TIME-PRESERVING STRUCTURAL STABILITY OF HYPERBOLIC DIFFERENTIAL DYNAMICS WITH NONCOMPACT PHASE SPACES

XIONGPING DAI

Abstract. Let $S : E \rightarrow \mathbb{R}^n$ where $T_w E = \mathbb{R}^n$ for all $w \in E$, be a $C^1$-differential system on an $n$-dimensional Euclidean $w$-space $E$, which naturally gives rise to a flow $\phi : (t, w) \mapsto t.w$ on $E$, and let $\Lambda$ be a $\phi$-invariant closed subset containing no any singularities of $S$. If $\Lambda$ is compact and hyperbolic, then Anosov’s theorem asserts that $S$ is structurally stable on $\Lambda$ in the sense of topological equivalence; that is, for any $C^1$-perturbation $V$ close to $S$, there is an $\varepsilon$-homeomorphism $H : \Lambda \rightarrow \Lambda_V$ sending orbits $\phi(\mathbb{R}, w)$ of $S$ into orbits $\phi_V(\mathbb{R}, H(w))$ of $V$ for all $w$ in $\Lambda$. In this paper, using Liao theory Anosov’s result is generalized as follows: Let $\psi_V : \mathbb{R} \times \Sigma \rightarrow \Sigma$ be the cross-section flow of $V$ relative to $S$ locally defined on the Poincaré cross-section bundle $\Sigma = \bigcup_{w \in \Lambda} \Sigma_w$ of $S$, where $\Sigma_w = \{ w' \in E | \langle S(w), w' - w \rangle = 0 \}$. If $S$ is hyperbolic on $\Lambda$ and $V$ is $C^1$-close to $S$, then there is an $\varepsilon$-homeomorphism $w \mapsto H(w) \in \Sigma_w$ from $\Lambda$ onto a closed set $\Lambda_V$ such that $\psi_V(t, H(w)) = H(t.w)$ for all $w \in \Lambda$, where $\Lambda$ need not be compact. Finally, an example is provided to illustrate our theoretical outcome.

1. Introduction

In [6, 7], professor S.-T. Liao established the theory of standard systems of differential equations for $C^1$-differential dynamical systems on compact Riemannian manifolds. Then he systematically applied methods in the qualitative theory of ODE to study stability problems of differentiable dynamical systems via his theory [8]. We in [2, 3] generalized in part Liao’s theory to differential systems on Euclidean spaces. Via the generalized, in turn we can apply the approaches of ergodic theory and differentiable dynamical systems to the study of the qualitative theory of ODE [3, 4]. In the present paper, we continue to perfect Liao theory and give a further application.

Assume, throughout this paper, that $S : E \rightarrow \mathbb{R}^n$ is a $C^1$-vector field on an $n$-dimensional Euclidean $w$-space $E$, where $n \geq 2$ and $T_w E = \mathbb{R}^n$ for all $w$, and the equation $\dot{w} = S(w)$ naturally induces a continuous-time dynamical system $\phi : \mathbb{R} \times E \rightarrow E; (t, w) \mapsto t.w$ on the phase-space $E$. Let

$$
\Sigma = \bigcup_{w \in E} \Sigma_w \quad \text{where} \quad \Sigma_w = \{ w' \in E | \langle S(w), w' - w \rangle = 0 \},
$$

be the cross-section bundle of $S$. Then, $S$ gives naturally rise to a formal (local) Poincaré cross-section flow

$$
\psi : \mathbb{R} \times \Sigma \rightarrow \Sigma; \quad (t, w + x) \mapsto t.w + \psi_{t,w}x,
$$

where $w' = w + x$ means $w' \in \Sigma_w$ and where $\psi_{t,w} : \Sigma_w - w \rightarrow \Sigma_t w - t.w$ is locally well defined for any $(t, w) \in \mathbb{R} \times E$ by $\psi_{t,w}x = \phi(t_0, w + x) - t.w$, where $t_0$ is the first $t' > 0$ when $t > 0$ or the first $t' < 0$ when $t < 0$ with $\phi(t', w + x) \in \Sigma_{t'w}$. Clearly, $\psi$ is a local skew-product flow based on $\phi$ satisfying $\psi_{t,w}0 = 0$. 

Date: May 17, 2008.

2000 Mathematics Subject Classification. Primary 37C10, 34D30; Secondary 37C20, 37D20.

Key words and phrases. Structural stability, hyperbolic differential system, Liao standard system.

This project was supported by NSFC (No. 10671088) and 973 project (No. 2006CB805903).
Let $\mathcal{X}^1(\mathbb{E})$ be the space of all $C^1$-vector fields on $\mathbb{E}$ endowed with the $C^1$-topology induced by the usual $C^1$-norm $\| \cdot \|_1$. Then, for any $V \in \mathcal{X}^1(\mathbb{E})$, on $\Sigma$ we may also naturally define a formal local skew-product flow

$$
\psi_V : \mathbb{R} \times \Sigma \to \Sigma; \ (t, w + x) \mapsto t.w + \psi_{V,t,w}x.
$$

Note here that $\psi_{V,t,w}0$ need not equal $0$ when $V \neq S$.

Let $T_w = T_0\Sigma_w$ be the $(n-1)$-dimensional tangent space to the hyperplane $\Sigma_w$ at $w + 0$ for all $w \in \mathbb{E}$ and $T = \bigcup_{w \in \mathbb{E}} T_w$ called the transversal tangent bundle to $S$ over $\mathbb{E}$. Clearly, $T_w = \Sigma_w - w = \{ x \in \mathbb{R}^n \mid \langle S(w), x \rangle = 0 \}$. Then, we can define naturally the linear skew-product flow transversal to $S$

$$
\Psi : \mathbb{R} \times T \to T; \ (t, (w, x)) \mapsto (t.w, \Psi_{t,w}x),
$$

where $\Psi_{t,w} : T_w \to T_{t.w}$ is defined as $\Psi_{t,w} = D_0\psi_{t,w}$ for any $(t, w) \in \mathbb{R} \times \mathbb{E}$, associated with $S$.

Recall that a $\phi$-invariant closed subset $\Lambda$ is said to be hyperbolic, provided that there exist constants $C \geq 1, \lambda < 0$ and a continuous $\Psi$-invariant splitting

$$
T_w = T_w^s \oplus T_w^u \quad w \in \Lambda
$$

such that

$$
\| \Psi_{t_0, t.w}x \| \leq C^{-1} \exp(\lambda t)\| \Psi_{t_0, w}x \| \quad \forall x \in T_w^s
$$

and

$$
\| \Psi_{t_0, t.w}x \| \geq C \exp(-\lambda t)\| \Psi_{t_0, w}x \| \quad \forall x \in T_w^u
$$

for any $t_0 \in \mathbb{R}$ and for all $t > 0$.

Then, Anosov’s structural stability theorem [11, 9] asserts that: If $\Lambda$ is a compact hyperbolic set for $S$, then for any $\varepsilon > 0$ there is a $C^1$-neighborhood $\mathcal{U}$ of $S$ in $\mathcal{X}^1(\mathbb{E})$ such that, if $V \in \mathcal{U}$ then there exists a $\varepsilon$-topological mapping $h$ from $\Lambda$ onto some subset $\Lambda_V$ of $\mathbb{E}$ which sends orbits of $S$ in $\Lambda$ into orbits of $V$ in $\Lambda_V$.

This important theorem was extended to axiom A differential systems [11, 13], and to $C^0$-perturbations by considering the so-called semi-structural stability independently by [5, 7]; for discrete versions, see [12, 14, 9, 15, 10]. On another direction, in this paper, we study the structural stability of noncompact hyperbolic set under time-preserving conjugacy between the induced cross-section flows. More precisely, using Liao theory we prove the following.

**Main Theorem.** Let $\Lambda$ be a hyperbolic set for $S$, not necessarily compact, satisfying the following conditions:

(U1) The first derivative $S'(w)$ is uniformly bounded on $\Lambda$;

(U2) $0 < \inf_{w \in \Lambda} \| S(w) \| \leq \sup_{w \in \Lambda} \| S(w) \| < \infty$;

(U3) $S'(w)$ is uniformly continuous at $\Lambda$; that is, to any $\varepsilon > 0$ there is some $\delta > 0$ so that for any $w \in \Lambda$, $\| S'(w) - S'(w') \| < \varepsilon$ whenever $\| w - w' \| < \delta$.

Then, for any $\varepsilon > 0$ there is a $C^1$-neighborhood $\mathcal{U}$ of $S$ in $\mathcal{X}^1(\mathbb{E})$ such that for any $V \in \mathcal{U}$ there exists a $\varepsilon$-topological mapping $H$ from $\Lambda$ onto some closed subset $\Lambda_V$ which sends orbits of $S$ in $\Lambda$ into orbits of $V$ in $\Lambda_V$, such that $H(w) \in \Sigma_w$ and $\psi_{V, t}(w, H(w)) = H(t.w)$ for all $w \in \Lambda$ and for any $t \in \mathbb{R}$.

Notice here that if $\Lambda$ is compact, then conditions (U1), (U2) and (U3) hold automatically. So our result is an extension of the classical one. Even for the compact case, the time-preserving property is still a new ingredient in our main theorem.
Finally, we prove the Main Theorem in §2 and recall Liao’s exponential dichotomy in §4.

2. Liao standard system of differential equations

Let \( S \) be any given \( C^1 \)-differential system on \( E \) and \( \Lambda \) a \( \phi \)-invariant closed subset in \( E \) satisfying conditions (U1), (U2) and (U3) as in the Main Theorem stated in §1. Around a regular orbit \( \phi(t, w) \) we defined in §2 the standard systems for perturbations of \( S \) itself. However, we will introduce below the standard systems for perturbations \( V \) of \( S \).

2.1. As usual in Liao theory §3, let \( \mathcal{F}^s_{n-1}(\Lambda) = \bigcup_{w \in \Lambda} \mathcal{F}^s_{n-1,w} \) be the bundle of transversal orthonormal \((n - 1)\)-frames, where the fiber over \( w \) is defined as

\[
\mathcal{F}^s_{n-1,w} = \{ \gamma = (\bar{u}_1, \ldots, \bar{u}_{n-1}) \in T_w \times \cdots \times T_w | \langle \bar{u}_i, \bar{u}_j \rangle = \delta_{ij} \text{ for } 1 \leq i, j \leq n - 1 \},
\]

endowed with the naturally induced topology. Then, \( S \) naturally generates a skew-product flow over \( \phi \)

\[
\chi^s: \mathbb{R} \times \mathcal{F}^s_{n-1}(\Lambda) \to \mathcal{F}^s_{n-1}(\Lambda); (t, (w, \gamma)) \mapsto (t, w, \chi^s_{t,w} \gamma),
\]

where \( \chi^s_{t,w}: \mathcal{F}^s_{n-1,w} \to \mathcal{F}^s_{n-1,t,w} \) is defined by the standard Gram-Schmidt orthonormalization process; cf. §2, §3 for the details.

Let \( e = \{ \bar{e}_1, \ldots, \bar{e}_{n-1} \} \) where \( \bar{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{R}^{n-1} \), be the standard basis of \( \mathbb{R}^{n-1} \) and we view \( y \in \mathbb{R}^{n-1} \) with components \( y^1, \ldots, y^{n-1} \) as a column vector \( (y^1, \ldots, y^{n-1})^T \) and \( \gamma \in \mathcal{F}^s_{n-1,w} \) as an \( n \)-by-(\( n - 1 \)) matrix with columns \( \text{col}_1 \gamma, \ldots, \text{col}_{n-1} \gamma \) successively.

Given any orthonormal \((n - 1)\)-frame \((w, \gamma) \in \mathcal{F}^s_{n-1}(\Lambda) \), sometimes written simply as \( \gamma_w \), we define by linear extension the linear transformation

\[
T^*: \mathbb{R}^{n-1} \to T_w
\]

in the way

\[
\bar{e}_j \mapsto \text{col}_j \gamma \quad (1 \leq j \leq n - 1).
\]

Since \( \gamma \) is an orthonormal basis of \( T_w \), \( T^*_\gamma \) is an isomorphism such that

\[
T^*_\gamma (y) = \gamma y = \sum_{j=1}^{n-1} y^j \text{col}_j \gamma \quad \forall y \in \mathbb{R}^{n-1}.
\]

Moreover, we now define

\[
C^*_\gamma (t) = T^*_{\chi^s t}(t, \gamma_w) \circ \psi_{t,w} \circ T^*_\gamma \quad \forall t \in \mathbb{R},
\]

where \( \chi^s: \mathbb{R} \times \mathcal{F}^s_{n-1}(\Lambda) \to \mathcal{F}^s_{n-1}(\Lambda) \) as in (2.1). Then the commutativity holds:

\[
\begin{align*}
\mathbb{R}^{n-1} & \xrightarrow{C^*_\gamma (t)} \mathbb{R}^{n-1} \\
\mathbb{R}^{n-1} & \xrightarrow{T^*_\gamma} T_w \\
\mathbb{R}^{n-1} & \xrightarrow{\psi_{t,w}} T_{t,w}.
\end{align*}
\]
We now think of \( C^*_w(t) \) as an \((n-1) \times (n-1)\)-matrix under the base \( e \) of \( \mathbb{R}^{n-1} \). Clearly, \( t \mapsto \frac{d}{dt} C^*_w(t) \) makes sense since \( S \) is of class \( C^1 \) and by \((2.4)\) we have
\[
C^*_w(t_1 + t_2) = C^*_{\chi^w(t_1, \gamma_w)}(t_2) \circ C^*_w(t_1) \quad \forall t_1, t_2 \in \mathbb{R}.
\]

Put
\[
R^*_w(t) = \left\{ \frac{d}{dt} C^*_w(t) \right\} C^*_w(t)^{-1} \quad \forall (w, \gamma) \in \mathcal{F}^*_n(\Lambda).
\]

**Definition 2.1.** The linear differential equation
\[
(R^*_w)
\]
for any \((w, \gamma) \in \mathcal{F}^*_n(\Lambda)\), is called the **reduced linearized system** of \( S \) under the moving frame \( \chi^w(t, \gamma_w) \). See [2, 3].

These reduced linearized systems of \( S \) possess the following properties.

**Lemma 2.2** ([2, 3]). The following statements hold:

1. **Uniform boundedness:** \( R^*_w(t) \) is continuous in \((t, (w, \gamma)) \) in \( \mathbb{R} \times \mathcal{F}^*_n(\Lambda) \) with
   \[
   \eta = \sup_{(w, \gamma) \in \mathcal{F}^*_n(\Lambda)} \left\{ \sum_{i,j} |R^*_{ij}(t)| \right\} < \infty.
   \]

2. **Upper triangularity:** \( R^*_w(t) \) is upper-triangular with
   \[
   R^*_w(t) = \begin{bmatrix}
   \omega^*_1(\chi^w(t, \gamma_w)) & \cdots & 0 \\
   \vdots & \ddots & \vdots \\
   0 & \cdots & \omega^*_n(\chi^w(t, \gamma_w))
   \end{bmatrix} \quad \forall t \in \mathbb{R}
   \]
   where \( \omega^*_k(w, \gamma), 1 \leq k \leq n-1 \), called the “Liao qualitative functions” of \( S \), are uniformly continuous in \((w, \gamma) \in \mathcal{F}^*_n(\Lambda)\).

3. **Geometrical interpretation:** Let \( \vec{v} = \gamma y \in T_w \) for \( y \in \mathbb{R}^{n-1} \). If \( y(t) = y(t, 0) \) is the solution of \((R^*_w)\) with \( y(0) = y \), then
   \[
   \Psi_{t,w}\vec{v} = T^*_{\chi^w(t, \gamma_w)} y(t) = (\chi^w_{\gamma_w}(\gamma)) y(t).
   \]
   Conversely, letting \( x(t) = (x^1(t), \ldots, x^{n-1}(t))^T \in \mathbb{R}^{n-1} \) be defined by
   \[
   x^i(t) = \left( \Psi_{t,w}, \text{col}_i \chi^w_{\gamma_w} \right)_{t,w} \quad i = 1, \ldots, n-1,
   \]
   we have \( \dot{x}(t) = R^*_w(t)x(t) \) and \( x(0) = y \). Particularly, \( C^*_w(t) \) is the fundamental matrix solution of \((R^*_w)\).

As a consequence of the above lemma, we have

**Corollary 2.3.** Let \( \Lambda \) be hyperbolic for \( S \) associated to \( \Psi \)-invariant splitting \( T_\Lambda = T^*_{\Lambda} \oplus T^\perp_{\Lambda} \). Then, there are two constants \( \eta > 0 \) and \( \delta > 0 \) such that: for any \((w, \gamma) \in \mathcal{F}^*_n(\Lambda)\), if \( \text{col}_i \gamma \in T^*_w \) for \( i = 1, \ldots, \dim T^*_w \) then
\[
\int_0^T \omega^*_k(\chi^w(t_0 + t, (w, \gamma))) \, dt \leq -\eta T, \quad 1 \leq k \leq \dim T^*_w
\]
and
\[
\int_0^T \omega^*_k(\chi^w(t_0 + t, (w, \gamma))) \, dt \geq \eta T, \quad \dim T^*_w + 1 \leq k \leq n-1
\]
for any \( t_0 \in \mathbb{R} \) and for all \( T \geq d \).

**Proof.** The statement comes immediately from Lemma 2.2 and Lemma 3.7. \(\square\)

### 2.2. For a constant \( c > 0 \), let \( \mathbb{R}^n_{c^{-1}} = \{ y \in \mathbb{R}^n; \| y \| < c \} \). Fix any \( w \in \Lambda \). For any \( \gamma \in \mathcal{F}_{n-1}^* \), we need the \( C^1 \)-mapping

\[
P_{w,\gamma}^*: \mathbb{R} \times \mathbb{R}^n_{c^{-1}} \to \mathbb{E}
\]

defined by

\[
P_{w,\gamma}^*(t, y) = t \cdot w + (\chi_{1,w,\gamma}^*)^y \in \Sigma_{t,w} \quad \forall (t, y) \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}}.
\]

It is known [3, Lemma 5.1] that there is a constant \( c > 0 \), such that \( P_{w,\gamma}^* \) is locally diffeomorphic on \( \mathbb{R} \times \mathbb{R}^n_{c^{-1}} \). In fact, according to [3] there is some \( \epsilon > 0 \) so that for any \( w \in \Lambda \), \( P_{w,\gamma}^* \) is diffeomorphism from \( (-\epsilon, \epsilon) \times \mathbb{R}^n_{c^{-1}} \) into \( \mathbb{E} \).

Given any \( (w, \gamma) \in \mathcal{F}_{n-1}^*(\Lambda) \). Define a \( C^0 \)-vector field on \( \mathbb{R} \times \mathbb{R}^n_{c^{-1}} \)

\[
\hat{S}_{w,\gamma}: \mathbb{R} \times \mathbb{R}^n_{c^{-1}} \to \mathbb{R}^n
\]

with \( \hat{S}_{w,\gamma}(t, 0) = (1, 0)^T \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}} \) in the following way:

\[
(D(t, y)P_{w,\gamma}^*) \hat{S}_{w,\gamma}(t, y) = S(t, y) \quad \forall (t, y) \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}}.
\]

Since \( P_{w,\gamma}^* \) is locally \( C^1 \)-diffeomorphism, \( \hat{S}_{w,\gamma}(t, y) \) is well defined. We now consider the autonomous system

\[
\frac{d}{dt} \begin{pmatrix} t \\ y \end{pmatrix} = \hat{S}_{w,\gamma}(t, y) \quad (t, y) \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}}
\]

and write

\[
\hat{S}_{w,\gamma}(t, y) = \left( \hat{S}_{w,\gamma}^{0}(t, y), \ldots, \hat{S}_{w,\gamma}^{n-1}(t, y) \right)^T \in \mathbb{R} \times \mathbb{R}^n.
\]

Next, put

\[
S_{w,\gamma}(t, y) = \left( \frac{\hat{S}_{w,\gamma}^{n-1}(t, y)}{\hat{S}_{w,\gamma}^{n-1}(t, y)} \right)^T \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}}
\]

**Definition 2.4 (3).** The non-autonomous differential equation

\[
(S_{w,\gamma}) \quad \hat{y} = S_{w,\gamma}^*(t, y) \quad (t, y) \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}}
\]

is called the reduced standard system of \( S \) under the base \( (w, \gamma) \in \mathcal{F}_{n-1}^*(\Lambda) \).

Is easy to see that

\[
S_{w,\gamma}^*(t + t_1, y) = S_{w,\gamma}^* \chi_{1,(t, (w, \gamma))}(t_1, y) \quad \forall (t, y) \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}}.
\]

For convenience of our later discussion, we write

\[
P_{w,\gamma}^*(t, y) = t \cdot w + \hat{P}_{w,\gamma}^*(t, y), \text{ where } \hat{P}_{w,\gamma}^*(t, y) \in T_{t,w}.
\]

The following is important for our later arguments.

**Lemma 2.5 (3).** Under the conditions \((U1), (U2) \) and \((U3)\), the following statements hold: for any \( (w, \gamma) \in \mathcal{F}_{n-1}^*(\Lambda) \)

1. \( S_{w,\gamma}^*(t, 0) = 0 \in \mathbb{R}^n_{c^{-1}} \) for all \( t \in \mathbb{R} \), and \( S_{w,\gamma}^*(t, y) \) is continuous with respect to \( (t, y) \in \mathbb{R} \times \mathbb{R}^n_{c^{-1}} \).
(2) For any \((\bar{t}, \bar{y}) \in \mathbb{R} \times \mathbb{R}_{\xi}^{n-1}\), let \(\bar{w} = \bar{t}, w + \bar{x} = P_{w,\gamma}(\bar{t}, \bar{y}) \in \Sigma_{\bar{t}, w}\) and

\[ y^*(t) = y_{w,\gamma}^*(t; \bar{t}, \bar{y}) \quad t \in (r', r'') \] where \(\bar{t} \in (r', r'')\),

be the solution of \((S_{w,\gamma}^*)\) with \(y^*(t) = \bar{y}\). Then

\[ \psi(t - \bar{t}, \bar{w}) = P_{w,\gamma}^*(t, y^*(t)) \in \Sigma_{t, w} \quad (r' < t < r''). \]

(3) \(S_{w,\gamma}^*(t, y)\) is of class \(C^1\) with respect to \(y \in \mathbb{R}_{\xi}^{n-1}\) such that

\[ \partial S_{w,\gamma}^*(t, y)/\partial y \rightarrow R_{w,\gamma}^*(t) \text{ as } y \rightarrow 0 \]

uniformly for \((t, (w, \gamma)) \in \mathbb{R} \times \mathcal{F}_{n-1}^*(\Lambda)\).

From here on, for any \(w \in \Lambda\) we will rewrite \((S_{w,\gamma}^*)\) as

\[ \dot{y} = R_{w,\gamma}^*(t) y + S_{\text{rem}(w,\gamma)}^*(t, y) \quad (t, y) \in \mathbb{R} \times \mathbb{R}_{\xi}^{n-1} \]

where

\[ S_{\text{rem}(w,\gamma)}^*(t, y) = S_{w,\gamma}^*(t, y) - R_{w,\gamma}^*(t) y. \]

Then, we have the following result.

**Lemma 2.6** ([3]). Under the conditions \((U1), (U2)\text{ and } (U3)\), to any \(\kappa > 0\), there is some \(\xi \in (0, \xi_0]\) so that

\[ \|S_{\text{rem}(w,\gamma)}^*(t, y) - S_{\text{rem}(w,\gamma)}^*(t, y')\| \leq \kappa \|y - y'\| \] whenever \(y, y' \in \mathbb{R}_{\xi}^{n-1}\)

holds uniformly for \((t, (w, \gamma)) \in \mathbb{R} \times \mathcal{F}_{n-1}^*(\Lambda)\).

2.3. In what follows, we let \(V: \mathbb{E} \rightarrow \mathbb{R}^n\) be an arbitrarily given another \(C^1\) vector field on \(\mathbb{E}\). Note here that \((\mathcal{F}_{n-1}^*(\Lambda), \mathcal{X}^f)\) still corresponds to \(S\).

In order to introduce the standard systems of \(V\) associated with \(S\), let us consider firstly a simple lemma.

**Lemma 2.7.** Let \(h: \hat{N} \rightarrow N\) be a map of class \(C^1\) from a \(C^1\) manifold \(\hat{N}\) into another \(C^1\) manifold \(N\). Let \(\hat{X}\) and \(X\) be \(C^0\) vector fields on \(\hat{N}\) and \(N\), respectively. If \((Dh)\hat{X} = X\) then for any \(\hat{p} \in \hat{N}\), \(h\) maps the integral curve \(\phi_{\hat{X}}(t, \hat{p})\) of \(\hat{X}\) into an integral curve \(\phi_{X}(t, h(\hat{p}))\) of \(X\) such that \(\phi_{\hat{X}}(t, h(\hat{p})) = h(\phi_{\hat{X}}(t, \hat{p}))\).

**Proof.** Let \(h(\hat{p}) = p\). Define a \(C^1\) curve in \(N\) by \(C: t \mapsto h(\phi_{\hat{X}}(t, \hat{p}))\). Since

\[ \frac{d}{dt} \phi_{\hat{X}}(t, \hat{p}) = \hat{X}(\phi_{\hat{X}}(t, \hat{p})) \] and \((Dh)\hat{X}(\phi_{\hat{X}}(t, \hat{p})) = X(C(t)) = \frac{d}{dt} C(t),\]

we get that \(C(t)\) is an integral curve of \(X\) satisfying the initial condition \(C(0) = p\). Now put \(\phi_{\hat{X}}(t, p) = C(t)\), which satisfies the requirement of Lemma 2.7. \(\square\)

Particularly, we will be interesting to the case where \(\hat{N} = \mathbb{R} \times \mathbb{R}_{\xi}^{n-1}, N = \mathbb{E}\) and \(h = P_{w,\gamma}^\ast\) and \(X = V\) for any given \((w, \gamma) \in \mathcal{F}_{n-1}^*(\Lambda)\). Correspondingly, there \(\hat{X}\) is right the so-called lifting system that we are going to define.

**Definition 2.8.** Given any \((w, \gamma) \in \mathcal{F}_{n-1}^*(\Lambda)\). Define a \(C^0\)-vector field

\[ \hat{V}_{w,\gamma}: \mathbb{R} \times \mathbb{R}_{\xi}^{n-1} \rightarrow \mathbb{R}^n \]

in the following way:

\[ (D(t, y) P_{w,\gamma}^\ast) \hat{V}_{w,\gamma}(t, y) = V(P_{w,\gamma}^\ast(t, y)) \quad \forall (t, y) \in \mathbb{R} \times \mathbb{R}_{\xi}^{n-1}. \]
Then, the autonomous differential equation
\begin{equation}
\frac{d}{dt} \begin{pmatrix} t \\ y \end{pmatrix} = \hat{V}_{w,\gamma}(t, y) \quad t \in \mathbb{R}, (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}_{c}
\end{equation}
is referred to as a \textit{lifting} of \( V \) under the moving frames \((\chi^{\ast i}_t(t, (w, \gamma)))_{t \in \mathbb{R}}\).

Write
\[
\hat{V}_{w,\gamma}(t, y) = \left(\hat{V}^0_{w,\gamma}(t, y), \ldots, \hat{V}^{n-1}_{w,\gamma}(t, y)\right)^T \in \mathbb{R} \times \mathbb{R}^{n-1}.
\]
Clearly, it follows from \( P^\ast_{w,\gamma}(t, y) = P^\ast_{\chi^{\ast i}_t(t, (w, \gamma))}(0, y) \) that
\begin{equation}
\hat{V}_{w,\gamma}(t, y) = \hat{V}^\ast_{\chi^{\ast i}_t(t, (w, \gamma))}(0, y) \quad \forall (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}.
\end{equation}
Although \( P^\ast_{w,\gamma} \) is only \( C^1 \), we can obtain more about the regularity of \( \hat{V}_{w,\gamma}(t, y) \) with respect to \( y \in \mathbb{R}^{n-1} \) as long as \( V \) is \( C^1 \).

**Lemma 2.9.** Given any \((w, \gamma) \in \mathcal{F}^{\ast\ast}_{n-1}(\Lambda)\), the lifting \( \hat{V}_{w,\gamma}(t, y) \) is of class \( C^1 \) in \( y \);
precisely, for \( 1 \leq i \leq n-1 \), \( \partial \hat{V}_{w,\gamma}(t, y)/\partial y_i \) makes sense and is continuous with respect to \((t, y, (w, \gamma)) \) in \( \mathbb{R} \times \mathbb{R}^{n-1} \times \mathcal{F}^{\ast\ast}_{n-1}(\Lambda) \).

**Proof.** The statement comes immediately from the regularity of \( P^\ast_{w,\gamma}(t, y) \), as the argument of \([3, \text{Lemma 5.3}]\). \( \square \)

Next, let
\[
\{S, V\}^1_{\Lambda} = \sup_{(t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}_{c}, (w, \gamma) \in \mathcal{F}^\ast_{n-1}(\Lambda)} \left\{ \| \hat{S}_{w,\gamma}(t, y) - \hat{V}_{w,\gamma}(t, y) \| + \left\| \frac{\partial}{\partial y} \hat{S}_{w,\gamma}(t, y) - \hat{V}_{w,\gamma}(t, y) \right\| \right\}.
\]
From (2.13) we get
\[
\{S, V\}^1_{\Lambda} = \sup_{y \in \mathbb{R}^{n-1}_{c}, (w, \gamma) \in \mathcal{F}^\ast_{n-1}(\Lambda)} \left\{ \| \hat{S}_{w,\gamma}(0, y) - \hat{V}_{w,\gamma}(0, y) \| + \left\| \frac{\partial}{\partial y} \hat{S}_{w,\gamma}(0, y) - \hat{V}_{w,\gamma}(0, y) \right\| \right\}.
\]
Then, we have

**Lemma 2.10.** There exists some constant \( \gamma_{\Lambda} > 0 \) such that
\[
\| S - V \|_1 \geq \gamma_{\Lambda} \{S, V\}^1_{\Lambda} \quad \forall V \in \mathcal{X}^1(\mathbb{E}).
\]

**Proof.** For any \((w, \gamma) \in \mathcal{F}^\ast_{n-1}(\Lambda)\) let
\[
J_{w,\gamma}(y) = \left. \frac{\partial P^\ast_{w,\gamma}(t, y)}{\partial(t, y)} \right|_{(0, y)}
\]
be the \( n \)-by-\( n \) Jacobi matrix of \( P^\ast_{w,\gamma}(t, y) \) at \((0, y) \in \mathbb{R} \times \mathbb{R}^{n-1}_{c}\). Then
\[
P^\ast_{w,\gamma}(0, y) = w + \gamma y
\]
and
\[
J_{w,\gamma}(y) = \left[ S(w) + \frac{d}{dt} \bigg|_{t=0} (\chi^\ast_{t, w}) y, \gamma \right]_{n \times n}.
\]
Thus, for any \( y \in \mathbb{R}^{n-1}_{c}\) we have
\[
\hat{S}_{w,\gamma}(0, y) - \hat{V}_{w,\gamma}(0, y) = J_{w,\gamma}(y)^{-1}(S - V)(w + \gamma y)
\]
Moreover, from condition (U1) we can prove by the argument of [3, Lemma 5.3] that \( \frac{1}{t!} \partial_t \chi_{w,\gamma} \), viewed as an \( n \)-by-\((n-1)\) matrix, is uniformly continuous and bounded for any \((w,\gamma) \in \mathcal{F}^{\mathbb{Z}}_{n-1}(\Lambda)\). Therefore, there is some constant \( b_{\Lambda} > 0 \) which satisfies the requirement of Lemma 2.10.

From Lemma 2.10 condition (U1) and \( \tilde{S}_{w,\gamma}^0(t,0) = 1 \), we may assume, without any loss of generality replacing \( c \) by a more small positive constant if necessary, that

- \( \exists \mathcal{N}_s \), a \( C^1 \)-neighborhood of \( S \) in \( \mathbb{X}(\mathbb{E}) \) such that: for any \( V \in \mathcal{N}_s \)
- \( \frac{1}{2} \leq \tilde{V}_{w,\gamma}^0(t,y) \leq \frac{3}{2} \) for any \((t,y,(w,\gamma)) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathcal{F}^{\mathbb{Z}}_{n-1}(\Lambda)\).

Thus, the following definition makes sense.

**Definition 2.11.** Given any \((V,(w,\gamma)) \in \mathcal{N}_s \times \mathcal{F}^{\mathbb{Z}}_{n-1}(\Lambda)\), set

\[
V_{w,\gamma}^*(t,y) = \left( \begin{array}{c} \tilde{V}_{w,\gamma}^1(t,y) \\ \tilde{V}_{w,\gamma}^2(t,y) \\ \vdots \\ \tilde{V}_{w,\gamma}^{n-1}(t,y) \\ \tilde{V}_{0,\gamma}(t,x) \end{array} \right) \in \mathbb{R}^{n-1} \quad \forall (t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}.
\]

The non-autonomous differential equation

\[
(V_{w,\gamma}^*) \quad \dot{y} = V_{w,\gamma}^*(t,y) \quad (t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}
\]

is referred to as the standard system of \( V \) associated to \((S,(w,\gamma))\).

From (2.13) we have

\[
V_{w,\gamma}^*(t+t',y) = V_{w,\gamma}^*((t,(w,\gamma)))(t',y) \quad \forall t,t' \in \mathbb{R} \text{ and } y \in \mathbb{R}^{n-1}.
\]

In what follows, we write \((V_{w,\gamma}^*)\) as

\[
(V_{w,\gamma}^*) \quad \dot{y} = R_{w,\gamma}^*(t)y + V_{w,\gamma}^*(t,y) \quad (t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}
\]

where

\[
(V_{w,\gamma}^*) \quad V_{w,\gamma}^*(t,y) = V_{w,\gamma}^*(t,y) - R_{w,\gamma}^*(t)y \quad \forall t,t' \in \mathbb{R}.
\]

Similar to Lemma 2.8 we obtain the following result.

**Theorem 2.12.** Given any \( V \in \mathcal{N}_s \), the following statements hold:

1. \( V_{w,\gamma}^*(t,y) \) and \( \partial V_{w,\gamma}^*(t,y)/\partial y \) are continuous in \((t,y,(w,\gamma)) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathcal{F}^{\mathbb{Z}}_{n-1}(\Lambda)\).
2. Given any \((w,\gamma) \in \mathcal{F}^{\mathbb{Z}}_{n-1}(\Lambda)\). If \( y^*(t) = y_{w,\gamma}^*(t;t_0,y) \) where \( t' < t, t_0 < t'' \), is the solution of \((V_{w,\gamma}^*)\) with \( y^*(t_0) = y \), then

\[
\psi_{\gamma}(t-t_0,P_{w,\gamma}^*(t_0,y)) = P_{w,\gamma}^*(t,y^*(t)) \in \Sigma w.
\]

Moreover, similar to Lemma 2.6 we have the following important result.

**Theorem 2.13.** The following three statements hold.

1. Given any \((V,(w,\gamma)) \in \mathcal{N}_s \times \mathcal{F}^{\mathbb{Z}}_{n-1}(\Lambda)\), there is some \( L > 0 \) such that

\[
\|V_{w,\gamma}^*(t,y) - V_{w,\gamma}^*(t,y')\| \leq L\|y - y'\|
\]

for any \( t \in \mathbb{R} \) and for any \( y,y' \in \mathbb{R}^{n-1} \).
To any \( \eta > 0 \) there exists a \( C^1 \)-neighborhood \( U'_s \subset N_s \) of \( S \) and \( \xi' \in (0, c] \) such that: \( \forall V \in U'_s \)

\[
\sup_{(t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}} \| V_{\text{rem}(w,\gamma)}^*(t,y) \| \leq \eta \xi' \quad \forall (w,\gamma) \in F^*_{n-1}(\Lambda).
\]

(3) To any given \( \kappa > 0 \) there corresponds a \( C^1 \)-neighborhood \( U''_s \subset N_s \) of \( S \) and a constant \( \xi'' \in (0, c] \) such that: \( \forall V \in U''_s \)

\[
\| V_{\text{rem}(w,\gamma)}^*(t,y) - V_{\text{rem}(w,\gamma)}^*(t,y') \| \leq \kappa \| y - y' \| \quad \forall y, y' \in \mathbb{R}^{n-1}
\]

uniformly for \( (t, (w,\gamma)) \in \mathbb{R} \times F^*_{n-1}(\Lambda) \).

Proof. By Lemma 2.10 and Lemma 2.2 and condition (U1)

\[
L := \sup_{(t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}} \left\{ \| \partial V_{w,\gamma}^*(t,y)/\partial y \| + \| R_{w,\gamma}^*(t) \| \right\} < +\infty
\]

which satisfies the requirement of the statement (1).

Given any \( \eta > 0 \). For any \( V \in N_s \) and for any \( (w,\gamma) \in F^*_{n-1}(\Lambda) \) one can write

\[
V_{\text{rem}(w,\gamma)}^*(t,y) = \left( V_{w,\gamma}^*(t,y) - S_{w,\gamma}^*(t,y) \right) + \left( S_{w,\gamma}^*(t,y) - R_{w,\gamma}^*(t) \right)
\]

for any \( (t,y) \in \mathbb{R} \times \mathbb{R}^{n-1} \). Then, from Lemma 2.10 and Lemma 2.6 there exists a \( C^1 \)-neighborhood \( U'_s \subset N_s \) of \( S \) and a constant \( \xi' \in (0, c] \) such that

\[
\sup_{(t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}} \| V_{w,\gamma}^*(t,y) \| \leq \eta \xi' \quad \forall (w,\gamma) \in F^*_{n-1}(\Lambda) \text{ and } V \in U'_s.
\]

This shows the statement (2).

Now given any \( \kappa > 0 \). Next, for any \( V \in N_s \) consider

\[
\frac{\partial}{\partial y} V_{\text{rem}(w,\gamma)}^*(t,y) = \frac{\partial}{\partial y} \left( V_{w,\gamma}^*(t,y) - S_{w,\gamma}^*(t,y) \right) + \left( \frac{\partial}{\partial y} S_{w,\gamma}^*(t,y) - R_{w,\gamma}^*(t) \right).
\]

From Lemma 2.10 we obtain that

\[
\| \frac{\partial}{\partial y} \left( V_{w,\gamma}^*(t,y) - S_{w,\gamma}^*(t,y) \right) \| \rightarrow 0 \text{ as } \| V - S \|_1 \rightarrow 0
\]

uniformly for \( (t,y,(w,\gamma)) \in \mathbb{R} \times \mathbb{R}^{n-1} \times F^*_{n-1}(\Lambda) \) and, from Lemma 2.2 there exists \( \xi'' \in (0, c] \) so that

\[
\| \frac{\partial}{\partial y} S_{w,\gamma}^*(t,y) - R_{w,\gamma}^*(t) \| \leq \frac{\kappa}{2} \quad \forall (t,(w,\gamma)) \in \mathbb{R} \times F^*_{n-1}(\Lambda) \text{ and } y \in \mathbb{R}^{n-1}.
\]

Hence, there is a \( C^1 \)-neighborhood \( U''_s \subset N_s \) of \( S \) such that: \( \forall V \in U''_s \)

\[
\| \frac{\partial}{\partial y} V_{\text{rem}(w,\gamma)}^*(t,y) \| \leq \kappa \quad \forall (t,(w,\gamma)) \in \mathbb{R} \times F^*_{n-1}(\Lambda) \text{ and } y \in \mathbb{R}^{n-1}.
\]

This implies the statement (3) by the mean value theorem.

Thus, Theorem 2.13 is proved.
3. Exponential dichotomy

In this section, we will introduce the exponential dichotomy due to Liao [7], by which we consider in part the relationship between the phase portraits of linear differential equations and their small perturbations on Euclidean spaces. Here we shall deal with families of ordinary differential equations, nor only a single equation itself.

Given a positive integer $p$. For convenience of our later discussion, let $M_{p \times p}^\Delta$ be the set of continuous matrix-valued functions $A: \mathbb{R} \to \text{gl}(p, \mathbb{R})$ such that

(a) $A(t)$ is triangular with $A_{ii}(t) = 0$ for $1 \leq i < p$;
(b) $A$ is uniformly bounded on $\mathbb{R}$ with $\eta_A := \sup_{t \in \mathbb{R}} \|A(t)\| < \infty$;
(c) $A$ is hyperbolic with index $p_-$ in the following sense:

$$
\xi_A := \sup_{t \in \mathbb{R}} \left\{ \sum_{k=1}^{p_-} \int_{-\infty}^t e^{\int_0^\tau A_{kk} \, d\tau} \, ds + \sum_{k=1}^{p} \int_t^{\infty} e^{\int_0^\tau A_{kk} \, d\tau} \, ds \right\} < \infty.
$$

In addition, let $M_{p \times 1}^\Delta$ be the set of continuous functions $f: \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$ such that

(d) $f(t, z)$ is bounded on $\mathbb{R} \times \mathbb{R}^p$ with $\eta_f := \sup_{(t, z) \in \mathbb{R} \times \mathbb{R}^p} \|f(t, z)\| < \infty$;
(e) $f(t, z)$ is Lipschitz in $z$ with a Lipschitz constant $L_f$:

$$
\|f(t, z) - f(t, z')\| \leq L_f \|z - z'\|
$$

for all $t \in \mathbb{R}$ and for any $z, z' \in \mathbb{R}^p$.

For any $(A, f) \in M_{p \times p}^\Delta \times M_{p \times 1}$, we will study the equations

$$
\dot{z} = A(t)z + f(t, z), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^p
$$

and

$$
\dot{z} = A(t)z, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^p.
$$

For any $(s, u) \in \mathbb{R} \times \mathbb{R}^p$, let $z_{A,f}(t; s, u)$ and $z_A(t; s, u)$ denote the solutions of (3.1) and (3.2) with $z_{A,f}(t; s, u) = u = z_A(t; s, u)$, respectively.

The following result is important for the proof of our main theorem.

**Theorem 3.1** ([7] Theorems 3.1 and 3.2). Let $(A, f) \in M_{p \times p}^\Delta \times M_{p \times 1}$ be any given. Then, there is a unique surjective mapping

$$
\Delta_{A,f} : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R} \times \mathbb{R}^p; \quad (s, u) \mapsto (s, \Delta_A(u))
$$

which possesses the following properties:

(i) $\Delta_{A,f}$ maps the phase-portraits of (3.1) onto that of (3.2). In fact,

$$
\Delta_{A,f}(t, z_{A,f}(t; s, u)) = (t, z_A(t; s, \Delta_A(u)));
$$

that is to say, the following commutativity holds:

$$
\begin{array}{ccc}
\mathbb{R}^p & \xrightarrow{z_{A,f}(t; s, \cdot)} & \mathbb{R}^p \\
\Delta_s & \downarrow & \Delta_t \\
\mathbb{R}^p & \xrightarrow{z_A(t; s, \cdot)} & \mathbb{R}^p.
\end{array}
$$

(ii) $\Delta_{A,f}$ is an $\varepsilon_{A,f}$-mapping, i.e., $\|(s, u) - \Delta_{A,f}(s, u)\| \leq \varepsilon_{A,f}$ for all $(s, u) \in \mathbb{R} \times \mathbb{R}^p$, where

$$
\varepsilon_{A,f} = \eta_f \xi_A (1 + 2\eta_A \xi_A)^p;
$$

(iii) For any $(s, u), (s, u') \in \mathbb{R} \times \mathbb{R}^p$, $z_{A,f}(t; s, u) - z_A(t; s, u')$ is bounded on $\mathbb{R}$ if and only if $\Delta_A(u) = u'$. 

(iv) If
\[ L_j \leq \frac{1}{\xi_A(1 + \eta_A \xi_A)^p}, \]
then \( \Delta_{A,f} \) is a self-homeomorphism of \( \mathbb{R} \times \mathbb{R}^p \).

Next, we endow \( M_{p \times p}^\Delta \times M_{p \times 1}^\Delta \) with the compact-open topology. Let \( (\mathbb{P},d) \) be a metric space with metric \( d \) and \( \eta_p > 0, \xi_p > 0, L_p > 0 \) constants with \( L_p \leq \frac{1}{\xi_p(1 + \eta_p \xi_p)^p} \). Let
\[
\mathcal{S} : \mathbb{P} \to M_{p \times p}^\Delta \times M_{p \times 1}^\Delta; \quad \lambda \mapsto (A_\lambda, f_\lambda)
\]
be a continuous mapping such that \( \eta_{A_\lambda} \leq \eta_p, \xi_{A_\lambda} \leq \xi_p, L_{f_\lambda} \leq L_p \), and
\[
\text{(3.2)} \lambda
\]
\[
\dot{z} = A_\lambda(t)z, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^p
\]
has no any nontrivial bounded solutions. We consider the bounded solutions of the equations with parameter \( \lambda \)
\[
\text{(3.1)} \lambda
\]
\[
\dot{z} = A_\lambda(t)z + f_\lambda(t,z), \quad (t,z) \in \mathbb{R} \times \mathbb{R}^p.
\]
Define
\[
\Delta^* : \mathbb{P} \to \mathbb{R}^p
\]
in the way: for any \( \lambda \in \mathbb{P} \)
\[
\Delta_\lambda(0, \Delta^*(\lambda)) = (0, 0) \in \mathbb{R} \times \mathbb{R}^p
\]
where \( \Delta_\lambda = \Delta_{A_\lambda,f_\lambda} : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R} \times \mathbb{R}^p \) is determined by Theorem 3.1 for \( \text{(3.1)} \lambda \) and \( \text{(3.2)} \lambda \).

We will need the following result, which will play a useful role in the later proof of our main theorem in \( \text{[4]} \).

**Theorem 3.2.** The mapping \( \Delta^* : \mathbb{P} \to \mathbb{R}^p \) is continuous.

**Proof.** Let \( \lambda_0 \in \mathbb{P} \) and \( \varepsilon > 0 \). Letting \( x_0 = \Delta^*(\lambda_0) \in \mathbb{R}^p \), we assert that there exists some \( \delta > 0 \) such that \( \|\Delta^*(\lambda) - x_0\| < \varepsilon \) whenever \( \lambda \in \mathbb{P} \) with \( d(\lambda, \lambda_0) < \delta \). If the assertion were not true, there would be a sequence \( \lambda_j \to \lambda_0 \) in \( \mathbb{P} \) satisfying \( \|\Delta^*(\lambda_j) - x_0\| \geq \varepsilon \) for all \( j \).

Since for all \( t \in \mathbb{R} \) we have
\[
\|z_{A_{\lambda_j},f_{\lambda_j}}(t;0,\Delta^*(\lambda_j))\| \leq \eta_p \xi_p(1 + 2\eta_p \xi_p)^p \quad j = 1, 2, \ldots
\]
by Theorem 3.1 we can assume \( \Delta^*(\lambda_j) \to x \) for some \( x \in \mathbb{R}^p \) and \( \|x - x_0\| \geq \varepsilon \). As \( \mathcal{S} \) is continuous, it follows from a basic theorem of ODE that
\[
\lim_{j \to \infty} z_{A_{\lambda_j},f_{\lambda_j}}(t;0,\Delta^*(\lambda_j)) = z_{A_{\lambda_0},f_{\lambda_0}}(t;0,x) \quad \forall t \in \mathbb{R}
\]
which implies that \( z_{A_{\lambda_0},f_{\lambda_0}}(t;0,x) \) is a bounded solution of \( \text{(3.1)} \lambda_0 \). So, \( x = x_0 \), it is a contradiction. \( \square \)

4. Structural stability of hyperbolic sets

In this section, we will prove our main theorem stated in the Introduction and construct an explicit example.

We assume that \( S : \mathbb{E} \to \mathbb{R}^n \) is a \( C^1 \)-vector field on the \( n \)-dimensional Euclidean \( w \)-space \( E, n \geq 2 \), which gives rise to a flow \( \phi : (t, w) \mapsto t.w \). Let \( \Lambda \) be a \( \phi \)-invariant closed subset, not necessarily compact, of \( \mathbb{E} \) such that
\[
\text{(U1)} \quad S'(w) \text{ is uniformly bounded on } \Gamma;
\]
\[
\text{(U2)} \quad 0 < \inf_{w \in \Lambda} \|S(w)\| \leq \sup_{w \in \Lambda} \|S(w)\| < \infty;
\]
\[
\text{(U3)} \quad S'(w) \text{ is uniformly continuous at } \Lambda; \text{ that is to say, to any } \varepsilon > 0 \text{ there is some } \delta > 0 \text{ so that for any } w, v \in \Lambda, \|S'(w) - S'(v)\| < \varepsilon \text{ whenever } \|w - v\| < \delta.
\]

Now we prove the following structural stability theorem by using Liao methods.
Theorem 4.1. Let $\Lambda$ be a hyperbolic set for $S$; that is to say, there exist constants $C \geq 1, \lambda < 0$ and a continuous $\Psi$-invariant splitting

$$T_w = T_w^s \oplus T_w^u, \quad \dim T_w^s = p_-(w) \quad w \in \Lambda$$

such that

$$\|\Psi_{t_0+t,w}x\| \leq C^{-1} \exp(\lambda t)\|\Psi_{t_0,w}x\| \quad \forall x \in T_w^s$$

and

$$\|\Psi_{t_0+t,w}x\| \geq C \exp(-\lambda t)\|\Psi_{t_0,w}x\| \quad \forall x \in T_w^u$$

for any $t_0 \in \mathbb{R}$ and for all $t > 0$. Then for any $\varepsilon > 0$ there is a $C^1$-neighborhood $U$ of $S$ in $\mathcal{F}(\mathbb{E})$ such that, if $V \in U$ then there exists a $\varepsilon$-topological mapping $H$ from $\Lambda$ onto some closed subset $\Lambda_V$ which sends orbits of $S$ in $\Lambda$ into orbits of $V$ in $\Lambda_V$, such that $H(w) \in \Sigma_w$ and $\psi_V(t,H(w)) = H(t,w) \in \Sigma_{t,w}$ for all $w \in \Lambda$ and for any $t \in \mathbb{R}$.

Proof. Let

$$A = \left\{ (w, \gamma) \in \mathcal{F}_{\mathbb{R}^{n-1}}^+ (\Lambda) \mid \text{col}_{k, \gamma} \in T_w^s \text{ for } 1 \leq k \leq p_-(w) \right\}.$$ 

Clearly, $A$ is a $\chi^{s+}$-invariant closed subset of $\mathcal{F}_{\mathbb{R}^{n-1}}^+ (\Lambda)$ with compact fibers $A_w$. For any $(w, \gamma) \in A$, we consider the reduced linearized equations

\begin{align*}
\dot{y} &= R^s_{w,\gamma}(t)y, \quad (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1},
\end{align*}

which is defined as Definition 2.1 and consider the reduced standard system

\begin{align*}
\dot{y} &= R^s_{w,\gamma}(t)y + V^s_{\text{rem}(w,\gamma)}(t, y), \quad (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}
\end{align*}

for any $V \in \mathcal{F}(\mathbb{E})$ defined as in Definition 2.11 associated with $S$. Then, we can take from Lemma 2.2 a constant $\eta_\Lambda > 0$ such that

$$\sup_{t \in \mathbb{R}, (w, \gamma) \in A} \|R^s_{w,\gamma}(t)\| \leq \eta_\Lambda < \infty.$$ 

Thus, it follows from Corollary 2.3 that there is another constant $\xi_\Lambda > 0$ such that

$$\xi_\Lambda = \sup_{(w, \gamma) \in A} \left\{ \sum_{k=1}^{p_-(w)} \int_{-\infty}^{t} e^{\int_{\tau}^{s} \omega^s_k(x^s(\tau, (w, \gamma))) \, d\tau} \, ds \right. \bigg\}$$

$$+ \left\{ \sum_{k=1}^{n-1} \int_{s}^{\infty} e^{\int_{t}^{s} \omega^s_k(x^s(\tau, (w, \gamma))) \, d\tau} \, ds \right\}$$

$$< \infty.$$ 

By Lemma 2.1, Theorem 2.13 and Theorem 3.1 there is no loss of generality in assuming that for any $(w, \gamma) \in A$ the reduced standard systems of $S$

\begin{align*}
\dot{y} &= R^s_{w,\gamma}(t)y + S^s_{\text{rem}(w,\gamma)}(t, y), \quad (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}
\end{align*}

has no any nontrivial bounded global solutions on $\mathbb{R}$.

Let $\mathcal{N}_\varepsilon(w) = \left\{ w + x \in \Sigma_w \mid \|x\| \leq \varepsilon \right\}$. Given any $\varepsilon > 0$ small enough to satisfy that for any $w \in \Lambda$ and any $w' \in \Lambda$ with $\|w - w'\| < \varepsilon$, we have $t.w' \in \mathcal{N}_\varepsilon(w)$ for some $|t| < 2\varepsilon \vartriangle^{-1}_\Lambda$, where $\vartriangle = \inf_{w \in \Lambda} |S(w)| > 0$. On the other hand, according to $\mathcal{N}_\varepsilon(w)$ we may assume that for any $w \in \Lambda, \mathcal{N}_\varepsilon(w) \cap \mathcal{N}_\varepsilon(t.w) = \emptyset$ for all $|t| < 2\varepsilon \vartriangle^{-1}_\Lambda$.

Denote

$$\rho_\xi = \frac{\xi}{4\xi_\Lambda(1 + 2\eta_\Lambda\xi_\Lambda)^{n-1}} \quad \text{and} \quad \kappa = \frac{1}{4\xi_\Lambda(1 + 2\eta_\Lambda\xi_\Lambda)^{n-1}}$$
for any $\xi \in (0, \varepsilon]$. Then, by Theorem 2.13 there exists a $C^1$-neighborhood $\mathcal{U}$ of $S$ in $\mathcal{X}(E)$ and a constant $\xi \in (0, \varepsilon]$ with $\xi < 1$ such that for any $V \in \mathcal{U}$ we have

\begin{equation}
(4.4a)
\sup_{(w, \gamma) \in \mathcal{A}} \| V_{\text{rem}}(w, \gamma)(t, y) \| \leq \varepsilon \rho \xi 
\end{equation}

and

\begin{equation}
(4.4b) \| V_{\text{rem}}(w, \gamma)(t, y) - V_{\text{rem}}(w, \gamma)(t, y') \| \leq \kappa \| y - y' \| \quad \forall y, y' \in \mathbb{R}^{n-1}
\end{equation}

uniformly for $(t, (w, \gamma)) \in \mathbb{R} \times \mathcal{A}$.

Fix some $C^\infty$ bump function $b : [0, \infty) \to [0, 1]$ with $b([0, 1/2]) \equiv 1$ and $b([1, \infty)) \equiv 0$. For any $V \in \mathcal{U}$ and any $(w, \gamma) \in \mathcal{A}$, let

$$
\bar{V}_{\text{rem}}(w, \gamma, \xi)(t, y) = \begin{cases} 
b(||y||/\xi)\bar{V}_{\text{rem}}(w, \gamma)(t, y) & \text{for } ||y|| \leq \varepsilon, \\
0 & \text{for } ||y|| \geq \varepsilon.
\end{cases}
$$

Next, we consider the adapted differential equations

\begin{equation}
(4.5) \quad \bar{y} = R^*_{w, \gamma}(t) \bar{y} + \bar{V}_{\text{rem}}(w, \gamma, \xi)(t, y), \quad (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}.
\end{equation}

It is easily seen that $R^*_{w, \gamma}(t) \in M_{(n-1) \times (n-1)}$ and $\bar{V}_{\text{rem}}(w, \gamma, \xi)(t, y) \in M_{(n-1) \times 1}$ as in §3 in the case $p = n - 1$ and $p_\gamma = p - (w)$ for any $(w, \gamma) \in \mathcal{A}$. Let $y_{v, \gamma}(w, \gamma)(t; s, u)$ be the solution of (4.6) such that $y_{v, \gamma}(w, \gamma)(t; s, u) = u$ for any $(s, u) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Given any $V \in \mathcal{U}$.

For any $(w, \gamma) \in \mathcal{A}$, it follows from Theorem 3.1 that there uniquely corresponds an $x \in \mathbb{R}^{n-1}$, writing $h_{v, \gamma}(w) = \gamma x \in \Sigma_w - w = T_w$, such that $y_{v, \gamma}(w, \gamma)(t; 0, x)$ is bounded on $\mathbb{R}$ with

\begin{equation}
(4.6) \quad \sup_{t \in \mathbb{R}} \| y_{v, \gamma}(w, \gamma)(t; 0, x) \| \leq \varepsilon \xi / 4.
\end{equation}

So, $y_{v, \gamma}(w, \gamma)(t; 0, x)$ is also the solution of (4.2). According to Theorem 2.12 we easily see that such $h_{v, \gamma}(w)$ is independent of the choice of $\gamma$ in $\mathcal{A}_w$ and is such that $\| h_{v, \gamma}(w) \| \leq \min\{\varepsilon, \xi\} / 4$ for $w \in A$. By Theorem 3.1 and Theorem 2.12(2) again we have easily

\begin{equation}
(4.7) \quad h_{v, \gamma}(w) = \psi_{v, t, w}(h_{v, \gamma}(w)) \in T_w \quad \forall t \in \mathbb{R},
\end{equation}

since $\psi_{v, t, w}(h_{v, \gamma}(w)) = \tilde{R}^*_{v, t, w}(t, y_{v, \gamma}(w, \gamma)(t; 0, x))$ for any $w \in A$. Moreover, we can assert that the mapping $w \mapsto w + h_{v, \gamma}(w)$ is injective. In fact, if $w + h_{v, \gamma}(w) = w' + h_{v, \gamma}(w')$ for some $w, w' \in A$, then $\| t \cdot w - t \cdot w' \| \leq \varepsilon / 2$ for all $t \in \mathbb{R}$. Since (3.3) has only one global bounded solution on $\mathbb{R}$, there is some $t'$ with $|t'| < 2 \varepsilon \Lambda^1$ such that $t' w' = w$. Thus, $t' = 0$. Otherwise $\mathcal{N}_w(w) \cap \mathcal{N}_w(w') \neq \emptyset$, it is a contradiction.

Let $\Lambda_V = \{ w + h_{v, \gamma}(w) \mid w \in A \}$ and $H_V : A \to \Lambda_V$, $w' = w + h_{v, \gamma}(w) \in \Sigma_w$. Clearly, $\| w - H_V(w) \| < \varepsilon$. It remains to prove that $\Lambda_V$ is closed in $\mathbb{E}$ and $H^{-1}_V : \Lambda_V \to A$ continuous as well. Let $w_j \to w'$ with $w_j \in \Lambda_V$ and $w_j = H^{-1}_V(w_j)$ for $j = 1, 2, \ldots$. We have to prove $w_j \to w$ for some $w \in A$ and $w' = H_V(w)$. By the definition of $H_V$, there is a sequence $(w_j, \gamma_j)$ in $A$ and a sequence $(x_j)$ in $\mathbb{R}^{n-1}$ such that

$$
w_j' = w_j + h_{v, \gamma}(w_j) = w_j + \gamma_j x_j \quad \text{for } j = 1, 2, \ldots
$$

Since $\gamma_j \in \mathcal{A}_w \subset \mathcal{F}^+_{n-1}$, $\| x_j \| \leq \varepsilon \xi / 4$ and $\mathcal{F}^+_{n-1}$ is compact, without loss of generality we may assume that $\gamma_j \to \gamma$ in $\mathcal{F}^+_{n-1}$ and $x_j \to x$ in $\mathbb{R}^{n-1}$. Let $w = w' - \gamma x$. Then $w_j \to w$ in $\Lambda$ and $w' = w + \gamma x$ and $(w, \gamma) \in A$. In order to prove $w' = H_V(w)$, it is
sufficient to prove that $h_{V,\xi}(w) = \gamma x$. In fact, from Theorem 2.12 \(1\) and a basic theorem of ODE, we have

$$\lim_{j \to \infty} \sup_{|t| < T, \|u\| \leq \epsilon} \|y_{V,(w_j,\gamma_j)}(t;0,u) - y_{V,(w,\gamma)}(t;0,u)\| = 0 \quad \forall T > 0.$$ 

Thus, for all $t \in \mathbb{R}$ we have

$$\lim_{j \to \infty} y_{V,(w_j,\gamma_j)}(t;0,x_j) = y_{V,(w,\gamma)}(t;0,x),$$

which means $\|y_{V,(w,\gamma)}(t;0,x)\| \leq \epsilon |x|/4$ for all $t \in \mathbb{R}$. So, $h_{V,\xi}(w) = \gamma x$, as desired.

We can show that $H_V$ is continuous by Theorem 3.2.

Thus, the theorem is proved. \(\square\)

**Remark 4.2.** If $\Lambda = E$, then $\Lambda_V = E$ by a standard topology argument. Indeed, letting $S^{n+1} = \mathbb{E} \cup \{\infty\}$, $H_V$ has a continuous extension from the topological sphere $S^{n+1}$ to itself which maps $\infty$ to $\infty$ and is homotopic to the identity. Thus, from differential topology we know that $H_V(E) = E$.

We conclude our arguments with an example.

**Example 4.3.** Let $S(x, y, z) = (1, y, -z)^T \in \mathbb{R}^3$ for any $(x, y, z) \in \mathbb{E}^3$, which is a differential system on the 3-dimensional Euclidean $(x, y, z)$-space $\mathbb{E}^3$. Let $\Lambda = \mathbb{E} \cup \{0\} \times \{0\}$. Then, $S$ gives rise to the $C^1$-flow $\phi: (t, (x, y, z)) \mapsto (x + t, y e^t, ze^{-t})$, and $S$ is hyperbolic with $T_{(x,0,0)} = T^s_{(x,0,0)} \oplus T^u_{(x,0,0)}$, where $T^s_{(x,0,0)} = \{0\} \times \mathbb{R}, T^u_{(x,0,0)} = \{0\} \times \mathbb{R} \times \{0\}$ for any $(x, 0, 0) \in \Lambda$, and $S$ satisfies conditions (U1), (U2) and (U3) on $\Lambda$. Thus, $S$ is structurally stable on $\Lambda$ from the Main Theorem. Particularly, for any $\epsilon > 0$, if $V \in \mathcal{X}(\mathbb{E}^3)$ is $C^1$-close to $S$, then $V$ has an integral curve which lies in the $\epsilon$-tubular neighborhood of $\mathbb{E} \times \{(0,0)\}$.

**References**

[1] D. V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature. *Proc. Steklov Math. Inst.*, 90 (1967), 1–235.

[2] X. Dai, Integral expressions of Lyapunov exponents for autonomous ordinary differential systems, *Science in China Series A: Mathematics*, 51 (2008), 000–000.

[3] X. Dai, Existence of contracting periodic orbit of autonomous differential systems. Preprint 2007.

[4] X. Dai, Transversal stable manifolds of $C^1$-differential systems having transversal dominated splitting. Preprint 2008.

[5] K. Kato and A. Morimoto, Topological stability of Anosov flows and their centralizers. *Topology*, 12 (1973), 255–273.

[6] S.-T. Liao, Applications to phase-space structure of ergodic properties of the one-parameter transformation group induced on the tangent bundle by a differential systems on a manifold I. *Acta Sci. Natur. Univ. Pekinensis*, 12 (1966), 1–43.

[7] S.-T. Liao, Standard systems of differential equations. *Acta Math. Sinica*, 17 (1974), 100–109; 175–196; 270–295.

[8] S.-T. Liao, *Qualitative Theory of Differential Equations*, Science Press, Beijing, 1996.

[9] J. Moser, On a theorem of Anosov. *J. Differential Equations*, 5 (1969), 411–440.

[10] Z. Nitecki, On semistability for diffeomorphisms. *Invent. Math.*, 14 (1971), 83–123.

[11] C. Pugh and M. Shub, The $\Omega$-stability theorem for flows. *Invent. Math.*, 11 (1970), 150–158.

[12] J. W. Robbin, A structural stability theorem. *Ann. of Math.*, 94 (1971), 447–493.

[13] C. Robinson, Structural stability of vector fields. *Ann. of Math.*, 99 (1974), 154–175.

[14] C. Robinson, Structural stability of $C^1$ diffeomorphisms. *J. Differential Equations*, 22 (1976), 28–73.

[15] P. Walters, Anosov diffeomorphisms are topologically stable. *Topology*, 9 (1970), 71–78.