Restricted Sum Formula of Multiple Zeta Values

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1 Introduction

For fixed positive integer \(d\) and \(d\)-tuple of positive integers \((s_1, \ldots, s_d)\) with \(s_1 > 1\), the multiple zeta value \(\zeta(s_1, \ldots, s_d)\) is defined by

\[
\zeta(s_1, \ldots, s_d) = \sum_{k_1 > \cdots > k_d > 0} k_1^{-s_1} \cdots k_d^{-s_d},
\]

where \(d\) is called the depth and \(s_1 + \cdots + s_d\) the weight. The double zeta values were studied by Euler [1] who derived many identities such as follows:

\[
\sum_{k=2}^{2n-1} (-1)^k \zeta(k, 2n-k) = \frac{1}{2} \zeta(2n),
\]

\[
\sum_{k=2}^{2n-1} \zeta(k, 2n-k) = \zeta(2n),
\]

from which we can easily get (see [2, Theorem 1])

\[
\sum_{k=1}^{n-1} \zeta(2k, 2n-2k) = \frac{3}{4} \zeta(2n).
\]

Using the stuffle relation \(\zeta(2k)\zeta(2n-2k) = \zeta(2k, 2n-2k) + \zeta(2n-2k, 2k) + \zeta(2n)\) we see immediately

\[
\sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k) = \frac{2n+1}{2} \zeta(2n).
\]

Recently, Hoffman [3] extended [2] to arbitrary depths. Moreover, similar formulas have been obtained for some special type Hurwitz-zeta values [4] and alternating Euler sums [5]. In this paper we consider the following restricted sum of multiple zeta values

\[
Q(4n, d) = \sum_{\substack{j_1, \ldots, j_d = n \\ j_1, \ldots, j_d > 0 \\ j_1 + \cdots + j_d = n}} \zeta(4j_1, \ldots, 4j_d).
\]

Our main theorem is
**Theorem 1.1.** For any positive integers $n \geq d \geq 3$,

\[
Q(4n, d) = \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=0}^{2k+1} \frac{2k+2(-1)^{\frac{j}{2}+j+d}}{(2k+1)!} \binom{2k+1}{j} \left(\frac{i-2}{d}\right) \zeta(4n-2k) \pi^{2k}
\]

\[
+ \sum_{k=0}^{\lfloor \frac{d-2}{4} \rfloor} \sum_{j=0}^{4k+2} \frac{2^{2k+5}(-1)^{k+j+d}}{(4k+2)!} \binom{4k+2}{j} \left(\frac{i-2}{4}\right) \left(Q(4n-4k, 2) - \frac{7}{8} \zeta(4n-4k)\right) \pi^{4k}.
\]

**Remark 1.2.** For $d = 2$, it’s easy to prove by stuffle relation that

\[
Q(4n, 2) = \frac{1}{2} \sum_{k=1}^{n-1} \zeta(4k) \zeta(4n-4k) - \frac{n-1}{2} \zeta(4n)
\]

for $n \geq 2$. However, it is an intriguing problem to find a compact formula similar to (3).

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## 2 The generating function of $Q(4n, d)$

Recall that the symmetric function of the infinitely many variables $x_1, x_2, \cdots$ form a subring $\text{Sym}$ of $\mathbb{Q}[x_1, x_2, \cdots]$ which is invariant under all the permutations of the variables. Let $e_j = \sum_{k_1 < \cdots < k_j} x_{k_1} \cdots x_{k_j}$ be the $j$-th elementary function. Following Hoffman [3] let’s consider its generating function

\[
E(t) = \prod_{j=1}^{\infty} (1 + tx_j) = \sum_{j=0}^{\infty} e_j t^j
\]

and define $\varepsilon : \text{Sym} \to \mathbb{R}$ to be the evaluation map such that $\varepsilon(x_j) = \frac{1}{j^4}$. Let

\[
F(s, t) = \prod_{j=1}^{\infty} (1 + tsx_j + ts^2x_j^2 + \cdots).
\]

Then it is not hard to see that the generating function of $Q(4n, d)$ is given by

\[
\varepsilon(F(s, t)) = \sum_{n=0}^{\infty} Q(4n, d) t^d s^n.
\]

First we need the following lemma.
Lemma 2.1. We have
\[ \varepsilon(F(s, t)) = \frac{\sin \pi \sqrt[4]{s(1-t)} \cdot \sinh \pi \sqrt{s(1-t)}}{\sqrt{1-t} \sin \pi \sqrt{s} \cdot \sinh \pi \sqrt{s}}. \]

Proof. We have
\[
\prod_{j=1}^{\infty} (1 + t \cdot s \cdot x_j + t \cdot s^2 \cdot x_j^2 + \cdots) = \prod_{j=1}^{\infty} \left(1 + t \cdot \frac{s \cdot x_j}{1 - s \cdot x_j}\right) = \prod_{j=1}^{\infty} \left(\frac{1 - s \cdot (1-t) \cdot x_j}{1 - s \cdot x_j}\right) = \frac{E(-s(1-t))}{E(-s)}.
\]
Further,
\[ \varepsilon(E(-t)) = \prod_{i=1}^{\infty} \left(1 - \frac{t}{i^2}\right) = \prod_{i=1}^{\infty} \left(1 - \frac{\sqrt{t}}{i^2}\right) \left(1 + \frac{\sqrt{t}}{i^2}\right) = \frac{\sin \pi \sqrt{t} \cdot \sinh \pi \sqrt{t}}{\pi^2 \sqrt{t}}. \]
The lemma follows immediately. \(\square\)

Let \(f(x) = \sin x \cdot \sinh x/(2x^2)\). The following lemma provides its series expansion.

Lemma 2.2. We have
\[ f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(4k+2)!} x^{4k}. \]

Proof. Using the well-known formula \(\sin x = (e^{ix} - e^{-ix})/(2i)\) we obtain
\[
f(x) = \frac{1}{2} \cdot \frac{e^{ix} - e^{-ix}}{2ix} \cdot \frac{e^x - e^{-x}}{2x} = \frac{e^{(i+1)x} + e^{-(i+1)x} - (e^{(i-1)x} + e^{-(i-1)x})}{8ix^2} = \frac{1}{4ix^2} \left(\sum_{n=0}^{\infty} \frac{(2i)^n x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-2i)^n x^{2n}}{(2n)!}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(4k+2)!} x^{4k},
\]
as desired. \(\square\)

3 Proof of Theorem 1.1

Let \(g(t) = f(\sqrt[4]{t})\). Then
\[
\frac{g(s(1-t))}{g(s)} = \varepsilon(F(s/\pi^4, t)) = \frac{1}{g(s)} \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(4k+2)!} s^k(1-t)^k.
\]
Write
\[ \frac{g(s(1-t))}{g(s)} = \sum_{d=0}^{\infty} G_d(s)t^d. \]

By the above expression, we have
\[ G_d(s) = \frac{(-s)^d}{g(s)d!}D^d g(s), \]
where \( D^d \) denotes the \( d \)-th derivative with respect to \( s \). Set
\[ G_d(s) = X_d(s)\sqrt[s]{s} \cot \sqrt[s]{s} + Y_d(s)\sqrt[s]{s} \coth \sqrt[s]{s} + Z_d(s) \cot \sqrt[s]{s} \coth \sqrt[s]{s} + W_d(s) \quad (4) \]
which yields easily
\[ \frac{(-1)^s D^d g(s)}{d!} = X_d(s)s^{-d+\frac{1}{4}} \cos \frac{s^{\frac{1}{4}}}{s} \sinh \frac{s^{\frac{1}{4}}}{s} + Y_d(s)s^{-d-\frac{1}{4}} \sin \frac{s^{\frac{1}{4}}}{s} \cosh \frac{s^{\frac{1}{4}}}{s} + Z_d(s)s^{-d-\frac{1}{4}} \cos s^{\frac{1}{4}} \cosh s^{\frac{1}{4}} + W_d(s)s^{-d-\frac{1}{4}} \sin s^{\frac{1}{4}} \sinh s^{\frac{1}{4}}. \]

To determine the coefficients \( X_d(s), Y_d(s), Z_d(s) \) and \( W_d(s) \) we differentiate the both sides of the above equation to get the following system of recursive differential equations
\[
\begin{cases}
(d + 1)X_{d+1}(s) = -sX'_d(s) + \left( d + \frac{1}{4} \right)X_d(s) - \frac{1}{4}Z_d(s) - \frac{1}{4}W_d(s), \\
(d + 1)Y_{d+1}(s) = -sY'_d(s) + \left( d + \frac{1}{4} \right)Y_d(s) + \frac{1}{4}Z_d(s) - \frac{1}{4}W_d(s), \\
(d + 1)Z_{d+1}(s) = -\frac{\sqrt{s}}{4}X_d(s) - \frac{\sqrt{s}}{4}Y_d(s) - sZ'_d(s) + \left( d + \frac{1}{2} \right)Z_d(s), \\
(d + 1)W_{d+1}(s) = \frac{\sqrt{s}}{4}X_d(s) - \frac{\sqrt{s}}{4}Y_d(s) + \left( d + \frac{1}{2} \right)W_d(s) - sW'_d(s),
\end{cases}
\]
with the initial conditions \( X_0(s) = Y_0(s) = Z_0(s) = 0 \) and \( W_0(s) = 1 \). Let \( x_d(u) = X_d(u^2), y_d(u) = Y_d(u^2), z_d(u) = Z_d(u^2) \) and \( w_d(u) = W_d(u^2) \). The above system is changed into the following system:
\[
\begin{cases}
(d + 1)x_{d+1}(u) = -\frac{u}{2}x'_d(u) + \left( d + \frac{1}{4} \right)x_d(u) - \frac{1}{4}z_d(u) - \frac{1}{4}w_d(u), \\
(d + 1)y_{d+1}(u) = -\frac{u}{2}y'_d(u) + \left( d + \frac{1}{4} \right)y_d(u) + \frac{1}{4}z_d(u) - \frac{1}{4}w_d(u), \\
(d + 1)z_{d+1}(u) = -\frac{u}{4}x_d(u) - \frac{u}{4}y_d(u) - \frac{u}{2}z'_d(u) + \left( d + \frac{1}{2} \right)z_d(u), \\
(d + 1)w_{d+1}(u) = \frac{u}{4}x_d(u) - \frac{u}{4}y_d(u) + \left( d + \frac{1}{2} \right)w_d(u) - \frac{u}{2}w'_d(u).
\end{cases}
\]

\[ (5) \]
Define

\[
\begin{align*}
\alpha(u, v) &= \sum_{d \geq 0} x_d(u)v^d = \sum_{d \geq 0} \tilde{x}_d(v)u^d, \\
\beta(u, v) &= \sum_{d \geq 0} y_d(u)v^d = \sum_{d \geq 0} \tilde{y}_d(v)u^d, \\
\gamma(u, v) &= \sum_{d \geq 0} z_d(u)v^d = \sum_{d \geq 0} \tilde{z}_d(v)u^d, \\
\delta(u, v) &= \sum_{d \geq 0} w_d(u)v^d = \sum_{d \geq 0} \tilde{w}_d(v)u^d.
\end{align*}
\]

(6)

Multiplying the system (5) by \(v^d\) and then taking the sum \(\sum_{d \geq 0}\) we get:

\[
\begin{align*}
\frac{\partial \alpha}{\partial v} &= v \frac{\partial \alpha}{\partial v} + \frac{1}{4} \alpha - u \frac{\partial \alpha}{\partial u} - \frac{1}{4} \gamma - \frac{1}{4} \delta, \\
\frac{\partial \beta}{\partial v} &= v \frac{\partial \beta}{\partial v} + \frac{1}{4} \beta - u \frac{\partial \beta}{\partial u} + \frac{1}{4} \gamma - \frac{1}{4} \delta, \\
\frac{\partial \gamma}{\partial v} &= v \frac{\partial \gamma}{\partial v} + \frac{1}{4} \gamma - u \frac{\partial \gamma}{\partial u} - \frac{1}{4} \alpha - \frac{1}{4} \beta, \\
\frac{\partial \delta}{\partial v} &= v \frac{\partial \delta}{\partial v} + \frac{1}{4} \delta - u \frac{\partial \delta}{\partial u} + \frac{1}{4} \alpha - \frac{1}{4} \beta.
\end{align*}
\]

Comparing the coefficients of \(u^n\) we get

\[
\begin{align*}
\tilde{x}'_n(v) &= v \tilde{x}'_n(v) + \frac{1}{2} \tilde{x}_n(v) - \frac{n}{2} \tilde{x}_n(v) - \frac{1}{4} \tilde{z}_n(v) - \frac{1}{4} \tilde{w}_n(v), \\
\tilde{y}'_n(v) &= v \tilde{y}'_n(v) + \frac{1}{2} \tilde{y}_n(v) - \frac{n}{2} \tilde{y}_n(v) + \frac{1}{4} \tilde{z}_n(v) - \frac{1}{4} \tilde{w}_n(v), \\
\tilde{z}'_n(v) &= v \tilde{z}'_n(v) + \frac{1}{2} \tilde{z}_n(v) - \frac{n}{2} \tilde{z}_n(v) - \frac{1}{4} \tilde{x}_{n-1}(v) - \frac{1}{4} \tilde{y}_{n-1}(v), \\
\tilde{w}'_n(v) &= v \tilde{w}'_n(v) + \frac{1}{2} \tilde{w}_n(v) - \frac{n}{2} \tilde{w}_n(v) + \frac{1}{4} \tilde{x}_{n-1}(v) - \frac{1}{4} \tilde{y}_{n-1}(v),
\end{align*}
\]

(7)

By definition (6), we see that the system has the following initial values: \(\tilde{x}_n(0) = 0, \tilde{y}_n(0) = 0, \tilde{z}_n(0) = 0\) for all \(n \geq 0\) and \(\tilde{w}_n(0) = 0\) for all \(n \geq 1\). But for \(\tilde{w}_0(v)\) we have from (5)

\[
w_0(0) = 1, \quad w_d(0) = \frac{2d - 1}{2d} \quad \forall d \geq 1.
\]

It follows that \(w_d(0) = \binom{2d}{d}/2^{2d}\) which yields easily

\[
\tilde{w}_0(v) = \sum_{d \geq 0} w_d(0)v^d = (1 - v)^{-\frac{1}{2}}.
\]

Similarly we see that \(\tilde{z}_0(v) = 0\). Solving (7) recursively starting from the first two equations in (7) we find the following functions are the unique solution satisfying the
Using (6) we can solve initial conditions:

\[
\begin{aligned}
\bar{x}_n(v) &= \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\frac{n+2}{2}+j}}{j!(2n+1-j)!} \left(1 - v\right)^{\frac{j-2}{4}}; \\
\bar{y}_n(v) &= \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\frac{n+2}{2}+j}}{j!(2n+1-j)!} \left(1 - v\right)^{\frac{j-2}{4}}; \\
\bar{z}_n(v) &= (1 - (-1)^n) \sum_{j=0}^{2n} \frac{2^{n-1}(-1)^{\frac{n-1}{2}+j}}{j!(2n-j)!} \left(1 - v\right)^{\frac{j-2}{4}}; \\
\bar{w}_n(v) &= (1 + (-1)^n) \sum_{j=0}^{2n} \frac{2^{n-1}(-1)^{\frac{n}{2}+j}}{j!(2n-j)!} \left(1 - v\right)^{\frac{j-2}{4}}.
\end{aligned}
\]

Using (6) we can solve \(x_n(v), y_n(v), z_n(v)\) and \(w_n(v)\) and get

\[
\begin{aligned}
x_d(u) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\frac{n+2}{2}+j+d}}{(2n+1)!} \left(2n+1\right) \left(\frac{j-2}{d}\right) u^n; \\
y_d(u) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\frac{n+2}{2}+j+d}}{(2n+1)!} \left(2n+1\right) \left(\frac{j-2}{d}\right) u^n; \\
z_d(u) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{2n} (1 - (-1)^n) \frac{2^{n-1}(-1)^{\frac{n+1}{2}+j+d}}{(2n)!} \left(\frac{j-2}{d}\right) u^n; \\
w_d(u) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{2n} (1 + (-1)^n) \frac{2^{n-1}(-1)^{\frac{n}{2}+j+d}}{(2n)!} \left(\frac{j-2}{d}\right) u^n.
\end{aligned}
\]

Thus

\[
\begin{aligned}
X_d(s) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\frac{n+2}{2}+j+d}}{(2n+1)!} \left(2n+1\right) \left(\frac{j-2}{d}\right) s^{\frac{n}{2}}; \\
Y_d(s) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\frac{n+2}{2}+j+d}}{(2n+1)!} \left(2n+1\right) \left(\frac{j-2}{d}\right) s^{\frac{n}{2}}; \\
Z_d(s) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{4n+2} \frac{2^{2n+1}(-1)^{n+j+d}}{(4n+2)!} \left(4n+2\right) \left(\frac{j-2}{d}\right) s^{n+1/2}; \\
W_d(s) &= \sum_{n=0}^{\lceil \frac{d-1}{2} \rceil} \sum_{j=0}^{4n} \frac{2^{2n}(-1)^{n+j+d}}{(4n)!} \left(4n\right) \left(\frac{j-2}{d}\right) s^n.
\end{aligned}
\]
By the well-known formulas
\[ z \cot z = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} z^{2n}, \quad z \coth z = -2 \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{\pi^{2n}} z^{2n}, \]
we obtain
\[ \sqrt{s} \cot \sqrt{s} = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} s^{\frac{n}{2}}, \quad \sqrt{s} \coth \sqrt{s} = -2 \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{\pi^{2n}} s^{\frac{n}{2}}, \]
and
\[ \sqrt{s} \cot \sqrt{s} \cdot \coth \sqrt{s} = 4 \sum_{k=0}^{\infty} \sum_{m+l=k} (-1)^m \frac{\zeta(2m)\zeta(2l)}{\pi^{2k}} s^{\frac{k}{2}} = 4 \sum_{k=0}^{\infty} \sum_{m+l=2k} (-1)^m \frac{\zeta(2m)\zeta(2l)}{\pi^{4k}} s^{k}. \]
Here by exchanging \( m \) and \( l \) we notice that the inner sum vanishes if \( k \) is odd. Hence the coefficient of \( s^n \) in \( G_d(\pi^4 s) \) is
\[ Q(4n, d) = 2 \sum_{k=0}^{\left\lfloor \frac{d}{2} \right\rfloor} \sum_{j=0}^{2k+1} \frac{2^k(-1)^{\frac{k}{2}+j+d}}{(2k+1)!} \binom{2k+1}{j} \left( \frac{j-2}{4} \right) \zeta(4n-2k)\pi^{2k} \]
\[ + 2 \sum_{k=0}^{\left\lfloor \frac{d-1}{2} \right\rfloor} \sum_{j=0}^{2k+1} (-1)^k \frac{2^k(-1)^{\frac{k+1}{2}+j+d}}{(2k+1)!} \binom{2k+1}{j} \left( \frac{j-2}{4} \right) \zeta(4n-2k)\pi^{2k} \]
\[ + 4 \sum_{k=0}^{\left\lfloor \frac{d-2}{4} \right\rfloor} \sum_{j=0}^{2k+1} \frac{2^{2k+1}(-1)^{k+j+d}}{(4k+2)!} \binom{4k+2}{j} \left( \frac{j-2}{4} \right) \left( \sum_{m+l=2n-2k} (-1)^m \zeta(2m)\zeta(2l) \right) \pi^{4k} \]
since \( W_d(s) \) has degree less than \( n \). Observe that the first two lines are the same and for any positive integer \( w \)
\[ \sum_{m,l \geq 0, m+l=2w} (-1)^m \zeta(2m)\zeta(2l) = 2 \sum_{l=1}^{w-1} \zeta(4l)\zeta(4w-4l) - \sum_{l=1}^{2w-1} \zeta(2l)\zeta(4w-2l) - \zeta(4w) \]
\[ = 4Q(4w, 2) + (2w-3)\zeta(4w) - \frac{4w+1}{2} \zeta(4w) \]
\[ = 4Q(4w, 2) - \frac{7}{2} \zeta(4w) \]
by stuffle relation \( \zeta(4m)\zeta(4l) = \zeta(4m, 4l) + \zeta(4l, 4m) + \zeta(4m+4l) + \zeta(4m+4l) \) and equation \( (3) \). Therefore we finally get
\[ Q(4n, d) = 4 \sum_{k=0}^{\left\lfloor \frac{d-2}{4} \right\rfloor} \sum_{j=0}^{2k+1} \frac{2^{2k+1}(-1)^{k+j+d}}{(4k+2)!} \binom{4k+2}{j} \left( \frac{j-2}{4} \right) \left( 4Q(4n-4k, 2) - \frac{7}{2} \zeta(4n-4k) \right) \pi^{4k}. \]
This concludes the proof of Theorem 1.1 and this paper.

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