Generalized Fibonacci and Lucas cubes arising from powers of paths and cycles

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Abstract
The paper deals with some generalizations of Fibonacci and Lucas sequences, arising from powers of paths, and cycles, respectively.

In the first part of the work we provide a formula for the number of edges of the Hasse diagram of the independent subsets of the $h^{th}$ power of a path ordered by inclusion. For $h = 1$ such a diagram is called a Fibonacci cube, and for $h > 1$ we obtain a generalization of the Fibonacci cube. Consequently, we derive a generalized notion of Fibonacci sequence, we call $h$-Fibonacci sequence. Then, we show that the number of edges of a generalized Fibonacci cube is obtained by convolution of an $h$-Fibonacci sequence with itself.

In the second part we consider the case of cycles. We evaluate the number of edges of the Hasse diagram of the independent subsets of the $h^{th}$ power of a cycle ordered by inclusion. For $h = 1$ such a diagram is called Lucas cube, and for $h > 1$ we obtain a generalization of the Lucas cube. We derive then a generalized version of the Lucas sequence, we call $h$-Lucas sequence. Finally, we show that the number of edges of a generalized Lucas cube is obtained by an appropriate convolution of an $h$-Fibonacci sequence with an $h$-Lucas sequence.

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1. Introduction

For a graph $G$ we denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its edges.

\textbf{Definition 1.1.} For $n,h \geq 0$,

(i) the $h$-power of a path, denoted by $P_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n, i \neq j$, $(v_i, v_j) \in E(P_n^{(h)})$ if and only if $|j - i| \leq h;$

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(ii) the \textit{h-power of a cycle}, denoted by $Q_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(Q_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$.

Thus, for instance, $P_n^{(0)}$ and $Q_n^{(0)}$ are the graphs made of $n$ isolated nodes, $P_n^{(1)}$ is the path with $n$ vertices, and $Q_n^{(1)}$ is the cycle with $n$ vertices. Figure 1 shows some powers of paths and cycles.

![Figure 1: Some powers of paths and cycles.](image)

**Definition 1.2.** An \textit{independent subset} of a graph $G$ is a subset of $V(G)$ not containing adjacent vertices.

Let $H_n^{(h)}$ and $M_n^{(h)}$ be the Hasse diagrams of the posets of independent subsets of $P_n^{(h)}$, and $Q_n^{(h)}$, respectively, ordered by inclusion. Clearly, $H_n^{(0)} \cong M_n^{(0)}$ is a Boolean lattice with $n$ atoms ($n$-cube, for short).

Before introducing the main results of the paper, we now provide some background on Fibonacci and Lucas cubes. Every independent subset $S$ of $P_n^{(h)}$ can be represented by a binary string $b_1b_2\cdots b_n$, where, for $i = 1, \ldots, n$, $b_i = 1$ if and only if $v_i \in S$. Specifically, each independent subset of $P_n^{(h)}$ is associated with a binary string of length $n$ such that the distance between any two 1’s of the string is greater than $h$. Following [10] (see also [7]), a Fibonacci string of order $n$ is a binary strings of length $n$ without consecutive 1’s. Recalling that the Hamming distance between two binary strings $\alpha$ and $\beta$ is the number $H(\alpha, \beta)$ of bits where $\alpha$ and $\beta$ differ, we can define the Fibonacci cube of order $n$, denoted $\Gamma_n$, as the graph $(V, E)$, where $V$ is the set of all Fibonacci strings of order $n$ and, for all $\alpha, \beta \in V$, $(\alpha, \beta) \in E$ if and only if $H(\alpha, \beta) = 1$.

One can observe that for $h = 1$ the binary strings associated with independent subsets of $P_n^{(h)}$ are Fibonacci strings of order $n$, and the Hasse diagram of the
set of all such strings ordered bitwise is \( \Gamma_n \). Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [3], and their combinatorial structure has been further investigated, e.g. in [3, 10]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, e.g., [4, 7]).

Remark. Consider the generalized Fibonacci cubes described in [4], i.e., the graphs \( B_n(\alpha) \) obtained from the \( n \)-cube \( B_n \) of all binary strings of length \( n \) by removing all vertices that contain the binary string \( \alpha \) as a substring. In this notation the Fibonacci cube is \( B_n(11) \). It is not difficult to see that \( H_n^{(h)} \) cannot be expressed, in general, in terms of \( B_n(\alpha) \). Instead we have:

\[
H_n^{(2)} = B_n(11) \cap B_n(101), \quad H_n^{(3)} = B_n(11) \cap B_n(101) \cap B_n(1001), \quad \ldots,
\]

where \( B_n(\alpha) \cap B_n(\beta) \) is the subgraph of \( B_n \) obtained by removing all strings that contain either \( \alpha \) or \( \beta \).

A similar argument can be carried out for cycles. Indeed, every independent subset \( S \) of \( Q_n^{(h)} \) can be represented by a circular binary string (i.e., a sequence of 0’s and 1’s with the first and last bits considered to be adjacent) \( b_1b_2\cdots b_n \), where, for \( i = 1, \ldots, n, \ b_i = 1 \) if and only if \( v_i \in S \). Thus, each independent subset of \( Q_n^{(h)} \) is associated with a circular binary string of length \( n \) such that the distance between any two 1’s of the string is greater than \( h \). A Lucas cube of order \( n \), denoted \( \Lambda_n \), is defined as the graph whose vertices are the binary strings of length \( n \) without either two consecutive 1’s or a 1 in the first and in the last position, and in which the vertices are adjacent when their Hamming distance is exactly 1 (see [9]). For \( h = 1 \) the Hasse diagram of the set of all circular binary strings associated with independent subsets of \( Q_n^{(h)} \) ordered bitwise is \( \Lambda_n \). A generalization of the notion of Lucas cubes has been proposed in [5].

Remark. Consider the generalized Lucas cubes described in [5], that is, the graphs \( B_n(\hat{\alpha}) \) obtained from the \( n \)-cube \( B_n \) of all binary strings of length \( n \) by removing all vertices that have a circular containing \( \alpha \) as a substring (i.e., such that \( \alpha \) is contained in the circular binary strings obtained by connecting first and last bits of the string). In this notation the Lucas cube is \( B_n(11) \). It is not difficult to see that \( M_n^{(h)} \) cannot be expressed, in general, in terms of \( B_n(\hat{\alpha}) \). Instead we have:

\[
M_n^{(2)} = B_n(\hat{11}) \cap B_n(101), \quad M_n^{(3)} = B_n(\hat{11}) \cap B_n(101) \cap B_n(1001), \quad \ldots
\]

To the best of our knowledge, our \( H_n^{(h)} \), and \( M_n^{(h)} \) are new generalizations of Fibonacci and Lucas cubes, respectively.

In the first part of this paper (which is an extended version of [1]) we evaluate \( p_n^{(h)} \), i.e., the number of independent subsets of \( P_n^{(h)} \), and \( H_n^{(h)} \), i.e., the number of edges of \( H_n^{(h)} \). We then introduce a generalization of the Fibonacci sequence, that we call \( h \)-Fibonacci sequence and denote by \( F^{(h)} \). Such integer sequence is based on the values of \( p_n^{(h)} \). Our main result (Theorem 3.4) is that, for \( n, h \geq 0 \), the sequence \( H_n^{(h)} \) is obtained by convolving the sequence \( F^{(h)} \) with itself.

In the second part we deal with power of cycles, and derive similar results for this case. Specifically, we compute \( q_n^{(h)} \), i.e., the number of independent subsets...
Remark. For $P$, the map $f$ establishes a bijection between independent $k$-subsets of $\mathbb{P}^{(h)}_n$. Further, we introduce a generalization of the Lucas sequence, that we call $h$-Lucas sequence and denote by $\mathcal{L}^{(h)}$. Such integer sequence is based on the values of $q^{(h)}_n$. The analogous of Theorem 3.4 in the Lucas case (Theorem 5.4) states that, for $n > h \geq 0$, the sequence $M^{(h)}_n$ is obtained by an appropriate convolution between the sequences $F^{(h)}$ and $\mathcal{L}^{(h)}$.

2. The independent subsets of powers of paths

For $n, h, k \geq 0$, we denote by $p^{(h)}_{n,k}$ the number of independent $k$-subsets of $\mathbb{P}^{(h)}_n$.

Remark. For $h = 1$, $p^{(h)}_{n,k}$ counts the number of binary strings $\alpha \in \Gamma_n$ with $k$ 1’s.

Lemma 2.1. For $n, h, k \geq 0$,

$$p^{(h)}_{n,k} = \binom{n - hk + h}{k}.$$ 

This is Theorem 1 of [2]. An alternative proof follows.

Proof. By Definition 1.2 any two elements $v_i$, $v_j$ of an independent subset of $\mathbb{P}^{(h)}_n$ must satisfy $|j - i| > h$. It is straightforward to check that whenever $n - hk - h < 0$, $p^{(h)}_{n,k} = 0 = \binom{n-hk+h}{k}$. It is also immediate to see that when $n = h = 0$ our lemma holds true.

Suppose now $n - hk - h \geq 0$. We can complete the proof of our lemma by establishing a bijection between independent $k$-subset of $\mathbb{P}^{(h)}_n$ and $k$-subsets of a set with $(n - hk + h)$ elements. Let $\mathcal{K}$ be the set of all $k$-subsets of a set $B = \{b_1, b_2, \ldots, b_{n-hk+h}\}$, and $\mathcal{I}_k$ the set of all independent $k$-subsets of $\mathbb{P}^{(h)}_n$. Consider the map $f : \mathcal{K} \to \mathcal{I}_k$ such that, for any $S = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \in \mathcal{K}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n - hk + h$,

$$f(\{b_{i_1}, b_{i_2}, \ldots, b_{i_j}, \ldots, b_{i_k}\}) = \{v_{i_1}, v_{i_2}+h, \ldots, v_{i_j+(j-1)h}, \ldots, v_{i_k+(k-1)h}\}.$$ 

Claim 1. The map $f$ associates an independent $k$-subset of $\mathbb{P}^{(h)}_n$ with each $k$-subset $S = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \in \mathcal{K}$.

To see this we first remark that $f(S)$ is a $k$-subset of $V(\mathbb{P}^{(h)}_n)$. Furthermore, for each pair $b_{i_j}, b_{i_j+t} \in S$, with $t > 0$, we have

$$i_{j+t} + (j + t - 1)h - (i_j + (j - 1)h) = i_{j+t} - i_j + th > h.$$ 

Hence, by Definition 1.1, $(f(b_{i_j}), f(b_{i_j+t})) = (v_{i_j+(j-1)h}, v_{i_j+(j+t-1)h}) \notin E(\mathbb{P}^{(h)}_n)$.

Thus, $f(S)$ is an independent subset of $\mathbb{P}^{(h)}_n$.

Claim 2. The map $f$ is bijective.
It is easy to see that $f$ is injective. Then, we consider the map $f^{-1} : \mathcal{I}_k \to \mathcal{K}$ such that, for any $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in \mathcal{I}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$,

$$f^{-1}(\{v_{i_1}, v_{i_2}, \ldots, v_{i_j}, \ldots, v_{i_k}\}) = \{b_{i_1}, b_{i_2-h}, \ldots, b_{i_{j-(j-1)h}}, \ldots, b_{i_k-(k-1)h}\}.$$ 

Following the same steps as for $f$, one checks that $f^{-1}$ is injective. Thus, $f$ is surjective.

We have established a bijection between independent $k$-subsets of $\mathbf{P}_n^{(h)}$ and $k$-subsets of a set with $(n - hk + h) \geq 0$ elements. The lemma is proved. \qed

Some values of $p_{n,k}^{(h)}$ are shown in Tables 1–3 (Section 6). The coefficients $p_{n,k}^{(h)}$ also enjoy the following property: $p_{n,k}^{(h)} = p_{n-k+1,k}^{(h-1)}$.

For $n, h \geq 0$, the number of independent subsets of $\mathbf{P}_n^{(h)}$ is

$$p_n^{(h)} = \sum_{k \geq 0} p_{n,k}^{(h)} = \sum_{k=0}^{\left\lceil n/(h+1) \right\rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\left\lceil n/(h+1) \right\rceil} \binom{n - hk + h}{k}.$$ 

**Remark.** Denote by $F_n$ the $n^{th}$ element of the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$ is the number of elements of the Fibonacci cube of order $n$.

The following, simple fact is crucial for our work.

**Lemma 2.2.** For $n, h \geq 0$,

$$p_n^{(h)} = \begin{cases} n + 1 & \text{if } n \leq h + 1, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h + 1. \end{cases}$$ 

A proof of this Lemma can also be obtained using the first part of [2] Proof of Theorem 1).

**Proof.** For $n \leq h + 1$, by Definition 1.2, the independent subsets of $\mathbf{P}_n^{(h)}$ have no more than 1 element. Thus, there are $n + 1$ independent subsets of $\mathbf{P}_n^{(h)}$.

Consider the case $n > h + 1$. Let $\mathcal{I}$ be the set of all independent subsets of $\mathbf{P}_n^{(h)}$, let $\mathcal{I}_{in}$ be the set of the independent subsets of $\mathbf{P}_n^{(h)}$ that contain $v_n$, and let $\mathcal{I}_{out} = \mathcal{I} \setminus \mathcal{I}_{in}$. The elements of $\mathcal{I}_{out}$ are in one-to-one correspondence with the $p_{n-1}^{(h)}$ independent subsets of $\mathbf{P}_{n-1}^{(h)}$, and those of $\mathcal{I}_{in}$ are in one-to-one correspondence with the $p_{n-h-1}^{(h)}$ independent subsets of $\mathbf{P}_{n-h-1}^{(h)}$. \qed

Tables 4 displays a few values of $p_n^{(h)}$. 

5
3. Generalized Fibonacci numbers and generalized Fibonacci cubes

Figure 2 shows a few Hasse diagrams $H_n^{(h)}$. Notice that, as stated in the introduction, for each $n$, $H_n^{(1)}$ is the Fibonacci cube $\Gamma_n$.

Let $H_n^{(h)}$ be the number of edges of $H_n^{(h)}$. Noting that in $H_n^{(h)}$ each non-empty independent $k$-subset covers exactly $k$ independent $(k-1)$-subsets, we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} h p_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \binom{n-hk+h}{k}.$$  \hspace{1cm} (1)

Remark. For $h = 1$, $H_n^{(h)}$ counts the number of edges of $\Gamma_n$.

Let now $T_{k,i}^{(n,h)}$ be the number of independent $k$-subsets of $P_n^{(h)}$ containing the vertex $v_i$, and let, for $h, k \geq 0$, $n \in \mathbb{Z}$, $p_{n,k}^{(h)} = \begin{cases} p_{0,k}^{(h)} & \text{if } n < 0, \\ p_{n,k}^{(h)} & \text{if } n \geq 0. \end{cases}$

Lemma 3.1. For $n, h, k \geq 0$, and $1 \leq i \leq n$,

$$T_{k,i}^{(n,h)} = \sum_{r=0}^{k-1} p_{i-h-1,r}^{(h)} \bar{p}_{n-i-h,k-1-r}^{(h)}.$$ 

Proof. No independent subset of $P_n^{(h)}$ containing $v_i$ contains any of the elements $v_{i-h}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+h}$. Let $r$ and $s$ be non-negative integers whose sum is $k-1$. Each independent $k$-subset of $P_n^{(h)}$ containing $v_i$ can be obtained by adding $v_i$ to a $(k-1)$-subset $R \cup S$ such that

(a) $R \subseteq \{v_1, \ldots, v_{i-h-1}\}$ is an independent $r$-subset of $P_n^{(h)}$;

(b) $S \subseteq \{v_{i+h+1}, \ldots, v_n\}$ is an independent $s$-subset of $P_n^{(h)}$.

Viceversa, one can obtain each of these pairs of subsets by removing $v_i$ from an independent $k$-subset of $P_n^{(h)}$ containing $v_i$. Thus, $T_{k,i}^{(n,h)}$ is obtained by counting independently the subsets of type (a) and (b). The remark that the subsets of type (b) are in bijection with the independent $s$-subsets of $P_n^{(h)}$ proves the lemma.

Remark. $T_{k,i}^{(n,1)}$ counts the number of strings $\alpha = b_1b_2 \cdots b_n \in \Gamma_n$ such that: (i) $H(\alpha, 00 \cdots 0) = k$, and (ii) $b_i = 1$. 

6
In order to obtain our main result, we prepare a lemma.

**Lemma 3.2.** For positive \(n\),

\[
\sum_{k=1}^{\left\lfloor n/(h+1) \right\rfloor} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = H_n^{(h)}.
\]

**Proof.** The inner sum counts the number of \(k\)-subsets exactly \(k\) times, one for each element of the subset. That is, \(\sum_{i=1}^{n} T_{k,i}^{(n,h)} = kp_n^{(h)}\). Hence the lemma follows directly from Equation (1). \(\square\)

Next we introduce a family of Fibonacci-like sequences.

**Definition 3.3.** For \(h \geq 0\), and \(n \geq 1\), we define the \(h\)-Fibonacci sequence \(F_n^{(h)} = \{F_n^{(h)}\}_{n \geq 1}\) whose elements are

\[
F_n^{(h)} = \begin{cases} 
1 & \text{if } n \leq h + 1, \\
F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h + 1.
\end{cases}
\]

The first values of the \(h\)-Fibonacci sequences, for \(h = 1, \ldots, 10\), are shown in Table 5. From Lemma 2.2, and setting for \(h \geq 0\), and \(n \in \mathbb{Z}\), \(p_n^{(h)} = \begin{cases} 
p_0^{(h)} & \text{if } n < 0, \\
p_n^{(h)} & \text{if } n \geq 0,
\end{cases}\) we have that,

\[
F_i^{(h)} = \overline{p}_{i-h-1}^{(h)}, \text{ for each } i \geq 1.
\]

Thus, our Fibonacci-like sequences are obtained by prepending \(h\) 1’s to the sequence \(p_0^{(h)}, p_1^{(h)}, \ldots\). Therefore, we have:

- \(F^{(0)} = 1, 2, 4, \ldots, 2^n, \ldots\);
- \(F^{(1)}\) is the Fibonacci sequence;
- more generally, \(F^{(h)} = 1, \ldots, 1, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots\).

In the following, we define the discrete convolution operation \(*\), as follows.

\[
\left( F^{(h)} * F^{(h)} \right)(n) \doteq \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)}
\]

**Theorem 3.4.** For \(n, h \geq 0\), the following holds.

\[
H_n^{(h)} = \left( F^{(h)} * F^{(h)} \right)(n).
\]
Proof. The sum \( \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} \) counts the number of independent subsets of \( \mathbf{P}_n^{(k)} \) containing \( v_i \). We can also obtain such a value by counting the independent subsets of both \( \{v_1, \ldots, v_{i-h-1}\} \) and \( \{v_{i+h+1}, \ldots, v_n\} \). Thus, we have:

\[
\sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.
\]

Using Lemma 3.2 we can write

\[
H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.
\]

By Equation (2) we have

\[
\sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)}.
\]

By (3), the theorem is proved. \( \square \)

We display some values of \( H_n^{(h)} \) in Table 6 (Section 6).

Remark. For \( h = 1 \), we obtain the number of edges of \( \Gamma_n \) by using Fibonacci numbers:

\[
H_n^{(h)} = \sum_{i=1}^{n} F_i F_{n-i+1}.
\]

The latter result is \cite[Proposition 3]{6}.

4. The independent subsets of powers of cycles

For \( n, h, k \geq 0 \), we denote by \( q_{n,k}^{(h)} \) the number of independent \( k \)-subsets of \( Q_n^{(h)} \).

Remark. For \( h = 1 \), \( n > 1 \), \( q_{n,k}^{(h)} \) counts the number of binary strings \( \alpha \in \Lambda_n \) with \( k \) 1’s.

Lemma 4.1. For \( n, h \geq 0 \), and \( k > 1 \),

\[
q_{n,k}^{(h)} = \binom{n-hk-1}{k-1}.
\]

Moreover, \( q_{n,0}^{(h)} = 1 \), and \( q_{n,1}^{(h)} = n \), for each \( n, h \geq 0 \).

Proof. Fix an element \( v_i \in V(Q_n^{(h)}) \), and let \( n > 2h \). Any independent subset of \( Q_n^{(h)} \) containing \( v_i \) does not contain the \( h \) elements preceding \( v_i \) and the \( h \) elements following \( v_i \). Thus, the number of independent \( k \)-subsets of \( Q_n^{(h)} \) containing \( v_i \) equals

\[
\bar{p}_{n-2h-1,k-1}^{(h)} = \binom{n-hk-1}{k-1}.
\]
The total number of independent \(k\)-subsets of \(Q_{n}^{(h)}\) is obtained by multiplying \(p_{n-2h-1,k-1}^{(h)}\) by \(n\), then dividing it by \(k\) (each subset is counted \(k\) times by the previous proceeding). The case \(n \leq 2h\), as well as the cases \(k = 0, 1\), can be easily verified.

\[\square\]

Some values of \(q_{n,k}^{(h)}\) are displayed in Tables 7–9.

For \(n, h \geq 0\), the number of all independent subsets of \(Q_{n}^{(h)}\) is

\[
q_{n}^{(h)} = \sum_{k \geq 0} q_{n,k}^{(h)} = \sum_{k=0}^{[n/(h+1)]} q_{n,k}^{(h)},
\]

(4)

**Remark.** Denote by \(L_n\) the \(n^{th}\) element of the Lucas sequence \(L_1 = 1, L_2 = 3,\) and \(L_i = L_{i-1} + L_{i-2}\), for \(i > 2\). Then, for \(n > 1\), \(q_{n}^{(1)} = L_n\) is the number of elements of the Lucas cube of order \(n\).

Some values of \(q_{n,k}^{(h)}\) are shown in Table 10. The coefficients \(q_{n,k}^{(h)}\) satisfy a recursion that closely resembles that of Lemma 2.2.

**Lemma 4.2.** For \(n, h \geq 0\),

\[
q_{n}^{(h)} = \begin{cases} 
  n + 1 & \text{if } n \leq 2h + 1, \\
  q_{n-1}^{(h)} + q_{n-h-1}^{(h)} & \text{if } n > 2h + 1.
\end{cases}
\]

(5)

**Proof.** The case \(n \leq 2h + 1\) can be easily checked. The case \(2h + 1 < n \leq 3h + 2\) is discussed at the end of this proof. Let \(n > 3h + 2\), and let \(\mathcal{I}\) be the set of the independent subsets of \(Q_{n}^{(h)}\). Let \(\mathcal{I}_n\) be the subset of these subsets that (i) do not contain \(v_n\), and that (ii) do not contain any of the following pairs: \((v_1, v_{n-h}), (v_2, v_{n-h+1}), \ldots, (v_h, v_{n-1})\). Let then \(\mathcal{I}_{out}\) be the subset of the remaining independent subsets of \(Q_{n}^{(h)}\).

It is easy to see that the elements of \(\mathcal{I}_n\) are exactly the independent subsets of \(Q_{n-1}^{(h)}\). Indeed, \(v_n\) is not a vertex of \(Q_{n-1}^{(h)}\) and the vertices of pairs \((v_1, v_{n-h}), (v_2, v_{n-h+1}), \ldots, (v_h, v_{n-1})\) are connected in \(Q_{n-1}^{(h)}\). On the other hand, to show that

\[|\mathcal{I}_{out}| = q_{n-h-1}^{(h)}\]

we argue as follows. First we recall (see the proof of Lemma 4.1) that the number of independent \(k\)-subsets of \(Q_{n}^{(h)}\) that contain \(v_n\) is \(p_{n-2h-1,k-1}^{(h)}\). Secondly we claim that the number of independent \(k\)-subsets of \(Q_{n}^{(h)}\) containing one of the pairs \((v_1, v_{n-h}), (v_2, v_{n-h+1}), \ldots, (v_h, v_{n-1})\) is \(hp_{n-3h-2,k-2}\). To see this, consider the pair \((v_1, v_{n-h})\). The independent subsets containing such a pair do not contain the \(h\) vertices from \(v_n - h + 1\) to \(v_n\), do not contain the \(h\) vertices from \(v_2\) to \(v_{h+1}\), and do not contain the \(h\) vertices from \(v_{n-2h}\) to \(v_{n-h-1}\). Thus, the removal of such vertices and of the vertices \(v_1\) and \(v_{n-h}\) turns \(Q_{n}^{(h)}\) into \(Q_{n-3h-2}^{(h)}\). Hence we can obtain all the independent \(k\)-subsets of \(Q_{n}^{(h)}\) that contain the pair \((v_1, v_{n-h})\) by simply adding these two vertices to one of the
independent \( k - 2 \)-subsets of \( D_{n-3h-2}^{(h)} \). Same reasoning can be carried out for any other one of the pairs: \((v_2, v_{n-h+1})\), \ldots, \((v_h, v_{n-1})\).

Using Lemmas 2.1 and 4.1 one can easily derive that

\[
p_{n-2h-1,k-1}^{(h)} + hp_{n-3h-2,k-2}^{(h)} = q_{n-h-1,k-1}^{(h)}.
\]

Hence, we derive the size of \( I_{\text{out}} \):

\[
|I_{\text{out}}| = q_{n-h-1}^{(h)} = \sum_{k \geq 1} p_{n-2h-1,k-1}^{(h)} + h \sum_{k \geq 2} p_{n-3h-2,k-2}^{(h)}.
\]

Summing up we have shown that \(|I| = |I_{\text{in}}| + |I_{\text{out}}|\), that is

\[
q_{n}^{(h)} = q_{n-1}^{(h)} + q_{n-h-1}^{(h)}.
\]

The proof of the case \( 2h + 1 < n \leq 3h + 2 \) is obtained in a similar way, observing that \(|I_{\text{out}}| = n - h\), and that \( n - h - 1 \leq 2h + 1 \).

5. Generalized Lucas cubes and Lucas numbers

Figure 3 shows a few Hasse diagrams \( M_n^{(h)} \). Notice that, as stated in the introduction, for each \( n \), \( M_n^{(1)} \) is the Lucas cube \( \Lambda_n \).

Let \( M_n^{(h)} \) be the number of edges of \( M_n^{(h)} \). As done in Section 3 for the case of paths, we immediately provide a formula for \( M_n^{(h)} \):

\[
M_n^{(h)} = \sum_{k=0}^{\lfloor n/h+1 \rfloor} k q_{n,k}^{(h)} = n \sum_{k=0}^{\lfloor n/h+1 \rfloor} \binom{n-hk-1}{k-1}.
\]  

(6)

Remark. For \( h = 1 \), \( n > 1 \), \( M_n^{(h)} \) counts the number of edges of \( \Lambda_n \). As shown in [9 Proposition 4(ii)], \( M_n^{(h)} = nF_{n-1} \).

As shown in the proof of Lemma 4.1, the value

\[
p_{n-2h-1,k-1}^{(h)} = \binom{n-hk-1}{k-1}
\]

is the analogue of the coefficient \( T_{k:i}^{(n,h)} \): in the case of cycles we have no dependencies on \( i \), because each choice of vertex is equivalent. We can obtain \( M_n^{(h)} \) in terms of a Fibonacci-like sequence, as follows.
Proposition 5.1. For $n > h \geq 0$, the following holds.

$$M_n^{(h)} = n F_{n-h}^{(h)}.$$

Proof. Using Equation (2) we obtain:

$$M_n^{(h)} = n \sum_{k=1}^{\lceil n/(h+1) \rceil} \bar{p}_{n-2h-1,k-1}^{(h)} = n F_{n-2h-1}^{(h)} = n F_{n-h}^{(h)}.$$

In analogy with Section 3, we introduce a family of Lucas-like sequences.

Definition 5.2. For $h \geq 0$, and $n \geq 1$, we define the $h$-Lucas sequence $L_n^{(h)} = \{L_n^{(h)}\}_{n \geq 1}$ whose elements are

$$L_n^{(h)} = \begin{cases} 
  h + 1 & \text{if } n = 1, \\
  1 & \text{if } 2 \leq n \leq h + 1, \\
  L_n^{(h)} + L_{n-h}^{(h)} & \text{if } n > h + 1.
\end{cases}$$

The first values of the $h$-Lucas sequences, for $h = 0, \ldots, 10$, are displayed in Table 11. We have that,

$$L_i^{(h)} = q_{i-1}^{(h)}, \text{ for each } i > h + 1. \quad (7)$$

To prove the main result of this section, the following lemma is needed.

Lemma 5.3. For $n > h \geq 0$, the following holds.

$$L_{n+1}^{(h)} = L_n^{(h)} + (h+1) F_{n-h}^{(h)}.$$

Proof. The result is proved by induction. Indeed, $L_{n+1}^{(h)} = L_n^{(h)} + L_{n-h}^{(h)}$. Applying the inductive hypothesis, we have

$$L_{n+1}^{(h)} = F_{n-1}^{(h)} + (h+1) F_{n-h-1}^{(h)} + F_{n-h-1}^{(h)} + (h+1) F_{n-2h-1}^{(h)} = F_n^{(h)} + (h+1) F_{n-h}^{(h)}.$$

Finally, our analogous of Theorem 3.4, for cycles, is the following.

Theorem 5.4. For $n > h \geq 0$, the following holds.

$$M_n^{(h)} = \left(F^{(h)} \ast L^{(h)}\right)(n-h).$$

Proof. By Proposition 5.1, the statement of the Theorem is equivalent to

$$\sum_{i=1}^{n-h} F_i^{(h)} L_{n-h+1-i}^{(h)} = n F_{n-h}^{(h)}. \quad (8)$$
Let $h = 0$. We have

$$
\sum_{i=1}^{n} F_i^{(0)} L_{n+1-i}^{(0)} = \sum_{i=1}^{n} 2^{i-1} 2^{n-i} = \sum_{i=1}^{n} 2^{n-1} = n 2^{n-1} = n F_n^{(0)}.
$$

Let $h = 1$. In this case the statement of the theorem reduces to the well known identity involving (classical) Fibonacci and Lucas sequences:

$$
\sum_{i=1}^{n} F_i L_{n-i+1} = (n + 1) F_n.
$$

Let $h \geq 2$. We prove (8) by induction on $n$. If $n = h + 1$, then

$$
\sum_{i=1}^{n-h} F_i^{(h)} L_{n-h+1-i}^{(h)} = F_1^{(h)} L_1^{(h)} = h + 1 = n F_1^{(h)}.
$$

Let $\bar{n} > h + 1$, and suppose (inductive hypothesis) that (8) holds for every $1 < n \leq \bar{n}$. Let $m = \bar{n} - h$ (and note that $m \geq 2$). We need to prove that

$$
\sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} = (m + h + 1) F_{m+1}^{(h)}.
$$

We find it convenient to define, for $h \geq 2$, the following integer sequences, which extend $F_n^{(h)}$ and $L_n^{(h)}$ to a range of negative integers.

$$
\bar{F}_n^{(h)} = \begin{cases} 
1 & \text{if } n = -h, \\
0 & \text{if } -h < n \leq 0, \\
\bar{F}_{n-1}^{(h)} + \bar{F}_{n-h-1}^{(h)} & \text{if } n > 0.
\end{cases}
$$

$$
\bar{L}_n^{(h)} = \begin{cases} 
\bar{h} + 1 & \text{if } n = -h, \\
-h & \text{if } n = -h + 1, \\
0 & \text{if } -h + 1 < n \leq 0, \\
\bar{L}_{n-1}^{(h)} + \bar{L}_{n-h-1}^{(h)} & \text{if } n > 0.
\end{cases}
$$

Using Definitions 3.3 and 5.2 one can easily check that, for $n > 0$, $F_n^{(h)} = \bar{F}_n^{(h)}$.
and \( L_n^{(h)} = \bar{L}_n^{(h)} \). Hence, applying the recurrences in (10) and (11), we obtain

\[
\sum_{i=1}^{m+1} F_i^{(h)} L_{m+1-i}^{(h)} = \sum_{i=1}^{m+1} \bar{F}_i^{(h)} \bar{L}_{m+1-i}^{(h)} = \\
= \sum_{i=1}^{m+1} \bar{F}_{i-1}^{(h)} \bar{L}_{m+1-i}^{(h)} + \\
\sum_{i=1}^{m+1} \bar{F}_{i-1}^{(h)} \bar{L}_{m+1-h-i}^{(h)} + \\
\sum_{i=1}^{m+1} \bar{F}_{i-h}^{(h)} \bar{L}_{m+1-i}^{(h)} + \\
\sum_{i=1}^{m+1} \bar{F}_{i-h}^{(h)} \bar{L}_{m+1-h-i}^{(h)}
\]  

We compute the sums (12)–(15) separately.

Direct computation shows that (12) equals

\[
\sum_{i=1}^{m-1} F_i^{(h)} L_{m-i}^{(h)} + \bar{F}_0^{(h)} \bar{L}_m^{(h)} + \bar{F}_m^{(h)} \bar{L}_0^{(h)}.
\]

Applying the inductive hypotheses to the first term of (16), and observing that \( \bar{L}_0^{(h)} = \bar{F}_0^{(h)} = 0 \), we obtain that the sum (12) is

\[
(m + h - 1) F_{m-1}^{(h)}.
\]

In order to compute (13), we distinguish two cases. If \( m > h + 1 \), direct computation shows that (13) equals

\[
\sum_{i=1}^{m-h-1} F_i^{(h)} L_{m-h-i}^{(h)} + \bar{F}_0^{(h)} \bar{L}_{m-h}^{(h)} + \bar{F}_m^{(h)} \bar{L}_m^{(h)} + \bar{F}_{m-h}^{(h)} \bar{L}_{m-h+1}^{(h)} + \bar{F}_{m-h+1}^{(h)} \bar{L}_{m-h+2}^{(h)} + \cdots + \bar{F}_{m-h+1}^{(h)} \bar{L}_h^{(h)},
\]

We note that \( m > h + 1 \geq 3 \). Applying the inductive hypotheses to the first term of (18), and observing that all other terms are zeroes, except \( \bar{F}_m^{(h)} \bar{L}_{m-h}^{(h)} = (h + 1) F_{m-1}^{(h)} \), and \( \bar{F}_{m-h}^{(h)} \bar{L}_{m-h+1}^{(h)} = -h F_{m-1}^{(h)} \), we obtain that (13) equals

\[
(m - 1) F_{m-h-1}^{(h)} + (h + 1) F_m^{(h)} - h F_{m-1}^{(h)}.
\]  

(for \( m > h + 1 \))  

(19)

If, on the other hand, \( m \leq h + 1 \), we observe that the summands of (13) vanish, with the exception of \( \bar{F}_m^{(h)} \bar{L}_{m-h}^{(h)} = (h + 1) F_m^{(h)} \), and \( \bar{F}_{m-h+1}^{(h)} \bar{L}_{m-h+1}^{(h)} = (h + 1) F_m^{(h)} \). Indeed, \( m \geq 2 \). Thus, the sum (13) is

\[
(h + 1) F_m^{(h)} - h F_{m-1}^{(h)}.
\]  

(for \( m \leq h + 1 \))  

(20)
A similar argument shows that \( \text{[14]} \) equals

\[
(m - 1)F_{m-h-1}^{(h)} + L_{m}^{(h)}, \quad \text{for } m > h + 1 \quad (21)
\]

\[
L_{m}^{(h)}. \quad \text{for } m \leq h + 1 \quad (22)
\]

To calculate \( \text{[15]} \) we distinguish the four cases \( m \leq h, m = h + 1, h + 1 < m \leq 2h + 1, \) and \( m > 2h + 1. \) If \( m > 2h + 1, \) \( \text{[15]} \) equals

\[
\sum_{i=1}^{m-2h-1} F_{i}^{(h)}L_{m-2h-i} + F_{0}^{(h)} + F_{m-2h}^{(h)} + \cdots + F_{m-1}^{(h)} + F_{m}^{(h)} + L_{m}^{(h)} + \cdots + L_{m-2h}^{(h)}.
\]

We note that \( m - h - 1 > h > 0. \) Applying the inductive hypotheses to the first term of the preceding sum, and observing that all other terms are zeroes, except \( L_{-h}^{(h)}F_{m-h}^{(h)} = (h + 1)F_{m-h}^{(h)}, F_{m-h}^{(h)} = -hF_{m-h-1}^{(h)}, \) and \( L_{-h}^{(h)}F_{m-h}^{(h)} = L_{m-h}^{(h)}, \) we obtain that \( \text{[15]} \) is

\[
(m - h - 1)F_{m-2h-1}^{(h)} + (h + 1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}. \quad \text{for } m > 2h + 1 \quad (23)
\]

To tackle the other cases, we expand the sum \( \text{[15]}, \) and we check how the non-zeroes terms change depending on the constraints on \( m. \) We obtain that \( \text{[15]} \) equals

\[
(h + 1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}^{(h)}, \quad \text{for } h + 1 < m \leq 2h + 1 \quad (24)
\]

\[
(h + 1)F_{m-h}^{(h)} + L_{m-h}^{(h)}, \quad \text{for } m = h + 1 \quad (25)
\]

\[
0. \quad \text{for } m \leq h \quad (26)
\]

Finally, we add up the four summands \( \text{[12]} - \text{[15]} \), distinguishing the four identified cases. If \( m > 2h + 1, \) we sum \( \text{[17]}, \text{[19]}, \text{[21]}, \) and \( \text{[23]}, \) obtaining

\[
\sum_{i=1}^{m+1} F_{i}^{(h)}L_{m+1-i}^{(h)} = (m + h - 1)F_{m-1}^{(h)} + 2(m - 1)F_{m-h-1}^{(h)} + (h + 1)F_{m}^{(h)} - hF_{m-1}^{(h)} +
\]

\[
+ F_{m}^{(h)} + (m - h - 1)F_{m-2h-1}^{(h)} + (h + 1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}^{(h)}.
\]

Repeatedly applying the recurrences \( F_{i}^{(h)} = F_{i-1}^{(h)} + F_{i-h-1}^{(h)} \) and \( L_{i}^{(h)} = L_{i-1}^{(h)} + \), and using Lemma \( \text{[5.3]} \), we obtain

\[
\sum_{i=1}^{m+1} F_{i}^{(h)}L_{m+1-i}^{(h)} = (m + h + 1)F_{m+1}^{(h)}
\]

which proves \( \text{[8]} \) in the case \( m > 2h + 1. \)
If \( h + 1 < m \leq 2h + 1 \), we sum (17), (19), (21), and (24), and we obtain

\[
\sum_{i=1}^{m+1} F_i^{(h)} F_{m+1-i}^{(h)} = (m+h-1)F_{m+1}^{(h)} + 2(m-1)F_{m-h-1}^{(h)} + (h+1)F_m^{(h)} + hF_{m-1}^{(h)} + L_m^{(h)} + (h+1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}^{(h)}.
\]

If \( m = h + 1 \), we sum (17), (20), (22), and (25), obtaining

\[
\sum_{i=1}^{m+1} F_i^{(h)} F_{m+1-i}^{(h)} = (m+h-1)F_{m+1}^{(h)} + (h+1)F_m^{(h)} + L_m^{(h)} + (h+1)F_{m-h}^{(h)} + L_{m-h}^{(h)}.
\]

In the last case, if \( m < h + 1 \), we sum (17), (20), (22), and (26), and we have

\[
\sum_{i=1}^{m+1} F_i^{(h)} F_{m+1-i}^{(h)} = (m+h-1)F_{m+1}^{(h)} + (h+1)F_m^{(h)} - hF_{m-1}^{(h)} + L_m^{(h)}.
\]

We display some values of \( M_n^{(h)} \) in Table 12.

### 6. Tables

We collect here some values obtained by computing the formula presented in the preceding sections.

Table 1: The number \( p_{n,k}^{(1)} \) of independent \( k \)-subsets of \( \mathbb{P}_n^{(1)} \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| \( k \) | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2: The number \( p_{n,k}^{(2)} \) of independent \( k \)-subsets of \( \mathbb{P}_n^{(2)} \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| \( k \) | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 2 | 2 | 0 | 0 | 0 | 0 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 | 78 | 91 |
| 3 | 3 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 10 | 20 | 35 | 56 | 84 | 120 | 165 | 220 | 286 |
| 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 5 | 15 | 35 | 70 | 126 | 210 | 330 | 495 |
| 5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 21 | 56 | 126 | 252 | 462 |
| 6 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 7 | 28 | 84 | 210 |
| 7 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

15
Table 3: The number $p_{n,k}^{(3)}$ of independent $k$-subsets of $P_n^{(3)}$

| $k$ | $n=0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ |
|-----|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1     | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 1   | 1     | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   |
| 2   | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
| 3   | 0     | 0   | 0   | 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  | 36  | 45  | 55  | 66  | 78  |
| 4   | 0     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 5   | 15  | 35  | 70  |
| 5   | 0     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   |

Table 4: The number $p_n^{(h)}$ of all independent subsets of $P_n^{(h)}$

| $h=0$ | $n=0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     | 1     | 2   | 4   | 8   | 16  | 32  | 64  | 128 | 256 | 512 | 1024 | 2048 | 4096 | 8192 |
| 1     | 1     | 2   | 3   | 5   | 8   | 13  | 21  | 34  | 55  | 89  | 144  | 233  | 377  | 610  |
| 2     | 1     | 2   | 3   | 4   | 6   | 9   | 13  | 19  | 28  | 41  | 60   | 88   | 129  | 189  |
| 3     | 1     | 2   | 3   | 4   | 5   | 7   | 10  | 14  | 19  | 26  | 36   | 50   | 69   | 95   |
| 4     | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 9   | 12  | 16  | 21   | 27   | 34   | 43   |
| 5     | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 9   | 12  | 16  | 21   | 27   | 34   | 43   |
| 6     | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 10  | 13  | 17   | 22   | 28   | 35   |
| 7     | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 11  | 14   | 18   | 23   |
| 8     | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 12   | 15   |
| 9     | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11   |
| 10    | 1     | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11   |

Table 5: Values of the $h$-Fibonacci sequence $F^{(h)} = \{F^{(h)}_n\}_{n \geq 1}$

| $h=0$ | $n=0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     | 1     | 1   | 4   | 12  | 32  | 80  | 192 | 448 | 1024 | 2304 | 5120 | 11264 | 24576 | 53248 |
| 1     | 0     | 1    | 5   | 10  | 20  | 38  | 71  | 130  | 235  | 420  | 744  | 1308  | 2285  |
| 2     | 0     | 1    | 2    | 3    | 6    | 11   | 18   | 30   | 50   | 81   | 130   | 208   | 330   |
| 3     | 0     | 1    | 2    | 3    | 4    | 7    | 12    | 19   | 28    | 42    | 64    | 97    | 144    |
| 4     | 0     | 1    | 2    | 3    | 4    | 5    | 8    | 13    | 20    | 29    | 40    | 56    | 80    |
| 5     | 0     | 1    | 2    | 3    | 4    | 5    | 6    | 9    | 14    | 21    | 30    | 41    |
| 6     | 0     | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 10    | 15    | 22    |
| 7     | 0     | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 11    | 16    |
| 8     | 0     | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 12    |
| 9     | 0     | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10    |
| 10    | 0     | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10    | 11    | 14    |

Table 6: The number $H_n^{(h)}$ of edges of $H_n^{(h)}$
Table 7: The number $q_{n,k}^{(1)}$ of independent $k$-subsets of $Q_n^{(1)}$

| n=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| k=0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|     | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|     | 2 | 0 | 0 | 0 | 0 | 2 | 5 | 9 | 14 | 20 | 27 | 35 | 44 | 54 | 65 | 77 | 90 | 104 |
|     | 3 | 0 | 0 | 0 | 0 | 0 | 2 | 7 | 16 | 30 | 50 | 77 | 112 | 156 | 210 | 275 | 352 |
|     | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 9 | 25 | 55 | 105 | 182 | 294 | 450 | 660 |
|     | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 11 | 36 | 91 | 196 | 378 | 672 |
|     | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 13 | 49 | 140 | 336 |
|     | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 15 | 64 |
|     | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |

Table 8: The number $q_{n,k}^{(2)}$ of independent $k$-subsets of $Q_n^{(2)}$

| n=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| k=0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|     | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|     | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 9: The number $q_{n,k}^{(3)}$ of independent $k$-subsets of $Q_n^{(3)}$

| n=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| h=0 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|     | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|     | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 10: The number $q_n^{(h)}$ of all independent subsets of $Q_n^{(h)}$

| n=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| h=0 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|     | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|     | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|     | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

17
Table 11: Values of the $h$-Lucas sequence $L^{(h)} = \{L_n^{(h)}\}_{n \geq 1}$

| $h=0$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $n=0$ |   | 1 |   |   |   |   |   |   |   |   | 32 | 64 | 128 | 256 | 512 | 1024 |
| 1     | 2 |   |   |   |   |   |   |   |   | 18 | 36 | 72 | 144 | 288 | 576 | 1152 |
| 2     | 3 | 4 |   |   |   |   |   |   | 24 | 40 | 64 | 128 | 256 | 512 | 1024 | 2048 |
| 3     | 5 |   |   |   |   |   |   |   | 20 | 32 | 52 | 104 | 208 | 416 | 832 | 1664 |
| 4     | 6 |   |   |   |   |   |   | 16 | 24 | 36 | 60 | 120 | 240 | 480 | 960 | 1920 |
| 5     | 7 |   |   |   |   |   |   | 12 | 18 | 27 | 45 | 90 | 180 | 360 | 720 | 1440 |
| 6     | 8 |   |   |   |   |   | 8  | 12 | 18 | 27 | 45 | 90 | 180 | 360 | 720 | 1440 |
| 7     | 9 |   |   |   |   | 6  | 9  | 12 | 18 | 27 | 45 | 90 | 180 | 360 | 720 | 1440 |
| 8     | 10|   |   |   | 5  | 7  | 9  | 12 | 18 | 27 | 45 | 90 | 180 | 360 | 720 | 1440 |
| 9     | 11|   |   | 4  | 6  | 8  | 10 | 12 | 18 | 27 | 45 | 90 | 180 | 360 | 720 | 1440 |
| 10    | 12|   | 3  | 5  | 7  | 9  | 11 | 12 | 18 | 27 | 45 | 90 | 180 | 360 | 720 | 1440 |

Table 12: The number $M_n^{(h)}$ of edges of $M_n^{(h)}$, for $n > h$

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