Curvature-dependent formalism, Schrödinger equation and energy levels for the harmonic oscillator on three-dimensional spherical and hyperbolic spaces

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Abstract

A nonlinear model representing the quantum harmonic oscillator on the three-dimensional spherical and hyperbolic spaces, $S^3_\kappa$ ($\kappa > 0$) and $H^3_\kappa$ ($\kappa < 0$), is studied. The curvature $\kappa$ is considered as a parameter and then the radial Schrödinger equation becomes a $\kappa$-dependent Gauss hypergeometric equation that can be considered as a $\kappa$-deformation of the confluent hypergeometric equation that appears in the Euclidean case. The energy spectrum and the wavefunctions are exactly obtained in both the three-dimensional sphere $S^3_\kappa$ ($\kappa > 0$) and the hyperbolic space $H^3_\kappa$ ($\kappa < 0$). A comparative study between the spherical and the hyperbolic quantum results is presented.

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1. Introduction

The study of quantum problems in curved spherical spaces (positive constant curvature) was initiated by Schrödinger [1], Infeld [2] and Stevenson [3], in 1940 and 1941. Infeld and Schild [4] considered in 1945 a similar problem but in a hyperbolic space (negative constant curvature). Later, Barut et al studied a path integral treatment for the Hydrogen atom in a curved space of constant curvature, first in the spherical case [5] and then in the hyperbolic case [6]. Since then other authors have studied similar problems on curved spaces with constant curvature making use of different approaches [7–28]. Most of these papers are concerned with fundamental problems (previously studied at the classical level) but some authors have proved that this matter is also important for the study of certain questions related to condensed matter physics as, for example, the existence of Landau levels for the motion of a charged particle in a curved space [29–32] and, more recently, the study of quantum dots [33–37].
It is clear that spherical and hyperbolic spaces are endowed with quite different geometrical properties and this is the main reason why the studies of physical systems on spherical and hyperbolic spaces are usually carried out in a separated way (see, e.g., most of the above-mentioned references) since their physical properties also turn out to be different. In spite of this, it has been proved that certain problems (in fact, those related to superintegrable potentials) can be studied by making use of a joint approach valid for the two types of spaces.

This paper is concerned with the study of the quantum harmonic oscillator on three-dimensional spherical and hyperbolic spaces making use of a set of coordinates \((r, \theta, \phi)\) obtained by introducing a small change in the radial part of the geodesic spherical coordinates. It can be considered as a new paper in a series devoted to the study of classical \([38–42]\) and quantum \([43–47]\) systems on Riemannian configuration spaces with constant curvature \(\kappa \neq 0\).

We follow an approach that can be summarized in the following two points.

(i) All the mathematical expressions will depend on the curvature \(\kappa\) as a parameter. So, the first step is to obtain general \(\kappa\)-dependent properties. Then, the second step is to particularize for the values \(\kappa > 0\), \(\kappa = 0\), or \(\kappa < 0\), and obtaining, in such a way, the corresponding property for the physical system on the sphere \(S^3\), on the Euclidean space \(E^3\), or on the hyperbolic space \(H^3\), respectively.

(ii) The idea is to formulate the results in explicit dependence of the curvature \(\kappa\) and to study the changes of the dynamics when \(\kappa\) varies.

We mention now two points that are important for the study presented in this paper. The first one is related to the geometric approach and the other to the dynamics.

- The differential element of distance \(ds_\kappa\), in the family \(M^3_\kappa = (S^3, E^3, H^3)\) of three-dimensional spaces with constant curvature \(\kappa\), can be written in some different but equivalent ways (this question is discussed in [46, 47]; see also [48]). For example, if we make use of the following \(\kappa\)-dependent trigonometric (either circular, parabolic or hyperbolic) functions:

\[
C_\kappa(x) = \begin{cases} 
\cos \sqrt{\kappa}x & \text{if } \kappa > 0, \\
1 & \text{if } \kappa = 0, \\
cosh \sqrt{-\kappa}x & \text{if } \kappa < 0, 
\end{cases}
\]

\[
S_\kappa(x) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa}x & \text{if } \kappa > 0, \\
x & \text{if } \kappa = 0, \\
\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa}x & \text{if } \kappa < 0, 
\end{cases}
\]

then it can be written as follows in geodesic spherical coordinates \((\rho, \theta, \phi)\)

\[
\begin{align*}
\text{d}s_\kappa^2 &= d\rho^2 + S_\kappa^2(\rho) \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right), \\
(1)
\end{align*}
\]

(note that this formalism is intrinsic and \(\rho\) denotes the distance along a geodesic on the manifold \(M^3_\kappa\) and not the radius of a sphere). Nevertheless, in the following, we will use a new radial variable \(r\) given by \(r = S_\kappa(\rho)\) so the expression of \(\text{d}s_\kappa^2\) in the coordinates \((r, \theta, \phi)\) becomes

\[
\begin{align*}
\text{d}s_\kappa^2 &= \frac{1}{1 - \kappa} \frac{dr^2}{r^2} + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2, \\
(2)
\end{align*}
\]

so it reduces to

\[
\begin{align*}
\text{d}s_1^2 &= \frac{dr^2}{1 - \kappa} + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2, \\
\text{d}s_0^2 &= dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right), \\
\text{d}s_{-1}^2 &= \frac{dr^2}{1 + \kappa} + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right),
\end{align*}
\]

in the three particular cases of the unit sphere, the Euclidean plane and the ‘unit’ Lobachevski plane.
The harmonic oscillator in the space of constant curvature $\kappa$ has a potential $U_\kappa$ which, when written in the system $(\rho, \theta, \phi)$, is given by

$$U_\kappa (\rho) = \frac{1}{2} \alpha^2 T^2_\kappa (\rho), \quad (3)$$

where $T_\kappa (\rho)$ denotes the $\kappa$-dependent tangent. It is ‘central’ in the sense that it depends only on the geodesic distance $\rho$ to a fixed center in $M^3_\kappa$. When the curvature $\kappa$ is positive, the potential tends to infinity at the sphere ‘equator’ (with the north pole placed in the center of forces), which corresponds to a finite value $\rho = \pi / (2 \sqrt{\kappa})$; the harmonic oscillator on the $\kappa > 0$ sphere splits the configuration space into two halves with an infinite potential wall on the equator; so the spherical harmonic oscillator motion is confined to just one of these halves. As stated above, we will use the coordinates $(r, \theta, \phi)$ wherein the potential $U_\kappa$ becomes

$$U_\kappa (r) = \frac{1}{2} \alpha^2 \left( \frac{r^2}{1 - \kappa r^2} \right). \quad (4)$$

Of course, for $\kappa > 0$, while the geodesic radial coordinate $\rho$ in the range $[0, \pi / \sqrt{\kappa}]$ allows covering the whole sphere (with coordinate singularities at both ends of the range), the range $[0, 1 / \sqrt{\kappa}]$ of the radial coordinate $r$ covers naturally only the upper-half of the sphere; this matches perfectly with the nature of the harmonic potential for the positive curvature case, which has a infinite wall at the boundary of the domain naturally covered by the coordinate $r$.

Figure 1 shows that the standard Euclidean potential ($\kappa = 0$) represents a borderline between two different behaviors. If $\kappa > 0$, then the potential tends to infinity when $r^2 \to 1 / \kappa$. Then, in the case $\kappa < 0$, the potential is well defined for all the values of $r$ and it is even bounded when $r \to \infty$.

The potential $U_\kappa (r)$ is interpreted as describing the harmonic oscillator in the spaces $M^3_\kappa = (S^3_\kappa, IE^3, H^3_\kappa)$ because of two reasons. First, it fulfils the Euclidean limit in the sense that when $\kappa \to 0$ it becomes the well-known potential of the isotropic harmonic oscillator in the Euclidean space (this is a necessary condition). Second, and even more important, this potential is singled out among other possibilities with the same Euclidean limit by the condition of being superintegrable (some details are provided in the next section).

The plan of the paper is as follows. In section 2, we obtain the expression of the $\kappa$-dependent quantum Hamiltonian $H(\kappa)$. In fact this section is mainly related to a previous study presented in [46]. In section 3, the $\kappa$-dependent Schrödinger equation is solved and then the properties of the spherical $\kappa > 0$ and the hyperbolic $\kappa < 0$ cases are studied in detail. Finally, in section 4, we make some final comments.
2. \(\kappa\)-dependent quantum Hamiltonian

The construction of the classical \(\kappa\)-dependent system and the transition from the classical \(\kappa\)-dependent system to the quantum one was studied in [46]. The main idea is to follow a method used in some previous references as [44–47] that considers the quantization of the Noether momenta as a first step; in this way the Hamiltonian \(H(\kappa)\) is obtained as a self-adjoint operator with respect to an appropriate \(\kappa\)-dependent measure.

The Lagrangian \(L\) of the geodesic motion (\(\kappa\)-dependent kinetic term \(T(\kappa)\) without a potential) on the three-dimensional spaces \((S^3_\kappa, \mathbb{E}^3_\kappa, H^3_\kappa)\) is given by

\[
L = T(\kappa) = \left(\frac{1}{2}\right) \left( \frac{v_r^2}{1 - \kappa r^2} + r^2 v_\theta^2 + r^2 \sin^2 \theta v_\phi^2 \right),
\]

where the parameter \(\kappa\) can be positive (spherical case), null (Euclidean space) and negative (hyperbolic space). In the spherical case, the study of the dynamics is restricted to the interior of the interval \(r^2 < 1/\kappa\) where the kinetic energy is a positive-definite function. As a consequence of the six-dimensional geometric symmetry of this system, encompassed by a group isomorphic to either \(SO(4), ISO(3), SO(1, 3)\) according to \(\kappa >, =, < 0\); this Lagrangian possesses a total of six Noether symmetries. Three of them are related to the common rotational \(SO(3)\) symmetry present in all spaces, and other are specific to each space.

The latter are the three \(\kappa\)-dependent Noether symmetries

\[
X_1(\kappa) = \sqrt{1 - \kappa r^2} \left( \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{1}{r} \left( \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \frac{\sin \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \right),
\]

\[
X_2(\kappa) = \sqrt{1 - \kappa r^2} \left( \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \left( \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \right),
\]

\[
X_3(\kappa) = \sqrt{1 - \kappa r^2} \left( \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \right),
\]

with associated constants of motion

\[
P_1(\kappa) = (\sin \theta \cos \phi) \sqrt{1 - \kappa r^2} v_r + (r \sqrt{1 - \kappa r^2}) \left[ (\cos \theta \cos \phi) v_\theta - (\sin \theta \sin \phi) v_\phi \right],
\]

\[
P_2(\kappa) = (\sin \theta \sin \phi) \sqrt{1 - \kappa r^2} v_r + (r \sqrt{1 - \kappa r^2}) \left[ (\cos \theta \sin \phi) v_\theta + (\sin \theta \cos \phi) v_\phi \right],
\]

\[
P_3(\kappa) = (\cos \theta) \sqrt{1 - \kappa r^2} v_r - (r \sqrt{1 - \kappa r^2}) \sin \theta v_\phi,
\]

while the former are the three \(\kappa\)-independent Noether symmetries that coincide with the Euclidean symmetries corresponding the space isotropy

\[
Y_1 = - \sin \phi \frac{\partial}{\partial \theta} - \left( \frac{\cos \phi}{\tan \theta} \right) \frac{\partial}{\partial \phi}, \quad Y_2 = \cos \phi \frac{\partial}{\partial \theta} - \left( \frac{\sin \phi}{\tan \theta} \right) \frac{\partial}{\partial \phi}, \quad Y_3 = \frac{\partial}{\partial \phi},
\]

leading to the three (\(\kappa\)-independent) components of the angular momentum

\[
J_1 = -r^2 (\sin \phi v_\theta + \sin \theta \cos \phi v_\phi),
\]

\[
J_2 = r^2 (\cos \phi v_\theta - \sin \theta \cos \phi v_\phi),
\]

\[
J_3 = r^2 \sin^2 \theta v_\phi.
\]

The \(\kappa\)-dependent Hamiltonian representing the harmonic oscillator on the spaces \((S^3_\kappa, \mathbb{E}^3_\kappa, H^3_\kappa)\) is given by

\[
H(\kappa) = \left(\frac{1}{2}\right) \left( (1 - \kappa r^2) p_r^2 + \frac{1}{r^2} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) \right) + \frac{1}{2} (ma^2) \left( \frac{r^2}{1 - \kappa r^2} \right),
\]
and it can also be written as follows:

\[
H(\kappa) = \left(\frac{1}{2m}\right) \left[ P_i^2 + P_j^2 + P_k^2 + \kappa \left( J_i^2 + J_j^2 + J_k^2 \right) \right] + \frac{1}{2} (m\alpha^2) \left( \frac{r^2}{1 - \kappa r^2} \right),
\]

(7)

where \( P_i \) and \( J_j, \ j = 1, 2, 3 \), denote now the Hamiltonian versions of the corresponding Noether momenta obtained above in the Lagrangian notation. Here, it is worth mentioning that the free part of the Hamiltonian is proportional to the quadratic Casimir operator in the Lie algebra of the geometric symmetries, reducing to the square of the linear momentum only in the case \( \kappa = 0 \), but including also a term with the square of the angular momentum otherwise.

The quantum Hamiltonian \( \hat{H}(\kappa) \) must be an operator obtained from \( \hat{H}(\kappa) \) that must be self-adjoint in the space \( L^2(\kappa) \) where \( d\mu_\kappa \) denotes the measure

\[
d\mu_\kappa = \left( \frac{r^2 \sin \theta \sqrt{1 - \kappa r^2}}{\sqrt{1 - \kappa r^2}} \right) dr d\theta d\phi,
\]

and the particular form of the Hilbert space \( L^2(\kappa) \) depends on \( \kappa \) as follows.

(i) In the hyperbolic \( \kappa < 0 \) case, the space \( L^2(\kappa) \) can be identified with \( L^2(\mathbb{R}^3, d\mu_\kappa) \).

(ii) In the spherical \( \kappa > 0 \) case, the space \( L^2(\kappa) \) can be identified with \( L^2(I_\kappa \times \mathbb{R}^2, d\mu_\kappa) \), where \( I_\kappa \) denotes the interval \([0, 1/\sqrt{\kappa}]\) and the subscript means that the functions must vanish at the endpoints \( r = 0 \) and \( r = 1/\sqrt{\kappa} \).

The first step is to obtain the expressions of the operators \( \hat{P}_1, \hat{P}_2 \) and \( \hat{P}_3 \), representing the quantum version of the Noether momenta \( P_1, P_2 \) and \( P_3 \), as self-adjoint operators in the space \( L^2(\kappa) \). They are given by

\[
\hat{P}_i = -i\hbar \sqrt{1 - \kappa r^2} \left( \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{1}{r} \left( \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \left( \frac{\sin \phi}{\sin \theta} \right) \frac{\partial}{\partial \phi} \right) \right),
\]

\[
\hat{P}_i = -i\hbar \sqrt{1 - \kappa r^2} \left( \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \left( \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \left( \frac{\cos \phi}{\sin \theta} \right) \frac{\partial}{\partial \phi} \right) \right),
\]

\[
\hat{P}_3 = -i\hbar \sqrt{1 - \kappa r^2} \left( \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \right).
\]

The quantum operators \( \hat{J}_i, i = 1, 2, 3 \), are \( \kappa \)-independent, and therefore, they coincide with the Euclidean ones

\[
\hat{J}_i = i\hbar \left[ \sin \phi \frac{\partial}{\partial \theta} + \left( \frac{\cos \phi}{\tan \theta} \right) \frac{\partial}{\partial \phi} \right], \quad \hat{J}_2 = -i\hbar \left[ \cos \phi \frac{\partial}{\partial \theta} - \left( \frac{\sin \phi}{\tan \theta} \right) \frac{\partial}{\partial \phi} \right], \quad \hat{J}_3 = -i\hbar \frac{\partial}{\partial \phi}.
\]

Then, we have that the quantum Hamiltonian \( \hat{H}(\kappa) \) that is given by

\[
\hat{H}(\kappa) = \left(\frac{1}{2m}\right) \left[ \hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2 + \kappa \left( \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 \right) \right] + \frac{1}{2} (m\alpha^2) \left( \frac{r^2}{1 - \kappa r^2} \right)
\]

(8)

is represented by the following differential operator:

\[
\hat{H} = -\frac{\hbar^2}{2m} \left[ (1 - \kappa r^2) \frac{\partial^2}{\partial r^2} + \frac{2 - 3\kappa r^2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} \right) + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \phi} \right]
\]

\[
+ \frac{1}{2} (m\alpha^2) \left( \frac{r^2}{1 - \kappa r^2} \right).
\]

(9)

that is self-adjoint with respect the measure \( d\mu_\kappa \), and it satisfies the appropriate Euclidean limit

\[
\lim_{\kappa \to 0} \hat{H}(\kappa) = -\frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \phi} \right] + \frac{1}{2} (m\alpha^2) r^2.
\]

We close this section with the following remarks and observations.
The requirement to have the correct Euclidean limit leaves a lot of possibilities open for a ‘harmonic oscillator potential in the curved space’. But the potential chosen is singled out if we require also that the curved potential be superintegrable. Indeed, for this potential $U_κ (r)$, there is a full set of constants of motion given by the $κ$-dependent functions $F_{ij}(κ)$ defined by

$$F_{ij}(κ) = P_i P_j + α^2 X_i X_j, \quad i, j = 1, 2, 3,$$

with $X_i, i = 1, 2, 3$, given by

$$X_1 = \frac{r \sin \theta \cos \phi}{\sqrt{1 - κr^2}}, \quad X_2 = \frac{r \sin \theta \sin \phi}{\sqrt{1 - κr^2}}, \quad X_3 = \frac{r \cos \theta}{\sqrt{1 - κr^2}}.$$

Of course, $F(κ)$ with components $F_{ij}(κ), i, j = 1, 2, 3$, represents the curved version of the Fradkin tensor [49]. Note that the expression of $F(κ)$ depends on the Noether momenta instead of the canonical momenta.

The measure $dμ_κ$, that was obtained as the unique measure (up to a multiplicative constant) invariant under the Killing vectors [46], coincides with the corresponding Riemannian volume in a space with curvature $κ$.

The (free part of the) quantum Hamiltonian we have obtained, i.e. $\hat{H}(κ)$, turns out to coincide with the one obtained by making use of the Laplace–Beltrami operator for the space under consideration. This is no surprise, of course. While the end result is the same, we want to emphasize that the logic in the argument is somewhat different to the usual because Laplace–Beltrami quantization procedure leads directly to the expression of the quantum Hamiltonian without a previous quantization of the momenta. The standard procedure in the Euclidean case is to first quantize the momenta (i.e. to identify them as self-adjoint operators) and then to obtain the quantum version of the Hamiltonian. We have translated this momentum approach to the spaces with curvature $κ$ but changing the quantization of the canonical momenta by the quantization of the Noether momenta which are taken as the basic objects.

One additional reason for the quantization via the Noether momenta is that it also seems appropriate for the quantization of systems with a position-dependent mass (PDM). In fact, there is a great interest in the study of the quantization of systems with a PDM not only for the applications to condensed matter physics, but also because there is an important problem at the starting level of quantization; since if the mass $m$ becomes a spatial function, then the quantum version of the mass no longer commutes with the momentum. A Hamiltonian system in a space with curvature $κ$ can also be considered as a very particular PDM system (in this case the mass $m$ is not an effective mass, but it becomes a spatial function as a consequence of the geometry). We think that the quantization, as a first step, of the Noether momenta is an appropriate method for the quantization of the PDM systems.

### 3. $κ$-dependent Schrödinger equation and wavefunctions

The Schrödinger equation $\hat{H}(κ) \Psi = E \Psi$ leads to the following $κ$-dependent differential equation:

$$\left[ -\frac{ℏ^2}{2m} \left( 1 - κr^2 \right) \frac{\partial^2}{\partial r^2} + \frac{2 - 3κr^2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} \right) \right] \Psi = E \Psi. \quad (10)$$
Thus, as $U_\kappa(r)$ is a central potential for all the values of $\kappa$, we can assume that $\Psi(r, \theta, \phi)$ can be factorized in the form

$$\Psi(r, \theta, \phi) = R(r)Y_{lm}(\theta, \phi),$$

where $R$ is a function of $r$ and $Y_{lm}(\theta, \phi)$ are the standard $\kappa$-independent spherical harmonics

$$\left(\frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta}\right)Y_{lm} = -l(l+1)Y_{lm}.$$

Then, we arrive at the following $\kappa$-dependent radial equation:

$$\begin{align*}
&\left[\frac{-\hbar^2}{2m} \left(1 - \kappa r^2\right) \frac{d^2}{dr^2} + \frac{2 - 3\kappa r^2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2}\right] + \frac{1}{2} \left(m\alpha^2\right)\left(\frac{r^2}{1 - \kappa r^2}\right)R = ER, \quad R = R(r).
\end{align*}$$

It is convenient to change the parameter $\alpha^2$ in the potential to the form $\alpha^2 \to \alpha^2 - (\kappa \hbar/m)\alpha$ (this change is done by similarity with the result obtained when using the Schrödinger factorization method for the one-dimensional nonlinear oscillator studied in \[43, 44\]) and introduce dimensionless variables $(\rho, \tilde{\kappa}, \tilde{E})$ defined by

$$\begin{align*}
&\rho = \sqrt{\frac{\hbar}{m \alpha^2}} r, \quad \kappa = \frac{m\alpha^2}{\hbar}, \quad E = \langle \alpha \rangle \tilde{E}, \quad \kappa r^2 = \tilde{\kappa} \rho^2,
\end{align*}$$

so that we obtain

$$\rho^2 \left(1 - \tilde{\kappa} \rho^2\right)R'' + \rho \left(2 - 3\tilde{\kappa} \rho^2\right)R' - \left(1 - \tilde{\kappa}\right) \left(\frac{\rho^4}{1 - \tilde{\kappa} \rho^2}\right)R + \left[2 \tilde{E} \rho^2 - l(l+1)\right]R = 0,$$

which represents a $\kappa$-dependent deformation of the Euclidean differential equation

$$R'' + \frac{2}{\rho} R' - \rho^2 R + \left[2 \tilde{E} - \frac{l(l+1)}{\rho^2}\right]R = 0.$$ 

We assume the following factorization for the function $R$:

$$R = f(\rho, \tilde{\kappa})(1 - \tilde{\kappa} \rho^2)^{(1/2)},$$

so that

$$\lim_{\rho \to 0} R(\rho, \tilde{\kappa}) = f(\rho) e^{-1/2)\rho^2}.$$ 

Then, the function $f(\rho)$ must be a solution of

$$\rho^2 \left(1 - \tilde{\kappa} \rho^2\right) f'' + \rho \left(2 - 2\rho^2 - 3\tilde{\kappa} \rho^2\right) f' + \left[2 \tilde{E} - 3 \rho^2 - l(l+1)\right] f = 0.$$ 

This equation can be solved by using the method of Frobenius. The solution near the regular singular point $\rho = 0$ can be written as follows:

$$f = \rho^\mu g(\rho, \tilde{\kappa}),$$

where $\mu$ is a solution of the indicial equation and $g$ is an analytical function with a $\kappa$-dependent power series

$$g = \sum_{n=0}^{\infty} g_n \rho^n = g_0 + g_1 \rho + g_2 \rho^2 + g_3 \rho^3 + \cdots \quad (g_0 \neq 0).$$
Then, it is proved that \( \mu \) must take one of the two values \( \mu_1 = l \) or \( \mu_2 = -l - 1 \). Considering \( \mu = l \), in order to have \( R \) well defined at the origin, we arrive at

\[
\rho (1 - \tilde{\kappa} \rho^2) g'' + [2(l + 1) - (2 + 3\tilde{\kappa} + 2\tilde{\kappa}l) \rho^2] g' + \left( (2E - 3) - (2 + 2\tilde{\kappa} + \tilde{\kappa}l)l \right) \rho g = 0,
\]

and then the \( \kappa \)-dependent recursion relation leads to the vanishing of all the odd coefficients, \( g_1 = g_3 = g_5 = g_7 = \cdots = 0 \), so that it is a series with only even powers of \( \rho \) and a radius of convergence \( R_c \) given by \( R_c = 1/\sqrt{|\kappa|} \) (determined by the presence of the second singularity). The even powers dependence suggests to introduce the new variable \( z = \rho^2 \) so that the equation becomes

\[
z (1 - \tilde{\kappa} z) g''_{\tilde{\kappa}} + \frac{1}{2} [(2l + 3) - 2(1 + 2\tilde{\kappa} + \tilde{\kappa}l)z] g'_{\tilde{\kappa}} + \frac{1}{4} [(2E - 3) - (2 + 2\tilde{\kappa} + \tilde{\kappa}l)l] g = 0.
\]

In the Euclidean case, this equation reduces to

\[
z g''_{\tilde{\kappa}} + [(l + \frac{3}{2}) - z] g'_{\tilde{\kappa}} - \frac{1}{2}(l + \frac{3}{2}) - E] g = 0,
\]

whose solution regular at \( z = 0 \) is a confluent hypergeometric function

\[g(\rho) = \, _1F_1(a; c; \rho^2), \quad a = \frac{1}{2} \left( l + \frac{3}{2} - E \right), \quad c = l + \frac{3}{2}.
\]

The boundary conditions at \( \rho = 0 \) and \( \rho = \infty \) (Sturm–Liouville problem) leads to the associated Laguerre polynomials.

In the general non-Euclidean \( \tilde{\kappa} \neq 0 \) case, it is convenient to introduce the change \( t = \tilde{\kappa} z \). Then, equation (14) reduces to

\[
t (1 - t) g''_{\tilde{\kappa}} + \left[ (l + \frac{3}{2}) - \frac{1}{\tilde{\kappa}}(1 + 2\tilde{\kappa} + \tilde{\kappa}l)l \right] g'_{\tilde{\kappa}} + \frac{1}{4\tilde{\kappa}} [(2E - 3 - 2l) - \tilde{\kappa}l(l + 2)] g = 0,
\]

that is a Gauss hypergeometric equation

\[
t (1 - t) g''_{\tilde{\kappa}} + [c - (1 + a_\kappa + b_\kappa)t] g'_{\tilde{\kappa}} - a_\kappa b_\kappa g = 0,
\]

with

\[c = l + \frac{3}{2}, \quad a_\kappa + b_\kappa = \frac{1}{\tilde{\kappa}}(1 + l + 1), \quad a_\kappa b_\kappa = -\frac{1}{4\tilde{\kappa}}[(2E - 3 - 2l) - \tilde{\kappa}l(l + 2)],
\]

and the solution regular at \( t = 0 \) is the hypergeometric function

\[g(t, \tilde{\kappa}) = \, _2F_1(a_\kappa, b_\kappa; c; t), \quad _2F_1(a_\kappa, b_\kappa; c; t) = 1 + \sum_{n=1}^{\infty} \frac{(a_\kappa)_n(b_\kappa)_n t^n}{(c)_n n!},
\]

with \( a_\kappa \) and \( b_\kappa \) given by

\[a_\kappa = \frac{1}{2\tilde{\kappa}}(A_\kappa \pm \sqrt{B_\kappa}), \quad b_\kappa = \frac{1}{2\tilde{\kappa}}(A_\kappa \mp \sqrt{B_\kappa})
\]

(where \( A_\kappa = 1 + \tilde{\kappa}(l + 1), \quad B_\kappa = 1 + (2E - 1)\tilde{\kappa} + \tilde{\kappa}^2 \).

Equation (15) has a singularity, when \( \tilde{\kappa} > 0 \), at \( t = 1 \) that corresponds to \( z = 1/\tilde{\kappa} \) (or \( r = 1/\sqrt{\tilde{\kappa}} \)). If the origin \( r = 0 \) is placed in the north pole of the sphere then this singularity is just placed at the equator. The property of regularity of the solutions leads us to analyze the existence of particular solutions well defined at this point. The polynomial solutions appear
when one of the two \( \kappa \)-dependent coefficients, \( a_\kappa \) or \( b_\kappa \), coincides with zero or with a negative integer number:

\[
a_\kappa = -n_\kappa \quad \text{or} \quad b_\kappa = -n_\kappa, \quad n_\kappa = 0, 1, 2, \ldots.
\]

Then, in this case, the coefficient \( \mathcal{E} \), that represents the energy, is restricted to one of the following values:

\[
\mathcal{E}_{n, l} = (2n_\kappa + l + \frac{3}{2}) + \frac{1}{\kappa} (2n_\kappa + l)(2n_\kappa + l + 2),
\]

and the hypergeometric series \( _2F_1(a_\kappa, b_\kappa, c; \kappa \rho^2) \) reduces to a polynomial of degree \( n_\kappa \).

The differential equation

\[
a_0 g'' + a_1 g' + \lambda \rho g = 0,
\]

with

\[
a_0 = \rho (1 - \kappa \rho^2), \quad a_1 = [2(l + 1) - (2 + 3\kappa + 2\kappa l)\rho^2], \quad \lambda = (2\mathcal{E} - 3) - (2 + 2\kappa + \kappa l)l,
\]
together with the boundary conditions at the points \( \rho_1 = 0 \) and \( \rho_2 = \rho_\kappa \), determine a singular Sturm–Liouville problem that is formally self-adjoint and if the boundary conditions are appropriately defined then the operator is symmetric. Then the eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function \( q = \rho^{2l+1}(1 - \kappa \rho^2)^{1/\kappa - 1/2} \); note that this involves the value of the curvature. More concretely, we have the following.

(i) In the spherical \( \kappa > 0 \) case, the function \( g \) must vanish in the point \( \rho_2 = \rho_\kappa \) and the eigenfunctions are orthogonal in the interval \([0, \rho_\kappa]\) with \( \rho_\kappa = 1/\sqrt{\kappa} \).

(ii) In the hyperbolic \( \kappa < 0 \) case, the function \( g \) must satisfy the property \( g \to 0 \) when \( \rho \to \infty \) and the eigenfunctions are orthogonal in the interval \([0, \infty)\).

This statement is just a consequence of the properties of the Sturm–Liouville problems. The \( \kappa \)-dependent differential equation for the function \( g(\rho) \) is not self-adjoint since \( a_0' \neq a_1 \), but it can be reduced to self-adjoint form by making use of the following integrating factor:

\[
\mu = \left( \frac{1}{a_0} \right) e^{\int \frac{a_0'}{a_0} d\rho} = \rho^{2l+1}(1 - \kappa \rho^2)^{1/\kappa - 1/2},
\]

so that the equation becomes

\[
\frac{d}{d\rho} \left[ p(\rho, \kappa) \frac{dg}{d\rho} \right] + \lambda q(\rho, \kappa) g = 0, \quad \lambda \text{ is a constant},
\]

where \( p(\rho, \kappa) = \mu a_0 \) and \( q(\rho, \kappa) \) is given by

\[
q(\rho, \kappa) = \rho^{2l+1}(1 - \kappa \rho^2)^{1/\kappa - 1/2} = (\rho^2(1 - \kappa \rho^2)^{1/\kappa}) \left( \frac{\rho^2}{\sqrt{1 - \kappa \rho^2}} \right).
\]

Note that this problem is singular in the two cases but in a different way. (i) If \( \kappa \) is positive because the function \( p(\rho, \kappa) \) vanishes at the boundary point \( \rho_2 = \rho_\kappa \). (ii) If \( \kappa \) is negative, then the problem is also singular since it is defined in the semi-infinite positive real line \( \mathbb{R}^+ \). Nevertheless, the properties of the Sturm–Liouville problems state that even in these cases the eigenfunctions of the problem are orthogonal with respect the function \( q(\rho, \kappa) \).

To sum up, the essential result we have obtained is the following: for either value of the curvature, the radial wavefunction \( R(\rho) \) which can appear together with the usual spherical harmonic \( Y_m(\theta, \phi) \) and which is regular at \( \rho = 0 \) is (a multiple of)

\[
r'(1 - k \rho^2)^{(1/2\kappa)}_2F_1 \left( \frac{1}{2\kappa} (A_\kappa \pm \sqrt{B_\kappa}), \quad \frac{1}{2\kappa} (A_\kappa \mp \sqrt{B_\kappa}), \quad l + 3/2; \ k \rho^2 \right).
\]

(17)
The associated wavefunctions of the oscillator on a space with constant curvature $k$ are
\[ \Psi_{n,l,m}(r, \theta, \phi; \kappa) = K_{\kappa} r^l (1 - \kappa r^2)^{(1/2k)} P_{n,l,m}(r; \kappa) Y_{lm}(\theta, \phi), \] (18)
where $P_{n,l,m}(r; \kappa)$ denotes the polynomial
\[ P_{n,l,m}(r; \kappa) = F_{l+1/2}(n - n_l, b_{n_r}; \kappa r^2), \]
with $b_{n_r} = n_r + l + 1 + 1/\kappa$ (the value of $b_0$ when $a_0 = -n_r$), $c = l + 3/2$ and $K_{\kappa}$ is a constant. These polynomials appear as a $k$-deformation of the $\kappa = 0$ associated Laguerre polynomials.

The set of wavefunctions $\Psi_{n,l,m}(r, \theta, \phi; \kappa)$ is a set of orthogonal functions with respect to the measure $d\mu_\kappa$ that is complete when $\kappa > 0$; in the hyperbolic case, there is, in addition to the discrete spectrum, also a continuous spectrum (the particular characteristics of the wavefunctions in the $\kappa < 0$ case are discussed below). The constant $K_{\kappa}$ is obtained from the normalization conditions which are given by
\[
\int_0^{2\pi} \int_0^{\pi} \int_0^{r_{\kappa}} |\Psi_{n,l,m}(r, \theta, \phi; \kappa)|^2 d\mu_\kappa d\theta d\phi = 1, \quad \kappa > 0,
\]
and
\[
\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} |\Psi_{n,l,m}(r, \theta, \phi; \kappa)|^2 d\mu_\kappa d\theta d\phi = 1, \quad \kappa < 0,
\]
where we have used the notation $r_{\kappa} = 1/\sqrt{\kappa}$ in the $\kappa > 0$ case. We have obtained the following values for the radial integrals:
\[
\int_0^{r_{\kappa}} r^{2l} (1 - \kappa r^2)^{(1/\kappa)} P_{n,l,m}(r; \kappa)^2 \left( \frac{r^2}{\sqrt{1 - \kappa r^2}} \right) dr = K_{\kappa} \frac{\Gamma(l + 3/2) \Gamma(n_r + l + 1 + 1/\kappa)}{\Gamma(n_r + l + 1 + 1/\kappa)}, \quad \kappa > 0,
\]
\[
\int_0^{\infty} r^{2l} (1 - \kappa r^2)^{(1/\kappa)} P_{n,l,m}(r; \kappa)^2 \left( \frac{r^2}{\sqrt{1 - \kappa r^2}} \right) dr = K_{\kappa} \frac{\Gamma(l + 3/2) \Gamma(1/|\kappa| - (2n_r + 1 + l))}{\Gamma(1/|\kappa| + 1/2 - n_r)} \quad \kappa < 0,
\]
where $\Gamma(\cdot)$ denotes the Gamma function and the coefficients $K_{\kappa}^+$ and $K_{\kappa}^-$ are given by
\[
K_{\kappa}^+ = \frac{\kappa^{-1/2+l} n_r!}{2(1 + \kappa (1 + l + 2n_r))(3/2 + l)n_r},
\]
and
\[
K_{\kappa}^- = \frac{|\kappa|^{-3/2+l} n_r!(1/|\kappa| - 2n_r - l)n_r}{2(3/2 + l)n_r},
\]
where $(a)_n$ denotes the Pochhammer symbol $(a)_n = a(a + 1) \cdots (a + n_r - 1)$.

The values of the energies are given by
\[
E_n = \left(n + \frac{1}{2}\right) + \frac{1}{2} \kappa n(n + 2), \quad n = 2n_r + l.
\] (19)

Two important properties are (i) $E_n$ depends only on $n$, so the energy levels are degenerate with respect to $n_r$ and $l$, and (ii) $E_n$ is the sum of the Euclidean value (corresponding to $\kappa = 0$).
plus an additional term proportional to $n^2$ and with a coefficient depending directly on the curvature.

In the Euclidean $\kappa = 0$ case, the three-dimensional harmonic oscillator is just the sum of three independent one-dimensional oscillators; and the energy level $E_n = (n + 3/2)\hbar\omega$ is $(n + 1)(n + 2)/2$-fold degenerate since this is the number of ways that $n$ can be written as the sum of three non-negative integers (usually denoted by $n_x$, $n_y$, and $n_z$). Now we have obtained that in the non-Euclidean $\kappa \neq 0$ case, the value of $n$, as a function of $n_r$ and $l$, is independent of $\kappa$; so the degeneracy of the energy levels is the same as in the Euclidean case (there exists accidental degeneracy in addition to the essential degeneracy of a central potential). Alternatively, the value $(n + 1)(n + 2)/2$ can also be directly calculated by using the relations

$$n = 2n_r + l, \quad n_r = 0, 1, 2, \ldots, \quad l = 0, 1, 2, \ldots, \quad m = -l, -l + 1, \ldots, l - 1, l,$$

in a similar way as in the Euclidean case (see e.g. [50] or [51]).

In the hyperbolic $\kappa < 0$ case, as the radial integral is defined on an infinite interval, the following property must be satisfied:

$$\lim_{r \to \infty} r f(r)(1 - \kappa r^2)^{(1/2)}_2F_1(\kappa r^2)\left(\frac{r^2}{\sqrt{1 - \kappa r^2}}\right) = 0.$$  

The consequence is that if $\kappa < 0$ then the quantum numbers $n_r$ and $l$ are limited by the condition

$$n = 2n_r + l < \frac{1}{|\kappa|} - 1, \quad (20)$$

and there are only $n_\kappa$ eigenvalues and eigenfunctions, where $n_\kappa$ denotes the greatest integer lower than $1/|\kappa| - 1$.

Figures 2 and 3 show the form of the radial functions $f(r, \kappa) (1 - \kappa r^2)^{(1/2)}_2F_1(\kappa r^2)$ for several values of $\kappa$ ($\kappa > 0$ in figure 2 and $\kappa < 0$ in figure 3).

The following two points summarize the main characteristics of the energies of the bound states.

(1) Spherical $\kappa > 0$ case. The Hamiltonian $\widehat{H}(\kappa)$ describes a quantum oscillator on the sphere $S^3$ ($\kappa > 0$). The oscillator possesses a countable infinite set of bound states $\Psi_{n_r, l, m}(r, \theta, \phi; \kappa)$, with $n_r, l = 0, 1, 2, \ldots$, and the energy spectrum is unbounded, not
equidistant and with a gap between every two consecutive levels that increases with $n$

$$E_0 < E_1 < E_2 < E_3 < \cdots < E_n < E_{n+1} < \cdots$$

$$E_{n+1} - E_n = 1 + \kappa \left( n + \frac{1}{2} \right).$$

The oscillations of the wavefunctions are reinforced and the values of the energies $E_n$ are higher than in the Euclidean $\kappa = 0$ case, i.e. $E_n(\kappa) > E_n(0)$.

(2) Hyperbolic $\kappa < 0$ case. The Hamiltonian $\hat{H}(\kappa)$ describes a quantum oscillator on the hyperbolic space $H^3_\kappa (\kappa < 0)$. The oscillator possesses only a finite number of bound states $\Psi_{n_r, l, m}(r, \theta, \phi; \kappa)$, with $n = 0, 1, 2, \ldots, n_\kappa$, $n_\kappa < 1/|k| - 1$, and the energy spectrum is bounded, not equidistant and with a gap between every two levels that decreases with $n$

$$E_0 < E_1 < E_2 < E_3 < \cdots < E_{n_\kappa}$$

$$E_{n_\kappa + 1} - E_{n_\kappa} = 1 - |\kappa| \left( n_\kappa + \frac{3}{2} \right).$$

The oscillations of the wavefunctions are smoothed down and the values of the energies $E_n$ are lower than in the Euclidean $\kappa = 0$ case, i.e. $E_n(\kappa) < E_n(0)$.

In this $\kappa < 0$ case, there is also, in addition to the discrete (quantized) spectrum, a continuous spectrum. Higher values of the energy $E$ such that $E > E_{n_\kappa}$ correspond to scattering solutions. These wavefunctions, related to (non-polynomial) hypergeometric functions, are characterized by a continuous index (continuous value of the energy) and with orthogonalization relations given by the Dirac delta. The total basis includes (as in a well with finite depth or in the Hydrogen atom) both functions labeled by a discrete index and functions labeled by a continuous index.

Figure 4 illustrates the two main characteristics of the energy levels. The first one is that the values are higher in the spherical case and lower in the hyperbolic plane, and the second one is that the number of bound states is finite in the hyperbolic case with the number increasing when $|\kappa|$ decreases. The plot clearly shows that when the absolute value $|\kappa|$ decreases the maximum of the curve moves into the upright, the number of bound states goes up and in the limit $\kappa \to 0$ the curve converges into the straight line parallel to the diagonal (dashed line) representing the Euclidean system.

The wavefunctions $\Psi_{n_r, n_m}(r, \theta, \phi; \kappa)$ and the energies $E_n$ show clear differences depending on the sign of $\kappa$ as it was expected. Nevertheless, if they are considered as functions of the curvature $\kappa$ then all the changes are presented in a smooth and continuous way.
4. Final comments and outlook

The Schrödinger equation is well defined for all the values of $\kappa$, but what introduces differences between the $\kappa > 0$ and the $\kappa < 0$ cases is that in the spherical $S^3_\kappa$ ($\kappa > 0$) case the space is compact and the oscillator possesses an infinite set of bound states; in the hyperbolic $H^3_\kappa$ ($\kappa < 0$) case, the potential $U_\kappa(r)$ is such that $U_\kappa(r) \to (1/2)(\alpha^2/|\kappa|)$ when $r \to \infty$ and the oscillator possesses only a finite number of bound states (for certain values of $|\kappa|$ only the fundamental level).

We finalize with two comments.

First, this paper is mainly concerned with the interface between geometry and quantum mechanics but it leads, in a natural way, to questions of functional analysis related to the theory of operators on Hilbert spaces. In some respects these problems are similar to those studied in the standard Euclidean case, but depending on the sign and the value of $\kappa$ these might go beyond and provide new aspects to the problem. The main point is the following: starting from the operator appearing in the (radial) Sturm–Liouville problem, which is symmetric in its natural domain, can it be extended to a self-adjoint operator in the $\kappa \neq 0$ case? Is this operator unique or are there a family of extensions depending on parameters entering into the boundary conditions? This is an open question in the $\kappa \neq 0$ case, and we think that it deserves to be studied in more detail.

Second, we point out once more that the existence of the harmonic oscillator is not a specific or special characteristic of the Euclidean space but it is a well-defined system in the three different spaces of constant curvature. In fact, we have proved that, making use of the curvature $\kappa$ as a parameter, there are not three different harmonic oscillators but only one defined, at the same time, in the three manifolds and endowed with properties depending smoothly on the curvature.

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