Lagrangian Formalism for Multiform Fields on Minkowski Spacetime

A. M. Moya,† V. V. Fernández,‡ and W. A. Rodrigues, Jr.§

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Abstract

We present an introduction to the mathematical theory of the Lagrangian formalism for multiform fields on Minkowski spacetime based on the multiform and extensor calculus. Our formulation gives a unified mathematical description for the main relativistic field theories including the gravitational field (which however will be discussed in a separate paper). We worked out several examples (including tricks of the trade), from simple to very sophisticated ones (like, e.g., the Dirac-Hestenes field on the more general gravitational background) which show the power and beauty of the formalism.

1 Introduction

It is now well known that the multiform and extensor fields over Minkowski spacetime provides an unifying language for expressing the main field equations of contemporary physics (Hestenes, 1966; Hestenes and Sobczyk, 1984; Rodrigues and de Souza, 1993, 1994; Moya 1999; Moya, Fernández, and Rodrigues, 2001), including gravitation. However, a comprehensive multiform

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‡Institute of Mathematics, Statistics and Scientific Computation, IMECC-UNICAMP CP 6065, 13083-970 Campinas, SP, Brazil
§Institute of Physics Gleb Wataghin, IFGW-UNICAMP, 13083-970 Campinas, SP, Brazil
§Institute of Mathematics, Statistics and Scientific Computation, IMECC-UNICAMP, CP 6065, 13081-970 Campinas, SP, Brazil and Center for Research and Technology, CPTEC-UNISAL, Av. A. Garret 267, 13087-290 Campinas, SP, Brazil; e-mail: walarod@ime.unicamp.br or walarod@cptec.br
1A complete review, including many new mathematical topics of the overall subject has been published in a special edition of Advances in Applied Clifford Algebras (Moya, Fernández and Rodrigues, 2001). See also (Moya, Fernández and Rodrigues, 2003).
Lagrangian formalism using rigorous mathematics is still lacking, despite some previous attempts (Lasenby, Doran and Gull, 1993; Rodrigues and de Souza, 1994; Rodrigues et al., 1995). In this paper we provides such a theory. In our formalism several different kinds of Lagrangians which occurs in well known physical theories are treated with the same mathematics. We include several examples, from simple to sophisticated ones, which show the power of the multiform calculus, together with the main identities (the tricks of trade) necessary for the derivation of several equations of motion in the different theories studied in this paper.

The power of the multiform calculus on Minkowski spacetime, \( (M, \eta, \tau, D^\eta) \), where \( \mathcal{M}^* \) the dual of \( \mathcal{M} (\approx R^4) \), is the vector space of the structure. In the rest of this section we introduce some necessary notations. Given a global coordinate system over \( M \), say \( M \ni x \leftrightarrow x^\mu (x) \in R^4 (\mu = 0, 1, 2, 3) \) associated to a inertial reference frame (Rodrigues and Rosa, 1989) at \( x \in M \).

\[
\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu,
\eta_{\mu\nu} = \eta(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) = \text{diag}(1,-1,-1,-1).
\]

**Definition 1** \( T_xM \ni v_x \) is said to be equipolent to \( v_x' \in T_{x'}M \) (written \( v_x = v_x' \)) if and only if

\[
\eta(x)(\frac{\partial}{\partial x^\mu} |_x, v_x) = \eta(x')(\frac{\partial}{\partial x'^\mu} |_{x'}, v_{x'}), \ (\mu = 0, 1, 2, 3).
\]

Note that \( \frac{\partial}{\partial x^\beta} |_{x'} = \frac{\partial}{\partial x'^\beta} |_{x'} \) \( (\beta = 0, 1, 2, 3) \).

**Definition 2** The set of equivalent classes of tangent vectors over the tangent bundle,
\[
\mathcal{M} = \{ C_{v_x} | \text{for all } x \in M \},
\]
has a natural structure of vector space, it is called Minkowski vector space.

Note that \( \langle C_{\frac{\partial}{\partial x^\mu}} \rangle \) is a natural basis for \( \mathcal{M} \) (dim \( \mathcal{M} = 4 \)). With the notations: \( \vec{v} \equiv C_{v_x} \) and \( \vec{e}_\mu \equiv C_{\frac{\partial}{\partial x^\mu}} \), we can write \( \vec{v} = v^\mu \vec{e}_\mu \).

\[\text{2In (Moya, Fernández and Rodrigues, 2000a,b) we present a Lagrangian formulation for the gravitational field, as a distortion field (an extensor field) on Minkowski spacetime. There, we show, that the formulation of the same problem by (Lasenby, Doran and Gull,1998) is non sequitur. A preliminary version of the gravitational field as a distortion field appears in (Rodrigues and de Souza, 1990).}

\[\text{3M is a 4-dimensional manifold oriented by } \tau (\text{the volume element 4-form}) \text{ and time oriented, which is diffeomorphic to } R^4, \eta \in \text{sec } T^*_0(M) \text{ is a Lorentzian flat metric and } D^\eta \text{ is the Levi-Civita connection of } \eta. \text{ For details, see (Sachs and Wu,1977).}\]
Definition 3 The 2-tensor over \( M \),
\[
\eta : M \times M \to R,
\]
(4)
such that for each \( \vec{v} = C_{v_x} \) and \( \vec{w} = C_{w_x} \in M : \eta(\vec{v}, \vec{w}) = \eta(x)(v_x, w_x), \) for all \( x \in M \), is called Minkowski metric tensor.

Note that, for each pair of basis vectors \( \vec{e}_\mu = C_{\partial / \partial x_\mu} \) and \( \vec{e}_\nu = C_{\partial / \partial x_\nu} \), it holds
\[
\eta(\vec{e}_\mu, \vec{e}_\nu) = \text{diag}(1, -1, -1, -1).
\]
(5)

Definition 4 The dual basis of \( \langle \vec{e}_\mu \rangle \) will be symbolized by \( \langle \gamma_\mu \rangle \), i.e., \( \gamma_\mu \in \Lambda^1(M) \) and \( \gamma_\mu(\vec{e}_\nu) = \delta^\mu_\nu \).

To continue, we observe the existence of a fundamental isomorphism between \( M \) and \( \Lambda^1(M) \) given by,
\[
M \ni \vec{a} \leftrightarrow a \in \Lambda^1(M),
\]
(6)
such that if \( \vec{a} = a^\mu \vec{e}_\mu \) then \( a = \eta_{\mu\nu} a_\mu \gamma^\nu \) and if \( a = a_\mu \gamma^\mu \) then \( \vec{a} = \eta^{\mu\nu} a_\mu \vec{e}_\nu \), where \( \eta_{\mu\nu} = \eta(\vec{e}_\mu, \vec{e}_\nu) \), \( \eta^{\mu\nu} = \eta_{\mu\nu} \).

Remark 5 To each basis vector \( \vec{e}_\mu \) correspond a basis form \( \gamma_\mu = \eta_{\mu\nu} \gamma^\nu \).

Definition 6 A scalar product of forms can be defined by
\[
\Lambda^1(M) \times \Lambda^1(M) \ni (a, b) \mapsto a \cdot b \in R,
\]
(7)
such that if \( \vec{a} \mapsto a \) and \( \vec{b} \mapsto b \) then \( a \cdot b = \eta(\vec{a}, \vec{b}) \).

Remark 7 \( \gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}, \gamma^\mu \cdot \gamma^\nu = \delta^\mu_\nu \) \( \langle \gamma_\mu \rangle \) is called the reciprocal basis of \( \langle \gamma^\mu \rangle \) and \( \gamma_\mu \cdot \gamma^\nu = \eta^{\mu\nu} \). Thus, \( \eta \) admits the expansions \( \eta = \eta_{\mu\nu} \gamma^\mu \otimes \gamma^\nu = \eta^{\mu\nu} \gamma_\mu \otimes \gamma_\nu \).

Remark 8 The oriented affine space \( (M, M^\ast) \) (oriented by \( \gamma^5 = \gamma^0 \wedge \gamma^1 \wedge \gamma^2 \wedge \gamma^3 \)) is a representation of the Minkowski manifold.

Remark 9 \( (M, M^\ast) \) equipped with the scalar product given by eq.\([7]\) is a representation of Minkowski spacetime.

Definition 10 Let \( \langle x^\mu \rangle \) be a global affine coordinate system for \( (M, M^\ast) \), relative to an arbitrary point \( o \in M \). A position form associated to \( x \in M \), is the form over \( M \) (designed by the same letter), given by the following correspondence
\[
M \ni x \mapsto x = x^\mu \gamma_\mu \in \Lambda^1(M).
\]
(8)

Remark 11 We denote by \( C(1,3) \cong H(2) \), the spacetime algebra, i.e., the Clifford algebra (Lounesto, 1997) of \( M^\ast \) equipped with the scalar product defined by eq.\([7]\).
Remark 12 As a vector space over the reals, we have $\mathcal{C}^\ell(M) = \sum_{p=0}^{4} \Lambda^p(M)$.

Definition 13 A smooth multiform field $A$ on Minkowski spacetime is a multiform valued function of position form,

$$\Lambda^1(M) \ni x \mapsto A(x) \in \Lambda(M).$$

Definition 14 Let $0 \leq p, q \leq 4$. A $(p,q)$-extensor $t$ is a linear mapping

$$t : \Lambda^p(M) \rightarrow \Lambda^q(M).$$

Remark 15 The set of all $(p,q)$-extensors is denoted by $\text{ext}(\Lambda^p(M), \Lambda^q(M))$.

Definition 16 A smooth $(p,q)$-extensor field $t$ on Minkowski spacetime is a differentiable $(p,q)$-extensor valued function of position form,

$$\Lambda^1(M) \ni x \mapsto t_x \in \text{ext}(\Lambda^p(M), \Lambda^q(M)).$$

Definition 17 The $a$-directional derivative ($a$ is an arbitrary form) of a smooth multiform field $X$, denoted as $a \cdot \partial X$, is defined by

$$a \cdot \partial X = \lim_{\lambda \rightarrow 0} \frac{X(x + \lambda a) - X(x)}{\lambda} = \frac{d}{d\lambda} X(x + \lambda a) \bigg|_{\lambda = 0}.$$  

Remark 18 The $\gamma_\mu$-directional derivative $\gamma_\mu \cdot \partial X$ coincides with the coordinate derivative $\partial_\mu X$. For short, we will use the notation $\partial_\mu \equiv \gamma_\mu \cdot \partial$.

Definition 19 The gradient, divergence and curl of a smooth multiform field $X$, respectively denoted by $\partial X$, $\partial \cdot X$ and $\partial \wedge X$, are defined by

$$\partial X = \gamma^\mu (\partial_\mu X).$$

$$\partial \cdot X = \gamma^\mu \cdot (\partial_\mu X).$$

$$\partial \wedge X = \gamma^\mu \wedge (\partial_\mu X).$$

Remark 20 For any $X$, it holds $\partial X = \partial \cdot X + \partial \wedge X$.

2 Lagrangian mappings $(X, \partial \ast X) \mapsto \mathcal{L}(X, \partial \ast X)$

Let $X$ be a smooth multiform field over the Minkowski spacetime $M$. Consider the ordinary multiform derivatives $\partial \ast X$, where $\ast$ means any one of the multiform products $(\cdot)$, $(\wedge)$ or (Clifford product) (i.e., the divergence, the curl or the gradient of $X$).  

\footnote{Called multivector derivatives in (Hestenes and Sobczyk, 1984), where this concept, for the best of our knowledge, has been introduced.}

\footnote{The dot product of defintion 1.5 is extended in a natural way to all $\mathcal{C}l(M)$ as follows: $\forall X, Y \in \Lambda(M), X \cdot Y = (XY)_0$. Note that it is an operation different from the left and right contractions. See (Rodrigues et al, 1995; Lounesto, 1999, Moya, Fernández and Rodrigues, 2000).}
Definition 21 A differentiable scalar-valued function of two multiform variables,
\[ \mathcal{L} : \text{Im} X \times \text{Im} \partial \ast X \to \mathbb{R}, \]  
(16)
(where ImX means image of the multiform field X, etc.) will be called Lagrangian mapping (LM) associated to X.

Definition 22 Let X be any smooth multiform field, then \( \hat{\mathcal{L}}[X] \) is a smooth scalar field defined by
\[ \Lambda^1(\mathcal{M}) \ni x \mapsto \hat{\mathcal{L}}[X](x) \in \mathbb{R}, \]  
(17)such that \( \hat{\mathcal{L}}[X](x) = \mathcal{L}[X(x), \partial \ast X(x)] \).

The operator \( \hat{\mathcal{L}} \) will be called Lagrangian operator and the smooth scalar field \( \hat{\mathcal{L}}[X] \) will be called Lagrangian scalar field associated to X.

Remark 23 For abuse of notation, in what follows, the Lagrangian mapping \( \mathcal{L} \) and the Lagrangian scalar field \( \hat{\mathcal{L}}[X] \) will be symbolized simply by \( (X, \partial \ast X) \mapsto \mathcal{L}(X, \partial \ast X) \).

Definition 24 To any LM \( (X, \partial \ast X) \mapsto \mathcal{L}(X, \partial \ast X) \), the action for the multiform field X (on \( U \subseteq \mathcal{M} \)) is the scalar (i.e., a real number)
\[ S = \int_U \hat{\mathcal{L}}[X](x) d^4x, \]  
(18)or, in abused notation \( S = \int_U \mathcal{L}(X, \partial \ast X) d^4x \).

Take an arbitrary smooth multiform field A, with the property A = \( \langle A \rangle_X \) (i.e., A and X are contains the same grades), such that it vanishes on the boundary \( \partial U \) (i.e., \( A|_{\partial U} = O \)), and take an open set \( S_0 \subset \mathbb{R} \), containing zero.

Definition 25 The \( \lambda \)-parametrized smooth scalar field
\[ \Lambda^1(\mathcal{M}) \times S_0 \ni (x, \lambda) \mapsto \hat{\mathcal{L}}[X + \lambda A](x) \in \mathbb{R}, \]  
(19)will be called varied Lagrangian.

Note that \( \hat{\mathcal{L}}[X + \lambda A](x) = \mathcal{L}[X(x) + \lambda A(x), \partial \ast X(x) + \lambda \partial \ast A(x)] \). Thus, for abuse of notation, the varied Lagrangian would be sometimes denoted by \( \mathcal{L}(X + \lambda A, \partial \ast X + \lambda \partial \ast A) \).

Definition 26 The \( \lambda \)-parametrized scalar (i.e., an ordinary scalar function of the real variable \( \lambda \))
\[ S_0 \ni \lambda \mapsto S(\lambda) \in \mathbb{R}, \]  
(20)such that
\[ S(\lambda) = \int_U \hat{\mathcal{L}}[X + \lambda A](x) d^4x, \]  
(21)(or \( S(\lambda) = \int_U \mathcal{L}(X + \lambda A, \partial \ast X + \lambda \partial \ast A) d^4x \), in abused notation) is called varied action.
Definition 27 Given any smooth multiform field $X$ and Lagrangian scalar field $\hat{L}[X]$, the variational operator $\delta_X$ is defined by $\Lambda^1(M) \ni x \mapsto \delta_X \hat{L}[X](x) \in \mathbb{R}$,

$$\delta_X \hat{L}[X](x) = \frac{d}{d\lambda} \hat{L}[X + \lambda A](x) \bigg|_{\lambda=0}$$

$$= \frac{d}{d\lambda} \mathcal{L}[X(x) + \lambda A(x), \partial^* X(x) + \lambda \partial^* A(x)] \bigg|_{\lambda=0}. \quad (22)$$

We simply write $\delta_X \mathcal{L}(X, \partial^* X) = \frac{d}{d\lambda} \mathcal{L}(X + \lambda A, \partial^* X + \lambda \partial^* A)|_{\lambda=0}$, in what follows.

In Lagrangian field theory, the dynamics of a multiform field $X$ is supposed to satisfy the so-called condition of stationary action, hereafter denoted \( AP \), i.e.,

$$S'(0) = 0, \text{ for all } A \text{ such that } A|_{\partial U} = 0. \quad (23)$$

The \( AP \) can also be written as

$$\int_U \delta_X \mathcal{L}(X, \partial^* X) d^4x = 0, \forall A \mid A|_{\partial U} = 0, \quad (24)$$

The \( AP \) implies the so-called Euler-Lagrange equation (ELE) for the multiform field $X$ (i.e., the field equation for $X$).

Proposition 28 Given a dynamical variable $X$ on $U \subseteq M$, and a LM $(X, \partial^* X) \mapsto \mathcal{L}(X, \partial^* X)$, where * is respectively: (a) $\cdot$, or (b) $\wedge$, or (c) the Clifford product, the \( AP \) implies for the cases (a), (b), (c) respectively the following ELEs.

(a) : $\partial_X \mathcal{L}(X, \partial X) - \partial \wedge \partial_{\partial X} \mathcal{L}(X, \partial X) = O,$ \quad (25)

(b) : $\partial_X \mathcal{L}(X, \partial \wedge X) - \partial \wedge \partial_{\partial X} \mathcal{L}(X, \partial \wedge X) = O,$ \quad (26)

(c) : $\partial_X \mathcal{L}(X, \partial X) - \partial \partial_X \mathcal{L}(X, \partial X) = O.$ \quad (27)

Proof.

Here we prove only case (c), leaving the proofs of (a) and (b) to the reader.

The $X$-variation of $\mathcal{L}(X, \partial X)$ yields

$$\delta_X \mathcal{L}(X, \partial X) = A \cdot \partial_X \mathcal{L}(X, \partial X) + \partial A \cdot \partial_{\partial X} \mathcal{L}(X, \partial X). \quad (28)$$

Using the identity (6.3), we have

$$\delta_X \mathcal{L}(X, \partial X) = A \cdot [\partial_X \mathcal{L}(X, \partial X) - \partial \partial_{\partial X} \mathcal{L}(X, \partial X)] + \partial \cdot [\partial_a (a A) \cdot \partial_{\partial X} \mathcal{L}(X, \partial X)], \quad (29)$$

and the \( AP \) yields

$$\int_U A \cdot (\partial_X \mathcal{L} - \partial \partial_{\partial X} \mathcal{L}) \, d^4x + \int_U \partial \cdot [\partial_a (a A) \cdot \partial_{\partial X} \mathcal{L}] \, d^4x = 0, \quad (30)$$

for all $A$ such that $A|_{\partial U} = O.$
Using the Gauss-Stokes theorem with the boundary condition $A|_{\partial U} = 0$, the second term gives

$$ \int_U \partial \cdot [\partial_u (aA) \cdot \partial_X L] \, d^4 x = \oint_{\partial U} \gamma^\mu \cdot [\partial_u (aA) \cdot \partial_X L] \, d^3 S_\mu $$

$$ = \oint_{\partial U} A \cdot (\gamma^\mu \partial_X L) \, d^3 S_\mu = 0. \quad (31) $$

Putting eq. (31) into eq. (30) gives

$$ \int_U A \cdot [\partial_X L(X, \partial X) - \partial \partial_{\partial X} L(X, \partial X)] \, d^4 x = 0, \text{ for all } A, \quad (32) $$

and due to the arbitrariness of $A$, we get

$$ \partial_X L(X, \partial X) - \partial \partial_{\partial X} L(X, \partial X) = 0. \quad (33) $$

3 Lagrangian mappings \((X, \mathcal{D} \ast X) \mapsto \mathcal{L}(X, \mathcal{D} \ast X)\)

Let $X$ be a smooth multiform field on \((U \subseteq M, M^*)\) and let $h$ be an invertible \((1,1)\)-extensor field. \((h_x : M^* \ni x \mapsto ext(\Lambda^1(M), \Lambda^1(M)))\) called the gauge metric extensor field (which is representation of the gravitational field in the most general possible gravitational theory over Minkowski spacetime). Also, define $h^* = (h^{-1})^\dagger = (h^\dagger)^{-1}$. Consider the operators $\mathcal{D}^\ast$ where $\ast$ means any multiform product ($\cdot$, $\wedge$) or the Clifford product acting on the set of smooth multivector fields. They are called, the $h$-divergence $\mathcal{D} \cdot X \equiv h^\ast(\partial_a) \cdot \mathcal{D} a \cdot X$, the $h$-curl $\mathcal{D} \wedge X \equiv h^\ast(\partial_a) \wedge \mathcal{D} a \cdot X$ and the $h$-gradient $\mathcal{D} X \equiv h^\ast(\partial_a) \mathcal{D} a \cdot X$.

$\mathcal{D} a \cdot X$ here is a directional covariant derivative, obtained from the Levi-Civita directional covariant derivative $D_a \cdot X = h(D_a h^{-1}(X))$ studied in the general geometrical algebraic theory of connections developed in (Fernández, Moya and Rodrigues, 2000b; Moya, Fernández and Rodrigues, 2000a,b)

$$ D_a \cdot X = a \cdot \partial X + \Omega(a) \times X. \quad (34) $$

We briefly recall that, $\Omega$ is called second connection extensor field, $\Omega_x : \Lambda^1(M) \rightarrow \Lambda^2(M)$, $\forall x \in M^*$, $\Omega_x(a) = -\frac{1}{2} \partial_n(n(x)) \wedge D_a n(x)$.

In theories which make use of the gauge covariant derivative concept, the action for the multiform field $X$ (on $U \subseteq M, M^*$), with dynamics given by an $a \mathcal{L}(X, \mathcal{D} \ast X) \mapsto \mathcal{L}(X, \mathcal{D} \ast X)$, the action is postulated to be the scalar

$$ S = \int_U \mathcal{L}(X, \mathcal{D} \ast X) d^4 x. \quad (35) $$

Take an arbitrary smooth multiform field $A$, with the property $A = (A)_X$, such that $A|_{\partial U} = 0$ and take an open set $S_0 \subset R$, containing zero.

**Definition 29** The $A$-varied action for the multivector field $X$ (on $U \subseteq M$) is the $\lambda$-parametrized scalar

$$ S(\lambda) = \int_U \mathcal{L}(X + \lambda A, \mathcal{D} \ast X + \lambda \mathcal{D} \ast A) \, d^4 x. \quad (36) $$
The dynamics of the multiform field $X$ is supposed to satisfy the $AP$
\[ S'(0) = 0, \text{ for all } A \text{ such that } A|_{\partial U} = O. \]

Equivalently, we have,
\[ \int_U \delta_X L(X, \mathcal{D} \ast X) \, d^4x = 0, \text{ for all } A \text{ such that } A|_{\partial U} = O, \]  \hspace{1cm} (37)

where $\delta_X L(X, \mathcal{D} \ast X) = \frac{d}{d\lambda} L(X + \lambda A, \mathcal{D} \ast X + \lambda \mathcal{D} \ast A)|_{\lambda=0}$ is the so-called $X$-variation of $L(X, \mathcal{D} \ast X)$.

**Proposition 30** Given a dynamical variable $X$ and a LM $(X, \mathcal{D} \ast X) \mapsto L(X, \mathcal{D} \ast X) = \det(h) \ell(X, \mathcal{D} \ast X)$ where $\ast$ is respectively: (a) $\ast$, or (b) $\wedge$, or (c) the Clifford product, the AP implies for the cases (a), (b), (c) the following ELEs,

\begin{align*}
(a) & : \partial_X \ell(X, \mathcal{D} \ast X) - \mathcal{D} \ast \partial_X \ell(X, \mathcal{D} \ast X) = 0, \hspace{1cm} (38) \\
(b) & : \partial_X \ell(X, \mathcal{D} \wedge X) - \mathcal{D} \ast \partial_X \ell(X, \mathcal{D} \wedge X) = 0, \hspace{1cm} (39) \\
(c) & : \partial_X \ell(X, \mathcal{D}X) - \mathcal{D} \partial_X \ell(X, \mathcal{D}X) = 0. \hspace{1cm} (40)
\end{align*}

**Proof.**

We prove only the case (b). Using the multiform identity (6.8) the $X$-variation of $L(X, \mathcal{D} \wedge X)$ yields
\[ \delta_X L(X, \mathcal{D} \wedge X) = \det(h)[A \cdot \partial_X \ell(X, \mathcal{D} \wedge X) + \mathcal{D} \wedge A \cdot \partial_{\mathcal{D} \wedge X} \ell(X, \mathcal{D} \wedge X)] \\
= \det(h) A \cdot [\partial_X \ell(X, \mathcal{D} \wedge X) - \mathcal{D} \ast \partial_{\mathcal{D} \wedge X} \ell(X, \mathcal{D} \wedge X)] + \partial \cdot [\det(h) \partial_a (h^* (a) \wedge A) \cdot \partial_{\mathcal{D} \wedge X} \ell(X, \mathcal{D} \wedge X)]. \]  \hspace{1cm} (41)

The $AP$ action yields,
\[ \int_U \det(h) A \cdot (\partial_X \ell - \mathcal{D} \ast \partial_{\mathcal{D} \wedge X} \ell) \, d^4x + \int_U \partial \cdot [\det(h) \partial_a (h^* (a) \wedge A) \cdot \partial_{\mathcal{D} \wedge X} \ell] \, d^4x = 0, \]  \hspace{1cm} (42)

for all $A$ such that $A|_{\partial U} = O$.

Using the Gauss-Stokes theorem with the boundary condition $A|_{\partial U} = O$, the second term gives,
\[ \int_U \partial \cdot [\det(h) \partial_a (h^* (a) \wedge A) \cdot \partial_{\mathcal{D} \wedge X} \ell] \, d^4x = \int_{\partial U} \det(h) \gamma^\mu \cdot [\partial_a (h^* (a) \wedge A) \cdot \partial_{\mathcal{D} \wedge X} \ell] \, d^3S^\mu \]  \hspace{1cm} (43)

Putting eq. (43) into eq. (42), we have
\[ \int_U \det(h) A \cdot [\partial_X \ell(X, \mathcal{D} \wedge X) - \mathcal{D} \ast \partial_{\mathcal{D} \wedge X} \ell(X, \mathcal{D} \wedge X)] \, d^4x = 0, \text{ for all } A. \]
and due to the arbitrariness of $A$, we finally get
\[ \partial_X \ell(X, D \wedge X) - D_J \partial_{D \wedge X} \ell(X, D \wedge X) = O. \tag{44} \]

**Remark 31** The proofs of (a) and (b) can be easily obtained by using the multiform identities (6.7) and (6.9).

### 4 Lagrangian Mapping \((\psi, D^s \psi) \mapsto \mathcal{L}(\psi, D^s \psi)\)

Let \(\psi\) be a smooth Dirac-Hestenes spinor field (DHSF) on \((U \subseteq M, M^*)\). We can take the gauge spinor derivative (Rodrigues et al., 1995; Fernández, Moya and Rodrigues, 2000b) \(D^s \psi = h^* (\partial_a) D^s_a \psi\) (recall that \(D^s_a \psi \equiv a \cdot \partial \psi + \frac{1}{2} \Omega(a) \psi\) is the directional spinor derivative) and consider a LM \((\psi, D^s \psi) \mapsto \mathcal{L}(\psi, D^s \psi)\).

**Definition 32** The action for a DHSF \(\psi\) (on \(U \subseteq M\)) is
\[ S = \int_U \mathcal{L}(\psi, D^s \psi) \, d^4x. \tag{45} \]

If we take an arbitrary smooth DHSF \(\eta\), such that \(\eta|_{\partial U} = O\), then, the so-called \(\psi\)-variation of \(\mathcal{L}(\psi, D^s \psi)\) is
\[ \delta_{\psi} \mathcal{L}(\psi, D^s \psi) = \left. \frac{d}{d\lambda} \mathcal{L}(\psi + \lambda \eta, D^s \psi + \lambda D^s \eta) \right|_{\lambda=0}. \tag{46} \]

**Proposition 33** Given a DHSF \(\psi\), as dynamical variable, and a LM \((\psi, D^s \psi) \mapsto \mathcal{L}(\psi, D^s \psi)\) the AP
\[ \int_U \delta_{\psi} \mathcal{L}(\psi, D^s \psi) \, d^4x = 0, \]
for all \(\eta\) such that \(\eta|_{\partial U} = O\), implies the ELE,
\[ \partial_{\psi} \ell(\psi, D^s \psi) - D^s \partial_{D^s \psi} \ell(\psi, D^s \psi) = O. \tag{47} \]

**Proof.**

The \(\psi\)-variation of \(\mathcal{L}(\psi, D^s \psi)\) is
\[ \delta_{\psi} \mathcal{L}(\psi, D^s \psi) = \frac{d}{d\lambda} \left[ \mathcal{L}(\psi + \lambda \eta, D^s \psi + \lambda D^s \eta) \right] \bigg|_{\lambda=0}, \tag{48} \]
and using the multiform identity (6.14) we have
\[ \delta_{\psi} \mathcal{L}(\psi, D^s \psi) = \partial(\mathcal{L}) \cdot \frac{d}{d\lambda} \left[ \mathcal{L}(\psi + \lambda \eta, D^s \psi + \lambda D^s \eta) \right] \bigg|_{\lambda=0}. \tag{49} \]

Thus, the AP can be written,
\[ \int_U \partial(\mathcal{L}) \cdot \frac{d}{d\lambda} \left[ \mathcal{L}(\psi + \lambda \eta, D^s \psi + \lambda D^s \eta) \right] \bigg|_{\lambda=0} \, d^4x = 0, \tag{50} \]
for all \(\eta\) such that \(\eta|_{\partial U} = O\).
The second term can be integrated using the Gauss-Stokes theorem with the boundary condition $\eta|_{\partial U} = 0$. We get,

$$\int_U \partial \cdot [\det(h) \partial_a (h^*(a) \eta) \cdot \partial_{D^* \psi} \ell] \ d^4x = \oint_{\partial U} \gamma^\mu \cdot [\det(h) \partial_a (h^*(a) \eta) \cdot \partial_{D^* \psi} \ell] \ d^3S_\mu = 0. \quad (51)$$

Putting eq. (51) into eq. (50), we get

$$\int_U \det(h) \eta \cdot [\partial_{\psi} \ell(\psi, D^* \psi) - D^* \partial_{D^* \psi} \ell(\psi, D^* \psi)] \ d^4x = 0, \text{ for all } \eta. \quad (52)$$

And since $\eta$ is arbitrary, it follows that

$$\partial_{\psi} \ell(\psi, D^* \psi) - D^* \partial_{D^* \psi} \ell(\psi, D^* \psi) = O. \quad (53)$$

5 Examples

5.1 Maxwell and Dirac-Hestenes Lagrangians on Minkowski spacetime

(a) The Lagrangian associated to the Maxwell field $A : M^* \to \Lambda^1(M)$ (i.e, the electromagnetic potential) generated by an electric charge current density $J : M^* \to \Lambda^0(M)$, is

$$\mathcal{L}(A, \partial \wedge A) = -\frac{1}{2\mu_0} (\partial \wedge A) \cdot (\partial \wedge A) - A \cdot J. \quad (54)$$

The Euler-Lagrange is then, according to previous results

$$\partial_A \mathcal{L}(A, \partial \wedge A) - \partial_\psi \partial_{\partial \wedge A} \mathcal{L}(A, \partial \wedge A) = 0. \quad (55)$$

Then, the Maxwell field $A$ and the Faraday field $F = \partial \wedge A$, satisfy the equations

$$\partial \wedge A = \mu_0 J, \text{ } \partial F = \mu_0 J. \quad (56)$$

The second equation in (56) is Maxwell equation in the spacetime calculus formalism (Hestenes, 1966).

(b) In quantum mechanics, the Lagrangian\(^6\) associated to the DHSF\(^7\) $\psi : M^* \to \Lambda^0(M) + \Lambda^2(M) + \Lambda^4(M)$, corresponding to a particle with mass $m$, electric charge $e$ and spin $\frac{1}{2}$ (i.e., a Dirac particle) in interaction with the Maxwell field $A$, is

$$\mathcal{L}(\psi, \partial \psi) = \hbar (\partial \psi i \gamma_3) \cdot \psi - e(A \psi \gamma_0) \cdot \psi - mc \psi \cdot \psi, \quad (57)$$

\(^6\) A thoughtful study of the Dirac-Hestenes Lagrangian which shows hidden assumptions in usual presentation can be found in (De Leo et al, 1999)

\(^7\) DHSF are certain equivalence classes of even sections of $\mathcal{C}(M)$. For details, see (Rodrigues et al, 1995).
where \( i = \gamma_0\gamma_1\gamma_2\gamma_3 \). To get the ELE we need

\[
\partial_\psi \mathcal{L}(\psi, \partial\psi) = \hbar \partial\psi \gamma_3 - 2eA\psi \gamma_0 - 2mc\psi,
\]

\[
\partial_{\partial\psi} \mathcal{L}(\psi, \partial\psi) = -\hbar \partial_{\partial\psi}(\partial\psi \cdot \psi \gamma_3) = -\hbar \psi \gamma_3.
\]

(58)

where the following multiform derivative formulas have been used,

\[
\partial_X (X \cdot X) = 2X, \quad \partial_X (X \cdot Y) = \langle Y \rangle_X, \quad \partial_X [(YXZ) \cdot X] = \left< YXZ + \tilde{Y}X\tilde{Z} \right> X.
\]

(59)

Thus, the DHSF \( \psi \) satisfies

\[
\hbar \partial\psi \sigma_3 - eA\psi = mc\psi \gamma_0, \quad \sigma_3 = \gamma_3\gamma_0
\]

(60)

which is the expression of the Dirac equation (called the Dirac-Hestenes equation (Hestenes, 1996)) in the spacetime calculus formalism.

### 5.2 Lagrangian for Maxwell and Dirac-Hestenes fields on a gravitational field background

In the flat spacetime formulation of the most general possible gravitational field theory (which includes curvature and torsion), this field is described by an invertible (1,1)- extensor field \( h \) (section 3).

(a) The dynamics of a Maxwell field \( A \) generated by a electric charge current density \( J \), moving in a background gravitational field is postulated to derived from the AP and the following Lagrangian

\[
\mathcal{L}(A, D \cdot A) = \det(h)[-\frac{1}{\mu_0}(D \cdot A) \cdot (D \cdot A) - A \cdot J].
\]

(61)

Then (using the identities in remark 4 in the Appendix), we obtain that \( A \) and \( F = D \cdot A \) satisfy

\[
D_A(D \cdot A) = \mu_0 J, \quad DF = \mu_0 J.
\]

(62)

(b) The dynamics of a DHSF \( \psi \) corresponding to a particle with mass \( m \), electric charge \( e \) and spin \( \frac{1}{2} \) (i.e., a Dirac particle), is supposed to be governed by the AP with Lagrangian

\[
\mathcal{L}(\psi, D^*\psi) = \det(h)[\hbar(D^*\psi \gamma_3) \cdot \psi - e(A\psi \gamma_0) \cdot \psi - mc\psi \cdot \psi],
\]

(63)

Then (using the identities mentioned in remark 5 of Appendix) we get that the DHSF \( \psi \) satisfies

\[
\hbar D^*\psi \sigma_3 - eA\psi = mc\psi \gamma_0.
\]

(64)
6 Appendix: fundamental identities used in Lagrangian formalism

Proposition 34 For all smooth multiform fields $X, Y$ and 1-form field $a$, it holds

$$ (\partial_a X) \cdot Y + X \cdot (\partial \wedge Y) = \partial \cdot [\partial_a (a \wedge X) \cdot Y]. \quad (65) $$

$$ (\partial \wedge X) \cdot Y + X \cdot (\partial Y) = \partial \cdot [\partial_a (a \wedge X) \cdot Y]. \quad (66) $$

$$ (\partial X) \cdot Y + X \cdot (\partial Y) = \partial \cdot [\partial_a (a X) \cdot Y]. \quad (67) $$

Proof.
In order to prove the first identity (65), we use the definitions of divergence and curl of a multiform field and the algebraic identity $(a \wedge B) \cdot C = B \cdot (a \wedge C)$, where $a$ is a 1-form and $B, C$ are multiforms. Then,

$$ (\partial_a X) \cdot Y + X \cdot (\partial \wedge Y) = (\gamma^\mu \cdot \gamma_\mu \cdot \partial X) \cdot Y + X \cdot (\gamma^\mu \cdot \gamma_\mu \cdot \partial Y) $$

$$ = \gamma_\mu \cdot \partial (\gamma^\mu \wedge X) \cdot Y + (\gamma^\mu \wedge \partial X) \cdot (\gamma_\mu \cdot \partial Y) $$

$$ = \gamma_\mu \cdot \partial (\gamma^\mu \wedge X) \cdot Y, \quad (68) $$

but, it is not difficult to transform the right side of (68) into a divergence of a 1-form field. Indeed,

$$ \gamma_\mu \cdot \partial (\gamma^\mu \wedge X) \cdot Y = \gamma_\beta \cdot \partial \gamma^\beta \cdot \gamma_\mu (\gamma^\mu \wedge X) \cdot Y $$

$$ = \gamma^\beta \cdot \partial \gamma_\beta \cdot \gamma_\mu (\gamma^\mu \wedge X) \cdot Y $$

$$ = \partial \cdot [\partial_a (a \wedge X) \cdot Y]. \quad (69) $$

Putting (68) into (69), complete the proof.

Remark 35 These identities are necessary, e.g., in the derivation from the AP of the ELE equations for a multiform field $X$ with dynamics governed by LM $(X, \partial_a X) \rightarrow L(X, \partial_a X), (X, \partial \wedge X) \rightarrow L(X, \partial \wedge X)$ or $(X, \partial X) \rightarrow L(X, \partial X)$.

The identity (69) can be proved by using (68) and, once again, the algebraic identity $(a \wedge B) \cdot C = B \cdot (a \wedge C)$. We have,

$$ (\partial \wedge X) \cdot Y + X \cdot (\partial Y) = (\partial_a Y) \cdot X + Y \cdot (\partial \wedge X) = \partial \cdot [\partial_a (a \wedge X) \cdot X] $$

$$ = \partial \cdot [\partial_a Y \cdot (a \wedge X)] = \partial \cdot [\partial_a (a \wedge X) \cdot Y]. \quad (70) $$

Identity (67) can be proved without difficulties by adding (65) and (66).

Proposition 36 For all smooth multiform fields $X, Y$ and 1-form field $a$, it holds

$$ (D_a \wedge X) \cdot Y + X \cdot (D \wedge Y) = \det(h^{-1}) \partial \cdot [\det(h) \partial_a (h^* (a \wedge X)) \cdot Y]. \quad (71) $$

$$ (D \wedge X) \cdot Y + X \cdot (D \wedge Y) = \det(h^{-1}) \partial \cdot [\det(h) \partial_a (h^* (a \wedge X)) \cdot Y]. \quad (72) $$

$$ (D X) \cdot Y + X \cdot (D Y) = \det(h^{-1}) \partial \cdot [\det(h) \partial_a (h^* (a X)) \cdot Y]. \quad (73) $$
Proof.

To prove the identity \( \text{(71)} \), we shall need to use two important multiform identities relating the gauge covariant divergence and the ordinary divergence, and the gauge covariant curl and the ordinary curl, respectively (Fernández, Moya and Rodrigues, 2000b)

\[
\mathcal{D} \wedge A = \det(h^{-1})h[\partial_j \det(h)h^{-1}(A)], \quad \mathcal{D} \wedge A = h^*[\partial \wedge h^1(A)]. \quad (74)
\]
and the multiform identity \( \text{(15)} \) above. We have

\[
(\mathcal{D} \wedge a) \cdot Y + X \cdot (\mathcal{D} \wedge Y) = \det(h^{-1})h[\partial_j \det(h)h^{-1}(X)] \cdot Y + X \cdot h^*[\partial \wedge h^1(Y)]
\]

\[
= \det(h^{-1})[\partial_j \det(h)h^{-1}(X)] \cdot h^1(Y)
\]

\[
+ \det(h)h^{-1}(X) \cdot [\partial \wedge h^1(Y)]
\]

\[
= \det(h^{-1})\partial \cdot [\partial_j \det(h)h^{-1}(X)] \cdot h^1(Y)
\]

\[
= \det(h^{-1})\partial \cdot [\det(h)\partial_j \partial_j a \cdot h^{-1}(X)] \cdot h^1(Y). \quad (75)
\]

Using the algebraic identity \( a \mathcal{D}(B) = \mathcal{D}(a \mathcal{D} B) \), where \( \mathcal{D} \) is the extension\(^8\) of a (1,1) extensor \( t \), \( a \) is a 1-form and \( B \) is a multiform, we can write

\[
[a \mathcal{D}^{-1}(X)] \cdot h^1(Y) = h^{-1}[\mathcal{D}(a \mathcal{D} X)] \cdot h^1(Y) = h^{-1}[\mathcal{D}(a \mathcal{D} X)] \cdot Y = h^{-1}[\mathcal{D}(a \mathcal{D} X)] \cdot Y. \quad (76)
\]

Putting \( \text{(76)} \) into the right side of \( \text{(75)} \), completes the proof.

The second identity \( \text{(72)} \) can be proved using \( \text{(71)} \), the algebraic identity \( (a \mathcal{D} B) \cdot C = B \cdot (a \wedge C) \), and following analogous steps just used in demonstrating the identity \( \text{(70)} \). The third identity \( \text{(73)} \) can be proved easily by adding \( \text{(71)} \) and \( \text{(72)} \).

Remark 37 These multiform identities are necessary in order to derive from the AP the ELEs for multiform fields \( X \) with Lagrangian mappings \( X, \mathcal{D}X \mapsto \mathcal{L}(X, \mathcal{D}X) \), or \( (X, \mathcal{D}X) \mapsto \mathcal{L}(X, \mathcal{D}X) \) or \( (X, \mathcal{D}X) \mapsto \mathcal{L}(X, \mathcal{D}X) \).

Proposition 38 For all smooth DHSF \( \psi \) and \( \varphi \), it holds

\[
(\mathcal{D}^*\psi) \cdot \varphi + \psi \cdot (\mathcal{D}^*\varphi) = (\mathcal{D}\psi) \cdot \varphi + \psi \cdot (\mathcal{D}\varphi). \quad (77)
\]

\[
(\mathcal{D}^*\psi) \cdot \varphi + \psi \cdot (\mathcal{D}^*\varphi) = \det(h^{-1})\partial \cdot [\det(h)\partial_j(h^*(a)\psi) \cdot \varphi]. \quad (78)
\]

To prove \( \text{(71)} \), note that for a smooth spinor field \( \psi \), we have

\[
\mathcal{D}\psi = \mathcal{D}^*\psi - \frac{1}{2}h^*(\partial_j a)\psi \Omega(a). \quad (79)
\]

Thus, by using \( \text{(79)} \) with the DHSF \( \psi \) and \( \varphi \), we can write

\[
(\mathcal{D}^*\psi) \cdot \varphi + \psi \cdot (\mathcal{D}^*\varphi) = [\mathcal{D}\psi + \frac{1}{2}h^*(\partial_j a)\psi \Omega(a)] \cdot \varphi + \psi \cdot [\mathcal{D}\varphi + \frac{1}{2}h^*(\partial_j a)\varphi \Omega(a)]
\]

\[
= (\mathcal{D}\psi) \cdot \varphi + \psi \cdot (\mathcal{D}\varphi)
\]

\[
- \frac{1}{2}h^*(\partial_j a) \cdot \varphi \Omega(a) \psi - \frac{1}{2}h^*(\partial_j a) \cdot \psi \Omega(a) \varphi,
\]

\[^8\text{Given a (1,1)-extensor} \ t \text{ over} \ \Lambda(\mathcal{M}), \text{its extension} \ \tilde{t} \text{, a general extensor over} \ \Lambda(\mathcal{M}), \text{is defined by} \ \tilde{t}(X) = 1 \cdot X + \sum_{i=1}^{n} \frac{1}{i+1} t(\gamma_i) \wedge \ldots \wedge t(\gamma_i) \wedge \ldots \wedge t(\gamma_i) \cdot X.\]
but, the last two terms yield zero, as can be seen from,

\[ \frac{1}{2} h^{*}(\partial_{a}) \cdot [\varphi \Omega(a) \bar{\psi} + \psi \Omega(a) \bar{\varphi}] = \frac{1}{2} h^{*}(\partial_{a}) \cdot [\varphi \Omega(a) \bar{\psi} - (\varphi \Omega(a) \bar{\psi})^{*}] = \frac{1}{2} h^{*}(\partial_{a}) \cdot 2 \left( \varphi \Omega(a) \bar{\psi} \right) = 0, \]  

which completes the proof. Eq. (78) follows directly from (77) and (6.9).

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