Contrapositionally Complemented Pseudo-Boolean Algebras and Intuitionistic Logic with Minimal Negation

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Abstract

The article is a study of two algebraic structures, the ‘contrapositionally complemented pseudo-Boolean algebra’ (ccpBa) and ‘contrapositionally ∨ complemented pseudo-Boolean algebra’ (c∨cpBa). The algebras have recently been obtained from a topos-theoretic study of categories of rough sets. The salient feature of these algebras is that there are two negations, one intuitionistic and another minimal in nature, along with a condition connecting the two operators. We study properties of these algebras, give examples, and compare them with relevant existing algebras. ‘Intuitionistic Logic with Minimal Negation (ILM)’ corresponding to ccpBAs and its extension ILM-∨ for c∨cpBAs, are then investigated. Besides its relations with intuitionistic and minimal logics, ILM is observed to be related to Peirce’s logic. With a focus on properties of the two negations, two kinds of relational semantics for ILM and ILM-∨ are obtained, and an intertranslation between the two semantics is provided. Extracting features of the two negations in the algebras, a further investigation is made, following logical studies of negations that define the operators independently of the binary operator of implication. Using Dunn’s logical framework for the purpose, two logics K_{im} and K_{im-∨} are presented, where the language does not include implication. K_{im}-algebras are reducts of ccpBAs. The negations in the algebras are shown to occupy distinct positions in an enhanced form of Dunn’s Kite of negations. Relational semantics for K_{im} and K_{im-∨} are given, based on Dunn’s compatibility frames. Finally, relationships are established between the different algebraic and relational semantics for the logics defined in the work.

Index terms— pseudo-Boolean algebras, contrapositionally complemented lattices, intuitionistic logic, minimal logic, compatibility frames

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1 Introduction

The work presents a study of two algebraic structures, ‘contrapositionally complemented pseudo-Boolean algebra’ (ccpBa) and ‘contrapositionally ∨ complemented pseudo-Boolean algebra’ (c∨cpBa). The salient feature of these algebras is that there are two negations amongst the operations defining the structures, one negation (¬) being intuitionistic in nature and another (∼), a minimal negation. The two operations are connected by the involutive property for a fixed element ∼1, with respect to the negation ∼. The algebraic structures came to the fore during a topos-theoretic investigation of categories formed by rough sets [33, 34]. Though the current article does not deal with category theory or rough sets, in order to give a motivation for our study, the work leading to the definitions of ccpBa and c∨cpBa is briefly recounted in the following paragraph.

Rough sets were introduced by Pawlak in 1982 to deal with situations where only partial or inadequate information may be available about objects of a domain of discourse. In such a situation, there may be some objects of the domain that are not distinguishable from others. Mathematically, the scenario is represented by a pair (U, R) called approximation space, where U is a set (domain of discourse) and R an equivalence relation on U. Let [x] denote the equivalence class of x (∈ U), giving all objects of the domain that are indiscernible from x. For any subset U of U, the upper approximation and the lower approximation of U in (U, R) are respectively defined as

\[ U^\uparrow := \{ x \in U \mid [x] \subseteq U \} \quad \text{and} \quad U^\downarrow := \{ x \in U \mid [x] \cap U \neq \emptyset \}. \]

U gives the objects of the domain that definitely belong to U, while \( U \) consists of all objects that are possibly in U. \( U \setminus U^\downarrow \) is called the boundary of U, giving the region of ‘uncertainty’. So the possible region of U includes its definite and boundary regions. U is called a rough set in (U, R), described by its lower and upper approximations. (There are other equivalent definitions of rough sets in literature, cf. [3].) In 1993, the study of categories of rough sets was initiated by Banerjee and Chakraborty [1, 2]. A category ROUGH was proposed with objects as triples of the form (U, R, U), (U, R) being an approximation space and U ⊆ U. A morphism between two ROUGH-objects is defined in such a way that it maps the possible and definite regions of the rough set in the domain, respectively into the possible and definite regions of the rough set in the range. The work was followed up by More and Banerjee in [34, 35]. A major objective in [11, 2] 34 35 was to study the topos-theoretic properties of ROUGH and associated categories. In [34], it was shown that ROUGH forms a quasitopos. Now any topos or quasitopos has an inherent algebraic structure: the collection of (strong) subobjects of any (fixed) object forms a pseudo-Boolean algebra. The work in [34, 35] explored the notion of negation in the set of strong subobjects of any ROUGH-object (U, R, U). This set forms a Boolean algebra \( M(U) := (\mathcal{M}, 1, 0, \lor, \land, \rightarrow, \neg) \), where \( \neg \) denotes the Boolean negation. However, as is well-known, algebraic structures formed by rough sets are typically non-Boolean, cf. [3]. Iwiński’s rough difference operator giving relative rough complementation was subsequently incorporated in \( M(U) \), in the form of a new negation ∼. Then, for any subobject (U, R, A) of ROUGH-object (U, R, U) and for C(⊆ U) such that ∼(U, R, A) = (U, R, C), the possible region of C contains the boundary region of A [33, 34] – which is meaningful, as the boundary is the region of uncertainty. But this is not the case if ∼ is replaced by ∼. An investigation of properties of the two negations ∼, ∼ and the enhanced structure \( (\mathcal{M}, 1, 0, \lor, \land, \rightarrow, \neg, \sim) \) resulted in the definition of ccpBa and c∨cpBa [35].
The study of algebraic structures with negation has a vast literature (cf. e.g. [43]). Some such structures that get directly related to \( ccpBa \) and \( c\lor cpBa \) are Boolean algebras, pseudo-Boolean algebras (also called Heyting algebras), contrapositionally complemented lattices and Nelson algebras (cf. [22, 39, 43, 44, 48]). An important direction of algebraic studies since the results obtained by Stone (cf. [43]), has been the investigation of representation theorems connecting algebras and topological spaces (cf. [5, 7, 10, 12]). A representation result for \( ccpBa \) with respect to topological spaces is given in [35]. Here, we give duality results for both \( ccpBa \) and \( c\lor ccpBa \) with respect to topological spaces that are certain restrictions of Esakia spaces (cf. [12, 27]), the topological spaces corresponding to pseudo-Boolean algebras.

A study of classes of algebras naturally leads to an investigation of corresponding logics. In [33, 35], the logic ILM - Intuitionistic logic with minimal negation, and ILM-\( \lor \), corresponding to the algebras \( ccpBa \) and its extension \( c\lor ccpBa \) respectively, have been defined. The language of ILM has two negations \( \neg, \sim \), and propositional constants \( \top, \bot \), apart from other propositional connectives. The nature of the two connectives of negation present in the logics expectedly yields relations of ILM with intuitionistic logic (IL) and minimal logic (ML). (Recall that ML is the logic corresponding to the class of contrapositionally complemented lattices [27].) Utilizing the notions of ‘interpretation’ as given in [6, 20, 21, 43], a further comparison is made between ILM and IL, ML in [35]. In continuation of the comparative study, in this work, we relate ILM with Peirce’s logic, which is an extension of ML obtained by adding Peirce’s law ((\( \alpha \to \beta \)) \( \to \) \( \alpha \)) \( \to \alpha \)) \( \to \alpha \)) and discussed in [37]. JP’ (also called Glivenko’s logic) is the weakest logic amongst the extensions of ML in which \( \sim \sim \alpha \) is derivable whenever \( \alpha \) is derivable.

A logic may be imparted multiple semantics, aside from an algebraic one. For propositional logics with negation, different relational semantics have been introduced, for instance in [13, 19, 24, 46, 49]. As is well-known, Kripke [31, 32] first studied relational semantics for IL, where frames are partially ordered sets (\( W, \leq \)) called normal frames. On these frames, Segerberg [46] added a hereditary set \( Y_0 \) of ‘queer’ worlds at each of which \( \bot \) (‘falsum’) holds. By adding conditions on such frames, natural relational semantics for ML and various extensions of ML are obtained [39, 38, 46]. One such semantics given by Woodruff [50], involving sub-normal frames, is used in our work to obtain relational semantics for ILM and ILM-\( \lor \).

As Došen remarked in [13], a drawback of the Segerberg-style semantics is that it cannot be used to characterize logics with negation weaker than minimal negation. This problem was addressed by Došen and Vakarelov [49] independently. In their work, taking motivation from Kripke frames in modal logic and in particular, the accessibility relations in the frames, negation is considered as an impossibility (modal) operator. This approach results in another relational semantics for ILM and ILM-\( \lor \). We give a Došen-style semantics for the two logics, and show that an inter-translation exists between the two (Segerberg-style and Došen-style) relational semantics in the lines of that given for ML in [13].

In the logical systems given by Došen, the alphabet of the language has the connectives of implication, disjunction, conjunction and negation. Studying negation independently, particularly in the absence of implication, constitutes an important area of work on logics. Dunn [15, 16, 17, 18, 19], Vakarelov [49] and others [45, 47] have
studied logics with negations and without implication. A basic feature of these logics is that the logical consequence, in the absence of implication, is defined through sequents: pairs of formulas of the form \((\phi, \psi)\), written as \(\phi \vdash \psi\). Properties are introduced in the logics as axioms or rules, defining pre-minimal, minimal, intuitionistic or other negations. To round up the study of negations defined through ILM and ILM-\(\lor\), we make an investigation adopting the approach of Dunn and Vakarelov. Extracting the features of the two negations, logics \(K_{im}\) and \(K_{im-\lor}\) are defined. \(K_{im}\)-algebras are reducts of \(ccpBa\), while \(K_{im-\lor}\)-algebras are reducts of \(c\lor cpBa\). The properties of the algebras show that Dunn’s Kite of negations can be enhanced to one accommodating pairs of negations, two distinct nodes in which are occupied by the pair defining the \(K_{im}\) and \(K_{im-\lor}\)-algebras. Dunn’s compatibility frames help in specifying the relational semantics for \(K_{im}\) and \(K_{im-\lor}\).

Connections between algebras and relational frames have been studied in literature, together with duality results (cf. e.g. [25, 28]). Kripke [31] demonstrated the connection between normal frames and pseudo-Boolean algebras (cf. [4, 8]). We observe here that this can easily be extended to obtain connections between sub-normal frames and \(ccpBa\). As compatibility frames provide a relational semantics for \(K_{im}\), we shall also establish this to and fro connection between relational and algebraic semantics for \(K_{im}\) and \(K_{im-\lor}\).

The paper is organised as follows. In the next section, we give the definitions of \(ccpBa\) and \(c\lor cpBa\), and some of their properties. A few examples of finite \(ccpBa\) and \(c\lor cpBa\) are studied in Section 2.2, while representation results are given in Section 2.3. We then move to the logics ILM and ILM-\(\lor\) in Section 3. Properties and relationships with IL, ML, and Peirce’s logic are discussed. In Sections 4 and 5, we give, respectively, the Došen-style and Segerberg-style semantics for the logics. To establish completeness results, an inter-translation between frames in the two semantics is used – this is given in Section 5.2. In Section 6 we study the negations in systems without implication. The logics \(K_{im}\) and \(K_{im-\lor}\) along with algebraic and relational semantics are discussed. Connections between algebraic and relational semantics of the logics are observed in Section 7. We conclude the work in Section 8. Hereafter, the symbols \(\forall\), \(\exists\), \(\Rightarrow\), \(\Leftrightarrow\), \(\&\) (and), \(\lor\), \(\land\), \(\neg\), \(\not\) will be used with the usual meanings in the metalanguage. For basic definitions and results, we refer to [13] (for algebra and logic), [12] (representation and duality), and [13, 19, 46] (relational semantics).

2 The algebras \(ccpBa\) and \(c\lor pBa\)

Let us first recall the definitions of the two algebraic structures.

**Definition 2.1.** [35]

An abstract algebra \(A := (A, 1, 0, \lor, \land, \to, \neg, \sim)\) is called a contrapositionally complemented pseudo-Boolean algebra \((ccpBa)\), if the reduct \((A, 1, 0, \lor, \land, \to)\) forms a bounded relatively pseudo-complemented \((rpc)\) lattice, and for all \(a \in A\),

\[-a = a \to 0 \text{ and } \sim a = a \to (\neg\neg\sim 1).\]

If, in addition, for all \(a \in A\), \(a \lor \sim a = 1\), we call \(A\) a contrapositionally \(\lor\) complemented pseudo-Boolean algebra \((c\lor cpBa)\).
Observation 2.2. For the negation \( \sim \), the condition \( \sim a = a \rightarrow (\sim \sim a) \) can be equivalently expressed as

1. \( \sim a = a \rightarrow \sim 1 \)
2. \( \sim 1 = (\sim \sim 1) \).

In any ccpBa \( \mathcal{A} := (A, 1, 0, \lor, \land, \rightarrow, \neg, \sim) \), the property \( \neg a = a \rightarrow 0 \) makes the bounded rpc lattice \((A, 1, 0, \lor, \land, \rightarrow)\) a pseudo-Boolean algebra (pBa) with respect to the negation \( \neg \). In any rpc lattice \((A, 1, \lor, \land, \rightarrow)\), if negation \( \neg \) is defined as in Observation 2.2(1), the resulting lattice \((A, 1, \lor, \land, \rightarrow, \sim)\) forms a contrapositionally complemented (cc) lattice \([43]\). Thus a ccpBa can be considered as an amalgamation of a cc lattice and a pBa satisfying Condition (2) in Observation 2.2—the reason for naming the algebraic structure ‘contrapositionally-complemented pseudo-Boolean algebra’.

In any cc lattice, the negation \( \sim \) is completely determined by \( \rightarrow \) and the element \( \sim 1 \). Moreover, the element \( \sim 1 \) need not be the bottom element of the lattice. Note that in a ccpBa, there is a bottom element \( 0 \), which defines the negation \( \neg \). Observation 2.2(2) then gives the involutive property for the element \( \sim 1 \) of a ccpBa, with respect to the negation \( \neg \). We shall observe through examples in Section 2.2 that a ccpBa \((a)\) need not have the involutive property for all elements, and \(b\) \( \sim 1 \) need not be the bottom element 0.

In a \( c\lor c\lor c \), the reduct lattice forms a contrapositionally \( \lor \) complemented \( (c\lor c) \) lattice \([43]\), and thus the nomenclature ‘contrapositionally \( \lor \) complemented pseudo-Boolean algebra’. Condition (2) of Observation 2.2 as we shall see in Section 2.2 is a distinctive property of ccpBAs that is not true in general for an arbitrary bounded cc lattice.

Let us now list some properties of ccpBAs. For properties of cc, rpc lattices and pBAs, we refer to [23, 43].

Proposition 2.3. In any ccpBa \( \mathcal{A} := (A, 1, 0, \lor, \land, \rightarrow, \neg, \sim) \), the following hold for all \( a, b \in A \).

1. \( a \rightarrow \sim b = b \rightarrow \sim a \)
2. \( a \leq \sim \sim a \)
3. \( \sim \sim (\sim 1 \rightarrow a) = 1 \)
4. \( \sim a = (a \land \sim \sim 1) \)
5. \( \sim a \leq \sim a \)
6. \( a \leq \sim a \)
7. \( \sim \sim a \leq \sim a \)
8. \( \sim a = \sim \sim a \)
9. \( \sim \sim a \leq \sim a \)

Proof.
(1) and (2) follow directly from properties of cc lattices and Definition 2.1
(4): In the rpc lattice \((A, 1, \lor, \land, \rightarrow)\), for any \( a, b, c \in A \),
(i) \( (a \rightarrow b) \land (a \rightarrow c) = (a \rightarrow (b \land c)) \) and (ii) \( a \rightarrow (b \rightarrow c) = (a \land b) \rightarrow c \).
Using (i), \( (\sim 1 \rightarrow 0) \land (\sim 1 \rightarrow a) = \sim 1 \rightarrow (0 \land a) = \sim 1 \rightarrow 0 = \sim 1 \).
Now, \( \sim (\sim 1 \rightarrow a) = (\sim 1 \rightarrow a) \rightarrow \sim \sim 1 \rightarrow \sim (\sim 1 \rightarrow a) = \sim \sim 1 \rightarrow ((\sim 1 \rightarrow a) \rightarrow 0) = (\sim \sim 1 \land (\sim 1 \rightarrow a)) \rightarrow 0) = ((\sim 1 \rightarrow 0) \land (\sim 1 \rightarrow a)) \rightarrow 0 \).
Using (ii), \( \sim (\sim 1 \rightarrow a) = \sim \sim 1 \rightarrow 0 = \sim \sim 1 \).
Using rpc property \( a \rightarrow b = 1 \leftrightarrow a \leq b \), we have \( \sim (\sim 1 \rightarrow a) \leq \sim \sim 1 \Rightarrow \sim (\sim 1 \rightarrow a) \rightarrow \sim \sim 1 = 1 \). Therefore, \( \sim (\sim 1 \rightarrow a) = 1 \).
(5): \( \sim a = a \rightarrow (\sim \sim 1) = a \rightarrow (\sim 1 \rightarrow 0) = (a \land \sim 1) \rightarrow 0 = \sim (a \land \sim 1) \).
(6): Using rpc property \( b \leq c \Rightarrow (a \rightarrow b) \leq (a \rightarrow c) \), we have \( 0 \leq \sim 1 \Rightarrow a \rightarrow 0 \leq a \rightarrow \sim 1 \). Therefore, \( \sim a \leq \sim a \).
We observe the following differences between Nelson algebras and ccpBAs.

1. The implication \( \land \rightarrow \) has a lower bound. So, \( (\sim a \land \sim a) = 0, \) i.e. \( (\sim a \land \sim a) \land \sim a = 0 \leq 1. \) Recall the rpc property \( a \land c \leq b \iff c \leq a \rightarrow b. \) So, \( \sim a \land (\sim a \land \sim a) \rightarrow \sim a \leq (\sim a \land \sim a) \rightarrow 1 = 1 \iff (\sim a \land \sim a) = \sim (\sim a). \)

2. The negation \( \sim \) satisfies the involution property \( \sim \sim a = \sim \sim \sim a \), which is a special case of Peirce’s law \[46\].

Observation 2.4. Using Observation 2.2, the property \( \sim \sim (\sim 1 \rightarrow a) = 1 \) in Proposition 2.3 can also be expressed as:
\[
((\sim 1 \rightarrow a) \rightarrow \sim 1) \rightarrow \sim 1 = 1,
\]
which is a special case of Peirce’s law \[46\] \((b \rightarrow a) \rightarrow b = 1, \) for \( b = \sim 1. \)

2.1 Comparison with other algebras

Based on the properties of ccp lattices and Proposition 2.3, the following is straightforward.

Proposition 2.5.
Consider \( A := (A, 1, 0, \lor, \land, \rightarrow, \sim) \) such that the reduct \( (A, 1, 0, \lor, \land, \rightarrow) \) is a pBa. Then the following are equivalent.
1. \( A \) is a ccpBa.
2. \( (A, 1, 0, \lor, \land, \rightarrow, \sim) \) is a cc lattice and \( \sim \sim 1 = 1 \).
3. \( (A, 1, 0, \lor, \land, \rightarrow, \sim) \) is a cc lattice and \( \sim \sim (\sim 1 \rightarrow a) = 1 \) for any \( a \in A. \)

Note that (3) in Proposition 2.5 does not have any occurrence of \( \sim. \) The following example shows that the condition in the statement of the proposition that the reduct \( (A, 1, 0, \lor, \land, \rightarrow) \) is a pBa, is necessary for \( A := (A, 1, 0, \lor, \land, \rightarrow, \sim) \) to be a ccpBa.

Example 2.6. Consider the linear lattice \( (L, 0, \lor, \land) \), where \( L \) consists of all negative integers including 0. Define an implication operator (cf. \[30\]) as
\[
a \rightarrow b := b \quad \text{if } b < a,
= 0 \quad \text{otherwise}.
\]

Here, \( (L, 0, \lor, \land, \rightarrow) \) is an rpc lattice. Define \( \sim \) as \( \sim a := a \rightarrow 0. \) Thus, trivially, \( (L, 0, \lor, \land, \rightarrow, \sim) \) satisfies condition (3) of Proposition 2.5. However, \( (L, \leq) \) does not have a lower bound. So, \( (L, 0, \lor, \land, \rightarrow) \) cannot be extended to a pBa, and hence the cc lattice \( (L, 0, \lor, \land, \rightarrow, \sim) \) cannot be extended to a ccpBa.

A familiar lattice with two distinct negation operators and having the same algebraic type as a ccpBa is the quasi-pseudo Boolean algebra, also called Nelson algebra \[43\]. We observe the following differences between Nelson algebras and ccpBAs.

1. The implication \( \sim \) is a relative pseudo-complement operator in a ccpBa, which may not be true for \( \rightarrow \) in a Nelson algebra.
2. The negation \( \sim \) satisfies the involution property \( \sim \sim a = a \) for all \( a \in A \) in a Nelson algebra \( A := (A, 1, 0, \lor, \land, \rightarrow, \sim) \). This may not be true in an arbitrary ccpBa, neither of the two negations in a ccpBa may satisfy this property.

The above points (1) and (2) may easily be verified using the 3-element Nelson algebra.
\( \mathcal{A} := (A, 1, 0, \lor, \land, \rightarrow, \neg, \sim) \) from \( \textbf{[43]} \), where \( A := \{0, a, 1\} \) with the ordering as \( 0 \leq a \leq 1 \). The operators \( \rightarrow, \neg \) and \( \sim \) are defined in Table \( \textbf{1} \). In \( \textbf{[43]} \), it has been observed that every 3-element Nelson algebra is isomorphic to \( \mathcal{A} \).

| \( \rightarrow \) | 0 | a | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| a | 1 | 1 | 1 |
| 1 | 0 | a | 1 |

| \( x \) | \( \neg x \) |
|---|---|
| 0 | 1 |
| a | 1 |
| 1 | 0 |

Table 1: 3-element Nelson algebra

Let us establish the claims made in (1) and (2) above.

(1): In \( \mathcal{A} \), \( a \leq a \rightarrow 0 = 1 \) holds, but \( a = a \land a \not\leq 0 \). Thus the operator \( \rightarrow \) is not a relative pseudo-complement in \( \mathcal{A} \), and \( \mathcal{A} \) is not a ccpBa.

(2): We shall see in the next section that there are only two 3-element ccpBAs up to isomorphism. However, none of the negations in these is involutive. So, no 3-element ccpBa forms a Nelson algebra.

Thus, we can conclude that neither of the two algebraic classes of ccpBAs and Nelson algebras is a sub-class of the other.

### 2.2 Examples of ccpBAs and \( c \lor pBAs \)

It was mentioned in Section \( \textbf{1} \) that the set of strong subobjects of any ROUGH-object \((U, R, U)\) gives rise to a ccpBa (in fact, a \( c \lor pBa \)). \( \textbf{[33]} \). It is shown in \( \textbf{[34, 35]} \) that on abstraction of the category-theoretic construction, entire classes of examples of ccpBAs and \( c \lor pBAs \) can be obtained by starting from an arbitrary pseudo-Boolean or Boolean algebra. Let us recall the construction and results from \( \textbf{[35]} \). Consider a pBa \( \mathcal{H} := (H, 1, 0, \lor, \land, \rightarrow, \neg) \), and the set \( \mathcal{H}^{[2]} := \{(a, b) : a \leq b, a, b \in H\} \) (cf. \( \textbf{[9]} \)). Fix any element \( u := (u_1, u_2) \in \mathcal{H}^{[2]} \). Define the following set \( A_u \) and operators on it:

\[
A_u := \{(a_1, a_2) \in \mathcal{H}^{[2]} : a_2 \leq u_2 \text{ and } a_1 = a_2 \land u_1\},
\]

\[
\sqcup : (a_1, a_2) \sqcup (b_1, b_2) := (a_1 \lor b_1, a_2 \lor b_2),
\]

\[
\sqcap : (a_1, a_2) \sqcap (b_1, b_2) := (a_1 \land b_1, a_2 \land b_2),
\]

\[
\rightarrow : (a_1, a_2) \rightarrow (b_1, b_2) := ((a_1 \rightarrow b_1) \land u_1, (a_2 \rightarrow b_2) \land u_2),
\]

\[
\neg : \sim(a_1, a_2) := (u_1 \land \neg u_1, u_2 \land \neg u_1), \text{ and}
\]

\[
\sim : \neg(a_1, a_2) := (a_1, a_2) \rightarrow (0, 0).
\]

**Proposition 2.7.** \( \textbf{[35]} \) \( A_u := (A_u, (u_1, u_2), (0, 0), \sqcup, \sqcap, \rightarrow, \neg, \sim) \) forms a ccpBa. Moreover, if \( \mathcal{H} \) is a Boolean algebra, then \( A_u \) forms a \( c \lor pBa \).

Consider the 6-element pBa \( \mathcal{H}_6 := (H_6, 1, 0, \lor, \land, \rightarrow, \neg) \), for which the Hasse diagram is given by Figure \( \textbf{1} \). Define \( (u_1, u_2) := (z, w) \in \mathcal{H}_6^{[2]} \). Proposition **2.7** implies that \( A_u \) forms a ccpBa (Figure \( \textbf{2} \)). Similarly, for other choices of \( (u_1, u_2) \), we get different ccpBAs. Proposition **2.7** also mentions that if \( \mathcal{H} \) is a Boolean algebra, then \( A_u \) is a \( c \lor pBa \). However, the converse is not always true, i.e. if \( A_u \) is a \( c \lor pBa \), then \( \mathcal{H} \) need not always be a Boolean algebra. This can be observed from Table \( \textbf{4} \) for the above example: \( A_u \) is a \( c \lor pBa \), however \( \mathcal{H}_6 \) is not a Boolean algebra.
Using Definition 2.1 and Observation 2.2, examples of ccpBa can also be obtained as follows: in any pBa $A := (A, 1, 0, \lor, \land, \rightarrow, \neg)$, choose an element '$\sim 1$' in $A$ such that $\neg \neg \sim 1 = \sim 1$, and define $\sim a := a \rightarrow \sim 1$. This results in a ccpBa $A' := (A, 1, 0, \lor, \land, \rightarrow, \neg, \sim)$.

(A) Consider the only 3-element pBa $(A, 1, 0, \lor, \land, \rightarrow, \neg)$ (Figure 3). There are two choices for $\sim 1$ such that $\neg \neg \sim 1 = \sim 1$, namely the elements 0 and 1.

1. $\sim 1 := 0$. This means $\sim x = \neg_1 x$ (Table 5). The resulting algebra $A' := (A, 1, 0, \lor, \land, \rightarrow, \neg_1)$ is a ccpBa, but not a c\lor\cap Ba because $a \lor \sim a \neq 1$.

2. $\sim 1 := 1$, mapping each element of $A$ to 1 (Table 6). The resulting algebra $B' := (A, 1, 0, \lor, \land, \rightarrow, \neg_1, \sim_1)$ is a c\lor\cap Ba.

Since $(A, 1, 0, \lor, \land, \rightarrow, \neg_1)$ is the only 3-element pBa up to isomorphism, $A'$ and $B'$ are the only 3-element ccpBAs' up to isomorphism.

(B) Consider the 5-element pBa $H_5 := (H_5, 1, 0, \lor, \land, \rightarrow)$ (Figure 4). There are four choices for $\sim 1$ such that $\neg \neg \sim 1 = \sim 1$ – these are the elements 0, 1, a, and b.

1. $\sim 1 := 0$, i.e. $\sim x = \neg x$. The resulting algebra $(H_5, 1, 0, \lor, \land, \rightarrow, \neg)$ is a ccpBa, but not a c\lor\cap Ba because $a \lor \sim a \neq 1$. 

| Table 4: Implication $\rightarrow_1$ | Table 5: Negation $\neg_1$ | Table 6: Negation $\sim_1$ |
|---|---|---|
| $\rightarrow_1$ | $\neg_1 x$ | $\sim_1 x$ |
| 0 1 1 1 | 0 1 | 0 1 |
| a 0 1 1 | a 0 | a 1 |
| 1 0 a 1 | 1 0 | 1 1 |
2. \( \sim 1 := a \). The resulting algebra \( (H_5, 1, 0, \lor, \land, \rightarrow, \sim) \) is a ccpBa, but again not a \( c \lor \text{cpBa} \) because \( b \lor \sim b \neq 1 \).

3. \( \sim 1 := b \). This case is the same as that for \( \sim 1 := a \), the resulting algebra is a ccpBa, but not a \( c \lor \text{cpBa} \).

4. \( \sim 1 := 1 \). The resulting algebra \( (H_5, 1, 0, \lor, \land, \rightarrow, \sim) \) is a \( c \lor \text{cpBa} \).

(C) For the 6-element \( pBa H_6 := (H_6, 1, 0, \lor, \land, \rightarrow, \sim) \) (Figure 3), the available choices for \( \sim 1 \) are the elements 0, 1, \( x \), and \( y \).

1. \( \sim 1 := 0 \), i.e. \( \sim a = \neg a \). The resulting algebra \( (H_6, 1, 0, \lor, \land, \rightarrow, \sim) \) is a ccpBa, but not a \( c \lor \text{cpBa} \) because \( z \lor \sim z \neq 1 \).

2. \( \sim 1 := y \). The resulting algebra \( (H_6, 1, 0, \lor, \land, \rightarrow, \sim) \) is a ccpBa, but again not a \( c \lor \text{cpBa} \) because \( z \lor \sim z \neq 1 \).

3. \( \sim 1 := x \). The resulting algebra \( (H_6, 1, 0, \lor, \land, \rightarrow, \sim) \) is a \( c \lor \text{cpBa} \).

4. \( \sim 1 := 1 \). The resulting algebra \( (H_6, 1, 0, \lor, \land, \rightarrow, \sim) \) is again a \( c \lor \text{cpBa} \).

### 2.3 Representation theorems for the algebras

The representation result for \( pB as \) in terms of pseudo-fields of open subsets of a topological space [35], can be directly extended to that for \( ccpB as \) [35]. ‘Contrapositionally complemented pseudo-fields’ are defined for the purpose. We recall the definition and state the result.

**Definition 2.8** (Contrapositionally complemented pseudo-fields).

Let \( \mathcal{G}(X) := (\mathcal{G}(X), X, \emptyset, \land, \lor, \rightarrow, \sim) \) be a pseudo-field of open subsets of a topological space \( X \). Choose and fix \( Y_0 \in \mathcal{G}(X) \). Define

\[
\sim X := \neg \neg Y_0, \quad \text{and} \quad \sim Z := Z \rightarrow (\neg \neg Z), \text{ for each } Z \in \mathcal{G}(X).
\]

Then the algebra \( (\mathcal{G}(X), X, \emptyset, \land, \lor, \rightarrow, \sim) \) is called a *contrapositionally complemented pseudo-field* (cc pseudo-field) of open subsets of \( X \).

**Proposition 2.9.** Any cc pseudo-field of open subsets of a topological space \( X \) forms a ccpBa. Moreover, for every ccpBa \( A := (A, 1, 0, \cap, \cup, \rightarrow, \sim) \), there exists a monomorphism \( h \) from \( A \) into a cc pseudo-field of all open subsets of a topological space \( X \).

In this work, we turn to representations of ccpB as and \( c \lor \text{cpB as} \) in terms of Priestley and Esakia spaces. Let us recall the basic definitions.

Consider a poset \( (X, \leq) \). \( Y \subseteq X \) is called an **upset** (downset) if for all \( x \in Y \) and \( y \in X \), \( x \leq y \) (\( y \leq x \)) implies \( y \in Y \). Let \( Up(X) \) denote the set of all upsets of \( X \). For
In this case, \( D \) is clopen implies

**Definition 2.10** (Priestley and Esakia spaces). (cf. [12] [4]) A Priestley space is a tuple \( (X, \tau, \leq) \), where \( (X, \leq) \) is a poset and \( \tau (\neq \emptyset) \) is a compact topological space on \( X \) satisfying the following property: for every \( x, y \in X \), if \( x \not\leq y \), then there exists a clopen (closed, as well as open) upset \( Y \) of \( X \) such that \( x \in Y \) and \( y \not\in Y \). Additionally, if a Priestley space \( (X, \tau, \leq) \) satisfies the property that for any \( U \subseteq X \), \( U \) is clopen implies \( \downarrow U \) is clopen, then it is called an Esakia space.

Let \( CpUp(X) \) be the set of clopen upsets of \( \tau \) in a Priestley space \( (X, \tau, \leq) \). Then \( \mathcal{D}(X) := (CpUp(X), X, \emptyset, \cup, \cap, \rightarrow, \neg) \) forms a bounded distributive lattice. Define the operator \( \rightarrow \) on \( CpUp(X) \), for any \( U, V \in CpUp(X) \):

\[
U \rightarrow V := X \downarrow (U \setminus V).
\]

(1) The operator \( \rightarrow \) is closed in \( CpUp(X) \) if and only if \( (X, \tau, \leq) \) is an Esakia space (cf. [7]). In this case, \( \mathcal{D}(X) := (CpUp(X), X, \emptyset, \cup, \cap, \rightarrow, \neg) \) forms a pBa, where \( \neg U := U \rightarrow \emptyset \).

Now consider a pBa \( A := (A, 1, 0, \lor, \land, \rightarrow, \neg) \), and let \( X_A \) denote the set of prime filters in \( A \). Define the topology \( \tau_A \) on \( X_A \) generated by the subbasis

\[
\{\sigma(a) \mid \sigma(a) \subseteq X_A \land a \in A\} \cup \{X_A \setminus \sigma(a) \mid \sigma(a) \subseteq X_A \land a \in A\},
\]

where \( \sigma(a) \) is the set of prime filters containing \( a \in A \). Then \( (X_A, \tau_A, \subseteq) \) forms an Esakia space. For \( a \in A \), \( \sigma(a) \) and \( X_A \setminus \sigma(a) \) are the only clopen upsets in \( \tau_A \). These definitions lead to the following representation result for pBAs (cf. [5]).

**Theorem 2.11** (Duality for pBAs). [5]

1. Given any pBa \( A := (A, 1, 0, \lor, \land, \rightarrow, \neg) \), there exist an Esakia space \( (X_A, \tau_A, \subseteq) \) and a pBa \( \mathcal{D}(X_A) := (CpUp(X_A), X_A, \emptyset, \cup, \cap, \rightarrow, \neg) \) such that \( A \) is isomorphic to \( \mathcal{D}(X_A) \), through the map \( \Phi : A \rightarrow CpUp(X_A) \) defined as \( \Phi(a) := \sigma(a) \), for any \( a \in A \).

2. Given any Esakia space \( (X, \tau, \leq) \), there exist a pBa \( \mathcal{D}(X) := (CpUp(X), X, \emptyset, \cup, \cap, \rightarrow, \neg) \) and an Esakia space \( (X_{CpUp(X)}, \tau_{\mathcal{D}(X)}, \subseteq) \) such that \( \tau \) is homeomorphic to \( \tau_{\mathcal{D}(X)} \) and the poset \( (X, \leq) \) is order-isomorphic to the poset \( (X_{CpUp(X)}, \subseteq) \).

The above duality result can be extended to ccpBAs and \( cc/cpBas \). Consider a ccpBa \( A := (A, 1, 0, \lor, \land, \rightarrow, \neg) \). Observation 2.2 implies the reduct \( (A, 1, 0, \lor, \land, \rightarrow, \neg) \) is a pBa. Using Theorem 2.11, we have an Esakia space \( (X_A, \tau_A, \subseteq) \) such that the pBa \( (A, 1, 0, \lor, \land, \rightarrow, \neg) \) is isomorphic to the pBa \( \mathcal{D}(X_A) := (CpUp(X_A), X_A, \emptyset, \cup, \cap, \rightarrow, \neg) \), through the map \( \Phi \). Define \( Y_0 := \sigma(\sim 1) \). Since \( \Phi \) is a homomorphism,

\[
s(\sim 1) = \Phi(\sim 1) = \Phi(\neg\neg\neg 1) = \neg\neg\Phi(\sim 1) = \neg\neg s(\sim 1)
\]

giving \( Y_0 = X_A \downarrow (X_A \downarrow Y_0) \). In particular, we have \( X_A \downarrow (X_A \downarrow Y_0) \subset Y_0 \). Expanding this,

\[
\forall x \in X_A (x \not\in (X_A \downarrow Y_0)) \Rightarrow x \in Y_0 \\
\forall x \in X_A (\forall y \in X_A (x \leq y \Rightarrow y \in Y_0)) \Rightarrow x \in Y_0 \\
\forall x \in X_A (\forall y \in X_A (x \leq y \Rightarrow \exists z \in X_A (y \subseteq z \land z \in Y_0))) \Rightarrow x \in Y_0.
\]
Thus, starting from a $ccpBa \ A$, we have obtained an Esakia space $(X_A, \tau_A, \subseteq)$ satisfying
\[
\forall x \in X_A(\forall y \in X_A(x \subseteq y \Rightarrow \exists z \in X_A(y \subseteq z & z \in Y_0)) \Rightarrow x \in Y_0).
\] (2)

Conversely, consider an Esakia space $(X, \tau, \subseteq)$ and $Y_0 \in X$ such that $Y_0$ is a clopen set in $\tau$ satisfying
\[
\forall x \in X(\forall y \in X(x \leq y \Rightarrow \exists z \in X(y \leq z & z \in Y_0)) \Rightarrow x \in Y_0).
\] (3)

Then, by Theorem 2.11 we have the $pBa \ D(X) := (CpUp(X), X, \emptyset, \cup, \cap, \rightarrow, \neg)$.

Now, define $\sim X := Y_0$ and $\sim U := U \rightarrow \sim X$ for all $U \in CpUp(X)$, thus making $(CpUp(X), X, \emptyset, \cup, \cap, \rightarrow, \neg)$ a cc lattice. Any $pBa$ has the property $a \leq \neg\neg a$, for all $a \in A$. Thus, $\sim X \subseteq \neg\neg\sim X$. Moreover, it can be observed that (11) and (3) imply $\neg\sim X \subseteq \sim X$. Therefore, $\sim X = \neg\neg\sim X$. By Proposition 2.5(2), $(CpUp(X), X, \emptyset, \cup, \cap, \rightarrow, \neg, \sim)$ is a $ccpBa$.

For simplicity, we use the same notation $D(X)$ for the $pBa$ and the $ccpBa$.

For $c\lor ccpBas$, note that the property $a \lor \neg a = 1$ corresponds to the following condition
\[
\forall x, y \in X(x, y \notin Y_0 \Rightarrow (x \leq y \Rightarrow y \leq x))
\] (4)
in the context of logic and relational semantics [46]. This correspondence can be replicated here to get the dual topological spaces for $c\lor ccpBas$. If $A$ is a $c\lor ccpBa$, we have $\sigma(a) \cup (\sigma(a) \rightarrow Y_0) = X_A$ for all $a \in A$, i.e.
\[
\sigma(a) \cup X_A \downarrow (\sigma(a) \setminus Y_0) = X_A \text{ for all } a \in A.
\] (5)

In this case, the Esakia space $(X_A, \tau_A, \subseteq)$ satisfies
\[
\forall x, y \in X_A(x, y \notin Y_0 \Rightarrow (x \subseteq y \Rightarrow y \subseteq x)).
\] (6)

For this, we have to show that for any two prime filters $F, G \in X_A$, if $F, G \notin Y_0$ and $F \subseteq G$ then $G \subseteq F$. Suppose not, i.e. there exist two prime filters $F, G$ such that $F, G \notin Y_0$, $F \subseteq G$ and there exists $x \in G$ such that $x \notin F$. Thus $F \notin \sigma(x)$ and $G \in \sigma(x)$. In Condition (5), since $F \in X_A$, we must have $F \in \sigma(x) \cup X_A \downarrow (\sigma(x) \setminus Y_0)$. We already have $F \notin \sigma(x)$. Therefore $F \in X_A \downarrow (\sigma(x) \setminus Y_0)$, i.e. $F \notin \downarrow (\sigma(x) \setminus Y_0)$. Since $F \subseteq G$, using the definition of $\downarrow (\sigma(x) \setminus Y_0)$, we have $G \notin \sigma(x) \setminus Y_0$, i.e. $G \notin \sigma(x)$ (because $G \notin Y_0$), a contradiction.

Conversely, consider an Esakia space $(X, \tau, \subseteq)$ and $Y_0 \in X$ such that $Y_0$ is a clopen set satisfying Conditions (3) and (4). We have to only show that for any $V \in CpUp(X)$, $V \cup (V \rightarrow X) = X$, i.e. $V \cup (X \downarrow (V \setminus Y_0)) = X$. Suppose not, i.e. there exists $x \in X$ such that $x \notin V$ and $x \in (V \setminus Y_0)$. Then there exists $y \in X$ such that $x \leq y$ and $y \in V \setminus Y_0$. We have $x \leq y$, $y \in V$ and $y \notin Y_0$, i.e. $x \notin Y_0$ (because $Y_0$ is an upset). Using $x \leq y$ and Condition (5), we have $y \leq x$. Since $y \in V$ and $V$ is an upset, we have $x \in V$, a contradiction.

We thus obtain

**Theorem 2.12 (Duality for $ccpBas$ and $c\lor ccpBas$).**

(1) Given any $ccpBa (c\lor ccpBa) \ A := (A, 1, 0, \lor, \land, \rightarrow, \neg, \sim)$, we have the following.
(a) The tuple \((X_A, \tau_A, \subseteq)\) forms an Esakia space; \(Y_0 := \sigma(1)\) is a clopen set in \(\tau_A\) satisfying Condition 3 (Conditions 3 and 7).
(b) The tuple \(D(X_A) := (CpUp(X_A), X_A, \emptyset, \cup, \cap, \to, \neg)\) is a ccpBa \((\text{c}\cap \text{ccpBa})\).
(c) \(A\) is isomorphic to \(D(X_A)\), through the map \(\Phi : A \to CpUp(X_A)\) defined as \(\Phi(a) := \sigma(a)\), for any \(a \in A\).

(2) Given any ordered topological space \((X, \tau, \leq, Y_0)\) such that \((X, \tau, \leq)\) is an Esakia space and \(Y_0\) is a clopen set satisfying Condition 3 (Conditions 3 and 11), we have the following.
(a) The tuple \(D(X) := (CpUp(X), X, \emptyset, \cup, \cap, \to, \neg)\) forms a ccpBa \((\text{c}\cap \text{ccpBa})\).
(b) \((X_{CpUp(X)}), \tau_{D(X)}, \subseteq)\) is an Esakia space.
Moreover, \(Y_0\) defined as the set of all prime filters containing \(\neg X\) in \(D(X)\), satisfies the following.
\[
\forall x \in X_{CpUp(X)}(\forall y \in X_{CpUp(X)}(x \subseteq y \Rightarrow \exists z \in X_{CpUp(X)}(y \subseteq z \land z \in Y_0)) \Rightarrow x \in Y_0).
\]
If \(Y_0\) satisfies Condition 3 then \(Y_0\) satisfies
\[
\forall x, y \in X_{CpUp(X)}(x, y \notin Y_0 \Rightarrow (x \subseteq y \Rightarrow y \subseteq x)).
\]
(c) \(\tau\) is homeomorphic to \(\tau_{D(X)}\) and \((X, \leq)\) is order-isomorphic to \((X_{CpUp(X)}, \subseteq)\).

Proof.
(1) From Theorem 2.11(1), we already have that \(\Phi\) is an isomorphism between the underlying \(\text{pBAs}\) - \((A, 1, 0, \lor, \land, \to, \neg)\) and \((CpUp(X_A), X_A, \emptyset, \cup, \cap, \to, \neg)\). We have to only show \(\Phi(\neg x) = \neg \Phi(x)\) for all \(x \in A\). Indeed, \(\Phi(\neg x) = \Phi(x \to \neg 1) = \Phi(x) \to \Phi(\neg 1) = \sigma(x) \to Y_0 = \sigma(x) \to \neg X_A = \neg \sigma(x) = \neg \Phi(x)\).
(2) This is direct from Theorem 2.11(2) and the discussion following it.

3 Intuitionistic logic with minimal negation

In this section, we present the logic corresponding to the class of \(\text{ccpBAs}\), called ‘Intuitionistic logic with minimal negation’ (ILM) [35]. As is well-known, positive logic (PL), minimal logic (ML), intuitionistic logic (IL) and classical logic (CL) are sound and complete with respect to the class of \(rcc\) lattices, \(cc\) lattices, \(pBAs\) and Boolean algebras respectively [33 39]. Since the \(\text{ccpBa}\) is an amalgamation of \(cc\) lattice and \(pBa\), it is then expected that ILM will be defined using IL and ML. We shall use the terminology and axiomatization of ML and IL as given in [33].

3.1 ILM and ILM-\(\lor\)

Definition 3.1 (Intuitionistic logic with minimal negation (ILM)). [35] The alphabet of the language \(L\) of ILM is that of IL, consisting of propositional constants \(\top\) and \(\perp\), a set \(PV\) of propositional variables, and logical connectives \(\to\) (implication), \(\lor\) (disjunction), \(\land\) (conjunction), \(\neg\) (negation). Additionally, there is a unary connective \(\sim\). The formulas are given by the scheme:
\[
\top, \perp, p, \alpha \land \beta, \alpha \lor \beta, \alpha \to \beta, \neg \alpha, \sim \alpha
\]
where \(p \in PV\). The set of all formulas is denoted by \(F\).

Axioms:
\(\text{(A1)}\) \(\alpha \rightarrow (\beta \rightarrow \alpha)\)  
\(\text{(A2)}\) \((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))\)  
\(\text{(A3)}\) (i) \(\alpha \rightarrow (\alpha \lor \beta)\),  
(ii) \(\beta \rightarrow (\alpha \lor \beta)\)  
\(\text{(A4)}\) \((\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \lor \beta) \rightarrow \gamma))\)  
\(\text{(A5)}\) (i) \((\alpha \land \beta) \rightarrow \alpha\),  
(ii) \((\alpha \land \beta) \rightarrow \beta\)  
\(\text{(A6)}\) \((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \land \gamma)))\)  
\(\text{(A7)}\) \(\alpha \rightarrow \top\)  
\(\text{(A8)}\) \(\bot \rightarrow \alpha\)  
\(\text{(A9)}\) \((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg \beta) \rightarrow \neg \alpha)\)  
\(\text{(A10)}\) \(\neg \alpha \rightarrow (\alpha \rightarrow \bot)\)  
\(\text{(A11)}\) \(\neg \alpha \leftrightarrow (\alpha \rightarrow \neg \neg \bot)\)  
\(\text{(A12)}\) \(\alpha \lor \neg \alpha\)

Modus ponens (MP) is the only rule of inference in ILM. The deduction procedure for ILM is specified in the usual manner, to give the relation of syntactic consequence \(\vdash_{\text{ILM}}\) and define \(\Gamma \vdash_{\text{ILM}} \alpha\), for any \(\Gamma \cup \{\alpha\} \subseteq F\). Addition of the following axiom gives the logic ILM-\(\lor\).

\(\text{(A12)}\) \(\alpha \lor \neg \alpha\)

Observe that axiom (A11) connects the negations \(\neg\) and \(\sim\).

3.1.1 Properties of ILM and comparison with IL and ML

It can be shown that the deduction theorem (DT) holds for ILM. Let us now give the algebraic semantics for the logics.

Consider any ccpBa \(\mathcal{A} := \langle A, 1, 0, \lor, \land, \rightarrow, \neg, \sim \rangle\). A valuation is a map \(v\) from PV to \(A\), and can be extended to \(F\) in the standard way [43]. For a formula \(\alpha \in F\), if for all valuations \(v\) on \(\mathcal{A}\), \(v(\alpha) = 1\), then we say \(\alpha\) is valid in \(\mathcal{A}\) (denoted as \(\models_{\mathcal{A}} \alpha\)). In the classical manner, we obtain

**Theorem 3.2 (Algebraic semantics).** For any \(\alpha \in F\), \(\vdash_{\text{ILM}} \alpha \Leftrightarrow \vdash_{\text{ILM-\(\lor\)}} \alpha\) if and only if \(\models_{\mathcal{A}} \alpha\) for every ccpBa (c\(\lor\)cpBa) \(\mathcal{A}\).

Thus, using the soundness part of the above theorem, Proposition [2.3] gets a version in terms of ILM-formulas.

**Proposition 3.3.**

\(\text{(a)}\) \(\vdash_{\text{ILM}} \neg \alpha \leftrightarrow (\alpha \rightarrow \bot)\)  
\(\text{(b)}\) \(\vdash_{\text{ILM}} \neg \neg \top \leftrightarrow \top\)  
\(\text{(c)}\) \(\vdash_{\text{ILM}} \sim \alpha \leftrightarrow (\alpha \rightarrow \sim \top)\)  
\(\text{(d)}\) \(\vdash_{\text{ILM}} \sim \sim \sim \top \rightarrow \alpha\)  
\(\text{(e)}\) \(\vdash_{\text{ILM}} \sim \alpha \leftrightarrow \neg (\alpha \land \neg \neg \top)\)  
\(\text{(f)}\) \(\vdash_{\text{ILM}} \neg \alpha \rightarrow \sim \alpha\)  
\(\text{(g)}\) \(\vdash_{\text{ILM}} \alpha \rightarrow \sim \alpha\)  
\(\text{(h)}\) \(\vdash_{\text{ILM}} \sim \alpha \rightarrow \sim \sim \alpha\)  
\(\text{(i)}\) \(\vdash_{\text{ILM}} \sim \sim \sim \alpha \leftrightarrow \neg \neg \sim \sim \sim \alpha\)  
\(\text{(j)}\) \(\vdash_{\text{ILM}} \sim \sim \sim \alpha \leftrightarrow \neg \sim \sim \sim \alpha\)

**Proof.** As \(\neg a = a \rightarrow 0\) in any ccpBa, Theorem [3.2] gives (a). Observation [2.2](2) and Theorem [3.2] give (b). (c) is obtained using (A11) and (b). Proposition [2.3](3)-(9) and Theorem [3.2] give (d)-(j) respectively.

**Remark 3.4.** The formulas in (a) and (c) in the above Proposition are the defining conditions for the negations \(\neg\) and \(\sim\) to be intuitionistic and minimal respectively. Intuitionistic negation can also be equivalently defined using formulas (A9) and (A10).
Recall that for any two logics $L$ and $L'$ with sets of formulas $F$ and $F'$ respectively, $L$ is *embedded* in $L'$ if there exists a map $r : F \to F'$ such that for any $\alpha \in F$, $\vdash_L \alpha$ if and only if $\vdash_{L'} r(\alpha)$ [34]. In case $r$ is the inclusion map, $L'$ is called an *extension* of $L$. $L$ is *equivalent* to $L'$, denoted as $L \equiv L'$, if the languages of $L$ and $L'$ are the same, that is $F = F'$, and $r$ is the identity map (cf. [39]). Another way to show equivalence between two logics over the same language is by comparing their corresponding classes of algebras: $L$ and $L'$ are equivalent if every $L$-algebra is isomorphic to some $L'$-algebra and conversely [43]. It is shown in [43] that both the definitions of equivalence coincide.

### Proposition 3.5. [35]

ML and IL are both embedded in ILM.

Comparison of two logics is also done by giving an ‘interpretation’ between them [41].

A map $r : F \to F'$ is an *interpretation of $L$ in $L'$ with respect to derivability*, if for any set $\Gamma \cup \{\alpha\} \subseteq F$, we have $\Gamma \vdash L \alpha$ if and only if $r(\Gamma) \cup \Delta_\alpha \vdash L' r(\alpha)$, where $\Delta_\alpha \subseteq F'$ is a finite set corresponding to $\alpha$. This yields

### Proposition 3.6. [35]

There exist interpretations of ILM in IL and in ML.

As observed in case of ccpBas (Observation 2.2), we get an equivalent axiomatization of ILM as follows.

### Theorem 3.7 (The logic ILM$_1$).

Consider a logic ILM$_1$ over the language $L$, with axioms (A1)-(A10), $\sim \alpha \leftrightarrow (\alpha \to \sim \top)$, and $\neg \neg \sim \top \leftrightarrow \sim \top$. MP is the only rule of inference. Then ILM $\equiv$ ILM$_1$.

#### Proof.

Axioms $\sim \alpha \leftrightarrow (\alpha \to \sim \top)$ and $\neg \neg \sim \top \leftrightarrow \sim \top$ of ILM$_1$ imply that (A11) is an ILM$_1$-theorem. On the other hand, we have $\vdash_{ILM} \neg \neg \sim \top \leftrightarrow \sim \top$ and $\vdash_{ILM} \sim \alpha \leftrightarrow (\alpha \to \sim \top)$ (Proposition 3.3(b) and (c)).

We shall use this equivalent version ILM$_1$ of ILM while discussing the two relational semantics for ILM in Sections 4 and 5.

### 3.1.2 Comparison of ILM with Peirce’s logic JP’

Consider the language $L'$ with formulas given by the scheme:

$\top \mid p \mid \alpha \lor \beta \mid \alpha \land \beta \mid \alpha \to \beta \mid \sim \alpha$

The axioms of ML are given as (A1)-(A7) of ILM and

(A13) $\alpha \to \beta \to ((\alpha \to \sim \beta) \to \sim \alpha)$,  

($\sim$ reductio ad absurdum)

and MP as the rule of inference. In [46], Segerberg defined various extensions of ML. One of these is the system JP, which is obtained by adding Peirce’s law (P) as an axiom to ML:

(P) $((\alpha \to \beta) \to \alpha) \to \alpha$

Let us consider the logic JP’ defined as below.
Definition 3.8 (JP′ [46]). The language of JP′ is \( \mathcal{L}' \). The axioms are (A1) – (A7), (A13) and
\[
(P') \quad \neg\neg(\neg\top \rightarrow \beta).
\]
MP is the only rule of inference.

One may remark here that in [39, 46], while defining (P′), a propositional constant ‘\( \perp \)’ that is logically equivalent to \( \neg\top \) is used in place of \( \neg\top \). We have used \( \neg\top \) instead, in order not to confuse with the propositional constant \( \perp \) that is already present in ILM, and which is not logically equivalent to \( \neg\top \) – as is established by the proof of Theorem 3.11 below.

Using the ML-theorem \( \vdash_{\text{ML}} \neg\alpha \leftrightarrow (\alpha \rightarrow \neg\top) \), one observes that (P′) is logically equivalent to the following formula (which we also refer to as (P′))
\[
((\neg\top \rightarrow \beta) \rightarrow \neg\top) \rightarrow \neg\top.
\]
Thus, (P′) may be regarded as a special case of (P) where \( \alpha \) is \( \neg\top \). Let us now compare the logics JP′ and ILM.

Theorem 3.9. ILM is an extension of JP′.

Proof. (P′) is a theorem in ILM (Proposition 3.3(d)). Axioms (A1)-(A7) are common to JP′ and ILM. Let us see the proof of (A13) in ILM. Given (A11), (A13) can be equivalently expressed as \( (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \neg\neg\top)) \rightarrow (\alpha \rightarrow \neg\neg\top)) \). Using MP, one can derive \( \{\alpha \rightarrow \beta, \alpha \rightarrow (\beta \rightarrow \neg\neg\top), \alpha\} \vdash_{\text{ILM}} \neg\neg\top \). Applying DT, we obtain the above equivalent expression for (A13) as a theorem in ILM.

We shall show in Theorem 3.11 below that JP′ is however, not equivalent to ILM. This is established by comparing the classes of algebras corresponding to the respective logics. Theorem 3.2 implies that any ILM-algebra is just a \( \text{ccpBa} \). For JP′, the corresponding class of algebras is given by

Definition 3.10 (JP′-algebra). A JP′-algebra \( (A, 1, \lor, \land, \rightarrow, \sim) \) is a cc lattice satisfying \( \sim\sim(\sim1 \rightarrow a) = 1 \) for all \( a \in A \).

Note that JP′-algebra is just the structure mentioned in (3) of Proposition 2.5.

Theorem 3.11. JP′ is not equivalent to ILM.

Proof. Suppose we had the equivalence. Then, the languages of JP′ and ILM would be the same, the axioms of JP′ would be theorems in ILM, and the axioms of ILM theorems in JP′. This would imply that we can define the connective \( \neg \) and propositional constant \( \perp \) in \( \mathcal{L}' \) such that (A8), (A9) and (A10) are theorems in JP′. In particular, \( \vdash_{\text{JP'}} \perp \rightarrow \alpha \). Now consider any JP′-algebra \( (A, 1, \lor, \land, \rightarrow, \sim) \). By soundness, any such algebra will have a bottom element 0 and an operator \( \neg \) such that \( \neg\alpha = a \rightarrow \neg\top \) for all \( a \in A \). However, in Example 2.6 we have encountered a JP′-algebra \( (L, 0, \lor, \land, \rightarrow, \sim) \) that does not have a bottom element and thus can never be extended to a \( \text{ccpBa} \).

It is then expected that if we add a new propositional constant \( \perp \) to the alphabet of \( \mathcal{L}' \) and consider JP′ enhanced with Axiom (A8), defining \( \neg \) as \( \neg\alpha := \alpha \rightarrow \perp \), the resulting system will be equivalent to ILM.
Definition 3.12 (The logic ILM$_2$). The formulas of ILM$_2$ are given by the scheme:

\[
\top | \bot | p | \alpha \land \beta | \alpha \lor \beta | \alpha \rightarrow \beta | \neg \alpha
\]

Axioms are (A1)-(A8), (A13) and (P'). MP is the only rule of inference.

We now show that ILM$_2$ is equivalent to ILM. Note that standard results like DT can be proved in ILM$_2$.

Theorem 3.13. ILM $\cong$ ILM$_2$.

Proof. Define $\neg \alpha := \alpha \rightarrow \bot$ in ILM$_2$. As shown for Theorem 3.9, (A13) and (P') are ILM-theorems.

One can show (A9), (A10) and (A11) are ILM$_2$-theorems, using following results.

(a) $\vdash_{ILM_2} \beta \leftrightarrow (\top \rightarrow \beta)$.
(b) $\vdash_{ILM_2} \neg \alpha \leftrightarrow (\alpha \rightarrow \neg \top)$.
(c) $\{\alpha, \neg \alpha\} \vdash_{ILM_2} \bot$.
(d) $\vdash_{ILM_2} ((\neg \top \rightarrow \bot) \rightarrow \neg \top)$.

\[\square\]

4  Došen semantics for ILM

In this section, we refer to the relational semantics given by Došen for minimal logic and its extensions [13, 14]. The class of ‘strictly condensed J-frames’ and that of ‘strictly condensed H-frames’ characterize ML and IL respectively [13], and these help us derive relational semantics for the logics ILM and ILM-$\lor$. Let us recall the definitions of these frames. Such a frame is essentially a structure based on a poset, equipped with a binary relation on the set that defines the semantics for negation.

Definition 4.1 (Strictly Condensed J-frame and H-frame). [13]

A triple $\mathcal{F} := (X, \leq, R_N)$ is a strictly condensed J-frame, if it satisfies the following.

1. $(X, \leq)$ is a poset.
2. $(\leq R_N \leq^{-1}) \subseteq R_N$.
3. $R_N$ is symmetric.
4. $\forall x, y \in X (xR_Ny \Rightarrow \exists z \in X (x \leq z \& y \leq z \& xR_Nz))$.

$\mathcal{F}$ is a strictly condensed H-frame when, in addition to the above conditions, the relation $R_N$ is reflexive.

Remark 4.2.

1. In a strictly condensed H-frame $\mathcal{F}$, reflexivity of $R_N$ implies its symmetry.
2. $\mathcal{F}$ satisfying (1) and (2) is a strictly condensed N-frame [13].

‘$\hat{N}$-frames’, giving the semantics for ILM, may now be defined.

4.1 $\hat{N}$-frames

There are two negations $\neg$ and $\sim$ in ILM. Therefore, to define the semantics for each negation, the frame should have two binary relations, one (say, $R_{N_1}$) corresponding to the negation $\neg$, and another (say, $R_{N_2}$) corresponding to the negation $\sim$. Recall Theorem 3.7 and Remark 3.4 (Section 3.1.1).

Observation 4.3.

1. As $\neg$ is an intuitionistic negation, $R_{N_1}$ must satisfy the properties corresponding to the relation $R_N$ in a strictly condensed H-frame $\mathcal{F} := (X, \leq, R_N)$. 

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We shall see in the sequel that Condition (3) of Definition 4.4 corresponds to the ILM formula

\[ \neg \neg \Top \leftrightarrow \neg \Top \]

axiom in the usual manner as defined in [13].

The class of all \( \hat{N} \)-frames is denoted by \( \mathfrak{F}_1 \).

An \( \hat{N} \)'-frame is an \( \hat{N} \)-frame \( \mathcal{F} := (X, \subseteq, R_{N_1}, R_{N_2}) \), satisfying the following conditions.

1. \( (X, \subseteq, R_{N_1}) \) is a strictly condensed \( H \)-frame.
2. \( (X, \subseteq, R_{N_2}) \) is a strictly condensed \( J \)-frame.
3. \( \forall x \in X \left( \forall y \in X \left( xR_{N_1}y \Rightarrow \exists z \in X (yR_{N_1}z \& \forall z' \in X (zR_{N_2}z')) \right) \Rightarrow \forall z'' \in X (xR_{N_2}z'') \right) \).

The class of all \( \hat{N} \)-frames is denoted by \( \mathfrak{F}_1 \).

We shall see in the sequel that Condition (3) of Definition 4.4 corresponds to the ILM-axiom \( \neg \neg \Top \leftrightarrow \Top \) (cf. Observation 4.3(3)). Moreover, the condition defining an \( \hat{N} \)'-frame, namely \( R_{N_2} \subseteq (\leq)^{-1} \) corresponds to the formula \( \alpha \lor \neg \neg \alpha \) [13], and thus, is the required condition as mentioned in Observation 4.3(4).

Let us now incorporate these points to define a new class of frames.

**Definition 4.4.** An \( \hat{N} \)-frame is a quadruple \( \mathcal{F} := (X, \subseteq, R_{N_1}, R_{N_2}) \), satisfying the following conditions.

1. \( (X, \subseteq, R_{N_1}) \) is a strictly condensed \( H \)-frame.
2. \( (X, \subseteq, R_{N_2}) \) is a strictly condensed \( J \)-frame.
3. \( \forall x \in X \left( \forall y \in X \left( xR_{N_1}y \Rightarrow \exists z \in X (yR_{N_1}z \& \forall z' \in X (zR_{N_2}z')) \right) \Rightarrow \forall z'' \in X (xR_{N_2}z'') \right) \).

The class of all \( \hat{N} \)-frames is denoted by \( \mathfrak{F}_1 \).

For an \( \hat{N} \)-frame \( \mathcal{F} := (X, \subseteq, R_{N_1}, R_{N_2}) \), a map \( v : PV \rightarrow \mathcal{P}(X) \) is called a valuation of \( \mathcal{L} \) on \( \mathcal{F} \) if \( v(p) \) is an upset for each \( p \in PV \). The pair \( \mathcal{M} := (\mathcal{F}, v) \) is called an \( \hat{N} \)-model on the \( \hat{N} \)-frame \( \mathcal{F} \).

**Definition 4.5 (Truth of a formula).** The truth of a formula \( \alpha \in F \) at a world \( x \in X \) in an \( \hat{N} \)-model \( \mathcal{M} \) (notation: \( \mathcal{M}, x \models \alpha \)) is defined by extending the valuation \( v : PV \rightarrow \mathcal{P}(X) \) to the set \( F \) of formulas in the standard way [13]. We only give the semantic clauses for the connectives \( \rightarrow, \neg \) and \( \sim \).

1. \( \mathcal{M}, x \models \alpha \rightarrow \beta \iff \text{for all } y \in X, \text{ if } x \leq y \text{ and } \mathcal{M}, y \models \alpha \text{ then } \mathcal{M}, y \models \beta \).
2. \( \mathcal{M}, x \models \neg \alpha \iff \text{for all } y \in X (xR_{N_1}y \Rightarrow \mathcal{M}, y \not\models \alpha) \).
3. \( \mathcal{M}, x \models \neg \neg \alpha \iff \text{for all } y \in X (xR_{N_2}y \Rightarrow \mathcal{M}, y \not\models \alpha) \).

A formula \( \alpha \) is true in a model \( \mathcal{M} \) (notation: \( \mathcal{M} \models \alpha \)) if \( \mathcal{M}, x \models \alpha \) for all \( x \in X \). A formula \( \alpha \in F \) is valid in the \( \hat{N} \)-frame \( \mathcal{F} \) (notation: \( \mathcal{F} \models \alpha \)) if \( \mathcal{M} \models \alpha \) for every model \( \mathcal{M} \) on the \( \hat{N} \)-frame \( \mathcal{F} \). A formula \( \alpha \in F \) is valid in a class \( \mathcal{C} \) of \( \hat{N} \)-frames (notation: \( \mathcal{C} \models \alpha \)) if for every \( \hat{N} \)-frame \( \mathcal{F} \in \mathcal{C} \), \( \mathcal{F} \models \alpha \).

The following can easily be observed.

**Proposition 4.6.** For any formula \( \alpha \in F \) and \( x \in X \),

1. \( \mathcal{M}, x \models \neg \Top \iff \forall y \in X \ (xR_{N_1}y) \),
2. \( \mathcal{M}, x \models \sim \Top \iff \forall y \in X \ (xR_{N_2}y) \),
3. \( \forall y \in X((M, x \models \alpha \land x \leq y) \Rightarrow M, y \models \alpha) \),
4. \( M, x \models \neg \alpha \iff \forall y \in X(\exists z \in X(x \leq z \land y \leq z) \Rightarrow y \not\models \alpha) \),
5. \( M, x \models \neg \alpha \iff \forall y \in X(xR_{N_{1}}y \Rightarrow (y \models \alpha \Rightarrow \exists z \in X(yR_{N_{1}}z \land \forall z' \in X(zR_{N_{2}}z'))) \).

Let us return to Condition (3) in Definition 4.3. Recall the ILM\(_{1}\)-axiom \( \neg \neg \top \iff \neg \top \). For any \( \mathcal{N} \)-model \( M \), expansion of \( M \models \neg \neg \top \iff \neg \top \) using Definition 4.5 gives

\[
\forall x \in X(\forall y \in X(xR_{N_{1}}y \Rightarrow \exists z \in X(yR_{N_{1}}z \land \forall z' \in X(zR_{N_{2}}z'))) \iff \forall z'' \in X(xR_{N_{2}}z'')
\]

The reverse implication in the above is always true. Arguing for its contrapose, let \( y \in X \) such that \( xR_{N_{1}}y \) and \( \forall z \in X(yR_{N_{1}}z \Rightarrow \exists z' \in X(zR_{N_{2}}z') \). \( xR_{N_{1}}y \) implies \( yR_{N_{1}}x \). Therefore, there exists \( z' \in X \) such that \( xR_{N_{2}}z' \). Take \( z'' = z' \).

Thus, one direction of the above bi-implication is always true. The other direction is exactly Condition (3) in Definition 4.3. Therefore this condition expresses the required property as mentioned in Observation 4.3.

A logic is said to be determined by a class of frames if it is complete with respect to the class of frames. Our aim now is to show that the logic ILM (ILM\(_{\lor}\)) is determined by the class \( \mathcal{F}_{1} (\mathcal{F}_{1}^{\lor}) \) of \( \mathcal{N} \)-frames (\( \mathcal{N}^{\lor} \)-frames).

### 4.2 Characterization results for ILM and ILM\(_{\lor}\)

Soundness, i.e., for any formula \( \alpha \in F \), \( \vdash_{\text{ILM}} \alpha \Rightarrow \mathcal{F}_{1} \models \alpha \) and \( \vdash_{\text{ILM} \lor} \alpha \Rightarrow \mathcal{F}_{1}^{\lor} \models \alpha \), can be obtained in the standard manner, using induction on the number of connectives of \( \alpha \). Let us sketch the proof of completeness for ILM and ILM\(_{\lor}\). The structure of the proof is similar to the cases of ML and IL given in [13]. Therefore, we shall only show the steps where there is a change or an extension to the proofs in [13]. We first require the concept of a theory [46, 13], that we extend to the context of ILM.

**Definition 4.7 (Theory).** A theory \( T \subseteq F \) with respect to an extension \( S \) of ILM, is a non-empty set of formulas in \( L \) such that, for formulas \( \alpha, \beta \in F \),

1. if \( \alpha \in T \) and \( \alpha \rightarrow \beta \in T \), then \( \beta \in T \) (closed under deduction),
2. \( \alpha \in T \), where \( \vdash_{S} \alpha \), and
3. if \( \alpha, \beta \in T \) then \( \alpha \land \beta \in T \) (closed under \( \land \)).

A theory is consistent if \( \bot \notin T \), otherwise inconsistent. A prime theory is a consistent theory such that for any two formulas \( \alpha, \beta \in F \), if \( \alpha \lor \beta \in T \) then either \( \alpha \in T \) or \( \beta \in T \).

Using Axiom (A8) namely \( \bot \rightarrow \alpha \), a theory \( T \) is consistent if and only if there exists a formula \( \alpha \) such that \( \alpha \notin T \). Now, for an arbitrary set \( \Delta \) of formulas, the intersection of all theories containing \( \Delta \) is also a theory; it is called the theory generated by \( \Delta \) and denoted by \( \text{Th}(\Delta) \). A set \( F' \subseteq F \) is closed under \( \lor \), if for any \( \alpha, \beta \in F' \), \( \alpha \lor \beta \in F' \); \( F' \) is then called disjunctive closed (or \( \lor \)-closed). In fact, any arbitrary \( \Delta \subseteq F \) can be extended to a \( \lor \)-closed set, called disjunctive closure of \( \Delta \) (denoted \( \text{dc}(\Delta) \)), as follows.

\[
\text{dc}(\Delta) := \bigcap\{\Delta' \subseteq F \mid \Delta \subseteq \Delta' \text{ and } \forall \alpha, \beta \in F (\alpha, \beta \in \Delta' \Rightarrow \alpha \lor \beta \in \Delta')\}.
\]

\( \text{dc}(\Delta) \) is \( \lor \)-closed. Moreover, for \( \alpha \in F \), if \( \beta \in \text{dc}(\{\alpha\}) \) then \( \vdash_{S} \alpha \iff \beta \).

**Lemma 4.8 (Extension lemma).** Let \( \Delta \) be a consistent theory and \( \Gamma \subseteq F \) be a \( \lor \)-closed set. If \( \Delta \cap \Gamma = \emptyset \) then there exists a prime theory \( P \) such that \( \Delta \subseteq P \) and \( P \cap \Gamma = \emptyset \).
An immediate corollary is obtained when $\Gamma := \text{dc}\{\alpha\}$ for any $\alpha \in F$.

**Corollary 4.9.** Let $\Delta$ be a consistent theory and $\alpha \in F$ be such that $\alpha \notin \Delta$. Then there is a prime theory $P$ such that $\Delta \subseteq P$ and $\alpha \notin P$.

Let us now define the ‘canonical’ frame $F^c := (X^c, \subseteq, R^c_{N_1}, R^c_{N_2})$ in the standard way.

**Definition 4.10** (Canonical frame). The canonical frame for any extension $S$ of ILM is the quadruple $F^c := (X^c, \subseteq, R^c_{N_1}, R^c_{N_2})$, where $X^c := \{P \subseteq F \mid P$ is a prime theory$,\}$, $PR^c_{N_1}Q$ if and only if (for all $\alpha \in F$, $\neg \alpha \in P \Rightarrow \alpha \notin Q$), and $PR^c_{N_2}Q$ if and only if (for all $\alpha \in F$, $\sim \alpha \in P \Rightarrow \alpha \notin Q$).

We shall now show that $F^c$ is indeed an $\hat{N}$-frame. For that, we shall require the following standard result.

**Lemma 4.11.** For any $\alpha \in F$ and any $P \in X^c$,
1. $\neg \alpha \in P$ if and only if for all $Q \in X^c$, $PR^c_{N_1}Q \Rightarrow \alpha \notin Q$, and
2. $\sim \alpha \in P$ if and only if for all $Q \in X^c$, $PR^c_{N_2}Q \Rightarrow \alpha \notin Q$.

Recall Definition 4.4 of an $\hat{N}$-frame, more specifically its distinctive feature Condition (3). We shall only show the following: for all $P \in X^c$,

$$\forall Q \in X^c (PR^c_{N_1}Q \Rightarrow \exists R \in X^c (QR^c_{N_1}R \& \forall R' \in X^c (R^c_{N_2}R'))) \Rightarrow \forall R'' \in X^c (P^c_{R^c_{N_2}R''})$$

Suppose $\forall Q \in X^c (PR^c_{N_1}Q \Rightarrow \exists R \in X^c (QR^c_{N_1}R \& \forall R' \in X^c (R^c_{N_2}R'))$. Using Lemma 4.11, $\neg \sim \top \in P$. Since $\vdash_{ILM} \neg \sim \top \iff \sim \top$, we have $\sim \top \in P$. Finally, Lemma 4.11 implies $\forall R'' \in X^c (P^c_{R^c_{N_2}R''})$. Thus we have

**Proposition 4.12.** $F^c := (X^c, \subseteq, R^c_{N_1}, R^c_{N_2})$ is an $\hat{N}$-frame.

The canonical valuation $v^c$ on the canonical frame $F^c := (X^c, \subseteq, R^c_{N_1}, R^c_{N_2})$, defined as $v^c(p) := \{P \in X^c \mid p \in P\}$ for all $p \in PV$, gives the canonical $\hat{N}$-model $M^c := (F^c, v^c)$. The truth lemma follows.

**Lemma 4.13** (Truth Lemma). For any $\alpha \in F$ and $P \in X^c$, $M^c, P \vdash \alpha$ if and only if $\alpha \in P$.

Finally, using the truth lemma, the completeness result for the logics ILM and ILM-$\forall$ is obtained.

**Theorem 4.14** (Completeness). For any formula $\alpha \in F$,
(i) $\mathfrak{F}_1 \vdash \alpha \Rightarrow \vdash_{ILM} \alpha$.
(ii) $\mathfrak{F}'_1 \vdash \alpha \Rightarrow \vdash_{ILM-\forall} \alpha$.

**Proof.**
(i) Let $\forall^c_{ILM} \alpha$. If $\alpha = \bot$, then by definition of valuation, $F \not\vdash \bot$ for any $\hat{N}$-frame $F$. Therefore, suppose $\alpha$ is not $\bot$. Then define $\Delta := Th(\top)$ and $\Gamma := \text{dc}(\{\alpha\})$. $\Delta$ is a consistent theory, because $\bot \notin \Delta$ using soundness. $\forall^c_{ILM} \alpha \Rightarrow \alpha \notin \Delta$, which implies, by Corollary 4.9 there is $P \in X^c$ such that $\Delta \subseteq P$ and $\alpha \notin P$. Using Lemma 4.13, $M^c, P \not\models \alpha$. 

\[19\]
(ii) For ILM-$\lor$, we have to show that the canonical frame for ILM-$\lor$ belongs to $\mathcal{F}_2$, and for all frames $\mathcal{F} \in \mathcal{F}_2$ and $\alpha \in F$, $\mathcal{F} \vDash \alpha \lor \lnot \alpha$. Consider the canonical frame $\mathcal{F}^c := (X^c, \subseteq, R^c_{N_1}, R^c_{N_2})$. We have already shown that it is an $\mathcal{N}$-frame. We have to show that for all $P, Q \in X^c$, if $PR^c_{N_2}Q$ then $Q \subseteq P$. Let $P$ and $Q$ be such that $PR^c_{N_2}Q$ and $Q \notin P$. Then there exists $\alpha \in Q$ such that $\alpha \notin P$. We have $\alpha \lor \lnot \alpha \in P$, as $\alpha \lor \lnot \alpha$ is an axiom in ILM-$\lor$. Therefore, $\lnot \alpha \in P$. Using the definition of $R^c_{N_2}$, we have $\alpha \notin Q$, a contradiction.

\[ \Box \]

5 \hspace{1em} Segerberg semantics for ILM

In this section, we shall see another relational semantics for ILM based on Segerberg’s semantics for ML and its extensions. The latter is defined through $j$-frames, which are triples of the form $(W, \leq, Y_0)$, where $(W, \leq)$ is a poset with $W \neq \emptyset$ and $Y_0$ is an upset of $W$. $Y_0$ is used to define the semantics for negation. When $Y_0 = \emptyset$, the $j$-frame is called a normal frame, denoted simply by the pair $(W, \leq)$. Segerberg showed that ML and IL are determined by the class of all $j$-frames and normal frames respectively \[50].

5.1 \hspace{1em} Sub-normal frames

We shall now study the class of $j$-frames that can characterize ILM. In Section 4.1, for the case of Došen semantics for ILM, two binary relations $R_{N_1}$ and $R_{N_2}$ were considered to define the semantics for the negations $\lnot$ and $\lnot$ respectively. In this case, we consider a poset $(W, \leq)$, and two upsets $Y_-$ and $Y_+$ (say), for the negations $\lnot$ and $\lnot$ respectively. Moreover, since $\lnot$ is intuitionistic, $Y_-$ must be the empty set, as for normal frames corresponding to $\lnot$.

Recall next the following class of frames, first given by Woodruff \[50], to characterize the extension $\text{JP}'$ (see Definition 3.8) of ML.

Definition 5.1 (Sub-normal frame).

A sub-normal frame \[50] is a $j$-frame $\mathcal{F} := (W, \leq, Y_0)$ satisfying the following condition:

$$\forall x \in W (\forall y \in W (x \leq y \Rightarrow \exists z \in W (y \leq z \& z \in Y_0) )) \Rightarrow x \in Y_0).$$

(D)

The class of all sub-normal frames is denoted by $\mathcal{F}_2$.

A sub-normal identity frame is a sub-normal frame $\mathcal{F} := (W, \leq, Y_0)$ where $\leq$ is an identity relation on $W \setminus Y_0$, i.e.

$$\forall x, y \in W \setminus Y_0 (x \leq y \Rightarrow y \leq x).$$

(E)

The class of all sub-normal identity frames is denoted by $\mathcal{F}'_2$.

Theorem 5.2. \[50] $\text{JP}'$ is determined by the class of all sub-normal frames.

As observed in Theorem 3.9, ILM is an extension of $\text{JP}'$. So it is expected that models of ILM and ILM-$\lor$, in this semantics, would be based on sub-normal frames. Let us give the basic definitions. A valuation of $\mathcal{L}$ on a sub-normal frame $\mathcal{F} := (W, \leq, Y_0)$ is a mapping $v : PV \rightarrow P(W)$ such that $v(p)$ is an upset for each $p \in PV$. A pair $\mathcal{M} := (\mathcal{F}, v)$ is called a sub-normal model on the sub-normal frame $\mathcal{F}$. The truth of a formula $\alpha \in F$ at a world $w \in W$ in the model $\mathcal{M}$ (notation: $\mathcal{M}, w \vDash \alpha$) for propositional variables, $\lor, \land, \bot$ and $\top$ is given in the standard way; for $\rightarrow$ as in Definition 4.5(1); and for $\lnot$ and $\lnot$, 

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1. \( M, w \models \neg \alpha \iff \) for all \( w' \in W \), if \( w \leq w' \) and \( M, w' \models \alpha \) then \( w' \in Y_0 \).
2. \( M, w \models \neg \alpha \iff \) for all \( w' \in X(w \leq w' \Rightarrow M, w' \not\models \alpha) \).

Standard definitions and notations give the notions of truth of a formula in a sub-normal model, its validity in a sub-normal frame and validity in a class of sub-normal frames. Some properties of a sub-normal model \( M \) are as follows – these can be derived in a straightforward manner.

**Proposition 5.3.**

1. For \( w \in W \), \( M, w \models \neg \top \iff w \in Y_0 \).
2. For all \( w' \in W \), if \( M, w \models \alpha \) and \( w \leq w' \), then \( M, w' \models \alpha \).
3. For all \( w' \in W \), \( M, w \models \neg \alpha \) if and only if \( \forall w'' \in W(\exists w'' \in W(w \leq w'' \& w' \leq w'') \Rightarrow w' \not\models \alpha) \).
4. For all \( w' \in W \), \( M, w \models \neg \alpha \) if and only if \( \forall w'' \in W(w \leq w' \Rightarrow (w \models \alpha \Rightarrow \exists w'' \in W(w \leq w'' \& w'' \in Y_0)) \).

Now, as done for \( \tilde{N} \)-frames in the previous section, let us consider the ILM\(_1\)-axiom \( \neg \neg \neg \top \iff \neg \top \). Let \( M \) be any sub-normal model. Expanding \( M \models \neg \neg \neg \top \iff \neg \top \), one obtains the following.

\[
\forall w \in W(\forall v \in W(v \leq v \Rightarrow \exists v' \in W(v \leq v' \& v' \in Y_0)) \iff w \in Y_0).
\]

The reverse direction of the implication in the above holds anyway. Indeed, consider the contraposition, and let \( v \in W \) be such that \( w \leq v \) and \( \forall v' \in W(v \leq v' \Rightarrow v' \not\in Y_0) \). If \( w \in Y_0 \) then \( v \in Y_0 \) (as \( Y_0 \) is an upset). However, \( v \leq v \) implies \( v \not\in Y_0 \), a contradiction. The forward direction of the implication is exactly Condition (D). Therefore, we expect that ILM will be complete with respect to the class \( \mathfrak{F}_2 \) of sub-normal frames. Moreover, Condition (E) on sub-normal identity frames corresponds to the formula \( \alpha \lor \neg \alpha \) \[46\]. Thus, we also expect that ILM-\( \lor \) will be complete with respect to the class \( \mathfrak{F}_2 \) of sub-normal identity frames. In the next section, we prove these completeness results by obtaining relationships between the classes of sub-normal (sub-normal identity) frames and \( \tilde{N} \)-frames (\( N' \)-frames).

### 5.2 Inter-translation between \( \tilde{N} \)-frames and sub-normal frames

Došen had shown that there is an inter-translation between strictly condensed \( J \)-frames and \( j \)-frames for ML, preserving the truth of any formula \( \alpha \in F \) at any world \( w \) of the respective frame. We observe that this inter-translation can be extended to the case of ILM, i.e. starting from a sub-normal frame, we can obtain an \( \tilde{N} \)-frame; and from an \( \tilde{N} \)-frame, we can obtain a sub-normal frame – preserving truth. We only give the highlights of the proofs.

**Theorem 5.4.** Let \( F := (W, \leq, Y_0) \) be a sub-normal frame, i.e. \( F \in \mathfrak{F}_2 \). Define the relations \( R_{N_1} \) and \( R_{N_2} \) over \( W \) for all \( x, y \in W \) as:

- \((A)\) \( xR_{N_1}y \) if and only if \( \exists z \in W(x \leq z \& y \leq z) \).
- \((B)\) \( xR_{N_2}y \) if and only if \( \exists z \in W(x \leq z \& y \leq z \& z \not\in Y_0) \).

Then we have the following.

1. \( Y_0 = \{ z \in W \mid \forall x \in W(zR_{N_2}x) \} \).
2. \( \Phi(\mathcal{F}) := (W, \leq, R_{N_1}, R_{N_2}) \) is an \( \hat{N} \)-frame. If \( \mathcal{F} \) is a sub-normal identity frame then \( \Phi(\mathcal{F}) \) is an \( \hat{N}' \)-frame.

3. If \( v \) is a valuation on \( \mathcal{F} \), then \( v \) is a valuation on \( \Phi(\mathcal{F}) \) such that for all \( \alpha \in F \) and \( x \in W \),
   (a) \((\mathcal{F}, v), x \vdash \alpha \equiv (\Phi(\mathcal{F}), v), x \vdash \alpha \),
   (b) \((\mathcal{F}, v) \vdash \alpha \equiv (\Phi(\mathcal{F}), v) \vdash \alpha \), and
   (c) \( \mathcal{F} \vdash \alpha \equiv \Phi(\mathcal{F}) \vdash \alpha \).

Proof.

2. We shall only prove that Condition (3) of Definition 5.3 is satisfied, i.e. \( \forall x \in W \left( \forall y \in W \left( xR_{N_1}y \Rightarrow \exists z \in W(yR_{N_2}z \& \forall z' \in W (z \vDash z')) \Rightarrow \forall z'' \in W (x \vDash z'') \right) \right) \).

Using the expression for \( Y_0 \) obtained in (1), this condition is equivalent to the following:
\[
\forall x \in W \left( \forall y \in W \left( xR_{N_1}y \Rightarrow \exists z \in W(yR_{N_1}z \& z \in Y_0) \Rightarrow x \in Y_0 \right) \right).
\]

To show \( x \in Y_0 \), we shall use Condition (D) of the sub-normal frame \((W, \leq, Y_0)\), i.e.
\[
\forall x \in W \left( \forall y \in W \left( x \leq y \Rightarrow \exists z \in W(y \leq z \& z \in Y_0) \Rightarrow x \in Y_0 \right) \right).
\]

Claim: \( \forall y \in W \left( x \leq y \Rightarrow \exists z \in W(y \leq z \& z \in Y_0) \right) \).

Indeed, let \( x \leq y \), for \( y \in W \). Using (A) and \( y \leq y \), we have \( xR_{N_1}y \). Then using (*), we have \( y \leq z \) such that \( yR_{N_1}z \) and \( z \in Y_0 \). Using (A) on \( yR_{N_1}z \), there exists \( z \in W \) such that \( y \leq z \) and \( z \leq z \). Since \( Y_0 \) is an upset and \( z \leq z \), we have the claim.

Therefore, using Condition (D), we have \( x \in Y_0 \).

Now let \( \mathcal{F} \in \mathcal{F}_2 \). We have to show that \( R_{N_2} \subseteq (\leq^{-1}) \). Let \( x, y \in W \) such that \( xR_{N_1}y \). Condition (B) implies that there exists \( z \in W \) such that \( x \leq z \), \( y \leq z \), and \( z \notin Y_0 \).

Since \( Y_0 \) is an upset, \( z \notin Y_0 \) implies \( x, y \notin Y_0 \). \( \mathcal{F} \in \mathcal{F}_2 \) and \( y \leq y \) imply that \( z \leq z \). Then, \( x \leq z \) and \( z \leq y \) imply \( x \leq y \). Again using \( \mathcal{F} \in \mathcal{F}_2 \), \( x, y \notin Y_0 \) and \( x \leq y \), we get \( y \leq x \).

**Theorem 5.5.** Let \( \mathcal{G} := (W, \leq, R_{N_1}, R_{N_2}) \) be an \( \hat{N} \)-frame. Define
\[
(C) \quad Y_0 := \{ z \in W \mid \forall x \in W (zR_{N_2}x) \}.
\]

Then we have the following.

1. (a) \( xR_{N_2}y \) if and only if \( \exists z \in W(x \leq z \& y \leq z \& z \notin Y_0) \), and
   (b) \( xR_{N_1}y \) if and only if \( \exists z \in W(x \leq z \& y \leq z) \).
2. \( \Psi(\mathcal{G}) := (W, \leq, Y_0) \) is a sub-normal frame. If \( \mathcal{G} \) is an \( \hat{N}' \)-frame then \( \Psi(\mathcal{G}) \) is a sub-normal identity frame.
3. If \( v \) is a valuation on \( \mathcal{G} \), then \( v \) is a valuation on \( \Psi(\mathcal{G}) \) such that for all \( \alpha \in F \) and \( x \in W \),
   (a) \((\mathcal{G}, v), x \vdash \alpha \equiv (\Psi(\mathcal{G}), v), x \vdash \alpha \),
   (b) \((\mathcal{G}, v) \vdash \alpha \equiv (\Psi(\mathcal{G}), v) \vdash \alpha \), and
   (c) \( \mathcal{G} \vdash \alpha \equiv \Psi(\mathcal{G}) \vdash \alpha \).

Proof.

2. We only show that the sub-normal frame satisfies Condition (D) of Definition 5.3, i.e.\((\forall y \in W(x \leq y \Rightarrow \exists z \in W(y \leq z \& z \in Y_0)) \Rightarrow x \in Y_0) \). Let \( x \in W \) such that \( \forall y \in W(x \leq y \Rightarrow \exists z \in W(y \leq z \& z \in Y_0)) \). Let \( x \in W \) be such that \( \forall y \in W(x \leq y \Rightarrow \exists z \in W(y \leq z \& z \in Y_0)) \).

To get \( x \in Y_0 \), we utilize Condition (3) of \( \hat{N} \)-frames and show:
\[
\forall y \in W(xR_{N_1}y \Rightarrow \exists z \in W(yR_{N_2}z \& \forall z' \in W (z \vDash z'))),
\]
and \( y \). Thus, as mentioned in Section 1, we now turn to an investigation of the two negation operators introduced through \( \text{ccpBa} \) to \( \hat{G} \in \mathcal{F} \). Now let \( xR_{N_1}y \). Using 1(b), there exists \( z \in W \) such that \( x \leq z \) and \( y \leq z \). Using \( x \leq z \) and (**), we have \( z' \in W \) such that \( z \leq z' \) and \( z' \in Y_0 \). Since \( R_{N_1} \) is reflexive, we have \( zR_{N_1}z' \). Using \( y \leq z \leq z' \) and \( (\leq R_{N_1} \leq^{-1}) \subseteq R_{N_1} \), we have \( yR_{N_1}z' \).

Thus, Condition (3) of \( \check{N} \)-frames gives \( \forall z'' \in W \ (xR_{N_2}z'') \), i.e. \( x \in Y_0 \).

Now let \( G \in \mathfrak{F}_1' \). We have to show that \( \forall x, y \in W \setminus Y_0 \ (x \leq y \Rightarrow y \leq x) \). Let \( x, y \in W \setminus Y_0 \) and \( x \leq y \). Using 1(a), \( x \leq y, y \leq y \) and \( y \not\in Y_0 \) imply \( xR_{N_2}y \). As \( G \in \mathfrak{F}_1', R_{N_2} \subseteq (\leq^{-1}) \).

Thus, \( y \leq x \).

**Observation 5.6.** In [13], for any strictly condensed \( J \)-frame \( (W, \leq, R_N) \), \( Y_0 \) is defined as follows.

\[
z \in Y_0 \Leftrightarrow \exists x, y \in W (x \leq z \& y \leq z \& xR_Ny).
\]

However, in Theorem 5.3, we have defined \( Y_0 := \{ x \in W \mid \forall z(xR_Nz) \} \), in order to give a simpler expression for \( Y_0 \). In fact, it can be easily seen that the two expressions are equivalent.

In Theorem 5.4, we have obtained a mapping \( \Phi : \mathfrak{F}_2 \rightarrow \mathfrak{F}_1 \) from sub-normal frames to \( \check{N} \)-frames. In fact, the restriction \( \Phi|_{\mathfrak{F}_2} \) of \( \Phi \) is also a map from the subclass \( \mathfrak{F}_2' \) of sub-normal identity frames to the subclass \( \mathfrak{F}_1' \) of \( \check{N} \)-frames. In the same way, in Theorem 5.5, \( \Psi : \mathfrak{F}_1 \rightarrow \mathfrak{F}_2 \) is a map from \( \check{N} \)-frames to sub-normal frames. The restriction \( \Psi|_{\mathfrak{F}_1} \) of \( \Psi \) is then a map from the subclass \( \mathfrak{F}_1' \) of \( \check{N} \)-frames to the subclass \( \mathfrak{F}_2' \) of sub-normal identity frames. We also get the following easily.

**Theorem 5.7.** Consider the maps \( \Phi \) and \( \Psi \) obtained in Theorems 5.4 and 5.5.

1. (a) \( \Phi\Psi \) and \( \Psi\Phi \) are identity maps on the classes \( \mathfrak{F}_1 \) and \( \mathfrak{F}_2 \) respectively.
2. (b) \( \Phi|_{\mathfrak{F}_2} \Psi|_{\mathfrak{F}_1} \) and \( \Psi|_{\mathfrak{F}_1} \Phi|_{\mathfrak{F}_2} \) are identity maps on the classes \( \mathfrak{F}_1' \) and \( \mathfrak{F}_2' \) respectively.
3. For any formula \( \alpha \in F \), (a) \( \mathfrak{F}_1 \models \alpha \Leftrightarrow \mathfrak{F}_2 \models \alpha \) and (b) \( \mathfrak{F}_1' \models \alpha \Leftrightarrow \mathfrak{F}_2' \models \alpha \).

We have already proved that the class \( \mathfrak{F}_1 \) of all \( \check{N} \)-frames (\( \check{N} \)-frames) determines ILM (ILM-\( \vee \)) in Theorem 4.11. Using Theorem 5.7(2), we obtain the following corollary.

**Corollary 5.8.** The classes \( \mathfrak{F}_2 \) of sub-normal frames and \( \mathfrak{F}_2' \) of sub-normal identity frames determine ILM and ILM-\( \vee \) respectively.

As pointed out in Theorem 5.2, the class of sub-normal frames determines the logic JP'. From Corollary 5.8, we have obtained that the same class of frames determines ILM. The logic JP' has the finite model property (FMP) with respect to the class of sub-normal frames, and being finitely axiomatizable, is decidable as well [23]. As a result, one can obtain FMP and decidability for ILM.

We have shown earlier in Theorem 3.11 JP' and ILM cannot be equivalent. The two logics have different alphabets, the latter having the extra propositional constant \( \bot \), not definable in JP' using other connectives. This indicates limitations of these relational frames – non-equivalent logics of the above kind cannot be differentiated through them.

## 6 The logics \( K_{im} \) and \( K_{im-\vee} \) without implication

As mentioned in Section 1, we now turn to an investigation of the two negation operators introduced through \( \text{ccpBas} \), adopting the approaches of Dunn and Vakarelov. Consider
an alphabet that has propositional variables $p, q, r, \ldots$, binary connectives ∨ and ∧, unary connectives ¬ and ∼, and constants ⊥ and ⊤. Define a language $\tilde{\mathcal{L}}$ with this alphabet and class $\tilde{\mathcal{F}}$ of well-formed formulas given by the scheme:

$$p \mid ⊤ \mid ⊥ \mid α \land β \mid α \lor β \mid -α \mid ∼α$$

The logical consequence is defined as a pair of formulas $(ϕ, ψ)$, written as $ϕ \vdash ψ$ and called a sequent. The rule expressing ‘if $α \vdash β$ then $γ \vdash δ$’ is written as $α \vdash β / γ \vdash δ$.

**Definition 6.1** (The logics $K_{im}$ and $K_{im-∨}$). The language of $K_{im}$ is $\tilde{L}$. The axioms and rules of $K_{im}$ are as follows:

A1. $α \vdash α$
A2. $α \vdash β$, $β \vdash γ / α \vdash γ$
A3. $α \land β \vdash α; α \land β \vdash β$
A4. $α \vdash β$, $α \vdash γ / α \vdash β \land γ$
A5. $α \vdash γ$, $β \vdash γ / α \lor β \vdash γ$
A6. $α \vdash α \lor β; β \vdash α \lor β$
A7. $α \land (β \lor γ) \vdash (α \land β) \lor (α \land β)$
A8. $α \vdash ⊤ (Top)$
A9. $⊥ \vdash α (Bottom)$
A10. $α \vdash β / -β \vdash -α$
A11. $-α \land -β \vdash -(α \lor β)$
A12. $⊥ \vdash -⊥$
A13. $α \vdash -α$
A14. $α \land β \vdash γ / α \land -γ \vdash -β$
A15. $α \land -α \vdash β$
A16. $-α \vdash -(α \land -⊥)$
A17. $-α \vdash -(α \land -⊥)$

The logic $K_{im-∨}$ is $K_{im}$ enhanced with the following axiom.

A18. $⊤ \vdash α \lor ∼α$

Derivability is defined in the standard manner. Following the nomenclature in [19], axioms A1-A7 give the Distributive Lattice Logic, while A1-A9 give the Bounded Distributive Lattice Logic (BDLL). A10-A12 are the axioms and rules that make the negation ¬ preminimal, and further, A13 and A14 make it minimal; adding A15 makes it intuitionistic. Note that minimal or intuitionistic negation defined here is different from that mentioned in Remark 3.4. BDLL, along with preminimal negation ¬, is denoted by $K_1$.

**Proposition 6.2.** The following can be proved in the system $K_{im}$:

P1. $α \vdash β$, $δ \vdash γ / α \land δ \vdash δ \land γ$
P2. ∼-Contraposition: $α \vdash β / ∼β \vdash ∼α$
P3. ∼-∨-Linearity: $∼α \land ∼β \vdash (∼α \lor ∼β)$
P4. ∼-Nor: $⊤ \vdash ∼⊥$
P5. $α \vdash -∞α$
P6. $α \land β \vdash γ / α \land ∼γ \vdash β$
P7. (DNE($∼⊤$)) $¬¬∼⊤ \vdash ∼⊤$

‘DNE’ in P7 stands for ‘double negation elimination’.

**Proof.** We only prove P7. $⊤ \land ∼¬⊥ \vdash ∼¬⊥$ (using A3). A10 implies $¬¬¬¬⊥ \vdash -⊥(⊥ \land ∼¬⊥)$. Finally using A17 and A2, we get $¬¬¬¬⊥ \vdash ∼¬⊥$. □

Here, P2, P3 and P4 make the negation ∼ a preminimal negation. Further, P5 and P6 make ∼ minimal. In fact, we have the following ‘equivalent’ version $K'_{im}$ of $K_{im}$.
Definition 6.3 (The logic $K'_{im}$). The language of $K'_{im}$ is $\mathcal{L}$. The axioms and rules are A1-A15, P2-P6 and P7 (DNE($\sim\top$)).

Theorem 6.4. For any $\alpha, \beta \in \hat{F}$, $\alpha \vdash_{K'_{im}} \beta$ if and only if $\alpha \vdash_{K_{im}} \beta$.

Proof. We shall only see the proofs of A16 and A17 in the logic $K'_{im}$.

A16: Using A3, $\alpha \wedge \top \vdash \alpha$. Then P6 implies $\alpha \wedge \sim \alpha \vdash \sim \top$. A14 then implies $\alpha \wedge \sim \top \vdash \sim \sim \alpha$. Finally, A10 gives $\sim \sim \sim \alpha \vdash \sim (\alpha \wedge \sim \top)$. A17: Using A1 and A14, $\alpha \wedge \sim (\alpha \wedge \sim \top) \vdash \sim \sim \top$. Using P7 and A2, $\alpha \wedge \sim (\alpha \wedge \sim \top) \vdash \sim \top$. P6 then implies $\sim \sim \top \wedge \sim (\alpha \wedge \sim \top) \vdash \sim \alpha$. We have $\top \vdash \sim \sim \top$ using P5. Now, using P1 and A2, $\top \wedge \sim (\alpha \wedge \sim \top) \vdash \sim \alpha$. A8 and P1 imply $\sim (\alpha \wedge \sim \top) \vdash \sim \alpha$. $\square$

Henceforth, we shall consider $K'_{im}$ instead of $K_{im}$.

6.1 Algebraic semantics for $K_{im}$ and $K_{im-\lor}$

Let us define the ‘$K_{im}$-algebras’. For that, let us note the terminologies related to algebras defined in [16]. A distributive lattice with preminimal negation is of the form $(A, 1, 0, \lor, \land, \sim)$, where the reduct $(A, 1, 0, \lor, \land)$ is a bounded distributive lattice, and $\sim$ satisfies the following properties, for all $a, b \in A$:

1. $a \leq b \Rightarrow \sim b \leq \sim a$,
2. $\sim a \land \sim b \leq \sim (a \lor b)$, and
3. $1 = \sim 0$.

If the preminimal negation satisfies $a \leq \sim \sim a$ and $a \land b \leq c \Rightarrow a \land c \leq \sim b$, for all $a, b, c \in A$, then it is called a minimal negation. If it further satisfies $a \land \sim a \leq b$ for all $a, b \in A$, it is called an intuitionistic negation. Moreover, intuitionistic negation satisfying $\sim \sim a \leq a$ is called an ortho negation.

Definition 6.5 ($K_{im}$-algebras).

A $K_{im}$-algebra $A$ is a tuple of the form $(A, 1, 0, \lor, \land, \sim)$ satisfying the following conditions:

1. the reduct $(A, 1, 0, \lor, \land)$ is a bounded distributive lattice,
2. the negation $\sim$ is an intuitionistic negation,
3. the negation $\sim$ is a minimal negation, and
4. $\sim \sim 1 = \sim 1$.

A $K_{im-\lor}$-algebra is a $K_{im}$-algebra $A$ satisfying

$$a \lor \sim a = 1 \quad \text{for all} \quad a \in A. \quad \text{(EM)}$$

EM stands for ‘excluded middle’.

Note that in (4) of Definition 6.5, $\sim 1 \leq \sim \sim 1$ holds anyway, as $\sim$ is minimal (being intuitionistic). We denote the other direction by DNE($\sim 1$), i.e.

$$\sim \sim 1 \leq \sim 1. \quad \text{(DNE($\sim 1$))}$$

Let us now give the algebraic semantics for $K_{im}$ ($K_{im-\lor}$). Consider any $K_{im}$-algebra ($K_{im-\lor}$-algebra) $A := (A, 1, 0, \lor, \land, \sim)$. A valuation is a map $v$ from PV to $A$, and can be extended to all formulas in the language $\mathcal{L}$ in the standard way [13]. For formulas $\alpha, \beta$ in $\mathcal{L}$, if for all valuations $v$ on $A$, $v(\alpha) = 1$ implies $v(\beta) = 1$, then we say $\alpha \vdash \beta$ is valid in $A$. If this is true for all $K_{im}$-algebras ($K_{im-\lor}$-algebras), then we denote it by $\alpha \vdash_{K_{im}} \beta$ ($\alpha \vdash_{K_{im-\lor}} \beta$). It is straightforward to see...
Theorem 6.6. For any $\alpha, \beta$ in $\mathcal{L}$, $\alpha \vdash_{K_{\imath}} \beta$ ($\alpha \vdash_{K_{\imath} \land \neg} \beta$) if and only if $\alpha \vdash_{K_{\imath}} \beta$ ($\alpha \vdash_{K_{\imath} \land \neg} \beta$), i.e. $K_{\imath}$ ($K_{\imath} \land \neg$) is sound and complete with respect to the class of $K_{\imath}$-algebras ($K_{\imath} \land \neg$-algebras).

Since the definition of $K_{\imath}$ is motivated through the logic ILM, one expects a relationship between $K_{\imath}$-algebras and $ccpBa$. The following is clear.

Proposition 6.7.
For any $ccpBa$ $(A, 1, 0, \lor, \land, \to, \neg, \sim)$, the reduct $(A, 1, 0, \lor, \land, \to, \neg)$ is a $K_{\imath}$-algebra.

Now the question is whether every $K_{\imath}$-algebra $(A, 1, 0, \lor, \land, \to, \neg, \sim)$ can be extended to a $ccpBa$ $(A, 1, 0, \lor, \land, \to, \neg, \sim)$? Consider the lattice $L := \mathbb{Z} \times \mathbb{Z}$, the set of pairs of integers, with the usual ordering $\leq$: $(m, n) \leq (r, s)$ if and only if $m \leq r$ and $n \leq s$. $L$ is a distributive lattice. Define $L' := L \cup \{(0, 1) \mid \hat{0} \neq \hat{1}\}$, such that $\leq$ is extended to $L'$ in the following way: $\hat{0} \leq (m, n) \leq \hat{1}$ for all $(m, n) \in L$. Addition of $\hat{0}$ and $\hat{1}$ makes the lattice bounded, i.e. $L'$ is a bounded distributive lattice. Define two negations $\neg$ and $\sim$ on $L'$ as follows:

1. $\neg(m, n) := \hat{0}$ for all $(m, n) \in L$, $\neg \hat{1} := \hat{0}$, and $\neg \hat{0} := \hat{1}$.
2. $\sim a := 1$, for all $a \in L'$.

One can easily check that $\neg$ and $\sim$ are intuitionistic and minimal negations respectively. Therefore, $\mathcal{L} := (L', \hat{1}, \hat{0}, \lor, \land, \to, \neg, \sim)$ is a $K_{\imath}$-algebra. For $\mathcal{L}$ to be extended to a $ccpBa$, we must be able to define an operator `→`, such that $(L', \hat{1}, \hat{0}, \lor, \land, \to)$ is an r-pc lattice, i.e the following holds: for all $a, b, x \in L'$, $a \land x \leq b \iff x \leq a \rightarrow b$.

Let $a := (1, 0)$ and $b := (0, 1)$. The possible choices of $x$ for which $a \land x \leq b$ are from the set $\{\hat{0}\} \cup \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m \leq 0\}$. Since $x \leq a \rightarrow b$ for all such $x$, the only possible choice for $a \rightarrow b$ is $\hat{1}$. However, this value would make the converse false, because $(1, 1) \leq \hat{1}$, but $(1, 0) \land (1, 1) = (1, 0) \not\leq (0, 1)$. Thus, we have the following.

Proposition 6.8.
There exists a $K_{\imath}$-algebra $A := (A, 1, 0, \lor, \land, \to, \neg)$ such that there is no binary operator $\rightarrow$ on $A$ that makes the algebra $(A, 1, 0, \lor, \land, \to, \neg, \sim)$ a $ccpBa$.

Propositions 6.7 and 6.8 demonstrate that even though there is a $K_{\imath}$-algebra that cannot be extended to a $ccpBa$, the properties of the two negation operators in a $K_{\imath}$-algebra are enough to capture the ‘non-implicative’ version of $ccpBa$.

Proposition 6.9.
1. There exists a bounded distributive lattice $A := (A, 1, 0, \lor, \land, \to, \neg)$ with minimal negation $\sim$ and intuitionistic negation $\neg$ such that $\neg \sim \sim 1 \neq \sim 1$.
2. There exists a $K_{\imath}$-algebra $A := (A, 1, 0, \lor, \land, \to, \neg)$ such that $\sim$ is not an intuitionistic negation.

Proof. 
1. Consider the pBa $H_6 := (H_6, 1, 0, \lor, \land, \to)$ with intuitionistic negation $\neg$ (Figure 1). $(H_6, 1, 0, \lor, \land)$ is thus a bounded distributive lattice. Define $\sim$ as follows: $\sim 1 := w$ and for all $a \in H_6$, $\sim a := a \rightarrow \sim 1$. Then it can be shown that $\sim$ is a minimal negation. But, $\sim 1 = w \neq \neg \sim 1 (= 1)$.
2. The $K_{\imath}$-algebra, $\mathcal{L} := (L', \hat{1}, \hat{0}, \lor, \land, \to, \neg)$, considered to establish Proposition 6.8 suffices. For $a := (m, n) \in L$, $a \land \sim a = a \land \hat{1} = a \neq \hat{0}$.
A comprehensive analysis of different negations in any bounded distributive lattice is given in [19], where a Kite-like figure is obtained with negations as nodes. Each node in the figure corresponds to a unique property of negation. A path connecting two nodes (that is two properties of negation) $A$ and $B$ where $A$ is above $B$, represents the fact that the negation at $B$ holds at $A$. Moreover, two paths starting at nodes $A$ and $B$ and meeting at a higher node $C$, implies that the property at $C$ can be derived from the properties at $A$ and $B$.

We enhance this Kite diagram to one where each node corresponds to a pair of negations $(\sim, \neg)$ with $\neg$ as intuitionistic, and only the property of $\sim$ is mentioned against a node. The enhanced kite is called a Kite with negation pair $(\sim, \neg)$, where $\neg$ is intuitionistic (Figure 4). In such a diagram, due to Proposition 6.9, we can place the negation $\sim$ of the $K_{im}$-algebra strictly in between the nodes of minimal and intuitionistic negations.

Let us now see the placement of the negation $\sim$ of $K_{im-\lor}$-algebras in the Kite with negation pair $(\sim, \neg)$. The following examples of bounded distributive lattices of the type $(A, 1, 0, \lor, \land, \neg, \sim)$ with two negations, where $\neg$ is intuitionistic, turn out to be useful for the purpose.

1. Consider the 3-element bounded distributive lattice $(A, 1, 0, \lor, \land)$ (Section 2.2), where $A := \{0, a, 1\}$ with ordering $0 \leq a \leq 1$. Define $\neg := \neg_1$ and $\sim := \sim_1$, where $\neg_1$ is as given in Table 3. Then $\sim$ is intuitionistic, but $(A, 1, 0, \lor, \land, \neg_1, \sim_1)$ is not a $K_{im-\lor}$-algebra (as $a \lor \neg_1 a = a \neq 1$). Thus, $\sim$ of $K_{im-\lor}$-algebras cannot lie between ‘Intuitionistic’ and $\text{DNE}(\sim 1)$.

2. For the same 3-element bounded distributive lattice, define $\neg := \neg_1$ and $\sim := \sim_1$, where $\neg_1$ and $\sim_1$ are as given in Tables 3 and 4 respectively. Then $(A, 1, 0, \lor, \land, \neg_1, \sim_1)$ is a $K_{im-\lor}$-algebra, but $\sim_1$ is not intuitionistic ($a \land \sim_1 a = a \neq 0$). Thus, $\sim$ of $K_{im-\lor}$-algebras cannot lie in the path above the node ‘Intuitionistic’.

3. For the same 3-element bounded distributive lattice, define $\neg := \neg_1$ and $\sim$ as identity map. Then $\sim$ is a De Morgan negation, but $(A, 1, 0, \lor, \land, \neg_1, \sim)$ is not a $K_{im-\lor}$-algebra. This implies that $\sim$ of $K_{im-\lor}$-algebras cannot be in the path below the node ‘De Morgan’.

4. Consider the 6-element $pBa$ $(H_6, 1, 0, \lor, \land, \rightarrow, \neg)$ (Figure 1). Recall the $c \lor cpBa$ $(H_6, 1, 0, \lor, \land, \rightarrow, \neg, \sim)$ mentioned in Example (C)(3) of Section 2.2. The reduct $(H_6, 1, 0, \lor, \land, \rightarrow, \sim)$ is a $K_{im-\lor}$-algebra. However, $\sim$ is not De Morgan ($\sim \wedge w = 1 \not\leq w$). So $\sim$ of $K_{im-\lor}$-algebras cannot be on the path connecting the nodes ‘De Morgan’ and ‘Ortho’.

Thus, $\sim$ of $K_{im-\lor}$-algebra has to be placed on a separate path connecting $\text{DNE}(\sim 1)$ and Ortho in the Kite with negation pair $(\sim, \neg)$ (Figure 4), marked by a node with property EM. Note also that $a \lor \sim a = 1$ and $a \land \sim a = 0$ for all $a \in A$ imply that $\sim$ is ortho [16].

6.2 Relational semantics for $K_{im}$ and $K_{im-\lor}$

One of the motivations of Dunn was to study negation as a modal impossibility operator in ‘compatibility’ frames, in line with the work on negation by Vakarelov [19]. Let us first see what ‘compatibility’ frames are, and define the semantics over such frames for the logic $K_i$. 

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Definition 6.10 (Compatibility frames [19]).
A tuple \( \mathcal{F} := (W, C, \leq) \) is called a compatibility frame if \((W, \leq)\) is a poset, and \(C\) is a binary relation on \(W\) satisfying the following condition for all \(x, x', y, y' \in W\):

\[
\text{If } x' \leq x, \ y' \leq y \text{ and } xCy \text{ then } x'Cy'.
\]  

(C)

Let us compare the strictly condensed \(N\)-frames mentioned in Remark 4.2 of Section 4 with compatibility frames.

Observation 6.11.
\((W, \leq, R_N)\) is a strictly condensed \(N\)-frame if and only if \((\leq R_N \leq^{-1}) \subseteq R_N\). So,

\[
\forall x, y, x', y' \in W \left((x' \leq x \& xR_Ny \& y' \leq y) \Rightarrow x'R_Ny'\right).
\]

According to Definition 6.10 therefore, \( (W, R_N, \leq) \) is a compatibility frame. Conversely, it is clear that any compatibility frame gives a strictly condensed \(N\)-frame. In other words, the class of strictly condensed \(N\)-frames is the same as that of compatibility frames.

The definitions of valuations and truth of a formula \(\alpha\) at \(x \in W\) under a valuation on a compatibility frame \(\mathcal{F} := (W, C, \leq)\) (denoted \(x \vDash \alpha\)) are then given in the standard way [19]. For the compatibility frame \(\mathcal{F} := (W, C, \leq)\), the pair \(\mathcal{M} := (\mathcal{F}, \vDash)\) is called a model of \(K_i\). A consequence pair \((\alpha, \beta)\) is valid in a compatibility frame \(\mathcal{F}\), denoted as \(\alpha \vDash_{\mathcal{F}} \beta\), when for every model \(\mathcal{M}\) on the frame \(\mathcal{F}\) and for all \(x \in W\), \(x \vDash \alpha \Rightarrow x \vDash \beta\).

It can be checked using induction on the number of connectives in a formula, that the following hereditary condition holds for any formula \(\alpha\) in \(K_i\):

for all \(x, y \in W\), \(x \leq y\) and \(x \vDash \alpha\) imply \(y \vDash \alpha\).

Theorem 6.12. [19] For any two formulas \(\alpha\) and \(\beta\), \(\alpha \vdash_{K_i} \beta\) if and only if \(\alpha \vDash_{\mathcal{F}} \beta\) for any compatibility frame \(\mathcal{F} := (W, C, \leq)\).

As \(K_{im}\) is an ‘extension’ of \(K_i\), it is then expected that the former will be sound and complete with respect to some class of compatibility frames. Let us now define sub-compatibility frames.
**Definition 6.13** (Sub-compatibility frame). Let \( F := (W, C, \leq) \) be a compatibility frame such that

1. \( C \) is symmetric,
2. \( \forall x, y \in W (xCy \Rightarrow \exists z \in W (x \leq z \& y \leq z \& xCz)) \), and
3. \( \forall x \in W (\forall y \in W (x \leq y \Rightarrow \exists z \in W (y \leq z \& \forall z' \in W (zCz'))) \Rightarrow \forall z'' \in W (xCz'')) \).

\( F \) is called a sub-compatibility frame. \( F \) is called a sub-compatibility identity frame. The class of sub-compatibility identity frames is denoted by \( \mathcal{F}_3 \).

Sub-compatibility frames satisfying the additional condition \( C \subseteq (\leq^{-1}) \) are called sub-compatibility identity frames. The class of sub-compatibility identity frames is denoted by \( \mathcal{F}_3 \).

The definitions of valuation, truth and validity remain the same. Let us note the definitions for the truth of a formula \( \alpha \) involving \( \neg \) and \( \sim \):

1. \( x \models \neg \alpha \) if and only if \( \forall y \in W, (x \leq y \Rightarrow y \not\models \alpha) \).
2. \( x \models \sim \alpha \) if and only if \( \forall y \in W, (xCy \Rightarrow y \not\models \alpha) \).

As done in previous sections, let us investigate the special conditions defining the frame here, namely Conditions (2) and (3) in Definition 6.13. In [19], it has been shown that Condition (2) is canonical to P6: \( \alpha \wedge \beta \vdash \gamma / \alpha \wedge \sim \gamma \vdash \sim \beta \) from Proposition 6.2, i.e. P6 is valid in a compatibility frame \( F := (W, C, \leq) \) if and only if \( C \) satisfies Condition (2).

Now recall the sequent \( P7 \) (DNE(\( \sim \Top \))): \( \neg \neg \sim \Top \vdash \sim \Top \) from Proposition 6.2. Validity of DNE(\( \sim \Top \)) in a sub-compatibility model \( M := ((W, C, \leq), \models) \) means the following.

\[
\forall x \in W (x \models \neg \neg \sim \Top \Rightarrow x \models \sim \Top ) \\
\Leftrightarrow \forall x \in W (\forall y \in W (x \leq y \Rightarrow y \not\models \sim \Top ) \Rightarrow x \models \sim \Top ) \\
\Leftrightarrow \forall x \in W (\forall y \in W (x \leq y \Rightarrow \exists z \in W (y \leq z \& z \models \sim \Top )) \Rightarrow x \models \sim \Top ) \\
\Leftrightarrow \forall x \in W (\forall y \in W (x \leq y \Rightarrow \exists z \in W (y \leq z \& \forall z' \in W (zCz'))) \Rightarrow \forall z'' \in W (xCz'')).
\]

The last condition is just Condition (3). Thus, we have the following.

**Proposition 6.14.** Condition (3) in Definition 6.13 is canonical to DNE(\( \sim \Top \)).

We also note here that the reverse implication in Condition (3) is always true. Consider the contraposition of the implication, i.e. for any \( x \in W \):

\[
\exists y \in W (x \leq y \& \forall z \in W (y \leq z \Rightarrow \exists z' \in W (zCz'))) \Rightarrow \exists z'' \in W (xCz'').
\]

Now let \( y \in W \) such that \( x \leq y \) and \( \forall z \in W (y \leq z \Rightarrow \exists z' \in W (zCz')). \( y \leq y \) implies there exists \( z' \in W \) such that \( yCz' \). Using condition (C), \( yCz' \), \( x \leq y \) and \( z' \leq z' \) imply \( xCz' \).

In Definition 6.3, we have given an equivalent version \( K_{im}' \) of \( K_{im} \). The axioms/rules involving the negation \( \sim \) are P2-P7. Based on the above points, the presence of Conditions (2) and (3) in sub-compatibility frames ensures the validity of P6 and P7 in the class. Moreover, any sub-compatibility frame is a compatibility frame, and P2-P5 are valid in any compatibility frame in the language of \( K_i \) (Theorem 6.12). So we have the soundness result for \( K_{im} \) with respect to sub-compatibility frames. One can obtain the completeness result by following similar steps as given in [19].
Theorem 6.15. For any formulas $\alpha, \beta \in \bar{F}$,
1. $\alpha \vDash_{\delta_2} \beta \iff \alpha \vDash_{K_{im}} \beta$, and
2. $\alpha \vDash_{\delta_3} \beta \iff \alpha \vDash_{K_{im-v}} \beta$.

7 Connections between relational and algebraic semantics

We have observed the duality between topological spaces and the algebras in Theorem 2.12. Let us now connect the frames studied in the previous sections and the algebras. Kripke [31] linked normal frames and $pBa$s (cf. [8, 4]): every $pBa$ $\mathcal{A}$ is embedded in the `complex algebra' of the `canonical frame' of $\mathcal{A}$; every normal frame $\mathcal{F}$ can be embedded into the `canonical' frame of the `complex algebra' of $\mathcal{F}$. Note that an embedding between two posets $(A, \leq)$ and $(A', \leq')$ is a map $\phi : A \to A'$ such that for all $a, b \in A$, $a \leq b \Rightarrow \phi(a) \leq' \phi(b)$ (Order-preserving) and $\phi(a) \leq' \phi(b) \Rightarrow a \leq b$ (Order-reflecting). Then, an embedding between two algebras or frames is just defined as an embedding between the underlying posets (preserving the operators/relations). We link sub-normal frames and $ccpBa$s similarly. Embeddings between sub-normal frames and canonical algebras of sub-normal frames are first defined.

Definition 7.1 (Embeddings between sub-normal frames).
Given two sub-normal frames $(W, \leq, Y_0)$ and $(W', \leq', Y'_0)$, a mapping $f : W \to W'$ is an embedding if it satisfies the following conditions for all $a, b \in W$:
1. $a \leq b$ if and only if $f(a) \leq' f(b)$ (poset embedding), and
2. $a \in Y_0$ if and only if $f(a) \in Y'_0$.

Definition 7.2 (Complex algebra of a sub-normal frame). Consider a sub-normal frame $\mathcal{F} := (W, \leq, Y_0)$. Define the following operators on $Up(W)$, the set of all upsets of $W$.
For any $U, V \subseteq Up(W)$,
1. $U \cup V := \{w \in W \mid \forall v \in W (w \leq v \Rightarrow (v \in U \Rightarrow v \in V))\}$,
2. $\neg U := U \rightarrow \emptyset$, i.e. $\neg U := \{w \in W \mid \forall v \in W (w \leq v \Rightarrow v \notin U)\}$, and
3. $\sim U := U \rightarrow Y_0$, i.e. $\sim U := \{w \in W \mid \forall v \in W ((w \leq v \land v \in U) \Rightarrow v \in Y_0)\}$.
The structure $Up(\mathcal{F}) := (Up(W), W, \emptyset, \cap, \cup, \rightarrow, \sim)$ is called the complex algebra of the sub-normal frame $\mathcal{F}$.

Definition 7.3 (Canonical frame of a $ccpBa$).
Consider a $ccpBa$ $\mathcal{A} := (A, 1, 0, \lor, \land, \rightarrow, \sim)$. Let $X_A$ denote the set of all prime filters in $\mathcal{A}$. Define a set $Y_0$ as follows: $Y_0 := \{P \in X_A \mid \sim 1 \in P\}$.
Then the triple $\mathcal{F}_A := (X_A, \subseteq, Y_0)$ is called the canonical frame of $\mathcal{A}$.

It is easy to check that for a sub-normal frame $\mathcal{F} := (W, \leq, Y_0) \in \mathfrak{F}_2$, the complex algebra $Up(\mathcal{F})$ of $\mathcal{F}$ forms a $ccpBa$. Further, if $\mathcal{F} \in \mathfrak{F}_2$ is a sub-normal identity frame then $Up(\mathcal{F})$ is a $c \lor cpBa$. On the other hand, given a $ccpBa$ $\mathcal{A} := (A, 1, 0, \lor, \land, \rightarrow, \sim)$, the canonical frame $\mathcal{F}_A := (X_A, \subseteq, Y_0)$ is a sub-normal frame. If $\mathcal{A}$ is a $c \lor cpBa$ then $\mathcal{F}_A$ is a sub-normal identity frame. This to and fro connection between sub-normal frames $\mathcal{F} \in \mathfrak{F}_2 (\mathfrak{F}_2')$ and $ccpBa$s ($c \lor cpBa$s) gives us the following result. The proof is similar to that in the case of $pBas$.

Theorem 7.4.
1. Every ccpBa (or c\(\lor\)cpBa) \(\mathcal{A}\) is embeddable into the complex algebra \(U_p(\mathcal{F}_\mathcal{A})\) of the canonical frame \(\mathcal{F}_\mathcal{A}\) of \(\mathcal{A}\).

2. Any sub-normal (sub-normal identity) frame \(\mathcal{F} \in \mathfrak{F}_2\) (or \(\mathfrak{F}_2^\prime\)) can be embedded into the canonical frame \(\mathcal{F}_{U_p(\mathcal{F})}\) of the complex algebra \(U_p(\mathcal{F})\) of \(\mathcal{F}\).

**Proof.** We just mention the maps involved.

1. Consider the complex algebra \(U_p(\mathcal{F}_\mathcal{A}) := (U_p(X_A), X_A, X_A, \emptyset, \cap, \cup, \rightarrow, \neg, \sim)\) of the canonical frame \(\mathcal{F}_\mathcal{A} := (X_A, \subseteq, Y_0)\) of a ccpBa \(\mathcal{A} := (A, 1, 0, \lor, \land, \rightarrow, \neg, \sim)\). The map \(h : A \to U_p(X_A)\) such that for all \(a \in A, h(a) := \{P \in X_A \mid a \in P\}\), is a monomorphism from the ccpBa (or c\(\lor\)cpBa) \((A, 1, 0, \lor, \land, \rightarrow, \neg, \sim)\) to the ccpBa (or c\(\lor\)cpBa) \((U_p(X_A), \emptyset, X_A, \cap, \cup, \rightarrow, \neg, \sim)\).

2. Consider the canonical frame \((X_{U_p(W)}, \subseteq, Y_{0_{U_p(W)}})\) of the complex algebra \(U_p(\mathcal{F}) := (U_p(W), W, \emptyset, \cap, \cup, \rightarrow, \neg, \sim)\) of the sub-normal frame (or sub-normal identity frame) \(\mathcal{F} := (W, \subseteq, Y_0)\). The map \(g : W \to X_A\) such that

\[
g(w) := \{U \in U_p(W) \mid w \in U\},
\]

is an embedding from \((W, \subseteq, Y_0)\) to \((X_{U_p(W)}, \subseteq, Y_{0_{U_p(W)}})\).

A similar result can also be obtained for \(K_{im}\) and \(K_{im, \lor}\). Let us first give the definition of embeddings between sub-compatibility frames and complex algebras of sub-compatibility frames.

**Definition 7.5** (Embeddings between sub-compatibility frames).

Given two sub-compatibility frames \((W, C, \subseteq)\) and \((W', C', \subseteq')\), a mapping \(f : W \to W'\) is an *embedding* if it satisfies the following conditions for all \(x, y \in W\):

1. \(x \leq y\) if and only if \(f(x) \leq f(y)\), and
2. \(xCy\) if and only if \(f(x)C'f(y)\).

**Definition 7.6** (Complex algebra of a sub-compatibility frame).

Given a sub-compatibility frame \(\mathcal{F} := (W, C, Y_0)\), the *complex algebra* of the sub-compatibility frame \(\mathcal{F}\) is the structure \(U_p(\mathcal{F}) := (U_p(W), W, \cap, \cup, \rightarrow, \neg, \sim)\), where the operators \(\neg, \sim\) on \(U_p(W)\) are defined as:

1. \(\neg U := \{w \in W \mid \forall v \in W (w \leq v \Rightarrow v \notin U)\}\), and
2. \(\sim U := \{w \in W \mid \forall v \in W (wCv \Rightarrow v \notin U)\}\).

**Definition 7.7** (Canonical frame of a \(K_{im}\)-algebra).

Given a \(K_{im}\)-algebra \(\mathcal{A} := (A, 1, 0, \lor, \land, \rightarrow, \neg, \sim)\), the triple \(\mathcal{F}_\mathcal{A} := (X_A, C, \subseteq)\) is called the *canonical frame* on \(X_A\), where the relation \(C\) on \(X_A\) is defined as follows. For any \(P, Q \in X_A\),

\[PCQ\text{ if and only if (for all }a \in A, \sim a \in P \Rightarrow a \notin Q)\].

We can check that the complex algebra of a sub-compatibility (identity) frame forms a \(K_{im}\)-algebra (\(K_{im, \lor}\)-algebra), and the canonical frame of a \(K_{im}\)-algebra (\(K_{im, \lor}\)-algebra) forms a sub-compatibility (identity) frame. Similar to Theorem 7.8, this to and fro connection between \(K_{im}\)-algebras and sub-compatibility frames gives the following result.

**Theorem 7.8.**

1. Every \(K_{im}\)-algebra (or \(K_{im, \lor}\)-algebra) \(\mathcal{A}\) can be embedded into the complex algebra \(U_p(\mathcal{F}_\mathcal{A})\) of the canonical frame \(\mathcal{F}_\mathcal{A}\) of \(\mathcal{A}\).

2. Any sub-compatibility (sub-compatibility identity) frame \(\mathcal{F} \in \mathfrak{F}_3\) (or \(\mathfrak{F}_3^\prime\)) can be embedded into the canonical frame \(\mathcal{F}_{U_p(\mathcal{F})}\) of the complex algebra \(U_p(\mathcal{F})\) of \(\mathcal{F}\).
8 Conclusions

The algebraic classes \( ccpBa \) and \( c\lor ccpBa \) are studied through examples, properties, representation theorems, and comparison with existing algebras. The corresponding logics ILM and ILM-\( \lor \) are defined, and ILM is compared with existing logics ML and IL. Further, ILM is shown to be equivalent to a logic which is an extension of J\( \mathcal{P} \)', a special case of Peirce’s logic, and hence observed to be decidable. A study of the features of the two negations is carried out next through the logics \( K_{im} \) and \( K_{im-\lor} \)-algebras is conducted. Dunn’s Kite of negations is enhanced to define a Kite with negation pair \((\neg,\neg)\), where \( \neg \) is intuitionistic. It is then shown that the negations in the algebras occupy distinct positions in this kite. The property \( DNE(\sim 1) (\neg\neg\sim 1 \leq \sim 1) \) leads to a position strictly between the Minimal and Intuitionistic nodes, while the property \( EM (a \lor \neg a = 1) \) yields a new path from \( DNE(\sim 1) \) to the Ortho node. Finally, relations between frames and algebras defined in the work are established through duality results.

The study of relational semantics of ILM and ILM-\( \lor \) helps in understanding the behaviour of the negation operators in the logics. We have given two different semantics for ILM through the classes \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of frames - \( \mathcal{F}_1 \) using Došen’s \( N \)-frames and \( \mathcal{F}_2 \) using Segerberg’s \( j \)-frames. The difference in both lies in the treatment of negation. For \( \mathcal{F}_1 \), both negations are considered as unary modal connectives and the semantics is defined using the modal accessibility relations \( R_{N_1} \) and \( R_{N_2} \). For \( \mathcal{F}_2 \), both negations are treated as unary connectives, their semantics being defined using the relation \( \leq \) and \( Y_0 \). This naturally gives the idea of constructing new frame classes and corresponding semantics, by considering one of the two negations of ILM as a unary connective, and the other negation as a unary ‘impossibility’ modal connective - of the types \((X, \leq, R_{N_2})\) and \((X, \leq, R_{N_1}, Y_0)\). In fact, we have seen the frames of the type \((X, \leq, R_{N_2})\) as sub-compatibility frames (where the relation \( R_{N_2} \) was represented by \( C \)). It is then naturally expected that, by suitably defining conditions on relations \( R_{N_2}, R_{N_1} \) and \( Y_0 \), ILM (ILM-\( \lor \)) can also be determined by the class of frames of the types \((X, \leq, R_{N_2})\) and \((X, \leq, R_{N_1}, Y_0)\). An inter-translation between these classes can also be obtained using the following relations.

1. \( xR_{N_1}y \iff \exists z(x \leq z \text{ and } y \leq z) \)
2. \( xR_{N_2}y \iff \exists z(x \leq z \text{ and } y \leq z \text{ and } z \notin Y_0) \), and
3. \( Y_0 := \{ x \mid \forall z(xR_{N_2}z) \} \).

\[
\begin{align*}
(X, \leq, Y_0) & \iff (X, \leq, R_{N_1}, R_{N_2}) \\
(X, \leq, R_{N_1}, Y_0) & \iff (X, \leq, R_{N_2})
\end{align*}
\]

Investigations of properties of negations have resulted in various schemes of logical systems. Some of the work in this direction may be found in \([14, 27, 44, 46]\) and more recently, in \([11, 19, 38, 39]\). A common approach adopted is that a ‘base logic with negation’ with minimum properties on negation is first defined, and new logics are obtained by adding axioms over the existing ones. This is followed by defining
relational semantics for the base logic, and obtaining canonical properties for various
negation properties. Odintsov [39] studied the class of extensions of minimal logic, and
presented them in a diagram, just like Dunn’s Kite. In the direction of logics with two
negations, the class of extensions of Nelson logic (denoted N4⊥) and its semantics is
also discussed by Odintsov [37, 39]. However, we have shown that negations in ILM
differ from those in Nelson logic. Another relevant and independent work on logics
with two negations is done in [19]. Taking a cue from the dual properties of negations
[47], Dunn defined logics with two negations obtained by merging two minimal systems
(dual to each other) [19]. Diagrammatically, it is represented by uniting the lopsided
kite of negations with the dual lopsided kite of negations. The base negation is kept
as preminimal (along with its dual) in the ‘United Kite’. The Kite of negation pair
(∼, ¬) presented in this work opens up a different direction of study. We have made
the properties of one negation (∼) of the pair vary, while the other (¬) is fixed to be
intuitionistic. ¬¬∼⊤ ⊢ ¬¬¬¬∼⊤ (DNE(¬¬¬¬⊤)) connects both the negations. This suggests
a line of work where one may start with a base logic with a pair (∼, ¬) of preminimal
negations connected by an appropriate condition. A Kite of the negation pair (∼, ¬)
may be developed, where one may wish to ensure that the base logic is extendable to
existing logics with two negations such as Nelson logic and its extensions and Dunn’s
logic with dual negations.

References

[1] M. Banerjee and M. K. Chakraborty. A category for rough sets. Foundations of
Computing and Decision Sciences, 18(3–4):167–180, 1993.

[2] M. Banerjee and M. K. Chakraborty. Foundations of vagueness: a category-
theoretic approach. Electronic Notes in Theoretical Computer Science, 82(4):10–19,
2003.

[3] M. Banerjee and M. K. Chakraborty. Algebras from rough sets. In S. K. Pal,
L. Polkowski, and A. Skowron, editors, Rough-Neural Computing: Techniques for
Computing with Words, pages 157–184. Springer, 2004.

[4] G. Bezhanishvili and W. H. Holliday. A semantic hierarchy for intuitionistic logic.
Indagationes Mathematicae, 30(3):403–469, 2019.

[5] N. Bezhanishvili. Lattices of intermediate and cylindric modal logics. PhD thesis,
Institute for Logic, Language and Computation, University of Amsterdam, 2006.

[6] W. A. Carnielli and I. M. L. D’Ottaviano. Translations between logical systems:
a manifesto. Logique et Analyse, 40(157):67–81, 1997.

[7] S. A. Celani and R. Jansana. Esakia duality and its extensions. In Leo Esakia on
Duality in Modal and Intuitionistic Logics, volume 4, pages 63–98. Springer, 2014.

[8] A. Chagrov and M. Zakharyaschev. Modal Logic. Clarendon Press Oxford, 1997.

[9] R. Cignoli. The algebras of Łukasiewicz many-valued logic: A historical overview.
In S. Aguzzoli and et al., editors, Algebraic and Proof-theoretic Aspects of Non-
classical Logics, volume 4460 of LNCS, pages 69–83. Springer, 2007.
[10] D. M. Clark and B. A. Davey. *Natural Dualities for the Working Algebraist*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[11] A. Colacito, D. de Jongh, and A. L. Vargas. Subminimal negation. *Soft computing*, 21(1):165–174, 2017.

[12] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2002.

[13] K. Došen. Negation as a modal operator. *Reports on Mathematical Logic*, 20:15–27, 1986.

[14] K. Došen. Negation in the light of modal logic. In D. M. Gabbay and H. Wansing, editors, *What is Negation?*, pages 77–86. Springer, 1999.

[15] J. M. Dunn. Gaggle theory: An abstraction of Galois connections and residuation, with applications to negation, implication, and various logical operators. In J. van Eijck, editor, *Logics in AI*, pages 31–51. Springer, 1991.

[16] J. M. Dunn. Star and perp: Two treatments of negation. *Philosophical Perspectives*, 7:331–357, 1993.

[17] J. M. Dunn. Positive modal logic. *Studia Logica*, 55(2):301–317, 1995.

[18] J. M. Dunn. Generalized ortho negation. In *Negation: A notion in focus*, pages 3–26. Walter de Gruyter, 1996.

[19] J. M. Dunn and C. Zhou. Negation in the context of gaggle theory. *Studia Logica*, 80(2-3):235–264, 2005.

[20] G. Ferreira and P. Oliva. On the relation between various negative translations. In *Logic, Construction, Computation*, volume 3 of *Ontos Math. Log.*, pages 227–258. Ontos Verlag, 2012.

[21] T. K. Fu and O. Kutz. The analysis and synthesis of logic translation. In *Proceedings of the Twenty-Fifth International FLAIRS Conference (AICogSem)*, pages 289–294. AAAI Press, 2012.

[22] J. Geisler and M. Nowak. Conditional negation on the positive logic. *Bulletin of the Section of Logic*, 23(3):130–136, 1994.

[23] R. I. Goldblatt. Decidability of some extensions of J. *Mathematical Logic Quarterly*, 20(13-18):203–206, 1974.

[24] Y. Gurevich. Intuitionistic logic with strong negation. *Studia Logica*, 36(1):49–59, 1977.

[25] C. Hartonas. Duality for lattice-ordered algebras and for normal algebraizable logics. *Studia Logica*, 58(3):403–450, 1997.

[26] T. B. Iwiński. Algebraic approach to rough sets. *Bull. Polish Acad. Sci. Math.*, 35:673–683, 1987.
[27] I. Johansson. Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus. Compositio Math., 4:119–136, 1937.

[28] P. T. Johnstone. *Stone Spaces*. Cambridge University Press, 1986.

[29] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*, volume 2. Oxford University Press, 2002.

[30] J. B. Kiszka, M. M. Gupta, and G. M. Trojan. Multivariable fuzzy controller under Gödel’s implication. Fuzzy Sets and Systems, 34(3):301–321, 1990.

[31] S. A. Kripke. Semantical analysis of modal logic I: Normal modal propositional calculi. Mathematical Logic Quarterly, 9(5-6):67–96, 1963.

[32] S. A. Kripke. Semantical analysis of intuitionistic logic I. In *Formal Systems and Recursive Functions (Proc. Eighth Logic Colloq., Oxford, 1963)*, pages 92–130. North-Holland, 1965.

[33] A. K. More. *A Study of Algebraic Structures and Logics based on Categories of Rough Sets*. PhD thesis, Indian Institute of Technology Kanpur, Kanpur, 2019.

[34] A. K. More and M. Banerjee. Categories and algebras from rough sets: new facets. Fundamenta Informaticae, 148(1-2):173–190, 2016.

[35] A. K. More and M. Banerjee. New algebras and logic from a category of rough sets. In L. Polkowski and et al., editors, *Rough Sets, IJCRS 2017*, volume 10313 of Lecture Notes in Computer Science, pages 95–108. Springer, 2017.

[36] M. Nowak. The weakest logic of conditional negation. Bulletin of the Section of Logic, 24(4):201–205, 1995.

[37] S. P. Odintsov. The class of extensions of Nelson’s paraconsistent logic. Studia Logica, 80(2-3):291–320, 2005.

[38] S. P. Odintsov. The lattice of extensions of the minimal logic. Siberian Adv. Math., 17(2):112–143, 2007.

[39] S. P. Odintsov. *Constructive negations and paraconsistency*. Springer, 2008.

[40] Z. Pawlak. Rough sets. International Journal of Computer and Information Sciences, 11(5):341–356, 1982.

[41] D. Prawitz and P. E. Malmnäs. A survey of some connections between classical, intuitionistic and minimal logic. Studies in Logic and the Foundations of Mathematics, 50:215–229, 1968.

[42] A. N. Prior. Peirce’s axioms for propositional calculus. Journal of Symbolic Logic, 23(2):135–136, 1958.

[43] H. Rasiowa. *An Algebraic Approach to Non-classical Logics*. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, 1974.
[44] H. Rasiowa and R. Sikorski. Algebraic treatment of the notion of satisfiability. *Fundamenta Mathematicae*, 40:62–95, 1953.

[45] G. Restall. Defining double negation elimination. *Logic Journal of IGPL*, 8(6):853–860, 2000.

[46] K. Segerberg. Propositional logics related to Heyting’s and Johansson’s. *Theoria*, 34:26–61, 1968.

[47] Y. Shramko. Dual intuitionistic logic and a variety of negations: the logic of scientific research. *Studia Logica*, 80(2-3):347–367, 2005.

[48] D. Vakarelov. Notes on \(N\)-lattices and constructive logic with strong negation. *Studia Logica*, 36(1–2):109–125, 1977.

[49] D. Vakarelov. Consistency, completeness and negation. In G. Priest, R. Routley, and J. Norman, editors, *Paraconsistent Logic: Essays on the Inconsistent*, pages 328–369. Philosophia Verlag, 1989.

[50] P. W. Woodruff. A note on JP’. *Theoria*, 36(2):183–184, 1970.