ESTIMATES FOR THE VARIABLE ORDER RIESZ POTENTIAL WITH APPLICATIONS

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Abstract. We study weak-type estimates and exponential integrability for the variable order Riesz potential. As an application we prove an exponential integrability result with respect to the Hausdorff content for functions from variable exponent Sobolev spaces. In particular, the earlier exponential integrability results are improved to a corresponding one with respect to the Choque integral whenever John domains are considered. Moreover, new exponential integrability results also for domains with outward cusps are obtained.

1. Introduction

If \( \Omega \) is a bounded, open set in the Euclidean space \( \mathbb{R}^n, n \geq 2 \), and \( \alpha \) is a continuous function which satisfies \( 0 < \alpha(x) < n \) for every \( x \in \Omega \), then we write for the operator \( I_{\alpha(x)} \) acting on locally integrable functions \( f \) in \( \Omega \)
\[
I_{\alpha(x)}(f)(x) := \int_{\Omega} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}} \, dy.
\]
This is called the variable order Riesz potential. By defining the variable dimensional Hausdorff content \( \mathcal{H}_{\infty}^{\alpha(x)} \) based on [15, 2.10.1, p. 169] in Definition 4.1 we prove that the level sets of the variable order Riesz potential \( I_{\alpha(x)}f \) are exponentially decaying with respect to the the variable dimensional Hausdorff content. We show

1.2. Theorem. Let \( \Omega \) be an open, bounded set in \( \mathbb{R}^n \) and let \( \alpha : \Omega \to [0, n) \) be a log-Hölder continuous function with

\[ 0 < \alpha^- := \text{ess inf}_{x \in \Omega} \alpha(x) \leq \alpha^+ := \text{ess sup}_{x \in \Omega} \alpha(x) < n. \]

Then, there exist constants \( c_1 \) and \( c_2 \) independent of the function \( f \) such that the inequality

\[
\mathcal{H}_{\infty}^{\alpha(x)}(\{x \in \Omega : I_{\alpha(x)}f(x) > t\}) \leq c_1 \exp(-c_2 t^\frac{\alpha^-}{n})
\]
holds for all \( f \in L^{n/\alpha(x)}(\Omega) \) with \( \|f\|_{L^{n/\alpha(x)}(\Omega)} \leq \frac{1}{2(1+|\Omega|)}. \)

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This result could be seen as a generalisation of the work of Ángel D. Martínez and Daniel Spector [31, Theorem 1.2] to the variable order case.

As an application of Theorem 1.2 we prove exponential integrability estimates for variable exponent Sobolev functions defined on bounded domains. This result is our second main theorem, Theorem 5.13 where we generalise the earlier known results of the exponential integrability which were with respect to the Lebesgue measure by taking the integration with respect to the Hausdorff content as the Choque integral as Martínez and Spector stated and proved for the usual Riesz potential in [31].

We state an important corollary of Theorem 5.13 which gives new results to the $s$-John domains. Examples of these domains are convex domains and domains with Lipschitz boundaries but also domains which are allowed to have outward cusps of the order $s$.

1.3. Corollary. Let $D$ be an $s$-John domain in $\mathbb{R}^n$, $n \geq 2$, with $1 \leq s < \frac{n}{n-1}$. Then, there exist positive constants $a$ and $b$ such that

$$\int_D \exp \left( a |u - u_B|^{\frac{s}{n-1}} \right) dH^{(n-1)}_\infty \leq b$$

for all $u \in L^1_{\frac{n}{n-s(n-1)}}(D)$ with $\|\nabla u\|_{L^{n/(n-s(n-1))}(D)} \leq 1$, $B := B(x_0, \text{dist}(x_0, \partial D))$.

In particular this corollary generalises [40, Theorem 2] and [12, Theorem 3.4] to the Choque integral case when $s = 1$, that is, for John domains. Corollary 1.3 gives a complete new result whenever $1 < s < n/(n-1)$.

We also prove a weak-type estimate for the variable order Riesz potential in Theorem 3.2. By this result Poincaré-type inequalities with respect to the Luxemburg norm on the variable exponent Lebesgue space are proved to be valid for $L^p_\lambda$-functions defined on a new class of domains in Theorems 5.6 and 5.8. The new class of domains suits well to the questions for variable exponent Lebesgue and Sobolev Spaces.

This paper is organized as follows. Definitions for variable exponent Lebesgue spaces and Sobolev spaces are recalled in Section 2. Estimates for the variable order Riesz potential acting on functions from the variable order Lebesgue spaces are given and proved in Section 3. The variable dimensional Hausdorff content is defined and its basic properties are proved in Section 4. The variable order maximal function is recalled there and point-wise estimates for the Riesz potential are proved there, too. Our main weak-type estimate Theorem 1.2 is proved in Section 4. Applications are considered in Section 5. We start by defining a new class of functions which includes $s$-John domains and then state and prove Poincaré-type inequalities of Theorems 5.6 and 5.8. there. Our main theorem in Section 5 is on the exponential integrability, Theorem 5.13 which yields Corollary 1.3.
2. Notations

Let $U$ in $\mathbb{R}^n$ be any bounded, open set. For any measurable function $g : U \to \mathbb{R}$ and measurable set $A \subset U$ we define

$$g_A^+ := \text{ess sup}_{x \in A} g(x) \quad \text{and} \quad g_A^- := \text{ess inf}_{x \in A} g(x).$$

In the case $A = U$ we write $g^+_U$ and $g^-_U$. We say that the function $g : U \to \mathbb{R}$ satisfies the log-Hölder continuity condition if there is a constant $c_1 > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_1}{\log(e + 1/|x - y|)}$$

for all $x, y \in U$, $x \neq y$. Note that $g$ is a log-Hölder continuous function if and only if $|B|^p g_B^{-p} g_B^p \leq 1$ for all open balls $B \cap U \neq \emptyset$.

By a variable exponent we mean a measurable function $p : U \to [1, \infty)$ such that $1 \leq p^- \leq p^+ < \infty$. The set of all variable exponents is denoted by $P(U)$. By $P_{\log}(U)$ we denote a subset consisting of all log-Hölder continuous variable exponents.

We define a modular on the set of Lebesgue measurable functions $f$ by setting

$$\varrho_{L^p(U)}(f) := \int_U |f(x)|^{p(x)} \, dx.$$

The variable exponent Lebesgue space $L^p(U)$ consists of all measurable functions $f : U \to \mathbb{R}$ for which the modular $\varrho_{L^p(U)}(f)$ is finite. We write $f \in L^p(U)$.

The Luxemburg–Nakano norm on this space is defined as

$$\|f\|_{L^p(U)} := \inf \left\{ \lambda > 0 : \varrho_{L^p(U)}(f/\lambda) \leq 1 \right\}.$$

Equipped with this norm, $L^p(U)$ is a Banach space. We use the abbreviation $\|f\|_{L^p(U)}$ to denote the norm in the whole space.

For open sets $U$, the variable exponent Sobolev space $L^{1,p}_{\text{loc}}(U)$ consists of functions $u \in L^{1}_{\text{loc}}(U)$ for which the absolute value of the distributional gradient $|\nabla u|$ belongs to $L^p(U)$:

$$L^{1,p}_{\text{loc}}(U) := \{ u \in W^{1,1}_{\text{loc}}(U) : \|\nabla u\| \in L^p(U) \},$$

where $W^{1,1}_{\text{loc}}(U)$ is the classical local Sobolev space.

More information and proofs for these facts can be found in [11, Chapters 2, 4, 8, and 9] or from [7].

3. Strong- and weak-type estimates

Let $\Omega$ be a bounded, open set in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. We assume that a continuous function $\alpha$ satisfies $0 < \alpha(x) < n$ for every $x \in \Omega$. We recall the definition (1.1) from Introduction. The potential in (1.1) is called the variable order Riesz potential of a function $f$. 

If $\alpha$ is a log-Hölder continuous function, then we have

$$I_{\alpha}f(x) \approx \int_{\Omega} \frac{|f(y)|}{|x - y|^{n-\alpha(y)}} \, dy,$$

we refer to [25, p. 270]. Let us define

$$\frac{1}{p^\#_\alpha(x)} := \frac{1}{p(x)} - \frac{\alpha(x)}{n} \quad \text{i.e.} \quad p^\#_\alpha(x) = \frac{np(x)}{n - \alpha(x)p(x)}.$$ 

The variable order Riesz potential has been studied extensively, see for example [13, 28, 32, 35, 36] and references therein. S. Samko has proved the following result.

3.1. **Lemma** (Theorem 3.2 of [36]). Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set. Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. Assume that $\alpha^- > 0$ and $(\alpha p)^+ < n$. Then $I_{\alpha} : L^{p}(\Omega) \to L^{p^\#}(\Omega)$ is bounded, where $p^\#_\alpha(x) = \frac{np(x)}{n - \alpha(x)p(x)}.$

Let us consider the case $p^- = 1$. It is well known that $I_1$, where $\alpha \equiv 1$, does not map $L^1(\Omega) \to L^1(\Omega)$. Thus instead of the strong-type estimate we can have only a weak-type estimate. For that we need to assume that $\alpha$ is log-Hölder continuous. The following theorem is a modification of [4, Theorem 4.3]. We consider a bounded set and obtain a better control for the extra term than in [4, Theorem 4.3].

3.2. **Theorem.** Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set. Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 \leq p^- \leq p^+ < \infty$. Assume that $\alpha$ is a log-Hölder continuous function with $\alpha^- > 0$ and $(\alpha p)^+ < n$. Let $p^\#_\alpha(x) = \frac{np(x)}{n - \alpha(x)p(x)}$, then there exists a constant $c$ such that for every $f \in L^{p}(\Omega)$ with $\|f\|_{p^\#} \leq 1$ and for every $t > 0$ the inequality

$$\int_{|x| \in \Omega : I_{\alpha}f(x) > t} p^\#_\alpha(x) \, dx \leq c \int_{\Omega} |f(y)|^{p^\#} \, dy + c |\{x \in \Omega : 0 < |f(x)| \leq 1\}|$$

holds. The constant $c$ depends only $n$, diam($\Omega$), $\alpha^-$, $(\alpha p)^+$, $p^+$ and log-Hölder constants of $p$ and $\alpha$.

**Proof.** As a part of the proof of [36, Theorem 3.2, p. 278] S. Samko proved the following point-wise inequality:

(3.3) $$|I_{\alpha}f(x)| \leq c|Mf(x)|^{\frac{\alpha}{p^\#_\alpha}}$$

for almost all $x \in \Omega$. Here $M$ is the Hardy–Littlewood maximal operator

$$Mf(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} |f| \, dy,$$

and the constant $c$ depends only on the dimension $n$, $\alpha^-$, $(\alpha p)^+$ and $p^+$. The proof is based on Hedberg’s trick, [24].

Thus we have

$$\{x \in \Omega : I_{\alpha}f(x) > t\} \subset \{x \in \Omega : c \left[Mf(x)\right]^{\frac{\alpha}{p^\#_\alpha}} > t\} =: E.$$
Let us write

\[ A_B f := \frac{1}{|B|} \int_{B \cap \Omega} |f| \, dy. \]

For every \( z \in E \) we choose \( B_z := B(z, r_z) \) such that \( c (A_B f)^{\frac{p(z)}{p_0(z)}} > t \) where \( c \) is the constant from (3.3). Let \( x \in B_z \) and let us raise this inequality to the power \( p_0(x) \). Assume first that \( \frac{p(z)}{p_0(z)} \geq p(x) \). From now on \( c \) may vary from line to line. Since \( \|f\|_{p(z)} \leq 1 \) we get \( A_B f \leq c |B_z|^{-1} \), and thus we obtain

\[
\begin{align*}
\| f \|_{p_0(z)} \leq c \left( A_B f \right)^{p_0(z)} & \leq c \left( A_B f \right)^{p_0(z)} |B_z|^{-1} p_0(z) \left( \frac{p(z)}{p_0(z)} \right) \\
= c \left( A_B f \right)^{p_0(z)} |B_z|^{-1} p_0(z) \left( \frac{p(z)}{p_0(z)} \right) & = c \left( A_B f \right)^{p_0(z)} |B_z|^{-1} p_0(z) \left( \frac{p(z)}{p_0(z)} \right).
\end{align*}
\]

If \( p(x) - p(z) > 0 \), then \( |B_z|^{p(x) - p_0(z)} \) is uniformly bounded by \( c(n) \text{diam}(\Omega)^{np^*} \). Otherwise we use log-Hölder continuity of \( p \) and obtain that \( |B_z|^{p(x) - p_0(z)} \) is uniformly bounded by [11, Lemma 4.1.6, p. 101]. Let us look at the last term in the previous estimate. A short calculation gives that

\[
p_0^\#(z) - p_0^\#(x) = \frac{n^2 (p(z) - p(x)) + np(z) (\alpha(z) - \alpha(x))}{(n - \alpha(z) p(z)) (n - \alpha(x) p(x))}.
\]

Since \( p \) and \( \alpha \) are bounded log-Hölder continuous functions and

\[
0 < c_1 \leq (n - \alpha(z) p(z)) (n - \alpha(x) p(x)) \leq n^2 < \infty,
\]

with some positive constant \( c_1 \), the term \( |B_z|^{\frac{p_0^\#(z)}{p_0(z)} (p_0^\#(z) - p_0^\#(x))} \) is uniformly bounded. By [11, Theorem 4.2.4, p. 108] we can move the exponent \( p(x) \) inside the integral and obtain

\[
\| f \|_{p_0^\#(z)} \leq c A_B \left( \| f \|_{p(z)} + \chi_{\{0 < |f| \leq 1\}} \right).
\]

Let us assume now that \( \frac{p(z)}{p_0(z)} p_0(z) < p_0(z) \). Again by [11, Theorem 4.2.4, p. 108] we obtain

\[
\begin{align*}
\| f \|_{p_0^\#(z)} & \leq c \left( A_B \left( \| f \|_{p(z)} + \chi_{\{0 < |f| \leq 1\}} \right) \right)^{\frac{p_0(z)}{p_0(z)}} \\
& \leq c A_B \left( \| f \|_{p(z)} + \chi_{\{0 < |f| \leq 1\}} \right),
\end{align*}
\]

where the last inequality follows from the estimates \( A_B \left( \| f \|_{p(z)} + \chi_{\{0 < |f| \leq 1\}} \right) \geq 1 \) and \( (p_0^\#(z) p(z))/(p_0^\#(z) p(x)) < 1 \).
By the Besicovitch covering theorem, \cite[2.7]{29}, there is a countable covering subfamily \((B_i)\), with the bounded overlapping-property. Thus we have

\[
\int_E \rho^p(x) \, dx \leq \sum_{i=1}^{\infty} \int_{B_i \cap \Omega} \rho^p(x) \, dx \\
\leq c \sum_{i=1}^{\infty} \frac{1}{|B_i|} \int_{B_i \cap \Omega} |f(y)|^{p(y)} + \chi_{\{0<|f|\leq 1\}}(y) \, dy \, dx \\
= c \sum_{i=1}^{\infty} \int_{B_i \cap \Omega} |f(y)|^{p(y)} + \chi_{\{0<|f|\leq 1\}}(y) \, dy \\
\leq c \int_{\Omega} |f(y)|^{p(y)} \, dy + c|\{x \in \Omega : 0 < |f| \leq 1\}|. \quad \square
\]

3.4. Remark. If \(\alpha\) is a constant and \(p \equiv 1\) in Theorem 3.2, then we obtain a weak type estimate. Indeed, in this case \(p^\alpha_a = \frac{a}{n-\alpha}\) is a constant and we have

\[
||x \in \Omega : I_{\alpha(t)} f > t||_{L^{\rho^\alpha}_a(\Omega)} = \int_{\{x \in \Omega : I_{\alpha(t)} f > t\}} \rho^\alpha_a \, dx \leq c(1 + |\Omega|)
\]

for some constant \(c\). Hence the quasinorm

\[
||f||_{L^{\rho^\alpha}_a(\Omega)} := \sup_{r>0} ||I_{\alpha(r)} f > t||_{L^{\rho^\alpha}_a(\Omega)}^{1/\rho^\alpha_a}
\]

is uniformly bounded for any function satisfying \(||f||_1 \leq 1\). Using a function \(f/||f||_1\) we obtain \(||f||_{L^{\rho^\alpha}_a(\Omega)} \leq ||f||_{L^{\rho^\alpha}_a(\Omega)} \leq c\). Thus, the weak-type estimate

\[
\sup_{r>0} ||I_{\alpha(r)} f > t||_{L^{\rho^\alpha}_a(\Omega)}^{1/\rho^\alpha_a} \leq c||f||_1
\]

holds.

3.5. Remark. In Section 5 we need the operator

\[
\tilde{I}_{\alpha(t)} f(x) := \int_{\Omega} \frac{|f(y)|}{|x - y|^{s(x)(n-1)}} \, dy
\]

where \(1 \leq s(x) < \frac{n}{n-1}\) for every \(x \in \Omega\). We have \(I_{\alpha(t)} = \tilde{I}_{\alpha(t)}\), where \(\alpha(x) := n - s(x)(n-1)\). We see that \(\alpha\) is log-Hölder continuous if and only if \(s\) is log-Hölder continuous. We recover the assumptions for \(s\): \(\alpha > 0\) if and only if \(s^+ < \frac{n}{n-1}\). \((\alpha p)^+ < n\) if and only if \(\sup_{s \in \Omega} (n - s(x)(n-1)) p(x) < n\). Moreover

\[
q(x) := p^\#_{n-s(n-1)} = n p(x) \frac{np(x)}{n - np(x) + s(x)p(x)(n-1)} = \frac{np(x)}{p(x) n (s(x) - 1) + n - s(x)p(x)}.
\]
4. Exponential integrability

In this section we study behaviour of the variable order Riesz potential $I_{\alpha(\cdot)}f$ in the limiting case $f \in L^{\frac{\alpha}{n}}$. We follow a recent article by Martínez and Spector [31] where they studied the usual Riesz potential, that is, the constant order Riesz potential in $\mathbb{R}^n$.

We define the variable dimensional Hausdorff content based on [15, 2.10.1, p. 169]. For the definition of the usual Hausdorff content we refer to [1], too.

4.1. Definition (Variable dimensional Hausdorff content). Let $\beta : \mathbb{R}^n \to (0, n]$ be a function and $E$ any set in $\mathbb{R}^n$. Then we write

$$H^{\beta(\cdot)}(\cdot)_{\infty}(E) := \inf \left\{ \sum_{i=1}^{\infty} r_i^{\beta(x_i)} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \right\}.$$

Here $H^{\beta(\cdot)}(\cdot)_{\infty}(E)$ is called the variable dimensional Hausdorff content of the set $E$, and in the next lemma we show that it is an outer capacity. Hence, based on [2] we could call $H^{\beta(\cdot)}(\cdot)_{\infty}$ also a variable Hausdorff capacity. We refer to [3], too. In the next lemma $\mathcal{P}(\mathbb{R}^n)$ is the power set.

4.2. Lemma. Let $\beta : \mathbb{R}^n \to (0, n]$. The set function $H^{\beta(\cdot)} : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ satisfies the following properties:

1. $H^{\beta(\cdot)}(\emptyset) = 0$;
2. if $A \subset B$ then $H^{\beta(\cdot)}(A) \leq H^{\beta(\cdot)}(B)$;
3. if $E \subset \mathbb{R}^n$ then

$$H^{\beta(\cdot)}(E) = \inf_{E \subset U \text{ and } U \text{ is open}} H^{\beta(\cdot)}(U);$$

4. if $(K_i)$ is a decreasing sequence of compact sets then

$$H^{\beta(\cdot)}\left( \bigcap_{i=1}^{\infty} K_i \right) = \lim_{i \to \infty} H^{\beta(\cdot)}(K_i);$$

Proof. The claim (1) is clear. The claim (2) follows since every cover of $B$ is also a cover of $A$.

Let us next prove (3). By (2) we have $H^{\beta(\cdot)}(E) \leq \inf_{U} H^{\beta(\cdot)}(U)$. Let $\varepsilon > 0$, and choose a covering that satisfies $E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$ and

$$\sum_{i=1}^{\infty} r_i^{\beta(x_i)} < H^{\beta(\cdot)}(E) + \varepsilon.$$

We choose that $V := \bigcup_{i=1}^{\infty} B(x_i, r_i)$ and note that it is open. Thus

$$\inf_{U} H^{\beta(\cdot)}(U) \leq H^{\beta(\cdot)}(V) \leq \sum_{i=1}^{\infty} r_i^{\alpha(x_i)} < H^{\beta(\cdot)}(E) + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we obtain $\inf_{U} H^{\beta(\cdot)}(U) \leq H^{\beta(\cdot)}(E)$. Hence the claim (3) is proved.
Let us then prove (4). By (2) we have $\mathcal{H}^{B(\cdot)}_{\infty}(\bigcap_{i=1}^{\infty} K_i) \leq \lim_{i \to \infty} \mathcal{H}^{B(\cdot)}_{\infty}(K_i)$, and the limit exists by (2) since the sequence of $(K_i)$ is decreasing. Let $\varepsilon > 0$, and choose by (3) an open set $V$ that satisfies $\bigcap_{i=1}^{\infty} K_i \subset V$ and

$$\mathcal{H}^{B(\cdot)}_{\infty}(V) < \mathcal{H}^{B(\cdot)}_{\infty}(\bigcap_{i=1}^{\infty} K_i) + \varepsilon.$$ 

Since $V$ is open we find $j_0 \in \mathbb{N}$ such that $K_j \subset V$ for all $j \geq j_0$. Thus for all $j \geq j_0$ we have

$$\mathcal{H}^{B(\cdot)}_{\infty}(K_j) \leq \mathcal{H}^{B(\cdot)}_{\infty}(V) < \mathcal{H}^{B(\cdot)}_{\infty}(\bigcap_{i=1}^{\infty} K_i) + \varepsilon,$$

and furthermore $\lim_{i \to \infty} \mathcal{H}^{B(\cdot)}_{\infty}(K_i) \leq \mathcal{H}^{B(\cdot)}_{\infty}(\bigcap_{i=1}^{\infty} K_i) + \varepsilon$. Since this holds for all $\varepsilon > 0$, we obtain the inequality for the other direction. Hence also the claim (4) is proved.

4.3. Remark. We do not know is $\mathcal{H}^{B(\cdot)}_{\infty}$ a capacity in the sense of Choquet [6]. More precisely we do not know does the following hold: if $(H_i)$ is an increasing sequence of sets then $\mathcal{H}^{B(\cdot)}_{\infty}(\bigcup_{i=1}^{\infty} H_i) = \lim_{i \to \infty} \mathcal{H}^{B(\cdot)}_{\infty}(H_i)$. If $\beta$ is a constant function, then this property has been proved in [8], see also [9, 37].

We recall the definition of the variable order fractional maximal function, [28].

4.4. Definition (Variable order fractional maximal function). Let $\alpha : \Omega \to [0, n)$ any measurable function and $f \in L^1_{\text{loc}}(\Omega)$. Then

$$\mathcal{M}_{\alpha(\cdot)} f(x) := \sup_{r>0} \frac{r^{\alpha(x)}}{|B(x, r)|} \int_{B(x,r) \cap \Omega} |f(y)| \, dy, \quad x \in \Omega.$$ 

We give a proof for the lower semicontinuity of this maximal function for the reader’s convenience.

4.5. Lemma. Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\alpha : \Omega \to (0, n]$ be continuous. Then $\mathcal{M}_{\alpha(\cdot)} f$ is lower-semicontinuous.

Proof. Let us write $E_t := \{x \in \Omega : \mathcal{M}_{\alpha(\cdot)} f(x) > t\}$. We need to show that $E_t$ is open in $\mathbb{R}^n$. So let $x \in E_t$. By the definition of $\mathcal{M}_{\alpha(\cdot)}$ for every $x \in E_t$ there exists a radius $r_x > 0$ such that

$$\frac{r_x^{\alpha(x)}}{|B(x, r_x)|} \int_{B(x,r_x) \cap \Omega} |f(y)| \, dy > t.$$ 

By the properties of the Lebesgue measure we obtain

$$\lim_{R \to r_x} \frac{R^{\alpha(x)}}{|B(x, R)|} \int_{B(x,R) \cap \Omega} |f(y)| \, dy = \frac{r_x^{\alpha(x)}}{|B(x, r_x)|} \int_{B(x,r_x) \cap \Omega} |f(y)| \, dy.$$

Hence there exists $R > r_x$ such that

$$\lambda := \frac{\frac{R^{\alpha(x)}}{|B(x, R)|} \int_{B(x,R) \cap \Omega} |f(y)| \, dy}{t} > 1.$$
Let \( x' \) be such that \(|x - x'| < R - r_x \) and assume that

\[
|\alpha(x') - \alpha(x)| < \frac{\log(t)}{\max\{1, |\log(R)|\}}.
\]

Then

\[
R^{\alpha(x') - \alpha(x)} \geq \begin{cases} 1, & \text{if } R \geq 1 \text{ and } \alpha(x') \geq \alpha(x), \\ R^{\frac{\log(t)}{\max\{1, |\log(R)|\}}} \geq R^{\frac{\log(t)}{\log(R)}} = \frac{1}{t}, & \text{if } R \geq 1 \text{ and } \alpha(x') < \alpha(x), \\ R^{\frac{\log(t)}{\log(R)}} \geq R^{\frac{\log(t)}{\log(R)}} = \frac{1}{t}, & \text{if } R < 1 \text{ and } \alpha(x') \geq \alpha(x), \\ 1, & \text{if } R < 1 \text{ and } \alpha(x') < \alpha(x). \end{cases}
\]

The condition (4.6) holds in some ball \( B(x, \xi) \cap \Omega \) by the continuity of \( \alpha \).

Thus for all \( x' \in B(x, \min\{|R - r_x, \xi|\}) \cap \Omega \) we have

\[
\mathcal{M}_{\alpha(x')} f(x') \geq \frac{R^{\alpha(x')}}{|B(x, R)|} \int_{B(x', R) \cap \Omega} |f(y)| \, dy \\
\geq \frac{R^{\alpha(x') - \alpha(x)}}{|B(x', R)|} \int_{B(x', R) \cap \Omega} |f(y)| \, dy \\
> \frac{\lambda t}{\lambda} = t.
\]

Hence the set \( E_t \) is open in \( \Omega \), and thus it is open also in \( \mathbb{R}^n \). \( \square \)

The following lemma is a generalisation of \cite[Lemma 3.5]{31} and \cite[Theorem (ii)]{34} to the case of the variational dimensional Hausdorff content.

4.7. Lemma. Let \( \Omega \subset \mathbb{R}^n \) be a bounded, open set. Let \( \alpha : \Omega \to [0, n) \) be a measurable function such that \( \alpha^+ < n \). Then there exists a constant \( c \), depending only on the dimension \( n \), such that the inequality

\[
\mathcal{H}^n_{\alpha^+}(\{x \in \Omega : \mathcal{M}_{\alpha(x)} f(x) > t\}) \leq \frac{c(n)}{t} \|f\|_{L^1(\Omega)}
\]

holds for all \( f \in L^1(\Omega) \).

Proof. Let us write \( E_t := \{x \in \Omega : \mathcal{M}_{\alpha(x)} f(x) > t\} \).

By the definition of \( \mathcal{M}_{\alpha(x)} \) for every \( x \in E_t \) there exists a radius \( r_x > 0 \) such that

\[
\frac{r_x^{\alpha(x)}}{|B(x, r_x)|} \int_{B(x, r_x) \cap \Omega} |f(y)| \, dy > t.
\]

This inequality yields that \( r_x^{\alpha(x)} < \frac{c(n)}{t} \|f\|_{L^1(\Omega)} \), and hence \( r_x \) is uniformly bounded provided that \( \alpha^+ < n \).

Then \( E_t \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_x) \), and hence by the 5-covering lemma \cite{39} or the simple Vitali covering lemma \cite[Theorem 1.4.1]{2} we find a countable subcollection of disjoint balls \( \{B(x_i, r_{x_i})\} \) such that

\[
E_t \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_x).
\]
Thus by the definition of the Hausdorff content, we obtain

\[ \mathcal{H}^{n-\alpha}_\infty(E_i) \leq \sum_{i=1}^{\infty} (5r_i)^{n-\alpha}(x_i) \]

\[ \leq \frac{5^n}{\omega_n} \sum_{i=1}^{\infty} \int_{B(x_i,r_i) \cap \Omega} |f(y)| \, dy \leq \frac{5^n}{\omega_n} \|f\|_{L^1(\Omega)}, \]

where \( \omega_n \) is the Lebesgue measure of the unit ball.

The proof of the next lemma follows the proof of [22, Proposition 2.1].

4.8. **Lemma.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded, open set. Let \( p \in \mathcal{P}^{\log}(\Omega) \) satisfy the inequalities \( 1 \leq p^- \leq p^+ < \infty \). Assume that \( \alpha : \Omega \to (0, n) \) satisfies \( \alpha^- > 0 \) and \((\alpha p)^+ < n\). Let \( f \in L^{p^+}(\mathbb{R}^n) \) with \( \|f\|_{p^+} \leq 1 \). Then there exists a constant \( c \) such that for every \( x \in \Omega \) and every \( r > 0 \) the inequality

\[ \int_{\Omega \cap B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha(y)}} \, dy \leq c \max \left\{ 1, \frac{p(x)}{n-\alpha(x)p(x)} \right\}^\frac{\alpha^-}{p^+} \]

holds. Here the constant \( c \) depends only on the dimension \( n \), the log-Hölder constant of \( p \) and \( \text{diam}(\Omega) \).

**Proof.** Let us denote by \( A(x, r) \) the annulus \((B(x, r) \setminus B(x, r/2)) \cap \Omega \) and write \( I := \{ i \in \mathbb{N} : r \leq 2^i \leq \text{diam}(\Omega) \} \).

Let us first note that Lemma 4.16 and Theorem 4.5.7 of [11] yield that

\[ \|1\|_{L^{p^+}(B)} \leq c |B|^{1/p(x)} \]

for all \( x \in B \cap \Omega \) and all ball \( B \) with \( \text{diam}(B) \leq \text{diam}(\Omega) \). Here the constant \( c \) depends only on \( n \), log-Hölder constant of \( p \) and \( \text{diam}(\Omega) \).

When we use in (4.9) Hölder’s inequality for the second inequality, for the norm of the constant one for the third inequality, and finally Hölder’s inequality again, we conclude that

\[ \int_{\Omega \cap B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}} \, dy \leq \sum_{i \in I} 2^{\frac{\alpha^-}{p(x)}} \int_{A(x,2^i)} |u(y)| \, dy \]

\[ \leq \sum_{i \in I} 2^{\frac{\alpha^-}{p(x)} \|u\|_{L^{p^+}(A(x,2^i))} \|1\|_{L^{p^+}(B(x,2^i) \cap \Omega)}} \]

\[ \leq \sum_{i \in I} 2^{\frac{\alpha^-}{p(x)} \|u\|_{L^{p^+}(A(x,2^i))}} \]

\[ \leq \left( \sum_{i \in I} 2^{\frac{\alpha^-}{p(x)} \|u\|_{L^{p^+}(A(x,2^i))}} \right)^{\frac{1}{p^+}} \]

for \( x \in \Omega \) where \( p^+_\alpha(x) = \frac{np(x)}{n-\alpha(x)p(x)} \). Since \( \|u\|_{p^+} \leq 1 \) we have \( \|u\|_{p^+} \leq q_{p^+}(u) \) by [11, Lemma 3.2.5. p. 75], and so

\[ \sum_{i \in I} \|u\|_{L^{p^+}(A(x,2^i))} \leq \sum_{i \in I} \int_{A(x,2^i)} |u(y)|^{p^+} \, dy \leq \int_{\Omega} |u(y)|^{p^+} \, dy \leq 1. \]
The first term on the last line of (4.9) is a geometric sum. Thus we obtain that
\[
\left(\sum_{i \in I} 2^{-in\left(p^+/p(x)\right)}\right)^{1/p^+} \leq r^{-n/p(x)/(p(x)-1)}(1 - 2^{-n\left(p^+/p(x)\right)})^{1/(p(x)-1)}.
\]

Let us write that \( k(x) := \max\{1, \frac{p(x)}{n-\alpha(x)p(x)}\} \). Now \( n/p_\alpha(x) \geq 1/k(x) \) and \( k(x) \geq 1 \). Thus by the inequality \( \alpha^x \leq \alpha x + 1 - a \) (which follows from Bernoulli’s inequality) with \( x = 2^{-\left(p^+\right)} \) and \( a = \frac{1}{k} \) we obtain
\[
(1 - 2^{-n\left(p^+/p(x)\right)})^{-1} \leq (1 - 2^{-\left(p^+\right)})^{-1} \leq (1 - 2^{-\left(p^+\right)})^{-1}k(x).
\]

Hence we have
\[
\left(\sum_{i \in I} 2^{-in\left(p^+/p(x)\right)}\right)^{1/p^+} \leq (1 - 2^{-\left(p^+\right)})^{-1/p^+}k(x)^{1/p^+}r^{-n/p(x)/(p(x)-1)}.
\]

Finally, when we note by \( (p^+) \in (1, \infty) \) that \( (1 - 2^{-\left(p^+\right)}) \in (\frac{1}{2}, 1) \). Hence we have the inequality \( (1 - 2^{-\left(p^+\right)})^{-1/p^+} \leq 2 \). \( \square \)

In the variable exponent case the Hedberg-type estimates are well known and variants have been used and proved for example in [10, Theorem 3.8], [11, Proposition 6.1.6], [23, (4.7)], [32, p. 429], [33, Lemma 4.6], [36, p. 279]. Here instead of the standard maximal operator we have the variable dimension fractional maximal operator, and we calculate how the constant depends on \( p \) and \( \alpha \).

4.10. Lemma (Hedberg-type estimate). Let \( \Omega \subset \mathbb{R}^n \) be a bounded, open set. Let \( p \in P^{log}(\Omega) \) satisfy the inequalities \( 1 \leq p^- \leq p^+ < \infty \). Assume that \( \alpha : \Omega \rightarrow (0, n) \) satisfies \( \alpha^- > 0 \) and \( (\alpha p)^+ < n \). If \( \delta(x) := \frac{n-\alpha(x)p(x)}{p(x)} \), then there exists a constant \( c \) such that for every \( \epsilon : \Omega \rightarrow (0, \infty) \), with \( \epsilon(x) \leq \alpha(x) \) for all \( x \in \Omega \), the inequality
\[
I_{\alpha}(f)(x) \leq c \max\left\{1, \frac{1}{\delta(x)}\right\}^{p^-} (M_{\alpha(x)-\epsilon(x)}f(x))^{\frac{\delta(x)}{\alpha(x)-\epsilon(x)}}
\]
holds for all \( f \in L^{p(\cdot)}(\Omega) \) with \( \|f\|_{p(\cdot)} \leq 1 \). Here \( c \) depends only on the dimension \( n \), \( \epsilon^- \), the log-Hölder constant of \( p \), and \( \text{diam}(\Omega) \).

Proof. For a ball \( B(x, r) \) we write
\[
I_{\alpha}(f)(x) = \int_{B(x,r)\cap \Omega} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}} \, dy + \int_{\Omega \setminus B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}} \, dy
\]
\[
= I + II.
\]
For the first term we obtain
\[ I \leq \sum_{j=0}^{\infty} \int_{B(x,r2^{-j}),B(x,r2^{-j+1})} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}} \, dy \]
\[ \leq c(n) \sum_{j=0}^{\infty} \frac{(r2^{-j})^\alpha (r2^{-j})^{\alpha(x)-\epsilon(x)}}{(r2^{-j})^{\alpha(x)-\epsilon(x)}} \int_{B(x,r2^{-j})} \frac{|f(y)|}{|y|^{n-\alpha(x)}} \, dy \]
\[ \leq c(n)r^{\epsilon(x)}2^{n-\alpha(x)} \sum_{j=0}^{\infty} 2^{-j\epsilon(x)} M_{\alpha(x)-\epsilon(x)}f(x) = \frac{c(n)2^{\epsilon(x)}}{2^{\epsilon(x)} - 1} r^{\epsilon(x)} M_{\alpha(x)-\epsilon(x)}f(x). \]

Next we estimate the second term. Since \(|f|\leq 1\) we use Lemma 4.8 in order to obtain that
\[ II \leq c \max \left\{ 1, \frac{p(x)}{n-\alpha(x)p(x)} \right\} \left( \frac{1}{p(x)} \right)^{n-\alpha(x)} \]
\[ = c \max \left\{ 1, \frac{1}{\delta(x)} \right\} \left( \frac{1}{p(x)} \right)^{\delta(x)}, \]
and the constant depends only on \(n\), the log-Hölder constant of \(p\), and \(\text{diam}(\Omega)\).

If
\[ \left( M_{\alpha(x)-\epsilon(x)}f(x) \right)^{-\frac{1}{\alpha(x)+\epsilon(x)}} < \text{diam}(\Omega) \]
we choose
\[ r(x) = \left( M_{\alpha(x)-\epsilon(x)}f(x) \right)^{-\frac{1}{\alpha(x)+\epsilon(x)}}. \]

Hence,
\[ I_{\alpha(x)}f(x) \leq c \max \left\{ 1, \frac{1}{\delta(x)} \right\} \left( M_{\alpha(x)-\epsilon(x)}f(x) \right)^{\delta(x)} \]
where the constant depends only on the dimension \(n\), the log-Hölder constant of \(p\), \(\epsilon^{-}\), and \(\text{diam}(\Omega)\).

If
\[ \left( M_{\alpha(x)-\epsilon(x)}f(x) \right)^{-\frac{1}{\alpha(x)+\epsilon(x)}} \geq \text{diam}(\Omega) \]
we choose \(r(x) = \text{diam}(\Omega)\). Thus we obtain
\[ I_{\alpha(x)}f(x) \leq I \leq c \text{diam}(\Omega)^{\delta(x)} M_{\alpha(x)-\epsilon(x)}f(x) \leq c \text{diam}(\Omega)^{\delta(x)} (M_{\alpha(x)-\epsilon(x)}f(x))^{\frac{\delta(x)}{\alpha(x)+\epsilon(x)}}, \]
where the constant depends only on the dimension \(n\), \(\epsilon^{-}\), and \(\text{diam}(\Omega)\). \(\square\)

Next we prove our main Theorem 1.2 which is a generalisation of [31, Theorem 1.2] to the variable order case. We clarify dependences of the final constants of the given parameters in Remark 4.13.

**Proof of Theorem 1.2.** Let \(p \in \mathfrak{P}_{\text{log}}(\Omega)\) satisfy the inequality \(p(x) < \frac{n}{\alpha(x)}\).
Then \(|f|\leq 1\) by the assumption \(|f| \leq 1\) and Corollary 3.3.4 of [11]. By Hedberg’s lemma, Lemma 4.10, we obtain
\[ I_{\alpha(x)}f(x) \leq c \max \left\{ 1, \frac{1}{\delta(x)} \right\} \left( M_{\alpha(x)-\epsilon(x)}f(x) \right)^{\frac{\delta(x)}{\alpha(x)+\epsilon(x)}}, \]
where \( c \) depends only on the dimension \( n, \varepsilon^- \), the log-Hölder constant of \( p \), and \( \text{diam}(\Omega) \). Let \( 1 < r < \min\{2, \frac{n}{\alpha^-}\} \). By Hölder’s inequality we obtain
\[
M_{\alpha^- - \varepsilon^-}(x) f(x) \leq (M_{\alpha^- - \varepsilon^-}(x)|f'(x)|)^{\frac{1}{r}}.
\]
Thus we have
\[
I_{\alpha^-}(x) f(x) \leq c \max\left\{1, \frac{1}{\delta(x)}\right\} \frac{1}{|\alpha|^r} (M_{\alpha^- - \varepsilon^-}(x)|f'(x)|)^{\frac{\delta(x)}{\text{diam}(\Omega)}}.
\]
These estimates yield
\[
(4.11)
\]
\[
H^{a^{-\alpha^{-\varepsilon^{-}}}}_{\alpha^{-\varepsilon^{-}}}(x_0 : I_{\alpha^-}(x) > t)
\leq H^{a^{-\alpha^{-\varepsilon^{-}}}}_{\alpha^{-\varepsilon^{-}}}(x_0 : M_{\alpha^- - \varepsilon^-}(x)|f'(x)| > \left(c \min\{1, \delta(x)\}\right)^{\frac{\delta(x)}{\text{diam}(\Omega)}})
\]
where \( c \) depends only on the dimension \( n, \varepsilon^- \), and the log-Hölder constant of \( p \), and \( \text{diam}(\Omega) \).

Let us recall that \( \delta(x) = \frac{n - \alpha(x)p(x)}{p(x)} \). Let \( \sigma \in (0, 1) \) be a small number and choose \( p_{\alpha}(x) := \frac{n}{\alpha(x)} - \sigma \). Note that the function \( p_{\alpha} \) is log-Hölder continuous provided that \( \alpha \) is log-Hölder continuous, and the log-Hölder constant of \( p_{\alpha} \) is independent of \( \sigma \). Moreover the values of the function \( p_{\alpha} \) can be chosen to be near the critical value \( n/\alpha \). With this choice we have
\[
\delta(x) = n - \alpha(x)p_{\alpha}(x) = \frac{\alpha(x)^2\sigma}{n - \alpha(x)\sigma} \in \left[\frac{\alpha^-}{n}, \frac{\alpha^+}{n} - \sigma\right] = \left[\frac{\alpha^-}{n}, \frac{\alpha^+}{n} - \sigma\right].
\]
Hence \( \delta(x) \to 0^+ \) uniformly as \( \sigma \to 0^+ \). Assume that \( t_0 \) is such that \( ct_0 = e \), where \( e \) is from (4.11). For every \( t > t_0 \) we choose \( \sigma \) so small that
\[
(4.12)\ c \min\{1, \delta(x)\}^{\frac{\sigma}{\alpha^-}} t \approx e,
\]
where we denote \( A \approx B \) if there exists a positive constant \( c \) independent of \( A \) and \( B \) such that \( c^{-1}A \leq B \leq cA \).

Thus we have
\[
H^{a^{-\alpha^{-\varepsilon^{-}}}}_{\alpha^{-\varepsilon^{-}}}(x_0 : |I_{\alpha^-}(x)| > t)
\leq H^{a^{-\alpha^{-\varepsilon^{-}}}}_{\alpha^{-\varepsilon^{-}}}(x_0 : M_{\alpha^- - \varepsilon^-}(x)|f'(x)| > c \exp\left(\frac{\sigma}{\delta(x)}\right))
\leq H^{a^{-\alpha^{-\varepsilon^{-}}}}_{\alpha^{-\varepsilon^{-}}}(x_0 : M_{\alpha^- - \varepsilon^-}(x)|f'(x)| > c \exp\left(\frac{\sigma^{-1}}{\sigma^+}\right)).
\]
Hence by Lemma 4.7 we obtain
\[
H^{a^{-\alpha^{-\varepsilon^{-}}}}_{\alpha^{-\varepsilon^{-}}}(x_0 : |I_{\alpha^-}(x)| > t) \leq c \exp\left(\frac{-\sigma^{-1}}{\sigma^+}\right).
\]
For the exponent we obtain
\[
\delta^+ = \sup_{x \in \Omega} \frac{n - \alpha(x)p_{\alpha}(x)}{p_{\alpha}(x)} = \sup_{x \in \Omega} \frac{\alpha(x)^2\sigma}{n - \alpha(x)\sigma} = \frac{(\alpha^-)^2}{n - \alpha^+\sigma^+}.
\]
and thus
\[ \frac{\delta(x)}{\delta^+} = \frac{\alpha(x)^2\sigma}{n - \alpha(x)\sigma} \cdot \frac{n - \alpha^+\sigma}{(\alpha^+)^2\sigma} \in \left[ \frac{(\alpha^-)^2(n - \alpha^+)}{n(\alpha^+)^2},\frac{n}{n - \alpha^+} \right]. \]

This yields that \(\delta(x) \approx \delta^+\) for every \(x \in \Omega\) and hence by (4.12) we have
\[ \frac{1}{\delta^+} \approx \frac{1}{\delta(x)} \approx \frac{|P_x^+|}{H} = t^{\frac{n-\alpha^-\gamma}{n(\alpha^-)^2\sigma}}. \]

This implies that
\[ \mathcal{H}_{\infty}^{n-r(\alpha^-)\gamma}(\{x \in \Omega : I_{\alpha}(f(x) > t)\}) \leq c_1 \exp \left(-c_2 r(r - 1)\alpha^- t^{\frac{n-\alpha^-\gamma}{n(\alpha^-)^2\sigma}}\right), \]

where \(c_1\) depend only on \(n, \alpha^-, \alpha^+\), and log-Hölder constant of \(\alpha\), and \(c_2\) depend only on \(n, \alpha^-, \alpha^+\), and log-Hölder constant of \(\alpha\).

Finally we choose \(\varepsilon(x) := (r - 1)\alpha(x)\) and obtain
\[ \mathcal{H}_{\infty}^{n-\alpha}(\{x \in \Omega : I_{\alpha}(f(x) > t)\}) \leq c_1 \exp \left(-c_2 r(r - 1)\alpha^- t^{\frac{n-\alpha^-\gamma}{n(\alpha^-)^2\sigma}}\right), \]

whenever \(t > t_0\). Since the left-hand side and the constants are independent of \(\sigma\), we take \(\sigma \to 0^+\) and obtain
\[ \mathcal{H}_{\infty}^{n-\alpha}(\{x \in \Omega : I_{\alpha}(f(x) > t)\}) \leq c_1 \exp \left(-c_2 r(r - 1)\alpha^- t^{\frac{n-\alpha^-\gamma}{n(\alpha^-)^2\sigma}}\right). \]

The claim holds also if \(0 < t \leq t_0\). Indeed, for \(0 < t \leq t_0\)
\[ \mathcal{H}_{\infty}^{n-\alpha}(\{x \in \Omega : I_{\alpha}(f(x) > t)\}) \leq \mathcal{H}_{\infty}^{n-\alpha}(\Omega) \leq \mathcal{H}_{\infty}^{n-\alpha}(\Omega) \exp \left(ct\frac{n}{\alpha^-}\right) \exp \left(-c_2 t^{\frac{n}{\alpha^-}}\right) \leq \mathcal{H}_{\infty}^{n-\alpha}(\Omega) \exp \left(c_2 t^{\frac{n}{\alpha^-}}\right) \exp \left(-c_2 t^{\frac{n}{\alpha^-}}\right), \]

where \(\mathcal{H}_{\infty}^{n-\alpha}(\Omega) \leq (1 + \text{diam}(\Omega))^n\) by the definition of the Hausdorff content. Hence, the theorem is proved.

4.13. Remark. The estimates in the previous proof yield Theorem 1.2 with a constant \(c_1\) which depends only on \(\text{diam}(\Omega), n, \alpha^-, \alpha^+\), and log-Hölder constant of \(\alpha\), and a constant \(c_2\) which depends only on \(n, \alpha^-, \alpha^+\), and log-Hölder constant of \(\alpha\).

5. Applications to non-smooth domains

The definition of a bounded John domain goes back to F. John [26, Definition, p. 402] who defined an inner radius and an outer radius domain, and later this domain was renamed as a John domain in [30, 2.1]. We generalise this definition so that the shape of the John cusp can depend on the point. If \(s\) is a constant function we have a so called \(s\)-John domain studied in [38]. For other studies and generalisations of John domains we refer to [20, 21].

5.1. Definition. Let \(D \subset \mathbb{R}^n, n \geq 2\), be a bounded domain, and \(s : D \to [1, \infty)\) a function. The domain \(D\) is an \(s(\cdot)\)-John domain if there exist constants \(0 < \alpha < \beta < \infty\) and a point \(x_0 \in D\) such that each point \(x \in D\) can be joined to \(x_0\) by a rectifiable curve \(\gamma_x : [0, \ell(\gamma_x)] \to D\), parametrized by its arc length, such that \(\gamma_x(0) = x, \gamma_x(\ell(\gamma_x)) = x_0, \ell(\gamma_x) \leq \beta\), and
\[ t^{s(t)} \leq \frac{\beta}{\alpha} \text{dist}(\gamma_x(t), \partial D) \quad \text{for all} \quad t \in [0, \ell(\gamma_x)]. \]
The point $x_0$ is called a John center of $D$ and $\gamma_x$ is called a John curve of $x$.

5.2. Example. We construct a mushrooms-type domain. Let $(r_m)$ be a decreasing sequence of positive real numbers converging to zero. Let $Q_m$, $m = 1, 2, \ldots$, be a closed cube in $\mathbb{R}^n$ with side length $2r_m$. Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function with $\lim_{t \to 0} \varphi(t) = \varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Let $P_m$, $m = 1, 2, \ldots$, be a closed rectangle in $\mathbb{R}^n$ which has side length $r_m$ for one side and $2\varphi(r_m)$ for the remaining $n - 1$ sides. Let $Q := [0, 12] \times [0, 12]$. We attach $Q_m$ and $P_m$ together creating ‘mushrooms’ which we then attach, as pairwise disjoint sets, to the side $\{(0, x_2, \ldots, x_n) : x_2, \ldots, x_n > 0\}$ of $Q$ so that the distance from the mushroom to the origin is at least 1 and at most 4, see Figure 1. We have to assume here also that $\varphi(r_m) \leq r_m$. We need copies of the mushrooms. By an isometric mapping we transform these mushrooms onto the side $\{(x_1, 0, \ldots, x_n) : x_1, x_3, \ldots, x_n > 0\}$ of $Q$ and denote them by $Q^*_m$ and $P^*_m$. So again the distance from the mushroom to the origin is at least 1 and at most 4. We define

\begin{equation}
D := \text{int} \left( Q \cup \bigcup_{m=1}^{\infty} \left( Q_m \cup P_m \cup Q^*_m \cup P^*_m \right) \right).
\end{equation}

See Figure 1.

We set that $\varphi(t) := t^{\frac{3}{2}}$. We define that $s(x) := 1$ in $Q$, $s(x) := \frac{3}{2}$ in $\bigcup_{m=1}^{\infty} (Q_m \cup Q^*_m)$, and grows lineary from 1 to $\frac{3}{2}$ in each $P_m$ and each $P^*_m$. Then $D$ is an $s(\cdot)$-John domain, which can be seen as in [21, Lemma 6.2].
Next we prove a chaining results for $s(\cdot)$-John domains. We refer to [17, Theorem 9.3] for the proof in the classical case, and [21, Lemma 3.5] and [19, 4.3. Lemma] for generalisations.

5.4. Lemma. Let $D \subset \mathbb{R}^n, n \geq 2$, be a $s(\cdot)$-John domain with John constants $\alpha$ and $\beta$. Let $x_0 \in D$ the John center. Then for every $x \in D \setminus B(x_0, \text{dist}(x_0, \partial D))$ there exists a sequence of balls $(B(x_i, r_i))$ such that $B(x_i, 2r_i)$ is in $D$ for each $i = 0, 1, \ldots$, and for some constants $K = K(\alpha, \beta, \text{dist}(x_0, \partial D))$, $N = N(n)$, and $M = M(n)$

1. $B_0 = B(x_0, \frac{1}{2} \text{dist}(x_0, \partial D))$;
2. $\text{dist}(x, B_i)^{(s(x))} \leq Kr_i$ and $r_i \to 0$ as $i \to \infty$;
3. no point of the domain $D$ belongs to more than $N$ balls $B(x_i, r_i)$; and
4. $|B(x_i, r_i) \cup B(x_{i+1}, r_{i+1})| \leq M|B(x_i, r_i) \cap B(x_{i+1}, r_{i+1})|.$

Proof. Let $\gamma$ be a John curve joining $x$ to $x_0$. Let us write

$$B'_0 := B(x_0, \frac{1}{4} \text{dist}(x_0, \partial D)).$$

Let us consider the balls $B'_0$ and

$$B(\gamma(t), \frac{1}{4} \text{dist}(\gamma(t), \partial D \cup \{x\})).$$

when $t \in (0, l)$, here $l$ stands for the length of $\gamma$. By the Besicovitch covering theorem, there is a sequence of closed balls $\overline{B'_1}, \overline{B'_2}, \ldots$ and $\overline{B'_0}$ that cover $\{\gamma(t) : t \in [0, l]\} \setminus \{x\}$ and have a uniformly bounded overlap depending on $n$ only, [29, 2.7]. Let us define open balls $B_i := 2B'_i$ with center at $x_i := \gamma(t_i)$ and radius $r_i := \frac{1}{2} \text{dist}(x_i, \partial D \cup \{x\})$, $i = 1, 2, \ldots$. That is $B_i = B(x_i, r_i)$, $i = 1, 2, \ldots$.

For a ball $B_0 := B(x_0, \frac{1}{2} \text{dist}(x_0, \partial D))$ we obtain by the definition of $s(\cdot)$-John domain

$$\text{dist}(x, B_0)^{(s(x))} \leq \ell(\gamma)_{(s(x))} \leq \frac{\beta}{\alpha} \text{dist}(x_0, \partial D) = \frac{\beta r_0}{2 \alpha}.$$

Assume then that $i \geq 1$. If $r_i = \frac{1}{2} \text{dist}(x_i, x)$, then $\text{dist}(x, B_i) \leq 2r_i$. If $r_i = \frac{1}{2} \text{dist}(x_i, \partial D)$, then the definition of a $s(\cdot)$-John domain gives that

$$\text{dist}(x, B_i)^{(s(x))} \leq \ell_i^{(s(x))} \leq \frac{\beta}{\alpha} \text{dist}(\gamma(t_i), \partial D) \leq \frac{\beta r_i}{2 \alpha}.$$

Thus, the first part of property (2) holds.

Let us reenumerate the balls. Let $B_0$ be as before. If we have chosen balls $B_i$, $i = 0, 1, \ldots, m$, then we choose that the ball $B_{m+1}$ is the ball for which $x_j \in B_m$ and $t_j < t_m$, by remembering that $\gamma(t_j) = x_j$ and $\gamma(t_m) = x_m$. Hence, $r_i \to 0$ and $x_i \to x$, as $i \to \infty$. Thus, the second part of property (2) holds.

The point $x$ does not belong to any ball. Let $x'$ be any other point in the domain $D$. The point $x'$ cannot belong to the balls $B_i$ with $3r_i < \text{dist}(x', x)$. If $x' \in B_i$, then

$$2r_i \leq \text{dist}(x, x_i) \leq \text{dist}(x, x') + r_i.$$
Thus, we obtain that \( x' \in B_i \) if and only if
\[
\frac{1}{3} \operatorname{dist}(x', x) \leq r_i \leq \operatorname{dist}(x, x').
\]

The Besicovitch covering theorem implies that the balls with radius of \( \frac{1}{3} \) of the original balls are disjoint. Hence \( x' \) belongs to less than or equal to
\[
N \frac{|B(x', 2r)|}{|B(0, \frac{1}{12} r)|} = 2^a N
\]
balls \( B_i \), where the constant \( N \) is from the Besicovitch covering theorem and depends on the dimension \( n \) only. Hence, property (3) holds.

If \( r_i = \frac{1}{2} \operatorname{dist}(x_i, \partial D) \) and \( r_{i+1} = \frac{1}{2} \operatorname{dist}(x_{i+1}, \partial D) \), then \( r_{i+1} \geq \frac{1}{2} r_i \) (since \( x_{i+1} \in B_i \)) and thus we obtain
\[
\frac{|B_i|}{|B_{i+1}|} \leq \left( \frac{r_i}{\frac{1}{2} r_i} \right)^a = 2^a.
\]

If \( r_i = \frac{1}{2} \operatorname{dist}(x_i, x) \) and \( r_{i+1} = \frac{1}{2} \operatorname{dist}(x_{i+1}, x) \), then \( r_{i+1} \geq \frac{1}{2} r_i \) and thus we obtain \( |B_i|/|B_{i+1}| \leq 2^a \). If \( r_i = \frac{1}{2} \operatorname{dist}(x_i, \partial D) \) and \( r_{i+1} = \frac{1}{2} \operatorname{dist}(x_{i+1}, x) \), then \( r_i \leq \frac{1}{2} \operatorname{dist}(x_i, x) \) and we obtain the same ratio as before. Similarly in the case when \( r_i = \frac{1}{2} \operatorname{dist}(x_i, x) \) and \( r_{i+1} = \frac{1}{2} \operatorname{dist}(x_{i+1}, \partial D) \). We have shown \( |B_i| \leq 2^n |B_{i+1}| \). In the same manner we obtain \( 2r_{i+1} \leq 3r_i \) and hence \( 2^n |B_i| \geq |B_{i+1}| \). These yield property (4). \( \square \)

The previous lemma yields the following lemma.

5.5. **Lemma.** Let \( D \subset \mathbb{R}^n, n \geq 2 \), be an \( s(\cdot) \)-John domain. Then
\[
\left| u(x) - u_{B(x_0, \operatorname{dist}(x_0, \partial D))} \right| \leq c \bar{I}_{s(y)} \nabla u \left( \begin{array}{c} x \\ y \end{array} \right)
\]
for all \( u \in L^1_1(D) \). Here the constant \( c \) depends only on \( n, \alpha, \beta, s^+ \), and \( \operatorname{dist}(x_0, \partial D) \).

**Proof.** If \( x \in B(x_0, \operatorname{dist}(x_0, \partial D)) \), then
\[
\left| u(x) - u_{B(x_0, \operatorname{dist}(x_0, \partial D))} \right| \leq \frac{\operatorname{diam}(B(x_0, \operatorname{dist}(x_0, \partial D)))^a}{n |B(x_0, \operatorname{dist}(x_0, \partial D))|} \frac{\int_{B(x_0, \operatorname{dist}(x_0, \partial D))} |\nabla u(y)| \, dy}{|x - y|^{n-1}}
\]
by \[14, \text{Lemma 7.16}\]. Since the domain is bounded we have
\[
|x - y|^{s(x)(n-1)} \leq \frac{\operatorname{diam}(D)^{\lambda}}{\operatorname{dist}(x_0, \partial D)} \frac{\operatorname{diam}(D)^{\lambda}}{\operatorname{dist}(x_0, \partial D)} |x - y|^{n-1}
\]
\[
\leq (1 + \operatorname{diam}(D))^{\lambda(n-1)} |x - y|^{n-1}
\]
\[
\leq (1 + 2\delta^{\lambda(n-1)} |x - y|^{n-1},
\]
which gives the claim.

Let us then assume that \( x \in D \setminus B(x_0, \operatorname{dist}(x_0, \partial D)) \) and let \( \{B_i\}_{i=0}^\infty \) be a sequence of balls constructed in Lemma 5.4. Property (2) gives that \( \operatorname{dist}(x, B_i) \rightarrow 0 \) as \( i \rightarrow \infty \). Thus, property (2) and the Lebesgue differentiation theorem \[39, \text{Section 1, Corollary 1}\] imply that \( u_{B_i} \rightarrow u(x) \) when
\[ i \to \infty \text{ for almost every } x. \] We obtain

\[ |u(x) - u_{B_i}| \leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \]

\[ \leq \sum_{i=0}^{\infty} \left( |u_{B_i} - u_{B_i \cap B_{i+1}}| + |u_{B_{i+1}} - u_{B_i \cap B_{i+1}}| \right) \]

\[ \leq \sum_{i=0}^{\infty} \left( \int_{B_i \cap B_{i+1}} |u(y) - u_{B_i}| \, dy + \int_{B_i \cap B_{i+1}} |u(y) - u_{B_{i+1}}| \, dy \right). \]

By property (4)

\[ |u(x) - u_{B_i}| \leq 2C \sum_{i=0}^{\infty} \int_{B_i} |u(y) - u_{B_i}| \, dy. \]

Using the \((1,1)\)-Poincaré inequality in a ball \(B_i\), \([14, \text{Section 7.8}]\), we obtain

\[ |u(x) - u_{B_i}| \leq C \sum_{i=0}^{\infty} r_i \int_{B_i} |\nabla u(y)| \, dy. \]

Thus, for each \(z \in B_i\) we obtain by property (2) that

\[ |x - z| \leq \text{dist}(x, B_i) + 2r_i \leq (Cr_i)^{\frac{1}{m}} + 2r_i \leq C r_i^\beta, \]

where in the last inequality we used that \(D\) is bounded. Hence, we have \(C|x - z|^{1/(x)} \leq r_i\). Using this we obtain by property (3) that

\[ |u(x) - u_{B_i}| \leq C \sum_{i=0}^{\infty} r_i \int_{B_i} |\nabla u(y)| \, dy \leq C \sum_{i=0}^{\infty} \int_{B_i} \frac{|\nabla u(y)|}{r_i^{n-1}} \, dy \]

\[ \leq C \sum_{i=0}^{\infty} \int_{B_i} \frac{|\nabla u(y)|}{|x - y|^{(n-1)(x(n-1))}} \, dy \leq C \int_D \frac{|\nabla u(y)|}{|x - y|^{(n-1)(x(n-1))}} \, dy. \]

Now Lemma 3.1 and Remark 3.5 yield the following theorem, where we do not need continuity of \(s\). Note that \(\alpha^+ > 0\) if and only if \(s^+ < \frac{n}{n-1}\) and \((\alpha p)^+ = (n - s^+ (n-1))p < n\) if and only if \(p < \frac{n}{n-s^+ (n-1)}\).

5.6. Theorem (Sobolev-Poincaré inequality in the case \(p > 1\)). Let \(D \subset \mathbb{R}^n, n \geq 2\), be an \(s(\cdot)\)-John domain. Assume that

\[ (5.7) \quad 1 \leq s^- \leq s^+ < \frac{n}{n-1} \quad \text{and} \quad 1 < p < \frac{n}{n-s^+ (n-1)}. \]

Then there exists a constant \(c\) such that

\[ ||u - u_B||_{L^p(D)} \leq c ||\nabla u||_{L^p(D)}, \]

for all \(u \in L^1_p(D)\). Here \(q(x) := \frac{np}{m(x) - 1 + n - s(x)p}\) and \(B := B(x_0, \text{dist}(x_0, \partial D))\). The constant \(c\) depends only on the dimension \(n, \alpha, \beta, s^+, \text{dist}(x_0, \partial D)\) and log-Hölder constant of \(p\).
Proof. By Lemma 5.5 we obtain
\[ |u(x) - u_{B(x_0, \text{dist}(x_0, \partial D))}| \leq c \tilde{I}_{s(x)} |\nabla u|(x). \]

Note that (5.7) yields assumption of Lemma 3.1 by Remark 3.5. Thus we obtain that
\[ \|u - u_{B(x_0, \text{dist}(x_0, \partial D))}\|_{L^{q+1}(D)} \leq c\|\nabla u\|_{L^p(D)}. \]

In the case \( p = 1 \) we use Theorem 3.2 and hence we need to assume that \( s \) is log-Hölder continuous.

5.8. Theorem (Sobolev-Poincaré inequality in the case \( p = 1 \)). Let \( D \subset \mathbb{R}^n, n \geq 2 \), be an \( s(\cdot) \)-John domain. Assume that \( s \in \mathcal{P}^{\log}(\Omega) \) satisfies
\[ 1 \leq s^- \leq s^+ < \frac{n}{n-1}. \]
Then there exists a constant \( c \) such that
\[ \|u - u_B\|_{L^{q+1}(D)} \leq c\|\nabla u\|_{L^1(D)}, \]
for all \( u \in L^1_1(D) \), where \( q(x) := \frac{n}{s(x)(n-1)} \), and \( B := B(x_0, \text{dist}(x_0, \partial D)) \). The constant \( c \) depends only on the dimension \( n \), \( \alpha \), \( \beta \), \( s^+ \), \text{dist}(x_0, \partial D) \), and the log-Hölder constant of \( \alpha \).

Proof. Let us first assume that \( \|\nabla u\|_1 \leq 1 \). We show that \( g_{L^{q+1}(D)}(u - u_B) \) is uniformly bounded. For every \( j \in \mathbb{Z} \) we set
\[ D_j := \{ x \in D : 2^j < |u(x) - u_B| \leq 2^{j+1} \} \] and \( v_j := \max\{0, \min\{|u - u_B| - 2^j, 2^j\}\} \).

From the lattice property of Sobolev functions it follows \( v_j \in L^1_1(D) \). By Lemma 5.5 we have
\[ |v_j(x) - (v_j)_B| \leq c \tilde{I}_{s(x)} |\nabla v_j|(x) \]
for almost every \( x \in D \).

We obtain by the pointwise inequality \( v_j \leq |u - u_B| \) and by the Poincaré inequality in a ball that
\[ v_j(x) \leq |v_j(x) - (v_j)_B| + (v_j)_B \leq c \tilde{I}_{s(x)} |\nabla v_j|(x) + \int_B |u - u_B| \, dx \]
\[ \leq c \tilde{I}_{s(x)} |\nabla v_j|(x) + c \text{diam}(B) \int_B |\nabla u| \, dx \]
\[ \leq c \tilde{I}_{s(x)} |\nabla v_j|(x) + c \frac{\text{diam}(B)}{|B|} \leq c_1 (\tilde{I}_{s(x)} |\nabla v_j|(x) + 1), \]
where in the second to last inequality we used that \( \|\nabla u\|_1 \leq 1 \), and in the last inequality we note that \( \text{diam}(B) \) and \( |B| \) both depends only on the distance from the John center to the boundary. For the rest of this proof we fix the constant \( c_1 \) to denote the constant on the last line. It depends only on \( n, \alpha, \beta, s^+ \) and \text{dist}(x_0, \partial D).

Using the definition of \( D_j \) we get
\[ \int_D |u(x) - u_B|^{q(x)} \, dx = \sum_{j=-\infty}^{\infty} \int_{D_j} |u(x) - u_B|^{q(x)} \, dx \leq \sum_{j=-\infty}^{\infty} \int_{D_j} 2^{(j+1)q(x)} \, dx. \]
For every \( x \in D_{j+1} \) we have \( v_j(x) = 2^j \) and thus obtain by \((5.9)\) the pointwise inequality \( c_1 \tilde{I}_{c_1} |\nabla v_j|(x) + c_1 > 2^j \) for almost every \( x \in D_{j+1} \). Note that if \( a + b > c \), then \( a > \frac{1}{2} c \) or \( b > \frac{1}{2} c \). Thus

\[
\sum_{j=-\infty}^{\infty} \int_{D_j} 2^{(j+1)q(x)} \, dx \leq \sum_{j=-\infty}^{\infty} \int_{\{x \in D_j : c_1 \tilde{I}_{c_1} |\nabla v_j|(x) + c_1 > 2^{j+1} \}} 2^{(j+1)q(x)} \, dx
\]

\[
\leq \sum_{j=-\infty}^{\infty} \int_{\{x \in D_j : c_1 \tilde{I}_{c_1} |\nabla v_j|(x) > 2^{j+2} \}} 2^{(j+1)q(x)} \, dx + \sum_{j=-\infty}^{\infty} \int_{\{x \in D_j : c_1 > 2^{j+2} \}} 2^{(j+1)q(x)} \, dx.
\]

Since \( \|\nabla u\|_1 \leq 1 \), we obtain by Theorem 3.2 and Remark 3.5 for the first term on the right-hand side that

\[
\sum_{j=-\infty}^{\infty} \int_{\{x \in D_j : c_1 \tilde{I}_{c_1} |\nabla v_j|(x) > 2^{j+2} \}} 2^{(j+1)q(x)} \, dx
\]

\[
\leq 2^{3n} \sum_{j=-\infty}^{\infty} \int_{\{x \in D_j : c_1 \tilde{I}_{c_1} |\nabla v_j|(x) > 2^{j+2} \}} 2^{(j-2)q(x)} \, dx
\]

\[
\leq c \sum_{j=-\infty}^{\infty} \left( \int_{D_j} |\nabla v_j| \, dy + |\{0 < |\nabla v_j| \leq 1\} \right)
\]

\[
\leq c \sum_{j=-\infty}^{\infty} \left( \int_{D_j} |\nabla u| \, dy + |D_j| \right) = c \int_{D} |\nabla u| \, dy + c |D|.
\]

Let \( j_0 \) be the largest integer satisfying \( c_1 > 2^{j_0-2} \). Hence

\[
\sum_{j=-\infty}^{\infty} \int_{\{x \in D_j : c_1 > 2^{j-2} \}} 2^{(j+1)q(x)} \, dx \leq \int_{D} \sum_{j=-\infty}^{j_0} 2^{(j+1)q(x)} \, dx \leq c |D|.
\]

Then we conclude the proof by the scaling argument: Since \( \|u - u_B\|_{q,(c)} \leq c \) for all \( \|\nabla u\|_1 \leq 1 \), we obtain the claim by applying this to \( u/\|\nabla u\|_1 \).

5.10. Remark. (1) If \( s \equiv 1 \) is chosen, then by Theorems 5.6 and 5.8 the classical Sobolev-Poincaré inequality is recovered for 1-John domains, [5].

(2) Let \( s \) be a constant function and \( 1 < s < n/(n-1) \). The target spaces \( L_{s,p}^{\alpha + n/(n-1)}(D) \) in Theorem 5.8 is optimal, while the target space \( L_{s,p}^{\alpha + n/(n-1)}(D) \) in Theorem 5.6 is not the best possible, see [16, 27].

(3) Let \( s \) be a constant function and \( 1 \leq s < n/(n-1) \), then the classical \((1,1)\)-Poincaré inequality in an \( s \)-John domain is recovered. And this yields the \((p,p)\)-Poincaré inequality for all \( 1 < p < \infty \). Recall that it has been known that the \((p,p)\)-Poincaré inequality holds for all \( p \in [1, \infty) \) whenever \( 1 \leq s < n/(n-1) \), [38].

Let us recall the following result, Theorem 5.11 for bounded John domains. The result was proved for domains with a fixed cone condition by N. S. Trudinger [40]. Domains with a fixed cone condition are examples
of John domains. But John domains form a strictly larger class of domains than domains with a fixed cone condition.

5.11. Theorem. [12] Let $D$ be a 1-John domain in $\mathbb{R}^n$, $n \geq 2$. There exists a constant $a > 0$ such that

$$\int_D \exp \left( a \frac{|u - u_D|}{\|\nabla u\|_{L_p(D)}} \right)^{\frac{n}{n-1}} \, dx < \infty$$

for all $u \in L^1_n(D)$.

Next we improve Theorem 5.11. In the next theorem the Hausdorff content is sharper than the Lebesgue measure in the following sense: the equation $|x| \in D : a|u - u_B|^{\frac{n}{n-1}} > t\}$ does not imply the equation $H^{(n-1)}(\{x \in D : a|u - u_B|^{\frac{n}{n-1}} > t\}) = 0$, but the latter equation implies the former one.

When the integration is with respect to the Hausdorff content, the integration is taken as a Choque integral, [1]. Let us define that

$$\int_D |u| \, dH^{\alpha}(\cdot) : = \int_0^\infty \mathcal{H}_n^{\alpha}(\{t \in D : |t(x)| > t\}) \, dt. \quad (5.12)$$

Note that $\mathcal{H}_n^{\alpha}(\cdot)$ is monotone by Lemma 4.2 i.e. if $A \subset B$ then $\mathcal{H}_n^{\alpha}(A) \leq \mathcal{H}_n^{\alpha}(B)$. Hence the function $t \mapsto \mathcal{H}_n^{\alpha}(\{t \in D : |t(x)| > t\})$ is a decreasing function $R \rightarrow [0, \infty)$ for every $u : D \rightarrow R$. By the decreasing property the function $t \mapsto \mathcal{H}_n^{\alpha}(\{t \in D : |t(x)| > t\})$ is measurable. Thus, $\int_0^\infty \mathcal{H}_n^{\alpha}(\{t \in D : |t(x)| > t\}) \, dt$ is well-defined as a Lebesgue integral. For the notion of Choque integral in applications we refer to [1].

5.13. Theorem (Sobolev-Poincaré inequality in the limit case). Let $D$ be an $s$-John domain in $\mathbb{R}^n$, $n \geq 2$. Assume that $s \in \mathcal{P}^\text{reg}(D)$ satisfies $1 \leq s^- \leq s^+ < \frac{n}{n-1}$. When $\alpha(x) : = n - s(x)(n - 1)$, then there exist positive constants $a$ and $b > 0$ such that

$$\int_D \exp \left( a \frac{|u - u_B|^{\frac{n}{n-1}}}{\|\nabla u\|_{L_p(D)}} \right)^{\frac{n}{n-1}} \, dx \leq b$$

for all $u \in L^1_n(D)$ with $\|\nabla u\|_{L_p(D)} \leq 1$, here $B : = B(x_0, \text{dist}(x_0, \partial D))$.

Theorem 5.13 yields Corollary 1.3 as a special case when $s$ is a constant function.

Proof of Theorem 5.13. Let $m > 0$ be a constant that we will fix later. By Lemma 5.5 there exists a constant $c_1$ such that

$$\int_D \exp \left( \left( \frac{m}{2(1 + |D|)} |u - u_B| \right)^{\frac{n}{n-1}} \right) \, dH^{\alpha}(\cdot) \leq b$$

$$\leq \int_D \exp \left( \left( \frac{mc_1}{2(1 + |D|)} \|\nabla u\|_1 \right)^{\frac{n}{n-1}} \right) \, dH^{\alpha}(\cdot).$$
Whenever we choose $m$ parts we obtain there exist constants $c_\alpha$ that

\[ \int_D \exp \left( \frac{m}{2(1 + |D|)} |u - u_D|^{\frac{\alpha}{n-1}} \right) d\mathcal{H}_\infty^{(n-1)} \]

\[ \leq \int_0^\infty \mathcal{H}_\infty^{(n-1)} \left( \left\{ x \in D : \exp \left( \left( \frac{m c_1}{2(1 + |D|)} I_{\alpha}(x) \right)^{\frac{\alpha}{n-1}} > t \right) \right\} \right) dt \]

\[ \leq \int_0^1 \mathcal{H}_\infty^{(n-1)} \left( \left\{ x \in D : \left( \frac{I_{\alpha}(x) \cdot |\nabla u|}{2(1 + |D|)} \right)^{\frac{\alpha}{n-1}} > \log(t) \right\} \right) dt 

+ \int_1^\infty \mathcal{H}_\infty^{(n-1)} \left( \left\{ x \in D : \left( \frac{I_{\alpha}(x) \cdot |\nabla u|}{2(1 + |D|)} \right)^{\frac{\alpha}{n-1}} > \log(t) \right\} \right) dt. \]

The integral over the unit interval is estimated by $\mathcal{H}_\infty^{\alpha-n^\alpha} (D)$. We estimate the second integral over the unbounded interval. Let us first note that $\left\| \frac{\nabla u}{2(1 + |D|)} \right\|_{\frac{\alpha}{n-1}} (D) \leq \frac{1}{2(1 + |D|)}$. By Theorem 3.5 we have $\tilde{I}_{\alpha}(x) = I_{\alpha}(x)$ where $\alpha(x) = n - s(x)(n - 1)$. The condition $\alpha^- > 0$ holds if and only if $s^+ < \frac{n}{n-1}$ and $\alpha^+ \leq 1 < n$ since $s^- \geq 1$. Moreover $\frac{n}{n-1} = \frac{n}{s^+(n-1)}$. By Theorem 1.2 there exist constants $c_2$ and $c_3$ such that

\[ \mathcal{H}_\infty^{\alpha-n^\alpha} \left( \left\{ x \in D : \tilde{I}_{\alpha}(x) \left( \frac{|\nabla u|}{2(1 + |D|)} \right) > \left( \frac{\log(t)}{\left(\frac{m c_1}{3(1 + |D|)} \right)^{\frac{\alpha}{n(s^+(n-1))}}} \right) \right\} \right) \]

\[ \leq c_2 \exp \left( - c_3 \frac{\log(t)}{\left(\frac{m c_1}{3(1 + |D|)} \right)^{\frac{\alpha}{n(s^+(n-1))}}} \right) = c_2 t^{-\frac{c_3}{(mc_1)^{\frac{\alpha}{n(s^+(n-1))}}}}. \]

Whenever we choose $m > 0$ to be so small that $\frac{c_3}{(mc_1)^{\frac{\alpha}{n(s^+(n-1))}}} > 1$, we obtain

\[ \int_1^\infty \mathcal{H}_\infty^{\alpha-n^\alpha} \left( \left\{ x \in D : \tilde{I}_{\alpha}(x) \left( \frac{|\nabla u|}{2(1 + |D|)} \right) > \left( \frac{\log(t)}{\left(\frac{m c_1}{3(1 + |D|)} \right)^{\frac{\alpha}{n(s^+(n-1))}}} \right) \right\} \right) dt \]

\[ \leq \int_1^\infty c_2 t^{-\frac{c_3}{(mc_1)^{\frac{\alpha}{n(s^+(n-1))}}}} dt =: c_4 < \infty. \]

Now the claim follows by choosing

\[ a = \left( \frac{m}{2(1 + |D|)} \right)^{\frac{\alpha}{n-1}} \quad \text{and} \quad b = \mathcal{H}_\infty^{\alpha-n^\alpha}(D) + c_4. \]

\[ \square \]

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