Eigenvalue statistics for generalized symmetric and Hermitian matrices

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Abstract
Random matrix theory predicts the level statistics of a Hamiltonian to exhibit either clustering or repulsion if the underlying dynamics is integrable or chaotic, respectively. In various physical systems it is also possible to observe intermediate spectral properties showing the transition between different symmetry classes. In this work, we study generalized random matrix ensembles by dropping the constraint of canonical invariance and considering different variances in the diagonal and off-diagonal elements. Tuning the relative value of the variances we show that the distributions of the level spacings exhibit intermediate level statistics. The nearest neighbour spacing (NNS) distributions can be computed for generalized symmetric $2 \times 2$ matrices exhibiting crossover from clustering to GOE-like repulsion. The analysis is extended to $3 \times 3$ matrices where the distributions of NNS as well as ratio of nearest neighbour spacing (RNNS) show similar crossovers. We show that it is possible to calculate NNS distributions for Hermitian matrices ($N = 2, 3$) where crossovers also take place between clustering and repulsion as in GUE. For large symmetric and Hermitian matrices we use interpolation between clustered and repulsive regimes to quantify the system size dependence of the crossover boundary.

Keywords: random matrix theory, level spacing distributions, Wigner surmise, generalized ensemble

(Some figures may appear in colour only in the online journal)

1. Introduction
Random matrix theory (RMT) has emerged as an important statistical tool to distinguish chaotic dynamics from the integrability of quantum systems [1]. Applications have been found in a variety of disciplines ranging from energy level fluctuations in nuclear physics [2], to chiral
phase transitions in quantum chromodynamics [3], to more recent studies on many body localization and thermalization in condensed matter physics [4, 5]. Random matrix theory (RMT) can predict some universal characteristics, without explicit knowledge of the Hamiltonian (or any suitable operator), solely dictated by the underlying symmetries of the dynamical system [6]. In this regard, the most investigated quantity is the nearest neighbour spacing (NNS) of energy levels [7]. If a quantum system follows regular dynamics, i.e. the system is in the integrable domain, then energy levels can cross each other, which implies level clustering [8]. On the other hand, level repulsion is seen in systems having time-reversal and rotational invariance (or time-reversal invariant systems with integer spin and broken rotational symmetry) and belong to Gaussian orthogonal ensemble (GOE) with a Dyson index of $\beta = 1$ [6]. Corresponding classical counterparts show chaos, hence level repulsion can be considered to be a signature of quantum chaos [1]. Two more ensembles are introduced following group theoretical arguments [9], namely Gaussian unitary ensemble (GUE) (broken time-reversal symmetry) for $\beta = 2$ and Gaussian symplectic ensemble (GSE) (time-reversal invariant systems with half-integer spin and broken rotational symmetry) for $\beta = 4$. Universal features in NNS distributions have been observed upon suitable normalization and unfolding of the spectrum [10]. Recently a simpler method, avoiding the numerical issues of the unfolding procedure, has been proposed to compute distributions of the ratio of nearest neighbour spacing (RNNS), which also show the universal features of random matrix ensembles [11].

All of the above ensembles are pure in the sense that they provide a description of dynamical systems that are either regular or chaotic. However, the level statistics found in many physical systems often indicate intermediate states that are neither purely integrable or chaotic [12, 13], and such intermediate statistics in level fluctuations have also been experimentally observed [14–17]. In order to describe the mixed dynamics, many phenomenological models have been suggested, e.g. the Brody distribution [18], the Izrailev distribution [13], the Lenz–Haake approach [19], the GOE–GUE transition [20], the cross-over between Poisson–GOE–GUE [21–23], etc. These models quantify the transition/crossover based on NNS, with a few recent studies based on the RNNS distributions [24]. Typical approaches in these set-ups are to consider additive random matrix models where a particular symmetry is broken perturbatively by tuning an interpolating parameter. But some limitations come with these approaches, e.g. the derivative of the Brody distribution diverges at zero energy [25], the absence of a scaling property [26], etc.

Along with the above limitations, the interpolation parameters in additive random matrix models lack any physical interpretation. A relevant way of exploring the mixed features is to consider generalized random matrices and investigate the possibility of any transition by tuning the statistical properties of the random matrix elements. In this regard, one can drop the constraint of canonical invariance, which leads to the freedom of choosing arbitrary variances for different matrix elements. One of the earliest instances of such a random matrix model is known as the Rosenzweig–Porter (RP) matrix ensemble [14]. It is known that such a matrix ensemble exhibits tunable spectral statistics ranging from Poissonian to GOE. In recent studies RP ensembles have gained some attention and the multifractal structure of the eigenfunctions of such ensembles might be indicative of non-ergodic extended states in the crossover from MBL to thermal phases [27, 28]. Generalization has also been possible for symmetric Gaussian matrices where diagonal and off-diagonal elements are drawn from normal distributions with different mean and variances [29, 30]. In this paper, we have studied the crossover between level repulsion and clustering by tuning the relative variance of the diagonal ($\sigma_d$) and the off-diagonal ($\sigma_o$) elements of symmetric random $2 \times 2$ matrices. The general belief is that $2 \times 2$ matrices are a good caricature of large matrices. We extend the analysis for $3 \times 3$ matrices by proposing an ansatz for the eigenvalue distribution and obtain analytical expressions
for the NNS as well as the RNNS distributions, which agree with the simulation results. To quantify different phases in the $\sigma_d$-$\sigma_o$ plane, we obtain interpolating functions, parametrized by tunable parameters which are numerically estimated. However, the qualitative feature for the $N = 2$ phase diagram is very different from that obtained for $N \geq 3$ matrices. We show that the analysis is also applicable to generalized Hermitian matrices and exact results are obtained for $N = 2$, again showing crossover with respect to the variances. For higher values of $N$ we have relied on the numerical data and demonstrate that the crossover from level clustering to repulsion is a generic feature of generalized random matrices.

2. Symmetric matrices

Let us consider a matrix $H$ composed of $m$-independent entries $x_1, x_2, \ldots, x_m$ each drawn from probability distributions $P_i(x_i)$ implying that,

$$P(H) = \prod_{i=1}^{m} P_i(x_i).$$

We are interested in the diagonalizing matrix $H = \Theta^{-1} \mathcal{E} \Theta$ and obtaining the joint probability distribution (JPDF) of the eigenvalues, $P(\mathcal{E}) = P(E_1, E_2, \ldots, E_N)$. If we consider that the matrix elements of eigenfunctions $\Theta$ are parametrized as $\{\theta_1, \theta_2, \ldots, \theta_M\}$ then the transformation from the matrix space to the eigenspace necessitates

$$P(x_1, \ldots, x_m) \prod_i dx_i = f(E_1, \ldots, E_N, \theta_1, \ldots, \theta_M) |J| \prod_j dE_j \prod_k d\theta_k$$

where $J$ is the Jacobian of the transformation. To find the JPDF of eigenvalues we need to integrate over $\theta_j$

$$P(\mathcal{E}) = |J| \int d\theta_1 \cdots d\theta_M f(E_1, \ldots, E_N, \theta_1, \ldots, \theta_M)$$

which is suitably normalized such that $\int d\mathcal{E} P(\mathcal{E}) = 1$. If $P(\mathcal{E})$ is a symmetric function of its arguments, i.e. $P(E_1, E_2, \ldots, E_N) = P(E_{i1}, E_{i2}, \ldots, E_{iN})$, where $\{i1, i2, \ldots, iN\}$ are arbitrary permutations of $\{1, 2, \ldots, N\}$, then the marginal PDF of the eigenvalue $P(E)$ is given by

$$P(E) = \int dE_2 \cdots \int dE_N P(\mathcal{E}).$$

In this section we consider $H$ as an $N \times N$ real symmetric matrix, $H_{ij} = H_{ji}$, with the diagonal and the off-diagonal entries drawn from normal distributions $H_{ii} \sim N(0, \sigma_0^2)$ and $H_{ij} \sim N(0, \sigma_0^2)$ for $i \neq j$ respectively. Then the density function of $H$ given in equation (1) can be written as

$$P(H) = C \exp \left( -\sum_{i=1}^{N} \frac{H_{ii}^2}{2\sigma_d^2} - \sum_{i<j}^{N} \frac{H_{ij}^2}{2\sigma_o^2} \right)$$

$$= C \exp \left( \frac{1}{4\sigma_o^2} \sum_{i}^{N} H_{ii}^2 - \frac{1}{2\sigma_d^2} \sum_{i}^{N} \frac{\text{Tr}(H_i^2)}{4\sigma_o^2} \right) \quad \text{where, } C = \frac{(2\pi)^{\frac{N(N+1)}{4}}}{\sigma_d \sigma_o^{\frac{N-1}{2}}}.$$  

(5)

For symmetric matrices, eigenvectors are orthogonal to each other and $\Theta$ becomes an orthogonal matrix, $O$. Then, we can do a similarity transformation, $H = O^T \mathcal{E} O$, which implies
\[ \text{Tr}(H^2) = \sum_{i} E_i^2. \] Moreover, the Jacobian \( J \) is given by the Vandermonde determinant, i.e.
\[ J(H \rightarrow \{ \mathcal{E}, O \}) = \prod_{i<j}^N (E_i - E_j) \] [31]. Using the above properties, the JPDF of the eigenvalues assumes the form
\[ P(\mathcal{E}) \propto \exp \left( -\frac{1}{4\sigma^2} \sum_{i} E_i^2 \right) \prod_{i<j}^N |E_i - E_j|. \] (6)

Symmetric matrices satisfying \( \sigma_o = \sqrt{2}\sigma_d \) belong to the GOE corresponding to \( \beta = 1 \) in Wigner’s surmise. For any arbitrary choice of \( \{\sigma_o, \sigma_d\} \), obtaining an analytical expression of \( P(\mathcal{E}) \) is difficult and becomes harder as \( N \) increases. We now illustrate the generalization of the Wigner surmise proposed for \( 2 \times 2 \) matrices [29] and show that by tuning \( \{\sigma_o, \sigma_d\} \), crossovers are possible between the level repulsion and clustering of the eigenvalue spectra.

2.1. NNS distributions for \( N = 2 \)

For \( N = 2 \), \( \Theta \) can be taken as the \( 2 \times 2 \) rotation matrix, i.e.
\[ O = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \] and equation (2) becomes
\[ \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} E_1 \cos^2 \theta + E_2 \sin^2 \theta & (E_1 - E_2) \sin \theta \cos \theta \\ (E_1 - E_2) \sin \theta \cos \theta & E_1 \sin^2 \theta + E_2 \cos^2 \theta \end{bmatrix}. \] (7)

Substituting \( H_{ij} \) in equation (5), integrating over \( \theta \) and normalizing yields a quadratic Rayleigh–Rice distribution [29]
\[ P(\mathcal{E}) = \frac{|E_1 - E_2|}{4\sqrt{2}\pi \sigma^2_d \sigma_o} \exp \left( f_E - \frac{1}{2} \frac{E_1^2}{\sigma^2_d} \right) I_0(f_E) \]
where, \( f_E = \left[ \frac{1}{8\sigma^2_d} - \frac{1}{16\sigma^2_o} \right] (E_1 - E_2)^2 \). (8)

\( I_0 \) is the modified Bessel function of a 1st kind of order 0. Using the above JPDF in equation (4), we get the marginal PDF of the eigenvalues,
\[ P(E) = \frac{1}{4\sqrt{2}\pi \sigma^2_d \sigma_o} \int_{-\infty}^{\infty} dx|x| \exp \left( -\frac{(y + 2E)^2}{4\sigma^2_d} - \frac{y^2}{8} \left( \frac{1}{\sigma^2_d} + \frac{1}{2\sigma^2_o} \right) \right) I_0 \left( \frac{y^2}{8} \left( \frac{1}{\sigma^2_d} - \frac{1}{2\sigma^2_o} \right) \right). \] (9)

The above expression can be numerically evaluated and is shown for different \( \{\sigma_d, \sigma_o\} \) in figure 1(a). For \( \sigma_d = 1 \) and \( \sigma_o = 1/\sqrt{2} \), the expression in equation (8) gives the PDF of eigenvalues for GOE
\[ P_{\text{GOE}}(E_1, E_2) = \frac{|E_1 - E_2|}{4\sqrt{\pi}} \exp \left( -\frac{E_1^2 + E_2^2}{2} \right). \] (10)

Equation (9) can be computed exactly for GOE [32] and we get
\[ P_{\text{GOE}}(E) = \frac{1}{2\sqrt{\pi}} e^{-E^2} + \frac{1}{2\sqrt{2}} E e^{-E^2/2} \text{Erf} \left( \frac{E}{\sqrt{2}} \right) \] (11)
which is shown by the red line in figure 1(a). Next we calculate the PDF of the NNS, \( s \), as
Note that to evaluate equation (12), considering the case $E_1 \geq E_2$ will suffice. Due to the symmetry of equation (8), the other possibility will yield the same results. $P(s)$ is normalized and unfolded by changing the unit of NNS such that $\langle s \rangle = 1$ to produce a universal expression [10] by demanding

$$\int_0^\infty P(s) \, ds = 1, \quad \langle s \rangle = \int_0^\infty P(s) \, s \, ds = 1. \quad (13)$$

Let us define $\tilde{\sigma} = \frac{\sigma_d}{\sqrt{2}\sigma_o}$. Using equation (8) in equation (12) and satisfying equation (13) we get

$$P(s) = \frac{2\sigma_o^2}{\pi \sigma} \exp \left( - \frac{1 + \sigma^2}{2\sigma^2} \right) I_0 \left( \frac{\sigma^2}{2\sigma^2} \right) \left[ f_o = \text{EllipticE}(1 - \sigma^2), \sigma = \min \left\{ \tilde{\sigma}, \frac{1}{\tilde{\sigma}} \right\} \right]$$

where $\text{EllipticE}$ is a complete elliptic integral [30]. If equation (14) is observed carefully, it shows the following symmetry:

$$P(s)_{\tilde{\sigma} = \frac{1}{x}} = P(s)_{\tilde{\sigma} = x} \quad \forall \ x > 0 \quad (15)$$

which implies that the NNS distributions are identical for $\sigma_d \gg \sigma_o$ as well as $\sigma_d \ll \sigma_o$. It is important to note that if $P(\mathcal{E})$ in equation (8) or $P(E)$ in equation (9) does not have the above symmetry. For GOE, i.e. $\sigma_d = \sqrt{2}\sigma_o$, it is easy to see that equation (14) reduces to the familiar Wigner surmise for $\beta = 1$, given by

$$P_{\text{GOE}}(s) = \frac{\pi}{2} s \exp \left( - \frac{\pi}{4} s^2 \right). \quad (16)$$

Thus level repulsion is evident for $\tilde{\sigma} \to 1$. On the other hand, level clustering can be shown for $\sigma_o \to 0$, where the off-diagonal elements $H_{ij} \approx 0 \ \forall \ i \neq j$, thus...
corresponding to clustering as $s \to 0$. However, for any small value of $\sigma_o$, the enumeration of $P(s)$ in equation (14) shows a sharp jump to maxima near $s = 0$, which then rapidly decays to 0. This decay is sharper with decreasing $\sigma_o$ such that for all practical purposes we consider this as level clustering described by equation (17). Due to the symmetry in equation (14) the observations for $\sigma_o \to 0$ i.e. $\tilde{\sigma} \to \infty$ are also applicable for the condition $\tilde{\sigma} \to 0$. This indicates a crossover between level repulsion ($\tilde{\sigma} \to 1$) and level clustering as the relative strengths of the variances of the diagonal and off-diagonal elements are varied. We would also like to mention that in integrable systems, the level clustering is associated with the Poisson distribution, $P(s) = e^{-s}$, which is obtained after a local transformation, $s = sNPX(s)$ [31], and is realizable for large $N$, while here we are considering matrices with size $N = 2$.

In order to characterize the crossover we need an empirical function for intermediate values of $\{\sigma_d, \sigma_o\}$ similar in spirit to the Brody distribution [18]. In our case, the limiting cases for level repulsion, $P(s) \sim s \exp (-s^2)$, and for level clustering, $P(s) \sim \exp (-s^2)$, suggest a transition function of the form $P(s) \sim s^\eta \exp (-s^2)$. Normalizing and unfolding this equation we get

$$P(\eta; s) = \frac{2\Gamma\left(1 + \frac{\eta}{2}\right)^{1+\eta}}{\Gamma\left(\frac{1+\eta}{2}\right)^{2+\eta}} s^\eta \exp \left(-\frac{4\Gamma\left(1 + \frac{\eta}{2}\right)^{2}}{\Gamma\left(\frac{1+\eta}{2}\right)^{2}} s^2\right).$$

The data obtained from enumeration of the expression in equation (14) are fitted to the above function and $\eta$ is estimated using the trust-region algorithm [33]. A value $\eta \approx 1$ will imply level repulsion while $\eta \approx 0$ will imply level clustering. In figure 1(b) we plot spacing distributions for different $\sigma_d$ while keeping $\sigma_o = 1/\sqrt{2}$. In the inset the fitted value of the parameter $\eta$ shows a crossover from clustering–repulsion–clustering with the symmetry ascertained by equation (15). Based on the values of $\eta$ we can construct the phase diagram as shown in figure 1(c), where the dark (white) region shows level clustering (repulsion). The GOE is represented at the centre of these PD. The phase diagram indicates that from the region where $\sigma_d = \sqrt{2}\sigma_o \ (\eta \to 1)$ changing the difference between $\sigma_d$ and $\sigma_o$ by one order of magnitude results in a crossover ($\eta \to 0$).

The choice of the distribution for elements of a symmetric matrix can be further generalized, where $H_{11} \sim N(\mu_1, \sigma_1), H_{12} = H_{21} \sim N(\mu_2, \sigma_2), H_{22} \sim N(\mu_3, \sigma_3)$ [30]. However, for $\mu_1 = \mu_2 = \mu_3 = 0$ we can redefine $\tilde{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2/2\sigma_2}$, then again $P(s)$ follow equation (14).

### 2.2. NNS distributions for $N = 3$

In the case of a $3 \times 3$ matrix, $O$ can be taken as the generalized $3 \times 3$ rotation matrix, i.e. $O = R_z(\theta)R_y(\phi)R_x(\psi)$, and the transformation rules can be obtained for $H_O$. However, subsequent calculations are tedious and obtaining simple expressions becomes impossible. Observing equations (6) and (8) for $N = 2$ we propose the following ansatz for generalized $N \times N$ symmetric matrices
\[ P(E) = C_0 \exp \left( -\sum_{i=1}^{N} \frac{E_i^2}{2\sigma_d^2} + \sum_{i<j} f_{E_i E_j} \right) \prod_{i<j} (|E_i - E_j|) \times \left( \frac{1}{\sigma_d^2} - \frac{1}{\sigma_o^2} \right) (E_i - E_j)^2 \]

(19)

where the unknown constant \( C \) and the normalization constant \( C_0 \) need to be determined numerically. When \( \sigma_d = \sqrt{2}\sigma_o \), equation (50) reduces to the known expression for the PDF of eigenvalues of GOE (i.e. \( \beta = 1 \)) [32], and is given by

\[ P(E_1, E_2, ..., E_n) = \frac{1}{Z_{n,\beta}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} E_i^2 \right) \prod_{j<k} |E_j - E_k|^\beta \quad \left( Z_{n,\beta} = (2\pi)^{n/2} \prod_{j=1}^{n} \Gamma(1+j\beta/2) \right). \]

(20)

For \( N = 3 \) the PDF of NNS is obtained from equation (50) and assumes the form

\[ P(s) = C_0 s I_0(c_1 s^3) \int_0^\infty dy I_0(c_1 (s + y)^2) I_0(c_1 y^2) \times \exp \left( (2c_1 - \frac{1}{3\sigma_d^2})(s^2 + sy + s^2) \right) y(s + y), \quad c_1 = \frac{1}{C} \left( \frac{1}{\sigma_d^2} - \frac{1}{2\sigma_o^2} \right). \]

(21)

The above expression needs to be normalized and unfolded numerically and gives an excellent match with the corresponding simulated data, with \( C \approx 10.5 \) for \( \sigma_d < \sqrt{2}\sigma_o \) and \( C \approx 8 \) for \( \sigma_d > \sqrt{2}\sigma_o \), and is shown in figure 2(a). It is important to note that equation (21) does not have the symmetry found in equation (15).

In the limit \( \sigma_d \gg \sigma_o \), as argued for the \( N = 2 \) case, the PDF of the eigenvalues can be approximated by the PDF of the matrix and an explicit calculation for \( N = 3 \) gives

\[ P_{\text{cluster}}(s) = \frac{9}{4\pi} \exp \left( -\frac{9s^2}{16\pi} \right) \text{Erfc} \left( \frac{\sqrt{3}s}{4\sqrt{\pi}} \right). \]

(22)

When \( \sigma_d = \sqrt{2}\sigma_o \), the PDF of the NNS is exactly calculated from equation (21) and is given by

\[ P_{\text{GOE}}(s) = \frac{243}{32\pi^2} s^3 \exp \left( -\frac{9}{4\pi} s^2 \right) - \frac{27}{32\pi} s \left( \frac{27}{4\pi} s^2 - 6 \right) \exp \left( -\frac{27}{16\pi} s^2 \right) \text{Erfc} \left( \frac{3}{4\sqrt{\pi}} s \right). \]

(23)

In the limit, \( \sigma_d \ll \sigma_o \), we have not been able to find an exact analytical expression, but the numerical data suggests that the corresponding NNS distribution is very close to that of the GOE. Hence, for intermediate statistics, we can again construct a Brody-like distribution similar to equation (18). Limiting cases of level clustering, \( P(s) \sim \exp \left( -3s^2 \right) \text{Erfc}(s) \) and level repulsion, \( P(s) \sim 2s^2 \exp \left( -4s^2 \right) - s(2s^2 - 1) \exp \left( -3s^2 \right) \text{Erfc}(s) \), suggests a transition function of the form \( P(s) \sim 2\pi s^2 \exp \left( -4s^2 \right) - s(2s^2 - 1) \exp \left( -3s^2 \right) \text{Erfc}(s) \). Normalized and unfolded interpolating function is given in appendix (equation (A.1)). Numerically estimated values of \( \eta \) are shown in figure 2(a) indicating a crossover from clustering to repulsion in spacing distributions as \( \sigma_o \) is varied keeping \( \sigma_d \) constant. The phase diagram obtained from the estimation of \( \eta \) is shown in figure 2(b). It is important to note that the \( N = 3 \) phase diagram is qualitatively different and lacks the symmetry evident for \( N = 2 \).

2.3. NNS distributions for \( N = 4 \)

From the ansatz in equation (50), for \( N = 4 \), we obtain the spacing distribution
$$P(s) = C_0 I_0(c_1 s^2) \int_0^\infty dy \int_0^\infty dz \ yz (s+y)(s+y+z) (s+y+z)(s+y+z) I_0(c_1 y^2)I_0(c_1 z^2)$$

$$I_0(c_1 y^2) I_0(c_1 (s+y+z)^2) \exp \left( \left(c_1 - \frac{1}{8\sigma_d^2}\right) \left(3s^2 + 4y^2 + 3z^2 + 2yz + 4y(s+z)\right) \right).$$

Even numerical evaluation of the above integral is difficult for any $\{\sigma_d, \sigma_o\}$, but the GOE limit, $\sigma_d = \sqrt{2}\sigma_o$, can be exactly calculated

$$P_{\text{GOE}}(s) = \frac{3\pi}{131 072\sqrt{2}} \exp \left(-\frac{27\pi^2}{128}s^2\right) \left(48\sigma(448 - 27\pi^2) + 9\sqrt{2}\exp \left(\frac{9\pi^2}{128}(3\pi s^2 - 64)(9\pi s^2 - 64)\right) \right.$$  

$$- 64) \text{Erfc} \left(\frac{3\sqrt{\pi}}{8\sqrt{2}}\right) + 512 \sqrt{6} \exp \left(\frac{3\pi}{128}s^2\right)(3\pi s^2 - 16) \text{Erfc} \left(\frac{1}{8}\sqrt{\frac{3\pi}{2}s}\right) \right).$$

Figure 2. Transition from level clustering to repulsion in the NNS of symmetric matrices: PDF of NNS for different $\sigma_o$ with $\sigma_d = 1$ for (a) $N = 3$, (c) $N = 4$. Markers in (a) denote the enumeration of equation (21). Insets in (a) and (c) show the fitted transition parameter $\eta$ versus $\log(\sigma_d)$ when $\sigma_o = 1/\sqrt{2}$. The corresponding regions are denoted by bold lines in the PD. The star in PD indicates the GOE. The PD constructed w.r.t. the $\eta$ values (b) $N = 3$, (d) $N = 4$. 

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In the clustering limit $\sigma_d \gg \sigma_o$, the normalized and unfolded expression is also exactly calculated

$$P_{\text{cluster}}(s) = \frac{\mu}{C} \left( \sqrt{\pi} \exp \left( -\frac{\mu^2}{4s^2} \right) \text{Erfc} \left( \frac{\mu}{2\sqrt{3}s} \right) - 2 \exp \left( -\frac{3\mu^2}{8s^2} \right) \sum_{j=0}^{\infty} \frac{H_j \left( \frac{\mu}{2\sqrt{3}s} \right)}{8^{(j+\frac{1}{2})} \Gamma (j + \frac{3}{2})} \right)$$

(26)

where $H_j(x)$ is the $j$th Hermite polynomial (details in the appendix equations (A.3) and (A.4)).

The normalization constant, $C \approx 1.047198$ and the mean of the normalized expression $\mu \approx 0.732364$ are obtained numerically. Here, also in the limit $\sigma_d \ll \sigma_o$, the simulations of the PDF of the NNS resemble that of the GOE. For intermediate statistics, constructing a Brody-like interpolating function is difficult as the limiting expressions are very complicated.

Thus, we can resort to the additive RMT approach [22] to form a crossover function like

$$P(\eta; s) = \eta P_{\text{GOE}}(s) + (1 - \eta) P_{\text{cluster}}(s)$$

(27)

from which we estimate $\eta$ to quantify the level statistics for any value of $\sigma_d, \sigma_o$, and the results are shown in figures 2(c) and (d). The spacing distributions behave identically for $N = 3$ and $N = 4$ and the numerical estimate of the transition parameter $\eta$ is indicative of the crossovers from repulsive to clustered states as $\bar{\sigma}$ increases.

2.4. RNNS distributions for $N = 3$

The numerical issues in the unfolding NNS distributions can be avoided by estimating the distribution of the ratio of level spacings between consecutive eigenvalues (RNNS) defined as

$$\bar{r}_i = \min(\tau_i, \tau_{i-1}) / \max(\tau_i, \tau_{i-1}) = \min \left( r_i, \frac{1}{r_i} \right)$$

where $r_i = \tau_i / \tau_{i-1}$ [34]. The RNNS can be defined for matrices of size $N \geq 3$, and for $N = 3$ the PDF of RNNS is

$$P(r) = \int_{-\infty}^{\infty} dE_2 \int_{E_2}^{E_1} dE_1 \int_{E_2}^{\infty} dE_3 P(\mathcal{E}) \delta \left( r - \frac{E_3 - E_2}{E_2 - E_1} \right).$$

(28)

Once again using $P(\mathcal{E})$ from equation (50) we can get the PDF of RNNS

$$P(r) = C_0 (r + r^2) \int_0^{\infty} dx \left[ x^4 I_0(c_1 (1 + r^2)x^2) I_0(c_1 r^2 x^2) I_0(c_1 x^2) \exp \left( \frac{2c_1 - \frac{1}{3\sigma_d^2}}{c_1} \right) (1 + r + r^2)\right].$$

(29)

As the RNNS is the ratio of two consecutive levels, in the level clustering regime $P(r)$ does not require any local transformation to bring out the universal features. For $N = 3$ we can calculate the RNNS distribution as

$$P_{\text{cluster}}(r) = \frac{3\sqrt{3}}{2\pi} \frac{1}{1 + r + r^2}.$$  

(30)

Again in the GOE limit $\sigma_d = \sqrt{2}\sigma_o$, the PDF of the RNNS has also been obtained [11]

$$P_{\text{GOE}}(r) = \frac{27}{8} \frac{r + r^2}{(1 + r + r^2)^{3/2}}.$$  

(31)

However, in the limit $\sigma_d \ll \sigma_o$, the numerical evaluation of equation (29) shows a deviation from the simulated data. In figure 3(a), $P(r)$ are shown for different values of $\sigma_d$ exhibiting level
For $\sigma_o > \sigma_d$. However, the corresponding functional form starts deviating from that of the GOE. This is in contrast to our findings for NNS distributions where linear level repulsion was observed for small spacings. For intermediate statistics we define an additive crossover function as in equation (27), and the transition parameter shows a crossover from repulsion to clustering (figure 3(a)). We systematically scan the $\sigma_o - \sigma_d$ plane and estimate $\eta$ to represent the phase diagram in figure 3(b). The phase diagrams from the NNS and RNNS distributions are qualitatively similar, establishing their equivalence and the universality of spectral properties.

2.5. RNNS distributions for $N = 4$

The PDF of the RNNS can be obtained for the clustering regime $\sigma_d \gg \sigma_o$

$$P_{\text{cluster}}(r) = \frac{3}{2\pi} \left( 2\sqrt{3} - \frac{2 + r}{\sqrt{3r^2 + 4r + 4}} - \frac{1 + 2r}{\sqrt{4r^2 + 4r + 3}} \right) \frac{1}{1 + r + r^2}.$$  

\[ (32) \]
For the GOE case, analytical calculation is straightforward but the final expression is very lengthy and has already been reported [35]. Interestingly, in the limit $\sigma_d \ll \sigma_o$, simulations show that the RNNS distributions have a stronger similarity to that of the GOE in contrast to the observed deviations for $N = 3$. Here we also use an interpolating function as in equation (27), identify the crossover with respect to the estimated values of $\eta$ (figure 3(c)) and identify the phase diagram (figure 3(d)).

### 3. Hermitian matrices

Let $H$ be an $N \times N$ Hermitian matrix, $H_{ij} = H_{ji}^*$. The diagonal elements are such that $H_{ii} \sim \mathcal{N}(0, \sigma_i^2)$ and the off-diagonal elements $H_{ij} = x_{ij} + iy_{ij}$ with $x_{ij}, y_{ij} \sim \mathcal{N}(0, \sigma_i^2) \forall i \neq j$.

Using equation (1), we get the PDF of such matrices

$$
P(H) = C \exp \left( -\sum_{i=1}^{N} \frac{H_{ii}^2}{2\sigma_i^2} - \sum_{i<j} \frac{x_{ij}^2 + y_{ij}^2}{2\sigma_o^2} \right)
$$

$$
= C \exp \left( \frac{1}{4\sigma_o^2} - \frac{1}{2\sigma_o^2} \sum_{i=1}^{N} H_{ii}^2 - \frac{\text{Tr}(H^2)}{4\sigma_o^2} \right)
$$

where, $C = \frac{(2\pi)^{N^2/2}}{\sigma_o^N N! \sqrt{\pi}}$.

(33)

We can use the similarity transformation, $H = U^\dagger E U$, by which we get

$$
\text{Tr}(H^2) = \sum_{i=1}^{N} E_i^2.
$$

But here going from matrix space to eigenspace requires the Jacobian to be $J(H \to \{E, U\}) = \prod_{i<k}^{N} (E_i - E_j)^2$ [31]. Using this in equation (33), then following equation (3), we get

$$
P(E) \propto \exp \left( -\frac{1}{4\sigma_o^2} \sum_{i=1}^{N} E_i^2 \right) \prod_{i<j}^{N} (E_i - E_j)^2.
$$

(34)

### 3.1. The NNS distributions for $N = 2$

For $N = 2$ we can take an arbitrary unitary matrix, i.e. $U = e^{\alpha \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}}$ giving us

$$
\begin{bmatrix}
H_{11} & x_{12} + iy_{12} \\
-x_{12} - iy_{12} & H_{22}
\end{bmatrix}
= \begin{bmatrix}
E_1 \cos^2 \theta + E_2 \sin^2 \theta & (E_1 - E_2) \sin \theta \cos (\phi + i\sin \phi) \\
(E_1 - E_2) \sin \theta \cos (\phi - i\sin \phi) & E_2 \cos^2 \theta + E_1 \sin^2 \theta
\end{bmatrix}.
$$

(35)

We can see that the transformation rules for the diagonal elements are the same as those of the symmetric matrix. Substituting equation (35) in equation (33) and using equation (3) along with the normalization we get

$$
P(E) = \frac{|E_1 - E_2|}{4\sigma_o^2 \sigma_d \sqrt{2\pi}} \exp \left( -\frac{2}{\sigma_d^2} \sum_{i=1}^{2} E_i^2 \right) \exp \left( \left( \frac{1}{4\sigma_o^2} - \frac{1}{8\sigma_o^2} \right) (E_1 - E_2)^2 \right) \times g\left( \frac{|E_1 - E_2|}{2} \right)
$$

$$
g(x) = \begin{cases}
\text{Erf} \left( \sqrt{\frac{x}{\sigma_o^2}} \right), & \sigma_d < \sqrt{2\sigma_o} \\
\text{Erf} \left( \frac{x}{\sqrt{2} \sigma_o} \right), & \sigma_d > \sqrt{2\sigma_o}
\end{cases}
$$

(36)

and the corresponding marginal PDF using equation (4) can be expressed as...
The numerical simulations agree well with the above expression and the distributions are shown in figure 4(a) for various values of $\sigma_o$ for a fixed value of $\sigma_d = 1$. The PDF of NNS is given exactly for generalized J. Phys. A: Math. Theor. 52 (2019) 395001

$$\begin{align*}
P(E) &= \frac{1}{4\sqrt{2\pi}\sigma_d\sqrt{[2\sigma_o^2 - \sigma|^2]}} \int_{-\infty}^{\infty} dy|y| \exp \left( -\frac{(y + 2E)^2}{4\sigma_d^2} - \frac{x^2}{8\sigma_o^2} \right) g(|y|).
\end{align*}$$

(37)

When $\sigma_d > \sigma_o$, then $H$ can be assumed to be a diagonal matrix with all real entries resulting in a similar symmetric matrix of the previous section, implying that the $P_{\text{cluster}}(s)$ is given by equation (17). However, equation (38) does not have the symmetry of equation (15) and clustered state does not exist for $\sigma_d < \sigma_o$.

The PDF in equation (36) correctly yields the expected results for GUE when $\sigma_d = 1, \sigma_o = 1/\sqrt{2}$ upon using the limits

$$\begin{align*}
\lim_{x \to 0} \frac{\text{Erf}(x)}{x} = \lim_{x \to 0} \frac{\text{Erfi}(x)}{x} = \frac{2}{\sqrt{\pi}}.
\end{align*}$$

(39)

The JPDF of the eigenvalues is given by

$$\begin{align*}
P_{\text{GUE}}(E_1, E_2) &= \frac{1}{4\pi} (E_1 - E_2)^2 \exp \left( -\frac{E_1^2 + E_2^2}{2} \right)
\end{align*}$$

(40)

with the corresponding marginal PDF

$$\begin{align*}
P_{\text{GUE}}(E) &= \frac{1 + E^2}{2\sqrt{2\pi}} \exp \left( -\frac{E^2}{2} \right).
\end{align*}$$

(41)

From equation (38) we get the PDF of NNS for GUE

$$\begin{align*}
P_{\text{GUE}}(s) &= \frac{32}{\pi^2} s^2 \exp \left( -\frac{4}{\pi} s^2 \right)
\end{align*}$$

(42)

showing quadratic level repulsion, i.e. the familiar form for $\beta = 2$ obtained by using the limits

$$\begin{align*}
\lim_{x \to 0} \frac{\cot^{-1}(1/x)}{x} = \lim_{x \to 0} \frac{\tanh^{-1}x}{x} = 1.
\end{align*}$$

(43)

Surprisingly in the limit $\sigma_d \ll \sigma_o$, using $\lim_{x \to \infty} \text{Erf}(x) = 1$ in equation (38) we get linear level repulsion as in equation (16). Thus it is possible to access three distinct phases as $\sigma$ is varied. This is an improvement over the crossover function described in [22], which uses one parameter to break integrability and another one to break anti-unitary symmetry.

The limiting cases for level repulsion, $P(s) \sim s^2 \exp(-s^2)$ and level clustering, $P(s) \sim \exp(-s^2)$, suggest a transition function of the form $P(s) \sim s^{2\eta} \exp(-s^2)$. Normalizing and unfolding this equation we get
The above equation has been used to fit the simulation data and estimate $\eta$ to characterize the crossover from clustered to repulsive states (figure 4(b)) and identify the corresponding phase diagram shown in figure 4(c). The different shades of the greyscale distinguish the three states possible in this case.

3.2. NNS distributions for $N = 3$

Observing the form of $P(E)$ in equation (36), here we also suggest an ansatz

$$P(E) = C_0 \exp \left( \sum_{i<j} \frac{1}{\sigma_d^2} \left( \frac{1}{2} \frac{1}{\sigma_o^2} \right) (E_i - E_j)^2 \right) \exp \left( - \sum_{i=1}^{N} E_i^2 \right) \prod_{i<j} \left( \frac{|E_i - E_j|}{\sqrt{C}} \right) g \left( \frac{|E_i - E_j|}{\sqrt{C}} \right)$$

(45)

where $g(x)$ is defined in equation (36). For $\sigma_d = 1$, $\sigma_o = 1/\sqrt{2}$, equation (45) reduces to equation (20), the known expression for the PDF of the eigenvalues of the GUE (i.e. $\beta = 2$) [32].

We can evaluate the PDF of NNS from

$$P(s) = C_0 g \left( \frac{s}{\sqrt{C}} \right) \int_{0}^{\infty} dy \left[ g \left( \frac{y}{\sqrt{C}} \right) g \left( \frac{s+y}{\sqrt{C}} \right) \exp \left( 2c_1 - \frac{1}{3\sigma_d^2} \right) \right]$$

(46)

Enumeration of this expression gives an excellent match with the corresponding simulated data with $C \approx 10.5$ for $\sigma_d < \sqrt{2}\sigma_o$ and $C \approx 4$ for $\sigma_d > \sqrt{2}\sigma_o$ (figure 5(a)).
In the limit $\sigma_d \gg \sigma_o$ we assume that the diagonal elements survive, similar to the case of symmetric matrices, and the PDF of the NNS will also be given by $P_{\text{cluster}}(s)$ given in equation (22). For the GOE limit, $\sigma_d = \sqrt{2}\sigma_o$, the PDF of the NNS is given by

$$P_{\text{GUE}}(s) = \frac{3^{12}}{2^{36}\pi^2}e^{-\frac{1}{4}\frac{\eta^2}{s^2}} \left(144\sqrt{3}(128\pi - 81s^2) + e^{\frac{1}{2}\frac{3\pi^2}{\eta^2}}(38s^4 - 283\pi^2s^2 + 214\pi^4)\text{Erfc} \left(\frac{9\sqrt{3}}{16\sqrt{\pi}}s\right)\right).$$

(47)

In the case $\sigma_d \ll \sigma_o$ exact analytical expressions are hard to obtain, as the $H_{ii}$ term survives in $P(H)$ and we have to rely on numerical data. Here we observe that the corresponding PDFs of the NNS have close resemblance to that of the GUE. Thus, similar to the analysis for symmetric matrices, for intermediate statistics we can construct a Brody-like distribution. The limiting case of level clustering, $P(s) \sim \exp(-3s^2)\text{Erfc}(s)$ and level repulsion, $P(s) \sim \frac{2}{3\sqrt{\pi}}s^3(3 - 2s^2)\exp(-4s^2) + s^2\left(\frac{4}{3}s^2(s^2 - 1) + 1\right)\exp(-3s^2)\text{Erfc}(s)$ suggest a transition function of the form $P(s) \sim \frac{2n}{3\sqrt{\pi}}s^3(3 - 2s^2)\exp(-4s^2) + s^2\eta \left(\frac{4n}{3}s^2(s^2 - 1) + 1\right)\exp(-3s^2)\text{Erfc}(s)$. Thus for $\eta = 0$ we will get clustering and $\eta = 1$ produces quadratic level repulsion. We need to normalize and unfold this function, and the resultant expression is given in the appendix equation (A.2). Figure 5(b) depicts the PD for $N = 3$ Hermitian matrices obtained from estimation of $\eta$ by numerical fitting. For $N = 4$, the NNS distributions are obtained by numerical simulations, showing qualitatively similar features to those of the $N = 3$ Hermitian matrices (figure 5(c)). Here we also numerically estimate the transition parameter and obtain the phase diagram, clearly identifying the clustered and GUE-like repulsion states (figure 5(d)).

3.3. The RNNS distribution for $N = 3$

From the ansatz in equation (45) we obtain the RNNS distribution as follows

$$P(r) = C_0(r + \rho) \int_0^{\infty} dx \left[ g \left(\frac{1 + r \mu}{\sqrt{C}}\right) g \left(\frac{x}{\sqrt{C}}\right) \exp \left(\left(2c_1 - \frac{1}{3\sigma_d^2}\right)(1 + r + \rho)\right)\right].$$

(48)

where $c_1$ is given in equation (46). For $N = 3$, the RNNS distribution for the GUE is known [35] and is given by

$$P_{\text{GUE}}(r) = \frac{81\sqrt{3}(r + \rho)^2}{4\pi(1 + r + \rho)^2}. (49)$$

In the limit $\sigma_d \gg \sigma_o$, a Hermitian matrix is similar to a diagonal symmetric matrix and the PDF of the RNNS is still given by equation (30). In the other limit, $\sigma_d \ll \sigma_o$, we could not find an analytical expression; however, the numerical data suggests a resemblance to the GUE. For $N = 4$, an exact expression for the PDF of the RNNS for the GUE has been exactly obtained [35]. In the limit $\sigma_d \gg \sigma_o$, the PDF of the RNNS is given by equation (32), while for $\sigma_d \ll \sigma_o$ the numerical data also shows repulsion as in the GUE. Based on these observations, we use an additive interpolating function, like equation (27), and estimate the transition parameter to obtain the phase diagrams shown in figures 6(b) and (d) for matrix sizes $N = 3, 4$.

4. Spectral properties for large $N$

The phase diagrams evident from our calculations for $N = 3$ matrices are also observed in the numerical analysis of the finite sized $N \times N$ symmetric and Hermitian matrices. If the relative
variances are chosen such that $\sigma_d \ll \sigma_o$ and $\sigma_d \gg \sigma_o$, the numerical analysis shows repulsion and clustering, respectively. In figure 7(a) we show the typical behavior of $P(s)$ for a large matrix dimension $N$. Symmetric and Hermitian matrices in the regime $\sigma_d \ll \sigma_o$ show GOE- and GUE-like repulsions, respectively. On the other hand for $\sigma_d \gg \sigma_o$, $P(s)$ shows clustering.

Similar behavior is observed for $P(r)$ as shown in figure 7(b). We would now like to explore how the repulsion-clustering boundary changes due to finite size effects. We numerically fit $P(r)$ distributions to an interpolating function (equation (A.9)) and estimate the parameter $\eta$ such that $\eta = 1$ indicates repulsion while $\eta = 0$ indicates clustering. In figure 7(c) we fix $\sigma_o = 1$, vary $\sigma_d$ and plot the estimated crossover parameters $\eta$ for different system sizes. The vertical dashed line indicates $\sigma_d = \sqrt{2}\sigma_o$, the GOE limit; note that the boundary shifts to the right as $N$ increases. If we approximate the critical value $\sigma_d$ as the value at $\eta = 0.5$, we find that $\sigma_d^c(N) \propto N$, implying that only the repulsive state will emerge in the thermodynamic limit, i.e. $N \to \infty$. While it can be naively argued that in the thermodynamic limit, the statistical properties of $O(N)$ diagonal elements are overshadowed by that of $O(N^2)$ off-diagonal

![Figure 5. The transition from level clustering to repulsion in the NNS of Hermitian matrices: the PDF of NNS for different $\sigma_o$ with $\sigma_d = 1$ for (a) $N = 3$, (c) $N = 4$. Markers in (a) denote the enumeration of equation (46). The insets in (a) and (c) show the fitted transition parameter $\eta$ versus $\log(\sigma_d)$ when $\sigma_o = 1/\sqrt{2}$. The corresponding regions are denoted by bold lines in the PD. The star in PD indicates GUE. The PD constructed w.r.t. the $\eta$ values (b) $N = 3$, (d) $N = 4$.](image-url)
matrix elements, we would like to test if our ansatz conforms to our numerical observations. Let us rewrite the ansatz for generalized $N \times N$ symmetric matrices

$$P(E) = \frac{1}{Z_N} \exp \left( - \sum_{i=1}^{N} E_i^2 / 2\sigma_d^2 + \sum_{i<j} f_{E_i,E_j} \prod_{i<j} (|E_i - E_j| I_0(f_{E_i,E_j})) \right)$$

Figure 6. The transition from level clustering to repulsion in the RNNS of the Hermitian matrices: the PDF of RNNS for different $\sigma_o$ with $\sigma_d = 1$ for (a) $N = 3$, (c) $N = 4$. The markers in (a) denote the enumeration of equation (48). Insets in (a) and (c) show the fitted transition parameter $\eta$ versus $\log(\sigma_d)$ when $\sigma_o = 1/\sqrt{2}$. Corresponding regions are denoted by bold lines in the PD. The star in the PD indicates the GUE. The PD constructed w.r.t. the $\eta$ values (b) $N = 3$, (d) $N = 4$. With $f_{E_i,E_j} = -\Lambda (E_i - E_j)^2$ such that all constants are clubbed in $\Lambda$. If we rescale eigenvalues $E_i \rightarrow \sqrt{N}E_i$, the normalization constant or the partition function can be given by

$$Z_N = C_0 \prod_{j=1}^{N} dE_j \exp \left( \frac{N}{2} \sum_{i<j} E_i^2 / 2\sigma_d^2 \right) \exp \left( \frac{1}{2} \sum_{i \neq j} \left( \ln |E_i - E_j| + \ln (I_0(f_{E_i,E_j})) + \frac{1}{2} \ln N \right) \right)$$

$$\sim \prod_{j=1}^{N} dE_j e^{-N^2\nu(E)}$$
where in analogy with the Coulomb gas [7] the potential takes the form
\[
V(E) = \frac{1}{2N\sigma_d^2} \sum_{i=1}^{N} E_i^2 - \frac{1}{2N} \sum_{i\neq j} \ln |E_i - E_j| - \frac{1}{2N} \sum_{i\neq j} f(E_i) - \frac{1}{2N} \sum_{i\neq j} \ln (I_0(Nf(E_i)))
\]
\[
= V_{GOE}(E) + \frac{\Lambda}{2N} \sum_{i\neq j} (E_i - E_j)^2 - \frac{1}{2N} \sum_{i\neq j} \ln (I_0(N\Lambda(E_i - E_j)^2))
\].

It is easy to see that in the limit \( N \to \infty \), the approximation \( I_0(x) \approx e^x \) results in the cancellation of the last two terms. We thus retrieve the potential \( V_{GOE} \), implying repulsion in the eigenvalue statistics of symmetric matrices for any arbitrary choices of \( \{\sigma_d, \sigma_o\} \). For Hermitian matrices, starting from the ansatz in equation (45), a similar explanation can be given for numerically observed repulsion for large matrix dimensions.

5. Conclusions

We studied symmetric and Hermitian matrices where the diagonal and the off-diagonal elements are drawn from normal distributions but with different variances. This enables us to define a parameter \( \tilde{\sigma} = \sigma_d / (\sqrt{2}\sigma_o) \) which has been tuned to show that the spacing distributions of the eigenvalues show a crossover from level clustering to level repulsion. In such a generalized setting the symmetric matrices are not invariant with respect to orthogonal transformations and do not belong to the GOE, which is realized only for \( \sigma_d = \sqrt{2}\sigma_o \). The analytical calculation of \( P(s) \) has been reported for \( 2 \times 2 \) symmetric matrices [29] and we also observe that the NNS statistics change from clustering to GOE-like repulsion. We have proposed an ansatz for the eigenvalue distributions for \( N \times N \) matrices, and for \( N = 3 \) we have derived an integral form of the PDF of the NNS as well as the RNNS. Here we also observe crossovers and obtain phase diagrams by invoking fitting functions with parameters that distinctly identify clustering and repulsion. However, the phase diagram obtained for \( N = 3 \) is qualitatively different from that of \( N = 2 \). For \( N = 2 \), the NNS distribution is symmetric for small and large values of \( \tilde{\sigma} \) and the symmetry is reflected in the corresponding phase diagram. In the \( \sigma_d = \sigma_o \) plane, the GOE is in the center of a narrow band which shows repulsion, while
away from the band in either direction clustering exists. On the other hand, for \( N = 3 \) the symmetry is broken and repulsion is observed only when \( \sigma_d < \sqrt{2}\sigma_o \), which is also evident from the PDFs of both the NNS and the RNNS.

In this work we have exactly solved for the eigenvalue and spacing distributions of similarly generalized \( 2 \times 2 \) Hermitian matrices and found the existence of crossover from clustering to repulsion. Interestingly, \( N = 2 \) generalized Hermitian matrices do not have symmetry in their NNS distribution. However, the phase diagram shows clustering and repulsion as in the GUE for \( \sigma_d \sim \sqrt{2}\sigma_o \), and GOE-like repulsion for \( \sigma_d \ll \sigma_o \). The introduction of relative variances thus results in spectral statistics to explore three distinct regimes. For \( N = 3 \) we also propose an ansatz and obtain an integral form of the PDFs for the NNS and the RNNS, and observe crossovers to GUE-like repulsion only. The competition between the relative strength of fluctuations in the on-site terms and the coupling terms can be possibly modeled as generalized random matrices resulting in intermediate statistics as observed in various theoretical studies and in experiments [14–17]. However, in the thermodynamic limit (\( N \to \infty \)), the numerical data shows repulsion irrespective of \( \tilde{\sigma} \), which can be understood from the large \( N \) asymptotics of our proposed ansatz. From a theoretical perspective it will be interesting to explore similar generalizations of matrices belonging to different symmetry classes and investigate the emergence of intermediate states and finite size effects on the transitions.

Appendix

Normalized and unfolded function used for interpolating \( P(s) \) of symmetric matrices, \( N = 3 \):

\[
P(\eta; s) = \frac{\mu^p}{C} f(\mu; s), \quad f(s) = \frac{2\eta}{\sqrt{\pi}} s^2 \exp\left(-4\eta^2\right) + s^2 \left(2\eta^2 - 1\right) \exp\left(-3\eta^2\right) \text{Erfc}(s)
\]

\[
C_\eta = \frac{1}{2^{4+\eta}} \left(8\Gamma(1+\eta)2\text{FR}_1\left(\frac{1+\eta}{2}, \frac{2}{2}, \frac{3+\eta}{2}, -3\right) + \eta \left(2^{\eta} - 4\Gamma(3+\eta)2\text{FR}_1\left(\frac{3+\eta}{2}, \frac{4+\eta}{2}, \frac{5+\eta}{2}, -3\right)\right)\right)
\]

\[
\mu_\eta = \frac{1}{\sqrt{\pi}} \left(8\Gamma(1+\eta)2\text{FR}_1\left(\frac{1+\eta}{2}, \frac{2+\eta}{2}, \frac{3+\eta}{2}, -3\right) + \eta \left(2^{\eta} - 4\Gamma(3+\eta)2\text{FR}_1\left(\frac{3+\eta}{2}, \frac{4+\eta}{2}, \frac{5+\eta}{2}, -3\right)\right)\right)
\]

(A.1)

where \( 2\text{FR}_1 \) is a regularized hypergeometric function.

Normalized and unfolded function used for interpolating \( P(s) \) of Hermitian matrices, \( N = 3 \):

\[
P(\eta; s) = \frac{\mu^p}{C} f(\mu; s), \quad f(s) = \frac{2\eta}{3\sqrt{\pi}} s^2 \left(3 - 2s^2\right) \exp\left(-4\eta^2\right) + s^2 \left(4\eta^2 - 1\right) \exp\left(-3\eta^2\right) \text{Erfc}(s)
\]

\[
C_\eta = \frac{1}{3\sqrt{2^{2+3\eta}}} \left(4^\eta + 24\sqrt{\pi}\Gamma(2\eta)2\text{F}_1\left(\frac{1+\eta}{2}, \frac{2}{2}, \frac{3}{2}, -3\right)
\]

\[
- 4\sqrt{\pi}\Gamma(3+2\eta)2\text{F}_1\left(\frac{3}{2} + \eta, 2 + \eta, \frac{5}{2} + \eta, -3\right) + \sqrt{\pi}\Gamma(5+2\eta)2\text{F}_1\left(\frac{5}{2} + \eta, 3 + \eta, \frac{7}{2} + \eta, -3\right)\right)
\]

\[
\mu_\eta = \frac{\sqrt{\pi} \left(189\eta 4^\eta + 128(27 - 4\eta + 4\eta^2)\Gamma(2 + 2\eta)2\text{F}_1\left(1 + \eta, \frac{3}{2} + \eta, 2 + \eta, -3\right) - 16\eta(1 + \eta)\Gamma\left(\frac{3}{2} + \eta\right)\right)}{1728\sqrt{\pi} 2^{2+3\eta} C_\eta}
\]

(A.2)

Symmetric matrix (\( N = 4 \)): in the clustering limit (\( \sigma_d \gg \sigma_o \)) the normalized and unfolded PDF is given by,
\[ P_{\text{cluster}}(r) = \frac{\mu}{C} \exp\left(-\frac{\mu^2}{4} r^2\right) \int_0^\infty dz \exp\left(-\frac{z^2}{4}\right) \text{Erfc}\left(\frac{\mu z}{2\sqrt{2}}\right), \quad \mu \approx 0.732364, \quad C \approx 1.047198. \]  

(A.3)

The normalization constant, \( C \) and mean, \( \mu \) are numerically determined using the result [36]

\[ I(a, b, \infty) = \frac{\sqrt{\pi}}{2} \int_0^\infty dx e^{-\beta x} \text{Erf} (at + b) = \frac{\pi}{4} \text{Erfc} \left( \frac{b}{\sqrt{1 + a^2}} \right) + \frac{\sqrt{\pi}}{2} e^{-\beta t} \sum_{j=0}^{\infty} \frac{(\beta^2)^{j+1}}{\Gamma(j + \frac{1}{2})} H_j(b) \]

where \( H_j(x) \) is the \( j \)th Hermite polynomial. Then, for \( N = 4 \), the PDF of the NNS in the clustering limit is given by equation (26).

Correction function for the RNNS of Hermite ensembles, \( N > 3 \): [11]

\[ \delta P_W(\kappa; r) = \frac{\kappa}{(1 + R)^2} \left( \frac{r + 1}{r} \right)^{-\beta} - c_\beta \left( \frac{r + 1}{r} \right)^{-(\beta+1)} \]

(A.5)

where, \( c_\beta \) is obtained from the normalization condition, \( \int_0^\infty \delta P_W(\kappa; r) dr = 0 \) \( \implies c_{\beta,GOE} = 2 \frac{\pi^2}{4 - \beta}, \quad c_{\beta,GUE} = 4 \frac{\pi^2}{3 \pi - 8} \). Hence, the PDF of the RNNS for arbitrary \( N \),

\[ P_W(\kappa; r) \approx P_W(r) + \delta P_W(\kappa; r), \quad \text{where} \quad P_W(r) \quad \text{is the PDF of the RNNS for} \quad N = 3 \ \text{Hermite ensembles given by} \]

\[ P_W(r) = \frac{1}{Z_\beta (1 + r^2)^{1+\frac{\beta}{2}}} \]

(A.6)

Thus, \( \kappa \) is the only free parameter in the above fitting.

Interpolation function for the RNNS of Poisson ensembles, \( N \geq 3 \): Exact expressions for the PDF of the RNNS for Poisson ensembles are

\[ P_{\text{cluster}} (r) = \begin{cases} \frac{3\sqrt{3}}{2\pi} - \frac{1}{(1+r)^2} & \text{N} = 3 \\ \frac{1}{1+r} & \text{N} \to \infty \end{cases} \]

(A.7)

Then, for arbitrary \( N \), we can empirically define the following interpolation function,

\[ P_{\text{cluster}}(\zeta; r) = \frac{3\sqrt{3}}{2\pi (1 - \zeta) + 3\sqrt{3}\zeta} \left( 1 - \frac{\zeta r}{(1 + r^2)} \right)^\frac{1}{1 + r + r^2}. \]

(A.8)

Thus by construction, \( \zeta = 0 \) denotes \( N = 3 \) and \( \zeta \to 1 \) for \( N \to \infty \).

Interpolation function for RNNS with arbitrary \( \{\sigma_d, \sigma_o\} \): for any \( N \), we know \( P_W(\kappa; r) \) (when \( \sigma_d = \sqrt{2}\sigma_o \)) and \( P_{\text{cluster}}(\zeta; r) \) (when \( \sigma_d \gg \sigma_o \)) from earlier fittings. Thus, for intermediate values of \( \{\sigma_d, \sigma_o\} \), we can use the additive RMT approach to create an interpolation function,

\[ P(\eta; r) = \eta P_W(\kappa; r) + (1 - \eta) P_{\text{cluster}}(\zeta; r). \]

(A.9)

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