Twisted Fock module of toroidal algebra via DAHA and vertex operators

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Abstract

We construct the twisted Fock module of quantum toroidal $\mathfrak{gl}_1$ algebra with a slope $n'/n$ using vertex operators of quantum affine $\mathfrak{gl}_n$. The proof is based on the $q$-wedge construction of an integrable level-one $U_q(\mathfrak{gl}_n)$-module and the representation theory of double affine Hecke algebra. The results are consistent with Gorsky-Negut conjecture (Kononov-Smirnov theorem) on stable integrable level-one $U$-vertex operators of quantum affine $\mathfrak{gl}_1$. We construct the twisted Fock module of quantum toroidal $\mathfrak{gl}_1$ algebra with a slope $n'/n$ using vertex operators of quantum affine $\mathfrak{gl}_n$. The proof is based on the $q$-wedge construction of an integrable level-one $U_q(\mathfrak{gl}_n)$-module and the representation theory of double affine Hecke algebra. The results are consistent with Gorsky-Negut conjecture (Kononov-Smirnov theorem) on stable integrable level-one $U$-vertex operators of quantum affine $\mathfrak{gl}_1$.
The space $\Lambda = \mathbb{C}[p_1, p_2, \ldots]$ is the space of polynomials in infinite number of variables (power sums). This space has a natural action of Heisenberg algebra with generators $a_k$, $k \in \mathbb{Z}_{\neq 0}$ and relations $[a_k, a_l] = k\delta_{k+l, 0}$. Namely, operators $a_{-k}$ act as multiplication by $p_k$ and $a_k$ act as $k\partial_{p_k}$, for $k > 0$. The action of $U_{q_1, q_2}(\mathfrak{gl}_1)$ on $\Lambda$ can be written via Heisenberg generators $a_k$, this construction is called bosonization. The formulas are written for the generators $P_{0,b}$, $P_{0,k}$, $P_{-1,b}$ for $b \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\neq 0}$, these generators are called Chevalley generators (contrary to all $P_{a,b}$ for $(a,b) \in \mathbb{Z}^2 \backslash \{(0,0)\}$, which could be called PBW generators). The formulas for the bosonization roughly look like

$$P_{0,k} = \# a_k, \quad \sum P_{1,b}z^{-b} = u^{-1}\# \exp \left( \sum \# a_{-k}z^k \right), \quad \sum P_{-1,b}z^{-b} = u\# \exp \left( \sum \# a_{-k}z^k \right), \quad (1.3)$$

see Section 5.3 for the precise formulas. The representation is called Fock module. It depends on the parameter $u$, we denote the representation by $\mathcal{F}_u$. The formula (1.3) is equivalent to the original construction in [FHH 09].

The algebra $U_{q_1, q_2}(\mathfrak{gl}_1)$ can be considered as a quantum group, it has topological coproduct as in Drinfeld current realization [DI97]. Using this coproduct we can construct $U_{q_1, q_2}(\mathfrak{gl}_1)$-representations of the form $\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n}$. It was shown in [FHS 10], [Neg18] that the image of (completed) toroidal algebra $U_{q_1, q_2}(\mathfrak{gl}_1)$ in the endomorphisms of this tensor product is the $q$-$W$-algebra for $\mathfrak{gl}_q$. Moreover, the algebra $U_{q_1, q_2}(\mathfrak{gl}_1)$ can be viewed as a central extension of the limit of such $W$-algebras [1] this was one of the first motivations to study $U_{q_1, q_2}(\mathfrak{gl}_1)$, see [Mik07].

**Stable bases and duality** The algebra $U_{q_1, q_2}(\mathfrak{gl}_1)$ plays an important role in geometric representation theory. Let $\text{Hilb}_k$ denotes Hilbert scheme of $k$ points in $\mathbb{C}^2$. There is a natural action of the torus $T = \mathbb{C}^{*}_{q_1} \times \mathbb{C}^{*}_{q_2}$ on $\text{Hilb}_k$. Let $K_T(\text{Hilb}_k)_{\text{loc}}$ denotes localized equivariant $K$-theory of $\text{Hilb}_k$. There is an action of $U_{q_1, q_2}(\mathfrak{gl}_1)$ on the sum $K = \bigoplus_{k=0}^{\infty} K_T(\text{Hilb}_k)_{\text{loc}}$ via correspondences. The obtained representation is isomorphic to the Fock module $\mathcal{F}_u$ [FTT11, SV13].

The torus fixed points of $\text{Hilb}_k$ are labeled by the partitions $\lambda$ with $|\lambda| = k$. Denote the corresponding point by $p_{\lambda} \in \text{Hilb}_k$. Due to localization theorem, there is a basis $\{p_{\lambda}\}$ in $K$. This basis corresponds to the Macdonald symmetric functions under the identification $K \cong \mathcal{F}_u \cong \Lambda$.

There is another remarkable basis in the space $K$, namely, $K$-theoretic stable envelope basis $\{s_{\lambda}^\omega\}$, see [Oko17]. The stable envelope $s_{\lambda}^\omega$ depends on the slope parameter $\omega \in \mathbb{R} \backslash \mathbb{Q}$. For fixed $k$, there is a chamber structure, namely, $s_{\lambda}^\omega \in K_T(\text{Hilb}_k)_{\text{loc}}$ is a piecewise constant function on $\omega$ with discontinuities at rational numbers $m/n$ with $n \leq k$. So for any rational number $m/n$ we can define elements $s_{\lambda}^{m/n+\epsilon}$ and $s_{\lambda}^{m/n-\epsilon}$ for sufficiently small $\epsilon$.

In the paper [GN17], Gorsky and Neguț conjectured that for coprime integer $m,n$ there is an identification between $\mathcal{F}_u$ and integrable level-one representation of quantum affine algebra $U_u(\mathfrak{gl}_n)$ such that bases $\{s_{\lambda}^{m/n}\}$ and $\{s_{\lambda}^{m/n-\epsilon}\}$ correspond to the standard and costandard bases respectively. This conjecture was supported by the consistency of the action of operators $P_{km,kn} \in U_{q_1, q_2}(\mathfrak{gl}_1)$ on stable basis [Neg16] and diagonal $v$-Heisenberg subalgebra in $U_u(\mathfrak{gl}_n)$ on standard basis [LT00].

This conjecture was proven by Kononov and Smirnov in [KS20] using geometric methods, in particular 3d-mirror symmetry. Algebraically, this relation could be viewed as a manifestation of $(\mathfrak{gl}_1, \mathfrak{gl}_n)$-duality in toroidal setting. The relation between 3d-mirror symmetry and $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-duality (in affine setting) appeared in [RSVZ19].

To state our main result, we need to reformulate the above “in the opposite way”. Let us start not from $U_{q_1, q_2}(\mathfrak{gl}_1)$ but from the integrable level-one representation of $U_u(\mathfrak{gl}_n)$. We construct an action of $U_{q_1, q_2}(\mathfrak{gl}_1)$ on this space such that the obtained representation is isomorphic to the Fock module

\footnote{The algebra $U_{q_1, q_2}(\mathfrak{gl}_1)$ is a central extension of the limit of $S\mathcal{H}^1_N$ for $N \to \infty$. The completed algebra $U_{q_1, q_2}(\mathfrak{gl}_1)$ is a central extension of the limit of $W$-algebras for $\mathfrak{gl}_1$ for $n \to \infty$. These statements are independent. E.g. non-completed $U_{q_1, q_2}(\mathfrak{gl}_1)$ with $c = v^{-1}$, $c' = 1$ acts faithfully on $\mathcal{F}_u$, though the image of the completion is isomorphic to (completed) Heisenberg algebra due to formula (1.3).}
twisted by \( \sigma \in \widetilde{SL}(2, \mathbb{Z}) \), here \( \sigma(m, n) = (0, 1) \). The action of Chevalley generators \( P_{1,b}, P_{-1,b} \) is given in terms of vertex operators for \( U_v(\mathfrak{gl}_n) \), and \( P_{0,k} \) are proportional to the generators of the diagonal \( v \)-Heisenberg subalgebra in \( U_v(\widehat{\mathfrak{gl}}_n) \).

**Our results**  The integrable level-one representation of \( U_v(\widehat{\mathfrak{gl}}_n) \) has an explicit construction in terms of semi-infinite \( v \)-wedges \cite{Ste95, KMS95}. We first consider the case of finite \( v \)-wedges and then take the limit. Let \( \mathbb{C}^n[Y^\pm] \) denotes the evaluation representation of \( U_v(\widehat{\mathfrak{gl}}_n) \), then its \( N \)-th tensor power can be written as

\[
(\mathbb{C}^n[Y^\pm])^\otimes N \simeq (\mathbb{C}^n)^\otimes N [Y_1^\pm, \ldots, Y_N^\pm].
\]

(1.4)

By the quantum affine version of Schur-Weyl duality, there is an action of the affine Hecke algebra \( H^Y \) with the generators \( T_i, Y_j \) on this space, moreover this action commutes with \( U_v(\widehat{\mathfrak{gl}}_n) \) (see \cite{GRV94, CP94} and references therein). In Theorem 3.1 we extend this action to the whole DAHA \( \mathcal{H}_N \). The construction depends on an integer number \( n_{tw} \). If \( n_{tw} \) is coprime to \( n \) then the obtained representation is isomorphic to Cherednik polynomial representation twisted by certain \( \sigma \in \widetilde{SL}(2, \mathbb{Z}) \), see Theorem 3.4. Below we will assume this coprimeness and denote \( n' = n_{tw} \).

Let \( S_- \) be antisymmetrizer in the (finite) Hecke algebra, \( \mathcal{S}_N^{-} = S_- \cdot \mathcal{H}_N S_- \) be the corresponding spherical DAHA. Above we discussed that the algebra \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \) surjects onto \( \mathcal{S}_N^{-} \). Analogously, \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \) surjects onto \( \mathcal{S}_N^{-} \) for \( q_1 = q, q_2 = v^{-2} \). Using this we obtain an action

\[
U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \to \mathcal{S}_N^{-} = \mathcal{S}_N^{-} (\mathbb{C}^n)^\otimes N [Y_1^\pm, \ldots, Y_N^\pm] = \mathcal{S}_N^{-} (\mathbb{C}^n[Y^\pm]).
\]

(1.5)

The space \( \mathbb{C}^n[Y^\pm] \) has a basis \( e_{ij+a} = Y^j e_a \) for \( a = 0, \ldots, n - 1 \) and \( j \in \mathbb{Z} \). Then the space \( \Lambda_N^N (\mathbb{C}^n[Y^\pm]) \) has a basis \( e_{k_1} \wedge \cdots \wedge e_{k_N} \) for \( k_1 < \cdots < k_N \). It appears that the action Chevalley generators \( P_{1,b}, P_{-1,b} \) on this space can be written in terms of vertex operators \( \Phi_k, \Psi_k, \Phi^*_k, \Psi^*_k \). The operators \( \Phi_k, \Psi_k \) are given by \( \Phi_k(w) = e_k \wedge w, \Psi_k(w) = w \wedge e_k \). The operators \( \Phi^*_k, \Psi^*_k \) are dual. Then the action of Chevalley generators is given by the formulas

\[
P_{1,b} \mapsto \sum_{k \in \mathbb{Z}} \#\Psi_{k+n'b} \Phi^*_{-k}, \quad P_{-1,b} \mapsto \sum_{k \in \mathbb{Z}} \#\Phi_{k-n'b} \Psi^*_{-k},
\]

(1.6)

see Section 6.1 for details. Let us remark that the operators \( \Phi_k, \Phi^*_k, \Psi_k, \Psi^*_k \) are components of natural intertwining operators

\[
\Phi : \mathbb{C}^n[Y^\pm] \otimes \Lambda_N^N (\mathbb{C}^n[Y^\pm]) \to \Lambda_{N+1}^N (\mathbb{C}^n[Y^\pm]), \quad \Phi^* : \Lambda_N^N (\mathbb{C}^n[Y^\pm]) \to \mathbb{C}^n[Y^\pm] \otimes \Lambda_{N-1}^N (\mathbb{C}^n[Y^\pm])
\]

\[
\Psi : \Lambda_N^N (\mathbb{C}^n[Y^\pm]) \otimes \mathbb{C}^n[Y^\pm] \to \Lambda_{N+1}^N (\mathbb{C}^n[Y^\pm]), \quad \Psi^* : \Lambda_N^N (\mathbb{C}^n[Y^\pm]) \to \Lambda_{N-1}^N (\mathbb{C}^n[Y^\pm]) \otimes \mathbb{C}^n[Y^\pm].
\]

Vertex operators can be considered in a more general context as such intertwiners \cite{FR92, JM95}.

The space \( \Lambda_N^{N/2} (\mathbb{C}^n[Y^\pm]) \) is defined as an inductive limit

\[
\mathbb{C}^n[Y^\pm] \to \Lambda_1^2 (\mathbb{C}^n[Y^\pm]) \to \cdots \to \Lambda_N^N (\mathbb{C}^n[Y^\pm]) \to \ldots
\]

(1.7)

Since \( U_v(\widehat{\mathfrak{gl}}_n) \) and \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \) act on \( \Lambda_N^N (\mathbb{C}^n[Y^\pm]) \), it is natural to ask for the limit of these actions. It was shown \cite{Ste95, KMS95} that the action of \( U_v(\widehat{\mathfrak{gl}}_n) \) has an (appropriately defined) limit, the obtained representation is an intertgradable level-one module of \( U_v(\widehat{\mathfrak{gl}}_n) \).

We take the limit of the \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \)-action in several steps. Introduce several subalgebras in \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \): \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1)^+, U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1)^- \), \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1)^\vee \), \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1)^\perp \) which are generated by \( c, c^* \), and \( P_{a,b} \) subject to the conditions \( a \geq 0, a \leq 0, an' + bn \leq 0, b < 0 \) respectively. These subalgebras are displayed on the Fig. 1. They are isomorphic due to \( \widetilde{SL}(2, \mathbb{Z}) \)-action on \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) \). Consider an element \( \sigma \in \widetilde{SL}(2, \mathbb{Z}) \) such that \( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1) = \sigma \left( U_{q_1,q_2}(\widehat{\mathfrak{gl}}_1)^+ \right) \). The element is particularly important since we will obtain the Fock module twisted by \( \sigma \).
Theorem 6.1 states that the action of operators from $U_{q,v}(\hat{gl}_1)^+$ converges for $|q^{-1}v^2| < 1$. Similarly, Theorem 6.2 states that the action of $U_{q,v}(\hat{gl}_1)^-$ converges for $|q^{-1}v^2| > 1$. Therefore we can not obtain the action of whole $U_{q,v}(\hat{gl}_1)$ by a straightforward limit argument. Though the obtained action of the generators $P_{\pm 1,b}$ is given by rational functions of $q,v$, hence it can be analytically continued for generic $q,v$. This gives actions $U_{q,v}(\hat{gl}_1)^\pm \simeq \Lambda_{v^2}^\infty (\mathbb{C}^n[Y^\pm 1])$ for generic $q,v$. The proofs use the expressions via vertex operators (1.6) and as a result we get analogous formulas after the limit, see formula (1.9) below.

On the other hand, we show in Section 7.1 that the action of $U_{q,v}(\hat{gl}_1)^\wedge$ converges for any $q,v$. Also, we prove that the obtained representation is isomorphic to the restriction of twisted Fock module $\mathcal{F}_u^\wedge$ to $U_{q,v}(\hat{gl}_1)^\wedge$. Using this we can glue the actions of $U_{q,v}(\hat{gl}_1)^+$ and $U_{q,v}(\hat{gl}_1)^-$ and obtain

**Theorem 1.1.** The following formulas determine an action of $U_{q,v}(\hat{gl}_1)$ on $\Lambda_{v^2}^\infty (\mathbb{C}^n[Y^\pm 1])$\[c \mapsto v^{-n}, \quad c' \mapsto v^{-n'}, \quad P_{0,j} \mapsto \#B_j, \eqno{(1.8)}\]

$$P_{1,b} \mapsto \sum_{k \in \mathbb{Z}} \#\hat{\Psi}_{k+n'+nb}\hat{\Psi}^*_k, \quad P_{-1,b} \mapsto \sum_{k \in \mathbb{Z}} \#\hat{\Psi}_{k-n'+nb}\hat{\Psi}^*_k, \tag{1.9}$$

for $q_1 = q$, $q_2 = v^{-2}$. The obtained representation is isomorphic to twisted Fock module $\mathcal{F}_u^\wedge$.

Here $B_k$ are the generators of the diagonal $v$–Heisenberg subalgebra in $U_v(\hat{gl}_n)$ and $\hat{\Psi}_k, \hat{\Phi}_k, \hat{\Psi}^*_k$ are vertex operators acting on the semi-infinite wedge. We discuss the meaning of the series (1.9) in Section 6.3 as an explicitly written analytic continuation. For the precise statement of the theorem above see Theorem 7.1.

Recall that our original motivation was the relation between standard and stable bases. By the definition [LT00], the standard basis consists of semi-infinite wedges $e_{-\lambda_1} \wedge e_{-\lambda_2+1} \wedge \cdots \in \Lambda_{v^2}^\infty (\mathbb{C}^n[Y^\pm 1])$. There is a combinatorial characterization of the stable basis given by certain three conditions [Neg16, Sect. 4.1]. We check two of these conditions, see Theorem 8.1. As above, the proof is based on the “finitization”, namely, we first prove similar properties for the basis $\{e_{i_1} \otimes \cdots \otimes e_{i_N}\}$ in the space (1.4), see Corollary 3.4 and Theorem 3.3. The most interesting “window” condition will be considered elsewhere. A verification of the condition would give a new proof of Gorsky-Neguţ conjecture (Kononov-Smirnov theorem).

**Further Questions**

- In the paper we proved Theorem 8.1 using the corresponding properties of the basis $\{e_{i_1} \otimes \cdots \otimes e_{i_N}\}$ in the space (1.4). It is natural to expect a finite non-symmetric analog of Gorsky-Neguţ conjecture. This means that apart from the proven properties, there is a “window” condition given in terms of nonsymmetric Macdonald polynomials.

- In the paper we constructed duality (action on the same space) of two algebras $U_{q,v}(\hat{gl}_1)$ and $U_v(\hat{gl}_n)$. The action of quantum affine algebra could be promoted to the toroidal algebra.
$U_{v,\nu}(\hat{\g}_n)$. This could be viewed as some twisted generalization of [FJM19], we expect that a
new presentation of toroidal algebra suggested in [Neg19] is relevant for this question.

Usually, the most interesting part of $(\g,\hat{\g}_n)$-duality is a relation between integrable systems, see e.g. [FJM19]. We do not know the corresponding results in our setting.

• It is interesting to find an interpretation of our construction in the framework of geometric representation theory. In particular, to relate our results to the results of [KS20].

• Usually, the algebra $U_{q_1,q_2}(\hat{\g}_1)$ acts on representations through a quotient which is isomorphic to a $q$-$W$ algebra. Our construction of the twisted Fock module should be related to realizations of twisted $q$-$W$ algebras. To the best of our knowledge, even the corresponding $q$-$W$ algebras are not known. See [BG21] for the $n = 2$ case and [BG20] for the case of any $n$ and $q_2 = 1$.\[
\text{Plan of the paper} \quad \text{In Section 2 we recall the definition of DAHA and the corresponding basic constructions.}
\]

In Section 2 we construct and study action of DAHA on the space $(\mathbb{C}^n) \otimes \mathcal{W}^{\pm 1}_N$. We prove that the obtained representation is isomorphic to twisted Cherednik representation (Theorem 3.4). Also, we prove Theorems 3.2 and 3.3 which state that certain matrices are upper triangular, the theorems are used in the proof of Theorem 3.4 in Sections 7 and 8.

In Section 3 we recall a presentation and some properties of the toroidal algebra $U_{q_1,q_2}(\hat{\g}_1)$. In particular, we describe its relation to spherical DAHA. We mainly follow [SV11].

In Section 4 we recall the definition of DAHA and the corresponding basic

In Section 5 we construct and study action of DAHA on the space $(\mathbb{C}^n) \otimes \mathcal{W}^{\pm 1}_N$. We prove that the obtained representation is isomorphic to twisted Cherednik representation (Theorem 3.4). Also, we prove Theorems 3.2 and 3.3 which state that certain matrices are upper triangular, the theorems are used in the proof of Theorem 3.4 in Sections 7 and 8.

In Section 5 we study the space of finite and semi-infinite wedges. We mainly follow [KMS95, LT00].

We slightly extend loc. cit. results since we need properties of all the types of the vertex operators $\Phi, \Phi^*, \Psi, \Psi^*$, but only $\Phi$ was considered in loc. cit.. The main results are Propositions 5.10, 5.12 on $N \to \infty$ stabilization of the vertex operators and Proposition 5.19 on commutation relations with the diagonal $\nu$-Heisenberg subalgebra of $U_{v}(\hat{\g}_n)$.

In Section 6 we study the limit for the action of Chevalley generators via the vertex operators formula. This gives the actions of $U_{q_1,q_2}(\hat{\g}_1)^+$ and $U_{q_1,q_2}(\hat{\g}_1)^-$ as a limit for $|q^{-1}v^2| < 1$ and $|q^{-1}v^2| > 1$ respectively. The formula allows us to obtain an analytic continuation of the actions. Also, we consider an example of the construction in the case of the (non-twisted) Fock module.

In Section 7 we prove that the action of $U_{q_1,q_2}(\hat{\g}_1)^{\nu'}$ converges for any $q$ and $\nu$. Then we show that the action of the subalgebras $U_{q_1,q_2}(\hat{\g}_1)^{\nu'}$ and $U_{q_1,q_2}(\hat{\g}_1)^{\nu'}$ are restrictions of an action of the whole $U_{q_1,q_2}(\hat{\g}_1)$. The obtained representation is isomorphic to the twisted Fock module. The action of Chevalley generators of $U_{q_1,q_2}(\hat{\g}_1)$ is given by the formulas via vertex operators (Theorem 7.1). This is the main result of the whole paper.

In Section 8 we study standard basis, the main result is Theorems 8.1 mentioned above.

In Appendix A we recall the action of $U_{v}(\g_n)$ on $\mathcal{W}^{\pm 1}_n(\mathbb{C}^n|Y^{\pm 1}|)$. Then we deduce intertwining properties for the operators $\Phi_k$, here the non-trivial part is a precise form of commutation relations with the diagonal $\nu$-Heisenberg algebra (Theorem A.1). This is used in the main part of the text (the proof of Proposition 5.19).

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5
2 Double Affine Hecke Algebra

In this section, we recall the definition and basic properties of double affine Hecke algebra (DAHA) [Che92, Kir97, Che05]. This section consists no new results.

Definition 2.1. Double affine Hecke algebra \( H_N \) is an algebra with generators \( T_1, \ldots, T_{N-1}, \pi^\pm, Y_1^\pm, \ldots, Y_N^\pm \) and relations

\[
(T_i - v)(T_i + v^{-1}) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad \text{for} \ |i - j| > 1, \quad (2.1)
\]

\[
T_i Y_i T_i = Y_{i+1}, \quad T_j Y_j = Y_j T_i \quad \text{for} \ j \neq i, i + 1, \quad (2.2)
\]

\[
\pi Y_i \pi^{-1} = q^{a_i} Y_{i+1}, \quad Y_i Y_j = Y_j Y_i, \quad (2.3)
\]

\[
\pi T_i \pi^{-1} = T_{i+1}, \quad \pi N_i = \pi N_i, \quad (2.4)
\]

here and below we use the convention \( Y_1 = Y_{N+1} \).

The operators \( T_1, \ldots, T_{N-1} \) generate finite Hecke algebra \( H \). The operators \( T_1, \ldots, T_{N-1}, Y_1, \ldots, Y_N \) generate affine Hecke algebra \( H^Y \). The operators \( T_1, \ldots, T_{N-1}, \pi^\pm \) generate another affine Hecke algebra denoted by \( H^X \). Here one can define

\[
X_i = T_i \cdots T_{N-1} \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1}. \quad (2.5)
\]

The relations on \( X_i \) are

\[
T_i X_i T_i^{-1} = X_{i+1}, \quad X_2^{-1} Y_1 X_2 Y_1^{-1} = T_1^2, \quad X_1 Y_2^{-1} X_1^{-1} Y_2 = T_1^2. \quad (2.6)
\]

Let \( \tilde{S}L(2, \mathbb{Z}) \) be the braid group on 3 stands. More precisely, \( \tilde{S}L(2, \mathbb{Z}) \) is generated by \( \tau_+ \) and \( \tau_- \) with the relation \( \tau_+ \tau_- \tau_+ = \tau_- \tau_+ \tau_- \). We use this notation since \( \tilde{S}L(2, \mathbb{Z}) \) is an extension of \( SL(2, \mathbb{Z}) \) by \( \mathbb{Z} \), the projection is given by

\[
\tau_+ \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_- \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (2.7)
\]

The kernel is generated by \( (\tau_+ \tau_- \tau_+)^4 \).

Proposition 2.2 (Cheh). There is an action of \( \tilde{S}L(2, \mathbb{Z}) \) on \( H_N \) determined by the following formulas

\[
\tau_+: \quad T_i \mapsto T_i, \quad X_i \mapsto X_i, \quad Y_i \mapsto Y_i T_i T_{i-1} \cdots T_1, \quad (2.8)
\]

\[
\tau_-: \quad T_i \mapsto T_i, \quad X_i \mapsto X_i T_{i-1} \cdots T_1^{-1} T_i^{-1} \cdots T_{i-1}^{-1}, \quad Y_i \mapsto Y_i. \quad (2.9)
\]

The algebra \( H_N \) is bigraded with the gradings \( \deg_X \) and \( \deg_Y \) defined by

\[
\deg_X \pi = -1, \quad \deg_X Y_i = 0, \quad \deg_X T_i = 0 \quad (2.10)
\]

\[
\deg_Y \pi = 0, \quad \deg_Y Y_i = 1, \quad \deg_Y T_i = 0. \quad (2.11)
\]

Lemma 2.3. Let \( n, n' \in \mathbb{Z}_{>0} \) be coprime integers. Let \( m, m' \in \mathbb{Z}_{>0} \) be unique pair of integers such that \( m < n, m' < n' \) and \( mn - n'm = 1 \). Then there is an \( \tilde{S}L(2, \mathbb{Z}) \) transformation mapping \( X_i^{-1} \) to \( B_i \) and \( Y_i \) to \( A_i \) such that

\[
\deg_X (B_i) = -n, \quad \deg_Y (B_i) = n', \quad \deg_X (A_i) = -m, \quad \deg_Y (A_i) = m'. \quad (2.12)
\]

Moreover, \( B_{i+1} = T_i \cdots T_1 B_i T_1 \cdots T_i, \quad A_{i+1} = T_i \cdots T_1 A_i T_1 \cdots T_i, \) and

\[
B_1 = Z_1 \cdots Z_{n+n'}, \quad A_1 = W_1 \cdots W_{m+m'}. \quad (2.13)
\]
where
\[
Z_j = Y_1 \text{ if } \left\lfloor \frac{jn}{n+n'} \right\rfloor = \left\lfloor \frac{(j-1)n}{n+n'} \right\rfloor, \quad Z_j = X_1^{-1} \text{ if } \left\lfloor \frac{jn}{n+n'} \right\rfloor = \left\lfloor \frac{(j-1)n}{n+n'} \right\rfloor + 1, \quad (2.14) \\
W_j = Y_1 \text{ if } \left\lfloor \frac{jm}{m+m'} \right\rfloor = \left\lfloor \frac{(j-1)m}{m+m'} \right\rfloor, \quad W_j = X_1^{-1} \text{ if } \left\lfloor \frac{jm}{m+m'} \right\rfloor = \left\lfloor \frac{(j-1)m}{m+m'} \right\rfloor + 1. \quad (2.15)
\]

Proof. The formulas for $B_1$ and $A_1$, in terms of $B_1$ and $A_1$, follows from the relations $X_{j+1}^{-1} = T_j X_j^{-1} T_j$, $Y_{j+1} = T_j Y_j T_j$. The formulas for $B_1$ and $A_1$ can be proven using Euclidean algorithm. For the step one uses the formulas $\tau_{-1}(X_1^{-1}) = Y_1 X_1^{-1} = \tau_{-1}(Y_1)$.

Euclidean algorithm used in the proof can be viewed as decomposition of the matrix $\begin{pmatrix} n & -m \\ -n' & m' \end{pmatrix}$ into a product of the matrices $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ which correspond to $\tau_1$, $\tau_{-1}$.

The formula for $B_1$ has also the following geometric interpretation. Draw the segment from $(0,0)$ to $(n',n)$. Draw the closest north-east lattice path below the segment. Then for each east step of the path we write $Y_1$ and for each north step we write $X_1^{-1}$.

**Example 2.4.** Let us take $n = 5$ and $n' = 3$. Then
\[
\tau_{-1}^{-1} \tau_{+}^{-1} \tau_{+}^{-1} \tau_{-1}(X_1^{-1}) = Y_1 X_1^{-1} Y_1 X_1^{-2} Y_1 X_1^{-2},
\]
\[
\tau_{-1}^{-1} \tau_{+}^{-1} \tau_{-1}^{-1}(Y_1) = Y_1 X_1^{-1} Y_1 X_1^{-2}.
\]

The formulas of $B_1$ agrees with the form of the sequence $\frac{jm}{m+m'}$ namely $0, \frac{5}{8}, \frac{10}{8}, \frac{15}{8}, \frac{20}{8}, \frac{25}{8}, \frac{30}{8}, \frac{35}{8}, \frac{40}{8}$ as well as the geometric description.

### 2.1 Cherednik representation

Algebra $H^X$ has one-dimensional representation $C_u$
\[
T_i \mapsto v, \quad \pi \mapsto u.
\]

Cherednik representation $C_u$ of $H_N$ is the induced representation $\text{Ind}_{H_N}^{H^X} C_u$. It can be identified with the space of Laurent polynomials $C[Y_1^{\pm1}, \ldots, Y_N^{\pm1}]$. The action of the generators $T_i$ and $\pi$ can be written as
\[
T_i = s_i^Y + (v-v^{-1}) \frac{s_i^Y - 1}{Y_i/Y_{i+1} - 1}, \quad \pi(Y_1^{\lambda_1} Y_2^{\lambda_2} \cdots Y_N^{\lambda_N}) = u q^{\lambda_N} \lambda_1 N_1 \lambda_2 N_2 \cdots Y_N^{\lambda_N - 1},
\]
here $s_i^Y$ is the permutation of $Y_i$ and $Y_{i+1}$.

For any composition $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ we denote $Y_\lambda = Y_1^{\lambda_1} Y_2^{\lambda_2} \cdots Y_N^{\lambda_N}$, such vectors form the standard monomial basis in Cherednik representation. We will write $\lambda \leq \mu$ if $\mu - \lambda \in \oplus \mathbb{Z}_{\geq 0} \alpha_i$, where $\alpha_i$ are positive simple roots of $s_i N$.

**Definition 2.5.** Let $\lambda, \mu \in \mathbb{Z}^N$. We write $\lambda \prec \mu$ if
1. $\lambda^+ < \mu^+$ where $\lambda^+$ is the dominant coweight lying in the orbit of $\lambda$, and analogously for $\mu^+$
2. $\lambda^+ = \mu^+$ and $\lambda < \mu$
For example, in case $N=2$ we have

$$(1,1) \prec (0,2) \prec (2,0).$$

The action of the operators $X_1, \ldots, X_N$ in the basis $Y^\lambda$ is triangular with respect to the order $\prec$. More explicitly

$$X_i Y^\lambda = u^{-1} q^{-\lambda_i} v^{-1_i(\lambda)} Y^\lambda + \sum_{\mu<\lambda} x_{\lambda,\mu} Y^\mu,$$  \hfill (2.16)

here

$$1_i^{(\lambda)} = \left| \{ j | \lambda_j > \lambda_i \} \right| + \left| \{ j | j < i, \lambda_j = \lambda_i \} \right| - \left| \{ j | j < i, \lambda_j > \lambda_i \} \right| - \left| \{ j | j > i, \lambda_j = \lambda_i \} \right|. \hfill (2.17)$$

The non-symmetric Macdonald polynomials $E_\lambda$ are defined as eigenvectors of $X_1, \ldots, X_N$ with the leading term $Y^\lambda$. Note that (2.16) implies

$$X_i E_\lambda = u^{-1} q^{-\lambda_i} v^{-1_i(\lambda)} E_\lambda.$$  \hfill (2.18)

\section{Representation}

In this section we introduce representation $\mathcal{C}^{(n,n_{tw})}_{a_0,\ldots,a_{n-1}}$, generalize (2.16) and interpret the obtained representation as twisted Cherednik representation. Also, we study a basis $A_\lambda$ of the representation $\mathcal{C}^{(n,n_{tw})}_{a_0,\ldots,a_{n-1}}$ in the case $\gcd(n,n_{tw}) = 1$.

\subsection{Explicit construction}

\paragraph{Action of affine Hecke algebra} Fix $n$ and let $\mathbb{C}^n$ be a vector space with the basis $e_0, \ldots, e_{n-1}$. Define an $R$-matrix acting on $\mathbb{C}^n \otimes \mathbb{C}^n$

$$R = v \sum_a E_{aa} \otimes E_{aa} + \sum_{a<b} \left( E_{ab} \otimes E_{ba} + E_{ba} \otimes E_{ab} + (v - v^{-1}) E_{aa} \otimes E_{bb} \right), \hfill (3.1)$$

here $E_{ab}$ is a matrix unit. Define an action of $H$ on $(\mathbb{C}^n)^{\otimes N}$ by the formula $T_i \mapsto R_{i,i+1}$, here the indices encodes factors on which $R$-matrix acts. One can induce an action of $H^Y$ on $(\mathbb{C}^n)^{\otimes N} [Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$.

Below we will introduce notation to distinguish different actions of permutation group on the space $(\mathbb{C}^n)^{\otimes N} [Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$. Let $s_{ij}^Y$ be an operator acting on $(\mathbb{C}^n)^{\otimes N} [Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$ which swaps $Y_i$ and $Y_j$. Let $s_{ij}^e$ be an operator acting on $(\mathbb{C}^n)^{\otimes N} [Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$ which swaps tensor factors number $i$ and $j$ (and commutes with all $Y_k$). Finally, let $s_{ij} = s_{ij}^Y s_{ij}^e$. Also, we will denote $s_i^Y = s_{i,i+1}^Y$, $s_i^e = s_{i,i+1}^e$, and $s_i = s_{i,i+1}$.

The action of $T_i$ is given by the following formula

$$T_i = s_i^Y R_{i,i+1} + (v - v^{-1}) \frac{s_i^Y - 1}{Y_i / Y_{i+1} - 1}. \hfill (3.2)$$

The obtained representation of affine Hecke algebra $H^Y$ is well-know [GRV94, CP94]. It appears in the context of quantum affine Schur-Weyl duality.

\paragraph{Action of DAHA} Below we will use the following identification

$$(\mathbb{C}^n[Y_i^{\pm 1}])^{\otimes N} \rightarrow (\mathbb{C}^n)^{\otimes N} [Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$$  \hfill (3.3)

$$(Y^{j_1} e_{i_1}) \otimes \cdots \otimes (Y^{j_N} e_{i_N}) \mapsto Y_1^{j_1} \cdots Y_n^{j_n} e_{i_1} \otimes \cdots \otimes e_{i_N}$$  \hfill (3.4)
Let us introduce \( e_i \in \mathbb{C}^n[Y^{\pm 1}] \) for \( i \in \mathbb{Z} \) by setting
\[
e_i = Y^{-1}e_i + n.
\]

Introduce operators \( \kappa \) and \( D \) acting on \( \mathbb{C}^n[Y^{\pm 1}] \) by \( \kappa e_i = e_{i-1}, \ D(Y^a e_i) = u_0 q^a Y^a e_i \) for \( a = 0, \ldots, n-1 \). Here \( u_0, \ldots, u_{n-1} \) are any non-zero numbers. By \( \kappa \) and \( D_i \) we denote the corresponding operators acting on \( i \)-th tensor factor.

**Theorem 3.1.** For any non-zero numbers \( u_0, \ldots, u_{n-1} \) and \( n_{tw} \in \mathbb{Z} \), there is an action of algebra \( \mathcal{H}_N \) on \( (\mathbb{C}^n)^{\otimes N}[Y^{\pm 1}] \) determined by the following conditions

- subalgebra \( H^Y \) acts as described above
- \( \pi = \kappa^{n_{tw}}D_1 s_1 \cdots s_{N-1} \)

Denote the obtained representation by \( C_{u_0, \ldots, u_{n-1}}^{(n,n_{tw})} \).

**Proof.** It is enough to check the relations involving \( \pi \). The relations \( \pi Y_i \pi^{-1} = q^{s_i} Y_{i+1} + \pi T_i \pi^{-1} = T_{i+1} \) are easy to see. Let us check that \( \pi^N \) commutes with \( T_i \). Since
\[
\pi^N(w_1 \otimes \cdots \otimes w_N) = (\kappa^{n_{tw}} D w_1) \otimes (\kappa^{n_{tw}} D w_2) \otimes \cdots \otimes (\kappa^{n_{tw}} D w_N),
\]

it is sufficient to consider only the \( N = 2 \) case. In this case we denote \( T = T_1 \) for brevity. Let \( l = h + nk + s \) for \( k \geq 0 \) and \( s = 0, \ldots, n - 1 \). For \( s = 0 \)
\[
T(e_h \otimes e_l) = v e_l \otimes e_h + (v - 1) \sum_{j=1}^k e_{l-nj} \otimes e_h + nj, \quad \text{for } k \geq 0,
\]
\[
T(e_l \otimes e_h) = v^{-1} e_h \otimes e_l - (v - 1) \sum_{j=1}^{k-1} e_{l-nj} \otimes e_h + nj, \quad \text{for } k > 0.
\]

For \( s > 0 \)
\[
T(e_h \otimes e_l) = e_l \otimes e_h + (v - 1) \sum_{j=0}^k e_{h+nj} \otimes e_{l-nj},
\]
\[
T(e_l \otimes e_h) = e_h \otimes e_l - (v - 1) \sum_{j=1}^k e_{l-nj} \otimes e_{h+nj}.
\]

Since the formulas (3.6)–(3.9) are invariant under the shift \( l \mapsto l - n_{tw}, \ h \mapsto h - n_{tw} \), we see that \( T \pi^2 = \pi^2 T \) for \( N = 2 \).

### 3.2 Triangularity of Macdonald operators

Introduce a grading on \( (\mathbb{C}^n[Y^{\pm 1}])^{\otimes N} \) as follows
\[
\deg e_{a_1} \otimes \cdots \otimes e_{a_N} = \sum a_i, \quad \deg Y_i = n.
\]

Then the operators \( T_i \) preserve the grading and \( \deg \pi = -n_{tw} \). Hence \( \deg X_i = n_{tw} \).

To simplify our notation, let us assume \( n, n_{tw} > 0 \). Let \( d = \gcd(n, n_{tw}) \), we use notations \( n'' = n/d, \ n' = n_{tw}/d \). Consider integers \( m, m' \) such that \( nm' - n_{tw} m = d, \ 0 \leq m' < n', \ 0 \leq m < n''. \) Hence there is \( \sigma \in SL(2, \mathbb{Z}) \) such that
\[
\sigma(X_i) = B_i^{-1}, \quad \deg_X(B_i) = -n'', \quad \deg_Y(B_i) = n', \quad \deg B_i = 0,
\]
\[
\sigma(Y_i) = A_i, \quad \deg_X(A_i) = -m, \quad \deg_Y(A_i) = m', \quad \deg A_i = d.
\]
The corresponding matrix in $SL(2,\mathbb{Z})$ is $\begin{pmatrix} n^n & -m' \\ -n' & m \end{pmatrix}$. Lemma 2.3 gives explicit formulas for $B_i, A_i$.

Consider operators

$$G_{ij} = R_{ij} S_{ij} + (v - v^{-1}) \frac{1 - s^y_{ij}}{Y_i/Y_j - 1} s^e_{ij}. \quad (3.13)$$

Denote $G_i = G_{i+i+1}$. It follows from (3.2) that $T_i = G_i s_i$. Using this, we can write the following formula

$$X_1^{-1} = \pi T_{N-1}^{-1} \cdots T_1^{-1} = \kappa_1^{n_{uw}} D_1 s_1 \cdots s_{N-1} s_{N-1} G_{N-1,N}^{-1} \cdots s_1 G_{1,2}^{-1} = \kappa_1^{n_{uw}} D_1 G_{1,N}^{-1} \cdots G_{1,2}^{-1}. \quad (3.14)$$

For any collection $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$, we can consider a vector $e_\lambda = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N}$. We are going to prove that the operators $B_i$ are triangular in the basis $e_\lambda$ with respect to order $\prec$. Basically, the proof is a computation, similar to [Kir97 Sect. 5]. Let us first explain the idea on the following example (cf. Example 2.4).

**Example 3.1.** Let us take $N = 3$, $n = 5$, $n_{tw} = 3$. Note that $Y_i = \kappa_i^{-5}$. Then we have

$$B_1 = Y_1 X_1^{-1} Y_1 X_1^{-2} Y_1 X_2^{-1} = \kappa_1^{-5} (\kappa_1^{-5} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-5} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-5} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-5} D_1 G_{1,3}^{-1} G_{1,2}^{-1})$$

$$= (\kappa_1^{-2} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-4} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-6} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-8} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-10} D_1 G_{1,3}^{-1} G_{1,2}^{-1}),$$

$$B_2 = T_1 B_1 T_1 = G_{1,2} s_1 \kappa_1^{-5} (\kappa_1^{-5} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-7} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-9} D_1 G_{1,3}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_1^{-11} D_1 G_{1,3}^{-1} G_{1,2}^{-1})$$

$$= G_{1,2} (\kappa_2^{-5} D_2 G_{2,3}^{-1} G_{2,1}^{-1} s_2) \kappa_1^{-5} (\kappa_2^{-7} D_2 G_{2,3}^{-1} G_{2,1}^{-1} s_2) \kappa_1^{-5} (\kappa_2^{-9} D_2 G_{2,3}^{-1} G_{2,1}^{-1} s_2) \kappa_1^{-5} (\kappa_2^{-11} D_2 G_{2,3}^{-1} G_{2,1}^{-1} s_2),$$

$$B_3 = T_2 T_1 B_1 T_2 \kappa_1^{-5} (\kappa_3^{-5} D_3 G_{3,1}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_3^{-7} D_3 G_{3,1}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_3^{-9} D_3 G_{3,1}^{-1} G_{1,2}^{-1}) \kappa_1^{-5} (\kappa_3^{-11} D_3 G_{3,1}^{-1} G_{1,2}^{-1}),$$

Using Proposition 3.3 below, we see that all these operators are triangular.

Now we proceed to the proof.

**Proposition 3.2.** The operators $G_i$ are triangular in the basis $e_\lambda$ with respect to order $\prec$.

**Proof.** It is sufficient to consider the case $N = 2$. In this case we will write simply $G$ omitting the index. The formulas below is just a reformulation of (3.6)–(3.9). Recall that $l = h + nk + s$ for $k \geq 0$ and $s = 0, \ldots, n-1$. For $s = 0$

$$G(e_l \otimes e_h) = ve_l \otimes e_h + (v - v^{-1}) \sum_{j=1}^{k} e_{l-nj} \otimes e_{h+nj} \quad \text{for } k \geq 0. \quad (3.15)$$

$$G(e_h \otimes e_l) = v^{-1} e_h \otimes e_l - (v - v^{-1}) \sum_{j=1}^{k-1} e_{l-nj} \otimes e_{h+nj} \quad \text{for } k > 0. \quad (3.16)$$

For $s > 0$

$$G(e_l \otimes e_h) = e_l \otimes e_h + (v - v^{-1}) \sum_{j=0}^{k} e_{h+nj} \otimes e_{l-nj}, \quad (3.17)$$

$$G(e_h \otimes e_l) = e_h \otimes e_l - (v - v^{-1}) \sum_{j=1}^{k} e_{l-nj} \otimes e_{h+nj}. \quad (3.18)$$

The formulas (3.15)–(3.18) imply that $G$ is triangular. \qed
Proposition 3.3. For \( i < j \), the operators \( \kappa_i^{-d}G_{i,j}G_k^d \), \( 0 \leq d < n \) and \( \kappa_j^{-d}G_{j,i}G_k^d \), \( 0 < d < n \) are triangular in the basis \( e_\lambda \) with respect to order \( < \). The operators \( \kappa_i^{-d}D_i \kappa_i^d \) are diagonal for any \( d \).

Proof. It is sufficient to consider \( N = 2 \) and the operators \( \kappa_i^{-d}G_k^d \), \( 0 \leq d < n \) and \( s_1\kappa_i^{-d}G_k^d s_1 \), \( 0 < d < n \). Everything follows from (3.15)-(3.18).

\[ \square \]

Theorem 3.2. The operators \( B_1, \ldots, B_N \) are triangular in the basis \( e_\lambda \) with respect to order \( < \).

Proof. Recall that \( n'' = n/d, n' = ntw/d \). Using Lemma 2.3 we can write

\[ B_i = T_{i-1} \cdots T_2 Z_1 \cdots Z_{n'' + n'} T_1 \cdots T_i. \]  
(3.19)

Now we substitute \( Y_1 = \kappa_1^{-n}, X_1^{-1} = \kappa_1^{ntw} D_1 G_{1,N}^{-1} \cdots G_{1,2}^{-1} \) and

\[ T_{i-1} \cdots T_1 = G_{i-1,i} \cdots G_{1,i} s_{i-1} \cdots s_1, \]
\[ X_1^{-1} T_i \cdots T_1 = \kappa_i^{ntw} D_1 s_1 \cdots s_{i-1} G_{i,N}^{-1} \cdots G_{i,i+1}^{-1}. \]

Hence we get

\[ B_i = G_{i-1,i} \cdots G_{1,i} \left( \prod_{j < n'' + n'}\kappa_j^{-d} D_j G_{i,N}^{-1} \cdots G_{i,1}^{-1} \right) D_i G_{i,N}^{-1} \cdots G_{i,i+1}^{-1}, \]  
(3.20)

for certain integers \( d_j \). Let \( \{x\} \) denote the fractional part of \( x \in \mathbb{R} \). One can observe that

\[ d_j = n \{s | s < j, Z_s = Y_1\} - ntw \{s | s \leq j, Z_s = X_1^{-1}\} = n \left( j - \left[ \frac{jn''}{n'' + n'} \right] \right) - ntw \left[ \frac{jn''}{n'' + n'} \right] = (n + ntw) \left( \frac{jn''}{n'' + n'} \right). \]  
(3.21)

Here \( j \) is such that \( Z_j = X_1^{-1} \), hence \( \left\{ \frac{jn''}{n'' + n'} \right\} < \frac{n''}{n'' + n'} \) by the condition (2.14). Hence \( 0 < d_j < n \). Therefore Proposition 3.3 implies, that the operator is triangular.

\[ \square \]

Corollary 3.4. If \( ntw = n' \) is coprime with \( n \), then there are eigenvectors \( \tilde{E}_\lambda = e_\lambda + \sum_{\mu<\lambda} \beta_{\lambda,\mu} e_\mu \) of \( B_1, \ldots, B_N \) in \( C_{n',n''} \) with eigenvalues given by

\[ B_i \tilde{E}_\lambda = u_0 \cdots u_{n-1} q^{1-n} q^{(\lambda)} q^{\lambda} \tilde{E}_\lambda \]  
(3.22)

Proof. Theorem 3.2 is equivalent to the following formula

\[ B_i e_\lambda = b_{\lambda,\alpha} e_\lambda + \sum_{\mu<\lambda} b_{\lambda,\mu} e_\mu \]  
(3.23)

It remains to compute \( b_{\lambda,\lambda} \) using (3.20). It follows from (3.21) that the numbers \( d_j \) are distinct and form a set \( \{1, \ldots, n-1\} \). Hence it remains to compute the diagonal terms in the action of the operators

\[ a) \ G_{j,i} \quad \text{for} \quad j < i; \quad b) \ \kappa_i^{-d} D_i \kappa_i^d \quad \text{for} \quad 0 \leq d < n; \]
\[ c) \kappa_i^{-d} G_{i,j} \kappa_i^d \quad \text{for} \quad 0 < d < n, j \neq i; \quad d) \ G_{i,j}^{-1} \quad \text{for} \quad j > i. \]  
(3.24)

Using formulas (3.15)-(3.18) we get

\[ b_{\lambda,\lambda} = u_0 \{j < (\lambda_j \geq \lambda_i \equiv \lambda_1)\} - \{j < i, \lambda_j \leq \lambda_i \equiv \lambda_1\} (u_0 \cdots u_{n-1}) q^{\lambda_i + 1 - n} q^{(\lambda)} q^{\lambda_i} \]
\[ \times \{j > (\lambda_j > \lambda_i \equiv \lambda_1)\} - \{j > i, \lambda_j \leq \lambda_i \equiv \lambda_1\} = u_0 \cdots u_{n-1} q^{1-n} q^{(\lambda)} q^{\lambda_i} \]

where \( \equiv \) stands for \( \equiv \ (\text{mod} \ n) \).
3.3 Monomial basis

Recall \( d = \gcd(n, n_{tw}) \).

**Lemma 3.5.** The operators \( A_i \) can be presented in the form \( A_i = A_i' \kappa_i^{-d} A_i'' \), where the operators \( A_i' \), \( A_i'' \) are compositions of operators of the form \( 3.24 \).

**Remark 3.6.** In particular, the operators \( A_i' \), \( A_i'' \) are triangular and the diagonal matrix entries are invertible in \( \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \) by Proposition 3.3. For technical reasons, we will need that operators the \( A_i' \), \( A_i'' \) are compositions of operators of the form \( 3.24 \) in the proof of Theorem 3.3. It will be important that the operators of the form \( 3.24 \) act at most on two tensor multiples.

The proof of the lemma is similar to the proof of Theorem 3.2.

**Proof.** Using (2.13) we get

\[
A_i = T_{i-1} \cdots T_1 W_1 \cdots W_{m+m'} T_1 \cdots T_{i-1}
\]

for certain integers \( c_j \). It remains to compute the numbers \( c_j \). We have

\[
c_j = n \left( \left\{ s \mid s < j, W_s = Y_1 \right\} \right) - n_{tw} \left( \left\{ s \mid s \leq j, W_s = X_1^{-1} \right\} \right)
\]

\[
= n \left( j - \left\lfloor \frac{jm}{m + m'} \right\rfloor \right) - n_{tw} \left( \frac{jm}{m + m'} \right) = d \left( n'' + n' \right) \left( \frac{jm}{m + m'} \right) + \frac{j}{m + m'},
\]

(3.26)

Note that \( c_{m+m'} = d \). Let \( j < m + m' \). Since \( W_j = X_1^{-1} \), we have \( 0 \leq \left\lfloor \frac{jm}{m + m'} \right\rfloor \leq \frac{m-1}{m+m'} \). Then

\[
(n'' + n') \left( \frac{jm}{m + m'} \right) \leq (n'' + n') \frac{m - 1}{m + m'} = \frac{n''m + n'm' - 1 - n'' - n'}{m + m'} < n'' - 1.
\]

Hence \( 0 < c_j < n \). \( \square \)

Let us now assume that \( d = 1 \), i.e. \( n = n'' \), \( n_{tw} = n' \). In this case, the operators \( A_i \) increase the grading by 1. For any \( \lambda \in \mathbb{Z}^N \) denote

\[
A_{\lambda} = A_{\lambda_1} \cdots A_{\lambda_N} e_0 \otimes \cdots \otimes e_0
\]

(3.27)

**Theorem 3.3.** The vectors \( A_{\lambda} \) form a basis of \( (\mathbb{C}^n[Y^{\pm 1}])^\otimes N \). The transition matrix from \( e_\lambda \) basis to \( A_{\lambda} \) basis is triangular. Moreover, we have

\[
A_{\lambda} = \alpha_{\lambda, \lambda} e_\lambda + \sum_{\mu < \lambda} \alpha_{\lambda, \mu} e_\mu
\]

(3.28)

where the coefficients \( \alpha_{\lambda, \mu} \in \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \) and \( \alpha_{\lambda, \lambda} \) is invertible in \( \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \).

**Proof.** Since \( A_i \) is expressed via the operators \( T_j^{\pm 1}, Y_j^{\pm 1}, \pi^{\pm 1} \), its matrix elements in \( e_\lambda \) basis belong to \( \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \). Hence, the vectors \( A_{\lambda} \) expand in \( e_\lambda \) basis with coefficients in \( \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \).

The product \( A_1 \cdots A_N \) is a combination of products \( Y_1 \cdots Y_N = \kappa_1^{-n} \cdots \kappa_N^{-n} \) and \( X_1^{-1} \cdots X_N^{-1} = \pi^N = \kappa_1^{n'} \cdots \kappa_N^{n'} D_1 \cdots D_N \). Notice that

\[
A_1 \cdots A_N e_{(\lambda_1, \ldots, \lambda_N)} = c_{\lambda} e_{(\lambda_1+1, \ldots, \lambda_N+1)} \quad A_1 \cdots A_N A_{(\lambda_1, \ldots, \lambda_N)} = A_{(\lambda_1+1, \ldots, \lambda_N+1)}
\]

(3.29)
for certain invertible \( c_\lambda \in \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \). Therefore it is sufficient to prove the formula (3.28) for the compositions such that \( \lambda_1, \ldots, \lambda_N \geq 0 \).

Let \( l = \max \lambda_j \). We proceed by induction on \( l \). For \( l = 0 \) there is nothing to prove. The induction step is \( l - 1 \rightarrow l \). Let \( 1 \leq i_1 < \cdots < i_k \leq N \) be a subset of indices such that \( \lambda_{i_1} = \cdots = \lambda_{i_k} = l \) and \( \lambda_j < l \), for \( j \notin \{i_1, \ldots, i_k\} \). Let \( \lambda(s) \), \( 0 \leq s \leq k \) be a composition such that

\[
\lambda_j(s) = \begin{cases} 
  l - 1 & \text{for } j = i_1, \ldots, i_s \\
  l & \text{for } j = i_{s+1}, \ldots, i_k \\
  \lambda_j & \text{for } j \notin \{i_1, \ldots, i_k\}.
\end{cases}
\]

For example, \( \lambda(0) = \lambda \). By the induction hypothesis we know that \( A_{\lambda(s)} \) is a linear combination of \( e_\mu \) with \( \mu \preceq \lambda(s) \) and the coefficient \( \alpha_{\lambda(s), \lambda} \) is invertible. Now we prove by induction on \( s \) that

\[
A_{\lambda(s)} = A_{i_{s+1}} \cdots A_{i_k} A_{\lambda(k)}
\]

satisfies condition (3.28) with an additional constraint on \( \mu \) appearing in the right-hand side

\[
\mu_j < l \quad \text{for } j < i_{s+1}.
\]

The induction base is \( s = k \), the induction step is \( s \mapsto s - 1 \). The step follows from Lemma 3.5 for the operator \( A_{i_k} \). Indeed, the triangular operators of the form (3.24) have invertible elements on the diagonal and cannot make \( \mu_j = l \) for \( j < i_s \).

**Corollary 3.7.** The transition matrix from \( A_\lambda \) basis to \( e_\lambda \) basis is triangular. Moreover, we have \( e_\lambda = \sum_{\mu \preceq \lambda} \tilde{\alpha}_{\lambda, \mu} A_\mu \) where the coefficients \( \tilde{\alpha}_{\lambda, \mu} \in \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \) and \( \tilde{\alpha}_{\lambda, \lambda} \) is invertible in \( \mathbb{Z}[q^{\pm 1}, v^{\pm 1}] \).

**Proof.** Recall the transition matrix \( \alpha \) defined by (3.28). It follows from Theorem 3.3 that \( \tilde{\alpha} = \alpha^{-1} \) satisfies the properties in question. \( \square \)

### 3.4 Twisted Cherednik representation

Below we assume \( d = 1 \), i.e. \( n_{tw} = n' \). In this case, the operators \( A_i \) increase the grading by 1. For any \( \mathcal{H}_N \)-module \( M \) denote by \( \rho_M : \mathcal{H}_N \to \text{End}_\mathbb{C}(M) \) the corresponding homomorphism.

**Definition 3.8.** For any \( \mathcal{H}_N \)-module \( M \) and \( \tau \in \tilde{SL}(2, \mathbb{Z}) \), let us define the representation \( M^\tau \) as follows. \( M \) and \( M^\tau \) are the same vector space with different actions, namely \( \rho_{M^\tau} = \rho_M \circ \tau^{-1} \).

We will refer to \( M^\tau \) as a twisted representation.

**Theorem 3.4.** The representation \( C_u^{(n, n')}_{u_0, \ldots, u_{n-1}} \) is isomorphic to twisted Cherednik representation \( C^\sigma_u \) for \( \sigma \) as in (3.11), (3.12) and \( u = u_0 \cdots u_{n-1} q^{1-n} \).

**Proof.** Let \( H^B \) be a copy of affine Hecke algebra generated by \( T_i \) and \( B_i \). Twisted Cherednik representation \( C^\sigma_u \) can be interpreted as a \( \mathcal{H}_N \)-representation induced from one-dimensional representation of \( H^B \). As a vector space, \( C^\sigma_u \) is isomorphic to the space of Laurent polynomials \( \mathbb{C}[A_1^{\pm 1}, \ldots, A_N^{\pm 1}] \).

The vector \( e_{(0)N} \in C_u^{(n, n')}_{u_0, \ldots, u_{n-1}} \) is an eigenvector of \( T_1, \cdots, T_{N-1} \). Moreover, due to Theorem 3.2, the vector \( e_{(0)N} \) is an eigenvector of \( B_1, \cdots, B_N \). Corollary 3.3 implies that the eigenvalues are given by

\[
B_i e_{(0)N} = u_0 \cdots u_{n-1} q^{1-n} u^{2i-1-N} e_{(0)N}.
\]

Comparing (2.18) with (3.31), we see that there is a homomorphism \( \psi : C^\sigma_u \to C_u^{(n, n')} \) determined by \( \psi(1) = e_{(0)N} \).

The twisted Cherednik representation \( C^\sigma_u \) has a basis \( A_{\lambda_1} \cdots A_{\lambda_N} \). On the other hand, it follows from Theorem 3.3 that their images form a basis of \( C_u^{(n, n')}_{u_0, \ldots, u_{n-1}} \). Hence the map \( \psi \) is an isomorphism. \( \square \)
Remark 3.9. There is another way to finish the proof without using Theorem 3.3. Namely, since Cherednik representation is irreducible, the map $\psi$ is injective. It remains to show that $\epsilon(0)N$ is a cyclic vector of $C_{(n,n')}_{u_0,\ldots,u_{n-1}}$ for the $H_N$-action.

Corollary 3.10. The isomorphism class of the representation $C_{(n,n')}_{u_0,\ldots,u_{n-1}}$ is determined by $n$, $n'$, and the product $u_0 \cdots u_{n-1}$.

4 Toroidal algebra

In this section, we recall presentations and certain properties of quantum toroidal $gl_1$ algebra denoted by $U_{q_1,q_2}(gl_1)$. In particular, we describe its connection with double affine Hecke algebra $H_N$. The section contains no new results.

PBW presentation and $\widetilde{SL}(2,\mathbb{Z})$-action The algebra $U_{q_1,q_2}(gl_1)$ is an algebra depending on parameters $q_1$ and $q_2$. Let us introduce a parameter $q_3$ such that $q_1q_2q_3 = 1$. The algebra has a presentation via generators $P_{a,b}$ for $(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and central elements $c^{\pm 1}$, $(c')^{\pm 1}$. We will not need explicit form of the relations, see [BS12], Def. 6.4 for a reference. Loc. cit. generators $u_{a,b}$ correspond to $P_{a,b}/(1 - q_1^{a})$ for $d = \gcd(a,b)$.

Proposition 4.1 (BS12). Group $\widetilde{SL}(2,\mathbb{Z})$ acts on $U_{q_1,q_2}(gl_1)$ via automorphisms.

Let us consider an element $\tau \in \widetilde{SL}(2,\mathbb{Z})$ such that under the projection (2.7) it is mapped

$$\tau \mapsto \begin{pmatrix} m' & m \\ n' & n \end{pmatrix}. \quad (4.1)$$

Let $\widetilde{SL}(2,\mathbb{R})$ be the universal covering of $SL(2,\mathbb{R})$. The group $\widetilde{SL}(2,\mathbb{Z})$ can be interpreted as the preimage of $SL(2,\mathbb{Z})$ in $\widetilde{SL}(2,\mathbb{R})$. Hence we can think about the element $\tau \in \widetilde{SL}(2,\mathbb{Z})$ as a path $\gamma$ in $SL(2,\mathbb{R})$ from the identity matrix to the matrix (4.1). The path $\gamma$ induces a path $\gamma(a,b)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ by action on $(a,b)$. The intersection number of $\gamma(a,b)$ and the line $a = 0$ is called winding number $n_\tau(a,b)$.

Then the action of $\tau$ is given by the following formulas

$$\tau(e) = e^m (e')^m, \quad \tau(e') = e^{m'} (e')^{m'}, \quad (4.2)$$

$$\tau(P_{a,b}) = \left((e')^{m'a+mb} e^{n'a+nb}\right)^{n_{\tau}(a,b)} P_{m'a+mb,n'a+nb}. \quad (4.3)$$

Remark 4.2. Another remarkable property of the generators $P_{a,b}$ is that an analog of PBW theorem holds with respect to the generators. This is the reason for the term PBW presentation. We will formulate an appropriate analog of PBW theorem (see Proposition 7.11) and use this in the proof of Theorem 7.1.

Chevalley presentation The algebra has another presentation, the equivalence between them was shown in [Sch12]. The generators are $P_{1,b}$, $P_{-1,b}$ for $b \in \mathbb{Z}$, $P_{0,k}$ for $k \in \mathbb{Z} \neq 0$, and central elements $c^{\pm 1}$, $(c')^{\pm 1}$. To describe the relations let us introduce currents (formal power series with coefficients in the algebra $U_{q_1,q_2}(gl_1)$)

$$E(z) = \sum_{b \in \mathbb{Z}} P_{1,b} z^{-b}, \quad F(z) = \sum_{b \in \mathbb{Z}} P_{-1,b} z^{-b}. \quad (4.4)$$
Define
\[ \sum_{k>0} \theta_{\pm k} z^{-k} = \exp \left( \sum_{k>0} \frac{(1 - q_1^k)(1 - q_3^k)}{k} P_{0, \pm k} z^{-k} \right). \] (4.5)

The relations are the following. For \( k, l \in \mathbb{Z} \)
\[ [P_{0,k}, P_{0,l}] = k \frac{(1 - q_1^{|k|})(c^{|k|} - c^{-|k|})}{(1 - q_2^k)(1 - q_3^k)} \delta_{k+l,0}. \] (4.6a)

For \( k \in \mathbb{Z}_{>0} \) and \( b \in \mathbb{Z} \)
\[ [P_{0,k}, P_{1,b}] = c^{-k}(q_1^k - 1) P_{1,b+k}, \quad [P_{0,-k}, P_{1,b}] = (1 - q_1^k) P_{1,b-k}, \] (4.6b)
\[ [P_{0,k}, P_{-1,b}] = (1 - q_3^{-k}) P_{-1,b+k}, \quad [P_{0,-k}, P_{-1,b}] = (q_3^k - 1) c^k P_{-1,b-k}. \] (4.6c)

\[ (z - q_1 w)(z - q_2 w)(z - q_3 w) E(z) E(w) = -(w - q_1 z)(w - q_2 z)(w - q_3 z) E(w) E(z), \] (4.6d)
\[ (z - q_1^{-1} w)(z - q_2^{-1} w)(z - q_3^{-1} w) F(z) F(w) = -(w - q_1^{-1} z)(w - q_2^{-1} z)(w - q_3^{-1} z) F(w) F(z). \] (4.6e)

For \( a + b > 0 \)
\[ [P_{1,a}, P_{-1,b}] = \frac{(1 - q_1^a c^b)}{(1 - q_2)(1 - q_3)} \theta_{a+b}, \quad [P_{1,-a}, P_{-1,-b}] = \frac{(1 - q_1) c^{-b}(c')^{-1}}{(1 - q_2)(1 - q_3)} \theta_{-a-b}. \] (4.6f)

For \( a \in \mathbb{Z} \)
\[ [P_{1,a}, P_{-1,-a}] = \frac{(1 - q_1)(c^a c' - c^{-a}(c')^{-1})}{(1 - q_2)(1 - q_3)}; \] (4.6g)
\[ [P_{1,a}, [P_{1,a-1}, P_{1,a+1}]] = 0; \] (4.6h)
\[ [P_{-1,a}, [P_{-1,a-1}, P_{-1,a+1}]] = 0. \] (4.6i)

**Definition 4.3.** Algebra \( U_{q_1,q_2}(\tilde{\mathfrak{gl}}_1) \) is an algebra generated by \( P_{1,b}, P_{0,b}, P_{-1,b} \) for \( b \in \mathbb{Z} \) and central elements \( c^{\pm 1}, (c')^{\pm 1} \) with the relations (4.6a)–(4.6i).

**Definition 4.4.** Algebra \( U_{q_1,q_2}(\tilde{\mathfrak{gl}}_1)^+ \) is an algebra generated by \( P_{1,b}, P_{0,b} \) for \( b \in \mathbb{Z} \) and central elements \( c^{\pm 1}, (c')^{\pm 1} \) with the relations (4.6a), (4.6b), (4.6d), (4.6h).

**Definition 4.5.** Algebra \( U_{q_1,q_2}(\tilde{\mathfrak{gl}}_1)^- \) is an algebra generated by \( P_{-1,b}, P_{0,b} \) for \( b \in \mathbb{Z} \) and central element \( c^{\pm 1}, (c')^{\pm 1} \) with the relations (4.6a), (4.6c), (4.6e), (4.6i).

**Proposition 4.6.** The algebras \( U_{q_1,q_2}(\tilde{\mathfrak{gl}}_1)^+ \) and \( U_{q_1,q_2}(\tilde{\mathfrak{gl}}_1)^- \) are subalgebras of \( U_{q_1,q_2}(\tilde{\mathfrak{gl}}_1) \).

**Connection with spherical DAHA.** Denote
\[ |i\rangle_v^\pm = \frac{v^{\pm i}}{v^{\pm 2} - 1}, \quad |i\rangle_v = \frac{v^i - v^{-i}}{v^i - v^{-i}}, \] (4.7)
\[ [k]_v^{\pm} = [1]_v^{\pm} \cdots [k]_v^{\pm}, \quad [k]_v = [1]_v \cdots [k]_v. \] (4.8)

In this paragraph we will need to consider double affine Hecke algebras for different parameters \( q \) and \( v \), therefore we will write \( \mathcal{H}_N(q, v) \). Let \( S_+ \) and \( S_- \) be the symmetrizer and the anti-symmetrizer in finite Hecke algebra
\[ S_+ = \frac{1}{[N]!_v} \sum v^{l(\sigma)} T_\sigma, \quad S_- = \frac{1}{[N]!_v} \sum (-v)^{-l(\sigma)} T_\sigma. \] (4.9)

The basic property of \( S_\pm \) is that for \( i = 1, \ldots, N - 1 \)
\[ T_i S_+ = S_+ T_i = v S_+, \quad T_i S_- = S_- T_i = -v^{-1} S_. \] (4.10)

Let \( S\mathcal{H}_N(q, v) = S_\pm \mathcal{H}_N(q, v) S_\pm \) be the corresponding spherical DAHA.
Proposition 4.7. There is an algebra isomorphism $S\mathcal{H}_N^{(-)}(q,v) \cong S\mathcal{H}_N^{(+)}(q,-v^{-1})$.

Proof. The relations \[ \text{[2.1] - [2.4]} \] imply that there is an isomorphism $\mu: \mathcal{H}_N(q,v) \xrightarrow{\sim} \mathcal{H}_N(q,-v^{-1})$ defined by $\mu(T_i) = T_i$, $\mu(Y_i) = Y_i$, $\mu(\pi) = \pi$. To finish the proof we note that $\mu(S_-) = S_+$. \qed

Theorem 4.1 (SV11). The following formulas determine a surjection of the algebra $U_{q_1,q_2}(\mathfrak{g}_1)$ onto $S\mathcal{H}_N^{(+)}(q,v)$ for $q_1 = q$, $q_2 = v^2$

\begin{align*}
P_{0,k}^{(N)} &= S_+(Y_{1}^{k} + \cdots + Y_{N}^{k})S_+, & P_{0,-k}^{(N)} &= q^kS_+(Y_{1}^{-k} + \cdots + Y_{N}^{-k})S_+, \\
P_{k,0}^{(N)} &= q^kS_+(X_{1}^{k} + \cdots + X_{N}^{k})S_+, & P_{-k,0}^{(N)} &= S_+(X_{1}^{-k} + \cdots + X_{N}^{-k})S_+, \tag{4.11a} \\
P_{1,b}^{(N)} &= q[N]_{a}^{-1}S_{+}X_{1}Y_{1}^{b}S_+, & P_{-1,b}^{(N)} &= [N]_{a}^{+}S_{-}Y_{1}^{b}X_{1}^{-1}S_-. \tag{4.11b} \end{align*}

Here $k \in \mathbb{Z}_{>0}$, $b \in \mathbb{Z}$, and the image of $P_{a,b}$ is denoted by $P_{a,b}^{(N)}$.

Remark 4.8. In [SV11] the authors prove that the quotient of $U_{q_1,q_2}(\mathfrak{g}_1)$ by the relations $c = c' = 1$ is a projective limit of the corresponding subalgebras of $S\mathcal{H}_N^{(+)}$. This deep result is one of the motivations for the limit $N \to \infty$ to be studied below. Though formally speaking, we will not use the mentioned result of loc. cit.

Corollary 4.9. Algebra $U_{q_1,q_2}(\mathfrak{g}_1)$ surjects onto $S\mathcal{H}_N^{(+)}(q,v)$ for $q_1 = q$, $q_2 = v^{-2}$. Moreover

\begin{align*}
P_{0,k}^{(N)} &= S_-(Y_{1}^{k} + \cdots + Y_{N}^{k})S_-, & P_{0,-k}^{(N)} &= q^kS_-((Y_{1}^{-k} + \cdots + Y_{N}^{-k})S_-, \\
P_{k,0}^{(N)} &= q^kS_-(X_{1}^{k} + \cdots + X_{N}^{k})S_-, & P_{-k,0}^{(N)} &= S_-(X_{1}^{-k} + \cdots + X_{N}^{-k})S_-, \tag{4.12a} \\
P_{1,b}^{(N)} &= q[N]_{a}^{-1}S_-X_{1}Y_{1}^{b}S_-, & P_{-1,b}^{(N)} &= [N]_{a}^{+}S_-Y_{1}^{b}X_{1}^{-1}S_-. \tag{4.12b} \end{align*}

5 Deformed exterior power

Spherical DAHA $S\mathcal{H}_N^{(-)}(q,v)$ acts on the subspace $S_- C_{n_{0},\ldots,n_{n-1}}^{(n_{n-1})} = S_- (\mathbb{C}^{n}[Y^{\pm 1}])^N \subset (\mathbb{C}^{[n]}Y^{\pm 1})^N$. The space $S_- (\mathbb{C}^{[n]}Y^{\pm 1})^N$ was considered in [KMS95, LT00]. In loc. cit., the authors considered only affine Hecke algebra action on $(\mathbb{C}^{[n]}Y^{\pm 1})^N$ but not DAHA. In this section, we will recall and extend their results. Spherical DAHA will be considered in the subsequent sections.

5.1 Finite $v$-wedge

The $v$-deformed exterior power can be defined as a subspace $S_- (\mathbb{C}^{[n]}Y^{\pm 1})^N$. On the other hand it can be identified with the quotient space via tautological projection

\begin{equation}
S_- (\mathbb{C}^{[n]}Y^{\pm 1})^N \xrightarrow{\sim} (\mathbb{C}^{[n]}Y^{\pm 1})^N / \sum_i \text{Im}(T_i + v^{-1}) \tag{5.1}
\end{equation}

The inverse map is induced by $S_-$. We will use both interpretations as the subspace and as the quotient. Denote by $e_{i_1} \wedge \cdots \wedge e_{i_n} = S_-(e_{i_1} \otimes \cdots \otimes e_{i_n})$.

Lemma 5.1 ([KMS95 eq. (41), (42)]) Let $l = h + nk + s$ for $k \geq 0$ and $s = 0, \ldots, n-1$. Then

\begin{align*}
e_{l} \wedge e_{k} &= - e_{l} \wedge e_{k} & \text{for } s = 0 \tag{5.2a} \\
e_{l} \wedge e_{h} &= - ve_{h} \wedge e_{l} & \text{for } k = 0 \tag{5.2b} \\
e_{l} \wedge e_{h} &= - ve_{h} \wedge e_{l} - e_{l-nk} \wedge e_{h+nk} - ve_{h+nk} \wedge e_{l-nk} & \text{otherwise} \tag{5.2c}
\end{align*}

The above identities can be used for vectors of the form $e_{i_1} \wedge \cdots \wedge e_{l} \wedge e_{h} \wedge \cdots \wedge e_{i_N}$. 16
Proposition 5.2 ([KMS95, Prop. 1.3]). The vectors $e_{i_1} \wedge \cdots \wedge e_{i_N}$ for $i_1 < i_2 < \cdots < i_N$ form a basis of $S_{-}(\mathbb{C}^{n}[Y^{\pm 1}])^{\otimes N}$.

Lemma 5.3 ([LT00, Lemma 7.6]). Let $k_1, \ldots, k_N$ be integers such that $\sum_{i=1}^{N} (i - m - k_i) < N$ and all $k_i < N - m$ for certain $m \in \mathbb{Z}$. Then $e_{k_1} \wedge \cdots \wedge e_{k_N} = 0$.

**Vertex operators** Considering a vector $w$ as an element of the subspace and the quotient space

$$w = \sum_{t_1, \ldots, t_N} y_{t_1, \ldots, t_N} e_{t_1} \otimes \cdots \otimes e_{t_N} \in S_{-}(\mathbb{C}^{n}[Y^{\pm 1}])^{\otimes N}, \quad w = \sum_{i_1 < \cdots < i_N} x_{i_1, \ldots, i_N} e_{i_1} \wedge \cdots \wedge e_{i_N}.$$ 

Let us define (modes of) vertex operators $\Phi_k, \Psi_k : S_{-}(\mathbb{C}^{n}[Y^{\pm 1}])^{\otimes N} \to S_{-}(\mathbb{C}^{n}[Y^{\pm 1}])^{\otimes (N+1)}$ by the formula

$$\Phi_k(w) = \sum_{i_1 < \cdots < i_N} x_{i_1, \ldots, i_N} e_k \wedge e_{i_1} \wedge \cdots \wedge e_{i_N}, \quad \Psi_k(w) = \sum_{i_1 < \cdots < i_N} x_{i_1, \ldots, i_N} e_{i_1} \wedge \cdots \wedge e_{i_N} \wedge e_k, \quad (5.3)$$

Note that here $w$ is considered as an element of the quotient. The vertex operators $\Phi_k, \Psi_k$ can also be defined in terms of the subspace.

$$\Phi_k(w) = S_{-}^{(N+1)} \sum_{i_1 < \cdots < i_N} x_{i_1, \ldots, i_N} e_k \otimes e_{i_1} \otimes \cdots \otimes e_{i_N} = S_{-}^{(N+1)} S_{-}^{(N)} \sum_{i_1 < \cdots < i_N} x_{i_1, \ldots, i_N} e_k \otimes e_{i_1} \otimes \cdots \otimes e_{i_N} = S_{-}^{(N+1)} \sum_{t_1, \ldots, t_N} y_{t_1, \ldots, t_N} e_k \otimes e_{t_1} \otimes \cdots \otimes e_{t_N},$$

where $S_{-}^{(N)}$ denotes anti-symmetrizer in $\mathcal{H}_N$.

Define (modes of) the dual vertex operators $\Phi^*_k, \Psi^*_k : S_{-}(\mathbb{C}^{n}[Y^{\pm 1}])^{\otimes N} \to S_{-}(\mathbb{C}^{n}[Y^{\pm 1}])^{\otimes (N-1)}$ by the formula

$$\Phi^*_k(w) = \sum_{t_2, \ldots, t_N} y_{-k, t_2, \ldots, t_N} e_{t_2} \otimes \cdots \otimes e_{t_N} = \sum_{t_1, \ldots, t_{N-1}} y_{t_1, \ldots, t_{N-1}, -k} e_{t_1} \otimes \cdots \otimes e_{t_{N-1}}. \quad (5.4)$$

Note that here $w$ is considered as an element of the subspace. It is easy to see that

$$(T_i + v^{-1}) \sum_{t_2, \ldots, t_N} y_{-k, t_2, \ldots, t_N} e_{t_2} \otimes \cdots \otimes e_{t_N} = (T_i + v^{-1}) \sum_{t_1, \ldots, t_{N-1}} y_{t_1, \ldots, t_{N-1}, -k} e_{t_1} \otimes \cdots \otimes e_{t_{N-1}} + 0$$

for any $1 \leq i \leq N - 2$. Hence the image of $\Phi^*_k$ and $\Psi^*_k$ indeed belongs to $S_{-}(\mathbb{C}^{n}[Y^{\pm 1}])^{\otimes N-1}$.

Consider operators

$$b_j = Y^j_i + \cdots + Y^j_N. \quad (5.5)$$

Lemma 5.4. The following relations hold for $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\neq 0}$

$$[b_j, \Phi_k] = \Phi_{k+nj}, \quad [b_j, \Psi_k] = \Psi_{k+nj}, \quad [b_j, \Phi^*_k] = - \Phi^*_{k+nj}, \quad [b_j, \Psi^*_k] = - \Psi^*_{k+nj}. \quad (5.6)$$

**Proof.** Follows directly from the following formulas

$$b_j \sum_{i_1 < \cdots < i_N} x_{i_1, \ldots, i_N} e_{i_1} \wedge \cdots \wedge e_{i_N} = \sum_{i_1 < \cdots < i_N} \sum_{r=1}^{N} x_{i_1, \ldots, i_N} e_{i_1} \wedge \cdots \wedge e_{i_r+nj} \wedge \cdots \wedge e_{i_N},$$

$$b_j \sum_{t_1, \ldots, t_N} y_{t_1, \ldots, t_N} e_{t_1} \otimes \cdots \otimes e_{t_N} = \sum_{t_1, \ldots, t_N} \sum_{r=1}^{N} y_{t_1, \ldots, t_N} e_{t_1} \otimes \cdots \otimes e_{t_r+nj} \otimes \cdots \otimes e_{t_N}.$$

\[\square\]
Involution In order to prove certain properties of vertex operators we will need bar involution. It is an antilinear map \( \bar{\psi} = v^{-1} \) and its action on \((C^n[Y^\pm 1])^\otimes N\) given by the formula [LT00] Prop. 5.5
\[
e_{i_1} \otimes \cdots \otimes e_{i_N} = v^{N(N-1)/2}p_t e_{i_1} \otimes \cdots \otimes e_{t_1}.
\] (5.7)

Here \( t = \{t_1, \ldots, t_N\} \), \( p_t \) is the number of pairs \( \{t_r, t_s\} \) such that \( t_r \neq t_s \) mod \( n \), \( w_0 \) is the longest element in Weyl group and \( T_{w_0} \) is the corresponding element of the Hecke algebra. This is an ad hoc definition, see [LT00] for more details.

Bar involution preserves the subspace \( S_- (C^n[Y^\pm 1])^\otimes N \) and acts by the formula [LT00] Prop. 5.9
\[
e_{i_1} \wedge \cdots \wedge e_{i_N} = (-1)^{N(N-1)/2} p_t e_{i_1} \wedge \cdots \wedge e_{i_N}.
\] (5.8)

**Lemma 5.5.** The following holds
\[
(-1)^N v^{-a(k)} \Psi \bar{e}_{i_1} \otimes \cdots \otimes e_{i_N} = \Phi e_{i_1} \otimes \cdots \otimes e_{i_N},
\] (5.9)
\[
(-1)^{N-1} v^{-a(k)} \Psi \bar{e}_{i_1} \otimes \cdots \otimes e_{i_N} = \Phi^* e_{i_1} \otimes \cdots \otimes e_{i_N}.
\] (5.10)

Here \( a(k) \) is the number of \( r \) such that \( i_r \neq k \) mod \( n \).

**Proof.** For the first relation we use (5.8) and obtain
\[
\Psi \bar{e}_{i_1} \otimes \cdots \otimes e_{i_N} = \Psi (-1)^{N(N-1)/2} v^{-p_t} e_{i_1} \wedge \cdots \wedge e_{i_N} = (-1)^{N(N-1)/2} v^{-p_t} e_{i_1} \wedge \cdots \wedge e_{i_N} = (-1)^N v^{-a(k)} e_{i_1} \wedge \cdots \wedge e_{i_N}.
\]
For the second relation we use notation \( e_{i_1} \wedge \cdots \wedge e_{i_N} = \sum_j y_j e_{j_1} \otimes \cdots \otimes e_{j_N} \). Then
\[
e_{i_1} \wedge \cdots \wedge e_{i_N} = (-v)^{-N(N-1)/2} T_{w_0} e_{i_1} \wedge \cdots \wedge e_{i_N} = (-1)^{N(N-1)/2} v^{-p_t} \sum_j y_j e_{j_1} \otimes \cdots \otimes e_{j_N}.
\] (5.11)

Here we used relations (5.7) and \( p_t = p_\bar{t} \). Therefore, using (5.11) twice, we obtain
\[
\Psi \bar{e}_{i_1} \otimes \cdots \otimes e_{i_N} = \sum_j (-1)^{N(N-1)/2} v^{-p_t} y_j \Psi \bar{e}_{j_1} \otimes \cdots \otimes e_{j_N}
\]
\[
= \sum_{j_2, \ldots, j_N} (-1)^{N(N-1)/2} v^{-p_t} y_{-k, j_2, \ldots, j_N} \bar{e}_{j_1} \otimes \cdots \otimes e_{j_N}
\]
\[
= (-1)^N v^{-a(k)} e_{i_1} \wedge \cdots \wedge e_{i_N} = (-1)^{N-1} v^{-a(k)} \Phi^* e_{i_1} \wedge \cdots \wedge e_{i_N}.
\] □

### 5.2 The limit

Let us consider an inductive system of vector spaces
\[
S_- \otimes C^n [Y^\pm 1] \xrightarrow{\varphi_2^{(s)}} \cdots \xrightarrow{\varphi_3^{(s)}} \xrightarrow{\varphi_{n+1}^{(s)}} S_- \otimes C^n [Y^\pm 1] \otimes \cdots \to \cdots
\] (5.12)

with the maps \( \varphi_r^{(s)} = \varphi_r^{(s)} : S_- \otimes C^n [Y^\pm 1] \to S_- \otimes C^n [Y^\pm 1] \). Also, let us define maps \( \varphi_r^{(s)}(w) = w \otimes e_{N-n} \otimes \cdots \otimes e_{N-n} \) for \( R > N \).

By \( \varphi_r^{(s)} = \varphi_r^{(s)} : S_- \otimes C^n [Y^\pm 1] \otimes \cdots \to \cdots \) we denote the canonical map.

**Definition 5.6.** A sequence of operators \( A^{(N)} : S_- \otimes C^n [Y^\pm 1] \to S_- \otimes C^n [Y^\pm 1] \) stabilizes if for any \( w \in S_- \otimes C^n [Y^\pm 1] \) there is \( M \) such that for any \( N > M \) we have
\[
\varphi_r^{(s)}(w) \circ A^{(N)}(w) = A^{(N)}(w) \circ \varphi_r^{(s)}(w).
\] (5.13)
If a sequence $A^{(N)}$ stabilizes, then it induces an operator $\hat{A}$: $\Lambda_{n,m}^{N/2}(\mathbb{C}^n[Y^{\pm 1}]) \to \Lambda_{n,m+\delta}^{N/2}(\mathbb{C}^n[Y^{\pm 1}])$. Actually, the operator $\hat{A}$ depends on the residue of $N$ modulo $n$. We will omit this dependence in our notation.

**Proposition 5.7.** Let $A^{(N)}_1$ and $A^{(N)}_2$ stabilize. Then the composition $A^{(N)}_1 A^{(N)}_2$ stabilizes and the induced operator equals to the composition of induced operators $A_1 A_2$.

**Proposition 5.8.** [LT00, Sect. 7.6] Action of bar involution stabilizes.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ be a partition with $l(\lambda) = r \leq N$. Consider a vector

$$|\lambda\rangle_{N,m} = e_{-\lambda_1 - m} e_{-\lambda_2 - 2m} \cdots e_{-\lambda_r - rm} e_{r-m} \cdots e_{N-1-m}. \quad (5.14)$$

We will abbreviate $|\lambda\rangle_{N,m} = |\lambda\rangle_N = |\lambda\rangle$ if the indices are clear from the context. We will write $|\lambda\rangle_{\infty,m} = \varphi^{(m)}_{\infty,N} |\lambda\rangle_{N,m}$.

**Lemma 5.9.** For $m - k + |\lambda| < N$ we have

$$[N]_v^+ (\Phi_k^* |\lambda\rangle_N) \wedge e_{N-m} = [N + 1]_v^+ \Phi_k^* |\lambda\rangle_{N+1,m}. \quad (5.15)$$

**Proof.** Let us introduce notation $|\lambda\rangle_{N,m} = \sum_{j_1, \ldots, j_N} y_{j_1, \ldots, j_N} e_{j_1} \otimes \cdots \otimes e_{j_N}$. Then we have

$$\text{LHS of (5.15)} = [N]_v^+ \left( \sum_{j_2, \ldots, j_N} y_{-k, j_2, \ldots, j_N} e_{j_2} \otimes \cdots \otimes e_{j_N} \right) \wedge e_{N-m}$$

$$= [N]_v^+ \sum_{j_2, \ldots, j_N} y_{-k, j_2, \ldots, j_N} e_{j_2} \wedge \cdots \wedge e_{j_N} \wedge e_{N-m}. \quad (5.16)$$

To compute the RHS of (5.15), we will use factorization formula

$$S_{-(N+1)}^{(N+1)} = \frac{1}{[N + 1]_v^2} \left( \sum_{p=1}^{N+1} (-v)^{p+N-1} T_p \cdots T_N \right) S_{-(N)}^{(N)}, \quad (5.17)$$

here for $p = N + 1$ we mean $T_p \cdots T_N = 1$. Using (5.17) we obtain that

$$|\lambda\rangle_{N,m} \wedge e_{N-m} = \frac{1}{[N + 1]_v^2} \left( \sum_{p=1}^{N+1} (-v)^{p+N-1} T_p \cdots T_N \right) \left( \sum_j y_j e_{j_1} \otimes \cdots \otimes e_{j_N} \otimes e_{N-m} \right).$$

Here and below we use multi-index notation $j = (j_1, \ldots, j_N)$. For the action of $T_p \cdots T_N$ we will use (3.6)–(3.9). Informally, (3.6)–(3.9) say that under the action of $T$, the vectors either remains the same, or permute, or approach to each other.

In the computation below, lower terms stands for linear combination of terms $e_{l_1} \otimes \cdots \otimes e_{l_{N+1}}$ where $\forall i, l_i < N - m$. Then we get

$$[N + 1]_v^+ |\lambda\rangle_{N,m} \wedge e_{N-m} = \left( \sum_j \left( \sum_{p=1}^{N} (-v)^{p+N-1} T_p \cdots T_{N-1} y_j \right. \right.$$

$$\left. \left( v^{d_{j,N-N-m}} e_{j_1} \otimes \cdots \otimes e_{j_{N-1}} \otimes e_{N-m} \otimes e_{j_N} + (v - v^{-1}) e_{j_1} \otimes \cdots \otimes e_{j_N} \otimes e_{N-m} \right) \right.$$

$$\left. + (-v)^{2N} y_j e_{j_1} \otimes \cdots \otimes e_{j_N} \otimes e_{N-m} \right) + \text{lower terms}$$

$$= \left( \sum_j \left( \sum_{p=1}^{N} (-v)^{p+N-1} v^{d_{j,N-N-m}} T_p \cdots T_{N-1} y_j e_{j_1} \otimes \cdots \otimes e_{j_{N-1}} \otimes e_{N-m} \otimes e_{j_N} \right. \right.$$

$$\left. \left. + (-v)^{2N} y_j e_{j_1} \otimes \cdots \otimes e_{j_N} \otimes e_{N-m} \right) + \text{lower terms} \right)$$

$$= \sum_{q=1}^{N+1} \sum_j y_j \sum_{r=1}^{q} d_{j,r-N} (-v)^{N-r+1} y_j e_{j_1} \otimes \cdots \otimes e_{j_{r-1}} \otimes e_{N-m} \otimes e_{j_r} \otimes \cdots \otimes e_{j_N} + \text{lower terms}$$

$$= \sum_{q=1}^{N+1} \sum_j y_j \sum_{r=q}^{N} d_{j,r-N} (-v)^{N-q+1} y_j e_{j_1} \otimes \cdots \otimes e_{j_{r-1}} \otimes e_{N-m} \otimes e_{j_r} \otimes \cdots \otimes e_{j_N} + \text{lower terms}.$$
In the computation we used \((3.6), (3.8)\) and relation \(T_p \sum_j y^j e_{j_1} \otimes \cdots \otimes e_{j_{q-1}} \otimes e_{N-m} e_{j_q} \otimes \cdots e_{j_N} = (-v)^{-1} \sum_j y^j e_{j_1} \otimes \cdots \otimes e_{j_{q-1}} \otimes e_{j_q} \otimes \cdots e_{j_N}\) for \(p < q - 1\).

In order to compute the RHS of \((5.15)\), we apply \(S^{(N)} \Phi_k^*\). By Lemma 5.3, the assumption \(m - k + |\lambda| < N\) implies that the lower terms vanish after the action of \(S^{(N)}\). Hence we get

\[
\text{RHS of } (5.15) = \sum_{j_2, \ldots, j_N} \sum_{q=2}^{N+1} v^{\sum_{r=q}^N \delta_{j_r} - N-m} ( -v )^{N-q+1} y^{j_q} e_{j_2} \wedge \cdots \wedge e_{j_{q-1}} \wedge e_{N-m} \wedge e_{j_q} \wedge \cdots \wedge e_{j_N} \\
= \sum_{j_2, \ldots, j_N} \sum_{q=2}^{N+1} ( -v )^{2N+2-2q} y^{j_q} e_{j_2} \wedge \cdots \wedge e_{j_N} \wedge e_{N-m} \\
= [N]_v \sum_{j_2, \ldots, j_N} y^{j_q} e_{j_2} \wedge \cdots \wedge e_{j_N} \wedge e_{N-m} \quad (5.18)
\]

where we used Lemma 5.1 to permute \(e_{N-m}\) to the right and Lemma 5.3 to cancel out additional lower terms. Comparing the formulas \((5.16)\) and \((5.18)\), we get the result.

\[\square\]

**Proposition 5.10.** Action of \(\Phi_{k}^*\) and \(\tilde{\Phi}_{k}^* = [N]_v \Phi_{k}^*\) stabilize.

**Proof.** Lemma 5.1 implies that action of \(\Phi_{k}^*\) stabilizes for \(N - m > k\). Lemma 5.9 implies that \(\tilde{\Phi}_{k}^*\) stabilizes.

\[\square\]

**Remark 5.11.** We used Lemma 5.9 in the proof above. But in Section 6 below, we will need a refinement of this result. Let us introduce the following notation

\[
\tilde{\Phi}_{k}^*|\lambda\rangle_{N,m} = \sum_{\mu} x_{\mu} e_{-\mu_1+N-m} \wedge \cdots \wedge e_{-\mu_{N-1}+N-m-1}, \quad (5.19)
\]

\[
\tilde{\Psi}_{k}^*|\lambda\rangle_{N+1,m} = \sum_{\mu} x_{\mu}' e_{-\mu_1+N-m} \wedge \cdots \wedge e_{-\mu_{N}+N-m}. \quad (5.20)
\]

Then for \(l(\mu) \leq N - 1\) we have \(x_{\mu} = x_{\mu}'\).

The proof is based on the same computation as the proof of Lemma 5.9. The lower term no longer vanish after the action of \(S^{(N)}\) since we have dropped the assumption \(m - k + |\lambda| < N\). Nevertheless, the lower terms do not contribute to \(x_{\mu}'\) for \(l(\mu) \leq N - 1\).

**Proposition 5.12.** The following operators stabilize

\[
\tilde{\Psi}_k = (-1)^{N-1} u^{\left[\frac{N-1}{m-k}\right]} N \Psi_k, \quad \tilde{\Psi}_k^* = (-1)^{N-1} [N]_v u^{N-1-\left[\frac{N-1}{m-k}\right]} \Psi_k^*. \quad (5.21)
\]

**Proof.** Follows from the previous proposition and Lemma 5.5

\[\square\]

**Remark 5.13.** We have chosen the factors in the definition of \(\tilde{\Phi}_{k}^*\), \(\tilde{\Psi}_{k}^*\), and \(\bar{\Psi}_{k}^*\) such that

\[
\Phi_{-m-1}|\emptyset\rangle_{N,m} = |\emptyset\rangle_{N+1,m+1}, \quad \Phi_{m}^*|\emptyset\rangle_{N,m} = |\emptyset\rangle_{N-1,m-1}, \quad (5.22)
\]

\[
\tilde{\Psi}_{-m-1}|\emptyset\rangle_{N,m} = |\emptyset\rangle_{N+1,m+1}, \quad \bar{\Psi}_{m}^*|\emptyset\rangle_{N,m} = |\emptyset\rangle_{N-1,m-1}. \quad (5.23)
\]

The statement for \(\Phi_{-m-1}\) is obvious. For \(\Phi_{m}^*\) one can argue by induction using Lemma 5.9. Formula (5.23) follows from (5.22) by Lemma 5.5.

**Remark 5.14.** Also, the operators \(\Phi_{k}^*, \tilde{\Phi}_{k}^*, \tilde{\Psi}_{k}^*, \text{ and } \bar{\Psi}_{k}^*\) satisfy certain intertwining properties \([FR92]\). We will discuss the properties for \(\Phi_{k}^*\) in Appendix A.2. The properties for other operators are analogous. The intertwining properties and the normalization conditions \((5.22), (5.23)\) determine the operators uniquely. In particular, the factors in the definition of \(\Phi_{k}^*, \tilde{\Psi}_{k}^*, \text{ and } \bar{\Psi}_{k}^*\) are determined uniquely. Also, note that we do not define \(\Phi_{k}\) since the operators \(\Phi_{k}\) satisfy all the properties.
Let us denote the induced operators as follows

\[ \Phi_k : \Lambda_{v,m}^{n/2}(C^n[Y^{±1}]) \rightarrow \Lambda_{v,m+1}^{n/2}(C^n[Y^{±1}]), \quad \Psi_k : \Lambda_{v,m}^{n/2}(C^n[Y^{±1}]) \rightarrow \Lambda_{v,m-1}^{n/2}(C^n[Y^{±1}]). \]  
(5.24)  
(5.25)

**Definition 5.15.** A sequence of operators \( A^{(N)} : S_-(C^n[Y^{±1}])^\otimes N \rightarrow S_-(C^n[Y^{±1}])^\otimes (N+\delta) \) weakly stabilizes if for any \( w \in S_-(C^n[Y^{±1}])^\otimes k \) there is \( M \) such that for any \( N > M \) we have

\[ \varphi^{(m+\delta)}_N \circ A^{(N)} \circ \varphi^{(m)}_N(w) = \varphi^{(m+\delta)}_{N+n+\delta} \circ A^{(N+n)} \circ \varphi^{(m)}_{N+n,k}(w). \]  
(5.26)

**Proposition 5.16.** ([KMS95]). The operators \( b_k = Y_1^k + \cdots + Y_N^k \) stabilize for \( k < 0 \) and weakly stabilize for \( k > 0 \). The induced operators \( B_k \) satisfy a deformed Heisenberg algebra relation

\[ [B_k, B_l] = k[n]_r^+ \delta_{k+l,0}. \]  
(5.27)

**Example 5.17.** The sequence of operators \( b_1 = Y_1 + \cdots + Y_N \) does not stabilize. It can be seen from the formula

\[ b_1 e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} = \sum_{k=0}^{n-1} (-v)^k e_0 \wedge e_1 \wedge \cdots \wedge \hat{e}_{N-1-k} \wedge \cdots \wedge e_{N-1} \wedge e_{N-1-k+n}. \]  
(5.28)

Definitely, the RHS of (5.28) does not belong to the image of \( \varphi^{(0)}_{N,N-n} \). Though the RHS of (5.28) belongs to the kernel of \( \varphi^{(0)}_N \). Hence \( B_1(\emptyset)_{∞,0} = B_1 \varphi^{(0)}_N(e_0 \wedge \cdots \wedge e_{N-1}) = 0 \). Moreover, note that

\[ b_1 e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} = [n]_r^+ (\emptyset)_{N,0} + \ker \varphi^{(0)}_N, \]  
(5.29)

\[ B_1 B_1(\emptyset)_{∞,0} = 0. \]  
(5.30)

This means that composition of induced operators does not have to be equal to the induced operator of the composition (if the second operator just weakly stabilizes). Also, this illustrates the fact that \( [B_1, B_{-1}] = [n]_r^+ \), though \( [b_1, b_{-1}] = 0 \).

**Proposition 5.18.** Let a sequence \( A_1^{(N)} \) weakly stabilizes and \( A_2^{(N)} \) stabilizes. Then the composition \( A_1^{(N)} A_2^{(N)} \) weakly stabilizes and the induced operator equals the composition of induced operators \( \hat{A}_1 \hat{A}_2 \).

**Proposition 19.** The following relations hold for any \( k \in \mathbb{Z} \) and \( j \in \mathbb{Z}_{>0} \)

\[ [B_{-j}, \Phi_k] = \Phi_{k-nj}, \quad [B_{-j}, \Psi_k] = v^{-j} \Psi_{k-nj}, \quad [B_{-j}, \Phi_k^±] = -\Phi_{k-nj}^±, \quad [B_{-j}, \Psi_k^±] = -v^{-j} \Psi_{k-nj}^±, \]  
(5.31)

\[ [B_j, \Phi_k] = \Phi_{k+nj}, \quad [B_j, \Psi_k] = v^{2j(n-1)} \Psi_{k+nj}, \quad [B_j, \Phi_k^±] = -v^{2jn} \Phi_{k+nj}^±, \quad [B_j, \Psi_k^±] = -v^{-2jn} \Psi_{k+nj}^±. \]  
(5.32)

**Proof.** Commutation relations (5.31) follows from (5.6) since \( b_{-j} \) stabilizes. Also, one can check that

\[ [B_j, \Phi_k] \varphi^{(m)}_N(w) = \varphi^{(m)}_N([b_j, \Phi_k]w) = \varphi^{(m)}_N(\Phi_k + n)w = \Phi_{k+n} \varphi^{(m)}_N(w). \]  
(5.33)

To prove the relation with \( \Psi_k \) we use Lemma 5.5 and [LT00 Prop. 7.8] that \( B_j = v^{-2j(n-1)} B_j \) for \( j > 0 \). Let \( a_\lambda(k) = a_\lambda(k) + [(N-1-m-k)/n] - N \) for sufficiently large \( N \). Then

\[ [B_j, \Psi_k] |\lambda\rangle_\infty = (-1)^N [v^{-2j(n-1)} B_j, v^{-a_\lambda(k) \Phi_k}] |\lambda\rangle_\infty \]

\[ = (-1)^N v^{-2j(n-1)} v^{-j a_\lambda(k+nj)} \Phi_{k+nj} |\lambda\rangle_\infty = v^{2j(n-1)} \Psi_{k+nj} |\lambda\rangle_\infty. \]  
(5.34)

\[^3\text{We use notation } a_\lambda(k) = a_i(k) \text{ for } i_j = -\lambda_j + j - 1 - m, \text{ see Lemma 5.5.} \]
The relation $[B_j, \hat{\Phi}_k^\ast] = -v^{2jn}\hat{\Phi}_{k+nj}^\ast$ is proven in Appendix A.3 (Theorem A.1). Finally, 

$$[B_j, \hat{\Phi}_k^\ast]|\lambda\rangle = (1)^{N-1}[v^{-2j(n-1)}B_j, v^{a_k^\ast(k)}\hat{\Phi}_k^\ast]|\lambda\rangle_{\infty} = (1)^{N}v^{-2j(n-1)+2jn} \times v^{-j+a_k^\ast(k+nj)}\hat{\Phi}_{k+jn}^\ast|\lambda\rangle_{\infty} = -v^{-j}\hat{\Phi}_{k+jn}^\ast|\lambda\rangle_{\infty},$$

(5.35)

here $a_k^\ast(k) = a_\lambda(k) + [(N - 1 - m + k)/n] - N + 1$ for sufficiently large $N$. 

\[\square\]

6 Semi-infinite construction of twisted Fock module I

The goal of this and the next sections is to provide an explicit construction for the action of $U_{q_1,q_2}(\hat{\mathfrak{g}}_1)$ on twisted Fock module $F_u^\nu$ (Theorem 7.1). This is the central result of the whole paper. Our method is semi-infinite construction. Namely, we will use the explicit realization of $C_u = C^{(n,n')}_u$ (Theorem 3.1) to derive an explicit construction of $F_u^\nu$ as a limit $N \to \infty$.

In Section 6.2 we will study the limit $N \to \infty$ for the Chevalley generators (after a rescaling) denoted by $P_{1,b}^{(N)}, P_{0,b}^{(N)},$ and $P_{-1,b}^{(N)}$. It turns out that the generators $P_{1,b}^{(N)}$ and $P_{-1,b}^{(N)}$ converge for $|q^{-1}v^2| < 1$ and $|q^{-1}v^2| > 1$ respectively. Therefore we can not obtain the action of whole $U_{q_1,q_2}(\hat{\mathfrak{g}}_1)$ by a straightforward limit argument. Though we prove that we do obtain actions of the subalgebras $U_{q_1,q_2}(\hat{\mathfrak{g}}_1)^+$ and $U_{q_1,q_2}(\hat{\mathfrak{g}}_1)^-$. We get explicit formulas for the limit of Chevalley generators. The formulas allow us to make analytic continuation for general $q$ and $v$ (subsection 6.3). Also, we consider the formulas in the case $n = 1, n’ = 0$ and show that the obtained operators give (non-twisted) Fock module of $U_{q_1,q_2}(\hat{\mathfrak{g}}_1)$ (subsection 6.4). We will prove for general $n$ and $n’$ that the obtained operators give twisted Fock module of whole $U_{q_1,q_2}(\hat{\mathfrak{g}}_1)$ in Section 7.

6.1 Finite case

It follows from the Corollary 4.9 that

$$P_{1,b}^{(N)} = (-1)^{N-1}[v]_n S_\pi^{-1} Y_1^b S_-, \quad P_{-1,b}^{(N)} = (-1)^{N-1}[v]_n S_ Y_1^b \pi S_-.$$ 

(6.1)

Recall the representation $C^{(n,n’)}_{u_0,\ldots,u_{n-1}}$, see Theorem 3.1. By the construction, we have

$$P_{0,b}^{(N)} = b_k, \quad P_{0,-k}^{(N)} = q^{kb}_{-k},$$

(6.2a)

$$P_{1,b}^{(N)} = (-1)^{N-1}[v]_n \sum_{a=0}^{n-1} \sum_{l \in \mathbb{Z}} u_a q^{-l} \Phi_{a+nl+n’+nb},$$

(6.2b)

$$P_{-1,b}^{(N)} = (-1)^{N-1}[v]_n \sum_{a=0}^{n-1} \sum_{l \in \mathbb{Z}} u_a q^l \Phi_{a+nl-n’+nb}. $$

(6.2c)

Note that for each vector $w \in S_-$ ($\mathbb{C}^{n}[Y^\pm1])^\otimes N$ only finitely many terms on the RHS of (6.2b)–(6.2c) have non-zero action.

To prepare to take the limit $N \to \infty$, let us rewrite the formulas above. Define $\tilde{P}_{\pm 1,b}^{(N)} = v^\pm \frac{N}{N} P_{\pm 1,b}^{(N)}$.

Then

$$\tilde{P}_{1,b}^{(N)} = q \sum_{a=0}^{n-1} \sum_{l \in \mathbb{Z}} u_a q^{-l} v^{\frac{N}{N}-\frac{N-1-s-a}{n}} \Phi_{a+nl+n’+nb},$$

(6.3a)

$$\tilde{P}_{-1,b}^{(N)} = \sum_{a=0}^{n-1} \sum_{l \in \mathbb{Z}} u_a q^l v^{-\frac{N-1-s-a}{n}} \Phi_{a+nl-n’+nb}. $$

(6.3b)
Remark 6.1. Let us write $\tilde{\Psi}^{(N,m)}_k$ instead of $\tilde{\Psi}_k$ to emphasize the dependence on $N$ and $m$. In the formula (6.3a) we should use $\tilde{\Psi}^{(N-1,m-1)}_k$. Indeed, the operator $\tilde{\Phi}^{s}_{a-nl+n'+nb}$ maps to $(\mathbb{C}^n[Y^{\pm 1}]) \otimes (N-1)$. Moreover, the inductive system must be taken for $m-1$. In the calculation above we use

$$
\tilde{\Psi}^{(N-1,m-1)}_a = (-1)^{N-1}u^{|\frac{N-1-m-a}{n}}}_{-N+1}v^{-1}u^{\frac{N-1-a}{n}}_a. 
$$

Recall that $u = u_0 \cdots u_{n-1} q^{1-n}$, see Theorem 3.4. Then we have

$$
\prod_{a=0}^{n-1} v^{|\frac{N-1-a}{n}}_{u_a} = v^{-n} \times q^n u. 
$$

(6.4)

We can choose any $u_0, \ldots, u_{n-1}$ such that (6.4) holds, see Corollary 3.10. Let us take

$$
u_a = v^{\frac{N}{n}} \cdot \left|\frac{N-1-a}{n}\right| \times v^\frac{k}{n} v^\frac{a}{n} \times v^\frac{a}{n} q^\frac{a}{n}. 
$$

(6.5)

As a corollary of the above discussion, we obtain the following proposition.

**Proposition 6.2.** The following formulas determine action of $S\mathcal{H}_N^{[-]}$ on $S_- (\mathbb{C}^n[Y^{\pm 1}]) \otimes N$

$$
P^{(N)}_{0, k} = b_k, 
$$

(6.6a)

$$
P^{(N)}_{0, -k} = q^k b_{-k}, 
$$

(6.6b)

$$
\tilde{P}^{(N)}_{1, b} = u^\frac{1}{n} v^\frac{a}{n} q \sum_{k \in \mathbb{Z}} q^{-k/n} \tilde{\Psi}^{\prime} \tilde{\Phi}^{s}_{-k+n'+nb}, 
$$

(6.6c)

$$
\tilde{P}^{(N)}_{-1, b} = u^\frac{1}{n} v^\frac{a}{n} q \sum_{k \in \mathbb{Z}} \tilde{\Psi}^{\prime} \tilde{\Phi}^{s}_{k-n'+nb} 
$$

here $\tilde{q} = v^{-1} q$. The obtained representation is isomorphic to $\mathbb{C}^\sigma_{k}$.

Denote the obtained representation by $\mathbb{C}^{(n,n')}_{k}$.

6.2 The limit for the right and left halves

Below we will construct an action of $U_{q_1, q_2}(\mathfrak{g}_1)$ on $\Lambda^\infty_{\nu, m} (\mathbb{C}^n[Y^{\pm 1}])$. Analogous results hold for $U_{q_1, q_2}(\mathfrak{g}_1)^-$. To simplify our notation, we will consider the case $u^{-\frac{1}{n}} v^\frac{1}{n} q = 1$ and $m = 0$. We will recover the parameter $u$ at the end. Recall $\tilde{q} = v^{-1} q$.

**Proposition 6.3.** The sequence $\tilde{P}^{(N)}_{1, b} = v^\frac{N}{n} P^{(N)}_{1, b}$ stabilizes for $n' + nb < 0$. The induced operator

$$
\tilde{P}^{(N)}_{1, b} = \sum_{k \in \mathbb{Z}} \tilde{q}^{-k(n'+nb)/n} \tilde{\Psi}^{s}_{k+n'+nb} \tilde{\Phi}^{s}_{-k}. 
$$

(6.7)

For any vector $\tilde{w} \in \Lambda^\infty_{\nu, 0} (\mathbb{C}^n[Y^{\pm 1}])$, only finitely many terms $\tilde{\Psi}^{s}_{k+n'+nb} \tilde{\Phi}^{s}_{-k} \tilde{w}$ are non-zero.

To prove the proposition we need certain preparations. Recall the notation $|\lambda|_{N, 0} = |\lambda|_N$ introduces in (5.11). We will need the following lemma.

**Lemma 6.4.** $\Psi_{-\Delta} \Phi^{s}_{-k} |\lambda\rangle_N = 0$ for $\Delta > 0$ and $k \geq |\lambda| + \Delta$.

**Proof.** Introduce notation $|\lambda|_N = \sum_j y_j e_j \otimes \cdots \otimes e_j$. Then

$$
\Psi_{-\Delta} \Phi^{s}_{-k} |\lambda\rangle_N = \sum_{j_1, \ldots, j_N} y_{k-j_1, \ldots, j_N} e_{j_1} \wedge \cdots \wedge e_{j_N} \wedge e_{k-\Delta}. 
$$

(6.8)
Consider the decomposition with respect to the basis $|\mu\rangle_N$ 

$$
\sum_{j_2,\ldots,j_N} y_{j_2} e_{j_2} \wedge \cdots \wedge e_{j_N} \wedge e_{k-\Delta} = \sum_{\mu_1,\ldots,\mu_N \geq \mu} x_{\mu_1,\ldots,\mu_N} e_{-\mu_1} \wedge e_{-\mu_2+1} \wedge \cdots \wedge e_{-\mu_N+N-1}. \tag{6.9}
$$

For the coefficients $x_{\mu_1,\ldots,\mu_N} \neq 0$ we have $|\mu\rangle = |\lambda\rangle + \Delta$.

There exists a number $t \in \{k, k+1, \ldots, N-1\}$ such that $t \notin \{j_2, \ldots, j_N\}$ for the terms of the RHS of (6.8). Then there is $t' \in \{k, k+1, \ldots, N-1\}$ such that $t' \notin \{-\mu_{k+1}+k, -\mu_{k+2}+k+1, \ldots, -\mu_N+N-1\}$. Hence there is $\mu_s > 0$ for $s \geq k+1$. Therefore $k+1 \leq |\mu| = |\lambda| + \Delta$, which contradicts the assumption of the lemma. \hfill \Box

\textbf{Proof of Proposition 6.3} Using Lemma 6.4 we can specify (6.6b) omitting zero terms and obtain

$$
P_{1,b}^{(N)}|\lambda\rangle_N = \sum_{k=-\lambda_1}^{|\lambda| - 1 - n'b} \hat{q}^{(-k-n'b)/n} \hat{\Psi}_{k+n'b} \hat{\Phi}^*_{-k} |\lambda\rangle_N. \tag{6.10}
$$

Propositions 5.10 and 5.12 imply that each term $\hat{q}^{(-k-n'b)/n} \hat{\Psi}_{k+n'b} \hat{\Phi}^*_{-k}$ stabilizes. \hfill \Box

\textbf{Convergence} Action of operators $\hat{P}_{1,b}^{(N)}$ does not weakly stabilize for $n' + nb > 0$. Therefore we will need the following notion.

\textbf{Definition 6.5.} Action of operators $A^{(N)}: S_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N} \to S_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ converges if for any $w \in S_- \left(\mathbb{C}^n[Y^{\pm 1}]\right)^{\otimes N}$ the following sequence converges for $R \to \infty$

$$
\varphi_{N+nR}^{(n)}(A^{(N+nR)} \circ \varphi_{N+nR}^{(n)}(w)). \tag{6.11}
$$

\textbf{Remark 6.6.} This is the first place in our article where we use that the base field is $\mathbb{C}$, but not a field of characteristic 0. Note that $\Lambda_{-\infty/2}^{n/2} \left(\mathbb{C}^n[Y^{\pm 1}]\right)$ is a graded vector space with finite-dimensional graded components. \textit{Convergence} of (6.11) is understood in sense of sequences in a finite-dimensional vector space over $\mathbb{C}$.

Actually, all the convergences below will follow from the convergence of infinite geometric series. Therefore all matrix elements in the limit will be rational functions.

\textbf{Proposition 6.7.} The operators $\hat{P}_{1,b}^{(N)} = v^{\frac{n}{2}} P_{1,b}^{(N)}$ converge for $|q^{-1}v^2| < 1$. Moreover, the induced operator $\hat{P}_{1,b}$ equals

$$
\hat{P}_{1,b} = \sum_{k \in \mathbb{Z}} q^{(-k-n'b)/n} \hat{\Psi}_{k+n'b} \hat{\Phi}^*_{-k}. \tag{6.12}
$$

In particular, the series on the RHS of (6.12) converges.

To prove the proposition we need certain preparations.

\textbf{Lemma 6.8.} $\hat{\Psi}_{k+\Delta+n} \hat{\Phi}^*_{-k-n}|\lambda\rangle_{\infty} = v \hat{\Psi}_{k+\Delta} \hat{\Phi}^*_{-k}|\lambda\rangle_{\infty}$ for $\Delta \geq 0$ and $k \geq |\lambda| - \Delta$.

\textbf{Proof.} Let $\Delta = nl - s$ for $s = 1, \ldots, n$. Lemma 6.4 implies that for $k + nl \geq |\lambda| + (nl - \Delta)$ we have $\hat{\Psi}_{k+\Delta} \hat{\Phi}^*_{-k-n}|\lambda\rangle_{\infty} = 0$, and for $k + nl \geq |\lambda| - nl + (nl - \Delta)$ we have $\hat{\Psi}_{k+\Delta} \hat{\Phi}^*_{-k-n} B_l |\lambda\rangle_{\infty} = 0$. Hence for $k \geq |\lambda| - \Delta$ we get using the Proposition 5.19

$$
0 = [B_l, \hat{\Psi}_{k+\Delta} \hat{\Phi}^*_{-k-n}|\lambda\rangle_{\infty} = v^{l(2n-1)} \left( \hat{\Psi}_{k+\Delta+l} \hat{\Phi}^*_{-k-l-n} - v^l \hat{\Psi}_{k+\Delta} \hat{\Phi}^*_{-k} \right) |\lambda\rangle_{\infty}. \tag{6.13}
$$

We argue by induction on $l$. For $l = 1$ the lemma follows from (6.13). Assuming induction hypothesis for $l - 1$, we have for $j + n \geq |\lambda| - (\Delta - n)$

$$
0 = [B_1, \hat{\Psi}_{j+\Delta+n} \hat{\Phi}^*_{-j-2n}|\lambda\rangle_{\infty} = v[B_1, \hat{\Psi}_{j+\Delta} \hat{\Phi}^*_{-j-n}]|\lambda\rangle_{\infty} = v^{2n-1} \left( \hat{\Psi}_{j+\Delta+2n} \hat{\Phi}^*_{-j-2n} - 2v \hat{\Psi}_{j+\Delta+n} \hat{\Phi}^*_{-j-n} + v^2 \hat{\Psi}_{j+\Delta} \hat{\Phi}^*_{-j} \right) |\lambda\rangle_{\infty}. \tag{6.14}
$$
Lemma 5.3. Let us consider the vector $\Psi_{k+\Delta+n}^\ast k-n - lv^{l-1}\Psi_{k+\Delta+n}\hat{\Phi}^\ast_{k-n} + (l-1)v^{l}\Psi_{k+\Delta}\hat{\Phi}^\ast_{k} |\lambda\rangle_\infty = 0. \tag{6.15}

Relations $\tag{6.13}$ and $\tag{6.15}$ imply $\Psi_{k+\Delta+n}\hat{\Phi}^\ast_{k-n} - v\Psi_{k+\Delta}\hat{\Phi}^\ast_{k} |\lambda\rangle_\infty = 0$, which completes the step of the induction. \hfill \square

This lemma implies that the series on the RHS of $\tag{6.12}$ converges (this boils down to the convergence of a geometric series). The next lemma is a finite analog valid before the limit $N \to \infty$.

Lemma 6.9. Relations $\tag{6.13}$ and $\tag{6.15}$ imply $\Psi_{k+\Delta+n}\hat{\Phi}^\ast_{k-n} - v\Psi_{k+\Delta}\hat{\Phi}^\ast_{k} |\lambda\rangle_\infty = 0$, which completes the step of the induction. \hfill \square

This lemma implies that the series on the RHS of $\tag{6.12}$ converges (this boils down to the convergence of a geometric series). The next lemma is a finite analog valid before the limit $N \to \infty$.

Lemma 6.9. $\Psi_{k+\Delta+n}\hat{\Phi}^\ast_{k-n} |\lambda\rangle_N = v\Psi_{k+\Delta+n}\hat{\Phi}^\ast_{k-n} |\lambda\rangle_N$ for $\Delta > 0$, and $N - \Delta > k \geq |\lambda| - \Delta + n$.

Proof. Let $\hat{\Phi}^\ast_{k-n} |\lambda\rangle_N = \sum_{\mu} \tilde{x}_\mu e_{-\mu_1 + 1} \wedge e_{-\mu_2 + 2} \wedge \cdots \wedge e_{-\mu_{N-1} + 1}$. We claim that

$$\Psi_{k+\Delta} \left( e_{-\mu_1 + 1} \wedge e_{-\mu_2 + 2} \wedge \cdots \wedge e_{-\mu_{N-1} + 1} \right) = 0, \quad \text{if } l(\mu) \neq k + \Delta. \tag{6.16}$$

Indeed, if $\mu_{k+\Delta} = 0$, then we get zero from the vanishing of the tale $e_{k+\Delta} \cdots \wedge e_{N-1} \wedge e_{k+\Delta}$ by Lemma 5.3.

If $\mu_{k+\Delta+1} > 0$ then there exists a number $t \in \{k + \Delta + 1, k + \Delta + 2, \ldots, N - 1\}$ such that $t \not\in \{-\nu_{k+\Delta+1} + k + \Delta, k + \Delta + 1, \ldots, -\nu_{N-1} + 1\}$ such that $t \not\in \{-\nu_{k+\Delta+2} + k + \Delta, 1, \ldots, -\nu_{N-1} + 1\}$. Consider the expansion

$$\Psi_{k+\Delta} \left( e_{-\mu_1 + 1} \wedge \cdots \wedge e_{-\mu_{N-1} + 1} \right) = \sum_{\nu_1 \geq \cdots \geq \nu_N} \tilde{x}_{\nu_1} \cdots \tilde{x}_{\nu_N} e_{-\nu_1} \wedge \cdots \wedge e_{-\nu_{N-1}}. \tag{6.17}$$

Then for each term on the RHS there is $t' \in \{k + \Delta + 1, k + \Delta + 2, \ldots, N - 1\}$ such that $t' \not\in \{-\nu_{k+\Delta+2} + k + \Delta, 1, \ldots, -\nu_{N-1} + 1\}$. Hence there is $\nu_0 > 0$ for $p > k + \Delta + 1$. Therefore $k + \Delta + 1 < |\nu| = |\lambda| - \Delta$, which contradicts the assumption of the lemma. Hence the expression $\tag{6.17}$ is zero.

Therefore

$$\Psi_{k+\Delta+n}\hat{\Phi}^\ast_{k-n} |\lambda\rangle_N = \Psi_{k+\Delta} \left( \sum_{\mu, l(\mu)=k+\Delta} \tilde{x}_\mu e_{-\mu_1 + 1} \wedge \cdots \wedge e_{-\mu_{k+\Delta} + k + \Delta} \wedge e_{k+\Delta+1} \wedge \cdots \wedge e_{N-1} \right) \tag{6.18}$$

It was shown in Remark 5.11 that the coefficients $\tilde{x}_\mu$ are stable. Hence

$$\Psi_{k+\Delta+n}\hat{\Phi}^\ast_{k-n} |\lambda\rangle_N = \Psi_{k+\Delta} \hat{\Phi}^\ast_{k} |\lambda\rangle_{N+n}. \tag{6.19}$$

Thus the lemma follows from Lemma 6.8. \hfill \square

Proof of Proposition 6.7. Due to Proposition 6.3, it is enough to consider the case $n' + nb > 0$. It follows from $\tag{6.6b}$ that

$$\tilde{P}^{(N)}_{1,b} = \sum_{k \in \mathbb{Z}} q^{-(k-n'-nb)/n} \Psi_{k+n'+nb}^* k-n. \tag{6.19}$$

We know that each term on the RHS of $\tag{6.19}$ stabilizes to the corresponding term on the RHS of $\tag{6.12}$. Let us consider the vector

$$w_{k,R} = \Psi_{N+nR}^* \left( q^{-(k-n'-nb)/n} \Psi_{k+n'+nb}^* k-n \right) |\lambda\rangle_{N+nR}. \tag{6.20}$$

The stabilization mentioned above means that $w_{k,R}$ stabilizes for any fixed $k$ and $R \to \infty$. Now the proposition follows from the following statements.
(i) \( w_{k,R} = 0 \) for \( k < -\lambda_1 \),
(ii) \( w_{k,R} = q^{-1}v^2w_{k-n,R} \) for \( N + nR - n' - nb > k \geq |\lambda| - n' - nb + n \),
(iii) \( w_{k,R} = 0 \) for \( k \geq N + nR - n' - nb \).

Statement (i) is obvious. Statement (ii) is equivalent to Lemma 6.9. For the statement (iii) note that terms containing \( e_{k+n'+nb} \) vanish after \( \varphi_{N+nR}^{(0)} \) for such \( k \).

In the rest of this subsection we will assume \(|q^{-1}v^2| < 1\). In order to study relations on \( \tilde{P}_{1,b} \), we will need the following proposition.

**Proposition 6.10.** The operators \( \tilde{P}_{1,b}^{(N)} \cdots \tilde{P}_{1,b_2}^{(N)} \tilde{P}_{1,b_1}^{(N)} \) converge to \( \tilde{P}_{1,b_1} \cdots \tilde{P}_{1,b_2} \tilde{P}_{1,b} \).

**Proof.** For partitions \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_r) \) and \( \nu = (\nu_1 \geq \cdots \geq \nu_s) \) such that \( r + s \leq N \), we denote
\[
|\mu, \nu\rangle_N = e_{-\mu_1} \cdots e_{-\mu_r} \nu_{s-1} \cdots \nu_1.
\]
In particular, \( |\mu, \emptyset\rangle_N = |\mu\rangle_N \). Let
\[
\tilde{P}_{1,b}^{(N)} |\lambda\rangle_N = \sum_{\mu} x^{(N)}_{\mu} |\mu\rangle_N + \sum_{\tilde{\mu}, \tilde{\nu}} y^{(N)}_{\tilde{\mu}, \tilde{\nu}} |\tilde{\mu}, \tilde{\nu}\rangle_N.
\]
In the first sum we have \(|\mu| = |\lambda| + n + n'b\). In the second sum we have \(|\tilde{\mu}| - |\tilde{\nu}| = |\lambda| + n' + nb\) and \(0 < |\tilde{\nu}| \leq n' + nb\). Note that only finitely many diagrams \( \mu, \tilde{\mu} \) and \( \tilde{\nu} \) satisfy these conditions.

**Lemma 6.11.** The coefficients \( y^{(N)}_{\tilde{\mu}, \tilde{\nu}} \) tend to 0 for \( N \to \infty \).

**Proof.** Note that
\[
\sum_{\tilde{\mu}, \tilde{\nu}} y^{(N)}_{\tilde{\mu}, \tilde{\nu}} |\tilde{\mu}, \tilde{\nu}\rangle_N = \sum_{k=N-n'-nb}^{N-1} q^{-k-n'-nb/n} \tilde{\Psi}_{k+n'+nb, \tilde{\mu}, \tilde{\nu}} |\lambda\rangle_N.
\]
Also, we can see that
\[
\tilde{\Phi}_{-k}^{*} |\lambda\rangle_N = \sum_{\Delta=0}^{N-1-k} y^{(N)}_{\tilde{\mu}, \Delta} e_{-\mu_1} \cdots e_{-\mu_\Delta} \cdots e_{-\mu_{k+\Delta}} \cdots e_{-\nu_{s-1}} |\lambda\rangle_N.
\]
Hence, all \( \tilde{\nu} \) appearing on the RHS of (6.22) are hook diagrams \( \tilde{\nu} = (k + n' + nb - N + 1, N-1-k-\Delta) \). Recall Remark 6.1. Denote \( c_{k,N} = (-1)^{N-1-k} v^{(N-1-k)/n-1} \). We have
\[
y^{(N)}_{\tilde{\mu}, \tilde{\nu}} = q^{-k-n'-nb/n} c_{k+n'+nb, \tilde{\nu}} y^{(k,N)}_{\tilde{\mu}, \tilde{\nu}} \quad \text{for} \quad \tilde{\nu} = (k + n' + nb - N + 1, N-1-k-\Delta).
\]

To study the coefficients \( y^{(k,N)}_{\tilde{\mu}, \tilde{\nu}} \) we will use the following trick. Let us act by \( \tilde{\Psi}_{k+\Delta} \) on (6.23). Using Lemma 5.1, we obtain
\[
\tilde{\Psi}_{k+\Delta} \tilde{\Phi}_{-k}^{*} |\lambda\rangle_N = (-1)^{N-1-k-\Delta} v^{N-1-k-\Delta-1} c_{k+\Delta, N} y^{(k,N)}_{\tilde{\mu}, \tilde{\nu}} |\tilde{\nu}\rangle_{N}.
\]
Denote \( y^{(k,N)}_{\tilde{\mu}, \tilde{\nu}} = (-1)^{N-1-k-\Delta} v^{N-1-k-\Delta-1} c_{k+\Delta, N} y^{(k,N)}_{\tilde{\mu}, \tilde{\nu}} \). Note that Lemma 6.9 and stabilization relation (6.18) imply
\[
y^{(k,n,N+n)}_{\tilde{\mu}, \tilde{\nu}} = v^{(k,n,N+n)}_{\tilde{\mu}, \tilde{\nu}} = v y^{(k,n)}_{\tilde{\mu}, \tilde{\nu}}.
\]
Then \((-1)^n v^n y^{(k,n,N+n)}_{\tilde{\mu}, \tilde{\nu}} = v y^{(k,n)}_{\tilde{\mu}, \tilde{\nu}} \). Thus, relation (6.24) implies \( y^{(N+n)}_{\tilde{\mu}, \tilde{\nu}} = q^{-1} v y^{(N)}_{\tilde{\mu}, \tilde{\nu}} = q^{-1} v^2 y^{(N)}_{\tilde{\mu}, \tilde{\nu}} \).
This finishes the proof since we have assumed \(|q^{-1}v^2| < 1\).
Let
\[ \tilde{P}_{1,b}^{(N)} |\alpha,\beta\rangle_N = \sum_{\gamma} \tilde{x}_{\gamma}^{(N)} |\gamma\rangle_N + \sum_{\epsilon,\delta} \tilde{y}_{\epsilon,\delta}^{(N)} |\epsilon,\delta\rangle_N. \] (6.27)

**Lemma 6.12.** The coefficients \( \tilde{x}_{\gamma}^{(N)} \) and \( \tilde{y}_{\epsilon,\delta}^{(N)} \) are bounded.

**Proof.** In case of empty \( \beta \), the lemma follows from Proposition [6.7] and Lemma [6.11]. Below we will deduce the general case from the case of empty \( \beta \).

Recall the operators \( b_j \), see [5.5]. Note that
\[ |\alpha,\beta\rangle_N = \sum_{j_1,\ldots,j_t} z_{j_1,\ldots,j_t,\beta} b_{j_1} \cdots b_{j_t} |\rho\rangle_N. \] (6.28)

Moreover, the coefficients \( z_{j_1,\ldots,j_t,\beta} \) do not depend on \( N \). Then Lemma follows from the commutation relation \( [b_j, \tilde{P}_{1,b}^{(N)}] = (q^j - 1) \tilde{P}_{1,b}^{(N)} \).

Let
\[ \tilde{P}_{1,b_1}^{(N)} \cdots \tilde{P}_{1,b_2}^{(N)} |\lambda\rangle_N = \sum_{\zeta} x_{\zeta,t-1}^{(N)} |\zeta\rangle_N + \sum_{\eta,\theta} y_{\eta,\theta,t-1}^{(N)} |\eta,\theta\rangle_N, \] (6.30)

\[ \tilde{P}_{1,b_1}^{(N)} \cdots \tilde{P}_{1,b_2}^{(N)} |\lambda\rangle_\infty = \sum_{\zeta} x_{\zeta,t-1}^{(N)} |\zeta\rangle_\infty. \] (6.31)

Note, that \( \varphi_N^{(0)} |\eta,\theta\rangle_N = 0 \). Hence the assumption of the induction says that \( x_{\zeta,t-1}^{(N)} \) tends to \( x_{\zeta,t-1} \). Then Lemmas [6.12] and [6.13] imply that
\[ \lim_{N \to \infty} \left( \varphi_N^{(0)} \tilde{P}_{1,b_1}^{(N)} \cdots \tilde{P}_{1,b_2}^{(N)} |\lambda\rangle_N - \tilde{P}_{1,b_1}^{(N)} \cdots \tilde{P}_{1,b_2}^{(N)} |\lambda\rangle_\infty \right) = \lim_{N \to \infty} \sum_{\eta,\theta} y_{\eta,\theta,t-1}^{(N)} \varphi_N^{(0)} \tilde{P}_{1,b_1}^{(N)} |\eta,\theta\rangle_N = 0. \]

Now let us drop the assumption \( u^{-\frac{1}{2}} v^\frac{1}{2} = q = 1 \). Also, recall \( \tilde{q} = v^{-1}q \).

**Theorem 6.1.** For \( |q^{-1}v^2| < 1 \) the following formulas determine an action of \( U_{q_1,q_2}(\hat{\mathfrak{g}}_1) \) on the space \( \Lambda_{\nu,0}^{\infty/2} \ (C^\infty[Y^\pm 1]) \)
\[ c \mapsto v^{-n}, \quad c' \mapsto v^{-n'}, \quad P_{0,-j} \mapsto q^j B_{-j}, \quad P_{0,j} \mapsto \frac{q^j - 1}{v^2 q^j - 1} v^{-jn} B_j, \] (6.32a)
\[ P_{1,b} \mapsto \tilde{P}_{1,b} = u^{-\frac{1}{2}} v^\frac{1}{2} \sum_{k \in \mathbb{Z}} \tilde{q}^{(-k-n'-nb)/n} \tilde{\Phi}_{k+n'+nb} \tilde{\Phi}_{-k}^*, \] (6.32b)

for \( q_1 = q, \) \( q_2 = v^{-2} \).
Proof. In this proof we will denote by $P_{0,j}$ and $P_{1,b}$ the images, prescribed by (6.32a) and (6.32b). The proof is a verification of the relations (4.6a), (4.6b), (4.6d), and (4.6h).

Relation (4.6a) follows from (5.27)

$$[P_{0,j}, P_{0,-j}] = \frac{q^j - 1}{v^{-2j} q^j - 1} v^{-jn} q^j [B_j, B_{-j}] = j \frac{1 - q^j}{(1 - q^{-jn})(1 - v^{-2j})}. \quad (6.33)$$

Equality in (6.32b) was proven in Proposition 6.7. For relation (4.6b) we will use the formula with vertex operators and Proposition 5.19. For example, for $j > 0$

$$[P_{0,j}, P_{1,b}] = u^{-\frac{1}{n} v^{-\frac{1}{n}} q \sum_{k \in \mathbb{Z}} q^{-(k-n^\prime-nb)/n} \left([P_{0,j}, \hat{\Psi}_{k+n^\prime+nb}] \hat{\Phi}^*_k + \hat{\Psi}_{k+n^\prime+nb}[P_{0,j}, \hat{\Phi}^*_k]\right)]$$

$$= \frac{q^j - 1}{v^{-2j} q^j - 1} \left(q^j v^j (2n-1) - v^{2jn}\right) P_{1,b+j} = (q^j - 1) v^{jn} P_{1,b+j}. \quad (6.34)$$

Relations (4.6d) and (4.6h) hold for $\hat{P}^{(N)}_{1,b}$ by Corollary 4.9. Proposition 6.10 implies that the relations hold for $P_{1,b} = \hat{P}^{(N)}_{1,b}$. Here we use the definition of $P_{1,b}$ as the limit of $\hat{P}^{(N)}_{1,b}$.

\[\square\]

Remark 6.14. Recall that $c'$ does not appear in the relations of $U_{q_1,q_2}(\mathfrak{gl}_1)^{+}$. Hence we do not have to specify $c'$ to formulate Theorem 6.1. The requirement $c' \mapsto v^{-n^\prime}$ will be essential in Section 7. For the same reason, we will specify $c'$ below.

Theorem 6.2. For $|qv^2| < 1$ the following formulas determine an action of $U_{q_1,q_2}(\mathfrak{gl}_1)^{-}$ on the space $\Lambda^{\infty/2}_{v,0}(\mathbb{C}^n[Y^{\pm 1}])$

$$c \mapsto v^{-n}, \quad c' \mapsto v^{-n^\prime}, \quad P_{0,-j} \mapsto q^j B_{-j}, \quad P_{0,j} \mapsto \frac{q^j - 1}{v^{-2j} q^j - 1} v^{-jn} B_j, \quad \text{(6.35a)}$$

$$P_{-1,b} \mapsto v^{-nb} \hat{P}_{-1,b} = u^{-\frac{1}{n} v^{-\frac{1}{n}} v^{-nb} \sum_{k \in \mathbb{Z}} q^{-k/n} \hat{\Phi}_{k-n^\prime+nb} \hat{\Phi}^*_k}. \quad \text{(6.35b)}$$

for $q_1 = q, q_2 = v^{-2}$.

Sketch of a proof. The results of this subsection have a counterpart for $P_{-1,b}$, the proofs are analogous. Relations (4.6e), (4.6h) hold for both $\hat{P}_{-1,b}$ and $v^{-nb} \hat{P}_{-1,b}$. Proposition 5.19 implies relation (4.6c) for the operators $P_{-1,b} = v^{-nb} \hat{P}_{-1,b}$.

\[\square\]

Currents For $\alpha = 0, 1, \ldots, n - 1$ let us define the following currents (i.e. operator-valued formal power series)

$$\hat{\Phi}_{(\alpha)}(z) = \sum_{k \in \mathbb{Z}} \hat{\Phi}_{\alpha+n^k} z^{-k}, \quad \hat{\Psi}_{(\alpha)}(z) = \sum_{k \in \mathbb{Z}} \hat{\Psi}_{\alpha+n^k} z^{-k}. \quad (6.36)$$

$$\hat{\Phi}^*_{(\alpha)}(z) = \sum_{k \in \mathbb{Z}} \hat{\Phi}^*_{-\alpha+n^k} z^{-k}, \quad \hat{\Psi}^*_{(\alpha)}(z) = \sum_{k \in \mathbb{Z}} \hat{\Psi}^*_{-\alpha+n^k} z^{-k}. \quad (6.37)$$

Then (6.32b) and (6.35b) can be reformulated as follows

$$E(z) = u^{-\frac{1}{n} v^{-\frac{1}{n}} q \sum_{\alpha \neq \pm n^\prime} \hat{q}^{-\frac{1}{n}} \hat{\Psi}_{(\alpha)}(\hat{q}z) \hat{\Phi}^*_{(\beta)}(z), \quad (6.38)$$

$$F(z) = u^{\frac{1}{n} v^{-\frac{1}{n}} \sum_{\alpha \neq \pm n^\prime} \hat{\Phi}^*_{(\alpha)}(v^n z) \hat{\Psi}_{(\beta)}(\hat{q}v^n z), \quad (6.39)$$

where $\equiv$ stands for $\equiv \pmod{n}$.
6.3 Analytic continuation

Consider \( \hat{P}_{1,\lambda} \lambda_{\infty} \) for any \( \lambda \) and \( n' + nb > 0 \). It follows from Lemma 6.8 that the series \((6.32b)\) applied to \( |\lambda|_{\infty} \) gives infinite geometric series, which equals to a rational function. This gives an analytic continuation from the region \( |q^{-1}v^2| < 1 \) to arbitrary \( q \) and \( v \).

Another way to say this is just rewrite

\[
\sum_{k \in \mathbb{Z}} q^{-\frac{k-n}{n}} \Psi_{k+n' + nb} \hat{\phi}^*_k = \frac{1}{1 - q^{-1}v^2} \sum_{k \in \mathbb{Z}} q^{-\frac{k-n}{n}} \left( \Psi_{k+n' + nb} \hat{\phi}^*_k - v \Psi_{k+n' + nb - 2n} \hat{\phi}^*_k \right). \tag{6.40}
\]

Lemma 6.8 implies that for any vector \( |\lambda|_{\infty} \), only finitely many terms of the RHS of \( (6.40) \) do not annihilate \( |\lambda|_{\infty} \). Hence the sum is well-defined without the assumption \( |q^{-1}v^2| < 1 \).

**Proposition 6.15.** The following formulas determine an action of \( U_{q_1,q_2}(\tilde{g}_l^1)^+ \) on \( \Lambda_{v,0}^{\infty/2} (\mathbb{C}[Y,\bar{Y}]) \)

\[
c \mapsto v^{-n}, \quad c' \mapsto v^{-n'}, \quad P_{0,-j} \mapsto q^{j}B_{-j}, \quad P_{0,j} \mapsto \frac{q^{j} - 1}{v^{-2j}q^{j} - 1}v^{-jn}B_{j}, \tag{6.41a}
\]

\[
P_{1,b}|\lambda|_{\infty} = \frac{u^{-\frac{1}{n}v^{-1}q}}{1 - q^{-1}v^2} \sum_{k \in \mathbb{Z}} q^{-\frac{k-n}{n}} \left( \Psi_{k+n' + nb} \hat{\phi}^*_k - v \Psi_{k+n' + nb - 2n} \hat{\phi}^*_k \right) |\lambda|_{\infty}. \tag{6.41b}
\]

for \( q_1 = q, q_2 = v^{-2} \).

**Proof.** Note that \( (6.41b) \) is an analytic continuation of \( (6.32b) \). The relations hold after an analytic continuation. \( \square \)

The above construction can be reformulated in the language of currents. Namely, the following current is well-defined

\[
\hat{\psi}_{(\alpha)}(qz) \hat{\phi}^*_{(\beta)}(z) \triangleq \frac{1}{1 - q^{-1}v^2} (1 - vz/w) \hat{\psi}_{(\alpha)}(w) \hat{\phi}^*_{(\beta)}(z) \bigg|_{w=\bar{q}z} \tag{6.42}
\]

This notation allows us to use \((6.38)\) without the assumption \( |q^{-1}v^2| < 1 \). Analogously, the following current is well-defined

\[
\hat{\phi}_{(\alpha)}(v^n z) \hat{\phi}^*_{(\beta)}(q^n z) \triangleq \frac{1}{1 - qv^{-2}} (1 - v^{-1}w/z) \hat{\phi}_{(\alpha)}(v^n z) \hat{\phi}^*_{(\beta)}(v^n w) \bigg|_{w=\bar{q}z} \tag{6.43}
\]

We prefer to write formulas via currents. The next proposition is a counterpart of Proposition 6.15 for \( U_{q_1,q_2}(\tilde{g}_l^1)^- \). We omit a version without the currents.

**Proposition 6.16.** The following formulas determine an action of \( U_{q_1,q_2}(\tilde{g}_l^1)^- \) on \( \Lambda_{v,0}^{\infty/2} (\mathbb{C}[Y,\bar{Y}]) \)

\[
c \mapsto v^{-n}, \quad c' \mapsto v^{-n'}, \quad P_{0,-j} \mapsto q^{j}B_{-j}, \quad P_{0,j} \mapsto \frac{q^{j} - 1}{v^{-2j}q^{j} - 1}v^{-jn}B_{j}, \tag{6.44a}
\]

\[
F(z) \mapsto u_{-\frac{1}{2}v^{-1}q^{-1}} \sum_{\alpha-\beta=-n'} q^{\frac{q^2}{n}v^{-2} - \alpha - n'} \frac{\hat{\phi}_{(\alpha)}(v^n z) \hat{\phi}^*_{(\beta)}(\bar{q}v^n z)}{z^{\alpha-\beta-n'}}. \tag{6.44b}
\]

for \( q_1 = q, q_2 = v^{-2} \).
6.4 Example $n = 1$

Consider an example $n = 1$ and $n' = 0$. In this case $[B_k, B_l] = k \delta_{k+l,0}$. The space $\Lambda_{e,m}^{\infty/2}(C[Y^\pm])$ is Fock space for the Heisenberg algebra. Namely, it has a cyclic vector $|\emptyset\rangle_{\infty,m} = e_m \wedge e_{m+1} \wedge \cdots$ such that $B_k |\emptyset\rangle_{\infty,m} = 0$ for $k > 0$. Let us consider an operator $e^{\pm Q} : \Lambda_{e,m}^{\infty/2}(C[Y^\pm]) \rightarrow \Lambda_{e,m}^{\infty/2}(C[Y^\pm])$ determined by $e^{\pm Q} |\emptyset\rangle_{\infty,m} = |\emptyset\rangle_{\infty,m+1}$ and $[B_k, e^{\pm Q}] = 0$.

**Proposition 6.17.** The operators $\hat{\Phi}, \hat{\Psi} : \Lambda_{e,m}^{\infty/2}(C[Y^\pm]) \rightarrow \Lambda_{e,m+1}^{\infty/2}(C[Y^\pm])$ are determined by

$$\hat{\Phi}(z) = \exp \left( \sum_{k=1}^\infty \frac{z^k}{k} B_{-k} \right) \exp \left( - \sum_{k=1}^\infty \frac{z^{-k}}{k} B_k \right) e^{Q z^{m+1}},$$

$$\hat{\Psi}(z) = \exp \left( \sum_{k=1}^\infty \frac{v^k z^k}{k} B_{-k} \right) \exp \left( - \sum_{k=1}^\infty \frac{v^{-k} z^{-k}}{k} B_k \right) e^{Q z^{m+1}}.$$  \hspace{1cm} (6.51)

The operators $\hat{\Phi}^*, \hat{\Psi}^* : \Lambda_{e,m}^{\infty/2}(C[Y^\pm]) \rightarrow \Lambda_{e,m-1}^{\infty/2}(C[Y^\pm])$ are determined by

$$\hat{\Phi}^*(z) = \exp \left( - \sum_{k=1}^\infty \frac{v^{2k} z^k}{k} B_{-k} \right) \exp \left( \sum_{k=1}^\infty \frac{z^{-k}}{k} B_k \right) e^{-Q z^{-m}},$$

$$\hat{\Psi}^*(z) = \exp \left( - \sum_{k=1}^\infty \frac{v^{-k} z^k}{k} B_{-k} \right) \exp \left( \sum_{k=1}^\infty \frac{v^{-k} z^{-k}}{k} B_k \right) e^{-Q z^{-m}}.$$  \hspace{1cm} (6.52)

**Proof.** Follows from Proposition 6.19 and Remark 5.13 \hfill \square

Then

$$\hat{\Psi}(w)\hat{\Phi}^*(z) = \frac{w^m z^{-n}}{1 - vz/w} \exp \left( \sum_{k=1}^\infty \frac{v^k w^k - v^{2k} z^k}{k} B_{-k} \right) \exp \left( \sum_{k=1}^\infty \frac{z^{-k} - v^{-k} w^{-k}}{k} B_k \right),$$

$$\hat{\Phi}(z)\hat{\Psi}^*(w) = \frac{z^m w^{-n}}{1 - v^{-1}w/z} \exp \left( \sum_{k=1}^\infty \frac{z^k w^k - v^{-k} z^k}{k} B_{-k} \right) \exp \left( \sum_{k=1}^\infty \frac{v^{-k} w^{-k} - z^{-k}}{k} B_k \right).$$  \hspace{1cm} (6.53)

Substituting this to (6.38) and (6.39), we obtain

$$E(z) = \frac{u^{-1}v^{1-n+q+1}}{1 - q^{-2}v^2} \exp \left( \sum_{k=1}^\infty \frac{q^{k} - q^{2k}}{k} B_{-k} z^k \right) \exp \left( \sum_{k=1}^\infty \frac{1 - q^{-k}}{k} B_k z^{-k} \right),$$

$$F(z) = \frac{u^{n-1}q^{-m}}{1 - q^{-2}v^{-2}} \exp \left( \sum_{k=1}^\infty \frac{q^{k} - q^{2k}}{k} B_{-k} z^k \right) \exp \left( \sum_{k=1}^\infty \frac{v^{-k}(q^{-k} - 1)}{k} B_k z^{-k} \right).$$  \hspace{1cm} (6.54)

The following proposition is Prop. A6.9]

**Proposition 6.18.** Formulas (6.51), (6.52), and

$$c \mapsto v^{-1}, \quad c' \mapsto 1, \quad P_{0,-j} \mapsto q^j B_{-j}, \quad P_{0,j} \mapsto \frac{q^j - 1}{v^{-2}q^j - 1} v^{-jn} B_j$$

determine an action of $U_{q_1,q_2}(\hat{\mathfrak{gl}}_1)$ for $q_1 = q$, $q_2 = v^{-2}$.

For $m = 0$ we will denote the representation by $\mathcal{F}_a$ and call it Fock module.

**Remark 6.19.** Note that Propositions 6.15 and 6.16 guarantee only existence of actions of $U_{q_1,q_2}(\hat{\mathfrak{gl}}_1)^+$ and $U_{q_1,q_2}(\hat{\mathfrak{gl}}_1)^-$ separately. Remarkably, the actions of $U_{q_1,q_2}(\hat{\mathfrak{gl}}_1)^\pm$ are restrictions of the action of whole $U_{q_1,q_2}(\hat{\mathfrak{gl}}_1)$. Below we will prove the same result for general $n$.

The obtained representation is celebrated Fock module of $U_{q_1,q_2}(\hat{\mathfrak{g}}_1)$. It was constructed in Prop. A6.9 via formulas (6.51), (6.52) (up to a different notation). Though, to the best of our knowledge, the interpretation via the operators $\Phi(z), \Psi(z), \Phi^*(z)$, and $\Psi^*(z)$ is new.
7 Semi-infinite construction of twisted Fock module II

In the previous section, we have obtained actions of \( U_{q_1,q_2}(\mathfrak{g}_1)^\pm \) and \( U_{q_1,q_2}(\mathfrak{g}_1)^- \) on \( \Lambda^\infty_{c,0}(\mathbb{C}^n[Y^\pm]) \).

In this section we prove that these actions (after a simple rescaling) give action of the whole algebra \( U_{q_1,q_2}(\mathfrak{g}_1) \). We do not check directly the relations (4.6f), (4.6g) due to technical difficulties.

We prove that the defined below operators \( \tilde{P}^{(N)}_{a,b} \) stabilize for \( an' + bn \leq 0 \) and general \( q \) and \( v \). Therefore we obtain a representation of the corresponding subalgebra \( U_{q_1,q_2}(\mathfrak{g}_1)^\prime \subset U_{q_1,q_2}(\mathfrak{g}_1) \) (subsection 7.1). We extend the action to the whole \( U_{q_1,q_2}(\mathfrak{g}_1) \), the obtained representation is isomorphic to twisted Fock module \( \mathcal{F}_u^\prime \) by construction. Then we compare the obtained action \( U_{q_1,q_2}(\mathfrak{g}_1)^\pm \) are determined by explicit formulas for Chevalley generators (Proposition 6.15 and 6.16). Hence we get explicit formulas for the action of Chevalley generators of \( U_{q_1,q_2}(\mathfrak{g}_1) \) on \( \Lambda^\infty_{c,0}(\mathbb{C}^n[Y^\pm]) \), the obtained representation is isomorphic to twisted Fock module \( \mathcal{F}_u^\prime \) (Theorem 7.1). This is the central result of the whole paper.

7.1 The limit for the bottom half

Existence of the limit  Below we will use the results of Sections 3.2 and 3.3. Recall that \( m, m' \) are integers such that \( mn' - n'm = 1 \) and \( 0 \leq m < n, 0 \leq m' < n' \). Also, recall the automorphism \( \sigma \in SL(2, \mathbb{Z}) \) defined in Section 3.2.

Lemma 7.1. a) The sequences of operators \( v^{kmN} P_{km,-km'}^{(N)} \) stabilize for \( k \in \mathbb{Z}_{>0} \).

b) The sequences of operators \( v^{kN} P_{km,-km'}^{(N)} - u^{-k} v^{-k} q^{2k} \sum_{i=1}^{N} (v^{2q^{-1}})^{ik} \) stabilize for \( k \in \mathbb{Z}_{>0} \).

c) The sequences of operators \( v^{-kN} P_{km,-km'}^{(N)} - u^{kN} v^{-kN} \sum_{i=1}^{N} (v^{-2q^{i}})^{ik} \) stabilize for \( k \in \mathbb{Z}_{>0} \).

Proof. a) The formula (4.12a) and \( SL(2, \mathbb{Z}) \)-transformation properties of the \( P_{a,b}^{(N)} \) generators imply

\[
P_{km,-km'}^{(N)} = \sigma \left( P_{0,-k}^{(N)} \right) = q^k S_- \sum_{i=1}^{N} \sigma (Y_i)^{-k} S_+ = q^k S_- \sum_{i=1}^{N} A_i^{-k} S_+. \tag{7.1}
\]

Note that \( \sum_{i=1}^{N} A_i^{-k} \) commutes with finite Hecke algebra and, in particular, with \( S_- \). Hence

\[
P_{km,-km'}^{(N)} |\lambda\rangle_N = q^k \sum_{i=1}^{N} S_- A_i^{-k} \left( e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1} \right). \tag{7.2}
\]

We decompose the proof into two steps.

Step 1. First we show that \( i \)-th term in (7.2) vanishes for \( i > |\lambda| + k \). Denote

\[
A_i^{-k} \left( e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1} \right) = \sum_{j_1, \ldots, j_N} c_{j_1, \ldots, j_N} e_{j_1} \otimes \cdots \otimes e_{j_N}. \tag{7.3}
\]

For a sequence \( \{j_1, \ldots, j_N\} \), we will say that a number \( r \) is a hole if \( 0 \leq r \leq N-1 \) and \( r \notin \{j_1, \ldots, j_N\} \). Let us prove that for each summand in (7.3) there is a hole \( i \geq 1 \).

We will use the formula (3.25). Note that the operators \( \kappa_i^{-1} G_{i,j_1}^\pm \kappa_i^\pm, \kappa_i^{-1} G_{j_1}^\pm \kappa_i^\pm\), and \( \kappa_i \) preserve the existence of a hole with position \( i - 1 \). If there is no such holes, operator \( \kappa_i \) must create one. Hence the operator \( A_i^{-k} \) must create a hole for \( i \geq 1 \).

Let us also denote

\[
S_- A_i^{-k} \left( e_{-\lambda_1} \otimes e_{-\lambda_2+1} \otimes \cdots \otimes e_{N-1} \right) = \sum_{\mu_1 \geq \mu_2 \geq \cdots} \tilde{c}_{\mu \mu_1} e_{-\mu_1} \wedge \cdots \wedge e_{N-1}. \tag{7.4}
\]
Note that \( |\mu| = |\lambda| + k \). Existence of a hole \( r \) implies \( \mu_r > 0 \). Hence \( |\mu| \geq r + 1 \geq i > |\lambda| + k \). Hence the sum runs over the empty set, i.e. \( S_\lambda A_i^{-k} (e_{-\lambda_i} \otimes e_{-\lambda_{i+1}} \otimes \cdots \otimes e_{N-1}) = 0 \).

Step 2. It remains to study the terms in (7.2) for small \( i \). If we replace \( N \mapsto N + n \) and use again formula (3.25), we get \( mk \) additional factors of the form \( \kappa_i^{-c} G_{i,j}^1 k_i \). Each of these factors acts diagonally with addition of the terms with holes \( r \geq N \). As before, such additional terms vanish after the action of \( S_\lambda \). The diagonal part acts by \( v^{-mk} \) by the formulas (3.16) and (3.18).

b) The proof is similar to the previous one. Using formula (4.12b) and \( SL(2, \mathbb{Z}) \)-transformation property, we obtain \( P_{kn, kn'}^{(N)} = q^k S_{\lambda} \sum_{i=1}^{N} B_i^k S_{\lambda} \). Then

\[
v^{kn} P_{kn, kn'}^{(N)} |\lambda\rangle_N = q^k v^{kn} \sum_{i=1}^{N} S_i B_i^k (e_{-\lambda_1} \otimes e_{-\lambda_2} \otimes \cdots \otimes e_{N-1}) . \tag{7.5}\]

We have two steps as in the proof above.

Step 1. Let \( i > |\lambda| \). Hence \( \lambda_i = 0 \). In order to compute the \( i \)-th term in the sum (7.5) we use the formula (3.20). Each triangular operator of the form \( \kappa_i^{-c} G_{i,j}^1 k_i \) acts on \( e_{-\lambda_1} \otimes e_{-\lambda_2} \otimes \cdots \otimes e_{N-1} \) diagonally with addition of the terms with holes \( r \geq i - 1 \). The terms with the holes vanish after the action of \( S_\lambda \). The diagonal contribution was computed in the proof of the Corollary 3.4. Hence the \( i \)-th term in the sum (7.5) is equal to

\[
q^k v^{kn} (u_0 \cdots u_{n-1} q^{1-i} v^{i\lambda} q^{1-n-i})^{-k} (e_{-\lambda_1} \otimes e_{-\lambda_2} \otimes \cdots \otimes e_{N-1}) = u^{-k} v^{-k} q^{2k} (v^2 q^{-1})^k (e_{-\lambda_1} \otimes e_{-\lambda_2} \otimes \cdots \otimes e_{N-1}) ,
\]

where we used \( L_i^{(\lambda)} = N + 1 - 2i \) and convention (6.5).

Step 2. Analogous to the above.

c) Using (4.12b) and \( SL(2, \mathbb{Z}) \)-transformation, we have \( P_{-kn, kn'}^{(N)} = S_{\lambda} \sum_{i=1}^{N} B_i^k S_{\lambda} \). The remaining part of the proof is similar to the proof of b).

\[\square\]

**Proposition 7.2.** The operators \( v^{\frac{an}{2}} P_{a,b}^{(N)} \) stabilize for \( an' + bn < 0 \).

**Proof.** It follows from the commutation relations that any \( P_{a,b} \in U_{q_1, q_2}(\hat{\mathfrak{g}}_1)^+ \) with \( a > 0 \) can be represented as algebraic combination of \( P_{1,0} \), its commutators with \( P_{0,b} \) for \( b \in \mathbb{Z} \), and also \( c, c' \). Using \( SL(2, \mathbb{Z}) \) symmetry we see that any element \( P_{a,b} \in U_{q_1, q_2}(\hat{\mathfrak{g}}_1) \) with \( an' + bn < 0 \) is algebraic combination of \( P_{m, -m'} \), its commutators with \( P_{bn, -bn'} \) for \( b \in \mathbb{Z} \), and \( c, c' \). To finish the proof we use Lemma (7.1). \[\square\]

**Remark 7.3.** a) This proposition gives another proof of Proposition 6.3.

b) The additional series \( \sum_{i=1}^{\infty} (v^2 q^{-1})^k \) which appears in Lemma (7.1) converges if \( |v^2 q^{-1}| < 1 \). This is in agreement with Theorem 6.1.

For \( an' + bn < 0 \), let \( \hat{P}_{a,b}^{(N)} = v^{\frac{an}{2}} P_{a,b}^{(N)} \). Denote the stable limit of \( \hat{P}_{a,b}^{(N)} \) by \( \hat{P}_{a,b} \). Similarly, consider operators

\[
\hat{P}_{kn, kn'}^{(N)} = v^{kn} P_{kn, kn'}^{(N)} - u^{-k} v^{-k} q^{2k} \sum_{i=1}^{N} (v^2 q^{-1})^k + \frac{u^{-k} v^{-k} q^{-k}}{1 - (v^2 q^{-1})^k} \tag{7.6}\]

\[
\hat{P}_{-kn, kn'}^{(N)} = -u^{-k} v^{-k} q^{2k} \sum_{i=1}^{N} (v^2 q^{-1})^k + \frac{u^{-k} v^{-k} q^{-k}}{1 - (v^2 q^{-1})^k} \tag{7.7}\]

Let \( \hat{P}_{kn, kn'} \) and \( \hat{P}_{-kn, kn'} \) denote the stable limits of \( P_{kn, kn'}^{(N)} \) and \( P_{-kn, kn'}^{(N)} \). Let \( U_{q_1, q_2}(\hat{\mathfrak{g}}_1)^c \) be a subalgebra of \( U_{q_1, q_2}(\hat{\mathfrak{g}}_1) \) generated by \( P_{a,b} \) for \( an' + bn \leq 0 \).
Proposition 7.4. There is an action of $U_{q_1,q_2}(\tilde{g}_1)^\vee$ on $\Lambda_{v,0}^{\infty/2} \left( \mathbb{C}^n[Y^\pm] \right)$ given by

$$c \mapsto v^{-n}, \quad c' \mapsto v^{-n'},$$

where $P_{a,b} = \tilde{P}_{a,b}$ for $a \geq 0$, and $P_{a,b} \mapsto v^{-n-a-nb} P_{a,b}$ for $a < 0$. (7.8a)

Proof. From the limit arguments we see that operators $\tilde{P}_{a,b}$ satisfy the relations of $U_{q_1,q_2}(\tilde{g}_1)^\vee$ for $c = c' = 1$. It remains to show that formulas (7.8b)–(7.8c) defines isomorphism of $U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=1}$ and $U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=-n,c'=v^{-n'}}$.

Let $\tilde{\sigma}$ be an element of $\tilde{S}L(2, \mathbb{Z})$ such that the corresponding matrix in $SL(2, \mathbb{Z})$ is

$$\begin{pmatrix} -n' & -n \\ m' & m \end{pmatrix}$$

and $n_\tilde{\sigma}(0,-1) = 0$. Using the formula (4.2), we see that action of $\tilde{\sigma}$ induces an isomorphism

$$U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=v^{-n},c'=v^{-n'}} \xrightarrow{\sim} U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=1}. \quad (7.9)$$

In the region $an' + bn \leq 0$ the winding number $n_\tilde{\sigma}(a,b) = 0$ for $a \geq 0$ and $n_\tilde{\sigma}(a,b) = 1$ for $a < 0$, hence the formula (4.3) for the action of $\tilde{\sigma}$ on $P_{a,b}$ generators gives

$$P_{a,b} \mapsto \begin{cases} P_{-n'a-nb,m'a+mb} & \text{for } a \geq 0 \\ v^{-n'-n} P_{-n'a-nb,m'a+mb} & \text{for } a < 0 \end{cases} \quad (7.10)$$

Similarly $\tilde{\sigma}$ induces an isomorphism between $U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=1}$ and $U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=1}$

$$P_{a,b} \mapsto P_{-n'a-nb,m'a+mb}. \quad (7.11)$$

Since the relations of $U_{q_1,q_2}(\tilde{g}_1)^\vee$ does not include $c'$, there is an isomorphism of $U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=v}$ and $U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=1}$ which acts $P_{a,b} \mapsto P_{a,b}$ for any $a \geq 0$ and $b$.

To sum up the above, we have obtained a chain of isomorphisms

$$U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=v^{-n},c'=v^{-n'}} \xrightarrow{\sim} U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=v} \xrightarrow{\sim} U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=1} \xrightarrow{\sim} U_{q_1,q_2}(\tilde{g}_1)^\vee |_{c=1,c'=1}. \quad \square$$

To finish the proof we notice that the composition indeed is given by (7.8b)–(7.8c).

Connection with twisted representation. Below we will give an alternative interpretation of $\mathcal{F}_u$ as a version of twisted representation. To do this we need to introduce the following notion.

Let $U_{q_1,q_2}(\tilde{g}_1)^\vee$ be a subalgebra of $U_{q_1,q_2}(\tilde{g}_1)$ generated by $P_{a,b}$ for $b \leq 0$. In Section 6.4 we have constructed action of $U_{q_1,q_2}(\tilde{g}_1)$ on $\Lambda_{v,0}^{\infty/2} \left( \mathbb{C}^n[Y^\pm] \right)$. Denote its restriction to $U_{q_1,q_2}(\tilde{g}_1)^\vee$ by $\rho_1$. On the other hand, let us consider particular case $n = 1$ and $n' = 0$ of Proposition 7.4. It gives a priori another action of $U_{q_1,q_2}(\tilde{g}_1)^\vee$ on $\Lambda_{v,0}^{\infty/2} \left( \mathbb{C}^n[Y^\pm] \right)$. Denote it by $\rho_2$.

Lemma 7.5. The actions $\rho_1$ and $\rho_2$ coincide.

Proof. Let us prove $\rho_1(P_{a,b}) = \rho_2(P_{a,b})$ for $a \geq 0$. The operator $v^{na} P_{a,b}$ converges to $\rho_1(P_{a,b})$ for $|v^2 q^{-1}| < 1$ by Proposition 6.10. On the other hand, operator $\tilde{P}_{a,b}^{(N)}$ stabilizes and the induced operator is $\rho_2(P_{a,b})$. Notice that for $|v^2 q^{-1}| < 1$, the limits of $v^{na} P_{a,b}^{(N)}$ and $\tilde{P}_{a,b}^{(N)}$ coincide (even for $an'+bn = 0$, see (7.6), (7.7)). Hence $\rho_1(P_{a,b}) = \rho_2(P_{a,b})$ for $|v^2 q^{-1}| < 1$. Since matrix coefficients of $\rho_1(P_{a,b})$ and $\rho_2(P_{a,b})$ are analytic (even rational) functions of $q$ and $v$, we have $\rho_1(P_{a,b}) = \rho_2(P_{a,b})$ for any values of $q$ and $v$.

The case $a < 0$ is analogous. Note that prefactors in formulas (7.8c) and (6.35b) coincide for $n = 1, n' = 0$. \hspace{1cm} \square
Let $\mathcal{F}_a^\dagger$ be the restriction of Fock module $\mathcal{F}_a$ to $U_{q_1,q_2}(\tilde{g}_1)^\dagger$. Recall that $\sigma$ is an element in $\tilde{S}L(2,\mathbb{Z})$ such that $U_{q_1,q_2}(\tilde{g}_1)^\dagger = \sigma \left( U_{q_1,q_2}(\tilde{g}_1)^\dagger \right)$. Recall the formulas (4.2), (4.3) for the action $\tilde{S}L(2,\mathbb{Z}) \curvearrowright U_{q_1,q_2}(\tilde{g}_1)$.

**Lemma 7.6.** For $b \leq 0$, the element $\sigma(P_{a,b})$ acts on $\mathcal{F}_a^\dagger$ as

$$v^{-n_\sigma(1,0)b} P_{na-mb,-n'a+m'b} \quad \text{for } a \geq 0 \text{ or } \frac{m}{n} b > a \quad (7.12)$$

$$v^{-(n_\sigma(1,0)+1)b} P_{na-mb,-n'a+m'b} \quad \text{otherwise} \quad (7.13)$$

**Proof.** In this proof we assume $b \leq 0$. One can check that the winding number is given by

$$n_\sigma(a,b) = \begin{cases} n_\sigma(1,0) & \text{for } a \geq 0 \text{ or } \frac{m}{n} b > a \\ n_\sigma(1,0) + 1 & \text{otherwise} \end{cases} \quad (7.14)$$

To finish the proof, we notice that the element $(e')^{na-mb} e^{-n'a+m'b}$ acts on $\mathcal{F}_a^\dagger$ as $v^{-b}$.

**Proposition 7.7.** There is an isomorphism of vector spaces $\hat{\psi}: \mathcal{F}_a^\dagger \rightarrow \mathcal{F}_a^\vee$ such that $\hat{\psi}$ intertwines the actions. More precisely, $\sigma(X) \hat{\psi} w = \hat{\psi}(Xw)$ for any $w \in \mathcal{F}_a^\dagger$ and $X \in U_{q_1,q_2}(\tilde{g}_1)^\dagger$.

**Proof.** Note that $\mathcal{F}_a^\dagger$ is a free cyclic module over the algebra generated by $P_{a,-k}$ for $k \in \mathbb{Z} > 0$ with cyclic vector $|\partial\rangle_\infty = e_0 \wedge e_1 \wedge \cdots$. Analogously, $\mathcal{F}_a^{\vee}$ is a free cyclic module over algebra generated by $\sigma(P_{a,-k})$ with the corresponding cyclic vector $|\partial\rangle_\infty$. Hence there is an isomorphism of vector spaces $\hat{\psi}: \mathcal{F}_a^\dagger \rightarrow \mathcal{F}_a^{\vee}$ such that $\hat{\psi}|\partial\rangle_\infty = |\partial\rangle_\infty$ and $\hat{\psi}$ satisfies the intertwining property for $X = P_{a,-k}$.

It remains to prove that the intertwining property holds for any $X \in U_{q_1,q_2}(\tilde{g}_1)^\dagger$. To do this we will need the following notation. Let $\mathcal{F}_a^{\dagger}[\leq k]$ and $\mathcal{F}_a^{\vee}[\leq k]$ be subspaces, spanned by $|\lambda\rangle_\infty$ for $|\lambda| \leq k$. Analogously, let $S_{-}(\mathbb{C}[Y^\pm])^\otimes N[\leq k]$ and $S_{-}(\mathbb{C}[Y^\pm])^\otimes N[\leq k]$ be subspaces, spanned by $|\lambda\rangle_N = e_{-\lambda_1} \wedge e_{-\lambda_2} \wedge \cdots e_{N-1}$ for $|\lambda| \leq k$. Let us consider the following diagram

$$\begin{array}{c}
\mathcal{S}_{-}(\mathbb{C}[Y^\pm])^\otimes N[\leq k] \\
\downarrow \psi
\end{array} \xrightarrow{\varphi^{[\lambda]}_N} \begin{array}{c}
\mathcal{F}_a^{\dagger}[\leq k] \\
\end{array} \xrightarrow{\varphi^{[\lambda]}_N} \begin{array}{c}
\mathcal{F}_a^{\vee}[\leq k] \\
\end{array} \quad (7.15)$$

Recall that we used notation $\varphi^{(m)}_N$ for the canonical map from the definition inductive limit (see Section 5.2). In this proof we omit superscript $[m]$ since we consider only the case $m = 0$. We denote the corresponding maps by $\varphi^{[\lambda]}_N$ to distinguish the first and the second rows of (7.15). Note that here we have used interpretation of $\mathcal{F}_a^{\dagger}$ via $\rho_2$, see Lemma 7.5.

Recall the isomorphism of irreducible representations $\psi: \mathcal{C}_a^\sigma \rightarrow \mathcal{C}_a^{(n_n')}$, see Theorem 3.4. Hence there exist unique map $\hat{\psi}$ satisfying

$$\hat{\psi}|\partial\rangle_N = |\partial\rangle_N \quad \quad \hat{\psi}^{-n_\sigma(1,0)+\frac{nN}{m}} P_{a,b}^{(N)} \hat{\psi} = \hat{\psi} P_{a,b}^{(N)} \quad (7.16)$$

Multiplying by $v^{aN}$, we can reformulate the intertwining property for $\hat{\psi}$ as follows

$$v^{-n_\sigma(1,0)b} P_{na-mb,-n'a+m'b} \hat{\psi} = \hat{\psi} P_{a,b}^{(N)} \quad (7.17)$$

Also, note that we abuse notation using the same symbols for maps with and without restriction to corresponding $[\leq k]$ subspace.

\footnote{Here and below we use the same notation for vectors in $\mathcal{F}_a^\dagger$ and $\mathcal{F}_a^\vee$. Hopefully, this will not lead to a confusion.}
Lemma 7.8. Diagram (7.15) is commutative for any (fixed) \( k \) and sufficiently large \( N \).

Proof. One can verify that the operators \( \varphi_N^{[n]} \hat{\psi} \) and \( \hat{\psi} \varphi_N^{[1]} \) satisfy the following properties for sufficiently large \( N \)

\[
\begin{align*}
\varphi_N^{[n]} \hat{\psi}(\emptyset)_N &= \emptyset_N \quad \sigma(P_{0,-l}) \varphi_N^{[n]} \hat{\psi}(\lambda)_N &= \varphi_N^{[n]} \hat{\psi} P_{0,-l}(\lambda)_N \\
\hat{\psi} \varphi_N^{[1]} (\emptyset)_N &= \emptyset_N \quad \sigma(P_{0,-l}) \hat{\psi} \varphi_N^{[1]}(\lambda)_N &= \hat{\psi} P_{0,-l}(\lambda)_N
\end{align*}
\]  

(7.18)

(7.19)

Actually, (7.18) and (7.19) are the same properties for the operators \( \varphi_N^{[n]} \hat{\psi} \) and \( \hat{\psi} \varphi_N^{[1]} \) respectively. To finish the proof we note that the maps \( \varphi_N^{[n]} \hat{\psi} \) and \( \hat{\psi} \varphi_N^{[1]} \) are determined by the properties.

Let us prove that \( \sigma(P_{a,b}) \hat{\psi}(\lambda)_\infty = \hat{\psi}(P_{a,b}) \hat{\psi}(\lambda)_\infty \) for \( b \leq 0 \). Let us take \( k \) large enough such that \( P_{a,b}(\lambda)_\infty \in \mathcal{F}_a \leq k \). Then we take sufficiently large \( N \) such that diagram (7.15) is commutative and

\[
P_{a,b} \varphi_N^{[n]}(\lambda)_N = \begin{cases} 
\varphi_N^{[n]} \hat{P}_{a,b}(\lambda)_N & \text{for } a \geq 0 \\
v^{-n-a-nb} \varphi_N^{[n]} P_{a,b}(\lambda)_N & \text{otherwise}
\end{cases}
\]

(7.20)

Also we take \( N \) large enough such that

\[
P_{a,b} \varphi_N^{[1]}(\lambda)_N = \begin{cases} 
\varphi_N^{[1]} \hat{P}_{a,b}(\lambda)_N & \text{for } a \geq 0 \\
v^{-n-a-nb} \varphi_N^{[1]} P_{a,b}(\lambda)_N & \text{otherwise}
\end{cases}
\]

(7.21)

Using Lemma 7.6 we obtain

\[
\sigma(P_{a,b}) \varphi_N^{[n]}(\lambda)_N = \begin{cases} 
v^{-na+1} \varphi_N^{[n]} \hat{P}_{a,b}(\lambda)_N & \text{for } a \geq 0 \\
v^{-na+1} \varphi_N^{[n]} P_{a,b}(\lambda)_N & \text{otherwise}
\end{cases}
\]

(7.22)

It follows from the above

\[
\sigma(P_{a,b}) \hat{\psi}(\lambda)_\infty = \sigma(P_{a,b}) \hat{\psi} \varphi_N^{[1]}(\lambda)_N = \sigma(P_{a,b}) \hat{\psi} \varphi_N^{[n]} \hat{\psi}(\lambda)_N = \cdots = \hat{\psi} P_{a,b} \varphi_N^{[1]}(\lambda)_N = \hat{\psi} P_{a,b}(\lambda)_\infty,
\]

here the dots stand for the omitted steps involving the cases (straightforward to write down using (7.22), (7.17), and (7.20)).

\( \Box \)

7.2 Action of the whole algebra

Recall that we write \( \equiv \) for \( \equiv \pmod{n} \). Also recall that the currents \( \hat{\Phi}_{(1)}(v^{-1}qz) \hat{\Phi}_{(\beta)}(z) \) and \( \hat{\Phi}_{(1)}(v^n z) \hat{\Phi}_{(\beta)}(v^{-1}qz) \) are defined for general \( q \) and \( v \) by (6.42) and (6.43).

Theorem 7.1. The following formulas determine an action of \( U_{q_1,q_2}(g\mathfrak{l}_1) \) on \( \Lambda_{v,0}^\infty \left( \mathbb{C}^n[Y^{\pm 1}] \right) \)

\[
\begin{align*}
&c \mapsto v^{-n}, \quad c' \mapsto v^{-n'}, \\
P_{0,j} \mapsto q^j B_{-j}, \quad P_{0,j} \mapsto \frac{q^j - 1}{-2iq^j} v^{-jn} B_{j}, \\
E(z) \mapsto u^{-\frac{1}{2}} v^{-\frac{1}{2}} \sum_{\alpha - \beta \equiv n'} q^{-\frac{1}{2} \alpha n z^{\beta - \alpha + n'}} \hat{\Phi}_{(1)}(v^{-1}qz) \hat{\Phi}_{(\beta)}(z), \\
F(z) \mapsto u^{\frac{1}{2}} v^{-\frac{1}{2}} \sum_{\alpha - \beta \equiv -n'} q^{\frac{1}{2} \alpha n z^{\beta - \alpha + n'}} \hat{\Phi}_{(1)}(v^n z) \hat{\Phi}_{(\beta)}(v^{-1}qz).
\end{align*}
\]

(7.23a)

(7.23b)

(7.23c)

(7.23d)

for \( q_1 = q, q_2 = v^{-2} \). The obtained representation is isomorphic to twisted Fock module \( \mathcal{F}_u^\alpha \).
Proof. There is an action of $U_{q_1,q_2}(\mathfrak{gl}_1)$ on $\Lambda_{c,0}^{\infty} (\mathbb{C}^{n}[Y^\pm])$ determined as follows. Recall that we have defined a map $\hat{\psi}$ from $\mathcal{F}_u^\mathfrak{a}$ to $\Lambda_{c,0}^{\infty} (\mathbb{C}^{n}[Y^\pm])$, see Proposition 7.7. But there is an action of the whole $U_{q_1,q_2}(\mathfrak{gl}_1)$ on $\mathcal{F}_u$, which coincides with $\mathcal{F}_u^\mathfrak{a}$ as a vector space. Hence for any $X \in U_{q_1,q_2}(\mathfrak{gl}_1)$ and $w \in \Lambda_{c,0}^{\infty} (\mathbb{C}^{n}[Y^\pm])$ we can define $\rho_{tw}(X)w := \hat{\psi} \circ \rho_{\mathcal{F}_u} (\sigma^{-1}(X)) \circ \hat{\psi}^{-1}w$. The representation obtained is isomorphic to twisted Fock module $\mathcal{F}^\mathfrak{a}$. In particular, formula (4.2) for $\tau = \sigma^{-1}$ implies $\rho_{tw}(c) = v^{-n}$ and $\rho_{tw}(c') = v^{-n'}$. It remains to prove that the action $\rho_{tw}$ is given by (7.23b)–(7.23d).

Note, that now we have two actions of $U_{q_1,q_2}(\mathfrak{gl}_1)^+$ on $\Lambda_{c,0}^{\infty} (\mathbb{C}^{n}[Y^\pm])$. The first one comes from Proposition 6.15 let us denote it by $\rho_+$. The second one comes from the restriction of $\rho_{tw}$ to $U_{q_1,q_2}(\mathfrak{gl}_1)^+$.

Lemma 7.9. $\rho_+(P_{a,b}) = \rho_{tw}(P_{a,b})$ for $n'a + nb \leq 0$ and $a \geq 0$.

Proof. Analogous to the proof of Lemma 7.5. \hfill \Box

Lemma 7.10. The actions $\rho_+$ and $\rho_{tw}|_{U_{q_1,q_2}(\mathfrak{gl}_1)^+}$ coincide.

Proof. We will simply write $P_{a,b} = \rho_+(P_{a,b}) = \rho_{tw}(P_{a,b})$ for $n'a + nb \leq 0$ and $b \geq 0$. Any vector of $\Lambda_{c,0}^{\infty} (\mathbb{C}^{n}[Y^\pm])$ is a linear combination of vectors $P_{a_1,b_1} \cdots P_{a_l,b_l}|\mathcal{O}_\infty$ for $n'a_i + nb_i < 0$ and $b_i \geq 0$. The following proposition is [BST12, Lemma 5.6].

Proposition 7.11. Algebra $U_{q_1,q_2}(\mathfrak{gl}_1)^+$ has a basis $P_{k_1,l_1} \cdots P_{k_l,l_l}$ for $\frac{1}{k_1} \leq \frac{1}{k_2} \leq \cdots \leq \frac{1}{k_l}$ over $\mathbb{C}^{[n^\pm1}, (\epsilon')^{\pm1}]$.

Hence the action of $P_{k,l} \in U_{q_1,q_2}(\mathfrak{gl}_1)^+$ for $n'k + nl > 0$ is determined by commutation relations in $U_{q_1,q_2}(\mathfrak{gl}_1)^+$ and the following conditions

$$P_{k,l}|\mathcal{O}_\infty = 0 \quad \text{for } n'k + nl > 0, \quad P_{kn,-kn'}|\mathcal{O}_\infty = \frac{u^{-k}v^k q^k}{1 - (v^2 q^{-1})^k}|\mathcal{O}_\infty \quad \text{for } k > 0.$$  

(7.24)

\hfill \Box

Let $\rho_-$ denotes the action of $U_{q_1,q_2}(\mathfrak{gl}_1)^-$, coming from Proposition 6.16. The following lemma is analogous to Lemma 7.10.

Lemma 7.12. It holds $\rho_{tw}(P_{a,b}) = v^{-an'}\rho_-(P_{a,b})$ for $a \leq 0$.

Proposition 6.15 implies that the current $\rho_{tw}(E(z)) = \rho_+(E(z))$ is given by (7.23c). Analogously, Proposition 6.16 implies that the current $\rho_{tw}(F(z)) = v^{n'} \rho_-(F(z))$ is given by (7.23d). To find the action of $\rho_{tw}(P_{0,0})$, one can use either $\rho_+$ or $\rho_-$. \hfill \Box

8 Standard basis

As was already mentioned in the introduction, one of the motivations of this paper is Gorsky-Neguț conjecture on stable envelope bases in the equivariant K-theory of Hilbert schemes of points in $\mathbb{C}^2$ [GN17]. Recall that there is the standard identification of the K-theory of Hilbert schemes and the space of symmetric functions $\Lambda$. Moreover, stable envelopes have a purely combinatorial characterization as symmetric functions [Neg16, Sect. 4.1]. This gives a reformulation of the conjecture to be given below.

We use notation $n,n',m,m'$ as before, see Section 7.1. The Fock module $\mathcal{F}_u$ can be identified as a vector space with the space of symmetric functions $\Lambda$ using the correspondence $p_k \leftrightarrow P_{0,-k}$ for $k > 0$. Hence the twisted Fock module $\mathcal{F}_u^\mathfrak{a}$ can be identified with $\Lambda$ using the correspondence $p_k \leftrightarrow P_{km,-kn'}$ for $k > 0$. 

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There are several classical bases in the space $\Lambda$, e.g. Schur basis $\{s_\mu\}$. Another important basis consists of Macdonald symmetric functions $P_\lambda$ and their renormalization $M_\lambda$, see [Neg16, Sect. 2.4] and the references therein.

**Remark 8.1.** The basis $\{P_\lambda\}$ can be defined as the eigenbasis for the action of the operators $P_{km, -km'} \in U_{q,v}(\mathfrak{gl}_1) \rtimes F_u^* \cong \Lambda$. Let $P_\lambda(q,t) \in \Lambda$ be Macdonald symmetric functions, here we use Macdonald notation $q,t$, see [Mac95]. Then $P_\lambda = P_\lambda(q,v^{-2})$.

There is an analogous (but different) construction to obtain Macdonald symmetric functions. One can consider the action $\mathcal{SH}_N^{(+)}(q,v) \simeq \text{S}_+ \mathbb{C}[A_1, \ldots, A_N]$. The corresponding eigenvectors of $\sum_i B_i$ are Macdonald symmetric polynomials with the parameters $q$, $t = v^{-2}$ [Che92]. In the limit $N \to \infty$ we obtain Macdonald symmetric functions $P_\lambda(q,v^{-2})$. On the contrary, Macdonald symmetric functions with the parameters $q$, $t = qv^{-2}$ are limits of the eigenvectors in $\text{S}_- \mathbb{C}[A_1, \ldots, A_N]$. This difference between the Macdonald parameters in $\text{S}_+ \mathbb{C}[A_1, \ldots, A_N]$ and $\text{S}_- \mathbb{C}[A_1, \ldots, A_N]$ was used in the proof Macdonald conjectures, see [Kir97, Sect. 7].

It was shown in Lemma 7.1 that the action of $P_{km, -km'}$ stabilizes in the basis $\{|\lambda\rangle_N\}$. Hence $|\lambda\rangle_\infty$ form a basis of $\Lambda$. Denote

$$|\lambda\rangle_\infty = \sum_\mu c^\mu_\lambda(q,v) M_\mu, \quad |\lambda\rangle_\infty = \sum_\mu r^\mu_\lambda(q,v) s_\mu. \quad (8.1)$$

**Conjecture 8.1.** The vectors $|\lambda\rangle_\infty$ (considered as elements of $\Lambda$) satisfy the following conditions

(i) $|\lambda\rangle_\infty$ are integral, i.e. the coefficients $r^\mu_\lambda(q,v) \in \mathbb{Z}[q^{\pm 1}, v^{\pm 1}]$.

(ii) The transition matrix $c^\mu_\lambda(q,v)$ is triangular with respect to the dominance order “$<$”.

(iii) The coefficients $c^\mu_\lambda(q,v)$ satisfy the “window” condition [Neg16, eq. (4.4)–(4.5)] for a certain slope.

Moreover, $|\overline{\lambda}\rangle_\infty$ satisfy the conditions (i)–(iii) for another slope.

We do not write the precise form of the window condition since we do not use it here.

Actually, the conjecture in [GN17] was formulated in a weaker form than Conjecture 8.1. Namely, it was stated that there exists an identification of $\Lambda^{\infty/2}$ ($\mathbb{C}^n[Y^{\pm 1}]$) with the space of symmetric functions $\Lambda$ such that image of the basis $\{|\lambda\rangle_\infty\}$ satisfies the properties (i)–(iii) and similarly for $\{|\overline{\lambda}\rangle_\infty\}$. In this form, the conjecture was proven in [KS20] using 3d-mirror symmetry. Above we conjectured additionally that this identification comes from our Theorem 7.1.

Below we prove the properties (i) and (ii) using analogous properties of the (non-symmetric, finite) basis $e_\lambda = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N}$ shown in Section 3. The proof consists of two steps: first we apply $\mathcal{S}_-$ and then take the limit $N \to \infty$. We believe that the property (iii) can be also deduced from an analog of the window condition for the basis $e_\lambda$.

**Theorem 8.1.** The vectors $|\lambda\rangle_\infty$ (considered as elements of $\Lambda$) satisfy the following conditions

(i) $|\lambda\rangle_\infty$ are integral, i.e. expands in terms of Schur functions with coefficients in $\mathbb{Z}[q^{\pm 1}, v^{\pm 1}]$.

(ii) The transition matrix from the basis $\{M_\mu\}$ to the basis $\{|\lambda\rangle_\infty\}$ is triangular with respect to the dominance order “$<$”.

The theorem precisely states the properties (i) and (ii) from Conjecture 8.1. To prove the theorem we need certain preparations. Similarly to the notations of Section 3, let us denote

$$e_{-\lambda - \rho,N} = e_{-\lambda_1} \otimes e_{-\lambda_2 + 2} \otimes \cdots \otimes e_{N-1}.$$  

(8.2)
Clearly, we have $|\lambda\rangle_N = S_- e_{-\lambda-\rho,N}$. Similarly, we use notation $A_{\lambda,N}$ for monomial basis in the space $(\mathbb{C}^n[Y^{\pm 1}])^{\otimes N}$, see formula (3.27). Let $s_{\mu,N}$ and $P_{\mu,N}$ be, respectively, Schur and Macdonald polynomial in $N$ variables. Also, we will need $v$-Vandermonde polynomial

$$A = \frac{1}{|N|v!} \prod_{j>i} (v^{-1}A_j - vA_i).$$

(8.3)

For a polynomial $f(A_1,\ldots,A_N)$ we denote $f(A^{-1}) = f(A_1^{-1},\ldots,A_N^{-1})$.

**Lemma 8.2.** a) The vector $S_- A_{\eta,N}$ is skew-symmetric in $\eta$.

b) If $\eta_i = -\mu_i + i - 1$ for $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ then we have $S_- A_{\eta,N} = s_{\mu,N}(A^{-1}) A$.

**Proof.** a) Introduce usual (not $v$-deformed) antisymmetrizer $S_{-v} = \frac{1}{N!} \sum w(-1)^{l(w)} w$, here the summation goes over all the permutations of the generators $A_i$. It was shown in [Kir97, Th. 8.2] that $\text{Ker} S_{-v} = \text{Ker} S_-$. Hence the vector $S_- A_{\eta,N}$ is skew-symmetric in $\eta$.

b) Any polynomial in the image of $S_-$ is a product of a symmetric polynomial and $A$, see [Kir97, proof of Th. 7.1]. Hence $S_- A_{\eta,N} = f_{\mu}(A^{-1}) A$ for some symmetric $f_{\mu}$. We claim that $f_{\mu}(A^{-1}) = s_{\mu}(A^{-1})$. This is obvious for $v = 1$. For generic $v$ it is easy to check that $f_{\mu} = 1$, and that multiplication of $f_{\mu}$ by $\sum A_i^{-k}$ is given by a formula which does not depend on $v$ (Murnaghan-Nakayama rule). Evidently, $f_{\mu}$ is determined by this. $\square$

**Proof of Theorem 8.1** Corollary 3.7 states that there is an expansion

$$e_{-\lambda-\rho,N} = \sum_{\eta \leq -\lambda-\rho} \alpha_{\lambda,\eta} A_{\eta,N},$$

(8.4)

where $\alpha_{\lambda,\eta} \in \mathbb{Z}[q^{\pm 1},v^{\pm 1}]$. We apply $S_-$ to both sides. Using Lemma 8.2(a), the sum can be rewritten as a sum over $\eta = (\eta_1 < \eta_2 < \cdots < \eta_N)$. Introduce $\mu = -\eta - \rho$ as in Lemma 8.2(a). The condition $-\mu - \rho = -\lambda - \rho$ implies $\mu_i \geq 0$ and $\mu \leq \lambda$. Due to Lemma 8.2 we obtain

$$|\lambda\rangle_N = \sum_{\mu \leq \lambda} r_{\lambda,\mu} s_{\mu,N}(A^{-1}) A.$$  

(8.5)

Using $\alpha_{\lambda,\eta} \in \mathbb{Z}[q^{\pm 1},v^{\pm 1}]$ and the definition of Macdonald polynomials we obtain the following.

(i) The coefficients $r_{\lambda,\mu}$ are integral.

(ii) The transition matrix from the Macdonald basis $\{ P_{\mu,N} \}$ to the basis $\{|\lambda\rangle_N\}$ is upper triangular.

To finish the proof we take the limit $N \to \infty$. $\square$

**Remark 8.3.** One can show that the costandard basis $\{|\lambda\rangle_\infty\}$ satisfies the properties [i] and [ii]. Indeed, the transition matrix from $\{|\lambda\rangle_\infty\}$ to $\{|\lambda\rangle_\infty\}$ is upper triangular with integral coefficients, see formula (8.8) and Lemma 5.1. One can deduce the properties of the costandard basis from the corresponding properties of the standard basis. This is also consistent with the conjecture of [GN17].

A Quantum affine algebra and its vertex operators

In this paper, we have used the space $\Lambda_{v,\infty}^{\otimes 2}(\mathbb{C}^n[Y^{\pm 1}])$ and the vertex operators $\Phi(z)$, $\Psi(z)$, $\Phi^*(z)$, and $\Psi^*(z)$. The space $\Lambda_{v,\infty}^{\otimes 2}(\mathbb{C}^n[Y^{\pm 1}])$ is known as an integrable level-one representation of quantum affine $U_v(\hat{\mathfrak{sl}}_n)$ [KMS95, LT00]. The vertex operators can be defined by intertwining properties.

Integrable level-one representations and the vertex operators have their counterpart for $U_v(\hat{\mathfrak{sl}}_n)$. Below we will study the connection between the $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ versions.

We will consider only the vertex operator $\Phi^*(z)$. The situation for the other operators is analogous. We will need the results concerning $\Phi^*(z)$ for the proof of Proposition 5.19.
A.1 Action of quantum affine algebra

Let \( \alpha_0, \ldots, \alpha_{n-1} \) be simple positive roots of \( \widehat{sl}_n \). We will use standard scale product \((\alpha_i, \alpha_j)\), and Cartan matrix \( \langle \alpha_i, \alpha_j \rangle = 2 (\alpha_i, \alpha_j) / (\alpha_j, \alpha_j) \). Note that for \( \widehat{sl}_n \) we have \( \langle \alpha_i, \alpha_j \rangle = (\alpha_i, \alpha_j) \). Algebra \( U_v(\widehat{sl}_n) \) is generated by \( E_i, K_i \) and \( F_i \) for \( i = 0, 1, \ldots, n - 1 \). The relations are

\[
K_i K_j = K_j K_i, \quad K_i E_i K_j^{-1} = \nu^{(\alpha_i, \alpha_j)} E_j, \quad K_i F_j K_j^{-1} = \nu^{-(\alpha_i, \alpha_j)} F_j, \tag{A.1}
\]

\[
[E_i, F_j] = \delta_{i,j} K_i - K_i^{-1}, \tag{A.2}
\]

\[
\sum_{k=0}^{b_{ij}} (-1)^k {b_{ij} \choose k} E_i E_j E_i^{b_{ij} - k} = 0, \quad \sum_{k=0}^{b_{ij}} (-1)^k {b_{ij} \choose k} E_i E_i E_i^{b_{ij} - k} = 0. \tag{A.3}
\]

where \( b_{ij} = 1 - \langle \alpha_i, \alpha_j \rangle \) and \( {b_{ij} \choose k} = [b_{ij} / (k!) [b_{ij} - k]!] \). There is an action of \( U_v(\widehat{sl}_n) \) on \( \mathbb{C}^n[Y^{\pm 1}] \) determined as follows

\[
E_i e_j = \delta_{i,j} e_{j+1}, \quad F_i e_j = \delta_{i,j-1} e_{j-1}, \quad K_i e_j = e^{\delta_{i,j-1} - \delta_{i,j}} e_j. \tag{A.4}
\]

Using the comultiplication

\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i, \tag{A.5}
\]

we can define action of \( U_v(\widehat{sl}_n) \) on \( (\mathbb{C}^n[Y^{\pm 1}])^\otimes N \). Recall the action of affine Hecke algebra \( H^Y \) on the space \( (\mathbb{C}^n[Y^{\pm 1}])^\otimes N \) determined in Section 3.1.

**Proposition A.1.** The actions of (affine) Hecke algebra and \( U_v(\widehat{sl}_n) \) commute.

Hence we have obtained an action \( U_v(\widehat{sl}_n) \curvearrowright S_- (\mathbb{C}^n[Y^{\pm 1}])^\otimes N. \)

**The limit** Let us use the inductive system \((5.12)\). Let \( E^{(N)}_i, K^{(N)}_i, F^{(N)}_i \) be the operators, coming from the defined above action \( U_v(\widehat{sl}_n) \curvearrowright (\mathbb{C}^n[Y^{\pm 1}])^\otimes N \). Consider a number \( \mathbf{r} = 0, 1, \ldots, n - 1 \) such that \( \mathbf{r} \equiv N - 1 - \mathbf{m} \ (\text{mod} \ n) \). Let us define the following operators

\[
\tilde{E}^{(N)}_i = E^{(N)}_i, \quad \tilde{K}^{(N)}_i = \nu^{\delta_i \mathbf{r}} K^{(N)}_i, \quad \tilde{F}^{(N)}_i = \nu^{\delta_i \mathbf{r}} F^{(N)}_i. \tag{A.6}
\]

Below we will need the following versions of Definitions 5.6 and 5.15.

**Definition A.2.** A sequence of operators \( A^{(N)}: S_- (\mathbb{C}^n[Y^{\pm 1}])^\otimes N \rightarrow S_- (\mathbb{C}^n[Y^{\pm 1}])^\otimes (N+\delta) \) 1-stabilizes if for any \( w \in S_- (\mathbb{C}^n[Y^{\pm 1}])^\otimes k \) there is \( M \) such that for any \( N > M \) we have

\[
\varphi_{N+\delta,1,N+\delta}^{(\mathbf{m}+\delta)} \circ A^{(N)} \circ \varphi_{N,k}^{(\mathbf{m})} (w) = A^{(N+1)} \circ \varphi_{N+1,k}^{(\mathbf{m})} (w). \tag{A.7}
\]

**Definition A.3.** A sequence of operators \( A^{(N)}: S_- (\mathbb{C}^n[Y^{\pm 1}])^\otimes N \rightarrow S_- (\mathbb{C}^n[Y^{\pm 1}])^\otimes (N+\delta) \) weakly 1-stabilizes if for any \( w \in S_- (\mathbb{C}^n[Y^{\pm 1}])^\otimes k \) there is \( M \) such that for any \( N > M \) we have

\[
\varphi_{N+\delta}^{(\mathbf{m}+\delta)} \circ A^{(N)} \circ \varphi_{N,k}^{(\mathbf{m})} (w) = \varphi_{N+1+\delta}^{(\mathbf{m}+\delta)} \circ A^{(N+1)} \circ \varphi_{N+1,k}^{(\mathbf{m})} (w). \tag{A.8}
\]

**Proposition A.4.** The operators \( \tilde{F}^{(N)}_i \) 1-stabilize.
Proposition A.5. The operators $\hat{E}_i^{(N)}$ and $\hat{F}_i^{(N)}$ 1-stabilize. The operators $\hat{E}_i^{(N)}$ weakly 1-stabilize. Moreover, for $i \neq x$ the operators $E_i^{(N)}$ stabilize. More explicitly, for any $w \in \mathbf{S}_- (\mathbb{C}^n[Y^\pm 1]) \otimes^k$ and sufficiently large $N \neq m + i + 1$ (mod $n$) we have

$$\varphi_{N+n,N}^{(N)} \circ \varphi_i^{(N)}(w) = \varphi_{N,n}^{(N)} \circ \varphi_N^{(N)}(w)$$  \hfill (A.9)

Denote the induced operators by $\hat{E}_i$, $\hat{K}_i$, and $\hat{F}_i$.

Proposition A.6 ([KMS95], [LT00]). The formulas $E_i \mapsto \hat{E}_i$, $K_i \mapsto \hat{K}_i$, $F_i \mapsto \hat{F}_i$ determine an action of $U_v(\mathfrak{sl}_n)$ on $\Lambda_{v,n}^{\infty/2} (\mathbb{C}^n[Y^\pm 1])$.

From $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$ Let $U_v(\text{Heis})$ be the algebra generated by $B_k$ for $k \in \mathbb{Z}$ and central $c^{\pm 1}$ with the relation

$$[B_k, B_l] = k \frac{e^{-2k} - 1}{e^{2k} - 1} \delta_{k+l,0}.$$  \hfill (A.10)

Remark A.7. We abuse notation since $B_k$ was defined as an operator on $\Lambda_{v,n}^{\infty/2} (\mathbb{C}^n[Y^\pm 1])$ and $c$ is an element of $U_{q_1,q_2}(\mathfrak{gl}_1)$.

Let $U_v(\mathfrak{gl}_n) = U_v(\mathfrak{sl}_n) \otimes U_v(\text{Heis})$. The algebra $U_v(\mathfrak{gl}_n)$ acts on $\Lambda_{v,n}^{\infty/2} (\mathbb{C}^n[Y^\pm 1])$ as follows. The action of $U_v(\mathfrak{sl}_n)$ comes from Proposition A.6. The generators $B_k$ act as the operators with the same name defined in Proposition 5.16. The central element $c$ acts as multiplication by $v^{-n}$. Denote the obtained representation by $F_n$.

Proposition A.8. The obtained representation $F_n$ is irreducible integrable level-one representation of $U_v(\mathfrak{gl}_n)$. There are $n$ non-isomorphic classes of such representations $F_0, \ldots, F_{n-1}$. The representations $F_n$ and $F_{n-1}$ are isomorphic for any $k \in \mathbb{Z}$.

Let $F_{0}^{sl}, F_{1}^{sl}, \ldots, F_{n-1}^{sl}$ be the irreducible integrable level-one representation of $U_v(\mathfrak{sl}_n)$. Let $F^n$ be Fock module of $U_v(\text{Heis})$ for $c = v^{-n}$.

Proposition A.9. The representation $F_n$ is isomorphic to $F_{n}^{sl} \otimes F^n$ as representations of $U_v(\mathfrak{gl}_n) = U_v(\mathfrak{sl}_n) \otimes U_v(\text{Heis})$ for $m = 0, 1, \ldots, n - 1$.

A.2 Vertex operators

Below we will study intertwining properties of $\hat{\Phi}_k^*$. Its analog defines vertex operators for $\mathfrak{sl}_n$ [FR92].

In this subsection we will recall basic properties of the vertex operators for $\mathfrak{sl}_n$ and start to study a connection between vertex operators of $\mathfrak{sl}_n$ and $\hat{\Phi}_k^*$. The connection will be made more precise in the next subsection.

Intertwining property Let us define an operator by

$$\hat{\Phi}^* : \Lambda_{v,n}^{\infty/2} (\mathbb{C}^n[Y^\pm 1]) \to \mathbb{C}^n[Y^\pm 1] \otimes \Lambda_{v,n-1}^{\infty/2} (\mathbb{C}^n[Y^\pm 1])$$  \hfill (A.11)

$$\hat{\Phi}^* w = \sum_{k \in \mathbb{Z}} e^{-k} \otimes \hat{\Phi}_k^* w.$$  \hfill (A.12)

Proposition A.10. $\hat{\Phi}^*$ is an $U_v(\mathfrak{sl}_n)$-intertwiner.

Evidently, Proposition [A.10] is equivalent to the following proposition.
Proposition A.11. The following relations hold
\[
\hat{\Phi}^*_k \hat{\Phi}_k = e^{\delta_i+k=-1-\delta_i+k=0} \hat{\Phi}_k^* \tag{A.13}
\]
\[
\hat{\Phi}^*_k \hat{E}_k = e^{\delta_i+k=-1} \hat{\Phi}_k^* \tag{A.14}
\]
\[
\hat{\Phi}^*_k \hat{F}_k = \delta_i+k=0 \hat{\Phi}_k^* + \hat{F}_k \tag{A.15}
\]

Proof. Analogously to (A.12), consider operator
\[
\Phi^*: S_+ (\mathbb{C}^n[Y\pm 1])^\otimes N \to (\mathbb{C}^n[Y\pm 1]) \otimes S_+ (\mathbb{C}^n[Y\pm 1])^\otimes N-1
\]
\[
\Phi^* w = \sum_{k \in \mathbb{Z}} e_{-k} \otimes \Phi^*_k w
\]

Evidently, the operator is an intertwiner. Equivalently, \(\Phi^*_k\) satisfy the counterparts of (A.13)–(A.15)
\[
\hat{\Phi}^*_k K_i^{(N)} = e^{\delta_i+k=-1-\delta_i+k=0} K_i^{(N-1)} \Phi^*_k \tag{A.18}
\]
\[
\hat{\Phi}^*_k E_i^{(N)} = e^{\delta_i+k=-1} \Phi^*_k \tag{A.19}
\]
\[
\hat{\Phi}^*_k F_i^{(N)} = \delta_i+k=0 K_i^{(N-1)} \Phi^*_k - 1 + F_i^{(N-1)} \Phi^*_k \tag{A.20}
\]

Note that in the relations above we can replace \(K_i^{(N)}, E_i^{(N)}, F_i^{(N)}\) by \(\hat{K}_i^{(N)}, \hat{E}_i^{(N)}, \hat{F}_i^{(N)}\) respectively. Since the operators \(\hat{K}_i^{(N)}, \hat{F}_i^{(N)}\) and \(\hat{E}_i^{(N)}\) stabilize for \(j \neq r,\) the corresponding relation hold for \(\hat{\Phi}^*_k\). It remains to check
\[
\hat{\Phi}^*_k \hat{E}_r = e^{\delta_i+k=-1} \hat{\Phi}_k^* + e^{\delta_i+k=0-\delta_i+k=-1} \hat{E}_r \hat{\Phi}_k^* \tag{A.21}
\]

Notice that \(E_r^{(N+1)}\) stabilizes. Hence (A.21) follows from
\[
\Phi^*_k E_r^{(N+1)} = e^{\delta_i+k=-1} \Phi^*_k + e^{\delta_i+k=0-\delta_i+k=-1} \hat{E}_r \Phi^*_k \tag{A.22}
\]

\[\Box\]

**Vertex operators** for \(U_v(\hat{sl}_n)\) Denote the highest vector of \(F_{n}^{sl}\) by \(|m\rangle\). Let us also define \(F_{n}^{sl}\) for any \(m \in \mathbb{Z}\) by \(F_{n} = F_{n}^{sl} \otimes F_{H}\).

Vertex operators for \(U_v(\hat{sl}_n)\) are defined as the intertwiners
\[
\tilde{\Phi}^{*,sl}: F_{n} \to (\mathbb{C}^n[Y\pm 1]) \otimes F_{n-1}
\]
satisfying the following normalization condition
\[
\tilde{\Phi}^{*,sl}|m\rangle = e_{-m} \otimes |m - 1\rangle + \sum_{k \leq m} e_{-k} \otimes w_k \tag{A.24}
\]
for certain vectors \(w_k \in F_{n-1}\).

**Proposition A.12** ([FR02]). There exists unique operator \(\tilde{\Phi}^{*,sl}\) determined by (A.23)–(A.24).

Also, we define operators \(\tilde{\Phi}^{*,sl}_k\) and currents \(\tilde{\Phi}^{*,sl}_{(\alpha)}(z)\) by
\[
\tilde{\Phi}^{*,sl}_k w = \sum_{k \in \mathbb{Z}} e_{-k} \otimes \tilde{\Phi}^{*,sl}_k w \quad \tilde{\Phi}^{*,sl}_{(\alpha)}(z) = \sum_{k \in \mathbb{Z}} \tilde{\Phi}^{*,sl}_{-\alpha+nk} z^{-k} \tag{A.25}
\]

Let us consider principal grading on \(F_{n}^{sl}\) given by \(\deg |m\rangle = -\frac{m(m+1)}{2}\), \(\deg E_i = 1\), \(\deg F_i = -1\). Note that \(\deg \tilde{\Phi}^{*,sl}_k = k\).
Lemma A.13. For any intertwiner $\phi^{*,sl} = \sum e_{-k} \otimes \phi_k^{*,sl} : F_n^{sl} \to (\mathbb{C}^n[Y^\pm1]) \otimes F_{n-1}^{sl}$ such that $\deg \phi_k^{*,sl} = k + \Delta$, we have $\phi_k^{*,sl} \gamma_{k+\Delta}$ for certain $\gamma \in \mathbb{C}$.

Proof. Note that $\deg \phi_k^{*,sl} |m| = \deg |m-1|$. Since the subspace of degree $\deg |m-1|$ is one-dimensional, we have $\phi_k^{*,sl} |m| = \gamma|m-1|$ for certain $\gamma \in \mathbb{C}$. Proposition A.12 implies $\sum e_{-k} \otimes \phi_k^{*,sl} = \gamma \Phi^{*,sl}$. □

Proposition A.14. There exist operators $\Xi_d \subset F^H$ such that for $\Xi(z) = \sum_{d \in \mathbb{Z}} \Xi_d z^{-d}$ we have

$$\Phi^{*,sl}_{(n-1)}(z) = \Phi^{*,sl}_{(n-1)}(z) \Xi(z) \tag{A.26}$$

Proof. We can extend the grading from $F_n^{sl}$ to $F_n$ by $\deg B_k = kn$. Note that $\deg \Phi_k = n$. We can present $\Phi_k = \sum_{d,v} \phi_{k,d,v} \otimes \Xi_{d,v}$ for linear independent operators $\Xi_{d,v}$ with $\deg \Xi_{d,v} = nd$ (e.g. take $\Xi_{d,v}$ to be matrix units for a homogeneous basis of $F^H$). Proposition A.10 and Lemma A.13 imply that $\phi_{k,d,v} = \gamma_{d,v} \Phi_{k-dv}$ for certain $\gamma_{d,v} \in \mathbb{C}$. Hence $\Phi_k = \sum_d \phi_{k-dv} \otimes (\sum_{d,v} \gamma_{d,v} \Xi_{d,v})$. □

Bosonization The operator $\phi^{*,sl}_{(n-1)}(z)$ was calculated in [Koy94 Thm. 3.4]. Note that the parameter $q$ used in the loc. cit. corresponds to our parameter $v^{-1}$. To write the answer, we recall notation of loc. cit.

Let $Q$ and $P$ be root and weight lattices for $\mathfrak{sl}_n$ respectively. We use notation $e^\beta$ for an element of group algebra $\mathbb{C}[P]$, corresponding to $\beta \in P$. Denote the fundamental weights of $\mathfrak{sl}_n$ by $\Lambda_1, \ldots, \Lambda_{n-1}$. There is Heisenberg algebra in $U_v(\mathfrak{sl}_n)$ generated by $a_j(k)$ for $k \in \mathbb{Z}$ and $j = 1, \ldots, n-1$. Let $F^n$ be Fock module for the Heisenberg algebra. Then $F_n$ can be naturally identified with $F^n \otimes \mathbb{C}[Q] e^{h_n}$, and the action of $U_v(\mathfrak{sl}_n)$ can be constructed explicitly, see [FJ88] or [Koy94 Sect. 2.4].

For $\alpha \in P$, let us introduce operator $\partial_{\alpha}(w \otimes \beta) = (\alpha, \beta) w \otimes \beta$. There exists subalgebra $a_\alpha^v(k)$ in the algebra generated by $a_j(k)$. In the representation $F_n$, the operators satisfy

$$[a_\alpha^v(k), a_\alpha^v(-k)] = -\frac{[n-1]!_v}{k[nk]_v} \tag{A.27}$$

Proposition A.15 ([Koy94]). Vertex operator $\hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) : F_n \to F_{n-2}$ is given by the following formula

$$\hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) = \exp \left( -\sum_{k=1}^{\infty} a_\alpha^v(k) \nu^{\frac{1}{2} k} z_1^k \right) \exp \left( -\sum_{k=1}^{\infty} a_\alpha^v(k) \nu^{\frac{1}{2} k} z_2^k \right) \times e^{\Lambda_1 \partial_{\Lambda_1} ((-1)^{n-1} v^{-1} z_1)} - \frac{v^{n+\frac{1}{2} n}}{n} \nu^{(-1)^{n+\frac{1}{2} n} \frac{1}{z_1^n+\frac{1}{z_2}}} \tag{A.28}$$

Normal ordered product $\hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) :$ is an operator from $F_n$ to $F_{n-2}$ defined by the following formula

$$\hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) := \exp \left( -\sum_{k=1}^{\infty} a_\alpha^v(k) \nu^{\frac{1}{2} k} (z_1^k + z_2^k) \right) \times e^{-2\Lambda_1} \prod_{j=1,2} ((-1)^{n-1} v^{-1} z_2) - \frac{v^{n+\frac{1}{2} n}}{n} \nu^{(-1)^{n+\frac{1}{2} n} \frac{1}{z_1^n+\frac{1}{z_2}}} \tag{A.29}$$

here $m_j = m + j - 2$. Note that the normal ordering is not symmetric, namely

$$z_1^{-\frac{1}{n}} \hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) := z_2^{-\frac{1}{n}} \hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) \tag{A.30}$$

The following relations can be checked directly

$$\hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) = ((-1)^{n-1} v^{-1} z_1)^{\frac{n+1}{n}} \frac{(v^2 z_2 / z_1, v^{2n})_{\infty}}{(v^{2n} z_2 / z_1, v^{2n})_{\infty}} \hat{\Phi}^{*,sl}_{(n-1)}(z_1) \hat{\Phi}^{*,sl}_{(n-1)}(z_2) \tag{A.31}$$
A.3 Factorization of the vertex operator

We continue to study connection between vertex operators for $\mathfrak{sl}_n$ and $\hat{\Phi}^*_\alpha(z)$. This subsection is devoted to a proof of Theorem A.1. The theorem is used for the proof of Proposition 5.19.

Theorem A.1. The following holds

$$\hat{\Phi}^*_\alpha(z) = \hat{\Phi}^*_{\alpha,\mathfrak{sl}_n}(z) \otimes \exp \left( - \sum_{j>0} \frac{v^2j}{j[n]_j}B_{-j}z^j \right) \exp \left( \sum_{j>0} \frac{1}{j[n]_j}B_jz^{-j} \right).$$

(A.32)

To prove the theorem we need certain preparations. Let us define

$$\Phi^*_\alpha(z) = \sum_{k \in \mathbb{Z}} \Phi^*_{-\alpha+nk}z^{-k}$$

(A.33)

Lemma A.16. The following relation holds

$$(v^2z_1 - z_2)\Phi^*_\alpha(z_1)\Phi^*_\alpha(z_2) = (v^2z_2 - z_1)\Phi^*_\alpha(z_2)\Phi^*_\alpha(z_1)$$

(A.34)

Proof. Consider operator

$$(1 \otimes \Phi^*) \circ \Phi^*: \mathcal{S}_-((\mathbb{C}^n[Y^{\pm1}])^N \to (\mathbb{C}^n[Y^{\pm1}])^N \otimes (\mathbb{C}^n[Y^{\pm1}])^N \otimes \mathcal{S}_-((\mathbb{C}^n[Y^{\pm1}])^N.$$ (A.35)

Also, consider operator

$$T_{12} \subset (\mathbb{C}^n[Y^{\pm1}])^N \otimes (\mathbb{C}^n[Y^{\pm1}])^N \otimes \mathcal{S}_-((\mathbb{C}^n[Y^{\pm1}])^N.$$ (A.37)

induced from action of $T$ on first two tensor multiples. Recall that $T$ is given by (4.6)–(4.9). The basic property of anti-symmetrizer (4.10) implies

$$T_{12} \circ (1 \otimes \Phi^*) \circ \Phi^* = -v^{-1} (1 \otimes \Phi^*) \circ \Phi^*.$$ (A.38)

Let $l > k$ and $l \equiv k \mod n$. Consider in (A.38) the coefficients in front of $e_l \otimes e_k$ and $e_{l+n} \otimes e_{k-n}$

$$\nu\Phi^*_{-l}\Phi^*_k - (v - v^{-1})\sum_{j=1}^{\infty} \Phi^*_{-k+nj}\Phi^*_{-l-nj} + (v - v^{-1})\sum_{j=1}^{\infty} \Phi^*_{-l-nj}\Phi^*_{-k+nj} = -v^{-1}\Phi^*_{-k}\Phi^*_{-l}$$

(A.39)

$$\nu\Phi^*_{-l-n}\Phi^*_k - (v - v^{-1})\sum_{j=1}^{\infty} \Phi^*_{-k+nj}\Phi^*_{-l-n-j} + (v - v^{-1})\sum_{j=1}^{\infty} \Phi^*_{-l-n-j}\Phi^*_{-k+nj} = -v^{-1}\Phi^*_{-k+n}\Phi^*_{-l-n}$$

(A.40)

Hence

$$\nu \left( \Phi^*_{-l}\Phi^*_k - \Phi^*_{-l-n}\Phi^*_k - \Phi^*_k \Phi^*_{-l-n} \right) - (v - v^{-1})\Phi^*_{-k+n}\Phi^*_{-l-n} + (v - v^{-1})\Phi^*_{-l-n}\Phi^*_{-k+n}$$

$$= -v^{-1}\left( \Phi^*_{-k}\Phi^*_l - \Phi^*_{-k+n}\Phi^*_l \right).$$ (A.41)

Equivalently

$$v\Phi^*_{-l}\Phi^*_k - v^{-1}\Phi^*_{-l-n}\Phi^*_k = -v^{-1}\Phi^*_{-k}\Phi^*_l + v\Phi^*_{-k+n}\Phi^*_l$$

(A.42)

Substituting $l - n$ instead of $l$ and multiplying by $v$, we obtain

$$v^2\Phi^*_{-l+n}\Phi^*_k - \Phi^*_{-l}\Phi^*_k - v^2\Phi^*_{-k+n}\Phi^*_l = v^2\Phi^*_{-k+n}\Phi^*_l - \Phi^*_{-k}\Phi^*_l$$

(A.43)

To finish the proof we notice that (A.43) is symmetric on $l$ and $k.$
Let us define \( \Xi = \Theta_2 - z_1 \Theta_1 \). Using (A.30), one can see that \( \hat{\Phi}^n = (z_1 - z_2) \hat{\Phi}^{n-1} \hat{\Phi}^1 \hat{\Phi}^1 \). The result is \( \hat{\Phi}^n \Theta_2 = \Theta_1 \Xi \). The relation 

\[
\Xi(z) = \Xi(z) \exp \left( \sum_{j>0} \frac{1}{j!n_{\nu_j}} B_{\nu_j} z_1^{-j} \right),
\]

is a formal power series, the coefficients are operators on \( F^H \), and \( [B_{\nu_j}, \Xi(z)] = 0 \). Equivalently, \( \Xi(z) = \sum \alpha_{\mu_1, \ldots, \mu_j} B_{-\mu_1} \cdots B_{-\mu_j} z_1^{\mu_1} \). Denote

\[
\Xi_0 = \sum_{\mu} \alpha_{\mu_1, \ldots, \mu_j} \left( B_{-\mu_1} + z_1^{\mu_1} \right) \cdots \left( B_{-\mu_j} + z_1^{\mu_j} \right) z_2^{\mu_2}.
\]

Substituting (A.47) to (A.46), we obtain

\[
\Xi_0 \equiv \Xi(z) \equiv \Xi(z) = \Xi(z) \exp \left( \sum_{j>0} \frac{1}{j!n_{\nu_j}} B_{\nu_j} z_1^{-j} \right),
\]

the relation \( [B_{-\nu_j}, \hat{\Phi}^1] = -\hat{\Phi}^1 \) implies

\[
[\Xi_0, z_2 \hat{\Phi}^1] = 0.
\]

Using (A.30), one can see that

\[
\Xi(z) = \Xi(z) \exp \left( \sum_{j>0} \frac{1}{j!n_{\nu_j}} B_{\nu_j} z_1^{-j} \right),
\]

is a formal power series, the coefficients are operators on \( F^H \), and \( [B_{-\nu_j}, \Xi(z)] = 0 \). Equivalently, \( \Xi(z) = \sum \alpha_{\mu_1, \ldots, \mu_j} B_{-\mu_1} \cdots B_{-\mu_j} z_1^{\mu_1} \). Denote

\[
\Xi_0 = \sum_{\mu} \alpha_{\mu_1, \ldots, \mu_j} \left( B_{-\mu_1} + z_1^{\mu_1} \right) \cdots \left( B_{-\mu_j} + z_1^{\mu_j} \right) z_2^{\mu_2}.
\]

Substituting (A.47) to (A.46), we obtain

\[
\Xi_0 \equiv \Xi(z) \equiv \Xi(z) \exp \left( \sum_{j>0} \frac{1}{j!n_{\nu_j}} B_{\nu_j} z_1^{-j} \right),
\]

the relation \( [B_{-\nu_j}, \hat{\Phi}^1] = -\hat{\Phi}^1 \) implies

\[
[\Xi_0, z_2 \hat{\Phi}^1] = 0.
\]

Using (A.30), one can see that
We can substitute (A.51) to (A.50). Note that the exponent from (A.51) is an invertible series. Since (A.50) has only positive degree in both $z_1$ and $z_2$, we can multiply both sides by the inverse to the exponents. We obtain

$$
\tilde{\Xi}_-(z_1)\tilde{\Xi}_-\left[B_{-k} + z_1^{-k}\right](z_2) = \tilde{\Xi}_-(z_2)\tilde{\Xi}_-\left[B_{-k} + z_2^{-k}\right](z_1)
$$

(A.52)

It is legitimate to divide by $\tilde{\Xi}_-(z_1)\tilde{\Xi}_-(z_2)$. *A priori*, the result is a series in $z_1$ and $z_2$ with coefficients in rational function in $B_{-j}$. We obtain

$$
\frac{\tilde{\Xi}_-\left[B_{-k} + z_1^{-k}\right](z_2)}{\tilde{\Xi}_-(z_2)} = \frac{\tilde{\Xi}_-\left[B_{-k} + z_2^{-k}\right](z_1)}{\tilde{\Xi}_-(z_1)}
$$

(A.53)

On the RHS we have only positive powers of $z_1$ and negative powers of $z_2$, and vice versa for the LHS. Hence, the expression is a constant. Therefore $\tilde{\Xi}(z)$ is a constant. Normalization condition (A.24) implies $\tilde{\Xi}(z) = 1$.

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