Singular lagrangians: some geometric structures along the Legendre map

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Abstract

New geometric structures that relate the lagrangian and hamiltonian formalisms defined upon a singular lagrangian are presented. Several vector fields are constructed in velocity space that give new and precise answers to several topics like the projectability of a vector field to a hamiltonian vector field, the computation of the kernel of the presymplectic form of lagrangian formalism, the construction of the lagrangian dynamical vector fields, and the characterisation of dynamical symmetries.

Key words: fibre derivatives, singular lagrangians, time-evolution operator, constraints, hamiltonian vector fields, presymplectic forms, lagrangian dynamics

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1 Introduction

The dynamics associated with a first-order time-independent variational principle on a configuration manifold \( Q \) can be formulated either in its tangent bundle \( TQ \) (lagrangian formalism) or in its cotangent bundle \( T^*Q \) (hamiltonian formalism). If the variational problem is defined by the lagrangian function \( L \), both formulations are related through the Legendre transformation, which is given by the fibre derivative of \( L \), \( \mathcal{F}L : TQ \to T^*Q \).

In the regular case, that is, when \( \mathcal{F}L \) is a local diffeomorphism (or when the fibre hessian is everywhere non-singular), the equivalence between both formulations is fairly simple. However, in the singular case, this correspondence between the lagrangian and the hamiltonian formalisms is far from trivial, and it is just this case which is the most relevant for the fundamental physical theories (as generally covariant theories, Yang-Mills theories and string theory), because the occurrence of gauge freedom is only possible within this framework. This explains the effort made since 1950 to define the lagrangian and hamiltonian formalisms in the singular case, to study the relations between them, their dynamics and symmetries, their quantisation, and so on. In contrast to the regular case, some specific features of the singular case include constraints, arbitrary functions, gauge invariance, gauge fixing, etc.

This development has benefitted from the introduction of differential-geometric methods in the study of dynamical systems — some books along this line are for instance [AM78] [Arn89] [God69] [JS98]. A great variety of tools from differential geometry — manifolds and bundles, differential forms, metrics, connections . . . — has been widely applied since the 70s to singular lagrangians, achieving a fair comprehension about the lagrangian and the hamiltonian formalisms and their relations.

The need of fine tools in the singular case is a direct consequence of the Legendre transformation \( \mathcal{F}L : TQ \to T^*Q \) being singular. For instance, if \( \mathcal{F}L \) is a diffeomorphism, a hamiltonian vector field \( Z \) in \( T^*Q \) (with respect to the canonical symplectic form \( \omega_Q \)) is directly converted into a hamiltonian vector field \( Y = \mathcal{F}L^*(Z) \) in \( TQ \) (with respect to the symplectic form \( \omega_L = \mathcal{F}L^*(\omega_Q) \)), which indeed can be used to describe the lagrangian dynamics). In the singular case, each part of this statement (which of course is not true) has to be scrutinised carefully.

The purpose of this paper is to introduce some as yet unveiled geometric structures that appear in these formalisms and that facilitate the connection between the lagrangian and the hamiltonian formulations in the singular case. Once the lagrangian function is fixed, a vector field \( Y_h \) in \( TQ \) will be defined from an arbitrary function \( h \) in \( T^*Q \); this is our main object. From it, once a hamiltonian and a basis for the primary hamiltonian constraints are chosen, another vector field \( \Delta_h \) will be defined; should the lagrangian be regular, the vector field \( \Delta_h \) would be the hamiltonian vector field of \( \mathcal{F}L^*(h) \) with respect to \( \omega_L \). These constructions, and other ones related to them, provide new connections between the dynamics in both pictures. Applications include the study of the projectability of a vector field in lagrangian formalism to a hamiltonian vector field, the construction of the lagrangian dynamical vector fields, the study of the relation between the arbitrary
functions of the lagrangian and hamiltonian dynamics, and the formulation of the dynamical symmetries (with special emphasis on the Noether symmetries); even the intrinsic construction of some structures as the kernel of the presymplectic form in tangent space will become almost trivial.

As for the geometric tools used in the paper, they are related with the fibred structure of the tangent and cotangent bundles. We use basically the fibre derivative (that is, the ordinary differentiation with respect to the fibre variables), the vertical lift (that is, the identification between points and tangent vectors in a vector space), and the canonical structures of the tangent bundle (vertical endomorphism, canonical involution) and of the cotangent bundle (the canonical differential forms).

The paper is organised as follows. Sections 2 and 3 provide some differential-geometric preliminaries concerning bundles and the fibre derivative. Section 4 contains a geometric description of lagrangian and hamiltonian formalisms in the singular case. The construction of the vector field $Y_h$ is presented in section 5, together with some of its properties. Two other vector fields, $R_h$ and $\Delta_h$, are also presented there. Section 6 uses the mentioned constructions to study the projectability to hamiltonian vector fields of $T^*Q$, and to give an explicit basis for the kernel of the presymplectic form $\omega_L$ of lagrangian formalism. In section 7 the preceding vector fields are used to construct the lagrangian dynamics and to relate the arbitrary functions of lagrangian and hamiltonian dynamics; the dynamical symmetries of hamiltonian formalism are also studied in a simple way. The case of regular lagrangians is studied in section 8. Section 9 contains a simple example. The final section is devoted to conclusions.

## 2 Some facts about bundles

Basic techniques concerning fibre bundles and vector bundles will be needed; in particular, the vertical vectors of a bundle and the tangent bundle of a bundle, as well as some canonical structures related to the tangent bundle. They may be found in many books, such as for instance [AM78] [AMR88] [Die70] [God69] [KMS93] [Sau89]. In this section we recall a few of these concepts and introduce some notation.

### Vertical vectors

Let $\pi: E \to B$ be a fibre bundle, with fibres $E_x = \pi^{-1}(x)$. The vertical bundle of $E$ is the vector subbundle $V(E) = \text{Ker} \, T(\pi) \subset T(E)$. Its fibre at a point $e_x \in E_x$ is the tangent space to the fibre of $E$ at $x$: $V_{e_x}(E) = T_{e_x}(E_x)$.

Let us consider a vector bundle $E \to B$. At each $x \in B$ we have a vector space $E_x$. The tangent space of $E_x$ at a point $e_x$ is naturally isomorphic to $E_x$ itself, $E_x \cong T_{e_x}(E_x)$; this isomorphism is constructed by sending $v_x$ to the tangent vector of the path $t \mapsto e_x + tv_x$ in $E_x$. Therefore $T(E_x) \cong E_x \times E_x$.

Globally this yields a canonical isomorphism $V(E) \cong E \times_B E$, called the vertical lift

$$E \times_B E \xrightarrow{\text{vl}_E} V(E) \subset T(E)$$  \hspace{1cm} (2.1)
Here $E \times_B E$ denotes the fibre product (its elements are the couples $(e, e') \in E \times E$ such that $\pi(e) = \pi(e')$), considered as a vector bundle over the first factor.

The vertical lift defines a natural bijection between fibre bundle maps $E \to E$ and vertical vector fields on $E$: if $\xi: E \to E$ is a fibre bundle map, then the map

$$\xi^v: E \to V(E) \subset \mathcal{T}(E), \quad \xi^v(e) = \text{vl}_E(e, \xi(e))$$

is a vertical vector field. This procedure applied to the identity map of $E$ yields a canonical vertical vector field, the Liouville's vector field, $\Delta_E(e) = \text{vl}_E(e, e)$. If $(x, a)$ are vector bundle coordinates of $E$—usually we will omit indices—then the local expression of $\Delta_E$ is $a^i \partial / \partial a^i$.

### Some structures of $T(TB)$

Given a vector bundle $\pi: E \to B$, the tangent bundle $TE$ has two vector bundle structures: $\tau_E: TE \to E$ and $T\pi: TE \to TB$. In the case of $E = TB$, we obtain two different vector bundle structures over the same base. Both structures are canonically isomorphic through the canonical involution, $\kappa_B: T(TB) \to T(TB)$. Its local expression in natural coordinates is

$$\kappa(x, v; u, a) = (x, u; v, a).$$

Another map in this manifold is the vertical endomorphism $J: T(TB) \to T(TB)$, whose local expression is

$$J(x, v; u, a) = (x, v; 0, u).$$

### Projectability

Let $\mathcal{F}: M \to N$ be a map between manifolds. A function $f: M \to \mathbb{R}$ is said to be projectable (through $\mathcal{F}$) if $f = \mathcal{F}^* g := g \circ \mathcal{F}$ for a certain function $g: N \to \mathbb{R}$. A vector field $X$ on $M$ is projectable if there exists a vector field $Y$ on $N$ such that $T(\mathcal{F}) \circ X = Y \circ \mathcal{F}$; one also says that $X$ and $Y$ are $\mathcal{F}$-related. Alternatively, one has $X \cdot \mathcal{F}^*(g) = \mathcal{F}^*(Y \cdot g)$ for any function $g$ on $N$.

When $\mathcal{F}$ has constant rank, one can use the rank theorem to obtain a characterisation of the local projectability of a function $f$: this condition is that $v \cdot f = 0$ for every $v \in \text{Ker} T(\mathcal{F})$. There are similar results for the local projectability of vector fields. However, let us just point out one result from the opposite side: a vector field $Y$ on $N$ is locally the projection of a vector field $X$ iff $Y$ is tangent to the image of $\mathcal{F}$.

### 3 Fibre derivatives

The fibre derivative will play an important role in our developments. Its definition can be found in many places such as, for instance, [GS 73] [AM 78], since it is a relevant structure when constructing the Legendre transformation that connects lagrangian and hamiltonian formalisms. In a recent article [Grà 00] the fibre derivative has been studied in detail, with
a view to application to singular lagrangian dynamics. In this section we summarise some of the results of this paper.

**Definition of the fibre derivative**

Our framework consists of two real vector bundles $E \to M$ and $F \to M$ over the same base, and a fibre $M$-bundle morphism $f: E \to F$, that is, a fibre-preserving map: for each $e_x \in E_x$, $f(e_x) \in F_x$. (In [Grà 00] the more general case of $E$ and $F$ being affine bundles is considered; this is especially interesting, for instance, when considering higher-order or time-dependent lagrangians, or field theory.)

The restriction of $f$ to a fibre defines a map $f_x: E_x \to F_x$ between vector spaces, whose ordinary derivative at a point $e_x \in E_x$ is a linear map $Df_x(e_x): E_x \to F_x$. In other words, we have defined an element

$$Ff(e_x) := Df_x(e_x) \in \text{Hom}(E_x, F_x)$$

for each $e_x \in E$. Globally, this defines a fibre-preserving map

$$Ff: E \longrightarrow \text{Hom}(E, F) \cong F \otimes E^*,$$

which is the *fibre derivative* of $f$.

If the local expression of $f$ is $(x^\mu, a^i) \mapsto (x^\mu, f^k(x, a))$, then the local expression of $Ff$ is

$$Ff(x^\mu, a^i) = \left( x^\mu, \frac{\partial f^k}{\partial a^i}(x, a) \right).$$

Since $Ff$ is also a fibre bundle map between vector bundles, the same procedure can be applied to compute its fibre derivative. The canonical isomorphism $\text{Hom}(E, \text{Hom}(E, F)) \cong \mathcal{L}^2(E; F)$ now yields the second fibre derivative, the *fibre hessian*, which is the map

$$F^2f: E \longrightarrow \mathcal{L}^2(E; F) \cong \text{Hom}(E \otimes E, F) \cong F \otimes E^* \otimes E^*,$$

whose local expression is

$$F^2f(x^\mu, a^i) = \left( x^\mu, \frac{\partial^2 f^k}{\partial a^i \partial a^j}(x, a) \right).$$

This can be readily generalised to higher order fibre derivatives.

**The case of a real function**

Let us notice the particular case where $F = M \times \mathbb{R}$. This corresponds indeed to considering a real function $f: E \to \mathbb{R}$ on a vector bundle $\pi: E \to M$. Then its fibre derivative is a map

$$Ff: E \longrightarrow \text{Hom}(E, M \times \mathbb{R}) =: E^*,$$

of which we shall study some properties.

First, there is a close relation between the tangent map

$$T(Ff): TE \longrightarrow TE^*$$
and the fibre hessian $\mathcal{F}^2 f$ of $f$,

$$\mathcal{F}^2 f = \mathcal{F}(\mathcal{F} f): E \to \text{Hom}(E, E^*) \cong E^* \otimes E^*.$$ 

Indeed, the restriction of $T_{e_x}(\mathcal{F} f)$ to vertical vectors is —thanks to the vertical lift— essentially the same map as the hessian considered as a map $\mathcal{F}^2 f(e_x): E_x \to E_x^*$. As a consequence, one has that

$$v_x \in \text{Ker} \mathcal{F}^2 f(e_x) \iff v|_{E_x}(e_x, v_x) \in \text{Ker} T_{e_x}(\mathcal{F} f),$$

and since $\text{Ker} T(\mathcal{F} f) \subset \mathcal{V}(E)$, in this way we obtain the whole subbundle $\text{Ker} T(\mathcal{F} f)$. Notice in particular that $\mathcal{F} f$ is a local diffeomorphism at $e_x \in E$ iff $\mathcal{F}^2 f(e_x)$ is a linear isomorphism.

These results can be also deduced from the local expressions of the maps; using as natural coordinates of $E$ and $E^*$ $(x, a)$ and $(x, \alpha)$ respectively, they are:

$$\mathcal{F} f: (x, a) \mapsto \left(x, \frac{\partial f}{\partial a}(x, a)\right),$$

$$T(\mathcal{F} f): (x, a; v, h) \mapsto \left(x, \frac{\partial f}{\partial a}(x, a); v, \frac{\partial^2 f}{\partial a \partial x} v + \frac{\partial^2 f}{\partial a \partial a} h\right),$$

$$\mathcal{F}^2 f: (x, a) \mapsto \left(x, \frac{\partial^2 f}{\partial a \partial a}(x, a)\right).$$

Finally we want to notice the following result. If $\xi: E \to E$ is a bundle map with associated vertical vector field $X = \xi^v$ on $E$, and $g: E \to \mathbb{R}$ is a function, then

$$X \cdot g = \langle \mathcal{F} g, \xi \rangle. \quad (3.7)$$

This can be applied in particular to the Liouville’s vector field, giving

$$(\Delta_E \cdot g)(e_x) = \langle \mathcal{F} g(e_x), e_x \rangle; \quad (3.8)$$

the fibre derivative of this expression can be computed by applying the Leibniz’s rule, and is

$$\mathcal{F}(\Delta_E \cdot g)(e_x) = \mathcal{F} g(e_x) + \mathcal{F}^2 g(e_x) \cdot e_x. \quad (3.9)$$

**Some useful structures: $\Gamma_h$ and $\Upsilon^g$**

Considering the fibre derivative $\mathcal{F} f: E \to E^*$ of $f$ as fixed data, we are going to derive several properties of a function $h: E^* \to \mathbb{R}$ and its fibre derivatives.

We use the notation

$$\gamma_h = \mathcal{F} h \circ \mathcal{F} f: E \to E \quad (3.10)$$

for the composition $E \xrightarrow{\mathcal{F} f} E^* \xrightarrow{\mathcal{F} h} E^{**} \cong E$. Recall that this map, through the vertical lift, defines a vertical vector field $\gamma_h^v$ on $E$:

$$\Gamma_h := \gamma_h^v = v|_E \circ (\text{Id}_E, \mathcal{F} h \circ \mathcal{F} f): E \to E \times_M E \to \mathcal{V} E \subset T E. \quad (3.11)$$

Their local expressions are

$$\gamma_h: (x, a) \mapsto \left(x, \frac{\partial h}{\partial \alpha} (\mathcal{F} f(x, a))\right) \quad \Gamma_h = (\mathcal{F} f)^* \left(\frac{\partial h}{\partial \alpha_i}\right) \frac{\partial}{\partial a^i}. $$
We can apply the chain rule to compute expressions like
\[ F(h \circ Ff) = F^2f \cdot \gamma_h, \tag{3.12} \]
\[ F(\gamma_h) = (F^2h \circ Ff) \cdot F^2f. \tag{3.13} \]
Here we have, for instance, \( F^2h \circ Ff: E \to E^* \to \text{Hom}(E^*, E^{**}) \cong \text{Hom}(E^*, E) \) and \( F^2f: E \to \text{Hom}(E, E^*) \); the symbol \( \cdot \) denotes the composition between the images of both maps — it is like the contraction of vector fields with differential forms.

Notice from (3.12) that if \( h \) vanishes on the image \( Ff(E) \subset E^* \) then \( \gamma_h \) is in the kernel of \( F^2f \). So we obtain the following result — see also [Grà00] [BGPR86]:

Suppose that \( Ff \) has constant rank; thus, locally the image of \( Ff \) is a submanifold of \( E^* \) that can be (locally) described by the vanishing of a set of independent functions \( \phi_\mu: E^* \to \mathbb{R} \). Then the vectors \( \gamma_{\phi_\mu}(e_x) \) are a basis for \( \text{Ker} F^2f(e_x) \), and the vertical vector fields \( \Gamma_{\phi_\mu} \) constitute a frame for \( \text{Ker} T(Ff) \).

As a byproduct, a function on \( E \) is (locally) projectable through \( Ff \) to \( E^* \) iff its Lie derivative with respect to the vector fields \( \Gamma_{\phi_\mu} \) is zero.

Now we present a construction dual to \( \Gamma_h \). Given a function \( g: E \to \mathbb{R} \), we can use its fibre derivative \( Fg: E \to E^* \) to construct a map
\[ \Upsilon^g = \text{vl}_{E^*} \circ (Ff, Fg): E \to E^* \times_M E^* \to V E^* \subset TE^*; \tag{3.14} \]
this is a vector field along the map \( Ff \), with local expression
\[ \Upsilon^g = \frac{\partial g}{\partial a^i} \left( \frac{\partial}{\partial \alpha_i} \circ Ff \right). \]
Recall that a section of a bundle \( \pi: E \to B \) along a map \( f: B' \to B \) is a map \( \sigma: B' \to E \) such that \( \pi \circ \sigma = f \). In particular, a section \( Z: B' \to TB \) of \( TB \) along \( f \) is called a vector field along \( f \); such a map derivates a function \( h: B \to \mathbb{R} \) giving a function \( Z \cdot h \) on \( B' \):
\[ (Z \cdot h)(y) = Z(y) \cdot h. \]

Notice finally that, as differential operators, \( \Gamma_h \) and \( \Upsilon^g \) are related by
\[ \Upsilon^g \cdot h = \Gamma_h \cdot g. \tag{3.15} \]
This follows from the fact that \( \Gamma_h \cdot g = \langle Fg, \gamma_h \rangle = \langle Fg, Fh \circ Ff \rangle = \Upsilon^g \cdot h. \]

4 Some structures of lagrangian and hamiltonian formalisms

The basic concepts about singular lagrangian and hamiltonian formalisms — Legendre map, energy, hamiltonian function, hamiltonian constraints ... — are well known and can be found in several papers, such as for instance [BGPR86] [BK86] [Car90] [GNH78] [MMS83] [MT78]. Now we will recall some of these concepts, introducing also some recent results from [Grà00].
Connection between the lagrangian and the hamiltonian spaces

Let us consider a first-order autonomous lagrangian on a configuration space \(Q\), that is to say, a map \(L: TQ \to \mathbb{R}\). Its fibre derivative (Legendre transformation) and fibre hessian are maps

\[
\mathcal{L}: TQ \to T^*Q, \\
\mathcal{F}^2L = \mathcal{F}(\mathcal{L}): TQ \to \text{Hom}(TQ, T^*Q) = T^*Q \otimes T^*Q.
\]

The local expression of \(\mathcal{L}\) is \(\mathcal{L}(q, \dot{q}) = (q, \hat{p})\), where \(\hat{p} = \frac{\partial L}{\partial \dot{q}}\) are the momenta. If the Legendre map is a local diffeomorphism —equivalently the hessian is everywhere nonsingular—the lagrangian \(L\) is called regular, otherwise it is called singular —this is our focus of interest.

We assume that the Legendre transformation of \(L\) has connected fibres and is a submersion onto a closed submanifold \(P_\circ \subset T^*Q\), the primary hamiltonian constraint submanifold—that is to say, \(L\) is an almost regular lagrangian in the terminology of [GN79]. This is the most basic technical requirement to develop a hamiltonian formulation from a singular lagrangian \(L\), though from a local viewpoint it suffices to have \(\mathcal{L}\) of constant rank. Locally \(P_\circ\) can be described by the vanishing of an independent set of functions \(\phi_\mu\), called the primary hamiltonian constraints. According to the preceding section, the vectors \(\gamma_\mu = \gamma_{\phi_\mu}\) constitute a basis for the kernel of \(\mathcal{F}^2L\), and the vertical vector fields \(\Gamma_\mu = \Gamma_{\phi_\mu}\) constitute a frame for \(\text{Ker} T(\mathcal{L})\).

The energy of \(L\) is defined by

\[E_L = \Delta_{TQ} \cdot L - L.\]

Due to the properties of the Liouville’s vector field (3.8) (3.9),

\[E_L(u_q) = \langle \mathcal{L}(u_q), u_q \rangle - L(u_q), \tag{4.1}\]

\[\mathcal{F}E_L(u_q) = \mathcal{F}^2L(u_q) \cdot u_q. \tag{4.2}\]

This shows at once that \(\Gamma_\mu \cdot E_L = \langle \mathcal{F}E_L, \gamma_\mu \rangle = 0\), that is to say, the energy is projectable (through \(\mathcal{L}\)) to a function \(H: T^*Q \to \mathbb{R}\) called a hamiltonian,

\[E_L = H \circ \mathcal{L},\]

which is unique on the primary hamiltonian constraint submanifold.

A resolution of the identity

Given an almost regular lagrangian \(L\), the choice of a hamiltonian and set of primary hamiltonian constraints yields a (local) resolution of the identity map of \(TQ\) as follows:

There exist functions \(v^\mu\) (defined on an open set of \(TQ\)) such that, locally,

\[\text{Id}_{TQ} = \gamma_H + \sum_\mu \gamma_\mu v^\mu. \tag{4.3}\]
Moreover,

\[ Id_{\text{Hom}(TQ, TQ)} = M \cdot F^2L + \sum_{\mu} \gamma_{\mu} \otimes Fv^{\mu}, \]  

(4.4)

where

\[ M = (F^2H \circ FL) + \sum_{\mu} (F^2\phi_{\mu} \circ FL) v^{\mu}. \]  

(4.5)

(Notice that \( F^2L \) is a map \( TQ \to \text{Hom}(TQ, T^*Q) \) and \( M \) is a map \( TQ \to \text{Hom}(T^*Q, TQ) = TQ \otimes TQ \).)

Since the functions \( v^{\mu} \) and their properties will be instrumental throughout the paper, we will recall the proof of this result [Grà 00]. Application of the chain rule (3.12) to the definition of \( H \) yields

\[ FEL(u_q) = F^2L(u_q) \cdot \gamma_H(u_q), \]

and so using (4.2) we obtain

\[ F^2L(u_q) \cdot (u_q - \gamma_H(u_q)) = 0. \]

The terms in parentheses are in \( \text{Ker} F^2L(u_q) \), thus there exist numbers \( v^{\mu}(u_q) \) such that

\[ u_q - \gamma_H(u_q) = \sum_{\mu} \gamma_{\mu}(u_q) v^{\mu}(u_q), \]

which is equation (4.3). Finally, using (3.13) and the Leibniz’s rule, one can compute the fibre derivative of (4.3); the result is equation (4.4).

The above results can be given a slightly different form, using the identification of bundle maps \( TQ \to TQ \) with vertical vector fields on \( TQ \). For instance, equation (4.3) can be rewritten as

\[ \Delta_{TQ} = \Gamma_H + \sum_{\mu} v^{\mu} \Gamma_{\mu}. \]  

(4.6)

Notice that application of (4.4) to \( \gamma_\nu \) yields \( \gamma_\nu = \sum_{\mu} \gamma_{\mu} \langle Fv^{\mu}, \gamma_\nu \rangle \). So we have

\[ \Gamma_{\nu} \cdot v^{\mu} = \langle Fv^{\mu}, \gamma_\nu \rangle = \delta_{\nu}^{\mu}, \]  

(4.7)

where we have applied equation (3.7). This shows that the functions \( v^{\mu} \) are not projectable; in a certain sense, they correspond to the velocities that cannot be retrieved from the momenta through the Legendre map.

Let us finally remark that the local expressions of equations (4.4) and (4.5) were initially deduced in [BGPR 86] by derivating the local expression of (4.3), which is

\[ \dot{q}^{\iota} = F^*L \left( \frac{\partial H}{\partial p^{\iota}} \right) + \sum_{\mu} F^*L \left( \frac{\partial \phi_{\mu}}{\partial p^{\iota}} \right) v^{\mu}. \]

(4.8)

The Euler-Lagrange equation

So far we have not considered the equations of motion. We will deal with them in several forms.

Let \( \omega_Q \) be the canonical 2-form of \( T^*Q \) (in coordinates \( dq^i \wedge dp_i \)). One defines the presymplectic form in \( TQ \)

\[ \omega_L = F^*L(\omega_Q) \]

— it is a symplectic form iff the lagrangian is regular. Then a path \( \gamma : I \to Q \) is a solution of the Euler-Lagrange equation iff

\[ i_{\dot{\gamma}} \omega_L = dE_L \circ \dot{\gamma}. \]  

(4.8)
A second representation of the equation of motion is

\[ \mathcal{E}_L \circ \dot{\gamma} = 0, \]  

(4.9)

where \( \mathcal{E}_L : T^2Q \rightarrow T^*Q \) is the Euler-Lagrange form of \( L \) —see for instance CLM91LM75; \( T^2Q \) denotes the second-order tangent bundle of \( Q \). \( \mathcal{E}_L \) is a 1-form along the projection \( T^2Q \rightarrow Q \), with local expression

\[ \mathcal{E}_L = [L]_i \, dq^i, \quad [L]_i = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right). \]  

(4.10)

A third version of the Euler-Lagrange equation can be written using the time-evolution operator \( K \) that connects lagrangian and hamiltonian formalisms. This operator was expressed in GP89 as a vector field along \( F_L \) satisfying certain properties that determine it completely. The local expression of \( K \) is

\[ K(q, \dot{q}) = (q, \hat{p}; \dot{q}, \frac{\partial L}{\partial q}). \]

In coordinates, \( K \) was first introduced BGPR86 as a differential operator —see also CL87Car90. Then its local expression reads

\[ K \cdot h = \mathcal{F}_L^* \left( \frac{\partial h}{\partial \dot{q}} \right) \dot{q} + \mathcal{F}_L^* \left( \frac{\partial h}{\partial p} \right) \frac{\partial L}{\partial q}. \]  

(4.11)

(In a time-dependent framework it would hold an additional piece, \( \mathcal{F}_L^*(\partial h/\partial t) \).) The operator \( K \) is a useful tool in the theory of singular lagrangians: it can be used —see below—to express the equations of motion GP88, to relate the lagrangian and the hamiltonian constraints BGPR86CL87Pon88, to study the symmetries of the equations of motion GP88BGGP89FP90GP92bGP94GP00 and, more recently, to study lagrangian systems with generic singularities PV00. See also GPR91GP95.

Using this operator, a path \( \xi : I \rightarrow TQ \) is the lift \( \dot{\gamma} \) of a solution of the Euler-Lagrange equation iff

\[ T(F_L) \circ \dot{\xi} = K \circ \xi. \]  

(4.12)

The following diagram shows all the objects involved:

\[ \begin{array}{ccc}
T(TQ) & \xrightarrow{T(F_L)} & T(T^*Q) \\
\xi & \downarrow & \downarrow K \\
I & \xrightarrow{\xi} & TQ & \xrightarrow{F_L} & T^*Q \\
\end{array} \]

The Hamilton-Dirac equation

In the singular case, hamiltonian dynamics was first studied by Dirac and Bergmann Dir50AB51Dir64. A path \( \eta : I \rightarrow P_o \) is a solution of the Hamilton-Dirac equation if there exist functions \( \lambda^\mu \) such that

\[ \dot{\eta} = Z_H \circ \eta + \sum_{\mu} \lambda^\mu Z_\mu \circ \eta. \]  

(4.13)
Here we denote by $Z_h$ the hamiltonian vector field defined by $h$: it satisfies
\[ i_{Z_h} \omega_Q = dh, \]
and, as a differential operator, it is related to the Poisson’s bracket by
\[ Z_h = \{-, h\}. \]
We have also put $Z_\mu = Z_\phi_\mu$.

Another geometric version of Dirac’s theory can be obtained by considering $j: P_o \hookrightarrow T^*Q$ and the presymplectic form $\omega_o = j^*(\omega_Q)$. Then the Hamilton-Dirac equation for a path $\eta: I \to P_o$ is
\[ i_{\dot{\eta}} \omega_o = dH_o \circ \eta, \tag{4.14} \]
where $H_o$ is the hamiltonian on $P_o$. 

Using the operator $K$, the Hamilton-Dirac equation can be written also as
\[ \dot{\eta} = K \circ T(\tau_Q^*) \circ \dot{\eta}, \tag{4.15} \]
for a path $\eta$ in $T^*Q$ —see also 

Of course, the hamiltonian dynamics is defined so as to be equivalent to the lagrangian dynamics, in the sense that if $\xi: I \to TQ$ is a solution of the Euler-Lagrange equation then $\eta: I \to T^*Q$ defined as $\eta = FL \circ \xi$ satisfies the Hamilton-Dirac equation, and conversely taking $\eta$ and defining $\xi = (\tau_Q^* \circ \eta)$ from it. We will say that such $\xi$, $\eta$ are a couple of related solutions.

Some further relations involving the operator $K$

Since the same dynamics is written in different ways, there are relations between the different structures involved. Let us point out first
\[ K \cdot h = \frac{d}{dt} FL^*(h) + (E_L, \gamma_h). \tag{4.16} \]
Here there is an abuse of notation that requires some explanation. On the right-hand side we have a function $FL^*(h)$ on $TQ$, whose total time-derivative —see for instance — is a function on $T^2Q$, and the contraction of $E_L$ with $\gamma_h$, considered as a function on $T^2Q$; however, the sum of both functions turns out to not depend on the acceleration, so it is a function on $TQ$, just as the left-hand side.

The local expression of (4.16) first appeared in

Though for singular lagrangians the lagrangian and the hamiltonian dynamics are not, in general, completely determined, equation (4.16) shows that, when considering solutions of Euler-Lagrange and Hamilton-Dirac equations, the evolution operator $K$ gives an unambiguous time-derivative of a function in hamiltonian space expressed in lagrangian terms. In particular, taking $h = \phi_\mu$, we obtain the primary lagrangian constraints
\[ \chi_\mu := K \cdot \phi_\mu = (E_L, \gamma_\mu): TQ \to \mathbb{R}; \tag{4.17} \]
notice that they also arise directly from (4.9) as a consistency condition —this is due to the fact that $\gamma_\mu$ are in the kernel of $\mathcal{F}^2L$. The vanishing of the primary lagrangian
constraints defines the primary lagrangian subset \( V_1 \subset TQ \), which we will assume to be a submanifold. Notice that the functions \( \chi_\mu \) are not necessarily independent, and indeed may vanish identically.

Now we can relate the operator \( K \) with the hamiltonian evolution. A very important result for our purposes is that

\[
K \cdot h = \mathcal{F}L^*\{h, H\} + \sum_\mu \mathcal{F}L^*\{h, \phi_\mu\} v^\mu,
\]

(4.18)

where there appear again the functions of equation (4.3). The proof can be found in [BGPR 86], and in [GPR 91] for higher-order lagrangians. This result can be expressed also as an equality between maps (in this case, vector fields along \( \mathcal{F}L \)) rather than as an equality of differential operators:

\[
K = Z_H \circ \mathcal{F}L + \sum_\mu v^\mu (Z_\mu \circ \mathcal{F}L),
\]

(4.19)

An immediate consequence of (4.18) is

\[
\Gamma_\mu \cdot (K \cdot h) = \mathcal{F}L^*\{h, \phi_\mu\}.
\]

(4.20)

This provides us with a test of projectability: the function \( K \cdot h \) is projectable iff \( h \) is a first-class function (with respect to \( P_0 \)). Recall that a function \( h: T^*Q \to \mathbb{R} \) is said to be first-class with respect to a submanifold \( P \subset T^*Q \) if the hamiltonian vector field \( Z_h \) is tangent to \( P \), which means that \( \{h, \phi\} \approx 0 \) for any constraint \( \phi \) defining the submanifold. (The notation \( f \approx 0 \) means that \( f(x) = 0 \) for all \( x \in M \) (Dirac’s weak equality); for instance \( \phi_\mu \approx 0 \) and \( \chi_\mu \approx 0 \).

5 Some canonical vector fields

The vector field \( Y_h \)

Let \( h: T^*Q \to \mathbb{R} \) be a function in phase space. Its fibre derivative is a map \( \mathcal{F}h: T^*Q \to TQ \), so we can define another map

\[
Y_h := \kappa \circ T(\mathcal{F}h) \circ K,
\]

(5.1)

where \( K \) is the time-evolution operator of \( L \) and \( \kappa: T(TQ) \to T(TQ) \) is the canonical involution of \( T(TQ) \). Let us show all this in a diagram:
Using the local expressions of all the objects involved, one obtains the local expression of $Y_h$:

$$Y_h(q, \dot{q}) = \left( q, \dot{q}, \frac{\partial h}{\partial p}(FL(q, \dot{q})), \dot{q} \frac{\partial^2 h}{\partial q \partial p}(FL(q, \dot{q})), \frac{\partial L}{\partial q} \frac{\partial^2 h}{\partial p^2}(FL(q, \dot{q})), \frac{\partial L}{\partial q} \frac{\partial^2 h}{\partial p \partial q}(FL(q, \dot{q})) \right).$$  \hfill (5.2)

**Proposition 1** The map $Y_h$ is a vector field on $TQ$, with local expression

$$Y_h = FL^* \{q, h\} \frac{\partial}{\partial q} + K \cdot \{q, h\} \frac{\partial}{\partial \dot{q}}.$$  \hfill (5.3)

It has the following properties:

$$J \circ Y_h = \Gamma_h,$$  \hfill (5.4)

$$Y_g \cdot (FL^* h) = FL^* \{h, g\} + \Gamma_h \cdot (K \cdot g),$$  \hfill (5.5)

$$Y_g \cdot (K \cdot h) = K \cdot \{h, g\} + Y_h \cdot (K \cdot g),$$  \hfill (5.6)

$$T(FL) \circ Y_g = Z_g \circ FL + T^K g.$$  \hfill (5.7)

**Proof.** The fact that $Y_h$ is a vector field is a direct consequence of its local expression (5.2). It follows also from

$$\tau_{TQ} \circ Y_h = \tau_{TQ} \circ \kappa \circ T(FLh) \circ K = T(\tau_Q) \circ T(FL) \circ K = T(\tau_Q) \circ K = Id_{TQ}.$$  \hfill (5.3)

The alternative (and more suggestive) local expression (5.3) of $Y_h$ is also clear from (5.2), as well as the fact that $J \circ Y_h = \Gamma_h$ — $J$ is the vertical endomorphism of $T(TQ)$.

The following two equations can be proved from their local expressions. This is simpler for the first one, (5.5): its left and right-hand sides read in coordinates

$$\left( \frac{\partial h}{\partial q} + \frac{\partial h}{\partial \dot{q}} \frac{\partial L}{\partial \dot{q}} \frac{\partial L}{\partial q} \right) \frac{\partial \dot{q}}{\partial p} + \frac{\partial h}{\partial p} \frac{\partial^2 L}{\partial \dot{q}^2} \left( \frac{\partial^2 \dot{q}}{\partial p \partial \dot{q}} \right) + \frac{\partial^2 \dot{q}}{\partial p \partial q} \frac{\partial \dot{q}}{\partial q}$$

(we have put $\tilde{h} = FL^* h$ to simplify the notation).

Regarding the second equation, (5.6), one has to prove $Y_g \cdot (K \cdot h) - Y_h \cdot (K \cdot g) = K \cdot \{h, g\}$. The terms remaining after the antisymmetrisation of $Y_g(K \cdot h)$ with respect to $(g, h)$ can be arranged to read

$$\left( \dot{q} FL^* \frac{\partial}{\partial q} + \frac{\partial L}{\partial q} FL^* \frac{\partial}{\partial p} \right) \left( \frac{\partial h}{\partial q} \frac{\partial \dot{q}}{\partial p} - \frac{\partial h}{\partial p} \frac{\partial \dot{q}}{\partial q} \right),$$

which is $K \cdot \{h, g\}$.

Finally, (5.7) is obtained by using relation (3.15) to express equation (5.3) as an equality between vector fields along $FL$. \hfill \blacksquare

**The vector fields** $R_h$ and $\Delta_h$

Equation (5.7) shows explicitly an obstruction for the projectability of $Y_g$ to the hamiltonian vector field $Z_g$. In the discussion of this issue it will be interesting to consider the vertical vector field

$$R_h = \Gamma_{\{h, H\}} + \nu^\mu \Gamma_{\{h, \phi_\mu\}},$$  \hfill (5.8)
defined from any function \( h \) on phase space — from now on we use the summation convention for the greek indices associated with the primary constraints. Notice that \( R_h \) depends on the choice of the hamiltonian \( H \) and the primary hamiltonian constraints \( \phi_\mu \). The action of \( R_h \) on projectable functions is

\[
R_g \cdot FL^* h = \Gamma_h \cdot (K \cdot g) - FL^* \{ g, \phi_\mu \} \Gamma_h \cdot v^\mu, \tag{5.9}
\]

which is a kind of generalisation of (4.20). To prove it, first we apply \( R_g \) to \( FL^* h \), then we use the symmetry property

\[
\Gamma_h \cdot FL^* (g) = FL^2 (\gamma_g, \gamma_h) = \Gamma_g \cdot FL^* (h), \tag{5.10}
\]

and finally we apply equation (4.18) to let \( K \) appear explicitly.

The interest of the vector field \( R_h \) comes from the fact that it appears when taking equation (5.6) and rewriting it using relations (4.18) and (5.5); after some cancellations one arrives at

\[
R_h \cdot (K \cdot g) + FL^* \{ h, \phi_\mu \} \cdot Y_g \cdot v^\mu = R_g \cdot (K \cdot h) + FL^* \{ g, \phi_\mu \} \cdot Y_h \cdot v^\mu. \tag{5.11}
\]

In other words, the left-hand side is symmetric in \((g, h)\). We can develop this further, applying equation (4.18) again to make \( K \) disappear from (5.11). A convenient organisation of the terms, together with some additional cancellations due to the symmetry property (5.10), finally yields another symmetric equation:

\[
FL^* \{ h, \phi_\mu \} (Y_g - R_g) \cdot v^\mu = FL^* \{ g, \phi_\mu \} (Y_h - R_h) \cdot v^\mu. \tag{5.12}
\]

This suggests to define, for any function \( g \) in phase space, the vector field

\[
\Delta_g = Y_g - R_g. \tag{5.13}
\]

**Proposition 2** The vector field \( \Delta_g \) has the following properties:

\[
J \circ \Delta_g = \Gamma_g, \tag{5.14}
\]

\[
\Delta_g \cdot v^\mu = -FL^* \{ g, \phi_\nu \} M \cdot \langle F v^\mu, F v^\nu \rangle, \tag{5.15}
\]

\[
\Delta_g \cdot (FL^* h) = FL^* \{ h, g \} + FL^* \{ g, \phi_\mu \} \Gamma_h \cdot v^\mu, \tag{5.16}
\]

\[
T(FL) \circ \Delta_g = Z_g \circ FL + FL^* \{ g, \phi_\mu \} Y^\nu v^\mu. \tag{5.17}
\]

**Proof.** The first property is a consequence of the same property of \( Y_g \) and the fact that \( R_g \) is vertical.

The second property gives the action of \( \Delta_g \) on the non-projectable functions \( v^\mu \). To prove it, we consider equation (5.12),

\[
FL^* \{ h, \phi_\mu \} \Delta_g \cdot v^\mu = FL^* \{ g, \phi_\mu \} \Delta_h \cdot v^\mu;
\]

taking for \( h \) the configuration variables \( h = q^i \), one gets

\[
(\Delta_g \cdot v^\mu) \gamma_\mu = -FL^* \{ g, \phi_\mu \} M \cdot F v^\mu,
\]

with \( M : TQ \to \text{Hom}(T^*Q, TQ) \) given by equation (4.3). Then contraction with \( F v^\nu \) and use of the property (4.7) finally yields equation (5.13).
Subtracting equations (5.5) and (5.9) yields (5.16).

Finally, using the relation (3.15) we can remove the function $h$ from the preceding equation to obtain an equality between vector fields along $\mathcal{F}L$, thus obtaining (5.17). ■

Some additional properties

The vector field on $TQ \Gamma_h$ and the vector field along $\mathcal{F}L \Upsilon^f$ are defined in terms of the fibre derivative, and a trivial application of Leibniz’s rule shows that

$$\Gamma_{h_1h_2} = \mathcal{F}L^*(h_1)\Gamma_{h_2} + \mathcal{F}L^*(h_2)\Gamma_{h_1},$$

(5.18)

$$\Upsilon^{f_1f_2} = f_1\Upsilon^{f_2} + f_2\Upsilon^{f_1}.$$  

(5.19)

Similarly one can compute

$$Y_{h_1h_2} = \mathcal{F}L^*(h_1)Y_{h_2} + \mathcal{F}L^*(h_2)Y_{h_1} + (K\cdot h_1)\Gamma_{h_2} + (K\cdot h_2)\Gamma_{h_1},$$

(5.20)

$$R_{h_1h_2} = \mathcal{F}L^*(h_1)R_{h_2} + \mathcal{F}L^*(h_2)R_{h_1} + (K\cdot h_1)\Gamma_{h_2} + (K\cdot h_2)\Gamma_{h_1},$$

(5.21)

$$\Delta_{h_1h_2} = \mathcal{F}L^*(h_1)\Delta_{h_2} + \mathcal{F}L^*(h_2)\Delta_{h_1}.$$  

(5.22)

The last equation, which is obtained immediately by subtracting the two previous ones, shows that the vector field $\Delta_h$ is also a first-order differential operator on $h$.

6 Applications to the kinematics

The projectability to a hamiltonian vector field

In equations (5.15), (5.16) and (5.17) there is a common piece $\mathcal{F}L^*\{g, \phi_\mu\}$ whose vanishing gives an answer to the question of projectability:

**Theorem 1** Let $L$ be an almost regular lagrangian. The necessary and sufficient condition for the hamiltonian vector field $Z_g$ in $T^*Q$ to be the projection (through the Legendre transformation) of a vector field in $TQ$ is that $g$ should be a first-class function with respect to the primary hamiltonian constraint submanifold $P_o \subset T^*Q$.

Then the vector field $\Delta_g$ projects to $Z_g$:

$$T(\mathcal{F}L) \circ \Delta_g = Z_g \circ \mathcal{F}L.$$  

(6.1)

Any other vector field projecting to $Z_g$ is obtained by adding to $\Delta_g$ any vector field in the kernel of the tangent map $T(\mathcal{F}L)$.

**Proof.** As we have said in section 2, the condition for a vector field in $T^*Q$ to be a projection is its tangency to $P_o = \mathcal{F}L(TQ)$. When this vector field is the hamiltonian vector field $Z_g$ this means that $g$ is a first-class function with respect to the primary constraint submanifold $P_o$, that is, $\mathcal{F}L^*\{g, \phi_\mu\} = 0$. Then (5.17) shows that $\Delta_g$ projects to $Z_g$.

The last assertion is obvious, since the vector fields that project to zero are those in $\text{Ker} T(\mathcal{F}L)$. ■
Comparing (5.17) and (5.7) one realises that the appropriate vector field candidate to project to $Z_g$ is $\Delta_g$. This is because the condition that $\mathcal{T}Kg = 0$, which is equivalent to $\mathcal{F}(Kg) = 0$, is more restrictive than $g$ being first-class. Indeed, $\mathcal{F}(Kg) = 0$ means that any vertical vector field acting on $Kg$ yields zero, then in particular $\Gamma^g_{\nu}(Kg) = \mathcal{F}L^*\{g, \phi_\mu\} = 0$ by (1.20). Of course, when $\mathcal{F}(K \cdot g) = 0$ we can say that also $Y_g$ projects to $Z_g$. This is also a consequence of the fact that if $\mathcal{F}(K \cdot g) = 0$ then $R_g$ is in Ker $T(\mathcal{F})$.

Equation (6.1) in the theorem is a direct consequence of equation (5.17) in proposition 2 when $g$ is first-class. Let us rewrite equations (5.15) and (5.16) accordingly:

**Proposition 3** Let $g: T^*Q \to \mathbb{R}$ be a first-class function with respect to the primary hamiltonian constraint submanifold $P_o \subset T^*Q$. Then the following results hold:

\[ \Delta_g \cdot v^\mu = 0, \quad (6.2) \]
\[ \Delta_g \cdot \mathcal{F}L^*h = \mathcal{F}L^*\{h, g\} \text{ for any function } h. \quad (6.3) \]

Recalling (4.7), $\Gamma^\nu_{\nu} \cdot v^\mu = \delta^\mu_{\nu}$, notice that equation (6.2) singles out $\Delta_g$, among the set of vector fields projecting to $Z_g$, as the only one whose action on the non-projectable functions $v^\mu$ is zero.

Now let us study some commutators among vector fields:

**Proposition 4** Let $\phi, \phi': T^*Q \to \mathbb{R}$ be primary hamiltonian constraints, and $g, g': T^*Q \to \mathbb{R}$ be first-class functions with respect to the primary hamiltonian constraint submanifold $P_o \subset T^*Q$. Then the following results hold:

\[ [\Gamma_\phi, \Gamma_{\phi'}] = 0, \quad (6.4) \]
\[ [\Delta_g, \Delta_g'] = -\Delta_{\{g, g'\}}, \quad (6.5) \]
\[ [\Delta_g, \Gamma_\phi] = -\Gamma_{\{g, \phi\}} - [R_g - \Gamma_{(g, H)}, \Gamma_\phi]. \quad (6.6) \]

**Proof.** The first result is well known, we include it for the sake of completeness, and it is readily proved in coordinates taking into account that $\Gamma_{\phi} \cdot \mathcal{F}L^*(h) = 0$ for any function $h$.

For the second result, to show the equality of both vector fields it is enough to prove that both coincide as differential operators when acting on projectable functions (this is a consequence of equation (4.8), together with $[Z_g, Z_{g'}] = Z_{\{g', g\}}$) and on the non-projectable functions $v^\mu$ (this is a trivial consequence of equation (6.2)).

One can proceed in the same way to prove the third commutator. To this end, we first prove that

\[ [\Delta_g, \Gamma_\mu] = 0. \quad (6.7) \]

On projectable functions the Lie bracket of the vector fields is zero; this is due to equation (4.3), and the fact that $\Gamma_\mu$ applied to any projectable function gives zero. On the non-projectable functions $v^\mu$, equation (5.2) and the fact that $\Gamma_\mu \cdot v^\nu$ is constant also yields zero.
Now let us deal with the general case. First, locally we can express \( \phi = a^\mu \phi_\mu \) for some functions \( a^\mu \). Then
\[
\Gamma_{a^\mu \phi_\mu} = \mathcal{F}L^\ast(a^\mu) \Gamma_\mu
\]
and \([\Delta_g, \Gamma_\phi] = [\Delta_g, \mathcal{F}L^\ast(a^\mu) \Gamma_\mu] = \Delta_g \cdot \mathcal{F}L^\ast(a^\mu) \Gamma_\mu\), thanks to (6.1). Using (6.3) we obtain
\[
[\Delta_g, \Gamma_\phi] = \mathcal{F}L^\ast \{a^\mu, g\} \Gamma_\mu.
\]
Considering \( \{g, \phi\} \) we have \( \Gamma_{\{g, \phi\}} = \mathcal{F}L^\ast(a^\mu) \Gamma_{\{g, \phi_\mu\}} + \mathcal{F}L^\ast \{g, a^\mu\} \Gamma_\mu \), and so we get
\[
[\Delta_g, \Gamma_\phi] + \Gamma_{\{g, \phi\}} = \mathcal{F}L^\ast(a^\mu) \Gamma_{\{g, \phi_\mu\}}.
\]
Finally, \( \Gamma_\phi \cdot v^\mu = \mathcal{F}L^\ast(a^\mu) \), so we arrive at
\[
[\Delta_g, \Gamma_\phi] + \Gamma_{\{g, \phi\}} = (\Gamma_\phi \cdot v^\mu) \Gamma_{\{g, \phi_\mu\}}. \tag{6.8}
\]
To obtain (6.6), notice that by definition \( R_g - \Gamma_{\{g, H\}} = v^\mu \Gamma_{\{g, \phi_\mu\}} \), and since by (6.4) the \( \Gamma \)'s of constraints commute, \([R_g - \Gamma_{\{g, H\}}, \Gamma_\phi] = [v^\mu \Gamma_{\{g, \phi_\mu\}}, \Gamma_\phi] = -(\Gamma_\phi \cdot v^\mu) \Gamma_{\{g, \phi_\mu\}} \). \( \blacksquare \)

Notice moreover that using the relation between \( Y_g \) and \( \Delta_g \) we can rewrite equation (6.3) as
\[
[\Delta_g + v^\mu \Gamma_{\{g, \phi_\mu\}}, \Gamma_\phi] = \Gamma_{\phi, g} = [Y_g - \Gamma_{\{g, H\}}, \Gamma_\phi]. \tag{6.9}
\]

The kernel of the presymplectic form in \( TQ \)

Here we will show that the vector fields \( \Delta_g \) provide an easy explicit construction of the kernel of the presymplectic form \( \omega_L = \mathcal{F}L^\ast \omega_Q \) of the lagrangian formalism.

If a vector field \( Y \) in \( TQ \) projects through \( \mathcal{F}L \) to a vector field \( Z \) in \( T^\ast Q \), we have
\[
i_Y \omega_L = \mathcal{F}L^\ast (i_Z \omega_Q).
\]
This shows trivially that \( \text{Ker} \, \mathcal{T}(\mathcal{F}L) \subset \text{Ker} \, \omega_L \) —indeed it is a well-known fact that \( \text{Ker} \, \mathcal{T}(\mathcal{F}L) = \text{Ker} \, \omega_L \cap V(TQ) \). So the vector fields \( \Gamma_\mu \) are part of a basis for \( \text{Ker} \, \omega_L \).

Now let us assume that the matrix of Poisson’s brackets \( \{\phi_\mu, \phi_\nu\} \) has constant rank. Then one can find an appropriate set \( \{\phi_\mu\} \) of independent primary hamiltonian constraints which are split into first-class \( \phi_{\mu_0} \) —their Poisson bracket with any primary hamiltonian constraint vanishes on \( P_o \) — and second-class \( \phi_{\mu_0} \) —see among others \( \text{DLGP'84} \). Being the functions \( \phi_{\mu_0} \) first-class, the corresponding vector field \( \Delta_{\mu_0} = \Delta_{\phi_{\mu_0}} \) projects to the hamiltonian vector field \( Z_{\mu_0} \), and since
\[
i_{\Delta_{\mu_0}} \omega_L = \mathcal{F}L^\ast (i_{Z_{\mu_0}} \omega_Q) = \mathcal{F}L^\ast (d\phi_{\mu_0}) = d\mathcal{F}L^\ast (\phi_{\mu_0}) = 0,
\]
we conclude that \( \Delta_{\mu_0} \) is also in \( \text{Ker} \, \omega_L \).

Notice that the vector fields \( \Delta_\mu \) are linearly independent, since application of the vertical endomorphism yields independent vector fields, \( J \circ \Delta_\mu = \Gamma_\mu \); moreover, they are also independent of the \( \Gamma_\mu \). Finally, the dimension of \( \text{Ker} \, \omega_L \) and the number of primary hamiltonian constraints plus the number of the first-class ones coincide —see for instance \( \text{MMS'83} \). So we have proved the following result:
The kernel of $\omega_L$ has a basis constituted by the vector fields $\Gamma_\mu$, associated with the primary hamiltonian constraints $\phi_\mu$, and the vector fields $\Delta_{\mu_o}$, associated with a basis of the first-class primary hamiltonian constraints $\phi_{\mu_o}$.

This kernel has been studied in the literature on singular Lagrangians for its interest in the classification of the constraints [CLR 88] [Car 90] [MR 92]. An explicit computation of the kernel was first presented in [PSS 99] (see equations (2.13a) and (2.13b) of that paper), but in a coordinate, rather than geometric, framework. In that paper the kernel was given in a slightly different basis, for $\Delta_{\mu_o}$ in that paper is the present $\Delta_{\mu_o}$ except for the term $\nu^r \Gamma_{\{\phi_{\mu_o},\phi_{\nu_o}\}}$, which is a combination of the vector fields $\Gamma_\mu$, also in the kernel. The present basis is preferable because it gives the commutation relations in their simplest form. Indeed, if

$$\{\phi_{\mu_o},\phi_{\nu_o}\} = B_{\mu_o\nu_o}^\rho \phi_\rho + O(\phi^2),$$

(the Poisson’s bracket of first-class constraints is first-class), then, taking into account proposition 4, the algebra reads

$$[\Gamma_\mu,\Gamma_\nu] = 0,$$

$$[\Gamma_\mu,\Delta_{\nu_o}] = 0,$$

$$[\Delta_{\mu_o},\Delta_{\nu_o}] = FL^*(B_{\nu_o\mu_o}^\rho) \Delta_\rho.$$

7 Applications to dynamics and symmetries

Lagrangian dynamics

Here we will give an explicit expression of the Lagrangian dynamics in terms of vector fields. Though in the case of a singular Lagrangian the Euler-Lagrange equation cannot be written in normal form, one can try to express its solutions in terms of integral curves of some dynamical vector fields. For instance, consider the Euler-Lagrange equation in the form (4.12): $T(FL) \circ \dot{\xi} = K \circ \xi$. Let $V \subset TQ$ be a submanifold and $X^L$ a second-order vector field in $TQ$ tangent to $V$. Then the integral curves of $X^L$ contained in $V$ are solutions of the Euler-Lagrange equation iff $X^L$ satisfies

$$T(FL) \circ X^L \approx K,$$

(the weak equality means equality on the points of the submanifold $V$).

As a first approximation to this problem, let us call $V_1$ the subset of points $u \in TQ$ where the linear equation —for the unknown vector $a_u - T_u(FL) \cdot a_u = K(u)$ is consistent, and assume it to be a submanifold, the primary Lagrangian constraint submanifold. Then the equation

$$T(FL) \circ X^L \approx K_{V_1}$$

has solutions, let us call them primary dynamical vector fields [GP 92a]. They are not unique on $V_1$, since they can be added vector fields in $\text{Ker} T(FL)$. On the other hand, one should find solutions that are tangent to $V_1$, and this is the beginning of an algorithm
that, under some regularity conditions, may give at the end all the solutions of the Euler-Lagrange equation. This is like the Dirac’s theory in lagrangian formalism —see a careful discussion in [GP 92a]; see also [BGPR 86, MR 92].

Notice that any integral curve of a primary dynamical field $X^L$ which is contained in $V_1$ is a solution of the Euler-Lagrange equation.

Our purpose now is to show that the choice of the hamiltonian function $H$ and the set of primary hamiltonian constraints $\phi_\mu$ yields a primary dynamical field $X^L$. Let us define the vector field

$$X^L_o = \Delta H + v^\mu \Delta_\mu.$$  

(7.3)

**Theorem 3** The vector field $X^L_o$ satisfies the second-order condition, and is a primary dynamical field. More precisely,

$$T(F^L) \circ X^L_o = K - \chi_\mu \uptriangledown_\mu \approx 0.$$  

(7.4)

**Proof.** A second-order vector field on $TQ$ can be characterised by the property that $J \circ X = \Delta_{TQ}$. We have

$$J \circ (\Delta_H + v^\mu \Delta_\mu) = \Gamma_H + v^\mu \Gamma_\mu = \Delta_{TQ},$$

by (5.14) and (4.6), so $X^L_o$ satisfies the second-order condition.

Now let us apply $T(F^L)$ to $X^L_o$, and use (5.7):

$$T(F^L) \circ X^L_o = Z_H \circ F^L + v^\mu Z_\mu \circ F^L + \left( F^L \{ H, \phi_\mu \} + v^\nu F^L \{ \phi_\nu, \phi_\mu \} \right) \uptriangledown_\nu.$$  

In this expression we recognise the operator $K$ —see equation (4.18)— and the primary lagrangian constraints $\chi_\mu = K \cdot \phi_\mu$, thus obtaining (7.4).

Before proceeding it will be interesting to notice some additional properties of $X^L_o$.

(We will use the notation $Y_\mu = Y_{\phi_\mu}$ and $R_\mu = R_{\phi_\mu}$.)

**Proposition 5** The vector field $X^L_o$ satisfies the following properties:

$$X^L_o = Y_H + v^\mu Y_\mu,$$  

(7.5)

$$X^L_o \cdot F^L(h) = K \cdot h - \chi_\mu \Gamma_h \cdot v_\mu,$$  

(7.6)

$$X^L_o \cdot v^\nu = \chi_\mu M\left(F v^\nu, F v^\mu\right) \approx 0,$$  

(7.7)

$$X^L_o \cdot (K \cdot h) = K \cdot \{ h, H \} + v^\mu K \cdot \{ h, \phi_\mu \} +$$

$$+ \chi_\nu \left(- R_h \cdot v^\nu + F^L \{ h, \phi_\mu \} M\left(F v^\mu, F v^\nu\right) \right).$$  

(7.8)

**Proof.** The first statement is an immediate consequence of the definition of $X^L_o$ and the fact that

$$R_H + v^\nu R_\nu = 0,$$  

(7.9)

whose proof is $R_H + v^\nu R_\nu = -v^\mu \Gamma_{\phi_\nu, H} + v^\nu \left( \Gamma_{\phi_\nu, H} + v^\mu \Gamma_{\phi_\nu, \phi_\mu} \right) = \Gamma_{\phi_\nu, \phi_\mu} v^\nu v^\mu = 0$, due to the antisymmetry of $\{ \phi_\nu, \phi_\mu \}$.

The second one is a direct consequence of equation (7.4): it tells us the action of $X^L_o$ (and indeed of any primary dynamical field $X^L$) on projectable functions.
The third equation gives the action of $X^L_o$ on the non-projectable functions $v^\mu$. It is obtained from (5.15) and the definition of the primary lagrangian constraints $\chi_\mu$:

$$X^L_o \cdot v^\nu = (\Delta_H + v^\mu \Delta_\mu) \cdot v^\nu = (\mathcal{F}^* \{ \phi_\mu, H \} + v^\nu \mathcal{F}^* \{ \phi_\mu, \phi_\rho \}) M(\mathcal{F} v^\nu, \mathcal{F} v^\mu)$$

$$= K \cdot \phi_\mu M(\mathcal{F} v^\nu, \mathcal{F} v^\mu) = \chi_\mu M(\mathcal{F} v^\nu, \mathcal{F} v^\mu).$$

The fourth equation is obtained from $K \cdot h = \mathcal{F}^* \{ h, H \} + \sum_\mu \mathcal{F}^* \{ h, \phi_\mu \} v^\mu$, (4.18), by applying (7.6) and (7.7).

As a consequence of the theorem we obtain the general form of a primary dynamical field in lagrangian formalism:

$$X^L = X^L_o + \varepsilon^\mu \Gamma_\mu.$$

On the other hand, according to (4.13), the primary dynamical fields in hamiltonian formalism are

$$X^H = Z_H + \lambda^\mu Z_\mu.$$

Both vector fields exhibit a set of arbitrary functions, $\varepsilon^\mu$ on $TQ$ and $\lambda^\mu$ on $T^*Q$, and we can relate the corresponding dynamics:

**Proposition 6** Let $\xi: I \to TQ$, $\eta: I \to T^*Q$ related solutions of the Euler-Lagrange and Hamilton-Dirac equations corresponding to the dynamical vector fields

$$X^L = X^L_o + \varepsilon^\mu \Gamma_\mu, \quad X^H = Z_H + \lambda^\mu Z_\mu.$$

Then the "arbitrary functions" $\varepsilon^\mu$, $\lambda^\mu$ are related by

$$\lambda^\mu(\eta(t)) = v^\mu(\xi(t)), \quad (7.10)$$

$$\varepsilon^\mu(\xi(t)) = (K \cdot \lambda^\mu)(\xi(t)). \quad (7.11)$$

**Proof.** We have

$$\dot{\eta} = Z_H \circ \eta + (\lambda^\mu \circ \eta) Z_\mu \circ \eta.$$

Since $\xi$ and $\eta$ are related, application of $T(\tau^*_Q)$ yields

$$\xi = \mathcal{F} H \circ \eta + (\lambda^\mu \circ \eta) \mathcal{F} \phi_\mu \circ \eta = \mathcal{F} H \circ \mathcal{F} \phi_\mu \circ \mathcal{F} \circ \xi + (\lambda^\mu \circ \eta) \mathcal{F} \phi_\mu \circ \mathcal{F} \circ \xi,$$

and from (4.13)

$$\xi = \gamma_H \circ \xi + (v^\mu \circ \xi) \gamma_\mu \circ \xi;$$

comparing both expressions we identify $\lambda^\mu$ with $v^\mu$.

Now we compute

$$(K \cdot \lambda^\mu)(\xi(t)) = \frac{d}{dt} \lambda^\mu(\eta(t)) = \frac{d}{dt} v^\mu(\xi(t))$$

$$= X^L_o \cdot v^\mu = (X^L_o + \varepsilon^\mu \Gamma_\mu) \cdot v^\mu$$

$$= \varepsilon^\mu(\xi(t)),$$

where we have used (7.10) and the properties $X^L_o \cdot v^\mu \approx 0$, $\Gamma_\nu \cdot v^\mu = \delta^\mu_\nu$. ■
Another application of the properties of $X^L_o$ is the relation between the lagrangian and the hamiltonian constraint algorithms. For instance, putting $\phi^1_\mu = \{\phi_\mu, H\}$ —this is a secondary hamiltonian constraint when $\phi_\mu$ is first-class—, from (7.8) we have

$$X^L_o(K \cdot \phi_\rho) = K \cdot \phi^1_\rho + v^\mu K \cdot \{\phi_\rho, \phi_\mu\} + \chi_\nu \left( - R_\rho \cdot v^\nu + FL^* \{\phi_\rho, \phi_\mu\} M (F v^\mu, F v^\nu) \right),$$

and so for first-class constraints we get

$$X^L_o(K \cdot \phi_{\mu_o}) \cong V_1 K \cdot \phi^1_{\mu_o},$$

which means that performing the first step of the hamiltonian stabilisation followed by application of $K$ is equivalent to applying $K$ and then performing the first step of the lagrangian stabilisation.

In a similar way from (7.6) we obtain

$$X^L_o FL^* \phi^1_{\mu_o} \cong V_1 K \cdot \phi^1_{\mu_o}.$$
Notice conversely that if a function $G$ satisfies (7.13) for every function $h$, then (5.6) implies that $Y_h \cdot (K \cdot G) \approx 0$ for each $h$, and so we obtain (7.12) again. We have thus obtained the following:

**Theorem 4** The necessary and sufficient condition for the hamiltonian vector field $Z_G$ to generate a symmetry of the Hamilton-Dirac equation of motion is

$$Y_G \cdot (K \cdot h) \approx K \cdot (Z_G \cdot h)$$

(7.14)

for all functions $h$.

One can also consider the more restrictive case of canonical Noether symmetries, whose infinitesimal generator $G$ can be characterised in a similar way [BGGP 89] as

$$K \cdot G = c.$$  

(7.15)

Then the same reasoning as above leads to the following:

**Theorem 5** The necessary and sufficient condition for the hamiltonian vector field $Z_G$ to generate a Noether symmetry in phase space is that

$$Y_G \cdot (K \cdot h) = K \cdot (Z_G \cdot h)$$

(7.16)

for all functions $h$.

Notice the remarkable fact that a weak (on-shell) equality or a standard equality is the only difference between the characterisation (7.14) for a symmetry of the Hamilton-Dirac equation of motion and the characterisation (7.16) for a canonical Noether symmetry. Since Noether symmetries exhibit a property of the action functional, it is clear that their characterisation must be, as we see, on-shell and off-shell. This characterisation (7.16) was first obtained in the paper [GP 00], which was instrumental in finding the new geometric structures that have been introduced in the present paper.

Notice also that, when $c \neq 0$ in (7.12) or (7.13), the conserved quantity associated to the symmetry is $G - ct$ rather than $G$.

8 The case of a regular lagrangian

In this section we will show what the preceding results become when the lagrangian is hyperregular, namely, when $\mathcal{FL}: TQ \to T^*Q$ is a diffeomorphism — in a local study, we might suppose only that the lagrangian is regular, namely, that $\mathcal{FL}$ is a local diffeomorphism.

Now the 2-form $\omega_L = \mathcal{FL}^*(\omega_Q)$ on $TQ$ is symplectic. Let us denote by $X_f$ the hamiltonian vector field of a function $f$ with respect to $\omega_L$. Recall that the lagrangian dynamics is now ruled by the hamiltonian vector field $X^L = X_{EL}$ of the energy function.
Proposition 7 Suppose that the lagrangian is hyperregular. Then:

\[ \Gamma_h = J \circ X_{FL^*(h)}, \]
\[ R_h = J \circ X_{FL^*(h)}^*(H), \]
\[ \Delta_h = X_{FL^*h}, \]
\[ Y_h = X_{FL^*(h)} + J \circ X_{FL^*(h)}^*(H). \]

Proof. The vertical vector fields in (8.1) correspond to bundle maps \( T_{\mathbb{T}Q} \rightarrow T\mathbb{Q} \). For the right-hand side the map is

\[ T(\tau_{\mathbb{T}Q}) \circ X_{FL^*(h)} = T(\tau_{\mathbb{T}Q}) \circ T(FL^{-1}) \circ Z_h \circ FL = T(\tau_{\mathbb{T}Q}) \circ Z_h \circ FL \]

which coincides with the map \( \gamma_h = \mathcal{F}_h \circ FL \) that corresponds to \( \Gamma_h \).

Definition (5.8) when there are no constraints yields \( R_h = \Gamma_{(h,H)} \). Then equation (8.2) follows immediately from (8.1). (Notice by the way that \( R_H = 0 \).)

Another consequence of the non existence of constraints is that, according to (5.17) or theorem 1, \( \Delta_h \) projects to the hamiltonian vector field \( Z_h \), and thus it is the hamiltonian vector field of \( FL^*(h) \), which is the contents of (8.3).

Finally, the last equation is an immediate consequence of the definition \( \Delta_h = Y_h - R_h \).

Given a second-order vector field \( D \) on \( T\mathbb{Q} \), a vector field \( X \) is called newtonoid with respect to \( D \) (see for instance [MM 86] [CLM 89] and references therein) if \( J \circ [X, D] = 0 \).

From any vector field \( X \) one can construct a newtonoid vector field —with respect to \( D \)— as \( X + J \circ [D, X] \). This construction, which has been used in several papers to study the symmetries of lagrangian dynamics, is a kind of generalisation of the complete lift of a vector field on \( Q \) to \( TQ \). From equation (8.4) it is then easy to deduce the following result:

Corollary 1 If the lagrangian is hyperregular then \( Y_h \) is a newtonoid vector field with respect to the dynamical vector field \( X_{L_o} \) of velocity space, and is the newtonoid vector field defined from the vector field \( X_{FL^*(h)} = \Delta_h \).

In the singular case, using (7.6) it is readily seen that \( Y_h \) satisfies the condition of being newtonoid with respect to \( X_{L_o} \) only on the primary lagrangian constraint submanifold \( V_1 \).

9 An example

As a simple example, let us consider the lagrangian of the conformal particle [Sie 88] [GR 93]

\[ L = \frac{1}{2}(\dot{x}^2 - \lambda x^2), \]

with configuration variables \((x, \lambda) \in Q = \mathbb{R}^n \times \mathbb{R}, \) and \( \mathbb{R}^n \) endowed with an indefinite scalar product. The Legendre transformation is given by

\[ FL(x, \lambda; \dot{x}, \dot{\lambda}) = (x, \lambda; \dot{p}, \dot{\pi}), \quad \dot{p} = \dot{x}, \quad \dot{\pi} = 0, \]
so the primary constraint submanifold $P_o \subset T^*Q$ has codimension 1, and is described by the primary hamiltonian constraint

$$\phi = \pi.$$  \hspace{1cm} (9.3)

As a hamiltonian we take

$$H = \frac{1}{2}(p^2 + \lambda x^2).$$  \hspace{1cm} (9.4)

Stabilization of $\phi^0 = \phi$ yields three additional generations of constraints $\phi^{i+1} = \{\phi^i, H\}$:

$$\phi^1 = -\frac{1}{2}x^2, \quad \phi^2 = -px, \quad \phi^3 = \lambda x^2 - p^2,$$

which are first-class. The lagrangian constraints are $\chi^i := K \cdot \phi^{-1}$:

$$\chi = \chi^1 = -\frac{1}{2}x^2, \quad \chi^2 = -\dot{x}x, \quad \chi^3 = \lambda x^2 - \dot{x}^2.$$

(The indeed $\chi^i = F_L^*(\phi^i)$, since the hamiltonian constraints are first-class.) Notice also that $K \cdot \phi^3 = -2\lambda \chi^1 - 4\lambda \chi^2$.

The kernel of $T(FL)$ is spanned by $\Gamma_{\phi} = \partial/\partial \dot{\lambda}$. From the identity $\text{Id} = \gamma_H + v \gamma_{\phi}$ we determine the function $v = \dot{\lambda}$. We also obtain

$$K \cdot g = \dot{x} a F_L^* \left( \partial g / \partial x^a \right) + \lambda F_L^* \left( \partial g / \partial \lambda \right) - \lambda x a F_L^* \left( \partial g / \partial p_a \right) - \frac{1}{2} \dot{x}^2 F_L^* \left( \partial g / \partial \pi \right)$$

$$= F_L^* \{ g, H \} + F_L^* \{ g, \pi \} \dot{\lambda}.$$

Now we can compute $Y_h = F_L^* \left( \partial h / \partial p \right) \partial / \partial x + F_L^* \left( \partial h / \partial \pi \right) \partial / \partial \lambda + (K \cdot \partial h / \partial p) \partial / \partial \lambda$, and in particular

$$Y_\phi = \partial / \partial \lambda, \quad Y_H = \dot{x} \partial / \partial x - \lambda x \partial / \partial \dot{x}.$$

Then, from $R_h = \Gamma_{\{ h, H \}} + \dot{\lambda} \Gamma_{\{ h, \pi \}}$ we get $R_\phi = \Gamma_{\phi} = 0$ and $R_H = \dot{\lambda} \Gamma_{-\phi} = 0$, from which $\Delta_{\phi} = Y_\phi$ and $\Delta_H = Y_H$.

According to our results, the kernel of the presymplectic form $\omega_L$ is spanned by $\Gamma_{\phi} = \partial / \partial \dot{\lambda}$ and $\Delta_{\phi} = \partial / \partial \lambda$. (In this case this is obvious since $\omega_L = dx \wedge d\dot{x}$.)

Finally we get the primary dynamical vector fields as $X_L = X_L^o + \epsilon \Gamma_{\phi}$, where

$$X_L^o = Y_H + \lambda Y_\phi = \dot{x} \partial / \partial x + \dot{\lambda} \partial / \partial \lambda - \lambda x \partial / \partial \dot{x}.$$

It is easily checked that $T(FL) \circ X_L^o - K = -\chi \partial / \partial \pi \approx 0$.

## 10 Conclusions

During the last two decades many papers have studied the close relations between lagrangian and hamiltonian formalisms when the lagrangian function is singular. One can expedite the lagrangian picture by using some results from the hamiltonian side.

In this paper we have added new objects to the geometric framework of these relations. First, for any function $h$ on phase space $T^*Q$ we have defined the vector field $Y_h$ on velocity space $TQ$. When looked in coordinates, this object reminds one of the definition of newtonoid vector fields; but instead of using a second-order dynamics on $Q$, which
is not well defined in general when the lagrangian is singular, we use the unambiguous
time-evolution operator $K$ that connects lagrangian and hamiltonian formalisms. Once
a hamiltonian $H$ and a set of primary hamiltonian constraints $\phi_\mu$ have been chosen, we
have also defined the vector fields $R_h$ and $\Delta_h$.

These objects give effective answers to several questions. The projectability of a vector
field to a hamiltonian vector field: we have shown that, when $h$ is a first-class function
on $T^*Q$, the vector field $\Delta_h$ projects to the hamiltonian vector field $Z_h$. The kernel of
the presymplectic form of lagrangian formalism: it can be computed as the subbundle
spanned by the vector fields $I_\mu$ associated with the primary hamiltonian constraints $\phi_\mu$
and the vector fields $\Delta_\mu$ associated with the first-class primary hamiltonian constraints.
The construction of the dynamical vector fields in lagrangian formalism: the vector field
$X^t_o = \Delta_H + v^\mu \Delta_\mu$ is a solution of the Euler-Lagrange equation on the primary lagrangian
constraint submanifold. Finally, the characterisation of dynamical symmetries: the fact
that $G$ is the generator of an infinitesimal symmetry can be expressed as a kind of com-
mutation relation between the time-evolution operator $K$ and the couple of vector fields
$Y_G, Z_G$.

In view of these results, we can say that the time-evolution operator $K$ still provides
one with new insights about the connections between singular lagrangian and hamiltonian
dynamics. The functions $v^\mu$, given by (4.3) as a kind of pseudo-inversion of the Legendre
transformation, and the fibre derivation, a seldom used operation in geometric mechanics,
complete, together with the usual structures of tangent and cotangent bundles, the set of
tools used in this paper.

As a final remark, let us point out that some of our expressions are also valid in the
time-dependent case, which is especially interesting for dealing with gauge symmetries.

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