WEINER-HOPF OPERATORS AND SPECTRAL PROBLEMS ON $L^2_\omega(\mathbb{R}^+)$

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Abstract. We study bounded operators $T$ on the weighted space $L^2_\omega(\mathbb{R}^+)$ commuting either with the right translations $S_t$, $t \in \mathbb{R}^+$, or left translations $P^+S_{-t}$, $t \in \mathbb{R}^+$, and we establish the existence of a symbol $\mu$ of $T$. We characterize completely the spectrum $\sigma(S_t)$ of the operator $S_t$ proving that $\sigma(S_t) = \{ z \in \mathbb{C} : |z| \leq e^{t\alpha_0} \}$, where $\alpha_0$ is the growth bound of $(S_t)_{t \geq 0}$. We obtain a similar result for the spectrum of $(P^+S_{-t})$, $t \geq 0$. Moreover, for a bounded operator $T$ commuting with $S_t$, $t \geq 0$, we establish the inclusion $\mu(\mathcal{O}) \subset \sigma(T)$, where $\mathcal{O} = \{ z \in \mathbb{C} : \text{Im} z < \alpha_0 \}$.

Key Words: translations, spectrum of Wiener-Hopf operator, semigroup of translations, weighted spaces, symbol

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1. Introduction

Let $\omega$ be a weight on $\mathbb{R}^+$. It means that $\omega$ is a positive, continuous function such that

$$0 < \inf_{x \geq 0} \frac{\omega(x+t)}{\omega(x)} \leq \sup_{x \geq 0} \frac{\omega(x+t)}{\omega(x)} < +\infty, \forall t \in \mathbb{R}^+.$$ 

Let $L^2_\omega(\mathbb{R}^+)$ be the set of measurable functions on $\mathbb{R}^+$ such that

$$\int_0^\infty |f(x)|^2 \omega(x)^2 dx < +\infty.$$ 

The space $H = L^2_\omega(\mathbb{R}^+)$ equipped with the scalar product

$$< f, g > = \int_{\mathbb{R}^+} f(x)\overline{g(x)}\omega(x)^2 dx, \quad f, g \in L^2_\omega(\mathbb{R}^+)$$

and the related norm $\| \cdot \|$ is a Hilbert space. Let $C^\infty_c(\mathbb{R})$ (resp. $C^\infty_c(\mathbb{R}^+)$) be the space of $C^\infty$ functions on $\mathbb{R}$ (resp. $\mathbb{R}^+$) with compact support in $\mathbb{R}$ (resp. $\mathbb{R}^+$). Notice that
$C_\infty^c(\mathbb{R}^+)$ is dense in $L^2_\omega(\mathbb{R}^+)$. For $t \in \mathbb{R}^+$, define the (right) shift operator $S_t$ on $H$ by

$$(S_t f)(x) = \begin{cases} f(x - t), & \text{a.e. if } x - t \geq 0, \\ 0, & \text{if } x - t < 0. \end{cases}$$

For simplicity $S_1$ will be denoted by $S$. Let $P^+$ be the projection from $L^2(\mathbb{R}^-) \oplus L^2_\omega(\mathbb{R}^+)$ into $L^2_\omega(\mathbb{R}^+)$. For $t > 0$ define the (left) shift operator $(P^+ S_{-t})f(x) = P^+ f(x + t)$ a.e. $x \in \mathbb{R}^+$. Let $I$ be the identity operator on $L^2_\omega(\mathbb{R}^+)$.\n
**Definition 1.** An operator $T$ on $L^2_\omega(\mathbb{R}^+)$ is called a Wiener-Hopf operator if $T$ is bounded and

$$P^+ S_{-t} T S_t f = T f, \quad \forall t \in \mathbb{R}^+, \quad f \in L^2_\omega(\mathbb{R}^+).$$

Every Wiener-Hopf operator $T$ has a representation by a convolution (see [5]). More precisely, there exists a distribution $\mu$ such that

$$T f = P^+ (\mu * f), \quad \forall f \in C_\infty^c(\mathbb{R}^+).$$

If $\phi \in C_\infty^c(\mathbb{R})$ then the operator

$$L^2_\omega(\mathbb{R}^+) \ni f \longrightarrow P^+ (\phi * f)$$

is a Wiener-Hopf operator and we will denote it by $T_\phi$.

A bounded operator $T$ commuting either with $S_t$, $\forall t > 0$ or with $P^+ S_{-t}$, $\forall t > 0$ is a Wiener-Hopf operator. On the other hand, every operator $\alpha P^+ S_{-t} + \beta S_t$ with $t > 0$, $\alpha, \beta \in \mathbb{C}$ is a Wiener-Hopf operator. It is clear that the set of Wiener-Hopf operators is not a sub-algebra of the algebra of the bounded operators on $L^2_\omega(\mathbb{R}^+)$.\n
Notice also that

$$(P^+ S_{-t} S_t)f = f, \quad \forall f \in L^2_\omega(\mathbb{R}^+), \quad t > 0,$$

but it is obvious that $(S_t P^+ S_{-t})f \neq f$, for all $f \in L^2_\omega(\mathbb{R}^+)$ with a support not included in $]-t, +\infty[$. The fact that $S$ is not invertible leads to many difficulties in contrast to the case when we deal with the space $L^2_\omega(\mathbb{R})$. The later space has been considered in [7] and [8] and the author has studied the operators commuting with the translations on $L^2_\omega(\mathbb{R})$ characterizing their spectrum. The group of translations on $L^2_\omega(\mathbb{R})$ is commutative and the investigation of its spectrum is easier. In this work, first we apply some ideas used in [7] and [8] to study Wiener-Hopf operators on $L^2_\omega(\mathbb{R}^+)$. For this purpose it is necessary to treat two semigroups of not invertible operators instead of a group of invertible operators. More precisely, we must deal with the semigroups $(S_t)_{t \in \mathbb{R}^+}$ and $(P^+ S_{-t})_{t \in \mathbb{R}^+}$ on $L^2_\omega(\mathbb{R}^+)$.\n
Consider the semigroup $(S_t)_{t \geq 0}$ on $L^2_\omega(\mathbb{R}^+)$ and let $A$ be its generator. We have the estimate

$$\|S_t\| \leq C e^{\alpha t}, \quad \forall t \in \mathbb{R}^+$$

and a similar estimate holds for the semigroup $(P^+ S_{-t})_{t \geq 0}$. This follows from the fact that the weight $\omega$ is equivalent to the special weight $\tilde{\omega}_0$ constructed in [5] following [1].
Denote by $\rho(B)$ (resp. $\sigma(B)$) the spectral radius (resp. the spectrum) of an operator $B$. Introduce the ground orders of the semigroups $(S_t)_{t \geq 0}$ and $(P^+S_{-t})_{t \geq 0}$ by

$$\alpha_0 = \lim_{t \to \infty} \frac{1}{t} \ln \|S_t\|, \quad \alpha_1 = \lim_{t \to \infty} \frac{1}{t} \ln \|P^+S_{-t}\|.$$ 

Then it is well known (see for example [2]) that we have

$$\rho(S_t) = e^{\alpha_0 t}, \quad \rho(P^+S_{-t}) = e^{\alpha_1 t}.$$ 

Let $f$ which yields the result. For a function $f$ and $\alpha \in \mathbb{C}$ we denote by $(f)_{\alpha}$ the function

$$(f)_{\alpha} : x \mapsto f(x)e^{\alpha x}.$$ 

Denote by $\mathcal{F}f = \hat{f}$ the usual Fourier transformation on $L^2(\mathbb{R})$. If a function $f \in L^2(\mathbb{R}^+)$, then we define $\hat{f}$ extending $f$ as $0$ on $\mathbb{R}^-$. Our first result is the following

**Theorem 1.** Let $a \in \mathcal{I} = [-\alpha_1, \alpha_0]$ and let $T$ be a Wiener-Hopf operator. Then for every $f \in L^2_a(\mathbb{R}^+)$ such that $(f)_{\alpha} \in L^2(\mathbb{R}^+)$, we have

$$(Tf)_{\alpha} = P^+\mathcal{F}^{-1}(h_{\alpha}(\hat{f}))$$

with $h_{\alpha} \in L^\infty(\mathbb{R})$ and

$$\|h_{\alpha}\|_{\infty} \leq C\|T\|,$$

where $C$ is a constant independent of $a$. Moreover, if $\alpha_1 + \alpha_0 > 0$, the function $h$ defined on $U = \{z \in \mathbb{C} : \text{Im} z \in [-\alpha_1, \alpha_0]\}$ by $h(z) = h_{\text{lim}}(\text{Re} z)$ is holomorphic on $U$.

**Definition 2.** The function $h$ defined in Theorem 1 is called the symbol of $T$.

A weaker result that Theorem 1 has been proved in [3] where the representation (1.1) has been obtained only for functions $f \in C^\infty_c(\mathbb{R}^+)$ which is too restrictive for the applications to the spectral problems studied in Section 3 and Section 4. On the other hand, in the proof in [3] there is a gap in the approximation argument. Guided by the approach in [8], in this work we prove a stronger version of the result of [5] applying other techniques based essentially on the spectral theory of semigroups. On the other hand, in many interesting cases as $\omega(x) = e^x$, $\omega(x) = e^{-x}$, we have $\alpha_0 + \alpha_1 = 0$ and the result of Theorem 1 is not satisfying since the symbol of $T$ is defined only on the line $\text{Im} z = \alpha_0$. To obtain more complete results we introduce the following class of operators.

**Definition 3.** Denote by $\mathcal{M}$ the set of bounded operators on $L^2_\omega(\mathbb{R}^+)$ commuting either with $S_t$, $\forall t > 0$ or $P^+S_{-t}$, $\forall t > 0$. 
For operators in $\mathcal{M}$ we obtain a stronger version of Theorem 1.

**Theorem 2.** Let $T$ be a bounded operator commuting with $(S_t)_{t>0}$ (resp. $(P^+ S_{-t})_{t>0}$). Let $a \in J = ]0, \alpha_0]$ (resp. $K = ]0, \alpha_1]$). Then for every $f \in L^2_\omega(\mathbb{R}^+)$ such that $(f)_a \in L^2(\mathbb{R}^+)$, we have

$$(Tf)_a = P^+ F^{-1}(h_a(f)_a)$$

with $h_a \in L^\infty(\mathbb{R})$ and

$$\|h_a\|_{\infty} \leq C\|T\|,$$

where $C$ is a constant independent of $a$. Moreover, the function $h$ defined on $O = \{z \in \mathbb{C} : \text{Im } z < \alpha_0\}$ (resp. $V = \{z \in \mathbb{C} : \text{Im } z > -\alpha_1\}$) by $h(z) = h_{\text{Im } z}(\text{Re } z)$ is holomorphic on $O$ (resp. $V$).

Our main spectral result is the following

**Theorem 3.** We have

(i) $\sigma(S_t) = \{z \in \mathbb{C}, |z| \leq e^{\alpha_0 t}\}, \forall t > 0.$ (1.2)

(ii) $\sigma(P^+ S_{-t}) = \{z \in \mathbb{C}, |z| \leq e^{\alpha_1 t}\}, \forall t > 0.$ (1.3)

Let $T \in \mathcal{M}$ and let $\mu_T$ be the symbol of $T$.

(iii) If $T$ commutes with $S_t$, $\forall t \geq 0$, then we have

$$\overline{\mu_T(O)} \subset \sigma(T).$$ (1.4)

(iv) If $T$ commutes with $P^+ S_{-t}$, $\forall t \geq 0$, then we have

$$\overline{\mu_T(V)} \subset \sigma(T).$$ (1.5)

It is important to note that for $T \in \mathcal{M}$ and $\lambda \in \mathbb{C}$, if the resolvent $(T - \lambda I)^{-1}$ exists, then this operator is also in $\mathcal{M}$. In general, this property is not valid for all Wiener-Hopf operators. The above result cannot be obtained from a spectral calculus which is unknown and quite difficult to construct for the operators in $\mathcal{M}$. On the other hand, our analysis shows the importance of the existence of symbols and this was our main motivation to establish Theorem 1 and Theorem 2.

The spectrum of the weighted right and left shifts on $l^2(\mathbb{R}^+)$ denoted respectively by $R$ and $L$ has been studied in [9]. It particular, it was shown that

$$\sigma(R) = \sigma(L) = \{z \in \mathbb{C}, |z| \leq \rho(R)\}.$$ (1.6)

In this special case the operators $R$ and $L$ are adjoint, while this property in general is not true for $S$ and $P^+ S_{-1}$.

The equalities [1.2], [1.3] are the analogue in $L^2_\omega(\mathbb{R}^+)$ of [1.6] however our proof is quite different from that in [9] and we use essentially Theorem 2. Moreover, these results agree with the spectrum of composition operators studied in [10] and the circular
symmetry about 0. In the standard case $\omega = 1$ the spectral results (1.2), (1.3) are well known (see, for example Chapter V, [2]). Their proof in this special case is based on the fact that the spectrum of the generator $A$ of $(S_t)_{t \geq 0}$ is in $\{z \in \mathbb{C}, \text{Re} z \leq 0\}$ and the spectral mapping theorem for semigroups yields $\sigma(S_t) = \{z \in \mathbb{C}, |z| \leq 1\}$. Notice also that in this case we have

$$s(A) = \sup\{\text{Re} \lambda : \lambda \in \sigma(A)\} = \alpha_0 = 0,$$

so the spectral bound $s(A)$ of $A$ is equal to the ground order and there is no spectral gap. In the general setting we deal with it is quite difficult to describe the spectrum of $A$. Consequently, we cannot obtain (1.2) from the spectrum of $A$ and our techniques are not based on $\sigma(A)$. Moreover, if for the semigroup $S_t$ on $L^2_{\omega}(\mathbb{R}^+)$ we can apply the spectral mapping theorem, since $S_t$ preserves positive functions (see [11], [12]), in general this is not true for other Hilbert spaces of functions and we could have a spectral gap $s(A) < \alpha_0$. This shows the importance of our approach which works also for more general Hilbert spaces $H$ of functions (see the conditions on $H$ listed below). To our best knowledge it seems that Theorem 3 is the first result in the literature giving a complete characterization of $\sigma(S_t)$ and $\sigma(P^+S_{-t})$ on the spaces $L^2_{\omega}(\mathbb{R}^+)$. On the other hand, for the weighted two-sided shift $S$ in $L^2_{\omega}(\mathbb{R})$ a similar result has been established in [8] saying that

$$\sigma(S) = \{z \in \mathbb{C} : \frac{1}{\rho(S^{-1})} \leq |z| \leq \rho(S)\}.$$

Following the arguments in [7], the results of this paper may be extended to a larger setup. Indeed, instead of $L^2_{\omega}(\mathbb{R}^+)$ we may consider a Hilbert space of functions on $\mathbb{R}^+$ satisfying the following conditions:

(H1) $C_c(\mathbb{R}^+) \subset H \subset L^1_{loc}(\mathbb{R}^+)$, with continuous inclusions, and $C_c(\mathbb{R}^+)$ is dense in $H$.

(H2) For every $x \in \mathbb{R}$, $P^+S_x(H) \subset H$ and $\sup_{x \in K} \|P^+S_x\| < +\infty$, for every compact set $K \subset \mathbb{R}$.

(H3) For every $\alpha \in \mathbb{R}$, let $T_{\alpha}$ be the operator defined by

$$T_{\alpha} : H \ni f \mapsto \left( \mathbb{R} \ni x \mapsto f(x)e^{i\alpha x} \right).$$

We have $T_{\alpha}(H) \subset H$ and moreover, $\sup_{\alpha \in \mathbb{R}} \|T_{\alpha}\| < +\infty$.

(H4) There exists $C_1 > 0$ and $a_1 \geq 0$ such that $\|S_x\| \leq C_1 e^{a_1|x|}, \forall x \in \mathbb{R}^+$.

(H5) There exists $C_2 > 0$ and $a_2 \geq 0$ such that $\|P^+S_{-x}\| \leq C_2 e^{a_2|x|}, \forall x \in \mathbb{R}^+$.

Taking into account (H3), without lost of generalities we may consider that in $H$ we have $\|fe^{i\alpha}\| = \|f\|$. For the simplicity of the exposition we deal with the case $H = L^2_{\omega}(\mathbb{R}^+)$. 
and the reader may consult [7] for the changes necessary to cover the more general setup.

2. Proof of Theorem 1

By using the arguments based on the spectral results for semigroups (see [3], [4]) we will prove the following

**Lemma 1.** Let \( \lambda \) be such that \( e^\lambda \in \sigma(S) \) and \( \text{Re} \lambda = \alpha_0 \). Then there exists a sequence \((n_k)_{k \in \mathbb{N}}\) of integers and a sequence \((f_{mk})_{k \in \mathbb{N}}\) of functions of \( H \) such that

\[
\lim_{k \to \infty} \|e^{tA} - e^{(\lambda + 2\pi in_k)t}\|f_{mk}\| = 0, \forall t \in \mathbb{R}^+, \|f_{mk}\| = 1, \forall k \in \mathbb{N}. \tag{2.1}
\]

**Proof.** We have to deal with two cases: (i) \( \lambda \in \sigma(A) \), (ii) \( \lambda \notin \sigma(A) \). In the case (i) \( \lambda \) is in the approximative point spectrum of \( A \). This follows from the fact that for any \( \mu \in \mathbb{C} \) with \( \text{Re} \mu > \alpha_0 \) we have \( \mu \notin \sigma(A) \), since \( s(A) \leq \alpha_0 \).

Let \( \mu_m \) be a sequence such that \( \mu_m \to \lambda, \text{Re} \mu_m \geq \lambda \). Then \( \|(\mu_m I - A)^{-1}\| \geq (\text{dist} (\mu_m, \text{spec}(A)))^{-1} \), hence \( \|(\mu_m I - A)^{-1}\| \to \infty \). Applying the uniform boundedness principle and passing to a subsequence \( \mu_{mk} \), we may find \( f \in H \) such that

\[
\lim_{k \to \infty} \|(\mu_{mk} I - A)^{-1} f\| = \infty.
\]

Introduce \( f_{mk} \in D(A) \) defined by

\[
f_{mk} = \frac{(\mu_{mk} I - A)^{-1} f}{\|(\mu_{mk} I - A)^{-1} f\|}.
\]

The identity

\[
(\lambda - A)f_{mk} = (\lambda - \mu_{mk})f_{mk} + (\mu_{mk} - A)f_{mk}
\]

implies that \((\lambda - A)f_{mk} \to 0 \) as \( k \to \infty \). Then the equality

\[
(e^{tA} - e^{\lambda t})f_{mk} = \left( \int_0^t e^{\lambda(t-s)}e^{As} ds \right) (A - \lambda)f_{mk}
\]

yields (2.1), where we take \( n_k = 0 \).

To deal with the case (ii), we repeat the argument in [7] and for the sake of completeness we present the details. We have \( e^\lambda \in \sigma(e^A) \setminus \sigma(e^A) \). Applying the results for the spectrum of a semigroup in Hilbert space (see [3], [4]), we conclude that there exists a sequence of integers \((n_k)\) such that \( |n_k| \to \infty \) and

\[
\|(A - (\lambda + 2\pi in_k)I)^{-1}\| \geq k, \forall k \in \mathbb{N}.
\]

We choose a sequence \((g_{mk}) \in H, \|g_{mk}\| = 1 \) so that

\[
\|(A - (\lambda + 2\pi in_k)I)^{-1} g_{mk}\| \geq k/2, \forall k \in \mathbb{N}
\]

and define

\[
f_{mk} = \frac{(A - (\lambda + 2\pi in_k)I)^{-1} g_{mk}}{\|(A - (\lambda + 2\pi in_k)I)^{-1} g_{mk}\|}.
\]
Next we have
\[ (e^{tA} - e^{(\lambda + 2\pi i n_k)t})f_{y_k} = \left( \int_0^t e^{(\lambda + 2\pi i n_k)(t-s)} e^{sA} ds \right) (A - (2\pi i n_k + \lambda)I)f_{y_k} \]
and we deduce (2.1). □

**Lemma 2.** Let \( \lambda \) be such that \( e^\lambda \in \sigma(S) \) and \( \Re \lambda = \alpha_0 \). Then, there exists a sequence \((n_k)_{k \in \mathbb{N}}\) of integers and a sequence \((f_{m_k})_{k \in \mathbb{N}}\) of functions of \( H \) such that for all \( t \in \mathbb{R} \),

\[ \lim_{k \to \infty} \left\| \left( P^+ S_t - e^{(\lambda + 2\pi i n_k)t} \right) f_{m_k} \right\| = 0, \quad \|f_{m_k}\| = 1, \quad \forall k \in \mathbb{N}. \]  

**Proof.** Clearly, for \( t \geq 0 \) we get (2.2) by (2.1) Moreover, we have

\[ \| (P^+ S_{-t} - e^{-(\lambda + 2\pi i n_k)t}) f_{m_k} \| = \| (P^+ S_{-t} - e^{-(\lambda + 2\pi i n_k)t} P^+ S_{-t} S_t) f_{m_k} \| \]

\[ \leq \| P^+ S_{-t} \| \| e^{-(\lambda + 2\pi i n_k)t} \| \| \left( e^{(\lambda + 2\pi i n_k)t} - S_t \right) f_{m_k} \|, \forall t \in \mathbb{R}^+. \]

Thus

\[ \lim_{k \to \infty} \| (P^+ S_{-t} - e^{-(\lambda + 2\pi i n_k)t}) f_{m_k} \| = 0. \]

and this completes the proof of (2.2). □

**Lemma 3.** For all \( \phi \in C_c^\infty(\mathbb{R}) \) and \( \lambda \) such that \( e^\lambda \in \sigma(S) \) with \( \Re \lambda = \alpha_0 \) we have

\[ |\hat{\phi}(i\lambda + a)| \leq \| T_\phi \|, \forall a \in \mathbb{R}. \]

**Proof.** Let \( \lambda \in \mathbb{C} \) be such that \( e^\lambda \in \sigma(S) \) with \( \Re \lambda = \alpha_0 \) and let \((f_{m_k})_{k \in \mathbb{N}}\) be the sequence satisfying (2.2). Fix \( \phi \in C_c^\infty(\mathbb{R}) \) and consider

\[ |\hat{\phi}(i\lambda + a)| = \left| \int_{\mathbb{R}} \langle \phi(t) e^{(\lambda - ia)t} f_{m_k}, f_{m_k} \rangle dt \right| \]

\[ \leq \left| \int_{\mathbb{R}} \langle \phi(t) \left( e^{(\lambda + 2\pi i n_k)t} - P^+ S_t \right) e^{-i(a + 2\pi n_k)t} f_{m_k}, f_{m_k} \rangle dt \right| \]

\[ + \left| \int_{\mathbb{R}} \langle \phi(t) P^+ S_t e^{-i(a + 2\pi n_k)t} f_{m_k}, f_{m_k} \rangle dt \right|. \]

The first term on the right side of the last inequality goes to 0 as \( k \to \infty \) since by Lemma 1, for every fixed \( t \) we have

\[ \lim_{k \to \infty} \| e^{-i(a + 2\pi n_k)t} \left( e^{(\lambda + 2\pi i n_k)t} - P^+ S_t \right) f_{m_k} \| = 0. \]

On the other hand,

\[ I_k = \left| \int_{\mathbb{R}} \langle P^+ S_t e^{-i(a + 2\pi n_k)t} f_{m_k}, f_{m_k} \rangle dt \right| \]

\[ = \left| \left[ \int_{\mathbb{R}} \phi(t) e^{-i(a + 2\pi n_k)t} P^+ f_{m_k}(\cdot - t) dt, f_{m_k}(\cdot) \right] \right| \]

\[ = \left| \langle P^+ \int_{\mathbb{R}} \phi(\cdot - y) e^{i(a + 2\pi n_k)y} f_{m_k}(y) dy, e^{i(a + 2\pi n_k)} f_{m_k}(\cdot) \rangle \right|. \]
and \(|I_k| \leq \|T_\phi\|\). Consequently, we deduce that
\[
|\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|. \quad \Box
\]

Notice that the property (2.3) implies that
\[
|\hat{\phi}(\lambda)| \leq \|T_\phi\|, \forall \lambda \in \mathbb{C}, \text{ provided } \text{Im } \lambda = \alpha_0.
\]

**Lemma 4.** Let \(\phi \in C_c^\infty(\mathbb{R})\) and let \(\lambda\) be such that \(e^{-\lambda} \in \sigma((P^+S_{-1})^*)\) with \(\text{Re } \lambda = -\alpha_1\). Then we have
\[
|\hat{\phi}(i\lambda + a)| \leq \|(T_\phi)\|, \forall a \in \mathbb{R}. \tag{2.4}
\]

**Proof.** Consider the semigroup \((P^+S_{-t})^*_{t \geq 0}\) and let \(B\) be its generator. We identify \(H\) and its dual space \(H'\). So the semigroup \((P^+S_{-t})^*, t \geq 0\) is acting on \(H\). Let \(\lambda \in \mathbb{C}\) be such that \(e^{-\lambda} \in \sigma((P^+S_{-1})^*)\) and \(|e^{-\lambda}| = \rho(P^+S_{-1}) = \rho((P^+S_{-1})^*) = e^{\alpha_1}\). Then, by the same argument as in Lemma 1, we prove that there exists a sequence \((n_k)_{k \in \mathbb{N}}\) of integers and a sequence \((f_{mk})_{k \in \mathbb{N}}\) of functions of \(H\) such that for all \(t \in \mathbb{R}^+\),
\[
\lim_{k \to \infty} \|(e^{tB} - e^{(-\lambda+i2\pi n_k)t})f_{mk}\| = 0
\]
and \(\|f_{mk}\| = 1\). Thus we deduce
\[
\lim_{k \to +\infty} \|(P^+S_{-t})^*f_{mk} - e^{(-\lambda-i2\pi n_k)t}f_{mk}\| = 0, \quad t \geq 0.
\]
Since for \(t \geq 0\) we have \(P^+S_{-t}S_t = I\), we get \((S_t)^*(P^+S_{-t})^* = I\). Then, for \(t \geq 0\) we get
\[
\|(S_t)^*f_{mk} - e^{(-\lambda-i2\pi n_k)t}f_{mk}\| = \|(S_t)^*f_{mk} - e^{(\lambda-i2\pi n_k)t}(S_t)^*(P^+S_{-t})^*f_{mk}\|
\]
\[
\leq \|(S_t)^*\|\|e^{(\lambda-i2\pi n_k)t}\|\|(e^{(-\lambda-i2\pi n_k)t}f_{mk} - (P^+S_{-t})^*f_{mk})\|.
\]
This implies that
\[
\lim_{k \to +\infty} \|(P^+S_{t})^* - e^{(-\lambda-i2\pi n_k)t})f_{mk}\| = 0, \forall t \in \mathbb{R}. \tag{2.5}
\]

We write
\[
\hat{\phi}(i\lambda + a) = \int_{\mathbb{R}} <\phi(t)e^{-i(a+2\pi n_k)t}f_{mk}, e^{\lambda t - 2\pi in_k t}f_{mk}> dt
\]
\[
= \int_{\mathbb{R}} <\phi(t)e^{-i(a+2\pi n_k)t}f_{mk}, e^{(\lambda t - 2\pi in_k t)(P^+S_t)^*}f_{mk}> dt
\]
\[
+ \int_{\mathbb{R}} <\phi(t)e^{-i(a+2\pi n_k)t}(P^+S_t)f_{mk}, f_{mk}> dt = J'_k + I'_k.
\]
From (2.5) we deduce that \(J'_k \to 0\) as \(k \to \infty\). For \(I'_k\) we apply the same argument as in the proof of Lemma 3 and we get
\[
|\hat{\phi}(i\lambda)| \leq \|T_\phi\|. \quad \Box
Lemma 5. For every function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ and for $z \in U = \{ z \in \mathbb{C}, \text{Im} z \in [-\alpha_1, \alpha_0] \}$ we have

$$|\hat{\phi}(z)| \leq \|T_\phi\|.$$  

Proof. We will use the Phragmén-Lindelöf theorem and we start by proving the estimations on the bounding lines. There exists $\alpha = e^{-iz} \in \sigma(S)$ such that $|\alpha| = e^{\text{Im} z} = e^{\alpha_0}$. Following (2.3), we obtain

$$|\hat{\phi}(z)| \leq \|T_\phi\|,$$

for every $z$ such that $\text{Im} z = \alpha_0$. Next notice that $\rho(P^+S_{-1}) = \rho((P^+S_{-1})^*)$. So there exists $\beta = e^{-iz} = e^{-(-iz)} \in \sigma((P^+S_{-1})^*)$ such that $|\beta| = e^{\alpha_1}$ and

$$-\text{Im} z = \ln |\beta| = \alpha_1.$$ 

Then taking into account (2.4), we get

$$|\hat{\phi}(z)| \leq \|T_\phi\|,$$

for every $z$ such that $\text{Im} z = -\alpha_1$. In the case $\alpha_1 + \alpha_0 = 0$ the result is obvious. So assume that $\alpha_0 + \alpha_1 > 0$. Since $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ we have

$$|\hat{\phi}(z)| \leq C\|\phi\|_\infty e^{k|\text{Im} z|} \leq K\|\phi\|_\infty, \quad \forall z \in U,$$

where $C > 0$, $k > 0$ and $K > 0$ are constants. An application of the Phragmén-Lindelöf theorem for the holomorphic function $\hat{\phi}(z)$, yields

$$|\hat{\phi}(\alpha)| \leq \|T_\phi\|$$

for $\alpha \in \{ z \in \mathbb{C} : \text{Im} z \in [-\alpha_1, \alpha_0] \}$. □

Combining the results in Lemma 3-5, we get

Lemma 6. For every $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ and for every $a \in [-\alpha_1, \alpha_0]$ we have

$$|\langle \hat{\phi} \rangle_a(x)| \leq \|T_\phi\|, \quad \forall x \in \mathbb{R}.$$

Proof of Theorem 1. The proof follows the approach in [5]. Let $T$ be a Wiener-Hopf operator. Then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset \mathcal{C}_c^\infty(\mathbb{R})$ such that $T$ is the limit of $(T_{\phi_n})_{n \in \mathbb{N}}$ with respect to the strong operator topology and we have $\|T_{\phi_n}\| \leq C\|T\|$, where $C$ is a constant independent of $n$ (see [5]). Let $a \in [-\alpha_1, \alpha_0]$. According to Lemma 6, we have

$$|\langle \hat{\phi} \rangle_a(x)| \leq \|T_{\phi_a}\| \leq C\|T\|, \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}$$  

(2.6)

and we replace $(\langle \hat{\phi} \rangle_a)_{n \in \mathbb{N}}$ by a suitable subsequence also denoted by $(\langle \hat{\phi} \rangle_a)_{n \in \mathbb{N}}$ converging with respect to the weak topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$ to a function $h_a \in L^\infty(\mathbb{R})$ such that $\|h_a\|_\infty \leq C\|T\|$. We have

$$\lim_{n \to +\infty} \int_{\mathbb{R}} \left( \langle \hat{\phi} \rangle_a(x) - h_a(x) \right) g(x) \, dx = 0, \quad \forall g \in L^1(\mathbb{R}).$$
Fix \( f \in L^2_\omega(\mathbb{R}^+) \) so that \( (f)_a \in L^2(\mathbb{R}^+) \). Then we get
\[
\lim_{n \to +\infty} \int_{\mathbb{R}} \left( (\phi_n)_a(x)(f)_a(x) - h_a(x)(f)_a(x) \right) g(x) \, dx = 0,
\]
for all \( g \in L^2(\mathbb{R}) \). We conclude that \( (\phi_n)_a(f)_a \) converges weakly in \( L^2(\mathbb{R}) \) to \( h_a(f)_a \).
On the other hand, we have
\[
(T\phi_a)_a = P^+(\phi_n)_a * (f)_a = P^+\mathcal{F}^{-1}(\phi_n)_a(f)_a
\]
and thus \( (T\phi_a)_a \) converges weakly in \( L^2(\mathbb{R}^+) \) to \( P^+\mathcal{F}^{-1}(h_a(f)_a) \). For \( g \in C^\infty_c(\mathbb{R}) \), we obtain
\[
\left| \int_{\mathbb{R}^+} (T\phi_a)_a(x) - (Tf)_a(x) \left| g(x) \right| \, dx \right| \leq C_{a,g} \|T\phi_a - Tf\|, \forall n \in \mathbb{N},
\]
where \( C_{a,g} \) is a constant depending only of \( g \) and \( a \). Since \( (T\phi_a)_n \) converges to \( Tf \) in \( L^2_\omega(\mathbb{R}^+) \), we get
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^+} (T\phi_a)_a(x)g(x) \, dx = \int_{\mathbb{R}^+} (Tf)_a(x)g(x) \, dx, \forall g \in C^\infty_c(\mathbb{R}).
\]
Thus we deduce that \( (Tf)_a = P^+\mathcal{F}^{-1}(h_a(f)_a) \). The symbol \( h \) is holomorphic on \( U \) following the same arguments as in [5].

3. Preliminary spectral result

As a first step to our spectral analysis in this section we prove the following

**Proposition 1.** Let \( T \in \mathcal{M} \) and suppose that the symbol \( \mu \) of \( T \) is continuous on \( U \). Then \( \mu(U) \subset \sigma(T) \).

**Proof of Proposition 1.** Let \( T \) be a bounded operator on \( H \) commuting with \( S_t, t \geq 0 \) or \( P^+S_{-t}, t \geq 0 \). For \( a \in [-\alpha_1, \alpha_0] \), we have
\[
(Tf)_a = P^+\mathcal{F}^{-1}(\mu_a(f)_a), \forall f \in L^2_\omega(\mathbb{R}^+),
\]
where \( \mu_a \in L^\infty(\mathbb{R}) \), provided \( (f)_a \in L^2(\mathbb{R}^+) \). Suppose that \( \lambda \notin \sigma(T) \). Then, it follows easily that the resolvent \( (T - \lambda I)^{-1} \) also commutes with \( (S_t)_{t \in \mathbb{R}^+} \) or \( (P^+S_{-t})_{t \in \mathbb{R}^+} \). Consequently, \( (T - \lambda I)^{-1} \) is a Wiener-Hopf operator and for \( a \in [-\alpha_1, \alpha_0] \) there exists a function \( h_a \in L^\infty(\mathbb{R}) \) such that
\[
((T - \lambda I)^{-1}g)_a = P^+\mathcal{F}^{-1}(h_a(g)_a),
\]
for \( g \in L^2_\omega(\mathbb{R}^+) \) such that \( (g)_a \in L^2(\mathbb{R}^+) \). If \( f \) is such that \( (f)_a \in L^2(\mathbb{R}^+) \), set \( g = (T - \lambda I)f \). Then following Theorem 1, we deduce that \( (Tf)_a \in L^2(\mathbb{R}^+) \) and \( (g)_a = ((T - \lambda I)f)_a \in L^2(\mathbb{R}^+) \). Thus applying once more Theorem 1, we get
\[
((T - \lambda I)^{-1}(T - \lambda I)f)_a = P^+\mathcal{F}^{-1}(h_a\mathcal{F}((T - \lambda I)f)_a) = P^+\mathcal{F}^{-1}\left(h_a\mathcal{F}[\mathcal{F}^{-1}(\mu_a - \lambda)(f)_a]\right).
\]
We have
\[ \| (f)_a \|_{L^2} \leq \| h_α F^+ F^{-1}(μ_α - λ)(f)_a \|_{L^2} \leq \| h_α \|_\infty \| F^+ F^{-1}(μ_α - λ)(f)_a \|_{L^2} \]
and we deduce
\[ \| (f)_a \|_{L^2} \leq C \| (μ_α - λ)(f)_a \|_{L^2}, \quad (3.1) \]
for all \( f \in L^2_α(\mathbb{R}^+) \) such that \((f)_a \in L^2(\mathbb{R}^+)\). Let \( λ = μ_α(η_0) = μ(η_0 + ia) \in μ(U) \) for \( a \in [-\ln ρ(\mathbb{P}^+ S_{-1}), \ln ρ(S)] \) and some \( η_0 \in \mathbb{R} \). Since the symbol \( μ \) of \( T \) is continuous, the function \( μ_α(η) = μ(η + ia) \) is continuous on \( \mathbb{R} \). We will construct a function \( f(x) = F(x)e^{-ax} \) with \( \text{supp}(F) \subset \mathbb{R}^+ \) for which \((3.1)\) is not fulfilled. Consider
\[ g(t) = e^{-b^2(t-t_0)^2}e^{i(t-t_0)η_0}, \quad b > 0, \ t_0 > 1 \]
with Fourier transform
\[ \hat{g}(ξ) = \frac{1}{b}e^{-(ξ-η_0)^2/4b^2}e^{-it_0ξ}. \]

Fix a small \( 0 < ε < \frac{1}{2}C^{-2} \), where \( C \) is the constant in \((3.1)\) and let \( δ > 0 \) be fixed so that \( |μ_α(ξ) - λ| \leq \sqrt{ε} \) for \( ξ \in Ω = \{ ξ \in \mathbb{R} : |ξ - η_0| ≤ δ \} \). Moreover, assume that
\[ |μ_α(ξ) - λ|^2 ≤ C_1, \text{ a.e. } ξ \in \mathbb{R}. \]
We have for \( 0 < b \leq 1 \) small enough
\[ \int_{\mathbb{R}\setminus Ω} |\hat{g}(ξ)|^2 dξ \leq \frac{1}{b^2} \int_{|ξ-η_0| \geq δ} e^{-(η_0^2/4b^2)} dξ \]
\[ \quad ≤ e^{-\frac{δ^2}{4b^2}} \frac{1}{b^2} \int_{|ξ-η_0| \geq δ} e^{-(η_0^2/4b^2)} dξ \leq C_0 b^{-1} e^{-\frac{δ^2}{4b^2}} ≤ ε \]
with \( C_0 > 0 \) independent of \( b > 0 \). We fix \( b > 0 \) with the above property and we choose a function \( φ \in C_c^∞(\mathbb{R}^+) \) such that \( 0 ≤ φ ≤ 1, \ φ(t) = 1 \) for \( 1 ≤ t ≤ 2t_0 - 1, \ φ(t) = 0 \) for \( t ≤ 1/2 \) and for \( t ≥ 2t_0 - 1/2 \). We suppose that \( |φ^{(k)}(t)| ≤ c_1, \ k = 1, 2, \forall t \in \mathbb{R} \). Set \( G(t) = φ(t) - 1)g(t) \). We will show that
\[ |(1 + ξ^2) \hat{G}(ξ)| ≤ \sqrt{\frac{C_2}{4πε}} \]
for \( t_0 \) large enough with \( C_2 > 0 \) independent of \( t_0 \). On the support of \((φ - 1)\) we have \( |t - t_0| > t_0 - 1 \) and integrating by parts in \( \int_\mathbb{R} (1 + ξ^2)G(t)e^{-itξ} dt \) we must estimate the integral
\[ \int_{|t-t_0| ≥ t_0-1} e^{-\frac{b^2(t-t_0)^2}{2}}(1 + |t - t_0| + (t - t_0)^2)dt \]
\[ \leq \left( \int_{-∞}^{1-t_0} (1 + |y| + y^2)e^{-by^2/2}dy + \int_{t_0-1}^{∞} (1 + y + y^2)e^{-by^2/2}dy \right). \]
Choosing \( t_0 \) large enough we arrange \((3.2)\).
We set $F = \varphi g \in C_c^\infty(\mathbb{R}^+)$ and we obtain

$$
\int_{\mathbb{R}\setminus V} |\hat{F}(\xi)|^2 d\xi \leq 2 \int_{\mathbb{R}\setminus V} |\hat{\varphi}(\xi)|^2 d\xi + 2 \int_{\mathbb{R}\setminus V} |\hat{G}(\xi)|^2 d\xi
\leq 2\varepsilon + \frac{C_2\varepsilon}{2\pi} \int_{\mathbb{R}} (1 + \xi^2)^{-2} d\xi \leq (2 + C_2)\varepsilon.
$$

Then

$$
\int_{\mathbb{R}} |(\mu_a(\xi) - \lambda)\hat{F}(\xi)|^2 d\xi \leq \int_{\mathbb{R}\setminus V} |(\mu_a(\xi) - \lambda)\hat{F}(\xi)|^2 d\xi + \int_{V} |(\mu_a(\xi) - \lambda)\hat{F}(\xi)|^2 d\xi
\leq C_1(2 + C_2)\varepsilon + (2\pi)^2 \|F\|_{L^2}^2\varepsilon.
$$

Now assume (3.1) fulfilled. Therefore

$$(2\pi)^2 \|F\|_{L^2}^2 \leq C^2 \|(\mu_a(\xi) - \lambda)\hat{F}(\xi)\|_{L^2}^2 \leq C^2 C_1(2 + C_2)\varepsilon + (2\pi)^2 \|F\|_{L^2}^2\varepsilon,$$

and since $C^2\varepsilon < \frac{1}{2}$, we conclude that

$$
\|F\|_{L^2}^2 \leq \frac{C^2 C_1}{4\pi^2}(2 + C_2)\varepsilon.
$$

On the other hand,

$$
\|F\|_{L^2}^2 \geq \frac{1}{2}\|g\|_{L^2}^2 - \|(\varphi - 1)g\|_{L^2}^2 \geq \frac{1}{2}\|g\|_{L^2}^2 - (2\pi)^{-2} \frac{C_2}{2}\varepsilon
$$

and

$$
\int_{\mathbb{R}} |g(t)|^2 dt \geq \int_{|t-t_0| \leq \frac{1}{b}} e^{-b^2(t-t_0)^2} dt \geq \frac{2e^{-1}}{b} \geq 2e^{-1}.
$$

For small $\varepsilon$ we obtain a contradiction, since $C_2$ is independent of $\varepsilon$. This completes the proof. $\square$

4. Spectra of $(S_t)_{t \in \mathbb{R}^+}$, $(P^+(S_{-t}))_{t \in \mathbb{R}^+}$ and bounded operators commuting with at least one of these semigroups

Observing that the symbol of $S_t$ is $z \rightarrow e^{-itz}$, an application of Proposition 1 to the operator $S_t$ yields

$$
\{z \in \mathbb{C}, e^{-\alpha t} \leq |z| \leq e^{\alpha t}\} \subset \sigma(S_t).
$$

This inclusion describes only a part of the spectrum of $S_t$. We will show that in our general setting we have (1.2). To prove this, for $t > 0$ assume that $z \in \mathbb{C}$ is such that $0 < |z| < e^{-\alpha t}$. Let $g \in H$ be a function such that $g(x) = 0$ for $x \geq t$ and $g \neq 0$. If the operator $(zI - S_t)$ is surjective on $H$, then there exists $f \neq 0$ such that $(z - S_t)f = g$. This implies $P^+S_{-t}g = 0$ and hence

$$(P^+S_{-t} - \frac{1}{z} I)f = 0$$

which is a contradiction. So every such $z$ is in the spectrum of $S_t$ and we obtain (1.2).
Next, it is easy to see that in our setup for the approximative point spectrum \( \Pi(S_t) \) of \( S \) we have the inclusion
\[
\Pi(S_t) \subset \{ z \in \mathbb{C} : e^{-\alpha t} \leq |z| \leq e^{\alpha t} \}. \tag{4.2}
\]
Indeed, for \( z \neq 0 \), we have the equality
\[
P^+S_{-t} - \frac{1}{z}I = \frac{1}{z}P^+S_{-t}(zI - S_t).
\]
If for \( z \in \mathbb{C} \) with \( 0 < |z| < e^{-\alpha t} \), there exists a sequence \((f_n)\) such that \( \|f_n\| = 1 \) and \( \|(zI - S_t)f_n\| \to 0 \) as \( n \to \infty \), then
\[
\left( P^+S_{-t} - \frac{1}{z}I \right)f_n \to 0, \quad n \to \infty
\]
and this leads to \( \frac{1}{z} \in \sigma(P^+S_{-t}) \) which is a contradiction. Next, if \( 0 \in \Pi(S_t) \), there exists a sequence \( g_n \in \hat{H} \) such that \( S_tg_n \to 0, \|g_n\| = 1 \). Then \( g_n = P^+S_{-t}S_tg_n \) and we obtain a contradiction.

Since the symbol of \( P^+S_{-t} \) is \( z \to e^{it\alpha} \), applying Proposition 1, we obtain
\[
\{ z \in \mathbb{C} : e^{-\alpha t} \leq |z| \leq e^{\alpha t} \} \subset \sigma(P^+S_{-t}).
\]
Passing to the proof of (1.3), notice that \( S_t^*(P^+S_{-t})^* = I \). Then for \( 0 < |z| < e^{-\alpha t} \) we have
\[
z\left( \frac{1}{z}I - S_t^* \right) = S_t^* \left( (P^+S_{-t})^* - z \right). \tag{4.3}
\]
It is clear that \( 0 \in \sigma_r(S_t) \), where \( \sigma_r(S_t) \) denotes the residual spectrum of \( S_t \). In fact, if \( 0 \notin \sigma_r(S_t) \), then \( 0 \) is in the approximative point spectrum of \( S_t \) and this contradicts (4.2).

Since \( 0 \in \sigma_r(S_t) \), we deduce that \( 0 \) is an eigenvalue of \( S_t^* \). Let \( S_t^*g = 0, \ g \neq 0 \). Assume that \( (P^+S_{-t})^* - zI \) is surjective. Therefore, there exists \( f \neq 0 \) so that \( ((P^+S_{-t})^* - z)f = g \) and (1.3) yields \( (\frac{1}{z} - S_t^*)f = 0 \). Consequently, \( \frac{1}{z} \leq \rho(S_t^*) = \rho(S_t) = e^{\alpha t} \) and we obtain a contradiction. Thus we conclude that \( z \in \sigma((P^+S_{-t})^*) \), hence \( \bar{z} \in \sigma(P^+S_{-t}) \) and the proof of (1.3) is complete.

To study the operators commuting with \( (S_t)_{t \in \mathbb{R}^+} \), we need the following

**Lemma 7.** Let \( \phi \in C_c^\infty(\mathbb{R}) \). The operator \( T_\phi \) commutes with \( S_t, \forall t > 0 \), if and only if the support of \( \phi \) is in \( \mathbb{R}^+ \).

**Proof.** First if \( \psi \in L_2^2(\mathbb{R}^+) \) has compact support in \( \mathbb{R}^+ \), it is easy to see that \( T_\psi \) commutes with \( S_t, \ t \geq 0 \). Now consider \( \phi \in C_c^\infty(\mathbb{R}) \) and suppose that \( T_\phi \) commutes with \( S_t, \ t \geq 0 \). We write \( \phi = \phi_{\mathbb{R}^+} + \phi_{\mathbb{R}^-} \). If \( T_\phi \) commutes with \( S_t, \ t \geq 0 \), then the operator \( T_{\phi_{\mathbb{R}^-}} \) commutes too. Let the function \( \psi = \phi_{\mathbb{R}^-} \) have support in \( [-a, 0] \) with \( a > 0 \). Setting \( \psi = \chi_{[0,a]} \), we get \( S_a\psi = \chi_{[a,2a]} \). For \( x \geq 0 \) we have
\[
P^+((\psi * S_a)f)(x) = \int_{-a}^0 \psi(t)x_{\{a \leq x-t \leq 2a\}}dt = \int_{\min(x-a,0)}^{\min(x-a,0)} \psi(t)dt.
\]
Since $P^{+}(\psi * S_{a}f) = S_{a}P^{+}(\psi * f)$, for $x \in [0,a]$, we deduce $P^{+}(\psi * S_{a}f)(x) = 0$ and
\[
\int_{-a}^{x-a} \psi(t) dt = 0, \quad \forall x \in [0,a].
\]
This implies that $\psi(t) = 0$, for $t \in [-a,0]$ and $\text{supp}(\phi) \subset \mathbb{R}^{+}$. □

**Lemma 8.** Let $\lambda$ be such that $e^{\lambda} \in \sigma(S)$. Then there exists a sequence $(n_{k})_{k \in \mathbb{N}}$ of integers and a sequence $(f_{m_{k}})_{k \in \mathbb{N}}$ of functions of $H$ such that
\[
\lim_{k \to \infty} <S_{t} - e^{(\lambda+2\pi i n_{k})t}\rangle f_{m_{k}}, f_{m_{k}} >= 0, \quad \forall t \in \mathbb{R}^{+}, \quad \|f_{m_{k}}\| = 1, \quad \forall k \in \mathbb{N}.
\]

**Proof.** Denote by $\sigma_{r}(A)$ the residual spectrum of $A$. If $\lambda \notin \sigma_{r}(A)$, or if $\lambda \notin \sigma(A)$, we obtain the sequences $(n_{k})_{k \in \mathbb{N}}$ and $(f_{m_{k}})_{k \in \mathbb{N}}$ as in the proof of Lemma 1. If $\lambda \in \sigma_{r}(A)$ then there exists $f \in H$ such that $A^{*}f = \overline{\lambda}f$ and $\|f\| = 1$. We set $f_{m_{k}} = f$ and $n_{k} = 0$, for $k \in \mathbb{N}$. □

**Lemma 9.** For all $\phi \in C_{c}^{\infty}(\mathbb{R}^{+})$ and $\lambda$ such that $e^{\lambda} \in \sigma(S)$ we have
\[
|\hat{\phi}(i\lambda)| \leq \|T_{\phi}\|.
\]

The proof is based on the equality
\[
\hat{\phi}(i\lambda) = \int_{\mathbb{R}^{+}} \phi(t)e^{\lambda t} dt = \int_{\mathbb{R}^{+}} \langle \phi(t)e^{(\lambda+2\pi i n_{k})t}f_{m_{k}}, e^{2\pi i n_{k}t}f_{m_{k}} \rangle dt
\]
\[
= \int_{\mathbb{R}^{+}} \langle \phi(t)\left(e^{(\lambda+2\pi i n_{k})t}I - S_{t}\right)f_{m_{k}}, e^{2\pi i n_{k}t}f_{m_{k}} \rangle dt + \int_{\mathbb{R}^{+}} \langle \phi(t)S_{t}f_{m_{k}}, e^{2\pi i n_{k}t}f_{m_{k}} \rangle dt.
\]

We apply Lemma 8 and we repeat the argument of the proof of Lemma 3. Notice that here the integration is over $\mathbb{R}^{+}$ and we do not need to examine the integral for $t < 0$.

Following [3], the operator $T$ is a limit of a sequences of operators $T_{\phi_{n}}$, where $\phi_{n} \in C_{c}^{\infty}(\mathbb{R})$ and $\|T_{\phi_{n}}\| \leq C\|T\|$. The sequence $(T_{\phi_{n}})$ has been constructed in [5] and it follows from its construction that if $T$ commutes with $S_{t}$, $t > 0$, then $T_{\phi_{n}}$ has the same property. Therefore, Lemma 7 implies that $\hat{\phi}_{n} \in C_{c}^{\infty}(\mathbb{R}^{+})$ and to obtain Theorem 2 for bounded operators commuting with $(S_{t})_{t>0}$, we apply Lemma 9 and the same arguments as in the proof of Theorem 1. Finally, applying Theorem 2 and the arguments of the proof of Proposition 1, we establish (1.3) and this completes the proof of iii) in Theorem 3.

Next we prove the following

**Lemma 10.** Let $\phi \in C_{c}^{\infty}(\mathbb{R})$. Then $T_{\phi}$ commutes with $P^{+}(S_{-t})$, $\forall t > 0$ if and only if $\text{supp}(\phi) \subset \mathbb{R}^{-}$.

The proof of Lemma 10 is essentially the same as that of Lemma 7. By using Lemma 10, we obtain an analogue of Lemma 9 and Theorem 2 for bounded operators commuting with $(P^{+}S_{-t})_{t>0}$ and applying these results we establish iv) in Theorem 3.
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