Moduli Spaces of Symmetric Cubic Fourfolds and Locally Symmetric Varieties

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Abstract

In this paper we realize the moduli spaces of cubic fourfolds with specified automorphism groups as arithmetic quotients of complex hyperbolic balls or type IV symmetric domains, and study their compactifications. Our results mainly depend on the well-known works about moduli space of cubic fourfolds, including the global Torelli theorem proved by Voisin ([Vo86]) and the characterization of the image of the period map, proved independently by Looijenga ([Loo09]) and Laza ([Laz09, Laz10]). The key input for our study of compactifications is the functoriality of Looijenga compactifications, which we formulate in the appendix (section A).

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1 Introduction

Cubic fourfold is an intensively studied object in algebraic geometry. The remarkable work by Voisin in 1986 ([Vo86]) showed the global Torelli theorem for smooth cubic fourfolds. Based on this, Alcock-Carlosn-Toledo ([ACT11]) and Looijenga-Swierstra ([LS07]) realized the moduli space of smooth cubic threefolds as an arrangement complement in an arithmetic ball quotient. Recently Laza-Pearlstein-Zhang ([LPZ17]) realized the moduli space of pairs consisting of a cubic threefold and a hyperplane as an arrangement complement in a type IV arithmetic quotient. In both cases, the authors studied compactifications of the moduli spaces. In this paper we characterize the moduli spaces of cubic fourfolds with specified automorphism groups,
and identify the GIT-compactifications with Looijenga compactifications. This generalizes the two results mentioned above.

Let $F$ be the normalization of the irreducible subvariety parameterizing smooth cubic fourfolds with specified action by finite group $A$ (see section 2 for the setup). Let $n = \text{dim } F$. Let $X$ be a cubic fourfold in $F$. Consider the induced action of $A$ on $H^4(X, \mathbb{C})$, and let $\zeta$ be the character corresponding to $H^3(X)$. Denote $H^4(X)_{\zeta}$ to be the $\zeta$-eigenspace, which admits a natural Hermitian form $h$ induced by the topological intersection pairing on $H^4(X, \mathbb{Z})$ (see section 4.1). Then $h$ has signature $(n', 2)$ if $\zeta = \overline{\zeta}$; $(n', 1)$ otherwise (see proposition 4.1). The first main theorem of the paper is the following:

**Theorem 1.1** (Main Theorem 1).  
(i) We have equality $n' = n$.  
(ii) The Hodge structure on $H^4(X)_{\zeta}$ gives an algebraic isomorphism $\mathcal{P}: F \cong \Gamma \setminus (\mathbb{D} - H_s)$. Here $\mathbb{D}$ is a complex hyperbolic ball if $h$ has signature $(n, 1)$; a type IV symmetric domain otherwise. The group $\Gamma$ is an arithmetic group acting properly discontinuously on $\mathbb{D}$ and $H_s$ is a $\Gamma$-invariant hyperplane arrangement in $\mathbb{D}$.

(iii) The period map $\mathcal{P}$ extends naturally to an algebraic isomorphism $F_1 \cong \Gamma \setminus (\mathbb{D} - H_s)$, where $F_1$ is a $\Gamma$-invariant partial completion of $F$, adding cubic fourfolds with at worst ADE-singularities, and $H_s$ is a $\Gamma$-invariant hyperplane arrangement contained in $H_s$.

Denote $\tilde{F}$ to be the GIT-compactification of $F$, see section 2.2. We characterize $\tilde{F}$ via:

**Theorem 1.2** (Main Theorem 2). There is an isomorphism between projective varieties $\tilde{F} \cong \Gamma \setminus \overline{\mathbb{D}^{H_s}}$.

Here $\Gamma \setminus \overline{\mathbb{D}^{H_s}}$ is the Looijenga compactification of $\Gamma \setminus (\mathbb{D} - H_s)$, see section A.5 in appendix.

Notice that in [GAL11], smooth cubic fourfolds with prime-order automorphisms are classified and form 13 irreducible subvarieties in the moduli of cubic fourfolds (see section 6.1). Two of the examples in the list are exactly the cases dealt in [ACT11], [LS07] and [LPZ17].

**Structure of the Paper:** We briefly introduce the main content of each section.

In section 2 we introduce the notion of symmetry type, and set up the geometric invariant theory of hypersurfaces with specified symmetry type.

In section 3 we review concepts about cubic fourfolds, and introduce the global Torelli theorem which was proved by Voisin ([Voi86]).

In section 4 we define the moduli of T-marked cubic fourfolds, and the local period map for those cubic fourfolds. We show that the local period map is an open embedding and characterize its image. Finally we discuss the global period maps by passing to certain quotients.

In section 5 we investigate the compactifications of both sides of the period map for symmetric cubic fourfolds, and identify them.

In section 6 we give some examples and relate them to the previous works.

In section A, we review Looijenga compactification of an arrangement complement in a complex hyperbolic ball or type IV domain. We prove functoriality of Looijenga compactifications.

**Convention:** All algebraic varieties are defined over the field of complex numbers. The adjectives open, closed refer to analytic topology and Zariski-open, Zariski-closed are used for Zariski topology.

**Notation:**

$(d, k)$: dimension and degree of a hypersurface  
$V$: complex vector space of dimension $k + 2$
2 General Setup: Symmetric Hypersurfaces

2.1 Space of Symmetric Polynomials

Let $V$ be a complex vector space of dimension $k + 2$. Denote $\text{Sym}^d(V^*)$ to be the space of degree $d$ polynomials on $V$. We have the natural action of $\text{SL}(V)$ on $\text{Sym}^d(V^*)$, namely, $g(F) = F \circ g^{-1}$ for $g \in \text{SL}(V)$ and $F \in \text{Sym}^d(V^*)$.

The center of $\text{SL}(V)$ is the group $\mu_{k+2}$ consisting of $(k + 2)$-th roots of unity. Let $A$ be a finite subgroup of $\text{SL}(V)$ containing $\mu_{k+2}$ and denote $\overline{A} = A/\mu_{k+2}$ the image of $A$ in $\text{PSL}(V)$. Then $\text{Sym}^d(V^*)$ is a representation of $A$. 
Note that for any $\xi \in \mu_{k+2}$ and $F \in \text{Sym}^d(V^*)$, we have $\xi(F) = \xi^{-d}F$. Let $\lambda: A \rightarrow \mathbb{C}^\times$ be a character of $A$ such that $\lambda|_{\mu_{k+2}}$ sends $\xi \in \mu_{k+2}$ to $\xi^{-d}$. Let $\mathcal{V}_\lambda$ be the $\lambda$-eigenspace of $\text{Sym}^d(V^*)$. We write $\mathcal{V} = \mathcal{V}_\lambda$ for short. Geometrically, an element in $\mathcal{V}$ determines a degree $d$ hypersurface (not necessarily smooth) in $\mathbb{P}V$, whose automorphism group contains $\overline{A}$.

Two pairs $(A_1, \lambda_1)$ and $(A_2, \lambda_2)$ are called equivalent if and only if there exists $g \in \text{SL}(V)$ such that $gA_1g^{-1} = A_2$ and $\lambda_1(a_1) = \lambda_2(ga_1g^{-1})$. We call an equivalence class a symmetry type, denoted by $T$. There is a poset structure on the space of symmetry types, namely, $T_2 \leq T_1$ if $T_1, T_2$ are represented by $(A_1, \lambda_1), (A_2, \lambda_2)$ respectively, such that $A_1 \subset A_2$ and $\lambda_1 = \lambda_2|_{A_1}$. Notice that the space $\mathcal{V}$ depends on the representative $(A, \lambda)$ of $T$.

For $F \in \mathcal{V}$, we denote $Z(F)$ to be the hypersurface determined by $F$ in $\mathbb{P}V$. For $X = Z(F)$, we denote $\text{Aut}(X)$ to be the group of elements in $\text{PSL}(V)$ preserving $X$, and $\text{Aut}(F)$ to be the preimage of $\text{Aut}(X)$ in $\text{SL}(V)$. From [MM64] (theorem 1 and theorem 2) we have:

**Theorem 2.1** (Matsumura-Monsky). When $X$ is smooth, $d \geq 3$, $k \geq 2$,

(i) the group $\text{Aut}(X)$ is finite,

(ii) if $(d,k) \neq (4,2)$, the group $\text{Aut}(X)$ contains all biregular automorphisms of $X$.

Apparently, the group $\overline{A}$ embeds into $\text{Aut}(X)$, for any $X = Z(F)$. We propose the following conditions on the symmetry type $T$:

**Condition 2.2.** The linear space $\mathcal{V}$ contains a point $F$ determining smooth hypersurface.

**Condition 2.3.** The linear space $\mathcal{V}$ contains a point $F$ with the determined hypersurface $X$ smooth and $\overline{A} = \text{Aut}(X)$.

For $T$ satisfying condition 2.2, a generic point in $\mathcal{V}$ determines a smooth hypersurface. We have similar result about condition 2.3:

**Proposition 2.4.** If $T = [(A, \lambda)]$ satisfies condition 2.3, then a generic element in $\mathcal{V}$ determines a smooth hypersurface $X$ with $\overline{A} = \text{Aut}(X)$.

**Proof.** Suppose $F \in \mathcal{V}$ with $X = Z(F)$ smooth, and $A = \text{Aut}(X)$. Then any small deformation $F_1$ of $F$ in $\mathcal{V}$ determines a smooth hypersurface $Z(F_1)$. By theorem 2.5 in [Zhe17], when $F_1$ is sufficiently close to $F$, there exists $g \in \text{PSL}(V)$ such that $g\text{Aut}(Z(F_1))g^{-1} \subset \text{Aut}(X) = \overline{A}$. Since $F_1 \in \mathcal{V}$, we have $\overline{A} \subset \text{Aut}(Z(F_1))$, hence $\overline{A} = \text{Aut}(Z(F_1))$. \hfill \Box

### 2.2 Geometric Invariant Theory for Symmetric Hypersurfaces

Now we assume that $d \geq 3$, $k \geq 2$. Given a symmetry type $T = [(A, \lambda)]$ satisfying condition 2.2, let

$C = \{g \in \text{SL}(V) | gag^{-1} = a, \forall a \in A\}$

and

$N = \{g \in \text{SL}(V) | gag^{-1} = A, \lambda(gag^{-1}) = \lambda(a), \forall a \in A\}$

be two reductive subgroups of $\text{SL}(V)$. For reductivity, see [LR79], lemma 1.1.

**Lemma 2.5.** There is a natural action of $N$ on $\mathcal{V}$, under which the points in $\mathcal{V}$ defining smooth hypersurfaces are stable.

**Proof.** For any $g \in N$ and $F \in \mathcal{V}$, we need to show $g(F) \in \mathcal{V}$. For any $a \in A$, we have:

$a(g(F)) = g(g^{-1}ag(F)) = g\lambda(g^{-1}ag)F = g\lambda(a)F = \lambda(a)g(F),$

which implies $g(F) \in \mathcal{V}$ by definition of $\mathcal{V}$. Therefore, there is a natural action of $N$ on $\mathcal{V}$. 

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Now take $F \in \mathcal{V}$ with $X = Z(F)$ smooth. Then $\text{Aut}(X)$ is finite by theorem 2.1. Since the stabilizer group of $F$ under action of $N$ is a subgroup of $\text{Aut}(F)$, hence also finite. Moreover, $NF$ is closed in $\text{SL}(V)F$, and the latter is closed in $\text{Sym}^d(V^*)$ since $F$ is smooth. Thus $NF$ is closed in $\text{Sym}^d(V^*)$, hence also closed in $\mathcal{V}$. We conclude that $F$ is stable under the action of $N$. \qed

Denote $\mathcal{V}^{sm} = \{ F \in \mathcal{V} | Z(F) \text{ smooth} \}$, $\mathcal{V}^{ss}$ the set of semi-stable elements in $\mathcal{V}$ under the action of $N$, and $\mathbb{P}\mathcal{V}^{sm}$, $\mathbb{P}\mathcal{V}^{ss}$ their projectivizations. By lemma 2.5, we can take $\mathcal{F} = N \backslash \mathbb{P}\mathcal{V}^{sm}$ to be the GIT quotient, with compactification $\overline{\mathcal{F}} = N \backslash \mathbb{P}\mathcal{V}^{ss}$. Different representatives of the symmetry type induce canonically isomorphic GIT-quotients. Define $\mathcal{M} = \text{SL}(V) \backslash \mathbb{P}\text{Sym}^d(V^*)^{ss}$ to be the moduli space of smooth degree $d$ hypersurfaces in $\mathbb{P}(V)$, with compactification $\overline{\mathcal{M}} = \text{SL}(V) \backslash \mathbb{P}\text{Sym}^d(V^*)^{ss}$. We have the following proposition:

**Proposition 2.6.** There is a natural morphism $j : \mathcal{F} \rightarrow \overline{\mathcal{M}}$ sending $[F] \in \mathcal{F}$ to $[F] \in \mathcal{M}$ for any $F \in \mathcal{V}^{sm}$. This morphism is finite. When $T$ satisfies condition 2.3, the morphism $j$ is a normalization of its image.

**Proof.** Here we use a projective version of the main theorem in [Lun75]. See the argument of proposition 8 in [Res10]. Since $A$ is a finite group, there exists certain symmetric power $\text{Sym}^d(V)$ on which the $A$-action is trivial. Consider the $\text{SL}(V)$-action on the coordinate ring $\bigoplus_m \text{Sym}^m(\text{Sym}^d(V^*))$ of $(\mathbb{P}(\text{Sym}^d(V^*)), \mathcal{O}(l))$. Notice that $N$ is of finite index in the normalizer of $A$ in $\text{SL}(V)$. By the main theorem in [Lun75], we have a finite morphism

$$\tilde{j} : \text{Spec}(\bigoplus_m \text{Sym}^m(\text{Sym}^d(V^*)))^N \rightarrow \text{Spec}(\bigoplus_m \text{Sym}^m(\text{Sym}^d(V^*)))^{\text{SL}(V)}$$

sends semi-stable points to semi-stable points, and preserving the cone structures. Thus $\tilde{j}$ does not contract any line, so descends to a finite morphism $j : \mathcal{F} \rightarrow \overline{\mathcal{M}}$. The morphism $j$ sends $[F] \in \mathcal{F}$ to $[F] \in \mathcal{M}$ for any $F \in \mathcal{V}^{sm}$.

We claim that when $T$ satisfies condition 2.3, the morphism $j$ is generically injective. Take generically $F_1, F_2 \in \mathcal{V}$ and assume $[F_1] = [F_2]$ in $\mathcal{M}$. Then there exists $g \in \text{SL}(V)$ with $g(F_1) = F_2$. By the calculation

$$g^{-1}a(g(F_1)) = g^{-1}a(F_2) = g^{-1}\lambda(a)F_2 = \lambda(a)F_1$$

we have that $g^{-1}a \in \text{SL}(V)$ is an automorphism of $Z(F_1)$. By the genericity of $F_1$, we have $A \cong \text{Aut}(F_1)$, which implies that $g^{-1}a \in A$. Then by equation (1) and $F_1 \in \mathcal{V}$, we have $\lambda(g^{-1}a) = \lambda(a)$. This implies that $g \in N$, hence $[F_1] = [F_2]$ in $\mathcal{F}$. Thus $j$ is generically injective.

Moreover, since $\mathcal{F}$ is normal and projective, $j$ is a normalization of its image. \qed

Let $T = [(A, \lambda)]$ be a symmetry type satisfying condition 2.2. Consider the automorphism groups $\text{Aut}(F)$ for all $F \in \mathcal{V}^{sm}$. There exists $F' \in \mathcal{V}^{sm}$ such that $\#\text{Aut}(F)$ is minimal. Fix this polynomial $F'$, and denote $A' = \text{Aut}(F')$. We have a symmetry type $T' = [(A', \lambda')]$, where $a(F') = \lambda'(a)F$ for all $a \in A'$. We have $T \geq T'$, and $T'$ satisfies condition 2.3. For $T'$, there are the corresponding $N', \mathcal{V}'$ and $\mathcal{F}'$. We have the following proposition:

**Proposition 2.7.** There exists a natural finite morphism $\mathcal{F} \rightarrow \mathcal{F}'$.

**Proof.** By proposition 2.6, we have two finite morphisms $j : \mathcal{F} \rightarrow \overline{\mathcal{M}}$ and $j' : \mathcal{F}' \rightarrow \overline{\mathcal{M}}$, and the latter one is a normalization of its image. We show that $j$ and $j'$ have the same image. Firstly, we have that $j'(\mathcal{F}') \subset j(\mathcal{F})$ since $\mathcal{V}' \subset \mathcal{V}$. By lemma 2.5 in [Zhe17], when $F'' \in \mathcal{V}$ is sufficiently close to $F'$, there exists $g \in \text{SL}(V)$, such that $g\text{Aut}(F'')g^{-1} \subset \text{Aut}(F') = A'$. By minimality of $\#A'$, we have $g\text{Aut}(F'')g^{-1} = A'$. This implies that $\text{Aut}(g(F'')) = A'$, hence $g(F'') \in \mathcal{V}'$. We then have that $\dim(j(\mathcal{F})) \leq \dim(j'(\mathcal{F}'))$. By irreducibilities of the two images, they are the same.

By universal property of normalization, the morphism $j$ factors through $j'$. Therefore, we have naturally a finite morphism $\mathcal{F} \rightarrow \mathcal{F}'$. \qed

**Remark 2.8.** The fiber of the finite morphism $\mathcal{F} \rightarrow \mathcal{F}'$ over $[F']$ is bijective to the orbit of $(A, \lambda)$ in the set of subdatas of $(A', \mathcal{X})$ under the action of $N'$.
2.3 Universal Deformation

We fix a type $T = [(A, \lambda)]$ satisfying condition 2.2, and assume $d \geq 3$ and $k \geq 2$. Next we use Luna’s étale slice theorem to describe the local structure of $\mathcal{F}$, and construct the universal family of smooth degree $d$ $k$-folds of type $T$. We follow the argument in [Zhe17] (section 2). For Luna’s étale slice theorem and its proof, one can refer to [Lun73] or [PV94].

Denote $G$ to be the centralizer of $\overline{A}$ in $\text{PSL}(V)$, which acts on the affine variety $\mathbb{P}^{V^\text{sm}}$. For any $x \in \mathbb{P}^{V^\text{sm}}$, we denote $G_x$ to be the orbit of $x$ and $G_z$ to be the stabilizer of $x$. By lemma 2.5, $G_x$ is closed in the affine variety $\mathbb{P}^{V^\text{sm}}$ and $G_x$ is finite. By Luna’s étale slice theorem, there exists a smooth, locally closed, $G_x$-invariant subvariety $S$ containing $x$, such that:

(i) The image of $\kappa: G \times G_x S \to \mathbb{P}^{V^\text{sm}}$, denoted by $U$, is Zariski-open and $G$-invariant, 

(ii) The morphism $\kappa: G \times G_x S \to U$ is étale,

(iii) The morphism $G \backslash \kappa: G_x S \to G \backslash U$ is étale,

(iv) The above two morphisms induce an isomorphism

$$G \times G_x S \cong U \times_{G \backslash U} G_x \backslash S.$$  \hspace{1cm} (2)

We can shrink $S$ in the analytic category such that:

(v) $S$ is $G_x$-invariant, contractible and contains $x$, with $U = \kappa(G \times G_x S)$ a $G$-invariant open subset of $\mathbb{P}^{V^\text{sm}}$,

(vi) the morphism between analytic spaces: $G_x \backslash S \to G \backslash U$ is an isomorphism.

From (2), we have an isomorphism between analytic spaces:

$$G \times G_x S \cong U,$$

by which we have a principal $G_x$-bundle $G \times S \to U$. In particular, $G \times S$ covers $U$.

**Definition 2.9.** For any symmetry type $T$, we define a category $\mathcal{C}_{d,k}^T$ as follows. The objects are families of degree $d$ $k$-folds of type $T$ with a specified central fiber, and the morphisms are holomorphic maps between families, sending central fiber to central fiber and compatible with the action of $\overline{A}$.

**Proposition 2.10.** The family $\mathcal{S}_S$ of degree $d$ $k$-folds of type $T$ over $S$ has the following universal property. For any subfamily $\mathcal{S}' \to S' \subset U$ of degree $d$ $k$-folds of type $T$ containing a central fiber $X'$ with isomorphism $f: X' \cong X$ commuting with $\overline{A}$, we have a unique morphism in the category $\mathcal{C}_{d,k}^T$:

$$\begin{array}{ccc}
\mathcal{S}' & \xrightarrow{\bar{f}} & \mathcal{S} \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}$$

such that the restriction of $\bar{f}$ to $X'$ is $f$.

Moreover, for any two fibers $X_1, X_2$ of $\mathcal{S}$ with an isomorphism $g: X_1 \to X_2$ commuting with $\overline{A}$, we can extend $g$ uniquely to a morphism $g: \mathcal{S} \to \mathcal{S}$ in $\mathcal{C}_{d,k}^T$.

**Proof of proposition 2.10.** The base $S'$ lies in $U$ and is covered by $G \times S$. Thus we have a unique lifting $S' \hookrightarrow G \times S$, sending $x'$ to $(f^{-1}, x)$. In other words, we have uniquely $f: \mathcal{S}' \to \mathcal{S}_S$, which restricts to $\bar{f}$ on $X'$.

Now suppose $X_1, X_2$ are two fibers of $\mathcal{S}$ with isomorphism $g: X_1 \cong X_2$. Denote $x_1, x_2$ the corresponding base points in $S$. Then $(g, x_1), (id, x_2) \in G \times S$ have the same image in $U$. Since $G \times S \to U$ is a principal $G_x$-bundle, the two pairs $(g, x_1)$ and $(id, x_2)$ are $G_x$-equivalent, hence $g \in G_x$. The corollary follows. 

\hfill $\square$
We have the following lemma, which will be used in the proof of proposition 4.8. Since it holds for general degree \( d \) \( k \)-folds, we state and prove it here.

**Lemma 2.11.** Let

\[
\mathcal{X} \longrightarrow S \times \mathbb{P}V \\
\downarrow \\
S
\]

be a family of smooth degree \( d \) \( k \)-folds, with the base \( S \) contractible. Suppose there is a group \( \tilde{A} \), such that for all \( s \in S \), the fiber \( \mathcal{X}_s \) admits a biregular action of \( \tilde{A} \), with induced actions on \( H^n(\mathcal{X}_s, \mathbb{Z}) \) compatible with respect to the local trivialization. Then there is an action of \( \tilde{A} \) on the whole family \( \mathcal{X} \longrightarrow S \) inducing on each fiber the existed action.

We need another lemma from [JL17] (proposition 2.12) and [MM64]:

**Lemma 2.12.** For \( d \geq 3 \), \( k \geq 2 \), and a smooth degree \( d \) \( k \)-fold \( X \), the induced action of \( \text{Aut}(X) \) on \( H^4(X, \mathbb{Z}) \) is faithful.

**Proof of lemma 2.11.** Take any \( s \in S \). By theorem 2.5 in [Zhe17], there is a universal hypersurface family \( \mathcal{X}' \) of \( \mathcal{X}_s \), such that any isomorphism between two fibers (may coincide) of \( \mathcal{X}' \) comes from an automorphism of the central fiber \( \mathcal{X}_s \). There exists an open neighbourhood \( U \) of \( s \) in \( S \), with a unique morphism \( \mathcal{X}' |_U \longrightarrow \mathcal{X}' \).

Then for any \( s' \in U \), the action of \( \tilde{A} \) on \( \mathcal{X}'_{s'} \) is induced by a subgroup \( \tilde{A}' \) of \( \text{Aut}(\mathcal{X}_s) \). By lemma 2.12, and compatibility of induced action of \( \tilde{A} \) on \( \mathcal{X}_s \) and \( \mathcal{X}'_{s'} \), we have that \( \tilde{A} = \tilde{A}' \) as subgroups of \( \text{Aut}(\mathcal{X}_s) \). Therefore, the actions of \( \tilde{A} \) on fibers of \( \mathcal{X} \longrightarrow S \) glue to an action of \( \tilde{A} \) on the whole family. \( \square \)

## 3 Review: Period Map for Smooth Cubic Fourfolds

In this section we recall some fundamental facts on period map for cubic fourfolds, the main references are [Voi86], [Has00a], [Loo09], [Laz09, Laz10].

Take \( (d, k) = (3, 4) \). Then we have \( \mathcal{M} \) the moduli of smooth cubic fourfolds, as a Zariski-open subset of its GIT compactification \( \overline{\mathcal{M}} \). Let \( X \) be a smooth cubic fourfold. We denote \( \varphi_X \) to be the intersection pairing on \( H^4(X, \mathbb{Z}) \). Then \( (H^4(X, \mathbb{Z}), \varphi_X) \) is an odd unimodular lattice of signature \((21, 2)\). Denote \( \eta_X \) to be square of the hyperplane class of \( X \). Then \( H^4_0(X, \mathbb{Z}) = \eta_X^2 \) is an even sublattice of discriminant \( 3 \). Now we define \( (\Lambda, \Lambda_0, \eta) \) to be an abstract data isomorphic to \((H^4(X, \mathbb{Z}), H^4_0(X, \mathbb{Z}), \eta_X)\), this does not depend on the choice of the cubic fourfold \( X \).

**Definition 3.1.** A marking of the cubic fourfold \( X \) is an isomorphism \( \Phi: H^4(X, \mathbb{Z}) \cong \Lambda \) sending \( \eta_X \) to \( \eta \).

Two marked cubic fourfolds \((X_1, \Phi_1)\) and \((X_2, \Phi_2)\) are called equivalent if there exists a linear isomorphism \( g: X_1 \longrightarrow X_2 \) such that \( \Phi_1 = g^* \Phi_2 \). Let \( \mathcal{M}^m \) be the set of equivalence classes of marked cubic fourfolds. From [Zhe17], section 3, we have:

**Proposition 3.2.** The set \( \mathcal{M}^m \) is a complex manifold in a natural way.

Next we define the period domain and period map for cubic fourfolds. Let

\[
\overline{\mathcal{D}} := \mathbb{P}\{x \in (\Lambda_0)_\mathbb{C}| \varphi(x, x) = 0, \varphi(x, \overline{\tau}) < 0\}.
\]

This is an analytically open subset of a quadric hypersurface in \( \mathbb{P}(\Lambda_0)_\mathbb{C} \), and has two connected components. We have naturally a holomorphic map

\[
\widetilde{\mathcal{P}}: \mathcal{M}^m \longrightarrow \overline{\mathcal{D}}
\]
Proposition 4.1. The Hermitian form on $A$ has signature $\lambda_0$.

Let $\hat{D}$ be one connected component of $D$ and $\hat{\Gamma}$ the index 2 subgroup of $\text{Aut}(\Lambda, \varphi, \eta)$ which respects the component $\hat{D}$. Then $\hat{\Gamma}$ is an arithmetic group acting on $\hat{D}$ and $\hat{\Gamma}$ descends to

$$\mathcal{P} : \mathcal{M} \rightarrow \hat{\Gamma} \backslash \hat{D},$$

which is called the (global) period map for cubic fourfolds.

Remark 3.3. The subgroup $\hat{\Gamma}$ consists of elements in $\Gamma$ with spinor norm 1. Since there exist vectors in $\Lambda_0$ with self intersection $-2$, the group $\hat{\Gamma}$ is of index 2 in $\text{Aut}(\Lambda, \varphi, \eta)$.

The global Torelli theorem is originally proved by Voisin ([Voi86]), with an erratum ([Voi08]) based on some work by Laza ([Laz09]):

Theorem 3.4 (Voisin). The period map $\mathcal{P}$ is an open embedding.

Remark 3.5. In fact, the period map $\mathcal{P}$ is algebraic, see discussion in [Has00b] (proposition 2.2.3).

We give a lemma which will be constantly used. See [Zhe17] (theorem 1.1).

Lemma 3.6. Take $X$ a smooth cubic fourfold, then $\text{Aut}(X) \cong \text{Aut}(H^4(X, \mathbb{Z}), \varphi_X, \eta_X, H^{3,1}(X))$.

We have a refined version of theorem 3.4:

Proposition 3.7 (Voisin, Hassett, Looijenga, Laza). The local period map $\tilde{\mathcal{P}}$ is an open embedding, with image being the complement of a hyperplane arrangement invariant under the action of $\text{Aut}(\Lambda, \eta)$ on $\tilde{D}$.

Proof. Combining theorem 3.4 and lemma 3.6 we have injectivity. The characterization of the image of $\tilde{\mathcal{P}}$ is due to Looijenga ([Loo90]) and Laza ([Laz10] (theorem 1.1), more precise version will be discussed in proposition 4.7.

4 Period Maps for Symmetric Cubic Fourfolds

4.1 Local Period Map for Symmetric Cubic Fourfolds

In this section we are going to discuss the local and global period maps for symmetric cubic fourfolds. Let $(d, k) = (3, 4)$, and fix a symmetry type $T = [(A, \lambda)]$ satisfying condition 2.2. We first introduce the local period domains with action of arithmetic groups. Let $X = Z(F)$ for a generic point $F \in \mathcal{V}$. Recall that the action of $A$ on $X$ induces an action of $A$ on $H^{3,1}(X)$. This action is a character $\zeta : A \rightarrow \mathbb{C}^\times$ with trivial restriction on $\mu_{k+2}$. We denote

$$H^4(X)_\zeta = \{ x \in H^4(X) \mid ax = \zeta(a)x, \forall a \in A \}.$$

Define a Hermitian form $h : H^4(X)_\zeta \times H^4(X)_\zeta \rightarrow \mathbb{C}$ by $h(x, y) = \varphi(x, \overline{y})$. Denote $\sigma_X$ to be the action of $A$ on $H^4(X, \mathbb{Z})$. Let $\sigma$ be a action of $A$ on $\Lambda$, making $(\Lambda, \eta, \sigma)$ isomorphic to $(H^4(X, \mathbb{Z}), \eta_X, \sigma_X)$. Denote $\Lambda_\zeta \subset \Lambda_0 \otimes \mathbb{C}$ to be the $\zeta$-eigenspace of the action of $A$ on $(\Lambda_0)_{\zeta}$.

Proposition 4.1. The Hermitian form $h$ has signature $(n', 2)$ if $\zeta = \overline{\zeta}$ (this is also equivalent to $\zeta(A) \subset \mu_2$); it has signature $(n', 1)$ otherwise. Here $n'$ is a non-negative integer independent of the choice of $X$.

Proof. Notice that the lattice $H^4(X, \mathbb{Z})$ has signature $(21, 2)$, with negative part $H^{3,1}(X) \oplus H^{1,3}(X)$. If $\zeta(A)$ is not contained in $\mu_2$, we have $\zeta = \overline{\zeta}$. Since $H^{1,3}$ lies in $\zeta$-eigenspace, the signature of $h$ is $(n', 1)$.

For the case $\zeta(A) \subset \mu_2$, both $H^{3,1}(X)$ and $H^{1,3}(X)$ are contained in $H_\zeta$, hence $h$ has signature $(n', 2)$.  \[\square\]
An isomorphisms \( \Phi: (H^4(X, \mathbb{Z}), \eta_X, \sigma_X) \cong (\Lambda, \eta, \sigma) \) is called a T-marking of \( X \). We consider pairs consisting of a smooth cubic fourfold and its T-marking. Two such pairs \((X_1, \Phi_1)\) and \((X_2, \Phi_2)\) are equivalent if there exists \( g \in G \) such that \( \Phi_1 = g^* \Phi_2 \). Let \( \mathcal{F}^m \) be the set of equivalence classes of such pairs, we have:

**Proposition 4.2.** The set \( \mathcal{F}^m \) is naturally a complex manifold.

**Proof.** First we describe the local charts on \( \mathcal{F}^m \). Take a point \((X, \Phi) \in \mathcal{F}^m\), and take a universal deformation \( \mathcal{X}_S \rightarrow S \) of \( X \) as in proposition 2.10. Since \( S \) is contractible, the local system \( R^1\pi_* (\mathbb{Z}) \) is trivializable over \( S \) and the T-marking \( \Phi \) of \( X \) naturally extends to T-marking of every fiber of \( \mathcal{X}_S \rightarrow S \). Thus we have a map

\[
\alpha : S \rightarrow \mathcal{F}^m.
\]

We first show that \( \alpha \) is injective. Suppose \( X_1, X_2 \) are two fibers of \( \mathcal{X}_S \), with \( \Phi_1, \Phi_2 \) the induced T-markings by \( \Phi \) respectively, such that \((X_1, \Phi_1)\) and \((X_2, \Phi_2)\) represent the same point in \( \mathcal{F}^m \). Then there exists \( g: X_1 \cong X_2 \) with \( \Phi_2 = \Phi_1 \circ g^* \). By proposition 2.10 we have \( g \in G_x \) and \( \Phi = \Phi \circ g^* \), hence \( g^* = id \). By lemma 3.6 we have \( g = id \). Thus \( \alpha \) is injective.

By definition, \( \mathcal{F}^m \) is covered by countably many such \( \alpha(S) \), and they form a basis of a topology. To show \( \mathcal{F}^m \) is a complex manifold, we need to prove that the topology is Hausdorff. Suppose not, then we have two non-separated points \((X, \Phi), (X', \Phi') \in \mathcal{F}^m \). Then \( X \) and \( X' \) are isomorphic (because \( \mathcal{F} \) is separated). Without loss of generality, we just assume \( X' = X \). Take \( \mathcal{X}_S \rightarrow S \) the universal family as in proposition 2.10, and

\[
\alpha, \alpha': S \rightarrow \mathcal{F}^m
\]

induced by \( \Phi \) and \( \Phi' \). Now since \((X, \Phi)\) and \((X', \Phi')\) are non-separated, we have \( \alpha(S) \cap \alpha'(S) \neq \emptyset \). Thus there exists \( x_1 \in S \) with corresponding cubic fourfold \( X_1 \), such that the two pairs \((X_1, \Phi)\) and \((X_1, \Phi')\) represent the same point in \( \mathcal{F}^m \). Then there is an automorphism \( g \) of \( X_1 \), such that \( \Phi' = \Phi \circ g^* \). Proposition 2.10 implies that \( g \) is also an automorphism of \( X \) and satisfies the above relation. Thus \((X, \Phi) = (X, \Phi')\) in \( \mathcal{F}^m \), contradiction. We showed the Hausdorff property, hence conclude that \( \mathcal{F}^m \) is naturally a complex manifold. \( \square \)

**Remark 4.3.** Proposition 4.2 can be generalized to degree \( d \) \( k \)-folds \((d \geq 3, k \geq 2)\) with specified automorphism group. The argument is the same.

When \( h \) has signature \((n', 1)\), we define \( \mathbb{D}_T = \mathbb{P}\{x \in \Lambda_\xi | \varphi(x, \overline{x}) < 0\} \), which is a hyperbolic complex ball of dimension \( n' \); when \( h \) has signature \((n', 2)\), define \( \mathbb{D}_T \) to be a component of \( \mathbb{P}\{x \in (\Lambda_0)_\xi | \varphi(x, x) = 0, \varphi(x, \overline{x}) < 0\} \), which is a type IV symmetric domain of dimension \( n' \).

We define local period map for symmetric cubic fourfolds of type \( T \) as the map from \( \mathcal{F}^m \) to \( \mathbb{D}_T \sqcup \overline{\mathbb{D}_T} \), sending \((X, \Phi)\) to \( \Pi(H^{1,1}(X)) \), still denoted by \( \mathcal{P} \). We make the choice of \( \mathbb{D}_T \) such that \( \mathcal{P} \) has nonempty image in \( \mathbb{D}_T \). Write \( \mathbb{D} = \mathbb{D}_T \) if there is no confusion.

### 4.2 Properties of Local Period Maps for Symmetric Cubic Fourfolds

We need to review basic works by Laza ([Laz09, Laz10]). In [Laz09] Laza classified stable and semistable cubic fourfolds. One of the main theorems is:

**Theorem 4.4 ([Laz09]).** A cubic fourfold with at worst ADE-singularities is stable.

Laza proved that the period map \( \mathcal{P}: \mathcal{M} \rightarrow \tilde{\mathcal{G}} \setminus \overline{\mathbb{D}} \) extends to the moduli space \( \mathcal{M}_1 \) of cubic fourfolds with at worst ADE singularities, and characterized its image. The results are gathered in the following theorem:
The local period map $\mathcal{P}: \mathcal{M} \rightarrow \hat{\Gamma}\backslash \hat{\mathbb{D}}$ has image $\hat{\Gamma}\backslash(\hat{\mathbb{D}} - \mathcal{H}_\infty - \mathcal{H}_\Delta)$, and extends holomorphically to

$$\mathcal{P}: \mathcal{M}_1 \rightarrow \hat{\Gamma}\backslash \hat{\mathbb{D}}$$

with image $\hat{\Gamma}\backslash(\hat{\mathbb{D}} - \mathcal{H}_\infty)$. Here $\mathcal{H}_\infty, \mathcal{H}_\Delta$ are two $\hat{\Gamma}$-invariant hyperplane arrangements in $\hat{\mathbb{D}}$, with the quotients $\hat{\Gamma}\backslash\mathcal{H}_\infty$ and $\hat{\Gamma}\backslash\mathcal{H}_\Delta$ irreducible.

**Remark 4.6.** This characterization of the image $\mathcal{P}(\mathcal{M})$ was conjectured by Hassett in [Has00b]. Hassett defined the special cubic fourfolds, some of which correspond to polarized K3 surfaces. The hyperplane arrangements $\mathcal{H}_\Delta$ and $\mathcal{H}_\infty$ are two particular ones, parameterizing nodal cubic fourfolds and secant lines of determinantal cubic fourfold, and corresponding to K3 surfaces of degree 6 and 2 respectively. See [Has00b], section 4.2 and 4.4.

We have also the following marked-version of theorem 4.5:

**Proposition 4.7.** The local period map $\tilde{\mathcal{P}}: \mathcal{M}^m \rightarrow \hat{\mathbb{D}}$ has image $\hat{\mathbb{D}} - \mathcal{H}_\infty - \mathcal{H}_\Delta - \mathcal{H}_\infty - \mathcal{H}_\Delta$.

**Proof.** By theorem 4.5, the image of $\tilde{\mathcal{P}}$ lies in $\hat{\mathbb{D}} - \mathcal{H}_\infty - \mathcal{H}_\Delta - \mathcal{H}_\infty - \mathcal{H}_\Delta$. Take any point $x$ in $\hat{\mathbb{D}} - \mathcal{H}_\infty - \mathcal{H}_\Delta - \mathcal{H}_\infty - \mathcal{H}_\Delta$. By theorem 4.5 the point $[x] \in \hat{\Gamma}\backslash(\hat{\mathbb{D}} - \mathcal{H}_\infty - \mathcal{H}_\Delta)$ lies in the image of $\mathcal{P}: \mathcal{M} \rightarrow \hat{\Gamma}\backslash \hat{\mathbb{D}}$. Thus the orbit $\text{Aut}(\Lambda, \eta)x$ intersects with $\tilde{\mathcal{P}}(\mathcal{M}^m)$. Notice that the set $\tilde{\mathcal{P}}(\mathcal{M}^m)$ is $\text{Aut}(\Lambda, \eta)$-invariant, hence contains the orbit $\text{Aut}(\Lambda, \eta)x$. We showed the surjectivity.

For a specified type $T$, we have a natural embedding $\hat{\mathbb{D}} \sqcup \hat{\mathbb{D}} \hookrightarrow \hat{\mathbb{D}}$. Denote $\mathcal{H}_s = \hat{\mathbb{D}} \cap (\mathcal{H}_\Delta \cup \mathcal{H}_\infty \cup \mathcal{H}_\Delta \cup \mathcal{H}_\infty)$ and $\mathcal{H}_a = \hat{\mathbb{D}} \cap (\mathcal{H}_\Delta \cup \mathcal{H}_\infty \cup \mathcal{H}_\Delta \cup \mathcal{H}_\infty)$. The local period map $\tilde{\mathcal{P}}: \mathcal{F}^m \rightarrow \hat{\mathbb{D}} \sqcup \hat{\mathbb{D}}$ has image contained in $\hat{\mathbb{D}} \sqcup \hat{\mathbb{D}} - \mathcal{H}_s - \mathcal{H}_a$. In particular, $n' = n$.

**Proposition 4.8.** The local period map $\tilde{\mathcal{P}}: \mathcal{F}^m \rightarrow \hat{\mathbb{D}} \sqcup \hat{\mathbb{D}}$ is an open embedding, with image either $\hat{\mathbb{D}} - \mathcal{H}_s$ or $\hat{\mathbb{D}} - \mathcal{H}_s - \mathcal{H}_a$. In particular, $n' = n$.

**Proof.** We have a closed embedding $\pi: \hat{\mathbb{D}} \sqcup \hat{\mathbb{D}} \hookrightarrow \hat{\mathbb{D}}$. There is a natural map $j: \mathcal{F}^m \rightarrow \mathcal{M}^m$. Suppose $(X_1, \Phi_1), (X_2, \Phi_2)$ represent the same point in $\mathcal{M}^m$, then there exists a linear isomorphism $g: X_1 \cong X_2$ such that

$$g^* = \Phi_1^{-1} \circ \Phi_2: H^4(X_2, \mathbb{Z}) \rightarrow H^4(X_1, \mathbb{Z})$$

Since $\Phi_1, \Phi_2$ are compatible with the action of $A$ on $H^4(X_1, \mathbb{Z}), H^4(X_2, \mathbb{Z})$, so is $g^*$. Lemma 3.6 implies that $g$ is compatible with the actions of $A$ on $X_1, X_2$. Thus $(X_1, \Phi_1), (X_2, \Phi_2)$ represent the same point in $\mathcal{F}^m$. We showed the injectivity of $j$.

Combining with the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{F}^m & \xrightarrow{j} & \hat{\mathbb{D}} \\
\downarrow{\tilde{\mathcal{P}}} & & \downarrow{\pi} \\
\mathcal{M}^m & \xrightarrow{\tilde{\mathcal{P}}} & \hat{\mathbb{D}}
\end{array}
$$

we obtain the injectivity of $\tilde{\mathcal{P}}: \mathcal{F}^m \rightarrow \hat{\mathbb{D}} \sqcup \hat{\mathbb{D}}$. In particular, $n \leq n'$.

Since the differential of $\tilde{\mathcal{P}}: \mathcal{M}^m \rightarrow \hat{\mathbb{D}}$ is injective everywhere, so is the differential of $\tilde{\mathcal{P}}: \mathcal{F}^m \rightarrow \hat{\mathbb{D}} \sqcup \hat{\mathbb{D}}$.

Take $(X, \Phi) \in \mathcal{F}^m$. Let $x = \Phi(H^3_!(X)) \in \hat{\mathbb{D}} \sqcup \hat{\mathbb{D}}$ and $y$ be any point in the component of $\hat{\mathbb{D}} \sqcup \hat{\mathbb{D}}$ containing $x$. Since both $\hat{\mathbb{D}} - \mathcal{H}_s$ and $\hat{\mathbb{D}} - \mathcal{H}_a$ are connected, there exists a path

$$\gamma: [0, 1] \rightarrow \hat{\mathbb{D}} \sqcup \hat{\mathbb{D}} - \mathcal{H}_s - \mathcal{H}_a$$

with $\gamma(0) = x$ and $\gamma(1) = y$. The path $\gamma$ has a unique lifting in $\mathcal{M}^m$. By proposition 3.7, we can choose a family $\mathcal{F} \rightarrow [0, 1]$ of cubic fourfolds, with marking $\Phi$ of every fiber, such that $(\mathcal{F}_0, \Phi) = (X, \Phi)$ and
\( \Phi(H^{3,1}(X_s)) = \gamma(s) \), for all \( s \in [0, 1] \). Since \( \gamma(s) \in \mathbb{D} \cup \overline{\mathbb{D}} \), the Hodge structure on \( H^4(X_s, \mathbb{Z}) \) has action of \( A \) induced by \( \Phi \). By lemma 3.6, there exists an action of \( A \) on \( X_s \) for any \( s \in [0, 1] \), inducing compatible action on \( H^4(X_s, \mathbb{Z}) \). By lemma 2.11, actions of \( A \) are of the same type \( T \). Thus we obtain a lifting of \( \gamma \) in \( F_m \), hence \( y \in \widetilde{\mathcal{P}}(F^m) \).

If \( \widetilde{\mathcal{P}}(F^m) \subset \mathbb{D} \), then \( \widetilde{\mathcal{P}}(F^m) = \mathbb{D} - H_s \); otherwise \( \widetilde{\mathcal{P}}(F^m) \) intersects with both \( \mathbb{D} \) and \( \overline{\mathbb{D}} \), which implies that \( \widetilde{\mathcal{P}}(F^m) = \mathbb{D} \cup \overline{\mathbb{D}} - H_s - \overline{H_s} \).

We introduce an involution on \( M_m \). Take any smooth cubic fourfold \( X = Z(F) \), and a marking \( \Phi: H^4(X, \mathbb{Z}) \rightarrow \Lambda \). Let \( X' = Z(\overline{F}) \). There exists a homeomorphism \( \tau \) from \( X \) to \( X' \) given by complex conjugation. Let \( \iota \) be the involution on \( M_m \) sending \( (X, \Phi) \) to \( (X', \Phi \circ \tau^*) \). Consider smooth cubic fourfold \( X = Z(F) \) such that \( F \) has real coefficients. Then \( \tau \) is a diffeomorphism of \( X \), and \( \tau^* \) sends \( H^{3,1}(X) \) to \( H^{1,3}(X) \). Therefore, choosing any marking \( \Phi \) of \( X \), the points \([X, \Phi]\) and \([X, \Phi \circ \tau^*]\) lie in different components of \( M_m \). This implies that the involution \( \iota \) exchanges the two components of \( M_m \).

Next we give criterions on number of connected components of \( F^m \). For a symmetry type \( T = [(A, \lambda)] \), we define the complex conjugate \( \overline{T} \) of \( T \) to be \( [(A, \overline{\lambda})] \), where \( \overline{A} \) is the complex conjugate of \( A \), and \( \overline{\lambda}(a) = \lambda(\overline{a}) \) for all \( a \in \overline{A} \). From definition, the involution \( \iota \) exchanges the two spaces \( F^m_\mathbb{R} \) and \( F^m_\mathbb{C} \).

**Proposition 4.9.** Given a symmetry type \( T = [(A, \lambda)] \).

(i) If \( \zeta \) is not real, then \( F^m \) is connected.

(ii) If \( T = \overline{T} \), then \( F^m \) has two components.

(iii) If \( T \) satisfies condition 2.3, and \( T \neq \overline{T} \), then \( F^m \) is connected.

**Proof.** Suppose \( \zeta \) is not real, then \( \widetilde{\mathcal{P}}(F^m) \) lies in the ball associated to \( (A_\zeta, h) \). Thus \( F^m \) is connected.

Suppose \( T = \overline{T} \), then \( F^m \) is preserved by \( \iota \). Thus \( F^m \) has two components.

Suppose \( F^m \) has two components, then \( \widetilde{\mathcal{P}}(F^m) = \mathbb{D} \cup \overline{\mathbb{D}} - H_s - \overline{H_s} \). Thus \( F^m \) is preserved by \( \iota \). Thus \( F^m_\mathbb{R} = F^m_\mathbb{C} \). This cannot happen if \( T \) satisfies condition 2.3 and \( T \neq \overline{T} \). The third part follows.

### 4.3 Global Period Map

In this section we are going to define the global period domain for symmetric cubic fourfolds of type \( T \) as an arithmetic quotient of \( \mathbb{D} \), and study the global period map.

Let \( (d, k) = (3, 4) \) and fix a symmetry type \( T = [(A, \lambda)] \) satisfying condition 2.2. Let \( \Gamma = \{ \rho \in \widehat{\mathbb{L}} | \rho \overline{\rho}^{-1} = \mathbb{A} \} \) be the normalizer of \( \mathbb{A} \) in \( \widehat{\mathbb{L}} \). Take \( \rho \in \widehat{\mathbb{L}} \) and a point \( x \in A_\zeta \). We claim that \( \rho x \in A_\zeta \). In fact, take any \( a \in A \), we have

\[
\rho a \rho x = \rho \rho^{-1} a \rho x = \rho (\rho^{-1} a \rho) x = (\rho^{-1} a \rho) x.
\]

Since \( \rho \in \widehat{\mathbb{L}} \), we have \( \rho [x] \in \overline{\mathbb{D}} \). The two characters \( \zeta \) and \( \rho^{-1} \zeta \rho \) both give non-definite eigensubspaces of \( A_\zeta \). We conclude that \( \zeta = \rho^{-1} \zeta \rho \), hence \( \rho x \in A_\zeta \). This gives a natural action of \( \Gamma \) on \( \mathbb{D} \).

Let \( N_A \) be the normalizer of \( A \) in \( \text{Aut}((\Lambda_0)_{\mathbb{Q}}, \phi) \), which is a reductive algebraic subgroup. The group \( \Gamma \) is an arithmetic subgroup of \( N_A \), see also appendix. The arithmetic quotient \( \Gamma \backslash \mathbb{D} \) is a quasi-projective variety thanks to the Baily-Borel compactification (see section A.3 in appendix). We denote \( (F^m)^1 \) to be the connected component of \( F^m \) such that \( \widetilde{\mathcal{P}}((F^m)^1) = \mathbb{D} - H_s \).

**Proposition 4.10.** The local period map \( \widetilde{\mathcal{P}}: (F^m)^1 \rightarrow \mathbb{D} - H_s \) descends to an algebraic isomorphism \( \mathcal{P}: F \cong \Gamma \backslash (\mathbb{D} - H_s) \).

**Proof.** There are natural analytic morphisms from \( F^m \) to \( F \), and \( \mathbb{D} - H_s \) to \( \Gamma \backslash (\mathbb{D} - H_s) \) respectively. We define the global period map \( \mathcal{P}: F \rightarrow \Gamma \backslash (\mathbb{D} - H_s) \) as follows. Take \( F \in \nu^\text{sym} \). We choose a \( T \)-marking \( \Phi \).
of $X = Z(F)$, such that $\Phi(H^{3,1}(X)) \in \mathbb{D}$ (this also means that $(F, \Phi) \in (\mathcal{F}^m)^1$). We define

$$\mathcal{P}([F]) = (\widehat{\mathcal{P}}(X, \Phi)).$$

We show this map is well-defined. Take $F_1, F_2 \in \mathcal{V}^{sm}$ with $T$-markings $\Phi_1, \Phi_2$ respectively. Suppose there exists $g \in N$, such that $g(F_1) = F_2$. We have an induced map

$$g^*: H^4(Z(F_2), \mathbb{Z}) \to H^4(Z(F_1), \mathbb{Z}).$$

Next we show $\rho = \Phi_1g^*\Phi_2^{-1} \in \Gamma$. Denote $\alpha' = gag^{-1}$. Since $g \in N$, we have $\alpha' \in A$. We have the following commutative diagram:

$$\begin{array}{ccc}
\Lambda & \xrightarrow{\Phi_2^{-1}} & H^4(Z(F_2), \mathbb{Z}) \\
\downarrow{a'} & & \downarrow{a'} \\
\Lambda & \xrightarrow{\Phi_1^{-1}} & H^4(Z(F_1), \mathbb{Z})
\end{array}$$

This implies that, as automorphisms of $\Lambda$, $a' = \rho^{-1}a\rho$. Thus $\rho \in \Gamma$. We then have a well-defined analytic morphism $\mathcal{P}: \mathcal{F} \to \Gamma \backslash (\mathbb{D} - \mathcal{H}_s)$.

By definition we have the following commutative diagram:

$$\begin{array}{ccc}
(F^m)^1 & \xrightarrow{\mathcal{P}} & \mathbb{D} - \mathcal{H}_s \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathcal{F} & \xrightarrow{\mathcal{P}} & \Gamma \backslash (\mathbb{D} - \mathcal{H}_s).
\end{array} \tag{3}$$

We next show $\mathcal{P}: \mathcal{F} \to \Gamma \backslash (\mathbb{D} - \mathcal{H}_s)$ is an isomorphism.

We first show injectivity. Suppose $(F_1, \Phi_1), (F_2, \Phi_2) \in \mathcal{F}^m$, with $\Phi_1(H^{3,1}(Z(F_1)))$ and $\Phi_2(H^{3,1}(Z(F_2)))$ representing the same point in $\Gamma \backslash (\mathbb{D} - \mathcal{H}_s)$. Then there exists $\rho \in \Gamma$, such that $\rho\Phi_1(H^{3,1}(Z(F_1))) = \Phi_2(H^{3,1}(Z(F_2)))$. The map

$$\Phi_2^{-1}\rho\Phi_1: H^4(Z(F_1), \mathbb{Z}) \to H^4(Z(F_2), \mathbb{Z})$$

preserves the polarized Hodge structures. By lemma 3.6, we have $g \in SL(V)$, with $gF_2$ equals to $F_1$ after rescaling of $F_2$, and $g^* = \Phi_2^{-1}\rho\Phi_1$. For any $a \in A$, we have $a^*: H^4(Z(F_1), \mathbb{Z}) \to H^4(Z(F_1), \mathbb{Z})$. We have $g^{-1}ag$ acting on $Z(F_2)$, which induces:

$$(g^{-1}ag)^* = g^*a^*g^{-1} = (\Phi_2^{-1}\rho\Phi_1)(\Phi_1^{-1}a\Phi_1)(\Phi_1^{-1}\rho^{-1}\Phi_2) = \Phi_2^{-1}\rho\rho^{-1}\Phi_2$$

Since $\rho \in \Gamma$, we have $\rho\rho^{-1} \in A$. Again by lemma 3.6, we have $g^{-1}ag \in A$. Since

$$g^{-1}agF_2 = g^{-1}aF_1 = \lambda(a)g^{-1}F_1 = \lambda(a)F_2,$$

we have $\lambda(g^{-1}ag) = \lambda(a)$. We conclude $g \in N$. Thus $\mathcal{P}$ is injective.

By proposition 4.8, the composition of

$$(\mathcal{F}^m)^1 \to \mathbb{D} - \mathcal{H}_s \to \Gamma \backslash (\mathbb{D} - \mathcal{H}_s)$$

is surjective. By commutativity of diagram (3), the composition of

$$(\mathcal{F}^m)^1 \to \mathcal{F} \to \Gamma \backslash (\mathbb{D} - \mathcal{H}_s)$$

is also surjective, hence $\mathcal{P}: \mathcal{F} \to \Gamma \backslash (\mathbb{D} - \mathcal{H}_s)$ is surjective.

The algebraicity of $\mathcal{P}$ can be deduced from its extension to certain compactifications on both sides, see theorem 5.3. An alternative argument follows the proof of proposition 2.2.3 in [Has00b] using Baily-Borel compactification and Borel extension theorem. \qed
5 Compactifications

In this section we are going to study the compactifications of both two sides of $\mathcal{P}: \mathcal{F} \rightarrow \Gamma(\mathcal{D} - \mathcal{H}_s)$. The essential ingredient is the identification of the GIT-compactification of the moduli space of cubic fourfolds and the Looijenga compactification of the global period domain, proved by Looijenga [Loo09] and Laza [Laz10] independently. Depending on this, we will prove theorem 1.2, and then deduce (iii) of theorem 1.1. In theorem 5.6, we give a criterion when the Looijenga compactification is actually Baily-Borel compactification.

Let $(d, k) = (3, 4)$. Recall that from theorem 4.5 we have isomorphism $\mathcal{P}: \mathcal{M}_1 \cong \hat{\Gamma}(\mathcal{D} - \mathcal{H}_s)$. From [Loo09] and [Laz10] we have:

**Theorem 5.1** (Looijenga, Laza). The period map $\mathcal{P}$ extends to $\mathcal{M} \rightarrow \Gamma(\mathcal{D} - \mathcal{H}_s)$.\end{equation}

Recall that $\mathcal{H}_s = \mathcal{D} \cap (\mathcal{H}_\infty \cup \mathcal{H}_\infty')$, which is a $\Gamma$-invariant hyperplane arrangement in $\mathcal{D}$. We have a morphism between locally symmetric varieties

$$\Gamma(\mathcal{D}) \rightarrow \text{Aut}(\Lambda, \eta),$$

We can construct the Looijenga compactification $\overline{\Gamma(\mathcal{D})^{H_s}}$ of $\Gamma(\mathcal{D} - \mathcal{H}_s)$ (see appendix A). From theorem A.14, we have:

**Proposition 5.2.** There exists finite morphism $\pi: \overline{\Gamma(\mathcal{D})^{H_s}} \rightarrow \overline{\Gamma(\mathcal{D})^{H_\infty}}$. If $T$ satisfies condition 2.3, then this morphism is a normalization of its image.

We now state our main theorem:

**Theorem 5.3.** The global period $\mathcal{P}: \mathcal{F} \rightarrow \Gamma(\mathcal{D} - \mathcal{H}_s)$ extends to an algebraic isomorphism $\mathcal{P}: \mathcal{F} \cong \overline{\Gamma(\mathcal{D})^{H_s}}$.

We need the following fact in algebraic geometry. We give the proof for reader’s convenience.

**Lemma 5.4.** Let $f_1: Z_1 \rightarrow Y$ and $f_2: Z_2 \rightarrow Y$ be finite morphisms between irreducible algebraic varieties. Suppose $Z_1, Z_2$ are normal. Moreover, there exists Zariski-open subset $U_i$ of $Z_i$, $i = 1$ or 2, with a biholomorphic map $g: U_1 \rightarrow U_2$, such that $f_1 = f_2 \circ g$. Then $g$ extends to an algebraic isomorphism $Z_1 \rightarrow Z_2$.

**Proof.** Let $\mathbb{C}(Z)$ be the field of rational functions on an irreducible algebraic variety $Z$, and $M(Z)$ the field of meromorphic functions. We claim $g^*\mathbb{C}(Z_2) = \mathbb{C}(Z_1)$. Let $x \in \mathbb{C}(U_2) = \mathbb{C}(Z_2)$. Since $\mathbb{C}(U_2)$ is a finite extension of $\mathbb{C}(Y)$, $g^*x$ is finite over $\mathbb{C}(U_1)$. We can find a Zariski-open subset $U_1^0$ of $U_1$, with a Galois covering $U \rightarrow U_1^0$, such that $g^*x \in \mathbb{C}(U_1)$. Thus $g^*x \in \mathbb{C}(U_1^0)$, it is invariant under the action of Deck transformations. Thus $g^*x \in \mathbb{C}(U_1^0) = \mathbb{C}(Z_1)$. The claim follows.

Without loss of generality, we assume $Y$ is affine. The coordinate ring $\mathbb{C}[Z_i]$ is the integral closure of $\mathbb{C}[Y]$ in $\mathbb{C}(Z_i)$. So $g^*\mathbb{C}[Z_2] = \mathbb{C}[Z_1]$. Thus $g$ extends to an algebraic isomorphism $Z_1 \cong Z_2$. \end{proof}

**Proof of theorem 5.3.** We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\cong} & \Gamma(\mathcal{D} - \mathcal{H}_s) \\
\downarrow & & \downarrow \\
\mathcal{F} & \rightarrow & \overline{\Gamma(\mathcal{D})^{H_s}} \\
\downarrow & & \downarrow \pi \\
\mathcal{M} & \xrightarrow{\mathcal{P}} & \overline{\Gamma(\mathcal{D})^{H_\infty}} \\
\end{array}
\]
with both $j, \pi$ finite morphisms. Since $F$ is Zariski-open in $\overline{F}$, the image $j(F)$ contains a Zariski-open subset of $j(\overline{F})$. Thus $j(\overline{F})$ is the closure of $j(F)$ in $\overline{M}$. The same argument shows that $\pi(\overline{\Gamma(\mathbb{D} - \mathcal{H}_s)})$ is the closure of $\Gamma(\mathbb{D} - \mathcal{H}_s)$ in $\overline{\Gamma(\mathbb{D} - \mathcal{H}_s)}$. By commutativity of diagram (4), the two images $j(F)$ and $\pi(\overline{\Gamma(\mathbb{D} - \mathcal{H}_s)})$ are identified via $\mathcal{P}$, so are $j(\overline{F})$ and $\pi(\overline{\Gamma(\mathbb{D} - \mathcal{H}_s)})$. By proposition 2.6, proposition 5.2 and lemma 5.4, we have an identification between $F$ and $\overline{\Gamma(\mathbb{D} - \mathcal{H}_s)}$, which extends $\mathcal{P} : F \cong \Gamma(\mathbb{D} - \mathcal{H}_s)$. This identification is the extended global period map $\mathcal{P} : \overline{F} \cong \overline{\Gamma(\mathbb{D} - \mathcal{H}_s)}$.

The proof of the above theorem does not use algebraicity of $\mathcal{P}$. Actually, we can deduce algebraicity of $\mathcal{P}$ from theorem 5.3. At this point, we already finish the proof of part (i), (ii) of theorem 1.1 and theorem 1.2. In the rest of this section, we prove part (iii) of theorem 1.1.

Let $\mathcal{V}_1$ be the subset of $\mathcal{V}$ consisting of cubic forms of type $T$ defining cubic fourfolds with at worst ADE-singularities. The points in $\mathcal{V}_1$ are stable with respect to the action of $\text{SL}(V)$ on $\text{Sym}^3(V^*)$, hence also stable with respect to the action of $N$ on $\mathcal{V}$. Define $\mathcal{F}_1 = N \backslash \mathbb{P}\mathcal{V}_1$ the moduli space of cubic fourfolds of type $T$ with at worst ADE-singularities. We have:

**Proposition 5.5.** The period map $\mathcal{P} : F \longrightarrow \Gamma(\mathbb{D} - \mathcal{H}_s)$ extends to an algebraic isomorphism $\mathcal{P} : \mathcal{F}_1 \cong \Gamma(\mathbb{D} - \mathcal{H}_s)$.

**Proof.** From definition we have $j(\mathcal{F}_1) = j(F) \cap \mathcal{M}_1$ and $j^{-1}(j(\mathcal{F}_1)) = \mathcal{F}_1$. From proposition 2.6, the morphism $j : \mathcal{F}_1 \longrightarrow \mathcal{M}_1$ is finite. On the other hand, we have

$$\pi(\Gamma(\mathbb{D} - \mathcal{H}_s)) = \pi(\overline{\Gamma(\mathbb{D} - \mathcal{H}_s)}) \cap \overline{\Gamma(\mathbb{D} - \mathcal{H}_s)}$$

and

$$\pi^{-1}(\pi(\Gamma(\mathbb{D} - \mathcal{H}_s))) = \Gamma(\mathbb{D} - \mathcal{H}_s).$$

From proposition 5.2, the morphism $\pi : \Gamma(\mathbb{D} - \mathcal{H}_s) \longrightarrow \overline{\Gamma(\mathbb{D} - \mathcal{H}_s)}$ is finite. By theorem 4.5 and theorem 5.3, the two images $j(\mathcal{F}_1)$ and $\pi(\Gamma(\mathbb{D} - \mathcal{H}_s))$ are identified via $\mathcal{P}$. By lemma 5.4, we have algebraic isomorphism $\mathcal{P} : \mathcal{F}_1 \cong \Gamma(\mathbb{D} - \mathcal{H}_s)$. \hfill $\square$

If the hyperplane arrangement $\mathcal{H}_s$ is empty, then the Looijenga compactification of $\Gamma(\mathbb{D})$ is actually the Baily-Borel compactification. In the rest of this section, we give a criterion of emptiness of $\mathcal{H}_s$ from the perspective of GIT. Following the notation in section 6 of [Laz09], there is a rational curve $\chi$ parametrizing certain semi-stable cubic fourfolds, given by:

$$g_{a,b}(x_0, \ldots, x_5) = \begin{vmatrix} x_0 & x_1 & x_2 + 2ax_5 & bx^2 \\ x_1 & x_2 - ax_5 & x_3 & x_4 \\ x_2 + 2ax_5 & x_3 & x_4 \\ \end{vmatrix}$$

where $(a : b) \in WP(1 : 3)$. We denote the cubic fourfold corresponding to $(a : b)$ to be $X_{(a:b)}$. When $b = 0$, the corresponding cubic fourfold $X_{(1:0)}$ is the determinantal cubic fourfold. The singular locus of $X_{(1:0)}$ is the image of the Veronese embedding $\mathbb{P}V_3 \hookrightarrow \mathbb{P}\text{Sym}^2(V_3) \cong \mathbb{P}V$. Here $V_3$ is a three dimensional complex vector space with $\text{Sym}^2(V_3) \cong V$. This induces a natural map from $GL(V_3)$ to $GL(V)$. Let $G_1$ be the intersection of $\text{SL}(V)$ with the image of $GL(V_3) \longrightarrow GL(V)$.

For $b \neq 0$, the singular locus of the cubic fourfold $X_{(a:b)}$ is the image of $\mathbb{P}V_2 \hookrightarrow \mathbb{P}(\text{Sym}^4(V_2) \oplus \mathbb{C}) \cong \mathbb{P}V$. Here $V_2$ is a two dimensional complex vector space with $\text{Sym}^4(V_2) \oplus \mathbb{C} \cong V$. Let $\tilde{G}_2$ be the subgroup of $GL(V_2) \times \mathbb{C}^* \cong V$ consisting of elements $(g, u)$ such that $(\det g)^2/u$ is a third root of unity. Let $G_2$ be the intersection of $\text{SL}(V)$ with the image of the natural map $\tilde{G}_2 \longrightarrow GL(V)$. The center of $\text{SL}(V)$ is contained in both $G_1$ and $G_2$.

The automorphism group $\text{Aut}(X_{(1:0)})$ is the image of $G_1$ in $\text{PSL}(V)$, and this induces a character $\lambda_1$ of $G_1$. Explicitly, for $g \in GL(V_3)$ representing an element in $G_1$, we have $\lambda_1(g) = \det(g)^4$. The automorphism
We have the following criterion:

**Theorem 5.6.** The following three statements are equivalent:

(i) The hyperplane arrangement $H_\ast$ is nonempty,

(ii) The space $\mathcal{V}_\chi$ intersects with $SL(V)\chi$,

(iii) The pair $(A, \lambda)$ factor through $(G_1, \lambda_1)$ or $(G_2, \lambda_2)$ defined as above.

**Proof.** The discussion above shows that (ii) and (iii) are equivalent.

We next show the first two statements are equivalent. Firstly we show that the image of $j: \mathcal{F} \to \overline{M}$ intersects with the image of $\chi$ in $\overline{M}$ if and only if (ii) holds. If (ii) holds, the intersection point survives after taking GIT quotients since the $SL(V)$ orbits of points in $\chi$ are closed. If $j(\mathcal{F})$ intersects with the image of $\chi$ at $[F]$ in $\overline{M}$, then we take the representative $F$ in $\mathcal{V}_\chi$ with closed $N$-orbit. According to the main theorem in [Lun75], the $SL(V)$-orbit of $F$ is also closed. So $F$ is contained in $SL(V)\chi$.

Secondly we recall that the blow-up and blow-down construction in Looijenga compactification $\overline{\Gamma \setminus \mathbb{D}}^{H_\ast}$ gives a strata of projective line $\mathbb{P}^1$ corresponding to $\chi$. We claim that $H_\ast$ is nonempty if and only if the image of $\overline{\Gamma \setminus \mathbb{D}}^{H_\ast}$ intersects with the $\mathbb{P}^1$. From the proof of functoriality of semi-toric compactification in appendix A.4, we know that $\mathbb{D}^{\Sigma}$ intersects with $\mathcal{H}_\infty$ if and only if $\mathbb{D}$ intersects with $H_\infty$. So the image of $\overline{\Gamma \setminus \mathbb{D}}^{H_\ast}$ intersects with the $\mathbb{P}^1$ if and only if $\mathbb{D}$ intersects with $H_\infty$. The equivalence of (i) and (ii) follows. □

We will apply this criterion to prime-order groups, see proposition 6.5.

### 6 Examples and Related Constructions

Now given a symmetry type $T = [(A, \lambda)]$ for cubic fourfolds, we can obtain a global period map $\mathcal{P}: \mathcal{F} \cong \overline{\Gamma \setminus \mathbb{D}}^{H_\ast}$. A closely related question is to classify automorphism groups of cubic fourfolds. There are 13 conjugacy classes of prime-order automorphisms of smooth cubic fourfolds (see [GAL11]). For two of them, our main theorems recover some of the main results in [ACT02, ACT11], [LS07] and [LPZ17]. We will discuss these examples in more details in sections 6.1 and 6.2.

Cubic fourfolds have very close relation with hyper-Kähler manifolds, see [BD85],[Has00b]. We briefly recall the story. For a smooth cubic fourfold $X$, consider its Fano scheme of lines $F_1(X)$. This is a hyper-Kähler fourfold of $K3^{[2]}$ type. Automorphism group of a smooth cubic fourfold $X$ is naturally identified with the automorphism group of the polarized hyper-Kähler manifold $F_1(X)$ (the polarization is from Veronese embedding), see [Fu16].

The classification of automorphisms and automorphism groups of hyper-Kähler manifolds have appealed a lot of interests recently. A celebrated result of Mukai ([Muk88]) says that there are 11 maximal finite groups of symplectic automorphisms of $K3$ surfaces. More precisely, these 11 groups are exactly those maximal subgroups of the Mathieu group $M_{23}$ with at least 5 orbits in their induced action on $\{1, 2, \ldots, 24\}$. The proof by Mukai was simplified by Kondo using Niemeier lattices ([Kondo98]). It turns out that using Leech lattice instead of Niemeier lattices, one can obtain a more uniform treatment (see [GHV12] and [Huy16]). About higher dimensional cases, there is a systematic study by Mongardi in his thesis ([Mon12, Mon13, Mon16]).

In [HM14], Höhn and Mason classified all maximal symplectic automorphism groups of hyper-Kähler fourfolds of $K3^{[2]}$ type. Those groups are all subgroups of the Conway group (automorphism group of the Leech lattice quotient by its center).

Another closely related problem is to characterize the moduli spaces of symmetric or lattice-polarized hyper-Kähler manifolds. There are works [DK07] (section 11), [AST11] (section 9), [BCS15], [Jou16], [Cam16] (section 3), [BCS16] (section 5) along this direction.
6.1 Prime-order Automorphisms of Smooth Cubic Fourfolds

The classification of prime-order automorphisms of smooth cubic fourfolds was given in [GAL11] (theorem 3.8). For readers’ convenience we present the result in this section. (There was a small mistake in [GAL11], theorem 3.8. The second example with \( p = 5 \) contains only singular cubic fourfolds. This is pointed out in [BCS16], remark 6.3).

**Theorem 6.1** ([GAL11]). Let \( \omega \) be a prime \( p \)-th root of unity and \( \rho = (m_0, \cdots, m_5) \) be the automorphism of \( V \cong \mathbb{C}^6 \) given by \((x_0, \cdots, x_5) \mapsto (\omega^{m_0}x_0, \cdots, \omega^{m_5}x_5)\). The list of smooth cubic polynomials \( F \) preserved by the action under \( \rho \) is as follows:

\begin{align*}
T_2^1: \rho &= (0, 0, 0, 0, 1), \ n = 14, \\
F &= L_3(x_0, \cdots, x_4) + x_3^2 L_1(x_0, \cdots, x_4). \\
T_2^2: \rho &= (0, 0, 0, 0, 1, 1), \ n = 12, \\
F &= L_3(x_0, \cdots, x_3) + x_3^2 L_1(x_0, \cdots, x_3) + x_4 x_5 M_1(x_0, \cdots, x_3) + x_3^2 N_1(x_0, \cdots, x_3). \\
T_2^3: \rho &= (0, 0, 0, 1, 1, 1), \ n = 10, \\
F &= L_3(x_0, x_1, x_2) + x_0 L_2(x_3, x_4, x_5) + x_1 M_2(x_3, x_4, x_5) + x_2 N_2(x_3, x_4, x_5). \\
T_3^1: \rho &= (0, 0, 0, 0, 0, 1), \ n = 10, \\
F &= L_3(x_0, \cdots, x_3) + x_3^2. \\
T_3^2: \rho &= (0, 0, 0, 0, 1, 1), \ n = 4, \\
F &= L_3(x_0, \cdots, x_3) + M_3(x_4, x_5). \\
T_3^3: \rho &= (0, 0, 0, 0, 1, 2), \ n = 8, \\
F &= L_3(x_0, \cdots, x_3) + x_3^2 + x_3 + x_4 x_5 M_1(x_0, \cdots, x_3). \\
T_3^4: \rho &= (0, 0, 0, 1, 1, 1), \ n = 2, \\
F &= L_3(x_0, x_1, x_2) + M_3(x_3, x_4, x_5). \\
T_3^5: \rho &= (0, 0, 0, 1, 1, 2), \ n = 7, \\
F &= L_3(x_0, x_1, x_2) + M_3(x_3, x_4) + x_3 x_5 L_1(x_0, x_1, x_2) + x_4 x_5 M_1(x_0, x_1, x_2).
\end{align*}
Remark 6.2. This classification offers 13 symmetry types with \(\#\mathcal{A}\) a prime number 2, 3, 5, 7 or 11. Those symmetry types may not satisfy condition 2.3.

By Griffiths residue calculus ([Gri69]), for a smooth cubic fourfold \(X = Z(F)\), the complex line \(H^{3,1}(X)\) is generated by \(\text{Res}_{\mathcal{X}}(\Omega/\pi^2)\). Here \(\Omega = \sum_{i=0}^{5}(-1)^i x_i dx_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge dx_5\). By direct calculation, we have:

**Proposition 6.3.** (i) For type \(T = T_2^2, T_3^3, T_4^3, T_5^3, T_7^3, T_{11}^3\), we have \(\zeta = 1\).

(ii) For type \(T = T_2^2, T_2^2\), we have \(\zeta = -1\).

(iii) For type \(T = T_3^3, T_3^3, T_3^3\), we have \(\zeta(\rho) = \omega \times \mathfrak{e}\).

We already proved that \(\mathcal{P}(\mathcal{F}^m)\) is either \(\mathbb{D} - \mathcal{H}_s\) or \(\mathbb{D} \cup \mathbb{D} - \mathcal{H}_s - \overline{\mathcal{H}_s}\). From proposition 4.9, we have:

**Proposition 6.4.** (i) If \(T = T_3^1, T_3^7, T_3^7\), then \(\mathbb{D}\) is a complex hyperbolic ball and \(\overline{\mathcal{P}(\mathcal{F}^m)} = \mathbb{D} - \mathcal{H}_s\).

(ii) If \(T = T_3^1, T_3^7, T_3^7, T_3^7\) or \(T_7^1\), then \(\mathbb{D}\) is a type IV domain and \(\overline{\mathcal{P}(\mathcal{F}^m)} = \mathbb{D} \cup \mathbb{D} - \mathcal{H}_s - \overline{\mathcal{H}_s}\).

Now we apply theorem 5.6 for prime-order cases.

**Proposition 6.5.** For \(T = T_2^1, T_3^3, T_3^3, T_3^3, T_3^3, T_3^3\), we obtain isomorphism between GIT compactification \(\mathcal{F}\) with Baily-Borel compactification \(\overline{\mathbb{D}\backslash \mathcal{D}^{bb}}\). For \(T = T_2^3, T_3^0, T_3^0, T_3^1, T_7^1\), we do not obtain Baily-Borel compactification.

**Proof.** We will only do the calculation for \(p = 2\), the other cases are similar. If \((A, \lambda)\) factors through \((G_1, \lambda_1)\), then there exists \(g \in \text{GL}(V_3)\) with order 2, such that the image of \(g\) generates \(\mathcal{A}\). We can choose basis of \(V_3\), such that the matrix corresponds to \(g = \text{diag}(1, -1, -1)\). The image of \(g\) in \(\text{GL}(V)\) is \(\text{diag}(1, 1, 1, 1, 1, -1, -1)\). If \((A, \lambda)\) factors through \((G_1, \lambda_1)\), then we can choose \((g, u) \in \text{GL}(V_2) \otimes \mathbb{C}^*\) such that \(g^2 = \text{id}\) and \((\det g^2)/u\) is a third root of unity. Under suitable basis, we have \(g = \text{diag}(1, -1)\). Then the image of \((g, u)\) in \(\text{SL}(V)\) is \(\text{diag}(1, 1, -1, -1, -1, 1, -1)\). In both two cases, the characters \(\lambda_1\) and \(\lambda_2\) are trivial. By theorem 5.6, the symmetry type \(T_2^3\) does not give Baily-Borel compactification and \(T_2^1, T_2^3\) give Baily-Borel compactifications.

6.2 Examples revisit

Take \(T = T_3^1\), then \(T = [\mathcal{A} = \mu_3, \lambda = 1]\) satisfies condition 2.3. The space \(\mathcal{F}\) can be identified with the moduli space of smooth cubic threefolds. The local period domain \(\mathbb{D}\) is a complex hyperbolic ball of dimension 10 with an action of an arithmetic group \(\Gamma\). Then theorem 1.1 and theorem 1.2 recover the main results in [LS07] and [ACT11]. By proposition 6.5, the hyperplane arrangement \(\mathcal{H}_s\) is nonempty. Actually, from [LS07] and [ACT11], the quotients \(\Gamma\backslash \mathcal{H}_s\) has two irreducible components, and \(\Gamma\backslash \mathcal{H}_s\) is irreducible.

Take \(T = T_3^1\), then \(T = [\mathcal{A} = \mu_2, \lambda = 1]\) satisfies condition 2.3. In this case, the moduli space \(\mathcal{F}\) turns out to be the moduli space of pairs consisting of a cubic threefold and a hyperplane section. This is recently studied in [LPZ17]. Denote \(\mathcal{W}_1 = H^0(\mathbb{P}^5, \mathcal{O}(3))\) the space of cubic forms in \(x_0, \ldots, x_4\) and \(\mathcal{W}_2 = H^0(\mathbb{P}^5, \mathcal{O}(1))\) to be the space of linear forms in \(x_0, \ldots, x_4\). We have an identification \(\mathcal{W}_1 \oplus \mathcal{W}_2 \cong \mathcal{W}\) sending \((L_3, L_1)\) to \(L_3 + x_5^2 L_1\). In their paper [LPZ17], the authors defined \(\mathcal{F}\) to be a GIT-quotient of \((\mathbb{P}\mathcal{W}_1 \times \mathbb{P}\mathcal{W}_2, \mathcal{O}(3) \boxtimes \mathcal{O}(1))\) by \(\text{SL}(5, \mathbb{C})\). Direct calculation shows that \(N = C = \text{SL}(5, \mathbb{C}) \times Z \subset \text{SL}(V)\), where \(Z = \{\text{diag}(u, u, u, u, u, u^5) \in \mathbb{C}^6\}\) is the center. The following proposition gives the relation of our constructions with that in [LPZ17]:

**Proposition 6.6.** We have identification between polarized projective varieties:

\[
\mathbb{Z} \backslash (\mathbb{P}\mathcal{V}, \mathcal{O}(1)) \cong (\mathbb{P}\mathcal{W}_1 \times \mathbb{P}\mathcal{W}_2, \mathcal{O}(3) \boxtimes \mathcal{O}(1)).
\]
Proof. It is equivalent to show

\[
\bigoplus_k (H^0(\mathcal{P}V, \mathcal{O}(k)))^Z \cong \bigoplus_k H^0(\mathcal{P}W_1 \times \mathcal{P}W_2, \mathcal{O}(3k) \boxtimes \mathcal{O}(k))
\]

as graded algebras. The action of $Z$ on $\mathcal{W}_1$ has weight 3, and on $\mathcal{W}_2$ weight $-9$.

We have the direct sum decomposition

\[
\text{Sym}^m(\mathcal{V}^*) = \bigoplus_{k+l=m} \text{Sym}^k \mathcal{W}_1^* \otimes \text{Sym}^l \mathcal{W}_2^*
\]

with $Z$-action of weight $-3k + 9l$. The weight zero part has $k = 3l$ and $m = 4l$. So we obtain identification of the two polarized varieties.

Moreover, by proposition 6.5, the hyperplane arrangement $\mathcal{H}_*$ is empty in this case, and we obtain identification between $\mathcal{F}$ and Baily-Borel compactification $\Gamma \backslash \mathcal{D}^{bb}$. This recovers the main result in [LPZ17].

A Locally Symmetric Varieties and Looijenga compactifications

It is well-known that the normalization of each stratum in the orbifold loci of a locally Hermitian symmetric variety is still a locally Hermitian symmetric variety. For reader’s convenience, we include a discussion of this fact in section A.1. In the rest of the appendix, we prove that similar result holds for Looijenga compactifications.

We will recall the construction of Looijenga compactifications of arithmetic quotients $X$ of complex hyperbolic balls or type IV domains. There are two steps. The first is the semi-toric blow up $X_\Sigma$, which is an intermediate compactification of arithmetic quotient $X$ sitting between Baily-Borel and toroidal compactifications. We will recall the geometric construction of Baily-Borel compactifications of complex hyperbolic balls and type IV domains in section A.3, and recall the semi-toric blow-up construction in section A.4. The second step is successive blow-up constructions along the hyperplane arrangement in $X_\Sigma$ and blow-down construction of certain induced strata (We will sketch this in section A.5).

A.1 Orbifold Loci of Locally Symmetric Varieties

In this section we show the normalization of an orbifold stratum of locally Hermitian symmetric variety is again locally Hermitian symmetric variety.

Let $G$ be a real reductive algebraic group with compact center. Let $K$ be a maximal compact subgroup of $G$. Let $\mathcal{D} = G/K$ be the corresponding symmetric space. Assume $\mathcal{D}$ is Hermitian symmetric and $G$ has a $\mathbb{Q}$-structure. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. For simplicity, we assume the action of $\Gamma$ on $\mathcal{D}$ is faithful. Denote $X = \Gamma \backslash \mathcal{D}$ to be the arithmetic quotient. This is naturally a quasi-projective variety due to Baily-Borel compactification (see [BB66]). There is a natural orbifold structure on $X$. We consider the orbifold locus indexed by certain finite subgroup $A \subset \Gamma$. More precisely, we take $A \subset \Gamma$ fixing some point $x \in \mathcal{D}$. Without loss of generality, we assume $K$ to be the stabilizer of $x \in \mathcal{D}$ under the action of $G$. Then $A \subset K$. Denote $G_A, K_A$ and $\Gamma_A$ to be the corresponding normalizers of $A$ in $G, K$ and $\Gamma$ respectively. Then $G_A$ is again a real reductive algebraic group with compact center and $K_A$ is a maximal compact subgroup (see [Loo16], page 37-38). There is a natural holomorphic embedding

\[
G_A/K_A \hookrightarrow \mathcal{D} = G/K.
\]

Define $\mathcal{D}_A := G_A/K_A$. This is a Hermitian symmetric subspace of $\mathcal{D}$. We have the following proposition:
Proposition A.1. The group $\Gamma_A$ is an arithmetic subgroup in $G_A(\mathbb{Q})$ and the map $\pi: \Gamma_A \backslash \mathbb{D}_A \rightarrow \Gamma \backslash \mathbb{D}$ is finite. Furthermore, if $A$ is the stabilizer of $x$ under the action of $\Gamma$, then this map gives a normalization of its image.

Proof. Due to the extension theorem of Baily-Borel compactifications (see theorem 2 in [KK72]), the map $\pi$ is algebraic and proper. We show $\pi$ is finite. It suffices to show $\pi$ is quasi-finite, namely, having finite fibers. Take any $y \in \mathbb{D}_A$. Suppose we have a point $y' = py$ for $\rho \in \Gamma$. Then $\rho^{-1}A\rho$ is contained in the stabilizer group of $y$. Actually, the $\Gamma_A$-orbits of such points $y'$ are one-to-one corresponding to subgroups with form $\rho^{-1}A\rho$ in the stabilizer group of $y$, hence finitely many.

If $A$ is the stabilizer group of $x$, a generic point in $X_A := \Gamma_A \backslash \mathbb{D}_A$ also has $A$ as stabilizer group. We first show that $\pi$ is generically injective in this case. Take generically $x_1, x_2 \in \mathbb{D}_A$, and assume they $[x_1] = [x_2]$ in $\Gamma \backslash \mathbb{D}$. Then there exists $\rho \in \Gamma$ such that $\rho x_1 = x_2$. Since both $x_1, x_2$ have stabilizer group $A$, we have $\rho A\rho^{-1} = A$, hence $\rho \in \Gamma_A$. This implies that $[x_1] = [x_2]$ in $\Gamma_A \backslash \mathbb{D}_A$. We have $\pi$ a finite and birational morphism from a normal variety to its image, hence a normalization of its image.

Remark A.2. The same construction also works for any finite volume locally Hermitian symmetric varieties. The difference from the arithmetic case is that $\Gamma_A$ is not automatically a lattice. We need to use the compactification in finite volume case (see theorem 1 in [MZ89]) to show that the orbifold locus also admits a compactification, which implies the finiteness of the volume by Yau’s Schwarz lemma ([Yau78]).

A.2 Orbifold Loci of Ball and Type IV Quotients

We fix the notation that will be used in the rest of the appendix. Let $(V_\mathbb{Q}, \varphi)$ be a vector space over $\mathbb{Q}$ with nondegenerate rational bilinear form $\varphi$ of signature $(2, N)$. Let $V = V_\mathbb{Q} \otimes \mathbb{C}$. Notice that here $V_\mathbb{Q}$ is not necessarily the middle cohomology of cubic fourfold. Similar as section 3, the type IV domain $\hat{\mathbb{D}}$ associated to $(V_\mathbb{Q}, \varphi)$ is a component of

$$\hat{\mathbb{D}} \sqcup \overline{\mathbb{D}} = \mathbb{P}\{x \in V|\varphi(x, x) = 0, \varphi(x, \overline{x}) > 0\}.$$  

Denote by $\hat{G}$ the subgroup of $\text{Aut}(\varphi)(\mathbb{R})$ (of index 2) respecting the component $\hat{\mathbb{D}}$. Let $\hat{\Gamma} \subset \hat{G}$ be an arithmetic subgroup. The corresponding locally Hermitian symmetric variety is $\hat{X} = \hat{\Gamma} \backslash \hat{\mathbb{D}}$. Let $A$ be a finite subgroup of $\hat{\Gamma}$. Let $\zeta$ be a character of $A$, such that there exists $x \in V$ with $\varphi(x, x) = 0$ and $\varphi(x, \overline{x}) > 0$, and $a(x) = \zeta(a)x$ for all $a \in A$. Denote $V_\zeta$ to be the $\zeta$-subspace of $V$. Then there is a natural Hermitian form $h$ on $V_\zeta$ defined by $h(x, y) = \varphi(x, \overline{y})$. If $\zeta = \overline{\zeta}$, this Hermitian form has signature $(2, n)$ and we obtain a type IV subdomain $\mathbb{D}$ of $\hat{\mathbb{D}}$. Otherwise the signature is $(1, n)$ and we obtain a complex hyperbolic ball $\mathbb{B}$ inside $\hat{\mathbb{D}}$. Indeed, let

$$G := \{g \in \hat{G}|gAg^{-1} = A\}$$

be an algebraic subgroup over $\mathbb{Q}$. The fixed locus of $A$ in $\mathbb{D}$ is $G(\mathbb{R})/K$, where $K$ is maximal compact subgroup of $G(\mathbb{R})$. Denote $\Gamma = \{\rho \in \hat{\Gamma}|\rho^{-1}A\rho = A\}$. The same as section 4, we have $\Gamma$ an arithmetic subgroup of $G(\mathbb{Q})$ acting on $\mathbb{B}$ or $\mathbb{D}$. Then we have a natural map $\Gamma \backslash \mathbb{D} \rightarrow \hat{\Gamma} \backslash \hat{\mathbb{D}}$ or $\Gamma \backslash \mathbb{B} \rightarrow \hat{\Gamma} \backslash \hat{\mathbb{D}}$. We consider the following condition:

Condition A.3. The group $A$ is the stabilizer of a generic point of $\mathbb{D}$ or $\mathbb{B}$.

If $A$ satisfies this condition, proposition A.1 implies that the morphism $\pi: \Gamma \backslash \mathbb{B} \rightarrow \hat{\Gamma} \backslash \hat{\mathbb{D}}$ or $\pi: \Gamma \backslash \mathbb{D} \rightarrow \hat{\Gamma} \backslash \hat{\mathbb{D}}$ is the normalization of its image.

We will consider a larger set of type IV subdomains. Take $W_\mathbb{Q}$ to be a $\mathbb{Q}$-subspace of $V_\mathbb{Q}$ with signature $(2, n)$, we have the associated type IV subdomain $\mathbb{D}$ inside $\hat{\mathbb{D}}$ with the action of an arithmetic group $\Gamma_W =$
\{ \rho \in \widehat{\Gamma} \mid \rho(W) = W \}. \) Take \( V_2 \) to be an integral structure on \( V_\mathbb{Q} \) such that \( \Gamma \subset \text{Aut}(V_2) \) has finite index. Denote \( W_2 := W_\mathbb{Q} \cap V_2 \). For \( x \in \mathbb{D} \), define \( \text{Pic}(x) := V_2^{1,1} \cap V_2 \) to be the Picard lattice of \( x \) where \( x \) is viewed as a weight two Hodge structure on \( V_2 \). Then for generic \( x \in \mathbb{D} \), we have \( \text{Pic}(x) = W_2^{1,1} \).

We have the following lemma:

**Lemma A.4.** For \( A \) satisfying condition A.3 and \( W = V_\zeta \), we have \( \Gamma_A = \Gamma_W \).

**Proof.** It is straightforward that \( \Gamma_A \subset \Gamma_W \), and they both act on \( \mathbb{D} \). Take any \( \rho \in \Gamma_W \) and a generic point \( x \) in \( \mathbb{D} \). Then \( \rho \) is contained in the stabilizer group of \( \rho x \). Thus both \( A \) and \( \rho^{-1}A\rho \) are contained in the stabilizer group of \( x \). Since \( x \) is generic, we have \( \rho^{-1}A\rho = A \) by condition A.3. So \( \rho \in \Gamma_A \). We showed that \( \Gamma_W \subset \Gamma_A \). \( \square \)

With this lemma, we will simply denote \( \Gamma \) to be the arithmetic group acting on \( \mathbb{D} \). We have:

**Proposition A.5.** For any \( \mathbb{Q} \)-subspace \( W_\mathbb{Q} \) (of \( V_\mathbb{Q} \)) with signature \((2, n)\), we have a morphism \( \pi : \Gamma \backslash \mathbb{D} \rightarrow \widehat{\Gamma} \backslash \widehat{\mathbb{D}} \), which is the normalization of its image.

**Proof.** Properness is by [KK72]. Take a generic point \( x \) in \( \mathbb{D} \). Suppose \( \rho \in \widehat{\Gamma} \) sends \( x \) to \( \rho x \in \mathbb{D} \). The Picard lattice \( \text{Pic}(\rho x) \) of \( \rho x \) contains \( W_\mathbb{D}^1 \), hence \( \rho^{-1}(W_\mathbb{D}^1) \subset \text{Pic}(x) \). Since \( x \) is generic, we have \( \text{Pic}(x) = W_\mathbb{D}^1 \). This implies that \( \rho(W_\mathbb{D}^1) = W_\mathbb{D}^1 \), hence \( \rho(W) = W \). Thus \( \rho \in \Gamma_W \).

Finally, we show finiteness. Take a point \( x \in \mathbb{D} \). For any \( \rho \in \widehat{\Gamma} \), we have \( \rho^{-1}(W_\mathbb{D}^1) \) contained in the Picard lattice \( \text{Pic}(x) \). The set \( \widehat{\Gamma} x \) is a disjoint union of some \( \Gamma \)-orbits, each of which corresponds to the image of certain primitive embedding of \( W_\mathbb{D}^1 \) into \( \text{Pic}(x) \). The orthogonal complement of \( W_\mathbb{D}^1 \) in \( \text{Pic}(x) \) is positive definite with discriminant at most \( \text{det}(W_\mathbb{D}^1) \text{det}(\text{Pic}(x)) \). By reduction theory of lattice, there are finitely many such primitive embeddings. \( \square \)

### A.3 Functoriality of Baily-Borel Compactification

In this section we recall Baily-Borel compactifications of arithmetic quotients of complex hyperbolic balls or type IV domains. See [BB66] and [Loo03a, Loo03b].

We deal with type IV domain \( \widehat{\mathbb{D}} \) first. The boundary components of Baily-Borel compactifications corresponds to \( \mathbb{Q} \)-isotropic planes \( J \) or \( \mathbb{Q} \)-isotropic lines \( I \). Let

\[
\pi_{J^\perp} : \mathbb{P}(V) - \mathbb{P}(J^\perp) \rightarrow \mathbb{P}(V/J^\perp)
\]

and

\[
\pi_{I^\perp} : \mathbb{P}(V) - \mathbb{P}(I^\perp) \rightarrow \mathbb{P}(V/I^\perp)
\]

be the natural projections. The image \( \pi_{J^\perp} \widehat{\mathbb{D}} \) is isomorphic to upper half plane. The image \( \pi_{I^\perp} \widehat{\mathbb{D}} \) is a point. Adding rational boundary components, we have

\[
\widehat{\mathbb{D}}^{bb} := \widehat{\mathbb{D}} \sqcup \bigsqcup_J \pi_{J^\perp} \widehat{\mathbb{D}} \sqcup \bigsqcup_I \pi_{I^\perp} \widehat{\mathbb{D}}
\]

with suitable topology and ringed space structure. The Baily-Borel compactification is the quotient \( \Gamma \backslash \widehat{\mathbb{D}}^{bb} \) as a projective variety.

Given \( W_\mathbb{Q} \subset V_\mathbb{Q} \) with signature \((2, n)\). Let \( \mathbb{D} \) be the corresponding type IV domain. We have a natural map from \( \mathbb{D} \) to \( \widehat{\mathbb{D}} \), inducing \( \Gamma \backslash \mathbb{D} \rightarrow \widehat{\Gamma} \backslash \widehat{\mathbb{D}} \). According to theorem 2 in [KK72], this holomorphic map can be extended to Baily-Borel compactifications, sending boundary components into boundary components.

**Proposition A.6** (type IV to type IV). There is a natural finite extension of \( \pi : \Gamma \backslash \mathbb{D} \rightarrow \widehat{\Gamma} \backslash \widehat{\mathbb{D}} \) to Baily-Borel compactifications

\[
\pi : \Gamma \backslash \mathbb{D}^{bb} \rightarrow \widehat{\Gamma} \backslash \widehat{\mathbb{D}}^{bb}
\]
If $A$ satisfies condition $A.3$, the map is a normalization of its image.

**Proof.** Let $W := V_\zeta$ in this proof. The boundary components of $D^{bb}$ correspond to rational isotropic planes $J$ and rational isotropic lines $I$ in $W$. From the natural embedding $W \hookrightarrow V$, they also have associated boundary components in $\overline{D}^{bb}$. Under the following natural commutative diagram

$$
P(W) - \mathbb{P}(J^\perp) \xrightarrow{\pi_{I^\perp}} \mathbb{P}(W/J^\perp)
$$

we have isomorphisms $\pi_{J^\perp} : \mathbb{P}(W) \rightarrow \mathbb{P}(J^\perp)$, and similar maps $\pi_{I^\perp} : \mathbb{P}(V) \rightarrow \mathbb{P}(I^\perp)$, which induce an extension $D^{bb} \rightarrow \overline{\mathbb{D}}^{bb}$ equivariant under the action of $\Gamma \rightarrow \Gamma$. After taking quotients, we have an extension map $X^{bb} \rightarrow \overline{X}^{bb}$. By proposition A.1, this map is generically injective and it is finite over $\overline{\Gamma}\backslash\mathbb{D}$. Let $\Gamma_J$ be the stabilizer of $J$ under the action of $\Gamma$. The projection of $\Gamma$ in $\text{GL}(J)$ (or equivalently $\text{GL}(V/J^\perp)$) is arithmetic. The boundary component corresponding to $J$ is the quotient of $\pi_{J^\perp} : \overline{\mathbb{D}}^{bb}$ by $\Gamma_J$, hence a modular curve. The restriction to the boundary component corresponding to each $J$ is a non-constant map between modular curves, hence finite. The restriction to boundary components corresponding to each $I$ is automatically finite. So we have an algebraic finite morphism between normal varieties $\overline{X}^{bb} \rightarrow \overline{X}^{bb}$. If $A$ satisfies condition $A.3$, then this morphism is generically injective by proposition A.1, hence a normalization of its image. \qed

We recall the Baily-Borel compactification of Ball quotient. Let $K$ be a CM field and $W_K$ a finite dimensional vector space over $K$ with

$$h_K : W_K \times W_K \rightarrow K$$

a $K$-valued Hermitian form. For each embedding $\iota : K \hookrightarrow \mathbb{C}$, we define $W_i := W_K \otimes_i \mathbb{C}$, then we have a Hermitian form $h_i : W_i \times W_i \rightarrow \mathbb{C}$. Assume the form $h_i$ has signature $(1, n)$ under embedding $\iota = \iota_1$ or $\iota_2$, and is definite otherwise. The complex ball $B$ is defined to be the set of positive lines in $W_{\iota_1}$. The boundary components of Baily-Borel compactification correspond to $K$-isotropic lines $I$ in $W_K$ and we denote $B^{bb} := B \cup \bigcup I \pi_I \mathbb{B}$. When the totally real part of $K$ is not $\mathbb{Q}$, there exists complex embedding $\iota$ such that $(W_i, h_i)$ is definite, which implies that any isotropic vector in $W_K$ must be zero. Thus in this case the boundary set is empty.

Now consider the action of $A$ on $V$ with $\zeta \neq \overline{\zeta}$. Let $K$ be the cyclotomic field generated by $\zeta(A)$. Take $W_K$ to be the $\zeta$-eigenspace of $V_K := V_\zeta \otimes K$ under the action of $A$.

**Lemma A.7.** The $K$-vector space $W_K$ is isotropic under $\varphi$.

**Proof.** Take any $x, y \in W_K$, we need to show $\varphi(x, y) = 0$. Take $a \in A$ such that $\zeta(a)$ is not real. Then $\zeta(a)^2 \neq 1$. By

$$\varphi(x, y) = \varphi(ax, ay) = \varphi(\zeta(a)x, \zeta(a)y) = \zeta(a)^2 \varphi(x, y),$$

we have $\varphi(x, y) = 0$. \quad \Box

There is natural Hermitian form $h$ of signature $(1, n)$ on $W_K$, given by $h(x, y) = \varphi(x, \overline{y})$ for all $x, y \in W_K$. The Galois conjugates of $K$ define eigenspaces of $V$ under the action of $A$. The sum of all those eigenspaces is a subspace of $V$ defined over $\mathbb{Q}$. Then we have the ball $B$ consisting of positive lines in $W$ and we denote $\overline{(\Gamma\backslash\mathbb{B})} := \Gamma\backslash\overline{\mathbb{B}}^{bb}$ the Baily-Borel compactification of $X = \Gamma\backslash\mathbb{B}$.

**Proposition A.8** (ball to type IV). There is a natural finite extension of $\pi : \Gamma\backslash\mathbb{B} \rightarrow \overline{\Gamma\backslash\mathbb{D}}$ to Baily-Borel compactifications

$$\pi : \overline{\Gamma\backslash\mathbb{B}}^{bb} \rightarrow \overline{\Gamma\backslash\mathbb{D}}^{bb}.$$

If $A$ satisfies condition $A.3$, the map is a normalization of its image.
Proof. Similar as the proof for type IV case, we need to identify the boundary components on both sides. The ball and its boundaries are defined as above by $W_K$. If $K$ is not a quadratic extension of $Q$, then the boundary set is empty, hence $\Gamma \setminus \mathbb{B}$ is already compact. If $K$ is, then each $K$-isotropic line $I$ together with its complex conjugate $\hat{I}$ defines a rational isotropic plane in $V_Q$. So there is a natural extension map $\mathbb{B}^{bb} \to \hat{\mathbb{B}}^{bb}$ which is equivariant under the action of $\Gamma \to \hat{\Gamma}$. After taking quotient on both sides, we have a finite algebraic map $\pi: X^{bb} \to \hat{X}^{bb}$. If $A$ satisfies condition A.3, then this morphism is generically injective by proposition A.1, hence a normalization of its image.

\begin{remark}
Similar construction of ball quotients appears in the arithmetic examples of Deligne-Mostow theory, see [DM86] and [Loo07]. In both constructions, if the cyclotomic field generated by the corresponding characters is not $Q(\sqrt{-1})$ or $Q(\sqrt{-3})$, then the Baily-Borel compactification is compact.
\end{remark}

A.4 Functoriality of Semi-toric Compactifications

We first briefly sketch the semi-toric blow-up constructions of complex hyperbolic balls and type IV domains with respect to certain hyperplane arrangements. See [Loo03a, Loo03b]. Semi-toric compactification with respect to a hyperplane arrangement is the minimal blow up of certain boundary components in Baily-Borel compactification, such that the closure of every hypersurface is Cartier at the boundary.

Let $\hat{H}$ be a hyperplane arrangement on $\hat{\mathbb{D}}$ defined by a set of negative vectors $v \in V_Q$, which form finitely many orbits under the action of $\hat{\Gamma}$. We recall some definitions and notation in [Loo03b]. Each rational isotropic line $I$ in $V_Q$ realizes $\hat{\mathbb{D}}$ as a tube domain, with real cone denoted by $C_{I} \subset (I^\perp/I \otimes I)(R)$.

Each rational isotropic plane $J$ determines a half line on the boundaries of the $C_{I}$ for any $I \subset J$. The union of these cones is called the conical locus of $\hat{\mathbb{D}}$. Let $C_{I,+}$ be the convex hull of $C_{I} \cap (I^\perp/I \otimes I)(Q)$, which is the union of $C_{I}$ with rational isotropic half lines corresponding to $J$ containing $I$. The hyperplane arrangement $\hat{H}$ determines an admissible decomposition $\Sigma(\hat{H})$ of the conical locus. More precisely, it is a $\Gamma$-invariant choice of locally rational cone decomposition of $C_{I,+}$ such that the support for isotropic half line corresponding to $J$ is is independent of those $I \subset J$. See section 6 of [Loo03b] for details. For each member $\sigma$ of $\Sigma(\hat{H})$ contained in $C_{I,+}$, we define a corresponding vector subspace $V_\sigma$ of $V$ as follows. When $\sigma$ is the half line corresponding to an isotropic plane $J$, then

$$V_\sigma := \bigcap_{H \subset \hat{H}} H \cap J^\perp.$$  

Otherwise $V_\sigma$ is the span of $\sigma$ in $I^\perp$, which is also the intersection among $I^\perp$ and those $H \in \hat{H}$ containing $I$. Here we identify $H \subset V$ with $H \in \hat{H}$. We have a projection $\pi_\sigma: \mathbb{D} \to \mathbb{F}(V/V_\sigma)$. The semi-toric compactifications is denoted by $\hat{X}^\Sigma := \Gamma \setminus \mathbb{D}^\Sigma$. Here $\mathbb{D}^\Sigma := \bigsqcup_{\sigma \in \Sigma} \pi_\sigma \mathbb{D}$. The space $\hat{X}^\Sigma$ has a natural map to $\hat{X}^{bb}$ respecting the stratifications. We have two different types of boundary components. One is finite quotient of abelian torsor over the modular curve $\hat{\Gamma}_J/\pi_{I^\perp} \hat{\mathbb{D}}$. The abelian torsor is modeled over vector group $J^\perp/V_\sigma$ quotient by a lattice. The other is algebraic torus torsor over a point $\pi_{I^\perp} \hat{\mathbb{D}}$, which is the boundary stratum in the quotient of an infinite-type toric variety induced by the cone decomposition of $C_{I,+}$. In particular, each cone of codimension $k$ corresponds to algebraic torus torsor of dimension $k$.

Given $W_Q \subset V_Q$ a sublattice of signature $(2, n)$, with $\mathbb{D}$ the associated type IV domain. We have the intersection $\mathbb{H} := \mathbb{D} \cap \hat{H}$ a $\Gamma$-invariant hyperplane arrangement in $\mathbb{D}$. We also have the semi-toric blowup of $\mathbb{D}$ with respect to $\mathbb{H}$.

\begin{theorem}[type IV to type IV]
There is a natural finite extension of $\pi: \Gamma \setminus \mathbb{D} \to \hat{\Gamma} \setminus \hat{\mathbb{D}}$ to semi-toric compactifications

$$\pi: \Gamma \setminus \mathbb{D}^\Sigma(\mathbb{H}) \to \hat{\Gamma} \setminus \hat{\mathbb{D}}^\Sigma(\hat{\mathbb{H}}).$$

\end{theorem}
If \( A \) satisfies condition A.3, the map is a normalization of its image.

Proof. We first show the existence of \( \pi : \Gamma \backslash \mathbb{D}^{\Sigma(\mathcal{H})} \to \Gamma \backslash \hat{\mathbb{D}}^{\Sigma(\hat{\mathcal{H}})} \) as a morphism between two projective varieties, then prove finiteness.

The subdomain is induced by \((W, \varphi)\). The isotropic lines and planes in \( W \) are naturally viewed as boundary data of both \( \mathbb{D} \) and \( \hat{\mathbb{D}} \). The conical locus of \( \mathbb{D} \) is naturally embedded into that of \( \hat{\mathbb{D}} \).

Suppose \( \sigma \in \Sigma(\mathcal{H}) \) does not correspond to a rational isotropic plane of \( W \). Then we have a rational isotropic line \( I \), such that \( \sigma \) is contained in \( C_{I,W,+} \) and intersects with \( C_{I,W} \). For each \( H \) containing \( I \), the intersection \( H \cap C_{I,W} \) being not empty is equivalent to \( H \cap \mathbb{D} \) being not empty. Then there exists \( \tau \in \Sigma(\hat{\mathcal{H}}) \) such that \( \sigma = \tau \cap W \). We denote \( \hat{\sigma} \) to be the minimal element among all such \( \tau \). Thus \( \sigma = C_{I,W} \cap \hat{\sigma} \), which implies \( W_\sigma = V_\hat{\sigma} \cap W \).

Let \( \sigma \in \Sigma(\mathcal{H}) \) correspond to an isotropic plane \( J \) contained in both \( W \) and a hyperplane \( H \). Suppose \( v \) is a normal vector of \( H \) and \( v = w + w^\perp \) the decomposition in \( V = W \oplus W^\perp \). We have \( \varphi(v,w) \leq 0 \). The hyperplane \( H \) intersects with \( \mathbb{D} \) if and only if \( \varphi(w,w) < 0 \). Since the orthogonal complement of \( w \) in \( W \) contains the isotropic plane \( J \), we have either \( \varphi(w,w) < 0 \) or \( \varphi(w,w) = 0 \). Suppose the latter case happens, then \( w \in J \) since otherwise \((J,w)\) is an isotropic subspace of rank 3 contained in \( W \), which is impossible. Thus in this case \( H \supseteq J^\perp \cap W \). The above argument holds for any \( H \in \mathcal{H} \) containing \( \sigma \), hence \( W_\sigma = V_\sigma \cap W \).

In this case we also denote \( \hat{\sigma} = \sigma \).

For \( \sigma = \{0\} \in \Sigma(\mathcal{H}) \), just take \( \hat{\sigma} = \{0\} \in \Sigma(\hat{\mathcal{H}}) \). Then for every \( \sigma \in \Sigma(\mathcal{H}) \), we have a natural holomorphic map \( \pi_\sigma \mathbb{D} \to \pi_\sigma \hat{\mathbb{D}} \) which is apparently injective. Taking union among \( \sigma \), we have

\[
\bigsqcup_{\sigma \in \Sigma(\mathcal{H})} \pi_\sigma \mathbb{D} \to \bigsqcup_{\sigma \in \Sigma(\mathcal{H})} \pi_\sigma \hat{\mathbb{D}} \to \bigsqcup_{\tau \in \Sigma(\hat{\mathcal{H}})} \pi_\tau \hat{\mathbb{D}}
\]

with the composition continuous. After taking quotients by the equivariant actions on both sides, we obtain a finite map between the boundary components. Actually, for those rational isotropic planes \( J \), we obtain finite morphisms between Abelian torsors; for those rational isotropic lines \( I \), we obtain finite morphisms between algebraic torus torsors. If \( A \) satisfies condition A.3, then \( \pi \) is generically injective by proposition A.1, hence a normalization of its image. \( \square \)

Remark A.11. The injectivity of \( \bigsqcup_{\sigma \in \Sigma(\mathcal{H})} \pi_\sigma \mathbb{D} \to \bigsqcup_{\tau \in \Sigma(\hat{\mathcal{H}})} \pi_\tau \hat{\mathbb{D}} \) is already known in [Loo03b] (the paragraph after lemma 7.1).

For \( \zeta \neq \bar{\zeta} \), we have ball \( \mathbb{B} \) associated to \( W = V_\zeta \). We next describe the semi-toric compactification of \( \mathbb{B} \) with respect to \( \mathcal{H} \). Here we identify elements in \( \mathcal{H} \) with hypersurfaces in \( W \). The cusp points correspond to isotropic lines \( I \) in \( W_K \). Let

\[
j(I) = \bigcap_{H \in \mathcal{H}, H \supseteq I} H \cap I^\perp_W
\]

and \( \pi_I : \mathbb{P}(W) - \mathbb{P}(j(I)) \to \mathbb{P}(W/j(I)) \). Define

\[
\mathbb{X}' = \Gamma \backslash (\mathbb{B} \sqcup \bigsqcup_I \pi_{j(I)} \mathbb{B}).
\]

It naturally maps to the Baily-Borel compatification. The boundary component over each cusp point is an abelian torsor modeled over the vector space \( I^\perp_W / j(I) \) quotient by a lattice.

Theorem A.12 (ball to type IV). There is a natural finite extension of \( \pi : \Gamma \backslash \mathbb{B} \to \Gamma \backslash \hat{\mathbb{D}} \) to semi-toric compactifications

\[
\pi : \Gamma \backslash \mathbb{B}^{\Sigma(\mathcal{H})} \to \Gamma \backslash \hat{\mathbb{D}}^{\Sigma(\hat{\mathcal{H}})}.
\]

If \( A \) satisfies condition A.3, the map is a normalization of its image.
Proof. If $K$ is not a quadratic extension of $\mathbb{Q}$, then $X$ is compact and the theorem holds. Now assume that $K$ is a quadratic extension of $\mathbb{Q}$. Namely, $K = \mathbb{Q}(\sqrt{-D})$ for certain positive integer $D$. Take any isotropic line $I$ in $W_K$. Suppose a nonzero generator of $I$ is $e + \sqrt{-D}f$, where $e, f \in V_K$. Then $\varphi(e + \sqrt{-D}f, e - \sqrt{-D}f) = 0$. From lemma A.7 we have $\varphi(e + \sqrt{-D}f, e + \sqrt{-D}f) = 0$. This implies that $J = (e, f)$ is an isotropic plane in $V_Q$.

We claim that $j(I) = W \cap V_J$. Take $H \in \hat{\mathcal{H}}$ with orthogonal vector $v \in V_Q$. Under the orthogonal decomposition $V_K = W_K \oplus W_K \oplus V'$, we can decompose $v$ as $v = v_W + v_W' + v'$. Then $\varphi(\text{Re}(v_W), J) = 0$. From lemma A.7 we have $\varphi(v_W, I) = 0$. Therefore, $\varphi(\text{Im}(v_W), I) = 0$ and hence $\varphi(\text{Im}(v_W), J) = 0$.

Since $(V_Q, \varphi)$ has signature $(2, N)$, the orthogonal complement of $J$ in $V_Q$ is negative semi-definite. Thus $\varphi(\text{Re}(v_W), \text{Re}(v_W)) \leq 0$ and $\varphi(\text{Im}(v_W), \text{Im}(v_W)) \leq 0$. We then have

$$\varphi(v_W, \overline{v_W}) = \varphi(\text{Re}(v_W), \text{Re}(v_W)) + \varphi(\text{Im}(v_W), \text{Im}(v_W)) \leq 0.$$ 

Suppose $\varphi(v_W, v_W') < 0$, then $H \cap B \neq \emptyset$. Suppose $\varphi(v_W, \overline{v_W}) = 0$, then $v_W$ is an isotropic line in $W_K$. The vectors $\text{Re}(v_W)$ and $\text{Im}(v_W)$ in $V_Q$ are then isotropic. These two vectors are orthogonal to $J$, hence they belong to $J$. We deduce that $H \supset I_{v_W}^\perp$. By the definition of $j(I)$ and $V_J$, we conclude the claim.

We now have naturally an injective map $\pi_{j(I)} : B \rightarrow \pi_{j(\hat{\mathcal{H}})}$. Taking the union among those isotropic lines $I$, we have an injective map

$$\mathbb{B} \sqcup \coprod_I \pi_{j(I)} : \mathbb{B} \rightarrow \coprod_{\sigma \in \Sigma(\hat{\mathcal{H}})} \pi_\sigma(\hat{\mathcal{H}}).$$

After taking quotients by the equivariant actions on both sides, we obtain a morphism $\pi : \Gamma \backslash \mathbb{B} \rightarrow \Gamma \backslash \mathbb{D}\Sigma(\hat{\mathcal{H}})$. Actually, the restriction of this $\pi$ to the boundary component corresponding to $I$ is a finite morphism between Abelian torsors. We conclude that there is natural extension $\pi : (\Gamma \backslash \mathbb{B})^* \rightarrow (\Gamma \backslash \mathbb{D})^{\Sigma(\hat{\mathcal{H}})}$ which is a finite morphism between projective varieties. If $A$ satisfies condition A.3, this $\pi$ is generically injective, hence a normalization of its image. \hfill $\square$

Remark A.13. By proposition 6.3, a non-symplectic prime-order automorphism of a smooth cubic fourfold has order 2 or 3. This is an evidence for us to conjecture that for all balls arising from symmetric cubic fourfolds, the corresponding field $K$ is either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.

A.5 Main theorem

In this section, we first describe the construction of Looijenga compactification $\overline{X^\mathcal{H}}$ of $X^\mathcal{H} := X - \Gamma \backslash \mathcal{H}$. We need to successively blow up non-empty intersections of components of $\Gamma \backslash \mathcal{H}$, and then contract the strict transformations of $\Gamma \backslash \mathcal{H}$ via a natural associated semi-ample line bundle on the blowup. We then prove existence and finiteness of morphism between Looijenga compactifications on both sides of $X \rightarrow \overline{X}$.

The blow-up and blow-down constructions with respect to hyperplane arrangement in any normal analytic variety with a properly given line bundle are discussed in the first 3 sections in [Loo03a]. Looijenga applied this general theory to $(\overline{X^{\Sigma(\mathcal{H})}}, \Gamma \backslash \mathcal{H}, L)$, where $X$ is either arithmetic quotient of type IV domain $\mathbb{D}$ or ball $\mathbb{B}$, and $L$ is the natural automorphic line bundle. See theorem 5.7 in [Loo03a] and theorem 7.4 in [Loo03b].

The blow-up and blow-down constructions before quotient by the arithmetic groups (and the Looijenga compactification can then be obtained by the last modified space quotient by the arithmetic group). We now describe this. Denote PO($\mathcal{H}$) to be the set of nonempty intersections of elements in $\mathcal{H}$ as hyperplanes in $\mathbb{D}$ (or $\mathbb{B}$). Let $L \in \text{PO}(\mathcal{H})$ also denote its closure in $\mathbb{D}^{\Sigma}$ (or $\mathbb{B}^\Sigma$). Denote $c(L) := \text{codim}(L) - 1$.

We first look at the semi-toric compactification $\mathbb{D}^{\Sigma}$ of $\mathbb{D}$. Denote $(\mathbb{D}^{\Sigma})^o$ to be the arrangement complement of $\mathcal{H}$ in $\mathbb{D}^{\Sigma}$. Choose $L \in \text{PO}(\mathcal{H})$ a minimal member. Blowing up along $L$ replaces $L$ by the projectivization of its normal bundle, which is isomorphic to $L \times \mathbb{P}^c(L)$. The modified space, denoted by
$\text{Bl}_L(\mathbb{D}^\Sigma)$, has natural topology, arrangement (the strict transform of the previous one) and automorphic line bundle. The strict transforms of those hypersurfaces passing through $L$ form a hyperplane arrangement in $\mathbb{P}^c(L)$, and we denote the complement by $(\mathbb{P}^c(L))^\circ$. The complement of the new arrangement in $\text{Bl}_L(\mathbb{D}^\Sigma)$ is the disjoint union $(\mathbb{D}^\Sigma)^\circ \sqcup L \times (\mathbb{P}^c(L))^\circ$. After blowing up successively until hypersurfaces disjoint, we obtain the final blowup $\widetilde{\mathbb{D}}$. This is a disjoint union of $(\mathbb{D}^\Sigma)^\circ$ with $L \times (\mathbb{P}^c(L))^\circ$ for all such minimal $L$ appearing in each step.

Now we can contract $L \times (\mathbb{P}^c(L))^\circ$ along the direction of $L$ for all such $L$, and obtain $\mathbb{D}^*$ with natural quotient topology. Set-theoretically, $L \times (\mathbb{P}^c(L))^\circ$ is contracted to $(\mathbb{P}^c(L))^\circ$. We have the following discription (see [Loo03b]):

$$\mathbb{D}^* = \bigsqcup_{L \in \mathbb{P}O(H)} \pi_L \mathbb{D}^\circ \sqcup \bigsqcup_{\sigma \in \Sigma(H)} \pi_\sigma \mathbb{D}^\circ. \quad (6)$$

Notice that for $\sigma$ being the vertex, $\pi_\sigma$ is identity and $\pi_\sigma \mathbb{D}^\circ = \mathbb{D}^\circ$.

The spaces $\mathbb{D}^\Sigma, \widetilde{\mathbb{D}}, \mathbb{D}^*$ constructed above all have natural ringed space structure. Namely, we have the structure sheaves consisting of continuous functions with analytic restriction to each stratum. The group $\Gamma$ naturally acts on those ringed spaces respecting the stratification. The topological quotient space $\overline{\mathbb{X}}^H := \Gamma \backslash \mathbb{D}^*$ has normal analytic structure respecting the stratification, see [Loo03b] (theorem 7.4). According to the Riemann extension theorem, the quotient ringed space structure and the analytic structure on $\overline{\mathbb{X}}^H$ coincide.

For the case of ball, parallel argument gives $\widetilde{\mathbb{B}}$ and $\mathbb{B}^*$. We have:

$$\mathbb{B}^* = \mathbb{D}^\circ \sqcup \bigsqcup_{L \in \mathbb{P}O(H)} \pi_L \mathbb{B}^\circ \sqcup \bigsqcup_{I} \pi_{I} \mathbb{B}^\circ. \quad (6)$$

This also has natural ringed structure, and $\overline{\mathbb{X}}^\mathbb{H} \cong \Gamma \backslash \mathbb{D}^*$ as analytic spaces.

**Theorem A.14** (Main Theorem). There is a natural finite extension of $\pi: \Gamma \backslash (\mathbb{D} - \mathcal{H}) \rightarrow \tilde{\Gamma} \backslash (\tilde{\mathbb{D}} - \tilde{\mathcal{H}})$ to Looijenga compactifications

$$\pi: \Gamma \backslash \mathbb{D}^H \rightarrow \Gamma \backslash \tilde{\mathbb{D}}^\mathbb{H}.$$ 

If $A$ satisfies condition A.3, the map is a normalization of its image. The same result holds for ball quotients.

**Proof.** From theorem A.10, we have natural morphisms from $\mathbb{D}^\Sigma$ to $\tilde{\mathbb{D}}^\Sigma$. Near each boundary component, there is a contraction map from a neighborhood to the boundary itself. The arrangement in total space is the pullback of smooth arrangement on the boundary. According to the map defined near the boundary components, we know that any $H \in \mathcal{H}$ not intersecting with $\mathbb{B}$ is still away from $\mathbb{D}^\Sigma$ after taking its closure.

From corollary 7.15 in chapter II in [Har77], we have injective map $\tilde{\mathbb{D}} \rightarrow \tilde{\mathbb{D}}$ respecting the ringed space structures. Notice that the automorphic line bundle on $\mathbb{D}^\Sigma$ is the pull back of that on $\tilde{\mathbb{D}}^\Sigma$, hence we have an injective map on the strata $L \times (\mathbb{P}^c(L))^\circ$ to $\tilde{L} \times (\mathbb{P}^c(L))^\circ$ which is linear on the second component. Here $\tilde{L}$ is a minimal member used in certain step of the successive blow-up construction of $\tilde{\mathbb{D}}$, and $L$ is the induced member on the smaller subspace by intersecting with $\tilde{L}$. After blowing down, we have a natural injective map $\mathbb{D}^* \rightarrow \tilde{\mathbb{D}}^*$ respecting the ringed space structures.

This morphism descends to a morphism $\pi: \Gamma \backslash \mathbb{D}^* \rightarrow \Gamma \backslash \tilde{\mathbb{D}}^*$, still in the category of ringed spaces. We then have an analytic morphism $\pi: \overline{\mathbb{X}}^H \rightarrow \overline{\mathbb{X}}^\mathbb{H}$. This analytic morphism extends $\pi: \mathbb{X}^\circ \rightarrow \mathbb{X}^\circ$, and sends boundary strata into boundary strata. Combining with theorem A.10, the extended morphism $\pi$ here is finite. If $A$ satisfies condition A.3, it is generically injective and hence a normalization of its image.

The same argument also holds for ball. 

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