On the Liouville type theorems for self-similar solutions to the Navier-Stokes equations

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Abstract

We prove Liouville type theorems for the self-similar solutions to the Navier-Stokes equations. One of our results generalizes the previous ones by Nečas-Růžička-Šverak and Tsai. Using the Liouville type theorem we also remove a scenario of asymptotically self-similar blow-up for the Navier-Stokes equations with the profile belonging to $L^{p,\infty}(\mathbb{R}^3)$ with $p > \frac{3}{2}$.

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1 Introduction

We consider the Navier-Stokes equation in the space time cylinder $\mathbb{R}^3 \times (-T, 0)$

\begin{equation}
\partial_t u + (u \cdot \nabla) u - \Delta u = -\nabla p, \quad \nabla \cdot u = 0,
\end{equation}

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$, $(x,t) \in \mathbb{R}^3 \times (-T, 0)$. The aim of the present paper is to exclude a possible self similar blow up at the point $(0,0)$ under more general assumptions than in [3]. More precisely, we assume that $u$ and $p$ respectively are given by a self similar profiles $U : \mathbb{R}^3 \to \mathbb{R}^3$ and $P : \mathbb{R}^3 \to \mathbb{R}$ such that

\begin{align}
(1.2) \quad u(x,t) &= \frac{1}{\sqrt{-2at}} U \left( \frac{x}{\sqrt{-2at}} \right), \\
(1.3) \quad p(x,t) &= \frac{1}{-2at} P \left( \frac{x}{\sqrt{-2at}} \right), \quad (x,t) \in \mathbb{R}^3 \times (-T, 0),
\end{align}
where \( a \) is a positive constant. Then \((U, P)\) solves the following system proposed by Leray (cf. [6]).

\[
-\Delta U + (U \cdot \nabla)U + ay \cdot \nabla U + aU = -\nabla P, \quad \nabla \cdot U = 0 \quad \text{in} \quad \mathbb{R}^3.
\]

It is already known that if \( U \in L^p(\mathbb{R}^3) \) for some \( p \in [3, +\infty] \), then \( U = 0 \) for \( p \in [3, \infty) \), while \( U = \text{const} \) for \( p = \infty \). The case \( p = 3 \) is proved in [7], while the case \( p > 3 \) has been proved by Tsai in [8]. In fact Tsai proved a more general result, namely that \( U = 0 \) if \( u \) satisfies a local energy bound

\[
\text{sup}_{t \in (-t_0,0)} \int_B |u(t)|^2 \, dx + \int_{-t_0}^0 \int_B |\nabla u|^2 \, dx dt < +\infty
\]

for some ball \( B \subset \mathbb{R}^3 \) and some \( t_0 > 0 \).

We extend the results mentioned above in different directions. Our first main result is the following

**Theorem 1.1.** Let \((U, P) \in C^\infty(\mathbb{R}^3)^3 \times C^\infty(\mathbb{R}^3)^3 \) be a solution to (1.4), and \( \Omega = \nabla \times U \).

Suppose that for some \( q > 0 \)

\[
\|U\|_{L^q(B_1(y_0))} + \|\Omega\|_{L^2(B_1(y_0))} = o(|y_0|^{\frac{q}{2}}) \quad \text{as} \quad |y_0| \rightarrow +\infty.
\]

Then, \( U \) is a constant function.

Below we remove the condition on \( \Omega \), and instead we restrict the range of \( q \) so that \( q > \frac{3}{2} \). Our second main result is the following

**Theorem 1.2.** Let \((U, P) \in C^\infty(\mathbb{R}^3)^3 \times C^\infty(\mathbb{R}^3)^3 \) be a solution to (1.4). Suppose that for some \( \frac{3}{2} < q < +\infty \) and \( \alpha > 0 \)

\[
\int_{B_1(y_0) \cap \{|U| > \alpha\}} |U|^q \, dx \rightarrow 0 \quad \text{as} \quad |y_0| \rightarrow +\infty.
\]

Then \( U \) is a constant function.

**Remark 1.3.** If \( U \in L^\infty(\mathbb{R}^3) \), then (1.7) is obviously satisfied with the choice of \( \alpha = \|U\|_{L^\infty} + 1 \). In general, if \( U \in L^{p,\infty}(\mathbb{R}^3) \) for \( p > q \) implies (1.7). Indeed, for \( \frac{2}{p} < \theta < 1 \) we have

\[
\int_{B_1(y_0) \cap \{|U| > \alpha\}} |U|^q \, dx = q \int_0^\infty \sigma^{q-1} \text{meas} \{B_1(y_0) \cap |U| > \sigma\} d\sigma
\]

\[
\leq q \|U\|_{L^{p,\infty}}^{p\theta} \text{meas} \{B_1(y_0) \cap |U| > \alpha\}^{1-\theta} \int_0^\infty \sigma^{q-p\theta-1} d\sigma
\]

\[
= q \|U\|_{L^{p,\infty}}^{p\theta} \text{meas} \{B_1(y_0) \cap |U| > \alpha\}^{1-\theta} \frac{\alpha^{q-p\theta}}{p\theta - q} \rightarrow 0
\]

as \( |y_0| \rightarrow +\infty \). Thus, Theorem 1.2 leads to the following Corollary
Corollary 1.4. Let \((U, P) \in C^\infty(\mathbb{R}^3)^3 \times C^\infty(\mathbb{R}^3)^3\) be a solution to (1.4). Suppose that for some \(\frac{3}{2} < p < +\infty\)

\[ (1.8) \quad U \in L^{p,\infty}(\mathbb{R}^3). \]

The above corollary shows clearly that Theorem 1.2 improves the previous results of [7, 8]. As an application the above result one can remove a scenario of asymptotically self-similar blow-up with a profile given by (1.8) as follows, which could viewed as an improvement of the corresponding result in [2].

Theorem 1.5. Let \(u \in C^2(\mathbb{R}^3 \times (0, t_\ast))\) be a solution to (1.1). Suppose there exists \(U\) satisfying (1.8) with \(\frac{3}{2} < p < +\infty\), and \(q \geq 2\) such that

\[ (1.9) \quad \lim_{t \uparrow t_\ast} \frac{(t_\ast - t)^{\frac{q-3}{2q}}}{\sqrt{2a(t_\ast - \tau)}} \left\| u(\cdot, \tau) - \frac{1}{\sqrt{2a(t_\ast - \tau)}} U \left( \frac{x - x_\ast}{\sqrt{2a(t_\ast - \tau)}} \right) \right\|_{L^q(B_r, \sqrt{t_\ast - t}(x_\ast))} = 0 \]

for all \(r > 0\). Then, \(U = 0\), and \(z_\ast = (x_\ast, t_\ast)\) is not a blow-up point.

2 Local \(L^\infty\) estimate for local suitable weak solutions to the Navier-Stokes equation without pressure

The aim of the present section is to provide a local \(L^\infty\) bound for local suitable weak solutions to the Navier-Stokes equations without pressure.

First, let us recall the notion of the local pressure projection \(E^*_G : W^{-1,s}_G \rightarrow W^{-1,s}(G)\) for a given bounded \(C^2\)-domain \(G \subset \mathbb{R}^n, n \in \mathbb{R}\), introduced in [11]. Ap-pealing to the \(L^p\)-theory of the steady Stokes system (cf. [3]), for any \(F \in W^{-1,s}_G\) there exists a unique pair \((v, p) \in W^{1,s}_0(G) \times L^s_0(G)\) which solves in the weak sense the steady Stokes system

\[
\nabla \cdot v = 0 \quad \text{in} \quad G, \quad -\Delta v + \nabla p = F \quad \text{in} \quad G, \\
v = 0 \quad \text{on} \quad \partial G.
\]

Then we set \(E^*_G(F) := \nabla p\), where \(\nabla p\) denotes the gradient functional in \(W^{-1,s}(G)\) defined by

\[
\langle \nabla p, \varphi \rangle = \int_G p \nabla \cdot \varphi dx, \quad \varphi \in W^{1,s}_0(G).
\]

Here we have denoted by \(L^s_0(G)\) the space of all \(f \in L^s(G)\) with \(\int_G f dx = 0\).

Remark 2.1. 1. The operator \(E^*_G\) is bounded from \(W^{-1,s}(G)\) into itself with \(E^*_G(\nabla p) = \nabla p\) for all \(p \in L^s_0(G)\). The norm of \(E^*_G\) depends only on \(s\) and the geometric properties of \(G\), and independent on \(G\), if \(G\) is a ball or an annulus, which is due to the scaling properties of the Stokes equation.
2. In case $F \in L^s(G)$ using the canonical embedding $L^s(G) \hookrightarrow W^{-1,s}(G)$ and the elliptic regularity we get $E^*_G(F) = \nabla p \in L^s(G)$ together with the estimate

$$(2.1) \quad \| \nabla p \|_{s,G} \leq c \| F \|_{s,G},$$

where the constant in (2.1) depends only on $s$ and $G$. In case $G$ is a ball or an annulus this constant depends only on $s$ (cf. [3] for more details). Accordingly the restriction of $E^*_G$ to the Lebesgue space $L^s(G)$ defines a projection in $L^s(G)$. This projection will be denoted still by $E^*_G$.

Below for a class of vector fields $X$ we denote by $X^\sigma$ the set of $u \in X$ such that $\nabla \cdot u = 0$ in the sense of distribution.

By using the projection $E^*_G$, we introduce the following notion of local suitable weak solution to the Navier-Stokes equations

**Definition 2.2 (Local suitable weak solution).** Let $Q = \mathbb{R}^3 \times (-T,0)$. A vector function $u \in L^2_{loc}(Q)$ is called a local suitable weak solution to (1.1), if

1. $u \in L^\infty_{loc}(-T,0; L^2_{loc}(\mathbb{R}^3)) \cap L^2(-T,0; W^{1,2}_{loc,\sigma}(\mathbb{R}^3))$.
2. $u$ is a distributional solution to (1.1), i.e. for every $\varphi \in C^\infty_c(Q)$ with $\nabla \cdot u = 0$

$$\int_Q - u \cdot \frac{\partial \varphi}{\partial t} - u \otimes u : \nabla \varphi + \nabla u : \nabla \varphi dxdt = 0.$$  

3. For every ball $B \subset \mathbb{R}^3$ the following local energy inequality without pressure holds for every nonnegative $\phi \in C^\infty_c(B \times (0, +\infty))$, and for almost every $t \in (-T,0)$

$$\frac{1}{2} \int_B |v_B(t)|^2 \phi dx + \int_{-T}^t |\nabla v_B|^2 \phi dxds$$

$$\leq \frac{1}{2} \int_{-T}^t \int_B |v_B|^2 \left(\Delta + \frac{\partial}{\partial t}\right) \phi + |v_B|^2 u \cdot \nabla \phi dxds$$

$$+ \int_{-T}^t \int_B (u \otimes v_B) : \nabla^2 p_{h,B} \phi dxdt + \int_{-T}^t \int_G p_{1,B}v_B \cdot \nabla \phi dxds$$

$$+ \int_{-T}^t \int_G p_{2,B}v_B \cdot \nabla \phi dxds,$$

(2.3)

where $v_B = u + \nabla p_{h,B}$, and

$\nabla p_{h,B} = -E^*_B(u)$,

$\nabla p_{1,B} = -E^*_B((u \cdot \nabla)u)$,

$\nabla p_{2,B} = E^*_B(\Delta u)$. 

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Remark 2.3. 1. Note that due to $\nabla \cdot u = 0$ the pressure $p_{h,B}$ is harmonic, and thus smooth in $x$. Furthermore, as it has been proved in [11] the pressure gradient $\nabla p_{h,B}$ is continuous in $B \times (-T,0)$.

2. The notion of local suitable weak solutions to the Navier-Stokes equations satisfying the local energy inequality (2.3) has been introduced in [10]. As it has been shown there such solutions enjoy the same partial regularity as the standard suitable weak solution as proved in the paper by Caffarelli-Kohn-Nirenberg [1]. Furthermore, the following $\varepsilon$-regularity criterion has been proved for solution satisfying (2.3):

There exists and absolute number $\varepsilon > 0$ such that if for any $Q_r = B_r(x_0) \times (t_0, t_0 - r^2)$ it holds

$$r^{-2} \int_{Q_r} |u|^3 dx dt \leq \varepsilon^3 \implies u \in L^\infty(Q_r/2)$$

(cf. also [10]).

Before turning to the statement of this result we will fix the notations used throughout this section. For $z_0 = (x_0, t_0) \in Q$ and $0 < r < \sqrt{-t_0}$ we define the parabolic cylinders

$$Q_r = Q_r(z_0) = B_r(x_0) \times (t_0 - r^2, t_0), \quad I_r = I_r(t_0) = (t_0 - r^2, t_0).$$

By $V^2_\sigma(Q_r)$ we denote the space $L^\infty(I_r; L^2(B_r)) \cap L^2(I_r; W^{1,2}_\sigma(B_r))$. Furthermore for $u \in V^2_\sigma(Q_r(z_0))$ we set

$$A_q(r, z_0) = r^{q-3} \text{ess sup} \int_{B_r(x_0)} |u(x, t)|^q dx^{1/q}, \quad G(r, z_0) = r^{-1} \int_{Q_r(z_0)} |
abla u|^2 dx dt,$$

$$E_q(r, z_0) = r^{-\frac{3q^2}{3q - 3}} \left( \int_{I_r(t_0)} \left( \int_{B_r(x_0)} |u|^q dx \right)^{\frac{3}{3q - 3}} dt \right)^{\frac{3q - 3}{3q}}.$$

Remark 2.4. According to Lemma 4.1[10] the following Caccioppoli-type inequality holds true

$$G\left(\frac{r}{2}, z_0\right) \leq C \left( E_3(r, z_0)^2 + E_3(r, z_0)^3 \right),$$

where $C > 0$ denotes an absolute constant.

Our main result of this section is the following $\varepsilon$-regularity criterion

Theorem 2.5. Let $u \in V^2_\sigma(Q)$ be a local suitable weak solution to (1.1). Let $\frac{3}{2} < q \leq 3$. There exist two positive constants $\varepsilon_q$ and $C_q$, both depending on $q$ only, such that if for $Q_r(z_0) \subset Q$, $z_0 = (x_0, t_0)$, the condition

$$A_q(r, z_0) \leq \varepsilon_q.$$
implies \( u \in L^\infty(Q_{\frac{3}{2}}(z_0)) \), and it holds

\[
(2.6) \quad \text{ess sup}_{Q_{\frac{3}{2}}(z_0)} |u| \leq C_q r^{-1} A_q(r, z_0).
\]

Before turning to the proof of Theorem 2.5, we provide some lemmas, which will be used in our discussion below. We begin with a Caccioppoli-type inequality similar to (2.4).

**Lemma 2.6.** Let \( u \in V^2_\sigma(Q_R) \) be local suitable weak solution to \((1.1)\). Then for every \( \frac{3}{2} < q < 3 \)

\[
(2.7) \quad E_3 \left(\frac{3}{4} R \right)^2 + G \left(\frac{3}{4} R \right) \leq C \left( E_q(R)^2 + E_q(R)^{\frac{2q}{q-2}} \right) \leq C \left( A_q(R)^2 + A_q(R)^{\frac{2q}{q-2}} \right),
\]

where \( C > 0 \) denotes a constant depending only on \( q \).

**Proof:** Let \( 0 < r < \rho \leq R \) be fixed. Set \( B = B_\rho \), and define \( v_B = u + \nabla p_{h,B} \), where \( \nabla p_{h,B} = -E_B^r(u) \). Let \( \phi \) denote a suitable cut off function for \( Q_r \subset Q_\rho \). As it has been proved in [10] (cf. estimate (4.4) therein), applying Hölder’s inequality, the following inequality holds

\[
\| \phi v_B \|_{L^2(\mathcal{I}_\rho;L^2(B_\rho))} + \| \nabla \phi v_B \|_{L^2(Q_\rho)} \leq c \rho (\rho - \rho)^{-2} \| u \|_{L^2(\mathcal{I}_\rho;L^2(B_\rho))}^2 + (\rho - \rho)^{-1} \| u \|_{L^3(Q_\rho)}^3 + \frac{1}{4} \| \nabla u \|_{L^2(Q_\rho)}^2
\]

\[
(2.8) \quad \leq c \rho^2 (\rho - \rho)^{-2} \| u \|_{L^2(Q_\rho)}^2 + (\rho - \rho)^{-1} \| u \|_{L^3(Q_\rho)}^3 + \frac{1}{4} \| \nabla u \|_{L^2(Q_\rho)}^2.
\]

By means of Sobolev’s inequality together with Hölder’s inequality, (2.1) and (2.8)

\[
\| \phi v_B \|_{L^3(\mathcal{I}_\rho;L^6(B_\rho))}^2 \leq \| \phi v_B \|_{L^3(\mathcal{I}_\rho;L^2(B_\rho))}^2 \| \nabla \phi v_B \|_{L^2(Q_\rho)}^2 \leq \| \phi v_B \|_{L^3(\mathcal{I}_\rho;L^2(B_\rho))}^2 \| \nabla \phi v_B \|_{L^2(Q_\rho)}^2 \leq \| \phi v_B \|_{L^3(\mathcal{I}_\rho;L^2(B_\rho))}^2 \left( (\rho - \rho)^{-1} \| v_B \|_{L^2(Q_\rho)}^2 + \| \nabla v_B \|_{L^2(Q_\rho)}^2 \right)^{\frac{3}{2}} \leq c \| \phi v_B \|_{L^3(\mathcal{I}_\rho;L^2(B_\rho))}^2 \left( (\rho - \rho)^{-1} \| v_B \|_{L^2(Q_\rho)}^2 + \| \nabla v_B \|_{L^2(Q_\rho)}^2 \right)^{\frac{3}{2}} \leq \frac{1}{16} \| \phi v_B \|_{L^2(Q_\rho)}^2 + \frac{1}{16} \| \nabla v_B \|_{L^2(Q_\rho)}^2.
\]

\[
(2.9) \quad \leq c \rho^2 (\rho - \rho)^{-2} \| u \|_{L^3(Q_\rho)}^2 + (\rho - \rho)^{-1} \| u \|_{L^3(Q_\rho)}^3 + \frac{1}{16} \| \nabla u \|_{L^2(Q_\rho)}^2.
\]

We recall the following Caccioppoli inequality for a harmonic function

\[
\int_{B_\rho} \phi^2 |\nabla h|^2 dx \leq \max_{B_\rho} |\Delta \phi| \int_{B_\rho} |h|^2 dx,
\]

which will be repeatedly used below. The proof is immediate from the formula \(-\Delta h^2 + 2|\nabla h|^2 = 0\), by multiplying \( \phi \), integrating over \( B_\rho \), and then using integration by part.
Recalling that $p_{h,B}$ is harmonic, by using (2.1) with $s = 3$ we get first
\[
\| \phi \nabla p_{h,B} \|_{L^3(B_{\rho})}^3 = c \left( \int_{B_{\rho}} \phi^{\frac{18}{5}} |\nabla p_{h,B}|^{\frac{18}{5}} dx \right)^{\frac{5}{6}} \leq c \left( \int_{B_{\rho}} \phi^3 |\nabla p_{h,B}|^3 dx \right)^{\frac{1}{3}} \rho
\]
\[
\leq c(\rho - r)^{-3} \left( \int_{B_{\rho}} |\nabla p_{h,B}|^2 dx \right)^{\frac{3}{2}} \rho + c \left( \int_{B_{\rho}} \phi^3 |\nabla^2 p_{h,B}|^2 dx \right)^{\frac{1}{4}} \rho
\]
(2.10)
\[
\leq c(\rho - r)^{-3} \left( \int_{B_{\rho}} |\nabla p_{h,B}|^2 dx \right)^{\frac{3}{2}} \rho,
\]
from which, integrating it over $I_{\rho}$, we obtain
\[
\| \phi \nabla p_{h,B} \|_{L^3(I_{\rho};L^\infty(B_{\rho}))}^2 \leq \rho^2 (\rho - r)^{-2} c \|u\|_{L^3(Q_{\rho})}^2.
\]
Using this estimate, we have
\[
\| \phi u \|_{L^3(I_{\rho};L^\infty(B_{\rho}))}^2 \leq 2 \| \phi v_B \|_{L^3(I_{\rho};L^\infty(B_{\rho}))}^2 + 2 \| \phi \nabla p_{h,B} \|_{L^3(I_{\rho};L^\infty(B_{\rho}))}^2 \leq 2 \| \phi v_B \|_{L^3(I_{\rho};L^\infty(B_{\rho}))}^2 + c \rho^2 (\rho - r)^{-2} c \|u\|_{L^3(Q_{\rho})}^2.
\]
(2.11)
Combining (2.9) with (2.11), we arrive at
\[
\| u \|_{L^3(I_{\rho};L^\infty(B_{\rho}))}^2 \leq c \rho^2 (\rho - r)^{-2} \|u\|_{L^3(Q_{\rho})}^2 + (\rho - r)^{-1} \|u\|_{L^3(Q_{\rho})}^3 + \frac{1}{8} \|\nabla u\|_{L^2(Q_{\rho})}^2.
\]
(2.12)
Once more using the fact that $p_{h,B}$ is harmonic applying integration by parts, Caccioppoli type inequality together (2.11), we evaluate for almost all $t \in I_{\rho}$
\[
\| \phi(t) \nabla u(t) \|_{L^2(B_{\rho})}^2 = \int_{B_{\rho}} |\nabla v_B(t)|^2 \phi^2(t) dx + \int_{B_{\rho}} (\nabla v_B(t) + \nabla u(t)) : (\nabla v_B(t) - \nabla u(t)) \phi^2(t) dx
\]
\[
= \int_{B_{\rho}} |\nabla v_B(t)|^2 \phi^2(t) dx + \int_{B_{\rho}} (\nabla v_B(t) + \nabla u(t)) : \nabla^2 p_{h,B} \phi^2(t) dx
\]
\[
= \int_{B_{\rho}} |\nabla v_B(t)|^2 \phi^2(t) dx - \int_{B_{\rho}} (v_B(t) + u(t)) \otimes \nabla \phi^2(t) : \nabla^2 p_{h,B} dx
\]
\[
\leq \| \phi(t) \nabla v_B(t) \|_{L^2(B_{\rho})}^2 + c(\rho - r)^{-2} \|u(t)\|_{L^2(B_{\rho})}^2.
\]
Integration of both side of the above inequality together with Hölder’s inequality gives
\[
\| \phi \nabla u \|_{L^2(Q_{\rho})}^2 \leq \| \phi \nabla v_B \|_{L^2(Q_{\rho})}^2 + c \rho^2 (\rho - r)^{-2} \|u\|_{L^3(Q_{\rho})}^2.
\]
(2.13)
Combining (2.8) with (2.13), we are led to
\[
\| \nabla u \|_{L^2(Q_{\rho})}^2 \leq c \rho^2 (\rho - r)^{-2} \|u\|_{L^3(Q_{\rho})}^2 + (\rho - r)^{-1} \|u\|_{L^3(Q_{\rho})}^3 + \frac{1}{4} \|\nabla u\|_{L^2(Q_{\rho})}^2.
\]
(2.14)
Thus, adding (2.12) to (2.14), we obtain
\[ \|u\|_{L^3(I_\rho;L^{3\over 2}(B_\rho))}^2 + \|\nabla u\|_{L^2(Q_\rho)}^2 \leq c\rho^{3\over 2}(\rho - r)^{-2}\|u\|_{L^3(Q_\rho)}^2 + (\rho - r)^{-1}\|u\|_{L^3(Q_\rho)}^2 + \frac{3}{8}\|\nabla u\|_{L^2(Q_\rho)}^2. \]
(2.15)

Let \( t \in I_\rho \) be chosen so that \( u(t) \in W^{1,2}(B_\rho) \). Applying H"older's inequality together with Poincaré-Sobolev's inequality, we see that
\[ \|u(t)\|_{L^3(B_\rho)} \leq c\|u(t)\|^{3\over 2} _{W^{1,2}(B_\rho)} \|u(t)\|^{1\over 2} _{L^6(B_\rho)} \leq c\|u(t)\|^{3\over 2} _{L^6(B_\rho)} \|\nabla u(t)\|_{L^2(Q_\rho)} + c\rho^{3\over 2} \|u(t)\|_{L^3(B_\rho)}^3. \]
Integrating this inequality over \( I_\rho \), and applying H"older's inequality, we are led to
\[ \|u\|_{L^3(Q_\rho)}^2 \leq c\|u\|^{3\over 2} _{W^{1,2}(I_\rho;L^6(B_\rho))} \|\nabla u\|_{L^2(Q_\rho)}^2 + c\rho^{3\over 2} \|u\|_{L^3(I_\rho;L^6(B_\rho))}^3. \]
(2.16)

We now estimate the right-hand side of (2.15) by the aid of (2.16), and applying Young's inequality. This gives
\[ \|u\|_{L^3(I_\rho;L^{3\over 2}(B_\rho))}^2 + \|\nabla u\|_{L^2(Q_\rho)}^2 \leq c\rho^{3\over 2} (\rho - r)^{-2}\|u\|_{L^3(I_\rho;L^{3\over 2}(B_\rho))}^2 + c(\rho - r)^{-1}\|u\|_{L^3(I_\rho;L^{3\over 2}(B_\rho))}^2 + \frac{1}{2}\|\nabla u\|_{L^2(Q_\rho)}^2. \]
(2.17)

By using a standard iteration argument (e.g. see [4]) we deduce from (2.17) together with Young's inequality that
\[ \|u\|_{L^3(I_{2\rho};L^{3\over 2}(B_{2\rho}))}^2 + \|\nabla u\|_{L^2(Q_{2\rho})}^2 \leq cR^{3\over 2} \|u\|_{L^{3\over 2}(I_{2\rho};L^6(B_{2\rho}))}^2 + cR^{-3\over 2}\|u\|_{L^{3\over 2}(I_{2\rho};L^6(B_{2\rho}))}^3. \]
(2.18)

Multiplying both sides of (2.18) by \( R^{-1} \), and applying H"older's inequality, we obtain the desired inequality (2.7). \( \blacksquare \)

We continue our discussion with some useful iteration lemmas. Let \( G \subset \mathbb{R}^n \) be a bounded \( C^2 \)-domain. By \( A^s(G) \), \( 1 < s < +\infty \), we denote the image of \( W^{2,s}_0(G) \) under the Laplacian \( \Delta \), which is a closed subspace of \( L^s(G) \). By \( B^s(G) \) we denote the complementary space, which contains all \( p \in L^s(\Omega) \) being harmonic in \( G \) such that
\[ L^s(G) = A^s(G) + B^s(G). \]
(2.19)

By using the well-known Calderón-Zygmund inequality, and the elliptic regularity of the Bi-harmonic equation we get the following
Lemma 2.7. 1. Let $A \in L^s(G; \mathbb{R}^n^2)$. Then there exists a unique $p_0 \in A^s(G)$ such that
\begin{equation}
\Delta p_0 = \partial_i \partial_j A_{ij} \quad \text{in} \quad G
\end{equation}
in the sense of distributions \footnote{1). Here (2.20) means $-\int_G \Delta \phi = \int_G A_{ij} \partial_i \partial_j \phi$ for all $\phi \in C_0^\infty(G)$.}. In addition, it holds
\begin{equation}
\|p_0\|_s \lesssim \|A\|_s.
\end{equation}

2. Let $h \in L^s(G; \mathbb{R}^n)$, $1 \leq s < n$. Then there exists a unique $p_0 \in A^s(G) \cap W^{1,s}(G)$ such that
\begin{equation}
\Delta p_0 = \partial_i h_i \quad \text{in} \quad G
\end{equation}
in the sense of distributions, and the following estimate holds true
\begin{equation}
\|p_0\|_s + \|\nabla p_0\|_s \lesssim \|h\|_s.
\end{equation}
The hidden constants in both (2.21) and (2.22) depend only on $s, n$, and the geometric property of $G$. In case $G$ equals a ball, these constants are independent of the radius.

Lemma 2.8. Let $f \in L^{\frac{n}{n+1}}(Q_1)$. Let $0 < r_0 < 1$. Suppose, there exists $4 \leq \lambda \leq 5$ and $C > 0$, such that for all $z_0 = (x_0, t_0) \in Q_\frac{1}{2}$ and $r_0 \leq r \leq \frac{1}{2}$.
\begin{equation}
\int_{Q_r(z_0)} |f - \tilde{f}_{B_r(x_0)}|^\frac{3}{2} \leq K_0 r^\lambda,
\end{equation}
where $\tilde{f}_{B_r}(t) = \int_{B_r} f(x, t) dx$. Let $\nabla p = E^r_{B_{\frac{3}{4}}} (\nabla \cdot f)$. Then for all $z_0 \in Q_\frac{1}{2}$ and $r_0 \leq r \leq \frac{1}{4}$ it holds
\begin{equation}
\int_{Q_r(z_0)} |p - \tilde{p}_{B_r(x_0)}|^\frac{3}{2} \leq CK_0 r^4.
\end{equation}

Proof: Let $z_0 \in Q_\frac{1}{2}$ and $r_0 \leq r \leq \frac{1}{8}$ be arbitrarily chosen, but fixed. Let $0 < \theta < \frac{1}{2}$, specified below. According to (2.19) there exist unique $p_{0,r}(t) \in A^\frac{3}{4}(B_r(x_0))$ and $p_{h,r}(t) \in B^\frac{3}{4}(B_r(x_0))$ such that $p(t) - \tilde{p}_{B_r(x_0)}(t) = p_{0,r}(t) + p_{h,r}(t)$. Noting that $p(t) - \tilde{p}_{B_{\theta r}(x_0)}(t) = p(t) - \tilde{p}_{B_r(x_0)}(t) - (p(t) - \tilde{p}_{B_r(x_0)}(t))_{B_{\theta r}(x_0)}$, it follows that
\begin{align*}
\int_{B_r(x_0)} |p(t) - \tilde{p}_{B_{\theta r}(x_0)}(t)|^\frac{3}{2} & \\
& \lesssim \int_{B_{\theta r}(x_0)} |p_{h,r}(t) - (p_{h,r}(t))_{B_{\theta r}(x_0)}|^\frac{3}{2} + \int_{B_r(x_0)} |p_{0,r}(t) - (p_{0,r}(t))_{B_{\theta r}(x_0)}|^\frac{3}{2} \\
& \lesssim \theta^{\frac{9}{2}} \int_{B_r(x_0)} |p_{h,r}(t)|^\frac{3}{2} + \int_{B_r(x_0)} |f - \tilde{f}_{B_r(x_0)}(t)|^\frac{3}{2} \\
& \lesssim \theta^{\frac{9}{2}} \int_{B_{\theta r}(x_0)} |p(t) - \tilde{p}_{B_{\theta r}(x_0)}(t)|^\frac{3}{2} + \int_{B_r(x_0)} |f(t) - \tilde{f}_{B_r(x_0)}(t)|^\frac{3}{2}.
\end{align*}
Integrating the above estimate over $I_{\theta r}(t_0)$, and observing the assumption \eqref{eq:2.22}, we arrive at

\begin{equation}
\int_{Q_{\theta r}(z_0)} |p - \tilde{p}_{B_{\theta r}(x_0)}|^{\frac{3}{2}} \leq C_1 \theta^\frac{3}{2} \int_{Q_{r}(z_0)} |p - \tilde{p}_{B_{r}(x_0)}|^{\frac{3}{2}} + C_2 K_0 r^4.
\end{equation}

By a standard iteration argument from \eqref{eq:2.25} we deduce that

\begin{equation}
\int_{Q_r(z_0)} |p - \tilde{p}_{B_{r}(x_0)}|^{\frac{3}{2}} \leq r^4 \int_{Q^{\frac{3}{2}}(z_0)} |p|^{\frac{3}{2}} + K_0 r^4.
\end{equation}

Noting that by the definition of $p$ having for almost every $t \in I_{\frac{1}{4}}(t_0)$

$$\|p(t)\|_{L^3(B_{\theta r}(x_0))} \leq c \|f(t) - \tilde{f}_{B_{\theta r}}(t)\|_{L^3(B_{\theta r})},$$

the assertion \eqref{eq:2.24} follows from \eqref{eq:2.26} together with \eqref{eq:2.22}. □

We are in a position to prove the following iteration lemma, based on the idea of \cite{1}.

**Proposition 2.9.** Let $u \in V^2(Q_1(0,0))$ be a local suitable weak solution to the Navier-Stokes equations. We define $v = u + \nabla ph$, where $\nabla ph = -E^u_{\hat{B}_{\frac{3}{4}}}(u)$. There exist absolute positive numbers $K_q$ and $\varepsilon_q$ such that if

\begin{equation}
A_q(1,0) \leq \varepsilon_q
\end{equation}

then for all $n \in \mathbb{N}, n \geq 2$, and for all $z_0 \in Q_{\frac{1}{2}}(0,0)$ it holds

\begin{equation}
\int_{Q_{r_n}(z_0)} |v|^3 dx dt \leq K_q^3 A_q(1,0)^3,
\end{equation}

where $r_n = 2^{-n}, n \in \mathbb{N}$.

**Proof:** From the definition of a local suitable weak solution the following local energy inequality holds true for every non negative $\phi \in C_\infty^\infty \left(B_{\frac{3}{4}} \times \left(-\frac{9}{16},0\right)\right)$, and for almost
all $t \in \left(-\frac{9}{16}, 0\right]$  

\[
\frac{1}{2} \int_{-r^3}^{t} |v(t)|^2 \phi(t) dx + \int_{-r^3}^{t} |\nabla v|^2 \phi dx ds \\
\leq \frac{1}{2} \int_{-r^3}^{t} \int |v|^2 \left( \frac{\partial \phi}{\partial t} + \Delta \phi \right) dx ds + \frac{1}{2} \int_{-r^3}^{t} \int |v|^2 |v \cdot \nabla \phi - |v|^2 \nabla p_h \cdot \nabla \phi| dx ds \\
+ \int_{-r^3}^{t} \int (v \otimes v - v \otimes \nabla p_h : \nabla^2 p_h) \phi dx ds + \int_{-r^3}^{t} \int p_1 v \cdot \nabla \phi dx ds \\
+ \int_{-r^3}^{t} \int p_2 v \cdot \nabla \phi dx ds.
\]

(2.24)

where

\[
\nabla p_1 = E^*_B (\Delta u), \quad \nabla p_2 = -E^*_B (\nabla \cdot (u \otimes u)),
\]

Note that by the definition of $v$ it holds almost everywhere in $Q_{\frac{3}{4}}(0, 0)$

(2.25) \quad \quad u \otimes u = v \otimes v - v \otimes \nabla p_h - \nabla p_h \otimes v + \nabla p_h \otimes \nabla p_h.

Proof of (2.23)$_n$ by induction: For $n = 2$ the inequality (2.23)$_2$ follows immediately from Lemma 2.6.

Let $K_q > 1$ be a constant specified below. Assume (2.23)$_k$ is true for $k = 1, \ldots, n$, for some $n \in \mathbb{N}$. This implies for all $z_0 \in Q_{\frac{3}{4}}(0, 0)$ and $r_n \leq r \leq \frac{1}{2}$

(2.26) \quad \quad \int_{Q_r(z_0)} |v|^3 dx dt \leq CK_q^3 A_q(1, 0)^3.

Let $r_{n+1} \leq r \leq r_3$ and $z_0 \in Q_{\frac{3}{4}}(0, 0)$ be arbitrarily chosen, but fixed. Using Cauchy-Schwarz’s inequality, (2.26), and recalling that $p_h$ is harmonic, we get

\[
\int_{Q_r(z_0)} |v|^3 |\nabla p_h|^3 dx dt \leq CK_q^3 A_q(1, 0)^3 r^{3-1} \left[ \int_{I_1} \left( \int_{B_{\frac{3}{4}}(z_0)} |\nabla p_h(t)|^q dx \right)^{\frac{3}{q}} \right]^\frac{1}{3} \\
\leq CK_q^3 A_q(1, 0)^3 r^{3-1} \left[ \int_{I_1} \|u(t)\|^3_{L^3(B_1)} dt \right]^\frac{1}{3} \\
\leq Cr^{-1} K_q^3 A_q(1, 0)^3.
\]

(2.27)
Furthermore, applying Poincaré’s inequality, and employing Lemma 2.6 we find

\[ \int_{Q_r(z_0)} |\nabla p_h \otimes \nabla p_h - (\nabla p_h \otimes \nabla p_h)_{B_r}|^\frac{3}{2} dxdt \]

\[ \leq C r^{-\frac{5}{2} + \frac{3}{2}} \int_{Q_r(z_0)} |\nabla p_h|^3 |\nabla^2 p_h|^\frac{3}{2} \leq C r^{-\frac{1}{2}} \int_{Q_r(z_0)} |\nabla p_h|^3 \]

\[ \leq C r^{-\frac{1}{2}} A_q(1, 0)^3. \]

By the aid of (2.26), (2.27) and (2.28) together with (2.25) we obtain for all \( r_{n+1} \leq r \leq 1 \)

\[ \int_{Q_r(z_0)} |u \otimes u - (\hat{u} \otimes u)_{B_r(x_0)}|^\frac{3}{2} \leq CK_q^3 A_q(1, 0)^3 r^4. \]

Applying the Lemma 2.8 we find that for all \( r_{n+1} \leq r \leq r_2 \)

\[ \int_{Q_r(z_0)} |p_2 - (\hat{p}_2)_{B_r(x_0)}|^\frac{3}{2} \leq CK_q^3 A_q(1, 0)^3 r^4. \]

By \( \Psi_{n+1} \) we denote the fundamental solution of the backward heat equation having its singularity at \((x_0, t_0 + r_{n+1}^2)\), more precisely,

\[ \Psi_{n+1}(x, t) = \frac{c_0}{(r_{n+1}^2 + t + t_0)^\frac{n}{2}} \exp \left\{ -\frac{|x - x_0|^2}{(r_{n+1}^2 - t + t_0)} \right\}, \quad (x, t) \in \mathbb{R}^3 \times (-\infty, t_0 + r_{n+1}). \]

Taking a suitable cut off function \( \chi \in C^\infty(\mathbb{R}^n) \) for \( Q_{r_{n+1}}(z_0) \subset Q_{r_{n}}(z_0) \), we may insert \( \phi = \Phi_{n+1} = \Psi_{n+1} \chi \) into the local energy inequality (2.24) to get for almost all \( t \in (t_0 - r_{n+1}^2, t_0) \)

\[ \frac{1}{2} \int_{B_{r_{n+1}}(x_0)} \Phi_{n+1}(t)|v(t)|^2 + \int_{t_0-r_{n+1}^2}^t \int_{B_{r_{n+1}}(x_0)} \Phi_{n+1}|\nabla v|^2 \]

\[ \leq \frac{1}{2} \int_{t_0-r_{n+1}^2}^t \int_{B_{r_{n+1}}(x_0)} |v|^2 \left( \frac{\partial \Phi_{n+1}}{\partial t} + \Delta \Phi_{n+1} \right) + \frac{1}{2} \int_{t_0-r_{n+1}^2}^t \int_{B_{r_{n+1}}(x_0)} |v|^2 v \cdot \nabla \Phi_{n+1} \]

\[ - \frac{1}{2} \int_{t_0-r_{n+1}^2}^t \int_{B_{r_{n+1}}(x_0)} |v|^2 \nabla p_h \cdot \nabla \Phi_{n+1} + \int_{t_0-r_{n+1}^2}^t \int_{B_{r_{n+1}}(x_0)} (v \otimes v - v \otimes \nabla p_h : \nabla^2 p_h) \Phi_{n+1} \]

\[ + \int_{t_0-r_{n+1}^2}^t \int_{B_{r_{n+1}}(x_0)} p_1 v \cdot \nabla \Phi_{n+1} + \int_{t_0-r_{n+1}^2}^t \int_{B_{r_{n+1}}(x_0)} p_2 v \cdot \nabla \Phi_{n+1}. \]

(2.30)
Arguing as in [11], the above inequality yields

\[
\text{ess sup}_{t \in (t_0 - r_{n+1}^3, t_0)} \int_{B_{r_{n+1}}(x_0)} |v(t)|^2 + r_{n+1}^{-3} \int_{Q_{r_{n+1}}(x_0)} |\nabla v|^2 \leq \int_{Q_{r_3}(x_0)} |v|^2 \frac{\partial \Phi_{n+1}}{\partial t} + \Delta \Phi_{n+1} \right. \\
\left. + \int_{Q_{r_3}(x_0)} |v|^3 |\nabla \Phi_{n+1}| + \int_{Q_{r_3}(x_0)} |v| |\nabla p_h| |\nabla \Phi_{n+1}| \\
\left. + \int_{Q_{r_3}(x_0)} |v|^2 |\nabla^2 p_h| \Phi_{n+1} \right. \\
\left. + \int_{Q_{r_3}(x_0)} |v| |\nabla \Phi_{n+1}| + \int_{Q_{r_3}(x_0)} p_1 v \cdot \nabla \Phi_{n+1} \\
+ \int_{Q_{r_3}(x_0)} p_2 v \cdot \nabla \Phi_{n+1}
\right)
\]

(2.31)

\[= I + II + III + IV + V + VI + VII.\]

(i) Obviously, as \(|\frac{\partial \Phi_{n+1}}{\partial t} + \Delta \Phi_{n+1}| \leq C \text{ in } Q_{r_3}(x_0),\) and using (2.7), we see that

\[I \leq C \|v\|_{L^2(Q_{r_3}(x_0))}^2 \leq CA_q(1, 0)^2.\]

(ii) As \(|\nabla \Phi_{n+1}| \leq Cr_k^{-4} \text{ in } Q_{r_k}(x_0) \setminus Q_{r_{k+1}}(x_0)\) for all \(k = 1, \ldots, n,\) observing (2.23)_k, and employing (2.30), we get

\[
II = \sum_{k=3}^n \int_{Q_{r_k}(x_0) \setminus Q_{r_{k+1}}(x_0)} |v|^3 |\nabla \Phi_{n+1}| + \int_{Q_{r_{n+1}}(x_0)} |v|^3 |\nabla \Phi_{n+1}|
\leq CK_q^3 A_q(1, 0)^3 \sum_{k=2}^n r_k^{-4} r_k^5 \leq CK_q^3 A_q(1, 0)^3.
\]

(iii) Similarly as in (ii),

\[
III = \sum_{k=3}^n \int_{Q_{r_k}(x_0) \setminus Q_{r_{k+1}}(x_0)} |v|^2 |\nabla p_h| |\nabla \Phi_{n+1}| + \int_{Q_{r_{n+1}}(x_0)} |v|^2 |\nabla p_h| |\nabla \Phi_{n+1}|
\leq CK_q^2 A_q(1, 0)^3 \sum_{k=1}^n r_k^{-4} r_k^4 \leq CK_q^3 A_q(1, 0)^3.
\]

(iv) As \(\Phi_{n+1} \leq Cr_k^{-3} \text{ in } Q_{r_k}(x_0) \setminus Q_{r_{k+1}}(x_0)\) for all \(k = 1, \ldots, n+1\) together with (2.23)_k and (2.30) we get

\[
IV = \sum_{k=3}^n \int_{Q_{r_k}(x_0) \setminus Q_{r_{k+1}}(x_0)} |v|^2 |\nabla^2 p_h| \Phi_{n+1} + \int_{Q_{r_{n+1}}(x_0)} |v|^2 |\nabla^2 p_h| \Phi_{n+1}|
\leq CK_q^2 A_q(1, 0)^3 \sum_{k=2}^n r_k^{-3} r_k^4 \leq CK_q A_q(1, 0)^3.
\]
(v) Similarly as in (vi) we estimate
\[ V = \sum_{k=3}^{n} \int_{Q_{rk}(z_0) \setminus Q_{rk+1}(z_0)} |v||\nabla p_h||\nabla^2 p_h| \Phi_{n+1} + \int_{Q_{rn+1}(z_0)} |v||\nabla p_h||\nabla^2 p_h| \Phi_{n+1} \]
\[ \leq CK_q A_q(1,0)^3 \sum_{k=2}^{n} r_k^{-3} r_k^{1/4} \leq CK_q A_q(1,0)^3. \]

(vi) To estimate \( V I \) we argue as in [1]. Let \( \chi_k \) denote cut off functions, suitable for \( Q_{rk+1}(z_0) \subset Q_{rk}(z_0) \), \( k = 2, \ldots, n + 1 \). Then
\[ V I = \int_{Q_{r2}(z_0)} p_1 v \cdot \nabla \Phi_{n+1} \]
\[ = \sum_{k=3}^{n} \int_{Q_{rk}(z_0) \setminus Q_{rk+2}(z_0)} p_1 v \cdot \nabla (\Phi_{n+1} (\chi_k - \chi_{k+1})) \]
\[ + \int_{Q_{r2}(z_0)} p_1 v \cdot \nabla (\Phi_{n+1} (1 - \chi_2)) + \int_{Q_{r2}(z_0)} p_1 v \cdot \nabla (\Phi_{n+1} \chi_{n+1}) \]
\[ = \sum_{k=3}^{n} \int_{Q_{rk}(z_0) \setminus Q_{rk+2}(z_0)} (p_1 - (\tilde{p}_1)_{B_1(x_0)}) v \cdot \nabla (\Phi_{n+1} (\chi_k - \chi_{k+1})) \]
\[ + \int_{Q_{r2}(z_0)} p_1 v \cdot \nabla (\Phi_{n+1} (1 - \chi_2)) \]
\[ + \int_{Q_{rn+1}(z_0)} (p_1 - (\tilde{p}_1)_{B_{n+1}(x_0)}) v \cdot \nabla (\Phi_{n+1} \chi_{n+1}). \]

As \( |\nabla (\Phi_{n+1} (\chi_k - \chi_{k+1}))| \leq Cr_k^{-4} \) for \( k = 1, \ldots, n \), applying Poincaré's inequality, using the fact that \( p_1 \) is harmonic, together with (2.23) and (2.7) we see that
\[ \int_{Q_{rk}(z_0) \setminus Q_{rk+2}(z_0)} (p_1 - (\tilde{p}_1)_{B_1(x_0)}) v \cdot \nabla (\Phi_{n+1} (\chi_k - \chi_{k+1})) \]
\[ \leq CK_q A_q(1,0)^3 \left( \int_{Q_{1/2}} p_1^2 \right)^{1/2} \]
\[ \leq CK_q A_q(1,0)^2 + A_q(1,0)^{3/2} r_k \]
\[ \leq CK_q A_q(1,0)^2 r_k. \]

Summation from \( k = 3 \) to \( n \) yields
\[ \sum_{k=3}^{n} \int_{Q_{rk}(z_0) \setminus Q_{rk+2}(z_0)} (p_1 - (\tilde{p}_1)_{B_1(x_0)}) v \cdot \nabla (\Phi_{n+1} (\chi_k - \chi_{k+1})) \lesssim CK_q A_q(1,0)^2. \]
Similarly, we find
\[\int_{Q_{r_2}(z_0)} p_1 v \cdot \nabla (\Phi_{n+1}(1 - \chi_2)) - \int_{Q_{r_{n+1}}(z_0)} (p_1 - (\bar{p}_1)_{B_{r_{n+1}}(z_0)}) v \cdot \nabla (\Phi_{n+1} \chi_{n+1}) \leq (1 + r_{n+1}) CK_q A_q(1, 0)^2.\]
Thus,
\[VI \leq CK_q A_q(1, 0)^2.\]
(vii) Finally, arguing as in (vi), and making use of (2.29), we estimate
\[VII \leq CK_q^3 A_q^3(1, 0).\]
Thus, inserting the estimates of I, II, III, IV, V, VI and VII into the right-hand side of (2.31), we get a constant \(C_q > 0\) independently of \(n\), such that
\[\text{ess sup}_{t \in (t_0 - r_{n+1}^2, t_0)} \int_{B_{r_{n+1}}(x_0)} |v(t)|^2 + r_{n+1}^{-3} \int_{Q_{r_{n+1}}(z_0)} |\nabla v|^2 \leq C_q \left( K_q^3 A_q(1, 0)^3 + K_q A_q(1, 0)^2 \right) = \left( C_q K_q A_q(1, 0) + \frac{C_q}{K_q} \right) K_q^2 A_q(1, 0)^2.\]
Note that \(C_q \to +\infty\) as \(q \to \frac{3}{2}\).

On the other hand, using a standard interpolation argument along with (2.32), we arrive at
\[\int_{Q_{r_{n+1}}(z_0)} |v|^3 \leq C_0 \left[ \text{ess sup}_{t \in (t_0 - r_{n+1}^2, t_0)} \int_{B_{r_{n+1}}(x_0)} |v(t)|^2 + r_{n+1}^{-3} \int_{Q_{r_{n+1}}(z_0)} |\nabla v|^2 \right]^{\frac{3}{2}} \leq \left( C_0 C_q K_q A_q(1, 0) + \frac{C_0 C_q}{K_q} \right) \frac{3}{2} K_q^2 A_q(1, 0)^2\]
with an absolute constant \(C_0 > 1\). Note that neither \(C_q\) nor \(C_0\) depend on the choice of \(K_q\). Thus we may set
\[K_q = 2C_q C_0, \quad \varepsilon_q = \frac{1}{4C_q^2 C_0^2}.\]
Accordingly, if \(A_q(1, 0) \leq \varepsilon_q\), (2.33) implies
\[\int_{Q_{r_{n+1}}(z_0)} |v|^3 \leq K_q^3 A_q(1, 0)^3.\]
Whence, by induction the assertion of the proposition is true.

Proof of Theorem 2.5: Proposition 2.9 implies for every Lebesgue point \(z_0 = (x_0, t_0) \in Q\) of \(|v|^3\), after letting \(n \to +\infty\) in (2.23), that the estimate following holds true
\[|v(x_0, t_0)| \leq K_q A_q(1, 0).\]
By using the triangular inequality and the mean value property of harmonic functions, we get from (2.34) for almost all \((x, t) \in Q_{\frac{1}{2}}(0, 0)\)
\[
|u(x, t)| \leq K_q A_q(1, 0) + |\nabla p_k(x, t)| \leq K_q A_q(1, 0) + c\|u(t)\|_{L^q(B_1)}.
\]
\[
\leq (K_q + c)A_q(1, 0).
\]
This leads to
\[
(2.35) \quad \|u\|_{L^\infty(Q_{\frac{1}{2}})} \leq c(K_q + 1) \text{ess sup}_{t \in I_1} \|u(t)\|_{L^q(B_1)} = c(K_q + 1)A_q(1, 0).
\]
Finally the assertion (2.6) follows from (2.35) respectively by using a routine scaling argument.

3 Proof of Theorem 1.1

Let \(y_0 \in \mathbb{R}^3\) be fixed. Let \(0 < r \leq 1\) be arbitrarily chosen. By means of Hölder’s inequality, Sobolev’s inequality and (A.1) (cf. Lemma A.1) we get
\[
(3.1) \quad r^{-2} \int_{B_r(y_0)} |U|^2 dy \leq C \int_{B_1(y_0)} |U|^2 + |\nabla U|^2 dy \leq C \left( \|U\|_{L^q(B_1(y_0))}^2 + \|\Omega\|_{L^2(B_1(y_0))}^2 \right).
\]
This implies for any \(1 \leq s_0 \leq \sqrt{1 + 2a}\)
\[
(3.2) \quad r^{-2} \int_{B_r(\frac{s_0}{2})} |U|^2 dy \leq C \sup_{1 \leq s \leq \sqrt{1 + 2a}} \left( \|U\|_{L^q(B_1(y_0))}^2 + \|\Omega\|_{L^2(B_1(y_0))}^2 \right) =: C\Psi(y_0).
\]
Recalling the definition of \(u\), and using (3.2), we get for almost all \(t \in \left(-\frac{1}{2a} - 1, -\frac{1}{2a}\right)\)
\[
\int_{B_r(y_0)} |u(x, t)|^2 dx = \frac{1}{-2at} \int_{B_r(y_0)} \left| U \left( \frac{x}{\sqrt{-2at}} \right) \right|^2 dx
\]
\[
= \sqrt{-2at} \int_{B_{\frac{s_0}{\sqrt{-2at}}}(\frac{y_0}{\sqrt{-2at}})} |U|^2 dy \leq \sqrt{1 + 2a} \int_{B_r(\frac{s_0}{\sqrt{-2at}})} |U|^2 dy \leq C_0 r^2 \Psi(y_0),
\]
Accordingly, setting \(z_0 = \left(y_0, -\frac{1}{2a}\right)\), the above estimate becomes
\[
(3.3) \quad A_2(r, z_0)^2 \leq C_0 r^2 \Psi(y_0).
\]
We take
\[
r = \min \left\{ 1, \frac{\varepsilon^2}{C_0 \Psi(y_0)} \right\}.
\]
where $\varepsilon_2$ denotes the constant in Theorem 2.5 for the case $q = 2$. By the choice of $r$ we infer from (3.3)
(A.4)

Accordingly, Theorem 2.5 together with (3.3) yields

\[
U(y_0) \leq \sup_{Q_r(z_0)} |u| \leq C_2 r^{-1} A_2(r, z_0) \leq C_2 C_0^\frac{1}{q} r^{-\frac{1}{2}} \Psi(y_0)^{\frac{1}{2}} 
\]

(3.5)

By the assumption (1.6) having $\Psi(y_0) = o(|y_0|)$, it follows from (3.5) that $U$ has sublinear growth. Furthermore, appealing to Lemma A.2, we see that $U$ has sublinear growth at infinity, we get $U = \text{const}$. 

4 Proof of Theorem 1.2

Let $y_0 \in \mathbb{R}^3$ be fixed. Let $\alpha > 0$ be chosen so that (1.8) holds true. Recalling the definition of $u$, we get for any $0 < r \leq 1$ and for almost all $t \in \left(-\frac{1}{2a} - 1, -\frac{1}{2a}\right)$

\[
\int_{B_r(y_0)} |u(x, t)|^q \, dx = \frac{1}{(\sqrt{-2at})^q} \int_{B_r(y_0)} \left| U\left(\frac{x}{\sqrt{-2at}}\right)\right|^q \, dx = (\sqrt{-2at})^{3-q} \int_{B_{\frac{\sqrt{-2at}}{\sqrt{2a}}}(\frac{y_0}{\sqrt{2a}})} |U|^q \, dy
\]

\[
\leq \alpha^q \text{meas}(B_1)r^3 + (1 + 2a)^{\frac{3-q}{2}} \int_{B_r(\frac{y_0}{\sqrt{2a}}) \cap \{|U| > \alpha\}} |U|^q \, dy
\]

(4.1)

\[
= C_0 r^3 + \Phi(y_0),
\]

where we set $\Phi(y_0) = (1 + 2a)^{\frac{3-q}{2}} \sup_{1 \leq s \leq \sqrt{1+2a}} \|U\|^q_{L^q(B_1(\frac{y_0}{\sqrt{2a}}))}$ and $C_0 = \alpha^q \text{meas}(B_1)$. Setting $z_0 = \left(y_0, -\frac{1}{\sqrt{2a}}\right)$ from the inequality above, we deduce that

\[
A_q(r, z_0)^q = \sup_{t \in (-\frac{1}{2a}, -r^2, -\frac{1}{2a})} \|u(t)\|^q_{L^q(B_1(y_0))} \leq C_0 r^q + r^{q-3} \Phi(y_0).
\]

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Without loss of generality we may assume that $C_0 > 1$. We now take $r$ such that

\begin{equation}
(4.3) \quad C_0 r = \frac{\varepsilon_q}{2},
\end{equation}

where $\varepsilon_q$ denotes the positive number in Theorem 2.5. On the other hand, our assumption (1.7) yields $\Phi(y_0) \to 0$ as $|y_0| \to +\infty$. Therefore, we may chose $R > 0$ such that for all $y_0 \in \mathbb{R}^3 \setminus B_R$

$$r^{-3} \Phi(y_0) \leq \frac{\varepsilon_q}{2},$$

Accordingly, Theorem 2.5 implies for all $y_0 \in \mathbb{R}^3 \setminus B_R$

\begin{equation}
(4.4) \quad |U(y_0)| \leq \sup_{Q_{r^{-3}}(y_0)} |u| \leq C q r^{-1} A_q(r, z_0) \leq C q r^{-1} \varepsilon_q.
\end{equation}

Therefore $U$ is bounded. According to Tsai’s result (cf. Lemma 5.1[8]), we conclude that $U = \text{const}$. This completes the proof of the theorem.

5 Proof of Theorem 1.5

As in [2] we consider the self-similar transform of the solution $(u, p)$ of (1.1) into $(U, \Pi)$ by

$$u(x, t) = \frac{1}{\sqrt{2a(t_* - t)}} V(y, s), \quad p(x, t) = \frac{1}{2a(t_* - t)} \Pi(y, s),$$

where

$$y = \frac{x - x_*}{\sqrt{2a(t_* - t)}}, \quad s = \frac{1}{2} \log \left( \frac{t_*}{t_* - t} \right).$$

Then, the system (1.1) is transformed into a system for $(V, \Pi) \in C^2(\mathbb{R}^3 \times (0, \infty))$

\begin{equation}
(4.5) \quad V_s - \Delta V + (V \cdot \nabla) V + a y \cdot \nabla V + aV = -\nabla \Pi, \quad \nabla \cdot V = 0 \quad \text{in} \quad \mathbb{R}^3.
\end{equation}

The condition (1.9) is transformed into

\begin{equation}
(4.6) \quad \lim_{s \to \infty} ||V(\cdot, s) - U||_{L^q\left( B_{r^{-3}}(0) \right)} = 0 \quad \forall r > 0.
\end{equation}

From the argument of Proof of [2, Theorem 1.2]) one can show from (4.6) that $U$ is a solution of (1.4) for a scalar function $P$. We include this part here for reader’s convenience. We choose $\xi \in C^1_c(0, 1)$ with $\int_0^1 \xi(s) ds = 1$, and $\varphi \in C^1_c(\mathbb{R}^3)$ with $\nabla \cdot \varphi = 0$. Then, multiplying (4.5) by $\xi(s - n)\varphi(y)$ and integrating it over $\mathbb{R}^3 \times [n, n + 1]$, then after integration by part we obtain

\begin{align}
&\int_0^1 \int_{\mathbb{R}^3} \xi'(s) \varphi(y) \cdot V(y, s + n) dy ds + \frac{a}{2} \int_0^1 \int_{\mathbb{R}^3} \xi(s) V(y, s + n) \cdot \varphi(y) dy ds \\
&= - \int_0^1 \int_{\mathbb{R}^3} \xi(s) (V \otimes V)(y, s + n) : \nabla \varphi(y) dy ds \\
&\quad - \int_0^1 \int_{\mathbb{R}^3} \xi(s) \{a V(y, s + n) \cdot (y \cdot \nabla) \varphi - V(y, s + n) \cdot \Delta \varphi\} dy ds.
\end{align}

(4.7)
Since (4.6) implies that $V(\cdot, s + n) \to U$ in $L^2_{loc}(\mathbb{R}^3)$ as $n \to \infty$, passing $n \to \infty$ in (4.7), using the fact $\int_0^1 \xi(s) ds = 1$, one has

$$
\int_{\mathbb{R}^3} \varphi(y) \cdot U\,dy = -\int_{\mathbb{R}^3} (U \otimes U)(y) : \nabla \varphi(y) \,dy - \int_{\mathbb{R}^3} \{aU(y) \cdot (y \cdot \nabla) \varphi - U(y) \cdot \Delta \varphi\} \,dy,
$$

which shows that $U \in L^2_{loc}(\mathbb{R}^3)$ is a weak solution of (1.4) for some scalar function $P = P(y)$. By a standard regularity theory $(U, P)$ is a smooth solution of (1.4). Now, applying Corollary 1.4 one can conclude that $U = 0$, and the condition (1.9) reduces to

$$
(4.9) \lim_{t \to t_*} (t_* - t)^{\frac{a-3}{2q}} \sup_{t < \tau < t_*} \|u(\cdot, \tau)\|_{L^q(B(\sqrt{t_* - t}(x_*))} = 0
$$

for each $r > 0$. Setting $r = 1$, $\rho = \sqrt{t_* - t}$, we find that

$$
\lim_{\rho \to 0} \left\{ \rho^{\frac{a-3}{2q}} \sup_{t_* - \rho^2 < \tau < t_*} \|u(\cdot, \tau)\|_{L^q(B(x_*, \rho))} \right\} = \lim_{\rho \to 0} A_q(z_*, \rho) = 0,
$$

where $z_* = (x_*, t_*)$. Thanks to Theorem 2.5 we find that $z_*$ is a regular point (cf. also the regularity criterion due to Gustafson, Kang and Tsai [5, Theorem 1.1]).

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A Gradient estimates and pressure estimate

Lemma A.1. Let $U \in L^q_{loc}(\mathbb{R}^3), 0 < q < +\infty$, with $\nabla \cdot U = 0$. Let $\Omega = \nabla \times U \in L^2_{loc}(\mathbb{R}^3)$. Then $\nabla u \in L^2_{loc}(\mathbb{R}^3)$, and there holds for all $y_0 \in \mathbb{R}^3$

$$
(A.1) \quad \|U\|_{L^2(B_1(y_0))} + \|\nabla U\|_{L^2(B_1(y_0))} \lesssim \|U\|_{L^q(B_2(y_0))} + \|\Omega\|_{L^2(B_2(y_0))},
$$

where the hidden constant depends only on $q$.

Proof: By means of a standard mollifying argument it suffice to verify the estimate (A.1) for smooth $U$.

Let $\alpha = \frac{6 - q}{2q}$. For $y_0 \in \mathbb{R}^3$ let $\zeta \in C^\infty_c(B_2(y_0))$ denote a suitable cut off function for
$B_1(y_0) \subset B_2(y_0)$. Using integration by parts, we find

$$\int_{B_2(y_0)} |\nabla U|^2 \zeta^{2\alpha} dx = \int_{B_2(y_0)} \nabla U : \nabla U \zeta^{2\alpha} dx$$

$$= - \int_{B_2(y_0)} U \cdot \Delta U \zeta^{2\alpha} dx - \alpha \int_{B_2(y_0)} |\nabla U|^2 \cdot \zeta^{2\alpha-1} \nabla \zeta dx$$

$$= \int_{B_2(y_0)} U \cdot \nabla \times \Omega \zeta^{2\alpha} dx + \alpha \int_{B_2(y_0)} |U|^2 \zeta^{2\alpha-1} \Delta \zeta dx$$

$$+ \alpha(2\alpha - 1) \int_{B} |U|^2 \zeta^{2\alpha-2} |\nabla \zeta|^2 dx.$$

Applying Young’s inequality, we get

$$(A.2) \quad \int_{B_2(y_0)} |\nabla U|^2 \zeta^{2\alpha} dx \lesssim \int_{B_2(y_0)} |\Omega|^2 \zeta^{2\alpha} dx + \int_{B_2(y_0)} |U|^2 \zeta^{2\alpha-2} dx.$$

Applying Hölder’s inequality along with Sobolev’s inequality we estimate

$$\|U \zeta^{\alpha-1}\|_{L^2(B_2(y_0))}$$

$$\leq \|U\|_{L^q(B_2(y_0))}^{\frac{6-q}{6-q}} \|U \zeta^{(\alpha-1)\frac{6-q}{6-q}}\|_{L^q(B_2(y_0))}^{\frac{12-6q}{6-q}} = \|U\|_{L^q(B_2(y_0))}^{\frac{6-q}{6-q}} \|U \zeta^{\alpha}\|_{L^q(B_2(y_0))}^{\frac{12-6q}{6-q}}$$

$$\lesssim \|U\|_{L^q(B_2(y_0))} \|
abla U \zeta^{\alpha}\|_{L^2(B_2(y_0))} + \|U\|_{L^q(B_2(y_0))} \|U \zeta^{\alpha-1}\|_{L^2(B_2(y_0))} dx.$$

Applying Young’s inequality, we are led to

$$(A.3) \quad \|U \zeta^{\alpha-1}\|_{L^2(B_2(y_0))}^2 \lesssim \|U\|_{L^q(B_2(y_0))} \|
abla U \zeta^{\alpha}\|_{L^2(B_2(y_0))} + \|U\|_{L^q(B_2(y_0))}^2.$$

Combining (A.2) and (A.3), and using once more Young’s inequality, we obtain (A.1).

**Lemma A.2.** Let $(U, P)$ be a smooth solution to (1.4). Assume that $|U(y)| = O(|y|)$, and $\|\nabla U\|_{L^2(B_1(y))} = O(|y|^{\frac{3}{2}})$ as $|y| \to \infty$. Then, we have

$$(A.4) \quad \|\nabla^k U\|_{L^2(B_1(y))} = O(|y|^{\frac{2k-1}{2}}),$$

$$(A.5) \quad |\nabla^k U(y)| = O(|y|^{\frac{2k+3}{2}}),$$

$$(A.6) \quad |P(y)| = O(|y|^{\frac{1}{2}})$$

as $|y| \to \infty$. 

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Proof: Applying $\nabla \times$ to both sides of (1.4), we obtain

$$\Delta \Omega + \nabla \times ((U \cdot \nabla)U) + ay \cdot \nabla \Omega + 2\Omega = 0. \tag{A.7}$$

Let $y_0 \in \mathbb{R}^3$, and let $\zeta \in C_c^\infty(B_2(y_0))$ denote a suitable cut off function for $B_1(y_0) \subset B_2(y_0)$. Multiplying (A.7) by $\Omega \zeta^2$, and applying integration by parts, using the formula $U \cdot \nabla U = \frac{1}{2} \nabla|U|^2 + \Omega \times U$, we get

$$\int_{B_2(y_0)} |\nabla \Omega|^2 \zeta^2 + \frac{a}{2} |\Omega|^2 \zeta^2 dx = \frac{1}{2} \int_{B_2(y_0)} |\Omega|^2 (\Delta \zeta^2 + ay \cdot \nabla \zeta^2) dx - \int_{B_2(y_0)} \Omega \times U \cdot \nabla \times (\Omega \zeta^2) dx. \tag{A.8}$$

Using Cauchy-Schwarz’s inequality and Young’s inequality, we find

$$\int_{B_2(y_0)} |\nabla \Omega|^2 \zeta^2 + 2a |\Omega|^2 \zeta^2 dx \leq C(1 + |y_0| + |y_0|^3). \tag{A.9}$$

With the aid of Lemma[A.1] together with Sobolev’s inequality we infer from (A.9)

$$\|\nabla^2 U\|_{L^2(B_1(y_0))} + \|\nabla U\|_{L^6(B_1(y_0))} \leq C(1 + |y_0|^\frac{3}{2}). \tag{A.10}$$

Differentiating (A.4), arguing as above together, and applying an inductive argument, we see that

$$\|\nabla^k U\|_{L^2(B_1(y_0))} + \|\nabla^{k-1} U\|_{L^6(B_1(y_0))} \leq C(1 + |y_0|^\frac{2k-1}{2}). \tag{A.11}$$

Whence, (A.4). Employing Sobolev’s embedding theorem, we get $H^{k+2}(B_1(y_0)) \hookrightarrow L^\infty(B_1(y_0))$. Thus (A.5) is an immediate consequence of (A.4) by using Sobolev’s inequality.

Observing (1.4), and making use of (A.5), we obtain

$$|\nabla P(y)| = O(|y|^\frac{3}{2}). \tag{A.12}$$

This immediately implies (A.6).

References

[1] L. Caffarelli, R. Kohn and L. Nirenberg, Partial Regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math., 35, (1982), pp. 771-831.

[2] D. Chae, Nonexistence of asymptotically self-similar singularities in the Euler and the Navier-Stokes equations, Math. Ann. 238, (2007), pp. 435-449.

[3] G. Galdi, C. Simader and H. Sohr, On the Stokes problem in Lipshitz domain, Annali di Mat. Pura Appl., (IV), 167 (1994), pp. 147-163.
[4] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Ann. Math. Stud. No. 105, Princeton Univ. Press, Princeton (1983).

[5] S. Gustafson, K. Kang and T.-P. Tsai, *Regularity criteria for suitable weak solutions of the Navier-Stokes equations near the boundary*, J. Diff. Eqns, 226, no. 2 (2006), pp. 594-618.

[6] J. Leray, *Sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Math., 63 (1934), pp. 193-248.

[7] J. Nečas, M. Růžička and V. Šverak, *On Leray’s self-similar solutions of the Navier-Stokes equations*, Acta Math., 176 (1996), pp. 283-294.

[8] T.-P. Tsai, *On Leray’s self-similar solutions of the Navier-stokes equations satisfying local energy estimates*, Arch. Rational Mech. Anal., 143 (1998), pp. 29-51.

[9] J. Wolf, *A new criterion for partial regularity of suitable weak solutions to the Navier-Stokes equations*, Advances in Mathematical Fluid Mechanics, A. S. R. Rannacher, ed. Springer, 2010, pp. 613-630.

[10] J. Wolf, *On the local regularity of suitable weak solutions to the generalized Navier-Stokes equations*, Ann. Univ. Ferrara, 61 (2015), pp. 149-171.

[11] J. Wolf, *On the local pressure of the Navier-Stokes equations and related systems*, to appear in Adv. Diff. Eqns., (2016).