A study on a class of generalized Schrödinger operators *

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Abstract: In this paper, we consider the pointwise convergence for a class of generalized Schrödinger operators with suitable perturbations, and convergence rate for a class of generalized Schrödinger operators with polynomial growth. We show that the pointwise convergence results remain valid for a class of generalized Schrödinger operators under small perturbations. As applications, we obtain the sharp convergence result for Boussinesq operator and Beam operator in $\mathbb{R}^2$. Moreover, the convergence result for a class of non-elliptic Schrödinger operators with finite-type perturbations is built. Furthermore, we proved that the convergence rate for a class of generalized Schrödinger operators with polynomial growth depends only on the growth condition of their phase functions. This result can be applied to all previously mentioned operators, and more operators.

Keywords: Schrödinger operator; Convergence; Polynomial growth; Perturbation.

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1 Introduction

Consider the generalized Schrödinger equation

\[
\begin{aligned}
\partial_t u(x, t) - iP(D)u(x, t) &= 0 \quad x \in \mathbb{R}^n, t \in \mathbb{R}^+, \\
u(x, 0) &= f
\end{aligned}
\]  

(1.1)

where $D = \frac{1}{i}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$, $P(\xi)$ is a real continuous function defined on $\mathbb{R}^n$, $P(D)$ is defined via its real symbol

\[
P(D)f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} P(\xi) \hat{f}(\xi) d\xi.
\]

The solution of (1.1) can be formally written as

\[
e^{itP(D)}f(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi + itP(\xi)} \hat{f}(\xi) d\xi,
\]

(1.2)

where $\hat{f}(\xi)$ denotes the Fourier transform of $f$.

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The convergence problem, that is, to determine the optimal $s$ for which
\[
\lim_{t \to 0^+} e^{itP(D)} f(x) = f(x)
\]
almost everywhere whenever $f \in H^s(\mathbb{R}^n)$, has been widely studied since the first work by Carleson ([5]), see [11], [16], [14], [15], [12] and references therein. Sharp results were derived in some cases, such as the elliptic case ([7, 8], when $n \geq 1$, $P(\xi) = |\xi|^2$); the non-elliptic case ([10], when $n \geq 1$, $P(\xi) = \xi_1^2 - \xi_2^2 \pm \cdots \pm \xi_n^2$) and the fractional case ([6], when $n \geq 1$ and $P(\xi) = |\xi|^\alpha$, $\alpha > 1$).

In this paper, we firstly consider the convergence problem for a class of generalized Schrödinger operators with small perturbations. We first establish the following general results.

**Theorem 1.1.** If there exist a real continuous function $Q(\xi)$ and a real number $s_0 > 0$ such that
\[
|P(\xi) - Q(\xi)| \lesssim 1, |\xi| \to +\infty,
\]
and for any $s > s_0$,
\[
\left\| \sup_{0 < t < 1} |e^{itQ(D)} f| \right\|_{L^p(B(0,1))} \lesssim \|f\|_{H^s(\mathbb{R}^n)}, \quad p \geq 1,
\]
then for all $s > s_0$ and $f \in H^s(\mathbb{R}^n)$,
\[
\left\| \sup_{0 < t < 1} |e^{itP(D)} f| \right\|_{L^p(B(0,1))} \lesssim \|f\|_{H^s(\mathbb{R}^n)}, \quad p \geq 1.
\]

Theorem 1.1 implies the Equivalence between the convergence property of operators with small perturbations. Theorem 1.1 is quite general and can be applied to a wide class of operators. In particular, we concentrate ourselves on $n = 2$, and consider the Boussinesq operator defined by
\[
P_B(\xi) = |\xi|\sqrt{1 + |\xi|^2},
\]
and obtain the following almost sharp result:

**Theorem 1.2.** (1) For each $s > 1/3$, if $f \in H^s(\mathbb{R}^2)$, then
\[
\left\| \sup_{0 < t < 1} |e^{itP_B(D)} f| \right\|_{L^3(B(0,1))} \lesssim \|f\|_{H^s(\mathbb{R}^2)}.
\]

(2) For each $s < 1/3$, there exists $f \in L^2(\mathbb{R}^n)$ and $\hat{f}$ supported in the annulus $\{\xi \in \mathbb{R}^2 : |\xi| \sim R\}$, such that
\[
\lim_{R \to +\infty} \frac{R^{-s}\|\sup_{0 < t < 1} |e^{itP_B(D)} f|\|_{L^1(B(0,1))}}{\|\hat{f}\|_{L^2}} = +\infty.
\]

By the same method, we can prove that the results in Theorem 1.2 also hold for operators such as the Beam operator $P(\xi) = \sqrt{1 + |\xi|^2}$. But we omit its proof here.

Recently, Buschenhenke, Müller and Vargas [2, 3] studied Fourier restriction estimate for finite-type perturbations of the hyperbolic paraboloid. We are also curious about how "finite-type perturbations"
works in the corresponding generalized Schrödinger equation. Next, we concentrate ourselves on \( n = 2, m \geq 1 \). Consider a class of operators with phase function

\[
P_m(\xi) = \xi_1 \xi_2 + h_m(\xi_1),
\]

where \( h_m(\xi_1) = \frac{1}{m} \xi_1^m \) when \( m \in \mathbb{N}^+ \). In this case, the corresponding equations are higher order dispersive equations, see [9] and its references for more information. When \( 1 < m < 2 \), \( h_m(\xi_1) = \frac{1}{m} |\xi_1|^m \), the corresponding equations are non-elliptic Schrödinger equations with fractional order perturbations. We obtained the following result.

**Theorem 1.3.** (1) For each \( m \in \mathbb{N}^+ \), \( s > 1/2 \), if \( f \in H^s(\mathbb{R}^2) \), then

\[
\left\| \sup_{0 < t < 1} |e^{itP_m(D)} f| \right\|_{L^2(B(0,1))} \lesssim \|f\|_{H^s(\mathbb{R}^2)}.
\]  

(2) The similarly convergence results hold for \( 1 < m < 2 \) and \( s > \frac{1}{2} \). In particular, for \( s < \frac{1}{2} \), there exists \( f \in L^2(\mathbb{R}^n) \) and \( \hat{f} \) supported in the annulus \( \{ \xi \in \mathbb{R}^2 : |\xi| \sim R \} \), such that

\[
\lim_{R \to +\infty} \sup_{0 < t < 1} \|e^{itP_m(D)} f\|_{L^2(B(0,1))} = +\infty.
\]

By [13], \( s > \frac{1}{2} \) is likely sharp for the convergence result to hold in the non-elliptic case up to the end point. Theorem 1.3 implies that the "finite-type perturbations" does not change the convergence result for \( s > \frac{1}{2} \). Moreover, for \( 1 < m < 2 \), our convergence result is sharp up to the end point.

Furthermore, it is interesting to seek the convergence speed of \( e^{itP(D)} f(x) \) as \( t \) tends to 0 if \( f \) has more regularity. The problem is, suppose that \( e^{itP(D)} f(x) \) converge to \( f \) for \( f \in H^s(\mathbb{R}^n) \) as \( t \) tends to 0, whether or not it is possible that, for \( f \in H^{s+\delta}(\mathbb{R}^n), \delta \geq 0 \),

\[
e^{itP(D)} f(x) - f(x) = o(t^{\theta(\delta)})
\]

almost everywhere for some \( \theta(\delta) \geq 0 \)? Cao, Fan and Wang [4] proved this property in the elliptic case when \( n \geq 1, P(\xi) = |\xi|^2, \theta(\delta) = \frac{2n}{\alpha}, 0 \leq \delta < 2, \) and in the fractional case when \( n = 1, P(\xi) = |\xi|^\alpha, \alpha > 1, \theta(\delta) = \frac{\delta}{\alpha}, 0 \leq \delta < \alpha. \)

In this paper, we obtain the convergence rate for a class of Schrödinger operators with polynomial growth:

**Theorem 1.4.** If there exist \( m > 0, s_0 > 0 \) such that

\[
|P(\xi)| \lesssim |\xi|^m, |\xi| \to +\infty,
\]

and for each \( s > s_0 \),

\[
\left\| \sup_{0 < t < 1} |e^{itP(D)} f| \right\|_{L^p(B(0,1))} \lesssim \|f\|_{H^s(\mathbb{R}^n)}, p \geq 1,
\]

then for all \( f \in H^{s+\delta}(\mathbb{R}^n), 0 \leq \delta < m \),

\[
e^{itP(D)} f(x) - f(x) = o(t^{\delta/m}), \ a.e. \ as \ t \to 0^+.
\]
Note that the convergence rate in Theorem 1.4 depends on the growth condition of the phase function, but independent of its gradient and the dimension of the spatial space. Theorem 1.4 is quite general and can be applied to a wide class of operators, such as the non-elliptic Schrödinger operators \( P(\xi) = \xi_1^2 - \xi_2^2 \pm \cdots \pm \xi_d^2 \), the fractional Schrödinger operators \( P(\xi) = |\xi|^\alpha, \alpha > 1 \) and the Boussinesq operator \( P(\xi) = |\xi|\sqrt{1 + |\xi|^2} \). It also generalized the previous result of [4].

2 Proof of Theorem 1.1

Proof of Theorem 1.1. In order to show (1.6), we decompose \( f \) as

\[
f = \sum_{k=0}^{\infty} f_k,
\]

where \( \text{supp} f_0 \subseteq B(0,1) \), \( \text{supp} f_k \subseteq \{ \xi : |\xi| \sim 2^k \} \), \( k \geq 1 \). Then we have

\[
\left\| \sup_{0 < t < 1} |e^{itP(D)} f| \right\|_{L^p(B(0,1))} \leq \sum_{k=0}^{\infty} \left\| \sup_{0 < t < 1} |e^{itP(D)} f_k| \right\|_{L^p(B(0,1))}.
\]

(2.1)

For \( k \lesssim 1 \), since for each \( x \in B(0,1) \),

\[
\left| e^{itP(D)} f_k(x) \right| \lesssim \| f_k \|_{L^2(\mathbb{R}^n)},
\]

it is obvious that

\[
\left\| \sup_{0 < t < 1} |e^{itP(D)} f_k| \right\|_{L^p(B(0,1))} \lesssim \| f \|_{H^s(\mathbb{R}^n)}.
\]

(2.2)

For \( k \gg 1 \), by Taylor’s formula, for each \( k \),

\[
\left| e^{itP(D)} f_k - e^{itQ(D)} f_k \right| \leq \sum_{j=1}^{\infty} \frac{t^j}{j!} \left| \int_{\mathbb{R}^n} e^{ix\xi + tQ(\xi)} [P(\xi) - Q(\xi)]^j \hat{f}_k(\xi) d\xi \right|.
\]

(2.3)

It is obvious that

\[
\left\| \sup_{0 < t < 1} |e^{itP(D)} f_k| \right\|_{L^p(B(0,1))} \leq \left\| \sup_{0 < t < 1} |e^{itP(D)} f_k - e^{itQ(D)} f_k| \right\|_{L^p(B(0,1))} + \left\| \sup_{0 < t < 1} |e^{itQ(D)} f_k| \right\|_{L^p(B(0,1))}.
\]

(2.4)

For \( \forall \epsilon > 0 \), from (1.5), for each \( g \) whose Fourier transform is supported in \( \{ \xi : |\xi| \sim 2^k \} \), we have

\[
\left\| \sup_{0 < t < 1} |e^{itQ(D)} g| \right\|_{L^p(B(0,1))} \lesssim 2^{(s_0 + \frac{\epsilon}{2})k} \| g \|_{L^2(\mathbb{R}^n)}.
\]

(2.5)

Let \( s_1 = s_0 + \epsilon \), then

\[
\left\| \sup_{0 < t < 1} |e^{itQ(D)} f_k| \right\|_{L^p(B(0,1))} \lesssim 2^{s_1 k} \| f \|_{H^{s_1}(\mathbb{R}^n)}.
\]

(2.6)
Inequalities (2.3), (2.5) and (1.4) imply
\[
\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f_k) - e^{itQ(D)}(f_k) \right| \right\|_{L^p(B(0,1))} \\
\leq \sum_{j=1}^\infty \frac{1}{j!} \left\| \sup_{0 < t < 1} \left| \int_{\mathbb{R}^n} e^{ix \cdot \xi + itQ(\xi)} [P(\xi) - Q(\xi)] f_k(\xi) d\xi \right| \right\|_{L^p(B(0,1))} \\
\leq \sum_{j=1}^\infty \frac{2^{(k_0 + \frac{1}{2})}}{j!} \left\| [P(\xi) - Q(\xi)] f_k(\xi) \right\|_{L^2(\mathbb{R}^n)} \\
\leq \sum_{j=1}^\infty \frac{C j^2 - k \epsilon}{j!} \left\| f \right\|_{H^{s_1}(\mathbb{R}^n)} \\
\lesssim 2^{-\frac{k}{4}} \left\| f \right\|_{H^{s_1}(\mathbb{R}^n)}. \tag{2.7}
\]

Inequalities (2.4), (2.6) and (2.7) yield for \( k \gg 1 \),
\[
\left\| \sup_{0 < t < 1} \left| e^{it\Delta f}(x) \right| \right\|_{L^3(B(0,1))} \lesssim 2^{-\frac{k}{4}} \left\| f \right\|_{H^{s_1}(\mathbb{R}^n)}. \tag{2.8}
\]

Combing (2.1), (2.2) and (2.8), inequality (1.6) holds true for \( s > s_0 \). By the arbitrariness of \( \epsilon \), in fact, we can get for any \( s > s_0 \), inequality (1.6) remains true.

3 Proof of Theorem 1.2

Proof of Theorem 1.2. (1) Inequality (1.7) follows directly form Theorem 1.1 and the following convergence result for Schrödinger operator ([7]).

\textbf{Theorem 3.1.} ([7]) For any \( s > 1/3 \), the following bounds hold: for any function \( \hat{f} \in H^s(\mathbb{R}^2) \),
\[
\left\| \sup_{0 < t < 1} \left| e^{it\Delta f}(x) \right| \right\|_{L^3(B(0,1))} \leq C_s \left\| f \right\|_{H^s}.
\]

(2) In [1], Bourgain actually showed that there exists \( f \),
\[
\hat{f}(\xi) = \chi_{A_R}(\xi),
\]
where \( A_R \) is the subset of \( \{ \xi \in \mathbb{R}^2 : |\xi| \sim R \} \) defined by
\[
A_R = \bigcup_{l \in \mathbb{N}^+ \cap R^{-1/3}} A_{R,l},
\]
\[
A_{R,l} = [R - R^{1/2}, R + R^{1/2}] \times [R^{2/3} l, R^{2/3} l + 1].
\]

And there exists a set \( S \) with positive measure such that for each \( x \in S \), there exists \( t, |l| \leq R^{-1} \),
\[
|e^{it\Delta f}(x)| \geq R^{3/4}. \tag{3.1}
\]

Hence,
\[
\sup_{0 < t < R^{-1}} |e^{it\Delta f}(x)| \geq R^{3/4}. \tag{3.2}
\]
By Taylor expansion,
\[ \sup_{0 < t < R^{-1}} |e^{it\Delta} f(x)| \leq \sup_{0 < t < R^{-1}} |e^{it\Delta} f(x) - e^{itP_\eta} f(x)| + \sup_{0 < t < R^{-1}} |e^{itP_\eta} f(x)| \]
\[ \leq \sum_{j=1}^{\infty} \frac{R^{-j}}{j!} \int_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi + \sup_{0 < t < 1} |e^{itP_\eta} f(x)| \]
\[ \leq R^{-1} R^{1/3} R^{1/2} + \sup_{0 < t < 1} |e^{itP_\eta} f(x)|. \] (3.3)

Inequalities (3.2) and (3.3) imply
\[ \| \sup_{0 < t < 1} |e^{itP_\eta} f| \|_{L^1(B(0,1))} \gtrsim R^{3/4}, \] (3.4)
which implies (1.8).

4 Proof of Theorem 1.3

We first prove the following Lemma 4.1.

Lemma 4.1. Assume that \( g \) is a Schwartz function whose Fourier transform is supported away from 0.

Then
\[ \left\| \sup_{0 < t < 1} |e^{itP_m(D)} g| \right\|_{L^2(B(0,1))} \leq \|g\|_{L^2(\mathbb{R}^2)} + \left( \int \frac{|P_m(\xi)|^2}{|\nabla P_m(\xi)|} |\hat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left( \int \frac{1}{|\nabla P_m(\xi)|} |\hat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \] (4.1)

Proof. For each \( x \in B(0,1) \),
\[ \sup_{0 < t < 1} |e^{itP_m(D)} g(x)|^2 \leq |g(x)|^2 + \left( \int_0^1 \int_{\mathbb{R}^2} e^{ix \cdot \xi + itP_m(\xi)} \hat{g}(\xi) d\xi dtdt \right)^{\frac{1}{2}} \]
\[ \times \left( \int_0^1 \int_{\mathbb{R}^2} e^{ix \cdot \xi + itP_m(\xi)} P_m(\xi) \hat{g}(\xi) d\xi dtdt \right)^{\frac{1}{2}}. \] (4.2)

By Hölder’s inequality,
\[ \left\| \sup_{0 < t < 1} |e^{itP_m(D)} g| \right\|_{L^2(B(0,1))} \leq \|g\|_{L^2(\mathbb{R}^2)} + \left( \int_{B(0,1)} \int_{\mathbb{R}^2} |e^{ix \cdot \xi + tP_m(\xi)} \hat{g}(\xi)| d\xi dtdx \right)^{\frac{1}{2}} \]
\[ \times \left( \int_{B(0,1)} \int_{\mathbb{R}^2} e^{ix \cdot \xi + tP_m(\xi)} P_m(\xi) |\hat{g}(\xi)|^2 d\xi dtdx \right)^{\frac{1}{2}}. \] (4.3)

By Theorem 4.1 in [10],
\[ \int_{B(0,1)} \int_0^1 \int_{\mathbb{R}^2} e^{ix_1 \cdot \xi_1 + ix_2 \cdot \xi_2 + tP_m(\xi_1, \xi_2)} \hat{g}(\xi_1, \xi_2) d\xi_1 d\xi_2 dtdx \]
\[ \leq \int_{B(0,1)} \int_0^1 \int_{\mathbb{R}^2} e^{i \eta_1 \cdot \xi_1 + i \eta_2 \cdot \xi_2 + tP_m(\eta_1, \eta_2)} \hat{g}(\eta_1 + \eta_2, \eta_1 - \eta_2) d\eta_1 d\eta_2 dtdy \]
\[ \leq \int_{B(0,2)} \int_0^1 \int_{\mathbb{R}^2} e^{i \eta_1 \cdot \xi_1 + i \eta_2 \cdot \xi_2 + tP_m(\eta_1, \eta_2)} \hat{g}(\eta_1 + \eta_2, \eta_1 - \eta_2) d\eta_1 d\eta_2 dtdy \]
\[ \leq \int_{B(0,1)} \int_0^1 \int_{\mathbb{R}^2} \frac{|\hat{g}(\eta_1, \eta_2)|^2}{|\nabla P_m(\eta_1, \eta_2)|} d\eta_1 d\eta_2 dtdy \]
\[ \leq \int_{\mathbb{R}^2} \frac{|\hat{g}(\xi_1, \xi_2)|^2}{|\nabla P_m(\xi_1, \xi_2)|} d\xi_1 d\xi_2. \] (4.4)
For the same reason,

\[ \int_{B(0,1)} \int_0^1 \int_{\mathbb{R}^2} e^{-it\xi_1 + it\xi_2 + itP_m(\xi_1, \xi_2)} P_m(\xi_1, \xi_2) \tilde{g}(\xi_1, \xi_2) d\xi_1 d\xi_2 dt dx \leq \int_{\mathbb{R}^2} |P_m(\xi_1, \xi_2)|^2 |\tilde{g}(\xi_1, \xi_2)|^2 |\nabla P_m(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2. \]  

(4.6)

Inequality (4.1) follows from (4.4), (4.5) and (4.6).

\[ \square \]

\textbf{Proof of Theorem 1.3.} (1) We decompose \( f \) as

\[ f = \sum_{k=0}^{\infty} f_k, \]

where \( \text{supp} f_0 \subset B(0,1) \), \( \text{supp} f_k \subset \{ \xi : |\xi| \sim 2^k \}, k \geq 1 \). Then we have

\[ \left\| \sup_{0 < t < 1} \left| e^{itP_m(D)} f \right| \right\|_{L^2(B(0,1))} \leq \sum_{k=0}^{\infty} \left\| \sup_{0 < t < 1} \left| e^{itP_m(D)} f_k \right| \right\|_{L^2(B(0,1))}. \]

(4.7)

For \( k \leq 1 \), since for each \( x \in B(0,1) \),

\[ \left| e^{itP_m(D)} f_k(x) \right| \lesssim \| f_k \|_{L^2(\mathbb{R}^2)}, \]

it is obvious that

\[ \left\| \sup_{0 < t < 1} \left| e^{itP_m(D)} f_k \right| \right\|_{L^2(B(0,1))} \lesssim \| f \|_{H^s(\mathbb{R}^2)}. \]

(4.8)

For \( k \gg 1 \), we decompose each \( f_k \) as

\[ f_k = \sum_{j=1}^{3} f_{k,j}, \]

where \( \text{supp} \tilde{f}_{k,j} \subset A_{k,j}, j = 1, 2, 3, \)

\[ A_{k,1} = \{ \xi : |\xi| \sim 2^k, |\xi_2| \gg |\xi_1|^m \}, \]

\[ A_{k,2} = \{ \xi : |\xi| \sim 2^k, |\xi_2| \sim |\xi_1|^m \}, \]

\[ A_{k,3} = \{ \xi : |\xi| \sim 2^k, |\xi_2| \ll |\xi_1|^m \}, \]

then

\[ \left\| \sup_{0 < t < 1} \left| e^{itP_m(D)} f_k \right| \right\|_{L^2(B(0,1))} \leq \sum_{j=1}^{3} \left\| \sup_{0 < t < 1} \left| e^{itP_m(D)} f_{k,j} \right| \right\|_{L^2(B(0,1))}. \]

(4.9)

By Lemma 4.1,

\[ \left\| \sup_{0 < t < 1} \left| e^{itP_m(D)} f_{k,1} \right| \right\|_{L^2(B(0,1))} \]

\[ \leq \| f, k \|_{L^2(\mathbb{R}^2)} + \left( \int \frac{|\xi_1 \xi_2 + \frac{1}{m} |\xi_1|^2}{|\xi_2 + |\xi_1|^m| + |\xi_1|} |\tilde{f}_{k,1}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^{\frac{1}{2}} \]

\[ \times \left( \int \frac{1}{|\xi_2 + |\xi_1|^m| + |\xi_1|} |\tilde{f}_{k,1}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^{\frac{1}{2}} \]

\[ \lesssim \min\{2^{\frac{k}{2}}, 2^{m-\frac{k}{2}}\} \| f, k \|_{L^2(\mathbb{R}^2)} \]

\[ \leq 2^{(-s+\frac{1}{2})k}\| f \|_{H^s(\mathbb{R}^2)}. \]

(4.10)
Analogously,
\[
\left\| \sup_{0 < t < 1} |e^{itP_m(D)} f_{k,3}| \right\|_{L^2(B(0,1))} \\
\leq \|f_{k,3}\|_{L^2(\mathbb{R}^2)} + \left( \int \frac{|\xi_1 \xi_2 + \frac{1}{m} \xi_m|^2}{|\xi_2 + \xi_1^{|m-1}| + |\xi_1|} |\widehat{f}_{k,3}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^\frac{1}{2} \\
\times \left( \int \frac{1}{|\xi_2 + \xi_1^{|m-1}| + |\xi_1|} |\widehat{f}_{k,3}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^\frac{1}{2} \\
\lesssim 2^{\frac{k}{2}} \|f_{k,3}\|_{L^2(\mathbb{R}^2)} \\
\leq 2^{l \frac{s}{2} + \frac{k}{4}} \|f\|_{H^s(\mathbb{R}^2)}. 
\]
(4.11)

In order to deal with $f_{k,3}$, we further decompose $A_{k,3} = \bigcup_{l=1}^k A_{k,3}^l$, where
\[
A_{k,3}^l = \{ \xi : |\xi| \sim 2^k, 2^{l-1} \leq |\xi_1| < 2^l, |\xi_2| \ll |\xi_1|^{m-1} \}, 
\]
(4.12)
and
\[
f_{k,3} = \sum_{l=1}^k f_{k,3}^l,
\]
such that $\text{supp} \, \widehat{f}_{k,3}^l \subset A_{k,3}^l$, $1 \leq l \leq k$. Then for each $f_{k,3}^l$,
\[
\left\| \sup_{0 < t < 1} |e^{itP_m(D)} f_{k,3}^l| \right\|_{L^2(B(0,1))} \\
\leq \|f_{k,3}^l\|_{L^2(\mathbb{R}^2)} + \left( \int \frac{|\xi_1 \xi_2 + \frac{1}{m} \xi_m|^2}{|\xi_2 + \xi_1^{|m-1}| + |\xi_1|} |\widehat{f}_{k,3}^l(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^\frac{1}{2} \\
\times \left( \int \frac{1}{|\xi_2 + \xi_1^{|m-1}| + |\xi_1|} |\widehat{f}_{k,3}^l(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^\frac{1}{2} \\
\lesssim 2^{\frac{k}{2}} \|f_{k,3}^l\|_{L^2(\mathbb{R}^2)}. 
\]
(4.13)

Due to (4.12) and (4.13), we have
\[
\left\| \sup_{0 < t < 1} |e^{itP_m(D)} f_{k,3}| \right\|_{L^2(B(0,1))} \leq \sum_{l=1}^k \left\| \sup_{0 < t < 1} |e^{itP_m(D)} f_{k,3}^l| \right\|_{L^2(B(0,1))} \\
\lesssim \sum_{l=1}^k 2^{\frac{k}{2}} \|f_{k,3}^l\|_{L^2(\mathbb{R}^2)} \\
\lesssim k 2^{\frac{k}{2}} \|f_{k,3}\|_{L^2(\mathbb{R}^2)} \\
\leq 2^{l \frac{s}{2} + \frac{k}{4}} k \|f\|_{H^s(\mathbb{R}^2)}. 
\]
(4.14)

Inequalities (4.10), (4.11) and (4.14) imply when $k \gg 1$,
\[
\left\| \sup_{0 < t < 1} |e^{itP_m(D)} f_k| \right\|_{L^2(B(0,1))} \lesssim 2^{l \frac{s}{2} + \frac{k}{4}} k \|f\|_{H^s(\mathbb{R}^2)}, 
\]
(4.15)
and then (1.9) follows.
(2) We can use the similar argument to give the proof of the positive result. Next we just show the counterexample for $1 < m < 2, s < \frac{1}{2}$.

Define the subset of $\{\xi \in \mathbb{R}^2 : |\xi| \sim R\}$ by

$$A_R = [R, R + 1] \times [R, \frac{3R}{2}]$$

and define the function $f$ by

$$\hat{f}(\xi) = \chi_{A_R}(\xi).$$

It is obvious that

$$\|f\|_{L^2(\mathbb{R}^2)} = R^{1/2}. \quad (4.16)$$

By Taylor expansion, for each $\eta_1 \in [0, 1]$, 

$$\frac{1}{m} |\eta_1 + R|^m = \frac{1}{m} R^m + R^{m-1} \eta_1 + \frac{m-1}{2} |\theta \eta_1 + R|^{m-2} \eta_1^2, \quad \theta \in [0, 1] \text{ depends on } \eta_1.$$ 

Hence by scaling and translating, we have 

$$|e^{itP_m(D)} f(x)| = \left| \frac{2\pi}{R} \int_{-R}^{R+1} \int_{-R}^{R} e^{ix_1 \xi_1 + ix_2 \xi_2 + it(\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2 \right|$$

$$= R^2 \left| \int_{0}^{1} \int_{0}^{1} e^{ix_1 + R(\eta_1 + \frac{1}{2} |\eta_1 + R|^m)} d\eta_1 d\eta_2 \right|$$

$$= R^2 \left| \int_{0}^{1} \int_{0}^{1} e^{ix_1 + R(\eta_1 + \frac{1}{2} |\eta_1 + R|^m - 1) \eta_1 + \frac{1}{2} (R\eta_1 + \frac{1}{2} |\eta_1 + R|^m) + (m-1) |\theta \eta_1 + R|^{m-2} \eta_1^2} d\eta_1 d\eta_2 \right|. \quad (4.17)$$

Therefore, if $(x_1, x_2) \in [-1/1000, 1/1000] \times [-1/2000, -1/1000], t = -x_2/R + 1/R^2$ and $R$ is sufficiently large, then the abstract value of the phase function 

$$|(x_1 + R t + R^{m-1} t) \eta_1 + \frac{R}{2} (x_2 + R t) \eta_2 + \frac{1}{2} (R \eta_1 + \frac{1}{2} |\eta_1 + R|^m) + (m-1) |\theta \eta_1 + R|^{m-2} \eta_1^2| \lesssim \frac{1}{1000},$$

it follows that if $(x_1, x_2) \in [-1/1000, 1/1000] \times [-1/2000, -1/1000], t = -x_2/R + 1/R^2$ and $R$ is sufficiently large

$$|e^{itP_m(D)} f(x)| \gtrsim R.$$ 

Hence

$$\left\| \sup_{0 < t < 1} |e^{itP_m(D)} f| \right\|_{L^1(B(0, 1))} \gtrsim R. \quad (4.18)$$

Inequalities (4.16) and (4.18) imply (1.10).

## 5 Proof of Theorem 1.4

**Proof of Theorem 1.4.** It is sufficient to show that for some $q \geq 1$ and $\forall \epsilon > 0, \forall x_0 \in \mathbb{R}^2, s_1 = s_0 + \epsilon$,

$$\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)} f - f|}{t^{s/q}} \right\|_{L^q(B(x_0, 1))} \lesssim \|f\|_{H^{s+\epsilon}(\mathbb{R}^n)}. \quad (5.1)$$
By translation, (5.1) can be reduced to
\[
\left\| \frac{\sup_{0<t<1} |e^{itP(D)}(f) - f|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \lesssim \|f\|_{H^{s+\delta}(\mathbb{R}^n)}. \tag{5.2}
\]
Concretely, if (5.2) holds for all \( f \in H^{s+\delta}(\mathbb{R}^n) \), take \( f_0 \),
\[
\hat{f}_0(\xi) = e^{ix_0\cdot \xi} \hat{f}(\xi)
\]
and insert \( f_0 \) into (5.2). Then (5.1) follows from simple computation.

Next we show (5.1) implies (1.14). In fact, if (5.1) holds, then fix \( \lambda > 0 \), for any \( \epsilon > 0 \), choose \( g \in C_c^\infty(\mathbb{R}^n) \) such that
\[
\|f - g\|_{H^{s+\delta}(\mathbb{R}^n)} \leq \frac{\lambda^{1/q}}{2}, \tag{5.3}
\]
it follows
\[
\left\| \left\{ x \in B(x_0,1) : \sup_{0<t<1} \frac{|e^{itP(D)}(f - g) - (f - g)|}{t^{\delta/m}} > \frac{\lambda}{2} \right\} \right\| \leq \frac{2^q}{\lambda^q} \left\| \sup_{0<t<1} \frac{|e^{itP(D)}(f - g) - (f - g)|}{t^{\delta/m}} \right\|_{L^q(B(x_0,1))} \tag{5.4}
\]
and
\[
\frac{|e^{itP(D)}(g)(x) - g(x)|}{t^{\delta/m}} \leq t^{1-\frac{\delta}{m}} \int_{\mathbb{R}^n} |P(\xi)\hat{g}(\xi)|d\xi \to 0, \quad \text{if} \ t \to 0^+
\tag{5.5}
\]
uniformly for \( x \in B(x_0,1) \). Then we have
\[
\left\| \left\{ x \in B(x_0,1) : \limsup_{t \to 0^+} \frac{|e^{itP(D)}(f)(x) - (f)(x)|}{t^{\delta/m}} > \lambda \right\} \right\| \leq \epsilon, \tag{5.6}
\]
which implies (1.14) for \( f \in H^{s+\delta}(\mathbb{R}^n) \) and \( x \in B(x_0,1) \). By the arbitrariness of \( \epsilon \) and \( x_0 \), in fact we can get (1.14) for all \( f \in H^{s+\delta}(\mathbb{R}^n) \), \( s > s_0 \) and \( x \in \mathbb{R}^n \). Next we will prove (5.2) for \( q = \min\{p,2\} \).

In order to prove (5.2), we decompose \( f \) as
\[
f = \sum_{k=0}^{\infty} f_k,
\]
where \( \text{supp} \hat{f}_0 \subset B(0,1) \), \( \text{supp} \hat{f}_k \subset \{ \xi : |\xi| \sim 2^k \} \), \( k \geq 1 \). It follows that
\[
\left\| \sup_{0<t<1} \frac{|e^{itP(D)}(f) - f|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \leq \sum_{k=0}^{\infty} \left\| \sup_{0<t<1} \frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \right\|_{L^q(B(0,1))}. \tag{5.7}
\]
By Taylor’s formula, for each \( k \),
\[
\frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \leq \sum_{j=1}^{\infty} \frac{t^{j-\delta/m}}{j!} \int_{\mathbb{R}^n} e^{ix\cdot \xi} P(\xi)^j \hat{f}_k(\xi)d\xi. \tag{5.8}
\]
For \( k \lesssim 1 \), because (5.8) and \( P(\xi) \) is continuous,

\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \leq \sum_{j=1}^{\infty} \frac{1}{j!} \left\| e^{ix\xi P(\xi)j\hat{f}_k(\xi)}d\xi \right\|_{L^q(B(0,1))} \\
\leq \sum_{j=1}^{\infty} \frac{1}{j!} \left\| e^{ix\xi P(\xi)j\hat{f}_k(\xi)}d\xi \right\|_{L^q(B(0,1))} \\
\leq \sum_{j=1}^{\infty} \frac{1}{j!} \right\| P(\xi)j\hat{f}_k(\xi)\right\|_{L^2(\mathbb{R}^n)} \\
\lesssim \|f\|_{H^{s+\delta}(\mathbb{R}^n)}. \tag{5.9}
\]

For \( k \gg 1 \),

\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \leq \left\| \sup_{0 < t < 2^{-mk}} \frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \\
+ \left\| \sup_{2^{-mk} \leq t < 1} \frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \right\|_{L^q(B(0,1))}. \tag{5.10}
\]

Inequalities (5.8) and (1.12) imply

\[
\left\| \sup_{0 < t < 2^{-mk}} \frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \leq \sum_{j=1}^{\infty} \frac{2^{-mkj+\delta k}}{j!} \left\| e^{ix\xi P(\xi)j\hat{f}_k(\xi)}d\xi \right\|_{L^q(B(0,1))} \\
\leq \sum_{j=1}^{\infty} \frac{2^{-mkj+\delta k}}{j!} \left\| e^{ix\xi P(\xi)j\hat{f}_k(\xi)}d\xi \right\|_{L^q(B(0,1))} \\
\leq \sum_{j=1}^{\infty} \frac{2^{-mkj+\delta k}}{j!} \right\| P(\xi)j\hat{f}_k(\xi)\right\|_{L^2(\mathbb{R}^n)} \\
\leq \sum_{j=1}^{\infty} \frac{2^{-mkj+\delta k}2^{-mkj}}{j!} \right\| \hat{f}_k(\xi)\right\|_{L^2(\mathbb{R}^n)} \\
\lesssim 2^{-s_1k} \|f\|_{H^{s+\delta}(\mathbb{R}^n)}. \tag{5.11}
\]

From (1.13) we have,

\[
\left\| \sup_{2^{-mk} \leq t < 1} \frac{|e^{itP(D)}f_k|}{t^{\delta/m}} \right\|_{L^p(B(0,1))} \lesssim 2^{(s_0+\frac{\delta}{2})k} \|f_k\|_{L^2(\mathbb{R}^n)}, \tag{5.12}
\]

hence,

\[
\left\| \sup_{2^{-mk} \leq t < 1} \frac{|e^{itP(D)}(f_k) - f_k|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \leq 2^{sk} \left\| \sup_{2^{-mk} \leq t < 1} \frac{|e^{itP(D)}(f_k)|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} \\
\leq 2^{sk} \left\{ \left\| \sup_{2^{-mk} \leq t < 1} \frac{|e^{itP(D)}(f_k)|}{t^{\delta/m}} \right\|_{L^q(B(0,1))} + \|f_k\|_{L^p(B(0,1))} \right\} \\
\lesssim 2^{sk} \left\{ \left\| \sup_{2^{-mk} \leq t < 1} \frac{|e^{itP(D)}(f_k)|}{t^{\delta/m}} \right\|_{L^p(B(0,1))} + \|f_k\|_{L^2(B(0,1))} \right\} \\
\lesssim 2^{sk}2^{(s_0+\frac{\delta}{2})k} \|f\|_{L^2(\mathbb{R}^n)} \\
\lesssim 2^{-sk} \|f\|_{H^{s+\delta}(\mathbb{R}^n)}. \tag{5.13}
\]

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Inequalities (5.11) and (5.13) yield for $k \gg 1$,
\[
\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f_k) - f_k \right| \right\|_{L^q(B(0,1))} \lesssim 2^{-\frac{4k}{3}} \| f \|_{H^{\frac{1}{2} + \epsilon}(\mathbb{R}^n)}. \tag{5.14}
\]

It is clear that (5.2) follows from (5.7), (5.9) and (5.14).

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