A REFINEMENT OF THE BEREZIN-LI-YAU TYPE INEQUALITY FOR NONLOCAL ELLIPTIC OPERATORS

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Abstract. In this paper, we prove a refinement of the Berezin-Li-Yau type inequality for a wider class of nonlocal elliptic operators including the fractional Laplacians $-(-\Delta^{\sigma/2})$ restricted to a bounded domain $D \subset \mathbb{R}^n$ for $n \geq 2$ and $\sigma \in (0, 2]$, which is optimal when $\sigma = 2$ in view of Weyl’s asymptotic formula. In addition, we describe the Berezin-Li-Yau inequality for the Laplacian $\Delta$ as the limit case of our result as $\sigma \to 2^-$.

1. Introduction

Let $K_\sigma$ be the class of all positive symmetric kernels $K$ satisfying the uniformly ellipticity assumption
\begin{equation}
K(y) = K(-y) \geq \frac{\lambda c_{n, \sigma}}{|y|^{n+\sigma}}, \quad 0 < \sigma < 2,
\end{equation}
for all $y \in \mathbb{R}^n \setminus \{0\}$ and $mK \in L^1(\mathbb{R}^n)$, where $m(y) = \min\{1, |y|^2\}$ and $c_{n, \sigma}$ is the constant given by
\begin{equation}
c_{n, \sigma} = \left( \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+\sigma}} d\xi \right)^{-1}.
\end{equation}
Then we consider the corresponding nonlocal elliptic operator $L_K$ given by
\begin{equation}
L_K u(x) = \frac{1}{2} \text{p.v.} \int_{\mathbb{R}^n} \mu(u, x, y)K(y) dy
\end{equation}
where $\mu(u, x, y) = u(x + y) + u(x - y) - 2u(x)$. In this paper, we consider the following eigenvalue problem
\begin{equation}
\begin{cases}
-L_K u = \nu u & \text{in } D \\
u = 0 & \text{in } \mathbb{R}^n \setminus D,
\end{cases}
\end{equation}
where $\sigma \in (0, 2)$, $n \geq 2$, $K \in K_\sigma$ and $D \subset \mathbb{R}^n$ is an open bounded set.

Let $X$ be the normed linear space of all Lebesgue measurable functions $v$ on $\mathbb{R}^n$ with the norm
\begin{equation}
\|v\|_X = \|v\|_{L^2(D)} + \left( \int_{C^0_D} |v(x) - v(y)|^2 K(x - y) dx dy \right)^{\frac{1}{2}} < \infty
\end{equation}
where $C^0_D = \mathbb{R}^{2n} \setminus (H^\infty \times H^\infty)$ for $H \subset \mathbb{R}^n$. Set $X_0 = \{ v \in X : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus D \}$. Since $C^0_D(D) \subset X_0$, we see that $X$ and $X_0$ are not empty. By [9], there is a constant $c > 1$ depending only on $n, \lambda, \sigma$ and $D$ such that
\begin{equation}
\int_{C^0_D} |v(x) - v(y)|^2 K(x - y) dx dy \leq \|v\|_X^2 \leq c \int_{C^0_D} |v(x) - v(y)|^2 K(x - y) dx dy
\end{equation}

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for any $v \in X_0$; that is, $\|v\|_{X_0} := \left( \int_{C_0^D} |v(x) - v(y)|^2 K(x-y) \ dx \ dy \right)^{1/2}$ is a norm on $X_0$ equivalent to (1.4). Moreover it is known [9] that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with inner product

\[
\langle u, v \rangle_{X_0} := \int_{C_0^D} (u(x) - u(y))(v(x) - v(y))K(x-y) \ dx \ dy.
\]

From simple computation, we note that $\langle u, v \rangle_{X_0} = -\langle L_K u, v \rangle_{L^2(D)}$ for all $u, v \in X_0$.

More precisely, we study the weak formulation of the problem (1.3) given by

\[
\begin{cases}
\langle u, v \rangle_{X_0} = \nu(u, v)_{L^2(D)}, \forall v \in X_0, \\
u \in X_0.
\end{cases}
\]

Then it is well-known [10] that there is a sequence $\{\nu_i^n(D)\}_{i \in \mathbb{N}}$ of eigenvalues of (1.6) with $0 < \nu_1^n(D) \leq \nu_2^n(D) \leq \cdots \leq \nu_i^n(D) \leq \cdots$ and $\lim_{n \to \infty} \nu_i^n(D) = \infty$ such that the set $\{e_i\}_{i \in \mathbb{N}}$ of eigenfunctions $e_i$ corresponding to $\nu_i^n(D)$ is an orthonormal basis of $L^2(D)$ and an orthogonal basis of $X_0$. Moreover, it turns out that $e_{i+1} \in P_{i+1}$ and

\[
\nu_i^n(D) = \|e_i\|_{X_0}^2 \quad \text{and} \quad \nu_i^{n+1}(D) = \|e_{i+1}\|_{X_0}^2
\]

for any $i \in \mathbb{N}$, where $P_{i+1} = \{u \in X_0 : \langle u, e_j \rangle_{X_0} = 0, \forall j = 1, 2, \cdots, i\}$.

Originally, Weyl’s asymptotic formula [12] for the Dirichlet eigenvalue problem of the Laplacian

\[
\begin{cases}
-\Delta u = \mu u \quad \text{in } D \\
u = 0 \quad \text{in } \partial D
\end{cases}
\]

asserts that

\[
\mu_k(D) \sim \frac{4\pi^2}{(|D| |B_1|)^{2/\pi}} k^\frac{\pi}{2} \quad \text{as } k \to \infty,
\]

where $|D|$ and $|B_1|$ denote the volumes of $D$ and the unit ball $B_1$ in $\mathbb{R}^n$, respectively. The relevant study on the eigenvalue problem for the Laplacian has been done along this line by Pólya [7] and Lieb [4]. P. Li and S. T. Yau [3] proved the following lower bound on the averages on the finite sums of eigenvalues

\[
\frac{1}{k} \sum_{j=1}^k \mu_j(D) \geq \frac{4n \pi^2}{(n+2)(|D| |B_1|)^{2/\pi}} k^\frac{\pi}{2}
\]

for any domain $D \subset \mathbb{R}^n$, which is sharp in terms of (1.10). P. Kröger [2] obtained an upper bound for the sums of the eigenvalues depending on geometric properties of $D$. A. Melas [5] improved their lower bound by using the moment of inertia of $D$. Using the method based on his argument, we obtain a lower bound on the averages on the finite sums of eigenvalues $\nu_i^n(D)$ of the eigenvalue problem (1.6).

**Theorem 1.1.** Let $D \subset \mathbb{R}^n$ be a bounded open set and $\sigma \in (0, 2)$. If $\{\nu_i^n(D)\}_{i \in \mathbb{N}}$ be the sequence of eigenvalues of the above eigenvalue problem (1.6) for the nonlocal elliptic operators $L_K$ with $K \in \mathcal{K}_\sigma$, then we have the estimate

\[
\frac{1}{k} \sum_{j=1}^k \nu_j^n(D) \geq \frac{\lambda n(2\pi)^{\sigma}}{(n+\sigma)(|B_1| |D|)^{2/\pi}} k^\frac{\pi}{2} + \frac{\lambda \sigma(2\pi)^{-\sigma-2}}{48(n+\sigma)(|B_1| |D|)^{2/\pi}} \frac{|D|}{k^{\frac{n-\sigma}{2}}} |D|
\]

where $|D| = \int_D |x|^2 \ dx$ is the moment of inertia of $D$ with mass center $0 \in \mathbb{R}^n$. 
In particular, if $K_0(y) = c_{n,\sigma}|y|^{-n-\sigma}$ with $\sigma \in (0, 2)$, then $L_{K_0} = -(\Delta^{\sigma/2})$ is the fractional Laplacian and it is well-known [6] that
\begin{equation}
\lim_{\sigma \to 2^{-}} - (\Delta^{\sigma/2}) u = \Delta u \quad \text{and} \quad \lim_{\sigma \to 0^+} - (\Delta^{\sigma/2}) u = u
\end{equation}
for any function $u$ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Also, the constant $c_{n,\sigma}$ satisfies the following property [6]:

\[ \lim_{\sigma \to 2^{-}} \frac{c_{n,\sigma}}{\sigma(2-\sigma)} = \frac{1}{|B_1|} \quad \text{and} \quad \lim_{\sigma \to 0^+} \frac{c_{n,\sigma}}{\sigma(2-\sigma)} = \frac{1}{2n|B_1|}. \]

If we consider the nonlocal operator $L_{K_0}$ corresponding to $K_0(y) = c_{n,\sigma}|y|^{-n-\sigma}$ with $\sigma \in (0, 2)$, our result makes it possible to recover the result obtained by S. Yildirim Yolcu and T. Yolcu [13] as follows.

**Theorem 1.2.** Let $D \subset \mathbb{R}^n$ be a bounded open set and $\sigma \in (0, 2)$. If $\{\nu_{0k}(D)\}_{k \in \mathbb{N}}$ be the sequence of eigenvalues of the above eigenvalue problem (1.6) for the fractional Laplacians $-(\Delta^{\sigma/2})$, then we have the estimate

\[ \frac{1}{k} \sum_{j=1}^{k} \nu_{0j}(D) \geq \frac{n(2\pi)^{\sigma}}{(n+\sigma)(|B_1||D|)^{\frac{\sigma}{2}}} k^{\frac{n}{\sigma}} + \frac{\sigma(2\pi)^{\sigma-2}}{48(n+\sigma)(|B_1||D|)^{\frac{\sigma}{2}}} \frac{|D|}{|D|^{1-\sigma}} \]

where $[D] = \int_D |x|^2 \, dx$ is the moment of inertia of $D$ with mass center $0 \in \mathbb{R}^n$.

As in (1.10), we could look on the Laplacian $\Delta$ as the limit of the fractional Laplacian $-(\Delta^{\sigma/2})$ as $\sigma \to 2^-$. Then our result implies an improvement (see A. D. Melas [5]) of the results proved by F. A. Berezin [1] and P. Li and S.-T. Yau [3] as follows.

**Theorem 1.3.** Let $D \subset \mathbb{R}^n$ be a bounded open domain. If $\{\mu_k(D)\}_{k \in \mathbb{N}}$ be the sequence of eigenvalues of the above eigenvalue problem (1.8) for the Laplacian $\Delta$, then we have the estimate

\[ \frac{1}{k} \sum_{j=1}^{k} \mu_j(D) \geq \frac{n(2\pi)^{2}}{(n+2)(|B_1||D|)^{\frac{2}{2}}} k^{\frac{n}{2}} + \frac{1}{24(n+2)} \frac{|D|}{|D|} \]

where $[D] = \int_D |x|^2 \, dx$ is the moment of inertia of $D$ with mass center $0 \in \mathbb{R}^n$.

## 2. Preliminaries

First of all, we furnish several fundamental lemmas which are useful in proving our main theorem. Our proof follows in part the argument of Melas [5], Li and Yau [3], and S. Yildirim Yolcu and T. Yolcu [13].

**Lemma 2.1.** If $\phi : [0, \infty) \to [0, 1]$ is a Lebesgue measurable function satisfying
\[ \int_{0}^{\infty} \phi(t) \, dt = 1 \quad \text{and} \quad 0 < \sigma < 2, \]
then there exists some $\eta > 0$ such that

\begin{equation}
\int_{\eta}^{\eta+1} t^{\sigma} \, dt = \int_{0}^{\infty} t^{\sigma} \phi(t) \, dt \quad \text{and} \quad \int_{\eta}^{\eta+1} t^{n+\sigma} \, dt \leq \int_{0}^{\infty} t^{n+\sigma} \phi(t) \, dt.
\end{equation}

**Proof.** First of all, we claim that $\int_{0}^{\infty} t^{\sigma} \phi(t) \, dt < \infty$. Indeed, if $\int_{0}^{\infty} t^{n+\sigma} \phi(t) \, dt < \infty$, then we easily obtain that $\int_{\eta}^{\eta+1} t^{\sigma} \phi(t) \, dt < \infty$ because $\int_{0}^{\infty} \phi(t) \, dt = 1$. In case that $\int_{0}^{\infty} t^{n+\sigma} \phi(t) \, dt = \infty$, we can derive that $\int_{\eta}^{\eta+1} t^{n+\sigma} \phi(t) \, dt = \infty$. We note that the
set $H = \{ t \in [2, \infty) : \phi(t) > 0 \}$ must not be Lebesgue measure zero; otherwise, it must be true that $\int_{2}^{\infty} t^{n+\sigma} \phi(t) \, dt = 0$, which is a contradiction. So we see that
\[
\int_{2}^{\infty} t^n \phi(t) \, dt = \int_{H} t^n \phi(t) \, dt < \int_{H} t^{n+\sigma} \phi(t) \, dt = \int_{2}^{\infty} t^{n+\sigma} \phi(t) \, dt = \infty.
\]
This implies that $\int_{0}^{\infty} t^n \phi(t) \, dt < \infty$.

Since $(t^n - 1)(\phi(t) - \mathbb{1}_{[0,1]}(t)) \geq 0$ for any $t \in [0, \infty)$, it follows from integrating the inequality on $[0, \infty)$ that $\int_{0}^{1} t^n \, dt \leq \int_{0}^{\infty} t^n \phi(t) \, dt < \infty$. We note that $g(s) = \int_{s}^{s+1} t^n \, dt$ is increasing on $[0, \infty)$ and $\lim_{s \to \infty} g(s) = \infty$. Thus there is an $\eta > 0$ such that $g(\eta) = \int_{0}^{\infty} t^n \phi(t) \, dt$.

We may also choose some $a, b \in (0, \infty)$ so that the function
\[ h(t) = t^{n+\sigma} - at^n + b \]
satisfies $h(\eta) = h(\eta + 1) = 0$. Indeed, the equation $h'(t) = 0$ has a unique solution $(\frac{a}{n+\sigma})^{1/\sigma}$ in $[0, \infty)$ at which $h$ has the minimum value $-\frac{n}{n+\sigma} (\frac{a}{n+\sigma})^{-n/\sigma} + b$. Since $h$ is convex in $[0, \infty)$, we can select such $a, b \in (0, \infty)$ satisfying the condition $h(\eta) = h(\eta + 1) = 0$. Thus we conclude that $h(t) < 0$ for any $t \in (\eta, \eta + 1)$ and $h(t) > 0$ for any $t \in [0, \infty) \setminus (\eta, \eta + 1)$, and hence $h(t) \phi(t) - \mathbb{1}_{(0, \eta+1)}(t)) \geq 0$ for any $t \in [0, \infty)$.

Integrating this inequality on $[0, \infty)$, we easily obtain the second result.

**Lemma 2.2.** The following inequality
\[
nt^{n+\sigma} - (n + \sigma) t^n s^\sigma + \sigma s^{n+\sigma} \geq \sigma s^{n+\sigma - 2} (t - s)^2
\]
always holds for any $s, t \in (0, \infty)$, $n \in \mathbb{N} + 1$ and $\sigma \in (0, 2]$.

**Proof.** If we set $\tau = t/s \in (0, \infty)$, then the inequality (2.2) becomes
\[
n\tau^{n+\sigma} - (n + \sigma) \tau^n + \sigma \geq \sigma (\tau - 1)^2.
\]
Consider the function $p(\tau) = n\tau^{n+\sigma} - (n + \sigma) \tau^n + \sigma - \sigma(\tau - 1)^2$. We write
\[ p(\tau) = \tau (n\tau^{n+\sigma-1} - (n + \sigma) \tau^{n-1} - \sigma \tau + 2\sigma) := \tau q(\tau).
\]
Then we have that $q'(\tau) = n(n + \sigma - 1)\tau^{n+\sigma - 2} - (n + \sigma)(n - 1)\tau^{n-2} - \sigma$ and
\[ q''(\tau) = \tau^{n-3} n(n + \sigma - 1)(n + \sigma - 2) (\tau^{\sigma} - \tau_0^{\sigma})
\]
where $\tau_0 := \left(\frac{-(n + \sigma)(n - 1)(n - 2)}{n(n + \sigma - 1)(n + \sigma - 2)}\right)^{1/\sigma}$. The equation $q''(\tau) = 0$ has a unique solution $\tau_0$ in $(0, \infty)$ at which the function $q'(\tau)$ has the minimum value
\[ q'(\tau_0) = -\frac{\sigma (n + \sigma)(n - 1)}{n + \sigma - 2} \left(\frac{n + \sigma}{n(n + \sigma - 1)(n + \sigma - 2)}\right)^{n-2} - \sigma < -\sigma < 0.
\]
Since $\lim_{\tau \to 0^+} q'(\tau) = -\sigma$ and $q'(1) = 0$, we see that the graph of $q'(\tau)$ is convex in $(0, \infty)$. Observing that $\lim_{\tau \to 0^+} q(\tau) = 2\sigma$ and $q(1) = 0$, this implies that the graph of $q(\tau)$ is starting at the point $(0, 2\sigma)$ and going down to the point $(1, 0)$ convexly, and going up convexly right after touching down to the point $(1, 0)$. Hence we conclude that $q(\tau) \geq 0$, and so $p(\tau) \geq 0$ for any $\tau \in (0, \infty)$.

**Lemma 2.3.** Let $n \in \mathbb{N} + 1$, $\varrho, \beta \in (0, \infty)$ and $\sigma \in (0, 2]$. If $\varphi : [0, \infty) \to (0, \infty)$ is a decreasing absolutely continuous function such that
\[
-\varrho \leq \varphi'(t) \leq 0 \quad \text{and} \quad \int_{0}^{\infty} t^{n-1} \varphi(t) \, dt := \beta,
\]
then for any $T > 0$ and $T' > 0$, we have
\[
\int_{0}^{T} t^n \varphi(t) \, dt \leq \int_{T'}^{\infty} t^n \varphi(t) \, dt = \beta.
\]
then we have that
\[
\int_0^\infty t^{n+\sigma-1}\varphi(t)\,dt \geq \frac{1}{n+\sigma} (n\beta)^{\frac{n+\sigma}{n}} \varphi(0)^{-\frac{\sigma}{n}} + \frac{\sigma}{12n(n+\sigma)\rho^2} (n\beta)^{\frac{n+\sigma-2}{n}} \varphi(0)^{\frac{2n-\sigma+2}{n}}.
\]  
(2.4)

Proof. By considering the function \(\varphi(0)^{-1}\varphi(t)\), we may assume that \(\varphi = 1\) and \(\varphi(0) = 1\). Without loss of generality, we assume that \(\alpha := \int_0^\infty t^{n+\sigma-1}\varphi(t)\,dt < \infty\); otherwise, we have already done. Set \(\phi(t) = -\varphi'(t)\) for \(t \in [0, \infty)\). Then we have that \(0 \leq \phi(t) \leq 1\) and \(\int_0^\infty \phi(t)\,dt = \varphi(0) = 1\), and moreover Lemma 2.1 and the integration by parts leads us to obtain
\[
\int_{\eta}^{\eta+1} t^{n+\sigma}\,dt \leq \int_0^\infty t^{n+\sigma} \phi(t)\,dt
\]  
(2.5)

Applying the integration by parts again, by (2.5) we see that
\[
0 \leq \lim_{t \to \infty} \left(-t^{n+\sigma}\varphi(t)\right) + (n+\sigma) \int_0^\infty t^{n+\sigma-1}\varphi(t)\,dt
\]  
(2.6)

Then we claim that \(\lim_{t \to \infty} t^{n+\sigma}\varphi(t) = 0\); indeed, if \(\gamma := \lim_{t \to \infty} t^{n+\sigma}\varphi(t) > 0\), then given any \(\varepsilon \in (0, \gamma)\) there is some large \(T > 0\) such that \(\gamma - \varepsilon < t^{n+\sigma}\varphi(t) < \gamma + \varepsilon\) for all \(t > T\), and thus we get that
\[
\infty = \int_T^\infty \frac{\gamma - \varepsilon}{t} \,dt \leq \int_0^\infty t^{n+\sigma-1}\varphi(t)\,dt < \infty,
\]
which gives a contradiction. Hence, it follows from Lemma 2.1 and the integration by parts that
\[
\int_\eta^{\eta+1} t^n\,dt = \int_0^\infty t^n \phi(t)\,dt = n \int_0^\infty t^{n-1}\varphi(t)\,dt = n\beta.
\]  
(2.7)

Integrating the inequality (2.2) on \([\eta, \eta+1]\], it follows from (2.6) and (2.7) that
\[
n(n+\sigma)\alpha - n(n+\sigma)s^\sigma \beta + \sigma s^{n+\sigma} \geq \sigma s^{n+\sigma-2} \int_\eta^{\eta+1} (t-s)^2\,dt
\]  
\[
\geq \sigma s^{n+\sigma-2} \int_{-1/2}^{1/2} t^2\,dt = \frac{\sigma}{2} s^{n+\sigma-2}.
\]

Selecting \(s = (n\beta)^{1/n}\), we obtain that
\[
\alpha \geq \frac{1}{n+\sigma} (n\beta)^{\frac{n+\sigma}{n}} + \frac{\sigma}{12n(n+\sigma)(n\beta)^{\frac{n+\sigma-2}{n}}},
\]

Therefore we complete the proof. \(\square\)
3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 by applying lemmas obtained in the previous section.

Let $D \subset \mathbb{R}^n$ be a bounded open domain and $D^*$ be its symmetric rearrangement given by

$$D^* = \{ x \in \mathbb{R}^n : |x| < (|D|/|B_1|)^{1/n} \}.$$  

That is, $D^*$ is the open ball with the same volume as $D$ and center $0 \in \mathbb{R}^n$. Since $|x|^2$ is radial and increasing, the moment of inertia of $D$ with mass center $0 \in \mathbb{R}^n$ has the lower bound as follows:

$$\int_D |x|^2 \, dx \geq \int_{D^*} |x|^2 \, dx = \frac{n|D|}{n+2} \left( \frac{|D|}{|B_1|} \right)^\frac{2}{n}. \quad (3.1)$$

Let $\{e_i\}_{i \in \mathbb{N}}$ be the set of eigenfunctions $e_i$ of (1.7) corresponding to eigenvalues $\nu_i^2(D)$ which is an orthonormal basis of $L^2(D)$ and an orthogonal basis of $X_0$. Then we consider the Fourier transform of each eigenfunction $e_i(x)$ given by

$$\hat{e}_i(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} e_i(x) \, dx = \langle e_i, e_i \rangle_{L^2(D)}$$

where $\xi = (2\pi)^{-n/2} e^{i\langle x, \xi \rangle}$. By Parseval’s formula and Plancherel theorem, we see that the set $\{\hat{e}_i\}_{i \in \mathbb{N}}$ is orthonormal in $L^2(\mathbb{R}^n)$. Since $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis of $L^2(D)$, it follows from Bessel’s inequality that

$$\sum_{i=1}^k |\hat{e}_i(\xi)|^2 \leq \|e_\xi\|_{L^2(D)} = \frac{|D|}{(2\pi)^n} \quad (3.2)$$

for any $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$. From standard analysis, we have that

$$\nabla \hat{e}_i(\xi) = \langle ix e_\xi, e_i \rangle_{L^2(D)} := \langle \{ix_1 e_\xi, e_i\}_{L^2(D)}, \cdots, \{ix_n e_\xi, e_i\}_{L^2(D)} \rangle.$$  

Applying Bessel’s inequality again, we obtain that

$$\sum_{i=1}^k |\nabla \hat{e}_i(\xi)|^2 \leq \|ix e_\xi\|_{L^2(D)} = \frac{|D|}{(2\pi)^n} \quad (3.3)$$

for any $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$. From (1.7) and Parseval’s formula, we have the estimate

$$\nu_i^2(D) = \|e_i\|^2_{X_0} = \langle -L_K e_i, e_i \rangle_{L^2(D)}$$

$$= \langle \hat{e}_i, \hat{e}_i \rangle_{L^2(D)} = \int_{\mathbb{R}^n} s(\xi)|\hat{e}_i(\xi)|^2 \, d\xi \quad (3.4)$$

where $s(\xi) = \int_{\mathbb{R}^n} (1 - \cos\langle y, \xi \rangle) K(y) \, dy$. Here we note that $1 - \cos\langle y, \xi \rangle \geq 0$. If we choose a matrix $M \in \mathcal{O}(n)$ such that $Me_1 = \xi/|\xi|$ where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$, then by (1.1) we get the estimate

$$s(\xi) \geq \lambda c_{n, \sigma} \int_{\mathbb{R}^n} \frac{1 - \cos \langle |y| \xi, \xi/|\xi| \rangle}{|y|^{n+\sigma}} \, dy$$

$$= \lambda c_{n, \sigma} |\xi|^\sigma \int_{\mathbb{R}^n} \frac{1 - \cos \langle \zeta, Me_1 \rangle}{|\zeta|^{n+\sigma}} \, d\zeta \quad (3.5)$$

$$= \lambda c_{n, \sigma} |\xi|^\sigma \int_{\mathbb{R}^n} \frac{1 - \cos \langle \zeta, e_1 \rangle}{|\zeta|^{n+\sigma}} \, d\zeta = \lambda |\xi|^\sigma.$$
If we set \( G_k(\xi) = \sum_{i=1}^{k} |\mathcal{F}_i(\xi)|^2 \), then by (3.2), (3.3) and Schwarz inequality, we have that \( 0 \leq G_k(\xi) \leq (2\pi)^{-n} |D| \). Also we observe that \( x^a e_i \in L^1(\mathbb{R}^n) \) for any \( i = 1, \cdots, k \) and multi-index \( a = (a_1, \cdots, a_n) \in (\mathbb{N} \cup \{0\})^n \), where \( x^a := x_1^{a_1} \cdots x_n^{a_n} \).

By Riemann-Lebesgue lemma and standard analysis, we see that \( \mathcal{F}_i \in C_0^\infty(\mathbb{R}^n) \) for any \( i = 1, \cdots, k \), and \( G_k \in C_0^\infty(\mathbb{R}^n) \). Thus we get that

\[
(3.6) \quad |\nabla G_k(\xi)| \leq 2 \left( \sum_{i=1}^{k} |\mathcal{F}_i(\xi)|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{k} |\nabla \mathcal{F}_i(\xi)|^2 \right)^{\frac{1}{2}} \leq \frac{2\sqrt{|D||D'|}}{(2\pi)^n}
\]

for any \( \xi \in \mathbb{R}^n \), and moreover \( \int_{\mathbb{R}^n} G_k(\xi) \, d\xi = k \) by Plancherel theorem and

\[
(3.7) \quad \sum_{i=1}^{k} \nu_i^2(D) \geq \lambda \int_{\mathbb{R}^n} |\xi|^\sigma G_k(\xi) \, d\xi
\]

by (3.4) and (3.5). Let \( G_k^*(\xi) = \varphi(|\xi|) \) be the symmetric decreasing arrangement of \( G_k \). Then it follows from Lemma 1.E. in [11] that \( \varphi \) is absolutely continuous in \([0, \infty)\). For \( \tau \geq 0 \), we set \( \omega(\tau) = |\{ \xi \in \mathbb{R}^n : G_k^*(\xi) \geq \tau \}| = |\{ \xi \in \mathbb{R}^n : G_k(\xi) \geq \tau \}|.

**Lemma 3.1.** If \( \varphi \) is differentiable at \( t \in (0, \infty) \), then \( \omega \) is differentiable at \( \varphi(t) \) and moreover \( \omega'(\varphi(t)) \varphi'(t) = n|B_1|t^{n-1} \).

**Proof.** Take any \( t \in (0, \infty) \) at which \( \varphi \) is differentiable. Then we have two possible cases; (i) there is an open interval \( I \subset (0, \infty) \) such that \( t \in I \) and \( \varphi' = 0 \) in \( I \), and (ii) there is an interval \( I \subset (0, \infty) \) such that \( t \in I \) and \( \varphi' < 0 \) in \( I \).

In case of (i), it is easy to check that \( \omega'(\varphi(t)) = 0 \). In case of (ii), by the property of the distribution function, we see that \( \omega \) is continuous at \( \varphi(t) \). We note that \( \omega(\varphi(t)) = |B_1|t^n \).

Write \( \Delta s = \varphi(t + \Delta t) - \varphi(t) \). Then we have that

\[
\frac{\omega(\varphi(t + \Delta s) - \omega(\varphi(t))}{\Delta s} = \frac{\omega(\varphi(t + \Delta t)) - \omega(\varphi(t))}{\Delta t} = |B_1| \sum_{i=0}^{n-1} (t + \Delta t)^{n-1-i}(\Delta t)^i.
\]

Taking the limit in the above because \( \varphi \) is continuous at \( t \), this implies the required result.

We continue the proof of Theorem 1.1. As in the above, there is nothing to prove it, because \( \varphi' = 0 \) in \( I \) in case of (i). So, without loss of generality, we may assume that we are now in the case (ii). By (3.1) and (3.7), we have that

\[
(3.8) \quad k = \int_{\mathbb{R}^n} G_k(\xi) \, d\xi = \int_{\mathbb{R}^n} G_k^*(\xi) \, d\xi = n|B_1| \int_0^\infty t^{n-1} \varphi(t) \, dt
\]

and

\[
(3.9) \quad \sum_{i=1}^{k} \nu_i^2(D) \geq \lambda \int_{\mathbb{R}^n} |\xi|^\sigma G_k(\xi) \, d\xi \geq \lambda \int_{\mathbb{R}^n} |\xi|^\sigma G_k^*(\xi) \, d\xi = \lambda n|B_1| \int_0^\infty t^{n+\sigma-1} \varphi(t) \, dt.
\]

Since \( \varphi : [0, \infty) \to [0, (2\pi)^{-n} |D|] \) is decreasing by (3.2), it follows from the coarea formula that

\[
(3.10) \quad \omega(\tau) = \int_\tau^{(2\pi)^{-n} |D|} \int_{S_t} \frac{1}{|\nabla G_k(\xi)|} \, d\sigma_s(\xi) \, ds
\]
where \( S_s = \{ \xi \in \mathbb{R}^n : G_k(\xi) = s \} \) and \( d\sigma_s \) is the surface measure on \( S_s \). Thus by (3.10) and Lemma 3.1, we obtain that

\[
-n|B_1|t^{n-1} = -\omega'(\phi(t))\phi'(t) = \left( \int_{S_{\phi(t)}} \frac{1}{|V G_k(\xi)|} d\sigma(\phi(t)) \right) \phi'(t) \\
\leq \frac{1}{\varrho} \sigma(\phi(t))(S_{\phi(t)}) \phi'(t) = \frac{1}{\varrho} n|B_1|t^{n-1} \phi'(t) \leq 0,
\]

where \( \varrho = 2(2\pi)^{-n} \sqrt{|D||D|} \). Thus this implies that \(-\varrho \leq \phi'(t) \leq 0\) for any \( t \geq 0 \).

If we set \( \beta = k/(n|B_1|) \) in (3.8), then by (3.9) and Lemma 2.3 we have that

\[
\frac{1}{k} \sum_{i=1}^{k} \nu_i^\sigma(D) \geq \frac{\lambda n}{n+\sigma} \left( \frac{k}{|B_1|} \right)^{\frac{n}{2}} \phi(0)^{-\frac{n}{2}}
\]

\[
+ \frac{\lambda \sigma}{12(n+\sigma)\varrho^2} \left( \frac{k}{|B_1|} \right)^{\frac{n}{2}} \phi(0)^{\frac{2n-\sigma+2}{2}}.
\]

For \( t \in [0, (2\pi)^{-n}|D|] \), we set

\[
h(t) = \frac{\lambda n}{n+\sigma} \left( \frac{k}{|B_1|} \right)^{\frac{n}{2}} t^{-\frac{n}{n+\sigma}} + \frac{\lambda \sigma}{12(n+\sigma)\varrho^2} \left( \frac{k}{|B_1|} \right)^{\frac{n}{2}} t^{-\frac{n-\sigma+2}{2}}.
\]

Differentiating \( h(t) \) once, we get that

\[
h'(t) = \frac{\lambda \sigma}{n+\sigma} \left( \frac{k}{|B_1|} \right)^{\frac{n}{2}} t^{-\frac{n}{n+\sigma} - 1} g(t)
\]

where \( g(t) = -1 + \frac{2n-\sigma+2}{2n(n+\sigma)\varrho^2} \left( \frac{k}{|B_1|} \right)^{-2/n} t^{\frac{n-\sigma+2}{n}} \). Since \( g \) is increasing on \([0, (2\pi)^{-n}|D|]\), we obtain that

\[
g(t) \leq g((2\pi)^{-n}|D|) = -1 + \frac{(n+2)(2n-\sigma+2)|B_1|^{4/n}}{192n^2\pi^2k^{2/n}} \leq -1 + \frac{20}{192} \leq 0
\]

from the fact that \( \sigma \in (0, 2], |B_1| = \frac{2n^{n/2}}{n!(\pi)^{n/2}} \) and \( \Gamma(\frac{3}{2}) \geq \Gamma(\frac{1}{2}) = \sqrt{\pi} \) for all \( n \in \mathbb{N} \). Thus \( h \) is decreasing on \([0, (2\pi)^{-n}|D|]\), a lower bound in (3.12) can be obtained by replacing \( \phi(0) \) by \((2\pi)^{-n}|D|\) as follows;

\[
\frac{1}{k} \sum_{i=1}^{k} \nu_i^\sigma(D) \geq h((2\pi)^{-n}|D|)
\]

\[
= \frac{\lambda n(2\pi)^{\sigma}}{(n+\sigma)(|B_1||D|)^{\frac{n}{2}}} k^{\frac{n}{2}} + \frac{\lambda \sigma(2\pi)^{\sigma-2}}{48(n+\sigma)(|B_1||D|)^{\frac{n-\sigma+2}{2}}} |D|^{\frac{2n-\sigma+2}{2}}.
\]

Therefore we complete the proof. \( \square \)

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