Elliptic Curves in Honeycomb Form

Melody Chan and Bernd Sturmfels

Abstract. A plane cubic curve, defined over a field with valuation, is in honeycomb form if its tropicalization exhibits the standard hexagonal cycle. We explicitly compute such representations from a given $j$-invariant with negative valuation, we give an analytic characterization of elliptic curves in honeycomb form, and we offer a detailed analysis of the tropical group law on such a curve.

1. Introduction

Suppose $K$ is a field with a nonarchimedean valuation $\text{val} : K^* \to \mathbb{R}$, such as the rational numbers $\mathbb{Q}$ with their $p$-adic valuation for some prime $p \geq 5$ or the rational functions $\mathbb{Q}(t)$ with the $t$-adic valuation. Throughout this paper, we shall assume that the residue field of $K$ has characteristic different from 2 and 3.

We consider a ternary cubic polynomial whose coefficients $c_{ijk}$ lie in $K$:

\begin{equation}
\begin{split}
f(x, y, z) &= c_{300}x^3 + c_{210}x^2y + c_{120}xy^2 + c_{030}y^3 + c_{021}y^2z \\
&\quad + c_{012}yz^2 + c_{102}xz^2 + c_{201}x^2z + c_{111}xyz.
\end{split}
\end{equation}

Provided the discriminant of $f(x, y, z)$ is non-zero, this cubic represents an elliptic curve $E$ in the projective plane $\mathbb{P}^2_K$. The group $\text{GL}(3, K)$ acts on the projective space $\mathbb{P}^8_K$ of all cubics. The field of rational invariants under this action is generated by the familiar $j$-invariant, which we can write explicitly (with coefficients in $\mathbb{Z}$) as

\begin{equation}
\begin{split}
j(f) &= \text{a polynomial of degree 12 in the } c_{ijk} \text{ having 1607 terms} \\
&\quad \text{a polynomial of degree 12 in the } c_{ijk} \text{ having 2040 terms}.
\end{split}
\end{equation}

The Weierstrass normal form of an elliptic curve can be obtained from $f(x, y, z)$ by applying a matrix in $\text{GL}(3, \overline{K})$. From the perspective of tropical geometry, however, the Weierstrass form is too limiting: its tropicalization never has a cycle. One would rather have a model for plane cubics whose tropicalization looks like the graphs in Figures 1, 3 and 5. If this holds then we say that $f$ is in honeycomb form. Cubic curves in honeycomb form are the central object of interest in this paper.

Honeycomb curves of arbitrary degree were studied in [17] §5; they are dual to the standard triangulation of the Newton polygon of $f$. For cubics in honeycomb form, by [9], the lattice length of the hexagon equals $-\text{val}(j(f))$. Moreover, by [2], a honeycomb cubic faithfully represents a subgraph of the Berkovich curve $E^{an}$.

A standard Newton subdivision argument [10] shows that a cubic $f$ is in honeycomb form if and only if the following nine scalars in $K$ have positive valuation:

\begin{equation}
\frac{c_{021}c_{102}}{c_{012}c_{120}}, \frac{c_{201}c_{012}}{c_{210}c_{021}}, \frac{c_{210}c_{021}}{c_{120}c_{021}}, \frac{c_{111}c_{012}}{c_{111}c_{021}}, \frac{c_{111}c_{021}}{c_{111}c_{021}}, \frac{c_{111}c_{021}}{c_{111}c_{021}}, \frac{c_{111}c_{021}}{c_{111}c_{021}}, \frac{c_{111}c_{021}}{c_{111}c_{021}}.
\end{equation}
If the six ratios in (3) have the same positive valuation, and also the three ratios in (4) have the same positive valuation, then we say that $f$ is in symmetric honeycomb form. So $f$ is in symmetric honeycomb form if and only if the lattice lengths of the six sides of the hexagon are equal, and the lattice lengths of the three bounded segments coming off the hexagon are also equal, as in Figure 1 on the right.

Our contributions in this paper are as follows. In Section 2 we focus on symmetric honeycomb cubics. We present a symbolic algorithm whose input is an arbitrary cubic $f$ with $\text{val}(j(f)) < 0$ and whose output is a $3 \times 3$-matrix $M$ such that $f \circ M$ is in symmetric honeycomb form. This answers a question raised by Buchholz and Markwig (cf. [4], §6). We pay close attention to the arithmetic of the entries of $M$. Our key tool is the relationship between honeycombs and the Hesse pencil [1, 12]. Results similar to those in Section 2 were obtained independently by Helminck [6].

Section 3 discusses the Tate parametrization [16] of elliptic curves using theta functions. Our approach is similar to that used by Speyer in [18] for lifting tropical curves. We present an analytic characterization of honeycomb cubics with prescribed $j$-invariant, and we give a numerical algorithm for computing such cubics.

Section 4 explains a combinatorial rule for the tropical group law on a honeycomb cubic $C$. Our object of study is the tropicalization of the surface $\{u, v, w \in C^3 | u \ast v \ast w = \text{id}\} \subset (\mathbb{P}^2)^3$. Here $\ast$ denotes multiplication on $C$. We explain how to compute this tropical surface in $(\mathbb{R}^2)^3$. See Corollary 11 for a concrete instance. Our results complete the partially defined group law found by Vigeland [19].

Practitioners of computational algebraic geometry are well aware of the challenges involved in working with algebraic varieties over a valued field $K$. One aim of this article is to demonstrate how these challenges can be overcome in practice, at least for the basic case of elliptic curves. In that sense, our paper can be read as a computational algebra supplement to the work of Baker, Payne and Rabinoff [2].

Many of our methods have been implemented in MATHEMATICA. Our code and the examples in this paper can be found at our supplementary materials website:

[www.math.berkeley.edu/~mtchan/honeycomb.html](http://www.math.berkeley.edu/~mtchan/honeycomb.html)
In our test implementations, the input data are assumed to lie in the field \( K = \mathbb{Q}(t) \), and scalars in \( K \) are represented as truncated Laurent series with coefficients in \( \mathbb{Q} \). This is analogous to the representation of scalars in \( \mathbb{R} \) by floating point numbers.

### 2. Symmetric Cubics

We begin by establishing the existence of symmetric honeycomb forms for elliptic curves whose \( j \)-invariant has negative valuation. Consider a symmetric cubic

\[
g = a \cdot (x^3 + y^3 + z^3) + b \cdot (x^2 y + x y^2 + x z^2 + y^2 z + y z^2 + x y z).
\]

The conditions in (3)-\( (\text{3}) \) imply that \( g \) is in symmetric honeycomb form if and only if

\[
\text{val}(a) > 2 \cdot \text{val}(b) > 0.
\]

Our aim in this section is to transform arbitrary cubics (1) to symmetric cubics (5) and (6). Note that \( a = 0 \) is allowed by the valuation inequalities (6), but \( b \) must be non-zero in (5). The classical Hesse normal form of \( \text{[1]} \), whose tropicalization was examined recently by Nobe \( \text{[12]} \), is therefore ruled out by the honeycomb condition.

**Proposition 1.** Given any two scalars \( \iota \) and \( a \) in \( K \) with \( \text{val}(\iota) < 0 \) and \( \text{val}(a) + \text{val}(\iota) > 0 \), there exist precisely six elements \( b \) in the algebraic closure \( \overline{K} \), defined by an equation of degree 12 over \( K \), such that the cubic \( g \) above has \( j \)-invariant \( j(g) = \iota \) and is in symmetric honeycomb form.

**Proof.** First, consider the case \( a = 0 \), so that \( \text{val}(a) = \infty \). By specializing (2), we deduce that the \( j \)-invariant of \( g = b(x^3 + y^3 + z^3) + x y z \) is

\[
j(g) = \frac{432b^{12} - 864b^{11} + 648b^{10} - (208\iota + 110592)b^9 + (15\iota + 165888)b^8 + (6\iota - 82944)b^7 - (\iota - 6912)b^6 + 6912b^5 - 1728b^4 - 144b^3 + 72b^2 - 1 = 0}.
\]

We examine the Newton polygon of this equation. It is independent of \( K \) because the characteristic of the residue field of \( K \) is not 2 or 3. Since \( \iota \) has negative valuation, we see that (8) has six solutions \( b \in \overline{K} \) with \( \text{val}(b) = 0 \) and six solutions \( b \) with \( \text{val}(b) = -\text{val}(\iota)/6 \). The latter six solutions are indexed by the choice of a sixth root of \( \iota^{-1} \). They share the following expansion as a Laurent series in \( \iota^{-1/6} \):

\[
b = \iota^{-1/6} + \iota^{-1/3} - 5\iota^{-1/2} - 7\iota^{-2/3} + 30\iota^{-5/6} + 43\iota^{-1} - 60\iota^{-7/6} - 15\iota^{-4/3} - 731\iota^{-3/2} - 1858\iota^{-5/3} + 11676\iota^{-11/6} + 22091\iota^{-2} - 30612\iota^{-13/6} + \cdots
\]

These six values of \( b \) establish the assertion in Proposition [1\text{]} when \( a = 0 \).

Now suppose \( a \neq 0 \). Then our equation \( j(g) = \iota \) has the more complicated form

\[
\frac{(6a - 1)^3(72ab^2 - 48b^3 - 36a^2 + 24b^2 - 6a - 1)^3}{(3a + 6b + 1)(3a - 3b + 1)^2(9a^3 - 3ab^2 + 2b^3 - 3a^2 - b^2 + a)^3} = \iota.
\]

Our hypotheses on \( K \) and \( a \) ensure that \( \text{val}(a) \) is large enough so as to not interfere with the lowest order terms when solving this equation for \( b \). In particular, the equation has 12 solutions \( b = b(\iota, a) \) that are scalars in \( \overline{K} \), and six of the solutions satisfy \( \text{val}(b) = 0 \) while the other six satisfy \( \text{val}(b) = -\text{val}(\iota)/6 \). The latter six establish our assertion. \( \square \)
We have proved the existence of a symmetric honeycomb form for any nonsingular cubic whose $j$-invariant has negative valuation. Our main goal in what follows is to describe an algorithm for computing a $3 \times 3$ matrix that transforms a given cubic into that form. Our method is to compute the nine inflection points of each cubic and find a suitable projective transformation that takes one set of points to the other. Computing the inflection points is a relatively easy task in the special case of symmetric cubics. The result of that computation is the following lemma.

**Lemma 2.** Let $C$ be a nonsingular cubic curve defined over $K$ by a symmetric polynomial $g$ as in (3), fix a primitive third root of unity $\xi$ in $\overline{K}$, and set

$$\omega = \frac{3a + 6b + 1}{-3a + 3b - 1}.$$  

Then the nine inflection points of $C$ in $\mathbb{P}^2$ are given by the rows of the matrix

$$A_\omega = \begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & -1 \\
1 + \omega^{1/3} & 1 + \xi \omega^{1/3} & 1 + \xi^2 \omega^{1/3} \\
1 + \xi \omega^{1/3} & 1 + \xi^2 \omega^{1/3} & 1 + \omega^{1/3} \\
1 + \xi^2 \omega^{1/3} & 1 + \omega^{1/3} & 1 + \xi \omega^{1/3} \\
1 + \omega^{1/3} & 1 + \xi \omega^{1/3} & 1 + \xi^2 \omega^{1/3} \\
1 + \xi \omega^{1/3} & 1 + \xi^2 \omega^{1/3} & 1 + \omega^{1/3} \\
1 + \xi^2 \omega^{1/3} & 1 + \omega^{1/3} & 1 + \xi \omega^{1/3}
\end{bmatrix}.$$  

The matrix $A_\omega$ has precisely the following vanishing $3 \times 3$-minors:

$$(12) \quad 123, 147, 159, 168, 249, 258, 267, 348, 357, 369, 456, 789.$$  

This list of triples is the classical *Hesse configuration* of 9 points and 12 lines.

Next, for an arbitrary nonsingular cubic $f$ as in (1), the nine inflection points can be expressed in radicals in the ten coefficients $c_{300}, c_{210}, \ldots, c_{111}$, since their Galois group is solvable [1] §4. How can we compute these inflection points? Consider the *Hesse pencil* $\text{HP}(f) = \{s \cdot f + s' \cdot H_f : s, s' \in K\}$ of plane cubics spanned by $f$ and its Hessian $H_f$. Each cubic in $\text{HP}(f)$ passes through the nine inflection points of $f$ since both $f$ and $H_f$ do, and in fact every such cubic is in $\text{HP}(f)$. In particular, the four systems of three lines through the nine points are precisely the four reducible members of $\text{HP}(f)$. Indeed, if $g = l \cdot h \in \text{HP}(f)$ where $l$ is a line passing through three inflection points, then $h$ passes through the remaining six and thus must itself be two lines by Bézout’s Theorem. So we may compute any two of the four such systems of three lines, and take pairwise intersections of their lines to obtain the nine desired inflection points. This algorithm was extracted from Salmon’s book [14], and it runs in exact arithmetic. We now make it more precise.

We introduce four unknowns $u, v, w, s$, and we consider the condition that a cubic $s \cdot f + H_f$ is divisible by the linear form $ux + vy + wz$. That condition translates into a system of polynomials that are cubic in the unknowns $u, v, w$ and linear in $s$. We derive this system by specializing the following universal solution, found by a Macaulay2 computation which is posted on our supplementary materials website.

**Lemma 3.** The condition that a linear form $ux + vy + wz$ divides a cubic (7) is given by a prime ideal in the polynomial ring $K[u, v, w, c_{300}, c_{210}, \ldots, c_{003}]$ in 13 unknowns. This prime ideal is of codimension 4 and degree 28. It has 96 minimal generators, namely 25 quartics, 15 quintics, 21 sextics and 35 octics.
Consider the polynomials in $u, v, w, s$ that are obtained by specializing the $c_{ijk}$ in the 96 ideal generators above to the coefficients of $s \cdot f + H_f$. After permuting coordinates if necessary, we may set $w = 1$ and work with the resulting polynomials in $u, v, s$. The lexicographic Gröbner basis of their ideal has the special form

$$\{ g^4 + \alpha_2 s^2 + \alpha_1 s + \alpha_0, \quad g^3 + \beta_2(s) v^2 + \beta_1(s) v + \beta_0(s), \quad u + \gamma(s, v) \},$$

where the $\alpha_i$ are constants in $K$, the $\beta_j$ are univariate polynomials, and $\gamma$ is a bivariate polynomial. These equations have 12 solutions $(s_i, u_{ij}, v_{ij}) \in (\overline{K})^3$ where $i = 1, 2, 3, 4$ and $j = 1, 2, 3$. The leading terms in the Gröbner basis reveal that the coordinates of these solutions can be expressed in radicals over $K$, since we need only solve a quartic in $s$, a cubic in $v$, and a degree 1 equation in $u$, in that order.

For each of the nine choices of $j, k \in \{1, 2, 3\}$, the two linear equations

$$u_{ij} x + v_{ij} y + z = u_{2k} x + v_{2k} y + z = 0$$

have a unique solution $(b_1^{jk} : b_2^{jk} : b_3^{jk})$ in the projective plane $\mathbb{P}^2$ over $\overline{K}$. We can write its coordinates $b_i^{jk}$ in radicals over $K$. Let $B$ denote the $9 \times 3$-matrix whose rows are the vectors $(b_1^{jk}, b_2^{jk}, b_3^{jk})$ for $j, k \in \{1, 2, 3\}$. While the entries of $B$ have been written in radicals over $K$, they can also be represented as formal series in the completion of $\overline{K}$, which we can approximate by a suitable truncation.

To summarize our discussion up to this point: we have shown how to compute the inflection points of a plane cubic, and we have written them as the rows of a $9 \times 3$-matrix $B$ whose entries are expressed in radicals over $K$. For the special case of symmetric cubics, the specific $9 \times 3$-matrix $A_\omega$ in (11) gives the inflection points.

Now, we return to our main goal. Suppose we are given a nonsingular ternary cubic $f$ whose $j$-invariant $\iota = j(f)$ has negative valuation. We then choose $a, b \in \overline{K}$ as prescribed in Proposition 1 and we define $\omega$ by the ratio in (10). The scalars $a$ and $b$ define a symmetric honeycomb cubic $g$ as in (5). Let $A_\omega$ and $B$ denote the sets of inflection points of the cubic curves $V(g)$ and $V(f)$ respectively. Thus $A_\omega$ and $B$ are unordered 9-element subsets of $\mathbb{P}^2$, represented by the rows of our matrices $A_\omega$ and $B$. There exists an automorphism $\phi$ of $\mathbb{P}^2$ taking $V(f)$ to $V(g)$, since their $j$-invariants agree. Clearly, any such automorphism $\phi$ takes $B$ to $A_\omega$.

We write $B = \{b_1, b_2, \ldots, b_9\}$, where the labeling is such that $b_1, b_2, b_3$ are collinear in $\mathbb{P}^2$ if and only if $ijk$ appears on the list (12). The automorphism group of the Hesse configuration (12) has order $9 \cdot 8 \cdot 6 = 432$. Hence precisely 432 of the 9! possible bijections $B \to A_\omega$ respect the collinearities of the inflection points. For each such bijection $\pi_i : B \to A_\omega, i = 1, 2, \ldots, 432$, we associate a unique projective transformation $\sigma_i : \mathbb{P}^2 \to \mathbb{P}^2$ by requiring that $\sigma_i(b_1) = \pi_i(b_1), \sigma_i(b_2) = \pi_i(b_2), \sigma_i(b_4) = \pi_i(b_4)$ and $\sigma_i(b_5) = \pi_i(b_5)$. We emphasize that $\sigma_i$ may or may not induce a bijection $B \to A_\omega$ on all nine points. We write $M_i$ for the unique (up to scaling) $3 \times 3$-matrix with entries in $K$ that represents the projective transformation $\sigma_i$.

The simplest version of our algorithm constructs all matrices $M_1, M_2, \ldots, M_{432}$. One of these matrices, say $M_j$, represents the automorphism $\phi$ of $\mathbb{P}^2$ in the second-to-last paragraph. The ternary cubics $f \circ M_j$ and $g$ are equal up to a scalar. To find such an index $j$, we simply check, for each $j \in \{1, 2, \ldots, 432\}$, whether $f \circ M_j$ is in symmetric honeycomb form. The answer will be affirmative for at least one index $j$, and we set $M = M_j$. This resolves the question raised by Markwig and Buchholz [4 §6]. The following theorem summarizes the problem and our solution.
Theorem 4. Let $f$ be a nonsingular cubic with $\operatorname{val}(j(f)) < 0$. If $M$ is the $3 \times 3$-matrix over $\overline{K}$ constructed above then $f \circ M$ is a symmetric honeycomb cubic.

Next, we discuss a refinement of the algorithm above that reduces the number of matrices to check from 432 to 12. It takes advantage of the detailed description of the Hessian group in [11]. Given a plane cubic $f$, the Hessian group $G_{216}$ consists of those linear automorphisms of $\mathbb{P}^2$ that preserve the pencil $\mathbb{H}P(f)$. This group was first described by C. Jordan in [12]. The elements of $G_{216}$ naturally act on the subset $A_\omega$ of $\mathbb{P}^2$ given by the rows $a_1, a_2, \ldots, a_9$ of $A_\omega$. Of the 432 automorphisms of $[12]$, precisely half are realized by the action of $G_{216}$. The group $G_{216}$ is isomorphic to the semidirect product $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \text{SL}(2,3)$. The first factor sends $f$ to itself and permutes $A_\omega$ transitively. The second factor, of order 24, sends $f$ to each of the 12 cubics in $\mathbb{H}P(f)$ isomorphic to it. The quotient of $\text{SL}(2,3)$ by the 2-element stabilizer of $f$ is isomorphic to $\text{PSL}(2,3)$, with 12 elements. Identifying $G_{216}$ with the subgroup of $S_9$ permuting $A_\omega$, a set of coset representatives for $\text{PSL}(2,3)$ inside $G_{216}$ consists of the following 12 permutations (in cycle notation):

\[
\text{id}, (456)(987), (654)(789), (2437)(5698), (246378)(59), (254397)(68), (249)(375), (258)(963), (2539)(4876), (852)(369), (287364)(59), (2836)(4975)
\]

With this notation, an example of an automorphism of the Hesse configuration $[12]$ that is not realized by the Hessian group $G_{216}$ is the permutation $\tau = (47)(58)(69)$.

Here is now our refined algorithm for the last step towards Theorem 4. Let $f, g, \mathcal{B}, A_\omega$ be given as above. For any automorphism $\rho$ of $[12]$, we denote by $\phi_\rho : \mathbb{P}^2 \to \mathbb{P}^2$ the projective transformation $(b_1, b_2, b_3, b_5) \mapsto (a_{\rho(1)}, a_{\rho(2)}, a_{\rho(4)}, a_{\rho(5)})$. To find a transformation from $f$ to $g$, we proceed as follows. First, we check to see whether $\phi_\text{id}$ maps $\mathcal{B}$ to $A_\omega$. (It certainly maps four elements of $\mathcal{B}$ to $A_\omega$, but maybe not all nine.) If $\phi_\text{id}(\mathcal{B}) = A_\omega$ then $\phi_\text{id}(f)$ is in the Hesse pencil $\mathbb{H}P(g)$. This implies that one of the 12 maps $\phi_\sigma$, where $\sigma$ runs over $[13]$, takes $f$ to $g$. If $\phi_\text{id}(\mathcal{B}) \neq A_\omega$ then $\phi_\tau$ must map $\mathcal{B}$ to $A_\omega$, since $G_{216}$ has index 2 in the automorphism group of the Hesse configuration and $\tau$ represents the nonidentity coset. Then one of the 12 maps $\phi_\sigma$, where $\sigma$ runs over $[13]$, takes $f$ to $g$. In either case, after computing $\phi_\text{id}$, we only have to check 12 maps, and one of them will work.

We close with two remarks. First, the set of matrices $M \in \text{GL}(3, \overline{K})$ that send a given cubic $f$ into honeycomb form is a rigid analytic variety, since the conditions on the entries of $M$ are inequalities in valuations of polynomial expressions therein. It would be interesting to study this space further.

The second remark concerns the arithmetic nature of the output of our algorithm. The entries of the matrix $M$ were constructed to be expressible in radicals over $K(\omega)$, with $\omega$ as in $[10]$. However, as it stands, we do not know whether they can be expressed in radicals over the ground field $K$. The problem lies in the application of Proposition 11. Our first step was to choose a scalar $a \in K$ whose valuation is large enough. Thereafter, we computed $b$ by solving a univariate equation of degree 12. This equation is generally irreducible with non-solvable Galois group. Perhaps it is possible to choose $a$ and $b$ simultaneously, in radicals over $K$, so that $(a, b)$ lies on the curve $[11]$, but at present, we do not know how to make this choice.

3. Parametrization and Implicitization

A standard task of computer algebra is to go back and forth between parametric and implicit representations of algebraic varieties. Of course, these transformations
are most transparent when the variety is rational. If the variety is not unirational then parametric representations typically involve transcendental functions. In this section, we use nonarchimedean theta functions to parametrize planar cubics, we demonstrate how to implicitize this parametrization, and we derive an intrinsic characterization of honeycomb cubics in terms of their nonarchimedean geometry.

In this section we assume that $K$ is an algebraically closed field which is complete with respect to a nonarchimedean valuation. Fix a scalar $\iota \in K$ with $\text{val}(\iota) < 0$. According to Tate’s classical theory \cite{Tate}, the unique elliptic curve $E$ over $K$ with $j(E) = \iota$ is analytically isomorphic to $K^*/q^\mathbb{Z}$, where $q \in K^*$ is a particular scalar with $\text{val}(q) > 0$, called the Tate parameter of $E$. The symbol $q^\mathbb{Z}$ denotes the multiplicative group generated by $q$. The Tate parameter of $E$ is determined from the $j$-invariant by inverting the power series relation

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

This relation can be derived and computed to arbitrary precision from the identity

$$j = \frac{(1 - 48a_4(q))^3}{\Delta(q)}.$$

where the invariant $a_4$ and the discriminant $\Delta$ are given by

$$a_4(q) = -5 \sum_{n \geq 1} \frac{n^3q^n}{1 - q^n}, \quad \Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

We refer to Silverman’s text book \cite{Tate} for an introduction to the relevant theory of elliptic curves, and specifically to \cite{Tate} Theorems V.1.1, V.3.1 for the above results.

Our aim in this section is to work directly with the analytic representation $E = K^*/q^\mathbb{Z}$, and to construct its honeycomb embeddings into the plane $P^2_K$. In our explicit computations, scalars in $K$ are presented as truncated power series in a uniformizing parameter. The arithmetic is numerical rather than symbolic. Thus, this section connects the emerging fields of tropical geometry and numerical algebraic geometry.

By a holomorphic function on $K^*$ we mean a formal Laurent series $\sum a_n x^n$ which converges for every $x \in K^*$. A meromorphic function is a ratio of two holomorphic functions; they have a well-defined notion of zeroes and poles as usual. A theta function on $K^*$, relative to the subgroup $q^\mathbb{Z}$, is a meromorphic function on $K^*$ whose divisor is periodic with respect to $q^\mathbb{Z}$. Hence theta functions on $K^*$ represent divisors on $E$. The fundamental theta function $\Theta : K^* \to K$ is defined by

$$\Theta(x) = \prod_{n > 0} (1 - q^n x) \prod_{n \geq 0} (1 - \frac{q^n}{x}).$$

Note that $\Theta$ has a simple zero at the identity of $E$ and no other zeroes or poles. Furthermore, given any $a \in K^*$, we define the shifted theta function

$$\Theta_a(x) = \Theta(x/a).$$

The function $\Theta_a$ represents the divisor $[a] \in \text{Div}E$, where $[a]$ denotes the point of the elliptic curve $E$ represented by $a$. One can also check that $\Theta_a(x/q) = -\frac{x}{q} \Theta_a(x)$.

Now suppose $D = n_1p_1 + \cdots + n_s p_s$ is a divisor on $E$ that satisfies $\text{deg}(D) = 0$ and $p_1^{n_1} \cdots p_s^{n_s} = 1$, as an equation in the multiplicative group $K^*/q^\mathbb{Z}$. We can use
theta functions to exhibit $D$ as a principal divisor, as follows. Pick lifts $\tilde{p}_1, \ldots, \tilde{p}_s \in K^*$ of $p_1, \ldots, p_s$, respectively, such that $\tilde{p}_1^{n_1} \cdots \tilde{p}_s^{n_s} = 1$ as an equation in $K^*$. Let

$$f(x) = \Theta_{\tilde{p}_1}(x)^{n_1} \cdots \Theta_{\tilde{p}_s}(x)^{n_s}.$$ 

This defines a function $f : K^* \to K$ that is $q$-periodic because

$$f(x/q) = \left(\frac{-x}{p_1^{n_1} \cdots p_s^{n_s}}\right)^{n_1+\cdots+n_s} f(x) = (-x)^{n_1+\cdots+n_s} f(x) = f(x).$$

The last equation holds because we assumed that $\deg(D) = n_1 + \cdots + n_s$ is zero. We conclude that $f$ descends to a meromorphic function on $K^*/q^\mathbb{Z}$ with divisor $D$.

We now present a parametric representation of plane cubic curves that will work well for honeycombs. In what follows, we write $(z_0 : z_1 : z_2)$ for the coordinates on $\mathbb{P}^2$. Fix scalars $a, b, c, p_1, \ldots, p_9$ in $K^*$ that satisfy the conditions

$$p_1 p_2 p_3 = p_4 p_5 p_6 = p_7 p_8 p_9 \quad \text{and} \quad p_i/p_j \notin q^\mathbb{Z} \quad \text{for} \quad i \neq j.$$ 

The following defines a map from $E = K^*/q^\mathbb{Z}$ into the projective plane $\mathbb{P}^2$ as follows:

$$x \mapsto (a \cdot \Theta_{p_1} \Theta_{p_2} \Theta_{p_3}(x) : b \cdot \Theta_{p_4} \Theta_{p_5} \Theta_{p_6}(x) : c \cdot \Theta_{p_7} \Theta_{p_8} \Theta_{p_9}(x)).$$

This map embeds the elliptic curve $E = K^*/q^\mathbb{Z}$ analytically as a plane cubic:

**Lemma 5.** If the image of the map \[16\] has three distinct intersection points with each of the three coordinate lines \(\{z_i = 0\}\), then it is a cubic curve in \(\mathbb{P}^2\). Every nonsingular cubic with this property and having Tate parameter $q$ arises this way.

**Proof.** By construction, the following two functions $K^* \to K$ are $q$-periodic:

$$f(x) = \frac{a \cdot \Theta_{p_1} \Theta_{p_2} \Theta_{p_3}(x)}{c \cdot \Theta_{p_4} \Theta_{p_5} \Theta_{p_6}(x)} \quad \text{and} \quad g(x) = \frac{b \cdot \Theta_{p_4} \Theta_{p_5} \Theta_{p_6}(x)}{c \cdot \Theta_{p_7} \Theta_{p_8} \Theta_{p_9}(x)}.$$ 

Hence $f$ and $g$ descend to meromorphic functions on the elliptic curve $E = K^*/q^\mathbb{Z}$. The map \[16\] can be written as $x \mapsto (f(x) : g(x) : 1)$ and this defines a map from $E$ into $\mathbb{P}^2$. The divisor $D = p_7 + p_8 + p_9$ on $E$ has degree 3. By the Riemann-Roch argument in [5] Example 3.3.3], its space of sections $L(D)$ is 3-dimensional. Moreover, the assumption about having three distinct intersection points implies that the meromorphic functions $f, g$ and 1 form a basis of the vector space $L(D)$. The image of $E$ in $\mathbb{P}(L(D)) \approx \mathbb{P}^2$ is a cubic curve because $L(3D)$ is 9-dimensional.

For the second statement, we take $C$ to be any nonsingular cubic curve with Tate parameter $q$ that has distinct intersection points with the three coordinate lines $\{z_i = 0\}$, then it is a cubic curve in $\mathbb{P}^2$. The map \[16\] can be written as $x \mapsto (f(x) : g(x) : 1)$ and this defines a map from $E$ into $\mathbb{P}^2$. The divisor $D = p_7 + p_8 + p_9$ on $E$ has degree 3. By the Riemann-Roch argument in [5] Example 3.3.3], its space of sections $L(D)$ is 3-dimensional. Moreover, the assumption about having three distinct intersection points implies that the meromorphic functions $f, g$ and 1 form a basis of the vector space $L(D)$. The image of $E$ in $\mathbb{P}(L(D)) \approx \mathbb{P}^2$ is a cubic curve because $L(3D)$ is 9-dimensional.

For the second statement, we take $C$ to be any nonsingular cubic curve with Tate parameter $q$ that has distinct intersection points with the three coordinate lines $\{z_i = 0\}$, then it is a cubic curve in $\mathbb{P}^2$. The map \[16\] can be written as $x \mapsto (f(x) : g(x) : 1)$ and this defines a map from $E$ into $\mathbb{P}^2$. The divisor $D = p_7 + p_8 + p_9$ on $E$ has degree 3. By the Riemann-Roch argument in [5] Example 3.3.3], its space of sections $L(D)$ is 3-dimensional. Moreover, the assumption about having three distinct intersection points implies that the meromorphic functions $f, g$ and 1 form a basis of the vector space $L(D)$. The image of $E$ in $\mathbb{P}(L(D)) \approx \mathbb{P}^2$ is a cubic curve because $L(3D)$ is 9-dimensional.

For the second statement, we take $C$ to be any nonsingular cubic curve with Tate parameter $q$ that has distinct intersection points with the three coordinate lines $\{z_i = 0\}$, then it is a cubic curve in $\mathbb{P}^2$. The map \[16\] can be written as $x \mapsto (f(x) : g(x) : 1)$ and this defines a map from $E$ into $\mathbb{P}^2$. The divisor $D = p_7 + p_8 + p_9$ on $E$ has degree 3. By the Riemann-Roch argument in [5] Example 3.3.3], its space of sections $L(D)$ is 3-dimensional. Moreover, the assumption about having three distinct intersection points implies that the meromorphic functions $f, g$ and 1 form a basis of the vector space $L(D)$. The image of $E$ in $\mathbb{P}(L(D)) \approx \mathbb{P}^2$ is a cubic curve because $L(3D)$ is 9-dimensional.
It is a natural ask to what extent the parameters in the representation \([16]\) of a plane cubic are unique. The following result answers this question.

**Proposition 6.** Two vectors \((a, b, c, p_1, \ldots, p_9)\) and \((a', b', c', p'_1, \ldots, p'_9)\) in \((K^*)^2\), both satisfying \([12]\), define the same plane cubic if and only if the latter vector can be obtained from the former by combining the following operations:

(a) Permute the sets \(\{p_1, p_2, p_3\}, \{p_4, p_5, p_6\}\) and \(\{p_7, p_8, p_9\}\).

(b) Scale each of \(a, b\) and \(c\) by the same multiplier \(\lambda \in K^*\).

(c) Scale each \(p_i\) by the same multiplier \(\lambda \in K^*\).

(d) Replace each \(p_i\) by its multiplicative inverse \(1/p_i\).

(e) Multiply each \(p_i\) by \(q^n\), for some \(n_i \in \mathbb{Z}\), where \(n_1+n_2+n_3 = n_4+n_5+n_6 = n_7+n_8+n_9\), and set \(a' = p'_1 p'_2 p'_3 a, b' = p'_4 p'_5 p'_6 b, c' = p'_7 p'_8 p'_9 c\).

**Proof.** Clearly the relabeling in (a) and the scaling in (b) preserve the curve \(C \subset \mathbb{P}^2\). For (c), we note that scaling each \(p_i\) by the same constant \(\lambda \in K^*\) produces a reparametrization of the same curve; only the location of the identity point changes. For part (e), note that \(\Theta_{aq'}(x) = \Theta_a(q \cdot x) = (-x/a) \Theta_a(x)\). Suppose \((a, b, c, p_1, \ldots, p_9)\) and \((a', b', c', p'_1, \ldots, p'_9)\) satisfy the conditions in (e). Then

\[
(a' \cdot \Theta_{p_1} \Theta_{p_2} \Theta_{p_3} : b' \cdot \Theta_{p_4} \Theta_{p_5} \Theta_{p_6} : c' \cdot \Theta_{p_7} \Theta_{p_8} \Theta_{p_9})
\]

\[
= \left(\frac{-x^n a' \Theta_{p_1} \Theta_{p_2} \Theta_{p_3}}{p_1^4 p_2^4 p_3^4}, \frac{-x^n b' \Theta_{p_4} \Theta_{p_5} \Theta_{p_6}}{p_4^4 p_5^4 p_6^4}, \frac{-x^n c' \Theta_{p_7} \Theta_{p_8} \Theta_{p_9}}{p_7^4 p_8^4 p_9^4}\right)
\]

\[
= \left(a \cdot \Theta_{p_1} \Theta_{p_2} \Theta_{p_3} : b \cdot \Theta_{p_4} \Theta_{p_5} \Theta_{p_6} : c \cdot \Theta_{p_7} \Theta_{p_8} \Theta_{p_9}\right),
\]

where \(n = n_1 + n_2 + n_3 = n_4 + n_5 + n_6 = n_7 + n_8 + n_9\). Finally, for (d), one may check the identity \(x \Theta(x^{-1}) = \Theta(x)\) directly from the definition of the fundamental theta function. In light of \([15]\), this implies

\[
(a \cdot \Theta_{p_1} \Theta_{p_2} \Theta_{p_3} : b \cdot \Theta_{p_4} \Theta_{p_5} \Theta_{p_6} : c \cdot \Theta_{p_7} \Theta_{p_8} \Theta_{p_9})
\]

\[
= \left(\frac{-x^n a \Theta(p_1) \Theta(p_2) \Theta(p_3)}{p_1^4 p_2^4 p_3^4}, \frac{-x^n b \Theta(p_4) \Theta(p_5) \Theta(p_6)}{p_4^4 p_5^4 p_6^4}, \frac{-x^n c \Theta(p_7) \Theta(p_8) \Theta(p_9)}{p_7^4 p_8^4 p_9^4}\right)
\]

\[
= \left(a \cdot \Theta_L\left(\frac{1}{p_1}\right) \Theta_L\left(\frac{1}{p_2}\right) \Theta_L\left(\frac{1}{p_3}\right) : b \cdot \Theta_L\left(\frac{1}{p_4}\right) \Theta_L\left(\frac{1}{p_5}\right) \Theta_L\left(\frac{1}{p_6}\right) : c \cdot \Theta_L\left(\frac{1}{p_7}\right) \Theta_L\left(\frac{1}{p_8}\right) \Theta_L\left(\frac{1}{p_9}\right)\right).
\]

This is the reparametrization of the elliptic curve \(E\) under the involution \(x \mapsto 1/x\) in the group law. We have thus proved the if direction of Proposition 6.

For the only-if direction, we write \(\psi\) and \(\psi'\) for the maps \(E \rightarrow C \subset \mathbb{P}^2\) defined by \((a, b, \ldots, p_9)\) and \((a', b', \ldots, p'_9)\) respectively. Then \((\psi')^{-1} \circ \psi\) is an automorphism of the elliptic curve \(E\). The \(j\)-invariant of \(E\) is neither of the special values 0 or 1728. By \([15]\) Theorem 10.1], the only automorphisms of \(E\) are the involution \(x \mapsto 1/x\) and multiplication by some fixed element in the group law. These are precisely the operations we discussed above, and they can be realized by the transformations from \((a, b, \ldots, p_9)\) to \((a', b', \ldots, p'_9)\) that are described in (c) and (d).

Finally, if \((\psi')^{-1} \circ \psi\) is the identity on \(E\), then plugging in \(x = p_i\) for \(i = 1, 2, 3\) shows that \(p_i\) is a zero of \(a \Theta_{p_1} \Theta_{p_2} \Theta_{p_3}\) and hence of \(a' \Theta_{p'_1} \Theta_{p'_2} \Theta_{p'_3}\). The same holds for \(i = 4, 5, 6\) and \(i = 7, 8, 9\). This accounts for the operations (a), (b) and (e).

Our main result in this section is the following characterization of honeycomb curves, in terms of the analytic representation of plane cubics in \([16]\). Writing \(S^1\) for the circle, let \(V : K^* \rightarrow S^1\) denote the composition \(K^* \text{ val} \rightarrow \mathbb{R} / \text{val}(q) \simeq S^1\).

**Theorem 7.** Let \(a, b, c, p_1, \ldots, p_9 \in K^*\) as in Lemma 5. Suppose the values \(V(p_1), \ldots, V(p_9)\) occur in cyclic order on \(S^1\), with \(V(p_3) = V(p_4), V(p_6) = V(p_7), V(p_8) = V(p_9),\) and all other values \(V(p_i)\) are distinct. Then the image of the
map (10) is an elliptic curve in honeycomb form. Conversely, any elliptic curve in honeycomb form arises in this manner, after a suitable permutation of the indices.

We shall present two alternative proofs of Theorem 7. These will highlight different features of honeycomb curves and how they relate to the literature. The first proof is computational and relates our study to the tropical theta functions studied by Mikhalkin and Zharkov [11]. The second proof is more conceptual. It is based on the nonarchimedean Poincaré-Lelong formula for Berkovich curves [2, Theorem 5.68]. Both approaches were suggested to us by Matt Baker.

First proof of Theorem 7. We shall examine the naive tropicalization of the elliptic curve $E = \mathbb{K}^*/q^\mathbb{Z}$ under its embedding (16) into $\mathbb{P}^2$. Set $Q = \text{val}(q) \in \mathbb{R}_{>0}$. If $a \in \mathbb{K}^*$ with $A = \text{val}(a) \in \mathbb{R}$ then the tropicalization of the theta function $\Theta_a : \mathbb{K}^* \to \mathbb{K}$ is obtained by replacing the infinite product of binomials in the definition of $\Theta(x)$ by an infinite sum of pairwise minima. The result is the function

$$\text{(18)} \quad \text{trop}(\Theta_a) : \mathbb{R} \to \mathbb{R}, \quad X \mapsto \sum_{n>0} \min(0, nQ+X-A) + \sum_{n\geq 0} \min(0, nQ+A-X).$$

For any particular real number $X$, only finitely many summands are non-zero, and hence $\text{trop}(\Theta_a)(X)$ is a well-defined real number. A direct calculation shows that

$$\text{(19)} \quad \text{trop}(\Theta_a)(X) = \min \left\{ \frac{m^2-m}{2} \cdot Q + m \cdot (A - X) : m \in \mathbb{Z} \right\}.$$

Indeed, the distributive law transforms the tropical product of binomials on the right hand side of (18) into the tropical sum in (19). The representation (19) is essentially the same as the tropical theta function of Mikhalkin and Zharkov [11].

The tropical theta function is a piecewise linear function on $\mathbb{R}$, and we can translate (19) into an explicit description of the linear pieces of its graph. We find

$$\text{(20)} \quad \text{trop}(\Theta_a)(X) = \frac{m^2-m}{2} \cdot Q + m \cdot (A - X)$$

where $m$ is the unique integer satisfying $mQ \leq A - X < (m+1)Q$. In particular, for arguments $X$ in this interval, the function is linear with slope $-m$.

The tropical theta function approximates the valuation of the theta function. These two functions agree unless there is some cancellation because the two terms in some binomial factor of $\Theta_a$ have the same order.

The gap between the tropical theta function and the valuation of the theta function is crucial in understanding the tropical geometry of the map $\mathcal{V} : \mathbb{K}^* \to S^1$. Our next definition makes this precise. If $x, y \in \mathbb{K}^*$ with $\mathcal{V}(x) = \mathcal{V}(y)$ then we set

$$\text{(21)} \quad \delta(x, y) := \text{val}(1 - \frac{x}{y}q^i)$$

where $i \in \mathbb{Z}$ is the unique integer satisfying $\text{val}(x) + \text{val}(q^i) = \text{val}(y)$. It is easy to check that the quantity defined in (21) is symmetric, i.e. $\delta(x, y) = \delta(y, x)$. With this notation, the following formula characterizes the gap between the tropical theta function and the valuation of the theta function. For any $a, x \in \mathbb{K}^*$, we have

$$\text{(22)} \quad \text{val}(\Theta_a(x)) - \text{trop}(\Theta_a)(\text{val}(x)) = \begin{cases} \delta(x, a) & \text{if } \mathcal{V}(a) = \mathcal{V}(x), \\ 0 & \text{otherwise}. \end{cases}$$
Consider now any three scalars \( x, y, z \in K^* \) that lie in the same fiber of the map from \( K^* \) onto the unit circle \( S^1 \). In symbols, \( V(x) = V(y) = V(z) \). Then

\[
(23) \quad \text{the minimum of } \delta(x, y), \delta(x, y) \text{ and } \delta(y, z) \text{ is attained twice.}
\]

This follows from the identity

\[
(i, j) \quad \text{where } \quad p_r \quad \text{for } \quad x(q^i - z) + (z - xq^i) = 0
\]

where \( i, j \in \mathbb{Z} \) are defined by \( \text{val}(x) + \text{val}(q^i) = \text{val}(y) + \text{val}(q^j) = \text{val}(z) \).

We are now prepared to prove Theorem 7. Set \( Q = \text{val}(y) \). For \( i = 1, 2, \ldots, 9 \), let \( p_i = \text{val}(p_i) \) and write \( p_i = n_i Q + r_i \) where \( n_i \in \mathbb{Z} \) and \( r_i \in [0, Q) \). Rescaling the \( p_i \)'s by a common factor and inverting them all does not change the cubic curve, by Proposition 6. After performing such operations if needed, we can assume

\[
(24) \quad 0 = r_9 = r_1 < r_2 < r_3 = r_4 < r_5 < r_6 = r_7 < r_8 < Q.
\]

The hypothesis (15) implies

\[
(25) \quad r_1 + r_2 + r_3 = r_4 + r_5 + r_6 = r_7 + r_8 + r_9 \quad \text{ (mod } Q \text{)}.
\]

Together with the chain of inequalities in (24), this implies

\[
(26) \quad r_1 + r_2 + r_3 = r_4 + r_5 + r_6 - Q = r_7 + r_8 + r_9 - Q,
\]

\[
n_1 + n_2 + n_3 = n_4 + n_5 + n_6 + 1 = n_7 + n_8 + n_9 + 1.
\]

We now examine the naive tropicalization \( \mathbb{R} \rightarrow \mathbb{TP}^2 \) of our map (16). It equals

\[
(27) \quad \text{val}(x) = \text{val}(p_1) = \text{val}(p_9). \quad \text{From (22) we have}
\]

\[
(28) \quad \text{we fix tropical affine coordinates with last coordinate 0. Similarly,}
\]

- the slope is \( (0 : 1 : 0) \) for \( X \in (r_2, r_3) \),
- the slope is \( (-1 : 0 : 0) \) for \( X \in (r_4, r_5) \),
- the slope is \( (-1 : -1 : 0) \) for \( X \in (r_5, r_6) \),
- the slope is \( (0 : -1 : 0) \) for \( X \in (r_7, r_8) \),
- the slope is \( (1 : 0 : 0) \) for \( X \in (r_8, Q) \).

These six line segments form a hexagon in \( \mathbb{TP}^2 \). The vertices of that hexagon are the images of the six distinct real numbers \( r_i \) in (24) under the map (26).

Finally, we examine the special values \( x \in K^* \) for which the naive tropicalization (26) does not compute the correct image in \( \mathbb{TP}^2 \). This happens precisely when some of the nine theta functions in (16) have a valuation gap when passing to (26).

Suppose, for instance, that \( V(x) = V(p_1) = V(p_9) \). From (22) we have

\[
(27) \quad \text{val}(x) = \text{val}(p_1) = \text{val}(p_9). \quad \text{From (22) we have}
\]

\[
(28) \quad \text{we fix tropical affine coordinates with last coordinate 0. Similarly,}
\]

- the slope is \( (0 : 1 : 0) \) for \( X \in (r_2, r_3) \),
- the slope is \( (-1 : 0 : 0) \) for \( X \in (r_4, r_5) \),
- the slope is \( (-1 : -1 : 0) \) for \( X \in (r_5, r_6) \),
- the slope is \( (0 : -1 : 0) \) for \( X \in (r_7, r_8) \),
- the slope is \( (1 : 0 : 0) \) for \( X \in (r_8, Q) \).

These six line segments form a hexagon in \( \mathbb{TP}^2 \). The vertices of that hexagon are the images of the six distinct real numbers \( r_i \) in (24) under the map (26).

Finally, we examine the special values \( x \in K^* \) for which the naive tropicalization (26) does not compute the correct image in \( \mathbb{TP}^2 \). This happens precisely when some of the nine theta functions in (16) have a valuation gap when passing to (26).

Suppose, for instance, that \( V(x) = V(p_1) = V(p_9) \). From (22) we have

\[
(27) \quad \text{val}(p_1(x)) - \text{val}(p_9(x)) = \begin{cases} \delta(x, p_1) & \text{if } i = 1 \text{ or } i = 9, \\ 0 & \text{if } i \neq 1, 9. \end{cases}
\]

We know from (23) that the minimum of \( \delta(p_1, p_9), \delta(x, p_1), \delta(x, p_9) \) is achieved twice. Moreover, by varying the choice of the scalar \( x \) with \( V(x) = r_1 \), the latter two quantities can attain any non-negative value that is compatible with this constraint.
This shows that the image of the set of such \( x \) under the tropicalization of the map \( (16) \) consists of one bounded segment and two rays in \( \mathbb{T} \mathbb{P}^2 \). The segment meets the hexagon described above at the vertex corresponding to \( r_1 = r_9 \), and consists of the images of the points \( x \) such that \( \delta(p_1, p_9) \geq \delta(p_1, x) = \delta(p_9, x) \). Since \( \Theta_{p_1} \) and \( \Theta_{p_9} \) occur in the first and third coordinates, respectively, of \( (16) \), the slope of the segment is \((1 : 0 : 1)\), and its length is \( \delta(p_1, p_9) \). Similarly, the image of the points \( x \) such that \( \delta(p_1, x) \geq \delta(p_1, p_9) = \delta(p_9, x) \) is a ray of slope \((1 : 0 : 0)\), and the image of the points \( x \) such that \( \delta(p_9, x) \geq \delta(p_1, p_9) = \delta(p_1, x) \) is a ray of slope \((0 : 0 : 1)\).

Note that these three slopes obey the balancing condition for tropical curves.

A similar analysis determines all six connected components of the complement of the hexagon in the tropical curve, and we see that the tropical curve is a honeycomb when the asserted conditions on \( a, b, c, p_1, p_2, \ldots, p_9 \) are satisfied.

The derivation of the converse direction, that any honeycomb cubic has the desired parametrization, will be deferred to the second proof. It seems challenging to prove this without \( [2] \). See also the problem stated at the end of this section. □

**Second proof of Theorem [4]** We work in the setting of Berkovich curves introduced by Baker, Payne and Rabinoff in \( [2] \). Let \( E^{an} \) denote the analytification of the elliptic curve \( E \), and let \( \Sigma \) denote the minimal skeleton of \( E^{an} \), as defined in \( [2] \, \S 5.14 \), with respect to the given set \( D = \{ p_1, p_2, \ldots, p_9 \} \) of nine punctures. Our standing assumption \( \text{val}(j) < 0 \) ensures that the Berkovich curve \( E^{an} \) contains a unique cycle \( S^1 \), and \( \Sigma \) is obtained from that cycle by attaching trees with nine leaves in total. In close analogy to \( [2] \, \S 7.1 \), we consider the retraction map onto \( S^1 \):

\[
E(K) \setminus D \hookrightarrow E^{an} \setminus D \to \Sigma \to \mathbb{R}/\text{val}(q)\mathbb{Z} \simeq S^1.
\]

The condition in Theorem [4] states that, under this map, the points of \( E(K) \) given by \( p_1, p_2, \ldots, p_9 \) retract to six distinct points on \( S^1 \), in cyclic order with fibers \( \{ p_9, p_1 \}, \{ p_2 \}, \{ p_3, p_4 \}, \{ p_5 \}, \{ p_6, p_7 \}, \{ p_8 \} \). This means that \( \Sigma \) looks precisely like the graph in Figure [2]. This picture is the Berkovich model of a honeycomb cubic. To
We do this by picking any $R$ of the logarithms of their norms: $F = -\log|f|$ and $G = -\log|g|$. Our tropical curve can be identified with its image in $\mathbb{R}^2$ under the map $(F,G)$. According to part (1) of [2] Theorem 5.69, this map factors through the retraction of $\mathcal{E}_\mathcal{C}/\mathcal{D}$ onto $\Sigma$. By part (2), the function $(F,G)$ is linear on each edge of $\Sigma$.

We shall argue that the graph $\Sigma$ in Figure 2 is mapped isometrically onto a tropical honeycomb curve in $\mathbb{R}^2$. Using part (5) of [2] Theorem 5.69, we can determine the slopes of the nine unbounded edges. Namely, $F$ has slope 1 on the rays in $\Sigma$ towards $p_7$, $p_8$ and $p_9$. Similarly, $G$ has slope 1 on the rays in $\Sigma$ towards $p_4$, $p_5$ and $p_6$, and slope $-1$ on the rays towards $p_7$, $p_8$ and $p_9$. By part (4), the functions $F$ and $G$ are harmonic, which means that the image in $\mathbb{R}^2$ satisfies the balancing condition of tropical geometry. This requirement uniquely determines the slopes of the nine bounded edges in the image of $\Sigma$. For the three edges not on the cycle this is immediate, and for the six edges on the cycle, this follows by solving a linear system of equations. The unique solution to these constraints is a balanced planar graph that must be a honeycomb cubic. Conversely, every tropical honeycomb cubic in $\mathbb{R}^2$ is trivalent with all multiplicities one. By [2] Corollary 6.27(1), the skeleton $\Sigma$ must look like Figure 2 and the corresponding map $(F,G)$ is an isometry onto the cubic. □

In the rest of Section 3, we discuss computational aspects of the representation of plane cubics given in Lemma 5 and Theorem 7. We begin with the implicitization problem. Given $a, b, c, p_1, \ldots, p_9 \in K^*$, how can we compute the implicit equation \((1)\)? Write $(f(x) : g(x) : h(x))$ for the analytic parametrization in (16). Then we seek to compute the unique (up to scaling) coefficients $c_{ijk}$ in a $K$-linear relation

\[
c_{300}f(x)^3 + c_{210}f(x)^2g(x) + c_{120}f(x)g(x)^2 + c_{030}g(x)^3 + c_{021}g(x)^2h(x) + c_{012}g(x)h(x)^2 + c_{003}h(x)^3 + c_{201}f(x)h(x)^2 + c_{111}f(x)g(x)h(x) = 0.
\]

Evaluating this relation at $x = p_i$, and noting that $\Theta_{p_i}(p_i) = 0$, we get nine linear equations for the nine $c_{ijk}$'s other than $c_{111}$. These equations are

\[
c_{300}f(p_i)^3 + c_{210}f(p_i)^2g(p_i) + c_{120}f(p_i)g(p_i)^2 + c_{030}g(p_i)^3 = 0 \quad \text{for } i = 7, 8, 9,
\]
\[
c_{003}h(p_i)^3 + c_{021}g(p_i)^2h(p_i) + c_{012}g(p_i)h(p_i)^2 + c_{003}h(p_i)^3 = 0 \quad \text{for } i = 4, 5, 6,
\]
\[
c_{030}g(p_i)^3 + c_{021}g(p_i)^2h(p_i) + c_{012}g(p_i)h(p_i)^2 + c_{003}h(p_i)^3 = 0 \quad \text{for } i = 1, 2, 3.
\]

The first group of equations has a solution $(c_{300}, c_{210}, c_{120}, c_{030})$ that is unique up to scaling. Namely, the ratios $c_{300}/c_{030}, c_{210}/c_{030}, c_{120}/c_{030}$ are the elementary symmetry functions in the three quantities

\[
\frac{b \cdot \Theta(p_4)}{a \cdot \Theta(p_1)} \frac{\Theta(p_5)}{\Theta(p_2)} \frac{\Theta(p_6)}{\Theta(p_3)}, \quad \frac{b \cdot \Theta(p_4)^2}{a \cdot \Theta(p_1)^2} \frac{\Theta(p_5)}{\Theta(p_2)} \frac{\Theta(p_6)}{\Theta(p_3)} \quad \text{and} \quad \frac{b \cdot \Theta(p_4)}{a \cdot \Theta(p_1)} \frac{\Theta(p_5)^2}{\Theta(p_2)^2} \frac{\Theta(p_6)}{\Theta(p_3)}.
\]

The analogous statements hold for the second and third group of equations.

We are thus left with computing the middle coefficient $c_{111}$ in the relation (28). We do this by picking any $v \in K$ with $f(v)g(v)h(v) \neq 0$. Then (28) gives

\[
c_{111} = -\frac{1}{f(v)g(v)h(v)}(c_{300}f(v)^3 + c_{210}f(v)^2g(v) + \cdots + c_{201}f(v)^2h(v)).
\]
Figure 3. A honeycomb cubic and its nine inflection points in groups of three

We have implemented this implicitization method in Mathematica, for input data in the field $K = \mathbb{Q}(t)$ of rational functions with rational coefficients.

The parameterization problem is harder. Here we are given the 10 coefficients $c_{ijk}$ of a honeycomb cubic that has three distinct intersection points with each coordinate line $z_0 = 0, z_1 = 0, z_2 = 0$. The task is to compute 12 scalars $a, b, c, p_1, \ldots, p_9 \in K^*$ that represent the cubic as in Lemma 5. The 12 output scalars are not unique, but the degree of non-uniqueness is characterized exactly by Proposition 6. This task amounts to solving an analytic system of equations. We shall leave it to a future project to design an algorithm for doing this in practice.

4. The Tropical Group Law

In this section, we present a combinatorial description of the group law on a honeycomb elliptic curve based on the parametric representation in Section 3. We start by studying the inflection points of such a curve. We continue to assume that $K$ is algebraically closed and complete with respect to a nonarchimedean valuation.

Let $E \rightarrow C \subset \mathbb{P}^2$ be a honeycomb embedding of the abstract elliptic curve $E = K^*/q\mathbb{Z}$. Let $v_1, v_2, v_3, v_4, v_5$ and $v_6$ denote the vertices of the hexagon in the tropical cubic trop$(C)$, labeled as in Figure 3. Let $e_i$ denote the edge between $v_i$ and $v_{i+1}$, with the convention $v_7 = v_1$, and let $\ell_i$ denote the lattice length of $e_i$. By examining the width and height of the hexagon, we see that the six lattice lengths $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$ satisfy two linearly independent relations:

$$\ell_1 + \ell_2 = \ell_4 + \ell_5 \quad \text{and} \quad \ell_2 + \ell_3 = \ell_5 + \ell_6.$$ 

We first prove the following basic fact about the inflection points on the cubic $C$.

**Lemma 8.** The tropicalizations of the nine inflection points of the cubic curve $C \subset \mathbb{P}^2$ retract to the hexagonal cycle of trop$(C) \subset \mathbb{R}^2$ in three groups of three.

**Proof.** This lemma is best understood from the perspective of Berkovich theory. The analytification $E^{an}$ retracts onto its skeleton, namely the unique cycle, which is isometrically embedded into trop$(C)$ as the hexagon. Thus every point of $E$ retracts onto a unique point in the hexagon. In fact, this retraction is given by

$$K^*/q\mathbb{Z} \cong E(K) \hookrightarrow E^{an} \twoheadrightarrow S^1 \cong \mathbb{R}/\text{val}(q)\mathbb{Z}$$

(29)
and is the natural map induced from the valuation homomorphism \((K^*, \cdot) \to (\mathbb{R}, +)\). We refer the reader unfamiliar with the map \((29)\) to [2] Theorem 7.2.

Now, after a multiplicative translation, we may assume that \(\psi\) sends the identity of \(E\) to an inflection point. Then the inflection points of \(C\) are the images of the 3-torsion points of \(E = K^*/q\mathbb{Z}\). These can be written as \(\omega^i \cdot q^{j/3}\) for \(\omega^3 = 1\), for \(q^{1/3}\) a cube root of \(q\), and for \(i, j = 0, 1, 2\). Note that \(\text{val}(\omega) = 0\) whereas \(\text{val}(q^{1/3}) = \text{val}(q)/3 > 0\). Hence the valuations of the scalars \(\omega^i \cdot q^{j/3}\) are 0, \(\text{val}(q)/3\), and \(2\text{val}(q)/3\), and each group contains three of these nine scalars. \(\square\)

Our next result refines Lemma 8. It is a very special case of a theorem due to Brugallé and de Medrano [3] which covers honeycomb curves of arbitrary degree.

**Lemma 9.** Let \(P \in \text{trop}(C)\) be the tropicalization of an inflection point on the cubic curve \(C \subset \mathbb{P}^2\). Then there are three possibilities, as indicated in Figure 3:

- The point \(P\) lies on the longer of \(e_1\) or \(e_2\), at distance \(|\ell_2 - \ell_1|/3\) from \(v_2\).
- The point \(P\) lies on the longer of \(e_3\) or \(e_4\), at distance \(|\ell_4 - \ell_3|/3\) from \(v_4\).
- The point \(P\) lies on the longer of \(e_5\) or \(e_6\), at distance \(|\ell_6 - \ell_5|/3\) from \(v_6\).

The nine inflection points fall into three groups of three in this way.

In the special case that \(\ell_1 = \ell_2\) (and similarly \(\ell_3 = \ell_4\) or \(\ell_5 = \ell_6\)), the lemma should be understood as saying that \(P\) lies somewhere on the ray emanating from \(v_2\).

**Proof.** Consider the tropical line whose node lies at \(v_2\), and let \(L\) be any classical line in \(\mathbb{P}^2_K\) that is generic among lifts of that tropical line. Then the three points in \(L \cap C\) tropicalize to the stable intersection points of trop(\(L\)) with trop(\(C\)), namely the vertices \(v_1, v_2, v_3\), and \(v_i\) denote the clockwise distance along the hexagon from \(v_2\) to \(P\). By applying a multiplicative translation in \(E\), we fix the identity to be an inflection point on \(C\) that tropicalizes to \(P\). Then \(v_1 + v_2 + v_3 = 0\) in the group \(S^1 \simeq \mathbb{R}/\text{val}(q)\mathbb{Z}\). This observation implies the congruence relation

\[(\ell_1 + x) + x + (x - \ell_2) \equiv 0 \mod \text{val}(q),\]

and hence \(3x \equiv \ell_2 - \ell_1\). One solution to this congruence is \(x = (\ell_2 - \ell_1)/3\). This means that the point \(P\) lies on the longer edge, \(e_1\) or \(e_2\), at distance \(|\ell_1 - \ell_2|/3\) from \(v_2\). The analysis is identical for the vertex \(v_4\) and for the vertex \(v_6\), and it identifies two other locations on \(S^1\) for retractions of inflection points. \(\square\)

Next, we wish to review some basic facts about the group structures on a plane cubic curve \(C\). The abstract elliptic curve \(E = K^*/q\mathbb{Z}\) is an abelian group, in the obvious sense, as a quotient of the abelian group \((K^*, \cdot)\). Knowledge of that group structure is equivalent to knowing the surface \(\{(r, s, t) \in E^3 : r \cdot s \cdot t = q^2\}\).

The group structure on \(E\) induces a group structure \((C, \ast)\) on the plane cubic curve \(C = \psi(E)\). While the isomorphism \(\psi\) is analytic, the group operation on \(C\) is actually algebraic. Equivalently, the set \(\{(u, v, w) \in C^3 : u \ast v \ast w = \text{id}\}\) is an algebraic surface in \((\mathbb{P}^2)^3\). However, this surface depends heavily on the choice of parametrization \(\psi\). Different isomorphisms from the abstract elliptic curve \(E\) onto the plane cubic \(C\) will result in different group structures in \(\mathbb{P}^2\).

The most convenient choices of \(\psi\) are those that send the identity element \(q\mathbb{Z}\) of \(E\) to one of the nine inflection points of \(C\). Such maps \(\psi\) are characterized by the condition that \(u \ast v \ast w = \text{id}\) if and only if \(u, v, w\) are collinear in \(\mathbb{P}^2\). If so, the data of the group law can be recorded in the surface

\[(30) \quad \{(u, v, w) \in C^3 : u, v \text{ and } w \text{ lie on a line in } \mathbb{P}^2\} \subset (\mathbb{P}^2)^3.\]
We emphasize, however, that we have no reason to assume a priori that the identity element $id = \psi(qZ)$ on the plane cubic $C$ is an inflection point: every point on $C$ is the identity in some group law. Indeed, we can simply replace $\psi$ by its composition with a translation $x \mapsto r \cdot x$ of the group $E$. Our analysis below covers all cases: our combinatorial description of the group law on $trop(C)$ is always valid, regardless of whether the identity is an inflection point or not.

A partial description of the tropical group law was given by Vigeland in [19]. Specifically, his choice of the fixed point $O$ corresponds to the point $trop(\psi(qZ))$. Vigeland’s group law is a combinatorial extension of the polyhedral surface $(31)$

$$\{(U,V,W) \in S^1 \times S^1 \times S^1 : U + V + W = O\}$$

This torus comes with a distinguished subdivision into polygons since $S^1$ has been identified with the hexagon. The polyhedral torus (31) is illustrated in Figure 4.

Our goal is to characterize the tropical group law as follows. For a given honeycomb embedding $\psi : E \to \mathbb{P}^2$, we define the tropical group law surface to be $TGL(\psi) = \{(\text{trop}(\psi(x)), \text{trop}(\psi(y)), \text{trop}(\psi(z))) : x, y, z \in E, x \cdot y \cdot z = \text{id}\} \subset (\mathbb{R}^2)^3$. If $\psi$ sends the identity to an inflection point, then this is the tropicalization of (30):

$$TGL(\psi) = \{(\text{trop}(u), \text{trop}(v), \text{trop}(w)) : u, v, w \in C \text{ lie on a line in } \mathbb{P}^2\} \subset (\mathbb{R}^2)^3.$$ 

The tropical group law surface is a tropical algebraic variety of dimension 2, and can in principle be computed, for $K = \mathbb{Q}(t)$, using the software $gfan$ [7]. This surface contains all the information of the tropical group law. We shall explain how $TGL(\psi)$ can be computed combinatorially, even if $\psi(id)$ is not an inflection point.

Our approach follows directly from Section 3. Let $\psi : E \to \mathbb{P}^2$ be a honeycomb embedding of an elliptic curve $E$, and let $V : K^* \to S^1$ denote the composition $\nu_{\text{val}} : \mathbb{R} \to \mathbb{R}/\text{val}(\varphi) \simeq S^1$. By Theorem 7 there exist $a, b, c, p_1, \ldots, p_9 \in K^*$ with $\nu(p_1), \ldots, \nu(p_9)$ occurring in cyclic order on $S^1$, with $\nu(p_3) = \nu(p_4), \nu(p_5) = \nu(p_7), \nu(p_8) = \nu(p_1)$, and all other values $\nu(p_i)$ distinct, such that $\psi$ is given by

$$x \mapsto (a \cdot \Theta_{p_1} \Theta_{p_2} \Theta_{p_3}(x) : b \cdot \Theta_{p_4} \Theta_{p_5} \Theta_{p_6}(x) : c \cdot \Theta_{p_7} \Theta_{p_8} \Theta_{p_9}(x)).$$
where \( i \in \mathbb{Z} \) is specified by \( \text{val}(x) + \text{val}(y) = \text{val}(y) \). We have seen that if \( x \in K^* \) satisfies \( \mathcal{V}(x) = \mathcal{V}(p_i) \), then trop(\( x \)) lies at distance \( \delta(x, p_i) \) from the hexagon along the tentacle associated to \( p_i \). By the tentacle of \( p_i \) we mean the union of the ray associated to \( p_i \) and the bounded segment to which that ray is attached.

The next proposition is our main result in Section 4. We shall construct the tropical group law surface by way of its projection to the first two coordinates

\[
\pi : \text{TGL}(\psi) \to \text{trop}(C) \times \text{trop}(C) \subset (\mathbb{R}^2)^2.
\]

As before, we identify the hexagon of trop(\( C \)) with the circle \( S^1 \simeq \mathbb{R}/\text{val}(q) \). Given \( U, V \in \text{trop}(C) \), let \( U \circ V \in S^1 \) denote the sum of the retractions of \( U \) and \( V \) to the hexagon. The location of \( U \circ V \) depends on the choice of \( \psi \) and in particular the location of \( O = \text{trop}(\psi(1)) \) on the hexagon \( S^1 \). Let \((U \circ V)^{-1} \) denote the inverse of \( U \circ V \), again under addition on \( S^1 \). By the distance of a point \( U \in \text{trop}(C) \) to the hexagon \( S^1 \) we mean the lattice length of the unique path in trop(\( C \)) from \( U \) to \( S^1 \). Finally, we say that \( u \in K^* \) is a lift of \( U \in \text{trop}(C) \) if \( \text{trop}(\psi(u \cdot q^i)) = U \). The following proposition characterizes the fiber of the map \( \pi \) over a given pair \((U, V)\).

**Proposition 10.** Let \( \psi : E \to C \subset \mathbb{P}^2 \) be a honeycomb embedding, with the operation \( \circ : \text{trop}(C) \times \text{trop}(C) \to S^1 \) defined as above, and “is a vertex” refers to the hexagon \( S^1 \). For any \( U \) and \( V \in \text{trop}(C) \), exactly one of the following occurs:

\begin{enumerate}
  \item If \( -(U \circ V) \) is not a vertex, then \( \pi^{-1}(U, V) \) is the singleton \( \{-(U \circ V)\} \).
  \item If \( -(U \circ V) \) is a vertex adjacent to a single unbounded ray \( R_i \) towards the point \( p_i \), then \( \pi^{-1}(U, V) \) is the set of points on \( R_i \) whose distance to \( S^1 \) equals \( \delta(u^{-1}v^{-1}, p_i) \) for some lifts \( u, v \in K^* \) of \( U \) and \( V \).
  \item If \( -(U \circ V) \) is a vertex adjacent to a bounded segment \( B \), along with two rays \( R_j \) and \( R_k \), toward the points \( p_j \) and \( p_k \), then \( \pi^{-1}(U, V) \) consists of
    \begin{itemize}
      \item the points on \( B \) whose distance to \( S^1 \) is equal to \( \delta(u^{-1}v^{-1}, p_j) = \delta(u^{-1}v^{-1}, p_k) \) for some lifts \( u, v \in K^* \) of \( U \) and \( V \),
      \item the points on \( R_j \) whose distance to \( S^1 \) is equal to \( \delta(u^{-1}v^{-1}, p_j) > \delta(u^{-1}v^{-1}, p_k) \) for some lifts \( u, v \in K^* \) of \( U \) and \( V \), and
      \item the points on \( R_k \) whose distance to \( S^1 \) is equal to \( \delta(u^{-1}v^{-1}, p_j) > \delta(u^{-1}v^{-1}, p_k) \) for some lifts \( u, v \in K^* \) of \( U \) and \( V \).
    \end{itemize}
\end{enumerate}

**Proof.** For any lifts \( u, v \in K^* \) of \( U, V \), we have \( \mathcal{V}(u^{-1}v^{-1}) + \mathcal{V}(u) + \mathcal{V}(v) = 0 \) in \( S^1 = \mathbb{R}/\text{val}(q) \mathbb{Z} \). This equation determines the retraction \( -(U \circ V) \) of the point trop(\( \psi(u^{-1}v^{-1}) \)) to the hexagon. If \( -(U \circ V) \) is not a vertex, then trop(\( \psi(u^{-1}v^{-1}) \)) must be precisely the point \( -(U \circ V) \), as no other points retract to it.

If instead \( -(U \circ V) \) is a vertex of the hexagon, then we either have \( \mathcal{V}(u^{-1}v^{-1}) = \mathcal{V}(p_i) \) for exactly one \( p_i \), or \( \mathcal{V}(u^{-1}v^{-1}) = \mathcal{V}(p_j) \) or \( \mathcal{V}(p_k) \) for exactly two points \( p_j \) and \( p_k \), depending on whether one or two rays emanate from \( -(U \circ V) \). We have seen (in our first proof of Theorem 10) that the distance of \( \text{trop}(\psi(u^{-1}v^{-1})) \) to the hexagon, measured along the tentacle associated to \( p_i \), is \( \delta(u^{-1}v^{-1}, p_i) \). In either case, these distances uniquely determine the location of \( \text{trop}(\psi(u^{-1}v^{-1})) \). \( \square \)
We demonstrate this method in a special example which was found in discussion with Spencer Backman. Pick \( r, s \in K^* \) with \( r^6 = q \) and \( \text{val}(1 - s) := \beta > 0 \). Let
\[
\begin{align*}
p_1 &= r^{-1} s^{-1}, & p_2 &= 1, & p_3 &= rs, \\
p_4 &= rs^{-1}, & p_5 &= r^2, & p_6 &= r^{-3} s, \\
p_7 &= r^{-3} s^{-1}, & p_8 &= r^{-2}, & p_9 &= r^{-1} s.
\end{align*}
\]
We also set \( a = b = c = 1 \) in (10). This choice produces a symmetric honeycomb embedding \( \psi : E \to C \subset \mathbb{P}^2 \). The parameter \( \beta \) is the common length of the three bounded segments adjacent to \( S^1 \). Note that \( p_1 p_2 p_3 = p_4 p_5 p_6 = p_7 p_8 p_9 = 1 \) in \( K^* \).

This condition implies that the identity of \( E \) is mapped to an inflection point in \( C \).

**Corollary 11.** For the elliptic curve in honeycomb form defined by (32), the tropical group law \( \text{TGL}(\psi) \) is a polyhedral surface consisting of 117 vertices, 279 bounded edges, 315 rays, 54 squares, 108 triangles, 279 flaps, and 171 quadrants.

Here, a “flap” is a product of a bounded edge and a ray, and a “quadrant” is a product of two rays. Note that the Euler characteristic is \( 117 - 279 + 54 + 108 = 0 \). This is consistent with the fact that the Berkovich skeleton of \((E \times E)^{an}\) is a torus.

**Proof.** Let \( X = \text{trop}(\psi(E)) \) and \( \pi : \text{TGL}(\psi) \to X \times X \) as in Proposition 10. We modify the tropical surface \( X \times X \) by attaching the fiber \( \pi^{-1}(U, V) \) to each point \((U, V) \in X \times X \). These modifications change the polyhedral structure of \( X \times X \). For example, the torus \( S^1 \times S^1 \) in \( X \times X \) consists of 36 squares, but the modifications subdivide each square into two triangles as in Figure 4 on the right.

We give three examples of explicit computations of fibers \( \pi^{-1}(U, V) \) but omit the full analysis. For convenience, say that a point \( U \in X \) prefers \( p_1 \) if it lies on the infinite ray associated to \( p_1 \). For our first example, suppose \( U \) prefers \( p_3 \) and \( V \) prefers \( p_6 \), and suppose \( u, v \in K^* \) are any lifts of \( U \) and \( V \). We set \( \rho = u/p_3 \) and \( \sigma = v/p_6 \). These two scalars in \( K^* \) satisfy \( u^{-1} v^{-1} = p_5 s^{-2} \rho^{-1} \sigma^{-1} \) and
\[
\text{val}(1 - \rho) = \delta(p_3, u) > \beta \quad \text{and} \quad \text{val}(1 - \sigma) = \delta(p_6, v) > \beta.
\]
It is a general fact that \( \text{val}(1 - xy) = \min\{\text{val}(1 - x), \text{val}(1 - y)\} \) if \( x, y \in K^* \) have valuation 0 and \( \text{val}(1 - x) \neq \text{val}(1 - y) \). Combining this fact with (33), we find
\[
\delta(u^{-1} v^{-1}, p_5) = \text{val}(1 - s^2 \rho \sigma) = \text{val}(1 - s^2) = \text{val}(1 - s) = \beta.
\]
We conclude that \( \text{trop}(\psi(u^{-1} v^{-1})) \) prefers \( p_5 \) and is at lattice distance \( \beta \) from the hexagon. Thus we do not modify \( X \times X \) above \((U, V) \) since the fiber is a single point.

As a second example, suppose \( U \) prefers \( p_1 \) and \( V \) prefers \( p_3 \). If \( u, v \in K^* \) are lifts of \( U \) and \( V \) then \( \text{trop}(\psi(u^{-1} v^{-1})) \) prefers \( p_2 \). Direct computation shows that the minimum of \( \delta(u, p_1), \delta(v, p_3), \delta(u^{-1} v^{-1}, p_2) \) is achieved twice, and this condition characterizes the possibilities for \( \text{trop}(\psi(u^{-1} v^{-1})) \). For example, if \( \delta(u, p_1) = \delta(v, p_3) = d \), then \( \text{trop}(\psi(u^{-1} v^{-1})) \) can be any point that prefers \( p_2 \) and is distance at least \( d \) from the hexagon. Thus, we modify \( X \times X \) at \((U, V) \) by attaching a ray representing the points on the ray of \( p_2 \) at distance \( \geq d \) from the hexagon.

Our third example is similar, but we display it visually. Let \( U, V \) be the points shown in blue in Figure 3 each at distance \( 2 \beta \) from the hexagon. Then \( \pi^{-1}(U, V) \) is the thick blue subray that starts at distance \( \beta \) from the node of the tropical line.

In this way, we modify the surface \( X \times X \) at each point \((U, V) \) as prescribed by Proposition 10. A detailed case analysis yields the \( f \)-vector in Corollary 11. \( \square \)
We note that the combinatorics of the tropical group law surface $\text{TGL}(\psi)$ depends very much on $\psi$. For example, if $\psi$ is a non-symmetric honeycomb embedding, then the torus in $\text{TGL}(\psi)$ can contain quadrilaterals and pentagons, as shown in Figure 4. For this reason, it seems that there is no “generic” combinatorial description of the surface $\text{TGL}(\psi)$ as $\psi$ ranges over all honeycomb embeddings.

**Acknowledgments**

This project grew out of discussions we had with Spencer Backman and Matt Baker. We are grateful for their contributions and help with the analytic theory of elliptic curves. MC was supported by a Graduate Research Fellowship from the National Science Foundation. BS was supported in part by the National Science Foundation (DMS-0968882) and the DARPA Deep Learning program (FA8650-10-C-7020).

**References**

[1] M. Artebani and I. Dolgachev: The Hesse pencil of plane cubic curves, *Enseign. Math.* (2) **55** (2009) 235–273.
[2] M. Baker, S. Payne and J. Rabinoff: Nonarchimedean geometry, tropicalization, and metrics on curves. [arXiv:1104.0320](http://arxiv.org/abs/1104.0320).
[3] E. Brugallé and L. L. de Medrano: Inflection points of real and tropical plane curves [arXiv:1102.2478](http://arxiv.org/abs/1102.2478).
[4] A. Buchholz: *Tropicalization of Linear Isomorphisms on Plane Elliptic Curves*, Diplomarbeit, University of Göttingen, Germany, April 2010.
[5] R. Hartshorne: *Algebraic Geometry*, Springer Verlag, New York, 2006.
[6] P.A. Helminck: *Tropical Elliptic Curves and $j$-Invariants*, Bachelor’s Thesis, University of Groningen, The Netherlands, August 2011.
[7] A. N. Jensen: Gfan, a software system for Gröbner fans and tropical varieties, available at [http://home.imf.au.dk/jensen/software/gfan/gfan.html](http://home.imf.au.dk/jensen/software/gfan/gfan.html).
[8] C. Jordan: Mémoire sur les équations différentielles linéaires à intégrale algébrique, *Journal für Reine und Angew. Math.* **84** (1877) 89–215.
[9] E. Katz, H. Markwig and T. Markwig: The j-invariant of a plane tropical cubic, *J. Algebra* **320** (2008) 3832–3848.
[10] D. Maclagan and B. Sturmfels: Introduction to Tropical Geometry, book manuscript, 2010, available at [http://www.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.pdf](http://www.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.pdf).
[11] G. Mikhalkin and I. Zharkov: Tropical curves, their Jacobians and theta functions, *Contemporary Mathematics* **465** (2007) 203–231.
[12] A. Nobe: The group law on the tropical Hesse pencil, [arXiv:1111.0131](http://arxiv.org/abs/1111.0131).
[13] P. Roquette: *Analytic Theory of Elliptic Functions over Local Fields*, Hamburger Mathematische Einzelschriften, Heft 1, Vandenhoeck & Ruprecht, Göttingen, 1970.
[14] G. Salmon: *A Treatise on the Higher Plane Curves: intended as a sequel to “A Treatise on Conic Sections”*, 3rd ed., Dublin, 1879; reprinted by Chelsea Publ. Co., New York, 1960.
[15] J. Silverman: *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, **106**, Springer Verlag, New York, second edition, 2009.
[16] J. Silverman: *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, **151**, Springer Verlag, New York, 1994.
[17] D. Speyer: Horn’s problem, Vinnikov curves, and the hive cone, *Duke Math. J.* **127** (2005) 395–427.
[18] D. Speyer: Uniformizing tropical curves: genus zero and one, [arXiv:0711.2677](http://arxiv.org/abs/0711.2677).
[19] M. Vigeland: The group law on a tropical elliptic curve, *Math. Scand.* **104** (2009) 188–204.

Department of Mathematics, University of California, Berkeley, CA 94720, USA

E-mail address: mtchan@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720, USA

E-mail address: bernd@math.berkeley.edu