BPS solutions and new phases of finite-temperature strings

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Abstract

All high-temperature phases of the known $N = 4$ superstrings in five dimensions can be described by the universal thermal potential of an effective four-dimensional supergravity. This theory, in addition to three moduli $s, t, u$, contains non-trivial winding modes that become massless in certain regions of the thermal moduli space, triggering the instabilities at the Hagedorn temperature. In this context, we look for exact domain wall solutions of first order BPS equations. These solutions preserve half of the supersymmetries, in contrast to the usual finite-temperature weak-coupling approximation, and as such may constitute a new phase of finite-temperature superstrings. We present exact solutions for the type-IIA and type-IIB theories and for a self-dual hybrid type-II theory. While for the heterotic case the general solution cannot be given in closed form, we still present a complete picture and a detailed analysis of the behaviour around the weak and strong coupling limits and around certain critical points. In all cases these BPS solutions have no instabilities at any temperature. Finally, we address the physical meaning of the resulting geometries within the contexts of supergravity and string theory.
1 Introduction

In the conventional description of string theory, where one has only very little information about the dynamics of all possible string states, the main framework is provided by an effective supergravity for the massless fields which is obtained by integrating out the massive modes. As a result, many stringy effects that are attributed to the massive modes cannot be addressed in a systematic way; we can only appreciate their relevance in certain regions of the moduli space where massive states can become massless. In those cases, the conventional supergravity approach has to be enlarged to include the massless as well as all the relevant would-be massless states on an equal footing, for otherwise the effective theory will break down due to the appearance of singularities. Put differently, the possibility to have extra massless states in string theory signals the limitations of the perturbative field theory description of string dynamics in all corners of the moduli space. Conversely, including the would-be massless fields into the effective field theory can teach us important non-perturbative lessons about string theory.

There are specific interesting problems which can be addressed systematically, based on supersymmetry, by isolating the dynamics of a few relevant modes that can become massless. An important example of this kind is provided by the conifold singularity in field theory and its string-theoretical resolution [1, 2]. In particular, in Calabi–Yau compactifications which admit a non-trivial 3-cycle, the moduli space develops a singularity that invalidates the applicability of the conventional low-energy effective theory when the cycle shrinks to zero size. Incorporating a D-brane that wraps the 3-cycle, with mass proportional to its period, resolves the problem of conifold singularities as the enlarged theory is appropriate for describing cycles of all sizes.

Superstrings at finite temperature provide another interesting example that has been studied on and off for quite some time, but admittedly has not been fully explored yet. In particular, it is known that string theory exhibits an exponential growth of the number of states at high energy, which gives rise to a limiting temperature, known as the Hagedorn temperature $T_H$ [3–11]. From the world-sheet point of view, where one uses a periodic Gaussian model to describe the propagation of strings on tori, there is a Kosterlitz–Thouless phase transition at a critical radius where vortices can condense [12, 13], thus leading to a limiting temperature for string thermodynamics (see also [8–11]) associated with a phase transition. It has been further realized that there are string states with non-trivial winding number that can become light close to this temperature, and then turn tachyonic beyond it, thus signaling thermal instabilities of string theory at very high temperatures [14]. Some implications of this effect have been examined in the context of strings in the very early universe, where one tries to find ways that avoid the initial singularity, and at the same time explain why the dimensionality of the physical space-time is four [15] (see also [10], and [17] for a more recent exposition that takes into account D-branes). In an recent development, the contribution of the relevant winding modes was explicitly described in terms of an effective four-dimensional supergravity using a universal thermal potential that incorporates all phases of $N = 4$
superstrings, taking into account the thermal deformation of the BPS mass formula in $N = 4$ supersymmetric theories [18, 19]. However, no solutions have been found so far with a concrete physical interpretation and the ability to resolve the problems of quantum cosmology or the subtle issues raised at the end point of the black-hole evaporation; these are situations where the temperature grows to infinity and one encounters singularities in the context of any perturbative field theory.

The present paper grew out of the attempt to construct explicit solutions of the effective theory of supergravity which was proposed to include the dynamics of the lightest relevant winding modes that could become massless at the Hagedorn temperature. We construct for the first time families of non-trivial solutions with varying winding fields in the type-II and the heterotic sectors of string theory. We focus attention primarily on domain wall solutions, one of the reasons being that they are technically the simplest ones to consider; all the fields in the bosonic sector of the theory, namely the metric and the collection of scalar fields, are taken to depend on a single spatial coordinate only. Ultimately, of course, it is necessary to search for other types of solutions, for instance solutions with spherical symmetry, in order to discuss models appropriate for quantum cosmology or black-hole physics. We hope to report on such generalized solutions of the effective supergravity elsewhere in the future.

There are also some other compelling reasons for being interested in the existence and the explicit construction of domain wall solutions of the effective supergravity that describes strings at finite temperature. The first is provided by the important rôle of domain walls, as space-time defects, in a variety of cosmological applications, though we do not focus here on these types of physical problems. The second is provided by the fact that the domain walls, by their nature, break the four-dimensional Poincaré invariance to an effective three-dimensional invariance. Then, one can have a working framework for studying the issue of supersymmetry breaking by non-perturbative effects, as it was done in [18, 19], taking into account the peculiarities of supersymmetry in three dimensions, namely that even if local supersymmetry is unbroken the massive multiplets will not necessarily exhibit mass degeneracy [20].

The domain wall ansatz allows us to consider solutions of certain first order differential equations derived from a prepotential. As we explicitly show, these solutions by construction preserve half of the supersymmetries, and in this sense they are BPS. The full set of equations allows consistent truncations to subsectors which correspond to the heterotic string, type-IIA or type-IIB, or the type-II string at the self-dual radius (which we call the hybrid type-II). For all type-II cases we can find the exact general solutions in closed form. For the heterotic sector, however, no solution in terms of known functions is available, and we instead give a detailed analysis of the solutions around the weak and the various strong coupling limits, as well as around certain critical points, which still allows us to obtain a reasonably complete overall picture.

It should be stressed that in the standard finite-temperature treatment of perturbative superstrings all supersymmetries are broken due to the different boundary conditions
on bosons and fermions one has to impose in the periodic imaginary time direction. For this reason, it is not surprising that some modes become tachyonic beyond certain temperatures. On the contrary, the solutions we find within the domain wall ansatz of the effective supergravity are non-perturbative, containing regions of strong coupling, and preserve half of the supersymmetries. As such they are expected to be stable solutions, and indeed we show that, although the temperature can be arbitrarily high, no tachyonic instabilities ever develop. Even though we are only considering solutions of an effective supergravity, they may well point to a new finite-temperature phase of superstrings which is BPS and has no thermal instabilities, i.e. no Hagedorn temperature.

To further probe the physical meaning of these domain wall solutions we first study the propagation of a quantum test particle in the corresponding supergravity backgrounds. This allows to discriminate between wave-regular and non-regular geometries, thus narrowing down the physically meaningful set of solutions. The analogous question could be addressed within string theory. The consistency of string thermodynamics seems to require the compactness of all spatial directions since otherwise one would encounter problems with negative specific heat (see, for instance, [15] for a delicate analysis, without making reference to the geometrical details of the space, and references therein). It is clear from our explicit solutions in the type-II cases, and also true for the heterotic case (with the possible exception of certain solutions that are never weakly coupled), that none of them is periodic in the spatial dimension that defines the domain walls. However, this does not mean that our solutions of the effective supergravity theory are insufficient to extrapolate to string theory. Indeed, the compactness criterion is established within the micro-canonical treatment of string thermodynamics, and this assumes that the theory remains weakly coupled throughout all space. Furthermore, it concerns the phase where supersymmetry is completely broken by the finite temperature. On the other hand, almost all our solutions always contain a region of strong coupling. Most important, our solutions are BPS, preserving half of the supersymmetries. These differences are enough to cast serious doubt on whether the arguments about compactness of all spatial dimensions should apply. Furthermore, as already noted, the fields corresponding to the winding modes of our solutions never become massless even if the temperature modulus approaches the would-be Hagedorn temperature, which is yet another way to see that these solutions belong to a different phase.

This paper is organized as follows: In section 2, we review the main aspects of the effective theory of supergravity that was proposed to describe the thermal phases of all $N = 4$ superstrings and present the form of the universal thermal potential for the six scalar fields that correspond to the $s, t, u$ moduli and the winding states $z_1, z_2, z_3$ that can become massless in the heterotic, type-IIA and type-IIB sectors. In section 3, we consider the domain wall ansatz for the metric and all the relevant scalar fields. We first discuss quite generally when a $D = 4, N = 1$ supergravity theory admits domain wall solutions that are derived from first order differential equations. Indeed, this follows from some simple reality assumptions about the scalar fields and the Kähler potential. We show that these solutions automatically preserve half (or all) of the supersymmetries and hence are
BPS. Then we specialise to the effective thermal supergravity and explicitly derive the coupled system of six first order non-linear differential equations for them. Consistent truncations of this system lead to various type-II or heterotic sectors. In section 4, this system is analysed explicitly in the type-IIA, type-IIB as well as in the hybrid type-II sectors where the general solutions can be derived in closed form. In section 5, we study the equations for the heterotic sector which cannot be solved analytically, apart from a very special solution. We are able to study the behaviour of the solutions in the vicinity of the weak and the strong coupling regions, as well as around certain critical points, and present a fairly complete general picture. The construction of domain wall solutions for the whole system of six equations is beyond the scope of the present paper and can probably only be done numerically. In section 6, we study the rôle of boundary conditions in selecting physical solutions in the effective theory of supergravity and comment on string thermodynamics. Finally, in section 7, we present our conclusions and outline some directions for future work. In an appendix, we address the issue of having periodic solutions using criteria from the general theory of dynamical systems. This may turn out to be useful when studying the complete system of six equations.

2 Thermal potential of effective supergravity

In this section we present the essential ingredients for constructing an effective $N = 1$ supergravity in four dimensions$^1$ that describes the thermal phases of all $N = 4$ superstrings in a universal way, following earlier work [18, 19, 21]. The construction takes into account the dynamics of the would-be tachyonic winding modes of the $N = 4$ superstrings which are responsible for inducing a phase transition at high temperatures in string thermodynamics. As such, the effective $N = 1$ supergravity theory provides a systematic framework for quantifying arguments about the phase transition occurring at the Hagedorn temperature due to the special form of the effective potential of the relevant winding modes $\omega$, which includes a trilinear coupling $\sigma \omega \bar{\omega}$ with the modulus $\sigma$ describing the temperature [14].

The starting point of the construction is provided by five-dimensional $N = 4$ theories that are effectively four-dimensional at finite temperature. The crucial observation is that $d$-dimensional superstrings at finite temperature look like $(d-1)$-dimensional strings with spontaneously broken supersymmetry. Concretely, putting strings at finite temperature $T$ amounts to compactifying the (Euclidean) time on a circle of inverse radius $R^{-1} = 2\pi T$. As the Euclidean time direction is compact, one imposes boundary conditions to take particle statistics into account; modular invariance of the thermal partition function dictates then specific phase factors in the related GSO projection [14]. Technically, this procedure is equivalent to a Scherk–Schwarz compactification from $d = 5$ to $d - 1 = 4$

$^1$We always count four-dimensional supersymmetries, i.e., $N = 1$ supersymmetry has four supercharges.
with a well-defined gauging associated to the temperature modification of the effective theory of gauged supergravity \[22, 18\]. This gauging of \(N = 4\) spontaneously breaks supersymmetry.

### 2.1 Thermal dyonic modes

We restrict ourselves to the study of \(N = 4\) strings, because in this case the supersymmetry algebra and its central extensions suffice to determine the masses of all BPS states. Recall that the states which become tachyonic above a certain Hagedorn temperature \(T_H\) are \(1/2\)-BPS states, at least in dimensions where there are no other BPS states with smaller fractions of supersymmetry. Then, in \(N = 4\) strings, one can identify completely all the perturbative and the non-perturbative BPS states that can induce thermal instabilities and construct from first principles an exact effective supergravity for these states, as it was done in refs. \[18, 19\]. It is useful to observe that the odd dyonic modes exhibit the same finite-temperature behaviour as the odd winding string states. This follows from the action of dualities on the \(N = 4\) strings, relating the dyonic modes to perturbative winding states of dual strings. These windings in turn induce thermal instabilities in the dual strings. Describing the complete set of thermal phases of \(N = 4\) theories requires then both perturbative and non-perturbative states from all string viewpoints.

We define the context of the present work, following \[18, 19\], by first considering string theories in six dimensions obtained by compactification on \(T^4\) (heterotic string) or \(K3\) (type-II strings). We further compactify one dimension \(S^1\), with radius \(R_6\), and so the resulting five-dimensional theory exhibits T-duality between type-IIA and type-IIB (under \(R_6 \to \alpha'_{\text{het}}/R_6\)) and S-dualities between heterotic and type-II strings. Moreover, since we are putting strings at finite temperature \(T\), the time dimension is taken Euclidean and compactified on \(S^1\) with radius \(R = 1/(2\pi T)\) with twisted fermionic boundary conditions that account for the thermal effects.

In order to obtain the resulting four-dimensional thermal mass formula and examine which states can become tachyonic above a certain temperature, thus inducing instabilities, we first consider the usual BPS mass formula in \(N = 4\) supersymmetric theories \[23, 24, 25, 26, 27\] written in heterotic variables:

\[
M^2 = \frac{1}{\alpha'_{\text{het}}tu} \left[ m + ntu + i(m'u + n't) + is[\tilde{m} + \tilde{n}tu - i(\tilde{m}'u + \tilde{n}'t)] \right]^2. \tag{2.1}
\]

Here \(s\), \(t\) and \(u\) are defined in terms of the compactification radii \(R_6\), \(R\) and the heterotic string coupling \(g_{\text{het}}\) as follows,

\[
s = \frac{1}{g_{\text{het}}^2}, \quad t = \frac{RR_6}{\alpha'_{\text{het}}}, \quad u = \frac{R}{R_6}, \tag{2.2}
\]

supplemented by the relation \(\alpha'_{\text{het}} = 4s\), in units where the four-dimensional gravitational coupling \(\kappa\) has been normalized to \(\sqrt{2}\). These expressions will help us later to understand the physical meaning of the various domain wall solutions in terms of the three
physical parameters in the problem $R_6, R$ and $g_{\text{het}}$. The integers $m, n, m', n'$ are the electric momentum and winding quantum numbers associated to the compactification on the 2-torus with radii $R_6$ and $R$. The tilded integers are the corresponding magnetic non-perturbative counterparts. Then, string dualities are simply described by the interchanges $s \leftrightarrow t$ (heterotic–IIA), $s \leftrightarrow u$ (heterotic–IIB) and $t \leftrightarrow u$ (IIA–IIB), which leave invariant the mass formula and the temperature radius $R^2 = 4stu$ which is common to all three string theories when measured in Planck units.

As discussed in refs. [18, 19], the thermal deformation of the mass formula (2.1) is very simple: the momentum quantum number $m$ is replaced by $m + Q' + n/2$, $Q'$ being the helicity charge. It also reverses the GSO projection, to account for the modified boundary conditions in the temperature deformation of the theory. For our present purposes, we may restrict our attention to only the states having $m' = n' = \tilde{m} = \tilde{n} = 0$, since the states that can become tachyonic first as the temperature increases are contained within this subset [19]. The thermal spectrum of these light states is then given by

$$M_2^2 = \left( \frac{1}{R} \left( m + Q' + \frac{1}{2} kp \right) + k T_{p,q,r} R \right)^2 - 2T_{p,q,r} \delta_{|k|,1} \delta_{Q',0}, \quad (2.3)$$

where we have defined the integers $k, p, q$ and $r$ by the condition $(n, \tilde{m}', \tilde{n}') = k(p, q, r)$, $p, q$ and $r$ being relatively prime numbers, and $T_{p,q,r}$ denotes the effective string tension

$$T_{p,q,r} = \frac{p}{\alpha'_{\text{het}}} + \frac{q}{\alpha'_{\text{IIA}}} + \frac{r}{\alpha'_{\text{IIB}}}, \quad (2.4)$$

which can be written in terms of $s, t$ and $u$ using the identifications

$$\alpha'_{\text{het}} = 4s, \quad \alpha'_{\text{IIA}} = 4t, \quad \alpha'_{\text{IIB}} = 4u, \quad (2.5)$$

again in the Planck units with normalization $\kappa = \sqrt{2}$. The mass formula (2.3) is an extension of the known perturbative finite-temperature mass formulas with the correct duality and zero temperature behaviour. In particular, the last term is generated in perturbative strings by the GSO projection related to the breaking of supersymmetry. The various constraints imposed among the quantum numbers require that $p, q, r$ are all positive and that $mk \geq -1$.

Analysing the four-dimensional mass formula for $M_2^2$, we note that tachyons can appear when $Q' = 0$ and $|k| = 1$ for certain values of the four quantum numbers $m, n, \tilde{m}', \tilde{n}'$. The critical temperature (and hence the critical radius $R$) can be found by locating the zeros of $M_2^2$. The first tachyons and the critical temperatures are:

$$(m, n, \tilde{m}', \tilde{n}') = \pm(-1, 1, 0, 0) \quad \text{with} \quad R = (\sqrt{2} + 1)\sqrt{\frac{\alpha'_{\text{het}}}{2}}, \quad (2.6)$$

$$(m, n, \tilde{m}', \tilde{n}') = \pm(0, 0, 1, 0) \quad \text{with} \quad R = \sqrt{2\alpha'_{\text{IIA}}}, \quad (2.7)$$

$$(m, n, \tilde{m}', \tilde{n}') = \pm(0, 0, 0, 1) \quad \text{with} \quad R = \sqrt{2\alpha'_{\text{IIB}}}. \quad (2.8)$$

Since the winding numbers in the heterotic, the type-IIA and the type-IIB strings are respectively $n, \tilde{m}'$ and $\tilde{n}'$, these three pairs of states have winding numbers $\pm 1$, as expected on general grounds, in their respective perturbative superstring theory. There
are also two other series of states occurring for $Q' = 0$ and $|k| = 1$, with quantum numbers $m = -1$ and $(p, q, r)$ arbitrary, which can become tachyonic when $p = 1$ or $p = 2$; however, in either series the critical temperatures that result from $\mathcal{M}_F^2 = 0$ are higher than at least one of the perturbative Hagedorn temperatures of the heterotic, type-IIA or type-IIB strings given above.

This analysis of the mass formula suggests the field content to be used in the effective field theory description. We certainly need the three moduli $s$, $t$ and $u$ and the six scalar fields (actually three conjugate pairs of fields) able to generate the thermal transitions. These states will be embedded in the generic scalar manifold of $N = 4$ supergravity,

$$\frac{Sl(2, R)}{U(1)} \times \frac{SO(6, m)}{SO(6) \times SO(m)},$$

where $m$ is the number of vector multiplets. To study the thermal phase structure however, it is consistent and sufficient to truncate this $N = 4$ theory to $N = 1$ and to retain only chiral $N = 1$ multiplets describing the relevant moduli and winding states. After truncation, the effective theory includes three complex moduli scalars $S$, $T$ and $U$, with scalar manifold

$$\left( \frac{Sl(2, R)}{U(1)} \right)_S \times \left( \frac{Sl(2, R)}{U(1)} \right)_T \times \left( \frac{Sl(2, R)}{U(1)} \right)_U,$$

(2.9)

and six complex windings $Z^\pm_A$, $A = 1, 2, 3$, living on

$$\left( \frac{SO(2, 3)}{SO(2) \times SO(3)} \right)_{Z^+_A} \times \left( \frac{SO(2, 3)}{SO(2) \times SO(3)} \right)_{Z^-_A}.$$  

(2.10)

The resulting scalar manifold is a Kähler manifold for chiral multiplets coupled to $N = 1$ supergravity that arises by two successive $Z_2$ projections applied on the $N = 4$ scalar manifold.

### 2.2 The effective supergravity

The construction of the effective $N = 1$ supergravity Lagrangian proceeds as follows [19]. A generic four-dimensional $N = 1$ supergravity theory is characterized by a Kähler potential $K$ and a holomorphic superpotential $W$. The bosonic sector of the theory is given by the following Lagrangian density:

$$e^{-1} \mathcal{L} = \frac{1}{4} R - \frac{1}{2} K_{IJ}(\partial^\mu \Phi^I)(\partial^\nu \bar{\Phi}^J) - V(\Phi, \bar{\Phi}),$$

(2.11)

where $K_{IJ} = \partial^2 K / \partial \Phi^I \partial \bar{\Phi}^J$ are the components of the Kähler metric for the collection of complex scalar fields $\{\Phi^I\}$ present in chiral multiplets. The scalar potential $V(\Phi, \bar{\Phi})$ assumes the special form

$$V = \frac{1}{4} e^K \left( K^{IJ} W_I W_J - 3W^2 \right),$$

(2.12)
using the notation of covariant derivatives
\[ W_{;I} = \frac{\partial W}{\partial \Phi^I} + \frac{\partial K}{\partial \Phi^I} W. \] (2.13)

The form of the Kähler potential \( K \) follows from the constraints defining the \( N = 4 \) scalar manifold, truncated to \( N = 1 \). One finds, in particular,
\[ K = -\ln(S + \bar{S}) - \ln(T + \bar{T}) - \ln(U + \bar{U}) - \ln Y_+ \ln Y_-, \] (2.14)

with
\[ Y_\pm = 1 - 2Z_\pm^A Z_{\bar{A}}^\pm + (Z_\pm^A Z_{\bar{A}}^\pm)(Z_\pm^B Z_{\bar{B}}^\pm), \] (2.15)

where summation over repeated indices \( A \) or \( B = 1, 2, 3 \) is implicitly assumed. The expression of the superpotential \( W \) follows from the gauging applied to the \( N = 4 \) theory. This gauging encodes the Scherk–Schwarz supersymmetry breaking mechanism generated by the introduction of a finite temperature [18, 19] and yields
\[ W = 2\sqrt{2} \left[ \frac{1}{2}(1 - Z_\pm^A Z_{\bar{A}}^\pm)(1 - Z_\pm^B Z_{\bar{B}}^\pm) + (TU - 1)Z_1^+ Z_1^- + SUZ_2^+ Z_2^- + STZ_3^+ Z_3^- \right]. \] (2.16)

The scalar potential of the theory defined by \( K \) and \( W \) is complicated, but focusing on the real directions defined by \( \text{Im } Z_\pm^A = \text{Im } S = \text{Im } T = \text{Im } U = \text{Re } (Z_\pm^A - Z_{\bar{A}}^-) = 0 \) leads to important simplifications. Furthermore, a study of the mass spectrum in the low-temperature limit shows that these directions correspond precisely to the possible phase transitions occurring in the theory. We are then led to consider only the relevant would-be tachyonic states using the field variables
\[ s = \text{Re } S, \quad t = \text{Re } T, \quad u = \text{Re } U, \quad z_A = \text{Re } Z_\pm^A = \text{Re } Z_{\bar{A}}^- . \] (2.17)

Let us introduce for later convenience the quantities
\[ x^2 = \sum_{A=1}^{3} z_A^2, \quad H_A = \frac{z_A}{1 - x^2}. \] (2.18)

Then, in this case, the components of the Kähler metric in the field space become
\[ K_{SS} = \frac{1}{4s^2}, \quad K_{TT} = \frac{1}{4t^2}, \quad K_{UU} = \frac{1}{4u^2}, \quad K_{A\bar{A} B\bar{B}} = \frac{2}{(1 - x^2)^2} \delta_A^B, \] (2.19)
i.e., the metric becomes diagonal in the directions \( Z_\pm^A \); in writing the kinetic terms for the fields \( z_A \) we have to take \( K_{AB} = 4\delta_A^B/(1 - x^2)^2 \), as there is a factor of 2 picked up by \( Z_\pm^A = Z_{\bar{A}}^- \) in this case.

In order to present the Lagrangian density for all remaining six scalar fields, we find it helpful to trade \( s, t \) and \( u \) for \( \phi_1, \phi_2 \) and \( \phi_3 \) as follows:
\[ s = e^{-2\phi_1}, \quad t = e^{-2\phi_2}, \quad u = e^{-2\phi_3}. \] (2.20)
Note that $\phi_1 \to +\infty$ corresponds to the strong coupling limit $s \to 0$, whereas the weak coupling limit is attained for $\phi_1 \to -\infty$. With these definitions, we obtain a simplified form of the effective supergravity with bosonic Lagrangian density

$$e^{-1} \mathcal{L} = \frac{1}{4} R - \frac{1}{2} \left( \sum_{i=1}^{3} (\partial_{\mu} \phi_i)^2 + \sum_{A=1}^{3} \frac{4}{(1 - x^2)^2} (\partial_{\mu} z_A)^2 \right) - V ,$$

(2.21)

where the scalar potential is now given by the sum

$$V = V_1 + V_2 + V_3 .$$

(2.22)

Each term $V_A$ is a function of the moduli $\phi_i$ and a polynomial in $H_A$. Explicitly, we have

$$V_1 = \frac{1}{2} e^{2\phi_1} H_1^2 \left( \cosh(2\phi_2 + 2\phi_3)(4H_1^2 + 1) - 3 \right) ,$$

(2.23)

$$V_2 = \frac{1}{4} e^{2\phi_2} H_2^2 \left( e^{-2\phi_1 - 2\phi_3}(4H_2^2 + 1) - 4 \right) ,$$

(2.24)

$$V_3 = \frac{1}{4} e^{2\phi_3} H_3^2 \left( e^{-2\phi_1 - 2\phi_2}(4H_3^2 + 1) - 4 \right) .$$

(2.25)

Our normalizations have been chosen so that the kinetic terms of the fields $\phi_i$ assume their canonical form, with an overall factor $1/2$ as coefficient, since the normalization $\kappa = \sqrt{2}$ has been used. Then, the scalar potential $V$ provides the universal thermal effective potential that describes all possible high-temperature instabilities of $N = 4$ superstrings.

At this point, it is very useful to observe that the scalar potential (2.22) for the six real fields $s, t, u$ and $z_A$ can also be cast into the special form

$$V = \frac{1}{4} \left( \sum_{i=1}^{3} \left( \frac{\partial \tilde{W}}{\partial \phi_i} \right)^2 + \frac{1}{4} \sum_{A=1}^{3} (1 - x^2)^2 \left( \frac{\partial \tilde{W}}{\partial z_A} \right)^2 - 3\tilde{W}^2 \right) .$$

(2.26)

This property only depends on the structure of the Kähler potential $K$. The ‘prepotential’ $\tilde{W}$ given by

$$\tilde{W} = \frac{1}{2} e^{\phi_1 + \phi_2 + \phi_3} - 2e^{\phi_1}\sinh(\phi_2 + \phi_3)H_1^2 + e^{-\phi_1} \left( e^{\phi_2 - \phi_3} H_2^2 + e^{\phi_1 - \phi_2} H_3^2 \right)$$

(2.27)

$$= \frac{1}{\sqrt{stu}} \left( \frac{1}{2} + (tu - 1)H_1^2 + suH_2^2 + stH_3^2 \right) .$$

The second expression indicates that the prepotential is actually given by

$$\tilde{W} = e^{K/2} W \big|_{\text{real directions}} ,$$

(2.28)

as can be inferred from the general form of the supergravity potential and from our specific Kähler potential$^2$ (2.14) and superpotential (2.16). As we will see in section 3.2, this is not an accident. The prepotential will be of particular importance when writing the system of first order differential equations for the domain wall configurations.

\footnote{In particular, $\exp(-K/2) = 2\sqrt{2stu} (1 - x^2)^2$ upon restriction to the real directions of the theory.}
of the theory. It should be emphasized that given the potential \( (2.22) \), we may view \( (2.26) \) as a differential equation for the prepotential \( \tilde{W} \). Hence, in general, there can be solutions other than \( (2.27) \). However, the latter choice is the one that leads to one-half supersymmetric solutions, as we will see in section 3.3. In general, another choice for \( \tilde{W} \) satisfying \( (2.26) \) would break supersymmetry completely.

2.3 Thermal phases of \( N = 4 \) strings

The thermal phase structure of \( N = 4 \) superstrings can be deduced from the form of the potential \( V \). First of all note that the low-temperature phase, i.e. large \( R \), or large \( stu \), or large and negative \( \phi_1 + \phi_2 + \phi_3 \), is common to all strings. It is characterized by the condition \( H_1 = H_2 = H_3 = 0 \), in which case \( V_1 = V_2 = V_3 = 0 \), and so the potential vanishes for all values of the moduli fields \( s, t \) and \( u \). Since the four-dimensional couplings of the three strings are \(^3\)

\[
\begin{align*}
    s &= \frac{\sqrt{2}}{g^2_{\text{het}}}, &
    t &= \frac{\sqrt{2}}{g^2_{\text{IIA}}} = \frac{\sqrt{2} R R_6}{\alpha'_{\text{het}}}, &
    u &= \frac{\sqrt{2}}{g^2_{\text{IIB}}} = \frac{\sqrt{2} R}{R_6},
\end{align*}
\]

we see that this phase exists in the perturbative regime of each string theory. Notice also that the configuration \( H_A = 0 \) and \( \phi_i \) constant (but arbitrary) is an exact solution of the theory, with all supersymmetries broken by the thermal deformation. The reason is that in this phase the supersymmetry transformation is proportional to

\[
e^{K/2} W_{iI} = \frac{1}{2\sqrt{stu}} \frac{\partial K}{\partial \Phi_I},
\]

which does not vanish in the moduli directions, except of course in the limit \( stu \to \infty \) (zero-temperature limit).

There are three high-temperature phases generated by each of the three contributions to the scalar potential \( (2.22) \), and which are characterized by a non-zero value of a winding state \( z_A \). Which phase is selected depends on the values of the moduli \( \phi_i \), corresponding to the coupling, the temperature radius \( R \) and the sixth-dimension radius \( R_6 \) in each string theory (with string units \( \alpha'_{\text{het}}, \alpha'_{\text{IIA}} \) or \( \alpha'_{\text{IIB}} \)). None of them is a solution to the theory with constant scalars. The detailed analysis of the thermal phase structure \(^4\) leads to the following classification.

\underline{Heterotic high-T phase:}

In heterotic-string units, the temperature modulus is \( tu = 1/(2\pi^2 \alpha'_{\text{het}} T^2) \), while it is \( stu = 1/(2\pi \kappa T)^2 \) in Planck units. A heterotic tachyon \( H_1 \) appears first if

\[
(\sqrt{2} - 1)^2 < tu < (\sqrt{2} + 1)^2, \quad su > 4, \quad st > 4,
\]

\(^3\)Notice that there is a factor of \( \sqrt{2} \) that rescales the right hand sides of \( (2.29) \) as compared with the corresponding equation for zero temperature \( (2.2) \). This is due to the new spin structure introduced by temperature. However, eq. \( (2.3) \) for the three string scales remains the same.
where the last two conditions guarantee the absence of type-II tachyons. Then, this implies for the four-dimensional heterotic coupling that

$$\sqrt{2} s^{-1} = g_{\text{het}}^2 < g_{\text{crit}}^2 = \frac{\sqrt{2} + 1}{2\sqrt{2}}. \quad (2.32)$$

The upper and lower critical temperatures in (2.31) are related by T-duality in the temperature radius. Within the interval (2.31), the theory has a (non-tachyonic) solution for $H_1 = 1/2$, $t_u = 1$. And, since the scalar potential reduces to

$$V = -\frac{1}{8} e^{2\phi_1}, \quad (2.33)$$

a linear dilaton background is a solution. This background breaks one half of the supersymmetries [19]. This particular solution will be rederived as a special case of a more general class of domain wall backgrounds that break half of the supersymmetry in the heterotic sector, in section 5.

In the strong coupling regime of the heterotic theory, where $s \to 0$, there are only type-II instabilities; the high-temperature heterotic phase cannot be reached for any value of the radius $R_6$ if $g_{\text{het}}^2 > g_{\text{crit}}^2$. Put differently, the high-temperature heterotic phase can only exist in the perturbative heterotic regime, which by the heterotic–type-II duality corresponds to the non-perturbative regime of the type-II string theory.

Type-IIA/B high-T phases:

The relevant temperature modulus is either $su = 1/(2\pi^2 \alpha'_{\text{IIA}} T^2)$ for type-IIA or $st = 1/(2\pi^2 \alpha'_{\text{IIB}} T^2)$ for type-IIB strings. The high-temperature phase for type-II strings is then defined by $su < 4$ (for IIA) or $st < 4$ (for IIB). Thus, type-II instabilities arise in the strong coupling region of the heterotic string. The analysis of the problem shows that the value $R_6/\sqrt{\alpha'_{\text{het}}}$ determines whether a type-IIA or a IIB tachyon comes first as the temperature increases. We have in particular the following two high-temperature type-II phases:

$$\text{IIA : } 2\pi T > \frac{1}{2g_{\text{het}}^2 R_6}, \quad g_{\text{het}}^2 > \frac{\sqrt{2} + 1}{2\sqrt{2}}, \quad R_6 < \sqrt{\alpha'_{\text{het}}}, \quad (2.34)$$

$$\text{IIB : } 2\pi T > \frac{1}{2g_{\text{het}}^2 \alpha'_{\text{het}}}, \quad g_{\text{het}}^2 > \frac{\sqrt{2} + 1}{2\sqrt{2}}, \quad R_6 > \sqrt{\alpha'_{\text{het}}}. \quad (2.35)$$

Note that the type-II high-temperature phases differ from the heterotic one by the absence of an exact solution with constant windings and/or moduli. In these phases, the theory depends on a non-zero, field-dependent winding and two non-trivial moduli, $t$ and $su$ in the IIA case, $u$ and $st$ for IIB strings. This concludes our general review of the thermal aspects of $N = 4$ strings at finite temperature.
3 BPS domain wall solutions

We are now in the position to look for solutions of the effective supergravity that describe
the thermal phases of $N = 4$ superstrings. We will use the domain wall ansatz for two
reasons. First, this probably is the simplest possible class of solutions one may construct
in supergravity theories whose bosonic sector is described by gravity coupled to a selection
of scalar fields with non-trivial potential terms, as the universal thermal effective potential
we have at our disposal. Second, with all fields depending only on a single variable one can
look for solutions that satisfy more stringent, but easier to solve, first order differential
equations rather than second order ones. We will show quite generally that all solutions
of these first order equations preserve (at least) half of the supersymmetries and hence
are BPS.

We will first discuss in a general setting when gravity coupled to scalars admits
solutions of first order equations. Then we will see under which circumstances this is
true for the bosonic sector of supergravity, and that in these cases the solutions are BPS.
Finally, we specialise to the case of present interest and derive the set of coupled first
order equations of the effective supergravity describing the thermal phases of $N = 4$
superstrings.

3.1 First order equations for gravity coupled to scalars

Consider the following $D$-dimensional Lagrangian density representing gravity coupled
to $N$ scalars

$$\frac{1}{\sqrt{g}} L = \frac{1}{4} R - \frac{1}{2} G_{IJ}(\varphi) g^{mn} \partial_m \varphi^I \partial_n \varphi^J - V(\varphi) ,$$

where summation over the repeated indices $I, J = 1, 2, \ldots, N$ and $m, n = 1, 2, \ldots, D$
is implied. The equations of motion that follow from varying the metric and the scalar
fields are:

$$\delta g : \quad \frac{1}{4} R_{mn} - \frac{1}{2} G_{IJ} \partial_m \varphi^I \partial_n \varphi^J = \frac{1}{D - 2} g_{mn} V ,$$

$$\delta \varphi : \quad D^2 \varphi^I + \Gamma^I_{JK} g^{mn} \partial_m \varphi^J \partial_n \varphi^K = G^{IJ} \partial_J V ,$$

where $\Gamma^I_{JK}$ is the Christoffel symbol formed using the scalar-field space metric $G_{IJ}$.
Let us assume that the potential $V(\varphi)$ is given in terms of a prepotential $\tilde{W}(\varphi)$ as:

$$V = \frac{\beta^2}{8} \left( G^{IJ} \partial_I \tilde{W} \partial_J \tilde{W} - 2 \frac{D - 1}{D - 2} \tilde{W}^2 \right) ,$$

where $\beta$ is a dimensionful constant.

Let us make the ansatz for a metric of the form

$$ds^2 = dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu , \quad \mu, \nu = 1, 2, \ldots, D - 1 ,$$

12
that preserves the flat space-time symmetries in $D - 1$ dimensions. We also assume that all scalars depend only on the variable $r$, i.e., $\varphi^I = \varphi^I(r)$. Then, it can be shown by a straightforward calculation that if the following first order equations hold
\[
\frac{d\varphi^I}{dr} = \pm \frac{\beta}{2} G^{Ij} \partial_j \tilde{W}, \quad \frac{dA}{dr} = \mp \frac{\beta}{D - 2} \tilde{W}, \tag{3.6}
\]
the second order equations are also satisfied. In proving this statement we use the fact that the non-vanishing components of the Riemann tensor for the metric (3.5) are
\[
R_{\mu\nu} = -\eta_{\mu\nu} e^{2A} \left( A'' + (D - 1)(A')^2 \right),
\]
\[
R_{rr} = -(D - 1) \left( A'' + (A')^2 \right), \tag{3.7}
\]
where the prime denotes the derivative with respect to $r$. Note that the freedom in the choice of sign in (3.6) is due to the fact that (3.4) does not determine the sign of $\tilde{W}$. Equivalently it expresses the freedom to change $r \to -r$.

Throughout the remainder of this paper we restrict to $D = 4$ and choose $\beta = \sqrt{2}$ (consistent with $\kappa = \sqrt{2}$).

### 3.2 First order equations for the bosonic sector of supergravity

Consider now the bosonic sector of a generic four-dimensional $N = 1$ supergravity theory as given by eqs. (2.11), (2.12) and (2.13). Assume that all $\text{Im}(\Phi^I)$ vanish and that the Kähler potential is such that
\[
2 \left. \frac{\partial K}{\partial \Phi^I} \right|_{\text{real directions}} = \frac{\partial}{\partial \text{Re} \, \Phi^I} \left( K \left|_{\text{real directions}} \right. \right). \tag{3.8}
\]
This is indeed the case for our thermal supergravity, but for the time being we do not want to consider any specific model. The definition of the covariant derivative (2.13) of the superpotential $W$ and the assumption (3.8) imply that if we define
\[
\tilde{W} = e^{K/2} W \left|_{\text{real directions}} \right. \tag{3.9}
\]
we will have
\[
\frac{\partial}{\partial \Phi^I} \tilde{W} = e^{K/2} W_{;I}. \tag{3.10}
\]
As a consequence, the scalar potential (2.12) of the supergravity can be rewritten using only $\tilde{W}$ as
\[
V = \frac{1}{4} \left( K' \sigma \partial^I \tilde{W} \partial^J \tilde{W} - 3 \tilde{W}^2 \right). \tag{3.11}
\]

\footnote{Recently, such first order equations with flat metric in the scalar-field space appeared in the context of supersymmetric solutions in five-dimensional gauged supergravity (for a first example, see [28]). For cases where the metric in the scalar-field space is non-trivial, five-dimensional examples have been given in [24].}
We see that this is precisely of the form \( (3.4) \) with \( D = 4 \) and \( \beta = \sqrt{2} \).

We conclude that if we find solutions to the domain wall ansatz \( (3.5) \) satisfying the first order equations

\[
\frac{d\Phi^I}{dr} = \pm \frac{1}{\sqrt{2}} K^{IJ} \frac{\partial \tilde{W}}{\partial \Phi^J}, \quad \frac{dA}{dr} = \mp \frac{1}{\sqrt{2}} \tilde{W},
\]

(3.12)

and take the fermions to vanish, then all equations of motion of the supergravity will be satisfied. We now proceed to show that these solutions of \( (3.12) \) precisely preserve half or all of the four-dimensional supersymmetries.

### 3.3 Supersymmetry of the solution

It is very simple to obtain the number of supersymmetries left unbroken by a solution of the first order equations. The only additional assumptions we need is that the Kähler potential satisfies \( (3.8) \), Im\( (\Phi^I) \) vanishes, and that the fields only depend on a single coordinate, say \( r \), with \( g^{rr} = 1 \).

Consider then the supersymmetry transformation of the (left-handed) fermionic component \( \chi^I_L \) of a chiral multiplet with scalar \( \Phi^I \):

\[
\delta \chi^I_L = \frac{1}{\sqrt{2}} \gamma^\mu (\partial_\mu \Phi^I) \epsilon_R - \frac{1}{2} \epsilon_L e^{K/2} K^{IJ} \tilde{W}_{;J} + \text{fermionic terms}.
\]

(3.13)

Since, by assumption, the solution depends only on a single coordinate \( r \) with metric \( g^{rr} = 1 \), we have \( \gamma^\mu \partial_\mu \Phi^I = \gamma^r \partial_r \Phi^I \) where \( \gamma^r \) satisfies \( (\gamma^r)^2 = 1 \). Then, using the first order equation \( (3.12) \) (and \( (3.10) \)) we have

\[
\gamma^\mu \partial_\mu \Phi^I = \pm \frac{1}{\sqrt{2}} \gamma^r K^{IJ} \frac{\partial \tilde{W}}{\partial \Phi^J} = \pm \frac{1}{\sqrt{2}} \gamma^r e^{K/2} K^{IJ} W_{;J}.
\]

(3.14)

It follows that for vanishing fermions

\[
\delta \chi^I_L = -\frac{1}{2} P_L (1 \mp \gamma^r) \epsilon e^{K/2} K^{IJ} W_{;J},
\]

(3.15)

where \( P_L \) is the left-handed chirality projector (of course there is a similar equation for \( \chi^I_R \)). Clearly, since \( (\gamma^r)^2 = 1 \), the solution will preserve one half of the supersymmetries if \( W_{;J} \neq 0 \), and all supersymmetries if \( W_{;J} = 0 \), as it should.

In order to compute the dependence of the Killing spinor \( \epsilon \) on the space-time coordinates, one in principle has to solve the gravitino equation. Equivalently, for static configurations (as in our domain wall ansatz \( (3.17) \)) one can use an argument based on the supersymmetry algebra and deduce that \( \epsilon = g^{1/4} \epsilon_0 \), where \( \epsilon_0 \) is a constant spinor subject to model-dependent projection conditions, which reduce the number of its independent components \( [30] \). Applying this to our case we find

\[
\epsilon(r) = e^{A(r)/2} \epsilon_0, \quad (1 \mp \gamma^r) \epsilon_0 = 0.
\]

(3.16)
This result should be interpreted with some care when considering strings at finite temperature. Formulating a theory at finite temperature requires a rotation to Euclidean compact time and, with a trivial background, supersymmetry is completely broken. As already discussed in section 2.3, this is the case of the low-temperature phase. Our domain wall solutions are, however, backgrounds depending on a spacelike variable $r$ with a manifest three-dimensional residual "space-time" symmetry, with signature $(3,0)$ at finite temperature. The supercharges preserved by these backgrounds are then supersymmetries inherited from the original five-dimensional supersymmetry algebra, with Minkowski signature $(4,1)$.

We conclude that the solutions of the first order equations (3.12) are $1/2$ BPS solutions, except of course if $\frac{\partial \tilde{W}}{\partial \Phi_i} = 0$, in which case the scalar fields are constant and supersymmetry is unbroken.

### 3.4 The first order equations for the thermal supergravity

Next, we specialise the discussion to the supergravity theory that describes the thermal phases of the $\mathcal{N} = 4$ superstrings, by applying the restrictions and identifications explained in section 2.2. The Kähler metric, the potential $V$ and prepotential $\tilde{W}$ are given in eqs. (2.21), (2.22) and (2.27). There are six real scalar fields and the Kähler metric is diagonal $K_{ij} = K_{(i) \delta_{ij}}$. For convenience we write the relevant equations again, for $D = 4$.

The metric is assumed to be conformally flat and written in the form (see, for instance, [31])

$$ds^2 = dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu , \quad \mu, \nu = 1, 2, 3$$

and the BPS domain wall configurations are solutions of the following system of first order differential equations:

$$\frac{d\phi_i}{dr} = -\frac{1}{\sqrt{2}} K_{(i)} \frac{\partial \tilde{W}}{\partial \phi_i} ; \quad \frac{dA}{dr} = \frac{1}{\sqrt{2}} \tilde{W} ,$$

taking into account the inverse of the diagonal metric in field space for each one of the components $\phi_i$. Note that we have used the freedom $r \rightarrow -r$ to make a definite choice of signs in these equations.

The six scalar fields are naturally separated into two groups: $\{ \phi_i ; i = 1, 2, 3 \}$ corresponding to $s$, $t$, $u$ and the three winding fields $\{ z_i ; i = 1, 2, 3 \}$. For the first set the metric in field space is flat, i.e., $K_{(i)}(\phi) = 1$, whereas for the second set all components are equal but non-trivial, namely $K_{(i)}(z) = 4/(1 - z_1^2 - z_2^2 - z_3^2)^2$, see eq. (2.27). The prepotential $\tilde{W}$ was given in (2.27). As a result, the domain walls of the theory correspond to solutions of the non-linear system

$$\sqrt{2} \frac{d\phi_1}{dr} = -\frac{1}{2} e^{\phi_1 + \phi_+} + 2e^{\phi_1} \sinh \phi_+ H_1^2 + e^{-\phi_1} \left( e^{\phi_+} H_2^2 + e^{-\phi_+} H_3^2 \right) ,$$

$$\sqrt{2} \frac{d\phi_2}{dr} = -\frac{1}{2} e^{\phi_1 + \phi_+} + 2e^{\phi_1} \cosh \phi_+ H_1^2 - e^{-\phi_1} \left( e^{\phi_+} H_2^2 - e^{-\phi_+} H_3^2 \right) ,$$

$$\sqrt{2} \frac{d\phi_3}{dr} = -\frac{1}{2} e^{\phi_1 + \phi_+} + 2e^{\phi_1} \cosh \phi_+ H_2^2 + e^{-\phi_1} \left( e^{\phi_+} H_1^2 - e^{-\phi_+} H_3^2 \right) ,$$

$$\sqrt{2} \frac{dA}{dr} = 1,$$

$$\sqrt{2} \frac{d\phi_4}{dr} = -\frac{1}{2} e^{\phi_1 + \phi_+} + 2e^{\phi_1} \sinh \phi_+ H_2^2 + e^{-\phi_1} \left( e^{\phi_+} H_1^2 + e^{-\phi_+} H_3^2 \right) ,$$

$$\sqrt{2} \frac{d\phi_5}{dr} = -\frac{1}{2} e^{\phi_1 + \phi_+} + 2e^{\phi_1} \cosh \phi_+ H_3^2 + e^{-\phi_1} \left( e^{\phi_+} H_1^2 + e^{-\phi_+} H_2^2 \right) ,$$

$$\sqrt{2} \frac{d\phi_6}{dr} = -\frac{1}{2} e^{\phi_1 + \phi_+} + 2e^{\phi_1} \sinh \phi_+ H_3^2 + e^{-\phi_1} \left( e^{\phi_+} H_1^2 - e^{-\phi_+} H_2^2 \right) .$$
\[ \sqrt{2} \frac{d\phi_3}{dr} = -\frac{1}{2} e^{\phi_1+\phi_+} + 2e^{\phi_1} \cosh \phi_+ H_1^2 + e^{-\phi_1} \left( e^{\phi_-} H_2^2 - e^{-\phi_-} H_3^2 \right), \] 

\[ \sqrt{2} \frac{d \ln H_1}{dr} = e^{\phi_1} \sinh \phi_+ (4H_1^2 + 1) - 2e^{-\phi_1} \left( e^{\phi_-} H_2^2 + e^{-\phi_-} H_3^2 \right), \] 

\[ \sqrt{2} \frac{d \ln H_2}{dr} = -\frac{1}{2} e^{-\phi_1+\phi_-} + 4e^{\phi_1} \sinh \phi_+ H_1^2 - 2e^{-\phi_1} \left( e^{\phi_-} H_2^2 + e^{-\phi_-} H_3^2 \right), \] 

\[ \sqrt{2} \frac{d \ln H_3}{dr} = -\frac{1}{2} e^{-\phi_1-\phi_-} + 4e^{\phi_1} \sinh \phi_+ H_1^2 - 2e^{-\phi_1} \left( e^{\phi_-} H_2^2 + e^{-\phi_-} H_3^2 \right), \]

where we found it more convenient to work with the fields \( H_i = z_i/(1 - z_1^2 - z_2^2 - z_3^2) \) instead of \( z_i \) and we defined \( \phi_{\pm} = \phi_2 \pm \phi_3 \). As soon as a solution has been found, the conformal factor of the metric can be obtained by a simple integration of the resulting expression for the prepotential \( \tilde{W} \), as it has been prescribed above.

Although these equations are first order, their explicit solution is still a difficult task because they are highly non-linear in the general case. We note that there are examples in the scalar sector of maximal gauged supergravity in 4, 5 and 7 space-time dimensions where the resulting system of equations can be linearized by introducing appropriate combinations of fields plus an algebraic constraint. This led to a systematic classification of the possible domain wall solutions in terms of Riemann surfaces with varying genus (for details, see the method presented in [32, 33]). However, things do not always work that way; as it turns out the above system of equations that describe strings at finite temperature can only be partially studied by similar methods.

It can be easily seen from the last three equations that the fields \( H_i \) can be solely expressed in terms of the fields \( \phi_i \), but one should avoid substituting their integral expressions into the first three equations because it would lead to a complicated system of integro-differential equations among the fields \( \phi_i \) alone. In either case, the system of equations at hand is rather complicated and difficult to solve in general. Instead, we will focus attention on subsectors obtained by consistent truncations of the field content. Each one of these consistent truncations results into a system of three first order equations that correspond to the various type-II and heterotic theories. Luckily, it will turn out that the general solutions for all type-II theories can be found explicitly. However, in the heterotic case a general solution cannot be given in closed form. Hence in this case, apart from extracting the behaviour of the fields in the strong and weak coupling regions as well as around certain critical points, one has unavoidably to rely on numerical studies. Combining these pieces of information, we can nevertheless give a rather complete picture of how the different solutions behave.
4 Type II string theories

The simplest truncations of the domain wall equations lead to different type-II sectors. We will first examine the type-IIA and IIB theories, which can be treated simultaneously using the appropriate field identifications, and then examine a hybrid type-II sector, which describes type-II strings at the self-dual radius, and turns out to be exactly solvable, too.

4.1 Type-IIA and IIB sector

According to the field identifications made in the effective supergravity, the type-IIA and type-IIB sectors of the theory are obtained by setting

\[ z_1 = z_2 = 0, \text{ for type IIB} \]  
\[ z_1 = z_3 = 0, \text{ for type IIA} \]

in which case \( H_1 = H_2 = 0 \) and \( H_1 = H_3 = 0 \) respectively. It follows from (3.19) and (3.20) or (3.21) that \( \phi_1 \) equals \( \phi_2 \) or \( \phi_3 \), respectively, up to an irrelevant additive constant which we will ignore. Then, it is convenient to set

\[ \phi_1 = \phi_2 = \frac{\phi}{\sqrt{2}}; \quad \phi_3 = \chi; \quad z_3 = z \quad \text{for IIB}, \]  
\[ \phi_1 = \phi_3 = \frac{\phi}{\sqrt{2}}; \quad \phi_2 = \chi; \quad z_2 = z \quad \text{for IIA} \]

and introduce in either case the field \( \omega \) for the winding field

\[ z = \tanh \left( \frac{\omega}{2} \right), \]

for which we have \( 2H = \sinh \omega \). The appropriate string coupling is \( g_{\text{II}} \sim e^\chi \) and the temperature field is \( T \sim e^{\sqrt{2}\phi} \), when it is normalized in type-II string units that remove the \( \chi \)-dependence. We treat both cases together because the (pre)potential assumes the same form for IIA and IIB, and the same is true for the truncated system of differential equations for the domain walls. Hence, no distinction will be made in the following between type-IIA or type-IIB. Note also that the kinetic terms for the fields \( \chi, \phi \) and \( \omega \) all assume their canonical form.

Explicit calculation shows that in terms of the new variables the prepotential (2.27) becomes

\[ \tilde{W}_{\text{II}} = \frac{1}{2} e^\chi \left( e^{\sqrt{2}\phi} + \frac{1}{2} e^{-\sqrt{2}\phi} \sinh^2 \omega \right), \]

whereas the corresponding potential (2.22) is

\[ V_{\text{II}} = \frac{1}{16} e^{2\chi} \sinh^2 \omega \left( e^{-2\sqrt{2}\phi} \cosh^2 \omega - 4 \right). \]
It is clear from this potential $V_{II}$ that $\omega$ becomes tachyonic if $e^{-2\sqrt{2}\phi} < 4$ at $\omega = 0$, in accordance with the discussion of section 2.3. Using the truncated prepotential $\tilde{W}_{II}$, the type-II domain walls obey the simpler system of first order equations

$$
\frac{d\chi}{dr} = -\frac{1}{2\sqrt{2}}e^{\chi} \left( e^{\sqrt{2}\phi} + \frac{1}{2}e^{-\sqrt{2}\phi}\sinh^2\omega \right),
$$

$$
\frac{d\phi}{dr} = -\frac{1}{2}e^{\chi} \left( e^{\sqrt{2}\phi} - \frac{1}{2}e^{-\sqrt{2}\phi}\sinh^2\omega \right),
$$

$$
\frac{d\omega}{dr} = -\frac{1}{4\sqrt{2}}e^{\chi-\sqrt{2}\phi}\sinh 2\omega .
$$

These equations can as well be obtained from the full set of first order equations (3.19)-(3.24) by imposing the above field identifications for the type-IIA or IIB strings. The equation for the conformal factor of the metric (3.17) is easily integrated and gives $A = -\chi$ (up to a constant that can be absorbed into a redefinition of the $x^\mu$'s). The system of equations (4.8) is invariant under $\omega \to -\omega$. Therefore we will consider the case $\omega \geq 0$ without any loss of generality.

We note for completeness that in the type-II sector one may fully absorb the $\chi$-dependence that appears on the right hand side of these differential equations by simply changing the coordinate variable $r$ to $\zeta$ defined as $d\zeta = e^\chi dr$. Put differently, $\chi$ plays the rôle of a Liouville field in that the “gravitational dressing” of an auxiliary massive model made out of the fields $\phi$ and $\omega$

$$
\mathcal{L}(\phi, \omega) = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\omega)^2 - \frac{1}{16} \sinh^2\omega \left( e^{-2\sqrt{2}\phi}\cosh^2\omega - 4 \right),
$$

yields the scalar field sector of the truncated type-II theory with all three fields $\phi$, $\omega$ and $\chi$ having kinetic terms in canonical form and a dressed potential that assumes the form $V_{II}$ above.\footnote{This interpretation is motivated by two-dimensional gravity coupled to massive $\sigma$-models. However, unlike other examples, where non-abelian Toda theories result in this fashion \cite{34}, the present model does not have a special meaning in integrable systems.}

It is an interesting finding that the domain wall equations can be completely integrated for the type-II sector, and one has for the first time a one-parameter family of explicit solutions with non-trivial winding $\omega$. It is convenient for this purpose to present the solution by treating the field $\omega$ as an independent variable, which is legitimate since the third equation in (4.8) implies that $\omega$ is a monotonous function of $r$. Then, the differential equation for $\phi$ can be easily integrated and also the equation for $\chi$. The final result is a family of BPS solutions, as shown in section 3.3, with

$$
e^{-2\sqrt{2}\phi} = 2\cosh^2\omega \left( \ln \coth^2\omega + c \right) - 2 ,
$$

$$
e^{2\chi} = \cosh^2\omega \ e^{\sqrt{2}\phi} ,
$$

parametrized by an arbitrary integration constant $c$. As we will see, the physical interpretation of the solutions depends crucially on whether $c$ is positive, negative or zero.
The behaviour of the functions $e^{2\sqrt{2}\phi}$ and $e^{2\chi}$ for the three choices of $c$ is sketched in Figure 1. We also note that we have omitted a multiplicative integration constant on the right hand side of the second equation in (4.10), since it can always be absorbed into trivial field redefinitions. We see that, independently of $c$, the behaviour of $T^{-2} \sim e^{-2\sqrt{2}\phi}$ as $\omega \to 0$ is $e^{-2\sqrt{2}\phi} \simeq 2 \ln \omega^{-2} \to \infty$. Hence, the winding mode cannot be tachyonic for all these solutions.

Figure 1: Qualitative behaviour of $e^{2\sqrt{2}\phi}$ (curves e) and $e^{2\chi}$ (curves f) as functions of $\omega$ for type-IIA and type-IIB.

Performing the integration of the conformal factor, we find that the metric takes the form

$$ds^2 = \frac{8e^{\sqrt{2}\phi}}{\sinh^2 \omega \cosh^4 \omega} d\omega^2 + \frac{e^{-\sqrt{2}\phi}}{\cosh^2 \omega} \eta_{\mu\nu} dx^\mu dx^\nu. \quad (4.11)$$

As for the relation between the variables $r$ and $\omega$ in (3.17) and (4.11) this turns out to be given by the relation of differentials

$$dr = -\frac{2\sqrt{2}e^{\phi}/\sqrt{2}}{\sinh \omega \cosh^2 \omega} d\omega. \quad (4.12)$$

Clearly, this integration cannot be performed in closed form and so we lack the explicit dependence of $\omega(r)$. However, the dependence can easily be spelled out in some limiting cases to which we now turn.
Universal behaviour for small windings: First consider the limit of vanishing winding field $\omega$. Then the asymptotic behaviour of the fields is

$$\omega \simeq e^{-(3r/8)^{4/3}}, \quad e^{-2\chi} \simeq e^{-\sqrt{2}\phi} \simeq \left(\frac{3}{\sqrt{8}} r\right)^{2/3}, \quad \text{as } r \rightarrow \infty.$$ (4.13)

Hence, in the limit of vanishing winding field $\omega$, the fields $\phi$ and $\chi$ approach $-\infty$ and the value of the constant $c$ plays no role. In addition, the metric becomes

$$ds^2 \simeq dr^2 + \left(\frac{3}{\sqrt{8}} r\right)^{2/3} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \text{as } r \rightarrow \infty.$$ (4.14)

The behaviour of the solution for larger values of the winding $\omega$ does depend on the parameter $c$. We distinguish the following three cases:

$c > 0$: In this case the reality conditions for the fields $\chi$ and $\phi$ allow for $0 \leq \omega < \infty$. Then, the string coupling $e^{2\chi}$ is a monotonically increasing function of $\omega$ in its full range of values. However, the temperature field squared $e^{2\sqrt{2}\phi}$ first increases until it reaches a maximum at $\omega = \omega_0$ and then decreases to zero as $\omega \rightarrow \infty$. In the limits of small and large $c$ the constant $\omega_0$ can be found analytically:

$$\omega_0 = \begin{cases} -\frac{1}{4} \ln \left(\frac{c}{8}\right) & \text{for } c \rightarrow 0^+, \\ \frac{1}{\sqrt{c}} & \text{for } c \rightarrow \infty. \end{cases}$$ (4.15)

For intermediate values of $c$, $\omega_0$ ranges between the above two extremes. The maximum value of $e^{2\sqrt{2}\phi}$ is given by

$$e^{2\sqrt{2}\phi}{\big|}_{\omega=\omega_0} = \frac{1}{2} \sinh^2 \omega_0.$$ (4.16)

The asymptotic behaviour of the various fields is found by noting that due to (4.12) we have the relation

$$r \simeq \frac{32 \cdot 3^{4/7}}{7 e^{1/4}} e^{-7\omega/2}, \quad \text{as } r \rightarrow 0^+ \quad \text{and} \quad \omega \rightarrow \infty,$$ (4.17)

up to an additive constant that has been fixed in an obvious way. Then,

$$e^{-\sqrt{2}\phi} \simeq \left(\frac{16c^{3/2}}{7 r}\right)^{2/7}, \quad e^{-2\chi} \simeq \left(\frac{7}{\sqrt{2}} c^2 r\right)^{2/7}, \quad \text{as } r \rightarrow 0^+.$$ (4.18)

Similarly, the asymptotic expression for the metric is

$$ds^2 \simeq dr^2 + \left(\frac{7}{\sqrt{2}} c^2 r\right)^{2/7} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \text{as } r \rightarrow 0^+.$$ (4.19)

$c = 0$: As before, the reality conditions for the fields $\chi$ and $\phi$ allow for $0 \leq \omega < \infty$. However, both fields $e^{2\chi}$ and $e^{2\sqrt{2}\phi}$ are now monotonically increasing functions of $\omega$ in

---

The exponential vanishing of the winding field $\omega$ is related to the fact that setting $\omega = 0$ from the very beginning is consistent with the system of eqs. (4.8). In fact, then the solution (4.13) for $\chi$ and $\phi$ and the metric in (4.14) below become exact (see also (5.7) below).
the entire range of values. The asymptotic behaviour of the various fields is found by noting that due to (4.12) we have the relation

\[ r \approx \frac{32}{5} e^{-5\omega/2} \quad \text{as} \quad r \to 0^+ \quad \text{and} \quad \omega \to \infty . \]  

(4.20)

Then,

\[ e^{-\sqrt{2}\phi} \approx \frac{1}{2} (5r)^{2/5} \quad \text{and} \quad e^{-2\chi} \approx \frac{1}{8} (5r)^{6/5} \quad \text{as} \quad r \to 0^+ . \]  

(4.21)

Similarly, the asymptotic expression for the metric is

\[ ds^2 \approx dr^2 + \frac{1}{8} (5r)^{6/5} \eta_{\mu\nu} dx^\mu dx^\nu , \quad \text{as} \quad r \to 0^+ . \]  

(4.22)

\( c<0 \): In this case the reality conditions for the fields \( \chi \) and \( \phi \) do not allow to exceed a maximum value for the winding field, i.e., \( 0 \leq \omega \leq \omega_{\text{max}} \). In this range, \( e^{2\chi} \) and \( e^{2\sqrt{2}\phi} \) are monotonically increasing functions of \( \omega \). Similarly, in the limit of small or large \( |c| \) the constant \( \omega_{\text{max}} \) can be found analytically, as before,

\[ \omega_{\text{max}} = \begin{cases} -\frac{1}{4} \ln \left( \frac{c}{8} \right) & \text{for} \quad c \to 0^- , \\ e^{c+1} & \text{for} \quad c \to -\infty . \end{cases} \]  

(4.23)

whereas for intermediate values of \( c \), \( \omega_{\text{max}} \) ranges between the two above extremes. The asymptotic behaviour of the various fields can be found by noting that due to (4.12) we have the relation

\[ r \approx \frac{8}{3^{3/4} \, c_m} \left( \omega_{\text{max}} - \omega \right)^{3/4} , \quad \text{as} \quad r \to 0^+ \quad \text{and} \quad \omega \to \omega_{\text{max}}^- . \]  

(4.24)

by introducing the notation

\[ s_m = \sinh \omega_{\text{max}} , \quad c_m = \cosh \omega_{\text{max}} . \]  

(4.25)

Then,

\[ e^{-\sqrt{2}\phi} \approx \left( \frac{3 c_m}{2\sqrt{2}} \right) r^{2/3} , \quad e^{-2\chi} \approx \left( \frac{3}{2\sqrt{2} \, c_m^2} \right) r^{2/3} , \quad \text{as} \quad r \to 0^+ . \]  

(4.26)

Similarly, the asymptotic expression for the metric is

\[ ds^2 \approx dr^2 + \left( \frac{3}{2\sqrt{2} \, c_m^2} \right) r^{2/3} \eta_{\mu\nu} dx^\mu dx^\nu , \quad \text{as} \quad r \to 0^+ . \]  

(4.27)

In all cases above, we note that the main qualitative difference is in the shape of the function \( e^{2\sqrt{2}\phi} \) representing the temperature field square: for \( c \leq 0 \) it starts from zero and evolves monotonically to \( +\infty \) following the range of \( r \) from \( \infty \) to 0, whereas for \( c > 0 \) we see a dramatic change in that the function starts and ends at zero, thus developing a maximum along the way. The end point at \( r = 0 \) corresponds to a curvature singularity. As we will see in section 6, there are physical ways to differentiate among the allowed values of \( c \) by requiring consistent propagation of a quantum test particle on the corresponding supergravity backgrounds which have a curvature singularity. Of course, string theory (and not supergravity alone) holds the ultimate answer for the physical relevance of our solutions in thermodynamics.
4.2 A hybrid type-II sector

There is another truncation of the domain wall equations with $H_1 = 0$, as in type-II, which is exactly solvable. We set for this purpose $H_2 = \pm H_3$ and observe that the full system of equations is consistent provided that $\phi_2 = \phi_3$. Actually, this sector can be viewed as a hybrid of type-IIB and IIA in that we impose the “diagonal” constraint $H_2 = \pm H_3$ instead of choosing the axes $H_2 = 0$ or $H_3 = 0$ respectively in $H$-field space; as such, it is a hybrid truncation. This truncation amounts to choosing the self-dual radius so that there is no distinction between the type-IIA and type-IIB theories.

We proceed further by setting

$$\phi_1 = \chi, \quad \phi_2 = \phi_3 = \frac{1}{\sqrt{2}} \phi, \quad H_2 = \pm H_3 = \frac{1}{2\sqrt{2}} \sinh \omega$$

and so $z_2 = \pm z_3 \equiv z$ can be chosen as

$$z = \frac{1}{\sqrt{2}} \tanh \frac{\omega}{2},$$

in order for the kinetic terms of the fields $\chi$, $\phi$ and $\omega$ to assume their canonical form. As before, the temperature field is $T \sim e^{\sqrt{2}\phi}$. Then, the prepotential (2.27) and the potential (2.22) become

$$\bar{W}_{hyb} = \frac{1}{2} \left( e^{\chi + \sqrt{2}\phi} + \frac{1}{2} e^{-\chi} \sinh^2 \omega \right)$$

and

$$V_{hyb} = \frac{1}{32} \sinh^2 \omega \left( e^{-2\chi} (1 + \cosh^2 \omega) - 8 e^{\sqrt{2}\phi} \right),$$

respectively. We see that $\omega$ will become tachyonic, if at $\omega = 0$ we have $e^{-\sqrt{2}\phi} - 2\chi < 4$.

The truncated system of equations is

$$\frac{d\chi}{dr} = -\frac{1}{2\sqrt{2}} \left( e^{\chi + \sqrt{2}\phi} - \frac{1}{2} e^{-\chi} \sinh^2 \omega \right),$$

$$\frac{d\phi}{dr} = -\frac{1}{2} e^{\chi + \sqrt{2}\phi},$$

$$\frac{d\omega}{dr} = -\frac{1}{4\sqrt{2}} e^{-\chi} \sinh 2\omega.$$
Luckily, the general solution can be found in parametric form by employing $\omega$ as an independent variable, as before. We find the following family of BPS solutions

$$e^{-2\sqrt{2}\phi} = b + 2a\left(\ln\left(\coth\frac{\omega}{2}\right) - \frac{1}{\cosh\omega}\right),$$

$$e^{2\chi} = \frac{a}{2\cosh\omega}e^{\sqrt{2}\phi},$$

(4.33)

where $a$ and $b$ are integration constants with $a$ being positive for reality reasons. For convenience, in the rest of this subsection, we set the positive constant $a = 2$, since its precise value can be adjusted by field redefinitions and hence it has no physical relevance.

The other integration constant $b$ cannot be fixed mathematically, but we will see later that physical considerations in the context of supergravity may impose some restrictions on it. Of course, as before, it is string theory that holds the answer for having an acceptable geometrical background for domain walls. The behaviour of the functions $e^{2\sqrt{2}\phi}$ and $e^{2\chi}$ for the three choices of $b$ is sketched in Figure 2.

Again, as $\omega \to 0$, the temperature field behaves as $T^{-2} \sim e^{-2\sqrt{2}\phi} \simeq 2\ln\omega^{-2} \to \infty$ and $e^{-2\chi} \sim e^{-\sqrt{2}\phi} \to \infty$ so that $\omega$ cannot be tachyonic.

![Figure 2: Qualitative behaviour of $e^{2\sqrt{2}\phi}$ (curves e) and $e^{2\chi}$ (curves f) as functions of $\omega$ for the hybrid type-II.](image)

After computing the conformal factor we may write the metric as

$$ds^2 = \frac{4ae^{\sqrt{2}\phi}}{\sinh^2\omega\cosh^3\omega}d\omega^2 + \frac{2e^{-\sqrt{2}\phi}}{a\cosh\omega}\eta_{\mu\nu}dx^\mu dx^\nu.$$  

(4.34)
The connection between the variables $r$ and $\omega$ in (3.17) and (4.34) is given by the relation of differentials

$$dr = -\frac{2\sqrt{ae^\phi / \sqrt{2}}}{\sinh \omega \cosh^{3/2} \omega} d\omega . \quad (4.35)$$

As before, the integration cannot be performed in closed form, apart from a few limiting cases, as we will see next.

**Universal behaviour for small windings:** Let us consider first the case of vanishing winding field $\omega$. It turns out that the asymptotic behaviour of the fields is

$$\omega \simeq 2e^{-(3r/8)^{4/3}}, \quad e^{-2\chi} \simeq e^{-\sqrt{2} \phi} \simeq \left( \frac{3}{\sqrt{8}} r \right)^{2/3} , \quad \text{as} \ r \to \infty . \quad (4.36)$$

Hence, in the limit of vanishing winding field $\omega$, the fields $\phi$ and $\chi$ approach $-\infty$ and the value of the constant $b$ plays no role. The metric becomes

$$ds^2 \simeq dr^2 + \left( \frac{3}{\sqrt{8}} r \right)^{2/3} \eta_{\mu \nu} dx^\mu dx^\nu , \quad \text{as} \ r \to \infty . \quad (4.37)$$

Note that this asymptotic behaviour is almost identical to the one found for the pure type-II case. A similar behaviour will also exist in the heterotic case to be discussed in the next section. There we will see that it is related to an exact solution with zero winding present in all cases.

The behaviour of the solution for larger values of the winding $\omega$, however, depends on the parameter $b$. As in the genuine type-II solution (1.10), we also distinguish here the following three cases:

**$b > 0$:** In this case the reality conditions for the fields $\chi$ and $\phi$ allow for $0 \leq \omega < \infty$. Then, the temperature field square $e^{2\sqrt{2} \phi}$ is a monotonically increasing function of $\omega$ until it reaches the asymptotic value $1/b$. However, the string coupling $e^{2\chi}$ first increases until it reaches a maximum at $\omega = \omega_0$, and then decreases to zero as $\omega \to \infty$. In the limits of small and large $b$, the constant $\omega_0$ can be found analytically

$$\omega_0 = \begin{cases} -\frac{1}{3} \ln \left( \frac{2}{16} \right) & \text{for} \ b \to 0^+ , \\ \sqrt{\frac{\pi}{b}} & \text{for} \ b \to \infty , \end{cases} \quad (4.38)$$

whereas for intermediate values of $b$, $\omega_0$ ranges between the above two extremes. The maximum value of $e^{2\chi}$ is given by

$$e^{2\chi} \big|_{\omega = \omega_0} = \frac{\sinh \omega_0}{\sqrt{2} \cosh \omega_0} . \quad (4.39)$$

The asymptotic behaviour of the various fields is extracted by noting that, due to (4.33), we have the relation

$$r \simeq \frac{32}{5b^{3/4}} e^{-5\omega/2} , \quad \text{as} \ r \to 0^+ \quad \text{and} \quad \omega \to \infty . \quad (4.40)$$

\footnote{A similar comment, as in footnote 6, applies here as well.}
Then, we find that
\[ e^{-2\sqrt{2}\phi} \simeq b + \frac{1}{6} (5b^{1/4}r)^{6/5}, \quad e^{2\chi} \simeq \frac{1}{2} \left( \frac{5r}{b} \right)^{2/5}, \quad \text{as} \quad r \to 0^+. \]  

Similarly, the asymptotic expression for the metric is
\[ ds^2 \simeq dr^2 + \frac{1}{2} (5b^{3/2}r)^{2/5} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \text{as} \quad r \to 0^+. \]

\( b = 0 \): As before, the reality conditions for the fields \( \chi \) and \( \phi \) allow for \( 0 \leq \omega < \infty \). However, both fields \( e^{2\chi} \) and \( e^{2\sqrt{2}\phi} \) are now monotonically increasing functions of \( \omega \) in the entire range of values. The asymptotic behaviour of the various fields is found using
\[ r \simeq \frac{32}{7} \left( \frac{3}{2} \right)^{1/4} e^{-7\omega/4}, \quad \text{as} \quad r \to 0^+ \quad \text{and} \quad \omega \to \infty, \]  

which is due to (4.33). Then, we find that
\[ e^{-2\sqrt{2}\phi} \simeq \frac{7}{24} \left( \frac{7^5}{54} \right)^{1/7} r^{12/7}, \quad e^{-2\chi} \simeq \left( \frac{98}{81} \right)^{1/7} r^{2/7}, \quad \text{as} \quad r \to 0^+. \]

Similarly, the asymptotic expression for the metric is
\[ ds^2 \simeq dr^2 + \frac{7}{24} \left( \frac{3 \cdot 7^3}{4} \right)^{1/7} r^{10/7} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \text{as} \quad r \to 0^+. \]

\( b < 0 \): In this case the reality conditions for the fields \( \chi \) and \( \phi \) do not allow to exceed a maximum value for the winding field, i.e., \( 0 \leq \omega \leq \omega_{\text{max}} \). In this range, \( e^{2\chi} \) and \( e^{2\sqrt{2}\phi} \) are monotonically increasing functions of \( \omega \). In the limit of small and large \( |b| \) we find for \( \omega_{\text{max}} \) the analytic expressions

\[ \omega_{\text{max}} = \begin{cases} 
-\frac{1}{3} \ln \left( \frac{3b}{32} \right) & \text{for} \quad b \to 0^-, \\
2e^{b/4-1} & \text{for} \quad b \to -\infty.
\end{cases} \]

For intermediate values of \( b \), \( \omega_{\text{max}} \) ranges between the two above extremes. The asymptotic behaviour of the various fields is extracted by noting that, due to (4.33), we have the relation
\[ r \simeq \frac{8}{3s_1^{3/4} c_m} (\omega_{\text{max}} - \omega)^{3/4}, \quad \text{as} \quad r \to 0^+ \quad \text{and} \quad \omega \to \omega_{\text{max}}, \]  

where we have used the same notation as in (4.25). Then,
\[ e^{-\sqrt{2}\phi} \simeq \left( \frac{3}{2\sqrt{2}c_m} r \right)^{2/3}, \quad e^{-2\chi} \simeq \left( \frac{3c_m}{2\sqrt{2}} r \right)^{2/3}, \quad \text{as} \quad r \to 0^+. \]

Similarly, the asymptotic expression for the metric is
\[ ds^2 \simeq dr^2 + \left( \frac{3}{2\sqrt{2}c_m^2} r \right)^{2/3} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \text{as} \quad r \to 0^+. \]
Note that in the hybrid type-II sector, as for the genuine type-II solution, the end point at $r = 0$ represents a curvature singularity. We will see in section 6 that the three different cases, which correspond to the values of $b > 0$, 0 or $< 0$, can be distinguished using certain physical criteria.

Finally, note that for all type-II sectors (IIA, IIB and the hybrid type-II), $\omega$ decreases monotonically from its value at $r = 0$ (which may be infinite) to $\omega = 0$ at $r = \infty$. In particular, this implies that $\omega(r)$ cannot be a periodic function of $r$, and hence $r$ necessarily is a non-compact coordinate.

## 5 Heterotic sector

The heterotic limit is obtained by setting $H_2 = H_3 = 0$, while keeping $H_1 \equiv H$ free to vary. It then follows from (3.20) and (3.21) that $\phi_2 = \phi_3$ up to an irrelevant additive constant that we will ignore. Trading the $z$ corresponding to $H$ for a winding field $\omega$ and introducing appropriate variables

$$
\begin{align*}
\phi_1 &= \chi, \quad \phi_2 = \phi_3 = \frac{\phi}{\sqrt{2}}, \quad z = \tanh \frac{\omega}{2},
\end{align*}
$$

for which $2H = \sinh \omega$, we obtain the expression for the truncated prepotential

$$
\tilde{W}_{\text{het}} = \frac{1}{2} e^\chi \left( e^{\sqrt{2} \phi} - \sinh(\sqrt{2} \phi) \sinh^2 \omega \right),
$$

and the potential

$$
V_{\text{het}} = \frac{1}{8} e^{2 \chi} \sinh^2 \omega \left( \cosh(2 \sqrt{2} \phi) \cosh^2 \omega - 3 \right).
$$

We note for completeness that $e^\chi$ is the string coupling and the temperature field is again $e^{\sqrt{2} \phi}$ (normalized in heterotic string units that remove the $\chi$-dependence) and the kinetic terms of the fields $\chi$, $\phi$ and $\omega$ assume their canonical form with our choice of normalizations. We see that $\omega$ can be tachyonic only if $\cosh(2 \sqrt{2} \phi) < 3$ at $\omega = 0$.

We remark that although the potential $V_{\text{het}}$ is invariant under $\omega \to -\omega$ and $\phi \to -\phi$, the prepotential $\tilde{W}_{\text{het}}$ is invariant under the first transformation only. The second transformation corresponds to T-duality in the temperature radius. Since the prepotential breaks this temperature T-duality one expects the same to be true for non-trivial solutions of our equations.

The truncated system of differential equations is

$$
\begin{align*}
\frac{d\chi}{dr} &= -\frac{1}{2 \sqrt{2}} e^\chi \left( e^{\sqrt{2} \phi} - \sinh(\sqrt{2} \phi) \sinh^2 \omega \right), \\
\frac{d\phi}{dr} &= -\frac{1}{2} e^\chi \left( e^{\sqrt{2} \phi} - \cosh(\sqrt{2} \phi) \sinh^2 \omega \right), \quad \text{(5.4)} \\
\frac{d\omega}{dr} &= \frac{1}{2 \sqrt{2}} e^\chi \sinh(\sqrt{2} \phi) \sinh 2\omega.
\end{align*}
$$
The equation for the conformal factor of the metric (3.17) is easily integrated and gives 
\( A = -\chi \), as in the case of the genuine type-IIA or type-IIB theories. Also, similarly to
this case, we may drop the \( \chi \)-dependence from the right hand side of the resulting first
order system by simply changing variables to \( \zeta \) defined by \( d\zeta = e^{\chi} dr \). Again, \( \chi \)
may be viewed as a Liouville field whose coupling provides the “gravitational dressing” of an
auxiliary massive model of the fields \( \phi \) and \( \omega \) with Lagrangian density
\[
L' (\phi, \omega) = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \omega)^2 - \frac{1}{8} \sinh^2 \omega \left( \cosh(2\sqrt{2}\phi) \cosh^2 \omega - 3 \right).
\]
(5.5)
This prescription produces naturally the canonical kinetic term for the field \( \chi \) and dresses
the potential into \( V_{\text{het}} \).

It turns out that it is not possible to solve the domain wall equations in closed form in
the heterotic sector. Nevertheless, we can still determine the leading small \( \omega \) asymptotics,
similarly to the analogous universal behaviour for the type-II theories. This corresponds
to large \( r \). As it turns out, \( \omega \) is again exponentially suppressed as \( r \to \infty \), while \( e^\chi \) and
\( e^{\sqrt{2}\phi} \) have a power-law behaviour. Hence, we can safely drop the \( \sinh^2 \omega \) part in the first
two equations in (5.4) and replace \( \sinh 2\omega \) by \( 2\omega \) in the third equation. The resulting
system is easily integrated and yields in this limit
\[
e^{-\sqrt{2}\phi} \simeq 2C r^{2/3} , \quad e^{-2\chi} \simeq \frac{9}{32C^2} r^{2/3} , \quad \omega \simeq \omega_0 e^{-C^2 r^{4/3}} , \quad \text{as } r \to \infty ,
\]
(5.6)
where \( C \) and \( \omega_0 \) are two positive constants of integration. Note that \( \cosh(2\sqrt{2}\phi) \simeq 2C^2 r^{4/3} \simeq 2 \ln \omega^{-2} \to \infty \) as \( \omega \to 0 \) and \( \omega \) cannot be tachyonic. As one decreases the
value of the constant \( \omega_0 \), the range of approximate validity of this asymptotic solution
extends to larger intervals of \( r \). Eventually, if \( \omega_0 = 0 \) one obtains a particular exact solution
\[
e^{-\sqrt{2}\phi} = 2C r^{2/3} , \quad e^{-2\chi} = \frac{9}{32C^2} r^{2/3} , \quad \omega = 0 ,
\]
(5.7)
which is valid for all values of \( r \). However, since we lack the most general solution, even in
some parametric form \( \phi(\omega) \), we will only provide results about the dominant behaviour
of the fields in the weak coupling (i.e., \( \chi \to -\infty \)) and the strong coupling (i.e., \( \chi \to +\infty \))
regions. In fact, we will succeed to identify these regions and perform an in depth analysis
in their vicinity using standard techniques from dynamical systems. The remaining part
of the two-dimensional space \((\phi, \omega)\) can only be studied numerically.

Before delving into details, we study a bit more the exact solution for \( \omega = 0 \). This
solution represents a straight orbit in the parameter space \((\phi, \omega)\) with \( \omega = 0 \) everywhere,
and extending from \( \phi = -\infty \) (identified as a weak coupling point) to \( \phi = +\infty \) (identified as
a strong coupling point), as \( r \) ranges from \(+\infty \) to \( 0 \). The Liouville coordinate \( \zeta \), which
trades \( r \) as \( d\zeta = e^{\chi} dr \), is given for this special solution by
\[
\zeta = 2\sqrt{2} C r^{2/3} ,
\]
(5.8)
\footnote{We note that this is an exact solution for the type-II sector as well. Indeed, when \( \omega = 0 \), the three
different truncated systems of equations (4.8), (4.32) and (5.4) become identical. Note that for
the corresponding type-II solution we had chosen the integration constant \( C = 3^{2/3}/4 \).}
where we absorbed an integration constant into $\zeta$.

Introducing small fluctuations $\delta \phi$ and $\delta \omega$ around this special solution, to test its stability, we find to first order that they satisfy the linearized system of equations

$$
\frac{d}{d\zeta} \delta \phi = -\frac{1}{\zeta} \delta \phi ,
$$

$$
\frac{d}{d\zeta} \delta \omega = \left( \frac{1}{2\zeta} - \frac{\zeta}{4} \right) \delta \omega ,
$$

(5.9)

which are written for convenience using $\zeta$. Therefore, we have the following solution of the linearized problem

$$
\delta \phi = \frac{A}{\zeta} , \quad \delta \omega = B\sqrt{\zeta}e^{-\zeta^2/8} ,
$$

(5.10)

where $A$, $B$ are some other constants. These fluctuations are small provided that $\zeta \to \infty$, otherwise the linearized approximation is not valid. We conclude this analysis by noting that the exact solution we found is stable against perturbations in the parameter space $(\phi, \omega)$ provided that we are in the weak coupling region where $\phi \to -\infty$. In fact, as we will see next, the point $(-\infty, 0)$ is an asymptotic critical point in the two-dimensional parameter space, and so we can talk about asymptotic stability in its vicinity.

The critical points: It is more convenient to work with the coordinate $\zeta$ instead of $r$ in order to drop the $\chi$-dependence from the right hand side of the equations. Then, as in the theory of dynamical systems, we have to study the solutions of the system

$$
\frac{d\phi}{d\zeta} = P(\phi, \omega) , \quad \frac{d\omega}{d\zeta} = Q(\phi, \omega) ,
$$

(5.11)

either as functions of $\zeta$, i.e., as $\phi(\zeta)$ and $\omega(\zeta)$, or in a parametric form $\phi(\omega)$. For the moment we have not prescribed a set of physical boundary conditions that select certain orbits among the infinite many that arise in the two-dimensional parameter space, and so we can talk about asymptotic stability in its vicinity.

Recall that the critical points are defined as the common zeros of the functions $P$ and $Q$. These occur in the present case either when $\phi = 0$ and $\sinh \omega = \pm 1$ or when $e^{\sqrt{2}\phi} = 0$ and $\omega = 0$. The second case has already been analyzed and lies in the weak coupling region of the exact solution that was described above with $\phi$ approaching asymptotically $-\infty$. Therefore, we focus on the other critical points

$$
\phi_0 = 0 , \quad \omega_0^\pm = \ln(\sqrt{2} \pm 1) ,
$$

(5.12)

which appear symmetrically on the $\omega$-axis because of the invariance of the equations under $\omega \to -\omega$. Then, the dilaton equation in (5.4) is easily integrated to give at these critical points

$$
\chi = -\frac{1}{2\sqrt{2}} \zeta = \ln \left( \frac{2\sqrt{2}}{r} \right) ,
$$

(5.13)
where we have absorbed integration constants into $\zeta$ and $r$ in an obvious way. Hence, the dilaton field exhibits a logarithmic dependence on $r$ and a linear in $\zeta$. Recall now that the string frame metric in four dimensions is obtained by (3.17) after multiplying with $e^{2\chi}$. Hence, the resulting background has a flat string metric and a linear dilaton, which is the exact solution preserving half of the supersymmetries found earlier in [18, 19]. Note that, in this case, it makes sense to pass from the Einstein to the string frame since, when $\phi$ and $\omega$ assume their critical values (5.12), we are left with only massless fields (graviton and dilaton) that couple to a 2-dimensional string world-sheet action.

Next, if we linearize around these critical points (5.12) by introducing small fluctuations $\delta \phi$ and $\delta \omega$, we will find the following evolution for them

$$
\frac{d}{d\zeta} \begin{pmatrix} \delta \phi \\ \delta \omega \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mp 2 \\ \mp 2 & 0 \end{pmatrix} \begin{pmatrix} \delta \phi \\ \delta \omega \end{pmatrix},
$$

(5.14)

whereas for the dilaton $d(\delta \chi)/d\zeta = 0$. Computing the two eigenvalues of the corresponding matrix we have in either case of $\omega_0^{\pm}$ that

$$
\lambda_{\pm} = \frac{1}{2\sqrt{2}}(-1 \pm \sqrt{17}),
$$

(5.15)

and since $\lambda_+ > 0$ and $\lambda_- < 0$, we see that the pair of critical points $(0, \omega_0^{\pm})$ are both saddle points. We may solve the linearized system of equations and obtain

$$
\delta \phi = \frac{A e^{\lambda_+ \zeta} + B e^{\lambda_- \zeta}}{\lambda_+ - \lambda_-},
$$

$$
\delta \omega = \pm \sqrt{2} \left( \frac{A}{\lambda_+} e^{\lambda_+ \zeta} + \frac{B}{\lambda_-} e^{\lambda_- \zeta} \right),
$$

(5.16)

with $\pm$ depending on the choice of critical point $\omega_0^{\pm}$, whereas the dilaton does not change, to linear order, from its value in (5.13).

If we demand that the critical points are reached as $\zeta \to +\infty$, we conclude for the integration constants that $A = 0$. Hence, the variations of $\phi$ and $\omega$ have a power law decay, namely

$$
\delta \phi = \mp \frac{\sqrt{17} + 1}{4} \delta \omega = B \left( \frac{2\sqrt{2}}{r} \right)^{\sqrt{17}+1}, \quad \text{as} \quad r \to \infty.
$$

(5.17)

On the other hand, if we demand instead, that the critical points are reached as $\zeta \to -\infty$, we conclude for the integration constants that $B = 0$. Now eq. (5.17) is replaced by

$$
\delta \phi = \pm \frac{\sqrt{17} - 1}{4} \delta \omega = A \left( \frac{r}{2\sqrt{2}} \right)^{\sqrt{17}-1}, \quad \text{as} \quad r \to 0^+.
$$

(5.18)

Note the different behaviour of the two solutions around the critical points (5.12) as given by (5.17) and (5.18). In particular, from (5.13), they correspond to weak and
strong string couplings, respectively. We will see that if trajectories in the entire $\phi-\omega$ plane are considered, both solutions will appear.

**Strong coupling points**: Having extracted the behaviour of the fields around the critical points, which can be either at weak or at strong coupling, we will examine next their behaviour in the vicinity of the other strong coupling points, where the dilaton field $\chi$ tends to $+\infty$. We have identified three such regions in the parameter space $(\phi, \omega)$, which of course have a mirror counterpart under $\omega \rightarrow -\omega$. They are (cf. figure 3 below):

- **region 1**: $\phi \rightarrow +\infty$ and $\omega \rightarrow 0$, (5.19)
- **region 2**: $\phi \rightarrow +\infty$ and $\omega \rightarrow \pm \infty$, (5.20)
- **region 3**: $\phi \rightarrow -\infty$ and $\omega \rightarrow \pm \infty$. (5.21)

To justify this claim and extract the details of the fields in these regions, we shall treat each case separately.

First, it is convenient to examine the heterotic equations in the limit $\phi \rightarrow +\infty$, which is common to both regions 1 and 2. Using the coordinate $\zeta$ (instead of $r$) we obtain in this limit the simplified system

\[
\frac{d\phi}{d\zeta} = \sqrt{2} \frac{d\chi}{d\zeta} = -\frac{1}{2} e^{\sqrt{2}\phi} \left(1 - \frac{1}{2} \sinh^2 \omega\right),
\]
\[
\frac{d\omega}{d\zeta} = \frac{1}{2\sqrt{2}} e^{\sqrt{2}\phi} \sinh \omega \cosh \omega
\]

and so we see immediately that $\sqrt{2}\chi = \phi$ (up to an additive constant). The remaining equations can be easily integrated to yield the expressions

\[
e^{-\sqrt{2}\phi} = \frac{3}{2a\sqrt{2}} (b - a\zeta)^{1/3} \left(1 - (b - a\zeta)^{2/3}\right),
\]
\[
\cosh^2 \omega = \frac{1}{(b - a\zeta)^{2/3}},
\]

in terms of some constants $a > 0$ and $b$. However, recall at this point that we are considering the behaviour of the equations for $\phi \rightarrow +\infty$, and so we should expand $\exp(-\sqrt{2}\phi)$ around zero. Naturally, we face two possibilities, which will be identified with regions 1 and 2.

Region 1 follows in this context by assuming $(b - a\zeta)^{2/3} \approx 1$. Defining the parameter $\zeta_1 = (b - 1)/a$ we have

\[
e^{-\sqrt{2}\phi} \approx \frac{1}{\sqrt{2}} (\zeta - \zeta_1), \quad \omega \approx \pm \sqrt{\frac{2a}{3}} (\zeta - \zeta_1)^{1/2}; \quad \text{as} \quad \zeta \rightarrow \zeta_1^+, \quad (5.24)
\]

and so $\omega \rightarrow 0$ as advertised for region 1. Since $\sqrt{2}\chi = \phi$ (up to a constant shift), we conclude that it is indeed a strong coupling region. One easily sees that $(\zeta - \zeta_1) \sim$
(r - r_1)^{2/3}$, for some constant $r_1$ which we henceforth absorb into $r$ so that the metric behaves like
\[ ds^2 \simeq dr^2 + (\text{const.}) r^{2/3} \eta_{\mu\nu} dx^\mu dx^\nu , \quad \text{as} \quad r \to 0^+ . \quad (5.25) \]

Region 2 follows by expanding the fields around $b - a \zeta \approx 0$. If we let $\zeta_2 = b/a$, we find the following behaviour
\[ e^{\sqrt{2} \phi} \simeq \frac{3}{2\sqrt{2} a^{2/3}} (\zeta_2 - \zeta)^{1/3} , \quad e^{\pm \omega} \simeq \frac{2}{a^{1/3}} (\zeta_2 - \zeta)^{1/3} , \quad \text{as} \quad \zeta \to \zeta_2^- . \quad (5.26) \]

In this vicinity we have $\omega \to \pm \infty$, depending on the branch in the upper or lower $\omega$ half-plane, and since $\phi \to \infty$ we also have $\chi \to +\infty$, which indeed describes the strong coupling region 2 as advertised. Using $(\zeta_2 - \zeta) \sim (r_2 - r)^{6/7}$, for some positive $r_2$, one finds that
\[ ds^2 \simeq dr^2 + (\text{const.}) (r_2 - r)^{2/7} \eta_{\mu\nu} dx^\mu dx^\nu , \quad \text{as} \quad r \to r_2^- . \quad (5.27) \]

Region 3 arises by first letting $\phi \to -\infty$. Then, the heterotic equations simplify in this limit to the following system:
\[ \frac{d\phi}{d\zeta} = -\sqrt{2} \frac{d\chi}{d\zeta} = \frac{1}{4} e^{-\sqrt{2} \phi} \sinh^2 \omega , \]
\[ \frac{d\omega}{d\zeta} = -\frac{1}{2\sqrt{2}} e^{-\sqrt{2} \phi} \sinh \omega \cosh \omega . \quad (5.28) \]

Clearly, $\sqrt{2} \chi = -\phi$ (up to a constant) and so this is also a strong coupling region. Since we have $\phi \to -\infty$, the dominant dependence on $\zeta$ is easy to extract. We find
\[ e^{\sqrt{2} \phi} \simeq \left( \frac{3}{2c^2 \sqrt{2}} \right)^{1/3} (\zeta - \zeta_3)^{1/3} , \quad e^{\pm \omega} \simeq 2 \left( \frac{2\sqrt{2}}{3c} \right)^{1/3} \frac{1}{(\zeta - \zeta_3)^{1/3}} , \quad \text{as} \quad \zeta \to \zeta_3^+ , \quad (5.29) \]

where $c$ and $\zeta_3$ are constants of integration. In this case we see that $\omega \to \pm \infty$ as it was initially stated. Finally, $\zeta - \zeta_3 \sim (r - r_3)^{6/7}$ for some constant $r_3$ which we again absorb into $r$. The behaviour of the metric near $r = 0$ is
\[ ds^2 \simeq dr^2 + (\text{const.}) r^{2/7} \eta_{\mu\nu} dx^\mu dx^\nu , \quad \text{as} \quad r \to 0^+ . \quad (5.30) \]

The behaviour of different trajectories in the $(\phi, \omega)$ plane is depicted in Figure 3. Because of the symmetry $\omega \to -\omega$ we have restricted to windings $\omega \geq 0$ without loss of generality, in particular solutions that start in the upper-half plane cannot end somewhere in the lower half-plane. The qualitative characteristics of the various trajectories can be deduced by treating $\omega$ (or $\phi$) as an independent variable and consider the differential equation for $d\phi/d\omega$. Before we examine the different trajectories in some detail, note by inspection of the differential equation $d\chi/d\omega$ and from the strong coupling behaviour that we already know, that there can be trajectories along which the string coupling $e^\chi$ develops a minimum at a point in the $\phi-\omega$ plane where
\[ e^{2\sqrt{2} \phi} = \frac{\sinh^2 \omega}{\sinh^2 \omega - 2} \quad \Rightarrow \quad \phi > 0 \quad \text{and} \quad \omega > \omega_1^+ \equiv \ln(\sqrt{3} + \sqrt{2}) . \quad (5.31) \]
Figure 3: Qualitative behaviour of different trajectories in the upper-half $\phi$-$\omega$ plane for the heterotic case.

For $\phi < 0$ there is no such minimum and the string coupling $e^\chi$ keeps increasing with increasing $\omega$. Let us now discuss the different types of trajectories as depicted in Figure 3:

**Trajectories with an extreme $\omega$:** These are trajectories where for some $\omega$ we have $\phi = 0$ and hence $d\omega/d\phi = 0$ showing that these trajectories intersect perpendicularly the axis $\phi = 0$. Since

$$
\frac{d^2 \omega}{d\phi^2} \bigg|_{\phi=0} = \frac{\sinh 2\omega}{\sinh^2 \omega - 1},
$$

we have for $\omega > \omega_0^+ = \ln(\sqrt{2} + 1) \ (< \omega_0^+)$ that this $\omega$ is a minimum (maximum) in the trajectory. Two such typical trajectories are drawn in Figure 3. The lower one connects the weak coupling region $(\phi, \omega) = (-\infty, 0)$ with the strong coupling region 1, after reaching a maximum for the winding $\omega_{\text{max}} \in (0, \omega_0^+)$. Since (5.31) can never be satisfied, it can be easily seen that $d\chi/d\phi > 0$ throughout the trajectory and hence the string coupling $e^\chi$ is progressively increasing with $\phi$. Another such trajectory connects the strong coupling regions 2 and 3 going through a minimum for the winding $\omega_{\text{min}} \in (\omega_0^+, \infty)$. For this trajectory there is of course a minimum for the string coupling at a point described by (5.31). We also note that for the trajectories with $\omega \leq \omega_{\text{max}}$ that connect the weak coupling region to region 1, the variable $\zeta$ and hence also $r$ decrease as $\phi$ increases, with $r$ going from $+\infty$ to 0. For the trajectories with $\omega \geq \omega_{\text{min}}$ that connect regions 3 and 2, $\zeta$ and $r$ increase as $\phi$ increases, with $r$ going from 0 to some $r_2 > 0$. Note that the value of $r_2$ depends on the chosen trajectory.

**Trajectories with an extreme $\phi$:** There are trajectories for which $d\phi/d\omega = 0$ and hence
there is an extreme value for $\phi$. We find that the condition for this is

$$
e^{2\sqrt{2}\phi} = \frac{\sinh^2 \omega}{2 - \sinh^2 \omega} \quad \Rightarrow \quad \begin{cases} 
\phi > 0, & \text{for } \omega_0^+ < \omega < \omega_1^+, \\
\phi < 0, & \text{for } 0 < \omega < \omega_0^+.
\end{cases} \quad (5.33)
$$

Since

$$
\left. \frac{d^2 \phi}{d\omega^2} \right|_{\omega=0} = \frac{\sqrt{2}}{\sinh^2 \omega - 1},
$$

we have for $\phi > 0$ that this $\phi$ is a minimum (indicated by $\phi_{\text{min}}$) occurring for $\omega \in (\omega_0^+, \omega_1^+)$, whereas for $\phi < 0$ it is a maximum (indicated by $\phi_{\text{max}}$) occurring for $\omega \in (0, \omega_0^+)$. Two such typical trajectories are drawn in Figure 3. The one with $\phi < 0$ connects the weakly coupled region having $(\phi, \omega) = (-\infty, 0)$ with the strongly coupled region 3. Since $\phi < 0$ there is no minimum for the string coupling, which increases with $\omega$ in accordance with (5.31). The trajectory with $\phi > 0$ connects the two strongly coupled regions 1 and 2. For such trajectories, besides the depicted minimum value for $\phi$, there is also a minimum value for the string coupling according to (5.31). Note that for the trajectory with $\phi > 0$ ($\phi < 0$) the variables $\zeta$ and $r$ increase (decrease) as $\omega$ increases, with $r$ ranging from 0 to some $r_2 > 0$ (from $\infty$ to 0). Again, the value of $r_2$ depends on the chosen trajectory.

We also note that region 2 contains trajectories that come either from region 1 or from region 3. The family of such trajectories is parametrized by the constant $a$ in (5.26). The values of $r_2$ and $a$ are clearly related, but we do not know exactly how. There exists a critical value, which is numerically found to be $a_{\text{crit}} \approx 0.9$, such that for $a > a_{\text{crit}}$ ($a < a_{\text{crit}}$) we connect to region 1 (3). Similarly, in region 3 the different trajectories are parametrized by the constant $c$ in (5.29) and there is also a critical value $c_{\text{crit}} \approx 1.2$; for $c > c_{\text{crit}}$ ($c < c_{\text{crit}}$) these trajectories end up in the weakly coupled region (region 2). Of course, for the trajectories that connects the regions 2 and 3 the constants $a$ and $c$ are related, but how precisely, cannot be answered without knowledge of the explicit solution.

**Trajectories ending at critical points:** These are the trajectories depicted with the dashed lines and occur when the constants $a$ and $c$ assume the critical values we mentioned. We emphasize that despite appearances, these trajectories have the critical points as their end-points, and they do not simply pass through them. In other words, they do not connect the strongly coupled regions 1 and 3 nor the weakly coupled region $(\phi, \omega) = (-\infty, 0)$ with the strongly coupled region 2. The critical point indicated in Figure 3, represents a weak coupling point for trajectories that come either from region 1 or from region 3. In these cases the solution in the vicinity of the critical point is given by (5.17) and the variables $\zeta$ or $r$ increase to $+\infty$ as we approach the critical point. In contrast, for trajectories that go either to region 2 or to the weakly coupled region with $(\phi, \omega) = (-\infty, 0)$, this critical point is at strong coupling. Then, the solution in its vicinity is given by (5.18) and the variables $\zeta$ or $r$ increase as we get away from the

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9In these regions we have the linear behaviour $\sqrt{2}\phi \simeq \omega + \ln(\sqrt{2}a/3)$ (for region 2) and $\sqrt{2}\phi \simeq -\omega + \ln(2/c)$ (for region 3). For region 1 we have instead $\sqrt{2}\phi \simeq -2\ln\omega - \ln(3/2\sqrt{2}a)$. 

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critical point. We note that the trajectory going to region 2 connects strong coupling to strong coupling so that $\chi$ must have a minimum along its way. This is consistent with the fact that such a minimum occurs if and only if $\phi > 0$ and $\omega > \omega^+_1$. This trajectory is the only one among the four for which $r$ takes values in a finite interval $[0, r_2]$.

In section 6, we will discuss certain criteria based on the propagation of a free quantum particle on the corresponding backgrounds that render some of these trajectories unphysical.

We conclude the discussion of the heterotic equations by mentioning the change of variables

$$x = \tanh \omega, \quad y = \cosh(\sqrt{2}\phi) \frac{\sinh \omega}{\cosh^2 \omega},$$  

(5.35)

which cast the differential equations for $\phi(\omega)$ into a simpler looking form for $y(x)$ (after taking its square)

$$x^2(y')^2 - y^2 = x^2(x^2 - 1).$$  

(5.36)

Unfortunately, we are still lacking its general solution in analytic form. Nevertheless, the form of the eq. (5.36) enables to perform a systematic study of the perturbative expansion around the critical points (5.12), which are located at $(x, y) = \pm \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ in the parametrization (5.35). We have the following perturbative expansion, with the upper (lower) signs refering to the corresponding signs in (5.12),

$$y = \pm \frac{1}{2} + \sum_{n=0}^{\infty} a_n \left(x \mp \frac{1}{\sqrt{2}}\right)^{n+2},$$  

(5.37)

where the coefficient $a_0$ of the expansion is given by

$$2a_0^2 \mp a_0 - 2 = 0.$$  

(5.38)

We also have the following recursive relations for $n \geq 1$:

$$(2(n + 2)a_0 \mp 1) a_n = \pm 2\sqrt{2}\delta_{n,1} - \delta_{n,2} - \frac{1}{2} \sum_{k=1}^{n-1} (k + 2)(n - k + 2)a_k a_{n-k}$$

$$\mp \sqrt{2} \sum_{k=0}^{n-1} (k + 2)(n - k + 1)a_k a_{n-k-1}$$

$$- \sum_{k=0}^{n-2} ((k + 2)(n - k) - 1)a_k a_{n-k-2}.$$  

(5.39)

Note that the freedom to choose the constant $a_0$ in (5.38) corresponds to the two independent solutions of (5.36) that arise from taking the square root. Since the constant $a_0$ determines the sign of $y'$, we have for $\omega \geq 0$ that $a_0 > 0$ for $\phi > 0$, whereas $a_0 < 0$ for $\phi < 0$. The first few terms of the expansion are given by

$$y = \pm \frac{1}{2} + a_0 \left(x \mp \frac{1}{\sqrt{2}}\right)^2 - \frac{2\sqrt{2}(a_0 \pm 1)}{6a_0 \mp 1} \left(x \mp \frac{1}{\sqrt{2}}\right)^3 + \cdots.$$  

(5.40)
One could develop a similar perturbative expansion around any of the strongly coupled regions 1, 2 and 3 or in the vicinity of the weakly coupled region \((-\infty, 0)\). The corresponding points, in the \(x-y\) plane, around which one should expand are \((x, y) = (0, +\infty), (1, \sqrt{2a}/3), (1, c/2)\) and \((0, 0)\) respectively.

Finally, let us comment on the possibility to have a compact coordinate \(r\) as suggested by the microcanonical analysis of refs. [15, 16, 17]. This would imply that all fields must come back to their initial value after some period in \(r\). We have seen that there are two types of trajectories. On the one hand, we have those for which \(r\) takes values in a semi-infinite interval. These are the trajectories that connect the weak-coupling region \((\phi, \omega) = (-\infty, 0)\) to any other region, as well as those connecting regions 3 or 1 to the critical point. Clearly, these solutions are not, and cannot be made periodic in \(r\). On the other hand, we have the trajectories ending in region 2 for which \(r\) takes values in the finite intervals \([0, r_2]\) with the value of \(r_2\) depending on the chosen trajectory. This is crucially different from what we found in the type-II cases. Note that trajectories in this second class always go from strong to strong coupling. It may well be that these solutions can be continued into periodic functions of \(r\) beyond these intervals, tracing the corresponding trajectories back and forth. The situation is exemplified by the parabola \(y = x^2\) in the \(x-y\) plane with \(x\) a function of \(r\) given by \(x(r) = \ln \frac{1-\cos r}{1+\cos r}\). This clearly is a periodic function in \(r\) and hence \(r\) can be taken as a compact coordinate with values on the circle \([0, 2\pi]\). On the other hand, if we only look at the interval \(r \in [0, \pi]\), \(x(r)\) goes from \(-\infty\) to \(+\infty\) and we trace the parabola just once. This is somewhat similar to our trajectories going into region 2. Without an explicit solution at our disposal, it is probably impossible to decide whether or not we can continue these solutions into periodic ones or not.

6 Physical boundary conditions

Next, we examine the boundary conditions which are physically relevant for the domain wall backgrounds. Note that until now the integration constants of the corresponding first order equations were left undetermined, and so any solution could be useful mathematically, as there is no preference among the different orbits in the parameter space, say \((\phi, \omega)\). We now apply some general criteria for restricting the physical range of the parameters within the context of supergravity using the propagation of a quantum test particle on the corresponding backgrounds. Then, we discuss the criteria that string theory may impose, although the situation is less clear in that case.

Curvature singularities lead to singular classical dynamics for test particles. If they persist at the quantum level, the theory is considered as unphysical. Unfortunately, we do not know enough about string theory in such backgrounds in order to answer this question definitely. Instead, we can ask the simpler question whether the propagation of a quantum test particle is well defined in the presence of a curvature singularity. A singular classical propagation, as indicated by an incomplete geodesic motion, does not necessarily
lead to a singular wave propagation since the space-time can produce an effective barrier that shields the classical singularity. In general terms, a singularity will be physical if the evolution of any state is uniquely defined for all times. The background is then called wave-regular. This criterion is identical to finding a unique self-adjoint extension of the wave-operator \[35, 36, 37\]. This can be quickly investigated by looking at the two solutions of the wave equation locally near the singularity. Note that the wave equation arises, for instance, when one considers linearized graviton fluctuations along the three-dimensional brane embedded into the four-dimensional domain wall ansatz (3.17), i.e., \(\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}\)\(^{10}\) In such cases, in addition to having a unique evolution of initial data, one has to ensure that these fluctuations never become strong, which would invalidate the linearized approximation.

A remark is in order. Since we are at finite temperature we have compact euclidean time and it is not clear right away what is meant by dynamics and evolution. Our coordinates \(r\) and \(x^\mu\) with \(\mu = 1, 2, 3\) are all spatial. To study the “adiabatic” dynamics of a single test particle one should add a test-particle time coordinate with \(g_{tt} = 1\), \(g_{t\mu} = g_{tr} = 0\). Indeed, a single test particle only propagates in the thermal background specified by the metric but does not interact with the thermodynamical system. Thus in the following, we use the five coordinates: test-particle time \(t\) and four space coordinates \(x^i\) split into \(r\) and \(x^\mu\), \(\mu = 1, 2, 3\).

The relevant part of the wave equation is

\[
\frac{d}{dr} \left( \sqrt{-g} g^{rr} \frac{d\psi}{dr} \right) = 0 .
\]

(6.1)

We will soon justify the need to keep only this term in the wave equation for our class of examples, where the metric behaves in general as in (6.4) below. In order to check the normalizability of the two independent solutions of the wave equation, we will use the Sobolev norm

\[
q^2 \int_{\Sigma} \sqrt{-g} g^{tt} \psi^\dagger \psi + \frac{1}{2} \int_{\Sigma} \sqrt{-g} g^{ij} \partial_i \psi^\dagger \partial_j \psi ,
\]

(6.2)

where \(q\) is a constant and the integrals are performed on a constant-time hypersurface \(\Sigma\). The Sobolev norm is bounded from above by the energy of the fluctuation associated with the wave, and guarantees that the back-reaction of the fluctuation is indeed small \[37\]. According to this criterion there exists a unique self-adjoint extension of the wave-operator and a unique evolution of the system, thus rendering the space wave-regular, if only one of the two independent solutions of (6.1) is normalizable near the singularity. This solution is then kept, whereas the other is discarded. We should note that this criterion is non-trivial for time-like singularities. For null singularities the space-time is globally hyperbolic and there should be a unique self-adjoint extension of the wave-operator since the evolution is ordinary Cauchy, and hence unique in this case.

\(^{10}\) This has been shown in the context of the AdS/CFT correspondence and in warped factor compactifications in \[38\] (for an exhaustive exposition see \[39, 40\]). The proof in the present case can also be given along similar lines.
The choice of Sobolev norm is compatible in our cases with an alternative criterion, namely that in geodesically incomplete space-times there should be no leak of conserved quantities through the singularity. For the translational isometries that remain unbroken by our metric ansatz (3.17), this amounts to the condition (see, for instance, [41])

$$\sqrt{-g} g^{rr} \partial_i \psi \partial_r \psi = 0$$  \hspace{1cm} (6.3)

at the singularity. In order to have a unique evolution of the system, only one of the two independent solutions of eq. (6.1) should satisfy (6.3). For the other solution, which is discarded, the right hand side of (6.3) should diverge.

Returning to our metrics, we have seen that they develop a curvature singularity as $r \to 0^+$ or $r \to r_2^-$. We will discuss the case of a singularity at $r = 0^+$, the other case being analogous. Then the asymptotic behaviour of the metric is of the form

$$ds^2 \simeq dr^2 + (\text{const.}) \, r^{2\nu} \eta_{\mu \rho} dx^\mu dx^\rho, \quad \text{as} \quad r \to 0^+,$$  \hspace{1cm} (6.4)

with $\nu$ being a positive constant that can be read from the appropriate expressions in sections 4 and 5. In the case of the strongly coupled solution (5.18) around the critical point $(\phi, \omega) = (0, \omega_0^+)$, we have that $\nu = 1$, and for all other cases $0 < \nu < 1$.

Applying the general formalism for a background behaving as in (6.4), we first see that the two independent solutions of (6.1) are $\psi_1 \sim 1$ and $\psi_2 \sim r^{1-3\nu}$ (or $\psi_2 \sim \ln r$ if $\nu = 1/3$). Note that if we had treated the full wave equation in (5.1), we would have added a term proportional to $r^\nu \psi$ on the left-hand side. Such term perturbs the two independent solutions that we have just mentioned by a term proportional to $r^{2-2\nu}$ and $r^{3-5\nu}$, respectively. Therefore, near the singularity occurring at $r = 0$, this extra term can be neglected if $\nu < 1$, or does not change the power of $r$ if $\nu = 1$.

Applying either one of the above criteria, we find that backgrounds with a single singularity behaving as in (6.4) are wave-regular for $\nu \geq 1/3$. Hence, for the genuine type-II, as well as for the hybrid type-II solutions, the backgrounds are wave-regular when the constants $c$ and $b$, respectively, are negative or zero, while positive $c$ or $b$ are unphysical.

For the heterotic case, we have to distinguish solutions with semi-infinite intervals of $r$ (those not going into region 2) and solutions with finite intervals of $r$ (those going into region 2). For the former there is only one singularity at $r = 0^+$ and we find that all solutions that contain in some limit the region 3 do not satisfy the criteria, and hence are unphysical. The only physical solutions in this class are those corresponding to the trajectories that connect region 1 to the weak coupling region, or the special trajectory that go from the critical point $\omega_0^+$ to region 1 or to the weak coupling region. Note that for all these trajectories $\omega$ remains bounded by $\omega_0^+$. Now consider the solutions that have a finite interval of $r$. They have two singularities, one at $r = 0^+$ and one at $r_2^-$. If at

11This condition also guarantees that the solution $\psi_2 \sim r^{1-3\nu}$ that diverges at the singularity occurring at $r = 0$, is discarded. Hence, if the wave equation that we are examining corresponds to linearized graviton fluctuations, these fluctuations will indeed remain small, thus not invalidating the approximation.
each singularity one of the two solutions to the wave equation is discarded, no solution will be left, as in general the solution discarded at one singularity will not be the same as the one discarded at the other. Such a background can be wave-regular only if one of the singularities discards one solution and the other singularity does not impose any condition, i.e. we need $\nu \geq 1/3$ at one singularity and $\nu < 1/3$ at the other. We have $\nu = 1/3$, $1/7$, $1/7$ and $1$ for regions 1, 2, 3 and the critical point. We conclude that solutions corresponding to trajectories from region 2 to region 1 or to the critical point are wave regular. The solution corresponding to the trajectory from region 2 to region 3 however is not wave-regular. It is evident that a better understanding of the string rather than the particle propagation is desirable on such backgrounds.

From the string thermodynamic point of view it has been argued \cite{15, 16, 17} that the exponential growth of the number of states forces one to work in the microcanonical ensemble, in which case the specific heat may turn negative unless all spatial dimensions are compact; of course some compact dimensions may well be large (see also \cite{43, 44}). Within our domain wall ansatz, compactness or not of the space coordinates $x^\mu$ is not an issue since the solutions do not depend on those directions. However, compactness of the $r$ coordinate is possible only if the solutions $\phi(r), \chi(r)$ and $\omega(r)$ at some $r$ equal their values at some other $r$. As we discussed in the previous section, this is not the case for the type-II solutions. For the heterotic case we have some solutions that are parametrized by a finite interval in $r$. However, without the explicit solution at our disposal, it is not clear whether they can be continued to periodic solutions or not. In any case, before jumping into conclusions about our solutions being physical or unphysical within string theory, one should bear in mind that the compactness criterion was established within the microcanonical treatment of string theory, which assumes that the theory remains weakly coupled everywhere.\textsuperscript{13} It is clear from our solutions that they always contain regions of arbitrarily strong coupling, thus invalidating the assumptions that could sentence some or all of them unphysical.

Another argument in favour of our solutions is their supersymmetry. As shown above, all solutions of the first order equations preserve half of the supersymmetries, by their construction, while the standard high-temperature phase of strings breaks all supersymmetries. Clearly, at this point we cannot say anything further as definite about the relevance and physical properties of our different domain wall solutions within string thermodynamics. Ideally, a string inspired criterion would render physical all or a proper subset of those backgrounds, which have already been named physical using the criteria of supergravity in this section.

\textsuperscript{12}The above criteria have been successfully applied within the AdS/CFT correspondence in many backgrounds related to the Coulomb branch of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ super Yang–Mills theories \cite{12}. In those cases the lack of information on string theory beyond the supergravity approximation is compensated by the information related to gauge theory expectations.

\textsuperscript{13}It is somewhat ironical that the only solutions that perhaps could be continued periodically actually are never weakly coupled.
7 Conclusions and discussion

In this paper we have constructed domain wall solutions of the effective supergravity which describes all possible high-temperature phases of the known $N = 4$ superstrings. We have used a universal thermal potential that contains the $s$, $t$, $u$ moduli and their couplings to the three (lightest) winding modes, which can become tachyonic above the Hagedorn temperature, thus triggering the thermal instabilities. Our solutions contain non-trivial winding fields, which vary with respect to the domain wall variable $r$, and exhibit a common property in that they always extend to regions of strong coupling. We have presented the exact solutions for the type-IIA and type-IIB sectors of the theory and a certain (self-dual) hybrid sector, and investigated the general structure of the solutions for the heterotic sector by extracting their behaviour in the vicinity of the weak and the strong coupling points. From a geometrical point of view we found that none of the type-II solutions has support on a compact spatial dimension, namely the direction parametrized by the variable $r$. For the heterotic case, the same is true for all solutions that contain a weakly coupled region, while solutions that never are weakly coupled have support only on finite intervals in the variable $r$. Whether or not these latter solutions can be continued into periodic functions of $r$ could not be answered without having an explicit solution at our disposal. From a supergravity point of view, for each type of theory, there are subclasses of domain wall backgrounds which are selected by imposing specific boundary conditions that lead to consistent propagation of a quantum test particle.

The main property of the domain wall solutions is their supersymmetry; they all satisfy first order differential equations, which arise as BPS conditions for gravitational backgrounds that preserve 1/2 of the supersymmetries. The supersymmetry of our solutions is the key to understand why, for all of them, tachyonic instabilities never occur, even though the temperature may become arbitrarily high. Of course, we are only considering solutions of an effective supergravity and not of the full string theory, but it may well be that they point to a new finite-temperature phase of superstrings which is supersymmetric and has no thermal instabilities, i.e., no Hagedorn temperature. One may then speculate that, as superstrings are heated up from zero temperature, they prefer to go into this more symmetric phase which is stable due to supersymmetry, and that the “ordinary” high-temperature phase with the Hagedorn instability is never reached.

Let us now address a few points one might object to our solutions. First, one might worry about the status of the domain wall ansatz in theories of gravity coupled to scalar fields and having a potential with run-away directions that account for the usual instabilities at the Hagedorn temperature. Note however, that this is not a priori forbidden by any general considerations alone. Recall that domain wall solutions with infinite tension (corresponding to infinite central charge of the supersymmetry algebra) have already found numerous applications in supersymmetric gauge theories, most notably in supersymmetric quantum chromodynamics with one massless flavor, where one has models with run-away vacua. In fact, these are models with rigid supersymmetry which
admit stable field configurations that restore one half of supersymmetry, as opposed to
the stable but non-supersymmetric ground state, and which are characterized by con-
stant positive energy density \([15]\). The domain wall solutions that we have constructed in
supergravity for the problem of string thermodynamics could be viewed as the gravita-
tional analogue of such supersymmetric configurations, having their own characteristic
properties. Their stability is guaranteed by the saturation of the BPS bound and can be
further supported by analysing the spectrum of the graviton fluctuations on the domain
wall backgrounds. Based on general principles, one can show that the graviton spectrum
is obtained by computing the energy levels of an equivalent non-relativistic problem in
supersymmetric quantum mechanics with a Schrödinger potential that is determined en-
tirely by the conformal factor of the metric. The details are not important as the only
relevant point here is that the spectrum is bounded from below (by zero) by the general
properties of supersymmetric quantum mechanics, thus rendering a physical spectrum for
the graviton fluctuations on the supersymmetric domain wall backgrounds that preserve
three-dimensional Poincare invariance.

Second, one should also study metric fluctuations taking us away from the class of
conformally flat metrics of the domain wall backgrounds. We have not investigated this
issue, but we expect that an analysis along the line that can e.g. be found in \([39, 40]\) is
possible.

Third, a well-known instability of hot flat space was analysed in \([16]\). In this simplest
example it was found that the classical Jeans instabilities arise as a tachyon in the graviton
propagator, using small fluctuations about hot flat space, and it was further suggested
that hot flat space will nucleate black holes. Inevitably, this is the fate of gravitational
systems due to the attractive nature of gravitation. In the case of a relativistic medium,
the Jeans instability implies that a thermal ensemble of sufficiently large volume will
collapse into a black hole. Of course, it remains to be seen how these results generalize
to domain wall backgrounds of our effective supergravity by performing calculations as
in ordinary quantum gravity. One might fear that most of our solutions will be afflicted
by Jeans instabilities because they do not have support in small volumes. However,
their defining BPS property ensures their stability quantum mechanically as well. Put
differently, the BPS equation which is interpreted as a “no force” condition, although
this interpretation is more appropriate for asymptotically flat backgrounds with a definite
Newtonian limit, ensures that no gravitational collapse will occur.

Clearly, spherically symmetric black hole type solutions of the effective supergravity
have to be analysed in the future before addressing the long standing problems of quan-
tum gravity within string thermodynamics. These general remarks put in perspective
future attempts to construct other non-trivial solutions of the effective supergravity for
the \(s, t, u\) moduli and the string winding modes beyond the supersymmetric domain wall
ansatz. An important lesson that we already learned, even from the simplest solutions
we have constructed here, is that they cannot be entirely confined in regions of weak
coupling only.
It would be interesting to study the most general domain wall solution of the effective supergravity that describes all possible phases of $N = 4$ superstrings at finite temperature and includes all six scalar fields. The resulting system of first order non-linear differential equations does not seem tractable by analytical methods alone, but a combination of numerical analysis and analytical calculations around certain points may well provide a reasonable global picture, similar to what we did in the heterotic case. In the appendix, below, we summarize for completeness some general remarks about the search for periodic orbits in order to appreciate the difficulty to establish analytic criteria for their existence in the general case.

Finally, it will also be interesting to study the higher dimensional interpretation of our domain wall configurations, as in other theories of gauged supergravity. This might be related to thermodynamics of string theory in D-brane backgrounds as has already been discussed in the weak-coupling limit in [48, 49]. One may also try to extend the formalism to construct domain wall junctions.

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A Remarks on periodic orbits

Although we have seen quite explicitly that all our type-II solutions are non-periodic in the domain wall coordinate $r$, and the same probably is true also for the heterotic solutions, it is much less clear whether this will still be true for the solutions of the full set of six equations \((3.19) - (3.24)\). Since for the latter one probably has to rely heavily on numerical methods, it would be nice to have at least some analytical tools at one’s disposal. One possibility is to try to establish the existence or non-existence of periodic orbits in the solution space of our non-linear system of first order differential equations for the domain wall configurations. Clearly if there are such periodic orbits in the solution space (e.g. in the $\phi - \omega$ plane in the heterotic case) then the corresponding solution will also be periodic in $r$. Note however that the converse is not necessarily true as would be exemplified by the heterotic trajectories going into region 2, should it turn out that they can be periodically continued in $r$. Of course, then one might still consider them as periodic orbits in the $\phi - \omega$ plane, although degenerate ones. Finally, for reasons already discussed at length in the introduction and discussion section, it is not clear whether periodicity may serve as a criterion within string thermodynamics to distinguish physical from unphysical solutions.

It is well known from the theory of dynamical systems that the existence of periodic solutions, i.e., closed loops in the two-dimensional parameter space $(\phi, \omega)$, of a first order system $\dot{\phi} = P(\phi, \omega)$ and $\dot{\omega} = Q(\phi, \omega)$ is a very important and delicate question that often can be established only numerically (unless the general solution is known in closed form). The boundary conditions select specific orbits $\phi(\omega)$ by fixing the integration constants, and so by singling out the periodic orbits, if there are any among the solutions, corresponds to specifying some physical boundary conditions.

To appreciate the difficulty in establishing the existence of periodic orbits analytically, we recall briefly some basic elements of Poincaré’s theory (see, for instance, \[17\] for an elementary account). There, one has the notion of a limit cycle, i.e. a stable closed curve in the parameter space, independent of initial conditions, towards which solutions tend in an asymptotic sense, or from which they unwind, as it were, as $\zeta \to \pm \infty$. A well known theorem states that if a limit cycle exists in a given system, then the existence of periodic orbits is guaranteed, even though their explicit construction could be a difficult task. On the other hand we have Bendixson’s theorem stating that if one considers the function

$$J = \frac{\partial P}{\partial \phi} + \frac{\partial Q}{\partial \omega} \quad (A.1)$$

in any given domain in $(\phi, \omega)$ which is bounded by a simple curve $C$, the system will have no limit cycles inside that region if $J$ has constant sign in it. Therefore, all we can learn on general grounds is for which regions there are no limit cycles; if $J$ changes sign in a bounded region (and hence it is bound to vanish somewhere in it), the system will not necessarily have limit cycles in it, but there is a good chance that it will. Furthermore, there are systems with periodic orbits that have no limit cycles.

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According to these results, we can already see a difference between the type-II and heterotic sectors concerning the possibility to have (or not to have) limit cycles in their parameter space. Computing Bendixson’s function for these two sectors (using the coordinate $\zeta$ that decouples the $\chi$-dependence from the flows of the fields $\phi$ and $\omega$) one finds the result

\[
J_{\text{II}} = -\frac{1}{2\sqrt{2}} \left( 2e^{\sqrt{2}\phi} + e^{-\sqrt{2}\phi}(1 + 3\sinh^2 \omega) \right),
\]

\[
J_{\text{hyb}} = -\frac{1}{2\sqrt{2}} \left( 2e^{\sqrt{2}\phi} + 1 + 2\sinh^2 \omega \right),
\]

\[
J_{\text{het}} = -\frac{1}{\sqrt{2}} \left( \cosh(\sqrt{2}\phi) - 3\sinh(\sqrt{2}\phi)\sinh^2 \omega \right).
\]

It is obvious that $J_{\text{II}}$ and $J_{\text{hyb}}$ are strictly negative everywhere in the parameter space $(\phi, \omega)$, and so according to Bendixson’s theorem we learn that there are no limit cycles in these cases. Of course, this by itself does not rule out the existence of periodic orbits, but we already know from our explicit solutions that for the pure type-II and hybrid type-II sectors there are none. On the other hand, the heterotic sector could support limit cycles, since $J_{\text{het}}$ vanishes along the curve

\[
\coth(\sqrt{2}\phi) = 3\sinh^2 \omega.
\]

This curve has two branches, one in the upper $(\phi, \omega)$ half-plane and the other in the lower, which are related to each other by the discrete symmetry $\omega \rightarrow -\omega$. Drawing the curve in the upper half-plane we see that it starts asymptotically from $\omega = +\infty$ at $\phi = 0$ and drops monotonically to the value $\omega = \ln \sqrt{3} < \omega^+_0$, which is approached asymptotically as $\phi \rightarrow +\infty$. Therefore, if one considers bounded domains in the parameter space which intersect with Bendixson’s curve $J_{\text{het}} = 0$, they will potentially contain limit cycles, although their existence is not at all guaranteed. Note that this curve is crossed precisely by the trajectories going to region 2. Of course, these trajectories are not contained in any bounded domain. Nevertheless, this is still suggestive that maybe these trajectories can indeed be periodically continued.

It will be interesting to look for closed trajectories in the higher dimensional parameter space with all the relevant fields turned on.
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