HOMOTOPY CLASSIFICATION OF LEAVITT PATH ALGEBRAS

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Abstract. In this paper we address the classification problem for purely infinite simple Leavitt path algebras of finite graphs over a field \( \ell \). Each graph \( E \) has associated a Leavitt path \( \ell \)-algebra \( L(E) \). There is an open question which asks whether the pair \((K_0(L(E)), [1_{L(E)}])\), consisting of the Grothendieck group together with the class \([1_{L(E)}]\) of the identity, is a complete invariant for the classification, up to algebra isomorphism, of those Leavitt path algebras of finite graphs which are purely infinite simple. We show that \((K_0(L(E)), [1_{L(E)}])\) is a complete invariant for the classification of such algebras up to polynomial homotopy equivalence. To prove this we further develop the study of bivariant algebraic K-theory of Leavitt path algebras started in a previous paper and obtain several other results of independent interest.

1. Introduction

A directed graph \( E \) consists of a set \( E^0 \) of vertices and a set \( E^1 \) of edges together with source and range functions \( r, s : E^1 \to E^0 \). This article is concerned with the Leavitt path algebra \( L(E) \) of a directed graph \( E \) over a field \( \ell \) ([11]). When \( \ell = \mathbb{C} \), \( L(E) \) is a normed algebra; its completion is the graph \( C^\ast \)-algebra \( C^\ast(E) \). A graph \( E \) is called finite or countable if both \( E^0 \) and \( E^1 \) are finite or countable. A result of Cuntz and Rørdam ([18, Theorem 6.5]) says that the purely infinite simple graph algebras associated to finite graphs, i.e. the purely infinite simple Cuntz-Krieger algebras, are classified up to (stable) isomorphism by the Grothendieck group \( K_0 \). It is an open question whether a similar result holds for Leavitt path algebras [3]. Here we prove that \( K_0 \) classifies simple Leavitt path algebras up to \((M_2\)-) homotopy equivalence. In the following theorem and elsewhere, we use the following notations. We write \( \iota_2 : R \to M_2R \) for the inclusion of an algebra into the upper left hand corner of the matrix algebra, \( \phi \approx \psi \) to indicate that two algebra homomorphisms \( \phi \) and \( \psi \) are (polynomially) homotopic and \( \phi \approx_{M_2} \psi \) to mean that \( \iota_2\phi \approx \iota_2\psi \). We also put \([1_R]\) for the \( K_0\)-class of the identity of a unital algebra \( R \). In Theorem 6.1 we prove the following.

Theorem 1.1. Let \( E \) and \( F \) be finite graphs. Assume that \( L(E) \) and \( L(F) \) are purely infinite simple. Let \( \xi : K_0(L(E)) \to K_0(L(F)) \) be an isomorphism of groups. Then there exist nonzero algebra homomorphisms \( \phi : L(E) \to L(F) : \psi \) such that \( K_0(\phi) = \xi, K_0(\psi) = \xi^{-1}, \psi\phi \approx_{M_2} \text{id}_{L(E)} \) and \( \phi\psi \approx_{M_2} \text{id}_{L(F)} \). If moreover \( \xi([1_{L(E)}]) = [1_{L(F)}] \) then \( \phi \) and \( \psi \) can be chosen to be unital homomorphisms such that \( \psi\phi \approx \text{id}_{L(E)} \) and \( \phi\psi \approx \text{id}_{L(F)} \).

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We also prove other results which we think are of independent interest. For example we have the following embedding theorem, proved in Corollary 3.2.

**Theorem 1.2.** Let $E$ be a graph such that $L(E)$ is simple and let $R$ be a unital purely infinite algebra.

i) If $E$ is countable then $L(E)$ embeds as a subalgebra of $M_\infty R$.

ii) If $E$ is finite and $[1_R] = 0$ in $K_0(R)$, then $L(E)$ embeds as a unital subalgebra of $R$.

iii) If $E$ is finite, then $L(E)$ embeds as a subalgebra of $R$.

For particular $R$, we have the following result on uniqueness up to homotopy for embeddings into $R$. In the next theorem and elsewhere, we write $[A, R]$ and $[A, R]_{M_2}$ for the set of homotopy classes and $M_2$-homotopy classes of homomorphisms $A \to R$. If moreover, $A$ and $R$ are unital, we write $[A, R]_1$ for the set of homotopy classes of unital homomorphisms $A \to R$. In the next theorem and elsewhere we use the notion of regular supercoherent ring from [14]. For example, $L(E)$ is regular supercoherent for every finite graph $E$ ([1, Lemma 6.4.16]). We write $L_n$ for the Leavitt path algebra of the one-vertex graph with $n$ loops.

**Theorem 1.3.** Let $E$ be finite graph such that $L(E)$ is simple and $R$ a purely infinite simple, regular supercoherent unital algebra. Then $[L(E), L_2]_1 = [L(E), L_2]_{M_2} \setminus \{0\}$, $[L(E), R \otimes L_2]_1 = [L(E), R \otimes L_2]_{M_2} \setminus \{0\}$, and both sets have exactly one element each.

In particular, Theorem 1.3 implies that if $d : L_2 \to L_2 \otimes L_2$, $d(x) = 1 \otimes x$ and $\phi : L_2 \to L_2 \otimes L_2$ is a nonzero homomorphism, then $\phi \approx_{M_2} d$ and that if $\phi$ is unital then $\phi \approx d$.

In [11] we introduced, for an algebra $A$ and a unital algebra $R$, an abelian monoid of homotopy classes of extensions of $A$ by $M_\infty R$, and considered its group completion $\mathcal{Ext}(A, R)$. We showed in [11, Remark 5.8] that if $E$ is a graph such that $E^0$ is finite and $E^1$ is countable, then for the bivariant algebraic $K$-theory group $kk_n(A, R)$ of [12], there is a natural map

$$\mathcal{Ext}(L(E), R) \to kk_{-1}(L(E), R).$$

Recall that a ring $R$ is $K_n$-regular if the canonical map $K_n(R) \to K_n(R[t_1, \ldots, t_m])$ is an isomorphism for every $m$. For example, every Leavitt path algebra is $K_n$-regular for all $n \in \mathbb{Z}$, by [11, Example 5.5].

**Theorem 1.5.** Let $E$ be a finite graph such that $L(E)$ is simple. Let $R$ be either a division algebra or a $K_0$-regular purely infinite simple unital algebra. Then the natural map (1.4) is an isomorphism

$$\mathcal{Ext}(L(E), R) \cong kk_{-1}(L(E), R).$$

Combining Theorem 1.5 with results from our previous paper [11] we are able to compute $\mathcal{Ext}(L(E), R)$ in some cases. For example, we get that if $E$ is as in Theorem 1.5, $\text{reg}(E) = E^0 \setminus \text{sink}(E)$ and $I$ and $A_E \in \mathbb{Z}^{\text{reg}(E) \times E^0}$ are the identity and the incidence matrices with the rows corresponding to sinks removed, then

$$\mathcal{Ext}(L(E), \ell) = \text{Coker}(I - A_E).$$

(1.6)
If moreover $K_0(L(E))$ is torsion, then for every $R$ as in Theorem 1.5 (in particular, for $R = \ell$ and for every purely infinite simple unital Leavitt path algebra $R$), we have

$$\text{Ext}(L(E), R) = \text{Ext}^1_{\mathbb{Z}}(K_0(L(E)), K_0(R)).$$  \hfill (1.7)

The following theorem is the main technical result of the paper; it is key for the proof of Theorems 1.1, 1.3 and 1.5.

**Theorem 1.8.** Let $E$ be a finite graph such that $L(E)$ is simple and $R$ a purely infinite simple unital algebra. Assume that $R$ is $K_1$-regular. Then the canonical map

$$[L(E), R]_{M_2} \setminus \{0\} \to kk(L(E), R)$$

is an isomorphism of monoids.

Thanks to Remark 5.11, we may view Theorem 1.8 as a generalization of the theorem of Ara, Goodearl and Pardo [5] which says that if $R$ is as in the theorem and $\mathcal{V}(R)$ is the monoid of Murray-von Neumann equivalence classes of idempotent matrices in $M_{\infty}R$, then $K_0(R) = \mathcal{V}(R) \setminus \{0\}$. In fact, the latter result is used in the proof of Theorem 1.8. The proof of Theorem 1.8 also uses results from our previous paper on $kk$ of Leavitt path algebras and an adaptation to the purely algebraic setting, developed in Sections 4 and 5, of several results proved for the $C^*$-algebra setting in Rørdam’s article [18].

The rest of this paper is organized as follows. In Section 2 we prove (Corollary 2.11) that if $R$ is a $K_1$-regular, purely infinite simple and unital algebra, then $K_1(R)$ is isomorphic to the group $\pi_0(U(R))$ of polynomially connected components of the group of invertible elements of $R$. The case of Theorem 1.8 when $L(E)$ is not purely infinite is contained in Proposition 2.18 (see Remark 2.19). Section 3 considers the problem of whether a given group homomorphism $K_0(L(E)) \to K_0(R)$ can be lifted to an algebra homomorphism. We show in Theorem 3.1 that if $R$ is purely infinite simple and unital and $E$ is countable, then any group homomorphism $\xi : K_0(L(E)) \to K_0(R)$ is induced by an algebra homomorphism $\psi : L(E) \to M_{\infty}R$, that if moreover $E^0$ is finite and $\xi$ is unital (i.e. $\xi([1_{L(E)}]) = [1_R]$) then $\xi$ is also induced by a unital homomorphism $\phi : L(E) \to R$, and that if $E$ is finite then any group homomorphism $\xi : K_0(L(E)) \to K_0(R)$ is induced by a nonzero algebra homomorphism $\phi : L(E) \to L(F)$. Theorem 1.2 is Corollary 3.2. If $E$ is a finite graph with reduced incidence matrix $A_E$ as above, we shall abuse notation and write $I - A'_E$ for the transpose of the matrix of (1.6). Section 4 is concerned with the question of whether, given a finite graph $E$ with reduced incidence matrix $A_E$, an algebra $R$ and a pair $(\xi_0, \xi_1)$ of group homomorphisms $\xi_0 : K_0(L(E)) \to K_0(R)$ and $\xi_1 : \text{Ker}(I - A'_E) \to K_1(R)$, there is an algebra homomorphism simultaneously inducing $\xi_0$ and $\xi_1$. We prove in Theorem 4.17 that if $L(E)$ is simple and $R$ is purely infinite, unital and $K_1$-regular, then there is an algebra homomorphism $\phi : L(E) \to R$ which induces both $\xi_0$ and $\xi_1$, and that if $\xi_0$ is unital, then $\phi$ can be chosen to be a unital homomorphism $L(E) \to R$. Section 5 is devoted to the proof of Theorem 5.8, which contains the case of Theorem 1.8 when $L(E)$ is purely infinite simple. Theorem 1.1 is proved in Section 6 (Theorem 6.1). Section 7 is devoted to algebra.
extensions. Theorem 1.5 is contained in Theorem 7.2; formulas (1.6) and (1.7) are proved in Corollary 7.3 and Example 7.5. Section 8 is concerned with maps to \( L_2 \) and \( R \otimes L_2 \); Theorem 1.3 is proved in Theorem 8.3.

Acknowledgements. A previous version of this article contained a proof of Proposition 2.3 for the particular case when \( R \) is a Leavitt path algebra. We are indebted to Pere Ara for pointing out that in fact the result holds for every unital purely infinite ring \( R \), and that the proof is immediate from results in [5].

2. Idempotents, Units and the Groups \( K_0 \) and \( K_1 \) in the purely infinite simple unital case

Let \( R \) be a ring; write \( \text{Idem}(R) \) for the set of idempotent elements. Let \( p,q \in \text{Idem}(R) \). We write \( p \sim q \) if \( p \) and \( q \) are Murray-von Neumann equivalent [5]; that is, if there exist elements \( x \in pRp \) and \( y \in qRq \) such that \( xy = p \) and \( yx = q \). We call such pair \( (x,y) \) an Mn equivalence from \( p \) to \( q \) and write \( (x,y) : p \sim q \).

Put \( \text{Idem}_n(R) = \text{Idem}(M_n(R)), \) \( 1 \leq n \leq \infty \). If \( R \) is unital, we write

\[
\mathcal{V}_n(R) = \text{Idem}_n(R)/\sim \quad (1 \leq n < \infty), \quad \mathcal{V}(R) = \text{Idem}_\infty(R)/\sim.
\]

Remark 2.1. One may also define \( \mathcal{V}(R) \) as the set of isomorphism classes of finitely generated projective right modules. The equivalence between the two definitions follows from [20, Theorem 1.2.3] and [8, Propositions 4.2.5 and 4.3.1]. One checks that if \( f : R \to S \) is a homomorphism and \( f(1) = p \), then under the identification, the map \( \mathcal{V}(R) \to \mathcal{V}(S) \) induced by \( M_\infty R \to M_\infty S \) corresponds to the scalar extension functor \( \otimes_R pS \).

If \( p,q \in \text{Idem}(R) \) and \( pq = qp = 0 \) we say that \( p \) and \( q \) are orthogonal and write \( p \perp q \) to indicate this. An idempotent \( p \) in a ring \( R \) is infinite if there exist orthogonal idempotents \( q,r \in R \) such that \( p = q + r \), \( p \sim q \) and \( r \neq 0 \). A ring \( R \) is said to be purely infinite simple if for every nonzero element \( x \in R \) there exist \( s,t \in R \) such that \( sxt \) is an infinite idempotent. If \( R \) is unital this is equivalent to asking that \( R \) not be a division ring and that for every \( x \in R \) there are \( a,b \in R \) such that \( axb = 1 \).

The following theorem describing \( K_0 \) and \( K_1 \) of purely infinite simple unital rings is due to Ara, Goodearl and Pardo. If \( R \) is a unital ring, write \( U(R) \) for the group of invertible elements of \( R \).

Theorem 2.2. [5, Corollary 2.3 and Theorem 2.4] If \( R \) is a purely infinite simple unital ring, then

\[
K_0(R) = \mathcal{V}(R) \setminus \{[0]\}
\]

\[
K_1(R) = U(R)^{ab}.
\]

Proposition 2.3. Let \( R \) be a purely infinite simple unital ring. Then the map \( \iota : \mathcal{V}_n(R) \to \mathcal{V}(R) \) is an isomorphism. Moreover, for every \( n \geq 1 \) and every element \( (q_1,\ldots,q_n) \in \text{Idem}_\infty(R)^n \) there exists \( (p_1,\ldots,p_n) \in \text{Idem}_1(R)^n \), such that \( p_i \sim q_i \) in \( \text{Idem}_\infty(R) \) and such that \( p_i \perp p_j \) for \( i \neq j \).

Proof. This is straightforward from [5, Proposition 1.5 and Lemma 1.1] \( \square \)
Combining Proposition 2.3 and Theorem 2.2 we obtain the following.

**Corollary 2.4.** Let $R$ be a purely infinite simple unital ring. Then

$$K_0(R) \cong \mathcal{V}_1(R)\langle[[0]]\rangle.$$ 

**Corollary 2.5.** Let $R$ be a purely infinite simple unital ring and let $e, f \in R$ be nonzero idempotents. Then the following are equivalent

1. $e \sim f$.
2. $[e] = [f]$ in $K_0(R)$.

If furthermore $e, f \in \text{Idem}_1(R)\{0, 1\}$ then the above conditions are also equivalent to the following:

3. There exists $u \in U(R)$ such that $f = ueu^{-1}$.
4. There exists a commutator $u \in [U(R) : U(R)]$ such that $f = ueu^{-1}$.

**Proof.** The equivalence of (1) and (2) follows from Corollary 2.4. By [8, Proposition 4.2.5], (3) is equivalent to having simultaneously $e \sim f$ and $1 - e \sim 1 - f$. Hence to prove that (1) implies (3) it only remains to show that $1 - e \sim 1 - f$. But

$$[e] + [1 - e] = [1] = [f] + [1 - f]$$

in $K_0(R)$ and $[e] = [f]$, implies $[1 - e] = [1 - f]$ in $K_0(R)$ and therefore in $\mathcal{V}_1(R)$. Hence $1 - e \sim 1 - f$. Next we show that (3) implies (4). Because $R$ is simple and $f \neq 1$, $1 - f$ is a full idempotent. Hence $(1 - f)L(E)(1 - f)$ is purely infinite simple (by [5, Corollary 1.7]) and the inclusion induces an isomorphism $K_1((1 - f)R(1 - f)) \rightarrow K_1(R)$. By Theorem 2.2, this implies that the induced map $U((1 - f)R(1 - f))^{ab} \rightarrow U(R)^{ab}$ is an isomorphism. Since the latter map sends $[\xi] \mapsto [\xi + f]$, there is an element $\omega \in U((1 - f)R(1 - f))$ such that $[\omega + f] = [u^{-1}]$. Then $(\omega + f)u \in [U(R) : U(R)]$ and $(\omega + f)ueu^{-1}(\omega^{-1} + f) = f$. To prove that (4) implies (1) take $x = eu^{-1}f$ and $y = fue$; we have $xy = e$ and $yx = f$. \qed

Let $G : \text{Alg}_\ell \rightarrow \mathfrak{Grp}$ be a functor from algebras to groups and let $A \in \text{Alg}_\ell$. The **connected component** of $G(A)$ is the subgroup

$$G(A) \supset G(A)^0 = \{g \mid (\exists u(t) \in G(A[t])) u(0) = 1, u(1) = g\}.$$ 

Observe that $G(A)^0$ is a normal subgroup. We write

$$\pi_0 G(A) = G(A)/G(A)^0.$$ 

The **Karoubi-Villamayor $K_1$-group** ([16]) is

$$KV_1(A) = \pi_0(GL(A)).$$ 

Observe that every elementary matrix is in $GL(A)^0$. It follows that we have a surjective homomorphism

$$K_1(A) \twoheadrightarrow KV_1(A).$$ \hspace{1cm} (2.6)

By [23, Proposition 1.5], the map (2.6) is an isomorphism whenever $A$ is $K_1$-regular.

**Lemma 2.7.** Let $R$ be a unital ring.

i) If $p \in \text{Idem}(R)$ and $u \in U(pRp)^0$, then $u + 1 - p \in U(R)^0$. 

ii) Let \( x_1, \ldots, x_n, y_1, \ldots, y_n \in R \) such that \( y_jx_jy_i = \delta_{i,j}y_i, x_iy_jx_i = \delta_{i,j}x_i \). Set \( p_i = x_iy_i, q_i = y_ix_i, P = \bigoplus_{i=1}^n p_i R, Q = \bigoplus_{i=1}^n q_i R \). Then the map

\[
c_{y,x} := \text{End}_R(P) = \bigoplus_{i,j} p_j R p_i \rightarrow \bigoplus_{i,j} q_j R q_i = \text{End}_R(Q),
\]

\[a \mapsto \text{diag}(y_1, \ldots, y_n) a \text{diag}(x_1, \ldots, x_n)\]
is an isomorphism which sends \( U(\text{End}_R(P))^0 \) isomorphically onto \( U(\text{End}_R(Q))^0 \).

**Proof.** Straightforward. \( \square \)

**Proposition 2.8.** Let \( R \) be a unital purely infinite simple ring. Then the canonical map \( \pi_0(U(R)) \rightarrow \pi_0(\text{GL}(R)) = KV_1(R) \) is an isomorphism.

**Proof.** We know from Theorem 2.2 and (2.6) that \( U(R) \rightarrow KV_1(R) \) is surjective. The kernel of this map is \( U(R) \cap \text{GL}(R)^0 \); it is clear that it contains \( U(R)^0 \). We have to show that

\[
U(R) \cap \text{GL}(R)^0 \subset U(R)^0. \tag{2.9}
\]

We claim that the argument of the proof that \([\text{GL}(R) : \text{GL}(R)] \cap U(R) \subset [U(R) : U(R)] \) in [5, Theorem 2.3] can be adapted to prove (2.9). The proof in loc.cit. has two parts. The first part shows that if \( 0 \neq p \in \text{Idem}(R) \) and \( u \in [\text{GL}(R) : \text{GL}(R)] \cap U(R) \) satisfies

\[
u = p + (1 - p)u(1 - p) \tag{2.10}
\]

\( u \in [U(R) : U(R)] \). Using the same argument and taking Lemma 2.7 into account, one shows that if (2.10) is in \( \text{GL}(R)^0 \), then it must be in \( U(R)^0 \). In the second part of the proof of [5, Theorem 2.3] it is observed that for adequately chosen idempotents \( e \) and \( f \in T = eRe \) and elements \( x_1, y_1, \ldots, x_n, y_n \in R \), the assignment \( a \mapsto \text{diag}(y_1, \ldots, y_n) a \text{diag}(x_1, \ldots, x_n) \) induces an isomorphism between \( R \) and the subring

\[M_n(T) \supseteq S = \{(a_{i,j}) : a_{i,i} \in T f, a_{n,j} \in f T \text{ for all } 1 \leq i \leq n\}.
\]

Let \( \mathcal{E} \subset U(R) \) be the image under the isomorphism \( U(S) \sim \rightarrow U(R) \) of the subgroup generated by the set of those elementary matrices \( 1 + a e_{i,j} i \neq j \) which are elements of \( S \). The authors then proceed, using the argument of the proof of [17, Theorem 2.2], to show that any \( u \in U(R) \) is congruent modulo \( \mathcal{E} \) to one of the form of (2.10). In view of Lemma 2.7 and of the fact that elementary matrices above are in \( U(S)^0 \), this shows that any \( u \in U(R) \) is congruent modulo \( U(R)^0 \) to one of the form (2.10). This finishes the proof. \( \square \)

**Corollary 2.11.** If \( R \) is unital, purely infinite simple and \( K_1 \)-regular then \( K_1(R) = \pi_0(U(R)) \).

Let \( A \) be an algebra. Identify \( \text{Hom}_{\text{Alg}}(\ell, A) = \text{Idem}_1(A) \) via the bijection \( \phi \mapsto \phi(1) \). We say that two idempotents \( p, q \in \text{Idem}_1(A) \) are homotopic, and write \( p \approx q \), if the corresponding homomorphisms \( \ell \rightarrow A \) are homotopic.

**Lemma 2.12.** Let \( A \) be an algebra and \( p \in \text{Idem}_1(A) \). Then \( p \approx 0 \) if and only if \( p = 0 \). If \( A \) is unital, then \( p \approx 1 \) if and only if \( p = 1 \).
Proof. The if part of both assertions is clear. One checks that if \( x \in \{0, 1\} \) and \( p(t) \in \text{Idem}_1(A[t]) \) satisfies \( p(0) = x \), then \( p = x \). The only if part of both assertions follows from this. \( \square \)

Recall (see [11, Section 2]) that a \( C_2 \)-algebra is a unital algebra \( R \) together with a unital homomorphism from the Cohn algebra \( C_2 \) to \( R \). Thus a \( C_2 \)-algebra is a unital algebra together with elements \( x_1, x_2, y_1, y_2 \in R \) such that \( y_1 x_j = \delta_{i,j} \). For example, if \( R \) is a purely infinite simple unital algebra then \( R \) is a \( C_2 \)-algebra (see [5, Proposition 1.5]). Put

\[
\boxplus : R \oplus R \to R, \quad a \boxplus b = x_1 ay_1 + x_2 by_2.
\]

(2.13)

Lemma 2.14. Let \( R_1 \) and \( R_2 \) be \( C_2 \)-algebras and let \( A_1 \triangleleft R_1 \) and \( A_2 \triangleleft R_2 \) ideals. Let \( \boxplus_i : A_i \oplus A_i \to A_i \) be the sum operation (2.13). Then the maps

\[
\boxplus_1 \otimes \text{id}_{A_2}, \text{id}_{A_1} \otimes \boxplus_2 : A_1 \otimes A_2 \oplus A_1 \otimes A_2 \to A_1 \otimes A_2
\]

are \( M_2 \)-homotopic.

Proof. Straightforward from [11, Lemma 2.3]. \( \square \)

Let \( C \) be an algebra, \( A, B \subset C \) subalgebras and \( x, y \in C \) satisfying \( xAy \subset B \) and \( ayx' = a' (a, a' \in A) \); then the following map is an algebra homomorphism

\[
ad(x, y) : A \to B, \quad \text{ad}(x, y)(a) = xay.
\]

(2.15)

If \( C \) is unital and \( y = x^{-1} \), then \( \text{ad}(x, y) = \text{ad}(x) \) is the usual conjugation map.

Lemma 2.16. Let \( A \) and \( R \) be algebras, with \( A \) finitely generated. Then:

i) The canonical map

\[
[A, M_\infty R] \to [A, M_\infty R]_{M_2}
\]

is bijective.

ii) If furthermore \( R \) is a \( C_2 \)-algebra then the canonical map

\[
[A, R]_{M_2} \to [A, M_\infty R]_{M_2}
\]

is an isomorphism of monoids.

Proof.

i) Because \( A \) is finitely generated,

\[
[A, M_\infty R] = \colim_n [A, M_2^n R] = \colim_n [A, M_2^n R]_{M_2} = [A, M_\infty R]_{M_2}.
\]

ii) Because \( R \) is a \( C_2 \)-algebra, the map \([A, R]_{M_2} \to [A, M_\infty R]_{M_2}\) is a monoid homomorphism by Lemma 2.14. We have to prove that it is bijective. Observe that \( M_2 R \) is again a \( C_2 \)-algebra. Hence in view of the proof of part i), it suffices to show that \([A, R]_{M_2} \to [A, M_2 R]_{M_2}\) is bijective. Let \( x = \varepsilon_{1,1} \otimes x_1 + \varepsilon_{1,2} \otimes x_2 \) and \( y = \varepsilon_{1,1} \otimes y_1 + \varepsilon_{2,1} \otimes y_2 \). By [11, Lemma 2.3], the following diagram is \( M_2 \)-homotopy commutative

\[
\begin{array}{ccc}
M_2 R & \xrightarrow{\text{ad}(x, y)} & M_2 (R) \\
\downarrow \text{inc} & & \downarrow \text{inc} \\
M_2 R & & \\
\end{array}
\]
It follows that the map of ii) is surjective. Injectivity follows similarly. □

**Lemma 2.17.** Let \( \phi, \psi : A \rightarrow R \) be algebra homomorphisms with \( R \) unital. Assume that there are \( n \geq 1 \) and \( u \in \text{GL}_n(R) \) such that \( \text{ad}(u)\iota_n \phi = \iota_n \psi \). Then there are elements \( x, y \in R \) such that \( \text{ad}(x,y) \phi = \psi \). If moreover \( A, \phi \) and \( \psi \) are unital, then we may choose \( x \) invertible and \( y = x^{-1} \).

**Proof.** Put \( v = u^{-1} \). It follows from the identity \( \text{ad}(u) \iota_n \phi = \iota_n \psi \) that for every \( a \in A \), \( u_{i1} \phi(a)v_{11} = \psi(a) \) and \( u_{i1} \phi(a) = \phi(a)u_{11} = 0 \) if \( i \neq 1 \). Hence \( x = u_{11} \) and \( y = v_{11} \) satisfy \( \text{ad}(x,y) \phi = \psi \) and if \( \phi \) and \( \psi \) are unital, then \( xy = yx = 1 \). □

**Proposition 2.18.** Let \( R \) be a unital, purely infinite simple, \( K_0 \)-regular algebra and \( n \geq 1 \). Then the natural monoid maps

\[
[M_n, R]_{M_2} \rightarrow [M_n, M_{\infty}R] \setminus \{0\} \rightarrow kk(M_n, R) \cong kk(\ell, R) \cong K_0(R)
\]

are bijective. Moreover, for nonzero algebra homomorphisms \( M_n \rightarrow M_{\infty}R \) as well as for unital algebra homomorphisms \( M_n \rightarrow R \), being homotopic is the same as being conjugate.

**Proof.** Because as explained above, any purely infinite simple unital algebra is a \( C_2 \)-algebra, the map \( [M_n, R]_{M_2} \rightarrow [M_n, M_{\infty}R] \) is an isomorphism of monoids by Lemma 2.16. Since \( (\iota_n)^{\ast} : kk(M_n, R) \rightarrow kk(\ell, R) = K_0R \) is an isomorphism, to prove that the map \( [M_n, M_{\infty}R] \setminus \{0\} \rightarrow kk(M_n, R) \) is surjective, it suffices, by Corollary 2.4, to show that the image of its composite with \( \iota_n \) contains the class of every nonzero idempotent in \( R \). Let \( p \in \text{Idem}_1 R \setminus \{0\} \); by Proposition 2.3 we may choose \( q \in \text{Idem}_1 R \), \( q \sim p \), and an embedding \( \theta : M_n \rightarrow R \) sending \( \epsilon_{1,1} \rightarrow q \). Hence the map of the proposition is surjective. If two homomorphisms \( \phi, \psi \in \text{Hom}_{\text{Alg}}(M_n, M_{\infty}R) \) induce the same \( K_0 \)-element then they are conjugate by the argument of the proof of [15, Lemma 15.23(b)], and therefore homotopic by Lemma 2.16 and [11, Lemma 2.3]. From what we have just proved and Lemma 2.17, it follows that if two unital homomorphisms \( M_n \rightarrow R \) are homotopic then they are conjugate. This finishes the proof. □

**Remark 2.19.** Let \( E \) be a finite graph such that \( L(E) \) is simple. If \( L(E) \) is not purely infinite, then it follows from [1, Lemma 2.9.5] and source elimination [1, Definition 6.3.26] that \( L(E) \cong M_n \) for some \( 1 \leq n < \infty \). Hence, since \( K_n \)-regularity implies \( K_{n-1} \)-regularity [21], Proposition 2.18 implies Theorem 1.8 in the case when \( L(E) \) is simple and not pure infinite.

3. **Lifting \( K \)-theory maps to algebra maps: \( K_0 \)**

Recall that a vertex \( v \in E^0 \) is **singular** if it is either a sink or an infinite emitter, and that it is **regular** otherwise. We write \( \text{reg}(E), \text{sink}(E), \text{sour}(E) \) and \( \text{inf}(E) \) for the sets of regular vertices, sinks, sources, and infinite emitters, and put \( \text{sing}(E) = \text{sink}(E) \cup \text{inf}(E) \).

Let \( R \) and \( S \) be unital algebras and \( \xi : K_0(R) \rightarrow K_0(S) \). We call \( \xi \) unital if \( \xi([1_R]) = [1_S] \).
Theorem 3.1. Let $E$ be a graph, $R$ a purely infinite simple unital algebra, and $\xi : K_0(L(E)) \to K_0(R)$ a group homomorphism. Set $\iota : R \to M_{\infty}(R), \iota(a) = \epsilon_{1,1} \otimes a$.

i) If $E$ is countable, then there exists a nonzero algebra homomorphism $\psi : L(E) \to M_{\infty}(R)$ such that $K_0(\psi) = K_0(\iota)\xi$.

ii) If $E$ is finite, then there exists a nonzero algebra homomorphism $\psi : L(E) \to R$ such that $K_0(\psi) = \xi$.

iii) If $E^0$ is finite, $E^1$ countable and $\xi$ unital, then there is a unital homomorphism $\phi : L(E) \to R$ such that $K_0(\phi) = \xi$.

Proof. Assume first that $E$ is countable and row-finite. By Theorem 2.2 there are orthogonal idempotents $\{p_e : e \in E^1 \} \cup \{p_v : v \in \text{sing}(E)\} \subset \text{Idem}_{\infty}(R) \setminus \{0\}$ such that $[p_e] = \xi[v]$ and $[p_v] = \xi[e\epsilon^*]$ in $K_0(R)$ ($v \in \text{sink}(E)$, $e \in E^1$). If $e \in E^1$ and $r(e) \in \text{reg}(E)$ then

$$[p_e] = \sum_{f \in E^1, s(f) = r(e)} p_f.$$ 

Hence for $\sigma_f = \sum_{f \in E^1, s(f) = r(e)} p_f$ there are elements $x_e, y_e \in M_{\infty}(R)$ implementing an MvN equivalence $p_e \sim \sigma_e$. Similarly if $e \in E^1$ and $r(e) = v \in \text{sink}(E)$, then there is an MvN equivalence $(x_e, y_v) : p_e \sim p_v$ with $x_e, y_e \in M_{\infty}(R)$. One checks that the prescriptions

$$\psi(e) = x_e, \psi(e^*) = y_e \quad (e \in E^1), \quad \psi(v) = p_v \quad (v \in \text{sink}(E))$$

define a nonzero algebra homomorphism $\psi : L(E) \to M_{\infty}(R)$. Let $\tau : M_{\infty}M_{\infty} \to M_{\infty}M_{\infty}, \tau(x \otimes y) = y \otimes x$; one checks that $\tau \otimes \text{Id}_{R}$ induces the identity of $K_0(M_{\infty}(R))$. By construction $K_0(\psi)$ agrees with $K_0(\tau \otimes 1)K_0(\iota)\xi = K_0(\iota)\xi$ on the classes of those vertices which are sinks and on those of elements of the form $ee^*$ ($e \in E^1$). Since the latter generate $K_0(L(E))$ (by [1, Theorem 3.2.5]), we have $K_0(\psi) = K_0(\xi)$.

For general countable $E$, let $E_\circ$ be a desingularization and $f : L(E) \to L(E_\circ)$ the canonical homomorphism [2, Section 5]; then $K_0(f)$ is an isomorphism. Hence by what we have just proved, there exists an algebra homomorphism $\psi' : L(E_\circ) \to M_{\infty}(R)$ such that $K_0(\psi') = K_0(\iota)\xi K_0(f)^{-1}$. Then $\psi = \psi' \circ f$ satisfies $K_0(\psi) = K_0(\iota)\xi$. This proves i). Next assume that $E^1$ is countable, that $E^0$ is finite and that $\xi[1_{L(E^1)}] = [1_R]$. Let $\psi : L(E) \to M_{\infty}(R)$ be a homomorphism such that $K_0(\iota)\xi = K_0(\psi)$. Set $p = \psi(1)$; then $\psi(L(E)) \subset pM_{\infty}(R)p$ and there is an MvN equivalence $(x, y) : p \sim \epsilon_{1,1}$. It follows that there is a unique unital homomorphism $\phi : L(E) \to R$ such that $\phi \phi = \text{ad}(y, x)\psi$. By [11, Lemma 2.3], $\phi$ satisfies the requirements of iii). Finally assume that $E$ is finite. By Corollary 2.4 and Proposition 2.3 there are orthogonal idempotents $\{p_e : e \in E^1\} \cup \{p_v : v \in \text{sink}(E)\} \subset \text{Idem}_1(R) \setminus \{0\}$ such that $[p_e] = \xi[v]$ and $[p_v] = \xi[e\epsilon^*] \quad (v \in \text{sink}(E), e \in E^1)$. If $e \in E^1$ and $r(e) \notin \text{sink}(E)$ then by Corollary 2.4, for $\sigma_e$, as in the proof of Theorem 3.1 there are elements $x_e \in p_eR\sigma_e$ and $y_e \in \sigma_eRp_e$ such that $p_e = x_ey_e$ and $\sigma_e = y_ex_e$. Similarly, if $e \in E^1$ and $r(e) = v \in \text{sink}(E)$, then there are $x_e \in p_eRp_v$ and $y_e \in p_vRp_e$ such that $y_ex_e = p_v$ and $x_ey_e = p_v$. One checks that the prescriptions

$$\psi(e) = x_e, \psi(e^*) = y_e \quad (e \in E^1), \quad \psi(v) = p_v \quad (v \in \text{sink}(E))$$

define a nonzero algebra homomorphism $\psi : L(E) \to R$ such that $K_0(\psi) = \xi$. □
Corollary 3.2. Let $R$ be a unital purely infinite algebra and $E$ a graph such that $L(E)$ is simple.

i) If $E$ is countable, then $L(E)$ embeds as a subalgebra of $M_{\infty}R$.

ii) If $E^1$ is countable, $E^0$ is finite and $\{1_R\} = 0$ in $K_0(R)$, then $L(E)$ embeds as a unital subalgebra of $R$.

iii) If $E$ is finite then $L(E)$ embeds as a subalgebra of $R$.

Proof. Apply Theorem 3.1 to the trivial homomorphism $\xi = 0$. □

Remark 3.3. It follows from Corollary 3.2 that any purely infinite algebra $R$ such that $\{1_R\} = 0$ contains $L_2$ as a unital subalgebra. Hence by [9, Theorem 4.1], if $E$ is countable (resp. finite), then $L(E)$ embeds as a subalgebra (resp. unital subalgebra) of $R$, independently of whether $L(E)$ is simple or not.

Corollary 3.4. Let $E$ be a countable graph with finite $E^0$. Assume that $K_0(L(E))$ is finite and let $d_1, \ldots, d_n, d_1^j, d_{i+1}$ be its invariant factors. Let $j : \text{Alg}_\ell \to kk$ be canonical functor (([12])). Then there is an algebra homomorphism $\psi : L(E) \to M_{\infty}(\bigoplus_{j=1}^n L_{d_{i+1}})$ such that $j(\psi)$ is an isomorphism in kk. If moreover $L(E)$ is purely infinite simple then there is an algebra homomorphism $\phi : \bigoplus_{j=1}^n L_{d_{i+1}} \to M_{\infty}L(E)$ such that $\psi^{-1}j(\phi)$ and $\phi^{-1}j(\psi)$ are inverse isomorphisms in kk. If $E$ is finite then the same holds with $L(E)$ substituted for $M_{\infty}(L(E))$.

Proof. Assume that $E$ is countable with finite $E^0$. By part ii) of Theorem 3.1, for each $1 \leq i \leq n$, there is a homomorphism $\psi_i : L(E) \to M_{\infty}L_{d_{i+1}}$ such that $K_0(\psi_i)$ is the projection from $K_0(L(E)) = \bigoplus_{j=1}^n \mathbb{Z}/d_j$ onto the copy of $\mathbb{Z}/d_j$. The map

$$\psi = (\psi_1, \ldots, \psi_n) : L(E) \to M_{\infty}(\bigoplus_{j=1}^n L_{d_{i+1}})$$

then induces an isomorphism in $K_0$. In view of [11, Lemma 7.2] and of the fact that, since $K_0(L(E))$ is finite, $\text{Ker}(I - A^\ell_{\xi}) = 0$, this implies that $K_1(\psi)$ is an isomorphism too. Hence $j(\psi)$ is an isomorphism by [11, Proposition 5.10]. Assume furthermore that $L(E)$ is purely infinite simple. Consider the graph

$$F = \bigsqcup_{j=1}^n R_{d_{i+1}}.$$

Then $L(F) = \bigoplus_{j=1}^n L_{d_{i+1}}$. The homomorphism $\phi$ of the corollary is obtained by applying Theorem 3.1 to $\xi = K_0(\psi)^{-1} : K_0(L(F)) \to K_0(L(E))$. This proves the first assertion of the corollary; the second, for finite $E$, is proved similarly, using part iii) of Theorem 3.1. □

Let $E$ be a finite graph; if $X \subset L(E)$, write span$(X)$ for the subspace generated by $X$. In the following proposition and elsewhere we consider the following “diagonal” subalgebra of $L(E)$

$$DL(E) = \text{span}(\text{sink}(E) \cup \{ee^* : e \in E^1\}) \subset L(E).$$

Proposition 3.5 below will be needed in the next section.
Proposition 3.5. Let $E$ and $R$ be as in part iii) of Theorem 3.1. Assume that $L(E)$ is simple and let $\phi, \psi : L(E) \to R$ be nonzero algebra homomorphisms such that $K_0(\phi) = K_0(\psi)$. Then there exists an algebra homomorphism $\psi' : L(E) \to R$ such that $j(\psi) = j(\psi')$ in $kk$ and $\psi'|_{DL(E)} = \phi|_{DL(E)}$.

Proof. First assume that $\phi(1) = \psi(1) = p$. For each $e \in E^1$ and each $v \in \sink(E)$ choose MvN equivalences $(x_v, y_v) : \phi(ee^*) \sim \psi(ee^*)$ and $(x_v, y_v) : \phi(v) \sim \psi(v)$. Define $x = \sum_{e \in E^1} x_e + \sum_{v \in \sink(E)} x_v$ and $y = \sum_{e \in E^1} y_e + \sum_{v \in \sink(E)} y_v$. Then $x, y \in pRp$ and $xy = p = yx$. Hence $\psi' : L(E) \to R, \psi'(a) = x\psi(a)y$ satisfies $\psi'|_{DL(E)} = \phi|_{DL(E)}$. Moreover $j(\psi) = j(\psi')$ by [11, Lemma 2.3]. Next assume that $\phi(1) \neq \psi(1)$ and that none of them is equal to 1. Then by Corollary 2.5, there is an element $u \in U(R)$ such that $u\phi(1)u^{-1} = \psi(1)$. Hence we can replace $\psi$ by $a \mapsto u\psi(a)u^{-1}$ and we are in the above case. Finally, if $\phi(1) \neq \psi(1)$ and one of them, say $\psi(1)$, is 1, we can replace $\phi$ by a unital homomorphism by Theorem 3.1 and we are again in the first case. $\square$

4. Lifting $K$-theory maps to algebra maps: $K_0$ and $K_1$

Let $E$ be a finite graph; below we will give a right inverse of the surjective map

$$\partial : K_1(L(E)) \to \Ker(I - A^t_E).$$ (4.1)

Observe that the analogue of the map (4.1) in the $C^*$-algebra setting is an isomorphism; an explicit formula for its inverse was given by Rørdam in [18, page 33] in the case when $E$ is regular. We shall show that in the purely algebraic case considered here, the same formula gives a right inverse of (4.1), even for singular $E$.

Let $I - B^t_E$ be as in [11, Remark 5.7]. Let

$$s^* : \mathbb{Z}E^0 \to \mathbb{Z}(E^1 \setminus \sink(E)), s^*(x_v) = \begin{cases} \sum (s(e) = v) X_e & v \in \reg(E) \\ X_v & v \in \sink(E) \end{cases}$$

By [4, formula 4.1], we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}E^0 & \xrightarrow{s^*} & \mathbb{Z}(E^1 \setminus \sink(E)) \\
\downarrow s & & \downarrow s^* \\
\mathbb{Z}\reg(E) & \xrightarrow{I - B^t_E} & \mathbb{Z}E^0
\end{array}$$

In particular, $s^*$ maps $\Ker(I - A^t_E) \to \Ker(I - B^t_E)$. Furthermore it is an isomorphism by the dual of [4, Lemma 4.3]. Let $x = (x_v) \in \Ker(I - A^t_E) \subseteq \mathbb{Z}\reg(E)$. Set $y = s^*(x) \in \Ker(I - B^t_E)$. Let

$$S = \{(e, j) : y_e \neq 0, 1 \leq j \leq |y_e|\}$$ (4.2)

Consider the diagonal matrix $V = V(x) \in M_3(L(E))$,

$$V(e, j), (e, j) = \begin{cases} e & \text{if } y_e > 0 \\ e^* & \text{if } y_e < 0 \end{cases}$$
Let \( p = 1 - VV^* \), \( q = 1 - V^*V \). Observe that \( p, q \in M_S(DL(E)) \). Moreover, for \( \Lambda = E^1 \bigcup \text{sink}(E) \), \( DL(E) \cong \ell^\Lambda \) and we may regard \( p = (p_\alpha) \) and \( q = (q_\alpha) \) as \( \Lambda \)-tuples of diagonal matrices in \( M_S \) whose entries are in \([0, 1] \). One checks, using that \( y \in \text{Ker}(I - B_E') \), that for each \( \alpha \in \Lambda \), \( p_\alpha \) and \( q_\alpha \) have the same number of nonzero coefficients. Hence we may choose for each \( \alpha \) a matrix \( W_\alpha \in p_\alpha M_S q_\alpha \) with coefficients in \([0, 1] \) such that \( W_\alpha W_\alpha^* = p_\alpha \) and \( W_\alpha^* W_\alpha = q_\alpha \). Further, we may even require that
\[
(W_\alpha)(e,i)(f,j) = 1 \Rightarrow (p_\alpha)(e,i)(e,i) = (q_\alpha)(f,j)(f,j) = 1. \tag{4.3}
\]
We shall use (4.3) in the proof of Lemma 4.12 below. Let \( W = W(x) \in M_S(DL(E)) \) be the matrix corresponding to \((W_\alpha); \) then
\[
WW^* = 1 - VV^*, \quad W^*W = 1 - V^*V \quad \text{and} \quad W^*V = V^*W = 0. \tag{4.4}
\]
Put
\[
U(x) = V(x) + W(x). \tag{4.5}
\]
It follows from (4.4) that \( U(x)U(x)^* = U(x)^*U(x) = 1. \)

**Proposition 4.6.** Let \( x \in \text{Ker}(I - A_E') \), \([U(x)] \in K_1(L(E)) \) the class of the element (4.5) and \( \hat{\partial} : K_1(L(E)) \to \text{Ker}(I - A_E') \) as in (4.1). Then \( \hat{\partial}(U(x)) = -x. \)

**Proof.** We keep the notation of the paragraph preceding the proposition. Let \( C(E) \) be the Cohn path algebra; consider the subalgebra
\[
DC(E) = \text{span}([q_v : v \in \text{reg}(E)]) \cup \text{sink}(E) \cup \{ee^* : e \in E^1 \} \subset C(E).
\]
Consider the diagonal matrix \( \hat{V} \) defined by the same prescription as \( V \) but regarded now as an element of \( M_S(C(E)) \). Let \( \hat{W} \in M_S(DC(E)) \) be the image of \( W \) under the map induced by the obvious inclusion \( DL(E) \subset DC(E) \); put \( \hat{U} = \hat{V} + \hat{W} \). Consider the matrix
\[
h = \begin{bmatrix}
2\hat{U} - \hat{U}^*\hat{U} & \hat{U}\hat{U}^* - 1 \\
1 - \hat{U}^*\hat{U} & \hat{U}^*
\end{bmatrix} \in M_{S \times S}(C(E)).
\]
By [10, Section 2.4] (see also [19, Definition 9.1.3]), \( h \) is invertible and
\[
\hat{\partial}([U]) = [h_1h^{-1}] - [1_S].
\]
Here \( 1_S \) is the \( S \times S \) identity matrix, located in the upper left corner.

One checks that \( \hat{U} = \hat{U}^*\hat{U} \), and that
\[
\hat{\partial}([U]) = [1 - \hat{U}^*(x_i)\hat{U}] - [1 - \hat{U}\hat{U}^*] \in K_0(\bigoplus_{v \in \text{reg}(E)} \ell q_v) \cong \ell^{\text{reg}(E)}. \tag{4.7}
\]

One checks, using (4.7) and the fact that \( x \in \text{Ker}(I - A_E') \), that
\[
\hat{\partial}([U]) = -\sum_{v \in E^0} x_v q_v.
\]
This finishes the proof. \( \square \)
In principle, the assignment \( \text{Ker}(I - A_E^I) \to K_1(L(E)), [x] \mapsto [U(x)] \) is just a set theoretic map. A group homomorphism with similar properties is obtained as follows. Choose a basis \( \mathcal{B} = \{x_i\} \) of the free abelian group \( \text{Ker}(I - A_E^I) \); let

\[
\gamma = \gamma_{\mathcal{B}} : \text{Ker}(I - A_E^I) \to K_1(L(E)), \quad \gamma(\sum_i n_i x_i) = \sum_i n_i [U(x_i)]. \tag{4.8}
\]

Let \( E \) be a finite graph such that \( L(E) \) is purely infinite simple. Then \( \text{sink}(E) = \emptyset \), by [1, Lemma 3.1.10 and Theorem 3.1.10]. Let \( \phi : L(E) \to R \) be a unital algebra homomorphism with \( R \) purely infinite simple. Set

\[
R_\phi = \{x \in R : \phi(e^*)x = x\phi(e^*), \quad \text{for all } e \in E^1\}. \tag{4.9}
\]

Note that \( R_\phi = \bigoplus_{e \in E} \phi(e^*)R\phi(e^*) \).

Because \( L(E) \) is simple, \( \phi(\alpha) \neq 0 (\alpha \in E^1) \), whence each of the inclusions \( \phi(\alpha^*)R\phi(\alpha^*) \subset R \) induces an isomorphism in \( K_1 \). Hence the direct sum \( R_\phi \subset R_{E^1} \) of those inclusions induces an isomorphism

\[
K_1(R_\phi) = \bigoplus_{e \in E^1} K_1(\phi(e^*)R\phi(e^*)) \xrightarrow{\sim} K_1(R)_{E^1}. \tag{4.10}
\]

Let \( \iota : K_1(R_\phi) \to K_1(R) \) be the map induced by the inclusion \( R_\phi \subset R \). Consider the bilinear map

\[
\langle \cdot, \cdot \rangle : \mathbb{Z}^{E^1} \times K_1(R_\phi) \to K_1(R), \quad \langle x, y \rangle = \sum_i x_i \iota(y_i). \tag{4.11}
\]

Observe that \( \langle \cdot, \cdot \rangle \) is a perfect pairing; indeed the adjoint homomorphism \( K_1(R_\phi) \to K_1(R)_{E^1} \) is the isomorphism (4.10).

**Lemma 4.12.** (cf.[18, Lemma 3.5]) Let \( E \) be a finite graph such that \( L(E) \) is purely infinite simple, \( R \) a purely infinite simple unital algebra, and \( \phi \) and \( \psi : L(E) \to R \) unital homomorphisms. Assume that \( \phi \) and \( \psi \) agree on DL(E). Let

\[
u = \sum_{\alpha \in E^1} \psi(\alpha)\phi(\alpha^*) \in R_\phi = R_\psi.
\]

Then

\[
K_1(\psi)(\gamma(x)) = \langle x, [u] \rangle + K_1(\phi)(\gamma(x)) \text{ for all } x \in \text{Ker}(I - A_E^I). \tag{4.13}
\]

**Proof.** Observe that \( \psi(e)\phi(e^*) \in U(\phi(e)R\phi(e^*)) (e \in E^1) \), whence \( u \in U(R_\phi) \). Let \( \{\chi_e : e \in I\} \) be the canonical basis of \( \mathbb{Z}^I \). One checks that

\[
\langle \chi_e, [u] \rangle = \psi(e)\phi(e^*) + 1 - \phi(ee^*). \tag{4.14}
\]

To prove the lemma, we may assume that \( x \) is an element of the basis \( \mathcal{B} \) of \( \text{Ker}(I - A_E^I) \) used in (4.8) to define \( \gamma \). Then taking (4.14) into account and adopting the notations and conventions used in the definition of \( U(x) \), one computes that the right hand side of equation (4.13) is

\[
\sum_{y_e \geq 0} y_e [\psi(e)\phi(e^*) + 1 - \phi(ee^*)] + [\phi(U(x))] - \sum_{y_e < 0} y_e [\psi(e)\phi(e^*) + 1 - \phi(ee^*)]. \tag{4.15}
\]
Let $S$ be as in (4.2). Consider the diagonal matrices $P, Q \in M_3 L(E)$ with diagonal entries as follows

$$
P_{(e, i), (e, j)} = \begin{cases} 
\phi(e)\phi(e^*) + 1 - \phi(ee^*) & \text{if } y_e > 0 \\
1 & \text{if } y_e < 0 
\end{cases}$$

$$
Q_{(e, i), (e, j)} = \begin{cases} 
1 & \text{if } y_e > 0 \\
\phi(e)\phi(e^*) + 1 - \phi(ee^*) & \text{if } y_e < 0
\end{cases}
$$

Observe that (4.15) is $[P\phi(U(x))Q]$. Hence it suffices to show that $K_1(\psi(U(x))) = [P\phi(U(x))Q]$; we shall show that in fact $\psi(U(x)) = P\phi(U(x))Q$. Recall that $U(x) = V(x) + W(x)$. It is immediate from the definition of $V(x)$ that $\psi(V(x)) = P\phi(V(x))Q$. Hence since $W$ has coefficients in $DL(E)$, it only remains to show that $\phi(W(x)) = P\phi(W(x))Q$. A tedious but straightforward calculation, using (4.3) shows that

$$
\phi(W(x)_{\alpha(e, i),(f, j)}) = (P\phi(W(x)_{\alpha}))_{(e, i),(f, j)} \quad \forall (e, i), (f, j) \in S, \quad \alpha \in \Lambda.
$$

This completes the proof.

Remark 4.16. Recall that if $L(E)$ is unital, we have an exact sequence

$$
0 \to K_0(L(E)) \otimes K_1(\ell) \to K_1(L(E)) \to \text{Ker}(I - A_\ell^1) \to 0.
$$

It follows from [11, Lemma 7.2] that if $R \in \text{Alg}_\ell$ is $K_1$-regular and $\xi \in kk(L(E), R)$, then $K_1(\xi)$ restricts to the composite of $K_0(\xi) \otimes \text{id}$ with the cup product $K_0(R) \otimes K_1(\ell) \to K_1(R)$.

Theorem 4.17. Let $E$ be a finite graph and $S$ an algebra. Assume that $L(E)$ is simple and that $S$ is unital, purely infinite simple and $K_1$-regular. Let $\xi_0 : K_0(L(E)) \to K_0(S)$ and $\xi_1 : \text{Ker}(I - A_\ell^1) \to K_1(S)$ be group homomorphisms. Then there exists a nonzero algebra homomorphism $\phi : L(E) \to S$ such that $K_0(\phi) = \xi_0$ and such that $K_1(\phi)\gamma = \xi_1$. If moreover $\xi_0$ is unital then we can choose $\phi$ to be a unital homomorphism $L(E) \to S$.

Proof: By Theorem 3.1, there exists a nonzero algebra homomorphism $\phi_0 : L(E) \to S$ such that $K_0(\phi_0) = \xi_0$, and if $\xi_0$ is unital then we may choose $\phi_0$ unital. If $L(E)$ is not purely infinite, then by Remark 2.19, $L(E) \cong M_n$ for some $1 \leq n < \infty$. Hence $\text{Ker}(I - A_\ell^1) = 0$ and $K_1(L(E)) = K_0(L(E)) \otimes U(\ell)$. Assume that $L(E)$ is purely infinite simple. Let $R = \phi_0(1)S\phi_0(1)$ and let $\tilde{\phi}_0 : L(E) \to R$ be the corestriction of $\phi_0$ and $\text{inc} : R \to S$ the inclusion. Since $\text{Ker}(I - A_\ell^1)$ is a direct summand of $\mathbb{Z}^{\text{reg}(E)}$ and $\langle \cdot, \cdot \rangle$ is a perfect pairing, there exists $\theta \in K_1(R_{\phi_0})$ such that

$$
\langle - , \theta \rangle = K_1(\text{inc})^{-1}\xi_1 - K_1(\tilde{\phi}_0)\gamma.
$$

Because $R_{\phi_0}$ is a direct sum of purely infinite simple algebras, by Theorem 2.2 there exists $g \in U(R_{\phi_0})$ such that $[g] = \theta$. Define $\phi : L(E) \to R$ by setting $\phi:\left| E^0 \right| = (\tilde{\phi}_0)\left| E^0 \right|$, $\phi(e) = g\phi_0(e)$, $\tilde{\phi}(e^*) = \tilde{\phi}_0(e^*)g^{-1}$. Observe that $\tilde{\phi}$ and $\tilde{\phi}_0$ agree on $DL(E)$; in particular, $\tilde{\phi}$ is unital. Hence by Lemma 4.12, we have

$$
K_1(\tilde{\phi})\gamma = K_1(\tilde{\phi}_0)\gamma + \langle - , [u] \rangle.
$$

But it follows from the formula defining $u$ in Lemma 4.12 and the definition of $\tilde{\phi}$ that $u = g$. Hence

$$
K_1(\tilde{\phi})\gamma = K_1(\text{inc})^{-1}\xi_1.
$$
Set $\phi = \text{inc} \tilde{\phi}$; then $K_1(\phi)y = \xi_1$. Further, $K_0(\phi) = K_0(\phi_0) = \xi_0$ because $\phi$ and $\phi_0$ agree on $E^0$. It is clear by construction that if $\phi_0$ is unital homomorphism, then $\phi$ is also unital. \hfill $\square$

5. Lifting $kk$-maps to algebra maps

Let $\phi, \psi : A \to B$ be algebra homomorphisms. Put $C_{\phi, \psi} = \{(a, f) \in A \oplus B[t] : f(0) = \phi(a), f(1) = \psi(a)\}$.

Let $\pi : C_{\phi, \psi} \to A$, $\pi(a,f) = a$; we have an algebra extension

$$\Omega B \to C_{\phi, \psi} \xrightarrow{\pi} A \quad (5.1)$$

Lemma 5.2. Let $j : \text{Alg}_\ell \to kk$ be the canonical functor. The sequence $(5.1)$ induces the following distinguished triangle in $kk$

$$j(\Omega B) \xrightarrow{0} j(C_{\phi, \psi}) \xrightarrow{j(\pi)} j(A) \xrightarrow{j(\phi) - j(\psi)} j(B).$$

Proof. By definition of $C_{\phi, \psi}$, we have a map of extensions

$$\Omega B \to C_{\phi, \psi} \xrightarrow{\pi} A \quad (5.3)$$

$$\Omega B \xrightarrow{\Delta} B[t] \xrightarrow{(ev_0, ev_1)} B \oplus B$$

Let $\Delta : B \to B \oplus B$, $\Delta(b) = (b, b)$. One checks that the $kk$-triangle associated to the bottom row of $(5.3)$ is isomorphic to

$$j(\Omega B) \xrightarrow{0} j(B) \xrightarrow{j(\Delta)} j(B) \oplus j(B) \xrightarrow{[\text{id}, -\text{id}]} j(B).$$

Let $\xi : j(A) \to j(B)$ be the boundary map in the triangle induced by $(5.1)$. It follows from $(5.3)$ that there is a commutative diagram

$$\begin{array}{c}
\begin{array}{ccc}
j(A) & \xrightarrow{\xi} & j(B) \\
j(B) & \xrightarrow{(j(\phi), j(\psi))} & j(B) \\
j(B) \oplus j(B) & \xrightarrow{[\text{id}, -\text{id}]} & j(B)
\end{array}
\end{array}$$

Hence $\xi = j(\phi) - j(\psi)$. \hfill $\square$

Let $R$ be a unital, purely infinite simple algebra, let $E$ be a finite graph such that $L(E)$ is simple and let $\phi, \psi : L(E) \to R$ be nonzero algebra homomorphisms which agree on $DL(E)$. Let $R_\phi$ be as in $(4.9)$. Put $p = \phi(1)$ and let $B = pRp$.

By corestriction, we may consider $\phi$ and $\psi$ as homomorphisms $L(E) \to B$. Let

$C = \{f \in B[t] \mid (\exists a \in L(E)) \phi(a) = f(0), \psi(a) = f(1)\}$.

Since $L(E)$ is simple, the map

$$C_{\phi, \psi} \to C, \quad (a, f) \mapsto f$$

is an isomorphism. We shall identify \( C = C_{\phi, \psi} \). Assume that \( R \) is \( K_1 \)-regular. Then \( B \) is \( K_1 \)-regular also, whence \( K_0(\Omega B) = KV_1(B) = K_1(B) \). Hence the extension (5.1) induces an exact sequence

\[
K_1(B) \xrightarrow{\partial'} K_0(C) \xrightarrow{\pi} K_0(L(E)) \xrightarrow{\phi-\psi} K_0(B) \tag{5.4}
\]

The following two lemmas adapt [18, Lemmas 3.2 and 3.3] to the purely algebraic case.

**Lemma 5.5.** Let \( u \) be as in Lemma 4.12, \( \partial' \) as in (5.4) and \( \langle \cdot, \cdot \rangle \) as in (4.11). Let \( \sigma \in K_0(C)^E \), \( \sigma_e = [\phi(ee^*)] \). Then for every \( x \in \mathbb{Z}^E \) we have

\[
\langle x, [u] \rangle = -\langle (I - A_E^i)x, \sigma \rangle
\]

**Proof.** Let \( u_e = u\phi(ee^*) + 1 - \phi(ee^*) \) \((e \in E^1)\). By Whitehead’s lemma there is \( U_e(t) \in GL(B(t)) \) such that \( U_e(0) = 1 \) and \( U_e(1) = \text{diag}(u_e, u_e^{-1}) \). Set \( V_e(t) = U_e(t) \text{diag}(\phi(e), 0), W_e(t) = \text{diag}(\phi(e^*), 0)U_e(t)^{-1} \). Now proceed as in the proof of [18, Lemma 3.2], substituting \( U_e(t)^{-1} \) and \( V_e(t) \) for \( U_e(t)^* \) and \( V_e(t)^* \).

**Lemma 5.6.** Let \( \lambda : R_\emptyset \to R_\emptyset \), \( \lambda(a) = \sum_{e \in E^1} \phi(e)a\phi(e^*) \). If \( j(\phi) = j(\psi) \in kk(L(E), B) \) then there is \( v \in U(R_\emptyset) \) such that \( [u] = [v^{-1}\lambda(v)] \in K_1(R_\emptyset) \).

**Proof.** The proof is the same as that of [18, Lemma 3.3].

Let \( S \) be an algebra, \( E \) a finite graph, and \( \phi, \psi : L(E) \to S \) algebra homomorphisms. We say that \( \phi \) and \( \psi \) are 1-step ad-homotopic if either

a) there is an MvN equivalence \((u, u') : \psi(1) \sim \phi(1) \) such that \( \text{ad}(u, u')\phi = \psi \), or

b) \( \phi \) and \( \psi \) agree on \( DL(E) \) and for \( B = \phi(1)S\phi(1) \) there is \( U(t) \in GL(B_\emptyset[t]) \) such that \( U(0) = 1 \) and \( \phi_{i+1}(e) = U(1)\phi(e), \psi(e^*) = \phi_i(e^*)U(1)^{-1} \).

We say that \( \phi \) and \( \psi \) are \( n \)-step ad-homotopic if there is a sequence of algebra homomorphisms \( \phi_i : L(E) \to S, 1 \leq i \leq n \), such that \( \phi_1 = \phi, \phi_n = \psi \), and \( \phi_i \) are \( 1 \)-step ad-homotopic for \( 1 \leq i \leq n - 1 \). Two unital homomorphisms \( \phi \) and \( \psi \) are \( n \)-step unitaly ad-homotopic if they are \( n \)-ad-homotopic and the \( \phi_i \) can be chosen to be unital for all \( 1 \leq i \leq n \). Call \( \phi \) and \( \psi \) (unitaly) ad-homotopic if they are \( n \)-step (unitaly) ad-homotopic for some \( n \).

**Remark 5.7.** Observe that if in a) above \( \phi \) and \( \psi \) are unital, then \( u \in U(S) \), so that \( \phi \) and \( \psi \) are conjugate in the usual, unital sense. Note also that in the situation b) above, \( \phi \) and \( \psi \) are homotopic. It follows that a unital homomorphism \( \phi : L(E) \to L(E) \) is unitaly ad-homotopic to the identity if and only if it is homotopic to \( \text{ad}(u) \) for some \( u \in U(L(E)) \).

**Theorem 5.8.** Let \( E \) be a finite graph and \( R \) a unital algebra. Assume that \( L(E) \) and \( R \) are purely infinite simple and that \( R \) is \( K_1 \)-regular. Then the canonical map

\[
j : [L(E), R]_{M_2} \setminus \{0\} \to kk(L(E), R) \tag{5.9}
\]

is an isomorphism of monoids. In particular, \([L(E), R]_{M_2} \setminus \{0\}\) is the group completion of \([L(E), R]_{M_2}\). Moreover, we have the following:
i) If \( \xi \in \text{kk}(L(E), R) \), then there is a nonzero algebra homomorphism \( \phi : L(E) \to R \) such that \( j(\phi) = \xi \). Moreover, \( \phi \) may be chosen to be unital if \( \xi \) is.

ii) Two nonzero (unital) algebra homomorphisms \( \phi, \psi : L(E) \to R \) satisfy \( j(\phi) = j(\psi) \) if and only if they are \( M_2 \)-homotopic if and only if they are (unital) ad-homotopic if and only if they are 3-step (unitally) ad-homotopic.

**Proof.** The map \( [L(E), R]_{M_2} \to \text{kk}(L(E), R) \) is a monoid homomorphism by the same argument as in Proposition 2.18.

Let \( \xi \in \text{kk}(L(E), R) \) and let \( \gamma : \text{Ker}(I - A_E^\gamma) \to K_1(L(E)) \) be as in (4.8). By Theorem 4.17 there exists a nonzero algebra homomorphism \( \psi : L(E) \to R \) such that \( K_0(\xi) = K_0(\psi) \) and \( K_1(\xi)\gamma = K_1(\psi)\gamma \). Let \( B = \psi(1)R\psi(1), \text{inc} : B \to R \) the inclusion and \( \tilde{\psi} : L(E) \to B \) the corestriction of \( \psi \). Then \( j(\text{inc}) \) is an isomorphism, and for \( \eta = j(\text{inc})^{-1}\xi \) we have \( \eta - j(\tilde{\psi}) \in \text{kk}(L(E), B) \cong \text{Ext}^2_B(K_0(L(E)), K_1(B)), \) by [11, Theorem 7.11]. To prove that the map of the theorem is surjective, it suffices to show that there exists an algebra homomorphism \( \phi(e) = \psi(e)e^{-1} \), we have \( \eta - j(\phi) = j(\phi) - j(\tilde{\psi}). \) The argument of the proof of [18, Theorem 3.1] shows this. Next we show that (5.9) is injective, and that the different notions of homotopy agree. It follows from Lemma 2.16, [11, Lemma 2.3] and the definition of ad-homotopy that ad-homotopic homomorphisms \( L(E) \to R \) are \( M_2 \)-homotopic, and from the universal property of \( \text{kk} \) that \( j \) sends homotopic maps to equal maps. Conversely, let \( \phi, \psi : L(E) \to R \) be algebra homomorphisms such that \( j(\phi) = j(\psi) \). Then \( K_0(\phi) = K_0(\psi) \), whence there exist for each \( e \in E \) elements \( u_e, \phi(e)R\phi(e)^{-1} \) and \( u_e' = \psi(e)e^{-1} \) such that \( u_e'u_e = \phi(e) \) and \( u_e'u_e = \psi(e) \). Thus \( u = \sum_{e \in E} u_e \in \text{Ker}(I - A_E^{\psi}) \) and \( u' = \sum_{e \in E} u_e' \in \text{Ker}(I - A_E^{\phi}) \), and \( \psi' = \text{ad}(u, u') \psi \) agrees with \( \phi \) on \( DL(E) \). Hence upon spending a 1-step ad-homotopy from \( \phi \) to \( \psi' \), if necessary, we may assume that \( \phi \) and \( \psi \) agree on \( DL(E) \). Let \( B = \phi(1)R\phi(1) \) and let \( u \in B_{\phi} \) be as in Lemma 4.12; we have

\[
\psi(e) = u\phi(e), \quad \psi(e^*) = \phi(e^*)u^{-1}.
\]

Observe that, because \( R \) is purely infinite and \( K_1 \)-regular, the same is true of \( B \). By Lemma 5.6 and \( K_1 \)-regularity of \( B \), there is \( v \in \text{GL}(B_{\phi}) \) and \( U(t) \in \text{GL}(B[t]) \) such that \( U(0) = 1 \) and \( U(1) = v^{-1}\lambda(v)u^{-1} \). Hence, upon using a second 1-step ad-homotopy, we may assume that \( u = v^{-1}\lambda(v) \). A calculation shows that \( \psi = \text{ad}(v)\phi \) is a third 1-step ad-homotopy. The proof of the nonunital part of the theorem. If \( \xi \) is unital, then by Theorem 4.17 there is a unital algebra homomorphism \( \psi : L(E) \to R \) such that \( K_0(\xi) = K_0(\psi) \) and \( K_1(\xi)\gamma = K_1(\psi)\gamma \). The argument used above to prove the surjectivity of (5.9) substituting \( \xi \) for \( \eta \) shows that there is a unital algebra homomorphism \( \phi : L(E) \to R \) such that \( j(\phi) = \xi \). Finally the same argument used above for nonunital homomorphisms shows that two unital homomorphisms \( L(E) \to R \) go to the same element in \( \text{kk} \) if and only if they are unital 3-step ad-homotopic.

**Remark 5.11.** By Lemma 2.16, we have that if \( R \) and \( L(E) \) are as in Theorem 5.8, then \([L(E), M_{oo}R] \) is an abelian monoid, with operation induced by the map (2.13), and the canonical homomorphism \([L(E), M_{oo}R] \setminus \{0\} \to \text{kk}(L(E), R) \) is an isomorphism of monoids.
6. Homotopy classification theorem

**Theorem 6.1.** Let $E$ and $F$ be finite graphs such that $L(E)$ and $L(F)$ are purely infinite simple. Let $\xi_0 : K_0(L(E)) \to K_0(L(F))$ be an isomorphism. Then there exist nonzero algebra homomorphisms $\phi : L(E) \to L(F)$ and $\psi : L(F) \to L(E)$ such that $K_0(\phi) = \xi_0$, $K_1(\psi) = \xi_0^{-1}$, $\phi \psi \approx_{M_2} \text{id}_{L(E)}$ and $\psi \phi \approx_{M_2} \text{id}_{L(F)}$. If $\xi_0$ is unital then we may choose $\phi$ and $\psi$ to be unital homomorphisms such that $\phi \psi$ and $\psi \phi$ are homotopic to the respective identity maps.

**Proof.** Because $\text{Ker}(I - A^1_E)$ and $\text{Ker}(I - A^1_F)$ are isomorphic to the quotients of $K_0(L(E))$ and $K_0(L(F))$ modulo torsion, the assumed isomorphism $\xi_0$ induces an isomorphism $\xi_1 : \text{Ker}(I - A^1_E) \to \text{Ker}(I - A^1_F)$. By [11, Corollary 7.19], there exists $\xi \in \text{kk}(L(E), L(F))$ such that for the injective homomorphism $\gamma_F : \text{Ker}(I - A^1_F) \to K_1(L(F))$ of (4.8), we have $K_0(\xi) = \xi_0$ and $K_1(\xi)\gamma_E = \gamma_F\xi_1$. Hence $\xi \in \text{kk}(L(E), L(F))$ is an isomorphism by [11, Proposition 5.10]. By Theorem 5.8 there are algebra homomorphisms $\phi : L(E) \to L(F)$ and $\psi : L(F) \to L(E)$ such that $j(\phi) = \xi$ and $j(\psi) = \xi^{-1}$, which may be chosen unital if $\xi_0$ is. Again by Theorem 5.8, $\phi \psi$ and $\psi \phi$ are $M_2$-homotopic to the respective identity maps. If moreover $\phi$ and $\psi$ are unital, then by Theorem 5.8, $\phi \psi$ and $\psi \phi$ are unitaly ad-homotopic to identity maps. Hence by Remark 5.7 there are $u \in U(L(E))$ and $v \in U(L(F))$ such that $\text{ad}(v)\phi \psi$ and $\psi \phi \text{ad}(u)$ are homotopic to identity maps. Hence $\psi$ is a homotopy equivalence. Upon replacing $\phi$ by the homotopy inverse of $\psi$, we get the last statement of the theorem. \hfill \square

Recall from [8, Chapter III, Section 6.2] that a *scaled ordered group* is an ordered group together with a choice of order unit. If $R$ is a unital algebra, then $K_0(R)$ has a natural structure of scaled ordered group whose positive cone is the image of $\mathcal{V}(R)$ and whose order unit is $[1_R]$.

We say that two unital algebras $R$ and $S$ are *unitally homotopy equivalent* if there are unital homomorphisms $\phi : R \to S$ and $\psi : S \to R$ such that $\psi \phi$ and $\phi \psi$ are homotopic to the respective identity maps.

**Corollary 6.2.** Let $E$ and $F$ be finite graphs such that $L(E)$ and $L(F)$ are simple. Assume that $K_0(L(E))$ and $K_0(L(F))$ are isomorphic as scaled ordered groups. Then either

i) there is $1 \leq n$ such that $L(E) \cong L(F) \cong M_n$

or

ii) $L(E)$ and $L(F)$ are purely infinite and unitaly homotopy equivalent.

**Proof.** By Remark 2.19 if $L(E)$ is simple but not purely infinite, then there is $n \geq 1$ such that $L(E) \cong M_n$. In this case $K_0(L(E)) \cong \mathbb{Z}$ with the usual order and $[1_{L(E)}]$ corresponds to $n$. On the other hand if $R$ is a purely infinite simple unital algebra, then every element of $K_0(R)$ is nonnegative, by Theorem 2.2. The proof is concluded using Theorem 6.1 and observing that the identity is the only automorphism of $\mathbb{Z}$ as an ordered group. \hfill \square
7. Algebra extensions

Let $R$ be an algebra. For $x \in R^N$, let $\text{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\}$. For a matrix $a \in R^{m \times n}$ and $i \in \mathbb{N}$, put $a_{i,*}$ and $a_{*,i}$ for the $i^{th}$ row and column, and set

$$\mathcal{S}(a) = \{a_{i,j} : i, j \in \mathbb{N}\} \subseteq R,$$

$$N(a) = \sup\{\# \text{supp}(a_{i,*}), \# \text{supp}(a_{*,i}) : i \in \mathbb{N}\}.$$

Consider the algebras

$$\Gamma R = \{a \in R^{m \times n} : \text{each row and column of } a \text{ is finitely supported}\},$$

$$\Gamma^\prime R = \{a \in \Gamma R : \# \mathcal{S}(a) < \infty \text{ and } N(a) < \infty\},$$

$$\Sigma R = \Gamma R / M_{\infty} R, \quad \Sigma^\prime R = \Gamma^\prime R / M_{\infty} R.$$

The algebras $\Sigma R$ and $\Sigma^\prime R$ are the Wagoner and Karoubi suspensions.

**Proposition 7.1.** Let $R$ be either a division algebra or a purely infinite simple unital algebra. Then $\Sigma R$ and $\Sigma^\prime R$ are purely infinite simple.

**Proof.** It suffices to show that if $M \in \Gamma R \setminus M_{\infty} R$ then there are $A, B \in \Gamma^\prime R$ such that $AMB = 1$. The conditions defining $\Gamma^\prime$ and $\Gamma$ imply that there are infinite, strictly increasing sequences $Y = \{y_1, y_2, \ldots\}$, $N = \{N_1 = 1, N_2, \ldots\} \subseteq \mathbb{N}$ such that for each $j$, $\emptyset \neq \text{supp}(m_{y_j}) \subseteq [N_j + 1, N_{j+1}]$. Let $B_1$ be the matrix whose $n^{th}$ column is the canonical basis element $\epsilon_{y_n}$. The support of the $j^{th}$-column of the matrix $MB_1$ is contained in $[N_j + 1, N_{j+1}]$. Choose, for each $j$, an element $x_j \in [N_{j+1}, N_{j+1}]$ such that $(MB_1)_{x_j,j} \neq 0$. Let $A_1$ be the matrix whose $j^{th}$ row is the basis element $\epsilon_{x_j}$. The matrix $A_1 MB_1$ is diagonal, and all its diagonal entries are nonzero. Hence by our hypothesis on $R$ there are diagonal matrices $A_2$ and $B_2$ such that $A_2 A_1 MB_1 B_2 = 1$. \hfill $\square$

Recall from [11, Lemma 2.8 and the paragraph below] that when $R$ is unital, every extension of an algebra $A$ by $M_{\infty} R$ is classified by a homomorphism $A \to \Sigma R$. By [11, Lemma 2.5], the sets $[A, \Sigma R]_{M_2}$ and $[A, \Sigma^\prime R]_{M_2}$ are abelian monoids with the sum induced by (2.13). Put

$$E\text{xt}(A, R) = [A, \Sigma R]_{M_2}, \quad E\text{xt}(A, R)_f = [A, \Sigma^\prime R]_{M_2}.$$

By definition, there is a canonical map $E\text{xt}(A, R)_f \to E\text{xt}(A, R)$; by [11, Remark 5.8] there is also a natural map $E\text{xt}(A, R) \to kk_{-1}(A, R)$.

**Theorem 7.2.** Let $R$ be either a division algebra or a $K_0$-regular purely infinite simple unital algebra and $E$ a finite graph such that $L(E)$ is simple. Then the canonical maps

$$E\text{xt}(L(E), R)_f \to E\text{xt}(L(E), R) \to kk_{-1}(L(E), R)$$

are isomorphisms. Moreover every nonzero element of each of these groups represents the $M_2$-homotopy class a nontrivial extension of $L(E)$ by $M_{\infty} (R)$.

**Proof.** Since $\ell$ is a field, $\Sigma$ and $\Sigma^\prime$ are models for the suspension functor. By Proposition 7.1, $\Sigma R$ and $\Sigma^\prime R$ are purely infinite simple. Now apply Theorem 5.8 to prove the first assertion. The second assertion follows from Theorem 5.8 and [11, Lemma 2.8]. \hfill $\square$
Corollary 7.3. \([\text{cf. [13, Theorem 5.3]}]\) For \(E\) as in the theorem above, we have
\[
\text{Ext}(L(E), \ell) = \text{Coker}(I - A_E).
\]

Proof. Immediate from Theorem 7.2 and the the fact that \(KH^1(L(E)) = \text{Coker}(I - A_E)\) [11, Formula 6.4].

\(\square\)

Corollary 7.4. Let \(E\) and \(R\) be as in Theorem 7.2. Then there is an exact sequence
\[
0 \rightarrow \text{Ext}_Z^1(K_0(L(E)), K_0(R)) \rightarrow \text{Ext}(L(E), R) \rightarrow \\
\text{Hom}_Z(\text{Ker}(I - A_E^0), K_0(R)) \oplus \text{Hom}_Z(\text{Ker}(L(E)), K_{-1}R) \rightarrow 0.
\]

Proof. Immediate from Theorem 7.2 and [11, Corollary 7.19].

\(\square\)

Example 7.5. If \(R\) is either \(\ell\) or a purely infinite simple unital Leavitt path algebra, then \(K_{-1}R = 0\), so the exact sequence of Corollary 7.4 becomes
\[
0 \rightarrow \text{Ext}_Z^1(K_0(L(E)), K_0(R)) \rightarrow \text{Ext}(L(E), R) \rightarrow \text{Hom}_Z(\text{Ker}(I - A_E^0), K_0(R)) \rightarrow 0.
\]
If furthermore \(K_0(L(E))\) is torsion, then \(\text{Ker}(I - A_E^0) = 0\), and we get a canonical isomorphism
\[
\text{Ext}(L(E), R) = \text{Ext}_Z^1(K_0(L(E)), K_0(R)).
\]

8. Maps into tensor products with \(L_2\)

Lemma 8.1. Let \(E\) be a graph and let \(\phi : L(E) \rightarrow R\) be an algebra homomorphism. Then \(\phi = 0 \iff \phi = 0\).

Proof. It suffices to show that if \(H : L(E) \rightarrow R[t]\) satisfies \(ev_0 H = 0\), and \(v \in L^0\), then \(H(v) = 0\). This follows from Lemma 2.12.

A unital algebra \(R\) is regular supercoherent if for every \(n \geq 0\), \(R[t_1, \ldots, t_n]\) is regular coherent in the sense of [14].

Lemma 8.2. Let \(E\) be graph and \(R\) a regular supercoherent algebra. Then \(L(E) \otimes R\) is \(K\)-regular. In particular, \(L(E) \otimes L(F)\) is \(K\)-regular for every finite graph \(F\).

Proof. By definition, \(R_n = R[t_1, \ldots, t_n]\) is regular supercoherent for every \(n \geq 0\). Hence by [11, Example 5.5] the canonical map \(K_n(R_n \otimes L(E)) \rightarrow KH_n(R_n \otimes L(E)) = KH_n(R_0 \otimes L(E))\) is an isomorphism for every \(n\), whence also \(K_n(R_0 \otimes L(E)) \rightarrow K_n(R_n \otimes L(E))\) is an isomorphism for all \(n\). The second assertion follows from the first, using [1, Lemma 6.4.16].

\(\square\)

Let \(R, S\) be unital algebras. Put
\[
[R, S] = [R, S]_1 = \{ [f] : f \text{ unital } \}.
\]

Theorem 8.3. Let \(E\) be finite graph such that \(L(E)\) is simple and \(R\) a purely infinite simple regular supercoherent algebra. Then \([L(E), L_2]_1 = [L(E), L_2]_{M_2} \setminus \{0\}, [L(E), R \otimes L_2]_1 = [L(E), R \otimes L_2]_{M_2}\), and both sets have exactly one element each.
Proof. By Remark 2.19, Proposition 2.18 and Theorem 1.8, \([L(E), L_2]_{M_2}\setminus \{0\}\) has exactly one element, since \(j(L_2) = 0\) in \(kk\); by Corollary 3.2 this element is the class of a unital homomorphism. Next let \(\phi, \psi : L(E) \to L_2\) be unital homomorphisms. If \(L(E)\) is not purely infinite, then by Proposition 2.18, \(\phi\) and \(\psi\) are conjugate, and therefore homotopic, since by Corollary 2.11, \(\pi_0(U(L_2)) = K_1(L_2) = 0\). If \(L(E)\) is purely infinite, then by part iii) of Theorem 5.8, \(\phi\) and \(\psi\) are 3-step unitally ad-homotopic. Hence by Remark 5.7 and the argument we have just used, \(\phi \approx \psi\). Thus the assertions about homomorphisms \(L(E) \to L_2\) are proved. It is well-known that the tensor product of a unital simple algebra with a unital central simple algebra is again simple. By [6, Theorem 4.2], \(L_2\) is central, so \(R \otimes L_2\) is simple. Moreover, \(R \otimes L_2\) is purely infinite by [7, Theorem 7.9]. Hence using that \(j(R \otimes L_2) = 0\) in \(kk\) and applying Lemmas 8.1 and 8.2, Proposition 2.18 and Theorem 5.8, we obtain

\[ [L(E), R \otimes L_2]_{M_2}\setminus \{0\} = kk(L(E), R \otimes L_2) = 0. \]

By Corollary 3.2 there is a unital homomorphism \(\phi : L(E) \to L(F) \otimes L_2\). If \(\psi\) is another, then \(\phi \approx \psi\) by Lemma 2.17 and the argument above. \(\square\)

Example 8.4. Let \(R\) be as in Theorem 8.3, let \(d : L_2 \to R \otimes L_2\), \(a \mapsto 1 \otimes a\) and let \(\phi : L_2 \to R \otimes L_2\) be any homomorphism. Setting \(L(E) = L_2\) in Theorem 8.3 we get that if \(\phi\) is nonzero then it is \(M_2\)-homotopic to \(d\) and that if \(\phi\) is unital then it is homotopic to \(d\).

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