BOUNDED EQUIDISTRIBUTION OF SPECIAL SUBVARIETIES II
KE CHEN

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Theorem 0.1. Assume GRH for CM fields. Let $M = M_K(P,Y)$ be a mixed Shimura variety, with $K \subset P(\bar{Q})$ a compact open subgroup of finite product type, and $E$ its reflex field. Fix an integer $N$, Then for $M'$ a pure special subvariety in $M$ defined by $(wG^t w^{-1}, wX; wX^+)$, we have
\[
\#\text{Gal}(\bar{Q}/E) \cdot M' \geq c_N D_N(T) \prod_{p \in \Delta(T,K_G(w))} \max\{1, I(T, K_G(w)_p)\}
\]
where
- $c_N$ is some absolute constant, independent of $K$, $M'$;
- $T$ is the connected center of $G'$, $D_N(T) := (\log(D(T)))^N$ with $D(T)$ the absolute discriminant of the splitting field of $T$;
- $K_G(w) = \{g \in K_G : wgw^{-1}g^{-1} \in K_W\}$ following the notations in [6], and $\Delta(T, K_G(w))$ is the set of rational primes such that $T(\bar{Q}_p) \cap K_G(w)_p \subset K_{T,p}^{\max}$, $K_{T,p}^{\max}$ being the maximal compact open subgroup of $T(\bar{Q}_p)$;
- $I(T, K_G(w)_p) = b \cdot [K_{T,p}^{\max} : T(\bar{Q}_p) \cap K_G(w)_p]$ with $b$ some absolute constant independent of $K$, $M'$.

For a general special subvariety $M' \subset M$ which is not pure, we introduce the notion of test invariants $\tau_M(M')$ as a substitute for the lower bound of Galois orbits, and we get

Theorem 0.2. Assume GRH for CM fields. Let $M$ be a mixed Shimura variety defined by $(P,Y)$ at some level $K$ of finite product type. Let $(M_n)$ be a sequence of special subvarieties in $M$, such that the sequence of test invariants $(\tau_M(M_n))$ is bounded. Then the sequence $(M_n)$ is bounded by some finite bounding set $B$ in the sense of [6]. In particular, the Zariski closure of $\bigcup_n M_n$ is a finite union of special subvarieties bounded by $B$.

Note that [21] has formulated their main lower bound via the intersection degree against the automorphic line bundle on pure Shimura varieties, which fits into the framework of [11]. We do not need this step yet in this paper, and we stick to the counting of Galois orbits.

1. LOWER BOUND OF THE GALOIS ORBIT OF A SPECIAL SUBVARIETY

In the pure case, Ullmo and Yafaev proved the following lower bound of Galois orbits of special subvarieties in a pure Shimura variety:
Theorem 1.1 (lower bound in the pure case, cf. [21] Theorem 2.19). Let $S = M_K(G, X)$ be a pure Shimura variety with reflex field $E$, with $K \subset G(\mathbb{Q})$ a level of product type. Assume the GRH for CM fields, and fix an integer $N > 0$. Then for $S' \subset S$ a $T$-special subvariety, we have
\[
\#	ext{Gal}(\bar{\mathbb{Q}}/E) \cdot S' \geq c_N \cdot D_N(T) \cdot \prod_{p \in \Delta(T, K)} \max \{1, I(T, K_p)\}
\]
with

- $D_N(T) = (\log D(T))^N$, where $D(T)$ is the absolute discriminant of the splitting field of the $\mathbb{Q}$-torus $T$;
- $\Delta(T, K)$ is the set of rational primes $p$ such that $K_{T,p} \subset K_{T,p}^{\text{max}}$, where $K_{T,p} = T(\mathbb{Q}_p) \cap K_p$,
- $K_{T,p}^{\text{max}}$ the unique maximal compact open subgroup of $T(\mathbb{Q}_p)$,
- $\Delta(T, K)$ is finite, i.e. $K_{T,p} = K_{T,p}^{\text{max}}$ for all but finitely many $p$'s.
- $I(T, K_p) = b[K_{T,p}^{\text{max}} : K_{T,p}]$
- and $c_N, b \in \mathbb{R}_{>0}$ are constants independent of $K, T$.

Remark 1.2 (dependence on levels). (1) The results in [21] was formulated for an ambient pure Shimura datum $(\mathfrak{G}, X)$, and a faithful algebraic representation $\rho: \mathfrak{G} \to \text{GL}_{n\mathbb{Q}}$.

The constants $c_N$ and $b$ are independent of $K$. This was not mentioned explicitly in [21], but one can verify through their arguments that $c_N$ and $b$ are determined by $(\mathfrak{G}, X)$ and $\rho$. $c_N$ does not depend on the prescribed integer $N$, but any fixed $N$ will suffice.

(2) The estimation depends on an embedding of $(G, X)$ into some ambient pure Shimura datum $(\mathfrak{G}, X)$, and a faithful algebraic representation $\rho: \mathfrak{G} \to \text{GL}_{n\mathbb{Q}}$.

(3) The quantity $D_N(T)$ is independent of $K$, while $I(T, K_p)$ describe the position of $T(\mathbb{Q})$ relative to $K_p$. Whether $p$ lies in $\Delta(T, K)$ or not is closely related to the integral structure of $T$ at $p$, and is also related to the reduction property of the special subvarieties, see [10] and [22] for details.

(4) The estimation in [21] was formulated using intersection degrees against the ample line bundle of top degree automorphic forms on $S = M_K(G, X)$. Actually the intersection degree of a single (connected) special subvariety only contributes as a real number greater than 1 in the lower bound. The formulation is used in further study of unbounded orbits in [11].

We can thus consider the lower bound of the Galois orbits of pure special subvariety in a given mixed Shimura variety.

Theorem 1.3 (orbit of a pure special subvariety). Let $(P, Y) = (U, V) \rtimes (G, X)$ be a mixed Shimura subdatum, defining a mixed Shimura variety $M$ at a level $K$ of product type. Write $E$ for the reflex field of $(P, Y)$, and $\pi$ for the natural projection $M \to S = M_K(G, X)$ with $\pi(0)$ the zero section.

Let $M'$ be a pure special subvariety of $M$ defined by a subdatum of the form $(wG'w^{-1}, wX')$ for some pure subdatum $(G', X') \subset (G, X)$ and $w \in W(\mathbb{Q})$. Then we have the following lower bound assuming the GRH for CM fields, using the same constants $c_N, b$, and notations in [11]:
\[
\#	ext{Gal}(\bar{\mathbb{Q}}/E) \cdot M' \geq c_N D_N(T) \prod_{p \in \Delta(T, K_{G}(w))} \max \{1, I(T, K_{G}(w)_p)\}
\]
where

- $T$ is the connected center of $G'$, and $D_N(T) = (\log D(T))^N$;
Lemma 1.5

where

Proof. In the statement above, \( K_G(w)_p = G(\mathbb{Q}_p) \cap K_G(w) \) is a compact open subgroup in \( K_{G,p} \), and the inclusion \( K_G(w)_p \subset K_{G,p} \) is an equality for all but finitely many rational primes \( p \)’s. In particular, the group \( K_G(w) \) is a compact open subgroup in \( K_G \) of fine product type, i.e. \( K_G(w) = \prod_p K_G(w)_p \).

Proof. For all but finitely many \( p \)’s, we have \( w \in W(\mathbb{Q}) \) lies in \( K_{W,p} \) and \( w g w^{-1} g^{-1} \in K_{W,p} \) for \( g \in K_{G,p} \).

When \( w \notin K_{W,p} \), write \( w = (u, v) \) for some \( u \in U(\mathbb{Q}) \) and \( v \in V(\mathbb{Q}) \), then by ?? we have \( w^n = (nu, nv) \) for any \( n \in \mathbb{Z} \), hence the subgroup \( K'_{W,p} \) generated by \( w \) and \( K_{W,p} \) is compact and open, containing \( K_{W,p} \) as a subgroup of finite index. \( K_{G,p} \) stabilizes \( K_{W,p} \), hence a compact open subgroup in \( K_{G,p} \) of finite index stabilizes \( K_{W,p} \).

The theorem is reduced to the following

Lemma 1.5 (reflex field). \( (P, Y) = W \rtimes (G, X) \) has the same reflex field as \( (G, X) \) does, and the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on \( M_{K_{G,w}}(w G w^{-1}, w X) \) is identified with its action on \( M_{K_{G,w}}(G, X) \) where \( K_{G,w} := w G(\overline{\mathbb{Q}}) w^{-1} \cap K_{W} \rtimes K_{G} \).

Proof. From the definition of reflex fields [16] Chapter 11, we know that for a morphism of mixed Shimura data \( (P, Y) \rightarrow (P', Y') \) we have \( E(P, Y) \supseteq E(G, X) \). Thus \( E(P, Y) = E(G, X) \) because we have the natural projection and the zero section. Conjugation by \( w \in W(\mathbb{Q}) \) gives the isomorphism \( (G, X) \cong (w G w^{-1}, w X) \) as maximal pure subdata of \( (P, Y) \). It also induces an isomorphism of pure Shimura varieties \( M_{K_{G,w}}(G, X) \cong M_{w K_{G,w}^{-1}}(w G w^{-1}, w X) \).

It is easy to verify that \( w K_{G,w}^{-1} \cap K_{W} \rtimes K_{G} = w K_{G}(w) w^{-1} \). Therefore when \( K_{G,w} = K_{G}(w) \), the conjugation by \( w \) gives an isomorphism \( M_{K_{G,w}^{-1}}(P, Y) \cong M_{K}(P, Y) \) sends the zero section \( M_{w K_{G,w}^{-1}}(w G w^{-1}, w X) \) with respect to \( (P, Y) = W \rtimes (w G w^{-1}, w X) \) to the zero section \( M_{K_{G,w}}(G, X) \) with respect to \( (P, Y) = W \rtimes (G, X) \). In particular, for any pure subdatum \( (G', X') \) of \( (G, X) \), the special subvariety defined by \( (w G' w^{-1}, w X'; w X'^{1}) \) is isomorphic to the one defined by \( (G', X'; X'^{1}) \), and the isomorphism respects the canonical models.

Hence the theorem holds trivially when \( K_{G,w} = K_{G}(w) \). When \( K_{G}(w) \subsetneq K_{G} \), it suffices to take a base change \( f : S' = M_{K_{G,w}}(G, X) \rightarrow S = M_{K_{G}}(G, X) \) which is a morphism of pure Shimura varieties defined over \( E(G, X) \). The base change is finite étale as we have taken \( K_{G} \) to be neat. It respects the special subvarieties of \( M = M_{K}(P, Y) \) and of \( MS' \) as well as their canonical models, hence the lemma.

Before we take a closer look at the term \( I(T, K_{G}(w)_p) \), we introduce the following

Notation 1.6. For \( w = (u, v) \in W(\mathbb{Q}) \), we have \( (u, v)^n = (nu, nv) \), hence it makes sense to talk about the order of \( w = (u, v) \) with respect to \( K_{W} \); it is the smallest positive integer \( m > 0 \) such that \( w^m \in K_{W} \), i.e. \( mw \in K_{U} \) and \( nv \in K_{V} \), which makes sense because \( u \) and \( v \) are in \( U(\mathbb{Q}) \) and \( V(\mathbb{Q}) \) respectively. We can also talk about the \( p \) order of \( w \) with respect to \( K_{W} \), namely the integer \( m \in \mathbb{N} \) such that \( w^n \in K_{W,p} \) if and only if \( p^m \) divides \( n \).

In the lower bound we have the set \( \Delta(T, K_{G}(w)) \) containing the subset \( \delta(T, K_{G}(w)) \) of primes \( p \) such that \( K_{T,p} \supseteq K_{T}(w)_p \). We want to show that for \( p \in \delta(T, K_{G}(w)) \), the inequality

\[
[K_{T,p}^{\text{max}} : K_{T}(w)_p] \geq c \cdot \text{ord}_p(w, K_{W})
\]
holds for some absolute constant \( c \) independent of \( K, w, T \). This is clear when \( T \cong \mathbb{G}_m \) acts on \( U \) and \( V \) by scaling \( g(u, v) = (gu, gv) \) using the central cocharacters \( \mathbb{G}_m \to \text{GL}_U \) and \( \mathbb{G}_m \to \text{GL}_V \). In fact, for the action on \( U(Q_p) \), \( K_{T,p} \) is a compact open subgroup of \( \mathbb{G}_m(Q_p) = \mathbb{Q}_p^* \) stabilizing \( K_{U,p} \) contained in the maximal compact open subgroup \( K_{T,p}^{\max} \cong \mathbb{Z}_p^* \), and \( K_T(u) \) is the stabilizer of the class \( u \) modulo \( K_{U,p} \). Since the automorphism by \( K_{T,p}^{\max} \cong \mathbb{Z}_p^* \) preserves the torsion order in \( U(Q_p)/K_{U,p} \) and leaves the line \( \mathbb{Q}_p u \) stable, we see that

\[
[K_{T,p}^{\max} : K_T(u) \] \geq (p - 1)p^{m-1}
\]

because \( (p - 1)p^{m-1} \) is the number of elements of order \( p^m \) in \( \mathbb{Q}_p u \) modulo \( K_{U,p} \). The case of \( T \) acting on \( V \) is similar under our assumption \( T \cong \mathbb{G}_m \), and it is obvious that

\[
\text{ord}_p(w, K_W) = \max\{\text{ord}_p(u, K_U), \text{ord}_p(v, K_V)\}
\]

hence

\[
[K_{T,p}^{\max} : K_T(u) \] \geq (1 - \frac{1}{p})\text{ord}_p(w, K_W) \geq \frac{1}{2}\text{ord}_p(w, K_W).
\]

In general, the \( \mathbb{Q} \)-torus does admit quotients isomorphic to split \( \mathbb{Q} \)-tori:

**Lemma 1.7** (split tori). Let \((P, Y) = (U, V) \times (G, X) \) be a mixed Shimura datum, with \( T \) the connected center of \( G \). Then for the actions of \( T \) on \( U \) and on \( V \),

1. there is a \( T \)-equivariant decomposition \( U = \bigoplus_{r=1}^{s_1} U_i \) such that \( T \) acts on \( U_i \) via the central scaling \( \mathbb{G}_m \to \text{GL}_U \);
2. there exists a \( T \)-equivariant decomposition \( V = \bigoplus_{j=1}^{s_2} V_j \) such that in the representation \( T \to \text{GL}_V \), the image of \( T \) contains the center of \( \text{GL}_V \).

**Proof.** \( G \) and \( T \) being reductive, it suffices to consider the case when \( U \) and \( V \) are irreducible as representations of \( G \).

1. This is clear because by [16] 2.16, \( G \), hence \( T \), acts on \( U \) through a split \( \mathbb{Q} \)-torus, hence the irreducible representation \( U \) is one-dimensional. Since \( U \) is of Hodge type \((-1, -1)\), the action of \( G \), hence the action of \( T \) on it is through the central scaling.

2. \( \rho : G \to \text{GL}_V \) is an irreducible representation of \( G \), such that for any \( x \in X \), the composition \( \rho \circ x \) is a rational Hodge structure of type \( \{(-1, 0), (0, -1)\} \). It thus follows from the definition of pure Shimura data [16] 1. ? that the Hodge structure is polarizable, namely \( G \) preserves some polarization \( \psi : V \times V \to \mathbb{Q}(-1) \) up to scalars, hence the representation factors through the Siegel datum, i.e. \((G, X) \to (\text{GSp}_V, \mathcal{H}_V)\).

It suffices to show that the image \( G \to \text{GSp}_V \) contains the center of \( \text{GSp}_V \). If it does not contain the center, then it is contained in \( \text{Sp}_V = \text{Ker}(\text{GSp}_V \xrightarrow{\det} \mathbb{G}_m) \). The construction in [21] ? shows that \( (\text{Sp}_V, X') \) is a pure Shimura subdatum of \( (\text{GSp}_V, \mathcal{H}_V) \) with \( X' \) the \( \text{Sp}_V(\mathbb{R}) \)-orbit of the image of \( X \) in \( \mathcal{H}_V \), which is ridiculous because \( x(S) \notin \text{Sp}_V(\mathbb{R}) \) for any \( x \in \mathcal{H}_V \) due to the \( \mathbb{R} \)-torus \( \mathbb{G}^{\text{mR}} \subset \mathbb{S} \).

We are thus led to

**Proposition 1.8** (torsion order). For \( p \in \delta(T, K_G(w)) \) as above, we have \( [K_{T,p}^{\max} : K_T(w) \] \geq cp^{\text{ord}_p(w, K_W)} \) for some constant \( c \) which is independent of \( K, w, T \).

**Proof.** Since the number of irreducible representations in \( U \) (resp. in \( V \)) is uniformly bounded by the dimension of \( U \) (resp. of \( V \)), we are reduced to the case when \( U \) and \( V \) are irreducible under \( G' \).

Thus \( U \) is one-dimensional, with \( T \) acts on it through the central scaling \( \mathbb{G}_m \cong \text{GL}_U \). It suffices to show that for each prime \( p \), the homomorphism \( T(Q_p) \to \mathbb{G}_m(Q_p) \) sends \( K_{T,p}^{\max} \) to a compact open subgroup of \( \mathbb{Z}_p^* \) whose index in \( \mathbb{Z}_p^* \) is uniformly bounded.
Recall that the splitting field $F = F_T$ is a number field, and $[F : \mathbb{Q}]$ is uniformly bounded by some constant $c_1$ that only depends on the dimension of $G$; in particular, we can rearrange $c_1 \in \mathbb{N}_{>0}$ such that for any connected center $T$ of pure Shimura subdatum $(G', X')$ of $(G, X)$, $[F_T : \mathbb{Q}]$ divides $c_1$ and $[F_T(p) : \mathbb{Q}_p]$ divides $c_1$ with $F_T(p)$ the splitting field of the $\mathbb{Q}_p$-torus $T_{\mathbb{Q}_p}$.

Fix $p$ a prime, $F$ the splitting field of $T_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$. Write $X$ for the group of characters $\text{Hom}(T_F, \mathbb{G}_m)$ with the natural action of $\Gamma = \text{Gal}(F/\mathbb{Q}_p)$. Then $T(F) = \text{Hom}(X, F^\times)$, and $T_q(p) = \text{Hom}(X, F^\times)$ is the $\Gamma$-fixed part of $T(F)$. The maximal compact open subgroup of $T(F)$ is $\text{Hom}(X_T, O_F^\times)$ with $O_F$ the integer ring of $F$, and the norm map $Nm : T(F) \rightarrow T_q(p)$ sends $\text{Hom}(X, O_F^\times)$ into the maximal compact open subgroup $K_{\mathbb{Q}_p}^{\text{max}}$ of $T(q_p)$.

From [1.7] we have homomorphism of $\mathbb{Q}$-tori $T \rightarrow \mathbb{G}_m$, where $\mathbb{G}_m$ is a $\mathbb{Q}$-subtorus in $\text{GL}_U$ (or $\text{GL}_V$) that acts as central scaling on direct summands of $U$ (or on $V$). The condition on Hodge types show that for the corresponding map of characters $\mathbb{Z}^d \rightarrow X$, the image of $\mathbb{Z}^d$ is of index at most 2 in a direct summand of $X$ on which $\Gamma$ acts trivially.

Now consider the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(X, O_F^\times) & \rightarrow & \text{Hom}(\mathbb{Z}^d, O_F^\times) \\
\downarrow Nm & & \downarrow Nm \\
K_{\mathbb{Q}_p}^{\text{max}} & \rightarrow & \text{Hom}(\mathbb{Z}^d, \mathbb{Z}_p^\times)
\end{array}
\]

where the horizontal maps are induced from the norm $Nm : F^\times \rightarrow \mathbb{Q}_p^\times$, the horizontal maps are induced by $T \rightarrow \mathbb{G}_m$, and the image of the upper horizontal map is of index at most $2^d$.

Since the degree $[\mathbb{Z}_p^\times : Nm(O_F^\times)] \leq [F : \mathbb{Q}_p]$ by local class field theory, we see that the image of the lower horizontal map $K_{\mathbb{Q}_p}^{\text{max}} \rightarrow \text{Hom}(\mathbb{Z}^d, \mathbb{Z}_p^\times)$ is of finite index, and the index is bounded by a constant that only depends on $c_1$ and the dimension of $T$.

We summarize the above computation into the following

**Corollary 1.9** (unipotent defects). There is some constant $c$, independent of $c, K, w$, such that in the expression $I(T, K_G(w))$ in [1.3] we have $I(T, K_G(w)) \geq cp^{\text{ord}_p(m, K \Gamma w)}$ for $p \in \delta(T, K_G(w))$.

For $p \in \Delta(T, K_G(w))$ such that $K_{\mathbb{Q}_p}^{\text{max}} \supseteq K_{\mathbb{Q}_p} = K_T(w_p)$ we still have $I(T, K_G(w)) \geq cp$ by [21].

As we have mentioned in ??, for a subdatum $(P', Y') = W' \times (wG'w^{-1}, wX')$, the choice of $w$ is unique up to translation by $W'(\mathbb{Q})$. In this case, we have:

**Corollary 1.10.** Let $M'$ be a special subvariety defined by a subdatum $(P', Y'; Y'^{+}) = W' \times (wG'w^{-1}, wX'; wX'^{+})$. Then the infimum

\[
\inf_{w' \in W'(\mathbb{Q})w} \prod_{p \in \Delta(T, K_G(w))} \max\{1, bI(T, K_G(w)_p)\}
\]

is reached at some $w' \in W'(\mathbb{Q})w$.

**Proof.** We first note that the representation of $T$ on $V$ (resp. $U$) does not admit any trivial subrepresentation. Otherwise we have some $\mathbb{Q}$-subspace $V' \subset V$ stabilized by $T$ and by $G^\text{der}$ because they commute with each other, hence

Write $w = (u, v)$ and $w' = (u', v') = (u + u' + \psi(u, v), v' + v)$ for $(u', v') \in W'(\mathbb{Q})$, and $\text{ord}_p(u', K_U)$ resp. $\text{ord}_p(v', K_V)$ for the $p$-order of $u'$ with respect to $K_U$ resp. of $v'$ with respect to $K_V$. 
We have $I(T, K_G(w')_p) \geq cp^m$ with $m = \max\{\text{ord}_p(u', \text{ord}_p(u', K_U)), \text{ord}_p(v', K_V)\}$ for $p \in \delta(T, K_G(w))$. $K_{T,p}$ is a compact open subgroup of $T(Q_p)$ stabilizing $K_{U,p}$ and $K_{V,p}$, hence $K_{T,w'} \subseteq K_{T,p}$ when either $\text{ord}_p(u', K_U)$ or $\text{ord}_p(v', K_V)$ are large. Combining with the estimation in [1], we see that the inferium is reached for some $w'$ such that $\text{ord}_p(w', K_W)$ is small.

For convenience we introduce the following:

**Definition 1.11.** Let $M$ be a mixed Shimura variety defined by $(P, Y) = W \ltimes (G, X)$ at some level $K$ of fine product type. For $M'$ a special subvariety defined by $(P', Y'; Y'^+) = W' \ltimes (wG'w^{-1}, wX', wX'^+)$, we define the test invariant of $M'$ in $M$ to be

$$\tau_M(M') := D(T) \min_{w' \in W^\prime(Q)w} \prod_{p \in \Delta(T, K_G(w))} \max\{1, b \cdot I(T, K_G(w)_p)\},$$

where $T$ is the connected center of $G'$, $D(T)$ is the absolute discriminant of the splitting field of $T$, and the minimum makes sense by the corollary above.

It is actually independent of the choice of subdata defining $M'$: by ??, if we pass to a second defining subdatum $(P'', Y''; Y''^+)$, then its image under the natural projection is a pure subdatum $(G'', X'', X''^+)$ of $(G, X; X^+)$ with $G'' = \gamma G' \gamma^{-1}$ for some $\gamma \in \Gamma_G$, and its connected center is $\gamma T \gamma^{-1}$, hence the absolute discriminant remains unchanged; the element $w$ could be replaced by a $\Gamma_w$-translation, which again leaves the set $\Delta(T, K_G(w))$ and the quantities $I(T, K_G(w)_p)$ unchanged.

In particular, when $M'$ is a pure special subvariety, then it is defined by some pure subdatum $(wG'w^{-1}, wX'; wX'^+)$ with $w$ unique up to translation by $\Gamma_w$. Different choices of $w$ give the same value of the test invariant, and we remove the minimum in this case.

We can thus transform the bounded equidistribution in Section 3 into:

**Proposition 1.12** (bounded test invariants). Assume GRH for CM fields. Let $M$ be a mixed Shimura variety defined by $(P, Y) = W \ltimes (G, X)$ at some level $K$ of fine product type, with $E$ its reflex field. Then a sequence $(M_n)$ of special subvarieties is bounded in the sense of ?? if and only if its sequence of associated sequence of test invariants $(\tau_M(M_n))$ is bounded, i.e. $t(M_n) \leq C$ for all $n$ with $C \in \mathbb{R}_{>0}$ some constant.

**Proof.** When the sequence is bounded by some $B = \{(T_1, w_1), \ldots, (T_r, w_r)\}$ then only finitely many values appear as test invariants of the sequence.

Conversely, assume that we are given a sequence of special subvarieties with test invariants uniformly bounded by some $C > 0$. The natural projection $M = M_K(P, Y) \to S = M_{K_G}(G, X)$ sends $(M_n)$ to a sequence of pure special subvarieties $(S_n)$ in $S$. If $M_n$ is $(T_n, w_n)$-special, defined by some subdatum $(P_n, Y_n; Y_n^+) = W_n \ltimes (w_n G_n w_n^{-1}, w_n X_n; w_n X_n^+)$ with $T_n$ the connected center of $G_n$ and $w_n$ chosen so that the minimum in the definition of test invariants of $M_n$ is reached at $w_n$.

Then $S_n$ is a $T_n$-special subvariety of $S$. From the definition of test invariants we have $\tau_S(S_n) \leq \tau_M(M_n)$ because the two invariants involve the same $Q$-torus $T_n$, and for the sets of primes of defects we have $\Delta(T_n, K_G) \subseteq \Delta(T_n, K_G(w_n))$. Now that $(S_n)$ is a sequence with bounded test invariants, we may apply [2] ?? under GRH for CM fields, which implies the existence of a finite set of $Q$-tori $\{C_1, \ldots, C_r\}$ in $G$ such that each $S_n$ is $C_i$-special for some $i$ (and it is clear that $T_n$ is conjugate to $C_i$ by some $\gamma_n \in \Gamma_G$). We may thus assume that $\{T_n : n \in \mathbb{N}\} = \{C_1, \ldots, C_r\}$. 


Therefore only finitely many values arise as $D(T_n)$ in the test invariants $\tau_M(M_n)$, and by the assumption we see that the sequence
\[
\prod_{p \in \Delta(T_n, K_G(w_n))} I(T_n, K_G(w_n)_p)
\]
is also uniformly bounded, hence by [LR] and ?? the classes of $w_n$'s modulo $\Gamma_W$ is finite, which means that the sequence $(M_n)$ is bounded.

\section{Bounded sequence and bounded Galois orbits}

Let $M$ be a mixed Shimura variety defined by $(P, Y) = W \times (G, X)$ at some level $K = K_W \times K_G$. If $M'$ is a special subvariety defined by $(P', Y'; Y'^+)$, then it contains the maximal special subvariety $S'$ defined by $(G', X'; X'^+)$, which is a section to the natural projection $M' \to S' \subset S$, and they have the same field of definition. In particular, the Galois conjugates of $M'$ are in natural bijection with those of $S'$, and in this case we have $\tau_M(M') = \tau_S(S')$, as $M'$ is $(T, 0)$-special, $T$ being the connected center of $G'$.

In general we do not have an explicit way to describe the Galois conjugates of $M'$ defined by $(P', Y'; Y'^+) = W' \rtimes (G'X'; X'^+)$ by the conjugates of some maximal pure special subvariety of it, unless we know a priori that $K_G = K_G(w)$. To remedy this we have the following two potential estimates:

\begin{proposition}
Assume GRH for CM fields. Let $M$ be a mixed Shimura variety defined by $(P, Y) = W \rtimes (G, X)$ at some level $K = K_W \rtimes K_G$ of fine product type, with $E$ its reflex field. Let $(M_n)$ be a sequence of special subvarieties defined by $(P_n, Y_n; Y_n^+) = W_n \rtimes (w_nG_nw_n^{-1}, w_nX_n; w_nX_n^+)$. If the sequence of test invariants $(\tau_M(M_n))$ is bounded, then there exists some compact open subgroup $K'_G \subset K_G$ of fine product type such that when we pass to $M' = M_{K'}(P, Y)$ for the level $K' = K_W \rtimes K'_G$, the sequence $(M'_n)$ with $M'_n$ defined by $(P_n, Y_n; Y_n^+) = \#Gal(\bar{Q}/E) \cdot M'_n \geq c_N D_N(T_n) \prod_{p \in \Delta(T_n, K'_G(w_n))} \max\{1, I(T_n, K'_G(w_n)_p)\}$
\end{proposition}

where $T_n$ is the connected center of $G_n$ and $w'_n \in W_n(\bar{Q})w_n$.

\begin{proof}
By [11.12] the sequence $(M_n)$ is bounded, namely we can choose the defining subdata to be $(P_n, Y_n; Y_n^+) = W_n \rtimes (w_nG_nw_n^{-1}, w_nX_n; w_nX_n^+)$, which are bounded by some finite set $B = \{ (T_\alpha, w_\alpha) : \alpha \in A \}$: $G_\alpha$ is of connected center $T_\alpha$ and $w_n = w_\alpha$ for some $\alpha \in A$ depending on $n$.

We thus take $K'_G = \bigcap_{\alpha \in A} K_G(w_\alpha)$, and consider the mixed Shimura variety $M' = M_{K'}(P, Y)$ with $K' = K_W \rtimes K'_G$. Now that $K_G = K'_G(w_\alpha)$ for all $\alpha \in A$, we have
\[
K_W \rtimes K'_G = w_\alpha(K_W \rtimes K'_G)w_\alpha^{-1} = K_W \rtimes w_\alpha K'_G w_\alpha^{-1}, \forall \alpha \in A
\]
and $K' \cap w_\alpha G w_\alpha^{-1}(\bar{Q}) = w_\alpha K'_G w_\alpha^{-1}, \forall \alpha \in A$. In particular, the natural projection $\pi : M' = M_{K'}(P, Y) \to S' = M_{K'_G}(G, X)$ has more pure sections than the one given by $(G, X) \to (P, Y)$: for each $\alpha \in A$ we have $(G, X) \cong (w_\alpha G w_\alpha^{-1}, w_\alpha X) \subset (P, Y)$, and the pure section it defines is
\[
S'(w_\alpha) := M_{w_\alpha K'_G w_\alpha}(w_\alpha G w_\alpha, w_\alpha X) \hookrightarrow M'
\]
which is isomorphic to $S'(0) := S'$ using the Hecke translate by $w_\alpha$, i.e. $M' \to M', [x, aK'] \mapsto [x, aw_\alpha K']$ because $w_\alpha K'_G w_\alpha^{-1} = K'$.

Therefore the Galois conjugates of pure special subvarieties in $S'(0)$ and in $S'(w_\alpha)$ are the same using the Hecke translate, and the Galois orbits of $M'_n$ is in bijection with the conjugates of its pure section $M'_n \cap S'(w_\alpha)$, as long as the original $M_n$ is $(T_\alpha, w_\alpha)$-special.
\end{proof}
The propositions above justify our use of test invariants as a substitute of the lower bound for the Galois orbit of a general special subvariety: it is "potentially" the correct one when we work with any bounded sequence of special subvarieties.

We also mention the following fact, as a complement to the notion of bounded sequences:

**Lemma 2.2 (upper bound).** Let $M$ be a mixed Shimura variety defined by $(P, Y) = W \times (G, X)$ with reflex field $E$ at some level $K = K_W \times K_G$ of fine product type. Let $M'$ be a $(T, w)$-special subvariety of $M$. Then we have an upper bound
\[
\#\text{Gal} (\bar{\mathbb{Q}}/E) \cdot M' \leq c_0 \cdot C(T, K_G) \text{ord}(w, K_W)^d
\]
where

- $c_0 > 0$ is some constant that only depends on $\dim G$;
- $C(T, K_G)$ is the class number $\#T(\hat{\mathbb{Q}})/T(\mathbb{Q})K_T$;
- $\text{ord}(w, K_W)$ is the order of the class $w$ in the sense of , and $d$ is the square of the dimension of $W$.

In particular, a sequence of special subvarieties bounded by some finite set $B = \{(T, w)\}$ is of uniformly bounded Galois orbits.

**Proof.** We first consider the case when $w = 0$, which is the same as the case of a $T$-special pure subvariety $S'$ in a given pure Shimura variety $S = M_K(G, X)$. It suffices to consider the $\text{Gal} (\bar{\mathbb{Q}}/E')$-orbit of $S'$ in $S$, with $E'$ the reflex field of the subdatum defining $S'$, because $[E' : E]$ is bounded by some constant that only depends on $\dim G$.

The size of $\text{Gal} (\bar{\mathbb{Q}}/E') \cdot S'$ is the size of the image of the reciprocity map describing the Galois action permuting connected components $\text{rec}_{G', X'} : \text{Gal} (\bar{\mathbb{Q}}/E') \to \pi_c(G')/K_{G'}$, which is reduced , up to some absolute constant that only depends on $\dim G$, to the image of $\text{rec}_N : \text{Gal} (\bar{\mathbb{Q}}/E') \to T(\hat{\mathbb{Q}})/T(\mathbb{Q})K_T$, $a \mapsto (\text{rec}_{G', X'}(a))^N$, $N \in \mathbb{N}$ being some absolute constant. Hence the image is bounded by the class number $T(\hat{\mathbb{Q}})/T(\mathbb{Q})K_T$ up to some constant $c_0$ that only depends on $\dim G$.

When $w \neq 0$, it suffices to shrink $K_G$ to $K_G(w)$ and replace $C(T, K_G)$ by $C(T, K_G(w))$. But
\[
[K_T : K_T(w)] \leq [K_G : K_G(w)] \leq \#(\text{Aut} (K_U[w]/K_U) \times \text{Aut} (K_V[w]/K_V))
\]
which is bounded by $\text{ord}(w, K_W)^d$ as is desired. \hfill $\Box$

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