Equivalence of Two Dimensional QCD and the $c = 1$ Matrix Model

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ABSTRACT

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We consider two dimensional QCD with the spatial dimension compactified to a circle. We show that the states in the theory consist of interacting strings that wind around the circle and derive the Hamiltonian for this theory in the large $N$ limit, complete with interactions. Mapping the winding states into momentum states, we express this Hamiltonian in terms of a continuous field. For a $U(N)$ gauge group with a background source of Wilson loops, we recover the collective field Hamiltonian found by Das and Jevicki for the $c = 1$ matrix model, except the spatial coordinate is on a circle. We then proceed to show that two dimensional QCD with a $U(N)$ gauge group can be reduced to a one-dimensional unitary matrix model and is hence equivalent to a theory of $N$ free nonrelativistic fermions on a circle. A similar result is true for the group $SU(N)$, but the fermions must be modded out by the center of mass coordinate.
1. Introduction

Two dimensional QCD (QCD$_2$) might prove to be a useful laboratory for exploring some properties of the confining phase of four dimensional QCD. This program was originally started by ’t Hooft[1] who computed the meson spectrum in the planar limit. He showed that asymptotically the states live on a Regge trajectory. Other researchers later demonstrated that this spectrum could be derived from a Nambu-Goto action[2,3].

More recently it was postulated that QCD$_2$ could be interpreted as a theory of maps of two dimensional world-sheets into a two dimensional target space[4]. This investigation was carried out further in [5] where it was shown that the free energy of QCD$_2$ was consistent with a sum over maps containing tubes and handles and it was also shown to low orders that the counting of branched surfaces was consistent. Finally in [6] it was proven completely that QCD$_2$ is described by a sum over branched maps with tubes into any two-dimensional target space, except for some anomalous terms that appear for target spaces with genus greater than one[7].

Since QCD$_2$ is a string theory, it is natural to ask how it compares with another well known two-dimensional string theory, the $c = 1$ matrix model. In [8] it was shown that the Weingarten model in two dimensions does indeed lead to this model, if the spatial dimension is compactified onto a vanishingly small circle. Hence by comparing QCD$_2$ with $c = 1$ matrix models we are also indirectly comparing it with the Weingarten model.

In this paper we consider $SU(N)$ and $U(N)$ QCD$_2$ on a cylinder with circumference $L$. In section two we construct the states of this system and argue that they are described by strings that wrap around the compactified dimension. The theory does not contain zero winding excitations. Using the rules developed in [6] for QCD$_2$ string theory, we derive the complete Hamiltonian for the theory which describes strings joining or breaking apart and also contains a potential term that describes a ferromagnetic-like interaction between the strings. This last term is ab-
sent in the $U(N)$ case. In section three we map the winding states to momentum states on a circle with circumference $4\pi/g^2L$, where $g/\sqrt{N}$ is the QCD coupling. At the large-$N$ limit we show that the Hamiltonian reduces to the Das-Jevicki Hamiltonian for a collective coordinate field, with the spatial coordinate living on the circle. The lack of zero winding excitations is important in this derivation. To reach the critical $c=1$ theory, it is necessary to turn on a background source of Wilson loops. In section four we show that QCD$_2$ on a cylinder reduces to the singlet sector of a one-dimensional unitary matrix model, the unitary matrix being the monodromy of the gauge field around the compact space dimension. Therefore, the spectrum can be reproduced by a theory of $N$ nonrelativistic free fermions on a circle. This gives a natural explanation to the appearance of the collective field Hamiltonian as well as to the origin of nonperturbative corrections. If the gauge group is $SU(N)$ then the theory is modded out by the center of mass coordinate. In the final section we present our conclusions.

2. Derivation of the Hamiltonian

Consider $SU(N)$ QCD$_2$ living on a torus with area $A$. Its partition function is given by[9,10]

$$Z = \sum_{\text{reps}} \exp(-Ag^2C_{2R}/N),$$  \hspace{1cm} (2.1)

where the sum is over all representations of $SU(N)$, $g/\sqrt{N}$ is the QCD coupling, and $C_{2R}$ is the quadratic Casimir of the representation. A given representation $R$ is associated with a Young tableau described by $m$ rows, with $n_i$ boxes in row $i$, which satisfy $n_i \geq n_j$ if $i < j$. $C_{2R}$ is then given by

$$C_{2R} = \frac{N}{2}(n + \frac{\bar{n}}{N} - \frac{n^2}{N^2}),$$  \hspace{1cm} (2.2)

where

$$n = \sum_{i=1}^{m} n_i, \quad \bar{n} = \sum_{i=1}^{m} n_i(n_i - 2i + 1).$$  \hspace{1cm} (2.3)
Let us describe the torus by two circles with circumferences $\beta$ and $L$ so that $A = \beta L$. $\beta$ can be thought of as the inverse temperature, therefore, the partition function describes QCD$_2$ at finite temperature with its spatial dimension compactified onto a circle of length $L$. From (2.1) it is clear that every representation of $SU(N)$ corresponds to a physical state of the theory, with energy $(g^2L/N)C_{2R}$. A state in representation $R$ is created and destroyed by a Wilson loop in representation $R$ that wraps once around the spatial dimension. To see this, we can consider a cylindrical surface with Euclidean length $\beta$ and Wilson loops with representations $R$ and $R'$ inserted at the two ends of the cylinder. This partition function is given by

$$Z = \int d\Omega d\Omega' \chi_R(\Omega) \chi_{R'}(\Omega'^{-1}) \sum_{R''} \chi_{R''}(\Omega) \chi_{R''}(\Omega'^{-1}) \exp(-g^2\beta LC_{2R''})$$

$$= \delta_{RR'} \exp(-g^2\beta LC_{2R}),$$

(2.4)

where $\Omega$ and $\Omega'$ are the $SU(N)$ elements around the circles at the ends of the cylinder and $\chi_R(\Omega)$ and $\chi_{R'}(\Omega'^{-1})$ are the corresponding characters. Clearly, the partition function in (2.4) represents the propagation of one state into itself over a Euclidean time $\beta$.

Recently it was shown that QCD$_2$ has a string theory interpretation[4-6]. That is, the partition function can be thought of as a set of maps of two-dimensional world-sheets into a two-dimensional target space. The maps can multiply cover the surface, and such maps can contain branch cuts or small tubes that connect the different sheets of the world-sheet.

For a given representation $R$, the expression $\exp[-(g^2A/2)(n + \tilde{n}/N - n^2/N^2)]$ can be expanded in powers of $1/N$. The leading term is $\exp(-g^2An/2)$, hence this representation describes an $n$-covered map, with the leading term coming from the integration of the Nambu-Goto action over the world-sheet. The expansion of $\exp[-(g^2A/2)(\tilde{n}/N)]$ is the contribution of the branch cuts connecting the sheets and the expansion of $\exp[-(g^2A/2)(-n^2/N^2)]$ gives the contributions of the tubes and small handles.
We also must consider the complex conjugate representations of $R, \bar{R}$. One can think of the sheets for this representation as having the opposite chirality to those of representation $R$. It is also possible to have representations which are tensor products of $R$ and $R'$. For such representations there are no branch points connecting sheets of opposite chirality and tubes that connect such sheets come with a minus sign[6].

Now consider the string picture for the cylinder with Wilson loops inserted at the ends. A chiral representation $R$, with $n$ boxes in its tableau is a linear combination of string states that wrap around the compact dimension a total of $n$ times, all in the same direction. Hence, there could be $n$ strings that wrap once, or one string that wraps around $n$ times. The total number of such states is $P(n)$, the number of partitions of $n$.

The branch points on the world-sheet correspond to interactions where two strings join to form one string or vice versa. The tubes correspond to interactions where two strings “kiss” at a point and break apart again, or a multiwound string which bumps into itself. Since there are no branch points joining sheets of opposite chirality, two strings of opposite winding will not join to form a single string, nor will a string break into strings with opposite winding. However, two states with opposite winding can have a pointlike interaction, but the sign is opposite to that of two strings with the same winding.

The almost triviality of the interactions for two strings with opposite winding essentially allows us to separate the two sectors. With this in mind, consider a state with $n$ windings in one direction. Following the work of Gross and Taylor[6], a string state can be described by an element of the permutation group for $n$ elements, $S_n$. At the spatial point $x = 0$, a label can be assigned to each of the $n$ strands of string. Tracing the strands form $x = 0$ to $x = L$, we find that some strands come back to themselves, but others are mapped to different strands. This mapping is described an element $s$, of $S_n$. So for example, the state with $n$ strings that wind once is given by the identity element. For any $t$ in $S_n$, the state $tst^{-1}$
corresponds to a relabeling of the strands, hence this state is equivalent to the state described by \( s \). Therefore, the inequivalent states are given by the conjugacy classes of \( S_n \).

We can define an inner product

\[
\langle s'|s \rangle = \delta_{s's},
\]

where \( s \) and \( s' \) are elements of \( S_n \), but it is more useful to define the unnormalized product

\[
\langle s'|s \rangle_{un} = \sum_{t \in S_n} \delta_{s', tst^{-1}} = \frac{n!}{C_s} \quad \text{if } s \simeq s' \]

\[
= 0 \quad \text{otherwise.}
\]

The symbol \( \simeq \) means that the elements are equivalent up to a conjugacy and \( C_s \) is the number of elements in the conjugacy class. Each conjugacy class is described by a partition of \( n \), where each element of the class is a cycle within the elements of the partition. If \( s = s' \), then the elements of \( S_n \) which commute with \( s \) are those elements which are cycles of \( s \), multiplied by those elements which exchange cycles with equal number of elements. If the partition is given by

\[
\prod_{l=1}^{n}(l)^{nl}, \quad \sum_{l=1}^{n} l n_l = n,
\]

then the order of the subgroup that commutes with \( s \) is

\[
\prod_{l=1}^{n}(l)^{nl} n_l!.
\]

Hence this particular state can be written as

\[
\prod_{l=1}^{n}(a_l^\dagger)^{n_l} |0\rangle,
\]

where \( a_l^\dagger \) is the creation operator for a string with winding \( l \) and \( |0\rangle \) is the vacuum state. We can also act on the vacuum with the operators \( a_{-l}^\dagger \) which are the creation operators for strings that wind in the opposite direction. The commutation
relations are given by

\[ [a_l, a_m^\dagger] = |l|\delta_{l,m}, \]  

(2.7)

thus the inner products of these states will reproduce the result in (2.5).

To leading order in \(1/N\), the energy of such a state is given by \(g^2L(n_l + n_r)/2\), where \(n_l\) is the number of left windings and \(n_r\) is the number of right windings. Hence, the leading order Hamiltonian is given by

\[ H_0 = \frac{g^2L}{2} \sum_{n \neq 0} a_n^\dagger a_n. \]  

(2.8)

Now consider the interactions among the strings. At a branch point two strings join or break apart. As far as the permutations of the strands are concerned, this corresponds to inserting an element of \(S_n\) which has one cycle of order 2 and \(n - 2\) cycles of order 1. One should sum over all possible branch points, which corresponds to summing over the entire conjugacy class of these elements. Therefore, the unnormalized matrix element describing this interaction is given by

\[ \sum_{p \in S_n} \langle s' | p | s \rangle_{un} = \sum_{t \in S_n, \; p \in S_{n_2}} \delta_{s'p,tst^{-1}}, \]  

(2.9)

where \(S_{n_2}\) are the elements of \(S_n\) in the conjugacy class with one 2-cycle and the rest 1-cycles. If \(s\) is comprised of two cycles of order \(n_1\) and \(n_2\) and \(s'\) is comprised of one cycle of order \(n_1 + n_2\), then there is a unique \(p\) such that \(s'p = s\). Consider the set of elements in \(S_n\) which are given by \(t = rq\), where \(qsq^{-1} = s\) and \(r^{-1}s'r = s'\). The elements \(q\) form a subgroup of order \(n_1n_2\), while the elements \(r\) form a subgroup of order \(n_1 + n_2\). Moreover, the conjugates of \(p, r^{-1}pr\) form \(n_1 + n_2\) distinct elements. Hence the sum in (2.9) is given by \((n_1 + n_2)n_1n_2\). \(s\) and \(s'\) could also have additional cycles, but these are basically spectators as far as \(p\) is concerned, so the matrix elements for these states can be determined using (2.5). Since each branch point comes with a factor \(g^2/2N\), and since the branch point
can occur anywhere along the circle of length $L$, then in terms of the creation and
annihilation operators, the operator that leads to the matrix element in (2.9) is

$$\frac{g^2 L}{2N} a_{n_1+n_2}^{\dagger} a_{n_1} a_{n_2}. \quad (2.10)$$

Including windings in both sectors, one then finds that the general Hamiltonian
describing this class of interactions is given by

$$H_b = \frac{g^2 L}{2N} \left( \sum_{n,n'>0} + \sum_{n,n'<0} \right) \left( a_{n+n'}^{\dagger} a_{n} a_{n'} + \text{c.c.} \right). \quad (2.11)$$

Finally, it is easy to see that the interaction term that describes the handles
and tubes on the world-sheet is given by

$$H_t = \frac{g^2 L}{2N^2} \left[ \sum_{n>0} \left( a_{n}^{\dagger} a_{-n} - a_{-n}^{\dagger} a_{n} \right) \right]^2. \quad (2.12)$$

The operator inside the square brackets counts the net winding number of the
state.

So far in this section we have been assuming that the gauge group is $SU(N)$
instead of $U(N)$. One disadvantage of $SU(N)$ is that the winding number is
actually only defined modulo $N$. That is, a state that has one string with winding
number $-1$ is equivalent to a linear combination of strings with winding number
$N-1$. If the group is enlarged to $U(N)$, then this is no longer the case. For a $U(N)$
representation the quadratic Casimir has the additional term $\frac{1}{2} n^2 (g'^2/g^2)$, where
$g'$ is the $U(1)$ coupling, $g$ is the $SU(N)$ coupling and $n$ is the total net winding of
left and right string states. Hence by considering $U(N)$, the interaction strength
of the term in (2.12) becomes an adjustable parameter, and for $g' = g/N$, it can
be eliminated entirely.
3. Derivation of the Das-Jevicki Hamiltonian

A striking feature of the Hamiltonian given in (2.8), (2.11) and (2.12) is that the creation and annihilation operators for the winding states look just like operators which create and destroy momentum states in one spatial dimension. With this in mind, define a new length $\tilde{L} = 4\pi/(g^2 L)$. We can then define a momentum variable as $k = 2\pi n/\tilde{L}$, where $n$ is the winding number. In order to consider $k$ as momentum excitations in a continuous space, it must take arbitrarily large values; therefore, $n$ must be very large. On the other hand, the interacting boson picture breaks down if $n \geq N$, since in this case some states end up being summed over that do not correspond to representations of $U(N)$ or $SU(N)$. Thus, the continuum limit is only valid in the large-$N$ limit.

Letting $a_k = a_n$, the commutation relation in (2.7) becomes

$$[a_k, a^\dagger_{k'}] = \frac{\tilde{L}}{2\pi} |k| \delta_{k,k'}.$$  \hspace{1cm} (3.1)

We can also define a field $\varphi(x)$ and its canonical conjugate field $\Pi(x)$, where $[\varphi(x), \Pi(y)] = i\delta(x-y)$. We can then write $a_k$ as

$$a_k = \frac{1}{2} \int dx e^{-ikx} [\varphi(x) + \frac{1}{\pi} \partial \Pi(x)], \quad k > 0$$

$$= \frac{1}{2} \int dx e^{-ikx} [\varphi(x) - \frac{1}{\pi} \partial \Pi(x)], \quad k < 0$$  \hspace{1cm} (3.2)

which one can easily show satisfies the commutation relations.

We now plug these expressions into the full Hamiltonian given in (2.8), (2.11) and (2.12). First substituting $k$ for $n$, we find that the complete Hamiltonian is given by

$$H = \frac{2\pi}{\tilde{L}} \sum_k a_k^\dagger a_k + \frac{2\pi}{LN} \left\{ \sum_{k,k'>0} (a^\dagger_{k+k'} a_k a_{k'} + c.c.) + \sum_{k,k'<0} (a^\dagger_{k+k'} a_k a_{k'} + c.c.) \right\}$$

$$- \frac{2\pi\alpha}{LN^2} \left( \sum_{k>0} (a^\dagger_k a_k - a^\dagger_{-k} a_{-k}) \right)^2.$$  \hspace{1cm} (3.3)
where \( \alpha \) is an adjustable parameter which depends on the \( U(1) \) coupling. Substituting the expression for \( a_k \) in (3.2) and performing the sums over momenta then gives

\[
H = \frac{1}{2\pi} \int dx (\pi^2 \varphi^2 + (\partial \Pi)^2) + \frac{\tilde{L}}{4\pi N} \int dx (\pi^2 \varphi^3 + 3\varphi \partial \Pi \partial \Pi)
\]

\[
- \frac{1}{4\pi L N} \left\{ \int dxdx^2 \varphi(x) \left[ \int dy \varphi(y) \cot \frac{\pi}{L} (x - y) \right]^2
\]

\[
+ \int dx \left[ \int dy \partial \Pi(y) \cot \frac{\pi}{L} (x - y) \right] \varphi(x) \left[ \int dz \partial \Pi(z) \cot \frac{\pi}{L} (x - z) \right]
\]

\[
+ 2 \int dx \partial \Pi(x) \int dy \varphi(y) \cot \frac{\pi}{L} (x - y) \int dz \partial \Pi(z) \cot \frac{\pi}{L} (x - z) \right\}
\]

\[
- \frac{\tilde{L} \alpha}{2\pi N^2} \left[ \int dx \varphi \partial \Pi \right]^2 + \Delta H,
\]

where \( \Delta H \) is the singular term,

\[
\Delta H = \frac{\tilde{L}}{4\pi N} \int dx dy \delta(x - y) \varphi(x) \varphi(y) \partial_x \partial_y \ln \left| \sin \frac{\pi}{L} (x - y) \right|.
\]

Let us now set \( \alpha \) to zero. The Hamiltonian we are left with is still non-local, but there is an important property of QCD\(_2\) which will improve this situation. QCD\(_2\) contains no zero winding excitations and in fact, no such terms appear in (2.8), (2.11) or (2.12). Therefore, \( \varphi(x) \) and \( \partial \Pi(x) \) can not contain zero modes. Thus, one must impose the constraints

\[
\int dx \varphi(x) = \int dx \partial \Pi(x) = 0.
\]

Since the integrals in (3.6) are finite, the non-local pieces can be expressed in terms of local terms by using a somewhat modified trick of collective coordinate
field theories[11]. Defining \( \tilde{f}(x) \) as

\[
\tilde{f}(x) = \frac{\pi}{L} \int dx \cot \frac{\pi}{L}(x - y)f(y),
\]

we have that

\[
\frac{\pi}{L} \int dyf(y) \cot \frac{\pi}{L}(x - y \pm i\epsilon) = \tilde{f}(x) \mp i\pi f(x),
\]

where \( f(x) \) needs to be reasonably smooth and \( \tilde{f}(x) \) must exist. Taking the identity

\[
\cot \frac{\pi}{L}(x - y + i\epsilon) \cot \frac{\pi}{L}(y - z + i\epsilon) + \cot \frac{\pi}{L}(y - z - 2i\epsilon) \cot \frac{\pi}{L}(z - x + i\epsilon) + \cot \frac{\pi}{L}(z - x - 2i\epsilon) \cot \frac{\pi}{L}(x - y - i\epsilon) = 1,
\]

multiplying it by \( f(x)g(y)h(z) \), and then integrating over \( x, y, \) and \( z \), we find that

\[
\int dx fgh = \int dx [\tilde{f}\tilde{g}\tilde{h} + \tilde{f}\tilde{g}\tilde{h}] + \frac{\pi^2}{L^2} \int dx f(x) \int dy g(y) \int dz h(z).
\]

Using (3.10) and the constraint (3.6), we are now able to rewrite the Hamiltonian in (3.4) as

\[
H = \frac{1}{2\pi} \int dx \left\{ \frac{\pi^2}{2} \phi^2 + (\partial \Pi)^2 + \frac{\tilde{L}}{N} \left[ \frac{\pi^2}{3} \phi^3 + \partial \Pi \phi \partial \Pi \right] \right\} + \Delta H.
\]

Shifting \( \phi \) to \( \phi + N/\tilde{L} \), the Hamiltonian becomes

\[
H = \frac{\tilde{L}}{2\pi N} \int dx \left\{ \partial \Pi \phi \partial \Pi + \frac{\pi^2}{3} \phi^3 - \left( \frac{\pi N}{\tilde{L}} \right)^2 \phi \right\} + \Delta H
\]

\[
= \frac{4}{g^2 LN} \int dx \left\{ \frac{1}{2} \partial \Pi \phi \partial \Pi + \frac{\pi^2}{6} \phi^3 - \left( \frac{g^2 LN}{4} \right)^2 \phi \right\} + \Delta H,
\]

up to a constant. Moreover, the new constraint becomes

\[
\int dx \phi(x) = \tilde{L}N/\tilde{L} = N.
\]

Except for a missing potential term and the fact that the fields live on a circle as opposed to in a box, the Hamiltonian in (3.12) and the constraint in
(3.13) are precisely those found by Das and Jevicki for the collective coordinate field of the $c = 1$ matrix model[12]. $\Delta H$ is the quantum correction to the free energy[12,13]. This calculation is also analogous to one in [14], but with different boundary conditions. From (3.12) we see that the bare string coupling constant is $4/(g^2LN)$, thus strong coupling QCD leads to a weak coupling string theory. The constraint in (3.13) can be imposed by adding the term

$$\int dx \mu_F (\varphi - \frac{N}{L})$$

to the Hamiltonian, where $\mu_F$ is a Lagrange multiplier which acts as the bare cosmological constant. Of course, the linear term in (3.12) will shift this value.

To complete the program, we need to have a potential term, $\int dx V(x)\varphi(x)$ in the Hamiltonian, so that theory can have some sort of critical behavior. Such a term in momentum space is given by

$$\sum_k V_k (a_k + a_k^\dagger).$$

(3.14)

Such terms can be produced by a background source of Wilson loops. For instance, if $V(x) = \cos \frac{\pi}{L}x$, then $V_k = a_1 + a_1^\dagger + a_{-1} + a_{-1}^\dagger$. Hence the QCD$_2$ action should contain the additional term

$$C \int dt [\chi_f(U(t)) + \chi_f(U(t))]$$

(3.15)

where $U(t)$ is the value of the $U(N)$ element around a closed loop at time $t$ and $\chi_f(U)$ and $\chi_f(U)$ are the characters for the fundamental representation and its complex conjugate. Note that $\chi_f(U)$ creates strings that wind to the right and annihilates strings that wind to the left. Critical behavior can now be found by tuning $C$. This has the same perturbative behavior as the $c = 1$ Hermitian matrix model, but its nonperturbative behavior is different because of the different boundary conditions.
4. Free Fermions

The fact that a collective coordinate field theory can be constructed using the rules derived from QCD$_2$ string theory suggests that there exists a free fermion picture of QCD$_2$. In this section we show that QCD$_2$ on a cylinder is equivalent to a theory of free fermions by showing that it can be reduced to a one-dimensional unitary matrix model.

To this end, consider QCD$_2$ in the gauge $A_0 = 0$. The Hamiltonian is then given as

$$H = \frac{1}{2} \int_0^L dx \tr F_{01}^2 = \frac{1}{2} \int_0^L dx \tr \dot{A}_1^2$$ \hspace{1cm} (4.1)$$

with the overdot denoting a time derivative. The $A_0$ equation of motion is now the constraint

$$D_1 F_{10} = \partial_1 \dot{A}_1 + ig [A_1, \dot{A}_1] = 0.$$ \hspace{1cm} (4.2)

Let us now define a new variable $V(x)$,

$$V(x) = W_0^x \dot{A}_1(x) W_x^L,$$ \hspace{1cm} (4.3)

where

$$W_a^b = Pe^{ig \int_a^b dx A_1}.$$ \hspace{1cm} (4.4)

Then (4.1) can be written as

$$\partial_1 V(x) = 0,$$ \hspace{1cm} (4.5)

so $V(x)$ is a constant. Thus $V(0) = V(L)$, which implies that

$$[W, \dot{A}_1(0)] = 0,$$ \hspace{1cm} (4.6)

where $W \equiv W_0^L$ and we have used the periodicity of $A_1$ in $x$. 


From the definitions (4.3) and (4.4), we find the relation

\[ \dot{W} = ig \int_{0}^{L} dx W_0^x \dot{A}_1(x) W_x^L = ig \int_{0}^{L} dx V(x), \quad (4.7) \]

and therefore using (4.5) and (4.6), we derive

\[ \dot{W} = ig L W \dot{A}_1(0) = ig L \dot{A}_1(0) W. \quad (4.8) \]

(4.8) then implies that

\[ [W, \dot{W}] = 0. \quad (4.9) \]

Because \( V(x) = V(0) \), \( \dot{A}_1(x) \) satisfies

\[ \dot{A}_1(x) = W_0^x \dot{A}_1(0) W_x^0. \quad (4.10) \]

Thus, using this relation along with (4.8), we can rewrite the Hamiltonian in (4.1) as

\[ H = -\frac{1}{2g^2 L} \text{tr}(W^{-1} \dot{W})^2. \quad (4.11) \]

If the gauge group is \( U(N) \), with the \( U(1) \) coupling given by \( g/N \), then (4.11) is the Hamiltonian for the one-dimensional unitary matrix model. The canonical structure of this Hamiltonian is also the standard matrix model one, as can be deduced from the fundamental brackets

\[ \{ A_1(x)_{ij}, \dot{A}_1(y)_{kl} \} = \delta_{ii} \delta_{jk} \delta(x - y) \quad (4.12) \]

and the definition of \( W \). The constraint in (4.9) reduces the space of states to singlets[15]. Hence, the problem is reducible to the eigenvalues of \( W \).
Upon quantization, this problem is equivalent to a system of \( N \) nonrelativistic fermions living on a circle, with the Hamiltonian given by

\[
H = -\left(\frac{g^2 L}{2}\right) \sum_{i=1}^{N} \frac{\partial^2}{\partial \theta_i^2}, \quad 0 \leq \theta_i < 2\pi.
\] (4.13)

The fermionization is achieved by the appearance of the Vandermonde determinant in the wavefunction of the states, which in the unitary matrix case reads

\[
\Delta = \prod_{i<j} \sin \frac{\theta_i - \theta_j}{2}.
\] (4.14)

Notice that each factor in (4.14) is antiperiodic on the circle. Thus, if \( N \) is even the fermions have antiperiodic boundary conditions. Likewise, if \( N \) is odd they have periodic boundary conditions. This can be understood in terms of transporting a fermion once around the circle, passing by \( N - 1 \) other fermions along the way and therefore picking up \( N - 1 \) minus signs. Hence, in either case, the ground state is built by filling all states with wave numbers between \(-N/2 + 1/2\) and \(N/2 - 1/2\), inclusive. Subtracting off the ground state energy, one easily sees that this spectrum reproduces that found for the different representations of \( U(N) \).

If the gauge group is \( SU(N) \), because \( A_1 \) is now traceless \( W \) will also obey the condition \( \det W = 1 \). Therefore the center of mass coordinate for the fermions is absent and we must mod it out of the theory. This means that we need to identify states in which all fermions have their momentum shifted by the same amount. (This is equivalent to identifying the antisymmetric tensor product of \( N \) copies of fundamental representations with the singlet representation.) Moreover, we must subtract the energy of the center of mass from the energy of each state in the theory.

The correspondence of the fermion states with the Young tableaux is as follows: since the center of mass coordinate drops out, we can always set the smallest wave number to zero (for odd \( N \)). The rest of the wave numbers are integer numbers.
greater than zero, with the largest number equal to $N - 1$ for the ground state. We can excite states by shifting the wave numbers up (except the smallest one). The size of the shift for the largest number gives the number of boxes in the first row of the tableau, the size of the next largest is the number of boxes in the second row, etc. If we denote the shift of the $i^{th}$ highest wave number as $n_i$, then the energy of the state minus the ground state energy and the center of mass energy is

$$E = \frac{g^2 L}{2} \left\{ \sum_i \left[ (n_i + N - i)^2 - (N - i)^2 \right] \right.$$  
$$- \frac{1}{N} \left[ \left( \frac{N(N-1)}{2} + \sum_i n_i \right)^2 - \left( \frac{N(N-1)}{2} \right)^2 \right] \right\}$$  

(4.15)

$$= \frac{g^2 L}{2} \left\{ N \sum_i n_i + \sum_i n_i(n_i - 2i + 1) - \frac{1}{N} \left( \sum_i n_i \right)^2 \right\} = g^2 LC_{2R}.$$  

After rescaling $g^2 \rightarrow g^2/N$, we recover the expected result. For even $N$ the argument is the same but with all the momenta shifted by $\frac{1}{2}$. Since this is a center of mass excitation, it does not affect the energy and the same result is obtained.

In the string picture the two chiral sectors of the QCD partition function are identified as excitations of left-moving or right-moving fermions. The factorization of the two sectors (that is, the fact that there are no states where a left-moving fermion is excited into a right-moving state) holds up to leading order in $1/N$ because the center of mass has been modded out. This factorization, of course, completely breaks down when a large number of quanta are excited (of order $N$) which signals the onset of nonperturbative effects.

If the gauge group is $U(N)$ but with $U(1)$ coupling $g' \neq g/N$, then the energies of the states are given by (4.13), but with a modified coefficient for the center of mass kinetic energy operator. Such a variable coefficient was discussed before in another context[16].
5. Discussion

We have found two main results. The first is that the rules derived from QCD$_2$ string theory lead to a collective coordinate theory which is the same perturbatively as the collective coordinate theory of $c = 1$ matrix models. The second, which is related to the first, is that QCD$_2$ is exactly equivalent to a theory of nonrelativistic fermions living on the circle. This theory differs from the usual $c = 1$ theory nonperturbatively, in that the fermions live on the circle instead of the real line.

The above results are quite encouraging and the extent to which they might apply to higher dimensional QCD is an interesting issue. Comparing the results of QCD$_2$ with those found for the Weingarten model, shows a qualitative difference between the two models. Unlike the Weingarten model it is not necessary to shrink $L \to 0$ to reach a $c = 1$ matrix model. The two theories also possess different nonperturbative behavior. Perhaps this will have some implications for higher dimensions.

These results might have more significance in understanding the anomalous terms in the free energy for QCD$_2$ on a higher genus target space. These terms can be reproduced by inserting special operators on the world-sheet surface[7], but their geometrical significance is yet to be understood.

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