Marginal integration $M$–estimators for additive models

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Abstract

Additive regression models have a long history in multivariate nonparametric regression. They provide a model in which each regression function depends only on a single explanatory variable allowing to obtain estimators at the optimal univariate rate. Beyond backfitting, marginal integration is a common procedure to estimate each component. In this paper, we propose a robust estimator of the additive components which combines local polynomials on the component to be estimated and marginal integration. The proposed estimators are consistent and asymptotically normally distributed. A simulation study allows to show the advantage of the proposal over the classical one when outliers are present in the responses, leading to estimators with good robustness and efficiency properties.

Key Words: Additive models; Local $M$–estimation; Kernel weights; Marginal integration; Robustness

AMS Subject Classification: MSC 62G35; 62G20, 62G05
1 Introduction

Several authors have dealt with the dimensionality reduction problem in non–parametric regression models. In particular, additive models allow the modelling of a response $Y$ as a sum of smooth functions of individual covariates $X = (X_1, \ldots, X_d)^T$. The advantage of additive models over general non–parametric regression models is that they allow to circumvent the so–called curse of dimensionality, which is caused by the fact that the expected number of observations in local neighbourhoods decreases exponentially as a function of the dimension $d$ of the covariates. More precisely, Stone (1985) defined the curse of dimensionality as “being that the amount of data required to avoid an unacceptably large variance increases rapidly with increasing dimensionality”. This results in the poor convergence rate of the estimators which, as it is well known, depends exponentially on the dimension and on the degree of smoothness of the regression function. To be more precise, let $(X^T, Y)$ be a random vector where $Y \in \mathbb{R}$ is the dependent variable and $X \in \mathbb{R}^d$ is the vector of explanatory variables. Consider the non–parametric regression model $Y = g(X) + \sigma(X)\varepsilon$ where the error $\varepsilon$ is independent of $X$ and centered at zero and $g : \mathbb{R}^d \to \mathbb{R}$ is the function to be estimated. Stone (1980, 1982) showed that the optimal rate for estimating $g$ is $n^{-\ell/(2\ell + d)}$ where $\ell$ is the degree of smoothness of $g$.

To face this problem, Stone (1985) and Hastie and Tibshirani (1990) considered additive models which generalize linear models, solve the problem of the curse of dimensionality and provide easily interpretable models. Additive models assume that $g(x) = \mu + \sum_{j=1}^{d} g_j(x_j)$ where $\mu$ is the location parameter and the additive components $g_j : \mathbb{R} \to \mathbb{R}$ satisfy some additional condition to be identifiable such as $\mathbb{E}[g_j(X_j)] = 0$. One of the advantages of additive models is that they allow for independent interpretation of the effect of each variable on the regression function $g$, as in linear regression models. Besides, as shown by Stone (1985), for such regression models the optimal rate for estimating $g$ is the one-dimensional rate of convergence $n^{-\ell/(2\ell + 1)}$ leading to dimensionality reduction through additive modelling.

Several estimation procedures to fit additive models have been proposed in the literature. The iterative method called backfitting proposed by Buja, Hastie and Tibshirani (1989) and Hastie and Tibshirani (1990) is one of the most popular procedures. Even if the procedure converges quickly, its iterative nature makes difficult to analyse its statistical properties. Besides, the backfitting algorithm does not allow to estimate derivatives since it does not give a closed form for the estimator. On the other hand, the marginal integration procedure proposed by Tjøstheim and Auestad (1994) and Linton and Nielsen (1995) and generalized by Chen et al. (1996) allows for the derivation of a closed form for the estimator and has been shown to work very well in simulation studies, see Sperlich et al. (1999). In particular, Severance–Lossin and Sperlich (1999) combine the integration procedure with a local polynomial approach to estimate simultaneously the additive components and its derivatives. When first moments exist, the idea beyond marginal integration is to estimate the marginal effects defined as the expectation of $Y$ with respect to the random error $\varepsilon$ and all the covariates except the $X_\alpha$ which is fixed. The marginal effect says how $Y$ varies in average when $X_\alpha$ varies. If the true multivariate function $g$ is additive, the marginal effects match with the additive components $g_\alpha$, except for the constant $\mu$, allowing for precise estimations under an additive model. The estimators are obtained estimating, in a first step, the multivariate function $g$ and then, using the marginal integration procedure to obtain the marginal effects.

As in other non–parametric settings, the estimators obtained through marginal integration can
be seriously affected by a relatively small proportion of atypical observations if the smoother chosen to estimate the multivariate function \( g \) is not resistant to outliers in the response variable. As is well known, in a non-parametric framework outlying observations can be even more dangerous than in a parametric model, since extreme points affect the scale and the shape of any estimate of the regression function based on local averaging, leading to possible wrong conclusions. This has motivated the interest in combining the ideas of robustness with those of smoothed regression, to develop procedures which will be resistant to deviations from the central model in non-parametric regression models. In this paper, we go further and we focus on robust estimators for additive models leading to reliable non-parametric regression estimators when atypical responses arise and which attain a univariate rate of convergence. Indeed, we seek for consistent estimators of the regression function \( g \) without requiring moment conditions on the errors \( \varepsilon \), so as to include the well-known \( \alpha \)-contaminated neighbourhood for the errors distribution. More precisely, in a robust framework, one looks for procedures that remain valid when \( \varepsilon_i \sim F_0 \in F_\alpha = \{ G : G(y) = (1 - \alpha)G_0(y) + \alpha H(y) \} \), with \( H \) any symmetric distribution and \( G_0 \) a central model with possible first or second moments. No moment conditions are required to the errors so that outliers correspond to deviations on the errors distribution.

In this framework, some resistant procedures for additive models based on \( M \)-smoothers have been considered previously in the literature. Bianco and Boente (1998) considered robust estimators for additive models using kernel regression. Their approach, which is a robust version of that considered in Baek and Wehrly (1993), has the drawback of assuming that \( Y - g_j(X_j) \) is independent from \( X_j \), which is difficult to justify or verify in practice. Robust estimators based on backfitting and penalized splines \( M \)-estimators have been proposed for generalized additive models by Alimadad and Salibian–Barrera (2012) and Wong et al. (2014). In the particular case of the non-parametric regression model \( Y = g(X) + \sigma(X)\varepsilon \), with \( g(X) = \mu + \sum_{j=1}^d g_j(x_j) \), the procedures considered in Alimadad and Salibian–Barrera (2012) and Wong et al. (2014) assume that the scale function is known. For generalized additive models with nuisance parameters, Croux et al. (2011) provides a robust fit using penalized splines, while recently Boente et al. (2015) combines the backfitting algorithm with robust univariate scale equivariant smoothers to provide robust estimators under an additive model with unknown scale. However, up to our knowledge, except for the estimators considered in Bianco and Boente (1998), the asymptotic distribution of the estimators mentioned above has not been obtained.

On the other hand, Li et al. (2012) introduced robust estimators of the additive components \( g_j \) using local linear regression and marginal integration and derived their asymptotic behaviour. Besides assuming that the scale is known, the main disadvantage of the procedure defined in Li et al. (2012) is that the estimators solve the curse of dimensionality only when \( d \leq 4 \), since the local multivariate polynomial considered is of order one. This effect has also been described for the classical estimators, based on a local least squares approach, by Hengartner and Sperlich (2005) and Kong et al. (2010) who noted that to solve the curse of dimensionality the order of the local polynomial approximation should increase with the dimension of the covariates, leading to a higher numerical complexity. To avoid this problem, Severance–Lossin and Sperlich (1999) modified the initial estimators used in the integration procedure, using higher order kernels and local polynomials that depend only on the covariate \( X_j \) related to the \( j \)-th additive component to be estimated.

In this paper, we introduce robust estimators of the additive components using local polynomials on the component to be estimated and marginal integration. In this sense, our approach can be viewed as a robust version of the estimators defined in Severance–Lossin and Sperlich (1999).
Besides, our proposal allows to provide also robust estimators of the derivatives of the marginal components. Taking into account that in some studies, specially in many biological situations, missing responses may arise, we will provide a unified approach for complete data sets and for data sets in which responses are missing at random. The rest of the paper is organized as follows. Section 2 introduce the family of estimators to be considered. Consistency results and the asymptotic distribution are derived in Sections 3 and 4, respectively. Finally, the results of a numerical experiment conducted to evaluate the performance of the proposed procedure with respect to its classical counterpart defined in Severance–Lossin and Sperlich (1999) are reported in Section 5. Proofs relegated to the Appendix.

2 The estimators

We will consider robust inference with an incomplete data set \((X_i^T, Y_i, \delta_i)^T, 1 \leq i \leq n\), where \(\delta_i = 1\) if \(Y_i\) is observed and \(\delta_i = 0\) if \(Y_i\) is missing. Let \((X^T, Y, \delta)^T\) be a random vector with the same distribution as \((X_i^T, Y_i, \delta_i)^T\) and assume that \((X^T, Y)^T\) satisfies the additive model \(Y = \mu + \sum_{j=1}^{d} g_j(X_j) + \sigma(X) \varepsilon\), where the error \(\varepsilon\) is independent of \(X\) with symmetric distribution \(F_0(\cdot)\), that is, we assume that the error’s scale equals 1 to identify the scale function. Hence, when second moments exist, we have that \(E(Y|X) = g(X) = \mu + \sum_{j=1}^{d} g_j(X_j)\) and \(\sigma^2(X) = E((Y-g(X))^2|X)\) is the conditional variance function. Some additional conditions to be discussed below on the marginal components need to be require in order to guarantee identifiability.

Our aim is to estimate the non–parametric regression components \(g_j\) and its derivatives in a robust way with the data set at hand. An ignorable missing mechanism will be imposed by assuming that \(\delta\) and \(Y\) are conditionally independent given \(X\), i.e.,

\[
P(\delta = 1|Y, X) = P(\delta = 1|X) = p(X).
\] (1)

To define the conditions needed for identifiability, we begin by fixing some notation. We will partition \(X_i\) and \(x\) into a scalar and a \((d-1)\)–dimensional sub–vectors. To avoid burden notation, we denote \(X_i = (X_{i,\alpha}, X_i^{T})^T\) and \(x = (x_\alpha, x_\omega)^T\), respectively where \(x_\alpha\) and \(x_\omega\) are the directions of interest and not of interest, respectively. As in Linton and Nielsen (1995) and Nielsen and Linton (1998), let \(Q\) be a given probability measure with density \(q(x)\). Denote as \(q_{\alpha}(x)\, dx = dQ_{\alpha}(x)\) and \(q_{\omega}dx_\omega = dQ_{\omega}(x_\omega)\) where \(Q_{\alpha}\) stands for the \(\alpha\)-th marginal of the measure \(Q\) and \(Q_{\omega}\) corresponds to the marginal of \(x_\omega\). From now on, the additive components will be identified using the condition

\[
\int g_{\alpha}(x)q_{\alpha}(x)\, dx = 0 \quad \text{for all} \quad \alpha = 1, \ldots, d.
\] (2)

In particular, when \(q = f_X\) the density of \(X\), equation (2) corresponds to \(Eg_{\alpha}(X_\alpha) = 0\) for \(\alpha = 1, \ldots, d\). However, to define the estimators we assume that the marginal \(q_\omega\) is known and so the choice \(q = f_X\) is not a valid one. It is worth noting that the location parameter \(\mu\) equals \(\int g(x)dQ(x)\), so it can be estimated with a root–\(n\) rate of convergence using a preliminary regression estimator. For that reason, throughout this paper, we assume that \(\mu = 0\), i.e., \(\int g(x)dQ(x) = 0\). Hence, the model to be considered throughout this paper is

\[
Y = \sum_{j=1}^{d} g_j(X_j) + \sigma(X) \varepsilon,
\] (3)
where the error \( \varepsilon \) is independent of \( \mathbf{X} \) and has a symmetric distribution \( F_0 \).

The estimators to be defined are based on initial local polynomial \( M \)-estimators of order \( q \) for the regression function \( g \), where the polynomial to be considered is expanded only on the component of interest. More precisely, if we are interested in estimating \( g_\alpha(x) \) the \( \alpha \)-th additive component, the estimator to be considered treat differently the covariate \( X_\alpha \) which corresponds to the direction of interest and the other ones, calculating a robust local polynomial of order \( q \) only on the \( \alpha \)-th direction. As for the estimators introduced in Severance–Losin and Sperlich (1999), the use of higher order kernels will allow to obtain resistant estimators of the additive component which achieve the optimal univariate rate of convergence.

To reduce the effect of outliers on the regression estimates, we replace the square loss function in Severance–Losin and Sperlich (1999) by a function \( \rho \) with bounded derivative. Usually, the loss function depends on a tuning constant \( c \) allowing to achieve a given efficiency, so that it can be written as \( \rho(u) = \rho_c(u) = c^2 p_1(u/c) \). Typical choices for the loss function are the Huber–loss function defined as \( p_1(u) = p_1(u) = (u^2/2) I_{|u|\leq 1} + ([u] - 1/2) I_{|u|>1} \) otherwise. The Tukey’s loss defined as \( p_1(u) = p_1(u) = \min(3u^2 - 3u^4 + u^6,1) \) provides an example of bounded loss function. The bounded derivative of the loss function controls the effect of outlying values in the responses. As it is well known, to obtain robust scale invariant estimators, the residuals must be standardized with a simple and robust nonparametric scale estimator which can be taken, for instance, as the local MAD defined in Boente and Fraiman (1989). If the additive model has homoscedastic errors, i.e., if \( \sigma(x) \equiv \sigma \) for all \( x \), the estimator \( \tilde{s}(x) \) stands for a preliminary robust consistent scale estimator which can be taken, for instance, as the local MAD of the residuals obtained with a simple and robust nonparametric regression estimator, such as the local median.

Assume that we are interested in estimating \( g_\alpha \) which is assumed to be a continuously differentiable function up to order \( q \). Denote as \( g_\alpha^{(\nu)}(x_\alpha) \) the derivative of order \( \nu \) of the component \( g_\alpha \) and let \( \mathbf{\beta}^{(\alpha)}(x) = \mathbf{\beta}(x) = (g(x), g_\alpha(x_\alpha), \ldots, g_\nu^{(\alpha)}(x_\alpha)/q) \). An estimator of \( \mathbf{\beta}^{(\alpha)}(x) \) can be defined as the value \( \widehat{\mathbf{\beta}}^{(\alpha)}(x) = \mathbf{\beta}(x) = (\widehat{\beta}_0(x), \widehat{\beta}_1(x), \ldots, \widehat{\beta}_q(x)) \) such that

\[
\widehat{\mathbf{\beta}}^{(\alpha)}(x) = \mathbf{\beta}(x) = \arg\min_{(\beta_0,\beta_1,\ldots,\beta_q)} \sum_{i=1}^n \delta_i \mathcal{K}_{\mathbf{H}_d}(X_i - x) \rho \left( Y_i - \left[ \beta_0 + \sum_{j=1}^q \beta_j (X_{i\alpha} - x_\alpha)^j \right] \right) / \tilde{s}(x)
\]

with \( \mathcal{K}_{\mathbf{H}_d}(X_i - x) = (\det(\mathbf{H}_d))^{-1} \mathcal{K}(\mathbf{H}_d^{-1}(X_i - x)) \), \( \mathcal{K}(x) = \prod_{j=1}^d K_j(x_j) \) with \( K_j : \mathbb{R} \rightarrow \mathbb{R} \) univariate kernels and \( \mathbf{H}_d = \text{diag}(h_1,\ldots,h_q) \) is diagonal bandwidth matrix. When there is no confusion, we will avoid the superscript \( \alpha \) to avoid burden notation.

The preliminary estimator of the regression function \( g(x) \) denoted \( \bar{g}_{\alpha\beta}(x) \) is defined as \( \bar{g}_{\alpha\beta}(x) = \widetilde{\beta}_0(x) \), where the letter \( M \) indicates that we are using a local \( M \)-estimator and the subscripts “\( q, \alpha \)” indicate the order of the local polynomial used on the \( \alpha \)-th component of \( x \).

Finally, the robust estimator of the \( \alpha \)-th component is obtained through the marginal integration procedure as

\[
\bar{g}_{\alpha\alpha}(x_\alpha) = \int \bar{g}_{\alpha\alpha}(x_\alpha, u_\alpha) q_\alpha(u_\alpha) \, du_\alpha = \int e_1^T \widehat{\mathbf{\beta}}(x_\alpha, u_\alpha) q_\alpha(u_\alpha) \, du_\alpha
\]

where \( e_j \in \mathbb{R}^{q+1} \) is the vector with its \( j \)-th coordinate equals 1 and the other ones equal 0. Moreover,
an estimator of the derivative of order $\nu$, $1 \leq \nu \leq q$ of $g_\alpha$ is given by

$$
\hat{g}_{\alpha,M,q,\alpha}(x_\alpha) = \nu! \int \beta(x_\alpha, \mathbf{u}_\alpha) q_\alpha(\mathbf{u}_\alpha) \, d\mathbf{u}_\alpha = \nu! \int e_{\nu+1}^T \beta(x_\alpha, \mathbf{u}_\alpha) q_\alpha(\mathbf{u}_\alpha) \, d\mathbf{u}_\alpha.
$$

Finally, the robust estimator of the multivariate regression function $g$ is defined as

$$
\hat{g}_{M,q,\alpha}(x) = \sum_{\alpha=1}^d \hat{g}_{\alpha,M,q,\alpha}(x_\alpha).
$$

When $\mu \neq 0$, in the expressions of the marginal component estimators an estimator $\hat{\mu}$ of $\mu$ should be subtracted in order to obtain consistent estimators, that is, the estimator of $g_\alpha$ equals $\hat{g}_{\alpha,M,q,\alpha}(x_\alpha) = \int \hat{g}_{M,q,\alpha}(x_\alpha, \mathbf{u}_\alpha) q_\alpha(\mathbf{u}_\alpha) \, d\mathbf{u}_\alpha - \hat{\mu}$, while the estimator of the multivariate regression function $g$ is $\hat{g}_{M,q,\alpha}(x) = \hat{\mu} + \sum_{\alpha=1}^d \hat{g}_{\alpha,M,q,\alpha}(x_\alpha)$. A possible choice for $\hat{\mu}$ is to compute a robust location estimator $\hat{a}$ of the residuals $Y_i - \sum_{\alpha=1}^d \int \hat{g}_{M,q,\alpha}(X_i, \mathbf{u}_\alpha) q_\alpha(\mathbf{u}_\alpha) \, d\mathbf{u}_\alpha$ and to define $\hat{\mu} = -\hat{a}/(d-1)$. The practitioner may also choose as location estimator $\hat{\mu} = (1/d) \sum_{j=1}^d \hat{\mu}_j$ where $\hat{\mu}_j = \int \hat{g}_{M,q,\alpha}(j) \, dQ(u)$. However, this estimator does not necessarily have a root--n order of convergence, a fact which has already been mentioned by Sperlich et al. (1999) for the classical estimators.

It is worth noting that when $\rho$ is continuously differentiable with derivative $\rho' = \psi$, $\hat{\beta}^{(a)}(x)$ satisfies the following system of equations

$$
\Psi_{n,a}(\hat{\beta}^{(a)}(x), x, \hat{s}(x)) = 0_{d+1},
$$

where $\Psi_{n,a}(\beta, x, \sigma) = (\Psi_{n,a,0}(\beta, x, \sigma), \ldots, \Psi_{n,a,q}(\beta, x, \sigma))$ and $\Psi_{n,a,\ell}(\beta, x, \sigma)$ is defined for $\ell = 0, 1, \cdots, d$ as

$$
\Psi_{n,a,\ell}(\beta, x, \sigma) = \sum_{i=1}^n \delta_i K_{Hd}(X_i - x) \sigma \left( \frac{Y_i - \beta_0 - \sum_{j=1}^q \beta_j (X_{i\alpha} - x_\alpha)^j}{\sigma} \right) (X_{i\alpha} - x_\alpha)^\ell.
$$

### 3 Consistency

In this section, we will show that the estimators defined in Section 2 are strongly consistent. Recall that the preliminary local $M$-estimator based on local polynomials of order $q$, $\hat{g}_{M,q,\alpha}(x)$, is adapted to the additive component $\alpha$ we want to estimate. Hence, we fix $\alpha = 1, \ldots, d$ and to derive strong consistency results for the estimator of the additive component $g_\alpha$, we state the conditions adapted to the choice of $\alpha$. The kernels to be used are also adapted to this framework. However, in order to allow more flexibility, we will not restrict the bandwidth choice to $h_{\alpha,n} = h_n$ and $h_{j,n} = h_n$ for $j \neq \alpha$ allowing different bandwidths for each component.

In what follows, $C$ stands for any compact set and for any function $m : \mathbb{R}^d \to \mathbb{R}$ we denote as $i(m) = \inf_{x \in C} m(x)$. Let $1 \leq \alpha \leq d$ be fixed and denote as $s_{i,j}^{(\alpha)} = \int u^{i+j} K_\alpha(u) \, du = \int u^{i+j} K_\alpha(u) \, du$, $0 \leq i, j \leq q$ with $u = (u_1, \ldots, u_d)^T$. The following set of assumptions will be needed.

**A0** The product measure $Q$ has compact support $S_Q$ contained in the support $S_f$ of $f_X$. 


A1 \((X_i^T, Y_i, \delta_i)^T, 1 \leq i \leq n\) are i.i.d. vectors satisfying (1). Moreover, \((X_i^T, Y_i)^T\) fulfill the additive model (3) where the functions \(g_\alpha\) verify (2).

A2 The density function, \(f_X(x)\), of \(X\) and the missingness probability \(p(x)\) are bounded over the compact \(C \subset S\) and such that \(i(p) > 0, i(f_X) > 0\). Moreover, \(p\) and \(f_X\) are continuous in a neighbourhood of \(C\).

A3 \(\sigma(x)\) and \(g(x)\) are continuous functions of \(x\) in a neighbourhood of \(C\) and \(i(\sigma) > 0\).

A4 a) For all \(j = 1, \ldots, d\), the marginal component \(g_j\) is continuously differentiable in a neighbourhood of the support, \(S_j\), of the density of \(X_j\) with derivative \(g_j' = g_j^{(1)}\) bounded.

b) \(g_\alpha\) is \((q + 1)\)-times continuously differentiable.

A5 a) The kernel function \(K : \mathbb{R}^d \to \mathbb{R}\) is such that \(K(x) = \prod_{j=1}^d K_j(x_j)\), where \(K_j : \mathbb{R} \to \mathbb{R}\) have bounded support, say \([-1, 1]\) and \(\int K_j(u) du = 1\). Besides, \(K_j : [-1, 1] \to \mathbb{R}\) are even, bounded functions and Lipschitz continuous of order one.

b) The matrix \(S^{(\alpha)}_{ij} = (S_{jk}^{(\alpha)})_{1 \leq i,j \leq q+1}\) is positive definite, where \(S_{ij}^{(\alpha)} = s_{i-1,j-1}^{(\alpha)}\) for \(1 \leq j, k, \leq q + 1\).

A6 The bandwidth sequences are such that \(h_{j,n} \to 0\) and \(n \prod_{j=1}^d h_{j,n}/\log n \to \infty\).

A7 The function \(\rho\) is an even and three times continuously differentiable function with bounded derivatives \(\psi = \rho', \psi''\) and \(\psi''\). Furthermore, \(\mathbb{E}(\psi'(\varepsilon)) > 0\) and \(\zeta(u) = u\psi''(u)\) and \(\zeta_2(u) = u\psi''(u)\) are bounded.

A8 The scale estimator \(\hat{s}(\cdot)\) satisfies that \(\sup_{x \in C} |\hat{s}(x) - \sigma(x)| \xrightarrow{a.s.} 0\).

Remark 3.1. Assumptions A3 to A6 are standard conditions to derive consistency results in nonparametric regression models. Assumption A1 establishes that the model is an additive one where the components are identifiable. On the other hand, A0 is a standard condition when using marginal integration procedures. It is worth noting that A2 implies that some response variables are observed for all \(x \in C\), which is a common assumption in the literature of missing data. Note that A5 implies that \(s_{0,0}^{(\alpha)} = 1\) and \(s_{i,j}^{(\alpha)} = 0\) if \(i + j\) is odd. Assumptions A1 and A7 imply that \(\mathbb{E}(\psi(\varepsilon)/\sigma) = 0\) for any \(\sigma > 0\). Assumption A7 is a standard condition on the score function when local polynomials and scale estimators are considered. Finally, A8 requires uniform consistency of the preliminary scale estimator which is needed to derive uniform consistency of the initial regression function. Note that A4 entails that the derivative of \(g_\alpha\) of order \(q + 1, g_\alpha^{(q+1)}\), is bounded in \(S_\alpha\).

Remark 3.2. It is easy to see that A3 and A8 imply that the robust scale estimator has upper and lower uniform bounds almost surely. More precisely, if \(A = \inf_{x \in C} \sigma(x)/2\) and \(B = (3/2)\sup_{x \in C} \sigma(x)\) we have that

\[
\mathbb{P}(\exists n_0 \text{ such that for all } n \geq n_0 \text{ and for all } x \in C \ A < \hat{s}(x) < B) = 1. \tag{7}
\]

On the other hand, if we denote as \(\hat{a}_\sigma(x) = \sigma(x)/\hat{s}(x)\), A3 and A8 imply

\[
\sup_{x \in C} |\hat{a}_\sigma(x) - 1| \xrightarrow{a.s.} 0. \tag{8}
\]
From now on, we denote as $H^{(\alpha)}$ the diagonal matrix given by $H^{(\alpha)} = \text{diag}(1, h_\alpha, h_\alpha^2, \ldots, h_\alpha^d)$.

**Proposition 3.1.** Let $C \subset S_f$ be a compact set such that **A2** is satisfied. Assume that **A1** to **A8** hold. Then, there exists a solution $\beta(x)$ of (6) such that $\sup_{x \in C} \|H^{(\alpha)}(\hat{\beta}(x) - \beta(x))\| \xrightarrow{a.s.} 0$ where $\beta(x) = (g(x), g_\alpha^{(1)}(x_\alpha), \ldots, g_\alpha^{(q)}(x_\alpha))^T$ and $g_\alpha^{(1)} = g_\alpha'$.

Theorem 3.1 shows the consistency of the marginal integration estimator of the regression function and its derivatives when using local polynomials of order $q$ in the direction $\alpha$. We omit the proof of Theorem 3.1 since it follows straightforwardly from Proposition 3.1 using similar arguments to those considered in the proof of Theorem 3.2.3 in Boente and Martínez (2015).

**Theorem 3.1.** Assume that **A0** to **A8** hold with $C = S_Q \subset S_f$ and some fixed $\alpha$. Denote as $C_\alpha$ the support of $q_\alpha$. Then, we have that

a) $\sup_{x \in C_\alpha} |\hat{g}_{\alpha,M_q}(x) - g_\alpha(x)| \xrightarrow{a.s.} 0$,

b) $\sup_{x \in C} h_\alpha^{(\nu)} |\hat{g}_{\alpha,M_q}^{(\nu)}(x) - g_\alpha^{(\nu)}(x)| \xrightarrow{a.s.} 0$.

Furthermore, if for any $\alpha = 1, \ldots, d$, **A0** to **A8** hold for the kernels used to define the $\alpha-$th additive component estimators and $C = S_Q$, then $\sup_{x \in C} |\hat{g}_{M_q}(x) - g(x)| \xrightarrow{a.s.} 0$, where $\hat{g}_{M_q}(x) = \sum_{j=1}^d \hat{g}_{j,M_q}(x_j)$.

### 4 Asymptotic distribution

In this section, we derive the asymptotic distribution of the $\alpha-$th additive component estimator. As in Severance–Lossin and Sperlich (1999), we will assume that to compute the preliminary estimator $\hat{g}_{M_q}(x)$ the diagonal bandwidth matrix $H_d$ is such that its $\alpha-$th diagonal element equals $h_\alpha$ and the remaining ones are $h$, i.e., we assume that $h_j = h$, for $j \neq \alpha$. Moreover, we will consider two different univariate even and bounded kernels, $K$ and $L$. The kernel $K$ is positive and used over the $\alpha-$th coordinate of $x$, i.e., $K_\alpha = K$. On the other hand, the kernel $L$ is used on the remaining components of $x$, that is, $K_j = L$, for $j \neq \alpha$. Furthermore, to obtain a univariate rate of convergence for the $\alpha-$th additive component estimator $L$ will be chosen as a kernel of order $\ell \geq 2$, that is, $\int L(u) du = 1$, $\int u^s L(u) du = 0$, for $s = 1, \ldots, \ell - 1$ and $\int u^\ell L(u) du \neq 0$. Clearly, the choice of kernel and bandwidth as well as the computation of the preliminary estimator need to be done for each additive component to be estimated, making the method computationally expensive. Thus, to gain in convergence rate some numerical complexity seems to be necessary.

Throughout this section, we will assume an homoscedastic model, that is, $\sigma(x) \equiv \sigma$ so that the additive model can be written as $Y = \sum_{j=1}^d g_j(X_j) + \sigma \varepsilon$ where the error $\varepsilon$ is independent of $X$ and has a symmetric distribution $F_0$ with scale 1, so as to identify $\sigma$. We will also assume that a robust root–$n$ convergent scale estimator $\tilde{s}$ of $\sigma$ is available.

Due to the kernels choice and the homoscedasticity assumption, assumptions **A1**, **A4** and **A5** will be replaced by the following ones.
N1 \((X_i^T, Y_i, \delta_i)^T, 1 \leq i \leq n\) are i.i.d. vectors satisfying (1). Moreover, \((X_i^T, Y_i)^T\) are such that \(Y_i = \sum_{j=1}^d g_j(X_{i,j}) + \sigma \varepsilon\) where the errors \(\varepsilon_i\) are independent of \(X_i\) with symmetric distribution \(F_0\) and the functions \(g_\alpha\) verify (2).

N2 For all \(j = 1, \ldots, d\) and \(j \neq \alpha\), the marginal component \(g_j\) is \(\ell\) times continuously differentiable in a neighbourhood of the support \(S_j\) of the density \(X_j\) and \(g_j(\ell)\) is bounded. Besides, \(g_\alpha\) is continuously differentiable until order \(q+1\) and the derivative \(q+1, g_\alpha^{(q+1)}\), is bounded in \(S_\alpha\).

N3 a) The kernel function \(K : \mathbb{R}^d \rightarrow \mathbb{R}\) is such that \(K_j = L\) for \(j \neq \alpha\). Moreover, \(K\) and \(L\) are bounded, even, compactly supported and Lipschitz continuous with \(\int K(u) du = \int L(u) du = 1\). Without loss of generality, we assume that the support of \(K\) and \(L\) is \([-1, 1]\).

b) The kernel \(K_\alpha\) is such that the matrix \(S^{(\alpha)} = (\int u^{i+j}K_\alpha(u) du)_{0 \leq i,j \leq q}\) defined in A5b) is non-singular.

c) The kernel \(L\) is a kernel of order \(\ell \geq 2\), that is, \(\int L(u) du = 1\), \(\int u L(u) du = 0\) if \(1 \leq j \leq \ell - 1\) and \(\int u^\ell L(u) du \neq 0\).

N4 The bandwidth sequences \(h_j = h_{j,n} > 0\) are such that \(h_{j,n} \rightarrow 0\) for \(j \neq \alpha\), \(h_\alpha = \beta n^{-\frac{1}{2(q+1)}}\). Moreover, \(\overline{h} = \bar{h}_n\) is such that \(\overline{h} = o\left(\left(n^{-\frac{q+1}{2(q+3)}}\right) + n^{\frac{q+1}{2q+3}}h^{d-1}/\log n \rightarrow \infty\right)\).

N5 The function \(q_d(u)\) is continuous and the functions \(f_X(u)\) and \(p(u)\) are continuously differentiable up to order \(\ell\). Furthermore, \(\sup_{x \in S_q} f_X(x) < \infty\), \(\inf_{x \in S} f_X(x) > 0\) and \(\inf_{x \in S} p(x) > 0\), where \(C \subset S_f\) stands for some compact neighbourhood of \(S_q\).

Assumptions N2 to N4 correspond to assumptions A3, A1 and A2 in Severance–Lossin and Sperlich (1999), respectively. Note that the order \(\ell\) of the kernel \(L\) is an even number, since \(L\) is an even function. Also, notice that N2 implies that \(\sup_{x \in S_q} |g(x)| < \infty\). The proof of the asymptotic distribution of the preliminary estimators \(\hat{g}_{m_\alpha}(x)\) can be found in Martínez (2014).

Denote as \(\lambda(\alpha) = \mathbb{E}\psi(\varepsilon + a)\) and \(\lambda_1(\alpha) = \mathbb{E}\psi'(\varepsilon + a)\). Given a symmetric matrix \(A \in \mathbb{R}^{m \times m}\), \(\nu_1(A) \leq \cdots \leq \nu_m(A)\) stand for the eigenvalues of \(A\).

**Theorem 4.1.** Assume that A0, A2, A7 and N1 to N5 hold and that the function \(\lambda(\alpha)\) has bounded Lipschitz continuous derivatives up to order \(\ell - 1\), in a neighbourhood of 0. Let \(\bar{s}\) be a consistent estimator of \(\sigma\) such that \(n(\bar{s} - \sigma) = O_P(1)\). Let \(x\) be an interior point of \(S_f\) and \(\hat{\beta}(x)\) be a solution of (6) with \(\bar{s}(x) = \bar{s}\), for all \(x\), such that \(\sup_{x \in S_q} \|H^{(\alpha)}(\hat{\beta}(x) - \beta(x))\| \overset{P}{\rightarrow} 0\), where \(\beta(x) = (g(x), g_1(x_\alpha), \ldots, g_q(x_\alpha)/q!)^T\) and \(g_1(x) = g'_\alpha\). Then, we have that

\[
\sqrt{n}h_\alpha [g_{m_\alpha}(x_\alpha) - g_\alpha(x_\alpha)] \overset{D}{\rightarrow} N\left(b_{q,\alpha}(x_\alpha), \sigma_{q,\alpha}^2(x_\alpha)\right)
\]

where

\[
b_{q,\alpha}(x_\alpha) = \beta^{\frac{q+1}{2}}(q+1)! g^{(q+1)}_\alpha(x_\alpha) e_1^T \left(S^{(\alpha)} \right)^{-1} s^{(\alpha)}_q,
\]

\[
\sigma_{q,\alpha}^2(x_\alpha) = \sigma^2 \left[\frac{\mathbb{E}\psi^2(\varepsilon)}{\mathbb{V}\psi'(\varepsilon)}\right]^2 \left(\int \frac{g^2_{\alpha}(x_\alpha) f_X(x_\alpha, x_\alpha) p(x_\alpha, x_\alpha) dx_\alpha}{\mathbb{E}^2 f_X(x_\alpha, x_\alpha) p(x_\alpha, x_\alpha)}\right) e_1^T \left(S^{(\alpha)} \right)^{-1} \mathbf{V}_\alpha \left(S^{(\alpha)} \right)^{-1} e_1
\]
with \( s_q^{(\alpha)} = (s_{q,1}^{(\alpha)}, \ldots, s_{q,q+1}^{(\alpha)})^T \) where \( s_q^{(\alpha)} = \int K_\alpha(t) t^{q+j} \, dt \) for \( j = 1, \ldots, q+1 \) and \( V_\alpha = (v_{sm}^{(\alpha)})_{1 \leq s,m \leq q+1} \) with \( v_{sm} = \int u^{s+m-2} K_\alpha^2(u) \, du \).

**Remark 4.1.** It is worth noting that as in other nonparametric settings, the asymptotic bias does not depend on the score function. Moreover, the score function appears in the asymptotic variance through the quantity

\[
V(\psi) = \frac{E\psi^2(\varepsilon)}{|E\psi'(\varepsilon)|^2}
\]

which is similar to that given in the location setting. Hence, to calibrate the estimators to attain a given efficiency it is enough to choose the same tuning constant as in a location model.

Assume that the smoothing parameter \( \tilde{h} \) in the directions not of interest is such that \( \tilde{h} = \gamma n^{-\tau} \). Then, \( \tilde{h} = o\left(n^{-\frac{q+1}{2(2q+3)}}\right) \) if and only if \( \tau > (q+1)/(2(2q+3)) \). On the other hand, \( n^{q+1}/\tilde{h}^{d-1}/\log n \to \infty \) when \( \tau < (q+1)/(2(2q+3)(d-1)) \). Hence, bandwidth rate of \( \tilde{h} \) must satisfy

\[
\frac{q+1}{\ell(2q+3)} < \tau < \frac{q+1}{(2q+3)(d-1)},
\]

which implies that the practitioner must choose a kernel \( L \) with order at least the dimension of the covariates, i.e., \( \ell \geq d \).

**Theorem 4.2.** Assume that A0, A2, A7 and N1 to N5 hold and that the function \( \lambda(a) \) has bounded Lipschitz continuous derivatives up to order \( \ell - 1 \), in a neighbourhood of 0. Let \( \tilde{s} \) be a consistent estimator of \( \sigma \) such that \( \sqrt{n}(\tilde{s} - \sigma) = O_p(1) \). Let \( x \) be an interior point of \( S_f \) and \( \tilde{\beta}(x) \) be a solution of (6) such that \( \sup_{x \in S_f} \| H^{(\alpha)}[\tilde{\beta}(x) - \beta(x)] \| \to 0 \) where \( \beta(x) = (g(x), g_1^{(1)}(x), \ldots, g_{q+1}^{(1)}(x)/q!)^T \) and \( g_1^{(1)} = g_1' \). Then, we have that for \( \nu = 1, \ldots, q \)

\[
\sqrt{n} h_\alpha \tilde{g}_\nu \left( g_{\nu,q+1}(x) - g_\nu(x) \right) \xrightarrow{D} N \left( b_{q,\alpha}^{(\nu)}(x), \sigma_{\nu,q,\alpha}^2(x) \right)
\]

where

\[
b_{q,\alpha}^{(\nu)}(x) = \nu! \beta_{2q+3}^{q+1} \left( g_\nu(x) e_{\nu+1} \left( S^{(\alpha)} \right)^{-1} s_q^{(\alpha)} \right),
\]

\[
\sigma_{\nu,q,\alpha}^2(x) = (\nu!)^2 \sigma^2 \frac{E\psi^2(\varepsilon)}{|E\psi'(\varepsilon)|^2} \left( \int \frac{g_\nu^2(x)}{f(x)p(x)} \, dx \right) e_{\nu+1} \left( S^{(\alpha)} \right)^{-1} V_\alpha \left( S^{(\alpha)} \right)^{-1} e_{\nu+1},
\]

with \( s_q^{(\alpha)} \) and \( V_\alpha \) given in Theorem 4.1.

It is worth noting that \( s_q^{(\alpha)} \) is such that its \( j \)-th component, \( 1 \leq j \leq q+1 \), equals 0 when \( q+j \) is odd since \( K_\alpha \) is an even function. Hence if \( q + \nu + 1 \) is odd, or equivalently, when \( q - \nu \) is even, the bias will be 0. Hence, the bias term in the estimation of \( g_\nu^{(\nu)} \) appears only when \( q - \nu \) is odd.

**5 Monte Carlo Study**

This section contains the results of a simulation study conducted with the aim of comparing the performance of estimator defined in Section 2 with that of its classical counterpart introduced
in Severance–Lossin and Sperlich (1999), which corresponds to the choice $\rho(u) = u^2$. We have performed $N = 500$ replications taking samples of size $n = 500$ when the dimension of the covariates is $d = 2$ and $d = 4$. We considered samples without outliers and also samples contaminated in different ways. For $d = 2$, we also included in our experiment cases where the response variable may be missing. All computations were carried out using an R implementation of our algorithm, which can be provided upon request.

To generate missing responses, we first generated observations $(X_i^T, Y_i)^T$ satisfying the additive model $Y = g_0(X) + u = \mu_0 + \sum_{j=1}^d g_{0,j}(X_j) + u$, where $u = \sigma_0 \varepsilon$. Then, we generate $\{\delta_i\}_{i=1}^n$ independent Bernoulli random variables such that $\mathbb{P}(\delta_i = 1|Y_i, X_i) = \mathbb{P}(\delta_i = 1|X_i) = p(X_i)$. When $d = 2$ we used two different missing probabilities: $p(x) \equiv 1$, which corresponds to the case where all the responses are observed, and $p(x) = p_2(x) = 0.4 + 0.5(\cos(x_1 + 0.2))^2$, which yields around 31.5% of missing responses. For $d = 4$, we only report the results for $p(x) \equiv 1$.

In all cases, we considered polynomials of order $q = 1$. The smoothers were computed using the Epanechnikov kernel $K_1(u) = K_2(u) = 0.75(1 - u^2)I_{[-1,1]}(u)$ when $d = 2$, while for $d = 4$ we choose $K_\alpha$ as the Epanechnikov kernel and $L$ the fourth order kernel $L(u) = (15/32)(1 - u^2)(3 - 7u^2)I_{[-1,1]}(u)$ when estimating $g_\alpha$. We compared the classical marginal integration estimator denoted $\hat{g}_C$ with the robust marginal integration estimator, denoted $\hat{g}_R$, using the Huber’s loss function with tuning constant $c = 1.345$. To identify the marginal component estimators, we added a subscript indicating the additive component label.

The performance of each estimator $\hat{g}_j$ of $g_j$, $1 \leq j \leq d$, was measured through the following approximated integrated squared error (ISE):

$$\text{ISE}(\hat{g}_j) = \frac{1}{\sum_{i=1}^n \delta_i} \sum_{i=1}^n \left( g_j(X_{ij}) - \hat{g}_j(X_{ij}) \right)^2 \delta_i,$$

where $X_{ij}$ is the $j$th component of $X_i$ and $\delta_i = 0$ if the $i$-th response was missing and $\delta_i = 1$ otherwise. A similar measure was used to compare the estimators of the regression function $g = \mu + \sum_{j=1}^d g_j$.

### 5.1 Monte Carlo study with $d = 2$ additive components

In this case, the covariates were generated from a uniform distribution on the unit square, $X_i = (X_{i,1}, X_{i,2})^T \sim U([0, 1]^2)$, the error scale was $\sigma_0 = 0.5$ and the overall location $\mu = 0$. We choose as measure in the integration procedure $Q = U([0, 1]^2)$ and the integral in (5) was approximated as the mean over 500 points generated according to $Q$.

The additive components were chosen to be $g_1(x_1) = 24(x_1 - 0.5)^2 - 2$, $g_2(x_2) = 2\pi \sin(\pi x_2) - 4$.

We have fixed both bandwidths $h_1$ and $h_2$ in 0.1. These are values close to the optimal ones with respect to the integrated mean square error for the bandwidth $h_\alpha = \beta n^{-1/5}$ given in N3 (see Severance–Lossin and Sperlich, 1999).

For the errors, we considered the following settings:

- $C_0$: $u_i \sim N(0, \sigma_0^2)$. 

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• $C_1$: $u_i \sim (1 - 0.15) N(0, \sigma_0^2) + 0.15 N(15, 0.01)$.

• $C_2$: $u_i \sim N(10, 0.01)$ for all $i$’s such that $X_i \in D_{0.99}$, where $D_\eta$ is as above.

• $C_3$: $u_i \sim (1 - 0.30) N(0, \sigma_0^2) + 0.30 N(15, 0.01)$ for all $i$’s such that $X_i \in D_{0.3}$.

Case $C_0$ corresponds to samples without outliers and they will illustrate the loss of efficiency incurred by using a robust estimator when it may not be needed. The contamination setting $C_1$ corresponds to a *gross-error model* where all observations have the same chance of being contaminated. On the other hand, case $C_2$ is pathological in the sense that all observations with covariates in the square $[0.2, 0.29] \times [0.2, 0.29]$ are severely affected. Note that we choose an area where the interval length is smaller than the bandwidth, otherwise, the initial estimator will be severely affected. Finally, case $C_3$ is a gross-error model with a higher probability of observing an outlier, but these are restricted to the square $[0.2, 0.05] \times [0.2, 0.5]$.

To summarize the values of $\text{ise}(\widehat{g}_j)$ and $\text{ise}(\widehat{g})$ over replications, we report an approximation of the mean integrated squared error, denoted MISE, which is obtained by averaging $\text{ise}$ over all replications, and a more robust measure, denoted MEDISE, that corresponds to the median over replications of the $\text{ise}$. The obtained results are given in Table 1, for the different errors distributions as well as for data sets with and without missing responses.

| $p(x) \equiv 1$ | $p_2(x) = 0.4 + 0.5 \cos^2(x_1 + 0.2)$ |
|-----------------|-----------------|
|                | $\widehat{g}_C$ | $\widehat{g}_{i.C}$ | $\widehat{g}_{2.C}$ | $\widehat{g}_R$ | $\widehat{g}_{1.R}$ | $\widehat{g}_{2.R}$ | $\widehat{g}_C$ | $\widehat{g}_{i.C}$ | $\widehat{g}_{2.C}$ | $\widehat{g}_R$ | $\widehat{g}_{1.R}$ | $\widehat{g}_{2.R}$ |
| **MISE**        |                |                |                |                |                |                |                |                |                |                |                |                |
| $C_0$           | 0.0188         | 0.0276         | 0.0174         | 0.0172         | 0.0200         | 0.0183         | 0.2506         | 0.1278         | 0.1451         | 0.2540         | 0.1270         | 0.1500         |
| $C_1$           | 6.4543         | 0.7739         | 0.5208         | 0.8348         | 0.4706         | 0.2517         | 16.3902        | 6.2902         | 4.8353         | 8.5197         | 4.5263         | 3.3571         |
| $C_2$           | 0.1005         | 0.0590         | 0.0532         | 0.0557         | 0.0374         | 0.0363         | 0.3353         | 0.1661         | 0.1823         | 0.2987         | 0.1470         | 0.1708         |
| $C_3$           | 0.8652         | 0.3662         | 0.3472         | 0.1557         | 0.0811         | 0.0792         | 1.1434         | 0.4921         | 0.4928         | 0.4734         | 0.2252         | 0.2443         |
| **MEDISE**      |                |                |                |                |                |                |                |                |                |                |                |                |
| $C_0$           | 0.0183         | 0.0113         | 0.0111         | 0.0108         | 0.0115         | 0.0115         | 0.0286         | 0.0220         | 0.0220         | 0.0307         | 0.0232         | 0.0234         |
| $C_1$           | 6.0024         | 0.4251         | 0.4179         | 0.5153         | 0.1474         | 0.1335         | 6.9604         | 0.8206         | 0.7700         | 1.8799         | 0.5898         | 0.5317         |
| $C_2$           | 0.0850         | 0.0477         | 0.0481         | 0.0300         | 0.0234         | 0.0264         | 0.1174         | 0.0641         | 0.0640         | 0.0708         | 0.0415         | 0.0454         |
| $C_3$           | 0.8030         | 0.3456         | 0.3249         | 0.0483         | 0.0338         | 0.0363         | 0.8886         | 0.3728         | 0.3476         | 0.1520         | 0.0798         | 0.0702         |

Table 1: $\text{MISE}$ and $\text{MEDISE}$ of the estimators of the regression functions $g$, $g_1$ and $g_2$ under different contaminations, for the complete data and for sets with missing responses.

As expected, when the data do not contain outliers or missing responses, the robust estimators shows larger MEDISE values than the classical estimators based on the square loss function. In a few cases, the MISE values of the robust estimators are slightly smaller than those of the classical ones. However, all these differences are well within the Monte Carlo margin of error. For contaminated errors, the behaviour of the classical and robust estimators are quite different. The contamination setting $C_1$ is the worst for the estimators defined in Severance–Lossin and Sperlich (1999), since a 15% of the observations are contaminated with a large residual. Effectively, under $C_1$, the MISE of the classical estimator of the regression function $g$ is more than 6 times larger than those of its robust counterpart, while the MEDISE is 10 times larger. This difference is smaller when estimating the additive components, but is still important. On the other hand, $C_2$ seems to affect less the classical estimator. Indeed, under $C_2$ the $\text{MISE}$ and $\text{MEDISE}$ of $\widehat{g}_C$ are twice those of $\widehat{g}_R$, while for each additive component, the $\text{MISE}$ and $\text{MEDISE}$ of the classical estimators are a 50% larger than those of the robust ones. Finally, contamination $C_3$ seems to be more harmful than $C_2$. Effectively,
the reported MEDISE values for the classical regression estimator $\hat{g}_c$ are more than 15 times larger than those of the robust estimator $\hat{g}_r$, while when estimating each additive component the classical estimators MEDISE is 10 times larger than those obtained with its robust counterpart. It is worth noting that the ratio between the classical and robust estimators MISE is smaller than when using the MEDISE, although large values are still obtained. This fact may be explained by the presence of a few samples where the estimators, specially the robust estimator, perform differently from the majority of the samples.

When missing responses arise, as one would expect, all estimators have larger MISE and MEDISE values than when $p \equiv 1$ due to the loss of about 31.5% of responses. Beyond this fact, similar conclusions can be drawn regarding the advantage of the robust procedure over the classical estimators.

### 5.2 Monte Carlo study with $d = 4$ additive components

For this model we generated covariates $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3}, X_{i4}) \sim U([-3, 3]^4)$, independent errors $\varepsilon_i \sim N(0, 1)$ and $\sigma_0 = 0.15$. Similarly to what we have set for $d = 2$, we chose as measure in the integration procedure $Q = U([-3, 3]^4)$ and, as in Section 5.1, the integral in (5) was also approximated as the mean over 500 points generated according to $Q$.

The additive components chosen are related to those in the numerical study in Severance–Lossin and Sperlich (1999) and correspond to

\[
g_{0,1}(x_1) = \frac{1}{12}x_1^3, \quad g_{0,2}(x_2) = \sin(-x_2),
\]

\[
g_{0,3}(x_3) = \frac{1}{2}x_3^2 - 1.5, \quad g_{0,4}(x_4) = \frac{1}{4}e^{x_4} - \frac{1}{24}(e^3 - e^{-3}).
\]

In this numerical experiment, the bandwidths were selected using a $K$–fold cross–validation procedure as follows. As usual, we first randomly partition the data set into $K$ disjoint subsets of approximately equal sizes $G_k, 1 \leq k \leq K$, so that $\bigcup_{k=1}^{K} G_k = \{1, \ldots, n\}$. For each fixed $(h, \tilde{h})$, let $h = (h, \tilde{h})$. Note that when estimating the $\alpha$–th additive component, the bandwidth used for the $\alpha$–th component is $h$, while on the nuisance directions we use $\tilde{h}$. Moreover, the kernels are also modified depending on the component to be estimated. More precisely, when estimating $g_\alpha$, for $\ell \neq \alpha$, $K_\ell = L$ the fourth order kernel described above, while $K_\alpha$ is the Epanechnikov kernel.

Denote as $\hat{g}_{c,h}^{(-k)}(\mathbf{x})$ and $\hat{g}_{r,h}^{(-k)}(\mathbf{x})$ the classical and robust marginal integration estimators computed with the bandwidths $h$ and $\tilde{h}$, without using the observations with indices in $G_k$. The classical $K$–fold cross–validation criterion given by

\[
L_{ls}(h, \tilde{h}) = \frac{1}{n} \sum_{k=1}^{K} \sum_{i \in G_k} (Y_i - \hat{g}_{c,h}^{(-k)}(\mathbf{X}_i))^2,
\]

is minimized over a set $\mathcal{H} \times \tilde{\mathcal{H}}$ of possible bandwidths $(h, \tilde{h})$.

On the other hand, as is well known, a robust cross–validation criterion needs to be considered when using robust estimators. The robust $K$–fold cross-validation method used in this numerical study is related to the procedure defined in Boente et al. (2010) and minimizes over $\mathcal{H} \times \tilde{\mathcal{H}}$ the
robust criterion

\[ L_R(h, \tilde{h}) = \sum_{k=1}^{K} \left\{ \left( \text{MED}_{i \in \tilde{U}_k} \{ Y_i - \tilde{g}^{-k}_{R,h}(X_i) \} \right)^2 + \left( \text{MAD}_{i \in \tilde{U}_k} \{ Y_i - \tilde{g}^{-k}_{R,h}(X_i) \} \right)^2 \right\}. \]

The number of folds \( K \) was set equal to \( K = 5 \). Due to the computational complexity involved, we only considered the contamination schemes \( C_0 \) and \( C_1 \) defined in Section 5.1.

To obtain bandwidths satisfying (9) with \( q = 1 \), the set \( \mathcal{H} \times \tilde{\mathcal{H}} \) of possible values for \((h, \tilde{h})\) was chosen satisfying \( h = C n^{-1/5} \) and \( \tilde{h} = C n^{-7} \) with \( \tau = 0.12 \). The constant \( C \) took initially five possible values leading to \( \tilde{\mathcal{H}} = \{1, 1.5, 2, 2.5, 3\} \). When the minimum, was attained at \( \tilde{h} = 3 \), the grid was enlarged to include values of \( \tilde{h} \in \{3.5, 4, 4.5, 5, 5.5\} \). Note that when \( \tilde{h} = 1, C \approx 2.11 \) and \( h = C n^{-1/5} \), so we expect in average 3 observations in each 4-dimensional neighbourhood. For that reason, to obtain a reliable estimate of the residual scale \( \sigma_0 \), independently of the choice of \((h, \tilde{h})\), a preliminary regression estimator was computed using as bandwidth \( h_0 = (0.93, 0.93, 0.93, 0.93) \). With these bandwidths, we expect an average of 5 points in each 4-dimensional neighbourhood. It is also worth noting that the optimal bandwidth \( h \) to estimate \( g_0 \) in this model lead to very small values and were not taken as possible values of the grid.

In this numerical study, the \( \text{ISE} \) of few samples was very different to most of the data sets, probably due to the fact that the bandwidth search was not exhaustive. Hence, to provide summary values for \( \text{ISE}(\tilde{g}_j) \) and \( \text{ISE}(\tilde{g}) \) over replications, we report the median over replications as well as the trimmed mean over replications of the \( \text{ISE} \) with 1% and 5% trimming. Note that the \( \text{MEDISE} \) corresponds to a 50% trimming. The results obtained under \( C_0 \) and \( C_1 \) are given in Table 2.

| \( \nu \)  | \( g_C \)  | \( g_{1.C} \) | \( g_{2.C} \) | \( g_{3.C} \) | \( g_{4.C} \) | \( g_{R} \)  | \( g_{1.R} \) | \( g_{2.R} \) | \( g_{3.R} \) | \( g_{4.R} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1%  | \( C_0 \) | 1.0969 | 0.2076 | 0.3019 | 0.1358 | 0.1824 | 1.4550 | 0.0962 | 0.1360 | 0.2527 | 0.0638 |
|     | \( C_1 \) | 10.2064 | 1.0854 | 1.1212 | 0.6874 | 0.5816 | 0.3268 | 0.0945 | 0.1080 | 0.1098 | 0.0653 |
| 5%  | \( C_0 \) | 0.3589 | 0.0788 | 0.1029 | 0.0916 | 0.0547 | 0.3612 | 0.0462 | 0.0498 | 0.0464 | 0.0370 |
|     | \( C_1 \) | 5.2254 | 0.2199 | 0.2324 | 0.2594 | 0.1994 | 0.3210 | 0.0929 | 0.1060 | 0.1087 | 0.0639 |
| 50% | \( C_0 \) | 0.1536 | 0.0577 | 0.0674 | 0.0808 | 0.0371 | 0.1526 | 0.0391 | 0.0415 | 0.0303 | 0.0277 |
|     | \( C_1 \) | 5.2118 | 0.1875 | 0.2033 | 0.2202 | 0.1738 | 0.3109 | 0.0926 | 0.1040 | 0.1072 | 0.0621 |

Table 2: Trimmed mean of the \( \text{ISE} \) for the estimators of the regression functions \( g \) and \( g_j \), \( 1 \leq j \leq 4 \) under different contaminations. The trimming values \( \nu \) considered equal 1%, 5% and 50%.

The numerical experiment for \( d = 4 \) yields similar conclusions regarding the advantage of the robust procedure over the classical one than in dimension \( d = 2 \). As expected, the robust marginal integration estimator is less efficient than the classical estimator for clean data. Under \( C_1 \), the \( \text{ISE} \) trimmed means of \( \tilde{g}_C \) are more than 15 times larger than those obtained with \( \tilde{g}_R \). Besides, when considering a 1% trimming, the classical estimators of \( g_1, g_2, g_3 \) and \( g_4 \) gives trimmed mean values more than 11, 10, 6 and 8 times larger than those corresponding to the robust estimator. When considering the 5% trimmed mean and the \( \text{MEDISE} \), the difference is not so noticeable as with a 1% trimming but it is still large, since in all cases the summary measure of classical estimator is at least the double of that corresponding to the robust estimator.

**Acknowledgements.** This research was partially supported by Grants PIP 112-201101-00339 from CONICET, PICT 0397 from ANPCyT and 20120130100279BA from the Universidad de Buenos Aires at Buenos Aires, Argentina.
A Appendix

We begin by fixing some notation which will be useful in the sequel. Denote as

\[
R_\alpha(X_{i,\alpha}, x_\alpha) = g_\alpha(X_{i,\alpha}) - g_\alpha(x_\alpha) - \sum_{j=1}^{q} g_\alpha(j) \frac{(X_{i,\alpha} - x_\alpha)^j}{j!}
\] (A.1)

and \( R(X_i, x) = \sum_{j \neq \alpha} (g_j(X_{i,j}) - g_j(x_j)) + R_\alpha(X_{i,\alpha}, x_\alpha) \). Furthermore, define

\[
\tilde{x}_{i,\alpha} = \left(1, \frac{X_{i,\alpha} - x_\alpha}{h_\alpha}, \frac{(X_{i,\alpha} - x_\alpha)^2}{h_\alpha^2}, \ldots, \frac{(X_{i,\alpha} - x_\alpha)^q}{h_\alpha^q}\right) = (\tilde{x}_{i,1,\alpha}, \ldots, \tilde{x}_{i,q+1,\alpha})^T.
\]

Then, \( Y_i - \tilde{x}_{i,\alpha}^T H^{(\alpha)} \theta(x) = \epsilon_i + R(X_i, x)/\sigma \). Let \( U_i = \sigma(X_i) \epsilon_i \) so that \( Y_i = g(X_i) + U_i \). Denote \( V_i = \sigma(X_i) \epsilon_i / \sigma(x) = U_i / \sigma(x) \) and for \( r = (\beta_0, h_\alpha \beta_1, h_\alpha^2 \beta_2, \ldots, h_\alpha^3 \beta_3)^T \) define

\[
\ell_n(r) = \frac{1}{n} \sum_{i=1}^{n} \delta_i \rho \left( \frac{Y_i - \beta_0 - \sum_{j=1}^{q} \beta_j (X_{i,\alpha} - x_\alpha)^j}{\hat{s}(x)} \right) K_{H_d}(X_i - x)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \delta_i \rho \left( \frac{Y_i - x^T \tilde{x}_{i,\alpha}}{\hat{s}(x)} \right) K_{H_d}(X_i - x),
\] (A.2)

\[
J_{n,\alpha}(x, a) = \frac{1}{n} \sum_{i=1}^{n} K_{H_d}(X_i - x) \xi_i \psi \left( \frac{\sigma(X_i)^{\epsilon_i}}{\sigma(x)} a \right) \tilde{x}_{i,\alpha}
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} K_{H_d}(X_i - x) \xi_i \psi (V_i a) \tilde{x}_{i,\alpha}.
\] (A.3)

Given a compact set \( C \subset \mathbb{R}^d \), we denote as \( N_\rho(C) \) the minimum number of balls of radius \( \rho \) needed to cover \( C \). Then, we have that \( C \subset \bigcup_{k=1}^{N_\rho(C)} B_d(x_k, \rho) \) where \( B_d(x_k, \rho) = \{y \in \mathbb{R}^d : \|y - x_k\| \leq \rho\} \) stands for the ball of center \( x_k \) and radius \( \rho \). It is well known that \( N_\rho(C) \leq A_1/\rho^d \) where the constant \( A_1 \) does not depend on \( \rho \). We also denote as \( B_{d, \rho} = B_d(0, \rho) \) the ball centered at \( 0 \) and as \( \mathcal{V}_{d, \rho} = \{y \in \mathbb{R}^d : \|y\| = \rho\} \) the sphere of center \( 0 \) and radius \( \rho \).

A.1 Proof of Proposition 3.1

We begin by proving some Lemmas that will be helpful in the sequel.

The following Lemma corresponds to the well known exponential inequality for bounded variables and can be seen, for instance, in Pollard (1984) or Ferraty and Vieu (2006). Lemma A.1.1 is needed to derive Lemma A.1.2 which is a previous step to prove Lemma A.1.3.

Lemma A.1.1. Let \( \{Z_i\}_{i \geq 1} \) be independent random variables such that \( \mathbb{E}Z_i = 0, |Z_i| \leq M \) and \( \sigma^2 = \mathbb{E}Z_i^2 < \infty \). Then, for all \( \epsilon > 0 \) we have that

\[
P \left( \left| \sum_{i=1}^{n} Z_i \right| > \epsilon n \right) \leq 2 \exp \left\{ -\frac{\epsilon^2 n}{2\sigma^2 (1 + \epsilon M/\sigma^2)} \right\}.
\]
Lemma A.1.2. Let $C \subset \mathbb{R}^d$ be a compact set with non-empty interior and $I_\delta = [1 - \delta, 1 + \delta]$ where $\delta \leq 1/2$. Let $W_i = W_i(a) = f(Y_i, X_i, \delta, a)$ be a sequence of random variables such that $|W_i| \leq M$, for all $i$ and $|W_i(a_1) - W_i(a_2)| \leq M_1|a_1 - a_2|$. Define $S_n(x, a) = (1/n)\sum_{i = 1}^n (G_i(x, a) - E G_i(x, a))$ where $G_i(x, a) = K_{H_d}(X_i - x)W_i(x, a)\tilde{\alpha}_{i,j,\alpha}^m \tilde{\gamma}_{i,\ell,\alpha}^n$ where $m, n = 0, 1$ and $1 \leq j, \ell \leq q + 1$ are fixed. Assume also that A2, A6, A5 hold and denote $A_h = 1/\min_{1 \leq j \leq d}h_j$.

a) Let $\theta_n$ and $\rho_n$ be non-negative numerical sequences converging to zero, such that $\theta_n - 1 A_h \rho \leq M_2$, for all $n \geq 1$ and $\rho_n \left\{ \prod_{j = 1}^d h_j \right\} \to 0$. Then, there exist $b_1 > 0$ and $b_2 > 0$ and a constant $C_0 > 0$ such that for all $C > C_0$ and for all $n \geq n_0$,

$$\mathbb{P}\left( \theta_n - 1 \sup_{x \in B(x_k, \rho) \cap C} \sup_{a \in I_i \cap I_\delta} |\tilde{S}_{k,s}(x, a)| > C \right) \leq 4 \exp \left\{ - \frac{C^2\theta_n^2 n \prod_{j = 1}^d h_j}{b_1 A_h^2 \rho^2 + b_2 C\theta_n A_h \rho} \right\}$$

where $\tilde{S}_{k,s}(x, a) = S_n(x, a) - S_n(x_k, a_n), C \subset \bigcup_{k = 1}^{N_n(C)} B_d(x_k, \rho)$ and $I_\delta = [1 - \delta, 1 + \delta] \subset \bigcup_{k = 1}^{N_n(I_\delta)} I_s$ with $I_s = [a_s - \rho, a_s + \rho]$.

b) Let $\theta_n$ and $\rho_n$ be non-negative numerical sequences converging to zero, such that $\theta_n - 1 A_h \rho \leq M_2$ and $\rho_n \left\{ \prod_{j = 1}^d h_j \right\} \to 0$. Then, there exist $b_j > 0$, $1 \leq j \leq 4$ and a constant $C_0 > 0$ such that for any $C > C_0$ and for all $n \geq n_0$,

$$\mathbb{P}\left( \sup_{x \in C} \sup_{a \in I_\delta} |S_n(x, a)| > C\theta_n \right) \leq 4 A_1 \exp \left\{ \exp \left\{ - \frac{C^2\theta_n^2 n \prod_{j = 1}^d h_j}{4b_3 + 2b_4 C\theta_n} \right\} + \exp \left\{ - \frac{C^2\theta_n^2 n \prod_{j = 1}^d h_j}{4b_1 A_h^2 \rho^2 + 2b_2 C\theta_n A_h \rho} \right\} \right\}$$

c) Let $\theta_n = \sqrt{\log n / (n \prod_{j = 1}^d h_j)}$. Then, there exists $C$ such that

$$\sum_{n \geq 1} \mathbb{P} \left( \theta_n - 1 \sup_{x \in C} \sup_{a \in I_\delta} |S_n(x, a)| > C \right) < \infty,$$

that is, $\sup_{x \in C} \sup_{a \in I_\delta} |S_n(x, a)| = O_{a.co.}(\theta_n)$.

Proof. a) For a fixed $1 \leq k \leq N_\rho(C)$, let

$$\tilde{x}_{k,i,\alpha} = \left(1, \frac{X_{i,\alpha} - x_{k,\alpha}}{h_\alpha}, \frac{(X_{i,\alpha} - x_{k,\alpha})^2}{h_\alpha^2}, \ldots, \frac{(X_{i,\alpha} - x_{k,\alpha})^q}{h_\alpha^q}\right)^T = (\tilde{x}_{k,i,1}, \ldots, \tilde{x}_{k,i,q+1})^T,$$

where we avoid the subscript $\alpha$ to simplify the notation. For $1 \leq j, \ell \leq d + 1$ define $K^{(j,\ell)}(u) = K(u)h_\alpha^{m(j-1)}h_\alpha^{m(\ell-1)}$. Then, we have that $K_{H_d}(X_i - x)\tilde{x}_{i,j,\alpha}^m \tilde{\gamma}_{i,\ell,\alpha}^n = K^{(j,\ell)}(X_i - x)$ and $K_{H_d}(X_i - x_k)\tilde{x}_{k,i,j,\alpha}^m \tilde{\gamma}_{k,i,\ell,\alpha}^n = K_{H_d}(X_i - x_k)$. Hence, using that the kernels $K_j$ have compact support in $[-1, 1]$ we obtain that

$$|\tilde{S}_{k,s}(x, a)| \leq \frac{1}{n} \sum_{i = 1}^n (G_i(x, a) - G_i(x_k, a_n)) + \frac{1}{n} \sum_{i = 1}^n |EG_i(x_k, a_s) - E G_i(x, a)|$$

$$\leq \bar{S}_k(x, a) + \tilde{S}_{k,s}(x, a)$$
Therefore, which leads to

\[ K \]

that for any \( K \), with \( K \)

Then, we have that

\[ \widetilde{S}_k(x, a) = \frac{1}{n} \left| \sum_{i=1}^{n} (G_i(x, a) - G_i(x_k, a)) \right| + \frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E} G_i(x_k, a) - \mathbb{E} G_i(x, a) \right| \]

\[ \widetilde{S}_{k,s}(x_k, a) = \frac{1}{n} \left| \sum_{i=1}^{n} (G_i(x_k, a) - G_i(x_k, a_s)) \right| + \frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E} G_i(x_k, a_s) - \mathbb{E} G_i(x_k, a) \right| \]

with \( h = (h_1, \ldots, h_d)^T \) and \( B(x, h) = \{ y \in \mathbb{R}^d : |y_j - x_j| \leq h_j \text{ for } 1 \leq j \leq d \} \). Using that the kernels \( K_j \) are Lipschitz of order one and that if \( K(X_i - x_k) \neq 0 \), then \( x_{k,i,j} \leq 1 \), we get easily that for any \( x \in B_d(x_k, \rho) \)

\[ \left| K^{(j\ell)}_{H_d}(X_i - x) - K^{(j\ell)}_{H_d}(X_i - x_k) \right| \leq c_1 \frac{\| H_d^{-1}(x - x_k) \|}{\prod_{j=1}^{d} h_j} \leq c_1 \frac{A_h \rho}{\prod_{j=1}^{d} h_j}, \]

which leads to

\[ \theta_n^{-1} \sup_{x \in B_d(x_k, h) \cap C} \sup_{a \in I_k \cap I_\delta} |\widetilde{S}_k(x, a)| \leq \theta_n^{-1} c_2 \frac{A_h \rho}{\prod_{j=1}^{d} h_j} \frac{1}{n} \sum_{i=1}^{n} \bar{z}_{B_d(x_k, h + \rho)}(X_i) = \tilde{A}_{k, n}, \]

where \( c_2 = 2M c_1 \).

On the other hand, since \( \rho/\prod_{j=1}^{d} h_j \to 0 \), we get that there exists \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1, \prod_{j=1}^{d} (h_j + \rho) \leq \prod_{j=1}^{d} h_j + c_3 \rho \leq 2 \prod_{j=1}^{d} h_j \). Observe that, since \( n \) is large enough, we may assume that \( h_j < 1 \) for all \( j \), so \( A_h \rho \leq \rho/\prod_{j=1}^{d} h_j \to 0 \).

Let \( Z_i = \mathbb{I}_{B_d(x_k, h + \rho)}(X_i) A_h \rho/\prod_{j=1}^{d} h_j \). Then, using that \( \mathbb{E} \mathbb{I}_{B_d(x_k, h + \rho)}(X_i) \leq c_4 \prod_{j=1}^{d} (h_j + \rho) \)

where \( c_4 = \| fX \|_\infty \), we obtain that for \( n \geq n_1 \),

\[ |Z_i| \leq \frac{A_h \rho}{\prod_{j=1}^{d} h_j} \leq c_4 \prod_{j=1}^{d} (h_j + \rho) \frac{A_h \rho}{\prod_{j=1}^{d} h_j} \leq 2c_4 A_h \rho \to 0. \]

Therefore, \( |Z_i - \mathbb{E} Z_i| \leq 2 A_h \rho/\prod_{j=1}^{d} h_j \) and

\[ \text{VAR}(Z_i) \leq \mathbb{E} Z_i^2 \leq c_4 \prod_{j=1}^{d} (h_j + \rho) \frac{A_h \rho}{\prod_{j=1}^{d} h_j} \left( \frac{A_h \rho}{\prod_{j=1}^{d} h_j} \right)^2 \leq 2c_4 \frac{A_h^2 \rho^2}{\prod_{j=1}^{d} h_j}. \]

Then, we have that

\[ \frac{A_h \rho}{\prod_{j=1}^{d} h_j} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{B_d(x_k, h + \rho)}(X_i) \right) = \frac{1}{n} \sum_{i=1}^{n} Z_i \leq \frac{1}{n} \sum_{i=1}^{n} (Z_i - \mathbb{E} Z_i) + 2c_4 A_h \rho. \]
On the other hand, the fact that \( \theta_n^{-1} A_h \rho \leq M_2 \), for any \( C > C_0 = 4c_2 c_4 M_2 \) lead us to

\[
\mathbb{P} \left( \theta_n^{-1} \sup_{x \in B(x_k, \rho)} |\tilde{S}_k(x) - E| > C \right) \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} |Z_i - E| > \frac{C \theta_n}{2c_2} \right)
\]

Finally, Lemma A.1.1 for all \( n \geq n_1 \) implies that, for all \( n \geq n_1 \), if \( C > C_0 \) we have that there exist \( b_1 \) and \( b_2 \)

\[
\mathbb{P} \left( \theta_n^{-1} \sup_{x \in B_d(x_k, \rho)} |\tilde{S}_k(x, a) - \tilde{S}_k(x_k) - E| > \frac{C}{2} \right) \leq \mathbb{P} \left( \theta_n^{-1} \tilde{A}_{k,n} > \frac{C}{2} \right) \leq 2 \exp \left\{ - \frac{C^2 \theta_n^2 n \prod_{j=1}^{d} h_j}{b_1 A_h^2 \rho^2 + b_2 C \theta_n A_h \rho} \right\}
\]

On the other hand, using that \( |W_i(a) - W_i(a_2)| \leq M_1 |a_1 - a_2| \) and the fact that \( |a - a_s| < \rho \), we get that

\[
|G_i(x_k, a) - G_i(x_k, a_s)| \leq M_1 \rho K_{H_d}(X_i - x_k) \leq \frac{M_1 \rho}{\prod_{j=1}^{d} h_j} \mathbb{P}_{B_d(x_k, h)}(X_i) \leq \frac{M_1 \rho}{\prod_{j=1}^{d} h_j} \mathbb{P}_{B_d(x_k, h + \rho)}(X_i)
\]

Then, if \( n \geq n_2 \) we have that

\[
\tilde{S}_{k,s}(x_k, a) \leq 2 M_1 \rho \mathbb{P}_{B_d(x_k, h + \rho)}(X_i) \leq \tilde{A}_{k,n}
\]

since \( A_h \to \infty \). Therefore, if \( n \geq \max \{ n_1, n_2 \} \)

\[
\mathbb{P} \left( \theta_n^{-1} \sup_{x \in B(x_k, \rho)} |\tilde{S}_k(x, a) - \tilde{S}_k(x_k) - E| > \frac{C}{2} \right) \leq \mathbb{P} \left( \theta_n^{-1} \tilde{A}_{k,n} > \frac{C}{2} \right) \leq 2 \exp \left\{ - \frac{C^2 \theta_n^2 n \prod_{j=1}^{d} h_j}{b_1 A_h^2 \rho^2 + b_2 C \theta_n A_h \rho} \right\}
\]

which together with (A.5) concludes the proof of a).

b) Recall that \( C \subset \bigcup_{k=1}^{N_{\rho}(C)} B_d(x_k, \rho) \) with \( N_{\rho}(C) \leq A_1 / \rho^d \) and \( \mathcal{I}_{\delta} = [1 - \delta, 1 + \delta] \subset \bigcup_{k=1}^{N_{\rho}(\mathcal{I}_{\delta})} \mathcal{I}_{\delta} \) with \( N_{\rho}(\mathcal{I}_{\delta}) \leq 2 / \rho \). Then, given \( x \in C \) and \( a \in \mathcal{I}_{\delta} \) there exist \( k, s \) such that \( x \in B(x_k, \rho) \), \( a \in \mathcal{I}_{\delta} \). Besides, for any \( x \in B(x_k, \rho) \), we have that \( |S_n(x, a)| \leq |S_n(x_k, a_s)| + |\tilde{S}_{k,s}(x, a)| \), so

\[
\sup_{x \in C} \sup_{a \in \mathcal{I}_{\delta}} |S_n(x, a)| \leq \max_{1 \leq k \leq N_{\rho}(C)} |S_n(x_k, a_s)| + \max_{1 \leq s \leq N_{\rho}(\mathcal{I}_{\delta})} \sup_{x \in B(x_k, \rho)} |\tilde{S}_{k,s}(x, a)|, 
\]

which entails that \( \mathbb{P} \left( \theta_n^{-1} \sup_{x \in C} \sup_{a \in \mathcal{I}_{\delta}} |S_n(x, a)| > C \right) \leq \beta_n + \gamma_n \) where

\[
\beta_n = \mathbb{P} \left( \max_{1 \leq k \leq N_{\rho}(C)} |S_n(x_k, a_s)| > \frac{C \theta_n}{2} \right) \leq N_{\rho}(C) N_{\rho}(\mathcal{I}_{\delta}) \sup_{x \in C} \mathbb{P} \left( |S_n(x, a)| > \frac{C \theta_n}{2} \right)
\]

\[
\gamma_n = \mathbb{P} \left( \max_{1 \leq k \leq N_{\rho}(C)} \sup_{1 \leq s \leq N_{\rho}(\mathcal{I}_{\delta})} |\tilde{S}_{k,s}(x, a)| > \frac{C \theta_n}{2} \right)
\]
Using Lemma A.1.1, straightforward calculations (see Martínez (2014) for details) allow to show that there exists $b_3, b_4 > 0$ such that

$$\beta_n \leq 2N_{\rho}(C)N_{\rho}(I_{\delta}) \exp \left\{- \frac{C^2\theta_n^2 n \prod_{j=1}^d h_j}{4 b_3 + 2 b_4 C \theta_n} \right\} \leq 2^{A_1} \rho^{\rho^d+1} \exp \left\{- \frac{C^2\theta_n^2 n \prod_{j=1}^d h_j}{4 b_3 + 2 b_4 C \theta_n} \right\}. \quad (A.6)$$

Using a), it follows that for all $C > 2 C_0 > 0$ and all $n \geq n_0$

$$\gamma_n \leq N_{\rho}(C)N_{\rho}(I_{\delta}) \mathbb{P} \left( \sup_{x \in B(x_k, \rho) \cap C} \sup_{a \in I_{\delta} \cap \mathbb{I}_{\delta}} |\hat{S}_{k,a}(x,a)| > \frac{C \theta_n}{2} \right) \leq 4^{A_1} \rho^{\rho^d+1} \exp \left\{- \frac{C^2\theta_n^2 n \prod_{j=1}^d h_j}{4 b_1 A_h^2 \rho^2 + 2 b_2 C \theta_n A_h \rho} \right\}. \quad (A.7)$$

The bound given in b) follows now from (A.6) and (A.7).

c) Observe that, **A6** implies that $\theta_n \to 0$. Define $\rho = \log n/n$. We will show that the conditions in b) are fulfilled. It is clear that $\rho \to 0$ and besides by **A6**, $\rho_n \left\{ \prod_{j=1}^n h_j \right\}^{-1} = \theta_n^2 \to 0$. On the other hand,

$$\left( \theta_n^{-1} A_h \rho \right)^2 = \left( \frac{1}{\min_{1 \leq j \leq d} \{ h_j \}} \frac{\log n}{n} \right)^2 \leq \frac{\log n}{n \prod_{j=1}^d h_j} \to 0,$$

so $\theta_n^{-1} A_h \rho \leq 1$ if $n$ is large enough. Noticing that $\theta_n^2 n \prod_{j=1}^d h_j = \log(n)$ and that there exists $A_1$ such that $N_{\rho}(C) \leq A_1 \rho^{-d}$, we have that b) implies that

$$\mathbb{P} \left( \theta_n^{-1} \sup_{x \in B(x_k, \rho) \cap C} \sup_{a \in I_{\delta} \cap \mathbb{I}_{\delta}} |S_n(x,a)| > C \right) \leq 4^{A_1} \rho^d \left[ \exp \left\{- \frac{C^2\theta_n^2 n \prod_{j=1}^d h_j}{4 b_3 + 2 b_4 C \theta_n} \right\} + \exp \left\{- \frac{C^2\theta_n^2 n \prod_{j=1}^d h_j}{4 b_1 A_h^2 \rho^2 + 2 b_2 C \theta_n A_h \rho} \right\} \right] \leq 4^{A_1} \rho^d \left[ \exp \left\{- \frac{C^2 \log(n)}{4 b_3 + 2 b_4 C \theta_n} \right\} + \exp \left\{- \frac{C^2 \log(n)}{4 b_1 A_h^2 \rho^2 + 2 b_2 C \theta_n A_h \rho} \right\} \right].$$

Finally, using that $\theta_n \to 0$ and $A_h \rho \to 0$, we obtain that there exists $n_1$ such that for all $n \geq n_1$, $2 b_1 C \theta_n \leq 4 b_3$ and $4 b_1 A_h^2 \rho^2 + 2 b_2 C \theta_n A_h \rho \leq 8 b_3$, then

$$\mathbb{P} \left( \theta_n^{-1} \sup_{x \in B(x_k, \rho) \cap C} \sup_{a \in I_{\delta} \cap \mathbb{I}_{\delta}} |S_n(x,a)| > C \right) \leq 8 A_1 \rho^{-d} \exp \left\{- \frac{C^2 \log(n)}{8 b_3} \right\} \leq 8 A_1 \left( \frac{1}{\log(n)} \right)^d \left( \frac{C^2}{8 b_3} \right)^{d \frac{C^2}{8 b_3}} \leq 84 A_1 n \left( \frac{C^2}{8 b_3} \right) n^{d \frac{C^2}{8 b_3}} \to \infty$$

concluding the proof.

**Remark A.1.1.** Taking $m = \tilde{m} = 0$ and $W_i \equiv 1$, Lemma A.1.2c) entails the uniform convergence for the kernel density estimator, that is, we obtain that

$$\sup_{x \in C} \frac{1}{n} \sum_{i=1}^n \left( \mathcal{K}_{H_d}(X_i - x) - \mathbb{E} \mathcal{K}_{H_d}(X_i - x) \right) = O_{n,co.}(\theta_n).$$
Besides, using that \( \sup_{x \in \mathcal{C}} |f_x(x)| < \infty \), \( \mathcal{K} \) has compact support and \( f_x \) is uniformly continuous in \( \mathcal{C} \), we get that

\[
\sup_{x \in \mathcal{C}} \left| \frac{1}{n} \sum_{i=1}^{n} \mathcal{E}_i \mathcal{K}_d(x_i - x) - f_x(x) \right| = \sup_{x \in \mathcal{C}} \left| \int \mathcal{K}(u) (f_x(H_d u + x) - f_x(x)) \right| \to 0
\]

which together with the above result implies that

\[
\sup_{x \in \mathcal{C}} \left| \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}_d(x_i - x) - f_x(x) \right| = o_{a.s.}(1) .
\]  \hspace{1cm} (A.8)

**Lemma A.1.3.** Assume that A1, A2 and A4 to A7 hold, then

\[
\sup_{x \in \mathcal{C}} |J_{n,j}(x, \tilde{a}_\sigma(x))| = O_{a.s.}(\theta_n)
\]

with \( \theta_n = \sqrt{\log n/(n \prod_{j=1}^{d} h_j)} \), where \( j = 1, \ldots, d + 1 \) and \( J_{n,j} \) is the \( j \)th component of vector \( J_n \) and \( \tilde{a}_\sigma(x) = \sigma(x)/\tilde{\sigma}(x) \).

**Proof.** By (8), taking \( \delta = 1/2 \), we get that there exists \( \mathcal{N} \) such that \( \mathbb{P}(\mathcal{N}) = 0 \) and for any \( \omega \in \mathcal{N} \), there exists \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1 \sup_{x \in \mathcal{C}} |\tilde{a}_\sigma(x) - 1| \leq \delta \). Therefore, for any \( \omega \in \mathcal{N} \) and \( n \geq n_1 \), we have that \( \sup_{x \in \mathcal{C}} |J_{n,j}(x, \tilde{a}_\sigma(x))| \leq \sup_{x \in \mathcal{C}} \sup_{a \in [1-\delta,1+\delta]} |J_{n,j}(x,a)| \), so to conclude the proof it is enough to see that \( \sup_{x \in \mathcal{C}} \sup_{a \in [1-\delta,1+\delta]} |J_{n,j}(x,a)| = O_{a.s.}(\theta_n) \). Recall that

\[
J_n(x,a) = \frac{1}{n} \sum_{i=1}^{n} \delta_i \mathcal{K}_d(x_i - x) \psi \left( \frac{\sigma_i(x) \bar{\varepsilon}_i}{\sigma(x)} a \right) \tilde{x}_{i,a} = \frac{1}{n} \sum_{i=1}^{n} \delta_i \mathcal{K}_d(x_i - x) \psi (V_i a) \tilde{x}_{i,a} .
\]

then, \( J_{n,j}(x,a) = (1/n) \sum_{i=1}^{n} \delta_i \mathcal{K}_d(x_i - x) \psi (V_i a) \tilde{x}_{i,j,a} \) and the proof follows now from Lemma A.1.2c) taking \( m = 1, \tilde{m} = 0 \) and \( W_i(a_1) = \delta_i \psi (V_i a) \) and noting that \( |W_i(a)| \leq \|\psi\|_{\infty}/i(t) \) and \( |W_i(a_1) - W_i(a_2)| \leq (\|\zeta\|_{\infty}/i(t)) |a_1 - a_2| \). \( \square \)

**Proof of Proposition 3.1.** Let \( r = (\beta_0, h_\alpha \beta_1, h^2_\alpha \beta_2, \ldots, h^d_\alpha \beta_q)^T \), \( \ell_n(r) \) be defined in (A.2) and \( r_0(x) = \left( g(x), h_\alpha g_\alpha^{(1)}(x_\alpha), \ldots, h^d_\alpha g_\alpha^{(q)}(x_\alpha) \right)^T \). For the sake of simplicity denote \( \nu = \nu_{q+1,\tau} = \{ r : \|r\| = \tau \} \). To prove Proposition 3.1, we will first show that it is enough to see that there exists \( \mathcal{N} \) such that \( \mathbb{P}(\mathcal{N}) = 0 \) and such that for all \( \omega \in \mathcal{N} \), given \( \nu > 0 \) there exists \( 0 < \tau_\nu < 1 \) small enough such that for any \( 0 < \tau < \tau_\nu \) and \( n \geq n_0 \),

\[
\inf_{r \in \mathcal{V}_\nu} \inf_{x \in \mathcal{C}} \{ \ell_n(r + r_0(x)) - \ell_n(r_0(x)) \} > 0 .
\]  \hspace{1cm} (A.9)

Indeed, in the set \( \{ \inf_{r \in \mathcal{V}_\nu} \inf_{x \in \mathcal{C}} |\ell_n(r + r_0(x)) - \ell_n(r_0(x))| > 0 \} \) for all \( x \in \mathcal{C} \) we have that \( \inf_{r \in \mathcal{V}_\nu} \ell_n(r + r_0(x)) > \ell_n(r_0(x)) \), which implies that the function \( L_n(r) = \ell_n(r + r_0(x)) - \ell_n(r_0(x)) \) has a local minimum \( \tilde{r}(x) \) in \( \tilde{B}_{q+1,\tau} \), where \( \tilde{B} \) stands for the interior of the set \( B \). Then, for all \( x \in \mathcal{C} \), \( \tilde{r}(x) + r_0(x) \) is a local minimum of \( \ell_n(r) \) and \( \tilde{r}(x) + r_0(x) \) belongs to \( \tilde{B}_{q+1}(r_0(x),\tau) = \{ r : \|r - r_0(x)\| < \tau \} \), as a result of which \( \tilde{\beta}(x) = \tilde{r}(x) + r_0(x) \) is a solution of (6). That is, with
probability 1, for all \( x \in C \), there exists a solution \( \tilde{\beta}(x) \) of (6) in the interior of \( B_{q+1}(r_0(x), \tau) \). Hence, for any \( \omega \notin \mathcal{N} \), given \( \nu > 0 \), and \( \tau > 0 \) small enough \( \sup_{x \in C} \| H^{(a)}[\tilde{\beta}(x) - \beta(x)] \| \leq \tau, n \geq n_0 \), which implies that \( \sup_{x \in C} \| H^{(a)}[\tilde{\beta}(x) - \beta(x)] \| \xrightarrow{a.s.} 0 \) as desired.

In order to prove (A.9), observe that \( Y_i - (r + r_0(x))^T \tilde{x}_{i, \alpha} = U_i + g(x_i) - r^T \tilde{x}_{i, \alpha} - r^T \tilde{x}_{i, \alpha} = U_i + R(x_i, x) \tilde{x}_{i, \alpha} - r^T \tilde{x}_{i, \alpha} \). Denote as \( \hat{\tilde{Z}}_i(x) = (U_i + R(x_i, x)) / \hat{s}(x) = \hat{V}_i(x) + R(x_i, x) \hat{s}(x)^{-1} \) with \( \hat{V}_i(x) = U_i / \hat{s}(x) = \sigma(x_i) \epsilon_i / \hat{s}(x) \) and \( \hat{\Delta}_i(x) = r^T \tilde{x}_{i, \alpha} / \hat{s}(x) \). Then, using that \( \rho(b) - \rho(a) = \int_a^b \psi(u) du \) we obtain that for all \( r \in V_\tau \)

\[
\ell_n(r + r_0(x)) - \ell_n(r_0(x)) = \frac{1}{n} \sum_{i=1}^{n} K_{H_d}(X_i - x) \xi_i \int_{\hat{Z}_i} \hat{Z}_i - \hat{\Delta}_i \psi(t) dt = K_{n1}(x) + K_{n2}(x) + K_{n3}(x), \quad (A.10)
\]

with

\[
K_{n1}(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_i K_{H_d}(x_i - x) \int_{\hat{Z}_i} \hat{Z}_i - \hat{\Delta}_i \psi(\hat{V}_i) dt = - \frac{1}{\hat{s}(x)} \sum_{i=1}^{n} \delta_i K_{H_d}(x_i - x) \psi(\hat{V}_i) \hat{s}_{i, \alpha},
\]

\[
K_{n2}(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_i K_{H_d}(x_i - x) \int_{\hat{Z}_i} \hat{Z}_i - \hat{\Delta}_i \psi'(\hat{V}_i)(t - \hat{V}_i) dt
\]

\[
= \frac{1}{2\hat{s}(x)} \frac{1}{n} \sum_{i=1}^{n} \delta_i K_{H_d}(x_i - x) \psi'(\hat{V}_i) \left[ (r^T \tilde{x}_{i, \alpha})^2 - 2R(x_i, x) r^T \tilde{x}_{i, \alpha} \right],
\]

\[
K_{n3}(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_i K_{H_d}(x_i - x) \int_{\hat{Z}_i} \hat{Z}_i - \hat{\Delta}_i \left[ \psi(t) - \psi(\hat{V}_i) - \psi'(\hat{V}_i)(t - \hat{V}_i) \right] dt.
\]

The proof of (A.9) will be done in several steps. Let us assume that the following approximations hold

\[
\sup_{r \in V_\tau} \sup_{x \in C} \| K_{n1}(x) \| = \tau \sqrt{\frac{\log n}{n \prod_{j=1}^{d} h_j}} \tilde{\gamma}_{1,n}, \quad (A.11)
\]

\[
K_{n2}(x) = \frac{1}{2\sigma^2(x)} \mathbb{E}(\psi'(\hat{\varepsilon})) f_X(x) p(x) r^T S^{(a)} r (1 + \tilde{\zeta}_n) + \tau \left( h_{\alpha}^{q+1} + \sum_{j \neq \alpha} h_j \right) \tilde{\gamma}_{2,n}, \quad (A.12)
\]

\[
\sup_{r \in V_\tau} \sup_{x \in C} \| K_{n3}(x) \| = \tau \left( \tau + h_{\alpha}^{q+1} + \sum_{j \neq \alpha} h_j \right)^2 \tilde{\gamma}_{3,n}, \quad (A.13)
\]

with \( \tilde{\gamma}_{1,n} = O_{a.co.}(1), \tilde{\gamma}_{3,n} = O_{a.co.}(1), \tilde{\gamma}_{1,n} \) and \( \tilde{\gamma}_{3,n} \) not depending on \( \tau \) and where \( \tilde{\zeta}_n = o_{a.co.}(1) \) and \( \tilde{\gamma}_{2,n} = O_{a.co.}(1) \) do not depend on \( x \in C \) nor \( r \) and, consequently, neither on \( \tau \).

We begin by showing that (A.11) to (A.13) imply (A.9).

Let us denote as \( \nu_1 > 0 \) the minimum eigenvalue of \( S^{(a)} \) which is a symmetric and positive definite matrix. Using that \( \mathbb{E}\psi'(\hat{\varepsilon}) > 0, i(f_X) > 0, i(p) > 0 \) and that the scale function \( \sigma \) is bounded over \( x \in C \), if \( M = \nu_1 \mathbb{E}\psi'(\hat{\varepsilon}) i(f_X) i(p) / (2 \sup_{x \in C} \sigma^2(x)) \), we obtain that \( x \in C \) and \( r \in V_\tau \)

\[
Q(r, x) = \frac{1}{2\sigma(x)} \mathbb{E}(\psi'(\hat{\varepsilon})) f_X(x) p(x) r^T S^{(a)} r \geq \frac{\nu_1}{2\sigma(x)} \mathbb{E}(\psi'(\hat{\varepsilon})) f_X(x) p(x) \tau^2 \geq M \tau^2 > 0.
\]
As \( \hat{Y}_{2,n} = O_{\text{a.co.}}(1) \) and \( \hat{\zeta}_n = o_{\text{a.co.}}(1) \), given \( \nu > 0 \) there exists \( \tilde{A}_1 \) such that

\[
\sum_{n \geq 1} \mathbb{P} \left( |\hat{Y}_{2,n}| > \tilde{A}_1 \right) < \infty \quad \sum_{n \geq 1} \mathbb{P} \left( |\hat{\zeta}_n| > \frac{1}{2} \right) < \infty.
\]

Let \( n_\tau \) be such that \( \hat{h}^{q+1}_\alpha + \sum_{j \neq \alpha} h_j \leq \tau \min\{M/(4\tilde{A}_1), 1\} \), for \( n \geq n_\tau \). Hence, there exists a set \( N_1 \) satisfying that \( \mathbb{P}(N_1^c) = 0 \) and for all \( \omega \notin N_1 \), there exists \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1 \), we have that \( |\hat{Y}_{2,n}| < A_1 \) and \( |\hat{\zeta}_n| < \frac{1}{2} \).

Since \( K_{n2}(x) \geq Q(r, x)(1 - |\hat{\zeta}_n|) - \tau \left( \hat{h}^{q+1}_\alpha + \sum_{j \neq \alpha} h_j \right) |\hat{Y}_{2,n}| \), if \( \omega \notin N_1 \) and \( n \geq n_{1,\tau} = \max(n_{\tau}, n_1) \) we have that

\[
\inf_{r \in V_r} \inf_{x \in \mathcal{C}} K_{n2}(x) > \frac{M}{4} \tau^2.
\]  

(A.14)

From (A.11) and the fact that \( n \prod_{j=1}^d h_j / \log n \to \infty \), we get easily that there exist a positive constant \( A_2 \) and a set \( N_2 \) such that \( \mathbb{P}(N_2^c) = 0 \) and for all \( \omega \notin N_2 \) there exists \( n_{2,\tau} \) such that

\[
|\hat{Y}_{1,n}| \leq \tilde{A}_2 \quad \text{and} \quad \sqrt{\frac{\log n}{n \prod_{j=1}^d h_j}} \leq \tau \min \left\{ \frac{M}{8A_2}, 1 \right\},
\]

for \( n \geq n_{2,\tau} \). Thus, using (A.14), we obtain that if \( \omega \notin N_1 \cup N_2 \) and \( n \geq \max(n_{1,\tau}, n_{2,\tau}) \)

\[
\inf_{r \in V_r} \inf_{x \in \mathcal{C}} (K_{n1}(x) + K_{n2}(x)) > \frac{M}{8} \tau^2.
\]  

(A.15)

On the other hand, \( K_{n3} \) satisfies (A.13) with \( \tilde{Y}_{3,n} = O_{\text{a.co.}}(1) \), then there exist a positive \( \tilde{A}_3 \) and a set \( N_3 \) such that \( \mathbb{P}(N_3^c) = 0 \) and for all \( \omega \notin N_3 \) there exists \( n_3 \) such that \( |\hat{Y}_{3,n}| \leq \tilde{A}_3 \), for \( n \geq n_3 \). Besides, there exists \( n_{3,\tau} \in \mathbb{N} \) such that \( \hat{h}^{q+1}_\alpha + \sum_{j \neq \alpha} h_j \leq \tau \), for \( n \geq n_{3,\tau} \). Therefore, we obtain that for all \( \omega \notin N_3 \) and for any \( n \geq n_{4,\tau} = \max\{n_{3,\tau}, n_{3,\tau}\} \), \( \sup_{r \in V_r} \sup_{x \in \mathcal{C}} |K_{n3}(x)| \leq \tilde{A}_3 \tau^3 \). Taking \( \tau_\nu < \min\{1, M/(16 \tilde{A}_3)\} \), we get that for \( \tau < \tau_\nu \), \( \omega \notin N_3 \) and \( n \geq n_{4,\tau} \)

\[
\sup_{r \in V_r} \sup_{x \in \mathcal{C}} |K_{n3}(x)| \leq \frac{M}{16} \tau^2.
\]  

(A.16)

Therefore, we have shown that for all \( 0 < \tau < \tau_\nu < 1, \omega \notin N_3 \) and \( n \geq \max(n_{1,\tau}, n_{2,\tau}, n_{4,\tau}) \), the assertions (A.16) and (A.15) hold which together with (A.10) lead us to (A.9).

It remains to show (A.11), (A.12) and (A.13).

• Let us begin by proving (A.11). Note that

\[
K_{n1}(x) = -\frac{1}{s(x)} r^T J_{n,\alpha}(x, \hat{\alpha}_\sigma(x)),
\]

where \( J_{n,\alpha}(x, a) \) is defined in (A.3). Note that (7) implies that

\[
\mathbb{P} \left( \exists n_0 \text{ tal que } \mathbb{P} \sup_{r \in V_r} \sup_{x \in \mathcal{C}} \|K_{n1}(x)\| < \frac{\tau}{A} \sup_{x \in \mathcal{C}} \|J_{n,\alpha}(x, \hat{\alpha}_\sigma(x))\| \right) = 1.
\]

Lemma A.1.3 entails that \( \sup_{x \in \mathcal{C}} \|J_{n,\alpha}(x, \hat{\alpha}_\sigma(x))\| = O_{\text{a.s.}}(\theta_n) \), which concludes the proof of (A.11).
• Let us show that (A.13) holds. Recall that \( \hat{Z}_i(x) = \hat{V}_i(x) + R(X_i, x) \hat{s}(x)^{-1} \) and \( \hat{\Delta}_i(x) = r^T \hat{x}_{i, \alpha} / \hat{s}(x) \).

By the integral mean value theorem,

\[
K_{n3}(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_i K_{H_d}(X_i - x) \int_{\hat{V}_i(x) + R(X_i, x) \hat{s}(x)^{-1}}^{\hat{V}_i(x) + R(X_i, x) \hat{s}(x)^{-1}} \left[ \psi(t) - \psi(\hat{V}_i(x)) - \psi'(\hat{V}_i(x))(t - \hat{V}_i(x)) \right] dt
= -r^T \frac{1}{n \hat{s}(x)} \sum_{i=1}^{n} \delta_i K_{H_d}(X_i - x) \left[ \psi'(\hat{V}_i(x) + \hat{\theta}_i(x)) - \psi'(\hat{V}_i(x)) \right] \hat{x}_{i, \alpha},
\]

where \( \hat{\theta}_i(x) \) is an intermediate point between \( R(X_i, x) \hat{s}(x) \) and \( \{ R(X_i, x) - r^T \hat{x}_{i, \alpha} \}/\hat{s}(x) \). By simplicity, if \( i \in \{ i : K_{H_d}(X_i - x) = 0 \} \), we define \( \hat{\theta}_i(x) = 0 \) since it does not change the sum. Note that \( |X_{i,j} - x_j| \leq h_j \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, d \), when \( K_{H_d}(X_i - x) \neq 0 \), thus A4 implies that

\[
\sup_{x \in \mathcal{C}} \max_{i : K_{H_d}(X_i - x) \neq 0} |R(X_i, x)| \leq A_g \left( \sum_{j \neq \alpha} h_j + h_{q+1}^{q+1} \right),
\]

where \( A_g \) is a constant only depending on \( \| g_j^{(l)} \|_\infty \) for \( j \neq \alpha \) and \( \| g_\alpha^{(q+1)} \|_\infty \). Then, using (7), we obtain that

\[
\mathbb{P} \left( \exists n_0 \text{ such that } \forall n \geq n_0 \sup_{x \in \mathcal{C}} \max_{1 \leq i \leq n} \left| \hat{\theta}_i(x) \right| \leq A_1 \left( \tau + h_{q+1}^{q+1} + \sum_{j \neq \alpha} h_j \right) \right) = 1.
\]

Let \( \mathcal{K}^*(u) = |\mathcal{K}(u)| \int |\mathcal{K}(u)| du \), then, Remark A.1.1 implies that the density estimator based on \( \mathcal{K}^* \), \( \hat{f}(x) = (1/n) \sum_{j=1}^{n} \mathcal{K}^*_H(x - X_j) \) converges uniformly and almost surely to \( f_X \) (see (A.8)). Hence, using A2, we obtain that \( \sup_{x \in \mathcal{C}} \hat{f}(x) = O_{a.s.}(1) \) which together with the fact that \( \psi'' \) is bounded and that each component of \( \hat{x}_{i, \alpha} \) is smaller or equal to 1 when \( K_{H_d}(X_i - x) \neq 0 \), leads to

\[
\sup_{r \in V_r} \sup_{x \in \mathcal{C}} \| \mathcal{K}_{n3}(x) \| \leq \frac{T \psi''(1)}{A_i(t)} \sup_{x \in \mathcal{C}} \max_{1 \leq i \leq n} \left| \hat{\theta}_i(x) \right|^2 \sup_{x \in \mathcal{C}} \frac{1}{n} \sum_{i=1}^{n} |\mathcal{K}_{H_d}(X_i - x)|,
\]

whenever \( A < \hat{s}(x) \) for all \( x \in \mathcal{C} \). Hence, from (7) we obtain that

\[
\mathbb{P} \left( \exists n_0 \text{ such that } \forall n \geq n_0 \sup_{r \in V_r} \sup_{x \in \mathcal{C}} \| \mathcal{K}_{n3}(x) \| \leq \frac{T \psi''(1)}{A_i(t)} A_1 \left( \tau + h_{q+1}^{q+1} + \sum_{j \neq \alpha} h_j \right)^2 \right) = 1.
\]

where \( \hat{\mathcal{T}}_{3,n} = \sup_{x \in \mathcal{C}} \frac{(1/n)}{\sum_{j=1}^{n} |\mathcal{K}_{H_d}(X_i - x)|} = O_{a.s.}(1) \) does not depend on \( \tau \), concluding the proof of (A.13).

• Finally, to conclude the proof we will obtain (A.12). We have that

\[
K_{n2}(x) = \frac{1}{2s^2(x)} \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}_{H_d}(X_i - x) \delta_i \psi'(\hat{V}_i(x)) \left[ (r^T \hat{x}_{i, \alpha})^2 - 2R(X_i, x) r^T \hat{x}_{i, \alpha} \right]
= \frac{1}{2s^2(x)} (r^T \hat{M}_{n1} r - 2r^T \hat{M}_{n2}). \tag{A.18}
\]
For $x \in C$ and $a \in I_{\delta} = [1 - \delta, 1 + \delta]$ (with $0 < \delta < 1$), define $M(x, a) = \mathbb{E}\psi'(\varepsilon a) f(x)p(x)S(\alpha)$ and

$$M_{n1}(x, a) = \frac{1}{n} \sum_{i=1}^{n} K_{H_d}(X_i - x) \delta_{i} \psi'(V_i(x)a) \hat{x}_{i,a} \hat{x}_{i,a}^T.$$ 

Then, $M(x, 1) = \mathbb{E}\psi'(\varepsilon) f(x)p(x)S(\alpha)$. We want to show that

$$\sup_{x \in C} \| \widehat{M}_{n1}(x) - \mathbb{E}\psi'(\varepsilon) f(x)p(x)S(\alpha) \| = o_{a.s.}(1) \quad (A.19)$$

$$\sup_{x \in C} \| \widehat{M}_{n2}(x) \| = O_{a.s.} \left( h_{\hat{\alpha}}^{q+1} + \sum_{j \neq \alpha} h_j \right). \quad (A.20)$$

Indeed, if (A.19) and (A.20) hold, using (7) and that $\sup_{x \in C} |\tilde{s}(x) - \sigma(x)| \xrightarrow{a.s.} 0$, $\sigma$ is bounded in $C$, $i(\sigma) > 0$ and replacing (A.19) and (A.20) in (A.18) we get that

$$K_{n2}(x) = \frac{1}{2\sigma^2(x)} \mathbb{E}(\psi'(\varepsilon)) f(x)p(x) r^T S(\alpha) r (1 + \hat{\zeta}_n) + \tau \left( h_{\hat{\alpha}}^{q+1} + \sum_{j \neq \alpha} h_j \right) \hat{\Upsilon}_{2,n},$$

where $\hat{\zeta}_n = o_{a.s.}(1)$ and $\hat{\Upsilon}_{2,n} = O_{a.s.}(1)$ do not depend on $r$ and therefore, neither on $\tau$, nor $x \in C$ since the convergences are uniform over $x$, which would conclude the proof of (A.12).

In order to prove (A.19) it is enough to show that for all $1 \leq j, k \leq d + 1$

$$\sup_{x \in C} | \widehat{M}_{n1,jk}(x) - M_{jk}(x, 1) | = o_{a.s.}(1), \quad (A.21)$$

where $\widehat{M}_{n1,jk}(x)$, $M_{n1,jk}(x, a)$ and $M_{jk}(x, a)$ are the components $(j, k)$ of matrices $\widehat{M}_{n1}(x, a)$, $M_{n1}(x, a)$ and $M(x, a)$, respectively.

Note that $\widehat{M}_{n1,jk}(x) = M_{n1,jk}(x, \tilde{\sigma}_\sigma(x))$ where $\tilde{\sigma}_\sigma(x) = \sigma(x)/\tilde{s}(x)$. Hence, from the bounds

$$\sup_{x \in C} | M_{n1,jk}(x, a) - M_{jk}(x, a) | \leq \sup_{x \in C} | M_{n1,jk}(x, a) - EM_{n1,jk}(x, a) | + \sup_{x \in C} | EM_{n1,jk}(x, a) - M_{jk}(x, a) |,$$

$$\sup_{x \in C} | \widehat{M}_{n1,jk}(x) - M_{jk}(x, 1) | \leq \sup_{x \in C} | M_{n1,jk}(x, \tilde{\sigma}_\sigma(x)) - M_{jk}(x, \tilde{\sigma}_\sigma(x)) | + \sup_{x \in C} | M_{jk}(x, \tilde{\sigma}_\sigma(x)) - M_{jk}(x, 1) |,$$

we obtain that in order to prove (A.21), it is enough to see that

(i) $\sup_{x \in C} \sup_{a \in I_{\delta}} | M_{n1,jk}(x, a) - EM_{n1,jk}(x, a) | = O_{a.s.} \left( \sqrt{\log n / (n \prod_{j=1}^{d} h_j) \right).$

(ii) $\sup_{x \in C} \sup_{a \in I_{\delta}} | EM_{n1,jk}(x, a) - M_{jk}(x, a) | = o(1)$

(iii) $\sup_{x \in C} | M_{n1,jk}(x, \tilde{\sigma}_\sigma(x)) - M_{jk}(x, \tilde{\sigma}_\sigma(x)) | = o_{a.s.}(1)$

(iv) $\sup_{x \in C} | M_{jk}(x, \tilde{\sigma}_\sigma(x)) - M_{jk}(x, 1) | = o_{a.s.}(1).$

(i) can be obtained immediately from Lemma A.1.2 taking $m = \hat{m} = 1$ and considering the sequence of independent random variables $W_i(a) = \psi' \left( \sigma(X_i) \varepsilon a / \sigma(x) \right) \delta_i$ and noting that $|W_i(a)| \leq \| \psi' \|_{\infty} / \bar{\iota}(t)$ for all $a$ and that $|W_i(a_1) - W_i(a_2)| \leq \| \zeta \|_{\infty} |a_1 - a_2|.$
To show (ii), let \(C_0\) be the compact neighbourhood of \(C\) given in assumptions \(A2\) and \(A3\). Then, \(f_x, p\) and \(\sigma\) are uniformly continuous functions in \(C_0\). Define

\[
\gamma_{x,a}(t) = \mathbb{E}\left( \psi' \left( \frac{\sigma(X_1)\varepsilon_1 a}{\sigma(x)} \right) | X_1 = t \right).
\]

Using that \(\varepsilon_i\) are independent from covariates, we obtain that

\[
|\gamma_{x,a}(t) - \mathbb{E}\psi'(\varepsilon_1 a)| \leq \mathbb{E}\left| \psi' \left( \frac{\sigma(t)}{\sigma(x)} \varepsilon_1 a \right) - \psi'(\varepsilon_1 a) \right| = \mathbb{E}|\zeta_2(\varepsilon_1 a \theta)| \left| \frac{\sigma(t)}{\sigma(x)} - 1 \right| \frac{1}{\theta},
\]

where \(\theta\) is an intermediate point between \(\sigma(t)/\sigma(x)\) and 1, so that

\[
\frac{1}{\theta} \leq \max \left\{ 1, \frac{\sigma(x)}{\sigma(t)} \right\}.
\]

Using that \(\zeta_2(u) = u \psi''(u)\) is bounded, we obtain the upper bound

\[
|\gamma_{x,a}(t) - \mathbb{E}\psi'(\varepsilon_1 a)| \leq \|\zeta_2\|_{\infty} \left| \frac{\sigma(t)}{\sigma(x)} - 1 \right| \max \left\{ 1, \frac{\sigma(x)}{\sigma(t)} \right\} = \|\zeta_2\|_{\infty} \|\sigma(t) - \sigma(x)\| \max \left\{ 1, \frac{\sigma(x)}{\sigma(t)} \right\}.
\]

Using that \(\inf_{t \in C_0} \sigma(t) > 0\) and \(\sup_{t \in C_0} \sigma(t) < \infty\), we have that there exists \(c_1\) such that for all \(x \in C\) and \(t \in C_0\),

\[
|\gamma_{x,a}(t) - \mathbb{E}\psi'(\varepsilon_1 a)| \leq c_1 |\sigma(t) - \sigma(x)|. \tag{A.22}
\]

Observe that

\[
\mathbb{E}M_{n1,jk}(x,a) = \mathbb{E}K_{H_d}(X_1 - x)p(X_1)\gamma_{x,a}(X_1)\tilde{x}_{1,a_j} \tilde{x}_{1,a_k} = \int K^{(\alpha)}_{H_d}(u - x)p(u)\gamma_{x,a}(u) \left[ \frac{u_\alpha - x_\alpha}{h_\alpha} \right]^{j+k-2} f_x(u) du.
\]

Changing variables \(y = H_d^{-1}(u - x)\), we obtain

\[
\mathbb{E}M_{n1,jk}(x,a) - M_{jk}(x,a) = \int \left( p(H_d y + x)\gamma_{x,a}(H_d y + x)f_x(H_d y + x)g^{j+k-2}\mathcal{K}(y) dy - \mathbb{E}\psi'(\varepsilon_1 a) f_x(x)p(x)S_{jk}^{(\alpha)} \right),
\]

where \(S_{jk}^{(\alpha)} = \int g^{j+k-2}\mathcal{K}(y) dy\) for \(1 \leq j, k, \leq q + 1\) is defined in \(A5\). Then, if we denote as \(r(y,x) = p(H_d y + x)\gamma_{x,a}(H_d y + x)f_x(H_d y + x) - \mathbb{E}\psi'(\varepsilon_1 a) f_x(x)p(x)\) we obtain that

\[
\mathbb{E}M_{n1,jk}(x,a) - M_{jk}(x,a) = \int r(y,x)g^{j+k-2}\mathcal{K}(y) dy.
\]

Using the uniform continuity of \(f_x, \sigma\) and \(p\) in \(C_0\), we obtain that given \(\epsilon > 0\) there exists \(\eta > 0\) such that for any \(x \in C, u \in C_0\) such that \(\|u - x\| < \eta\) implies \(|f_x(u)p(u) - f_x(x)p(x)| < \epsilon\) and \(|\sigma(u) - \sigma(x)| < \epsilon\). The fact that \(K_j\) has compact support \([-1, 1]\), entails that \(\|y\| \leq \sqrt{d}\) for any \(y\) such that \(\mathcal{K}(y) \neq 0\). Therefore, using that \(\max_{1 \leq j \leq d} h_{jn} \to 0\), we obtain that there exists \(n_0\) such that if \(n \geq n_0\), \(\|H_d y\| < \eta\) for all \(y\) such that \(\mathcal{K}(y) \neq 0\) and \(H_d y + x \in C_0\). Hence, if \(n \geq n_0\) for all \(x \in C\) and for all \(y\) such that \(\mathcal{K}(y) \neq 0\), we have that \(|f_x(H_d y + x)p(H_d y + x) - f_x(x)p(x)| < \epsilon|
and $|\sigma(H dy + x) - \sigma(x)| \leq \varepsilon$. Using that $\gamma_{x,a}(x) = \mathbb{E}\psi'(\varepsilon_1 a)$, (A.22) and $|\gamma_{x,a}(x)| \leq \|\psi'\|_{\infty}$, we obtain that for all $x \in \mathcal{C}$, $a \in \mathcal{I}_{\delta}$ and $y$ such that $\mathcal{K}(y) \neq 0$

$$|r(y, x)| \leq c_1 \sup_{u \in \mathcal{C}_0} p(u) f_x(u) \left|\sigma(H dy + x) - \sigma(x)\right| + \|\psi'\|_{\infty} \left|p(H dy + x) f_x(H dy + x) - f_x(p(x))\right|$$

$$\leq \left(c_1 \sup_{u \in \mathcal{C}_0} p(u) f_x(u) + \|\psi'\|_{\infty}\right) \varepsilon = c_2 \varepsilon.$$

Then, for $n \geq n_0$, we have that

$$\sup_{x \in \mathcal{C}} \sup_{a \in \mathcal{I}_{\delta}} \left|\mathbb{E}M_{1,jk}(x, a) - M_{jk}(x, a)\right| \leq c_2 \varepsilon \int y^{j+k-2} \mathcal{K}(y) \, dy = c_3 \varepsilon,$$

concluding the proof of (ii).

Note that from (i) and (ii) it follows that

$$\sup_{x \in \mathcal{C}} \sup_{a \in \mathcal{I}_{\delta}} \left|M_{1,jk}(x, a) - M_{jk}(x, a)\right| = O_{a.s.} \left(\sqrt{\frac{\log n}{n \prod_{j=1}^{d} h_j}}\right) + o(1) = o_{a.s.}(1), \quad (A.23)$$

which in particular leads to

$$\sup_{x \in \mathcal{C}} \left|M_{1,jk}(x, 1) - M_{jk}(x, 1)\right| = O_{a.s.} \left(\sqrt{\frac{\log n}{n \prod_{j=1}^{d} h_j}}\right) + o(1) = o_{a.s.}(1). \quad (A.24)$$

Let us prove (iii). By (8) and (A.23), given $\eta > 0$, there exists a set $\mathcal{N}$ such that $\mathbb{P}(\mathcal{N}) = 0$ and for any $\omega \notin \mathcal{N}$, there exists $n_1$ satisfying that for all $n \geq n_1$, $|\hat{a}_\sigma(x) - 1| < 1/2$ and $\sup_{x \in \mathcal{C}} \sup_{a \in \mathcal{I}_{\delta}} \left|M_{1,jk}(x, a) - M_{jk}(x, a)\right| < \eta$, with $\delta = 1/2$. Then, for all $\omega \notin \mathcal{N}$ and $n \geq n_1$ we get that

$$\sup_{x \in \mathcal{C}} \left|M_{1,jk}(x, \hat{a}_\sigma(x)) - M_{jk}(x, \hat{a}_\sigma(x))\right| \leq \sup_{x \in \mathcal{C}} \sup_{a \in \mathcal{I}_{\delta}} \left|M_{1,jk}(x, a) - M_{jk}(x, a)\right| < \eta,$$

which concludes the proof of (iii).

Let now prove (iv). Denote $c_4 = \sup_{x \in \mathcal{C}} f_x(x)p(x) \max(1, s_{1,1}^{(a)})$. Using that $f_x$ and $p$ are bounded in $\mathcal{C}$, we get that

$$\sup_{x \in \mathcal{C}} \left|M_{jk}(x, \hat{a}_\sigma(x)) - M_{jk}(x, 1)\right| \leq c_4 \sup_{x \in \mathcal{C}} \left|\mathbb{E}\psi'(\varepsilon\hat{a}_\sigma(x)) - \mathbb{E}\psi'(\varepsilon)\right|.$$ 

Using similar arguments to those considered to bound $\gamma_{x,a}$ in (ii), we obtain that

$$|\lambda_1(a) - \lambda_1(1)| = \left|\mathbb{E}\left[\psi'(\varepsilon a) - \psi'(\varepsilon)\right]\right| = \left|\mathbb{E}\psi''(\varepsilon\theta)(a - 1)\right| \leq \|\zeta_2\|_{\infty} \frac{1}{\theta}(a - 1),$$

where $\lambda_1(a) = \mathbb{E}\psi'(\varepsilon a)$ and $\theta$ is an intermediate point between $a$ and 1. Hence,

$$|\lambda_1(a) - \lambda_1(1)| \leq \|\zeta_2\|_{\infty} \left(a - 1\right) \max\left(1, \frac{1}{a}\right),$$

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which implies that
\[
\sup_{x \in C} |M_{jj}(x, \tilde{a}_\sigma(x)) - M_{jj}(x, 1)| \leq c_4 \| \zeta_2 \|_\infty \sup_{x \in C} |\tilde{a}_\sigma(x) - 1| \sup_{x \in C} \left( 1 + \frac{1}{a_\sigma(x)} \right).
\]

Now (iv) follows from the fact that \( \sup_{x \in C} |\tilde{a}_\sigma(x) - 1| \xrightarrow{a.s.} 0 \). That is, we have concluded the proof of (A.19).

By (A.17) and using that \( \psi' \) is bounded and that \( i(p) > 0 \), we obtain that
\[
\sup_{x \in C} \| \tilde{M}_{n2}(x) \| \leq \frac{\| \psi' \|_\infty}{i(t)} A_{\rho} \left( h_{\alpha}^{q+1} + \sum_{j \neq \alpha} h_j \right) \sup_{x \in C} \frac{1}{n} \sum_{i=1}^{n} |\mathcal{K}_{H_3}(X_i - x)|.
\]

Using that \( \tilde{f}(x) = \frac{1}{n} \sum_{j=1}^{n} \mathcal{K}_{H_3}^*(x - X_j) \) converges to \( f_X \) uniformly and \( A2 \), we obtain that \( \sup_{x \in C} \tilde{f}(x) = O_{n.s.}(1) \), so we obtain (A.20) and the proof is concluded. \( \square \)

### A.2 Proof of Theorem 4.1.

We begin by proving the following Lemma which will be useful in the proof of Theorem 4.1.

**Lemma A.2.1.** Assume that \( A0, A2, A7 \) and \( N1 \) to \( N5 \) hold and that the function \( \lambda(u) \) has bounded Lipschitz continuous derivatives up to order \( \ell - 1 \), in a neighbourhood of 0. Let \( x \) be an interior point of \( S_f \). Define
\[
\tilde{A}_{1,n}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}_{H_3}(X_i - x) p(X_i) \lambda \left( \frac{R(X_i, x)}{\sigma} \right) \tilde{x}_{i, \alpha},
\]
where \( R(X_i, x) = \sum_{j \neq \alpha} (g_j(X_{i,j}) - g_j(x_j)) + R_\alpha(X_{i,\alpha}, x_\alpha) \) and \( R_\alpha(X_{i,\alpha}, x_\alpha) \) is given in (A.1).

Then, we have that
\[
\mathbb{E} \tilde{A}_{1,n}(x) = \frac{A_0(\psi)}{\sigma} h_{\alpha}^{q+1} p(x) f_X(x) \frac{1}{(q + 1)!} g_{\alpha}^{(q+1)}(x_\alpha) s_\alpha^{(\alpha)} + \nu_n(x),
\]
where \( \sup_{x \in S_\psi} \| \nu_n(x) \| = h_{\alpha}^{q+1} o(1) \).

**Proof.** Using that \( \lambda \) is \( \ell - 1 \) times differentiable, a \( (\ell - 1) \)th order Taylor’s expansion together with the facts that \( \lambda(0) = 0, \lambda'(0) = \mathbb{E} \psi'(\varepsilon_1) = A_0(\psi) \) entail that
\[
\lambda \left( \frac{R(u, x)}{\sigma} \right) = \lambda(0) + \sum_{k=1}^{\ell-1} \frac{1}{k!} \lambda^{(k)}(0) \left( \frac{R(u, x)}{\sigma} \right)^k + \lambda^{(\ell-1)}(\theta(u)) - \lambda^{(\ell-1)}(0) \frac{R(u, x)}{\ell - 1} \left( \frac{R(u, x)}{\sigma} \right)^{\ell-1},
\]
where \( \tilde{\lambda}(u, x) = \lambda^{(\ell-1)}(\theta(u)) - \lambda^{(\ell-1)}(0) \) with \( \theta(u) \) an intermediate point between 0 and \( R(u, x)/\sigma \).

On the other hand, we also have that \( |\tilde{\lambda}(u, x)| \leq C|\theta(u)| \leq C|R(u, x)|/\sigma \) since \( \lambda^{(\ell-1)} \) is Lipschitz.
Note that $N^2$ and the fact that $h_j = \bar h$ if $j \neq \alpha$ and $R(x, x) = 0$ imply that

$$R(x + H_d u, x) = \sum_{j\neq\alpha} \sum_{s=1}^{\ell} \tilde h^s j_{\alpha}^{(s)}(x_j) u_j^s + \tilde h^\ell \sum_{j\neq\alpha} \frac{g_j(\xi_j)g_j^{(\ell)}(x_j)}{\ell!} u_j^\ell \, + \, h_{q+1} \frac{g_{q+1}(\alpha)}{(q + 1)!} u_{q+1} + h_{q+1} \frac{g_{q+1}(\alpha) - g_{q+1}(\alpha)}{(q + 1)!} u_{q+1}^2. \quad (A.25)$$

Then, we have that

$$\mathbb{E}\left( \bar A_{1,n}(x) \right) = \mathbb{E}\left[ K_{H_d}(X_1 - x) p(X_1) \lambda \left( \frac{R(X_1, x)}{\sigma} \right) \hat{x}_{1,\alpha} \right]$$

$$= \frac{A_0(\psi)}{\sigma^2} \mathbb{A}_{11,n} + \sum_{k=2}^{\ell-1} \frac{\lambda^{(k)}(0)}{k! \sigma^k} \mathbb{A}_{1k,n} + \frac{1}{(\ell - 1)! \sigma^{\ell-1}} \mathbb{A}_{1\ell,n}$$

where for $k = 1, \ldots, \ell - 1$

$$\mathbb{A}_{1k,n} = \frac{1}{h_\alpha h^{d-1}} \mathbb{E}\left[ \prod_{s=1}^{d} K_s \left( \frac{X_{1,s} - x_s}{h_s} \right) p(X_1) R^k(X_1, x) \hat{x}_{1,\alpha} \right]$$

$$= \int K(u) v(x + H_d u) R^k(x + H_d u, x) \hat{u}_\alpha du$$

$$\mathbb{A}_{1\ell,n} = \frac{1}{h_\alpha h^{d-1}} \mathbb{E}\left[ \prod_{s=1}^{d} K_s \left( \frac{X_{1,s} - x_s}{h_s} \right) p(X_1) \lambda(X_1, x) R^{\ell-1}(X_1, x) \hat{x}_{1,\alpha} \right]$$

$$= \int K(u) v(x + H_d u) \lambda(x + H_d u, x) R^\ell(x + H_d u, x) \hat{u}_\alpha du,$$

where $v(u) = p(u) f_X(u)$ and $\hat{u}_\alpha = (1, u_\alpha, \ldots, u_q^q)^T \in \mathbb{R}^{q+1}$. Using an $\ell$-th order Taylor’s expansion of $v(x + H_d u)$ around $x$, we get that

$$v(x + H_d u) = v(x) + \sum_{0 < |m| \leq \ell} D^m v(x) h^m u^m + \sum_{|m| = \ell} \frac{1}{m!} \left[ D^m v(\xi^{(1)}) - D^m v(x) \right] h^m u^m \quad (A.26)$$

where we have used the notation of Bourbaki for the expansion, $h = (h_1, \ldots, h_d)^T$, where $h_j = \bar h$ for $j \neq \alpha$, $m = (m_1, \ldots, m_d)$ with $m_i \in \mathbb{N}$, $|m| = \sum_{j=1}^{d} m_j$, $u^m = \prod_{j=1}^{d} u_j^{m_j}$, $m! = m_1! \ldots m_d!$ and $D^m v = \partial^{|m|} v / \partial u_1^{m_1} \ldots \partial u_d^{m_d}$.

Using (A.25) and (A.26) in $A_{1k,n}$, $k = 1, \ldots, \ell$, we obtain that $A_{11,n}$ can be written as $A_{11,n} = \sum_{j=1}^{18} A_{11,j,n}$ where

$$A_{11,1,n} = v(x) \left\{ \sum_{j\neq\alpha} g_j'(x_j) \bar h \int \prod_{t=1}^{d} K_t(u_t) u_j \hat{u}_\alpha du \right\}$$

$$A_{11,2,n} = v(x) \frac{1}{(q + 1)!} g_{q+1}(x_\alpha) h_{q+1}^{q+1} \int \prod_{t=1}^{d} K_t(u_t) u_{q+1}^{q+1} \hat{u}_\alpha du$$

$$A_{11,3,n} = \sum_{k=1}^{\ell-1} \sum_{|m| = k} \frac{1}{m!} D^m v(x) \sum_{j\neq\alpha} g_j'(x_j) \bar h \int \prod_{t=1}^{d} K_t(u_t) u_j^{m} \hat{u}_\alpha du$$

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\[ A_{11.4,n} = \sum_{k=1}^{\ell-1} \sum_{|m|=k} \frac{1}{m!} D^m v(x) \frac{1}{(q+1)!} g_{\alpha}^{(q+1)}(x_\alpha) h_{\alpha}^{q+1} h^{m} \int \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.5,n} = \sum_{|m|=\ell} \frac{1}{m!} D^m v(x) \sum_{j \neq \alpha} \frac{1}{h^{(q+1)}} g_{\alpha}(x_\alpha) h_{\alpha}^{q+1} h^{m} \int \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.6,n} = \sum_{|m|=\ell} \frac{1}{m!} D^m v(x) \frac{1}{(q+1)!} g_{\alpha}^{(q+1)}(x_\alpha) h_{\alpha}^{q+1} h^{m} \int \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.7,n} = \sum_{|m|=\ell} \sum_{j \neq \alpha} g_j(x_j) \frac{1}{m!} h^{m} \int \left[ D^m v(\xi^{(1)}) - D^m v(x) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.8,n} = \sum_{|m|=\ell} \frac{1}{m!} \frac{1}{(q+1)!} g_{\alpha}(x_\alpha) h_{\alpha}^{q+1} h^{m} \int \left[ D^m v(\xi^{(1)}) - D^m v(x) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.9,n} = v(x) \left\{ \sum_{j \neq \alpha} \sum_{m=2}^{\ell-1} \frac{1}{m!} g_j^{(m)}(x_j) h^{m} \int \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \right\} \]

\[ A_{11.10,n} = v(x) \left\{ \sum_{j \neq \alpha} \frac{1}{\ell!} g_j^{(\ell)}(x_j) h^{\ell} \int \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \right\} \]

\[ A_{11.11,n} = v(x) \left\{ \frac{1}{(q+1)!} h_{\alpha}^{q+1} h^{m} \int \left[ g_{\alpha}^{(\ell)}(\xi \alpha) - g_{\alpha}(x_\alpha) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \right\} \]

\[ A_{11.12,n} = \sum_{k=1}^{\ell-1} \sum_{|m|=k} \frac{1}{m!} D^m v(x) \sum_{j \neq \alpha} \sum_{m=2}^{\ell-1} \frac{1}{m!} g_j^{(m)}(x_j) h^{m} \int \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.13,n} = \sum_{k=1}^{\ell-1} \sum_{|m|=k} \frac{1}{m!} D^m v(x) \sum_{j \neq \alpha} h^{m} \int \left[ g_j^{(\ell)}(\xi_\alpha) - g_j^{(\ell)}(x_j) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.14,n} = \sum_{|m|=\ell} \frac{1}{m!} h^{m} \int D^m v(\xi^{(1)}) \left[ g_j^{(\ell)}(\xi_\alpha) - g_j^{(\ell)}(x_j) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.15,n} = \sum_{k=1}^{\ell-1} \sum_{|m|=k} \frac{1}{m!} D^m v(x) \frac{1}{(q+1)!} h_{\alpha}^{q+1} h^{m} \int \left[ g_{\alpha}^{(q+1)}(\xi_\alpha) - g_{\alpha}(x_\alpha) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.16,n} = \sum_{|m|=\ell} \frac{1}{m!} \frac{1}{(q+1)!} h_{\alpha}^{q+1} h^{m} \int D^m v(\xi^{(1)}) \left[ g_{\alpha}^{(q+1)}(\xi_\alpha) - g_{\alpha}(x_\alpha) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.17,n} = \sum_{|m|=\ell} \frac{1}{m!} D^m v(x) \sum_{j \neq \alpha} \sum_{m=2}^{\ell-1} \frac{1}{m!} g_j^{(m)}(x_j) h^{m} \int \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du \]

\[ A_{11.18,n} = \sum_{|m|=\ell} \sum_{j \neq \alpha} \sum_{m=2}^{\ell} \frac{1}{m!} g_j^{(m)}(x_j) h^{m} \int \left[ D^m v(\xi^{(1)}) - D^m v(x) \right] \prod_{t=1}^{d} K_t(u_t) u_\alpha^{q+1} u_m u_\alpha^{m} \, du , \]
with \( \xi^{(1)} \) an intermediate point between \( \mathbf{x} \) and \( \mathbf{x} + \mathbf{H}_d \mathbf{u} \). The fact that \( \int K_j(t) t \; dt = 0 \) entails that \( A_{11,1,n} = 0 \). On the other hand, using that \( K_j = L \) is a kernel of order \( \ell \), if \( j \neq \alpha \), we get that 
\[
\int K(u) u_j u^m \hat{u}_\alpha \; du = 0 \text{ for } |m| \leq \ell - 1.
\]
Moreover, we also have that \( \int K(u) u_j u^m \hat{u}_\alpha \; du = 0 \) if \( |m| = \ell - 1 \) and \( m \neq (\ell - 1) e_j \). On the other hand, using again that \( L \) is a kernel of order \( \ell \), we obtain that 
\[
\int K(u) u_j u^m \hat{u}_\alpha \; du = 0 \text{ for all } m \text{ with at least one component } m_j \neq 0, \text{ for } j \neq \alpha.
\]
Thus, \( A_{11,9,n} = 0 \).

On the other hand, we have that
\[
A_{11,2,n} = h_\alpha^{q+1} v(\mathbf{x}) \frac{1}{(q + 1)!} g_\alpha^{(q+1)}(x_\alpha) \int K_\alpha(u_\alpha) u^\alpha \hat{u}_\alpha \; du_\alpha
\]
\[
= h_\alpha^{q+1} v(\mathbf{x}) \frac{1}{(q + 1)!} g_\alpha^{(q+1)}(x_\alpha) \partial_{\hat{u}_\alpha}^q.
\]
Since \( D^m v(\mathbf{u}) \) with \( |m| = k \) and \( g_\alpha^{(q+1)} \) are continuous and bounded functions and \( g_j \) is \( \ell \) times differentiable, with bounded derivatives for all \( j \neq \alpha \), it follows that
\[
\sup_{x \in S_Q} \| A_{11,3,n} \| = \tilde{h}^\ell O(1), \quad \sup_{x \in S_Q} \| A_{11,4,n} \| = h_\alpha^{q+2} O(1) \quad \text{and} \quad \sup_{x \in S_Q} \| A_{11,10,n} \| = \tilde{h}^\ell O(1).
\]

Similarly, using that the kernels are even we have that \( \int \prod_{j=1}^d L_{\alpha j}(u_j) u^m \hat{u}_\alpha \; du = 0 \) when \( m \) has a component different from \( \alpha \) and \( j \) different from 0. Moreover, when \( m_s = 0 \) for \( s \neq j, \alpha \), using that \( L \) is a kernel of order \( \ell \) we get that the integral equals 0 except when \( m_j \neq \ell - 1 \) and \( m_\alpha = 1 \). Arguing similarly with \( \int \prod_{j=1}^d L_{\alpha j}(u_j) u^m \hat{u}_\alpha \; du \), we get that
\[
A_{11,5,n} = \sum_{j \neq \alpha} g_j(x_j) \frac{1}{(\ell - 1)!} \frac{\partial^\ell v(\mathbf{u})}{\partial u_j^{\ell-1} \partial u_\alpha} \bigg|_{u=\mathbf{x}} \tilde{h}^\ell \int K(u) u_j^\alpha \hat{u}_\alpha \; du
\]
\[
A_{11,6,n} = \frac{1}{(q + 1)!} g_\alpha^{(q+1)}(x_\alpha) h_\alpha^{q+1} \sum_{j \neq \alpha} \frac{1}{\ell!} \frac{\partial^\ell v(\mathbf{u})}{\partial u_j^\ell} \bigg|_{u=\mathbf{x}} \bigg( h_\alpha^\ell \int K(u) u_j^\alpha \hat{u}_\alpha \; du + \frac{1}{\ell!} g_\alpha^{(q+1)}(x_\alpha) h_\alpha^{q+1} \frac{1}{\ell!} \frac{\partial^\ell v(\mathbf{u})}{\partial u_\alpha^\ell} \bigg|_{u=\mathbf{x}} \bigg) \tilde{h}^\ell \int K(u) u_j^\alpha \hat{u}_\alpha \; du,
\]
which implies that
\[
\sup_{x \in S_Q} \| A_{11,5,n} \| = \tilde{h}^\ell h_\alpha O(1) \quad \sup_{x \in S_Q} \| A_{11,6,n} \| = h_\alpha^{q+1} (\tilde{h}^\ell + h_\alpha^\ell) O(1).
\]

On the other hand, \( D^m v(\mathbf{u}) \) for \( |m| = k \leq \ell \) is uniformly continuous, so using that \( K_\alpha \) and \( L \) have compact support in \([-1, 1]\) and that \( \xi^{(1)} \) is an intermediate point between \( \mathbf{x} \) and \( \mathbf{x} + \mathbf{H}_d \mathbf{u} \), we have that
\[
\sup_{x \in S_Q} \left| D^m v(\xi^{(1)}) - D^m v(\mathbf{x}) \right| = o(1)
\]
which leads to
\[
\sup_{x \in S_Q} \| A_{11,7,n} \| = \tilde{h}^\ell o(1), \quad \sup_{x \in S_Q} \| A_{11,8,n} \| = h_\alpha^{q+1} o(1) \quad \text{and} \quad \sup_{x \in S_Q} \| A_{11,18,n} \| = \tilde{h}^\ell O(1).
\]
Similarly, using that $g_j^{(\ell)}$ is uniformly continuous and bounded, we get that $\sup_{x \in S_Q} |g_j^{(\ell)}(x_j) - g_j^{(\ell)}(x_j)| = O(1)$ which implies that

$$\sup_{x \in S_Q} \| A_{11,1,n} \| = h_{\alpha}^{q+1}o(1) \quad \sup_{x \in S_Q} \| A_{11,12,n} \| = h^{\ell}O(1)$$

$$\sup_{x \in S_Q} \| A_{11,13,n} \| = \tilde{h}^{\ell}(h + h_{\alpha})o(1) \quad \sup_{x \in S_Q} \| A_{11,14,n} \| = h^{\ell}o(1)$$

$$\sup_{x \in S_Q} \| A_{11,15,n} \| = h_{\alpha}^{q+2}o(1) \quad \sup_{x \in S_Q} \| A_{11,16,n} \| = h_{\alpha}^{q+1}o(1)$$

$$\sup_{x \in S_Q} \| A_{11,17,n} \| = \tilde{h}^{\ell}O(1) \quad \sup_{x \in S_Q} \| A_{11,18,n} \| = h^{\ell}o(1).$$

Using (A.25) and the fact that, for $j \neq \alpha$, $K_j = L$ is a kernel of order $\ell$ and that $\tilde{h}^{\ell} = o(h_{\alpha}^{q+1})$, using analogous arguments, we obtain that for all $k = 2, \ldots, \ell - 1$

$$\sup_{x \in S_Q} \| A_{1k,n} \| = h_{\alpha}^{q+1}o(1).$$

Let $A_{1\ell,n,s}$ indicate the $s$th coordinate of $A_{1\ell,n}$. Using that $|\tilde{\lambda}(u,x)| \leq C|R(u,x)|/\sigma$, $K_j$ has support in $[-1,1]$, $v$ is bounded and (A.25), we get that, for $s = 1, \ldots, q + 1$,

$$\sup_{x \in S_Q} |A_{1\ell,n,s}| \leq \sup_{x \in S_Q} \int |K(u)||v(x + H_d u)||\tilde{\lambda}(x + H_d u, x)||R(x + H_d u, x)|^{\ell-1}|u_\alpha|^{s-1} \, du$$

$$\leq \frac{C}{\sigma} \int |K(u)||v(x + H_d u)||R(x + H_d u, x)|^{\ell} \, du \leq c_2(\tilde{h} + h_{\alpha}^{q+1})^{\ell} = o\left(h_{\alpha}^{q+1}\right),$$

thus,

$$\sup_{x \in S_Q} \| A_{1\ell,n} \| = h_{\alpha}^{q+1}o(1).$$

Hence, using that $h_{\alpha} \to 0$ and $\tilde{h} \to 0$, we get that

$$\mathcal{E} \tilde{A}_{1,n}(x) = \frac{A_0(\psi)}{\sigma} A_{11,n} + \sum_{k=2}^{\ell-1} \frac{\lambda^{(k)}(0)}{k! \sigma^k} A_{1k,n} + \frac{1}{(\ell - 1)!} \tau^{\ell - 1} A_{1\ell,n}$$

$$= \frac{A_0(\psi)}{\sigma} h_{\alpha}^{q+1} v(x) \frac{1}{(q + 1)!} g_{\alpha}^{(q+1)}(x_\alpha) s_q^{(\alpha)} + \nu_n(x),$$

where $\sup_{x \in S_Q} \| \nu_n(x) \| = \tilde{h}^{\ell}O(1) + h_{\alpha}^{q+1}o(1) = h_{\alpha}^{q+1}o(1)$ and the proof is concluded. $\square$

**Proof of Theorem 4.1.** The proof will be carried out in several steps. In a first step, we will show that it is enough to assume that, since the scale estimator has a root $-n$ rate of convergence, it is enough to prove the result in the situation in which scale is known to obtain the conclusion of Theorem 4.1. In a second step, we obtain an expansion for the estimator computed when scale is known into two terms. The first one will converge to the asymptotic bias and the second one to a centered normal distribution from which the conclusion follows. To obtain these two last results some intermediate approximations will be needed.
**Step 1.** For any $s > 0$, define $\Psi_{n,\alpha}^*(\beta, x, s) = (\Psi_{n,\alpha,0}^*(\beta, x, s), \ldots, \Psi_{n,\alpha,q}^*(\beta, x, s))$ where

$$
\Psi_{n,\alpha}^*(b, x, s) = \frac{1}{n} \sum_{i=1}^n \psi \left( \frac{Y_i - b_0 - \sum_{m=1}^q b_m (X_i, \alpha - x, \alpha)^m}{s} \right) \mathcal{K}_{H_d}(X_i - x) \delta_i \tilde{x}_{i,\alpha} \\
= \frac{1}{n} \sum_{i=1}^n \psi \left( \frac{Y_i - \tilde{x}_{i,\alpha}^T H_d b}{s} \right) \mathcal{K}_{H_d}(X_i - x) \delta_i \tilde{x}_{i,\alpha}.
$$

Using that $\sqrt{n}(\bar{s} - \sigma) = O_P(1)$, $\psi$ is Lipschitz and $\zeta(u) = u \psi'(u)$ is bounded, it is easy to see that for $j = 0, \ldots, q$, $\hat{D}_{n,j} = \sup_{x \in S_Q} \sup_{b} |\bar{s} \Psi_{n,\alpha,j}^*(b, x, \bar{s}) - \sigma \Psi_{n,\alpha,j}^*(b, x, \sigma)| = O_P(1/\sqrt{n})$. On the other hand, $\beta(x)$ is a solution of (6) with $\bar{s}(x) = \bar{s}$, that is, $\Psi_{n,\alpha}^*(\beta(x), x, \bar{s}) = 0_{q+1}$, which implies that

$$
\Psi_{n,\alpha}^*(\beta(x), x, \sigma) = O_P(1/\sqrt{n}) \quad (A.27)
$$

Denote as $\hat{\beta}(x)$ the solution of $\Psi_{n,\alpha}^*(b, x, \sigma) = 0_{q+1}$. Then, Proposition 3.1 entails that $\sup_{x \in S_Q} \left\| H^{(\alpha)} \left[ \hat{\beta}(x) - \beta(x) \right] \right\| = o_P(1)$, so using that $\sup_{x \in S_Q} \left\| H^{(\alpha)} \left[ \beta(x) - \beta(x) \right] \right\| = o_P(1)$, we get that $\hat{D}_n = \sup_{x \in S_Q} \left\| H^{(\alpha)} \left[ \hat{\beta}(x) - \beta(x) \right] \right\| = o_P(1)$. We will further show that

$$
\hat{D}_n = O_P(1/\sqrt{n}). \quad (A.28)
$$

To prove (A.28), denote as

$$
\hat{D}_{1,n}(x, \xi) = -\frac{1}{\sigma n} \sum_{i=1}^n \psi' \left( \frac{Y_i - \tilde{x}_{i,\alpha}^T H^{(\alpha)} \xi}{\sigma} \right) \mathcal{K}_{H_d}(X_i - x) \delta_i \tilde{x}_{i,\alpha} \tilde{x}_{i,\alpha}^T
$$

Then, a first order Taylor expansion and the fact that $\Psi_{n,\alpha}^*(\hat{\beta}(x), x, \sigma) = 0$ lead us to

$$
\Psi_{n,\alpha}^*(\hat{\beta}(x), x, \sigma) = \Psi_{n,\alpha}^*(\beta(x), x, \sigma) + \hat{D}_{1,n}(x) H^{(\alpha)}(\hat{\beta}(x) - \beta(x)) = \hat{D}_{1,n}(x, \xi_n) H^{(\alpha)}(\hat{\beta}(x) - \beta(x)), \quad (A.29)
$$

where $\xi_n = \xi_n(x)$ stands for an intermediate point between $\hat{\beta}(x)$ and $\beta(x)$, so $\sup_{x \in S_Q} \| H^{(\alpha)}[\xi_n(x) - \beta(x)]\| = o_P(1)$. Denote as $A_0(\psi) = E(\psi'(\varepsilon))$ and

$$
D_0(x) = -\frac{1}{\sigma} A_0(\psi)p(x) f_x(x) S^{(\alpha)}.
$$

Then, using that from N3b) $S^{(\alpha)}$ is non-singular, $\inf_{x \in C} f_x(x) > 0, \inf_{x \in C} p(x) > 0$ and $A_0(\psi) \neq 0$ we get that $\inf_{x \in C} \nu_1(D_0(x)) > 0$, with $\nu_1(A)$ the smallest eigenvalue of the matrix $A$. Hence, (A.27) and (A.29) implies that to show (A.28) it is enough to see that

$$
\sup_{x \in S_Q} \| \hat{D}_{1,n}(x, \xi_n) - D_0(x) \| = o_P(1). \quad (A.30)
$$

We get that $\sigma \hat{D}_{1,n}(x, \xi_n) = \hat{D}_{11,n}(x, \xi_n) + \hat{D}_{12,n}(x) + \hat{D}_{13,n}(x)$ where

$$
\hat{D}_{11,n}(x, \xi_n) = \frac{1}{n} \sum_{i=1}^n \mathcal{K}_{H_d}(X_i - x) \delta_i \left[ \psi' \left( \frac{Y_i - \tilde{x}_{i,\alpha}^T H^{(\alpha)} \beta(x)}{\sigma} \right) - \psi' \left( \frac{Y_i - \tilde{x}_{i,\alpha}^T H^{(\alpha)} \xi_n}{\sigma} \right) \right] \tilde{x}_{i,\alpha} \tilde{x}_{i,\alpha}^T
$$

$$
\hat{D}_{12,n}(x) = \frac{1}{n} \sum_{i=1}^n \mathcal{K}_{H_d}(X_i - x) \delta_i \left[ \psi' \left( \frac{Y_i - \tilde{x}_{i,\alpha}^T H^{(\alpha)} \beta(x)}{\sigma} \right) - \psi' \left( \frac{Y_i - \tilde{x}_{i,\alpha}^T H^{(\alpha)} \beta(x)}{\sigma} \right) \right] \tilde{x}_{i,\alpha} \tilde{x}_{i,\alpha}^T
$$

$$
\hat{D}_{13,n}(x) = -\frac{1}{n} \sum_{i=1}^n \mathcal{K}_{H_d}(X_i - x) \delta_i \psi \left( \frac{Y_i - \tilde{x}_{i,\alpha}^T H^{(\alpha)} \beta(x)}{\sigma} \right) \tilde{x}_{i,\alpha} \tilde{x}_{i,\alpha}^T.
$$
Using that \(\psi\) is Lipschitz, \(\sup_{x \in S_Q} \|H^{(a)}[\xi_n(x) - \beta(x)]\| = o_P(1), \sup_{x \in S} \sum_{i=1}^n |\mathcal{K}_{H,q}(X_i - x)|/n = O_P(1), \sup_{x \in S} |\mathcal{K}_{H,1}(x_i - x)| \neq 0 |x_i, \alpha| \leq 1\) we obtain that, for \(1 \leq j, m \leq q + 1, \sup_{x \in S_Q} |\tilde{D}_{11,n,j,m}(x, \xi_n)| = o_P(1)\), where \(\tilde{D}_{11,n,j,m}(x, \xi_n)\) is the \((j, m)\)-th element of matrix \(\tilde{D}_{11,n}(x, \xi_n)\).

On the other hand, from the bound \(\sup_{x \in S} |\mathcal{K}_{H,q}(x_i - x)| = C(h + h_{a}^{q+1})\), for \(1 \leq j, m \leq q + 1\) we get that \(\sup_{x \in S_Q} |\tilde{D}_{12,n,j,m}(x)| = o_P(1)\).

Finally, Lemma A.1.2 entails that, for \(1 \leq j, m \leq q + 1, \sup_{x \in S_Q} |\tilde{D}_{13,n,j,m}(x) - \tilde{E}\tilde{D}_{13,n,j,m}(x)| = o_P(1)\), while standard arguments allow to show that \(\sup_{x \in S_Q} |\tilde{E}\tilde{D}_{13,n,j,m}(x) - D_{0,j,m}(x)| = o_P(1)\), concluding the proof of (A.30) and so that of (A.28).

Observe that since the first element of the diagonal matrix \(H^{(a)}\) equals 1, we have that

\[
\tilde{g}_{\alpha,m,q,a}(x_\alpha) = \int e_1^T \tilde{\beta}_{\alpha}(x, u_\alpha) q_{\alpha}(u_\alpha) d\mathbf{u}_{\alpha} = \int e_1^T \beta^{(a)}(x, u_\alpha) q_{\alpha}(u_\alpha) d\mathbf{u}_{\alpha}.
\]

On the other hand, \(\beta(x) = (g(x), g_1^{(1)}(x), \ldots, g_q^{(q)}(x))\), so using (2) we get that

\[
\int e_1^T \beta^{(a)}(x) q_{\alpha}(x_\alpha) d\mathbf{x}_{\alpha} = \int g(x) q_{\alpha}(x_\alpha) d\mathbf{x}_{\alpha} = g_\alpha(x_\alpha),
\]

which implies that

\[
\left| \sqrt{n h_\alpha} (\tilde{g}_{\alpha,m,q,a}(x_\alpha) - g_\alpha(x_\alpha)) - \sqrt{n h_\alpha} \int e_1^T H^{(a)}[\tilde{\beta}(x) - \beta(x)] q_{\alpha}(x_\alpha) d\mathbf{x}_{\alpha} \right| \\
= \left| \sqrt{n h_\alpha} \int e_1^T H^{(a)}[\tilde{\beta}(x) - \beta(x)] q_{\alpha}(x_\alpha) d\mathbf{x}_{\alpha} \right| \leq \sqrt{h_\alpha} \sqrt{n} \tilde{D}_n \overset{p}{\rightarrow} 0. \tag{A.31}
\]

Let us denote as \(\tilde{g}_{\alpha}(x_\alpha) = \int e_1^T H^{(a)}[\tilde{\beta}(x) q_{\alpha}(x_\alpha) d\mathbf{x}_{\alpha}.\) Then, (A.31) implies that to obtain the asymptotic distribution of \(\sqrt{n h_\alpha} (\tilde{g}_{\alpha,m,q,a}(x_\alpha) - g_\alpha(x_\alpha))\) it is enough to derive that of

\[
\sqrt{n h_\alpha} (\tilde{g}_{\alpha}(x_\alpha) - g_\alpha(x_\alpha)) = \sqrt{n h_\alpha} \int e_1^T H^{(a)}[\tilde{\beta}(x) - \beta(x)] q_{\alpha}(x_\alpha) d\mathbf{x}_{\alpha},
\]

that is, we have reduced the problem to obtain the conclusion of Theorem 4.1, when the scale is known.

**Step 2.** Using that \(\Psi_{\alpha,m,q,a}(\tilde{\beta}(x), x, \sigma) = 0_{q+1}\) and a first order Taylor’s expansion of \(\Psi_{\alpha,m,q,a}(\beta, x, \sigma)\) around \(\beta(x)\), it is easy to see that

\[
H^{(a)}[\tilde{\beta}(x) - \beta(x)] = \sigma \tilde{A}_{\alpha,0,n}(x) \tilde{A}_{\alpha,1,n}(x). \tag{A.32}
\]
where \( \tilde{A}_{0,n}(x) = \tilde{A}_{0,1,n}(x) + \tilde{A}_{0,2,n}(x) \) with

\[
\tilde{A}_{0,1,n}(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_i \mathcal{K}_{H_d}(X_i - x) \psi'(Y_i - \tilde{x}_{i,a}^T H^{(\alpha)}(x) \beta(x) - \frac{x_i}{\sigma}) \tilde{x}_{i,a}^T \tilde{x}_{i,a} \n
= \frac{1}{n} \sum_{i=1}^{n} \delta_i \mathcal{K}_{H_d}(X_i - x) \psi'(\varepsilon_i + \frac{R(X_i, x)}{\sigma}) \tilde{x}_{i,a}^T \tilde{x}_{i,a} \n
\tilde{A}_{0,2,n}(x) = -\frac{1}{2n} \sum_{i=1}^{n} \delta_i \mathcal{K}_{H_d}(X_i - x) \psi''(Y_i - \tilde{x}_{i,a}^T \beta(x) - \frac{x_i}{\sigma}) \tilde{x}_{i,a}^T (\tilde{x}_{i,a}^T H^{(\alpha)}(\tilde{\beta}(x) - \beta(x))) \tilde{x}_{i,a} \n
\tilde{A}_{1,n}(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_i \mathcal{K}_{H_d}(X_i - x) \psi(Y_i - \tilde{x}_{i,a}^T H^{(\alpha)}(x) \beta(x) - \frac{x_i}{\sigma}) \tilde{x}_{i,a} \n
\]

where \( \tilde{\theta}(x) \) is a midpoint between \( H^{(\alpha)}(x) \beta(x) \) and \( H^{(\alpha)} \tilde{\beta}(x) \). Denote as \( v(u) = p(u)f_X(u) \) and \( A_0(u) = v(u)A_0(\psi)S^{(\alpha)} \). Lemma A.1.2 allow to show that \( \sup_{x \in S_Q} |\tilde{A}_{0,1,n}(x) - A_0(x)| = o_P(1) \).

On the other hand, the fact that \( \psi'' \) is bounded, \( \sup_{x \in S_Q} \| H^{(\alpha)}(\tilde{\beta}(x) - \beta(x)) \| = o_P(1) \) and that each component of \( \tilde{x}_{i,a} \) is smaller or equal to 1 when \( \mathcal{K}_{H_d}(X_i - x) \neq 0 \), imply that \( \sup_{x \in S_Q} |\tilde{A}_{0,2,n}(x)| = o_P(1) \), so \( \sup_{x \in S_Q} |\tilde{A}_{0,0}(n) - A_0(x)| = o_P(1) \).

In **Step 2.1**, we study the asymptotic behaviour of

\[
\hat{B}_n = \sigma \sqrt{n h_\alpha} \int \tilde{A}_{0,1,n}(x) q_2(x, 2) \, dx \n
\]

and we show that

\[
\hat{B}_n \xrightarrow{D} N_{q+1}(b_{q,a}(x_a), \Sigma_{q,a}(x_a)) \quad (A.33) \n
\]

\[
b_{q,a}(x_a) = \frac{\beta^{q+1}}{(q + 1)!} g_0^{(q+1)}(x_a)(S^{(\alpha)})^{-1} s_0^{(\alpha)} \quad (A.34) \n
\]

\[
\Sigma_{q,a}(x_a) = \sigma^2 \frac{\varepsilon \psi'(\varepsilon)}{A_0^2(\psi)} \int \frac{g_0^2(x_a)}{f_X(x_a, x_2)p(x_a, x_2)} \, dx_2 (S^{(\alpha)})^{-1} V_{\alpha}(S^{(\alpha)})^{-1}. \quad (A.35) \n
\]

We will then show, in **Step 2.2**, that

\[
\sqrt{n h_\alpha} [\tilde{g}_a(x_a) - g_a(x_a)] - e_1^T \hat{B}_n = \sigma \sqrt{n h_\alpha} \int e_1^T (\tilde{A}_{0,n}^{-1}(x) - A_0^{-1}(x)) \tilde{A}_{0,1,n}(x) q_2(x, 2) \, dx_2 = o_P(1), \n
\]

which together with (A.33) concludes the proof.

**Step 2.1.** Recall that \( Y_i - \tilde{x}_{i,a} H^{(\alpha)}(x) \beta(x) = \sigma \varepsilon_i + R(X_i, x) \), so that

\[
\mathbb{E} \left\{ \psi \left( \frac{Y_i - \tilde{x}_{i,a} H^{(\alpha)}(x) \beta(x)}{\sigma} \right) | X_i \right\} = \lambda \left( \frac{R(X_i, x)}{\sigma} \right). \quad (A.36) \n
\]

Define \( \tilde{A}_{1,n}(x) \) as in Lemma A.2.1, i.e.,

\[
\tilde{A}_{1,n}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}_{H_d}(X_i - x)p(X_i)\lambda \left( \frac{R(X_i, x)}{\sigma} \right) \tilde{x}_{i,a}. \]
and note that (A.36) entails that $\mathbb{E}\tilde{A}_{1,n}(x) = \mathbb{E}\tilde{A}_{1,n}(x)$. Moreover, we have that $\tilde{B}_n = \tilde{B}_{n,1} + \tilde{B}_{n,2}$ where

\[
\begin{align*}
\tilde{B}_{n,1} &= \sigma \sqrt{nh_\alpha} \int A_0^{-1}(x) \tilde{A}_{1,n}(x) q_\alpha(x_\alpha) \, dx_\alpha, \\
\tilde{B}_{n,2} &= \sigma \sqrt{nh_\alpha} \int A_0^{-1}(x) \left[ \tilde{A}_{1,n}(x) - \tilde{A}_{1,n}(x) \right] q_\alpha(x_\alpha) \, dx_\alpha.
\end{align*}
\]

Then, to derive (A.33), we have to show that

a) $\tilde{B}_{n,1} \xrightarrow{p} b_{q,\alpha}(x_\alpha)$ with $b_{q,\alpha}(x_\alpha)$ given in (A.34).

b) $\tilde{B}_{n,2} \xrightarrow{D} N_{q+1}(0, \Sigma_{q,\alpha}(x_\alpha))$ where $\Sigma_{q,\alpha}(x_\alpha)$ is defined in (A.35).

a) To show that $\tilde{B}_{n,1} \xrightarrow{p} b_{q,\alpha}$, it is enough to see that $\mathbb{E}\tilde{B}_{n,1} \rightarrow b_{q,\alpha}$ and that for all $1 \leq j \leq q + 1$, $\text{VAR}(\tilde{B}_{n,1,j}) \rightarrow 0$.

Lemma A.2.1 together with the fact that $A_0(u) = v(u)A_0(\psi)S^{(a)}$ and $\sqrt{nh_\alpha} h_\alpha^{q+1} = \beta(2q+3)^2$ entail that $\mathbb{E}\tilde{B}_{n,1} = \sigma \sqrt{nh_\alpha} \int A_0(x)^{-1} \mathbb{E}A_{1,n}(x)q_\alpha(x_\alpha) \, dx_\alpha = B_{11,n} + B_{12,n}$, where

\[
\begin{align*}
B_{11,n} &= \beta(2q+3)/2 \frac{1}{(q+1)!} g_\alpha^{(q+1)}(x_\alpha)(S^{(a)})^{-1} s^{(a)}_q, \\
B_{12,n} &= \sqrt{nh_\alpha} \int A_0^{-1}(x) \mathbb{E}\{\nu_n(x)\} q_\alpha(x_\alpha) \, dx_\alpha.
\end{align*}
\]

Hence, $\mathbb{E}\tilde{B}_{n,1} \rightarrow b_{q,\alpha}$, since $\sup_{x \in S_Q} \|\nu_n(x)\| = h_\alpha^{q+1} o(1)$ and $\sqrt{nh_\alpha} h_\alpha^{q+1} = \beta(2q+3)^2$ entail that $B_{12,n} \rightarrow 0$.

We will now show that $\text{VAR}(\tilde{B}_{n,1,j}) \rightarrow 0$, for $1 \leq j \leq q + 1$. denote as $\tilde{B} = A_0(\psi)S^{(a)}\tilde{B}_{n,1}/\sigma$ and $\tilde{B}_j$ its $j$-th component. Then, it will be enough to show that the variance of $\tilde{B}_j$ converges to 0. Note that $A_0(u) = v(u)A_0(\psi)S^{(a)}$ implies that

\[
\tilde{B}_j = \sqrt{nh_\alpha} \int \frac{1}{v(x)} \tilde{A}_{1,n,j}(x) q_\alpha(x_\alpha) \, dx_\alpha,
\]

\[
= \frac{1}{\sqrt{nh_\alpha}} \sum_{i=1}^n K_\alpha \left( \frac{X_{1,i} - x_\alpha}{h_\alpha} \right) p(X_i) \zeta(H_{d-1}, X_i, x_\alpha) \left( \frac{X_{1,i} - x_\alpha}{h_\alpha} \right)^{j-1}
\]

with

\[
\zeta(H_{d-1}, X_i, x_\alpha) = \frac{1}{h_{d-1}} \int \prod_{s \neq \alpha} K_s \left( \frac{X_{i,s} - x_s}{h_s} \right) q_\alpha(x_\alpha) \lambda \left( \frac{R(X_i, x)}{\sigma} \right) \, dx_\alpha
\]

\[
= \int \prod_{s \neq \alpha} K_s(u_s) q_\alpha(x_{i\alpha} + H_{d-1}u_\alpha) \lambda \left( \frac{R(X_i, (x_\alpha, X_{i\alpha} + H_{d-1}u_\alpha))}{\sigma} \right) \, du_\alpha
\]

where $H_{d-1} = \text{diag}(\vec{h}, \ldots, \vec{h}) \in \mathbb{R}^{(d-1) \times (d-1)}$. Thus, using that $\psi$ is bounded, $q_\alpha$ is continuous and bounded, we get that $|\zeta(H_{d-1}, X_i, x_\alpha)| \leq C$, for all $i$. Since $p \leq 1$ and $|X_{1,i} - x_\alpha| \leq h_\alpha$ if
\( K_{\alpha} ( (X_{1,\alpha} - x_{\alpha})/h_{\alpha} ) \neq 0 \), we conclude that

\[
\text{VAR}(\tilde{B}_j) = \frac{1}{h_{\alpha}} \text{VAR} \left( K_{\alpha} \left( \frac{X_{1,\alpha} - x_{\alpha}}{h_{\alpha}} \right) p(X_i) \zeta(\mathbf{H}_{d-1}, \mathbf{X}_i, x_{\alpha}) \left( \frac{X_{1,\alpha} - x_{\alpha}}{h_{\alpha}} \right)^{j-1} \right) 
\leq \frac{1}{h_{\alpha}} \mathbb{E} \left[ K_{\alpha}^2 \left( \frac{X_{1,\alpha} - x_{\alpha}}{h_{\alpha}} \right) p^2(X_i) \zeta^2(\mathbf{H}_{d-1}, \mathbf{X}_i, x_{\alpha}) \left( \frac{X_{1,\alpha} - x_{\alpha}}{h_{\alpha}} \right)^{2(j-1)} \right] 
\leq \frac{1}{h_{\alpha}} \int K_{\alpha}^2 \left( \frac{v_{\alpha} - x_{\alpha}}{h_{\alpha}} \right) \zeta^2(\mathbf{H}_{d-1}, \mathbf{v}, x_{\alpha}) f_{\mathbf{X}}(\mathbf{v}) \, d\mathbf{v} 
\leq \int K_{\alpha}^2 (u_{\alpha}) \zeta^2(\mathbf{H}_{d-1}, (x_{\alpha} + h_{\alpha} u_{\alpha}, v_{\alpha}), x_{\alpha}) f_{\mathbf{X}}(x_{\alpha} + h_{\alpha} u_{\alpha}, v_{\alpha}) \, du_{\alpha} \, dv_{\alpha}.
\]

Then, from the dominated convergence theorem it follows that \( \text{VAR}(\tilde{B}_j) \to 0 \) since \( \mathbf{H}_d \to 0 \) when \( n \to \infty \) and \( \zeta^2(0_{d-1}, (x_{\alpha}, v_{\alpha}), x_{\alpha}) = 0 \), since \( R(\mathbf{x}, \mathbf{x}) = 0 \) and \( \lambda(0) = 0 \), concluding the proof of a).

b) Let \( \mathbf{B}_{n,2} = (A\theta(\psi)/\sigma) \mathbf{S}(\alpha) \tilde{\mathbf{B}}_{n,2} \). To obtain b) it is enough to see that, for any \( \mathbf{c} \in \mathbb{R}^{q+1}, \mathbf{c} \neq 0 \),

\[
\mathbf{c}^T \mathbf{B}_{n,2} = \sum_{i=1}^{n} \mathbf{c}^T \mathbf{W}_{i,n} \overset{D}{\to} N(0, \mathbf{c}^T \Sigma_{11}(x_{\alpha}) \mathbf{c}),
\]

where

\[
\Sigma_{11}(x_{\alpha}) = \mathbb{E} \psi^2(\varepsilon) \int \frac{q_{\alpha}^2(x_{\alpha})}{f_{\mathbf{X}}(x_{\alpha}, x_{\alpha})} p(x_{\alpha}, x_{\alpha}) \, dx_{\alpha} \mathbf{V}_{\alpha}.
\]

Denote as \( \mathbf{H}_{d-1} = \text{diag}(\tilde{h}, \ldots, \tilde{h}) \in \mathbb{R}^{(d-1) \times (d-1)} \) and

\[
V(\varepsilon_i, \mathbf{X}_i, \mathbf{x}) = \delta_{\alpha} \psi \left( \varepsilon_i + \frac{R(\mathbf{X}_i, \mathbf{x})}{\sigma} \right) - p(\mathbf{x}) \lambda \left( \frac{R(\mathbf{X}_i, \mathbf{x})}{\sigma} \right),
\]

\[
\gamma(\varepsilon_i, \tilde{h}, \mathbf{X}_i, x_{\alpha}) = \frac{1}{h_{d-1}} \int \frac{1}{v(x)} \prod_{j \neq \alpha} K_j \left( \frac{X_{i,j} - x_j}{\tilde{h}} \right) q_{\alpha}(x_{\alpha}) V(\varepsilon_i, \mathbf{X}_i, (x_{\alpha}, x_{\alpha^2})) \, dx_{\alpha}
\]

\[
= \int \frac{q_{\alpha}(X_{i,\alpha} + \mathbf{H}_{d-1} u_{\alpha})}{v(x, X_{i,\alpha} + \mathbf{H}_{d-1} u_{\alpha})} \prod_{j \neq \alpha} K_j(u_j) V(\varepsilon_i, \mathbf{X}_i, (x_{\alpha}, X_{i,\alpha} + \mathbf{H}_{d-1} u_{\alpha})) \, du_{\alpha}.
\]

\[
\mathbf{W}_{i,n} = \frac{1}{\sqrt{nh_{\alpha}}} K_{\alpha} \left( \frac{X_{i,\alpha} - x_{\alpha}}{h_{\alpha}} \right) \mathbf{x}_{i,\alpha} \gamma(\varepsilon_i, \tilde{h}, \mathbf{X}_i, x_{\alpha}).
\]

Note that \( \gamma(\varepsilon, 0, \mathbf{X}_i, x_{\alpha}) \) is well defined as

\[
\gamma(\varepsilon, 0, \mathbf{X}_i, x_{\alpha}) = \frac{q_{\alpha}(X_{i,\alpha})}{v(x, X_{i,\alpha})} V(\varepsilon, \mathbf{X}_i, (x_{\alpha}, X_{i,\alpha})).
\]

(A.37)

It is clear that

\[
\tilde{\mathbf{A}}_{1,n}(\mathbf{x}) - \mathbf{A}_{1,n}(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{K}_{\mathbf{H}_d}(\mathbf{X}_i - \mathbf{x}) \mathbf{x}_{i,\alpha} V(\varepsilon_i, \mathbf{X}_i, \mathbf{x}),
\]

hence,

\[
\mathbf{B}_{n,2} = \sqrt{nh_{\alpha}} \int v^{-1}(\mathbf{x}) \left[ \tilde{\mathbf{A}}_{1,n}(\mathbf{x}) - \mathbf{A}_{1,n}(\mathbf{x}) \right] q_{\alpha}(x_{\alpha}) \, dx_{\alpha}
\]

\[
= \frac{1}{\sqrt{nh_{\alpha}}} \sum_{i=1}^{n} K_{\alpha} \left( \frac{X_{i,\alpha} - x_{\alpha}}{h_{\alpha}} \right) \mathbf{x}_{i,\alpha} \gamma(\varepsilon_i, \tilde{h}, \mathbf{X}_i, x_{\alpha}) = \sum_{i=1}^{n} \mathbf{W}_{i,n}.
\]
Let $c \in \mathbb{R}^{d+1}$, $c \neq 0$. Since $\mathbb{E}(V(\epsilon_i, X_i, x)|X_i) = 0$, for all $x$, we have that $\mathbb{E}W_{i,n} = 0$ so $\mathbb{E}^T W_{i,n} = 0$. Besides, as $\psi$ and $p$ are continuous functions and $|\delta_i| \leq 1$ we have that $|V(\epsilon_i, X_i, x)| \leq C$ for some constant $C > 0$ which entails that $\gamma(\epsilon_i, \tilde{h}, X_i, x_0)$ is bounded since $\inf_{x \in S_0} v(x) > 0$ and $q_\alpha$ is bounded on its support. Therefore, using that $|\tilde{x}_{i,j}| \leq 1$ when $K_{H_d}(X_i - x) \neq 0$, we obtain that, for some general constant $C_1 > 0$,

$$\sum_{i=1}^{n} \mathbb{E}|c^T W_{i,n}|^3 \leq C_1 n \frac{1}{(nh_\alpha)^{3/2}} \mathbb{E} \left[ K_\alpha \left( \frac{X_{1,n} - x_\alpha}{h_\alpha} \right) \right]^3 = C_1 \frac{1}{\sqrt{nh_\alpha}} \int \left| K_\alpha \left( \frac{u - x_\alpha}{h_\alpha} \right) \right| f_{X_\alpha}(u) \, du$$

$$\leq C_2 \frac{1}{\sqrt{nh_\alpha}} \rightarrow 0.$$  

Hence, applying the Lyapunov’s central limit theorem to the triangular array of independent variables $\{c^T W_{i,n}\}_{i=1}^{n}$ the proof of b) follows if we show that $\lim_{n \to \infty} \mathbb{V} \mathbb{A} \mathbb{R}(c^T \sum_{i=1}^{n} W_{i,n}) = c^T \Sigma_{11}(x_\alpha)c$ or equivalently that $\lim_{n \to \infty} \mathbb{V} \mathbb{A} \mathbb{R}(\sum_{i=1}^{n} W_{i,n}) = \Sigma_{11}(x_\alpha)$.

Using that $W_{1,1}, \ldots, W_{n,n}$ are independent and that $\mathbb{E}W_{1,n} = 0$, we get that $\mathbb{V} \mathbb{A} \mathbb{R}(\sum_{i=1}^{n} W_{i,n})=n\mathbb{V} \mathbb{A} \mathbb{R}(W_{1,n})=n\mathbb{E}(W_{1,n} W_{1,n})$. Given $1 \leq s, m \leq q+1$, denote as $E_{sm} = n\mathbb{E}(W_{1,n} W_{1,n,m})$ where $W_{1,n,m}$ is the $m$-th component of $W_{1,n}$. We have to show that $E_{sm}$ converges to the $(s,m)$-th element of $\Sigma_{11}(x_\alpha)$.

Let $M(\tilde{h}, u, x_\alpha) = \mathbb{E} \left[ \gamma^2(\epsilon_1, \tilde{h}, u, x_\alpha) | X_1 = u \right]$, then we have that

$$E_{sm} = \frac{1}{h_\alpha} \mathbb{E} \left[ K_\alpha \left( \frac{X_{1,s} - x_\alpha}{h_\alpha} \right) \gamma^2(\epsilon_1, \tilde{h}, X_1, x_\alpha) \left( \frac{X_{1,s} - x_\alpha}{h_\alpha} \right)^{s+m-2} \right]$$

$$= \frac{1}{h_\alpha} \int K_\alpha \left( \frac{u - x_\alpha}{h_\alpha} \right) M(\tilde{h}, u, x_\alpha) \left( \frac{u - x_\alpha}{h_\alpha} \right)^{s+m-2} f_X(u_\alpha, u_\alpha) \, du$$

$$= \int K_\alpha(u_\alpha)^{s+m-2} M(\tilde{h}, (u_\alpha h_\alpha + x_\alpha, u_\alpha), x_\alpha) f_X(u_\alpha h_\alpha + x_\alpha, u_\alpha) \, du . \quad \text{(A.38)}$$

Note that (A.37) implies that $M(0, u, x_\alpha)$ is well defined and equals

$$M(0, u, x_\alpha) = \frac{q_\alpha^2(u_\alpha)}{v_\alpha^2(x_\alpha, u_\alpha)} \mathbb{E} \left[ V^2(\epsilon_1, u, (x_\alpha, u_\alpha)) | X_1 = u \right] \quad \text{(A.39)}$$

Hence, taking limit in (A.38) and using the dominated convergence theorem, we get that

$$\lim_{n \to \infty} E_{sm} = \int K_\alpha(u_\alpha)^{s+m-2} M(0, (x_\alpha, u_\alpha), x_\alpha) f_X(x_\alpha, u_\alpha) \, du .$$

Using that

$$\mathbb{E} \left[ \delta_1 \psi \left( \frac{X_1 + R(\mathbf{X}_1, \mathbf{x})}{\sigma} \right) | X_1 \right] = p(X_1) \lambda \left( \frac{R(\mathbf{X}_1, \mathbf{x})}{\sigma} \right) ,$$

we get that

$$\mathbb{E} \left[ V^2(\epsilon_1, u, (x_\alpha, u_\alpha)) | X_1 = u \right] = p(u) \lambda_2 \left( R(u, (x_\alpha, u_\alpha)) \right) - p^2(u) \lambda^2 \left( \frac{R(u, (x_\alpha, u_\alpha))}{\sigma} \right) . \quad \text{(A.40)}$$
where \( \bar{\lambda}_2(a) = \mathbb{E}\psi^2(\varepsilon + a) \). Then, the fact that \( R((x_a, u_\alpha), (x_a, u_\alpha)) = 0 \), \( \lambda(0) = 0 \) and \( \bar{\lambda}_2(0) = \mathbb{E}\psi^2(\varepsilon) \), together with (A.39) and (A.40) entail that

\[
M(0, (x_a, u_\alpha), x_\alpha) = \frac{q_{\alpha}^2(u_\alpha)}{v^2(x_a, u_\alpha)} p(x_a, u_\alpha) \mathbb{E}\psi^2(\varepsilon)
\]

Hence, we have

\[
\lim_{n \to \infty} E_{sm} = \int K_\alpha^2(u_\alpha) u_\alpha^{a+m-2} f(x_a, u_\alpha) \frac{q_{\alpha}^2(u_\alpha)}{v^2(x_a, u_\alpha)} p(x_a, u_\alpha) \mathbb{E}\psi^2(\varepsilon)
\]

\[
= \mathbb{E}\psi^2(\varepsilon) \int f(x_a, u_\alpha) p(x_a, u_\alpha) d\mu_{\alpha}^{(a)}
\]

where \( v_{sm}^{(a)} \) is the \((s, m)\)th element of the matrix \( V_\alpha \), concluding the proof of b).

**Step 2.2** To conclude the proof, we have to show that

\[
\sqrt{nh_a} [\hat{g}_\alpha(x_a) - g_\alpha(x_a)] - \mathbf{e}_1^T \hat{B}_n = \sigma \sqrt{nh_a} \int \mathbf{e}_1^T \left( \hat{A}_{0,n}(x) - A_0^{-1}(x) \right) \hat{A}_{1,n}(x) q_\alpha(x_\alpha) dx_\alpha = o_p(1),
\]

(A.41)

Note that \( (\hat{A}_{0,n}(x) - A_0^{-1}(x))\hat{A}_{1,n}(x) = \hat{D}_1(x) + \hat{D}_2(x) \) with

\[
\hat{D}_1(x) = (\hat{A}_{0,n}(x) - A_0^{-1}(x))\mathbb{E}\hat{A}_{1,n}(x)
\]

\[
\hat{D}_2(x) = (\hat{A}_{0,n}(x) - A_0^{-1}(x)) \left( \hat{A}_{1,n}(x) - \mathbb{E}\hat{A}_{1,n}(x) \right).
\]

We will show that, for all \( 1 \leq j \leq q + 1 \)

\[
\sqrt{nh_a} \sup_{x \in S_q} |\hat{D}_{1,j}(x)| \xrightarrow{p} 0 \quad (A.42)
\]

\[
\sqrt{nh_a} \sup_{x \in S_q} |\hat{D}_{2,j}(x)| \xrightarrow{p} 0 \quad (A.43)
\]

where \( \hat{D}_{\ell,j}(x) \) is the \( j \)th coordinate of \( \hat{D}_\ell(x) \), \( \ell = 1, 2 \), which entails that (A.41) holds concluding the proof.

Fix \( 1 \leq j \leq q + 1 \). In order to prove (A.42), observe that Lemma A.2.1, the fact that \( \mathbb{E}\hat{A}_{1,n,j}(x) = \mathbb{E}\hat{A}_{1,n,j}(x) \) and the Cauchy-Schwartz inequality entail that

\[
\sup_{x \in S_q} |\hat{D}_{1,j}(x)| \leq \sup_{x \in S_q} \left\| \mathbf{e}_1^T (\hat{A}_{0,n}(x) - A_0^{-1}(x)) \right\| o(h_\alpha^{q+1}),
\]

where the term \( o(h_\alpha^{q+1}) \) does not depend on \( x \) since \( v \) and \( g_\alpha^{(q+1)} \) are bounded. On the other hand, since \( S^{(a)} \) is non-singular, \( \inf_{x \in S_q} |v(x)| > 0, A_0(\psi) \neq 0 \) and \( \sup_{x \in S_q} \left\| \hat{A}_{0,n}(x) - A_0(x) \right\| \xrightarrow{p} 0 \)

we get that \( \sup_{x \in S_q} \left\| \hat{A}_{0,n}(x) - A_0^{-1}(x) \right\| \xrightarrow{p} 0 \). Hence, since \( \sqrt{nh_a} h_\alpha^{q+1} = \beta^{(2q+3)/2} \) we have that

\[
\sqrt{nh_a} \sup_{x \in S_q} |\hat{D}_{1,j}(x)| \xrightarrow{p} 0 \text{ for all } 1 \leq j \leq q + 1\), so the proof of (A.42) is concluded.
To prove (A.43), we will use Lemma A.1.2 with \( \theta_n = \sqrt{\log n/\mathcal{H}_a} \) applied to each coordinate of vector \( \hat{A}_{1,n}(x) = (\hat{A}_{1,n,1}(x), \ldots, \hat{A}_{1,n,q+1}(x))^T \) obtaining that, for \( 1 \leq j \leq q + 1 \),

\[
\sup_{x \in S_Q} \left| \hat{A}_{1,n,j}(x) - E \hat{A}_{1,n,j}(x) \right| = O_P \left( \frac{\log n}{\mathcal{H}_a} \right)^{1/2}.
\]

On the other hand, as above, from Lemma A.2.1, we get that \( \sup_{x \in S_Q} E \hat{A}_{1,n,j}(x) = o(h_a^{q+1}) \).

Then, using \( \sup_{x \in S_Q} \left| \hat{A}_{0,n}^{-1}(x) - A_0^{-1}(x) \right| \xrightarrow{p} 0 \) and \( \inf_{x \in S_Q} \nu_1(A_0(x)) > 0 \), we conclude that \( \sup_{x \in S_Q} \nu_{q+1}(\hat{A}_{0,n}^{-1}(x)) = O_P(1) \). Hence, using (A.32), we obtain that

\[
\sup_{x \in S_Q} \left| H^{(\alpha)}(\beta(x) - \beta(x)) \right| \leq o_P(h_a^{q+1}) + O_P \left( \frac{\log n}{\mathcal{H}_a} \right)^{1/2}.
\]

Let \( \mathcal{K}^*(u) = |\mathcal{K}(u)|/\int|\mathcal{K}(u)|du \), then, Remark A.1.1 implies that \( \hat{f}(x) = (1/n) \sum_{j=1}^{n} \mathcal{K}_H^*(x - X_j) \) converges uniformly and almost surely to \( f_x \) (see (A.8)). Hence, using \( A2 \), we obtain that

\[
\sup_{x \in S_Q} \|\hat{A}_{02,n}(x)\| \leq C \sup_{x \in S_Q} \|H^{(\alpha)}(\beta(x) - \beta(x))\| \sup_{x \in S_Q} \hat{f}(x) \leq o_P(h_a^{q+1}) + O_P \left( \frac{\log n}{\mathcal{H}_a} \right)^{1/2},
\]

which together with the fact that \( h_a = \beta n^{-1/2q+1} \) and \( n^{q+1} \mathcal{H}_a^{d-1}/\log n \to \infty \) implies that

\[
\sup_{x \in S_Q} \|\hat{A}_{02,n}(x)\| O_P \left( \frac{\log n}{\mathcal{H}_a} \right)^{1/2} \leq \frac{\beta n^{-1/2q+1}}{\mathcal{H}_a} + \frac{\log n}{\mathcal{H}_a} O_P(1)
\]

Recall that \( \sup_{x \in S_Q} \nu_{q+1}(A_0(x)) < \infty \), since \( \inf_{x \in S_Q} \nu_1(A_0(x)) > 0 \). Therefore, using the Cauchy-Schwartz inequality, the fact that \( A_0^{-1}(x) - \hat{A}_0^{-1}(x) = \hat{A}_0^{-1}(x)(\hat{A}_0^{-1}(x) - A_0(x)) = \hat{A}_0^{-1}(x) + \hat{A}_2^{-1}(x) \) and \( \sup_{x \in S_Q} \nu_{q+1}(\hat{A}_0^{-1}(x)) = O_P(1) \), we get that

\[
\sqrt{\mathcal{H}_a} \sup_{x \in S_Q} \|\hat{D}_{2,j}(x)\| \leq C_1 \sqrt{n} \sup_{x \in S_Q} \|\hat{A}_0^{-1}(x) - A_0(x)\| O_P \left( \frac{\log n}{\mathcal{H}_a} \right)^{1/2}
\]

\[
+ C_2 \sup_{x \in S_Q} \|\hat{A}_2^{-1}(x)\| O_P \left( \frac{\log n}{\mathcal{H}_a} \right)^{1/2}
\]

\[
\leq C_2 \sup_{x \in S_Q} \|\hat{A}_0^{-1}(x) - A_0(x)\| O_P \left( \frac{\log n}{\mathcal{H}_a} \right)^{1/2} + o_P(1) \tag{A.46}
\]

where the last inequality follows from (A.45).
Recall that \( \lambda_1(t) = \mathbb{E}\psi'(\varepsilon_1 + t) \) and \( \lambda_1(0) = A_0(\psi) \). Denote 
\[
\Lambda_{1,n}(x) = \mathbb{E}\hat{A}_{01,n}(x) = \mathbb{E}\left[ K_{\mathbf{H}_d}(X_1 - x)r(x_1)\lambda_1 \left( \frac{R(x_1, x)}{\sigma} \right) \hat{x}_{1,\alpha}^T \right] 
\]
\[
= \int K(u)v(x + H_d u)\lambda_1 \left( \frac{R(x + H_d u, x)}{\sigma} \right) \hat{u}_{\alpha}^T du .
\]

Let \( \hat{A}_{01,m}(x) \) be the \((j, m)\)th element of matrix \( \hat{A}_{01,n} \). Then, analogous arguments to those considered in the proof of Lemma A.1.2 allow to show that, for \( 1 \leq j, m \leq q + 1 \)
\[
\sup_{x \in S_Q} \left| \hat{A}_{01,j,m}(x) - A_{1,j,m}(x) \right| = O_P \left( \left( \frac{\log n}{n h^d} \right)^{1/2} \right) .
\]

Hence,
\[
\sup_{x \in S_Q} \left\| \hat{A}_{01,n}(x) - \Lambda_{1,n}(x) \right\|_{OP} \left( \left( \frac{\log n}{h^{d-1}} \right)^{1/2} \right) \leq n^{-2q+2} \left( \frac{\log n}{h^{d-1}} \right) O_p(1) = o_P(1) \quad (A.47)
\]

Hence, (A.46) and (A.47) entail that to conclude the proof of (A.43) we only have to show that
\[
\sup_{x \in S_Q} \left\| \Lambda_{1,n}(x) - A_0(x) \right\|_{OP} \left( \left( \frac{\log n}{h^{d-1}} \right)^{1/2} \right) = o_P(1) \quad (A.48)
\]

Denote as \( A_{0,j,m}(x) \) and \( \Lambda_{1,n,j,m}(x) \) the \((j, m)\) element of \( A_0(x) \) and \( \Lambda_{1,n}(x) \), respectively. Then, using that \( v \) is \( \ell \) times differentiable, \( \lambda_1 = \lambda' \) is \( \ell - 1 \) times differentiable, the kernel \( L \) is of order \( \ell \) and that \( R(x + H_d u, x) = \sum_j u_j^T g_j^{(\ell)}(\xi_j) + \sum_r u_r^T g_j^{(r)}(x_j) + h^{q+1} u_{\alpha}^{(q+1)}(\xi_\alpha) \), since \( g_j \) and \( g_\alpha^{(\ell)} \) are continuously differentiable functions, we obtain that, for \( 1 \leq j, m \leq q + 1 \)
\[
\sup_{x \in S_Q} \left| \Lambda_{1,n,j,m}(x) - A_{0,j,m}(x) \right| = \sup_{x \in S_Q} \left| \int K(u) v(x + H_d u)\lambda_1 \left( \frac{R(x + H_d u, x)}{\sigma} \right) \right.
\]
\[
\left. - v(x)\lambda_1 \left( \frac{R(x, x)}{\sigma} \right) \right| \hat{u}_{j,\alpha} \hat{u}_{m,\alpha} du \right| \leq C_2 \left( h^{\ell} + h^{q+1} \right) \leq C_3 h^{q+1}
\]

which allow to conclude that, for \( 1 \leq j, m \leq q + 1 \),
\[
\sup_{x \in S_Q} \left| \Lambda_{1,n,j,m}(x) - A_{0,j,m}(x) \right|_{OP} \left( \left( \frac{\log n}{h^{d-1}} \right)^{1/2} \right) \leq h^{q+1} \leq O_P \left( \left( \frac{\log n}{h^{d-1}} \right)^{1/2} \right)
\]

which combined with the fact that \( \log n/(n^{(q+1)/(2q+3)}h^{d-1}) \rightarrow 0 \) conclude the proof of (A.48) and also that of the Theorem 4.1. \( \square \)

**Proof of Theorem 4.2.** The proof follows using similar arguments to those considered in the proof of Theorem 4.1, noting that
\[
\hat{g}_{\alpha,M,q,\alpha}^{(\nu)}(x) - g_\alpha^{(\nu)}(x) = \nu! \int_{-1}^{1} e_{\nu+1}^{T} \left[ \hat{\beta}(x, \alpha) - \beta(x, \alpha) \right] q_\alpha(\alpha) d\alpha
\]
\[
= \nu! \frac{1}{h^{\nu}} \int_{-1}^{1} e_{\nu+1}^{T} H^{(\alpha)} \left[ \hat{\beta}(x, \alpha) - \beta(x, \alpha) \right] q_\alpha(\alpha) d\alpha . \quad \square
\]
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