Mathematical modelling of contact of ruled surfaces: theory and practical application

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Abstract. In the theory of ruled surfaces there are well known researches of contact of ruled surfaces along their common generator line (Klein image is often used [1]). In this paper we propose a study of contact of nondevelopable ruled surfaces via the dual vector calculus. The advantages of this method have been demonstrated by E. Study, W. Blaschke and D. N. Zeiliger in differential geometry studies of ruled surfaces in space \( \mathbb{R}^3 \) over the algebra of dual numbers. A practical use of contact is demonstrated by the example modeling of the working surface of the progressive tool for tillage.

1. Fundamentals of theory
The equation of a ruled surface can be represented as a unit dual vector-function with real parameter \( t \):

\[
A_1(t) = a_{01}(t) + \alpha a_{11}(t), |A_1(t)| = 1, \alpha^2 = 0,
\]
where \( a_{01}(t) \) – a unit generator vector of the ruled surface, \( a_{11}(t) \) – a momentum vector of the generator with respect to the origin. \( A_1(t) \) is \( n+1 \) times continuously differentiable. It is also known that a trihedron of the ruled surface consists of dual unit vectors:

\[
A_1; A_2 = \frac{A_1'}{|A_1'|}; A_3 = A_1 \times A_2,
\]
where \( A_1' = dA_1/dt; H_A = |A_1'|; A_2 \) – a central normal vector; \( A_3 \) – a central tangent vector.

And we also used an inner geometric parameter of the ruled surface - an element of a dual arc:

\[
d\omega = d\omega_0 + \omega \cdot d\omega_0 = H_A dt, \omega^2 = 0,
\]
which means a dual angle between its two infinitesimally close generators.

A distribution parameter of the ruled surface is deduced from the element of the dual arc:

\[
p = \frac{d\omega}{d\omega_0}.
\]

Trihedron derivation equations of W. Blaschke [2] are as follows:

\[
A_1' = H_A \cdot A_2; A_2' = -H_A \cdot A_1 + Q_A \cdot A_3; A_3' = -Q_A \cdot A_2,
\]
where \( H_A = h_{00} + \omega h_{01}, Q_A = q_{00} + \omega q_{01} = (A_1 A_1') A_1'')/H_A^2, \omega^2 = 0.\)

Let us consider the contact of two nondevelopable ruled surfaces parameterized by \( A_1(t) \) and \( B_1(t) \) with common ruling \( a_{01}(t) = b_{01}(t) \), when parameter value \( t = t_0 \). Initially we will introduce the dual vector of divergence \( G(t) \) in their common ruling. Since each unit dual vector function \( A_1(t) \) and \( B_1(t) \) can be represented as a Taylor series it can be written as:
\[ G(t) = A_i(t_0) - B_i(t_0) + [A_i'(t_0) - B_i'(t_0)] \cdot \Delta t + [A_i''(t_0) - B_i''(t_0)] \cdot \frac{\Delta t^2}{2!} + \ldots \] (1)

We will define the concept of an order of contact of two ruled surfaces in their common generator line. Each of the considered dual vector functions has the following coordinate representation:

\[ A_i(t) = i \cdot x_A(t) + j \cdot y_A(t) + k \cdot z_A(t), \]
\[ B_i(t) = i \cdot x_B(t) + j \cdot y_B(t) + k \cdot z_B(t), \] (2)

where \( x_A, y_A, z_A; x_B, y_B, z_B \) – dual scalar functions. It allows one to write an expression for dual vector \( G(t) \):

\[ G(t) = A_i(t) - B_i(t) = i \cdot x_G(t) + j \cdot y_G(t) + k \cdot z_G(t) = i(x_A(t) - x_B(t)) + j(y_A(t) - y_B(t)) + k(z_A(t) - z_B(t)). \] (3)

Dual vector \( G(t) \) characterizes the closeness of ruled surfaces \( A_i(t) \) and \( B_i(t) \) in infinitesimal neighborhood for their common generator at parameter \( t = t_0 \). It is defined by two generators \( a'_{0i}(t) \) and \( b'_{0i}(t) \), each of which is shifted in ruled surface along the same dual arc \( ds_A = ds_B \) relatively their common generator \( a_{0i}(t) = b_{0i}(t) \). Let us introduce the notion of the order contact of two ruled surfaces in their common generator. We will estimate an order of infinitesimality of module \( |G(t)| \) relatively dual infinitesimal \( \Delta s \). If \( n \) – positive integer then the following condition holds:

\[ \lim_{\Delta s \to 0} \left| \frac{G(t)}{\Delta s^n} \right| = 0, \] (4)

thus, we will assume that ruled surfaces in their common generator \( a_{0i}(t_0) = b_{0i}(t_0) \) are not lower than the \( n \)-th order contact. But if \( n \) - maximal then ruled surfaces are exactly in the \( n \)-th order contact.

Because of \( ds = ds_0 + ods_1 = h_0 dt + o_0 h_1 dt, o^2 = 0 \), where \( ds_0 = h_0 dt, ds_1 = h_1 dt \), so condition \( \Delta s \to 0 \) can be replaced by the following two conditions: \( \Delta s_0 \to 0 \) and \( \Delta s_1 \to 0 \), that in case of \( h_0 \neq 0 \) and \( h_1 \neq 0 \) leads to \( \Delta t \to 0 \). Indeed \( ds_0 \) and \( dt, ds_1 \) and \( dt \) – pairs of equivalent infinitesimals. As a result we obtain the following condition for estimating the order contact of ruled surfaces in their common generator:

\[ \lim_{\Delta t \to 0} \left| \frac{G(t)}{\Delta t^n} \right| = 0. \] (5)

**Theorem.** To fulfill the condition of (5) it is necessary and sufficiently that the following equalities are carried out

\[ x_G(t_0) = 0, x_G'(t_0) = 0, x_G''(t_0) = 0, \ldots x_G^n(t_0) = 0; \]
\[ y_G(t_0) = 0, y_G'(t_0) = 0, y_G''(t_0) = 0, \ldots y_G^n(t_0) = 0; \]
\[ z_G(t_0) = 0, z_G'(t_0) = 0, z_G''(t_0) = 0, \ldots z_G^n(t_0) = 0. \] (6)

**Sufficiency.** Assume that equalities (6) hold. Then from (1) it follows that:

\[ x_G(t) = x_A(t) - x_B(t) = x^{n+1}(\delta_1) \frac{A^{n+1}}{(n+1)!}; y_G(t) = y_A(t) - y_B(t) = y^{n+1}(\delta_2) \frac{A^{n+1}}{(n+1)!}; \]
\[ z_G(t) = z_A(t) - z_B(t) = z^{n+1}(\delta_3) \frac{A^{n+1}}{(n+1)!}. \]

where \( \delta_1, \delta_2, \delta_3 \) – values of parameter \( t \), which satisfy the following conditions: \( t_0 < \delta_1 < t, t_0 < \delta_2 < t, t_0 < \delta_3 < t \).

Thus, the condition of (5) holds.

**Necessity.** Let us assume that at least one of equalities (6) does not hold, for example \( x_G^k(t_0) = x_A^k(t_0) - x_B^k(t_0) \neq 0; k < n \).

Thus, \( x_G(t) = x_A(t) - x_B(t) = \left[ x_A^k(t_0) - x_B^k(t_0) \right] \frac{A^k}{k!} + \ldots, \)

hence \( \lim_{\Delta t \to 0} \left| \frac{G(t)}{\Delta t^n} \right| \neq 0, k < n. \)
So the condition of (5) does not hold.
Let us consider the primary orders of contact of ruled surfaces.

2. The orders of contact
2.1. Order \( n=0 \) of contact of ruled surfaces

\[ A_i(t_0) = B_i(t_0) \cdot A_i'(t_0) \neq B_i'(t_0). \] (7)

In this case ruled surfaces are intersecting along their common generator.

2.2. Order \( n=1 \) of contact of ruled surfaces

\[ A_i(t_0) = B_i(t_0) \cdot A_i'(t_0) = B_i'(t_0) \cdot A_i''(t_0) \neq B_i''(t_0). \] (8)

Ruled surfaces are exactly in the first order of contact in their common generator.

From equalities of dual arcs

\[ ds_A = ds_B \]

it follows that additionally to the result of contact of order \( n=0 \) we obtain that in the common generator the following equalities hold:

\[ A_1 = B_1; A_2 = B_2; A_3 = B_3; H_A = H_B. \] (9)

For order \( n = 1 \) of contact ruled surfaces have coincided with trihedrons \((a_01, a_02, a_03)\) and \((b_01, b_02, b_03)\) at the center point of their common generator (see Figure 1, a).

2.3. Order \( n=2 \) of contact of ruled surfaces

\[ A_i(t_0) = B_i(t_0) \cdot A_i'(t_0) = B_i'(t_0) \cdot A_i''(t_0) = B_i''(t_0). \] (10)

Ruled surfaces are exactly in the second order of contact in their common generator \( a_{01}(t_0) = b_{01}(t_0) \) (see Figure 1,b). From relations:

\[ A_i^* = (H_A \cdot A_2)' = -H_A^2 \cdot A_1 + H_A' \cdot A_2 + H_A \cdot Q_A \cdot A_3; \]

\[ B_i^* = (H_B \cdot B_2)' = -H_B^2 \cdot B_1 + H_B' \cdot B_2 + H_B \cdot Q_B \cdot B_3; \]

it follows that additionally to the result of contact of order \( n = 1 \) the following equalities are carried out:

\[ H_A' = H_B' \cdot Q_A = Q_B. \] (11)

The element of the dual arc of ruled surface \( A_3(t) \) on the basis of derivation equation \( A_3' = -Q_A \cdot A_2 \) may be expressed as \( ds_{A3} = [A_3'] \cdot dt = Q_A' \cdot dt \). On the basis of the second equation of (11) it follows that \( ds_{A3} = ds_{B3} \). Thus, we obtain the equality of elements of dual arcs of ruled surfaces \( A_3(t) \) and \( B_3(t) \).

And, similarly, on the basis of equations (11) and \( A_2 = -H_A \cdot A_1 + Q_A \cdot A_3 \) it follows that \( ds_{A2} = ds_{B2} \). We obtain the equality of elements of dual arcs of ruled surfaces \( A_2(t) \) and \( B_2(t) \).

If order \( n \) of contact of initial surfaces is equal 2 then the elements of dual arcs in pairs of surfaces \( A_i(t) \) and \( B_i(t) \), where \( i = 1, 2, 3 \), are equal and calculated from common generators in these pairs.

The equality of elements of dual arcs is equivalent to the equality of distribution parameters in the common generators of these pairs of surfaces. In theory of ruled surfaces [3] it is known that

\[ \frac{ds_{A2}}{ds_A} = \frac{1}{\rho_A}, \]

where \( \rho_A \) - a dual curvature radius of ruled surface \( A_1(t) \) at centre point M of its ruling.

Since order \( n = 2 \) of contact is characterized by equality \( ds_A = ds_B \) and \( ds_{A2} = ds_{B2} \) it follows that at this contact ruled surfaces in their common generator \( a_{01}(t_0) = b_{01}(t_0) \) have equal dual curvature radius \( \rho_A = \rho_B \). Furthermore, since \( \rho = \sin R \), where \( R = R_0 + \omega R_1 \) is a dual angle between the generator of ruled surface \( A_1(t) \) and the generator of its first evolute (see Figure 1, b), it follows from equation \( \rho_A = \rho_B \) that \( R_A = R_B \) and the trihedrons of the first order evolutes coincide.
2.4. Order \( n=3 \) of contact of ruled surfaces

\[
\begin{align*}
\mathbf{A}''(t_0) &= \mathbf{B}''(t_0), \\
\mathbf{A}'(t_0) &= \mathbf{B}'(t_0), \\
\mathbf{A}(t_0) &= \mathbf{B}(t_0) \quad \text{and} \quad \mathbf{A}(t_0) = \mathbf{B}(t_0).
\end{align*}
\]  

Ruled surfaces are exactly in the third order of contact in their common generator

\[
\mathbf{a}_0 = \mathbf{b}_0.
\]

From relations

\[
\frac{ds_{Al}}{dt} = H_A = H_B = \frac{ds_{Bl}}{dt}; \quad \frac{d^2 s_{Al}}{dt^2} = H_A' = H_B' = \frac{d^2 s_{Bl}}{dt^2}; \quad \frac{d^3 s_{Al}}{dt^3} = H_A'' = H_B'' = \frac{d^3 s_{Bl}}{dt^3};
\]

it follows that additionally to the results of order \( n = 2 \) of contact the following equalities are carried out:

\[
H_A'' = H_B''; \quad Q_A = Q_B.
\]  

Central tangent vector \( \mathbf{A}_3(t) \) has coordinates \( \mathbf{A}_3(t) = \{x_3(t), y_3(t), z_3(t)\} \).

We define the derivatives of this vector based on the existence of functional relationship \( s_{A1}(t) \):

\[
\mathbf{A}' = \frac{d\mathbf{A}_3}{ds_{A1}} \frac{ds_{A1}}{dt} = \left\{ \frac{dx_3}{ds_{A1}} \frac{ds_{A1}}{dt}, \frac{dy_3}{ds_{A1}} \frac{ds_{A1}}{dt}, \frac{dz_3}{ds_{A1}} \frac{ds_{A1}}{dt} \right\}
\]

Proceeding from equations (15) for vector \( \mathbf{A}_3(t) \) we can write:

\[
\begin{align*}
\frac{dx_3}{ds_{A1}} &= -x_2 \cdot \cot \mathbf{R}_1, \\
\frac{dy_3}{ds_{A1}} &= -y_2 \cdot \cot \mathbf{R}_1, \\
\frac{dz_3}{ds_{A1}} &= -z_2 \cdot \cot \mathbf{R}_1,
\end{align*}
\]
A'_2(t) = \left\{ \frac{-x_{1(2)}}{\rho_{A1}} \cot g R + \frac{x_{2}}{\sin R} \cdot \frac{1}{r_{A1}} \right\} ds_{A1} + \left\{ -x_{1(2)} \cot g R \right\} \frac{d^{2}s_{A1}}{dt^{2}}; \ldots; \ldots \quad (16)

where \( A'_{1(2)} = \{ x_{1(2)}, y_{1(2)}, z_{1(2)} \} \), \( \frac{dR}{ds_{A1}} \), \( \rho_{A1} \) and \( r_{A1} \) are dual radii of curvature and torsion of ruled surface \( A(t) \) in the generator of contacting. The analysis of the components of vector \( A'(t) \) in view of equations (14) and \( dR + ds_{A1} = 0 \), which is peculiar to the evolute of ruled surface \( A(t) \) [3], allows making certain conclusions. For order \( n = 3 \) of contact of the nondevelopable ruled surfaces the elements of dual arcs in pairs of surfaces \( A_{1(i)}(t) \) and \( B_{1(i)}(t) \), \( i = 1, 2, 3 \) are equal. Hence, the evolute of the first order has order \( n=2 \) of contact. In addition, the trihedrons of the evolute of the second order coincide and dual radii of curvature \( r_{A1} = r_{B1} \) of \( A_1(t) \) and \( B_1(t) \) with parameter \( t = t_0 \) are equal.

3. Theoretical examples

As an example we will consider contact of ruled surface \( A_1 \rightarrow R_A(t,u) \) with elliptical hyperboloid \( B_1 \rightarrow R_B(t,u) \) (Figure 2):

\[
A_1 \rightarrow R_A(t,u) = \left\{ A(\cos t \sin t) + \frac{uc\cos}{m}; A(\sin t - t \cos t) + \frac{us\sin t}{m}; 1 + Bt + \frac{u}{m} \right\},
\]

\[
B_1 \rightarrow R_B(t,u) = \left\{ c_1 \sin t + \frac{uc\cos}{\sqrt{1 + e_3^2}}; c_2 \cos t + \frac{us\sin t}{\sqrt{1 + e_3^2}}; \frac{uc_3}{\sqrt{1 + e_3^2}} \right\},
\]

where \( n = 1 \):

\[
c_1 = \frac{1 - 2t_0}{2c_3 (3\cos^2 t_0 - 4)}; c_2 = \frac{-4(1 - 2t_0)}{2c_3 (3\cos^2 t_0 - 4)}; c_3 = 2; t_0 = 2.46.
\]

where \( n = 2 \):

\[
c_1 = \frac{2(1 - t_0) - \cos^2 t_0 \cdot (3 - 4t_0)}{2(2\cos^2 t_0 - 1)}; c_2 = \frac{-1 - 2t_0 - \cos^2 t_0 \cdot (3 - 4t_0)}{2(2\cos^2 t_0 - 1)}; c_3 = 1; t_0 = 4.3.
\]

\[ a \quad \text{and} \quad b \]

**Figure 2.** Theoretical examples of orders \( n = 1 \) (a) and \( n = 2 \) (b) of contact.
4. Practical results
The results of these researches were applied to designing of the working surfaces of the progressive tools for cultivation of the soil. In accordance with the mathematical model of tillage process ‘soil-chisel plow’ a ruled surface of the chisel plow’s rack was designed, which increases the wear resistance and improves the quality of the soil loosening (Figure 3, chisel – 1, rack – 2, knife – 3, working surface – 4).

The developed mathematical theory allows designing the constituent ruled surfaces of technical products with pre-assigned geometric and technical conditions. The mathematical toolkit allows a flexible change of the shape of the surface depending on the type of soil to be treated.

Figure 3. The working surface of the progressive tillage tool with order $n = 2$ of contact.

5. Conclusion
Order $n$ of contact of nondevelopable ruled surfaces is considered. The notion of a dual vector of divergence on the common generator line of two contacting nondevelopable ruled surfaces is introduced via the dual vector calculus, which describes the manifolds of line geometry. To determine this vector we proved necessary and sufficient conditions for order $n$ of contact on the common generator line. Thus, the primary orders of contact have been researched. The results of these researches can be applied in the design of the constituent ruled shells and in CAD systems via approximating issues.

References
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