The Cauchy problem of non-local space-time reaction-diffusion equation involving fractional $p$-Laplacian

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For the non-local space-time reaction-diffusion equation involving fractional $p$-Laplacian

\[
\begin{align*}
\frac{\partial^\alpha u}{\partial t^\alpha} + (-\Delta)^s_p u &= \mu u^2(1 - kJ^s u) - \gamma u, \quad (x,t) \in \mathbb{R}^N \times (0,T), \\
\quad u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N,
\end{align*}
\]

where $\mu, k > 0$, $\gamma \geq 1$, $\alpha \in (0,1), s \in (0,1), 1 < p$, we consider for $N \leq 2$ the problem of finding a global boundedness of the weak solution by virtue of Gagliardo-Nirenberg inequality and fractional Duhamel’s formula. Moreover, we prove such weak solution converge to 0 exponentially or locally uniformly as $t \to \infty$ for small $\mu$ values with the comparison principle and local Lyapunov type functional. In those cases the problem is reduced to fractional $p$-Laplacian equation in the non-local reaction-diffusion range which is treated with the symmetry and other properties of the kernel of $(-\Delta)^s_p$. Finally, a key element in our construction is a proof of global bounded weak solution with the fractional nonlinear diffusion terms $(-\Delta)^s_p u^{m}(2 - \frac{2}{N} < m \leq 3, 1 < p < \frac{N}{2})$ by using Moser iteration and fractional differential inequality.

I. INTRODUCTION

In recent years, the study of differential equations using non-local fractional operators has attracted a lot of interest. The space-time fractional equations could be applied to a wide range of applications such as: continuum mechanics, phase transition phenomena, population dynamics, image process, game theory and Lévy processes, see ² ³ and the references therein. In particular, by ² ³, we know that the fractional Laplacian operator is defined for functions $u(x), x \in \mathbb{R}^N$, as

\[
(-\Delta)^s u(x) = C_{n,s} pV \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,
\]

where $s \in (0,1), C_{n,s} = \frac{4s\Gamma(\frac{n}{2} + s)}{\Gamma(1+s)\pi^\frac{n}{2}}$, and $pV$ is the principal value of Cauchy. In the special case when $p = 2$, Then problem (1) reduces to the space-time fractional non-local reaction-diffusion equation.

We are concerned here with the nonlinear version given by the fractional $p$-Laplacian operator $(-\Delta)_p^s$ defined by Definition ² ⁴. To be precise, it can be called the $s$-fractional $p$-Laplacian operator. Following our previous papers ² ³, we will continue the study of the cauchy problem of weak solution, i.e., the space-time reaction-diffusion equation involving fractional $p$-Laplacian

\[
\begin{align*}
\frac{\partial^\alpha u}{\partial t^\alpha} + (-\Delta)^s_p u &= \mu u^2(1 - kJ^s u) - \gamma u, \quad \text{in } \mathbb{R}^N \times (0,T), \\
\quad u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N,
\end{align*}
\]

where $\mu, k > 0$, $\gamma \geq 1$, $s \in (0,1), 1 < p$. And furthermore, for $\alpha = 1, s = 1$ we recover the classical non-local reaction-diffusion equation, whose theory is well known etc. ² ³. We also assume that we are given the competition kernel $J(x)$ with

\[
0 \leq J \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} J(x)dx = 1, \quad \inf_{B(0,\delta_0)} J > \eta, \quad \text{for some } \delta_0 > 0, \eta > 0 \text{ and where } B(0,\delta_0) = (-\delta_0, \delta_0)^N \text{ and }
\]

\[
J * u(x,t) = \int_{\mathbb{R}^N} J(x - y)u(y,t)dy.
\]

The related equations of this model are currently of great interest to researchers. There, the case $p \geq 2$, usually called nonlinear time-space fractional diffusion equation, has been treated accurately. We pursue in this paper the analysis of such a space-time diffusion equation involving fractional $p$-Laplacian devote much attention to settle Cauchy problem of equation (1)-(2), like blow-up, global boundedness and asymptotic behavior. Moreover, denoting $J = k = \mu = \gamma = 1$, we verify existence and global boundedness of weak solution for problem (2)-(3) in fractional Sobolev space.

Outline of result. As a starting novelty, the present paper recall some necessary definitions and useful properties of the fractional Sobolev space. See whole details in Section ² ³. Note that for $1 < p$, we prove the blow-up, global boundedness and asymptotic behavior in equation (1)-(2). However, in the range $1 < p < \frac{N}{2}$, we need the extra condition, $sp < 1$, that seems to play a major role in the proof for Theorem ² ³. Indeed, in order to proof the existence of equation (2)-(3), we introduces a basic tool called the weighted $L^1$ estimate that will play an important role in the existence theory for general classes of data in Appendix ² ³.
In section III, we aim to prove the global boundedness for equation (1.1)-(1.2). The proof consists of several steps. First, by reference 10, we obtain the global existence of weak solution. Further, we prove the blow-up of the problem by reference 11. Two key steps in our proofs are briefly listed in the following. Let $\delta_0, \mu > 0$ be as in (1.3) and $x \in \mathbb{R}^N$ fixed. For any $0 < \delta \leq 1/\delta_0$, using the symmetry of the kernel of $u \in W^{s,p}(B(x,\delta))$ and the observation: $\int_{B(x,\delta)} u(-\Delta)^s_p u \, dx = \frac{1}{\Gamma(1-s)} \left[ \int_{B(x,\delta)} u \right]^{p+s-1}_p$, we obtain a key differential inequality on $\int_{B(x,\delta)} u^2 \, dx$ (see (III.2)).

$$\frac{\partial^\alpha}{\partial t^\alpha} \int_{B(x,\delta)} u^2 \, dy + \left[ \int_{B(x,\delta)} u \right]^{p+s-1}_p + 2\gamma \int_{B(x,\delta)} u^2 \, dy$$

$$\leq 2\mu \int_{B(x,\delta)} u^3 \, dy - 2\mu \eta k \int_{B(x,\delta)} u^3 \, dy \, u \, dy$$

with further controlling of the term $\mu \int_{B(x,\delta)} u^3 \, dy$ by the Gagliardo-Nirenberg inequality. Then we can obtain the uniform boundedness of $\int_{B(x,\delta)} u^2 \, dy$ by using Sobolev embedding inequality and fractional differential equation. Based on the these, by applying fractional Duhamel’s formula representation in time spitted intervals $[0,T)$, for example, let $f(u) = \mu u^\alpha (1 - k \xi u - \gamma u)$, we have the solution

$$u(t) = \mathcal{I}_\alpha (u_0) + \int_0^t (t-s)^{-\alpha} \mathcal{X}_\alpha (t-s) f(u(s)) \, ds,$$

where $\mathcal{I}_\alpha, \mathcal{X}_\alpha$ is defined by Lemma A.10. Finally, we can obtain the global uniform boundedness of $u$ by a series of careful analyses of the heat kernel.

In section IV, in order to study the asymptotic behavior of weak solutions, we use the global boundedness of $u$ of Section III to obtain $0 \leq u(x,t) < a$ for any $(x,t) \in \mathbb{R}^N \times [0,\infty)$. Then, we can derive a Lyapunov type functional on $B(x,\delta)$ in Proposition IV.1

$$\frac{\partial^\alpha}{\partial t^\alpha} H(x,t) \leq -\left( -\Delta \right)^\alpha_p H(x,t) + \int_{B(x,\delta)} |\nabla u(y,t)|^2 \, dy - D(x,t)$$

(1.4)

with

$$D(x,t) = \frac{1}{2} (A - a) \mu k \int_{B(x,\delta)} u^2 (y,t) \, dy.$$

This Lyapunov function plays key roles in studying the long time behavior. Thus, with the help of comparison principle of equation (1.1)-(1.2) and using fractional Duhamel’s formula to (1.3), we can proceed to deduce the long time behavior of the solution.

Section V studies the global boundedness of equation (V.2)-(V.3) relying on fractional differential inequality, Gagliardo-Nirenberg inequality and other common inequalities. Before the proof we used weighted $L^1$ estimates to prove the existence and uniqueness of the weak solution in Appendix CA. CB. Firstly, due to Sobolev inequality, one has a key formula that

$$\frac{1}{k} \left( \int_{\mathbb{R}^N} u^2 \, dx \right) + \frac{c_\delta S}{2} \left( \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} \frac{|u(y)|^{p+1}}{|x-y|} \, dy \right]^{p+s} \, dx \right)^{\frac{1}{p+s}}$$

$$\leq \int_{\mathbb{R}^N} u^{k+1} \, dx(1 - \int_{\mathbb{R}^N} u \, dx) - \int_{\mathbb{R}^N} u^2 \, dx.$$  (1.5)

And by using fractional differential inequality (see Lemma A.6), we obtain the $L^1$ estimates for equation (V.2)-(V.3). Next, in order to extend $L^\infty$ estimation to $L^\infty$ estimation, by taking $k = \frac{q_k}{2} + 2$ in (1.5), we obtain (V.3). Furthermore, through Moser iteration and $L^1$ estimation, the global boundedness of the solution for (V.2)-(V.3) is obtained in fractional Sobolev space.

As already mentioned above, we collect some useful lemmas about fractional derivative, fractional Duhamel’s formula and blow-up for equation (1.1)-(1.2) in Appendix A.3. Furthermore, the Appendix B gives some useful results for proof of theorem. Finally, based on weighted $L^1$ estimates in Appendix CA, we study the Appendix CB to obtain the existence of the weak solution for (V.2)-(V.3).

NOTATIONS: In the whole paper we fix Caputo derivative

$$\frac{\partial^\alpha}{\partial t^\alpha}, \alpha \in (0,1) \text{ models memory effects in time. And we also replace fractional } p\text{-Laplacian } (-\Delta)^s_p, s \in (0,1), 1 < p < \infty \text{ with Neumann Laplacian operator } \Delta \text{ in section III IV. In contrast, under the conditions of } 1 < p < \frac{N}{\gamma_3}, \text{ the nonlinear fractional term } -(-\Delta)^s_p u \text{ satisfies } 2 - \frac{\gamma_3}{\gamma_3} < m \leq 3.$$

II. PRELIMINARIES

The fractional Sobolev space. According to, let $\Omega$ be an open set in $\mathbb{R}^N$. For any $\max \{ \frac{2N}{N+s}, 1 \} < p$ and any $0 < s < 1$, let us denote by

$$[u]_{W^{s,p} (\Omega)} = \left( \int_{\Omega} \left( \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dxdy \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}$$

the $(s,p)$-Gagliardo seminorm of a measurable function $u$ in $\Omega$. We consider the fractional Sobolev space

$$W^{s,p} (\Omega) = \left\{ u \in L^p (\Omega) : [u]_{W^{s,p} (\Omega)} < \infty \right\},$$

which is a Banach space with respect to the norm

$$\|u\|_{W^{s,p} (\Omega)} := [u]_{W^{s,p} (\Omega)} + \|u\|_{L^p (\Omega)}.$$

Definition II.1 \(^{12}\)

(i) Assume that $X$ is a Banach space and let $u : [0,T] \to X$. The Caputo fractional derivative operators of $u$ is defined by

$$\xi_t^\alpha D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} u(s) \, ds,$$

(II.1)

$$\xi_t^\alpha D_t^\alpha u(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T (s-T)^{-\alpha} \frac{d}{ds} u(s) \, ds,$$

where $\Gamma(1-\alpha)$ is the Gamma function. The above integrals are called the left-sided and the right-sided the Caputo fractional derivatives.

(ii) For $u : [0,\infty) \to \mathbb{R}^N$, the left Caputo fractional derivative with respect to time $t$ of $u$ is defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d}{ds} u(x,s) (t-s)^{-\alpha} \, ds, \quad 0 < \alpha < 1$$

(II.2)
Definition II.2 The left and right Riemann-Liouville fractional integrals of order \( \alpha \in (0, 1) \) for an integrable function \( u(t) \) are given by

\[
\mathcal{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in (0, T],
\]

and

\[
\mathcal{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^0 (s-t)^{\alpha-1} u(s) ds, \quad t \in [0, T).
\]

Definition II.3 The Mittag-Leffler function \( E_\alpha(z) \) is defined as

\[
E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)},
\]

where \( \beta, z \in \mathbb{C}, R(\alpha) > 0, \mathbb{C} \) denote the complex plane.

Definition II.4 The fractional \( p \)-Laplacian operator for \( s \in (0, 1), p > 1 \) and \( u \in W^{s,p}(\mathbb{R}^N) \), is defined by

\[
(-\Delta)_p^s u = PV \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x-y|^{N+sp}} dy
\]

\[
= \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x-y|^{N+sp}} dy.
\]

where \( PV \) is the principal value of Cauchy.

Theorem II.1 (Existence) Let \( u_0 \in W^{s,p}_0(\Omega), u_0 \geq 0, sp < N, \) then there exists \( T > 0 \) such that the problem \( \text{II.1-II.2} \) has a local real-valued unique weak solution \( u \in \Pi \) on \( (0, T) \), where

\[
\Pi = \left\{ u, \frac{\partial u}{\partial t} \in W^{s,p}(\Omega; L^\infty(0, T)) \cap L^2(\Omega; L^\infty(0, T)) \right\}.
\]

III. GLOBAL BOUNDEDNESS OF WEAK SOLUTIONS

Theorem III.1 Assume \( \text{\textbf{II.1-II.2}} \) holds and \( 0 < u_0 \in X_0, T > 0. \)

When \( k > k^* \), denoting \( k^* = 0 \) for \( N = 1 \) and \( k^* = (\mu C_{GN}^2 + 1) \eta^{-1} \) for \( N = 2 \), where \( C_{GN} \) is the constant appears in Gagliardo-Nirenberg inequality in Lemma \( \text{\textbf{II.7}} \), there exists a non-negative weak solution to equation \( \text{\textbf{II.1-II.2}} \) and it is globally bounded in time, that is, there exists

\[
K = \begin{cases} 
(\|u_0\|_{L^\infty(\mathbb{R}^N)} \mu, \eta, k, C_{GN}), & N = 1, \\
(\|u_0\|_{L^\infty(\mathbb{R}^N)} \mu, C_{GN}), & N = 2,
\end{cases}
\]

such that

\[
0 \leq u(x,t) \leq K, \quad \forall (x,t) \in \mathbb{R}^N \times (0, T).
\]

Proof 1 First, we denote

\[
B(x_0, \delta) := \left\{ x \triangleq (x_1, \cdots, x_N) \in \mathbb{R}^N \mid |x_i - x_i^0| \leq \delta, 1 \leq i \leq N \right\}
\]

where \( x_0 \triangleq (x_1^0, \cdots, x_N^0) \in \mathbb{R}^N \) and \( 0 < \delta \leq \frac{1}{3} \delta_0 \). And we choose \( \phi_{\delta}(\cdot) \in C_0^\infty(B(x, \delta)), \) and \( \phi_{\delta}(\cdot) \to 1 \) locally uniformly in \( B(x, \delta) \) as \( \varepsilon \to 0 \). For any \( x \in \mathbb{R}^N \), we multiply \( \text{\textbf{II.1}} \) by \( 2u\phi_{\delta} \) and integrate by parts over \( B(x, \delta) \), we obtain

\[
\int_{B(x, \delta)} 2u\phi_{\delta} \frac{\partial u}{\partial t} dy + \int_{B(x, \delta)} 2u\phi_{\delta} (-\Delta)^{\frac{s}{2}} u dy
\]

\[
= \int_{B(x, \delta)} 2u^3 \mu \phi_{\delta} (1 - k \ast u) dy - 2\gamma \int_{B(x, \delta)} u^2 \phi_{\delta} dy.
\]

By Lemma \( \text{\textbf{I.13}} \) then we can get

\[
\int_{B(x, \delta)} 2u\phi_{\delta} \frac{\partial u}{\partial t} dy \geq \frac{\partial^\alpha}{\partial t^\alpha} \int_{B(x, \delta)} u^2 \phi_{\delta} dy.
\]

And we assume that the integral in the definition of \( (-\Delta)^{\frac{s}{2}} \) exists, then for the symmetry of the kernel of \( u \in W^{s,p}(B(x, \delta)) \), we obtain the following formula for the integration by parts

\[
\int_{B(x, \delta)} u^2 \phi_{\delta} dy
\]

\[
= \int_{B(x, \delta)} \int_{B(x, \delta)} \frac{|u(x, t) - u(y, t)|^{p-2} (u(x, t) - u(y, t))}{|x-y|^{N+sp}} dy u(x, t) dx dy
\]

\[
= \int_{B(x, \delta)} \int_{B(x, \delta)} |u(x, t) - u(y, t)|^{p-2} (u(x, t) - u(y, t)) u(y, t) dy dx
\]

\[
= - \int_{B(x, \delta)} \int_{B(x, \delta)} \frac{|u(x, t) - u(y, t)|^{p-2} (u(x, t) - u(y, t))}{|x-y|^{N+sp}} u(y, t) dy dx
\]

\[
= \frac{1}{2} \int_{B(x, \delta)} \int_{B(x, \delta)} |u(x, t) - u(y, t)|^p |x-y|^{N+sp} dx dy
\]

\[
= \frac{1}{2} \left[ |u|^p \right]_{W^{s,p}(B(x, \delta))}.
\]

By \( \text{\textbf{II.1-II.2}} \) and \( \varepsilon \to 0 \), we have

\[
\frac{\partial^\alpha}{\partial t^\alpha} \int_{B(x, \delta)} u^2 dy + [u]^p_{W^{s,p}(B(x, \delta))}
\]

\[
\leq 2\mu \int_{B(x, \delta)} u^3 (1 - k \ast u) dy - 2\gamma \int_{B(x, \delta)} u^2 dy,
\]

where we applied the following facts that for any \( y \in B(x, \delta) \), then \( J(y - \cdot) \geq \eta \) and for any \( z \in B(\gamma, 2\delta) \), \( B(x, \delta) \subset B(\gamma, 2\delta) \), then

\[
J \ast u(y, t) \geq \eta \int_{B(\gamma, 2\delta)} u(z, t) dz \geq \eta \int_{B(x, \delta)} u(y, t) dy.
\]

Therefore, we get

\[
\frac{\partial^\alpha}{\partial t^\alpha} \int_{B(x, \delta)} u^2 dy + [u]^p_{W^{s,p}(B(x, \delta))}
\]

\[
\leq 2\mu \int_{B(x, \delta)} u^3 dy - 2\mu \eta k \int_{B(x, \delta)} u^3 dy \int_{B(x, \delta)} u dy
\]

\[
- 2\gamma \int_{B(x, \delta)} u^2 dy.
\]

(III.2)
Next, in order to estimate the term $\int_{\mathbb{R}^n} u^2 \, dy$, by using Gagliardo-Nirenberg inequality in Lemma [B.7] we combine (III.5) + (III.7) to obtain

$$2\mu \int_{B(x,\delta)} u^2 \, dy \leq 2 \int_{B(x,\delta)} |\nabla u|^2 \, dy + 2\mu \int_{B(x,\delta)} u^3 \, dy \, dy + 2\mu \int_{B(x,\delta)} u^2 \, dy$$

From this, we will consider the following $N = 1$ and $N = 2$ cases respectively.

Case 1. $N = 1$. From (III.3), by Young’s inequality, let $\varepsilon = \eta k, p = \frac{6}{5}, q = 6$, we get

$$2\mu \int_{B(x,\delta)} u^2 \, dy \leq 2 \int_{B(x,\delta)} |\nabla u|^2 \, dy + 2\mu \int_{B(x,\delta)} u^3 \, dy \, dy + 2\mu \int_{B(x,\delta)} u^2 \, dy$$

where $s_2 \geq p$ will be determined later. Here we set $s_2 = p = 2, then we have

$$\|\nabla u\|_{L^2(B(\delta,\delta))} \geq C_1 \|u\|_{L^2(B(\delta,\delta))}$$

where $s_2 \geq p$ will be determined later. Here we set $s_2 = p = 2, then we have

$$\|\nabla u\|_{L^2(B(\delta,\delta))} \geq C_2 \|u\|_{L^2(B(\delta,\delta))}$$

By (B.8), we have

$$[u]_{W^{p,\delta}(B(\delta,\delta))} \geq C_{s-p} \|u\|_{L^2(B(\delta,\delta))}$$

Then, we get

$$Q_1 = 2\mu \left( \mu^{\frac{1}{2}} C_{GN}^4(1, \delta) + 1 \right)^\frac{5}{2} (\eta k)^{-5} + C_{GN}^4(1, \delta)$$

and the equation (III.5) is satisfied above, we can get

$$\frac{\partial^\alpha}{\partial t^\alpha} \int_{B(x,\delta)} u^2 \, dy + C_{\delta-p} \left[ [u]_{L^2(B(\delta,\delta))} \right] + 2\gamma \int_{B(x,\delta)} u^2 \, dy \leq C_2 \int_{B(x,\delta)} u^2 \, dy + Q_1.$$
In summary, for any $(x,t) \in \mathbb{R}^N \times [0,T_{\text{max}}]$, we have
\begin{equation}
\|u\|_{L^2(B(x,\delta))} \leq M = \begin{cases} \frac{\sqrt{M_1}}{\sqrt{M_2}}, & N = 1, \\ \sqrt{N}M, & N = 2. \end{cases} \tag{III.8}
\end{equation}

Then, we obtain
\begin{equation}
\|u\|_{L^1(B(x,\delta))} \leq (2\delta)^{\frac{N}{2}}M. \tag{III.9}
\end{equation}

In order to improve the $L^2$ boundedness of $u$ to $L^\infty$, by using fractional Duhamel formula to equation (1.1)-(1.2) and let $f(u) = \mu u^2 - (1 - k \ast u) - \gamma u$, for all $(x,t) \in \mathbb{R}^N \times [0,T)$, we have the solution
\begin{equation}
 \begin{aligned}
 u(t) &= \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{K}_\alpha(t-s) f(u(s)) ds.
 \end{aligned}
\end{equation}

From Lemma A.10(A.7) and Proposition A.2 we have
\begin{equation}
0 \leq u(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^t (t-s)^{\alpha-1} \|\mathcal{K}_\alpha(t-s) f(u(s))\|_{L^\infty(\mathbb{R}^N)} ds
\leq C_4 \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu C_4 \int_0^t (t-s)^{\alpha-1} \|f(u(s))\|_{L^\infty(\mathbb{R}^N)} ds
\leq C_4 \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu C_4 \int_0^t (t-s)^{\alpha-1} \|u(s)\|^2_{L^\infty(\mathbb{R}^N)} ds
\leq C_4 \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu C_4 M^2 \int_0^t (t-s)^{\alpha-1} ds
\leq C_4 \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu C_4 M^2 T^\alpha \frac{1}{\alpha}.
\end{equation}

So there is
\begin{equation}
0 \leq u(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu M^2 C_2(N,T,\alpha), \tag{III.10}
\end{equation}
with $M$ defined in (III.8) and where
\begin{equation}
Q_1 = 2\mu \left( (\mu^2 + \frac{4}{C_G^2})^2 + (\eta k)^{-5} + C_G^{10} \right)
\end{equation}
and
\begin{equation}
\lambda^* = (C_2 - 2\gamma) + \frac{p}{2-p} \varepsilon, \varepsilon > 0.
\end{equation}

From Lemma A.13 we know $T_{\text{max}} = +\infty$. As a conclusion, we have shown that understanding $u$ in time is the existence of weak solution in Theorem III.1 and the blow-up criterion in Lemma A.13 shows that $u$ is the unique weak solution for (1.1)-(1.2) on $(x,t) \in \mathbb{R}^N \times [0,T)$. Thus, the global boundedness of $u$ is proved.

**Remark III.1** Firstly, on the basis of the existence of the solution, Literature uses the principle of comparison to obtain the expression of the solution, and this paper uses the fractional Duhamel formula (see Appendix A.10B) to construct the solution. Secondly, in the process of proof, reference mainly uses local energy estimation (Gagliardo-Nirenberg inequality) to reconcile thermonuclear decomposition and so on to prove the bounded nature of the solution. In addition to local energy estimation (Gagliardo-Nirenberg inequality), this paper also uses fractional differential equation, the symmetry of the kernel of $(-\Delta)^{\frac{\alpha}{2}}$ and other properties to verify. Finally, compared with reference, the blow-up for the solution is also presented in Appendix A.C.

**IV. ASYMPTOTIC BEHAVIOR OF WEAK SOLUTIONS**

Now, we consider the asymptotic behavior of the weak solution of (1.1)-(1.2).

**Proposition IV.1** Based on Theorem III.1 we have that
\begin{equation}
\|u(x,t)\|_{L^\infty(\mathbb{R}^N \times (0,T))} < a
\end{equation}
holds, denoting $A,a$ in Appendix B. Assume that the function
\begin{equation}
H(x,t) = \int_{B(x,\delta)} h(u(y,t)) dy
\end{equation}
with
\begin{equation}
h(u) = Aln \left( 1 - \frac{u}{A} \right) - a ln \left( 1 - \frac{u}{a} \right)
\end{equation}
is nonnegative and satisfies
\begin{equation}
\frac{\partial^2 u H(x,t)}{\partial t^\alpha} \leq \int_{B(x,\delta)} \|\nabla u(y,t)\|^2 dy - D(x,t) \tag{IV.1}
\end{equation}
with
\begin{equation}
D(x,t) = \frac{1}{2} (A-a) \mu k \int_{B(x,\delta)} u^2(y,t) dy.
\end{equation}

**Proof 2** Let $K = \|u(x,t)\|_{L^\infty(\mathbb{R}^N \times (0,T))}$, then $0 < K < a$. From the definition of $h(\cdot)$, it is easy to deduce that
\begin{equation}
h'(u) = \frac{a}{a-u} - \frac{A}{A-u} = \frac{(A-a)u}{(A-a)(u-a)}, \tag{IV.2}
\end{equation}
and
\begin{equation}
h''(u) = \frac{a}{(a-u)^2} - \frac{A}{(A-a)^2} = \frac{(A-a)^2(A-a)}{(a-u)^2(a-a)}. \tag{IV.3}
\end{equation}

By Lagrange mean value theorem and let $|h'(u(\xi))|^p - 1 < 1$, we will multiply (1.1) by $h'(u) \Phi_\varepsilon$ and integrate by parts over $B(x,\delta)$, where $\Phi_\varepsilon(\cdot) \in C^0_0(B(x,\delta))$, $\Phi_\varepsilon(\cdot) \rightarrow 1$ in $B(x,\delta)$ as $\varepsilon \rightarrow 0$. Then, we obtain
\begin{equation}
(-\Delta)^{\frac{\alpha}{2}} h(u)
\end{equation}
\begin{equation}
= \int_{B(x,\delta)} \frac{|h(u(x)) - h(u(y))|^p}{|x-y|^{N+sp}} dy
\end{equation}
\begin{equation}
= \int_{B(x,\delta)} |h'(u(\xi))|^p - 1 |u(x) - u(y)|^{N+sp} (u(x) - u(y)) h'(u(x)) dy
\end{equation}
\begin{equation}
\leq |h'(u(\xi))|^p - 1 h'(u) (-\Delta)^{\frac{\alpha}{2}} u \leq h'(u) (-\Delta)^{\frac{\alpha}{2}} u,
\end{equation}
where \( \xi, \eta \in B(x, \delta) \), and

\[
(-\Delta)^p B_{B(x, \delta)} h(u) dy = \int_{B(x, \delta)} \frac{|h(u(x)) - h(u(y))|}{|u(x) - u(y)|^{N+2\alpha}} dy.
\]

Then

\[
\int_{B(x, \delta)} \frac{|h(u(x)) - h(u(y))|}{|u(x) - u(y)|^{N+2\alpha}} dy.
\]

Taking \( \varepsilon \to 0 \), we obtain

\[
\frac{\partial^\alpha}{\partial t^\alpha} \int_{B(x, \delta)} h(u) dy = \int_{B(x, \delta)} \frac{|h(u(x)) - h(u(y))|}{|u(x) - u(y)|^{N+2\alpha}} dy.
\]

Next, we have

\[
\int_{B(x, \delta)} h(u) dy \leq \int_{B(x, \delta)} (-\Delta)^p h(u) dy.
\]

By (B.1) in Appendix B, we obtain

\[
\int_{B(x, \delta)} h(u) dy \leq \int_{B(x, \delta)} (-\Delta)^p h(u) dy.
\]

From (IV.3) and we obtain

\[
h''(u) > \frac{(A-a)^2}{A^2}.
\]

inserting (IV.3) into (IV.4), we obtain

\[
\frac{\partial^\alpha}{\partial t^\alpha} H(x, t) \leq -(-\Delta)^p H(x, t) + \frac{(A-a)k^4 \mu k(2\delta)^2}{2(A-k)^2(a-k)} \int_{B(x, \delta)} |\nabla u(y, t)|^2 dy - \frac{1}{2} (A-a) \mu k \int_{B(x, \delta)} u^2(y, t) dy.
\]

By choosing a sufficiently small \( \delta \) such that

\[
\frac{(A-a)k^4 \mu k(2\delta)^2}{2(A-k)^2(a-k)} \leq 1.
\]

Then, we obtain

\[
\frac{\partial^\alpha}{\partial t^\alpha} H(x, t) \leq -(-\Delta)^p H(x, t) + \int_{B(x, \delta)} |\nabla u(y, t)|^2 dy - D(x, t),
\]

with

\[
D(x, t) = \frac{1}{2} (A-a) \mu k \int_{B(x, \delta)} u^2(y, t) dy.
\]

Remark IV.1 This proposition is based on Theorem [III.7] comparing with Literature, we constructed the Lyapunov type function using the related properties of fractional p-Laplace and using [B.2] - [B.4] in Appendix B.

Theorem IV.1 Denote \( u(x, t) \) the globally bounded solution of (1) - (2).

(i). For any \( \gamma \geq 1 \), there exist \( \mu^* > 0 \) and \( m^* > 0 \) such that for \( \mu \in (0, \mu^*) \) and \( \|u_0\|_{L^\infty(\mathbb{R}^N)} < m^* \), we have

\[
\|u(x, t)\|_{L^\infty(\mathbb{R}^N \times [0, T])} < \frac{\gamma}{\mu},
\]

and therefore

\[
\|u(x, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} e^{-\frac{\gamma}{\mu} t},
\]

where \( \sigma := \gamma - \mu \|u_0\|_{L^\infty(\mathbb{R}^N)} > 0 \).

(ii). if \( 1 \leq \gamma < \frac{\mu}{4\mu^*} \), there exist \( \mu^{**} > 0 \) and \( m^{**} > 0 \) such that for \( \mu \in (0, \mu^{**}) \) and \( \|u_0\|_{L^\infty(\mathbb{R}^N)} < m^{**} \), we obtain

\[
\|u(x, t)\|_{L^\infty(\mathbb{R}^N \times [0, T])} < \alpha,
\]

and therefore

\[
\lim_{t \to \infty} u(x, t) = 0
\]

locally uniformly in \( \mathbb{R}^N \).

Proof 3 From Theorem [III.7] for any \( K' > 0 \) and the definition of \( M, \) there exist \( \mu^*(K') > 0 \) and \( m_0(K') > 0 \) such that for \( \mu \in (0, \mu^*(K')) \) and \( \|u_0\|_{L^\infty(\mathbb{R}^N)} < m_0(K') \), there is \( M \) which is sufficiently small such that

\[
\|u(x, t)\|_{L^\infty(\mathbb{R}^N \times [0, T])} < K'.
\]
(i) The case: \( \|u(x,t)\|_{L^r(\mathbb{R}^N \times [0,T])} < K' = \frac{\gamma}{\mu} \). Noticing \( \sigma_i(t) = \gamma - \mu \), we consider the function \( v(x,t) := v(t) > 0 \) for all \( x \in \mathbb{R}^N \). Then it follows that

\[
\frac{\partial u}{\partial t} - (-\Delta)^s_p v + \mu v^2(1 - k_J * v) - \gamma v \\
= (-\Delta)^s_p v + \mu v(1 - k_J * v) - \gamma v \\
\leq -(-\Delta)^s_p v - \gamma v + \mu v(1 - k_J * v) \\
\leq -(-\Delta)^s_p v - \gamma v + \mu v
\]

which ensures that \( (-\Delta)^s_p v(t) \geq 0 \). The last expression can be rewritten in the following form

\[
\left\{ \begin{array}{l}
C_0D^\alpha_{\tau}v(t) = -\sigma_1 v(t), \\
v(0) = \|v_0\|_{L^r(\mathbb{R}^N)},
\end{array} \right.
\]

or equivalently

\[
\lim_{t \to \infty} \int_{B(x,\delta)} u^2(z,s)dz = 0,
\]

Together with the fact that the heat kernel converges to a delta function at \( r \to t \), we can conclude that for any \( x \in \mathbb{R}^N \),

\[
\lim_{t \to \infty} \int_{B(x,\delta)} u^2(y,t)dy = 0.
\]

Moreover, due to the uniform boundedness of \( u \) on \( \mathbb{R}^N \times [0,T] \), by Lemma B.4, we can obtain the global boundedness of \( \|\nabla u(x,t)\|_{L^r(\mathbb{R}^N)} \). From this last point and Lemma B.7, we let \( \Omega = B(x,\delta), p = q = \infty, r = 2 \) and the convergence of \( \|u(\cdot,t)\|_{L^r(\mathbb{R}^N)} \). That is, when \( t \to 0 \),

\[
\|u(t)\|_{L^r(\mathbb{R}^N)} \to 0
\]

as \( t \to 0 \). Thus, for any compact set in \( \mathbb{R}^N \), by finite coverage, we get that \( u \) converges uniformly to \( 0 \) in that compact set, which means that when \( t \to \infty \), we have that \( u \) converges locally and uniformly to \( 0 \) in \( \mathbb{R}^N \). The proof is complete.

Remark IV.2 First, a local Lyapunov-type functional (Proposition IV.1) will be constructed before the proof on the sense of fractional derivative. And this paper verifies that (IV.1) in Proposition IV.1 uses the fractional Duhamel formula again. Secondly, we validate the comparison principle (see Lemma B.9) of equation (I.1) and (I.2) more than in literature\(^2\). Finally, based on the asymptotic convergence of the thermoneutral, the solution to the differential equation is changed to the fractional differential equation.

V. GLOBAL BOUNDEDNESS OF SOLUTIONS FOR (V.2)-(V.3)

In the section, we will make \( J = k = \mu = \gamma = 1 \) and replace fractional diffusion \( (-\Delta)^s_p u \) as nonlinear fractional diffusion \( (-\Delta)^s_p u^m \) in (I.1)-(I.2) to get equation (V.2)-(V.3). And we make \( f(x,t) = u^2(1 - \int_{\mathbb{R}^N} u dx) - u \), then equation (V.2)-(V.3) can write the following form

\[
\left\{ \begin{array}{l}
\frac{\partial^\alpha u}{\partial t^\alpha} + (-\Delta)^s_p (u^m) = f(x,t), \quad x \in \mathbb{R}^N \times (0,T] \\
u(x,0) = u_0(x),
\end{array} \right.
\]

We take the announced values of the parameters \( s, p \) with the condition \( sp < 1 \). Let \( u \geq v \) be two ordered solutions of the equation (V.1) defined in a strip \( Q_T = \mathbb{R}^N \times (0,T) \).

Theorem V.1

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + (-\Delta)^s_p u^m = u^2(1 - \int_{\mathbb{R}^N} u dx) - u \\
u(x,0) = u_0(x),
\]

with \( 2 - \frac{2}{N} < m \leq 3, N \geq 3, 0 < \alpha, s < 1 \) and \( 1 < p < \frac{4}{3} \). Then the problem (V.2)-(V.3) has a unique weak solution which is global bounded in fractional Sobolev space. Then there exist \( M > 0, T > 0 \) such that

\[
0 \leq u(x,t) \leq M, \quad (x,t) \in \mathbb{R}^N \times (0,T].
\]
Proof 4 Step 1. The $L^p$ estimates.
For any $x \in \mathbb{R}^N$, multiply (4.2) by $u^{k-1}, k > 1$ and integrating by parts over $\mathbb{R}^N$, by the proof of Theorem 5.2 in \textsuperscript{1}, we obtain
\[
\int_{\mathbb{R}^N} u^{k-1}(\Delta)^{\frac{p}{2}} u^m \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u^m(x,t) - u^m(y,t)) |u^m(x,t) - u^m(y,t)|^{p-2} \\
\cdot (u^{k-1}(x,t) - u^{k-1}(y,t)) \, dx \, dy,
\]
from Lemma B.5 and equation (B.11), let $a = u^m(x,t), b = u^m(y,t), \alpha = \frac{m-1}{m}$, then we have
\[
|u^m(x,t) - u^m(y,t)|^{p-2} (u^m(x,t) - u^m(y,t)) \\
\cdot (u^{k-1}(x,t) - u^{k-1}(y,t)) \\
\geq c_3 \left| \frac{u^{m(p-1)+k-1}}{p^p}(x,t) - \frac{u^{m(p-1)+k-1}}{p^p}(y,t) \right|^p.
\]
By Sobolev inequality (Lemma A.7), we obtain that
\[
\int_{\mathbb{R}^N} u^{k-1}(\Delta)^{\frac{p}{2}} u^m \, dx \\
\geq c_3 \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \frac{u^{m(p-1)+k-1}}{p^p}(x,t) - \frac{u^{m(p-1)+k-1}}{p^p}(y,t) \right|^p \\
\cdot |x-y|^{N+sp} \, dx \, dy \\
\geq c_3 S^\frac{s}{2} \left( \int_{\mathbb{R}^N} \left| \frac{u^{m(p-1)+k-1}}{p^p}(x,t) \right|^{p^*} \, dx \right)^{\frac{p}{p^*}},
\]
where $p^* = \frac{Np}{N-1}$. From Lemma A.4, we get
\[
\int_{\mathbb{R}^N} u^{k-1}(\Delta)^{\frac{p}{2}} u \, dx \geq \frac{1}{k} (\Delta)^{\frac{p}{2}} \int_{\mathbb{R}^N} u^k \, dx.
\]
In order to estimate $\int_{\mathbb{R}^N} u^{k+1} \, dx$, when $m \leq 3$ and $k$ satisfy \textbf{(B.13)-(B.16)} into (V.4) to get
\[
\int_{\mathbb{R}^N} u^{k+1} \, dx + \frac{c_3 S^k}{2} \left( \int_{\mathbb{R}^N} \left| \frac{u^{m(p-1)+k-1}}{p^p}(x,t) \right|^{p^*} \, dx \right)^{\frac{p}{p^*}} + k \int_{\mathbb{R}^N} u^k \, dx \\
\leq \frac{2k(k-1)}{(m+k-1)^2} \left\| \nabla u^{m+1} \right\|^2_{L^2(\mathbb{R}^N)} + C_2(N,k,m).
\]
Let $k \to \infty$, then $\frac{2k(k-1)}{(m+k-1)^2} \to 2$, so $\frac{2k(k-1)}{(m+k-1)^2} \leq 2$. And from Lemma B.6 and $N \geq 3$, we obtain
\[
\frac{1}{S_N^\frac{1}{p}} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^{m+1/2/N} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^2 + C_2(N,k,m).
\]
Let $k = \frac{(N+1)(m-1)}{p}, m \frac{m(p-1)+k-1}{p^p},$, then $\frac{m(p-1)+k-1}{p^p} = k, p^p = \frac{k(p-1)-1}{k}$. Therefore, we can get
\[
\int_{\mathbb{R}^N} u^k \, dx \\
\leq \frac{2}{S_N^\frac{1}{p}} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^{m+1/2/N} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^2 + C_2(N,k,m).
\]
Let $y(t) = \int_{\mathbb{R}^N} u^k \, dx$ and $a = 1 + m+2/1$, $f(t) = \frac{2}{S_N^\frac{1}{p}} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^{m+1/2/N} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^2 + C_2(N,k,m)$.

Then, the above-mentioned inequality can be written as
\[
\frac{c_3 S^k}{2} y(t) + \frac{c_3 S^k}{2} y^\beta(t) + ky(t) \leq f(t) y^\beta(t) + C_2(N,k,m)
\]
By Lemma A.7 and $0 < \beta < \alpha < 1, t \in [0,T)$, fractional differential inequality \textbf{(V.6)} has following solution
\[
y(t) \leq y(0) + (1-a) \frac{T^\alpha}{\alpha^\beta} \frac{T^\alpha}{\alpha^\beta} f^T \frac{f^T}{f^T}(t) \\
\left[ \frac{\lambda_k y^\beta(0) + (C_2(N,k,m) - \frac{c_3 S^k}{2})(1-\beta)}{\alpha \Gamma(\alpha)} \right]^\frac{1}{\beta}.
\]
Therefore, we have
\[
y(t) = \int_{\mathbb{R}^N} u^k \, dx \\
\leq y(0) + \left[ \frac{\lambda_k y^\beta(0) + (C_2(N,k,m) - \frac{c_3 S^k}{2})(1-\beta)}{\alpha \Gamma(\alpha)} \right]^\frac{1}{\beta} \\
+ (1-a) \frac{T^\alpha}{\alpha^\beta} \frac{T^\alpha}{\alpha^\beta} f^T \frac{f^T}{f^T}(0).
\]
Step 1. The $L^p$ estimates.

On account of the above arguments, our last task is to give the uniform boundedness of solution for any $t > 0$. Denote $q_k = 2^k + 2$, by taking $k = q_k$ in (V.3), we have

$$\frac{1}{q_k} D_t^\alpha \int_{\mathbb{R}^N} u^{q_k} dx + \frac{c_k \lambda_k}{2} \left( \int_{\mathbb{R}^N} \left| \frac{q_k + m(p-1)-1}{p} (x,t) \right|^p \right)^{\frac{p}{p^*}} dx \
\leq \int_{\mathbb{R}^N} u^{q_k+1} dx (1 - \int_{\mathbb{R}^N} u dx) - \int_{\mathbb{R}^N} u^{q_k} dx.$$

By Lemma B.8 we substitute (B.19) into (V.3) and with notice that $\frac{1}{q_k} (q_k - 1)^p \geq 2$. It follows

$$\frac{c_k}{q_k} D_t^\alpha \int_{\mathbb{R}^N} u^{q_k} dx + \frac{c_k \lambda_k}{2} \left( \int_{\mathbb{R}^N} \left| \frac{q_k + m(p-1)-1}{p} (x,t) \right|^p \right)^{\frac{p}{p^*}} dx \leq C(N)q_k^{\frac{\delta}{q_k-1}} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_k} + \frac{1}{2} \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^2 \\leq C(N)q_k^{\frac{\delta}{q_k-1}} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_k} + C_3(N) \\leq \max \{C(N), C_3(N)\} q_k^{\frac{\delta}{q_k-1}} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_k} + \frac{1}{2} \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^2$$

and

$$\int_{\mathbb{R}^N} u^{q_k} dx \leq \left\{ \frac{1}{q_k} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_k} + \frac{1}{2} \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^2 \right\}^{-\gamma_k}$$

Let

$$K_0 = \max \left\{ 1, \|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^m(\mathbb{R}^N)}, \left\| \nabla u_0^{\frac{m+q_k-1}{2}} \right\|_{L^2(\mathbb{R}^N)} \right\} \gamma_k$$

we have the following inequality for initial data

$$\int_{\mathbb{R}^N} u^{q_k} dx \leq \left\{ \frac{1}{q_k} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_k} + \frac{1}{2} \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^2 \right\}^{-\gamma_k} \leq K_0.$$
Therefore we finally have
\[ \|v(x,t)\|_{L^\infty(\mathbb{R}^N)} \leq C(N, \|v_0\|_{L^1(\mathbb{R}^N)}, \|v_0\|_{L^\infty(\mathbb{R}^N)}) \|\nabla u_0\|_{L^2(\mathbb{R}^N)}^{m+2}, T^\alpha = M. \]

The proof of this theorem is complete.

**Remark V.1** Firstly, compared with Literature\textsuperscript{o}, it proves the global boundedness of solution in arbitrary dimension space, while this paper mainly proves in fractional Sobolev space. Secondly, Literature\textsuperscript{o} mainly uses L’ estimation and Young inequality, Holder inequality, Gagliardo-Nirenberg inequality and other common inequalities to verify the equation. This paper is also based on the above-mentioned inequalities and the application of various fractional differential inequalities in Appendix A. Finally, the existence of the solution of equation (V.2) is discussed in Appendix C B.

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**Appendix A DEFINITIONS, RELATED LEMMA, AND COMPLEMENTS**

A Some useful Lemma about fractional derivative

In the section, let us recall some necessary Lemma and useful properties of fractional derivative.

**Lemma A.1** If 0 < α < 1, u ∈ AC\textsuperscript{1}[0, T] or u ∈ C\textsuperscript{1}[0, T], then the equality
\[ \partial_t^\alpha_0 D^\alpha_0 u(t) = u(t) - u(0) \]
and
\[ \partial_t^\alpha D^\alpha u(t) = u(t), \]
hold almost everywhere on [0, T]. In addition,
\[ \partial_t^\alpha_1 - \alpha \int_0^t \partial_t^\alpha D^\alpha u(\tau)d\tau = \left(\partial_t^{\alpha_1} - \alpha \frac{d}{dt} u\right)(t) = u(t) - u(0). \]

**Lemma A.2** If 0 < α < 1, η > 0, then there is 0 ≤ E\textsubscript{α, a}(−η) ≤ \frac{1}{\Gamma(\alpha)}. In addition, for η > 0, E\textsubscript{α, a}(−η) is a monotonically decreasing function.

**Lemma A.3** Let 0 < α < 1 and \( v \in C([0, T], \mathbb{R}^N), v' \in L^1(0, T; \mathbb{R}^N) \) and u be monotone. Then
\[ v(t)\partial_t^\alpha v(t) \geq \frac{1}{2}\partial_t^\alpha v^2(t), \quad t \in (0, T). \]  

**Lemma A.4** Let 0 < α < 1 and u ∈ C([0, T], \mathbb{R}^N), u' ∈ L\textsuperscript{1}(0, T; \mathbb{R}^N) and u be monotone. And when n ≥ 2, the Caputo fractional derivative with respect to time t of u is defined by \( \Gamma(1 + \alpha) t^{-\alpha} \). Then there is
\[ u^{n-1} C_0 D_t^\alpha u \geq \frac{1}{n} (C_0 D_t^\alpha u)^n. \]

**Lemma A.5** Suppose that a nonnegative function u(t) ≥ 0 satisfies
\[ \partial_t^\alpha C_0 D_t^\alpha u(t) + c_1 u(t) ≤ f(t) \]
for almost all t ∈ [0, T], where c_1 > 0, and the function f(t) is nonnegative and integrable for t ∈ [0, T]. Then
\[ u(t) ≤ u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds. \]

**Lemma A.6** Assume the function y_\text{κ}(t) is nonnegative and exists the Caputo fractional derivative for t ∈ [0, T] satisfying
\[ \partial_t^\alpha C_0 D_t^\alpha y_\text{κ}(t) ≤ -C_0(y_\text{κ}(t))^{k-\frac{1}{k}} - y_\text{κ} + a_\text{κ} y_\text{κ}^{\frac{1}{k-1}}(t) + y_\text{κ}^{\frac{1}{k-1}}(t), \]
where a_κ = \tilde{a}\bar{r}_\text{κ} > 1 with \( \tilde{a}, r \) are positive bounded constants and 0 < \( \gamma_2 < \gamma_1 \leq 3 \). Assume also that there exists a bounded constant \( K > 1 \) such that \( y_\text{κ}(0) \leq K^{\gamma_1} \), then
\[ y_\text{κ}(t) ≤ (2\tilde{a})^{\frac{k}{k-1}} 3^{\frac{k}{k-1}} + \frac{1}{\gamma_2} \max \left\{ \sup \limits_{t \in [0, T]} y_\text{κ}^{\gamma_2}(t) \right\} \left( \frac{T^\alpha}{\alpha \Gamma(\alpha)} \right)^\frac{1}{\gamma_2} \]
for almost all t ∈ [0, T], then
\[ y(t) ≤ y(0) + \left[ (k\gamma_2)^k(0) + (c_4 c_\alpha - (1-k)\Gamma(\alpha) \right] \frac{T^\gamma_1}{\alpha \Gamma(\alpha)} \left( \frac{T^\gamma_1}{\gamma_1 \Gamma(\gamma_1)} \right)^\frac{1}{\gamma_2} \epsilon^\frac{1}{\gamma_2} \bar{b}^\frac{1}{\gamma_2}, \]
where \( \lambda_\text{κ} = -\frac{m_\text{κ}}{\epsilon} - \beta(1-k) \) and \( \alpha, \beta, c_4, \epsilon > 0 \) are all constants.
Lemma A.8 Let us consider the fractional differential equation
\[
\begin{aligned}
\frac{\partial}{\partial t}^\alpha u(t) &= -w u(t), \quad 0 < \alpha < 1, w > 0, \\
u(0) &= u_0.
\end{aligned}
\] (A.7)

Then, the solution of (A.7) can be obtained by applying the Laplace transform technique which implies:
\[
u(t) = u_0 E_\alpha(-w t^\alpha), \quad t > 0. \tag{A.8}
\]

Lemma A.9 If $0 < \alpha < 1, t > 0, w > 0$, for Mittag-Leffler function $E_{\alpha,1}(w t^\alpha)$, then there is a constant $C$ such that
\[
E_\alpha(w t^\alpha) = E_{\alpha,1}(w t^\alpha) \leq C e^{w t}. \tag{A.9}
\]

Theorem A.1 (Fractional Sobolev inequality) Assume that $0 < s < 1$ and $p > 1$ such that $sp < N$. Then, there exist a positive constant $S = S(N,s,p)$ such that for all $u \in C^0_0(\mathbb{R}^N)$,
\[
\left[ v \right]^p_{W^{s,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dxdy \\
\geq S \left( \int_{\mathbb{R}^N} |v(x)|^{p_s} \right)^{\frac{p}{p_s}},
\]
where $p_s = \frac{sp}{N-sp}$.

B Fractional Duhamel’s formula

According to section 10, we define the operator $A := -(-\Delta)^s_p$ on $L^\infty(\Omega)$ by
\[
\begin{aligned}
D(A) &= \{ u \in X_0 \cap L^\infty(\Omega) : A u \in L^\infty(\Omega) \}, \\
Au &= -(-\Delta)^s_p u \quad \text{for} \quad u \in D(A).
\end{aligned}
\] (A.10)

It is well known that $A$ is m-accretive in $L^\infty(\Omega)$. Then, we convert Equation (1.1)-(1.2) to the following abstract form:
\[
\begin{aligned}
\frac{d}{dt} \phi(t) &= Au + f(u), \quad (x,t) \in \Omega \times (0,T), \\
u(x,0) &= u_0(x), \quad \forall x \in \Omega,
\end{aligned}
\] (A.11)

where $f(u(x,t)) = \mu u^2(1-k \mu * u) - \mu u$. Let $X_1$ be a Hilbert space and the function $f : [0,T] \to X_1$ and $A$ is self-adjoint on $X_1$. Then there exists a measure space $(\Gamma, \mu)$ and a Borel measurable function $a$ and a unitary map $U : L^\infty(\Gamma, \mu) \to X$ such that
\[
U^{-1} A U = T_a, \quad \text{where} \quad T_a \varphi(\xi) = a(\xi) \varphi(\xi), \quad \xi \in \Gamma.
\]

Lemma A.10 If $u$ satisfies Equation (1.1) and $t \in [0,T]$, then $u$ also satisfies
\[
u(t) = U(E_\alpha(a(\xi)t^\alpha)U^{-1}u_0) \\
+ \int_0^t (t-s)^{\alpha-1} U(E_\alpha,\alpha ((t-s)^\alpha a(\xi))U^{-1}f(u(s))) ds.
\]

Remark A.1 By define two maps
\[
J_A(t) = U(E_\alpha(a(\xi)t^\alpha)U^{-1}U^{-1} \varphi)
\]
and
\[
J_{\alpha}(t) = U(E_\alpha,\alpha ((t)^\alpha a(\xi))U^{-1}U^{-1} \varphi), \quad \varphi \in X.
\]

Then through Lemma A.12 we can get the following fractional Duhamel’s formula.

Lemma A.11 (Fractional Duhamel’s formula) If $u \in C([0,T];X)$ satisfies Equation (1.1)-(1.2), then $u$ satisfies the following integral equation:
\[
u(t) = J_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1} J_{\alpha}(t-s)f(u(s)) ds.
\]
The solution operators $J_A(t)$ and $J_{\alpha}(t)$ are linear and bounded operators, for any $\varphi \in X$,
\[
\|J_A(t)\varphi\|_X \leq C_4 \|\varphi\|_X, \quad \|J_{\alpha}(t)\varphi\|_X \leq C_4 \|\varphi\|_X, \tag{A.12}
\]
where $C_4$ is a constant.

Lemma A.12 Assume $1 \leq p < N$, then $\forall f \in C^0_0(\mathbb{R}^N)$, there is
\[
\|f\|_{L^p(\mathbb{R}^N)} \leq C_1 \|\nabla f\|_{L^p(\mathbb{R}^N)}.
\] (A.13)

Which is $q = \frac{Np}{N-p}$, and $C_1$ only dependent on $p, N$.

C Blow-up for equation (1.1)-(1.2)

Definition A.1 Suppose eigenfunction $e_1 > 0$ associated to the first eigenvalue $\lambda_1 > 0$ that satisfies the fractional eigenvalue problem
\[
(-\Delta)^s_p e_1(x) = \lambda_1 e_1(x), \quad x \in \Omega, \tag{A.14}
\]

normalized such that $\int_\Omega e_1(x) dx = 1$.

Lemma A.13 (Blow-up) Assume the initial data $0 < u_0 \in X_0 < 1$. And if $1 + \lambda_1 \leq \int_\Omega u_0(x) e_1(x) dx = H_0$, and $\lambda_1, e_1(x)$ is the value in Definition A.7 then the solution of problem (1.1)-(1.2) blow-up in a finite time $T_{\max}$ that satisfies the bi-lateral estimate
\[
\left( \frac{\Gamma(\alpha + 1)}{4(H_0 + 1/2)} \right)^{\frac{1}{\alpha}} \leq T_{\max} \leq \left( \frac{\Gamma(\alpha + 1)}{H_0} \right)^{\frac{1}{\alpha}}.
\]

Furthermore, if $T_{\max} < \infty$, then
\[
\lim_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty.
\]
Proof 5 Multiplying equations (1.1) by $e_1(x)$ and integrating over $\Omega$, we obtain
\[
\begin{align*}
\partial_t^a \int_{\Omega} u(x,t)e_1(x)dx + \int_{\Omega} (-\Delta)^a p u(x,t)e_1(x)dx &= \int_{\Omega} [\mu (1 - kJ^* u) u(x,t) - \gamma u(x,t)]e_1(x)dx, \\
= \int_{\Omega} [\mu (1 - kJ^* u) u(x,t) - \gamma u(x,t)]e_1(x)dx.
\end{align*}
\] (A.15)

By (A.14), we have
\[
\begin{align*}
\int_{\Omega} (-\Delta)^a p u(x,t)e_1(x)dx &= \int_{\Omega} u(x,t)(-\Delta)^a p u(x,t)dx \\
&= \lambda_1 \int_{\Omega} u(x,t)e_1(x)dx,
\end{align*}
\] as $u = 0, e_1(x) = 0, x \in \mathbb{R}^N \setminus \Omega$ and
\[
\left( \int_{\Omega} u(x,t)e_1(x)dx \right)^2 \leq \int_{\Omega} u^2(x,t)e_1(x)dx,
\]
let function $H(t) = \int_{\Omega} u(x,t)e_1(x)dx$, then satisfies
\[
\partial_t^a H(t) + (\gamma + \lambda_1) H(t) \geq \mu H^2(t). \quad (A.16)
\]
The proof process later can refer to the reference [10, Theorem 2.1] and [11, Theorem 3.2]. The proof of the lemma is complete.

Remark A.2 The proof of the lemma is based on references [10] and [11], but the difference is that the properties of eigenfunction for fractional $p$-Laplacian and the correlation inequality are used to verify them.

Appendix B SOME USEFUL RESULTS

To study the long time behavior of solutions for (1.1)-(1.2), by $\mu$, we denote
\[
F(u) := \mu u^2 (1 - kJ^* u) - \gamma u.
\]
For $1 \leq \gamma < \frac{4k}{4k}$, there are three constant solutions for $F(u) = 0$: $0, a, A$, where
\[
a = 1 - \frac{\sqrt{1 - 4k^2 \mu}}{2k}, \quad A = 1 + \frac{\sqrt{1 - 4k^2 \mu}}{2k}, \quad (B.1)
\]
and satisfy $1 < \frac{\gamma}{\mu} < a < A$.

Remark B.1 By (B.1), we can get
\[
\mu u^2 (1 - ku) - \gamma u = k\mu u(A - u)(u - a),
\]
and
\[
\begin{align*}
\int_{B(x, \delta)} h'(u)[\mu u^2 (1 - ku) - \gamma u]dy &= \int_{B(x, \delta)} h'(u)[\mu u^2 (1 - ku) - \gamma u]dy \\
&+ \mu k \int_{B(x, \delta)} h'(u)u^2(u - J^* u)dy \\
&= - (A - a) \mu k \int_{B(x, \delta)} u^2(y,t)dy \\
&+ \mu k \int_{B(x, \delta)} h'(u)u^2(u - J^* u)dy.
\end{align*}
\] (B.2)

Noticing that when $0 \leq u \leq k$, there is
\[
0 \leq h'(u)u \leq \frac{(A - a)^2}{(A - k)(a - k)}.
\]
From Young’s inequality and the median value theorem, we can get
\[
\begin{align*}
\mu k \int_{B(x, \delta)} h'(u)u^2(u - J^* u)dy &\leq \mu k \int_{B(x, \delta)} h'(u)u^2(u - J^* u)dy \\
&\leq \frac{(A - a)^2}{(A - k)(a - k)} \mu k \int_{B(x, \delta)} u(y,t)[(u(y,t) - u(z,t))J(y,z)zdy \\
&\leq \frac{(A - a)^2}{(A - k)(a - k)} \mu k \int_{B(x, \delta)} u(y,t)[(u(y,t) - u(z,t))J(y,z)zdy \\
&\leq \frac{(A - a)^2}{(A - k)(a - k)} \mu k \int_{B(x, \delta)} u(y,t)^2J(z - y)zdy \\
&\leq \frac{(A - a)^2}{(A - k)(a - k)} \mu k \int_{B(x, \delta)} u(y,t)^2J(z - y)zdy \\
&\leq \frac{(A - a)^2}{(A - k)(a - k)} \mu k \int_{B(x, \delta)} u(y,t)^2J(z - y)zdy \\
&\leq \frac{(A - a)^2}{(A - k)(a - k)} \mu k \int_{B(x, \delta)} u(y,t)^2J(z - y)zdy.
\end{align*}
\] (B.3)

changing the variables $y' = y + \theta(z - y), z' = z - y$, then
\[
\begin{align*}
\left| \frac{\partial h'}{\partial x} \right| \leq \left| \frac{1 - \theta}{1 - \theta} \right| = 1 - \theta + \theta = 1.
\end{align*}
\] (B.4)

Lemma B.1 \[\square\] Let $\Omega$ be an open subset of $\mathbb{R}^N$, assume that $1 \leq p, q \leq \infty$ with $(N - q)p < Nq$ and $r \in (0, p)$. Then there exists constant $C_{GN} > 0$ only depending on $q, r$ and $\Omega$ such that for any $u \in W^{1, q} (\Omega) \cap L^p (\Omega)$
\[
\int_{\Omega} u^p dx \leq C_{GN} (||\nabla u||_{L^q(\Omega)}) ||u||^{(1 - \lambda^*)p} ||u||^p \quad (B.4)
\]
holds with
\[
\lambda^* = \frac{\frac{N}{1 - \frac{N}{q} + \frac{N}{r}}}{1 - \frac{N}{q} + \frac{N}{r}} \in (0, 1).
\]
**Remark B.2** By using Gagliardo-Nirenberg inequality in Lemma B.1, let $p = 3, q = r = 2, \lambda^* = \frac{1}{\mu}$, there exists constant $C_G > 0$, such that

$$\int_{\mathbb{R}^N} u^3 \, dy \leq C_G(N)(\|\nabla u\|_{L^2(\mathbb{R}^N)}^N\|u\|_{L^3(\mathbb{R}^N)}^{3-N} + \|u\|_{L^2(\mathbb{R}^N)}^3).$$  \hfill (B.5)

On the one hand, by Young’s inequality, we can obtain

$$C_G(N)\|\nabla u\|_{L^2(\mathbb{R}^N)}^N\|u\|_{L^3(\mathbb{R}^N)}^{3-N} \leq \frac{1}{\mu}\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \mu C_G(N)\|u\|_{L^2(\mathbb{R}^N)}^{2(N-1)},$$

and

$$C_G(N)\|u\|_{L^2(\mathbb{R}^N)}^2 \leq \|u\|_{L^2(\mathbb{R}^N)}^{2(N-1)} + C_G(N).$$ \hfill (B.6)

By interpolation inequality, we obtain

$$\|u\|_{L^2(\mathbb{R}^N)}^{2(N-1)} \leq \left(\|u\|_{L^1(\mathbb{R}^N)}^{\frac{2}{3}}\right)^{2(N-1)} = \left(\int_{\mathbb{R}^N} u^3 \, dy \right)^{\frac{2}{3}N}.$$ \hfill (B.7)

**Lemma B.2** The embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous, that is

$$\|u\|_{L^r} \leq C_s\|u\|_{W^{s,p}}, \quad \forall u \in W_0$$ \hfill (B.8)

where $1 \leq r < p^*_s, C_s = C_s(s, p, r, N, \mathbb{R}^N)$ is optimal embedded constant. And $W_0$ is defined as

$$W_0 := \left\{ u \in W^{s,p}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} u \, dx = 0 \right\}.$$

**Lemma B.3** Suppose that a nonnegative function $y_k(t) \geq 0, \bar{a}, \bar{b} > 0$ be a solution of the fractional differential inequality

$$\frac{d}{dt}\lambda_k y_k(t) \leq -\bar{a}y_k(t) + \beta b_k(t) y_k^{1-\beta}(t).$$ \hfill (B.9)

Then, the solution of (B.9) can be estimated as

$$y_k(t) \leq y_k(0) + \lambda_k y_k(0) \int_0^t (t-s)^\alpha - 1 E_{\alpha, \alpha}(\lambda_k(t-s)^\alpha) ds + \varepsilon \lambda_k^{\frac{1}{\beta}} \int_0^t (t-s)^\alpha - 1 E_{\alpha, \alpha}(\lambda_k(t-s)^\alpha) b_k^{\frac{1}{\beta}}(s) ds,$$

where $\lambda_k = -\bar{a} + \frac{1-\beta}{\beta \varepsilon^{1-\beta}}$ and $\varepsilon > 0$

**Definition B.1** Let $X$ be a Banach space, $z_0, \ell \in X$, and $f \in L^1(0, T; X)$. The function $z(\ell, t) \in C([0, T]; X)$ given by

$$z(t) = e^{-t} \ell z_0 + \int_0^t e^{-(t-s)} e^{-(t-s)} \lambda f(s) ds, 0 \leq t \leq T,$$

is the mild solution of (B.10) on $[0, T]$, while $(e^\lambda f)(\ell, t) = \int_{\mathbb{R}^N} G(x - y, t)f(y) dy$ and $G(x, t)$ is the heat kernel by $G(x, t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right)$.

**Lemma B.4** Let $0 \leq q \leq p < \infty\frac{1}{q} - \frac{1}{p} < \frac{1}{4}$ and suppose that $\tau$ is the function given by (B.11) and $z_0 \in W^{1,p}(\mathbb{R}^N)$. If $f \in L^m(0, \infty; L^q(\mathbb{R}^N))$, then

$$\|\tau(z)\|_{L^p(\mathbb{R}^N)} \leq \|z_0\|_{L^p(\mathbb{R}^N)} + C \Gamma(\gamma) \sup_{0 < c < \ell} \|f(s)\|_{L^q(\mathbb{R}^N)},$$

$$\|\nabla \tau(z)\|_{L^p(\mathbb{R}^N)} \leq \|\nabla z_0\|_{L^p(\mathbb{R}^N)} + C \Gamma(\gamma) \sup_{0 < c < \ell} \|f(s)\|_{L^q(\mathbb{R}^N)},$$

for $t \in [0, \infty)$, where $C$ is a positive constant independent of $p, \Gamma(\gamma)$ is the gamma function, and $\gamma = 1 - \frac{m}{4} - \frac{1}{p}, \sqrt{\gamma} = \frac{1}{2} - \frac{1}{4} - \frac{1}{p} - \frac{1}{2}.

**Lemma B.5** Assume that $(a, b) \in (\mathbb{R}^+)^2, 0 < \alpha < 1$, then there exist $c_1, c_2, c_3 > 0$ such that

$$(a + b)\alpha \leq c_1 a^{\alpha} + c_2 b^{\alpha}$$

and

$$|a - b|^{p-2} (a - b)(a^{\alpha} - b^{\alpha}) \geq c_3 \left| a^{\frac{p-1}{p}} - b^{\frac{p-1}{p}} \right|^p.$$ \hfill (B.11)

**Lemma B.6** Let $N \geq 3, q > 1, m > 1 - 2/N$, assume $u \in L^1(\mathbb{R}^N)$ and $u^{\frac{m-1}{2}} \in H^1(\mathbb{R}^N)$, then

$$\|u\|_{L^q(\mathbb{R}^N)}^{q + \frac{m-1}{2}} \leq S_{\frac{N}{q}} \|u\|_{H^1(\mathbb{R}^N)}^{q + \frac{m-1}{2}} \|u\|_{L^q(\mathbb{R}^N)}^{2q/N + m - 1}.$$ \hfill (B.12)

**Lemma B.7** When the parameters $p, q, r$ meet any of the following conditions:

(i) $q > N \geq 1, r \geq 1$ and $p = \infty$;

(ii) $q > max\{\frac{N}{m + 1}, 2\}$, $1 \leq r < \sigma$ and $p < \sigma + 1$ in

$$\sigma := \left\{\begin{array}{ll}
\frac{(q-1)N+q}{N-q}, & q < N, \\
\infty, & q \geq N.
\end{array}\right.$$ \hfill (B.13)

Then the following inequality is established

$$\|u\|\|u\|^{\lambda^*}_{\mathcal{L}^\infty(\mathbb{R}^N)} \leq C_G \|u\|^{\lambda^*}_{\mathcal{L}^{\mathcal{L}^\infty(\mathbb{R}^N)}} \|u\|_{L^\infty(\mathbb{R}^N)}$$

among

$$\lambda^* = \frac{qN(p-r)}{pN(q-r)+qr}.$$ \hfill (B.14)

**Remark B.3** The following estimate $k \int_{\mathbb{R}^N} u^{k+1} \, dx$. when $m \leq 3$ and

$$k > max\left\{\frac{(3-m)(N-2)}{4} - (m-1), \frac{2-m}{2}(N-1), (N-2)(2-m)\right\},$$

from Lemma B.7 we obtain

$$k \int_{\mathbb{R}^N} u^{k+1} \, dx = k \left\| u^{k+1} \right\|_{L^{k+1}(\mathbb{R}^N)} \left\| u \right\|_{L^k(\mathbb{R}^N)} \left\| u \right\|_{L^\infty(\mathbb{R}^N)} \leq k \left\| u^{k+1} \right\|_{L^{k+1}(\mathbb{R}^N)} \left\| u \right\|^{k+1}_{L^{k+1}(\mathbb{R}^N)} \left\| u \right\|_{L^\infty(\mathbb{R}^N)}$$

where $\lambda^* = \frac{(3-m)(N-2)}{4} - (m-1), \frac{2-m}{2}(N-1), (N-2)(2-m)$. 

\hfill (B.15)
where
\[ \lambda^* = 1 + \frac{(k+m-1)N}{2(k+1)} - \frac{N}{2} \in \left\{ \max \left\{ 0, \frac{2-m}{k+1} \right\}, 1 \right\}, \]
using Young’s inequality, there are
\[ k \int_{\mathbb{R}^N} u^{k+1} dx \leq k \left\| \frac{u^{k+1}}{2} \right\|_{L^2(k+1)}^{2(k+1)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)} \leq 2k \frac{k-1}{(m+k-1)^2} \left\| \nabla u \right\|^2_{L^2(\mathbb{R}^N)} + C_1(N,k,m) \left\| u^{k+1} \right\|_{L^2(m+1)(\mathbb{R}^N)}, \]
where is \( Q_2 = \frac{2(k+1)\lambda^*}{m+2(k+1)\lambda^*} \). Next estimate
\[ \left\| \frac{u^{k+1}}{2} \right\|_{L^2(k+1)}^{2(k+1)} \right\|_{L^2(m+1)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)} \]
\[ \leq \left( \left\| \frac{u^{k+1}}{2} \right\|_{L^2(k+1)}^{2(k+1)} \right\|^2 \left\| \frac{u^{k+1}}{2} \right\|_{L^2(m+1)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)} \]
in \( \lambda = \frac{k+1}{k+2} \) and
\[ Q_2(1-\lambda - \frac{\lambda}{k+1}) = 0. \]
Then
\[ C_1(N,k,m) \left\| u^{k+1} \right\|_{L^2(m+1)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)} \]
\[ \leq \left( \left\| \frac{u^{k+1}}{2} \right\|_{L^2(m+1)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)} \right) \left( \frac{2(k+1)}{m+2(k+1)} \right). \]
Noticing that when \( m > 2 - \frac{2}{k} \), it is easy to verify
\[ Q_2\lambda(k+m-1) = \frac{(k+1)(m-2)+\lambda^*(k+1)(3-m)}{(k+1)(k+m-1)\lambda^*} < 1, \]
using Young’s inequality, then
\[ C_1(N,k,m) \left\| u^{k+1} \right\|_{L^2(m+1)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)} \]
\[ \leq k \left( \left\| \frac{u^{k+1}}{2} \right\|_{L^2(m+1)} \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)} \right) \left( \frac{2(k+1)}{m+2(k+1)} \right) + C_2(N,k,m). \]
\[ v = \frac{m+2q-1}{2}, q = 2, r = \frac{2q^p-1}{m+2q-1}, c_0 = 1, c_1 = \frac{1}{2q}, \]
\[ u \leq u^{m+q-1}, q = 2, r = \frac{2q^p-1}{m+2q-1}, c_0 = 1, c_1 = \frac{1}{2q}, \]
noticing \( q_k = \frac{(q+1)+1}{2} \), and using Young’s inequality, we obtain
\[ \frac{1}{2} \left\| u \right\|_{m+q-1}^2 \leq \frac{1}{2} \int_{\mathbb{R}^N} u^{m+q-1} dx \]
\[ \leq c_2(N) \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^2 + \frac{1}{2} \left\| \nabla u \right\|^2_{L^2(\mathbb{R}^N)} \]
\[ + \frac{1}{2} \left\| \nabla u \right\|^2_{L^2(\mathbb{R}^N)}, \]
where
\[ q_k = 1 + \frac{m+q-1}{q_{k-1}} < 2. \]
Lemma B.9 (Comparison principle) Let \( 0 < s < 1, p > 1, \Omega \subset \mathbb{R}^N \) and let \( u, v \in \Omega \) be a real-valued weak subsolution and supersolution of \((B.21), (B.22)\), respectively, with \( u_0(x) \leq v_0(x) \) for \( x \in \Omega \). Based on the following fractional differential inequality

\[
\int_0^1 \int_\Omega \mu^p D_\tau^p [v-(u-v)] (u-v) \cdot dx \cdot d\tau \\
\leq \left( \mu L(2) - k\mu \eta^{\frac{q}{2}} C_4 - \gamma \right) \int_0^1 \int_\Omega (u-v)(u-v) \cdot dx \cdot d\tau,
\]

then \( u \leq v \ a.e. \ in \ \Omega \), where \( \Omega_T = \Omega \times (0, T) \) and \( L(m) = C(m) \max(\|u\|_{L^m(\Omega)}, \|v\|_{L^m(\Omega)}). \)

**Proof** We select the function \( \varphi = (u-v)_+ \), where \( \varphi = (u-v)_+ \) is a positive real part of the real number \( \varphi = u-v \) is max \( \{u-v, 0\} \). Followed by \( \varphi(x, 0) = 0, \varphi(x, t) \big |_{t=0} = 0 \), then we obtain for \( t \in (0, T) \)

\[
\int_0^1 \int_\Omega \mu^p \Delta_T^p [u-v] \varphi \cdot dx \cdot d\tau + \int_0^1 \int_\Omega \Gamma_T^p [u-v] \varphi \cdot dx \cdot d\tau \\
\leq \mu \int_0^1 \int_\Omega (u^2-v^2) \varphi \cdot dx \cdot d\tau - k\mu \int_0^1 \int_\Omega (u^2 J * u - v^2 J * v) \varphi \cdot dx \cdot d\tau \\
- \gamma \int_0^1 \int_\Omega (u-v) \varphi \cdot dx \cdot d\tau \quad \text{(B.21)}
\]

By \( \text{(III.3)} \), we can write the last term on the left of the inequality \( \text{(B.21)} \) as

\[
\int_0^1 \int_\Omega (-\Delta_T^p u) \varphi \cdot dx \cdot d\tau + \int_0^1 \int_\Omega (-\Delta_T^p v) \varphi \cdot dx \cdot d\tau \\
= \int_0^1 \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \cdot dx \cdot d\tau \\
- \int_0^1 \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^{p-2} (v(x) - v(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \cdot dx \cdot d\tau \\
+ \int_0^1 \int_\Omega \frac{\mathcal{M}(u,v)(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \cdot dx \cdot d\tau, \quad \text{(B.22)}
\]

where

\[
\mathcal{M}(u,v) = |u(x) - u(y)|^{p-2} (u(x) - u(y)) \\
- |v(x) - v(y)|^{p-2} (v(x) - v(y)) (\varphi(x) - \varphi(y)) \quad \text{(B.23)}
\]

Thus, we can show that

\[
\mathcal{M}(u,v)(\varphi(x) - \varphi(y)) = |(u(x) - u(y))^{p-2} (u(x) - u(y)) \\
- |v(x) - v(y)|^{p-2} (v(x) - v(y)) (\varphi(x) - \varphi(y))| \\
\times |(u(x) - u(y)) - (v(x) - v(y))|_+ \\
is nonnegative for any \( p > 1 \), thanks to the two inequalities \( \text{(see\,P.99-100)} \)

\[
|a|^{p-2} a - |b|^{p-2} b \leq \left( |a|^{p-2} a - |b|^{p-2} b, a - b \right), \quad p \geq 2
\]

and

\[
(p-1) |b-a|^2 \int_0^1 |a+t(b-a)|^{p-2} dt \\
\leq \left( |a|^{p-2} a - |b|^{p-2} b, a - b \right), \quad 1 \leq p \leq 2
\]

with \( a := u(x) - u(y), b := v(x) - v(y) \) in \( \text{(B.23)} \). Now, we’ll zoom in on the right side of \( \text{(B.21)} \).

By \( \text{Theorem 3.2 p12} \), we take into account the following inequalities for \( m \geq 2 \)

\[
\|u\|_{m-1} - \|v\|_{m-1} \leq C(m) \|u-v\|_{m-1} + \|v\|_{m-1} \\
\leq L(m) \|u-v\|,
\]

where \( L(m) = C(m) \max(\|u\|_{L^m(\Omega)}, \|v\|_{L^m(\Omega)}). \)

Specially, when \( m = 2 \), then

\[
\int_0^1 \int_\Omega (u^2-v^2) \varphi \cdot dx \cdot d\tau \leq L(2) \int_0^1 \int_\Omega (u-v) \varphi \cdot dx \cdot d\tau, \quad \text{(B.24)}
\]

where we used the commonly used inequality \( \text{[22, Theorem 8.2]} \) for any \( u \in L^p(\Omega) \) such that

\[
\|u\|_{L^p(\Omega)} \leq \|u\|_{C(\Omega)} \|u\|_{L^p(\Omega)}, \beta = (sp-N)/p,
\]

which gives the boundness of \( \max(\|u\|_{C(\Omega)}, \|v\|_{C(\Omega)}) \).

In addition, by \( \text{Theorem III.7} \) let

\[
C_4 = \min \left\{ \left\| \frac{\partial}{\partial t} \right\|_{L^1(\Omega)}, \left\| v \right\|_{L^1(\Omega)} \right\},
\]

then we have

\[
k\mu \int_0^1 \int_\Omega (u^2 J * u - v^2 J * v) \varphi \cdot dx \cdot d\tau \\
= k\mu \int_0^1 \int_\Omega (u \int_\Omega J(x-y) u(y,t) dy)^{\frac{1}{2}} - v \left( \int_\Omega J(x-y) v(y,t) dy \right)^{\frac{1}{2}} \cdot dx \cdot d\tau \\
\times (u \int_\Omega J(x-y) u(y,t) dy)^{\frac{1}{2}} + v \left( \int_\Omega J(x-y) v(y,t) dy \right)^{\frac{1}{2}} \cdot dx \cdot d\tau \\
\geq k\mu \eta^{\frac{2}{2}} \int_0^1 \int_\Omega (u \int_\Omega J(x-y) u(y,t) dy)^{\frac{1}{2}} - v \left( \int_\Omega J(x-y) v(y,t) dy \right)^{\frac{1}{2}} \cdot dx \cdot d\tau \\
\geq k\mu \eta^{\frac{2}{2}} C_4 \int_0^1 \int_\Omega (u-v) \varphi \cdot dx \cdot d\tau. \quad \text{(B.25)}
\]

Combining \( \text{(B.22)}, \text{(B.24)} \) and \( \text{(B.25)} \), we can rewrite the inequality \( \text{(B.21)} \) as

\[
\int_0^1 \int_\Omega \left( \frac{\partial}{\partial t} [u-v] \right) (u-v) \cdot dx \cdot d\tau \\
\leq \left( \mu L(2) - k\mu \eta^{\frac{q}{2}} C_4 - \gamma \right) \int_0^1 \int_\Omega (u-v) (u-v) \cdot dx \cdot d\tau. \quad \text{(B.26)}
\]
Using the lemma A.7, we can rewrite the inequality (B.26) as follows
\[
\frac{1}{2} \int_0^1 \int_\Omega C_D^e (u-v)^2 dx \, d\tau \\
\leq \left( \mu L(2) - k \mu \eta C_4 - \gamma \right) \int_0^1 \int_\Omega (u-v)^2 dx \, d\tau. 
\text{(B.27)}
\]
Because \( k, \mu, \eta > 0, \gamma \geq 1, \) applying left Caputo fractional differential operator \( C_D^e \) to both sides of (B.27) and using the lemma A.7 we get
\[
\frac{1}{2} \int_\Omega (u-v)^2 \, dx \leq \frac{\mu L(2)}{\eta C_4} \int_\Omega (u-v)^2 \, dx. 
\]
Then, from the weakly singular Gronwall’s inequality [see33 Lemma 7.1.1]
\[
\int_\Omega (u-v)^2 \, dx = 0 \Leftrightarrow (u-v)_+ = 0, \quad x \in \Omega.
\]
Finally, using the fact \((u-v)_+ = \max \{u-v, 0\},\) it follows that \( u \leq v \) almost everywhere for \((x,t) \in \Omega_r.\)

Remark B.5 The method of proving this Lemma is similar to [2, Theorem 3.2]. But the biggest difference between the two papers is that the range of \((-\Delta)^p\) value is different. This paper is \((-\Delta)^p, p > 1,\) and the literature4 is \(p \geq 2.\) Not only that, but also slightly different about the shrinking part of the non-local items for space-time fractional diffusion equation.

Appendix C. EXISTENCE AND UNIQUE WEAK SOLUTION FOR (V.2), (V.3)

A. Weighted \(L^1\) estimates in the nonlinear fractional diffusion range

In order to state the main estimate we introduce that concept of weighted mass at time \(0 \leq t < T\)
\[
X(t; u,v, \varphi) = \int_{\mathbb{R}^N} (u(t) - v(t)) \varphi dx,
\text{(C.1)}
\]
where the weight \(\varphi\) is a positive function to be specified next. From34, we introduce the operator \(\mathcal{M}_f\) by the formula
\[
(\mathcal{M}_f \varphi)(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{|\varphi(x,t) - \varphi(y,t)|}{|x-y|^{N+2p'}} \, dy.
\text{(C.2)}
\]
We remark that when \(0 < 2p' < 1\) this operator is well-defined and bounded for bounded and uniformly Lipschitz continuous function since the singularity at \(x = y\) is integrable.

The class \(C = \mathcal{C}(s,p,m)\). The class of suitable weight function for our main estimate is formed by the smooth and positive functions \(\varphi\) defined in \(\mathbb{R}^N\) such that \(\mathcal{M}_{sp/2}(\varphi)\) is locally bounded and
\[
C(\varphi) = \int_{\mathbb{R}^N} \frac{|\mathcal{M}_{sp/2}(\varphi)(x)|^{1-m(p-1)}}{\varphi(x)^{1-m(p-1)}} \, dx < \infty. 
\text{(C.3)}
\]
Note that this class depends on \(s, p, m\). The conditions \(s' = sp/2 < 1\) ensures that the class contains a large class of uniformly Lipschitz functions depending on our choice of \(s, p\) and \(m\). The value of \(C(\varphi)\) only depends on the positivity, smoothness and behaviour of \(\varphi(x)\) as \(|x| \to \infty.\)

Admissible decay rates. From35, there are many smooth, bounded and positive function \(\varphi\) decaying at infinity like a power \(\varphi \sim O(|x|^{-(N+\gamma)})\) with \(\gamma > 2s'\) such that \((-\Delta)^p \varphi\) decays like \(O(|x|^{-(N+2s')})\) as \(|x| \to \infty,\) assuming that \(0 < s' < 1\). We can check that \(\mathcal{M}_p \varphi\) decays in the same way if \(2s' < 1.\)

We have
\[
|\mathcal{M}_{sp/2}(\varphi)(x)|^{1-m(p-1)} dx \sim |x|^{-\mu}
\]
with
\[
\mu = \frac{(N+2s')}{1-m(p-1)} - \frac{(N+\gamma)(m(p-1))}{1-m(p-1)}. 
\]
The expression is integrable if \(\mu > N\). Working out the details we find that \(C(\varphi)\) is finite if \(\gamma < \frac{2s'}{m(p-1)}\).

Theorem C.1 (Weighted \(L^1\) estimates). Let \(0 < s < 1, 1 < p < 2, 0 < m < \frac{1}{p-1}, \varepsilon > 0\) with \(sp < 1\). Let \(u \geq v\) be two nonnegative semigroup solution of (V.1) in a strip \(Q_T = \mathbb{R}^N \times (0,T)\) with \(T > 0\) and \(X(t)\) be as in (C.1). Then for all \(\varphi \in \mathcal{C}(s,p,m)\) there is a finite constant \(K > 0\) depending on \(\varphi, \varepsilon\) such that we have
\[
|X^{1-m(p-1)}(t_1) - X^{1-m(p-1)}(t_2)| \\
\leq K(\varphi) |t_1 - t_2|^{\alpha[1-m(p-1)]}. 
\text{(C.4)}
\]
Actually, we may take \(K(\varphi) = \varepsilon \left( \frac{C(\varphi)}{C(\varphi)} \right)^{1-m(p-1)} \) with \(C(\varphi)\) given by (C.3).

Proof 7 We multiply an \(L^2\) solution by a smooth and positive
test function $\phi(x), u \geq v$, and from Lemma [A.4] we have
\[ \left| \int_{\mathbb{R}^N} \left( \frac{\partial}{\partial t} D^p u(x,t) - D^p (x, t) \phi(x) \right) dx \right| \]
\[ = \left| \int_{\mathbb{R}^N} (u(x,t) - v(x,t)) \phi(x) dx \right| \]
\[ \leq \left| \int_{\mathbb{R}^N} \left( (\Delta)^p u_m - (\Delta)^p v_m \right) \phi(x) dx \right| + \left| \int_{\mathbb{R}^N} (u^2 - v^2) \phi dx \right| \]
\[ \leq \left| \int_{\mathbb{R}^N} \left( (u(x,t) - v(x,t)) \phi(x) \right) dx \right| + \left| \int_{\mathbb{R}^N} \left( (u^2 - v^2) \phi dx \right) \right| \]
\[ = \int_{\mathbb{R}^N} \left( (u(x,t) - v(x,t)) \phi(x) \right) dx \]
\[ 
\]
By [C.7] we can get
\[ \left| \frac{\partial}{\partial t} D^p X(t) \right| \leq C(\phi)^{1-m(p-1)} X(t)^{m(p-1)} + L(2) X(t). \]

From Young’s inequality, let $a = C(\phi)^{1-m(p-1)}, b = X(t)^{m(p-1)}, q = \frac{1}{m(p-1)}, p = \frac{1}{m(p-1)}$, then we have
\[ C(\phi)^{1-m(p-1)} X(t)^{m(p-1)} \leq \frac{1}{e^{m(p-1)}} C(\phi) + \frac{m(p-1)}{e^{m(p-1)}} X(t). \]

So we obtain
\[ \frac{\partial}{\partial t} D^p X(t) \leq \frac{1}{e^{m(p-1)}} C(\phi) + \frac{m(p-1)}{e^{m(p-1)}} X(t). \] (C.5)

and by Lemma [A.3] and $a^\alpha - b^\alpha \leq (a - b)^\alpha, 0 < \alpha < 1$, the above fractional differential inequality [C.5] on $(t_1, t_2)$ with $t_1, t_2$ $\geq 0$ gives the result [C.4].

**Remark C.1** The main references for the proof of this lemma are [A.2] and [A.3]. The first difference from above two literature is that we replace the partial derivative $\partial u$ in the equation with the Caputo fractional derivative $D^p u$ and the nonlinear fractional diffusion term $(\Delta)^p u_m$. Secondly, there are no partial terms in the above two literature, and the main references for the treatment of non-local terms is scaled down. Finally, we also take advantage of the scaling of fractional differential inequalities in the proof process.

**B Proof of Theorem C.2**

**Definition C.1** A function $u$ is a weak solution to the problem (V.1) if:

- $u \in L^2((0, T]; H^1_0(\mathbb{R}^N)) \cap C((0, T]; L^1(Q_T))$, where $Q_T = \mathbb{R}^N \times (0, T)$ and $|u|^{m-1} u \in L^2_{loc}((0, T]; W^{s-p}(\mathbb{R}^N))$;
- identity
\[ \int_0^T \int_{\mathbb{R}^N} u^\alpha \phi dxdt \]
\[ + \int_0^T \int_{\mathbb{R}^N} \left( (u^m(x,t) - u^m(y,t))^{p-1} (\phi(x,t) - \phi(y,t)) \right) \]
\[ \frac{1}{|x-y|^{N+sp}} dx dy dt \]
\[ = \int_0^T \int_{\mathbb{R}^N} f \phi dx dt \]
holds for every $\phi \in C^0(\mathbb{R}^N \times [0, T])$;
- $u(\cdot, 0) = u_0 \geq 0$ and $\int u_0(x) \phi(x) dx < \infty$ for some admissible test function $\phi \geq 0$ in the class $C(s, p, m)$.

**Definition C.2** We say that the problem (V.1) with initial data $u_0 \in L^2(\mathbb{R}^N)$ has a unique strong solution $u \in C((0, T]; L^2(\mathbb{R}^N))$, if moreover
\[ \partial^\alpha u \in L^m((\tau, \infty); L^1(\mathbb{R}^N)) \cap L^2(\mathbb{R}^N), \]
for every $t, \tau > 0$. 

\[ (-\Delta)^p u \in L^2(\mathbb{R}^N) \]
Remark C.2 The above two definitions are mainly combined with reference\textsuperscript{38,39}. On the one hand\textsuperscript{38,37}, gives the definition of strong solution for time-space fractional nonlinear diffusion equations with $\partial_t^\alpha u \in L^\infty((\tau, \infty); L^1(\mathbb{R}^N))$, $\tau > 0$. On the other hand\textsuperscript{38}, gives the definition of existence of strong solution for fractional $p$-Laplacian evolution equation with $u_t$ and $(-\Delta)^\alpha u \in L^2(\mathbb{R}^N)$ for every $t > 0$.

Theorem C.2 [Existence] Let $sp < 1, 0 < m < \frac{1}{p-1}$ and let us consider the problem (\ref{eq:boundary}) if $\alpha$ is a weak solution of viscous fusion equations with $u$. Let $\partial_t u \in L^2(\mathbb{R}^N)$ by $V$ and define

$$ D(L) \subset \left\{ u \in V : \frac{\partial^\alpha u}{\partial t^\alpha} \in V^* \right\}. $$

We note that the operator $L$ is linear, densely defined and $m$-accretive, see\textsuperscript{39}. Next, by\textsuperscript{32}, let $\phi \in C^0$ and $\phi \alpha$ the scaling of $\phi$. Let $0 \leq \phi \leq L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a non-decreasing sequence of initial data $u_{0,n}$ converging monotonically to $u_0 \in L^1(\mathbb{R}^N, \phi dx)$. By the Monotone Convergence Theorem, it follows that $\int_{\mathbb{R}^N} u_{0,n} \phi dx \to 0$ as $n \to \infty$.

We prove the existence of the monotone limit of the approximating solutions. From Definition\textsuperscript{C.2} considering the unique strong solution $u_n(x,t)$ of Eq.\textsuperscript{\ref{eq:boundary}} with initial data $u_{0,n}$. By the comparison results of\textsuperscript{39}, we know that the sequence of solutions is a monotone sequence. The weighted estimates of previous section imply that the sequence is bounded in $L^1(\mathbb{R}^N, \phi dx)$ uniformly in $t \in [0, T]$.

$$ \left( \int_{\mathbb{R}^N} u_n(x,t) \phi(x) dx \right)^{1-m(p-1)} \leq \left( \int_{\mathbb{R}^N} u_n(x,0) \phi(x) dx \right)^{1-m(p-1)} + K(\phi) \alpha^{1-m(p-1)} \tag{C.6} $$

By the Monotone Convergence Theorem in $L^1(\mathbb{R}^N, \phi dx)$, we know that the solution $u_n(x,t)$ converge monotonically as $n \to \infty$ to a function $u(x,t) \in L^\infty((0, T); L^1(\mathbb{R}^N, \phi dx))$. We also have

$$ \left( \int_{\mathbb{R}^N} u_n(x,t) \phi(x) dx \right)^{1-m(p-1)} \leq \left( \int_{\mathbb{R}^N} u_n(x,0) \phi(x) dx \right)^{1-m(p-1)} + K(\phi) \alpha^{1-m(p-1)}. $$

(3). Next, we show that the function $u(x,t)$ is a weak solution to Eq.\textsuperscript{\ref{eq:boundary}} in $\mathbb{R}^N \times [0, T]$. We know that each $u_n$ is a bounded strong solution according to Definition\textsuperscript{C.2} since the initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Therefore, for all $\psi \in C^0_c(\mathbb{R}^N \times [0, T])$ we have

$$ \int_0^T \int_{\mathbb{R}^N} u_n(x,t) \frac{\partial^\alpha \psi}{\partial t^\alpha} dx dt = \int_0^T \int_{\mathbb{R}^N} \left( u_n^m(x,t) - u_n^m(y,t) \right)^{p-1} (\psi(x,t) - \psi(y,t)) dx dy dt \tag{C.7} $$

Taking the limit $n \to \infty$ in the first line is easy:

$$ \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^N} u_n(x,t) \frac{\partial^\alpha \psi}{\partial t^\alpha} dx dt = \int_0^T \int_{\mathbb{R}^N} u(x,t) \frac{\partial^\alpha \psi}{\partial t^\alpha} dx dt $$

since $\psi$ is compactly supported and we already know that $u_n(x,t) \to u(x,t)$ in $L^\infty$. (4). On the other hand, the integral in the second line $I(u_n)$, is well defined and can be estimated uniformly. We argue as before, by using the inequality

$$ \left| u_n^m(x,t) - u_n^m(y,t) \right|^{p-1} \leq u_n(x,t)^{m(p-1)} + u_n(y,t)^{m(p-1)}, $$

and bounding the first of the two ensuring integrals by

$$ I_1(u_n) = \int_{\mathbb{R}^N} dt \left( \int_{\mathbb{R}^N} u_n(x,t)^{m(p-1)} M(x,t) dx \right), $$

where

$$ M(x,t) = \int_{\mathbb{R}^N} \left| \frac{\psi(x,t) - \psi(y,t)}{|x-y|^{N+sp}} \right| dy. $$

Due to the regularity of $\psi$, the last integral is bounded above by some $M_\psi(x)$ that behaves like $C(1+|x|)^{-(N+sp)}$ independently of $t$. Hence,

$$ I_1(u_n) \leq \int_{\mathbb{R}^N} dt \left( \int_{\mathbb{R}^N} u_n(x,t) \phi dx \right)^{m(p-1)} \left( \int_{\mathbb{R}^N} \frac{M_\psi(x)^{1-m(p-1)}}{\phi^{1-m(p-1)}} dx \right)^{1-m(p-1)}, $$

so that finally

$$ I_1(u_n) \leq C(\psi, \phi) \int_{\mathbb{R}^N} dt \left( \int_{\mathbb{R}^N} u_n(x,t) \phi dx \right)^{m(p-1)} \leq CT $$

The second integral $I_2(u_n)$ is treated similarly by exchanging $x$ and $y$.

(5). We call that $u \geq u_n$, and we may apply the argument of the previous paragraph to $v_n = u - u_n$ as follows

$$ I(u) - I(u_n) = \int_0^T \int_{\mathbb{R}^N} A(x,y,t)(\psi(x,t) - \psi(y,t)) dx dy dt \tag{C.8} $$

with

$$ A(x,y,t) = (u^m(x,t) - u^m(y,t))^{p-1} - (u_n^m(x,t) - u_n^m(y,t))^{p-1}. $$

Using the numerical inequality \( |a^m - b^m| \leq 2^{1-m} |a - b|^m \) and (a) we conclude that for all possible values of \((u(x, t) - u(y, t))\) and \((u_n(x, t) - u_n(y, t))\) we have

\[
|A(x, y, t)| \leq \left( \int (u(x, t) - u(y, t))^{m(p-1)} - (u_n(x, t) - u_n(y, t))^{m(p-1)} \right) \leq c(m, p) \left| (u(x, t) - u(y, t)) - (u_n(x, t) - u_n(y, t)) \right|^{m(p-1)},
\]

so that by virtue of the previous estimates on the weighted convergence of \(u_n(x, t) \to u(x, t)\), then we obtain

\[
|I_1(u) - I_1(u_0)| \to 0 \quad \text{as} \quad n \to \infty.
\]

(6) The solutions constructed above for \(0 \leq u_0 \in L^1(\mathbb{R}^N, \phi dx)\) satisfies the weighted estimates C.2 so that

\[
\left| \int_{\mathbb{R}^N} u(x, t) \phi(x) dx - \int_{\mathbb{R}^N} u(x, \tau) \phi(x) dx \right| \leq C_1(2e)^{\frac{1}{2}} \frac{1}{\alpha \Gamma(\alpha)} \left| t - \tau \right|^\alpha
\]

which gives the continuity in \(L^1(\mathbb{R}^N, \phi dx)\). Therefore, the initial trace of this solution is given by \(u_0 \in L^1(\mathbb{R}^N, \phi dx)\).

**Remark C.3** The proof of the above solution mainly refers to the proof process outline of the Theorem 3.2 of [43]. Through the above two papers, the uniqueness of the solution for the problem (V.1) mainly proves that solution \(v \leq u\), and then the same argument can prove \(u \leq v\).

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