Theoretical framework for estimating a product’s reliability using a variable-amplitude loading spectrum and a stress-based approach

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Abstract
An innovative approach for predicting the reliability of a structure that is subject to a variable-amplitude dynamic load is presented. In this approach, a Gassner durability curve with its scatter is modelled using a 2-parametric Weibull’s probability density function (PDF). The trend of the Gassner durability curve is modelled with a general hyperbola equation in a log-log scale. The hyperbola equation is applied to represent the durability curve for the 63.2% probability of fatigue failure that describes the dependency of the Weibull’s scale parameter on the loading spectrum’s maximum stress. Equations are derived to link the parameters of the hyperbola curve to the material’s S-N curve and the loading spectrum. The Weibull’s shape parameter is estimated from the scatter of the material’s S-N curve. The proposed Gassner-curve model is applied to calculate the fatigue reliability from the PDF of the loading spectrum’s maximum stress and the PDF of the durability-curve’s amplitude stress for the selected number of loading-cycles-to-failure. Method for predicting reliability of the structures with a spectrum loading is presented. Durability curve and its scatter are modelled with a 2-parametric Weibull’s PDF. Scale parameter of the Weibull’s PDF is modelled using a general hyperbola equation. Reliability is calculated on the basis of the modelled durability curve with its scatter.

KEYWORDS
durability curve, fatigue life reliability, S-N curve, variable-amplitude loading, 2-parametric Weibull PDF

Nomenclature: $a_0$, $\ldots$ constant term in a S-N curve equation; $a_1$, $\ldots$ scale coefficient in a S-N curve equation; $a, b, c, \ldots$ characteristics distances of a hyperbola; $f$, $\ldots$ probability density function; $i$, index of a sample point in a sample set; $j$, index of a sample point in a sample set; $k$, $\ldots$ Basquin’s intercept of the S-N curve; $l$, $\ldots$ number of stress-amplitude levels; $n$, $\ldots$ number of counted load cycles; $m$, $\ldots$ number of data points in a sample set; $p, p_i$, $\ldots$ probability of failure; $x$, $\ldots$ independent variable in PP plots; $y$, $\ldots$ dependent variable in PP plots; $A, B, C, D, E, F$, $\ldots$ constant terms of the hyperbola general equation; $Dmg$, $\ldots$ fatigue damage; $F$, $\ldots$ cumulative density function; $F_0, F_2$, $\ldots$ focus points of a hyperbola; $\delta$, $\ldots$ Index representing loading-spectrum; $N$, $\ldots$ number of load-cycles to failure; $R$, $\ldots$ reliability; $S_a$, $[\text{MPa}]$ stress amplitude related to the material’s fatigue life; $S_{aL, \max}$, $[\text{MPa}]$ maximum stress amplitude related to the structure’s loading spectrum; $S_{aL}$, $[\text{MPa}]$ stress amplitude related to the structure’s dynamic load; $\alpha$, $[\text{°}]$ angle between the hyperbola’s asymptotes; $\alpha_{EC}$, $[\text{°}]$ angle of the slope of the high-cycle fatigue domain line in the S-N curve; $\phi$, $[\text{°}]$ angle of the slope of the endurance-cycle fatigue damage line in the S-N curve; $\beta$, $[\text{°}]$ shape parameter of a Weibull’s probability distribution; $\eta$, $[\text{°}]$ scale parameter of a Weibull’s probability distribution; $\delta$, $[\text{°}]$ complementary angle of the hyperbola angle of rotation; $\theta$, $[\text{°}]$ angle of rotation of a hyperbola; $\Delta$, $[\ldots]$ increment or interval width

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INTRODUCTION

To assess a product’s structural integrity under real operating conditions, its fatigue life should be calculated if the product is loaded with a dynamic load. The fatigue life of a structure depends on its geometry, the applied material, and the operating conditions. Due to different random effects during the material’s processing, its fatigue strength, which is usually represented by the material’s S-N curve, is scattered around some central tendency. To make the prediction of reliability more complex, the statistical uncertainty is often not only linked to the material’s strength, but also to the loads that result from the operating conditions, because they cannot be controlled most of the time—see Zhu et al.\(^1\) This means that it is impossible to calculate or predict an exact value of the fatigue life for the given product’s geometry and operating conditions, but only its reliability, i.e., the probability that the product will survive a certain number of loading-cycles-to-failure—see Ebeling,\(^2\) O’Connor and Kleyner,\(^3\) Shen et al.,\(^4\) Tovo,\(^5\) Franko and Nagode.\(^6\)

Zhu et al.\(^7,8\) derived a probabilistic model for predicting the low-cycle fatigue that combines new information with the prior information on the variability of the parameters related to the applied material, stress, strain etc. For the high-cycle fatigue, Klemenc\(^9\) showed how the product’s reliability as a function of the number of loading-cycles-to-failure can be estimated, if a family of products is loaded with a load that can be considered as a constant-amplitude dynamic loading for each individual member of the population, but the load-cycle amplitudes differ between different products in the population. In his approach, he calculated the reliability for a selected fatigue life (i.e., the number of loading-cycles-to-failure \(N\)) as a cross section of a probability density function (PDF) of the load-cycle amplitudes \(f_L(S_{a, L})\) and a PDF of the dynamic strength of the material \(f_S(S_{a|N})\). This approach works well for simple structural parts, e.g., a vehicle’s brake pedal, hand-brake carrier frame, etc.

However, it is not often the case in practice that more complex products are subjected to loads that could be approximated with constant-amplitude loading. This means that the true loading spectrum of loading cycles with variable amplitudes should be considered for estimating the product’s fatigue life and its reliability. If both the scatter of the material’s dynamic strength and the scatter of the loading spectrum are to be considered, it is very difficult to estimate the product’s reliability. In general, there are two approaches for estimating the fatigue reliability of structures and components in this case. One is based on a fracture-mechanics concept, and the other is based on a fatigue-damage accumulation combined with the S-N curves and their scatter. The fracture-mechanics approach enables prediction of a crack-growth evaluation, but it is computationally demanding, and a lot of detailed inputs are needed for its application. A thorough overview of the fracture-mechanics approaches for estimating the fatigue reliability was presented by Altamura and Straub.\(^10\) The damage-accumulation approach is simpler and therefore better suited for general engineering applications, but its prediction capabilities often do not reach the accuracy of the fracture-mechanics approach. However, due to its importance in the engineering applications, we decided to follow the damage-accumulation approach.

In most of the cases, the damage-accumulation approach is based on the Palmgren-Miner’s damage accumulation rule. In the case of the variable-amplitude loading, a fatigue damage is usually calculated from the loading spectrum, and the reliability is then calculated from the statistical distribution of the damage that is influenced by variations of the material’s S-N curve(s), loading factors, and/or loads themselves.\(^1,11\) To calculate reliability from the fatigue damage, Monte-Carlo simulations and/or FORM methods are frequently applied.\(^12-14\) Often, the influence of the loading spectrum is considered through the equivalent stress amplitude or range.\(^14,15\) Some researcher also follow a stochastic approach to calculate damage as opposed to the application of the material’s S-N curve with scatter, but it was shown by Paolino and Cavatorte\(^12\) that the former approach yields better reliability predictions only 10% of the cases.

Although significant research contributions in the field of fatigue reliability were made in the past years, one of the most direct approaches is still the one proposed by Haibach.\(^16\) Following his idea, the structure’s reliability is estimated from the cross section of the 2 probability distributions: the PDF \(f_{LS, \text{max}}(S_{a, L, \text{max}})\) of the load spectrum’s maximum amplitudes \(S_{a, L, \text{max}}\) and the PDF \(f_{LS}(S_{a, \text{max}|N})\) of the amplitude strength \(S_{a}\) at the selected number of loading-cycles-to-failure \(N\) for the durability curve that corresponds to that particular loading spectrum.\(^\ast\) Although phenomenologically correct, this approach is difficult to apply in practice, because the durability curves for the different loading spectra are rarely known. On the other hand, this approach can be applied in the early phases of the product’s R&D process, if it is possible to model the Gassner durability curve and its scatter on the basis of the structural loading spectrum and the scattered material’s S-N curve.

\(^\ast\)A durability curve that corresponds to a particular loading spectrum is often called a Gassner curve after the researcher who made the first experimental effort to characterise these durability curves in practice. In the article, we will use the term Gassner curve to relate the fatigue life with the corresponding loading spectrum.
In this article, a phenomenologically correct method is presented that determines the composition of the Gassner durability curve and its scatter on the basis of the structural loading spectrum and the material’s S-N curve. The trend of the Gassner durability curve is described with a general equation of the hyperbola. Its scatter is modelled with a 2-parametric Weibull’s PDF of the maximum stress-amplitude strength \( S_{a,\text{max}} \) at the given number-of-load-cycles-to-failure \( f_s(S_{a,\text{max}}|N) \):

\[
f_s(S_{a,\text{max}}|N) = \frac{dF_S(S_{a,\text{max}}|N)}{dS_{a,\text{max}}};
\]

\[
F_S(S_{a,\text{max}}|N) = 1 - e^{-\left(\frac{S_{a,\text{max}}}{\eta_S}\right)^{\beta_S}}; N, \beta_S, \eta_S > 0.
\]

The Weibull’s shape parameter \( \beta_S \) is constant over the observed domain of the number of loading-cycles-to-failure \( N \), while the scale parameter \( \eta_S \) follows the hyperbolic trend of the Gassner curve. For this purpose, the Palmgren-Miner’s fatigue damage accumulation rule is used first to estimate the unknown parameters of the hyperbola on the basis of the loading spectrum and the constant-amplitude S-N curve with the Haibach’s knee point.\(^{16}\)

Then, the scatter of the Gassner curve is estimated on the basis of the scatter of the S-N curve from which it originates. This enables us to finally estimate the product’s reliability as a function of the number of loading-cycles-to-failure \( N \) in the same manner as shown by Klemenc\(^{9}\) for the case of constant-amplitude dynamic loading.

The article is structured as follows. After the introductory section, the theoretical background is given for calculating the trend and scatter of the Gassner durability curve for the given loading spectrum and the S-N curve of the material and for estimating the product’s reliability. Then, the experimental dataset is presented together with the simple case that was used for the application of the method. This is followed by a discussion of the results, the concluding section, the list of references, and the annex.

2 | THEORETICAL BACKGROUND

2.1 | Origin of the Gassner durability curve

The Gassner durability curve is a loading-spectrum-dependent fatigue-life curve that is associated with the S-N curve of a selected material when the load amplitude is not constant (ie, variable-amplitude loading). The scatter of the S-N curve, from which the Gassner curve originates, is based on the S-N curve model that assumes the Woehler equation in the high-cycle fatigue domain—see Haibach\(^{16}\) or Dowling\(^{17}\):

\[
\frac{N_{f_1}}{N_{f_2}} = \left(\frac{S_{a_1}}{S_{a_2}}\right)^{-k}, k > 0.
\]

\( S_{a,1} \) and \( S_{a,2} \) are 2 arbitrary amplitude-stress levels in the high-cycle fatigue domain, \( N_{f1} \) and \( N_{f2} \) are the corresponding numbers of load-cycles-to-failure, and \( k \) is the exponent of the S-N curve. In Klemenc and Fajdiga,\(^{18}\) an integral procedure for estimating the S-N curve and its scatter was developed based on the Equation 2 and the 2-parametric Weibull conditional PDF (Weibull\(^{19}\)):

\[
f_N(S_a|N) = \frac{dF_N(N|S_a)}{dN};
\]

\[
F_N(N|S_a) = 1 - e^{-\left(\frac{N}{\eta_N}\right)^{\beta_N}}; N, \beta_N, \eta_N > 0
\]

where \( N \) is the number of loading-cycles-to-failure, \( S_a \) is the selected amplitude-stress level, \( \beta_N \) is the shape parameter that defines the generic shape of the Weibull function, and \( \eta_N \) is the scale parameter that defines the trend of the S-N curve, because the probability of failure \( F(N) \) is equal to \( p = 0.632 \) if \( N = \eta_N \):

\[
\eta = N = \eta_N(S_a) = 10^{a_0 + a_1 \cdot \log S_a}; a_0 > 0, a_1 < 0
\]

\( a_0 \) and \( a_1 \) are characteristics of the material and describe the trend of the S-N curve. Equation 4 is just another (ie, Basquin’s) form of the Equation 2 with \( a_1 \) being equal to \( -k \). Therefore, the S-N curve for any probability of failure \( p \) can be expressed as follows:

\[
N_p = 10^{\frac{\log(-\log(1-p))}{a_1}} + a_0 + a_1 \cdot \log S_a.
\]

The loading spectrum for variable-amplitude loading is obtained from the original loading history by counting the load cycles with a counting method. For this purpose, a rainflow counting method is often applied—see Haibach,\(^{16}\) Dowling,\(^{17}\) ASTM E1049,\(^{20}\) or Amzallag et al.\(^{21}\) In this way, an empirical probability density of load-cycle amplitudes is obtained that can be represented by an arbitrary analytical function, which fulfils the necessary conditions for the PDF. We decided to model the probability distribution of the stress-amplitude \( S_{a,L} \) for the counted loading cycles using a 2-parametric Weibull’s PDF \( f_L(S_{a,L}) \). By using this function, the normalised loading spectrum (NLS) is then obtained by plotting the normalised stress amplitudes \( S_{a,L}^{\text{norm}} = S_{a,L}/S_{a,L,\text{max}} \) of the counted load cycles against the function \( 1–F_L\left(S_{a,L}^{\text{norm}}\right) \), where \( F_L\left(S_{a,L}^{\text{norm}}\right) \) represents the cumulative density function of the corresponding Weibull’s PDF—see Equation 6. The \( S_{a,L}^{\text{norm}} \)
ranges from 0 to 1, and the $1 - F_L(S_{\text{a.L}}^{\text{norm}})$ spans 6 decades from 1 to $10^{-6}$ (Haibach16).

$$F_L(S_{\text{a.L}}^{\text{norm}}) = \int_0^{S_{\text{a.L}}^{\text{norm}}} f_L(S_{\text{a.L}}^{\text{norm}}) \, dS_{\text{a.L}}^{\text{norm}} = 1 - e^{-\left(\frac{S_{\text{a.L}}^{\text{norm}}}{\eta_{\text{L, norm}}}\right)^{1/\beta_L}}$$  \hspace{1cm} (6)

In order to ensure that the combination of the Weibull's parameters $\beta_L$ and $\eta_{\text{L, norm}}$ results in a NLS that is bound by the value 1.0 in both axes, they are related as follows:

$$\eta_{\text{L, norm}}(\beta_L) = \left(-\ln(10^{-6})\right)^{1/\beta_L}.$$  \hspace{1cm} (7)

Finally, to obtain the Gassner curve from the material’s S-N curve and the loading spectrum, the modified Palmgren-Miner’s damage-accumulation rule is applied that links the loading cycles extracted by the rainflow cycle counting with the S-N curve to estimate the accumulated fatigue-life damage $D_{\text{mg}}$:

$$\frac{n_1}{N_{f_1}} + \frac{n_2}{N_{f_2}} + \frac{n_3}{N_{f_3}} + ... = \sum_{j=1}^l \frac{n_j}{N_{f_j}} = D_{\text{mg}} = 1.$$  \hspace{1cm} (8)

In theory, the fatigue-life damage should occur when $D_{\text{mg}}$ is equal to 1. This means, if each stress-amplitude load $S_{\text{a.L}, j}$ at the j-th stress-amplitude level in the loading spectrum is applied $n_j$ times, the failure will occur at the number of loading-cycles-to-failure obtained from the material's S-N curve $N_{f_j}(S_{\text{a.L}, j})$, where the load amplitude is constant. However, when the load-cyle amplitude is variable, the failure for the corresponding variable-amplitude loading spectrum will occur at $N_{\text{tot}}$ number of load cycles. This corresponds to the total number of loading-cycles-to-failure in the Gassner curve for a given maximum stress amplitude of the loading spectrum $S_{\text{a, max}, \text{L}}$, (see also Heuler and Klaetschke\(^2\)):

$$N_{\text{tot}} = \frac{1}{\int_0^{S_{\text{a.L}, \text{max}}} \frac{f_L(S_{\text{a.L}}^{\text{norm}})}{N_f(S_{\text{a.L}}^{\text{norm}})} \, dS_{\text{a.L}}^{\text{norm}}} \rightarrow N_{\text{tot}} $$ \hspace{1cm} (9)

$$= \frac{1}{\sum_{j=1}^l \frac{f_L(S_{\text{a.L}, j}) \cdot \Delta S_{\text{a.L}, j}}{N_{f,j}(S_{\text{a.L}, j})}}.$$

PDF $f_L^*$ is a discrete version of the PDF $f_L$. The left-hand side of Equation 9 is valid if the PDF of the counted loading cycles is a continuous function and the right-hand side is valid if it is a discrete function. With Equation 9, 1 point $(N_{\text{tot}, j}, S_{\text{a.L, max}, j})$ in the Gassner durability curve is obtained. If this procedure is applied for different maximum stresses of the loading spectra {\{S_{\text{a.L}, \text{max}}, j\} = 1, ..., m}, the complete Gassner curve is represented. Hence, for different random loading spectra and the same material's S-N curve, different Gassner curves can be obtained in this manner.

### 2.2 Analytical Gassner curve—hyperbola equation

When comparing the material's S-N curves and the corresponding Gassner durability curves on a log-log scale, it is clear that the Gassner curve has the same shape trend as the material's S-N curve, but it is shifted horizontally to the right by a certain number of loading-cycles-to-failure $N$. This holds even in the case where a modified Palmgren-Miner rule (see Haibach\(^1\)\(^6\) for details) is applied for calculating the Gassner durability curve. However, the trend of the Gassner curve surrounding the area of the knee point (ie, the region where the extensions of the high-cycle and endurance-cycle fatigue lines in the log-log scale intersect) is not as straight as it is in the material's S-N curve. Instead, an arc radius is present in the transition area. This implies that a hyperbolic trend could be fitted to the Gassner durability curve. This hyperbola has, on a log-log scale, the 2 straight-line asymptotes, the slopes of which correspond to the high-cycle ($k$) and endurance-cycle ($2k - I$) fatigue domains of the material's S-N curve. Thus, by developing the general equation of the hyperbola on a log-log scale that is not centred on the origin and at the same time rotated (see Weisstein\(^2\), Hilbert and Cohn-Vossen\(^2\)), the model of the Gassner durability curve can be obtained by applying trigonometric relationships and relating it to the durability curve's properties—see Figure 1. The hyperbola parameters depend on the selected probability of fatigue failure $p$ and are calculated, as explained in the flow chart in Figure 2 and in the appendix.

Second, by analysing the influence of the loading spectrum, it was found that the lower the value of its characteristic parameters ($\beta_L$ and $\eta_{\text{L, norm}}$), the emptier the
rectangle of the NLS is, which means that the loading history is composed of loading cycles with more variable stress amplitudes (see Figure 3A). This affects the Gassner curve by shifting it more to the right and increasing its transition area, as shown in Figure 3B.

The parameters $\beta_L$ and $\eta_{L,\text{norm}}$ depend purely on the loading spectrum and were combined to quantify the percentage of fullness of the NLS rectangle (so the variability of the load-cycle amplitudes can be visually estimated) by creating the loading-spectrum factor (LSF). It was calculated by integrating the area under the curve of the $i$-th NLS, as follows:

$$LSF_i = \int_0^1 \left[ 1 - F_{L,i} \left( \frac{S_{\text{norm},a_L}}{S_{\text{norm},a_L,i}} \right) \right] dS_{\text{norm},a_L,i} \rightarrow LSF_i \quad (10)$$

We draw a $\beta_L$-LSF diagram for 40 points from $\beta_L = 2$ to $\beta = 600$, thus obtaining a sample set with an appropriate number of sample points, from which the LSF values corresponding to the $\beta_L$ parameter (or vice versa) can be extracted with a linear interpolation—see the example in Figure 9A.

To fit the equation of the hyperbola to the Gassner durability curve, we introduced 2 additional factors, which are explained in the flow chart in Figure 2: the right-shifting-scale factor (SSF) and the arc-curve factor (ACF) that represents the transition area of the Gassner durability curve. Examples of the diagrams from which they are extracted are presented in Figure 9B,C, respectively.

### 2.3 Scatter of the Gassner curve

To estimate the reliability, the scatter of the Gassner durability curve along the stress amplitudes $S_a$ should also be known. To do this, the scatter of the Gassner durability

![Flowchart for the composition of the analytical Gassner curve hyperbola](Image)
durability curve, it follows from Equations 5 and 9 that the Gassner durability curve for an arbitrary probability of fatigue failure \( p \) is just a linear transformation of the material’s S-N curve for the same probability of fatigue failure:

\[
N_{\text{tot},p} = \frac{1}{C_{\text{LS}}} 10^{a_1} \log S_a + \frac{\log(-\ln(1-p))}{\eta} + a_0 \tag{12}
\]

where \( C_{\text{LS}} \) is a loading-spectrum-dependent constant:

\[
C_{\text{LS}} = \int_0^\infty f(S_{a,L}) \cdot 10^{a_1} \log S_a \, dS_{a,L}. \tag{13}
\]

Because we applied the same functional form for modelling the scatter of the Gassner durability curve, but calculated it using the modified Palmgren-Miner’s rule, which accounts for the Haibach’s knee point, the objective is to demonstrate that the Weibull’s scale parameter \( \beta_S \) remains constant along the whole Gassner durability curve, even in this case. In any case, it should be equal to the value obtained with Equation 11.

The procedure to demonstrate this involved calculating the scatter for 33 values of the number-of-cycles-to-failure \( N \) for a family of Gassner curves that were obtained for the same loading-spectrum shape from the material’s S-N curves, representing different probabilities of fatigue failure. The \( \beta_S \) values for the Gassner curve family that corresponded to the selected values of the number-of-cycles-to-failure \( N \) were calculated by fitting the Weibull’s PDF to the corresponding PP plots. First, the Gassner curves for 11 probabilities of failure, ranging from \( P = 0.05 \) to \( P = 0.95 \) for the same loading spectrum, were calculated. Second, the stress-amplitude values \( S_{a,i} \) that correspond to each probability of failure \( p_i \) were calculated for each level of the number-of-cycles-to-failure \( N \). Third, the \((x,y)\) values for the PP plot for each level of the number-of-cycles-to-failure \( N \) were calculated as \((x, y) = [ \ln (S_{a,i}), \ln (-\ln (1 - p_i))] ; j = 1 \ldots n_F \). Thus, the slope of the linear regression line of the PP plot represents the parameter \( \beta_S \) and the intercept represents the quantity \(-\beta_S \ln (\eta(N))\)—see Equation 14b that follows from the Weibull cumulative density function \( F_S(S_a|N) \):

\[
F_S(S_a|N) = p_i = 1 - e^{-\left(\frac{S_a}{\xi(N)}\right)^{\beta_S}} \tag{14a}
\]

\[
\ln(-\ln(1-p_i)) = \beta_S \ln(S_{a,i}) - \beta_S \ln(\eta_S(N)) \tag{14b}
\]

An example of the procedure for 1 level of the number-of-cycles-to-failure \( N \) is shown in Figure 4, Figure 5, and Table 1. The 33 \( \beta_S \) values corresponding to different values of \( N \) are shown in Figure 11.

For the case of the material’s S-N curve, the maximum stress-amplitude \( S_{a,\text{max}} \) in the Gassner durability curve case is replaced by the constant amplitude stress \( S_a \).
## Calculation of reliability

The reliability of a structure can be estimated as the cross section of the PDF describing the scatter of the loading spectrum's maximum $F_{LS, \text{max}}(S_{a, \text{max}})$ and the scatter of the stress amplitude at a certain number of loading-cycles-to-failure from the Gassner curve that corresponds to the loading spectrum—see also Ebeling, O’Connor and Kleyner, Klemenc, and Haibach:

$$R(N) = \int_0^\infty F_{LS, \text{max}}(S_{a, \text{max}}) \cdot f_S(S_{a, \text{max}}|N) dS_{a, \text{max}} \quad (15a)$$

$$F_{LS, \text{max}}(S_{a, \text{max}}) = \Pr\{S_{a, L, \text{max}} < S_{a, \text{max}}\} = \int_0^{S_{a, \text{max}}} f_{LS, \text{max}}(S_{a, L, \text{max}}) dS_{a, L, \text{max}} \quad (15b)$$

where $R(N)$ is the reliability for a given number of cycles-to-failure $N$, $S_{a, L, \text{max}}$ is the maximum amplitude stress of the loading spectrum, $S_{a, \text{max}}$ is the maximum amplitude stress from the Gassner durability curve, $F_{LS, \text{max}}(S_{a, \text{max}})$ is the cumulative density function of the loading spectrum’s maximum amplitudes, $f_{LS, \text{max}}(S_{a, L, \text{max}})$ is the PDF of the loading spectrum’s maximum amplitudes, and $f_S(S_{a, \text{max}}|N)$ is the conditional PDF of the stress amplitude $S_{a, \text{max}}$ for the given number of cycles-to-failure $N$ in the Gassner durability curve.

### EXPERIMENTAL DATASET AND CASE-STUDY EXAMPLE

To apply the theory presented in Section 2, the fatigue-life data of 3 and 4-mm-thick steel plates, which were made from S420MC steel, were used. This fatigue-life dataset comprises $m = 79$ data points $(S_{a, i}; N_i; i = 1, ..., m = 79)$. Sixty-five specimens experienced fatigue failure and 14 survived 2 million loading cycles without fatigue failure. The data are presented in Figure 6 together with the S-N curves for 5%, 50%, and 95% probability of
rupture. The specimens were shaped according to the ASTM E606-92 standard.  

All the fatigue-life experiments were performed on a SCHENK PHQ-49 resonant pulsating machine. The S-N curves for the high-cycle fatigue domain were determined according to Equation 2—see Klemenc, Klemenc, and Fajdiga for details. Thus, for our material properties, the parameters of the material’s S-N curve are \( \{ a_0 = 27.1861, a_1 = -8.9725, \beta_N = 2.8980 \} \). At 5 million loading cycles, the Haibach’s knee point of the S-N curves was considered, as in the case of the EUROCODE 3-1.9 standard.  

This means that the slope of the S-N curves beyond 5 million loading cycles changes from the value of \( k \) in the high-cycle fatigue domain to the Haibach’s slope of \( (2k - 1) \).

For calculating the reliability as a function of 2 different loading spectra, the same example of a linear beam was applied as in the article of Klemenc—see Figure 7.

The maximum amplitude force is \( F_a = 1900 \) N, which results in approximately 420 Nm of maximum amplitude bending moment and 200 MPa of maximum amplitude normal bending stress \( S_{a. L. max} \) in a critical cross section of the beam. We decided (similar to the case in the article of Klemenc) to apply a Gaussian probability distribution to model the PDF of the loading spectrum’s maximum stress amplitude \( f_{LS, max}(S_{a. L. max}) \). The mean value of the Gaussian PDF was \( S_{a. L. max, avg} = 200 \) MPa in our case, and we choose a value of 20 MPa for its standard deviation \( \sigma \).

4 | RESULTS AND DISCUSSION

4.1 Analytical calculation and representation of a Gassner durability curve

The calculations of the 3 factors—LSF, SSF, and ACF—and the knee point are connected by their interdependency (see Figure 2) and were applied for calculating the hyperbolic model of the Gassner durability curve. To validate our approach, 2 different Gassner curves were determined that correspond to 2 loading spectra with \( LSF_1 = 0.35 \) and \( LSF_2 = 0.7 \) (see Figure 8) for the material properties of the S420MC steel (see Section 3) and a probability of failure \( p_{50} = 50\% \)—see Table 2. The resulting \( \beta \) of the 2 normalised loading spectra together with the SSF and ACF parameters were extracted from the trends presented in Figure 9. Therefore, the hyperbola parameters of the Gassner durability curve from Eq. (10) were calculated and are shown in Table 2.

In order to make a final assessment of the goodness-of-fit, the 2 Gassner-curve hyperbolas were plotted together with the corresponding numerically determined Gassner durability curves, which were calculated using the modified Palmgren-Miner rule—see Figure 10. It is clear from Figure 10 that the hyperbolic model of the Gassner durability curve fits almost perfectly to the

![FIGURE 7](image-url) Simple supported linear beam, loaded with a force at the end of the cantilever section.

![FIGURE 8](image-url) Normalised loading spectrum (NLS) for LSF = 0.35 and LSF = 0.7 [Colour figure can be viewed at wileyonlinelibrary.com]

![TABLE 2](image-url) Parameters of the Gassner-curve hyperbolic model for the 2 examples

| \( GC \) for LSF = 0.3 | \( GC \) for LSF = 0.7 |
|---|---|
| \( p_{50} \) | 50% | 50% |
| Material | Steel S420MC | Steel S420MC |
| LSF | 0.35 | 0.7 |
| \( \beta \) | 2.8133 | 8.8147 |
| \( \eta \) | 0.3932 | 0.7424 |
| SSF | 576.1837 | 15.1249 |
| ACF | 1.0530 | 1.0254 |
| \( A \) | 0.07769 | 0.0183 |
| \( B \) | 2.0137 | 0.4748 |
| \( C \) | 11.8127 | 2.7851 |
| \( D \) | -6.1510 | -1.3830 |
| \( E \) | -73.7041 | -16.5403 |
| \( F \) | 114.4330 | 24.4632 |
reference, numerically determined, Gassner curves, which can be generalised for any loading spectrum and material, the dynamic strength of which follows Equation 2.

### 4.2 Scatter of the Gassner curve

The scatter of the stress amplitudes was demonstrated to be constant, as shown in Figure 11. The corresponding data points can be found in Table 3. From these data points, it follows that the average value is \( \beta_{\text{avg}}^{S} = 26,008 \) and the true value obtained from Equation 11 is \( \beta_{S} = 26,002 \). The 2 values can be considered equal for the engineering accuracy, especially if we take into account the fact that the data points in Figure 11 and Table 3 were calculated using a linear interpolation from the family of Gassner durability curves. Based on this, a sensitivity analysis was carried out, which showed that the scatter of the data points around the mean value of the \( S \) parameter is reduced if the Gassner durability curves are sampled with a larger number of points. Then, because the trend and the scatter of the Gassner curve are known, we can proceed to calculate the reliability.

### 4.3 Reliability calculation

The reliability was calculated for the 2 cases of the loading spectra from Figure 8 in Section 4.1. First, the Gaussian
probability distribution from Eq. (18), which describes the scatter of the loading spectrum maximum stress amplitude \( S_{a,L,max} \), was calculated for a maximum stress amplitude average of \( S_{a,L,max,\text{avg}} = 200 \) MPa, and a standard deviation of \( \sigma = 10\% \cdot S_{a,L,max,\text{avg}} \). The range of the stress amplitude’s maximum \( S_{a,max} \) for a numerical calculation of the reliability from Equations 17a and 17b spans the interval 0 to 600 MPa with 3000 divisions having an interval of \( \Delta S_{a,max} = 0.2 \) MPa. Then, the 2 Weibull PDFs, which describe the scatter of the 2 Gassner durability curves, were calculated for the same stress-amplitude range as in the load PDF. The shape parameter \( \beta_S \) corresponds to the demonstrated value \( \beta_S = 26 \) 002. The scale parameter \( \eta_S(N) \) was obtained from the corresponding hyperbolic model of the Gassner durability curve at a probability of failure \( p = 63.2\% \) for the selected number of loading-cycles-to-failure \( N \). For the selected number of loading-cycles-to-failure \( N = 10 \) 000 000 and \( N_2 = 1 \) 220 000 000, respectively. Thus, the more variable

\[
\begin{array}{ccccccc}
N_r & \beta_S & \eta_S(N) & N_r & \beta_S & \eta_S(N) \\
4.0E + 04 & 26.12 & 430.03 & 6.0E + 08 & 25.67 & 183.82 \\
6.0E + 04 & 26.09 & 411.10 & 9.0E + 08 & 25.92 & 179.36 \\
9.0E + 04 & 26.08 & 392.98 & 1.0E + 09 & 25.78 & 178.25 \\
1.0E + 05 & 26.11 & 388.34 & 3.0E + 09 & 26.05 & 167.02 \\
3.0E + 05 & 25.77 & 343.70 & 6.0E + 09 & 26.07 & 160.28 \\
6.0E + 05 & 25.69 & 317.99 & 9.0E + 09 & 25.88 & 156.51 \\
9.0E + 05 & 26.07 & 303.86 & 1.0E + 10 & 25.99 & 155.58 \\
1.0E + 06 & 25.99 & 300.31 & 3.0E + 10 & 26.27 & 145.81 \\
3.0E + 06 & 26.00 & 266.77 & 6.0E + 10 & 26.02 & 140.01 \\
6.0E + 06 & 26.07 & 249.10 & 9.0E + 10 & 26.17 & 136.71 \\
9.0E + 06 & 25.99 & 240.33 & 1.0E + 11 & 26.07 & 135.87 \\
1.0E + 07 & 25.95 & 238.17 & 3.0E + 11 & 26.15 & 127.34 \\
3.0E + 07 & 25.97 & 219.86 & 6.0E + 11 & 26.03 & 122.25 \\
6.0E + 07 & 26.00 & 210.57 & 9.0E + 11 & 26.25 & 119.37 \\
9.0E + 07 & 26.11 & 205.50 & 1.0E + 12 & 26.18 & 118.62 \\
1.0E + 08 & 25.93 & 204.25 & 2.5E + 12 & 25.95 & 112.40 \\
3.0E + 08 & 25.87 & 191.42 &
\end{array}
\]

and \( 10^{14} \) loading-cycles-to-failure, the corresponding reliability curves, illustrated in Figure 13, were obtained.

It is clear from Figure 13 that the reliability curve corresponding to the lower loading-spectrum factor is shifted to the right-hand side in the same manner as the corresponding Gassner durability curve from which it originates. To compare the reliability of both cases, we set a reliability value \( R(N) = 0.95 \) that may delimit the useful life of the structure in each case and obtained the useful lives for the \( LSF_1 \) and \( LSF_2 \) of \( N_1 = 11 \) 100 000 and \( N_2 = 1 \) 220 000 000, respectively. Thus, the more variable

\[
\begin{array}{cc}
\text{FIGURE 12} & \text{PDFs of the load and the Gassner curve for a load amplitude of 200 MPa and a variable amplitude load at 10 000 000 loading-cycles-to-failure for the 2 case studies [Colour figure can be viewed at wileyonlinelibrary.com]} \\
\text{FIGURE 13} & \text{Reliability curves of a steel S420MC subjected to an average load amplitude of 200 MPa and the 2 variable-amplitude loading spectra case studies}
\end{array}
\]
are the load-cycle amplitudes in the loading spectrum (ie, the lower the LSF value) for the same maximum amplitude stress, the same material applied, and the same geometric detail, the longer is the useful life of the structure.

5 CONCLUSION

In this article, a model for predicting the reliability of a structure subjected to a variable-amplitude stress loading is presented. After an integral study of the Gassner durability curve’s properties, it was found that its trend resembles a hyperbola on a log-log scale for the given generic shape of the loading spectrum, which cannot be found in the references from this field. For this reason, the hyperbola equation and its general applicability shape and trigonometric relationships were developed and related to the properties of the durability curve.

In order to fit the hyperbolic function to the Gassner durability curve, the influencing factors were considered and assessed: the loading spectrum that defines the variability of the load and the material’s S-N curve. The equations were derived to account for: (1) a right shifting of the durability curve with a respect to the material’s S-N curve and (2) a variable transition area with an arc radius around the knee point of the durability curve. These properties were related to the general equation of the hyperbola on a log-log scale. In order to calculate its parameters, 2 factors that regulate the variability of the Gassner durability curve’s trend were defined anew, ie the loading-spectrum factor (LSF), and the arc-curve factor (ACF), while the shifting-scale factor (SSF) was defined similar to the existing literature.22 By connecting all 3 factors together, the hyperbolic model of the Gassner durability curve on a log-log scale was obtained. After the trend of the Gassner durability curve is estimated, its scatter along the stress amplitudes Sa needs to be calculated. The modelling of the S-N curve’s scatter with a conditional 2-parametric Weibull’s PDF was extended to the case of Gassner durability curves. It was found out from the extensive numerical simulations that the scatter along the amplitude-stress axis remains constant regardless of the variability of the slope for both the material’s S-N curve and the Gassner durability curve. This finding is important, because it enables modelling of the PDF of the Gassner curve’s scatter in the direction of the stress-amplitude values without additional information. The newly proposed Gassner-curve model was compared with the numerically calculated Gassner durability curves. For 2 different loading spectra, the parameters of the corresponding Gassner curves and their scatter were calculated. It was found that the modelled and the numerically calculated durability curves fit almost perfectly.

Furthermore, the reliability curve for a simple structure was determined on the basis of the proposed Gassner-curve model. Individual points of the reliability curve were calculated from the cross section of the PDF related to the loading spectrum’s maximum stress and the PDF related to the scatter of the Gassner durability curve for the selected number of loading-cycles-to-failure. By repeating this procedure for different numbers of loading-cycles-to-failure, the complete reliability curves for the 2 loading spectra were obtained. By doing this, we presented that the fatigue reliability of the structure, which is subjected to the variable-amplitude loading, can be estimated very effectively even in the early phases of product design, if the loading spectrum, the material’s S-N curve, and the geometric detail are known.

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APPENDIX—DETERMINATION OF HYPERBOLA PARAMETERS

The Gassner durability curve can be modelled with a general hyperbola equation:

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \]  \hspace{1cm} (A.1)

Its parameters are related to the durability curve’s properties on Figure 1 using the following relations:

\[ A = 4(F_{2x} - F_{1x})^2 - 16a^2 \]  \hspace{1cm} (A.2a)

\[ B = 8(F_{2x} - F_{1x})(F_{2y} - F_{1y}) \]  \hspace{1cm} (A.2b)

\[ C = 4(F_{2y} - F_{1y})^2 - 16a^2 \]  \hspace{1cm} (A.2c)

\[ D = 4(F_{2x} - F_{1x})\left( \left( F_{1x} - F_{2x} + F_{1y} - F_{2y} \right)^2 - 4a^2 \right) \]  \hspace{1cm} (A.2d)

\[ + 32F_{2x}a^2 \]

\[ E = 4(F_{2y} - F_{1y})\left( \left( F_{1x} - F_{2x} + F_{1y} - F_{2y} \right)^2 - 4a^2 \right) \]  \hspace{1cm} (A.2e)

\[ + 32F_{2y}a^2 \]

\[ F = \left( \left( F_{1x} - F_{2x} + F_{1y} - F_{2y} \right)^2 - 4a^2 \right)^2 - 16a^2 F_{2x} - 16a^2 F_{2y} \]  \hspace{1cm} (A.2f)

where the coordinates of the 2 hyperbola focal points \( F_1 \) and \( F_2 \) as follows:

\[ F_1(F_{1x}, F_{1y}) = (c \cdot \cos(\delta) + X_{KP}^{GC}, c \cdot \sin(\delta) + Y_{KP}^{GC}) \]  \hspace{1cm} (A.3)

\[ F_2(F_{2x}, F_{2y}) = (X_{KP}^{GC} - c \cdot \cos(\delta), Y_{KP}^{GC} - c \cdot \sin(\delta)) \]  \hspace{1cm} (A.4)

\( X_{KP}^{GC} \) and \( Y_{KP}^{GC} \) are the coordinates of the virtual knee point \( O \) of the Gassner curve \( \left( N_{KP}^{GC} = X_{KP}^{GC}, S_{u,KP}^{GC} = Y_{KP}^{GC} \right) \) on a log-log scale that coincides with the origin of the hyperbola.

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