Traversable wormholes satisfying the weak energy condition in third-order Lovelock gravity

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In this paper, we consider third order Lovelock gravity with a cosmological constant term in an n-dimensional spacetime \( M^4 \times \mathcal{K}^{n-4} \), where \( \mathcal{K}^{n-4} \) is a constant curvature space. We decompose the equations of motion for four and higher dimensional ones and find wormhole solutions by considering a vacuum \( \mathcal{K}^{n-4} \) space. Applying the latter constraint, we determine the second and third order Lovelock coefficients and the cosmological constant in terms of specific parameters of the model, such as the size of the extra dimensions. Using the obtained Lovelock coefficients and \( \Lambda \), we obtain the 4-dimensional matter distribution threading the wormhole. Furthermore, by considering the zero tidal force case and a specific equation of state, given by \( \rho = (\gamma p - \tau)/|\omega(1 + \gamma)| \), we find the exact solution for the shape function which represents both asymptotically flat and non-flat wormhole solutions. We show explicitly that these wormhole solutions in addition to traversibility satisfy the energy conditions for suitable choices of parameters and that the existence of a limited spherically symmetric traversable wormhole with normal matter in a 4-dimensional spacetime, implies a negative effective cosmological constant.

I. INTRODUCTION

Wormholes are tunnel-like objects, connecting two different regions of spacetime, which include a throat-like region with a minimum radius. The concept can be traced back as far as the 1916 paper by Flamm [2], where he considered a “tunnel-shaped” nature of space near the Schwarzschild radius, which is possibly analogous to the modern concept of the wormhole “throat”. However, one can consider the 1935 paper of Einstein and Rosen as the first serious work on wormhole physics. The so-called Einstein-Rosen bridge was a construction of an elementary particle model represented by a “bridge” connecting two identical sheets [2]. Nevertheless, it was later shown that the Einstein-Rosen bridge is unstable, as it collapses before a photon has had time to pass through [3]. The term ‘wormhole’ was coined for the first time in 1957 by John Wheeler [4] and a few wormhole solutions were investigated [5], before the seminal Morris-Thorne paper in 1988, in which a static traversable wormhole solution was presented for the first time. It was shown that the energy-momentum distribution threading these solutions violated the energy conditions, in particular, it violates the null energy condition (NEC), and which has been denoted exotic matter [6, 7]. In fact, it was shown that the exoticity of the matter is a generic and universal property of static wormholes within general relativity. This analysis was presented taking into account the local geometry at the throat, or in its vicinity, where the assumption of asymptotic flatness is not a necessary requirement [8–10].

Recently, the late-time accelerated expansion of the Universe [11] has also motivated research in wormhole physics, due to the possible presence of an exotic cosmic fluid responsible for this cosmic acceleration, i.e., phantom energy, that violates the null energy condition [12]. Observational features of wormholes physics have also been extensively analysed in the literature [13]. Relative to the energy condition violations, some attempts have been done in the literature to construct wormholes with normal matter [14]. Indeed, as the violation of the energy conditions is a particularly problematic issue [15], it is necessary to minimize its usage. In fact, to this aim, wormholes have been substantially examined from different points of view in the literature [16–23]. In [24], a class of static and spherically symmetric solutions in a vacuum brane was obtained where it was shown that bulk Weyl effects support the wormhole. In fact, as it is the effective total stress energy tensor that violates the energy conditions, one may impose that the energy-momentum tensor confined on the brane, threading the wormhole, satisfies the NEC. Furthermore, the analysis carried out in [24] was generalized in [25], by showing that in addition to the nonlocal corrections from the Weyl curvature in the bulk, the local high-energy bulk effects may induce a NEC violating signature on the brane. Thus, braneworld gravity seems to provide a natural scenario for the existence of traversable wormholes.

From the beginning of introducing higher dimensional spacetimes, physicists faced questions on the size of these non-observable extra dimensions. The problem was first solved by compactifying the extra dimension on small enough regions, as the non-observability of the extra dimensions was due to their smallness. This idea has recently been substituted by another in that higher dimen-

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sions can be large but they are not observable as no observable matter and fields can exist there, and it is only gravity that propagates in these extra dimensions \[26\]. The latter approach is designated by the “braneworld” scenario \[27\]. The general formalism of the braneworld is to consider the metric of the observable world on a “brane”, which is embedded in a higher dimensional space denoted by the “bulk”. In this paper, although we do not use the typical formalism of the braneworld scenario, we apply its idea to obtain wormhole solution. We consider a theory consisting of third order Lovelock gravity \[28\] in addition to a cosmological constant. The constants of the theory are fixed so that there is no energy-momentum tensor except for the observable world. Applying this, we first present both asymptotically flat and non-flat novel wormhole solutions in third-order Lovelock gravity, and investigate the physical properties of these solutions, showing that it is possible that these traversable wormholes may be constructed from normal matter.

In fact, it is interesting to note that it has recently been shown that exact traversable wormholes in \((3 + 1)\)-dimensional general relativity can be constructed, in which the only “exotic source” is a negative cosmological constant (which can hardly be defined as exotic) \[29\]. Based on the techniques developed in Refs. [30], the authors found the first self-gravitating, analytic and globally regular Skyrmion solution of the Einstein-Skyrme system, in the presence of a cosmological constant. More specifically, the equations admit analytic bouncing cosmological solutions in which the universe contracts to a minimum non-vanishing size, and then expands. A non-trivial byproduct of this solution is that a minor modification of the construction gives rise to a family of stationary, regular configurations in general relativity with a negative cosmological constant supported by an SU(2) nonlinear sigma model. These solutions represent traversable AdS wormholes with a NUT parameter in which the only “exotic matter” required for their construction is a negative cosmological constant. Thus, for instance, one can have that a cloud of interacting Pions (described with the usual nonlinear sigma model) in \((3 + 1)\)-dimensional general relativity can support a traversable AdS wormhole with NUT charge. Moreover, this wormhole is likely to be stable as the Pions are in a configuration with a non-trivial topological charge and so they cannot collapse to the trivial Pions vacuum. As a last remark, at a first glance, if one deals with the NUT parameter as it is usually done, then there are closed time-like curves. However, it was recently found that closed time-like curves can be avoided in spacetimes with a NUT parameter when these are regular \[31\], as is the case with the solutions found in [29], so that one may, in principle, get rid of the closed time-like curves in the latter case, as well.

This paper is outlined in the following manner: In Section II, we present the field equations in third order Lovelock gravity in the presence of a cosmological constant term, which will be used in the subsequent sections. In Section III, by considering a 4-dimensional spacetime \(\mathcal{M}^4\) which is a part of an \(n\)-dimensional \(\mathcal{M}^4 \times K^{n-4}\) spacetime where \(K^{n-4}\) is a constant curvature space which we consider empty, we decompose the equations of motion to four and higher dimensional ones, and consequently write out the general four-dimensional gravitational field equations. In Section IV, by imposing a constant redshift function and considering a specific equation of state, we find specific wormhole solutions, i.e., asymptotically flat and general wormhole spacetimes, and show that these solutions satisfy the energy conditions and are traversable. Furthermore, for the non-asymptotically flat spacetimes, these interior geometries are matched to an exterior spherically-symmetric spacetime, in an AdS Universe.

II. THIRD ORDER LOVELock GRAVITY: FIELD EQUATIONS

The generalized form of the gravitational field equation which maintains the properties of the Einstein equation is the Lovelock equation. Thus, we consider third order Lovelock gravity \[28\] with a cosmological constant term, where the gravitational field equations are given by

\[
G_{\mu\nu} = G^{(1)}_{\mu\nu} + \alpha_2 G^{(2)}_{\mu\nu} + \alpha_3 G^{(3)}_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu},
\]

where \(G^{(1)}_{\mu\nu}\) is the Einstein tensor, the constants \(\alpha_i\)’s are the Lovelock coefficients, \(g_{\mu\nu}\) is the metric, the coupling constant of matter and gravity \(\kappa\) is set equal to 1 \((\kappa = 1)\), and \(\mathcal{T}_{\mu\nu}\) is the energy-momentum tensor. \(G^{(2)}_{\mu\nu}\) and \(G^{(3)}_{\mu\nu}\) are the second and third order Lovelock tensors provided by

\[
G^{(2)}_{\mu\nu} = \frac{1}{2}(R_{\mu\sigma\kappa\lambda} R_{\nu}^{\sigma\kappa\tau} - 2 R_{\mu\nu\rho\sigma} R^{\rho\sigma} - 2 R_{\mu\nu} R^{\rho}_{\nu} + R R_{\mu\nu}) - \frac{1}{2} \mathcal{L}_2 g_{\mu\nu},
\]

and

\[
G^{(3)}_{\mu\nu} = -3(4 R^\tau_{\rho\sigma\kappa\lambda} R_{\sigma\kappa\rho\tau} R^\lambda_{\tau\mu\nu} - 8 R^\tau_{\rho\sigma\kappa} R^{\rho}_{\lambda\sigma} R^\lambda_{\tau\mu\nu} + 2 R^\tau_{\nu\sigma\rho\lambda} R^{\rho\sigma}_{\tau\mu\nu} - 2 R^\tau_{\rho\sigma\kappa\lambda} R^{\rho\sigma}_{\tau\mu\nu} + 8 R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu}) + 8 R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} + 4 R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} - 8 R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} + 8 R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} - 8 R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} + 4 R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} - 2 R R_{\mu\nu} R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} - 2 R R_{\mu\nu} R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} + 4 R R_{\mu\nu} R^\tau_{\nu\sigma\rho\lambda} R_{\nu\sigma\rho\tau} R^\lambda_{\tau\mu\nu} - 2 R^2 R_{\mu\nu} - \frac{1}{2} \mathcal{L}_2 g_{\mu\nu},
\]

respectively, where

\[
\mathcal{L}_2 = R_{\mu\nu\rho\delta} R^{\mu\nu\rho\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2,
\]
is the Gauss-Bonnet Lagrangian and
\[
L_3 = 2R^\mu\nu\sigma\rho R_{\sigma\rho\mu\nu} + 8 R^\mu\nu R^\kappa\nu R_{\mu\nu\kappa} + 24 R^\mu\nu\sigma R_{\sigma\nu\mu} + 3 R R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + 24 R^\mu\nu R_{\sigma\nu} + 16 R^\mu\nu R_{\mu\nu} R_{\rho\sigma}\rho\sigma + 12 R R_{\mu\nu} + R^2, \tag{5}
\]
is the third order Lovelock Lagrangian.

It is worthwhile to mention that due to the absence of derivatives of the curvatures in Eq. (1), it does not include the derivatives of the metric higher than two. It is also remarkable to note that the dimension of the spacetime should be equal or larger than seven so that all the terms in (1) contribute in the field equation[28].

III. WORMHOLE SOLUTIONS: METRIC AND FIELD EQUATIONS

In this section, we aim to present the field equations of a 4-dimensional Morris and Thorne wormhole which is a part of an \( n \)-dimensional spacetime in third order Lovelock gravity. More specifically, we will investigate the properties and characteristics of wormholes in a 4-dimensional spacetime \( M^4 \) with the following metric
\[
g_{ab}dx^a dx^b = -e^{2\phi(r)} dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Sigma^2_{2(k)} , \tag{6}
\]
where \( a, b = 0, 1, 2, 3 \), and \( d\Sigma^2_{2(k)} \) represents the line element of a 2-dimensional hypersurface with constant curvature \( 2\hat{k} \), with \( \hat{k} = 0, \pm 1 \). The functions \( \phi(r) \) and \( b(r) \) are denoted the redshift function and shape function, respectively. We consider metric (6) as part of an \( n \)-dimensional spacetime \( M^4 \times K^{n-4} \) with the metric
\[
ds^2 = g_{ab}dx^a dx^b + r_0^2 \gamma_{ij} d\theta^i d\theta^j. \tag{7}
\]
where \( i, j = 4...(n-1) \), \( r_0 \) is a constant and \( \gamma_{ij} \) is the metric of \( (n-4) \)-dimensional \( K^{n-4} \) space which has constant curvature. The constant curvature of \( K^{n-4} \) is \( (n-4)(n-5)k \), where \( k = 0, \pm 1 \).

The metric (6) represents a traversable wormhole provided the function \( \phi(r) \) is finite everywhere and the shape function \( b(r) \) satisfies the following two conditions:
\[
d(r) \equiv b(r) - r \leq 0 \quad r \in [a_0, a), \tag{8}
\]
\[
F(r) \equiv rb' - b(r) < 0 \quad r \in (a, a_0), \tag{9}
\]
respectively, where \( a_0 \) is the throat radius and the prime denotes a derivative with respect to the radial coordinate \( r \). Note that the equality in Eq. (8) only occurs at \( r = a_0 \), i.e., for \( b(a_0) = a_0 \). The parameter \( a \) could, in principle, be as large as \(+\infty\) but it is typically set to be finite and resides at the throat neighbourhood. The first condition is due to the fact that the proper radial distance \( l(r) = \int_{a_0}^r [1 - b(r)/r]^{-1/2} dr \), should be real and finite for \( r > a_0 \). The second condition arises from the flaring-out condition [6].

The mathematical analysis and the physical interpretation will be simplified using a set of orthonormal basis vectors
\[
e_i = e^{-\phi(r)} \frac{\partial}{\partial t} , \quad e_r = \left(1 - \frac{b(r)}{r}\right)^{1/2} \frac{\partial}{\partial r} , \quad e_\theta = r^{-1} \frac{\partial}{\partial \theta} , \quad e_\phi = \begin{cases} \frac{1}{r} \frac{\partial}{\partial \varphi} & \text{for } \hat{k} = 1 \\ (r \sin \theta)^{-1} \frac{\partial}{\partial \varphi} & \text{for } \hat{k} = 0 \\ (r \sinh \theta)^{-1} \frac{\partial}{\partial \varphi} & \text{for } \hat{k} = -1 \end{cases} , \quad e_{ij} = (r_0 \sqrt{\gamma_{ij}})^{-1} \frac{\partial}{\partial \theta^i \theta^j}. \tag{10}
\]
Thus, Eq. (1) finally provides the following decomposition
\[
\mathcal{G}_{\hat{ab}} = \left\{ \frac{-(n-4)(n-5)k}{2r_0^2} \left[ 1 + \frac{k\alpha_2(n-6)(n-7)}{r_0^4} \right] + \frac{\alpha_3(n-7)(n-8)(n-9)}{r_0^4} \right\} \tilde{g}_{\hat{ab}}, \tag{11}
\]
\[
\mathcal{G}_{\hat{ij}} = -\frac{1}{2} \left\{ \frac{(n-5)(n-6)k}{r_0^2} \left[ 1 + \frac{k\alpha_2(n-7)(n-8)}{r_0^2} \right] + \frac{\alpha_3(n-7)(n-8)(n-9)(n-10)}{r_0^4} \right\} \tilde{R} + \frac{2\Lambda r_0^2}{(n-5)(n-6)k} \left[ \frac{1}{r_0^2} + \frac{3k^2\alpha_3(n-5)(n-6)(n-7)(n-8)}{r_0^4} \right] \tilde{L}_2 \tag{12}
\]
where the superscripts “tilde” represent quantities on \( M^4 \).
In the following, we construct a 4-dimensional wormhole solution where $\mathcal{K}^{n-4}$ space is empty. In general, $\hat{R}$ and $\hat{L}_2$ do not vanish, however, if the three brackets in Eq. (12) vanish, we have $\hat{G}_{ij} = 0$. Thus, setting the quantities in the three square brackets in Eq. (12) equal to zero, for $k \neq 0$, one obtains for the specific cases of $k = \pm 1$, the following relations
\begin{equation}
\alpha_2 = \frac{-r_0^2}{\Xi k}, \quad (13)
\end{equation}
\begin{equation}
\alpha_3 = \frac{r_0^4}{3(n - 5)(n - 6)\Xi}, \quad (14)
\end{equation}
\begin{equation}
\Lambda = \frac{k}{6\Xi r_0^2} \{ 6(n - 5)^2(n - 6)^2 - (n - 7)(n - 8)
\times [6(n - 5)(n - 6) - (n - 9)(n - 10)] \}, \quad (15)
\end{equation}
where, for notational simplicity, we have defined
\begin{equation}
\Xi = 2(n - 5)(n - 6) - (n - 7)(n - 8). \quad (16)
\end{equation}
Note that in order to have a finite value for $\alpha_3$, we need $n \geq 7$ and therefore $\Xi > 0$. Now inserting the above values of $\Lambda$, $\alpha_2$ and $\alpha_3$ in Eq. (11), one obtains
\begin{equation}
\hat{G}_{\hat{a}\hat{b}} = \frac{8}{\Xi} \left( \hat{G}_{\hat{a}\hat{b}} + g_{\hat{a}\hat{b}}\Lambda_{\text{eff}} \right), \quad (17)
\end{equation}
where
\begin{equation}
\Lambda_{\text{eff}} = \left( \frac{2n - 13}{r_0^2} \right). \quad (18)
\end{equation}
One should note that $\hat{G}_{\hat{a}\hat{b}} \neq 0$ even if $\hat{G}_{\hat{a}\hat{b}} = 0$. Therefore, one may consider the effects of higher curvature terms of Lovelock gravity in 4-dimensional solutions as a cosmological constant term with an effective constant $\Lambda_{\text{eff}}$, given by Eq. (18). Furthermore, $\Lambda_{\text{eff}}$ can be positive or negative for $k = 1$ or $k = -1$, respectively (note that $n \geq 7$).

Next, we calculate the energy density, radial tension and pressure associated to the matter of the 4-dimensional manifold $\mathcal{M}^4$, in order to study the properties and characteristics of the solution. Using the orthonormal basis (10), the components of the energy-momentum tensor $\hat{T}_{\hat{a}\hat{b}}$, carry a simple physical interpretation, i.e.,
\begin{equation}
\hat{T}_{\hat{t}\hat{t}} = \hat{G}_{\hat{t}\hat{t}} \equiv \rho(r), \quad \hat{T}_{\hat{r}\hat{r}} = \hat{G}_{\hat{r}\hat{r}} \equiv -\tau(r), \quad \hat{T}_{\hat{\theta}\hat{\theta}} = \hat{G}_{\hat{\theta}\hat{\theta}} \equiv p(r), \quad \hat{T}_{\hat{\phi}\hat{\phi}} = \hat{G}_{\hat{\phi}\hat{\phi}} \equiv \rho(r), \quad (19)
\end{equation}
where $\rho$ is the energy density, $\tau$ is the radial tension, and $p$ is the pressure measured in the tangential directions orthogonal to the radial direction. The radial tension is $\tau = -p_r$, where $p_r$ is the radial pressure. Calculating the components of $\hat{G}_{\hat{a}\hat{b}}$ for the metric (6), we obtain the following energy-momentum profile
\begin{equation}
\rho(r) = \frac{8}{\Xi} \left( \frac{b'}{r^2} + \frac{k - 1}{r^2} - \Lambda_{\text{eff}} \right), \quad (20)
\end{equation}
\begin{equation}
\tau(r) = -\frac{8}{\Xi} \left( \frac{b'}{r^3} - 2 \left( 1 - \frac{b}{r} \right) \frac{\phi'}{r} + \frac{k - 1}{r^2} - \Lambda_{\text{eff}} \right), \quad (21)
\end{equation}
\begin{equation}
p(r) = -\frac{8}{\Xi} \left( \frac{1 - b}{r} \left( \frac{\phi''}{r} + \frac{b'r - b}{2r(r - b)} \phi' \right)
- \frac{b'r - b}{2r^2(r - b)} \phi' + \frac{\phi'}{r} + \Lambda_{\text{eff}} \right). \quad (22)
\end{equation}
Applying the conservation of the energy momentum tensor $\hat{T}_{\hat{\mu}\hat{\nu}}$ for $\hat{\mu} = r$, the relativistic Euler equation is given by
\begin{equation}
\tau' + (\tau - \rho)\phi' + \frac{2}{r} (p + \tau) = 0. \quad (23)
\end{equation}

In the next section, we study the physical properties of wormhole solutions by applying an equation of state between components of energy-momentum tensor.

**IV. PHYSICAL PROPERTIES OF THE WORMHOLE SOLUTIONS**

In this section we study the physical properties of the wormhole solutions, by considering a specific class of solutions corresponding to the choice of $\phi(r) = \text{const} = \Phi$. In order to solve Eqs. (20)-(22) for $b(r)$, we assume the equation of state (EOS) given by $\rho = (\gamma p - \tau)/[\omega(1 + \gamma)]^\gamma$ [32], so that one finally obtains the following solution
\begin{equation}
b(r) = \left( \tilde{k} - a_0^2 \beta \right) a_0^{1-\eta} r^n + \beta r^3 - (\tilde{k} - 1) r, \quad (24)
\end{equation}
where
\begin{equation}
\beta = \frac{(\omega + 1)(2n - 13)k}{r_0^2(3\omega + 1)}, \quad \eta = \frac{\gamma - 2}{2(1 + \gamma)\omega + \gamma}. \quad (25)
\end{equation}
It is easy to check that Eq. (24) satisfies the condition $b(a_0) = a_0$. One can see from (24) that the wormhole solution is asymptotically flat, i.e., $b(r)/r \to 0$ as $r \to \infty$, provided that $\beta = 0$, $\tilde{k} = 1$ and $\eta < 1$. Since $k \neq 0$ [we refer the reader to the discussion above Eq. (13)], $\beta = 0$ is equivalent to $\omega = -1$ and therefore $\eta$ reduces to $(2 - \gamma)/(2 + \gamma)$ for this case. Thus, the condition $\eta < 1$ for the asymptotically flat solution is satisfied provided $\gamma > 0$ or $\gamma < -2$. With the shape function (24) in hand, we now investigate the energy conditions and the physical properties of the wormhole, for both the asymptotically flat and general cases.

\footnote{It is worth noting that for an isotropic matter i.e. $-\tau = p$, the EOS reduces to the well-known isotropic EOS $p = \omega \rho$.}
A. Energy conditions

Here, we analyse the energy conditions for the energy-momentum tensor profile described by the system of equations (20)-(22). In particular, we consider the weak energy condition (WEC), which states that \( T_{\mu\nu}u^\mu u^\nu \geq 0 \) where \( u^\mu \) is the timelike velocity of the observer. In terms of the non-zero components of the diagonal energy-momentum tensor, we have the following inequalities: \( \rho \geq 0, \rho - \tau \geq 0 \) and \( \rho + p \geq 0 \). Note that the last two inequalities are referred to as the null energy condition (NEC).

1. Asymptotically flat case

In the asymptotically flat case where \( \beta = 0 \) (or \( \omega = -1 \)), \( \hat{k} = 1 \) and \( \eta < 1 \), Eqs. (20)-(22) and solution (24) yield the following relations

\[
\rho = \frac{8}{\Xi} \left( -\eta a_0^{-1-\eta} r^{-3} + \frac{2n - 13k}{r_0^2} \right) \geq 0, \quad (25)
\]

\[
\rho - \tau = \frac{8}{\Xi} (1 - \eta) a_0^{-1-\eta} r^{-3} \geq 0, \quad (26)
\]

\[
\rho + p = -\frac{4}{\Xi} (1 + \eta) a_0^{-1-\eta} r^{-3} \geq 0. \quad (27)
\]

For \( \eta \leq -1 \), the inequalities (26) and (27) are readily satisfied, and consequently the NEC is also obeyed (note that \( \Xi > 0 \)). One can also see that the inequality (25) is satisfied for suitable choices of the parameters. For instance, for \( \eta \leq 0 \) and \( k = 1 \), we have \( \rho \geq 0 \) (note that \( k \neq 0 \) and \( n \geq 7 \); see discussions above Eq. (13) and below Eq. (15)). In conclusion, for the asymptotically flat case, the WEC is satisfied for suitable choice of parameters, for example, \( \eta \leq -1 \) and \( k = 1 \).

2. General case

For the general case, by taking into account Eqs. (20)-(22) and (24), one arrives at the following inequalities

\[
\rho = \frac{8}{\Xi} \left[ (\hat{k} - a_0^2 \beta - \hat{k}) \eta a_0^{-1-\eta} r^{-3} - \frac{2\beta}{1 + \omega} \right] \geq 0, \quad (28)
\]

\[
\rho - \tau = \frac{8}{\Xi} \left[ (\hat{k} - a_0^2 \beta)(1 - \eta) a_0^{-1-\eta} r^{-3} - 2\beta \right] \geq 0, \quad (29)
\]

\[
\rho + p = \frac{8}{\Xi} \left[ \frac{1}{2} (a_0^2 \beta - \hat{k})(1 + \eta) a_0^{-1-\eta} r^{-3} - 2\beta \right] \geq 0, \quad (30)
\]

from which it is clear that satisfying the WEC relies on the values of \( r, a_0 \) and \( r_0 \), in general. However, independently of the specific choice of the latter parameters, i.e., \( r, a_0 \) and \( r_0 \), one finds that the WEC is satisfied for \( k = 0,1 \) provided that

\[
\beta < 0, \quad \text{and} \quad \eta \leq -1. \quad (31)
\]

These inequalities can be expressed in terms of \( \omega \) and \( \gamma \) as follows:

1. \( k = -1 \): In this case, the WEC is satisfied provided

a. \( \omega < -1 \) and \( \frac{1 - \omega}{1 + \omega} \leq \gamma < 2 \)

b. \( \omega > \frac{1}{3} \) and \( \gamma \leq \frac{1 - \omega}{1 + \omega} \). \quad (32)

2. \( k = 1 \): In this case, inequality (31) is satisfied provided

\[
-1 < \omega < \frac{1}{3} \quad \text{and} \quad \gamma \geq \frac{1 - \omega}{1 + \omega} \quad (33)
\]

B. Conditions imposed on the shape function

In this section, we analyse the conditions (8) and (9) for both the asymptotically flat and general wormhole solutions considered above.

1. Asymptotically flat case

The conditions (8) and (9) for the asymptotically flat solution \([\beta = 0 \text{ or } \omega = -1], k = 1 \text{ and } \eta < 1\) reduce to the following

\[
d(r) = a_0^{\frac{1}{1-\eta}} r^{-3} - r \leq 0, \quad (34)
\]

\[
F(r) = (\eta - 1)a_0^{\frac{1}{1-\eta}} r^{\eta} < 0. \quad (35)
\]

Note that the inequality (35) is readily satisfied. In order to analyse (34), one can rewrite it as \((a_0/r)^{1-\eta} \leq 1\) which is obviously satisfied since \( r \geq a_0 \) (note that \( 1 - \eta > 0 \) for the asymptotically flat solution).

2. General case

The conditions (8) and (9) for the general wormhole solution (24) take the form

\[
d(r) = \left( \hat{k} - a_0^2 \beta \right) a_0^{\frac{1}{1-\eta}} r^{\eta} + \beta r^3 - \hat{k} r \leq 0, \quad (36)
\]

\[
F(r) = \left( \hat{k} - a_0^2 \beta \right)(\eta - 1)a_0^{\frac{1}{1-\eta}} r^{\eta} + 2\beta r^3 < 0, \quad (37)
\]

respectively. For \( \hat{k} = 0 \), the above two conditions reduce to

\[
\beta \left[ 1 - \left( \frac{r}{a_0} \right)^{\frac{1}{\eta} - 3} \right] \leq 0. \quad (38)
\]
FIG. 1: Left: $d(r)$ and $F(r)$ versus $r$; Right: $\rho$, $\rho - \tau$ and $\rho + p$ versus $r$ for $n = 8$, $a_0 = 1$, $r_0 = 0.001$, $\omega = 0.3$, $k = -1$, $\gamma = -0.3$ and $\hat{k} = 1$. The scale of the vertical axes of both figures is divided by $10^7$. Note that the local maximum of the $F(r)$ corresponds to the change of concavity of the $d(r)$ or similarly $b(r)$.

FIG. 2: Left: $d(r)$ and $F(r)$ versus $r$; Right: $\rho$, $\rho - \tau$ and $\rho + p$ versus $r$ for $n = 8$, $a_0 = 1$, $r_0 = 0.001$, $\omega = 1$, $k = -1$, $\gamma = 0.5$ and $\hat{k} = 1$. Note that the scale of the vertical axes of both figures is divided by $10^7$.

\[
\beta \left[ 1 - \frac{\eta - 1}{2} \left( \frac{r}{a_0} \right)^{-3} \right] < 0, \tag{39}
\]

which are satisfied if

$\beta > 0$ and $\eta \geq 3$, or $\beta < 0$ and $\eta \leq 3$. \tag{40}

The intersection of the two conditions (31) and (40) is simply condition (31). Note that condition (31) in terms of $\omega$ and $\gamma$ is given by the inequalities (32) and (33).

Since $a_0$ is the root of $d(r)$, one may discuss the negativity of $d(r)$ in the case of $\hat{k} = \pm 1$ by expanding $d(r)$ around $r = a_0$. Thus, one obtains

\[
d(r)|_{r=a_0+\epsilon} \approx \epsilon \left[ \beta a_0^2(3 - \eta) + \hat{k}(\eta - 1) \right], \tag{41}
\]

where $\epsilon$ is an infinitesimal positive parameter. Using Eq. (41), condition (8) reduces to

\[
\beta a_0^2(3 - \eta) + \hat{k}(\eta - 1) \leq 0
\]
or equivalently

\[
2a_0^2\beta + (\eta - 1)(\hat{k} - a_0^2\beta) \leq 0, \tag{42}
\]

which may be satisfied for specific choice of $a_0$ and $r_0$. More specifically, the conditions under which (42) is satisfied for arbitrary values of $a_0$ and $r_0$ are given by:

\[
\begin{align*}
\text{for } \hat{k} &= 1 \Rightarrow \beta < 0 \text{ and } \eta \leq 1, \\
\text{for } \hat{k} &= -1 \Rightarrow \begin{cases} 
\beta > 0 \text{ and } \eta \geq 3 \\
\beta < 0 \text{ and } 1 \leq \eta \leq 3
\end{cases}. \tag{43}
\end{align*}
\]

Using (42), one deduces the following inequality:

\[
F(r) = (\hat{k} - a_0^2\beta)(\eta - 1)a_0^{1-\eta}r^n + 2\beta r^3 \leq 2\beta(r^3 - a_0^{3-\eta}r^n). \tag{44}
\]
The parameter \( a_0 \) is the root of rhs of inequality (44) and the condition (9) is satisfied provided the rhs is negative

\[
\text{rhs} \big|_{r=a_0+\epsilon} \approx 2a_0^2 \beta (3 - \eta) \leq 0, \tag{45}
\]

which is satisfied if

\[
\beta > 0 \quad \text{and} \quad \eta \geq 3, \quad \text{or} \quad \beta < 0 \quad \text{and} \quad \eta \leq 3.
\tag{46}
\]

Note that the intersection of the two sets (46) and (43) is simply (43). Therefore, the conditions (8) and (9) are satisfied if and only if (43) are satisfied. Furthermore, the intersection of the two sets (31) and (43) is condition (31). Thus, conditions (8) and (9), and the energy conditions are satisfied if and only if (31) is satisfied. (As mentioned before, (31) is expressed in terms of \( \omega \) and \( \gamma \) by (32) and (33)).

The above discussions are depicted in Figs. (1) and (2). The left hand side plots of Figs. (1) and (2) show the negativity of the functions \( d(r) \) and \( F(r) \) for two appropriate settings of parameters. The right hand side plots show that for these sets of parameters the WEC is also satisfied. Therefore, it is shown that by considering appropriate parameters, it is possible to construct a wormhole with physically reasonable properties and with normal matter.

\section{C. Spherically-symmetric wormholes in an Ads universe}

In this subsection, we consider a wormhole solution in the spacetime \( r \in [a_0, a] \). In the exterior region \( r > a \), the metric may satisfy the field equation \( G_{\hat{a}\hat{b}} = 0 \), where \( G_{\hat{a}\hat{b}} \) is given in Eq. (17). Thus, the metric of a spherically symmetric spacetime for \( r > a \) may be written as [18]

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{2(k)},
\tag{47}
\]

where

\[
f(r) = \hat{k} - \frac{2m}{r} - \frac{\Lambda_{\text{eff}}}{3} r^2.
\tag{48}
\]

The background shows an asymptotically de–Sitter (dS) or anti de-Sitter (AdS) spacetime for \( k = 1 \) or \( k = -1 \), respectively (note that \( \Lambda_{\text{eff}} = (2n - 13)k/r_0^2 \)). Using the wormhole metric (6), the continuity of the metric at the boundary \( r = a \) leads to

\[
f(a) = e^{2\Phi},
\tag{49}
\]

\[
b(a) = a - ae^{2\Phi}.
\tag{50}
\]

In the case of \( \hat{k} = 1 \), by setting \( \Phi = 0 \) one has

\[
\frac{2m}{a} + \frac{\Lambda_{\text{eff}}}{3} a^2 = 0 \quad \Rightarrow \quad a = \left( \frac{6m}{\Lambda_{\text{eff}}} \right)^{1/3},
\]

and \( b(a) = 0 \), where \( m \) may be set equal to the mass inside the region \( r \in [a_0, a] \), which can be calculated by using Eq. (20) as

\[
m = \int_{a_0}^{a} \left( \frac{6m}{\Lambda_{\text{eff}}} \right)^{1/3} 4\pi \rho(r) r^2 dr
\]

\[
= \frac{32\pi}{\Xi} \left[ \frac{a_0^3 \Lambda_{\text{eff}}}{3} + 2m - a_0 \right].
\tag{51}
\]

Note that in above calculations we do not use any specific solution for \( b(r) \), namely, in order to determine Eq. (51), only the conditions \( b(a) = 0 \) and \( b(a_0) = a_0 \) were used. Using Eq. (51), \( m \) can be calculated as

\[
m = \frac{32\pi a_0}{\Xi + 64\pi} \left[ 1 - \frac{a_0^2 \Lambda_{\text{eff}}}{3} \right]
\]

\[
= \frac{32\pi a_0}{\Xi + 64\pi} \left[ 1 - \frac{a_0^2 (2n - 13)k}{3r_0^2} \right]
\tag{52}
\]

Since \( a = (-6m/\Lambda_{\text{eff}})^{1/3} \) and \( m \) should be positive, one can find that (note \( n \geq 7 \))

\[
\Lambda_{\text{eff}} < 0 \quad \text{or equivalently} \quad k = -1.
\]

Therefore \( m \) is given by

\[
m = \frac{32\pi a_0}{\Xi + 64\pi} \left[ 1 + \frac{a_0^2 (2n - 13)}{3r_0^2} \right] > 0.
\]

It is remarkable to note that the possibility of an accelerating expansion in theories with negative cosmological constant has been shown recently [33].

\section{V. SUMMARY AND DISCUSSION}

In this paper, we first decomposed the field equations of third order Lovelock gravity in the presence of cosmological constant and matter field in an \( n \)-dimensional spacetime, \( M^4 \times K^{n-4} \), into two gravitational field equations. We fixed the Lovelock coefficients and the cosmological constant in terms of the size of the extra dimensions \( r_0 \) and the dimension of the spacetime by applying the idea that the \( K^{n-4} \) space is empty. This idea also implies that \( n \geq 7 \).

Furthermore, the energy density, the radial and tangential pressures threading the wormhole solution were obtained using fixed Lovelock coefficients and \( \Lambda \). As an interesting example, the equation of state \( \rho = (\gamma p - \tau)/[\omega(1 + \gamma)] \) [32] was used to find the exact solution for the shape function in the case of a zero tidal force, i.e., of a constant redshift function. Next, the energy conditions were substantially investigated and the conditions under which the normal matter could sustain a wormhole structure were analysed. Then, physical properties and characteristics of the wormhole solutions were investigated by imposing specific conditions on the shape function.
Indeed, it was shown that it is possible to have asymptotically flat and non-flat traversable wormhole structure satisfying the WEC (see Figs. 1 and 2). We considered a limited wormhole in a 4-dimensional asymptotically non-flat spacetime, and found the relationship between the wormhole parameters and the cosmological constant corresponding to the 4-dimensional spacetime. It was shown that if a spherically symmetric wormhole is constructed with normal matter, this implies a negative cosmological constant. In this context, it is worth mentioning that the possibility of accelerating expansion in the presence of a negative cosmological constant has been shown recently [33].

In the analysis outlined in this paper, we have followed the second order formalism, by assuming torsion to be zero. However, in the first order approach, where the independent dynamical variables are the vielbein and the spin connection, which obey first order differential field equations, the equations of motion do not imply the vanishing of torsion, which is a propagating degree of freedom [34]. In addition to this, an advantage of the first order formalism is that it can be written out entirely in terms of differential forms and their exterior derivatives, without the introduction of the inverse vielbein. Several exact solutions with non-trivial torsion have been extensively investigated in the literature [35–37]. More specifically, in Lovelock gravity using the first order formalism, the equations of motion do not imply the vanishing of torsion in vacuum as in general relativity, so that torsion may also have propagating degrees of freedom, as mentioned above. Thus, an interesting future project would be to extend the analysis carried out in this work, in the first order formalism, by considering the approach outlined in [37].

The latter approach is particularly interesting in light of the vacuum static compactified wormholes found in 8-dimensional Lovelock theory [36]. Indeed, the exact static vacuum 8-dimensional solutions were constructed with the structure of $M_5 \times \Sigma_3$, where $M_5 = M_2 \times F(r)N_3$ is a 5-dimensional manifold, where $M_2$ plays the role of an $r-t$ plane, $N_3$ is a constant curvature manifold, $F(r)$ is a warp factor, and $\Sigma_3$ is a compact constant curvature manifold that plays the role of a compactified 3-dimensional space. The role of torsion is of particular interest as it was shown that these wormholes solutions with torsion act as a geometrical filter, which may distinguish between scalars and spinors and also between the helicities of particles. More specifically, it was also shown that a very large torsion may increase the traversability properties for scalars and a “polarizator” on spinning particles. In comparing these latter results with those outlined in the present paper, we have been essentially interested in the analysis of the energy conditions, and our results differ radically from those found in [36]. However, it is of interest to extend our analysis, in third-order Lovelock theory, to include the effects of torsion, and not only investigate the traversability conditions but to analyse the energy conditions. Work along these lines is presently under consideration.

An interesting and important issue is related to the stability of these solutions. One may ask which family of compactified wormholes is more stable? Indeed, the stability issue in Lovelock theory is very complicated and certainly lies outside the scope of this paper. However, as a first approach one may analyze just the Euclidean actions of the different configurations. At lowest order, the Euclidean action provides the free energy and so one could compare the novel solutions presented in this work with some of the already known solutions by comparing the corresponding Euclidean actions. In this way, one could find the range of parameters in which the novel solutions are favored with respect to previous ones. However, this stability analysis lies completely outside the scope of this paper and will be addressed in future work, in addition to a full-fledged stability analysis. Work along these lines are currently underway.

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