COVARIANT PHOTON QUANTIZATION IN THE SME

D. Colladay

New College of Florida,
Sarasota, FL 34234, USA

+E-mail: colladay@ncf.edu

The Gupta Bleuler quantization procedure is applied to the SME photon sector. A direct application of the method to the massless case fails due to an unavoidable incompleteness in the polarization states. A mass term can be included into the photon lagrangian to rescue the quantization procedure and maintain covariance.

1. Introduction

The fermion sector of the SME was quantized consistently during the first stages of its theoretical development, at least in theories with a significant nonzero mass parameter.\(^1\) The photon sector has remained largely unaddressed due to several factors that make it more complicated to deal with. For example, there is no simple linear Hamiltonian arising from the equation of motion that can be used for a complete set of orthogonal states. In addition, the modified equation of motion has implications for the gauge states that are nontrivial to incorporate. Addition of a mass term to the lagrangian makes the problem more similar to the fermion case and generates a tractable problem, so this is the approach used in this talk. An alternative, perturbative approach in the non-birefringent case has also been implemented.\(^2\)

2. Gupta Bleuler method applied to the SME

The starting point is the Stuckelberg Lagrangian including a CPT-conserving Lorentz-violating term as well as a mass term

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \kappa_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} + \frac{1}{2} m^2 A_{\mu} A^{\mu} - \frac{\lambda}{2} (\partial_{\mu} A^{\mu})^2. \tag{1}
\]

Note that the CPT-violating term has been omitted since it can cause instabilities even at tree level.\(^3\) The gauge condition \(\lambda = 1\) is also chosen for
simplicity of the commutation relations. The starting assumptions are the standard covariant commutation relations for the field and the conjugate momenta

\[ \pi^j = F^{j0} + k_F^{j0} \alpha^\beta F_{\alpha\beta}, \quad \pi^0 = -\partial_\mu A^\mu. \]  

(2)

Imposing equal-time canonical commutation rules

\[ [A_\mu(t, \vec{x}), \pi^\nu(t, \vec{y})] = i\delta_\mu^\nu \delta^3(\vec{x} - \vec{y}), \]  

(3)

along with

\[ [A_\mu(t, \vec{x}), A_\nu(t, \vec{y})] = [\pi^\mu(t, \vec{x}), \pi^\nu(t, \vec{y})] = 0, \]  

(4)

implements the standard canonical quantization in a covariant manner as is done in the conventional Gupta-Blued method. This implies that the time derivatives of the spatial components \( A^i \) satisfy the modified commutation relations

\[ \{ \dot{A}^i(t, \vec{x}), A^j(t, \vec{y}) \} = -iR^{ij} \delta^3(\vec{x} - \vec{y}) \]  

(5)

where \( R^{ij} \) is the inverse matrix of \( \delta^{ij} - 2 (k_F)^{i0} \). In any concordant frame where \( k_F \) is reasonably small, this inverse exists. The commutation relations involving \( \dot{A}^0 \) and \( A^i \) are the same as in the usual case, so it is convenient to define a covariant-looking tensor \( \tilde{\eta}_{\mu\nu} \) by setting \( \tilde{\eta}_{00} = 1, \tilde{\eta}_{0i} = 0, \) and \( \tilde{\eta}_{ij} = -R^{ij} \). The commutation relations are expressed as

\[ \{ \dot{A}^\mu(t, \vec{x}), A^\nu(t, \vec{y}) \} = i\tilde{\eta}_{\mu\nu} \delta^3(\vec{x} - \vec{y}). \]  

(6)

This matrix is also the inverse of \( \tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - 2k_F^{\mu0}k_F^{\nu0} \) as \( \tilde{\eta}_{\mu\nu}\tilde{\eta}_{\nu\alpha} = \eta_{\mu\alpha} \). Note that the time derivatives of \( A \) do not commute, rather

\[ \{ \dot{A}^\mu(t, \vec{x}), \dot{A}^\nu(t, \vec{y}) \} = -2i\tilde{\eta}_{\mu\alpha}(k_F^{\alpha0}\beta + k_F^{\beta0}\alpha)\tilde{\eta}_{\beta\nu}\frac{\partial}{\partial x^i}\delta^3(\vec{x} - \vec{y}), \]  

(7)

involving the spatial derivatives of the delta function.

The equation of motion in momentum space is

\[ (p^2 - m^2)\epsilon^\mu + 2(k_F)^{\mu\lambda\nu}p_\lambda p_\nu \epsilon^\nu = 0, \]  

(8)

where \( \epsilon^\mu \) is the polarization vector. One implication of this equation is found by dotting with \( p_\mu \) yielding the condition

\[ p^2 = m^2 \quad \text{or} \quad \epsilon \cdot p = 0, \]  

(9)

A key observation is the modified orthogonality relation for the polarization vectors that follows from the equation of motion

\[ \epsilon^{(\lambda)}_\mu(\vec{p}) \left( (p_0(\lambda) + p_0(\lambda')) - 2 \left( k_F^{0\lambda\nu} + k_F^{0\lambda\mu} \right) p_\lambda \right) \epsilon^{(\lambda)}_\nu(\vec{p}) = 2p_0(\lambda)\eta^{\lambda\lambda'}, \]  

(10)
which holds whenever \( p_0^{(\lambda)} \neq p_0^{(\lambda')} \). The normalization is chosen so that the \( \lambda = 0 \) polarization vector is timelike while the others are spacelike. This is possible due to the presence of a sufficiently large mass term which generically protects the normalization of the polarization vectors from vanishing. One of the issues of taking the \( m \to 0 \) limit is that the above orthogonality condition can fail due to some polarization vectors becoming light-like.

### 3. Momentum-Space Expansion

The fields can be expanded in a standard Fourier expansion using

\[
A_\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_\lambda \frac{1}{2p_0^{(\lambda)}} \left( a^{\lambda} (\vec{p}) \epsilon^{(\lambda)}_\mu (\vec{p}) e^{-ip \cdot x} + a^{\lambda'} (\vec{p}) \epsilon^{(\lambda')}_\mu (\vec{p}) e^{ip \cdot x} \right).
\]

The modified orthogonality relation for the polarization vectors can be used to invert this transform and solve for the raising and lowering operators. A straightforward computation then yields the standard relations

\[
[a^{\lambda} (\vec{p}), a^{\lambda'} (\vec{q})] = -(2\pi)^3 2p_0^{(\lambda)} \eta^{\lambda \lambda'} \delta^3 (\vec{p} - \vec{q}),
\]

as well as

\[
[a^{\lambda} (\vec{p}), a^{\lambda'} (\vec{q})] = 0,
\]

demonstrating that the raising and lowering operators obey conventional statistical relations. There are subtle issues associated with the above expansion. Although the mass term is not explicitly present it turns out to be crucial for generating a complete set of polarization states required for the quantization procedure. When the mass is set to zero it turns out that the conjugate momentum \( \pi^0 \) is identically zero for most directions in momentum space. This creates a serious problem for Gupta-Bleuler as the gauge term initially added into the Lagrangian is not sufficient to produce a generically nontrivial conjugate momenta for \( A^0 \). This indicates that the standard Gupta-Bleuler method in fact fails in the massless case, at least when there is birefringence present.

### 4. Explicit Example

As an explicit example of the issue with \( m \to 0 \), consider the single parameter model \( k_F^{0103} = k/2 \) (along with required nonzero symmetric components to make it anti-self-dual and therefore pure birefringent). When \( p_1 = p_2 = 0 \),
the matrix for $K$ in the equation of motion is

$$K^{\mu\nu}(p) = k \begin{pmatrix} 0 & p_0p_3 & 0 & 0 \\ p_0p_3 & 0 & 0 & -p_0^2 \\ 0 & 0 & 0 & 0 \\ 0 & -p_0^2 & 0 & 0 \end{pmatrix}. \quad (14)$$

Searching for zero eigenvalues (with $p^2 = 0$, candidates for the gauge modes...) yields two eigenvectors, one with the polarization vector proportional to the momentum and another with

$$\epsilon = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (15)$$

Both of these modes satisfy $\epsilon \cdot p = 0$ indicating that there is in fact no nontrivial mode corresponding to $\pi^0 = -\partial \cdot A$. Making the momentum more general does not help as the rank of the $K$ matrix is generally increased to three indicating the same fundamental problem.

5. Summary

The standard Gupta-Bleuler method seems to work well when there is a mass term present in the Lagrangian, but there are serious impediments to implementing this method when the mass is identically zero. The most serious issue appears to be the vanishing of $\pi^0$ implied by the equation of motion, something that is not an issue in the conventional case. In addition, certain directions in momentum space yield a set of polarization vectors that is strictly less than four-dimensional.

Acknowledgments

I would like to thank New College of Florida for summer funding that aided in the completion of this project.

References

1. D. Colladay and A. Kostelecky, Phys. Rev. D 55 6760 (1997); Phys. Rev. D 58 16002 (1998).
2. M. Hohensee, D. Phillips, and R. Walsworth, quant-ph/1210.2683 (2012).
3. S. Carroll, G. Field, and R. Jackiw, Phys. Rev. D 41, 1231 (1990).
4. S. Gupta, Proc. Phys. Soc. A63 681 (1950); K. Bleuler, Helv. Phys. Acta. 23 567 (1950).