On coloring numbers of graph powers

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Abstract

The weak \( r \)-coloring numbers \( \text{wcol}_r(G) \) of a graph \( G \) were introduced by the first two authors as a generalization of the usual coloring number \( \text{col}(G) \), and have since found interesting theoretical and algorithmic applications. This has motivated researchers to establish strong bounds on these parameters for various classes of graphs.

Let \( G^p \) denote the \( p \)-th power of \( G \). We show that, all integers \( p > 0 \) and \( \Delta \geq 3 \) and graphs \( G \) with \( \Delta(G) \leq \Delta \) satisfy \( \text{col}(G^p) \in O(p \cdot \text{wcol}_\lceil p/2 \rceil(G)(\Delta - 1)^{\lfloor p/2 \rfloor}) \); for fixed tree width or fixed genus the ratio between this upper bound and worst case lower bounds is polynomial in \( p \). For the square of graphs \( G \), we also show that, if the maximum average degree \( 2k - 2 < \text{mad}(G) \leq 2k \), then \( \text{col}(G^2) \leq (2k - 1)\Delta(G) + 2k + 1 \).

Keywords: graph power, square of graphs, coloring number, weak coloring number, maximum average degree, Harmonious Strategy.

1 Introduction

Let \( G = (V, E) \) be a graph. For two vertices \( x \) and \( y \) in the same component of \( G \), the distance \( \text{dist}_G(x, y) \) between \( x \) and \( y \) is the length of a shortest \( x, y \)-path in \( G \). The \( k \)-th open neighborhood \( N^k_G(v) \) and \( k \)-th closed neighborhood \( N^k_G[v] \) of a vertex \( v \in V \) are defined by

\[
N^k_G(v) = \{ w \in V : \text{dist}_G(v, w) = k \} \quad \text{and} \quad N^k_G[v] = \{ w \in V : \text{dist}_G(v, w) \leq k \}.
\]

As usual, we set \( N_G(v) = N^1_G(v) \), \( N_G[v] = N^1_G[v] \) and \( d_G(v) = |N_G(v)| \). Finally, we drop the subscripts \( G \) in the above notations when \( G \) is clear from the context.

The \( p \)-th power of \( G \) is the graph \( G^p = (V, E^p) \), where \( E^p = \{ xy : 1 \leq \text{dist}_G(x, y) \leq p \} \). Then \( N_{G^p}(x) = N^p_G[x] - x \). Here we are concerned with the problem of bounding the chromatic number and the list chromatic number of the \( p \)-th powers of graphs from various classes, particularly for fixed maximum degree \( \Delta \) and arbitrary \( p \). Although more general, our results improve on the known bounds for the chromatic number of graph powers of graphs excluding some fixed minor.

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1.1 Generalized coloring numbers

For a graph $G = (V, E)$, let $\Pi := \Pi(G)$ be the set of total orderings of the vertex set $V$. For $\sigma \in \Pi$ and $x \in V$, set

1. $V^l_{\sigma}(x) = \{ y \in V : y <_{\sigma} x \}$, $V^r_{\sigma}[x] = V^l_{\sigma}(x) + x$; and
2. $V^r_{\sigma}(x) = \{ y \in V : x <_{\sigma} y \}$, $V^r_{\sigma}[x] = V^r_{\sigma}(x) + x$.

Thus $\{V^l_{\sigma}(x), \{x\}, V^r_{\sigma}(x)\}$ partitions $V$ into the left set of $x$, singleton $x$, and the right set of $x$. The coloring number of $G$, denoted col($G$), is defined by

$$\text{col}(G) = \min_{\sigma \in \Pi} \max_{x \in V} |N[x] \cap V^l_{\sigma}[x]|.$$ \hspace{1cm} \text{(1.1)}

Greedily coloring the vertices of $G$ in an order that witnesses its coloring number shows that

$$\chi(G) \leq \chi_l(G) \leq \text{col}(G),$$ \hspace{1cm} \text{(1.2)}

where $\chi(G)$ is the chromatic number of $G$, and $\chi_l(G)$ is the list chromatic number of $G$.

**Generalized coloring numbers** were first introduced by Kierstead and Yang in \[18\] after similar notions were explored by various authors \[4, 15, 16, 17\] in the cases $k = 2, 4$. Let $k \in \mathbb{Z}^+ \cup \{\infty\}$. A vertex $y$ is weakly $k$-reachable from $x$ with respect to $\sigma$ if $y \in V^l_{\sigma}[x]$ and there is an $x, y$-path $P$ with $|P| \leq k$ and $V(P) \subseteq V^r_{\sigma}[y]$. Let $W^k_{\sigma}[x]$ be the set of vertices that are weakly $k$-reachable from $x$ with respect to $\sigma$. The weak $k$-coloring number, denoted $wcol_k(G)$, of $G$ is defined by:

$$wcol_k(G) = \min_{\sigma \in \Pi} \max_{x \in V} |W^k_{\sigma}[x]|.$$ \hspace{1cm} \text{(1.3)}

Observe that $\text{col}(G) = wcol_1(G)$.

The weak coloring numbers have found many important and diverse applications (cf. \[4, 7, 11\]). As shown by Nešetřil and Ossona de Mendez \[23, \text{Lemma 6.5}\], they also provide a gradation between the coloring number and the tree-depth $td(G)$ of a graph $G$ as follows:

$$\text{col}(G) = wcol_1(G) \leq wcol_2(G) \leq \cdots \leq wcol_{\infty}(G) = td(G).$$ \hspace{1cm} \text{(1.4)}

Graph classes with bounded expansion (a notion extending graph classes excluding a minor or topological minor) were first introduced by Nešetřil and Ossona de Mendez \[23, 24\]. Zhu \[28\] (also see \[26\]) characterized graph classes with bounded expansion as those classes $\mathcal{C}$ for which there is a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that all graphs $G \in \mathcal{C}$ and all integers $k \in \mathbb{N}$ satisfy

$$wcol_k(G) \leq f(k).$$ \hspace{1cm} \text{(1.5)}

The following theorem gives upper bounds on the weak $k$-coloring numbers for various graph classes. Items (2–4) below are essentially due to \[12\], where all are proved using the same technique; here by using an observation in \[11\], their results are improved a bit by adding the last negative term. Item 5 is Proposition 28 of \[13\].

**Theorem 1.1** All positive integers $k$ and graphs $G$ satisfy:

1. \[8\] $wcol_k(G) \leq \binom{k+t}{t}$, if $tw(G) \leq t \leq \Delta + 1$, where $tw(G)$ is the tree-width of $G$, and this is sharp;
2. [11,12] \( \text{wcol}_k(G) \leq \binom{k+t-2}{t-2}(t-3)(2k+1) - k(t-3), \) if \( t \geq 4 \) and \( G \) has no \( K_t \) minor;

3. [11,12] \( \text{wcol}_k(G) \leq (2g + \binom{k+2}{2})(2k+1) - k, \) if \( G \) has genus \( g; \)

4. [11,12] \( \text{wcol}_k(G) \leq \binom{k+2}{2}(2k+1) - k, \) if \( G \) is planar;

5. [13] \( \text{wcol}_k(G) \leq s(t-1)\binom{k+s}{s}(2k+1), \) if \( G \) is \( K^*_s,t \)-minor-free, where \( K^*_s,t \) is the complete join of \( K_s \) and \( \overline{K_t}. \)

1.2 Parameters for measuring density

The coloring number is closely related to various parameters for measuring the local density of a graph. The arboricity of a graph \( G \), denoted \( \text{arb}(G) \), is the minimum number of forests required to cover the edges of \( G \). By Nash-Williams’ Theorem [22], \( \text{arb}(G) = \max_{H \subseteq G, |H| \geq 2} \left\lceil \frac{|H|}{|H|-1} \right\rceil. \)

The maximum average degree of \( G \) is \( \text{mad}(G) = \max_{\emptyset \neq H \subseteq G} \frac{2|H|}{|H|-1}. \) The following proposition is well known and easy to prove.

**Proposition 1.2** Every graph \( G \) satisfies

\[ \chi(G) \leq \chi_t(G) \leq \text{col}(G) \leq \lfloor \text{mad}(G) \rfloor + 1 \leq 2\text{arb}(G). \]

For a graph \( G = (V,E) \), let \( \vec{E} = \{ \vec{e}, \vec{e} : e \in E \} \) be the set of orientations of its edges. Define a weak orientation of \( G \) to be a function \( w : \vec{E} \rightarrow \mathbb{R} \) such that \( w(\vec{u}v) + w(\vec{uv}) = 1 \) and \( w(\vec{uv}), w(\vec{vu}) \geq 0 \) for all \( uv \in E \). We say that \( G \) is weakly oriented if it has been assigned a weak orientation. Observe that ordinary (unoriented) graphs can be interpreted as weakly oriented graphs whose edges have weight 1/2 in both directions, and oriented graphs can be interpreted as weakly oriented graphs whose weights are 0,1-valued. Define the out-weight \( w^+(u) \) of \( u \) and the maximum out-weight \( \Delta^+_w(G) \) of \( G \) by

\[ w^+_G(u) := w^+(u) := \sum_{uv \in E} w(\vec{uv}) \quad \text{and} \quad \Delta^+_w(G) := \max_{u \in V} w^+_G(u). \]

Using standard notation, let \( \Delta^+(\vec{G}) \) denote the maximum outdegree of an oriented graph \( \vec{G} \).

**Proposition 1.3** Every graph \( G \) satisfies both:

1. \( 2\min_w \Delta^+_w(G) = \text{mad}(G) \), where \( w \) runs over all weak orientations of \( G \) and

2. (cf. Hakimi [23]) \( 2\min_G \Delta^+(\vec{G}) = \lfloor \text{mad}(G) \rfloor \).

**Proof.** First we prove item 1. For any subgraph \( H \subseteq G \), and weak orientation \( w, \)

\[ \|H\| = \sum_{\vec{e} \in E(H)} (w(\vec{e}) + w(\vec{e})) = \sum_{v \in V(H)} w^+_H(v) \leq |H| \cdot \Delta^+_w(H) \leq |H| \cdot \Delta^+_w(G). \]

Setting \( m = \text{mad}(G) = \max_{H \subseteq G} \|H\| / |H|, \) we have \( m \leq 2\Delta^+_w(G) \).
Now we find a weak orientation $w$ with $2\Delta^+(G) \leq m$. Fix $H$ witnessing $m$, and let $n = |H|$. Pick $w : \vec{E} \rightarrow \{k/n : k = 0, \ldots, n\}$ so that the excess weight

$$b(w) := \sum \{2w^+(x) - m : x \in V \text{ and } 2w^+(x) > m\}$$

is minimum. This is possible since there are only $(n+1)||G||$ choices for $w$. It suffices to show $b(w) = 0$. Suppose not. Then there is a vertex $x$ with $2w^+(x) > m$. By the choice of $w$, $2w^+(x) - m \geq 1/n$. Let $S$ be the set of vertices $v \in V$ for which there is a path $P_v = v_0 \ldots v_s$ with $x = v_0$ and $v = v_s$ such that every forward oriented edge has positive weight (and so weight at least $1/n$). Set $H' = G[S]$. If $v \in S$ and $w \in N(v) \setminus S$ then $w(v\bar{w}) = 0$, so $w_{H'}^+(v) = w_G^+(v)$. Since $2\|H'\|/\|H\| \leq m$, there is $v \in S$ with $2w_{H'}^+(v) = 2w_{H'}^+(v) < m$, and so $2w^+(v) \leq m - 1/n$. Define a new weak orientation $w'$ by decreasing (increasing) the weight of each forward (backward) edge of $P_v$ by $1/n$. Now $b(w') < b(w)$, a contradiction.

For the proof of item 2 replace “weak orientation” with “0,1-orientation”, set $m = \lceil \text{mad}(G) \rceil = \max_{H \subseteq G} \|H\|/\|H\|$, and set $n = 1$ in the proof of item 1. [1]  

In Section 2, for general $p$, we study the coloring number of the $p$-th power of graphs $G$. We show that, all positive integers $p$ and $\Delta$ and graphs $G$ with $\Delta(G) \leq \Delta$ satisfy $\text{col}(G^p) \leq O(p \cdot \text{wcol}_{\lceil p/2\rceil}(G)(\Delta - 1)^{\lceil p/2 \rceil})$; for fixed tree width or fixed genus the ratio between this upper bound and worst case lower bounds is polynomial in $p$. In Section 3, we study the coloring number of the square of graphs $G$; we show that, if the maximum average degree $2k - 2 < \text{mad}(G) \leq 2k$, then $\text{col}(G^2) \leq (2k - 1)\Delta(G) + 2k + 1$.

## 2 Coloring numbers of graph powers

### 2.1 Previous results

If $G$ is a connected graph with diameter at most $p$ then $G^p = K_{|G|}$. As observed in [1], if $T$ is a maximum tree of height $\lceil p/2 \rceil$ and $\Delta(T) \leq \Delta$ then

$$\chi(T^p) \geq 1 + \Delta \sum_{i=0}^{\lceil p/2 \rceil - 1} (\Delta - 1)^i = 1 + \Delta \frac{(\Delta - 1)^{\lceil p/2 \rceil} - 1}{\Delta - 2} = \frac{\Delta(\Delta - 1)^{\lceil p/2 \rceil} - 2}{\Delta - 2} \approx L. \quad (1)$$

For the square of planar graphs, Agnarsson and Halldórsson [1] proved that if $G$ is a planar graph with $\Delta = \Delta(G) \geq 750$, then $\text{col}(G^2) \leq \lceil \frac{5}{8} \Delta \rceil$; and this is sharp. An upper bound on the coloring number of $G^p$ is provided by the following theorem.

**Theorem 2.1 (Agnarsson and Halldórsson [1])** For all $p, \Delta \in \mathbb{Z}^+$ and graphs $G$ with $\Delta(G) \leq \Delta$,

$$\text{arb}(G^p) \leq 2^{p+1}(\text{arb}(G))^{\lceil p/2 \rceil} \Delta^{\lceil p/2 \rceil}.$$ 

This upper bound was improved for chordal graphs in [21].

**Theorem 2.2 (Král’ [21])** For all $p, \Delta \geq 2 \in \mathbb{Z}^+$ and chordal graphs $G$ with $\Delta(G) \leq \Delta$,

$$\text{col}(G^p) \leq \left\lfloor \sqrt{\frac{91p - 118}{384}}(\Delta + 1)^{(p+1)/2} \right\rfloor + \Delta + 1.$$
2.2 New result

In this subsection we improve the known bounds on $\text{col}(G^p)$ for graph classes, including planar and chordal graphs, whose weak coloring numbers grow subexponentially.

Theorem 2.3 All integers $p > 0$ and $\Delta \geq 3$ and graphs $G = (V, E)$ with $\Delta(G) \leq \Delta$ satisfy

$$\text{mad}(G^p) \leq \frac{\Delta}{\Delta - 2} \left[ \frac{p + 1}{2} \right] \text{wcol}_{\lfloor p/2 \rfloor}(G)(\Delta - 1)^{\lfloor p/2 \rfloor}.$$ 

Proof. Suppose $G$ is a graph with $\Delta(G) \leq \Delta$ and $\sigma \in \Pi(G)$ witnesses that $\text{wcol}_{\lfloor p/2 \rfloor}(G) = q$.

By Proposition 1.3.1, it suffices to construct a weak orientation $w$ such that

$$w^+(G) \leq \frac{\Delta}{\Delta - 2} \left[ \frac{p + 1}{2} \right] q(\Delta - 1)^{\lfloor p/2 \rfloor}.$$ 

Consider any edge $e = uv$ in $G^p$. Then there is a path of length at most $p$ that connects $u$ and $v$. Choose such a $u,v$-path $Q_e = u_0\ldots u_s \subseteq G$ with minimum length $s = ||Q_e|| \leq p$. Let $l_e$ be the $\sigma$-least vertex in $Q_e$. If $e$ has a unique end, say $u$, with the distance $||uQ_el_e|| \geq s/2$, then set $w(u\bar{v}) = 1$ and $w(v\bar{u}) = 0$; else set $w(u\bar{v}) = \frac{1}{2} = w(v\bar{u})$.

Consider any vertex $u \in V$, and suppose $e = uv \in E$ with $w(u\bar{v}) > 0$. Then $Q_e$ has the form $u_0Q_eu_hQ_eu_s$, where $l_e = u_i$, $h = 0$ if $i \leq \lfloor s/2 \rfloor$ and $h = i - \lfloor s/2 \rfloor$ else. Then

$$0 \leq h \leq \left\lfloor \frac{p}{2} \right\rfloor, u_h \in N^h(u), l_e = u_i \in W^{[s/2]}(u_h) \text{ and } v \in N^{s-i}(l_e). \quad (2)$$

Moreover, if $h = 0$ then $w(u\bar{v}) = \frac{1}{2} = w(v\bar{u})$.

Thus $w^+(u)$ is at most the number of ways to pick $h > 0, u_i, v$ satisfying (2) plus one half the number ways to pick $h = 0, u_i, v$ satisfying (2). By the definition of the $h$-th open neighborhood, $|N^h(u)| \leq \Delta(\Delta - 1)^{h-1}$ and $|N^{s-i}(l)| \leq (\Delta - 1)^{s-i}$; also $|W^{[p/2]}(u_h)| \leq q$.

Noticing that the special case $h = 0$ accounts for the first term on the RHS of (3), we have

$$w^+(u) \leq \frac{1}{2} \left\lfloor \frac{p}{2} \right\rfloor \sum_{j=0}^{\left\lfloor \frac{p}{2} \right\rfloor} (\Delta - 1)^j + \sum_{h=1}^{\left\lfloor \frac{p}{2} \right\rfloor} \Delta(\Delta - 1)^{h-1} q \sum_{j=0}^{\left\lfloor \frac{p}{2} \right\rfloor - h} (\Delta - 1)^j$$

$$\leq \frac{1}{2} \frac{\Delta(\Delta - 1)^{\left\lfloor \frac{p}{2} \right\rfloor}}{\Delta - 2} + \sum_{h=1}^{\left\lfloor \frac{p}{2} \right\rfloor} \Delta(\Delta - 1)^{h-1} q \frac{(\Delta - 1)^{\left\lfloor \frac{p}{2} \right\rfloor - h + 1}}{\Delta - 2}$$

$$\leq \frac{\Delta}{\Delta - 2} \left[ \frac{p + 1}{2} \right] q(\Delta - 1)^{\lfloor p/2 \rfloor}. \quad (3)$$

The ratio obtained by dividing the bound of Theorem 2.1 by the lower bound $L$ from eq. (1), is clearly exponential. Dividing the bound of Theorem 2.2 by $L$ we get:

$$\sqrt[\Delta\Delta - 2]{\frac{91p - 118}{384} (\Delta + 1)^{\left(\frac{p+1}{2}\right)^2}} + 1 \geq \sqrt[\Delta\Delta - 2]{\frac{91p - 118}{384} (\Delta + 1)^{\left(\frac{p+1}{2}\right)^2}} \cdot \frac{(\Delta - 1)^{\left(\frac{p}{2}\right)}}{(\Delta - 1)^{\left(\frac{p}{2}\right)}}$$

$$\geq \sqrt[\Delta\Delta - 2]{\frac{91p - 118}{384} \cdot \frac{\Delta - 2}{\Delta} \cdot \left(1 + \frac{2}{\Delta - 1}\right)^{\frac{p}{2}}},$$
which is also exponential. But the ratio obtained by dividing the bound of Theorem 2.3 by \( L \) is polynomial in \( p \) whenever \( \text{wcol}_p(G) \) is polynomial in \( p \). In particular this is the case for graphs with bounded tree width and graphs with no \( K_t \) minor, including graphs with bounded genus; and graphs with no \( K_{s,t}^* \)-minor (where \( K_{s,t}^* \) is the complete join of \( K_s \) and \( K_t \)).

3 On the coloring number of the square of graphs

3.1 Previous results

The study of \( \chi(G^2) \) was initiated by Wegner in [25], and has been actively studied ever since. In [3], Charpentier made the following conjectures.

**Conjecture 3.1** ([3]) There exists an integer \( D \) such that every graph \( G \) with \( \Delta(G) \geq D \) has \( \chi(G^2) \leq 2\Delta(G) \).

**Conjecture 3.2** ([3]) For each integer \( k \geq 3 \), there exists an integer \( D_k \) such that every graph \( G \) with \( \Delta(G) \geq D_k \) has \( \chi(G^2) \leq k\Delta(G) - k \).

In [3], some examples are given to show that Conjectures 3.1 and 3.2 are best possible if they are true. In [20], Kim and Park disproved Conjectures 3.1 and 3.2 by showing that, for any positive integer \( D \), there is a graph \( G \) with \( \Delta(G) \geq D \) and \( \text{mad}(G) < 4 \) such that \( \chi(G^2) = 2\Delta(G) + 2 \); for any integers \( k \) and \( D \) with \( k \geq 3 \) and \( D \geq k^2 - k \), there exists a graph \( G \) with \( \text{mad}(G) < 2k \) and \( \Delta(G) \geq D \), such that \( \chi(G^2) \geq k\Delta(G) + k \).

For the upper bounds, the following result is [14, Theorem 4] by Hocquard, Kim and Pierron (very recently); a similar version was given by Charpentier in [3]. The version in [14] is proved by using a variant of discharging, and fixed some errors and inaccuracies of the original proof.

**Theorem 3.3** ([3, 13]) Let \( k \) be an integer and \( G \) be a graph with \( \text{mad}(G) < 2k \). Then
\[
\chi(G^2) \leq \max\{(2k-1)\Delta(G) - k^2 + k + 1, (2k-2)\Delta(G) + 2k^3 + k^2 + 2, (k-1)\Delta(G) + k^4 + 2k^3 + 2\}.
\]

Kim and Park [20] proved the following theorem.

**Theorem 3.4** ([20]) Let \( c \) be an integer such that \( c \geq 2 \). If a graph \( G \) satisfies \( \text{mad}(G) < 4 - \frac{1}{c} \) and \( \Delta(G) \geq 14c - 7 \), then \( \chi_l(G^2) \leq 2\Delta(G) \).

Bonamy, Lévêque, and Pinlou in [2] proposed the following question.

**Question 3.5** ([2]) What is, for any \( C \geq 1 \), the maximum \( m(C) \) such that any graph \( G \) with \( \text{mad}(G) < m(C) \) satisfies \( \chi_l(G^2) \leq \Delta(G) + C \).

As a natural generalization of Question 3.5, the following question seems interesting, especially by taking Conjectures 3.1, 3.2 and their recent developments into considerations.

**Question 3.6** What is, for a given integer \( k \geq 1 \) and any \( C \) (if \( k = 1 \), then \( C \geq 1 \)), the minimum \( m(C) \) such that any graph \( G \) with \( \text{mad}(G) \leq 2k - m(C) \) satisfies \( \chi_l(G^2) \leq k\Delta(G) + C \).
3.2 New result

The techniques we use in this section have their roots in the study of coloring games on graphs, in particular, the Harmonious Strategy introduced in [19]. In fact, a more recent game theoretic result of Yang [27, Theorem 4.5], already yields the following corollary.

**Corollary 3.7** ([27]) Let \( k \) be a positive integer, and let \( G \) be a graph with \( \text{mad}(G) \leq 2k \) and \( \Delta(G) \geq 2k - 2 \). Then \( \text{col}(G^2) \leq (3k - 2)\Delta(G) - k^2 + 4k + 2 \).

The next theorem is our result on the coloring number of the square of graphs. In its proof we construct an ordering of the vertices of a graph \( G \) to witness the given bound on \( \text{col}(G^2) \). This is done by iteratively adding new vertices to the end of the initial segment of already ordered vertices. (Contrast this with the usual method of adding new vertices at the front of the final segment already constructed.) There is a natural tension between adding a vertex too late and thus giving it too many earlier neighbors, and adding it too soon, and thus giving too many other vertices an earlier neighbor. The Harmonious Strategy provides a scheme for balancing these considerations by ensuring that no vertex is chosen before its out-neighbors and distance-2 out-neighbors have been considered. See [10] for another application of the Harmonious Strategy to a non-game problem.

**Theorem 3.8** Let \( k > 0 \) be an integer. If \( G \) is a graph with \( 2k - 2 < \text{mad}(G) \leq 2k \), then

\[
\text{col}(G^2) \leq (2k - 1)\Delta(G) + 2k + 1.
\]

**Proof.** Suppose \( G = (V, E) \) is a graph with \( 2k - 2 < \text{mad}(G) \leq 2k \). Thus

\[
2k - 1 \leq \Delta := \Delta(G).
\]

By Proposition [132] \( G \) has an orientation \( \vec{G} = (V, \vec{E}) \) with \( \Delta^+ := \Delta^+(\vec{G}) \leq k \). Let \( L \in \Pi(G) \).

Given a path \( P = x_0 \ldots x_l \subseteq G \), define the sign-sequence of \( P = x_0 \ldots x_l \) to be the sequence \( s(P) \) with length \( l \) whose \( i \)-th symbol is “+” if \( v_{i-1}v_i \in \vec{E} \) and “-” if \( v_iv_{i-1} \in \vec{E} \). For any \( x \in V \) and sign-sequence \( s \in \{+, -, ++, +-, --, -+, ---\} \), let \( N^s(x) \) denote the set of vertices \( y \) such that there is a shortest \( x, y \)-path \( P \subseteq G \) with \( s(P) = s \). Put \( d^s(x) = |N^s(x)| \). Also put \( N^{++}(x) = N^+(x) \cup N^{++}(x) \), \( d^{++}(x) = d^+(x) + d^{++}(x) \), \( N^{--}(x) = N^-(x) \cup N^{--}(x) \) and \( d^{--}(x) = d^-(x) + d^{--}(x) \).

Our task is to construct \( \sigma \in \Pi(G) \) witnessing [1]. To do this, we design an algorithm that collects vertices one at a time. Each time a vertex is collected, it is deleted from the set \( U \) of uncollected vertices, and put at the end of the initial segment of \( \sigma \) that has already been constructed. We maintain a set \( S_x \subseteq N^{++}(x) \) for each \( x \in V \). The vertex sets \( U \) and \( S_x \) are dynamic—they are updated as the algorithm runs.

We start without any collected vertices, so \( U := V \). For all \( x \in V \), set \( S_x := N^{++,++}(x) \). Then we run Algorithm [1] (see below).

When vertex \( u \in U \) with \( S_u \cap U \neq \emptyset \) is assigned to variable \( x \) at Line 2 or at Line 9 and then \( v \in S_u \cap U \) is immediately assigned to variable \( y \) at Line 4, we say that \( u \) contributes to \( v \) and \( v \) receives a contribution from \( u \).
Algorithm 1

1: while $U \neq \emptyset$ do
2:    $x := L\text{-min } U$
3:    while $S_x \cap U \neq \emptyset$ do
4:        $y := L\text{-min } S_x \cap U$
5:        $S_x := S_x \setminus \{y\}$
6:        if $S_x \cap U = \emptyset$ then
7:            collect $x$
8:        end if
9:    end while
10:    collect $x$
11: end while

When a vertex $w$ assigned to variable $x$ is collected at Line 7 or Line 11, we have $S_w \cap U = \emptyset$, so every vertex $v \in N^{+,++}(w)$ has received a contribution from $w$ or has been collected. Thus:

When $w \in V$ is collected, it has contributed to all $y \in N^{+,++}(w) \cap U$. (6)

When a vertex $v \in V$ receives a contribution at Line 4, it is still uncollected. It is immediately assigned to variable $x$ at Line 9. If $S_v \cap U = \emptyset$ then the inner while-loop ends, and $v$ is collected at Line 11; else $v$ contributes to some $u \in S_v \cap U$ at Line 4, and $|S_v \cap U|$ is reduced by 1 at Line 5. If now $S_v \cap U = \emptyset$ then $v$ is immediately collected at Line 7. Thus:

Each $v \in V$ receives at most $d^{+,++}(v)$ contributions. (7)

Consider any uncollected $v \in U$. It suffices to prove that $v$ has at most $(2k - 1)\Delta + 2k$ collected neighbors in $G^2$, i.e.,

$$|N^2[v] \setminus U| \leq (2k - 1)\Delta + 2k.$$ For all $w \in N^{+,--}(v) \setminus U$, we have $v \in N^{+,++}(w)$ and $w$ is collected before $v$. By (6), $w$ has contributed to $v$. By (7), $v$ has received at most $d^{+,++}(v)$ contributions. Thus:

$$|N^{-,--}(v) \setminus U| \leq d^{+,++}(v).$$ (8)

Now we have:

$$|N^2[v] \setminus U| \leq |N^{+,++}(v) \cup (N^{--,--}(v) \setminus U) \cup N^{++}(v) \cup N^{--}(v)|$$

$$\leq d^{+,++}(v) + |N^+(v) \cup N^{++}(v) \cup N^{--}(v)| + d^{--}(v)$$ (by (8))

$$\leq (k^2 + k) + d^+(v)\Delta + (\Delta - d^+(v))(k - 1)$$

$$= (k^2 + k) + d^+(v)(\Delta - k + 1) + \Delta(k - 1)$$

$$\leq (k^2 + k) + k(\Delta - k + 1) + \Delta(k - 1)$$ (by (5))

$$= (2k - 1)\Delta + 2k.$$

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\end{itemize}
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