On Bit Commitment and Oblivious Transfer in Measurement-Device Independent settings

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Among the most studied tasks in Quantum Cryptography one can find Bit Commitment (BC) and Oblivious Transfer (OT), two central cryptographic primitives. In this paper we propose for the first time protocols for these tasks in the measurement-device independent (MDI) settings and analyze their security. We analyze two different cases: first we assume the parties have access to perfect single photon sources (but still in the presence of noise and losses), and second we assume that they only have imperfect single photon sources. In the first case we propose a protocol for both BC and OT and prove their security in the Noisy Quantum Storage model. Interestingly, in the case where honest parties do not have access to perfect single photon sources, we find that BC is still possible, but that it is “more difficult” to get a secure protocol for OT: We show that there is a whole class of protocols that cannot be secure. All our security analyses are done in the finite round regime.

I. INTRODUCTION

Oblivious Transfer (OT) – [Rab81] – and Bit Commitment (BC) – [GMW91, Nao89, CDVdG87] – are two central, and well studied cryptographic primitives. In fact it has been shown that OT is universal [Kil88] in the sense that all two-party secure function evaluation tasks can be reduced to OT. This means that if one is given a secure implementation of OT, one can construct a protocol using OT (and classical communications) that implements any secure function evaluation protocol.

OT and BC are related to each other. In particular since OT is universal, it is possible to implement BC given an OT routine. The converse is not true if we limit the parties to classical communication [MN05]. However if the parties have access to quantum communication it can be shown that OT can be reduced to BC [BBCS92, FS09], and therefore OT and BC are equivalent in the quantum settings.

Unfortunately it is now well known that neither OT nor BC can be implemented when no restriction (other than following the laws of (quantum) physics) is made on the power of the adversary [May97, LC97, LC98]. This motivated the search for realistic assumptions that could be made on the adversary’s power.

Inspired by the classical Bounded Storage Model [Mau92], Ref. [DFSS05] proposed protocols assuming that the adversary can only store a limited amount of qubits, i.e. that the quantum memory of the adversary is bounded. The assumption is called the Bounded Quantum Storage Model. Note that the Bounded Quantum Storage Model has the advantage over computational assumptions that it allows for everlasting security, meaning that the assumption only needs to be satisfied during the execution of the protocol. No additional resources or power gained after the execution of the protocol can allow the adversary to break security. This contrasts with computational assumption for which giving more computational power to the adversary after the execution of the protocol is a threat to the security of the protocol. The Bounded Quantum Storage Model – as well as the more general Noisy Quantum Storage Model – allows to prove the security of protocols for OT and BC [DFSS05, DFSS07, WCSL10, KWW12, NJC+12, ENG+14].

However, these security proofs rely implicitly on the assumption that the devices used by the honest parties are sufficiently well characterized and will always work as expected. This assumption might not always be satisfied in practice. In particular, in the context of Quantum Key Distribution (QKD), attacks performed by tampering with the measurement devices exist [MAS06, SRK+15].

In the context of quantum key distribution, in order to present protocols that are not subject to these types of attack, Refs. [MY98, ABG+07, PAB+09] propose a security proof that is independent of the inner working of the quantum devices used during the protocol. In fact the devices are considered as black boxes, and the security only relies on the ability of the devices to demonstrate certain “non-local” statistics for their inputs and outputs. More precisely the authors show that if the devices are able to violate the CHSH inequality [CHSH69] then there is a secure protocol for quantum key distribution. This result has been later generalized to include a more powerful adversary [VVT14, MST17, MS16, AFDF+18, RMW18]. The model in which the devices are considered as black boxes is called device independence.

Following this idea of device independence Refs. [KW16, RTK+18] proved security of BC and OT in the bounded/noisy quantum storage model in device independent settings. However it is important to note that the authors assume that, even if the devices may behave in an arbitrary way, they do so in the same fashion in every
use of the devices independently of the past. In other words they assume that the devices are memoryless. Other protocols \cite{AMPS16,SCA+11} are secure against a more powerful adversary but require different settings where they only achieve an imperfect bit commitment scheme. In general it is quite hard to prove device independent security of protocols. In particular there is no known security proof for device independent OT and BC in the settings presented in Refs. \cite{KW16,RTK+18} without the memoryless assumption. Experimental implementations of device independent protocols are also a lot more demanding as discussed in Ref. \cite{MvR+18} for quantum key distribution. In fact it is so demanding that, while many quantum key distribution and some (quantum) BC protocols have been implemented, there has not been any device independent implementation of these protocols so far, even not assuming that the devices are memoryless.

These difficulties together with the fact that many attacks on the non device independent protocols \cite{MAS06,SRK+15} are tampering with the measurement devices and not with the photon sources (or quantum state sources), has led Refs. \cite{LCQ12} to introduce a weaker but more practical notion of device independence called measurement-device independence (MDI). Here only the measurement devices are treated as black boxes, not the sources of photons (states) that are still trusted. Since then many measurement-device independent protocols have been implemented \cite{LCW+13,POS+15,TLX+14,TYC+14,FdSVX+13} for QKD. Typically, in a measurement-device independent protocol, all the measurement devices are in a measurement station in between the parties (see Fig. 1). Having the measurement station in between the two parties is very natural if one considers that it is part of the network infrastructure also used for QKD. The parties will send BB84 type states to the measurement station which will perform a joint Bell measurement on the incoming qubits. As there is no assumption on these devices we will here always assume that the dishonest party can control the station (see Fig. 2).

This situation is different from MDI QKD, where the dishonest party is always a third party who only controls the measurement station, but never the sources of Alice and Bob. In particular, in QKD, Alice and Bob can always trust each other, which is not the case for BC or OT.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{MDI.png}
  \caption{Schematic of a MDI protocol.}
\end{figure}

However, almost all the work on measurement-device independence is focused on quantum key distribution \cite{LCQ12,LCW+13,POS+15,TLX+14,TYC+14}, and as far as we know there is no proposed protocol for BC or OT in the measurement-device independent settings. In this work we present protocols for BC and OT and analyze their security. Importantly, all our security proofs hold in the finite rounds regime and can be implemented with current state-of-the-art quantum technologies. We first analyze the situation where the honest parties have perfect single photon sources. Interestingly, in the case where honest parties do not have access to perfect single photon sources, we find that BC is still possible, but that it is “more difficult” to get a secure protocol for OT: We show that there is a whole class of protocols that cannot be secure. We present in the next section a detailed overview of our results.

1. Notation

In this paper we will denote quantum states by the Greek letters $\rho, \sigma$. For the purpose of this analysis quantum states can be taken to be positive linear operators of trace equal to 1 acting on a Hilbert space. We use the ket notation (e.g. $|\Psi\rangle$) to denote pure quantum states. Quantum measurements are described by Positive Operator Valued Measures (POVMs) which are finite sets of positive operators that sum up to the identity, e.g. $\{P_x, x \in \mathcal{X} \} : P_x \geq 0 \& \sum_x P_x = 1\}$, where $\mathcal{X}$ is a finite set of indices. If for some measurement the operators $P_x$ are mutually orthogonal projectors, then we say that the measurement is projective. The probability of observing outcome $x \in \mathcal{X}$ when measuring the state $\rho$ with the measurement described by $\{P_x\}_{x \in \mathcal{X}}$ is given by $p_x := \text{tr}(P_x \rho)$. We use $X^n_j$ as a shorthand for the string $X_j, \ldots, X_n$ ($j \leq n$). The symbol $\approx_{\varepsilon}$ will be used to express that two states are $\varepsilon$-close in the trace distance (e.g. $\sigma \approx_{\varepsilon} \rho$ for two states $\rho, \sigma$). In several occasions we will denote $\mathcal{R}$ for a family (not necessarily specified) of 2-universal hash functions. We will use $X \in \mathcal{R} \mathcal{E}$ to say that $X$ is picked uniformly at random from set $\mathcal{E}$.
Alice has control over the measurement station, therefore we will treat Alice and the measurement station as one party.

Bob has control over the measurement station, therefore we will treat Bob and the measurement station as one party.

Figure 2: Schematic MDI protocol with dishonest Alice or dishonest Bob.

$[n]$ is a shorthand notation for $\{1, \ldots, n\}$.

II. RESULTS

In this section, we will present the results of our work. Formal statements and their proofs will be given in the Methods Section.

- We start by presenting the MDI protocols for OT and BC for the case where the honest parties have access to perfect single photon sources.
- Then we present and analyze the security of a protocol for BC where the honest parties only have imperfect single photon sources, i.e. multiphoton emissions are possible.
- Finally we show that there is a family of protocols that cannot be secure for OT in MDI settings when the honest parties are using imperfect single photon sources.

A. Bit Commitment (BC) with perfect single photon sources

In this section we explain Bit Commitment, present a protocol that implements it when the honest parties have access to a perfect single photon source (Protocol II.1), and state the security of this protocol in the Noisy Quantum Storage Model.

Bit Commitment is a two-phase task between two parties, Alice and Bob, where in the first phase Alice commits to a bit of her choice to Bob. Later they can run the second phase (the “Open” phase) where Alice reveals the bit to which she committed. Importantly, Alice should not be able to open a bit different than the one to which she committed. Also we require that Bob cannot learn the value of the committed bit before Alice opens it. The case in which Alice commits to a bit-string rather than a single bit is called String Commitment. In the following we give a formal definition for a randomized version of String Commitment, where Alice does not get to choose the string she commits to. This string will be produced uniformly at random by the protocol. Note that a Randomized String Commitment can be turned into a String Commitment scheme as explained in [KWW12].

In this paper we will use the security definition of Bit Commitment from [KWW12] informally stated below. The reader can find the formal Definition C.1 in Appendix C.

Definition II.1 (Randomized String Commitment (informal)).
A protocol implements an $(l, \epsilon)$-Randomized String Commitment if it satisfies the following three conditions:

Correctness: If both Alice and Bob are honest, the protocol outputs a classical state $\rho_{C_1|C_1,F}$ such that $\rho_{C_1,F}$ is $\epsilon$-close to $\tau_{C_1} \otimes |\text{accept}\rangle\langle\text{accept}|_F$, where $\tau_{C_1} := \frac{1}{2^l}$ is maximally mixed and $C_1$ is an $l$-bit-string.
Security for Bob: If Bob is honest, then there exists a string $C_1^l$ after the Commit phase, such that the probability that Alice opens to another string $C_1^l \neq C_1^l$, and Bob accepts is smaller than $\epsilon$.

Security for Alice: If Alice is honest, then after the Commit phase and before the Open phase Bob is “$\epsilon$-ignorant” about the string $C_1^l$ that Alice has received during the Commit phase.

$$C_1^l \in_R \{0,1\}^l$$

Figure 3: Ideal Randomized String Commitment. In the first part Alice gets a random $l$-bit string $C_1^l \in \{0,1\}^l$, and Bob is notified that the string is committed. In the second phase, Alice asks “the box” to reveal the string to Bob.

In this work we show that the protocol below implements a secure String Commitment scheme.

**Theorem** (Security of Protocol [II.1] (Informal)). Let $0 < \epsilon < 1$, let $l$ be the length of the string to be committed, let $\epsilon_{\text{err}}$ be the expected error rate between the outcomes of honest Alice and honest Bob in the preparation phase of the protocol [II.1] and let $D$ be an upper-bound on the size of dishonest Bob’s quantum memory expressed in qubits. If honest players have access to perfect single photon sources, then Protocol [II.1] implements an $(l, 3\epsilon)$–Randomized String Commitment according to the above definition. In particular it does so using $n$ rounds of quantum communication, where $n$ is a positive integer solution to $(\lambda - h(\delta)) n \geq l + 2 \log(1/2\epsilon) + \ln(\epsilon^{-1})$, where $\lambda := f(-D/n) - 1/n$ ($f$ is defined in eq. [I]), and $\delta = 2\epsilon_{\text{err}} + 2\alpha_2$, where $\alpha_2$ is a term that accounts for statistical fluctuations $\alpha_2 = O(n^{-1/2})$.

The reader can find a formal version of this theorem in the Methods Section together with its proof, see Theorem IV.6 Intuitively – in the MDI settings with perfect single photon sources – the only difference for the security analysis as compare to the analysis of the protocols presented in Refs. [KWW12, NJC+12] is that honest Bob sends information to malicious Alice. However since we are guarantied (by assumption) that Bob sends BB84 states on single photons, we can use a purification argument in order to reduce the MDI situation to the one of Refs. [KWW12, NJC+12] (see Figure 7) where only Alice sends information to Bob.

**Remark II.1** (Bell measurement). In the MDI protocols we will describe in this paper we use measurements that we call “Bell measurement”. Usually the terminology “Bell measurement” designates a two-qubits-projective measurement described by the four projections onto the Bell states $X^a Z^b \frac{(|00\rangle + |11\rangle)}{\sqrt{2}}$, $(a, b) \in \{0,1\}^2$, where $X$ and $Z$ denote the Pauli $X$ and $Z$ operators. In general, a measurement whose operators are projections onto four orthogonal maximally entangled state is called a “deterministic Bell measurement”. However, in this paper the expression “Bell measurement” refers to a more general type of measurements sometimes called “probabilistic Bell measurements”. A probabilistic Bell measurement is a two-qubit measurement where one or two of its operators are projections onto orthogonal maximally entangled states, the other operators being arbitrary (on the condition that the set of operators considered describes a valid measurement). The outcomes corresponding to operators that are not projections onto maximally entangled states will be considered as “failure” outcomes. This notion of probabilistic Bell measurement arises naturally when considering linear optical implementation of such measurements. Indeed linear optics does not allow to implement deterministic Bell measurements [CL01]. Furthermore, it is sometimes possible to detect when the qubits were lost before reaching the measurement device. This will also be considered as a “failure” outcome. The overall probability of obtaining a failure outcome is denoted $p_{\text{fail}}$.

We present below a protocol for Randomized String Commitment adapted from [KWW12] to the measurement-device-independent case. In this protocol Alice and Bob will start with a preparation phase in which they send $n$ states randomly chosen from the set $\{0, 1, +, -\}$ to the measurement station which will perform a Bell measurement on these qubits and broadcast the outcome. For the rounds in which Alice and Bob have used the same basis to encode their states, the Bell measurement outcome tells Bob whether he has encoded the same bit as Alice in his qubit or the opposite bit. If they have used a different basis then the Bell measurement outcome does not give any information on their correlation. In order to force any dishonest party to store quantum information, both parties will wait a certain time $\Delta t$ before Alice reveals to Bob which bases she has used to prepare her qubits. This allows
Bob to compute the set of rounds $I \subseteq [n]$ where they have used the same bases. Bob will discard the rounds that do not belong to $I$. From there, they will only use classical communication to extract a random committed string $C^e_i$ in the Commit phase, and to reveal this string in the Open phase.

For the following protocol, we will use a randomly generated $[n, k, d]$-linear code $C \subseteq \{0, 1\}^n$ with fixed rate $R := k/n$ to describe Protocol II.1 and to analyze its security. This does not affect the efficiency of the protocol since the honest parties do not need to decode: We only need to use this code to impose that two strings with the same syndrome have Hamming distance at least $d$. We denote $\text{Syn} : \{0, 1\}^n \rightarrow \{0, 1\}^{n-k}$ for the function that outputs the parity-check syndrome of the code $C$. In this protocol we use the two following shorthand notations $\alpha_1 := \sqrt{\frac{\ln \epsilon^{-1}}{2n}}$, $\alpha_2 := \frac{\ln \epsilon^{-1}}{2(1/2 - \alpha_1)n}$. Let $f(\cdot)$ be the function defined as follows.

$$f(x) := \begin{cases} 
0 & \text{if } x < -1 \\
\frac{1}{2} \log(1/2x + \ln(\epsilon^{-1})) & \text{if } -1 \leq x < 1/2 \\
x & \text{if } 1/2 \leq x \leq 1,
\end{cases}$$

where $g(x) := h(x) + x - 1$ and $h(x) := -x \log(x) - (1 - x) \log(1 - x)$ is the binary entropy.

**Protocol II.1 (Randomized String Commitment).**

**Inputs:** security parameter $\epsilon > 0$, length of the committed string $l > 0$, bound on the size of the adversary’s quantum memory $D$, $e_{\text{err}}$ is the expected error rate that should be observed between Alice’s an Bob strings $X^e_i$ and $\hat{X}^e_i$ (see below).

**Preparation phase:**

Choose the number $n$ of rounds that click, such that $n \geq \frac{l + 2 \log(\lambda/2) + \ln(\epsilon^{-1})}{\lambda - h(\delta)}$, where $\lambda := f(-D/n) - 1/n$, and $\delta = 2e_{\text{err}} + 2\alpha_2$.

1. For round $i$ (until the number of rounds in which the measurement station has clicked is higher than $n$):
   - Alice chooses $X_i \in_R \{0, 1\}$ and $\Theta_i \in_R \{0, 1\}$ uniformly at random, and prepares and sends the state $|X_i\rangle_{\Theta_i}$ (where $|0\rangle := |0\rangle, |1\rangle := |1\rangle, |0\rangle := |+, |1\rangle := |\rangle)$ to the measurement station.
   - Bob chooses $\hat{X}_i \in_R \{0, 1\}$ and $\hat{\Theta}_i \in_R \{0, 1\}$ uniformly at random and prepares and sends the state $|\hat{X}_i\rangle_{\hat{\Theta}_i}$ (where $|0\rangle := |0\rangle, |1\rangle := |1\rangle, |0\rangle := |+, |1\rangle := |\rangle)$ to the measurement station.
   - The measurement station performs a Bell measurement on the two states it receives, and broadcasts the outcome, or whether the measurement failed (see Remark II.1). Depending on the outcome, Bob chooses whether he should flip his bit or not.

2. Alice and Bob discard all the rounds where a failure has been announced. Let’s call $n$ the remaining number of rounds. Alice has strings $X^n_i$ and $\Theta^n_i \in \{0, 1\}^n$, and Bob has strings $\hat{X}^n_i$ and $\hat{\Theta}^n_i \in \{0, 1\}^n$.

3. Both parties wait for a time $\Delta t$.

4. Alice sends $\hat{\Theta}^n_i$ over to Bob.

5. Bob computes the set $I \subseteq [n]$ of rounds $i$ where $\Theta_i = \hat{\Theta}_i$. Bob discards all the rounds $j \notin I$. Let’s then call $\hat{X}_I$ the string formed by all the remaining bits $\hat{X}_i$ with $i \in I$.

Note that when there is no noise we have that $\forall i \in I X_i = \hat{X}_i$. In practice there are always errors: We will call $e_{\text{err}}$ the expected error rate between $X_i$ and $\hat{X}_i$ (for $i \in I$), in other words $e_{\text{err}}$ is the expected fraction of error between $X^e_i$ and $\hat{X}^e_i$.

**Commit Phase:**

1. Bob checks whether $m := |I| \geq 1/2 \cdot n - \alpha_1$. If it is not the case Bob aborts the protocol.

2. Alice chooses a random $[n, k, d]$-linear code $C$ (for fixed $n$ and $k$) and computes $w = \text{Syn}(X^n_I)$ and sends it to Bob.

3. Alice picks a random 2-universal hash function $r \in_R \mathcal{R}$ and sends it to Bob.
4. Alice outputs $C'_1 := \text{Ext}(X^R_1, r)$ where $\text{Ext}(\cdot, \cdot)$ is a randomness extractor from the 2-universal family of functions.

Open phase:

1. Alice sends $X^R_1$ to Bob.
2. Bob computes its syndrome and checks if it agrees with $w$ he received from Alice in the Commit phase. If they disagree Bob aborts the protocol.
3. Bob checks that the number of rounds $i \in I$ where $X^R_1$ and $\tilde{X}^R_i$ do not agree lies in the interval $|e_{\text{err}} - \alpha_2, e_{\text{err}} + \alpha_2|$. If not, Bob aborts the protocol, otherwise Bob accepts, and he outputs $C'_1 := \text{Ext}(X^R_1, r)$ where $\text{Ext}(\cdot, \cdot)$ is a randomness extractor from the 2-universal family of function.

In order to satisfy the security definition for Randomized String Commitment, when the protocol aborts, the honest parties will continue the protocol as if they were not aborting – in particular they do not announce the abort event until the end of the protocol – and in the end honest Bob always rejects the commitment and output a uniformly random value to $C'_1$, and honest Alice outputs a uniformly random value for $C'_1$.

B. Oblivious Transfer (OT) with perfect single photon sources

In this section we explain OT, present a protocol that implements it when the honest parties have access to a perfect single photon source (Protocol II.3), and we state the security of this protocol in the Bounded Quantum Storage Model.

![Figure 4: In a Randomized 1-out-2 Oblivious String Transfer, Alice should get two random $l$-bit strings $(S_0, S_1)$ and Bob should receive a random bit $C$ together with $S_C$ which is one of the two strings Alice has received. Alice should never learn $C$ and Bob should remain ignorant about at least one of the two bit-strings Alice receives.](image)

Figure 4: In a Randomized 1-out-2 Oblivious String Transfer, Alice should get two random $l$-bit strings $(S_0, S_1)$ and Bob should receive a random bit $C$ together with $S_C$ which is one of the two strings Alice has received. Alice should never learn $C$ and Bob should remain ignorant about at least one of the two bit-strings Alice receives.

OT, or rather its variant called Randomized 1-out-2 Oblivious String Transfer, is a task where Alice receives two random strings $(S_0, S_1)$ and Bob receives one of this string $S_C$ together with its corresponding index $C$ (see Fig. 4). We will use the definition of the Randomized 1-out-2 Oblivious String Transfer from [KWW12] which is informally stated below. The reader can find the formal Definition C.2 in Appendix C.

**Definition II.2 (Randomized String Transfer (informal)).** A protocol implements an $(l, \epsilon)$-Randomized 1-out-2 Oblivious String Transfer if it satisfies the following three conditions:

**Correctness:** If Alice and Bob are honest the protocol’s output state $\rho_{(S_0, S_1), (S_C, C)}$ is such that the reduced state $\rho_{S_0, S_1, C}$ is $\epsilon$-close to $\tau_{S_0} \otimes \tau_{S_1} \otimes \tau_C$, where $\tau_R$ denotes the maximally mixed state on register $R$, and $S_0, S_1$ are two $l$-bit-strings.

**Security for Alice:** If Alice is honest, then Alice should get two $l$-bit-strings $S_0$ and $S_1$ such that there exists a binary random variable $C$ such that Bob is “$\epsilon$-ignorant” about the bit string $S_{1-C}$. We say that the protocol is $\epsilon$-hiding.

**Security for Bob:** If Bob is honest then he should receive a random bit $C$ and an $l$-bit-string $S_C$, such that Alice is “$\epsilon$-ignorant” about $C$. We say that the protocol is $\epsilon$-binding.

In this work we show that Protocol II.2 presented below implements a secure Randomized Oblivious Transfer.

**Theorem (Randomized 1-out-2 OT (Informal)).** Let $0 < \epsilon < 1$, let $l \geq 0$ be an integer, let $e_{\text{err}} \in [0, 1/2]$ be the expected error rate between the outcomes of honest Alice and honest Bob in the preparation phase of Protocol II.2 and let $D$ be an upper-bound on the size of dishonest Bob’s quantum memory expressed in qubits. When honest parties have access to perfect single photon sources, Protocol II.2 implements an $(l, \epsilon)$-Randomized 1-out-2 Oblivious String Transfer according to the above definition in the Bounded Quantum Storage Model. In particular it does so using a
linear (in the length $l$ of Alice’s strings $|S_0| = |S_1| = l$) number of rounds of quantum communication. More precisely, the number $n$ of quantum communication rounds must satisfy $n \geq 2^{l + D + 1 - 2 \log(1 - \sqrt{\frac{1}{2\pi\epsilon'}})}$, where $\lambda := 1/2 - \delta'$ with $\delta' = 2 \log(\sqrt{(32 \ln \epsilon^{-1})/n})/(32 \ln \epsilon^{-1})/n$.

The reader can find a formal version of this theorem in the Methods Section together with its proof, see Theorem IV.11. Intuitively – in the MDI settings with perfect single photon source – using a purification argument on the states sent by Bob, we can essentially reduce the security proof of our protocol to the security proofs of the trusted device protocol presented in Ref. [KWW12] in which all devices are trusted. However we need to be careful because we also want to take into account noise which has not been done in Ref. [KWW12].

The protocol presented below is also adapted from [KWW12]. For the following Protocol, let $\alpha_1 := \sqrt{\frac{\ln \epsilon^{-1}}{2n}}$ be a term accounting for statistical fluctuations.

### Protocol II.2 (Randomized 1-out-2 OT).

**Inputs:** security parameter $\epsilon > 0$, the length $l$ of the strings Alice receives, the bound (expressed in qubits) on the adversaries memory $D$, expected error rate $\epsilon_{err}$ between Alice’s and Bob’s strings $X_I$ and $\hat{X}_I$ defined below.

**Preparation phase:** They first choose the number of rounds $n$ in which the station clicks, such that $n \geq 2^{l + D + 1 - 2 \log(1 - \sqrt{\frac{1}{2\pi\epsilon'}})}$. Then Alice and Bob do the same as in the preparation phase of Protocol II.1. At this point Alice has a string $X^n_I$, and Bob has a string $\hat{X}_I$ and the set $I \subseteq [n]$.

**Post Processing:**

1. Bob checks whether $|I| \geq (1/2 - \alpha_1)n =: m$. If this is the case he randomly truncates $I$ such that $|I| = m$. Otherwise he aborts.
2. Bob picks a random subset of $I^c$ of size $m$ called $I_{Bad}$. Bob chooses a bit $C$ uniformly at random. He then renames $(I, I_{Bad})$ into $(I_C, I_I-C)$. Bob sends $(I_0, I_1)$ to Alice.
3. Alice sends Bob error correction information for the strings $X_{I_0}$ and $X_{I_1}$.
4. Bob uses the error correction information $O$ to correct his string $\hat{X}_I$.
5. Alice chooses two 2-universal hash functions $r_0, r_1 \in R$ uniformly at random and sends them to Bob.
6. Alice outputs $(S_0, S_1) := (\text{Ext}(X_{I_0}, r_0), \text{Ext}(X_{I_1}, r_1))$, and Bob outputs $(\hat{S}_C, C) := (\text{Ext}(X_{I_C}, r_C), C)$.

In order to satisfy the security definition for OT, when an honest party aborts the protocol, the aborting party will continue the protocol as if they were not aborting – in particular, they do not announce the abort event until the end of the protocol – except that in the end, when the abort event is announced all honest parties assign to their outputs uniformly random values.

### C. Bit Commitment with imperfect single photon sources

In this section we present a protocol that implements String Commitment when the honest parties do not have access to perfect single photon sources (Protocol II.3), and we state the security of this protocol in the Noisy Quantum Storage Model. In this situation, the multiphoton emissions can leak – to dishonest Alice – information about the bases Bob used in his encoding. As a consequence, malicious Alice could take advantage of that by selectively announcing all single photon emissions as “lost”, and keep only the rounds where she has information on the bases used by Bob. Malicious Bob can do the same to get some advantage over honest Alice. To prevent this, and make sure that most of the rounds that are kept in the end correspond to single photon emission rounds we will use the decoy states technique [LC05] similar to [WCSL10]. This will allow the honest party to estimate an upper-bound on the number of rounds that are kept in the end and which correspond to multiphoton emissions.

Examples of photon sources are lasers. They produce coherent states that can be written in the Fock basis as


\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{a^n}{\sqrt{n!}} |n\rangle, 
\]

where \(|n\rangle\) is the photon number eigenstate associated to photon number \(n\), and \(\alpha \in \mathbb{C}\). The intensity is the average number of photons of such a state, and is given by \(|\alpha|^2\). As in some MDI QKD experiments \(\text{LCQ12, LCW}^{+13}\), one can use a randomized phase coherent state in order to turn the laser into an imperfect single photon source \(\text{WCSL10}\). A randomized phase coherent state is a coherent state where \(\alpha = re^{i\phi}\) with \(r > 0\) and where \(\phi\) is chosen uniformly at random in \([0, 2\pi]\). To anyone that does not know which phase has been picked, this state is equivalent to the mixed state \(\rho_{|\alpha|^2} = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} |n\rangle\langle n|\). When one wants to produce single photons, one can use an attenuated laser that produces states with a low average number of photon, i.e. with small \(|\alpha|^2\). For example for \(|\alpha|^2 = 0.1\), the state \(\rho_{|\alpha|^2}\) is essentially a mixture of \(|0\rangle\langle 0|\) with probability \(\approx 0.905\), \(|1\rangle\langle 1|\) with probability \(p_1 \approx 0.0905\), and multiphoton emissions with probability \(p_{\geq 2} \approx 0.0045\), which gives a fraction of \(\approx 5\%\) of multiphoton emissions conditioned on emitting at least one photon, which means that the source mostly (95\% of non 0 emissions) emits single photons and and emits a small amount of multiphoton states (about 5\% of non 0 photon emissions). In a protocol like MDI BC we encode the state in some degree of freedom like polarization. This is a problem for the rounds where multiple photons have been emitted. When only one photon is emitted the possible states Bob can encode are \(\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}\), and therefore the state sent from Bob to Alice conditioned on a choice of basis, \(\theta = 0\) or \(\theta = 1\) are \(\rho_{\theta=0} = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|) = 1/2 = 1/2(|+\rangle\langle +| + |-\rangle\langle -|) = \rho_{\theta=1}\), meaning that Alice cannot guess which basis Bob has used to encode his state. On the contrary, if for example two photons have been emitted the states are \(\rho_{\theta=0} = 1/2(|00\rangle\langle 00| + |11\rangle\langle 11|) \neq 1/2(|++\rangle\langle ++| + |-\rangle\langle -|) = \rho_{\theta=1}\), meaning that Alice can guess the basis used with non 0 advantage. This is a problem since security against dishonest Alice relies on her being ignorant about Bob’s basis information. In particular we want to avoid the case where dishonest Alice measures the photon number of the incoming state from Bob, and chooses to announce failure only if she receives single photon. This is why we use decoy states: They will allow us to estimate how many single photon rounds have been reported as failure.

For BC in the case where honest parties use imperfect single photon sources, Protocol \(\text{II.3}\) can be used. The main difference as compare to Protocol \(\text{II.1}\) is the use of \(q\) additional decoy states in the “Preparation phase”. Alice (Bob) can use different intensities \(\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}\) for the state she (he) sends. Among these intensities one will correspond to the “signal” state and will be denoted \(a_s\) (\(b_s\)), while the others will be the “decoy” states with intensities \(a \in \{a_1, \ldots, a_q\}\) (\(b \in \{b_1, \ldots, b_q\}\)). In Protocol \(\text{II.3}\) we will call \(n_A^1 + n_B^1\) (\(n_A^{\geq 2} + n_B^{\geq 2}\)) the number of rounds where Alice (Bob) has used a “signal” state – i.e. a state with intensity \(a_s\) (\(b_s\)) – and where the measurement station reported the measurement as successful. \(n_A^1\) (\(n_B^1\)) is the number of these states where Alice (Bob) has sent 1 photon, and \(n_A^{\geq 2}\) (\(n_B^{\geq 2}\)) is the number of these rounds where Alice (Bob) has sent \(\geq 2\) photons. Note that at the end of Step 1 of the “preparation phase”, and because we do not consider dark counts in this work, Alice (Bob) knows the value of \(n_A^1 + n_B^1\) (\(n_A^{\geq 2} + n_B^{\geq 2}\)). However even if she (he) knows the sum \(n_A^1 + n_B^1\) (\(n_A^{\geq 2} + n_B^{\geq 2}\)), she (he) does not know the individual terms \(n_A^1\) (\(n_A^1\)) and \(n_B^1\) (\(n_B^1\)) of this sum. Alice (Bob) will only be able to estimate a lower-bound \(L_A\) (\(L_B\)) on \(n_A^1\) (\(n_B^1\)) by using the decoy states. Since \(n_A^1 + n_B^1\) (\(n_A^{\geq 2} + n_B^{\geq 2}\)) is known to Alice (Bob), this lower-bound gives automatically an upper-bound \(U_A = n_A^1 + n_B^1 - L_A\) (\(U_B = n_A^{\geq 2} + n_B^{\geq 2} - L_B\)) on \(n_A^{\geq 2}\) (\(n_B^{\geq 2}\)).

In the following we will write \(p_a\) (\(p_b\)) for the probability that Alice (Bob) prepares a signal of intensity \(a \in \{a_1, a_2, \ldots, a_q\}\) (\(b \in \{b_1, b_2, \ldots, b_q\}\)). When the identity of the emitter is not determined, the intensity will be denoted \(i\) (meaning that \(i = a\) is the emitter is Alice or \(i = b\) is the emitter is Bob). The probability that an emitter emits \(k\) photons will be denoted \(p_k\) (e.g. if \(k = 1\) then we will write \(p_1\) etc.). The probability that the emitter emits more than \(k\) photons will be denoted \(p_{\geq k}\) (e.g. \(p_{\geq 2}\)). We will also mix the two above notations when talking about conditional events. For example, the probability that Alice emits 2 photons conditioned on choosing signal intensity \(a\) will be denoted \(p_{2|a}\).

In this paper we show that Protocol \(\text{II.3}\) below is secure, in particular we show the following. Here we state this lemma in the case in which Alice is honest. A similar statement would apply for honest Bob.

**Lemma** (Single-photon emission round number estimation (Informal)). Let \(q \geq 2\) be the number of different intensities honest Alice can use for the decoy states in Protocol \(\text{II.3}\). Let \(\{p_{a_1}, \ldots, p_{a_q}\}\) be the (known) probabilities that Alice’s
source emits a state with (known) intensities \( \{a_d, \ldots, a_d\} \). Let \( p_a \) be the (known) probability that Alice’s source emits a state with (known) intensity \( a_s \) corresponding to the signal state (i.e. non-decay state). Let \( x^i \) be the observed number of non-discarded rounds where Alice has prepared a signal of intensity \( i \in \{a_s, a_d, \ldots, a_d\} \). Then with high probability, the number of non-discarded signal rounds \( n_s^i \) in which Alice’s source has emitted exactly 1 photon is lower-bounded by \( L_{A1} \), where \( L_{A1} \) is a function of the intensities \( \{a_s, a_d, \ldots, a_d\} \), the probabilities \( \{p_{a_s}, p_{a_d}, \ldots, p_{a_d}\} \), and the observations \( \{x^s, x^a, \ldots, x^d\} \). The analytical expression for \( L_{A1} \) is given in the formal version of the lemma, Lemma IV.13 in the case \( q = 2 \). Its proof is given in Appendix B. For \( q > 2 \) one can compute \( L_{A1} \) numerically as explained in Appendix B.

The above lemma is an essential ingredient to prove the following security theorem.

**Theorem (Security of Protocol II.3 (Informal)).** Let \( 0 < \epsilon < 1 \), let \( l \) be the length of the committed string, let \( p \) be the probability that – in the preparation phase of an honest execution of Protocol II.3 – a given round \( i \) is not discarded, let \( \epsilon_{err} \) be the expected error rate between the outcomes of honest Alice and honest Bob in the Preparation phase of the protocol, and let \( D \) be a bound on the size of dishonest Bob’s quantum memory measured in qubits.

Protocol II.3 implements an \((l, \epsilon)\)-Randomized String Commitment as defined in Definition II.7. In particular it does so using a number \( N \) of quantum communication rounds that is linear in \( l \). More precisely \( N \) must satisfy \( (p - \sqrt{\ln(\epsilon^{-1})/2N}) N \geq n^* \) where \( n^* \) is smallest positive integer solution to \( (\lambda - h(\delta)n) n \geq l + 2\log(1/2\epsilon) + \ln(\epsilon^{-1}) \), \( \lambda := f(D/n) - (\gamma + \alpha_2^A) - 1/n \) with \( n \) being the length of honest Alice’s string \( X^l_2 \) produced at the end of the Preparation phase, \( \alpha_2^A \) is a term accounting for statistical fluctuations, and \( \delta \) is a function of the expected error rate \( \epsilon_{err} \). The exact expression of \( \delta \) and \( \alpha_2^A \) are given in the formal version of this theorem: Theorem IV.14.

A formal version of this theorem together with its proof are given in the Methods Section: Theorem IV.14.

In Protocol II.3 and its security analysis we will use the following notations: \( \in [0,1] \), and \( f_{a_s}, f_{a_d}, \in [0,1] \) are fractions defined in Step 2 of Protocol II.3, \( \alpha_1, \alpha_2 \) are the same as in Section II.4 and as for the terms \( \beta^A, \beta^B, \alpha_4^A, \alpha_4^B \), they account for statistical fluctuation. They all are \( O(1/\sqrt{N}) \) where \( N \) is the number of rounds of the protocol.

Their exact expressions are given in Theorem IV.14. As in Protocol II.1, \( C \) is an random \([n,k,d]\)-linear code and \( \text{Syn} : \{0,1\}^n \rightarrow \{0,1\}^{n-k} \) is the function that outputs the parity-check syndrome of code \( C \).

In order to satisfy the security definition for Randomized String Commitment (Def. II.1), when the protocol aborts, the honest parties will continue the protocol as if they were not aborting – in particular they do not announce the abort event until the end of the protocol – and in the end honest Bob always rejects the commitment and assigns a uniformly random value to his output \( \hat{C}_1 \), and honest Alice assigns a uniformly random value to her output \( \hat{C}_1 \).

**Protocol II.3 (Randomized String Commitment with decoy states).**

**Inputs:** The security parameter \( \epsilon > 0 \), the parameter \( \gamma \in [0,1/2] \) that essentially measures how good the single photon sources are, the length \( l \) of the string that will be produced by the protocol, the maximum size (expressed in qubits) of the adversary’s quantum memory \( D \), the expected error rate \( \epsilon_{err} \) between Alice’s and Bob’s string \( X^l_2 \) and \( \hat{X}^l_2 \), the probability distributions \( (p_{a_s}, p_{a_d}, \ldots, p_{a_d}) \) and \( (p_{b_s}, p_{b_d}, \ldots, p_{b_d}) \) that Alice and Bob use intensities \( \{a_s, a_d, \ldots, a_d\} \) and \( \{b_s, b_d, \ldots, b_d\} \) respectively.

**Preparation phase:**

Alice and Bob agree on a number \( N \) of rounds. \( N \) must satisfy \( (p - \sqrt{\ln(\epsilon^{-1})/2N}) N \geq n^* \), where \( n^* \) the smallest positive integer solution to the inequality eq. (37), and where \( p \) is the probability that any given round \( i \in [N] \) is not discarded in the preparation phase when both parties are honest.

1. For round \( i \in [N] \):
   - Alice chooses \( X^i \in \{0,1\} \) and \( \Theta^i \in \{0,1\} \) uniformly at random, and chooses intensity \( a \in \{a_s, a_d, \ldots, a_d\} \) with some probability distribution \( p_a \). Alice prepares a quantum signal of intensity \( a \), encoding \( X^i \) in the basis \( \Theta^i \), and sends it over to the measurement station.
   - Bob chooses \( \hat{X}^i \in \{0,1\} \) and \( \hat{\Theta}^i \in \{0,1\} \) uniformly at random, and chooses intensity \( b \in \{b_s, b_d, \ldots, b_d\} \) with some probability distribution \( p_b \). Bob prepares a quantum signal of intensity \( b \), encoding \( X^i \) in the basis \( \Theta^i \), and sends it over to the measurement station.
   - The measurement station performs a Bell measurement on the two states it receives, and publicly reveals the outcome, or whether the measurement failed (see Remark II.4).

2. Alice and Bob publicly announce the intensities they have used for all the rounds \( i \in [N] \) (the order in which this is announced is not important). Alice checks that among the rounds where she has used intensity \( a_s \) and the measurement succeeded, the fraction \( f_{a_s} \) of rounds where Bob has used intensity

\[ \text{Lemma IV.15 in the case q = 2. Its proof is given in Appendix B. For q > 2 one can compute L_{A1} numerically as explained in Appendix B.} \]
b_s is higher than \( p_{a_s} - \beta^A \). Bob checks that among the rounds where he has used intensity b_s and the measurement succeeded, the fraction \( f_{a_s} \) of rounds where Alice has used intensity a_s is higher than \( p_{a_s} - \beta^B \). If this is not the case, Alice or Bob abort the protocol.

3. Using the decoy states Alice estimates a lower-bound \( L_{A1} \) for \( n_1^A \) (this is given by Lemma IV.13), the number of rounds where the Bell measurement has not been announced as a failure and where Alice emitted 1 photon with intensity a_s. If \( \frac{L_{A2}}{f_{a_s}(n_1^A + n_2^A) + n_2^A} \geq \gamma + \alpha^A \) Alice aborts the protocol.

4. Using the decoy states Bob estimates a lower-bound for \( n_1^B \) (this is given by Lemma IV.13), the number of rounds where the Bell measurement has not been announced as a failure and where Bob emitted 1 photon with intensity b_s. If \( \frac{L_{B2}}{f_{a_s}(n_1^A + n_2^A) + n_2^B} \geq \gamma + \alpha^B \) Bob aborts the protocol.

5. Alice and Bob discard all the rounds where a failure has been announced, and where the intensities used by Alice and Bob are not \( a_s \) and \( b_s \). Let’s call the remaining number of rounds \( n \). Alice has strings \( X_1^a \) and \( \Theta_1^a \in \{0,1\}^n \), and Bob has strings \( X_1^b \) and \( \Theta_1^b \in \{0,1\}^n \). Note that \( n = f_{a_s} \times (n_1^A + n_2^A) = f_{a_s} \times (n_1^B + n_2^B) \). Alice and Bob check that \( n \geq \frac{l+2\log(1/2e)+\ln(\epsilon^{-1})}{\lambda-h(\delta)} \), and otherwise abort the protocol.

6. Both parties wait for a time \( \Delta t \).

7. Alice sends \( \Theta_1^a \) over to Bob.

8. Bob computes the set \( I \subseteq [n] \) of rounds \( i \) where \( \Theta_i = \Theta_i^a \). Bob discards all the rounds \( j \notin I \). Let’s then call \( X_I \) the string formed by all the remaining bits \( X_i \) with \( i \in I \).

Note that when there is no noise we have that \( \forall i \in I \) \( X_i = X_i^a \). In practice there are always errors: We will call \( \epsilon_{\text{err}} \) the expected errors rate between \( X_i \) and \( \hat{X}_i \) (for \( i \in I \)).

**Commit Phase:**

1. Bob checks whether \( m := |I| \in [1/2 \cdot n - \alpha_1, 1/2 \cdot n + \alpha_1] \). If this is not the case Bob aborts.

2. Alice chooses a random \( [n,k,d] \)-linear code \( C \) (for fixed \( n \) and \( k \)) and computes \( w = \text{Syn}(X_1^a) \) and sends it to Bob.

3. Alice picks a random 2-universal hash function \( r \in_R \mathcal{R} \) and sends it to Bob.

4. Alice outputs \( C_1^d := \text{Ext}(X_1^a, r) \) where \( \text{Ext}(\cdot, \cdot) \) is a randomness extractor from the 2-universal family of function.

**Open phase:**

1. Alice sends \( X_1^a \) to Bob.

2. Bob computes its syndrome and checks if it agrees with \( w \) he received from Alice in the Commit phase. If they disagree Bob aborts.

3. Bob checks that the number of rounds \( i \in I \) where \( X_i^a \) and \( \hat{X}_I \) do not agree lies in the interval \( [\epsilon_{\text{err}} - \alpha_2, \epsilon_{\text{err}} + \alpha_2] \). If not, Bob aborts the protocol, otherwise he outputs \( C_1^d := \text{Ext}(X_1^a, r) \) where \( \text{Ext}(\cdot, \cdot) \) is a randomness extractor from the 2-universal family of function.

We require that \( (p - \sqrt{\ln(\epsilon^{-1})/2N}) \geq n^* \), for \( n^* \) satisfying \( (\lambda - h(\delta))n \geq \frac{l+2\log(1/2e)+\ln(\epsilon^{-1})}{\lambda-h(\delta)} \) only to make sure there are enough rounds to produce \( l \)-bits final strings in a secure way. \( p \) is the probability that a round is not discarded in the honest scenario, and it can be expressed a function of the experimental parameters: The round won’t be discarded if both players sent a signal state for this round, which happens with probability \( p_{a_s} \times p_{b_s} \), and if the measurement station did not report this round as failure (see Remark II.1) which happens with probability \( 1 - p_{\text{fail}_{a_s,b_s}} \), so \( p = p_{a_s} \times p_{b_s} \times (1 - p_{\text{fail}_{a_s,b_s}}) \).

\(^\dagger\) Note that \( \lambda \) and \( \delta \) implicitly depend on \( n \), therefore one cannot solve the inequality analytically.
D. OT with an imperfect single photon sources

In this section we will prove that MDI Oblivious Transfer is “not easy” in practical settings. Indeed in practice photon sources are not perfect i.e. they have some probability $p_{>2}$ to emit more than one photon. If now one considers a protocol containing a preparation phase similar to the one of Protocol II.2 but where now Bob has an imperfect single photon source, it becomes possible for a malicious Alice to deduce from the states she receives from Bob, some of the bases $\Theta_i$ that have been used in Bob’s encoding. As we will explain below this is due to the fact that when more than one photon are emitted by Bob’s source, a dishonest Alice can distinguish states encoded in the standard and the Hadamard basis, which is not possible to do when a single photon is emitted. This is a leakage of information that has heavy consequences on the feasibility of an OT protocol as explained below.

We will illustrate how this leakage of information can break security of a protocol, by describing what happens to Protocol II.2 when Alice is malicious and Bob holds an imperfect single photon source. After this we will generalize the reasoning.

Dishonest Alice’s end goal is to guess correctly the value of bit $C$ that Bob will get at the end of the protocol. Moreover, Alice being malicious implies that Alice has full control over the measurement station, and therefore everything Bob sends to the measurement station can be considered in Alice’s possession. Let us now start with the preparation phase of Protocol II.2. In this phase of the protocol, Bob sends BB84 state $C$ to the measurement station, or equivalently to dishonest Alice. But contrary to section IIB Bob now holds an imperfect single photon source. This means that in some of the rounds, more than one photon are sent to Alice. This becomes a problem because if, for example, the source has emitted two photons, then the state Alice receives conditioned on Bob preparing it in the standard basis is $1/2(\langle 00|00\rangle + \langle 11|11\rangle)$, while if we condition the state on being prepared in the Hadamard basis it is $1/2(\langle + +| + +\rangle + \langle + -| - -\rangle)$. These two states are not the equal, and therefore Alice can use these states to guess the basis $\Theta$, that Bob has used to encode the state. When a single photon is used this is not a problem since $1/2(\langle 0|0\rangle + \langle 1|1\rangle) = 1/2(\langle +|+\rangle + \langle -|-\rangle)$: the two cases – Bob prepares the state in the standard or the Hadamard basis – are perfectly indistinguishable. Moreover, the more photons are emitted by the source, the easier it is for Alice to guess correctly which basis Bob has used. To be conservative, for each round in which multiple photons have been emitted we will consider that malicious Alice knows exactly Bob’s choice of basis $\Theta_i$.

At the end of the preparation phase malicious Alice sends a string $\Theta^{|i|}_i$ to Bob. Bob uses the string $\Theta^{|n|}_i$ he received from Alice and his own choice of bases described by the string $\hat{\Theta}^{|n|}_i$ to compute the set $I := \{i \in [n] : \Theta_i = \hat{\Theta}_i\}$, which is the set of rounds in which Bob’s choice of bases matches the value of the bit malicious Alice has sent to him, and where $n$ denotes the total number of rounds. He also erases all the bits $X_i$ he has used to encode the states he has sent to the station for all $i$ such that $i \notin I$. At this point Bob holds the set $I$ and the string $\hat{X}_I$ which is formed by all the bits $X_i$ he has used in the round $i \in I$. Remember that Malicious Alice knows the value of $\hat{\Theta}_i$ in some of the rounds, and therefore knows whether these rounds correspond to rounds in $I$ or not. We call $I_G$ the set of rounds for which Alice knows that they are in $I$ and $I_B$ the set of rounds for which she knows that they are not in $I$. The choice bit $C$ that has heavy consequences on the feasibility of an OT protocol as explained below.

The way Alice chooses the value for $\Theta^{|i|}_i$ has no importance, and we will therefore consider $\Theta^{|n|}_i$ as a fully random string in this argument.

The statement we will make is expressed in in terms of asymptotic security, i.e. we will say that Alice can cheat if she has a non-negligible advantage in guessing Bob’s bit $C$ (see Theorem IV.20 below). A function is said to be

\[ |X_i|_{\Theta_i} \]

which correspond to encoding in the basis $\hat{\Theta}_i$, where $\hat{\Theta}_i = 0$ corresponds to the standard basis and $\hat{\Theta}_i = 1$ corresponds to the Hadamard basis.
negligible (in some variable \( n \)) if it is smaller than \( 1/n^a \) (for any \( a > 0 \) and for \( n \) large enough). Similarly we will say that a probability \( p \) is overwhelmingly large if \( 1 - p \) is negligible.

In order to generalize the attack on Protocol II.2 as we have seen above, we work in a model (see Fig. 3), where Alice and Bob have already run a quantum phase of a protocol, that has given registers \( X^n \) to honest Alice and \( X_t, I \) to Bob. \( X^n \) is a bit string and \( X_t \) is a substring of \( X^n \) whose bits are the ones corresponding to the set of indices \( I \subseteq \{ n \} \). One can typically think of a “quantum phase” as being the preparation phase of Protocols II.2 & II.3 for example.

If Alice is dishonest we assume that she has recorded — during this quantum phase — information leaked by the imperfection of Bob’s source. We model this leakage of information by giving dishonest Alice two extra registers \((I_G, I_B)\) that correspond to two sets of indices correlated with \( I \). When Bob is dishonest we simply assume that he holds the cq-registers \( KQ \) such that his min-entropy on Alice’s string \( X^n \) is smaller than honest Bob’s one. Since we work in the bounded storage model we assume \( \log \text{dim} Q \leq D \).

After this quantum phase of the protocol, we assume that Alice and Bob perform a classical post-processing. One such post-processing is the post-processing of Protocol II.2. When a party is dishonest we assume he will in fact be semi-honest during the post processing, meaning that he will run the post-processing honestly but record all the information he has received or sent. We prove that if such a protocol is correct and secure against dishonest Bob, then Protocol II.4 gives dishonest Alice a (semi-honest) strategy to use her extra input registers to guess honest Bob’s output bit \( C \) with non-negligible advantage.

We will describe the set of messages going from Bob to Alice by the random variable \( M_{BA} \). The messages from Alice to Bob will be described by the random variable \( M_{AB} \). The random variable composed of these two variables will be called \( M \). In other words \( M := (M_{AB}, M_{BA}) \).

The output of honest Alice is \((S_0, S_1) := (f_0(X^n, M), f_1(X^n, M)) \in \{0, 1\} \times \{0, 1\}\), where \( f_0 \) and \( f_1 \) are two functions determined by the protocol. Typically, these functions are the composition of error correction with a randomness extractor. The output of honest Bob is \((C, S_C) := (g(X_t, I, M), \tilde{g}(X_t, I, M))\), where \( g \) and \( \tilde{g} \) are two other functions determined by the protocol. These four functions model the operations that honest Alice and Bob have to perform according to the protocol they are running.

We construct an attack where Alice is semi-honest (or equivalently “honest but curious”), that is, she will execute the post-processing part of the protocol honestly but keep all the information that she has exchanged with Bob so that she can in the end compute whatever she is interested in, which in this case is \( C \). Our result holds under two assumptions stated below. This restricts the applicability of our theorem. However, we argue in the Discussion Section that these assumptions should still be sufficiently general for many practical settings.

**Assumptions II.1** (Informal). In order to prove the theorem below we need two assumptions. Let \( f_0 \) and \( f_1 \) be the functions that map honest Alice’s available information \((X^n, M)\) to her outputs \( S_0 \) and \( S_1 \): \( S_0 := f_0(X^n, M) \) \& \( S_1 := f_1(X^n, M) \). Let \( (I_G, I_B) \) be the sets of indices that dishonest Alice gets before the execution of the post-processing procedure due to the imperfection of Bob’s photon source (see Fig. 3).

1. There exists a computable function \( F \) than maps \((X^n, M)\) to the pair of sets \((I_0, I_1)\) corresponding to the positions of the bits of \( X^n \) on which the functions \( f_0 \) and \( f_1 \) depend.

2. There is a non-negligible probability that,

\[
\begin{align*}
\text{the intersection between the set } I_G \cup I_B \text{ and } I_0 \setminus I_1 & \text{ is not empty} \\
\text{the intersection between the set } I_G \cup I_B \text{ and } I_1 \setminus I_0 & \text{ is not empty.}
\end{align*}
\]

If we define \( \kappa \) being the minimum size of the two intersections above, we can rephrase this condition by saying that, there is a non-negligible probability that \( \kappa \geq 1 \).

The sets \( I_G, I_B \) are the sets dishonest Alice gets from the leakage of the quantum part of the protocol. The sets \( I_0, I_1 \) are the sets correlated to set \( I \) and bit \( C \) that do not reveal value of bit \( C \) as long as \( I \) is completely unknown from Alice. Of course since dishonest Alice has extra information \( I_G, I_B \) correlated to \( I \), Alice is not ignorant about \( I \): She therefore has some information about the bit \( C \), which as we will see allows her to cheat. The reader can find a more formal version of these assumptions in the Methods Section: Assumption IV.1.

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4 Remember that in Protocol II.2 sets \((I_0, I_1)\) correspond to the renaming of the sets \((I, I')\) that Bob sends to Alice.
Some quantum protocol with imperfect photon source

Figure 5: Schematic view of the classical post-processing between (dishonest) Alice and (dishonest) Bob. Before the post-processing Alice and Bob have run an unspecified quantum protocol which gave them their inputs: $X_i^n + (I_G, I_B)$ for (dishonest) Alice, and $(X_T, X)$, or KQ for (dishonest) Bob.

When Alice is dishonest, we will consider that she is “honest but curious” at the post-processing level, meaning that Alice will run the post-processing honestly with Bob, but she will record all communication $M_{AB}, M_{BA}$ and use them at the end together with her extra-input $(I_G, I_B)$, to extract more information than what she should get out of the protocol. To do so she will use the strategy described in Protocol II.4.

Theorem (Dishonest Alice cheating (Informal)). If a quantum protocol between Alice and Bob that implements OT is such that it leaks some information $(I_G, I_B)$ to dishonest Alice in the quantum phase (before the classical post-processing), and if this protocol is correct and secure against dishonest Bob, then there exists a strategy for dishonest Alice that allows her to cheat, i.e. she can guess Bob’s bit $C$ with non-negligible advantage. This strategy runs as follows: Dishonest Alice runs honestly the post processing phase with Bob, but records all messages sent and received during this post-processing. At the end of the post-processing she will use all this messages together with her extra information $(I_G, I_B)$ in order to locally run the procedure described in Protocol II.4. This procedure outputs her guess for Bob’s bit $C$.

The reader can find a formal version of this theorem in the Methods Section, together with its proof: Theorem IV.20.

We recall that when Alice is dishonest she holds some extra set $I_G$ (and $I_B$), which in a protocol like Protocol II.2 would typically correspond to the multiphoton rounds where dishonest Alice has inferred that Bob has used the same basis as she did (or a different basis for $I_B$). So at the end of the post-processing she will execute the strategy detailed in Protocol II.4 where she starts by computing the two sets $I_0$ and $I_1$. She will then choose uniformly at random – thanks to random bit $r$ – whether she later wants to sample at random an index in $S_0 := I_0 \setminus I_1 \cap (I_G \cup I_B)$ or in $S_1 := I_1 \setminus I_0 \cap (I_G \cup I_B)$. At this point Alice samples uniformly at random an index in $S_r$, and checks whether this round is in $I_G$ or in $I_B$. If it is in $I_G$ then Alice’s guess for Bob’s bit $C$ will be $r$, and otherwise she guesses $1 - r$. More formally Alice proceeds as follows.

**Protocol II.4** (Dishonest Alice’s strategy).

**Inputs:** $x_i^n, m, I_G, I_B$.

**Outputs:** $b$.

- Alice computes $(I_0, I_1) = F(x_i^n, m)$, where $F$ is given by Assumptions II.2.

- Alice checks that $I_0 \setminus I_1 \cap (I_G \cup I_B) \neq \emptyset$ and $I_1 \setminus I_0 \cap (I_G \cup I_B) \neq \emptyset$. If this is not the case Alice outputs $b \in_R \{0, 1\}$ uniformly at random, otherwise she continues with the protocol.
• Alice sample a bit r uniformly at random.
• Alice chooses an index \( i_r \in I_r \setminus I_{1-r} \cap (I_G \cup I_B) \) uniformly at random.
• Alice checks whether \( i_r \in I_G \) or \( i_r \in I_B \). If \( i_r \in I_G \) then Alice outputs \( b = r \) and she outputs \( b = 1 - r \) otherwise.

Alice’s output bit \( b \) represents Alice’s guess for the bit \( C \) that honest Bob got from the protocol.

Intuitively the sets \( I_0, I_1 \) carry information about the correlations Alice and Bob share at the beginning of the post-processing, but not about Bob’s final output \( C \). In particular these sets say that if their initial (honest) inputs are such that Bob knows the bits of \( X^n_1 \) on positions given by \( I_0 \) then \( C = 0 \), and if he initially knows the bits of \( X^n_1 \) on positions given by \( I_1 \) then \( C = 1 \). However since honest Alice does not know which bits of \( X^n_1 \) Bob knows (she is ignorant about \( I \)), it does not say anything about the actual value of Bob’s output \( C \). Dishonest Alice however gets extra inputs \((I_G, I_B)\) that precisely gives her information about which are the bits Bob knows. As a consequence by cross-referencing these two pieces of information dishonest Alice can get some advantage in guessing bit \( C \).

III. DISCUSSION

In the previous section we show that all protocols that satisfy the two assumptions given in Assumptions II.1 (or more formally Assumption IV.1) cannot be secure against dishonest Alice. We believe that the class of protocols that satisfy these conditions is general enough to encompass many of the protocols that are currently implementable with current technology. In this section we argue in this direction.

We first point out that it should not be possible to get a fully general impossibility theorem, since we have shown that when having a sufficiently good single photon source it is possible to devise a secure protocol (see Theorem IV.1). As a consequence one can only prove statements about more restrictive classes of protocols. This is what we have done in the previous section. However we have analyzed these protocols under Assumptions II.1 and it is not clear how restrictive Assumptions II.1 are.

First, let us spell out some of the implicit assumptions made for our theorem that necessarily limit the range of its applicability. In the model we use (see Fig. 5), it is clear that the classical post-processing operated by Alice and Bob runs on bit strings \( (X^n_1, X^n_2, \ldots) \) and of sets of indices \( "I_0, I_1, \ldots" \), however we think that the reasoning used for our theorem can be extended to more general inputs. In this model, we also only start by looking at the attack directly at a post-processing part of the protocols. This is convenient since it allows our theorem to be valid for various quantum implementations that could have run before the post-processing. Of course this assumes that the protocols end with a fully classical post-processing phase. As a consequence our proof only applies for such protocols. However, even though these implicit assumptions limit the applicability of our theorem, we believe that this is enough for any practical implementation.

Let us now go to the core of our assumptions, i.e. let us look at conditions given in Assumptions II.1. The first assumption is, informally, that there exists a way for dishonest Alice to compute, from \( X^n_1 \) and \( M \), sets of indices \( (I_0, I_1) \) that correspond to the positions in the string \( X^n_1 \) where the functions \( f_0 \) and \( f_1 \) are dependent on the value of the bits located at these positions.

The second assumption can be reformulated as follows. If, for a fraction of rounds, some information is leaked then there is a non-negligible probability that \( \kappa \geq 1 \) (see Assumptions II.1).

We now argue that these two condition are not very restrictive.

- Indeed we conjecture that the first assumption should always hold in protocols where the basis choice relates to Bob’s output \( C \): In order for the protocol to be correct, intuitively the set of messages exchanged in the protocol represented by the random variable \( M \) should contain the information that “tells” the functions \( f_0 \) and \( f_1 \) how they should act on the bits of \( X^n_1 \), and on which of these bits they should operate. This suggests that Alice can also retrieve this information, i.e. compute \( (I_0, I_1) \). We do not give a formal proof of this statement, that is why it is taken as an assumption. In a protocol like Protocol II.2 it is clear that this condition is satisfied since Bob explicitly sends the pair \( (I_0, I_1) \) to Alice.

- If the second assumption was not satisfied then – at least intuitively – Bob is able to know (with overwhelming probability) which rounds leak information (multiphoton emission rounds) and therefore choose the sets \( I_0 \) and \( I_1 \) (or sufficiently influence the protocol) such that \( \kappa = 0 \). But then Bob could effectively get an almost perfect
Figure 6: Schematic representation of the “Feasibility” of OT and BC depending on the resources/model used in the protocol. In the first column neither BC nor OT are possible, but since OT can be used in order to get BC but not the contrary, OT is somewhat a harder problem, which is why it is below BC. When quantum communication are possible then OT and BC are equivalent (represented at the same level) but still impossible. In the third column we add the Bounded Storage assumption, which makes both protocol possible. They are still equivalent. In the last column we add that quantum communication between the parties are made in the MDI settings, and we assume that the parties do not have perfect single photon source. In this case BC is possible (see Theorem IV.14) but OT is not (see Theorem IV.20).

(except with negligible probability) single photon source, by preventing any multiphoton emission from leaving his lab. In a protocol like the ones we have presented in the previous sections, Bob does not know in which round his source has emitted multiple photons, therefore there will be in the end with very high probability multiphoton rounds that are kept.

For these reasons we believe that our impossibility result applies to most (if not all) currently implementable OT protocols.

In the presence of quantum communication, it is known that OT and BC are equivalent [BBS92, FS09], meaning that from one of these tasks one can build a secure protocol for the other. However the construction used, implicitly assumes a trusted device setting, and as a consequence this construction does not necessarily prove equivalence between OT and BC in MDI settings. Since we prove in this work that MDI BC is secure (in the bounded/noisy quantum storage model), if our impossibility result for OT generalizes, the MDI settings (without a single photon source) would be the first quantum setting where one can prove security for BC but not for OT with the same adversarial model (see Fig. 6), i.e. it would be a quantum setting in which OT and BC are not equivalent.

IV. METHODS

In this section we present and prove security statement for the protocols presented in the results sections. We start by stating theorems and lemmas that will be useful in our proofs. Then we prove security for BC and OT when the honest players have perfect photon sources. We continue by giving the security proof for BC when the honest parties only have imperfect single photon sources. We finally prove that a class of protocols cannot be secure for OT when using imperfect single photon sources.

Remark IV.1. For simplicity, all our statements and proofs are expressed in the Bounded Storage model, but can easily be extended to the Noisy Storage Model as explained in [BBCW13, RTK+18].
A. Useful Lemmas and Theorems

Here, we give useful theorems that we will use as tools for our proofs. Before stating these theorems, we need to define (smoothed) min-entropies. The “smoothness” of the smooth-min entropy is defined relatively to the purified distance defined as follows.

**Definition IV.1 (see [Tom16]).** Let $\rho$ and $\sigma$ be two non-normalized quantum states. Their purified distance is given by,
\[\nabla(\sigma, \rho) := \sqrt{1 - F(\sigma, \rho)},\]
where $F$ is the fidelity defined as,
\[F(\sigma, \rho) := \left(\|\sqrt{\sigma} \sqrt{\rho}\|_1 + \sqrt{(1 - \text{tr} \rho)(1 - \text{tr} \sigma)}\right)^2,\]
where $\| \cdot \|_1$ is the Schatten 1-norm: If $A$ is a linear operator acting on a finite dimensional Hilbert space, then $\|A\|_1 := \text{tr} \sqrt{A^\dagger A}$.

We define the ball $B(\rho, \epsilon)$ of radius $\epsilon$ centered in $\rho$, as being the set of non-normalized quantum states whose purified distance to $\rho$ is less or equal to $\epsilon$. One can now define the smooth min-entropy.

**Definition IV.2.** Let $\rho_{AB}$ be a quantum state, and let $\epsilon \geq 0$. The $\epsilon$-smoothed min-entropy on $A$ conditioned on $B$ is defined as
\[H^\epsilon_{\min}(A|B)_\rho := \sup_{\rho \in B(\rho, \epsilon)} \left(\inf_{\sigma_B} - \inf_{\eta \in \mathbb{R}} \{\hat{\rho} : \hat{\rho} \leq 2^n \mathbb{1}_A \otimes \sigma_B\}\right),\]
where $\sigma_B$ ranges over the density matrices, and where $B(\rho, \epsilon)$ is the ball of radius $\epsilon$ centered in $\rho$. The $(\epsilon = 0)$-smoothed min-entropy is simply called min-entropy and is denoted $H_{\min}(A|B)_\rho$.

**Theorem IV.1** (Leftover Hash Lemma with smooth min-entropy [Ren05, TL17]). Let $\rho_{A^n E}$ be a classical-quantum state and let $\text{Ext}(\cdot, \cdot) : \{0, 1\}^n \times \mathcal{R} \mapsto \{0, 1\}^l$ be an extractor based on a 2-universal family of hash functions $\mathcal{R}$ from $\{0, 1\}^n$ to $\{0, 1\}^l$, that maps the classical $n$-bit string $A^n$ into $K_A$. Then
\[\|\rho_{K_A RE} - \tau_{K_A} \otimes \rho_{RE}\|_1 \leq 2^{-\frac{1}{2}H^\epsilon_{\min}(A^n|E)_{\rho^{-1}}} + 2\epsilon,\]
where $\tau$ denotes the maximally mixed state, and $\| \cdot \|_1$ is the Schatten 1-norm.

We will use many times a chain rule on min-entropy stating that a conditioning quantum register cannot decrease the entropy more than by its size expressed in qubits.

**Theorem IV.2** (min-entropy chain rule ([Ren05]). Let $\rho_{XKQ}$ be a classical on $XK$, and $\epsilon \geq 0$. Then we have
\[H^\epsilon_{\min}(X|KQ) \geq H^\epsilon_{\min}(X|K) - \log \text{dim}(Q).\]

Using the ideas from [KWW12, NJC+12] we will use random codes to prove the security of Bit Commitment. We give here one useful property of these random codes, which can be viewed as a tradeoff between the minimal distance $d$ of the code and its rate $R$.

**Theorem IV.3** (Gal62). For a randomly generated $[n, k, d]$ binary linear code with rate $R := k/n$, the minimum distance $d$ satisfies,
\[\Pr(d \leq \delta n) \leq 2^{(R - C_3) n}, \text{ for } 0 \leq \delta \leq 1,\]
where $C_3 := 1 - h(\delta)$, $h(x) := -x \log(x) - (1 - x) \log(1 - x)$ is the binary entropy, and where the probability is taken uniformly over all the codes with fixed parameters $k$ and $n$.

The following min-entropy splitting lemma intuitively states that for a classical distribution $P_{X_0X_1Z}$, if the min-entropy (conditioned on $Z$) on $(X_0, X_1)$ is large then it must be the case that the random variable $X_{1-C}$ has high min-entropy too, where $C$ is a binary random variable.
Lemma IV.4 (Min-entropy splitting [Wul07, DFR+07]). Let $X_0, X_1, Z$ be three random variables with distribution $P_{X_0X_1Z}$. Let $1 > \epsilon > 0$. If

$$H^e_{\min}(X_0X_1|Z) \geq K,$$

then there exists a binary random variable $C$ such that,

$$H^e_{\min}(X_1-c|CZ) \geq K/2 - 1 + 2\log(1 - \sqrt{1 - \epsilon^2}).$$

Very often we will use a concentration bound called the Hoeffding inequality.

Theorem IV.5 (Hoeffding inequality [Hoe63]). Let $X_1, \ldots, X_n$ be $n$ identically and independently distributed random variables. If $\forall i \in [n], a \leq X_i \leq b$, then

$$\Pr \left( \frac{1}{n} \sum_{i} X_i - \mathbb{E} \left( \frac{1}{n} \sum_{i} X_i \right) \geq t \right) \leq \exp \left( -\frac{2t^2n}{(b-a)^2} \right),$$

and

$$\Pr \left( \mathbb{E} \left( \frac{1}{n} \sum_{i} X_i \right) - \frac{1}{n} \sum_{i} X_i \geq t \right) \leq \exp \left( -\frac{2t^2n}{(b-a)^2} \right).$$

As a consequence, by taking $t = \sqrt{\frac{(b-a)^2 \ln \epsilon^{-1}}{2n}}$ for some $\epsilon \in [0, 1]$, we get,

$$\Pr \left( \frac{1}{n} \sum_{i} X_i - \mathbb{E} \left( \frac{1}{n} \sum_{i} X_i \right) \geq \sqrt{\frac{(b-a)^2 \ln \epsilon^{-1}}{2n}} \right) \leq \epsilon,$$

and

$$\Pr \left( \mathbb{E} \left( \frac{1}{n} \sum_{i} X_i \right) - \frac{1}{n} \sum_{i} X_i \geq \sqrt{\frac{(b-a)^2 \ln \epsilon^{-1}}{2n}} \right) \leq \epsilon.$$

B. Bit Commitment (BC) with perfect single photon sources

In this section we present the security proof for Protocol [11], which implements BC when honest parties have perfect single photon sources. In particular we prove Theorem IV.6 below. The security proof is mostly the same as in [NJC+12, KWW12], the only differences are that in our Protocol [11], we are guaranteed that the sources emit single photons, so we do not need to care about multiphoton emissions, and that because we want the security to hold even in the presence of noise, we adapt the simulator argument of [KWW12]. More over we use a more recent lower bound [DFW15] on the min-entropy.

Theorem IV.6 (Security of Protocol [11]). Let $\epsilon > 0$ be a security parameter, $e_{err} \in [0, 1/2]$ is the expected error rate of the protocol [11], and let $l \in \mathbb{N}$, $l > 0$ be the length of the string we want to commit. Let us call $n$ the number of quantum communication rounds in which the measurement station has clicked in Protocol [11], and let $\alpha_2 := \sqrt{\frac{\ln \epsilon^{-1}}{2l}}, \alpha_1 := \sqrt{\ln \epsilon^{-1}}$, which account for statistical fluctuations. Let $Q$ be dishonest Bob’s quantum register, $K$ his classical register, and $D$ be such that $\log \dim(Q) \leq D$. Let $C$ be a randomly generated-$[n, k, d]$ linear code with fixed $n$ and $k$ and rate $R := k/n$. We choose the rate of code $C$ to be $R = \ln(\epsilon)/n + 1 - h(\delta)$, where $\delta := 2e_{err} + 2\alpha_2$. Let $\lambda := f(-D/n) - 1/n$ be a lower-bound on the $\epsilon$-smooth min-entropy rate of honest Alice’s string $X^\delta$, conditioned on (malicious) Bobs information $KQ$, where $f$ is defined in eq. (18).

If $n$ satisfies

$$(\lambda - h(\delta)) n \geq l + 2\log(1/2\epsilon) + \ln(\epsilon^{-1}),$$

then Protocol [11] implements a $(l, 3\epsilon)$–Randomized String Commitment.

---

5 Since $\delta$ and $\lambda$ implicitly depend on $n$, one cannot analytically solve the inequality.
Proof. When the two parties are honest and conditioned on not aborting, one can check that the protocol is correct. When the two parties are honest, they can abort in two places. Either they abort in the first step of the Commit phase or in the third phase of the Open phase. In the first case Bob aborts if \( t < 1/2n - \alpha_1 \). By the definition of \( \alpha_1 \) and the Hoeffding inequality (see Theorem IV.5), this happens with probability at most \( \epsilon \). Similarly in step 3 of the Open phase Bob aborts the protocol if he observes an error rate that does not lie in the interval \( [e_{err} - \alpha_2, e_{err} + \alpha_2] \), which by Hoeffding inequality happens with probability at most \( 2\epsilon \). Putting this two potential abort events together, the honest parties have a probability at most \( 3\epsilon \) to abort, which proves correctness.

Lemma IV.8 proves that Protocol II.1 is 3\( \epsilon \)-hiding. Lemma IV.10 together with Theorem IV.3 show that Protocol II.1 is 2\( \epsilon \)-binding.

In the following we will prove Lemmas IV.8 and IV.10 which state security for honest Alice and for honest Bob respectively.

Security for Alice: When Bob is dishonest we will assume that he controls the measurement station and Bob as one single party (Fig. 2). Note that this reduces to the trusted device scenario in which Bob is dishonest [KWW12, DFW15, NJC12]. As a consequence several results from Refs. [DFW15, NJC12] can be reused here.

In fact, the situation in this section is even simpler in the sense that we consider that the honest party (Alice) has access to a perfect single photon source. This, together with the fact that we use a lower bound [DFW15] on the min-entropy that does not depend on the specifics of the state but only on the structure of Alice’s measurements, prevents Bob from gaining any advantage by (selectively) discarding rounds. We discuss this in more details in Appendix A.

Let \( f(\cdot) \) be the following function.

\[
f(x) := \begin{cases} 
0 & \text{if } x < -1 \\
g^{-1}(x) & \text{if } -1 \leq x < 1/2 \\
x & \text{if } 1/2 \leq x \leq 1,
\end{cases}
\]

where \( g(x) := h(x) + x - 1 \) and \( h(x) := -x \log(x) - (1 - x) \log(1 - x) \) is the binary entropy.

**Lemma IV.7** (from [DFW15]). Let \( \epsilon \geq 0 \). If Alice is honest, and Bob has a bounded quantum memory \( Q \) (his quantum register \( Q \) has dimension at most \( 2^D \)) then at the end of the preparation phase, the smooth min-entropy of Bob on Alice string is

\[
H_{\min}^{\epsilon}(X_1^n | QK)_\rho \geq \lambda n,
\]

where \( \lambda = f(-D/n) - 1/n - \log(2/e^2)/n \), and \( K \) is Bob’s classical register.

Since in the protocol Alice sends the syndrome of her string \( X_1^n \) to Bob, we need this syndrome to be sufficiently small in order to keep the entropy relatively high so that the protocol is secure against dishonest Bob. On the other hand, we need the distance of the code to be sufficiently large in order to tolerate errors that might occur between honest Alice and honest Bob. As in Ref. [NJC12] we use a random code: They have sufficiently small syndrome with high distance for our purpose, and since the honest party are not using any decoding we do not need an efficiently decodable code.

**Lemma IV.8** (Security against Dishonest Bob, similar as in Ref. [NJC12]). Let \( \epsilon \in ]0, 1[ \). Let \( Q \) be Bob’s quantum memory such that \( \log \dim(Q) \leq D \). Let \( C \) be a random \([n, k, d] \)-linear code with rate \( R := k/n \). If \( n \) satisfies

\[
\begin{align*}
\lambda - 1 + R &> 0 \\
\text{and,} \\
n &\geq \frac{l + 2 \log(1/2\epsilon)}{\lambda - 1 + R}.
\end{align*}
\]

If Alice is honest, then the protocol is 3\( \epsilon \)-hiding.

Proof. Using Lemma IV.7 we obtain that after the Commit phase, Bob’s entropy on Alice’s string \( X_1^n \) is,

\[
H_{\min}^{\epsilon}(X_1^n | QK \text{Syn}(X_1^n))_\rho \geq (\lambda - 1 + R)n,
\]

where \( R \) is the rate of the code \( C \), i.e., and the length of the syndrome being \( n - k = (1 - R)n \). This together with the leftover hash Lemma IV.1 leads us to

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\rho_{C_1^n \otimes QK \text{Syn}(X_1^n)} \approx \epsilon \cdot \tau_{C_1^n} \otimes \rho_{QK \text{Syn}(X_1^n)},
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In fact, the situation in this section is even simpler in the sense that we consider that the honest party (Alice) has access to a perfect single photon source. This, together with the fact that we use a lower bound [DFW15] on the min-entropy that does not depend on the specifics of the state but only on the structure of Alice’s measurements, prevents Bob from gaining any advantage by (selectively) discarding rounds. We discuss this in more details in Appendix A.

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Since in the protocol Alice sends the syndrome of her string \( X_1^n \) to Bob, we need this syndrome to be sufficiently small in order to keep the entropy relatively high so that the protocol is secure against dishonest Bob. On the other hand, we need the distance of the code to be sufficiently large in order to tolerate errors that might occur between honest Alice and honest Bob. As in Ref. [NJC12] we use a random code: They have sufficiently small syndrome with high distance for our purpose, and since the honest party are not using any decoding we do not need an efficiently decodable code.

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Proof. Using Lemma IV.7 we obtain that after the Commit phase, Bob’s entropy on Alice’s string \( X_1^n \) is,

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\[
\rho_{C_1^n \otimes QK \text{Syn}(X_1^n)} \approx \epsilon \cdot \tau_{C_1^n} \otimes \rho_{QK \text{Syn}(X_1^n)},
\]
where $\tau_{C_i}$ is the maximally mixed state on $C_i$, and

$$
\epsilon' = 2\epsilon + \frac{1}{2}2^{-\frac{1}{2}\left(H_{\min}^*(X_i^1 | Q \text{Syn}(X_i^1))-l\right)}.
$$

(23)

If $\lambda - 1 + R > 0$, then by choosing $n$ sufficiently large we can have $\epsilon' \leq 3\epsilon$, meaning that Protocol [II.1] is $3\epsilon$-hiding. \hfill \Box

Security for Bob:

Figure 7 tells us that the protocol where it is dishonest Alice that sends half of an EPR pair to Bob produces the exact same state as Protocol [II.1] when Alice is dishonest. We can therefore adapt the analysis of [KWW12] to the presence of noise similarly to the analysis performed in [WCSL10], which leads us to the following lemma.

**Lemma IV.9** (Similar to Theorem III.5 of [KWW12] and [WCSL10]). If Bob is honest, then at the end of the preparation phase, there exists an ideal state $\sigma_{A^X_\bar{I}T}$ between (dishonest) Alice and Bob such that:

- $\sigma_{A^X_\bar{I}T} = \sigma_{A^X_\bar{I}} \otimes \tau_I$
- $\rho_{AB} = \sigma_{A(X_\bar{I})}$,

where $\tau_I$ is the maximally mixed state on $I$, $\rho_{AB}$ is the real state produced by the protocol between (dishonest) Alice and Bob, and where the registers $(A, B)$ are identified with $(A, X_\bar{I})$.

**Proof (Sketch).** We will place ourselves in the virtual scenario of Figure 7 where Alice sends the states to Bob. Here, contrary to [KWW12] we want to take care of the noise that might affect the quantum signal and measurements, therefore the simulator introduced in Ref. [KWW12] has to be slightly modified.

In order to prove the existence of an ideal state $\sigma$, in Ref. [KWW12] the authors introduce a virtual protocol where a simulator lies between dishonest Alice and honest Bob. This simulator will measure the states sent from Alice to Bob, thus creating the register $X_\bar{I}$ and then send an “honest” state to Bob. Then they show that the ideal state $\sigma$ created by this virtual protocol satisfies the two relations of Lemma IV.9 with the real state $\rho$ of the real protocol.

In our case Fig. 7 tells us that the noise will only be on the half of the EPR pair kept by Alice, and that the qubit sent to Bob is not affected by any noise. Therefore if the simulator measures it and re-encodes it honestly (and without noise) a qubit corresponding to its outcome and choice of measurement basis, the two relations of this lemma will be satisfied. \hfill \Box

From here on, reusing the argument in Refs. [KWW12, NJC+12] we get the final statement for Bob’s security.

**Lemma IV.10.** Let $\epsilon > 0$. Let $\mathcal{C}$ be an $[n, k, d]$-code with minimum distance $d$ that satisfies,

$$
d \geq 2(e_{\text{err}} + 2\alpha_2) n \sim \frac{2e_{\text{err}} n}{n \to \infty},
$$

(24)

with $\alpha_2 := \sqrt{\frac{\ln e^{-1}}{2(1/2-\alpha_1)n}}, \alpha_1 := \sqrt{\frac{\ln e^{-1}}{2n}}$, then Protocol [II.1] either aborts before the open phase or is $\epsilon$-binding according to definition C.7. Note that the protocol specifies what the honest parties have to do when aborting. What they do during an abort event enforces security definition C.7 to be also satisfied when the protocol aborts.

**Proof.** We again follow the reasoning from [KWW12, NJC+12]. According to Lemma IV.9 there exists a random variable $X_\hat{I}$ such that Bob knows $X_\hat{I}$ and $I$. Now if Alice wants to cheat she needs to send to Bob a string $X_n^1 \neq X_\hat{I}$ such that $\text{Syn}(X_n^1) = w$ which implies that $d_H(X_n^1, X_\hat{I}) \geq d/2$ (see [KWW12] Lemma IV.4), where $d_H(\cdot)$ is the hamming distance. Therefore Alice has to flip at least $d/2$ bits from $X_n^1$ in such a way that $d_H(X_n^1, X_\hat{I}) \leq (e_{\text{err}} + \alpha_2)m$. However Alice is ignorant about which bits Bob knows. As a consequence the situation is equivalent to where $I$ is chosen after that Alice has chosen which bits she wanted to flip. This is a sampling problem, which means that we can use Hoeffding’s inequality (see Theorem IV.5) to estimate the number $W$ of bits in $I$ that Alice will flip:

$$
\text{Pr}(W \leq m(d/2n - \alpha_2)) \leq \exp(-m \alpha_2^2) \leq \exp\left(-m \sqrt{\frac{\ln e^{-1}}{2d}} \right) \leq \epsilon.
$$

(25)

Therefore if,

$$
d \geq 2(e_{\text{err}} + 2\alpha_2)n
$$

(26)

$$
\Rightarrow m(d/2n - \alpha_2) \geq (e_{\text{err}} + \alpha_2)m
$$

(27)

then by using eqs. (25) and (27) we get $\text{Pr}(W < (e_{\text{err}} + \alpha_2)m) \leq \epsilon$ meaning that Alice’s attempt in cheating is detected (and Bob will not accept) with probability $\geq 1 - \epsilon$. \hfill \Box
Figure 7: When honest Bob has access to a single photon source and Alice is dishonest, the three situations depicted are equivalent: In the first Bob chooses the bases $\Theta_1$ and $\hat{X}_1$ uniformly at random, and sends a BB84 type state, as described in Protocol [I.1]. The second picture depicts the equivalent scenario where he sends half of an EPR pair to Alice, and gets $\hat{X}$ and $\Theta$ by measuring the other half. This scenario itself is equivalent to the last fictitious scenario where it is dishonest Alice who sends half of the EPR pair. Note that in the last fictitious scenario, sending EPR pairs might not be the optimal cheating strategy for dishonest Alice, but it is the one that makes this fictitious scenario equivalent to the real scenario represented in the first picture. If some noise acts on the qubit sent by Bob to Alice in the first scenario, this can be seen as Alice applying a noise map on the half of the EPR pair she keeps before applying a measurement in the third scenario. In this virtual scenario, the other half of the EPR pair is assumed to be sent (measured) to (by) Bob without any noise.

C. Oblivious Transfer (OT) with perfect single photon sources

In this section we present and prove Theorem [IV.11] stating security for Protocol [II.2] which implements a Randomized Oblivious String Transfer when the honest parties have access to single photon sources. The security proof closely follows the security proofs from [ENG+14, KWW12]. Indeed the main difference in our case is simply to show that security of our protocol can be reduced to the security of [ENG+14]. This is the case because when Bob is dishonest, he controls the measurement station so we are in a situation where Alice sends BB84 states to dishonest Bob, which is exactly the same situation as in [ENG+14], and therefore the security immediately follows from [ENG+14] when Bob is dishonest. When Alice is dishonest we use the fact that sources emit single photons together with a purification argument in order to reduce the security of our protocol to the one of [ENG+14].

**Theorem IV.11.** Let $\epsilon > 0$ and let $l = |S_0| = |S_1|$, and $\alpha_1 := \sqrt{\frac{\ln \epsilon^{-1}}{2n}}$. If the number of $n$ of quantum communication rounds in which the measurement station has clicked satisfies condition (28), then the Protocol [II.2] implements an 1-out-2 Randomized $(l, 8\epsilon)$—Oblivious String Transfer (see Def. C.2).

**Proof.** Let’s first check correctness with honest Alice and honest Bob. Note that conditioned on not aborting the protocol is $\epsilon$—correct. Indeed the only case where the protocol is not correct conditioned on not aborting is when the error correction procedure fails to correct Bob’s string which happens with probability at most $\epsilon$. We then prove that when both parties are honest, the protocol aborts with probability at most $2\epsilon$. Indeed an abort event happens either if $|Z| < m$ which happens with probability at most $\epsilon$, or if the error correction procedure aborts which happens with probability at most $\epsilon$. As a consequence the protocol aborts with probability at most $2\epsilon$, and since conditioned on not aborting it is $\epsilon$—correct, it implies that overall the protocol is $3\epsilon$—correct. According to Lemma [IV.12] the protocol is $8\epsilon$—secure for honest Alice. According to Lemma [IV.13] the protocol is ($\epsilon = 0$)—secure for honest Bob.

In the following we state and prove Lemmas [IV.12] and [IV.13] which state security for honest Alice and for Honest Bob respectively.

**Security for Alice:** Since the preparation phase of Protocols [I.1] and [II.2] are the same, we will use similar bounds as in Lemma [IV.7] [KWW12] to lower bound the entropy on $X_1$. However we will not use the exact same bounds because we afterwards want to use the min-entropy splitting lemma that is valid only on purely classical states. As a consequence we will first use a chain rule (Theorem [IV.3]) to get rid of Bob’s quantum memory and then lower bound the entropy.
Lemma IV.12. Let Bob be dishonest with a bounded quantum memory denoted $Q$ such that $\log \text{dim}(Q) \leq D$ for some $D$. Let $l := |S_0| = |S_1|$ be the length of the two strings $S_0$ and $S_1$. If

$$n \geq 2 + \frac{l + 2 \log(1 - \sqrt{1 - \epsilon^2})}{1 - \text{leak}_O - 2\alpha_1}$$

(28)

where $\text{leak}_O := |Q|$ is the size of the error correction information Alice sends to Bob, then Protocol II.2 is $8\epsilon$-secure for Alice, with $\lambda = 1/2 - 2\delta'$, $\delta' = (2 - \log(\sqrt{(32 \ln \epsilon^{-1})/n})/\sqrt{(32 \ln \epsilon^{-1})/n})$ \cite{KWW12} eq. (19).

Proof. Protocol II.2 is designed in such a way that it is sufficient to prove that there exists a binary random variable $C$ such that the entropy $H_{\text{min}}(X_{i_0-c}|K\text{QCO})$ at the end of the preparation phase is sufficiently high. Indeed after the preparation phase Alice and Bob will use a randomness extractor on $X_{i_0}$ and on $X_{i_1}$, meaning that if the above mentioned entropy is high enough then Bob will be ignorant of at least one of the two “extracted” strings, which is what we want from the security definition. In order to bound this entropy, we will start by bounding $H_{\text{min}}^c(X_{i_0}X_{i_1}|KO)$ where the quantum register $Q$ is not used, and we will reintroduce it later using a min-entropy chain rule (Theorem IV.2).

Note that $X_i^n = X_I X_{\text{Bad}} X_{\text{remaining}} = X_{i_0} X_{i_1} X_{\text{remaining}}$. By definition of $I$ and $I_{\text{Bad}}$, we have that $|X_{\text{remaining}}| = n - 2m = 2\alpha_1 n$. Therefore

$$H_{\text{min}}^c(X_{i_0}X_{i_1}|KO) = H_{\text{min}}^c(X_I X_{\text{Bad}}|KO) \geq H_{\text{min}}^c(X_I^n|KO) - 2\alpha_1 n.$$  

(29)

By using the previous bound together with the min-entropy splitting lemma (Lemma IV.4), we get that there exists a binary random variable $C$ such that,

$$H^{4\epsilon}_{\text{min}}(X_{1-c}|K\text{OC}) \geq (H_{\text{min}}^c(X_I^n|KO) - 2\alpha_1 n)/2 - 2 \log(1 - \sqrt{1 - \epsilon^2}).$$

(30)

Using the min-entropy chain rule (Theorem IV.2) on the register $O$ ($|Q| \leq D$) and combining it with eq. (30) we conclude that

$$H^{4\epsilon}_{\text{min}}(X_{1-c}|K\text{COQ}) \geq H^{4\epsilon}_{\text{min}}(X_{i_0-c}|KOC) - |Q|$$

$$\geq (H_{\text{min}}^c(X_I^n|KO) - 2\alpha_1 n)/2 - 2 \log(1 - \sqrt{1 - \epsilon^2}) - D,$$

(31)

(32)

where $C$ is defined by the use of the min-entropy splitting lemma in eq. (30).

We will now again use the chain rule (Theorem IV.2) to get rid of the register $O$, and we will call $\text{leak}_O := |Q|$ the maximum leakage due to error correction, and we get

$$H_{\text{min}}^c(X_I^n|KO) \geq H_{\text{min}}^c(X_I^n|K) - \text{leak}_O.$$  

(33)

inserting this into the previous inequality gives,

$$H^{4\epsilon}_{\text{min}}(X_{1-c}|K\text{COQ}) \geq (H_{\text{min}}^c(X_I^n|K) - \text{leak}_O - 2\alpha_1 n)/2 - 2 \log(1 - \sqrt{1 - \epsilon^2}) - D.$$  

(34)

The amount of error correction information $\text{leak}_O$ sent during the protocol can be predetermined by considering the necessary amount of error correction information the parties need when they are both honest, i.e. when both parties (and the measurement station) act in an identically and independently distributed (IID) and trusted manner, and where all the errors come from an i.i.d. noise – an “honest noise”. Indeed if the parties are honest – and if $\text{leak}_O$ is sufficiently large – they will be able to correct their string with probability $(\geq 1 - \epsilon)$, making the protocol correct. If Bob is not honest, since the amount of error correction information is fixed, then the leakage of information is also fixed no matter what strategy he uses. The question is now, how large is “sufficiently large” to allow honest Alice and Bob to correct their string with high probability? This question has been answered in Refs. \cite{KWW05} II.31 where it is shown that one can take

$$\text{leak}_O = H_{\text{max}}^c(X_{i_0}|\hat{X}_{i_0}C = 0)_{\text{phonest}} + H_{\text{max}}^c(X_{i_1}|\hat{X}_{i_1}C = 1)_{\text{phonest}} = 2H_{\text{max}}^c(X_{i_0}|\hat{X}_{i_0}C = 0)_{\text{phonest}},$$

where the entropies are evaluated on the state $\rho_{\text{honest}}$ produced by the protocol when both parties are honest.

One can then lower-bound $H_{\text{min}}^c(X_{i_1}|K)$ using \cite{KWW12} eq. (19) (see also \cite{NBW12}),

$$H_{\text{min}}^c(X_{i_1}|K) \geq \lambda n,$$

with $\lambda = 1/2 - 2\delta$, $\delta = (2 - \log(\sqrt{(32 \ln \epsilon^{-1})/n})/\sqrt{(32 \ln \epsilon^{-1})/n})$. Since $H_{\text{max}}^c(X_{i_0}|\hat{X}_{i_0}C = 0)_{\text{phonest}}$ is evaluated on honest i.i.d parties we can upper-bound the max-entropy using the equipartition Theorem \cite{TCK09}, getting
If the single photon sources used by honest parties are sufficiently good, i.e. let \( \epsilon \) be a randomly generated \( [n, k, d] \) linear code with rate \( R := k/n \). We choose this code such that the rate is \( R = \frac{\ln(\epsilon)}{n + 1} - h(\delta) \), where \( \delta := 2 \left[ (1/2 + \alpha_1')(\gamma + \alpha_2') + \alpha_3 + \frac{4\epsilon}{\ln(\epsilon)} \right] \). Let \( Q \) be Bob’s quantum register, and let \( D \) be such that \( \log \dim(Q) \leq D \). Let \( \lambda := f(-D/n) - (\gamma + \alpha_2') - 1/n \) lower-bound the \( \epsilon \)-smooth min-entropy rate \( (H_{\min}(X_1^m|QK)_c)/n \) except with probability \( 16(\epsilon + \varepsilon + \hat{\varepsilon}) + 8\epsilon_1 \), where \( f \) is defined in eq. (14).

If the single photon sources used by honest parties are sufficiently good, i.e.

\[
\begin{align*}
p_{\geq 2|a_s}(1 - p_{0|a_s}) & \leq p_b, \\
\text{and} & \\
p_{\geq 2|b_s}(1 - p_{0|b_s}) & \leq p_a, \\
\end{align*}
\]
and if \((p - \sqrt{\ln(2)/2N}) N \geq n^*\), where \(n^*\) the smallest positive integer solution to the following inequality\(^6\)

\[
n \geq \frac{l + 2 \log(1/2e) + \ln(\epsilon^{-1})}{\lambda - h(\delta)},
\]

(37)

and where \(p\) is the probability that a round \(i \in [N]\) is not discarded in the preparation phase when both parties are honest, then Protocol [IV.3] implements a \((1, 9e + 32(\epsilon + \varepsilon + \hat{\varepsilon}) + 16\varepsilon_1)\) - 1-out-2-Randomized String Commitment.

Proof. Let’s start with correctness. First of all note that conditioned on not aborting the protocol is correct. We now show that when both parties are honest the protocols aborts with probability smaller than \(9e + 32(\epsilon + \varepsilon + \hat{\varepsilon}) + 16\varepsilon_1\), which implies that the protocol is \((9e + 32(\epsilon + \varepsilon + \hat{\varepsilon}) + 16\varepsilon_1)\) - correct. Using the Hoeffding inequality (see Theorem [IV.5]) it is easy to check that the honest parties will abort with probability at most \(2\varepsilon\) at step 2 of the preparation phase.

If the two parties are honest with sources such that \(p_{23} \geq 2\|p_{\varepsilon,\theta} \| - p_{b,\gamma}\) (for Alice) and \(p_{23} \geq 2\|p_{\varepsilon,\theta} \| - p_{a,\gamma}\) (for Bob), then the probability to abort at step 3 is at most \(\epsilon + 16(\epsilon + \varepsilon + \hat{\varepsilon}) + 8\varepsilon_1\) and at most \(\epsilon + 16(\epsilon + \varepsilon + \hat{\varepsilon}) + 8\varepsilon_1\) at step 4. Indeed in step 3, using the Hoeffding inequality one can check that with probability at most \(\epsilon\), we have

\[
n_{1}^{2} \geq \frac{p_{a,\gamma} + 1}{p_{b,\gamma}^{2}} \leq p_{b,\gamma} + \beta^{A}.
\]

By dividing the expression by \(f_{b}\) and using that conditioned on not aborting in the previous steps \(f_{b} \geq p_{b,\gamma} - \beta^{A}\) we get

\[
n_{1}^{2} \leq \frac{p_{b,\gamma}^{2} + 1/f_{b}^{2}}{p_{b,\gamma}^{2}} \leq p_{b,\gamma}^{2} + 1/f_{b}^{2} \beta^{A}.
\]

Using that except with probability \(16(\epsilon + \varepsilon + \hat{\varepsilon}) + 8\varepsilon_1\) we have \(U_{A2} \geq n_{1}^{2}\) and that for \(1/p_{b,\gamma} \leq 1/2\) we have \(1/(1 - 1/p_{b,\gamma}) \leq 1 + 2/p_{b,\gamma} \beta^{A}\) we get the desired result. An analog proof holds for step 4. Again by the Hoeffding inequality, there is a probability at most \(\epsilon\) to abort at step 5.

Using again the Hoeffding inequality one can check that Bob will abort the protocol with probability at most \(2\varepsilon\) at step 1 of the Commit phase and with probability at most \(2\varepsilon\) at phase 3 of the open phase. Over all the protocol aborts with probability at most \(9e + 32(\epsilon + \varepsilon + \hat{\varepsilon}) + 16\varepsilon_1\).

Security for honest Alice is given in Lemma [IV.17] and for honest Bob in Lemma [IV.19].

Proof. Before proving security for honest Alice (Lemma [IV.17]) and for honest Bob (Lemma [IV.19]), we need to prove that the honest party \(H \in \{Alice, Bob\}\) can always find a lower-bound \(L_{H1}\) on \(n_{1}^{H}\), the number rounds where \(H\) has emitted a single photon and has sent a “signal” state. This is what the following lemma shows. You can find its proof in Appendix B.

Lemma IV.15. Let \(x_{i,\theta}\) be the “observed” number of rounds where \(H\) has prepared a signal of intensity \(i\) in the bias \(\theta\) and where the measurement station (or the dishonest party) reported outcome \(o\) failure. Let \(e, \varepsilon, \hat{\varepsilon}\) such that \(\forall (o, \theta), (2e^{-1})^{1/\zeta_{o,\theta,L}} \leq \exp(3/(4\sqrt{2}))\) and \((\varepsilon^{-1})^{1/\zeta_{o,\theta,L}} < \exp(1/3), \) with \(\zeta_{o,\theta,L} := x_{i,\theta} - \sqrt{\sum_{i} x_{i,\theta}^{2} \ln(1/\epsilon)}\). Let \(\Delta_{i,o,\theta} := g(x_{i,\theta}, \varepsilon^{2}/16), \hat{\Delta}_{i,o,\theta} := g(x_{i,\theta}, \varepsilon^{3}/2),\) and \(g(x, y) := 2x \ln(y^{-1})\). Then if \(q = 2\) (\(q\) is the number of decay states used during the protocol i.e. \(i \in \{i_{1}, i_{2}, i_{3}\}\) ) we have,

\[
n_{1}^{H} \geq L_{H1} := \sum_{o, \theta} \left[ p_{i,\theta} | k = 1 | S_{1, o, \theta} | \min - g(p_{i,\theta} | k = 1 | S_{1, o, \theta} \mid \min, \varepsilon_{1} ) \right]
\]

(38)

except with probability \(16(\epsilon + \varepsilon + \hat{\varepsilon}) + 8\varepsilon_1\), where \(| S_{1, o, \theta} \mid \min \) is given by,

\[
| S_{1, o, \theta} \mid \min := \min(1, V_{2}, V_{3}, V_{4}),
\]

(39)

with

\[
V_{1} = \frac{p_{i,\theta} | k = 2 (x_{i,\theta}^{2} + \Delta_{i,\theta, o, \theta}) - p_{i,\theta} | k = 2 (x_{i,\theta}^{2} + \Delta_{i,\theta, o, \theta})}{p_{i,\theta} | k = 1 p_{i,\theta} | k = 2 - p_{i,\theta} | k = 2 p_{i,\theta} | k = 1}
\]

(40)

\[
V_{2} = \frac{p_{i,\theta} | k = 2 (x_{i,\theta}^{2} - \Delta_{i,\theta, o, \theta}) - p_{i,\theta} | k = 2 (x_{i,\theta}^{2} + \Delta_{i,\theta, o, \theta})}{p_{i,\theta} | k = 1 p_{i,\theta} | k = 2 - p_{i,\theta} | k = 2 p_{i,\theta} | k = 1}
\]

(41)

\[
V_{3} = \frac{p_{i,\theta} | k = 2 (x_{i,\theta}^{2} + \Delta_{i,\theta, o, \theta}) - p_{i,\theta} | k = 2 (x_{i,\theta}^{2} - \Delta_{i,\theta, o, \theta})}{p_{i,\theta} | k = 1 p_{i,\theta} | k = 2 - p_{i,\theta} | k = 2 p_{i,\theta} | k = 1}
\]

(42)

\[
V_{4} = \frac{p_{i,\theta} | k = 2 (x_{i,\theta}^{2} - \Delta_{i,\theta, o, \theta}) - p_{i,\theta} | k = 2 (x_{i,\theta}^{2} - \Delta_{i,\theta, o, \theta})}{p_{i,\theta} | k = 1 p_{i,\theta} | k = 2 - p_{i,\theta} | k = 2 p_{i,\theta} | k = 1}
\]

(43)

One can compute tighter bounds using more decay states (i.e. for \(q > 2\)). For more details see Appendix B.

\(^6\) Remember that the parameters like \(\lambda, \delta\) etc. depend on \(n\)
For simplicity we will, in the following, continue the security analysis for the case where \( q = 2 \). The above lemma will allow us to prove the following security lemmas: Lemma \([IV.17]\) proves security for honest Alice, and Lemma \([IV.19]\) proves security for honest Bob.

**Security for Alice:** When Alice is honest almost nothing changes except that Bob’s entropy about Alice’s string is smaller by roughly \( \gamma n \) bits. As a consequence lemma \([IV.7]\) has to be changed.

Let \( f(\cdot) \) be the following function.

\[
f(x) := \begin{cases} 
0 & \text{if } x < -1 \\
g^{-1}(x) & \text{if } -1 \leq x < 1/2 \\
x & \text{if } 1/2 \leq x \leq 1, 
\end{cases}
\]

(44)

where \( g(x) := h(x) + x - 1 \) and \( h(x) := -x \log(x) - (1-x) \log(1-x) \) is the binary entropy.

**Lemma IV.16.** Let \( \epsilon, \delta, \epsilon_1 \) be as defined in lemma \([IV.15]\). If Alice is honest (but uses a non-perfect photon source), and Bob has a bounded quantum memory \( Q \) (his quantum register \( Q \) has dimension at most \( D \)) then at the end of the preparation phase, and if Alice did not abort, the smooth min-entropy of Bob on Alice string is,

\[
H_{\min}^0(X^n_t | QK)_\rho \geq \lambda n
\]

(45)

with probability higher than \( 1 - 16(\epsilon + \delta + \delta) - 8\epsilon_1 \), where \( \lambda := f(-D/n) - (\gamma + \alpha_4^\delta) - 1/n \) \((DFW15)\), and \( K \) is Bob’s classical register.

Here we have used that – as proven in Theorem \([IV.15]\) – with probability higher than \( 1 - 16(\epsilon + \delta + \delta) - 8\epsilon_1 \) dishonest Bob gets at most \( (\gamma + \alpha_4^\delta)n \) extra bits of information due to the leakage information on the bases used by Alice.

We can then reuse Lemma \([IV.8]\) with the only difference that we have to include the probability that Alice emits 2 or more photons in more than \( (\gamma + \alpha_4^\delta)n \) “non-failure” rounds.

**Lemma IV.17 (Security against Dishonest Bob).** Let \( \epsilon, \delta, \epsilon_1 \) be as defined in Lemma \([IV.15]\). Let \( Q \) be Bob’s quantum memory such that \( \dim(Q) \leq D \), and the rate \( \rho \) of the code \( C_f \) be such that,

\[
n \geq \frac{l + 2 \log(1/2\epsilon)}{\lambda - 1 + R}.
\]

(46)

If Alice is honest, then Protocol \([II.3]\) (with \( q = 2 \)) either aborts or is \([3\epsilon + 16(\epsilon + \delta + \delta) + 8\epsilon_1\])-hiding. Note that when the honest Alice aborts she is required to output uniformly random strings, so that the security definition \([C.f]\) is also satisfied when the protocol aborts. In fact when aborting the ideal and the real state are equal.

**Proof.** The proof is exactly the same as in Lemma \([IV.8]\) except that we add \( 16(\epsilon + \delta + \delta) + 8\epsilon_1 \) to the failure probability, which corresponds to the probability that there are more than \( (\gamma + \alpha_4^\delta)n \) rounds where at least 2 photons have been emitted (see Theorem \([IV.15]\) ), and where \( \lambda \) has value given by Lemma \([IV.16]\).

**Security for Bob:**

We will start by stating a lemma similar to Lemma \([IV.9]\) adapted to the case of an imperfect single photon source.

**Lemma IV.18.** When Bob is honest, at the end of the preparation phase, there exist a state \( \sigma_{X^n_{AI}} \) such that

- \( \sigma_{X^n_{AI}} = \sigma_{A\tilde{X}_{Im}^t} \otimes \tau_{Im} \)
- \( \rho_{AB} = \sigma_{A(\tilde{X}_tI)} \)

where \( \tau \) denotes the maximally mixed state, \( I' \) is the register encoding the set of rounds where Alice got extra information from the emission of multiple photons, and \( I'' \) is the register encoding the set of rounds in \( \bar{I} \) where Alice did not get any information. Formally the registers \( I' \) and \( I'' \) are such that \( I' \otimes I'' = \bar{I} \). \( \rho_{AB} \) is the real state produced by the protocol between (dishonest) Alice and Bob, and where the registers \( (A, B) \) are identified with \( (A, \tilde{X}_tI) \).

In the following we will use the same reasoning as in Lemma \([IV.10]\) adapting it to the case where the multiphoton emissions are possible.

Intuitively when Bob is honest but uses a non-perfect single photon source dishonest Alice basically knows, for a fraction \( \gamma \) of the rounds, whether they belong to \( \bar{I} \) or not. Using similar notations as in Lemma \([IV.10]\) this knowledge will help dishonest Alice when she will have to flip \( d/2 \) bits from \( X^n_t \). Indeed she can flip the \( \approx (\gamma/2)n \) bits that she knows not to be in \( \bar{I} \). For the \( \approx d/2 - (\gamma/2)n \) remaining bits, she will flip bits that are not the \( \approx (\gamma/2)n \) she knows to be in \( \bar{I} \).
Lemma IV.19 (see [NJC+12]). Let $\epsilon, \varepsilon, \bar{\varepsilon}, \epsilon_1$ be as defined in Lemma IV.15. Let $\alpha_1, \alpha_2$ be the same as in Lemma IV.16. and $\alpha'_d$ as defined in Protocol II.3. Let $\alpha''_d := \sqrt{\frac{\ln \frac{\alpha''_1}{\alpha''_2}}{2(1 - \gamma - \alpha''_1)n}}$. Then when Bob is honest, Protocol II.1 either aborts or is $[\epsilon + 16(\epsilon + \varepsilon + \bar{\varepsilon}) + 8\epsilon_1]$-binding according to definition C.1. Since when honest Bob aborts he is required to reject the opening and output a random string $\overline{C'_1}$, the security definition is automatically satisfied when honest Bob aborts the protocol.

Proof (Sketch). From Lemma IV.15 we know that except with probability $16(\epsilon + \varepsilon + \bar{\varepsilon}) + 8\epsilon_1$, dishonest Alice gets information on at most $(\gamma + \alpha'_{d,1})n$ bits.

Except with probability $\epsilon$, at most a fraction $(1/2 + \alpha'_1)$ of them are not in $I$, so Alice can flip them without Bob being able to detect this. We can compute this fraction by noticing that on rounds where 1 photon has been emitted (there are at least $(1 - \gamma - \alpha''_1)n$ of them), the probability of each of these rounds to be in $I$ is 1/2 and is independent of Alice’s information. Therefore, by Hoeffding inequality (see Theorem IV.5), the number of these rounds being in $I$ should be $\leq 1/2 + \alpha'_1$, except with probability $\epsilon$. Moreover, if the protocol does not abort then the total number of rounds in $I$ is $\leq (1/2 + \alpha'_1)n$. Combining this with the fact that $1/2 + \alpha'_1 \leq 1$ gives the expression for $\alpha'_{d,1}$.

Let $d/2 - (1/2 + \alpha'_1)(\gamma + \alpha''_d)n$ bits remains for Alice to flip. However she knows that she should flip these remaining bits on the position on which she did not get any information during the preparation phase. There are $\geq (1 - \gamma - \alpha''_1)n$ such positions. Therefore Alice’s choice of bit flip is equivalent to uniformly sampling without replacement $d/2 - (1/2 + \alpha'_1)(\gamma + \alpha''_d)n$ positions out of $\geq (1 - \gamma - \alpha''_1)n$ to estimate the number $W$ of bits that Alice chooses to flip while being in a position in the set $I$. As for Lemma IV.16 this is equivalent to first fixing Alice’s bit flip and then choosing the position that are in $I$ among the $(1 - \gamma - \alpha''_1)n$ available positions. Using Hoeffding inequality we get that,

$$\Pr \left( W < n \left[1/2 - \alpha''_d - (1/2 + \alpha'_1)(\gamma + \alpha''_d)\right](d/2n - (1/2 + \alpha'_1)(\gamma + \alpha''_d) - \alpha_3) \right) \leq \exp \left( -2n \left[1/2 - \alpha''_d - (1/2 + \alpha'_1)(\gamma + \alpha''_d)\right]^2 \right).$$

(48)

Now if

$$d \geq 2 \left[1/2 + \alpha'_1\right](\gamma + \alpha''_d) + \alpha_3 + (\epsilon_{err} + \alpha_2)(1/2 + \alpha'_1)(1 - \gamma - \alpha''_1)n,$$

(50)

then with probability $\geq 1 - \epsilon - 16(\epsilon + \varepsilon + \bar{\varepsilon}) - 8\epsilon_1$,

$$W \geq n \left[1/2 - \alpha''_d - (1/2 + \alpha'_1)(\gamma + \alpha''_d)\right](d/2n - (1/2 + \alpha'_1)(\gamma + \alpha''_d) - \alpha_3) \geq (\epsilon_{err} + \alpha_2)(1/2 + \alpha'_1)n \geq (\epsilon_{err} + \alpha_2)m.$$

(51)

(52)

This means that if eq. (50) is satisfied there is a probability at most $\epsilon + 16(\epsilon + \varepsilon + \bar{\varepsilon}) + 8\epsilon_1$ that Alice can cheat and make Bob accept.

E. OT with an imperfect single photon source

In this section we state more formally our impossibility result for a secure Oblivious Transfer protocol. In particular we show that if a protocol satisfy Assumption IV.1 then Protocol II.4 allows dishonest Alice to cheat.

1. Informal description of the settings

We recall that dishonest Alice’s goal is to guess correctly the bit $C$ that is given to honest Bob by the protocol (the protocol gives him an random bit $C$ and a bit string $S_C$). In section II.D we already give a simple example on how an attack could work on a protocol like Protocol II.2. Here we explain informally what is the general form of the protocols to which our impossibility result applies. In the next section we will make this setup definition more precise. Our impossibility result applies to protocols of the following form.
**First phase:** In a first phase, called the quantum phase, Alice and Bob can used classical and quantum communication. This phase outputs string $X^*_I$ to Alice and a string $X_T$ and set of indices $I$ to Bob, where $X_T$ is a string formed by the bits of the string $X^*_I$ that are placed at indices in $I$. In order to model the leakage of information due to the multiphoton emissions (see section I[1D]), we assume that Alice receives two extra sets of indices $I_G$ and $I_B$. This two sets are correlated to the set $I$. In particular we will consider elements of $I_G$ are more likely to belong to $I$ than elements in $I_B$. In the simple example of Section I[1D], dishonest Alice could compute these sets from the leakage information concerning the bases Bob has used in this phase. Moreover, in this specific example we had that $I_G \subseteq I$ and $I_G \subseteq I^c$, where $I^c$ is the complement of $I$.

**Second phase:** The second phase of the protocol is purely classical, that is they only send classical messages. Alice and Bob should use the data they got from the first phase in order to compute the desired strings $(S_0, S_1)$ and bit $C$ that the OT protocol should produce (see Definition I[1.2]).

Note that we don’t specify the specific form for the first phase, we simply require that it outputs the strings $X^*_I$, $X_T$ and the set $I$ with some probability distribution, as well as the extra sets $I_G$ and $I_B$ when Alice is dishonest. The strategy we use to break security of MDI OT protocols is a semi-honest strategy. This means that Alice will essentially run the protocol honestly but record all the information from the communication between her and Bob. In particular, in the quantum phase Alice extracts – from the quantum signal Bob sends to the measurement station – the set of indices $I_G$ and $I_B$, which is the set of indices she is more likely to get from the quantum signal instead of the classical signal. Of course our attack depend on the fact that the set $I_G$ and $I_B$ are sufficiently large so that Alice gets enough statistics to have a good guess of Bob’s bit $C$. In other words we need that Bob’s photon source leaks enough information. This is captured in the second equation of eq. (53) in Assumptions IV.1.

After this quantum phase, we assume that Alice and Bob can post process the data they received from the quantum phase, by using purely classical communication. Since we assume that dishonest Alice is semi-honest, we will assume that she runs the post-processing honestly but records all the information she receives from, or sends to Bob.

In the post-processing of Protocol I[1.2] Bob chooses uniformly at random the bit $C$, and then renames the sets $I$ and $I^c$ into $I_0$ and $I_1$ in such a way that $I = I_C$. Bob then sends $(I_0, I_1)$ to Alice. The information $(I_0, I_1)$ sent by Bob to Alice, should not by itself reveal bit $C$. But because Alice holds the extra sets $I_G$ and $I_B$, she can determine which set from $(I_0, I_1)$ corresponds to set $I$, and therefore she learns the value of bit $C$. In the general settings, we will only assume that from all the information Alice has she can compute two sets $I_0$ and $I_1$ such that $I_C \subseteq I$ and $I_{1-C} \not\subseteq I$. This is the second assumption in Assumptions IV.1.

In the following sections, we describe in details how we generalize this idea of attack to a more general settings.

### 2. Settings Definition

In this section we defined the settings in which our theorem holds. Theorem IV.20 states that any protocol that has the form we describe below, and that is correct, and secure against Bob, can be attacked by dishonest Alice. That is, it is always possible for Alice to correctly guess Bob’s bit $C$ with sufficiently high probability. Dishonest Alice’s cheating strategy is given in Protocol I[1.3]. In order to generalize the discussion of Section IV.E, all the random variables mentioned in Section IV.E, $X^*_I, I, I_0, I_1, C, \ldots$ will be redefined in a more abstract manner.

In order to prove our result we will forget about the quantum part of the OT protocol, and start directly in a scenario, in which Alice and Bob share from the start the type of correlation they would have had by running a preparation phase similar to Protocol I[1.2].

In particular we will assume that the Preparation phase gives the following to Alice and Bob:

**Honest Alice:** Alice gets a random bit string $X^*_I$ with probability distribution $P_{X^*_I}$.

**Honest Bob:** Bob gets a random subset $I \subseteq [n]$ with probability distribution $P_I$, and the string $X_T$, whose bits are the bits of $X^*_I$ that are indexed by $i \in I$.

When one of the parties is dishonest we will assume they have the following additional information as input:

**Dishonest Alice:** Dishonest Alice gets the same $X^*_I$ as when she was honest, plus the sets $I_G, I_B \subseteq [n]$, which are sets of indices satisfying the following:

- $I_G \cap I_B = \emptyset$ and $|I_G \cup I_B| = \gamma n$ for some $\gamma \in [0, 1]$.
- $\forall i \in I_G \cup I_B$

She still has full control over the measurement station. In particular everything that Bob sends to the measurement station is considered to be in dishonest Alice’s possession.
- If $i \in \mathcal{I}$ then $i \in I_C$ with probability $1/2(1 + \mu)$ or $i \in I_B$ with probability $1/2(1 - \mu)$.
- If $i \notin \mathcal{I}$ then $i \in I_C$ with probability $1/2(1 - \mu)$ or $i \in I_B$ with probability $1/2(1 + \mu)$.

$\gamma$ represents the fraction of rounds in which more than two photons have been emitted (we should have $\gamma \approx p_{c/2}(1 - p_0)$, the probability that more than two photons are emitted when at least one is emitted). $\mu \in [0, 1]$ models Alice’s probability of guessing Bob’s basis conditioned on receiving several photons from Bob.

Note that this definition can be seen as first giving $I_G \cup I_B$ to Alice and then giving her $I_G$ and $I_B$ through the probabilistic process described above.

**Dishonest Bob:** When Bob is dishonest we will assume that he holds a classical register $K$ and a quantum register $Q$ such that his min-entropy rate $\min_{n}(X^n | KQ)_{\frac{1}{n}}$ is smaller than the one of honest Bob.

Let $M_{BA}$ be the random variable that describes the set of the messages sent from Bob to Alice, and $M_{AB}$ be the random variable that describes the messages sent from Alice to Bob. The random variable composed of these two variables will be called $M$, in other words $M := (M_{AB}, M_{BA})$.

The output of honest Alice is $(S_0, S_1) := (f_0(X^n_1, M), f_1(X^n_1, M)) \in \{0, 1\}^l \times \{0, 1\}^l$, where $f_0$ and $f_1$ are two functions. The output of honest Bob is $(C, S_C) := (g(X_T, I, M), \tilde{g}(X_T, I, M))$, where $g$ and $\tilde{g}$ are two other functions. These four functions model the operations that honest Alice and Bob have to perform according to the protocol they are running.

Before estimating Alice’s cheating probability (see Theorem [IV.20]), we will need the following definition.

**Definition IV.3.** Let $J \subseteq [n]$ be a set of indices. Let $f_0, f_1$ be the functions defined above. We will say that $J$ stabilizes a function $f_a$ ($a \in \{0, 1\}$) with respect to (w.r.t.) random string $X^n_1$ and random variable $M$ when the value $(x^n_1, m)$ of random variable $(X^n_1, M)$ is such that $J$ stabilizes $f_a$ w.r.t. $x^n_1$ and $m$. We will say that $J$ stabilizes the function $f_a$ w.r.t. $x^n_1$ and $m$ if $f_{x_j} \rightarrow f_a((x_{f_j}, x_j), m)$ is constant for all $x_{f_j}$ s.t. $\Pr((X^n_1, M) = ((x_{f_j}, x_j), m)) \neq 0$, where $(x_{f_j}, x_j)$ denotes the string composed of the bits $x_{f_j}$ and $x_j$ at the positions corresponding to the sets $J$ and $J^c$.

Intuitively this definition captures the notion of a function $f$ depending only on the values of the bits of $X^n_1$ at positions indexed by the set $J \subseteq [n]$.

### 3. Assumptions and main Theorem

In this section we state the assumptions we make to prove our theorem and prove Theorem [IV.20]. Since we assume Alice is semi-honest her cheating strategy consists in making her guess on Bob’s bit $C$ using all the information she has collected during the protocol. Therefore we can consider that her cheating strategy is an algorithm she runs at the end of the protocol on all her data. The cheating strategy we use is described in Protocol [I.4]. The basic idea of the protocol is the following. At the end of the protocol Alice has the two sets $I_0$ and $I_1$ that are correlated to bit $C$ and set $\mathcal{I}$ in the following way. If $C = 0$ then $I_0 \subseteq \mathcal{I}$ and $I_1 \not\subseteq \mathcal{I}$. If $C = 1$ the situation is reversed (see previous section). In themselves, these sets do not reveal the value of bit $C$ since Alice should not know anything about set $\mathcal{I}$. However, since there has been information leakage during the protocol, she does know something about set $\mathcal{I}$. She knows that indices in $I_C$ are more likely to belong to $\mathcal{I}$ than the ones in $I_B$, and this allows her to guess with some probability which set $I_0$ or $I_1$ is a subset of $\mathcal{I}$, and therefore it allows her to guess the value of bit $C$. Let us state more precisely the assumptions we use to prove Theorem [IV.20].

Let $\mathfrak{P}_F$ be the following statement: $\exists F(\cdot, \cdot)$ such that $F(X^n_1, M) =: (I_0, I_1)$ where $I_0, I_1 \subseteq [n]$ are such that $I_C$ stabilizes $f_C$ but not $f_{1-C}$ (w.r.t. $(X^n_1, M)$), and $I_{1-C}$ stabilizes $f_{1-C}$ but not $f_C$ (w.r.t. $(X^n_1, M)$), where $C := g(X^n_1, I, M)$.

If $\mathfrak{P}_F$ is true then one can define $\alpha \in [0, 1]$ such that $|I_{1-C} \cap I_C \cap (I_G \cup I_B)| = (1 - \alpha)|I_{1-C} \setminus I_C \cap (I_G \cup I_B)|$, i.e. $\alpha$ is the fraction of rounds in $I_{1-C} \setminus I_C \cap (I_G \cup I_B)$ that are not in $\mathcal{I}$.

**Assumptions IV.1.** Let $I_0, I_1, C, M, X^n_1, \mathcal{I}, I_G, I_B, \alpha$, and $\mu$ be as defined above. Let $\delta \in [0, 1/2]$. Let $\kappa := \min \{|I_0 \cap (I_G \cup I_B)|, |I_1 \cap (I_G \cup I_B)|\}$. Let $\Omega_\kappa$ be the event: "$\kappa \geq 1$". We assume in Theorem [IV.20] that:

\[
\begin{cases}
\mathfrak{P}_F \text{ is true,} \\
\Pr(\Omega_\kappa) \text{ is non-negligible in } n,
\end{cases}
\]

\[
(53)
\]

8 They have to be correlated to $C$ in this way for the OT protocol to be correct, and secure against dishonest Bob.
Now we can state and prove our theorem that shows that Protocol II.4 is a strategy that allows dishonest Alice to cheat.

**Theorem IV.20.** Let $I_0, I_1, C, M, X^n, \mathcal{I}, I_G, I_B, \alpha$, and $\mu$ be as defined above. Let $\delta \in [0, 1/2]$. Let $\kappa := \min \left( |I_0 \setminus I_1 \cap (I_G \cup I_B)|, |I_1 \setminus I_0 \cap (I_G \cup I_B)| \right)$. Let $\Omega_\kappa$ be the event: “$\kappa \geq 1$”. Let $P_{\text{guess}}$ be the maximum probability that Alice correctly guesses Bob’s bit $C$.

If Assumptions [V.1] are satisfied by the protocol run between Alice and Bob, and if this protocol is correct, and secure against dishonest Bob, then dishonest Alice’s strategy presented in Protocol II.4 allows Alice to guess $C$ with probability $P_{\text{guess}} = 1/2 + \delta$, where $\delta$ satisfies

$$\delta \geq \Pr \left( \Omega_\kappa \right) \times \alpha \mu.$$

We can also prove that $\alpha \geq 1/n$, which is not negligible in $n$.

**Proof.** In order to prove the theorem we will lower bound Alice’s guessing probability $P_{\text{guess}}$, for a protocol satisfying Assumptions [V.1]. In particular we want to show that $P_{\text{guess}}$ is larger than $1/2$ by a non-negligible amount. Before doing that let us spell out important consequences of a protocol being correct and secure against Bob.

Because we assume that $\mathcal{F}_\mathcal{P}$ is true, the sets $(I_0, I_1) := F(X^n, M)$ are well defined. In order to get correctness we should have that $I_C \subseteq I$, and for having security against Bob it is necessary that $I_{1-C} \not\subseteq I$, where $C$ is the bit held by honest Bob that Alice tries to guess. Let us call $b$ the bit that corresponds to dishonest Alice’s guess of Bob’s bit $C$. We can then write,

$$P_{\text{guess}} = \Pr(b = C) = \Pr(\Omega_\kappa) \Pr(b = C|\Omega_\kappa) + (1 - \Pr(\Omega_\kappa)) \Pr(b = C|\neg \Omega_\kappa),$$

$$\geq \Pr(\Omega_\kappa) \Pr(b = C|\Omega_\kappa) + (1 - \Pr(\Omega_\kappa)) \times 1/2.$$

From Assumptions [V.1] we have that $\Pr(\Omega_\kappa)$ is not negligible. Intuitively, saying that $\Pr(\Omega_\kappa)$ is not negligible ensures that there has been information leakage during the quantum phase of the protocol. If $\Pr(\Omega_\kappa)$ were negligible we already know by Theorem [V.11] that a protocol like Protocol II.2 would be secure. As a consequence, we will focus on computing $\Pr(b = C|\Omega_\kappa)$.

In Protocol II.4 Alice chose uniformly at random an index $i_r \in I_r I_{1-r} \cap (I_G \cup I_B)$ and check whether $i_r$ ends up in $I_G$ or $I_B$. The idea is that if $r = C$ the probability that $i_r$ ends up in $I_G$ is slightly higher than the one of ending up in $I_B$. If $r = 1 - C$ it biased towards ending up in $I_B$. Therefore, if she outputs $b = r$ when $i_r \in I_G$, and outputs $b = 1 - r$ if $i_r \in I_B$ she will have a probability of guessing correctly bit $C$ slightly higher than $1/2$, which is what we are trying to prove.

Note that the event $\Omega_\kappa$ depends on the “value” of the set $I_G \cup I_B$, but is completely independent on how $I_G \cup I_B$ is partitioned into the sets $I_G$ and $I_B$. In particular the probability for a round in $I_G \cup I_B$ to be in $I_G$ is independent of $\Omega_\kappa$.

Let us now write Alice’s guessing probability conditioned on $\Omega_\kappa$, with $r$ and $i_r$ as defined by Protocol II.4.

$$P_{\text{guess}}|\Omega_\kappa = \Pr(b = C|\Omega_\kappa) = \Pr(r = C|\Omega_\kappa) \Pr(i_r \in I_G|r = C, \Omega_\kappa) + \Pr(r = 1 - C|\Omega_\kappa) \Pr(i_r \in I_B|r = 1 - C, \Omega_\kappa),$$

where $r$ is a uniformly random bit chosen by Alice in Protocol II.4. As a consequence, $\Pr(r = C|\Omega_\kappa) = \Pr(r = 1 - C|\Omega_\kappa) = 1/2$. From the definition of $I_G$ and $I_B$, and their independence from $\Omega_\kappa$ we get that,

$$\Pr(i_r \in I_G|r = C, \Omega_\kappa) = \Pr(i_r \in I_G|r = C) = 1/2(1 + \mu),$$

$$\Pr(i_r \in I_B|r = 1 - C, \Omega_\kappa) = \Pr(i_r \in I_B|r = 1 - C) = 1 - 1/2\alpha(1 - \mu) - 1/2(1 - \alpha)(1 + \mu).$$

Plugging this into the expression for $P_{\text{guess}}|\Omega_\kappa$ we get that:

$$P_{\text{guess}}|\Omega_\kappa = 1/2(1/(1 + \mu) + 1 - 1/2\alpha(1 - \mu) - 1/2(1 - \alpha)(1 + \mu)) = 1/2(1 + \alpha \mu).$$

As expected the probability that Alice correctly guesses the value of bit $C$ is a bit higher than $1/2$.

Combining this with the fact that $\kappa \geq 1$ is true with probability $\Pr(\Omega_\kappa)$ leads to eq. (54). That is, Alice’s overall probability of guessing correctly bit $C$ is still slightly higher than $1/2$, namely it is higher than,

$$1/2 + \Pr(\Omega_\kappa) \times \alpha \mu.$$

As we stated earlier $I_{1-C} \not\subseteq \mathcal{I}$, meaning that at least one index in $I_{1-C}$ is not in $\mathcal{I}$, and since $I_{1-C} \setminus I_C \cap (I_G \cup I_B)$ cannot be larger than the total length of the string $X^n$ (which is obviously $n$), we must have $\alpha \geq 1/n$. 

\[\square\]
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Appendix A: Why doesn’t dishonest Bob get any advantage by selectively discarding rounds when Alice uses a perfect single photon source?

In this section we explain why for our proof we can consider that we can simply evaluate the min-entropy bound of Lemma IV.7 as if Bob were honest in choosing which rounds he announces to be lost. In other words we explain why dishonest Bob can’t get any advantage by selectively discarding rounds.

In Protocol II.1 Alice sends $n'$ BB84 states to Bob using a perfect single photon source. This, by purification of the states she sends, is equivalent as to Alice preparing $n'$ EPR pairs, and sending half of each pairs to dishonest Bob, and randomly measuring her halves of EPR pairs in the $X$ or $Z$ basis. This allows us to delay Alice’s measurements to the end of the preparation phase.

The bound we use for the min-entropy is independent of the details of the state. Indeed the bound works as follows. For any state $\rho_{A^B^n,E}$ (for some $n \in \mathbb{N}$), if Alice’s measurements (modeled by the CPTP map $\mathcal{M}_{A^B^n \to X^n}$) on the systems $A^n_B$ (outputting bit string $X^n_B$) satisfy some condition (that is indeed satisfied when Alice randomly measures in the $X$ or $Z$ basis [DFW15]), then $H_{\min}(X^n_B|E)_{\mathcal{M}(\rho)} \geq B(H_{\min}(A^n_B|E)_\rho/n) \cdot n$, where $B(\cdot)$ is some function that bounds the min-entropy rate.

Since the bounds applies to any state, one can then choose $\rho_{A^B^n,E}$ to be the state of the protocol after that Bob (holding register $E = KQ$, where $K$ is classical and $Q$ is the quantum state in his memory) has stored quantum information and after he has announced which rounds are kept and which are not, but before Alice has measured. Using the bounded storage assumption ($\log \dim(Q) \leq D$) we can bound $H_{\min}(A^n_B|E)_\rho \geq -D$. This leads us to $H'_{\min}(X^n_B|E)_{\mathcal{M}(\rho)} \geq B(-D/n) \cdot n$ as stated in Lemma IV.7. Note that this bound is evaluated on the state conditioned on Bob keeping some particular rounds, but the bound does not depend on the strategy he uses for choosing which rounds he keeps and which he discards.

For Protocol II.2 the same reasoning apply. Indeed even though we use a different bound, the bound we use is also independent of the details states on which the entropy is evaluated.

Appendix B: Proof of Lemma IV.15

In this section we will explain how the honest party $H \in \{A,B\}$ can use the decoy states in order to estimate a lower-bound $L_{H1}$ on $n_H^1$. To do so we will use techniques inspired by [CXC+14]. In the following we will detail the analysis considering that Alice is honest. The case when Bob is honest follows the same structure.

First we can observe that Protocol II.3 is equivalent to a virtual protocol where Alice first chooses the number $k$ of photons she is sending according to a probability distribution $p_a$, and the encoding basis with probability $p_o$, and only after the station reveals the measurement outcome $o$ she chooses the signal intensity $a \in \{a_d, a_{d_1} \ldots a_{d_r}\}$ according to probability distribution $p_{a|k}$ (this choice in independent from $\theta$ and outcome $o$). The probability distribution $p_k$ and $p_{a|k}$ in the virtual protocol can be deduced from the distribution $p_a$, and $p_{b|a}$ of Protocol II.3 via Bayes’ rule.

As a consequence for any set $S^A_{k,o,a}$ of rounds where Alice has emitted $k$ photons encoded in the basis $\theta (\theta = 0$ for the standard basis, and $\theta = 1$ for the Hadamard basis) and the measurement station (or dishonest Bob) reported measurement outcome $o$ (with $o \neq$ failure), each subset of $S^A_{k,o,a}$ corresponding to intensity $a$ can be seen as a random sample of $S^A_{k,o,a}$. Therefore we can use (classical) random sampling theory to estimate $L_{A1}$, like Chernoff’s bound for example. In particular we will use the following lemma proven in Ref. [CXC+14],

**Lemma B.1.** Let $X_1, \ldots, X_n$ be $n$ independent Bernoulli random variables such that $\Pr(X_i = 1) = p_i$, and let $X := \sum_i X_i$ and $\zeta := \mathbb{E}(X) = \sum_i p_i$. Let $x$ be the observed outcome of $X$ for a certain trial and $\Gamma := x - \sqrt{n/2 \ln(1/\epsilon)}$ for a certain $\epsilon > 0$. If $\epsilon, \hat{\epsilon} > 0$ are such that $(2\hat{\epsilon}^{-1})^{1/\zeta} \leq \exp(3/(4\sqrt{2}))^2$ and $(\hat{\epsilon}^{-1})^{1/\zeta} < \exp(1/3)$ then $x$ satisfies,

$$x = \zeta + \delta,$$

except with probability $\epsilon + \hat{\epsilon} + \delta$, where $\delta \in [-\Delta, \hat{\Delta}]$, with $\Delta := g(x, \epsilon^4/16)$, $\hat{\Delta} := g(x, \epsilon^2/3)$ and $g(x, y) := \sqrt{2x \ln(y^{-1})}$. Here $\epsilon(\hat{\epsilon})$ denotes the probability that $x < \zeta - \Delta$ ($x > \zeta + \hat{\Delta}$).

This lemma is a variation of the Chernoff’s bound, where the bounds on the fluctuations $\Delta(\hat{\Delta})$ do not depend on the expectation value $\zeta := \mathbb{E}(X)$ of the random variable $X$, but only on the observed value $x$ of $X$ (and the epsilon).

Let $S^A_{k,o,a}$ be the set of rounds as defined above, and let $X^a_{i|k,o,a}$ be 1 if the $i$th element of $S^A_{k,o,a}$ corresponds to an
emission of a state (from honest Alice) with intensity \( a \), and 0 otherwise. Let

\[
X^a_{\alpha, \theta} = \sum_k \sum_i S^A_{k, i, \alpha, \theta} |S^A_{k, i, \alpha, \theta}|
\]

(B2)

with \( \zeta^a_{\alpha, \theta} := \mathbb{E}(X^a_{\alpha, \theta}) = \sum_k \sum_i |S^A_{k, i, \alpha, \theta}| \). Let \( x^a_{\alpha, \theta} \) be an observed outcome of \( X^a_{\alpha, \theta} \). Then applying Lemma 1.4.1 we have that for some \( (2\varepsilon^{-1})^{1/4/\gamma_{\alpha, \theta}} \leq \exp(3/(4\sqrt{2}))^2 \), \( (\varepsilon^{-1})^{1/4/\gamma_{\alpha, \theta}} < \exp(1/3) \) with

\[
\Gamma^a_{\alpha, \theta} = x^a_{\alpha, \theta} - \sqrt{\sum_a x^a_{\alpha, \theta}^2/2 \ln(1/\varepsilon)},
\]

(B3)

the following must be satisfied:

\[
x^a_{\alpha, \theta} = \sum_k p_k |S^A_{k, \alpha, \theta}| + \delta^{a, \alpha, \theta},
\]

(B4)

except with probability \( \epsilon + \varepsilon + \bar{\varepsilon} \), where \( \delta^{a, \alpha, \theta} \in [\Delta_{a, \alpha, \theta}, \bar{\Delta}_{a, \alpha, \theta}] \), with \( \Delta_{a, \alpha, \theta} = g(x^a_{\alpha, \theta}, \varepsilon^4/16) \) and \( \bar{\Delta}_{a, \alpha, \theta} = g(x^a_{\alpha, \theta}, \varepsilon^3/2) \).

Since \( n^A_{1} = \sum_{\alpha, \theta} n^A_{1, \alpha, \theta} \) it is enough to find a lower bound on \( n^A_{1, \alpha, \theta} \) for all values of \( (\alpha, \theta) \) in order to find a lower bound \( L_{A1} \) on \( n^A_{1} \). Then using concentration bounds one can write that for each value of \( (\alpha, \theta) \)

\[
n^A_{1, \alpha, \theta} \geq p_{\alpha, \theta} = |S^A_{1, \alpha, \theta}| - g(p_{\alpha, \theta} |S^A_{1, \alpha, \theta}|, \epsilon_1),
\]

(B5)

except with probability \( \epsilon_1 \). For a fixed value of \( (\alpha, \theta) \) one can find a lower-bound on \( |S^A_{1, \alpha, \theta}| \) by minimizing \( |S^A_{1, \alpha, \theta}| \) under the constraints given by eq. 1.4.1. This can be solved by using linear programming [Van14], or we can use a simplified version of this reasoning to find analytical (but looser) bounds. This is what we will be doing in the following section.

1. Simple Analytical Bound

In this section we propose to find a simple analytical bound on \( n^A_{1} \), using the reasoning and methods of the previous section. To do so we will minimize \( |S^A_{1, \alpha, \theta}| \) for a fixed value for \( (\alpha, \theta) \). Moreover we will restrict ourselves to the use of only 2 decoy states and one signal state, i.e. \( \alpha \in \{a_s, a_{d1}, a_{d2}\} \).

In the previous section we have split the rounds into many sets \( S_{k, \alpha, \theta} \) (1 set for each value of \( k \)). Here we split the round into two sets \( S^A_{1, \alpha, \theta} \) and \( S^A_{2, \alpha, \theta} \).

With this in mind we can rewrite equation [1.4.1] as the following system of inequalities,

\[
\begin{cases}
X^a_{\alpha, \theta} + \Delta_{a, \alpha, \theta} & \geq p_{a, \alpha, \theta} |S^A_{1, \alpha, \theta}| + p_{a, \alpha, \theta} |S^A_{2, \alpha, \theta}| \\
X^a_{\alpha, \theta} - \Delta_{a, \alpha, \theta} & \leq p_{a, \alpha, \theta} |S^A_{1, \alpha, \theta}| + p_{a, \alpha, \theta} |S^A_{2, \alpha, \theta}| \\
X^a_{\alpha, \theta} + \Delta_{a, \alpha, \theta} & \geq p_{a, \alpha, \theta} |S^A_{1, \alpha, \theta}| + p_{a, \alpha, \theta} |S^A_{2, \alpha, \theta}| \\
X^a_{\alpha, \theta} - \Delta_{a, \alpha, \theta} & \leq p_{a, \alpha, \theta} |S^A_{1, \alpha, \theta}| + p_{a, \alpha, \theta} |S^A_{2, \alpha, \theta}|
\end{cases}
\]

(B6)

Each of the four inequalities represents half a space delimited by a straight line in \( \mathbb{R}^2 \). The two first inequalities define a region delimited by two parallel lines, and the two last inequalities define another region delimited by two other parallel lines. The set of four inequalities is then the intersection of these two regions, see Fig. 8. Since we are minimizing a linear function with linear constraints the minimum is reached for one of the extreme points of this region. Each of these points corresponds to the solution of the system of equations formed by two of the inequalities from [B6] (one for decoy state 1 and one for decoy state 2) by changing symbols \( \leq, \geq \) into \( = \). Since there are two equations for each decoy state, the number of extreme points must be 4. They can be found analytically by solving this system of equations. In the end the lower-bound \( L_{A1} \) is given by,

\[
L_{A1} = \sum_{\alpha, \theta} \left[ p_{\alpha, \theta} |S^A_{1, \alpha, \theta}|_{\min} - g(p_{\alpha, \theta} |S^A_{1, \alpha, \theta}|_{\min}, \epsilon_1) \right],
\]

(B7)

where \( |S^A_{1, \alpha, \theta}|_{\min} \) is given by,

\[
|S^A_{1, \alpha, \theta}|_{\min} = \min(V_1, V_2, V_3, V_4),
\]

(B8)
Appendix C: Formal Security Definitions for OT and BC

In this section you can find the formal definitions for Randomized String Commitment and for Randomized 1-out-2 \((l, \epsilon)\)-Oblivious String Transfer. These definitions come directly from Refs. [KWW12].

Remark C.1 (on the abort events). The careful reader will see that the definitions below do not mention any abort event. In fact our protocols specify the action a party has to take when he wants to abort. In particular we ask the aborting party to output uniformly random outcomes, so that even when aborting the security definitions are satisfied.

Definition C.1 (Randomized String Commitment). Let \(\tau_R\) denote the maximally mixed state on a register \(R\). An \((l, \epsilon)\)-Randomized String commitment scheme is a protocol between Alice and Bob that satisfies the following three properties.

Correctness: When both parties are honest, then there exists a state \(\sigma_{C_1C_1F}\), called the ideal state that is defined as:

\[
V_1 = \frac{p_{a_{d_1}|k \geq 2}(x_{a_{d_2}} + \Delta_{a_{d_2},o,\theta}) - p_{a_{d_2}|k \geq 2}(x_{a_{d_1}} + \Delta_{a_{d_1},o,\theta})}{p_{a_{d_1}|k=1}p_{a_{d_2}|k \geq 2} - p_{a_{d_1}|k \geq 2}p_{a_{d_2}|k=1}} \quad (B9)
\]

\[
V_2 = \frac{p_{a_{d_1}|k \geq 2}(x_{o,\theta} - \hat{\Delta}_{a_{d_2},o,\theta}) - p_{a_{d_2}|k \geq 2}(x_{a_{d_1}} + \Delta_{a_{d_1},o,\theta})}{p_{a_{d_1}|k=1}p_{a_{d_2}|k \geq 2} - p_{a_{d_1}|k \geq 2}p_{a_{d_2}|k=1}} \quad (B10)
\]

\[
V_3 = \frac{p_{a_{d_1}|k \geq 2}(x_{o,\theta} - \hat{\Delta}_{a_{d_2},o,\theta}) - p_{a_{d_2}|k \geq 2}(x_{a_{d_1}} + \Delta_{a_{d_1},o,\theta})}{p_{a_{d_1}|k=1}p_{a_{d_2}|k \geq 2} - p_{a_{d_1}|k \geq 2}p_{a_{d_2}|k=1}} \quad (B11)
\]

\[
V_4 = \frac{p_{a_{d_1}|k \geq 2}(x_{o,\theta} - \hat{\Delta}_{a_{d_2},o,\theta}) - p_{a_{d_2}|k \geq 2}(x_{a_{d_1}} + \Delta_{a_{d_1},o,\theta})}{p_{a_{d_1}|k=1}p_{a_{d_2}|k \geq 2} - p_{a_{d_1}|k \geq 2}p_{a_{d_2}|k=1}} \quad (B12)
\]
• \( \sigma_{C_1 F} := \tau_{C_1} \otimes |\text{accept}\rangle \langle \text{accept}|_F \),

• The real state produced by the protocol \( \rho_{C_1 \tilde{C}_1 F} \) is \( \epsilon \)-close to the ideal state \( \sigma_{C_1 \tilde{C}_1 F} \),

\[
\rho_{C_1 \tilde{C}_1 F} \approx \epsilon \sigma_{C_1 \tilde{C}_1 F}.
\]

Security for Alice (against dishonest Bob): When Alice is honest, Bob is ignorant about \( C_1 \) before the Open phase:

\[
\rho_{C_1 B} \approx \epsilon \tau_{C_1} \otimes \rho_B.
\]

The protocol is then said to be \( \epsilon \)-hiding.

Security for Bob (against dishonest Alice): After the Commit phase and before the Open phase, there exists an ideal state \( \sigma_{C_1 AB} \) such that for any Open algorithm, described by the CPTP maps \( O_{AB} \), in which Bob is honest, we have:

• Bob almost never accepts \( \tilde{C}_1 \neq C_1 \):

\[
\Pr(C_1 \neq \tilde{C}_1 \text{ and } F = \text{accept}) \leq \epsilon.
\]

• The real state produced by the commitment phase is close to the ideal state:

\[
\rho_{AB} \approx \epsilon \sigma_{AB}.
\]

The protocol is then said to be \( \epsilon \)-binding.

Definition C.2 (Randomized 1-out-2 \((l, \epsilon)\)-Oblivious String Transfer (OST)).

Let \( \tau_R \) denote the maximally mixed state on register \( R \).

A fully randomized 1-out-2 \((l, \epsilon)\)-Oblivious String Transfer scheme is a protocol between two parties, Alice and Bob, that satisfies the following three conditions.

Correctness: If both parties are honest there exists an ideal state \( \sigma_{S_0 S_1 C_S} \), where \( S_1, S_1 \in \{0, 1\}^l \) and \( C \in \{0, 1\} \), such that:

• The distribution over \( S_0, S_1 \) and \( C \) is uniform:

\[
\sigma_{S_0 S_1 C} = \tau_{S_0} \otimes \tau_{S_1} \otimes \tau_C
\]

• The real state \( \rho \) produced by the protocol is \( \epsilon \)-close to the ideal state:

\[
\rho_{S_0 S_1 C_S} \approx \epsilon \sigma_{S_0 S_1 C_S}
\]

Security for Bob: If Bob is honest, there exists an ideal state \( \sigma_{AS_0 S_1 C} \) such that:

• Alice is ignorant about \( C \):

\[
\sigma_{AS_0 S_1 C} = \sigma_{AS_0 S_1} \otimes \tau_C
\]

• The real state \( \rho \) produced by the protocol is close to the ideal state:

\[
\rho_{AC_S} \approx \epsilon \sigma_{AC_S}
\]

Security for Alice: If Alice is honest, there exists an ideal state \( \sigma_{S_0 S_1 BC} \) such that:

• Bob is ignorant about \( S_1 - C \):

\[
\sigma_{S_0 S_1 BC} = \sigma_{S_0 S_1 BC} \otimes \tau_{S_1 - C}
\]

• The real state \( \rho \) is close to the ideal state:

\[
\rho_{S_0 S_1 B} \approx \epsilon \sigma_{S_0 S_1 B}
\]