Analytical solution for vibration characteristics of rotating graphene nanoplatelet-reinforced plates under rub-impact and thermal shock

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Abstract
This article presents an analytical investigation on vibration characteristics of rotating graphene nanoplatelet (GPL)-reinforced plates subjected to rub-impact and thermal shock. The effective material properties are assumed to vary continuously and smoothly along the thickness direction of the plate and are determined via the Halpin–Tsai micromechanics model together with the rule of mixture. Considering the gyroscopic effect, the equations of motion are derived by adopting the Hamilton’s principle based on the Kirchhoff’s plate theory. Then, the Galerkin method and the small parameter perturbation method are utilized to obtain the free and forced vibration results for the rotating plate. A detailed parametric study is conducted to examine the effects of the GPL weight fraction, GPL distribution pattern, length-to-thickness ratio and length-to-width ratio of GPLs, and the rotating speed on free vibration characteristics of the nanocomposite plate. Attention is also given to the influences of the GPL weight fraction, thermal flow, and friction coefficient on forced vibration responses of the plate. The obtained results can play a role in the design of a rotating GPL-reinforced plate structure to achieve significantly improved mechanical performance.

Keywords
graphene nanoplatelet, analytical solution, rotating plates, rub-impact, thermal shock

Introduction
Rotating plates are considerably used as main components in various applications, such as blades with low aspect ratios. Compared to beam models, plate models are more accurate to describe the deformation along their width direction. As two-dimensional structures, rotating plates have been investigated by many scholars.¹⁻⁹

Young and Liao¹⁰ presented a method to investigate the Coriolis effect on the vibration of a cantilever rotating plate. Cote et al.¹¹ examined the effects of shear deformation and rotary inertia on the free vibration of a rotating annular plate. Yoo and Pierre¹² presented a dynamic modeling method of a rotating cantilever plate and studied its modal characteristics. Hashemi et al.¹³ developed a finite element formulation for vibration analysis of rotating thick plates. Younesian et al.¹⁴ presented an analytical analysis on vibrations of a hollow circular plate subjected to a rotating peripheral force. Li and Zhang¹⁵ carried out a dynamic model of a
functionally graded (FG) rectangular plate undergoing large overall motions and investigated its free vibrations. Rostami et al.16 presented a study on in-plane vibrations of rotating orthotropic cantilever plates. Adopting the absolute nodal coordinate formulation, Chen et al.17 carried out the dynamic analysis of a rotating plate with a setting angle.

As the modern industry develops rapidly, more and more structural components are demanding high strength performance. Owing to the exceptional mechanical properties, graphene nanoplatelets (GPLs) have tremendous application potentials as nanofillers to enhance structural strength and stiffness. Recently, some researchers have paid their attention to the mechanical behavior of GPL-reinforced structures.

Yang and his colleagues conducted extensive research on vibration behaviors of GPL-reinforced structures.18–22 Within the framework of the first-order shear deformation plate theory, Song et al.23 carried out free and forced vibrations of FG multilayer GPL reinforced composite plates. Yang et al.24 presented the thermoelastic bending behavior of novel FG polymer nanocomposite rectangular plate reinforced with GPLs. Bahaadini and Saidi25 investigated aeroelastic characteristics of FG multilayer GPL-reinforced composite rotating plates under supersonic flow. Ebrahimi et al.26 dealt with the thermal vibration analysis of GPL-reinforced nanocomposite plates embedded on the viscoelastic substrate. Considering the cantilever boundary conditions, Niu et al.27 investigated free vibrations of the rotating FG composite cylindrical panels reinforced with GPLs. Nguyen et al.28 presented a Bezier finite element formulation for the bending and transient analysis of FG porous plates reinforced by GPLs embedded in piezoelectric layers.

The main issue treated in this article is to present an analytic method to investigate free and forced vibrations of a rotating plate with GPL reinforcement subject to rub-impact and thermal shock. The plate is modeled by adopting the Kirchhoff’s plate theory, and the equations of motion are derived by using the Hamilton’s principle. Then, the Galerkin method and the small parameter perturbation method are utilized to obtain the analytical solution for the nanocomposite plate. Moreover, a parametric study is conducted to examine the effects of the rotating speed, GPL weight fraction, GPL distribution pattern, length-to-thickness ratio and length-to-width ratio of GPLs, thermal flow, and friction coefficient on vibration characteristics of the rotating plate.

**Modeling**

A rotating plate reinforced with GPLs, subjected to a combined action of rub-impact and thermal shock, is established in Figure 1. To describe its motion accurately, the fixed coordinate system $o_1\{x_1y_1z_1\}$ and the rotating coordinate system $o-xyz$ are proposed. The plate rotates by a fixed distance $R_0$ at an angular velocity $\Omega$ along the $y_1$ axis, which is the axial direction. $x_1$ axis and $z_1$ axis are the corresponding radial directions. The origin $o$ of the rotating coordinate system $o-xyz$ is at the corner of the plate. The $x$ axis, $y$ axis, and $z$ axis are the length direction, width direction, and thickness direction of the plate, respectively. The corresponding sizes of the plate along these three directions are $a$, $b$, and $h$.

**Figure 1.** GPL-reinforced rotating plate subjected to rub-impact and thermal shock.

**Description of effective material properties**

The rotating plate, in this article, is considered as a polymer nanocomposite structure reinforced with GPLs. According to the Halpin–Tsai model,29,30 the effective Young’s modulus can be predicted as follows:

$$E(z) = E_M \left\{ \frac{3}{8} \left[ 1 + \xi_l \left( \frac{E_{GPL}/E_M - 1}{E_{GPL}/E_M + \xi_l} \right) \right] V_{GPL} \right\} \left\{ 1 - \xi_l \left( \frac{E_{GPL}/E_M - 1}{E_{GPL}/E_M + \xi_l} \right) \right\}$$

$$= \frac{5}{8} \left[ 1 + \xi_w \left( \frac{E_{GPL}/E_M - 1}{E_{GPL}/E_M + \xi_w} \right) \right] V_{GPL} \right\} \left\{ 1 - \xi_w \left( \frac{E_{GPL}/E_M - 1}{E_{GPL}/E_M + \xi_w} \right) \right\}$$

where $E_M$ and $E_{GPL}$ are Young’s modulus of the polymer matrix and GPLs, respectively; $\xi_l$ and $\xi_w$ are GPL’s geometry factors in the form of

$$\left\{ \begin{array}{l}
\xi_l = 2l/h \\
\xi_w = 2w/h
\end{array} \right.$$  

in which $l$, $w$, and $h$ are GPL’s average length, width, and thickness, respectively.

The volume fraction of GPLs is defined as

$$V_{GPL}(z) = \frac{g_{GPL}}{g_{GPL} + \rho_{GPL}(1 - g_{GPL})/\rho_M}$$

where $g_{GPL}$ is the weight fraction of GPLs; $\rho_M$ and $\rho_{GPL}$ are the mass density of polymer matrix and GPL nanofillers, respectively.

Due to the nonuniform distribution of GPLs in the polymer matrix, the weight fraction of GPLs is position
dependent. Five different GPL distribution patterns, shown in Figure 2, enter in consideration.

The expressions of GPL distribution patterns can be written as

\[ g_{GPL}(z) = \begin{cases} 
\lambda_1 g_0 \left( \frac{1}{2} + \frac{z}{h} \right) & \text{pattern I} \\
\frac{4}{h^2} \lambda_2 g_0 z^2 & \text{pattern II} \\
\lambda_3 g_0 & \text{pattern III} \\
\lambda_4 g_0 \left( 1 + \frac{z}{h} \right) & \text{pattern IV} \\
\lambda_5 g_0 \left( 1 - \frac{4}{h^2} z^2 \right) & \text{pattern V} 
\end{cases} \]

where \( g_0 \) is a characteristic value of weight fraction; \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \) and \( \lambda_5 \) are the weight fraction indices.

According to the rule of mixture, the effective Poisson’s ratio, mass density, thermal expansion coefficient, and specific heat capacity of the composite are expressed as

\[ \begin{aligned}
u(z) &= V_{GPL} \nu_{GPL} + V_M \nu_M \\
\rho(z) &= V_{GPL} \rho_{GPL} + V_M \rho_M \\
\alpha(z) &= V_{GPL} \alpha_{GPL} + V_M \alpha_M \\
C(z) &= V_{GPL} C_{GPL} + V_M C_M 
\end{aligned} \]

where \( \nu_{GPL}, \alpha_{GPL}, \) and \( C_{GPL} \) are GPL’s Poisson’s ratio, thermal expansion coefficient, and specific heat capacity, respectively; \( \nu_M, \alpha_M, \) and \( C_M \) are Poisson’s ratio, thermal expansion coefficient, and specific heat capacity of the matrix, respectively.

The volume fraction of matrix is determined by

\[ V_M = 1 - V_{GPL} \]

Based on the micromechanics model, the effective thermal conductivity coefficient of nanocomposite is

\[ k(z) = k_M + k_{GPL} \frac{V_{GPL}}{3} \left[ \frac{2}{L + 1 / \left( k_{GPL}/k_M \right) - 1} + \frac{1}{(1 - L)/2 + 1 / \left( k_{GPL}/k_M \right) - 1} \right] \]

where \( R_s \) is an average interfacial thermal resistance between the GPL and the matrix; \( k_M \) and \( k_{GPL} \) are the intrinsic thermal conductivity coefficients of the polymer matrix and GPLs, respectively; the GPL geometric parameter \( L \) is obtained as

\[ L = \frac{8 \ln(\xi + \sqrt{\xi^2 - 4}) - 16 \ln 2 - \frac{4}{\xi^2 - 4}}{\sqrt{(\xi^2 - 4)^3}} \]

**Description of rub-impact and thermal shock**

In actual engineering, rub-impact fault occurs frequently because the clearance between blades and
casing needs to be designed as small as possible. In this article, the impact force is regarded as an approximate periodic impact load, illustrated in Figure 3 and expressed as

\[
(n - 1)T_{\text{impact}} + t_{\text{act}} < t < nT_{\text{impact}}
\]

\[
(n - 1)T_{\text{impact}} < t < nT_{\text{impact}} + t_{\text{act}}
\]

where, \( n = 1, 2, 3, \ldots; t_{\text{act}} \) is the impact time of one period; \( T_{\text{impact}} \) is one periodic time; \( F_M \) is the impact force distribution patterns in the form of

\[
F_M(y) = \frac{F_z - F_1}{b} y + F_1
\]

in which \( F_1 \) and \( F_2 \) are the minimum and maximum amplitudes of impact force, respectively.

To facilitate subsequent calculation, equation (9) needs to be expanded as

\[
F_{\text{impact}}(y, t) = a_m + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n\pi}{T_{\text{impact}}} t + b_n \sin \frac{2n\pi}{T_{\text{impact}}} t \right)
\]

where

\[
a_m = \frac{1}{T_{\text{impact}}} \int_{0}^{T_{\text{act}}} F_M \sin \left( \frac{\pi t}{t_{\text{act}}} \right) \, dt
\]

\[
a_n = \frac{2}{T_{\text{impact}}} \int_{0}^{T_{\text{act}}} F_M \sin \left( \frac{\pi t}{t_{\text{act}}} \right) \cos \left( \frac{2n\pi}{T_{\text{impact}}} t \right) \, dt
\]

\[
b_n = \frac{2}{T_{\text{impact}}} \int_{0}^{T_{\text{act}}} F_M \sin \left( \frac{\pi t}{t_{\text{act}}} \right) \sin \left( \frac{2n\pi}{T_{\text{impact}}} t \right) \, dt
\]

The corresponding rub force is related by

\[
F_{\text{rub}}(y, t) = \mu F_{\text{impact}}(y, t)
\]

in which \( \mu \) is the friction coefficient.

A large number of spinning blades in an aeroengine always work in high-temperature environment. In this article, the thermal flow \( q_T \) is applied to the surface of rotating plate.

The temperature of an arbitrary point on the plate is assumed as

\[
T_{\text{shock}} = T_{xy} T_z
\]

where \( T_{xy} \) is the in-plane temperature distribution; \( T_z \) can be derived from

\[
k \frac{\partial^2 T_z}{\partial z^2} = C \rho \frac{\partial T_z}{\partial t} \tag{15}
\]

The boundary and initial conditions are

\[
\left\{ \begin{array}{l}
z = \frac{h}{2}, \quad k \frac{\partial T_z}{\partial z} = q_T; \quad z = -\frac{h}{2}, \quad \frac{\partial T_z}{\partial z} = 0 \\
t = 0, \quad T_z = 0
\end{array} \right.
\]

By solving equation (15), one can get

\[
T_z = \frac{q_T h}{k} \left[ \frac{\beta t}{\pi^2} + \frac{z^2 + h^2}{2h^2} \right] - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-n^2 \beta t} \cos \frac{n \pi}{2} \left( \frac{2h}{h} + 1 \right)
\]

in which \( \beta = kH/C_{ph} \).

**Theoretical formulations**

**Analytic solution for free vibration**

The kinetic energy of the plate is given by

\[
T = \frac{1}{2} \int_{-h/2}^{h/2} \int_{-b/2}^{b} \rho \left[ w^2 \Omega^2 + \left( \frac{\partial w}{\partial t} \right)^2 + (R_0 + x)^2 \Omega^2 \right] - 2 \left( \frac{\partial w}{\partial t} \right) (R_0 + x) \Omega \, dx \, dy \, dz
\]

(18)

The deformation potential energy is

\[
U_1 = \frac{1}{2} \int_{-h/2}^{h/2} \int_{-b/2}^{b} D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] z^2 \, dx \, dy \, dz
\]

(19)

The centrifugal potential energy caused by rotation is written as
\[ U_2 = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^b \rho \Omega^2 \left\{ R_0(a-x) + \frac{1}{2} (a^2 - x^2) \right\} \left( \frac{\partial w}{\partial x} \right)^2 \, dx \, dy \]  

From this, the total potential energy of the plate is

\[ U = U_1 + U_2 \]

\[ = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^b D \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-v) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} z^2 \, dx \, dy \]  

\[ + \frac{1}{2} \int_{-h/2}^{h/2} \int_0^b \rho \Omega^2 \left\{ R_0(a-x) + \frac{1}{2} (a^2 - x^2) \right\} \left( \frac{\partial w}{\partial x} \right)^2 \, dx \, dy \]  

where \( D(z) = E(z)/[1 - u(z)^2] \).

The transversal deformation \( w \) is assumed as

\[ w(x, y, t) = W(x, y)\sin(\omega t + \phi) \]  

in which \( W(x, y) \) is the mode function that satisfies the geometric boundary condition and takes the form of

\[ W(x, y) = \sum_{m=1}^M \sum_{n=1}^N A_{mn} \phi_m(x) \phi_n(y) \]  

where

\[ \phi_m(x) = \cosh(\alpha_m x) - \cos(\alpha_m x) - c_m[\sinh(\alpha_m x) - \sin(\alpha_m x)] \]  

\[ \phi_n(y) = \cosh(\beta_n y) + \cos(\beta_n y) - d_n[\sinh(\beta_n y) + \sin(\beta_n y)] \]  

Substituting equations (18), (21), and (22) into the Hamilton’s principle

\[ \int_{t_0}^{t_f} \delta(T - U) \, dt = 0 \]  

in which gives

\[ \int_{-h/2}^{h/2} \int_0^b \left\{ \rho(\Omega^2 - \omega^2) W \delta W - \frac{1}{2} \rho \Omega^2 (a^2 - x^2) \frac{\partial W}{\partial x} \delta \left( \frac{\partial W}{\partial x} \right) - D \left( \frac{\partial^2 W}{\partial x^2} \right) \delta \left( \frac{\partial^2 W}{\partial x^2} \right) + \frac{\partial^2 W}{\partial y^2} \delta \left( \frac{\partial^2 W}{\partial y^2} \right) - \frac{\partial^2 W}{\partial x \partial y} \delta \left( \frac{\partial^2 W}{\partial x \partial y} \right) \right\} \, dx \, dy \, \delta \]  

\[ + D(1-v)z^2 \left[ \frac{\partial^2 W}{\partial x^2} \delta \left( \frac{\partial^2 W}{\partial x^2} \right) + \frac{\partial^2 W}{\partial y^2} \delta \left( \frac{\partial^2 W}{\partial y^2} \right) - 2 \frac{\partial^2 W}{\partial x \partial y} \delta \left( \frac{\partial^2 W}{\partial x \partial y} \right) \right] \, dx \, dy \]  

Substituting equation (23) into equation (28), we have
where

\[
E \int_{-h/2}^{h/2} \rho(\omega^2 - \Omega^2)dz + (H + G - 2K) \int_{-h/2}^{h/2} Dv^2 dz + (I + F + 2K) \int_{-h/2}^{h/2} Dz^2 dz + \frac{\Omega^2}{2} L \int_{-h/2}^{h/2} \rho dz = 0
\] (30)

where \(E, F, G, H, I, K\) and \(L\) follow the same form of

\[
X = \begin{bmatrix}
X_{11} & \cdots & X_{1j} & \cdots & X_{1(M \times N)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
X_{i1} & \cdots & X_{ij} & \cdots & X_{i(M \times N)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
X_{(M \times N)1} & \cdots & X_{(M \times N)j} & \cdots & X_{(M \times N)(M \times N)}
\end{bmatrix}
\] (31)

in which \(X_{ij}\) is the corresponding element in each matrix, expressed as

\[
\begin{align*}
E_{[(i-1)N+j][(m-1)N+n]} &= \int_0^B \int_0^a \phi_m(x)\phi_n(y)\phi_i(x)\phi_j(y) dx dy \\
F_{[(i-1)N+j][(m-1)N+n]} &= \int_0^B \int_0^a \phi_m''(x)\phi_n'(y)\phi_i''(x)\phi_j(y) dx dy \\
G_{[(i-1)N+j][(m-1)N+n]} &= \int_0^B \int_0^a \phi_m(x)\phi_n''(y)\phi_i''(x)\phi_j(y) dx dy \\
H_{[(i-1)N+j][(m-1)N+n]} &= \int_0^B \int_0^a \phi_m''(x)\phi_n(y)\phi_i''(x)\phi_j''(y) dx dy \\
I_{[(i-1)N+j][(m-1)N+n]} &= \int_0^B \int_0^a \phi_m(x)\phi_n'(y)\phi_i''(x)\phi_j(y) dx dy \\
K_{[(i-1)N+j][(m-1)N+n]} &= \int_0^B \int_0^a \phi_m'(x)\phi_n(y)\phi_i'(x)\phi_j'(y) dx dy \\
L_{[(i-1)N+j][(m-1)N+n]} &= \int_0^B \int_0^a (a^2 - x^2)\phi_m''(x)\phi_n'(y)\phi_i'(x)\phi_j(y) dx dy
\end{align*}
\] (32)
Thus, the natural frequencies $\omega$ of the rotating GPL reinforced plate can be calculated from equation (30) directly.

In addition, the backward traveling wave frequency $\omega_b$ and forward traveling wave frequency $\omega_f$ are obtained as

$$
\begin{align*}
\omega_b &= \omega + \Omega \\
\omega_f &= \omega - \Omega 
\end{align*}
$$

(33)

**Analytic solution for forced vibration**

In accordance to the linear thermoelastic principle, the relationship between stress and strain is

$$
\sigma_x = \frac{E}{1 - \nu^2} \left[ (\varepsilon_x + \nu\varepsilon_y) - (1 + \nu)\alpha T_{\text{shock}} \right] \\
\sigma_y = \frac{E}{1 - \nu^2} \left[ (\varepsilon_x + \nu\varepsilon_y) - (1 + \nu)\alpha T_{\text{shock}} \right] \\
\tau_{xy} = G\gamma_{xy} 
$$

(34)

Based on the Kirchhoff plate theory, the constitutive relations are

$$
\varepsilon_x = -\frac{\partial^2 w}{\partial x^2}, \varepsilon_y = -\frac{\partial^2 w}{\partial y^2}, \gamma_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} 
$$

(35)

The thermal deformation potential energy of the plate is determined by

$$
U_T = U_2 + U_3 
$$

(37)

$$
U_T = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^a \int_0^b D \left\{ Dz^2(1 + \nu)\alpha T_{\text{shock}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - 2Dz^2(1 - \nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dxdydz 
$$

Thus, the total potential energy of the plate under thermal shock can be obtained as

$$
\delta W_T = q(x, y, t)\delta w 
$$

(38)

where

$$
q(x, y, t) = \mu F_{\text{impact}}(y, t)\delta(x - a) 
$$

(39)

Applying the Hamilton’s principle

$$
\int_{t_0}^{t_1} (\delta L + \delta W_T)dt = 0 \ (L = T - U_T) 
$$

(40)

and substituting equations (18), (37), and (38) into equation (40) yield

$$
= \int_{-h/2}^{h/2} \left\{ \rho w\Omega^2 - \rho w + \rho\Omega^2 \left[ R_0(a - x) + \frac{1}{2}(a^2 - x^2) \right] \frac{\partial^2 w}{\partial x^2} \right\} dz \\
- \rho\Omega^2(R_0 + x) \frac{\partial w}{\partial x} - Dz^2 \left[ \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] \\
= \int_{-h/2}^{h/2} \left\{ Dz^2(1 + \nu)\alpha \left( \frac{\partial^2 T_{\text{shock}}}{\partial x^2} + \frac{\partial^2 T_{\text{shock}}}{\partial y^2} \right) \right\} dz - q(x, y, t) 
$$

(41)
The solution of equation (41) can be considered as a superposition solution of two parts in the form of

\[ w = w_{\text{static}} + w_{\text{dynamic}} \]  \hspace{1cm} (42)

where \( w_{\text{static}} \) is the static solution which satisfies

\[ \int_{-h/2}^{h/2} Dz^2 \left( \frac{\partial^4 w_{\text{static}}}{\partial x^4} + 2 \frac{\partial^2 w_{\text{static}}}{\partial x^2 \partial y^2} + \frac{\partial^4 w_{\text{static}}}{\partial y^4} \right) dz + \int_{-h/2}^{h/2} D(1 + \nu)z^2 \alpha \left( \frac{\partial^2 T_{\text{shock}}}{\partial x^2} + \frac{\partial^2 T_{\text{shock}}}{\partial y^2} \right) dz = 0 \]  \hspace{1cm} (43)

In addition, \( w_{\text{dynamic}} \) is the dynamic solution which can be derived from

\[ \int_{-h/2}^{h/2} \left\{ \rho w_{\text{dynamic}} \frac{\partial}{\partial x} - \rho \dot{w}_{\text{dynamic}} + \rho \Omega^2 \left[ R_0(a - x) + \frac{1}{2}(a^2 - x^2) \right] \frac{\partial^2 w_{\text{dynamic}}}{\partial x^2} \right\} dz \]

\[ = -q(x,y,t) - \int_{-h/2}^{h/2} \left\{ \rho w_{\text{static}} \frac{\partial}{\partial x} - \rho \dot{w}_{\text{static}} - \rho \Omega^2 (R_0 + x) \frac{\partial w_{\text{static}}}{\partial x} + \rho \Omega^2 R_0(a - x) + \frac{1}{2}(a^2 - x^2) \frac{\partial^2 w_{\text{static}}}{\partial x^2} \right\} dz \]  \hspace{1cm} (44)

**Static solution. Setting**

\[ w_{\text{static}} = \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn} \phi_m(x) \phi_n(y) \]  \hspace{1cm} (45)

and applying the Ritz method

\[ \frac{\partial U_T}{\partial B_{ij}} = 0 \]  \hspace{1cm} (46)

give

\[ \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn} \int_{-h/2}^{h/2} \int_{0}^{a} \int_{0}^{b} Dz^2 \left\{ \phi_m''(x) \phi_n''(y) \phi_i'(x) \phi_j(y) + \phi_m'(x) \phi_n''(y) \phi_i(x) \phi_j'(y) + \phi_m'(x) \phi_n'(y) \phi_i(x) \phi_j'(y) \right. \left. + 2(1 - \nu) \phi_m'(x) \phi_n'(y) \phi_i'(x) \phi_j(y) \right\} dx dy dz \]

\[ = \int_{-h/2}^{h/2} \int_{0}^{a} \int_{0}^{b} -Dz^2(1 + \nu) \alpha T_{xy} \left( \frac{\partial^2 T_{xy}}{\partial x^2} + \frac{\partial^2 T_{xy}}{\partial y^2} \right) \phi_i(x) \phi_j(y) dx dy dz \]  \hspace{1cm} (47)

where

\[ T_{xy} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{\pi^2 mn} \left( 1 - \cos m\pi \right) \left( 1 - \cos n\pi \right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  \hspace{1cm} (48)

Further, equation (47) can be changed to
By substituting equation (51) into equation (45), the solution for equation (53) can be obtained as

\[
\sum_{m=1,3,5n=1,3,5} \left\{ \frac{16}{\pi^2 mn} \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right) \right\} \int_0^a \frac{\sin \frac{m\pi x}{a} \phi_i(x) dx}{\phi_j(x)} \int_0^b \frac{\sin \frac{n\pi y}{b} \phi_j(y) dy}{\phi_j(y)}
\]

and substituting equation (52) into equation (44), we have

\[
\ddot{Q}_ij(t) + \omega_i^2 Q_{ij}(t) = \frac{P_{ij}(t)}{M_{ij}}
\]

where \( P_{ij} \) is determined by

\[
P_{ij}(t) = \int_0^b \int_0^a q'(x,y,t) W_{ij}(x,y) dx dy
\]

and

\[
M_{ij} = \int_{-h/2}^{h/2} \int_0^a \int_0^b \rho(z) W_{ij}(x,y) dx dy dz
\]

Suppose

\[
B_{mn}(t) = b_{mn} \int_{-h/2}^{h/2} [-Dz^2(1 + v)\alpha T^2] dz
\]

where, \( B_{mn} \) can be determined by substituting equation (51) into equation (49).

By substituting equation (51) into equation (45), the static solution can be obtained.

**Dynamic solution.** Setting

\[
w_{\text{dynamic}} = \sum_{i=1}^{l} \sum_{j=1}^{J} Q_{ij}(t) W_{ij}(x,y)
\]

and

\[
q'(x,y,t) = q(x,y,t) + \int_{-h/2}^{h/2} \left\{ \rho w_{\text{static}} \dot{Q}^2 - \rho \ddot{w}_{\text{static}} - \rho Q^2 \frac{\partial w_{\text{static}}}{\partial x} \right\} dz
\]

According to the vibration theory of single freedom system, the solution for equation (53) can be obtained as

\[
Q_{ij}(t) = a_{ij} \sin \omega_{ij} t + b_{ij} \cos \omega_{ij} t + \frac{1}{M_{ij} \omega_{ij}^2} \int_0^\tau P_{ij}(\tau) \sin \omega_{ij}(t-\tau) d\tau
\]

where \( a_{ij} \) and \( b_{ij} \) are determined by initial conditions; \( P_{ij} \) is determined by

\[
Q_{ij}(t)
\]
\[
P_{ij}(\tau) = \int_0^h \int_0^a q(x, y, \tau) W_{ij}(x, y) \, dx \, dy
\]

\[- \Omega^2 \int_{-h/2}^{h/2} \int_0^a \left[ \rho D z^2 (1 + v) \alpha T_z \right] dz \int_0^a \int_0^a \left\{ \frac{w_{\text{static}} - (R_0 + x) \partial w_{\text{static}}}{\partial x} + \left[ R_0 - \frac{1}{2} (a^2 - x^2) \right] \frac{\partial^2 w_{\text{static}}}{\partial x^2} \right\} W_{ij}(x, y) \, dx \, dy \]

\[+ \frac{d^2}{d\tau^2} \int_{-h/2}^{h/2} \int_0^a \left[ \rho D z^2 (1 + v) \alpha T_z(\tau) \right] dz \int_0^a \int_0^a w_{\text{static}} W_{ij}(x, y) \, dx \, dy \]

Thus, the dynamic solution can be calculated by substituting equation (56) into equation (52).

### Results and discussion

#### Validation study

To validate the accuracy of modeling in this paper, the theoretical solution, given by Piovan and Sampaio,\(^{32}\) is provided for a direct comparison. The dimension and material parameters are the following: plate length \(a = 152.40\) mm, plate width \(b = 22.12\) mm, plate thickness \(h = 2.66\) mm, Young’s modulus \(E = 214\) GPa, Poisson’s ratio \(v = 0.3\), and mass density \(\rho = 7800\) kg m\(^{-3}\).

Another validation example is presented in Table 2, where the numerical results\(^{33}\) are calculated by employing the finite element method. The selected parameters are defined as plate length \(a = 80\) mm, width \(b = 80\) mm, thickness \(h = 30\) mm, Young’s modulus \(E = 70.2\) GPa, Poisson’s ratio \(v = 0.33\), and mass density \(\rho = 2850\) kg m\(^{-3}\).

It can be seen from Tables 1 and 2 that the present results agree well with the two literature studies. The maximum error is less than 2\%, which indicates the proposed model is sufficiently accurate.

#### Parametric analysis

In this section, vibration characteristics of the rotating plate reinforced with GPL under rub-impact and thermal shock is investigated. Unless otherwise stated, the dimension parameters of the plate are the following: \(a = 0.45\) m, \(b = 0.45\) m, and \(h = 0.045\) m; the material parameters of the plate are the following: \(E_{\text{GPL}} = 1050\) GPa, \(E_M = 2.76\) GPa, \(\nu_{\text{GPL}} = 0.186, \nu_M = 0.35, \rho_{\text{GPL}} = 1060\) kg m\(^{-3}\), \(\rho_M = 1200\) kg m\(^{-3}\), \(\alpha_{\text{GPL}} = 2.0 \times 10^{-5}\) K\(^{-1}\), \(\alpha_M = 8.1 \times 10^{-5}\) K\(^{-1}\), \(C_{\text{GPL}} = 650\) J kg\(^{-1}\) K\(^{-1}\), \(C_M = 1912.55\) J kg\(^{-1}\) K\(^{-1}\), \(k_{\text{GPL}} = 2000\) W mK\(^{-1}\), \(k_M = 0.2\) W mK\(^{-1}\), \(R_1 = 1.0 \times 10^{-8}\) m\(^2\) K W\(^{-1}\), 
\(g_{\text{GPL}} = 1\%\), \(l/h = 10^5\) and \(l/w = 2\); the load parameters are: \(T_{\text{impact}} = 2\Pi /\Omega, \Omega = 300\) rad s\(^{-1}\), \(f_{\text{act}} = 0.1\) T, \(F_1 = 1000\) N, \(F_2 = 20F_1, \mu = 0.3\) and \(q_0 = 0\). In addition, pattern II of GPL distributions are considered if not specified in the following analysis.

Figure 4 depicts the variations of first two natural frequencies of the plate with rotating speed for different GPL distribution patterns, where \(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \) and \(\hat{\lambda}_5\) represent pattern I, pattern II, pattern III, pattern IV, and pattern V, respectively. It can be seen obviously that the natural frequencies of forward traveling waves decrease monotonically with a corresponding increase in the rotating speed, while those of backward traveling waves increase gradually. Another observation is that pattern II and pattern V of GPLs provide the highest and lowest natural frequencies, respectively. Thus, it is demonstrated that adding a larger number of GPL nanofillers near the top and bottom surfaces of the plate is an effective way to enhance the structural stiffness.

To examine the effect of GPL’s geometrical dimension, Figure 5 displays the variations of first two natural frequencies of the plate with rotating speed for different GPL length-to-thickness ratios. Results show that the natural frequencies are increased significantly with the increase.

### Table 1. Comparison between theoretical and experimental results.

| Frequency | Present | Ref\(^{32}\) | Error (%) |
|-----------|---------|--------------|-----------|
| 1st       | 97.81   | 97.00        | 0.835     |
| 2nd       | 611.95  | 611.00       | 0.320     |
| 3rd       | 1711.71 | 1693.00      | 1.105     |

### Table 2. Comparison between theoretical and numerical results.

| Frequency | Present | Ref\(^{33}\) | Error (%) |
|-----------|---------|--------------|-----------|
| First     | 39.21   | 39.56        | 0.88      |
| Second    | 94.86   | 96.78        | 1.98      |
| Third     | 239.32  | 242.76       | 1.42      |
| Fourth    | 306.73  | 310.92       | 1.35      |
| Fifth     | 346.98  | 352.93       | 1.69      |
of the GPL length-to-thickness ratio. For the same content of GPLs, the higher value of $l/h$ represents thinner GPLs, namely, GPLs with less graphene layers. It is worth noting that adopting thinner GPLs would lead to the better reinforcing effect for the rotating plate.

Furthermore, the variations of first two natural frequencies of the plate with rotating speed for different GPL length-to-width ratios are presented in Figure 6, where the GPL length remains constant. As can be seen, the natural frequencies increase markedly as the GPL length-to-width ratio decreases. Virtually, a smaller $l/w$ means that each individual GPL has a larger surface area. This suggests that GPLs with larger surface areas would be more effective in improving mechanical performance, which is caused by better load transfer capability.

Figure 7 illustrates the variations of first two natural frequencies of the plate with rotating speed for different GPL weight fractions. As shown, the natural frequencies with $g_{GPL} = 0$ are significantly lower than those with $g_{GPL} \neq 0$, which implies that dispersing a few GPL reinforcements into the matrix can considerably improve the mechanical behavior of the rotating plate. In addition, it is obvious that an increase in GPL weight fraction can lead to increased natural frequencies.

Moreover, to study the influence of material property on the forced vibration responses, Figure 8 plots the variations of forced vibration responses of the rotating plate for different GPL weight fractions. Results show that the forced vibration responses decrease significantly with the increase of the GPL weight fraction. This implies that adding more GPL nanofillers into polymer matrix is an efficient way to improve the mechanical behavior of the rotating plate. It can be told that the similar conclusions are drawn by the free and forced vibration results with different GPL weight fractions.
Figure 6. Effects of GPL length-to-width ratios on natural frequencies of the rotating plate: (a) first frequency and (b) second frequency.

Figure 7. Effects of GPL weight fractions on natural frequencies of the rotating plate: (a) first frequency and (b) second frequency.

Figure 8. Effects of GPL weight fractions on forced vibration responses of the rotating plate: (a) time-domain response and (b) frequency-domain response.
fractions. Besides, for the case of different GPL distribution patterns, different GPL length-to-thickness ratios or different GPL length-to-width ratios, the forced vibration results also give the conclusions which are similar with those obtained by the free vibration results. Thus, they are omitted to avoid repetition.

Figure 9 gives the variations of forced vibration responses of the rotating plate for different thermal flow. It is obvious that increasing thermal flow tends to give higher vibration response. This implies that thermal shock can aggravate the dynamic vibration of rotating plates and should be avoided in actual engineering.

Figure 10 illustrates the variations of forced vibration responses of the rotating plate for different friction coefficients. It can be seen that the vibration amplitudes increase steadily with the increase of friction coefficients, which indicates higher friction coefficient would exacerbate the vibration caused by rub-impact. For the purpose of preventing damage, the friction can be decreased by reducing the surface roughness between the tip of the plate and the casing during the production.

Conclusions
This article investigates the free and forced vibration characteristics of a nanocomposite rotating plate reinforced with GPLs subject to rub-impact and thermal shock. In accordance to the Kirchhoff plate theory, the equations of motion are derived by the Hamilton’s principle. Then, the Galerkin method and the small parameter perturbation method are employed to obtain the analytical solutions for free and forced vibration.

Results imply the following: (1) Adding a larger number of GPL reinforce fillers near the surfaces of the plate is an effective way to enhance the structural stiffness; (2) Adopting thinner GPLs would lead to better reinforcing effect; (3)
GPLs with larger surface areas would be more effective in improving mechanical performance; (4) Dispersing more GPL nanofillers into polymer matrix is an efficient way to improve the mechanical behavior of the rotating plate. (5) Thermal shock can aggravate the dynamic vibration of rotating plates and should be avoided in actual engineering; (6) Higher friction coefficient would exacerbate the vibration caused by rub-impact. For the purpose of preventing damage, the friction can be decreased by reducing the surface roughness between the tip of the plate and the casing during production.

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