Many-body localization: construction of the emergent local conserved operators via block real-space renormalization

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Abstract. A fully many-body localized (FMBL) quantum disordered system is characterized by the emergence of an extensive number of local conserved operators that prevents the relaxation towards thermal equilibrium. These local conserved operators can be seen as the building blocks of the whole set of eigenstates. In this paper, we propose to construct them explicitly via block real-space renormalization. The principle is that each renormalization group step diagonalizes the smallest remaining blocks and produces a conserved operator for each block. The final output for a chain of $N$ spins is a hierarchical organization of the $N$ conserved operators with $\frac{\ln N}{\ln 2}$ layers. The system size nature of the conserved operators of the top layers is necessary to describe the possible long-range order of the excited eigenstates and the possible critical points between different FMBL phases. We discuss the similarities and the differences with the strong disorder RSRG-X method that generates the whole set of the $2^N$ eigenstates via a binary tree of $N$ layers. The approach is applied to the long-range quantum spin-glass Ising model, where the constructed excited eigenstates are found to be exactly like ground states in another disorder realization, so that they can be either in the paramagnetic phase, in the spin-glass phase or critical.

Keywords: disordered systems (theory), spin chains, ladders and planes (theory), quantum phase transitions (theory), renormalisation group
1. Introduction

The field of many-body localization (MBL) has attracted a lot of interest recently (see the recent reviews [1, 2] and references therein). The general goal is to understand the unitary dynamics of isolated random interacting quantum systems and to determine how they can avoid thermalization and remain non-ergodic.

One of the most important characterizations of the many-body localized phase is that excited eigenstates display an area-law entanglement [3] instead of the volume-law entanglement of thermalized eigenstates. This property has been used numerically to identify the MBL phase [4, 5] and to show consistency with the other criteria of MBL [5]. The fact that excited eigenstates in the MBL phase are similar to ground-states from the point of view of entanglement suggests that various approaches that have been developed for ground states in the past can actually be adapted to study the MBL phase. One example is the efficient representation via density-matrix-renormalization group (RG) or matrix product states [6–9] and tensor networks [10]. Another example is the strong disorder real-space RG approach (see [11, 12] for reviews) developed by Ma–Dasgupta–Hu [13] and Daniel Fisher [14, 15] to construct the ground states of random quantum spin chains, with its extension in higher dimensions \( d = 2, 3, 4 \) [16–26]. This approach has been extended into the strong disorder RG procedure for the unitary dynamics [32, 33], and into the
RSRG-X procedure in order to construct the whole set of excited eigenstates \[27–31\]. It should be stressed that these two strong disorder RG procedures, based on the spin variables, are limited to the MBL phase, whereas the current RG descriptions of the MBL transition towards delocalization are based on RG rules for the entanglement \[34\] or for the resonances \[35\].

Another recent essential idea in the field is the claim that a fully many-body localized system (FMBL) can be characterized by an extensive number of emergent localized conserved operators \[36–44\]. More precisely for a random quantum chain containing \(N\) spins \(\sigma\) described by Pauli matrices, it should be possible to introduce \(N\) localized pseudo-spin operators \(\tau_i^z\) that commute with each other and with the Hamiltonian. In terms of these localized pseudo-spins \(\tau_i\), the Hamiltonian can be rewritten as:

\[
H_{\text{FMBL}} = \sum_{k=1}^{N} \sum_{1 \leq h_1 < h_2 < \ldots < h_k \leq N} J^{(k)}_{h_1, h_2, \ldots, h_k} \prod_{q=1}^{k} \tau_i^z
\]

This contains \(2^N\) independent couplings \(J^{k}_{h_1, \ldots, h_k}\). It should be stressed that the diagonalization of any Hamiltonian with \(2^N\) energy levels can be rewritten as equation (1) since the \(2^N\) couplings are sufficient to reproduce the \(2^N\) energies. So the non-trivial statement in the FMBL Hamiltonian of equation (1) is that the pseudo-spins \(\tau_i^z\) are obtained from the real spins \(\sigma_i^z\) by a quasi-local unitary transformation, and that the couplings \(J^{k}_{h_1, \ldots, h_k}\) decay exponentially with the distance at a sufficient rate (see \[36–44\] and the reviews \[1, 2\] for more details). On the contrary, in the delocalized phase, the pseudo-spins \(\tau_i^z\) that diagonalize the Hamiltonian are delocalized over the whole chain.

Since the essential property of these conserved pseudo-spins \(\tau_i^z\) in the FMBL phase is their locality, it seems natural to try to generate them locally and homogeneously in space. In the present paper we thus propose to construct them via some block real-space renormalization procedures that generalize previous procedures introduced for ground states. The principle is that at each RG step, instead of always projecting onto the lowest energy subspace, one keeps the projection onto the lowest energy subspace and also the projection onto the highest energy subspace, so that the choice between the two subspaces corresponds to a conserved pseudo-spin. In spirit, the idea is thus very close to the RSRG-X method \[27–31\] where each decimation produces a bifurcation in the spectrum and identifies a conserved pseudo-spin. In practice, however, the output is somewhat different. For instance, for a chain of \(N\) spins with \(2^N\) eigenstates:

(i) the RSRG-X method produces a binary branching tree with \(N\) layers, corresponding to the \(N\) successive decimations, so that at the end of the procedure, the \(2^N\) leaves of the tree are the eigenstates.

(ii) the present block real-space approach produces a hierarchical tree with \((\log N / \log 2)\) layers for the conserved operators themselves: the first RG step produces in parallel \(N/2\) conserved operators associated to the blocks of \(L = 2\) sites, the second RG step
produces $\frac{N}{4}$ conserved operators associated to the blocks of $L = 4$ sites, and so on, i.e. the $k$ RG step produces $\frac{N}{2^k}$ conserved operators associated to the blocks of $L = 2^k$ sites. So the first RG steps produce an extensive number of very local conserved operators as expected. However, the few conserved operators associated to the last RG steps are system size and no longer ‘local’. Their system size nature, however, is necessary to describe the possible long-range order of the excited eigenstates and the possible critical points between different FMBL phases [27, 45].

The paper is organized as follows. We consider the random quantum Ising chain with random transverse fields and with random couplings that are either short-range in section 2, or long-range in section 3. Our conclusions are summarized in section 4.

2. Random quantum Ising chain

The quantum Ising chain

$$H = -\sum_i h_i \sigma_i^z - \sum_i J_{i,i+1} \sigma_i^x \sigma_{i+1}^x$$

with random transverse fields $h_i$ and random nearest-neighbor couplings $J_{i,i+1}$ is ‘trivially’ in the fully many-body localized phase, as a consequence of the Jordan–Wigner mapping onto free fermions. Nevertheless, we feel that it is useful as a toy model to see how the present approach works in the simplest possible case. In this section, we thus describe how the Fernandez–Pacheco self-dual block-RG used to construct first the ground state of the pure chain [46, 47] and then the ground state of the random chain [48–51] can be extended to construct the whole set of eigenstates.

2.1. Diagonalization of the intra-block Hamiltonian

For each block of two spins $(2i-1, 2i)$, the intra-block Hamiltonian chosen by Fernandez–Pacheco [46–51]

$$H_{(2i-1,2i)}^{\text{intra}} = -h_{2i-1} \sigma_{2i-1}^x - J_{2i-1,2i} \sigma_{2i-1}^z$$

commutes with $\sigma_{2i}^z$. So for each eigenvalue $S_{2i} = \pm 1$, the diagonalization of the effective Hamiltonian for the remaining odd spin $\sigma_{2i-1}$

$$h_{2i-1}^{\text{eff}}(S_{2i}) = -h_{2i-1} \sigma_{2i-1}^x - J_{2i-1,2i} S_{2i} \sigma_{2i-1}^z$$

leads to the two eigenvalues (each is independent of the value $S_{2i} = \pm 1$ as it should by symmetry)

$$\lambda_{2i-1}^{\pm}(S_{2i}) = \pm \sqrt{h_{2i-1}^2 + J_{2i-1,2i}^2}$$

with the following corresponding eigenvectors.
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$$\lambda_{2i-1}(S_{2i}) = \sqrt{\frac{1 + \frac{J_{2i-1}S_{2i}}{\sqrt{h_{2i-1} + J_{2i-1,2i}}}}{2}} |S_{2i-1} = +1, S_{2i} > + \sqrt{\frac{1 - \frac{J_{2i-1}S_{2i}}{\sqrt{h_{2i-1} + J_{2i-1,2i}}}}{2}} |S_{2i-1} = -1, S_{2i} >$$

$$\lambda_{2i-1}(S_{2i}) = - \sqrt{\frac{1 - \frac{J_{2i-1}S_{2i}}{\sqrt{h_{2i-1} + J_{2i-1,2i}}}}{2}} |S_{2i-1} = +1, S_{2i} > + \sqrt{\frac{1 + \frac{J_{2i-1}S_{2i}}{\sqrt{h_{2i-1} + J_{2i-1,2i}}}}{2}} |S_{2i-1} = -1, S_{2i} >$$

To construct the ground state [46–51], one chooses to keep only the two degenerate lowest states $|\lambda^- (S_{2i} = \pm) >$. Here, as in the RSRG-X method [27–31], we also wish to keep the two degenerate highest levels $|\lambda^+ (S_{2i} = \pm) >$. It is thus convenient to introduce a pseudo-spin $\tau_{R(2i)}$ for the choice between these two possible energy levels and a spin $\sigma_{R(2i)}$ to distinguish the two degenerate states within each energy level. More precisely, the four eigenstates are re-labeled as follows:

$$\lambda_{2i-1}^-(+) \equiv |\tau_{R(2i)}^z = +, \sigma_{R(2i)}^z = + >$$
$$\lambda_{2i-1}^-(-) \equiv |\tau_{R(2i)}^z = +, \sigma_{R(2i)}^z = - >$$
$$\lambda_{2i-1}^+(+) \equiv |\tau_{R(2i)}^z = -, \sigma_{R(2i)}^z = + >$$
$$\lambda_{2i-1}^+(+) \equiv |\tau_{R(2i)}^z = -, \sigma_{R(2i)}^z = - >$$

so that the intra-block Hamiltonian of equation (3) depends only on the pseudo-spin $\tau_{R(2i)}$

$$H_{(2i-1,2i)} = -\Omega_{R(2i)} \tau_{R(2i)}^z$$

with the field

$$\Omega_{R(2i)} = \sqrt{h_{2i-1}^2 + J_{2i-1,2i}^2}$$

Since this procedure is applied independently to all blocks $(2i-1, 2i)$, the total intra-block Hamiltonian reads in this new basis

$$H_{\text{intra}} = \sum_i H_{(2i-1,2i)}^{\text{intra}} = - \sum_i \Omega_{R(2i)} \tau_{R(2i)}^z$$

For a chain of $N$ spins, this intra-block Hamiltonian depends only on the $\frac{N}{2}$ pseudo-spins $\tau_{R(2i)}^z$, the flipping of which is associated with the 'large' fields $\Omega_{R(2i)}$, but is completely independent of the $\frac{N}{2}$ spins $\sigma_{R(2i)}$ where each flipping has 'no cost' yet when one takes into account only $H_{\text{intra}}$. In the following, we thus make the approximation that the pseudo-spins $\tau_{R(2i)}^z$ for $i = 1, 2, ..., \frac{N}{2}$ are the first $\frac{N}{2}$ emergent local conserved operators. This approximation is possible only when the initial transverse fields $h_i$ and/or the initial couplings $J_{i,i+1}$ are distributed with continuous probability distributions, so that there isn’t any exact degeneracy between the fields $\Omega_{R(2i)}$ of equation (9). (For instance for the pure chain where all fields $\Omega_{R(2i)}$ coincide, it is clear that this approximation would be meaningless.)
2.2. Renormalization of the extra-block Hamiltonian

The extra-block Hamiltonian defined in terms of the initial spins

$$H^\text{extra} = H - H^\text{intra} = - \sum_i h_{2i} \sigma_{2i}^x - \sum_i J_{2i-2} \sigma_{2i-2}^x \sigma_{2i-1}^z$$

(11)

has to be rewritten in terms of the new spin variables $\tau_{R(2i)}$, $\sigma_{R(2i)}$ introduced in equation (7). Within the approximation that the pseudo-spins $\tau_{R(2i)}$ are emergent local conserved operators, one neglects all terms involving the flip operators $\sigma_{R(2i)}^x$, and one obtains the following effective Hamiltonian for the remaining spins

$$H^\text{extra}_{\text{eff}} = - \sum_i h_{R(2i)} \sigma_{R(2i)}^x - \sum_i J_{R(2i-2), R(2i)} (\tau_{R(2i)}^x \sigma_{R(2i-2)}^x \sigma_{R(2i)}^z)$$

(12)

where the renormalized transverse field

$$h_{R(2i)} = \frac{h_{2i-1}}{\sqrt{h_{2i-1}^2 + J_{2i-1,2i}^2}}$$

(13)

has decreased $h_{R(2i)} \leq h_{2i}$ and where the renormalized coupling

$$J_{R(2i-2), R(2i)} (\tau_{R(2i)}^x) = \tau_{R(2i)}^x J_{2i-2,2i-1} \frac{J_{2i-1,2i}}{\sqrt{h_{2i-1}^2 + J_{2i-1,2i}^2}}$$

(14)

has also decreased in absolute value $|J_{R(2i-2), R(2i)}| \leq |J_{2i-2,2i-1}|$. The renormalized fields $h_{R(2i)}$ and couplings $J_{R(2i-2), R(2i)}$ are thus typically smaller than the initial fields and couplings, and thus also typically smaller than the fields $\Omega_{R(2i)}$ of equation (9) associated to the conserved pseudo-spins $\tau_{R(2i)}^x$.

Equations (13) and (14) are the same as in the RG rules for the ground state [48–50] except for the presence of the pseudo-spin $\tau_{R(2i)}^x$ that can change the sign of the effective coupling between the spins $\sigma_{R(2i-2)}^x$ and $\sigma_{R(2i)}^z$.

2.3. Comparison with the RSRG-X method

The differences with the RSRG-X method are as follows:

(i) here one makes an a priori arbitrary choice with the couplings $(h_{2i-1}, J_{2i-1,2i})$ concerning the odd spins, while the RSRG-X method selects the biggest coupling of the whole chain at each step.

(ii) here the renormalized field $\Omega_{R(2i)} = \sqrt{h_{2i-1}^2 + J_{2i-1,2i}^2}$ that also appear in the eigenstates of equation (6), and in the RG rules of Equations (13) and (14), take into account any ratio $h_{2i-1}/J_{2i-1,2i}$ between the transverse field $h_{2i-1}$ and the coupling $J_{2i-1,2i}$, whereas in the RSRG-X method it is approximated by the maximal coupling, namely either $h_{2i-1}$ or $J_{2i-1,2i}$.

As for the RSRG method for the ground state [14, 15], the RSRG-X method [27] is expected to become asymptotically exact at large RG steps when the RG flow is towards...
an infinite disorder fixed point, which happens at the critical point between two many-body-localized phases [27]. However, for the present goal to construct the extensive local conserved operators of the MBL phase, it is clear that the majority of the pseudo-spins are produced by the first RG steps on small scales, and not by the asymptotic RG flow on large scales. In a previous work [50] concerning the Shannon and Rényi entropies of the ground state which are also dominated by the first RG steps on small scales, it was found that the block RG approach gives a better approximation of the multifractal dimensions than the RSRG. So here also we expect that the better RG rules of (ii) overcompensate the drawback of the arbitrary choice of (i) and that the block RG will produce a better approximation for the whole set of local conserved operators.

2.4. Iteration of the renormalization process

In summary, at the end of the first RG step, the spectrum of the $2^N$ levels of the chain containing $N$ spins has been decomposed into $2^N$ groups of $2^N$ levels each. A given group is labeled by the values of the $\frac{N}{2}$ pseudo-spins $\tau^z_{R(2i)} = \pm 1$ and for reference energy has the contribution of $H^{\text{intra}}$ equation (10)

$$ E_R^\text{e}(\{\tau^z_{R(2i)}\}) = -\sum_{i=1}^{2^{N-1}} \Omega_{R(2i)} \tau^z_{R(2i)} $$

(15)

With respect to this reference energy, the $2^N$ levels are those of the quantum Ising chain described by the effective Hamiltonian $H^{\text{extra}}_{\text{eff}}$ of equation (12) for the $\frac{N}{2}$ spins $\sigma^z_{R(2i)}$ with the smaller renormalized fields $h_{R(2i)} \leq h_2$, and the smaller renormalized couplings $|J_{R(2i-2),R(2i)}| \leq |J_{2i-2,2i-1}|$.

The RG procedure may be iterated as follows. Each group of $2^N$ levels will be subdivided into $2^N$ sub-groups of $2^N$ levels each. Each subgroup will be labeled by the $\frac{N}{2}$ pseudo-spins $\tau^z_{R(2i)} = \pm 1$ of the first generation that define the group and by the subsequent choice of the $\frac{N}{2}$ pseudo-spins $\tau^z_{R(i)} = \pm 1$ of the second RG step, so that the reference energy of this subgroup is now

$$ E_{R'}^\text{e}(\{\tau^z_{R(2i)}\};\{\tau^z_{R(i)}\}) = -\sum_{i=1}^{2^{N-1}} \Omega_{R(2i)} \tau^z_{R(2i)} - \sum_{i=1}^{2^{N-2}} \Omega_{R'(4i)} \tau^z_{R'(4i)} $$

(16)

where the fields of the second generation

$$ \Omega_{R'(4i)} = \sqrt{h^2_{R(2i-2),R(2i)} + J^2_{R(2i-2),R(2i)}} $$

(17)

are actually independent of the multiplicative sign $\tau^z_{R(2i)}$ appearing in the renormalized coupling in equation (14). However, the second generation pseudo-spin $\tau^z_{R'(4i)}$ itself depends on the two pseudo-spins values $\tau^z_{R(4i-2)} = \pm 1$ and $\tau^z_{R(4i)} = \pm 1$ of the first generation. Of course, if one insists on reproducing the form of equation (1), it is always possible to rewrite any function of these two pseudo-spins as the polynomial

$$ f(\tau^z_{R(4i-2)},\tau^z_{R(4i)}) = f(0,0) + f(1,0)\tau^z_{R(4i-2)} + f(0,1)\tau^z_{R(4i)} + f(1,1)\tau^z_{R(4i-2)}\tau^z_{R(4i)} $$

(18)
where the four coefficients are chosen to reproduce the only four possible values $f(\pm 1, \pm 1)$. But in the present framework, it is simpler to stick to the pseudo-spin $\tau^z\tau^z\tau^z\tau^z$ as defined by the renormalization process in each sector, and to keep in mind that it depends on the lower-generation pseudo-spins within its block.

With respect to the reference energy of equation (16), the $2^N$ levels of this subgroup are those of the quantum Ising chain of $2^N/4$ spins $\sigma^z\sigma^z\sigma^z\sigma^z$ with the appropriate renormalized transverse field and coupling obtained by the next iteration of the RG rules of equations (13) and (14):

$$h_{R^{(4i)}} = h_{R^{(4i)}} \frac{h_{R^{(4i-2)}}}{\sqrt{h_{R^{(4i-2)}} + J_{R^{(4i-2)}, R^{(4i)}}^2}}$$

$$J_{R^{(4i-4)}, R^{(4i)}} = \tau^z_{R^{(4i)}}J_{R^{(4i-4)}, R^{(4i-2)}} \frac{J_{R^{(4i-2)}, R^{(4i)}}}{\sqrt{h_{R^{(4i-2)}} + J_{R^{(4i-2)}, R^{(4i)}}^2}}$$

(19)

For a chain of $N = 2^n$ spins, the procedure ends after the $n$th RG step: the choice of $2^{n-1}$ pseudo-spins $\tau_{R^{(2i)}}$ of the first generation has been followed by the choice of $2^{n-2}$ pseudo-spins $\tau_{R^{(4i)}}$ of the second generation, then by the choice of $2^{n-3}$ pseudo-spins $\tau_{R^{(8i)}}$ of the third generation, etc up to the choice of $2^0 = 1$ pseudo-spin $\tau_{R^{(2^n)}}$ of the $n$th last generation; there remains a single spin $\sigma_{R^{(2^n)}}$ with its renormalized field $h_{R^{(2^n)}}$ so that the Hamiltonian is fully diagonalized and reads

$$H_{R^{(2^n)}} = -\sum_{i=1}^{2^{n-1}} \Omega_{R^{(2i)}} \tau_{R^{(2i)}}^z - \sum_{i=1}^{2^{n-2}} \Omega_{R^{(4i)}} \tau_{R^{(4i)}}^z - \sum_{i=1}^{2^{n-3}} \Omega_{R^{(8i)}} \tau_{R^{(8i)}}^z$$

$$- \Omega_{R^{(2^n)}} \tau_{R^{(2^n)}}^z - h_{R^{(2^n)}} \sigma_{R^{(2^n)}}^z$$

(20)

i.e. $\sigma_{R^{(2^n)}}^z$ is the last $N$th conserved operator.

Note that equation (20) contains only $2^n = N$ parameters (namely $2^n - 1$ parameters $\Omega_{R^{(k)}}$ of the various generations $1 \leq k \leq n$ and one parameter $h_{R^{(2^n)}}$) instead of the $2^N$ coefficients of equation (1) to parametrize the $2^N$ energies: this is a consequence of the free-fermion nature of the quantum Ising chain of equation (2). For models that cannot be reduced to free-fermions, the diagonalized Hamiltonian will be more complicated than equation (20), as shown explicitly in section 3.

### 2.5. Properties of the eigenstates

In terms of the ratio

$$K_{i-1} = \frac{J_{i-1,i}}{h_i}$$

(21)

the RG rules of equations (13) and (14) reduce to the simple multiplicative rule

$$K_{R^{(2i-2)}} \equiv \frac{J_{R^{(2i-2)}, R^{(2i)}}}{h_{R^{(2i)}}} = \tau_{R^{(2i)}}^z K_{2i-2} K_{2i-1}$$

(22)
The conclusions for the amplitudes $|K|$ are thus exactly the same as for the ground state case corresponding to the choice $\tau_{R(2n)} = +1$ discussed in detail in previous works [48–50]:

(i) in the region $\ln h_i > \ln J_{i,i+1}$, the flow is attracted towards $K \to 0$ and the eigenstates are paramagnetic.

(ii) in the region $\ln h_i < \ln J_{i,i+1}$, the flow is attracted towards $|K| \to +\infty$ and the eigenstates display a long-range order adapted to the signs $\tau^i = \pm 1$. In particular, in the middle of the spectrum, as a consequence of the random signs $\tau^i = \pm 1$, the eigenstates are in the spin-glass phase for any initial distribution of the couplings (even if the initial model is ferromagnetic $J_{i,i+1} > 0$).

(iii) at the phase transition $\ln h_i = \ln J_{i,i+1}$, the eigenstates are critical and described by the infinite disorder fixed point with the activated exponent $\psi = 1/2$, with the typical correlation length exponent $\nu_{\text{typ}} = 1$ and the finite size correlation exponent $\nu_{\text{FS}} = 2$, in agreement with the exact Fisher-Strong disorder results [15].

(iv) the eigenstates also have exactly the same Shannon–Rényi entropies as the ground state studied in [50], since the signs $\tau^i$ completely disappear from the RG rules for the Shannon–Rényi entropies.

The entanglement properties of the excited eigenstates are also exactly the same as for the ground state, namely an area-law (i.e. a constant here in $d = 1$) outside criticality, and a logarithmic violation at criticality, the exact behavior of which has been computed via the strong disorder RG approach [52, 53].

3. Long-range Dyson quantum Ising model

Many-body localization has been studied for various long-range power-law interactions [54–62]. Any long-range model has a Dyson hierarchical analog where real-space renormalization procedures are easier to define and to study: since its introduction for the classical ferromagnetic Ising model [63–72], this framework has been used recently for many long-range disordered models, either classical like random fields Ising models [73–75] and spin-glasses [76–82], or quantum like Anderson localization models [83–90] and quantum spin models [91, 92]. In this section, we consider the Dyson hierarchical version of the long-range quantum Ising chain

$$ H = -\sum_i h_i \sigma_i^z - \sum_{i<j} J_{i,j} \sigma_i^x \sigma_j^x $$

(23)

where the properties of the long-range couplings $J_{i,j}$, namely their decay with distance $(j - i)$ and their signs, have to be specified.
3.1. Dyson hierarchical version of the long-range quantum Ising chain

The Dyson quantum Ising Hamiltonian for \( N = 2^n \) spins is defined as a sum over the generations \( k = 0, 1, \ldots, n-1 \)

\[
H_{(1,2^k)} = \sum_{k=0}^{n-1} H_{(1,2^k)}^{(k)}
\]

The Hamiltonian of lowest generation \( k = 0 \) contains the transverse fields \( h_i \) and the lowest order couplings \( J_{2i-1,2i}^{(0)} \) associated to the length \( L_0 = 2^0 = 1 \)

\[
H_{(1,2^k)}^{(k=0)} = -\sum_{i=1}^{2^k} h_i \sigma_i^z - \sum_{i=1}^{2^{k-1}} J_{2i-1,2i}^{(0)} \sigma_{2i}^x \sigma_{2i}^z
\]

The Hamiltonian of next generation \( k = 1 \) contains couplings \( J^{(1)} \) associated to the length \( L_1 = 2^1 = 2 \)

\[
H_{(1,2^k)}^{(k=1)} = -\sum_{i=1}^{2^{k-2}} \left[ J^{(1)}_{4i-3,4i-1} \sigma_i^x \sigma_{4i-1}^x + J^{(1)}_{4i-3,4i-1} \sigma_i^z \sigma_{4i-1}^z + J^{(1)}_{4i-2,4i-1} \sigma_i^x \sigma_{4i-1}^z + J^{(1)}_{4i-2,4i-1} \sigma_i^z \sigma_{4i-1}^x \right]
\]

and so on up to the last generation \( k = n-1 \) associated to the length \( L_{n-1} = 2^{n-1} = \frac{N}{2} \) that couples all pairs of spins between the two halves of the system

\[
H_{(1,2^k)}^{(n-1)} = -\sum_{i=1}^{2^{k-1}} \sum_{j=2^{k-1}+1}^{2^k} J_{i,j}^{(n-1)} \sigma_i^x \sigma_j^x
\]

To mimic a power-law decay of the non-hierarchical model of equation (23)

\[
J(r) \propto \frac{1}{r^a}
\]

one chooses the same dependence for the scale of the Dyson coupling \( J^{(k)} \) of generation \( k \) as a function of the length \( L_k = 2^k \)

\[
J^{(k)} \propto \frac{1}{L_k^a} = 2^{-ka}
\]

Within a given generation \( k \), the couplings \( J_{i,j}^{(k)} \) may be chosen as uniform \( J_{i,j}^{(k)} = J^{(k)} = 2^{-ka} \) as in the Dyson quantum ferromagnetic Ising model with uniform or random transverse fields discussed in [91] or may be taken as random variables as in the Dyson quantum spin-glass discussed in [92]. In the following, we extend the real-space RG procedure introduced to construct the ground state of these models [91, 92] in order to also construct the excited states.

3.2. Diagonalization of the lowest generation \( k = 0 \)

The elementary renormalization step concerns the block two-spin Hamiltonian of generation \( k = 0 \)

\[
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\]
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\[ H_{(2i-1,2i)} = -h_{2i-1} \sigma_{2i-1}^z - h_{2i} \sigma_{2i}^z - J_{2i-1,2i}^{(0)} \sigma_{2i-1}^x \sigma_{2i}^x \]  

(30)

The diagonalization within the symmetric sector

\[ H_{(2i-1,2i)} |++> = -(h_{2i-1} + h_{2i}) |++> - J_{2i-1,2i}^{(0)} |--> \]
\[ H_{(2i-1,2i)} |--> = -J_{2i-1,2i}^{(0)} |++> + (h_{2i-1} + h_{2i}) |--> \]  

(31)

yields the two eigenvalues

\[ \lambda_{2i}^{S\pm} = \pm \sqrt{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} + h_{2i})^2} \]  

(32)

corresponding to the eigenvectors

\[ |\lambda_{2i}^{S-}> = \cos \theta_{2i}^S |++> + \sin \theta_{2i}^S |--> \]
\[ |\lambda_{2i}^{S+}> = -\sin \theta_{2i}^S |++> + \cos \theta_{2i}^S |--> \]  

(33)

in terms of the angle \( \theta_{2i}^S \) defined by

\[ \cos(\theta_{2i}^S) = \sqrt{\frac{1 + \frac{h_{2i-1} + h_{2i}}{\sqrt{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} + h_{2i})^2}}}{2}} \]
\[ \sin(\theta_{2i}^S) = \text{sgn}(J_{2i-1,2i}^{(0)}) \sqrt{\frac{1 - \frac{h_{2i-1} + h_{2i}}{\sqrt{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} + h_{2i})^2}}}{2}} \]  

(34)

Similarly the diagonalization within the antisymmetric sector

\[ H_{(2i-1,2i)} |+-> = -(h_{2i-1} - h_{2i}) |+-> - J_{2i-1,2i}^{(0)} |--> \]
\[ H_{(2i-1,2i)} |--> = -J_{2i-1,2i}^{(0)} |+-> + (h_{2i-1} - h_{2i}) |--> \]  

(35)

leads to the two eigenvalues

\[ \lambda_{2i}^{A\pm} = \pm \sqrt{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} - h_{2i})^2} \]  

(36)

with the corresponding eigenvectors

\[ |\lambda_{2i}^{A-}> = \cos \theta_{2i}^A |+-> + \sin \theta_{2i}^A |--> \]
\[ |\lambda_{2i}^{A+}> = -\sin \theta_{2i}^A |+-> + \cos \theta_{2i}^A |--> \]  

(37)

in terms of the angle \( \theta_A \) defined by

\[ \cos(\theta_{2i}^A) = \sqrt{\frac{1 + \frac{h_{2i-1} - h_{2i}}{\sqrt{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} - h_{2i})^2}}}{2}} \]
\[ \sin(\theta_{2i}^A) = \text{sgn}(J_{2i-1,2i}^{(0)}) \sqrt{\frac{1 - \frac{h_{2i-1} - h_{2i}}{\sqrt{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} - h_{2i})^2}}}{2}} \]  

(38)
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To construct the ground states, one chooses to keep only the two negative energy levels $\lambda_{2i}^S$ and $\lambda_{2i}^A$ [91, 92]. Here we also wish to keep the two positive energy levels $\lambda_{2i}^S$ and $\lambda_{2i}^A$. As in the previous section, it is thus convenient to introduce a pseudo-spin $\tau_{R(2i)} = \pm$ for the choice between the subspace of positive or negative eigenvalues, and another spin $\sigma_{R(2i)}$ for the subsequent choice between the symmetric and the anti-symmetric sectors. More precisely, we re-label the four eigenstates as follows:

$$\begin{align*}
\Lambda_{2i}^S^- &\equiv |\tau_{R(2i)}^+\sigma_{R(2i)}^+\rangle > \equiv = + > \\
\Lambda_{2i}^A^- &\equiv |\tau_{R(2i)}^+\sigma_{R(2i)}^-\rangle > \equiv = - > \\
\Lambda_{2i}^S^+ &\equiv |\tau_{R(2i)}^-\sigma_{R(2i)}^+\rangle > \equiv = + > \\
\Lambda_{2i}^A^+ &\equiv |\tau_{R(2i)}^-\sigma_{R(2i)}^-\rangle > \equiv = - > 
\end{align*}$$

The block two-spin Hamiltonian of equation (30) then reads:

$$H_{(2i-1,2i)} = - \Omega_{R(2i)} \tau_{R(2i)}^z - h_{R(2i)} \sigma_{R(2i)}^z$$

where the renormalized field for $\tau_{R(2i)}^z$

$$\Omega_{R(2i)} = \frac{\sqrt{(J_{2i-1,2i}^{0})^2 + (h_{2i-1} + h_{2i})^2} + \sqrt{(J_{2i-1,2i}^{0})^2 + (h_{2i-1} - h_{2i})^2}}{2}$$

is bigger than the renormalized field for $\sigma_{R(2i)}^z$

$$h_{R(2i)} = \frac{\sqrt{(J_{2i-1,2i}^{0})^2 + (h_{2i-1} + h_{2i})^2} - \sqrt{(J_{2i-1,2i}^{0})^2 + (h_{2i-1} - h_{2i})^2}}{2h_{2i-1}h_{2i}}$$

In the following, we thus make the approximation that the pseudo-spins $\tau_{R(2i)}^z$ are the first $n = 2^{n-1}$ emergent local conserved operators. As in the previous section, this approximation is possible only when the initial transverse fields $h_i$ and/or the initial couplings $J_{ij}^{(0)}$ are distributed with continuous probability distributions, so that there is not any exact degeneracy between the fields $\Omega_{R(2i)}$ of equation (41).

3.3. Renormalization of the couplings of higher generations $k > 0$

We need to compute the initial operators $\sigma_{i}^x$ on the new basis of equation (39). Within the approximation that the pseudo-spins $\tau_{R(2i)}^z$ are emergent local conserved operators, one neglects all terms involving the flip operators $(\tau_{R(2i)}^x, \tau_{R(2i)}^y)$, and one obtains

$$\begin{align*}
\sigma_{2i-1}^x &\approx c_{2i-1} \tau_{R(2i)}^z \sigma_{R(2i)}^z \\
\sigma_{2i}^x &\approx c_{2i} \sigma_{R(2i)}^z
\end{align*}$$

in terms of the constants

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\[ e_{2i-1} = (\cos \theta_{2i}^S \sin \theta_{2i}^A + \sin \theta_{2i}^S \cos \theta_{2i}^A) = \text{sgn}(J_{2i-1,2i}^{(0)}) \sqrt{1 + \frac{(J_{2i-1,2i}^{(0)})^2 - h_{2i-1}^2 + h_i^2}{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} - h_i)^2}} \]

\[ e_{2i} = (\cos \theta_{2i}^S \cos \theta_{2i}^A - \sin \theta_{2i}^S \sin \theta_{2i}^A) = \sqrt{1 + \frac{(J_{2i-1,2i}^{(0)})^2 + h_{2i-1}^2 - h_i^2}{(J_{2i-1,2i}^{(0)})^2 + (h_{2i-1} - h_i)^2}} \] (44)

As a consequence, the renormalized coupling between the spins \( \sigma_{R(2i)} \) and \( \sigma_{R(2j)} \) is given by the following linear combination of the four initial couplings of generation \( k \) associated to the positions \((2i - 1, 2i)\) and \((2j - 1, 2j)\):

\[ J_{R(2i),R(2j)} = J_{2i,2j}^{(k)} e_{2i} e_{2j} + \tau_{R(2i)}^z J_{2i-1,2i}^{(k)} e_{2i-1} e_{2i} + \tau_{R(2j)}^z J_{2j-1,2j}^{(k)} e_{2j-1} e_{2j} + \tau_{R(2i)R(2j)}^z J_{2i-1,2i}^{(k)} J_{2j-1,2j}^{(k)} e_{2i-1} e_{2j-1} \] (45)

So here, in contrast with the short-range RG rule of equation (14), the choice of the pseudo-spin values \( (\tau_{R(2i)}^z = \pm 1; \tau_{R(2j)}^z = \pm 1) \) will produce different amplitudes for the renormalized coupling in the different sectors.

3.4. Renormalization procedure for the non-hierarchical long-range model

The block renormalization procedure explained above can be applied to the non-hierarchical long-range model of equation (23). The only difference is that in the RG rule of equation (45), the initial couplings do not have exactly the same properties: \( J_{2i,2j} \) and \( J_{2i-1,2j-1} \) are associated with the distance \((2j - 2i)\), whereas \( J_{2i,2j-1} \) is associated with the slightly smaller distance \((2j - 2i - 1)\) and \( J_{2i-1,2j} \) is associated with the slightly bigger distance \((2j - 2i + 1)\). Whenever the long-range nature dominates over the short-range case, one expect that the initial power-law model of equation (23) and its Dyson hierarchical analog have the same scaling properties at a large scale. However, the RG flows for the Dyson analog are usually easier to study analytically and clearer to analyze numerically.

3.5. Iteration of the renormalization process

When the renormalization process is iterated as explained in section 2.4 with the appropriate renormalization rules of equations (41), (42) and (45), it should be stressed that not only the pseudo-spins of higher generation depend on the lower pseudo-spins, but also the amplitudes of the renormalized fields and couplings. More precisely, we may write the Hamiltonian after the first RG step

\[ H_{R(1,2)^{1}} = - \sum_{i=1}^{2^{n-1}} \Omega_{R(2i)} \tau_{R(2i)}^z - \sum_{i=1}^{2^{n-1}} h_{R(2i)} \sigma_{R(2i)}^z - \sum_{i<j} J_{R(2i),R(2j)}(\tau_{R(2i)}^z, \tau_{R(2j)}^z) \sigma_{R(2i)}^z \sigma_{R(2j)}^z \] (46)

and after the second RG step

\[ H_{R(1,2)^{2}} = - \sum_{i=1}^{2^{n-1}} \Omega_{R(2i)} \tau_{R(2i)}^z - \sum_{i=1}^{2^{n-1}} h_{R(2i)} \sigma_{R(2i)}^z - \sum_{i<j} J_{R(2i),R(2j)}(\tau_{R(2i)}^z, \tau_{R(2j)}^z) \sigma_{R(2i)}^z \sigma_{R(2j)}^z \]

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\[
H_{(1,2')}^{R} = -\sum_{i=1}^{2^{n-1}} \Omega_{R(2i)} \tau_{R(2i)}^z - \sum_{i=1}^{2^{n-2}} \Omega_{R^{(4i)}} (\tau_{R(4i-2)}^z \tau_{R(4i)}^z \tau_{R^{(4i)}}^z) - \sum_{i=1}^{2^{n-2}} h_{R^{(4i)}} (\tau_{R(4i-2)}^z \tau_{R(4i)}^z)
\]

\[
\sigma_{R(4i)} = -\sum_{i<j} J_{R(4i),R(4j)} (\tau_{R(4i-2)}^z \tau_{R(4i)}^z \tau_{R(4j-2)}^z \tau_{R(4j)}^z) \tau_{R^{(4i)}}^z \tau_{R^{(4j)}}^z \tau_{R^{(4i)}} \tau_{R^{(4j)}} \sigma_{R(4i)} \sigma_{R(4j)}
\]

and so on up the last \( n \)th RG step

\[
H_{(1,2')}^{R} = -\sum_{i=1}^{2^{n-1}} \Omega_{R(2i)} \tau_{R(2i)}^z - \sum_{i=1}^{2^{n-2}} \Omega_{R^{(4i)}} (\tau_{R(4i-2)}^z \tau_{R(4i)}^z \tau_{R^{(4i)}}^z) - \sum_{i=1}^{2^{n-3}} \Omega_{R^{(8i)}} (\tau_{R^{(8i-4)}}^z \tau_{R^{(8i)}}^z \tau_{R^{(8i-6)}}^z \tau_{R^{(8i-4)}}^z \tau_{R^{(8i-2)}}^z \tau_{R^{(8i)}}^z \tau_{R^{(8i-2)}}^z \tau_{R^{(8i)}}^z) - \cdots
\]

\[
- \Omega_{R^{(2')}} (\{\tau_{R^{(2')}}^z\}) \tau_{R^{(2')}}^z - h_{R^{(2')}} (\{\tau_{R^{(2')}}^z\}) \sigma_{R^{(2')}}^z
\]

(47)

To make the link with equation (1), one needs to expand the renormalized fields \( \Omega_{R^{(2')}} \) and \( h_{R^{(2')}} \) in terms of the pseudo-spins of lower generations using the principle of equation (18): each renormalized field of the second generation can be expanded into \( 2^2 = 4 \) terms

\[
\Omega_{R^{(4i)}} (\tau_{R(4i-2)}^z \tau_{R(4i)}^z) = \Omega_{R^{(4i)}}^{(0,0)} + \Omega_{R^{(4i)}}^{(1,0)} \tau_{R(4i-2)}^z \tau_{R(4i)}^z + \Omega_{R^{(4i)}}^{(0,1)} \tau_{R(4i-2)}^z \tau_{R(4i)}^z + \Omega_{R^{(4i)}}^{(1,1)} \tau_{R(4i-2)}^z \tau_{R(4i)}^z
\]

(49)

and so on. Each renormalized field \( \Omega_{R^{(8i)}} (\tau_{R^{(8i)}}^z) \) of the third generation can be expanded into \( 26 \) terms and so on. This rewriting will thus generate all possible products containing an arbitrary number of pseudo-spins as in equation (1).

As a consequence of the random sum structure of equation (48) over all blocks and all scales, one expects that at the middle of the spectrum near zero energy, two energy levels that happen to be consecutive (i.e. that have an exponentially small energy difference of order \( N^{-\frac{1}{2}} \)) with respect to the number \( N = 2^n \) spins) have completely different wavefunctions labelled by completely different values of the pseudo-spins.

### 3.6. Properties of the eigenstates for the Dyson quantum spin-glass model

To be more specific, let us now focus on the case of the Dyson quantum spin-glass model, where the initial couplings \( J_{i,j}^{(k)} \) of generation \( k \) are random Gaussian variables of zero mean and of variance

\[
\overline{(J_{i,j}^{(k)})^2} = 2^{-2\sigma} \frac{1}{L_{2^k}}
\]

(50)

The parameter \( \sigma \) governing the decay of the distance has to be in the region \( \sigma > 1/2 \) in order to have an extensive energy. The initial transverse fields are taken as uniform \( h_i = h \) (but of course the renormalized transverse fields are random as a consequence of the RG rule of equation (42)). For an excited state, the random choice of the pseudo-spins \( \tau_{R(2i)}^z = \pm \) in the RG rule of equation (45) simply amounts to randomly changing

\[
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\]

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the signs of the couplings that are already of random sign. As a consequence, the RG
flow is exactly the same as for the ground state in another disorder realization. We can
thus directly use the results concerning the RG flows for the ground state studied in
[92] as a function of the control parameter $h$: the constructed excited eigenstates are
similarly either in the paramagnetic phase, in the spin-glass phase or critical with a
finite dynamical exponent $z(\sigma)$.

4. Conclusion

In this paper, we have proposed to construct the extensive number of local conserved
operators characterizing a fully many-body localized quantum disordered system via
some block real-space renormalization. The general idea is that each RG step diagonal-
izes the smallest remaining blocks and produces a conserved pseudo-spin for each block:
for a chain of $N$ spins, the first RG step produces $N/2$ conserved operators associated
to the blocks of $L = 2$ sites, the second RG step produces $N/4$ conserved operators associated
to the blocks of $L = 4$ sites, and so on. The final output for a chain of $N$ spins is a hier-
archical organization of the conserved operators with $\text{int}(\log_2 N)$ layers. We have explained
why the system size nature of the conserved operators of the top layers is necessary to
describe the possible long-range spin-glass order of the excited eigenstates and the pos-
sible critical points between different FMBL phases. We have discussed the similarities
and the differences with the strong disorder RSRG-X method [27–31] that constructs
the whole set of the $2^N$ eigenstates via a binary tree of $N$ layers. The block RG construc-
tion of the whole spectrum has been described for the random quantum Ising chain,
first with short-range couplings and then with long-range couplings. In the spin-glass
models, we have established that the excited eigenstates are exactly like ground states
in another disorder realization, so that they can be either in the paramagnetic phase,
in the spin-glass phase or critical.

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