MORPHISMS FROM QUINTIC THREEFOLDS TO CUBIC THREEFOLDS ARE CONSTANT

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Abstract. We show that every morphism from a quintic threefold in $P^4$ to a nonsingular cubic threefold in $P^4$ is constant in characteristic zero. In the process, we classify morphisms from $P^2$ to nonsingular cubic hypersurfaces in $P^4$ given by degree 3 polynomials.

Introduction

The author shows in [14] that if $f: X \to Y$ is a morphism of hypersurfaces in $P^4$ such that $\text{deg} Y = 3$ and $\text{deg} X \leq 4$, then $f$ is either constant or $\text{deg} X = 3$ and $f$ is an isomorphism. The purpose of this paper is to extend this result by proving the following theorem.

Theorem 1. If $f: X \to Y$ is a morphism of hypersurfaces in $P^4$ over an algebraically closed field of characteristic zero such that $\text{deg} X = 5$, $\text{deg} Y = 3$, and $Y$ is nonsingular, then $f$ is constant.

The motivation for investigating such morphisms to cubic hypersurfaces is the expectation that if $f: X \to Y$ is a morphism between hypersurfaces in $P^n$, $n \geq 4$, such that $Y$ is nonsingular of degree at least 2, then $\text{deg} Y$ divides $\text{deg} X$ with quotient $q$, and $f$ is given by polynomials of degree $q$, i.e. $f^*O_Y(1) = O_X(q)$. This result is proven in [14] when $\text{deg} Y \geq n + 2$. It is also proven in some cases where $3 \leq \text{deg} Y \leq n + 1$. The fact that morphisms from quintic threefolds to cubic threefolds are necessarily constant is the first nontrivial case of morphisms to cubics.

1. Outline of Proof

Throughout the paper, the base field will be algebraically closed of characteristic zero, and $f: X \to Y$ will denote a nonconstant morphism of hypersurfaces in $P^4$, such that $\text{deg} X = 5$, $\text{deg} Y = 3$, and $Y$ is nonsingular.

The Grothendieck-Lefschetz Theorem, [5, Theorem 4.3.2], states that $\text{Pic} X$ is generated by $O_X(1)$. So $f^*O_Y(1) = O_X(m)$ for some positive integer $m$. In terms of $m$, [14, Theorem 1] states that $\text{deg} f^*_{C^3} (\Omega^1_Y(2)) \leq \text{deg} c_3 (\Omega^1_X(2m))$. By computing the Chern classes in this inequality, one checks that $m \leq 3$.

We claim there is a rational map $F: P^4 \dasharrow P^4$ undefined at only finitely many points disjoint from $X$ such that $f = F|_X$. To see this, consider the short exact sequence

$$H^0(P^4, O_{P^4}(m)) \to H^0(X, O_X(m)) \to H^1(P^4, O_{P^4}(m - 5))$$

Since $H^1(P^4, O_{P^4}(m - 5)) = 0$, the sections of $O_X(m)$ that define $f: X \to P^4$ lift to $P^4$ and define a rational map $F: P^4 \dasharrow P^4$, as claimed.
We introduce a hypersurface $H$ in $\mathbb{P}^4$ that will play a key role. Lazarsfeld proves in [10] that $\mathbb{P}^k$ is the only smooth $k$-dimensional variety that is the image of a morphism from $\mathbb{P}^k$. In particular, $Y$ is not the image of a morphism from $\mathbb{P}^3$. It follows by considering a general hyperplane in $\mathbb{P}^4$ that the image of $F : \mathbb{P}^4 \to \mathbb{P}^4$ is not $Y$. So $F$ is dominant because the image of $F$ is irreducible and contains $Y$. Hence $F^{-1}(Y) = X + H$ for some hypersurface $H$ of degree $3m - 5$.

This shows $m \neq 1$. We see that $m \neq 2$ as follows. Suppose $m = 2$. Then $H$ is a hyperplane in $\mathbb{P}^4$. Therefore $F|_H : H \to Y$ is not a morphism. Let $p \in H$ be a point of indeterminacy of $F$. This means that $F = (F_0, \ldots, F_4)$ for homogeneous polynomials $F_i$ of degree $2$ that all vanish at $p$. If $Y$ is defined by the homogeneous cubic polynomial $G = G(y_0, \ldots, y_4)$, then $F^{-1}(Y) = X \cup H$ is defined by the homogeneous sextic polynomial $G(F_0, \ldots, F_4)$. Since the $F_i$ all vanish at $p$ and $G$ is homogeneous of degree $3$, $G(F_0, \ldots, F_4)$ vanishes to order at least $3$ at $p$. In other words, $p$ is a triple point of $F^{-1}(Y) = X \cup H$. This is impossible because $p$ is contained in the hyperplane $H$ but not contained in $X$. Therefore, $m \neq 2$.

It is considerably more difficult to check that $m = 3$ is also not possible. That task will occupy us for the remainder of the paper. We focus our attention on the degree $4$ hypersurface $H$ and the rational map $F|_H : H \to Y$.

We prove Theorem 1 by considering the various possibilities for $H$. If $H$ contains many copies of $\mathbb{P}^2$, then we will use the resulting maps $\mathbb{P}^2 \to Y$, which would be given by degree $3$ polynomials. So our first task is to classify such morphisms from $\mathbb{P}^2$. This is done in Section 2. Then we return to the map $H \to Y$ in Section 3 and work through several cases using the geometry of rational maps from threefolds to three-dimensional cubics.

2. Morphisms from $\mathbb{P}^2$ to Cubic Threefolds

Let $g : \mathbb{P}^2 \to Y$ be a morphism given by degree $3$ polynomials, i.e. $g^*O_Y(1) = O_{\mathbb{P}^2}(3)$. The purpose of this section is to prove the following result about $g$.

**Theorem 2.** We can choose coordinates $x_0, x_1, x_2$ on $\mathbb{P}^2$ and coordinates on $\mathbb{P}^4$ such that $g = (x_0^3, x_1^3, x_2^3, x_0x_1x_2, 0) : \mathbb{P}^2 \to Y \subset \mathbb{P}^4$.

Let $S$ be the image of $g$. Since $Y$ is nonsingular, $S \subset Y$ is a Cartier divisor. So by the Grothendieck-Lefschetz Theorem, [3] Theorem 4.3.2, $S$ is the zero locus $V(s)$ of a section $s \in H^0(Y, O_Y(a))$ for some $a > 0$. To see that $S$ is the scheme-theoretic intersection of $Y$ with another hypersurface $Y'$ in $\mathbb{P}^4$, consider the following piece of a long exact sequence of cohomology groups.

$$H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(a)) \to H^0(Y, O_Y(a)) \to H^1(\mathbb{P}^4, O_{\mathbb{P}^4}(a - 3))$$

Since $H^1(\mathbb{P}^4, O_{\mathbb{P}^4}(a - 3)) = 0$, $s$ is the image of some $s' \in H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(a))$. So $S = V(s') \cap Y$. Take $Y' := V(s')$.

Calculate $g^*c_1(O_Y(1))^2$ to be $9 = \deg g \cdot \deg S$ to see that $3 = \deg g \cdot \deg Y'$. Hence we will break the proof of Theorem 2 into two cases depending on whether $\deg Y' = 3$ or $\deg Y' = 1$. First, we recall the following result from [4], which we will use repeatedly.

**Proposition 2.1.** Let $\Delta$ be the space of lines on the smooth cubic threefold $Y$. Then $\Delta$ is a complete nonsingular surface that does not contain a rational curve.
Case 1. Assume $Y'$ has degree 3. We will derive a contradiction.

By assumption, $g : \mathbb{P}^2 \to S$ has degree 1, so it is a finite birational morphism. Our strategy is to analyze the double point class $\mathbb{D}(g)$ of $g : \mathbb{P}^2 \to Y$. See [6, Section 9.3] for the construction and computation of $\mathbb{D}(g)$. The construction will be used implicitly in the proof of Proposition 2.6 and Corollary 2.7. Following [6], we calculate the double point class of $g$:

\[
\mathbb{D}(g) = g^*g_*[\mathbb{P}^2] - c_1(g^*T_V) + c_1(T_{\mathbb{P}^2})
\]

\[
= g^*c_1(\mathcal{O}_Y(3)) - g^*c_1(\mathcal{O}_Y(2)) + c_1(\mathcal{O}_{\mathbb{P}^2}(3))
\]

\[
= c_1(\mathcal{O}_{\mathbb{P}^2}(6)).
\]

An important fact for our purposes is that the construction of the cycle $\mathbb{D}(g)$ not only gives a 1-cycle modulo rational equivalence, it actually constructs a Weil divisor on $\mathbb{P}^2$. We will denote this Weil divisor by $D(g)$ and consider it as a closed subscheme of $\mathbb{P}^2$. This notation differs slightly from [6] in that we use $D(g)$ to denote a divisor, not just a set. Roughly speaking, $D(g)$ is the curve in $\mathbb{P}^2$ consisting of the closed points $x$ such that either $g(x) = g(y)$ for some $y \neq x$ or such that $T_x \mathbb{P}^2 \to T_{g(x)}Y$ is not injective. The scheme structure of $D(g)$ comes from the fact that an integral curve $D$ in $D(g)$ will appear with multiplicity if for a general point $x \in D$ there is more than one other point $y \in \mathbb{P}^2$ with $g(x) = g(y)$ or if $g$ ramifies to high order along $D$.

The following result tells us that the image of $D(g)$ under $g$ is equal to the singular locus of $S$ as a set.

Lemma 2.2. Let $g : V \to W$ be a finite surjective birational morphism of varieties with $V$ regular. Then for $w \in W$ a closed point, $w$ is a nonsingular point of $W$ if and only if the scheme-theoretic preimage $g^{-1}(w)$ is a single reduced point.

Proof. Suppose $g^{-1}(w)$ is a single reduced point $v$. Let $D \subset W$ be a general curve in $W$ through $w$ such that $D$ is a local complete intersection in $W$ at $w$ and

\[
\dim T_wW - \dim T_wD = \dim W - 1.
\]

By assumption, the pullback of the maximal ideal of $w$ generates the maximal ideal of $v$. So the curve $C := g^{-1}(D)$ is nonsingular at $v$ because $C$ is cut out near $v$ by the dim $V - 1$ equations that define $D$ near $w$. The morphism $g|_C$ is an isomorphism in a neighborhood of $v$ because $g^{-1}(w) = v$ and $g|_C$ is birational in a neighborhood of $v$. So $D$ is regular at $w$, whence $W$ is too by the above equation.

Conversely, suppose $w$ is a regular closed point of $W$. Let $A$ be the ring of regular functions on an open affine $U$ of $W$ containing $w$, and let $B$ be the ring of regular functions on the open affine $g^{-1}(U)$. Let $m \subset A$ be the maximal ideal corresponding to the point $w$, and let $M$ denote the multiplicative set $A \setminus m$. Then $M^{-1}B$ is the integral closure of $A_m = M^{-1}A$ in its field of fractions. But $A_m$ is integrally closed because it is regular. Therefore $M^{-1}B = A_m$, whence $M^{-1}B$ is a local ring whose maximal ideal is generated by $m$. In other words, $g^{-1}(w)$ is a single reduced point.

Now let $C \subset S$ denote the image of $D(g)$ with its reduced structure:

$$C := g(D(g))_{\text{red}}$$

By Lemma 2.2, $C$ is the maximal reduced curve contained in the singular locus of $S$. We will rule out the case $\deg Y' = 3$ by comparing $D(g)$ to $g^{-1}(C)$, where
$g^{-1}(C)$ denotes the scheme-theoretic preimage. So our strategy is to compare the multiplicity of integral curves in $D(g)$ and $g^{-1}(C)$.

**Lemma 2.3.** There is a cubic hypersurface $Y''$ in $\mathbb{P}^4$ such that $S = Y \cap Y''$, and $Y''$ is singular along $C$.

**Proof.** Let $\{Y_t\}$ be the pencil of cubic hypersurfaces spanned by $Y$ and $Y'$. Since $Y_t \cap Y = S$ so long as $Y_t \neq Y$, it suffices to show that one of the $Y_t$ is singular along $C$.

Let $C_0$ be an irreducible component of $C$. For every point $p \in C_0$ we have $T_pS = T_pY$ because $S$ is singular at $p$. Therefore $T_pS = T_pY_t$ for $t$ general, and there is a unique $t$, say $t_p$, such that $Y_{t_p}$ is singular at $p$. Only finitely many of the $Y_t$ are singular, so one of the $Y_t$, say $Y_{t_0}$, is singular at infinitely many points of $C_0$. Hence $Y_{t_0}$ is singular along $C_0$.

If $C_1$ is another irreducible component of $C$, then $C_0$ and $C_1$ meet at some point $p$ because $C_0$ and $C_1$ are both images of curves in $\mathbb{P}^2$. If $Y_{t_0}$ is singular along $C_1$, then $Y_{t_0}$ and $Y_{t_1}$ are both singular at $p$, so $Y_{t_0} = Y_{t_1}$. The Lemma follows. \(\square\)

We now assume $Y'$ to be singular along $C$.

**Lemma 2.4.** The scheme-theoretic preimage $g^{-1}(C)$ does not contain a curve of degree 6 or more.

**Proof.** Let $G' = G'(y_0, \ldots, y_4)$ be the homogeneous equation for $Y'$ in $\mathbb{P}^4$. Set $C_i := S \cap V(\frac{\partial G'}{\partial y_i})$ and $D_i := g^{-1}(C_i)$ for $i = 0, \ldots, 4$. The $D_i$ are all degree 6 plane curves containing $g^{-1}(C)$. So it suffices to show that the $D_i$ are not all equal.

Suppose they were. Then the equations for the $D_i$ in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ are all scalar multiples of each other. These equations are the images of the partials $\frac{\partial G'}{\partial y_i}$ under the composition

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \xrightarrow{\rho} H^0(S, \mathcal{O}_S(2)) \xrightarrow{g^*} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$$

where $\rho$ is restriction. This composition is an injection because $S$ is not contained in a quadric. So the $\frac{\partial G'}{\partial y_i}$ are also scalar multiples of each other, say $\frac{\partial G'}{\partial y_i} = \alpha_i \frac{\partial G'}{\partial y_0}$ for scalars $\alpha_i$. By Euler’s formula,

$$3G' = \sum_{i=0}^4 y_i \frac{\partial G'}{\partial y_i} = \frac{\partial G'}{\partial y_0} \sum_{i=0}^4 \alpha_i y_i$$

which implies $Y'$ is not integral. So $S = Y \cap Y'$ is not integral. This contradiction finishes the proof. \(\square\)

Now we make several observations, which we list in the following Lemma.

**Lemma 2.5.**

1. If $\Lambda \subset \mathbb{P}^4$ is any 2-plane, then $g^{-1}(\Lambda \cap S)$ does not contain a curve of degree 3 or more.
2. For $C' \subset C$ any curve, the secant variety of $C'$ is contained in $Y'$.
3. Every curve in $C$ that lies in a hyperplane is a plane curve.
4. $C$ does not contain a conic curve.
5. For every $x \in \mathbb{P}^2$, $g$ induces a nonzero map $T_x \mathbb{P}^2 \to T_{g(x)}Y$. 
Proof. (1) follows from the fact that the preimage of every hyperplane containing \( \Lambda \) is a distinct degree 3 curve in \( \mathbb{P}^2 \). The preimages are distinct because \( S \) is not contained in a hyperplane, which is our assumption for Case 1. (2) holds because every line meeting \( Y' \) at two singular points is contained in \( Y' \), and \( Y' \) is singular along \( C' \). (3) follows from (2) because \( Y' \) does not contain a hyperplane. This is because \( \deg Y' = 3 \) and \( Y \cap Y' = S \) is integral. (4) Suppose \( C' \subset C \) is a conic curve. Then \( g^{-1}(C') \) contains a curve of degree at least 2 because lines in \( \mathbb{P}^2 \) map to either lines or cubics. If \( \Lambda \) is the 2-plane containing \( C' \), then \( \Lambda \subset Y' \) by (2). So \( \Lambda \cap S = \Lambda \cap Y \) is a degree 3 plane curve consisting of \( C' \) and a line. This line is the image of a curve in \( \mathbb{P}^2 \). Therefore, \( g^{-1}(\Lambda) \) contains a curve of degree at least 3, contradicting (1). (5) If \( T_x \mathbb{P}^2 \to T_{g(x)}Y \) is the zero map, then every line in \( \mathbb{P}^2 \) through \( x \) maps to either a line or a cuspidal plane cubic with cusp at \( g(x) \). In particular, the image of every line in \( \mathbb{P}^2 \) through \( x \) is contained in \( T_{g(x)}Y \), which is impossible by the assumption for this subsection that \( S \) is not contained in a hyperplane. □

We will describe the integral curves \( D \) that might occur in \( D(g) \) in terms of their multiplicity in \( D(g) \) and in the scheme-theoretic preimage \( g^{-1}g(D) \). For this purpose, we recall the following definition.

Definition. If \( D \) is an integral curve contained in a one dimensional scheme \( Z \), then \( \text{mult}_D Z \) will denote the length of the Artin ring obtained by localizing \( O_Z \) at the generic point of \( D \).

Proposition 2.6. If \( D \subset D(g) \) is an integral curve with \( n := \deg g|_D \), then one of the following four cases holds:

1. \( n = 1, \text{mult}_D g^{-1}g(D) = 2, \text{mult}_D D(g) = 1 \), \( D \) is a nonsingular conic, and \( D \) is the only integral curve in \( g^{-1}g(D) \).
2. \( n = 1, \text{mult}_D g^{-1}g(D) = 1, \text{mult}_D D(g) = 1, \) and \( g^{-1}g(D) \) contains two distinct nonsingular conics \( D, D' \).
3. \( n = 2, \text{mult}_D g^{-1}g(D) = 1, \text{mult}_D D(g) = 1 \), \( D \) is a nonsingular conic, and \( g(D) \) is a singular plane cubic.
4. \( n = 3, \text{mult}_D g^{-1}g(D) = 1, \text{mult}_D D(g) = 2, \) and \( D, g(D) \) are both lines.

Proof. We calculate \( \text{mult}_D D(g) \) as follows. First suppose that \( g \) is unramified along every curve in \( g^{-1}g(D) \) in the sense that \( T_x \mathbb{P}^2 \to T_{g(x)}Y \) is injective for \( x \) a general point on any irreducible curve in \( g^{-1}g(D) \). In this case, if \( D_1, \ldots, D_r \) are the distinct integral curves contained in \( g^{-1}g(D) \), then

\[
\text{mult}_D D(g) = -1 + \sum_{i=1}^r \deg g|_{D_i},
\]

because the number of preimage points of a general point in \( g(D) \) is the sum of the \( \deg g|_{D_i} \). On the other hand, if \( D \) is the only integral curve contained in \( g^{-1}g(D) \) and \( g \) is simply ramified along \( D \) in the sense that \( 2D \) is contained in \( g^{-1}g(D) \) but \( 3D \) is not, then

\[
\text{mult}_D D(g) = 1.
\]

These are the only cases that will arise in our discussion here. See [6, Section 9.3] for the construction of the Weil divisor \( D(g) \) representing \( D(g) \).
We claim that if $D \subset D(g)$ is a line, then $g(D)$ is a line too. So suppose $D \subset D(g)$ is a line. By Lemma 2.5(3), $g(D)$ is not a twisted cubic. So it suffices to show that $g(D)$ is not a plane cubic. If it were, then by Lemma 2.5(2), $Y'$ contains the 2-plane $\Lambda$ spanned by $g(D)$. Choose any hyperplane $\Gamma$ containing $g(D)$, and let $\Sigma := \Gamma \cap Y'$. If $p \in \Gamma \cap Y'$ is a singular point of $Y'$ away from $\Lambda$, then $\Gamma \cap Y'$ contains the cone $\Sigma_p$ over $g(D)$ with vertex $p$. Since $\Lambda \cup \Sigma_p \subset \Sigma$ and $\Sigma$ and $\Sigma_p$ both have degree 3, no such $p$ can exist. In particular, $C \subset \Lambda$. By Lemma 2.5(1), $g^{-1}(C)$ is either $D$, $\Lambda$ spanned by $D$, or $D$ and another line. So $D(g)$ will have respectively degree $0$, $1$, or $2$, according to the formulas for $\text{mult}_D D(g)$ above. However, $\deg D(g) = 6$. So $g(D)$ is not a plane cubic. Therefore, if $D \subset D(g)$ is a line, then $g(D)$ is too.

Now let $D \subset D(g)$ be any integral curve. Using $(n - 1)D \subset D(g)$ and taking degrees, we obtain the inequality $(n - 1)\deg D \leq 6$. The projection formula yields $3 \deg D = n \deg g(D)$. Therefore, if $n \geq 2$, then

$$\frac{n}{3} \deg D = \deg D \leq \frac{6}{n - 1}.$$  

This shows $n \leq 3$ because the left hand side is an integer. Now we analyze what happens for each value $n = 1, 2, 3$.

Suppose $n = 3$, so that $\deg D = \deg g(D) \leq 3$. If $\deg D = 3$, then Lemma 2.5(1) and (3) imply that $g(D)$ is neither a plane cubic nor a twisted cubic. So $\deg g(D) \leq 2$. By Lemma 2.5(4), $\deg D \neq 2$. So $D$, $g(D)$ are lines, as claimed in Case (4).

To see $\text{mult}_D g^{-1}(D) = 1$, suppose $2D \subset g^{-1}(D)$. By Lemma 2.5(5), for $x \in D$ a general point, the map $T_x \mathbb{P}^2 \to T_{g(x)}Y$ has rank 1. Choose $p \in g(D)$ general so that $(g|D)^{-1}(p)$ consists of three distinct points $x_1, x_2, x_3 \in D$. Through each $x_i$ there is a unique line $l_i$ in $\mathbb{P}^2$ such that $T_{x_i} l_i \to T_p S$ is the zero map. Each $g(l_i)$ is either a line through $p$ or a cuspidal plane cubic with cusp at $p$. In both cases, $g(l_i)$ is contained in $T_p Y$. Now the degree 3 curve $g^{-1}(T_p Y)$ contains $2D + l_1 + l_2 + l_3$. This contradiction proves $\text{mult}_D g^{-1}(D) = 1$.

To show $\text{mult}_D D(g) = 2$, it suffices by formula (2.1) to show there is no integral curve $D' \subset g^{-1}(D)$ other than $D$. If $D'$ were such a curve, then $\deg g|D' = 3 \deg D'$ because $g(D)$ is a line. So for $p \in g(D)$ a general point, $g^{-1}(p)$ would contain 3 points on $D$ and $3 \deg D'$ points on $D'$. There would be $9 \deg D'$ lines joining one of the points on $D$ to one of the points on $D'$, and these lines would map to either lines through $p$ or nodal plane cubics with node at $p$. These image curves would be contained in $T_p Y$, so we would have the impossible situation that these $9 \deg D'$ lines would all be contained in the degree 3 plane curve $g^{-1}(T_p Y)$. So there is no $D'$, and $\text{mult}_D D(g) = 2$. This gives Case (4).

Suppose $n = 2$. If $2D \subset g^{-1}(D)$, then $2D \subset g^{-1}C$, which implies $2 \deg D < 6$ by Lemma 2.4. By formula (2.3), $g(D)$ can not be a line. So $D$ is not a line, from above. Hence $\deg D = 2$ and $\deg g(D) = 3$ by (2.3) and the fact $\deg D < 3$. Since $g(D)$ has degree 3 and can not be a twisted cubic, $g(D)$ is a singular plane cubic. By Lemma 2.5(1), $2D$ can not be contained in $g^{-1}(D)$, so $\text{mult}_D g^{-1}(D) = 1$. Likewise, $D$ is the only integral curve contained in $g^{-1}(D)$, so $\text{mult}_D D(g) = 1$ by formula (2.1). This gives Case (3).

Suppose $n = 1$. By (2.3), $\deg g(D) = 3 \deg D$. Since $g(D)$ is not a line, $D$ is not a line. So $\deg g(D) \geq 6$. If $D'$ is another integral curve contained in $g^{-1}(D)$, then $D' \subset D(g)$. By Cases (3) and (4), $\deg g|D' = 1$ because $\deg g(D') \geq 6$. Therefore, $\deg D' = 3 \deg g(D) = \deg D$. 


If $D$ is the only integral curve in $g^{-1}g(D)$, then $2D \subset g^{-1}g(D)$ because $D \subset D(g)$ and $n = 1$. By Lemma 2.4, $g^{-1}g(D)$ does not contain a curve of degree 6. So the only possibility is $\text{mult}_D g^{-1}g(D) = 2$ and $\deg D = 2$ because $\deg D > 1$. Since $D$ is integral, $D$ is a nonsingular conic. This gives Case (1) by formula (2.2).

The same reasoning shows that if $D'$ is another integral curve in $g^{-1}g(D)$, then the maximal curve contained in $g^{-1}g(D)$ is $D + D'$, and $D, D'$ are both nonsingular conics. This gives Case (2) by formula (2.1).

**Corollary 2.7.** There are distinct lines $l_1, l_2, l_3$ in $\mathbb{P}^2$ such that

$$D(g) = 2l_1 + 2l_2 + 2l_3$$

and the $g(l_i)$ are distinct lines in $S$.

**Proof.** In cases (1), (2), (3) of Proposition 2.6, $D$ has multiplicity 1 in $D(g)$. So if $D(g)$ does not contain a line, then $D(g)$ is reduced and $D(g) \subset g^{-1}(C)$. This contradicts Lemma 2.4. So $D$ contains a line $D$. If $L = g(D)$ is the image line of $D$, then $D$ is the only curve contained in $g^{-1}L$ by Case (4) of Proposition 2.6.

Now suppose $D' \neq D$ is another integral curve in $D(g)$. We will show that $D'$ is also a line. So suppose $\deg D' > 1$. Then $g(D')$ is not a line by Proposition 2.6. Since $D$ and $D'$ meet at a point, so do $L = g(D)$ and $g(D')$. If $g(D')$ is contained in a hyperplane, then by Lemma 2.5(3), $g(D')$ is contained in a 2-plane. So either $L$ and $g(D')$ are contained in the same 2-plane, which is impossible by Lemma 2.5(1), or the curve $L \cup g(D')$ spans a hyperplane, which is impossible by Lemma 2.5(3).

So $g(D')$ is not contained in a hyperplane.

Let $p \in g(D')$ be a general point, and let $\Lambda_p$ be the 2-plane spanned by $p$ and $L$. Since the cubic $Y'$ is singular along $L$ and at $p$, $\Lambda_p$ is contained in $Y'$. Therefore, $\Lambda_p \cap S = \Lambda_p \cap Y$ is equal to $L + Q_p$ for some conic curve $Q_p$ contained in $\Lambda_p$.

If $L$ is contained in $Q_p$, then $2L$ is contained in $\Lambda_p \cap Y$. So for every point $q \in L$, $T_q \Lambda_p = T_q(2L) \subset T_qY$. Since $g(D')$ is not contained in any hyperplane, there is no hyperplane that contains $\Lambda_p$ for every $p \in g(D')$. Therefore, $\dim T_qY = 4$ for every $q \in L$ because $T_q\Lambda_p$ is contained in $T_qY$ for $p \in g(D')$ a general point. Since $Y$ is nonsingular, one concludes that $L$ is not contained in $Q_p$.

If $Q_p$ is not a double line, then $g^{-1}Q_p$ contains a curve of degree at least 2. This is impossible by Lemma 2.5(1), because $\Lambda_p$ contains $L$ and $Q_p$. So for every $p \in g(D')$, $Q_p = 2L_p$ for some line $L_p \neq L$.

Therefore, $g(D')$ parametrizes a one-dimensional family of lines on $Y$. By Proposition 2.6, every curve in $D(g)$ is rational, so $g(D')$ is rational. However, the space of lines on $Y$ does not contain a rational curve by Proposition 2.1.

This proves that every integral curve in $D(g)$ is a line, and every line in $D(g)$ occurs with multiplicity 2 by Proposition 2.6. □

**Proposition 2.8.** The case $\deg Y' = 3$ does not occur.

**Proof.** We will use the $l_i$ from Corollary 2.7 to give an explicit formula for $g$ and derive a contradiction using this formula.

Let $L_i := g(l_i)$. Each of the $L_i$ intersect the other two, but they are not all contained in a plane by Lemma 2.5(1). So they all meet at some point $p$ in $S$, and $T_pY$ is the unique hyperplane containing all the $L_i$.

I claim the $l_i$ do not all meet at a point $x$. So suppose there were such a point $x$, and note $g(x) = p$. By Lemma 2.5(5), at most one of the $g|_{l_i}$, say $g|_{l_1}$, ramifies at $x$. Therefore there are points $x_2 \in l_2$ and $x_3 \in l_3$ different from $x$ such that
g(x_2) = g(x_3) = p, and g maps the line l_{23} containing x_2 and x_3 to either a line or a nodal plane cubic with node at p. In particular, g(l_{23}) is contained in T_pY. Now the degree 3 curve \( g^{-1}(T_pY) \) contains \( l_1 + l_2 + l_3 + l_{23} \), which is impossible. So the \( l_i \) do not all meet at \( x \).

Let \( x_{ij} := l_i \cap l_j \), and note \( g(x_{ij}) = p \). If \( g|_{l_i} \) is not ramified at either \( x_{12} \) or \( x_{13} \), then there is a point \( x' \) on \( l_i \) different from \( x_{12}, x_{13} \) such that \( g(x') = p \), and the line from \( x' \) to \( x_{23} \) gives a contradiction just as the line \( l_{23} \) did above. Hence each \( g|_{l_i} \) ramifies at exactly one of the \( x_{ij} \). So the scheme-theoretic preimage \( g^{-1}(p) \) consists of three copies of \( \text{Spec } k[[\epsilon]]/((\epsilon^2)) \), one supported at each of the \( x_{ij} \). The scheme \( g^{-1}(p) \) is contained in the scheme \( l_1 + l_2 + l_3 \), but with no two copies of \( \text{Spec } k[[\epsilon]]/((\epsilon^2)) \) contained in the same \( l_i \) because \( (g|_{l_i})^{-1}(p) \) has length 3, as \( g|_{l_i} \) is a morphism of degree 3 of nonsingular curves, and is therefore flat. Hence, we can choose homogeneous coordinates \( x_0, x_1, x_2 \) on \( \mathbb{P}^2 \) such that \( l_i = V(x_{i-1}) \) and

\[
g^{-1}(p) = V(x_0^2x_1, x_0^2x_2, x_0^2x_0, x_0x_1x_2).
\]

There is a 3 dimensional space of hyperplanes in \( \mathbb{P}^4 \) containing \( p \), and these hyperplanes pull back to the linear system of cubics on \( \mathbb{P}^2 \) spanned by

\[
x_0^2x_1, x_1^2x_2, x_2^2x_0, x_0x_1x_2.
\]

Set \( p = (0, 0, 0, 0, 1) \) so that for a suitable choice of coordinates \( y_0, \ldots, y_4 \) on \( \mathbb{P}^4 \) we have

\[
g = (x_0^2x_1, x_1^2x_2, x_2^2x_0, x_0x_1x_2, y_4) : \mathbb{P}^2 \to \mathbb{P}^4
\]

for some homogeneous polynomial \( g_4 \) of degree 3.

Since \( g^{-1}(y_3) = l_1 + l_2 + l_3 \), one sees that \( T_pY = V(y_3) \). So if \( G \) is the homogeneous equation for \( Y \), then

\[
G = y_3^2 + y_4G_2 + G_3
\]

where \( G_i = G_i(y_0, \ldots, y_3) \) is homogeneous of degree \( i \).

By equation \( (2.4) \), the fact that \( g \) is defined at \( (1, 0, 0) \in \mathbb{P}^2 \) tells us that \( x_0^3 \) has nonzero coefficient in \( g_4 \). Considering equation \( (2.4) \), we see that \( x_0^2x_1x_2 \) has nonzero coefficient in \( g^*(y_3^2) \). Since \( g(\mathbb{P}^2) \subset Y \), \( g^*G \) is the zero polynomial, so \( x_0^2x_1x_2 \) has nonzero coefficient in \( g^*(y_4G_2 + G_3) \) by equation \( (2.5) \). The highest power of \( x_0 \) that can appear in \( g^*G_3 \) is \( x_0^3 \) by \( (2.4) \). Therefore, \( x_0^2x_1x_2 \) has nonzero coefficient in \( g^*(y_4G_2) \).

The highest power of \( x_0 \) that can appear in a monomial of \( g^*G_2 \) is \( x_0^4 \), and this necessarily occurs in the monomial \( y_4^2x_0^2 \). So if \( x_0^2 \) appears in a monomial of \( g^*(y_4G_2) = g_4g^*G_2 \), then that monomial is \( x_0^2x_1^2 \), not \( x_0^2x_1x_2 \). This contradiction shows that \( \deg Y' \neq 3 \).

\[\square\]

**Case 2.** By Proposition 2.8, the image of \( g : \mathbb{P}^2 \to Y \) is the surface \( S := Y \cap Y' \), where \( Y' \) is a hyperplane in \( \mathbb{P}^4 \). We will derive a contradiction, thus proving Theorem 2.

The morphism \( \delta : Y \to Y' \) sending a point \( p \) to the hyperplane \( T_pY \) is given by an ample invertible sheaf on \( Y \), since \( O_Y(1) \) generates \( \text{Pic } Y \) by the Grothendieck-Lefschetz Theorem, \[\text{[S Theorem 4.3.2].} \] So \( \delta \) is a finite morphism. In other words, no hyperplane in \( \mathbb{P}^4 \) can be tangent to \( Y \) at infinitely many points. Therefore \( S \) is a cubic surface with isolated singularities.
For $s \in S$ a general closed point, let $\{E_t\}$ be a general pencil of hyperplane sections of $S$ containing $s$, and let $E_t' := g^{-1}(E_t)$. Then $E_t$ and $E_t'$ are both smooth plane cubics for $t \in \mathbb{P}^1$ general.

Since $g : \mathbb{P}^2 \to S$ has degree 3, we can set $g^{-1}(s) = \{a, b, c\}$. Then $(E_t', a) \to (E_t, s)$ is an isogeny of elliptic curves of degree 3 for $t$ general. Note that $b$ is a 3-torsion point of $E_t'$.

Let $E' \subset \mathbb{P}^1 \times \mathbb{P}^2$ and $E \subset \mathbb{P}^3 \times S$ be the total spaces of the families $\{E_t\}$ and $\{E_t\}$ over the base $\mathbb{P}^1$, and consider the diagram

$$
\begin{array}{ccc}
E' & \to & \mathbb{P}^2 \\
\downarrow g' & & \downarrow g \\
E & \to & S
\end{array}
$$

where $g'$ is the restriction of $\text{id}_{\mathbb{P}^1} \times g$ to $E' \subset \mathbb{P}^1 \times \mathbb{P}^2$.

Let $\zeta$ be the generic point of $\mathbb{P}^1$, and let $E_0' \subset E'$ be the fibers of $E'$, $E$ over $\zeta \in \mathbb{P}^1$. Then $A := \zeta \times a$ and $B := \zeta \times b$ are $\zeta$-valued points of $E_0'$, and $B$ is a 3-torsion point of the elliptic curve $(E_0', A)$. Hence $P \mapsto P + B$ gives an automorphism of $E_0'$ over $\mathbb{P}^1$ of order 3 that extends to some birational self-map $\psi : E' \dashrightarrow E'$ of $E'$ over $E$. In other words, $g' = g' \circ \psi$ as rational maps. The horizontal arrows in the diagram above are birational morphisms. So $\psi$ induces an order $3$ birational self-map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $g = g \circ \phi$ as rational maps. Therefore $\phi$ is an automorphism of $\mathbb{P}^2$ by the following lemma.

**Lemma 2.9.** If $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational map and $g : \mathbb{P}^2 \to S$ is a dominant morphism to a surface $S$ such that $g \circ \phi = g$ as rational maps, then $\phi$ extends to an automorphism of $\mathbb{P}^2$.

**Proof.** Let $U \subset \mathbb{P}^2$ be the domain of definition of $\phi$. Then $U$ is the complement of finitely many points in $\mathbb{P}^2$, and $\text{Pic} U$ is generated by $\mathcal{O}_U(1)$, the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$ to $U$. From $g \circ \phi = g$ it follows that $\phi^* g^* \mathcal{O}_{\mathbb{P}^2}(1) = g^* \mathcal{O}_S(1)|_U$. In other words, $\phi^* \mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_U(1)$. Therefore $\phi^* \mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_U(1)$.

Since $\phi$ is dominant, $\phi : U \to \mathbb{P}^2$ is given by three linearly independent sections $\sigma_0, \sigma_1, \sigma_2$ of $\mathcal{O}_U(1)$, and these sections are the restriction to $U$ of sections $\tau_0, \tau_1, \tau_2$ of $\mathcal{O}_{\mathbb{P}^2}(1)$. The $\tau_i$ are linearly independent, and hence there is no point where they all vanish. So $\phi$ extends to the automorphism $(\tau_0, \tau_1, \tau_2)$ of $\mathbb{P}^2$. \qed

Since $\phi$ has order 3, its matrix is one of the following after being placed in Jordan form and scaling:

$\phi = \begin{bmatrix} 1 & \rho \\ \rho & \rho^2 \end{bmatrix}$

or $\phi = \begin{bmatrix} 1 & \rho \\ \rho \end{bmatrix}$

where $\rho^3 = 1$, $\rho \neq 1$. Fix an identification of $Y'$ with $\mathbb{P}^3$ so that we can write $g : \mathbb{P}^2 \to S \subset Y'$ as $(g_0, \ldots, g_3)$ for some degree 3 polynomials $g_i$. Then $(g_0, \ldots, g_3) = (\phi^* g_0, \ldots, \phi^* g_3)$. So each of the $g_i$ are eigenvectors of $\phi^* : H^0(\mathbb{P}^2, \mathcal{O}(3)) \to H^0(\mathbb{P}^2, \mathcal{O}(3))$ with the same eigenvalue.

Suppose

$\phi = \begin{bmatrix} 1 & \rho \\ \rho & \rho \end{bmatrix}$

Then the following are a basis for each of the eigenspaces of $\phi^*$:
Since $S$ is not a 2-plane in $\mathbb{P}^3$, the eigenspace containing the $g_i$ has dimension at least 4. So the $g_i$ all have eigenvalue 1.

Now consider the morphism

$$h = (x_0^3, x_1^3, x_2^3, x_1x_2, x_1^2x_2, x_0^2).$$

The image surface $S'$ of $h$ is a cone over a twisted cubic curve $T'$ with vertex $(1, 0, 0, 0)$. The morphism $g : \mathbb{P}^2 \to Y' \cong \mathbb{P}^3$ is the morphism $h$ followed by projection $\pi_p$ from some point $p$. Projection maps lines to lines, so the image $S$ of $g$ is a cone over some cubic plane curve $T$. The curve $T$ is singular because it is the image of a rational twisted cubic curve. Therefore, $S$ is singular along a line. We already saw that $S$ has only finitely many singularities. This contradiction rules out the first possibility for $\phi$.

Therefore

$$\phi = \begin{bmatrix} 1 \\ \rho \\ \rho^2 \end{bmatrix}.$$ 

So the following are bases for the eigenspaces of $\phi^*$:

1 : $x_0^3, x_1^3, x_2^3, x_0x_1x_2$

$\rho : x_0^2x_1, x_0x_2^2, x_1^2x_2$

$\rho^2 : x_0x_2, x_0x_1^2, x_1^2x_2$

The eigenspace containing the $g_i$ has dimension at least 4. So $g : \mathbb{P}^2 \to Y' = \mathbb{P}^3$ is the morphism $(x_0^3, x_1^3, x_2^3, x_0x_1x_2)$. Embed $Y'$ in $\mathbb{P}^4$ as $V(y_4)$ so that $g : \mathbb{P}^2 \to \mathbb{P}^4$ has the form claimed in Theorem 2.

3. Morphisms from Quintic to Cubic Threefolds

In Section 3.1 we discuss preimages of lines on $Y$ under the rational map $H \dashrightarrow Y$. Section 3.2 gives some information about $H$ in case it has a component that does not map dominantly onto $Y$. In Section 3.3, we consider the various possibilities for $H$ and rule them out case by case. When $H$ is integral, the results of Section 3.1 will be the main tool. When $H$ is more degenerate, Theorem 2 will play a central role.

3.1. Preimages of Lines on $Y$. First we will prove a basic fact about how the dualizing sheaf of a curve behaves under normalization. Then we will discuss the family of lines on $Y$ and what can be said about the preimage in $H$ of a general line in $Y$. The main result is Corollary 3.6.

Lemma 3.1. Let $C$ be a projective Gorenstein scheme of pure dimension 1 that is reduced at the generic point of one of its irreducible components $C'$. If $\nu : \tilde{C} \to C'$ is the normalization map of the reduced structure on $C'$ and $N$ is the number of points $p \in \tilde{C}$ such that the map of local rings $O_{C',\nu(p)} \to O_{\tilde{C},\bar{p}}$ fails to be an isomorphism, then

$$\deg \nu^* \omega_{\tilde{C}}^p \geq \deg \omega_{\tilde{C}} + N.$$
If $N = 0$, then equality holds.

Proof. Following [9 Ex. 3.7.2], we compute
\[
\omega_C \cong \text{Hom}_C(\nu_* \mathcal{O}_C, \omega_C^\sigma) \\
\cong \text{Hom}_C(\nu_* \mathcal{O}_C, \mathcal{O}_C) \otimes_{\mathcal{O}_C} \omega_C^\sigma
\]
as $\mathcal{O}_C$-modules. Since $\omega_C$ and $\omega_C^\sigma$ are invertible, $\text{Hom}_C(\nu_* \mathcal{O}_C, \mathcal{O}_C)$ is an invertible sheaf of $\mathcal{O}_C$-modules. So it is enough to show that $\text{Hom}_C(\nu_* \mathcal{O}_C, \mathcal{O}_C)$ is isomorphic to an ideal sheaf in $\mathcal{O}_C$ corresponding to a closed subscheme of $\tilde{C}$ supported at the points where $\nu : \tilde{C} \to C$ fails to be an isomorphism.

Consider the map of local rings
\[
A := \mathcal{O}_{C, \nu(p)} \to B := \mathcal{O}_{\tilde{C}, p}
\]
for some closed point $p \in \tilde{C}$. We will show that $\text{Hom}_A(B, A)$ is an ideal in $B$ and is the unit ideal if and only if $A \to B$ is an isomorphism.

Note that $B$ is the normalization of $A/P$ in its field of fractions, where $P \subset A$ is the prime ideal of $C' \subset C$. Consider the map of $B$-modules
\[
\Phi : \text{Hom}_A(B, A) \to B
\]
that sends $\psi \in \text{Hom}_A(B, A)$ to the equivalence class $\psi(1)$ of $\psi(1)$ in $A/P$, which injects into $B$.

If $\Phi(\psi) = 0$, then $\psi(1) \in P$, whence $\psi(B)$ is contained in the ideal $P \subset A$. In other words, $\psi$ is a local section of the $\mathcal{O}_C$-module $\text{Hom}_C(\nu^* \mathcal{O}_C, \mathcal{O}_C)$ defined in a neighborhood of $\nu(p)$ in $C$ such that $\psi$ vanishes on the reduced structure of $C'$. If we consider $\text{Hom}_C(\nu^* \mathcal{O}_C, \mathcal{O}_C)$ as an $\mathcal{O}_{C'}$-module, then $\psi$ is a local section defined on a neighborhood of $p$ in $\tilde{C}$, and $\psi$ vanishes at all but finitely many points in the neighborhood. Since $\text{Hom}_C(\nu^* \mathcal{O}_C, \mathcal{O}_C)$ is an invertible $\mathcal{O}_{C'}$-module, $\psi = 0$. Therefore $\Phi$ is an injection of $B$-modules, whence $\text{Hom}_A(B, A)$ is realized as an ideal in $B$. It remains to check that $A \to B$ is an isomorphism if and only if $\Phi$ is.

If $A \to B$ is an isomorphism, then $\Phi$ is clearly an isomorphism. Conversely, suppose $\Phi$ is an isomorphism. Then there is some $\psi \in \text{Hom}_A(B, A)$ such that $\Phi(\psi) = 1$, i.e. $\psi(1) - 1 \in P$. Therefore $\psi(1)$ is a unit in $A$ because $A$ is local. Hence $\Psi : \text{Hom}_A(B, A) \to A$ given by $\psi \mapsto \psi(1)$ is surjective because it is a morphism of $A$-modules. Note that $\Phi$ factors as $\Psi$ followed by $A \to B$. Since $\Psi$ is surjective and $\Phi$ is injective, we conclude that $A \to B$ is injective. And since $\Phi$ is surjective, $A \to B$ is surjective. \hfill $\Box$

Lemma 3.2. There are only finitely many closed points $p \in Y$ such that there are infinitely many lines on $Y$ through $p$.

Proof. Suppose not. Then there is a curve $C \subset Y$ such that for every point $p \in C$ there are infinitely many lines on $Y$ through $p$. So for every $p \in C$ there is an irreducible component $\Sigma_p$ of $Y \cap T_pY$ such that $\Sigma_p$ is a cone over a plane curve with vertex $p$.

Recall from [17] or [13] that the family $\Delta_1$ of lines on $Y$ is an irreducible surface, so a dimension count shows that every line on $Y$ lies on one of the surfaces $\Sigma_p$ for some $p \in C$. Also recall from [13] that a general line $L$ in $Y$ has normal bundle $N_{L/Y} = \mathcal{O}_L \oplus \mathcal{O}_L$.

Fix a general line $L$ on $Y$, lying on $\Sigma_p$. There is some 2-plane $\Lambda$ that is tangent to $\Sigma_p$ at every point of $L$ on account of $\Sigma_p$ being a cone. Therefore $T_q\Lambda \subset T_qY$ for
every \( q \in L \). It follows from Nakayama’s Lemma that these pointwise inclusions give an injection \( T_\Lambda|_L \to T_Y|_L \) of \( \mathcal{O}_L \)-modules. Note that \( T_\Lambda|_L = \mathcal{O}_L(1) \oplus \mathcal{O}_L(2) \), and consider the normal bundle sequence
\[
0 \to T_L \to T_Y|_L \overset{\phi}{\to} \mathcal{O}_L \oplus \mathcal{O}_L \to 0.
\]
By the above description of \( T_\Lambda|_L \), the composition
\[
T_\Lambda|_L \to T_Y|_L \overset{\phi}{\to} \mathcal{O}_L \oplus \mathcal{O}_L
\]
is the zero morphism. So \( \ker \phi \) has rank at least 2, contradicting \( \ker \phi = T_L \). \( \square \)

We will need the following modification of [13, Lemma 2.1].

**Lemma 3.3.** Let \( g : Z \to Y \) be a morphism from a purely 3 dimensional separated scheme of finite type over the ground field. If \( L \) is a general line on \( Y \), then \( g^{-1}(L) \) has pure dimension 1.

Moreover, if \( Z \) is integral and \( g \) is dominant, then \( g^{-1}(L) \) is singular at only finitely many points. In other words, \( g^{-1}(L) \) is reduced at the generic point of each irreducible component.

**Proof.** Let \( \mathcal{F} \) be the total space of the family of lines on \( Y \) with base \( \Delta \). From [13], \( \Delta \) is a smooth surface, and there is some open subscheme \( \Delta_0 \in \Delta \) with preimage \( F_0 \in \mathcal{F} \) such that \( F_0 \) is a locally trivial fiber bundle over \( \Delta_0 \), whose fibers are lines in \( Y \).

Let \( Z_0 \) be the union of the surfaces in \( Z \) that are mapped to points in \( Y \), let \( Z_1 \) be the union of the curves on \( Z \) that are mapped to points in \( Y \), and let \( Z_2 \) be the set of points in \( Z \) at which \( g \) fails to induce an injection on tangent spaces. Note that \( Z_2 \) contains the singular locus of \( Z \).

Note that \( g(Z_0) \) has dimension at most 1, and \( g(Z_1) \) has dimension at most 2. Since the canonical morphism \( F_0 \to Y \) is dominant, a general line will meet \( g(Z_1) \) in only finitely many points. Also, if a general line meets \( g(Z_0) \), then \( g(Z_0) \) is a curve such that for every point \( p \in g(Z_0) \) there are infinitely many lines on \( Y \) through \( p \). This contradicts Lemma 3.2. So a general line does not meet \( g(Z_0) \). Therefore, \( g^{-1}(L) \) has pure dimension 1 for \( L \) a general line on \( Y \).

Now suppose that \( Z \) is integral and \( g \) is dominant. Then \( \dim g(Z_1) \leq i \) for \( i = 0, 1, 2 \). We use the characteristic zero assumption of this section to get \( \dim g(Z_2) \leq 2 \). So a general line will meet \( g(Z_2) \) in only finitely many points, and will not meet \( g(Z_0) \) and \( g(Z_1) \) at all. Hence \( g^{-1}(L) \) is nonsingular away from the preimage of \( L \cap g(Z_2) \), which consists of only finitely many points. \( \square \)

**Definition.** If \( Z \) is any scheme and \( F : Z \to \mathbb{P}^n \) is a rational map given by sections \( F_0, \ldots, F_n \) of some line bundle on \( Z \), then let \( \operatorname{indet}(F) \) denote the scheme of common vanishing of the \( F_i \) in \( Z \):
\[
\operatorname{indet}(F) := V(F_0, \ldots, F_n) \subset Z.
\]

**Lemma 3.4.** Take \( F \) as in the definition, and let \( \pi : \tilde{Z} \to Z \) be the blowup of \( Z \) in \( \operatorname{indet}(F) \). Then there is a canonical morphism \( \tilde{F} : \tilde{Z} \to \mathbb{P}^n \) such that \( \tilde{F} = F \circ \pi \) as rational maps. Moreover, if \( p \in \operatorname{indet}(F) \) is a closed point, then \( \tilde{F} \) induces a closed immersion of \( \pi^{-1}(p) \) into \( \mathbb{P}^n \).
Proof. Recall that \( \tilde{Z} \) is isomorphic to the closure of the graph of \( F, \Gamma_F \subset Z \times \mathbb{P}^n \). Thus projection onto \( \mathbb{P}^n \) induces the desired morphism \( \tilde{F} : \tilde{Z} \to \mathbb{P}^n \). If \( p \in \text{indet}(F) \) is a closed point, then \( \pi^{-1}(p) \) is a closed subscheme of \( p \times \mathbb{P}^n \), which maps isomorphically onto \( \mathbb{P}^n \) by projection.

Lemma 3.5. If \( p \in \text{indet}(F) \) is a closed point, then \( p \in H \) is a point of order at least 3.

Proof. Let \( F = (F_0, \ldots, F_4) \), and let \( Y \) have homogeneous equation \( G \). Since the \( F_i \) vanish at \( p \) and \( G \) has degree 3, \( G(F_0, \ldots, F_4) \) vanishes to order at least 3 at \( p \). In other words, \( p \in F^{-1}(Y) \) is a point of order at least 3. But \( p \) is contained in \( H \) and not \( X \). The lemma follows.

Lemma 3.6. With \( C \) as above, \( \omega_C^0 = \mathcal{O}_C(-1) \).

Proof. Since \( F^{-1}(L) \) is the complete intersection of three cubic hypersurfaces in \( \mathbb{P}^4 \),
\[
\omega_{F^{-1}(L)}^0 = \mathcal{O}_{F^{-1}(L)}(-5 + 3 + 3) = \mathcal{O}_{F^{-1}(L)}(4).
\]

Following [9, Ex. 3.7.2], compute
\[
\omega_C^0 = \text{Hom}_{F^{-1}(L)}(\mathcal{O}_C, \omega_{F^{-1}(L)}^0) = \text{Hom}_{F^{-1}(L)}(\mathcal{O}_C, \mathcal{O}_{F^{-1}(L)}) \otimes \mathcal{O}_{F^{-1}(L)}(4).
\]
Therefore, if \( I_C, I_D \subset \mathcal{O}_{F^{-1}(L)} \) denote the ideals of \( C, D \subset F^{-1}(L) \), then it suffices to carry out the following computation
\[
\text{Hom}_{F^{-1}(L)}(\mathcal{O}_C, \mathcal{O}_{F^{-1}(L)}) \cong (0 : I_C)
\]
\[
\cong I_D
\]
\[
\cong \mathcal{O}_C(-C \cap D)
\]
\[
\cong \mathcal{O}_C(-5).
\]
The isomorphism (3.1) is given by \( \psi \mapsto \psi(1) \).

To see (3.2), note that \( D \) is Cohen-Macaulay since it is a local complete intersection in \( X \) and \( X \) is C.M., as \( L \) is a l.c.i. in \( Y \). It follows that \( C \) and \( D \) are linked because \( F^{-1}(L) \) is Gorenstein, cf. [5, Theorem 21.23].

To see (3.3), we will show that
\[
I_D \to I_D + I_C
\]

\[
\cong \frac{I_C}{I_C}
\]
is an isomorphism of $\mathcal{O}_C$-modules, i.e. $I_D \cap I_C = 0$. So let $a$ be a local section of $I_D \cap I_C$, which is necessarily supported on $C \cap D$. We will show $a = 0$.

Let $A = \mathcal{O}_{F^{-1}(L), p}$ for some closed point $p \in C \cap D$ with maximal ideal $m$. Let $b$ denote the image of $a$ in $A$. Some power of $m$ annihilates $b$ because $b$ is supported at $p$. Therefore $m$ is the annihilator of some nonzero multiple of $b$, provided $b \neq 0$. So $m$ is an associated prime ideal of $A$. But $A$ is C.M. and one dimensional, so every associated prime is minimal. Hence $b = 0$ after all. Therefore $a = 0$ because its localization is zero at every point.

The isomorphism (3.4) follows from $C \cap D = C \cap X$. □

**Corollary 3.7.** If $C := F^{-1}(L) \cap H$ is reduced at the generic point of an irreducible component $C'$, then $C'$ is either a smooth plane conic disjoint from the rest of $C$, or $C'$ is a line meeting the rest of $C$ at only one point.

**Proof.** This is immediate from Lemma 3.1 and Lemma 3.5 because the degree of the dualizing sheaf of a smooth curve is at least $-2$. □

### 3.2. A Multiplicity Result.

The following result holds in arbitrary characteristic and will be used in the next subsection.

**Lemma 3.8.** Let $Z$ be the reduced structure on an irreducible component of $H$. Assume $m \leq d$. If $F|x : Z \rightarrow Y$ is not dominant, then $2Z \subset H$ as divisors in $\mathbb{P}^n$.

**Proof.** Suppose $F$ does not map $Z$ dominantly onto $Y$. Then $Z$ is covered by curves that are mapped to points under $F$. So suppose that $C \subset Z$ is an integral curve with $F(C) = (1, 0, \ldots, 0)$ for simplicity, and let $I \subset \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of $C$. If $F = (F_0, \ldots, F_n)$, then $F_i \in H^0(\mathbb{P}^n, I(m))$ for $i > 0$. We show that $H$ is singular along $C$.

Let $K \in H^0(\mathbb{P}^n, I(em - d))$ be the homogeneous equation for $H$. Let $I^{(2)} \subset \mathcal{O}_{\mathbb{P}^n}$ be the ideal such that $I/I^{(2)} = (I/I^2)/\text{torsion}$. Then $K$ induces a section $\overline{K} \in H^0(\mathbb{P}^n, I/I^{(2)}(em - d))$. We will show $\overline{K} = 0$.

If $G = G(y_0, \ldots, y_n)$ is the homogeneous equation of $Y$, then we can write $G = y_0^{-1}G_1 + \cdots + y_0G_{e-1} + G_e$ where the $G_i = G_i(y_1, \ldots, y_n)$ are homogeneous of degree $i$. So the homogeneous equation of $F^{-1}(Y) = X + H$ is

$$F^*G = F_0^{-1}F^*G_1 + F_0^{-2}F^*G_2 + \cdots$$

where $F^*G = G(F_0, \ldots, F_n)$.

Let $D$ be the Cartier divisor $V(F_0^{-1}) \cap C$ on $C$. Note that $D$ is disjoint from $X$ because $D$ is supported on $C \cap V(F_0, \ldots, F_n)$ and $V(F_0, \ldots, F_n)$ is disjoint from $X$. So from (3.5) it follows that $\overline{K}$ restricts to the zero section on $D$ because $F_0^{-1}F^*G_i$ restricts to zero on $C$ for $i \geq 2$, and $F_0^{-1}$ restricts to zero on $D$.

Tensor the exact sequence

$$0 \rightarrow \mathcal{O}_C(-D) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_D \rightarrow 0$$

with $I/I^{(2)}(em - d)$ and use $\mathcal{O}_C(-D) = \mathcal{O}_C(-m(e - 1))$ to obtain

$$I/I^{(2)}(m - d) \rightarrow I/I^{(2)}(em - d) \rightarrow I/I^{(2)} \otimes \mathcal{O}_D(em - d) \rightarrow 0$$

To see that $\tau$ is an injection, note that $\tau$ is multiplication by $F_0^{-1}$, which is a unit in the local ring of almost all the points of $C$. Therefore $\tau$ could only have torsion elements in its kernel. But $I/I^{(2)}(em - d)$ is a torsion-free sheaf. So $\tau$ is injective.
Since $\rho(\mathcal{K}) = 0$, $\mathcal{K}$ is the image of a section

$$\mathcal{K} \in H^0(C, I/I^{(2)}(m-d)).$$

From the conormal sequence of $C$ in $\mathbb{P}^n$, we get a morphism $I/I^{(2)}(m-d) \to \Omega_{\mathbb{P}^n|C}^1(m-d)$ that is an injection on the regular locus of $C$, and is therefore an injection because $I/I^{(2)}(m-d)$ is torsion-free.

The sheaf $\Omega_{\mathbb{P}^n|C}^1(m-d)$ has no nonzero global sections because of the injection $\Omega_{\mathbb{P}^n|C}^1(m-d) \to \mathcal{O}_{\mathbb{P}^n}(m-d-1)^{\otimes n+1}$ from the Euler sequence. Therefore, $I/I^{(2)}(m-d)$ has no nonzero global sections. So $\mathcal{K} = 0$, whence $\mathcal{K} = 0$. Therefore $K \in H^0(\mathbb{P}^n, I^{(2)})$, which implies $H$ is singular along $C$. Since $Z$ is covered by such curves, $H$ is singular at every point of $Z$. This is only possible if $2Z \subset H$ as divisors.

3.3. Morphisms from Quintic to Cubic Threefolds. Recall the decomposition $F^{-1}(Y) = X + H$ from Section 1. We will consider the various possibilities for $H$ and rule them out one at a time, thus proving Theorem 1.

**Proposition 3.9.** $H$ does not contain a hyperplane in $\mathbb{P}^4$ that maps dominantly onto $Y$.

**Proof.** Suppose $Z \subset H$ is a hyperplane that maps dominantly onto $Y$. If we restrict $F$ to any $\mathbb{P}^2$ contained in $Z$, then $F|_{\mathbb{P}^2}$ is described by Theorem 2. In particular, $F(\mathbb{P}^2)$ is the intersection of $Y$ with a hyperplane tangent to $Y$ at 3 points. The family of 2-planes contained in $Z$ and the family of tangent planes to $Y$ both have dimension 3, so a general tangent plane to $Y$ is tangent to $Y$ at 3 points. However, since $Y$ is a nonsingular hypersurface, a general tangent plane to $Y$ is tangent at only one point, cf. [4, Lemma 5.15].

**Corollary 3.10.** If $K$ is a hyperplane contained in $H$, then $2K \subset H$, and we can choose coordinates $x_0, \ldots, x_4$ on the $\mathbb{P}^4$ containing $K$ such that $K = V(x_4)$ and $F|_K$ is given by the formula

$$F|_K = (x_0^3, x_1^3, x_2^3, x_0 x_1 x_2, 0)$$

**Proof.** By Lemma 3.7 and Proposition 3.8, $2K \subset H$.

Since $F|_K$ is not dominant and $K \cong \mathbb{P}^3$, $F|_K$ can not be a morphism. Choose a point $p$ in the indeterminacy locus of $F|_K$. Let $\pi : \mathcal{K} \to K$ be the blowup of $K$ at the indeterminacy scheme $\text{indet}(F|_K)$, so that the rational map $F|_K$ extends to a morphism $\Phi : \mathcal{K} \to S$, where $S$ is the image surface of $K$ under $F$. By Lemma 3.4, $\pi^{-1}(p)$ maps isomorphically onto $S$ under the morphism $\Phi$. So for $s \in S$ a general point, $\Phi^{-1}(s)$ is a curve in $\mathcal{K}$ that meets $\pi^{-1}(p)$. Therefore the preimage of $s$ under $F|_K$ is a curve in $K$ through $p$.

If $\Lambda$ is any 2-plane in $K$ disjoint from $\text{indet}(F|_K)$, then $F|_\Lambda$ is described by Theorem 2. In particular, for $s \in S$ any nonsingular point, the preimage of $s$ in $\Lambda$ consists of three reduced points. Therefore the preimage of $s$ in $K$ consists of three distinct lines that are reduced away from $\text{indet}(F|_K)$. Indeed, if the preimage of $s$ in $K$ contained a curve other than a line, we could choose $\Lambda$ such that $F|_\Lambda^{-1}(s)$ was not 3 reduced points by choosing $\Lambda$ to be tangent to $F|_\Lambda^{-1}(s)$ at a point but not contain any component of $F|_\Lambda^{-1}(s)$.

So there is a two parameter family of lines in $K$ that are mapped by $F$ to points in $S$. Every such line meets $\text{indet}(F|_K)$, which consists of finitely many points. So
there is some point \( p \in \text{indet}(F|_K) \) such that there is a two parameter family of lines in \( K \) through \( p \) that are each mapped to a point in \( S \) under \( F \).

A line \( L \) is mapped to a point by \( F \) exactly when the scheme \( L \cap \text{indet}(F) \) has length 3 because \( F^\ast \mathcal{O}(1) = \mathcal{O}(3) \). Since a general line in \( \mathbb{P}^4 \) through \( p \) meets \( \text{indet}(F) \) in a scheme of length 3, the same holds for every line through \( p \). So every line through \( p \) is mapped to a point. Therefore, \( F|_K \) is determined by \( F|_\Lambda \) for any 2-plane \( \Lambda \) in \( K \) not containing \( p \). Theorem 2 determines \( F|_\Lambda \). If one takes \( p = V(x_0, x_1, x_2) \) and \( \Lambda = V(x_3) \), where \( x_0, \ldots, x_3 \) are homogeneous coordinates on \( K \), then \( F|_K \) has the desired form. \( \square \)

**Proposition 3.11.** \( H \) is not equal to \( Q + 2K \) for an integral quadric \( Q \) and a hyperplane \( K \).

**Proof.** Suppose \( H = Q + 2K \). By Corollary 3.9, \( F|_K \) factors through projection from some point \( p \in K \). By Lemma 3.5, \( H \) has order at least 3 at every point in \( \text{indet}(F) \). So every point in \( \text{indet}(F) \) is in \( K \cap Q \). Hence, \( p \) is the only point in \( \text{indet}(F) \) because it is the only point on \( K \) where \( F \) is undefined.

The map \( F|_Q \) is dominant by Lemma 3.7. Let \( \pi : 
\tilde{Q} \to Q \) be the blowup of \( Q \) at the indeterminacy scheme \( \text{indet}(F|_Q) \), and let \( q : 
\tilde{Q} \to Y \) be the resulting morphism extending \( F|_Q \). Apply Lemma 3.3 to \( q \) to see that for \( L \) a general line, \( q^{-1}(L) \) is reduced at the generic point of each of its irreducible components, whence the same is true for \( C_Q := F^{-1}(L) \cap Q \).

By Lemma 3.4, \( q \) is a closed immersion when restricted to \( \pi^{-1}(p) \). Since \( q(\pi^{-1}(p)) \) is an effective divisor on \( Y \), it is ample by the Grothendieck-Lefschetz Theorem, [N Theorem 4.3.2]. So every line in \( Y \) meets \( q(\pi^{-1}(p)) \). Therefore, \( C_Q \) has an irreducible component \( C_1 \) containing the point \( p \). By the same argument, \( F^{-1}(L) \cap K \) has an irreducible component \( C_2 \) that contains \( p \). Since \( L \) is a general line on \( Y \), which is covered by lines, \( L \) is not contained in the image of \( K \cap Q \). So \( C_1 \neq C_2 \). By Lemma 3.6, \( \omega^2_C = \mathcal{O}_C(-1) \). Since \( C \) has more than one irreducible component that contains \( p \), Lemma 3.1 shows that every irreducible component of \( C_Q \) that contains \( p \) is a line because the dualizing sheaf of every smooth curve is at least \(-2\). So \( Q \) is covered by lines through \( p \), and these lines are parametrized by a general hyperplane section \( Q \cap \mathbb{P}^3 \) of \( Q \). Because the line \( L \subset Y \) is general, \( \dim Q \cap \mathbb{P}^3 = 2 \), and \( \dim \Delta = 2 \), where \( \Delta \) is the space of lines on \( Y \), we conclude that every general line on \( Q \) through \( p \) maps to a line on \( Y \). Hence the rational map \( F|_Q : Q \dashrightarrow Y \) induces a rational map \( Q \cap \mathbb{P}^3 \dashrightarrow \Delta \). However, \( \Delta \) does not contain a rational curve by Proposition 2.1. This contradiction finishes the proof. \( \square \)

**Lemma 3.12.** If \( L \) is a line in \( \mathbb{P}^4 \) such that \( L \cap \text{indet}(F) \) is nonempty and is not a single reduced point, then \( L \) is contained in \( H \).

**Proof.** Suppose \( L \) intersects \( \text{indet}(F) \). Then \( L \cap \text{indet}(F) \) is zero dimensional, so its structure sheaf has finite dimension \( \lambda \) over the ground field \( k \). So the rational map \( F|_L \) is given by \( \mathcal{O}_{\mathbb{P}^4}(3-\lambda) \) after \( F|_L \) is extended over the points of indeterminacy. The intersection \( L \cap \text{indet}(F) \) is a single reduced point exactly when \( \lambda = 1 \).

If \( \lambda = 2 \), then \( F \) maps \( L \) isomorphically onto a line. This is impossible if \( L \) is not contained in \( H \) because \( L \) would meet \( X \) in a scheme of length 5 while \( F(L) \) would meet \( Y \) in a scheme of length 3. If \( \lambda = 3 \), then \( F(L) \) is a point in \( Y \), so \( L \subset H \). This completes the proof because \( \lambda \leq 3 \). \( \square \)

**Proposition 3.13.** \( H \) is not \( 2K_1 + 2K_2 \) for distinct hyperplanes \( K_1 \) and \( K_2 \).
Proof. Suppose $H = 2K_1 + 2K_2$. According to the formula of Corollary 3.10, there are points $p_i \in K_i$ such that $F|_{K_i}$ factors through projection from $p_i$, and the tangent space to the indeterminacy scheme $\text{indet}(F|_{K_i})$ at $p_i$ is equal to the tangent space of $K_i$ at $p_i$.

We claim $p_1 \neq p_2$. Indeed, if $p = p_1 = p_2$, then $T_pK_1 \neq T_pK_2$ because the $K_i$ are distinct hyperplanes. So $\text{indet}(F)$ would have a four dimensional tangent space at $p$. But now every line in $\mathbb{P}^4$ through $p$ meets $\text{indet}(F)$ in a scheme that is nonreduced at $p$. So Lemma 3.12 implies that every line in $\mathbb{P}^4$ through $p$ is contained in $H$. This is impossible because $H$ is not all of $\mathbb{P}^4$, so $p_1 \neq p_2$.

Let $L$ be the line containing $p_1, p_2$. By Lemma 3.5, $H$ has order at least 3 at both $p_i$, so $p_1, p_2 \in K_1 \cap K_2$. Therefore, $L \subset K_1 \cap K_2$, so that $L$ meets $\text{indet}(F)$ in a scheme of length 6, which is impossible because $L \cap \text{indet}(F)$ can have length at most 3. \hfill \Box

**Proposition 3.14.** $H$ does not contain a hyperplane.

**Proof.** The only case left to rule out is $H = 4K$. So suppose $H = 4K$. We will lift the polynomials that give $F|_K$ from $K$ to its second infinitesimal neighborhood $2K$ and then derive a contradiction.

Using Corollary 3.10, we choose homogeneous coordinates $x_0, \ldots, x_4$ on $\mathbb{P}^4$ such that $K = V(x_4)$ and

\begin{equation}
F = (x_0^3 + x_4 q_0, x_1^3 + x_4 q_1, x_2^3 + x_4 q_2, x_0 x_1 x_2 + x_4 q_3, x_4 q_4)
\end{equation}

for some homogeneous polynomials $q_i$ of degree 2 in the $x_i$. Let $y_0, \ldots, y_4$ be homogeneous coordinates on the target $\mathbb{P}^4$. Since $Y \cap Y'$ is the image of $K = V(x_4)$ we see $Y' = V(y_4)$, and $Y \cap Y' = V(y_4, y_0 y_1 y_2 - y_3^2)$. So $Y$ is given by the equation

$$G := y_0 y_1 y_2 - y_3^2 + y_4 G_2$$

for some homogeneous $G_2$ of degree 2 in the $y_i$.

Since $2K \subset H$, $x_4^2$ divides the pullback

$$F^*G = (x_0^3 + x_4 q_0)(x_1^3 + x_4 q_1)(x_2^3 + x_4 q_2) - (x_0 x_1 x_2 + x_4 q_3)^3 + x_4 q_4 F^*G_2$$

$$= x_4 (q_0 x_1^3 x_2^2 + q_1 x_0^3 x_2^2 + q_2 x_0^3 x_1^3 - 3 q_3 x_0^2 x_1^2 x_2^2 + q_4 F^*G_2)$$

$$+ x_4^2 \text{(other terms)}$$

Therefore $x_4$ divides

\begin{equation}
q_0 x_1^3 x_2^2 + q_1 x_0^3 x_2^2 + q_2 x_0^3 x_1^3 - 3 q_3 x_0^2 x_1^2 x_2^2 + q_4 F^*G_2
\end{equation}

Using the notation $\partial_i := \frac{\partial}{\partial y_i}$, the partial derivatives of $G$ are

$$\partial_0 G = y_1 y_2 + y_4 \partial_0 G_2$$
$$\partial_1 G = y_0 y_2 + y_4 \partial_1 G_2$$
$$\partial_2 G = y_0 y_1 + y_4 \partial_2 G_2$$
$$\partial_3 G = -3 y_3^2 + y_4 \partial_3 G_2$$
$$\partial_4 G = G_2 + y_4 \partial_4 G_2$$

It follows that $V(y_0 y_1, y_0 y_2, y_1 y_2, y_3, y_4, G_2)$ is empty because it is contained in the singular locus of $Y$. Hence $V(G_2)$ does not contain the point $(1 : 0 : 0 : 0 : 0)$.\end{document}
In other words, \( y_0^2 \) appears with nonzero coefficient in \( G_2 \). Therefore by equation (3.6), \( x_0^6 \) has nonzero coefficient in \( F^*G_2 \).

Moreover, \( q_iF^*G_2 \) is the only term in formula (3.7) in which \( x_0^6 \) can occur because the \( q_i \) have degree 2. Since (3.7) is zero (mod \( x_4 \)), the \( x_0^6 \) term in (3.7) disappears when considered (mod \( x_4 \)). This can only happen if \( x_4 \) divides \( q_4 \). Further consideration of (3.7) shows that there are scalars \( a_0, a_1, a_2 \) such that the following equations hold (mod \( x_4 \)):

\[
\begin{align*}
q_0 &\equiv a_0x_0^2 \\
q_1 &\equiv a_1x_1^2 \\
q_2 &\equiv a_2x_2^2 \\
q_3 &\equiv \frac{1}{3}(a_0x_1x_2 + a_1x_0x_2 + a_2x_0x_1) \\
q_4 &\equiv 0
\end{align*}
\]

Therefore, equation (3.6) yields

\[
\begin{align*}
F_0 &= x_0^3 + a_0x_4x_0^2 + x_4^3h_0 \\
F_1 &= x_1^3 + a_1x_4x_1^2 + x_4^3h_1 \\
F_2 &= x_2^3 + a_2x_4x_2^2 + x_4^3h_2 \\
F_3 &= x_0x_1x_2 + \frac{1}{3}x_4(a_0x_1x_2 + a_1x_0x_2 + a_2x_0x_1) + x_4^3h_3 \\
F_4 &= x_4^3h_4
\end{align*}
\]

for some homogeneous linear polynomials \( h_i \) in \( x_0, \ldots, x_4 \).

As \( H = 4K \) is contained in \( F^{-1}(Y) \), \( x_4 \) divides

\[
F^*G = F_0F_1F_2 - F_3^3 + F_4F^*G_2.
\]

Since \( F^*G \) has degree 9, the monomial \( x_0^6x_4^3 \) can not appear with nonzero coefficient in \( F^*G \). On the other hand, the monomial \( x_0^6 \) does have nonzero coefficient in \( F^*G_2 \). So \( x_0^6 \) must appear in a monomial with nonzero coefficient in

\[
F^*G - F_4F^*G_2 = F_0F_1F_2 - F_3^3.
\]

However, by the formulas for \( F_0, \ldots, F_3 \) given above, the highest power of \( x_0 \) that can appear is \( x_0^5 \). This contradiction finishes the proof.

**Lemma 3.15.** Every point \( p \in \text{indet}(F) \) is a point of order 3 on \( H \).

**Proof.** Let \( T_p \) denote the tangent space of the indeterminacy scheme \( \text{indet}(F) \) at some closed point \( p \in \text{indet}(F) \). If \( L \) is a line in \( \mathbb{P}^4 \) tangent to \( \text{indet}(F) \) at \( p \), then \( L \) is contained in \( H \) by Lemma 3.12. Proposition 3.14 asserts that \( H \) does not contain a hyperplane, so \( T_p \) can have dimension at most 2.

By Lemma 3.5, \( H \) has order at least 3 at \( p \). So we only have to rule out the case \( \text{ord}_p \) \( H \) = 4.

Suppose \( p \in \text{indet}(F) \) is a point of order 4 in \( H \), so that \( H \) is a cone over \( p \). Let \( \pi: \mathbb{P}^4 \to \mathbb{P}^4 \) be the blowup of \( \mathbb{P}^4 \) at \( \text{indet}(F) \), and let \( \bar{F} : \mathbb{P}^4 \to Y \) be the resulting morphism that extends \( F \). Let \( E_p = \pi^{-1}(p) \) be the preimage of the reduced point \( p \). Since the dimension of \( T_p \) is at most 2, if \( l_1 \) and \( l_2 \) are two general lines in \( \mathbb{P}^4 \) through \( p \), then the scheme \( l_1 \cup l_2 \), which is contained in a 2-plane, meets \( \text{indet}(F) \) in the reduced point \( p \). So the strict transform of \( l_1 \cup l_2 \) is the blowup of \( l_1 \cup l_2 \) at
Proof. By [14] Proposition 2.2(2)], Y is not the image of a morphism from a hypersurface in $\mathbb{P}^4$ of degree 2, so $F|_Q$ is not a morphism. Let $p \in \text{indet}(F)$ be a closed point. If $H = 2Q$, then every closed point in $Q$ has order 2 or 4 in $H$. By Lemma 3.5, $p$ is a point on $H$ of order at least 3. So $p$ has order 4, which is impossible by Lemma 3.15. We conclude $H \neq 2Q$. □

Proposition 3.17. $H$ is not $Q_1 + Q_2$ for distinct integral quadrics $Q_1$ and $Q_2$.

Proof. Suppose $H = Q_1 + Q_2$. By Lemma 3.8, the $Q_i$ map dominantly onto $Y$. Therefore, by Lemma 3.3 applied to the blowups of $Q_1$ and $Q_2$ at the indeterminacy schemes of $F|_{Q_1}$ and $F|_{Q_2}$, for $L$ a general line in $Y$, $C := F^{-1}(L) \cap H$ is reduced at the generic point of each of its irreducible components.

By Lemma [14] Proposition 2.2(2)], $F|_{Q_1}$ can not be a morphism because $\text{deg}Y > 2$. Pick $p \in \text{indet}(F|_{Q_1})$. By Lemma 3.15, $p$ is a point of order 3 in $H$, so $Q_2$ contains $p$. I claim that $C$ has an irreducible component contained in each of the $Q_i$ through $p$. Indeed, if $\pi_i : \tilde{Q}_i \to Q_i$ is the blowup at $\text{indet}(F|_{Q_i})$, and $q_i : \tilde{Q}_i \to Y$ is the resulting morphism, then a general line $L$ in $Y$ will meet $q_i(\pi_i^{-1}(p))$. So the preimage of $L$ in $Q_i$ passes through $p$.

Therefore, by Corollary 3.7, the connected component of $C$ containing $p$ is the union of two lines through $p$, one line contained in $Q_1$ and the other in $Q_2$. But now each of the $Q_i$ contain a two parameter family of lines through $p$ because $Y$ is covered by a two parameter family of lines. So both the $Q_i$ are cones over $p$, so that $p$ is a point of order 4 in $H$. This contradicts Lemma 3.15. □
The following Lemma will be needed to analyze the preimage in \( H \) of a general line in \( Y \) in the case where \( H \) is integral.

**Lemma 3.18.** Let \( \pi : F \to B \) be a projective morphism with \( B \) integral. Suppose \( \sigma : B \to F \) is a section of \( \pi \), and for each \( b \in B \) the connected component \( F_b \) of \( \pi^{-1}(b) \) that contains \( \sigma(b) \) is irreducible. Then \( \bigcup_b F_b \) is an irreducible component of \( F \).

**Proof.** Let \( B \xrightarrow{\pi} B' \xrightarrow{\sigma} B \) be the Stein factorization of \( \pi \). So \( \pi' \) has connected fibers, and \( g \) is finite. Then \( \pi' \circ \sigma \) is a section of \( g \), and \( (\pi')^{-1}(\pi' \circ \sigma)(b) = F_b \). Notice that \( (\pi' \circ \sigma)(B) \) is an irreducible component of \( B' \) because they have the same dimension. Since \( (\pi')^{-1}(\pi' \circ \sigma(B)) \) has irreducible fibers \( F_b \) and \( B \) is irreducible, \( (\pi')^{-1}(\pi' \circ \sigma(B)) = \bigcup_b F_b \) is irreducible. \( \square \)

The only case left to rule out is when \( H \) is integral.

**Theorem 3.19.** \( H \) is not integral.

**Proof.** Suppose \( H \) is integral. By Lemma 3.8, \( F|_H \) is dominant because \( H \) has multiplicity 1 in \( F^{-1}(Y) \). However, \( F|_H \) cannot be a morphism, because if it were then Table 1 in the Appendix would imply that the polynomial degree \( m \) of \( F \) would be at most 2, not 3. Choose \( p \in \text{indet}(F) \) a reduced point.

Let \( L \) be a general line in \( Y \), cut out by the linear forms \( \xi_1, \xi_2, \xi_3 \) in \( \mathbb{P}^4 \). Define

\[
F^{-1}(L) := V(F^*\xi_1, F^*\xi_2, F^*\xi_3) \\
C(L) := F^{-1}(L) \cap H \\
D(L) := F^{-1}(L) \cap X
\]

By Lemma 3.3, \( C = C(L) \) and \( D = D(L) \) are reduced at the generic point of each of their irreducible components, and

Since the linear forms \( F^*\xi_i \) vanish on the indeterminacy scheme \( \text{indet}(F) := V(F_0, \ldots, F_4) \), \( \text{indet}(F) \) is contained in \( F^{-1}(L) = C \cup D \). Therefore, \( \text{indet}(F) \) is contained in \( C \) because \( \text{indet}(F) \) does not intersect \( D \), as \( D \) is contained in \( X \). By Corollary 3.7, the connected component of \( C \) that contains \( p \) is either a smooth quadric curve or the union of two lines meeting at some point. According to Lemma 3.15, \( p \) is triple point on \( H \), so that \( H \) is not a cone over \( p \). By Corollary 3.7, the connected component of \( C \) that contains \( p \) is a plane conic. Since \( H \) is not a cone over \( p \), \( Y \) has a two dimensional family of lines, and \( L \) is a general line on \( Y \), the connected component of \( C \) that contains \( p \) is a smooth plane conic.

Let \( B \) be an open subscheme of the space of lines in \( Y \) such that for every \( L \in B \), every component of \( C(L) \) that meets \( \text{indet}(F) \) is a smooth plane conic. For \( L \in B \), \( F^{-1}(L) \) has degree 27, and \( D(L) \) has degree 15. So \( C(L) \) has degree 12 and is the disjoint union of 6 reduced plane conics.

Choose a general \( L_0 \in B \), and let \( C_1(L_0), \ldots, C_6(L_0) \) be the connected components of \( C(L_0) \). Let \( \zeta_i := C_i(L_0) \cap \text{indet}(F) \), and let \( \lambda(\zeta_i) \) denote the length of the zero dimensional scheme \( \zeta_i \). The polynomial degree of \( F \) is \( m = 3 \), \( C_i(L_0) \) has degree 2, and \( L_0 \) has degree 1. So the restriction \( C_i(L_0) \to L_0 \) of \( F \) to \( C_i(L_0) \) has degree 6 - \( \lambda(\zeta_i) \). Therefore \( \deg F|_H \) is the sum of the \( \deg F|_{C_i(L_0)} \):

\[
\deg F|_H = \sum_{i=1}^{6} 6 - \lambda(\zeta_i).
\]
Now number the $C_i(L_0)$ so that $\lambda(\zeta_1) \geq \cdots \geq \lambda(\zeta_6)$, and let $p$ be a closed point of $\zeta_1$. Let $F \subset B \times H$ be the total space of the family $\pi : F \to B$ whose fiber over $L \in B$ is $C(L)$. The closed subscheme $B \times p \subset F$ is a section of $\pi$. So

$$F_1 := \bigcup_{L \in B} C_1(L)$$

is an irreducible component of $F$ by Lemma 3.18.

Notice that there is a morphism $F \to B_H$, where $B_H$ is the space of quadric plane curves in $H$ that contain $p$, given by sending a point $(L, x) \in F \subset B \times H$ to the irreducible component of $C(L)$ that contains $x$. The fibers of $F \to B_H$ are one dimensional and $\dim F = 3$. So the image in $B_H$ of the intersections of the various irreducible components of $F$ has dimension at most 1. Hence there is at most a one dimensional space of lines in $Y$ such that $C(L)$ has a component corresponding to a point in $B_H$ whose fiber in $F$ lies in more than one irreducible component. So by generality of $L_0 \in B$, $F_1$ is the only irreducible component of $F$ that has nonempty fiber over the point $C_1(L_0) \subset B_H$. Therefore, the irreducible component $F_1$ of $F$ did not depend on the choice of $p \in \zeta_1$ because for any $p \in \zeta_1$ the connected component of $C(L_0)$ that contains $p$ is $C_1(L_0)$.

Since $F_1$ did not depend on the choice of $p \in \zeta_1$, for every line $L \in B$, the component $C_1(L) := \pi^{-1}(|L|) \cap F_1$ of $C(L)$ that is contained in $F_1$ has the property that $C_1(L) \cap \text{indet}(F) = \zeta_1$. This is because for every $q \in \zeta_1$, $B \times q$ is contained in $F_1 \subset B \times H$.

Consider the composite morphism $\phi : F_1 \to H \rightarrow Y$. Let $y \in Y$ be a general point. There are six lines $L_1, \ldots, L_6$ on $Y$ through $y$, and $\phi^{-1}(L_i) = C_1(L_i)$. By counting preimage points of $y$ we see that $\deg \phi$ is the sum of the degrees of the $C_1(L_i) \rightarrow L_i$:

$$\deg \phi = \sum_{i=6}^{0} 6 - \lambda(\zeta_1).$$

Since $\deg \phi \geq \deg F_{|H}$, equations (3.8) and (3.9) show that all the $\lambda(\zeta_i)$ are equal by maximality of $\lambda(\zeta_1)$. Therefore $F_1 \to H$ is a birational morphism.

Let $p \in \text{indet}(F)$. By Lemma 3.15, $p$ is a triple point on $H$, and so $H$ is rational. Therefore $F_1$ is rational and dominates the surface $B$. However, $B$ does not contain a rational curve by Proposition 2.1. This contradiction shows that $H$ can not be integral. 

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