NEW ASYMPTOTIC ANALYSIS METHOD FOR PHASE FIELD MODELS IN MOVING BOUNDARY PROBLEM WITH SURFACE TENSION

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Abstract. In this paper, we give an asymptotic analysis of the phase field Allen-Cahn and Cahn-Hilliard models of free surfaces with surface tension. Unlike the traditional approach that approximates the solution by the so-called matched asymptotic expansion involving outer expansion, inner expansion and matching, our new approach utilizes a uniform double asymptotic expansion to expand the whole phase field function directly. Although the main result is not new, we would like to emphasize that we derive the result under a uniform double asymptotic expansion. Thus, in this paper the detailed structure of the phase field functions in the equilibrium state is obtained, and the consistency of the phase field models with the corresponding sharp interface models is discussed, including the free surface Allen-Cahn model, Cahn-Hilliard model, and the Allen-Cahn model with volume constraint. The explicit asymptotic expansion of the phase field function reveals rich details of its structures. Moreover, it nicely explains some unusual phenomena we observed in numerical experiments. The theory introduced in this paper can be applied to guide the future modeling and simulation of other moving boundary problems by phase field models.

1. Introduction. Phase field models, due to their advantages of handling topological changes, complex geometry, and ease for implementation, have been widely used in modeling free surface or moving surface problems. The mostly modeled problems are those driven by surface tension. Specifically, consider a system consisting of two components (or phases) \( \Omega_1, \Omega_2 \), the free energy is given by

\[
E = \sigma |\Gamma|, \quad (1)
\]

where \( \sigma \) is the surface tension between the two phases, and \( \Gamma \) is the free surface between the two components. We expect there is a minimum surface when the system reaches the equilibrium state.

Instead of a sharp separation between the two phases, an alternative way is to introduce a thin layer with a smooth transition from one phase to the other. Van
nder Waals and Cahn-Hilliard [45, 11] proposed the following phase field method to model the separation. By introducing a phase field function $\phi$ defined on a physical domain $\Omega$, we are able to label the two phases separated by the surface $\Gamma$, which is given by the zero level set $\{x : \phi(x) = 0\}$. We denote one phase by $\{x : \phi(x) > 0\}$ and the other $\{x : \phi(x) < 0\}$. The surface tension energy is formulated by:

$$W(\phi) = \int_{\Omega} \frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{4\epsilon}(\phi^2 - 1)^2 \, dx,$$

where the parameter $\epsilon$ is the length scale of the transition region from one phase to the other. The second term $(\phi^2 - 1)^2$ is a double-well potential, which vanishes at $-1$ and $+1$. When minimizing $W(\phi)$, the phase field function $\phi$ tends to $-1$ and $+1$ in different regions, and the first term $\frac{\epsilon}{2} |\nabla \phi|^2$ penalizes the spatial inhomogeneity of $\phi$.

There are already a lot of analysis to bridge the sharp interface model (1) and the phase field model (2). For example, Caginalp, X. Chen and etc. showed the connection in [6, 7, 8, 10, 12, 9] by using formal asymptotic analysis. Modica [37] proved the $\Gamma$-convergence of phase field model to sharp interface model. The concept of the $\Gamma$-convergence was introduced by E. De Giorgi and T. Franzoni in 1975 [21]. We refer to [38, 43] for further information on this topic.

Consider the phase field model (2), where $W(\phi)$ is the surface tension energy. Minimizing $W(\phi)$ will result in a minimum surface $\Gamma$ that is the zero level set of $\phi$. The minimization process is driven by the classic Allen-Cahn equation

$$\phi_t = -\frac{1}{\epsilon} \frac{\partial W(\phi)}{\partial \phi} = \Delta \phi - \frac{1}{\epsilon^2}(\phi^2 - 1)\phi$$

with boundary condition

$$\frac{\partial \phi}{\partial n} = 0 \text{ in } \partial \Omega, \quad \phi(x, 0) = \phi_0(x) \text{ in } \Omega;$$

and the classic Cahn-Hilliard equation

$$\phi_t = \Delta \frac{\partial W(\phi)}{\partial \phi} = -\Delta(\epsilon \Delta \phi - \frac{1}{\epsilon}(\phi^2 - 1)\phi)$$

with boundary condition

$$\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n}(\epsilon \Delta \phi - \frac{1}{\epsilon}(\phi^2 - 1)\phi) = 0 \text{ in } \partial \Omega, \quad \phi(x, 0) = \phi_0(x) \text{ in } \Omega.$$ 

Note that $\phi_t$ is the time derivative of $\phi$. The Allen-Cahn equation (3) was originally introduced by Allen-Cahn [2] to study the antiphase boundaries in crystalline solids. It is the gradient flow of the phase field energy $W(\phi)$. The Cahn-Hilliard equation (5) was originally introduced by Cahn-Hilliard [11] to describe the complicated phase separation and coarsening phenomena in melted alloy. Same as Allen-Cahn equation, Cahn-Hilliard equation also results in energy decreasing of $W(\phi)$ because

$$W(\phi) = -\int_{\Omega} |\nabla \frac{\partial W(\phi)}{\partial \phi}|^2 \, dx.$$ 

It is well-known that Cahn-Hilliard equation preserves the inside volume of surface $\Gamma$, due to the fact that

$$\frac{d}{dt} \int_{\Omega} \phi(x) \, dx = \int_{\Omega} \Delta \frac{\partial W(\phi)}{\partial \phi} \, dx = 0,$$

with boundary condition (6).
The Allen-Cahn and Cahn-Hilliard equations have been extensively studied in the past few decades. For Allen-Cahn equation, please refer to [2, 15, 35, 31, 36, 17, 39]. It was first formally shown [2, 33] that, as $\epsilon \to 0$, the zero level set $\Gamma(t)$ of $\varphi(t)$ approaches to a surface $\sigma(t)$ which evolves following the mean curvature flow

$$V_{\sigma}(x) = H(x),$$

where $V_{\sigma}$ is the normal velocity of $\sigma(t)$ and $H$ is its mean curvature. In [31], Evans, Soner and Souganidis, and later Ilmanen in [36] rigorously proved that the limit is actually one of the Brakke’s motion by mean curvature solution [4]. Besides the theoretical advances, extensive numerical simulations have been done to approximate the interfaces driven by curvature and the solution of Allen-Cahn equation in recent years. Numerous algorithms are proposed together with the convergence and error analysis of the numerical interfaces to the curvature driven surfaces [3, 18, 26, 40, 41, 32].

For the Cahn-Hilliard equation, it was first shown by Pego [42] the relationship with the Hele-Shaw (Mullins-Sekerka) problem, which was raised from the study of the pressure of immiscible fluids in the air [42, 1, 14, 13]. Denote

$$f_{\epsilon} = -\epsilon \Delta \varphi + \frac{1}{\epsilon} (\varphi^2 - 1) \varphi.$$

As $\epsilon \to 0$, $f_{\epsilon}$ tends to a limit $f$ which satisfies

$$\Delta f = 0 \quad \text{in } \Omega \setminus \Gamma(t), \quad \frac{\partial f}{\partial n} = 0 \quad \text{on } \partial \Omega,$$

and

$$f = \mu H \quad \text{on } \Gamma(t), \quad V_{\sigma} = \frac{1}{2} [\frac{\partial f}{\partial n}] \quad \text{on } \Gamma(t),$$

with initial $\Gamma_0$ is given at $t = 0$. Here $\mu = \int_{-1}^{1} \sqrt{s(s^2 - 1)/2} ds$, $V_{\sigma}$ is the normal velocity of surface $\Gamma(t)$ and $H$ is its mean curvature, $[\frac{\partial f}{\partial n}]$ is the jump of $\frac{\partial f}{\partial n}$ from the exterior to the interior of $\Gamma(t)$. The rigorous justification of this limit was done by Alikakos, Bates, and Chen [1] with assumption that the above Hele-Shaw problem has a classic solution. Later Chen [14, 9] proved the convergence of $f_{\epsilon}$ to a weak solution of the Hele-Shaw problem. Meanwhile, extensive research focused on the numerical analysis of the Cahn-Hilliard equation is done and its limiting behavior to the Hele-Shaw problem has been verified [5, 19, 20, 25, 27, 28, 29, 30, 34].

As the Allen-Cahn and Cahn-Hilliard equations become two very popular approaches in various free boundary models, it is desirable to know the detailed structure of the phase field function. It is well-known that the phase field function is roughly like a tanh function. Sometimes, the phase field function is directly assumed to be a tanh function in numerical simulations [44]. However, it is uncertain how much difference the real $\varphi$ is from the tanh function, and how the difference affects the models. The objective of this paper is to reveal the more subtle structure of the phase field function $\varphi$ in the equilibrium state for both of the Allen-Cahn models and the Cahn-Hilliard models. Moreover, we build the consistency of the phase field models with the sharp interface models via a different approach.

Traditional approach to approximate the solution of Allen-Cahn and Cahn-Hilliard equations is the so-called “matched asymptotic expansion”. Three ingredients are involved: outer expansion, inner expansion, and matching. In the bulk region, i.e., region far away from the interface $\Gamma$, outer expansion is applied and the
problem can be treated as a regular perturbation problem. In a thin neighborhood of the interface $\Gamma$, inner expansion and the method of stretched coordinates are applied [8]. In this case, the problem is usually treated as a singular perturbation problem. Finally, the matching condition is used such that the outer and inner expansions are consistent with each other in the overlapping region. Although it is quite easy to compute the leading order term by this traditional approach, the computations for the matching conditions are quite complicated for higher order terms [9]. In this paper we use a different approach that uses a uniform so-called “double asymptotic expansion” to approximate the whole phase field function directly. An appealing feature of this approach is that no matching conditions are needed.

The idea of this paper follows our previous asymptotic analysis paper on Willmore problems [46]. The most widely used asymptotic expansion to the phase field functions is as follows

$$
\varphi(x) = q_0\left(\frac{d(x)}{\epsilon}\right) + \epsilon q_1\left(\frac{d(x)}{\epsilon}\right) + \epsilon^2 q_2\left(\frac{d(x)}{\epsilon}\right) + o(\epsilon^2),
$$

where the functions $q_0, q_1, q_2$ are assumed to be smooth functions independent of $\epsilon$ and $d(x)$ is a signed distance function. This assumption is too strong. In [46], we use the following double expansion, which is weaker than previous expansion, but it can be shown to be more accurate to describe the phase field function:

$$
\varphi(x) = q_0\left(\frac{d(x)}{\epsilon}, x|\Gamma\right) + \epsilon q_1\left(\frac{d(x)}{\epsilon}, x|\Gamma\right) + \epsilon^2 q_2\left(\frac{d(x)}{\epsilon}, x|\Gamma\right) + o(\epsilon^2),
$$

where $q_i$ are functions from $\mathbb{R} \times \Gamma$ to $\mathbb{R}$, and $x|\Gamma$ is the projection of $x$ onto $\Gamma$.

The minimum surface is defined by the surface with mean curvature $H = 0$. In this paper, we prove that in the equilibrium state the mean curvature of $\Gamma$ is zero for the Allen-Cahn equation. The further detailed analysis of the equilibrium state of the Cahn-Hilliard equation reveals how the phase field function represents a free surface minimizing the surface area while preserving the inside volume. This explains some unusual phenomena observed from our numerical experiments. Similar analysis is carried out for the Allen-Cahn equation with volume constraint. We believe that the theory introduced in this paper can be used to guide future phase field modeling and simulation of moving boundary or free surface problems with surface tension.

We organize this paper as follows. In the second section, some necessary assumptions on the asymptotic expansions of the phase field function are given. We discuss the Allen-Cahn equation in the third section. The exact formulations of the leading and second order terms are given for the cases with and without volume preserving constraint. With the help of the first two terms, we derive the higher order terms and prove the consistency of the phase field model with the sharp interface model. Then, similar analysis of the Cahn-Hilliard equation is given in the fourth section. In the fifth section, we provide some numerical experiments to verify our theory. Concluding remarks and some further considerations are given in the sixth section.

2. Geometric notations and asymptotic double expansion. In this section, we first give all the necessary geometric notations and assumptions for the phase field models. They are the cornerstones of this paper. Then some frequently used lemmas for computations will be introduced. With these preparations, we are ready to derive the asymptotic expansions term by term in the next section.
2.1. Geometric notations. Suppose the zero level surface $\Gamma$ is smooth and compact. Denote $D_\Gamma = \{ x : \text{dist}(x, \Gamma) < \delta \}$, where $\delta$ is a small constant. Within domain $D_\Gamma$, we can define a special set of smooth and compact surfaces $\{ \Gamma_l \}_{l \in A}$ that are parallel to surface $\Gamma$.

**Definition 2.1.** Suppose that surface $\Gamma$ is smooth and compact. We construct another surface $\Gamma_l$ by

$$\Gamma_l = \Gamma + l \vec{n} = \{ x + l \vec{n}(x) : \text{for all } x \in \Gamma \},$$

where $l$ is a real number and $\vec{n}(x)$ denotes the unit normal of $\Gamma$ at $x \in \Gamma$ pointing to the inside of $\Gamma$. The resulting surface $\Gamma_l$ is called a **parallel surface** of $\Gamma$.

**Remark.** In definition (2.1), we always assume $|l|$ is small enough such that $\Gamma_l \in D_\Gamma$ and $\Gamma_l$ preserves the topological structure of $\Gamma$. Moreover, by assuming that $|l|$ is small enough, the existence of the parallel surfaces can be guaranteed by the smoothness and compactness of surface $\Gamma$.

The following lemma shows that the normal direction of $\Gamma$ is actually the normal direction of all its parallel surfaces.

**Lemma 2.2.** Suppose that surface $\Gamma$ is compact and smooth. Then, its normal vector is perpendicular to all of the parallel surfaces $\Gamma_l$.

**Proof.** Given surface $\Gamma$ a local coordinates $(u, v)$, we can write the parallel surface as:

$$\Gamma_l(u, v) = \Gamma(u, v) + l \vec{n}(u, v).$$

Therefore

$$\frac{\partial \Gamma_l(u, v)}{\partial u} \cdot \vec{n}(u, v) = \frac{\partial \Gamma(u, v)}{\partial u} \cdot \vec{n}(u, v) + \frac{\partial \vec{n}(u, v)}{\partial u} \cdot \vec{n}(u, v) = 0$$

Similarly, we have

$$\frac{\partial \Gamma_l(u, v)}{\partial v} \cdot \vec{n}(u, v) = 0.$$

All together, we conclude that $\vec{n}(u, v)$ is normal to $\Gamma_l$ as well. \hfill \Box

It is evident that parallel surfaces do not intersect with each other. Moreover, for any $x \in D_\Gamma$, there is a unique parallel surface $\Gamma_l$ going through $x$. Rigorously, we have

$$D_\Gamma = \bigcup_{l \in A} \Gamma_l, \quad \Gamma_i \cap \Gamma_j = \phi \text{ if } i \neq j.$$

**Remark.** In this paper, we will frequently use some geometric quantities, like the mean curvature $H$ and the Gaussian curvature $K$. For any point $x \in D_\Gamma$, and $x \in \Gamma_l$, the mean curvature $H(x)$ and the Gaussian curvature $K(x)$ are the mean curvature and Gaussian curvature of $\Gamma_l$ at $x$, respectively.

Given $\Gamma$ and the set of its parallel surfaces $\{ \Gamma_l \}_{l \in A}$, the so called **integral curve** (or **trajectory curve**) $\gamma$ is defined as the curve tangential to all of the normal directions $\vec{n}(x)$ at points $x \in \gamma$. Denote the integral curve going through point $x$ by $\gamma_x$. Lemma 2.2 shows that the integral curve $\gamma_x$ is a straight line. In addition, for any point $x \in D_\Gamma$, we call $x|_{\Gamma} = \gamma_x \cap \Gamma$ the projection of $x$ onto surface $\Gamma$. and we
can see that \( \vec{n}(x) = \vec{n}(x|_\Gamma) \). Therefore, we have an immediate corollary of lemma (2.2) as follows.

**Corollary 1.** Suppose that surface \( \Gamma \) is compact and smooth enough such that there exists a set of parallel surfaces \( \{\Gamma_l\}_{l \in A} \) of \( \Gamma \). For any point \( x \in D_\Gamma \), the integral curve \( \gamma_x \) takes the form of

\[
\gamma_x(l) = x|_\Gamma + l\vec{n}(x|_\Gamma). \tag{12}
\]

From the definition of parallel surfaces, the real number \( l \) in equations (10) and (12) can be positive or negative, which can be viewed as a **signed distance**. Because of its importance in this paper, the signed distance deserves a specific notation. For the rest of this paper, we denote by \( d(x, \Gamma) \) the signed distance from point \( x \) to surface \( \Gamma \); or \( d(\Gamma_a, \Gamma_b) \), the signed distance between two parallel surfaces of \( \Gamma \). Clearly, the sign of \( d(x, \Gamma) \) is determined by the direction of \( \vec{n}(x) \). The absolute value of \( d(x, \Gamma) \) measures the length of \( \gamma_x \) from \( x \) to \( x|_\Gamma \). Because the normal direction \( \vec{n}(x) \) coincides with the tangential direction of \( \gamma_x \) at \( x \), we have

\[
\vec{n}(x) = \nabla d(x, \Gamma).
\]

For simplicity, we denote \( d(x, \Gamma) \) by \( d(x) \).

Moreover, we know the mean curvature can be expressed as

\[
H(x) = -\frac{1}{2}\Delta d(x).
\]

Under the above geometric setting, we can write \( D_\Gamma = (-\delta, \delta) \oplus \Gamma \). In other words, for a smooth function \( f(x) \) defined on \( D_\Gamma \), we can write \( f \) as \( f(d(x), x|_\Gamma) \).

### 2.2. Asymptotic double expansion assumptions.

Before we construct the asymptotic double expansion, we need to make some assumptions on the regularity of the phase field function. Assume that \( \varphi \) is the phase field function defined on the computational domain \( \Omega \). For any positive \( \epsilon \), let

\[
\mathcal{F}^\epsilon(\varphi) = \frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{4\epsilon} (\varphi^2 - 1)^2 \tag{13}
\]

Our goal is to minimize

\[
W^\epsilon(\varphi) = \int_\Omega \mathcal{F}^\epsilon(\varphi) dx. \tag{14}
\]

Let the minimizer of (14) be \( \varphi^* \) and \( \Gamma \) be \( \{x : \varphi^*(x) = 0\} \).

With the geometric notations defined above, we give some assumptions:

(A1) \( \Gamma \) is smooth and compact, and there exists a set of surfaces \( \Gamma_l \) that are parallel to \( \Gamma \).

(A2) \( \varphi^* \) has an asymptotic expansion as

\[
\varphi^*(x) = \sum_{n=0}^{\infty} \epsilon^n q_n(\frac{d(x)}{\epsilon}, x|_\Gamma),
\]

where \( q_n \in C^\infty(\mathbb{R} \times \Gamma) \) are bounded smooth functions and independent of \( \epsilon \).

(A3) \( \Phi(\epsilon, z, x) = \sum_{n=0}^{\infty} \epsilon^n q_n(z, x) \in C^\infty(\mathbb{R}^2 \times \Gamma) \), where \( z \in \mathbb{R} \), \( x \in \Gamma \).

(A4) If \( d(x) > 0 \), then \( \lim_{\epsilon \to 0} \varphi^*(x) = 1 \); otherwise, \( \lim_{\epsilon \to 0} \varphi^*(x) = -1 \). Moreover, \( \nabla^n \varphi^*(x) = 0 \) for \( x \in \partial \Omega \) and \( n > 0 \). For each \( q_n(z, x|_\Gamma) \) and integers \( k \geq 0 \), \( m > 0 \), \( \lim_{z \to \pm \infty} z^k \frac{\partial^m q_n(z, x|_\Gamma)}{\partial z^m} \) are bounded.
We note that, assumption (A1) is natural based on the geometric notions we introduced before. (A2) is a key assumption that reveals the structure of the asymptotic expansion of the phase field function. As parameter $\epsilon$ appears inside and outside of $q_n$, it is called a double asymptotic expansion. The boundness of every partial derivatives of the phase field function is guaranteed by assumption (A3). (A4) posts the boundary conditions for computational convenience. According to different applications, (A4) can be adjusted or dropped.

2.3. Useful lemmas. We list some lemmas that are very helpful in our calculations.

**Lemma 2.3.**

\[
\nabla H \cdot \nabla d = 2H^2 - K
\]
\[
\nabla K \cdot \nabla d = 2HK
\]
\[
\Delta H = \Delta_T H + 4H(H^2 - K)
\]

The detailed proof of Lemma 2.3 can be found in [46].

Suppose that function $f(z, x|\Gamma) \in C^\infty(\mathbb{R} \times \Gamma)$ is an arbitrary smooth function of $z$ and $x|\Gamma$. For simplicity, we denote

\[ f^{(k)} = (\partial^k_{z}) f(z, x|\Gamma), \]

which is the $k$-th partial derivative of $f$ with respect to its first variable. For the $k$-th partial derivative of $f$ with respect to the second variable $x|\Gamma$, we denote it by $\nabla^k_T$. The subscript $T$ represents the tangential plane $T_x(\Gamma_l)$ of parallel surface $\Gamma_l$ that goes through $x$. If $f$ is independent of $x|\Gamma$, then we have $\nabla_T f = 0$. Notice that, $\nabla_T f$ is in fact the projection of gradient of $f$ onto the tangential plane $T_x(\Gamma_l)$, and thus we have

\[ \nabla_T f \cdot \nabla d = 0. \]

If $x \in \Gamma$, then $\nabla_T f = \nabla_T f$, which is the projection of $\nabla f$ onto the tangential plane $T_x(\Gamma)$.

In the following lemma, we give the explicit expression of $\nabla^n f(\frac{d(x)}{\epsilon}, x|\Gamma)$ up to $n = 4$. These formulæ are extremely useful through this paper.

**Lemma 2.4.** Based on the above notations, we have

\[
\nabla f(\frac{d(x)}{\epsilon}, x|\Gamma) = \frac{1}{\epsilon} f' \nabla d + \nabla_T f = \frac{1}{\epsilon} f' \nabla d + O(1),
\]

\[
\Delta f(\frac{d(x)}{\epsilon}, x|\Gamma) = \frac{1}{\epsilon^2} f'' + \frac{1}{\epsilon} f' \Delta d + \frac{1}{\epsilon} \nabla_T f' \cdot \nabla d + \Delta_T f
\]

\[
= \frac{1}{\epsilon^2} f'' - \frac{2H}{\epsilon} f' + \Delta_T f,
\]

\[
\nabla^3 f(\frac{d(x)}{\epsilon}, x|\Gamma) = \frac{1}{\epsilon^2} \left( \frac{1}{\epsilon} f''' \nabla d + \nabla_T f'' \right)
\]

\[
- \frac{2}{\epsilon} \left[ \frac{1}{\epsilon} (H' \nabla d + \nabla_T H) f' + H(\frac{1}{\epsilon} f'' \nabla d + \nabla_T f') \right]
\]
Proof. Similarly, we have the following expansions:

\[ K = \frac{1}{\epsilon} f'' + \frac{1}{\epsilon^2} (\nabla x f'' + \nabla \partial H) + \frac{1}{\epsilon^3} \left( \nabla x f'' + \nabla H f'' \right) + 2 \Delta f \]  

Substituting \( d = \epsilon z \) into equation (27), we get equation (24).

We omit the proofs for the other results since they are mostly the same. \( \square \)
3. Asymptotic analysis of Allen-Cahn models. In many real problems, we need to add some constraints to the original problems. In this section, we will derive the asymptotic expansion $q_n$ of the phase field function $\varphi$ that minimizes (14) with and without the volume constraint.

3.1. Asymptotic analysis of Allen-Cahn models without volume preserving. We first consider the Allen-Cahn equation without the volume preserving constraint. Following the Allen-Cahn gradient flow, as time goes to infinity, $\varphi$ tends to an energy minimizer of (14). The following theorem describes the leading order term of this minimizer.

Theorem 3.1. Suppose that by following the Allen-Cahn gradient flow $\varphi$ reaches the minimizer of (14) as time goes to infinity, and $\varphi$ satisfies assumptions (A1) to (A4). Then, we have

$$q_0(z) = \tanh\left(\frac{z}{\sqrt{2}}\right).$$

Note that, although assumption (A2) says $q_0 \in C^\infty(\overline{\mathbb{R}} \times \Gamma)$, the expression of $q_0$ implies that $q_0$ only depends on the first parameter.

Proof. Recall that the surface tension energy functional is

$$W_\epsilon^s(\varphi) = \int_\Omega \frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{4\epsilon}(\varphi^2 - 1)^2 dx.$$  \hspace{1cm} (28)

The first variation of $W_\epsilon^s(\varphi)$ is:

$$\frac{\delta W_\epsilon^s(\varphi)}{\delta \varphi} = -\epsilon \Delta \varphi + \frac{1}{\epsilon}(\varphi^2 - 1)^2 \varphi.$$  \hspace{1cm} (29)

Since $\varphi$ is the minimizer of $W_\epsilon^s(\varphi)$, we have $\varphi_t = -\frac{1}{\epsilon} \frac{\delta W_\epsilon^s(\varphi)}{\delta \varphi} = 0$, which leads to $\frac{\delta W_\epsilon^s(\varphi)}{\delta \varphi} = 0$. Therefore,

$$-\epsilon \Delta \varphi + \frac{1}{\epsilon}(\varphi^2 - 1)\varphi = 0.$$  \hspace{1cm} (30)

Let $\varphi = q_0 + \epsilon e_1$, where $e_1 = q_1 + O(\epsilon) = O(1)$. By lemma (2.4), we have

$$\frac{\delta W_\epsilon^s(\varphi)}{\delta \varphi} = -\epsilon \Delta (q_0 + \epsilon e_1) + \frac{1}{\epsilon}[(q_0 + \epsilon e_1)^2 - 1](q_0 + \epsilon e_1)$$

$$= -\epsilon \left( \frac{1}{\epsilon^2} q_0'' - \frac{2H}{\epsilon} q_0' + \Delta_T q_0 + \frac{1}{\epsilon} q_1'' - 2H e_1' + \epsilon \Delta_T e_1 \right)$$

$$+ \frac{1}{\epsilon}[(q_0^2 - 1) + 2\epsilon q_0 e_1 + \epsilon^2 e_1^2](q_0 + \epsilon e_1)$$

$$= \frac{1}{\epsilon}[-q_0'' + (q_0^2 - 1)q_0] + O(1).$$  \hspace{1cm} (31)

Therefore, $q_0$ should satisfy

$$-q_0'' + (q_0^2 - 1)q_0 = 0.$$  \hspace{1cm} (32)
As assumption (A4) implies that \( q_0(\infty) = 1 \) and \( q_0(-\infty) = -1 \), we have
\[
q_0(z) = \tanh\left(\frac{z}{\sqrt{2}}\right).
\]

Theorem 3.2. Suppose that by following the Allen-Cahn gradient flow \( \varphi \) reaches the minimizer of (14) as time goes to infinity, and \( \varphi \) satisfies assumptions (A1) to (A4). Then we have
\[
q_1 = 0 \quad \text{and} \quad H_{\Gamma} = 0,
\]
i.e., \( \Gamma = \{ x : \varphi(x) = 0 \} \) is a minimal surface when the system is in the equilibrium state and \( q_1 \) vanishes constantly.

Proof. Similar to the proof of Theorem (3.1), by letting \( e_1 = q_1 + O(\epsilon) \), \( \varphi = q_0 + \epsilon e_1 \) and substitute it into equation (30), we have
\[
\frac{\delta W(\varphi)}{\delta \varphi} = -\epsilon \Delta (q_0 + \epsilon e_1) + \frac{1}{\epsilon} [(q_0 + \epsilon e_1)^2 - 1] (q_0 + \epsilon e_1)
\]
\[
= -\epsilon \left( \frac{1}{\epsilon^2} q_0'' - \frac{2H}{\epsilon} q_0' + q_0 + \frac{1}{\epsilon^2} e_1'' - 2 H e_1' + \epsilon \Delta_T e_1 \right)
\]
\[
+ \frac{1}{\epsilon} [(q_0^2 - 1) + 2 \epsilon q_0 e_1 + \epsilon^2 e_1^2] (q_0 + \epsilon e_1)
\]
\[
= \frac{1}{\epsilon} [-q_0'' + (q_0^2 - 1) q_0] + [-e_1'' + (3 q_0^2 - 1) e_1 + 2 H q_0'] + O(\epsilon)
\]
\[
= -q_1'' + (3 q_0^2 - 1) q_1 + 2 H q_0' + O(\epsilon), \tag{33}
\]
which implies that,
\[
q_1'' - (3 q_0^2 - 1) q_1 = 2 H q_0' \tag{34}
\]
Moreover, by lemma (2.5), \( H \) can be expanded along the trajectory curve as
\[
H = H_{\Gamma} + \epsilon (2 H_{\Gamma}^2 - K_{\Gamma}) z + z^2 O(\epsilon^2) \tag{35}
\]
From theorem (3.1), we know that \( q_0 = \tanh\left(\frac{z}{\sqrt{2}}\right) \), and thus \( z q_0' \) and \( z^2 q_0'' \) are all bounded. Therefore equation (34) becomes:
\[
q_1'' - (3 q_0^2 - 1) q_1 = 2 H_{\Gamma} q_0'. \tag{36}
\]
On the other hand, we have
\[
q_0''' = (3 q_0^2 - 1) q_0'. \tag{37}
\]
By multiplying \( q_0' \) to both sides of equation (36) and integrating from \(-\infty\) to \( \infty \) with respect to \( z \), we have
\[
2 H_{\Gamma} \int_{-\infty}^{\infty} (q_0')^2 dz = \int_{-\infty}^{\infty} (q_1'' - (3 q_0^2 - 1) q_1) q_0' dz
\]
\[
= \int_{-\infty}^{\infty} [(q_0')'' - (3 q_0^2 - 1) q_0'] q_0' dz
\]
\[
= 0, \tag{38}
\]
which implies that
\[
H_{\Gamma} = 0 \tag{39}
\]
Thus, (36) becomes
\[
q_1'' - (3 q_0^2 - 1) q_1 = 0. \tag{40}
\]
From assumption (A4), \( q_1(-\infty) \) and \( q_1(\infty) \) are bounded, which leads to \( q_1 = 0 \).

\[ \square \]

**Remark.** Note that Theorem 3.2 actually proves the consistency of the Allen-Cahn Model to the sharp interface model for the equilibrium case, i.e., the interface between the two phases in the equilibrium state is a minimal surface.

Follow the same procedure, we can easily obtain the equations for \( q_2 \) and \( q_3 \).

**Theorem 3.3.** Suppose that by following the Allen-Cahn gradient flow \( \varphi \) reaches the minimizer of (14) as time goes to infinity, and \( \varphi \) satisfies assumptions (A1) to (A4). Then we have

\[
q''_2 - (3q_0^2 - 1)q_2 = 2z(2H_f^2 - K_f)q_0^2 + 2H_f q'_1 + 3q_0 q_1^2,
\]

and

\[
q''_3 - (3q_0^2 - 1)q_3 = 2z^2 H_f(4H_f^2 - 3K_f)q_0^2 + 2H_f q'_2 + 2z(2H_f^2 - K_f) q_0^2 + q_1^3 - \Delta_T q_1 + 6q_0 q_1 q_2.
\]

**Proof.** Let \( \varphi = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \epsilon^2 q_3 + O(\epsilon^4) \) and we substitute it into equation (41). Then, we have

\[
\frac{W^\epsilon(\varphi)}{\delta \varphi} = -\epsilon \Delta [q_0 + \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + O(\epsilon^4)] + \frac{1}{\epsilon}[(q_0 + \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + O(\epsilon^4)) - \epsilon q_0^2 q''_0 - (3q_0^2 - 1)q_0 - \epsilon q_1^2 q''_1 + \epsilon^2 q_2^2 + \epsilon^3 q_3^2 + O(\epsilon^4)]
\]

By lemma (2.5), we have

\[
H(z, x|\Gamma) = H_f + \epsilon z(2H_f^2 - K_f) + \epsilon^2 z^2 H_f(4H_f^2 - 3K_f) + z^3 O(\epsilon^3).
\]

By substituting it into equation (41) and considering assumption (A4), we have

\[
\frac{W^\epsilon(\varphi)}{\delta \varphi} = \frac{1}{\epsilon} \left[ -q''_0 + (3q_0^2 - 1) q_0 \right] + \left[ -q''_1 + (3q_0^2 - 1) q_1 + 2H_f q'_0 \right]
\]

\[
+ \epsilon \left[ -q''_2 + (3q_0^2 - 1) q_2 + 2z(2H_f^2 - K_f)q_0^2 + 2H_f q'_1 + 3q_0 q_1^2 \right]
\]

\[
+ \epsilon^2 \left[ -q''_3 + (3q_0^2 - 1) q_3 + 2z^2 H_f(4H_f^2 - 3K_f)q_0^2 + 2z(2H_f^2 - K_f) q_0^2 \right.
\]

\[
+ q_1^3 - \Delta_T q_1 + 2H_f q'_2 + 6q_0 q_1 q_2 \right] + O(\epsilon^3).
\]

Note that, from Theorem (3.1), we know \( q_0 \) only depends on \( z \), and thus \( \Delta_T q_0 = 0 \). Moreover, from theorem (3.2), we have \( q_1 = 0 \). Therefore all terms involve \( q_1 \) would vanish.
3.2. Asymptotic analysis of Allen-Cahn equation with volume preserving constraint. The classic Allen-Cahn model does not preserve the volume. Simply by using the Lagrange multiplier technique, we are able to develop the modified Allen-Cahn model that preserves the volume.

It is easy to see that the volume preserving condition can be formulated by:

\[ \int_{\Omega} \varphi dx = C \quad (43) \]

where \( C \) is a constant.

Therefore the Lagrange of equation (14) can be written as:

\[ L^*(\varphi) = W^*(\varphi) + \lambda \int_{\Omega} \varphi dx \]

\[ = \int_{\Omega} \frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{4\epsilon} (\varphi^2 - 1)^2 + \lambda \varphi dx. \quad (44) \]

Our goal is to find \( \varphi \) that minimizes (44) and \( \lambda \) satisfies equation (43). In deed we have the following theorem.

**Theorem 3.4.** Suppose that by following the Allen-Cahn gradient flow, \( \bar{\varphi} \) reaches the minimizer of (44) as time goes to infinity, and \( \bar{\varphi} \) satisfies assumptions (A1) to (A4). Then, \( H_F \) is a constant and

\[ \bar{q}_0(z) = \tanh\left(\frac{z}{\sqrt{2}}\right), \quad \bar{q}_1(z, x) = \frac{\sqrt{2}}{3} H_F \bar{q}_0^2. \]

**Proof.** The Euler-Lagrange equation of (44) is:

\[ -\epsilon \Delta \varphi + \frac{1}{\epsilon} (\varphi^2 - 1) \varphi + \lambda. \quad (45) \]

Suppose that \( \bar{\varphi} \) is the minimizer of equation (44). Then, we have:

\[ -\epsilon \Delta \bar{\varphi} + \frac{1}{\epsilon} (\bar{\varphi}^2 - 1) \bar{\varphi} + \lambda = 0. \quad (46) \]

Same as before, let \( \bar{\varphi} = \bar{q}_0 + \epsilon \bar{q}_1 + \epsilon^2 \bar{q}_2 + \epsilon^3 \bar{q}_3 + O(\epsilon^4) \) and we substitute it into equation (46). Then, we have

\[
\frac{L^*(\bar{\varphi})}{\delta \bar{\varphi}} = -\epsilon \Delta [\bar{q}_0 + \epsilon \bar{q}_1 + \epsilon^2 \bar{q}_2 + \epsilon^3 \bar{q}_3 + O(\epsilon^4)] + \frac{1}{\epsilon} \left[ (\bar{q}_0 + \epsilon \bar{q}_1 + \epsilon^2 \bar{q}_2 + \epsilon^3 \bar{q}_3 + O(\epsilon^4))^2 - 1 \right] \bar{q}_0 + \epsilon \bar{q}_1 + \epsilon^2 \bar{q}_2 + \epsilon^3 \bar{q}_3 + O(\epsilon^4) + \lambda
\]

\[= -\epsilon \left( \frac{1}{\epsilon^2} \bar{q}_0'' - \frac{2H}{\epsilon} \bar{q}_0 + \Delta \bar{q}_0 \right) - \epsilon^2 \left( \frac{1}{\epsilon^2} \bar{q}_1'' - \frac{2H}{\epsilon} \bar{q}_1 + \Delta \bar{q}_1 \right) + \epsilon^3 \left( \frac{1}{\epsilon^2} \bar{q}_2'' - \frac{2H}{\epsilon} \bar{q}_2 + \Delta \bar{q}_2 \right) + \epsilon^4 \left( \frac{1}{\epsilon^2} \bar{q}_3'' - \frac{2H}{\epsilon} \bar{q}_3 + \Delta \bar{q}_3 \right) \]

\[+ \frac{1}{\epsilon} \left[ (\bar{q}_0^2 - 1) \bar{q}_0 + \epsilon (3\bar{q}_0^2 - 1) \bar{q}_1 + \epsilon^2 (3\bar{q}_0^2 - 1) \bar{q}_2 + \epsilon^3 (3\bar{q}_0^2 - 1) \bar{q}_3 + 3\epsilon^2 \bar{q}_0 \bar{q}_1^2 + \epsilon^3 \bar{q}_1^3 + 6\epsilon^2 \bar{q}_0 \bar{q}_1 \bar{q}_2 + O(\epsilon^4) \right] + \lambda \]
\[
\begin{align*}
&= \frac{1}{\epsilon} \left[ -\tilde{q}_0'' + (\tilde{q}_0' - 1)\tilde{q}_0 \right] + \left[ -\tilde{q}_1'' + (3\tilde{q}_0' - 1)\tilde{q}_1 + 2H\tilde{q}_0' \right] \\
&\quad + \epsilon \left[ -\tilde{q}_2'' + (3\tilde{q}_0' - 1)\tilde{q}_2 + 3\tilde{q}_0\tilde{q}_1\tilde{q}_1' + 2H\tilde{q}_1 - \Delta T\tilde{q}_0 \right] \\
&\quad + \epsilon^2 \left[ -\tilde{q}_3'' + (3\tilde{q}_0' - 1)\tilde{q}_3 - \Delta T\tilde{q}_1 + 2H\tilde{q}_2 + \tilde{q}_1 + 6\tilde{q}_0\tilde{q}_1\tilde{q}_2 \right] \\
&\quad + \lambda + O(\epsilon^3).
\end{align*}
\]

Note that, \( \lambda \) may depend on \( \epsilon \). From the equation above, we write \( \lambda \) as \( \lambda_0/\epsilon + \lambda_1 + \lambda_2\epsilon + \lambda_3\epsilon^2 + O(\epsilon^3) \). By setting the \( O(\frac{1}{\epsilon}) \) and \( O(1) \) terms equal to 0, we have the following set of equations:

\[
\tilde{q}_0'' = (\tilde{q}_0' - 1)\tilde{q}_0 + \lambda_0, \quad (48)
\]

\[
\tilde{q}_1'' - (3\tilde{q}_0' - 1)\tilde{q}_1 = 2H\tilde{q}_0' + \lambda_1. \quad (49)
\]

Since \( \tilde{q}_0(\pm \infty) = \pm 1 \), and \( \tilde{q}_0''(\pm \infty) = 0 \) from equation (48), \( \lambda_0 = 0 \), and (48) is exactly the same as equation (32), we conclude that

\[ \tilde{q}_0 = \tanh\left( \frac{z}{\sqrt{2}} \right). \]

Next, let us consider equation (49). By lemma (2.5), we know that

\[ H(z, x) = H_{1\epsilon} + \epsilon (2H_{2\epsilon} - K_{r\epsilon}) + \epsilon^2 z^2 H_{r\epsilon} + z^3 O(\epsilon^3). \]

From \( \tilde{q}_0 \), we know that \( z^n\tilde{q}_0 \) is bounded for any integer \( n \geq 0 \). Therefore, \( H\tilde{q}_0'' = H_{1\epsilon}\tilde{q}_0'' + O(\epsilon) \). Equation (49) can be rewritten as:

\[
\tilde{q}_1'' - (3\tilde{q}_0' - 1)\tilde{q}_1 = 2H_{1\epsilon}\tilde{q}_0' + \lambda_1. \quad (50)
\]

By multiplying \( \tilde{q}_0' \) to both sides of equation (50) and integrating from \(-\infty\) to \( \infty \) with respect to \( z \), we obtain:

\[
2H_{1\epsilon} \int_{-\infty}^{\infty} (\tilde{q}_0')^2 dz + \lambda_1 \int_{-\infty}^{\infty} \tilde{q}_0' dz = \int_{-\infty}^{\infty} [\tilde{q}_1'' - (3\tilde{q}_0' - 1)\tilde{q}_1]\tilde{q}_0' dz
\]

\[
= \int_{-\infty}^{\infty} [(\tilde{q}_0' - 1)\tilde{q}_0' - (3\tilde{q}_0' - 1)\tilde{q}_0']\tilde{q}_0' dz
\]

\[ = 0. \quad (51)
\]

Moreover, we have

\[
\int_{-\infty}^{\infty} (\tilde{q}_0')^2 dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} (1 - \tilde{q}_0') d\tilde{q}_0 = \frac{1}{\sqrt{2}} \left( \tilde{q}_0 \bigg|_{-\infty}^{\infty} - \frac{\tilde{q}_0}{3} \bigg|_{-\infty}^{\infty} \right) = \frac{2\sqrt{2}}{3} \quad (52)
\]

\[
\int_{-\infty}^{\infty} \tilde{q}_0' dz = \tilde{q}_0 \bigg|_{-\infty}^{\infty} = 2 \quad (53)
\]

Combine equations (52) and (53), we can see that

\[ \frac{4\sqrt{2}}{3} H_{1\epsilon} + 2\lambda_1 = 0. \]

Therefore, we conclude

\[ H_{1\epsilon} = -\frac{3\sqrt{2}}{4} \lambda_1. \quad (54) \]

Equation (54) tells us that the zero level set of \( \tilde{\varphi} \) is in fact a surface with constant mean curvature when the system is in the steady state.
On the other hand, equation (49) can be rewritten as:

\[ \tilde{q}_1'' - (3\tilde{q}_0^2 - 1)\tilde{q}_1 = 2H_1\tilde{q}_0 - \frac{2\sqrt{3}}{3}H_1. \]

(55)

As \( \tilde{q}_0 = \frac{1}{\sqrt{2}}(1 - \tilde{q}_0^2) \), the right hand side of equation (55) is in the function space spanned by \( \{1, \tilde{q}_0, \tilde{q}_0^2, \ldots\} \). By comparing both sides of equation (55), we must have \( \tilde{q}_1 \in \text{span}\{1, \tilde{q}_0, \tilde{q}_0^2, \ldots\} \) as well, i.e., \( \tilde{q}_1 = \sum_{k=0}^{\infty} a_k \tilde{q}_0^k \). Substituting the series expression of \( \tilde{q}_1 \) into (55), we have

\[
\tilde{q}_1' = \sum_{k=2}^{\infty} \frac{k(k-1)a_k}{\sqrt{2}} \tilde{q}_0^{k-2} \tilde{q}_1' - \sum_{k=1}^{\infty} \frac{k(k+1)a_k}{\sqrt{2}} \tilde{q}_0^k \\
= a_2 + (3a_3 - a_1)\tilde{q}_0 + (6a_4 - 4a_2)\tilde{q}_0^2 + (a_1 - 9a_3 + 10a_5)\tilde{q}_0^3 \\
+ \sum_{k=4}^{\infty} \left[ \frac{(k-1)(k-2)}{2}a_{k-2} - k^2a_k + \frac{(k+1)(k+2)}{2}a_{k+2} \right] \tilde{q}_0^k,
\]

(56)

and

\[
(3\tilde{q}_0^2 - 1)\tilde{q}_1 = -a_0 - a_1\tilde{q}_0 + \sum_{k=2}^{\infty} (3a_{k-2} - a_k)\tilde{q}_0^k.
\]

(57)

All above together implies that

\[
a_0 + a_2 = \frac{\sqrt{3}}{3}H_1, \quad \text{and} \quad a_k = 0 \quad \text{if} \quad k \neq 0, 2
\]

Because \( \tilde{q}_1(0) = 0 \), we have \( a_0 = 0 \), and thus

\[
\tilde{q}_1 = \frac{\sqrt{3}}{3}H_1\tilde{q}_0^2.
\]

Remark. By \( \tilde{q}_0 \) and \( \tilde{q}_1 \), we know \( z\tilde{q}_0^2 \), \( z^2\tilde{q}_0^4 \) and \( z\tilde{q}_1' \) are all bounded. By expanding \( 2H_1\tilde{q}_0^2 \) and \( 2H_1\tilde{q}_1' \) with lemma (2.5) and setting the \( O(\epsilon) \) term of equation (47) to 0, we obtain the following equations of \( \tilde{q}_2 \)

\[
\tilde{q}_2'' - (3\tilde{q}_0^2 - 1)\tilde{q}_2 = 2z(2H_1^2 - K_1)\tilde{q}_0 + 2H_1\tilde{q}_1' + 3\tilde{q}_0\tilde{q}_1^2 + \lambda_2.
\]

(58)

By multiplying \( \tilde{q}_0^2 \) to both sides of equation (58) and integrating it with respect to \( z \) from \( -\infty \) to \( \infty \), we have

\[
\int_{-\infty}^{\infty} [\tilde{q}_2'' - (3\tilde{q}_0^2 - 1)\tilde{q}_2] \tilde{q}_0^2 dz = \int_{-\infty}^{\infty} [2z(2H_1^2 - K_1)\tilde{q}_0 + 2H_1\tilde{q}_1' + 3\tilde{q}_0\tilde{q}_1^2 + \lambda_2] \tilde{q}_0 dz,
\]

\[
\int_{-\infty}^{\infty} [\tilde{q}_0'' - (3\tilde{q}_0^2 - 1)\tilde{q}_0] \tilde{q}_0^2 dz = \lambda_2 \int_{-\infty}^{\infty} \tilde{q}_0^2 dz,
\]

\[
0 = 2\lambda_2,
\]

(59)

as first three terms on the right hand side of equation (58) are all odd functions with respect to \( z \). Thus, \( \lambda_2 \) has to be 0, and equation (58) reduces to

\[
\tilde{q}_2'' - (3\tilde{q}_0^2 - 1)\tilde{q}_2 = 2z(2H_1^2 - K_1)\tilde{q}_0 + 2H_1\tilde{q}_1' + 3\tilde{q}_0\tilde{q}_1^2.
\]

(60)
4. Asymptotic analysis of Cahn-Hilliard models. In this section, we present the asymptotic expansion $\hat{q}_n$ of the phase field function $\hat{\varphi}$ that minimizes (14) along the Cahn-Hilliard flows. Due to the volume preserving property of Cahn-Hilliard flow, we expect a different asymptotic expansion from the Allen-Cahn flow without volume preserving constraint. However, if we follow the Allen-Cahn flow with volume preserving constraint, the structure of the phase field function in the steady state is rather similar with the Cahn-Hilliard flows. In fact, we can show at least to the order of $O(\epsilon^2)$, the structure of the phase field function is exactly the same for the Cahn-Hilliard flow and Allen-Cahn flow with volume preserving constraint. Furthermore, like the Allen-Cahn flow with volume preserving constraint, the interface between the two phases in the equilibrium state along the Cahn-Hilliard flow is a constant mean curvature surface as well. Error estimate of the surface tension energy $W^\epsilon$ will be given at the end of this section.

4.1. Asymptotic expansion. Recall that the surface tension energy is defined by:

$$W^\epsilon(\varphi) = \int_\Omega \frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{4\epsilon}(\varphi^2 - 1)^2 dx.$$  

The first variation of $W^\epsilon$ with respect to $\varphi$ is:

$$\frac{\delta W^\epsilon(\varphi)}{\delta \varphi} = -\epsilon \Delta \varphi + \frac{1}{\epsilon}(\varphi^2 - 1) \varphi. \quad (61)$$

The Cahn-Hilliard flow is given by:

$$\varphi_t = \Delta \frac{\delta W^\epsilon(\varphi)}{\delta \varphi}. \quad (62)$$

Suppose that $\hat{\varphi}$ is corresponding to the equilibrium state of the system following the Cahn-Hilliard flow, and $\Gamma = \{ x : \hat{\varphi}(x) = 0 \}$. Then $\hat{\varphi}$ satisfies

$$\hat{\varphi}_t = \Delta \frac{\delta W^\epsilon(\hat{\varphi})}{\delta \hat{\varphi}} = 0,$$

i.e.,

$$\epsilon \Delta^2 \hat{\varphi} - \frac{1}{\epsilon} \Delta ((\hat{\varphi}^2 - 1)\hat{\varphi}) = 0. \quad (63)$$

Then, by using Lemma (2.4), equation (63) becomes

$$0 = \frac{1}{\epsilon^3} \hat{\varphi}^{(4)} - \frac{2}{\epsilon^2} (2H\hat{\varphi}'' + H''\hat{\varphi} + 2H'\hat{\varphi}'') + \frac{2}{\epsilon} (\Delta T \hat{\varphi}'' + 2HH'\hat{\varphi}' + 2H^2\hat{\varphi}'') - 2(2H\Delta T \hat{\varphi}' + 2\nabla T H \cdot \nabla T \hat{\varphi}' + \hat{\varphi}' \Delta T H) + \epsilon \Delta T \hat{\varphi}' + \frac{1}{\epsilon^3} [6(\hat{\varphi}'^2 + 3\hat{\varphi}' \hat{\varphi}'' - \hat{\varphi}''') + \frac{2}{\epsilon^2} (3\hat{\varphi}^2 - 1)\hat{\varphi}' - \frac{1}{\epsilon} \Delta T [(\hat{\varphi}^2 - 1)\hat{\varphi}]

= \frac{1}{\epsilon^3} [\hat{\varphi}'' - (\hat{\varphi}^2 - 1)\hat{\varphi} + 2\nabla T H (-2\hat{\varphi}'' + (3\hat{\varphi}^2 - 1)\hat{\varphi}') - 2H'\hat{\varphi}'' - H''\hat{\varphi}']

+ \frac{2}{\epsilon} \{ \Delta T [\hat{\varphi}'' - \frac{1}{2}(\hat{\varphi}^2 - 1)\hat{\varphi} + 2H H' \hat{\varphi}' + 2H^2 \hat{\varphi}''] 

- 2(2H\Delta T \hat{\varphi}' + 2\nabla T H \cdot \nabla T \hat{\varphi}' + \hat{\varphi}' \Delta T H) + \epsilon \Delta T \hat{\varphi}'. \quad (64)$$

Let $\varphi = \hat{q}_0 + O(\epsilon)$ and we substitute it into equation (64). Then, the $O(\frac{1}{\epsilon^3})$ term becomes:

$$\frac{1}{\epsilon^3} [\hat{\varphi}'' - (\hat{\varphi}^2 - 1)\hat{\varphi}]'' = \frac{1}{\epsilon^3} [\hat{q}_0'' - (\hat{q}_0^2 - 1)\hat{q}_0]'' + O(\frac{1}{\epsilon^2}). \quad (65)$$
From equation (64), it is easy to see that \( \frac{1}{\epsilon^3} \left[ \dot{q}_0'' - (\dot{q}_0^2 - 1)\dot{q}_0 \right]'' \) is the only \( O(\frac{1}{\epsilon^3}) \) term. Therefore we must have:

\[
\dot{q}_0'' - (\dot{q}_0^2 - 1)\dot{q}_0 = 0.
\]  

(66)

It is clear that equation (66) is exactly the same with equation (32). Therefore, we can conclude that

\[
\dot{q}_0(z) = \tanh\left( \frac{z}{\sqrt{2}} \right).
\]  

(67)

In fact, we have the following theorem.

**Theorem 4.1.** Suppose that following the Cahn-Hilliard flow we get \( \dot{\varphi} \) as time goes to infinity. With assumptions (A1) to (A4) we have

\[
\dot{q}_0(z) = \tanh\left( \frac{z}{\sqrt{2}} \right).
\]

(67)

Note that, although assumption (A2) assumes that \( \dot{q}_0 \in C^\infty(\bar{\mathbb{R}} \times \Gamma) \), the expression of \( \dot{q}_0 \) implies that \( \dot{q}_0 \) only depends on the first parameter.

**Proof.** Please refer to theorem 3.1. \( \square \)

Note that, \( \dot{q}_0 \) is the same as \( \ddot{q}_0 \) as well, which is the leading order term of the phase field function in the equilibrium state along the Allen-Cahn flow with volume preserving constraint.

To derive \( \ddot{q}_1 \), let \( \varphi = \dot{q}_0 + \epsilon \ddot{q}_1 + O(\epsilon^2) \) and we substitute it into equation (64). Then, the \( O(\frac{1}{\epsilon^3}) \) term becomes:

\[
2 \epsilon^2 \left[ H\left[ -2\dot{q}_0'' + (3\dot{q}_0^2 - 1)\dot{q}_0 \right] + O(\frac{1}{\epsilon}) \right] = 2 \epsilon^2 H[ -2\dot{q}_0'' + (3\dot{q}_0^2 - 1)\dot{q}_0 ] + O(\epsilon). \]

(69)

From lemma (2.5), we know that

\[
H(z, x|_\Gamma) = H_\Gamma + zO(\epsilon).
\]

In addition, assumption (A4) tells us that \( \lim_{z \to \pm \infty} z^k q_0^{(n)} \) are bounded for all integers \( k \geq 0 \) and \( n > 0 \) (in fact, from the expression of \( \dot{q}_0 \), we have \( \lim_{z \to \pm \infty} z^k q_0^{(n)} = 0 \) for all integers \( k \geq 0 \) and \( n > 0 \)). Thus,

\[
H \left[ -2\dot{q}_0'' + (3\dot{q}_0^2 - 1)\dot{q}_0 \right] = H_\Gamma \left[ -2\dot{q}_0'' + (3\dot{q}_0^2 - 1)\dot{q}_0 \right] + O(\epsilon).
\]

Therefore, the \( O(\frac{1}{\epsilon^2}) \) term of equation (64) becomes:

\[
\frac{2}{\epsilon^2} \left[ H(-2\ddot{q}'' + 3\ddot{q}' - 1)\ddot{q}'') - 2H'(\ddot{q}' - H''\ddot{q}') \right] = \frac{2}{\epsilon^2} H_\Gamma \left[ -2\dot{q}_0'' + (3\dot{q}_0^2 - 1)\dot{q}_0 \right] + O(\frac{1}{\epsilon}). \]

(69)

Combine equations (68) and (69), we have

\[
[\ddot{q}_1'' - (3\dot{q}_0^2 - 1)\dot{q}_1]'' + 2H_\Gamma(-2\ddot{q}_0'' + (3\dot{q}_0^2 - 1)\dot{q}_0) = 0,
\]

which leads to the following theorem.
Theorem 4.2. Suppose that following the Cahn-Hilliard flow we get $\hat{\varphi}$ as time goes to infinity. Then, with assumptions (A1) to (A4) we have

$$\hat{q}_1(z, x|\Gamma) = \frac{\sqrt{2}}{3} H_\Gamma \hat{q}_0^2.$$ 

Proof. From $\hat{q}_0$, we have

$$\hat{q}_0''' = (3\hat{q}_0^2 - 1)\hat{q}_0'. \quad (71)$$

Integrating both sides of equation (70) with respect to $z$ yields

$$[\hat{q}_1''' - (3\hat{q}_0^2 - 1)\hat{q}_1] = 2H_\Gamma \hat{q}_0' + Az + B. \quad (72)$$

Let us multiply $\hat{q}_0'$ to both sides of equation (72) and integrate it from $-\infty$ to $\infty$ with respect to $z$. Then, the left hand side of (72) becomes:

$$\int_{-\infty}^{\infty} [\hat{q}_1''' - (3\hat{q}_0^2 - 1)\hat{q}_1] \hat{q}_0' dz = \int_{-\infty}^{\infty} [\hat{q}_0''' - (3\hat{q}_0^2 - 1)\hat{q}_0] \hat{q}_1 dz = 0. \quad (73)$$

Similarly, the right hand side of (72) becomes:

$$\int_{-\infty}^{\infty} 2H_\Gamma (\hat{q}_0')^2 dz + \int_{-\infty}^{\infty} Az \hat{q}_0' dz + \int_{-\infty}^{\infty} B \hat{q}_0' dz = \frac{4\sqrt{2}}{3} H_\Gamma + 0 + 2B = 0. \quad (74)$$

Therefore, we have

$$B = -\frac{2\sqrt{2}}{3} H_\Gamma.$$ 

Furthermore, according to assumption (A4), $q_1(-\infty)$ and $q_1(\infty)$ are bounded. Therefore, $A$ has to be 0. By substituting $A = 0$ to equation (72), we have:

$$[\hat{q}_1''' - (3\hat{q}_0^2 - 1)\hat{q}_1] = 2H_\Gamma \hat{q}_0' - \frac{2\sqrt{2}}{3} H_\Gamma. \quad (75)$$

Note that equation (75) is exactly the same as (55), and $\hat{q}_0$ is the same as $\tilde{q}_0$ as well. Therefore, we have:

$$\hat{q}_1 = \frac{\sqrt{2}}{3} H_\Gamma \hat{q}_0^2.$$ 

Theorems 4.1 and 4.2 validate the interesting fact we stated above: the leading order and $O(\epsilon)$ terms of the phase field function in the equilibrium state of the Cahn-Hilliard Models are the same as that of the Allen-Cahn Models with the volume preserving constraint. Note that, in both cases, the volume is preserved. In fact, we can rigorously show that, up to the order of $O(\epsilon^2)$, the structure of the phase field function in the equilibrium state of Cahn-Hilliard Models is the same as that of Allen-Cahn Models with volume preserving constraint. Meanwhile, the zero level set of the phase field function following the Cahn-Hilliard flow in the equilibrium state can be shown as a constant mean curvature surface, that is the same as the Allen-Cahn flow with volume preserving as well. To show this, we need higher order terms of $\hat{\varphi}$. 

\[ \square \]
Let \( \hat{\varphi} = \hat{q}_0 + \epsilon \hat{q}_1 + \epsilon^2 \hat{q}_2 + \epsilon^3 \hat{q}_3 + O(\epsilon^4) \) and we substitute it into equation (64). Then, the \( O(\frac{1}{\epsilon^2}) \) term becomes

\[
\frac{1}{\epsilon^2} \left[ (\hat{\varphi}'' - (\hat{\varphi}^2 - 1)\hat{\varphi})'' \right] = \frac{1}{\epsilon^2} \left[ (\hat{q}_0'' - (\hat{q}_0^2 - 1)\hat{q}_0)'' \right] + \frac{1}{\epsilon^2} \left[ (\hat{q}_1'' - (3\hat{q}_0^2 - 1)\hat{q}_1)'' \right] + \frac{1}{\epsilon^2} \left[ (\hat{q}_2'' - (3\hat{q}_0^2 - 1)\hat{q}_2 - 3\hat{q}_0\hat{q}_1)'' \right] + \left[ (\hat{q}_3'' - (3\hat{q}_0^2 - 1)\hat{q}_3 - 6\hat{q}_0\hat{q}_1\hat{q}_2 - \hat{q}_1^3)'' \right] + O(\epsilon). \tag{76}
\]

From lemma (2.5) we know that

\[
H' = \epsilon(2H_1^2 - K_T) + zO(\epsilon^2), \tag{77}
\]

\[
K' = 2\epsilon H_1 K_T + zO(\epsilon^2), \tag{78}
\]

\[
H'' = 2\epsilon^2 H_1 (4H_1^2 - 3K_T) + zO(\epsilon^3). \tag{79}
\]

Together with assumption (A4), i.e., \( \lim_{z \to \pm \infty} z^k \hat{q}_n^{(m)} \) is bounded for all integers \( k \geq 0, \ n \geq 0, \) and \( m > 0, \) the \( O(\frac{1}{\epsilon^2}) \) term of equation (64) becomes:

\[
\frac{2}{\epsilon^2} \left[ H(-2\hat{\varphi}''' + (3\hat{\varphi}^2 - 1)\hat{\varphi}') - 2H'\hat{\varphi}'' - H''\hat{\varphi}' \right] = \frac{2}{\epsilon^2} H_1 \left[ -2\hat{q}_0''' + (3\hat{q}_0^2 - 1)\hat{q}_0' \right] + \frac{2}{\epsilon} \left\{ H_1 \left[ -2\hat{q}_1''' + (3\hat{q}_0^2 - 1)\hat{q}_1' + 6\hat{q}_0\hat{q}_0'\hat{q}_1 \right] \right.
\]

\[
\left. + (2H_1^2 - K_T) \left[ z(-2\hat{q}_0''' + (3\hat{q}_0^2 - 1)\hat{q}_0') - 2\hat{q}_0'' \right] \right\} + \frac{2}{\epsilon} \left\{ H_1 \left[ -2\hat{q}_2''' + (3\hat{q}_0^2 - 1)\hat{q}_2' + 6\hat{q}_0\hat{q}_1\hat{q}_2 + 3\hat{q}_0\hat{q}_1' \right] \right.
\]

\[
\left. + (2H_1^2 - K_T) \left[ z(-2\hat{q}_0''' + (3\hat{q}_0^2 - 1)\hat{q}_0') + 6\hat{q}_0\hat{q}_0'\hat{q}_1 \right] - 2\hat{q}_1'' \right] \right.
\]

\[
\left. + H_1 (4H_1^2 - 3K_T) \left[ z^2(-2\hat{q}_0''' + (3\hat{q}_0^2 - 1)\hat{q}_0') - 4z\hat{q}_0'' - 2\hat{q}_0'' \right] \right\} + O(\epsilon). \tag{80}
\]

Following the same strategy, the \( O(\frac{1}{\epsilon}) \) and \( O(1) \) terms of equation (64) can be derived as:

\[
\frac{2}{\epsilon} \left\{ \Delta_T \left[ \hat{\varphi}''' - \frac{1}{2}(\hat{\varphi}^2 - 1)\hat{\varphi} \right] + 2HH'\hat{\varphi}' + 2H^2\hat{\varphi}'' \right\} = \frac{2}{\epsilon} \left\{ \Delta_T \left[ \hat{q}_0''' - \frac{1}{2}(\hat{q}_0^2 - 1)\hat{q}_0 \right] + 2H_1^2\hat{q}_0'' \right\}
\]

\[
+ \frac{2}{\epsilon} \left\{ \Delta_T \left[ \hat{q}_1''' - \frac{1}{2}(\hat{q}_0^2 - 1)\hat{q}_1 - \hat{q}_0\hat{q}_1' \right] + 2H_1^2\hat{q}_1'' \right. \)

\[
\left. + 2H_1 (2H_1^2 - K_T)(\hat{q}_0' + 2z\hat{q}_0'') \right\} + O(\epsilon). \tag{81}
\]

\[-2(2H\Delta_T\hat{\varphi}' + 2\nabla_T H \cdot \nabla_T \hat{\varphi}' + \hat{\varphi}' \Delta_T H) = -2(\Delta_T H_T)\hat{q}_0' + O(\epsilon). \tag{82}\]
By substituting equations (76), (80), (81), and (82) to (64), we have the following set of equations by setting the \(O(\frac{1}{\epsilon})\) and \(O(1)\) terms to 0:

\[
[q''_2 - (3q''_0 - 1)q_2 - 3q_0q''_1]'' = -2 \left\{ H_T \left[ -2q'''_1 + (3q''_0 - 1)q'_1 + 6q_0q''_0q_1 \right] + (2H^2_T - K_T) \left[ z(-2q'''_1 + (3q''_0 - 1)q'_1 + 6q_0q''_0q_1) - 2q''_1 \right] + H_T(4H^2_T - 3K_T) \left[ z^2(-2q'''_1 + (3q''_0 - 1)q'_1) - 4zq''_1 - 2q''_1 \right] \right\} - 2 \left\{ \Delta_T \left[ q''_0 - \frac{1}{2}(q''_0 - 1)q_0 \right] + 2H^2_Tq''_0 \right\} + 2(\Delta_TH_T)q''_0 \quad (83)
\]

Solving equations (83) and (84) results in the following theorems.

**Corollary 2.** Suppose that the phase field model is driven by the Cahn-Hilliard flow. With assumptions (A1) to (A4), we have

\[
q''_2 - (3q''_0 - 1)q_2 = 2z(2H^2_T - K_T)q''_0 + 2H_Tq'_1 + 3q_0q''_1. \quad (85)
\]

**Proof.** Corollary 2 implies that equation (83) can be reduced to (85). To simplify (83), we first show some useful identities.

\[
-2q'''_1 + (3q''_0 - 1)q'_1 + 6q_0q''_0q_1 = -2q'''_1 + [(3q''_0 - 1)q_1]' \quad = -q'''_1 + [-q''_1 + (3q''_0 - 1)q_1]' \quad = -q'''_1 + [-2H_Tq''_0 + \frac{2\sqrt{2}}{3}H_T]' \quad = -q'''_1 - 2H_Tq''_0. \quad (86)
\]

By equation (71), we have:

\[
-2q'''_0 + (3q''_0 - 1)q'_0 = -q'''_0 - [q'''_0 + (3q''_0 - 1)q_0] \quad = -q'''_0. \quad (87)
\]

In addition, as \(q_0\) does not depend on \(x_1\), we have

\[
\Delta_T \left[ q''_0 - \frac{1}{2}(q''_0 - 1)q_0 \right] = 0 \quad (88)
\]
Therefore, the right hand side of equation (83) becomes:
\[
2H_T(z_0'') + 2z(2H_T^2 - K_T)q_0'' + 4(2H_T^2 - K_T)q_0''' - 4H_T^2q_0'' \\
= 2(2H_T^2 - K_T)(z_0'') + 2H_Tq_0'' \\
= 2(2H_T^2 - K_T)(z_0'') + 2H_Tq_0'' .
\]
(89)

Clearly, equation (83) can be written by:
\[
[q_2'' - (3q_0^2 - 1)q_2 - 3\hat{q}_0\hat{q}_1'']' = 2(2H_T^2 - K_T)(z_0'') + 2H_Tq_0'',
\]
(90)

which is equivalent to
\[
q_2'' - (3q_0^2 - 1)q_2 = 2z(2H_T^2 - K_T)q_0' + 2H_Tq_0' + 3q_0\hat{q}_1 + Az + B.
\]
(91)

A and B are two constants.

Let us multiply \( q_0' \) to both sides of equation (91) and integrate it from \(-\infty \) to \( \infty \) with respect to \( z \). Then, the left hand of (91) becomes:
\[
\int_{-\infty}^{\infty} [q_2'' - (3q_0^2 - 1)q_2]q_0'dz = \int_{-\infty}^{\infty} [q_0'' - (3q_0^2 - 1)q_0']q_2dz = 0.
\]
(92)

Note that, the first four terms on the right hand side of equation (91) are all odd functions. Because \( q_0' \) is an even function, the integration of the first four terms will vanish, i.e.
\[
\int_{-\infty}^{\infty} [2z(2H_T^2 - K_T)q_0' + 2H_Tq_0' + 3q_0\hat{q}_1 + Az]q_0'dz = 0.
\]
(93)

Therefore, we have
\[
B\int_{-\infty}^{\infty} q_0'dz = 2B = 0.
\]
(94)

which implies that
\[
B = 0.
\]

In addition, assumptions (A1) to (A4) imply that \( q_i^{(j)} \) is bounded for all \( i \) and \( j \). In other words, the left hand side and the first three terms of the right hand side of equation (91) are bounded. However, if \( A \neq 0 \), \( Az \) will be unbounded when \( z \to \pm \infty \). Hence, we can see that
\[
A = 0.
\]

Therefore, equation (91) reduces to (85), which completes the proof.

\[\Box\]

**Theorem 4.3.** Suppose that by following the Cahn-Hilliard flow we get \( \hat{\varphi} \) as time goes to infinity. With assumptions (A1) to (A4), we have

\[H_T = \text{const}\]
on the zero level set \( \Gamma \) of \( \hat{\varphi} \), i.e., \( \Gamma \) is a constant mean curvature surface.

**Proof.** First let us prove some useful identities.
\[
2q_2'' - (3q_0^2 - 1)q_2 - 6q_0\hat{q}_1q_1' - 6q_0\hat{q}_0q_2 - 3q_1^2q_0'' \\
= 2q_2'' - [(3q_0^2 - 1)q_2]' - (3q_0\hat{q}_1)^' \\
= [q_2'' - (3q_0^2 - 1)q_2]' - 3q_0\hat{q}_1^2 \\
= q_2'' + [2z(2H_T^2 - K_T)q_0' + 2H_Tq_0'']' \\
= q_2'' + 2(2H_T^2 - K_T)q_0'' + 2z(2H_T^2 - K_T)q_0'' + 2H_Tq_0''
\]
(95)
\[ \Delta_T \left[ \frac{1}{2} (\theta_0^2 - 1) \dot{q}_1 - \theta_0^2 \dot{q}_1 \right] \]
\[ = \Delta_T \left[ \frac{1}{2} (3\theta_0^2 - 1) \dot{q}_1 \right] = \Delta_T \left\{ \frac{1}{2} \dot{q}_1' + \frac{1}{2} \left[ \dot{q}_1'' - (3\theta_0^2 - 1) \dot{q}_1 \right] \right\} \]
\[ = \frac{1}{2} \Delta_T \dot{q}_1'' + (\Delta_T H_T) \dot{q}_0' - \sqrt{\frac{2}{3}} \Delta_T H_T \]

As \( H_T \) only depends on \( x|\Gamma \), it is easy to see \( \Delta_T H_T \) only depends on \( x|\Gamma \) either, and thus \( \frac{\partial \Delta_T H_T}{\partial d} = 0 \). Suppose that \( x \in D_T \) and \( T \) is the tangential plane through \( x \).

Let the distance from \( x \) to \( \Gamma \) be \( l \). We have
\[ \Delta_T H_T = \Delta_T H_T + \frac{\partial \Delta_T H_T(\theta l, x|\Gamma)}{\partial d} \cdot l = \Delta_T H_T, \]
where \( 0 \leq \theta \leq 1 \).

Therefore, we can see that
\[ \Delta_T \dot{q}_1'' = \Delta_T \left( \frac{\sqrt{2}}{3} H_T \theta_0^2 \right)'' = \Delta_T \left[ \frac{\sqrt{2}}{3} H_T \theta_0^2 \right]'' \]
\[ = \left( \Delta_T \frac{\sqrt{2}}{3} H_T \theta_0^2 \right)'' = \left( \Delta_T \frac{\sqrt{2}}{3} H_T \theta_0^2 \right)'' \]
\[ = \left[ \Delta_T \frac{\sqrt{2}}{3} H_T \theta_0^2 \right]'' = \Delta_T \left( \frac{\sqrt{2}}{3} H_T \theta_0^2 \right)'' \]
\[ = (\Delta_T \dot{q}_1)''. \]

Equation (96) becomes:
\[ \Delta_T \left[ \frac{1}{2} (\theta_0^2 - 1) \dot{q}_1 - \theta_0^2 \dot{q}_1 \right] \]
\[ = \frac{1}{2} (\Delta_T \dot{q}_1)''' + (\Delta_T H_T) \dot{q}_0' - \sqrt{\frac{2}{3}} \Delta_T H_T. \]

Combining all of the above together, equation (84) can be simplified to
\[ [\dot{q}_3'' - (3\theta_0^2 - 1) \dot{q}_3 - \dot{q}_1^3 - 6\theta_0 \dot{q}_1 \dot{q}_2]'' \]
\[ = 2H_T(4\theta_0^2 - 3K_T)(z^2 \dot{q}_0'' + 4\dot{q}_0'' + 2\dot{q}_0') + 2(2H_T^2 - K_T)(z \dot{q}_0'' + \dot{q}_0'') \]
\[ - (\Delta_T \dot{q}_1)''' + 2H_T \dot{q}_2'' + \frac{2\sqrt{2}}{3} \Delta_T H_T \]
\[ = 2H_T(4\theta_0^2 - 3K_T)(z \dot{q}_0'' + 2(2H_T^2 - K_T)(z \dot{q}_0'') \]
\[ - (\Delta_T \dot{q}_1)''' + 2H_T \dot{q}_2'' + \frac{2\sqrt{2}}{3} \Delta_T H_T. \]

Integrate both sides of equation (99) from \(-\infty \) to \( \infty \) with respect to \( z \), the left hand side becomes:
\[ \int_{-\infty}^{\infty} [\dot{q}_3'' - (3\theta_0^2 - 1) \dot{q}_3 - \dot{q}_1^3 - 6\theta_0 \dot{q}_1 \dot{q}_2]'' \, dz \]
\[ = \left[ [\dot{q}_3'' - (3\theta_0^2 - 1) \dot{q}_3 - \dot{q}_1^3 - 6\theta_0 \dot{q}_1 \dot{q}_2] \right]_{-\infty}^{\infty} \]
\[ = 0 \] (100)
because of the boundary conditions from assumption (A4). For the same reason, we have
\begin{align*}
&\int_{-\infty}^{\infty} 2\Delta \Gamma (4H^2 - 3K\Gamma)(z^2q_0'') + 2(2H^2 - K\Gamma)(zq_1'') - (\Delta \Gamma q_1)'' + 2H\Gamma q_2'' \, dz \\
&= 2H(4H^2 - 3K\Gamma)(z^2q_0') + 2(2H^2 - K\Gamma)(zq_1') - (\Delta \Gamma q_1)' + 2H\Gamma q_2' \\
&\bigg|_{-\infty}^{\infty} = 0.
\end{align*}
(101)

Therefore, we can see that
\begin{align*}
\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{3}} \Delta \Gamma H \, dz = 2\sqrt{\frac{2}{3}} \Delta \Gamma H \int_{-\infty}^{\infty} dz = 0
\end{align*}
(102)
which implies that \( \Delta \Gamma H \) has to be 0, and thus \( H \) is a harmonic function on the surface \( \Gamma \). By the maximum principle, we have
\[ H = \text{const} \]
all over \( \Gamma \) (note that, we assume periodic boundary condition). Therefore, \( \Gamma \) is a constant mean curvature surface.

**Remark.** Theorem 4.3 in fact shows the consistency of the phase field models following the Cahn-Hilliard models and the sharp interface models, i.e., the Hele-Shaw models. It is easy to see that \( H = \text{const} \) is consistent with the equilibrium condition of the Hele-Shaw models, i.e., if \( \Gamma \) is closed, compact and complete, both of the resulting surfaces are sphere with minimum surface area for prescribed volume.

**4.2. Error estimate.** Since we have already obtained the explicit expressions of the first two terms \( q_0 \) and \( q_1 \), we will give error estimate of the surface tension energy \( W(\varphi) \). Before we derive the error, we give the following lemma that will be used in estimating integration over the whole domain \( \Omega \).

**Lemma 4.4.** Suppose that \( f \in C^2(\Omega) \) and \( p \in L^1(\mathbb{R}) \) satisfy the bound
\[ \max_{|z|<s} |p(z)z^3| \leq \frac{C}{s^m}, \quad m > 1. \] (103)

Then
\begin{align*}
&\frac{1}{\epsilon} \int_{-\infty}^{\infty} p \left( \frac{d(x)}{\epsilon} \right) f(x) \, dx \\
&= \int_{-\infty}^{\infty} p(z)dz \int_{\Gamma} f(x)\,dS + \epsilon \int_{-\infty}^{\infty} p(z)zdz \int_{\Gamma} -2fH + \nabla f \cdot \nabla d \, dS \\
&+ \epsilon^2 \int_{-\infty}^{\infty} p(z)z^2dz \int_{\Gamma} fK - 2(\nabla f \cdot \nabla d)H + \frac{1}{2} \nabla(\nabla f \cdot \nabla d) \cdot \nabla d \, dS \\
&+ O(\epsilon^3).
\end{align*}
(104)

**Proof.** Please refer to [46] for detailed proof.

Here comes the error estimation of the surface tension energy \( W^\epsilon(\varphi) \).
Lemma 4.5. Suppose that the dynamic system evolves according to the Cahn-Hilliard equation and arrives at the equilibrium state. If the surface tension energy is given by

\[ W^\varepsilon(\varphi) = \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{4\varepsilon}(\varphi^2 - 1)^2 \, dx, \]

and \( \hat{\varphi} \) is corresponding to the equilibrium state, then we have

\[ W^\varepsilon(\hat{\varphi}) = \frac{2\sqrt{2}}{3} |\Gamma| + \frac{2\varepsilon}{9} H^2 |\Omega| + 2\varepsilon \left[ \int_{-\infty}^{\infty} (\hat{q}_0)^2 \, dz \int_{\Gamma} KdS + \int_{\Gamma} \hat{q}_2^d \, dS - \frac{\sqrt{2}}{9} H^2 |\Gamma| \right] + O(\varepsilon). \]

Proof. When the system reaches its equilibrium state, the phase field function is corresponding to the minimizer of the surface tension energy \( W^\varepsilon(\varphi) \). Let \( \hat{\varphi} \) be the minimizer, and it can be expanded by

\[ \hat{\varphi} = \hat{q}_0 + \varepsilon \hat{q}_1 + \varepsilon^2 \hat{q}_2 + O(\varepsilon^3), \]

where \( \hat{q}_0 = \tanh \left( \frac{d(x)}{2\varepsilon} \right) \) and \( \hat{q}_1 = \frac{\sqrt{2}}{3} H^2 \hat{q}_0^2 \) by Theorems 4.1 and 4.2. In addition, Theorem 4.3 implies that the mean curvature \( H \) of \( \hat{\varphi} \)'s zero level set \( \Gamma \) is a constant, i.e., \( H_{\Gamma} = \text{const.} \)

By substituting the above expansion of \( \hat{\varphi} \) into the equation of \( W^\varepsilon(\varphi) \), we have

\[
\begin{align*}
W^\varepsilon(\hat{\varphi}) &= \int_{\Omega} \frac{\varepsilon}{2} |\nabla (\hat{q}_0 + \varepsilon \hat{q}_1 + \varepsilon^2 \hat{q}_2 + O(\varepsilon^3))|^2 + \frac{1}{4\varepsilon} [(\hat{q}_0 + \varepsilon \hat{q}_1 + \varepsilon^2 \hat{q}_2 + O(\varepsilon^3))^2 - 1]^2 \, dx \\
&= \int_{\Omega} \frac{1}{2\varepsilon} (\hat{q}_0^2)^2 + \hat{q}_0^2 \hat{q}_1^2 + \frac{\varepsilon}{2} \left[ |\nabla \hat{q}_0|^2 + (\hat{q}_1)^2 + 2\hat{q}_0 \hat{q}_2 + O(\varepsilon^2) \right] \, dx \\
&\quad + \int_{\Omega} \frac{1}{4\varepsilon} (\hat{q}_0^2 - 1)^2 + (\hat{q}_0^2 - 1) \hat{q}_0 \hat{q}_1 \\\n&\quad \quad + \frac{\varepsilon}{2} [2\hat{q}_0 \hat{q}_1^2 + (\hat{q}_0^2 - 1)(\hat{q}_1^2 + 2\hat{q}_0 \hat{q}_2)] + O(\varepsilon^2) \, dx.
\end{align*}
\]

By Lemma 4.4 and assumption (A4), we have the following identities

\[
\frac{1}{2\varepsilon} \int_{\Omega} (\hat{q}_0^2) \, dx = \frac{1}{2} \left[ \int_{-\infty}^{\infty} (\hat{q}_0^2) \, dz \int_{\Gamma} dS + \varepsilon \int_{-\infty}^{\infty} (\hat{q}_0^2) \, dz \int_{\Gamma} -2HdS \right. \\
&\quad + \varepsilon^2 \int_{-\infty}^{\infty} (\hat{q}_0^2) \, dz \int_{\Gamma} KdS \left. \right] + O(\varepsilon^3) \\
= \frac{\sqrt{2}}{3} |\Gamma| + \varepsilon^2 \int_{-\infty}^{\infty} (\hat{q}_0^2) \, dz \int_{\Gamma} KdS + O(\varepsilon^3),
\]

which is due to

\[
\int_{-\infty}^{\infty} (\hat{q}_0^2) \, dz = \int_{-\infty}^{\infty} \hat{q}_0^2 \, d\hat{q}_0 = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} (1 - \hat{q}_0^2) \, d\hat{q}_0 = \frac{1}{\sqrt{2}} \left( \hat{q}_0 - \frac{\hat{q}_0^3}{3} \right) \bigg|_{-\infty}^{\infty} = \frac{2\sqrt{2}}{3}
\]

and \( (\hat{q}_0^2)z \) is an odd function with respect to \( z \). Note that \( |\Gamma| \) denotes the surface area of \( \Gamma \).

Similarly, we can see that

\[
\int_{\Omega} \hat{q}_0 \hat{q}_1 \, dx = \int_{-\infty}^{\infty} \hat{q}_0 \hat{q}_1 \, dz \int_{\Gamma} dS + \varepsilon \int_{-\infty}^{\infty} \hat{q}_0 \hat{q}_1 \, dz \int_{\Gamma} -2HdS + O(\varepsilon^3) \\
= -2\varepsilon H_{\Gamma} |\Gamma| \int_{-\infty}^{\infty} \hat{q}_0 \hat{q}_1 \, dz + O(\varepsilon^3),
\]

(107)
as $\hat{q}_0'\hat{q}_1'$ is an odd function with respect to $z$.

Because $\nabla_{\Gamma}\hat{q}_0 = 0$, we have

$$
\frac{\epsilon}{2} \int_{\Omega} (|\nabla_{\Gamma}\hat{q}_0|^2 + (\hat{q}_1')^2 + 2\hat{q}_0'\hat{q}_2')dx
= \frac{\epsilon}{2} \int_{\Omega} ((\hat{q}_1')^2 + 2\hat{q}_0'\hat{q}_2')dx
= \epsilon^2 \int_{-\infty}^{\infty} \frac{1}{2} (\hat{q}_1')^2 dz \int_{\Gamma} dS + \epsilon^2 \int_{-\infty}^{\infty} \hat{q}_0' dS + \int_{\Gamma} dS + O(\epsilon^3)
= \epsilon^2 |\Gamma| \int_{-\infty}^{\infty} \frac{1}{2} (\hat{q}_1')^2 dz + 2\epsilon^2 \int_{\Gamma} \hat{q}_2' dS + O(\epsilon^3).
$$

(108)

Note that, by assumptions (A1) to (A3), $\int_{\Gamma} \hat{q}_2' dS$ must be bounded. Therefore, we can see that $2\epsilon^2 \int_{\Gamma} \hat{q}_2' dS \sim O(\epsilon^2)$.

Moreover, we have

$$
\frac{1}{4\epsilon} \int_{\Omega} (\hat{q}_0^2 - 1)^2 dx = \frac{1}{2\epsilon} \int_{\Omega} (\hat{q}_0')^2 dx = \frac{\sqrt{\epsilon}}{3} |\Gamma| + \epsilon^2 \int_{-\infty}^{\infty} |\hat{q}_0'|^2 z^2 dz \int_{\Gamma} KdS + O(\epsilon^3),
$$

(109)

$$
\frac{\epsilon}{2} \int_{\Omega} 2\hat{q}_0'\hat{q}_1' + (\hat{q}_0^2 - 1)(\hat{q}_1^2 + 2\hat{q}_0\hat{q}_2) dx
= \frac{\epsilon}{2} \int_{\Omega} (3\hat{q}_0^2 - 1)(\hat{q}_1^2 + 2\hat{q}_0'\hat{q}_2) dx
= \frac{\epsilon}{2} \int_{\Omega} \hat{q}_0''\hat{q}_1 + 2\hat{q}_0'\hat{q}_1 + 2\hat{q}_0''\hat{q}_2 dx
= \epsilon^2 \int_{-\infty}^{\infty} \frac{1}{2} \hat{q}_0''\hat{q}_1 dz \int_{\Gamma} dS + \epsilon^2 \int_{-\infty}^{\infty} \hat{q}_0'' dS \int_{\Gamma} dS
- \epsilon^2 |\Gamma| \int_{-\infty}^{\infty} \hat{q}_0''\hat{q}_1 dz + \frac{\sqrt{\epsilon}}{3} H_{\Gamma} \int_{\Omega} \hat{q}_1 dx + O(\epsilon^3).
$$

(111)

It is evident that

$$
\int_{-\infty}^{\infty} \hat{q}_0' dz = \hat{q}_0' \bigg|_{-\infty}^{\infty} = 0,
$$

$$
\int_{-\infty}^{\infty} \hat{q}_0'\hat{q}_1 dz = \frac{\sqrt{\epsilon}}{3} H_{\Gamma} \int_{-\infty}^{\infty} \hat{q}_0^2 d\hat{q}_0 = \frac{\sqrt{\epsilon}}{3} H_{\Gamma},
$$

$$
\int_{\Omega} \hat{q}_1 dx = \frac{\sqrt{\epsilon}}{3} H_{\Gamma} \int_{\Omega} \hat{q}_0^2 dx = \frac{\sqrt{\epsilon}}{3} H_{\Gamma} \left[ \int_{\Omega} (\hat{q}_0^2 - 1) dx + \int_{\Omega} dx \right]
= \frac{\sqrt{\epsilon}}{3} H_{\Gamma} \left[ - \sqrt{\epsilon} \int_{\Omega} \hat{q}_0' dx + |\Omega| \right].
$$
Because the first order term \( q_1 \) for detailed description of the numerical schemes of gradient flow.

fourier spectral method is used to get a high accuracy of the result. One may refer numerical experiments to test this result.

while preserving the inside volume can be expanded as

\[
\phi(x) = \tanh \left( \frac{d(x)}{\epsilon \sqrt{2}} \right) + \frac{\sqrt{2}}{3} H_\Gamma \tanh^2 \left( \frac{d(x)}{\epsilon \sqrt{2}} \right) \epsilon + O(\epsilon^2).
\]

Because the first order term \( q_1(x) = \frac{\sqrt{2}}{3} H_\Gamma \tanh^2 \left( \frac{d(x)}{\epsilon \sqrt{2}} \right) \neq 0 \), we can expect that away from the surface \( \Gamma \), i.e., the zero level set of \( \phi \), the two phases will not be exactly +1 and −1, but instead, be \( \pm 1 + \frac{\sqrt{2}}{3} H_\Gamma \epsilon \). In this section, we perform numerical experiments to test this result.

Our experiments are performed in three dimensional axis-symmetrical case. The fourier spectral method is used to get a high accuracy of the result. One may refer [23, 24, 22] for detailed description of the numerical schemes of gradient flow.
The numerical simulations presented in this paper are in full 3-D cases. The computational domain is taken to be $[-\pi, \pi]^3$, which is divided into a $64^3$ mesh. Our experiment starts from a sphere with the initial phase field function selected to be $q_0:\quad \phi_0(x) = q_0(d(x)/\epsilon) = \tanh(0.5\pi - \sqrt{x_1^2 + x_2^2 + x_3^2}/(\sqrt{2}\epsilon))$.

The phase field function evolves following the Cahn-Hilliard flow (5).

We set $\epsilon = 1.25h = 0.123$, where $h$ is the mesh size. The cross section picture of the initial data is shown by the left image of Figure 1. Note that, the initial data is very close to the final result. In the middle of the sphere, the initial value is +1. For the outer region near the boundary of the domain, the initial value is −1. Following the Cahn-Hilliard flow, the phase field function $\varphi$ gradually arrives at an equilibrium state (see the right picture of Figure 1).

From Figure 1, we observe that the maximum value of the final $\varphi$ exceeds +1. The exact range is illustrated in Table 5.

The second and the third columns of Table 5 show the numerical results of the resulting phase field function. If we ignore the second and higher order terms, the
Table 1. Range of phase field function compared with the theoretical value

| $\epsilon$ | min   | max   | $-1 + H\epsilon\sqrt{2}/3$ | $+1 + H\epsilon\sqrt{2}/3$ |
|------------|-------|-------|-----------------------------|-----------------------------|
| 1.25h      | -0.957| 1.039 | -0.963                      | 1.037                       |
| 2.00h      | -0.941| 1.065 | -0.941                      | 1.059                       |
| 2.50h      | -0.939| 1.074 | -0.926                      | 1.074                       |

Theoretical minimum and maximum are given in the last two columns. Comparing our theoretical values with the numerical results, we can see that they are very close to each other. The remaining differences are due to the ignorance of the higher order terms and our numerical iterations do not fully converge.

The experiments justify Theorem 4.1 and Theorem 4.2 that $\varphi = \tanh\left(\frac{d(x)}{\epsilon\sqrt{2}}\right) + \frac{\sqrt{2}}{4} H \Gamma \tanh^2\left(\frac{d(x)}{\epsilon\sqrt{2}}\right) + O(\epsilon^2)$. Note that, in many phase field models, the inside volume is formulated by

$$\text{Vol} = \frac{1}{2} \int_{\Omega} \varphi(x) + 1 \, dx$$

with the assumption that the inside and outside value of $\varphi$ are very close to $\pm 1$ respectively. With our derivation, the volume formulation here has an error $O(\epsilon)$. If $\epsilon$ is not small enough, we may not be able to ignore this error.

6. Conclusion. Based on some relaxed assumptions, explicit asymptotic expansions of the solution of Allen-Cahn and Cahn-Hilliard equations are derived in this paper. The assumptions and asymptotic approximations are verified by the numerical experiments. On the other hand, the unusual phenomenon we observed in our former numerical experiments has been nicely explained by the asymptotic series. The consistency of the phase field models with the sharp interface model is well established. Meanwhile, the geometric features of the interface in the equilibrium state are illustrated by elegant formulae with respect to its mean curvature.

The analysis in this paper clearly shows the detailed structure of the phase field function resulting from the Allen-Cahn and Cahn-Hilliard equations. This knowledge helps us to better understand the phase field models with surface tension, and thus can be used to guide the future modeling and simulations of other moving boundary problems by phase field models.

Our future work is to explore the asymptotic expansions for the time dependent evolutionary phase field model, in contrast to the current steady case studied in this paper. In [1, 31], some classic convergence analysis has been made. We will try to use our double asymptotical expansion method to show the convergence of the time dependent phase field models to the corresponding sharp interface models, e.g., the mean curvature flow and Hele-Shaw models. We expect some subtle structure of the time dependent phase field function, which will help us to further understand the evolution of the phase field function in those phase field models.

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