Upon \(h\)-normal \(\Gamma\)-linear connections
on \(J^1(T, M)\)

Mircea Neagu

Abstract

Section 1 introduces the notion of \(h\)-normal \(\Gamma\)-linear connection on the 1-jet fibre bundle \(J^1(T, M)\), and studies its local components. Section 2 analyses the main local components of torsion and curvature \(d\)-tensors attached to an \(h\)-normal \(\Gamma\)-linear connection \(\nabla\). Section 3 presents the local Ricci identities induced by \(\nabla\). The identities of the local deflection \(d\)-tensors are also exposed. Section 4 is dedicated to the writing of the local Bianchi identities of \(\nabla\).

Mathematics Subject Classification (1991): 53C07, 53C43, 53C99.

Key Words: 1-jet fibre bundle, nonlinear connection, \(\Gamma\)-linear connection, \(h\)-normal \(\Gamma\)-linear connection, Ricci and Bianchi identities.

1 Components of \(h\)-normal \(\Gamma\)-linear connections

Let \(T\) (resp. \(M\)) be a "temporal" (resp. "spatial") manifold of dimension \(p\) (resp. \(n\)) which is coordinated by \((t^\alpha)_{\alpha=1}^p\) (resp. \((x^i)_{i=1}^n\)). Let us consider the 1-jet fibre bundle \(J^1(T, M)\) → \(T \times M\), naturally coordinated by \((t^\alpha, x^i, x^j_\alpha)\). The coordinate transformations on the product manifold \(T \times M\), induce the following coordinate transformations (gauge group) on \(J^1(T, M)\),

\[
\begin{align*}
\tilde{t}^\alpha &= \tilde{t}^\alpha(t^\beta) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}^j_\alpha &= \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\alpha}{\partial t^k} x^j_\beta.
\end{align*}
\]

(1.1)

Note that, throughout this paper, the indices \(\alpha, \beta, \gamma, \ldots\) run from 1 to \(p\) while the indices \(i, j, k, \ldots\) run from 1 to \(n\).

On \(E = J^1(T, M)\), we fix a nonlinear connection \(\Gamma\) defined by the temporal components \(M^{(i)}_{(\alpha)\beta}\) and the spatial components \(N^{(i)}_{(\alpha)j}\). We recall that the transformation rules of the local components of the nonlinear connection \(\Gamma\) are expressed by

\[
\begin{align*}
\tilde{M}^{(i)}_{(\beta)\mu} \frac{\partial \tilde{t}^\mu}{\partial t^\alpha} &= M^{(k)}_{(\gamma)\alpha} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial t^\beta} - \frac{\partial \tilde{x}^j_\beta}{\partial t^\gamma} \\
\tilde{N}^{(i)}_{(\beta)k} \frac{\partial \tilde{x}^k}{\partial x^i} &= N^{(k)}_{(\gamma)i} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial t^\beta} - \frac{\partial \tilde{x}^j_\beta}{\partial x^i}.
\end{align*}
\]

(1.2)
Example 1.1 Let $h_{\alpha\beta}(t^i)$ (resp. $g_{ij}(x^k)$) a semi-Riemannian metric on the temporal (resp. spatial) manifold $T$ (resp. $M$), and $H^\gamma_{\alpha\beta}$ (resp $\gamma^k_{ij}$) its Christoffel symbols. Studying the transformation rules of the local components

\[
\begin{align*}
M_{(\beta)\alpha}^{(j)} &= -H^\gamma_{\alpha\beta} x^i_j, \\
N_{(\beta)\alpha}^{(j)} &= \gamma^k_{ij} x^i_k,
\end{align*}
\]

we conclude that $\Gamma_0 = (M_{(\beta)\alpha}, N_{(\beta)\alpha})$ represents a nonlinear connection on $E$. This is called the canonical nonlinear connection attached to the semi-Riemannian metrics $h_{\alpha\beta}$ and $\varphi_{ij}$.

Let us consider $\{\delta_{\delta t^\alpha}, \delta_{\delta x^i}, \partial_{\partial x^i_{\alpha}}\} \subset \mathcal{X}(E)$ and $\{dt^\alpha, dx^i, \delta x^i_{\alpha}\} \subset \mathcal{X}^*(E)$ the adapted bases of the nonlinear connection $\Gamma$, where

\[
\begin{align*}
\frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} - M_{(\beta)\alpha}^{(j)} \frac{\partial}{\partial x^i_j}, \\
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_{(\beta)\alpha}^{(j)} \frac{\partial}{\partial x^i_j}, \\
\delta x^i_{\alpha} &= dx^i_{\alpha} + M_{(\alpha)\beta}^{(i)} dt^\beta + N_{(\alpha)j}^{(i)} dx^j.
\end{align*}
\]

These bases will be used in the description of geometrical objects on $E$, because their transformation laws are very simple [10]:

\[
\begin{align*}
\frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha}, \\
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i}, \\
\frac{\delta}{\delta x^i_{\alpha}} &= \frac{\partial}{\partial x^i_{\alpha}}.
\end{align*}
\]

In order to develop the theory of $\Gamma$-linear connections on the 1-jet space $E$, we need the following

Proposition 1.1 i) The Lie algebra $\mathcal{X}(E)$ of vector fields decomposes as

\[\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}),\]

where

\[\mathcal{X}(\mathcal{H}_T) = \text{Span}\left\{ \frac{\delta}{\delta t^\alpha} \right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span}\left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(\mathcal{V}) = \text{Span}\left\{ \frac{\partial}{\partial x^i_{\alpha}} \right\}.
\]

ii) The Lie algebra $\mathcal{X}^*(E)$ of covector fields decomposes as

\[\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}),\]

where

\[\mathcal{X}^*(\mathcal{H}_T) = \text{Span}\{ dt^\alpha \}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span}\{ dx^i \}, \quad \mathcal{X}^*(\mathcal{V}) = \text{Span}\{ \delta x^i_{\alpha} \}.
\]
Let us consider $h_T$, $h_M$ (horizontal) and $v$ (vertical) as the canonical projections of the above decompositions. In this context, we have

**Definition 1.1** A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E)$ is called a $\Gamma$-linear connection on $E$ if $\nabla h_T = 0$, $\nabla h_M = 0$ and $\nabla v = 0$.

In order to describe in local terms a $\Gamma$-linear connection $\nabla$ on $E$, we need nine unique local components,

\[
\nabla \Gamma = (\bar{G}^\alpha_{\beta\gamma}, G^k_{\alpha\gamma}, C^{(i)(\beta)}_{(\alpha)(j)\gamma}, \bar{L}^\alpha_{\beta j}, L^k_{(\alpha)(j)k}, \bar{C}^{(\gamma)(\gamma)}_{\beta(k)}, C^{(i)(\beta)(\gamma)}_{\gamma(k)}, C^{(i)(\beta)(\gamma)}_{(\alpha)(j)(k)}),
\]

which are locally defined by the relations

\[
\begin{align*}
\langle h_T \rangle & \left. \frac{\delta}{\delta \epsilon} \right| = \bar{G}^\alpha_{\beta\gamma} \frac{\delta}{\delta \epsilon^\alpha} - \frac{\delta}{\delta \epsilon^\beta} - \delta x^\gamma \\
\langle h_M \rangle & \left. \frac{\delta}{\delta \epsilon} \right| = L^k_{\gamma(j)k} \frac{\delta}{\delta \epsilon^k} - \frac{\delta}{\delta \epsilon^k} - \delta x^\gamma \\
\langle v \rangle & \left. \frac{\delta}{\delta \epsilon} \right| = C^{(i)(\gamma)}_{\alpha(i)\gamma} \frac{\delta}{\delta \epsilon^i} - \frac{\delta}{\delta \epsilon^i} - \delta x^\gamma
\end{align*}
\]

**Remark 1.1** The transformation rules of the above connection coefficients are completely described in \([12]\).

**Example 1.2** Let us consider $\Gamma_0 = (M^{(i)}_{(\alpha)(j)}, N^{(i)}_{(\alpha)(j)})$ the canonical nonlinear connection on $E$, attached to the semi-Riemannian metrics pair $(h_{\alpha\beta}, \varphi_{ij})$. In these conditions, the following local coefficients \([12]\)

\[
B\Gamma_0 = (\bar{G}^\alpha_{\beta\gamma}, 0, G^{(k)(\beta)}_{(\alpha)(i)\gamma}, 0, L^k_{(\alpha)(i)j}, 0, 0, 0),
\]

where $\bar{G}^\gamma_{\alpha\beta} = H^{\gamma}_{\alpha\beta}$, $G^{(k)(\beta)}_{(\alpha)(i)\gamma}$, $L^k_{(\gamma)(i)j}$, and $L^{(k)(\beta)}_{(\gamma)(i)j} = \delta^{k}_{\gamma} \delta^{\beta}_{\gamma}$, verify the transformation rules of the local coefficients of a $\Gamma_0$-linear connection. This is called the Berwald $\Gamma_0$-linear connection of the metrics pair $(h_{\alpha\beta}, \varphi_{ij})$.

Now, let $\nabla$ be a $\Gamma$-linear connection on $E$, locally defined by \([12]\). The linear connection $\nabla$ induces a natural linear connection on the d-tensors set of the jet fibre bundle $E = J^1(T, M)$, in the following fashion: starting with a vector field $X$ and a d-tensor field $D$ locally expressed by

\[
\begin{align*}
X &= X^\alpha \left. \frac{\delta}{\delta t^\alpha} \right| + X^m \left. \frac{\delta}{\delta x^m} \right| + X^{(m)}_{(\alpha)} \left. \frac{\partial}{\partial x^m} \right| \\
D &= D^{\alpha(i)(\beta)(\gamma)}_{\gamma(k)(\beta)(\gamma)} \left. \frac{\delta}{\delta t^\alpha} \right| \otimes \left. \frac{\delta}{\delta x^\beta} \right| \otimes \left. \frac{\partial}{\partial t^\gamma} \right| \otimes dx^k \otimes dx^l \ldots
\end{align*}
\]

we introduce the covariant derivative

\[
\nabla X = X^\alpha \nabla_{\alpha} + X^m \nabla_{m} + X^{(m)}_{(\alpha)} \nabla_{t^m},
\]

\[
\nabla D = \left[ X^{\alpha} D^{\alpha(i)(\beta)(\gamma)}_{\gamma(k)(\beta)(\gamma)} \right] \left. \frac{\delta}{\delta t^\alpha} \right| \otimes \left. \frac{\delta}{\delta x^\beta} \right| \otimes \left. \frac{\partial}{\partial t^\gamma} \right| \otimes dx^k \otimes dx^l \ldots.
\]
where

\[
\begin{align*}
(h_T) & \quad D_{\gamma k(\beta)}(\delta)_{/\varepsilon} = \frac{\delta D^{\alpha i(j)(\delta)}}{\delta t^e} + D^{\mu i(j)(\delta)}_{\gamma k(\beta)}(l)\cdots \bar{G}^\alpha_{\mu e} + \\
D^{\alpha i(j)(\delta)}_{\gamma k(\beta)}(l)\cdots G^\alpha_{m e} + D^{\alpha i(m)(\delta)}_{\gamma k(\mu)}(l)\cdots G^{(m)(\delta)}_{(\mu)(l)}e + \ldots - \\
- D^{\alpha i(j)(\delta)}_{\mu k(\beta)}(l)\cdots G^\alpha_{\gamma e} - D^{\alpha i(j)(\delta)}_{\gamma m(\beta)}(l)\cdots G^\alpha_{k e} - D^{\alpha i(j)(\mu)}_{\gamma k(\beta)}(m)\cdots G^{(m)(\delta)}_{(\mu)(l)}e - \ldots,
\end{align*}
\]

\[
(h_M) & \quad D^{\alpha i(j)(\delta)}_{\gamma k(\beta)}(l)\cdots |_p = \frac{\delta D^{\alpha i(j)(\delta)}}{\delta x^p} + D^{\mu i(j)(\delta)}_{\gamma k(\beta)}(l)\cdots f^\alpha_{\mu p} + \\
D^{\alpha i(j)(\delta)}_{\gamma k(\beta)}(l)\cdots l^i_m + D^{\alpha i(m)(\delta)}_{\gamma k(\mu)}(l)\cdots l^{(m)(\delta)}_{(\mu)(m)p} + \ldots - \\
- D^{\alpha i(j)(\delta)}_{\mu k(\beta)}(l)\cdots l^m_\gamma - D^{\alpha i(j)(\delta)}_{\gamma m(\beta)}(l)\cdots l^m_{(\mu)(l)p} - D^{\alpha i(j)(\mu)}_{\gamma k(\beta)}(m)\cdots l^{(m)(\delta)}_{(\mu)(l)p} - \ldots.
\]

\[
(v) & \quad D^{\alpha i(j)(\delta)}_{\gamma k(\beta)}(l)\cdots |_(e) = \frac{\partial D^{\alpha i(j)(\delta)}}{\partial x^e} + D^{\mu i(j)(\delta)}_{\gamma k(\beta)}(l)\cdots C^\alpha_{\mu e} + \\
D^{\alpha i(m)(\delta)}_{\gamma (\beta)(m)(p)} + D^{\alpha i(m)(\delta)}_{\gamma k(\mu)}(l)\cdots C^{(m)(\delta)}_{(\mu)(l)p} + \ldots - \\
- D^{\alpha i(j)(\delta)}_{\mu k(\beta)}(l)\cdots C^\alpha_{\gamma p} - D^{\alpha i(j)(\delta)}_{\gamma m(\beta)}(l)\cdots C^{(m)(\delta)}_{k p} - D^{\alpha i(j)(\mu)}_{\gamma k(\beta)}(m)\cdots C^{(m)(\delta)}_{(\mu)(l)p} - \ldots.
\]

The local operators ""/"", ""|_p"" and ""|_(e)"" are called the T-horizontal covariant derivative, M-horizontal covariant derivative and vertical covariant derivative of the \(\Gamma\)-linear connection \(\nabla\).

**Remarks 1.2** i) In the particular case of a function \(f(t^\gamma, x^k, x^\gamma)\) on \(J^1(T, M)\), the above covariant derivatives reduce to

\[
\begin{align*}
(1.7) & \quad f_{/\varepsilon} = \frac{\delta f}{\delta t} = \frac{\partial f}{\partial t^e} - M^{(k)(\gamma)e} \frac{\partial f}{\partial x^e} \\
f_{|_p} = \frac{\delta f}{\delta x^p} = \frac{\partial f}{\partial x^p} - N^{(k)(\gamma)p} \frac{\partial f}{\partial x^\gamma} \\
f_{|(e)} = \frac{\partial f}{\partial x^e}.
\end{align*}
\]

ii) Particularly, starting with a d-vector field \(X\) on \(J^1(T, M)\), locally expressed by

\[
X = X^\alpha \frac{\partial}{\partial t^\alpha} + X^i \frac{\partial}{\partial x^i} + X^{(i)} \frac{\partial}{\partial x^i},
\]

the following expressions of above covariant derivatives hold good:

\[
\begin{align*}
(h_T) & \quad X^\alpha_{/\varepsilon} = \frac{\delta X^\alpha}{\delta t^\varepsilon} + X^\mu \bar{G}^\alpha_{\mu e} \\
X^i_{/\varepsilon} = \frac{\delta X^i}{\delta t^\varepsilon} + X^m G^i_{me} \\
X^{(i)}_{(\alpha)/\varepsilon} = \frac{\delta X^{(i)}}{\delta t^\varepsilon} + X^{(m)}_{(\mu)} C^{(i)(\mu)}_{(m)e},
\end{align*}
\]
Theorem 1.2

The coefficients of an equivalent to quantities The first three relations come from the definition of an "connection."

Remark 1.3

Taking into account the local covariant derivatives associated to the Berwald $\Gamma_0$-linear connection, will be denoted by " $\|/\|$ " and " $\|(\epsilon)\|".

Now, let $h_{\alpha\beta}$ be a fixed pseudo-Riemannian metric on the temporal manifold $T$, $H_{\alpha\beta}^\gamma$ its Christoffel symbols and $J = J_{(\alpha)(\beta)\gamma} = \delta_{\alpha\beta}\delta_{\gamma}^\delta \otimes dt^\delta \otimes dx^\gamma$, where $J_{(\alpha)(\beta)\gamma} = h_{\alpha\beta}\delta_{\gamma}^\delta$, the normalization d-tensor $[\Box]$ attached to the metric $h_{\alpha\beta}$. The big number of coefficients which characterize a $\Gamma$-linear connection on $E$, determines us to consider the following

Definition 1.2 A $\Gamma$-linear connection $\nabla$ on $J^1(T, M)$, defined by the local coefficients

$\nabla = (\hat{G}_{\alpha\gamma}^\delta, H_{\alpha\beta}^\gamma, C_{\alpha\beta(\gamma)}^\delta, \hat{L}_{\alpha\beta}^\delta, I_{\alpha\beta(\gamma)}^\delta, \bar{C}_{\alpha\beta(\gamma)}^\delta, J_{(\alpha)(\beta)(\gamma)}^\delta, C_{(\alpha)(\beta)(\gamma)}^\delta),$

that verify the relations $\hat{G}_{\alpha\gamma}^\delta = H_{\alpha\beta}^\gamma, \hat{L}_{\alpha\beta}^\delta = 0, \bar{C}_{\alpha\beta(\gamma)}^\delta = 0$ and $\nabla J = 0$, is called an $h$-normal $\Gamma$-linear connection.

Remark 1.3 Taking into account the local covariant T-horizontal " $\|/\|$ " and M-horizontal " $\|/\|$ " and vertical " $\|(\epsilon)\|$ " covariant derivatives induced by $\nabla$, the condition $\nabla J = 0$ is equivalent to $J_{(\alpha)(\beta)\gamma} = 0, J_{(\alpha)(\beta)\gamma} = 0$.

In this context, we can prove the following

Theorem 1.2 The coefficients of an h-normal $\Gamma$-linear connection $\nabla$ verify the identities

$\hat{G}_{\alpha\beta}^\gamma = H_{\alpha\beta}^\gamma, \hat{L}_{\beta\gamma}^\delta = 0, \bar{C}_{\beta(\gamma)}^\delta = 0, G_{(\alpha)(\gamma)\gamma}^{(\beta)} = \delta_{\alpha\beta}C_{(\gamma)\gamma}^\delta, L_{(\alpha)(\gamma)\gamma}^{(\beta)} = \delta_{\alpha\beta}L_{(\gamma)\gamma}^{(\beta)}$, $C_{(\alpha)(\gamma)\gamma}^{(\beta)(\gamma)} = \delta_{\alpha\beta}C_{(\gamma)\gamma}^{(\beta)(\gamma)}$.

Proof. The first three relations come from the definition of an $h$-normal $\Gamma$-linear connection.
The condition $\nabla J = 0$ implies locally that

$$
\begin{cases}
    h_{\beta\mu}G^{(i)}_{(\alpha)(j)\gamma} = h_{\alpha\beta}g^{i}_{j} + \delta^{i}_{j} \left[ -\frac{\partial h_{\alpha\beta}}{\partial \gamma} + H_{\beta\gamma\alpha} \right], \\
    h_{\beta\mu}L^{(i)}_{(\alpha)(j)} = h_{\alpha\beta}L_{ij}, \\
    h_{\beta\mu}C^{(i)(\gamma)}_{(\alpha)(j)(k)} = h_{\alpha\beta}C_{ij}^{(\gamma)}(k),
\end{cases}
$$

where $H_{\beta\gamma\alpha} = H_{\beta\gamma}^{\mu}h_{\mu\alpha}$ represent the Christoffel symbols of the first kind attached to the pseudo-Riemannian metric $h_{\alpha\beta}$. Contracting the above relations by $h_{\beta\epsilon}$, one obtains the last three identities of the theorem.

**Remarks 1.4**

i) The preceding theorem implies that an $h$-normal $\Gamma$-linear on $E$ is determined just by four effective coefficients

$$
\nabla \Gamma = (H_{\gamma}^{\beta\alpha}, G_{ij}^{k}, L_{ij}^{k}, C_{ij}^{(\gamma)}).
$$

ii) In the particular case $(T, h) = (R, \delta)$, a $\delta$-normal $\Gamma$-linear connection identifies to the notion of $N$-linear connection used in Lagrangian geometry [5].

**Example 1.3** The canonical Berwald $\Gamma_{0}$-linear connection associated to the metrics pair $(h_{\alpha\beta}, \varphi_{ij})$ is an $h$-normal $\Gamma_{0}$-linear connection, defined by the local coefficients $B_{0} = (H_{\gamma}^{\beta\alpha}, 0, \gamma_{ij}^{k}, 0)$.

## 2 Components of torsion and curvature d-tensors

The study of the torsion $T$ and curvature $R$ d-tensor of an arbitrary $\Gamma$-linear connection $\nabla$ on $E$ was made in [12]. In that context, we proved that the torsion d-tensor is determined by twelve effective local torsion d-tensors, while the curvature d-tensor of $\nabla$ is determined by eighteen local d-tensors.

Let us start with an $h$-normal $\Gamma$-linear connection $\nabla$. Following the formulas described in [12] and using the properties of $\nabla$, it follows that the torsion d-tensors $\bar{T}_{\alpha\beta}^{\mu}$, $\bar{T}_{\alpha j}^{\mu}$ and $P_{\alpha(j)}^{\mu(\beta)}$ vanish. Moreover, we deduce that the following theorem holds good.

**Theorem 2.1** The torsion d-tensor $T$ of an $h$-normal $\Gamma$-linear connection $\nabla$, is determined by nine effective local d-tensors,

\[
\begin{array}{|c|c|c|}
\hline
h_{T} & h_{M} & v \\
\hline
h_{T}h_{T} & 0 & R^{(m)}_{(\mu)\alpha\beta} \\
\hline
h_{M}h_{T} & 0 & T^{m}_{\alpha j} \\
\hline
h_{M}h_{M} & 0 & T^{m}_{ij} \\
\hline
v_{h_{T}} & 0 & 0 \\
\hline
v_{h_{M}} & 0 & P^{m(\beta)}_{(\mu)\alpha(j)} \\
\hline
vv & 0 & S^{(m)(\beta)}_{(\mu)(i)(j)} \\
\hline
\end{array}
\]

(2.1)
where \( P^{(m)}_{(\mu)\alpha(j)} = \frac{\partial M_{(\mu)\alpha}}{\partial x^\beta} - \delta^\beta_\mu G_{j\alpha} + \delta^m_\beta H_{\mu\alpha}, \) \( P^{(m)}_{(\mu)\beta(i)} = \frac{\partial N_{(\mu)i}}{\partial x^\beta} - \delta^\beta_\mu L^m_{ji}, \)

\[
R^{(m)}_{(\mu)\alpha\beta} = \frac{\delta M^{(m)}_{(\mu)\beta}}{\delta t^\alpha}, \quad R^{(m)}_{(\mu)\alpha\beta} = \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta t^\beta}, \quad R^{(m)}_{(\mu)ij} = \frac{\delta N^{(m)}_{(\mu)i}}{\delta x^j} - \frac{\delta N^{(m)}_{(\mu)j}}{\delta x^i}, \quad S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)} = \delta^\alpha_\mu C^{m(\beta)}_{(\mu)\beta(i)} - \delta^\beta_\mu C^{m(\alpha)}_{(\mu)\alpha(i)}.
\]

**Remark 2.1** For the Berwald \( \Gamma_0 \)-linear connection associated to the metrics \( h_{\alpha\beta} \) and \( \varphi_{ij} \), all torsion d-tensors vanish, except

\[
R^{(m)}_{(\mu)\alpha\beta} = -H^\gamma_{\mu\alpha\beta} x^\gamma, \quad R^{(m)}_{(\mu)\alpha\beta} = r^m_{ij \alpha \beta}.
\]

where \( H^\gamma_{\mu\alpha\beta} \) (resp. \( r^m_{ij \alpha \beta} \)) are the curvature tensors of the metric \( h_{\alpha\beta} \) (resp. \( \varphi_{ij} \)).

The form of expressions of local curvature d-tensors from the general case of a \( \Gamma \)-linear connection \([12]\), and again the properties of the \( h \)-normal \( \Gamma \)-linear connection \( \nabla \), imply a reduction (from eighteen to seven) of the number of the effective curvature d-tensors attached to an \( h \)-normal \( \Gamma \)-linear connection. Consequently, we obtain

**Theorem 2.2** The curvature d-tensor \( R \) of an \( h \)-normal \( \Gamma \)-linear connection \( \nabla \), is characterized by seven effective local d-tensors,

\[
\begin{array}{|c|c|c|c|c|}
\hline
     & h_T & h_M & v & \\
\hline
h_T h_T & H^\alpha_{\eta\beta\gamma} & R^{(l)}_{\alpha(ij)} & R^{(l)(\alpha)}_{(\eta)(i)(j)} = \delta^\alpha_\eta R^{(l)}_{\alpha(ij)} + \delta^l_\eta H^\alpha_{\eta\beta\gamma} & \\
\hline
h_M h_T & 0 & R^{(l)}_{\alpha(ij)} & R^{(l)(\alpha)}_{(\eta)(i)(j)} = \delta^\alpha_\eta R^{(l)}_{\alpha(ij)} & \\
\hline
h_M h_M & 0 & 0 & R^{(l)}_{\alpha(ij)} & \\
\hline
v h_T & 0 & 0 & P^{(l)}_{\alpha(ij)} & \\
\hline
v h_M & 0 & 0 & P^{(l)}_{\alpha(ij)} & \\
\hline
v v & 0 & 0 & S^{(l)(\alpha)(\beta)}_{(i)(j)} & \\
\hline
\end{array}
\]

where

\[
H^\alpha_{\eta\beta\gamma} = \frac{\partial H^\alpha_{\eta\beta\gamma}}{\partial x^\gamma} - \frac{\partial H^\alpha_{\eta\beta\gamma}}{\partial x^\beta} + H^\alpha_{\eta\beta\gamma} H^\alpha_{\eta\beta\gamma} - H^\mu_{\eta\beta\gamma} H^\nu_{\eta\beta\gamma},
\]

\[
R^{(l)}_{\alpha(ij)} = \frac{\delta G^{\alpha}_{\eta\beta\gamma}}{\delta x^\gamma} - \frac{\delta G^{\alpha}_{\eta\beta\gamma}}{\delta x^\beta} + G^{\alpha}_{\eta\beta\gamma} H^\alpha_{\eta\beta\gamma} - G^{\alpha}_{\eta\beta\gamma} H^\alpha_{\eta\beta\gamma} - C^{(l)}_{(\mu)(\alpha)} R^{(m)}_{(\mu)\beta\gamma},
\]

\[
R^{(l)}_{\alpha(ij)} = \frac{\delta L^{(l)}_{\alpha(ij)}}{\delta x^\gamma} - \frac{\delta L^{(l)}_{\alpha(ij)}}{\delta x^\beta} + L^{(l)}_{\alpha(ij)} L^{(l)}_{\alpha(ij)} - L^{(l)}_{\alpha(ij)} L^{(l)}_{\alpha(ij)} + C^{(l)}_{(\mu)(\alpha)} R^{(m)}_{(\mu)\beta\gamma},
\]

\[
P^{(l)}_{\alpha(ij)} = \frac{\delta C^{(l)}_{\alpha(ij)}}{\delta x^\gamma} - C^{(l)}_{\alpha(ij)} C^{(l)}_{\alpha(ij)} + C^{(l)}_{(\mu)(\alpha)} P^{(m)}_{(\mu)\beta(ij)},
\]

\[
P^{(l)}_{\alpha(ij)} = \frac{\delta C^{(l)}_{\alpha(ij)}}{\delta x^\gamma} - C^{(l)}_{\alpha(ij)} C^{(l)}_{\alpha(ij)} + C^{(l)}_{(\mu)(\alpha)} P^{(m)}_{(\mu)\beta(ij)}.
\]
\[
S_{i(j)(k)}^{(\beta)(\gamma)} = \frac{\partial C_{i(j)}^{(\beta)}}{\partial x_\gamma} - \frac{\partial C_{i(k)}^{(\gamma)}}{\partial x_\beta} + C_{i(k)}^{m(\beta)} C_{m(k)}^{(\gamma)} - C_{i(k)}^{m(\gamma)} C_{m(k)}^{(\beta)},
\]

**Remark 2.1** In the case of the Berwald \(\Gamma_h\)-linear connection associated to the metric pair \((h_{\alpha\beta}, \varphi_{ij})\), all curvature d-tensors vanish, except \(H_{\alpha\beta\gamma}^h\) and \(R_{ijk}^h = r_{ijk}\), where \(r_{ijk}\) are the curvature tensors of the metric \(\varphi_{ij}\).

### 3 Ricci identities. Deflection d-tensors identities

The Ricci identities of a \(\Gamma\)-linear connection are described in [2]. In the particular case of an \(h\)-normal \(\Gamma\)-linear connection, these simplify because the number and the form of the torsion and curvature d-tensors reduced. A meaningful reduction of these identities can be obtained, considering the more particular case of an \(h\)-normal \(\Gamma\)-linear connection \(\nabla\) of Cartan type, (i. e., \(L_{jk}^i = L_{kj}^i\) and \(C_{i(j)}^{m(\gamma)} = C_{k(j)}^{m(\gamma)}\)). In that case, the condition \(L_{jk} = L_{kj}^i\) implies \(T_{jk}^i = 0\). Consequently, we have

**Theorem 3.1** The following Ricci identities of an \(h\)-normal \(\Gamma\)-linear connection of Cartan type, are true:

\[
\begin{align*}
\{h_T\} & \quad X^\alpha_{/\gamma} - X^\alpha_{/\gamma /\beta} = X^\mu H^\alpha_{/\mu /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma} \\
& \quad X^\alpha_{/\beta k} - X^\alpha_{/k /\beta} = -X^\alpha_{/m} T^m_{/\beta k} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\beta k} \\
& \quad X^\alpha_{/\gamma k} - X^\alpha_{/k /\gamma} = -X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma k} \\
& \quad X^\alpha_{/\gamma /\gamma} - X^\alpha_{/\gamma /\beta} = -X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta} \\
X^\alpha_{/\beta} - X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta}, \\
& \quad X^\alpha_{/\gamma /\beta} - X^\alpha_{/k /\beta} = -X^\alpha_{/m} T^m_{/\beta k} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\beta k} \\
& \quad X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta}, \\
X^\alpha_{/\gamma /\beta} - X^\alpha_{/k /\beta} = -X^\alpha_{/m} T^m_{/\beta k} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\beta k} \\
& \quad X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta}, \\
& \quad X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta}.
\end{align*}
\]

\[
\{h_M\} & \quad X^\alpha_{/\gamma} - X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta} \\
& \quad X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta} \\
& \quad X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta} \\
& \quad X^\alpha_{/\gamma /\beta} = X^m P^\alpha_{m, \beta /\gamma} - X^\alpha_{(\mu)} P^{(m)}_{(\mu) /\gamma /\beta}.
\]

8
\[
\begin{align*}
\mathcal{D}^{(i)}_{(\alpha)\beta} &= x^{i}_{\alpha/\beta}, \\
\mathcal{D}^{(i)}_{(\alpha)\beta} &= x^{i}_{\alpha|\beta}, \\
d^{(i)}_{(\alpha)(\beta)} &= x^{i}_{\alpha(\beta)}.
\end{align*}
\]

where "\(\alpha\)" , "\(\beta\) " and "\(\gamma\) " are the local covariant derivatives induced by \(\nabla\). By a
direct calculation, the deflection d-tensors get the expressions

\[
\begin{align*}
\mathcal{D}^{(i)}_{(\alpha)\beta} &= -M^{(i)}_{(\alpha)\beta} + \mathcal{G}^{m}_{\alpha\beta} x^{m} - H^{\mu}_{(\alpha)\beta} x^{\mu}, \\
D^{(i)}_{(\alpha)\gamma} &= -N^{(i)}_{(\alpha)\gamma} + L^{i}_{mj} x^{m}, \\
d^{(i)}_{(\alpha)(\beta)} &= \delta^{i}_{j} \delta^{\beta}_{\alpha} + C^{(\beta)}_{mj} x^{m}.
\end{align*}
\]

Applying the \((v)\)-set of the Ricci identities, attached to the h-normal \(\Gamma\)-linear connection \(\nabla\), verify the identities:

\[
\begin{align*}
\mathcal{D}^{(i)}_{(\alpha)\beta}/(\gamma) - \mathcal{D}^{(i)}_{(\alpha)\gamma}/(\beta) &= x^{m}_{\alpha/\beta} R^{m}_{\gamma\beta} - x^{i}_{\mu} H^{\mu}_{\alpha\beta\gamma} - d^{(i)}_{(\alpha)(\mu)} R^{(m)}_{(\mu)\beta\gamma}, \\
\mathcal{D}^{(i)}_{(\alpha)\beta} - \mathcal{D}^{(i)}_{(\alpha)\beta}/(\gamma) &= x^{m}_{\alpha/\beta} R^{m}_{\gamma\beta} - D^{(i)}_{(\alpha)m} T^{m}_{\beta\gamma} - d^{(i)}_{(\alpha)(\mu)} R^{(m)}_{(\mu)\beta\gamma}, \\
\mathcal{D}^{(i)}_{(\alpha)\beta}/(\gamma) - \mathcal{D}^{(i)}_{(\alpha)\beta}/(\gamma) &= x^{m}_{\alpha/\beta} R^{m}_{\gamma\beta} - D^{(i)}_{(\alpha)m} T^{m}_{\beta\gamma} - d^{(i)}_{(\alpha)(\mu)} R^{(m)}_{(\mu)\beta\gamma}, \\
\mathcal{D}^{(i)}_{(\alpha)(\beta)/(\gamma)} - \mathcal{D}^{(i)}_{(\alpha)(\gamma)/(\beta)} &= x^{m}_{\alpha/(\beta)} P^{m}_{\gamma/(\beta)} - d^{(i)}_{(\alpha)(\gamma)} P^{(m)}_{(\mu)/(\beta)}, \\
\mathcal{D}^{(i)}_{(\alpha)(\beta)/(\gamma)} - \mathcal{D}^{(i)}_{(\alpha)(\gamma)/(\beta)} &= x^{m}_{\alpha/(\beta)} P^{m}_{\gamma/(\beta)} - D^{(i)}_{(\alpha)m} C^{m(\gamma)}_{(\beta)(\mu)} - d^{(i)}_{(\alpha)(\mu)} P^{(m)}_{(\mu)/(\beta)}, \\
d^{(i)}_{(\alpha)(\beta)/(\gamma)} - d^{(i)}_{(\alpha)(\gamma)/(\beta)} &= x^{m}_{\alpha/(\beta)} P^{m}_{\gamma/(\beta)} - D^{(i)}_{(\alpha)m} C^{m(\gamma)}_{(\beta)(\mu)} - d^{(i)}_{(\alpha)(\mu)} P^{(m)}_{(\mu)/(\beta)}.
\end{align*}
\]

**Remark 3.1** The importance of the deflection d-tensors identities is emphasized in
[7, 8], where is developed the (generalized) metrical multi-time Lagrangian geometry.
of physical fields on $J^1(T,M)$. In that context, the deflection d-tensors identities are used in the description of the Maxwell equations which govern the electromagnetic field of a (generalized) metrical multi-time Lagrange space \[7, 11\].

## 4 Bianchi identities

From the general theory of linear connections on a vector bundle $E$, it is known that the torsions $T$ and the curvature $R$ of a linear connection $\nabla$ are not independent. They verify the Bianchi identities, whose expressions, in a local basis $(X_A)$ of $X(E)$, are $$\sum_{A,B,C} \{ R^F_{ABC} - T^F_{ABC} - T^G_{AB} T^F_{CG} \} = 0,$$

$$\sum_{A,B,C} \{ R^F_{DABC} + T^G_{AB} R^F_{DAG} \} = 0,$$

where $R(X_A, X_B)X_C = R^D_{CBA} X_D$, $T(X_A, X_B) = T^D_{BA} X_D$ and "))_\cdot\) \cdots\) represents the local covariant derivative induced by $\nabla$.

In our context, we have $E = J^1(T,M)$. Let $\Gamma = (M^{(i)}_{(\alpha)j}, N^{(i)}_{(\alpha)j})$ be a fixed nonlinear connection on $E$, and $(X_A) = (\frac{\delta}{\delta x^A}, \frac{\delta}{\partial x^B}, \frac{\partial}{\partial x^\alpha})$ its adapted basis. In the sequel, we try to rewrite the above Bianchi identities for an $h$-normal $\Gamma$-linear connection $\nabla$ of Cartan type on $E$. In this sense, taking into account that the indices $A, B, \ldots$ are of type $\{\alpha, i, (i)\}$, it follows that the covariant derivative "))_\cdot\) \cdots\) becomes one of the covariant derivatives "))\cdots\) or "))_\cdot\) \cdots\). Consequently, we deduce

**Theorem 4.1** The following thirty effective Bianchi identities of the $h$-normal $\Gamma$-linear connection $\nabla$ of Cartan type are true: $\sum_{(\alpha, \beta, \gamma)} R^\delta_{\alpha\beta\gamma} = 0$, $\sum_{(\alpha, \beta, \gamma)} R_{\alpha\beta\gamma}^{(1)} T^l_{\alpha\beta\gamma} = 0$, $\sum_{(\alpha, \beta, \gamma)} R_{\alpha\beta\gamma}^{(1)} T^l_{\alpha\beta\gamma} = 0$, $\sum_{(\alpha, \beta, \gamma)} R_{\alpha\beta\gamma}^{(1)} T^l_{\alpha\beta\gamma} = 0$.
\[
\begin{align*}
(3) & \quad T^{(l)}_{\alpha k \beta (p)} - c^{(c)}_{m(p)} T^{(l)}_{\alpha k} + P^{(l)}_{\alpha (p)} - C^{(c)}_{k(p)/\alpha} - C^{(l)}_{k(m) \mu} P^{(m)}_{(\mu) \alpha (p)} = 0 \\
(4) & \quad A_{(j,k)} \left\{ C^{(l)}_{j(k)} + C^{(m)}_{k(m) \beta (p)} + P^{(l)}_{j(k)} \right\} = 0,
\end{align*}
\]
\[
\begin{align*}
(4) & \quad A_{(\alpha, \beta)} \left\{ P^{(l)}_{(\beta)(\alpha)(p)} + P^{(l)}_{(\beta)(\mu) \alpha (p)} \right\} = R^{(l)}_{(\beta) \alpha (p)}, \\
& \quad - R^{(l)}_{(\beta) \alpha (p)} + S^{(l)}_{(\beta)(\mu) \alpha (p)} R^{(l)}_{(\mu) \alpha (p)} \\
(4) & \quad A_{(\alpha, k)} \left\{ P^{(l)}_{(\beta) \alpha (p)} + P^{(l)}_{(\beta) k(m) \alpha (p)} \right\} = R^{(l)}_{(\beta) k(m) \alpha (p)}, \\
& \quad - R^{(l)}_{(\beta) k(m) \alpha (p)} + S^{(l)}_{(\beta)(\mu) \alpha (p)} R^{(l)}_{(\mu) \alpha (p)} - T^{m}_{\alpha (p)} R^{(l)}_{(\mu) \alpha (p)} \\
(4) & \quad A_{(j, k)} \left\{ P^{(l)}_{(\beta) j(k)} + P^{(l)}_{(\beta) k(m) \mu} \right\} = R^{(l)}_{(\beta) k(m) \mu}, \\
& \quad - R^{(l)}_{(\beta) k(m) \mu} + S^{(l)}_{(\beta)(\mu) \alpha (k)} R^{(l)}_{\alpha (k) \mu} \\
(5) & \quad A_{(\beta)(\gamma)(\iota)(\iota)} \left\{ C^{(c)}(\gamma)_{(\iota)(\iota)} + C^{(m)(\gamma)}_{(\iota)(\iota)} \right\} = S^{(c)}(\gamma)_{(\iota)(\iota)} - C^{(m)(\gamma)}_{(\iota)(\iota)} S^{(m)(\beta)(\gamma)}_{(\mu)(\iota)(\iota)},
\end{align*}
\]
\[
\begin{align*}
(6) & \quad A_{(\beta)(\gamma)(\iota)(\iota)} \left\{ P^{(l)}_{(\beta)(\gamma)(\iota)(\iota)} + P^{(m)}_{(\beta)(\iota)(\iota) \mu} \right\} = R^{(l)}_{(\beta) \gamma (\iota)(\iota) \mu}, \\
& \quad - S^{(l)}_{(\beta)(\gamma)(\iota)(\iota) \mu} - S^{(m)(\beta)(\iota)(\iota) \mu} P^{(l)}_{(\iota)(\iota) \mu} \\
(6) & \quad A_{(\beta)(\gamma)(\iota)(\iota)} \left\{ P^{(l)}_{(\beta)(\gamma)(\iota)(\iota)} + P^{(m)}_{(\beta)(\iota)(\iota) \mu} \right\} = R^{(l)}_{(\beta)(\iota)(\iota) \mu}, \\
& \quad - S^{(l)}_{(\beta)(\gamma)(\iota)(\iota) \mu} - S^{(m)(\beta)(\iota)(\iota) \mu} P^{(l)}_{(\iota)(\iota) \mu} \\
(7) & \quad \sum_{(\alpha, \beta, \gamma)} \left\{ S^{(l)(\alpha)(\beta)(\gamma)}_{(\iota)(\iota)(\iota)} \right\} = 0,
\end{align*}
\]
\[
\begin{align*}
(8) & \quad \sum_{(\alpha, \beta, \gamma)} H^{(\delta)}_{(\alpha)(\beta)(\gamma)} = 0 \\
(8) & \quad H^{(\delta)}_{(\alpha)(\beta)} = 0 \\
(8) & \quad \sum_{(\iota)(\iota)} R^{(m)}_{(\mu)(\mu) \iota} = 0 \\
(8) & \quad \sum_{(\alpha, \beta, \gamma)} \left\{ R^{(l)}_{(\alpha)(\beta)(\gamma)} \right\} = 0 \\
(8) & \quad A_{(\alpha, \beta)} \left\{ R^{(l)}_{(\alpha)(\beta)(\gamma)} + R^{(m)}_{(\alpha)(\beta)(\gamma)} \right\} = R^{(l)}_{(\alpha)(\beta)(\gamma)} + R^{(m)}_{(\alpha)(\beta)(\gamma)} P^{(\mu)}_{(\mu) \gamma (\iota)(\iota)}, \\
(8) & \quad A_{(\alpha, \beta)} \left\{ R^{(l)}_{(\alpha)(\beta)(\gamma)} + R^{(m)}_{(\alpha)(\beta)(\gamma)} \right\} = R^{(l)}_{(\alpha)(\beta)(\gamma)} + R^{(m)}_{(\alpha)(\beta)(\gamma)} P^{(\mu)}_{(\mu) \gamma (\iota)(\iota)}, \\
(8) & \quad \sum_{(\iota)(\iota)} \left\{ R^{(l)}_{(\mu)(\mu) \iota} - R^{(m)}_{(\mu)(\mu) \iota} \right\} = 0.
\end{align*}
\]
The set of the Bianchi identities reduces to the one of the classical twelve Bianchi identities.

\[ A_{\alpha,\beta} \left\{ P^{l}(z)_{\alpha(p)/\beta} - P^{(m)}_{(\mu)(\rho)} P^{l}(\mu)_{\beta}(m) \right\} = R^{l}_{\alpha(\beta)}(z)_{(p)} + R^{(m)}_{(\mu)(\rho)} S^{l}(z)(m) \]

\[ A_{\alpha,k} \left\{ P^{l}(z)_{\alpha(p)k} - P^{(m)}_{(\mu)(\rho)} P^{l}(\mu)_{\beta}(m) \right\} = R^{l}_{\alpha(k)}(z)_{(p)} + R^{(m)}_{(\mu)(\rho)} S^{l}(z)(m) - C^{m}(z)_{k(p)} R^{k}_{\alpha(m)} + T^{m}_{k(p)} P^{l}(z) \]

\[ A_{[j,k]} \left\{ P^{l}(z)_{(i)(p)j} - P^{(m)}_{(\mu)(j)(p)} P^{l}(\mu)(k) - C^{m}(z)_{j(k)} R^{k}_{(p)l} \right\} = R^{l}_{(i)(p)j}(z) + R^{(m)}_{(\mu)(j)(p)} S^{l}(z)(m) \]

9.1 \[ \sum_{\alpha, i} A_{\alpha, i} \left\{ P^{l}(z)_{\alpha(i)j} - P^{(m)}_{(i)(j)(\mu)} P^{l}(\mu)(m) \right\} = S^{l}(z)(j)(m) \]

9.2 \[ \sum_{\alpha, i} A_{\alpha, i} \left\{ P^{l}(z)_{\alpha(i)j} - P^{(m)}_{(i)(j)(\mu)} P^{l}(\mu)(m) \right\} = S^{l}(z)(j)(m) \]

9.3 \[ \sum_{\alpha, i} A_{\alpha, i} \left\{ P^{l}(z)_{\alpha(i)j} - P^{(m)}_{(i)(j)(\mu)} P^{l}(\mu)(m) \right\} = S^{l}(z)(j)(m) \]

where, if \( \{A, B, C\} \) are indices of type \( \{(\alpha, i), (\beta, j), (\gamma, k)\} \) then \( \sum_{\{A, B, C\}} \) means the cyclic sum and \( A_{\{A, B\}} \) means alternate sum.

Remarks 4.1 i) In the particular case \( (T, h) = (R, \delta) \), the last identity of every above set of the Bianchi identities reduces to the one of the classical twelve Bianchi identities of an \( N \)-linear connection from the Lagrange geometry.

ii) The Bianchi identities of an \( h \)-normal \( \Gamma \)-linear connection of Cartan type are used in the description of the Maxwell equations and the conservation laws of the Einstein equations of the gravitational potentials from the background of the (generalized) metrical multi-time Lagrange geometry of physical fields.

Acknowledgements. The author would like to thank to the reviewers of Journal of the Mathematical Society of Japan for their valuable comments upon a previous version of this paper.

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University POLITEHNICA of Bucharest
Department of Mathematics I
Splaiul Independentei 313
77206 Bucharest, Romania
e-mail: mircea@mathem.pub.ro
Fax: (401)411.53.65.