EXTENDED DERIVATIVE DISPERSION RELATIONS

R.F. ÁVILA
Instituto de Matemática, Estatística e Computação Científica
Universidade Estadual de Campinas, 13083-970 Campinas, SP, Brazil

M.J. MENON
Instituto de Física Gleb Wataghin
Universidade Estadual de Campinas, 13083-970 Campinas, SP, Brazil

It is shown that, for a wide class of functions with physical interest as forward scattering amplitudes, integral dispersion relations can be replaced by derivative forms without any high-energy approximation. The applicability of these extended derivative relations, in the investigation of forward proton-proton and antiproton-proton elastic scattering, is exemplified by means of a Pomeron-Reggeon model with totally nondegenerate trajectories.

Contribution to "Sense of Beauty in Physics"
Miniconference in Honor of Adriano Di Giacomo on his 70th Birthday
Pisa, Italy, Jan. 26-27, 2006

1 Introduction

Dispersion relations constitute a traditional and important mathematical tool in several areas of physics, not only as a formal theoretical result, but also as a powerful phenomenological framework. In particular, for elastic hadron-hadron scattering, analyticity, unitarity and crossing lead to dispersion relations, which connect the real and the imaginary parts of the amplitude as function of the energy, allowing simultaneous investigation of particle-particle and antiparticle-particle scattering. Originally introduced in integral forms\textsuperscript{1}, these Integral Dispersion Relations (IDR) have two kinds of limitations: their nonlocal character (in order to evaluate the real part, the imaginary part must be known over all the integration space) and the restricted class of functions that allows analytical integration. The first limitation (and, in part, the second too) can be avoided by means of Derivative Dispersion Relations (DDR), which have been established only for the region of high and asymptotic energies\textsuperscript{2}. In a recent work\textsuperscript{3} we present a critical review on the replacement of IDR by DDR and also references to outstanding results. However, despite the important developments provided by this derivative approach, the high-energy condition (specifically, center-of-mass energies above 10 - 20 GeV) turns out difficult any attempt to perform global fits to the experimental data, connecting information from low and high energy regions.

In a previous paper\textsuperscript{4} we have shown that, for a class of functions of physical interest as forward elastic scattering amplitudes, the IDR can be replaced by derivative forms without the high-energy approximation, which we call Extended Derivative Dispersion Relations (EDDR). In this communication we first review the replacement of IDR by the EDDR and then discuss a novel example on the practical applicability of both forms in the context of a Pomeron-Reggeon model, with totally nondegenerate secondary trajectories. In Sec. 2 we recall the main formulas related with the IDR and the standard DDR (high-energy approximation assumed). In Sec. 3 we shortly review the replacement of IDR by
the EDDR and then discuss the applicability of these dispersion relations in simultaneous
description of the experimental data on the total cross section and the ratio $\rho$ of the real to
imaginary parts of the forward amplitude, from proton-proton ($pp$) and antiproton-proton
($\bar{p}p$) scattering. The conclusions are the contents of Sec. 4.

2 Integral and Derivative Dispersion Relations

For an elastic process, $m + m \rightarrow m + m$, in the forward direction, IDR are expressed in
terms of a crossing symmetric variable, which corresponds to the energy of the incident
particle in the laboratory system, $E$. For elastic $pp$ and $\bar{p}p$ scattering, polynomial bound-
edness demands one subtraction and for crossing even (+) and odd (−) amplitudes, in
the physical region ($E : m \rightarrow \infty$), the IDR read

$$
\text{Re} F_+(E) = \frac{2E^2}{\pi} \int_m^{+\infty} \frac{1}{E'(E'^2 - E^2)} \text{Im} F_+(E') dE',
$$

(1)

$$
\text{Re} F_-(E) = \frac{2E}{\pi} \int_m^{+\infty} \frac{1}{E'(E'^2 - E^2)} \text{Im} F_-(E') dE',
$$

(2)

where we have omitted the subtraction constant.

Basically, at high energies, the replacement of IDR by DDR is analytically performed
by considering the limit $m \rightarrow 0$ in Eqs. (1) and (2). Expansion of the integrand and then
integration term by term lead to the standard DDR

$$
\text{Re} F_+(E) = E \tan \left[ \frac{\pi}{2} \frac{d}{d \ln E} \frac{\text{Im} F_+(E)}{E} \right],
$$

(3)

$$
\text{Re} F_-(E) = \tan \left[ \frac{\pi}{2} \frac{d}{d \ln E} \frac{\text{Im} F_-(E)}{E} \right].
$$

(4)

Conditions on the convergence of the above tangent series will be specified in what fol-

3 Extended Derivative Relations

3.1 Analytical Results

Let us consider the even amplitude, Eq. (1). Details on the calculation can be found in
our previous work [4]; here we only summarize the four main steps: (a) integration of Eq.
(1) by parts; (b) change of variable $E \rightarrow me^\xi$; (c) expansion of the integrand in power
series; (d) integration term by term, under the assumption of uniform convergence of the
series associated with the function

$$
\frac{d}{d \ln E} \frac{\text{Im} F(E)}{E}.
$$

(5)

With this procedure and returning to the variable $E$ we obtain
\[ \text{Re } F_+(E) = -\frac{E}{\pi} \ln \left| \frac{m - E}{m + E} \right| \frac{\text{Im } F_+(m)}{m} \]
\[ + \frac{4E}{\pi} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{d^{2k+1}}{d(\ln E)^{2k+1}} \left( \frac{\text{Im } F_+(E)}{E} \right) \frac{1}{(2p+1)^{2k+2}} \]
\[ + \frac{2E}{\pi} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{d^{2k+1}}{d(\ln E)^{2k+1}} \left( \frac{\text{Im } F_+(E)}{E} \right) \frac{(-1)^{k+1} \Gamma(k+1, (2p+1) \ln(E/m))}{(2p+1)^{k+2}k!} \]

which can be put in the final form

\[ \text{Re } F_+(E) = E \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) \frac{\text{Im } F_+(E)}{E} + \Delta^+(E, m), \quad (6) \]

where the factor \( \Delta^+ \) is given by

\[ \Delta^+(E, m) = -\frac{E}{\pi} \ln \left| \frac{m - E}{m + E} \right| \frac{\text{Im } F_+(m)}{m} \]
\[ + \frac{2E}{\pi} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{d^{2k+1}}{d(\ln E)^{2k+1}} \left( \frac{\text{Im } F_+(E)}{E} \right) \frac{(-1)^{k+1} \Gamma(k+1, (2p+1) \ln(E/m))}{(2p+1)^{k+2}k!} \]

With analogous procedure for the odd relation we obtain

\[ \text{Re } F_-(E) = \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) \frac{\text{Im } F_-(E)}{E} + \Delta^-(E, m), \quad (7) \]

where

\[ \Delta^-(E, m) = -\frac{1}{\pi} \ln \left| \frac{m - E}{m + E} \right| \frac{\text{Im } F_-(m)}{m} \]
\[ + \frac{2}{\pi} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{d^{2k+1}}{d(\ln E)^{2k+1}} \left( \frac{\text{Im } F_-(E)}{E} \right) \frac{(-1)^{k+1} \Gamma(k+1, (2p+1) \ln(E/m))}{(2p+1)^{k+2}k!} \]

Equations (6) and (7) are the novel EDDR, which are valid, in principle, for any energy above the physical threshold \( E > m \). We note that the factors \( \Delta^\pm \to 0 \) as \( E \to \infty \), leading, in this case, to the standard DDR, Eqs. (3) and (4).

Necessary and sufficient conditions for the convergence of the tangent series have been established by Kolář and Fischer\(^6\), in particular through the following theorem:

**Theorem 1** Let \( f : \mathbb{R}^1 \to \mathbb{R}^1 \). The series

\[ \tan \left[ \frac{\pi}{2} \frac{d}{dx} \right] f(x) \]
converges at a point $x \in \mathbb{R}$ if and only if the series

$$\sum_{n=0}^{\infty} f^{(2n+1)}(x)$$

is convergent.

Since this Theorem insures the uniform convergence of the series expansion associated with (5), the above condition defines the class of functions for which the EDDR hold. Other conditions are discussed by Kolár and Fischer.6

### 3.2 Applicability in Elastic Hadron Scattering

An important practical use of the derivative relations concerns simultaneous investigations on the total cross section (Optical Theorem) and the ratio $\rho$ of the real to imaginary parts of the forward amplitude. In terms of the symmetrical variable $E$ these physical quantities are given, respectively, by

$$\sigma_{\text{tot}} = \frac{4\pi}{\sqrt{E^2 - m^2}} \text{Im} F(E, \theta_{\text{lab}} = 0), \quad \rho(E) = \frac{\text{Re} F(E, \theta_{\text{lab}} = 0)}{\text{Im} F(E, \theta_{\text{lab}} = 0)}, \quad (8)$$

where $\theta_{\text{lab}}$ is the scattering angle in the laboratory system.

In order to check the consistences between the IDR and the EDDR in a specific practical example, we consider, as a framework, a Pomeron-Regge parametrization for the scattering amplitude, in which all the associated secondary reggeon contributions are nondegenerate. In this case, for $pp$ and $\bar{p}p$ scattering, the even ($+$) contributions comes from the $a_2$ and $f_2$ trajectories and the odd ($-$) contributions from the $\rho$ and $\omega$ trajectories; the full parametrization also includes a simple pole Pomeron contribution:

$$\text{Im} F(E) = X E^{\alpha_{\text{PP}}(0)} + Y_{a_2} E^{\alpha_{a_2}(0)} + Y_{f_2} E^{\alpha_{f_2}(0)} + \tau \left[ Y_{\rho} E^{\alpha_{\rho}(0)} + Y_{\omega} E^{\alpha_{\omega}(0)} \right], \quad (9)$$

where $\tau = +1$ for $pp$ and $\tau = -1$ for $\bar{p}p$. As usual, the Pomeron and the Reggeon intercepts are expressed by

$$\alpha_{\text{PP}}(0) = 1 + \epsilon, \quad \alpha_{\text{i}}(0) = 1 - \eta_i, \quad (10)$$

where $i = a_2, f_2, \rho$ and $\omega$.

The point is to treat simultaneous fits to the total cross section and the $\rho$ parameter from $pp$ and $\bar{p}p$ scattering and compare the results obtained with IDR, EDDR and also standard DDR. Schematically, with parametrization (9-10) for $pp$ and $\bar{p}p$ we determine $\text{Im} F_{+/-}(E)$ through the usual definitions

$$F_{+} = \frac{F_{pp} + F_{\bar{p}p}}{2}, \quad F_- = \frac{F_{pp} - F_{\bar{p}p}}{2} \quad (11)$$

and then $\text{Re} F_{+/-}(E)$ by means of the IDR, Eqs. (1-2), DDR, Eqs. (3-4) and EDDR, Eqs. (6-7). Returning to Eq. (11) we obtain $\text{Re} F_{pp}(E)$ and $\text{Re} F_{\bar{p}p}(E)$ and, at last, Eq. (8) leads to the analytical connections between $\sigma_{\text{tot}}(E)$ and $\rho(E)$ for both reactions.
Table 1: Simultaneous fits to $\sigma_{\text{tot}}$ and $\rho$, from $pp$ and $\bar{p}p$ scattering, for $\sqrt{s}_{\text{min}} = 3$ GeV (308 data points), using Integral Dispersion Relations (IDR), standard Derivative Dispersion Relations (DDR) and the Extended Derivative Dispersion Relations (EDDR)

|        | IDR             | DDR             | EDDR            |
|--------|-----------------|-----------------|-----------------|
| $X$ (mb) | 1.6634 ± 0.0093 | 1.7586 ± 0.0080 | 1.6634 ± 0.0093 |
| $Y_{a_2}$ (mb) | -16.779 ± 0.038  | -19.600 ± 0.055 | -16.779 ± 0.038 |
| $Y_{f_2}$ (mb) | 20.792 ± 0.038   | 23.063 ± 0.054  | 20.792 ± 0.038  |
| $Y_{\rho}$ (mb) | -0.334 ± 0.038   | 0.441 ± 0.056   | -0.334 ± 0.038  |
| $Y_{\omega}$ (mb) | -0.334 ± 0.014  | 0.574 ± 0.047   | -0.334 ± 0.014  |
| $\epsilon$  | 0.08869 ± 0.00065| 0.08402 ± 0.00061| 0.08869 ± 0.00065|
| $\eta_{a_2}$ | 0.37679 ± 0.00091| 0.5873 ± 0.0014  | 0.37679 ± 0.00091|
| $\eta_{f_2}$ | 0.37681 ± 0.00074| 0.5389 ± 0.0010  | 0.37681 ± 0.00074|
| $\eta_{\rho}$ | 0.334 ± 0.014    | 0.574 ± 0.047   | 0.334 ± 0.014   |
| $\eta_{\omega}$ | 0.700 ± 0.014    | 0.5746 ± 0.0085 | 0.700 ± 0.014   |
| $\chi^2$     | 478.8           | 405.7           | 478.8           |
| $\chi^2/F$   | 1.61            | 1.36            | 1.61            |

For $\sigma_{\text{tot}}$ and $\rho$, we have compiled all the experimental data available above the physical threshold. The fits were performed through the CERN-Minuit code, using the variable $s = 2(m^2 + mE)$. However, with the present model (intended for the high-energy region), the large number of experimental points just above this threshold allows reasonable statistical results only for an energy cutoff at $\sqrt{s}_{\text{min}} = 3$ GeV. The numerical results and statistical information on the fits are displayed in Table 1 and the corresponding curves together with the experimental data are shown in Fig. 1.

These results demonstrate the complete equivalence between the IDR and the EDDR; moreover, the high-energy approximation (DDR) indicates a slower increase for the total cross section at the highest energies than that obtained with the exact results (IDR and EDDR) and different behavior for $\rho(s)$ at low energies. We note that the high values for $\chi^2/F$, in all the cases, are consequences of the particular model considered (intended for high energies) and the fact that we have neglected the subtraction constant as a free fit parameter. The important role played by this parameter is discussed in our previous work.

### 4 Conclusions

We have obtained novel analytical expressions for the derivative dispersion relations, without high-energy approximations. These EDDR are intended for any energy above the physical threshold and their applicability is restricted to the class of functions specified by Theorem 1. However, since the experimental data on the total cross sections indicate a smooth variation with the energy (and a smooth systematic increase above $\sqrt{s} \approx 20$ GeV), this class includes the majority of functions of physical interest. Using as a framework a Pomeron-Reggeon model without degenerate trajectories, we have
demonstrated the numerical equivalence between the results obtained with the IDR (finite lower limit $m$) and the EDDR.

**Acknowledgments**

It is our pleasure to dedicate this work to Prof. Adriano Di Giacomo, on the occasion of his 70th birthday. M.J.M. is deeply grateful to Prof. Di Giacomo for all the support and encouragement, since 1987, in particular for the hospitality at the Università di Pisa (1991 - 1993). *Parabéns, caro Di Giacomo e muitas felicidades!*

We are thankful to the organizers for the invitation to contribute to this Volume and to FAPESP for financial support (contracts No. 03/00228-0 and No. 04/10619-9).

**References**

1. M.L. Goldberger, Y. Nambu and R. Oehme, *Ann. Phys.* 2, 226 (1957); P. Söding, *Phys. Lett.* 8, 285 (1964).
2. V.N. Gribov and A.A. Migdal, *Sov. J. Nucl. Phys.* 8, 583 (1969); J.B. Bronzan, G.L. Kane and U.P. Sukhatme, *Phys. Lett.* B 49, 272 (1974); Kang and Nicolescu, *Phys. Rev. D* 11, 2461 (1975).
3. R.F. Ávila and M.J. Menon, *Nucl. Phys. A* 744, 249 (2004).
4. R.F. Ávila and M.J. Menon, [hep-ph/0512166](http://arxiv.org/abs/hep-ph/0512166)
5. J.R. Cudell, E. Martynov, O. Selyugin, [hep-ph/0307254], E. Martynov J.R. Cudell, O. Selyugin, Eur. Phys. J. C 33, s533 (2004); hep-ph/0311019.

6. J. Fischer, P. Kolář, Phys. Lett. B 64, 45 (1976); Phys. Rev. D 17, 2168 (1978); P. Kolář, J. Fischer, J. Math. Phys. 25, 2538 (1984); J. Fischer, P. Kolář, Czech. J. Phys. B 37, 297 (1987).

7. S. Eidelman et al., Phys. Lett. B 592, 1 (2004) and 2005 partial update for the 2006 edition available on the PDG www pages (URL http://pdg.lbl.gov); E811 Collaboration, C. Avila et al., Phys. Lett. B 537, 42 (2002).