac{1}{2} g_{ij} (x_0(\tau) + x_{qu}(\tau)) \left[ (\dot{x}_0^i(\tau) + \dot{x}_{qu}^i(\tau)) (\dot{x}_0^j(\tau) + \dot{x}_{qu}^j(\tau)) + b^i(\tau)c^j(\tau) + a^i(\tau)a^j(\tau) \right] \quad (8.6) \]

where \( \tau = t/\epsilon \). Here \( a, b, c \) are anticommuting ghosts due to exponentiating the measure. They were first introduced by Bastianelli in [3]. The present form was obtained in [4]. Due to these ghosts all loop contributions become completely finite. We contract all quantum fields, using (see (A.7))

\[
\begin{align*}
<x_{qu}^i(\sigma)x_{qu}^j(\tau)>&= -\epsilon \hbar g^{ij} \Delta(\sigma, \tau) \\
b^i(\sigma)c^j(\tau)&= -2\epsilon \hbar g^{ij} \partial_a^2 \Delta(\sigma, \tau) \\
a^i(\sigma)a^j(\tau)&= \epsilon \hbar g^{ij} \partial_a^2 \Delta(\sigma, \tau)
\end{align*}
\]

(8.7)

where

\[ \Delta(\sigma, \tau) = -2 \sum_{n=1}^{\infty} \frac{\sin(n\pi\sigma)\sin(n\pi\tau)}{n^2\pi^2} \quad (8.8) \]

The great advantage of the formulation in (8.6) is that it allows a regularization which treats \( x \) and the ghosts in a uniform manner, namely by mode cut-off: in all graphs one first uses \( \Delta(\sigma, \tau) \) in (8.8) where the sum runs up to \( N \), and only at the end takes \( N \) to infinity. Of course, \( x_0(t) = x + (y - x)t/\epsilon \) is only a solution for \( S_0 \) and not a solution of the full equation of motion, hence there are terms linear in \( x_{qu}(t) \) in the action. The path integral expectation value of a single \( x_{qu}(t) \) vanishes (since \( L_0 \) is symmetric under \( x_{qu}(t) \leftrightarrow -x_{qu}(t) \)), but contractions of the form \( x_{qu}(s)x_{qu}(t) \) contribute.

We found that the total set of 2 loop diagrams in the bosonic case precisely reproduces all terms in the propagator through order \( \epsilon \), except for one term (which vanishes in normal coordinates). Namely, if one uses the covariant action (omitting the \( \frac{1}{8} \Gamma^2 \) term), the path integral result is equal to the propagator plus the following remainder

\[-\frac{1}{32} \epsilon \hbar g^{ij} g^{kl} g^{mn} (\partial_i g_{km}) (\partial_j g_{ln}) + \frac{1}{48} \epsilon \hbar g^{ij} g^{kl} g^{mn} (\partial_i g_{km}) (\partial_l g_{jn}) = -\frac{1}{24} \epsilon \hbar g_{kl} \Gamma^k_i \Gamma^l_j \quad (8.9)\]

If one adds the \( \frac{1}{8} \Gamma^2 \) term, part of this remainder is canceled, but we are still left with another extra term

\[-\frac{1}{24} \epsilon \hbar g^{ij} g^{kl} g^{mn} (\partial_i g_{km}) (\partial_l g_{jn}) \quad (8.10)\]

We believe that a proper treatment of the Jacobian for the transformation from normal to general coordinates will cancel (8.9). Our conclusion for the bosonic case is that in the Hamiltonian sector we have a complete understanding, but in the path integral sector, we believe we know the correct \( S_{\text{conf}} \) (namely the covariant action with the \( \frac{1}{8} \Gamma^2 \) term but without \( \Gamma^2 \) terms) but we have no proof of this rule.

We now turn to the fermionic sector where we have not only found similar results, but also new results for fermionic phase space path integrals. In the \( N=2 \) supersymmetric case, there are also fermionic variables \( \psi^a_\alpha(t) \) present, with \( \alpha = 1, 2 \) and \( a = 1 \ldots n \). The Hamiltonian for
this system is not obtained from the regulator of a corresponding quantum field theory, but we
fixed its ordering by requiring the supersymmetry relation $2\delta_{ab}\hat{H} = \{\hat{Q}_a, \hat{Q}_b\}$ where the operator
orderings in the supersymmetry charge operator $\hat{Q}_\alpha$ in (5.10) were fixed by requiring hermiticity.
We then computed the propagator between states $|y, \xi\rangle$ and $|x, \eta\rangle$. The states $|\xi\rangle$ and $|\eta\rangle$ are
fermionic coherent states, and using general theory of coherent states, suitably adapted to the
fermionic case, we straightforwardly obtained the propagator to order $\epsilon$. Of course, we must now
also take into account the commutators $\{\hat{\psi}^i_a, \hat{\psi}^b_b\} = \delta_{ab}$. The final result in (6.11) was then written
as $\exp\left(-\frac{1}{8}(S_B + S_F) - \delta_{ab}\bar{\eta}^a\psi^{b}_{cl}(0)\right)$ times a factor containing only terms with curvatures. The
extra term $-\delta_{ab}\bar{\eta}^a\psi^{b}_{cl}(0)$ (where $\psi^{b}_{cl}(t)$ is a solution of the field equations with boundary value
$\psi^{b}_{cl}(t=-\epsilon) = \chi^b$) is well-known from the theory of path integrals for coherent states, and can be
found in textbooks, see section 3. The curvature terms should describe the Van Vleck determinant
and a term with $R$ which would have been the trace anomaly of a quantum field theory if $\hat{H}$ could
have been identified with its regulator. However, in supersymmetric theories it is more natural
to use superdeterminants, and hence we considered the double superderivative of the action. The
super Van Vleck determinant absorbed then all terms with Ricci curvatures, just as in the bosonic
case. We were left with a term involving the scalar curvature, which is exactly the same as in the
bosonic case. (Using the ordinary Van Vleck determinant, one is left with a term containing the
Ricci curvature). Thus, the propagator factorizes exactly as in the bosonic case, but now with a
super Van Vleck determinant.

To reproduce these operator results for the $N=2$ supersymmetric case from a path integral,
we took as action $S_{\text{conf}}$ the classical $N=2$ action in (6.19) with the boundary term, including the
$R\psi^4$ term but without any further term proportional to $R$. The reason we did not include an $R$
term is that, applying the time-ordering prescription to the $N=2$ Hamiltonian, we have found that
the term $-\frac{\hbar^2}{8}R$ of the bosonic action is precisely canceled by a similar term obtained by time-
ordering the $R\psi^4$ term! (In the $N=1$ case a similar cancellation of $\frac{1}{3}R$ terms occurs for different
reasons). Time-ordering however, also produces the $\Gamma^2$ term of the bosonic sector discussed above,
together with an $\omega^2$ term if one treats $\hat{p}, \hat{x}, \hat{\psi}$ and $\hat{\psi}$ as the independent variables. (However, with
$\hat{p} - \omega_{iab}(\hat{x})\hat{\psi}^i_a\hat{\psi}^b_b, \hat{x}, \hat{\psi}$ and $\hat{\psi}$ as independent variables, no $\omega^2$ term is produced).

With the fermionic propagator given by the mode expansion (and with the propagator for the
bosonic and ghost fields mentioned before) we then may compute all 2-loop graphs using (8.6)
and (8.7) together with (A.15) and (A.16). The result of these calculations can be summarized
as follows: to order $\epsilon$, the path integral produces the propagator plus two extra terms. The first
term comes from the bosonic sector and is equal to (8.9), the second one is due to contracting two
$\int dx \dot{x}^{i}_{qu} \omega_{iab}(x) \bar{\psi}^a_q \psi^b_u$ vertices. Writing the three propagators in a mode expansion, and truncating
the bosonic and fermionic propagators at the same $n = N$, we have found (by a numerical
evaluation) a finite but nonzero result

$$-\frac{1}{24} \epsilon \hbar^2 g_{kl} \Gamma^k_{ij} \Gamma^l_{lj} - \frac{1}{12} \epsilon \hbar^2 g^{ij} \omega^{a}_{ia} \omega^{b}_{ja}$$

(8.11)

In normal coordinates $\omega_{iab}(x)$ vanishes, and hence in normal coordinates we obtain complete agreement
between the operator approach and path integrals. In a future extension to general coordi-
nates, the contribution in (8.11) should be taken into account.

Finally, we considered the supersymmetric $N=1$ case. This case is more complicated than the $N=2$ case, because there are only real fermions present, whereas one needs complex fermions, both in the Hamiltonian approach (for operators $\psi$ and $\psi^\dagger$) and in the path integrals (to define separate boundary conditions on the left and the right). We resolved this problem by adding a set of free fermions in both cases; this should not alter the model. To preserve local Lorentz invariance, we required that these free fermions are inert under these transformations.

In their fundamental paper on chiral and gravitational anomalies\cite{1}, Alvarez-Gaumé and Witten did not need to introduce these extra fermions because they only considered traces of the propagator, so they did not need to specify separate boundary conditions on the left and the right. Moreover, the Jacobian for chiral transformations of the quantum fields led in the corresponding quantum mechanical model to periodic boundary conditions for the $N=1$ case, and half-periodic and half-antiperiodic or completely periodic for the $N=2$ case. In all these cases, there is a zero mode $\psi_0$ in the expansion of $\psi(t)$ in terms of eigenfunctions of the operator $\frac{\partial}{\partial t}$ (corresponding to our $S_0$). The Grassmann integration then requires a minimal number of $\epsilon R \psi^2 \psi_0^2$ vertices to saturate the integration over these zero modes, and this already produces enough powers of $\epsilon$ to cancel the $\epsilon^{-n/2}$ in the measure of the path integral. Consequently, for them the harmonic approximation is sufficient. For us, however, this is not sufficient. For example, for the trace anomaly we need completely antiperiodic boundary conditions for the fermions, so that no $\epsilon$-producing zero modes $\psi_0$ are present. In addition, we considered the full propagator, and not only its traces.

The actual computations in the $N=1$ case became very similar to those of the $N=2$ case, once the free fermions were added. The Hamiltonian which corresponds to the general coordinate invariant regulator of a spin-$\frac{1}{2}$ field, was the sum of the Laplace-Beltrami operator for fermions plus a term $-\frac{1}{8} \hbar^2 R$ (twice as much as truncation of the $N=2$ Hamiltonian would give). This result was consistent with the requirement that $\hat{H}$ be equal to the square of a hermitian supersymmetry charge. Again we found that the propagator factorized into the classical action (with boundary term), the super Van Vleck determinant, and a scalar curvature term which yields now the trace anomaly for the quantum field theory of a (Dirac or Majorana) fermion minimally coupled to gravity.

Then we considered the path integral approach to the propagator of the $N=1$ case. In this case we took as action $S_{\text{conf}}$ the action in (7.8), which is the classical supersymmetric action (with boundary term). No extra $R$ term was present because we showed that the $R$ term due to time-ordering in the bosonic sector cancels the explicit $R$ present in the Hamiltonian. Hence, for different reasons, both in the $N=1$ and $N=2$ cases, the action $S_{\text{conf}}$ equals the classical supersymmetric action. This result one might have taken anyhow on the basis of symmetry arguments, but it is gratifying that if follows rigorously from our Hamiltonian approach. We computed all graphs through two loops which contribute through order $\epsilon$. In normal coordinates we found again complete agreement with the operator approach, but in general coordinates we found again an extra term which a future extension of the path integral approach using arbitrary coordinates should explain.

We conclude this paper with a general solution for the path integral in terms of an action
$S_{\text{conf}}$, which corresponds to any, covariant or noncovariant, Hamiltonian of the form $\hat{H} = a(\hat{x})\hat{p}^2 + b(\hat{x})\hat{p} + c(\hat{x})$ with $n$ operators $\hat{x}^i$ and $\hat{p}_i$. First rewrite $\hat{H}$ into the form

$$\hat{H} = g^{-1/4}(\hat{x})(\hat{p} - A_i(\hat{x}))g^{1/2}(\hat{x})g^{ij}(\hat{x})(\hat{p}_j - A_j(\hat{x}))g^{-1/4}(\hat{x}) + V(\hat{x})$$

(8.12)

where $g = \det g_{ij}$. Then use the metric $g_{ij}(x)$ to construct normal coordinates. In these normal coordinates one has

$$S_{\text{conf}} = \int_{-\Delta t}^{0} \left[ \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + A_i(x)\dot{x}^i + V(x) - \frac{1}{8} \hbar^2 R(x) \right] \, dt$$

(8.13)

The path integral is evaluated by splitting off a kinetic term $g_{ij}(x(0))\dot{x}^i(t)\dot{x}^j(t)$, and adding the ghost action in (8.6). Propagators and vertices are now defined, and the loop expansion of the propagator in normal coordinates is obtained. Using mode cut-off $[3,4]$ to regulate divergent graphs, it is found that at any loop level the final result is completely finite. For general coordinate invariant $S_{\text{conf}}$, this result also holds in general coordinates. In other cases one must transform back from normal to general coordinates. We have checked that this procedure reproduces the propagator obtained from $\hat{H}$ through order $\epsilon$, and trace anomalies through order $\epsilon^2$, for the point particle, and $N=1$ and $N=2$ supersymmetry. Summarizing: whereas the heat kernel or the Hamiltonian approach yield the propagator directly, we have found the corresponding action $S_{\text{conf}}$ which exactly reproduces this propagator from a path integral.

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**A. Appendix: Evaluation of path integrals.**

The path integral in the bosonic case (section 4) has already been evaluated in normal coordinates in $[3,4]$, by expanding fields in Fourier series and converting the path integral to an integral over the Fourier coefficients. An important detail is that we must introduce a ghost system to exponentiate the factors $\sqrt{g(x_j)}$ at the $N-1$ intermediate points $x_j$ in the measure, hereby making the remainder of the measure translationally invariant. This makes loop corrections finite. We will now first evaluate the bosonic path integral in an arbitrary coordinate frame, and afterwards we will compute the fermionic path integrals.

We follow the approach of $[3,4]$, and start with the action in configuration space (note that extra terms of the type $R + \Gamma^2$ are already of order $\epsilon$, and can, to the order we are considering, simply be replaced by their classical values)

$$S_{\text{conf}} = \frac{1}{\epsilon} \int_{-1}^{0} d\tau \frac{1}{2} g_{ij}(x(\tau)) \left[ \dot{x}^i(\tau)\dot{x}^j(\tau) + b^i(\tau)c^j(\tau) + a^i(\tau)a^j(\tau) \right]$$

(A.1)
where we have rescaled \( \tau = t/\epsilon \), and introduced a ghost system to exponentiate the factors \( \sqrt{g(x(t))} \) appearing in the measure. The boundary conditions are given by \( x^i(-1) = y^i, x^i(0) = x^i \). All ghosts vanish at both endpoints, \( b^i(-1) = b^i(0) = c^i(-1) - c^i(0) = a^i(-1) = a_i(0) = 0 \), because there are only \( \sqrt{g} \) factors at the intermediate points. We will now make a decomposition in background and quantum fields, where we choose the background fields to satisfy the boundary conditions. We take vanishing background fields in the ghost sector, whereas for the bosonic fields we take

\[
x^i(\tau) = x^i_0(\tau) + x^i_{\text{qu}}(\tau) \quad ; \quad x_0^i(\tau) = x^i + (x - y)^i \tau
\]

We now evaluate the path integral by expanding the term \( g_{ij}(x(\tau)) \) around its value at \( x(\tau) = x \), and write \( S = S^{\text{kin}} + S^{\text{int}} \), where

\[
S^{\text{kin}} = \frac{1}{\epsilon} \int_{-1}^{0} d\tau \frac{1}{2} g_{ij} [\dot{x}^i \dot{x}^j + b^i c^j + a^i a^j]
\]

and

\[
S^{\text{int}} = \frac{1}{\epsilon} \int_{-1}^{0} d\tau \left( \frac{1}{2} \partial_s g_{ij} x^k + \frac{1}{4} \partial_l \partial_l g_{ij} x^k x^l \right) [\dot{x}^i \dot{x}^j + b^i c^j + a^i a^j]
\]

Now all functions \( g, \partial g \) are evaluated at the endpoint \( x \). Since all quantum fields vanish at \( \tau = 0 \) and at \( \tau = -1 \), we can decompose them in a mode expansion, yielding a natural regularization scheme for the propagators. The propagator was obtained in reference [4] by using the mode expansion, but one can also obtain it (more easily) from canonical methods. Expanding the Heisenberg operator \( \hat{x} \) in a complete set of zero modes of the kinetic operator \( -g_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} \) we get \( \hat{x}(t) = \hat{x}^i + g^{ij}(x) \hat{p}_j t \), with \( [\hat{x}^i, \hat{p}_j] = i \hbar \delta^i_j \). Imposing the boundary conditions \( x_{\text{qu}}(t=-\epsilon) = x_{\text{qu}}(t=0) = 0 \) as conditions on the bra and ket vacua, \( \hat{x}(t=-\epsilon)|0\rangle = 0 \) and \( \langle 0|\hat{x}(t=0) = 0 \), we find \( \langle 0|\hat{x}^i = 0 \) and \( \langle \hat{x}^i - \epsilon \hat{p}_j |0\rangle = 0 \). Hence

\[
\Delta^{ij}(s, t) \equiv \langle 0|T \hat{x}^i(s) \hat{x}^j(t)|0\rangle = \theta(s - t)\langle 0|\hat{x}^i + \hat{p}^j t|0\rangle \quad + \quad (s \leftrightarrow t)
\]

\[
= \theta(s - t)\langle 0|\hat{p}^j s \hat{x}^i(1 + t/\epsilon)|0\rangle \quad + \quad (s \leftrightarrow t)
\]

\[
= \theta(s - t)\hbar g^{ij} s(1 + t/\epsilon) \quad + \quad (s \leftrightarrow t)
\]

The Euclidean version of this result, after we make the rescaling \( \tau = t/\epsilon \), is given by

\[
\Delta^{ij}(\sigma, \tau) = -\epsilon h g^{ij} [\sigma(\tau + 1)\theta(\sigma - \tau) + \tau(\sigma + 1)\theta(\tau - \sigma)]
\]

Using the mode expansion and (A.3) we find the same propagator, but now with a regularized expression for \( \Delta(\sigma, \tau) \) (see [4])

\[
\Delta(\sigma, \tau) = -2 \sum_{n=1}^{\infty} \frac{\sin(n\pi\sigma) \sin(n\pi\tau)}{n^2 \pi^2}
\]

The simplest term comes about when we only consider the background fields \( x_0(\tau) \). This yields

\[
S_0 = \frac{1}{\epsilon} \left[ \frac{1}{2} g_{ij} + \frac{1}{4} \partial_k g_{ij} (y - x)^k + \frac{1}{12} \partial_k \partial_l g_{ij} (y - x)^k (y - x)^l \right] (y - x)^i (y - x)^j
\]

\[37\]
which already reproduces most terms in the classical action (compare (4.11)); note that since we are not expanding around the full solution to the equation of motion, we cannot expect to reproduce the full classical action in $S_0$. We will see that the remainder of the classical action is produced by quantum corrections.

Next we consider contributions from terms in which we contract one pair of quantum fields. These come about either when we just take $S^{\text{int}}$ by itself, or when we take $(S^{\text{int}})^2$. The terms of the first type yield

$$\frac{1}{12} \partial_k \partial_l g_{ij} \left[ \frac{1}{2} g^{ij} (x-y)^k (x-y)^l - g^{jl} (x-y)^i (x-y)^k + \frac{1}{2} g^{kl} (x-y)^i (x-y)^j \right]$$

(A.9)

which is part of the $R_{ij} (x-y)^i (x-y)^j$ term. Furthermore, the terms of the second type yield

$$\frac{1}{12} \frac{1}{24} g_{mn} \Gamma^m_{ij} \Gamma^n_{kl} (x-y)^i (x-y)^j (x-y)^k (x-y)^j$$

(A.10)

which we recognize as the remainder of the classical action.

The next set of terms comes from contracting two pairs of quantum fields. Again there are two types of contributions, from $S^{\text{int}}$ by itself and from its square. The first terms are

$$\frac{1}{24} \frac{1}{24} \bar{\psi}^a \gamma^i \psi^b \gamma^j g_{ij} \left[ g^{kl} g^{mn} (\partial_i g_{km}) (\partial_j g_{ln}) + \frac{1}{48} \epsilon g^{ij} g^{kl} g^{mn} (\partial_i g_{km}) (\partial_j g_{ln}) \right]$$

(A.11)

which we can identify as part of the scalar curvature term, whereas the terms coming from $(S^{\text{int}})^2$ combine with the terms in (A.9) into $-\frac{1}{12} R_{ij} (x-y)^i (x-y)^j$.

Finally, we should contract three pairs of quantum fields, now only in $(S^{\text{int}})^2$. Part of these terms we need to covariantize the expression in (A.11) into $\frac{1}{24} \epsilon h R$. The remaining terms are given by

$$-\frac{1}{32} \epsilon \bar{\psi}^a \gamma^i g^{kl} g^{mn} (\partial_i g_{km}) (\partial_j g_{ln}) + \frac{1}{48} \epsilon g^{ij} g^{kl} g^{mn} (\partial_i g_{km}) (\partial_j g_{ln})$$

(A.12)

which is equal to

$$\frac{1}{24} \epsilon \bar{\psi}^a \gamma^i \Gamma^k_{ij} \Gamma^l_{kj}$$

(A.13)

The significance of this term is discussed in the conclusions.

We now evaluate the fermionic (supersymmetric) path integrals. In these cases, the bosonic and ghost sector of the path integral can be handled in exactly the same way as before, again yielding the action $S_B$ together with the contribution $\frac{1}{24} \epsilon h R - \frac{1}{12} R_{ij} (x-y)^i (x-y)^j$. We now consider the fermionic sector.

Similarly to [3,4], we rescale $\tau = t/\epsilon$ to make the $\epsilon$-dependence more explicit and facilitate keeping track of the order in the expansion in $\epsilon$. We now first note that we can write

$$S = S_0 + S^{\text{kin}}_{\text{fer}} + S^{\text{int}}_{\text{fer}}$$

(A.14)

Here $S_0$ contains all terms with only bosonic or ghost fields. The other two terms are given by

$$S^{\text{kin}}_{\text{fer}} = \int_0^\tau d\tau \delta_{ab} \bar{\psi}^a \gamma^i \psi^b$$

(A.15)
for the $N=2$ supersymmetric nonlinear sigma model considered in section 6, whereas for the $N=1$ model discussed in section 7 we have

\begin{equation}
S_{\text{kin}}^{\text{int}} = \int_{-1}^{0} dt \left[ \frac{1}{2} \epsilon R_{abcd} \tilde{\psi}^{a} \psi^{b} \tilde{\psi}^{c} \psi^{d} \right]
\end{equation}

In a background field approach we decompose $\psi(\tau) = \psi_{0}(\tau) + \psi_{\text{qu}}(\tau)$, and $\tilde{\psi}(\tau) = \tilde{\psi}_{0}(\tau) + \tilde{\psi}_{\text{qu}}(\tau)$. We could have used fermionic normal coordinates and added a term proportional to $x_{\text{qu}}$ as in [2], but the result should not depend on the split between classical and quantum fields, and our choice is quite simple. We choose $\tilde{\psi}_{0}(\tau)$ and $\psi_{0}(\tau)$ to be the solutions the equations of motion of the kinetic part of the action, (A.15), which satisfy the boundary conditions, i.e. we take $\tilde{\psi}_{0}^{a}(\tau) = \tilde{\eta}^{a}$ and $\psi_{0}^{a}(\tau) = \chi^{a}$. Since we require $\tilde{\psi}_{0}^{a}(0) = \tilde{\eta}^{a}$ and $\psi_{0}^{a}(-1) = \chi^{a}$, this implies that the quantum fields need to satisfy the boundary conditions $\tilde{\psi}_{\text{qu}}^{a}(0) = 0$ and $\psi_{\text{qu}}^{a}(-1) = 0$. The Green function for the action (A.15) of the quantum fields then reads

\begin{equation}
G_{\alpha\beta}(\sigma, \tau) = \langle \psi_{\text{qu}, \alpha}(\sigma) \psi_{\text{qu}, \beta}(\tau) \rangle = \frac{1}{2} \hbar \delta^{\alpha\beta} \delta_{\alpha\beta} \left( \theta(\sigma - \tau) - \theta(\tau - \sigma) \right) + \frac{i}{2} \hbar \delta^{\alpha\beta} \epsilon_{\alpha\beta}
\end{equation}

One can easily check that this Green function is a solution to the equations of motion of (A.15) and that the boundary conditions are indeed satisfied. In particular, at $\sigma = \tau$ we define $< A(\tau) B(\tau) > = \lim_{\delta \rightarrow 0} \frac{1}{2} [A(\tau + \delta) B(\tau - \delta) + A(\tau - \delta) B(\tau + \delta)]$, and at $\sigma = \tau = 0$ or $\sigma = \tau = -1$ one gets the correct result using $\theta(0) = \frac{1}{2}$.

The propagator for the fields $\tilde{\psi}(t)$ and $\psi(t)$ can also easily be found from the mode expansion or from canonical methods. In the canonical approach $\psi(t) = \hat{\psi}$ and $\tilde{\psi}(t) = \hat{\tilde{\psi}}$ with boundary conditions on the vacua given by $\langle 0 | \hat{\psi}(t=0) = 0$ and $\hat{\tilde{\psi}}(t=-\epsilon)|0 \rangle = 0$. In the mode expansion, we write

\begin{equation}
\psi(t) = \sum_{n} b_{n} \cos((n + \frac{1}{2}) \pi t / \epsilon)
\end{equation}

and

\begin{equation}
\tilde{\psi}(t) = \sum_{n} \tilde{b}_{n} \sin((n + \frac{1}{2}) \pi t / \epsilon)
\end{equation}

and the kinetic matrix $\int_{-\epsilon}^{0} dt \tilde{\psi} \bar{\psi}$ becomes $\frac{1}{2} \pi (n + \frac{1}{2}) \tilde{b}_{n} b_{n}$, yielding the propagator

\begin{equation}
< T \tilde{\psi}^{a}(s) \psi^{b}(t) > = \frac{2}{\pi} \delta^{ab} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2}} \sin((n + \frac{1}{2}) \pi s / \epsilon) \cos((n + \frac{1}{2}) \pi t / \epsilon)
\end{equation}
This is the $\theta$-function $< T \tilde{\psi}^a(s) \psi^b(t) > = \delta^{ab} \theta(s - t)$ as follows from a Fourier analysis. For our purposes we prefer the expression (A.21) for the propagators in terms of modes, since this yields a convenient regularization.

We can now straightforwardly compute the transition element by expanding $\exp \left( -\frac{1}{\hbar} S^{\text{inf}}_{\text{fer}} \right)$ and contracting the quantum fields, using the above Green functions for the fermions. For the bosonic and ghost fields, the Green functions have been computed in [4]. The propagator for the bosonic fields is given in (A.6).

We will first consider the $N=2$ case, with the interaction given in (A.16). When we expand $\exp \left( -\frac{1}{\hbar} S^{\text{inf}}_{\text{fer}} \right)$ we will for the first term in this expansion only need the contraction at equal time $< \tilde{\psi}^a(\tau) \psi^b(\tau) > = -\frac{1}{2} \hbar \delta^{ab}$. Since this is symmetric in $a$ and $b$, the first term in (A.16) will yield no contribution, whereas the second term contributes \( \frac{1}{2} \epsilon R^{ab} \eta^a \chi^b - \frac{1}{2} \hbar R \). Next consider the term \( \frac{1}{2} \left( \frac{1}{\hbar} S^{\text{inf}}_{\text{fer}} \right)^2 \). Terms involving the $R \psi^4$ term are of higher order in $\epsilon$ (for example, the cross term is of order $(x - y) \epsilon \sim \epsilon^{3/2}$), so we only need to find the contribution from the square of the $\hat{x} \psi^2$ term. When we contract only the four fermionic fields, using (A.19), one finds zero. The contraction of the two $\dot{x}$ fields and two fermionic fields (using (A.19) and (A.6)) vanishes because the fermionic contractions yield $\theta(\sigma - \tau) + \theta(\tau - \sigma) = 1$, and the integral of the bosonic part vanishes. Finally, we can contract four fermionic and two bosonic fields, which yields a contribution \( -\frac{1}{12} \epsilon h g^{ij} \omega^a_{ia} \omega^b_{jb} \chi^a \). In order to obtain this result, we used the regularized version of the propagator from the mode expansion, evaluated the integrals using the same number of bosonic and fermionic modes, and only afterwards let the number of modes go to infinity. Contractions involving the other bosonic fields are again of higher order and need not be considered.

We should be careful since we are in fact expanding the fermionic fields $\tilde{\psi}^a(\tau)$ and $\psi^a(\tau)$ around the constant fields $\bar{\eta}^a$ and $\chi^a$ respectively, which are not solutions to the equations of motion (6.15) of the full action. Since $< \psi_{\text{qu}}(\tau) >= < \tilde{\psi}_{\text{qu}}(\tau) >= 0$ due to the fact that the action is even in the number of Grassmann fields, the term from $S^{\text{inf}}_{\text{fer}}$ which is linear in quantum fields vanishes. However, we do have to take into account the contribution when we contract only one pair of fermions, each field from a different factor $S^{\text{inf}}_{\text{fer}}$ in \( \frac{1}{2} \left( \frac{1}{\hbar} S^{\text{inf}}_{\text{fer}} \right)^2 \). This yields the contribution \( \frac{1}{2} (x^i - y^i)(x^j - y^j) \omega_{ia} \omega_{ja} \bar{\eta}^a \chi^b \). When we were expanding around the solutions to the equations of motion, this term was part of the classical action (6.13). Since we are now expanding around the constant fields $\bar{\eta}^a$ and $\chi^a$, this term is not part of the expansion of the action but instead appears as an additional quantum correction.

Taking the contributions from the bosonic and ghost sector, together with the above computed contributions from the fermionic sector, we arrive at (6.11), plus a remainder $-\frac{1}{12} \epsilon h g^{ij} \omega_{ia} \omega_{jb} - \frac{1}{24} \epsilon h g_{kl} \Gamma^k_{ja} \Gamma^l_{ia}$. Similarly to the bosonic case, this term vanishes in normal coordinates, so that we obtain complete agreement between the Hamiltonian and path integral approach. In general coordinates, we again expect that a proper treatment of the Jacobian generated by a change of variables from normal to general coordinates will lead to cancellation of this term.

Finally we consider the $N=1$ case. Our aim is to reproduce the result in (7.3) from a path integral with (7.8) as action, and with boundary conditions $\tilde{\psi}^a(0) = \bar{\eta}^a$ and $\psi^a(-1) = \chi^a$. As explained in the beginning of this appendix we should obtain from the fermionic sector the terms
in \( \tilde{S}_F \) in (7.4) but also the last term in (7.3). In this case it is easier to work with the \( \psi_i \) propagator given in (A.18) then with the \( \tilde{\psi} \), \( \psi \) propagators, since the fermionic sector of the interactions, given in (A.17), only depends on \( \psi_i \). We again decompose \( \psi_i^a \) into a sum \( \psi_{i,0}^a + \psi_{i,1qu}^a \) with \( \psi_{i,0}^a = \frac{1}{\sqrt{2}} (\bar{\eta}^a + \chi^a) \) and \( \psi_{i,1qu}^a \) appearing in the propagators. The first term in \( \tilde{S}_F \) is of course due to the last term in (7.8). The next set of contributions comes from the expectation value of \( -\frac{1}{\hbar} S_{\text{int}}^d \). The classical part is given by

\[
-\frac{1}{\hbar} \int_{-1}^{0} d\tau \left[ \frac{1}{4} x_0^i(\tau) \omega_{iab}(x_0(\tau)) (\bar{\eta}^a + \chi^a)(\bar{\eta}^b + \chi^b) \right] \quad (A.22)
\]

where \( x_0^i(\tau) \) is a solution of the bosonic part of the action. Expanding \( x_0^i(\tau) = x^i + (x - y)^i \tau + O(x - y)^2 \), one recovers the last two terms in (7.4). The equal time contraction of \( \psi_i^a \psi_i^b \) in (A.17) vanishes by itself (it anyhow is proportional to \( \delta^{ab} \)). Also the contribution from the contraction in \( \dot{x}^i \omega_{iab} \) in (A.17) vanishes, as it is proportional to \( \int_{-1}^{0} \tau < \dot{x}^i(\tau) x_j(\tau) > \) which is zero according to (A.6). We are left with the last term in (7.3), which should come from the term \( \frac{1}{2} \left( \frac{1}{\hbar} S_{\text{int}}^d \right)^2 \). The classical part of this term is of course contained in the expansion of \( \exp \left( -\frac{1}{\hbar} \tilde{S}_F \right) \). The contraction of \( \dot{x}^i(\sigma) \) with \( \dot{x}^j(\tau) \) is of order \( \epsilon \), and would contribute, but its integral over \( \sigma \) and \( \tau \) vanishes. One \( \dot{x}^i(\sigma) \) with \( x^j(\tau) \) contraction is also of order \( \epsilon \), but would leave a factor \( \dot{x}_0^i \) which is of order \( \epsilon^{1/2} \), so this term is of higher order. The contraction of all four fermions is nonvanishing, and indeed reproduces the term \( \frac{1}{4!} \epsilon^{ijkl} (x^i - y^i)(x^j - y^j) \omega_{ia}^b \omega_{jb}^a \), in agreement with (7.3). Finally, we can contract four fermionic fields and two bosonic fields \( \dot{x}^i \). This yields \( -\frac{1}{24} \epsilon^{ijkl} \omega_{ia}^b \omega_{ja}^b \), where, as in the \( N=2 \) case, we take the same number of bosonic and fermionic modes to regularize the integrals. Adding all contributions we find the propagator for the \( N=1 \) case as given in (7.3), plus a remainder which now equals \( -\frac{1}{24} \epsilon^{ijkl} \omega_{ia}^b \omega_{ja}^b - \frac{1}{24} \epsilon^{ijkl} \Gamma_{kl}^i \Gamma_{ij} \). Again, for general coordinates we expect this term to cancel against a Jacobian; in normal coordinates we find complete agreement with the Hamiltonian result.

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