Strichartz and smoothing estimates for dispersive equations with magnetic potentials

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Abstract. We prove global smoothing and Strichartz estimates for the Schrödinger, wave, Klein-Gordon equations and for the massless and massive Dirac systems, perturbed with singular electromagnetic potentials. We impose a smallness condition on the magnetic part, while the electric part can be large. The decay and regularity assumptions on the coefficients are close to critical.

1. Introduction

Strichartz estimates have become a standard tool in the study of linear and nonlinear evolution equations. They are available for a large class of constant coefficient equations, by the methods of [17] and [25]. In a sense, they represent the modern energy estimates, and are especially effective for problems of low regularity and global existence for nonlinear equations.

Using the notations $L^p L^q = L^p (\mathbb{R}^n; L^q(\mathbb{R}^n))$, $\| f \| \lesssim \| g \|$ to mean $\| f \| \leq C \| g \|$, and $H^s_q$ and $\dot{H}^s_q$ to denote the spaces with norms $\| f \|_{H^s_q} = \| \langle D \rangle^s f \|_{L^q}$, $\| f \|_{\dot{H}^s_q} = \| |D|^s f \|_{L^q}$.

where $\langle D \rangle = (1 - \Delta)^{1/2}$, $|D| = (-\Delta)^{1/2}$, the Strichartz estimates for the Schrödinger equation take the following form: for $n \geq 2$,

$$\| e^{it\Delta} f \|_{L^p L^q} \lesssim \| f \|_{L^2},$$

provided the couple $(p, q)$ is Schrödinger admissible:

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad 2 \leq p \leq \infty, \quad \frac{2n}{n-2} \geq q \geq 2, \quad q \neq \infty.$$

The couple $(p, q) = (2, 2n/n - 2)$ is called the endpoint and is allowed when $n > 2$.

For the wave equation the estimates can be written as follows: for $n \geq 3$,

$$\| e^{it|D|} f \|_{L^p \dot{H}^{s+1/p - 1/2}_q} \lesssim \| f \|_{L^2},$$

provided the couple $(p, q)$ is wave admissible:

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \quad 2 \leq p \leq \infty, \quad \frac{2(n-1)}{n-3} \geq q \geq 2, \quad q \neq \infty.$$

The wave equation endpoint is $(p, q) = (2, 2(n-1)/(n-3))$ and is allowed in dimension $n > 3$.

Finally for the Klein-Gordon equation we have: for $n \geq 2$,

$$\| e^{it\sqrt{\cdot}} f \|_{L^p H^{s+1/p - 1/2}_q} \lesssim \| f \|_{L^2},$$

provided $(p, q)$ is Schrödinger admissible (see the Appendix for a proof of the last estimate, for which a reference is not immediately available).
We shall also be interested in the decay properties of the Dirac equation, which is a $4 \times 4$ constant coefficient system of the form

$$iu_t + Du = 0$$

in the massless case, and

$$iu_t + Du + \beta u = 0$$

in the massive case. Here $u : \mathbb{R} \times \mathbb{R}^3_x \to \mathbb{C}^4$, the operator $D$ is defined as

$$D = \frac{1}{i} \sum_{k=1}^{3} \alpha_k \partial_k$$

and the $4 \times 4$ Dirac matrices can be written

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad k = 1, 2, 3$$

in terms of the Pauli matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the solution $u(t, x) = e^{itD}f$ of the massless Dirac system with initial value $u(0, x) = f(x)$ satisfies the Strichartz estimate:

$$\|e^{itD}f\|_{L^p_x \dot{H}^{\frac{1}{2} + \frac{n}{2}}_t} \lesssim \|f\|_{L^2_x}, \quad n = 3,$$

for all wave admissible $(p, q)$, while in the massive case we have

$$\|e^{it(D+\beta)}f\|_{L^p_x \dot{H}^{\frac{1}{2} + \frac{n}{2}}_t} \lesssim \|f\|_{L^2_x}, \quad n = 3,$$

for all Schrödinger admissible $(p, q)$ (see the Appendix for more details).

In view of the applications, it is an important problem to extend Strichartz estimates to more general equations with variable coefficients, possibly of low regularity in order to retain the advantages over classical energy methods. Indeed, in recent years a large number of works have investigated this kind of problem. In the case of potential perturbations like

$$iu_t - \Delta u + V(x)u = 0, \quad \Box u + V(x)u = 0,$$

Strichartz estimates are now fairly well understood. We mention among the many works [7], [19], [18], [29], [31] and the survey [30] for the Schrödinger equation, and [9], [15], [13] for the wave equation. We also mention the wave operator approach of Yajima ([36], [37], [38], [2]), which was recently optimized in dimension 1 in [11].

Results are much less complete in the case of first order perturbations i.e. magnetic potentials

$$iu_t + \Delta u + a \cdot \nabla u + bu = 0, \quad \Box u + a \cdot \nabla u + bu = 0.$$

Concerning Strichartz estimates for the Schrödinger equation with small potentials $a, b$ we recall at least the papers [33], [16]; in 3D the recent work [14] handles for the first time the case of large magnetic potentials. For the wave equation with small magnetic potentials, partial Strichartz estimates were obtained in 3D in [10] in the case of smooth, rapidly decaying coefficients. The dispersive estimate in 3D was proved in [12] for the magnetic wave equation with small singular potentials and for the massless Dirac system with a small singular matrix potential. We must also mention the papers [32], [28], [35] containing some local estimates in the fully variable coefficient case. Only in the one dimensional case the optimal dispersive estimates for the case of fully variable singular coefficients have been proved in [11].

A method of proof which is very efficient in the case of electric potentials was introduced in [29] and further developed in [7]. The main idea is to combine Strichartz
estimates for the free equation with Kato smoothing estimates for the perturbed equation. The same method is used in [14] for the 3D Schrödinger equation with a large magnetic potential.

Our goal here is to apply a suitable modification of this method in a systematic way to several equations perturbed with magnetic potentials: Schrödinger, wave and Klein-Gordon equations, and the Dirac system with and without mass.

Thus consider a magnetic Schrödinger operator

$$H = -(\nabla + iA(x))^2 + B(x),$$

which is selfadjoint under the following assumptions: $A_j$ and $B$ are real valued, and

$$\|B\|_{L^{n/2},\infty} < \infty, \quad \|B_\tau\|_{L^{n/2},\infty} < \delta, \quad \|A\|_{L^{n},\infty} < \delta$$

for some $\delta$ sufficiently small (see Lemma 2.2 below). Here $L^{p,\infty} = L^{p}_w$ denotes the Lorentz or weak Lebesgue space. However, in order to state our results, it is more convenient to represent the operator in the form

$$H = -\Delta + W(x, D) = -\Delta + a(x) \cdot \nabla + b(x)$$

and to make the abstract assumption that $H$ is selfadjoint. In view of (1.4), the following explicit conditions on $a, b$ are sufficient (but not necessary) for the selfadjointness of $H$:

$$a(x) \text{ is pure imaginary, } \exists b = -i \nabla \cdot a$$

and

$$\|\nabla a\|_{L^{n/2},\infty} + \|b\|_{L^{n/2},\infty} < \infty, \quad \|\Re b_\tau\|_{L^{n/2},\infty} < \delta, \quad \|a\|_{L^{n},\infty} < \delta$$

for a small enough $\delta$.

Our first result concerns smoothing estimates of Kato-Yajima type for the scalar Schrödinger, wave and Klein-Gordon equations. Besides being a necessary tool to prove the Strichartz estimates, they have also an independent interest (see e.g. [3], [23], [24]). Notice in particular that we allow a singularity at 0 in the coefficient, and that the electric potential can be large, while the magnetic term must satisfy a smallness condition. We shall use the following weight functions:

$$\tau_{\epsilon}(x) = \begin{cases} |x|^{\frac{1}{2} - \epsilon} + |x| & \text{if } n \geq 3, \\ |x|^{\frac{1}{2} - \epsilon} + |x|^{1+\epsilon} & \text{if } n = 2 \end{cases}$$

and

$$w_\sigma(x) = |x|(1 + |\log |x||)^{\sigma}, \quad \sigma > 1.$$ 

Then we have:

**Proposition 1.1** (Smoothing estimates for scalar equations). Let $n \geq 2$. Assume the operator

$$-\Delta + W(x, D) = -\Delta + a(x) \cdot \nabla + b_1(x) + b_2(x)$$

is selfadjoint with

$$|a(x)| \leq \frac{\delta}{\tau_{\epsilon}^2 |x|^{n/2}}, \quad |b_1(x)| \leq \frac{\delta}{\tau_{\epsilon}^2}, \quad 0 \leq b_2(x) \leq C \frac{\tau_{\epsilon}}{\tau_{\epsilon}^2}$$

for some $\delta, \epsilon > 0$ sufficiently small and some $\sigma > 1, C > 0$. Moreover assume that 0 is not a resonance for $-\Delta + b_2$.

Then the following smoothing estimates hold: for the Schrödinger equation

$$\|\tau_{\epsilon}^{-1}e^{it(-\Delta + W)}f\|_{L^2} \lesssim \|f\|_{L^2}$$

while for the wave and Klein-Gordon equations

$$\|\tau_{\epsilon}^{-1}e^{it\sqrt{-\Delta + W}}f\|_{L^2} \lesssim \|f\|_{L^2}$$

$$\|\tau_{\epsilon}^{-1}e^{it\sqrt{-\Delta + W}}f\|_{L^2} \lesssim \|f\|_{L^2}.$$
The assumption that $0$ is not a resonance for $-\Delta + b_2(x)$ here means: if $(-\Delta + b_2)f = 0$ and $\langle x \rangle^{-1} f \in L^2$ then $f \equiv 0$.

We can then prove Strichartz estimates for the perturbed scalar equations as a consequence of the above smoothing properties. Notice that we must require some additional regularity on the magnetic coefficient $a(x)$. Moreover, the use of the Christ-Kiselev lemma (see Section 3 for details) prevents us from reaching the endpoint.

**Theorem 1.2** (Strichartz for Schrödinger). Let $n \geq 2$, $-\Delta + W$ be as in Proposition 1.1 and assume in addition that
\begin{equation}
(1.9) \quad \langle x \rangle^{1+3\epsilon} \chi \langle x \rangle a_j(x) \in C^{4+2\epsilon}
\end{equation}
for some function $\chi \gtrsim w_\sigma^{1/2}$.

Then, for any non-endpoint Schrödinger admissible couple $(p, q)$, the following Strichartz estimate holds:
\begin{equation}
(1.10) \quad \| e^{it(-\Delta+W)} f \|_{L^p L^q} \lesssim \| f \|_{L^2}.
\end{equation}

**Theorem 1.3** (Strichartz for wave). Let $n \geq 3$, $-\Delta + W$ be as in Proposition 1.1 and assume in addition that
\begin{equation}
(1.11) \quad |a(x)| \leq \frac{C}{\tau_\epsilon^2}, \quad |b_1 + b_2 - \nabla \cdot a| \leq \frac{C}{|x| \tau_\epsilon}.
\end{equation}

Then, for any non-endpoint wave admissible couple $(p, q)$ the following Strichartz estimate holds:
\begin{equation}
(1.12) \quad \| e^{it\sqrt{-\Delta+1+W}} f \|_{L^p H^\frac{1}{2} + \frac{1}{q} \cdot \frac{1}{2} \cdot \frac{1}{q}} \lesssim \| f \|_{L^2}.
\end{equation}

**Theorem 1.4** (Strichartz for Klein-Gordon). Let $n \geq 2$, $-\Delta + W$ be as in Proposition 1.1 and assume in addition that
\begin{equation}
(1.13) \quad |a(x)| \leq \frac{C}{\tau_\epsilon^2}, \quad |b_1 + b_2 - \nabla \cdot a| \leq \frac{C}{\langle x \rangle \tau_\epsilon}.
\end{equation}

Then, for any non-endpoint Schrödinger admissible couple $(p, q)$, the following Strichartz estimate holds:
\begin{equation}
(1.14) \quad \| e^{it\sqrt{-\Delta+1+W}} \|_{L^p H^\frac{1}{2} + \frac{1}{q} \cdot \frac{1}{2} \cdot \frac{1}{q}} \leq C \| f \|_{L^2}.
\end{equation}

Our final results concern the Dirac system:

**Theorem 1.5** (Massless Dirac). Let $n = 3$, and let $V(x) = V(x)^*$ be a $4 \times 4$ complex valued matrix such that
\begin{equation}
(1.15) \quad |V(x)| \leq \frac{\delta}{w_\sigma(x)}
\end{equation}
for some $\delta$ sufficiently small and some $\sigma > 1$. Then the following smoothing estimate holds:
\begin{equation}
(1.16) \quad \| w_\sigma^{-1/2} e^{it(D+V)} f \|_{L^2 L^2} \lesssim \| f \|_{L^2}
\end{equation}
and, for any non-endpoint wave admissible couple $(p, q)$, the following Strichartz estimate holds:
\begin{equation}
(1.17) \quad \| e^{it(D+V)} f \|_{L^p H^\frac{1}{2} + \frac{1}{q} \cdot \frac{1}{2} \cdot \frac{1}{q}} \lesssim \| f \|_{L^2}.
\end{equation}

**Theorem 1.6** (Massive Dirac). Let $n = 3$, and let $V(x) = V(x)^*$ be a $4 \times 4$ complex valued matrix such that
\begin{equation}
(1.18) \quad |V(x)| \leq \frac{\delta}{\tau_\epsilon(x)}
\end{equation}
for some $\delta, \epsilon > 0$ sufficiently small. Then the following smoothing estimate holds:
\begin{equation}
(1.19) \quad \| \tau_\epsilon^{-1} e^{it(D+\beta+V)} f \|_{L^2 L^2} \lesssim \| f \|_{L^2}
\end{equation}
and, for any non-endpoint Schrödinger admissible couple \((p, q)\), the following Strichartz estimate holds:

\[
\|e^{it(D + \beta + V)}f\|_{L^p_t L^\frac{2p}{p-1}\times R^d} \lesssim \|f\|_{L^2}.
\]

The paper is organized as follows: in Section 2 we prove resolvent estimates for the perturbed operator, which are equivalent to smoothing estimates for the corresponding flow via Kato theory, while Section 3 is devoted to the proof of the main theorems. A short Appendix collects the estimates for the free Klein-Gordon and Dirac equations; these can be obtained by a standard application of the Ginibre-Velo and Keel-Tao methods, and we decided to include a sketch of the proof for the sake of completeness.

2. Resolvent Estimates

In this section we shall prove the basic resolvent estimates for the perturbed operators, which are the crucial step in the proof. As an immediate consequence we shall obtain smoothing estimates for the corresponding evolution operators, by a standard application of the well-known result of Kato (see [27]):

**Theorem 2.1** (Kato smoothing Theorem, [23]). Let \(X, Y\) be Hilbert spaces, let \(H : X \rightarrow X\) be a self-adjoint operator whose resolvent we denote by \(R(\lambda) = (H - \lambda)^{-1}\), and let \(A : X \rightarrow Y\) be a closed, densely defined operator, which may be unbounded. Assume that

\[
\|AR(\lambda)A^*g\|_Y \leq M\|g\|_Y \quad \forall g \in D(A^*), \; \lambda \notin \mathbb{R}.
\]

Then the operator \(A\) is \(H\)-smooth, i.e., \(e^{itH}f \in D(A)\) for all \(f \in X\) and a.e. \(t\), and

\[
\int_{-\infty}^{\infty} \|Ae^{-itH}f\|_Y^2\, dt \leq \frac{2}{\pi} M^2 \|f\|_X^2 \quad \forall f \in X.
\]

2.1. The magnetic Schrödinger operator. The following lemma gives sufficient conditions for the magnetic Schrödinger operator \(H = -(\nabla + iA(x))^2 + B(x)\) to be selfadjoint. We sketch a proof since the assumptions on the coefficients are not completely standard:

**Lemma 2.2.** Let \(A_j(x), \; A = (A_1, \ldots, A_n)\) and \(B(x)\) be real valued functions satisfying

\[
\|B_+\|_{L^{n/2, \infty}} < C, \quad \|B_-\|_{L^{n/2, \infty}} < \delta, \quad \|A\|_{L^{n, \infty}} < \delta
\]

for some \(C, \delta > 0\). Then, if \(\delta\) is sufficiently small, the operator

\[
H = -(\nabla + iA(x))^2 + B(x)
\]

can be uniquely defined as a selfadjoint nonnegative operator in \(L^2\), with form domain \(H^1(\mathbb{R}^n)\). Moreover we have

\[
\|H^{1/2}g\|_{L^2} \simeq \|g\|_{H^1}.
\]

**Proof.** The quadratic form

\[
g(\phi, \psi) = \langle (\nabla + iA(x))\phi, (\nabla + iA(x))\psi \rangle_{L^2} + \langle B(x)\phi, \psi \rangle_{L^2}
\]

is well defined on \(H^1 \times H^1\) under assumptions (2.3). Indeed, using the embedding \(H^1 \subset L^{2n/(n-2), 2}\), Hölder’s inequality in Lorentz spaces [26] and assumptions (2.3), we have

\[
|g(\varphi, \psi)| \leq \|\nabla \varphi\|_{L^2}^2 + 2\|A\|_{L^{n, \infty}}\|\nabla \varphi \cdot \nabla \psi\|_{L^{\frac{n}{n-2}, 1}} + \|A\|_{L^{\frac{n}{n-2}, \infty}} \|\varphi^2\|_{L^{\frac{2n}{n-2}, 1}} \lesssim \|\nabla \varphi\|_{L^2}^2.
\]

The form \(g\) is symmetric since \(A\) and \(B\) are real valued. By standard results (see e.g. [27], Theorem VIII.15), \(g\) is the form associated to a unique defined self-adjoint
operator provided the form is closed, i.e. its domain $H^1(\mathbb{R}^n)$ is complete under the norm
\begin{equation}
\|\varphi\|^2 = q(\varphi, \varphi) + C\|\varphi\|_{L^2}^2,
\end{equation}
for some $C > 0$, and it is semibounded, i.e.
\begin{equation}
q(\varphi, \varphi) \geq -C\|\varphi\|_{L^2}^2,
\end{equation}
for some $C > 0$. To prove this we estimate the form from below as follows
\begin{align*}
q(\varphi, \varphi) &= \|\nabla\varphi\|_{L^2}^2 + 2\Im(A + \nabla \varphi, \varphi) + ((|A|^2 + B_+ + B_-)\varphi, \varphi)_{L^2} - (B_-, \varphi)_{L^2} \\
&\geq \|\nabla\varphi\|_{L^2}^2 + 2\Im(A + \nabla \varphi, \varphi)_{L^2} - (B_-, \varphi)_{L^2}.
\end{align*}
Proceeding as for the upper bound we obtain
\begin{equation}
q(\varphi, \varphi) \geq \|\nabla\varphi\|_{L^2}^2 - C\|\nabla\varphi\|_{L^2}^2 \geq \|\nabla\varphi\|_{L^2}^2
\end{equation}
for $\delta$ small enough. This proves the semiboundedness of the form and \eqref{eq:2.15}, which implies that the norm \eqref{eq:2.6} is equivalent to the norm of $H^1$ and hence the form is closed.

We now investigate in some detail the properties of the resolvent operators
\begin{equation}
R(z) = (-\Delta + W - z)^{-1}
\end{equation}
\begin{align*}
R_0(z) &= (-\Delta - z)^{-1}, \\
R_{b_1}(z) &= (-\Delta + b_2(x) - z)^{-1}.
\end{align*}
The following weight functions will appear in our resolvent estimates ($\epsilon > 0, \sigma > 1$):
\begin{equation}
\langle x \rangle = (1 + |x|^2)^{\frac{\sigma}{2}}, \quad w_\sigma(x) = |x|(1 + |\log |x||)^\epsilon,
\end{equation}
and
\begin{equation}
\tau_\epsilon(x) = \begin{cases}
|x|^{\frac{1}{2} - \epsilon} + |x| & \text{if } n \geq 3, \\
|x|^{\frac{1}{2} - \epsilon} + |x|^{1+\epsilon} & \text{if } n = 2.
\end{cases}
\end{equation}
Notice that
\begin{align*}
|x| &\leq \tau_\epsilon(x), \\
w_\sigma(\epsilon) &\leq C\tau_\epsilon(x),
\end{align*}
and
\begin{align*}
\tau_\epsilon(x) &\leq C\langle x \rangle, \text{ for } n \geq 3, \\
\tau_\epsilon(x) &\leq C\langle x \rangle^{1+\epsilon}, \text{ for } n = 2
\end{align*}
for some constant $C = C(\epsilon, \sigma)$.

In order to estimate the resolvent $R$ we shall use the formal identity
\begin{equation}
R = R_0(I + bR_0)^{-1}(I + (a \cdot \nabla + b_1)R_{b_2})^{-1}.
\end{equation}
Our first goal will be to prove that the operators $(I + bR_0)^{-1}$ and $(I + (a \cdot \nabla + b_1)R_{b_2})^{-1}$ are well defined and uniformly bounded in suitable weighted $L^2$ spaces. In the following lemma, the assumption that 0 is not a resonance of $-\Delta + b(x)$ means that the only distribution solution $f$ of the equation $-\Delta f + bf = 0$ belonging to $L^2((x)^{-2}dx)$ is $f \equiv 0$.

**Lemma 2.3.** Let $b(x)$ be real valued and such that, for some $\epsilon, \delta > 0$ small enough (recall \eqref{eq:2.11}),
\begin{equation}
\|\tau_\epsilon^2 b_+\|_{L^\infty} < \infty, \quad \|\tau_\epsilon^2 b_-\|_{L^\infty} < \delta.
\end{equation}
Assume that 0 is not a resonance for $-\Delta + b(x)$. Then $I + bR_0(z)$ is invertible with a uniformly bounded inverse on $L^2(\tau_\epsilon^2dx)$:
\begin{equation}
\|\tau_\epsilon(I + bR_0(z))^{-1}f\|_{L^2} \leq C\|\tau_\epsilon f\|_{L^2}.
\end{equation}
Proof. We recall the following estimates for the free resolvent \( R_0 \): fix any \( \sigma > 1 \), then for all \( z \in \mathbb{C} \)

\[
\| w_\sigma^{-\frac{1}{2}} R_0(z) f \|_{L^2} \leq \frac{C}{|z|^{n/2}} \| w_\sigma^{-\frac{1}{2}} f \|_{L^2},
\]

\[
\| w_\sigma^{-\frac{1}{2}} \nabla R_0(z) f \|_{L^2} \leq C \| w_\sigma^{-\frac{1}{2}} f \|_{L^2},
\]

\[
\| |x|^{-1} R_0 f \|_{L^2} \leq C \| x f \|_{L^2}, \quad n \geq 3
\]

\[
\| |x|^{-1+\epsilon} |D|^\epsilon R_0 f \|_{L^2} \leq C \| |x|^{-\epsilon} |D|^{-\epsilon} f \|_{L^2}, \quad n = 2 \quad (0 < \epsilon < 1/2)
\]

(see \[5\], \[12\] for \[(2.15), (2.16)\], and \[24\] for \[(2.17)-(2.18)\]). As usual, for \( \lambda \in \mathbb{R}^+ \) the resolvent \( R_0(z) \) must be replaced with the limit operators \( R_0(\lambda \pm i0) \). By the elementary inequalities \( |x| \leq \tau_\epsilon(x) \), \( w_\sigma^{-\frac{1}{2}}(x) \leq C \tau_\epsilon(x) \), we can condense the estimates \[(2.15) \text{ and } (2.17)\] in the following (weaker) one for \( n \geq 3 \):

\[
\| \tau_\epsilon^{-1} R_0(z) f \|_{L^2} \leq \frac{C}{\sqrt{|z|}} \| \tau_\epsilon f \|_{L^2}, \quad \text{for all } z \in \mathbb{C}.
\]

In dimension \( n = 2 \) we deduce by duality from \[(2.18)\] the following

\[
\| |D|^\epsilon |x|^{-1+\epsilon} R_0 f \|_{L^2} \leq C \| |D|^{-\epsilon} |x|^{-\epsilon} f \|_{L^2},
\]

which implies, via Sobolev embedding and Hölder inequality,

\[
\| \langle x \rangle^{-\epsilon} |x|^{-1+\epsilon} R_0 f \|_{L^2} \leq C \| \langle x \rangle^\epsilon |x|^{-\epsilon} f \|_{L^2}, \quad \sigma > \epsilon
\]

and hence \[(2.19)\] follows also for \( n = 2 \) (recall \[(2.11)\]).

Now, using assumption \[(2.13)\], we have

\[
\| \tau_\epsilon b R_0(z) f \|_{L^2} \leq \| \tau_\epsilon^2 b \|_{L^\infty} \| \tau_\epsilon^{-1} R_0(z) f \|_{L^2} \leq \frac{C}{\sqrt{|z|}} \| \tau_\epsilon^2 b \|_{L^\infty} \| \tau_\epsilon f \|_{L^2},
\]

with \( C \) as in \[(2.19)\]; hence, if \( z \) is sufficiently large, namely so large that

\[
\langle z \rangle > C^2 \| \tau_\epsilon^2 b \|_{L^\infty}^2,
\]

we can invert the operator \( I + b R_0 \) by a Neumann series in the weighted space \( L^2(\tau_\epsilon^2 dx) \), with a uniform bound on the norm of the inverse.

In the low frequency case

\[
\langle z \rangle \leq C^2 \| \tau_\epsilon^2 b \|_{L^\infty}^2,
\]

the family of operators \( (I + b R_0)(z) \) is uniformly bounded in \( L^2(\tau_\epsilon^2 dx) \) by \[(2.20)\]. We also notice that \( b R_0 \) is a compact operator on \( L^2(\tau_\epsilon^2 dx) \); indeed, \( R_0 \) is a compact operator from \( L^2(\tau_\epsilon^2 dx) \) to \( L^2(\tau_\epsilon^{-2} dx) \) (see \[(2.15), (2.16)\]), while multiplication by \( b \) is bounded from \( L^2(\tau_\epsilon^{-2} dx) \) to \( L^2(\tau_\epsilon^2 dx) \). Thus by standard analytic Fredholm theory we can invert \( I + b R_0(z) \) uniformly in \( z \), provided \( I + b R_0(z) \) is injective on \( L^2(\tau_\epsilon^2 dx) \) for each fixed \( z \). This is obvious for \( z \) outside \( \mathbb{R}^+ \), since by our assumptions the operator \( -\Delta + b \) is nonnegative and selfadjoint, and is true by assumption for \( z = 0 \), hence we need only check the case \( z = \lambda > 0 \).

Thus let \( \lambda \geq 0 \) and \( f \in L^2(\tau_\epsilon^2 dx) \) such that \( f + b(x) R_0(\lambda + i0) f = 0 \) (the \(-i0\) case is identical). We notice that estimate \[(2.16)\] implies that \( R_0(\lambda) f \in H^1_{loc} \) and hence in particular \( R_0(\lambda) f \) is in \( L^{2n/(n-1)} \) locally. Since \( |b| \lesssim \tau_\epsilon^{-2} \) which is locally in \( L^n \), we conclude that \( f = -b R_0(\lambda) f \) is locally in \( L^2 \). Recalling that \( f \in L^2(\tau_\epsilon^2 dx) \) this implies \( f \in L^2(\langle x \rangle^2 dx) \). Thus we are in the framework of the standard Agmon theory and we deduce that \( \lambda \) is an eigenvalue of \( -\Delta + b(x) \); but this is excluded under our assumptions on \( b \), for instance by the results of \[22\] (Theorem 2.1).

In conclusion, we can invert \( (I + b R_0)(z) \) in \( L^2(\tau_\epsilon^2 dx) \) with an uniform bound for the inverse \((I + b R_0)^{-1}\), and this completes the proof.
The preceding lemma allows us to construct the resolvent operator
\[(2.22) \quad R_0(z) = R_0(z)(I + bR_0(z))^{-1},\]
which, in view of \((2.14)\) and \((2.19)\), is a bounded operator from \(L^2(\tau_\varepsilon dx)\) to \(L^2(\tau_\varepsilon^{-2}dx)\) for all \(z \in \mathbb{C}\).

We have next:

**Lemma 2.4.** Consider the operator \(-\Delta + a(x) \cdot \nabla + b_1(x) + b_2(x)\) under the following assumptions: the operator is selfadjoint, \(b_2\) is real valued and nonnegative, and for some \(\delta, \epsilon > 0\) small enough, \(\sigma > 1\),
\[(2.23) \quad \|\tau_\varepsilon \cdot a \|_{L^\infty} + \|\tau_\varepsilon^2 b_1\|_{L^\infty} < \delta, \quad \|\tau_\varepsilon^2 b_2\|_{L^\infty} < \infty.\]
Moreover assume that \(0\) is not a resonance for \(-\Delta + b_2(x)\). Then \(I + (a \cdot \nabla + b_1)R_{b_2}\) is invertible with a bounded inverse on \(L^2(\tau_\varepsilon^2 dx)\):
\[(2.24) \quad \|\tau_\varepsilon(I + (a \cdot \nabla + b_1)R_{b_2})^{-1}f\|_{L^2} \leq C\|\tau_\varepsilon f\|_{L^2}.
\]

**Proof.** Using assumptions \((2.23)\), Hölder inequality and estimate \((2.16)\), we can write
\[
\|\tau_\varepsilon a \cdot \nabla R_{b_2}f\|_{L^2} \leq \|\tau_\varepsilon a \cdot \nabla R_0(I + b_2 R_0)^{-1}f\|_{L^2} \leq \|\tau_\varepsilon \cdot a\|_{L^\infty} \|\nabla R_0(I + b_2 R_0)^{-1}f\|_{L^2} \leq \delta \cdot \|\tau_\varepsilon (I + b_2 R_0)^{-1}f\|_{L^2} \leq \delta \cdot \|\tau_\varepsilon f\|_{L^2},
\]
and Lemma \((2.23)\) gives finally
\[
\|\tau_\varepsilon a \cdot \nabla R_{b_2}f\|_{L^2} \leq \delta \cdot \|\tau_\varepsilon f\|_{L^2}.
\]

On the other hand, by \((2.23)\) and estimate \((2.19)\)
\[
\|\tau_\varepsilon b_1 R_{b_2}f\|_{L^2} \leq \|\tau_\varepsilon^2 b_1\|_{L^\infty} \|\tau_\varepsilon^{-1} R_0(I + b_2 R_0)^{-1}f\|_{L^2} \leq \delta \cdot \|\tau_\varepsilon (I + b_2 R_0)^{-1}f\|_{L^2} \leq \delta \cdot \|\tau_\varepsilon f\|_{L^2}.
\]

Thus, if \(\delta\) is sufficiently small, we can invert \(I + (a \cdot \nabla + b_1)R_{b_2}\) via a Neumann series, and we obtain \((2.24)\).
\[\square\]

We collect and complete the above estimates in the following

**Proposition 2.5.** Consider the operator \(-\Delta + W(x, D) \equiv -\Delta + a(x) \cdot \nabla + b_1(x) + b_2(x)\) under the assumptions: the operator is selfadjoint, \(b_2\) is real valued and nonnegative, and for some \(\delta, \epsilon > 0\) small enough, \(\sigma > 1\),
\[(2.25) \quad \|\tau_\varepsilon \cdot a \|_{L^\infty} + \|\tau_\varepsilon^2 b_1\|_{L^\infty} < \delta, \quad \|\tau_\varepsilon^2 b_2\|_{L^\infty} < \infty.\]
Moreover assume that \(0\) is not a resonance for \(-\Delta + b_2(x)\). Then the resolvent operator \(R(z) = (\tau_\varepsilon^2 + W - z)^{-1}\) satisfies the following estimates for all \(z \in \mathbb{C}\):
\[(2.26) \quad \|\tau_\varepsilon^{-1} R(z)f\|_{L^2} \leq \frac{C}{\sqrt{(z)}} \|\tau_\varepsilon f\|_{L^2},
\]
\[(2.27) \quad \|\tau_\varepsilon^{-1} \nabla R(z)f\|_{L^2} \leq C\|\tau_\varepsilon f\|_{L^2}.
\]
and
\[(2.28) \quad \|(x)^{-1} R(z)f\|_{H^1} \leq C\|\tau_\varepsilon f\|_{L^2}, \quad n \geq 3;
\]
replace the weights \( \langle x \rangle^{-1} \), \( \langle x \rangle \) by \( \langle x \rangle^{-1-\epsilon}, \langle x \rangle^{1+\epsilon} \) respectively in dimension 2. As a consequence, the Schrödinger flow \( e^{it(-\Delta+ W)} \) has the smoothing property
\[
\| \tau_\epsilon^{-1} e^{it(-\Delta+ W)} f \|_{L^2 L^2} + \| \tau_\epsilon^{-1} D^{1/2} e^{it(-\Delta+ W)} f \|_{L^2 L^2} \leq C \| f \|_{L^2}.
\]

Remark 2.1. For the following applications it will be convenient to rewrite the (second) smoothing estimate above in the equivalent form
\[
\| \tau_\epsilon^{-1} \nabla D^{1/2} e^{it(-\Delta+ W)} f \|_{L^2 L^2} \leq C \| f \|_{L^2}.
\]

This follows immediately from the fact that \( \partial_j |D|^{-1/2} = i R_j |D|^{1/2} \), where \( R_j \) is the \( j \)-th Riesz operator, and on the other hand \( \tau_\epsilon^{-1} \) is an \( A_2 \) weight, as proved in Lemma 2.6 below.

**Proof.** Estimates (2.26) and (2.27) are immediate consequences of (2.12), (2.16) and of Lemmas 2.3, 2.4. Moreover, (2.26) implies in particular
\[
\| \tau_\epsilon^{-1} R(z)f \|_{L^2} \leq C \| f \|_{L^2},
\]
and the Kato smoothing theorem with the choices \( A = \tau_\epsilon^{-1}, \ X = Y = L^2 \) gives the first estimate in (2.29).

To prove (2.28), write
\[
\| \langle x \rangle^{-1} Rf \|_{H^1} \lesssim \| \langle x \rangle^{-1} Rf \|_{L^2} + \| \langle x \rangle^{-1} Rf \|_{L^2} + \| \langle x \rangle^{-1} \nabla Rf \|_{L^2}.
\]

The first two terms can be estimated by (2.24)
\[
\| \langle x \rangle^{-1} Rf \|_{L^2} + \| \langle x \rangle^{-1} Rf \|_{L^2} \leq C \| \tau_\epsilon^{-1} Rf \|_{L^2} \leq C \| \tau_\epsilon f \|_{L^2} \leq C \| \langle x \rangle f \|_{L^2},
\]
while the third term is bounded using (2.27):
\[
\| \langle x \rangle^{-1} \nabla Rf \|_{L^2} \leq \| \tau_\epsilon^{-1} Rf \|_{L^2} \leq C \| \tau_\epsilon f \|_{L^2} \leq C \| \langle x \rangle f \|_{L^2}
\]
and this proves (2.28).

Now write (2.28) in the equivalent forms
\[
\| \langle D \rangle \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} f \|_{L^2} \leq C \| f \|_{L^2}
\]
and, by duality,
\[
\| \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} \langle D \rangle f \|_{L^2} \leq C \| f \|_{L^2}.
\]
The last two estimates state that the operator \( \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} \) is bounded, uniformly in \( z \in \mathbb{C} \), from \( L^2 \) to \( H^1 \) and from \( H^{-1} \) to \( L^2 \). By complex interpolation this implies that it is also bounded from \( H^{-1/2} \) to \( H^{1/2} \), i.e.,
\[
\| \langle D \rangle \langle x \rangle^{-1/2} R(z) \langle x \rangle^{-1} \langle D \rangle^{1/2} f \|_{L^2} \leq C \| f \|_{L^2}
\]
Then by Kato smoothing we obtain also the second estimate in (2.29).

The proof for the case \( n = 2 \) is completely analogous.

\[
\square
\]

2.2. The wave and Klein-Gordon generators. We consider now the operator \( \sqrt{-\Delta + W} \), where as usual
\[
W = W(x,D) = a \cdot \nabla + b, \quad b = b_1 + b_2
\]
which generates the flow \( e^{it\sqrt{-\Delta + W}} \) of the perturbed wave equation. The free operator \( |D| := \sqrt{-\Delta} \) is self-adjoint and nonnegative on \( L^2 \), and can be handled as follows. If we denote its resolvent by \( R_{|D|}(z) = (|D| - z)^{-1} \), we have
\[
(2.33) \quad R_{|D|}(z) = (|D| + z)R_0(z^2).
\]
This simple identity allows us to estimate \( R_{|D|} \) using some standard techniques from harmonic analysis. We need a lemma:
Lemma 2.6. Let \( n \geq 2 \). For any \( \sigma > 1 \), the weight \( w_\sigma = |x|(1 + |\log |x||)^\sigma \) is an \( A_2 \) weight, i.e., there exist a constant \( A \) such that, for any ball \( B = B(x_0, R) \),

\[
A(x_0, R) \equiv \left[ \frac{1}{|B|} \int_B w_\sigma dx \right] \cdot \left[ \frac{1}{|B|} \int_B w_\sigma^{-1} dx \right] \leq A < \infty.
\]

Obviously, we have also \( w_\sigma^{-1} \in A_2 \). The same property holds for the weights \( \tau_\epsilon \), \( \tau_\epsilon^{-1} \) defined in (2.11).

Proof. The bound for the function \( A(x_0, R) \) is trivial if \( R \leq |x_0|/2 \), indeed it is sufficient to write

\[
A(x_0, R) \leq C \max_B w_\sigma \cdot \max_B w_\sigma^{-1} \leq C'
\]
since the ball \( B \) is at a distance greater than \( |x_0|/2 \) from the origin.

If, on the other hand, \( R \geq |x_0|/2 \), it is easy to check that \( A(x_0, R) \) is bounded by a constant (depending only on the space dimension \( n \)) times \( A(0, 3R) \). Thus we are reduced to the case of balls \( B(0, R) \) centered in 0.

For small \( R \leq 10 \) the function \( A(0, R) \) is bounded. Indeed, Hôpital’s theorem gives

\[
\lim_{\epsilon \to 0} \int_0^\epsilon \frac{r^{n-2} dr}{(1 + |\log r|)^\sigma} \cdot \frac{(1 + |\log \epsilon|)^\sigma}{\epsilon^{n-1}} = \frac{1}{n-1}
\]

which implies for small \( R \)

\[
\int_0^R \frac{r^{n-2} dr}{(1 + |\log r|)^\sigma} \sim \frac{R^{n-1}}{(1 + |\log R|)^\sigma}
\]

and similarly

\[
\int_0^R r^n (1 + |\log r|)^\sigma dr \sim R^{n+1}(1 + |\log R|)^\sigma
\]

whence we get \( A(0, R) \leq C \).

For large \( R > 10 \) we rescale and obtain

\[
A(0, R) = \int_0^{\sqrt[2]{R}} \frac{\tau^{n-2} d\tau}{(1 + |\log R + \log \tau|)^\sigma} \cdot \int_0^{\sqrt[2]{R}} \tau^n (1 + |\log R + \log \tau|)^\sigma \, d\tau
\]

The second integral is clearly bounded by \( C(\log R)^\sigma \). The first integral can be split into

\[
\int_0^{\sqrt[2]{R}} \frac{\tau^{n-2} d\tau}{(1 + |\log R + \log \tau|)^\sigma} \leq \int_0^{\sqrt[2]{R}} \frac{\tau^{n-2} d\tau}{(1 + |\log \tau|)^\sigma} \sim \frac{R^{-\frac{n-1}{2}}}{(1 + \frac{1}{2} \log R)^\sigma} \leq R^{-\frac{n-1}{2}}
\]

where we used again (2.35), and

\[
\int_0^1 \frac{\tau^{n-2} d\tau}{(1 + |\log R + \log \tau|)^\sigma} \leq \int_0^1 \frac{\tau^{n-2} d\tau}{(1 + \frac{1}{2} \log R)^\sigma} \leq C(\log R)^{-\sigma}.
\]

Putting everything together, we obtain the required bound also for large \( R \), and this concludes the proof of the Lemma.

The proof for \( \tau_\epsilon \) is much simpler. We reduce as above to the case of spheres \( B(0, R) \) centered in the origin. For \( R \leq 1 \) we can use the equivalence \( \tau_\epsilon \simeq |x|^{1/2-\epsilon} \) and the bound follows from the well-known fact that \( |x|^{1/2-\epsilon} \) is an \( A_2 \) weight. For \( R > 1 \) we use the estimate

\[
A(0, R) \lesssim \frac{1}{|B|} \int_B (1 + |x|) dx \cdot \frac{1}{|B|} \int_B \frac{dx}{|x|}
\]

(replace \( |x| \) with \( |x|^{1+\epsilon} \) for \( n = 2 \)) whence the bound follows easily.
Knowing that $w^{-1}_\sigma \in A_2$, we see that the Riesz operators
\[ R_j = i^{-1} \frac{\partial_j}{|D|} \]
are bounded on the space $L^2(w^{-1}_\sigma \, dx)$ by standard results (see e.g. the Corollary to Theorem 2.4 of [34]). Writing $|D| = i \sum R_j \partial_j$, we have
\[ \| w^{-1/2}_\sigma |D| g \|_{L^2} \leq \sum_j \| w^{-1/2}_\sigma R_j \partial_j g \|_{L^2} \leq C \| w^{-1/2}_\sigma \nabla g \|_{L^2}. \]
Thus estimate (2.38) implies
\[ \| w^{-\frac{1}{2}} R_{|D|}(z) f \|_{L^2} \leq C \| w^{-\frac{1}{2}} \nabla R_0(z) f \|_{L^2} + C|z| \cdot \| w^{-\frac{1}{2}} R_0(z) f \|_{L^2}. \]
Then, inequalities (2.15) and (2.10) yield immediately the following estimate for the free resolvent: for any fixed $\sigma > 1$,
\[ \| w^{-\frac{1}{2}} R_{|D|}(z) f \|_{L^2} \leq C \| w^{\frac{1}{2}} f \|_{L^2}, \]
uniformly in $z \in \mathbb{C}$.

We are ready to prove a corresponding estimate for the resolvent of the perturbed operator
\[ R(z) = (\sqrt{-\Delta + W} - z)^{-1}, \quad W = a(x) \cdot \nabla + b(x), \]
following the same approach as in the preceding cases.

**Lemma 2.7.** Consider the operator $-\Delta + W(x, D) \equiv -\Delta + a(x) \cdot \nabla + b_1(x) + b_2(x)$ under the assumptions: the operator is selfadjoint, $b_2$ is real valued and nonnegative, and for some $\delta, \epsilon > 0$ small enough, $\sigma > 1$,
\[ \| \tau_{\epsilon} w^{\frac{1}{2}} a \|_{L^\infty} + \| \tau^{2}_{\epsilon} b_1 \|_{L^\infty} < \delta, \quad \| \tau_{\epsilon}^2 b_2 \|_{L^\infty} < \infty. \]
Moreover assume that $0$ is not a resonance for $-\Delta + b_2(x)$. Then the resolvent operator $R(z) = (\sqrt{-\Delta + W} - z)^{-1}$ satisfies
\[ \| \tau_{\epsilon}^{-1} R(z) f \|_{L^2} \leq C \| \tau_{\epsilon} f \|_{L^2}. \]
As a consequence, the perturbed wave flow $e^{it\sqrt{-\Delta + W}}$ satisfies the smoothing estimate
\[ \| \tau_{\epsilon}^{-1} e^{it\sqrt{-\Delta + W}} f \|_{L^2} \leq C \| f \|_{L^2}. \]

**Proof.** We write for brevity
\[ |D_W| = \sqrt{-\Delta + W(x, D)}. \]
By the (Phragmén-Lindelöf) maximum principle, it is sufficient to prove estimate (2.39) for real $z = \lambda$. We notice that by the same arguments used in the proof of Lemma 2.2, we have
\[ \| |D_W| g \|_{L^2} \simeq \| g \|_{H^1}; \]
thus for $\lambda \leq 0$ we can write
\[ \| (|D_W| - \lambda) g \|_{L^2}^2 = \| |D_W| g \|_{L^2}^2 + \lambda^2 \| g \|_{L^2}^2 - 2\lambda (|D_W| g, g)_{L^2} \gtrsim \| g \|_{H^1}, \]
by the nonnegativity of $|D_W|$. This implies for all $\lambda \leq 0$
\[ \| R(\lambda) g \|_{H^1} \lesssim \| g \|_{L^2}, \]
whence by duality we have also
\[ \| R(\lambda) g \|_{L^2} \lesssim \| g \|_{H^{-1}}, \]
and interpolating we obtain
\[ \| R(\lambda) g \|_{H^{1/2}} \lesssim \| g \|_{H^{-1/2}}, \quad \lambda \leq 0. \]
Now, using the Hardy’s inequalities
\[ \| |x|^{-1/2} f \|_{L^2} \lesssim \| f \|_{H^{1/2}} \quad \text{or equivalently} \quad \| f \|_{H^{-1/2}} \lesssim \| |x|^{1/2} f \|_{L^2} \]

we obtain the estimate
\[(2.41) \quad \|x\|^{-1/2} R(\lambda) g \|_{L^2} \lesssim \|x\|^{1/2} g \|_{L^2}, \quad \lambda \leq 0 \]
which implies (2.39) for \( z = -\lambda \leq 0 \) (and is actually stronger).

Consider now \( R(\lambda), \lambda \geq 0 \); we use the identity
\[ R(\lambda) = (|D_W| - \lambda)^{-1} = 2\lambda R_W(\lambda^2) + (|D_W| + \lambda)^{-1} \]
where \( R_W(\lambda) = (-\Delta + W - \lambda)^{-1} \). The second term at the right hand side has already been estimated, while the first one can be estimated using (2.29), and this concludes the proof of (2.39). The last inequality (2.40) is an application of Kato’s theorem as usual.

We conclude this section with a study of the operator \( \sqrt{-\Delta + 1 + W} \) associated with the perturbed Klein-Gordon flow \( e^{it\sqrt{-\Delta + 1 + W}} \). In the free case \( W = 0 \) the operator reduces to \( (D) = (1 - \Delta)^{1/2} \) and its resolvent
\[ R_{(D)}(z) = ((D) - z)^{-1} \]
can be handled in a similar way as \( R_{(D)} \).

We start from estimates (2.16) and (2.19) which imply
\[ \langle z \rangle^{1/2} \| \tau_{\epsilon}^{-1} R_0(z) \|_{L^2} + \| w_{\sigma}^{-1/2} \nabla R_0(z) \|_{L^2} \lesssim \| \tau_{\epsilon} f \|_{L^2}. \]
As above, using the fact that \( w_{\sigma}^{-1} \) is an \( A_2 \) weight, we can replace \( \nabla \) with \( |D| \) in the left hand side and hence (recalling that \( w_{\sigma} \lesssim \tau_{\epsilon} \)) we arrive at
\[ \langle z \rangle^{1/2} \| \tau_{\epsilon}^{-1} R_0(z) \|_{L^2} + \| w_{\sigma}^{-1/2} (D) R_0(z) \|_{L^2} \lesssim \| \tau_{\epsilon} f \|_{L^2} \]
which implies
\[(2.42) \quad \langle z \rangle^{1/2} \| \tau_{\epsilon}^{-1} R_0(z) \|_{L^2} + \| \tau_{\epsilon}^{-1} (D) R_0(z) \|_{L^2} \lesssim \| \tau_{\epsilon} f \|_{L^2}. \]
Then using the identity
\[ R_{(D)}(z) = ((D) + z) \cdot R_0(1 - z^2) \]
we obtain from (2.42) the estimate
\[(2.43) \quad \| \tau_{\epsilon}^{-1} R_{(D)}(z) f \|_{L^2} \lesssim \| \tau_{\epsilon} f \|_{L^2}. \]
For the perturbed operator we have:

**Lemma 2.8.** Consider the operator \( -\Delta + W(x, D) \equiv -\Delta + a(x) \cdot \nabla + b_1(x) + b_2(x) \) under the assumptions: the operator is selfadjoint, \( b_2 \) is real valued and nonnegative, and for some \( \delta, \epsilon > 0 \) small enough, \( \sigma > 1 \),
\[(2.44) \quad \| \tau_{\epsilon} w_{\sigma}^{1/2} a \|_{L^\infty} + \| \tau_{\epsilon}^2 b_1 \|_{L^\infty} < \delta, \quad \| \tau_{\epsilon}^2 b_2 \|_{L^\infty} < \infty. \]
Moreover assume that \( 0 \) is not a resonance for \( -\Delta + b_2(x) \). Then the resolvent operator \( R(z) = (\sqrt{1 - \Delta + W} - z)^{-1} \) satisfies
\[(2.45) \quad \| \tau_{\epsilon}^{-1} R(z) f \|_{L^2} \leq C \| \tau_{\epsilon} f \|_{L^2}. \]
As a consequence, the perturbed Klein-Gordon flow \( e^{it\sqrt{-\Delta + 1 + W}} \) satisfies the smoothing estimate
\[(2.46) \quad \| \tau_{\epsilon}^{-1} e^{it\sqrt{-\Delta + 1 + W}} f \|_{L^2 L^2} \leq C \| f \|_{L^2}. \]

**Proof.** Writing
\[ |D_W| = \sqrt{-\Delta + W(x, D)}, \quad \langle D_W \rangle = \sqrt{1 - \Delta + W(x, D)} \]
we notice that
\[ \| (D_W) f \|_{L^2} \simeq \| f \|_{L^2} + \| |D_W| f \|_{L^2} \simeq \| f \|_{H^1}. \]
by the same arguments used in the proof of Lemma 2.2 and the identity
\[ \| (D_W) f \|_{L^2} = (\langle -\Delta + W \rangle f, f). \]
Proceeding as in the proof of Lemma 2.7, we arrive at
\[ \| R(\lambda) g \|_{H^{1/2}} \lesssim \| g \|_{H^{-1/2}}, \quad \lambda \leq 0 \]
for the resolvent \( R = (\langle D_W \rangle - z)^{-1} \), and by Hardy inequality as before we obtain half of (2.40).

For positive \( \lambda \) we write
\[ R(\lambda) = (\langle D_W \rangle - \lambda)^{-1} = 2\lambda R_W(\lambda^2 - 1) + (\langle D_W \rangle + \lambda)^{-1} \]
where \( R_W(z) = (-\Delta + W - z)^{-1} \), and by (2.20) and the first part of the proof we obtain (2.45). Kato’s theorem gives (2.46) as usual.

2.3. The magnetic Dirac operators. We now consider the resolvent of a perturbed Dirac operator \( D + V(x) \). The proofs here will be short since we shall rely on a few results proved in [12]; in particular, we recall that if \( V = V^* \) has a sufficiently small \( L^3, \infty \) norm, hence under the assumptions of Theorem 1.5, the operator \( D + V \) is self-adjoint on \( L^2(\mathbb{R}^3, \mathbb{C}^4) \), with form domain \( H^1(\mathbb{R}^3, \mathbb{C}^4) \) and spectrum \( \mathbb{R} \). The same holds for the operator with nonzero mass \( D + \beta + V \), but the spectrum is \( \mathbb{R} \setminus [-1, 1] \).

Let us consider the massless case first. We shall use the notations
\[ R_D(z) = (-D - zI_4)^{-1}, \quad R(z) = (-D + V - zI_4)^{-1} \]
where \( I_4 \) denotes the identity \( 4 \times 4 \) matrix. The following result is contained in Proposition 3.6 of [12], apart from the smoothing estimate which is a standard consequence of Kato’s theorem as above:

**Proposition 2.9.** Assume that the \( 4 \times 4 \) matrix \( V(x) = V^*(x) \) satisfies
\[ \| w_\sigma V \|_{L^\infty} < \delta, \]
for some \( \delta \) sufficiently small and some \( \sigma > 0 \). Then \( D + V \) satisfies the limiting absorption principle, i.e., the limit operators \( R(\lambda \pm i0) \) exist in the topology of bounded operators from \( L^2(w_\sigma^{1/2} dx) \) to \( H^1(w_\sigma^{1/2} dx) \). Moreover the resolvent operator \( R = (\langle D + V - zI_4 \rangle)^{-1} \) satisfies the estimate
\[ \| w_\sigma^{-1/2} R(z) f \|_{L^2} \leq C \| w_\sigma^{1/2} f \|_{L^2}, \quad z \in \mathbb{C}. \]
As a consequence, the Dirac flow satisfies the smoothing estimate
\[ \| w_\sigma^{-1/2} e^{t(D + V)} f \|_{L^2} \lesssim C \| f \|_{L^2}. \]

We consider now the operators with mass \( D + \beta \) and \( D + \beta + V \). We shall use the notations
\[ R_\beta(z) = (\langle D + \beta - zI_4 \rangle)^{-1}, \quad R(z) = (\langle D + \beta + V - zI_4 \rangle)^{-1}. \]
From the identities
\[ D^2 = -\Delta I_4, \quad (D + \beta)^2 = (1 - \Delta) I_4, \]
we obtain the following representations in terms of \( R_0(z) = (-\Delta - z)^{-1} \)
\[ R_D(z) = R_0(z^2) (D + zI_4), \quad R_\beta(z) = R_0(z^2 - 1) (D + \beta + zI_4); \]
and hence we can write
\[ R_\beta(z) = R_0(z^2 - 1) D + R_0(z^2 - 1) (\beta + zI_4). \]
Then a straightforward application of estimate (2.42) gives
\[ \| \tau^{-1}_\epsilon R_\beta(z) f \|_{L^2} \leq C \| \tau f \|_{L^2}. \]
uniformly in \( z \in \mathbb{C} \).
In the perturbed case we can prove

**Proposition 2.10.** Assume that the $4 \times 4$ matrix $V(x) = V^*(x)$ satisfies

$$
\|\tau_2^p V\|_{L^\infty} < \delta,
$$

for some $\delta$ sufficiently small and $\epsilon > 0$. Then the perturbed resolvent operator $R(z) = (D + \beta + V - z I_4)^{-1}$ satisfies

$$
\|\tau_e^{-1} R(z) f\|_{L^2} \leq C \|\tau_e f\|_{L^2}.
$$

As a consequence, the flow $e^{it(D+\beta+V)}$ satisfies the smoothing estimate

$$
\|\tau_e^{-1} e^{it(D+\beta+V)} f\|_{L^2 L^2} \leq C \|f\|_{L^2}.
$$

**Proof.** The operator $VR_\beta(z)$ is bounded on $L^2(\tau_2^p dx)$ with norm bounded by $C\delta$ since

$$
\|\tau VR_\beta(z)\|_{L^2} \leq \|\tau_2^p V\|_{L^\infty} \|\tau_e^{-1} R_\beta(z)\|_{L^2} \leq C\delta \|\tau_e f\|_{L^2}
$$

by (2.53) and (2.52). Thus for $\delta$ small a Neumann expansion shows that $(I + VR_\beta(z))^{-1}$ is well defined and uniformly bounded on $L^2(\tau_2^p dx)$. Hence the usual representation

$$
R(z) = R_\beta(z)(I + VR_\beta(z))^{-1}
$$

together with (2.52) gives (2.53), and (2.55) follows. \qed

3. Proof of the Strichartz Estimates

The method we shall follow is inspired by [29], [6] and consists in mixing Strichartz and smoothing estimates for the free operator with smoothing estimates for the perturbed operator. The main tool will be the well-known Christ-Kiselev lemma [8], which can be stated as follows: given two Banach spaces $X,Y$ perturbed operator. The main tool will be the well-known Christ-Kiselev lemma [8], which can be stated as follows: given two Banach spaces $X,Y$ and a bounded integral operator $T f = \int_k K(t,s) f(s)ds$ from $L^p(\mathbb{R}, X)$ to $L^q(\mathbb{R}, Y)$, then its truncated version $S f = \int_k^t K(t,s) f(s)ds$ is also bounded on the same spaces, provided $p < \tilde{p}$ (the Hilbert transform being a trivial counterexample for $p = \tilde{p}$). Thus to prove an estimate of the form

$$
\left\| \int_0^t e^{i(t-s)A} F(s)ds \right\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^\tilde{p} L_x^\tilde{q}}
$$

it is sufficient to prove the untruncated estimate

$$
\left\| \int_\mathbb{R} e^{i(t-s)A} F(s)ds \right\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^\tilde{p} L_x^\tilde{q}}
$$

but only if $p < \tilde{p}$, which in particular excludes endpoint-endpoint estimates where $p = \tilde{p} = 2$.

3.1. Schrödinger equation: proof of Theorem 1.2. Notice that $u(t,x) = e^{it(\Delta + W)} f$ satisfies the equation $iu_t - \Delta u = -Wu$, hence we can write

$$
e^{it(\Delta - W)} f = e^{it\Delta} f - \int_0^t e^{i(t-s)\Delta} W(x,D)u ds = I - II - III
$$

with

$$
I = e^{it\Delta} f, \quad II = \int_0^t e^{i(t-s)\Delta} b(x) u ds, \quad III = \int_0^t e^{i(t-s)\Delta} a(x) \cdot \nabla u ds.
$$

The first term $I$ can be estimated directly with standard Strichartz estimates:

$$
\|e^{it\Delta} f\|_{L_t^p L_x^q} \lesssim C \|f\|_{L^2}
$$

for any admissible couple \((p, q)\). In order to estimate the second term we resort to the Christ-Kiselev lemma and we are reduced to estimate the untruncated integral
\[
II_1 = e^{it\Delta} \int e^{-is\Delta} b(x)u \, ds.
\]
To this end we apply first the Strichartz estimates for the free group, then the dual of the smoothing estimate from Proposition 2.5 in the special case \(W = 0\), i.e.,
\[
\left\| \int e^{-is\Delta} F(s) ds \right\|_{L^2} \lesssim \|\tau F\|_{L^2L^2}
\]
obtaining
\[
\|II_1\|_{L^pL^q} \lesssim \left\| e^{-is\Delta} b u \right\|_{L^2} \lesssim \|\tau b u\|_{L^2L^2} \leq \|\tau b\|_{L^\infty} \|\tau u\|_{L^2L^2}.
\]
Then by assumption (2.25) and again the smoothing estimate (2.29) we conclude (3.2)
\[
\|II\|_{L^pL^q} \lesssim \|f\|_{L^2}
\]
for any non-endpoint admissible couple \((p, q)\).

The last term \(III\) is more delicate. We reduce it as above to the untruncated form
\[
III_1 = e^{it\Delta} \int e^{-is\Delta} a \cdot \nabla u \, ds
\]
and we apply to it the free Strichartz estimate and then the following dual smoothing estimate:
(3.3)
\[
\left\| \int e^{-is\Delta} F(s) ds \right\|_{L^2} \lesssim \|D\|^{-1/2} \|F\|_{L^2L^2},
\]
valid for any function \(\chi(x) \gtrsim w_\sigma(x)^{1/2}\). Estimate (3.3) is proved as follows: from (2.16) we deduce, using the fact that \(w_\sigma\) is an \(A_2\) weight, the equivalent property
\[
\|w_\sigma^{-1/2} |D|^{1/2} R_0(z) f \|_{L^2} \leq C \|w_\sigma^{1/2} |D|^{-1/2} f \|_{L^2}
\]
which implies, via Kato smoothing,
\[
\|w_\sigma^{-1/2} |D|^{1/2} e^{it\Delta} f \|_{L^2L^2} \leq \|f\|_{L^2}.
\]
Since \(\chi \gtrsim w_\sigma^{1/2}\) this gives also
\[
\|\chi^{-1} |D|^{1/2} e^{it\Delta} f \|_{L^2L^2} \leq \|f\|_{L^2}
\]
and by duality we get (3.3). Thus we arrive at
(3.4)
\[
\|III_1\|_{L^pL^q} \lesssim \|D\|^{-1/2} \chi a(x) \cdot \nabla u\|_{L^2L^2}
\]
Now assume we can prove the inequality
(3.5)
\[
\|\chi^{-1} |D|^{1/2} a(x) \cdot \nabla g\|_{L^2} \lesssim \|\tau^{-1} \nabla |D|^{-1/2} g\|_{L^2};
\]
then from (3.4) and the smoothing estimate (2.30) we finally obtain
(3.6)
\[
\|III_1\|_{L^pL^q} \lesssim \|\tau^{-1} \nabla |D|^{-1/2} u\|_{L^2} \lesssim \|f\|_{L^2}
\]
which, together with (3.1) and (3.2), concludes the proof of the Theorem.

It remains to check inequality (3.5). We rewrite it in the equivalent form
\[
\|D\|^{-1/2} \chi a(x) |D|^{1/2} \tau h\|_{L^2} \lesssim \|h\|_{L^2},
\]
i.e., we need to prove that the operator
(3.7)
\[
T = |D|^{-1/2} \chi a(x) |D|^{1/2} \tau_e
\]
is bounded on \(L^2\). We shall use the following lemma, where we shall make use of several properties of Lorentz spaces \(L^{p,q}\) (see [26]).
Lemma 3.1. Let $\alpha(x), \beta(x)$ be measurable functions on $\mathbb{R}^n$ such that for some $0 < \delta < 1/2$, some $\rho \in (0, n/2 - \delta]$, and a radial function $\gamma(|x|)$, with $\gamma(s)$ decreasing, we have

(i) $|\alpha(x) - \alpha(y)| \lesssim |x - y|^{1/2 + \delta} (\gamma(|x|) + \gamma(|y|))$ and $\gamma \in L^{\frac{n}{2(1+\delta)}}$

(ii) $\alpha \beta \in L^\infty$, $|x|^{-\rho} \beta(x) \in L^\infty$ and $|x|^{\rho} \gamma(|x|) \in L^{\frac{2n}{n+\rho-\delta}}$

Then the operator $T = |D|^{-1/2} \alpha(x)|D|^{1/2} \beta(x)$ is bounded on $L^2$.

The same result holds in the range $\rho \in [0, n/2 + \delta]$ if we replace (i) with

(i') $|\alpha(x) - \alpha(y)| \lesssim (x - y)^{-2\delta} |x - y|^{1/2 + \delta} (\gamma(|x|) + \gamma(|y|))$ and $\gamma \in L^{\frac{2n}{2(1+\delta)}}$.

Proof. Since $\alpha \beta$ is bounded, we can equivalently prove that the modified operator

$\tilde{T} = T - \alpha \beta = |D|^{-1/2} \cdot |D|^{1/2} \cdot \beta$

is bounded on $L^2$. Moreover, by the Sobolev embedding in Lorentz spaces (proved e.g. by real interpolation)

$$\| |D|^{-1/2} y \|_{L^2} \lesssim \| g \|_{L^{\frac{2n}{n+2}}}$$

it is sufficient to prove that the following reduced operator $S$ satisfies

$$S = [\alpha, |D|^{1/2}] \cdot \beta : L^{\frac{2n}{n+2}} \rightarrow L^2.$$

Now we observe that the commutator $[\alpha, |D|^{1/2}]$ admits an explicit representation of the form

$$[\alpha, |D|^{1/2}]f = c(n) \int_{\mathbb{R}^n} \frac{\alpha(x) - \alpha(y)}{|x - y|^{n+1/2}} f(y)dy$$

for a constant $c(n)$ depending only on the space dimension. Indeed, by standard Fourier transform techniques we see that

$$[\alpha, |D|^2]f = c(n) \int_{\mathbb{R}^n} \frac{\alpha(x) - \alpha(y)}{|x - y|^{n+2}} f(y)dy$$

and this formula is valid for $\Re z < 0$ under quite general assumptions on $\alpha$; moreover our assumptions show that the right hand side is a well defined and analytic function of $z$ for $\Re z < 1/2 + \delta$ (as proved below), hence by analytic continuation the representation is valid also in this larger region and in particular for $z = 1/2$.

In order to estimate $S$ we split it as $S = S_1 + S_2$ with

$$S_1 f = c \int_{|y| \geq 2|x|} \frac{\alpha(x) - \alpha(y)}{|x - y|^{n+1/2}} \beta(y) f(y)dy$$

$$S_2 f = c \int_{|y| \leq 2|x|} \frac{\alpha(x) - \alpha(y)}{|x - y|^{n+1/2}} \beta(y) f(y)dy$$

In the region $|y| \geq 2|x|$ we deduce by assumption (i) that

$$|\alpha(x) - \alpha(y)| \lesssim 2|x - y|^{1/2 + \delta} \gamma(|x|)$$

since $\gamma$ is decreasing; moreover we have $|x - y| \simeq |y|$, hence

$$\left| \frac{\alpha(x) - \alpha(y)}{|x - y|^{n+1/2}} \beta(y) f(y) \right| \lesssim \gamma(|x|) \frac{\beta(y)}{|y|^{\rho}} \frac{|f(y)|}{|x - y|^{n-\rho - \delta}} \lesssim \gamma(|x|) \frac{|f(y)|}{|x - y|^{n-\rho - \delta}}$$

using (ii). Thus, by Hölder inequality in Lorentz spaces, we get

$$\|S_1 f\|_{L^{\frac{2n}{n+2}}} \lesssim \|\gamma\|_{L^{\frac{2n}{2(n+1/2)}}} \left\| \int \frac{|f(y)|}{|x - y|^{n-\rho - \delta}} dy \right\|_{L^{\frac{2n}{n-\rho - \delta}}},$$

(provided $\rho < n/2 - \delta$) and by (i) and Young inequality we arrive at

$$\|S_1 f\|_{L^{\frac{2n}{n+2}}} \lesssim \|y|^{-1/2 + \delta}\|_{L^{\frac{n}{n+2}}} \|f\|_{L^{2}}$$

which concludes the estimate of the first piece $S_1$. 

In the region \(|y| \leq 2|x|\), on the other hand, we can write

\[
\left| \frac{\alpha(x) - \alpha(y)}{|x - y|^{n+1/2}} \beta(y) f(y) \right| \lesssim \frac{\beta(y) \rho |\gamma|(|y|/2) f(y)}{|x - y|^{n-\delta}} \lesssim \frac{|y|^\rho |\gamma|(|y|/2) f(y)}{|x - y|^{n-\delta}}
\]

so that by Young inequality

\[
\|S_2 f\|_{L^{2^*\infty}} \lesssim \left\| \frac{|\beta(y)| |\gamma|(|y|/2) f(y)}{|x - y|^{n-\delta}} \right\|_{L^{2^*\infty}} \lesssim \|y|^\delta \|y^\rho f\|_{L^{\frac{2n}{n-\rho-3\delta}}}^2
\]

and by Hölder inequality we get

\[
\|S_2 f\|_{L^{2^*\infty}} \lesssim \|y|^\rho |\gamma| \|f\|_{L^{\frac{2n}{n-\rho-3\delta}}}
\]

and this concludes the proof under assumptions (i)-(ii).

The case of assumptions (i')-(ii) is almost identical. No change is necessary in the estimate of \(S_2 f\), while for \(S_1 f\) it is sufficient to write

\[
\|S_1 f\|_{L^{2^*\infty}} \lesssim \|y|^\rho \gamma \|f\|_{L^{\frac{2n}{n-\rho-3\delta}}}
\]

which is true if \(\rho < n/2 + \delta\), and then proceed as above. \(\Box\)

Notice that if we restrict to the special choice \(\beta = |x|^\rho, \gamma(x) = \langle x \rangle^{-\lambda}, \alpha(x) = \chi(x) a(x)\), the following conditions imply that (i), (ii), (i') are all satisfied:

(3.8) \[0 < \delta < \frac{1}{2}, \quad 0 \leq \rho < \frac{n}{2} + \delta, \quad \lambda \geq \frac{1}{2} + \rho + \delta\]

and

(3.9) \[\langle x \rangle^\lambda \chi(x) a(x) \in C^{1/2+\delta}\]

(recall that \(\|f\|_{C^\alpha} = \|f\|_{L^\infty} + \sup_{x \neq y} |x - y|^{-\alpha} |f(x) - f(y)|\)). All conditions in (i), (ii), (i') are trivial to check apart from Hölder continuity; actually we shall now see that the following stronger inequality holds:

(3.10) \[|\alpha(x) - \alpha(y)| \lesssim \min\{1, |x - y|\}^{1/2+\delta} (\langle x \rangle^{-\lambda} + \langle y \rangle^{-\lambda})\]

Indeed, when \(|x - y| \geq 1\) condition \((3.10)\) follows from \(\langle x \rangle^\lambda \chi(x) a(x) \in L^\infty\) which is contained in \((3.9)\). When \(|x - y| \leq 1\), we write

\[|\alpha(x) - \alpha(y)| \leq A + B,\]

where

\[A = \chi(x) a(x) \langle x \rangle^\lambda \langle x \rangle^{-\lambda} - \langle y \rangle^{-\lambda},\]

and

\[B = \langle y \rangle^{-\lambda} \langle x \rangle^\lambda \chi(x) a(x) - \langle y \rangle^\lambda \chi(y) a(y)\].

Then we have directly from \((3.9)\)

\[B \lesssim \langle y \rangle^{-\lambda} |x - y|^{1/2+\delta} \lesssim (\langle x \rangle^{-\lambda} + \langle y \rangle^{-\lambda}) |x - y|^{1/2+\delta}\]

while for \(A\) we use the elementary inequality

\[|\langle x \rangle^{-\lambda} - \langle y \rangle^{-\lambda}| \lesssim \sup_{\xi \in [x,y]} |\nabla \langle z \rangle^{-\lambda}|_{z=\xi} |x - y| \lesssim (\langle x \rangle^{-\lambda} + \langle y \rangle^{-\lambda}) |x - y|^{1/2+\delta}\]

together with the bound \(\langle x \rangle^\lambda \chi(x) a(x) \in L^\infty\).

We can finally apply the lemma to the operator \((3.7)\); since \(\tau_\epsilon = |x|^{1/2-\epsilon} + |x|\) for \(n \geq 3\) and \(\tau_\epsilon = |x|^{1/2-\epsilon} + |x|^{1+\epsilon}\) for \(n = 2\), by the above computation it is sufficient to check conditions \((3.8), (3.9)\) for \(\rho = 1/2 - \epsilon\) and \(\rho = 1\ (\rho = 1/2 - \epsilon\) and \(\rho = 1 + \epsilon\) in dimension 2). We see that the choices \(\delta = 2\epsilon\) and \(\lambda = 1 + 3\epsilon\) work in all cases, thus it is sufficient to assume \(\langle x \rangle^{1+3\epsilon} \chi(x) a(x) \in C^{1/2+2\epsilon}\) i.e. assumption \((1.9)\). The proof is concluded.
3.2. Wave and Klein-Gordon equations: proof of Theorems 1.3, 1.4

Since $u(t,x) = e^{it\sqrt{-\Delta + W}} f$ solves the Cauchy problem
\begin{equation}
\begin{cases}
  u_{tt} - \Delta u = -W u \\
  u(0,x) = f(x) \\
  u_t(0,x) = i \left( \sqrt{-\Delta + W} \right) f(x),
\end{cases}
\end{equation}
we have the alternative representation
\begin{equation}
  e^{it\sqrt{-\Delta + W}} f = \cos(t|D|) f + i \frac{\sin(t|D|)}{|D|} \sqrt{-\Delta + W} f - \int_0^t \frac{\sin((t-s)|D|)}{|D|} Wuds.
\end{equation}

The first two terms satisfy the standard Strichartz estimates for the free wave equation (see [1.2] in the Introduction, and recall also [21]). For the third term we apply as usual the Christ-Kiselev lemma and we are reduced to the untruncated equation (see (1.2) in the Introduction, and recall also (2.5)). For the third term we have the alternative representation
\begin{equation}
  e^{it\sqrt{-\Delta + W}} f = \cos(t|D|) f + i \frac{\sin(t|D|)}{|D|} \sqrt{-\Delta + W} f - \int_0^t \frac{\sin((t-s)|D|)}{|D|} Wuds.
\end{equation}

By duality is equivalent to
\begin{equation}
|\tau^{-1} e^{it|D|} f|_{L^2} \lesssim ||f||_{L^2}
\end{equation}
valid for any wave admissible couple $(p,q)$. Moreover, the smoothing estimate (2.40) holds also in the free case $W \equiv 0$
\begin{equation}
|\tau^{-1} e^{it|D|} f|_{L^2} \lesssim ||f||_{L^2}
\end{equation}
and by duality is equivalent to
\begin{equation}
\left\| \int e^{-is|D|} F(s) ds \right\|_{L^2} \lesssim ||a||_{L^2}.
\end{equation}

Applying (3.13) and (3.14) to $I_1$ we obtain, since the Riesz operators are bounded in all $L^p$ with $1 < p < \infty$,
\begin{equation}
|\tau^{-1} a(x) u|_{L^2} \lesssim ||\tau^{-1} a(x)||_{L^2} ||u||_{L^2}.
\end{equation}

Using again the smoothing estimate (2.40) and assumption (1.11), we conclude
\begin{equation}
|\tau^{-1} a(x) u|_{L^2} \lesssim ||f||_{L^2}.
\end{equation}

Consider now the second term $II$, or more generally
\begin{equation}
II_1 = e^{it|D|} \int |D|^{-1} e^{-is|D|} c(x)uds.
\end{equation}

Proceeding as in [7], we shall use the following estimate from [9] (see also [21])
\begin{equation}
|\tau^{-1} |D|^{-1} e^{it|D|} f|_{L^2} \lesssim ||f||_{H^{-1/2}}
\end{equation}
in the dual form:
\begin{equation}
\int |D|^{-1} e^{-is|D|} F(s) ds \lesssim ||x||_{L^2}.
\end{equation}
Then, applying the Strichartz estimate for the wave equation (3.13) in the form
\[ \| e^{itD} f \|_{L^p H^s_{\frac{1}{p} - \frac{1}{2}}} \lesssim \| f \|_{H^s_{\frac{1}{2}}} \]
followed by (3.16), we obtain
\[ \| II_1 \|_{L^p H^s_{\frac{1}{p} - \frac{1}{2}}} \lesssim \| x \| c(x) u \|_{L^2 L^2} \lesssim \| x \| \tau_r c(x) \|_{L^\infty} \| \tau_r^{-1} u \|_{L^2 L^2}. \]
Recalling assumption (1.11) and the smoothing estimate (2.40) we finally obtain
\[ \| II_1 \|_{L^p H^s_{\frac{1}{p} - \frac{1}{2}}} \lesssim \| f \|_{L^2} \]
which concludes the proof of Theorem 1.3.

The proof of Theorem 1.4 is completely analogous, using the Strichartz estimate for the free equation
\[ \| e^{it(D)} f \|_{L^p H^s_{\frac{1}{p} - \frac{1}{2}}} \lesssim \| f \|_{L^2}, \]
which is valid for all Schrödinger admissible couple \((p, q)\), and the following estimate from [3]:
\[ \| \langle x \rangle^{-1} e^{it(D)} f \|_{L^2 L^2} \lesssim \| f \|_{L^2} \]
which implies by duality
\[ \left\| \int e^{-is(D)} F(s) ds \right\|_{L^2} \lesssim \| \langle x \rangle F \|_{L^2 L^2} \]
and hence also
\[ \left\| \int (D)^{-1} e^{-is(D)} F(s) ds \right\|_{L^{1/2}} \lesssim \| \langle x \rangle F \|_{L^2 L^2}. \]
This estimate replaces (3.16) in the above computation.

3.3. Dirac equation: proof of Theorems 1.5, 1.6. As proved in the Appendix, the Strichartz estimate for the free massless Dirac equation is the following:
\[ \| e^{itD} f \|_{L^p H^s_{\frac{1}{p} - \frac{1}{2}}} \lesssim \| f \|_{L^2} \]
for any wave admissible couple \((p, q)\). On the other hand, as a special case of the smoothing estimate (2.50), we have
\[ \| w_\sigma^{-\frac{1}{2}} e^{itD} f \|_{L^2 L^2} \lesssim \| f \|_{L^2} \]
and by duality we obtain
\[ \left\| \int e^{-isD} F(s) ds \right\|_{L^2} \lesssim \| w_\sigma^{\frac{1}{2}} F \|_{L^2 L^2}. \]

Consider now the perturbed Dirac flow \( u = e^{it(D + V)} f \). An alternative representation of \( u \) is the following:
\[ u(t, x) = e^{itD} f - e^{itD} \int_0^t e^{-isD} V u(s) ds. \]
The term \( e^{itD} f \) satisfies the free Strichartz estimates (A.1), in order to estimate the Duhamel term as usual we apply the Christ-Kiselev lemma and switch to the untruncated integral. Then, using (1.20), (3.19) and Hölder inequality, we have
\[ \| e^{itD} \int e^{-isD} V u ds \|_{L^p H^s_{\frac{1}{p} - \frac{1}{2}}} \lesssim \left\| \int e^{-isD} V u ds \right\|_{L^2} \]
\[ \lesssim \| w_\sigma^{\frac{1}{2}} V u \|_{L^2 L^2} \leq \| w_\sigma V \|_{L^\infty} \cdot \| w_\sigma^{-\frac{1}{2}} u \|_{L^2 L^2}. \]
Recalling the smoothing estimate (2.50) we obtain
\[
\left\| e^{it\mathcal{D}} \int e^{-ix\mathcal{D}} V u ds \right\|_{L^p H^q} \lesssim \| f \|_{L^2}
\]
and this completes the proof of 1.5.

The proof of Theorem 1.6 is completely analogous.

APPENDIX A. STRICHARTZ ESTIMATES FOR THE FREE FLOWS

Strichartz estimates for the free Schrödinger and wave equations are well known, see the Introduction for the precise statements. It is less easy to find in the literature optimal results for Klein-Gordon and Dirac equations. Hence we devote this appendix to a quick proof of the estimates in these cases.

The massless Dirac flow is trivial since it can be reduced to the wave equation:

**Proposition A.1.** Let \( n = 3 \). The following Strichartz estimates hold:
\[
\left\| e^{it\mathcal{D}} f \right\|_{L^p H^q} \lesssim \| f \|_{L^2}
\]
for any wave admissible couple \( (p, q) \).

**Proof.** By the identity
\[
(i\partial_t + \mathcal{D})(i\partial_t - \mathcal{D}) = -\Box,
\]
we obtain that \( u(t, x) = e^{it\mathcal{D}} f \) satisfies the Cauchy problem
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta I_4 u &= 0 \\
u(0, x) &= f(x) \\
u_t(0, x) &= i\mathcal{D} f(x)
\end{aligned}
\]
and hence each component of \( u \) satisfies the same Strichartz estimates as for the 3D wave equation. \(\square\)

The Klein-Gordon and massive Dirac equations need some work. We begin by the free Klein-Gordon flow \( u = e^{it(D)} f \). We shall apply a precise stationary phase result due to Hörmander [20]:

**Lemma A.2.** Assume that \( \phi : \mathbb{R}^n \to \mathbb{R} \) has a Fourier transform \( \hat{\phi} \in C^\infty \) with the decay property
\[
\left| D^\alpha \hat{\phi}(\xi) \right| \leq C_\alpha \langle \xi \rangle^{-\frac{n}{2} - 1 - |\alpha|} \quad \forall \xi \in \mathbb{R}^n, \ \alpha \in \mathbb{N}^n.
\]
Then the following estimate holds: for some \( C > 0 \),
\[
\left| e^{it(D)} \hat{\phi} \right| \leq C(|t| + |x|)^{-\frac{n}{2}}.
\]

Now, using an inhomogeneous dyadic decomposition \( \{ \psi_0, \varphi_j(D) \}_{j \geq 1} \) with the usual properties: \( \psi_0(\xi) \) supported in \( B(0, 1) \), \( \varphi_0(\xi) = \psi_0(\xi/2) - \psi_0(\xi), \varphi_j(\xi) = \varphi_j(2^{-j} \xi), \psi_0 + \sum_{j \geq 1} \varphi_j = 1 \),
we can localize the estimate as follows:

**Lemma A.3.** The flow \( e^{it(D)} f \) satisfies the localized dispersive estimate
\[
\left| e^{it(D)} \varphi_j(D) f \right| \leq C|t|^{-\frac{n}{2} - 2j(\frac{n}{2} + 1)} \left\| \varphi_j(D) f \right\|_{L^1},
\]
for each \( t \in \mathbb{R}, x \in \mathbb{R}^n, j \geq 0 \) and some \( C > 0 \); here \( \varphi_j \) denotes \( \varphi_{j-1} + \varphi_j + \varphi_{j+1} \), with \( \varphi_{-1} = 0 \).
Proof. We can write
\[ e^{it(D)} \varphi_j(D)f = e^{it(D)} \langle -\xi^{-\frac{2}{n}} - \xi^{-\frac{2}{n}}(\varphi_j(D)f) = e^{it(D)} F^{-1} \left( \langle -\xi^{-\frac{2}{n}} \right) * (\varphi_j(D)f) \right), \]
where \( F^{-1} \) denotes the inverse Fourier transform. Then, applying Lemma \( \text{A.2} \) with \( \phi = F^{-1} \left( \langle -\xi^{-\frac{2}{n}} \right) \), we obtain
\[(A.6) \quad \left| e^{it(D)} \varphi_j(D)f \right| \leq C |t|^{-\frac{n}{2}} \| \langle D \rangle^{\frac{2}{n}+1} \varphi_j(D)f \|_{L^1}. \]

Since
\[(A.7) \quad \| \langle D \rangle^{\frac{2}{n}+1} \varphi_j(D)f \|_{L^1} \leq \| F^{-1} \left( \langle -\xi^{-\frac{2}{n}} \right) * (\varphi_j(D)f) \|_{L^1}. \]

Young inequality gives
\[(A.8) \quad \| F^{-1} \left( \langle -\xi^{-\frac{2}{n}} \right) * (\varphi_j(D)f) \|_{L^1} \leq C 2^{j(\frac{2}{n}+1)}. \]

Using the scaling operators \( S_\lambda \phi(x) = \phi(\lambda x) \), we can write
\[ F^{-1} \left( \langle -\xi^{-\frac{2}{n}} \right) = F^{-1} \left( \langle -\xi^{-\frac{2}{n}} \right) S_{-\xi^{-\frac{2}{n}}} \varphi_0(\xi) \right) \]
and hence
\[ \| F^{-1} \left( \langle -\xi^{-\frac{2}{n}} \right) \|_{L^1} \leq 2^{j(\frac{2}{n}+1)} \| F^{-1} \left( (2^{-\frac{j}{2}} + |\xi|^2)^{\frac{n+1}{2}} \varphi_0(\xi) \right) \|_{L^1}. \]

Moreover, multiplying and dividing by \( \langle x \rangle^{2m} \) for some integer \( m \), we obtain
\[(A.9) \quad \| F^{-1} \left( (2^{-\frac{j}{2}} + |\xi|^2)^{\frac{n+1}{2}} \varphi_0(\xi) \right) \|_{L^1} \leq C \| \langle x \rangle^{2m} F^{-1} \left( (2^{-\frac{j}{2}} + |\xi|^2)^{\frac{n+1}{2}} \varphi_0(\xi) \right) \|_{L^2} \]

provided
\[(A.10) \quad m > \frac{n}{4}. \]

We shall choose \( m \) as the smallest integer satisfying \( \text{A.10} \). We are interested in the growth with respect to \( j \) of the quantity
\[ I := (1 - \Delta)^m \left( (2^{-\frac{j}{2}} + |\xi|^2)^{\frac{n+1}{2}} \varphi_0(\xi) \right) \]

When \( n \) is even, \( (2^{-\frac{j}{2}} + |\xi|^2)^{\frac{n+1}{2}} \) is a polynomial, and hence we obtain
\[ \| I \|_{L^2} \leq C \| \varphi_0 \|_{L^2} \]
with \( C \) independent of \( j \). When \( n \) is odd, it is clear that almost all the terms in the expansion of \( I \) are uniformly bounded in \( j \), apart from the (possibly) worst one
\[ II = \Delta^m \left( (2^{-\frac{j}{2}} + |\xi|^2)^{\frac{n+1}{2}} \varphi_0(\xi) \right). \]

We have the two possibilities
\[ n = 4k + 3 \quad \text{or} \quad n = 4k + 1, \]
with \( m = k + 1 \). If \( n = 4k + 3 \), we have
\[ |II| \sim \left| D^{2k+2} \left( (2^{-\frac{j}{2}} + |\xi|^2)^{2k+\frac{2}{2}} \right) \right| \]
which expands in a sum of bounded terms. If \( n = 4k + 1 \), we have
\[ |II| \sim \left| D^{2k+2} \left( (2^{-\frac{j}{2}} + |\xi|^2)^{2k+\frac{2}{2}} \right) \right| \lesssim (2^{-\frac{j}{2}} + |\xi|^2)^{-1/2} |\xi|^{2k+2} + \text{ bounded terms}, \]
and also in this case we have a uniform bound in \( j \). In conclusion, we have proved that
\[ \| (1 - I)^m \left( (2^{-\frac{j}{2}} + |\xi|^2)^{\frac{n+1}{2}} \varphi_0(\xi) \right) \|_{L^2} \leq C, \]
for some $C > 0$, which implies (A.8), and the proof is complete. □

Remark A.1. By interpolation between estimate (A.5) and the localized $L^2$ conservation
(A.11) $\|e^{it(D)}\varphi_j(D)f\|_{L^2} \leq \|\varphi_j(D)f\|_{L^2},$
we obtain the following $L^q - L^{q'}$ decay estimates:
(A.12) $\|e^{it(D)}\varphi_j(D)f\|_{L^q} \leq C |t|^{-\frac{n}{2} + \frac{n}{2}q + \frac{2n}{2}q'(\frac{n}{2}+1)(1-\frac{n}{2})}\|\varphi_j(D)f\|_{L^{q'}}$
for any $q \geq 2$ with $1/q + 1/q' = 1$.

Starting from estimates (A.12) and using the standard techniques of [17], [25],
in particular the abstract Theorem 10.1 of [25], we obtain the full set of estimates including the endpoint case:

Theorem A.4. The Klein-Gordon flow $u = e^{it(D)}f$ satisfies the Strichartz estimates
(A.13) $\|e^{it(D)}f\|_{L^p H^{\frac{1}{2} - \frac{1}{2p}}_n} \lesssim \|f\|_{L^2}$
for any Schrödinger admissible couple $(p, q)$.

Finally, the Dirac equation with mass can be handled in a similar way to Proposition A.1.

Proposition A.5. Let $n = 3$. The following Strichartz estimates hold:
(A.14) $\|e^{it(D)+\beta}f\|_{L^p H^{\frac{1}{2} - \frac{1}{2p}}_3} \lesssim \|f\|_{L^2},$
for any Schrödinger admissible couple $(p, q)$.

Proof. As in the proof of Proposition A.1 by the identity

$$(i\partial_t + (D + \beta))(i\partial_t - (D + \beta)) = (-\Box - 1)I_4$$

we obtain that each component of $u$ solves a Klein-Gordon equation with initial data $f$ and $(D + \beta)f$. Thus estimate (A.14) follows immediately from the Strichartz estimates for the Klein-Gordon equation in space dimension $n = 3$. □

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