Avalanche size distribution in a random walk model

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Abstract. We introduce a simple model for the size distribution of avalanches based on the idea that the front of an avalanche can be described by a directed random walk. The model captures some of the qualitative features of earthquakes, avalanches and other self-organized critical phenomena in one dimension. We find scaling laws relating the frequency, size and width of avalanches and an exponent $4/3$ in the size distribution law.

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1 Introduction

Driven dissipative systems arise in modelling many different phenomena in Nature, e.g. avalanches, forest fires, earthquakes and Darwinian evolution. Such systems frequently exhibit long range temporal and spatial correlations and scaling laws analogous to statistical mechanical systems at a critical point. The concept of self-organized criticality was introduced \[1\] to describe systems where critical behaviour arises without the fine tuning of any parameter.

The study of self-organized criticality is essentially a study of a family of dynamical systems which evolve in time in small or large steps which have a power law distribution and we shall call “avalanches” but they may have a different interpretation. The canonical example of a self-organized critical system is the abelian sandpile model \[1, 2\] where an integer variable \(z_i\) is assigned to each site \(i\) in a lattice and the dynamics is given by choosing a random site \(i\) in the lattice, increasing \(z_i\) by 1, \(z_i \rightarrow z_i + 1\), and if \(z_i\) exceeds a threshold value \(z_c\), then an avalanche start and the value of \(z_i\) is decreased by redistributing some of \(z_i\) among the neighbours. This in turn may push some of the \(z_j\) over threshold, where \(j\) is a neighbour of \(i\), and so on. These rules are then supplemented by appropriate boundary conditions. See \[3\] for a recent detailed discussion.

It is natural to introduce a mean field theory for self-organized criticality \[4\]. This has been done in many different ways for different models yielding identical critical exponents \[5, 6, 7, 8\]. One of the approaches is to study sandpiles on a Bethe lattice \[5\]. In this case different parts of an avalanche front are uncorrelated and one obtains the scaling law

\[N_w(k) \sim k^{-3/2}\]  \hspace{1cm} (1)

where \(N_w(k)\) is the frequency of avalanches involving \(k\) sites. We shall refer to \(k\) as the width of the avalanche.

In this paper we introduce a simple model for the propagation of a one-dimensional avalanche front. This model was suggested by studying earthquakes in the Burridge–Knopoff model \[9, 10, 11, 12, 13\]. The basic idea is to assume that neighbouring parts of an avalanche are correlated in the following way. If we label elements of the avalanche by integers \(i\) and an element \(i\) moves a distance \(h_i\) then the displacement of its neighbour, labelled by \(i + 1\), is distributed with a probability distribution \(P_i(h_i+1) = \phi(h_i+1-h_i)\) which is centered on \(h_i\) but otherwise independent of \(i\). This distribution is modified by an appropriate boundary condition at \(h_{i+1} = 0\) amounting to a certain killing probability for the avalanche. In this paper we shall in fact work with the simplest possible probability distribution \(\phi\) which corresponds to the avalanche front performing a Bernoulli random walk on the positive integers and

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terminating once it returns to 0. In this case one can perform explicit calculations and we find an exponent $3/2$ in the scaling law relating the frequency of avalanches to their width.

If we define the size of an avalanche as

$$A = \sum_i h_i,$$

then we find that the frequency of avalanches of size $A$ is given by

$$N(A) \sim A^{-4/3}.$$  

Sometimes, e.g. in earthquake models, it is natural to place an upper bound on the maximal allowed displacement $h_i$ and in that case the power law turns into an exponential decay for large $A$. The relation to earthquakes is discussed in more detail in [15].

2 The random walk model

In this section we define the random walk model. We introduce a class of random walks (or paths) and each path will correspond to an avalanche of a particular form. Questions about the size, shape and frequency of avalanches can therefore be translated into questions about the number of paths satisfying the appropriate conditions. In this model we are not able to address questions about the correlations between different avalanches nor discuss the self-organized critical spatial structure that arises in realistic models.

We shall refer to the discrete elements participating in an avalanche as blocks. Consider a semi-infinite chain of blocks $M_i$, lying along the $x$-axis, labelled by the non-negative integers which can be regarded as the $x$ coordinates of the blocks. We assume that the blocks move in integer steps in the direction of the $y$ axis with the $x$ coordinate unchanged. In applications to many one-dimensional systems, e.g. earthquakes and sandpiles, the $x$ coordinate would be the $x$ coordinate of the blocks at time 0 while the $y$ coordinate, in our present terminology, would be the addition to the $x$ coordinate in an avalanche.

We imagine that the block labelled by $i = 1$ is the first one to move in an avalanche and the next block to move is $M_2$ etc. For convenience we assume that the block at the origin $M_0$ is fixed at all times, but this assumption is not at all essential. We could easily have another avalanche moving to the left and this second avalanche would be uncorrelated to the one moving to the right. We assume that $M_1$ moves a distance 1 along the $y$-axis. This is the initial condition for an avalanche.
We now assume that \( M_2 \) moves a distance 2 along the \( y \)-axis with probability \( \frac{1}{2} \) and a distance 0 with probability \( \frac{1}{2} \). In general, given that the block \( M_n \) has moved a distance \( h_n \), block \( M_{n+1} \) moves a distance \( h_n \pm 1 \) with equal probabilities. By definition, the avalanche stops once the next block to move in fact does not move. The minimal avalanche is the one where \( M_2 \) does not move and this is the most likely event with probability \( \frac{1}{2} \). It is sometimes natural to place an upper bound \( h \) on how far the blocks can move so we modify the rules by requiring \( M_{n+1} \) to move a distance \( h - 1 \), if \( M_n \) has moved a distance \( h \). Of course it would be natural to allow neighbouring blocks to move the same distance. This would not affect any of the conclusions but make some of our equations more complicated. In this formulation an avalanche can be regarded as a directed random walk on the non-negative integers which is reflected from \( h \) and stops when it returns to 0.

We can write
\[
  h_n = \sum_{i=1}^{n} \sigma_i, \tag{4}
\]
where \( \sigma_1 = 1 \) and \( \sigma_i \) for \( i \geq 2 \) is a random variable which takes the values \( \pm 1 \) with equal probabilities. The duration \( T \) of the walk is defined by
\[
  h_T = 0, \quad h_n > 0, \quad 0 < n < T. \tag{5}
\]
The size (moment in seismology) \( A \) of the avalanche is given by the area under the graph of the function \( h_n \), i.e.
\[
  A = \sum_{n=1}^{T} h_n. \tag{6}
\]
We are interested in calculating the frequency of avalanches with size \( A \) and the relation between \( A \) and the width \( T \).

### 3 Short time behaviour

We begin by considering the model in the absence of an upper cutoff \( h \), i.e. we put \( h = \infty \) and return to the case of finite \( h \) later. A directed random walk (or path) is one where the \( x \) coordinate increases by 1 in each step. Let \( \mathcal{W} \) denote the set of all directed walks in the positive quadrant of the \( xy \)-plane which start from \((0,0)\) and return to the \( x \)-axis. Such a walk is located at \((1,1)\) after one step. Let \( N(A,T) \) denote the number of paths in \( \mathcal{W} \) which return to the \( x \)-axis after \( T \) steps and whose graph, together with the \( x \) axis, encloses an area \( A \) given by Eq. (6), see Fig. 1. Let
\[
  N(T) = \sum_{A} N(A,T). \tag{7}
\]
The probability that the walk returns to the origin after \(T\) steps is given by

\[ P(T) = 2^{-T+1}N(T). \quad (8) \]

We divide by \(2^{T-1}\) since the first step in the walk is given. Similarly, the conditional probability \(P(A|T)\) that a walk covers an area \(A\), given that it lasts a time \(T\), can be written as

\[ P(A|T) = \frac{N(A,T)}{N(T)}. \quad (9) \]

It follows that the probability that an avalanche has size \(A\) is given by

\[ P(A) = 2^{-T+1} \sum_{T} N(A,T). \quad (10) \]

It is well known from the solution of the classical gambler’s ruin problem, see e.g. [16], that

\[ P(T) = 2^{-T+1} \left( \frac{T-1}{T/2} \right) \quad (11) \]

if \(T\) is even and 0 otherwise. For large \(T\)

\[ P(T) \sim T^{\frac{3}{2}}. \quad (12) \]

Since we are discussing random walks it is natural to expect the average height \(\langle h_n \rangle\) of a walk which lasts a time \(T\) to scale like \(\sqrt{T}\) for large \(T\). The average area \(A\) should therefore grow as \(T^{3/2}\) for such walks. A priori one would expect entropic repulsion to play a role here so \(3/2\) should be a lower bound on the “average area exponent” but, as demonstrated below, there is in fact no shift away from the naive value of this exponent.

If we consider directed random walks that are allowed to cross the \(x\) axis and define the area under the walks to be positive if the walk is in the upper half plane and negative when it is in the lower half plane, we can use (4) and (6) to express the area as a sum of independent but non-identical random variables. The generalized central limit theorem [16] applies to this sum and we find that asymptotically the area is normally distributed around zero with a variance \(T^3\). Assuming that \(P(A|T)\) is normally distributed around its average value with a variance \(T^3\) we expect that for large \(A\) we have

\[ P(A) = \sum_{T} P(A|T)P(T) \sim \int_{0}^{\infty} T^{-3} \exp \left( \frac{(A-T^{3/2})^2}{T^3} \right) dT \sim A^{-4/3}. \quad (13) \]
Below we shall verify that the asymptotic behaviour of $P(A)$ is indeed given by Eq. (13) even though one can prove that $P(A|T)$ is in fact not normally distributed by computing its first few moments.

Let $\tilde{W}$ denote the class of directed walks in $W$ which avoid the line $y = 1$ until they return to $y = 0$, i.e. if $w \in \tilde{W}$ and $w$ returns at time $T$ then $w(x) > 1$ for $1 < x < T - 1$, where $w(t)$ denotes the $y$-coordinate of the path for $x = t$. Let us denote by $\tilde{N}(A, T)$ the number of paths in $\tilde{W}$ which return to 0 at time $T$ and cover an area $A$. Then

$$\tilde{N}(A, T) = N(A - T + 1, T - 2),$$

see Fig. 2. Now consider any directed walk $w \in W$ which lasts a time $T > 2$ and covers an area $A$. Let $T_1$ denote the smallest integer $> 1$ such that $w(T_1) = 1$. The largest possible value of $T_1$ is of course $T_1 = T - 1$. If we cut the path $w$ in two pieces at the point $(T_1, 1)$ then we can associate uniquely to $w$ two paths, $\tilde{w} \in \tilde{W}$ and $w_1 \in W$, of duration $T_1 + 1$ and $T - T_1 + 1$, respectively, see Fig. 3. In the extreme case $T_1 = T - 1$ the second walk is the trivial one of length 2. If we denote the area under the first walk by $A_1$ then the area under the second one equals $A - A_1 + 1$ and we find that

$$N(A, T) = \delta_{A1} \delta_{T2} + \sum_{T_1=1}^{T-1} \sum_{A_1=1}^{A} \tilde{N}(A_1, T_1 + 1) N(A - A_1 + 1, T - T_1 + 1)$$

$$= \delta_{A1} \delta_{T2} + \sum_{T_1=1}^{T-1} \sum_{A_1=1}^{A} N(A_1 - T_1, T_1 - 1) N(A - A_1 + 1, T - T_1 + 1),$$

by Eq. (14). The first term on the right side of Eq. (13) corresponds to the two step path. We define the generating function $f(z, u)$ for the numbers $N(A, T)$ by

$$f(z, u) = \sum_{T=2}^{\infty} \sum_{A=1}^{\infty} N(A, T) z^A u^T,$$

which is convergent for $|z| < 1$ and $|u| \leq \frac{1}{2}$. Eq. (15) can now be rewritten as

$$f(z, u) = zu^2 + f(z, u) f(z, uz).$$

For $z = 1$ we can easily solve this equation and find

$$f(1, u) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4u^2},$$

which for $u = \frac{1}{2}$ takes the value $\frac{1}{2}$ in accordance with Eq. (10).

For general values of $z$ and $u$ Eq. (17) is not explicitly soluble. We note however that the equation can be rearranged to read

$$f(z, u) = \frac{zu^2}{1 - f(z, uz)},$$

(19)
which, upon iteration, yields a continued fraction expansion\footnote{It is perhaps of interest to note that the continued fraction (20) appeared in a letter from Ramanujan to Hardy written in 1913 \cite{17} where it was used to express some remarkable identities, one of which can be written, in our notation,\[ f(e^{-\pi}, ie^{\pi}/2) = -\left(\sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2}\right)e^{2\pi/5}.\] However, the Ramanujan identities seem unrelated to the statistical properties of the random walk model we are interested in.} for \( f(z, u) \)

\[
f(z, u) = \frac{zu}{1 - \frac{z^2u}{1 - \frac{z^4u}{1 - \cdots}}}. \tag{20}
\]

Looking at Eq. (13) we expect the average size of an avalanche to diverge, i.e. we expect

\[
\lim_{u \uparrow \frac{1}{2}} \left. \frac{\partial}{\partial z} f(z, u) \right|_{z=1} = \infty. \tag{21}
\]

Indeed, we shall prove that

\[
\left. \frac{\partial^n}{\partial z^n} f(z, u) \right|_{z=1} \sim (1 - 4u^2)^{-\frac{n}{4}} \tag{22}
\]

as \( u \uparrow \frac{1}{2} \), for any \( n \geq 1 \). Let us define

\[
P_u(A) = \sum_T u^{T-1} N(A, T). \tag{23}
\]

Then

\[
\sum_A A^n P_u(A) = \left( z \left. \frac{\partial}{\partial z} \right| f(z, u) \right)_{z=1} ^n. \tag{24}
\]

For any path of duration \( T \), covering an area \( A \), it is easy to see that

\[
\frac{3}{2} T - 2 \leq A \leq \frac{1}{4} T^2. \tag{25}
\]

The lower bound is obtained by considering the path which zigzags between 1 and 2 and covers the smallest possible area while the upper bound corresponds to the triangular path that climbs to height \( T/2 \) in time \( T/2 \) and then descends to zero in time \( T/2 \). It follows that for \( u \) smaller than but close to \( \frac{1}{2} \) we have the bounds

\[
e^{-c_1(1-2u)A} P(A) \leq P_u(A) \leq e^{-c_2(1-2u)\sqrt{\pi}} P(A) \tag{26}
\]
where \( c_1 \) and \( c_2 \) are positive constants. It is natural to regard \( u \) as a temperature-like parameter and \( u = \frac{1}{2} \) as a critical point so we expect scaling as this point is approached. Assuming that

\[
P_u(A) \sim A^{-\gamma} F(\sqrt{1 - 4u^2} A^\beta)
\]

(27)

for large \( A \), where \( F \) is a function decaying more rapidly than any power, we find that

\[
\sum_A A^n P_u(A) \sim (1 - 4u^2)^{\frac{-n-1}{2\beta}}.
\]

(28)

Since this is valid for any \( n \) it follows from Eq. (22) that

\[
\beta = \frac{1}{3} \quad \text{and} \quad \gamma = \frac{4}{3}.
\]

(29)

This is of course consistent with the inequalities (26) which imply that \( \frac{1}{4} \leq \beta \leq \frac{1}{2} \).

In order to verify Eq. (22) let us denote the derivatives of the generating function \( f \) with respect to the first and second argument by \( \partial_1 f \) and \( \partial_2 f \), respectively. We begin by considering the case \( n = 1 \). Differentiating Eq. (17) with respect to \( z \) we obtain

\[
\partial_1 f(z, u) = u^2 + f(z, uz) \partial_1 f(z, u) + f(z, u)(\partial_1 f(z, uz) + u \partial_2 f(z, uz)).
\]

(30)

Putting \( z = 1 \), rearranging and using Eq. (18) we obtain

\[
\partial_1 f(1, u) = \frac{u^2 + u \partial_2 f(1, u)}{1 - 2f(1, u)} = \frac{u^2 + 2u^2(1 - 4u^2)^{-\frac{1}{2}}}{\sqrt{1 - 4u^2}} \sim \frac{1}{1 - 4u^2}.
\]

(31)

Assume now that Eq. (22) holds for \( n \leq N - 1 \) where \( N \geq 2 \). Differentiating Eq. (17) \( N \) times with respect to \( z \) yields

\[
\partial_1^N f(1, u) = \sum_{k=0}^{N} \binom{N}{k} \partial_1^k f(1, u) \sum_{j=0}^{N-k} \binom{N-k}{j} u^{N-j-k} \partial_1^j \partial_2^{N-j-k} f(1, u).
\]

(32)

Rearranging we find that

\[
\partial_1^N f(1, u) = \frac{1}{1 - 2f(1, u)} \sum_{k=0}^{N-1} \binom{N}{k} \partial_1^k f(1, u)
\]

\[
\times \sum_{j=0, j\neq N}^{N-k} \binom{N-k}{j} u^{N-j-k} \partial_1^j \partial_2^{N-j-k} f(1, u).
\]

(33)
By the inductive hypothesis the most singular terms on the right hand side of Eq. (33) correspond to \( j + k = N \) if \( k > 0 \) and \( j = N - 1 \) when \( k = 0 \). The desired result follows.

Working slightly harder we can determine the coefficient of the leading divergence of the \( N \)th moment of \( P_u(A) \) and this allows us to place a further restriction on the function \( F \) introduced above. In view of Eq. (22) one can write

\[
\partial_1^N f(1, u) = C_N (1 - 4u^2)^{\frac{1}{2} - \frac{3N}{2}} + O((1 - 4u^2)^{1 - \frac{3N}{2}}). \tag{34}
\]

It is straightforward to check that Eq. (33) determines the following recursion relation for the coefficients \( C_N \)

\[
C_N = \sum_{k=1}^{N-1} \binom{N}{k} C_k C_{N-k} + (3N - 4)NC_{N-1}. \tag{35}
\]

It follows that up to power corrections

\[
C_N \sim (2N)! \tag{36}
\]

for large \( N \).

Suppose now that the function \( F \) in (27) is an exponential function of a power, i.e.

\[
F(x) = e^{-\alpha x^q} \tag{37}
\]

for some constants \( \alpha, q > 0 \). Using the ansatz (27) to calculate the \( N \)th moment of the area distribution we find that \( q \) is fixed to equal \( 3/2 \) by the asymptotic formula (36). We therefore expect that

\[
P_u(A) \sim A^{-4/3} e^{-\alpha (1-4u^2)^{3/4} A^{1/2}} \tag{38}
\]

in agreement with the bound (26).

This completes our discussion of the frequency-size distribution of avalanches where we do not need to take the upper cutoff \( h \) into account, i.e. the exponent 4/3 governs the size distribution of small avalanches in the presence of a cutoff. We now turn to the study of the tail of the size distribution and will see that for large \( A \) the probability \( P(A) \) falls off exponentially when the upper bound \( h \) is taken into account. We also estimate the area for which we have a transition from a power law to an exponential decay.

4 Long time behaviour

We now consider a directed random walk with a reflecting barrier at height \( y = h \). Let \( p_i(t) \) be the probability of the walk being at height \( i \) after \( t \) steps. Then the
initial condition is
\[ p_i(1) = \delta_{i1}. \] (39)
We let \( p(t) \) denote the column vector whose \( i \)th entry is \( p_i(t) \). Then
\[ p(t) = \left( \frac{1}{2} M \right)^{t-1} p(1) \] (40)
where \( M \) is the matrix
\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1 & 0 & 0 \\
0 & \ldots & 0 & 1 & 0 \\
\end{pmatrix}.
\] (41)
The probability that a walk lasts exactly \( T \) steps is evidently
\[ P(T) = \frac{1}{2} p_1(T-1). \] (42)
Let \( \langle \cdot, \cdot \rangle \) denote the standard inner product on \( \mathbb{R}^{h+1} \) and \( e_i, i = 0, \ldots, h \), the standard orthonormal basis. Then we can write
\[ P(T) = \langle e_0, \left( \frac{1}{2} M \right)^{T-1} e_1 \rangle. \] (43)
Let \( D \) denote the matrix whose elements are defined by
\[ D_{ij} = z^i \delta_{ij} \] (44)
i, \( j = 0, \ldots, h \). If \( P(A, T) \) denotes the probability that a walk lasts a time \( T \) and covers an area \( A \) then \( P(A, T) \) is given by the coefficient of \( z^A \) in the matrix element
\[ \langle e_0, (MD)^{T-1} e_1 \rangle. \] (45)
The generating function for the probabilities \( P(A, T) \) can therefore be expressed as
\[ Q(z, u) = \langle e_0, \frac{u^2 MD}{1 - uMD} e_1 \rangle, \] (46)
since the Neumann series for the inverse of \((1 - uMD)\) is easily seen to converge for \( |u| \leq \frac{1}{2} \) and \( |z| \leq 1 \).
Eq. (46) allows us in principle to calculate \( P(A, T) \). However, the interesting feature of \( P(A, T) \) is that it falls exponentially with \( T \) and \( P(A, T) = 0 \) unless \( A \leq hT \). It follows that \( P(A) = \sum_T P(A, T) \) falls exponentially with \( A \) provided \( A \)}
is large enough. In order to establish this exponential decay it suffices to show that $Q(1, u)$ is finite for some $u > \frac{1}{2}$. We can write

$$Q(1, u) = \frac{1}{2} \langle e_0, \frac{1}{\lambda - \frac{1}{2} M} e_1 \rangle$$  \hspace{1cm} (47)$$

where $\lambda = (2u)^{-1}$. Evaluating the matrix element in Eq. (47) is an elementary calculation and we find

$$Q(1, u) = \frac{1}{4} \cos((h - 2)\theta) \cos((h - 1)\theta)$$  \hspace{1cm} (48)$$

where

$$e^{i\theta} = \lambda + i\sqrt{1 - \lambda^2}$$  \hspace{1cm} (49)$$

and we are assuming $\lambda \leq 1$. The first singularity of $Q(1, u)$ as $u$ moves beyond $\frac{1}{2}$ is encountered for the smallest $\theta \in [0, 2\pi)$ for which the denominator in Eq. (48) vanishes, i.e.

$$\theta = \frac{\pi}{2(h - 1)}.$$  \hspace{1cm} (50)$$

It follows that the radius of convergence of $Q(1, u)$ is

$$r = \frac{1}{2} \sqrt{1 + \tan^2 \frac{\pi}{2(h - 1)}},$$  \hspace{1cm} (51)$$

and for large $A$ we find

$$P(A) \leq Ce^{-c A / h^2}$$  \hspace{1cm} (52)$$

where $c$ and $C$ are positive constants and $c$ can be taken to be independent of $h$.

The exponential decay of $P(A)$ takes over from the power law found in the previous section for $A \approx h^3$ since a random walk must have at least $h^2$ steps in order to feel the effect of the reflecting barrier at $y = h$. In order to prove this note that we can write

$$P(T) = \frac{2^{1-T}}{2\pi i} \oint \frac{Q(1, u)}{u^{T+1}} du,$$  \hspace{1cm} (53)$$

where the contour encloses the unit disc in the complex plane. Calculating the residues we find that

$$P(T) = 2h^{-1} \sum_{n=0}^{2(h-1)} \sin^2 \frac{\pi(1 + 2n)}{h - 1} \cos^{-1} \frac{\pi(1 + 2n)}{h - 1} \hspace{1cm} (54)$$

and consideration of this formula at large $h$ shows that $P(T)$ crosses over from exponential decay to a $T^{-3/2}$ decay for $T \approx h^2$. 

5 Discussion

By universality, we do not expect the principal results we have obtained to change if we replace the simple Bernoulli random walk, considered in this paper, by a random walk with any rapidly decaying transition function. The exponent value $4/3$ ought to be universal as will be exponential decay for large $A$ in the presence of an upper cutoff $h$.

We do, however, expect to be able to change the power law decay by considering strongly correlated random walks. In earthquake models, for example, the size distribution exponent for small and intermediate size events varies from around $2/3$ to values greater than $1$. It is clear that by looking at the statistics of avalanches in a realistic system one can concoct a random walk model with an identical avalanche distribution.

The more interesting problem of understanding correlations between different avalanches cannot be studied in the random walk framework unless one introduces different interacting random walks. The principal virtue of the model we have discussed is that it gives us a qualitative and quantitative insight into the genesis of power law distribution for avalanches without introducing any complicated dynamics.

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Fig. 1 A directed random walk of duration $T = 12$. 
Fig. 2. This figure illustrates the one to one correspondence between paths in $\mathcal{W}'$ of duration $T$ and paths in $\mathcal{W}$ of duration $T - 2$. 
Fig. 3. This figure illustrates how one can uniquely decompose any directed path into a pair of paths in $\mathcal{W}'$ and $\mathcal{W}$. 