Gearhart–Koshy Acceleration for Affine Subspaces

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Abstract
The method of cyclic projection is an iterative scheme which can be used to find nearest points in the intersection of finitely many affine subspaces, having access only to the orthogonal projectors onto the individual subspaces. In an attempt to accelerate its convergence, Gearhart & Koshy (1989) proposed a modification of the scheme which, in each iteration, performs an exact line search based on minimising the distance to the desired nearest point. In the special case when the affine subspaces are linear subspaces, this line search can be made explicit by using the fact that the zero vector is always feasible. In this work, we derive an alternate (but nevertheless still explicit) implementation of this line search which does not require knowledge of a feasible vector, thus providing an efficient implementation of the scheme for affine subspaces.

Keywords. method of cyclic projections · acceleration schemes · linear systems

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1 Introduction
Our setting is a real Hilbert space $\mathcal{H}$ equipped with inner-product $\langle \cdot , \cdot \rangle$ and induced norm $\| \cdot \|$. Let $M_1, \ldots , M_n \subseteq \mathcal{H}$ be closed affine subspaces, and suppose

$$M := \bigcap_{i=1}^{n} M_i \neq \emptyset .$$

Given $x_0 \in \mathcal{H}$, we consider the best approximation problem

$$\min_{x \in \mathcal{H}} \| x - x_0 \|^2 \text{ subject to } x \in M. \quad (1)$$

In this work, our focus is the case in which the nearest point projectors onto the individual spaces, $M_1, \ldots , M_n$, are accessible. Recall that the projector onto $M_i$ is the operator $P_{M_i} : \mathcal{H} \to M_i$ given by

$$P_{M_i}(x) := \arg \min_{z \in M_i} \| x - z \| .$$

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The method of cyclic projections is an iterative procedure for solving (1), that is for computing \( P_M(x_0) \), by using only the individual projection operators \( P_{M_1}, \ldots, P_{M_n} \). Although originally studied when \( M_1, \ldots, M_n \) are linear subspaces \([14, 11]\), the following affine variant readily follows from translation properties of the projector.

**Theorem 1.1** (The method of cyclic projections). Let \( M_1, \ldots, M_n \) be closed affine subspaces of \( \mathcal{H} \) with \( M = \bigcap_{i=1}^n M_i \neq \emptyset \). Then, for each \( x_0 \in \mathcal{H} \),

\[
\lim_{k \to \infty} (P_{M_n} P_{M_{n-1}} \cdots P_{M_1})^k(x_0) = P_M(x_0).
\]

The convergence rate of the sequence in Theorem 1.1 can be related to the angle between the subspaces. Recall that the (Friederichs) angle between two closed subspaces \( A \) and \( B \) is the angle in \([0, \pi/2]\) whose cosine is given by

\[
c(A, B) := \sup \left\{ |\langle a, b \rangle| : a \in A \cap (A \cap B)^\perp, b \in B \cap (A \cap B)^\perp, \|a\| = \|b\| = 1 \right\}.
\]

(2)

The following result provides a bound on the convergence rate based on this quantity.

**Theorem 1.2** ([9, Corollary 9.34]). Let \( M_1, \ldots, M_n \) be closed affine subspaces of \( \mathcal{H} \) with \( M = \bigcap_{i=1}^n M_i \neq \emptyset \). For \( i \in \{1, \ldots, n\} \), let \( M'_i \) denote the linear subspace parallel to \( M_i \).

Then, for each \( x_0 \in \mathcal{H} \),

\[
\|(P_{M_n} P_{M_{n-1}} \cdots P_{M_1})^k(x_0) - P_M(x_0)\| \leq c^k \|x_0 - P_M(x_0)\|,
\]

(3)

where the constant \( c \in [0, 1] \) is given by

\[
c := \left(1 - \prod_{i=1}^{n-1} (1 - c_i^2)\right)^{1/2} \quad \text{with} \quad c_i := c \left(M'_i, \bigcap_{j=i+1}^n M'_j\right).
\]

(4)

When \( c < 1 \), Theorem 1.2 establishes \( R \)-linear convergence of the method of cyclic projections. This is easily seen to be the case, for instance, when there exists an \( i \in \{1, \ldots, n\} \) such that \( c_i < 1 \). In the setting with \( n = 2 \), this characterisation can be further refined: \( c < 1 \) if and only if \( M_1 + M_2 \) is closed (which always holds in finite dimensions) in which case convergence is linear, else \( c = 1 \) and the rate of convergence is arbitrarily slow \([2, 5]\).

Let \( Q : \mathcal{H} \to \mathcal{H} \) denotes the cyclic projections operator given by \( Q := P_{M_n} \cdots P_{M_1} \). In an attempt to accelerate the method of cyclic projections, Gearhart & Koshy \([10]\) proposed the following scheme which iterates by performing an exact line search to choose to nearest point to \( P_M(x_0) \) in the affine span of \( \{x_k, Q(x_k)\} \). When \( M_1, \ldots, M_n \) are linear subspaces, it can be shown (see Section 3) that the step size \( t_k \) can be computed using the expression

\[
t_k = \frac{\langle x_k - Q(x_k), x_k \rangle}{\|x_k - Q(x_k)\|^2}.
\]

(5)

Since it only requires vector arithmetic, evaluating of this expression comes with relatively low computational cost. Moreover, Gearhart–Koshy’s scheme gives the following refinement of the upper-bound provided by Theorem 1.2 in (3).
Algorithm 1: Gearhart–Koshy acceleration for (1).

**Initialisation.** An initial point $x_0 \in \mathcal{H}$.

for $k = 0, 1, 2, \ldots$ do
  1. Compute the step size $t_k$ by solving the (quadratic) minimisation problem
     \[
     \min_{t \in \mathbb{R}} \| x_k + t(Q(x_k) - x_k) - P_M(x_0) \|^2.
     \]
  2. Compute $x_{k+1}$ according to
     \[
     x_{k+1} := x_k + t_k(Q(x_k) - x_k).
     \]

**Theorem 1.3** (Gearhart–Koshy [10]). Let $M_1, \ldots, M_n$ be closed affine subspaces of $\mathcal{H}$ with $M = \cap_{i=1}^n M_i \neq \emptyset$. For each sequence $(x_k)$ generate by Algorithm 1, there exists a sequence $(f_k) \subseteq [0, 1]$ such that
\[
\| (P_{M_n}P_{M_{n-1}} \ldots P_{M_1})^k(x_0) - P_M(x_0) \| \leq c^k \left( \prod_{i=1}^k f_i \right) \| x_0 - P_M(x_0) \|,
\]
where the constant $c \in [0, 1]$ is given by (4).

Although Theorem 1.3 still holds for affine subspaces, the efficient expression for $t_k$ provided by (5) is only valid for linear subspaces (this will be explained more precisely Section 3). Thus, in the affine case, it is no longer obvious to apply the scheme.

In this work, we address the aforementioned problem by deriving an alternative expression for (5) which still holds in the affine case and still only requires vector arithmetic for its evaluation. Our key insight is the observation that (5) implicitly relies on the fact that the zero vector is always feasibility for linear subspaces. The remainder of this work is structure as follows. In Section 2, we collect the necessary preliminaries. In Section 3, we discuss Gearhart & Koshy’s derivation of (5) and, in Section 4, we provide an alternative formula which still holds in the affine setting. Finally, in Section 5, we discuss some implications for nonlinear fixed iterations.

## 2 Preliminaries

Let $S \subseteq \mathcal{H}$ be a non-empty subset of $\mathcal{H}$. Recall that the (nearest point) projector onto $S$ is the operator $P_S : \mathcal{H} \to S$ defined by
\[
P_S(x) := \arg\min_{z \in S} \| x - z \|.
\]
It is well-known (see, for instance, [9, 3.5]) that $P_S$ is a well-defined operator whenever $S$ closed and convex. Further, the definition in (7) also implies the translation formula
\[
P_S(x) = P_{S-y}(x-y) + y \quad \forall x, y \in \mathcal{H},
\]
where $S - y := \{s - y \in H : s \in S\}$. The following proposition collects important properties of the projectors for use in the subsequence sections.

**Proposition 2.1 (Properties of projectors).** Let $S \subseteq H$ be a nonempty, closed set.

(a) Suppose $S$ is convex. Then $p = P_S(x)$ if and only if
\[
p \in S \text{ and } \langle x - p, s - p \rangle \leq 0 \quad \forall s \in S.
\]

(b) Suppose $S$ is an affine subspace. Then $p = P_S(x)$ if and only if
\[
p \in S \text{ and } \langle x - p, s - p \rangle = 0 \quad \forall s \in S.
\]

(c) Suppose $S$ is a linear subspace. Then $P_S$ is a bounded, self-adjoint linear operator.

**Proof.** See, for instance, [9, 4.1] for (a), [9, 9.26] for (b), and [9, 5.13] for (c).

Let $S$ be a nonempty closed affine subspace and let $S'$ denote the associated linear subspace parallel to $S$. Then $S'$ can be expressed as $S' = S - y$ for any $y \in S$. In this case, the translation formula (8) implies
\[
P_S(x) = P_{S'}(x - s) + s \quad \forall x \in H.
\]

By using Proposition 2.1(c), this formula allows us related the affine projector $P_S$ to the self-adjoint operator $P_{S'}$. Furthermore, the characterisation in Proposition 2.1(b) is also equivalent to the condition $x - p \in (S')^\perp$ where the superscript "⊥" orthogonal complement of a subspace (see, for instance, [9, 9.26]).

### 3 Gearhart–Koshy Acceleration for Linear Subspaces

In this section, we recall the derivation of Gearhart & Koshy’s scheme for linear subspaces [10]. This serves to both introduce the scheme, and to highlighting the immediate difficulty with extending the result to affine spaces.

Denote $Q := P_{M_n} \cdots P_{M_1}$. Using this notation, the method of cyclic projection (as discussed in Theorem 1.1) generates a sequence $(x_k)$ according to fixed-point iteration
\[
x_{k+1} := Q(x_k) \quad \forall k \in \mathbb{N}.
\]

Gearhart & Koshy’s scheme attempts to accelerate convergence by instead defining the sequence $(x_k)$ given by
\[
x_{k+1} := x_k + t_k(Q(x_k) - x_k),
\]
where $t_k$ is chosen such that $x_{k+1}$ is the point in the affine span of $\{x_k, Q(x_k)\}$ closest to $P_M(x_0)$. In order words, $t_k$ is a solution to the (quadratic) minimisation problem
\[
\min_{t \in \mathbb{R}} \|x_k + t(Q(x_k) - x_k) - P_M(x_0)\|^2.
\]
Using the first-order optimality condition, we deduce that a solution to \((10)\) is given by
\[
t_k = \begin{cases} 
\frac{\langle x_k - Q(x_k), x_k - P_M(x_0) \rangle}{\|x_k - Q(x_k)\|^2} & \text{if } Q(x_k) \neq x_k \\
1 & \text{otherwise.}
\end{cases}
\]
(11)

It is worth noting that so-far the derivation of \((11)\) has not replied any properties of the sets other than the fact that \(Q\) is well-defined. However, as written, \((11)\) does not provide a useful expression for computing \(t_k\) since, when \(Q(x_k) \neq x_k\), it requires knowledge of \(P_M(x_0)\) (i.e., the solution to the problem we are trying to solve).

In the case when \(M_1, \ldots, M_n\) are linear subspaces, this difficulty can be overcome by using self-adjointess of the projectors (Proposition 2.1(c)). Indeed, since \(Q^* P_M = P_{M_1} \ldots P_{M_n} P_M = P_M\), we have
\[
\langle x_k - Q(x_k), P_M(x_0) \rangle = \langle x_k, P_M(x_0) \rangle - \langle x_k, Q^* P_M(x_0) \rangle = 0.
\]
(12)

Combining \((11)\) with \((12)\) then gives
\[
t_k = \begin{cases} 
\frac{\langle x_k - Q(x_k), x_k \rangle}{\|x_k - Q(x_k)\|^2} & \text{if } Q(x_k) \neq x_k \\
1 & \text{otherwise.}
\end{cases}
\]
(13)

This expression no longer requires knowledge of \(P_M(x_0)\), and can be evaluated using vector arithmetic and the current iterate. Explicitly, we have the following fully-explicit version Algorithm 1 for linear subspaces.

**Algorithm 2:** Gearhart–Koshy acceleration with linear subspaces.

**Initialisation.** An initial point \(x_0 \in \mathcal{H}\).

for \(k = 0, 1, 2, \ldots\) do

1. Compute the step size \(t_k\) using the formula \((13)\).
2. Compute \(x_{k+1}\) according to

\[x_{k+1} := x_k + t_k \left( Q(x_k) - x_k \right).\]

We now highlight the difficulty in using \((13)\) for affine subspace. To this end, assume that \(M_1, \ldots, M_n\) are affine subspaces with parallel linear subspaces denotes \(M'_1, \ldots, M'_n\). Further let \(M'\) denote the linear subspaces parallel to the affine subspace \(M\). As a consequence of translation formula \((9)\), for \(m_i \in M_i\) and \(m \in M\) respectively, we have
\[
P_{M_i}(x) = P_{M'_i}(x - m_i) + m_i \quad \text{and} \quad P_M(x) = P_{M'}(x - m) + m \quad \forall x \in \mathcal{H}.
\]
(14)

We now attempt an argument analogous to \((12)\) by reduction to the linear case using the translation formulae \((14)\). To this end, let \(m \in M = \cap_{i=1}^n M_i\) and denote

\[Q' := P_{M'_1} \ldots P_{M'_n}.
\]

5
Then, for all \( x \in \mathcal{H} \), (14) implies
\[
Q(x) = (P_{M_n} \ldots P_{M_1} P_{M_1})(x - m) + m \\
= (P_{M_n} \ldots P_{M_1})(P_{M_2} P_{M_1})(x - m) + m \\
= \vdots \\
= (P_{M_n} \ldots P_{M_3} P_{M_2} P_{M_1})(x - m) + m \\
= Q'(x - m) + m.
\]

Noting that \( (Q')^* P_{M'} = P_{M'} \), we may express the term involving \( P_M(x_0) \) in (11) as
\[
\langle x_k - Q(x_k), P_M(x_0) \rangle = \langle (x_k - m) - Q'(x_k - m), P_{M'}(x_0 - m) + m \rangle \\
= \langle x_k - m, P_{M'}(x_0 - m) \rangle - \langle x_k - m, (Q')^* P_{M'}(x_0 - m) \rangle \\
+ \langle (x_k - m) - Q'(x_k - m), m \rangle = \langle x_k - Q(x_k), m \rangle.
\]

When \( Q(x_k) \neq x_k \), substituting this expression into (11) gives
\[
t_k = \frac{\langle x_k - Q(x_k), x_k - m \rangle}{\|x_k - Q(x_k)\|^2}.
\]

Thus \( t_k \) could be computed using (17) whenever an intersection point \( m \in M \) is known. In particular, when \( M \) is a linear subspace, taking \( m = 0 \in M \) recovers the original Gearhart–Koshy formula (13). In this sense, the derivation of (13) implicitly used the fact that linear subspaces always contain the zero vector. For the general problem however, finding an intersection point \( m \in M \) is as hard as solving the best approximation problem (1) itself. Thus in practice, (17) is generally not of much use.

### 4 Gearhart–Koshy Acceleration for Affine Subspaces

In this section, we derive an alternate expression for the step size \( t_k \) in Gearhart & Koshy’s scheme which is still valid for affine subspaces and which can explicitly computed without knowledge of an intersection point (unlike expression in (17)). To this end, let \( Q_i : \mathcal{H} \to \mathcal{H} \) denote the operator
\[
Q_i := \begin{cases} 
P_{M_i} \ldots P_{M_1} & \text{if } i \in \{1, \ldots, n\} \\
1 & \text{if } i = 0.
\end{cases}
\]

**Lemma 4.1.** Let \( M_1, \ldots, M_n \) be closed affine subspaces of \( \mathcal{H} \) with \( M = \cap_{i=1}^n M_i \neq \emptyset \). For \( i \in \{1, \ldots, n\} \), let \( M'_i \) denote the linear subspace parallel to \( M_i \). If \( Q(x_k) \neq x_k \), then the solution of (10) is given by
\[
t_k = \frac{1}{2} + \frac{\sum_{i=1}^n \|Q_{i-1}x_k - Q_ix_k\|^2}{2\|x_k - Q(x_k)\|^2}.
\]
Proof. Let \( m \in M = \cap_{i=1}^{n} M_i \). By the argument in (16), we have
\[
\langle x_k - Q(x_k), P_M(x_0) \rangle = \langle x_k - Q(x_k), m \rangle.
\] (19)
Since \( Q_i(x_k) - m \in M_i' \) and \( \text{range}(I - P_{M_i}) \subseteq (M_i')^\perp \), for all \( i \in \{1, \ldots, n\} \), we have
\[
\langle Q_i(x_k) - m, (I - P_{M_i})Q_{i-1}x_k \rangle = 0.
\] (20)
By combining (19) and (20), we therefore obtain
\[
\langle x_k - Q(x_k), P_M(x_0) \rangle = \langle m, x_k - Q(x_k) \rangle
\]
\[
= \sum_{i=1}^{n} \langle m, (I - P_{M_i})Q_{i-1}(x_k) \rangle
\]
\[
= \sum_{i=1}^{n} \langle Q_i(x_k), (I - P_{M_i})Q_{i-1}(x_k) \rangle
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} (\|Q_{i-1}(x_k)\|^2 - \|Q_i(x_k)\|^2 - \|Q_{i-1}(x_k) - Q_i(x_k)\|^2)
\]
\[
= \frac{1}{2} \|x_k\|^2 - \frac{1}{2} \|Q(x_k)\|^2 - \frac{1}{2} \sum_{i=1}^{n} \|Q_{i-1}(x_k) - Q_i(x_k)\|^2.
\]
Together with (11), this yields
\[
2t_k\|x_k - Q(x_k)\|^2 = 2\langle x_k - Q(x_k), x_k \rangle - 2\langle x_k - Q(x_k), P_M(x_0) \rangle
\]
\[
= (\|x_k\|^2 + \|x_k - Q(x_k)\|^2 - \|Qx_k\|^2) - 2\langle x_k - Q(x_k), P_M(x_0) \rangle
\]
\[
= \|x_k - Q(x_k)\|^2 + \sum_{i=1}^{n} \|Q_{i-1}x_k - Q_i x_k\|^2,
\]
from which the claimed result follows. \( \square \)

Theorem 4.1 (Gearhart–Koshy acceleration for affine subspaces). Let \( M_1, \ldots, M_n \) be closed affine subspaces of \( \mathcal{H} \) with \( M = \cap_{i=1}^{n} M_i \neq \emptyset \). For each sequence \((x_k)\) generated by Algorithm 3, there exists a sequence \((f_k) \in [0, 1] \) such that
\[
\|x_k - P_M(x_0)\| \leq \|x_0 - P_M(x_0)\| \left( \prod_{i=1}^{k} f_i \right)^c \forall k \in \mathbb{N},
\] (22)
where the constant \( c \in [0, 1] \) is given by (4).

Proof. Let \( m \in M = \cap_{i=1}^{n} M_i \) and denote \( x_k' := x_k - m \) for all \( k \in \mathbb{N} \). Then (21) together with (15) implies
\[
x_{k+1}' = x_k' + t_k(Q'(x_k') - x_k') \quad \forall k \in \mathbb{N}.
\] (23)
By Lemma 4.1, \( t_k \) given by (18) is the solution to (10). Hence, the iteration (23) coincides with Gearhart & Koshy’s scheme applied to the linear subspaces \( M_1', \ldots, M_n' \). The claimed result thus follows from Theorem 1.3, noting that \( x_k = x_k' + m \) for all \( k \in \mathbb{N} \). \( \square \)
Algorithm 3: Gearhart–Koshy acceleration with affine subspaces.

**Initialisation.** An initial point $x_0 \in H$.

for $k = 0, 1, 2, \ldots$ do

1. Compute the step size $t_k$ using the formula

\[
t_k = \begin{cases} 
\frac{1}{2} + \frac{\sum_{i=1}^{n} \|Q_{i-1}x_k - Q_i x_k\|^2}{2\|x_k - Q(x_k)\|^2} & \text{if } Q(x_k) \neq x_k \\
1 & \text{otherwise.}
\end{cases}
\]

2. Compute $x_{k+1}$ according to

\[
x_{k+1} := x_k + t_k(Q(x_k) - x_k).
\]

We note that although Gearhart & Koshy’s scheme is attempt to accelerate convergence, Theorems 1.3 and 4.1 does not necessarily imply that the sequence $(x_k)$ converges faster. Rather, the theorem implies that the scheme improves on the upper bound on the rate of convergence provided by (3) and (22), respectively. Nevertheless, when $n = 2$, the scheme does indeed accelerate convergence, see [6, Theorem 3.23]. On the other hand, when $n \geq 3$, the scheme can actually be slower, see [6, Example 3.24].

To overcome this, Bauschke, Deutsch, Hundal and Park studied a symmetrised version of the method of cyclic projections based on the operator $S : H \rightarrow H$ given by

\[
S := P_{M_1}P_{M_2} \cdots P_{M_{n-1}}P_{M_n}P_{M_{n-1}} \cdots P_{M_1}.
\]

The method of symmetric cyclic projections is the corresponding fixed point iteration given by $x_{k+1} := S(x_k)$ for all $k \in \mathbb{N}$. When the sets are linear subspaces, the operator $S = Q^*Q$ has better properties than $Q$. For instance, $S$ is self-adjoint and nonnegative (i.e., $\langle Sx, x \rangle \geq 0$ for all $x \in H$) whereas the operator $Q$ is usually not.

For $i \in \{1, \ldots, 2n\}$, define the operator $S_i : H \rightarrow H$ by

\[
S_i := \begin{cases} 
P_{M_i}P_{M_{i-1}} \cdots P_{M_1} & \text{if } i \in \{1, \ldots, n\} \\
P_{M_{2n-i}} \cdots P_{M_{n-1}}P_{M_n}S & \text{if } i \in \{n+1, \ldots, 2n-1\} \\
I & \text{if } i = 0.
\end{cases}
\]

The following theorem, which extends [6, Corollary 3.21] to the affine case, shows that the Gearhart–Koshy-type acceleration of method of symmetric cyclic projections is at least as fast as method of symmetric cyclic projections. The resulting algorithm is summarised in Algorithm 4.

**Theorem 4.2** (Accelerated symmetric cyclic projections). Let $M_1, \ldots, M_n$ be closed affine subspaces of $H$ with $M = \cap_{i=1}^{n} M_i \neq \emptyset$. Then the sequence $(z_k)$ generated by Algorithm 4 satisfies

\[
\|z_k - P_M(x_0)\| \leq \|S_{k+1}(x_0) - P_M(x_0)\| \quad \forall k \in \mathbb{N}.
\]
Thus, the accelerated sequence \((z_k)\) converges at least as fast as the unaccelerated symmetric cyclic projection sequence.

**Proof.** First note that the symmetrised operator \(S\) coincides with its non-symmetric counterpart \(Q\) applied to the \(2n-1\) sets \(M_1\), \(M_2\), \(M_3\), \(M_4\) and \(M_5\). Consequently, Lemma 4.1 implies that \(s_k\) in (24) is a solution to the problem

\[
\min_{s \in \mathbb{R}} \|z_k + s_k(S(z_k) - z_k) - P_M(z_0)\|^2.
\]

The result then follows by a translation argument together with [6, Corollary 3.21]. □

**Algorithm 4:** Accelerated symmetric cyclic projections with affine subspaces.

**Initialisation.** Given an initial point \(x_0 \in H\), set \(z_0 := S(x_0)\).

**for** \(k = 0, 1, 2, \ldots\) **do**

1. Compute the step size \(t_k\) using the formula

\[
s_k = \begin{cases} 
\frac{1}{2} + \sum_{i=1}^{2n} \frac{\|S_{i-1}(z_k) - S_i(z_k)\|^2}{\|z_k - S_i(z_k)\|^2} & \text{if } S(z_k) \neq z_k \\
1 & \text{otherwise.}
\end{cases} \tag{24}
\]

2. Compute \(z_{k+1}\) according to

\[
z_{k+1} := z_k + s_k(S(z_k) - z_k).
\]

## 5 Extensions to Firmly Nonexpansive Operators

The orthogonality condition (20) was a key ingredient in the proof of Lemma 4.1. In this section, we investigate what remains true without this property. Our focus will be the following class of operators which generalise affine projectors.

**Definition 5.1.** An operator \(T: H \to H\) is **firmly quasi-nonexpansive** if

\[
\|T(x) - y\|^2 + \|x - T(x)\|^2 \leq \|x - y\|^2 \quad \forall x \in H, \forall y \in \text{Fix } T, \tag{25}
\]

where \(\text{Fix } T := \{y \in H : T(y) = y\}\) denotes its set of fixed points.

It is straightforward to check that the inequality (25) is equivalent to requiring

\[
0 \leq \langle T(x) - y, x - T(x) \rangle \quad \forall x \in H, \forall y \in \text{Fix } T. \tag{26}
\]

As a consequence of Proposition 2.1(a), projectors onto convex sets are firmly quasi-nonexpansive. And, in particular, Proposition 2.1(b) shows that (26) holds with equality when \(T\) is a projector onto an affine set. More generally, it can be seen that (26) (as
well as \((25)\) holds with equality when \(2T - I\) preserves distances to fixed points in the sense that
\[
\|(2T - I)(x) - y\| = \|x - y\| \quad \forall x \in \mathcal{H}, \forall y \in \text{Fix} \, T.
\]

Another example of quasi-firmly nonexpansive operator satisfying this problem is the 
Douglas–Rachford operator \(T_{C_1,C_2}: \mathcal{H} \to \mathcal{H}\) defined by
\[
T_{C_1,C_2} := \frac{1}{2} (I + R_{C_2} R_{C_1}),
\]
when the sets \(C_1, C_2 \subseteq \mathcal{H}\) are closed affine subspaces and \(R_{C_i} := 2P_{C_i} - I\) denotes the reflector with respect to \(C_i\). This can be verified by noting that \(2T_{C_1,C_2} - I = (2P_{C_2} - I)(2P_{C_1} - I)\) and applying Proposition 2.1(b).

Let \(T_1, \ldots, T_n: \mathcal{H} \to \mathcal{H}\) be firmly quasi-nonexpansive operators with \(\cap_{i=1}^n \text{Fix} \, T_i \neq \emptyset\). Denote \(Q := T_n \cdots T_2 T_1\). Given an initial point \(x_0 \in \mathcal{H}\), the iteration
\[
x_{k+1} := Q(x_k) \quad \forall k \in \mathbb{N},
\]
can be shown to converge weakly to a solution of the common fixed point problem
\[
\text{find } x \in \bigcap_{i=1}^n \text{Fix} \, T_i.
\]

In an attempt to accelerate \((28)\), we consider schemes of the form
\[
x_{k+1} = x_k + t_k (Q(x_k) - x_k),
\]
where \(m_k \in \cap_{i=1}^n \text{Fix} \, T_i\) and \(t_k\) is the solution to the problem
\[
\min_{t \in \mathbb{R}} \|x_k + t (Q(x_k) - x_k) - m_k\|^2.
\]
In other words, when \(x_k \notin \cap_{i=1}^n \text{Fix} \, T_i\), \(t_k\) is given by
\[
t_k = \frac{\langle x_k - Q(x_k), x_k - m_k \rangle}{\|x_k - Q(x_k)\|^2}.
\]

The following propositions, which can be viewed as a firmly nonexpansive analogue of Lemma 4.1, provides a lower bound for the value of \(t_k\). In worth noting that this lower bound is independent of the choice of intersection point \(m_k \in \cap_{i=1}^n \text{Fix} \, T_i\).

**Proposition 5.1 (Acceleration step size lower bound).** Let \(T_1, \ldots, T_n: \mathcal{H} \to \mathcal{H}\) be firmly quasi-nonexpansive operators and let \(m \in \cap_{i=1}^n \text{Fix} \, T_i \neq \emptyset\). If \(Q(x_k) \neq x_k\), then the solution to the minimisation problem
\[
\min_{t \in \mathbb{R}} \|x_k + t (Q(x_k) - x_k) - m\|^2
\]
satisfies
\[
t \geq \frac{1}{2} + \frac{\sum_{i=1}^n \|Q_{i-1}(x_{k-1}) - Q_i(x_{k-1})\|^2}{2\|x_{k-1} - Qx_{k-1}\|^2}.
\]

Furthermore, if \(T_1, \ldots, T_n\) satisfy \((25)\) with equality, then \((31)\) also holds with equality.
Proof. Since \( m \in \cap_{i=1}^{n} \text{Fix} T_i \), (26) implies

\[
\langle x_k - Q(x_k), m \rangle = \sum_{i=1}^{n} (m, (I - T_i)Q_{i-1}(x_k)) \\
\leq \sum_{i=1}^{n} (T_i Q_{i-1}(x_k), (I - T_i)Q_{i-1}(x_k)) \\
= \frac{1}{2} \sum_{i=1}^{n} (\|Q_{i-1}(x_k)\|^{2} - \|Q_i(x_k)\|^2 - \|Q_{i-1}(x_k) - Q_i(x_k)\|^2) \\
= \frac{1}{2} \|x_k\|^2 - \frac{1}{2} \|Q(x_k)\|^2 - \frac{1}{2} \sum_{i=1}^{n} \|Q_{i-1}(x_k) - Q_i(x_k)\|^2. 
\]

Using the optimality conditions for (30), followed by applying (32) yields

\[
2\|x_k - Q(x_k)\|^2 = 2\langle x_k - Q(x_k), x_k \rangle - 2\langle x_k - Q(x_k), m \rangle \\
= (\|x_k\|^2 + \|x_k - Q(x_k)\|^2 - \|Q(x_k)\|^2) - 2\langle x_k - Q(x_k), m \rangle \\
\leq \|x_k - Q(x_k)\|^2 + \sum_{i=1}^{n} \|Q_{i-1}(x_k) - Q_i(x_k)\|^2. 
\]

The claimed result then follows by rearranging this expression. Furthermore, when \( T_1, \ldots, T_n \) satisfy (25) with equality, (32) hold with equality and hence so does (31). \( \square \)

This observation allows us to apply the acceleration technique to affine settings beyond projectors including the Douglas–Rachford variants studied in [7, 8, 13, 3]. The simplest realisation is the symmetrised Douglas–Rachford algorithm consider below.

**Proposition 5.2.** Let \( M_1, M_2 \subseteq \mathcal{H} \) be closed affine with \( M_1 \cap M_2 \neq \emptyset \) with parallel linear subspaces denoted \( M'_1 \) and \( M'_2 \), respectively. Consider the operators \( T, T' : \mathcal{H} \to \mathcal{H} \) given by

\[
T := T_{M_2,M_1}T_{M_1,M_2} \quad \text{and} \quad T' := T'_{M'_2,M'_1}T'_{M'_1,M'_2}. 
\]

Then the following assertions hold.

(a) \( T(x) = T'(x - m) + m \) for all \( x \in \mathcal{H} \) and \( m \in M_1 \cap M_2 \).

(b) \( (T_{M'_1,M'_2})^* = T_{M'_2,M'_1} \) and \( (T_{M'_2,M'_1})^* = T_{M'_1,M'_2} \).

(c) \( T' \) is self-adjoint and nonnegative (i.e., \( \langle x, T'(x) \rangle \geq 0 \) for all \( x \in \mathcal{H} \)).

(d) \( \text{Fix} T = \text{Fix} T_{M_1,M_2} \cap \text{Fix} T_{M_2,M_1} = M_1 \cap M_2 + (M_1 - M_2)^\perp \cap (M_2 - M_1)^\perp \).

(e) \( P_{M_1}P_{\text{Fix} T} = P_{M_2}P_{\text{Fix} T} = P_{M_1 \cap M_2} \).
Proof. (a): See the proof of [7, Theorem 4.1]. (b): By linearity of adjoints and self-adjointness of the projectors onto linear subspaces (Proposition 2.1(c)), we have $R^*_M = R^*_M$, $R^*_M = R^*_M$ and

$$\left( T'_{M, M'} \right)^* = \frac{I + R^*_M R^*_M}{2} = \frac{I + R^*_M R^*_M}{2} = T'_{M, M}.$$  

(c): Using (b), we deduce that $(T') = (T'_{M, M'})^*(T'_{M, M'}) = T'_{M, M} T'_{M, M} = T'$ and

$$\langle x, T'(x) \rangle = \langle (T'_{M, M'})^*(x), T'_{M, M'}(x) \rangle = \| T'_{M, M'} x \|^2 \geq 0 \quad \forall x \in \mathcal{H}.$$  

(d): For the first equality, see the proof of [13, Theorem 2.4.5]. For the second equality, see [4, Corollary 3.9]. (e): Let $x \in \mathcal{H}$ and denote $p := P_{\text{Fix}T}(x)$. By [13, Lemma 2.4.4], we have $P_{M_1}(p) = P_{M_2}(p) \in M_1 \cap M_2$. Let $m \in M_1 \cap M_2 \subseteq \text{Fix} T$ be arbitrary. By Proposition 2.1(b) applied to $P_{\text{Fix}T}$, $P_{\text{Fix}T}$ and $P_{M_1}$, respectively, we have

$$\langle x - P_{M_1}(p), m - P_{M_1}(p) \rangle = \langle x - p, m - p \rangle + \langle x - p, p - P_{M_1}(p) \rangle + \langle p - P_{M_1}(p), m - P_{M_1}(p) \rangle = 0.$$  

This shows that $P_{M_1}(p) = P_{M_1 \cap M_2}(x)$ and hence completes the proof. □

**Theorem 5.1** (Accelerated symmetric Douglas–Rachford). Let $M_1, M_2 \subseteq \mathcal{H}$ be closed affine subspaces with $M := M_1 \cap M_2 \neq \emptyset$. Let $T : \mathcal{H} \to \mathcal{H}$ denote the symmetric Douglas–Rachford operator given by

$$T := T_{M_2, M_1} T_{M_1, M_2}.$$  

Then the sequence $(z_k)$ generated by Algorithm 5 satisfies

$$\| z_k - P_{\text{Fix} T}(x_0) \| \leq \| T^{k+1}(x_0) - P_{\text{Fix} T}(x_0) \| \quad \forall k \in \mathbb{N}. \quad (33)$$  

Thus, the accelerated sequence $(z_k)$ converges at least as fast as the unaccelerated symmetric Douglas–Rachford sequence. Moreover, we have

$$\max \{ \| P_{M_1}(z_k) - P_M(x_0) \|, \| P_{M_2}(z_k) - P_M(x_0) \| \} \leq \| z_k - P_{\text{Fix} T}(x_0) \|. \quad (34)$$  

Proof. According to the discussion after (27), the operators $T_{M_2, M_1}$ and $T_{M_1, M_2}$ both satisfy (25) in Definition 5.1 with equality and thus Proposition 5.1 implies that $t_k$ given by (35) satisfies

$$t_k = \arg \min_{t \in \mathbb{R}} \| z_k + t(T(z_k) - z_k) - P_{\text{Fix} T}(x_0) \|^2.$$  

Let $m \in M \subseteq \text{Fix} T$, denote $T' := T_{M_2, M_1} T_{M_1, M_2}$ and denote $z_k' = z_k - m$ for all $k \in \mathbb{N}$. By Proposition 5.2(c), $T'$ is self-adjoint and nonnegative. By Proposition 5.2(a), we have

$$z_{k+1}' = z_k' + t_k(T'(z_k') - z_k') \quad \forall k \in \mathbb{N}.$$  

Applying [6, Theorem 3.20], followed by a translation argument, yields (33). Inequality (34) then follows from firm quasinonexpansivity of the $P_{M_1}$ and $P_{M_2}$ combined with Proposition 5.2(e). □
Algorithm 5: Accelerated symmetric Douglas–Rachford with affine subspaces.

**Initialisation.** given an initial point $x_0 \in H$, set $z_0 := T(x_0)$.

```
for k = 0, 1, 2, . . . do
  1. Compute the step size $t_k$ using the formula
     \[
     t_k = \begin{cases} 
     \frac{1}{2} + \frac{\|z_k - T_{M_1 \cdot M_2}(z_k)\|^2 + \|T_{M_1 \cdot M_2}(z_k) - T(z_k)\|^2}{2\|z_k - T(z_k)\|^2} & \text{if } T(z_k) \neq z_k \\
     1 & \text{otherwise.} 
     \end{cases} \tag{35}
     
  2. Compute $z_{k+1}$ according to
     \[
     z_{k+1} = z_k + t_k(T(z_k) - z_k). \tag{36}
     
```

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