Analysis of dynamic response of axially loaded pile using nodal exact finite element model

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Abstract. This research work presents the adoption of proposed displacement based finite element model so-called nodal exact finite element to solve dynamic response of axially loaded pile. The pile is assumed to be circular cross section embedded in elastic soil. The shape functions satisfy homogeneous differential equation of uniform elasto-dynamic bar was constructed. The stiffness and mass matrices for axially loaded pile are then formulated using energy principle, incorporate with proposed shape function. The dynamic response parameters, such as dynamic pile-head stiffness and damping of pile’s axial vibration were computed from proposed finite element model by iterative procedure. Illustrative examples show fairly accurate results and demonstrate an effectiveness of proposed finite element compared with available analytical solutions from literatures.

1. Introduction
Modelling of static and dynamic pile-soil interaction has been significant research attention over the past four decades. One of the most efficient way to model pile-soil interaction problem is to replace the surrounding soil medium by a series of distributed Winkler springs along the pile axis. Winkler models are widely accepted in the analysis of both axially and laterally loaded piles subjected to static and dynamic loads [1–4]. The most fundamental problem in the implementation of Winkler models lies in the procedure to assess the moduli of Winkler springs. The current methods for determining the moduli of the Winkler spring can be classified into three main groups as described in [1, 2]: (a) experiment methods; (b) calibration with rigours numerical solutions; and (c) simplified theoretical models. Notwithstanding the significance of these three methods, they can all be criticised for certain drawbacks. Hence, in this study, the settlement of axially loaded pile embedded in elastic soil is solved via proposed finite element procedure. The Winkler moduli are assumed to be constant along the depth of homogeneous elastic soil, and pile section are assumed to be remain a rigid plane. The nodal exact shape function concept suggested in [5–9] are used to construct the stiffness matrix and equivalent nodal force incorporate with fixed-point iteration algorithm to solve nonlinear algebraic equations. In each iteration step, the coefficients of differential equation describe pile settlement behaviour were estimated easily via component of stiffness matrix and nodal displacement obtained from previous iteration step. Examples of end-bearing pile embedded in elastic soil subjected to vertical static and harmonic point loads on top soil level were analysed. The results of pile-head stiffness and damping from proposed element are compared with analytical solution obtained from [1, 2] to verify the accuracy of proposed pile element.
2. Mathematical formulation

In this section, the physical parameter of axisymmetric pile embedded in elastic soil will be explained. The dynamic equilibrium equation of pile and soil subjected to harmonic point load at top of pile head are also derived from the variational principle.

2.1. Problem definition

The analysis considers a single circular cross section pile [2], with radius $r_p$ and total length $L$ embedded in a homogeneous soil layer as shown in figure 1. The pile is subjected to an axial harmonic force of magnitude $P$ with cyclic frequency $\omega$ at the pile head which is flush with the ground surface. The soil layer is assumed to extend to infinity in the radial direction, and the bottom is assumed to be fixed on the firm ground. The soil medium is assumed to be elastic and isotropic material, with elastic properties described by shear modulus $G_s$, Poisson’s ratio $\nu_s$, and mass density $\rho_s$. Pile radius is denoted by $r_p$. Pile behave as an elastic column with Young’s modulus $E_p$, Poisson’s ratio $\nu_p$ and mass density $\rho_p$. Perfect contact or no-slip condition is assumed at the pile-soil interface. The positive notation for coordinates $r$ and $z$, vertical displacement and stresses is illustrated in figure 1.

![Figure 1. Pile subject to harmonic point load, modified form][2]

2.2. Governing differential equation

Since the cylindrical pile settlement problem in Figure 1 is axisymmetric. Hence, we use the system of cylindrical coordinates ($r$-$z$ coordinate) to indicate any position in pile and soil bodies. The origin of cylindrical coordinate coincide with the center of pile cross section at the pile head level. The vertical (positive in downward direction) coordinate $z$-axis is coincide with pile axis. The vertical displacement $u_z(r,z,t)$ at any point in the soil is represented as product of three functions as follows:

$$u_z(r,z,t) = w(z)\phi(r)\exp(i\omega t)$$  \hspace{1cm} (1)

where $w(z)$ is the vertical displacement of pile at any point along pile axis, and $\phi(r)$ is the soil displacement decay function in the radial direction. Assume that normal stress $\sigma$ and shear stress $\tau$, as
shown in figure 1, are controlled exclusively by the vertical displacement $u_z$ in equation (1); the influence of radial displacement $u_r$ on these stresses is considered negligibly small [1]. Based on this assumption, the simplified stress-displacement relations for $\sigma_z$ and $\tau_{rz}$ are written as [2]:

$$\sigma_z = -\eta_s^* G_s' \frac{\partial u_z}{\partial z}$$

(2)

$$\tau_{rz} = -G_s' \frac{\partial u_z}{\partial r}$$

(3)

where $G_s' = G_s(1 + 2i\zeta_s)$ is the complex shear modulus of soil, and $\eta_s$ is a dimensionless compression parameter that depend on Poisson’s ratio. In present work, $\eta_s = \frac{2}{1 - v_s}$

(4)

which conforms to the assumption $\varepsilon_0 = 0$ and $\sigma_0 = 0$. Then, the calculus of variations are employed to obtain the governing differential equation in pile and surrounding soil by define the strains and stresses from displacement functions in equation (1), i.e. equations (2) and (3). The governing differential equations are obtained by prescribe the variation of total potential energy with respect to $w$ and $\phi$ equal to zeros with the assumption that pile cross-sections remain plane [4, 9]. The governing differential equation for the pile displacement is in the following form:

$$\omega^2 \rho w + \left( E_p A_p + 2t_s \right) \frac{d^2 w}{dz^2} - k_w w = 0$$

(5)

where

$$k_s = 2\pi G_s' \int_0^r r \left( \frac{d\phi}{dr} \right)^2 dr$$

(6)

$$t_s = \pi \eta_s^* G_s' \int_0^r r \phi^2 dr$$

(7)

$$\bar{m} = \rho_p A_p + 2\pi \rho_s \int_0^r r \phi^2 dr$$

(8)

The governing differential equation for the soil surrounding the pile is expressed as:

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \gamma^2 \phi = 0$$

(9)

where

$$\gamma = \sqrt{\frac{n_s}{m_s}}$$

(10)

and $m_s$ and $n_s$ are given by

$$m_s = G_s' \int_0^L w^2 dz$$

(11)

$$n_s = \eta_s^* G_s' \int_0^L \left( \frac{dw}{dx} \right)^2 dz - \rho_s \omega^2 \int_0^L w^2 dz$$

(12)

and the solution of Eq. (9) with boundary conditions $\phi(r) = 0$ at $r \to \infty$, and $\phi(r) = 1$ at $r = r_p$ is a zero-
order modified Bessel function of the second kind:

\[ \phi(r) = \frac{K_0(\gamma r)}{K_0(\gamma r_p)} \text{ for } r_p \leq r \leq \infty \] (13)

The general solution of pile displacement from equation (5) is given by following hyperbolic functions

\[ w(z) = A \cosh(\alpha z) + B \sinh(\alpha z) \text{ for } 0 \leq z \leq L_p \] (14)

where \( A \) and \( B \) are integration constant. The characteristic parameter of pile and soil interaction is expressed in the following form:

\[ \alpha = \sqrt{k_s - \bar{m}\omega^2} \sqrt{E_p A_p + 2t_s} \] (15)

This parameter was already defined in previous work [9] for static case (\( \omega = 0 \)). Note that the dimension of parameter \( \alpha \) is an inversion of length. The amplitude of axial force, as shown in figure 1, at any depth \( z \) is expressed as:

\[ N(z) = -\left(E_p A_p + 2t_s\right) \frac{dw}{dz} = -a \left[ A \sinh(\alpha z) + B \cosh(\alpha z) \right] \] (16)

where

\[ a = \alpha \left(E_p A_p + 2t_s\right) = \sqrt{k_s \left(E_p A_p + 2t_s\right)} \] (17)

The particular solution of equation (14) can be determined from prescribing boundary conditions at both ends directly to obtain integration constants \( A \) and \( B \) [4]. However, in the case of multi-layer soils, this method have to take care of displacement continuity conditions at interface between beneath soil layers. Therefore, the stiffness matrix method, which is derived from force-displacement relation from exact solution (14) is proposed in this work. Note that this method satisfy the continuity condition between soil layers automatically.

3. Finite element formulation

Consider the one-dimensional element in figure 2, which represents the portion of pile embedded in any one layer of surrounding soil governed by equation (5). Pile element in figure 2 compose of two nodes at top and bottom levels, numbering with node number 1 and 2, respectively. Shear resistance of soil is represented by equivalent soil spring coefficient \( k_s \). The force-displacement relations at two-end nodes of figure 2 are expressed as:

\[ P_1 = N(0) = -\left(E_p A_p + 2t_s\right) \frac{dw(0)}{dz} \] (18)

\[ P_2 = -N(L) = \left(E_p A_p + 2t_s\right) \frac{dw(L)}{dz} \] (19)

Then, the stiffness matrix, which represent the relation between nodal forces and nodal displacements \( w_1 \) and \( w_2 \) according to equations (18) and (19) will be derived from solution in equation (14). The results of stiffness matrix obtained from this derivation can then easily extended to solve the problem of pile embedded in multi-layer soil. Due to the exactness of solution in equation (14), the weighted residual scheme described in [10] also yield the same result of stiffness matrix components.
3.1. Derivation of stiffness matrix
To construct the relation between nodal forces and nodal displacements of figure 2, the solution in equation (14) is rearranged in the following form:

\[ w(z) = \varphi_1(z)w_1 + \varphi_2(z)w_2 \]  \hspace{1cm} (20)

The shape functions in equation (20) are expressed as follows:

\[ \varphi_1(z) = \frac{\sinh[\alpha(L-z)]}{\sinh \beta} \]  \hspace{1cm} (21)

\[ \varphi_2(z) = \frac{\sinh(\alpha z)}{\sinh \beta} \]  \hspace{1cm} (22)

where the dimensionless parameter \( \beta = \alpha L \). Employ trial solution from equation (20) into the force-displacement relations, equations (18) and (19), the stiffness matrix can be expressed as:

\[ k = a \begin{bmatrix} \coth \beta & -\csc \beta \\ -\csc \beta & \coth \beta \end{bmatrix} \]  \hspace{1cm} (23)

The component of stiffness matrix in equation (23) depends on decay parameter \( \gamma \) in equation (10). This decay parameter evaluated from pile displacement \( w(z) \) according to equations (11) and (12).

3.2. Calculation of decay parameter
The value of decay parameter can be evaluated from nodal displacement \( w = \{w_1, w_2\} \). Substitutes the displacement function, equation (20), into equations (11) and (12), yields

\[ m_i = \frac{G' \alpha}{2} \left[ (\coth \beta - \beta \csc \beta)(w_1^2 + w_2^2) - 2w_1w_2 \csc \beta (1 - \beta \coth \beta) \right] \]  \hspace{1cm} (24)
\[ n_i = \frac{\eta_i^* G_i^*}{2} \left[ \left( \beta \cosh^2 \beta + \coth \beta \right) \left( W_i^2 + W_i^2 \right) - 2W_i W_i \cosh \beta \left( 1 + \beta \cosh \beta \right) \right] - m \left( \frac{\omega}{V_s^*} \right)^2 \]  

(25)

where the variable \( V_s^* = \sqrt{G_i^* / \rho_s} \) is the complex-valued propagation velocity of shear waves in the soil. The values of \( m_i \) and \( n_i \) are then substituted into equation (10) to evaluate the value of decay parameter.

3.3. Iteration scheme

The shape parameter of decay function, \( \gamma \), in equation (10) depends on pile displacement \( w(z) \). This leads to the nonlinear algebraic equations as follow:

\[ k(w)w = f \]  

(26)

where \( f \) is the external nodal load. To solve the nodal solution from equation (26), the fixed point iteration technique is employed:

\[ k(w^{(i)})w^{(i+1)} = f \]  

(27)

where the symbols \( i \) and \( i+1 \) represent the previous and current iteration, respectively. The convergence criteria of nodal solution \( \mathbf{w} \) is dictated by satisfy the following condition:

\[ \left| \gamma^{(i+1)} - \gamma^{(i)} \right| r_p \leq 10^{-5} \]  

(28)

The convergence criteria in equation (28) was used in [4, 9], and also guarantee the convergence of nodal solution \( \mathbf{w} \) in equation (26).

4. Numerical examples

The proposed numerical examples are tested using the value of Poisson’s ratio \( v_s = 0.4 \) and soil damping \( \zeta_s = 0 \). Table 1 compares results for the static stiffness of end-bearing piles obtained from proposed model and from available solutions in the literature [2], where circular frequency \( \omega \) is set to be zero. The results are presented in terms of the normalized static pile-head stiffness \( K_p/E_d \) where diameter \( d = 2r_p \). The performance of the proposed model is fairly accurate. The maximum deviations from the referenced solution [2], as shown in last column of table 1, is not exceeding approximately 9.0%. For the dynamic stiffness and damping with \( \omega > 0 \) and the ratio \( E_p/E_s = 100 \), figures 3 and 4 shown the comparison of dynamic pile impedance normalized by the static value \( K_p = \text{Re}(K^*)/K_p \) and damping factor \( \zeta = \text{Im}(K^*)/2\text{Re}(K^*) \) against dimensionless frequency parameter \( \omega_0 = \omega d / V_s \), respectively. In figure 4, the increase in damping is stronger for long piles.

### Table 1. Comparison of normalized static pile-head stiffness from available results [2] and proposed model, for end-bearing pile.

| \( L/d \) | \( E_p/E_s \) |\( \begin{array}{c} \text{Normalized static pile head stiffness, } K_p/E_d \\
\text{Anoyatis (A)} \text{ Proposed (P)} \text{ (P - A)/A} \% \end{array} \) |
|---|---|---|
| 10 | 100 | 11.17 | 11.34 | 1.52 |
|   | 1000 | 82.08 | 82.18 | 0.12 |
| 20 | 100 | 8.42 | 8.77 | 4.16 |
|   | 1000 | 44.58 | 44.74 | 0.36 |
| 30 | 100 | 7.80 | 8.33 | 6.36 |
|   | 1000 | 32.99 | 33.21 | 0.67 |
| 40 | 100 | 7.56 | 8.24 | 8.99 |
|   | 1000 | 27.74 | 28.03 | 1.05 |
Fig. 3. Comparison of dynamic pile-head stiffness obtained with proposed model and from literature [2], $E_p/E_s = 100$.

Fig. 4. Comparison of dynamic pile-head damping obtained with proposed model and from literature [2], $E_p/E_s = 100$.

5. Conclusions

The stiffness method to solve dynamic pile-soil interaction was investigated. The proposed model yields solutions along axially loaded end-bearing piles resting on firm ground, embedded in elastic soil stratum. The result from proposed model were compared with available results from literatures and shown fairly accurate in prediction of pile-head stiffness and damping ratio. The damping of pile head stiffness is increase with higher value of pile length.
6. References

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